New Approaches to Robust Inference on Market (Non-)Efficiency, Volatility Clustering and Nonlinear Dependence†

Rustam Ibragimov

Imperial College Business School

Rasmus Søndergaard Pedersen††

Department of Economics, University of Copenhagen

Anton Skrobotov

Russian Presidential Academy of National Economy and Public Administration (RANEPA)
and St. Petersburg University (Centre for Econometrics and Business Analytics)

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†† Corresponding author. Mailing address: Oster Farimagsgade 5, Building 26, 1353 Kobenhavn K, Denmark. E-mail: rsp@econ.ku.dk Tel.: +45 40 51 78 62.
Abstract

We present novel, robust methods for inference on market (non-)efficiency, volatility clustering, and nonlinear dependence in financial return series. In contrast to existing methodology, our proposed methods are robust against non-linear dynamics and tail-heaviness of returns. Specifically, our methods only rely on return processes being stationary and weakly dependent (mixing) with finite moments of a suitable order. This includes robustness against power law distributions associated with non-linear dynamic models such as GARCH and stochastic volatility. The methods are easy to implement and perform well in realistic settings. We revisit a recent study by Baltussen et al. (2019, Journal of Financial Economics, vol. 132, pp. 26-48) on autocorrelation in major stock indexes. Using our robust methods, we document that the evidence of presence of negative autocorrelation is weaker, compared to the conclusions of the original study.

Key words: robust inference, t-test, market efficiency, volatility clustering, nonlinear dependence, GARCH.

JEL Classification: C12, C14, G12, G14
1 Introduction

Many studies argue that time series of financial returns, $R_t$, and other key economic and financial variables and indicators like foreign exchange rates exhibit several common statistical properties, often referred to as stylized facts; see e.g. Campbell et al. (1997, Ch. 2), Cont (2001), Taylor (2008 Ch. 1-2), Tsay (2010 Ch. 1-3), Christoffersen (2012 Ch. 1), McNeil et al. (2015 Ch. 3), and references therein. The following three properties are the most important stylized facts that much of the empirical literature agrees upon, together with the standard mean-zero property, $E(R_t) = 0$, implying the absence of systematic gains or losses:

(i) Absence of linear dependence and linear autocorrelations: $\text{Corr}(R_t, R_{t-h}) \approx 0$, even for small lags $h = 1, 2, ...$

(ii) The presence of nonlinear dependence and volatility clustering, captured by significant positive autocorrelation in simple nonlinear functions of the returns and different measures of volatility, such as squared returns: $\text{Corr}(R^2_t, R^2_{t-h}) >> 0$, even for large lags $h > 0$.

(iii) Heavy tails: The (unconditional) returns distributions exhibit heavy power-law tails, $\lim_{x \to +\infty} x^\zeta P(|R_t| > x) = C$, with a constant $C > 0$ and the tail index $\zeta > 0$.

The properties (i)(ii) are related to different forms of efficiency hypotheses for financial markets. Property (i) is often cited as support for the weak market efficiency/martingale hypothesis as it is implied by the assumption/property that $R_t$ is a martingale difference sequence. On the other hand, property (ii) implies that returns $R_t$ are not independent and thus the strong market efficiency/random walk hypothesis does not hold (see e.g., Ch. 2 in Campbell et al., 1997 Cont, 2001 and references therein).

A standard way of testing properties (i)(ii) for a given sample $(R_t)_{t=1,\ldots,T}$ is to compute the sample counterparts of the correlations and rely on limiting Gaussian distributions of these
(suitably scaled and standardized) statistics together with some “robust” (heteroskedasticity and autocorrelation consistent, HAC-type) estimates of their asymptotic variances; see, e.g., Baltussen et al. (2019) for a recent application in the analysis of linear autocorrelations in returns on major world stock indices. Such approaches may not be reliable under heavy-tailedness and nonlinear dependence (see, e.g., Granger and Orr 1972, Davis and Mikosch 1998, Mikosch and Stărică 2000). Properties (i)-(ii) are commonly modelled using the much celebrated class of GARCH-type processes. Depending on the tail index $\zeta$, characterizing the degree of heavy-tailedness in property (iii) of a given (GARCH-type) process, the correlations may not be defined and/or the sample correlations may have limiting distributions given by functions of multivariate non-Gaussian stable distributions, or may be inconsistent. For instance, $\text{Corr}(R_t, R_{t-h})$ ($\text{Corr}(R_t^2, R_{t-h}^2)$) is defined only if $\zeta > 2$ ($\zeta > 4$), and asymptotic normality of the sample counterpart requires that $\zeta > 4$ ($\zeta > 8$). The latter condition is hardly justified empirically, as it is typically found that $\zeta \in (2, 4)$ for financial returns in developed markets, whereas emerging markets’ returns may even have $\zeta < 2$ and, hence, infinite variances (e.g., Loretan and Phillips 1994, Cont 2001). Note that the applicability of HAC inference approaches typically relies on moments of even higher order to be finite. Moreover, HAC-based inference methods often have poor finite sample properties, even in rather standard inference problems, especially with data with pronounced dependence and heterogeneity; see, among others, Andrews (1991), Andrews and Monahan (1992), den Haan and Levin (1997), and Ibragimov and Müller (2010) [IM (2010), henceforth].

In this paper, we present new robust approaches for dealing with the issue of heavy tails in testing for (non-)efficiency, volatility clustering and nonlinear dependence in financial return distributions.

As kindly pointed out by a referee, some studies have found that certain asset returns have heavier tails than the Gaussian distribution, which is more general than the power law tail property (iii). We emphasize that our proposed methodology and results do not rely on power law tailed return distributions, but only that return distributions have finite moments of a suitable order.
series. We exploit the property that if $R_t$ has power-law tails with the index $\zeta > 0$, as in (iii) then for any $p > 0$, $|R_t|^p$ has the tail index $\zeta/p$. This suggests that, even under pronounced heavy tails, correlations are well-defined and Gaussian limiting distributions for sample correlations can be obtained under suitable power transformations of the original return time series. Specifically, as a natural analogue to nonlinear dependence property (ii), we consider the property

(ii') $\text{Corr}(|R_t|^p, |R_{t-h}|^p) >> 0$, even for large lags $h > 0$ for some $p > 0$.

Under heavy-tailedness property (iii), the correlations in (ii') are well-defined whenever $2p < \zeta$, and, as is shown in this paper, under general conditions, asymptotic normality of the corresponding sample correlations holds if $4p < \zeta$.

We further propose the correlations $\text{Corr}(R_t, |R_{t-h}|^s \text{sign}(R_{t-h}))$, $s > 0$, of ‘signed’ powers of absolute returns as measures of market (non-)efficiency. Similar to the power transformations in (ii') these measures lead to formulation of natural analogues of property (i) under heavy-tailed and conditionally heteroskedastic time series:

(i') $\text{Corr}(R_t, |R_{t-h}|^s \text{sign}(R_{t-h})) \approx 0$, even for small lags $h = 1, 2, \ldots$ for some $s > 0$.

Similar to property (ii') the correlations in (i') are well-defined for $\zeta > 2 \max(1, s)$, and, under suitable conditions, asymptotic normality of the corresponding sample correlations holds if $\zeta > 2(1 + s)$.

To the best of our knowledge, property (ii') for powers of absolute returns was originally considered by Ding et al. (1993) in relation to detecting long-memory in returns. The purpose of power transformations in the present manuscript is different in the sense that power transformations of returns serve as a necessary step for making the corresponding correlations well-defined and for carrying out reliable inference in the presence of heavy-tailedness and conditional heteroskedasticity in returns.

The main contribution of this paper is the development of robust approaches to inference
on measures of market (non-)efficiency, nonlinear dependence, and volatility clustering, such as the correlations in (i') and (ii'). Firstly, we establish asymptotic normality of sample auto(cross)correlations of arbitrary transformations of a time series under general mixing conditions for the data-generating process (DGP). Further, in order to avoid (HAC-based) estimation of limiting variances of the statistics, we propose robust $t$–statistic inference approaches in the spirit of IM (2010, 2016), and prove their asymptotic validity for general classes of DGPs, including GARCH-type time series. A similar approach was recently considered in Pedersen (2020) in relation to inference about an autoregressive coefficient in linear autoregressive models in the presence of heavy-tailed symmetric GARCH-type errors. In contrast, the robust inference approaches proposed in this paper do not impose any symmetry restriction on the DGPs. This is a desirable feature, as it is quite common for financial time series to have skewed marginal distributions (gain/loss asymmetry) as well as leverage effects (e.g., Cont 2001).

We provide a numerical analysis that demonstrates appealing finite sample properties of the robust $t$–statistic inference approaches. Lastly, we revisit the aforementioned study by Baltussen et al. (2019) and illustrate the applicability of the approaches in relation to inference on properties (i') and (ii') in major stock market indices. Importantly, we document that all the associated return series are likely to have tail indices $\zeta < 4$, i.e. infinite fourth moments. Moreover, when applying our robust approaches, taking into account return heavy-tailedness, we find weak evidence of negative serial dependence in returns, in contrast to the conclusions made by Baltussen et al. (2019).

The $t$–statistic approaches to robust inference complement and are related to inference approaches based on self-normalization (see the review in de la Peña et al. 2009, Shao 2015). Moreover, it is related to the fixed-smoothing (fixed-$b$) heteroskedasticity and autocorrelation robust (HAR) methods that do not rely on consistency of limiting variance estimators and use nonstandard “fixed-$b$” asymptotics or Student-$t$ or $F$ distributional approximations (e.g.
Kiefer et al., 2000, Kiefer and Vogelsang, 2002, 2005, Jansson, 2004, Müller, 2007, 2014, Sun et al., 2008, Sun, 2013, 2014a, 2014b, Lazarus et al., 2018, Lazarus et al., 2021). The methodology presented in this paper is easy to implement and does not rely on non-standard asymptotic theory. Moreover, as is discussed in the paper, in contrast to the aforementioned approaches, the methodology is valid even in certain non-standard settings, where limiting distributions of estimators are heavy-tailed, that is, non-Gaussian symmetric stable.

