BIVARIATE TRANSMUTED EXPOIENTIATED GUMBEL DISTRIBUTION (BTEGD) AND CONCOMITANTS OF ITS ORDER STATISTICS

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Abstract: In this article we have studied bivariate transmuted exponentiated Gumbel distribution using Morgenstern approach (Morgenstern [4]). We have also studied the shape behavior of the pdf and cdf of the bivariate transmuted exponentiated Gumbel distribution. The distribution of the concomitants of \( r^{th} \) order statistics, the moment generating function (mgf) and moments of the concomitants of \( r^{th} \) order statistics are obtained. Numerical computations have been done for the moments of the concomitants.

Keywords: bivariate transmuted exponentiated Gumbel distribution; Morgenstern family; concomitants of order statistics; moment generating function; moments of concomitants.

2021 AMS Subject Classification: 62E10.

1. INTRODUCTION

The Gumbel distribution is a very popular statistical distribution due to its extensive applicability in several areas and its wide applications have been reported by Kotz and Nadarajah [16]. The

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Received March 17, 2021
applicability of Gumbel distribution in the area of climate modeling, for example: global warming problems, offshore modeling, rainfall and wind speed modeling have been discussed by Nadarajah [18]. In several areas of engineering such as: flood frequency analysis, network space, software reliability, structural and wind engineering, the applicability of Gumbel Distribution has been reported by Cardeiro, Ortega and Cunha [5]. Due to its wide applicability, several works aimed at extending the Gumbel distribution becomes important. Some examples are mentioned in: Nadarajah and Kotz [17], Cardeiro, Ortega and Cunha [5], Andrade, Rodrigues, Bourguignon and Cordeiro [22] and Deka, Das and Baruah [3]. Thus, the interest in theory and methods about the Gumbel distribution is progressive.

A bivariate distribution $F(x,y)$ for a pair of random variables $(X,Y)$ expresses the dependence between $X$ and $Y$ as embedded in its functional form and parameters. The Morgenstern family of distribution is a flexible system of distribution for constructing bivariate distribution using marginal probability density function. Morgenstern [4] introduced a family of bivariate distribution functions having a representation of the form

$$F_{XY}(x,y) = F_X(x)F_Y(y)[1 + \rho[1 - F_X(x)][1 - F_Y(y)]]$$  \hspace{1cm} (1)

where $F_X(x)$ and $F_Y(y)$ are two univariate distribution functions and the association parameter $\rho$ is constrained to lie in the interval $[-1, 1]$.

The corresponding pdf is

$$f_{XY}(x,y) = f_X(x)f_Y(y)[1 + \rho(1 - 2F_X(x))\{1 - 2F_Y(y)\}]; -1 \leq \rho \leq 1$$  \hspace{1cm} (2)

Concomitants of order statistics have been used extensively by several authors using the concept of Morgenstern approach. Shahbaz and Shahbaz [19] have studied concomitants of generalized order statistics for a bivariate Weibull distribution. Tahmaseb and Jafari [21] have studied concomitants of order statistics and record values from Morgenstern type bivariate generalized exponential distribution. Khan and Kumar [14] have studied concomitants of order statistics from Weighted Marshall-Olkin Bivariate Exponential distribution. Chacko and Thomas [13] have studied estimation of a parameter of Morgenstern type bivariate exponential distribution by ranked set sampling. Athar and Nayabuddin [10], have studied concomitants of dual generalized
order statistics from Farlie Gumbel Morgenstern Type Bivariate Inverse Rayleigh distribution. Concomitants are useful accompaniments in statistical modeling that may stand as random variables of interest in connection with order statistics of random samples. For such a situation, order statistics and their concomitants are taken under consideration in the current paper for the Morgenstern Type Bivariate Transmuted Exponentiated Gumbel distribution.

In the following sections, order statistics and concomitants are mentioned first and then the new Bivariate Transmuted Exponentiated Gumbel distribution is presented. A next step is the construction of the distribution of the concomitants of order statistics for the presented Bivariate Transmuted Exponentiated Gumbel distribution.

2. ORDER STATISTICS AND CONCOMITANTS

A random sample of \((X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)\) from a bivariate distribution with bivariate distribution function \(F(x, y)\) yields order statistics of the first coordinate as \((X_{1:n}, X_{2:n}, \ldots, X_{n:n})\) such that \(X_{i:n} \leq X_{j:n}\) for \(i < j\). If the pairs \((X_i, Y_i)\), \(i = 1, 2, \ldots, n\), are ordered by their \(X\) variates according to \((X_{1:n}, X_{2:n}, \ldots, X_{n:n})\); then the \(Y\) variate associated with the \(r^{th}\) order statistic \(X_{r:n}\) of \(X\), denoted by \(Y_{[r:n]}\), \(1 \leq r \leq n\), and is called the concomitant of the \(r^{th}\) order statistic. For a detailed overview of concomitants, we refer to David and Nagaraja [8], [9].

Concomitants of order statistics have several applications in statistics. Concomitants are used in many applied areas where a population characteristic \(Y\) is investigated with respect to another characteristic \(X\) of the same population. For example in selection procedures, Yeo and David [23] considered the problem of choosing the best \(k\) objects out of \(n\) candidates on the basis of auxiliary measurements \(X\), while the measurements of primary interest \(Y\) are not available. The authors are interested in the probability that the \(m\) subjects with the largest \(X\)-values consists of the \(k\) objects with the largest \(Y\)-values. Recently, it becomes a very important research topic and discussed in several literature. For a detailed description of the theoretical aspects of concomitants of order statistics, one may refer David and Nagaraja [9], Chacko and Thomas [12], Scaria and Nair [11], Nair and Scaria [15] and Tahmasebi and Behboodian [20] etc. The distribution of \(r^{th}\)
concomitants is given by David and Nagaraja [8] as
\[
g_{[r:n]}(y) = \int f(y|x) f_{r:n}(x) \, dx, \tag{3}
\]
Where \( f_{r:n}(x) \) is the pdf of \( X_{r:n} \).

The density function of an \( r^{th} \) order statistic \((X_{r:n})\), is defined by Arnold et al. [2] as,
\[
f_{r:n}(x) = \frac{n!}{(r - 1)! (n - r)!} (F(x))^{r-1} [1 - F(x)]^{n-r} f(x), -\infty < x < \infty \tag{4}
\]

The general expressions given here are used in the following sections where a new Bivariate Transmuted Exponentiated distribution is presented.

