The Fate of the Universe Evolution in the Quadratic Form of Ricci–Gauss–Bonnet Cosmology

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Abstract—We investigate the possibility of a future singularity due to accelerating expansion of the Universe in a gravitational theory that comprises the Ricci scalar $R$ and the Gauss–Bonnet invariant $G$, known as $F(R, G)$ gravity, which can be viewed in the quadratic form. Three models are presented using Hubble parameters to represent a finite and infinite future. The model parameters are analyzed on the basis of their physical and geometrical properties. This study also explores the properties of the modified gravitational theory, and neither a future singularity nor a little or pseudo-rip are posed as threats to the fate of the Universe. We present scalar perturbation approaches to perturbed evolution equations and demonstrate their stability.

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1. INTRODUCTION

It is well known that modified theories of gravity have been accepted to address the recent accelerated expansion of the Universe. The changing fraction of the mass–energy budget of the Universe and the presence of dark energy are attributed to this late-time acceleration. This behavior of the Universe has been confirmed by cosmological observations [1–4]. The antigravity effect of a matter field with negative pressure resulted in the violation of strong energy conditions, and hence the role of General Relativity (GR) has been restricted. In addition to some early Universe issues like an initial singularity, flatness, cosmic horizon and, at present, the issue of late-time cosmic acceleration, GR has certain limitations to address. Therefore, modifying geometry or matter components in GR has become necessary. Geometrically extended gravity models are used to add more variables to the geometrical elements of the model. The $F(R, G)$ gravity is such a modified theory of gravity, where $R$ and $G$ denote the Ricci scalar and the Gauss–Bonnet invariant [5, 6]. We shall discuss some of the important results of the cosmological and astrophysical aspects done in this theory: the late-time acceleration behavior [7–12] the energy conditions [13–15], gravastars [16, 17], dynamics of inflation and dark energy [18–20], bouncing [21, 22] and so on. Most recently, Martino et al. [23] traced the cosmic history and demonstrated that it might lead gravity from ultraviolet to infrared scales. Also, the ghost-free issue has been resolved in $F(R, G)$ gravity as in [24].

The Wilkinson Microwave Anisotropy Probe (WMAP) observations indicated that the Universe is dominated by phantom energy, and in the presence of phantom energy, there would be some fascinating physical events, such as the Big Rip (BR) scenario [25]. Also, the mass of black holes reduces due to phantom energy accretion [26], and the emergence of a new type of wormhole is possible. This phenomenon can be explained if dark energy exists with a negative pressure, which can be described using a barotropic fluid with the equation of state $\omega = p/\rho$ with $\omega = -1.10 \pm 0.14$ [27]. The equation $p = \omega \rho$ with $\omega < -1$ shows that a Universe with dark energy leads to a classic future singularity known as a BR [25, 28].

In this type of singularity, the size of the Universe, its expansion and acceleration all diverge [29]. Also, because of the phantom or quintessence dark energy,
the evolution of the Universe often results in a finite-time future singularity with a parameter $\omega \approx -1$. Recently, an elegant solution to this problem was given by [30], known as a Little Rip (LR) singularity. Another type of singularity is the pseudo-Rip (PR), as an example of an intermediate case between LR and the cosmological constant. The structure disintegration in a PR depends on the model parameters [31].

In modified theories of gravity, several rip cosmological scenarios are given in the literature. We discuss here some of the important findings on rip cosmology. Sami [32] discussed the nature of the future Universe evolution or the ultimate fate of the Universe and commented that the evolution depends on the steepness of the phantom potential. Brevik and Elizalde [33] explained that a viscous fluid can produce an LR scenario as a pure viscosity effect. In $f(T)$ gravity, where $T$ is the torsion scalar, Bamba et al. [34] have shown inflation in the early Universe, the $\Lambda$CDM model, LR and PR scenarios of the Universe. Among the BR, LR and PR scenarios, PR models can generate inertial forces that do not rise monotonically [31], however, they will diminish at some point after reaching a high value in the future. Due to the intensity of the expansion, Saez-Gomez [35] demonstrated the likelihood of LR and PR singularities. In modified $F(R, G)$ gravity, Makarenko et al. [36] have shown that the effective phantom-type model does not lead to a future singularity. Saez-Gomez [35] discussed that $f(R)$ gravity theory provides useful information for the occurrence of cosmological evolution, future singularities, LR and PR in viable $f(R)$ theories. Brevik et al. [37] described the LR and PR phenomena in coupled dark energy cosmological models. Mishra and Tripathy [38] presented a LR model in an anisotropic background. Ray et al. [39] showed the nonoccurrence of BR or PR singularities in $f(R, T)$ theory of gravity. Other recent modified theories of gravity are based on nonmetricity [40, 41]. These theories showed some significant results addressing the cosmic expansion phenomena [42, 43]. Pati et al. [44] obtained cosmological models with LR, BR, and PR scenarios in the non-metricity gravity.

