A FREE BOUNDARY ISOPERIMETRIC PROBLEM IN
THE HYPERBOLIC SPACE BETWEEN PARALLEL
HOROSPHERES

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ABSTRACT. In this work we investigate the following isoperimetric problem: to find the regions of prescribed volume with minimal boundary area between two parallel horospheres in hyperbolic 3-space (the area of the part of the boundary contained in the horospheres is not included). We reduce the problem to the study of rotationally invariant regions and obtain the possible isoperimetric solutions by studying the behaviour of the profile curves of the rotational surfaces with constant mean curvature in the hyperbolic space.

1. Introduction

Geometric isoperimetric problems, (upper) estimates for the volume of regions of a given fixed boundary volume, or the dual problems, play an important role in Analysis and Geometry. There are both isoperimetric inequalities, common in Analysis, and actual classification of optimal geometric objects, like the round ball in Euclidean Geometry. We will be interested in the study of a relative free-boundary isoperimetric problem in Hyperbolic 3-space between two parallel horospheres. A survey of recent results about the geometric isoperimetric problems is [13].

For a Riemannian manifold $M^n$, the classical isoperimetric problem assumes the following formulation: to classify, up to congruency by the isometry group of $M$, the (compact) regions $\Omega \subseteq M$ enclosing a fixed volume that have minimal boundary volume. The relevant concepts of volume involved are those of Geometric Measure Theory: regions and their boundaries are $n$-rectifiable (resp. $(n-1)$-rectifiable) subsets of $M$ (cf. [11]).

If $M$ has boundary, the part of $\partial \Omega$ included in the interior of $M$ will be called the free boundary of $\Omega$, the other part will be called the fixed boundary. One may specify how the fixed boundary of $\Omega$ is included in the computation of the boundary volume functional. In this paper, we will assume that the volume of the fixed boundary of $\Omega$ is not considered in the boundary volume functional. We will see in Section 3 that this implies that the angle of contact between the interior boundary of $\Omega$ and $\partial M$ is $\pi/2$ (when
such contact occurs). Such relative problems are related to the geometry of stable drops in capillarity problems (the angle of contact depends, as mentioned, on how one considers the volume of the fixed boundary in the computation of the boundary volume functional). For a discussion about that, we refer to [7].

The motivation for our work is the well-known result of Athanassenas [2] and Vogel [14] which implies that between two parallel planes in the Euclidean space $\mathbb{R}^3$, a (stable) soap-bubble touching both walls perpendicularly is a straight cylinder perpendicular to the planes (enclosing a tube), and may only exist down to a certain minimal enclosing volume depending on the distance between the planes. Below that value, only half-spheres touching one of the planes or whole spheres not touching either plane occur, the cylinders becoming unstable. A newer proof can be found in [12], where the authors study the analogous problem in higher-dimensional Euclidean spaces.

In this paper we study the analogous relative isoperimetric problem between two parallel horospheres in hyperbolic space $\mathbb{H}^3(-1)$. We will use the upper halfspace model $\mathbb{R}^3_+$. The parallel horospheres are then represented by the horizontal Euclidean 2-planes of $\mathbb{R}^3_+$. We will present in this paper a detailed classification of the possible isoperimetric solutions.

The existence of isoperimetric regions in the manifold with boundary $(B, g)$, the slab composed by the two horospheres and the region between them, may be obtained by adapting a result due to Morgan [11] ($B/G$ is a compact space, where $G$ is the subgroup of the isometry group of $\mathbb{H}^3(-1)$ leaving $B$ invariant, so that Morgan’s result applies). Regarding the regularity of the free boundary, well-known results about the lower codimension bounds of the singular subset imply that it must be regular, in fact analytic.

In Section 2 we give some basic definitions in the model $\mathbb{R}^3_+$ like geodesics, totally geodesic surfaces, umbilical surfaces and rotational surfaces. We also give a more precise formulation for the isoperimetric problem considered using area and volume functionals.

In Section 3 we discuss briefly the rotationally invariance of isoperimetric regions and some of their basic geometric properties, since their (free) boundaries must have constant mean curvature and, when touching the bounding horospheres, the contact angle must be $\pi/2$. We will also discuss in more detail the existence of isoperimetric regions and the regularity of the free boundary part.

In Section 4 we investigate the tangency of profile curves for the rotational surfaces with constant mean curvature to determine the possible isoperimetric regions between the two parallel horospheres in $\mathbb{R}^3_+$. We define what we mean by catenoid, equidistant and onduloid type surfaces and prove the following result.

Let $c_1, c_2$ be positive real constants, $c_1 < c_2$, and $\mathcal{F}_{c_1, c_2} = \{(x, y, z) \in \mathbb{R}^3_+ : c_1 \leq z \leq c_2\}$. Let $V > 0$ and $\mathcal{C}_{c_1, c_2, V}$ be the set of $\Omega \subset \mathcal{F}_{c_1, c_2}$ with
volume $|\Omega| = V$ and boundary volume (area) $A(\Omega \cap \mathcal{F}_{c_1,c_2}^O) < \infty$, where we suppose $\Omega$ to be connected, compact, 3-rectifiable in $\mathcal{F}_{c_1,c_2}$, having as boundary (between the horospheres) an embedded, orientable, 2-rectifiable surface.

**Theorem 1.1.** Let $A_{c_1,c_2,V} = \inf \{ A(\Omega \cap \mathcal{F}_{c_1,c_2}^O) : \Omega \in \mathcal{C}_{c_1,c_2,V} \}$. Then

1. there exists $\Omega \in \mathcal{C}_{c_1,c_2,V}$ such that $A(\Omega \cap \mathcal{F}_{c_1,c_2}^O) = A_{c_1,c_2,V}$. As already mentioned, the free boundaries are actually analytic surfaces;
2. if $\Omega$ has minimal boundary volume, between the horospheres, the free boundary of $\Omega$, is either
   a) of catenoid type or umbilical with $H = 1$, or
   b) of equidistant type or umbilical with $0 < H < 1$, or
   c) of onduloid type or umbilical with $H > 1$.

The details of the description above are included in Section 4. We observe here that this result shows how the situation in hyperbolic geometry is different from the one in the Euclidean 3-space, where we also have rotationally invariant surfaces of catenoid and of onduloid type, but they cannot appear as boundaries of optimizing tubes (even though, in higher dimensions, hypersurfaces generated by onduloids in Euclidean spaces are known to occur as boundaries of optimal tubes connecting two parallel hyperplanes (cf. [12])).

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2. **Preliminaries**

In this section we will introduce some basic facts and notations that will appear along the paper.

Let $\mathcal{L}^4 = (\mathbb{R}^4, g)$ the 4-dimensional Lorentz space endowed with the metric $g(x, y) = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$ and the 3-dimensional hyperbolic space $\mathcal{H}^3(-1) := \{ p = (x_1, x_2, x_3, x_4) \in \mathcal{L}^4 : g(p, p) = -1, \ x_4 > 0 \}$.