The paper is organized as follows. Section 2 introduces and discusses measures of serial and nonlinear dependence based on autocorrelations of powers of absolute returns, and provides asymptotic theory for estimators of these measures. We further propose robust \( t \)-statistic approaches for reliable inference on the measures, and show that the approaches are asymptotically valid under general conditions. Section 3 investigates the finite-sample properties of the inference methods. Section 4 provides an empirical illustration of the inference methods. Section 5 concludes and discusses suggestions for future research. The Online Appendix contains a discussion of the relationship between the robust \( t \)-statistic inference approaches and existing methods based on self-normalization and HAR-based inference. In addition, the Online Appendix contains proofs and additional simulation results.

2 Inference on measures of market (non-)efficiency, nonlinear dependence and volatility clustering

2.1 Autocovariances and -correlations for transformed returns

The results in this paper hold for a wide class of stationary time series processes that satisfy mixing and moment conditions stated below. (Throughout, “stationarity” refers to the notion of strict stationarity.) This includes heavy-tailed GARCH\((p, q)\) time series, generalized GARCH processes (e.g., Pedersen, 2020), and heavy-tailed stochastic volatility processes (e.g., Davis and Mikosch, 2001), among others. To present the main ideas, we focus on the GARCH\((1,1)\) process as an ongoing example.
Let $Z = \{..., -2, -1, 0, 1, 2, ...\}$. A GARCH(1, 1) process, $(R_t)_{t \in \mathbb{Z}}$, is given by

$$R_t = \sigma_t Z_t, \quad t \in \mathbb{Z},$$

(1)

where $(Z_t)_{t \in \mathbb{Z}}$ is a sequence of i.i.d. random variables (r.v.’s) with mean zero and unit variance, $E(Z_t) = 0$ and $\text{Var}(Z_t) = 1$, and $(\sigma^2_t)_{t \in \mathbb{Z}}$ is a conditional volatility process,

$$\sigma^2_t = \omega + \alpha R^2_{t-1} + \beta \sigma^2_{t-1}, \quad \omega > 0, \quad \alpha, \beta \geq 0.$$  

(2)

As is well-known, the process in (1)-(2) has a stationary and ergodic version if and only if $E[\log(\alpha Z^2_t + \beta)] < 0$ (e.g., Nelson 1990). In addition, under mild conditions on the distribution of $Z_t$, e.g., if it has a Lebesgue density, the GARCH process is $\beta$-mixing with geometric rate; e.g., Francq and Zakoïan 2006, Thm. 3). This implies that the process is also $\alpha$-mixing with geometric rate (see, e.g., Rio 2017 for additional details on mixing processes). Under, essentially, the conditions listed above, the stationary solution to (1)-(2) satisfies Kesten’s theorem (e.g., Mikosch and Stàricà 2000). Specifically, the unconditional distribution of $R_t$ has power-law tails as in (iii) with the tail index $\zeta > 0$ given by the unique positive solution to the equation

$$E[(\alpha Z^2_t + \beta)^{\zeta/2}] = 1.$$  

(3)

For instance, $\zeta \in (2, 4)$ if $1 - (\kappa_Z - 1)\alpha^2 < (\alpha + \beta)^2 < 1$, where $\kappa_Z \equiv E[Z^4_t]$.

Given a stationary process, $(R_t)_{t \in \mathbb{Z}}$, we consider the following population autocovariance and autocorrelation functions of order $h$ for measuring nonlinear dependence and volatility clustering in the process. We emphasize that the measures are non-zero if the process is
conditionally heteroskedastic. For \( p > 0 \) and \( E[|R_t|^{2p}] < \infty \), let

\[
\gamma_{|R|^p}(h) = \text{Cov}(|R_t|^p, |R_{t-h}|^p), \quad h = 0, 1, \ldots, \tag{4}
\]
\[
\rho_{|R|^p}(h) = \text{Corr}(|R_t|^p, |R_{t-h}|^p) = \frac{\gamma_{|R|^p}(h)}{\gamma_{|R|^p}(0)}, \quad h = 1, 2, \ldots, \tag{5}
\]

To quantify the degree of efficiency, i.e. if \( R_t \) is predictable with respect to its lagged values, we define, for \( s > 0 \) and \( E[|R_t|^{1+s}] < \infty \),

\[
\gamma'_{R,|R|^{s}\text{sign}(R)}(h) = \text{Cov}(R_t, |R_{t-h}|^s\text{sign}(R_{t-h})), \quad h = 0, 1, \ldots, \tag{6}
\]

and for \( \max\{E[|R_t|^{2s}], E[|R_t|^2]\} < \infty \), denote

\[
\rho'_{R,|R|^{s}\text{sign}(R)}(h) = \text{Corr}(R_t, |R_{t-h}|^s\text{sign}(R_{t-h})) = \frac{\gamma'_{R,|R|^{s}\text{sign}(R)}(h)}{\sqrt{\gamma_R(0)\gamma'_{R,|R|^{s}\text{sign}(R)}(0)}}, \quad h = 1, 2, \ldots, \tag{7}
\]

where \( \gamma_R(0) = \text{Var}(R_t) \) and \( \gamma'_{R,|R|^{s}\text{sign}(R)}(0) = \text{Var}(|R|^s\text{sign}(R)) \).

**Example 2.1** As indicated in the introduction, in the presence of heavy tails, e.g., when \((R_t)_{t \in \mathbb{Z}} \) follows a GARCH(1,1) process with the tail index \( \zeta > 0 \), the quantities in (4) and (5) are defined if \( \zeta > 2p \). Likewise, the covariances in (6) are defined if \( \zeta > 1+s \), and the correlations in (7) are defined if \( \zeta > 2\max\{1,s\} \).

**Remark 2.1** For \( s = 1 \), (6) and (7) are identical to the usual linear autocovariances and autocorrelations, respectively. For \( s \neq 1 \), (6) and (7) are still able to detect market (non-)efficiency. If \((R_t)_{t \in \mathbb{Z}} \) is a martingale difference sequence, e.g. if it is a GARCH process, \( \text{Corr}(g(R_t), f(R_{t-h}) = 0 \) for any linear function \( g(\cdot) \) and any function \( f(\cdot) \) (providing that the correlation is well-defined), and in particular, the quantities in (6) and (7) are equal to zero; see, e.g., (Campbell et al., 1997, Section 2.1) for a discussion. In contrast, we note that the martingale difference property does not necessarily imply that \( \text{Corr}(|R|^s\text{sign}(R)|R|^s\text{sign}(R_{t-h})|R_{t-h}|^s) = 0 \) under asymmetric conditional distributions.

In the next section, we consider estimation of the dependence measures and provide large sample theory for their estimators.
2.2 Limit theory for sample dependence measures

Let \((R_t)_{t=1,\ldots,T}\) be a sample of observations. Denote by \(\hat{\mu}_R\), \(\hat{\mu}_{|R|^p}\) and \(\hat{\mu}_{|R|^s\text{sign}(R)}\), respectively, the sample means of \(R_t\), \(|R_t|^p\), and \(|R_t|^s\text{sign}(R_t)\), for \(p, s > 0\), i.e.

\[
\hat{\mu}_R = \frac{1}{T} \sum_{t=1}^{T} R_t, \quad \hat{\mu}_{|R|^p} = \frac{1}{T} \sum_{t=1}^{T} |R_t|^p, \quad \hat{\mu}_{|R|^s\text{sign}(R)} = \frac{1}{T} \sum_{t=1}^{T} |R_t|^s\text{sign}(R_t).
\] (8)

The sample versions of (4) and (5) are given, respectively, by

\[
\hat{\gamma}_{|R|^p}(h) = \frac{1}{T} \sum_{t=h+1}^{T} (|R_t|^p - \hat{\mu}_{|R|^p})(|R_{t-h}|^p - \hat{\mu}_{|R|^p}),
\] (9)

\[
\hat{\rho}_{|R|^p}(h) = \frac{\hat{\gamma}_{|R|^p}(h)}{\hat{\gamma}_{|R|^p}(0)}.
\] (10)

Likewise, the sample versions of (6) and (7) are

\[
\hat{\gamma}_{R,|R|^s\text{sign}(R)}(h) = \frac{1}{T} \sum_{t=h+1}^{T} (R_t - \hat{\mu}_R)(|R_{t-h}|^s\text{sign}(R_{t-h}) - \hat{\mu}_{|R|^s\text{sign}(R)}),
\] (11)

\[
\hat{\rho}_{R,|R|^s\text{sign}(R)}(h) = \frac{\hat{\gamma}_{R,|R|^s\text{sign}(R)}(h)}{\sqrt{\hat{\gamma}_R(0)\hat{\gamma}_{|R|^s\text{sign}(R)}(0)}},
\] (12)

where \(\hat{\gamma}_R(0)\) and \(\hat{\gamma}_{|R|^s\text{sign}(R)}(0)\) denote the sample variances of \(R_t\) and \(|R_t|^s\text{sign}(R_t)\) defined in the usual way similar to \(\hat{\gamma}_{|R|^p}(0)\) in (9).

The following Lemmas 2.1 and 2.2 provide a basis for asymptotic inference on the properties (ii') and (i'), respectively. The lemmas follow from the general results in the Online Appendix for sample autocovariances and autocorrelations of arbitrary functions of \(\alpha\)-mixing processes.

