Online Learning of Combinatorial Objects
via Extended Formulation

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Abstract
The standard on-line learning techniques for combinatorial objects perform multiplicative updates followed by projections into the convex hull of all the objects. However, this framework can be expensive if the convex hull contains many facets. For example, the convex hull of $n$-symbol Huffman trees is known to have exponentially many facets (Maurras et al., 2010). We get around this problem by exploiting extended formulations (Kaibel, 2011), which encode the polytope of combinatorial objects into a higher dimensional space with only polynomially many facets. We develop a general framework for converting extended formulations into efficient on-line algorithms with good relative loss bounds. We present applications of our framework to on-line learning of Huffman trees and permutations. The resulting algorithms have regret bounds within a factor of $\sqrt{\log(n)}$ of the state-of-the-art for permutations, and depending on the loss regimes, are better than or within a factor of $\log(n)$ for Huffman trees. Our method is general and can be applied to other combinatorial objects. Furthermore, we believe this technique provides a promising approach for problems with non-additive losses as well as the bandit setting.

Keywords: online learning, extended formulation, combinatorial object

1. Introductions
This paper introduces a general methodology for developing efficient and effective on-line learning algorithms over combinatorial structures. Examples include learning the best permutation of a set of elements for scheduling or assignment problems, or learning the best Huffman tree for compressing sequences of symbols. On-line learning algorithms are being successfully applied to an increasing variety of problems, so it is important to have good tools and techniques for creating good algorithms that match the particular problem at hand.

The on-line learning setting proceeds in a series of trials where the algorithm makes a prediction or takes an action associated with a combinatorial object in the space and then receives the loss of its choice in such a way that the loss of any of the possible combinatorial objects can be easily computed. The algorithm can then update its internal representation based on this feedback and the process moves on to the next trial. Unlike batch learning settings, there is no assumed distribution from which losses are randomly drawn. Instead the losses are drawn adversarially. In general, an adversary can force arbitrarily large loss on the algorithm. So instead of measuring the algorithm’s performance by the total loss incurred, the algorithm is measured by its regret, the amount of loss.
the algorithm incurs above that of the single best predictor in some comparator class. Usually the comparator class is the class of predictors defined by the combinatorial space being learned. To make the setting concrete, consider the case of learning Huffman trees. In each trial, the algorithm would (perhaps randomly) predict a Huffman tree, and then obtain a sequence of symbols to be encoded. The loss of the algorithm on that trial is the average code length of the sequence. More generally, the loss is defined as the inner product of any loss vector from the unit cube and the sequence of code lengths of the symbols. Over a series of trials, the regret of the algorithm is the difference between the combined average code lengths for the on-line algorithm and for the single best Huffman tree chosen in hindsight. Therefore the regret of the algorithm can be viewed as the cost of not knowing the best combinatorial object ahead of time. With proper tuning, the regret is typically logarithmic in the number of combinatorial objects.

One way to create algorithms for these combinatorial problems is to use one of the well-known so-called “experts algorithms” like Randomized Weighted Majority (Littlestone and Warmuth, 1994) or Hedge (Freund and Schapire, 1997) with each combinatorial object treated as an “expert”. However, this requires explicitly keeping track of one weight for each of the exponentially many combinatorial objects, and thus results in an inefficient algorithm. Furthermore, it also causes an additional loss range factor in regret bounds as well. There has been much work on creating efficient algorithms that implicitly encode the weights over the set of combinatorial objects using a concise representations. For example, many distributions over the \(2^n\) subsets of \(n\) elements can be encoded by the probability of including each of the \(n\) elements. In addition to subsets, such work includes permutations (Helmbold and Warmuth, 2009; Yasutake et al., 2011), paths (Takimoto and Warmuth, 2003; Kuzmin and Warmuth, 2005), and \(k\)-sets (Warmuth and Kuzmin, 2008).

There have also been more general tools for learning combinatorial concepts. Suehiro et al. (2012) introduced efficient online learning algorithms with good regret bounds for structures that can be formulated by submodular functions\(^1\). The Component Hedge algorithm of Koolen et al. (2010) is a powerful generic technique when the implicit encodings are suitably simple.

The Component Hedge algorithm works by performing multiplicative updates on the parameters of its implicit representation. However, the implicit representation is typically constrained to lie in a convex polytope. Therefore Bregman projections are used after the update to return the implicit representation to the desired polytope. The technique for proving good relative loss bounds is to, on each trial, relate the excess loss of the algorithm to its movement (from both update and projection) towards the implicit representation of an arbitrary comparator in the class. Note that the projection step can only be efficient when there are a small (polynomial) number of constraints on the implicit representations.

The problem of concisely specifying the convex hulls of complicated combinatorial structures (e.g. permutations and Huffman trees) using few constraints has been well studied in the combinatorial optimization literature. A powerful technique – namely extended formulations – has been developed to represent these polytopes as a linear projection of a higher-dimensional polyhedron so that the polytope description has far fewer (polynomial instead of exponential) constraints (Kaibel and Pashkovich, 2013; Kaibel, 2011; Conforti et al., 2010).

Our main contribution is a general methodology exploiting these extended formulations to create efficient and effective on-line learning algorithms over combinatorial structures. This methodology creates new on-line leaning algorithms over structures like permutations that already have efficient

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\(^1\) For instance, permutations belong to such classes of structures (see Suehiro et al. (2012)); but Huffman trees do not as the sum of the code lengths of the symbols is not fixed.
algorithms, as well as efficient algorithms over structures like Huffman Trees that are not suitable for component hedge techniques.

Our methodology with extended formulations includes a novel way of producing predictions directly from the algorithm’s implicit representation. Previous techniques require that the algorithm perform a potentially expensive decomposition of its implicit representation into a convex combination of corners of its convex polytope, and then samples one of these corners to use as its prediction. In contrast, we can sample directly from the algorithm’s implicit representation without require a decomposition step.

The remainder of the paper is organized as follows. In Section 2, we discuss existing work in the area of online learning generally, and over structured concepts specifically, as well as extended formulation techniques. Section 3 explains the extended formulation used in our setting by reviewing the work by Kaibel and Pashkovich (2013). We then describe our algorithm in Section 4. Finally, Section 5 concludes with contrasting our approach to existing ones and describing directions for future work.

2. Related Work

On-line learning is a rich and vibrant area, see Cesa-Bianchi and Lugosi (2006) for a textbook treatment. The implicit representations for structured concepts (sometimes called ‘indirect representations’) have been used for a variety of problems (Helmbold et al., 2002; Helmbold and Schapire, 1997; Maass and Warmuth, 1998; Takimoto and Warmuth, 2002, 2003). In the case of permutations (Helmbold and Warmuth, 2009; Yasutake et al., 2011) and Component Hedge (Koolen et al., 2010) the implicit representations seem to be better matched with the combinatorial structure than the explicit representation, allowing not only decreased running time but also the proof of better bounds. Our more general methodology with extended formulations also has this advantage, although perhaps to a lesser extent than the permutation-specific algorithms.

As in Yasutake et al. (2011), the loss family provided in our approach is linear over the first order representation of the objects (Diaconis, 1988). Concretely, for permutations of \( n \) items, we work with vectors \( v \in \mathbb{R}^n \) in which each of the elements of \( \{1, 2, \ldots, n\} \) appears exactly once. Also for Huffman trees of \( n \) symbols, we work with vectors \( v \in \mathbb{R}^n \) in which \( v_i \) indicates the depth of the node corresponding to symbol \( i \) in the coding tree. So the loss is \( v \cdot \ell \) in which \( \ell \) is a loss vector in the unit cube \([0, 1]^n\). In contrast, Helmbold and Warmuth (2009) work with the second order representation (i.e. Birkhoff polytope), and consequently losses, which is a more general loss family (see Yasutake et al. (2011) for comparison).

There have been several works aimed at efficiently describing the polytope of different combinatorial objects like permutations (Goemans, 2015) and Huffman trees (Maurras et al., 2010). Our results rely on extended formulation, a general methodology for nicely describing combinatorial polyhedra discovered by the combinatorial optimization community (Kaibel and Pashkovich, 2013; Kaibel, 2011; Conforti et al., 2010).

