1. Introduction

Let $\mathbb{F}_q$ be the finite field of order $q$ and $V$ be a vector space of dimension $n$ over $\mathbb{F}_q$. Since $V$ is isomorphic to $\mathbb{F}_q^n$, we will assume $V = \mathbb{F}_q^n$ in the following. By $\mathcal{G}_q(n,k)$ we denote the set of all $k$-dimensional subspaces of $\mathbb{F}_q^n$, where $0 \leq k \leq n$. The projective space of order $n$ over $\mathbb{F}_q$ is given by $\mathcal{P}_q(n) = \cup_{0 \leq k \leq n}\mathcal{G}_q(n,k)$. It is well known that

$$d_S(U,W) := \dim U + \dim W - 2 \dim(U \cap W)$$

is a metric on $\mathcal{P}_q(n)$ [1]. Thus, one can define codes on $\mathcal{P}_q(n)$ and $\mathcal{G}_q(n,k)$, which are called subspace codes and constant dimension codes, respectively. The distance function $d_S$ is known as subspace distance and one of the two distance functions that can be motivated by an information-theoretic analysis of the so-called Silva-Kschischang-Kötter model [24]. The second distance function is the so-called injection distance $d_I(U,V) := \max\{\dim U, \dim V\} - \dim(U \cap V)$. For two subspaces of the same dimension we have $d_S(U,V) = 2d_I(U,V)$, i.e., the two metrics are equivalent on $\mathcal{G}_q(n,k)$, and $d_I(U,V) \leq d_S(U,V) \leq 2d_I(U,V)$ in general. We say that $\mathcal{C} \subseteq \mathcal{P}_q(n)$ is an $(n,M,d)_q$ code in the projective space if $|\mathcal{C}| = M$ and $d(U,V) \geq d$ for all $U,V \in \mathcal{C}$. If $\mathcal{C} \subseteq \mathcal{G}_q(n,k)$ for some $k$, we speak of an $(n,M,d;k)_q$ code. The minimum distance of a code $\mathcal{C} \subseteq \mathcal{P}_q(n)$ is denoted by $D_S(\mathcal{C}) := \min_{U \neq V \in \mathcal{C}}d_S(U,V)$. One major problem is the determination of the maximum size $A_q(n,d)$ of an $(n,M,d)_q$ code in $\mathcal{P}_q(n)$ and the maximum size $A_q(n,d;k)$ of an $(n,M,d;k)_q$ code in $\mathcal{G}_q(n,k)$. Bounds for $A_q(n,d)$ and $A_q(n,d;k)$ have been heavily studied, see e.g. the survey [13] or the new online database at http://subspacecodes.uni-bayreuth.de [15].

With respect to lower bounds on $A_q(n,d;k)$, an asymptotically optimal construction is given by lifted maximum-rank-distance codes [14, 24]. To be more precise, the rate of the transmission $\log_q|\mathcal{C}|$ is asymptotically optimal up to a constant factor [17]. A rough estimation between $|\mathcal{C}|$ and the Singleton bound yields an approximation factor of at most 4. The concept of maximum-rank-distance codes was generalized from arbitrary rectangular matrices to matrices with a (structured)

1991 Mathematics Subject Classification. Primary 05B25, 51E20; Secondary 51E22, 51E23.

Key words and phrases. Constant dimension codes, subspace codes, subspace distance, Echelon-Ferrers construction.

The work of the authors was supported by the grant KU 2430/3-1 and WA 1666/9-1 “Integer Linear Programming Models for Subspace Codes and Finite Geometry” from the German Research Foundation and the COST Action IC1104 “Random Network Coding and Designs over GF(q)”.

COSET CONSTRUCTION FOR SUBSPACE CODES

DANIEL HEINLEIN AND SASCHA KURZ

Abstract. One of the main problems of the research area of network coding is to compute good lower and upper bounds of the achievable cardinality of so-called subspace codes in $\mathcal{P}_q(n)$, i.e., the set of subspaces of $\mathbb{F}_q^n$ for a given minimal distance. Here we generalize a construction of Etzion and Silberstein to a wide range of parameters. This construction, named coset construction, improves or attains several of the previously best-known subspace code sizes and attains the MRD bound for an infinite family of parameters.
set of prescribed zeros in [11] and used to combine several maximum-rank-distance codes to generate a constant dimension code — the so-called multilevel or Echelon-Ferrers construction. Many of the best-known lower bounds on $A_q(n,d;k)$ arise from this construction. However, it is rather general and involves several search spaces or optimization problems in order to be evaluated optimally. For special subclasses explicit variants of the construction and indeed explicit formulas for the sizes of the corresponding codes have been obtained, see [25]. We remark that additional refinements of the Echelon-Ferrers construction have been proposed recently, see [10, 12, 23].

An improvement beyond the Echelon-Ferrers construction was Construction III in [12] giving $A_2(8,4;4) \geq 4797$. The authors conjecture that the underlying idea can be generalized to further parameters assuming the existence of a corresponding parallelism. In Theorem 9 we will show that this is indeed the case. Moreover, there is a more general underlying construction for $(n,M,d)_q$ codes that is capable of improving some of the so far best-known lower bounds on $A_q(n,d;k)$, which is the core of this paper. To this end, we will give several infinite, parametric families of constructions as well as sporadic examples.

The remaining part of the paper is organized as follows. In Section 2 we collect some facts about representations of subspaces, MRD codes, parallelisms, and the Echelon-Ferrers construction. The main idea of the coset construction is described in Section 3. Since this construction has several degrees of freedom, we present some first insights on the choice of “good” parameters in Section 4. After listing some examples improving or attaining several lower bounds on $A_q(n,d;k)$ in Section 5, we conclude with open questions in Section 6.

2. Preliminaries

In this section we summarize some notation and well-known insights that will be used in the later parts of the paper.

2.1. Gaussian elimination and representations of subspaces. Let $A \in F_q^{k \times n}$ be a matrix of (full) rank $k$. The row-space of $A$ forms a $k$-dimensional subspace of $F_q^n$. The matrix $A$ is called generator matrix of a given element of $G_q(n,k)$. Since the application of the Gaussian elimination algorithm on a generator matrix $A$ does not change the row-space, we can restrict ourselves on generator matrices which are in reduced row echelon form (RRE form), i.e., the matrix has the shape resulting from Gaussian elimination. The representation is unique and does not depend on the elimination algorithm. This well-known connection is indeed a bijection, which we denote by

$$\tau : G_q(n,k) \to \{ A' \in F_q^{k \times n} : \text{rk}(A') = k, A' \text{ in RRE form} \}.$$  (1)

This observation is capable of easily explaining many properties of $G_q(n,k)$ so that we commonly identify the elements of $G_q(n,k)$ with their corresponding generator matrices in reduced row echelon form.

Given a matrix $A \in F_q^{k \times n}$ of full rank we denote by $p(A) \in F_q^n$ the binary vector whose 1-entries coincide with the pivot columns of $A$. For each $v \in F_q^n$ let $EF_q(v)$ denote the set of all $k \times n$ matrices over $F_q$ that are in reduced row echelon form with pivot columns described by $v$, where $k$ is the weight of $v$.

**Example 1.** For $v = (1,0,1,1,0)$ we have

$$EF_q(v) = \left\{ \begin{pmatrix} 1 & * & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \end{pmatrix} \right\},$$

where the *s represent arbitrary elements of $F_q$, i.e., $|EF_q(v)| = q^4$. 
In general we have
\[
\left| EF_q \left( v_1, \ldots, v_n \right) \right| = q^n \sum_{i=1}^{n} (1-v_i) \cdot \sum_{j=1}^{i} v_j.
\]
and the structure of the corresponding matrices can be read off from the corresponding (Echelon)-Ferrers diagram
\[
\bullet \quad \bullet \quad \bullet,
\]
where the pivot columns and zeros are omitted and the stars are replaced by solid black circles. A Ferrers diagram represents partitions as patterns of dots, with the ith row having the same number of dots as the ith term of the partition \( n' = s_1 + \cdots + s_l \), where \( s_1 \geq \cdots \geq s_l \) and \( s_j \in \mathbb{N}_{>0} \), cf. [3]. Usually a Ferrers diagram is depicted in such a way that it is the vertically mirrored version of the above constructed (Echelon)-Ferrers diagram. In the special case of Echelon-Ferrers diagrams, we have
\[
n' = \sum_{i=1}^{n} (1-v_i) \cdot \sum_{j=1}^{i} v_j.
\]
By summing over all binary vectors of weight \( k \) in \( \mathbb{F}_2^n \) one can compute
\[
|G_q(n,k)| = \left[ \begin{array}{c} n \\ k \end{array} \right]_q := \prod_{i=1}^{k} \frac{q^{n-k+i} - 1}{q^i - 1},
\]
where \( \left[ \begin{array}{c} n \\ k \end{array} \right]_q \) is called Gaussian binomial coefficient.

