Strong conciseness of Engel words in profinite groups

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Abstract
A group word \( w \) is said to be strongly concise in a class \( \mathcal{C} \) of profinite groups if, for any group \( G \) in \( \mathcal{C} \), either \( w \) takes at least continuum many values in \( G \) or the verbal subgroup \( \omega(G) \) is finite. It is conjectured that all words are strongly concise in the class of all profinite groups. Earlier Detomi, Klopsch, and Shumyatsky proved this conjecture for multilinear commutator words, as well as for some other particular words. They also proved that every group word is strongly concise in the class of nilpotent profinite groups, as well as that 2-Engel words are strongly concise (but their approach does not seem to generalize to \( n \)-Engel words for \( n > 2 \)).

In this paper, we prove that for any \( n \), the \( n \)-Engel word \([x, \ldots, [x, y], \ldots y]\) (where \( y \) is repeated \( n \) times) is strongly concise in the class of finitely generated profinite groups.

KEYWORDS
Engel word, finite groups, Lie ring method, profinite groups, pro-\( p \) groups, strongly concise word

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1 | INTRODUCTION

A group word \( w \) is said to be strongly concise in a class \( \mathcal{C} \) of profinite groups if, for any group \( G \) in \( \mathcal{C} \), either \( w \) takes at least continuum many values in \( G \) or the verbal subgroup \( \omega(G) \) is finite. It is conjectured that all words are strongly concise in the class of all profinite groups. Earlier Detomi, Klopsch, and Shumyatsky [3] proved this conjecture for multilinear commutator words. They also proved that every group word is strongly concise in the class of nilpotent profinite groups.

In this paper, we prove that for any \( n \), the \( n \)-Engel word \([x, \ldots, [x, y], \ldots y]\) is strongly concise in the class of finitely generated profinite groups. Henceforth, we use the left-normed simple commutator notation \([a_1, a_2, a_3, \ldots, a_r] := \ldots [[a_1, a_2], a_3], \ldots, a_r\) and the abbreviation \([a, k b] := [a, b, b, \ldots, b]\) where \( b \) is repeated \( k \) times.

Theorem 1.1. For any \( n \), the \( n \)-Engel word \([x, \ldots, [x, y], \ldots y]\) is strongly concise in the class of finitely generated profinite groups.

By the Detomi–Klopsch–Shumyatsky theorem [3, Theorem 1.2] on strong conciseness of all words in the class of nilpotent profinite groups, Theorem 1.1 is an immediate consequence of the following result.

Theorem 1.2. Let \( n \) be a positive integer, and suppose that \( G \) is a profinite group in which the word \([x, \ldots, [x, y], \ldots y]\) has strictly less than \( 2^{2^{2^0}} \) values. Then, \( G \) has a finite normal subgroup \( N \) such that \( G/N \) is locally nilpotent.
Indeed, let \( G \) be a finitely generated profinite group in which the word \([x, n y]\) has strictly less than \(2^{\aleph_0}\) values. Assuming that Theorem 1.2 holds, we obtain a finite normal subgroup \( N \) of \( G \) such that \( G/N \) is locally nilpotent. Since \( G \) is finitely generated, \( G/N \) is nilpotent. By [3, Theorem 1.2], the verbal subgroup \( \langle [x, y] \mid x, y \in G/N \rangle \) is finite, and since \( N \) is finite, the verbal subgroup \( \langle [x, y] \mid x, y \in G \rangle \) is also finite.

It is therefore the proof of Theorem 1.2 that occupies the rest of the paper. Much of the technique used in this proof was developed by the authors earlier for studying profinite (and more generally compact) groups with finite or countable Engel sinks. An Engel sink of an element \( g \) of a group \( G \) is a set \( \mathcal{E}(g) \) such that for every \( x \in G \) all sufficiently long commutators \([x, g, g, \ldots, g] \) belong to \( \mathcal{E}(g) \), that is, for every \( x \in G \), there is a positive integer \( n(x, g) \) such that \([x, n g] \in \mathcal{E}(g)\) for all \( n \geq n(x, g) \). (Thus, \( g \) is an Engel element precisely when we can choose \( \mathcal{E}(g) = \{1\} \), and \( G \) is an Engel group when we can choose \( \mathcal{E}(g) = \{1\} \) for all \( g \in G \).) In [10, 11], we considered finite, profinite, and compact groups in which every element has a finite or countable Engel sink and proved the following theorem (the version with finite sinks was proved in [10, Theorem 1.1]):

**Theorem 1.3** [11, Theorem 1.2]. If every element of a compact group \( G \) has a countable Engel sink, then \( G \) has a finite normal subgroup \( N \) such that \( G/N \) is locally nilpotent.

Here, “countable” stands for “finite or denumerable.” This result is a generalization of a theorem of Wilson and Zel’manov [25] saying that any Engel profinite group is locally nilpotent (which was later also extended to Engel compact groups by Medvedev [15]).

It is easy to see that every element of a group \( G \) in Theorem 1.2 has an Engel sink of cardinality strictly less than \(2^{\aleph_0}\). Therefore, Theorem 1.2 follows from Theorem 1.3 under the Continuum Hypothesis. However, we aim at proving Theorem 1.2 without assuming this additional axiom. The condition on the cardinalities being strictly less than \(2^{\aleph_0}\) rather than countable presents certain challenges. For example, as shown by Abért [1], the Baire Category Theorem (saying that if a compact Hausdorff group is a countable union of closed subsets, then one of these subsets has nonempty interior) cannot be proved in the version where the union of closed subsets is taken over a set of cardinality less than \(2^{\aleph_0}\).

The proof of Theorem 1.2 is in many respects similar to the proof of Theorem 1.3 in [11], albeit with certain modifications. First, the case of pro-\( p \) groups is considered, where powerful Lie ring methods are applied including Zelmanov’s theorem on Lie algebras satisfying a polynomial identity and generated by elements all of whose products are ad-nilpotent [26–28]. Then, the case of prosoluble groups is settled by using properties of Engel words and Engel sinks in coprime actions and a Hall–Higman–type theorem. The general case of profinite groups is dealt with by bounding the nonsoluble length of the group, which enables induction on this length. (We introduced the nonsoluble length in [12], although bounds for nonsoluble length had been implicitly used in various earlier papers, for example, in the celebrated Hall–Higman paper [7], or in Wilson’s paper [23]; more recently, bounds for the nonsoluble length were studied in connection with verbal subgroups in finite and profinite groups in [4, 6, 13, 20, 22].)

