On the $d$-cluster generalization of Erdős-Ko-Rado

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Abstract

If $2 \leq d \leq k$ and $n \geq dk/(d-1)$, a $d$-cluster is defined to be a collection of $d$ elements of $\binom{[n]}{k}$ with empty intersection and union of size no more than $2k$. Mubayi [6] conjectured that the largest size of a $d$-cluster-free family $F \subset \binom{[n]}{k}$ is $\binom{n-1}{k-1}$, with equality holding only for a maximum-sized star. Here we prove two results. The first resolves Mubayi’s conjecture and proves a stronger result, thus completing a new generalization of the Erdős-Ko-Rado Theorem. The second shows, by a different technique, that for a slightly more limited set of parameters only a very specific kind of $d$-cluster need be forbidden to achieve the same bound.

1 Introduction

For any $m, n \in \mathbb{Z}$ with $m < n$, we use $[m, n]$ to denote $\{m, m+1, \ldots, n\}$, and define $[n] := [1, n]$. Furthermore, we use $\binom{X}{k}$ to denote the set of $k$-element subsets of a set $X$, and will henceforth use $n, k$ and $d$ to represent positive integers. We recall the definition of a $d$-cluster:

**Definition 1.** Let $2 \leq d \leq k$ with $n \geq dk/(d-1)$ and suppose $F \subset \binom{[n]}{k}$. Then, if we have $B = \{B_1, \ldots, B_d\} \subset \binom{[n]}{k}$ such that $|B_1 \cup \cdots \cup B_d| \leq 2k$ and $B_1 \cap \cdots \cap B_d = \emptyset$, we say that $B$ is a $d$-cluster. Furthermore, if $F$ contains no such $B$, we say that $F$ is $d$-cluster-free.

If all elements of $F$ contain some $x \in [n]$, then we say that $F$ is a star centered at $x$. Note that stars are naturally $d$-cluster-free for all $d \geq 2$ and that the maximum size of a star is $\binom{n-1}{k-1}$. We furthermore wish to introduce the following special kind of $d$-cluster:

**Definition 2.** Let $2 \leq d \leq k$ and $n \geq 2k-d+2$. Then, if we have $B = \{B, B', B_1, \ldots, B_{d-2}\} \subset \binom{[n]}{k}$ such that $B \cap B' = \{a_1, \ldots, a_{d-2}\}$ and $B - B_i = \{a_i\}$ for all $i \in [d-2]$, then we say that $B$ is a simple $d$-cluster.

Every simple $d$-cluster has empty intersection and union at most $2k$ and is thus a $d$-cluster. Furthermore, there is no distinction between 2-clusters and simple 2-clusters. However, for $d \geq 3$ we note that the set of simple $d$-clusters constitutes a very small fraction of the total $d$-clusters for a given set of parameters. In particular, for any $k \geq 2d$, the number of simple $d$-clusters depends only on $d$, while the number of regular $d$-clusters increases as some power of $k$ (up to isomorphism). This is especially apparent in small values: for any $k \geq 4$ there are only 2 non-isomorphic simple 3-clusters and 5 non-isomorphic simple 4-clusters. We recall the following conjecture of Mubayi [6]:

**Conjecture 1** (Mubayi). Let $2 \leq d \leq k$ and $n \geq dk/(d-1)$. Furthermore, suppose that $F \subset \binom{[n]}{k}$ is $d$-cluster-free. Then,

$$|F| \leq \binom{n-1}{k-1}$$

Furthermore, except when both $d = 2$ and $n = 2k$, equality implies $F$ is a maximum-sized star.
The classical Erdős-Ko-Rado (EKR) Theorem \[1\] is the case of \(d = 2\) (the union condition holds automatically, so a 2-cluster-free family is simply “pairwise intersecting”). Later, Katona proposed a version of the \(d = 3\) case, and Frankl and Füredi \[3\] obtained a result for \(n \geq k^2 + 3k\). In \[6\], Mubayi completely resolved the \(d = 3\) case and proposed the conjecture for general \(d\). This was then resolved for \(d = 4\) and sufficiently large \(n\) by Mubayi in \[7\], and later for all \(d \geq 3\) and sufficiently large \(n\) both by Mubayi and Ramadurai in \[8\] and independently by Özkahya and Füredi in \[4\]. In \[5\], Keevash and Mubayi solved another case of this problem, namely where both \(k/n\) and \(n/2 - k\) are bounded away from zero. The case of \(n < 2k\) (where, again, the union condition holds automatically) has also been resolved, first by Frankl in \[2\] (where the bound was established), and later by Mubayi and Verstraëte in \[9\] (where equality was characterized).