Later on we will use the inverse operation of deleting the pivot columns of a matrix in RRE form:

**Definition 2.** Let \( B \in \mathbb{F}_q^{k \times n} \) be a full-rank matrix in RRE form and \( F \in \mathbb{F}_q^{k' \times (n-k)} \) be arbitrary, where \( k, k', n \in \mathbb{N} \) and \( k \leq n \). Let further \( f^i \) denote the ith column of \( F \) for \( i \in \{1, \ldots, n\} \). Then, \( G = \varphi_B(F) \) denotes the \( k' \times n \) matrix over \( \mathbb{F}_q \) whose columns are given by \( g^i = 0 \in \mathbb{F}_q^{k'} \) if \( v_i = 1 \) and \( g^i = f^{i-s_i} \), otherwise, where \((v_1, \ldots, v_n) = p(B) \) and \( s_i = \sum_{j=1}^{i} v_j \), for all \( 1 \leq i \leq n \).

**Example 3.** For
\[
B = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\]
we have \( p(B) = (0,1,0,1,0,1) \), \( s_1 = 0, s_2 = s_3 = 1, s_4 = s_5 = 2, s_6 = 3 \), and
\[
\varphi_B(F) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.
\]

**2.2. MRD codes and the Echelon-Ferrers construction.** For matrices \( A,B \in \mathbb{F}_q^{m \times n} \) the rank distance is defined via \( d_R(A,B) := \text{rk}(A-B) \). It is indeed a metric, as observed in [14]. The maximum possible cardinality of a rank-metric code with given minimum rank distance is exactly determined in all cases.

**Theorem 4.** (see [14]) Let \( m,n \geq d \) be positive integers, \( q \) a prime power, and \( C \subseteq \mathbb{F}_q^{m \times n} \) be a rank-metric code with minimum rank distance \( d \). Then, \( |C| \leq q^{\max(m,n)-(\min(m,n)-d+1)} \). Codes attaining this upper bound are called maximum-rank distance (MRD) codes. They exist for all (suitable) choices of parameters.
If \( m < d \) or \( n < d \), then only \(|C| = 1\) is possible, which can be combined to give a single upper bound \(|C| \leq \left[ q^{\max\{n,m\} (\min\{n,m\} - d + 1)} \right] \). Using an \( n \times m \) identity matrix as a prefix, one obtains the corresponding subspace codes known as lifted MRD codes.

**Theorem 5.** (see [24]) For positive integers \( k, d, n \) with \( k \leq n \), \( d \leq 2 \min(k, n-k) \), and \( d \equiv 0 \pmod 2 \), the size of a lifted MRD code in \( G_q(n, k) \) with subspace distance \( d \) is given by

\[
M(q, k, n, d) := q^{\max\{k, n-k\} (\min\{k, n-k\} - d/2 + 1)}.
\]

If \( d > 2 \min(k, n-k) \), then we have \( M(q, k, n, d) = 1 \).

The subspace distance between two subspaces with the same pivots can be computed by the rank distance of the corresponding generator matrices. Using \( \tau \) from (1), we have:

**Lemma 1.** ([22, Corollary 3]) Let \( v \in \mathbb{F}_q^n \) and \( \tau(U), \tau(W) \in EF_q(v) \), then

\[
d_S(U, W) = 2 \cdot d_R(\tau(U), \tau(W)).
\]

So, in order to construct an \((n, M, 2\delta; k)\) code, it suffices to select a subset of \( EF_q(v) \) with minimum rank distance \( \delta \). Additionally, we can further expand such a code by introducing codewords with different pivot columns as long as the sets of pivot columns are sufficiently apart. Let \( d_H(v, v') := \{|1 \leq i \leq n : v_i \neq v'_i\}| \) denote the Hamming distance for two binary vectors \( v, v' \in \mathbb{F}_q^n \).

**Lemma 2.** ([11, Lemma 2]) Let \( v, v' \in \mathbb{F}_q^n \), \( U \in EF_q(v) \), and \( W \in EF_q(v') \), then \( d_S(U, W) \geq d_H(v, v') \).

Having Lemma 1 and Lemma 2 at hand, the Echelon-Ferrers construction from [11] works as follows: For two integers \( k \) and \( \delta \) choose a binary constant weight code \( S \) of length \( n \), weight \( k \), and minimum Hamming distance \( 2\delta \) as a so-called skeleton code. For each \( s \in S \) construct a code \( C_s \subseteq EF_q(s) \) having a minimum rank distance of at least \( \delta \). Setting \( C = \bigcup_{s \in S} C_s \) yields an \((n, M, 2\delta; k)\) code, where \( M = \sum_{s \in S} |C_s| \). We remark that Lemma 2 does not need two binary vectors \( v, v' \) of the same weight, i.e., the very same approach can be used to construct general subspace codes in which the codewords may have different dimensions. The only necessary modification is to choose a general binary code \( S \) of length \( n \) and minimum Hamming distance \( d \) as skeleton code. The codes \( C_s \) need to have a rank distance of at least \( d/2 \).

For a given binary vector \( v \in \mathbb{F}_q^n \) and an integer \( 1 \leq \delta \leq n \) let \( q^{\dim(v, \delta)} \) be the largest cardinality of a linear rank-metric code over \( EF_q(v) \) with rank distance at least \( \delta \).

**Theorem 6.** ([11, Theorem 1]) For a given \( i \), \( 0 \leq i \leq \delta - 1 \), if \( \nu_i \) is the number of dots in the Echelon-Ferrers diagram corresponding to \( v \), which are not contained in the first \( i \) rows and not contained in the rightmost \( \delta - 1 - i \) columns, then \( \min_i \{\nu_i\} \) is an upper bound of \( \dim(v, \delta) \).

The conjecture that the upper bound of Theorem 6 can be obtained for all parameters is still unrefuted and valid in many cases, see [10]. Several of the currently best known lower bounds for constant dimension codes are obtained via the Echelon-Ferrers construction. We remark that for the special binary vector \( v = (1, \ldots, 1, 0, \ldots, 0) \) of length \( n \) and weight \( k \), the rank-metric codes of maximum cardinality in \( EF_q(v) \) are given by lifted MRD codes, see Theorem 5. So, the Echelon-Ferrers construction uses building blocks that are lifted MRD codes with a prescribed structure. It is possible to improve the best currently known upper bounds on \( A_q(n, d; k) \) for constant dimension codes that contain a lifted MRD code.
Theorem 7. (see [12, Theorems 10 and 11]) Let \( C \subseteq G_q(n, k) \), where \( n \geq 2k \), with minimum subspace distance \( d \) contain a lifted MRD code.

- If \( d = 2(k - 1) \) and \( k \geq 3 \), then \( |C| \leq q^{2(n-k)} + A_q(n-k, 2(k-2); k-1) \);
- If \( d = k \), where \( k \) is even, then \( |C| \leq q^{(n-k)(k/2+1)} + \binom{n-k}{k/2} q^{\frac{q^k-q^{k-1}}{q^k-q}} + A_q(n-k, k, k) \).

2.3. Parallelisms and packings of \( G_q(n, k) \). Let \( X \) be a set. A packing \( P = \{P_1, \ldots, P_l\} \) of \( X \) is a set of subsets \( P_i \subseteq X \) such that \( P_i \cap P_j = \emptyset \) for all \( 1 \leq i < j \leq l \), i.e., the subsets \( P_i \) are pairwise disjoint. A point is an element of \( G_q(n, 1) \) and a spread is a subset of \( G_q(n, k) \) that partitions the corresponding set of points, i.e., the elements have a pairwise trivial intersection. Counting the points yields that the size of a spread is \( \left\lceil \frac{n}{k} \right\rceil \) = \( \frac{q^n-1}{q^k-1} \). A spread is a special constant dimension code with subspace distance \( d = 2 \cdot k \). Spreads exist if and only if \( k \) divides \( n \), see [2].

With this, a parallelism in \( G_q(n, k) \) is a packing of spreads such that it partitions \( G_q(n, k) \).

Parallelisms in \( G_q(n, k) \) are known to exist for:

1. \( q = 2, k = 2 \) and \( n \) even;
2. \( k = 2, \) all \( q \) and \( n = 2^m \) for \( m \geq 2 \);
3. \( n = 4, k = 2, \) and \( q \equiv 2 \mod 3 \);
4. \( q = 2, k = 3, n = 6, \)

see e.g. [13].

3. The coset construction

Construction III in [12] gives \( A_2(8, 4; 4) \geq 4797 \). While this specific construction does not involve parameters, the authors conjecture that the underlying idea can be generalized to further parameters assuming the existence of a corresponding parallelism. In Theorem 9 in Subsection 5.1 we will show that this is indeed the case. Moreover, there is a more general underlying construction, introduced as coset construction in this paper, that yields improvements of the best-known lower bounds for constant dimension codes, see Section 5.

