On canonical transformations between equivalent Hamiltonian formulations of General Relativity

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Abstract

Two Hamiltonian formulations of General Relativity, due to Pirani et al. and Dirac, are considered. Both formulations, despite having different expressions for constraints, allow to derive four-dimensional diffeomorphism invariance. The relation between these two formulations at all stages of the Dirac approach to the constrained Hamiltonian systems is analyzed. It is shown that the complete sets of their phase-space variables are related by a transformation which satisfies the ordinary condition of canonicity known for unconstrained Hamiltonians and, in addition, converts one total Hamiltonian into another, thus preserving form-invariance of generalized Hamiltonian equations for constrained systems.

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I. INTRODUCTION

The Hamiltonian formulation of General Relativity (GR) is an old subject which is still plagued by some long-standing questions. One of the most important problems, related to essence of Einstein’s General Relativity, is the disappearance of four-dimensional diffeomorphism

\[ \delta g_{\mu\nu} = -\xi_{\mu;\nu} - \xi_{\nu;\mu} \]  

in the Hamiltonian formulation of GR “that has worried many people working in geometrodynamics for so long” [1]. According to some authors, in the Hamiltonian formulation of GR, it is possible to restore only spatial diffeomorphism [2, 3] or, according to others, the so-called “special diffeomorphism”, for which a non-covariant and field-dependent redefinition of gauge parameters is needed, can be derived [4, 5, 6].

In fact, the two Hamiltonian formulations that preserve four-dimensional diffeomorphism have been known for a long time. They are the first Hamiltonian formulations of GR due to Pirani, Schild, and Skinner (PSS) [7] and Dirac [8], both of which allow one to derive the full diffeomorphism from their constraint structure [9, 10]. These two formulations both lead to the expected gauge invariance (1). At the same time, they provide an example that allows us to discuss the conditions under which different Hamiltonian formulations of GR are equivalent. The study of the conditions for which a change of phase-space variables preserves the properties of an original Hamiltonian system is of great importance for constrained dynamical systems. It is especially important in the Hamiltonian formulations of General Relativity where it is customary to perform changes of variables or to introduce new variables. The legitimacy of these changes must be verified.

To the best of our knowledge, the equivalence of Hamiltonian formulations of GR which differ from each other by a change of phase-space variables was never been analyzed. What is only known to us is a brief statement of DeWitt which he made for PSS formulation: “four so-called primary constraints could, by a phase transformations, be changed into pure momenta” (see [2] where the author refers to his unpublished report). The connection

\[ \text{1 The word “diffeomorphism” is often used as equivalent to the transformation (1) (the semicolon “;” means a covariant derivative); in this Letter the same meaning is employed.} \]
between the linearized versions of the two formulations of [7] and [8] was analyzed in [11] where it was demonstrated that the two formulations of linearized GR are connected by a change of phase-space variables which is, in fact, a canonical transformation in the sense of ordinary Classical Mechanics. Moreover, these formulations, despite having different forms of Hamiltonians and constraints, give an equivalent description, i.e. the corresponding generators built from the first-class constraints allow one to derive the same gauge invariance (the linearized version of diffeomorphism invariance).

The main goal of this Letter is to extend this analysis to the two Hamiltonian formulations of full GR [7, 8, 9, 10]. We investigate the relation between the corresponding phase-space variables in both formulations and discuss the effects of such a change of variables at all stages of the Dirac procedure. Another aim is to formulate some general conditions that should be imposed on transformations of phase-space variables for singular systems to preserve the equivalence of different Hamiltonian formulations.

II. COMPARISON OF THE TWO HAMILTONIAN FORMULATIONS OF GR

A starting point of the Hamiltonian formulations of GR in both the approaches of [7] and [8] is the “gamma-gamma” part of the Einstein-Hilbert (EH) Lagrangian which is quadratic in first-order derivatives of the metric tensor (for more details see, e.g., [12])

\[
L = \sqrt{-g}g^{\alpha\beta}(\Gamma^\mu_{\alpha\nu}\Gamma^\nu_{\beta\mu} - \Gamma^\nu_{\alpha\beta}\Gamma^\mu_{\nu\mu}) = \frac{1}{4}\sqrt{-g}B^{\alpha\beta\gamma\mu\nu\rho}g_{\alpha\beta,\gamma}g_{\mu\nu,\rho} \quad (2)
\]

where

\[
B^{\alpha\beta\gamma\mu\nu\rho} = g^{\alpha\beta}g^{\gamma\rho}g_{\mu\nu} - g^{\alpha\mu}g^{\beta\nu}g_{\gamma\rho} + 2g^{\alpha\nu}g^{\beta\gamma}g_{\mu\rho} - 2g^{\alpha\beta}g^{\gamma\mu}g_{\nu\rho}.
\]  