In this paper, we investigate the possible occurrence of a future singularity scenario in the context of the modified theory of gravity that includes the Gauss–Bonnet invariant. The paper is organized as follows. A brief description of $F(R, G)$ gravity and its field equations are presented in Section 2. Three singularity-free models based on LR, BR, and PR scale factors, along with their dynamical parameters, are discussed in Section 3, and the energy conditions are discussed in Section 4. The stability analysis under linear homogeneous and isotropic perturbations of the models is presented in Section 5, and finally the results and a conclusion are given in Section 6.

2. $F(R, G)$ GRAVITY FIELD EQUATIONS AND DYNAMICAL PARAMETERS

A modified gravity theory that contains both the Ricci scalar $R$ and the Gauss–Bonnet invariant $\mathcal{G}$ is $F(R, \mathcal{G})$ gravity [6, 45, 46]. This gravitational theory has evolved to justify the evolution of the Universe in the context of dark energy and an initial singularity. The action for $F(R, \mathcal{G})$ gravity is

$$S = \int \sqrt{-g} \frac{1}{2\kappa} F(R, \mathcal{G}) d^4x + \int \sqrt{-g} \mathcal{L}_m d^4x,$$  \hspace{1cm} \text{(1)}

where $\kappa = 8\pi G = c = 1$, $G$ and $\mathcal{L}_m$ denote the Newtonian gravitational constant and the matter Lagrangian, respectively. The Gauss–Bonnet invariant can be expressed as $\mathcal{G} \equiv R^2 - 4R^\mu\nu R_{\mu\nu} + R^\mu\nu\alpha\beta R_{\mu\nu\alpha\beta}$. Now, varying the action (1) with respect to the metric tensor $g_{\mu\nu}$, the field equations of $F(R, \mathcal{G})$ gravity can be presented as

$$F_R G_{\mu\nu} = \kappa T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} [F(R, \mathcal{G}) - FR] + \nabla_\mu \nabla_\nu F_R - g_{\mu\nu} \Box F_R + 2(\nabla_\mu \nabla_\nu F_\mathcal{G}) R - 2g_{\mu\nu}(\Box F_\mathcal{G}) R$$
$$+ 4(\Box F_\mathcal{G}) R_{\mu\nu} - 4(\nabla_k \nabla_\mu F_\mathcal{G}) R^k_{\mu\nu} + F_\mathcal{G}(2RR_{\mu\nu} + 4R_{\mu\nu} R^k - 2R^k_{\mu\nu} R_{\nu k l m} + 4g^{kl} g^{mn} R_{\mu k l m} R_{\nu n})$$
$$- 4(\nabla_\mu \nabla_\nu F_\mathcal{G}) R^k_{\mu\nu} + 4g_{\mu\nu}(\nabla_k \nabla_l F_\mathcal{G}) R^{k l} - 4(\nabla_l \nabla_n F_\mathcal{G}) g^{k l} g^{mn} R_{\mu k l m}.$$

\hspace{1cm} \text{(2)}

Here, the subscripts $R$ and $\mathcal{G}$ denote partial derivatives with respect to the Ricci scalar $R$ and the Gauss–Bonnet invariant $\mathcal{G}$; $G_{\mu\nu}$ is the conventional Einstein tensor, $\nabla_\mu$ is the covariant derivative operator associated with $g_{\mu\nu}$. Also, $\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$ is the covariant d’Alembert operator. $T_{\mu\nu}$ is the energy momentum tensor of the matter field. Here, we consider $T_{\mu\nu} = (\rho + p) u_\mu u_\nu + pg_{\mu\nu}$, with $\rho$ and $p$ representing the matter energy density and pressure, respectively. $u^\mu$ is the four-velocity vector of the cosmic fluid that sat-
satisfies \( u_{a}w^{a} = -1 \). Now, we shall derive the field equations in an isotropic and homogeneous Friedmann–Lemaître–Robertson–Walker (FLRW) space-time,

\[
ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2),
\]

where the scale factor \( a(t) \) measures the expansion rate of the Universe, and, as it appears in FLRW space-time, the expansion is uniform in all spatial directions. Using Eq. (3), the Ricci scalar \( R \) and the Gauss–Bonnet term \( \mathcal{G} \) can be expressed in terms of the Hubble parameter \( H = \dot{a}/a \) as \( R = 6(\dot{H} + 2H^2) \) and \( \mathcal{G} = 24H^2(\dot{H} + H^2) \). With this background, the \( F(R, \mathcal{G}) \) gravity field equations [Eq. (2)] can be reduced to

\[
3H^2 FR = \kappa p + \frac{1}{2}[RF_R + \mathcal{G}F_{\mathcal{G}} - F(R, \mathcal{G})]
- 12H^3 \dot{F}_{\mathcal{G}} - 3H \ddot{F}_{R},
\]

\[
2\dot{H}F_R + 3H^2 F_R = -\kappa p + \frac{1}{2}[RF_R + \mathcal{G}F_{\mathcal{G}}
- F(R, \mathcal{G})] - 8H \dot{\dot{H}}F_{\mathcal{G}} - 2\dot{H} \dot{F}_{R} - \ddot{F}_{R}
- 8H^3 \ddot{F}_{\mathcal{G}} - 4H^2 \dddot{F}_{\mathcal{G}}.
\]

An overdot represents an ordinary derivative with respect to cosmic time \( t \). The energy density and pressure can be obtained if the functional \( F(R, \mathcal{G}) \) has some explicit form.