The main idea of the coset construction is to use a collection of codewords which will be part of a subspace code such that \( \tau \) form (1), i.e., the corresponding RRE form, of each element of this collection is of the form

\[
\begin{pmatrix}
A & \varphi_B(F) \\
0 & B
\end{pmatrix}.
\]

Here, \( A \) is the RRE form of a \( k' \)-dimensional subspace in \( \mathbb{F}_q^n \) and \( B \) is the RRE form of a \( k - k' \)-dimensional subspace in \( \mathbb{F}_q^{n-n'} \), so that we obtain a RRE form of a \( k \)-dimensional subspace \( C(A, B, F) \) of \( \mathbb{F}_q^n \). Note that the integers \( k' \) and \( n' \) are respectively the same for any codeword in this collection although Lemma 6 allows to combine multiple such collections. \( F \) is an arbitrary \( k' \times (n-n'-k+k') \) matrix over \( \mathbb{F}_q \), in which \( \varphi_B \) inserts zero columns at the pivot positions of \( B \), see Subsection 2.1 for the precise definition of \( \varphi_B \) and an example. In \( C(A, B, F) \), the vectors have the shape \( (\lambda \cdot A, \lambda \cdot F + \mu \cdot B) \). So \( \lambda \cdot F \) is the offset for the coset of the suffixes, i.e., the vector \( \lambda \cdot A \) is prefix for every vector in the coset \( \lambda \cdot F + B \), explaining the naming of our construction. In order to obtain a constant dimension code with large minimum subspace distance, the matrices \( A, B, \) and \( F, \) as well as their combinations, are chosen from certain sets. Using \( \tau \) from (1), we have:

Lemma 3. (Coset construction) Let \( q \) be a prime power and \( n, k, n', k' \in \mathbb{N} \) satisfy \( 1 \leq k \leq n/2, 1 \leq k' \leq n', \) and \( 1 \leq k - k' \leq n - n' \). Let further \( A = \cup_{1 \leq i \leq l} A_i, \)
with arbitrary matrices $M, N$ in RRE form. For each matrix $B$ there exists a subset $\mathcal{C}((A_i), (B_i), F) := \left\{ \tau^{-1}\begin{pmatrix} A & \varphi_B(F) \\ 0 & B \end{pmatrix} : \tau^{-1}(A) \in A_i, \tau^{-1}(B) \in B_i, 1 \leq i \leq l, F \in F \right\}$ is a subset of $G_q(n, k)$, i.e., a constant dimension code where the codewords have dimension $k$.

**Proof.** For an arbitrary but fixed index $1 \leq i \leq l$ let $A$, $B$ be matrices with $\tau^{-1}(A) \in A_i$ and $\tau^{-1}(B) \in B_i$. We can easily check that $A \in F_q^{k' \times n'}$ is a full-rank matrix in RRE form. Similarly, $B \in F_q^{(k'-k) \times (n-n')}$ is a full-rank matrix in RRE form. For each matrix $F \in F$ we have $F \in F_q^{k' \times (n-n'-k+k')}$, so that $\varphi_B(F) \in F_q^{k' \times (n-n')}$. The dimensions fit so that

$$M := \begin{pmatrix} A & \varphi_B(F) \\ 0 & B \end{pmatrix} \in F_q^{k \times n}.$$ 

Moreover $\varphi_B(F)$ has zero columns at the positions of the pivot columns of $B$. Since $A$ has $k'$ and $B$ has $k-k'$ pivot columns, $M$ has exactly $k$ pivot columns and full rank. Thus, $\tau^{-1}(M) \in G_q(n, k)$. \hfill $\square$

The number $l$ of disjoint subsets for $A$ and $B$ is called the length of the specific coset construction. We remark that we have excluded the ranges for the parameters $k', n'$ where the construction would be degenerated in the sense that either $A$ or $B$ have to be empty matrices. Nevertheless, the degenerate case $k' = k$ has a nice interpretation. Here $B$ is an empty matrix and $A$ is a $k \times n'$ matrix. If additionally $n' = k$ then $A$ is an identity matrix and we are in the case of lifted MRD codes. Using $\tau$ from (1), we have:

**Lemma 4.** Let $q, n, k, n', k'$ be parameters satisfying the conditions from Lemma 3, $A, A' \in F_q^{k' \times n'}$ and $B, B' \in F_q^{(k-k') \times (n-n')}$ be full-rank matrices in RRE form. Let further $d$ be a positive integer and $F, F' \in F_q^{k' \times (n-n'-k+k')}$. If

$$d_S(\tau^{-1}(A), \tau^{-1}(A')) + d_S(\tau^{-1}(B), \tau^{-1}(B')) \geq d \quad (2)$$

or $d_R(F, F') \geq d/2$ then

$$d_S\left(\tau^{-1}\begin{pmatrix} A & \varphi_B(F) \\ 0 & B \end{pmatrix}, \tau^{-1}\begin{pmatrix} A' & \varphi_B(F') \\ 0 & B' \end{pmatrix}\right) \geq d.$$ 

The proof is rather technical and can be found in the appendix.

We remark that condition (2) of Lemma 4 is trivially satisfied for the special case of distance $d = 4$, if $A \neq A'$ and $B \neq B'$.

Next we demonstrate that the coset construction from Lemma 3 can in general not be obtained by an application of the Echelon-Ferrers construction. (For a more explicit example, see Theorem 13 in Subsection 5.3.) It is easy to construct a family of examples with subspace distance $d$ but whose pivot vectors have Hamming distance 2, so that they cannot be used in the Echelon-Ferrers construction. To this end, let $q$ be an arbitrary prime power, $d$ an even integer $\geq 2$, and $n, k, n', k' \in \mathbb{N}$ such that $\frac{d}{4} \leq k', n' - k', k - k', n - n' - k + k'$. For the sake of this example we use:

$$A_1 := \begin{pmatrix} I_{k'-1} & 0 & M \\ 0 & 1 & 0 \end{pmatrix} \quad A_2 := \begin{pmatrix} I_{k'-1} & 0 & M + N \\ 0 & 0 & 0 \end{pmatrix}$$

$$B_1 := \begin{pmatrix} I_{k-k'} & M' \end{pmatrix} \quad B_2 := \begin{pmatrix} I_{k-k'} & M' + N' \end{pmatrix}$$

with arbitrary matrices $M, N \in F_q^{(k'-1) \times (n'-k'-1)}$ of full rank,
$M', N' \in \mathbb{F}_q^{(k-k') \times (n-n'-k+k')}$, where $I_*$ denotes the identity matrix. Then, for arbitrary $F_1, F_2 \in \mathbb{F}_q^{k' \times (n-n'-k+k')}:
\begin{align*}
    d_H \left( p \left( \begin{pmatrix} A_1 & F_1 \\ 0 & B_1 \end{pmatrix} \right), p \left( \begin{pmatrix} A_2 & F_2 \\ 0 & B_2 \end{pmatrix} \right) \right) &= 2
\end{align*}

but
\begin{align*}
    d_S \left( \begin{pmatrix} A_1 & F_1 \\ 0 & B_1 \end{pmatrix}, \begin{pmatrix} A_2 & F_2 \\ 0 & B_2 \end{pmatrix} \right) &\geq 2 \left( \text{rk} \left( \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \right) + \text{rk} \left( \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \right) - k \right) \\
    &= 2((k' + 1 + \text{rk}(N)) + (k - k' + \text{rk}(N')) - k) \\
    &= 2(\min\{k', n' - k'\} + \min\{k - k', n - n' - k + k'\}) \\
    &\geq d.
\end{align*}

3.1. A multilevel coset construction. In this subsection we want to use the coset construction in combination with other constructions. At first we show that the pivot vector $p_* \in k d$ then
\begin{align*}
    \sum_i \lambda_i x_i \geq |s - k'| + |\tilde{k} - s - k + k'| = 2 |s - k'|.
\end{align*}

Applying Lemma 2 yields the stated lower bound on the subspace distance.

**Lemma 5.** For a prime power $q$ and $n, k, n', k', \tilde{k} \in \mathbb{N}$ satisfying $1 \leq k \leq n/2, 1 \leq k' \leq n', 1 \leq k - k' \leq n - n'$, and $0 \leq \tilde{k} \leq n$, let $U \in \mathcal{G}_q(n', k')$, $V \in \mathcal{G}_q(n-n', k-k')$, $F \in \mathbb{F}_q^{k' \times (n-n'-k+k')}$, and $X \in \mathcal{G}_q(n, \tilde{k})$. Let $s$ be the sum of the first $n'$ entries in the pivot vector $p(X)$ of $X$, i.e., $s := \sum_{i=1}^{n'} p(X)_i$. If $d \leq |s - k'| + |\tilde{k} - s - k + k'|$ then $d_S(X, W) \geq d$, where
\begin{align*}
    W = \tau^{-1} \left( \begin{pmatrix} \tau(U) & \varphi_\tau(V)(F) \\ 0 & \tau(V) \end{pmatrix} \right).
\end{align*}

**Proof.** Let $x := p(X)$ and $w := p(W)$ be the pivot vectors of $X$ and $W$, respectively. From the construction we know $\sum_{i=1}^{n'} x_i = s, \sum_{i=1}^{n'} w_i = k', \sum_{i=n'+1}^{n} x_i = \tilde{k} - s$, and $\sum_{i=n'+1}^{n} w_i = k - k'$, so that
\begin{align*}
    d_H(x, w) &\geq |s - k'| + |\tilde{k} - s - k + k'| \\
    &\geq d.
\end{align*}

Applying Lemma 2 yields the stated lower bound on the subspace distance.