3. BIVARIATE TRANSMUATED EXPONENTIATED GUMBEL DISTRIBUTION

Deka et al. [3] have studied the Transmuted Exponentiated Gumbel Distribution (TEGD) along with several statistical properties and applied it to model water quality parameters data set. The cdf of the TEGD is
\[
F(x) = 1 - \left\{ 1 - \exp \left( -\exp \left( -\frac{x - \mu}{\sigma} \right) \right) \right\}^\alpha \left( 1 - \lambda + \lambda \left\{ 1 - \exp \left( -\exp \left( -\frac{x - \mu}{\sigma} \right) \right) \right\} \right\}^\alpha, -\infty < x, \mu < \infty, \quad \sigma, \alpha > 0, \quad |\lambda| \leq 1 \tag{5}
\]
And its corresponding pdf is
\[
f(x) = \frac{\alpha}{\sigma} \left\{ 1 - \exp \left( -\exp \left( -\frac{x - \mu}{\sigma} \right) \right) \right\}^{\alpha-1} \left\{ \exp \left( -\exp \left( -\frac{x - \mu}{\sigma} \right) \right) \right\} \left( \exp \left( -\frac{x - \mu}{\sigma} \right) \right) \left[ 1 - \lambda + 2\lambda \left\{ 1 - \exp \left( -\exp \left( -\frac{x - \mu}{\sigma} \right) \right) \right\} \right\] \left\{ 1 - \lambda + \lambda \left\{ 1 - \exp \left( -\exp \left( -\frac{x - \mu}{\sigma} \right) \right) \right\} \right\}, -\infty < x, \mu < \infty, \quad \sigma, \alpha > 0, \quad |\lambda| \leq 1 \tag{6}
\]
Using the marginal Transmuted Exponentiated Gumbel density functions for the random variable \( X \) and \( Y \) where \( X \sim TEGD(\mu_1, \sigma_1, \alpha_1, \lambda_1) \) and \( Y \sim TEGD(\mu_2, \sigma_2, \alpha_2, \lambda_2) \) in equation (1) we get the cdf for MTBTEGD as
\[
F_{XY}(x,y) = \left\{ 1 - \exp \left( -\exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right) \right\}^{\alpha_1} \left[ 1 - \lambda_1 + \lambda_1 \left\{ 1 - \exp \left( -\exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right) \right\} \right] \left[ 1 - \lambda_2 + \lambda_2 \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \right], -\infty < x, y < \infty, \quad \sigma_1, \sigma_2, \alpha_1, \alpha_2, \lambda_1, \lambda_2 > 0, \quad |\lambda_1|, |\lambda_2| \leq 1 \tag{7}
\]
\[
\begin{align*}
\exp \left( -\exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right)^{\alpha_1} \left\{ 1 - \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \right\} & - \frac{\alpha_2}{\sigma_1 \sigma_2} \left\{ 1 - \exp \left( -\exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right) \right\} \left\{ \exp \left( -\exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right) \right\} \left\{ \exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right\} \left\{ 1 - \lambda_1 \right\} \\
& + 2\lambda_1 \left\{ 1 - \exp \left( -\exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right) \right\} \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ \exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right\} \left\{ 1 - \lambda_2 \right\} \\
& + 2\lambda_2 \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ \exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right\} \left\{ 1 - \lambda_2 \right\} \\
& + \rho \left\{ \left\{ 1 - \exp \left( -\exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right) \right\} \left\{ 1 - \exp \left( -\exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right) \right\} \left\{ 1 - \exp \left( -\exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right) \right\} \left\{ 1 - \exp \left( -\exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right) \right\} \right\} \\
& - 1 \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ 1 - \lambda_2 \right\} \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ 1 - \lambda_2 \right\} \right\} \\
& - 1 \right\} \\
\end{align*}
\] (7)

And the corresponding pdf is obtained by using (2) as

\[
f_{X,Y}(x, y) = \frac{\alpha_1 \alpha_2}{\sigma_1 \sigma_2} \left\{ 1 - \exp \left( -\exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right) \right\} \left\{ \exp \left( -\exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right) \right\} \left\{ \exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right\} \left\{ 1 - \lambda_1 \right\} \\
+ 2\lambda_1 \left\{ 1 - \exp \left( -\exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right) \right\} \left\{ 1 - \exp \left( -\exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right) \right\} \left\{ \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ \exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right\} \left\{ 1 - \lambda_2 \right\} \\
+ 2\lambda_2 \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ \exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right\} \left\{ 1 - \lambda_2 \right\} \\
+ \rho \left\{ \left\{ 1 - \exp \left( -\exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right) \right\} \left\{ 1 - \exp \left( -\exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right) \right\} \left\{ 1 - \exp \left( -\exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right) \right\} \left\{ 1 - \exp \left( -\exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right) \right\} \right\} \\
- 1 \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ 1 - \lambda_2 \right\} \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ 1 - \lambda_2 \right\} \right\} \\
- 1 \right\} \\
\] (8)

It can be shown that
\[ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{XY}(x, y) \, dx \, dy = 1 \]

4. **Graphical Representation of Bivariate Transmuted Exponentiated Gumbel Distribution (BTEGD)**

**Fig1:** pdf of Bivariate Transmuted Exponentiated Gumbel Distribution
5. DISTRIBUTION OF CONCOMITANTS FOR BIVARIATE TRANSMUTED EXPONENTIATED GUMBEL DISTRIBUTION (BTEGD)

5.1. Distribution of \( r^{th} \) Concomitants for BTEGD

In this section, we obtain the distribution of the concomitants of the \( r^{th} \) order statistics for the Bivariate Transmuted Exponentiated Gumbel distribution, given in Eq. (7). To obtain the distribution of the concomitants of \( r^{th} \) order statistics we need the conditional distribution of \( Y \) given \( X \) and the distribution of \( r^{th} \) order statistics \( X_{rX} \). The conditional distribution of \( Y \) given \( X \) is obtained as follows:

We have,

\[
f_Y(y|X) = \frac{f_{XY}(x,y)}{f_X(x)}
\]  

(9)

Using equation (6) and (8), in equation (9) we get the conditional distribution of \( Y \) given \( X \) for BTEGD as
\[
f_{(Y|X)}(y|x)
\]
\[
= \frac{\alpha_2}{\sigma_2} \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{\alpha_2 - 1} \left\{ \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}\left\{ \exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right\} \left[ 1 - \lambda_2 \right]
\]
\[
+ 2\lambda_2 \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{\alpha_2} \right\} \left[ 1 \right]
\]
\[
+ \rho \left\{ \left\{ 1 - \exp \left( -\exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right) \right\}^{\alpha_1} \left[ 1 - \lambda_1 + \lambda_1 \left\{ 1 - \exp \left( -\exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right) \right\}^{\alpha_1} \right] \right.
\]
\[
- 1 \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{\alpha_2} \left[ 1 - \lambda_2 + \lambda_2 \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{\alpha_2} \right] \right.
\]
\[
- 1 \left\} \right\}
\]

Using Eq. (12) in Eq. (4), we get the expression for \( f_{1:n}(x) \) as
\[
f_{1:n}(x) = \sum_{j=0}^{\infty} (-1)^j \binom{n-1}{j} n[F(x)]^j f(x)
\]

Using Eq. (5), (6), (12) and (13); in Eq. (11), we get the expression for \( g_{[1:n]}(y) \) as

Putting \( r = 1 \), in the equation (3), we get the distribution of the concomitants of 1\(^{st}\) order statistics as
\[
g_{[1:n]}(y) = \int f_{(Y|X)}(y|x)f_{1:n}(x) \, dx
\]
\[
g_{[1:n]}(y) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} \binom{n-1}{j} \binom{j}{k} \lambda_1^j \lambda_2^k \frac{\alpha_1 \alpha_2}{\sigma_1 \sigma_2} \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{\alpha_2 - 1} \left\{ \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ \exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right\} \left[ 1 - \lambda_2 \right] \\
+ 2\lambda_2 \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{\alpha_2} \int_{-\infty}^{+\infty} \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{\alpha_2} \left\{ \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ \exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right\} \left[ 1 - \lambda_2 \right] \\
- \exp \left( -\exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right) \left\{ \exp \left( -\exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right) \right\} \left\{ \exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right\} \left[ 1 - \lambda_1 \right] \\
+ 2\lambda_1 \left\{ 1 - \exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right\}^{\alpha_1} \left\{ 1 - \lambda_1 + \lambda_1 \left\{ 1 - \exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right\}^{\alpha_1} \right\] \\
+ \rho \left\{ \left\{ 1 - \exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right\}^{\alpha_1} \left[ 1 - \lambda_1 + \lambda_1 \left\{ 1 - \exp \left( -\frac{x - \mu_1}{\sigma_1} \right) \right\}^{\alpha_1} \right\] \\
- 1 \right\} \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{\alpha_2} \left\{ 1 - \lambda_2 + \lambda_2 \left\{ 1 - \exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right\}^{\alpha_2} \right\] \\
- 1 \right\} \right\} dx \\
\] (14)

