Geometric model for the electron spin correlation

Ana María Cetto

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Instituto de Física, Universidad Nacional Autónoma de México, Mexico

Abstract

The quantum formula for the spin correlation of the bipartite singlet spin state, $C_Q(a, b)$, is derived on the basis of a probability distribution $\rho(\phi)$ that is generic, i.e., independent of $(a, b)$. In line with a previous result obtained within the framework of the quantum formalism, the probability space is partitioned according to the sign of the product $A = \alpha \beta$ of the individual spin projections $\alpha$ and $\beta$ onto $a$ and $b$. A specific partitioning and a corresponding set of realizations $\{\phi\}$ are associated with every measurement setting $(a, b)$; this precludes the transfer of $\alpha$ or $\beta$ from $C_Q(a, b)$ to $C_Q(a, b')$, for $b' \neq b$. A geometric model that reproduces the spin correlation serves to validate our approach, giving a concrete meaning to the quantum result in terms of a (local random variable) probability distribution.

1 Introduction

In a recent paper [1] an analysis was made of the spin projection operator correlation function $C_Q(a, b) = \langle (\hat{\sigma} \cdot a) (\hat{\sigma} \cdot b) \rangle$ for the bipartite singlet spin state. The analysis, conducted strictly within the framework of the quantum formalism, led to an unequivocal probabilistic reading. Specifically, the calculation of $C_Q(a, b)$ was shown to entail a partitioning of the probability space, which is dependent on the directions $(a, b)$. This result is the outcome of a purely theoretical analysis; however, it can be readily translated to the laboratory language, meaning that the series of values ($\pm 1$) obtained for the projections $\alpha$ and $\beta$ leading to the experimental correlation $C(a, b)$, cannot be mixed or combined with those obtained for $\alpha$ and $\beta'$ and leading to $C(a, b')$, if $b' \neq b$.

In the present paper we elaborate on the previous results and take them further to construct the quantum formula for the spin correlation on the basis of a probability distribution $\rho(\phi)$. The distribution function is independent of $(a, b)$; the dependence on the directions resides exclusively in the subdivision of the entire probability space into four mutually orthogonal subspaces and
the realization of the set of variables \( \{ \phi \} \) specific to this subdivision. In other words, for a given pair \((a, b)\), the entire set \( \{ \phi \} \) is formed by four complementary subsets \( \{ \phi \}_{ab}^k \), leading respectively to the eigenvalues \( A_k = \alpha_k \beta_k \), with \( \alpha_k, \beta_k = \pm 1 \). Further, given the degeneracy of eigenvalues \( A_k \), the four probability subspaces can be merged pairwise to form two mutually exclusive subsets \( \{ \phi \}_{\pm ab} \), corresponding to \( A = \pm 1 \). A distribution function \( \rho(\phi) \) that reproduces the quantum result for \( C_Q(a, b) \) is obtained on this basis, and its application is illustrated by means of a specific geometric model.

The paper is organized as follows. Section 2 contains a brief introduction to the quantum description of the bipartite singlet state, followed by a discussion of the disaggregation of the correlation \( C_Q(a, b) \) on the basis of the spin projection eigenfunctions associated with the directions \((a, b)\). The correlation operator is thus expressed in terms of the projection operators in the product space of the individual spin spaces. In Section 3 a generic distribution function \( \rho(\phi) \) is obtained that reproduces the quantum spin correlation for the entangled state. A simple geometric model for the spin orientations serves to give concrete meaning to the quantum result.