3. Extended Formulation

There are many classes of combinatorial objects whose description as a polytope requires exponentially many facets in the original space (e.g. see Maurras et al. (2010)). In order to have more efficient algorithms, there have been several efforts in the field of combinatorial optimization to-
wards describing these polytopes in some other spaces. In recent years, the idea of representing these polytopes as a linear projection of a higher-dimensional polyhedron – known as extended formulation – has received significant attention. There are many combinatorial objects whose associated polyhedra can be described as the linear projection of a simpler, but higher dimensional, polyhedra (see Figure 1). See Kaibel (2011) for some of the tools for constructing such extended formulations. In the following subsections, we first overview the work by Kaibel and Pashkovich (2013), and then adapt the formulation to the on-line learning setting.

3.1. Constructing Extended Formulation from Reflection Relations

One framework for constructing polynomial size extended formulations was developed by Kaibel and Pashkovich (2013) using reflection relations. The basic idea is to start with a canonical corner of the polytope (e.g. a particular permutation) and then create a sequence of reflections through hyperplanes (e.g. swapping a pair of elements) so that any corner of the polytope can be generated by applying a subsequence of the reflections to the canonical corner. The convex hull is then generated by allowing “partial reflections” (i.e. containing the entire line segment connecting the original point and its reflection). Any point in the convex hull is then created by a sequence of partial reflections (Figure 2), and can be encoded by a sequence of variables indicating how much of each reflection was used (Figure 3). In essence, applying each additional reflection relation creates a new expanded polytope and generates new corners.

In Figure 3, we start with the single point \( P_0 \) (which is a polytope). Polytope \( P_1 \) (shown in green) is obtained by applying the reflection relation associated with hyperplane (1) to \( P_0 \). Similarly, \( P_2 \) (shown in violet) is generated by applying the reflection relation corresponding to hyperplane (2) to \( P_1 \). Observe that applying reflection relations (1) and (2) to \( P_0 \) and \( P_1 \) results in generating 1 and 2 new corners, respectively.
For each reflection relation, there will be one additional variable indicating the extent to which the reflection occurs, and two additional inequalities indicating the two extreme cases of complete reflection and remaining unchanged (see Figure 2). Therefore, if polynomially many reflection relations are used, then we can construct an extended formulation of polynomial size with polynomially many constraints. Appendix B provides more details about the type of results shown by Kaibel and Pashkovich (2013).

3.2. Extended Formulation of Objects Closed under Re-Ordering

Assume we want to construct an extended formulation for the class of combinatorial objects which is closed under any re-ordering. Note that Huffman trees and – trivially – permutations belong to such class of objects. In these cases, the reflection relations use hyperplanes going through the origin with normal vectors of the form $e_r - e_s$, whose reflections implement swapping the $r$th and $s$th elements. Now let us figure out the form of extended formulation in this particular case. First, we find the additional variable along with two additional inequalities associated with this reflection relation. Concretely, assume $v \in \mathbb{R}^n$ is going through this reflection relation and $v' \in \mathbb{R}^n$ is the output. $v'$ is in the convex combination of $v$ and its reflection. Often this would be expressed as $v' = \alpha v + (1 - \alpha) v_{\text{reflected}}$. However, it will be more convenient to parameterize $v'$ by its absolute distance $x$ from $v$, rather than the relative distance $\alpha \in [0, 1]$. Using this parameterization, we have $v' = v + x (e_r - e_s)$ constrained by $(e_r - e_s) \cdot v \leq (e_r - e_s) \cdot v' \leq -(e_r - e_s) \cdot v$. With these two properties, one can obtain the relation between $v'$ and $v$ via a linear transformation given the additional variable $x$ as well as the constraints enforced on $x$:²

\[
v' = m x + v, \quad m_i = \begin{cases} 1 & i = r \\ -1 & i = s \\ 0 & \text{otherwise} \end{cases}
\]

² In general $v$ (and thus $v_s$ and $v_r$) may be functions of the variables for previous reflection relations.
\[ 0 \leq x \leq v_s - v_r \tag{2} \]

Notice that \( x \) indicates the value that is being swapped between \( r \)th and \( s \)th elements which can go from zero (remaining unchanged) to the maximum swap capacity (complete swap). Now we can express the extended formulation using the results above. Suppose we are using \( m \) reflection relations in total. Then starting from an anchor point \( c \) and applying Equation (1) successively, we obtain the affine transformation connecting the extended formulation space \( \mathcal{X} \) and original space \( \mathcal{V} \):

\[
v = M x + c, \quad v, c \in \mathcal{V} \subset \mathbb{R}^n, \quad x \in \mathcal{X} \subset \mathbb{R}^m, \quad M \in \{-1, 0, 1\}^{n \times m}
\]

The anchor point \( c \) is \([1, 2, \ldots, n]^T\) and \([1, 2, \ldots, n - 2, n - 1, n - 1]^T\) in the permutation and Huffman tree cases, respectively. Also using all the inequalities from the reflection relations as in (2), the extended formulation space \( \mathcal{X} \) can be described as \( \mathcal{X} = \{x \in \mathbb{R}^m : Ax \leq b \text{ and } x \geq 0\} \) in which \( A \in \mathbb{R}^{m \times m} \) and \( b \in \mathbb{R}^m \). See the Appendix C for some theoretical results regarding the structure of \( A \) and \( b \).

Now the natural question is: what should these \( m \) reflection relations be so that we have a valid extended formulation for permutation/Huffman tree of size \( n \)? Kaibel and Pashkovich (2013) show that if the \( m \) reflection relations correspond to the \( m \) comparators in an arbitrary sorting network with \( n \) inputs, then the result is a valid extended formulation for permutations of size \( n \) (see Figure 4). Similarly, by using an arbitrary sorting network along with \( \tilde{O}(n \log n) \) additional comparators and simple linear maps, an extended formulation for Huffman trees can also be built (see Section 2.24 in Pashkovich (2012) for more details). Note that the order of reflection relations is reversed compared to the network of comparators. Additionally, the extended formulation of a Huffman tree/permutation is the vector of swap values of the comparators in the corresponding network. Moreover, mixtures of Huffman trees/permutations can be represented by partial swap values in the comparators (see Figure 4).

Figure 4 illustrates an example of sorting network coupled with the extended formulation for permutations of \( n = 3 \) items. The vectors \( v \) (shown in blue) indicate the elements of the original space \( \mathcal{V} \) (i.e. mixtures of permutations). The vectors \( x \) (shown in red) represent the elements of the extended formulation space \( \mathcal{X} \). Each component of \( x \) can be interpreted as the net value transferred between wires by the corresponding comparator as an input is passed through the sorting network\(^3\). The green values are the intermediate values of the wires through the sorting network.

### 3.3. Combinatorial Polytope Description in Augmented Formulation

Even though the polytope can now be described using polynomial number of facets in extended formulation \( x \in \mathcal{X} \), it is not natural to define linear loss over the elements of \( x \). However, one can concatenate the original formulation \( v \in \mathcal{V} \) to the extended formulation \( x \in \mathcal{X} \), so that it is possible to not only efficiently describe the polytope, but also provide meaningful losses such as average code length and sum of completion times in Huffman trees and permutations, respectively. So given a loss vector \( \ell \in [0, 1]^n \) with the same dimension as \( \mathcal{V} \), we can work with \((\mathcal{V}, \mathcal{X})\) as shown in the left column of Table 1.

In addition, in order to have the comfort of working with affine subspaces\(^4\), we add a positive slack vector \( \lambda \) to turn all inequalities into equalities. As a result, we define the augmented form-

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\(^3\) When sorting in ascending order, the value transferred is always subtracted from the upper wire and added to the lower one.

\(^4\) The reader will see in Section 4.2 that having affine subspaces allows us to use iterative Bregman projections with convergence guarantees.
Figure 4: An example of sorting network illustrating the extended formulation in pure and mixture permutations of \( n = 3 \) items.

| (\( \mathcal{V}, \mathcal{X} \)) | \( \mathcal{W} \) |
|-------------------------------|-----------------|
| \((v, x) \in \mathbb{R}^{n+m}, \quad L = (\ell, 0) \in \mathbb{R}^{n+m}\) | \((v, x, \lambda) \in \mathbb{R}^{n+2m}, \quad L = (\ell, 0, 0) \in \mathbb{R}^{n+2m}\) |
| s.t. \( x \geq 0, \quad A x \leq b, \quad v = M x + c \) | s.t. \( x, \lambda \geq 0, \quad A x + \lambda = b, \quad v = M x + c \) |

Table 1: Concatenating original space \( \mathcal{V} \) and extended formulation \( \mathcal{X} \) (left), and augmented formulation space \( \mathcal{W} \) (right).
lation space $\mathcal{W}$ (see the right column of the Table 1). Observe that there exists a duality between $x$ and $\lambda$: $x_i + \lambda_i$ indicates the swap value capacity at the $i$th reflection relation and comparator. Also $x_i \lambda_i = 0$ for all $i \in [m]$ for arbitrary pure Huffman trees/permutations since each comparator either passes through or completely swaps its input elements.