**Lemma 2.1** Let \((R_t)_{t\in\mathbb{Z}}\) be a stationary \(\alpha\)-mixing process. For \(p > 0\), assume that there exists a value \(\delta > 0\) such that \(E[|R_t|^{4p+\delta}] < \infty\), and such that the mixing coefficients \(\alpha(n)\) satisfy \(\sum_{n=1}^{\infty} \alpha(n)^{\delta/(2+\delta)} < \infty\). Then, with \(\hat{\gamma}_{|R|^p}(h)\) and \(\hat{\rho}_{|R|^p}(h)\) defined in (9) and (10), respectively, for a fixed integer \(m\), one has

\[
\sqrt{T}(\hat{\gamma}_{|R|^p}(h) - \gamma_{|R|^p}(h))_{h=0,1,\ldots,m} \rightarrow_d (G_{h,p})_{h=0,\ldots,m},
\] (13)
\[ \sqrt{T}(\hat{\rho}_{|R|^p}(h) - \rho_{|R|^p}(h))_{h=1,\ldots,m} \to_d (H_{h,p})_{h=1,\ldots,m}, \]  

(14)

where the limits are multivariate Gaussian with mean zero.

**Lemma 2.2** Let \((R_t)_{t \in \mathbb{Z}}\) be a stationary \(\alpha\)-mixing process. For \(s > 0\), assume that there exists a value \(\delta > 0\) such that \(E[|R_t|^{2(1+s)+\delta}] < \infty\), and such that the mixing coefficients \(\alpha(n)\) satisfy \(\sum_{n=1}^{\infty} \alpha(n)^{\delta/(2+\delta)} < \infty\). Then, with \(\hat{\gamma}'_{R,|R|^s\text{sign}(R)}(h)\) defined in (11), one has

\[ \sqrt{T}(\hat{\gamma}'_{R,|R|^s\text{sign}(R)}(h) - \gamma'_{R,|R|^s\text{sign}(R)}(h))_{h=0,1,\ldots,m} \to_d (G'_{h,s})_{h=0,1,\ldots,m}, \]  

(15)

where \((G'_{h,s})_{h=0,1,\ldots,m}\) is multivariate Gaussian with mean zero.

If \(\gamma'_{R,|R|^s\text{sign}(R)}(h)_{h=1,\ldots,m} = (0, \ldots, 0)\), with \(\hat{\rho}'_{R,|R|^s\text{sign}(R)}(h)\) defined in (12), one has

\[ \sqrt{T}(\hat{\rho}'_{R,|R|^s\text{sign}(R)}(h) - \rho'_{R,|R|^s\text{sign}(R)}(h))_{h=1,\ldots,m} \to_d ((\gamma R(0)\gamma_{|R|^s\text{sign}(R)}(0))^{-1/2}G'_{h,s})_{h=1,\ldots,m}. \]  

(16)

If \(\max\{E[|R_t|^{4+\delta}], E[|R_t|^{4s+\delta}]\} < \infty\), then

\[ \sqrt{T}(\hat{\rho}'_{R,|R|^s\text{sign}(R)}(h) - \rho'_{R,|R|^s\text{sign}(R)}(h))_{h=1,\ldots,m} \to_d (H'_{h,s})_{h=1,\ldots,m}, \]  

(17)

where \((H'_{h,s})_{h=1,\ldots,m}\) is multivariate Gaussian with mean zero.

In Lemma 2.2 the moment conditions for asymptotic normality of sample covariances in (15) are weaker than those of sample correlations in (17). The reason is that asymptotic normality of sample correlations relies on joint asymptotic normality of sample covariances and variances of \(R_t\) and \(|R_t|^s\text{sign}(R_t)\), if the true correlations are non-zero. If the true correlations are zero, as in (16), the convergence of the sample correlations only relies on asymptotic normality of the sample covariances and consistency of the sample variances, which in turn requires the same moment conditions as for convergence of sample covariances.

**Example 2.2** As discussed in Section 2.1, under suitable conditions, GARCH(1,1) processes are \(\alpha\)-mixing with geometric decay, and hence satisfy the conditions on the mixing coefficients in Lemmas 2.1 and 2.2. In particular, Lemma 2.1 holds if the tail index \(\zeta > 4p\). Whenever \(\zeta > 2(1+s)\), the normal asymptotics in (15) and (16) hold for the GARCH(1, 1) processes. If the moment conditions in Lemmas 2.1 and 2.2 are not satisfied, the rates of convergence of the sample covariances and correlations are slower than \(\sqrt{T}\) and the limits are given by functions of r.v.’s with non-Gaussian (in general, asymmetric) stable distributions. Importantly, the rates of convergence and the limiting distributions depend on the
(unknown) tail index $\zeta$ and the powers $p$ and $s$. E.g., from the results in \cite{DavisMikosch1998} and \cite{MikoschStaicu2000} (see also \cite{DavisMikosch2009}) it follows that, in the case $p = 1$ and $\zeta \in (2, 4)$, $T^{1-2/\zeta}(\hat{\gamma}_{|R|}(h) - \gamma_{|R|}(h))$ has an infinite variance asymmetric stable limiting distribution with the index of stability given by $\zeta/2$. A similar result applies to $T^{1-2/\zeta}(\hat{\gamma}_{|R|^2}(h) - \gamma_{|R|^2}(h))$ when $\zeta \in (4, 8)$, where the index of stability of the limiting asymmetric stable distribution is $\zeta/4$. Moreover, for the cases \{ $p = 1$ and $\zeta \in (0, 2)$ \} and \{ $p = 2$ and $\zeta \in (0, 4)$ \}, $\hat{\gamma}_{|R|^p}(h)$ is inconsistent and has a non-Gaussian asymmetric stable limit. As the rate of convergence and the limiting distributions depend on the unknown value of the tail index, $\zeta$, the results on convergence of full-sample covariance and correlation estimators are not directly applicable in terms of hypothesis testing and other inference problems. However, as discussed in Remark 2.5 and Example 2.3 below, the stable limit theory may be used (under suitable conditions) in inference using the robust $t$-statistic approaches considered in the next section.

The formulas for the covariance matrices of the limiting Gaussian variables in Lemmas 2.1 and 2.2 are provided in the Online Appendix for the case of covariances and correlations, $\text{Cov}(f(R_t), g(R_{t-h}))$ and $\text{Corr}(f(R_t), g(R_{t-h}))$, for general functions $f$ and $g$. The asymptotic covariance matrices have a complicated structure: For instance, for the case of (13), the asymptotic covariance matrix depends on autocovariances of any order of the time series of the products $(|R_t|^p|R_{t-h}|^p)_{h=0,\ldots,m}$. Under suitable conditions, including more restrictive moment conditions, the limiting covariance matrices may be estimated by HAC-type estimators, as discussed in the introduction. For instance, following \cite{NeweyWest1987} Theorem 2, one has to assume that $E[|R_t|^{2p(4+\epsilon)}] < \infty$ for some $\epsilon > 0$ for Newey-West-type standard errors to be applicable. Such conditions are restrictive for financial applications as, e.g., letting $p = 1$ requires that $R_t$ has finite eighth-order moments, i.e., $\zeta > 8$ in terms of the tail index (see also the discussion in the Online Appendix on self-normalization and HAR approaches that typically may be used under more relaxed moment conditions as compared to HAC). Importantly, HAC-based inference methods often have poor finite sample properties, even in rather standard inference problems; see the discussion and references in the introduction.

In the next section, we propose robust approaches to inference on covariances and correlations.
of the form [4]−[7] using $t$-statistics in estimates of these quantities computed over groups of time series observations. The main advantage of these approaches is that no estimation of limiting covariance matrices is needed. We establish asymptotic validity of the robust inference approaches by relying on the large-sample results in Lemmas 2.1 and 2.2 and new results on asymptotic independence of the group-based estimators.

### 2.3 Robust inference on market (non-)efficiency, volatility clustering and nonlinear dependence

Following IM (2010, 2016), we consider robust $t$-statistic inference on a parameter $\beta$ of a general stationary process $(R_t)_{t\in\mathbb{Z}}$. In the following, the parameter $\beta$ of interest may be the population covariance $\beta = \gamma_{|R|^p}(h), \gamma'_{R,|R|^s\text{sign}(R)}(h)$ or correlation $\beta = \rho_{|R|^p}(h), \rho'_{R,|R|^s\text{sign}(R)}(h)$. Let $(R_t)_{t=1,...,T}$ be a sample of observations. Consider a partition of the sample into a fixed number $q \geq 2$ of (approximately) equal sized groups of consecutive observations, i.e. the observations in group $j = 1, \ldots, q$ have time indexes $(j-1)[T/q] < t \leq j[T/q]$, where $[x]$ is the integer part of $x \in \mathbb{R}$. The robust $t$-statistic inference on $\beta$ is conducted using its group estimators, $(\hat{\beta}_j)_{j=1,...,q}$, given by the sample covariances/correlations in (4)-(7) based on the observations in group $j$, e.g., $\hat{\beta}_j$ may equal

$$\hat{\gamma}_{j,|R|^p}(h) = \frac{1}{[T/q]} \sum_{t=(j-1)[T/q]+h+1}^{j[T/q]} (|R_t|^p - \hat{\mu}_{j,|R|^p})(|R_{t-h}|^p - \hat{\mu}_{j,|R|^p}),$$

or

$$\hat{\gamma}'_{j,R,|R|^s\text{sign}(R)}(h) = \frac{1}{[T/q]} \sum_{t=(j-1)[T/q]+h+1}^{j[T/q]} (R_t - \hat{\mu}_{j,R})(|R_{t-h}|^s\text{sign}(R_{t-h}) - \hat{\mu}_{j,R}^{|R|^s\text{sign}(R)}),$$

where $\hat{\mu}_{j,|R|^p}$ is the group-based version of $\hat{\mu}_{|R|^p}$ in (8) based on the observations in group $j$, and similar for $\hat{\mu}_{j,R}$ and $\hat{\mu}_{j,R}^{|R|^s\text{sign}(R)}$.

