CONFORMAL WELDING PROBLEM, FLOW LINE PROBLEM, 
AND MULTIPLE SCHRAMM–LOEWNER EVOLUTION

MAKOTO KATORI AND SHINJI KOSHIDA

Abstract. A quantum surface (QS) is an equivalence class of pairs \((D, H)\) of simply connected domains \(D \subseteq \mathbb{C}\) and random distributions \(H\) on \(D\) induced by the conformal equivalence for random metric spaces. This distribution-valued random field is extended to a QS with \(N + 1\) marked boundary points (MBPs) with \(N \in \mathbb{Z}_{\geq 0}\). We propose the conformal welding problem for it in case of \(N \in \mathbb{Z}_{\geq 1}\). If \(N = 1\), it is reduced to the problem introduced by Sheffield, who solved it by coupling the QS with the Schramm–Loewner evolution (SLE). When \(N \geq 3\), this problem is well posed only if the configuration of MBPs is randomized, and hence a new problem arises how to determine the probability law of the configuration. We report that the multiple SLE in \(\mathbb{H}\) driven by the Dyson model on \(\mathbb{R}\) helps us to fix the problems and makes them solvable for any \(N \geq 3\).

We also propose the flow line problem for an imaginary surface with boundary condition changing points (BCCPs). In the case when the number of BCCPs is two, this problem was solved by Miller and Sheffield. We address the general case with an arbitrary number of BCCPs in a similar manner to the conformal welding problem. We again find that the multiple SLE driven by the Dyson model plays a key role to solve the flow line problem.

1. Introduction

1.1. Preliminaries.

Gaussian free field. Let \(D \subseteq \mathbb{C}\) be a simply connected domain and let \(W_{\text{Dir}}(D)\) be the Hilbert space completion of the space \(C^\infty_0(D)\) of smooth functions supported in \(D\) with respect to the Dirichlet inner product

\[
(f, g)_{\nabla} = \frac{1}{2\pi} \int_D (\nabla f)(z) \cdot (\nabla g)(z) d\mu(z),
\]

where \(\mu\) is the Lebesgue measure on \(D \subset \mathbb{C};\ d\mu(z) = dzd\overline{z}\). The Gaussian free field (GFF) with Dirichlet boundary condition on \(D\) is defined as an isotopy \(H_D^{\text{Dir}}: W_{\text{Dir}}(D) \to L^2(\Omega_D^{\text{Dir}}, \mathcal{F}_D^{\text{Dir}}, \mathbb{P}_D^{\text{Dir}})\) where \((\Omega_D^{\text{Dir}}, \mathcal{F}_D^{\text{Dir}}, \mathbb{P}_D^{\text{Dir}})\) is a probability space such that each \((H_D^{\text{Dir}}(\rho), \rho \in W_{\text{Dir}}(D))\) is a mean-zero Gaussian random

2010 Mathematics Subject Classification. 60D05, 60J67, 82C22.

Key words and phrases. Conformal welding problem, Flow line problem, Gaussian free field, Quantum surface with marked boundary points, Imaginary surface with boundary condition changing points, Multiple Schramm–Loewner evolution, Dyson model.
variable [She07]. One can construct such an isotopy relying on the Bochner-
Minlos theorem that is an analogue of Bochner’s theorem applicable to the case
when the source Hilbert space is infinite dimensional [Hid80, Chapter 3]. It is also
known that, in this construction, the sigma field when the source Hilbert space is infinite dimensional [Hid80, Chapter 3]. It is also
Minlos theorem that is an analogue of Bochner’s theorem applicable to the case
variable [She07]. One can construct such an isotopy relying on the Bochner-
...
**Liouville quantum gravity.** GFF plays a relevant role in constructing the Liouville quantum gravity (LQG) \[\text{[DS11]}\]. Following the original idea by Polyakov \[\text{[Pol81a,Pol81b]}\], it is expected that the object \(e^{\gamma H^\epsilon_D(z)}d\mu(z)\) is the desired random area measure on \(D\), where \(\gamma \in (0, 2]\). This does not work, however, because each realization of \(H^\epsilon_D\), \(h(\cdot) = H^\epsilon_D(\cdot, \omega)\), \(\omega \in \Omega^\epsilon_D\), is not a function but a distribution on \(D\), thus its exponentiation has to be verified in some sense. This difficulty is overcome by a certain regularization. Let us fix a realization \(h \in C^\infty(\overline{D})\) and let \(h_\epsilon(z)\) be the mean value of \(h\) on the circle \(\partial B_\epsilon(z)\) of radius \(\epsilon\) centered at \(z \in D\). Then the required area measure is obtained by

