Well-posedness of the Viscous Boussinesq System in Besov Spaces of Negative Order Near Index $s = -1$

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Abstract

This paper is concerned with well-posedness of the Boussinesq system. We prove that the $n$ ($n \geq 2$) dimensional Boussinesq system is well-posed for small initial data $(\vec{u}_0, \theta_0)$ ($\nabla \cdot \vec{u}_0 = 0$) either in $(B^{-1}_{\infty,1} \cap B^{-1,1}_{\infty}) \times B^{-1,1}_p$ or in $B^{-1,1}_\infty \times B^{-1,1}_{p,r}$ if $r \in [1, \infty], \varepsilon > 0$ and $p \in (\frac{2}{\varepsilon}, \infty)$, where $B_{s,\varepsilon}^{p,q}$ ($s \in \mathbb{R}, 1 \leq p, q \leq \infty, \varepsilon > 0$) is the logarithmically modified Besov space to the standard Besov space $B_{s,p,q}$. We also prove that this system is well-posed for small initial data in $(B^{-1}_{\infty,1} \cap B^{-1,1}_{\infty}) \times B^{-1,1}_p$.

Keywords: Boussinesq system; Navier-Stokes equations; well-posedness; Besov spaces.

Mathematics Subject Classification: 76D05, 35Q30, 35B40.

1 Introduction

In this paper we will discuss the Cauchy problem for the normalized $n$-dimensional viscous Boussinesq system which describes the natural convection in a viscous incompressible fluid as follows:

$$\begin{align*}
\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} + \nabla P &= \Delta \vec{u} + \theta \vec{a} \quad \text{in } \mathbb{R}^n \times (0, \infty), \\
\text{div} \vec{u} &= 0 \quad \text{in } \mathbb{R}^n \times (0, \infty), \\
\theta_t + \vec{u} \cdot \nabla \theta &= \Delta \theta \quad \text{in } \mathbb{R}^n \times (0, \infty), \\
(\vec{u}(\cdot, t), \theta(\cdot, t))|_{t=0} &= (\vec{u}_0(\cdot), \theta_0(\cdot)) \quad \text{in } \mathbb{R}^n,
\end{align*}$$

where $\vec{u} = (u_1(x,t), u_2(x,t), \ldots, u_n(x,t)) \in \mathbb{R}^n$ and $P = P(x,t) \in \mathbb{R}$ denote the unknown vector velocity and the unknown scalar pressure of the fluid, respectively. $\theta = \theta(x,t) \in \mathbb{R}$ denotes the density or the temperature. $\theta \vec{a}$ in (1.1) takes into account the influence of the gravity and the stratification on the motion of the fluid. The whole system is considered under initial condition $(\vec{u}_0, \theta_0) = (\vec{u}_0(x), \theta_0(x)) \in \mathbb{R}^{n+1}$.

The Boussinesq system is extensively used in the atmospheric sciences and oceanographic turbulence (cf. [15] and references cited therein). Due to its close relation to fluids, there are
a lot of works related to various aspects of this system. Among the fruitful results we only cite papers on well-posedness. In 1980, Cannon and DiBenedetto in [3] established well-posedness of the full viscous Boussinesq system in Lebesgue space within the framework of Kato semigroup. Around 1990, Mirimoto, Hishida and Kagei have investigated weak solutions of this system in [16], [11] and [13]. Well-posedness results in pseudomeasure-type space and weak $L^p$ space, etc. can be found in [10] and references cited therein. Recently, the two dimensional Boussinesq system with partial viscous terms has drawn a lot of attention, see [1, 5, 9, 12] and references cited therein.

In this paper, we aim at achieving the lowest regularity results of the full viscous Boussinesq system with dimension $n \geq 2$. Though it is hard to deal with the coupled term $\bar{u}\nabla \theta$, we succeed in finding a suitable product space with regular index being almost $-1$ in which the Boussinesq system is well-posed. More precisely, we prove that if $(\bar{u}_0, \theta_0) \in (B^{-1,-1}_{\infty,1} \cap B^{-1,-1}_{\infty,\infty}) \times (B^{-1,-1}_{\infty,1} \cap B^{-1,-1}_{\infty,\infty})$ satisfying $\text{div} \bar{u}_0 = 0$, where $B^{-1,-1}_{p,\infty} (1 \leq p \leq \infty)$ is the logarithmically modified Besov space to the standard Besov space $B^{-1}_{p,\infty}$ (see definition 1.1 below), then there exists a local solution to Eqs. (1.1)∼(1.4). We also prove that if $\theta_0$ belongs to $B^{-1,-1}_{p,r}$ with $p \in (\frac{3}{2}, \infty)$ and $r \in [1, \infty]$ and $\bar{u}_0$ belongs to $B^{-1,-1}_{\infty,1} \cap B^{-1,-1}_{\infty,\infty}$ satisfying the divergence free condition, then there exists a local solution to Eqs. (1.1)∼(1.4). The method we use here is essentially frequency localization.

As usual, we use the well-known fixed point arguments and hence we invert Eqs. (1.1)∼(1.4) into the corresponding integral equations:

\[ \bar{u} = e^{t\Delta} \bar{u}_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P}(\bar{u} \cdot \nabla) \bar{u} ds + \int_0^t e^{(t-s)\Delta} \mathbb{P}(\theta \bar{u}) ds, \]  

\[ \theta = e^{t\Delta} \theta_0 - \int_0^t e^{(t-s)\Delta} (\bar{u} \cdot \nabla \theta) ds, \]

where $\mathbb{P}$ is the Helmholtz projection operator given by $\mathbb{P} = I + \nabla (-\Delta)^{-1} \text{div}$ with $I$ representing the unit operator. In what follows, we shall regard Eqs. (1.5) and (1.6) as a fixed point system for the map

\[ \mathcal{J} : (\bar{u}, \theta) \mapsto \mathcal{J}(\bar{u}, \theta) = (\mathcal{J}_1(\bar{u}, \theta), \mathcal{J}_2(\bar{u}, \theta)), \]

where $\mathcal{J}_1(\bar{u}, \theta)$ and $\mathcal{J}_2(\bar{u}, \theta)$ denote the right-hand sides of (1.5) and (1.6), respectively.

Before showing our main results of this paper, let us first recall the nonhomogeneous littlewood-Paley decomposition by means of a sequence of operators $(\Delta_j)_{j \in \mathbb{Z}}$ and then we define the Besov type space $B^{s,\alpha}_{p,r}$ and the corresponding Chemin-Lerner type space $\tilde{L}^p(B^{s,\alpha}_{p,r})$.

To this end, let $\gamma > 1$ and $(\varphi, \chi)$ be a couple of smooth functions valued in $[0, 1]$, such that $\varphi$ is supported in the shell $\{x \in \mathbb{R}^n; \gamma^{-1} \leq |x| \leq 2\gamma\}$, $\chi$ is supported in the ball $\{x \in \mathbb{R}^n; |x| \leq \gamma\}$ and

\[ \chi(x) + \sum_{q \in \mathbb{N}} \varphi(2^{-q} x) = 1, \quad \forall x \in \mathbb{R}^n. \]

For $u \in S'(\mathbb{R}^n)$, we define nonhomogeneous dyadic blocks as follows:

\[ \Delta_q u := 0 \quad \text{if} \quad q \leq -2, \]
\[ \triangle_{-1} u := \chi(D)u = \tilde{h} * u \quad \text{with} \quad \tilde{h} := \mathcal{F}^{-1}\chi, \]
\[ \triangle_q u := \varphi(2^{-q}D)u = 2^m \int h(2^q y)u(x - y)dy \quad \text{with} \quad h := \mathcal{F}^{-1}\varphi \quad \text{if} \quad q \geq 0. \]

One can prove that
\[ u = \sum_{q \geq -1} \triangle_q u \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n) \]
for all tempered distribution \( u \). The right-hand side is called nonhomogeneous Littlewood-Paley decomposition of \( u \). It is also convenient to introduce the following partial sum operator:
\[ S_q u := \sum_{p \leq q - 1} \triangle_p u. \]

Obviously we have \( S_0 u = \triangle_{-1} u \). Since \( \varphi(\xi) = \chi(\xi/2) - \chi(\xi) \) for all \( \xi \in \mathbb{R}^n \), one can prove that
\[ S_q u = \chi(2^{-q}D)u = \int \tilde{h}(2^q y)u(x - y)dy \quad \text{for all} \quad q \in \mathbb{N}. \]

Let \( \gamma = 4/3 \). Then we have the following result, i.e. for any \( u \in \mathcal{S}'(\mathbb{R}^n) \) and \( v \in \mathcal{S}'(\mathbb{R}^n) \), there holds
\[ \triangle_k \triangle_q u = 0 \quad \text{for} \quad |k - q| \geq 2, \quad (1.7) \]
\[ \triangle_k (S_{q-1} u \triangle_q v) = 0 \quad \text{for} \quad |k - q| \geq 5, \quad (1.8) \]
\[ \triangle_k (\triangle_q u \triangle_{q+l} v) = 0 \quad \text{for} \quad |l| \leq 1, \quad k \geq q + 4. \quad (1.9) \]

**Definition 1.1.** Let \( T > 0 \), \(-\infty < s < \infty \) and \( 1 \leq p, r, \rho \leq \infty \).

1. We say that a tempered distribution \( f \in B_{s,0}^{p,0} \) if and only if
\[ \left( \sum_{q \geq -1} 2^{qrs} (3 + q)^{\alpha r} \left\| \triangle_q f \right\|_{L^p}^r \right)^{\frac{1}{r}} < \infty \]
(1.10)
(with the usual convention for \( r = \infty \)).

