A Taxonomy of Morphic Sequences

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Abstract

In this note we classify sequences according to whether they are morphic, pure morphic, uniform morphic, pure uniform morphic, primitive morphic, or pure primitive morphic, and for each possibility we either give an example or prove that no example is possible.
1 Introduction

Sequences (also known as right-infinite words; we use the terms interchangeably) that arise from the iteration of a morphism appear in many different places in the mathematical and theoretical computer science literature. To name just a few examples, they play an important role in the theory of words avoiding patterns [39, 40, 9], in the theory of Sturmian sequences [11], in L-systems used in computer graphics [34], and in the theory of algebraic series in positive characteristic [17, 18].

Different kinds of morphisms give rise to different kinds of sequences, with different properties. In this paper we consider some of these variations and properties and classify sequences according to whether they do, or do not, satisfy these properties. The goal is to create a relatively complete taxonomy of the different kinds of behavior exhibited by morphic sequences, as well as to illustrate the many different techniques that can be used to show that a sequence exhibits, or does not exhibit, a certain property.

Let us fix our notation. Let $\Sigma^*$ denote the set of all finite words over the alphabet $\Sigma$, including the empty word $\epsilon$, and let $\Sigma^\omega$ denote the set of all right-infinite words over $\Sigma$. We write $\Sigma^\infty = \Sigma^* \cup \Sigma^\omega$. Let $h : \Sigma^* \to \Sigma^*$ be a morphism, that is, a map obeying $h(xy) = h(x)h(y)$ for all words $x, y \in \Sigma^*$. From the definition, it suffices to define $h$ on each element of $\Sigma$. A morphism $h : \Sigma \to \Delta$ is said to be non-erasing if $h(a) \neq \epsilon$ for all $a \in \Sigma$. A letter $a$ is said to be growing for $h$ if $\lim_{n \to \infty} |h^n(a)| = \infty$; otherwise it is non-growing.

We can define $h$ on $\Sigma^\omega$ in the obvious way: $h(a_1a_2\cdots) = h(a_1)h(a_2)\cdots$.

We can iterate $h$, writing $h^2$ for the composition $h \circ h$, $h^3$ for $h \circ h \circ h$, etc. If there exists a letter $a \in \Sigma$ and $x \in \Sigma^*$ such that

1. $h(a) = ax$; and
2. $h^i(x) \neq \epsilon$ for all $i \geq 0$,

then $h$ is said to be prolongable on $a$. In this case, iterating $h$ on $a$ produces a sequence of words of increasing length,

$a, h(a), h^2(a), \ldots$

where each word is a proper prefix of the word that follows. In the limit, this tends to the infinite word

$h^\omega(a) := ax h(x) h^2(x) h^3(x) \cdots \in \Sigma^\omega$,

which is a fixed point of $h$; that is,

$h(h^\omega(a)) = h^\omega(a)$.

We say that the infinite word $h^\omega(a)$ is generated by $h$.

If $w$ is a finite word, then $|w|$ denotes the length of $w$. If $w$ is a finite or infinite word, then $w[n]$ denotes the $n$’th symbol of $w$, and $w[i..j]$ represents the factor of $w$ beginning at position $i$ and ending at position $j$.

If $x = uvw$, then we say that $u$ is a prefix of $x$, that $w$ is a suffix of $x$, and $v$ is a factor of $x$. 

2
Let $\Sigma_{k} = \{0, 1, \ldots, k - 1\}$. If $w \in \Sigma_{k}^{*}$, we let $[w]_{k}$ denote the integer represented by $w$ in base $k$.

We start by discussing different kinds of morphisms.

1.1 Pure morphic words

If $w$ is an infinite word over $\Sigma$ and there exists a morphism $h$, prolongable on $a$, such that $w = h^{\omega}(a)$, then $w$ is said to be pure morphic.

**Example 1.** One of the most famous pure morphic words is the Fibonacci word

$$f = 01001010 \cdots,$$

which is generated by the morphism $\varphi$ defined by

$$0 \to 01; \quad 1 \to 0.$$ 

See, e.g., [7, 8].

1.2 Pure uniform morphic words

A morphism $h$ is said to be $k$-uniform if $|h(a)| = k$ for all $a \in \Sigma$. It is said to be uniform if it is uniform for some $k \geq 2$. If an infinite word is generated by a uniform prolongable morphism, then it is said to be pure uniform morphic.

**Example 2.** One of the most famous pure uniform morphic words is the Thue-Morse word

$$t = 01101001 \cdots,$$

which is generated by the morphism $\mu$ defined by

$$0 \to 01; \quad 1 \to 10.$$ 

See, for example, [39, 40, 10, 3].

1.3 Morphic words

We can also apply a coding (a 1-uniform morphism from $\Sigma$ to a possibly different alphabet $\Delta$) to a morphic word. This has the effect of renaming the symbols. One way to think about this is to give letters in $\Sigma$ subscripts, and then the effect of $\tau$ is to erase the subscripts. We use the shorthand $a_1 a_2 \cdots a_n \to b_1 b_2 \cdots b_n$ to represent the coding $a_i \to b_i$ for $i = 1, 2, \ldots, n$.

