Thin-shell wormholes from black holes with dilaton and monopole fields

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We provide a new type of thin-shell wormhole from the black holes with dilaton and monopole fields. The dilaton and monopole that built the black holes may supply fuel to construct the wormholes.

Several characteristics of this thin-shell wormhole have been discussed. Finally, we discuss the stability of the thin-shell wormholes with a phantom-like equation of state for the exotic matter at the throat.

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I. INTRODUCTION

In a pioneering work, Morris and Thorne \[1\] have shown that wormholes are the solutions of the Einstein field equations and are supported by exotic matter that violates the null energy condition. It is a topological feature of spacetime that connects widely separated regions by a throat that allows to travel from one region to the other. Since, it is not possible to get wormhole like geometry with normal matter in Einstein theory, several alternative theories, such as Brans-Dicke theory, Brain world, C-field theory, Kalb-Ramond, Einstein-Maxwell theory etc. are studied time to time \[2, 3, 4, 5, 6, 7, 8\].

Since matter source plays the crucial role for constructing wormholes, several proposals have been proposed in literature \[9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\].

Visser \[22\] proposed a method to construct Wormholes by surgically grafting two black hole spacetimes together in such a way that no event horizon is permitted to form. Usually here, wormholes are generated from exotic three-dimensional thin shell.

Visser’s approach was adopted by various authors as it is the most simple to construct theoretically, and perhaps also practically because it minimizes the amount of exotic matter required \[23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47\].

More recently, Kyriakopoulos \[48\] discovered a new black-hole solution from an action that besides gravity contains a dilaton field and a pure ( magnetic ) monopole field. These solutions are characterized by three free parameters namely, the dilaton field, the monopole charge and the ADM mass.

He considered a generalized action as

\[
I = \int d^4x \sqrt{-g} \left[ R - \frac{1}{2} \partial_{\mu} \psi \partial^{\mu} \psi - f(\psi) F_{\mu\nu} F^{\mu\nu} \right],
\]

where

\[
f(\psi) = g_1 e^{(c+\sqrt{c^2+1})\psi} + g_2 e^{(c-\sqrt{c^2+1})\psi}
\]

with \(c\), \(g_1\) and \(g_2\) are real constants and \(R\) the Ricci scalar, \(\psi\) representing a dilaton field and \(F_{\mu\nu}\) corresponding to a pure ( magnetic ) monopole field described by

\[
F = Q \sin \theta d\theta \wedge d\varphi
\]

where \(Q\) is the magnetic charge. Above action readily gives the following equations on motion:

\[
(\partial^\mu \psi) ; \alpha = \frac{df}{d\psi} F_{\mu\nu} F^{\mu\nu} = 0
\]

(4)

(\(f F^{\mu\nu}) ; \mu = 0
\]

(5)

\[
R_{\mu\nu} = \frac{1}{2} \partial_{\mu} \psi \partial_{\nu} \psi + 2f \left( F_{\sigma\nu} F_\sigma^\nu - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right)
\]

(6)

After some straight forward calculations, Kyriakopoulos \[48\] obtained the following expressions in terms of the integration constants \(A, B, \alpha\) and \(\psi_0\) and generalized black hole solutions:

\[
g_1 = \frac{AB}{2Q^2} e^{-\psi_0}, \quad g_2 = \frac{(\alpha - A)(\alpha - B)}{2Q^2} e^{\psi_0}, \quad e^{\psi} = e^{\psi_0} \left( 1 + \frac{\alpha}{r} \right)
\]

(7)

where \(\psi_0\) is the asymptotic value of \(\psi\).

\[
ds^2 = -f(r) dt^2 + f(r)^{-1} dr^2 + h(r)(d\theta^2 + \sin^2 \theta d\phi^2),
\]

(8)

where

\[
f(r) = \frac{(r + A)(r + B)}{r(r + \alpha)} \left( \frac{r}{r + \alpha} \right)^{\sqrt{c^2+1}}
\]

(9)

and

\[
h(r) = r(r + \alpha) \left( \frac{r + \alpha}{r} \right)^{\sqrt{c^2+1}}
\]

(10)
This black hole solution is asymptotically flat and has two horizons at \( r = -A \) and \( r = -B \). The Arnowitt-Deser-Misner (ADM) mass \( M \) is given by

\[
M = \frac{1}{2} \left[ \alpha \left( 1 + \frac{c}{\sqrt{c^2 + 1}} \right) - (A + B) \right]
\] (11)

In this paper, we present a new kind of thin-shell wormhole employing such a class of black holes by means of the cut-and-paste technique [22]. The dilaton and monopole that built the black holes may supply fuel to construct the wormholes.