2. We say that a tempered distribution \( u \in \tilde{L}_T^p(B_{s,0}^{p,0}) \) if and only if
\[ \| u \|_{\tilde{L}_T^p(B_{s,0}^{p,0})} := \left( \sum_{q} 2^{qrs} (3 + q)^{\alpha r} \left\| \triangle_q u \right\|_{L^p(0,T; L^r_\rho)}^r \right)^{\frac{1}{r}} < \infty. \]
(1.11)

**Remarks.** (i) The definition (1) is essentially due to Yoneda [19] where he considered the homogeneous version of the space \( B_{s,0}^{p,0} \) (see also remarks there). Note that by using the heat semigroup characterization of these spaces (see Lemma 4.1 in Section 4), we see that \( B_{-1,1}^{0,\infty} \) coincides with the space \( B_{\infty,\infty}^{1}(\mathbb{R}^n) \) considered by the second author in his recent work [7]. The definition (2) in the case \( \alpha = 0 \) (note that \( B_{s,0}^{0,0} = B_{s,0}^{s} \)) is due to Chermin etc. (cf. [6 8]).
(ii) Similar to the case $\alpha = 0$ (see [S] and references cited therein), by using the Minkowski inequality we see that for $0 \leq \alpha \leq \beta < \infty$,

$$\|f\|_{L^p_t(B^{\alpha,p}_r)} \leq \|f\|_{L^p_t(B^{\beta,p}_r)} \text{ if } r \geq \rho, \quad \|f\|_{L^p_t(B^{\alpha,p}_r)} \leq \|f\|_{L^p_t(B^{\beta,p}_r)} \text{ if } r \leq \rho.$$  

We now state the main results. In the first two results we consider the case where the first component $\vec{u}_0$ of the initial data lies in the space $B^{-1,1}_{\infty,1} \cap B^{-1,1}_{0,1}$. In the third result we consider the case where $\vec{u}_0$ lies in the less regular space $B^{-1,1}_{\infty,\infty}$. As we shall see, in this case we need the second component $\theta_0$ of the initial data to lie in a more regular space.

**Theorem 1.2.** Let $n \geq 2$. Given $T > 0$, there exist $\mu_1, \mu_2 > 0$ such that for any $(\vec{u}_0, \theta_0) \in (B^{-1,1}_{\infty,1} \cap B^{-1,1}_{0,1}) \times (B^{-1,1}_{\infty,1} \cap B^{-1,1}_{0,1})$ satisfying

$$\left\{ \begin{array}{l}
\|\vec{u}_0\|_{B^{-1,1}_{\infty,1}} + \|\vec{u}_0\|_{B^{-1,1}_{0,1}} \leq \mu_1 \quad \text{and} \quad \text{div}\vec{u}_0 = 0, \\
\|\theta_0\|_{B^{-1,1}_{0,1}} \leq \mu_2,
\end{array} \right.$$  

the Boussinesq system has a unique solution $(\vec{u}, \theta)$ in $(\tilde{L}^2_t(B^0_{\infty,1}) \cap \tilde{L}^2_t(B^0_{0,1})) \times (\tilde{L}^2_t(B^0_{\infty,1}) \cap \tilde{L}^2_t(B^0_{0,1}))$ and $C_w([0,T]; B^{-1,1}_{\infty,1} \cap B^{-1,1}_{0,1}) \times C([0,T]; B^{-1,1}_{\infty,1} \cap B^{-1,1}_{0,1})$ satisfying

$$\|\vec{u}\|_{\tilde{L}^2_t(B^0_{\infty,1})} + \|\vec{u}\|_{\tilde{L}^2_t(B^0_{0,1})} \leq 2\mu_1 \quad \text{and} \quad \|\theta\|_{\tilde{L}^2_t(B^0_{\infty,1})} + \|\theta\|_{\tilde{L}^2_t(B^0_{0,1})} \leq 2\mu_2.$$  

**Theorem 1.3.** Let $n \geq 2$, $1 \leq r \leq \infty$ and $p \in (\frac{2}{r}, \infty)$. Given $T > 0$, there exist $\mu_1, \mu_2 > 0$ such that for any $(\vec{u}_0, \theta_0) \in (B^{-1,1}_{\infty,1} \cap B^{-1,1}_{0,1}) \times B^{-1,r}_{p_r}$ satisfying

$$\left\{ \begin{array}{l}
\|\vec{u}_0\|_{B^{-1,1}_{\infty,1}} + \|\vec{u}_0\|_{B^{-1,1}_{0,1}} \leq \mu_1 \quad \text{and} \quad \text{div}\vec{u}_0 = 0, \\
\|\theta_0\|_{B^{-1,1}_{0,1}} \leq \mu_2,
\end{array} \right.$$  

the Boussinesq system has a unique solution $(\vec{u}, \theta)$ in $(\tilde{L}^2_t(B^0_{\infty,1}) \cap \tilde{L}^2_t(B^0_{0,1})) \times \tilde{L}^2_t(B^0_{p_r})$ and $C_w([0,T]; B^{-1,1}_{\infty,1} \cap B^{-1,1}_{0,1}) \times C((0,T]; B^{-1,1}_{p_r})$ or $C_w([0,T]; B^{-1,1}_{\infty,1} \cap B^{-1,1}_{0,1}) \times C_w([0,T]; B^{-1,1}_{0,1})$ satisfying

$$\|\vec{u}\|_{\tilde{L}^2_t(B^0_{\infty,1})} + \|\vec{u}\|_{\tilde{L}^2_t(B^0_{0,1})} \leq 2\mu_1 \quad \text{and} \quad \|\theta\|_{\tilde{L}^2_t(B^0_{\infty,1})} \leq 2\mu_2.$$  

**Theorem 1.4.** Let $n \geq 2$, $\varepsilon > 0$ and $p \in (\frac{2}{r}, \infty)$. Given $0 < T \leq 1$, there exist $\mu_1 = \mu_1(\varepsilon)$, $\mu_2 = \mu_2(\varepsilon) > 0$ such that for any $(\vec{u}_0, \theta_0) \in B^{-1,1}_{\infty,\infty} \times B^{-1,1,\varepsilon}_{p,\infty}$ satisfying

$$\left\{ \begin{array}{l}
\|\vec{u}_0\|_{B^{-1,1,\varepsilon}_{\infty,\infty}} \leq \mu_1 \quad \text{and} \quad \text{div}\vec{u}_0 = 0, \\
\|\theta_0\|_{B^{-1,1,\varepsilon}_{p,\infty}} \leq \mu_2,
\end{array} \right.$$  

the Boussinesq system has a unique solution $(\vec{u}, \theta)$ in $C_w([0,T]; B^{-1,1}_{\infty,\infty}) \times C_w([0,T]; B^{-1,1,\varepsilon}_{p,\infty})$ satisfying

$$\sup_{0 < t < T} t^\frac{1}{r} \|\ln(t)\| \|\vec{u}\|_{\infty} \leq 2\mu_1 \quad \text{and} \quad \sup_{0 < t < T} t^\frac{1}{r} |\ln(t)|^r \|\theta\|_{p} \leq 2\mu_2.$$
There is no inclusion relation between the spaces $\mathcal{F}$ and $f$ stand for Fourier transform of $f$ with respect to space variable and $\mathcal{F}^{-1}$ stands for the inverse Fourier transform. We denote $A \leq CB$ by $A \lesssim B$ and $A \lesssim B \lesssim A$ by $A \sim B$. For any $1 \leq p, q \leq \infty$, we denote $L^q(\mathbb{R}^n)$, $L^p(0, T)$ and $L^\rho(0, T; L^q(\mathbb{R}^n))$ by $L^q_T$, $L^\rho_T$ and $L^\rho_T L^q_T$, respectively. We denote $\|f\|_{L^p_T}$ by $\|f\|_p$ for short. In what follows we will not distinguish vector valued function space and scalar function space if there is no confusion.