Concluding this introduction, we highlight two open problems. The proof of Theorem 1.1 uses essentially the condition that the group is finitely generated, so the question remains open for arbitrary profinite groups.

**Problem 1.4.** Is the \( n \)-Engel word strongly concise in the class of all profinite groups?

Obtaining an affirmative answer would be helped by a stronger version of Theorem 1.3, obtaining which is of independent interest and is stated as the following problem.

**Problem 1.5.** Suppose that every element of a compact group \( G \) has an Engel sink of cardinality less than \(2^{\aleph_0}\). Without the assumption of the Continuum Hypothesis, does it follow that \( G \) has a finite normal subgroup \( N \) such that \( G/N \) is locally nilpotent?

## 2 PRELIMINARIES

In this section, we recall some notation and terminology and establish some important properties of Engel words and Engel sinks in finite and profinite groups.
Our notation and terminology for profinite groups are standard; see, for example, [16, 24]. A subgroup (topologically) generated by a subset $S$ is denoted by $\langle S \rangle$. Recall that centralizers are closed subgroups, while commutator subgroups $[B,A] = \langle [b, a] \mid b \in B, a \in A \rangle$ are the closures of the corresponding abstract commutator subgroups.

For a group $A$ acting by automorphisms on a group $B$, we use the usual notation for commutators $[b, a] = b^{-1}ba$ and commutator subgroups $[B,A] = \langle [b, a] \mid b \in B, a \in A \rangle$, as well as for centralizers $C_B(A) = \{ b \in B \mid b^a = b \text{ for all } a \in A \}$.

We record for convenience the following simple lemma.

**Lemma 2.1** (see, e.g., [11, Lemma 2.1]). Suppose that $\varphi$ is a continuous automorphism of a compact group $G$ such that $G = [G, \varphi]$. If $N$ is a normal subgroup of $G$ contained in $C_G(\varphi)$, then $N \leq Z(G)$.

We denote by $\pi(k)$ the set of prime divisors of $k$, where $k$ may be a positive integer or a Steinitz number, and by $\pi(G)$ the set of prime divisors of the orders of elements of a (profinite) group $G$. Let $\sigma$ be a set of primes. An element $g$ of a group is a $\sigma$-element if $\pi(|g|) \subseteq \sigma$, and a group $G$ is a $\sigma$-group if all of its elements are $\sigma$-elements. We denote by $\sigma'$ the complement of $\sigma$ in the set of all primes. When $\sigma = \{ p \}$, we write $p$-element, $p'$-element, etc.

Recall that a pro-$p$ group is an inverse limit of finite $p$-groups, a pro-$\sigma$ group is an inverse limit of finite $\sigma$-groups, a pronilpotent group is an inverse limit of finite nilpotent groups, a prosoluble group is an inverse limit of finite soluble groups.

We denote by $\gamma_\infty(G) = \bigcap \gamma_i(G)$ the intersection of the lower central series of a group $G$. A profinite group $G$ is pronilpotent if and only if $\gamma_\infty(G) = 1$.

Profinite groups have $p$-Sylow subgroups and satisfy analogs of the Sylow theorems. Prosoluble groups similarly have $\pi$-Hall subgroups satisfying analogous theorems. We refer the reader to the corresponding chapters in [16, chapter 2] and [24, chapter 2]. We add a simple folklore lemma (see, e.g., [11, Lemma 2.2]).

**Lemma 2.2.** A profinite group $G$ that is an extension of a prosoluble group $N$ by a prosoluble group $G/N$ is prosoluble.

Recall that the definition of an Engel sink was given in the introduction. Clearly, the intersection of two Engel sinks of a given element $g$ of a group $G$ is again an Engel sink of $g$, with the corresponding function $n(x,g)$ being the maximum of the two functions. Therefore, if $g$ has a finite Engel sink, then $g$ has a unique smallest Engel sink. If $\mathcal{E}(g)$ is a smallest Engel sink of $g$, then the restriction of the mapping $x \mapsto [x,g]$ to $\mathcal{E}(g)$ must be surjective, which gives the following characterization.

**Lemma 2.3** [10, Lemma 2.1]. If an element $g$ of a group $G$ has a finite Engel sink, then $g$ has a smallest Engel sink $\mathcal{E}(g)$ and for every $s \in \mathcal{E}(g)$ there is $k \in \mathbb{N}$ such that $s = [s, kg]$.

The following well-known fact is a straightforward consequence of the Baire Category Theorem (see [9, Theorem 34]).

**Theorem 2.4.** If a compact Hausdorff group is a countable union of closed subsets, then one of these subsets has nonempty interior.

As shown by Abért [1], an analog of this theorem does not hold in the version where the union of closed subsets is taken over a set of cardinality less than $2^{\aleph_0}$ (without the Continuum Hypothesis). But a certain special case of such a generalization was recently obtained in [3].

**Proposition 2.5** [3, Proposition 2.1]. Let $\varphi : X \to Y$ be a continuous map between nonempty profinite spaces such that the cardinality of the image $|\varphi(X)|$ is strictly smaller than $2^{\aleph_0}$. Then, there exists a nonempty open subset $U$ of $X$ such that the restriction $\varphi|_U$ is constant.

This proposition was used for deriving the following fact.

**Lemma 2.6** [3, Lemma 2.2]. Let $G$ be a profinite group and let $x \in G$. If the conjugacy class $\{ x^g \mid g \in G \}$ contains less than $2^{\aleph_0}$ elements, then it is finite.
We now use Proposition 2.5 for proving two important technical lemmas about values of $n$-Engel words in profinite groups, which will be crucial in the proof of Theorem 1.2.