Various aspects of this thin-shell wormhole are analyzed, particularly the equation of state relating pressure and density. Also it has been discussed the attractive or repulsive nature of the wormhole. Our final topic is to search whether this wormhole is stable or not.

II. THIN-SHELL WORMHOLE CONSTRUCTION

From the Kyriakopoulos black hole, we can take two copies of the region with \( r \geq a \):

\[
M^\pm = (x \mid r \geq a)
\]

and paste them at the hypersurface

\[
\Sigma = \Sigma^\pm = (x \mid r = a)
\]

Here we take \( a > Max(-A, -B) \) to avoid horizon and this new construction produces a geodesically complete manifold \( M = M^+ \cup M^- \) with a matter shell at the surface \( r = a \), where the throat of the wormhole is located. We shall use the Darmois-Israel formalism to determine the surface stress at the junction boundary.

The induced metric on \( \Sigma \) is given by

\[
d\tilde{s}^2 = -d\tau^2 + a(\tau)^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\] (12)

where \( \tau \) is the proper time on the junction surface. Using the Lanczos equations [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42], one can obtain the surface stress energy tensor \( T^j_i = \text{diag}(-\sigma, p_\theta, p_\phi) \), where \( \sigma \) is the surface energy density and \( p_\theta \) and \( p_\phi \) are the surface pressures. The Lanczos equations now yield [28]

\[
\sigma = -\frac{1}{4\pi} \frac{h'(a)}{h(a)} \sqrt{f(a)} + \dot{a}^2
\] (13)

and

\[
p_\theta = p_\phi = p = \frac{1}{8\pi} \frac{h'(a)}{h(a)} \sqrt{f(a)} + \dot{a}^2 + \frac{1}{8\pi} \frac{2\dot{a} + f'(a)}{\sqrt{f(a)} + \dot{a}^2}
\] (14)

To understand the dynamics of the wormhole, we assume the radius of the throat to be a function of proper time, or \( a = a(\tau) \). Also, overdot and prime denote, respectively, the derivatives with respect to \( \tau \) and \( a \). For a static configuration of radius \( a \), we need to assume \( \dot{a} = 0 \) and \( \ddot{a} = 0 \) to get the respective values of the surface energy density and the surface pressures which are given by

\[
\sigma = \left[ \frac{\sqrt{\alpha^2} - (2a + \alpha)}{4\pi a(a + \alpha)} \right] \sqrt{\frac{(a + A)(a + B)}{a(a + \alpha)}} \left( \frac{a}{a + \alpha} \right) \frac{1}{\sqrt{\alpha^2 + 1}}
\] (15)

and

\[
p = \frac{1}{8\pi} \sqrt{\frac{\sqrt{\alpha^2} - (2a + \alpha)}{a(a + A)(a + B)(a + \alpha)(2a + A + B)}}
\] (16)

Case -1 : \( c \to \infty \):
The above expressions read

\[
\sigma = -\sqrt{\frac{(a + B)(a + A)}{2\pi(a + \alpha)^2}}
\] (17)

and

\[
p_\theta = p_\phi = p = \frac{1}{8\pi} \sqrt{\frac{1}{a(a + A)(a + B)}} \left( \frac{2a + A + B}{a + \alpha} \right)
\] (18)

Case -2 : \( \alpha = B = -\frac{c^2}{8M} e^{-\psi_0}, A = -2M, c=0 \):
The above expressions read

\[
\sigma = -\frac{1}{4\pi a} \left( \frac{2a - \frac{c^2}{8M} e^{-\psi_0}}{a - \frac{c^2}{8M} e^{-\psi_0}} \right) \sqrt{1 - \frac{2M}{a}}
\] (19)

and

\[
p_\theta = p_\phi = p = \frac{1}{8\pi} \sqrt{\frac{1}{1 - \frac{2M}{a}}} \left[ \frac{2a - \frac{c^2}{8M} e^{-\psi_0} - 2M}{a(a - \frac{c^2}{8M} e^{-\psi_0})} \right]
\] (20)