4. Algorithm

In this section, we describe the new Extended-Learn algorithm and prove its regret bounds. Let $\mathcal{H}$ be the discrete set of all instances of the combinatorial object (e.g. the set of all $n!$ permutations).

The main idea of our algorithm is to maintain a distribution over $\mathcal{H}$ by keeping track of an evolving point $w = (v^t, x^t, \lambda^t)$ in the augmented formulation space $W$ through trials $t = 1, \ldots, T$. The main structure of Extended-Learn is shown in Algorithm 1. Similar to (Koolen et al., 2010), our algorithm consists of three main steps:

1. **Prediction**: Predict with an instance $\gamma^{t-1} \in \mathcal{H}$ such that $\mathbb{E}[\gamma^{t-1}] = v^{t-1}$

2. **Update**: Multiplicatively update the mixture $w^{t-1}$ to $\hat{w}^{t-1}$ according to the incurred loss.

3. **Projection**: Project the updated mixture $\hat{w}^{t-1}$ back to the polytope $W$ and obtain $w^t$.

In the prediction step, which is discussed in 4.1, we draw an instance probabilistically from the distribution latent in $w^{t-1} = (v^{t-1}, x^{t-1}, \lambda^{t-1}) \in W$ such that it has the same expected value as $v^{t-1} \in \mathcal{V}$. For the update step, having defined $L^t = (\ell^t, 0, 0)$, the updated $\hat{w}^{t-1}$ is obtained from a trade-off between the linear loss and the unnormalized relative entropy (Koolen et al., 2010):

$$\hat{w}^{t-1} = \arg \min_{w \in \mathbb{R}^{n+2m}} \Delta(w || w^{t-1}) + \eta w \cdot L^t$$

It is fairly straight-forward to see:

$$\forall i \in [n+2m], \hat{w}^{t-1}_i = w^{t-1}_i e^{-\eta \ell^t_i} \quad \Rightarrow \quad \begin{cases} \forall i \in [n], \hat{v}^{t-1}_i = v^{t-1}_i e^{-\eta \ell^t_i} \\ \forall i \in [m], \hat{x}^{t-1}_i = x^{t-1}_i \\ \forall i \in [m], \hat{\lambda}^{t-1}_i = \lambda^{t-1}_i \end{cases}$$

In the projection step, we obtain $w^t$, the Bregman projection of $\hat{w}^{t-1}$ back to the augmented formulation space $W$,

$$w^t = \arg \min_{w \in W} \Delta(w || \hat{w}^{t-1})$$

for which we propose an iterative approach in Section 4.2.
Algorithm 1 Extended-Learn

1: \( w^0 \leftarrow q \in \mathcal{W} \) - a proper prior distribution discussed in 4.3
2: For \( t = 1, \ldots, T \)
3: \( w^{t-1} = (v^{t-1}, x^{t-1}, \lambda^{t-1}) \in \mathcal{W} \)
4: Execute Prediction\((w^{t-1})\) and get a random instance \( \gamma^{t-1} \in \mathcal{H} \) s.t. \( \mathbb{E}[\gamma^{t-1}] = v^{t-1} \)
5: Incur a loss \( \gamma^{t-1} \cdot \ell^t \)
6: Update:
7: Set \( \tilde{x}^{t-1} \leftarrow x^{t-1} \), and \( \tilde{\lambda}^{t-1} \leftarrow \lambda^{t-1} \)
8: Set \( \tilde{v}_i^{t-1} \leftarrow v_i^{t-1} e^{-\eta \ell_i^t} \) for all \( i \in [n] \)
9: Set \( \tilde{w}^{t-1} = (\tilde{v}^{t-1}, \tilde{x}^{t-1}, \tilde{\lambda}^{t-1}) \)
10: Execute Projection\((\tilde{w}^{t-1})\) and obtain \( w^t \) which is \( w^t = \arg \min_{w \in \mathcal{W}} \Delta(w || \tilde{w}^{t-1}) \)

4.1. Prediction

In this subsection, we describe how to probabilistically predict with an instance such that it has the same expected value as the mixture vector \( v^t \) of which we are keeping track. First we propose an algorithm for decomposing any mixture point into convex combination of instances. Despite its inefficiency, it leads us to another algorithm which does efficiently produce proper predictions.

**Inefficient Decomposition**  The decomposition algorithm is shown in Algorithm 2 in which \( T_{rs} \) is row-switching matrix that is obtained from switching \( r \)th and \( s \)th row of identity matrix. The main idea of the algorithm is to exploit the notion of partial swaps in the comparators corresponding to reflection relations in our extended formulation. In other words, at each comparator, we want to decompose based on the extent to which we used swap capacity. One can show that Algorithm 2, despite its inefficiency, results in valid convex combination of instances (see the Appendix D for the proof):

**Lemma 1**  (i) Given \( (v, x, \lambda) \in \mathcal{W} \), Algorithm 2 generates a decomposition set \( S \) of \( (\gamma, p_\gamma) \) pairs with \( \gamma \in \mathcal{H} \) such that \( v = \sum_{(\gamma, p_\gamma) \in S} p_\gamma \gamma \). (ii) The time complexity of Algorithm 2 is \( O(2^m) \).
Algorithm 2 Extended-Decomposition

1: Input: \((v, x, \lambda) \in W \subseteq \mathbb{R}^{n+2m}\)
2: Output: A decomposition set \(S\) of (instance, probability)-pairs – \((\gamma, p_\gamma)\)
3: \(S(0) \leftarrow \{(c, 1)\}\)
4: for \(i = 1\) to \(m\) do
5: \((r_i, s_i) \leftarrow\) components associated with the \(i\)-th comparator
6: if \(x_i = 0\) then
7: \(S(i) \leftarrow S(i-1)\) – no reflection
8: else if \(\lambda_i = 0\) then
9: \(S(i) \leftarrow \{(T_{r_i} s_i, p_\gamma) \mid (\gamma, p_\gamma) \in S(i-1)\}\) – full reflection
10: else
11: \(S(i) \leftarrow \emptyset\) and \(p_i \leftarrow \frac{x_i}{x_i + \lambda_i}\)
12: for \((\gamma, p_\gamma) \in S(i-1)\) do
13: Add \((\gamma, (1 - p_i)p_\gamma)\) and \((T_{r_i} s_i, p_\gamma, p_\gamma)\) to \(S(i)\)
14: end for
15: end if
16: end for
17: return \(S(m)\)

Efficient Prediction

In order to achieve efficiency, despite the vastly common idea in the literature (Helmbold and Warmuth, 2009; Koolen et al., 2010; Yasutake et al., 2011; Warmuth and Kuzmin, 2008), one can avoid decomposition and do prediction directly. To this purpose, in Algorithm 2, we replace the idea of partial swaps with probabilistic swaps. Algorithm 3 describes this idea in concrete terms. It can be shown (see the proof in the Appendix E) that Algorithm 3 does proper prediction efficiently:

Lemma 2  (i) For all \(x, \lambda \geq 0\), the distribution produced by Algorithm 2 with inputs \(x\) and \(\lambda\) is the distribution sampled by Algorithm 3 with inputs \(x\) and \(\lambda\). (ii) The time complexity of Algorithm 3 is \(O(m)\).