**Lemma 2.7.** Suppose that for an element $g$ of a profinite group $G$ the cardinality of the set of $n$-Engel word values $\{[h, g] \mid h \in G\}$ is strictly smaller than $2^{\aleph_0}$. Then, there is an element $s \in G$ and a coset $Nb$ of an open normal subgroup $N$ such that 

$$[xb, ng] = s \quad \text{for all} \quad x \in N.$$  

*Proof.* The mapping 

$$\varphi : h \mapsto [h, ng], \quad h \in G,$$

is continuous. Hence the result follows from Proposition 2.5. □

**Lemma 2.8.** Suppose that for an element $g$ of a profinite group $G$, the cardinality of the set of $n$-Engel word values $\{[h, g] \mid h \in G\}$ is strictly smaller than $2^{\aleph_0}$. Then, there is a positive integer $k$ and a coset $Nb$ of an open normal subgroup $N$ such that 

$$[[xb, ng], g^k] = 1 \quad \text{for all} \quad x \in N.$$  

*Proof.* By Lemma 2.7, there is an element $s \in G$ and a coset $Nb$ of an open normal subgroup $N$ such that 

$$[xb, ng] = s \quad \text{for all} \quad x \in N.$$  

Since $G/N$ is a finite group, the coset $Nb$ is invariant under conjugation by some power $g^k$. Then, 

$$s^k = [b, ng]^k = [b^{g^k}, ng] = [xb, ng] \quad \text{for some} \quad x \in N = s.$$  

In other words, $g^k$ commutes with $s$, so that 

$$[[xb, ng], g^k] = [s, g^k] = 1 \quad \text{for all} \quad x \in N.$$ □

**Remark 2.9.** The condition that the word $[x, ny]$ has strictly less than $2^{\aleph_0}$ values in a group $G$ is inherited by every section of $G$, and we shall freely use this property without special references.

### 3 | PRONILPOTENT GROUPS

When $G$ is a pro-$p$ group, or more generally a pronilpotent group, the conclusion of Theorem 1.2 is equivalent to $G$ being locally nilpotent, and this is what we prove in this section.

**Theorem 3.1.** Suppose that $G$ is a pronilpotent group in which the word $[x, ny]$ has strictly less than $2^{\aleph_0}$ values. Then, $G$ is locally nilpotent.

The bulk of the proof is about the case where $G$ is a pro-$p$ group. First, we remind the reader of important Lie ring methods in the theory of pro-$p$ groups.

For a prime number $p$, the Zassenhaus $p$-filtration of a group $G$ (also called the $p$-dimension series) is defined by 

$$G_i = \langle g^{p^k} \mid g \in \gamma_j(G), \; j p^k \geq i \rangle \quad \text{for} \quad i = 1, 2, \ldots$$
This is indeed a filtration (or an N-series, or a strongly central series) in the sense that

\[ [G_i, G_j] \leq G_{i+j} \quad \text{for all} \quad i, j. \quad (3.1) \]

Then, the Lie ring \( D_p(G) \) is defined with the additive group

\[ D_p(G) = \bigoplus G_i / G_{i+1}, \]

where the factors \( Q_i = G_i / G_{i+1} \) are additively written. The Lie product is defined on homogeneous elements \( xG_{i+1} \in Q_i, yG_{j+1} \in Q_j \) via the group commutators by

\[ [xG_{i+1}, yG_{j+1}] = [x, y]G_{i+j+1} \in Q_{i+j} \]

and extended to arbitrary elements of \( D_p(G) \) by linearity. Condition (3.1) ensures that this product is well-defined, and group commutator identities imply that \( D_p(G) \) with these operations is a Lie ring. Since all the factors \( G_i / G_{i+1} \) have prime exponent \( p \), we can view \( D_p(G) \) as a Lie algebra over the field of \( p \) elements \( \mathbb{F}_p \). We denote by \( L_p(G) \) the subalgebra generated by the first factor \( G / G_2 \).

A group \( G \) is said to satisfy a coset identity if there is a group word \( w(x_1, \ldots, x_m) \) and cosets \( a_1H, \ldots, a_mH \) of a subgroup \( H \leq G \) such that \( w(a_1h_1, \ldots, a_mh_m) = 1 \) for any \( h_1, \ldots, h_m \in H \). We shall use the following result of Wilson and Zelmanov [25] about coset identities.

**Theorem 3.2** (Wilson and Zelmanov [25, Theorem 1]). If a group \( G \) satisfies a coset identity on cosets of a subgroup of finite index, then for every prime \( p \), the Lie algebra \( L_p(G) \) satisfies a polynomial identity.

Theorem 3.2 was used in the proof of the above-mentioned theorem on profinite Engel groups, which we state here for convenience.

**Theorem 3.3** (Wilson and Zelmanov [25, Theorem 5]). Every profinite Engel group is locally nilpotent.

The proof of Theorem 3.3 was based on the following deep result of Zelmanov [26–28], which is also used in our paper.

**Theorem 3.4** (Zelmanov [26–28]). Let \( L \) be a Lie algebra over a field and suppose that \( L \) satisfies a polynomial identity. If \( L \) can be generated by a finite set \( X \) such that every commutator in elements of \( X \) is ad-nilpotent, then \( L \) is nilpotent.

We now consider the case of pro-\( p \) groups.

**Proposition 3.5.** Suppose that \( P \) is a finitely generated pro-\( p \) group in which the word \( [x_n, y] \) has strictly less than \( 2^{N_0} \) values. Then, \( P \) is nilpotent.

**Proof.** We shall first prove that the Lie algebra \( L_p(P) \) is nilpotent, using Theorem 3.4. The next lemma confirms that the hypotheses in Theorem 3.4 are satisfied.

**Lemma 3.6.** The Lie algebra \( L_p(P) \) satisfies a polynomial identity and is generated by finitely many elements all commutators in which are ad-nilpotent.

**Proof.** The proof of this lemma is obtained by repeating word-for-word the proofs of Lemmas 3.6 and 3.7 in [11], where Lemma 2.7 in [11] is replaced with Lemma 2.8 in this paper, which provides exactly the same result as in [11] under the hypotheses of Proposition 3.5.

We now resume the proof of Proposition 3.5. Lemma 3.6 together with Theorem 3.4 show that \( L_p(P) \) is nilpotent. The nilpotency of the Lie algebra \( L_p(P) \) of the finitely generated pro-\( p \) group \( P \) implies that \( P \) is a \( p \)-adic analytic group. This
result goes back to Lazard [14]; see also [19, Corollary D]. Furthermore, by a theorem of Breuillard and Gelander [2, Theorem 8.3], a $p$-adic analytic group satisfying a coset identity on cosets of a subgroup of finite index is soluble.

Thus, $P$ is soluble, and we prove that $P$ is nilpotent by induction on the derived length of $P$. By induction hypothesis, $P$ has an abelian normal subgroup $U$ such that $P/U$ is nilpotent. We aim to show that $P$ is an Engel group. Since $P/U$ is nilpotent, it is sufficient to show that every element $a \in P$ is an Engel element in the product $U\langle a \rangle$.