We will prove here two main results. Our first theorem resolves Mubayi’s conjecture and in particular proves something stronger by bounding the size of families that are not necessarily \(d\)-cluster-free, but where the \(d\)-clusters use only a specific subset of the total family. This proof uses a method of induction introduced in \[7\]. Our second result shows that we may make the weaker assumption that a family \(\mathcal{F}\) is free of simple \(d\)-clusters and still achieve the same bound. This second proof is valid on a slightly more limited set of parameters but is also of note for its brevity. Formally, these results are as follows:

**Theorem 1.** Let \(2 \leq d \leq k \leq n/2\). Furthermore, suppose \(\mathcal{F}^* \subset \mathcal{F} \subset \binom{[n]}{k}\) have the property that any \(d\)-cluster in \(\mathcal{F}\) is contained entirely in \(\mathcal{F}^*\). Then:

\[
|\mathcal{F}^*| + \frac{n}{k}|\mathcal{F} - \mathcal{F}^*| \leq \binom{n}{k}
\]

furthermore, excepting the case where both \(d = 2\) and \(n = 2k\), equality implies one of the following:

(i) \(\mathcal{F}^* = \emptyset\) and \(\mathcal{F}\) is a maximum-sized star

(ii) \(\mathcal{F} = \mathcal{F}^* = \binom{[n]}{k}\)

**Theorem 2.** Let \(2 \leq d \leq k\) and \(n \geq 3k - 2d + 4\) and suppose \(\mathcal{F}\) contains no simple \(d\)-clusters. Then:

\[
|\mathcal{F}| \leq \binom{n - 1}{k - 1}
\]

Note that Theorem \[1\] implies Conjecture \[1\] for all \(n \geq 2k\) if we set \(\mathcal{F}^* = \emptyset\). We will divide the remainder of this paper into two sections, dedicated to the proofs of Theorems \[1\] and \[2\] respectively. Each proof will use a different technique, but they share the following common notation - for any \(\mathcal{F} \subset \binom{[n]}{k}\) and \(x \in [n]\), we define:

\[
\triangle_x(\mathcal{F}) = \{D \in \binom{[n]}{k - 1} : (D \cup \{x\}) \in \mathcal{F}\}
\]

and for \(D \in \binom{[n]}{k - 1}\), we define

\[
\nabla_\mathcal{F}(D) = \{B \in \mathcal{F} : D \subset B\}
\]

Note that Theorem \[1\] implies Conjecture \[1\] for all \(n \geq 2k\) if we set \(\mathcal{F}^* = \emptyset\). We will divide the remainder of this paper into two sections, dedicated to the proofs of Theorems \[1\] and \[2\] respectively. Each proof will use a different technique, but they share the following common notation - for any \(\mathcal{F} \subset \binom{[n]}{k}\) and \(x \in [n]\), we define:

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and for \(D \in \binom{[n]}{k - 1}\), we define

\[
\nabla_\mathcal{F}(D) = \{B \in \mathcal{F} : D \subset B\}
\]
2 \textit{d-clusters}

In this section we give a proof of Theorem 1. To start, we prove a result that is itself strengthening of classical EKR, and will serve as a base case for Theorem 1.

\textbf{Theorem 3.} Let $2 \leq k \leq n/2$. Suppose furthermore that $\mathcal{F}' \subset \mathcal{F} \subset \binom{[n]}{k}$ has the property that for any $B, B' \in \mathcal{F}$ with $B \cap B' = \emptyset$ we have $B, B' \in \mathcal{F}'$. Then:

$$|\mathcal{F}'| + \frac{n}{k}|\mathcal{F} - \mathcal{F}'| \leq \binom{n}{k}$$

where, for $n > 2k$, equality is achieved only if $\mathcal{F}' = \mathcal{F} = \binom{[n]}{k}$ or if $\mathcal{F}' = \emptyset$ and $\mathcal{F}$ is a maximum sized star.

