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ABSTRACT

A two parameter family of general relativistic shock wave solutions to the Einstein field equations with a perfect fluid source are constructed. The solutions are obtained by matching a self-similar perturbation of a Friedmann-Lemaître-Robertson-Walker spacetime to a self-similar static spacetime across a spherical shock surface. These shock wave solutions model a general relativistic explosion within a static isothermal sphere and extend the one parameter family of general relativistic shock waves constructed by Smoller and Temple. Such an extension partially resolves a long standing problem posed by Cahill and Taub by determining a subset of the self-similar spacetimes that may be matched to a self-similar static spacetime to form a general relativistic shock wave. The original problem is posed for a pure radiation equation of state, however the shock waves that are constructed resolve the problem for general barotropic equations of state either side of the shock. These shock waves are stable in the Lax sense and a formal existence proof is provided in the pure radiation case, as self-similar perturbations of FLRW spacetimes are not known explicitly. These spacetimes are of particular interest as they have an accelerated expansion similar to the accelerated expansion found in the Standard Model of Cosmology, but solve the Einstein field equations in the absence of a cosmological constant. It is conjectured by Temple that a vast primordial shock wave, with a perturbed FLRW interior, could provide the mechanism for the accelerated expansion observed today without the need for dark energy.

Keywords  General Relativity · Shock Wave · Cosmology · Dark Energy

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1 Introduction

The analysis of spherically symmetric self-similar perfect fluid spacetimes originated with [Cahill and Taub [1971]]. In this analysis, solutions of the Einstein field equations with a perfect fluid source are assumed to be spherically symmetric and self-similar in the variable:

$$\xi = \frac{r}{t}$$

These two assumptions mean that the Einstein field equations, a system of nonlinear partial differential equations, reduce to a system of nonlinear ordinary differential equations in the single variable $\xi$. It is in this setting that [Cahill and Taub [1971]] establish criteria for the uniqueness of solutions to these equations along with a method to form strong shock waves. The flat Friedmann-Lemaître-Robertson-Walker (FLRW) and Tolman-Oppenheimer-Volkoff (TOV) spacetimes are both solutions to the spherically symmetric self-similar equations when admitting barotropic equations of state. The former spacetime is central to the Standard Model of Cosmology and the latter is a static model for the interior of a star. In fact, the TOV spacetimes are the unique family of static spherically symmetric self-similar spacetimes, and play a
central role in the construction of certain families of general relativistic shock waves. The first Friedmann-static shock wave was constructed by Cahill and Taub [1971] by matching a pure radiation FLRW spacetime to a TOV spacetime across a spherical shock surface. Cahill and Taub [1971] claimed the existence of a two parameter family of self-similar pure radiation spacetimes that could be matched to the TOV spacetime to form a shock wave in a subsequent paper that was not published and possibly never completed. The construction of this two parameter family of shock waves remains an open problem.

Friedmann-static shock waves are considered again by Smoller and Temple [1994], where a number of theorems concerning the regularity of spherically symmetric shock waves are proved. Smoller and Temple [1995] generalise the previous Friedmann-static shock wave to a one parameter family of Friedmann-static shock waves and introduce a criteria for determining the Lax stability of these shock waves, that is, stability in the gas dynamical sense. In addition, Smoller and Temple [2012] derive a two parameter family of exact self-similar perturbations of the FLRW spacetime, opening up the possibility of forming new Friedmann-static shock waves from these spacetimes.

Unbeknown to Smoller and Temple, Carr and Yahil [1990] knew the asymptotic form of these self-similar perturbations and referred to them as asymptotically Friedmann spacetimes. The complete classification of spherically symmetric self-similar solutions to the Einstein field equations with a perfect fluid source was completed by Carr and Coley [2000]. In addition to determining the number of free parameters present in each family of solutions, Carr and Coley [2000] provides a detailed discussion of the cosmological relevance of each of these families.

In particular, all Friedmann-static shock waves model a general relativistic explosion within a static isothermal sphere. This isothermal sphere could model a star or possibly the early universe, with the explosion then analogous to a supernova or Big Bang respectively. In either case, Friedmann-static shock waves offer the simplest two state model that incorporates conservation of mass-energy and momentum across the shock surface. The two parameter family of asymptotically Friedmann spacetimes derived by Smoller and Temple [2012] are of particular interest as they have an accelerated expansion similar to the accelerated expansion found in the Standard Model of Cosmology, but solve the Einstein field equations in the absence of a cosmological constant. Temple conjectures that a Friedmann-static shock wave, constructed by matching an asymptotically Friedmann spacetime to a TOV spacetime, is a possible candidate for a cosmological model with an accelerated expansion but without a cosmological constant, and thus, without dark energy.

The objective is the construction of these Friedmann-static shock waves, which in turn are a two parameter generalisation of the one parameter family of Friedmann-static shock waves constructed by Smoller and Temple [2012]. The extra parameter corresponds to the magnitude of acceleration, with the original two parameters corresponding to the equations of state either side of the shock. For an interior pure radiation equation of state, these shock waves form a one parameter subset of the two parameter family of shock waves sought by Cahill and Taub [1971] in the aforementioned open problem, thus partially resolving it.

There is a fair amount of preliminary theory that is introduced prior to the construction of the two parameter family of Friedmann-static shock waves and this is the topic of Section 2. The ODE derived by Cahill and Taub [1971] and used by Carr et al. are derived using self-similar comoving coordinates, whereas the ODE derived by Smoller and Temple [2012] are derived using self-similar Schwarzschild coordinates. There are advantages to both approaches, but the latter approach is more useful in the construction of shock wave solutions, so this approach is used. Once Smoller and Temple’s system of ODE is introduced, the TOV, FLRW, and asymptotically FLRW solutions are derived in self-similar Schwarzschild coordinates. Proceeding these derivations, the shock wave construction process is introduced, followed by important regularity and stability results.

Section 3 begins with an alternative derivation of the explicit one parameter family of Friedmann-static shock waves originally derived by Smoller and Temple [1995]. This warm-up derivation introduces a lemma that is central to the construction of the more general two parameter family of Friedmann-static shock waves. This explicit construction is then followed by an additional lemma that generalises the Lax characteristic conditions to a broad family of general relativistic shock waves. Since the asymptotically Friedmann spacetimes are not known explicitly, numerical approximations are used to construct the two parameter family of Friedmann-static shock waves. The acceleration parameter and shock position are then approximated in the pure radiation case. Following this construction, an analysis of the Rankine-Hugoniot jump conditions results in a theorem that establishes the Lax stability of a broad family of general relativistic shock waves.

Section 4 introduces a lemma that broadly bounds the full two parameter family of asymptotically Friedmann spacetimes and then provides a formal proof of the existence of a Friedmann-static pure radiation shock wave. This is the unique Friedmann-static shock wave that models a perfect fluid with a pure radiation equation of state both sides of the shock.

Finally, Section 5 introduces a conjecture regarding the formal existence of the full two parameter family of Friedmann-static shock waves and briefly discusses the cosmological applications of these shock waves.
2 Preliminaries

2.1 Spherically Symmetric Self-Similar Field Equations

Consider first the Einstein field equations:

\[ G = \frac{8\pi G}{c^4} T \]  

(1)

where \( G \) is the Einstein curvature tensor, \( T \) is the stress-energy-momentum tensor, \( c \) is the speed of light and \( G \) is the gravitational constant. When modelling a perfect fluid, the stress-energy-momentum tensor takes the form:

\[ T = \left( \rho + \frac{p}{c^2} \right) u \otimes u + pg \]  

(2)

where \( g \) is the metric tensor, \( \rho \) is the fluid density, \( p \) is the fluid pressure and \( u \) is the fluid four-velocity. From now onwards, the constants \( c \) and \( G \) are set to unity and solutions of the Einstein field equations with a perfect fluid source are referred to simply as solutions. In addition, all solutions are assumed to be spherically symmetric and self-similar of the first kind, that is, the self-similar variable takes the form:

\[ \xi = \frac{r}{t} \]

where \( t \) and \( r \) are the temporal and radial coordinates respectively. It is shown by Cahill and Taub [1971] that under these two assumptions, any metric tensor may be written, without loss of generality, in the following self-similar Schwarzschild coordinate form:

\[ ds^2 = -B(\xi)dt^2 + \frac{1}{A(\xi)} dr^2 + r^2 d\Omega^2 \]  

(3)

where \( A, B > 0 \) and \( d\Omega^2 \) denotes the standard metric on the unit two-sphere, that is:

\[ d\Omega^2 = d\theta^2 + \sin^2(\theta)d\phi^2 \]

Under the assumption of spherical symmetry the fluid four-velocity may also be written without loss of generality as:

\[ u = (u^0, u^1, 0, 0) \]

and under the normalisation condition:

\[ g(u, u) = -1 \]

the fluid four-velocity has only one independent component. The normalisation condition means that the fluid four-velocity can be fully specified through a single variable.

Definition 2.1. The \textit{Schwarzschild coordinate velocity} is defined by:

\[ v = \frac{1}{\sqrt{AB}} \frac{u^1}{u^0} \]

Together with \( A, B, \rho \) and \( p \), the Schwarzschild coordinate velocity \( v \) is one of five unknown variables that completely specify a spherically symmetric self-similar solution. As there are only four independent components of the Einstein field equations for our metric ansatz, an equation of state is required to close the system. In this light we assume that all solutions have a barotropic equation of state, that is, one of the form:

\[ p = p(\rho) \]

Under this assumption, and in addition to the assumptions of spherical symmetry and self-similarity, it is demonstrated by Cahill and Taub [1971] that all barotropic equations of state take the more restricted form:

\[ p = \sigma \rho \]  

(4)

for some constant \( \sigma \). Physically, \( \sigma \) represents the square root of the sound speed in the fluid, so for a strictly positive pressure and subluminal sound speed we require:

\[ 0 < \sigma < 1 \]

Definition 2.2. The special case \( \sigma = \frac{1}{3} \) corresponds to the extreme relativistic limit of free particles and the state of matter known as \textit{pure radiation}. 

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A perfect fluid stress-energy-momentum tensor with a pure radiation equation of state models the matter in the Radiation Dominated Epoch of the universe. This equation of state also yields a trace free stress-energy-momentum tensor. Now the self-similar assumption reduces the complexity of the Einstein field equations, which without such an assumption form a system of nonlinear PDE when written in Schwarzschild coordinates. By substituting (3) into (2) Smoller and Temple [2012] demonstrate that the Einstein field equations reduce to the following system of nonlinear ODE:

\[
\xi \frac{dA}{d\xi} = -\frac{(3 + 3\sigma)(1 - A)v}{\{\cdot\}_S} \quad (5)
\]

\[
\xi \frac{dG}{d\xi} = -G \left[ \left( \frac{1 - A}{A} \right) \frac{(3 + 3\sigma)(1 + v^2)G - 2v}{2\{\cdot\}_S} - 1 \right] \quad (6)
\]

\[
\xi \frac{dv}{d\xi} = \frac{1 - v^2}{2\{\cdot\}_D} \left[ 3\sigma\{\cdot\}_S + \left( \frac{1 - A}{A} \right) \frac{(3 + 3\sigma)^2\{\cdot\}_N}{4\{\cdot\}_S} \right] \quad (7)
\]

in addition to the constraint:

\[
\rho = \frac{3(1 - v^2)(1 - A)G}{8\pi r^2\{\cdot\}_S} \quad (8)
\]

where \( G \), not to be confused with the Einstein tensor, is defined by:

\[
G = \frac{\xi}{\sqrt{AB}} \quad (9)
\]

and:

\[
\{\cdot\}_S = 3(G - v) - 3\sigma v(1 - Gv)
\]

\[
\{\cdot\}_N = -3(G - v)^2 + 3\sigma v^2(1 - Gv)^2
\]

\[
\{\cdot\}_D = \frac{3}{4}(3 + 3\sigma) [(G - v)^2 - \sigma(1 - Gv)^2]
\]

Note that \( \sigma = \frac{1}{3} \) is used in the derivation provided by Smoller and Temple [2012]. It is not difficult to modify this derivation to yield equations (5)-(8) for general \( \sigma \), although this derivation will not be given here. Note also that under the change of variable:

\[
\xi = e^s \quad (10)
\]

the equations become explicitly autonomous, since:

\[
\xi \frac{d}{d\xi} = \frac{d}{ds} \quad (11)
\]

The autonomous nature of these equations distinguish them from the self-similar ODE derived by Cahill and Taub [1971], which are derived in self-similar comoving coordinates. It is worth noting that in self-similar comoving coordinates the variable \( G \), denoted by \( V \) in Carr and Coley [2000], represents surfaces of constant \( \xi \) relative to the fluid.

### 2.2 Tolman-Oppenheimer-Volkoff Spacetimes

The Tolman-Oppenheimer-Volkoff (TOV) spacetimes are the family of static spherically symmetric spacetimes. It was first demonstrated by Cahill and Taub [1971] that the self-similar subset of TOV spacetimes which solve the Einstein field equations with a perfect fluid source and barotropic equation of state form the unique family of spherically symmetric self-similar static spacetimes. In the context of equations (5)-(8) these spacetimes are distinguished by having a Schwarzschild coordinate velocity that is identically zero.

**Proposition 2.3.** Spherically symmetric self-similar perfect fluid spacetimes are static if and only if \( v \equiv 0 \).

**Proof.** Since it is known that the family of static spherically symmetric self-similar perfect fluid spacetimes are unique, it is sufficient to demonstrate that solutions with zero Schwarzschild coordinate velocity are static. In this light, substituting \( v \equiv 0 \) into equation (5) implies that \( A \equiv A_0 \) for some constant \( A_0 \). Furthermore, substituting \( v \equiv 0 \) into equation (7) requires:

\[
9\sigma - \frac{1}{4}(3 + 3\sigma)^2 \left( \frac{1 - A_0}{A_0} \right) = 0
\]
to ensure the derivative of the Schwarzschild coordinate velocity remains zero. This means $A_0$ can be given as a function of $\sigma$ as so:

$$A_0(\sigma) = 1 - 2M(\sigma)$$

where:

$$M(\sigma) = \frac{2\sigma}{1 + 6\sigma + \sigma^2}$$

Now substituting both $v \equiv 0$ and $A \equiv A_0$ into equation (6) and solving for $G$ yields:

$$G(\xi) = C_1 \xi^{1-\sigma}$$

for some positive constant $C_1$. Putting these results together yields the following metric in self-similar Schwarzschild coordinates:

$$ds^2 = -C_2 \xi^{\frac{4\sigma}{1+\sigma}} dt^2 + \frac{1}{1 - 2M(\sigma)} dr^2 + r^2 d\Omega^2$$

for some positive constant $C_2$. The density is given by:

$$\rho = \frac{M(\sigma)}{4\pi r^2}$$

Note that because $v \equiv 0$, this coordinate frame is also comoving with the fluid. Finally, making the temporal transformation:

$$\tilde{t} = \frac{1 + \sigma}{1 - \sigma} t^{1-\sigma}$$

$$\tilde{r} = r$$

puts the metric in the following explicitly static form:

$$d\tilde{s}^2 = -C_2 \tilde{r}^{\frac{4\sigma}{1+\sigma}} d\tilde{t}^2 + \frac{1}{1 - 2M(\sigma)} d\tilde{r}^2 + \tilde{r}^2 d\Omega^2$$

noting that the density also remains static.

**Definition 2.4.** A scale transformation is a transformation of the form:

$$\tilde{t} = \mathcal{T}_0 t$$

$$\tilde{r} = \mathcal{R}_0 r$$

where $\mathcal{T}_0$ and $\mathcal{R}_0$ are constants. A parameter that appears in a solution is essential if it cannot be removed by a scale transformation and inessential if it can.