2 Quantum description of the bipartite singlet spin correlation

We consider a system of two 1/2–spin particles in the (entangled) singlet state

\[
|\Psi^0\rangle = \frac{1}{\sqrt{2}} (|+r\rangle |-r\rangle - |-r\rangle |+r\rangle),
\]

in terms of the simplified (standard) notation \(|\phi\rangle|\chi\rangle = |\phi\rangle \otimes |\chi\rangle\), with \(|\phi\rangle\) a vector in the Hilbert space of particle 1, and \(|\chi\rangle\) a vector in the Hilbert space of particle 2. The individual state vectors

\[
|+r\rangle = \cos \frac{\theta_r}{2} |+z\rangle + e^{i\varphi_r} \sin \frac{\theta_r}{2} |-z\rangle, \quad (2a)
\]

\[
|-r\rangle = -e^{-i\varphi_r} \sin \frac{\theta_r}{2} |+z\rangle + \cos \frac{\theta_r}{2} |-z\rangle, \quad (2b)
\]

form an orthogonal basis, with \( 0 \leq \theta_r \leq \pi \) and \( 0 \leq \varphi_r \leq 2\pi \), \( \theta_r \) and \( \varphi_r \) being the zenithal and azimuthal angles that define the Bloch vector \( r = i \sin \theta_r \cos \varphi_r + j \sin \theta_r \sin \varphi_r + k \cos \theta_r \) (see, e.g., Ref. [2]).

In Eq. (1) the direction of \( r \) is arbitrary, since the singlet state is spherically symmetric. The projection of the first spin operator along an arbitrary direction \( a \) is described by \((\hat{\sigma} \cdot a) \otimes I\), and the projection of the second spin operator along \( b \) is described by \( I \otimes (\hat{\sigma} \cdot b)\). With the purpose of carrying out a detailed calculation of the correlation

\[
C_Q(a, b) = \langle \Psi^0 | (\hat{\sigma} \cdot a) \otimes (\hat{\sigma} \cdot b) |\Psi^0\rangle,
\]

(3)
we use Eqs. (2) to obtain

\begin{equation}
\langle \pm r | \hat{\sigma} \cdot a | \pm r \rangle = \pm r \cdot a = \pm \cos \theta ra
\end{equation}

and

\begin{equation}
\langle -r | \hat{\sigma} \cdot a | +r \rangle = \langle +r | \hat{\sigma} \cdot a | -r \rangle^* = e^{i\varphi} (\theta + i \varphi) \cdot a,
\end{equation}

whence

\begin{equation}
| \langle \mp r | \hat{\sigma} \cdot a | \pm r \rangle |ig| = | r \times a |.
\end{equation}

In terms of the complete set of vectors in the composite Hilbert space,

\begin{align*}
| \Psi^1 \rangle &= | +r \rangle | -r \rangle, & | \Psi^2 \rangle &= | -r \rangle | +r \rangle, \\
| \Psi^3 \rangle &= | +r \rangle | +r \rangle, & | \Psi^4 \rangle &= | -r \rangle | -r \rangle,
\end{align*}

we get, with the help of Eqs. (4),

\begin{equation}
C_Q(a, b) = \langle \Psi^0 | (\hat{\sigma} \cdot a) \left( \sum_{k=1}^{4} | \Psi^k \rangle \langle \Psi^k | \right) (\hat{\sigma} \cdot b) | \Psi^0 \rangle = \sum_{k=1}^{4} F_k,
\end{equation}

with

\begin{align*}
F_1 &= -\frac{1}{2} (r \cdot a)(r \cdot b) = F_2, \\
F_3 &= -\frac{1}{2} [(r \times a) \cdot (r \times b) - i r \cdot (a \times b)] = F_4^*.
\end{align*}

These equations are greatly simplified by making \( r \) lie on the plane formed by \( a \) and \( b \), i.e., \( \varphi_r = \varphi_a = \varphi_b = 0 \); with \( \theta ra = \theta_r - \theta_a \) and \( \theta rb = \theta_r - \theta_b \) they become

\begin{align*}
F_1 &= F_2 = -\frac{1}{2} \cos \theta ra \cos \theta rb, \\
F_3 &= F_4 = -\frac{1}{2} \sin \theta ra \sin \theta rb.
\end{align*}