Using Prudnikov et al. [1] after integration equation (14) becomes
\[ g_{[1:n]}(y) \]

\[ = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} \binom{n-1}{j} \binom{n}{k} \frac{n \lambda_1^j \alpha_2}{\sigma_2} \{ 1 \]

\[ - \exp \left( - \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \right) \}^{\alpha_2-1} \left\{ \exp \left( - \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \right) \} \left\{ \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \right\} [1 - \lambda_2 \]

\[ + 2 \lambda_2 \left\{ 1 - \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \}^{\alpha_2} \left\{ \frac{(1 - \rho)(1 - \lambda_1)}{n + k} - \frac{\rho \lambda_1 (\rho \lambda_1 - 3 \rho + 1)}{n + k + 1} \]

\[ - \frac{2 \rho \lambda_1 (1 - \lambda_1)}{n + k + 2} - \frac{\rho \lambda_1}{n + k + 3} \right\} \]

\[ + \left\{ \frac{\rho \lambda_2^2}{n + k + 3} + \frac{\rho \lambda_1 (2 - \lambda_1 - \lambda_2)}{n + k + 2} + \frac{\rho (1 - 2 \lambda_1 - \lambda_2 + \rho \lambda_1)}{n + k + 1} \]

\[ - \frac{\rho (1 - \lambda_1)}{n + k} \left\{ 1 - \exp \left( - \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \right) \}^{\alpha_2} \left\{ 1 - \lambda_2 \]

\[ + \lambda_2 \left\{ 1 - \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \}^{\alpha_2} \right\} \right\} \right]\]

(15)

According to David [7] the \textit{cdf} of the order statistics connected by the relation

\[ F_{r:n}(x) = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i} F_{1:i}(x); \ 1 \leq r \leq n \dots \dots \dots \dots \ (\ast) \]

Therefore the relation (\ast) is also true for \textit{pdf} of order statistics also. Thus we can obtained the \textit{pdf}

of \( Y_{[r:n]} \) from the following relation

\[ g_{[r:n]}(y) = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i} g_{[1:i]}(y) \]

(16)

Now using equation (15) in equation (16) we get the \textit{pdf} of the concomitants of \( rth \) order statistics as
\[ g_{[r:n]}(y) = \sum_{i=n-r+1}^{n} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{i-n+r+j+k-1} \binom{n}{i-r} \binom{n-i}{j} \binom{n-i-j}{k} \frac{i \lambda_1 \alpha_2}{\sigma_2} \{1 - \exp \left( - \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \right) \} \{ \exp \left( - \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \right) \} \{ \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \} \]

\[ - \exp \left( - \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \right) \]

\[ \left[ 1 - \lambda_2 + 2 \lambda_2 \left\{ 1 - \exp \left( - \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{\alpha_2} \right] \left\{ (1 - \rho)(1 - \lambda_1) - \frac{\rho \lambda_1 (\rho \lambda_1 - 3 \rho + 1)}{n+k} \right\} \left( 1 - \frac{2 \rho \lambda_1 (1 - \lambda_1)}{n+k+2} - \frac{\rho \lambda_1}{n+k+3} \right) \]

\[ + \frac{\rho \lambda_1^2}{n+k+3} + \frac{\rho \lambda_1 (2 - \lambda_1 - \lambda_2)}{n+k+2} + \frac{\rho (1 - 2 \lambda_2 - \lambda_2 + \rho \lambda_1)}{n+k+1} \]

\[ - \frac{\rho (1 - \lambda_1)}{n+k} \left\{ 1 - \exp \left( - \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{\alpha_2} \left[ 1 - \lambda_2 \right] \]

\[ + \lambda_2 \left\{ 1 - \exp \left( - \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{\alpha_2} \]  

(17)

**5.2. Moment Generating Function of the Concomitants of Order Statistics for BTEGD**

In this section, we have obtained the exact expression for the moment generating function (mgf) of the concomitants of \( r^{th} \) order statistics when random variables \( (X_i, Y_i); (i = 1, 2, ..., n) \) are i.i.d. and follows TEGD. The mgf of the concomitants of \( r^{th} \) order statistics \( Y_{[r:n]} \) is given by

\[ M_{Y_{[r:n]}}(t) = E(e^{tY_{[r:n]}}) \]

\[ = \int_{-\infty}^{+\infty} e^{ty} g_{Y_{[r:n]}}(y) dy \]
\[ M_{Y_{i:n}}(t) = \sum_{i \geq n-r+1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{i-n+r+j+k-1} \frac{(i-1) (n-1) (i-j) j ! \lambda_1^j \lambda_2^j}{\sigma_2^2} \int_{-\infty}^{+\infty} e^{ty} \left\{ 1 - \exp\left( -\exp\left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{a_2-1} \left\{ \exp\left( -\exp\left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ \exp\left( -\exp\left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left( 1 - \lambda_2 \right) + 2\lambda_2 \left( 1 - \exp\left( -\exp\left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right) \right\} \left\{ \left( 1 - \frac{\rho}{n+k} \right) - \frac{\rho \lambda_1 (\rho \lambda_1 - 3 \rho + 1)}{n+k+1} \right\} \\
- \rho \lambda_1 (1 - \lambda_1) \left( 1 - \exp\left( -\exp\left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right) \right\} \left( 1 - \lambda_2 \right) + \lambda_2 \left( 1 - \exp\left( -\exp\left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right) \right\} \right\} dy \\
M_{Y_{i:n}}(t) = \sum_{i \geq n-r+1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{i-n+r+j+k-1} \frac{(i-1) (n-1) (i-j) j ! \lambda_1^j \lambda_2^j}{\sigma_2^2} \int_{-\infty}^{+\infty} e^{ty} \left\{ 1 - \exp\left( -\exp\left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{a_2-1} \left\{ \exp\left( -\exp\left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ \exp\left( -\exp\left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left( 1 - \lambda_2 \right) + 2\lambda_2 \left( 1 - \exp\left( -\exp\left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right) \right\} \right\} \left\{ \left( \frac{(1 + \rho)(1 - \lambda_1)}{n+k} - \frac{\rho + \lambda_1 (\rho \lambda_1 - 3 \rho + 1)}{n+k+1} - \frac{\rho \lambda_1}{n+k+3} - \frac{2\rho \lambda_1 (1 - \lambda_1)}{n+k+2} \right) \right\} \\
\int_{-\infty}^{+\infty} e^{ty} \left\{ 1 - \exp\left( -\exp\left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{a_2-1} \left\{ \exp\left( -\exp\left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ \exp\left( -\exp\left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left( 1 - \lambda_2 \right) + 2\lambda_2 \left( 1 - \exp\left( -\exp\left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right) \right\} dy + \left\{ \left( \frac{\rho \lambda_2^2}{n+k+3} + \frac{\rho \lambda_1 (2 - \lambda_1 - \lambda_2)}{n+k+2} \right) \right\} \\
\int_{-\infty}^{+\infty} e^{ty} \left\{ 1 - \exp\left( -\exp\left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{a_2-1} \left\{ \exp\left( -\exp\left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ \exp\left( -\exp\left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left( 1 - \lambda_2 \right) + 2\lambda_2 \left( 1 - \exp\left( -\exp\left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right) \right\} dy + \left\{ \left( \frac{\rho (1 - 2\lambda_1 - \lambda_2 + \rho \lambda_1)}{n+k+1} - \frac{\rho (1 - \lambda_1)}{n+k+1} \right) \right\} \\
\int_{-\infty}^{+\infty} e^{ty} \left\{ 1 - \exp\left( -\exp\left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{2a_2-1} \left\{ \exp\left( -\exp\left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ \exp\left( -\exp\left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left( 1 - \lambda_2 \right) + 2\lambda_2 \left( 1 - \exp\left( -\exp\left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right) \right\} dy \]
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\[
\left( \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right) \left( \exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \left[ 1 - \lambda_2 \right] \\
\quad + 2\lambda_2 \left[ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right]^{\alpha_2} \left[ 1 - \lambda_2 + \lambda_2 \left( 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right) \right]^{\alpha_2} \\
\left[ 1 - \lambda_2 + \lambda_2 \left( 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right) \right] dy \]  

(18)

Let \( A = \int_{-\infty}^{+\infty} e^{ty} \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{\alpha_2-1} \left( \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right) \left( \exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \left[ 1 - \lambda_2 + \lambda_2 \left( 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right) \right]^{\alpha_2} dy \)