Essential and inessential parameters are discussed in more detail by Cahill and Taub [1971]. In counting the number of essential parameters in a solution, Cahill and Taub [1971] and Carr and Coley [2000] first fix a value for $\sigma$ and count the remaining essential parameters, however this is not be the convention adopted here.

**Proposition 2.5.** The one parameter family of TOV spacetimes, denoted by $\text{TOV}(\sigma)$, are given in self-similar comoving Schwarzschild coordinates as:

$$ds^2 = -\alpha^2 \xi^{\frac{4\sigma}{1+\sigma}} dt^2 + \frac{1}{1 - 2M(\sigma)} dr^2 + r^2 d\Omega^2$$

$$\rho = \frac{M(\sigma)}{4\pi r^2}$$

$$p = \sigma \rho$$

where $\alpha$ is an inessential parameter and:

$$M(\sigma) = \frac{2\sigma}{1 + 6\sigma + \sigma^2}$$

Proof. The proof follows from Proposition 2.3. □
2.3 Friedmann-Lemaître-Robertson-Walker Spacetimes

The flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes are the family of spatially homogenous spherically symmetric spacetimes. Following Carr and Yahil [1990], the self-similar subset of FLRW spacetimes which solve the Einstein field equations with a perfect fluid source and barotropic equation of state take the following form in self-similar comoving coordinates:

\[ ds^2 = -e^{2\varphi} dt^2 + e^{2\psi} dr^2 + \mathcal{R}^2 d\Omega^2 \]

\[ \rho = \hat{\xi}^2 \]

\[ p = \sigma \rho \]

where:

\[ e^{2\varphi} = \beta^2 \]

\[ e^{2\psi} = \gamma^{-2} \hat{\xi}^{-\frac{4}{3+3\sigma}} \]

\[ \mathcal{R}^2 = \hat{\xi}^{-\frac{2}{3+3\sigma}} \]

and:

\[ \beta = \frac{\sqrt{6}}{3+3\sigma} \]

\[ \gamma = \frac{3+3\sigma}{1+3\sigma} \]

Note that the density is independent of \( r \) and that the metric can also be put into an explicitly spatially homogenous form through the purely radial transformation:

\[ \hat{t} = \hat{t} \]

\[ \hat{r} = r \frac{1+3\sigma}{3+3\sigma} \]

to yield:

\[ ds^2 = -\beta^2 d\hat{t}^2 + \hat{t} \frac{1}{3+3\sigma} \left( d\hat{r}^2 + \hat{r}^2 d\Omega^2 \right) \]

noting that the density also remains spatially homogenous.

**Proposition 2.6.** The one parameter family of self-similar perfect fluid FLRW spacetimes with barotropic equations of state, denoted by FLRW\((0, \sigma, 1)\), are given in self-similar Schwarzschild coordinates as:

\[ ds^2 = -\delta^{-2} \left[ 1 + \frac{1}{3} (1 + 3\sigma) \xi^{\frac{2+6\sigma}{3+3\sigma}} \right]^{-\frac{1+3\sigma}{3+3\sigma}} \left[ 1 - \frac{2}{3} \xi^{\frac{2+6\sigma}{3+3\sigma}} \right]^{-1} dt^2 + \left[ 1 - \frac{2}{3} \xi^{\frac{2+6\sigma}{3+3\sigma}} \right]^{-1} dr^2 + r^2 d\Omega^2 \]

\[ v = \frac{2}{\sqrt{6}} \xi^{\frac{1+3\sigma}{3+3\sigma}} \]

\[ \rho = \frac{3v^2}{8\pi r^2} \]

\[ p = \sigma \rho \]

where \( \delta \) is an inessential parameter and:

\[ \xi = \frac{1}{\sqrt{6}} \delta^{-1} (3 + 3\sigma) \xi^{\frac{1+3\sigma}{3+3\sigma}} \left[ 1 + \frac{1}{3} (1 + 3\sigma) \xi^{\frac{2+6\sigma}{3+3\sigma}} \right]^{-\frac{1+3\sigma}{3+3\sigma}} \quad (12) \]

The zero in FLRW\((0, \sigma, 1)\) corresponds to the flat subset of FLRW spacetimes, that is, those with \( k = 0 \) in reduced circumference Schwarzschild coordinates. The one corresponds to the lack of perturbation, which is defined in the next subsection.

**Proof.** To change FLRW\((0, \sigma, 1)\) from self-similar comoving coordinates to self-similar Schwarzschild coordinates, the following coordinate transformation employed by Cahill and Taub [1971] is used:

\[ dt = e^{-\mu} \left( e^{\varphi} \cosh \omega \, d\hat{t} + e^{\psi} \sinh \omega \, d\hat{r} \right) \quad (13) \]

\[ dr = e^{-\nu} \left( e^{\varphi} \sinh \omega \, d\hat{t} + e^{\psi} \cosh \omega \, d\hat{r} \right) \quad (14) \]
where:
\[
\tanh \omega = e^{\psi - \phi} \frac{\partial \mathcal{R}}{\partial (\mathcal{R}T)} \quad (15)
\]
\[
e^{-2\nu} = e^{-2\psi} [\partial_{t}(\mathcal{R}T)]^{2} - e^{-2\phi} [\partial_{t}(\mathcal{R}T)]^{2} \quad (16)
\]
and \( \mu \) is such that \( dt \) is a perfect differential. Relations [(15) and (16)] come from setting \( r = \mathcal{R}T \). The resulting self-similar Schwarzschild form of the metric is then given by:
\[
ds^{2} = -e^{2\nu}dt^{2} + e^{2\nu}dr^{2} + r^{2}d\Omega^{2}
\]
To begin, the \( \tanh \omega \) term is computed as so:
\[
\tanh \omega = e^{\psi - \phi} \frac{\partial \mathcal{R}}{\partial (\mathcal{R}T)}
\]
\[
= \beta^{-1} \gamma^{-1} \xi^{-\frac{2}{3+3\sigma}} - \xi \partial_{t} \frac{\mathcal{R}}{\mathcal{R}T} + \gamma \partial_{t} \mathcal{R}
\]
\[
= \beta^{-1} \gamma^{-1} \xi^{-\frac{2}{3+3\sigma}} - \xi \left( \frac{2}{3+3\sigma} \right) \xi^{-\frac{2}{3+3\sigma}-1} \psi
\]
\[
= 2(1 + 3\sigma)^{-1} \beta^{-1} \gamma^{-1} \xi^{-\frac{2}{3+3\sigma}}
\]
and this yields:
\[
cosh \omega = (1 - \tanh^{2} \omega)^{-\frac{1}{2}} = \left[ 1 - 4(1 + 3\sigma)^{-2} \beta^{-2} \gamma^{-2} \xi^{\frac{2+6\sigma}{3+3\sigma}} \right]^{-\frac{1}{2}}
\]
\[
sinh \omega = \tanh \omega (1 - \tanh^{2} \omega)^{-\frac{1}{2}} = 2(1 + 3\sigma)^{-1} \beta^{-1} \gamma^{-1} \xi^{-\frac{1+3\sigma}{3+3\sigma}} \left[ 1 - 4(1 + 3\sigma)^{-2} \beta^{-2} \gamma^{-2} \xi^{\frac{2+6\sigma}{3+3\sigma}} \right]^{-\frac{1}{2}}
\]
The \( e^{-2\nu} \) term is computed similarly:
\[
e^{-2\nu} = e^{-2\psi} (\partial_{t}(\mathcal{R}T))^{2} - e^{-2\phi} (\partial_{t}(\mathcal{R}T))^{2}
\]
\[
= \eta^{2} \xi^{\frac{2+6\sigma}{3+3\sigma}} \left[ 2 \xi^{-\frac{2}{3+3\sigma} - 1} \psi - \beta^{-2} \left( \frac{2}{3+3\sigma} \right) \xi^{-\frac{2}{3+3\sigma}-1} \psi \right]^{2}
\]
\[
= (1 + 3\sigma)^{2} (3 + 3\sigma)^{-2} \xi^{-2} \left[ 1 - 4(1 + 3\sigma)^{-2} \beta^{-2} \gamma^{-2} \xi^{\frac{2+6\sigma}{3+3\sigma}} \right]^{-\frac{1}{2}}
\]
and this results in:
\[
dt = \beta e^{-\mu} \left[ 1 - 4(1 + 3\sigma)^{-2} \beta^{-2} \gamma^{-2} \xi^{\frac{2+6\sigma}{3+3\sigma}} \right]^{-\frac{1}{2}} \left[ dt + 2(1 + 3\sigma)^{-1} \beta^{-2} \gamma^{-2} \xi^{-1} \xi^{\frac{2+6\sigma}{3+3\sigma}} \frac{d\mathcal{R}}{dt} \right]
\]
\[
\frac{d\mathcal{R}}{dt} = 2(3 + 3\sigma)^{-1} \xi^{\frac{2+6\sigma}{3+3\sigma}} \left[ dt + \frac{1}{2} (1 + 3\sigma) \xi^{-1} \frac{d\mathcal{R}}{dt} \right]
\]
Now given that \( \mu \) is such that the right hand side of [(13)] is a perfect differential, it must be the case that:
\[
\frac{\partial}{\partial \mathcal{R}} e^{-\eta} = \frac{\partial}{\partial t} \left[ 2(1 + 3\sigma)^{-1} \beta^{-2} \gamma^{-2} \xi^{-1} \xi^{\frac{2+6\sigma}{3+3\sigma}} e^{-\eta} \right]
\]
where:
\[
e^{-\eta} = e^{-\mu} \left[ 1 - 4(1 + 3\sigma)^{-2} \beta^{-2} \gamma^{-2} \xi^{\frac{2+6\sigma}{3+3\sigma}} \right]^{-\frac{1}{2}}
\]
The equation for \( \eta \) can be solved as so:
\[
\frac{\partial}{\partial \mathcal{R}} e^{-\eta} = \frac{\partial}{\partial t} \left[ 2(1 + 3\sigma)^{-1} \beta^{-2} \gamma^{-2} \xi^{-1} \xi^{\frac{2+6\sigma}{3+3\sigma}} e^{-\eta} \right]
\]
\[
\overset{\times}{\frac{d}{dt} \frac{d}{d\mathcal{R}} e^{-\eta}} = \frac{1}{\mathcal{R}} \frac{d}{d\xi} \left[ 2(1 + 3\sigma)^{-1} \beta^{-2} \gamma^{-2} \xi^{-1} \xi^{\frac{2+6\sigma}{3+3\sigma}} e^{-\eta} \right]
\]
\[
\overset{\times}{\eta' = 2(1 + 3\sigma)^{-1} \beta^{-2} \gamma^{-2} \xi^{-1} \xi^{\frac{2+6\sigma}{3+3\sigma}} \frac{d}{d\mathcal{R}} e^{-\eta} + (3\sigma - 1)(3 + 3\sigma)^{-1} \xi^{-\frac{2}{3+3\sigma}} e^{-\eta} \right]
\]
\[
\overset{\times}{\eta' = 2(1 + 3\sigma)^{-1} (3\sigma - 1)(3 + 3\sigma)^{-1} \beta^{-2} \gamma^{-2} \xi^{-1} \xi^{\frac{2+6\sigma}{3+3\sigma}} \left[ 1 + 2(1 + 3\sigma)^{-1} \beta^{-2} \gamma^{-2} \xi^{\frac{2+6\sigma}{3+3\sigma}} \right]^{-1}}
\]
\[
\overset{\times}{\eta = (3\sigma - 1)(2 + 6\sigma)^{-1} \log \left[ 1 + 2(1 + 3\sigma)^{-1} \beta^{-2} \gamma^{-2} \xi^{\frac{2+6\sigma}{3+3\sigma}} + C_{3} \right]}
\]
where $C_3$ is a constant. This then yields:

$$e^{-\eta} = \delta \left[ 1 + 2(1 + 3\sigma)^{-1}\beta^{-2}\gamma^{-2}\xi^{2+6\sigma}\xi^{\frac{1-3\sigma}{2+6\sigma}} \right]$$

for some positive constant $\delta$, thus:

$$dt = \delta \beta \left[ 1 + 2(1 + 3\sigma)^{-1}\beta^{-2}\gamma^{-2}\xi^{2+6\sigma}\xi^{\frac{1-3\sigma}{2+6\sigma}} \right]^{\frac{1-3\sigma}{2+6\sigma}} \left[ dt + 2(1 + 3\sigma)^{-1}\beta^{-2}\gamma^{-2}\xi^{-1}\xi^{2+6\sigma}\delta \right]$$

Because the transformation is taking the metric from one self-similar form to another, let:

$$t = \mathcal{T}(\xi)\hat{t}$$

so that:

$$\frac{\partial t}{\partial \hat{t}} = \mathcal{T}(\xi) - \xi \frac{\partial \mathcal{T}}{\partial \xi} = \delta \beta \left[ 1 + 2(1 + 3\sigma)^{-1}\beta^{-2}\gamma^{-2}\xi^{2+6\sigma}\xi^{\frac{1-3\sigma}{2+6\sigma}} \right]^{\frac{1-3\sigma}{2+6\sigma}}$$

$$\frac{\partial t}{\partial r} = \frac{\partial \mathcal{T}}{\partial \xi} = 2\delta(1 + 3\sigma)^{-1}\beta^{-1}\gamma^{-1}\xi^{-1}\xi^{2+6\sigma}\left[ 1 + 2(1 + 3\sigma)^{-1}\beta^{-2}\gamma^{-2}\xi^{2+6\sigma}\xi^{\frac{1-3\sigma}{2+6\sigma}} \right]^{\frac{1-3\sigma}{2+6\sigma}}$$

Solving these equations yields the same function for $\mathcal{T}(\xi)$ only when the integration constant is zero, thus:

$$\mathcal{T}(\xi) = \delta \beta \left[ 1 + 2(1 + 3\sigma)^{-1}\beta^{-2}\gamma^{-2}\xi^{2+6\sigma}\xi^{\frac{1-3\sigma}{2+6\sigma}} \right]^{\frac{1-3\sigma}{2+6\sigma}}$$

Now the fluid four-velocity $u$ is given in self-similar comoving coordinates as:

$$u = (\hat{u}^0, \hat{u}^1, \hat{u}^2, \hat{u}^3) = (e^{-\nu}, 0, 0, 0) = (\beta^{-1}, 0, 0, 0)$$

and in self-similar Schwarzschild coordinates as:

$$u = (u^0, u^1, u^2, u^3) = \left( \hat{u}^0 \frac{\partial}{\partial \hat{t}}, \hat{u}^0 \frac{\partial}{\partial \hat{t}}, 0, 0 \right) = \left( \beta^{-1} \frac{\partial}{\partial \hat{t}}, \beta^{-1} \frac{\partial}{\partial \hat{t}}, 0, 0 \right)$$

$$= \left( \delta \left[ 1 + 2(1 + 3\sigma)^{-1}\beta^{-2}\gamma^{-2}\xi^{2+6\sigma}\xi^{\frac{1-3\sigma}{2+6\sigma}} \right]^{\frac{1-3\sigma}{2+6\sigma}}, 2(3 + 3\sigma)^{-1}\beta^{-1}\gamma^{-1}\xi^{2+6\sigma}\xi^{\frac{1-3\sigma}{2+6\sigma}}, 0, 0 \right)$$

Therefore, by Definition 2.1

$$v = e^{\nu \hat{u}^1} u^1 = 2(3 + 3\sigma)^{-1}\beta^{-1}\gamma^{-1}\xi^{2+6\sigma}\xi^{\frac{1-3\sigma}{2+6\sigma}}$$

Finally, by substituting in $\beta$ and $\gamma$ and noting that:

$$\xi = \frac{r}{\hat{t}} = \frac{\mathcal{R}(\xi)^\hat{t}}{\mathcal{T}(\xi)^\hat{t}} = \frac{\mathcal{R}(\xi)^\hat{r}}{\mathcal{T}(\xi)^\hat{r}}$$

the rest follows.