If an infinite word is expressible as the coding of a pure morphic word, it is said to be morphic.
Example 3. The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. The characteristic word $\chi_F = (\chi(n))_{n \geq 0}$ of the Fibonacci numbers is defined by

$$\chi(n) = \begin{cases} 1, & \text{if } n = F_i \text{ for some } i \geq 0; \\ 0, & \text{otherwise}. \end{cases} \quad (1)$$

Thus

$$\chi = 11110100100001 \cdots.$$ 

This word is morphic, as it is generated by the morphism sending

$$
\begin{align*}
c_0 &\rightarrow c_0e_0 \\
e_0 &\rightarrow c_1 \\
c_1 &\rightarrow c_1e_1 \\
e_1 &\rightarrow c_2 \\
c_2 &\rightarrow c_2e_1
\end{align*}
$$

followed by the coding $c_0e_0c_1e_1c_2 \rightarrow 11100$. See, for example, [20, Example 7, p. 168].

1.4 Uniform morphic words

If an infinite word is expressible as the coding of a pure uniform morphic word, it is said to be uniform morphic. From Cobham’s theorem [20], we know that an infinite word is uniform morphic if and only if it is automatic. (A $k$-automatic word is one where the $n$’th term is computed by a finite automaton reading the base-$k$ expansion of $n$ as input and producing an output associated with the last state reached; a word is said to be automatic if it is $k$-automatic for some integer $k \geq 2$.)

Example 4. Consider the uniform morphic word $s$ generated by the morphism

$$
\begin{align*}
a &\rightarrow ab \\
b &\rightarrow ac \\
c &\rightarrow db \\
d &\rightarrow dc.
\end{align*}
$$

Apply the coding sending $abcd \rightarrow 0011$ to $s$; the result is

$$r = 0001001000011101 \cdots,$$

the Golay-Rudin-Shapiro sequence. (See, for example, [28, 29, 38, 37].) It is not hard to see that $r[n]$ equals the parity of the number of (possibly overlapping) occurrences of 11 in the binary expansion of $n$. 

1.5 Pure primitive morphic words

A morphism \( h : \Sigma^* \to \Sigma^* \) is called \textit{primitive} if there exists an integer \( n \geq 1 \) such that for all \( a \in \Sigma \), the word \( h^n(a) \) contains at least one occurrence of each symbol in \( \Sigma \). For example, the Fibonacci morphism \( \varphi \) is primitive, since \( \varphi^2(0) = 010 \) and \( \varphi^2(1) = 01 \). An infinite word \( w \) is \textit{pure primitive morphic} if it is the fixed point of some primitive morphism.

**Example 5.** Consider the morphism \( \gamma \) defined by

\[
\begin{align*}
\gamma(0) &= 03 \\
\gamma(1) &= 43 \\
\gamma(3) &= 1 \\
\gamma(4) &= 01.
\end{align*}
\]

Then

\[
\begin{align*}
\gamma^6(0) &= 031430110343430310110111 \\
\gamma^6(1) &= 03143011031011031031011011 \\
\gamma^6(3) &= 03143034303434343 \\
\gamma^6(4) &= 03143011034343031430343430343343,
\end{align*}
\]

so \( \gamma \) is primitive. The infinite word 031430110343430310110111\ldots generated by \( \gamma \) is thus pure primitive morphic. It was studied in [16] and has the interesting property that it avoids additive cubes, that is, three consecutive blocks of the same length and same sum.

1.6 Primitive morphic words

An infinite word is \textit{primitive morphic} if it is the image, under a coding, of some primitive morphism.

**Example 6.** An interesting example of a primitive morphic word appears in [22]. Consider the morphism \( g \) and coding \( \rho \) defined as follows:

\[
\begin{align*}
g(a) &= abcab & \rho(a) &= 0 \\
g(b) &= cda & \rho(b) &= 0 \\
g(c) &= cdacd & \rho(c) &= 1 \\
g(d) &= abc & \rho(d) &= 1
\end{align*}
\]

and form the infinite word \( R = \rho(g^\omega(a)) \). By considering \( g^2 \) we see that \( g \) is primitive and so \( R \) is primitive morphic. The word \( R \), sometimes called the Rote-Fibonacci word, is an aperiodic word that avoids the pattern \( xxx^R \), where \( x^R \) denotes the reversal of \( x \).
1.7 Uniform primitive morphic words

Finally, an infinite word is pure uniform primitive morphic if it is the fixed point of a primitive morphism that is also uniform, and uniform primitive morphic if it is the image, under a coding of such a word.

1.8 Recurrence and uniform recurrence

An infinite word $w$ is said to be recurrent if every finite factor $x$ appearing in $w$ appears infinitely often in $w$. If, in addition, for each factor $x$ there is a constant $c(x)$ such that every two consecutive occurrences of $x$ in $w$ are separated by no more than $c(x)$ symbols, then we say that $w$ is uniformly recurrent. Synonyms for uniformly recurrent in the literature include “almost periodic” [20] and “minimal” [25].

A basic result connecting uniform recurrence with other properties is the following theorem of Cobham [20, Theorem 5, p. 178]:

**Theorem 7.**

(a) If $w$ is a primitive morphic word, then it is uniformly recurrent.

(b) If $w$ is uniformly recurrent and uniform morphic, then it is uniform primitive morphic.