We use the upper halfspace model $\mathbb{R}_+^3 := \{ (x, y, z) \in \mathbb{R}^3 ; z > 0 \}$ for $\mathcal{H}^3(-1)$, endowed with the metric $<,> = ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}$.

Let $\phi : \Sigma \to \mathbb{R}_+^3$ be an isometric immersion of a compact manifold $\Sigma$ with boundary $\partial \Sigma \neq \emptyset$ and $\Gamma$ be a curve in $\mathbb{R}_+^3$. If $\phi$ is a diffeomorphism of $\partial \Sigma$ onto $\Gamma$, we say that $\Gamma$ is the boundary of $\phi$ and if $\phi$ has constant mean curvature $H$, we say that $\Sigma$ is an $H$-surface with boundary $\Gamma$. We identify $\Sigma$ with its image by $\phi$ and $\partial \Sigma$ with the curve $\Gamma$.

The plane $z = 0$ is called the infinity boundary of $\mathbb{R}_+^3$ and we denote it by $\partial_\infty \mathbb{R}_+^3$. The geodesics of $\mathbb{R}_+^3$ are represented by vertical Euclidean lines and half-circles orthogonal to $\partial_\infty \mathbb{R}_+^3$, contained in $\mathbb{R}_+^3$. The totally geodesic
surfaces have constant mean curvature $H = 0$ and are represented by vertical Euclidean planes and hemispheres orthogonal to $\partial_\infty \mathbb{R}^3_+$, contained in $\mathbb{R}^3_+$.

The horizontal Euclidean translations and the rotations around a vertical geodesic are isometries of $\mathbb{R}^3_+$. We have two family of isometries associated to one point $p_0 \in \partial_\infty \mathbb{R}^3_+$: the Euclidean homotheties centered in $p_0$ with factor $k > 0$, called hyperbolic translations through a geodesic $\alpha$ perpendicular to $\partial_\infty \mathbb{R}^3_+$ in $p_0$, and the hyperbolic reflections with respect to a totally geodesic surface $P$.

When $P$ is a hemisphere orthogonal to $\partial_\infty \mathbb{R}^3_+$ centered in $p_0$ and radius $r > 0$, the hyperbolic reflections are Euclidean inversions centered in $p_0$ that fix $P$ and, when $P$ is a vertical Euclidean plane, they are Euclidean reflections with respect to $P$.

The umbilical surfaces of $\mathbb{R}^3_+$ are described as follows (see for example [8]):

1. **Totally geodesics**: These surfaces were already described ($H = 0$).
2. **Geodesic spheres**: represented by Euclidean spheres entirely contained in $\mathbb{R}^3_+$, they have $H > 1$ (mean curvature vector points to the interior). If $\rho$ is the hyperbolic radius of a geodesic sphere then $H = \coth \rho$.
3. **Horospheres**: represented by horizontal Euclidean planes of $\mathbb{R}^3_+$ and Euclidean spheres of $\mathbb{R}^3_+$ which are tangent to $\partial_\infty \mathbb{R}^3_+$, they have $H = 1$ and the mean curvature vector points upward in the case of horizontal planes and to the interior in the case of spheres.
4. **Equidistant surfaces**: represented by the intersection of $\mathbb{R}^3_+$ with the planes of $\mathbb{R}^3$ that are neither parallel nor perpendicular to the plane $z = 0$ and by (pieces of) Euclidean spheres that are not entirely contained in $\mathbb{R}^3_+$ and are neither parallel tangent nor perpendicular to the plane $z = 0$. They have $0 < H < 1$ and the mean curvature vector points to the totally geodesic surface they are equidistant to.

In our study, the (spherical) rotational surfaces of $\mathbb{R}^3_+$ play an important role since the solutions of the isoperimetric problem must be rotationally invariant regions. They are defined as surfaces invariant by a subgroup of isometries whose principal orbits are (Euclidean) circles.

Let $\Pi_1$ and $\Pi_2$ be horospheres represented by distinct parallel horizontal Euclidean planes, $\Pi = \Pi_1 \cup \Pi_2$, $\mathcal{F} = \mathcal{F}(\Pi_1, \Pi_2)$ the closed slab between them and $\phi : \Sigma \to \mathcal{F}$ an immersion of a compact, connected, embedded and orientable $C^2$-surface with boundary $\Gamma = \partial \Sigma$ and such that $\phi(\Gamma) \subset \Pi$ (we will see later that the image by $\phi$ of the interior of the surface $\Sigma$ will not be allowed to touch $\Pi$ if $\Sigma$ is the boundary of an optimal domain in our variational problem, but this is not part of the general situation at this point).

We fix, now, the notation for some well-known geometric invariants related to isometric immersions. We identify (locally) $\Sigma$ with $\phi(\Sigma)$ and $X(p) \in T_p \Sigma$ with $d\phi_p(X(p)) \subset \mathbb{R}^3_+$. We have, as usual, the decomposition $T_p(\mathbb{R}^3_+) = \mathbb{R}^3_+ = \mathbb{R}^3_+$. We have, as usual, the decomposition $T_p(\mathbb{R}^3_+) = \mathbb{R}^3_+ = \mathbb{R}^3_+$. We have, as usual, the decomposition $T_p(\mathbb{R}^3_+) = \mathbb{R}^3_+ = \mathbb{R}^3_+$. We have, as usual, the decomposition $T_p(\mathbb{R}^3_+) = \mathbb{R}^3_+ = \mathbb{R}^3_+$. We have, as usual, the decomposition $T_p(\mathbb{R}^3_+) = \mathbb{R}^3_+ = \mathbb{R}^3_+. \]
$T_p(\Sigma) \oplus N_p(\Sigma)$ into the tangent and normal spaces to $\Sigma$ in $p$, respectively. Choose an orientation for $\Sigma$ and let $N$ be the (positive) unitary normal field along the immersion $\phi$. If $X(p) \in T_p(\mathbb{R}^3)$, we may write $X(p) = X(p)^T + X(p)^N = X(p)^T + \alpha N(p)$, where $\alpha \in \mathbb{R}$.

Let $<\cdot, \cdot>$ be the metric induced on $\Sigma$ by the immersion $\phi$, making it isometric, $\nabla$ be the Riemannian connection of the ambient space $\mathbb{R}^3$ and $\nabla$ the induced Riemannian connection of $\Sigma$. Let $X, Y \in \mathfrak{X}(\Sigma)$ be $C^\infty$-vector fields, then $\nabla_X Y = ((\nabla_X Y)^T$ and $\mathcal{B}(X, Y) = (\nabla_X Y)^N$ are, as usual, the induced connection on $\Sigma$ and the second fundamental form of the immersion given by $\mathcal{B}$. We also have the Weingarten operator $A_N(Y) = -(\nabla Y . N)^T$ so that $<A_N(X), Y> = \mathcal{B}(X, Y), N>$. Finally, the mean curvature of the immersion $\phi$ is $H = 1/2 \text{trace}(A_N)$.