To find the momenta \(\pi^{\alpha\beta}\), conjugate to the ten components of the metric tensor \(g_{\alpha\beta}\), we rewrite Eq.\(\text{(2)}\) in a form which explicitly contains the time derivatives of the metric tensor, i.e. in terms of “velocities”

\[
L = \frac{1}{4}\sqrt{-g}B^{\alpha\beta\gamma\mu\nu\rho}g_{\alpha\beta,0}g_{\mu\nu,0} + \frac{1}{2}\sqrt{-g}B^{(\alpha\beta)\mu\nu\rho}g_{\alpha\beta,0}g_{\mu\nu,k} + \frac{1}{4}\sqrt{-g}B^{\alpha\beta\gamma\mu\nu\rho\gamma}g_{\alpha\beta,k}g_{\mu\nu,t}, \quad (4)
\]

where the Latin alphabet is used for spatial components and 0 for a temporal one. The brackets \((\alpha,\beta)\) indicate symmetrization in two indices, while the notation \((... | ...)\) is used for symmetrization in two groups of indices, i.e.

\[
B^{(\alpha\beta\gamma)\mu\nu\rho} = \frac{1}{2}(B^{\alpha\beta\gamma\mu\nu\rho} + B^{\mu\nu\rho\alpha\beta\gamma}).
\]
Moments conjugate to the metric tensor are defined in standard way, and (4) gives
\[ \pi^{\gamma\sigma} = \frac{\delta L}{\delta g_{\gamma\sigma,0}} = \frac{1}{2} \sqrt{-g} B((\gamma\sigma)0) g_{\mu\nu,0} + \frac{1}{2} \sqrt{-g} B((\gamma\sigma)0)\mu\nu k) g_{\mu\nu,k}. \] (5)

By using (3) one finds the explicit form of the first term of (5)
\[ B((\gamma\sigma)0) g_{\mu\nu,0} = g^{00} E^{\mu\nu\gamma\sigma} g_{\mu\nu,0} \] (6)

where
\[ E^{\mu\nu\gamma\sigma} = e^{\mu\nu} e^{\gamma\sigma} - e^{\mu\gamma} e^{\nu\sigma}, \quad e^{\mu\nu} = g^{\mu\nu} - g^{00} g_{\mu0} g_{\nu0}. \] (7)

Note that both \( e^{\mu\nu} \) and \( E^{\mu\nu\gamma\sigma} \) are zero unless all of the \( \mu, \nu, \gamma, \) and \( \sigma \) indices differ from 0.

The notation \( e^{km} \) designates the inverse of the spatial components of the metric tensor, i.e. \( g_{nk} e^{km} = \delta^{m}_{n} \), and \( \frac{\delta}{\delta g_{00}} e^{\mu\nu} = \frac{\delta}{\delta g_{00}} E^{\mu\nu\gamma\sigma} = 0 \). From (6-7) it follows that we cannot express some of the velocities in (5) in terms of momenta, therefore, \( d \) primary constraints arise (here \( d \) is the dimension of space-time); they are
\[ \phi^{0\sigma} = \pi^{0\sigma} - \frac{1}{2} \sqrt{-g} B^{((0\sigma)0)\mu\nu\kappa} g_{\mu\nu,\kappa} \approx 0. \] (8)

If \( \gamma \) and \( \delta \) are space-like, then (5) is invertible and we find
\[ g_{mn,0} = I^{mpq}_{mn} \frac{1}{g^{00}} \left( \frac{2}{\sqrt{-g}} \pi^{pq} - B^{((pq)0)\mu\nu\kappa) g_{\mu\nu,\kappa} } \right) \] (9)

where
\[ I^{mpq}_{mn} = \frac{1}{d-2} g_{mn} g_{pq} - g_{mp} g_{nq}, \quad I^{mpq}_{mn} E^{pqkl} = \delta^{k}_{m} \delta^{l}_{n}. \] (10)

The appearance of a singularity in (10) for \( d = 2 \) corresponds to the fact that in two dimensions none of the components of (5) can be solved for velocities. The number of primary constraints (three) in this case equals the number of independent components of the metric tensor in two dimensions.

The Hamiltonian is defined by \( H = \pi^{\alpha\beta} g_{\alpha\beta,0} - L \). After using (9) to eliminate all the velocities \( g_{mn,0} \), one finds the following total Hamiltonian,
\[ H_T = H_c + g_{00,0} \phi^{00} + 2 g_{0k,0} \phi^{0k}, \] (11)
where the ‘canonical part’\(^2\) of the total Hamiltonian is
\[
H_c = \frac{1}{\sqrt{-g}g^{00}}I_{mnpq}\pi^{mn}\pi^{pq} - \frac{1}{g^{00}}I_{mnpq}\pi^{mn}B(pq0|\mu\nu)g_{\mu\nu,k}
\]
\[
+ \frac{1}{4}\sqrt{-g}\left[\frac{1}{g^{00}}I_{mnpq}B(mn0|\mu\nu)B(pq0|\alpha\beta) - B^{\mu\nu\alpha\beta}g_{\mu\nu,k}g_{\alpha\beta,t}\right].
\]
(12)

For the detailed analysis of (11), including the constraint structure and derivation of the corresponding generators and gauge transformations, see [9].