We now mention some other useful results:

**Theorem 8.** Let $u$ be a fixed point of a morphism $\sigma$ “without useless letters” (i.e., the alphabet on which $\sigma$ is defined is exactly the set of letters occurring in $u$). Furthermore suppose that for each letter $a$ in $u$ the length of the iterates $|\sigma^k(a)|$ tends to infinity. If the sequence $u$ is uniformly recurrent, then $\sigma$ is primitive.

**Proof.** By the hypothesis on the letters being all useful, it follows that for every letter $a$ that $\sigma^k(a)$ is a factor of $u$ for all $k \geq 0$. Since $u$ is uniformly recurrent, for every letter $b$ there exists a length $\ell_b$ such that every factor of $u$ of length $\geq \ell_b$ contains at least one $b$. Taking $\ell = \max_{b \in \Sigma} \ell_b$, every factor of $u$ of length $\geq \ell$ contains at least one copy of each letter. On the other hand, for each $a$ we know that $|\sigma^k(a)|$ is unbounded. Hence there exists $k_a$ such that for all $k \geq k_a$ one has $|\sigma^k(a)| \geq \ell$. Taking $K = \max k_a$, we see that for each $k \geq K$ and for each letter $a$, the word $\sigma^k(a)$ contains at least one copy of each letter. In other words, $\sigma$ is primitive.

**Corollary 9.** If a sequence is pure uniform morphic and uniformly recurrent, then the sequence is pure uniform primitive morphic.

**Proof.** Clean out the useless letters.

**Corollary 10.** If a sequence is pure uniform morphic and primitive morphic, then the sequence is pure uniform primitive morphic.
Proof. The sequence is uniformly recurrent. Apply the previous corollary.

We recall a result from Queffélec [35, Prop. 5.5, p. 130]:

**Theorem 11.** Let \( h : \Sigma^* \to \Sigma^* \) be a morphism, prolongable on \( a \), and suppose all letters of \( \Sigma \) are growing. Then \( h^\omega(a) \) is uniformly recurrent if and only if \( h \) is primitive.

Finally, we also mention a recent important result of Durand [23, Thm. 3, p. 124]:

**Theorem 12.** Let \( u \) be a morphic sequence that is uniformly recurrent. Then \( u \) is primitive morphic.

**Remark 13.** More precisely, Durand proves in [23] that a sequence is uniformly recurrent and morphic if and only if it is “primitive substitutive”, but his theorem also implies the statement above (F. Durand, private communication, June 2017).

## 2 The classification

From the preceding section, a word can be classified in ten different ways:

- P1: pure morphic
- P2: morphic
- P3: pure uniform morphic
- P4: uniform morphic
- P5: pure primitive morphic
- P6: primitive morphic
- P7: pure uniform primitive morphic
- P8: uniform primitive morphic
- P9: uniformly recurrent
- P10: recurrent

However, these ten properties are clearly not independent. We have the following trivial implications:

- \( P1 \implies P2 \)
- \( P3 \implies P1, P2, P4 \)
- \( P4 \implies P2 \)
• \( P_5 \implies P_1, P_2, P_6 \)
• \( P_6 \implies P_2 \)
• \( P_7 \implies P_1, P_2, P_3, P_4, P_5, P_6, P_8 \)
• \( P_8 \implies P_2, P_4, P_6 \)
• \( P_9 \implies P_{10} \)

Theorem 7 (a) tells us that \( P_6 \implies P_9 \), and Theorem 7 (b) tells us \( (P_6 \text{ and } P_4) \implies P_8 \). Corollary 10 tells us that \( (P_3 \text{ and } P_9) \implies P_7 \). Theorem 12 tells us that \( (P_2 \text{ and } P_9) \implies P_6 \). All these restrictions lower the total number of possibilities from 1024 to 20:

(a) Neither morphic nor recurrent.

(b) Recurrent, but neither morphic nor uniformly recurrent.

(c) Uniformly recurrent, but not morphic.

(d) Morphic; but neither pure morphic, uniform morphic, primitive morphic, nor recurrent.

(e) Morphic and recurrent; but neither pure morphic, uniform morphic, primitive morphic, nor uniformly recurrent.

(f) Primitive morphic; but neither pure morphic nor uniform morphic.

(g) Uniform morphic; but neither pure morphic, primitive morphic, nor recurrent.

(h) Uniform morphic and recurrent; but neither pure morphic nor primitive morphic.

(i) Uniform primitive morphic; but not pure morphic.

(j) Pure morphic; but neither uniform morphic, primitive morphic, nor recurrent.

(k) Pure morphic and recurrent; but neither uniform morphic, primitive morphic, nor uniformly recurrent.

(l) Pure morphic and primitive morphic; but neither uniform morphic nor pure primitive morphic.

(m) Pure primitive morphic; but not uniform morphic.

(n) Pure morphic and uniform morphic, but neither pure uniform morphic, primitive morphic, nor recurrent.

(o) Pure morphic and uniform morphic and recurrent, but neither pure uniform morphic nor primitive morphic.
(p) Pure morphic and uniform primitive morphic; but neither pure uniform morphic nor pure primitive morphic.

(q) Pure primitive morphic and uniform primitive morphic; but not pure uniform morphic.

(r) Pure uniform morphic; but neither primitive morphic nor recurrent.

(s) Pure uniform morphic and recurrent; but not primitive morphic.

(t) Pure uniform primitive morphic.