So, here we consider a quadratic form of \( f_1(R) \) and a quadratic form of \( f_2(\mathcal{G}) \) for the functional \( F(R, \mathcal{G}) = f_1(R) + f_2(\mathcal{G}) \) [12, 19, 47]. The linear component in \( f_1(R) \) is included to generate the correct weak field limit. We have analyzed an \( R^2 \) model with a correction that introduces extra degrees of freedom due to the inclusion of the Gauss–Bonnet component. Because the linear one does not contribute, the term \( \mathcal{G}^2 \) is the first important term in the above Lagrangian,

\[
F(R, \mathcal{G}) = R + \alpha R^2 + \beta \mathcal{G}^2,
\]

where \( \alpha \) and \( \beta \) are coupling constants. The energy density and pressure in terms of the Hubble parameter are obtained by substituting Eq. (6) in Eqs. (4) and (5):

\[
\rho = \frac{1}{k^2}(3H^2 + 108\alpha \dot{H}H^2 - 18\alpha \ddot{H}^2
+ 1728\beta \dot{H}H^6 + 864\beta \dot{H}^2H^4 + 36\alpha H\dddot{H}
+ 576\beta \dot{H}H^5 - 288\beta H^8),
\]

\[
p = \frac{1}{k^2}(-2\dot{H} - 3H^2 - 54\alpha \dot{H}^2 - 108\alpha \dddot{H}H^2
- 960\beta \dot{H}H^6 - 4320\beta \dot{H}^2H^4 - 72\alpha \dddot{H}H - 12\dot{H}\dddot{H}
- 1152\beta \dot{H}H^5 - 1152\beta \dot{H}^2H^3 - 1536\beta \dot{H}H^2H^3
- 192\beta \dot{H}^4H + 288\beta H^8).
\]

The model parameters influence the evolution of the pressure and energy density of the model. However, we may modify the values to examine the behavior of dynamical parameters. On the other hand, the EoS parameter allows us to analyze the late-time acceleration issue, which may be determined by using Eqs. (7) and (8).

Now, the dynamical and EoS parameters are expressed in Hubble terms, and to study its behavior, the Hubble parameter is to be expressed in cosmic time. We intend to study the possible occurrence of a singularity in a finite or infinite future. Therefore, we consider some similar forms of the Hubble parameter to find the future evolution of the Universe.

### 3. THE MODELS

In this section, we discuss three future singularities scenarios, such as LR, BR, and PR, as three cosmological models. As the cosmological observations have confirmed the Universe expansion, their is a possibility that the Universe may explode in future with phantom energy accretion. We intend to provide potential consequences of the hypothetical scenarios and their implications for the future Universe.

To study the geometric and dynamic parameters of the rip models, the following parameters are required:

\[
q = \frac{a\ddot{a}}{a^2}, \quad j = \frac{\dot{a}}{aH^2}, \quad s = \frac{r - 1}{3(q - 1/2)}.
\]

For better clarity, we will discuss the physical behavior of the parameters in terms of redshift, which can be related to the scale factor as \( z + 1 = 1/a \).

#### 3.1. Model I (Little Rip)

The LR model is derived from scenarios where the Universe expands gradually, eventually completely dissolving the bound structures. The LR scale factor can be presented as \( a(t) = e^{\lambda/\nu}(e^{\nu t} - e^{\nu t_0}) \), where \( \lambda, \nu, \) and \( t_0 \) are parameters. The equivalent Hubble parameter, which measures the expansion rate of the Universe, and the deceleration parameter which determines whether the Universe accelerates or decelerates, can be written as

\[
H = \lambda e^{\nu t}, \quad q = -\frac{\lambda + \nu e^{-\nu t}}{\lambda}.
\]

From Eq. (10), we find that for positive values of \( \lambda \) and \( \nu \), the deceleration parameter remains negative throughout the evolution. It rises from a lower negative value to \(-1\) at the late evolution time. Since \( e^{-\nu t} \) is positive, the deceleration parameter is always negative for positive values of \( \nu \). As a result, its negative value may correspond to an accelerating Universe. However, the sign of the scale factor parameter
determines whether the Universe is accelerating or decelerating. The present values of \( H \) and \( q \) are given in Table 1.