Putting \( \exp \left( -\frac{y - \mu_2}{\sigma_2} \right) = u \) and using series representation for \((1 + z)^a\); in \( A \) we get

\[
A = (1 - \lambda_2)\sigma_2 e^{t\mu_2} \sum_{l_1=0}^{\infty} (-1)^{l_1} \left( \alpha_2 - 1 \right) \left( \frac{1}{l_1 + 1} \right) \int_0^{+\infty} u^{(1-t\sigma_2)-1} \exp(-(l_1 + 1)u) du + \\
2\lambda_2\sigma_2 e^{t\mu_2} \sum_{l_2=0}^{+\infty} (-1)^{l_2} \left( \frac{2\alpha_2 - 1}{l_2} \right) \left( \frac{1}{l_2 + 1} \right) \int_0^{+\infty} u^{(1-t\sigma_2)-1} \exp(-(l_2 + 1)u) du
\]

After integration we get,

\[
A = (1 - \lambda_2)\sigma_2 e^{t\mu_2} \sum_{l_1=0}^{\infty} (-1)^{l_1} \left( \alpha_2 - 1 \right) \left( \frac{1}{l_1 + 1} \right) \frac{\Gamma(1-t\sigma_2)}{\Gamma(1-t\sigma_2)} + \\
2\lambda_2\sigma_2 e^{t\mu_2} \sum_{l_2=0}^{+\infty} (-1)^{l_2} \left( \frac{2\alpha_2 - 1}{l_2} \right) \left( \frac{1}{l_2 + 1} \right) \frac{\Gamma(1-t\sigma_2)}{\Gamma(1-t\sigma_2)}
\]  

(19)

Also let
\[ B = \int_{-\infty}^{+\infty} e^{ty} \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{2\alpha_2 - 1} \left( \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right) \left( \exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \]

\[ \left[ 1 - \lambda_2 + 2\lambda_2 \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{\alpha_2} \right] [1 - \lambda_2 \]

\[ + \lambda_2 \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{\alpha_2} \] dy

Putting \( \left( \exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) = u \), and using series representation for \((1 + z)^a\); in \( B \) we get

\[ B = (1 - \lambda_2)^2 \sigma_2 e^{\mu_2} \sum_{l_3=0}^{\infty} (-1)^{l_3} \left( \frac{2\alpha_2 - 1}{l_3} \right) \int_0^{+\infty} u^{(1-t\sigma_2) - 1} \exp(-l_3 + 1)(l_3 + 1)u) \] du +

\[ 3\lambda_2(1 - \lambda_2)\sigma_2 e^{\mu_2} \sum_{l_4=0}^{\infty} (-1)^{l_4} \left( \frac{3\alpha_2 - 1}{l_4} \right) \int_0^{+\infty} u^{(1-t\sigma_2) - 1} \exp(-l_4 + 1)(l_4 + 1)u) \] du +

\[ 2\lambda_2^2 \sigma_2 e^{\mu_2} \sum_{l_5=0}^{+\infty} (-1)^{l_5} \left( \frac{4\alpha_2 - 1}{l_5} \right) \int_0^{+\infty} u^{(1-t\sigma_2) - 1} \exp(-l_5 + 1)(l_5 + 1)u) \] du

After integration we get,

\[ B = (1 - \lambda_2)^2 \sigma_2 e^{\mu_2} \sum_{l_3=0}^{\infty} (-1)^{l_3} \left( \frac{2\alpha_2 - 1}{l_3} \right) \frac{\Gamma(1-t\sigma_2)}{(l_3 + 1)^{1-t\sigma_2}} \]

\[ + 3\lambda_2(1 - \lambda_2)\sigma_2 e^{\mu_2} \sum_{l_4=0}^{\infty} (-1)^{l_4} \left( \frac{3\alpha_2 - 1}{l_4} \right) \frac{\Gamma(1-t\sigma_2)}{(l_4 + 1)^{1-t\sigma_2}} \]

\[ + 2\lambda_2^2 \sigma_2 e^{\mu_2} \sum_{l_5=0}^{+\infty} (-1)^{l_5} \left( \frac{4\alpha_2 - 1}{l_5} \right) \frac{\Gamma(1-t\sigma_2)}{(l_5 + 1)^{1-t\sigma_2}} \] (20)

Using (19) and (20), in equation (18) we get the expression for the moment generating function of the concomitants of \( r^{th} \) order statistics as
$$M_{Y_{[r:n]}}(t) = \sum_{i=n-r+1}^{n} \sum_{j=0}^{\infty} (-1)^{i-n+r+j+k-1} \binom{i-1}{n-r} \binom{n}{i} \binom{t}{j} \frac{i \lambda_1^j \alpha_2}{\sigma_2}$$

$$\left\{ \frac{(1-\rho)(1-\lambda_1)}{n+k} - \frac{\rho + \lambda_1(\rho \lambda_1 - 3 \lambda_1 + 1)}{n+k+1} - \frac{2 \rho \lambda_1(1-\lambda_1)}{n+k+2} - \frac{\rho \lambda_1}{n+k+3} \right\}$$

$$\left\{ (1-\lambda_2) \sigma_2 e^{t \mu_2} \sum_{l_1=0}^{\infty} (-1)^{l_1} \left( \frac{\alpha_2 - 1}{l_1} \right) \frac{\Gamma(1-t \sigma_2)}{(l_1 + 1)^{1-t \sigma_2}} + 2 \lambda_2 \sigma_2 e^{t \mu_2} \sum_{l_2=0}^{\infty} (-1)^{l_2} \left( \frac{2 \alpha_2 - 1}{l_2} \right) \frac{\Gamma(1-t \sigma_2)}{(l_2 + 1)^{1-t \sigma_2}} \right\}$$

$$\left\{ \frac{\rho \lambda_1^2}{n+k+3} + \frac{\rho \lambda_1(2-\lambda_1 - \lambda_2)}{n+k+2} + \frac{\rho(1-2 \lambda_1 - \lambda_2 + \rho \lambda_1)}{n+k+1} - \frac{\rho(1-\lambda_1)}{n+k} \right\}$$

$$\left\{ (1-\lambda_2)^2 \sigma_2 e^{t \mu_2} \sum_{l_3=0}^{\infty} (-1)^{l_3} \left( \frac{2 \alpha_2 - 1}{l_3} \right) \frac{\Gamma(1-t \sigma_2)}{(l_3 + 1)^{1-t \sigma_2}} + 3 \lambda_2(1-\lambda_2) \sigma_2 e^{t \mu_2} \sum_{l_4=0}^{\infty} (-1)^{l_4} \left( \frac{3 \alpha_2 - 1}{l_4} \right) \frac{\Gamma(1-t \sigma_2)}{(l_4 + 1)^{1-t \sigma_2}} + 2 \lambda_2^2 \sigma_2 e^{t \mu_2} \sum_{l_5=0}^{\infty} (-1)^{l_5} \left( \frac{4 \alpha_2 - 1}{l_5} \right) \frac{\Gamma(1-t \sigma_2)}{(l_5 + 1)^{1-t \sigma_2}} \right\}$$

(21)