The sum of the four terms gives of course \( C_Q(a, b) = -a \cdot b \). The fact that the result depends only on the angle formed by \( a \) and \( b \) is due to the spherical symmetry of the singlet spin state. Looking at the terms separately, however, we observe that \( F_1 + F_2 \), involving intermediate states (\( | \Psi^1 \rangle \) and \( | \Psi^2 \rangle \)) of antiparallel spins (along the arbitrary direction \( r \)), gives the product of the projections of \( a \) and \( b \) onto \( r \), whilst \( F_3 + F_4 \), involving intermediate states (\( | \Psi^3 \rangle \) and \( | \Psi^4 \rangle \)) of parallel spins, contains their vector products. In other words, the two spin projection operators \( \hat{\sigma} \cdot a \), \( \hat{\sigma} \cdot b \) establish a correlation not just through the intermediate states representing antiparallel spins—as one might naïvely suppose for the entangled spin-zero state—but also through the intermediate states of parallel spins, \( | +r \rangle | +r \rangle \) and \( | -r \rangle | -r \rangle \).

We now propose an alternative calculation, by resorting to the individual eigenvalue equations

\begin{align*}
\hat{\sigma} \cdot a | \pm a \rangle &= \alpha | \pm a \rangle, & \alpha &= \pm 1, \\
\hat{\sigma} \cdot b | \pm b \rangle &= \beta | \pm b \rangle, & \beta &= \pm 1.
\end{align*}
to construct a new orthonormal basis for the bipartite system:

\[ |\phi^1\rangle_{ab} = |+a\rangle |-b\rangle, \quad |\phi^2\rangle_{ab} = |a\rangle |+b\rangle, \]
\[ |\phi^3\rangle_{ab} = |+a\rangle |+b\rangle, \quad |\phi^4\rangle_{ab} = |a\rangle |-b\rangle, \quad (10) \]

and write as before

\[ C_Q(a, b) = \langle \Psi^0 | (\hat{\sigma} \cdot a) \left( \sum_{k=1}^{4} |\phi^k\rangle_{ab} \langle \phi^k|_{ab} \right) (\hat{\sigma} \cdot b) |\Psi^0 \rangle. \quad (11) \]

In view of (9) and (10), the terms that contribute to \( C_Q \) are

\[ (\hat{\sigma} \cdot a) \otimes I |\phi^k\rangle_{ab} \langle \phi^k|_{ab} \otimes (\hat{\sigma} \cdot b) = A_k |\phi^k\rangle_{ab} \langle \phi^k|_{ab}, \quad (12) \]

where

\[ A_k = \alpha_k \beta_k \quad (13) \]

are the eigenvalues of the spin correlation operator

\[ \hat{C}_Q(a, b) = (\hat{\sigma} \cdot a \otimes \hat{\sigma} \cdot b) \quad (14) \]

corresponding to the bipartite states \( |\phi^k\rangle_{ab} \) given according to Eqs. (9) and (10) by

\[ A_1 = A_2 = -1 \equiv A^-, \quad A_3 = A_4 = +1 \equiv A^+, \quad (15) \]

and \( \alpha_k, \beta_k \) are the individual eigenvalues corresponding to \( |\phi^k\rangle_{ab} \). Thus from Eqs. (11) and (12) we get

\[ C_Q(a, b) = \sum_{k=1}^{4} A_k(a, b) C_k(a, b), \quad (16) \]

with

\[ C_k(a, b) = |\langle \phi^k|_{ab} |\Psi^0 \rangle|^2. \quad (17) \]

It is clear from this expression that the coefficients \( C_k \) are nonnegative and add to give

\[ \sum_{k=1}^{4} C_k(a, b) = \sum_{k=1}^{4} |\langle \phi^k|_{ab} |\Psi^0 \rangle|^2 = 1. \quad (18) \]

Notice that the operators

\[ \hat{P}^k(a, b) = |\phi^k\rangle_{ab} \langle \phi^k|_{ab} \quad (19) \]

appearing in Eqs. (11), (12) and (18) are the projection operators in the product space of the individual spin spaces, \( S = S_1 \otimes S_2 \), with respective eigenvalues given by \( A_k \). Equation (19) is therefore the appropriate spectral decomposition
of the spin correlation. In terms of the projection operators, the spin correlation operator (14) takes the form

\[ \hat{C}_Q(a, b) = \sum_{k=1}^{4} A_k(a, b) \hat{P}_k(a, b) \equiv \sum_{k=1}^{4} \hat{C}_k(a, b), \]  

(20)
each term in the sum projecting onto one and only one of the four mutually orthogonal subspaces \( U_k(a, b) \) that add to form space \( S \).