In this note we give examples of all 20 possibilities. Of course, examples of some of these cases are very well-known; it is our point to collect these examples in one place and to illustrate each of the 20 classes. Inexplicably, the authors of [4] failed to do this explicitly.

Before we get to the examples, we recall several more useful results.

The first few involve frequency of letters. Define the frequency of a symbol $a$ in an infinite word $w$ to be the quantity $\lim_{n \to \infty} \frac{|w[0, n-1]|_a}{n}$, if it exists.

**Theorem 14.** Suppose the frequency $\alpha$ of the letter $a$ in the word $w$ exists. Then

(a) If $w$ is morphic then $\alpha$ is algebraic.

(b) If $w$ is uniform morphic then $\alpha$ is rational.

For a proof, see [4, Theorem 8.4.5, p. 268].

**Proposition 15.** Let $w$ be a morphic sequence and let $a$ be a letter occurring infinitely often in $w$. Then the number of occurrences of $a$ in a prefix of length $n$ of $w$ is $\Omega(\log n)$.

**Proof.** This follows from the matrix representation of the morphism. See, for example, [4, Cor. 8.2.4, p. 249].

The last results involve subword complexity (or factor complexity), the number of distinct factors of length $n$. We recall the following theorem of Pansiot [33]:

**Theorem 16.** If $w$ is a pure morphic word, then the subword complexity of $w$ is in $\Theta(1), \Theta(n), \Theta(n \log \log n), \Theta(n \log n), \text{ or } \Theta(n^2)$.

We also recall the following theorem:

**Theorem 17.** If $w$ is a uniform morphic word or a primitive morphic word that is not ultimately periodic, then the subword complexity of $w$ is $\Theta(n)$.

**Proof.** For uniform morphic words, see [20, Thm. 2, p. 171]. For primitive morphic words, see [31, 32, 33].
3 The examples

We now turn to providing examples of all of the 20 possibilities listed in the previous section.

Example 18. (a) A word that is neither morphic nor recurrent.

Consider the binary word 01100100000000000000000000010 · · · that is the characteristic sequence of the factorials 1, 2, 6, 24, . . . . This word has \( n = O((\log N)/ (\log \log N)) \) 1’s in a prefix of length \( N = n! + 1 \), and so by Proposition 15 it cannot be morphic. It is evidently not recurrent because the factor 11 appears only once.

Example 19. (b) A word that is recurrent, but neither morphic nor uniformly recurrent.

Consider the binary word

\[ b = 11011100101110111 \cdots \]

formed by the concatenation of the binary expansions of 1, 2, 3, . . . in order. This word clearly has \( 2^n \) distinct factors of length \( n \), and hence by Theorem 16 cannot be morphic. Nor is \( b \) uniformly recurrent, because it contains arbitrarily long blocks of 0’s. However, it is recurrent.

Example 20. (c) A word that is uniformly recurrent, but not morphic.

Consider the Sturmian characteristic word \( s_\alpha = s_0 s_1 s_2 \cdots \) defined by \( s_n = \lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor \) for \( n \geq 0 \). It is well-known that all such words are uniformly recurrent, see, e.g., [36, Proposition 3.17, p. 186].

However, for \( s_\alpha \) to be morphic, the number \( \alpha \) must be a quadratic irrational [13, Prop. 2.11]. So take \( \alpha = \pi \), for example.

Example 21. (d) A word that is morphic; but neither pure morphic, uniform morphic, primitive morphic, nor recurrent.

Take the Fibonacci word mentioned above in Example 1, and change the first two symbols to 2, giving the word

\[ f' = 22001010 \cdots \]

Now \( f' \) cannot be pure morphic, because if it were, then we would be able to write it as \( h^\omega(2) \) where \( h(2) \) begins with 22. Then iterating \( h \) would produce infinitely many 2’s, a contradiction. This word cannot be uniform morphic by Theorem 14, because the frequency of the symbol 0 is the same as that in the Fibonacci word, namely \( (\sqrt{5} - 1)/2 \), which would contradict Theorem 14. Finally, \( f' \) cannot be primitive morphic by Theorem 7 because the symbol 2 only occurs twice, and so \( f' \) is not recurrent.

However, the word \( f' \) is morphic. It is generated by the morphism \( a \rightarrow ab, b \rightarrow c, c \rightarrow cd, d \rightarrow c \), followed by the coding \( abcd \rightarrow 2201 \).

Example 22. (e) A word that is morphic and recurrent; but neither pure morphic, uniform morphic, primitive morphic, nor uniformly recurrent.
Consider the fixed point $x$ of the morphism $a \rightarrow ababb, b \rightarrow bc, c \rightarrow c$ coded by $\tau$ sending $a \rightarrow 0$ and $b, c \rightarrow 1$. An easy induction gives
\[ x = 0101111011011111110110111101 \cdots = \prod_{n \geq 1} 01^{a(n)} \]
where $a(n) = (\nu_2(n) + 1)^2$ and $\nu_2(n)$ is the exponent of the highest power of 2 dividing $n$.

Then $01^n0$ occurs in $x$ if and only if $n$ is a perfect square.

Following the construction in [21], it can be shown that $x$ has subword complexity $\Theta(n \sqrt{n})$, so by Theorem 17 $x$ is not uniform morphic or primitive morphic. By Theorem 16 it is not pure morphic. It is recurrent but not uniformly recurrent.