5.3. Moments of Concomitants of Order Statistics

In this section, we have deduced the expression for the moment of concomitants of $r^{th}$ order statistics when random variables $(X_i, Y_i); (i = 1, 2, 3, ..., n)$ are i.i.d. and follows TEGD. Utilizing these results, we can compute means and variates of concomitants of $r^{th}$ order statistics. According to David [6], the $j^{th}$ moment about origin of concomitants of $r^{th}$ order statistics $Y_{[r:n]}$ is given by

$$\mu_{Y_{[r:n]}}^{(j)} = \int_{-\infty}^{+\infty} y^{j} g_{[r:n]}(y) dy$$

Thus we get the $p^{th}$ moment about origin for the concomitants of $r^{th}$ order statistics $Y_{[r:n]}$ for TEGD as
\[
\begin{align*}
\therefore \mu_{Y_{[r:n]}}^{p/} &= \int_{-\infty}^{+\infty} y^p g_{[r:n]}(y) \, dy \\
\therefore \mu_{Y_{[r:n]}}^{p/} &= \sum_{i=n-r+1}^{n} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{i-n+r+j+k-1} \binom{i-1}{n-r} \binom{n}{i} \binom{j-1}{k} \frac{i \lambda_1 \alpha_2}{\sigma_2} \int_{-\infty}^{+\infty} y^p \\ &\cdot \left[ 1 - \exp \left( - \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \right) \right] \left\{ \exp \left( - \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \right\} \left[ 1 - \lambda_2 \right] \\
&\cdot 2 \lambda_2 \left\{ 1 - \exp \left( - \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{a_2} \left[ \frac{(1 - \rho)(1 - \lambda_1)}{n + k} - \frac{\rho + \lambda_1 (\rho \lambda_1 - 3 \rho + 1)}{n + k + 1} \right] \\
&\cdot \frac{2 \rho \lambda_1 (1 - \lambda_1)}{n + k + 2} - \frac{\rho \lambda_1}{n + k + 3} \\
&\cdot \left\{ \frac{\rho \lambda_1^2}{n + k + 3} + \frac{\rho \lambda_1 (2 - \lambda_1 - \lambda_2)}{n + k + 2} + \frac{\rho (1 - 2 \lambda_1 - \lambda_2 + \rho \lambda_1)}{n + k + 1} \right\} \\
&\cdot \left\{ 1 - \exp \left( - \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{a_2} \left[ 1 - \lambda_2 \right] \\
&\cdot \lambda_2 \left\{ 1 - \exp \left( - \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{a_2} \right] dy
\end{align*}
\]
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\[ \mu_{Y[r:n]} = \sum_{i=n-r+1}^{n} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{i-n+r+j+k-1} \binom{n}{i} \binom{j}{k} \frac{i\lambda_1}{\sigma_2} \left( \frac{(1-\rho)(1-\lambda_1)}{n+k} \right) \]

\[ = \frac{\rho + \lambda_1(\rho \lambda_1 - 3 \rho + 1)}{n+k + 1} - \frac{2 \rho \lambda_1(1 - \lambda_1)}{n+k + 2} \]

\[ - \frac{\rho \lambda_1}{n+k+3} \int_{-\infty}^{+\infty} y^p \left( \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right) \left\{ \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}\left\{ \exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right\} \left[ 1 - \lambda_2 \right] \]

\[ + 2 \lambda_2 \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{\alpha_2} \left[ 1 - \lambda_2 \right] \]

\[ + \lambda_2 \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{\alpha_2} \left[ 1 - \lambda_2 \right] \]

\[ + \frac{\rho \lambda_1^2}{n+k+3} + \frac{\rho \lambda_1(2 - \lambda_1 - \lambda_2)}{n+k+2} + \frac{\rho(1 - 2\lambda_1 - \lambda_2 + \rho \lambda_1)}{n+k+1} \]

\[ - \frac{\rho(1 - \lambda_1)}{n+k} \int_{-\infty}^{+\infty} y^p \left( \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right) \left\{ \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}\left\{ \exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right\} \left[ 1 - \lambda_2 \right] \]

\[ + 2 \lambda_2 \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{\alpha_2} \left[ 1 - \lambda_2 \right] \]

\[ + \lambda_2 \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{\alpha_2} \left[ 1 - \lambda_2 \right] \]

\[ + \frac{\rho \lambda_1^2}{n+k+3} + \frac{\rho \lambda_1(2 - \lambda_1 - \lambda_2)}{n+k+2} + \frac{\rho(1 - 2\lambda_1 - \lambda_2 + \rho \lambda_1)}{n+k+1} \]

\[ - \frac{\rho(1 - \lambda_1)}{n+k} \int_{-\infty}^{+\infty} y^p \left( \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right) \left\{ \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}\left\{ \exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right\} \left[ 1 - \lambda_2 \right] \]

\[ + 2 \lambda_2 \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{\alpha_2} \left[ 1 - \lambda_2 \right] \]

\[ + \lambda_2 \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{\alpha_2} \left[ 1 - \lambda_2 \right] \]

\[ = \int_{-\infty}^{+\infty} y^p \left( 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right) \left\{ \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}\left\{ \exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right\} \left[ 1 - \lambda_2 \right] \]

\[ + 2 \lambda_2 \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{\alpha_2} \left[ 1 - \lambda_2 \right] \]

\[ + \lambda_2 \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{\alpha_2} \left[ 1 - \lambda_2 \right] \]

\[ \{ \exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \} \left[ 1 - \lambda_2 + 2 \lambda_2 \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{\alpha_2} \} \int_{-\infty}^{+\infty} dy \]
\[ A = (1 - \lambda_2) \int_{-\infty}^{+\infty} y^p \left\{ 1 - \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \right\}^{\alpha_2 - 1} \{ \exp \left( - \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \right) \}^p \left\{ \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \right\} dy + 2 \lambda_2 \int_{-\infty}^{+\infty} y^p \left\{ 1 - \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \right\}^{2\alpha_2 - 1} \{ \exp \left( - \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \right) \}^p \left\{ \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \right\} dy \]

Let
\[ u = \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \]
\[ => - \frac{y - \mu_2}{\sigma_2} = \log u \]
\[ => y = \mu_2 - \sigma_2 \log u \]

Also
\[ \{ \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) \} \left\{ \frac{1}{\sigma_2^p} \right\} dy = du \]
\[ => \exp \left( - \frac{y - \mu_2}{\sigma_2} \right) dy = -\sigma_2 du \]

When
\[ y = -\infty, u = \infty \]
\[ y = \infty, u = 0 \]

Therefore we can write
\[ A = (1 - \lambda_2) \sigma_2 \sum_{k_1=0}^{+\infty} \sum_{l_1=0}^{+\infty} (-1)^{k_1+l_1} \left( \begin{array}{c} p \\ k_1 \end{array} \right) \left( \begin{array}{c} \alpha_2 - 1 \\ l_1 \end{array} \right) \mu_2^{p-k_1} \sigma_2^{k_1} \int_{0}^{+\infty} (\log u)^{k_1} \{ \exp(-l_1 + 1)u \} du + 2 \lambda_2 \sigma_2 \sum_{k_2=0}^{p} \sum_{l_2=0}^{+\infty} (-1)^{k_2+l_2} \left( \begin{array}{c} p \\ k_2 \end{array} \right) \left( \begin{array}{c} 2\alpha_2 - 1 \\ l_2 \end{array} \right) \mu_2^{p-k_2} \sigma_2^{k_2} \int_{0}^{+\infty} (\log u)^{k_2} \{ \exp(-l_2 + 1)u \} du \]
After integration we get;

\[
A = (1 - \lambda_2) \sigma_2 \sum_{k_1=0}^{p} \sum_{l_1=0}^{\infty} (-1)^{k_1+l_1} \binom{p}{k_1} \binom{\alpha_2 - 1}{l_1} \mu_2^{p-k_1} \sigma_2^{k_1} \left( \frac{\partial}{\partial \alpha} \right)^{k_1} [l_1 + 1 - \alpha] |_{\alpha=1} + \\
2\lambda_2 \sigma_2 \sum_{k_2=0}^{p} \sum_{l_2=0}^{\infty} (-1)^{k_2+l_2} \binom{2\alpha_2 - 1}{l_2} \mu_2^{p-k_2} \sigma_2^{k_2} \left( \frac{\partial}{\partial \alpha} \right)^{k_2} [(l_2 + 1 - \alpha] |_{\alpha=1}
\]

Equation (23) is obtained by using Prudnikov, Brychkov and Marichev [1]