**Proposition 2.7.** FLRW($0, \sigma, 1$) is given implicitly by:

$$A = 1 - u^2$$

$$G = \frac{1}{2} (3 + 3\sigma) v \left( 1 + \frac{1}{2} (1 + 3\sigma) v^2 \right)^{-1}$$

$$v = \frac{2}{\sqrt{6}} \xi^\frac{1+3\sigma}{2+6\sigma}$$

**Proof.** First note that (19) is immediately obtained from Proposition 2.6. Then by definition:

$$A = e^{-2\nu} = 1 - \frac{2}{3} \xi^\frac{1+3\sigma}{2+6\sigma}$$

$$G = \xi v^{\nu - \mu} = \frac{1}{\sqrt{6}} (3 + 3\sigma) \xi^\frac{1+3\sigma}{2+6\sigma} \left[ 1 + \frac{1}{3} (1 + 3\sigma) \xi^\frac{2+6\sigma}{2+6\sigma} \right]^{-1}$$

which yields (17) and (18).
To check that (17)-(19) satisfies equations (5)-(7), it is recommended to first show:

$$\xi \frac{d}{d\xi} = \xi \frac{d\hat{\xi}}{d\xi} \frac{d}{d\hat{\xi}} = \frac{(3 + 3\sigma)^2}{2 + 6\sigma} \frac{v}{AG} \xi \frac{d}{d\xi}$$

and secondly show:

$$\xi \frac{dA}{d\xi} = -\frac{2 + 6\sigma}{3 + 3\sigma} v^2$$

$$\xi \frac{dG}{d\xi} = \frac{2 + 6\sigma}{(3 + 3\sigma)^2} \left(1 - \frac{1}{2}(1 + 3\sigma)v^2\right) \frac{G^2}{v}$$

$$\xi \frac{dv}{d\xi} = \frac{1 + 3\sigma}{3 + 3\sigma} v$$

Then by Proposition 2.7 it is not difficult to confirm that (17)-(19) solves equations (5)-(7).

**Corollary 2.8.** FLRW(0, $\frac{1}{3}, 1$) is given in self-similar Schwarzschild coordinates as:

$$ds^2 = -\frac{1}{2} \delta^{-2} \left(1 + \frac{1}{\sqrt{1 - \delta^2 \xi^2}}\right) dt^2 + \frac{1}{2} \left(1 + \frac{1}{\sqrt{1 - \delta^2 \xi^2}}\right) dr^2 + r^2 d\Omega^2$$

$$v = \frac{2}{\sqrt{6}} \hat{\xi}$$

$$\rho = \frac{3v^2}{8\pi r^2}$$

$$p = \sigma \rho$$

where $\delta$ is an inessential parameter and:

$$\hat{\xi} = \frac{3}{2} \left(\frac{2 - \delta^2 \xi^2 - 2\sqrt{1 - \delta^2 \xi^2}}{\delta^2 \xi^2}\right)$$

(20)

**Proof.** From Proposition 2.6 in the case $\sigma = \frac{1}{3}$, relation (12) can be inverted to yield (20). The metric then follows from using (20) and some algebraic manipulation.

**Corollary 2.9.** FLRW(0, $\frac{1}{3}, 1$) is given implicitly by:

$$A = 1 - v^2$$

(21)

$$G = \frac{2v}{1 + v^2}$$

(22)

$$G = \delta \xi$$

(23)

**Proof.** Relations (21) and (22) follow immediately from Proposition 2.7 and relation (23) follows from (9) and the identity $B = \delta^{-2} A^{-1}$ from Corollary 2.8.

### 2.4 Perturbations of Friedmann-Lemaître-Robertson-Walker Spacetimes

Let $(A, G, v)$ denote a solution of equations (5)-(7). Since these equations are autonomous, solutions can be represented by non-intersecting trajectories in $(A, G, v)$-space. The FLRW(0, $\sigma, 1$) spacetimes solve equations (5)-(7) and constraint (8) with the trajectories emanating from the point:

$$(A, G, v) = (1, 0, 0)$$

The nature of equations (5)-(7) suggests that to analyse this point, it is helpful to rewrite these equations as functions of $v, A$ and $H$, where $H$ is defined as the ratio:

$$H = \frac{G}{v}$$
This is completed in Smoller and Temple [2012] for \( \sigma = \frac{1}{3} \), however it is not difficult to reproduce these equations for general \( \sigma \), especially when working from equations (5)-(7), although this will not be done here. Recalling (10) and (11), equations (5)-(8) are given in autonomous form as functions of \( v, A \) and \( H \) as so:

\[
\begin{align*}
\frac{dv}{ds} &= -v \left( 1 - \frac{2v^2}{3(3 + 3\sigma)} \right) \left[ 3\sigma \{ \cdot \}_S + \left( \frac{1 - A}{A} \right) \frac{(3 + 3\sigma)^2 \{ \cdot \}_S}{4\{ \cdot \}_S} \right] \quad (24) \\
\frac{dA}{ds} &= -\left( 3 + 3\sigma \right) (1 - A) \\
\frac{dH}{ds} &= -H \left[ \left( \frac{1 - A}{A} \right) \left( 2\{ \cdot \}_S \right) - 1 \right] - \frac{H dH}{v ds}
\end{align*}
\]

with:

\[
\rho = \frac{3(1 - v^2)(1 - A)H}{8\pi r^2 \{ \cdot \}_S} \quad (27)
\]

and where:

\[
\begin{align*}
\{ \cdot \}_S &= -(3 + 3\sigma) + (3 + 3\sigma v^2)H \\
\{ \cdot \}_N &= -(3 - 3\sigma) + 2(3 - 3\sigma v^2)H - (3 - 3\sigma v^2)H^2 \\
\{ \cdot \}_D &= -\frac{1}{4}(3 + 3\sigma)(3 - 3\sigma v^2) - \frac{3}{2}(3 - 3\sigma v^2)H v^2 + \frac{1}{4}(3 + 3\sigma)(3 - 3\sigma v^2)H^2 v^2
\end{align*}
\]

In variables \( v, A \) and \( H \), the point of interest is given by:

\[
(v, A, H) = \left( 0, 1, \frac{1}{2}(3 + 3\sigma) \right)
\]

and it is not difficult to check that this is a fixed point of the system of equations (24)-(26). Following Smoller and Temple [2012], a linear analysis of this fixed point is achieved by first representing equations (24)-(26) as:

\[
\begin{align*}
\begin{bmatrix} v' \\ A' \\ H' \end{bmatrix} &= \begin{bmatrix} F_1(v, A, H) \\ F_2(v, A, H) \\ F_3(v, A, H) \end{bmatrix}
\end{align*}
\]

and then denoting these equations by:

\[
U' = F(U)
\]

where:

\[
U = (v, A, H)^T \\
F = (F_1(U), F_2(U), F_3(U))^T
\]

Next, the Jacobian of \( F \) at the fixed point is calculated. Note that the Jacobian is calculated by Smoller and Temple [2012], however there is a small error in this calculation, so a brief new derivation will be produced. To begin, denote the fixed point by \( U_0 \) and note that:

\[
\begin{align*}
dF_2(U_0) &= \begin{bmatrix} \frac{\partial F_2}{\partial v} \\ \frac{\partial F_2}{\partial A} \\ \frac{\partial F_2}{\partial H} \end{bmatrix}_{U_0} \\
&= (0, 2, 0)
\end{align*}
\]

Neglecting terms second order in \( v \) and second order in terms that vanish at \( U_0 \) on the right hand side of (24) gives:

\[
\begin{align*}
dF_1(U_0) &= d \left[ -v \left( -\frac{2}{3\sigma(3 + 3\sigma)} \right) [9\sigma H - 3\sigma(3 + 3\sigma)] \right]_{U_0} \\
&= (1, 0, 0)
\end{align*}
\]
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and similarly for (26) gives:

\[ \frac{dF_3(U_0)}{dU_0} = d \left[ H - H \left( \frac{1 - A}{A} \right) \frac{(3 + 3\sigma)(H - 2)}{2[3H - (3 + 3\sigma)]} \right] \]

\[ + \left[ \begin{array}{c} dH - \frac{2}{3\sigma(3 + 3\sigma)} \left[ 9\sigma H - 3\sigma(3 + 3\sigma) + \left( \frac{1 - A}{A} \right) \frac{(3 + 3\sigma)(1 - H)H}{6\sigma} \right] \left( \frac{(3 + 3\sigma)^2}{4[3H - (3 + 3\sigma)]} \right) \right] \]

\[ = d \left[ 3H - \frac{6H^2}{3 + 3\sigma} + \left( \frac{1 - A}{A} \right) \frac{(3 + 3\sigma)(1 - H)H}{6\sigma} \right] \]

\[ \left( 0, 0, -\frac{(1 + 3\sigma)(3 + 3\sigma)^2}{24\sigma} \right) \]

Thus the Jacobian is given by:

\[ \frac{dF(U_0)}{dU_0} = \left( \begin{array}{ccc} \frac{dF_1(U_0)}{dU_0} & \frac{dF_2(U_0)}{dU_0} & \frac{dF_3(U_0)}{dU_0} \end{array} \right) = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & N(\sigma) & -3 \end{array} \right) \]

where:

\[ N(\sigma) = -\frac{(1 + 3\sigma)(3 + 3\sigma)^2}{24\sigma} \]

This means \( U_0 \) is a hyperbolic rest point of the system of equations (24)-(26) with eigenvalues:

\[ \lambda_1 = 1 \]
\[ \lambda_2 = 2 \]
\[ \lambda_3 = -3 \]

Therefore the solutions:

\[ U(s) = U_0 + V(s) \]

where \( V(s) \) solves the linearised equations:

\[ V' = dF(U_0) \cdot V \]

lie in the two-dimensional unstable manifold \( \mathcal{M}_0 \) of \( U_0 \), given by:

\[ \mathcal{M}_0 = \left( \begin{array}{c} 0 \\ \frac{1}{2}(3 + 3\sigma) \end{array} \right) + \text{Span} \left\{ \left( \begin{array}{c} 1 \\ 0 \end{array} \right) e^s + \left( \begin{array}{c} 0 \\ 1 \end{array} \right) e^{2s} \right\} \]

In particular:

\[ U(s) = \left( \begin{array}{c} C_4e^s \\ 1 + C_5e^{2s} \\ \frac{1}{2}(3 + 3\sigma) \end{array} \right) \]

for arbitrary constants \( C_4 \) and \( C_5 \). In the variable \( \xi \), the solutions are given by:

\[ A_1(\xi) = 1 + C_5\xi^2 \]
\[ G_1(\xi) = \frac{1}{2}(3 + 3\sigma)C_4\xi \]
\[ v_1(\xi) = \xi \xi \]

with the subscript denoting the fact that \( (A_1, G_1, v_1) \) represents solutions to the linearised equations (5)-(7). Now for small \( \xi \), functions \( A, G \) and \( v \) of FLRW\((0,0,1)\) are given to leading order as:

\[ A \approx 1 - \frac{4}{(3 + 3\sigma)^2} \delta^2 \xi^2 \]
\[ G \approx \delta \xi \]
\[ v \approx \frac{2}{3 + 3\sigma} \delta \xi \]
Comparing (28)-(30) to (31)-(33) suggests setting $C_4$ and $C_5$, without loss of generality, as so:

$$C_4 = \frac{2}{3 + 3\sigma} \delta$$

$$C_5 = -\frac{4}{(3 + 3\sigma)^2} \delta^2 a^2$$

where $a$ is an essential parameter. In other words, (28)-(30) is a two parameter family of solutions originating from the fixed point $U_0$ with the leading order approximations of FLRW$(0, \sigma, 1)$ as a one parameter subset. The FLRW$(0, \sigma, 1)$ spacetimes correspond to $a = 1$ and any other value of $a$ represents a self-similar perturbation from FLRW$(0, \sigma, 1)$. From this point onwards the value of $\delta$ is fixed as:

$$\delta = \frac{1}{4}(3 + 3\sigma)$$

so as to simplify calculations and match the notation and proceeding definition found in [Smoller and Temple 2012].

**Definition 2.10.** The asymptotically Friedmann spacetimes, denoted by FLRW$(0, \sigma, a)$, are defined as the two parameter family of solutions to (5)-(8) with the following leading order form as $\xi \to 0$:

$$A(\xi) \approx 1 - \frac{1}{4} a^2 \xi^2$$

$$G(\xi) \approx \frac{1}{4}(3 + 3\sigma)\xi$$

$$v(\xi) \approx \frac{1}{2} \xi$$

Furthermore, the parameter $a$ is referred to as the acceleration parameter.

It is demonstrated by [Smoller and Temple 2012] that the FLRW$(0, \sigma, a)$ spacetimes are distinct from the FLRW$(k, \sigma, 1)$ spacetimes, and in particular, that the FLRW$(k, \sigma, 1)$ spacetimes are not self-similar in the variable $\xi$. As Definition 2.10 suggests, the FLRW$(0, \sigma, a)$ spacetimes are what [Carr and Yahil 1990] classify as asymptotically Friedmann spacetimes. Unlike the asymptotic solutions given in [Carr and Yahil 1990] and [Carr and Coley 2000], the FLRW$(0, \sigma, a)$ solutions derived by [Smoller and Temple 2012] are exact, even though they are not known explicitly. Despite this, we can still give a leading order approximation of the FLRW$(0, \sigma, a)$ solutions local to the centre of expansion.

**Proposition 2.11.** The FLRW$(0, \sigma, a)$ spacetimes are given in self-similar Schwarzschild coordinates to leading order as $\xi \to 0$ as so:

$$ds^2 \approx -\frac{16}{(3 + 3\sigma)^2} \left(1 + \frac{1}{4} a^2 \xi^2\right) dt^2 + \left(1 + \frac{1}{4} a^2 \xi^2\right) dr^2 + r^2 d\Omega^2$$

$$v \approx \frac{1}{2} \xi$$

$$\rho \approx \frac{3a^2 \xi^2}{32\pi r^2}$$

$$p = \sigma \rho$$

**Proof.** This follows from Proposition 2.6 and Definition 2.10 by noting that:

$$B = \frac{\xi^2}{AG^2}$$

As a cosmological model, the closer the acceleration parameter is to one, the more spatially homogeneous the associated universe is, with homogeneity increasing the closer an observer is to centre of expansion. Despite this, any FLRW$(0, \sigma, a)$ spacetime with $a \neq 1$ is still inhomogeneous and thus violates the Cosmological Principle. The acceleration parameter is also responsible for the rate of acceleration of the spacetime. In the Radiation Dominated Epoch, observational data suggests cosmic acceleration was small, which corresponds to an acceleration parameter only slightly larger than one. An FLRW$(0, \frac{1}{3}, a)$ universe with an acceleration parameter close to one is thus close to the Standard Model of Cosmology in the Radiation Dominated Epoch, since the spacetime appears homogeneous close to the centre of expansion and induces a small acceleration. A more in-depth consideration of the cosmological applications of FLRW$(0, \sigma, a)$ spacetimes is given in [Smoller and Temple 2012].
2.5 Shock Wave Construction

Suppose that we have two spherically symmetric, although not necessarily self-similar, solutions of the Einstein field equations with a perfect fluid source. Let us denote these solutions by the triples \((g, \rho, u)\) and \((\bar{g}, \bar{\rho}, \bar{u})\) and assume also that these solutions have equations of state \(p = p(\rho)\) and \(\bar{p} = \bar{p}(\bar{\rho})\) respectively. Since we are assuming spherical symmetry, when specifying a set of coordinates \((t, r, \theta, \phi)\) it is sufficient to only consider the coordinates \((t, r)\). In this light, let metrics \(g\) and \(\bar{g}\) be given in Schwarzschild coordinates \((t, r, \theta, \phi)\) and \((\bar{t}, \bar{r})\) respectively as so:

\[
\begin{align*}
    ds^2 &= -B(t, r)dt^2 + \frac{1}{A(t, r)}dr^2 + r^2d\Omega^2 \\
    d\Omega^2 &= d\theta^2 + \sin^2\theta d\phi^2
\end{align*}
\]

where the coordinate variables \((\theta, \phi)\) and \((\bar{\theta}, \bar{\phi})\) have been identified.