The Hubble parameter of the cosmological models constructed with the assumed form of the scale factor may lead to a divergence of the comoving Hubble radius \( r_{\text{H}} = 1/aH \) as the Hubble parameter vanishes, e.g., at a bouncing scenario. At the same time, the accelerating or decelerating behavior of the Universe can also be assessed through the asymptotic behavior of the comoving Hubble radius. Whenever the Hubble radius reduces monotonically, before asymptotically shrinking to zero, it leads to an accelerating behavior of the Universe. In the bouncing scenario, the Hubble horizon becomes infinite in size near the bouncing point. At late time, the Hubble horizon shrinks to zero. The Hubble parameter increases over time, and at present, \( H_0 = 73.03 \text{ km s}^{-1} \text{ Mpc}^{-1} \) for \( \lambda = 25.11 \) [Fig. 1 (left panel)]. This situation arises unlike a bouncing scenario. The Gauss–Bonnet invariant, depending on the Hubble parameter, increases gradually and is infinitely large at late times [Fig. 1 (right panel)].

Using the LR scale factor \( H = \lambda e^{\nu t} \) and \( \dot{H} = \lambda \nu e^{\nu t}, \ddot{H} = \lambda \nu^2 e^{\nu t} \) in Eqs. (7)–(8), we can obtain expressions for the energy density (\( \rho \)), pressure (\( p \)), and the EoS (\( \omega \)), respectively. Figure 2 shows the behavior of the jerk and snap parameters versus redshift for the LR model. The jerk parameter decreases whereas the snap parameter increases over the cosmic time.

To keep the Hubble and deceleration parameters in the range suggested by the cosmological observations, we constrained the LR scale factor parameter as \( \nu = 0.39 \). Next, we have appropriately adjusted the model parameters \( \alpha \) and \( \beta \), so that the energy density remains positive throughout, and the EoS parameter exhibits an accelerating behavior. We have assumed three representative values of the other parameter of the scale factor, \( \lambda \). The energy density remains positive and increases over time, and at a sufficiently late time it becomes very high (Fig. 3, left panel).

The EoS parameter remains negative throughout, and at present (\( z \approx 0 \)) it remains in a phantom phase. At \( z = 0 \), \( \omega_{01} \) is observed to be \(-1.0110, -1.0114, -1.01137 \), respectively, for \( \lambda = 25.11, 25.18, 25.25 \). In the literature, it has been mentioned that there are three significant classes of scalar field dark-energy models available to investigate the theoretical aspects of dark-energy models. These are a phantom phase \( \omega < -1 \) [48], a quintessence phase \(-1/3 < \omega < -1 \) [49], and a quintom \( \omega \) crossing \(-1 \), moving from a phantom region to a quintessence region, that performs the quintom scenario. Here, all curves remain in a phantom phase at the present time.

### 3.2. Model II (Big Rip)

In a BR, the expansion of the Universe accelerates to such an extreme that all structures are disrupted, including galaxies, stars, atoms and fundamental particles. During a finite time, the expansion rate diverges to infinity as the Universe scale factor increases. In this case, we consider the scale factor for a BR singularity as \( a(t) = a_0(t) + 1/(t_s - t)^\gamma \), where \( a_0(t) = c \) is an integration constant. The scale factor \( a(t) \to \infty \) as \( t \to t_s \), and when \( t \to \infty, a(t) \to a_0(t) \). Also, \( t_s \) is when a BR occurs, and the cosmic derivative and Hubble rate blow up at \( t = t_s \). Consequently, the curvature is ill-defined at \( t = t_s \). The physical properties of the model depend on the free parameters \( t_s \) and \( \gamma \), which hence must be defined using some physical basis. The Hubble and deceleration parameters are, respectively, \( H = \gamma \frac{\gamma}{t_s - t} \) and \( q = -\frac{\gamma + 1}{\gamma} \) if the integrating constant \( a_0(t) \) vanishes. We can observe that to keep \( q \) negative, the scale factor parameter \( \gamma > -1 \), whereas \( \gamma < -1 \) provides a positive \( q \) that leads to a decelerating behavior. The present value of the Hubble and the deceleration parameters of the model are provided in Table 1. The Hubble parameter increases with time, and at present, \( H_0 = 73.26 \text{ km s}^{-1} \text{ Mpc}^{-1} \) for \( \gamma = 74.1 \) (Fig. 4, left panel), the Gauss–Bonnet invariant increases gradually and, at a late time, approaches a high positive value (Fig. 4, left panel).

### Table 1. Estimated results for the EoS parameter and cosmological parameters at the current era

| Parameters | LR (\( \lambda = 25.11 \)) | BR (\( \gamma = 74.1 \)) | PR (\( \eta = 0.3011 \)) | Present observational values |
|------------|-----------------------------|-----------------------------|-----------------------------|-------------------------------|
| \( H \) (\( \text{km s}^{-1} \text{ Mpc}^{-1} \)) | 73.03 | 73.26 | 73.29 | 73.55 ± 1.68 [51] |
| \( q \) | -1.004 | -1.028 | -1.0006 | -1.08 ± 0.29 [51] |
| \( \omega \) | -1.011 | -1.1008 | -1.0015 | -1.006 ± 0.045 [3] |
| \( j \) | 1.012 | 1.040 | 1.0017 | - |
| \( s \) | -0.002 | -0.0089 | -0.0003 | - |
Fig. 1. The Hubble parameter (left panel) and the Gauss–Bonnet invariant (right panel) versus redshift for the LR model with the parameters $\nu = 0.3122$, $t_0 = 3.42$.