Algorithm 3 Extended-Prediction

1: Input: \((x, \lambda) \in \mathbb{R}^{2m}_+\)
2: Output: A prediction \(\gamma \in \mathcal{H}\)
3: \(\gamma \leftarrow c\)
4: for \(i = 1\) to \(m\) do
5: \((r_i, s_i) \leftarrow\) components associated with the \(i\)-th comparator
6: if \(x_i = 0\) then
7: continue
8: else
9: \(\gamma \leftarrow\) \(T_{r_i} s_i \gamma\) with probability \(x_i/(x_i + \lambda_i)\)
10: \(\gamma\) with probability \(\lambda_i/(x_i + \lambda_i)\)
11: end if
12: end for
13: return \(\gamma\)
4.2. Projection
Formally, the last step is to find the \( \Delta \)-projection of the multiplicatively-updated \( \hat{w}^t \) back onto set \( W \):

\[
P^\Delta_W(\hat{w}^{t-1}) := \arg \min_{w \in W} \Delta(w||\hat{w}^{t-1})
\]

where \( \Delta(w_1||w_2) = \sum_i w_{1,i} \log \frac{w_{1,i}}{w_{2,i}} + w_{2,i} - w_{1,i}. \)

\( \Delta(\cdot||\cdot) \) is known as the unnormalized relative entropy. Observe that \( W \) is an intersection of affine subspaces defined by the \( m + n \) equality constraints in \( Ax + x = b \) and \( Mx + c = v \). Call these constraints \( C_1, \ldots, C_{m+n} \). Since the non-negativity constraints are already enforced by the definition of \( \Delta \), it is possible to solve (3) by simply using iterative \( \Delta \)-projections\(^5\) (Bregman, 1967). Starting from \( p_0 = \hat{w}^{t-1} \), we iteratively compute:

\[
\forall k > 0, \quad p_k = P^\Delta_{C_k}(p_{k-1})
\]

in which the index \( k \) repeatedly cycles through the constraints. In Appendix F, we discuss how one can efficiently project onto each hyperplane \( C_k \) for all \( k \in [n+m] \). It is known that \( p_k \) converges in norm to the unique solution of (3) (Bregman, 1967; Bauschke and Borwein, 1997). Appendix J provides an analysis showing that the additional loss incurred by Algorithm I due to working with approximate projection is negligible.

4.3. Regret Bounds
At each trial \( t \), the algorithm predicts a random instance \( \gamma_t \in H \) from the current \( w_t \) in the augmented space \( W \), and incurs loss \( \gamma_t \cdot \ell_t \). We prove the following bound on the algorithm’s cumulative expected loss relative to any fixed comparator \( \gamma \in H \) starting with initial point \( w^0 \in W \).

**Lemma 3** For all \( \gamma \in H \), we have \( \mathbb{E} \left[ \sum_{t=1}^{T} \gamma_t \cdot \ell_t \right] \leq \frac{\eta \sum_{t=1}^{T} \gamma \cdot \ell_t + \Delta(\theta||w^0)}{1 - e^{-\eta}} \) where \( \theta \in W \) is the augmented formulation corresponding to the instance \( \gamma \in H \).

The proof is standard in the online learning literature (see, e.g., Koolen et al. (2010)) and is shown in the Appendix G. Now, we prove that there exists a good initial point \( w^0 \) in \( W \) such that \( \Delta(\theta||w^0) \) is appropriately bounded (shown in the Appendix H).

**Lemma 4** Assume we are working with \( m \) reflection relations. Also assume that associated network does not have any redundant comparator. Then given \( n \leq m \leq n^2 \), there exists \( q \in W \) such that for all \( p \in W \), we have \( \Delta(p||q) \leq 16m n \log n \).

By Lemmas 3 and 4, we can now show the main regret bound result (see the Appendix I for the proof):

**Theorem 5** Assume the number of reflection relations used in the extended formulation is \( m \in [n, n^2] \)\(^7\) and none of the corresponding comparators in the network are redundant. Then, given

\(^5\) In Helmbold and Warmuth (2009) Sinkhorn balancing is used for projection which is also a special case of iterative Bregman projection.

\(^6\) The construction of \( q \) can be done as a pre-processing step and may depend on the network of comparators. For instance, if we use the bubble-sort sorting network in case of permutation, the average of the augmented formulation of the two permutation \([1,2,\ldots,n]\) and \([n,n-1,\ldots,1]\) will be a good choice.

\(^7\) We still have \( \Delta(p||q) = O(m n \log n) \) and the same bounds asymptotically with \( m = \text{poly}(n) \).
Table 2: Comparing the regret bounds of Extended-Learn with other existing algorithms in different problems and different loss regimes.

| Algorithm            | Permutation                          | Huffman Tree                        |
|----------------------|--------------------------------------|-------------------------------------|
|                      | \(\ell^t \in \text{Unit Cube} \)    | \(\ell^t \in \text{Unit Simplex} \) |
| Extended-Learn       | \(O(n^2 \log(n) \sqrt{T})\)         | \(O(n^2 \log(n) \sqrt{T})\)       |
| Permutation and       |                                      |                                     |
| PermutahedLearn       | \(O(n^2 \log(n) \frac{3}{2} \sqrt{T})\) | \(-\)                               |
| FPL                  | \(O(n^{2.5} \sqrt{T})\)             | \(O(n^{1.5} \sqrt{T})\)           |
| Hedge Algorithm      | \(O(n^{2.5} (\log n)^{\frac{1}{2}} \sqrt{T})\) | \(O(n^{1.5} (\log n)^{\frac{1}{2}} \sqrt{T})\) |

\(\ell^t \in [0, 1]^n\) for all \(t = 1 \ldots T\), we have:

\[
\mathbb{E} \left[ \sum_{t=1}^{T} \gamma^{t-1} \cdot \ell^t \right] \leq \frac{\eta \min_{\gamma \in \mathcal{H}} \sum_{t=1}^{T} \gamma \cdot \ell^t + 16 \, m \, n \, \log n}{1 - e^{-\eta}}
\]

Furthermore, in both cases of permutations and Huffman trees, by choosing a sorting network of size \(m = O(n \log n)\) and tuning \(\eta\) appropriately, the expected regret of Extended-Learn is at most \(O(n^2 \log(n) \sqrt{T})\).

5. Conclusion and Future Work

Table 2 contains a comparison of the regret bounds for the new Extended-Learn algorithm, previous algorithms for permutations, Follow the Perturbed Leader (FPL) (Kalai and Vempala, 2005), and the Hedge algorithm (Freund and Schapire, 1997) which inefficiently maintains an explicit weight for each of the exponential in \(n \log n\) permutations or Huffman trees. For permutations, using loss vectors from the general unit cube leads to the scheduling loss from Yasutake et al. (2011) that has range \([0, n^2]\) per trial. When compared with the state of the art implicit algorithms PermE-Learn (Helmbold and Warmuth, 2009) and PermutahedLearn (Yasutake et al., 2011), the general Extended-Learn methodology has a small additional regret bound penalty of \(\sqrt{\log(n)}\). When compared with the generic explicit Hedge algorithm (which is not computationally efficient) and FPL, Extended-Learn has a better loss bound by a factor of \(\sqrt{n/ \log n}\) and \(\sqrt{n/ \log n}\), respectively.

When comparing Extended-Learn with explicit Hedge and FPL on Huffman trees, we consider two loss regimes: (i) one where the loss vectors are from the general unit cube, and consequently, the per-trial losses are in \([0, n^2]\) (like permutations), (ii) and one where the loss vectors represent frequencies and lie on the unit simplex so the per-trial losses are in \([0, n]\). In the first case, Extended-Learn, FPL, and Hedge have the same asymptotic bounds as with permutations. In the second case, the lower loss range favors Hedge and FPL, and their bounds are slightly better by a factor of \(\sqrt{\log n}\) and \(\log n\), respectively.

In traditional on-line learning settings, projections are exact (i.e. renormalizing a weight vector). In contrast, for combinatorial objects iterative Bregman projections are often used (Koolen et al., 2010; Helmbold and Warmuth, 2009). These methods are known to converge to the exact projection theoretically (Bregman, 1967; Bauschke and Borwein, 1997) and are reported to be empirically very efficient (Koolen et al., 2010). However, the iterative nature of the projection step necessitates an
analysis such as the one in Appendix J to bound the additional loss incurred due to stopping short of full convergence.