**Definition 1.** A variation of $\phi$ is a smooth map $F : (-\epsilon, \epsilon) \times \Sigma \rightarrow \mathbb{R}^3_+$, such that for all $t \in (-\epsilon, \epsilon)$, $\phi_t : \Sigma \rightarrow \mathbb{R}^3_+$, defined by $\phi_t(p) = F(t, p)$,

(a) is an immersion;

(b) $\phi_0 = \phi$.

For $p \in \Sigma$, $X(p) = \partial \phi_t(p)/\partial t|_{t=0}$ is the variation vector field of $F$ and the normal variation function of $F$ is given by $f(p) = <X(p), N(p)>$. We say that the variation $F$ is normal if $X$ is normal to $\phi$ at each point and that $F$ has compact support if $X$ has compact support. For a variation with compact support and small values of $t$ we have that $\phi_t$ is an immersion of $\Sigma$ in $\mathbb{R}^3_+$. In this case we define the (variation) area function $A : (\epsilon, \epsilon) \rightarrow \mathbb{R}$ by

$$A(t) = \int_{\Sigma} dA_t = \int_{\Sigma} \sqrt{\text{det}((d\phi_t)^*(d\phi_t))} \ dA,$$

where $dA$ is the element of area of $\Sigma$. $A(t)$ is the area of $\Sigma$ with the metric induced by $\phi_t$. We also define the (variation) volume function $V : (-\epsilon, \epsilon) \rightarrow \mathbb{R}$ by

$$V(t) = -\int_{[0, t] \times \Sigma} F^*d(\mathbb{R}^3_+),$$

where $d(\mathbb{R}^3_+)$ is the element of volume of $\mathbb{R}^3_+$ and $F^*d(\mathbb{R}^3_+)$ is the pull-back of $d(\mathbb{R}^3_+)$ by $F$. $V(t)$ is not, actually, the volume of some region with $\phi_t(\Sigma)$ as boundary, but of a “tubular neighborhood” along $\phi(\Sigma)$ between $\phi(\Sigma)$ and $\phi_t(\Sigma)$. The sign is related to the net change with respect to the normal field defining the orientation (for example, contracting a sphere in $\mathbb{R}^3$, which means moving it in the direction of the mean curvature vector, gives a negative sign for $V(t)$, as expected).

**Definition 2.** Let $F : (-\epsilon, \epsilon) \times \Sigma \rightarrow \mathbb{R}^3_+$ a variation of $\phi$.

(i) $F$ preserves volume if $V(t) = V(0)(= 0), \forall t \in (-\epsilon, \epsilon)$;

(ii) $F$ is admissible if $F(\partial \Sigma) \subset \Pi, \forall t \in (-\epsilon, \epsilon)$.

**Definition 3.** We say that the immersion $\phi$ is stationary when $A'(0) = 0$, for all admissible variations that preserve volume.
Remark 2.1. If $\Omega$ is a (compact) regular region in the slab $\mathcal{F}$ between the horospheres $\Pi$, taking $\Sigma$ in Definition 2 as the (embedded regular) free boundary of $\Omega$, we may extend the above variational approach to produce a variation $\Omega(t)$ of $\Omega$ by embedded domains (for small $t$), such that the condition $V(t) = 0$ in Definition 2 is equivalent to holding the measure (volume) $|\Omega(t)|$ of $\Omega(t)$ constant (the same as that of $\Omega = \Omega(0)$) along the variation. This justifies saying that the variation “preserves volume” in the above definition.

We end this section by stating again our problem. Let $\Pi_1$ and $\Pi_2$ be two parallel horospheres in $\mathbb{R}^3$ and $\mathcal{F} = \mathcal{F}(\Pi_2, \Pi_2)$ the (closed) slab between them.

**Isoperimetric problem for $\mathcal{F}(\Pi_1, \Pi_2)$**: fix a volume value and study the domains $\Omega \subset \mathcal{F}$ with the prescribed volume which have minimal free boundary area.

**Definition 4.** A (compact) minimizing region $\Omega$ for this problem will be called an isoperimetric domain or region in $\mathcal{F}$.

In more detail, one is looking for the classification and geometric description of isoperimetric regions (as a function of the volume value), in as much detail as possible, aiming at the determination of the isoperimetric profile (minimal free boundary area as a function of prescribed volume) for $\mathcal{F}$.

### 3. First Results about the Isoperimetric Solutions

Our main goal in this section is to characterize the stationary immersions according to Definition 3. The following formulae for the first variations of the area and volume functions are well known. For an immersed surface with boundary, the exterior conormal is the vector field along the boundary given as follows: in the tangent plane of $\Sigma$ in $p \in \Gamma$, take the perpendicular vector to the tangent vector to $\Gamma$ in $p$ along the boundary $\Gamma$.

**Proposition 3.1.** Let $\mathcal{F}$ be a variation of $\phi$ with variational field $X$ and compact support in $\Sigma$. Then

1. $A'(0) = -2 \int_{\Sigma} H f \, dA + \int_{\Gamma} <X, \nu> \, d\Gamma$, where $\nu$ is the unitary exterior conormal, $dA$ is the element of area of $\Sigma$ and $d\Gamma$ is the element of length of $\Gamma$ induced by $\phi$.
2. $V'(0) = -\int_{\Sigma} f \, dA$, where $f(p) = <X(p), N(p)>$, as before.

**Proof:** Although the formula of the variation of the area functional is well known (see [4]), we show here a different proof to get it. From the
definition of $A(t)$ we obtain

$$A'(t) = \int_{\Sigma} \left[ \frac{1}{2\sqrt{\det((d\phi_t)^{*}d\phi_t)}} \left( \det((d\phi_t)^{*}d\phi_t) \right) \times \right.$$

$$\left. \frac{\partial}{\partial t} \left( (d\phi_t)^{*}d\phi_t \right) \right] dA.$$

As $\phi_0$ is the inclusion of $\Sigma$ in $\mathbb{R}^3_+$, $d\phi_0$ is the inclusion of the respective tangent spaces and $d\phi_0^\perp$ is the orthogonal projection on $T\Sigma$

Evaluating $A'(t)$ for $t = 0$ we get

$$A'(0) = \int_{\Sigma} \frac{1}{2} \operatorname{trace} \left( \frac{d}{dt} \bigg|_{t=0} [(d\phi_t)^{*}d\phi_t] \right) dA.$$

Using the Symmetry Lemma for the connection $\nabla^\phi$ along the immersion we have

$$\frac{d}{dt} \bigg|_{t=0} (d\phi_t) = \nabla^\phi \frac{\partial \phi_t}{\partial t} \bigg|_{t=0} = \nabla^\phi X.$$