Suppose that one seeks to test the null hypothesis $H_0 : \beta = \beta_0$, e.g. that the covariance $\gamma'_{R,|R|^s\text{sign}(R)}(h) = 0$, against the two-sided alternative $H_a : \beta \neq \beta_0$. Let $t_\beta$ denote the
t-statistic in the group estimators, $(\hat{\beta}_j)_{j=1,...,q}$, i.e.

$$t_\beta = \sqrt{q} \frac{\hat{\beta} - \beta_0}{s_\beta}, \quad (20)$$

with $\bar{\beta} = q^{-1} \sum_{j=1}^{q} \hat{\beta}_j$ and $s^2_\beta = (q-1)^{-1} \sum_{j=1}^{q} (\hat{\beta}_j - \bar{\beta})^2$. The robust t-statistic approaches rely on rejecting the null hypothesis $H_0$ in favor of the two-sided alternative $H_a$ at commonly used significance levels $\tilde{\alpha}$ (to be specified below), if the absolute value of the t-statistic, $|t_\beta|$, exceeds the $(1-\tilde{\alpha}/2)$ quantile of a Student’s t-distribution with $q-1$ degrees of freedom.

According to Theorem 2.1 below, the t-statistic approaches to inference on properties (i') and (ii') are asymptotically valid and have asymptotically correct size. The theorem follows from, firstly, asymptotic normality and asymptotic independence of the group estimators $(\hat{\beta}_j)_{j=1,...,q}$, implied by Lemmas 2.1 and 2.2 in the previous section and Lemma 2.3 below, and, secondly, a small sample result on the conservativeness property of the t-statistic in heterogeneous normal r.v.’s originally proved by Bakirov and Székely (2005); see also IM (2010, Theorem 1).

The following lemma states that the group estimators, $(\hat{\beta}_j)_{j=1,...,q}$, are asymptotically independent under the assumptions in Lemmas 2.1 and 2.2.

**Lemma 2.3** Suppose that $(R_t)_{t \in \mathbb{Z}}$ is stationary and $\beta$-mixing. Under the assumptions of Lemma 2.1, the centered and scaled group sample covariances $\sqrt{T/q} (\hat{\gamma}_{i,j,R|\gamma|h} - \gamma_{R|\gamma|h})$ and $\sqrt{T/q} (\hat{\gamma}'_{i,j,R|\gamma^*\text{sign}(R)}(h) - \gamma'_{R|\gamma^*\text{sign}(R)})$, are asymptotically independent for $i, j = 1, 2, ..., q$, with $i \neq j$. Likewise, under the conditions of Lemma 2.2, $\sqrt{T/q} (\hat{\gamma}_{i,j,R|\gamma|h} - \gamma_{R|\gamma|h})$ and $\sqrt{T/q} (\hat{\gamma}'_{i,j,R|\gamma^*\text{sign}(R)}(h) - \gamma'_{R|\gamma^*\text{sign}(R)})$ are asymptotically independent for $i, j = 1, 2, ..., q$, with $i \neq j$. The asymptotic independence property also holds for the group sample correlations $\hat{\rho}_{i,j,R|\gamma|h}(h)$ and $\hat{\rho}'_{i,j,R|\gamma^*\text{sign}(R)}$, $j = 1, 2, ..., q$, under the assumptions in Lemmas 2.1 and 2.2 respectively.

**Remark 2.2** The proof of Lemma 2.3 in the case of covariances and correlations of general functions of the process $(R_t)$ is given in the Online Appendix and relies on exact coupling properties for $\beta$-mixing processes. A similar argument was recently used in Pedersen (2020) in the context of inference on the autoregressive coefficient in linear autoregressive models in the presence of heavy-tailed GARCH-type errors under symmetry. The proof is general
also in the sense that the arguments hold for arbitrary limiting distributions of the group estimators. The limiting distributions may be non-Gaussian, e.g., stable (with the index of stability less than 2), as discussed in Example 2.2.

The following main result follows by Lemmas 2.1-2.3 and IM (2010, Theorem 1).

Theorem 2.1 Consider, as above, testing the hypothesis $H_0 : \beta = \beta_0$ against $H_a : \beta \neq \beta_0$ for covariances/correlations $\beta = \gamma |R|^p(h), \gamma' |R|^s \text{sign}(R)(h), \rho |R|^p(h), \rho' |R|^s \text{sign}(R)(h)$. Suppose that the assumptions of Lemma 2.3 are satisfied. Let $T_{q-1}$ denote a r.v. with a Student’s $t$-distribution with $q-1$ degrees of freedom, and let $cv(q, \tilde{\alpha})$ satisfy $P(T_{q-1} > cv(q, \tilde{\alpha})) = \tilde{\alpha}/2$. With $t_\beta$ defined in (20) using group estimators, $\hat{\beta}_j$, of $\beta$ one has, under $H_0$,

$$\limsup_{T \to \infty} P(|t_\beta| > cv(q, \tilde{\alpha})|H_0) \leq \tilde{\alpha}, \quad (21)$$

for any $\tilde{\alpha} \leq 2\Phi(-\sqrt{3}) = 0.08326 \ldots$, where $\Phi(\cdot)$ is the standard normal cdf. The inequality (21) also holds for $2 \leq q \leq 14$ if $\tilde{\alpha} \leq 0.1$. Moreover, for any $x \geq 0$ and $q \geq 2$,

$$\limsup_{T \to \infty} P(|t_\beta| > x|H_0) \leq \max_{R<k \leq q} P\left(|T_{k-1}| > \sqrt{\frac{R(k-1)}{k-R}}\right), \quad (22)$$

where $R = R(x) = \frac{qx^2}{x^2+q-1}$.

Theorem 2.1 demonstrates the asymptotic validity of robust $t$-statistic approaches for inference on the properties (i’)-(ii’) under appropriate conditions on the degree of heavy-tailedness (the tail index $\zeta$) for the return process $(R_t)$ and powers $p$ and $s$ in (i’)-(ii’) like $p < \zeta/4$, or $s < \zeta/2 - 1$. The asymptotic size control in (21) holds for any choice of level $\tilde{\alpha} \leq 2\Phi(-\sqrt{3}) = 0.08326 \ldots$, and is hence applicable in most practical settings; see Remark 2.6 below for additional intuition for this upper level bound. We also note that the method is simple to implement and does not require new tables (or simulations) of critical values. In addition, the method does not rely on estimation of long-run asymptotic variances, in contrast to HAC- and HAR-based methods. We refer to the Online Appendix for additional discussion about the relation between the $t$-statistic approaches and existing methods.

Remark 2.3 The asymptotic size control in (21) holds for any (fixed) choice of group numbers $q \geq 2$. Importantly, as noted by IM (2010), if one is only willing to assume that the group-based estimators are asymptotically independent and (mixed) Gaussian, then it is not
possible to apply data-driven methods for determining an optimal number \( q \) of groups. Further, the numerical results in IM (2010) indicate that in many settings with presence of dependence and heterogeneity considered in the literature, and typically encountered in practice for time series, choosing \( q = 8 \) or \( q = 16 \) leads to robust tests with attractive finite sample performance. This is also supported by the results in our simulation experiments in Section 3. Specifically, for practical purposes, based on our results we recommend using \( q = 8 \) or \( q = 12 \) groups for the \( t \)-statistic.

**Remark 2.4** The property of the \( t \)-statistic in (22) implies that the asymptotic \( p \)-value of the test of \( H_0 : \beta = 0 \) against \( H_a : \beta \neq 0 \) is given by \( p = \max_{R \leq k \leq q} P \left( T_{k-1} > \sqrt{\frac{R(k-1)}{k-R}} \right) \), where \( R = R(t_q(\tau)) = \frac{q(t_q(\tau))^2}{(t_q(\tau))^2 + q-1} \). Hence, one may conduct asymptotically valid robust \( t \)-statistic tests of any level. As discussed in IM (2010), asymptotic validity of the robust \( t \)-statistic approaches further implies that the confidence intervals

\[
\bar{\beta} \pm \text{cv}_s \beta, \quad (23)
\]

where \( \text{cv} \) is the usual \((1 + C)/2\) quantile of the Student’s \( t \)-distribution with \( q - 1 \) degrees of freedom, have asymptotic coverage of at least \( C \) for all \( C \geq 1 - 2\Phi(-\sqrt{3}) = 0.917 \ldots \). Further, for \( 2 \leq q \leq 14 \), confidence intervals (23) with \( \text{cv} \) being the usual 0.95 quantile of the Student-\( t \) distribution with \( q - 1 \) degrees of freedom, have asymptotic coverage of at least 0.9.

**Remark 2.5** The \( t \)-statistic approaches are applicable whenever the group-based estimators converge to scale mixtures of normal distributions and are asymptotically independent (see Bakirov and Székely, 2005, IM, 2010, and Remark 2.2). Hence, the limiting distributions are, in principle, not required to be Gaussian, as further discussed in Pedersen (2020) in the symmetric case as well as in Example 2.3 below.

**Remark 2.6** The origin of the particular threshold for the level, \( \tilde{\alpha} \leq 2\Phi(-\sqrt{3}) = 0.08326 \ldots \), (and the implied coverage in confidence intervals (23)) is due to theoretical results derived by Bakirov and Székely (2005). More precisely, let \( X_j, j = 1, \ldots, q \), with \( q \geq 2 \), be independent Gaussian r.v.’s with common mean \( E(X_j) = \mu \) and variances \( \text{Var}(X_j) = \sigma_j^2 \). Consider the usual \( t \)-statistic \( t = \sqrt{qX/sX} \) for the hypothesis test \( H_0 : \mu = 0 \) against the alternative \( \mu \neq 0 \), where \( X = q^{-1} \sum_{j=1}^q X_j \) and \( s_X^2 = (q-1)^{-1} \sum_{j=1}^q (X_j - \bar{X})^2 \). From Corollary 2 in Bakirov and Székely (2005) it follows that \( P(|T_{q-1}| > \sqrt{3}) > 2\Phi(-\sqrt{3}) \) for all \( q \geq 4 \), and, by direct checking, this inequality also holds for \( q = 2, 3 \). This implies that, for \( \tilde{\alpha} \leq 2\Phi(-\sqrt{3}) \), one has \( \text{cv}(q, \tilde{\alpha}) > \sqrt{3} \) and, thus, by Theorem 1 in that paper, \( \sup_{\sigma_1^2, \ldots, \sigma_q^2} P(|t| > \text{cv}(q, \tilde{\alpha})) = P(|T_{q-1}| > \text{cv}(q, \tilde{\alpha})) \), as is stated in IM (2010, Theorem 1). To provide some intuition on why this result holds, suppose that the independent variables \( X_j \)
are heavy-tailed symmetric stable (which is a special case of Gaussian scale mixtures). In this case, the \( t \)-statistic has a bimodal probability density function (see, e.g., Fiorio et al., 2010 and the references therein). This shifts its probability mass from the tails to the center and leads to the density being smaller in the tails than the (Student’s \( t \)) density of the \( t \)-statistic in i.i.d. Gaussian r.v.’s.