Applying Lemma 2.7 to $U\langle a \rangle$, we obtain a coset $Nb$ of an open normal subgroup $N$ of $U\langle a \rangle$ and an element $s \in U$ such that
\[ [xb, na] = s \quad \text{for all} \quad x \in N. \]

Since $[a'u, a] = [ux, a]$ for any $u \in U$, $x \in N$, we can assume that $b \in U$. Then, for any $m \in U \cap N$ we have
\[ s = [mb, na] = [m, na] \cdot [b, na] = [m, na] \cdot s, \]

since $U$ is abelian. Hence, $[m, na] = 1$ for any $m \in U \cap N$. Since $U \cap N$ has finite index in $U$ and $U\langle a \rangle$ is a pro-$p$ group, it follows that $a$ is an Engel element of $U\langle a \rangle$.

Thus, $P$ is an Engel group and therefore, being a finitely generated pro-$p$ group, $P$ is nilpotent by Theorem 3.3.

**Proof of Theorem 3.1.** By Theorem 3.3, it is sufficient to prove that $G$ is an Engel group. For each prime $p$, let $G_p$ denote the $p$-Sylow subgroup of $G$, so that $G$ is a Cartesian product of the $G_p$, since $G$ is pronilpotent. Given any two elements $a, g \in G$, we write $g = \prod_p g_p$ and $a = \prod_p a_p$, where $a_p, g_p \in G_p$. Clearly, $[g_q, a_p] = 1$ for $q \neq p$.

By Lemma 2.8, for the element $a \in G$, there is a positive integer $k$ and a coset $Nb$ of an open normal subgroup $N$ such that
\[ [(xb, na), a^k] = 1 \quad \text{for all} \quad x \in N. \quad (3.2) \]

Let $l$ be the (finite) index of $N$ in $G$. Then, $N$ contains all $q$-Sylow subgroups of $G$ for $q \notin \pi(l)$. Hence, we can choose $b$ to be a $\pi(l)$-element. Let $\pi = \pi(l) \cup \pi(k)$; note that $\pi$ is a finite set of primes.

We claim that
\[ [g_q, n+1a_q] = 1 \quad \text{for} \quad q \notin \pi. \]

Indeed, since $b$ commutes with elements of $G_q$ and $G_q \leq N$, by (3.2) we have
\[ 1 = [g_q b, na]a^k = [(g_q, na), a^k] = [[g_q, na], a^k] = [[g_q, na], a^k] = [g_q, na_q]a^k, \quad (3.3) \]

Thus, $a_q^k$ centralizes $[g_q, na_q]$. Since $k$ is coprime to $q$, we have $\langle a_q^k \rangle = \langle a_q \rangle$. Therefore, (3.3) implies that $[g_q, na_q], a_q = 1$, as claimed.

For every prime $p$, the group $G_p$ is locally nilpotent by Proposition 3.5, so there is $k_p$ such that $[g_p, k_p a_p] = 1$. Now for $m = \max\{n + 1, \max_{p \in \pi} \{k_p}\}$, we have $[g_p, m a_p] = 1$ for all $p$, which means that $[g, na] = 1$. Thus, $G$ is an Engel group and therefore it is locally nilpotent by Theorem 3.3.

## 4 COPRIME ACTIONS

In this section, first we list several profinite analogs of the properties of coprime automorphisms of finite groups. Then, we prove two lemmas on coprime automorphisms in relation to Engel sinks and values of Engel words.

If $\varphi$ is an automorphism of a finite group $H$ of coprime order, that is, such that $(|\varphi|, |H|) = 1$, then we say for brevity that $\varphi$ is a coprime automorphism of $H$. This definition is extended to profinite groups as follows. We say that $\varphi$ is a coprime
automorphism of a profinite group $H$ meaning that a procyclic group $\langle \varphi \rangle$ faithfully acts on $H$ by continuous automorphisms and $\pi(\varphi) \cap \pi(H) = \emptyset$. Since the semidirect product $H \langle \varphi \rangle$ is also a profinite group, $\varphi$ is a coprime automorphism of $H$ if and only if for every open normal $\varphi$-invariant subgroup $N$ of $H$ the automorphism (of finite order) induced by $\varphi$ on $H/N$ is a coprime automorphism.

In the following lemma, we collect some folklore results on coprime automorphisms.

**Lemma 4.1.** Suppose that $\varphi$ is a coprime automorphism of a profinite group $G$.

(a) For every prime $q \in \pi(G)$, there is a $\varphi$-invariant $q$-Sylow subgroup of $G$. If $G$ is in addition prosoluble, then for every subset $\sigma \subseteq \pi(G)$, there is a $\varphi$-invariant $\sigma$-Hall subgroup of $G$.

(b) If $N$ is a $\varphi$-invariant closed normal subgroup of $G$, then every fixed point of $\varphi$ in $G/N$ is an image of a fixed point of $\varphi$ in $G$, that is, $C_{G/N}(\varphi) = C_G(\varphi)N/N$.

(c) $[G, \varphi] = [G, \varphi]$.

Part (a) follows from the Sylow theory for profinite groups and an analog of the Schur–Zassenhaus theorem (see, e.g., [11, Lemma 4.1]). Part (b) is a special case of [16, Proposition 2.3.16], and part (c) follows from part (b).

The following lemma is a consequence of [18, Lemma 2.4].

**Lemma 4.2** [11, Lemma 4.6]. Let $\varphi$ be a coprime automorphism of a pronilpotent group $G$. Then, the restriction of the mapping

$$\theta : x \mapsto [x, \varphi]$$

to the set $K = \{ [g, \varphi] \mid g \in G \}$ is injective.

We now prove two lemmas where the condition of the word $[x, n y]$ having less than $2^{\aleph_0}$ values appears.

**Lemma 4.3.** Let $\varphi$ be a coprime automorphism of a pronilpotent group $G$. Suppose that, for some $n \in \mathbb{N}$, the set $\mathcal{E}_{G,n}(\varphi) = \{ [g, n \varphi] \mid g \in G \}$ has less than $2^{\aleph_0}$ elements. Then, the set $K = \{ [g, \varphi] \mid g \in G \}$ is a finite smallest Engel sink of $\varphi$ in the semidirect product $G \langle \varphi \rangle$.

**Proof.** Since the mapping $\theta : x \mapsto [x, \varphi]$ is injective on the set $K$ by Lemma 4.2, the sets $\mathcal{E}_{G,n}(\varphi)$ and $K$ have the same cardinality, which is less than $2^{\aleph_0}$ by hypothesis. The set $K = \{ [g, \varphi] \mid g \in G \}$ is in a one-to-one correspondence with the set of (say, right) cosets of the centralizer $C_G(\varphi)$. But this set of cosets cannot be infinite of cardinality less than $2^{\aleph_0}$ by Lemma 2.6. Therefore, it is finite, and so is the set $K$.