**Example 23. (f) A word that is primitive morphic, but neither pure morphic nor uniform morphic.**

From a theorem of Yasutomi [41, 12], we know that the Sturmian word $s_{\alpha,\rho} = s_0s_1s_2 \cdots$ defined by $s_n = \lfloor (n + 1)\alpha + \rho \rfloor - \lfloor n\alpha + \rho \rfloor$ is pure morphic if and only if $\alpha$ is a quadratic irrational, $\rho \in \mathbb{Q}(\alpha)$, and either $\alpha' > 1$, $1 - \alpha' \leq \rho' \leq \alpha'$ or $\alpha' < 0$, $\alpha' \leq \rho' \leq 1 - \alpha'$ where $\alpha'$ is the (algebraic) conjugate of $\alpha$.

Now consider the case where $\alpha = (3 - \sqrt{5})/2$. Then $\alpha' = (3 + \sqrt{5})/2 > 1$. Take $\rho = 2\alpha = 3 - \sqrt{5}$. Then $\rho' = 3 + \sqrt{5} > \alpha'$, so $s_{\alpha,\rho}$ is not pure morphic. However, this word is just the shift of the Fibonacci word $f$, and hence is morphic. In fact, it is easy to see that $s_{\alpha,\rho} = \tau(h^\omega(a))$, where $h : a \rightarrow ac, c \rightarrow b, b \rightarrow ac$ and $\tau(abc) = 100$. Then $h$ is primitive, as $h^3$ applied to each letter contains every letter. So $s_{\alpha,\rho}$ is primitive morphic. It cannot be uniform morphic by Theorem 14.

**Example 24. (g) A word that is uniform morphic; but neither pure morphic, primitive morphic, nor recurrent.**

As is well known, the morphism $g$ defined by $2 \rightarrow 210, 1 \rightarrow 20, 0 \rightarrow 1$ generates a squarefree word $g^\omega(2) = 210201 \cdots$; see [6]. Now consider the morphism $h$ defined by
\[
\begin{align*}
a &\rightarrow ab \\
b &\rightarrow ca \\
c &\rightarrow cd \\
d &\rightarrow ac
\end{align*}
\]
and the coding $\tau$ defined by $\tau(abcd) = 2101$. Then it is known that $\tau(h^\omega(a)) = g^\omega(2)$; see [6]. Hence $g^\omega(2)$ is 2-automatic and hence uniform morphic.

Now take the word $g^\omega(2)$ and change the first 1 to 2, obtaining the word $w = 220201210120 \cdots$. The resulting word $w$ is 2-automatic since $g^\omega(2)$ is. (In fact, it is generated by iterating the morphism
\[
\begin{align*}
0 &\rightarrow 01 \\
2 &\rightarrow 24 \\
4 &\rightarrow 32 \\
1 &\rightarrow 23 \\
3 &\rightarrow 35 \\
5 &\rightarrow 23
\end{align*}
\]
followed by the coding 012345 → 220211.)

But \( w \) is not pure morphic. Suppose \( w = \xi^\omega(2) \). Then \( \xi \) maps 2 to a word beginning with 22, which means that 22 occurs infinitely often in \( w \), a contradiction.

Finally, \( w \) is neither primitive morphic nor recurrent because 22 only occurs once.

Example 25. (h) A word that is uniform morphic and recurrent; but neither pure morphic nor primitive morphic.

The idea is to construct a word that contains a nonzero but finite number of \( k \)-th powers. Such a word cannot be pure uniform morphic, and it can only be pure morphic if the \( k \)-th powers are made up of non-growing letters.

Consider \( v \), the fixed point of the morphism \( a \to abba, b \to bccb, c \to cbbc. \) Apply the coding \( abc \to 010 \) to get

\[ x = 0000110001100000011010011001101001100101100000 \cdots. \]

Clearly \( x \) is uniform morphic and recurrent. However, 000 occurs with unbounded gaps between occurrences, since arbitrarily long factors of Thue-Morse occur, so \( x \) is not uniformly recurrent.

If \( x \) were pure morphic, it would be fixed by some morphism \( f \) prolongable on 0, then it would start with \( f(0)^5 \), since \( f(0) \) starts with 000001. But this is not possible, because there is no factor 000001u000001u0000. To see this, note that 000001 occurs only at positions that are a multiple of 4, and once 000001u is synchronized modulo 4, it can be factored into \( \{0000, 0110, 1001\} \) and decoded. An occurrence of 000001u000001u0000 then corresponds to an overlap in \( v \), which is impossible since \( v \) is overlap-free (applying the coding \( abc \to 010 \) gives the Thue-Morse sequence).

An alternative construction creates a word with unbounded powers, but not of the right kind: Consider the fixed point of \( a \to abba, b \to bccb, c \to cbbc. \) Apply the coding \( a, b \to 0, c \to 1 \), generating the word \( y = 000110001111111110011100110 \cdots. \)

As in the previous example, \( y \) is uniform morphic, recurrent, and not uniformly recurrent. If it were pure morphic, fixed by some \( f \), then it would start with \( f(0)^3 \), since \( f(0) \) starts with 000. But 000u000u000 does not occur: if \( k \) is the size of the largest block of ones in \( u \) (it is a power of 3), then we have two occurrences of 01k0 without a larger block of ones between them, which is always the case. In other words: our word is the coding of 010201030102 \cdots (the “ruler sequence” \( (\nu_2(n))_{n \geq 1} \) under \( i \to 001^{3+1} \). An occurrence of 000u000u000 could be decoded into a square in 010201030102 \cdots, which is square-free. This ends our second construction for this case.

