Index Theorem in Finite Noncommutative Geometry

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Index theorem is formulated in noncommutative geometry with finite degrees of freedom by using Ginsparg-Wilson relation. It is extended to the case where the gauge symmetry is spontaneously broken. Dynamical analysis about topological aspects in gauge theory is also shown.

§1. Introduction

Noncommutative (NC) geometry is interesting since it is related to string theories and matrix models,\textsuperscript{1)–3)} and it may capture some nature of quantum gravity. It is also a candidate for a new regularization of quantum field theory.\textsuperscript{4)} Study about topological aspects of gauge theory on it is important since compactification of extra dimensions with nontrivial index in string theory can realize chiral gauge theory in our spacetime. Ultimately, we hope to realize such a mechanism dynamically, for instance, in IIB matrix model\textsuperscript{5)} where spacetime structure has been studied intensively.\textsuperscript{6), 7)} Unusual properties of NC geometry may also provide a solution of strong CP problem and baryon asymmetry of the universe.

Index theorem plays a key role in these studies. While it can be proved in theories with infinite degrees of freedom,\textsuperscript{8), 9)} it becomes a nontrivial issue in finite cases. This problem was solved in lattice gauge theory by using Ginsparg-Wilson (GW) relation,\textsuperscript{10)–13)} and this idea has been successfully extended to the NC geometry. We have provided a general prescription to construct a GW Dirac operator with coupling to non-vanishing gauge field backgrounds on general finite NC geometries.\textsuperscript{14)} Owing to the GW relation, an index theorem can be proved even for finite NC geometries. The index takes only integer values by construction, and it is shown to become the corresponding topological charge as the number of degrees of freedom is properly taken to infinity. While explicit construction has been provided for the fuzzy 2-sphere\textsuperscript{14), 15)} and for the NC torus,\textsuperscript{16), 17)} it is possible for other cases, which will be reported in future publication.

As a topologically nontrivial configuration on the fuzzy 2-sphere, we constructed 't Hooft-Polyakov (TP) monopole configuration.\textsuperscript{18), 19)} We further presented a mechanism for dynamical generation of a nontrivial index, by showing that the TP monopole configurations are stabler than the topologically trivial sector in the Yang-Mills-Chern-Simons matrix model.\textsuperscript{20)} However, in order to obtain non-zero indices for these configurations, one needs to introduce a projection operator in the definition of the index. We gave an interpretation for the projection operator, and extended the index theorem to general configurations which do not necessarily satisfy the equation of motion.\textsuperscript{21)} Since the $U(2)$ gauge theory on the fuzzy sphere
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is generally broken down to $U(1) \times U(1)$ gauge theory through Higgs mechanism, this generalization shows that the configuration space on the fuzzy sphere can be classified into the topological sectors. We take this subject in section 2. In section 3 we show dynamical analysis about topological aspects in gauge theory on the NC torus.

§2. Index theorem in spontaneously symmetry-broken gauge theory on the fuzzy 2-sphere

We first give an interpretation for the projection operator. TP monopole configuration breaks the $SU(2)$ gauge symmetry down to $U(1)$, and the matter field in the fundamental representation contains two components, corresponding to $+1/2$ and $-1/2$ electric charge of the unbroken $U(1)$ gauge group. Since these two components cancel the index, we need to introduce the projection operator to prevent the cancellation.

We thus generalize the electric charge operator to

$$T' = \frac{(A_i)^2 - \frac{n^2 - 1}{4}}{\sqrt{(A_i)^2 - \frac{n^2 - 1}{4}}}.$$  

This definition is valid for general configurations $A_i = L_i + \rho a_i$ unless the denominator has zero-modes. Here $L_i$ are $n$-dimensional representation of $SU(2)$ algebra and correspond to coordinates of the fuzzy 2-sphere. $\rho$ is sphere radius. $a_i$ are gauge field in three dimensions, and contain gauge field and scalar field on the sphere.

We next define modified chirality operators and GW Dirac operator as

$$\Gamma' = \frac{\{T', \Gamma^R\}}{2} = T' \Gamma^R, \quad \hat{\Gamma}' = \frac{\{T', \hat{\Gamma}\}}{\sqrt{T' \Gamma'^2}}, \quad D'_{GW} = -\frac{n}{2} \Gamma'(1 - \Gamma' \hat{\Gamma}').$$  