The mapping $[g, \varphi] \mapsto [g, \varphi, \varphi]$ is injective on $K$ by Lemma 4.2, and therefore it is also surjective, since $K$ is finite. Hence this set is a smallest finite Engel sink of $\varphi$. \[\Box\]

**Lemma 4.4.** Let $\varphi$ be a coprime automorphism of a pronilpotent group $G$. Suppose that, for some $n \in \mathbb{N}$, the word $[x, n y]$ has strictly less than $2^{\aleph_0}$ values in the semidirect product $G \langle \varphi \rangle$. Then, $\gamma_\infty(G(\varphi))$ is finite and $\gamma_\infty(G(\varphi)) = [G, \varphi]$.

**Proof.** The group $G$ is locally nilpotent by Theorem 3.1. By Lemma 4.3, the set $K = \{ [g, \varphi] \mid g \in G \}$ is finite. Therefore, the subgroup $[G, \varphi] = \langle K \rangle$ is nilpotent. By Lemma 4.1(c),

$$[[G, \varphi], \varphi] = [G, \varphi]. \tag{4.1}$$

Let $V$ be the quotient of $[G, \varphi]$ by its derived subgroup. For any $u, v \in V$, we have $[u \varphi, \varphi] = [u, \varphi][v, \varphi]$, since $V$ is abelian, and $[V, \varphi] = V$ by (4.1). Hence, $V$ consists of the images of elements of $K$, and therefore is finite. Then, the nilpotent group $[G, \varphi]$ is also finite (see, e.g., [17, 5.2.6]).

The quotient $G(\varphi)/[G, \varphi]$ is obviously the direct product of the images of $G$ and $\langle \varphi \rangle$ and therefore is pronilpotent. Hence, $\gamma_\infty(G(\varphi)) \leq [G, \varphi]$, so $\gamma_\infty(G(\varphi))$ is finite. Since the set of commutators $\{ [g, \varphi] \mid g \in G \}$ is the smallest Engel sink of $\varphi$ by Lemma 4.3, it follows that $\gamma_\infty(G(\varphi)) = [G, \varphi]$. \[\Box\]
In this section, we prove Theorem 1.2 for prosoluble groups. First, we consider the case of prosoluble groups of finite Fitting height. Recall that by Theorem 3.1 a pronilpotent group in which the word \([x, ny]\) has strictly less than \(2^{\aleph_0}\) values is locally nilpotent. Therefore, if \(G\) is a profinite group in which the word \([x, ny]\) has strictly less than \(2^{\aleph_0}\) values, then the largest pronilpotent normal subgroup \(F(G)\) is also the largest locally nilpotent normal subgroup, and we call it the Fitting subgroup of \(G\). Then, further terms of the Fitting series are defined as usual by induction: \(F_1(G) = F(G)\) and \(F_{i+1}(G)\) is the inverse image of \(F(G/F_i(G))\). A group has finite Fitting height if \(F_k(G) = G\) for some \(k \in \mathbb{N}\).

Proposition 5.1. Let \(G\) be a prosoluble group in which the word \([x, ny]\) has strictly less than \(2^{\aleph_0}\) values. If \(G\) has finite Fitting height, then \(\gamma_\infty(G)\) is finite.

Proof. It is sufficient to prove the result for the case of Fitting height 2. Then, the general case will follow by induction on the Fitting height \(k\) of \(G\). Indeed, then \(\gamma_\infty(G/\gamma_\infty(F_{k-1}(G)))\) is finite, while \(\gamma_\infty(F_{k-1}(G))\) is finite by the induction hypothesis, and as a result, \(\gamma_\infty(G)\) is finite.

Thus, we assume that \(G = F_2(G)\). By Theorem 1.3, it is sufficient to show that every element \(a \in G\) has a finite Engel sink. Since \(G/F(G)\) is locally nilpotent, an Engel sink of \(a\) in \(F(G)\langle a \rangle\) is also an Engel sink of \(a\) in \(G\).

For a prime \(p\), let \(P\) be a \(p\)-Sylow subgroup of \(F(G)\), and write \(a = a_p a_{p'}\), where \(a_p\) is a \(p\)-element, \(a_{p'}\) is a \(p'\)-element, and \([a_p, a_{p'}] = 1\). Then \(P(a_p)\) is a normal \(p\)-Sylow subgroup of \(P(a)\), on which \(a_{p'}\) induces by conjugation a coprime automorphism. By Lemma 4.4, the subgroup \(\gamma_\infty(P(a)) = [P, a_{p'}]\) is finite. Since the pronilpotent group \(P(a)/\gamma_\infty(P(a))\) is locally nilpotent by Theorem 3.1, we can choose a finite smallest Engel sink \(\mathcal{E}_p(a) \subseteq \gamma_\infty(P(a))\) of \(a\) in \(P(a)\).

Note that

\[
\text{if } \mathcal{E}_p(a) = \{1\}, \text{ then } \gamma_\infty(P(a)) = 1. \tag{5.1}
\]

Indeed, if \(\mathcal{E}_p(a) = \{1\}\), then, in particular, the image \(\bar{a}\) of \(a\) in \(\langle a \rangle/C_{\langle a \rangle}\langle P, a_{p'}\rangle\) is an Engel element of the finite group \([P, a_{p'}]\langle \bar{a} \rangle\) and therefore \(\bar{a}\) is contained in its Fitting subgroup by Baer’s theorem [8, Satz III.6.15]. Then,

\[
\gamma_\infty(P(a)) = [P, a_{p'}] = [[P, a_{p'}], a_{p'}] = ([P, a_{p'}], \bar{a}_{p'}) = 1.
\]

By Lemma 2.3, for every \(s \in \mathcal{E}_p(a)\), we have \(s = [s, k]a\) for some \(k \in \mathbb{N}\), and then also

\[
s = [s, k]a \quad \text{for any } t \in \mathbb{N}. \tag{5.2}
\]

We claim that \(\mathcal{E}_p(a) = \{1\}\) for all but finitely many primes \(p\). Suppose the opposite: \(\mathcal{E}_p(a) \neq \{1\}\) for each prime \(p\) in an infinite set of primes \(\pi\). Choose a nontrivial element \(s_p \in \mathcal{E}_p(a)\) for every \(p \in \pi\). For any subset \(\sigma \subseteq \pi\), consider the (infinite) product

\[
s_\sigma = \prod_{p \in \sigma} s_p.
\]