Example 26. (i) A word that is uniform primitive morphic; but not pure morphic.

Here our example is \( r \), the Rudin-Shapiro sequence. The morphism and coding given in Example 4 show that \( r \) is uniform morphic. Assume it is pure morphic, generated by a morphism \( g \). Since \( r \) starts with 000, it must be that \( g(r) = r \) contains arbitrarily large cubes (namely, \( g^n(0)g^n(0)g^n(0) \) for all \( n \geq 1 \)). But from a well-known result [2] (also see [30]), the only cubes in \( r \) are 000 and 111, a contradiction.
Example 27. (j) A word that is pure morphic; but neither uniform morphic, primitive morphic, nor recurrent.

Take the Fibonacci sequence \( f \) discussed above in Example 1 and change the first symbol from 0 to 2. The resulting sequence \( u = 21001010 \cdots \) is pure morphic, since it is generated by the morphism that sends \( 2 \to 21, 1 \to 0, 0 \to 01 \). However, it is not automatic by Theorem 14 since 0 occurs with frequency \((\sqrt{5} - 1)/2\), which is irrational. It is neither recurrent nor primitive morphic since 2 occurs only once in \( u \).

Example 28. (k) A word that is pure morphic and recurrent; but neither uniform morphic, primitive morphic, nor uniformly recurrent.

Consider the morphism \( h \) defined by \( h(0) = 010 \) and \( h(1) = 11 \). Then \( h^\omega (0) \) is evidently pure morphic. It is evidently recurrent because any block that appears must appear in \( h^n(0) \) for some \( n \), and then that block appears twice in \( h^n(010) = h^{n+1}(0) \).

It is not uniformly recurrent because there are arbitrarily long blocks of 1’s. So it is also not primitive morphic.

Suppose it is \( k \)-automatic for some \( k \). Using the “logical” approach to automatic sequences [14], the sequence \( u = (u_n)_{n \geq 0} \) defined by

\[
u_n = \begin{cases} 
1, & \text{if the position of the first occurrence of a block of } n \text{ consecutive 1’s in } h^\omega (0) \\
0, & \text{otherwise;}
\end{cases}
\]

is also \( k \)-automatic. But this sequence (it is easy to see) is the characteristic sequence of powers of 2. So we can assume \( k = 2 \).

Again, using the “logical” approach, the function \( f(n) \) computing the starting position of the first occurrence of a block of \( n \) consecutive 1’s in \( h^\omega (0) \) is \( k \)-synchronized [27], and hence by a theorem about \( k \)-synchronized sequences, we have \( f(n) = O(n) \). But it is not hard to see that in fact \( f(n) = g(\lceil \log_2 n \rceil) \), where \( g(n) = (n+2) \cdot 2^{n-1} + 1 \). So \( f(n) \notin O(n) \), a contradiction.

Example 29. (l) A word that is pure morphic and primitive morphic; but neither uniform morphic nor pure primitive morphic.

The Chacon morphism is defined by the map \( c : 0 \to 0012, 1 \to 12, 2 \to 012 \) (see [24] and [25, p. 133, §5.5.1]). Iterating \( c \) on 0 gives the infinite word \( C := c^\omega (0) = 0012001212012 \cdots \). The morphism \( c \) is primitive, as \( c^2 \) applied to each letter contains every letter. Now consider the coding \( \tau : 012 \to 010 \) Applying \( \tau \) to \( C \) gives the word \( D = \tau(C) = 0010001001010 \cdots \), which satisfies \( D = \delta^\omega(0) \), where \( \delta : 0 \to 0010, 1 \to 1 \). It follows that \( D \) is pure morphic and primitive morphic.

We first show that \( D \) is not pure primitive morphic. Let \( h \) be any morphism such that \( h(D) = D \). If \( h(0) = 0 \), we will show that \( h(1) = 1 \). Suppose \( h(1) = u \). If \( u = \epsilon \), then \( D \) would be periodic, which it is not. Otherwise, if \( u \) is nonempty and not equal to 1, then \( u \) begins with 1000 (since \( D \) begins 001000 \cdots) and must end with 1 (since \( u \) can be followed by 000). Now if \( u \) ends in 001, then since \( u \) can be followed by 00u and hence by 001 we get a contradiction, since 001001 is not a factor of \( D \).
If $u$ ends in 101, then since $u$ can be followed by $0u$ and hence by 01 we get a contradiction, since 10101 is not a factor of $D$. So $u = 1$.

Now let $h$ be a primitive morphism with $h(D) = D$. If $uu$ is a prefix of $D$ then either $u = 0$ or $u = t^n(0)$. This is easy to see by induction on $|u|$. If $|u| > 1$, then $u$ begins with 0010, and since every occurrence of 0010 in $D$ comes from $t(0)$, it follows that $u = t(u')$ for some prefix $u'$ of $D$. So $D$ begins with $t(u'u')$ and hence begins with $u'u'$.