\textit{Proof.} We will proceed by the Katona cycle method. First, we let $C(n)$ denote the set of all cyclic permutations on $n$ elements. Then, if we have \{a_0, \ldots, a_{n-1}\} = \sigma \in C(n)$ and $\mathcal{G} \subset \binom{[n]}{k}$, we define (with all subscripts henceforth taken mod $n$):

$$A_\sigma(\mathcal{G}) = \{B \in \mathcal{G} : B = \{a_i, a_{i+1}, \ldots, a_{i+(k-1)}\} \text{ for some } i \in [0, n-1]\}$$

Observe trivially that $|A_\sigma(\mathcal{G})| \leq n$. Furthermore, for any such $B = \{a_i, \ldots, a_{i+(k-1)}\}$, we say that $B$ has “starting point” $i$ in $\sigma = \{a_0, \ldots, a_{n-1}\}$. Now, we wish to prove the following:

(i) $|A_\sigma(\mathcal{F} - \mathcal{F}'')| \leq k$ for all $\sigma \in C(n)$

(ii) if $A_\sigma(\mathcal{F} - \mathcal{F}'') \neq \emptyset$, then $|A_\sigma(\mathcal{F} - \mathcal{F}'')| \leq 2(k - |A_\sigma(\mathcal{F} - \mathcal{F}'')|)$ for all $\sigma \in C(n)$

Let $\sigma = \{a_0, \ldots, a_{n-1}\}$ as before, suppose $|A_\sigma(\mathcal{F} - \mathcal{F}'')| \geq 1$ and take $B \in A_\sigma(\mathcal{F} - \mathcal{F}'')$. Furthermore, suppose without loss of generality that $B = \{a_0, \ldots, a_{k-1}\}$. Then, let $B' \in (A_\sigma(\mathcal{F}) - \{B\})$ have starting point $i'$ in $\sigma$, and observe that since $B \cap B' = \emptyset$, we must have either $i' \in [n - (k-1), n-1]$ or $i' \in [1, k-1]$. Suppose then that we have $B_1, B_2 \in (A_\sigma(\mathcal{F}) - \{B\})$ with starting points $i_1 \in [n - (k-1), n-1]$ and $(i_1 + k) \in [1, k-1]$ in $\sigma$ respectively. Since $n \geq 2k$ this implies $B_1 \cap B_2 = \emptyset$ and thus that $B_1, B_2 \in \mathcal{F}'$. Combining these facts we get both (i) and (ii). Now, we define the following subsets of $C(n)$:

$$C_j = \{\sigma \in C(n) : |A_\sigma(\mathcal{F} - \mathcal{F}'')| = j\}$$

and using (i) we observe that $C_0, C_1, \ldots, C_k$ partition $C(n)$. Then, again using (i) and (ii), and since every $B \in \mathcal{F}$ is in $A_\sigma(\mathcal{F})$ for precisely $k!(n-k)!$ different $\sigma \in C(n)$ we get:

$$|\mathcal{F} - \mathcal{F}'|k!(n-k)! = \sum_{\sigma \in C(n)} |A_\sigma(\mathcal{F} - \mathcal{F}'')| = \sum_{1 \leq i \leq k} |C_i|i$$

$$|\mathcal{F}'|k!(n-k)! = \sum_{\sigma \in C(n)} |A_\sigma(\mathcal{F}'')| \leq n|C_0| + \sum_{1 \leq i \leq k} |C_i|2(k - i)$$

And combining these:

$$|\mathcal{F}'| + \left(\frac{n}{k}\right)|\mathcal{F} - \mathcal{F}'| \leq \frac{n|C_0| + \sum_{i=1}^{k} 2(k - i)|C_i|}{k!(n-k)!} + \frac{n/k}{k!(n-k)!} = \frac{n|C_0| + n|C_k| + \sum_{i=1}^{k-1} \frac{i+2k(k-i)}{k}|C_i|}{k!(n-k)!}$$
A quick calculation gives us that, for all $1 \leq i \leq k - 1$:

\[ \frac{in + 2k(k - i)}{k} \leq \frac{in + n(k - i)}{k} = n \]

(3)

with equality only if $n = 2k$. Combining (2) and (3), since $|C_0| + \cdots + |C_k| = |C(n)| = (n - 1)!$, we get:

\[ |F^*| + \binom{n}{k}|F - F^*| \leq \frac{n(|C_0| + \cdots + |C_k|)}{k!(n-k)!} \]

(4)

\[ = \frac{n!}{k!(n-k)!} \]

(5)

\[ = \binom{n}{k} \]

(6)

Now, suppose $n > 2k$ and we have equality. Note that in this case we do not have equality in (3) and so $C(n) = C_0 \cup C_k$. Furthermore, for any $B, B' \in \mathcal{F}$, we can easily construct $\sigma \in C(n)$ such that $B, B' \in \mathcal{F}^*$ or $B, B' \in (\mathcal{F} - \mathcal{F}^*)$. By extension, we get that either $\mathcal{F} \subset \mathcal{F}^*$ or $\mathcal{F} = (\mathcal{F} - \mathcal{F}^*)$. If we assume the former then $|\mathcal{F}| = |\mathcal{F}^*| = \binom{n}{k}$ in which case $\mathcal{F} = (\mathcal{F}^*)$. For the latter, we get that $|\mathcal{F}| = |\mathcal{F} - \mathcal{F}^*| = \binom{n-1}{k-1}$ and $\mathcal{F}$ is pairwise intersecting, in which case classical EKR tells us that $\mathcal{F}$ is a star. This completes the proof.