Fig. 2. The jerk parameter (left panel) and the snap parameter (right panel) versus redshift for the LR model with the parameters $\nu = 0.3122$, $t_0 = 3.42$.

Fig. 3. The energy density (left panel) and the EoS parameter (right panel) versus redshift for the LR model with the parameters $\alpha = 0.3$, $\beta = -0.15$, $\nu = 0.3122$, $t_0 = 3.42$. 
The Hubble parameter (left panel) and the Gauss–Bonnet invariant (right panel) versus redshift for the BR model.

The jerk parameter (left panel) and the snap parameter (right panel) versus redshift for the BR model.

The energy density (left panel) and EoS parameter (right panel) versus redshift for a BR model. The parameters are $\alpha = 0.3$, $\beta = -0.15$.

right panel). Figure 5 shows the behavior of the jerk and snap parameters versus redshift for a BR model. Interestingly, the behavior of these parameters remains the same throughout the evolution.

The energy density, pressure, and EoS parameter for the BR case can be obtained by using the following expressions in Eqs. (7), (8):

\[ \dot{H} = \frac{\gamma}{(t_s - t)^2}, \quad \ddot{H} = \frac{\gamma}{(t_s - t)^3}, \]
in the same region. At late times, it approaches the epoch, progressively increases, and then remains evolves in the phantom phase at the beginning of value (Fig. 6, left panel). The EoS parameter curve positive and, at a late time, it attains a very large indicates that the chosen parameters keep it entirely within the structures of future singularities, including a BR, the fate of phantom-driven Universes and discussed it is worth noting that Nojiri et al. [50] explored the evolutionary behavior of a parameter can be observed for representative values of \( \gamma \) at early and present times only (Fig. 4, right panel). In this context, the change in the EoS parameter needs to be doubled in value for a continuous transition from quintessence to a phantom phase after evaluating a BR evolution model based on the EoS parameter.

\[ \hat{H} = \frac{\gamma}{(\eta_0 - \chi_1 e^{-\eta t})^2}. \] (11)

The graphical behavior of the energy density indicates that the chosen parameters keep it entirely positive and, at a late time, it attains a very large value (Fig. 6, left panel). The EoS parameter curve evolves in the phantom phase at the beginning of the epoch, progressively increases, and then remains in the same region. At late times, it approaches the \( \Lambda CDM \) line. The BR model records the values of the EoS parameter at the current cosmic epoch, \( \omega_0 = -1.10, -1.18, -1.27 \), respectively, for \( \gamma = 74.1, 74.4, 74.7 \). This result is in good agreement with the range \( \omega(t_0) = -1.10 \pm 0.14 \) as determined by a recent observation [27]. However, the EoS curves evolve from sufficiently different phases initially, and at late times merge together. So, the change in the evolutionary behavior of a parameter can be observed for representative values of \( \gamma \) at early and present times only (Fig. 4, right panel). In this context, it is worth noting that Nojiri et al. [50] explored the fate of phantom-driven Universes and discussed the structures of future singularities, including a BR, within finite time \( (t_0) \). Nojiri et al. [50] found that the EoS parameter needs to be doubled in value for a continuous transition from quintessence to a phantom phase after evaluating a BR evolution model based on the EoS parameter.

### 3.3. Model III (Pseudo-Rip)

A PR is a scenario between BR and LR. Compared to the BR scenario, the expansion of the Universe accelerates more slowly, but the growth of the scale factor and divergence of the expansion rate occur more gradually. In some cases, the Universe appears to be heading towards a rip-like behavior, but the effects are not as drastic as in a BR. The Hubble parametrization suggests another phantom behavior without singularity at finite time, \( H = \chi_0 - \chi_1 e^{-\eta t} \), where \( \eta, \chi_0, \) and \( \chi_1 \) are positive constants, and \( \chi_0 > \chi_1 \). The Hubble parameter \( H \to \chi_0 \) as \( t \to \infty \). Asymptotically, this model leads to a de Sitter Universe [37]. The deceleration parameter \( q \) becomes \( q = -1 - \frac{\chi_1 \eta e^{\eta t}}{(\chi_1 - \chi_0 e^{\eta t})^2} \).

With time, the Hubble parameter increases in value, and the current value is \( \approx 73.29 \) (km/s)/Mpc. As \( t \to 0 \), \( q = -1 - \eta \chi_1 / (\chi_1 - \chi_0)^2 \), and as \( t \to \infty \), \( q \) approach \(-1\). The parameters \( \chi_1 > 0 \) and \( \eta > 0 \) were restricted to maintain the current value of the deceleration parameter \( q_0 = -1.00006 \), which is within the preferred range of recent observations \( (q_0 = -1.08 \pm 0.29) \) [51]. The present value of the Hubble parameter and deceleration parameter of this model are given in Table 1. The present value of the Hubble parameter is achieved for \( \eta = 0.3011 \) (Fig. 7, left panel). The Gauss–Bonnet invariant remains the same as in the LR and BR cases (Fig. 7, right panel). Figure 8 shows the behavior of the jerk and snap parameters versus redshift for a PR model, which remain similar to that of the LR model.