Note that the elements \(s_p\) commute with one another belonging to different normal Sylow subgroups of \(F(G)\). If \(\mathcal{E}(a)\) is any Engel sink of \(a\) in \(G\), then for some \(k \in \mathbb{N}\) the commutator \(s_\sigma, k\) belongs to \(\mathcal{E}(a)\). Because of the properties (5.2), all the components of \([s_\sigma, k]a\) in the \(p\)-Sylow subgroups of \(F(G)\) for \(p \in \sigma\) are nontrivial, while all the other components in \(q\)-Sylow subgroups for \(q \notin \sigma\) are trivial by construction. Therefore, for different subsets \(\sigma \subseteq \pi\), we thus obtain different elements of \(\mathcal{E}(a)\). The infinite set \(\pi\) has \(2^{\aleph_0}\) different subsets, whence \(\mathcal{E}(a)\) has cardinality at least \(2^{\aleph_0}\). But we can choose

\[
\mathcal{E}(a) = \bigcup_{i \in \mathbb{N}} \{[g, n]a \mid g \in G\},
\]

which is a countable union of sets each having cardinality less than \(2^{\aleph_0}\) by hypothesis. This Engel sink \(\mathcal{E}(a)\) therefore also has cardinality less than \(2^{\aleph_0}\), a contradiction.
Thus, for all but finitely many primes $p$, we have $\mathcal{C}_p(a) = \{1\}$, which is the same as $\gamma_\infty(P(a)) = 1$ by (5.1). Therefore, the subgroup

$$\gamma_\infty(F(G)(a)) = \prod_p \gamma_\infty(P(a))$$

is finite. The quotient $F(G)(a)/\gamma_\infty(F(G)(a))$ is pronilpotent and therefore locally nilpotent by Theorem 3.1. Hence, we can choose a finite Engel sink for $a$ in $G$ as a subset of $\gamma_\infty(F(G)(a))$.

Thus, every element of $G$ has a finite Engel sink, and therefore $\gamma_\infty(G)$ is finite by Theorem 1.3.

**Lemma 5.2.** Let $\varphi$ be a coprime automorphism of a prosoluble group $G$ such that the set of primes $\pi(G)$ is finite. If the word $[x, n y]$ has strictly less than $2^{\aleph_0}$ values in the semidirect product $G \varphi$, then the subgroup $\langle G, \varphi \rangle$ is finite.

**Proof.** By Lemma 4.1(c), we can assume that $G = [G, \varphi]$. For every prime $q \in \pi(G)$, there is a $\varphi$-invariant $q$-Sylow subgroup $G_q$ of $G$ by Lemma 4.1(a). By Lemma 4.3, the set $\{[g, \varphi] : g \in G_q\}$ is finite. Since $\pi(G)$ is finite, there is an open normal subgroup $N$ of $G$ that intersects trivially with every set $\{[g, \varphi] : g \in G_q\}$, which implies that $\varphi$ centralizes every $q$-Sylow subgroup $N \cap G_q$ and therefore $[N, \varphi] = 1$. Since $N$ is normal and $G = [G, \varphi]$, we obtain $[N, G] = 1$ by Lemma 2.1. Thus, $G/Z(G)$ is finite and, in particular, the Fitting height of $G$ is finite. Then, $\gamma_\infty(G(\varphi))$ is finite by Proposition 5.1, and therefore $[G, \varphi]$ is also finite, since $[G, \varphi] \leq \gamma_\infty(G(\varphi))$ by Lemma 4.4 applied to $G(\varphi)/\gamma_\infty(G(\varphi))$. □

**Lemma 5.3.** Let $a$ be a coprime automorphism of prime order $p$ of a prosoluble group $H$ such that $2 \notin \pi(H)$. If the word $[x, n y]$ has strictly less than $2^{\aleph_0}$ values in the semidirect product $H \langle a \rangle$, then the subgroup $\langle H, a \rangle$ is finite.

**Proof.** The proof of this lemma is obtained by repeating word-for-word the proof of Lemma 5.3 in [11], where Lemma 2.6 in [11] is replaced with Lemma 2.7 in this paper, Proposition 5.1 in [11] is replaced with Proposition 5.1 in this paper, and Theorem 3.1 in [11] is replaced with Theorem 3.1 in this paper. The corresponding lemmas and proposition provide exactly the same results as in [11] under the hypotheses of Lemma 5.3. □

**Proposition 5.4.** If the word $[x, n y]$ has strictly less than $2^{\aleph_0}$ values in a prosoluble group $G$, then $F(G) \neq 1$.

**Proof.** The proof of this proposition is obtained by repeating word-for-word the proof of Proposition 5.4 in [11], where Proposition 5.1 in [11] is replaced with Proposition 5.1 in this paper, and Theorem 3.1 in [11] is replaced with Theorem 3.1 in this paper. The corresponding theorems provide exactly the same results as in [11] under the hypotheses of Proposition 5.4. □

We are now ready to prove the main result of this section.

**Theorem 5.5.** Suppose that $G$ is a prosoluble group in which the word $[x, n y]$ has strictly less than $2^{\aleph_0}$ values. Then, $G$ has a finite normal subgroup $N$ such that $G/N$ is locally nilpotent.

**Proof.** By Theorem 3.1, it is sufficient to prove that $\gamma_\infty(G)$ is finite. By Proposition 5.1, we obtain that $\gamma_\infty(F_2(G))$ is finite and the quotient $F_2(G)/\gamma_\infty(F_2(G))$ is locally nilpotent by Theorem 3.1. Then, the subgroup $C = C_{F_2(G)}(\gamma_\infty(F_2(G)))$ has finite index in $F_2(G)$ and is locally nilpotent. Indeed, for any finite subset $S \subseteq C_{F_2(G)}(\gamma_\infty(F_2(G)))$, we have $\gamma_k(S) \leq \gamma_\infty(F_2(G))$ for some $k$, and then

$$\gamma_{k+1}(S) = [\gamma_k(S), S] \leq [\gamma_\infty(F_2(G)), C] = 1.$$ 

As a normal locally nilpotent subgroup, $C$ is contained in $F(G)$. Hence, $F_2(G)/F(G)$ is finite. We claim that the quotient $G/F(G)$ is finite. Let the bar denote the images in $\tilde{G} = G/F(G)$. Then, $F(\tilde{G}) = F_2(\tilde{G})$ is finite by the above. There is an open normal subgroup $N$ of $\tilde{G}$ such that $N \cap F(\tilde{G}) = 1$. If $N \neq 1$, then $F(N) \neq 1$ by Proposition 5.4. But $F(N) \leq N \cap F(\tilde{G}) = 1$; hence we must have $N = 1$, so $\tilde{G}$ is finite.
Thus, \( G/F(G) \) is finite, and therefore \( G \) has finite Fitting height. By Proposition 5.1, we obtain that \( \gamma_\infty(G) \) is finite, as required. \[\square\]

A proof of the next result can be obtained as in [11, Corollary 5.6] with only obvious modifications, so we omit details.