Observe that the energy density \( \sigma \) is negative. The pressure \( p \) may be positive, however. This would depend on the position of the throat and hence on the physical parameters \( \alpha \), \( A \), and \( B \) and \( c \) defining the wormhole. Similarly, \( p + \sigma, \sigma + 2p \), and \( \sigma + 3p \), obtained by using the above equations, may also be positive under certain conditions, in which case the strong energy condition is satisfied.
III. THE GRAVITATIONAL FIELD

We now turn our attention to the attractive or repulsive nature of our wormhole. To perform the analysis, we calculate the observer’s four-acceleration $a^u = u^\nu \, u^\nu$, where $u^\nu = dx^\nu/d\tau = (1/\sqrt{f(r)}, 0, 0, 0)$. In view of the line element, Eq. (8), the only non-zero component is given by

$$a^r = \Gamma^r_{tt} \left( \frac{dt}{d\tau} \right)^2 = \frac{1}{2} \frac{1}{(a^2 + a\alpha)^2} \frac{\alpha}{a + \alpha} \sqrt{\frac{e}{c^2}} \left[ Fa^2 - Da - H \right]$$

where,

$$F = \alpha - A - B + \frac{c\alpha}{\sqrt{c^2 + 1}}$$

$$D = 2AB - c\alpha(A + B) / \sqrt{c^2 + 1}$$

and

$$E = AB\alpha - c\alpha AB$$

**Case -1 :** $c \to \infty$:

The above expression reads

$$F = -A - B ; \quad D = 2AB - \alpha(A + B) ; \quad E = 0$$

**Case -2 :** $\alpha = B = -\frac{Q^2}{M} e^{-\psi_0}, A = -2M, c=0$:

The above expression reads

$$F = 2M ; \quad D = 2Q^2 e^{-\psi_0} ; \quad E = -2MQ^4 e^{-2\psi_0}$$

A radially moving test particle initially at rest obeys the equation of motion

$$\frac{d^2r}{dr^2} = -\Gamma^r_{tt} \left( \frac{dt}{d\tau} \right)^2 = -a^r.$$  

(22)

If $a^r = 0$, we obtain the geodesic equation. Moreover, a wormhole is attractive if $a^r > 0$ and repulsive if $a^r < 0$. These characteristics depend on the parameters $\alpha$, $A$, and $c$, the conditions on which can be conveniently expressed in terms of the coefficients $F$, $D$, and $E$. To avoid negative values for $r$, let us consider only the root $r = (D + \sqrt{D^2 - 4FE})/(2F)$ of the quadratic equation $Fr^2 - Dr - E = 0$. It now follows from Eq. (21) that $a^r = 0$ whenever

$$\left( r - \frac{D}{2F} \right)^2 = \frac{D^2 - 4FE}{4F^2}.$$ 

For the attractive case, $a^r > 0$, the condition becomes

$$\left( r - \frac{D}{2F} \right)^2 > \frac{D^2 - 4FE}{4F^2}.$$ 

For the repulsive case, $a^r < 0$, the sense of the inequality is reversed.

IV. THE TOTAL AMOUNT OF EXOTIC MATTER

In this section we determine the total amount of exotic matter for the thin-shell wormhole. This total can be quantified by the integral

$$\Omega_\sigma = \int |\rho + p| \sqrt{-g} d^3x.$$ 

(23)

By introducing the radial coordinate $R = r - a$, we get

$$\Omega_\sigma = \int_0^{2\pi} \int_0^\pi \int_{-\infty}^\infty |\rho + p| \sqrt{-g} R d\theta d\phi.$$ 

Since the shell is infinitely thin, it does not exert any radial pressure. Moreover, $\rho = \delta(R)\sigma(a)$. So

$$\Omega_\sigma = \int_0^{2\pi} \int_0^\pi \int_{-\infty}^\infty |\rho - g| \left| \frac{d\theta}{d\phi} = 4\pi h(a)\sigma(a) \right.$$ 

$$= - \left( (2a + \alpha) - \frac{c\alpha}{\sqrt{c^2 + 1}} \right) \times$$

$$\left( a + \alpha \right)^{2\sqrt{c^2 + 1}} \sqrt{\frac{(a + A)(a + B)}{a(a + \alpha)}}$$

(24)