**Definition 2.12.** We say that two metrics can be matched on a spherical surface \(\Sigma = \Phi(t)\) if there exists a common set of coordinates \((\bar{t}, \bar{r})\) such that the coefficients of the metrics agree on this surface when written in these coordinates.

It is not required that the metrics be given in Schwarzschild coordinates in order to be matched, but it does provide a convenient set of coordinates from which the metrics can be compared. For metrics \(g\) and \(\bar{g}\), we may simply take \((t, r)\) as our common set of coordinates and ask which transformation of the form:

\[
\begin{align*}
    \bar{t} &= \bar{t}(t, r) \\
    \bar{r} &= \bar{r}(t, r)
\end{align*}
\]

is required in order to match these metrics. The reason Schwarzschild coordinates are so useful is because we automatically match the \(d\Omega^2\) coefficients through the identification \(\bar{r} = r\). This identification means that in order to avoid introducing \(dt dr\) cross terms, the most general transformation that can be applied takes the form:

\[
\bar{t} = \bar{t}(t)
\]

Thus for two metrics given in Schwarzschild coordinates, the process of matching these metrics reduces to the existence of a spherical surface \(r = \Phi(t)\) and a coordinate transformation \(\bar{t} = \bar{t}(t)\) that satisfy the following algebraic-differential equations:

\[
\begin{align*}
    B(t, \Phi(t)) &= B(\bar{t}(t), \Phi(t)) \left| d\bar{t} \right|^2 \\
    A(t, \Phi(t)) &= A(\bar{t}(t), \Phi(t))
\end{align*}
\]

If these equations can be solved, then metrics \(g\) and \(\bar{g}\) can be matched along the surface \(r = \Phi(t)\). However, such a matching does not automatically imply that mass-energy and momentum are conserved across the surface. With this in mind, let us assume that there exists a set of coordinates \((t, r)\) for which the metrics match on the spherical surface \(r = \Phi(t)\) and define:

\[
\Sigma = \{(t, r, \theta, \phi) : r = \Phi(t), t > 0\}
\]

**Definition 2.13.** We say that the spacetime given by the matched metric \(g\cup\bar{g}\), along with the associated hydrodynamic variables, forms a shock wave solution of the Einstein field equations with a perfect fluid source if the Rankine-Hugoniot jump conditions hold across the surface \(\Sigma\). Furthermore, the spherical surface \(\Sigma\) is known as the shock surface or simply the shock.

As like in classical shock wave theory, the Rankine-Hugoniot jump conditions express the weak form of the conservation of mass-energy and momentum across the shock-surface.

**Proposition 2.14.** Let \(p \in \Sigma\) and \(U\) be a neighbourhood of \(p\), then the weak form of the conservation of mass-energy and momentum across \(\Sigma \cap U\) is given by:

\[
\int_U T^{\mu\nu} \nabla_\nu \phi \, dx = 0 \quad \forall \phi \in C^\infty_c(U)
\]

or equivalently:

\[
\int_U G^{\mu\nu} \nabla_\nu \phi \, dx = 0 \quad \forall \phi \in C^\infty_c(U)
\]
Proof. If each component of the stress-energy-momentum tensor $T$ is differentiable in $U$, then the conservation of mass-energy and momentum in $U$ is given by:

$$\nabla_\nu T^{\mu\nu} = 0$$

These conditions are equivalent to:

$$\int_U \varphi \nabla_\nu T^{\mu\nu} \, dx = 0 \ \forall \ \varphi \in C^\infty_c(U)$$

and by using the identity:

$$\varphi \nabla_\nu T^{\mu\nu} = \nabla_\nu (\varphi T^{\mu\nu}) - T^{\mu\nu} \nabla_\nu \varphi$$

are then equivalent to:

$$\int_U \nabla_\nu (\varphi T^{\mu\nu}) \, dx - \int_U T^{\mu\nu} \nabla_\nu \varphi \, dx = 0 \ \forall \ \varphi \in C^\infty_c(U)$$

Now since $\varphi$ is compactly supported within $U$, the divergence theorem implies:

$$\int_U \nabla_\nu (\varphi T^{\mu\nu}) \, dx = \int_{\partial U} \varphi T^{\mu\nu} n_\nu \, dy = 0 \ \forall \ \varphi \in C^\infty_c(U)$$

where $n$ denotes the outward normal vector to $\Sigma$. Thus \ref{34} yields the weak form of the conservation of mass-energy and momentum across $\Sigma \cap U$. Conditions \ref{35} then follow from equation \ref{1}.

Proposition 2.15. The Rankine-Hugoniot jump conditions are given by:

$$[G^{\mu\nu}] n_\nu = 0$$

where:

$$[G^{\mu\nu}] n_\nu = G^{\mu\nu}(g) n_\nu - G^{\mu\nu}(\bar{g}) n_\nu$$

Proof. Let $U = U_1 \cup U_2$ where $\partial U_1 \cap \partial U_2 = \Sigma \cap U$ and assume that $g$ and $\bar{g}$ are sufficiently regular on their respective side of $\Sigma$, then:

$$\int_U G^{\mu\nu} \nabla_\nu \varphi \, dx = \int_{U_1} G^{\mu\nu} (g) \nabla_\nu \varphi \, dx + \int_{U_2} G^{\mu\nu} (\bar{g}) \nabla_\nu \varphi \, dx$$

$$= \int_{U_1} \nabla_\nu (\varphi G^{\mu\nu} (g)) \, dx - \int_{U_1} \varphi \nabla_\nu G^{\mu\nu} (g) \, dx$$

$$+ \int_{U_2} \nabla_\nu (\varphi G^{\mu\nu} (\bar{g})) \, dx - \int_{U_2} \varphi \nabla_\nu G^{\mu\nu} (\bar{g}) \, dx$$

$$= \int_{\partial U_1} \varphi G^{\mu\nu} (g) n_\nu \, dy - \int_{\partial U_2} \varphi G^{\mu\nu} (\bar{g}) n_\nu \, dy$$

$$= \int_{\Sigma} \varphi G^{\mu\nu} (g) n_\nu \, dy - \int_{\Sigma} \varphi G^{\mu\nu} (\bar{g}) n_\nu \, dy$$

$$= \int_{\Sigma} \varphi [G^{\mu\nu}] n_\nu \, dy \ \forall \ \varphi \in C^\infty_c(U)$$

Thus:

$$[G^{\mu\nu}] n_\nu = 0 \iff \int_U G^{\mu\nu} \nabla_\nu \varphi \, dx = 0 \ \forall \ \varphi \in C^\infty_c(U)$$
2.6 Regularity

As like in the previous subsection, consider the solution triples \((g, \rho, u)\) and \((\bar{g}, \bar{\rho}, \bar{u})\). Assume that these solutions can be matched Lipschitz continuously along a spherical surface \(\Sigma\) with a spacelike normal vector \(n\) to form the matched metric \(g \cup \bar{g}\). Furthermore, let \(g \cup \bar{g}\) satisfy the Rankine-Hugoniot jump condition across \(\Sigma\) so that \(g \cup \bar{g}\) forms a shock wave solution. For the rest of this subsection, the matched metric is to be referred to simply as the metric.

It is reasonable to be concerned with the regularity of such a solution, since a Lipschitz continuous shock wave has discontinuities in the first order derivatives of the metric and delta function sources in the second order derivatives. The Einstein tensor comprises second order derivatives of the metric, so this too is expected to harbour delta function sources. On the other side of the Einstein field equations, the hydrodynamic variables \(\rho, p\) and \(u\), along with the metric, form the stress-energy-momentum tensor, and since the hydrodynamic variables are expected to be at worst discontinuous at the shock, so too is the stress-energy-momentum tensor. This is problematic, since the Einstein field equations cannot have different levels of regularity on the left and right hand sides of the equation. However, it turns out that even though delta function sources may appear in the second order derivatives of the metric at the shock, with such being coordinate dependent, the Einstein tensor does not have any delta function sources, that is, the delta function sources cancel in the Einstein tensor. This result is summarised in the following theorem of Smoller and Temple [1995].

**Theorem 2.16.** Let \(\Sigma\) denote a smooth, three-dimensional surface with a spacelike normal vector \(n\). Assume that the components of the metric are continuous on both sides of \(\Sigma\) and Lipschitz continuous across \(\Sigma\) in some fixed coordinate system. Then the following statements are equivalent:

1. \(K = 0\) at each point of \(\Sigma\), where \(K\) is the second fundamental form of the metric.
2. The Riemann curvature and Einstein tensors, viewed as second order operators on the metric components, produce no delta function sources on \(\Sigma\).
3. For each point \(p \in \Sigma\) there exists a \(C^{1,1}\) coordinate transformation defined in a neighbourhood of \(p\) such that in the new coordinates, which can be taken to be the Gaussian normal coordinates for the surface, the metric components are \(C^{1,1}\) functions of these coordinates.
4. For each point \(p \in \Sigma\) there exists a coordinate frame that is locally Lorentzian at \(p\) and can be reached from the original coordinates by a \(C^{1,1}\) coordinate transformation.

Moreover, if any one of these equivalences hold, then the Rankine-Hugoniot jump conditions:

\[
(G^{\mu\nu})n_\nu = 0
\]

hold at each point of \(\Sigma\).

This theorem provides a criterion for the removal of the delta function sources and also a coordinate system for which the shock wave solution can achieve optimal regularity, that is, when the metric has a Lipschitz continuous derivative at the shock. The following theorem, also from Smoller and Temple [1995], provides convenient criteria for satisfying one of the equivalences of Theorem 2.16.

**Theorem 2.17.** Assume the following:

1. That \(g\) and \(\bar{g}\) are two spherically symmetric metrics that match across a three-dimensional surface \(\Sigma\) to form the matched metric \(g \cup \bar{g}\).
2. The matched metric is Lipschitz continuous across \(\Sigma\).
3. The normal \(n\) to \(\Sigma\) is non-null.

Then the following are equivalent:

1. \((G^{\mu\nu})n_\nu = 0\)
2. \((G^{\mu\nu})n_\mu n_\nu = 0\)
3. \((K) = 0\) at each point of \(\Sigma\), where \(K\) is the second fundamental form of the metric.
4. The components of the matched metric in any Gaussian-normal coordinate system are \(C^{1,1}\) functions of these coordinates across \(\Sigma\).
If the conditions of Theorem 2.17 are satisfied, then it is clear that the weak form of mass-energy and momentum conservation across the shock surface is equivalent to the single condition:

\[ [T^{\mu\nu}]_{\eta_{\mu}\eta_{\nu}} = 0 \]

Thus the Rankine-Hugoniot jump conditions reduce to the single equivalent condition:

\[ [G^{\mu\nu}]_{\eta_{\mu}\eta_{\nu}} = 0 \]

Therefore a shock wave solution, which satisfies the Rankine-Hugoniot jump conditions by definition, only requires the

\[(\tilde{g}, \tilde{\rho}, \tilde{\psi})\]

Choose Minkowskian system at \( p \). In a locally Minkowskian coordinate frame, a speed at \( p \) transforms according to the special relativistic velocity transformation law when a Lorentz transformation is performed. The shock speed at a point \( p \) on the shock in a locally Minkowskian frame that is comoving with the interior fluid will now be determined. To this end, let \( \tilde{\tau} = \Phi(\tilde{t}) \) be the position of the shock in \((\tilde{t}, \tilde{\tau})\) coordinates and let \((\tilde{t}, \tilde{\tau})\) coordinates correspond to a locally Minkowskian system at \( p \) obtained from \((\tilde{t}, \tilde{\tau})\) by a transformation of the form:

\[ \tilde{t} = \tilde{t}(\tilde{t}) \]
\[ \tilde{\tau} = \Phi(\tilde{t}) \]

so that, in \((\tilde{t}, \tilde{\tau})\) coordinates:

\[ d\tilde{s}^2 = -e^{2\psi}\left(\frac{dt}{d\tilde{t}}\right)^2 d\tilde{t}^2 + e^{2\psi}\left(\frac{d\tilde{\tau}}{d\tilde{t}}\right)^2 d\tilde{\tau}^2 + \mathcal{R}^2\tilde{\tau}^2 d\Omega^2 \]

Choose \((\tilde{t}, \tilde{\tau})\) so that:

\[ \frac{dt}{d\tilde{t}} = e^{\varphi} \]
\[ \frac{d\tilde{\tau}}{d\tilde{t}} = e^{\psi} \]
Then in \((\tilde{t}, \tilde{r})\) coordinates at \(p\) the metric takes the form of (37). The \((\tilde{t}, \tilde{r})\) coordinates represent the class of locally Minkowskian coordinate frames that are fixed relative to the fluid particles of the interior spacetime at the point \(p\), that is, any two members of this class of coordinate frames differ only by higher order terms that do not affect the calculation of radial velocities at \(p\). Thus the speed \(\tilde{r}\) of a particle in \((\tilde{t}, \tilde{r})\) coordinates gives the value of the speed of the particle relative to the interior fluid in the special relativistic sense. If the speed of a particle in \((\tilde{t}, \tilde{r})\) coordinates is \(\dot{\tilde{r}}\), then its geometric speed relative to observers fixed with the interior fluid, and hence also fixed relative to the radial coordinate \(\tilde{r}\) of the metric \(g\) because the fluid is comoving, is equal to:

\[ e^{\psi - \varphi} \dot{\tilde{r}} \]

since:

\[ \frac{d\tilde{r}}{dt} = \frac{d\tilde{r}}{d\tilde{t}} \frac{d\tilde{t}}{dt} = e^{\varphi} \frac{d\tilde{r}}{d\tilde{t}} \]

(38)

Now considering the shock wave moves with speed \(\dot{\Phi}\), therefore by (38) the speed of the shock relative to the interior fluid particles must be given by (36), which completes the proof. \(\square\)

Let \(\dot{\lambda}_{int}^+\) and \(\dot{\lambda}_{int}^-\) denote the speeds of the interior characteristics in \((\tilde{t}, \tilde{r})\) coordinates. Since the characteristic speeds on the interior side of the shock equal the sound speeds in locally Minkowskian coordinates, we have:

\[ \dot{\lambda}_{int}^+ = \pm \sqrt{\frac{dp}{d\rho}} \]

The +,- characteristics refer to the 1,2-characteristic families respectively. In the 1+1 dimensional theory of conservation laws, the Lax characteristic conditions state that the characteristic curves in the family of the shock impinge upon the shock from both sides. Since we are considering shocks that are outward moving with respect to \(\tilde{r}\) and \(\tilde{r}\), it follows that on the interior side, only the 2-characteristic can impinge on the shock, and thus the shock must be identified as a 2-shock. For more details on \(n\)-shocks, see [Smoller 1994]. Let \(\dot{\lambda}_{Ext}^+\) and \(\dot{\lambda}_{Ext}^-\) denote the speeds of the exterior characteristics in \((\tilde{t}, \tilde{r})\) coordinates. Since the shock has been identified as a 2-shock, the Lax characteristic conditions are given as the following inequalities:

\[ \dot{\lambda}_{Ext}^+ < s < \dot{\lambda}_{int}^+ \quad (39) \]

where \(s\) is the speed of the shock in \((\tilde{t}, \tilde{r})\) coordinates.