For the special case $\tilde{k} = k$, i.e., the constant dimension case, we have $|s - k'| + |\tilde{k} - s - k + k'| = 2 |s - k'|$. There is also an easy-to-check sufficient criterion to determine whether the union of two codes constructed by the coset construction have a subspace distance of at least $d$.

**Lemma 6.** Let $C_i$ be codes having subspace distance at least $d$ and that are obtained from the coset construction with suitable parameters $n, k_i, n'_i,$ and $k'_i$ for $i = 1, 2$, where we assume $n'_1 \leq n'_2$. Let $f(m) = |m - k'_1| + |m - \gamma|$ and
\begin{align*}
    K &= \begin{cases} 
        f(\beta) & \text{if } \beta \leq \lambda \\
        f(\lambda) & \text{if } \beta > \lambda < \overline{\beta} \\
        f(\overline{\beta}) & \text{else}
    \end{cases},
\end{align*}

where $\beta = \max\{k'_2 - n'_2 + n'_1, 0\}$, $\overline{\beta} = \min\{n'_1, k'_2\}$, $\gamma = k'_1 + k_2 - k_1$, and $\lambda = \max\{\gamma, k'_1\}$. If $d \leq K$, then $D_S(C_1 \cup C_2) \geq d$.

**Proof.** At first we observe that we have $d_H(u, v) \geq |a - b|$ for $u, v \in \mathbb{F}_q^2$ with $\|u\|_1 = a$ and $\|v\|_1 = b$. 
\vspace{1cm}
We set $x := p(W_1)$ and $y := p(W_2)$, where the $W_i$ are matrices corresponding to an arbitrary but fixed codeword from $C_i$, see the formulation of Lemma 5.

Let $x^1$ consist of the first $n'_1$ entries of $x$, $y^1$ consist of the first $n'_1$ entries of $y$, $x^2$ consist of the last $n-n'_1$ entries of $x$, and $y^2$ consist of the last $n-n'_1$ entries of $y$. For $m := \|y^1\|_1$, where

$$\beta = \max\{k'_2 - n'_2 + n'_1, 0\} \leq m \leq \min\{n'_1, k'_2\} = \beta,$$

we have $d_H(x^1, y^1) \geq |m - k'_1|$ and $d_H(x^2, y^2) \geq |m - \gamma|$. Thus $f(m) \leq d_H(x, y)$ is minimized for $K$. Applying Lemma 2 yields the stated lower bound on the subspace distance.

Considering the exemplary parameters $n = 6$, $k_1 = k_2 = 3$, $n'_1 = n'_2 = 2$, $k'_1 = 1$ and $k'_2 = 2$ for the codes $C_1 = \{(100000)\}$ and $C_2 = \{(100000)\}$.

Lemma 6 uses $\beta = \beta = 2$, $\gamma = \lambda = 1$, $f(m) = 2|m - 1|$ and $K = f(\beta) = 2$. This lower bounds any two codewords from two coset constructed parts having these parameters, whereas the Hamming distance of the depicted pivot vectors is 2.

We remark that Lemma 6 is best possible in the sense that the estimations on the Hamming distance of two binary vectors with known weights and weights of two suffixes, of possibly different lengths, is tight. Performing similar analyses on generalized structures like

$$\begin{pmatrix} A & \varphi_B(F) & \varphi_C(G) \\ 0 & B & \varphi_C(H) \\ 0 & 0 & C \end{pmatrix}$$

may have the potential to yield stronger bounds.

4. Optimal choices for the parameters of the coset construction

4.1. General reasoning. Like the Echelon-Ferrers construction, the coset construction from the previous section is far from being explicit, i.e., there are several degrees of freedom. In this section we give several lower and upper bounds for the sizes of the codes obtained from the coset construction, which allow to minimize the range of choices of the parameters that can lead to improvements of the best-known bounds.

The cardinality of a subspace code obtained from the coset construction with length $l$ is given by

$$|C((A_i), (B_i), F)| = |F| \cdot \sum_{i=1}^l |A_i| \cdot |B_i|. \quad (3)$$

Given $q$, $n$, and the desired even subspace distance $d$, the aim is to maximize (3) under the restrictions of Lemma 4. Obviously, this term is maximal if both $|F|$ and the sum are maximal. Thus, we may choose an MRD code, with appropriate parameters, for $F$, so that

$$|F| = \left[ q^{\max(k', n-n' - k+k')} \cdot (\min(k', n-n' - k+k') - d/2 + 1) \right]$$

is optimal by Theorem 4.

The sets $A_i$ and $B_i$ need to have additional structure.

Lemma 7. For a code obtained from the construction of Lemma 3 with $d := D_S(C((A_i), (B_i), F))$, length $l$, and parameters $q, n, k, n', k'$ we have $D_S(A_i) \geq d$ and $D_S(B_i) \geq d$ for all $1 \leq i \leq l$. 


Proof. If $U \neq U' \in \mathcal{A}_i$, then there exists $V \in \mathcal{B}_i$ such that Condition (2) yields $d \leq d_S(U, U') + d_S(V, V) = d_S(U, U')$. A similar conclusion can be drawn for the elements in $\mathcal{B}_i$. \hfill \Box

From this we can conclude an upper bound on $\Lambda$.

**Corollary 1.** Using the notation from Lemma 3 and Equation (3) we have

\[
\Lambda \leq \min \left\{ \frac{n'}{k'} q \cdot A_q(n, n'; d, k - k'), \frac{n - n'}{k - k'} q \cdot A_q(n', n'; d, k') \right\}.
\]

**Proof.** Due to Lemma 7 we have $|\mathcal{A}_i| \leq A_q(n', d; k')$, so that

\[
\sum_{i=1}^{l} |\mathcal{A}_i| \cdot |\mathcal{B}_i| \leq A_q(n', d; k') \cdot \sum_{i=1}^{l} |\mathcal{B}_i| \leq A_q(n', d; k') \cdot \frac{n - n'}{k - k'} q.
\]

Interchanging the roles of the $\mathcal{A}_i$ and $\mathcal{B}_i$ yields the other stated upper bound. \hfill \Box

**Corollary 2.** The upper bound of Corollary 1 can be attained if $d \leq 4$ and both $G_q(n', k')$ and $G_q(n - n', k - k')$ admit parallelisms, e.g., the corresponding parameters are in the list in Subsection 2.3.

The dependency between the cardinalities of the $\mathcal{A}_i$ and $\mathcal{B}_i$ in optimal solutions of (3) is already decoupled to some extent, but we can even do more.

**Lemma 8.** For a code obtained from the construction of Lemma 3 with $d := D_S\left( \mathcal{C} (\{\mathcal{A}_i\}, \{\mathcal{B}_i\}, \mathcal{F}) \right)$, length $l$, and parameters $q, n, k, n', k'$, there exists an integer $d'$ such that $D_S(\mathcal{A}) \geq d'$ and $D_S(\mathcal{B}) \geq d - d'$, where $\mathcal{A} = \cup_i \mathcal{A}_i$ and $\mathcal{B} = \cup_i \mathcal{B}_i$.

**Proof.** Let $U, U' \in \mathcal{A}$ with $d_S(U, U') = D_S(\mathcal{A}) := d'$ and $V, V' \in \mathcal{B}$ with $d_S(V, V') = D_S(\mathcal{B}) := d''$. W.l.o.g. we can assume that $\mathcal{F}$ contains the zero matrix, since the rank distance is invariant with respect to translations. Choosing $F = F' = 0$ we can conclude $d'' \geq d - d'$ from Inequality (2). \hfill \Box

In later applications we will commonly assume $2 \leq d' \leq d - 2$, since the other values lead to trivial cases where either $|\mathcal{A}| = 1$ or $|\mathcal{B}| = 1$.

**Lemma 9.** For a code obtained from the construction of Lemma 3 with $d := D_S\left( \mathcal{C} (\{\mathcal{A}_i\}, \{\mathcal{B}_i\}, \mathcal{F}) \right)$, length $l$, and parameters $q, n, k, n', k'$, then for each permutation $\sigma : \{1, \ldots, l\} \rightarrow \{1, \ldots, l\}$ we have $D_S\left( \mathcal{C} (\{\mathcal{A}_i\}, \{\mathcal{B}_{\sigma(i)}\}, \mathcal{F}) \right) = d$.

**Proof.** Apply Lemma 4. \hfill \Box

The question which permutation $\sigma$ of Lemma 9 maximizes the crucial parameter $\Lambda$ can be answered easily.