Then

$$A'(0) = \int_{\Sigma} \frac{1}{2} \operatorname{trace} \left( (\nabla^\phi X)^{*} \bigg|_{T\Sigma} + \operatorname{proj}_{T\Sigma} \nabla^\phi X \right) dA = \int_{\Sigma} \operatorname{trace} \left( \operatorname{proj}_{T\Sigma} \nabla^\phi X \right) dA,$$

where $\operatorname{proj}_{T\Sigma}$ denotes the projection on $T\Sigma$

Decomposing the variational field as $X = X^T + X^N$, we have that the projections of the tangent and normal components of $\nabla^\phi(X)$ on $T\Sigma$ are

$$\operatorname{proj}_{T\Sigma} \nabla^\phi(X^T) = \nabla(X^T)$$

$$\operatorname{proj}_{T\Sigma} \nabla^\phi(X^N) = -AX^N.$$

So

$$A'(0) = \int_{\Sigma} \left( \operatorname{div}X^T - 2 < X^N, H \ N > \right) dA.$$

Using the Stokes theorem we get

$$A'(0) = \int_{\Gamma} <X^T, \nu> d\Gamma - 2 \int_{\Sigma} <X^N, H \ N > dA =$$

$$= -2 \int_{\Sigma} Hf \ dA + \int_{\Gamma} <X, \nu> d\Gamma.$$

The first variation of volume given in (2) is the standard one and will be omitted (cf.[1]).

From the next result we deduce that the boundary of the isoperimetric region we are studying must be an $H$-surface that makes a contact angle $\pi/2$ with the horospheres $\Pi_1$ and $\Pi_2$.

**Theorem 3.1.** An immersion $\phi : \Sigma \rightarrow \mathbb{R}^3_+$ is stationary if and only if $\phi$ has constant mean curvature and meets the horospheres $\Pi = \Pi_1 \cup \Pi_2$ that contains its boundary $\Gamma = \partial \Sigma$, perpendicularly along the boundary (if the intersection is non-empty).
Proof: Adapting the proof of Proposition 2.7 of [3] we show that if \( \phi \) has constant mean curvature and meets the horospheres \( \Pi = \Pi_1 \cup \Pi_2 \) that contains its boundary \( \Gamma = \partial \Sigma \), perpendicularly along the boundary then \( \phi \) is stationary; and that if \( \phi \) is stationary then \( \phi \) has constant mean curvature. To show that if \( \phi \) is stationary then \( \phi \) meets \( \Pi \) perpendicularly along its boundary \( \Gamma = \partial \Sigma \), we take an admissible variation \( \Phi \) that preserves volume with variational field \( X \) and \( p_0 \in \partial \Sigma \). Suppose, by contradiction, \( \langle X(p_0), \nu(p_0) \rangle \neq 0 \). By continuity there is a neighbourhood \( U = W_1 \cup \partial \Sigma \) of \( p_0 \) such that \( \langle X(p), \nu(p) \rangle > 0 \), \( \forall p \in U \), where \( W_1 \) is a neighbourhood of \( p_0 \) in \( \Sigma \). We take \( q \in \Sigma \setminus W_1 \), \( W_2 \) a neighbourhood of \( q \) disjoint to \( W_1 \) and a partition of unity on \( W_1 \cup W_2 \). There exists a differentiable function \( \xi_1 : W_1 \to \mathbb{R} \) such that \( \xi_1(W_1) \subset [0,1] \) with support \( \text{supp} \xi_1 \subset W_1 \). We may consider also a differentiable map \( \xi_2 : W_2 \to \mathbb{R} \) such that \( \xi_2(W_2) \subset [0,1] \), \( \text{supp} \xi_2 \subset W_2 \) and 

\[
\int_{W_1} \xi_1 f \ dW_1 + \int_{W_2} \xi_2 f \ dW_2 = 0.
\]

We consider the variation \( \Phi_\xi : (-\epsilon, \epsilon) \times \Sigma \to \mathbb{R} \) with compact support on \( W_1 \cup W_2 \) such that 

\[
\Phi_\xi(t, p) = \begin{cases} 
\Phi(\xi_1 t, p), & p \in W_1, \\
\Phi(\xi_2 t, p), & p \in W_2.
\end{cases}
\]

Notice that \( \Phi_\xi \) is admissible because \( \Phi \) is admissible.

If \( f_\xi(p) \) denotes the normal component of the variation vector we have 

\[
\int_{\Sigma} f_\xi \ dA = \int_{W_1} \xi_1 \langle p, \nu \rangle \ dW_1 + \int_{W_2} \xi_2 \langle p, \nu \rangle \ dW_2 = 0,
\]

and \( \Phi_\xi \) preserves volume.

For this variation we have 

\[
0 = A'(0) = -2H \int_{\Sigma} f_\xi \ dA + \int_{W_1} \xi_1 \langle X, \nu \rangle \ d\Gamma = \int_{W_1} \xi_1 \langle X, \nu \rangle \ d\Gamma > 0,
\]

which is a contradiction. Then \( \forall p \in \partial \Sigma \) it follows that \( \langle X, \nu \rangle(p) = 0 \). 

Next we show that the boundary of the isoperimetric solutions are rotationally invariant regions.

We need some symmetrization principle for \( H \)-surfaces. Taking the version of Alexandrov’s Principle of Reflection for the hyperbolic space (for further references and details see [1]) and using [3] as reference to take the suitable objects in our case as the reflection planes, we get the next result (a detailed proof may be found in [7]).

Theorem 3.2. Let \( \Sigma \) be a compact connected orientable and embedded \( H \)-surface of class \( C^2 \), between two parallel horospheres \( \Pi_1, \Pi_2 \) in \( \mathbb{R}^3_+ \) and with boundary \( \partial \Sigma \subset \Pi_1 \cup \Pi_2 \) (possibly empty). Then \( \Sigma \) is rotationally symmetric around an axis perpendicular to \( \Pi_1 \) and \( \Pi_2 \).
We observe that the intersection of $\Sigma$ with a horosphere $H$ (represented by a horizontal Euclidean plane) is just an Euclidean circle. In fact, if they were two concentric circles and the isoperimetric region was delimited by these circles we would get a totally geodesic symmetry plane $P$ that would not contain the axis of symmetry.

4. ISOPERIMETRIC REGIONS BETWEEN HOROSPHERES IN $\mathbb{R}^3_+$

In this section we classify the rotational $H$-surfaces of $\mathbb{R}^3_+$ between two parallel horospheres with boundary contained in the horospheres and that intersects the horospheres perpendicularly. Therefore we get the possible solutions for the isoperimetric problem in the hyperbolic space since they must be regions delimited by those surfaces. We start with some important results obtained by Barrientos [5] in his PhD thesis which will be useful for our task.