**Proposition 2.19.** For an expanding shock wave, the Lax characteristic conditions are given as the following inequalities:

\[ \frac{\dot{\bar{w}}} + \sqrt{\frac{dp}{d\rho}} e^{\psi - \varphi} \dot{\Phi} < \sqrt{\frac{dp}{d\rho}} \]

\[ 1 + \bar{w} \sqrt{\frac{dp}{d\rho}} \quad (40) \]

where:

\[ \bar{w} = e^{\psi - \varphi} \frac{\partial \tilde{r}}{\partial t} \left( \frac{\partial t}{\partial \tilde{t}} \right)^{-1} \]

**Proof.** This proof largely follows an analogous proof provided by [Smoller and Temple 1995]. Since the shock wave is expanding, it is a 2-shock, so the Lax characteristic conditions are given by (39), and by Lemma 2.18, \(s\) is given by (36). As we are working in \((\tilde{t}, \tilde{r})\) coordinates, \(\dot{\lambda}_{int}^+\) is already known, so it remains to determine \(\dot{\lambda}_{Ext}^+\). Let \(\bar{v}, \bar{v}\) and \(\bar{v}\) denote the exterior fluid four-velocity given in interior comoving, exterior comoving and interior locally Minkowskian coordinates respectively. Since the aim is to compute the characteristic speed, which is a ratio of two vector components, a tangent vector of any length is sufficient. By writing \(\bar{x} = (\tilde{t}, \tilde{r})\) and \(\bar{x} = (\tilde{t}, \tilde{r})\), then:

\[ \bar{\dot{v}}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \bar{v}^\nu = \frac{\partial \tilde{x}^\mu}{\partial \tilde{x}^\nu} \bar{v}^\nu = \frac{\partial \tilde{x}^\mu}{\partial \tilde{x}^\nu} \]

In light of this, the speed of the exterior fluid as measured in the interior coordinates \((\tilde{t}, \tilde{r})\) is given by:

\[ \bar{\dot{w}} = \frac{\bar{\dot{v}}^1}{\bar{\dot{v}}^0} = \frac{\partial \tilde{x}^1}{\partial \tilde{x}^0} \left( \frac{\partial \tilde{x}^0}{\partial \tilde{x}^0} \right)^{-1} = \frac{\partial \tilde{r}}{\partial \tilde{t}} \left( \frac{\partial t}{\partial \tilde{t}} \right)^{-1} \]
and so, by (38):
\[ \ddot{w} = e^{\psi - \varphi} \dot{w} \]
This gives the exterior fluid speed in $(\tilde{t}, \tilde{r})$ coordinates, and since the sound speed in the exterior spacetime is given by:
\[ \sqrt{\frac{dp}{d\rho}} \]
the relativistic addition of velocities formula yields:
\[ \lambda^+_{\text{Ext}} = \frac{\dot{w} + \sqrt{\frac{d\rho}{dp}}}{1 + \ddot{w} \sqrt{\frac{d\rho}{dp}}} \]
which completes the proof. \[\square\]

3 Friedmann-Static Shock Waves

3.1 Explicit Construction

The objective of this section is the construction of a particular family of general relativistic shock waves.

**Definition 3.1.** A shock wave solution with an FLRW $(0, \sigma, a)$ spacetime on the interior and a TOV $(\bar{\sigma})$ spacetime on the exterior is referred to as a *Friedmann-static shock wave* and denoted by FLRW $(0, \sigma, a)$-TOV $(\bar{\sigma})$.

Since all Friedmann-static shock waves share a TOV $(\bar{\sigma})$ exterior, the following lemma is of great utility in their construction.

**Lemma 3.2.** Let $(A, G, v)$ denote a spherically symmetric self-similar solution to the Einstein field equations with a perfect fluid source and equation of state $p = \sigma \rho$. If there exists a $\xi_0 > 0$ such that:
\[ A(\xi_0) = 1 - 2M(\bar{\sigma}) \]
then $(A, G, v)$ can be matched to TOV $(\bar{\sigma})$ on the surface $\xi = \xi_0$, and the Rankine-Hugoniot jump condition is given by:
\[ \frac{[\sigma + v^2(\xi_0)]G(\xi_0) - (1 + \sigma)G^2(\xi_0)v(\xi_0)}{[1 + \sigma v^2(\xi_0)]G(\xi_0) - (1 + \sigma)v(\xi_0)} = \bar{\sigma} \]

**Proof.** Let the metric of the $(A, G, v)$ solution in self-similar Schwarzschild coordinates be given by:
\[ ds^2 = -B(\xi)dt^2 + \frac{1}{A(\xi)}dr^2 + r^2d\Omega^2 \]
and recall by Proposition 2.5 that TOV $(\bar{\sigma})$ is given in self-similar comoving Schwarzschild coordinates as:
\[ ds^2 = -\xi^\frac{4\sigma}{1 + 2M(\bar{\sigma})} dt^2 + \frac{1}{1 - 2M(\bar{\sigma})} dr^2 + \bar{r}^2 d\Omega^2 \]
\[ \bar{\rho} = \frac{M(\bar{\sigma})}{4\pi \bar{r}^2} \]
\[ \bar{\rho} = \bar{\sigma} \bar{\rho} \]

where the inessential parameter has been set to one and:
\[ M(\bar{\sigma}) = \frac{2\bar{\sigma}}{1 + 6\bar{\sigma} + \bar{\sigma}^2} \]

Because both metrics are specified in Schwarzschild coordinates, the $d\Omega^2$ components automatically match under the identification $\bar{r} = r$. Matching the $dr^2$ components implies that the shock surface is defined by $\xi = \xi_0$, with the constant $\xi_0$ given implicitly by (41). This also implies that $B$ is constant on the surface. A temporal rescaling of the form:
\[ \tilde{t} = \alpha t \]
With the matching in place, recall that the Rankine-Hugoniot jump condition is equivalent to:

$$\alpha \xi = \xi = \xi_0$$

and matches the $dt^2$ coefficients providing $\alpha$ satisfies:

$$\alpha^2 \frac{d\xi}{\xi_0} = B(\xi_0)$$

(43)

With the matching in place, recall that the Rankine-Hugoniot jump condition is equivalent to:

$$[T^{\mu\nu}] n_\mu n_\nu = T^{\mu\nu}(g, \rho, p, u_{FLRW}) n_\mu n_\nu - T^{\mu\nu}(\bar{g}, \bar{\rho}, \bar{p}, u_{TOV}) n_\mu n_\nu = 0$$

where $n$ is the outward normal to the shock surface. Using (2) and (4) we obtain:

$$(1 + \sigma)\mu_{\mu\nu_{FLRW}} u_{FLRW}^\nu n_\mu n_\nu + \sigma \rho |n|^2 - (1 + \bar{\sigma})\bar{\rho} u_{\mu\nu_{TOV}} n_\mu n_\nu - \bar{\rho} |n|^2 = 0$$

Now since the surface is defined by $\xi = \xi_0$, which is equivalent to:

$$r - \xi_0 t = 0$$

then the components of the normal satisfy:

$$n_\mu dx^\mu = d(r - \xi_0 t) = -\xi_0 dt + dr$$

and so:

$$n_0 = -\xi_0$$

$$n_1 = 1$$

Noting that the metric components are identified on the surface, the following identities are obtained:

$$|n|^2 = A(\xi_0) - \xi_0^2 B^{-1}(\xi_0)$$

$$u_{FLRW}^0 = [1 - v^2(\xi_0)]^{-\frac{1}{2}} B^{-\frac{1}{2}}(\xi_0)$$

$$u_{FLRW}^1 = v(\xi_0)[1 - \sigma^2(\xi_0)]^{-\frac{1}{2}} A^\frac{1}{2}(\xi_0)$$

$$u_{FLRW}^\mu u_{FLRW}^\nu n_\mu n_\nu = [1 - v^2(\xi_0)]^{-1} \left[v(\xi_0)A^\frac{1}{2}(\xi_0) - \xi_0 B^{-\frac{1}{2}}(\xi_0)\right]^2$$

$$u_{TOV}^\mu u_{TOV}^\nu n_\mu n_\nu = \frac{\xi_0^2}{3} B^{-1}(\xi_0)$$

Applying these identities puts the Rankine-Hugoniot jump condition in the following form:

$$0 = (1 + \sigma)[1 - v^2(\xi_0)]^{-1} \left[v(\xi_0)A^\frac{1}{2}(\xi_0) - \xi_0 B^{-\frac{1}{2}}(\xi_0)\right]^2 \rho - (1 + \bar{\sigma})\xi_0^2 B^{-1}(\xi_0)\bar{\rho} + [A(\xi_0) - \xi_0^2 B^{-1}(\xi_0)](\sigma \rho - \bar{\sigma} \bar{\rho})$$

Dividing by $A(\xi_0)$ and substituting $B(\xi_0)$ for $G(\xi_0)$ then yields:

$$0 = (1 + \sigma)[1 - v^2(\xi_0)]^{-1}[v(\xi_0) - G(\xi_0)]^2 \rho - (1 + \bar{\sigma})G^2(\xi_0)\bar{\rho} + [1 - G^2(\xi_0)](\sigma \rho - \bar{\sigma} \bar{\rho})$$

Finally, applying (8) and (41) gives (42), which completes the proof.  

As FLRW$(0, \sigma, 1)$ is known explicitly, it is possible to construct an explicit FLRW$(0, \sigma, 1)$-TOV$(\bar{\sigma})$ shock wave. Such a construction has been achieved by [Cahill and Taub (1971)] for the case $\sigma = \frac{1}{2}$ and in full generality by [Smoller and Temple (1975)]. The result can instead be derived directly from Lemma 3.2.

Theorem 3.3. For each $0 < \sigma < 1$, FLRW$(0, \sigma, 1)$ can be matched to TOV$(\bar{\sigma})$ to form a general relativistic shock wave providing:

$$\bar{\sigma} = H(\sigma)$$

where:

$$H(\sigma) = \frac{1}{2} \sqrt{9\sigma^2 + 54\sigma + 49} - \frac{3}{2} \sigma - \frac{7}{2}$$

(44)
\textbf{Proof.} The matching follows similarly to the matching completed in the proof of Lemma 3.2 but with (41) and (43) replaced with:

\[ 1 - \frac{2}{3} \bar{\xi}_0^{\frac{2+6\sigma}{2+6\rho}} = 1 - 2M(\bar{\sigma}) \]

and:

\[ \alpha^2 \bar{\xi}_0^{\frac{2+6\sigma}{2+6\rho}} = \frac{16}{(3 + 3\sigma)^2} \left[ 1 + \frac{1}{3} (1 + 3\sigma) \bar{\xi}_0^{\frac{2+6\sigma}{2+6\rho}} \right] \left[ 1 - \frac{2}{3} \bar{\xi}_0^{\frac{2+6\sigma}{2+6\rho}} \right]^{-1} \]

respectively, where the inessential parameter has been set to the value given in Section 2.4 and:

\[ \xi = \frac{4}{\sqrt{6}} \bar{\xi}_0^{\frac{1+3\sigma}{2+6\sigma}} \left[ 1 + \frac{1}{3} (1 + 3\sigma) \bar{\xi}_0^{\frac{2+6\sigma}{2+6\rho}} \right]^{-\frac{3+3\sigma}{2+6\sigma}} \]

Note that this matching is Lipschitz continuous, as any $0 < \rho < 1$ and $0 < \bar{\sigma} < 1$ imply that the components of the interior and exterior metrics are continuous in a neighborhood of the surface when given in $(t, r)$ coordinates. Thus it remains to show that the condition $\bar{\sigma} = H(\sigma)$ is equivalent to the Rankine-Hugoniot jump condition, which we know by Lemma 3.2 is given by:

\[ [\sigma + v^2(\xi_0)]G(\xi_0) - (1 + \sigma)G^2(\xi_0)v(\xi_0) = \bar{\sigma} \]

By Proposition 2.7, $G(\xi_0)$ can be substituted for $v(\xi_0)$, which in turn can be substituted for $A(\xi_0)$ to yield:

\[ \frac{(3\sigma + 3[1 - A(\xi_0)])(2 + (1 + 3\sigma)[1 - A(\xi_0)]) - (3 + 3\sigma)^2[1 - A(\xi_0)]}{A(\xi_0)(2 + (1 + 3\sigma)[1 - A(\xi_0)])} = \bar{\sigma} \]

Finally, substituting $A(\xi_0)$ for $1 - 2M(\bar{\sigma})$ yields:

\[ \sigma = \bar{\sigma}(7 + \bar{\sigma}) \]

which is equivalent to $\bar{\sigma} = H(\sigma)$. \hfill \Box

\textbf{Lemma 3.4.} Let $(A, G, v)$ denote a spherically symmetric self-similar solution to the Einstein field equations with a perfect fluid source and equation of state $p = \sigma \rho$. If there exists a $\xi_0 > 0$ such that $(A, G, v)$ can be matched to TOV($\bar{\sigma}$) to form a shock wave solution, then the Lax characteristic conditions are given by:

\[ \frac{\sqrt{\sigma} - v(\xi_0)}{1 - \sqrt{\sigma}v(\xi_0)} < \frac{G(\xi_0) - v(\xi_0)}{1 - G(\xi_0)v(\xi_0)} < \frac{\sqrt{\sigma}}{1 - \sqrt{\sigma}v(\xi_0)} \] (45)

\textbf{Proof.} As a reverse to the coordinate transformation introduced in the proof of Proposition 2.6, we begin by transforming a general solution given in self-similar Schwarzschild coordinates, to a solution given in self-similar comoving coordinates. Noting that $B$ and $u$ are given implicitly by the triple $(A, G, v)$, we can write this solution in self-similar Schwarzschild coordinates as so:

\[ ds^2 = -B(\xi)dt^2 + \frac{1}{A(\xi)} dr^2 + r^2 d\Omega^2 \]

where $p$ and $\rho$ are determined by $\xi$ and $\xi$ respectively. In self-similar comoving coordinates, the solution can be written as:

\[ ds^2 = -e^{2\tau} dt^2 + e^{2\psi} dr^2 + \rho^2 r^2 d\Omega^2 \]

\[ u = (u^0, u^1, 0, 0) \]

To eliminate the radial component of the four-velocity, the transformation from Schwarzschild to comoving coordinates must satisfy:

\[ \ddot{u}^1 = u^0 \frac{\partial \ddot{r}}{\partial t} + u^1 \frac{\partial \dot{r}}{\partial r} = 0 \]
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which is equivalent to:

\[ \frac{\partial v}{\partial t} = -\frac{\xi v}{G} \frac{\partial r}{\partial t} \] (46)

Now given that:

\[ d\hat{t} = \frac{\partial \hat{t}}{\partial t} dt + \frac{\partial \hat{t}}{\partial r} dr \]
\[ d\hat{r} = \frac{\partial \hat{r}}{\partial t} dt + \frac{\partial \hat{r}}{\partial r} dr \]

then:

\[ dt = \left( \frac{\partial \hat{t}}{\partial t} \frac{\partial \hat{t}}{\partial r} - \frac{\partial \hat{t}}{\partial t} \frac{\partial \hat{t}}{\partial t} \right)^{-1} \left( \frac{\partial \hat{r}}{\partial r} \frac{\partial \hat{t}}{\partial t} - \frac{\partial \hat{t}}{\partial r} \frac{\partial \hat{t}}{\partial t} \right) \]
\[ dr = \left( \frac{\partial \hat{t}}{\partial t} \frac{\partial \hat{t}}{\partial r} - \frac{\partial \hat{t}}{\partial t} \frac{\partial \hat{t}}{\partial t} \right)^{-1} \left( -\frac{\partial \hat{t}}{\partial t} \frac{\partial t}{\partial t} + \frac{\partial \hat{t}}{\partial r} \frac{\partial t}{\partial r} \right) \]

Thus to keep the metric diagonal, the following condition is also needed:

\[ B \frac{\partial \hat{t}}{\partial t} \frac{\partial \hat{t}}{\partial r} = \frac{1}{A} \frac{\partial \hat{r}}{\partial t} \frac{\partial \hat{t}}{\partial t} = 0 \]

which by (46) is equivalent to:

\[ \frac{\partial \hat{t}}{\partial t} = -\frac{G v}{\xi} \frac{\partial \hat{t}}{\partial t} \] (47)