The following result will give us the tools to induct on $d$ in the proof of Theorem [1]. It is a stronger version of a proposition from [7] that was later stated more clearly in [8].

**Proposition 1.** Let $3 \leq d \leq k$ and $n \geq dk/(d-1)$. Furthermore, suppose $\mathcal{F}^* \subset \mathcal{F} \subset \binom{n}{k}$ has the property that any $d$-cluster in $\mathcal{F}$ is contained in $\mathcal{F}^*$. Then, if $\{D_1, \ldots, D_{d-1}\} \subset \Delta_x(\mathcal{F})$ is a $(d-1)$-cluster, it follows that either $|\nabla_{\mathcal{F}}(D_i)| = 1$ or $D_i \in \Delta_x(\mathcal{F}^*)$ for all $i \in [d-1]$.

**Proof.** Suppose for the sake of contradiction that there exists $i_0 \in [d-1]$ such that $|\nabla_{\mathcal{F}}(D_{i_0})| \geq 2$ and $D_{i_0} \notin \Delta_x(\mathcal{F}^*)$. Then, we know both that $(D_{i_0} \cup \{x\}) \in (\mathcal{F} - \mathcal{F}^*)$ and that $(D_{i_0} \cup \{y\}) \in \mathcal{F}$ for some $y \in [n] \setminus \{x\}$. Then:

\[ |(D_1 \cup \{x\}) \cup (D_2 \cup \{x\}) \cup \cdots \cup (D_{d-1} \cup \{x\}) \cup (D_{i_0} \cup \{y\})| \leq |D_1 \cup \cdots \cup D_{d-1}'| + |\{x, y\}| \leq 2(k-1) + 2 = 2k \]

and since, $x, y \notin D_{i_0}$, we get:

\[ (D_1 \cup \{x\}) \cap \cdots \cap (D_{d-1} \cup \{x\}) \cap (D_{i_0} \cup \{y\}) \subset D_1 \cap \cdots \cap D_{d-1} = \emptyset \]

Furthermore, since $x \notin D_{i_0}$, we know that $(D_{i_0} \cup \{y\}) \neq (D_j \cup \{x\})$ for any $j \in [d-1]$. Thus, $\mathcal{F}$ contains a $d$-cluster not contained entirely in $\mathcal{F}^*$, which is a contradiction. 

After the following (in)equalities, we will begin with a proof of Theorem [1]

**Proposition 2.** Let $2 \leq k < n$ and $\mathcal{F} \subset \binom{n}{k}$. Then, the following hold:

(i) $\sum_{x \in [n]} |\Delta_x(\mathcal{F})| = k|\mathcal{F}|$

(ii) $|\{D \in \binom{n}{k-1} : |\nabla_{\mathcal{F}}(D)| = 1\}| \leq \frac{n(n-1)\ldots(n-k+1)}{n-k}$
\textbf{Proof.} The proof of \((i)\) is straightforward from the definitions, so we will focus on \((ii)\). First, we denote by \(\mathcal{F}^C\) the complement of \(\mathcal{F}\) in \((\binom{n}{k})\), and observe that for a given \(D \in (\binom{n}{k-1})\), we have:

\[
\sum_{x \in [n]} |\triangle_x(\mathcal{F})| + |\triangle_x(\mathcal{F}^C)| = n\binom{n-1}{k-1}
\]

using this along with \((i)\), we get:

\[
|\{D \in \left(\binom{n}{k-1}\right) : |\nabla_{\mathcal{F}}(D)\} = 1\}| = |\{D \in \left(\binom{n}{k-1}\right) : |\nabla_{\mathcal{F}^C}(D)\} = n - k\}|
\]

\[
\leq \frac{\sum_{x \in [n]} |\triangle_x(\mathcal{F})|}{n - k}
\]

\[
= \frac{n_{\binom{n-1}{k-1}} - \sum_{x \in [n]} |\triangle_x(\mathcal{F})|}{n - k}
\]

\[
= \frac{n_{\binom{n-1}{k-1}} - k|\mathcal{F}|}{n - k}
\]

\[
\text{(10)}
\]

We now begin with the proof of our main result.