**Case -1 :** $c \to \infty$:

The above expression reads

$$\Omega_\sigma = -2\sqrt{(a + A)(a + B)}$$

**Case -2 :** $\alpha = B = -\frac{Q^2}{M} e^{-\psi_0}, A = -2M, c=0$:

The above expression reads

$$\Omega_\sigma = -\left( 2a - \frac{Q^2}{M} e^{-\psi_0} \right) \sqrt{1 - \frac{2M}{a}}$$

This NEC violating matter can be reduced by taking the value of $a$ closer to $r_+ = Max(-A, -B)$, the location of the outer event horizon. The closer $a$ is to $r_+$, however, the closer the wormhole is to a black hole: incoming microwave background radiation would get blueshifted to an extremely high temperature. It is interesting to note that total amount of exotic matter needed to support traversable wormhole can be reduced with the suitable choice of the parameters $Q$ and $\psi_0$. One can see that total amount of exotic matters will be reduced with the increasing of magnetic charge $Q$ as well as with decrease of the asymptotic value of the dilaton field $\psi_0$. 
V. AN EQUATION OF STATE

Taking the form of the equation of state (EoS) to be \( p = w \sigma \), we obtain from Eqs. (19) and (20),
\[
p = w = \frac{1}{2} \left( \frac{a}{a + \alpha} \right) \sqrt{a + \alpha} \left[ 2a^2 + (A + B)(a + \alpha) + 2a\alpha \right] \]

\( \frac{p}{\sigma} = w = -\frac{1}{4} \frac{(2a + A + B)(a + \alpha)}{(a + \alpha) (2a + \alpha)} \]

**Case -1:** \( c \to \infty \):

The above expression reads
\[
p = w = -\frac{1}{2} \frac{(2a + A + B)(a + \alpha)}{(2a + \alpha) - \frac{c\alpha}{\sqrt{c^2 + 1}}} \]

**Case -2:** \( \alpha = B = -\frac{Q^2}{M^2} e^{-\psi_0} \), \( A = -2M \), \( c=0 \):

The above expression reads
\[
p = w = -\frac{1}{2} \frac{(2a - \frac{Q^2}{M^2} e^{-\psi_0} - 2M)}{(2a - \frac{Q^2}{M^2} e^{-\psi_0})} \]

Observe that if the location of the wormhole throat is very large, i.e., if \( a \to +\infty \), then \( w \to -\frac{1}{2} \). So the distribution of matter in the shell is of the phantom-energy type.

VI. CASIMIR EFFECT

Another property worth checking is the traceless surface stress-energy tensor \( S^i_i = 0 \), i.e., \( -\sigma + 2p = 0 \). The reason is that the Casimir effect with a massless field is of the traceless type. From this equation we find that

\[
C = \frac{2(2a + \alpha) - \frac{c\alpha}{\sqrt{c^2 + 1}}}{a(a + \alpha)} \times \\
\sqrt{\frac{(a + A)(a + B)}{a(a + \alpha)}} \left( \frac{a}{a + \alpha} \right) \sqrt{a^2 + 1} \\
+ \sqrt{\frac{\frac{a}{a + \alpha}}{(a + A)(a + B)}} \left[ \frac{(a - A + B)a^2 - AB(2a + \alpha)}{(a^2 + aa)^2} \right] = 0
\]

**Case -1:** \( c \to \infty \):

The above expression reads
\[
C = \frac{\sqrt{(a + B)(a + A)}}{2\pi(a + \alpha)^2} \\
+ \frac{1}{\sqrt{(a + A)(a + B)}} \frac{2a + A + B}{a + \alpha} = 0
\]

**Case -2:** \( \alpha = B = -\frac{Q^2}{M^2} e^{-\psi_0} \), \( A = -2M \), \( c=0 \):

The above expression reads
\[
C = \frac{\left( 2a - \frac{Q^2}{M^2} e^{-\psi_0} \right)^2}{a \left( a - \frac{Q^2}{M^2} e^{-\psi_0} \right)} \sqrt{1 - \frac{2M}{a}} \\
+ \frac{1}{\sqrt{1 - \frac{2M}{a}}} \left[ \frac{2a - \frac{Q^2}{M^2} e^{-\psi_0} - 2M}{a(a - \frac{Q^2}{M^2} e^{-\psi_0})} \right] = 0
\]

On can note that the no real values of \( a \) exist which satisfy these equations. This ensures that this situation could not occur when dealing with thin-shell wormholes. This result is rather unfortunate as we expected Casimir effect would be associated with massless fields confined in the throat.