If $(\rho, \theta, z)$ are the cylindrical coordinates of a point $p$ in $\mathbb{R}^3_+$ then the cartesian coordinates are given by

$$ (x, y, z) = e^z(tanh \rho \cos \theta, tanh \rho \sin \theta, sech \rho). $$

For a spherical rotational surface $\Sigma$ of $\mathbb{R}^3_+$ we can provide the so called natural parametrization, whose metric is

$$ d\sigma^2 = ds^2 + U^2(s) \, dt^2, $$

where $U = U(s)$ is a positive function. Supposing that its profile curve is locally a graphic $z = \lambda = \lambda(\rho)$ in the plane $\theta = 0$, the natural parameters are given by

$$ ds = \sqrt{1 + \dot{\lambda}^2(\rho)} \cos^2 \rho \, d\rho \, e \, dt = d\varphi, $$

and the following relations hold

$$ U^2(s) = \sinh^2 \rho(s) \text{ and } \dot{\lambda}^2(\rho) = \frac{1 + U^2(s) - \dot{U}^2(s)}{(1 + U^2(s))^2}. $$

Then the natural parametrization for a rotational surface in cylindrical coordinates is

$$ \begin{align*}
\sinh^2 \rho(s) &= U^2(s), \\
\lambda(s) &= \int_0^s \frac{\sqrt{1 + U^2(t) - \dot{U}^2(t)}}{1 + U^2(t)} \, dt, \\
\varphi(t) &= t.
\end{align*} $$

In [5], the rotational were studied $H$-surfaces of $\mathbb{R}^3_+$. By replacing $z(s) = U^2(s)$, he got the differential equation for a rotational $H$-surface in $\mathbb{R}^3_+$

$$ \frac{z^2}{4} = (1 - H^2)z^2 + (1 + 2aH)z - a^2, $$
where \( a \) is a constant of integration, and showed that the behaviour of their profile curves is determined by the constant \( a \). Choosing the orientation for the surfaces in order to \( H \geq 0 \), there are three cases to study: \( H = 1 \), \( 0 \leq H < 1 \) and \( H > 1 \).

The natural parametrization for a rotational \( H \)-surface in \( \mathbb{R}^3_+ \), \( H = 1 \), generated by a curve \( c(s) = (\rho(s), \lambda(s)) \) is given by

\[
\begin{align*}
\sinh^2 \rho(s) &= \frac{a^2 + (1 + 2a)^2s^2}{1 + 2a}, \\
\lambda(s) &= \int_0^s \frac{\sqrt{1 + 2a(-a(1 + a) + (1 + 2a)t^2 + (1 + 2a)^2t^2)}}{(-a(1 + a) + (1 + 2a)t^2)^2 + (1 + 2a)^4t^2} \, dt, \\
\varphi(t) &= t
\end{align*}
\] (4.3)

where \( A = 1 + 2aH \), \( B = \sqrt{1 + 4aH + 4a^2} \) and \( \alpha = \sqrt{1 - H^2} \).

The natural parametrization for a rotational \( H \)-surface in \( \mathbb{R}^3_+ \), \( 0 \leq H < 1 \), generated by a curve \( c(s) = (\rho(s), \lambda(s)) \) is given by

\[
\begin{align*}
\sinh^2 \rho(s) &= \frac{-A + B \cosh(2\alpha s)}{2\alpha^2}, \\
\lambda(s) &= \int_0^s \frac{\sqrt{2a(-2aH(-1 + B \cosh(2\alpha t)))}}{(-2aH(-1 + B \cosh(2\alpha t)))^2 + \alpha^2 a^2 \sin^2(2\alpha t)} \, dt, \\
\varphi(t) &= t
\end{align*}
\] (4.4)

Next we present some notations and definitions we will use throughout this section. From (4.1) the profile curve of a rotational \( H \)-surface in \( \mathbb{R}^3_+ \) is given by

\[
c_+(s) = e^{\lambda(s)(\tanh \rho(s), \sech \rho(s))}.
\] (4.6)

If \( H = 1 \), \( \rho(s) \) and \( \lambda(s) \) are given by (4.3) and \( a > -\frac{1}{2} \). When \( -\frac{1}{2} < a < 0 \) we call the family of such of rotational as catenoid’s cousin type surfaces and when \( a = 0 \) we have the umbilical surfaces with \( H = 1 \).
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If $0 \leq H < 1$, $\rho(s)$ and $\lambda(s)$ are given by (4.4) and $a \in \mathbb{R}$. When $a < 0$ such rotational surfaces are called *equidistant type surfaces* and when $a = 0$ we get umbilical surfaces, with $0 \leq H < 1$.

If $H > 1$, $\rho(s)$ and $\lambda(s)$ are given by (4.5) and $a \geq -\frac{1}{4H} \sqrt{H^2 - 1}$. When $-\frac{1}{4H} < a < 0$ we obtain the so called *onduloid type surfaces* and if $a = 0$ we get the umbilical surfaces with $H > 1$.

We observe that the profile curve $c_+(s)$ depends on the parameters $H$, $a$, and $r$, where $r$ is the geodesic radius, that is, the radius of a geodesic perpendicular to the $z$-axis. Then for $H$ and $r$ fixed we get a family of rotational $H$-surfaces. From the Euclidean homothety $\mathcal{H}_r$, with factor $r$,

$$\mathcal{H}_r(c_+(s)) = e^{\lambda(s)}(r \tanh \rho(s), r \text{sech} \rho(s)),$$

we get also other families of profile curves of rotational $H$-surfaces, so that $r = 1$.

According to Theorem 3.1 and Theorem 3.2 the boundary of the isoperimetric solutions must be rotational $H$-surfaces that meet the horospheres $\{z = c_1\}$ and $\{z = c_2\}$ perpendicularly. Our goal is to determine the vertical tangency points of the profile curves of the rotational surfaces.

**Definition 5.** Let $c_+(s)$ be a curve parametrized by (4.6). We say that a point $c_+(s)$ is a vertical tangency point if the tangent vector in $c_+(s)$ satisfies $\dot{c}_+(s) = (0, b)$, where $b \in \mathbb{R}^*$, that is,

(4.7) \[ e^{\lambda(s)}(\tanh \rho(s) \dot{\lambda}(s) + \text{sech}^2 \rho(s) \dot{\rho}(s)) = 0, \]

(4.8) \[ e^{\lambda(s)}(\text{sech} \rho(s) \dot{\lambda}(s) - \text{sech} \rho(s) \tanh \rho(s) \dot{\rho}(s)) = b. \]

As $e^{\lambda(s)} > 0$, (4.7) implies that

(4.9) \[ \tanh \rho(s) \dot{\lambda}(s) + \text{sech}^2 \rho(s) \dot{\rho}(s) = 0. \]

By (4.9) we obtain the points where the vertical tangency occurs and by (4.8) we get the direction of the vertical tangency (upward or downward).

Replacing (4.2) in (4.9) we see that if $p$ is a vertical tangency point for which $U(s) \neq 0$ then

(4.10) \[ U^2(s) = \tilde{U}^2(s), \]

and the roots of (4.10) give us the vertical tangency points.

Next we study the behaviour of the profile curve of rotational $H$-surfaces determining the possible vertical tangency points.

**First Case:** $H = 1$.

The geodesic radius is given by $\{\lambda = 0\}$ in cylindrical coordinates. By (4.6) we get the curve $c_g(s) = (\tanh \rho(s), \text{sech} \rho(s))$. 