\textbf{Proof of Theorem 3.} Let \(\mathcal{F}^* \subset \mathcal{F}\) be as described and note that the \(d = 2\) case is taken care of by Theorem 3 so we suppose \(d \geq 3\). We define the following subset of \(\triangle_x(\mathcal{F})\), for any \(x \in [n]\)

\[
\triangle_x^*(\mathcal{F}) = \{D \in \triangle_x(\mathcal{F}) : |\nabla_{\mathcal{F}}(D)\} = 1\} \cup \triangle_x(\mathcal{F}^*)
\]

Observe first that the sets \(\{D \in \triangle_x(\mathcal{F}) : |\nabla_{\mathcal{F}}(D)\} = 1\}\) over \(x \in [n]\) partition \(\{D \in (\binom{n}{k-1}) : |\nabla_{\mathcal{F}}(D)\} = 1\}\). Using this and proposition 2, we get:

\[
\sum_{x \in [n]} |\triangle_x^*(\mathcal{F})| \leq |\{D \in \left(\binom{n}{k-1}\right) : |\nabla_{\mathcal{F}}(D)\} = 1\}| + \sum_{x \in [n]} |\triangle_x(\mathcal{F}^*)|
\]

\[
\leq \frac{n_{\binom{n-1}{k-1}} - k|\mathcal{F}|}{n - k} + k|\mathcal{F}^*|
\]

\[
\text{(12)}
\]

Furthermore, by proposition 1, we know that any \((d - 1)\)-cluster in \(\triangle_x(\mathcal{F})\) is contained in \(\triangle_x^*(\mathcal{F})\). Since \(\triangle_x(\mathcal{F}) \subset (\binom{n}{k-1})\) and \((d - 1) \leq (k - 1) < (n - 1)/2\), we may apply induction on \(d\) to get:

\[
\binom{n-1}{k-1} \geq |\triangle_x^*(\mathcal{F})| + \frac{n - 1}{k - 1}|\triangle_x(\mathcal{F}) - \triangle_x^*(\mathcal{F})| = \frac{n - 1}{k - 1}|\triangle_x(\mathcal{F})| - \frac{n - k}{k - 1}|\triangle_x^*(\mathcal{F})|
\]

\[
\text{(13)}
\]

Then, summing over all \(x \in [n]\) and using proposition 2 in combination with (12), we get:

\[
n_{\binom{n-1}{k-1}} \geq \frac{n - 1}{k - 1}\sum_{x \in [n]} |\triangle_x(\mathcal{F})| - \frac{n - k}{k - 1}\sum_{x \in [n]} |\triangle_x^*(\mathcal{F})|
\]

\[
\geq \frac{n - 1}{k - 1}|\mathcal{F}| - \frac{n - k}{k - 1}|\mathcal{F}^*| - \frac{(n - k)\binom{n-1}{k-1} - k|\mathcal{F}|}{n - k}
\]

\[
= \frac{nk}{k - 1}|\mathcal{F}| - \frac{(n - k)k|\mathcal{F}^*| - n\binom{n-1}{k-1}}{k - 1}
\]

\[
\text{(16)}
\]
and therefore:

$$\frac{nk}{k-1} \binom{n-1}{k-1} \geq \frac{nk}{k-1} |F| - \frac{(n-k)k}{k-1} |F^*|$$

(17)

finally, multiplying both sides by $\frac{k-1}{k}$ we get:

$$\left(\frac{n}{k}\right) \geq \frac{n}{k} |F| - \frac{n-k}{k} |F^*| = |F^*| + \frac{n}{k} |F - F^*|$$

(18)

**Equality**: Suppose now that we have equality - that is, that $|F^*| + \binom{n}{k} |F - F^*| = \binom{n}{k}$. This implies a couple of things. First, we get that $|F| \geq \binom{n-1}{k-1}$ with $|F| = \binom{n-1}{k-1}$ only if $F^* = \emptyset$. Furthermore, we must have equality in (13) for all $x \in [n]$ - that is:

$$|\Delta_x^*(F)| + \frac{n-1}{k-1} |\Delta_x(F) - \Delta_x^*(F)| = \binom{n-1}{k-1}$$

where since $(n-1) > 2(k-1)$, we get by induction that either $\Delta_x^*(F) = \Delta_x(F) = \binom{n-1}{k-1}$ or $\Delta_x(F)$ is a maximum sized star and $\Delta_x^*(F) = \emptyset$. Suppose for the sake of contradiction that the latter is true for all $x \in [n]$. Then $|\Delta_x(F)| = \binom{n-2}{k-2}$ for all $x \in [n]$ and using proposition 2

$$|F| = \frac{\sum_{x \in [n]} |\Delta_x(F)|}{k} = \frac{n}{k} \binom{n-2}{k-2} < \frac{n-1}{k-1} \binom{n-2}{k-2} = \binom{n-1}{k-1}$$