**Lemma 10.** Let $a_1 \geq \cdots \geq a_l$ and $b_1 \geq \cdots \geq b_l$ positive integers. For each permutation $\sigma : \{1, \ldots, l\} \rightarrow \{1, \ldots, l\}$, we have

\[
\sum_{i=1}^{l} a_i \cdot b_i \geq \sum_{i=1}^{l} a_i \cdot b_{\sigma(i)}.
\]

**Proof.** For integers $a > a'$ and $b < b'$ we have

\[
(ab + a'b') - (ab' + a'b) = (a - a') \cdot (b - b') < 0.
\]

Having these ingredients at hand we can generalize and improve the upper bound from Corollary 1 using the analytical solution of another optimization problem.
Lemma 11. Let $\alpha, \beta, \overline{\alpha}, \overline{\beta}$, and $l$ be positive integers with $\alpha, \beta \geq l$. An optimal solution of the non-linear integer programming problem

$$\max \sum_{i=1}^{l} a_i \cdot b_i$$

subject to

$$\sum_{i=1}^{l} a_i \leq \overline{\alpha} \quad \forall 1 \leq i \leq l$$

$$\sum_{i=1}^{l} b_i \leq \overline{\beta} \quad \forall 1 \leq i \leq l$$

$$a_i, b_i \in \mathbb{N}_{\geq 0} \quad \forall 1 \leq i \leq l$$

is given by

1. $a_i^* = \overline{\alpha}, b_i^* = \overline{\beta}$ for all $1 \leq i \leq l$ if $\overline{\alpha} \cdot l \leq \alpha$ and $\overline{\beta} \cdot l \leq \beta$;

2. $a_i^* = \overline{\alpha}, b_i^* = 1 + \min\{\overline{\beta} - 1, \max\{0, \beta - l - (i - 1) \cdot (\overline{\beta} - 1)\}\}$ for all $1 \leq i \leq l$ if $\overline{\alpha} \cdot l \leq \alpha$ and $\overline{\beta} \cdot l > \beta$;

3. $a_i^* = 1 + \min\{\overline{\alpha} - 1, \max\{0, \alpha - l - (i - 1) \cdot (\overline{\alpha} - 1)\}\}, b_i^* = \overline{\beta}$ for all $1 \leq i \leq l$ if $\overline{\alpha} \cdot l > \alpha$ and $\overline{\beta} \cdot l \leq \beta$;

4. $a_i^* = 1 + \min\{(\overline{\alpha} - 1), \max\{0, \alpha - l - (i - 1) \cdot (\overline{\alpha} - 1)\}\}, b_i^* = 1 + \min\{\overline{\beta} - 1, \max\{0, \beta - l - (i - 1) \cdot (\overline{\beta} - 1)\}\}$ for all $1 \leq i \leq l$ if $\overline{\alpha} \cdot l > \alpha$ and $\overline{\beta} \cdot l > \beta$.

Proof. W.l.o.g. we can additionally assume $a_1 \geq \cdots \geq a_l$ and $b_1 \geq \cdots \geq b_l$ without decreasing the maximal target value of the optimization problem. Let us allow $a_i, b_i \in \mathbb{R}$ for a moment, i.e., we consider the standard relaxation, and denote a corresponding optimal solution by $\bar{a}_i, \bar{b}_i \in \mathbb{R}_{\geq 1}$.

For non-negative real numbers $a' \geq a''$ and $b' \geq b''$ we have

$$(a'b' + a''b'') \cdot 2 \cdot \frac{a' + a''}{2} \cdot \frac{b' + b''}{2} = \frac{(a' - a'') \cdot (b' - b'')}{2} \geq 0,$$

so that we can assume $\bar{a}_i = \bar{a}_j =: \bar{a}$ and $\bar{b}_i = \bar{b}_j =: \bar{b}$, for all $1 \leq i, j \leq l$, w.l.o.g.

Either we have $\bar{l} \bar{a} = \alpha$ or $\bar{a} = \overline{\alpha}$, since otherwise we could slightly increase $\bar{a}$ and improve the target value. The same reasoning applies to $\bar{b}$.

If $\bar{a} = \overline{\alpha}$ and $\bar{b} = \overline{\beta}$, then we are in case (1). Next we consider the case where $\bar{a} = \overline{\alpha}$ and $\bar{b} < \overline{\beta}$ so that $\bar{b} = \beta/l$. Since $\sum_{i=1}^{l} \bar{b}_i \overline{\alpha} = \overline{\alpha} \cdot \sum_{i=1}^{l} b_i$ it suffices to determine integers $1 \leq b_i^* \leq \overline{\beta}$ with $\sum_{i=1}^{l} b_i^* = \beta$. This is done in the formula of case (2). The underlying idea is the following: Start with $b_i^* = 1$ for all $1 \leq i \leq l$; observe $\beta \geq l$. Then fill up the $b_i^*$ with increasing indices up to $\overline{\beta}$ as long as the sum does not violate $\beta$. Observe that every (integer) vector $(b_i)$ with $\sum_{i=1}^{l} b_i = \beta$ gives the same target value. Case (3) describes the symmetric situation. It remains to assume $\overline{\alpha} \cdot l > \alpha$ and $\overline{\beta} \cdot l > \beta$. Let $\bar{a}_i, \bar{b}_i$ be an optimal solution of our initial optimization problem where we assume $\bar{a}_1 \geq \cdots \geq \bar{a}_l$ and $\bar{b}_1 \geq \cdots \geq \bar{b}_l$. Let further $f$ be the smallest index such that $\bar{a}_f < \overline{\alpha}$ and $r$ be the largest index such that $\bar{a}_r > 1$. If either $r$ does not exist or $f = r$ ($f$ exists due to $\overline{\alpha} \cdot l > \alpha$), then the solution $\bar{a}_i$ has the shape described in case (4). But, for $f < r$ we could improve the target value by

$$(\bar{a}_f + 1) \cdot \bar{b}_f + (\bar{a}_r - 1) \cdot \bar{b}_r - \bar{a}_f \cdot \bar{b}_f - \bar{a}_r \cdot \bar{b}_r = \bar{b}_f - \bar{b}_r \geq 0,$$

so that such a case could never produce an optimal value and so our solution must have the shape described in case (4). The same reasoning applies for the $\bar{b}_i$. \qed

Lemma 12. Using the notation from Lemma 3 and Equation (3) we have

$$\Lambda \leq \max_{d^* \in \mathbb{Z}: 0 < d^* < d} \max_{1 \leq k \leq \min\{A_q(n', d'; k'), A_q(n-n', d-d'; k-k')\}} \sum_{i=1}^{l} a_i \cdot b_i,$$
where the $a_i$, $b_i$ are given by Lemma 11 for

\[
\begin{align*}
\alpha &= A_q(n', d'; k'), \\
\beta &= A_q(n - n', d - d'; k - k'), \\
\overline{\alpha} &= A_q(n', d; k'), \\
\overline{\beta} &= A_q(n - n', d; k - k').
\end{align*}
\]

**Proof.** From Lemma 8 we conclude $|A| \leq A_q(n', d'; k')$ and $|B| \leq A_q(n - n', d - d'; k - k')$. The possible values for the length $l$ are part of the stated optimization formulation. For each index $1 \leq i \leq l$ we have $|A_i| \leq A_q(n, d; k')$ and $|B_i| \leq A_q(n - n', d; k - k')$ due to Lemma 7. It remains to check that we can apply Lemma 11.

Fixing the parameter $d'$ from Lemma 8 one can state a lower bound on the maximal value of $\Lambda$ in terms of the sizes of lifted MRD codes (cf. Theorem 5).

**Lemma 13.** Let $d' \in 2\mathbb{Z}$ with $2 \leq d' \leq d - 2$, then we have

\[
\Lambda \geq M(q, k', n', d) \cdot M(q, k - k', n - n', d) \cdot l
\]

with

\[
l = \min \left\{ \frac{M(q, k', n', d')}{M(q, k', n', d)}, \frac{M(q, k - k', n - n', d)}{M(q, k - k', n - n', d)} \right\}.
\]

for, with respect to Lemma 3, feasible parameters $q, n, k, n', k, d$. Proof.

Similar to the proof of [12, Lemma 5], we consider $A$ as a linear MRD code with parameters $k' \times n'$ with distance $d'$ and $B$ as a linear MRD code with parameters $(k - k') \times (n - n')$ with distance $d - d'$. Let $S_A$ be a linear MRD code with parameters $k' \times n'$ with distance $d > d'$ and $S_B$ be a linear MRD code with parameters $(k - k') \times (n - n')$ with distance $d > d - d'$. We choose the $A_i$ as the cosets of $S_A$ in $A$ and $B_i$ as the cosets of $S_B$ in $B$. For $S_A$ there are exactly $M(q, k', n', d')$ cosets and for $S_B$ there are exactly $M(q, k - k', n - n', d)$ cosets. Since $d_A(A + C, B + C) = d_B(A, B)$ for all suitable matrices $A, B, C \in F_q^{k \times t}$, we have $D_S(A_i), D_S(B_i) \geq d$ for all $1 \leq i \leq l$.