Let

\[
B = \int_{-\infty}^{+\infty} y^p \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{2\alpha_2-1} \left\{ \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ \exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right\} dy
\]

\[
B = (1 - \lambda_2)^2 \int_{-\infty}^{+\infty} y^p \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\}^{2\alpha_2-1} \left\{ \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} dy + 2\lambda_2 (1 - \lambda_2) \int_{-\infty}^{+\infty} y^p \left\{ 1 - \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} \left\{ \exp \left( -\exp \left( -\frac{y - \mu_2}{\sigma_2} \right) \right) \right\} dy
\]

Let \( \exp \left( -\frac{y - \mu_2}{\sigma_2} \right) = u \)

\[
=> y = \mu_2 - \sigma_2 \log u
\]

When \( y = -\infty, u = \infty \)
\[ y = \infty, u = 0 \]

and
\[ \exp\left(-\frac{y - \mu_2}{\sigma_2}\right) dy = -\sigma_2 du \]

Thus we get
\[ B = (1 - \lambda_2)^2 \sigma_2 \int_0^\infty (\mu_2 - \sigma_2 \log u)^p (1 - \exp(-u))^{2\alpha_2 - 1} \exp(-u) \, du + \lambda_2 (1 - \lambda_2) \sigma_2 \]
\[ \int_0^\infty (\mu_2 - \sigma_2 \log u)^p (1 - \exp(-u))^{3\alpha_2 - 1} \exp(-u) \, du + 2\lambda_2 (1 - \lambda_2) \sigma_2 \int_0^\infty (\mu_2 - \sigma_2 \log u)^p (1 - \exp(-u))^{4\alpha_2 - 1} \exp(-u) \, du \]

Using binomial expression for \((a + bz)^n\) and series representation for \((1 + z)^a\) above expression can be written as
\[ B = (1 - \lambda_2)^2 \sigma_2 \sum_{k_3=0}^p \sum_{l_3=0}^\infty (-1)^{k_3+l_3} \left(\frac{p}{k_3}\right) \left(\frac{2\alpha_2 - 1}{l_3}\right) \mu^{p-k_3} \sigma_2^{k_3} \int_0^\infty (\log u)^{k_3} (\exp(-l_3 + 1) u) \, du \]
\[ + \lambda_2 (1 - \lambda_2) \sigma_2 \sum_{k_3=0}^p \sum_{l_3=0}^\infty (-1)^{k_3+l_3} \left(\frac{p}{k_3}\right) \left(\frac{3\alpha_2 - 1}{l_3}\right) \mu^{p-k_3} \sigma_2^{k_3} \int_0^\infty (\log u)^{k_3} (\exp(-l_3 + 1) u) \, du \]
\[ + 2\lambda_2 (1 - \lambda_2) \sigma_2 \sum_{k_3=0}^p \sum_{l_3=0}^\infty (-1)^{k_3+l_3} \left(\frac{p}{k_3}\right) \left(\frac{4\alpha_2 - 1}{l_3}\right) \mu^{p-k_3} \sigma_2^{k_3} \int_0^\infty (\log u)^{k_3} (\exp(-l_3 + 1) u) \, du \]
\[ + 2\lambda_2^2 \sigma_2 \sum_{k_3=0}^p \sum_{l_3=0}^\infty (-1)^{k_3+l_3} \left(\frac{p}{k_3}\right) \left(\frac{4\alpha_2 - 1}{l_3}\right) \mu^{p-k_3} \sigma_2^{k_3} \int_0^\infty (\log u)^{k_3} (\exp(-l_3 + 1) u) \, du \]

Finally, by using (2.6.21.1) from Prudnikov et al. [1] in equation (24), and after integration we get
\[ B = (1 - \lambda_2)^2 \sigma_2 \sum_{k_3=0}^p \sum_{l_3=0}^\infty (-1)^{k_3+l_3} \left(\frac{p}{k_3}\right) \left(\frac{2\alpha_2 - 1}{l_3}\right) \mu^{p-k_3} \sigma_2^{k_3} \left(\frac{\partial}{\partial a}\right)^{k_3} \left[(l_3 + 1)^{-a} \right]_{a=1} + \]
\[ \lambda_2 (1 - \lambda_2) \sigma_2 \sum_{k_3=0}^p \sum_{l_3=0}^\infty (-1)^{k_3+l_3} \left(\frac{p}{k_3}\right) \left(\frac{3\alpha_2 - 1}{l_3}\right) \mu^{p-k_3} \sigma_2^{k_3} \left(\frac{\partial}{\partial a}\right)^{k_3} \left[(l_3 + 1)^{-a} \right]_{a=1} + \]
\[ \lambda_2 (1 - \lambda_2) \sigma_2 \sum_{k_3=0}^p \sum_{l_3=0}^\infty (-1)^{k_3+l_3} \left(\frac{p}{k_3}\right) \left(\frac{4\alpha_2 - 1}{l_3}\right) \mu^{p-k_3} \sigma_2^{k_3} \left(\frac{\partial}{\partial a}\right)^{k_3} \left[(l_3 + 1)^{-a} \right]_{a=1} + \]
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\[ 2\lambda_2(1 - \lambda_2)\sigma_2 \sum_{k_5=0}^{p} \sum_{l_5=0}^{\infty} (-1)^{l_5+k_5} \left( \frac{p}{k_5} \right) \left( \frac{3\alpha_2 - 1}{l_5} \right) \mu_2^{p-k_5}\sigma_2^{k_5} \left( \frac{\partial}{\partial a} \right)^{k_5} [(l_5 + 1)^{-\alpha} \left[ a \right]]_{a=1} + \]

\[ 2\lambda_2^2 \sigma_2 \sum_{k_6=0}^{p} \sum_{l_6=0}^{\infty} (-1)^{l_6+k_6} \left( \frac{p}{k_6} \right) \left( \frac{4\alpha_2 - 1}{l_6} \right) \mu_2^{p-k_6}\sigma_2^{k_6} \left( \frac{\partial}{\partial a} \right)^{k_6} [(l_6 + 1)^{-\alpha} \left[ a \right]]_{a=1} \]

Using (23) and (25) in (22), we get the expression for the \( p^{th} \) moment of concomitants of \( r^{th} \) order statistics as

\[ \mu_{[r;n]}^{p/} = \sum_{i=n-r}^{n} \sum_{j=0}^{\infty} \sum_{k_1=0}^{l_1=0} \sum_{k_2=0}^{l_2=0} (-1)^{i+l_1+k_1} \left( \frac{p}{k_1} \right) \left( \frac{\alpha_2 - 1}{l_1} \right) \mu_2^{p-k_1}\sigma_2^{k_1} \left( \frac{\partial}{\partial a} \right)^{k_1} [(l_1 + 1)^{-\alpha} \left[ a \right]]_{a=1} \]

\[ \mu_{[r;n]}^{p/} = \sum_{i=n-r}^{n} \sum_{j=0}^{\infty} \sum_{k_1=0}^{l_1=0} \sum_{k_2=0}^{l_2=0} (-1)^{i+l_1+k_1} \left( \frac{p}{k_1} \right) \left( \frac{\alpha_2 - 1}{l_1} \right) \mu_2^{p-k_1}\sigma_2^{k_1} \left( \frac{\partial}{\partial a} \right)^{k_1} [(l_1 + 1)^{-\alpha} \left[ a \right]]_{a=1} \]

\[ + 2\lambda_2 \sum_{k_2=0}^{l_2=0} (-1)^{k_2+l_2} \left( \frac{p}{k_2} \right) \left( \frac{2\alpha_2 - 1}{l_2} \right) \mu_2^{p-k_2}\sigma_2^{k_2} \left( \frac{\partial}{\partial a} \right)^{k_2} [(l_2 + 1)^{-\alpha} \left[ a \right]]_{a=1} \]

\[ + \left\{ \frac{\rho \lambda_2}{n+k+3} + \frac{\rho \lambda_3(2 - \lambda_1 - \lambda_2)}{n+k+2} + \frac{\rho(1 - 2\lambda_1 - \lambda_2 + \rho \lambda_1)}{n+k+1} \right\} \]