In this Letter we want to compare the Hamiltonian formulation of GR given by (11) with that of Dirac [8, 10]. Dirac’s main idea was based on the fact that the Lagrangian, (2) (it is called below \(L_{PSS}\)) can be modified in order to simplify the primary constraints by adding a non-covariant combination of spatial and temporal derivatives that does not affect the equations of motion. This modification leads to the following Lagrangian
\[
L_D = L_{PSS} - L^*
\]
(13)

where \(L^*\) is taken by Dirac to be
\[
L^* = \left[\frac{\sqrt{-g}g^{00}}{0} \frac{g_{0k}}{0} \right] - \left[\frac{\sqrt{-g}g^{00}}{0} \frac{g_{0k}}{0} \right].
\]
(14)

The explicit form of (14) can be found using the identity \(F_{,\gamma} = \frac{\delta F}{\delta g_{\mu\nu}}g_{\mu\nu,\gamma}\) for the metric-dependent functional and rewriting the contravariant components of the metric tensor in terms of \(e^{\alpha\beta}\) (see (7)). Finally, we find
\[
L^* = \frac{1}{2}\sqrt{-g}A^{\alpha\beta0\mu0}g_{\alpha\beta,0}g_{\mu\nu,k},
\]
(15)

where we have introduced the following notation
\[
A^{\alpha\beta0\mu0k} = e^{\alpha\beta}e^{k\mu}g^{00} - e^{\mu\nu}e^{k\alpha}g^{0\beta} + e^{k\alpha}g^{0\mu}g^{00}g_{\nu0} - e^{k\mu}g^{0\alpha}g^{0\nu}g^{00}.
\]
(16)

This relation is obtained by taking into account symmetries \(\alpha\beta \Leftrightarrow \beta\alpha\) and \(\mu\nu \Leftrightarrow \nu\mu\) in (15) due to the presence of \(g_{\alpha\beta,0}g_{\mu\nu,k}\). The important property of \(A^{\alpha\beta0\mu0k}\) is its antisymmetry with respect to interchange of the two pairs of indices
\[
A^{\alpha\beta0\mu0k} = -A^{\mu0\alpha\beta k}.
\]
(17)

\(^2\) We shall use this standard terminology, or alternatively ‘canonical Hamiltonian’, both of which however, can be misleading because for the canonical treatment of singular systems the total Hamiltonian, \(H_T\), not its individual parts, is needed to provide the complete description.
Using the explicit form of (3) we can rewrite the coefficients \( B^{(\alpha \beta \lambda \mu \nu \kappa)} \) in terms of the \( A^{\alpha \beta \lambda \mu \nu \kappa} \) and \( E^{\alpha \beta \mu \nu} \)

\[
B^{(\alpha \beta \lambda \mu \nu \kappa)} = A^{\alpha \beta \lambda \mu \nu \kappa} + g^{0k} E^{\alpha \beta \mu \nu} - 2g^{0\mu} E^{\alpha \beta k \nu}.
\] (18)

Now, the relation between Dirac’s Lagrangian \( L_D \) and the Lagrangian of PSS, \( L_{PSS} \), takes the form

\[
L_D = L_{PSS} - \frac{1}{2} \sqrt{-g} A^{\alpha \beta \lambda \mu \nu \kappa} g^{\alpha \beta, \mu \nu, k}.
\] (19)

Note that at the Lagrangian level the difference between the PSS and Dirac formulations does not affect the equations of motion and, in this sense, the two formulations are equivalent.

Now, we analyze this difference from the point of view of the Hamiltonian formulation. If we have the two Lagrangians, then we can introduce the two corresponding Hamiltonians which, as we know, give the same gauge invariance [9, 10]. Let us find the relation between their phase-space variables and constraints. This will provide a clue about the changes which can be performed at the Hamiltonian level in a constrained system that will preserve its properties.

The two Lagrangians in (19) differ from each other by the terms linear in time derivatives of the metric tensor; this will affect the expression for conjugate momenta in these two Hamiltonian formulations. For PSS we have

\[
\pi^{\gamma \sigma} = \frac{\delta L_{PSS}}{\delta g_{\gamma \sigma, 0}},
\] (20)

while for the Dirac formulation the momentum is

\[
p^{\gamma \sigma} = \frac{\delta L_D}{\delta g_{\gamma \sigma, 0}} = \frac{\delta L_{PSS}}{\delta g_{\gamma \sigma, 0}} + \frac{\delta L^*}{\delta g_{\gamma \sigma, 0}}.
\] (21)

To obtain the relation between these two momenta, we subtract the last two equations, which gives

\[
\pi^{\gamma \sigma} - p^{\gamma \sigma} = \frac{\delta}{\delta g_{\gamma \sigma, 0}} \left( \frac{1}{2} \sqrt{-g} A^{\alpha \beta \lambda \mu \nu \kappa} g_{\alpha \beta, \mu \nu, k} \right)
\] (22)

or

\[
p^{\gamma \sigma} = \pi^{\gamma \sigma} - \frac{1}{2} \sqrt{-g} A^{(\gamma \sigma) \lambda \mu \nu \kappa} g_{\mu \nu, k}.
\] (23)

Equation (23) represents the transformation of phase-space variables for two Hamiltonian formulations of GR, [7] and [8].