The most general transformation that preserves self-similarity takes the form:

\[ \hat{t} = T(\xi) t \]
\[ \hat{r} = R(\xi) r \]

and conditions (46) and (47) determine the functions \( T(\xi) \) and \( R(\xi) \). In self-similar Schwarzschild coordinates the shock speed is given by \( \xi = \xi_0 \), so in self-similar comoving coordinates the shock speed is given by \( \hat{\xi} = \hat{\xi}_0 \), where:

\[ \hat{\xi} = \frac{\hat{r}}{\hat{t}} = \frac{R(\xi) r}{T(\xi) t} = \frac{R(\xi)}{T(\xi)} \xi \]

Thus by Lemma 2.18 the shock speed is given in interior locally Minkowskian coordinates by:

\[ e^{\psi - \varphi} \xi_0 \]

By Proposition 2.19 it remains to determine \( e^{\psi - \varphi} \) and \( \hat{\psi} \). In this light:

\[ e^{2\varphi} = \frac{1}{A} \left( \frac{\partial \hat{t}}{\partial t} \right)^2 - \frac{\xi^2}{AG^2} \left( \frac{\partial \hat{t}}{\partial r} \right)^2 \]
\[ = \xi^2 \left( 1 - \nu^2 \right) \left( \frac{\partial \hat{t}}{\partial t} \right)^2 \]

and:

\[ e^{2\psi} = \frac{1}{A} \left( \frac{\partial \hat{t}}{\partial t} \right)^2 - \frac{\xi^2}{AG^2} \left( \frac{\partial \hat{t}}{\partial r} \right)^2 \]
\[ = \frac{1 - \nu^2}{A} \left( \frac{\partial \hat{t}}{\partial t} \right)^2 \]

Now:

\[ \frac{\partial \hat{r}}{\partial t} = -\xi^2 R'(\xi) \]
\[ \frac{\partial \hat{r}}{\partial r} = R(\xi) + \xi R'(\xi) \]
so (46) yields:

\[-\xi^2 R' = -\frac{\xi v}{G} (R + \xi')\]
\[\iff \xi R' = \frac{v}{G-v} R\]

Similarly:

\[\frac{\partial \tilde{t}}{\partial t} = T(\xi) - \xi T'(\xi)\]
\[\frac{\partial \tilde{t}}{\partial r} = T'(\xi)\]

to which (47) yields:

\[T' = -\frac{Gv}{\xi} (T - \xi T')\]
\[\iff \xi T' = -\frac{Gv}{1 - Gv} T\]

Therefore the shock speed is given in interior locally Minkowskian coordinates by:

\[e^{\psi - \phi} \xi_0 = G(\xi_0) \frac{\partial \tilde{t}}{\partial t} \left( \frac{\partial \tilde{r}}{\partial r} \right)^{-1} \frac{R(\xi_0)}{T(\xi_0)}\]
\[= \frac{\xi_0}{1 - G(\xi_0)v(\xi_0)}\]

By Proposition 2.5, TOV(\bar{\sigma}) is comoving in Schwarzschild coordinates, and given that TOV(\bar{\sigma}) is matched to (A, G, v) in (t, r) coordinates, then the (\bar{t}, \bar{r}) coordinates of Proposition 2.19 are identified with (t, r), so:

\[\tilde{w} = e^{\psi - \phi} \tilde{x}_0 \frac{\partial \tilde{r}}{\partial \tilde{t}} \left( \frac{\partial \tilde{t}}{\partial t} \right)^{-1}\]
\[\tilde{w} = e^{\psi - \phi} \frac{\partial \tilde{r}}{\partial t} \left( \frac{\partial \tilde{t}}{\partial t} \right)^{-1}\]
\[\tilde{w} = \frac{G(\xi_0)}{\xi_0} \frac{\partial \tilde{r}}{\partial t} \left( \frac{\partial \tilde{r}}{\partial r} \right)^{-1}\]
\[\tilde{w} = -v(\xi_0)\]

Finally, substituting \(e^{\psi - \phi} \xi_0\), \(\tilde{w}\) and the equations of state into (40) yields (45).

The following theorem was first proved by Smoller and Temple [1995], but can instead be obtained directly from Lemma 3.4. In Smoller and Temple’s proof, the value of \(\sigma_1\) is approximated, however it is now possible to obtain an exact value.

**Theorem 3.5.** The FLRW(0, \sigma, 1)-TOV(\bar{\sigma}) shock wave solutions are stable for:

\[0 < \sigma < \sigma_1\]

where:

\[\sigma_1 = \frac{1 + \sqrt{10}}{9} \approx 0.462\]

**Proof.** By Lemma 3.2 and Proposition 2.7 we know that FLRW(0, \sigma, 1) satisfies (42) and:

\[G(\xi_0) = \frac{1}{2} (3 + 3\sigma) v(\xi_0) \left( 1 + \frac{1}{2} (1 + 3\sigma) v^2(\xi_0) \right)^{-1}\] (48)
at the point of intersection with the shock surface. Solving (42) and (48) for \( G(\xi_0) \) and \( v(\xi_0) \) yields:

\[
G(\xi_0) = \frac{1}{2}(3 + \bar{\sigma})v(\xi_0) \\
v(\xi_0) = \sqrt{\frac{2(3\sigma - \bar{\sigma})}{(1 + 3\sigma)(3 + \bar{\sigma})}}
\]

Thus using (44), (49) and (50), the left hand inequality of (45) is found to be satisfied for \( 0 < \sigma < 1 \) and the right hand inequality is found to be satisfied for \( 0 < \sigma < \sigma_1 \).

**Lemma 3.6.** Let \((A, G, v)\) denote a spherically symmetric self-similar solution to the Einstein field equations with a perfect fluid source and equation of state \( p = \sigma \rho \). If there exists a \( \xi_0 > 0 \) such that \((A, G, v)\) can be matched to TOV(\(\bar{\sigma}\)) to form a shock wave solution, then the shock speed is subluminal if:

\[
G(\xi_0) < 1
\]

and in such a case the Lax characteristic conditions reduce to:

\[
G(\xi_0) > \sqrt{\bar{\sigma}}
\]

\[
\{ \cdot \} D(\xi_0) < 0
\]

**Proof.** By Lemma 3.4, the shock speed is subluminal if:

\[
\frac{G(\xi_0) - v(\xi_0)}{1 - G(\xi_0)v(\xi_0)} < 1
\]

which for \( 0 < v < 1 \) is equivalent to (51). For \( G < 1 \) it is then not difficult to check that the left hand inequality of (45) is equivalent to (52). Thus it remains to demonstrate that the right hand inequality is equivalent to (53). In this light, we have:

\[
\{ \cdot \} D = \frac{3}{4}(3 + 3\sigma) [(G - v)^2 - \sigma(1 - Gv)^2]
\]

\[
= \frac{3}{4}(3 + 3\sigma) [G - v + \sqrt{\sigma}(1 - Gv)] [G - v - \sqrt{\sigma}(1 - Gv)]
\]

and for \( 0 < v < G < 1 \) we see that \( \{ \cdot \} D = 0 \) is equivalent to:

\[
\frac{G - v}{1 - Gv} = \sqrt{\sigma}
\]

which completes the proof.

The following theorem, also first proved by Smoller and Temple [1995], demonstrates that even though FLRW\((0, \sigma, 1)\)-TOV(\(\bar{\sigma}\)) shock waves can be constructed mathematically, their physical applicability is limited for \( \sigma > \sigma_2 \).

**Theorem 3.7.** The FLRW\((0, \sigma, 1)\)-TOV(\(\bar{\sigma}\)) shock wave solutions have subluminal shock speeds for:

\[
0 < \sigma < \sigma_2
\]

where:

\[
\sigma_2 = \frac{\sqrt{5}}{3} \approx 0.745
\]

**Proof.** This follows directly from Lemma 3.6 and relations (44), (49) and (50).

### 3.2 Numerical Construction

We are now in a position to extend the one parameter family of FLRW\((0, \sigma, 1)\)-TOV(\(\bar{\sigma}\)) shock waves to the two parameter family of FLRW\((0, \sigma, a)\)-TOV(\(\bar{\sigma}\)) shock waves. Even though the FLRW\((0, \sigma, a)\) spacetimes are exact solutions, they are not known explicitly away from \( \xi = 0 \), so these solutions need to be approximated numerically. One way of describing FLRW\((0, \sigma, a)\) solutions, is to numerically generate their trajectories in \((A, G, v)\)-space, such as in Figure 1.
A physically important Friedmann-static shock wave is the one for which the equation of state both sides of the shock line (Figure 1). The reason considering solutions in FLRW models pure radiation, since these shock waves may have been present during the Radiation Dominated Epoch. As demonstrated in Figure 2, for \( \sigma = \bar{\sigma} = \frac{1}{3} \), the value of \( \alpha \) can be varied in order to achieve an intersection and thus form the Friedmann-static pure radiation shock wave.

**Definition 3.8.** The Rankine-Hugoniot curve, denoted by:

\[
v = \Gamma_{\text{RH}}(G; \sigma, \bar{\sigma})
\]

is the curve in \((A, G, v)\)-space generated by constraints \((42)\) and \((41)\).

The Rankine-Hugoniot curve is represented by the dashed curve in Figure 1 and lives in the plane \( A = 1 - 2M(\bar{\sigma}) \). The dotted curve represents the explicitly known FLRW\((0, \frac{1}{3}, 1)\) trajectory. For this particular solution, the trajectory obeys the implicit relationship given by Corollary \(2.9\) that is, as \( \xi \) increases from zero, \( G \) increases linearly with \( \xi \), \( v \) increases according to \((23)\) and \( A \) decreases according to \((21)\). General FLRW\((0, \sigma, a)\) trajectories are similar to the FLRW\((0, \frac{1}{3}, 1)\) trajectory for small \( \xi \) but differ as \( \xi \) increases. One characteristic that remains similar for larger \( \xi \) is the close to linear dependence \( G \) has on \( \xi \). Note that because equations \((5) - (7)\) are autonomous, all trajectories, the Rankine-Hugoniot curve and surfaces \( \{ \cdot \}_S = 0 \) and \( \{ \cdot \}_D = 0 \) are all independent of \( \xi \). Because of this, it is often easier to think of \( G \) as the independent variable, and consider the trajectory as a function of \( G \).

The TOV\((\bar{\sigma})\) trajectories are simple to represent in \((A, G, v)\) solution space as they are the lines defined by \( A = 1 - 2M(\bar{\sigma}) \) and \( v = 0 \). Now because:

\[
\min_{0 \leq \sigma \leq 1} \{ 1 - 2M(\bar{\sigma}) \} = \frac{1}{2}
\]

the TOV\((\bar{\sigma})\) trajectories span the surface:

\[
\frac{1}{2} < A < 1 \quad \text{and} \quad v = 0
\]

The reason considering solutions in \((A, G, v)\)-space is so useful, is the immediate implication that any trajectory that crosses the \( A = 1 - 2M(\bar{\sigma}) \) plane Lipschitz continuously can be matched to the TOV\((\bar{\sigma})\) solution. Furthermore, if the solution trajectory crosses the \( A = 1 - 2M(\bar{\sigma}) \) plane and intersects the Rankine-Hugoniot curve, which lies in this plane, then the solution can be matched to the TOV\((\bar{\sigma})\) solution to form a general relativistic shock wave. In the case of FLRW\((0, \sigma, a)\) trajectories, changing the parameters \( \sigma \) and \( a \) changes the trajectory, so certain combinations of \( \sigma \) and \( a \) result in an intersection with the Rankine-Hugoniot curve, and thus the formation of an FLRW\((0, \sigma, a)\)-TOV\((\bar{\sigma})\) shock wave. We already know from Theorem \(3.3\) that for \( a = 1 \) the relationship between \( \sigma \) and \( \bar{\sigma} \) obeys \( \bar{\sigma} = H(\sigma) \). For \( a \neq 1 \), trajectories can be generated numerically and the parameters \( \sigma \), \( a \) and \( \bar{\sigma} \) can be adjusted to achieve the intersection. Since the intersection imposes a single constraint on the parameters \( a \), \( \sigma \) and \( \bar{\sigma} \), we conclude that the family of FLRW\((0, \sigma, a)\)-TOV\((\bar{\sigma})\) shock waves is a two parameter family. Fixing \( \sigma = \frac{1}{3} \), the resulting one parameter family partially answers a claim given in C-E- and Taub \(1971\) by determining a subset of the self-similar pure radiation spacetimes that can be matched to TOV\((\bar{\sigma})\) to form a general relativistic shock wave.

Figure 1}

Figure 1 represents a side view of \((A, G, v)\)-space and depicts the most important features. The left and right bold curves represent the surfaces \( \{ \cdot \}_S = 0 \) and \( \{ \cdot \}_D = 0 \) respectively. These surfaces have no dependence on \( A \) and so remain the same in any constant \( A \) plane.
Unlike in Figure 1, the trajectories given in Figure 2 are terminated once they reach the $A = 1 - 2M(\sigma)$ plane. The leftmost trajectory overshoots the curve and rightmost trajectory undershoots it. The leftmost, centre and the rightmost trajectories are generated for:

$$a = 2.8$$
$$a = 2.58$$
$$a = 2.4$$

respectively. Therefore, the value of the acceleration parameter for the Friedmann-static pure radiation shock wave is approximated by:

$$a \approx 2.58$$

with the corresponding point of intersection approximated by:

$$\xi_0 \approx 0.706$$

It is conjectured by Temple that a vast primordial shock wave, with an FLRW$((0, \sigma, a)$ interior, could provide the mechanism for the accelerated expansion observed today without the need for dark energy. As a cosmological model in the Radiation Dominated Epoch, the Friedmann-static pure radiation shock wave has an associated cosmic acceleration many orders of magnitude larger than what is observed today, and observational data suggests that cosmic acceleration has only increased since the Radiation Dominated Epoch. This rules out the Friedmann-static pure radiation shock wave as a cosmological model, but does not rule out a shock wave cosmological model consisting of an interior FLRW$((0, \sigma, a)$ spacetime matched to a non-TOV$(\bar{\sigma})$ exterior spacetime, especially if the resulting acceleration parameter is close to one. For $a = 1$, one such shock wave is constructed by Smoller and Temple [2004]. It remains an open problem to generalise this construction to the $a \neq 1$ case.

**Definition 3.9.** The singular surface and sonic surface are defined in $(A, G, v)$-space by $\{\cdot\}_S = 0$ and $\{\cdot\}_D = 0$ respectively.