We end this section with following example that states that Theorem 2.1 applies to GARCH processes under mild conditions.

**Example 2.3** As discussed in Section 2.1 under suitable conditions, the GARCH(1,1) process is \( \beta \)-mixing (with geometric rate) and, hence, satisfies the assumptions of Lemmas 2.1-2.3 under the conditions on the tail index \( \zeta \) and powers \( p, s \) discussed in Example 2.2. Hence, Theorem 2.1 holds. Thus, for instance, the (centered and scaled) group estimators \( \hat{\gamma}_j, R^p(h) \) and \( \hat{\rho}_j, R^p(h) \) are asymptotically independent and normal if \( 4p < \zeta \). Likewise, the group estimators \( \hat{\gamma}_{j, R^s} (R^{\text{sign}(R)}(h)) \) are asymptotically independent and normal if \( 2(1+s) < \zeta \). Similar conclusions hold for the group estimators \( \hat{\rho}_{j, R^s} (R^{\text{sign}(R)}(h)) \). Suppose instead that \( 2(1+s) > \zeta > 1+s \). Then the group estimators \( \hat{\gamma}'_{j, R^s} (R^{\text{sign}(R)}(h)) \) are asymptotically independent and stable; see also Remark 2.2. If the innovations, \( Z_t \), are symmetric, then the limiting stable distributions are symmetric, and given by scale mixtures of normal distributions. In such situations, as pointed out in Remark 2.5, the \( t \)-statistic approaches are applicable for inference on property (i’). These aspects of the \( t \)-statistic approaches are further investigated in the simulation experiments in the next section.

### 3 Finite-sample properties

In this section, we present numerical results on finite-sample properties of the \( t \)-statistic approaches and compare them with those of HAC-based approaches in inference on properties (i’)(ii’). We consider the AR-ARCH DGPs given by

\[
R_t = \phi R_{t-1} + \varepsilon_t, \quad t = 1, \ldots, T, \tag{24}
\]

\[
\varepsilon_t = \sigma_t Z_t, \tag{25}
\]

\[
\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2, \tag{26}
\]
where $\omega = 0.1$, $0 < \alpha < 1$, $0 \leq \phi < 1$, and $(Z_t)_{t=1,...,T}$ are i.i.d. r.v.’s with $E(Z_t) = 0$ and $\text{Var}(Z_t) = 1$. In terms of the distribution of the innovations $Z_t$, we consider:

(a) *Symmetric light-tailed distribution:* $Z_t$ is standard normal, $Z_t \sim \mathcal{N}(0, 1)$.

(b) *Asymmetric light-tailed distribution:* $Z_t$ has an asymmetric (skewed) t-distribution with 50 degrees of freedom and the skewness parameter of 0.5: $Z_t \sim t(50, 0.5)$.

(c) *Asymmetric heavy-tailed distribution:* $Z_t$ has an asymmetric (skewed) t-distribution with 3 degrees of freedom and the skewness parameter of 0.5: $Z_t \sim t(3, 0.5)$.

The densities of the asymmetric $t$-distributions in (b) and (c) are given in Hansen (1994, equations (10)-(13)) with $\lambda = 0.5$ and $\eta = 50, 3$, respectively.

In order to keep the discussion focused, we here present results for the simple case of ARCH(1) errors, $\varepsilon_t$, as the tail index $\zeta$ is easier to determine in terms of $\alpha$ (rather than as a combination of $\alpha$ and $\beta$ for GARCH processes). In the Online Appendix, we present simulations for GARCH(1,1) errors, and we emphasize that the numerical results for the ARCH(1) cases are very similar to the results for GARCH errors and asymmetric GJR-GARCH errors (not reported in the paper). All tests considered have a 5% nominal level. We use a sample size of 5,000 observations and 10,000 Monte Carlo replications. All computations were done in MATLAB (v. 2020a).

### 3.1 Testing for linear (in)dependence and market (non-)efficiency

For investigating the finite-sample properties of inference methods, we consider the processes in (24)-(26) with $\alpha = \pi^{1/3}/2$ and innovations $Z_t$ with distributions in (a)-(c). By Kesten’s equation (3), when $Z_t \sim \mathcal{N}(0, 1)$ as in (a), the processes $R_t$ and $\varepsilon_t$ in (24)-(26) have heavy-tailed power-law distributions as in (iii) with the tail index $\zeta = 3$. Likewise, solving Kesten’s equation (3) numerically, we obtain that $R_t$ and $\varepsilon_t$ have heavy-tailed power-law distributions with the tail indices $\zeta \approx 2.89$ and $\zeta \approx 2.24$, respectively, in cases (b) and (c). Therefore,
under distributions \([a],[c]\) for innovations \(Z_t\), the tail indices \(\zeta\) of the processes \(R_t\) and \(\varepsilon_t\) lie in the interval \((2, 4)\), as is typically the case for financial returns in developed markets (see the discussion in the introduction). Further, as \(\zeta \in (2, 3]\), the absolute third moments of the processes are infinite.

We consider the finite-sample size and power properties of tests of the null hypotheses \(H_0 : \beta = 0\) against the two-sided alternative \(H_a : \beta \neq 0\) for \(\beta = \rho'_R,|R|^s\text{sign}(R)(1) = \text{Corr}(R_t, |R_{t-1}|^s\text{sign}(R_{t-1}))\) in \([i']\) with the lag \(h = 1\) and different powers \(s > 0\). Note that, under \(H_0\), the autoregressive coefficient in (24) is zero, \(\phi = 0\). For \(s = 1\), \(H_0\) corresponds to the standard property of absence of linear autocorrelations, \(\text{Corr}(R_t, R_{t-1}) = 0\), as in \([i]\). In simulations below, we consider the powers \(s = 0.1, 0.25, 0.5\).

Based on the stated values of the tail index, \(\zeta = 3, 2.89, 2.24\), Example 2.2 implies asymptotic normality of the full-sample estimator \(\hat{\beta} = \hat{\rho}'_{R,|R|^s\text{sign}(R)}(1)\) of \(\beta = \rho'_R,|R|^s\text{sign}(R)(1)\) in \((16)\) whenever \(s < 0.5\), \(s < 0.445\), and \(s < 0.12\), respectively, for the cases \([a],[b]\) and \([c]\). Likewise, from Example 2.3 it follows that, under \(H_0\) and the same conditions on powers \(s\), asymptotic normality and asymptotic independence hold for the group estimators \(\hat{\beta}_j = \hat{\rho}'_{j,R,|R|^s\text{sign}(R)}(1)\), implying, by Theorem 2.1, asymptotic validity of robust \(t\)–statistic inference approaches.

The first class of tests we consider is based on the HAC-based \(t\)–statistic of (full-)sample correlations \(\hat{\beta} = \hat{\rho}'_{R,|R|^s\text{sign}(R)}(1)\), with a long-run variance estimator based on a QS kernel with automatic bandwidth selection (Andrews 1991; see also Section 2.2), and the critical values based on the standard normal distribution. The second class of tests is based on \(t\)–statistics in the group estimators for \(q = 4, 8, 12\) and 16 groups.

### 3.1.1 Size properties

The results on size properties of HAC-based and robust \(t\)–statistic approaches for testing \(H_0 : \beta = \rho'_R,|R|^s\text{sign}(R)(1) = 0\), as in property \([i']\) against \(H_a : \beta \neq 0\) are provided in
Table 1. We note that the standard HAC-based tests are oversized. In particular, in the case of asymmetric heavy-tailed innovations $Z_t$ (case [c]), the size distortions are severe in the case where $s = 1$, i.e., when one carries out the usual test for the absence of linear autocorrelations as in property [1]. For the robust $t$-statistic approaches, the size control is good even for the case $s = 1$, except for the case of asymmetric heavy-tailed innovations. For the latter case, the limit of the full-sample and group estimators $\hat{\beta} = \hat{\rho}_{R,R|R|s\text{sign}(R)}(1)$ and $\hat{\beta}_j = \hat{\rho}_{j,R,R|R|s\text{sign}(R)}(1)$ is asymmetric stable, invalidating the use of both the HAC and robust $t$-statistic approaches (Example 2.2). In contrast, in the case where $s = 1$ and the distribution of $Z_t$ is standard normal (case [a]), one has that the group estimators $\hat{\beta}_j = \hat{\rho}_{j,R,R|R|s\text{sign}(R)}(1)$ are asymptotically independent and symmetric stable (Example 2.3), implying that the $t$-statistic robust tests of $H_0$ are asymptotically valid. This is reflected in the attractive size properties of the robust $t$-statistic approaches, e.g., in contrast to those of the standard HAC-approaches. The same conclusions hold in the case $s = 0.5$. The robust $t$-statistic approaches to testing the hypotheses $H_0$ under [a], [b] with $s = 0.1, 0.25$ and under [c] with $s = 0.1$ – where estimators are asymptotically normal – are slightly over-sized, with reasonable size control for $q = 4$ and $q = 8$. Quite remarkably, the tests with the most desirable size properties are the ones for testing $H_0$ with $s = 0.1$ based on the $t$-statistic approaches with $q = 4$ or $q = 8$ number of groups. The reason might be that the asymptotic (Gaussian) distributions in (16) in Lemma 2.2 provide relatively good approximations to the distributions of the group estimators for small choices of powers $s$, whereas the quality of the approximations worsens for larger values of $s$.