Thus, we have two Hamiltonians with two sets of phase-space variables, \((g_{\alpha \beta}, \pi^{\alpha \beta})\) and \((g_{\alpha \beta}, p^{\alpha \beta})\); the momenta of these two sets are connected by the transformation of (23) and
the components of the metric tensor are identical in both formulations. The two sets of fundamental Poisson brackets (PB) are:

\[
\{g_{\alpha\beta}, \pi^{\mu\nu}\} = \frac{1}{2} (\delta^{\mu}_{\alpha} \delta^{\nu}_{\beta} + \delta^{\nu}_{\alpha} \delta^{\mu}_{\beta}) = \Delta^{\mu\nu}_{\alpha\beta}, \quad \{g_{\alpha\beta}, g_{\mu\nu}\} = \{\pi^{\alpha\beta}, \pi^{\mu\nu}\} = 0
\]  

(24)

and

\[
\{g_{\alpha\beta}, p^{\mu\nu}\} = \Delta^{\mu\nu}_{\alpha\beta}, \quad \{g_{\alpha\beta}, g_{\mu\nu}\} = \{p^{\alpha\beta}, p^{\mu\nu}\} = 0.
\]  

(25)

Note that the conjugate momenta have to be introduced for all generalized coordinates irrespective of whether or not the corresponding time derivatives (velocities) for particular fields are present in the Lagrangian. (In fact, in the first-order, metric-affine, formulations of GR [13] some fields enter the Lagrangian with no derivatives, but nevertheless momenta, conjugate to all of fields, are needed [14].)

For regular, i.e. non-singular, systems, the two sets of phase-space variables are related to each other by a canonical transformation if and only if the following relations are fulfilled

\[
\{g_{\alpha\beta}, p^{\mu\nu}\}_{g,p} = \{g_{\alpha\beta}, g^{\mu\nu}(\pi, g)\}_{g,\pi} = \Delta^{\mu\nu}_{\alpha\beta},
\]

\[
\{g_{\alpha\beta}, g_{\mu\nu}\}_{g,p} = \{g_{\alpha\beta}, g_{\mu\nu}\}_{g,\pi}, \quad \{p^{\alpha\beta}, p^{\mu\nu}\}_{g,p} = \{p^{\alpha\beta}, p^{\mu\nu}\}_{g,\pi}.
\]  

(26)

In our case, which is based on the use of phase-space variables of [7] and [8], we have to explicitly check in detail only one PB (the rest of PBs is obviously fulfilled) to find

\[
\{p^{\gamma\sigma}, p^{\delta\rho}\}_{g,p} = \left\{\pi^{\gamma\sigma} - \frac{1}{2} \sqrt{-g} A^{(\gamma\sigma)0\mu\nu} g_{\mu\nu,k}, \pi^{\delta\rho} - \frac{1}{2} \sqrt{-g} A^{(\delta\rho)0\alpha\beta} g_{\alpha\beta,m}\right\}_{g,\pi} = 0.
\]  

(27)

Note that for pairs with at least one temporal index this PB was calculated in [7, 9], where it is just the PB between primary constraints

\[
\{\phi^{\sigma0}, \phi^{\gamma0}\} = 0.
\]  

(28)

However, this result is also valid for all indices. The PB of (27) shows that Dirac’s modification of \(L_{PSS}\) at the Hamiltonian level leads to a Hamiltonian formulation in which the phase-space variables are canonically related to those of PSS.

Our next goal is to consider the effect of such a canonical change of phase-space variables on all steps of the Dirac procedure. We can utilize the PB of (27), and, by rearranging terms, present the canonical part of the PSS Hamiltonian in a different, but equivalent form
by explicitly creating (extracting) combinations that correspond to a canonical change of variables

$$\pi^{pq} = \phi^{pq} + \frac{1}{2}\sqrt{-g}A^{(pq)0\mu\nu}g_{\mu\nu,k}$$  \hspace{1cm} (29)$$

where

$$\phi^{pq} = \pi^{pq} - \frac{1}{2}\sqrt{-g}A^{(pq)0\mu\nu}g_{\mu\nu,k}.$$  \hspace{1cm} (30)$$

This simple rearrangement is very convenient to study canonical transformations and allows us to present two Hamiltonians of [7] and [8] as one expression that will make transparent the effect of such changes on all steps of the Dirac procedure. By substituting Eq. (30) into the canonical part of the PSS Hamiltonian, (12), and using (16) we obtain the total Hamiltonian, (11), where $H_c$ written in terms of $\phi^{pq}$ takes the form

$$H_c = \frac{1}{\sqrt{-g}}g^{00}I_{mnpq}\phi^{mn}\phi^{pq} - \frac{1}{g^{00}}\phi^{mn}(g^{0t}g_{mn,t} - 2g^{0\alpha}g_{\alpha,m})$$

$$+ \frac{1}{4}\sqrt{-g} \left[ \frac{1}{g^{00}}(g^{0k}E^{(mn)\mu\nu} - 2g^{0\mu}E^{(mn)k\nu}) \left( g^{0t}\delta_m^\alpha \delta_n^\beta - 2g^{0\alpha}\delta_m^t \delta_n^\beta \right) - B^{\mu\nu\delta\kappa\lambda t} \right] g_{\mu\nu,k}g_{\delta\kappa\lambda,t}. \hspace{1cm} (31)$$