We will simplify the energy density and EoS parameter expressions for the PR model by substituting the following expressions in Eqs. (7), (8):

\[ \hat{H} = \eta \chi_1 e^{-\eta t}, \quad \ddot{H} = -\eta^2 \chi_1 e^{-\eta t}, \quad \dot{H} = \eta^3 \chi_1 e^{-\eta t}. \] (12)

Throughout the evolution, the energy density remains positive and increases from early to late times (Fig. 9, left panel). Since the current value of the EoS parameter is very close to \(-1\), the model appears to be
Fig. 8. The jerk parameter (left panel) and the snap (right panel) versus redshift for a PR model. The parameters are $\chi_0 = 74.31$, $\chi_1 = 1$.

Fig. 9. The energy density (left panel) and the EoS parameter (right panel) versus redshift for a PR model. The parameters are $\alpha = 0.3$, $\beta = -0.15$, $\chi_0 = 74.31$, $\chi_1 = 1$.

aligned with the concordant ΛCDM model (Fig. 9, right panel). Because at finite time a future singularity is not observable, it avoids a BR.

To summarize, in Table 1 we have listed the present values of the Hubble parameter, the deceleration parameter and the EoS parameter for all three rip models discussed above. Also, the results of cosmological observations are mentioned against the parameters.

A recent study showed that $H(z)$ and the SNeIa data could help in constraining the cosmic parameters: $H_0 = 75.35 \pm 1.68 \text{ km s}^{-1} \text{ Mpc}^{-1}$ in the most recent Pantheon sample, with a 2.2% uncertainty, close to the 1.9% error found by the SH0ES Collaboration. The deceleration parameter demonstrated in [52] that a competitive limit on the Hubble constant could be obtained using the broad (truncated) Gaussian prior $q_0 = 0.5 \pm 1$. Without high-redshift Type Ia supernovae, the limit on a constant dark energy equation of state parameter from WMAP + BAO + $H_0$ is $\omega = -1.10 \pm 0.14 \ (68 \% \text{ CL})$ [27]. Several data sources were used, which suggest the limit for $\omega$ as, (i) Planck collaboration: $-1.03 \pm 0.03$ [4]; (ii) Supernovae cosmology project, $-1.035^{+0.055}_{-0.069}$ [53]; and (iii) WMAP + SN Ia, $-1.084 \pm 0.063$ [54]. The limit of $\omega$ obtained here is in the prescribed limit from different observation sources.

4. ENERGY CONDITIONS

This study is to know the possible occurrence of a rip in the evolution process of the Universe at its late-time expansion, and this expansion issue can be addressed by modified theories of gravity. Some of the properties of these modified theories need to be verified; prominent among them is the behavior of energy conditions assigning the causal and geodesic structure of space-time. So, a modified theory of
gravity, here \( F(R, G) \) gravity, must confront to the energy conditions. Basically, the energy conditions are boundary conditions to maintain positive energy density \([55, 56]\). But the effect of dark energy, additional limits on cosmic models are imposed by energy conditions \([57]\). For example, dark energy models violate strong energy conditions.

The energy conditions include the Null Energy Condition (NEC), \( \rho + p \geq 0 \); the Weak Energy Condition (WEC), \( \rho \geq 0 \) and \( \rho + p \geq 0 \); the Strong Energy Condition (SEC), \( \rho + 3p \geq 0 \) and \( \rho + p \geq 0 \); and the Dominant Energy Condition (DEC), \( \rho \geq 0 \) and \( \rho + p \geq 0 \). Since violation of the strong energy requirement has become so crucial in modified gravity theories, its survival is now in jeopardy. For this \( F(R, G) \) gravity model, the quantities involved in the NEC, WEC, SEC, and DEC can be shown as follows:

\[
\rho + p = -72\alpha \ddot{H}^2 - 3456\beta H^4 - 36\alpha H \dddot{H} - 2\dddot{H} - 576\beta H^5 \dddot{H} + 12\alpha H^3 - 1536\beta H^3 \dddot{H} \dddot{H} - 192\beta H^7 + 768\beta H^6 \dddot{H} - 1152\beta H^2 \dddot{H} \dddot{H},
\]