As a consequence of Lemma 3.6, the sonic surface serves as a convenient indicator for the Lax stability of a general relativistic shock wave with a TOV$(\bar{\sigma})$ exterior. This is particularly useful for numerical approximations, since if the intersection with the Rankine-Hugoniot curve is to the left of the sonic surface and to the right of the $G = \sqrt{\sigma}$ plane, the resulting shock wave is stable in the Lax sense. For $\sigma \neq \bar{\sigma}$, condition (52) is not automatically satisfied, since the Rankine-Hugoniot jump condition curve does not intersect the $v = 0$ plane at $G = \sqrt{\sigma}$, as Figure 3 demonstrates.
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Figure 3 depicts the singular and sonic surfaces as unbroken curves and three Rankine-Hugoniot curves by dashed curves, all for $\sigma = \frac{1}{3}$. The leftmost, centre and rightmost dashed curves correspond to:

\[
\begin{align*}
\sigma < \bar{\sigma} &= \frac{2}{3} \\
\sigma &= \bar{\sigma} = \frac{1}{3} \\
\sigma > \bar{\sigma} &= \frac{1}{6}
\end{align*}
\]

respectively. In the $\sigma < \bar{\sigma}$ case, the Rankine-Hugoniot curve always touches the singular surface at $(G, v) = (0, 0)$ and $(G, v) = (1, 1)$. In the $\sigma = \bar{\sigma}$ case, the Rankine-Hugoniot curve always touches the sonic surface at $(G, v) = (\sqrt{\sigma}, 0)$ and $(G, v) = (1, 1)$. In the $\sigma > \bar{\sigma}$ case, the Rankine-Hugoniot curve also touches the sonic surface at $(G, v) = (1, 1)$ and intersects it at:

\[
\begin{align*}
G &= \sqrt{\sigma(1 + \bar{\sigma})} + \sqrt{(\sigma - \bar{\sigma})(1 - \sigma \bar{\sigma})} \\
v &= \sqrt{\frac{1 - \sigma}{1 - \sigma \bar{\sigma}}}
\end{align*}
\]

Thus for a Friedmann-static shock wave to be unstable in the Lax sense, the solution trajectory must either hit the Rankine-Hugoniot curve before the $G = \sqrt{\sigma}$ plane or after passing through the sonic surface. Since conditions \[52\] and \[53\] are always satisfied for $\sigma = \bar{\sigma}$, then the Friedmann-static shock waves for which $\sigma = \bar{\sigma}$ are always stable in the Lax sense, as the following theorem summarises.

**Theorem 3.10.** Let $(A, G, v)$ denote a spherically symmetric self-similar solution to the Einstein field equations with a perfect fluid source and equation of state $p = \sigma \rho$. If there exists a $\xi_0 > 0$ such that $(A, G, v)$ can be matched to TOV($\bar{\sigma}$) to form a shock wave solution with a subluminal shock speed, then the Lax characteristic conditions are satisfied if:

1. $\sigma = \bar{\sigma}$ or
2. $\sigma < \bar{\sigma}$ and $G(\xi_0) > \sqrt{\sigma}$ or
3. $\sigma > \bar{\sigma}$ and $\{ \cdot \}_D(\xi_0) < 0$.

**Proof.** This is an immediate consequence of Lemma 3.6 and the discussion proceeding Definition 3.9.

**4 Formal Existence**

In the previous subsection, the Friedmann-static pure radiation shock wave is constructed numerically. This section provides a formal proof of this construction. We know from Proposition 2.7 that FLRW(0, $\sigma$, 1) solutions have a certain structure that allow us to determine if and when the solution trajectory crosses the singular or sonic surfaces. However, even though FLRW(0, $\sigma$, $\alpha$) solutions can be expected to behave similar to FLRW(0, $\sigma$, 1) solutions for $\alpha \approx 1$, there is no guarantee that they remain similar as $\xi$ increases, or for larger perturbations of $\alpha$. We know from Figure 2 that the FLRW(0, $\frac{1}{3}$, 2.4) trajectory differs significantly from the FLRW(0, $\frac{1}{3}$, 1) trajectory, in particular, the FLRW(0, $\frac{1}{3}$, 2.4)
trajectory encounters a singularity in equation (7) by hitting the sonic surface. The following lemma helps to predict the behaviour of FLRW\((0, \sigma, a)\) trajectories.

**Lemma 4.1.** Let \(0 < \sigma < 1\) and \(a > 0\). Then so long as FLRW\((0, \sigma, a)\) satisfies:

\[
A > 1 - 2M(\sigma)
\]

it also satisfies:

\[
\{\cdot\}_D < 0
\]

\[
A' < 0
\]

\[
G' > 0
\]

\[
v > 0
\]

\[
\{\cdot\}_S > 0
\]

Proof. Note from Figure 1 that \(\{\cdot\}_D < 0\) implies the trajectory remains to the left of the sonic surface and \(\{\cdot\}_S > 0\) implies that the trajectory remains below the singular surface. The monotonicity of \(A\) and \(G\) implies that the trajectory advances to the right whilst simultaneously approaching the \(A = 1 - 2M(\sigma)\) surface. Now because the FLRW\((0, \sigma, a)\) trajectory begins by satisfying inequalities (55)- (58), it is sufficient to show that each one of the four inequalities is implied by the other three. In this light, assume \(A < 1\) implies inequality (55). For inequality (56), assume \(v > 0\) and \(\{\cdot\}_S > 0\) and note that \(\{\cdot\}_S > 0\) and \(\{\cdot\}_D < 0\) imply \(v < 1\). Given these constraints, equation (6) implies:

\[
\xi \frac{dG}{d\xi} = -G \left[ \frac{1 - A}{A} \right] \frac{(3 + 3\sigma)(1 + v^2)G - 2G - 2\{\cdot\}_S}{2\{\cdot\}_S} - 1
\]

\[
= G \left[ 1 - \frac{1 - A}{A} \right] \frac{(3 + 3\sigma)(1 + v^2)G - (6 + 6\sigma)v}{(6 + 6\sigma v^2)G - (6 + 6\sigma)v}
\]

\[
> G \left[ 1 - \frac{1 - A}{A} \right]
\]

\[
> 0
\]

with the last line following from (54) and the semi-initial condition \(G > 0\). Now it is sufficient to demonstrate inequality (57) in the interval \(0 < G < \sqrt{\sigma}\), since the sonic surface intersects the \(v = 0\) plane at \(G = \sqrt{\sigma}\) and we are assuming that the trajectory stays off the sonic surface. In this light, assume \(A' < 0\), \(G' > 0\) and \(\{\cdot\}_S > 0\) and note that \(A' < 0\) implies \(A < 1\) and \(G' > 0\) implies \(G > 0\). By equation (7), the sign of \(v^2\) on the plane \(v = 0\) in the region bounded by \(1 - 2M(\sigma) < A < 1\) and \(0 < G < \sqrt{\sigma}\) is strictly positive, since:

\[
\xi \frac{dv}{d\xi} = -\left( \frac{1 - \nu^2}{2\{\cdot\}_D} \right) \left[ 3\sigma\{\cdot\}_S + \left( \frac{1 - A}{A} \right) \frac{(3 + 3\sigma)^2\{\cdot\}_S}{4\{\cdot\}_S} \right]
\]

\[
= \left( \frac{2G}{3(3 + 3\sigma)(\sigma - G^2)} \right) \left[ 9\sigma - \frac{(3 + 3\sigma)^2}{4} \left( \frac{1 - A}{A} \right) \right]
\]

\[
> 0
\]

Thus any trajectory that begins above the \(v = 0\) plane remains above the plane. Therefore the semi-initial condition \(v > 0\) then implies inequality (57). Note that this result still holds when \(1 - 2M(\tilde{\sigma}) < A < 1\) for \(0 < \tilde{\sigma} \leq \sigma < 1\), since:

\[
9\sigma - \frac{(3 + 3\sigma)^2}{4} \left( \frac{1 - A}{A} \right) > 9\sigma - \frac{(3 + 3\sigma)^2}{4} \left( \frac{2M(\tilde{\sigma})}{1 - 2M(\tilde{\sigma})} \right)
\]

\[
= 9\sigma - \frac{9\tilde{\sigma}(3 + 3\sigma)^2}{(3 + 3\tilde{\sigma})^2}
\]

\[
= (3 + 3\sigma)^2 \left( \frac{9\sigma}{(3 + 3\sigma)^2} - \frac{9\tilde{\sigma}}{(3 + 3\sigma)^2} \right)
\]

\[
\geq 0
\]

Finally, inequality (58) is demonstrated in a similar manner to inequality (57) by showing that trajectories stay away from the surface \(\{\cdot\}_S = mv\) for some \(0 < m < \frac{3}{\tilde{\sigma}}\). The upper bound for \(m\) ensures FLRW\((0, \sigma, a)\) trajectories initially
satisfy inequality (58). Now assume inequalities (55)-(57) and note that the inequalities additionally imply \(A < 1\) and \(G > 0\). Since \(\{ \cdot \}_S = mv\) is equivalent to:

\[
G = \frac{(3 + 3\sigma + m)v}{3 + 3\sigma v^2}
\]  
(59)

then by equation (6) and (59), we have:

\[
q_A(v; \sigma, m) = \xi \frac{d}{d\xi} \left( G - \frac{(3 + 3\sigma + m)v}{3 + 3\sigma v^2} \right) \Big|_{\{ \cdot \}_S = mv} = \left( \xi \frac{dG}{d\xi} - \frac{(3 + 3\sigma + m)(3 - 3\sigma v^2)}{3 + 3\sigma v^2} \xi \frac{dv}{d\xi} \right) \Big|_{\{ \cdot \}_S = mv} = \frac{(3 + 3\sigma + m)v}{3 + 3\sigma v^2} \left[ 1 - \left( \frac{1 - A}{A} \right) \frac{(3 + 3\sigma)(3 + 3\sigma + m)(1 + v^2) - 2(3 + 3\sigma v^2)}{2m(3 + 3\sigma v^2)} \right] + \frac{(3 + 3\sigma + m)(3 - 3\sigma v^2)}{(3 + 3\sigma v^2)^2} \left[ \frac{1 - v^2}{2\{ \cdot \}_B} \right] \frac{3\sigma mv + \left( \frac{1 - A}{A} \right) \frac{(3 + 3\sigma^2)\{ \cdot \}_N}{4mv} }{\{ \cdot \}_A + \{ \cdot \}_B + \{ \cdot \}_C} \\
\]

where:

\[
\{ \cdot \}_A = \frac{(3 + 3\sigma)(3 - 3\sigma - m)(1 - v^2)}{2m} \\
\{ \cdot \}_B = \frac{(3 + 3\sigma^2)(1 - v^2)(3 - 3\sigma v^2)\{ \cdot \}_N}{8mv^2\{ \cdot \}_D} \\
\{ \cdot \}_C = -(3 + 3\sigma)v^2
\]

The objective for this part is to find an \(m\) such that \(q_A(v; \sigma, m) > 0\) for all \(0 < \sigma < 1\) and \(0 < v < v_*\) for arbitrary \(v_* < 1\). Note that it is always possible to choose an \(m\) small enough to ensure \(v_* < v_I(\sigma, m)\), where \(v_I(\sigma, m)\) is the intersection of surfaces (59) and \(\{ \cdot \}_D = 0\), since:

\[
\lim_{m \to 0} v_I(\sigma, m) = 1
\]

Now even though it can be shown that \(\{ \cdot \}_A + \{ \cdot \}_B + \{ \cdot \}_C > 0\) for a certain interval of \(v\), it is easier to show \(\{ \cdot \}_A + \{ \cdot \}_B > 0\) for the whole interval \(0 < v < v_I\). This case since:

\[
\{ \cdot \}_A + \{ \cdot \}_B = \frac{(3 + 3\sigma)(1 - v^2)}{2m} \left[ 3 - 3\sigma - m + \frac{(3 + 3\sigma)(3 - 3\sigma v^2)\{ \cdot \}_N}{4v^2\{ \cdot \}_D} \right] = \frac{(3 + 3\sigma)(1 - v^2)}{8m(-\{ \cdot \}_D)v^2} \left[ 4(3 - 3\sigma - m)(-\{ \cdot \}_D)v^2 - (3 + 3\sigma)(3 - 3\sigma v^2)\{ \cdot \}_N \right] = \frac{3\sigma(3 + 3\sigma^2)(1 - v^2)(3 - 3v^2 + n)}{8(-\{ \cdot \}_D)(3 + 3\sigma v^2)^2} \left[ 3 - 3v^2 + \sigma(9 + n)v^2 - \sigma(9 + n\sigma)v^4 \right] > 0
\]

where \(m = n\sigma\) for some \(0 < n < \frac{3}{2}\). With \(\{ \cdot \}_A + \{ \cdot \}_B > 0\) and \(\{ \cdot \}_C < 0\), then for \(\frac{1}{2} < A < 1\) we have:

\[
\left( \frac{1 - A}{A} \right) \left( \{ \cdot \}_A + \{ \cdot \}_B + \{ \cdot \}_C \right) > \left( \frac{1 - A}{A} \right) \{ \cdot \}_C > \{ \cdot \}_C
\]

Thus for any \(0 < \sigma < 1\) and \(0 < v < v_*\):

\[
\lim_{m \to 0} q_A(v; \sigma, m) = \lim_{n \to 0} q_A(v; \sigma, n\sigma) > \lim_{n \to 0} \frac{(3 + 3\sigma + n\sigma)v}{(3 + 3\sigma v^2)^2} \left[ 3 + 3\sigma v^2 + \frac{3n\sigma^2(1 - v^2)(3 - 3\sigma v^2)}{2\{ \cdot \}_D} + \{ \cdot \}_C \right] = \lim_{n \to 0} \frac{(3 + 3\sigma + n\sigma)(1 - v^2)v}{(3 + 3\sigma v^2)^2} \left[ 3 - \frac{2n(3 + 3\sigma)^{-1}(3 - 3\sigma v^2)(3 + 3\sigma v^2)^2}{(3 - 3v^2 - n\sigma v^2)^2 - \sigma v^2(3 - 3v^2 + n)^2} \right] > 0
\]
Therefore, for any interval \(0 < v < v_\ast\) with \(v_\ast < 1\), there exists an \(0 < n < \frac{3}{2}\) such that the surface \(\{ \cdot \}_S = n\sigma v\) cannot be crossed. Now assume for contradiction that a trajectory crosses the \(\{ \cdot \}_S = 0\) surface. Because \(v_f(\sigma, 0) = 1\) and we assume \(\{ \cdot \}_D < 0\), the trajectory cannot cross the surface \(\{ \cdot \}_S = 0\) at \(v = 1\), so it must intersect at some point \(0 < v_\ast < 1\). Given that FLRW\((0, \sigma, a)\) satisfies \(\{ \cdot \}_S > n\sigma\) initially for any \(0 < n < \frac{3}{2}\) and we can pick a \(v_\ast\) such that \(v_\ast < v_\ast < 1\), we know that the surface \(\{ \cdot \}_S = n\sigma\) cannot be crossed in the interval \(0 < v < v_\ast\), which is a contradiction. Thus under our assumptions, FLRW\((0, \sigma, a)\) satisfies inequality (58) and completes the proof.

With Lemma 4.1 in place, we are now in a position to prove the main result.

**Theorem 4.2.** There exists an \(a > 1\) such that FLRW\((0, \frac{1}{3}, a)\) can be matched to TOV\((\frac{1}{3})\) to form a general relativistic shock wave that satisfies the Lax characteristic conditions.