In conclusion, we have presented a general methodology for creating on-line learning algorithms from extended formulations constructed by reflection relations. Because these extended formulations are designed to describe complex polytopes using a manageable number of constraints, they allow efficient learning on complicated structures, like the polytope of Huffman trees. Several important areas remain for potentially fruitful future work:

5.1. Extended Formulation for other Combinatorial Objects

In this paper, we focused on the extended formulations using reflection relations and sorting-type networks. However, the underlying ideas are more widely applicable. We will still have efficient prediction as long as the extended formulation is coupled with an algorithm which can efficiently traverse the space of the combinatorial object in polynomial time. There is a rich literature on extended formulation for different combinatorial objects (Conforti et al., 2010; Kaibel, 2011; Pashkovich, 2012; Afshari Rad and Kakhki, 2017; Fiorini et al., 2013). For example, given a dynamic programming algorithm for a combinatorial optimization problem (e.g. longest common subsequence problem), one can often derive an extended formulation for the associated polytope (Kaibel, 2011), and hence, an efficient online learning algorithm using the underlying ideas of Extended-Learn.

5.2. Non-Additive Losses

In Algorithm 1, we assign zero loss to the extended formulation components ($x$) and its associated slack variables ($\lambda$). However, if these additional variables indicate some meaningful parameters in the combinatorial object, one can define losses over these variable as well. For instance, in many natural language processing applications such as speech recognition, or in optical character recognition, an edit-distance with non-uniform edit costs is used as the loss. Since there is a dynamic programming solution for edit-distance, one can potentially derive an extended formulation for the set of strings (Kaibel, 2011), define losses over insert, delete and edit operations, and consequently, develop an online learning algorithm similar to Extended-Learn.

5.3. Bandit Setting

With the help of extended formulation, we can now describe ill-behaved polytopes using polynomially many constraints in the extended space, and as a result, have efficient projection. Now that we can project efficiently onto these polytopes, perhaps we can also solve the problem in the bandit setting as well using algorithms similar to those introduced in Audibert et al. (2011).

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References

Maria Afshari Rad and Hossein Taghizadeh Kakhki. Two extended formulations for cardinality maximum flow network interdiction problem. *Networks*, 2017.

Miklós Ajtai, János Komlós, and Endre Szemerédi. Sorting inc logn parallel steps. *Combinatorica*, 3(1):1–19, 1983.

Jean-Yves Audibert, Sébastien Bubeck, and Gábor Lugosi. Minimax policies for combinatorial prediction games. In *COLT*, volume 19, pages 107–132, 2011.

Heinz H Bauschke and Jonathan M Borwein. Legendre functions and the method of random bregman projections. *Journal of Convex Analysis*, 4(1):27–67, 1997.

Lev M Bregman. The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming. *USSR computational mathematics and mathematical physics*, 7(3):200–217, 1967.

Nicolo Cesa-Bianchi and Gábor Lugosi. *Prediction, learning, and games*. Cambridge university press, 2006.

Michele Conforti, Gérard Cornuéjols, and Giacomo Zambelli. Extended formulations in combinatorial optimization. *4OR: A Quarterly Journal of Operations Research*, 8(1):1–48, 2010.

Frank Deutsch. Dykstras cyclic projections algorithm: the rate of convergence. In *Approximation Theory, Wavelets and Applications*, pages 87–94. Springer, 1995.

Inderjit S Dhillon and Joel A Tropp. Matrix nearness problems with bregman divergences. *SIAM Journal on Matrix Analysis and Applications*, 29(4):1120–1146, 2007.

Persi Diaconis. Group representations in probability and statistics. *Lecture Notes-Monograph Series*, 11:i–192, 1988.

Samuel Fiorini, Volker Kaibel, Kanstantsin Pashkovich, and Dirk Oliver Theis. Combinatorial bounds on nonnegative rank and extended formulations. *Discrete mathematics*, 313(1):67–83, 2013.

Yoav Freund and Robert E Schapire. A decision-theoretic generalization of on-line learning and an application to boosting. *Journal of computer and system sciences*, 55(1):119–139, 1997.

Michel X Goemans. Smallest compact formulation for the permutahedron. *Mathematical Programming*, 153(1):5–11, 2015.

David P Helmbold and Robert E Schapire. Predicting nearly as well as the best pruning of a decision tree. *Machine Learning*, 27(1):51–68, 1997.

David P Helmbold and Manfred K Warmuth. Learning permutations with exponential weights. *The Journal of Machine Learning Research*, 10:1705–1736, 2009.

David P Helmbold, Sandra Panizza, and Manfred K Warmuth. Direct and indirect algorithms for on-line learning of disjunctions. *Theoretical Computer Science*, 284(1):109–142, 2002.
Mark Herbster and Manfred K Warmuth. Tracking the best linear predictor. *The Journal of Machine Learning Research*, 1:281–309, 2001.

Volker Kaibel. Extended formulations in combinatorial optimization. *arXiv preprint arXiv:1104.1023*, 2011.

Volker Kaibel and Kanstantsin Pashkovich. Constructing extended formulations from reflection relations. In *Facets of Combinatorial Optimization*, pages 77–100. Springer, 2013.

Adam Kalai and Santosh Vempala. Efficient algorithms for online decision problems. *Journal of Computer and System Sciences*, 71(3):291–307, 2005.

Wouter M Koolen, Manfred K Warmuth, and Jyrki Kivinen. Hedging structured concepts. 2010.

Dima Kuzmin and Manfred K Warmuth. Optimum follow the leader algorithm. In *Learning Theory*, pages 684–686. Springer, 2005.

Nick Littlestone and Manfred K Warmuth. The weighted majority algorithm. *Information and Computation*, 108(2):212–261, 1994.

Wolfgang Maass and Manfred K Warmuth. Efficient learning with virtual threshold gates. *Information and Computation*, 141(1):66–83, 1998.

Jean-François Maurras, Thanh Hai Nguyen, and Viet Hung Nguyen. On the convex hull of Huffman trees. *Electronic Notes in Discrete Mathematics*, 36:1009–1016, 2010.

Kanstantsin Pashkovich. *Extended formulations for combinatorial polytopes*. PhD thesis, Otto-von-Guericke-Universität Magdeburg, 2012.

Daiki Suehiro, Kohei Hatano, Shuji Kijima, Eiji Takimoto, and Kiyohito Nagano. Online prediction under submodular constraints. In *International Conference on Algorithmic Learning Theory*, pages 260–274. Springer, 2012.

Eiji Takimoto and Manfred K Warmuth. Predicting nearly as well as the best pruning of a planar decision graph. *Theoretical Computer Science*, 288(2):217–235, 2002.

Eiji Takimoto and Manfred K Warmuth. Path kernels and multiplicative updates. *The Journal of Machine Learning Research*, 4:773–818, 2003.

Manfred K Warmuth and Dima Kuzmin. Randomized online pca algorithms with regret bounds that are logarithmic in the dimension. *Journal of Machine Learning Research*, 9(10):2287–2320, 2008.

Shota Yasutake, Kohei Hatano, Shuji Kijima, Eiji Takimoto, and Masayuki Takeda. Online linear optimization over permutations. In *Algorithms and Computation*, pages 534–543. Springer, 2011.
Appendix A. Table of Notations

| Symbol | Description |
|--------|-------------|
| $n$    | dimensionality of the object |
| $H$    | the set of all the objects $- H \subset \mathbb{R}^n$ |
| $\gamma$ | a particular object $- \gamma \in H$ |
| $V$    | the convex hull of all the objects $- V \subset \mathbb{R}^n$ |
| $v$    | a mixture of the objects $- v \in V$ |
| $\ell$ | the loss vector $- \ell \in [0, 1]^n$ |
| $m$    | the size of the extended formulation |
| $X$    | the space of extended formulation $- X \subset \mathbb{R}^m$ |
| $x$    | a particular extended formulation for a mixture of the objects $- x \in X$ |
| $e_r$  | the $r$-th unit vector in $\mathbb{R}^n$ |
| $M$    | the $n \times m$ matrix representing the affine transformation corresponding to $m$ reflection relations |
| $M_k$  | the $k$-th column of $M$ |
| $c$    | the anchor point in $H$ e.g. $[1, 2, \ldots, n]^T$ for permutations |
| $A, b$ | the $m \times m$ matrix of coefficients and $m$-dimensional vector of constant terms specifying $X$ along $x \geq 0$ i.e. $Ax \leq b$ |
| $\lambda$ | the vector of slack variables in $\mathbb{R}^m$ |
| $\mathcal{W}$ | the augmented space of $(v, x, \lambda) \in \mathbb{R}^{n+2m}$ |
| $w$    | an element in the augmented space $\mathcal{W}$ |
| $T_{rs}$ | row-switching matrix obtained from switching $r$-th and $s$-th row of $I_{n \times n}$ |
| $\Delta(|| \cdot ||)$ | the unnormalized relative entropy i.e. $\Delta(w_1||w_2) = \sum_i w_{1,i} \log \frac{w_{1,i}}{w_{2,i}} + w_{2,i} - w_{1,i}$ |
| $P_C^\Delta(\cdot)$ | $\Delta$-projection into the set $C$ i.e. $P_C^\Delta(p) := \arg\min_{q \in C} \Delta(q||p)$ |
| $\theta$ | the augmented formulation corresponding to a particular instance $\gamma \in H$ |
| $T$    | the total number of trials |

Table 3: Table of notations in the order of appearance in the paper.