Remark 3.1 We note that for some of the DGPs, and for some powers $s$, the size distortions of the HAC-based approach are not overly severe, although the approach is not theoretically justified, in contrast to $t$-statistic inference. The reason may be two-fold. Firstly, finite sample distributions of the HAC-based $t$-statistics may be fairly well approximated by a standard Gaussian distribution, if the tails of the DGPs are not too heavy. Secondly, for any fixed bandwidth, the HAC-based $t$-statistic may be viewed as having a self-normalized structure. Further, as discussed in the Online Appendix time series inference approaches based on self-normalization typically exhibit robustness to infinite higher-order moments.
3.1.2 Power properties

To investigate the power properties of the HAC-based and robust $t$–statistic tests of $H_0 : \beta = 0$ against $H_a : \beta \neq 0$ for $\beta = \rho' R_t |R_{t-1}|^s \text{sign}(R_{t-1})$ as in (i'), we consider the alternatives where the autoregressive parameter $\phi$ in (24) ranges from 0 to 0.5. Figure 1 provides the size-adjusted rejection frequencies for the HAC-based and robust $t$–statistic (with $q = 8$ groups) tests of $H_0$ for the case of normal innovations (case (a)) and asymmetric heavy-tailed innovations (case (c)). When $Z_t$ are standard normal, the power curves for the HAC and the robust $t$–statistic tests are very close to each other. We note that the rejection frequencies are generally lower when $s = 1$, i.e., when testing the classical hypothesis of no linear dependence/absence of linear autocorrelations as in (i). As in the previous section, we note that, under $s = 1$ and standard normal innovations, the robust $t$–statistic tests remain asymptotically valid, in contrast to the HAC-based tests. Similar conclusions hold in the case of asymmetric heavy-tailed innovations (case (c)), with the only exception that the robust $t$–statistic tests have much better power properties than the HAC-based tests for the case of $s = 1$, although, as discussed in the previous section, the use of HAC-based and the robust $t$–statistic tests are not theoretically justified for this case.

For the sake of brevity, we do not report the results for the case of light-tailed asymmetric innovations (case (b)), where the power properties of the tests are similar to those for the Gaussian case in (a). Overall, the tests with $s = 0.1, 0.25$ are the most powerful.

Figure 2 contains rejection frequencies for the tests under standard normal innovations (case (a)) for $s = 1$ and different number $q$ of groups in robust $t$–statistic approaches. In this case, as mentioned, the robust $t$–statistic approaches are asymptotically valid, in contrast to HAC-based tests. Note that the rejection frequencies are increasing in $q$, and that the tests based on $q = 8, 12, 16$ groups appear to be more powerful than the HAC-based tests.
Similar conclusions hold for other powers, $s$, and other distributions of the innovations $Z_t$ (see Online Appendix).

[Figure 2 about here.]

To conclude, the numerical results indicate that in order to conduct reliable inference on property (i'), one may choose $s > 0$ small and rely on the robust t-statistic approaches with $q = 4$ or $q = 8$ groups. This ensures very reasonable size control as well as quite attractive power properties, e.g., in comparison with the widely used HAC-based approaches. The robust $t-$statistic approaches to inference may thus be viewed as useful complements to the traditional HAC-based methods.

3.2 Testing for nonlinear dependence and volatility clustering

Next, we turn to the finite sample properties of the HAC-based and robust $t-$statistic approaches to inference on property (ii') with the lag order of $h = 1$ and powers $p \in \{0.1, 0.25, 1, 2\}$. The DGPs considered follow (24)-(26) with $\phi = 0$ and the ARCH parameter $\alpha \in (0,1)$. For the sake of brevity, we present the results for the case of standard normal innovations $Z_t$ in (a). The results in the Online Appendix for asymmetric/heavy-tailed cases (b) and (c) reveal the same qualitative conclusions.

We investigate the relative performance of the $t$-statistic and HAC-based approaches by comparing the coverage levels of the corresponding $t$-statistic and HAC-based confidence intervals for the unknown population correlation $\beta = \rho_{|R|^p}(1)$. Specifically, the confidence intervals based on the $t$-statistic approaches are constructed as in (23), and the HAC-based confidence intervals are computed by standard methods, relying on asymptotic normality of the full-sample estimator $\hat{\beta} = \hat{\rho}_{|R|^p}(1)$. We compare the coverage of the confidence intervals instead of comparing the rejection frequencies of the tests for illustrative purposes. From a practical point of view, the non-linear dependence may almost always be prevalent, that is, the null hypothesis of no non-linear dependence is typically rejected, and it may instead be of
particular interest to consider a confidence interval for the degree of non-linear dependence. Note that for the present data-generating process, the null of no non-linear dependence is false for $\alpha > 0$. Note that with $\alpha \in (0,1)$, the processes $R_t$ and $\varepsilon_t$ have heavy-tailed power-law distributions as in (iii), with the tail index $\zeta > 2$. From Example 2.2 it follows that the sample correlation $\hat{\beta} = \hat{\rho}_{|R|^p}(1)$, is asymptotically normal if $\zeta > 4p$, and hence whenever $p \leq 0.5$. From Example 2.3 under the same conditions on powers $p$, it holds that the group-based estimators are asymptotically normal and asymptotically independent, implying asymptotic validity of the robust $t-$statistic inference approaches.

Figure 3 contains coverage levels of the confidence intervals for the HAC-based and $t-$statistic inference approaches for different choices of powers $p$ and different number of groups $q$ in the $t$-statistic approaches. As expected, the coverage is very unstable for the tests based on the powers $p = 2$ and $p = 1$, due to the loss of asymptotic normality for the full-sample and group estimators $\hat{\beta} = \hat{\rho}_{R|p}$ and $\hat{\beta}_j = \hat{\rho}_{j|R|p}$ and their convergence to functions of r.v.’s with asymmetric stable distributions for sufficiently large values of $\alpha$ (Example 2.2). Specifically, this holds if $\zeta \leq 4p$, and, hence, by Kesten’s equation (3), whenever $\alpha \geq 3^{-1/2} \approx 0.5574$ ($\alpha \geq 105^{-1/4} \approx 0.3124$) for $p = 1$ ($p = 2$).

The coverage improves for smaller powers, in particular for $p = 0.1, 0.25$. The best coverage across all values of $\alpha$ is observed for the robust $t-$statistic tests with $p = 0.1$ and $q = 4,8$ number of groups. For $p \leq 0.5$, the coverage levels of the HAC-based approaches are comparable to those of the $t-$statistic approaches with $q = 8$ groups, although the coverage for $q = 4$ groups is always better and closer to the correct 95%.

We emphasize that for the case where $p = 0.5$, the HAC-based methods are not theoretically justified due to infinite moments (as this requires $\zeta > 8p$). This is in contrast to the robust $t-$statistic approaches that are asymptotically valid under $p = 0.5$. Again, the reasonable performance of the HAC-based approach may be due to self-normalized structure of the HAC-based $t$-statistic, as discussed in Remark 3.1.
Similar to the findings in Section 3.1 to make reliable inference on property (ii'), one may choose \( p > 0 \) small in the robust \( t \)-statistic approaches with \( q = 4 \) or \( q = 8 \) groups. Similar to the discussion in Section 3.1, the latter approaches may be viewed as useful complements to HAC-based inference methods in the analysis of nonlinear dependence/volatility clustering property (ii').

4 Illustration: Revisiting Baltussen et al. (2019)

In this section, we revisit a recent study by Baltussen et al. (2019) that (among other contributions) tests for linear dependence in returns on the world’s major stock market indices. Specifically, relying on HAC-based inference applied to the first-order autocorrelations, \( \rho_{R,R}(1) \), Baltussen et al. (2019) state that serial dependence in daily returns on 20 major market indices covering 15 countries in North America, Europe, and Asia was significantly positive until the end of the 1990s, and switched to being significantly negative since the early 2000s. In light of the discussion in the introduction and Examples 2.2 and 2.3 with \( s = 1 \), asymptotic normality of sample linear autocorrelations requires finite fourth-order moments of the (GARCH-type) return process. In the case where such moment conditions are not satisfied, the sample linear autocorrelations weakly converge (under suitable conditions) to functions of non-Gaussian stable variables, invalidating the HAC-based inference approaches based on asymptotic normality.

We consider daily percentage returns on the major stock indexes from March 3, 1999 to December 31, 2016 as in Baltussen et al. (2019). The second and third columns of Table 2 provide, respectively, the (bias-corrected) log-log rank-size regression estimates of the tail indices for the return time series and their 95% confidence intervals (see Gabaix and Ibragimov, 2011). Importantly, for 19 out of 20 series, the estimates of the tail index are smaller than 4. The left end-points of the confidence intervals vary from 2.49 to 3.41 across the
return series, and several of the intervals lie to the left of 4. This indicates that standard HAC-based inference on linear autocorrelations, $\rho'_{R,R}(1)$, is invalid for several of the data series.