Theorem 4.1. If \( c_+(s) = e^{\lambda(s)}(\tanh \rho(s), \text{sech}\rho(s)) \) with \( \rho(s), \lambda(s) \) given by (4.3) is the parametrization of the profile curve of a rotational \( H \)-surface in \( \mathbb{R}^3_+ \), with \( H = 1 \), then \( c_+(s) \) is symmetric to the geodesic radius \( c_g \).

Proof: By (4.3) we have that \( \lambda(0) = 0 \). So \( c_+(0) \in c_g \). If \( I \) denotes the Euclidean inversion through \( c_g \), we have that \( I(c_+(s)) = c_+(-s) \), since \( \rho(s) \) is an even function and \( \lambda(s) \) is odd.

By (4.3) we have that
\[
\sinh \rho(s) = 0 \iff a = 0 \text{ and } s = 0.
\]

So \( \tanh \rho(s) > 0 \) if \( a \) and \( s \) were not both equal to zero. Furthermore \( s = 0 \) is the unique minimum point of \( \rho(s) \).

If \( H = 1 \) we have from (4.3) that
\[
U^2(s) = \frac{a^2 + (1 + 2a)^2 s^2}{1 + 2a}, \quad \ddot{U}(s) = \frac{(1 + 2a)^3 s^2}{a^2 + (1 + 2a)^2 s^2}.
\]

Replacing (4.12) in (4.10) we get
\[
(1 + 2a)^4 s^4 + \left(2a^2(1 + 2a)^2 - (1 + 2a)^4\right)s^2 + a^4 = 0.
\]

Making \( t = s^2 \) in (4.13) we get a second degree equation whose discriminant is
\[
\Delta = (1 + 2a)^6(4a + 1).
\]

Since \( 1 + 2a > 0 \) in case \( H = 1 \) we have that
- if \( -\frac{1}{2} < a < -\frac{1}{4} \), (4.13) has no real roots. Then there are no vertical tangency points in this case;
- if \( a = -\frac{1}{4} \), there are at most two vertical tangency points

\[
s = \pm \frac{1}{2};
\]

- if \( a > -\frac{1}{4} \), there are at most four vertical tangency points given by
\[
s_1 = \frac{1 + 2a + \sqrt{1 + 4a}}{2(1 + 2a)}, \quad s_2 = -s_1, \quad s_3 = \frac{1 + 2a - \sqrt{1 + 4a}}{2(1 + 2a)}, \quad s_4 = -s_3.
\]

Besides these informations we study the vertical tangencies according to the variation of the parameter \( a \).

(1) If \( -\frac{1}{4} < a < 0 \) we have \( \dot{\lambda}(s) > 0 \). If \( s \geq 0 \) then
\[
\tanh \rho(s)\dot{\lambda}(s) + \text{sech}^2 \rho(s)\dot{\rho}(s) > 0
\]
and (4.10) is not possible. As \( 1 + 2a > 0 \) the roots \( s_1 \) and \( s_3 \) of (4.10) given by (4.16) are strictly positives. So they do not give vertical
tangency points. The other roots \( s_2, s_4 < 0 \) give us the vertical tangency points that point out upward for \( b > 0 \) in (4.8). In Figure 1 we have the profile curve for \( H = 1 \) and \( a = -0.2 \) and the horocycles that pass through the vertical tangency. In Figure 2 we see two parallel horospheres and the rotational surface, between the horospheres, that meets them perpendicularly.

![Figure 1. Profile curve for \( H = 1 \) and \( a = -0.2 \).](image1)

![Figure 2. Rotational surface with \( H = 1 \) and \( a = -0.2 \).](image2)

In particular if \( a = -\frac{1}{4} \), the positive root \( s = \frac{1}{2} \) of (4.10) given by (4.15) does not give a vertical tangency point. There exists only one vertical tangency point corresponding to \( s = -\frac{1}{2} \). From the informations about \( \rho(s) \) and \( \lambda(s) \) we get

\[
\lim_{s \to -\infty} e^{\lambda(s)} \tanh \rho(s) = 0, \quad \lim_{s \to -\infty} e^{\lambda(s)} \text{sech} \rho(s) = 0,
\]

\[
\lim_{s \to \infty} e^{\lambda(s)} \tanh \rho(s) = \infty, \quad \lim_{s \to \infty} e^{\lambda(s)} \text{sech} \rho(s) = \infty.
\]

(4.17)

Therefore this case is not suitable for the isoperimetric problem.

(2) If \( a = 0 \) we have two pieces of horocycles tangent at \((0,1)\) that generate the umbilical surfaces with \( H = 1 \). They are represented by the Euclidean plane \( \{z = 1\} \) or the Euclidean sphere with radius \( \frac{1}{2} \) tangent to \( \partial \mathbb{R}^3_+ \) at \((0,0,0)\). In the last case it occurs only one vertical tangency point and in this case the surface meets only one of the horospheres (boundary) perpendicularly. In fact, taking the upper Euclidean half sphere that represents the horosphere we get the possible isoperimetric solution for the umbilical case with \( H = 1 \). In Figure 3 we see the pair of profile curves for umbilical surfaces with \( H = 1 \) and the horocycle that pass through the vertical tangency point. Figure 4 illustrates the possible isoperimetric solution for the umbilical case with \( H = 1 \).
(3) If \( a > 0 \) the profile curves have only one self-intersection. From (4.6), if \( c_+(s_i) = c_+(s_j) \) is a self-intersection then \( s_i = \pm s_j \) and since the curves are symmetric with respect to \( c_g \), the self-intersections must occur on \( c_g \). So \( \lambda(s_i) = \lambda(s_j) = 0 \). From (4.3) we deduce that \( \lambda(s) \) has a maximum point in \( \pm \sqrt{a(1+a)} \) and a minimum point in \( \sqrt{a(1+a)} \). Furthermore \( \lim_{s \to \infty} \lambda(s) = \infty \) (see [5]), \( \lambda(0) = 0 \) and \( \lambda(s) \) is an odd function.

We also have that \( \rho(s) \) has only one minimum point in \( s = 0 \). So if \( s > \sqrt{a(1+a)} > 0 \) then \( \dot{\rho}(s), \dot{\lambda}(s) > 0 \) and

\[
\tanh \rho(s) \dot{\lambda}(s) + \text{sech}^2 \rho(s) \dot{\rho}(s) > 0;
\]

if \( -\sqrt{a(1+a)} < s < 0 \) we have \( \dot{\rho}(s), \dot{\lambda}(s) < 0 \), which implies that

\[
\tanh \rho(s) \dot{\lambda}(s) + \text{sech}^2 \rho(s) \dot{\rho}(s) < 0.
\]

In both cases (4.9) is not verified. As \( a > 0 \) the roots \( s_1, s_2, s_3, s_4 \) of (4.10) given by (4.16) satisfy

\[
s_1 > \frac{\sqrt{a(1+a)}}{1+2a}, \quad s_2 < -\frac{\sqrt{a(1+a)}}{1+2a},
\]

\[
0 < s_3 < \frac{\sqrt{a(1+a)}}{1+2a}, \quad -\frac{\sqrt{a(1+a)}}{1+2a} < s_4 < 0.
\]

Therefore the vertical tangency is possible only for the positive roots \( s_2 \) and \( s_3 \). As \( \dot{\rho}(s_2) < 0 \) and \( \dot{\lambda}(s_2) > 0 \), the vertical tangency in \( s_2 \) points out upward. However \( \dot{\rho}(s_3) > 0 \) and \( \dot{\lambda}(s_3) < 0 \), which implies that the vertical tangency in \( s_3 \) points out downward. The isoperimetric solution is not possible in this case because if the vertical tangencies have not occurred in the same height, then a piece of the rotational surface would be out of the region between the horospheres (see Figure 5 and 6).
Even if the vertical tangency occurred in the same height, the intersection of the rotational $H$-surface with the parallel horospheres would be two concentric circles which is not possible due to Theorem 3.2.