Which is a contradiction. Thus, there exists $x_0 \in [n]$ such that $\Delta_{x_0}(F) = \binom{n-1}{k-1}$, and therefore $F$ contains a maximum-sized star centered at $x_0$. If $|F| = \binom{n-1}{k-1}$ this implies that $F$ is, itself, a maximum size star centered at $x_0$ and that $F^* = \emptyset$. Now, suppose $|F| > \binom{n-1}{k-1}$ and take $B, B' \in F$ such that exactly one of $B, B'$ contains $x_0$ (note that there must exist at least one element not containing $x_0$). Now, take $Z \subseteq [n]$ such that $|Z| = 2k$ and $(B \cup B') \subseteq Z$. Then, since $|B \cap B'| \leq k-1$ we know that $|Z - \{x_0\} - (B \cap B')| \geq k$ and thus there exist distinct:

$$D_1, \ldots, D_{d-2} \in \binom{Z - \{x_0\} - (B \cap B')}{k-1}$$

such that $(D_i \cup \{x_0\}) \neq B, B'$ for all $i \in [d-2]$. Furthermore, since $F$ contains a maximum-sized star centered at $x_0$, we get $(D_i \cup \{x_0\}) \in F$ for all $i \in [d-2]$ and:

$$|B \cup B' \cup (D_1 \cup \{x_0\}) \cup \cdots \cup (D_{d-2} \cup \{x_0\})| \leq |Z| = 2k$$

and, because $x_0$ is not in one of $B$ or $B'$

$$B \cap B' \cap (D_1 \cup \{x_0\}) \cap \cdots \cap (D_{d-2} \cup \{x_0\}) = \emptyset$$

Thus, every element of $F$ is part of a $d$-cluster. Since all $d$-clusters in $F$ are contained in $F^*$, we get that $F = F^*$ and thus that $|F| = \binom{n}{k}$ and $F = \binom{[n]}{k}$. This completes the proof.

\[\square\]

### 3 Simple $d$-clusters

In this section we give a short proof of Theorem 2. We start with a few definitions:
**Definition 3.** Let $2 \leq d \leq k < n$ and suppose $\mathcal{F} \subset \binom{[n]}{k}$. Then, we define, for any $B \in \mathcal{F}$ we define the following subset of $B$:

$$\alpha_{\mathcal{F}}(B) = \{ y \in B : \exists B' \in \mathcal{F} \text{ such that } B - B' = \{y\} \}$$

Furthermore, for any $x \in [n]$, we let $\mathcal{F}^x := \{ B \in \mathcal{F} : x \in B \}$ and $\mathcal{F}^{-x} := \mathcal{F} - \mathcal{F}^x$ and define the following:

$$\mathcal{R}_x(\mathcal{F}) = \{ D \in \binom{[n] - \{x\}}{k-1} : (D \cap B) \in \left( \frac{\alpha_{\mathcal{F}}(B)}{d-2} \right) \text{ for some } B \in \mathcal{F}^{-x} \}$$

$$\mathcal{S}_x(\mathcal{F}) = \{ D \in \binom{[n] - \{x\}}{k-1} : D \cup \{y\} = B \text{ for some } B \in \mathcal{F}^{-x} \text{ and } y \notin \alpha_{\mathcal{F}}(B) \}$$

We now make the following observation:

**Proposition 3.** Let $2 \leq d \leq k$ and $n \geq 2k - d + 2$, and suppose $\mathcal{F} \subset \binom{[n]}{k}$ contains no simple $d$-clusters. Then, if $B, B' \in \mathcal{F}$ are such that $|\alpha_{\mathcal{F}}(B) \cap B'| \geq d - 2$, it follows that $|B \cap B'| \geq d - 1$.