For details, please refer to ref.21) These operators are weighted by the electric charge operator $T'$, which prevent the cancellation of the index. By the definition, the GW relation

$$\Gamma' D'_{GW} + D'_{GW} \hat{\Gamma}' = 0$$  

is satisfied, and thus an index theorem

$$\frac{1}{2} \text{index}(D'_{GW}) = \frac{1}{4} Tr[\Gamma' + \hat{\Gamma}']$$  

can be proved. In the commutative limit, where one takes $n$ to infinity and the noncommutativity parameter $\alpha$ to zero simultaneously fixing $\rho \sim \alpha n$, it turns out that

$$\frac{1}{4} Tr[\Gamma' + \hat{\Gamma}'] - \frac{\rho^2}{8\pi} \int_{S^2} d\Omega \epsilon_{ijk} n_i \left( \phi'^{\alpha a} F_{jk}^a - \epsilon_{abc} \phi'^{\alpha a} (D_j \phi'^b)(D_k \phi'^c) \right),$$  

where $n_i$ are a unit vector in the normal direction of the sphere, and $\phi'$ is a normalized scalar field. This is precisely the topological charge for the unbroken $U(1)$ gauge symmetry given by ’t Hooft.22)
This index theorem is valid if the gauge symmetry is spontaneously broken to $U(1)$, that is, when the scalar field takes non-vanishing vacuum expectation values on any points of the sphere. This is assured by the condition $[(A_i)^2 - \frac{n^2-1}{4}]^2 \sim O(n^2)$, which means that all of the eigenvalues are of order $n^2$. Smaller eigenvalues may invalidate the definition of the index, while larger eigenvalues may alter the structure of sphere. Since this condition has both upper and lower bounds, it gives an extension of the admissibility condition in the lattice gauge theory.\(^{23}\)

Since the formulation is valid for general configurations, the configuration space can be classified into the topological sectors.

§3. **Dynamical analysis about topological aspects in gauge theory on the NC torus**

It is interesting to study opposite cases where the projection operator is not necessary. It is also important to investigate dynamics of topological properties in gauge theory for the aim of the studies mentioned at the beginning in section 1.

We studied these problems using a lattice formulation of gauge theory on the NC torus\(^{24}\) based on the twisted reduced model.\(^{25},^{26}\) For the 2 dimensional $U(1)$ case, general classical solutions are known.\(^{27}\) We computed the index of the GW Dirac operator for these classical solutions and compared the results with the topological charge which is obtained as a naive discretization of the 1st Chern character.\(^{28}\) The two quantities agree when the action is small, but they take only multiple integer values of $N$. $N$ is the size of the matrices, and corresponds to the size of the 2d lattice. The action for these configurations is of order $\beta$, where $\beta \sim 1/g^2$ is the bare coupling constant. By interpolating the classical solutions, we constructed explicit configurations for which the index is of order 1, but the action becomes of order $\beta N$.

We further performed Monte Carlo simulation and obtained the probability distribution of the index $\nu$.\(^{29}\) In the strong coupling region, the distribution has a form of Gaussian with a width of order $N$, which confirms the existence of $\nu \neq 0$ configurations. Some examples of such configurations are obtained in the above interpolation, and reported in ref.\(^{30}\) Surprisingly, it turns out that the distribution of $\nu$ is asymmetric under $\nu \mapsto -\nu$, which reflects the parity violation of NC geometry.

In the weak coupling region, however, the probability for $\nu \neq 0$ decreases rapidly for increasing $N$ and for increasing $\beta$. This is consistent with the above analysis with the classical solutions. In the continuum limit, we have to send $N$ and $\beta$ to infinity simultaneously fixing the ratio $\beta/N$, and thus the distribution approaches to the Kronecker delta $\delta_{\nu,0}$. This result is consistent with the instanton calculation in the continuum theory.\(^{31}\) However, it differs drastically from the results in the commutative case obtained from the lattice simulation,\(^{32}\) where the distribution is Gaussian with the width of the physical extent of the space $N/\sqrt{\beta} \sim aN$. $a$ is lattice spacing. Intuitively, this situation can be understood as follows. In the commutative lattice case, singular configurations contribute to $\nu \neq 0$ sectors, which are not realized in the NC geometry by some smoothing effect.

This property may provide a solution to the strong CP problem, though some
care must be taken when we define the $\theta$ vacuum. As another property of NC geometry, we found that in general the probability distribution of $\nu$ becomes asymmetric under $\nu \mapsto -\nu$. One can also twist the boundary condition to make a topologically nontrivial sector dominate in the continuum limit.\textsuperscript{33} We expect that these unusual properties of NC geometry may provide a dynamical mechanism for realizing chiral fermions in string theory compactifications, or a mechanism for generating baryon asymmetry of the universe. Related works are performed using fuzzy spheres in the extra dimensions.\textsuperscript{20,34}

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