\[- \frac{\rho (1 - \lambda_1)}{n+k} \left\{ \sum_{k_3=0}^{l_3=0} \sum_{k_4=0}^{l_4=0} (-1)^{l_3+k_3} \left( \frac{p}{k_3} \right) \left( \frac{2\alpha_2 - 1}{l_3} \right) \mu_2^{p-k_3}\sigma_2^{k_3} \left( \frac{\partial}{\partial a} \right)^{k_3} [(l_3 + 1)^{-\alpha} \left[ a \right]]_{a=1} \right\} \]

\[ + \lambda_2 (1 - \lambda_2) \sum_{k_4=0}^{l_4=0} (-1)^{l_4+k_4} \left( \frac{p}{k_4} \right) \left( \frac{3\alpha_2 - 1}{l_4} \right) \mu_2^{p-k_4}\sigma_2^{k_4} \left( \frac{\partial}{\partial a} \right)^{k_4} [(l_4 + 1)^{-\alpha} \left[ a \right]]_{a=1} \]

\[ + 2\lambda_2 (1 - \lambda_2) \sum_{k_5=0}^{l_5=0} (-1)^{l_5+k_5} \left( \frac{p}{k_5} \right) \left( \frac{3\alpha_2 - 1}{l_5} \right) \mu_2^{p-k_5}\sigma_2^{k_5} \left( \frac{\partial}{\partial a} \right)^{k_5} [(l_5 + 1)^{-\alpha} \left[ a \right]]_{a=1} \]

\[ + 2\lambda_2^2 \sum_{k_6=0}^{l_6=0} (-1)^{l_6+k_6} \left( \frac{p}{k_6} \right) \left( \frac{4\alpha_2 - 1}{l_6} \right) \mu_2^{p-k_6}\sigma_2^{k_6} \left( \frac{\partial}{\partial a} \right)^{k_6} [(l_6 + 1)^{-\alpha} \left[ a \right]]_{a=1} \left\} \right\} \]
Here we compute some numerical values of the first four moments for some selected values of the parameters which are given in the following tables:

**Table 1:** 1st moment of the concomitants of Order Statistics from BTEGD for selected values of the parameters

| n  | r  | $\mu_1 = 5, \mu_2 = -5, \sigma_1 = 2.5, \sigma_2 = 2.5, \alpha_1 = 2, \alpha_2 = 2$ | $\mu_1 = -10, \mu_2 = -10, \sigma_1 = 3, \sigma_2 = 3, \alpha_1 = 2.5, \alpha_2 = 2.5$ |
|----|----|---------------------------------------------------------------------------------|----------------------------------------------------------------------------------|
| 5  | 1  | 2.653007                                                                        | 2.335468                                                                         |
|    | 2  | 2.792302                                                                        | 3.096762                                                                         |
|    | 3  | 2.931492                                                                        | 3.858057                                                                         |
|    | 4  | 3.070682                                                                        | 4.619358                                                                         |
|    | 5  | 3.209878                                                                        | 5.380615                                                                         |
| 6  | 1  | 2.633099                                                                        | 2.226722                                                                         |
|    | 2  | 2.752538                                                                        | 2.879249                                                                         |
|    | 3  | 2.871844                                                                        | 3.531787                                                                         |
|    | 4  | 2.991145                                                                        | 4.184326                                                                         |
|    | 5  | 3.110454                                                                        | 4.836864                                                                         |
|    | 6  | 3.229755                                                                        | 5.489441                                                                         |
| 7  | 1  | 2.618163                                                                        | 2.145143                                                                         |
|    | 2  | 2.722713                                                                        | 2.716116                                                                         |
|    | 3  | 2.827102                                                                        | 3.287086                                                                         |
|    | 4  | 2.931492                                                                        | 3.858081                                                                         |
|    | 5  | 3.035889                                                                        | 4.429027                                                                         |
|    | 6  | 3.140276                                                                        | 5.000003                                                                         |
|    | 7  | 3.244664                                                                        | 5.570966                                                                         |
|    |     |     |
|----|-----|-----|
| 8  | 1   | 2.606542 |
|    | 2   | 2.699518 |
|    | 3   | 2.792308 |
|    | 4   | 2.865098 |
|    | 5   | 2.977891 |
|    | 6   | 3.070682 |
|    | 7   | 3.162295 |
|    | 8   | 3.257058 |
|    |     | 2.081703 |
| 9  | 1   | 2.597241 |
|    | 2   | 2.680958 |
|    | 3   | 2.764468 |
|    | 4   | 2.847981 |
|    | 5   | 2.931498 |
|    | 6   | 3.015001 |
|    | 7   | 3.098516 |
|    | 8   | 3.182033 |
|    | 9   | 3.265541 |
|    |     | 2.030949 |
| 10 | 1   | 2.589626 |
|    | 2   | 2.665775 |
|    | 3   | 2.741691 |
|    | 4   | 2.817613 |
|    | 5   | 2.893536 |
|    | 6   | 2.969456 |
|    | 7   | 3.045377 |
|    | 8   | 3.121293 |
|    | 9   | 3.197217 |
|    | 10  | 3.273134 |
Table 2: $2^{nd}$ moment of the Concomitants of Order Statistics from BTEGD for selected values of
the parameters