**Proof.** Let $\{b_1, \ldots, b_{d-2}\} \subseteq (\alpha_{\mathcal{F}}(B) \cap B)$. Then, by the definition of $\alpha_{\mathcal{F}}$, there exist $B_1, \ldots, B_{d-2} \in \mathcal{F}$ such that $B - B_i = \{a_i\}$ for all $i \in [d - 2]$. However, $\mathcal{F}$ contains no simple $d$-clusters, so this implies $\{b_1, \ldots, b_{d-2}\} \neq (\alpha_{\mathcal{F}}(B) \cap B)$ and thus $|B \cap B'| \geq d - 1$. 

We will furthermore be needing the following combinatorial identity, which is a straightforward consequence of the fact that \( \binom{r_1}{i} + \binom{r_2}{i} \geq \binom{r_1-1}{i} + \binom{r_2+1}{i} \) for any $r_1,r_2,l \in \mathbb{N}$ with $r_1 > r_2$:

**Proposition 4.** Let $l,r_1,\ldots,r_m \in \mathbb{N}$, and let $r := \left\lceil \frac{r_1+r_2+\cdots+r_m}{m} \right\rceil$. Then:

$$\sum_{1 \leq i \leq m} \binom{r_i}{l} \geq m \binom{r}{l}$$

and we now begin with the proof of our second main result:

**Proof of Theorem.** Let $\mathcal{F}$ contain no simple $d$-clusters and take $x \in [n]$. We suppose first that $|\mathcal{S}_x(\mathcal{F})| \geq |\mathcal{F}^{-x}|$. Note that for every $D \in \mathcal{S}_x(\mathcal{F})$, we have $D \cup \{y\} = B$ for some $B \in \mathcal{F}^{-x}$ and $y \notin \alpha_{\mathcal{F}}(B)$. Thus, $|\bigtriangledown_{\mathcal{F}}(D)| = 1$, or else we would have $y \in \alpha_{\mathcal{F}}(B)$. Therefore $\Delta_x(\mathcal{F}) \cap \mathcal{S}_x(\mathcal{F}) = \emptyset$, and since $\Delta_x(\mathcal{F}),\mathcal{S}_x(\mathcal{F}) \subset \binom{[n]-\{x\}}{k-1}$ and $|\mathcal{F}^x| = |\Delta_x(\mathcal{F})|$, we get:

$$|\mathcal{F}| = |\mathcal{F}^x| + |\mathcal{F}^{-x}| \leq |\Delta_x(\mathcal{F})| + |\mathcal{S}_x(\mathcal{F})| \leq \binom{n-1}{k-1}$$

We now suppose that $|\mathcal{S}_x(\mathcal{F})| < |\mathcal{F}^{-x}|$ and wish to show that in this case we have $|\mathcal{R}_x(\mathcal{F})| \geq |\mathcal{F}^{-x}|$. Observe first that for any $B \in \mathcal{F}$ we have $|\alpha_{\mathcal{F}}(B)| + |B - \alpha_{\mathcal{F}}(B)| = |B| = k$. Summing over all $B \in \mathcal{F}^{-x}$, we get:

$$k|\mathcal{F}^{-x}| = \sum_{B \in \mathcal{F}^{-x}} |\alpha_{\mathcal{F}}(B)| + \sum_{B \in \mathcal{F}^{-x}} |B - \alpha_{\mathcal{F}}(B)|$$

As noted before, we have $|\bigtriangledown_{\mathcal{F}}(D)| = 1$ for all $D \in \mathcal{S}_x(\mathcal{F})$. Thus, there exists a bijection between pairs $(B,y)$ with $B \in \mathcal{F}^{-x}$ and $y \in (B - \alpha_{\mathcal{F}}(B))$ and elements $D \in \mathcal{S}_x(\mathcal{F})$ as given by $D = B - \{y\}$. This gives us $\sum_{B \in \mathcal{F}^{-x}} |B - \alpha_{\mathcal{F}}(B)| = |\mathcal{S}_x(\mathcal{F})|$, and since we assumed $|\mathcal{S}_x(\mathcal{F})| < |\mathcal{F}^{-x}|$, we get:

$$\sum_{B \in \mathcal{F}^{-x}} |\alpha_{\mathcal{F}}(B)| = k|\mathcal{F}^{-x}| - \sum_{B \in \mathcal{F}^{-x}} |B - \alpha_{\mathcal{F}}(B)| > (k-1)|\mathcal{F}^{-x}|$$

...
and thus

$$\sum_{B \in \mathcal{F}_x} |\alpha_x(B)| > (k - 1)$$

(21)