Note that (11) and (31) simultaneously represent the total Hamiltonians for both formulations, [7] and [8]. In the Dirac case $\phi^{\alpha\beta} = p^{\alpha\beta}$; while for PSS, $\phi^{\alpha\beta}$ is given by (30). Both equations (11) and (31) manifestly demonstrate the effect of canonical transformations for the total Hamiltonians:

$$H_T^{\text{PSS}}(g,\pi)|_{g_{\mu\nu} = g_{\mu\nu}, p^{\gamma\sigma} = p^{\gamma\sigma} - \frac{1}{4}\sqrt{-g}A^{(\gamma\sigma)0\mu\nu}g_{\mu\nu,k}} = H_T^D(g,p); \hspace{1cm} (32)$$

and for the generalized Hamiltonian equations:

$$g_{\alpha\beta,0} = \{g_{\alpha\beta}, H_T^{\text{PSS}}\}, \hspace{1cm} \pi^{\alpha\beta,0} = \{\pi^{\alpha\beta}, H_T^{\text{PSS}}\} \implies g_{\alpha\beta,0} = \{g_{\alpha\beta}, H_T^D\}, \hspace{0.5cm} p^{\alpha\beta,0} = \{p^{\alpha\beta}, H_T^D\}. \hspace{1cm} (33)$$

In [9] the PSS formulation was analyzed by considering the combinations of different orders in $\pi^{\alpha\beta}$. Here we will work in orders of $\phi^{\alpha\beta}$ which, due to the simple relation $\{\phi^{\alpha\beta}(g,\pi), \phi^{\mu\nu}(g,\pi)\}_{(g,\pi)} = 0$, makes the amount of calculations absolutely the same for the PSS and Dirac formulations. It also makes transparent the effect of the considered canonical transformation.

Now we calculate the time development of the primary constraints,

$$\phi^{0\sigma,0} = \{\phi^{0\sigma}, H_c\}. \hspace{1cm} $$
By working with combinations $\phi^{\alpha\beta}$, we can use associative properties of PB, and therefore $\{\phi^{0\sigma}, H_c\} = -\frac{\delta H_c}{\delta g^{0\sigma}}$, where the variation is not performed inside the expression for $\phi^{\alpha\beta}$, since $\{\phi^{\alpha\beta}, \phi^{\mu\nu}\} = 0$. The variation $-\frac{\delta H_c}{\delta g^{0\sigma}}$ leads to the following secondary constraint

$$\chi^{0\sigma} = -\frac{1}{2} \left\{ \chi^{0\sigma}, g_{00,0} \phi^{00} + 2g_{0k,0} \phi^{0k} \right\}.$$ (34)

This expression coincides with the secondary constraint in Dirac’s formulation [10], where $\phi^{mn} = p^{mn}$. In order to show the equivalence of (34) to the secondary constraint of PSS [9], one has to rewrite this result in terms of $\pi^{km}$ using (30).

Let us continue the Dirac procedure and consider the time development of the secondary constraint

$$\chi^{0\sigma}_{,0} = \left\{ \chi^{0\sigma}, H_c \right\} + \left\{ \chi^{0\sigma}, g_{00,0} \phi^{00} + 2g_{0k,0} \phi^{0k} \right\}.$$ (35)

We have found it more convenient to perform the calculations in different orders of momenta $\phi^{\mu\nu}$, which are indicated by the numbers in brackets. We start from the PB of $\chi^{0\sigma}$ with the primary constraint for which the highest order contribution gives

$$\left\{ \chi^{0\sigma} (2), \phi^{0\gamma} \right\} = -\frac{1}{2} g^{0\sigma} \left( \chi^{0\sigma} \right) I_{mnpq} \phi^{mn} \phi^{pq} = -\frac{1}{2} g^{\gamma\sigma} \chi^{00} (2).$$ (36)

Using this higher order result, (36), as a guide, we have to verify by calculation that to all orders of $\phi^{km}$ this structure is preserved. The explicit calculation confirms that the following PB is valid to all orders of $\phi^{km}$

$$\left\{ \chi^{0\sigma}, \phi^{0\gamma} \right\} = -\frac{1}{2} g^{\gamma\sigma} \chi^{00}.$$ (37)

We obtained this relation for both formulations, [9] and [10], and it demonstrates the form-invariance of the PB among the primary and secondary constraints for canonically related formulations. Now we proceed and find the PB of the secondary constraints with the canonical part of the Hamiltonian $\{\chi^{0\sigma}, H_c\}$. As before, we start from the highest order contribution, which for this part is of third order in $\phi^{ab}$,

$$\left\{ \chi^{0\sigma}, H_c \right\} (3) = \left\{ \chi^{0\sigma} (2), H_c (2) \right\} = -\frac{2}{\sqrt{-g} g^{0\sigma}} g^{\sigma\delta} \phi^{ab} I_{abcd} g_{00} \left( -\frac{1}{2} \frac{1}{\sqrt{-g}} I_{mnpq} \phi^{mn} \phi^{pq} \right);$$ (38)