Then for $H = 1$ the boundary of the region $\Omega$ may be a catenoid’s cousin type surface (see Figure 2) or umbilical with $H = 1$ (see Figure 4).

We proceed in the analogous way to study the other cases. We give here only the main equations and results for them.

**Second Case:** $0 \leq H < 1$.

**Theorem 4.2.** If $c_+(s) = e^{\lambda(s)}(\tanh \rho(s), \operatorname{sech} \rho(s))$, with $\rho(s), \lambda(s)$ given by (4.4), is the parametrization of the profile curve of a rotational $H$-surface in $\mathbb{R}^3_+$ with $0 \leq H < 1$ then

1. $c_+(s)$ is symmetric with respect to the geodesic radius $c_g$;
2. the assintotic boundary of the profile curves consists of one or two points.

**Proof:** The proof of (a) is similar Theorem 4.1. In [5] it is shown that $\rho(s)$ is illimited but $\lambda(s)$ is limited and has finite limit. Then

$$\lim_{|s| \to \infty} e^{\lambda(s)} \operatorname{sech} \rho(s) = 0,$$

and the assintotic boundary of the profile curves consists of one or two points.

As $\sinh \rho(s) \geq 0$, we have from (4.4) that

$$\sinh \rho(s) = 0 \iff a = 0 \text{ e } s = 0.$$ 

If $0 \leq H < 1$, from (4.4) it follows that

$$U^2(s) = \frac{-A + B \cosh(2\alpha s)}{2\alpha^2}, \quad \dot{U}^2(s) = \frac{B^2 \sinh^2(2\alpha s)}{2(-A + B \cosh(2\alpha s))}.$$ 

Replacing (4.18) in (4.19) we get

$$B^2 H^2 \cosh^2(2\alpha s) - 2AB \cosh(2\alpha s) + A^2 + \alpha^2 B^2 = 0.$$ 

**Figure 5.** Profile curve for $H = 1$ and $a = 1$.

**Figure 6.** Rotational surface with $H = 1$ and $a = 1$ (excluded).
Making \( t = \cosh(2\alpha s) \) in (4.19) we get a second degree equation whose discriminant is
\[
\Delta = 4B^2(1 - H^2)^2(1 + 4aH).
\]
As \( B > 0 \) in case \( 0 < H < 1 \) and \( a \) is defined for any real, we have that
\begin{itemize}
  \item if \( a < -\frac{1}{4H} \), there are no vertical tangency points;
  \item if \( a = -\frac{1}{4H} \), there are at most two vertical tangency points in
\end{itemize}
(4.20)
\[ s = \pm \frac{1}{2\alpha} \text{arccosh} \left( \frac{1}{H} \right); \]
\begin{itemize}
  \item if \( a > -\frac{1}{4H} \), there are at most four vertical tangency points
\end{itemize}
\[
s_1 = \frac{1}{2\alpha} \text{arccosh} \left( \frac{A + (1 - H^2)\sqrt{1 + 4aH}}{BH^2} \right),
\]
(4.21)
\[
s_3 = \frac{1}{2\alpha} \text{arccosh} \left( \frac{A - (1 - H^2)\sqrt{1 + 4aH}}{BH^2} \right),
\]
\[
s_2 = -s_1, \quad s_4 = -s_3.
\]
In particular if \( H = 0 \) equation (4.19) is written as
\[
2B \cosh(2s) - 1 - B^2 = 0,
\]
whose solutions are
\[
s = \pm \frac{1}{2} \text{arccosh} \left( \frac{B^2 + 1}{2B} \right).
\]
First, let us suppose \( 0 < H < 1 \).

(1) If \( -\frac{1}{4H} \leq a < 0 \), we see that only the roots \( s_2, s_4 < 0 \) give us the vertical tangency points pointing out upward. In Figure 7 we see the profile curve for \( H = 0.5 \) and \( a = -0.25 \) and the horocycles that pass through the vertical tangencies. We observe that the mean curvature vector for the part of the rotational surface in the interior of the totally geodesic (symmetry plane of the surface) points out to the rotation axis so determining the isoperimetric region illustrated in Figure 8.

**Figure 7.** Profile curve for \( H = 0.5 \) and \( a = -0.25 \).

**Figure 8.** Rotational surface with \( H = 0.5 \) and \( a = -0.25 \).
In particular if \( a = -\frac{1}{4H} \) there is only one vertical tangency point 
\[ s = -\frac{1}{2\alpha} \arccosh\left(\frac{1}{H}\right) < 0. \]
Although the profile curve intersects the horocycle in another point, it is not a vertical tangency point.

(2) If \( a = 0 \) we have two pieces of equidistant curves tangent at \((0,1)\) that generate the umbilical surfaces with \( 0 < H < 1 \). They are represented by pieces of Euclidean spheres tangent at \((0,0,1)\). We observe that the vertical tangency occurs only for the equidistant profile curve nearest the rotation axis. As the mean curvature vector of this umbilical surface points out to the rotation axis it determines a isoperimetric region.

(3) If \( a > 0 \) only the roots \( s_2 \) and \( s_3 \) correspond to vertical tangencies pointing out upward in \( s_2 \) and downward in \( s_3 \), which implies that this possibility is not suitable for the isoperimetric problem.

If \( H = 0 \) then for \( a < 0 \) or \( a > 0 \) we get only one vertical tangency point which it is not suitable for the isoperimetric problem (due to the behaviour of the surface). If \( a = 0 \), the rotational surface is a totally geodesic plane which is not suitable for the problem.

Finally we conclude that for \( 0 \leq H < 1 \) the boundary of the region \( \Omega \) may be an equidistant type surface (see Figure 8) or an umbilical surface with \( 0 < H < 1 \).

**Third Case: \( H > 1 \).**

**Theorem 4.3.** If \( c_+(s) = e^{\lambda(s)}(\tanh \rho(s), \sech \rho(s)) \) with \( \rho(s), \lambda(s) \) given by (4.5) is the parametrization of the profile curve of a rotational \( H \)-surface in \( \mathbb{R}^3 \), with \( H > 1 \), then \( c_+(s) \) is a periodic curve with period \( \pi/\alpha \).