Later on, we shall use $C$ and $c$ to denote positive constants which depend on dimension $n$, $|a|$ and might depend on $p$ and may change from line to line. $\mathcal{F}f$ and $\hat{f}$ stand for Fourier transform of $f$ with respect to space variable and $\mathcal{F}^{-1}$ stands for the inverse Fourier transform.

We use two different methods which are used by Chermin, etc. and Kato, etc. respectively to prove Theorems 1.2 and Theorem 1.3. Therefore we write their proofs in separate sections. In Sect. 2 we introduce the paradifferential calculus results, while in Sect. 3 we prove Theorems 1.2 and 1.3. Finally, in Sect. 4 we prove Theorem 1.4.

## 2 Paradifferential calculus

In this section, we prove several preliminary results concerning the paradifferential calculus. We first recall some fundamental results.

**Lemma 2.1.** (Bernstein) Let $k$ be in $\mathbb{N} \cup \{0\}$ and $0 < R_1 < R_2$. There exists a constant $C$ depending only on $R_1, R_2$ and dimension $n$, such that for all $1 \leq a \leq b \leq \infty$ and $u \in L^a$, we have

\[
\sup \{u \subset B(0, R_1 \lambda), \sup_\{a=|r|\} \|\partial^a u\|_b \leq C^{k+1} x^{k+n(1/a-1/b)} \|u\|_a, \tag{2.1}
\]

\[
\sup \{u \subset C(0, R_1 \lambda, R_2 \lambda), \sup_\{a=|r|\} \|\partial^a u\|_a \sim C^{k+1} x^k \|u\|_a. \tag{2.2}
\]

**Lemma 2.2.** [8, 18] Let $1 \leq p < \infty$. Then we have the following assertions:

1. $B^0_1 \hookrightarrow C \cap L^\infty_x \hookrightarrow L^\infty_x \hookrightarrow B^0_{\infty, \infty}$.  
2. $B^0_{p, 1} \hookrightarrow L^p_x \hookrightarrow B^0_{p, \infty}$.

**Lemma 2.3.** (1) Let $1 \leq p \leq \infty$, $0 \leq \beta \leq \alpha < \infty$, $-\infty < s < \infty$ and $1 \leq r_1 < r_2 \leq \infty$. Then

\[
B^\beta_{p, r_1} \hookrightarrow B^\alpha_{p, r_2} \hookrightarrow B^\beta_{p, r_2}.
\]

(2) Let $1 < \tilde{\rho} \leq \infty$, $1 \leq p \leq \infty$ and $-\infty < s < \infty$. For any $\epsilon > 0$, we have

\[
B^\beta_{p, \tilde{\rho}} \hookrightarrow B^\alpha_{p, \tilde{\rho}} \hookrightarrow B^\beta_{p, \tilde{\rho}}, \quad B^\beta_{p, \tilde{\rho}} \hookrightarrow B^\epsilon_{p, \tilde{\rho}} \hookrightarrow B^\beta_{p, \tilde{\rho}}. \tag{2.3}
\]

(3) There is no inclusion relation between the spaces $B^0_{\infty, 1}$ and $B^0_{\infty, \infty}$.

**Proof.** It suffices to prove (3). Similar to [19], we set

\[
f = \sum_{j=-1}^{\infty} a_j \delta_{2^j} \text{ for } \{a_j\}_{j=-1}^{\infty} \subset \mathbb{R},
\]

where $\delta_{2^j}$ is the characteristic function of the interval $[2^j, 2^{j+1})$. Then $f \in B^0_{\infty, \infty}$, but $f \not\in B^0_{\infty, 1}$. This completes the proof.
Therefore, \( B_{\infty}^{0,1} \) is not included in \( B_{\infty,1}^{0,1} \). Next, let \( \delta_{jk} \) be Kronecker’s delta. For fixed \( k \in \mathbb{N} \), if we take \( a_j = \frac{\delta_{kj}}{3+j} \) for \( j \geq 0 \) and \( a_j = 0 \) for \( j < 0 \), then we have \( \| f \|_{B_{\infty,1}^{0,1}} \simeq 1 \) and \( \| f \|_{B_{\infty,1}^{0,1}} = \frac{1}{k} \). Since \( k \) is arbitrary, \( B_{\infty,1}^{0,1} \) is not included in \( B_{\infty,\infty}^{0,1} \). \( \square \)

We now begin our discussion on paradifferential calculus.

**Lemma 2.4.** For any \( p, p_1, p_2, r \in [1, \infty] \) satisfying \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \), the bilinear map \( (u, v) \mapsto uv \) is bounded from \( (B_{p_1,1}^{0,1} \cap B_{p_2,1}^{0,1}) \times B_{p,r}^{0,1} \) to \( B_{p,r}^{0,1} \), i.e.

\[
\| uv \|_{B_{p,r}^{0,1}} \lesssim \| u \|_{B_{p_1,1}^{0,1} \cap B_{p_2,1}^{0,1}} \| v \|_{B_{p,r}^{0,1}}. \tag{2.4}
\]

**Proof.** Following Bony [2] we write

\[
u v = \mathcal{T}(u, v) + \mathcal{T}(v, u) + \mathcal{R}(u, v),
\]

where

\[
\mathcal{T}(u, v) = \sum_{q \geq 1} S_{q-1} u \Delta_q v, \quad \mathcal{R}(u, v) = \sum_{l=-1}^{1} \sum_{q \geq 1} \Delta_q u \Delta_{q+l} v.
\]

The estimate of \( \mathcal{T}(u, v) \) is simple. Indeed, by Proposition 1.4.1 (i) of [8] we know that for any \( p, r \in [1, \infty] \), \( \mathcal{T} \) is bounded from \( L_{x}^{\infty} \times B_{p,r}^{0,1} \) to \( B_{p,r}^{0,1} \). By slightly modifying the proof of that proposition, we see that for any \( p, p_1, p_2, r \in [1, \infty] \) satisfying \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \), \( \mathcal{T} \) is also bounded from \( L_{x}^{p_1} \times B_{p_2,1}^{0,1} \) to \( B_{p,r}^{0,1} \). Thus for any \( p, p_1, p_2, r \in [1, \infty] \) we have

\[
\| \mathcal{T}(u, v) \|_{B_{p,r}^{0,1}} \lesssim \| u \|_{L_{x}^{p_1}} \| v \|_{B_{p_2,1}^{0,1}} \lesssim \| u \|_{B_{p_1,1}^{0,1}} \| v \|_{B_{p_2,1}^{0,1}}. \tag{2.5}
\]

In what follows we estimate \( \mathcal{T}(v, u) \) and \( \mathcal{R}(u, v) \). By interpolation, it suffices to consider the two end point cases \( r = 1 \) and \( r = \infty \).

To estimate \( \| \mathcal{T}(v, u) \|_{B_{p_1,1}^{0,1}} \), we use (1.3) to deduce

\[
\| \mathcal{T}(v, u) \|_{B_{p_1,1}^{0,1}} = \sum_{k \geq 1} \| \Delta_k ( \sum_{q \geq 1} S_{q-1} v \Delta_q u) \|_{p_1} \lesssim \sum_{k \geq 1} \sum_{|q-k| \leq 4, q \geq 1} \| S_{q-1} v \|_{p_2} \| \Delta_q u \|_{p_1}
\]
\[
\lesssim \| v \|_{p_2} \sum_{k \geq 1} \| \Delta_k u \|_{p_1} \lesssim \| u \|_{B_{p_1,1}^{0,1}} \| v \|_{B_{p_2,1}^{0,1}}. \tag{2.6}
\]
To estimate $\|T(v, u)\|_{B^0_{p, \infty}}$ we first note that

$$\|S_{q-1}v\|_{p_2} \lesssim \sum_{j=-1}^{q-2} \|\Delta_j v\|_{p_2} \lesssim (q + 3)\|v\|_{B^0_{p_2, \infty}}.$$  

Using this inequality and (1.8) we see that

$$\|T(v, u)\|_{B^0_{p, \infty}} = \sup_{k \geq -1} \|\Delta_k \left( \sum_{q \geq -1} S_{q-1} v \Delta_q u \right)\|_p \lesssim \sup_{k \geq -1} \sum_{|q - k| \leq 4, q \geq -1} \|S_{q-1} v\|_{p_2} \|\Delta_q u\|_{p_1} \lesssim \|v\|_{B^0_{p_2, \infty}} \sup_{k \geq -1} \sum_{|q - k| \leq 4, q \geq -1} (q + 3)\|\Delta_q u\|_{p_1} \lesssim \|u\|_{B^{0,1}_{p_1, \infty}} \|v\|_{B^0_{p_2, \infty}}.$$  