\[
d\mu_\epsilon^h(z) := \lim_{\epsilon \to 0} e^{\gamma^2/2}e^{\gamma h_\epsilon(z)}d\mu(z), \quad z \in D.
\]

In a similar way, one can construct a linear measure on the boundary

\[
d\nu_\epsilon^h(x) := \lim_{\epsilon \to 0} e^{\gamma^2/4}e^{\gamma h_\epsilon(x)/2}d\nu(x), \quad x \in \partial D,
\]

where \(\nu\) is the Lebesgue measure on the boundary, while, in this case, \(h_\epsilon(x)\) is the average over the semi-circle centered at \(x \in \partial D\) of radius \(\epsilon\) included in \(D\).

Let \(\tilde{D} \subseteq \mathbb{C}\) be another simply connected domain, and \(\psi : \tilde{D} \to D\) be a conformal equivalence. Then an area measure is induced on \(\tilde{D}\) by pulling back the measure \(\mu_\epsilon^h\) on \(D\). Namely, for a measurable set \(A \subseteq \tilde{D}\), its area is computed as \(\psi^*\mu_\epsilon^h(A) := \mu_\epsilon^h(\psi(A))\). When we closely look at the pulled-back measure, we find that it can also be realized as \(\mu_\epsilon^h\) built from a distribution \(\hat{h}\) on \(\tilde{D}\). Indeed, by changing integration variables, the area of \(A \subseteq \tilde{D}\) with respect to the pulled-back measure \(\psi^*\mu_\epsilon^h\) becomes

\[
\lim_{\epsilon \to 0} \int_{\psi(A)} e^{\gamma^2/2}e^{\gamma h_\epsilon(z)}d\mu(z) = \lim_{\epsilon \to 0} \int_A (|\psi'(w)|e)^{\gamma^2/2}e^{\gamma(h_\epsilon \circ \psi)(w)}|\psi'(w)|^2d\mu(w),
\]

where \(\psi'(w) = \frac{d\nu}{d\omega}(w)\). Note that, in the right hand side, in which the integral is taken over \(A \subseteq \tilde{D}\), the regularization parameter \(\epsilon\) has to be rescaled by \(|\psi'(w)|\).

This implies that if we introduce a distribution on \(\tilde{D}\) by \(\bar{h} = h \circ \psi + Q \log |\psi'|\) with the parametrization \(Q = (\frac{\pi}{\gamma} + 2)/\gamma = \frac{2}{\gamma} + \frac{2}{\gamma}\), then the corresponding area measure \(\mu_\epsilon^\bar{h}\) agrees with the pulled-back measure \(\psi^*\mu_\epsilon^h\). It can be verified that the boundary measure also behaves correctly: \(\nu_\epsilon^\bar{h}(I) = \nu_\epsilon^h(\psi(I))\) for a measurable \(I \subseteq \partial \tilde{D}\). Motivated by this, we make the following definition:

**Definition 1.1.** Let \(\gamma \in (0, 2]\). Pairs \((D_1, h_1)\) and \((D_2, h_2)\) of simply connected domains \(D_i \subseteq \mathbb{C}\) and distributions \(h_i \in C^\infty(\overline{D_i})\), \(i = 1, 2\), are said to be \(\gamma\)-equivalent if there exists a conformal equivalence \(\psi : D_1 \to D_2\) such that

\[
h_1 = h_2 \circ \psi + Q \log |\psi'|
\]

holds, where \(Q = \frac{2}{\gamma} + \frac{2}{\gamma}\).