Appendix B. Construction of Extended Formulation Using Reflection Relations

Instead of starting with a single corner, one could also consider passing an entire polytope as an input through the sequence of (partial) reflections to generate a new polytope. Using this fact, Theorem 1 in Kaibel and Pashkovich (2013) provides an inductive construction of higher dimensional polytopes via sequences of reflection relations. Concretely, let $P_{\text{obj}}^n$ be the polytope of a given combinatorial object of size $n$. The typical approach is to properly embed $P_{\text{obj}}^n \subset \mathbb{R}^n$ into $\tilde{P}_{\text{obj}}^n \subset \mathbb{R}^{n+1}$, and then feed it through an appropriate sequence of reflection relations as an input polytope in order to obtain an extended formulation for $\tilde{P}_{\text{obj}}^{n+1} \subset \mathbb{R}^{n+1}$. Theorem 1 in Kaibel and Pashkovich (2013) provides sufficient conditions for the correctness of this procedure. Again, if polynomially many reflection relations are used to go from $n$ to $n + 1$, then we can construct an extended formulation...
of polynomial size for $P_{\text{obj}}$ with polynomially many constraints. In this paper, however, we work with batch construction of the extended formulation as opposed to the inductive construction.

### Appendix C. Structure of Inequality Constraints in Extended Formulation

**Lemma 6** Let $M$ be the matrix representing the affine transformation corresponding to $m$ reflection relations and

$$A = \text{Tri}(M^T M) + I, \quad b = -M^T c$$

in which $\text{Tri}(\cdot)$ is a function over square matrices which zeros out the upper triangular part of the input including the diagonal. Then the extended formulation space $\mathcal{X}$ is

$$Ax \leq b, \quad x \geq 0$$

or equivalently with slack variables $\lambda$

$$Ax + \lambda = b, \quad x, \lambda \geq 0$$

**Proof** Let $v^k$ be the vector in $\mathcal{V}$ after going through the $k$th reflection relation. Also denote the $k$th column of $M$ by $M_k$. Observe that $v^0 = c$ and $v^k = c + \sum_{i=1}^{k} M_i x_i$. Let $M_k = e_r - e_s$. Then, using (2), the inequality associated with the $k$th row of $Ax \leq b$ will be obtained as below:

$$x_k \leq v^k_s - v^{k-1}_s = -M^T_k v^{k-1} = -M^T_k \left( c + \sum_{i=1}^{k-1} M_i x_i \right)$$

$$\rightarrow x_k + \sum_{i=1}^{k-1} M_i^T M_i x_i \leq -M^T_k c = b_k$$

Thus:

$$\forall i, j \in [m]\quad A_{ij} = \begin{cases} M^T_i M_j & i > j \\ 1 & i = j \\ 0 & i < j \end{cases}, \quad \forall k \in [m] \quad b_k = -M^T_k c$$

which concludes the proof.

### Appendix D. Proof of Lemma 1

**Proof** Let $x = [x_1, x_2, \ldots, x_m]^T$. Using induction, we prove that by the end of the $i$th iteration of Algorithm 2, $S^{(i)}$ is the correct decomposition for $x^{(i)} = [x_1 \ldots x_i 0 \ldots 0]$. Concretely, $\sum_{(\gamma, p\gamma) \in S^{(i)}} p\gamma M = M x^{(i)} + c$. The desired result is obtained by setting $i = m$ as $v = M x + c$. The base case $i = 0$ (i.e. before the first iteration of the loop) is indeed true, since $S^{(0)}$ is initialized to $\{(c, 1)\}$, and $x^{(0)} = 0$, thus we have $v^{(0)} = M x^{(0)} + c = c$. Now assume that by the end of the $(k-1)$st iteration we have the right decomposition namely $v^{(k-1)} = \sum_{(\gamma, p\gamma) \in S^{(k-1)}} p\gamma \gamma$. Also
The $k$th comparator is applied on the $r$th and $s$th element. Thus the $k$th column of $M$ will be $M_k = e_r - e_s$. Now, according to (2) the swap capacity at $k$th comparator is:

$$x_k + \lambda_k = v_k^{k-1} - v_r^{k-1}$$

$$= \sum_{(\gamma, p\gamma) \in S^{(k-1)}} p_\gamma (\gamma_s - \gamma_r)$$

$$= -\sum_{(\gamma, p\gamma) \in S^{(k-1)}} p_\gamma M_k^T \gamma$$

$$= -M_k^T \sum_{(\gamma, p\gamma) \in S^{(k-1)}} p_\gamma \gamma$$

(5)

Now observe:

$$v_k = Mx_k + c$$

$$= x_kM_k + Mx_k^{k-1} + c$$

$$= x_kM_k + v_k^{k-1}$$

$$= x_kM_k + \sum_{(\gamma, p\gamma) \in S^{(k-1)}} p_\gamma \gamma$$

$$= p_kM_k (x_k + \lambda_k) + \sum_{(\gamma, p\gamma) \in S^{(k-1)}} p_\gamma \gamma$$

$$= -p_kM_kM_k^T \sum_{(\gamma, p\gamma) \in S^{(k-1)}} p_\gamma \gamma + \sum_{(\gamma, p\gamma) \in S^{(k-1)}} p_\gamma \gamma$$

According to (5)

$$= (I - p_kM_kM_k^T) \sum_{(\gamma, p\gamma) \in S^{(k-1)}} p_\gamma \gamma$$

$$= (I - p_kI + p_kT_{rs}) \sum_{(\gamma, p\gamma) \in S^{(k-1)}} p_\gamma \gamma$$

$$= \sum_{(\gamma, p\gamma) \in S^{(k-1)}} (1 - p_k) p_\gamma \gamma + p_k p_\gamma T_{rs} \gamma$$

$$= \sum_{(\gamma, p\gamma) \in S^{(k)}} p_\gamma \gamma$$

in which $T_{rs}$ is row-switching matrix that is obtained from switching $r$th and $s$th row from identity matrix. This concludes the inductive proof. It is easy to see that the time complexity of the algorithm is $O(2^m)$ since $|S^{(i)}|$ at most doubles each iteration and we have to go through all elements of it in each iteration.

---

8. Note that $s > r$ as in sorting networks the swap value is propagated to lower wires.
Appendix E. Proof of Lemma 2

**Proof** For each comparator $i \in [m]$ in the network, the Algorithm 3 defines the distribution below for the action of the comparator $i$:

$$P(\text{action}_i) = \begin{cases} p_i & \text{action}_i = \text{swap} \\ 1 - p_i & \text{action}_i = \text{pass} \end{cases}$$

It is easy to see from Lemma 1 that the distribution of a given instance is drawn as below:

$$P(\gamma) = \prod_{i=1}^{m} P(\text{action}_i)$$

That is, the distribution over instances $\gamma \in \mathcal{H}$ is decomposed into individual actions of swap/pass through the network of comparators independently. Thus one can draw an instance according to the distribution by simply doing independent Bernoulli trials associated with the comparators. It is also easy to see that the time complexity of the algorithm is $O(m)$ since one just needs to do $m$ Bernoulli trials. ■

Appendix F. Projection onto Each Constraint

Each constraint of the polytope in the augmented formulation is of the form $a^T w = a_0$. Formally, the projection $w^*$ of a give point $w$ to this constraint is solution to the following:

$$\arg \min_{a^T w = a_0} \sum_i w_i^* \log \left( \frac{w_i^*}{w_i} \right) + w_i - w_i^*$$