To estimate $\|R(u, v)\|_{B^0_{p_1}}$, we write

$$\|R(u, v)\|_{B^0_{p_1}} \lesssim \sum_{l = -1}^1 \sum_{k \geq -1} \|\Delta_k \left( \sum_{q \geq -1} \Delta_q u \Delta_{q+l} v \right)\|_p \lesssim \sum_{l = -1}^3 \|\Delta_k \left( \sum_{q \geq -1} \Delta_q u \Delta_{q+l} v \right)\|_p + \sum_{l = -1}^3 \sum_{k \geq 4} \|\Delta_k \left( \sum_{q \geq -1} \Delta_q u \Delta_{q+l} v \right)\|_p =: I_1 + I_2.$$  

For $I_1$ we have

$$I_1 \lesssim \sum_{l = -1}^1 \sum_{q \geq -1} \|\Delta_q u \Delta_{q+l} v\|_p \lesssim \sum_{l = -1}^1 \sum_{q \geq -1} \|\Delta_q u\|_{p_1} \|\Delta_{q+l} v\|_{p_2} \lesssim \|u\|_{B^{0,1}_{p_1, \infty}} \|v\|_{B^0_{p_2, 1}}.$$  

For $I_2$, by using (1.9) we deduce

$$I_2 \lesssim \sum_{l = -1}^1 \sum_{k \geq 4} \sum_{q \geq -3} \|\Delta_q u\|_{p_1} \|\Delta_{q+l} v\|_{p_2} \lesssim \sum_{l = -1}^1 \sum_{q \geq 1} \sum_{4 \leq k < l + 3 + q} \|\Delta_q u\|_{p_1} \|\Delta_{q+l} v\|_{p_2} \lesssim \sum_{l = -1}^1 \sum_{q \geq 1} (q + 3)\|\Delta_q u\|_{p_1} \|\Delta_{q+l} v\|_{p_2} \lesssim \|u\|_{B^{0,1}_{p_1, \infty}} \|v\|_{B^0_{p_2, 1}}.$$  

Hence

$$\|R(u, v)\|_{B^0_{p_1}} \lesssim \|u\|_{B^{0,1}_{p_1, \infty}} \|v\|_{B^0_{p_2, 1}}.$$  

Similarly we have

$$\|R(u, v)\|_{B^0_{p_2, \infty}} \lesssim \sum_{l = -1}^1 \sup_{k \geq -1} \|\Delta_k \left( \sum_{q \geq -1} \Delta_q u \Delta_{q+l} v \right)\|_p \lesssim \|u\|_{B^{0,1}_{p_1, \infty}} \|v\|_{B^0_{p_2, 1}}.$$  

(2.7)
\[ \lesssim \sum_{l=1}^{1} \left( \sum_{q \geq -1} \| \Delta q u \|_{p_1} \| \Delta q v \|_{p_2} \right) \]
\[ \lesssim \| u \|_{B_{p_1,1}^0} \| v \|_{B_{p_2,\infty}^0}. \]  \tag{2.9}

From (2.5) ~ (2.9) and interpolation, we obtain the desired estimate. This completes the proof of Lemma 2.4.

\begin{lemma}
For any \( p, p_2, p_2 \in [1, \infty] \) satisfying \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \), the bilinear map \((u, v) \mapsto uv\) is bounded from \((B_{p_1,1}^0 \cap B_{p_1,\infty}^0) \times (B_{p_2,1}^0 \cap B_{p_2,\infty}^0)\) to \(B_{p_1,1}^0 \cap B_{p_1,\infty}^0\), i.e.
\[ \|uv\|_{B_{p_1,1}^0 \cap B_{p_1,\infty}^{0,1}} \lesssim \|u\|_{B_{p_1,1}^0 \cap B_{p_1,\infty}^{0,1}} \|v\|_{B_{p_2,1}^0 \cap B_{p_2,\infty}^{0,1}}. \]  \tag{2.10}

In particular, \(B_{\infty,1}^0 \cap B_{\infty,\infty}^{0,1}\) is a Banach algebra.
\end{lemma}

\textbf{Proof}. By Lemma 2.4 we only need to prove that
\[ \|uv\|_{B_{p_1,1}^0 \cap B_{p_1,\infty}^{0,1}} \lesssim \|u\|_{B_{p_1,1}^0 \cap B_{p_1,\infty}^{0,1}} \|v\|_{B_{p_2,1}^0 \cap B_{p_2,\infty}^{0,1}}. \]  \tag{2.11}

As before we decompose \(uv\) into the sum of \(\mathcal{T}(u, v), \mathcal{T}(v, u)\) and \(\mathcal{R}(u, v)\). To estimate \(\|\mathcal{T}(u, v)\|_{B_{p_1,\infty}^{0,1}}\), we use (1.8) to deduce
\[ \|\mathcal{T}(u, v)\|_{B_{p_1,\infty}^{0,1}} = \sup_{k \geq -1} (k + 3) \| \Delta k \left( \sum_{q \geq -1} S_{q-1} u \Delta q v \right) \|_p \]
\[ \lesssim \sup_{k \geq -1} (k + 3) \| \Delta k \left( \sum_{|q-k| \leq 4, q \geq -1} S_{q-1} u \Delta q v \right) \|_p \]
\[ \lesssim \sup_{k \geq -1} (k + 3) \sum_{|q-k| \leq 4, q \geq -1} \| S_{q-1} u \|_{p_1} \| \Delta q v \|_{p_2} \]
\[ \lesssim \| u \|_{p_1} \sup_{q \geq -1} (q + 3) \| \Delta q v \|_{p_2} \]
\[ \lesssim \| u \|_{B_{p_1,1}^0} \| v \|_{B_{p_2,\infty}^{0,1}}. \]  \tag{2.12}

The estimate of \(\|\mathcal{T}(v, u)\|_{B_{p_1,\infty}^{0,1}}\) is similar, with minor modifications. Indeed,
\[ \|\mathcal{T}(v, u)\|_{B_{p_1,\infty}^{0,1}} = \sup_{k \geq -1} (k + 3) \| \Delta k \left( \sum_{q \geq -1} S_{q-1} v \Delta q u \right) \|_p \]
\[ \lesssim \sup_{k \geq -1} (k + 3) \sum_{|q-k| \leq 4, q \geq -1} \| S_{q-1} v \|_{p_2} \| \Delta q u \|_{p_1} \]
\[ \lesssim \| v \|_{p_2} \sup_{q \geq -1} (q + 3) \| \Delta q u \|_{p_1} \]
\[ \lesssim \| u \|_{B_{p_1,\infty}^{0,1}} \| v \|_{B_{p_2,1}^0}. \]  \tag{2.13}
To estimate \( \| R(u, v) \|_{B_{p, \infty}^{0, 1}} \) we write

\[
\| R(u, v) \|_{B_{p, \infty}^{0, 1}} \leq \sum_{l=1}^{1} \sup_{k \geq -1} (3 + k) \| \triangle_k \left( \sum_{q \geq -1} \triangle_q u \triangle_{q+l} v \right) \|_p
\]

\[
\leq \sum_{l=1}^{1} \sup_{l \geq -1} (3 + k) \| \triangle_k \left( \sum_{q \geq -1} \triangle_q u \triangle_{q+l} v \right) \|_p
\]

\[
+ \sum_{l=1}^{1} \sup_{q+3 \geq k} (3 + k) \| \triangle_k \left( \sum_{q \geq -1} \triangle_q u \triangle_{q+l} v \right) \|_p
\]

\[
=: I_3 + I_4.
\]

For \( I_3 \) we have

\[
I_3 \lesssim \sum_{l=1}^{1} \sum_{q \geq -1} \| \triangle_q u \|_{p_1} \| \triangle_{q+l} v \|_{p_2} \lesssim \| u \|_{B_{p_1, \infty}^0} \| v \|_{B_{p_2, \infty}^0} \lesssim \| u \|_{B_{p_1, \infty}^0} \| v \|_{B_{p_2, \infty}^0}.
\]

For \( I_4 \) we have

\[
I_4 = \sum_{l=1}^{1} \sup_{q \geq -1} \sum_{q+3 \geq k \geq 8} (3 + k) \| \triangle_k \left( \sum_{q \geq -1} \triangle_q u \triangle_{q+l} v \right) \|_p
\]

\[
\lesssim \sum_{l=1}^{1} \sum_{q \geq 4} (3 + q) \| \triangle_q u \|_{p_1} \| \triangle_{q+l} v \|_{p_2} \lesssim \| u \|_{B_{p_1, \infty}^0} \| v \|_{B_{p_2, \infty}^0}.
\]

Hence

\[
\| R(u, v) \|_{B_{p, \infty}^{0, 1}} \lesssim \| u \|_{B_{p_1, \infty}^0} \| v \|_{B_{p_2, \infty}^0}.
\]  \hspace{1cm} (2.14)

Combining (2.12)~(2.14), we see that (2.11) follows. This prove Lemma 2.5 \( \square \)