**Proof.** By Lemma 3.2 and Definition 3.8, for FLRW\((0, \frac{1}{3}, a)\) to match with TOV\((\frac{1}{3})\) to form a general relativistic shock wave, then FLRW\((0, \frac{1}{3}, a)\) must satisfy:

\[
A(\xi_0) = \frac{4}{7}
\]

\[
v(\xi_0) = \Gamma_{RH} \left( G(\xi_0); \frac{1}{3}, \frac{1}{3} \right)
\]

for some positive constant \(\xi_0\). We know from Theorem 3.3 that FLRW\((0, \frac{1}{3}, 1)\) cannot form a general relativistic shock wave with TOV\((\frac{1}{3})\), since \(\sigma = \bar{\sigma} = \frac{1}{3}\) is not a solution of \(\bar{\sigma} = H(\sigma)\). Instead, when the FLRW\((0, \frac{1}{3}, 1)\) trajectory hits the \(A = \frac{4}{7}\) plane, then:

\[
v(\xi_0) < \Gamma_{RH} \left( G(\xi_0); \frac{1}{3}, \frac{1}{3} \right)
\]

That is, the FLRW\((0, \frac{1}{3}, 1)\) trajectory passes under the Rankine-Hugoniot curve. Note that the explicitly known FLRW\((0, \frac{1}{3}, 1)\) solution is able to cross the sonic surface without becoming singular due to the cancellation of \(\{ \cdot \}_D\) in equation (7) when on the sonic surface. General FLRW\((0, \sigma, a)\) solutions typically become singular at the point of intersection with the sonic surface. Now suppose that there exists a \(b > 1\) such that the FLRW\((0, \frac{1}{3}, b)\) trajectory hits the plane \(A = \frac{4}{7}\) with:

\[
v(\xi_0) > \Gamma_{RH} \left( G(\xi_0); \frac{1}{3}, \frac{1}{3} \right)
\]

then providing the transition of the FLRW\((0, \frac{1}{3}, 1)\) trajectory to the FLRW\((0, \frac{1}{3}, b)\) trajectory crosses the Rankine-Hugoniot curve, there exists an \(1 < a < b\) such that (60) and (61) are satisfied. An example of this process is demonstrated numerically in Figure 2. Lemma 4.1 establishes the fact that if the FLRW\((0, \frac{1}{3}, a)\) trajectory does not cross the sonic surface, then it must eventually hit the \(A = \frac{4}{7}\) plane. The continuous dependence of FLRW\((0, \frac{1}{3}, a)\) on the parameter \(a\) means that there is a continuous transition from FLRW\((0, \frac{1}{3}, \frac{1}{5})\) to FLRW\((0, \frac{1}{3}, b)\), at least up until the trajectory hits the \(A = \frac{4}{7}\) plane or hits the sonic surface. This continuous transition, along with Lemma 4.1 guarantees the crossing of the Rankine-Hugoniot curve in the \(\sigma = \bar{\sigma} = \frac{1}{3}\) case, since the transition from hitting the sonic surface to hitting the \(A = \frac{4}{7}\) plane occurs on the intersection of the sonic surface with the \(A = \frac{4}{7}\) plane, which lies under the Rankine-Hugoniot curve. Thus it is sufficient to demonstrate the formal existence of an FLRW\((0, \frac{1}{3}, b)\) solution that satisfies (60) and (62). We know from Figure 2 that a numerical approximation of the FLRW\((0, \frac{1}{3}, \frac{14}{5})\) trajectory passes above the Rankine-Hugoniot curve, so existence is considered for \(b = \frac{14}{5}\).

Because the FLRW\((0, \frac{1}{3}, a)\) trajectories originate from the fixed point of an unstable manifold, the vector field generated by the system of equations (3)-(7) points toward the FLRW\((0, \frac{1}{3}, a)\) trajectories when moving away from the fixed point. This fact allows for the construction of a trapping region around the trajectory and this is how the FLRW\((0, \frac{1}{3}, \frac{14}{5})\) trajectory is shown to overshoot the Rankine-Hugoniot curve. Since Lemma 4.1 establishes the monotonicity of \(G\) as a
function of $\xi$, $A$ and $v$ can be considered as functions of $G$, with equations (5)–(7) becoming:

$$
\frac{dA}{dG} = -\left(\frac{dG}{d\xi}\right)^{-1} \frac{(3 + 3\sigma)(1 - A)v}{\{\cdot\}_S} 
$$

$$
\frac{dG}{d\xi} = -G \left[ \frac{1 - A}{A} \right] \frac{(3 + 3\sigma)[(1 + v^2)G - 2v] - 1}{2\{\cdot\}_S} 
$$

$$
\frac{dv}{dG} = -\left(\frac{dG}{d\xi}\right)^{-1} \frac{1 - v^2}{2\{\cdot\}_D} \left[ 3\sigma\{\cdot\}_S + \left(1 - \frac{A}{A}\right) \frac{(3 + 3\sigma)^2\{\cdot\}_N}{4\{\cdot\}_S} \right] 
$$

In this sense, the trajectory of FLRW($0, \frac{1}{3}, \frac{14}{3}$) can be represented as $(A(G), v(G))$, with $G$ parameterising the progress of the trajectory towards the $A = \frac{4}{7}$ plane. This step provides a considerable simplification, since the trapping region now only needs to contain $A$ and $v$. The next step is to construct a trapping region using the Taylor polynomials of $A$ and $v$ about $G = 0$. In this light, define:

$$
P_{2N+1}(G) = \sum_{n=0}^{N} \frac{A^{(2n)}(0)}{(2n)!} G^{2n}
$$

$$
Q_{2N+1}(G) = \sum_{n=0}^{N} \frac{v^{(2n+1)}(0)}{(2n + 1)!} G^{2n+1}
$$

noting that $A$ and $v$ have even and odd expansions respectively. Furthermore, define:

$$
A_M(G) = P_{2N-1}(G) + M_A G^{2N}
$$

$$
A_m(G) = P_{2N-1}(G) + m_A G^{2N}
$$

$$
v_M(G) = Q_{2N-1}(G) + M_v G^{2N+1}
$$

$$
v_m(G) = Q_{2N-1}(G) + m_v G^{2N+1}
$$

where $M_A, m_A, M_v$ and $m_v$ are chosen so that:

$$
m_A < \frac{A^{(2N)}(0)}{(2N)!} < M_A
$$

$$
m_v < \frac{v^{(2N+1)}(0)}{(2N + 1)!} < M_v
$$

The functions $A_M$ and $A_m$ are used to bound $A$ from above and below respectively, with $v_M$ and $v_m$ providing analogous bounds for $v$. The objective is to show:

$$
v_m(G_0) > \Gamma_{RH} \left( G_0; \frac{1}{3}, \frac{1}{3} \right) 
$$

where $G_0$ is found implicitly through:

$$
A_M(G_0) = \frac{4}{7}
$$

This is so the lowest point of the Taylor trapping region of $v$ remains above the Rankine-Hugoniot curve for the most conservative value of $G$, which is given by the intersection of the highest point of the Taylor trapping region of $A$ with the $A = \frac{4}{7}$ plane. For large enough $N$, it is possible to find values for $M_A, m_A, M_v$ and $m_v$ such that (65) and (66) are satisfied and inequalities:

$$
A_m(G) < A < A_M(G)
$$

$$
v_m(G) < v < v_M(G)
$$

hold for $0 < G < G_0$. This can be done through extensive trial and error, using a numerical approximation of $A$ and $v$ as a guide. Note that the Taylor expansions of $A$ and $v$ converge quicker for smaller values of $a$, but larger values of $a$ allow for (65) to be more easily satisfied, this is why $a = \frac{14}{7}$ is chosen, as it provides a good compromise. In this light, and using a numerical approximation of $A$ and $v$ as a guide, it is found that $N = 16$ and the following values satisfy
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\[ M_A = (1 + 2^{-7}) A^{(32)}(0) \]
\[ m_A = 2^{-1} A^{(32)}(0) \]
\[ M_v = 2^{-1} v^{(33)}(0) \]
\[ m_v = 2^5 v^{(33)}(0) \]

noting that \( M_A \) and \( m_A \) are chosen in the knowledge that \( v^{(33)}(0) \) is negative. With \( M_A, m_A, M_v \) and \( m_v \) specified, the Taylor polynomials of \( A \) and \( v \) can be computed and \( A_M, A_m, v_M \) and \( v_m \) become known explicitly. The plots of these bounding functions are given in Figure 4.

![Figure 4](image)

Even at 33rd order, Figure 4 shows that \( v_M \) and \( v_m \) noticeably diverge after passing the Rankine-Hugoniot curve. This is due to the trajectory approaching the sonic surface, where the solution is likely to become singular, resulting in a slower convergence of the Taylor polynomials. With \( A_M \) known explicitly, equation (66) can be solved, at least approximately, to yield:

\[ G_0 \approx 0.601 \]

and this results in inequality (65) being satisfied, since \( v_m \) is also known explicitly. The final, and most difficult step, is to show that inequalities (67) and (68) hold in the interval \( 0 < G < G_0 \). To do this, the structure of equations (63) and (64) can be exploited, that is, it is possible to show:

\[ \frac{\partial}{\partial v} \frac{dA}{dG} < 0 \] (69)
\[ \frac{\partial}{\partial A} \frac{dv}{dG} > 0 \] (70)

within the region given by (67) and (68). Starting with (69), we have:

\[ \frac{\partial}{\partial v} \frac{dA}{dG} = -\frac{4(1 - A)v}{\{\} S} \frac{\partial}{\partial v} \left( \frac{dG}{d\xi} \right)^{-1} - \left( \frac{dG}{d\xi} \right)^{-1} \frac{\partial}{\partial v} \left( \{\} S \right) \]
\[ = -\frac{4(1 - A)G^2}{\{\} S} \left( \frac{dG}{d\xi} \right)^{-2} \left[ 4v^2(3 - v^2) \left( \frac{1 - A}{A} \right) + (3 - v^2)\{\} S - 2(1 - v^2)\{\} S \left( \frac{1 - A}{A} \right) \right] \]
\[ < 0 \]
which holds in the more general region described by \( \frac{2}{5} < A < 1, v > 0, \{ \cdot \}_S > 0 \) and \( \{ \cdot \}_D < 0 \). For (70) we have:

\[
\frac{\partial}{\partial A} \frac{dv}{dG} = - \left( \frac{1-v^2}{2\{ \cdot \}_D} \right) \left[ \{ \cdot \}_S + 4 \left( \frac{1-A}{A} \right) \{ \cdot \}_N \right] \frac{\partial}{\partial A} \left( \xi \frac{dG}{d\xi} \right)^{-1}
\]

\[
- \left( \frac{\partial}{\partial \xi} \right)^{-1} \left( \frac{1-v^2}{2\{ \cdot \}_D} \right) \frac{\partial}{\partial A} \left[ \{ \cdot \}_S + 4 \left( \frac{1-A}{A} \right) \{ \cdot \}_N \right]
\]

\[
\frac{G}{A^2\{ \cdot \}_S} \left( \frac{\partial}{\partial \xi} \right)^{-2} \left( \frac{1-v^2}{2\{ \cdot \}_D} \right) \left[ (2(1+v^2)G - 4v)\{ \cdot \}_S + 4\{ \cdot \}_N \right]
\]

> 0

which holds in the region described by \( v > 0, \{ \cdot \}_D < 0 \) and:

\[
(2(1+v^2)G - 4v)\{ \cdot \}_S + 4\{ \cdot \}_N < 0
\]

This region is slightly smaller than the region described by \( v > 0, \{ \cdot \}_D < 0 \) and \( \{ \cdot \}_S > 0 \), but includes the region given by (67) and (68) nonetheless. Now by construction, we know that (67) and (68) are satisfied in the interval \( 0 < G < G_r \) for some small \( G_r > 0 \), so to demonstrate (67) and (68) in the interval \( 0 < G < G_0 \), it is sufficient to demonstrate:

\[
\frac{d}{dG} (A_M - A) |_{A=A_m} \geq 0
\]

(71)

\[
\frac{d}{dG} (A - A_m) |_{A=A_m} \geq 0
\]

(72)

\[
\frac{d}{dG} (v_M - v) |_{v=v_m} \geq 0
\]

(73)

\[
\frac{d}{dG} (v - v_m) |_{v=v_m} \geq 0
\]

(74)

in the interval \( G_r \leq G < G_0 \). Note that the left hand sides of (71)-(74) are functions of \( A, v \) and \( G \), so (69) can be used to determine the most conservative value of \( v \) in (71) and (72), and (70) can be used to determine the most conservative value of \( A \) in (73) and (74). In particular, the most conservative choice out of \( v_M \) and \( v_m \), for (71) is \( v_m \) and the most conservative choice for (72) is \( v_M \), Likewise, the most conservative choice out of \( A_M \) and \( A_m \), for (73) is \( A_M \) and the most conservative choice for (74) is \( A_m \). This can be interpreted as remaining within the right wall of the trapping region implies remaining below the ceiling, and remaining below the ceiling implies remaining within the left wall and so on. Such an interpretation can be summarised as so:

\[
A < A_M \Rightarrow v < v_M
\]

\[
\uparrow \quad \downarrow
\]

\[
v > v_m \Leftarrow A > A_m
\]

Now using these conservative choices, the left hand sides of (71)-(74) become explicitly known functions of \( G \) and thus the interval for which they remain positive can be calculated, at least approximately.
Figure 5 depicts the plots of the left hand sides of (71)-(74). The diagram on the left depicts the plot of (71) as an unbroken line and (72) as a dashed line. Similarly, the diagram on the right depicts the plot of (73) as an unbroken line and (74) as a dashed line. The intervals for which (71)-(74) hold are given by:

\begin{align*}
0 &< G < G_1 \\
0 &< G < G_2 \\
0 &< G < G_3 \\
0 &< G < G_4
\end{align*}

respectively, where:

\begin{align*}
G_1 &> G_I \\
G_2 &\approx 0.627 \\
G_3 &> G_I \\
G_4 &\approx 0.612
\end{align*}

and $G_I$ is the value of $G$ for which $v_m$ intersects the sonic surface. Since (65) has already been established, then $G_0 < G_I$ and thus:

\[ G_0 < \min\{G_1, G_2, G_3, G_4\} \]

Therefore (67) and (68) hold in the interval $0 < G < G_0$ and since the Lax characteristic conditions follow by Theorem 3.10 the proof is complete.

5 Concluding Remarks

Given that it is possible to formally demonstrate the existence of a Friedmann-static pure radiation shock wave, the obvious follow-up question is whether it is possible to formally demonstrate the existence of the whole two parameter family of Friedmann-static shock waves. For a certain range of values of $\sigma$ and $\bar{\sigma}$ there is no reason to suspect that this would not be possible.

**Conjecture 5.1.** For $0 < \bar{\sigma} \leq \sigma \leq \frac{1}{3}$, there exists an $a > 0$ such that $\text{FLRW}(0, \sigma, a)$ can be matched to $\text{TOV}(\bar{\sigma})$ to form a general relativistic shock wave.

The resolution of this conjecture is the topic of future research. The continuous dependence of the solution trajectories on the parameters means that formal existence is all but guaranteed for $\bar{\sigma}, \sigma \approx \frac{1}{3}$ and $\bar{\sigma} \approx H(\sigma)$. Moreover, the existence proof in the pure radiation case is readily modified to demonstrate the formal existence for any fixed pair of $0 < \bar{\sigma} \leq \sigma \leq \frac{1}{3}$. The difficulty arises when generalising the proof from fixed parameter values to two dimensional parameter spaces, since conservative estimates need to be satisfied for all values of $\sigma$ and $\bar{\sigma}$ in such spaces. It is likely possible to construct such a proof by patching together many subproofs demonstrating existence in small two dimensional parameter spaces, although this method may be rather tedious.

Regarding the cosmological applications of Friedmann-static shock waves, it is found in Section 3 that a Friedmann-static pure radiation shock wave yields an acceleration parameter value of $a \approx 2.58$. For reference, the acceleration parameter that would be expected in the Radiation Dominated Epoch, according to Smoller and Temple [2012], would likely satisfy $a \approx 1$. In addition, Smoller and Temple [2004] demonstrate that Friedmann-static shock waves have shock positions that would already be observable, as the shock surface would lie within our current Hubble radius. Just one of these implications rules out Friedmann-static shock waves as cosmological models in the Radiation Dominated Epoch, but there remains an interesting modification to these shock waves that keeps Temple’s conjecture open.

Smoller and Temple [2004] demonstrate that it is possible to construct a shock wave beyond the Hubble radius by modelling the entire universe as a finite mass explosion within the Schwarzschild radius of a time-reversed black hole. Such a shock wave would consist of an asymptotically Friedmann spacetime on the interior and a modification of the TOV solution on the exterior. This modification is not known explicitly but incorporates the swapping of the temporal and radial variables in the metric to account for being within the Schwarzschild radius of a black hole.

The possibility remains to construct general relativistic shock waves in which the shock surface lies beyond the Hubble radius and determine the associated rate of expansion. If the predicted rate of expansion lies within current estimates then such a model offers a mathematically independent derivation for the cosmic acceleration observed today, without the need for dark energy.
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