Finding the solution to the projection above for general hyperplanes and Bregman divergence can be found in Section 3 of Dhillon and Tropp (2007). Nevertheless, for the sake of completeness, we also provide the solution in our particular case as well. Using the method of Lagrange multipliers, we have:

$$L(w^*, \mu) = \sum_i w_i^* \log \left( \frac{w_i^*}{w_i} \right) + w_i - w_i^* - \mu \left( \sum_{j=1}^{2m+n} a_i w_i^* - a_0 \right)$$

$$\frac{\partial L}{\partial w_i^*} = \log \left( \frac{w_i^*}{w_i} \right) - \mu a_i = 0, \quad \forall i \in [n + 2m]$$

$$\frac{\partial L}{\partial \mu} = \sum_{j=1}^{2m+n} a_i w_i^* - a_0 = 0$$

Replacing $\beta = e^{-\mu}$, we have $w_i^* = w_i \beta^{a_i}$. By enforcing $\frac{\partial L}{\partial \mu} = 0$, one needs to find $\beta > 0$ such that:

$$\sum_{i=1}^{n+2m} a_i w_i \beta^{a_i} - a_0 = 0 \quad (6)$$
Observe that due to the structure of matrices $M$ and $A$ (see Lemma 6), $a_i \in \mathbb{Z}$ and $a_i \geq -1$ for all $i \in [n + 2m]$, and furthermore $a_0 \geq 0$. Thus we can re-write the equation (6) as the polynomial below:

$$f(\beta) = \alpha_k \beta^k + \ldots + \alpha_2 \beta^2 - \alpha_1 \beta - \alpha_0 = 0$$

in which all $\alpha_i$’s are positive real numbers. Note that $f(0) < 0$ and $f(\beta) \to +\infty$ as $\beta \to +\infty$. Thus $f(\beta)$ has at least one positive root. However, it can not have more than one positive roots and we can prove it by contradiction. Assume that there exist $0 < r_1 < r_2$ such that $f(r_1) = f(r_2) = 0$. Since $f$ is convex on positive real line, using Jensen’s inequality, we can obtain the contradiction below:

$$0 = f(r_1) = f\left(\frac{r_2 - r_1}{r_2} \times 0 + \frac{r_1}{r_2} \times r_2\right) < \frac{r_2 - r_1}{r_2} f(0) + \frac{r_1}{r_2} f(r_2) = \frac{r_2 - r_1}{r_2} f(0) < 0$$

Therefore $f$ has exactly one positive root which can be found by Newton’s method starting from a sufficiently large initial point. Note that if the constraint belongs to $v = Mx + c$, since all the coefficients are in $\{-1, 0, 1\}$, the $f$ will be quadratic and the positive root can be found through the closed form formula.

**Appendix G. Proof of Lemma 3**

**Proof** Assuming $w = (v, x, \lambda)$, $\theta = (\gamma, x, \lambda)$ and $L = (\ell, 0, 0)$:

$$(1 - e^{-\eta})w^{t-1} \cdot \ell^t = (1 - e^{-\eta})w^{t-1} \cdot L^t \leq \sum_i w^{t-1}_i (1 - e^{-\eta} L^t_i)$$

$$= \Delta(\theta || w^{t-1}) - \Delta(\theta || \hat{w}^{t-1}) + \eta \theta \cdot \ell^t$$

$$= \Delta(\theta || w^{t-1}) - \Delta(\theta || \hat{w}^{t-1}) + \eta \gamma \cdot \ell^t$$

$$\leq \Delta(\theta || w^{t-1}) - \Delta(\theta || w^T) + \eta \gamma \cdot \ell^t$$

The first inequality is obtained using $1 - e^{-nx} \geq (1 - e^{-n})x$ for $x \in [0, 1]$ as done in Littlestone and Warmuth (1994). The second inequality is a result of the Generalized Pythagorean Theorem (Herbster and Warmuth, 2001), since $w^t$ is a Bregman projection of $\hat{w}^{t-1}$ into the convex set $W$ which contains $\theta$. By summing over $t = 1 \ldots T$ and using the non-negativity of divergences, we obtain:

$$(1 - e^{-\eta}) \sum_{t=1}^T w^{t-1} \cdot \ell^t \leq \Delta(\theta || w^0) - \Delta(\theta || w^T) + \eta \sum_{t=1}^T \gamma \cdot \ell^t$$

$$\rightarrow \mathbb{E} \left[ \sum_{t=1}^T \gamma^{t-1} \cdot \ell^t \right] \leq \frac{\eta \sum_{t=1}^T \gamma \cdot \ell^t + \Delta(\theta || w^0)}{1 - e^{-\eta}}$$


Appendix H. Proof of Lemma 4

Proof  According to the definition:

\[
\Delta(p||q) = \sum_i p_i \log \frac{p_i}{q_i} + q_i - p_i = \sum_i p_i \log p_i - p_i \log q_i + q_i - p_i
\]

We will bound each term of the expression above. First observe that, given \( S_p = \sum_i p_i \), we have:

\[
\sum_i \frac{p_i}{S_p} \log \frac{p_i}{S_p} \leq \log(n + 2m)
\]

\[
\rightarrow \sum_i p_i \log p_i \leq S_p \log(n + 2m) + S_p \log S_p
\]

Note that for an arbitrary point \((v, x, \lambda) \in \mathcal{W}\), for all \(i \in [m]\), \(x_i + \lambda_i \leq n\) because it indicates the maximum swap value at reflection relation \(i\). Also since \(v_i \leq n\) for all \(i \in [n]\), thus \(S_q, S_p \leq n^2 + mn \leq 2mn\). Therefore:

\[
\sum_i p_i \log p_i \leq S_p \log(n + 2m) + S_p \log S_p
\]

\[
\leq 4mn \log n + 6mn \log n
\]

\[
= 10mn \log n
\]

Now we only need to bound \(\sum_i -p_i \log q_i\). To do so, we choose \(q\) to be the average of some instances \(\theta^{(j)}\) \(j \in J \subset \mathcal{W}\) such that for all \(i \in [n + 2m]\), \(q_i\) is sufficiently large. Trivially \(v_\theta \geq 1\). Also note that, since we do not have any redundant comparator in our network, we can assume for all \(i \in [m]\), there exists a witness instance in \(\mathcal{W}\) such that \(x_i \geq 1\). Same argument can be made for \(\lambda_i\)’s. Now let \(q\) be the average of all these \(2m\) instances. Therefore for all \(i \in [n + 2m]\), \(q_i \geq \frac{1}{2m}\) and consequently \(- \log q_i \leq \log 2m\).

Putting everything together, we obtain:

\[
\Delta(p||q) = \sum_i p_i \log p_i - p_i \log q_i + q_i - p_i
\]

\[
\leq 10mn \log n + 2mn \log 2m + 2mn
\]

\[
\leq 16mn \log n
\]

Appendix I. Proof of Theorem 5

Proof  The first part is the immediate consequence of Lemmas 3 and 4:

\[
\mathbb{E} \left[ \sum_{t=1}^{T} \gamma^{t-1} \cdot \ell^t \right] \leq \frac{\eta \min_{\gamma \in \mathcal{H}} \sum_{t=1}^{T} \gamma \cdot \ell^t + 16mn \log n}{1 - e^{-\eta}}
\]
Let $L_{\text{best}} = \min_{\gamma \in \mathcal{H}} \sum_{t=1}^{T} \gamma \cdot \ell^t$. We can tune $\eta$ as instructed in Lemma 4 in Freund and Schapire (1997):

$$E \left[ \sum_{t=1}^{T} \gamma^{t-1} \cdot \ell^t \right] - \min_{\gamma \in \mathcal{H}} \sum_{t=1}^{T} \gamma \cdot \ell^t \leq \sqrt{2L_{\text{best}} 16 m n \log n} + 16 m n \log n$$

Applying $L_{\text{best}} \leq T n^2$ and $m = O(n \log n)$ (by choosing an appropriate sorting network as in Ajtai et al. (1983)) into inequality above, we will obtain the desired result.

\[ \blacksquare \]