it can be presented as a term proportional to $\chi^{00} (2)$ or $\chi^{0c} (2)$, or as a linear combination of both. So, we have many possible and non-unique ways to present this result, which requires
us to investigate all combinations, to all orders. Such an approach involves a considerable amount of calculation. The wrong choice can lead to the erroneous conclusion that the time development of the secondary constraint $\chi_0^0$ is not proportional to the secondary constraints and gives rise to tertiary constraints, etc. The approach that allows one to perform unambiguous calculations (sort out the contributions uniquely in terms of secondary constraints) is presented in the Appendix and here we give only the final result:

$$\{\chi_0^0, H_c\} = -\left[ \frac{2}{\sqrt{-g}} I_{pqmk} g^{0m} g^{0n} \phi^{pq} + g^{0\sigma} g_{00,k} + 2 g^{\sigma p} g_{0p,k} + g^{0\sigma} g^{0q} \left( g_{pq,k} + g_{qk,p} - g_{pk,q} \right) \right] \chi_k^{0k}$$

$$- \delta_0^\sigma \chi_k^{0k} + \frac{1}{2} g^{\sigma k} g_{00,k} \chi^{00}, \quad (39)$$

which is easy to compare with the results obtained for the Dirac and PSS formulations in \[9\] and \[10\]. Note that again the constraints and structure functions are different for the two formulations; but the whole structure of (39) is form-invariant. In fact, (39) can be presented in the following compact form $\{\chi_0^0, H_c\} = V_\gamma^\sigma (g, \phi, \partial) \chi^{0\gamma}$ where upon a canonical transformation not only the constraints, but also the structure functionals $V_\gamma^\sigma$ of one formulation transforms into another independently. Equation (39) proves at the same time the closure of the Dirac procedure for both formulations. This equation, along with (37) and (28), is sufficient to find the gauge generators and derive the gauge transformations for both formulations. We do not want to repeat such calculations here, since they are given in detail in \[9\] and \[10\] using two different methods described in \[15\] and \[16\]. The result of such calculations is the four-dimensional diffeomorphism invariance (1) (for both formulations) that follows directly using each formalism, without any non-covariant and field-dependent redefinition of the gauge parameters; and the gauge transformation can be written in the covariant form (1) for all components of the metric tensor. For completeness we provide the expression for the canonical part of the Hamiltonian

$$H_c = -2 g_{0\sigma} \chi_0^0 + (2 g_{0m} \phi^{mk}) \right|_k - \left[ \sqrt{-g} E^{mnki} g_{mn,i} - \sqrt{-g} g^{\mu\nu,i} \frac{\partial}{\partial \mu} \left( g^{\nu k} g^{0i} - g^{\nu i} g^{0k} \right) \right] \right|_k. \quad (40)$$

In both formulations, $H_c$ is the sum of the term proportional to the secondary constraints, $-2 g_{0\sigma} \chi_0^0$, and the total spatial derivatives, despite having different expressions for $\chi_0^0$ and $\phi^{mk}$ (for details see \[9\] and \[10\]).
III. CONCLUSION

We have analyzed the relation between the two Hamiltonian formulations of GR \cite{7} and \cite{8} which allow one to derive four-dimensional diffeomorphism invariance \cite{9}, \cite{10}. It is shown that the full sets of phase-space variables for these two formulations are related to each other by a transformation of (23), which satisfies the condition of canonicity (26) known for the Hamiltonian formulations of non-singular systems. It also preserves the form-invariance of the expressions for the total Hamiltonians (32). These properties are well known for Hamiltonian formulations of systems with the regular (i.e., non-singular) Lagrangians. Despite these similarities with regular systems, the non-singular and singular cases are not completely equivalent. In the former, condition (26) is necessary and sufficient for equivalence of the two formulations, whereas in the latter case it is a necessary, but not sufficient condition, as was demonstrated in \cite{10} by an example of a canonical transformation which is in the ordinary sense (see (26)), transformations that nevertheless destroys the form-invariance of the total Hamiltonian.

For non-singular systems the canonicity condition, (26), is actually independent of the particular form of the unconstrained Hamiltonian and the canonical transformations automatically convert the Hamiltonian written in terms of one set of phase-space variables into another Hamiltonian. Whereas for singular systems the Hamiltonian (that is, the total Hamiltonian) consists of two distinct parts namely the ‘canonical Hamiltonian’ and a linear combination of primary constraints both of which play different roles. In particular, the number of primary constraints corresponds to the number of velocities that cannot be solved for terms of momenta and, for systems with first-class constraints, it defines the number of gauge parameters, an important intrinsic characteristic of a theory. Thus, in the case of singular systems, the explicit form of the total Hamiltonian becomes crucially important and only transformations that preserve this form (as in (32)) keep different formulations equivalent. This additional condition makes the canonical transformations for singular systems significantly more restrictive in comparison to non-singular systems.

We have considered two equivalent Hamiltonian formulations of GR related by a relatively simple transformation which involves only a change of momenta. Neither more complicated transformations of the phase-space variables, nor more general cases, such as one which also includes the presence of second-class constraints, have been studied. But it is possible to
make a conjecture that the equivalence of different Hamiltonian formulations of singular systems (at least restricted to systems with only first-class constraints) is preserved if the complete set of their phase-space variables are related by a canonical, in ordinary sense, transformations which, in addition, must preserve the form-invariance of the total Hamiltonian. The transformations that do not satisfy these conditions do not lead to Hamiltonian formulations equivalent to the original theory (Einstein GR, as well as any other singular system).