| $n$ | $r$ | $\mu_1 = 5, \mu_2 = -5, \sigma_1 = 2.5, \sigma_2 = 2.5, \alpha_1 = 2, \alpha_2 = 2, \rho = -0.5, \lambda_1 = -0.5, \lambda_2 = -0.5$ | $\mu_1 = -10, \mu_2 = -10, \sigma_1 = 3, \sigma_2 = 3, \alpha_1 = 2.5, \alpha_2 = 2.5, \rho = -1, \lambda_1 = 1, \lambda_2 = 1$ |
|-----|-----|-------------------------------------------------|-------------------------------------------------|
| 5   | 1   | 16.74247                                        | 26.44601                                        |
|     | 2   | 17.93063                                        | 34.17772                                        |
|     | 3   | 19.11811                                        | 41.9094                                         |
|     | 4   | 20.3056                                         | 49.6411                                         |
|     | 5   | 21.49317                                        | 57.37285                                        |
| 6   | 1   | 16.57269                                        | 25.34148                                        |
|     | 2   | 17.59136                                        | 31.96866                                        |
|     | 3   | 18.6092                                         | 38.59582                                        |
|     | 4   | 19.62704                                        | 45.22538                                        |
|     | 5   | 20.64487                                        | 51.85032                                        |
|     | 6   | 21.66273                                        | 58.47742                                        |
| 7   | 1   | 16.44532                                        | 24.5131                                         |
|     | 2   | 17.33691                                        | 30.31188                                        |
|     | 3   | 18.22751                                        | 36.11063                                        |
|     | 4   | 19.11811                                        | 41.90942                                        |
|     | 5   | 20.00877                                        | 47.70822                                        |
|     | 6   | 20.89933                                        | 53.50699                                        |
|     | 7   | 21.78997                                        | 59.30575                                        |
| 8   | 1   | 16.34621                                        | 23.86878                                        |
|     | 2   | 16.72082                                        | 29.02325                                        |
|     | 3   | 17.13901                                        | 34.17772                                        |
|     | 4   | 17.93062                                        | 39.33222                                        |
|   |   |   |   |
|---|---|---|---|
| 5 | 18.72229 | 44.48665 |
| 6 | 19.51398 | 49.64116 |
| 7 | 20.30566 | 54.79555 |
| 8 | 21.88893 | 59.95018 |
| 1 | 16.26691 | 23.35333 |
| 2 | 16.98067 | 27.99243 |
| 3 | 17.69313 | 32.63138 |
| 4 | 18.40563 | 37.27003 |
| 5 | 19.11814 | 41.90948 |
| 6 | 19.83066 | 46.54841 |
| 7 | 20.5431  | 51.18753 |
| 8 | 21.25557 | 55.82651 |
| 9 | 21.9681  | 60.46554 |
| 1 | 16.20199 | 22.93162 |
| 2 | 16.85113 | 27.14891 |
| 3 | 17.49882 | 31.36619 |
| 4 | 18.14655 | 35.58346 |
| 5 | 18.79427 | 39.80078 |
| 6 | 19.44201 | 44.01819 |
| 7 | 20.08974 | 48.23534 |
| 8 | 20.73741 | 52.45268 |
| 9 | 21.38508 | 56.66997 |
| 10| 22.03288 | 60.88725 |
| $n$ | $r$ | $\mu_1 = 5, \mu_2 = -5, \sigma_1 = 2.5, \sigma_2 = 2.5, \alpha_1 = 2, \alpha_2 = 2, \rho = -0.5, \lambda_1 = -0.5, \lambda_2 = -0.5$ | $\mu_1 = -10, \mu_2 = -10, \sigma_1 = 3, \sigma_2 = 3, \alpha_1 = 2.5, \alpha_2 = 2.5, \rho = -1, \lambda_1 = 1, \lambda_2 = 1$ |
|-----|-----|-------------------------------------------------|-------------------------------------------------|
| 5   | 1   | 46.21034                                       | 304.2023                                       |
|     | 2   | 46.36356                                       | 384.429                                        |
|     | 3   | 46.51494                                       | 464.5551                                       |
|     | 4   | 46.66646                                       | 540.791                                        |
|     | 5   | 46.81796                                       | 542.4345                                       |
| 6   | 1   | 46.18826                                       | 292.7411                                       |
|     | 2   | 46.3203                                        | 361.5084                                       |
|     | 3   | 46.45004                                       | 430.2701                                       |
|     | 4   | 46.5799                                        | 498.8401                                       |
|     | 5   | 46.70976                                       | 561.7638                                       |
|     | 6   | 46.83957                                       | 538.5677                                       |
| 7   | 1   | 46.17162                                       | 284.1452                                       |
|     | 2   | 46.28786                                       | 344.3163                                       |
|     | 3   | 46.40136                                       | 404.4554                                       |
|     | 4   | 46.51493                                       | 464.6494                                       |
|     | 5   | 46.62895                                       | 524.4826                                       |
|     | 6   | 46.74226                                       | 576.6806                                       |
|     | 7   | 46.8558                                        | 532.2156                                       |
| 8   | 1   | 46.15857                                       | 277.4594                                       |
|     | 2   | 46.26278                                       | 330.9449                                       |
|     | 3   | 46.3635                                        | 384.4287                                       |
|     | 4   | 46.46446                                       | 437.915                                        |
|   |   |   |   |
|---|---|---|---|
|   |   |   |   |
| 5 | 46.56582 | 491.384 |
| 6 | 46.66684 | 544.3416 |
| 7 | 46.75886 | 587.4596 |
| 8 | 46.86842 | 524.3235 |
| 9 |   |   |   |
| 1 | 46.14808 | 272.1111 |
| 2 | 46.24251 | 320.2479 |
| 3 | 46.3332  | 368.3824 |
| 4 | 46.42407 | 416.5217 |
| 5 | 46.51513 | 464.6573 |
| 6 | 46.6062  | 512.7646 |
| 7 | 46.69968 | 560.13  |
| 8 | 46.78758 | 595.2691 |
| 9 | 46.87883 | 515.4633 |
|10 |   |   |   |
| 1 | 46.13938 | 267.735 |
| 2 | 46.22629 | 311.4938 |
| 3 | 46.30853 | 355.2529 |
| 4 | 46.39105 | 399.0171 |
| 5 | 46.47393 | 442.7771 |
| 6 | 46.55676 | 486.537 |
| 7 | 46.63924 | 530.2497 |
| 8 | 46.72145 | 572.9357 |
| 9 | 46.80406 | 600.8519 |
|10 | 46.88709 | 505.9661 |
Table 4: $4^{th}$ moment of the Concomitants of Order Statistics from BTEGD for selected values of the parameters

| $n$ | $r$ | $\mu_1 = 5, \mu_2 = -5, \sigma_1 = 2.5, \sigma_2 = 2.5, \alpha_1 = 2, \alpha_2 = 2, \rho = -0.5, \lambda_1 = -0.5, \lambda_2 = -0.5$ | $\mu_1 = -10, \mu_2 = -10, \sigma_1 = 3, \sigma_2 = 3, \alpha_1 = 2.5, \alpha_2 = 2.5, \rho = -1, \lambda_1 = 1, \lambda_2 = 1$ |
|-----|-----|-------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------|
| 5   | 1   | 120.7733                                                                                         | 355.6867                                                                                         |
|     | 2   | 139.8303                                                                                         | 440.4934                                                                                         |
|     | 3   | 157.3197                                                                                         | 525.1884                                                                                         |
|     | 4   | 174.7976                                                                                         | 605.5808                                                                                         |
|     | 5   | 191.3078                                                                                         | 603.488                                                                                          |
| 6   | 1   | 117.9617                                                                                         | 343.571                                                                                          |
|     | 2   | 134.8312                                                                                         | 416.265                                                                                          |
|     | 3   | 149.8283                                                                                         | 488.9521                                                                                         |
|     | 4   | 164.811                                                                                          | 561.427                                                                                          |
|     | 5   | 179.791                                                                                          | 627.6618                                                                                         |
|     | 6   | 193.6115                                                                                         | 598.6534                                                                                         |
| 7   | 1   | 115.7753                                                                                         | 334.4848                                                                                         |
|     | 2   | 131.0799                                                                                         | 398.0905                                                                                         |
|     | 3   | 144.2097                                                                                         | 461.6975                                                                                         |
|     | 4   | 157.3208                                                                                         | 525.2939                                                                                         |
|     | 5   | 170.4295                                                                                         | 588.5272                                                                                         |
|     | 6   | 183.5355                                                                                         | 643.3157                                                                                         |
|     | 7   | 195.2905                                                                                         | 591.2097                                                                                         |
| 8   | 1   | 114.0061                                                                                         | 327.4168                                                                                         |
|     | 2   | 128.1599                                                                                         | 383.9558                                                                                         |
|     | 3   | 139.8398                                                                                         | 440.4935                                                                                         |
|     | 4   | 151.4931                                                                                         | 497.0336                                                                                         |
From the above tables we can see that the numerical values of moments of concomitants of order statistics of BTEGD shown an increasing trend. For fixed values of $n$ and for the fixed values of the parameters, the values of moments increase for increasing values of $r$. 

|   |   |   |
|---|---|---|
| 5 | 163.1463 | 553.5519 |
| 6 | 174.7995 | 609.5107 |
| 7 | 186.4476 | 654.5829 |
| 8 | 196.5538 | 582.1563 |
| 1 | 112.5292 | 321.763 |
| 2 | 125.8216 | 372.648 |
| 3 | 136.3437 | 423.5304 |
| 4 | 146.8319 | 474.4184 |
| 5 | 157.3202 | 525.3023 |
| 6 | 167.8075 | 576.1526 |
| 7 | 178.2954 | 626.1872 |
| 8 | 188.7768 | 662.6969 |
| 9 | 197.5259 | 572.0886 |
| 1 | 111.2651 | 317.1369 |
| 2 | 123.9062 | 363.3961 |
| 3 | 133.4833 | 409.6512 |
| 4 | 143.0181 | 455.9151 |
| 5 | 152.5525 | 502.174 |
| 6 | 162.0869 | 548.4318 |
| 7 | 171.6213 | 594.6379 |
| 8 | 181.1557 | 639.7085 |
| 9 | 190.682 | 668.4436 |
| 10 | 198.2868 | 561.3813 |
6. Conclusion

In this paper, we have studied the Morgenstern Type Bivariate Transmuted Exponentiated Gumbel Distribution and their concomitants of order statistics when a sample is available from a Bivariate Transmuted Exponentiated Gumbel Distribution. We have obtained the distribution of the concomitants of \( r^{th} \) order statistics, moment generating function and the distribution of moments of the concomitants from BTEGD. Numerical computations have been done for the first four moments of the concomitants of order statistics for some selected values of the parameters, which show an increasing trend.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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