**Lemma 2.6.** Let \( p, p_i, r, \rho, \rho_i \in [1, \infty] \) \( (i = 1, 2) \) be such that \( \frac{1}{\rho} = \frac{1}{p_1} + \frac{1}{p_2} \) and \( \frac{1}{\rho} = \frac{1}{p_1} + \frac{1}{p_2} \). Then we have

\[
\| uv \|_{\dot{L}_{T}^{\rho, r}(B_{p, r}^0)} \lesssim \| u \|_{\dot{L}_{T}^{\rho_1, r}(B_{p_1, r}^0)} \| v \|_{\dot{L}_{T}^{\rho_2, r}(B_{p_2, r}^0)}.
\]  \hspace{1cm} (2.15)

**Proof.** The proof is similar to that of Lemma 2.4. Indeed, as in the proof of Lemma 2.4 we decompose \( uv \) into the sum of \( \mathcal{T}(u, v) \), \( \mathcal{T}(v, u) \) and \( \mathcal{R}(u, v) \). To estimate \( \| \mathcal{T}(u, v) \|_{\dot{L}_{T}^{\rho, r}(B_{p, r}^0)} \), we use (1.8) to write

\[
\| \mathcal{T}(u, v) \|_{\dot{L}_{T}^{\rho, r}(B_{p, r}^0)} = \sup_{k \geq -1} \| \triangle_k \left( \sum_{q \geq -1} S_{q-1} u \triangle_q v \right) \|_{L_{T}^{\rho, r}}
\]

\[
\lesssim \sup_{k \geq -1} \| \triangle_k \left( \sum_{|q-k| \leq 4, q \geq -1} S_{q-1} u \triangle_q v \right) \|_{L_{T}^{\rho, r}}.
\]
\[
\lesssim \sup_{k \geq -1, |q-k| \leq 4, q \geq -1} \|S_{q-1}u\|_{L^p_T L^q_x} \|\Delta_q v\|_{L^p_T L^q_x}
\lesssim \|u\|_{L^p_T(B^0_{p,1})} \|v\|_{L^p_T(B^0_{p,\infty})}.
\]

The estimates of \(\|T(v, u)\|_{L^p_T(B^0_{p,\infty})}\) and \(\|\mathcal{R}(u, v)\|_{L^p_T(B^0_{p,\infty})}\) are similar and we omit the details here.

\[\square\]

**Lemma 2.7.** Let \(p, p_i, \rho, \rho_i \in [1, \infty] (i = 1, 2)\) be such that \(\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}\) and \(\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}\). Then we have

\[
\|uv\|_{L^p_T(B^0_{p,\infty})} \lesssim \|u\|_{L^p_T(B^0_{p,1})} \|v\|_{L^p_T(B^0_{p,\infty})}.
\]

**Proof.** The proof is similar to that of Lemma 2.5, we thus omit it.

\[\square\]

We note that results obtained in Lemmas 2.4–2.7 still hold for vector valued functions.

### 3 Proofs of Theorems 1.2 and 1.3

In this section, we give the proofs of Theorems 1.2 and 1.3. We need the following preliminary result:

**Lemma 3.1.** (\cite{4}, p.189, Lemma 5) Let \((X \times Y, \|\cdot\|_X + \|\cdot\|_Y)\) be an abstract Banach product space. \(B_1 : X \times X \to X, B_2 : X \times Y \to Y\) and \(L : Y \to X\) are respectively two bilinear operators and one linear operator such that for any \((x_i, y_i) \in X \times Y (i = 1, 2)\), we have

\[
\|B_1(x_1, x_2)\|_X \leq \lambda \|x_1\|_X \|x_2\|_X, \quad \|L(y_i)\|_X \leq \eta \|y_i\|_Y, \quad \|B_2(x_i, y_i)\|_Y \leq \lambda \|x_i\|_X \|y_i\|_Y,
\]

where \(\lambda, \eta > 0\). For any \((x_0, y_0) \in X \times Y\) with \(\|(x_0, c_0 y_0)\|_{X \times Y} < 1/(16\lambda)\) \((c_* = \max\{2\eta, 1\})\), the following system

\[
(x, y) = (x_0, y_0) + \left(B_1(x, x), B_2(x, y)\right) + \left(L(y), 0\right)
\]

has a solution \((x, y)\) in \(X \times Y\). In particular, the solution is such that

\[
\|(x, c_* y)\|_{X \times Y} \leq 4 \|(x_0, c_0 y_0)\|_{X \times Y}
\]

and it is the only one such that \(\|(x, c_* y)\|_{X \times Y} < 1/(4\lambda)\).

For \(n \geq 2, p \in (\frac{n}{2}, \infty)\) and \(r \in [1, \infty]\), let \(X_T\) and \(Z_T\) respectively be the spaces

\[
X_T = \tilde{L}^2_T(B^0_{\infty,1}) \cap \tilde{L}^2_T(B^0_{\infty,\infty}) \quad \text{and} \quad Z_T = \tilde{L}^2_T(B^0_{p,r})
\]

with norms

\[
\|\vec{u}\|_{X_T} := \|\vec{u}\|_{\tilde{L}^2_T(B^0_{\infty,1})} + \|\vec{u}\|_{\tilde{L}^2_T(B^0_{\infty,\infty})} \quad \text{and} \quad \|\theta\|_{Z_T} := \|\theta\|_{\tilde{L}^2_T(B^0_{p,r})}.
\]
Let \( \mathcal{Y}_T \) be the space
\[
\mathcal{Y}_T = \tilde{L}^2_T(B^0_{\frac{3}{4}, 1}) \cap \tilde{L}^2_T(B^{0, 1}_{\frac{3}{2}, \infty})
\]
with norm
\[
\|\theta\|_{\mathcal{Y}_T} := \|\theta\|_{\tilde{L}^2_T(B^0_{\frac{3}{4}, 1})} + \|\theta\|_{\tilde{L}^2_T(B^{0, 1}_{\frac{3}{2}, \infty})}.
\]
Recall that
\[
\begin{aligned}
&\mathcal{J}_1(\bar{u}, \theta) = e^t \Delta \bar{u}_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P}(\bar{u} \cdot \nabla) \bar{u} ds + \int_0^t e^{(t-s)\Delta} \mathbb{P}(\theta \bar{u}) ds , \\
&\mathcal{J}_2(\bar{u}, \theta) = e^t \Delta \theta_0 - \int_0^t e^{(t-s)\Delta} (\bar{u} \cdot \nabla \theta) ds.
\end{aligned}
\]

In what follows, we prove several bilinear estimates.

**Lemma 3.2.** Let \( T > 0 \), \( n \geq 2 \) and \( \bar{u}_0 \in B_{\infty, 1}^{-1} \cap B_{\infty, \infty}^{-1, 1} \). We have the following two assertions:

1. For \( \theta \in \mathcal{Y}_T \) we have
\[
\|\mathcal{J}_1(\bar{u}, \theta)\|_{X_T} \lesssim (1 + T) \left( \|\bar{u}_0\|_{B_{\infty, 1}^{-1} \cap B_{\infty, \infty}^{-1, 1}} + \|\bar{u}\|_{X_T}^2 + \|\theta\|_{Y_T} \right). \tag{3.2}
\]

2. For \( \theta \in \mathcal{Z}_T \), \( r \in [1, \infty) \) and \( p \in (\frac{n}{2}, \infty) \) we have
\[
\|\mathcal{J}_1(\bar{u}, \theta)\|_{X_T} \lesssim (1 + T) \left( \|\bar{u}_0\|_{B_{r, \infty}^{-1} \cap B_{r, \infty}^{-1, 1}} + \|\bar{u}\|_{X_T}^2 + \|\theta\|_{Z_T} \right). \tag{3.3}
\]

**Proof.** We divide the proof of the \( \mathcal{J}_1(\bar{u}, \theta) \) into two subcases \( q \geq 0 \) and \( q = -1 \). Since when \( q \geq 0 \), the symbol of \( \triangle_q \) is supported in dyadic shells and the symbol of \( \mathbb{P} \) is smooth in the corresponding dyadic shells we have
\[
\triangle_q \mathcal{J}_1(\bar{u}, \theta) = e^t \Delta \triangle_q \bar{u}_0 - \int_0^t e^{(t-s)\Delta} \triangle_q (\mathbb{P} \nabla \cdot (\bar{u} \otimes \bar{u}) (s)) ds.
\]