Combining a lifted MRD code with a code constructed from Lemma 13 yields a $(9, 1032, 6; 4)_2$ code, which improves on the previously best-known codes, see Subsection 5.2.

We can formulate the following greedy-type algorithm to construct sequences $A_i$ and $B_i$ that yield a “reasonable” lower bound on $\Lambda$.

**Algorithm 8.**

\[
\begin{align*}
\mathcal{R}_A &\leftarrow G_q(n', k') \\
i &\leftarrow 0 \\
\textbf{while } \mathcal{R}_A \neq \emptyset \textbf{ do} \\
&\text{ select constant dimension code } A_i \text{ of maximum } \\
&\text{ cardinality in } \mathcal{R}_A \text{ with } D_S(A_i) \geq d \\
&\mathcal{R}_A \leftarrow \mathcal{R}_A \setminus \{ V \mid D_S(A_i \cup \{ V \}) \leq d' - 1 \} \\
&i &\leftarrow i + 1 \\
\textbf{end while} \\
\mathcal{I}_A &\leftarrow i \\
\mathcal{R}_B &\leftarrow G_q(n - n', k - k') \\
i &\leftarrow 0 \\
\textbf{while } \mathcal{R}_B \neq \emptyset \textbf{ do} \\
&i &\leftarrow i + 1
\end{align*}
\]
select constant dimension code \( B_i \) of maximum cardinality in \( \mathcal{R}_B \) with \( D_S(B_i) \geq d \)
\[ \mathcal{R}_B \leftarrow \mathcal{R}_B \setminus \{V \mid D_S(B_i \cup \{V\}) \leq d - d' - 1\} \]
end while

\( l_B \leftarrow i \)
\( l \leftarrow \min\{l_A, l_B\} \)

Unfortunately, this algorithm is not capable of determining the optimal \( \Lambda \) in general. If we use

\[ E := \{\text{all constant dimension codes in } \mathcal{G}_q(\tilde{n}, \tilde{k}) \text{ with subspace distance } d\} \]

as ground set and \( I := \{\text{disjoint subsets of } E\} \) as independent sets, then this does not form a matroid and hence a greedy algorithm will not yield an optimal solution in general, see e.g. [8]. To be more precise, the independent set exchange property fails: Use for example \( U \neq V \in \mathcal{G}_q(\tilde{n}, \tilde{k}) \) with \( d_S(U, V) \geq d, A := \{\{U\}, \{V\}\} \in I \) and \( B := \{\{U, V\}\} \in I \). Although \( A \) is larger than \( B \) we cannot add an element of \( A \) to \( B \) without losing the independence.

4.2. Decomposing constant dimension codes. Due to Lemma 8 we can construct the necessary parts of the coset construction of Lemma 3 starting from constant dimension codes \( A \) and \( B \) with \( D_S(A) \geq d' \) and \( D_S(B) \geq d - d' \). The aim is to partition the codewords of \( A \) into subcodes \( A_i \) for \( 1 \leq i \leq l_A \) in such a way that \( D_S(A_i) \geq d \). Simultaneously, we aim to partition the codewords of \( B \) into subcodes \( B_i \) for \( 1 \leq i \leq l_B \) in such a way that \( D_S(B_i) \geq d \). Setting the length \( l \) of the coset construction to \( l := \min\{l_A, l_B\} \), we observe that trying to maximize the cardinalities \( |A_i| \) or \( |B_i| \) for \( i > l \) has no benefit, so that we may simply complete a given packing by singletons. Or, in other words, we directly start from packings within \( A \) and \( B \).

However, the design of suitable \( A_i \) is not that obvious since the \( \Lambda \)-part of the target function (3) comprises a non-linear integer optimization problem. Ignoring almost all of the geometric restrictions from \( P_q(n) \), we are able to exactly solve the mentioned optimization problem in Lemma 11. In general this gives us an upper bound only. To obtain tighter bounds one has to go a bit more into the details. In Lemma 12 we have only used the implication \( |A_i| \leq A_q(n', d'; k') \) from \( D_S(A_i) \geq d \), which is valid for all \( \cup_{i=1}^l A_i \subseteq A \subseteq \mathcal{G}_q(n', k') \). For a given \( A \) we may be able to determine tighter bounds on the cardinalities of the \( A_i \)s. Since the only change in the setting is the exclusion of the possible codewords in \( \mathcal{G}_q(n', k') \setminus A \) this subproblem can be formulated as an independent set problem and be solved using several algorithmic approaches, see e.g. [18]. We will present an explicit example of this technique in Subsection 5.3.

Having candidates for the \( A_i \) at hand, it still remains to select a subset of the candidates that are pairwise disjoint. This subproblem can also be formulated as a (restricted) independent set problem of \( a \), possibly large, graph \( G = (V, E) \). To this end, let \( \kappa \) be a suitable upper bound on the cardinalities of the \( |A_i| \) and \( S \) be the set of subsets of \( A \) of cardinality \( i \) having a subspace distance of at least \( d \). Setting \( S = \cup_{1 \leq i \leq \kappa} S_i \) one can consider the optimization problem

\[
\max \sum_{s \in S} |s| \cdot x_s \tag{4}
\]
\[
\sum_{s \in S} x_s = l
\]
\[
x_a + x_b \leq 1 \quad \forall a \neq b \in S : a \cap b \neq \emptyset
\]
\[
x_s \in \{0, 1\} \quad \forall s \in S
\]
for a given number \( l \) of parts of the desired packing. Notwithstanding that the target function of ILP formulation (4) completely ignores the correlation with the sizes of the items of the second packing on \( \Lambda \), it can be used to determine the exact value of \( \Lambda \) in special cases, see Subsection 5.3. Setting the vertex set of our graph \( G \) to \( V = S \) and taking edges \( e = \{s_1, s_2\} \in E \) if \( s_1 \cap s_2 \neq \emptyset \), this corresponds to a vertex-weighted independent set problem with an additional restriction on the number of chosen vertices. The algorithmic approaches described in [18] can be adopted easily for these extra requirements.

Since the two subproblems from this subsection on their own even might be too hard, we may apply heuristic approaches only. The very successful approach of prescribing automorphisms can also be applied here. Here the prescribed subgroup of automorphisms has to be a subgroup of the automorphism group of \( \mathcal{A} \) which typically is much smaller than \( \text{GL}(n, q) \). However, “good” codes often have non-trivial automorphism groups.

5. Examples

In this section we describe the details of the coset construction for some parameters where we were able to attain or improve the best known constructions. See Table 1 for an overview of code sizes where the Echelon-Ferrers construction from [11] uses sizes of Echelon-Ferrers diagrams from [10] which was developed later.

| parameters \((n, \cdot, d; k)\) | old largest known code | coset construction |
|-----------------------------|------------------------|--------------------|
| \((8, \cdot, 4; 4)_q\)        | \(q^{12} + \frac{[4]}{2}_q(q^2 + 1)^2 + 1\) | cf. [12]          |
| \((3k - 3, \cdot, 2k - 2; k)_q\) for \(k \geq 4\) | \(q^{4k - 6} + q^{k - 1} + 1\) | \(q^{4k - 6} + q^{k - 1} + 1\) |
| \((10, \cdot, 6; 4)_2\)       | 4167, cf. [11]         | 4173               |

Table 1. Improved or attained code sizes by the coset construction.

5.1. \( n = 8, d = 4, k = 4, \) and \( q = 2 \) revisited. We apply the coset construction with \( n' = 4, k' = 2, d' = 2 \) and use a parallelism in \( G_2(4, 2) \) for the \( \mathcal{A}_i \) and \( \mathcal{B}_i \). Here we have \( l = 7 \) and \( |\mathcal{A}_i| = |\mathcal{B}_i| = 5 \) for all \( 1 \leq i \leq 7 \). Thus, \( \Lambda = 7 \cdot 5 \cdot 5 = 175 \). Since \( F \) is an MRD code of shape \( 2 \times 2 \) and rank distance \( 2 \), its cardinality is 4, hence the corresponding code obtained from the coset construction has cardinality 700. The rank distance between the two pivot vectors \( v_1 = (1, 1, 1, 1, 0, 0, 0, 0) \) and \( v_2 = (0, 0, 0, 0, 1, 1, 1, 1) \), as well as \( v_1 \) and any pivot vector of any codeword of the coset construction for \( i = 1, 2 \) is 4, cf. Lemma 5. So the Echelon-Ferrers construction applied to \( v_1 \) and \( v_2 \) and combined with the coset construction yields a feasible subspace code for our parameters, i.e., \( A_2(8, 4; 4) \geq 4096 + 700 + 1 = 4797 \). This is Construction III in [12]. A different technique was applied in [7, Theorem 4.1] to find a code of this size. Here, the MRD bound from Theorem 7 is attained. Recently, an \((8, 4801, 4; 4)_2\) code has been found by a heuristic computer search [6].