**Corollary 5.6.** Suppose that \( G \) is a virtually prosoluble group in which the word \([x, n y]\) has strictly less than \( 2^{\aleph_0} \) values. Then, \( G \) has a finite normal subgroup \( N \) such that \( G/N \) is locally nilpotent.

### 6 PROFINITE GROUPS

The proof of Theorem 1.2 uses induction on nonprosoluble length; we recall the relevant definitions. The *nonsoluble length* \( \lambda(H) \) of a finite group \( H \) is defined as the minimum number of nonsoluble factors in a normal series in which every factor either is soluble or is a direct product of nonabelian simple groups. (In particular, the group is soluble if and only if its nonsoluble length is 0.) Clearly, every finite group has a normal series with these properties, and therefore its nonsoluble length is well defined. It is easy to see that the nonsoluble length \( \lambda(H) \) is equal to the least positive integer \( l \) such that there is a series of characteristic subgroups

\[
1 = L_0 \leq R_0 < L_1 \leq R_1 < \cdots \leq R_l = H
\]

in which each quotient \( L_i/R_{i-1} \) is a (nontrivial) direct product of nonabelian simple groups, and each quotient \( R_i/L_i \) is soluble (possibly trivial).

We shall use the following result of Wilson [23], which we state in the special case of \( p = 2 \) using the terminology of nonsoluble length.

**Theorem 6.1** (see [23, Theorem 2*]). Let \( K \) be a normal subgroup of a finite group \( G \). If a 2-Sylow subgroup \( Q \) of \( K \) has a coset \( tQ \) of exponent dividing \( 2^k \), then the nonsoluble length of \( K \) is at most \( k \).

It is natural to say that a profinite group \( G \) has finite *nonprosoluble length* at most \( l \) if \( G \) has a normal series

\[
1 = L_0 \leq R_0 < L_1 \leq R_1 < \cdots \leq R_l = G
\]

in which each quotient \( L_i/R_{i-1} \) is a (nontrivial) Cartesian product of nonabelian finite simple groups, and each quotient \( R_i/L_i \) is prosoluble (possibly trivial).

As a special case of a general result in Wilson’s paper [23], we have the following.

**Lemma 6.2** (see [23, Lemma 2]). If, for some positive integer \( m \), all continuous finite quotients of a profinite group \( G \) have nonsoluble length at most \( m \), then \( G \) has finite nonprosoluble length at most \( m \).

We are now ready to prove the key proposition.

**Proposition 6.3.** Suppose that \( G \) is a profinite group in which the word \([x, n y]\) has strictly less than \( 2^{\aleph_0} \) values. Then, \( G \) has finite nonprosoluble length.

**Proof.** Let \( H = \bigcap G^{(i)} \) be the intersection of the derived series of \( G \). Then, \( H = [H, H] \). Indeed, if \( H \neq [H, H] \), then the quotient \( G/[H, H] \) is a prosoluble group by Lemma 2.2, whence \( \bigcap G^{(i)} = H \leq [H, H] \), a contradiction. Since the quotient \( G/H \) is prosoluble, it is sufficient to prove the proposition for \( H \). Thus, we can assume from the outset that \( G = [G, G] \).

Let \( T \) be a 2-Sylow subgroup of \( G \). By Theorem 3.1, the group \( T \) is locally nilpotent. Consider the subsets of the direct product \( T \times T \)

\[
S_i = \{(x, y) \in T \times T \mid \text{the subgroup } \langle x, y \rangle \text{ is nilpotent of class at most } i \}\]
Note that each subset $S_i$ is closed in the product topology of $T \times T$, because the condition defining $S_i$ means that all commutators of weight $i + 1$ in $x, y$ are trivial. Since every 2-generator subgroup of $T$ is nilpotent, we have

$$\bigcup_i S_i = T \times T.$$

By Theorem 2.4, one of the sets $S_i$ contains an open subset of $T \times T$. This means that there are cosets $aN$ and $bN$ of an open normal subgroup $N$ of $T$ and a positive integer $c$ such that

$$\langle x, y \rangle \text{ is nilpotent of class at most } c \text{ for any } x \in aN, \ y \in bN. \quad (6.1)$$

Let $K$ be an open normal subgroup of $G$ such that $K \cap T \leq N$. If we replace $N$ by $K \cap T$, then (6.1) still holds with the same $a, b$. Hence, we can assume that $N$ is a 2-Sylow subgroup of $K$.

By [16, Lemma 2.8.15], there is a subgroup $H \leq G$ such that $G = KH$ and $K \cap H$ is pronilpotent. Since $H$ is virtually pronilpotent and every element has a countable Engel sink, by Corollary 5.6 the subgroup $\gamma_\infty(H)$ is finite. Recalling our assumption that $G = [G, G]$, we obtain

$$G = [G, G] = \gamma_\infty(G) \leq \gamma_\infty(HK) \leq \gamma_\infty(H)K.$$

Thus, $G = \gamma_\infty(H)K$, where $\gamma_\infty(H)$ is a finite subgroup.

Hence, we can choose the coset representative $a$ satisfying (6.1) in a conjugate of a 2-Sylow subgroup of $\gamma_\infty(H)$, and therefore having finite order, say, $|a| = 2^m$.

For any $y \in bN$, the 2-subgroup $\langle a, y \rangle$ is nilpotent of class at most $c$, while $a^{2^m} = 1$. Then,

$$[a, y^{2^m(c-1)}] = 1. \quad (6.2)$$

This follows from well-known commutator formulae (and for any $p$-group); see, for example, [21, Lemma 4.1].

In particular, for any $z \in N$, by using (6.2) we obtain

$$[z, c y^{2^m(c-1)}] = [az, c y^{2^m(c-1)}] = 1, \quad (6.3)$$

since $\langle az, y^{2^m(c-1)} \rangle$ is a subgroup of $\langle az, y \rangle$, which is nilpotent of class $c$ by (6.1).