**Appendix J. Additional Loss with Approximate Projection**

Each iteration of Bregman Projection is described in Appendix F. Since it is basically finding a positive root of a polynomial (which $n/(n + m)$ of the time is quadratic), each iteration is arguably efficient. Now suppose, using iterative Bregman projections, we reached at $\hat{w} = (\hat{v}, \hat{x}, \hat{\lambda})$ which is $\epsilon$-close to the exact projection $w = (v, x, \lambda)$, that is $\|w - \hat{w}\|_2 < \epsilon$. In this analysis, we work with a two-level approximation: 1) approximating mean vector $v$ by the mean vector $\tilde{v} := M\hat{x} + c$ and 2) approximating the mean vector $\tilde{v}$ by the mean vector $v(\tilde{p})$ obtained from Algorithm 3 with $\hat{x}$ and $\lambda$ as input. First, observe that:

\[
\|v - \tilde{v}\|_2 = \|M (x - \hat{x})\|_2 \\
\leq \|M\|_F \|x - \hat{x}\|_2 \\
\leq (\sqrt{2/ \pi}) \epsilon \quad (7)
\]

Now suppose we run Algorithm 3 with $\hat{x}$ and $\hat{\lambda}$ as input. Similar to Appendix D, let $M_k$ be the $k$-th column of $M$, and let $T_{r,s}$ be the row-switching matrix that is obtained from switching $r$-th and $s$-th row in identity matrix. Since both Algorithms 3 and 2 will result in the same mean vector, for the sake of analysis, we can work with the mean vector obtained from Algorithm 2 with $\hat{x}$ and $\hat{\lambda}$ as input. More precisely, let $v^{(k)}(\tilde{p})$ be the mean vector associated with the distribution $S^{(k)}$ obtained by the end of $k$-th iteration of the Algorithm 2 i.e. $v^{(k)}(\tilde{p}) := \sum_{(\gamma, p) \in S^{(k)}} p \gamma$ (so $v^{(m)}(\tilde{p}) = v(\tilde{p})$). Also for all $k \in [m]$ define $\tilde{v}^{(k)} := c + \sum_{i=1}^{k} M_i \tilde{x}_i$ (thus $\tilde{v}^{(m)} = \tilde{v}$). Furthermore, let $\delta^{(k)} := v^{(k)}(\tilde{p}) - \tilde{v}^{(k)}$. Now we can write:
\[
v^{(k)}(\hat{p}) = \sum_{(\gamma,p) \in S^{(k)}} p_i \gamma = \sum_{(\gamma,p) \in S^{(k-1)}} (1 - \hat{p}_k) p_i \gamma + \hat{p}_k p_i Trs \gamma
\]

\[
= ((1 - \hat{p}_k) I + \hat{p}_k Trs) \sum_{(\gamma,p) \in S^{(k-1)}} p_i \gamma
= (I - \hat{p}_k M_k M_k^T) v^{(k-1)}(\hat{p})
= (I - \hat{p}_k M_k M_k^T) u^{(k-1)} + (I - \hat{p}_k M_k M_k^T) \delta^{(k-1)}
= (I - \hat{p}_k M_k M_k^T) \left( c + \sum_{i=1}^{k-1} M_i \hat{x}_i \right) + (I - \hat{p}_k M_k M_k^T) \delta^{(k-1)}
= \left( c + \sum_{i=1}^{k-1} M_i \hat{x}_i \right) - \hat{p}_k M_k M_k^T \left( c + \sum_{i=1}^{k-1} M_i \hat{x}_i \right) + (I - \hat{p}_k M_k M_k^T) \delta^{(k-1)}
= \tilde{v}^{(k)} - M_k \hat{x}_k - \hat{p}_k M_k M_k^T \left( c + \sum_{i=1}^{k-1} M_i \hat{x}_i \right) + (I - \hat{p}_k M_k M_k^T) \delta^{(k-1)}
\]

Now let \(err_k := -M_k^T \left( c + \sum_{i=1}^{k-1} M_i \hat{x}_i \right) - (\hat{x}_k + \hat{\lambda}_k)\), which is – according to Lemma 6 – the error in the \(k\)-th row of \(Ax + \lambda = b\) using \(\hat{x}\) and \(\hat{\lambda}\) i.e. amount by which \((A\hat{x} + \hat{\lambda})_k\) falls short of \(b_k\), violating the \(k\)-th constraint of \(Ax + \lambda = b\). Thus we obtain:

\[
\delta^{(k)} = -M_k \hat{x}_k + \hat{p}_k M_k (\hat{x}_k + \hat{\lambda}_k + err_k) + (I - \hat{p}_k M_k M_k^T) \delta^{(k-1)}
= \hat{p}_k M_k \text{err}_k + (I - \hat{p}_k M_k M_k^T) \delta^{(k-1)}
\]

Observe that \(\delta^{(0)} = c - c = 0\). Thus, by unrolling the recurrence relation above, we have:

\[
v(\hat{p}) - \tilde{v} = \delta^{(m)} = \sum_{k=1}^{m} \hat{p}_k M_k \text{err}_k \prod_{i=k+1}^{m} (I - \hat{p}_i M_i M_i^T)
\]

Note that since \(I - \hat{p}_i M_i M_i^T\) is a \(n \times n\) doubly-stochastic matrix, \(\prod_{i=1}^{k-1} (I - \hat{p}_i M_i M_i^T)\) is also a \(n \times n\) doubly-stochastic matrix, and consequently, its Frobenius norm is at most \(\sqrt{n}\). Thus we have:

\[
\|v(\hat{p}) - \tilde{v}\|_2 = \|\delta^{(m)}\|_2 \leq \sum_{k=1}^{m} |\hat{p}_k| \|M_k\|_2 |\text{err}_k| \sqrt{n} \leq \sqrt{2n} \sum_{k=1}^{m} |\text{err}_k|
= \sqrt{2n} \|\text{err}\|_1
\]

\[
\leq \sqrt{2n} m \|\text{err}\|_2
\]

\[
\text{err} := (\text{err}_1, \ldots, \text{err}_m)
\]

(8)
Observe that we can bound the 2-norm of the vector $\text{err}$ as follows:

$$\|\text{err}\|_2 = \| - A\hat{x} - \hat{\lambda} + b\|_2$$
$$= \| A(x - \hat{x}) + (\lambda - \hat{\lambda})\|_2$$
$$= \| A\|_F \| x - \hat{x}\|_2 + \| \lambda - \hat{\lambda}\|_2$$
$$\leq \| M^T M + I \|_F \| A \|_F + \epsilon$$
$$\leq (\| M^T \|_2 \| M \|_2 + \| I \|_2) \epsilon + \epsilon$$
$$= (2n + \sqrt{n} + 1) \epsilon$$

(9)

Therefore, if we perform Algorithm 3 with inputs $\hat{x}$ and $\hat{\lambda}$, combining the inequalities (7), (8), and (9), the generated mean vector $v(\hat{p})$ can be shown to be close to the mean vector $v$ associated with the exact projection:

$$\| v - v(\hat{p})\|_2 \leq \| v - \tilde{v}\|_2 + \| \tilde{v} - v(\hat{p})\|_2$$
$$\leq (\sqrt{2n}) \epsilon + \sqrt{2n} m (2n + \sqrt{n} + 1) \epsilon$$
$$= \sqrt{2n} (1 + \sqrt{m} (2n + \sqrt{n} + 1)) \epsilon$$

Now we can compute the total expected loss using approximate projection:

$$\left| \sum_{t=1}^{T} v^{t-1}(\hat{p}) \cdot \ell^t \right| = \left| \sum_{t=1}^{T} (v^{t-1} + (v^{t-1} - v^{t-1}(\hat{p}))) \cdot \ell^t \right|$$
$$= \left| \sum_{t=1}^{T} v^{t-1} \cdot \ell^t + \sum_{t=1}^{T} (v^{t-1} - v^{t-1}(\hat{p})) \cdot \ell^t \right|$$
$$\leq \left| \sum_{t=1}^{T} v^{t-1} \cdot \ell^t \right| + \sum_{t=1}^{T} \| v^{t-1} - v^{t-1}(\hat{p})\|_2 \| \ell^t\|_2$$
$$\leq \left| \sum_{t=1}^{T} v^{t-1} \cdot \ell^t \right| + T \left( \sqrt{2n} (1 + \sqrt{m} (2n + \sqrt{n} + 1)) \epsilon \right) \sqrt{n}$$

Setting $\epsilon = \frac{1}{(\sqrt{2n}) (1 + \sqrt{m} (2n + \sqrt{n} + 1)) T}$, we add at most one unit to the expected cumulative loss with exact projections.