**Proof:** We show that the hyperbolic length of the segment between the points \( c_+(s) \) and \( c_+(s + \pi/\alpha) \) keeps constant for any \( s \). In [5] it was shown that
\[
\rho(s + \pi/\alpha) = \rho(s), \quad \lambda(s + \pi/\alpha) = \lambda(s) + \lambda(\pi/\alpha),
\]
which implies from (4.6) that
\[
c_+(s + \pi/\alpha) = e^{\lambda(\pi/\alpha)}c_+(s).
\]
We fix \( s_0 \) and parametrize the segment between the points \( c_+(s_0) \) and \( c_+(s_0 + \pi/\alpha) \) by
\[
\beta(t) = (t, \frac{1}{\sinh \rho(s_0)} t), \quad \text{com} \ e^{\lambda(s_0)} \tanh \rho(s_0) \leq t \leq e^{\lambda(s_0)} e^{\lambda(\pi/\alpha)} \tanh \rho(s_0).
\]
So its hyperbolic length is
\[
L(\beta(t)) = \lambda(\frac{\pi}{\alpha}) \cosh \rho(s_0).
\]
The length of the segment depends only on the function \( \rho(s) \) with period \( \pi/\alpha \), given in (4.22). So \( L(\beta(t)) \) is constant for any \( s_0 \).

If \( H > 1 \) we have from (4.23) that

\[
U^2(s) = \frac{A + B \sin(2\alpha s)}{2\alpha^2}, \quad \dot{U}^2(s) = \frac{B^2 \cos^2(2\alpha s)}{2(A + B \sin(2\alpha s))}.
\]

Replacing (4.23) in (4.10) we get

\[
B^2 H^2 \sin^2(2\alpha s) + 2AB \sin(2\alpha s) + A^2 - \alpha^2 B^2 = 0.
\]

Making \( t = \sin(2\alpha s) \) in (4.24), the discriminant of equation (4.24) is

\[
\Delta = 4B^2(H^2 - 1)^2(1 + 4aH).
\]

From (4.25) we have the following analysis.

As \( H > 1 \) and \( B > 0 \),

- if \( a < -\frac{1}{4H} \) there are no vertical tangency points;
- if \( a = -\frac{1}{4H} \) the possible vertical tangency points are

\[
s_k = s_0 + \frac{k\pi}{\alpha}, \quad k \in \mathbb{Z},
\]

(4.26)

\[
\tilde{s}_k = \tilde{s}_0 + \frac{k\pi}{\alpha}, \quad k \in \mathbb{Z},
\]

where

\[
s_0 = \frac{1}{2\alpha} \arcsin\left(\frac{1}{H}\right), \quad -\frac{\pi}{2} < 2\alpha s_0 < \frac{\pi}{2},
\]

\[
\tilde{s}_0 = \frac{1}{2\alpha} \arcsin\left(\frac{1}{H}\right), \quad \frac{\pi}{2} < 2\alpha \tilde{s}_0 < \frac{3\pi}{2}.
\]

- if \( a > -\frac{1}{4H} \) the possibilities are

\[
S_k = S_0 + \frac{k\pi}{\alpha}, \quad k \in \mathbb{Z},
\]

\[
\tilde{S}_k = \tilde{S}_0 + \frac{k\pi}{\alpha}, \quad k \in \mathbb{Z},
\]

\[
s_k = s_0 + \frac{k\pi}{\alpha}, \quad k \in \mathbb{Z},
\]

\[
\tilde{s}_k = \tilde{s}_0 + \frac{k\pi}{\alpha}, \quad k \in \mathbb{Z},
\]
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where

$$S_0 = \frac{1}{2\alpha} \arcsin \left( \frac{-A + (H^2 - 1)(1 + 4aH)}{BH^2} \right), \quad \frac{-\pi}{2} < 2\alpha S_0 < \frac{\pi}{2},$$

$$\tilde{S}_0 = \frac{1}{2\alpha} \arcsin \left( \frac{-A + (H^2 - 1)(1 + 4aH)}{BH^2} \right), \quad \frac{\pi}{2} < 2\alpha \tilde{S}_0 < \frac{3\pi}{2},$$

$$s_0 = \frac{1}{2\alpha} \arcsin \left( \frac{-A - (H^2 - 1)(1 + 4aH)}{BH^2} \right), \quad \frac{-\pi}{2} < 2\alpha s_0 < \frac{\pi}{2},$$

$$\tilde{s}_0 = \frac{1}{2\alpha} \arcsin \left( \frac{-A - (H^2 - 1)(1 + 4aH)}{BH^2} \right), \quad \frac{\pi}{2} < 2\alpha \tilde{s}_0 < \frac{3\pi}{2}.$$

Now we determine when the vertical tangency really occurs, depending on the geometry of the profile curve.

1. If $-\frac{1}{4H} \leq a < 0$ only the roots $\tilde{S}_k$, $\tilde{s}_k$ furnish the vertical tangency points pointing out upward. In Figure 9 we see a hyperbolic onduloid.

![Figure 9. Hyperbolic onduloid with $H = 3$ and $a = -0.05$.](image)

2. If $a = 0$ we have tangent geodesic half circles along de rotation axis. Each geodesic half circle generates a geodesic sphere in $\mathbb{R}^3_+$ which are umbilical surfaces with $H > 1$ and isoperimetric regions.

3. If $a > 0$ we analyse the behaviour of the profile curve in the interval $\left[ -\frac{\pi}{4\alpha}, 3\frac{\pi}{4\alpha} \right]$ since according to Theorem 4.3 the profile curves are periodic with period $\pi/\alpha$. It is easy to see that only the roots $s_0$ and $\tilde{S}_0$ furnish the vertical tangency pointing out downward in $s_0$ and upward in $\tilde{S}_0$. This case corresponds to the so called hyperbolic nodoids, which are not embedded surfaces.
We conclude that for $H > 1$ the boundary of the region $\Omega$ can be an onduloid’s type surface (see Figure 9) or an umbilical surface with $H > 1$.

Next we prove Theorem 1.1. We start with the existence and then we obtain the possible minimizing regions.

**Proof of Theorem 1.1:** By Theorem 3.2, the solutions to the isoperimetric problem have as boundaries rotationally invariant surfaces which have constant mean curvature where they are regular. But they must be regular (actually analytic), since the singularities along such boundaries must have, by well-known results, (Hausdorff) codimension at least 7, which is not possible for (2-dimensional) surfaces. Now, from [10] the existence of the isoperimetric solutions follows from the fact that $F_{c_1,c_2}/G$ is compact, where $G$ is the group of isometries of $\mathbb{R}^3_+$ whose elements let invariant the region $F_{c_1,c_2}$ between the horospheres, that is, the rotations around a vertical geodesic and the horizontal translations. The second part of Theorem 1.1 follows from the analysis of vertical tangencies done in the First, Second and Third Cases above.

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