Now, we wish to make a double counting argument to institute a lower bound on the size of $\mathcal{R}_x(\mathcal{F})$. In particular, we shall be counting pairs $(B, D)$, where $B \in \mathcal{F}_x$ and $D \in \left(\binom{[n]-\{x\}}{k-1}\right)$ and $(B \cap D) \in \left(\alpha_x(B)\right)$. Note that these are precisely the $D$ in $\mathcal{R}_x(\mathcal{F})$. We will count these pairs two ways. First, take $B \in \mathcal{F}_x$, and note that there are $\left(\binom{\alpha_x(B)}{d-2}\right)$ choices for $B \cap D$. Furthermore, $D - B$ is a $(k - d + 1)$-subset of $[n] - B - \{x\}$, giving us a total of $\left(\binom{n-k-1}{k-d+1}\right)$ possibilities. Thus, using (21) and Proposition 4 we get the total number of pairs to be:

$$\sum_{B \in \mathcal{F}_x} \left(\binom{|\alpha_x(B)|}{d-2}\right) \left(\binom{n-k-1}{k-d+1}\right) \geq |\mathcal{F}_x| \left(\binom{k-1}{d-2}\right) \left(\binom{n-k-1}{k-d+1}\right)$$

Now the question is: for each $D \in \mathcal{R}_x(\mathcal{F})$, for how many $B \in \mathcal{F}_x$ can we have counted the pair $(B, D)$ above? Observe that there are $\left(\binom{k-1}{d-2}\right)$ ways to pick $B \cap D$. Then, suppose $B_1, \ldots, B_s \in \mathcal{F}_x$ are such that $B_1 \cap D = B_2 \cap D = \cdots = B_s \cap D$ and $B_i \cap D \in \left(\alpha_x(B_i)\right)$ for all $i \in [s]$. Then, by Proposition 3 we get that $|B_i \cap B_j| \geq d - 1$ for all $i, j \in [s]$, and thus that $|(B_i - D) \cap (B_j - D)| \geq 1$. Thus, $(B_1 - D), \ldots, (B_s - D)$ constitutes a pairwise-intersecting family of $(k - d + 2)$-subsets of $[n] - \{x\} - D$. By assumption $n \geq 3k - 2d + 4$ and thus $n - k \geq 2(k - d + 2)$, so we may apply classical EKR to get:

$$s \leq \left(\binom{n-k-1}{k-d+2} - 1\right) = \binom{n-k-1}{k-d+1}$$

Note that in the case of $d = 2$ we are technically inducting here, but because this is the case of classical EKR we will not bother proving the base cases. Combining our two methods of counting, we get:

$$|\mathcal{R}_x(\mathcal{F})| \geq \frac{|\mathcal{F}_x| \left(\binom{k-1}{d-2}\right) \left(\binom{n-k-1}{k-d+1}\right)}{\left(\binom{k-1}{d-2}\right) \left(\binom{n-k-1}{k-d+1}\right)} = |\mathcal{F}_x|$$

(22)

Now, suppose there exists $D \in (\Delta_x(\mathcal{F}) \cap \mathcal{R}_x(\mathcal{F}))$. Then, it follows both that $(D \cup \{x\}) \in \mathcal{F}$ and that there exists $B \in \mathcal{F}_x$ such that $B \cap D \in \left(\alpha_x(B)\right)$. However, because $B \in \mathcal{F}_x$, this implies $|B \cap (D \cup \{x\})| = |B \cap D| = d - 2$, which is a contradiction by Proposition 3. Thus, $\Delta_x(\mathcal{F}) \cap \mathcal{R}_x(\mathcal{F}) = \emptyset$, and using (22) and the fact that $\Delta_x(\mathcal{F}), \mathcal{R}_x(\mathcal{F}) \subset \left(\binom{[n]}{k-1}\right)$, we get:

$$|\mathcal{F}| = |\mathcal{F}_x| + |\mathcal{F}_x| = |\Delta_x(\mathcal{F})| + |\mathcal{R}_x(\mathcal{F})| \leq \binom{n-1}{k-1}$$

□

In conclusion, we would like to note that there is quite a bit of flexibility built in to the proof of Theorem 2. In particular, we at no point leverage our choice of $x \in [n]$, nor how many $B' \in \mathcal{F}$ are such that $B - B' = \{y\}$ for a given $B \in \mathcal{F}$ and $y \in \alpha_x(B)$. One wonders, then, how far it is possible to push these results, and if the ideas above could be used to achieve a characterization of equality or perhaps to prohibit an even more restrictive kind of $d$-cluster and still achieve the same bound.

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