Our aim is to show that there is a uniform bound, in terms of $|G : K|$, $c$, and $m$, for the nonsoluble length of all finite quotients of $G$ by open normal subgroups. Let $M$ be an open normal subgroup of $G$ and let the bar denote the images in $\bar{G} = G/M$. It is clearly sufficient to obtain a required bound for the nonsoluble length of $\bar{K}$.

Let $R_0$ be the soluble radical of $\bar{K}$, and $L_1$ the inverse image of the generalized Fitting subgroup of $\bar{K}/R_0$, so that

$$L_1/R_0 = S_1 \times S_2 \times \cdots \times S_k \quad (6.4)$$

is a direct product of nonabelian finite simple groups. Note that $R_0$ and $L_1$ are normal subgroups of $\bar{G}$. The group $\bar{G}$ acting by conjugation induces a permutational action on the set $\{S_1, S_2, \ldots, S_k\}$. The kernel of the restriction of this permutational action to $\bar{K}$ is contained in the inverse image $R_1$ of the soluble radical of $\bar{K}/L_1$:

$$\bigcap_i N_{\bar{K}}(S_i) \leq R_1. \quad (6.5)$$

This follows from the validity of Schreier’s conjecture on the solubility of the outer automorphism groups of nonabelian finite simple groups, confirmed by the classification of the latter, because $L_1/R_0$ contains its centralizer in $K/R_0$.

Let $e$ be the least positive integer such that $2^e \geq c$, and let $t = m(c-1) + e$. We claim that for any $y \in bN$ the element $y^{2^t}$ normalizes each factor $S_i$ in (6.4). Arguing by contradiction, suppose that the element $y^{2^t}$ has a nontrivial orbit on the set of the $S_i$. Then, the element $y^{2^m(c-1)}$ has an orbit of length $2^{t} \geq 2^{e+1}$ on this set; let $\{T_1, T_2, \ldots, T_{2^{e+1}}\}$ be such an orbit cyclically permuted by $y^{2^m(c-1)}$. Since nonabelian finite simple groups have even order (by the Feit–Thompson theorem [5]) and the subgroups $S_i$ are subnormal in $\bar{K}/R_0$, each subgroup $S_i$ contains a nontrivial element of $N_{R_0}/R_0$. If $x$ is a
nontrivial element of $T_1 \cap \overline{N}R_0 / R_0$, then the commutator

$$[x, \epsilon^{2m(c-1)}],$$

written as an element of $T_1 \times T_2 \times \cdots \times T_{2^s}$, has a nontrivial component in $T_{c+1}$ since $2^s \geq 2^{c+1} > c$. This, however, contradicts (6.3).

Thus, for any element $y \in \overline{b}\overline{N}$, the power $y^{2^u}$ normalizes each factor $S_i$ in (6.4). Let $2^d$ be the highest power of 2 dividing $[G : K]$, and let $u = \max\{t, d\}$. Then, $y^{2^u} \in R_1$ by (6.5), since $y^{2^u} \in K$ and $y^{2^u}$ normalizes each $S_i$ in (6.4) by the choice of $u$.

As a result, in the quotient $\overline{G} / R_1$ all elements of the coset $\overline{b}\overline{N}R_1 / R_1$ of the 2-Sylow subgroup $\overline{N}R_1 / R_1$ of $\overline{K} / R_1$ have exponent dividing $2^u$. We can now apply Theorem 6.1, by which the nonsoluble length of $\overline{K} / R_1$ is at most $u$. Then, the nonsoluble length of $\overline{G} / \overline{K}$ is bounded in terms of $[G : K]$. As a result, since the number $u$ depends only on $[G : K]$, $m$, and $c$, the nonsoluble length of $G$ is bounded in terms of these parameters only. Since this holds for any quotient of the profinite group $G$ by a normal open subgroup, the group $G$ has finite nonprosoluble length by Lemma 6.2. This completes the proof of Proposition 6.3.

We are now ready to handle the general case of profinite groups using Corollary 5.6 on virtually prosoluble groups and induction on the nonprosoluble length. First, we eliminate infinite Cartesian products of nonabelian finite simple groups.

**Lemma 6.4.** Suppose that $G$ is a profinite group that is a Cartesian product of nonabelian finite simple groups. If the word $[x, n^y]$ has strictly less than $2^{\aleph_0}$ values in $G$, then $G$ is finite.

**Proof.** Suppose the opposite: then $G$ is a Cartesian product of infinitely many nonabelian finite simple groups $G_i$ over an infinite set of indices $i \in I$. Since finite Engel groups are nilpotent, each $G_i$ contains a nontrivial value $[a_i, n^b_i] \neq 1$. For any subset $J \subseteq I$, the product

$$\prod_{j \in J}[a_j, n^b_j]$$

is also a value of the word $[x, n^y]$, and these values are different for different subsets $J \subseteq I$. Since the infinite set $I$ has at least $2^{\aleph_0}$ different subsets, we obtain a contradiction with the hypothesis. □

We now finish the proof of Theorem 1.2, which also completes the proof of the main Theorem 1.1, as explained in the introduction.

**Proof of Theorem 1.2.** Recall that $G$ is a profinite group in which the word $[x, n^y]$ has strictly less than $2^{\aleph_0}$ values. We need to show that $G$ has a finite normal subgroup $N$ such that $G / N$ is locally nilpotent.

By Proposition 6.3, the group $G$ has finite nonprosoluble length $l$. This means that $G$ has a normal series

$$1 = L_0 \leq R_0 \leq L_1 \leq R_1 \leq L_1 \leq \cdots \leq R_l = G$$

in which each quotient $L_i / R_{i-1}$ is a (nontrivial) Cartesian product of nonabelian finite simple groups, and each quotient $R_i / L_i$ is prosoluble (possibly trivial). We argue by induction on $l$. When $l = 0$, the group $G$ is prosoluble, and the result follows by Theorem 5.5.

Now let $l \geq 1$. By Lemma 6.4, each of the nonprosoluble factors $L_i / R_{i-1}$ is finite. In particular, the subgroup $L_1$ is virtually prosoluble, and therefore $\gamma_\infty(L_1)$ is finite by Corollary 5.6. The quotient $R_1 / \gamma_\infty(L_1)$ is prosoluble by Lemma 2.2. Hence the nonprosoluble length of $G / \gamma_\infty(L_1)$ is $l - 1$. By the induction hypothesis, we obtain that $\gamma_\infty(G / \gamma_\infty(L_1))$ is finite, and therefore $\gamma_\infty(G)$ is finite. By Theorem 3.1, the quotient $G / \gamma_\infty(G)$ is locally nilpotent, and the proof is complete. □

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