Columns 4 and 7 in Table 2 contain full-sample estimates $\hat{\rho}'_{R,|R|*\text{sign}(R)}$ of the correlations $\rho'_{R,|R|*\text{sign}(R)}(1)$ (as in (i')) for different values of $s$, and column 10 contains estimates of the multi-period (auto)correlation MAC(5) (a weighted sum of the correlation coefficients $\rho'_{R,|R|*\text{sign}(R)}(h)$ of order $h = 1, \ldots, 5$) used in Baltussen et al. (2019). Column 5 provides, similar to Baltussen et al. (2019), HAC-based $t$-statistics for the nullity of the linear autocorrelation $\rho'_{R,R}(1)$ (as in (i)), whereas column 6 contains the $t$-statistics in (20) based on $q = 8$ group estimates of $\rho'_{R,R}(1)$. In column 7, for each return time series, the power $s$ is chosen based on the left end-points of the confidence intervals for the tail index of the time series (column 3 of Table 2). Specifically, following Examples 2.2 and 2.3, if the left end-point exceeds 3, we set $s = 0.5$; if the end-point lies between 2.5 and 3.0, we set $s = 0.25$; and if the end-point lies between 2.2 and 2.5, we set $s = 0.1$. Under the above values of powers $s$ (and assuming that the true tail index $\zeta$ belongs to the reported confidence intervals), the moment conditions for asymptotic normality of $\hat{\rho}'_{R,|R|*\text{sign}(R)}(1)$ and asymptotic independence and normality of the group estimators (under the hypothesis $H_0 : \rho'_{R,|R|*\text{sign}(R)}(1) = 0$ in consideration) in Lemmas 2.2 and 2.3 are satisfied. Consequently, by Theorem 2.1, the robust $t$-statistic approaches for testing $H_0 : \rho'_{R,|R|*\text{sign}(R)}(1) = 0$ are asymptotically valid.

The reported HAC $t$-statistics and $t$-statistics in group estimates as in (20) in columns 5 and 6 suggest that the hypothesis of zero linear autocorrelation, $H_0 : \rho'_{R,R}(1) = 0$, cannot be rejected for most of the series at conventional significance levels. However, the HAC and robust $t$-statistic approaches are not theoretically justified in the case $s = 1$.

Further, based on the theoretically justified $t$-statistics in group estimates (as in (20)) in column 9, the hypothesis $H_0 : \rho'_{R,|R|*\text{sign}(R)}(1) = 0$ is rejected only for six of the series. Hence, based on the theoretically justified robust $t$-statistic approaches, we find evidence of zero
correlations in most of the series in contrast to the conclusions in Baltussen et al. (2019). On the other hand, similar to Baltussen et al. (2019), we find somewhat stronger evidence for non-zero weighted autocorrelations (MAC(5)), based on the robust $t$-statistics in group estimates reported in column 12, where the null of MAC(5) = 0 is rejected for 11 out of 20 series.

Table 3 contains the results on testing for nonlinear dependence and volatility clustering in the return time series considered using the 95% confidence intervals constructed on the base of robust $t$-statistic approaches applied to inference on the autocorrelations $\rho_{|R|^p}(5)$ with $p = 0.1, 0.5, 1, 2$. The table also presents the results on tail index estimation and HAC-based confidence intervals for the dependence measures considered. The $t$-statistic approaches are theoretically justified for the powers $p = 0.1, 0.5$ (but not for $p = 1, 2$) provided that the true tail indices belong to the confidence intervals reported in the table. One should further note that, for the above tail indices, the HAC approaches are theoretically justified only under $p = 0.1$. Overall, the results in the table confirm the presence of nonlinear dependence and volatility clustering in the returns on the most of the financial indices. Exceptions are the ASX 200 and Russell 2000 indices, where $t$-statistic approaches with $q = 8$ applied to $\rho_{|R|^{0.1}}(5)$ indicate, somewhat surprisingly, absence of volatility clustering.

5 Conclusion and suggestions for further research

The paper proposes new approaches to inference on measures of market (non-)efficiency, volatility clustering and nonlinear dependence in the case of general heavy-tailed dependent time series, including GARCH-type processes. We provide the results that motivate the use of measures of market (non-)efficiency and volatility clustering based on (small) powers of absolute returns and their signed versions.
The inference approaches dealt with in the paper are based on robust \( t \)-statistic tests and several new results on their applicability in the settings considered. Theoretical and numerical results and empirical applications in the paper confirm validity, appealing finite sample properties, and wide applicability of the proposed inference approaches.

The future research may focus on the development of the two-sample analogues of the approaches to robust inference on market (non-)efficiency and volatility clustering dealt with in the paper using the results in IM (2016). The results in this direction may be used in testing for structural breaks in the dynamics of key economic and financial time series, including financial returns and foreign exchange rates, and comparisons of the properties of the dynamics of different economic and financial markets. The research on the above inference problems is currently under way by the authors, and will be presented elsewhere.
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| q      | HAC | t–statistic approach |
|-------|-----|---------------------|
|       |     | $\rho_{R,R}(h)$ | $\rho'_{R,R}(h)$ | $\rho'_{R,|R|^{0.5}\text{sign}(R)}(h)$ | $\rho'_{R,|R|^{0.25}\text{sign}(R)}(h)$ | $\rho'_{R,|R|^{0.1}\text{sign}(R)}(h)$ |
|       |     | $\rho_{R,R}(h)$ |             |                     |                     |                    |
|       |     | $\rho_{R,|R|^{0.5}\text{sign}(R)}(h)$ |             |                     |                     |                    |
|       |     | $\rho_{R,|R|^{0.25}\text{sign}(R)}(h)$ |             |                     |                     |                    |
|       |     | $\rho_{R,|R|^{0.1}\text{sign}(R)}(h)$ |             |                     |                     |                    |
|       |     | $\rho'_{R,R}(h)$ |             |                     |                     |                    |
|       |     | $\rho'_{R,|R|^{0.5}\text{sign}(R)}(h)$ |             |                     |                     |                    |
|       |     | $\rho'_{R,|R|^{0.25}\text{sign}(R)}(h)$ |             |                     |                     |                    |
|       |     | $\rho'_{R,|R|^{0.1}\text{sign}(R)}(h)$ |             |                     |                     |                    |
| ARCH(1), $N(0,1)$ | 7.9 | 6.1 | 6.2 | 6.1 | 4.6 | 4.5 | 4.8 | 4.8 | 4.9 | 4.9 | 5.1 | 5.2 | 5.3 | 5.1 | 5.2 | 5.4 | 5.0 | 5.2 | 5.4 |
| ARCH(1), $t(50,0.5)$ | 8.6 | 5.9 | 5.8 | 5.8 | 4.4 | 5.1 | 5.6 | 6.2 | 4.6 | 5.1 | 5.3 | 6.1 | 4.9 | 5.4 | 5.5 | 5.8 | 5.0 | 5.3 | 5.7 |
| ARCH(1), $t(3,0.5)$ | 17.3 | 9.7 | 8.2 | 7.6 | 6.6 | 9.5 | 11.5 | 13.6 | 5.8 | 7.3 | 8.2 | 9.3 | 5.5 | 6.5 | 7.1 | 8.0 | 5.2 | 5.9 | 6.3 | 7.3 |
| GARCH(1,1), $N(0,1)$ | 7.6 | 6.2 | 6.1 | 5.9 | 4.3 | 4.5 | 4.8 | 5.2 | 4.5 | 5.0 | 5.1 | 5.2 | 4.6 | 5.0 | 5.1 | 5.3 | 4.5 | 5.1 | 5.2 | 5.1 |
| GARCH(1,1), $t(50,0.5)$ | 8.6 | 6.7 | 6.5 | 6.4 | 4.6 | 5.6 | 6.1 | 6.4 | 4.8 | 5.6 | 5.7 | 6.0 | 4.8 | 5.4 | 5.4 | 5.6 | 4.8 | 5.4 | 5.3 | 5.4 |
| GARCH(1,1), $t(3,0.5)$ | 14.5 | 8.4 | 7.6 | 7.2 | 5.3 | 7.5 | 9.0 | 9.9 | 4.9 | 6.2 | 6.8 | 7.5 | 5.2 | 5.9 | 6.3 | 7.0 | 5.5 | 5.8 | 6.1 | 6.5 |

The DGPs for the AR-ARCH simulations are given in (24)-(26).

The DGPs for the AR-GARCH simulations are given in (S.22)-(S.24) in the Online Appendix.
Table 2: Empirical results, testing for efficient market hypothesis, $h = 1$