As already observed in [12], the crucial ingredient for the feasibility of the above construction is the existence of a parallelism in \( G_q(4, 2) \). Performing the above cardinality computations for arbitrary \( q \) we obtain \( A_q(8, 4; 4) \geq q^{12} + \frac{[4]}{2}_q(q^2 + 1)^2 + 1 \), which also attains the MRD bound from Theorem 7.

The authors of [12] have remarked that they believe that their construction from their Construction III can be generalized to further parameters assuming the existence of a corresponding parallelism. This is indeed the case.
Theorem 9. If $\mathcal{P}_1$ is a parallelism in $\mathcal{G}_q(n', k')$ and $\mathcal{P}_2$ a parallelism in $\mathcal{G}_q(n - n', k - k')$, then we can choose $A = \mathcal{P}_1$, $B = \mathcal{P}_2$, and $d = 4$ in the coset construction. The corresponding code $C$ attains the upper bound of Corollary 1. If additionally $k - k' \geq 2$ and $n' - k' \geq 2$, then $C$ is compatible with the lifted MRD code having pivot vector $(1, \ldots, 1, 0, \ldots, 0)$.

5.2. $n = 9$, $d = 6$, $k = 4$, and general field sizes $q$. Since the combination of the MRD code $C_1$ with pivot vector $v = (1, 1, 1, 1, 0, 0, 0, 0, 0)$ and cardinality 1024 with the code $C_2$ obtained from the explicit construction of Lemma 13 of cardinality 8 yields a $(9, 1032, 6; 4)_2$ constant dimension code whose cardinality is one less than the MRD bound from Theorem 7, we were motivated to look for a coset construction yielding a larger addendum than 8.

Theorem 10. $A_q(9, 6; 4) \geq q^{10} + q^3 + 1$.

Proof. We choose $n' = 4$, $k' = 1$, and $d' = 2$ in the coset construction. For the choice of $A$ and $B$ we observe $A_q(4, 2; 1) = q^3 + q^2 + q + 1$ and $A_q(5, 4; 3) = A_q(5, 4; 2) = q^3 + 1$, see e.g. [5]. Choose $A$ and $B$ as arbitrary codes attaining the mentioned upper bounds. Choosing a trivial packing of $B$ into singletons yields a code $C$ of cardinality $q^3 + 1$. Adding the lifted MRD code of size $q^{10}$ gives the stated upper bound.

We remark that the codes from Theorem 10 meet the MRD bound from Theorem 7. The underlying construction can be generalized even more.

Theorem 11. For each $k \geq 4$ and arbitrary $q$ we have

$$A_q(3k - 3, 2k - 2; k) \geq q^{4k-6} + q^{k-1} + 1.$$  

Proof. We choose $n' = k$, $k' = 1$, and $d' = 2$ in the coset construction. For the choice of $A$ and $B$ we observe $A_q(k, 2; 1) = \begin{bmatrix} k \\ q \end{bmatrix}$ and

$$A_q(2k - 3, 2k - 4; k - 1) = A_q(2k - 3, 2k - 4; k - 2)$$

$$\begin{bmatrix} k \\ q \end{bmatrix} \frac{q^{2k-3} - q}{q^{k-2} - 1} - q + 1 = q^{k-1} + 1 < \begin{bmatrix} k \\ 1 \end{bmatrix},$$

where the first equality is true by considering the so-called complementary subspace code $C^\perp = (U^\perp | U \in C)$, cf. [17]. Choose $A$ and $B$ as arbitrary codes attaining the mentioned upper bounds. Choosing a trivial packing of $B$ into singletons yields a code $C$ of cardinality $q^{k-1} + 1$. Adding a $(k \times (3k - 3))$ lifted MRD code gives the stated lower bound.

We remark that the codes from Theorem 11 meet the MRD bound from Theorem 7.

5.3. $n = 10$, $d = 6$, $k = 4$, and $q = 2$. For the coset construction we choose $n' = 4$ and $k' = 1$. Since $A \subseteq \mathcal{G}_q(4, 1)$ we can only have $D_S(A_i) = 2$, so that we must choose $d' = 2$. Then, we can choose $A = \mathcal{G}_q(4, 1)$ and $\begin{bmatrix} 4 \\ 1 \end{bmatrix} = 15$ singletons $A_i$, which is obviously best possible. For $B \subseteq \mathcal{G}_q(6, 3)$ we have the condition $D_S(B) \geq 4$. Reasonable candidates for $B$ might be the five isomorphism types of $(6, 77, 4; 3)_2$ codes attaining the maximum cardinality $A_2(6, 4; 3) = 77$, see [16]. Using the first subproblem from Subsection 4.2 we computationally obtain the upper bound $|B_i| \leq 5 =: \kappa$ for four out of the five isomorphism types. This information is enough to conclude the upper bound $|A(B)| \leq 15 \cdot 5 = 75$. For the remaining isomorphism type, i.e., the self-dual code having 168 automorphisms which was labeled as “type $A$”, we have $|B_i| \leq 7 =: \kappa$. So, we solve the optimization problem (4) for $l = 15$. The sizes of the requested sets $S_l$ are stated in Table 2. The optimal target value
is 76 and there exists a solution where the sizes of the elements in the packing are
given by 4, 4, 4, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 7, 7. Since in our situation we have
\[ |A_i| = 1 \]
for all \( i \), the target function of (4) coincides with the expression for \( \Lambda \).
Also the predefinition of \( l = 15 \) results in the maximum possible value, since we
have \( l \leq 15 \) from the \( A \)-part and the existence of a packing of \( B \) into \( l' \) sets implies
the existence of packings into \( l \geq l' \) sets. In general it is far from being obvious that
we obtain the best possible codes from the coset construction by choosing codes for
\( B \) that have the maximal possible cardinality \( A_q(n-n', d; k-k') \). However, in our
situation each choice for \( B \) different from the five considered isomorphism types of
\( (6, q^6+2q^2+2q+1, 4; 3) \) codes has a cardinality of at most 76, so that
\[ \sum_i |A_i| \cdot |B_i| \leq 76. \]
Theorem 12. For \( n = 10 \), \( k = 4 \), \( n' = 6 \), \( k' = 3 \), \( q = 2 \), and \( d = 6 \), the maximum
achievable \( \Lambda \) of the coset construction is given by 76.

\[
\begin{array}{cccccccc}
 i & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
|S_i| & 77 & 840 & 2240 & 1792 & 560 & 112 & 16 \\
\end{array}
\]

Table 2. Sizes of \( S_i \) for \( 1 \leq i \leq 7 = \kappa \).

For general field sizes \( q \) we may choose \( A = G_q(4,1) \) and \( [4]_{1,q} = q^3 + q^2 + q + 1 \)
singletons \( A_i \). For \( B \) one may choose a \( (6, q^6+2q^2+2q+1, 4; 3) \) code, see [16].
Can one analytically describe packings of \( (6, q^6+2q^2+2q+1, 4; 3) \) codes into
\( q^3 + q^2 + q + 1 \) parts of large cardinality?

Theorem 13. \( A_2(10, 6; 4) \geq 4173 \).

Proof. Let \( C_2 \) be the code from the coset construction as outlined above. There is
exactly one pivot vector \( v = (1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0) \) satisfying the condition from
Lemma 5. The corresponding code \( C_1 \) is the MRD code of size \( \left\lceil 2^{6(4-3+1)} \rightceil = 4096 \),
so that \( |C_1 \cup C_2| = 4172 \). By a computer search we found a single codeword that
can be added to \( C_1 \cup C_2 \).

We remark that the code from Theorem 13 meets the MRD bound from Theorem 7. By an exhaustive search we have verified that the general Echelon-Ferrers
construction yields only codes with \( |C| \leq 4167 \).

6. Conclusion

The arguably most successful generally applicable construction for both constant
dimension and subspace codes of large minimum subspace distance is the Echelon-
Ferrers construction from [11]. Here, we have introduced a generalization of [12,
Construction III], which we call coset construction. It turned out that the new
construction is provably superior to the Echelon-Ferrers construction for some pa-
rameters, see Subsection 5.3. We were able to apply the coset construction to an
infinite family of constant dimension codes that attain the MRD bound from [12,
Theorem 11]. So far all improvements include the usage of a lifted MRD code of
maximal shape, so that these approaches are all limited by the MRD bound from
Theorem 7. For the relatively small addendums constructed by the coset construc-
tion, we may utilize subcodes that have a larger cardinality than the corresponding
value of the MRD bound, see Subsection 5.3. The constructions of subspace codes
based on the coset construction typically should yield many non-isomorphic codes,
since there are already many non-isomorphic MRD codes, see e.g. [4, 20]. In Sec-
tion 4 we have obtained some first insights on the optimal choice of parameters for
the coset construction and related optimization problems. However, we are rather
far away from a clear assessment of the capabilities of the coset construction. This can be seen for example through the following facts. While the coset construction is principally applicable for general subspace codes, we so far have not found a single example improving one of the currently known lower bounds, significantly contrasting the situation for constant dimension codes.