In operational terms \([4]\), Ch. 2), this means that the result of every (joint) measurement falls under one and only one of these (eigen)subspaces. Further, the coefficient \( C_k \), which in \([16]\) appears as the relative weight of eigenvalue \( A_k \) contributing to the expectation value \( C_Q(a, b) \), is identified with the probability measure, i. e., the probability of obtaining \( A_k \) as the result of a measurement, in accordance with the Born rule \([5]\), Ch. 1). We have thus completed the elements used to describe in quantum theory the measurement statistics obtained through experiment.

Let us now consider the observable \( C_Q(a, b') \) with \( b' \neq b \). The corresponding projection operators are

\[ \hat{P}_k(a, b') = |\phi^k\rangle_{ab'} \langle \phi^k|_{ab'}, \]  

(22)
where \( |\phi^k\rangle_{ab'} \) is defined as in \([10]\) with \( b \) replaced by \( b' \). Therefore, instead of the partitioning of \( S \) given by (21) the spectral decomposition involves now the partitioning into four mutually orthogonal subspaces \( U_k(a, b') \), such that every (joint) measurement falls under one and only one of these subspaces. In other words, the probability subspaces are specific to the observable being measured, i. e., to the measurement setting.

This assigns a clear meaning to the term measurement dependence that has been introduced in the context of the Bell-type inequalities (see e. g. \([6]\)) according to the present discussion, it refers to the dependence of the probability subspaces on the measurement setting.

3 Probability distribution for the bipartite singlet spin state

In order to arrive at a probability distribution for our problem we need to calculate the coefficients \( C_k \) given by (17). To simplify the calculation one may, without loss of generality, select the vector \( r \) on the plane defined by the directions \( a \) and \( b \), so that Eqs. (2) reduce to

\[ |+r\rangle = \cos \frac{\theta_r}{2} |+_z\rangle + \sin \frac{\theta_r}{2} |-_z\rangle, \quad |-_r\rangle = -\sin \frac{\theta_r}{2} |+_z\rangle + \cos \frac{\theta_r}{2} |-_z\rangle. \]  

(23)
This gives, using Eqs. (1) and (10), with $\theta_{ab} = \theta_a - \theta_b$,

$$C_1(a, b) = C_2(a, b) = \frac{1}{2} \cos^2 \frac{\theta_{ab}}{2}, \quad (24a)$$

$$C_3(a, b) = C_4(a, b) = \frac{1}{2} \sin^2 \frac{\theta_{ab}}{2}, \quad (24b)$$

for the relative weights of the four eigenvalues $A_k$ given by (15). Inserted into Eq. (16) they reproduce the quantum result

$$C_Q(a, b) = -\cos \theta_{ab}, \quad (25)$$

as expected. The contributions due to different signs of $\alpha_k$ and $\beta_k$ contained in $A_k = \alpha_k \beta_k$, pertain to mutually exclusive, complementary probability subspaces, as discussed above.

Let us call $\Phi$ the entire probability space and $\Phi_{ab}^k$ the four complementary subspaces. Assuming there exists an associated probability distribution $\rho(\phi)$ that is a function of a continuous random variable $\phi$ spanning the entire probability space, such that $\int \rho(\phi)d\phi = 1$, the contributions to $C_Q(a, b)$ stemming from the four distinct measurement results $A_k$ are

$$\int_{\Phi_{ab}^1} \rho(\phi)d\phi = \int_{\Phi_{ab}^2} \rho(\phi)d\phi = \frac{1}{2} \cos^2 \frac{\theta_{ab}}{2}, \quad (26a)$$

$$\int_{\Phi_{ab}^3} \rho(\phi)d\phi = \int_{\Phi_{ab}^4} \rho(\phi)d\phi = \frac{1}{2} \sin^2 \frac{\theta_{ab}}{2}, \quad (26b)$$