| Series       | Estimate of tail index | CI for tail index           | $\hat{\rho}_{R,R}(h)$ | t-statistic for $\hat{\rho}_{R,R}(h) = 0$ (HAC) | $t$ for $\hat{\rho}_{R,R}(h) = 0$ ($q = 8$) | Estimate of MAC(5) | t-statistic for MAC(5) = 0 (HAC) | $t$ for MAC(5) = 0 ($q = 8$) |
|--------------|------------------------|-----------------------------|------------------------|-------------------------------------------------|-------------------------------------------------|------------------------|-------------------------------|-------------------------------|
| S&P 500      | 3.26                   | [2.66,3.86]                 | -0.074                 | -3.44***                                         | -3.12***                                         | -0.056                 | -3.74***                      | -3.32***                      | -0.085                         | -4.41***                         | -5.02***                         |
| FTSE 100     | 3.48                   | [2.84,4.12]                 | -0.040                 | -2.24***                                         | -1.75                                           | -0.029                 | -1.96**                       | -1.69                          | -0.075                         | -4.31***                         | -4.12***                         |
| DJESI 50     | 3.73                   | [3.05,4.42]                 | -0.029                 | -1.66**                                          | -2.1**                                          | -0.027                 | -1.71**                       | -2.03**                       | -0.057                         | -3.59***                         | -3.31***                         |
| TOPIX        | 3.31                   | [2.60,3.93]                 | 0.014                  | 0.66                                            | 1.63                                            | 0.036                  | 2.26***                       | 2.05**                         | -0.014                         | -0.87                            | 1.42                             |
| ASX 200      | 3.41                   | [2.78,4.04]                 | -0.034                 | -1.77**                                          | -1.23                                           | -0.018                 | -1.16                          | -0.71                          | -0.038                         | -1.86**                          | -0.96                            |
| TSE 60       | 3.13                   | [2.55,3.70]                 | -0.039                 | -1.7**                                           | -0.15                                           | -0.003                 | -0.2                           | 0.06                           | -0.058                         | -2.93***                         | -1.55                            |
| CAC 40       | 3.64                   | [2.97,4.30]                 | -0.027                 | -1.51                                           | -1.77                                           | -0.025                 | -1.65                          | -1.72                          | -0.061                         | -3.57***                         | -2.76***                         |
| DAX          | 3.62                   | [2.96,4.29]                 | -0.013                 | -0.8                                            | -0.64                                           | -0.008                 | -0.53                          | -0.6                           | -0.026                         | -1.51                            | -1.48                            |
| IBEX 35      | 3.77                   | [3.07,4.46]                 | 0.004                  | 0.23                                            | -0.09                                           | 0.005                  | 0.29                           | -0.04                          | -0.036                         | -2.12***                         | -1.91***                         |
| MIB          | 3.75                   | [3.06,4.44]                 | -0.024                 | -1.48                                           | -1.91**                                          | -0.027                 | -1.78**                        | -1.91**                        | -0.031                         | -1.79**                          | -2.93***                         |
| AEX Index    | 3.32                   | [2.71,3.93]                 | -0.007                 | -0.37                                           | -0.19                                           | 0.002                  | 0.12                           | 0.14                           | -0.026                         | -1.58                            | -1.14                            |
| OMX Stockholm| 4.18                   | [3.44,4.95]                 | 0.001                  | 0.06                                            | -0.86                                           | 0.013                  | 0.83                           | -0.03                          | -0.030                         | -2.07***                         | -1.59                            |
| SMI          | 3.19                   | [2.60,3.78]                 | 0.029                  | 1.49                                            | 0.89                                            | 0.017                  | 1.15                           | 0.69                           | -0.016                         | -0.78                            | -1.24                            |
| Nikkei 225   | 3.32                   | [2.70,3.95]                 | -0.042                 | -1.95**                                          | -2.05**                                          | -0.024                 | -1.48                          | -1.54                          | -0.046                         | -2.96***                         | -2.26**                          |
| HSI          | 3.41                   | [2.78,4.05]                 | -0.004                 | -0.2                                            | 0.51                                            | 0.026                  | 1.69**                         | 1.88                           | -0.014                         | -0.9                            | 0.76                             |
| Nasdaq 100   | 3.64                   | [2.97,4.32]                 | -0.059                 | -3.17**                                          | -2.18**                                          | -0.036                 | -2.6**                         | -2.59**                        | -0.087                         | -4.72***                         | -3.36***                         |
| NYSE         | 3.05                   | [2.49,3.62]                 | -0.057                 | -2.57**                                          | -1.56                                           | -0.025                 | -1.75**                        | -1.23                          | -0.067                         | -3.54***                         | -2.42***                         |
| Russell 2000 | 3.51                   | [2.86,4.16]                 | -0.058                 | -2.62**                                          | -1.41                                           | -0.026                 | -1.58                          | -1.24                          | -0.060                         | -3.48***                         | -2.37***                         |
| S&P 400      | 3.38                   | [2.76,4.01]                 | -0.029                 | -1.34                                           | 0.05                                            | -0.005                 | 0.3                            | 0.59                           | -0.053                         | -2.93**                          | -2.4**                           |
| KOSPI 200    | 3.88                   | [3.16,4.61]                 | 0.023                  | 1.23                                            | 2.05**                                          | 0.031                  | 1.88**                         | 2.71**                         | -0.014                         | -0.96                            | 0.17                             |

Note: *, ** and *** denote the significance at 10, 5 and 1% levels respectively.
Table 3: Empirical results, testing for non-linearity, $h = 5$

| Series          | $\hat{\rho}_Q(h)$ | CI for $\hat{\rho}_Q(h)$ (q = 5) | $\hat{\rho}_R(h)$ | CI for $\hat{\rho}_R(h)$ (q = 5) | $\hat{\rho}_{Q^0}(h)$ | CI for $\hat{\rho}_{Q^0}(h)$ (q = 5) | $\hat{\rho}_{Q^0^1}(h)$ | CI for $\hat{\rho}_{Q^0^1}(h)$ (q = 5) |
|-----------------|-------------------|----------------------------------|-------------------|----------------------------------|------------------------|----------------------------------|------------------------|----------------------------------|
| S&P 500         | 0.316             | [0.17, 0.47]                     | 0.331             | [0.28, 0.38]                     | 0.261                  | [0.21, 0.31]                     | 0.261                  | [0.21, 0.31]                     |
| FTSE 100        | 0.330             | [0.17, 0.49]                     | 0.302             | [0.23, 0.37]                     | 0.240                  | [0.20, 0.28]                     | 0.090                  | [0.06, 0.12]                     |
| DJESI 50        | 0.259             | [0.16, 0.35]                     | 0.255             | [0.20, 0.31]                     | 0.203                  | [0.16, 0.24]                     | 0.126                  | [0.09, 0.16]                     |
| TOPIX           | 0.203             | [0.11, 0.3]                      | 0.219             | [0.14, 0.29]                     | 0.170                  | [0.12, 0.22]                     | 0.109                  | [0.08, 0.14]                     |
| ASX 200         | 0.228             | [0.12, 0.34]                     | 0.243             | [0.18, 0.3]                      | 0.188                  | [0.14, 0.23]                     | 0.020                  | [0.01, 0.09]                     |
| TSE 60          | 0.350             | [0.22, 0.48]                     | 0.354             | [0.26, 0.44]                     | 0.272                  | [0.22, 0.33]                     | 0.035                  | [0.01, 0.09]                     |
| CAC 40          | 0.262             | [0.15, 0.37]                     | 0.248             | [0.19, 0.3]                      | 0.201                  | [0.16, 0.24]                     | 0.142                  | [0.11, 0.17]                     |
| DAX             | 0.246             | [0.16, 0.34]                     | 0.274             | [0.23, 0.32]                     | 0.234                  | [0.20, 0.27]                     | 0.080                  | [0.05, 0.14]                     |
| IBEX 35         | 0.159             | [0.05, 0.26]                     | 0.217             | [0.16, 0.27]                     | 0.191                  | [0.15, 0.23]                     | 0.041                  | [0.01, 0.09]                     |
| MIB             | 0.212             | [0.12, 0.3]                      | 0.219             | [0.20, 0.3]                      | 0.220                  | [0.18, 0.26]                     | 0.041                  | [0.01, 0.09]                     |
| AEX Index       | 0.377             | [0.25, 0.51]                     | 0.325             | [0.26, 0.39]                     | 0.248                  | [0.20, 0.29]                     | 0.152                  | [0.12, 0.18]                     |
| OMX Stockholm   | 0.232             | [0.14, 0.32]                     | 0.240             | [0.19, 0.29]                     | 0.208                  | [0.17, 0.24]                     | 0.071                  | [0.05, 0.13]                     |
| SMI             | 0.283             | [0.19, 0.38]                     | 0.302             | [0.23, 0.37]                     | 0.244                  | [0.20, 0.29]                     | 0.090                  | [0.05, 0.13]                     |
| Nikkei 225      | 0.214             | [0.11, 0.32]                     | 0.219             | [0.14, 0.3]                      | 0.174                  | [0.13, 0.22]                     | 0.126                  | [0.09, 0.16]                     |
| HSI             | 0.177             | [0.10, 0.25]                     | 0.235             | [0.18, 0.29]                     | 0.195                  | [0.15, 0.24]                     | 0.143                  | [0.11, 0.17]                     |
| Nasdaq 100      | 0.224             | [0.11, 0.34]                     | 0.346             | [0.27, 0.39]                     | 0.310                  | [0.27, 0.35]                     | 0.163                  | [0.14, 0.19]                     |
| NYSE            | 0.366             | [0.19, 0.54]                     | 0.437             | [0.27, 0.43]                     | 0.261                  | [0.21, 0.31]                     | 0.178                  | [0.15, 0.21]                     |
| Russell 2000    | 0.315             | [0.22, 0.41]                     | 0.288             | [0.22, 0.36]                     | 0.206                  | [0.15, 0.26]                     | 0.113                  | [0.06, 0.16]                     |
| S&P 400         | 0.338             | [0.22, 0.46]                     | 0.326             | [0.26, 0.4]                      | 0.245                  | [0.19, 0.29]                     | 0.161                  | [0.13, 0.19]                     |
| KOSPI 200       | 0.218             | [0.13, 0.3]                      | 0.263             | [0.22, 0.31]                     | 0.238                  | [0.20, 0.27]                     | 0.051                  | [0.02, 0.12]                     |

Note: * denote the significance at 5% level.
Figure 1: Size-adjusted power for ARCH(1)

\[
\begin{align*}
\rho'_{R,R}(h), \ HAC : & \quad \rho'_{R,|R|^{0.5}\text{sign}(R)}(h), \ HAC : \quad , \\
\rho'_{R,|R|^{0.25}\text{sign}(R)}(h), \ HAC : & \quad \rho'_{R,|R|^{0.1}\text{sign}(R)}(h), \ HAC : \quad , \\
\rho'_{R,R}(h), q = 8 : & \quad , \quad \rho'_{R,|R|^{0.5}\text{sign}(R)}(h), q = 8 : \quad \triangle , \\
\rho'_{R,|R|^{0.25}\text{sign}(R)}(h), q = 8 : & \quad , \quad \rho'_{R,|R|^{0.1}\text{sign}(R)}(h), q = 8 : \quad \square .
\end{align*}
\]
Figure 2: Size-adjusted power for ARCH(1) with $N(0,1)$ noise, $\rho'_{R,R}(h)$. HAC: $q = 4$: □, $q = 8$: △, $q = 12$: ▽, $q = 16$: ◊.
Figure 3: Coverage level for ARCH(1) with $N(0,1)$ noise

HAC: $q = 4$ $q = 8$ $q = 12$ $q = 16$