(13)

\[
\rho + 3p = -216\alpha \dddot{H} H^2 + 4608\beta H^3 \dddot{H} \dddot{H} + 6\dddot{H}.
\]
+ 1152βH^2H^6 - 30αH^2 - 2016βH^4H^2 \\
- 30αH^2H - 480βH^5H - 6αH^3 - 6H^2 \\
- 3456βH^2H^3 + 576βH^8 - 96βH^7, \quad (14)
\rho - p = 36αH^2 + 216αH^2 + 1536βH^3H^3 \\
+ 2688βH^4H^2 + 2H + 5184βH^4H^2 + 6H^2 \\
+ 108αH^4H + 1728βH^5H - 576βH^8 \\
+ 12αH^3 + 192βH^7 + 1152βH^2H^3. \quad (15)

Because all models evolve in a phantom phase, except the DEC, all other energy conditions are predicted to be violated. The behavior of the energy conditions for Models I, II, and III are presented in Figs. 10–12. In all models, the DEC is satisfied in the suitable range; however, as expected, both the NEC and SEC are violated. For better visibility, we show the violation of NEC in the figures. The NEC quantity decreases and keeps falling to a negative value in the negative cosmic time domain. Hence, the models validate the behavior in \( F(R, G) \) gravity. Summarizing, we can say that in all models under consideration, the NEC, WEC and SEC are violated while the DEC is satisfied both at \( z \simeq -1 \) and \( z \gg 1 \).

5. SCALAR PERTURBATIONS

Considering scalar perturbations for stability analysis in modified theories of gravity has the advantage that they are dominant in structure formation, simplify the analysis, compare with observations, and are consistent with GR. It is possible to gain insights into the stability and dynamics of the theory while capturing the essence of perturbation evolution by analyzing scalar perturbations. Under linear homogeneous and isotropic perturbations, we shall investigate the stability of the rip cosmological models obtained in \( F(R, G) \) gravity [58]. We will use the FLRW pressureless dust background with a general explanation of \( H(t) = H_0(t) \). The matter is fluid in the form of a perfect fluid with constant EoS such that \( p_m = \omega p_m \), and the matter-energy density \( \rho_m \) obeys the standard continuity equation

\[ \dot{\rho}_m + 3H(1 + \omega)\rho_m = 0. \quad (16) \]

Solving Eq. (16), the evolution of the matter energy density can be described as

\[ \rho_{m0}(t) = \rho_0 \exp \left[ -3(1 + \omega_m) \int H_0(t) dt \right]. \quad (17) \]

An isotropic deviation of the Hubble baseline parameter and the matter density is represented by \( \delta(t) \) and \( \delta_m(t) \), respectively. Accordingly, we define their perturbations as

\[ H(t) = H_0(t)[1 + \delta(t)], \]
\[ \rho_{m}(t) = \rho_{m0}[1 + \delta_{m}(t)]. \quad (18) \]

We consider the Hubble parameter and the energy density around arbitrary solutions \( H_0(t) \) as perturbations [58] and perform a perturbation analysis on the solution \( H(t) = H_0(t) \), so that the function \( F(R, G) \) may be represented in powers of \( R \) and \( G \) as

\[ F(R, G) = f_0 + f_{R0}(R - R_0) \]
\[ + f_{G0}(G - G_0) + \mathcal{O}^2, \quad (19) \]

where the subscript 0 means values of \( F(R, G) \) and its derivatives \( f_R \) and \( f_G \) evaluated at \( R = R_0 \) and \( G = G_0 \). Although only the linear terms of the induced perturbations are examined, the \( \mathcal{O}^2 \) term contains all terms proportional to the square of \( R \) and \( G \) or any higher powers that will be included in the equation. For brevity, we ignore terms other than the linear one in Eq. (19). Thus, by substituting Eqs. (18) and (19) in the FLRW background equation (4) and the continuity equation (16), we obtain the perturbation equation in terms of \( \delta(t) \) and \( \delta_{m}(t) \) in the form of the differential equation

\[ c_2\ddot{\delta}(t) + c_1\dot{\delta}(t) + c_0\delta(t) = c_m\delta_{m}(t). \quad (20) \]

The coefficients \( c_0, c_1, c_2, \) and \( c_m \) depend explicitly on the background of \( F(R, G) \) solution and its derivatives.

We consider models based on the function \( F(R, G) = R + \alpha R^2 + \beta G^2 \). Using a perturbative approach in the FLRW equations, we obtain

\[ -18H_0(t)^2 \left( 16F_{GG}^0 H_0(t)^4 + F_{RR}^0 \right) \delta(t) - 18H_0(t) \left( -48F_{GG}^0 H_0(t)^6 - 80F_{GG}^0 H_0(t)^4 \dot{H}_0(t) - 3F_{RR}^0 H_0(t)^2 \right) \\
- F_{RR}^0 \dot{H}_0(t) \delta(t) + 6 \left[ 12H_0(t)^4 (F_{RR}^0 - 36F_{GG}^0 H_0(t)^2) + 192F_{GG}^0 H_0(t)^8 - 1008F_{GG}^0 H_0(t)^6 \dot{H}_0(t) \right] \\
- 288F_{GG}^0 H_0(t)^5 \ddot{H}_0 - H_0(t)^2 (F_{R}^0 + 21F_{RR}^0 \dot{H}_0) - 6F_{RR}^0 H_0(t)^2 \dot{H}_0(t) + 3F_{RR}^0 H_0(t)^2 \delta(t) \\
= \kappa^2 \rho_{m0} \delta_{m}(t). \quad (21) \]
In addition, the matter continuity equation (16) is disturbed, and a second perturbed equation is formed from Eq. (18). Thus,

\[ \dot{\delta}_m(t) + 3H_0(t)\delta(t) = 0. \]  