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V. APPENDIX

In this Appendix we describe the calculation of \( \{\chi^{0\sigma}, H_c\} \), that proves the closure of the Dirac procedure. This result is also needed to find generators and gauge transformations which are discussed in detail for both formulations in [9], [10].

As we have already pointed out, it is more convenient to perform the calculations in different orders of momenta \( \phi^{\mu\nu} \), which are indicated by the numbers in brackets. Comparison of contributions of the highest order in \( \phi^{km} \) into the constraints \( \chi^{0\sigma} \), leads to a simple relation: \( \chi^{0m}(2) - \frac{g^{0m}}{g^{00}} \chi^{00}(2) = 0 \). A generalization of this result to all orders gives

\[
\chi^{0m} - \frac{g^{0m}}{g^{00}} \chi^{00} = \psi^{0m}
\]

where

\[
\psi^{0m} = \phi^{mk}_{,k} + \left( \phi^{pk} e^{qm} - \frac{1}{2} \phi^{pq} e^{km} \right) g_{pq,k}.
\]

If the Dirac procedure is closed in terms of the constraints \( \chi^{00}, \chi^{0k} \), then it is also closed in terms of \( \chi^{00}, \psi^{0k} \), and vice versa, which follows directly from (41). However, working with the \( \chi^{00}, \psi^{0k} \) pair allows us to sort terms unambiguously because we have only the following non-zero contributions in both constraints: \( \chi^{00}(2), \chi^{00}(0), \) and \( \psi^{0k}(1) \),
so separating terms of different order in $\phi^{km}$ simplifies calculations (note that in [9] this procedure could not be simplified to such an extent).

Let us start with $\{\chi^{00}, H_c\}$. In the highest order, this PB, (38), unambiguously gives (there are no derivatives of $\phi^{km}$ in this expression, so it cannot be proportional to $\psi^{0k}$):

$$
\{\chi^{00}, H_c\}(3) = \{\chi^{00}(2), H_c(2)\} = -\frac{2}{\sqrt{-g}} \frac{g^{0d}g^{0c}}{g^{00}g^{00}} \phi^{ab} I_{abcd} \chi^{00}(2).
$$

(43)

In the next order, we have contributions with and without derivatives of the momenta $\phi^{km}$:

$$
\{\chi^{00}, H_c\}(2) = \{\chi^{00}(2), H_c(1)\}(\phi \partial \phi) + \{\chi^{00}(2), H_c(1)\}(\phi),
$$

(44)

which we consider separately starting from the terms proportional to $\phi \partial \phi$. Such terms might be presented as proportional to the corresponding orders of the $(\chi^{00}, \psi^{0k})$ pair through either derivatives of $\chi^{00}(2)$ or $\phi^{\nu k}$, both of which have a particular structure in the indices. Again this allows us to sort such terms uniquely:

$$
\{\chi^{00}(2), H_c(1)\}(\phi \partial \phi) = -\frac{g^{0k}}{g^{00}} \chi^{00,k}(2)(\phi \partial \phi) - \frac{2}{\sqrt{-g}} I_{mnqy} \phi^{mn} g^{0p} \psi^{0q}(\partial \phi)
$$

(45)

where

$$
\chi^{00,k}(2) = \chi^{00,k}(2)(\phi \partial \phi) + \chi^{00,k}(2)(\phi) = -\frac{1}{\sqrt{-g}} I_{mnqy} \phi^{mn} \phi^{pq} \left( \frac{1}{\sqrt{-g}} I_{mnqy} \right)_k,
$$

$$
\psi^{0m} = \psi^{0m}(\partial \phi) + \psi^{0m}(\phi) = \phi^{\nu k} e^{\nu m} \left( \phi^{\nu k} e^{\nu m} - \frac{1}{2} \phi^{\nu k} e^{km} \right) g_{pq,k}.
$$

(46)

(47)

By performing the completion of (45) to full expressions $\chi^{00,k}(2)$ and $\psi^{0q}(1)$, and combining them with the second term of (44), we obtain

$$
\{\chi^{00}(2), H_c(1)\}(\phi) + \frac{g^{0k}}{g^{00}} \chi^{00,k}(2)(\phi) + \frac{2}{\sqrt{-g}} I_{mnqy} \phi^{mn} g^{0p} \psi^{0q}(\phi)
$$

that, in case of closure, can be proportional only to $\chi^{00}(2)$. For the second order, we finally have

$$
\{\chi^{00}, H_c\}(2) = -\frac{g^{0k}}{g^{00}} \chi^{00,k}(2) - \frac{2}{\sqrt{-g}} I_{mnqy} \phi^{mn} g^{0p} \psi^{0q} + \left[ \frac{g^{0k}g^{0\alpha}g^{0\beta}}{(g^{00})^2} g_{\alpha\beta,k} + \frac{1}{2} g^{0k} g^{00,k} - \left( \frac{g^{0k}}{g^{00}} \right)_k \right] \chi^{00}(2)
$$

(48)

Next, the first order

$$
\{\chi^{00}, H_c\}(1) = \{\chi^{00}(2), H_c(0)\} + \{\chi^{00}(0), H_c(2)\} - (Eq. 43) \left( \chi^{00}(2) \rightarrow \chi^{00}(0) \right)
$$

(49)
can be proportional to only $\psi^{0k}$, where the last term comes from the completion of (47) to
the full (all orders) constraint $\chi^{00}$. Here and in what follows the equations (Eq.(#)) inside
the formulae are used to indicate that the right-hand side of (Eq.(#)) must be substituted
with the change indicated by “→”.