Now since $D$ begins with $h(0)h(0)$ it follows that $h(0) = t^n(0)$ for some $n > 0$ (if $h(0) = 0$, then $h$ would not be primitive). So $h(0)$ begins with 0010 and so $h(1) = t(u)$ for some $u$. Note that $u$ must contain 0 and 1. We can suppose that $|h(1)|$ is minimal among all primitive morphisms $h$ fixing $D$. So $D = t^n(0)t^n(0)t(u)t^n(0)t^n(0)t^n(0)t(u) \cdots$ so $D = t^{n-1}(0)t^{n-1}(0)ut^{n-1}(0)t^{n-1}(0)t^{n-1}(0)u \cdots$. By minimality of $|h(1)|$ it follows that the morphism $h': 0 \to t^{n-1}(0)$ and $1 \to u$ is not primitive. Since $u$ contains both 0 and 1 the only way this can fail to be primitive is if $n = 1$, i.e., $h(0) = 0$. But now the argument above completes the proof.

Finally, we prove that for all $k \geq 2$, the word $D$ is not $k$-automatic. By the argument in [1, Thm. 3.1], it suffices to prove that $D = d(0)d(1)d(2) \cdots$ is not $3$-automatic. Suppose it were. Then, by a well-known result (e.g., [4, Cor. 5.3.3]) the sequence $(d(x_n))_{n \geq 0}$ is ultimately periodic, where $x_n = [|(20)^n|]$. We will show it is not.

Note that $x_n = (3^{2n+1} - 3)/4$. An easy induction shows that $|\delta^n(0)| = (3^{n+1} - 1)/2$ and hence $|\delta^{n-1}(0)\delta^{2n-2}(0) \cdots \delta^3(0)\delta(0)0| = \sum_{0 \leq i \leq 2n-1} |\delta^i(0)| = \sum_{0 \leq i \leq 2n-1} (3^{i+1} - 1)/2 = (3^{2n+1} - 3)/4 - n = x_n - n$. Now another easy induction gives

$$\delta^{2n}(0) = \delta^{2n-1}(0)\delta^{2n-2}(0) \cdots \delta(0)00101\delta(0)1\delta^2(0)1 \cdots 1\delta^{2n-1}(0).$$

It follows that $d(x_n)$ is the $n$’th symbol of the infinite word $w = 0101\delta(0)1\delta^2(0)1\delta^3(0) \cdots$. However, $w$ is not ultimately periodic. If it were, then its subword complexity $\rho$ would satisfy $\rho(n) \leq n$ for some $n$; however $w$ contains every prefix of $D$ as a factor, and the subword complexity of $D$ is well-known to be $2n + 1$, a contradiction [24].

**Example 30. (m) A word that is pure primitive morphic; but not uniform morphic.**

The Fibonacci word $f$ discussed above in Example 1 is generated by the morphism $0 \to 01; 1 \to 0$ and hence is pure primitive morphic. But it is not automatic, as already mentioned above in Example 1, and hence not uniform morphic.

**Example 31. (n) A word that is pure morphic and uniform morphic, but neither pure uniform morphic, nor primitive morphic, nor recurrent.**

Consider the morphism $h$ defined by

$$
\begin{align*}
3 & \to 32 \\
2 & \to 102012 \\
1 & \to 1012 \\
0 & \to 02
\end{align*}
$$
An easy induction shows that \( h^{n+1}(3) = 3g^{2n}(2)g^{2n-2}(2) \cdots g^2(2)2 \), where \( g \) is the morphism defined in Example 24. Letting \( n \to \infty \), we see that \( h^n(3) = 3g^n(2) \). Define \( w = h^n(3) \); then we claim \( w \) has the desired properties. It is clearly pure morphic, and it is 2-automatic because \( g^n(2) \) is (as remarked above in Example 24), and automatic sequences are closed under shift (see, e.g., [4, Theorem 6.8.4]). In fact, \( w \) is the image under the coding \( \rho \) of the fixed point, starting with \( a \), of \( \delta \), where \( \delta(a) = ab, \delta(b) = cd, \delta(c) = bd, \delta(d) = eb \), and \( \delta(e) = db \), and \( \rho(abcede) = 32101 \).

However, \( w \) is not primitive morphic because if it were, it would be uniformly recurrent. But 3 only appears once, so it is not even recurrent, a contradiction.

Finally, \( w \) is not pure uniform morphic. Suppose it is generated by a \( k \)-uniform morphism \( f \). If \( k \) is multiplicatively independent of 2, then \( w \) is both 2-automatic and \( k \)-automatic, and so by Cobham’s theorem [19] it is ultimately periodic, a contradiction.

Therefore \( k \) is multiplicatively dependent on 2, and hence \( k = 2^n \) for some \( n \geq 1 \). But now \( w[2] = 1 \) and \( w[6] = 1 \). If \( w \) were the fixed point of the \( 2^n \)-uniform morphism \( f \) we would have the image of \( w[2] \) under \( f \), which is \( w[2 \cdot 2^n \cdots 3 \cdot 2^n - 1] \), equal to the image of \( w[6] \) under \( f \), which is \( w[6 \cdot 2^n \cdots 7 \cdot 2^n - 1] \). However, from our description above we have \( w = \rho(\delta^n(a)) \). Since \( \delta^n(a) \) begins with \( abcdede \), it follows that \( w[2 \cdot 2^n \cdots 3 \cdot 2^n - 1] = \rho(\delta^n(c)) \) and \( w[6 \cdot 2^n \cdots 7 \cdot 2^n - 1] = \rho(\delta^n(e)) \). However \( \rho(\delta^n(c)) \) begins with 20 if \( n \) is odd and 10 if \( n \) is even, whereas \( \rho(\delta^n(e)) \) begins with 02 if \( n \) is odd and 12 if \( n \) is even, a contradiction.