(22)

Now, we carry out a stability analysis for the models discussed. If we assume that GR is retrieved from the current model (i.e., \( \alpha = 0, \beta = 0 \)), we may ignore contributions from the higher derivatives of the function \( F(R, \mathcal{G}) \). Ignoring such contributions, we obtain

\[ \dot{\delta}(t) = -\frac{c_m}{6H_0^2}\delta_m(t), \]  

(23)

where \( c_m = \kappa^2 \rho_{m0} \). Equation (23) is an algebraic relation between geometric and matter perturbations, from which one may infer that matter perturbations ultimately dictate the whole perturbation surrounding a cosmological solution in GR. Substituting the above relation between geometrical and matter perturbations into (22) and integrating, we get

\[ \delta_m = \exp\left(\frac{c_m}{2}\int H_0^{-1} dt\right). \]  

(24)

Obviously, the matter perturbations decay out for a negative value of the integral \( I = \int H_0^{-1} dt \). In the LR case, we have \( H = \lambda e^{\nu t} \), and the integral becomes \( I = -(1/\nu)\lambda e^{\nu t} \), which is a negative quantity. Therefore, in this case, the magnitude of matter and geometry perturbations decays as time grows, thereby ensuring stability of the model.

In the BR case, we have the Hubble parameter expressed as \( H = \gamma/(t_s - t) \), so that the integral becomes \( I = (1/\gamma)(t_s - t)/2t \). In the very large limit of \( t \), the integral becomes a negative quantity, leading to a decrease in matter and geometric perturbations. Given the situation, the stability of this model can also be ascertained.

However, in the PR case, we have the Hubble parameter \( H = \chi_0 - \chi_1 e^{-\eta t} \). To get a fair idea about perturbations at late times, we may approximate the Hubble parameter by \( H \approx \chi_0 \). This leads to a positive value of the integral \( I \), which poses a question mark on the stability of the model at late times of cosmic evolution as the magnitude of matter and geometry perturbations grows with time.

6. CONCLUSION

Modified gravity theories have emerged as promising options for addressing the challenges concerning the accelerating cosmic expansion and predicting the ultimate fate of the Universe. The \( F(R, \mathcal{G}) \) is a generic modified gravity theory based on curvature-matter coupling that gives an alternate explanation for the present cosmic acceleration without introducing an additional spatial dimension or an exotic component of dark energy. Because of the expansion of the Universe, the possible occurrence of a singularity at finite or infinite future has been examined through three Rip scenarios such as the LR, BR, and PR cases. For all models, we first studied the dynamic behavior of the Hubble parameter \( H \) and the deceleration parameter \( q \).

As determined by the LR, BR, and PR models, the jerk and snap parameters illustrate evolutionary trajectories coming from a Chaplygin gas regime at the present times and approaching the concordant \( \Lambda \)CDM point, and the diagnostic pair \((j, s)\) becomes very close to \((1, 0)\). Also, we obtained the values of these geometric parameters at the present epoch and presented them in Table 1. The behavior and current values of these parameters agree with the results of recent cosmological observations. As constructed, all models show an accelerating behavior. Based on the behavior of the EoS parameter, the models seem to be in a phantom phase at present \((z = 0)\). At late times, though the EoS parameter in the LR model exhibits a phantom-like behavior, it remains exceptionally near the \( \Lambda \)CDM line, whereas, in the BR model, it remains precisely just below the \( \Lambda \)CDM line. The PR model exhibits an identical behavior to that of the LR model, except that it remains in a smaller range. Violations of the NEC and SEC and satisfaction of the DEC in all three models have been asserted. These results are as expected in the context of the behavior of the EoS parameters within modified theories of gravity. Further, the NEC appears immediately below the zero line, which is intriguing. It shows that in these models, a NEC contribution essentially does not exist.

Implementing a linear homogeneous perturbative approach to the Hubble parameter and the energy density, we investigated the stability of the LR, BR, and PR solutions within the modified gravity theory. We have established that, in the used linear perturbative approach, matter perturbations dictate geometry perturbations through a linear equation. The matter perturbations in the three models depend on the behavior of the Hubble parameter. In the LR and BR cases, linear geometry and matter perturbations are obtained to decay smoothly at large cosmic times, ensuring stability of these models. However, in the PR case, because of the behavior of the Hubble parameter where the first term dominates at most of the evolutionary period, linear geometry and matter perturbations appear to grow with time. This brings the PR model stability doubtful. A more detailed investigation of the PR case can therefore be helpful.
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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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