Alternatively, in view of the degeneracy indicated in Eq. (15), one may integrate the subspaces $\Phi_{ab}^1$ and $\Phi_{ab}^2$ into a common subspace $\Phi_{ab}^-$, corresponding to $A^- = -1$, and $\Phi_{ab}^3$ and $\Phi_{ab}^4$ into the complementary subspace $\Phi_{ab}^+$, corresponding to $A^+ = +1$, so that

$$\int_{\Phi_{ab}^-} \rho(\phi)d\phi = \cos^2 \frac{\theta_{ab}}{2}, \quad \int_{\Phi_{ab}^+} \rho(\phi)d\phi = \sin^2 \frac{\theta_{ab}}{2}. \quad (27)$$

It is essential to note that the distribution $\rho(\phi)$ is the same function of $\phi$ regardless of the directions $(a, b)$; only the separate domains of integration depend on the angle formed by $a$ and $b$. Changing the measurement setting (i.e. from $(a, b)$ to $(a, b')$) means using a new set of variables $\Phi$ that is partitioned accordingly. To make this distinction clear, we denote with $\phi_{ab}$ the variables $\phi$ spanning the complementary probability spaces $\Phi_{ab}^\pm$, so that

$$C_Q(a, b) = -\int_{\Phi_{ab}^-} \rho(\phi_{ab})d\phi_{ab} + \int_{\Phi_{ab}^+} \rho(\phi_{ab})d\phi_{ab}. \quad (28)$$

It should be stressed that the notation $\phi_{ab}$ does not imply a functional dependence of the random variable $\phi$ on the measurement setting $(a, b)$; it is simply meant to remind us that the realization $\phi$ pertains to the set of realizations carried out under this measurement setting.
3.1 General probability distribution function

As noted above, we are looking for a probability distribution function \( \rho(\phi) \) that complies with Eqs. (27) and therefore reproduces the quantum correlation (25). Such a function can be readily found by observing that

\[
\cos^2 \frac{\theta_{ab}}{2} = \frac{1}{2} (1 + \cos \theta_{ab}) = \frac{1}{2} \int_{\theta_{ab}}^{\pi} \sin \phi d\phi,
\]

\[
\sin^2 \frac{\theta_{ab}}{2} = \frac{1}{2} (1 - \cos \theta_{ab}) = \frac{1}{2} \int_{0}^{\theta_{ab}} \sin \phi d\phi.
\]

Therefore, the distribution function

\[
\rho(\phi) = \frac{1}{2} \sin \phi, \quad 0 \leq \phi \leq \pi
\]  

is a general solution to our problem. With Eq. (30) the quantum correlation (28) is given by

\[
C_Q(a, b) = \left( \int_{0}^{\theta_{ab}} - \int_{\theta_{ab}}^{\pi} \right) \rho(\phi_{ab}) d\phi_{ab} = -\cos \theta_{ab},
\]  

(31)

where the notation \( \phi_{ab} \) reminds us that \( \phi \) pertains to the set of realizations carried out under the measurement setting \( (a, b) \). It is interesting to note that the same formula for the distribution, Eq. (30), has been previously obtained by Oaknin (7), see also (8), also within the standard framework of quantum mechanics. By giving up the assumption implicit in the proof of Bell’s inequalities that there exists an absolute reference frame of angular coordinates for the entangled bipartite system, Oaknin concludes that the probability distribution is necessarily given by a function of the form of Eq. (30). The variable of integration can of course be changed to \( x_{ab} = \cos \theta_{ab} \) \((-1 \leq x_{ab} \leq 1)\), in which case \( \rho(x_{ab}) = \frac{1}{2} \) and Eq. (31) becomes

\[
C_Q(a, b) = \frac{1}{2} \left( \int_{\cos \theta_{ab}}^{1} - \int_{-1}^{\cos \theta_{ab}} \right) dx = -\cos \theta_{ab}.
\]  

(32)

3.2 A geometric model for the spin correlation

Given that we have found a general probability distribution and an appropriate separation of the probability space that accounts for the positive and negative outcomes contributing to the spin correlation, we now explore a possible geometric explanation for this result.