**Definition 1.2** (Pre-quantum surface). Let \(\gamma \in (0, 2]\). A \(\gamma\)-pre-quantum surface is an \(\gamma\)-equivalence class of pairs \((D, h)\) of simply connected domains \(D \subseteq \mathbb{C}\) and
distributions \( h \in C^\infty(\overline{D})' \). We denote the \( \gamma \)-equivalence class of \((D, h)\) by \([D, h]_\gamma\) and write the totality of \( \gamma \)-pre-quantum surfaces as

\[
S_\gamma := \{ [D, h]_\gamma | D \subseteq \mathbb{C}, h \in C^\infty(\overline{D})' \}.
\]

We will give the construction of \( S_\gamma \) as an orbifold in Appendix A.

The quantization of \( \gamma \)-pre-quantum surfaces is carried out by randomizing them:

**Definition 1.3 (Quantum surface).** Let \( \gamma \in (0, 2] \). A \( \gamma \)-quantum surface is a probability measure on \( S_\gamma \). Equivalently, a \( \gamma \)-quantum surface is a collection of pairs \((D, H_D)\), where \( D \subseteq \mathbb{C} \) is a simply connected domain and \( H_D \) is a \( C^\infty(\overline{D})' \)-valued random field subject to the condition that, for all simply connected domains \( D_1, D_2 \subseteq \mathbb{C} \) and conformal equivalences \( \psi : D_1 \to D_2 \), the equality in probability law

\[
H_{D_1} \overset{\text{law}}{=} H_{D_2} \circ \psi + Q \log |\psi'| \tag{1.2}
\]

holds, where \( Q = \frac{2}{\gamma} + \frac{2}{\gamma} \). We write this collection as \([D, H_D]_\gamma\).

**Remark 1.4.** If one has a pair \((D, H_D)\) of a simply connected domain \( D \subseteq \mathbb{C} \) and \( C^\infty(\overline{D})' \)-valued random field \( H_D \), then it uniquely extends to a \( \gamma \)-quantum surface \([D, H_D]_\gamma\).

**Example 1.5.** A relevant example of a \( \gamma \)-quantum surface arises from the free boundary GFF. A pair \((D, H^\text{Fr}_D)\) of a simply connected domain \( D \subseteq \mathbb{C} \) and the free boundary GFF \( H^\text{Fr}_D \) on \( D \) defines a quantum surface. Indeed, the assignment \( \omega \mapsto [D, H^\text{Fr}_D(\omega)]_\gamma \) gives an \( S_\gamma \)-valued random field on \( \Omega_D^\text{Fr} \) and induces a probability measure on \( S_\gamma \).

**Quantum surface with marked boundary points.** In order to address the conformal welding problem, we define a quantum surface with marked boundary points. It is a refined version of a quantum surface decorated by data of boundary points. Let \( D \subseteq \mathbb{C} \) be a simply connected domain. For \( N \in \mathbb{Z}_{\geq 0} \), we define the configuration space for \( N + 1 \) ordered boundary points of \( D \) as

\[
\text{Conf}_{N+1}^<(\partial D) = \left\{ (x_1, \ldots, x_{N+1}) \in (\partial D)^{N+1} |_{\text{aligned counterclockwisely}} \right\},
\]

where \( \partial D = \overline{D} \setminus D \) is the boundary of \( D \).

**Definition 1.6.** Let \( \gamma \in (0, 2] \), \( N \in \mathbb{Z}_{\geq 0} \) and let \((D, h_D, (x_D^{(1)}, \ldots, x_D^{(N+1)}))\) be a triple consisting of

- A simply connected domain \( D \subseteq \mathbb{C} \).
- A distribution \( h_D \in C^\infty(\overline{D})' \).
- An \((N+1)\)-tuple of ordered boundary points \((x_D^{(1)}, \ldots, x_D^{(N+1)})\) in \( \text{Conf}_{N+1}^<(\partial D) \).