In \( 3.4 \) we have \( n \) scalar equations and each of the \( n \) components shares the same estimate. By making use of \( 2.2 \) twice we obtain
\[
\|\triangle_q \mathcal{J}_1(\bar{u}, \theta)\|_{\infty} \lesssim e^{-\kappa 2^q t} \|\triangle_q \bar{u}_0\|_{\infty} + \int_0^t e^{-\kappa 2^q (t-s)} \left( 2q \|\triangle_q (\bar{u} \otimes \bar{u})\|_{\infty} + \|\triangle_q \theta\|_{\infty} \right) ds
\]
\[
\lesssim e^{-\kappa 2^q t} \|\triangle_q \bar{u}_0\|_{\infty} + \int_0^t e^{-\kappa 2^q (t-s)} 2q \|\triangle_q (\bar{u} \otimes \bar{u})\|_{\infty} ds
\]
\[
+ \int_0^t e^{-\kappa 2^q (t-s)} \min\{2q \|\triangle_q \theta\|_{\frac{q}{2}}, 2 \|\triangle_q \theta\|_{p}\} ds.
\]
Applying convolution inequalities to the above estimate with respect to time variable we get
\[
\|\triangle_q \mathcal{J}_1(\bar{u}, \theta)\|_{L_T^2 L_x^\infty} \lesssim \left( 1 - \frac{e^{-2\kappa T 2^q}}{2\kappa} \right)^{\frac{1}{2}} \left( 2^{-q} \|\triangle_q \bar{u}_0\|_{\infty} + \|\triangle_q (\bar{u} \otimes \bar{u})\|_{L_T^1 L_x^\infty} \right)
\]

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\[
+ \min \left\{ \frac{1 - e^{-2\kappa T 2^q}}{2\kappa} \| \triangle_q \theta \|_{L^2_T L^r_p}, \frac{1 - e^{-2\kappa T 2^q}}{2\kappa 2^{q(2 - \frac{n}{p})}} \| \triangle_q \theta \|_{L^2_T L^p} \right\}. \tag{3.5}
\]

Considering
\[
\sum_{q \geq 0} 2^{-q(2 - \frac{n}{p})} < \infty \quad \text{for} \quad p \in \left( \frac{n}{2}, \infty \right) \cap [1, \infty),
\]
from (3.5) and Definition 1.1 we see that
\[
\sum_{q \geq 0} \| \triangle_q \mathfrak{J}_1(\vec{u}, \theta) \|_{L^2_T L^\infty} \lesssim \sum_{q \geq 1} \left( 2^{-q} \| \triangle_q \vec{u}_0 \|_\infty + \| \triangle_q (\vec{u} \otimes \vec{u}) \|_{L^1_T L^\infty} \right.
\]
\[
+ \min \left\{ \| \triangle_q \theta \|_{L^2_T L^\infty}, 2^{-q(\frac{n}{p} - 2)} \| \triangle_q \theta \|_{L^2_T L^p} \right\}
\]
\[
\lesssim \| \vec{u}_0 \|_{L^1_{-1, 1}} + \| \vec{u} \otimes \vec{u} \|_{L^1_T L^\infty} + \min \{ \| \theta \|_{L^1_T L^0_{\infty, 1}}, \| \theta \|_{L^1_T L^0_{p, \infty}} \}. \tag{3.6}
\]

Considering
\[
\sup_{q \geq -1} \frac{q + 3}{2^{q(2 - \frac{n}{p})}} \leq C(p, n) < \infty \quad \text{for} \quad p > \frac{n}{2},
\]
from (3.5), Definition 1.1 and a similar argument as before we see that
\[
\sup_{q \geq 0} (q + 3) \| \triangle_q \mathfrak{J}_1(\vec{u}, \theta) \|_{L^2_T L^\infty} \lesssim \| \vec{u}_0 \|_{B_{\infty, 1}^{1, 1}} + \| \vec{u} \otimes \vec{u} \|_{L^1_T L^\infty} + \min \{ \| \theta \|_{L^1_T L^0_{\infty, 1}}, \| \theta \|_{L^1_T L^0_{p, \infty}} \}. \tag{3.7}
\]

Next we consider the case \( q = -1 \). We recall the decay estimates of Oseen kernel (cf., Chapter 11, [14]), by interpolating we observe that \( e^{\Delta \mathbb{P}(-\Delta)^{-\frac{1}{2} + \delta}} \nabla \) (for any \( \delta \in (0, \frac{1}{2}) \)) is \( L^1_T \) bounded. Similar to (3.4) we get
\[
S_0 \mathfrak{J}_1(\vec{u}, \theta) = e^{\Delta} S_0 \vec{u}_0 - \int_0^t e^{(t - \tau) \Delta} S_0 [\mathbb{P} \nabla \cdot (\vec{u} \otimes \vec{u})] d\tau + \int_0^t e^{(t - \tau) \Delta} S_0 \mathbb{P} \theta \vec{u}(\tau) d\tau.
\]

Applying decay estimates of heat kernel and Lemma 2.1 of supp\( \mathcal{F} S_0 \subset B(0, \frac{4}{3}) \) we see that
\[
\| S_0 \mathfrak{J}_1(\vec{u}, \theta) \|_\infty \lesssim \| e^{\Delta} S_0 \vec{u}_0 \|_\infty + \int_0^t (t - \tau)^{-\frac{n}{2p}} \| \mathbb{P} S_0 \theta \|_{2p} d\tau
\]
\[
+ \int_0^t \| e^{(t - \tau) \Delta} S_0 \mathbb{P} \nabla \cdot (\vec{u} \otimes \vec{u}) \|_\infty d\tau
\]
\[
\lesssim \| S_0 \vec{u}_0 \|_\infty + \int_0^t (t - \tau)^{-\frac{n}{2p}} \min \{ \| S_0 \theta \|, \| S_0 \theta \|_{2p} \} d\tau
\]
\[
+ \int_0^t \| S_0 (\vec{u} \otimes \vec{u}) \|_\infty d\tau.
\]

In the above estimate we have used the following fact (see (5.29) of [17]):
\[
\| S_0 \mathbb{P} \nabla \cdot (\vec{u} \otimes \vec{u}) \|_\infty \lesssim \| \mathbb{P} \nabla \vec{h} \|_1 \| S_0 (\vec{u} \otimes \vec{u}) \|_\infty \lesssim \| \nabla \vec{h} \|_{B^0_{1, 1}} \| S_0 (\vec{u} \otimes \vec{u}) \|_\infty \lesssim \| S_0 (\vec{u} \otimes \vec{u}) \|_\infty.
\]

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Applying convolution inequalities to time variable we obtain that

\[
\|S_0\mathcal{J}_1(\bar{u}, \theta)\|_{L_T^2 L_\infty^\infty} \lesssim T^{\frac{1}{2}} \|S_0\bar{u}_0\|_\infty + T^\frac{1}{2} \|S_0(\bar{u} \otimes \bar{u})\|_{L_T^1 L_x^\infty} + T^{4p-n} \min\{\|S_0\theta\|_{L_T^2 L_x^2}, \|S_0\theta\|_{L_T^2 L_x^p}\}
\]

\[
\lesssim C_T \left[\|\bar{u}_0\|_{B^{1,\infty}_{\infty,\infty}} + \|\bar{u}\|_{L_T^1(B^{0,1}_{\infty,\infty})} + \min\{\|\theta\|_{L_T^1(B^{0,1}_{\infty,\infty})}, \|\theta\|_{L_T^1(B^{0,0}_{\infty,\infty})}\} \right]
\]  

(3.8)

\[
\lesssim C_T \left[\|\bar{u}_0\|_{B^{1,\infty}_{\infty,\infty}} + \|\bar{u}\|_{L_T^1(B^{1,1}_{\infty,\infty})} + \min\{\|\theta\|_{L_T^1(B^{1,0}_{\infty,\infty})}, \|\theta\|_{L_T^1(B^{1,1}_{\infty,\infty})}\} \right],
\]

(3.9)

where \(C_T = (T^{4p-n} + T^\frac{1}{2})\).

By applying (3.6) \(~\) (3.9) and Definition 1.1 as well as Lemmas 2.6 \(~\) 2.7 we prove (3.2) and (3.3) and we complete the proof of Lemma 3.2.