A more systematic analysis of “good” choices of parameters is needed. To this end we propose some strongly related open research questions. As a benchmark, it would be very valuable to generalize the MRD bound of Theorem 7 to a larger class of parameters. As it is an open problem whether the MRD bound of Theorem 7 can be attained in all cases, it seems promising to look at the corresponding open cases with more effort. Going along the lines of Theorem 7, one may study upper bounds on $(n,M,d; k)_q$ codes that contain $(n,M', d'; k)_q$ subcodes where $d' > d$, since such results would give upper bounds on the achievable parameters $\kappa$, see Subsection 4.2. This might give another hint which constant dimension codes may be appropriate for $A$ and $B$ within the coset construction.

Our analysis of the “optimality” of the example from Subsection 5.3 heavily relies on the classification of $(6, 77, 4; 3)_2$ codes. Since it possibly was only a matter of coincidence that we did not need to look at codes of smaller cardinalities we would like to classify all codes attaining cardinality $A_q(n, d; k)$ up to isomorphism extendability results, see e.g. [21], at least for moderate parameters. In order to generalize this example for field sizes $q > 2$, packings of $(6, q^6 + 2q^2 + 2q + 1, 4; 3)_q$ codes into $q^3 + q^2 + q + 1$ parts of large cardinality have to be studied.

The construction of Theorem 11 can easily by generalized to parameters $n = n' + 2k - 3$, where $k \geq 3$ and $n' \geq 3$. For $k' = 1$, $d' = 2$, we can choose $A = G_q(n', 1)$ and $B$ as a (maximal) partial $(k - 2)$-spread $P$ in $F_{q^n}$. Then, a packing of $P$ into $[\gamma]_q$ parts is needed. For the parameters of Theorem 11 this packing trivially exists. We remark that the maximum size of partial $k$-spreads in $F_{\tilde{n}}$ is known for $\tilde{n} \equiv 0, 1 \pmod{\tilde{k}}$ for arbitrary $q$, see e.g. [5], and for $\tilde{n} \equiv 2 \pmod{\tilde{k}}$ and $q = 2$, see [9, 19]. So it seems useful to study packings of the known best constructions for partial spreads into $[\gamma]_q$ parts of large cardinality for different values of $m$.

Acknowledgement

The authors would like to thank the editor and the anonymous referees for their remarks significantly improving the presentation of our results.

Appendix: Proof of Lemma 4

Proof. For $U, V \in G_q(n,k)$ and $\tau$ from (1), we have

\[ d_S(U, V) = 2(\dim(U + V) - k) = 2 \left( \text{rk} \left( \begin{array}{c} U \\ V \end{array} \right) - k \right). \]
In the case when \( A = A' \) and \( B = B' \) we conclude

\[
d_{S}\left(\tau^{-1}\left(\begin{array}{l}
A \\
0 \\
B
\end{array}\right),\tau^{-1}\left(\begin{array}{l}
A' \\
0 \\
B'
\end{array}\right)\right)
\]

\[
= 2 \left( \begin{array}{c}
\text{rk} \\
A \\
0 \\
B
\end{array}\right)
\]

\[
\geq 2 \left( \begin{array}{c}
\text{rk} \\
A \\
0 \\
B
\end{array}\right)
\]

Since the pivot columns of \( B \) in \( \varphi_{B}(F') - \varphi_{B}(F) \) consists solely of zeros, we have

\[
2 \left( \begin{array}{c}
\text{rk} \\
A \\
0 \\
B
\end{array}\right)
\]

\[
= 2(\text{rk}(A) + \text{rk}(B) - k)
\]

\[
= 2(k' + \text{rk}(F' - F) + k - k' - k)
\]

\[
= 2\text{rk}(F' - F) = 2d_{R}(F, F').
\]

For \( A \neq A' \) or \( B \neq B' \) we similarly conclude

\[
d_{S}\left(\tau^{-1}\left(\begin{array}{l}
A \\
0 \\
B
\end{array}\right),\tau^{-1}\left(\begin{array}{l}
A' \\
0 \\
B'
\end{array}\right)\right)
\]

\[
= 2 \left( \begin{array}{c}
\text{rk} \\
A \\
0 \\
B
\end{array}\right)
\]

\[
\geq 2 \left( \begin{array}{c}
\text{rk} \\
A \\
0 \\
B
\end{array}\right)
\]

using the fact that \( \text{rk}(X Y Z) \geq \text{rk}(X) + \text{rk}(Z) \) with equality if \( Y \) is zero and swapping rows or columns, respectively, does not change the rank. We continue with

\[
2 \left( \begin{array}{c}
\text{rk} \\
A' \\\n0 \\
B'
\end{array}\right)
\]

\[
= 2 \left( \begin{array}{c}
d_{S}(A, A') \\\n\frac{k'}{2} + \frac{d_{S}(B, B')}{2} + k - k' - k
\end{array}\right)
\]

\[
= d_{S}(A, A') + d_{S}(B, B').
\]

\[\square\]

REFERENCES

[1] R. Ahlswede, H.K. Aydinian, and L.H. Khachatrian, On perfect codes and related concepts, Designs, Codes and Cryptography 22 (2001), no. 3, 221–237.

[2] J. André, Über nicht-desarguessche Ebenen mit transitiver Translationsgruppe, Mathematische Zeitschrift 60 (1954), no. 1, 156–186.

[3] G.E. Andrews, The theory of partitions, no. 2, Cambridge university press, 1998.

[4] T.P. Berger, Isometries for rank distance and permutation group of gabidulin codes, IEEE Transactions on Information Theory 49 (2003), no. 11, 3016–3019.

[5] A. Beutelspacher, Partial spreads in finite projective spaces and partial designs, Mathematische Zeitschrift 145 (1975), no. 3, 211–229.
M. Braun, P.R.J. Östergård, and A. Wassermann, *New lower bounds for binary constant-dimension subspace codes*, Experimental Mathematics (2016), 1–5.

A. Cossidente and F. Pavese, *Subspace codes in PG(2n-1,q)*, Combinatorica (2016), 1–23.

J. Edmonds, *Matroids and the greedy algorithm*, Mathematical programming 1 (1971), no. 1, 127–136.

S. El-Zanati, H. Jordon, G. Seelinger, P. Sissoko, and L. Spence, *The maximum size of a partial 3-spread in a finite vector space over GF(2)*, Designs, Codes and Cryptography 54 (2010), no. 2, 101–107.

T. Etzion, E. Gorla, A. Ravagnani, and A. Wachter-Zeh, *Optimal Ferrers diagram rank-metric codes*, IEEE Trans. Inform. Theory 62 (2016), no. 4, 1616–1630. MR 3480069

T. Etzion and N. Silberstein, *Error-correcting codes in projective spaces via rank-metric codes and Ferrers diagrams*, IEEE Transactions on Information Theory 55 (2009), no. 7, 2909–2919.

T. Etzion and L. Storme, *Galois geometries and coding theory*, Designs, Codes and Cryptography (2015), 1–40.

D. Heinlein, M. Kiermaier, S. Kurz, and A. Wassermann, *Tables of subspace codes*, University of Bayreuth, 2015, available at [http://subspacecodes.uni-bayreuth.de](http://subspacecodes.uni-bayreuth.de).

T. Honold, M. Kiermaier, and S. Kurz, *Optimal binary subspace codes of length 6, constant dimension 3 and minimum distance 4*, Contemp. Math. 632 (2015), 157–176.

R. Koetter and F.R. Kschischang, *Coding for errors and erasures in random network coding*, IEEE Transactions on Information Theory 54 (2008), no. 8, 3579–3591.

A. Kohnert and S. Kurz, *Construction of large constant dimension codes with a prescribed minimum distance*, Mathematical methods in computer science, Springer, 2008, pp. 31–42.

S. Kurz, *Improved upper bounds for partial spreads*, Designs, Codes and Cryptography (to appear), 1–10.

K. Morrison, *Equivalence for rank-metric and matrix codes and automorphism groups of Gabidulin codes*, IEEE Transactions on Information Theory 60 (2014), no. 11, 7035–7046.

A. Nakić and L. Storme, *On the extendability of particular classes of constant dimension codes*, Designs, Codes and Cryptography (2015), 1–16.

N. Silberstein and T. Etzion, *Large constant dimension codes and lexicodes*, Adv. Math. Commun. 5 (2011), no. 2, 177–189.

N. Silberstein and A.-L. Trautmann, *Subspace codes based on graph matchings, ferrers diagrams, and pending blocks*, IEEE Transactions on Information Theory 61 (2015), no. 7, 3937–3953.

D. Silva, F.R. Kschischang, and R. Koetter, *A rank-metric approach to error control in random network coding*, IEEE Transactions on Information Theory 54 (2008), no. 9, 3951–3967.

V. Skachek, *Recursive code construction for random networks*, IEEE Transactions on Information Theory 56 (2010), no. 3, 1378–1382.

E-mail address: daniel.heinlein@uni-bayreuth.de

E-mail address: sascha.kurz@uni-bayreuth.de