Example 32. (o) A word that is pure morphic and uniform morphic and recurrent, but neither pure uniform morphic nor primitive morphic.

Consider the fixed point of \( f: a \to abca, b \to bced, \) and \( c, d, e \to eeeee \), followed by the coding by \( g: abcede \to 01123 \). The resulting word \( q \) starts

\[
01120112333333333333011201123^{62}0 \cdots
\]

It is the fixed point of \( h \) defined by \( 0 \to 01120, 1 \to 1, 2 \to 2333333333333, 3 \to 33333 \).

(observe that \( hqf = gf^2 \)). If \( q \) were fixed by a non-trivial uniform morphism \( j \), then by Cobham’s theorem [19] the length of \( j \) would be \( 5^k \) for some \( k > 0 \). Then we would have \( jg = gf^k \). But \( (gf^k)(b) \) starts with 1, while \( (gf^k)(c) \) starts with 3, a contradiction.

The word \( q \) is not uniformly recurrent, because it has arbitrarily long blocks of 3’s.

Example 33. (p) A word that is pure morphic and uniform primitive morphic, but neither pure uniform morphic nor pure primitive morphic.

Let \( u = acbecacbacabc \cdots \) be the image of the Thue-Morse word \( t \) under the morphism \( 0 \to ac \) and \( 1 \to bc \). Then \( u \) is pure morphic, because it is generated by the morphism \( a \to acb, b \to bca, c \to c \).

Now \( u \) is uniform primitive morphic because it is the image, under the coding \( 0123 \to acbc \) of the fixed point of the word generated by the morphism \( \eta \) defined by \( 0 \to 01, 1 \to 23, 2 \to 23, 3 \to 01 \). Note that \( \eta^2 \) applied to each letter contains every letter.

However, \( u \) is neither pure uniform morphic, nor pure primitive morphic. To see this, assume \( u = f^\omega(a) \) for some morphism \( f \). If \( f \) is primitive or \( k \)-uniform, with \( k \geq 2 \), then \( f(c) \) is neither \( c \) nor \( \epsilon \). Since \( u \) contains no occurrence of the factor \( cc \), it must be that \( f(c) \)
contains an occurrence of $a$ or $b$. Now $u$ can be factored over $\{f(acbc), f(bcac)\}$, which are words of the same length that have this $a$ or $b$ occurring at the same position. This implies that there is an arithmetic sequence of indices on which $u$ is constantly $a$ or constantly $b$, so the Thue-Morse word $t$ has the same property, which is not true: any sequence extracted from Thue-Morse by indexing from an arithmetic progression contains both $a$’s and $b$’s [26].

**Example 34.** (q) A word that is pure primitive morphic and uniform primitive morphic; but not pure uniform morphic.

Let $T$ be the word generated by the morphism $g$ mentioned in Example 24: $g$ maps $2 \rightarrow 210; 1 \rightarrow 20; 0 \rightarrow 1$. Then, as in Example 24, the word $T$ is 2-automatic, and the underlying morphism is primitive, so it is uniform primitive morphic.

Suppose $T$ is pure uniform morphic. Then it is generated by iterating a $s$-uniform morphism for some $s \geq 2$. If $s$ is not a power of 2, then $T$ is both 2-automatic and $s$-automatic where 2 and $s$ are multiplicatively independent. Hence by Cobham’s theorem [19], $T$ is ultimately periodic. But in fact $T$ is a well-known squarefree word arising from the Thue-Morse sequence [6]. So $T$ must be generated by iterating a morphism $h$ that is $2^k$-uniform for some $k \geq 1$. In this case, Berstel has shown that this is impossible [6], because then $T$ and $g(T)$ differ at the position $5 \cdot 2^k$.

**Example 35.** (r) A word that is pure uniform morphic; but neither primitive morphic nor recurrent.

The word generated by the morphism $a \rightarrow ab; b \rightarrow bc; c \rightarrow cc$ iterated on $a$. This is clearly pure uniform morphic. However, since $a$ only appears once, it is not recurrent and thus cannot be primitive morphic.

**Example 36.** (s) A word that is pure uniform morphic and recurrent; but not primitive morphic.

Here we can take the word that is the fixed point of the morphism $0 \rightarrow 010$ and $1 \rightarrow 111$, as in [15]. This is evidently pure uniform morphic and recurrent, but as there are arbitrarily long blocks of 1’s, it cannot be uniformly recurrent, and hence it is not primitive morphic.

**Example 37.** (t) A word that is pure uniform primitive morphic. The Thue-Morse word $t$.

4 Final remarks

None of the 20 examples we provided are ultimately periodic.

One might ask whether every morphic word can be generated by a coding applied to a non-uniform morphism. The answer is yes: it suffices to prove this for uniform morphic words, which is done in [5].

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