**Lemma 3.3.** Let \(T > 0, n \geq 2\) and \(\bar{u} \in X_T\). We have the following assertions:

1. For \(\theta_0 \in B^{-1}_{\frac{1}{2},1} \cap B^{-1,1}_{\frac{1}{2},\infty}\) we have

\[
\|\mathcal{J}_2(\bar{u}, \theta)\|_{Y_T} \lesssim (1 + T) \left(\|\theta_0\|_{B^{-1}_{\frac{1}{2},1} \cap B^{-1,1}_{\frac{1}{2},\infty}} + \|\bar{u}\|_{X_T} \|\theta\|_{Y_T}\right).
\]

(3.10)

2. For \(\theta_0 \in B^{-1}_{p,r}, r \in [1, \infty]\) and \(p \in (\frac{1}{2}, \infty)\) we have

\[
\|\mathcal{J}_2(\bar{u}, \theta)\|_{Z_T} \lesssim (1 + T) \left(\|\theta_0\|_{B^{-1}_{p,r}} + \|\bar{u}\|_{Z_T} \|\theta\|_{Z_T}\right).
\]

(3.11)

**Proof.** Similar as before, we divide the proof of the \(\mathcal{J}_2(\bar{u}, \theta)\) into two subcases \(q \geq 0\) and \(q = -1\). In the case \(q \geq 0\) we have

\[
\Delta_q \mathcal{J}_2(\bar{u}, \theta) = e^{t\Delta} \Delta_q \theta_0 - \int_0^t e^{(t-\tau)\Delta} \Delta_q \nabla \cdot (\bar{u}(\tau)\theta(\tau)) d\tau.
\]

Applying Lemma 2.1, using convolution inequalities to time variable and following a similar argument as before we see that

\[
\|\Delta_q \mathcal{J}_2(\bar{u}, \theta)\|_{L_T^2 L_x^\frac{q}{2}} \lesssim 2^{-q} \|\Delta_q \theta_0\|_{\frac{q}{2}} + \|\Delta_q (\bar{u}\theta)\|_{L_T^1 L_x^\frac{q}{2}},
\]

(3.12)

which yields

\[
\sum_{q \geq 0} \|\Delta_q \mathcal{J}_2(\bar{u}, \theta)\|_{L_T^2 L_x^\frac{q}{2}} \lesssim \sum_{q \geq -1} (2^{-q} \|\Delta_q \theta_0\|_{\frac{q}{2}} + \|\Delta_q (\bar{u}\theta)\|_{L_T^1 L_x^\frac{q}{2}})
\]

\[
\lesssim \|\theta_0\|_{B^{-1}_{\frac{1}{2},1}} + \|\bar{u}\theta\|_{L_T^1(B^{0,1}_{\frac{1}{2},1})}
\]

(3.13)

and

\[
\sup_{q \geq 0} (q + 3) \|\Delta_q \mathcal{J}_2(\bar{u}, \theta)\|_{L_T^2 L_x^\frac{q}{2}} \lesssim \|\theta_0\|_{B^{-1,1}_{\frac{1}{2},\infty}} + \|\bar{u}\theta\|_{L_T^1(B^{1,1}_{\frac{1}{2},\infty})},
\]

(3.14)
where we have used Definition 1.1. Now we consider the case $q = -1$. Similarly, we have

$$S_0 \mathfrak{J}_2(\vec{u}, \theta) = e^{t \Delta} S_0 \theta_0 - \int_0^t e^{(t - \tau) \Delta} S_0 \nabla \cdot (\vec{u} \theta)(\tau) d\tau.$$  

Applying Lemma 2.1 and convolution inequality to time variable we obtain

$$\|S_0 \mathfrak{J}_2(\vec{u}, \theta)\|_{L^1_T L^\infty_x} \leq T^{\frac{1}{2}} \|S_0 \theta_0\|_{\frac{4}{3}} + T^{\frac{1}{2}} \|S_0(\vec{u} \theta)(\tau)\|_{L^1_T L^\infty_x}$$

which yields

$$\|S_0 \mathfrak{J}_2(\vec{u}, \theta)\|_{L^1_T L^\infty_x} \leq T^{\frac{1}{2}} (\|\theta_0\|_{B^{-1}_p, \infty} + \|\vec{u} \theta\|_{L^p_t(B^{0}_{\frac{4}{3}, \infty})}) \quad (3.15)$$

$$\leq T^{\frac{1}{2}} (\|\theta_0\|_{B^{-1}_p, \infty} + \|\vec{u} \theta\|_{L^p_t(B^{0}_{\frac{4}{3}, \infty})}). \quad (3.16)$$

The desired results (3.10) and (3.11) follows from (3.13) $\sim$ (3.16) and Definition 1.1 as well as Lemmas 2.6 and 2.7. This proves Lemma 3.3. \hfill \Box

**Proofs of Theorems 1.2 and 1.3:** From Lemmas 3.2, 3.3 and 3.1 as well as a standard argument, we see that Theorems 1.2 and 1.3 follow.

### 4 Proof of Theorem 1.4

In this section, we give the proof of Theorem 1.4. We first prove the following heat semigroup characterization of the space $B^s_{p,r}$:

**Lemma 4.1.** Let $p, r \in [1, \infty]$, $s < 0$ and $\sigma \geq 0$. The following assertions are equivalent:

1. $f \in B^s_{p,r}$,

2. For all $t \in (0, 1)$, $e^{t \Delta} f \in L^p_t$ and $t^{\frac{1}{2}} \|\ln(\frac{t}{e^2})\|^{\sigma} \|e^{t \Delta} f\|_p \in L^r((0, 1), \frac{dt}{t})$.

**Proof.** The idea of the proof mainly comes from [14] and the proof is quite similar. But for readers convenience, we give the details as follows. We denote by $C$ the constant depends on $n$ and might depend on $s$, $\sigma$ and $r$ in the proof of this Lemma.

1. $\Rightarrow$ (2). We write $f = S_0 f + \sum_{j \geq 0} \Delta_j f$ with

$$\|S_0 f\|_p = \varepsilon_{-1}, \quad \|\Delta_j f\|_p = 2^j |s|(3 + j)^{-\sigma} \varepsilon_j \quad \text{and} \quad (\varepsilon_j)_{j \geq -1} \in \ell^r.$$

We estimate the norm $t^{\frac{|s|}{2}} \|\ln(\frac{t}{e^2})\|^{\sigma} \|e^{t \Delta} f\|_p$ by

$$t^{\frac{|s|}{2}} \|\ln(\frac{t}{e^2})\|^{\sigma} \|e^{t \Delta} S_0 f\|_p \leq t^{\frac{|s|}{2}} \|\ln(\frac{t}{e^2})\|^{\sigma} \|e^{t \Delta} \mathfrak{J}_2 S_0 f\|_p \leq C t^{\frac{|s|}{2}} \|\ln(\frac{t}{e^2})\|^{\sigma} \|S_0 f\|_p.$$  

Similarly, for $j \geq 0$, we have

$$t^{\frac{|s|}{2}} \|\ln(\frac{t}{e^2})\|^{\sigma} \|e^{t \Delta} \Delta_j f\|_p \leq C t^{\frac{|s|}{2}} \|\ln(\frac{t}{e^2})\|^{\sigma} \|\Delta_j f\|_p.$$
Moreover, when \( j \geq 0 \) and \( N \geq 0 \), from the \( L^1_x \) integrability of heat kernel and Lemma 2.1 we have
\[
\frac{1}{t^{\frac{3}{2}}} \left| \ln \left( \frac{t}{e^2} \right) \right|^\sigma \| e^{t \Delta_j f} \|_p = \frac{1}{t^{\frac{3}{2}}} \left| \ln \left( \frac{t}{e^2} \right) \right|^\sigma \| e^{t \Delta j} ( -t \Delta )^{\frac{N}{2}} ( -t \Delta )^{-\frac{N}{2}} \Delta_j f \|_p \\
\leq C t^{-\frac{3}{2}} \left| \ln \left( \frac{t}{e^2} \right) \right|^\sigma 2^{-jN} \| \Delta_j f \|_p.
\]
Combining the above estimates, for some \( N \geq 2 \) and any \( 0 < t < 1 \) we have
\[
\| e^{t \Delta} f \|_p \leq C \left( \varepsilon_{-1} + \sum_{j \geq 0} \min \left\{ 2^j (s) (j + 3)^{-\sigma} \varepsilon_j, \ t^{-\frac{3}{2}} 2^{-jN} (j + 3)^{-\sigma} \varepsilon_j \right\} \right) \\
\leq C \| (\varepsilon_j)_{j \geq 1} \|_p t^{-\frac{3}{2}} < \infty.
\]
Let \( I_k = (2^{-2k-2}, 2^{-2k}) \). For any \( t \in (0, 1] = \cup_{k \geq 0} I_k \), there exists an integer \( j_0 \) such that \( t \in I_{j_0} \). And for \( t \in I_{j_0} \), we have
\[
\frac{1}{t^{\frac{3}{2}}} \left| \ln \left( \frac{t}{e^2} \right) \right|^\sigma \| e^{t \Delta} f \|_p \leq C \left( \varepsilon_{-1} + \sum_{j = 0}^{j_0} \frac{(3 + j_0)^{\sigma}}{(3 + j)^{\sigma}} 2^{-j_0} (s) (j_0 + 3)^{-\sigma} \varepsilon_j + \sum_{j = j_0 + 1} 2^{j_0} (s) (|s| - N)^{-\sigma} \varepsilon_j \right) \geq C \eta_{j_0}.
\]
From the above estimate we see that
\[
\left\| \frac{1}{t^{\frac{3}{2}}} \left| \ln \left( \frac{t}{e^2} \right) \right|^\sigma e^{t \Delta} f \|_{L^r((0, 1), \frac{dt}{t^3})} \leq C \| (\eta_j)_{j \geq 1} \|_{L^r} \leq C \| (\varepsilon_j)_{j \geq 1} \|_{L^r}.
\]
Indeed, by using Young’s inequality we have
\[
\sum_{j_0 \geq 1} \left( \sum_{j = 0}^{j_0} \frac{(3 + j_0)^{\sigma}}{(3 + j)^{\sigma}} 2^{j_0} (s) (|s| - N)^{-\sigma} \varepsilon_j \right) \leq C \sum_{j_0 \geq 1} \left( \sum_{j = 0}^{j_0} 2^{j_0} (s) (|s| - N)^{-\sigma} \varepsilon_j \right) \leq C \| (\varepsilon_j)_{j \geq 1} \|_{L^r}.
\]
and
\[
\sum_{j_0 \geq 1} \left( \sum_{j = j_0 + 1} 2^{j_0} (s) (|s| - N)^{-\sigma} \varepsilon_j \right) \leq C \| (\varepsilon_j)_{j \geq 1} \|_{L^r}.
\]
\((2) \Rightarrow (1)\). We get that \( S_0 f = e^{-\frac{1}{4} \Delta} S_0 e^{\frac{1}{4} \Delta} f \in L^p_x \) since the kernel of \( e^{-\frac{1}{4} \Delta} S_0 \) is \( L^1_x \) bounded. Similarly, when \( j \geq 0 \), we write \( \Delta_j f = e^{-t \Delta} \Delta_j e^{t \Delta} f \). For any \( j \geq 0 \), we choose \( t \) such that \( 2^{-2j} < t < 2^{-2j} \). Then we have
\[
2^{j_0} (3 + j)^{\sigma} \| \Delta_j f \|_p \leq C t_\frac{1}{|t|} (2 - \ln t) \| e^{-t \Delta} \Delta_j e^{t \Delta} f \|_p \leq C \frac{1}{t^{\frac{3}{2}}} \left| \ln \left( \frac{t}{e^2} \right) \right|^\sigma \| e^{t \Delta} f \|_p.
\]
Consequently, we have
\[
\| f \|_{L^r_{\tilde{B}_{p, r}}} = \sum_{j \geq 1} 2^{j r} (3 + j)^{\sigma r} \| \Delta_j f \|_p^r \leq C \left( \sup_{0 < t < 1} t_\frac{1}{|t|} \left| \ln \left( \frac{t}{e^2} \right) \right|^\sigma \| e^{t \Delta} f \|_p \right)^r \\
\leq C \| t^{\frac{3}{2}} \left| \ln \left( \frac{t}{e^2} \right) \right|^\sigma \| e^{t \Delta} f \|_p \|_{L^r((0, 1), \frac{dt}{t^3})},
\]
where the last inequality follows from a similar argument as in Lemma 16.1 of [14].
Next we prove a bilinear estimate.