VII. STABILITY

We analyze the stability taking specific equation of state at the throat.

We re-introduce an equation of state between the surface pressure \( p \) and surface energy density \( \sigma \) as
\[
 p = w\sigma
\]
with \( w < 0 \).
This is analogous to dark energy equation of state. Here $p$ and $\sigma$ obey the conservation equation
\[
{\frac{d}{d\tau}}[\sigma h(a)] + p {\frac{d}{d\tau}}[h(a)] = 0 \quad (31)
\]
or
\[
\dot{\sigma} + \frac{\dot{h}}{h}(p + \sigma) = 0. \quad (32)
\]
In the above equations, the overdot denotes, the derivative with respect to $\tau$.

Using the above specific equation of state, the equation (27) yields
\[
\sigma(a) = \sigma_0 \left( \frac{h_0}{h} \right)^{(1+w)} \quad (33)
\]
where, $a_0$ being initial position of the throat with $\sigma_0 = \sigma(a_0)$ and $h_0 = h(a_0)$.

Rearranging equation (13), we obtain the thin shell’s equation of motion
\[
\dot{a}^2 + V(a) = 0. \quad (34)
\]
Here the potential $V(a)$ is defined as
\[
V(a) = f(a) - \left[ \frac{4\pi \sigma(a) h(a)}{h'(a)} \right]^2. \quad (35)
\]

Now substituting the value of $\sigma(a)$ in the above equation, we obtain the following form of potential as
\[
V(a) = f(a) - \left[ \frac{4\pi \sigma_0 h(a)}{h'(a)} \left( \frac{h_0}{h} \right)^{(1+w)} \right]^2 \quad (36)
\]
The explicit expression for $V(a)$ is given by
\[
V(a) = \left( \frac{a + A}{a(a + \alpha)} \right) \left( \frac{a + \alpha}{\alpha + a} \right)^{-\frac{\alpha}{\sqrt{c^2+1}}} -
\frac{L \{a(a + \alpha) \left( \frac{a + \alpha}{\sqrt{c^2+1}} \right)^{-\frac{\alpha}{\sqrt{c^2+1}}} \}^{-\frac{\alpha}{\sqrt{c^2+1}}}}{\{(2a + \alpha) - \frac{\alpha \sigma_0}{\sqrt{c^2+1}} \left( \frac{a + \alpha}{\alpha + a} \right)^{-\frac{\alpha}{\sqrt{c^2+1}}} \}^{-\frac{\alpha}{\sqrt{c^2+1}}} \}^2 \quad (37)
\]
where $L = 4\pi \sigma_0 h_0^{(1+w)}$.

**Case -1 :** $c \rightarrow \infty$:
The above expression reads
\[
V(a) = \frac{(a + A)(a + B)}{(a + \alpha)^2} - \left[ \frac{L_1 \left( a + \alpha \right)^{-2w-1}}{2} \right]^2 \quad (38)
\]

**Case -2 :** $\alpha = B = -\frac{\alpha^2}{M} e^{-\psi_0}$, $A = -2M$, $c = 0$:
The above expression reads

We analysis stability by means of the figures. The plot (fig 2) indicates that $V(a)$ has a local minimum at some $a$. In other words, it is stable in case 1. However, the plot (fig 3) indicates that $V(a)$ has a local maximum at some $a$. In other words, it is unstable in case 2. Thus for some specific choices of the parameters, the thin-shell wormholes constructed from the black holes with dilaton and monopole fields are stable.
VIII. FINAL REMARKS

An exact black hole solution with a dilaton and a pure monopole field and its generalization were developed by Kyriakopoulos [48]. Our aim in this article is to search the new type of thin shell wormholes from the dilaton and monopole fields that built the black holes may supply fuel in the throat to construct the wormholes. We analyzed various aspects of this wormhole, such as the amount of exotic matter required, the attractive or repulsive nature of the wormhole, and a possible equation of state for the exotic matter at the throat. This approach to the stability analysis is different from the other method [23] of the stability of the configuration under small perturbations around a static solution at $a_0$. It has been shown that for some specific choices of the parameters, the thin-shell wormholes constructed from the black holes with dilaton and monopole fields are stable unlike the wormholes constructed from two Schwarzschild spacetimes [48].

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