With this purpose in mind, let us take a pair of entangled spins and consider the situation in which the sign of the projection of spin 1 onto \( a \) has been determined, say \( a = +1 \); for simplicity in the discussion take the \(+z\) axis along the direction \( a \), and the \( x \) axis perpendicular to it. If the bipartite system is in the singlet state, we know for sure that the projection of spin 2 onto the \(+z\)
axis would give -1. This means that spin 2 lies in the lower half plane, forming any angle $\phi$ such that $0 \leq \phi \leq \pi$, with the origin of $\phi$ along the $-x$ axis and $\phi$ increasing counterclockwise. Conversely, if the sign of the projection of spin 1 is $\alpha = -1$, the second spin lies in the upper half plane, forming any angle $\phi$ such that $0 \leq \phi \leq \pi$, with the origin of $\phi$ along the $x$ axis. In both cases, $A = -1$. (The argument is of course reversible, in the sense that the sign of the projection of spin 2 can be defined first, in which case the angle variable $\phi$ refers to spin 1.) In summary, any series of measurements along parallel directions gives perfect anticorrelation, $C_Q(a, a) = C_Q(b, b) = -1$.

Consider now a series of measurements carried out to determine the correlation of the spin projections onto directions $(a, b)$ with the $+z$ axis again along $a$, and $b \neq a$. Take first the case $\alpha = +1$ for spin 1: when spin 2, lying in the lower half plane, is projected onto the direction $b$ forming an angle $\theta_{ab}$ with the $+z$ axis, $A$ will still be negative for any angle $\phi$ such that $\theta_{ab} \leq \phi \leq \pi$, whilst it will become positive for $0 \leq \phi \leq \theta_{ab}$. This gives a concrete meaning to Eq. (31).

What is it that determines in each instance the specific value of the (random) variable $\phi$ is unknown; we only know its probability distribution. When the direction $b$ is changed to $b'$, a different series of measurements is carried out, with the probability space subdivided accordingly:

$$C_Q(a, b') = \left( \int_{0}^{\theta_{ab'}} d\phi_{ab'} - \int_{\theta_{ab'}}^{\pi} d\phi_{ab'} \right) \rho(\phi_{ab'}) = -\cos \theta_{ab'}.$$  

The subdivision depends on the range of values of the random variable $\phi_{ab}$ for which the sign of the product $A = \alpha \beta$ is either positive or negative. This means that neither $\alpha$ nor $\beta$ may be transferred from (31) to (33); not even if the direction of $a$ remains fixed. Precisely herein lies the essence of the correlation.

Incidentally, a similar reasoning can be applied to the spin correlation for a single electron,

$$C(a, b) = \langle \psi | (\vec{\sigma} \cdot a) (\vec{\sigma} \cdot b) | \psi \rangle.$$  

In this case, when the spin projection onto $a$ (taken again along the $+z$ axis) is +1, its projection onto $b$ is +1 (i.e., $A = +1$) for any angle $\phi$ such that $\theta_{ab} \leq \phi \leq \pi$, whilst it is -1 (i.e., $A = -1$) for $0 \leq \phi \leq \theta_{ab}$, and inversely if the spin projection onto $a$ is negative. The two contributions taken together give the quantum result

$$C(a, b) = \left( -\int_{0}^{\theta_{ab}} + \int_{\theta_{ab}}^{\pi} \right) \rho(\phi_{ab}) d\phi_{ab} = \cos \theta_{ab},$$  

with $\rho(\phi)$ given by (30). We observe that in the one-particle case the first measurement (say onto $a$ along the $z$ direction) is equivalent to a preparation of the system for a measurement of the second projection onto $b$. In the bipartite case the measurement of the two spin projections counts as a single event (i.e., it is a joint measurement); yet having chosen the result of the projection of spin 1 (say onto $a$ along the $z$ direction) can be considered equivalent to a
‘preparation’. In both cases illustrated here, \( \rho(\phi_{ab}) \) plays the role of a probability density conditioned by the outcome of the projection onto \( a \).

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