**Lemma 4.2.** Let $n \geq 2$, $\varepsilon > 0$, $p \in (\frac{1}{2}, \infty)$ and $\mathcal{J}_1, \mathcal{J}_2$ be as in (3.1). For $0 < T \leq 1$, there exists $0 < \nu < \nu(p) = \frac{2n-n}{2p}$ such that

\[
\begin{dcases}
\sup_{t \in (0,T)} \frac{1}{t} \ln \left( \frac{t}{e^2} \right) \| \mathcal{J}_1(\vec{u}, \theta) \|_B^{1,1} \leq \| \vec{u}_0 \|_{B^{1,1}} + \left( \sup_{t \in (0,T)} \frac{1}{t} \ln \left( \frac{t}{e^2} \right) \| \vec{u} \|_\infty \right)^2 + T^{\nu} \sup_{t \in (0,T)} \frac{1}{t} \ln \left( \frac{t}{e^2} \right) \| \theta \|_p \\
\sup_{t \in (0,T)} \frac{1}{t} \ln \left( \frac{t}{e^2} \right) \| \mathcal{J}_2(\vec{u}, \theta) \|_p \leq \| \theta_0 \|_{B^{1,1}} + \frac{1}{\varepsilon} \sup_{t \in (0,T)} \frac{1}{t} \ln \left( \frac{t}{e^2} \right) \| \vec{u} \|_\infty \sup_{t \in (0,T)} \frac{1}{t} \ln \left( \frac{t}{e^2} \right) \| \theta \|_p.
\end{dcases}
\]

**Proof.** The term $\int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (\vec{u} \otimes \vec{u}) d\tau$ is already treated in Lemma 2.5 of [7]. From Lemma 4.1 we see that

\[
\sup_{t \in (0,1)} \frac{1}{t} \ln \left( \frac{t}{e^2} \right) \| e^{\Delta \vec{u}_0} \|_\infty \sim \| \vec{u}_0 \|_{B^{1,1}}, \quad \sup_{t \in (0,1)} \frac{1}{t} \ln \left( \frac{t}{e^2} \right) \| e^{\Delta \theta_0} \|_p \sim \| \theta_0 \|_{B^{1,1}}.
\]

Therefore, it remains to estimate

\[
\int_0^t e^{(t-\tau)\Delta} \mathbb{P}(\theta \vec{u}) d\tau, \quad \int_0^t e^{(t-\tau)\Delta} \nabla \cdot (\vec{u} \theta) d\tau.
\]

By using the decay estimates of the Oseen kernel (cf. [14], Proposition 11.1) we see that

\[
t^\frac{1}{2} \frac{1}{t} \ln \left( \frac{t}{e^2} \right) \| e^{(t-\tau)\Delta} \nabla \cdot (\theta \vec{u}) d\tau \|_\infty \leq t^\frac{1}{2} \frac{1}{t} \ln \left( \frac{t}{e^2} \right) \| e^{(t-\tau)\Delta} \theta \|_p d\tau
\]

\[
\leq t^\frac{1}{2} \frac{1}{t} \ln \left( \frac{t}{e^2} \right) \| e^{(t-\tau)\Delta} \theta \|_p \leq t^\nu(p) \sup_{\tau \in (0,T)} \tau^\frac{1}{2} \left( \frac{\tau}{e^2} \right)^2 \| \theta \|_p
\]

\[
\leq t^\nu(p) \sup_{\tau \in (0,T)} \tau^\frac{1}{2} \| \theta \|_p,
\]

where $\nu(p)$ and $\lim_{\varepsilon \to 0} \nu(p)-\varepsilon |\ln(\frac{t}{\varepsilon})|^{1-\varepsilon} = 0$. By applying the decay estimates of the heat kernel (is Oseen kernel too) (cf. [14], Proposition 11.1) we see that

\[
t^\frac{1}{2} \frac{1}{t} \ln \left( \frac{t}{e^2} \right) \| e^{(t-\tau)\Delta} \nabla \cdot (\vec{u} \theta) d\tau \|_p \leq t^\frac{1}{2} \frac{1}{t} \ln \left( \frac{t}{e^2} \right) \| e^{(t-\tau)\Delta} \theta \|_p \| \vec{u} \|_\infty d\tau
\]

\[
\leq t^\frac{1}{2} \frac{1}{t} \ln \left( \frac{t}{e^2} \right) \| e^{(t-\tau)\Delta} \theta \|_p \| \vec{u} \|_\infty \| \vec{u} \|_\infty \| \theta \|_p
\]

\[
\leq 1 \sup_{\tau \in (0,T)} \tau^\frac{1}{2} \left( \frac{\tau}{e^2} \right)^2 \| \theta \|_p \| \vec{u} \|_\infty \| \vec{u} \|_\infty,
\]

where $t^\frac{1}{2} \frac{1}{t} \ln \left( \frac{t}{e^2} \right) \| e^{(t-\tau)\Delta} \theta \|_p \| \vec{u} \|_\infty \| \vec{u} \|_\infty \| \theta \|_p \leq \frac{1}{\varepsilon}$. Indeed, it suffices to show

\[
t^\frac{1}{2} \frac{1}{t} \ln \left( \frac{t}{e^2} \right) \| e^{(t-\tau)\Delta} \theta \|_p \| \vec{u} \|_\infty \| \vec{u} \|_\infty \| \theta \|_p \leq \frac{1}{\varepsilon}.
\]
which is equivalent to
\[
\left| \ln(t^2) \right|^\varepsilon \int_0^{t/2} \tau^{-1} \left| \ln(t^2) \right|^{-1-\varepsilon} d\tau \lesssim \frac{1}{\varepsilon} \Leftrightarrow \int_{\left| \ln(t^2) \right|}^{\infty} x^{-1-\varepsilon} dx \lesssim \frac{1}{\varepsilon} \left| \ln(t^2) \right|^{-\varepsilon}.
\]

\[\square\]

**Proof of Theorem 1.4.** Applying Lemmas 3.1 and 4.2 and following similar arguments as in [7], we prove Theorem 1.4. We omit the details here.

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