In the last order, by using a similar compensation from the second order, we have

$$\{\chi^{00}, H_c\} (0) = \{\chi^{00} (0), H_c (1)\} - (Eq.(48)) (\chi^{00} (2) \to \chi^{00} (0), \psi^{0k} \to 0) \quad (50)$$

which, in the case of closure should give zero. This is confirmed by explicit calculation.

By calculating (49) and combining it with the results of (43) and (48), we obtain:

$$\{\chi^{00}, H_c\} = -\frac{2}{\sqrt{-g} I_{mnpq} \phi^{0m} g_{00}^{0m} g_{00}^{0n} \chi^{00}} - \frac{g^{0k}}{g_{00}^{00} \chi^{00} k} - \frac{2}{\sqrt{-g} I_{mnpq} \phi^{0m} g_{00}^{0p} \psi^{0q}}$$

$$+ \left[ -\frac{g^{0k} g_{00}^{03} g_{00}^{03}}{(g_{00}^{00})^2} g_{\alpha \beta, k} + \frac{1}{2} g^{0k} g_{00, k} - \left(\frac{g^{0k}}{g_{00}^{00}}\right)_k \right] \chi^{00} - \psi^{0k}_{,k} - \frac{g^{0k} g_{00}^{03}}{g_{00}^{00}} g_{\alpha \beta, t} \psi^{0t}, \quad (51)$$

and in terms of $\chi^{00}$ and $\chi^{0k}$ equation (51) gives:

$$\{\chi^{00}, H_c\} = -\left(\frac{2}{\sqrt{-g} I_{mnpq} \phi^{0m} g_{00}^{0p} g_{00}^{0q}} + \frac{g^{0k} g_{00}^{03} g_{00}^{03}}{g_{00}^{00}} g_{\alpha \beta, k}\right) \chi^{0k} - \chi^{0k} + \frac{1}{2} g^{0k} g_{00, k} \chi^{00}. \quad (52)$$

In a similar way, order by order, we consider the PB of $\psi^{0k}$ with $H_c$, which leads to

$$\{\psi^{0m}, H_c\} = -\frac{2}{\sqrt{-g} I_{pqkn} \phi^{pq} g_{0m}^{0n} \psi^{0k}} - \frac{g^{0m} g_{00}^{0k} g_{00}^{00} \chi^{00}}{g_{00}^{00} \chi^{00} k} + \frac{g^{0m} g_{00}^{0k} \psi^{0k}}{g_{00}^{00} \psi^{0k} k} + 2 \left(\frac{g^{0k}}{g_{00}^{00}}\right)_k \psi^{0k}$$

$$- \frac{g^{0p} g_{00}^{0m}}{g_{00}^{00}} (g_{pk, q} - g_{pq, k} - g_{qk, p}) \psi^{0k} +$$

$$\left[ \frac{g^{0m} g_{00}^{0q} g_{0k, q} + \frac{1}{2} (g_{0q}^{0m} g_{00}^{0k} - g_{0k}^{0m} g_{00}^{0q} - g_{0m}^{0q} g_{0k, p}) g_{0q, k} + \frac{1}{2} e_{km}^{0m} g_{00, k} \right] \chi^{00}. \quad (53)$$

Now, using Eq.(41) and Eqs.(51) - (53) we find $\{\chi^{0m}, H_c\}$ in terms of $\chi^{00}$ and $\chi^{0k}$

$$\{\chi^{0m}, H_c\} = \left\{ \frac{g^{0m} g_{00}^{00} \chi^{00} + \psi^{0m}}{g_{00}^{00}}, H_c \right\} = \frac{g^{0m} g_{00}^{0m} \chi^{00}}{g_{00}^{00}} \left\{ \chi^{00}, H_c \right\} + \chi^{00} \left\{ \frac{g^{0m} g_{00}^{00}}{g_{00}^{00}}, H_c \right\} + \left\{ \psi^{0m}, H_c \right\}$$

$$= -\left[ \frac{2}{\sqrt{-g} I_{pqkn} \phi^{pq} g_{mn}^{0m} + g_{00}^{0m} g_{00, k} + g_{00, p}^{0m} g_{00}^{0p} g_{0m}^{0p} (g_{pk, q} + g_{pq, k} - g_{qk, p}) \right] \chi^{0k} + \frac{1}{2} g^{0m} g_{00, m} \chi^{00}. \quad (54)$$
Finally, Eqs. (52) and (54) which are written in terms of \((\chi^{00}, \chi^{0k})\) can be combined into one covariant expression given in the main text.

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