Triples \((D_1, h_{D_1}, (x_{D_1}^{(1)}, \ldots, x_{D_1}^{(N+1)}))\) and \((D_2, h_{D_2}, (x_{D_2}^{(1)}, \ldots, x_{D_2}^{(N+1)}))\) are said to be \( \gamma \)-equivalent if there exists a conformal equivalence \( \psi : D_1 \to D_2 \) such that
ψ(x_D^{(i)}) = x_{D_2}^{(i)}, i = 1, \ldots, N + 1 and the following identity among distributions holds:

\[ h_{D_1} = h_{D_2} \circ \psi + Q \log |\psi'|, \]

where \( Q = \frac{2}{\gamma} + \frac{2}{\gamma} \).

**Definition 1.7** (Pre-quantum surface with marked boundary points). Let \( \gamma \in (0, \gamma_0] \), \( N \in \mathbb{Z}_{\geq 0} \). A \( \gamma \)-pre-quantum surface with \( N + 1 \) marked boundary points is a \( \gamma \)-equivalence class of triples \((D, h_D, (x_D^{(1)}, \ldots, x_D^{(N+1)})\) of simply connected domains \( D \subseteq \mathbb{C} \), distributions \( h_D \in C^\infty(D') \), and ordered boundary points \((x_D^{(1)}, \ldots, x_D^{(N+1)}) \in \text{Conf}^{\leq N+1}_D(\partial D) \). We denote the \( \gamma \)-equivalence class of \((D, h_D, (x_D^{(1)}, \ldots, x_D^{(N+1)})\) as \([D, h_D, (x_D^{(1)}, \ldots, x_D^{(N+1)})]_\gamma \) and write the totality of \( \gamma \)-pre-quantum surfaces with \( N + 1 \) marked boundary points as \( S_{\gamma, N+1} \).

We will give the construction of \( S_{\gamma, N+1} \) as an orbifold in Appendix \( \Delta \).

**Definition 1.8** (Quantum surface with marked boundary points). Let \( \gamma \in (0, \gamma_0] \) and \( N \in \mathbb{Z}_{\geq 0} \). A \( \gamma \)-quantum surface with \( N + 1 \) marked boundary points is a probability measure on \( S_{\gamma, N+1} \). Equivalently, a \( \gamma \)-quantum surface with \( N + 1 \) marked boundary points is a collection of triples \((D, H_D, (X_D^{(1)}, \ldots, X_D^{(N+1)})\), where \( D \subseteq \mathbb{C} \) is a simply connected domain and \((H_D, (X_D^{(1)}, \ldots, X_D^{(N+1)})\) is a \((C^\infty(D') \times \text{Conf}^{\leq N+1}_D(\partial D))\)-valued random field subject to the condition that, for all simply connected domains \( D_1, D_2 \subseteq \mathbb{C} \) and conformal equivalences \( \psi: D_1 \to D_2 \),

\[
(H_{D_1}, (X_{D_1}^{(1)}, \ldots, X_{D_1}^{(N+1)})) \overset{(\text{law})}{=} (H_{D_2} \circ \psi + Q \log |\psi'|, (\psi^{-1}(X_{D_2}^{(1)}), \ldots, \psi^{-1}(X_{D_2}^{(N+1)})))
\]

holds, where \( Q = \frac{2}{\gamma} + \frac{2}{\gamma} \). We write this collection as \([D, h_D, (X_D^{(1)}, \ldots, X_D^{(N+1)})]_\gamma \).

The relevant example of \( \gamma \)-quantum surfaces with marked boundary points in the present paper is of the standard type defined as follows: We consider the following space:

\[
\tilde{S}^{\text{Rot}_{\gamma,N+1}}(\mathbb{H}) := C^\infty(\mathbb{H})' \times \text{Conf}^{\leq N}_\gamma(\mathbb{R}),
\]

where \( \text{Conf}^{\leq N}_\gamma(\mathbb{R}) := \{ (x_1, \ldots, x_N) \in \mathbb{R}^N | x_1 < \cdots < x_N \} \). Another presentation of this space contained in Appendix \( \Delta \) will motivate the superscript “Rot”. Suppose that a probability space \((\text{Conf}^{\leq N}_\gamma(\mathbb{R}), \mathcal{F}_N, \mathbb{P}_N)\) is given. It is equivalent to a random \( N \)-point configuration \( X = (X_1, \ldots, X_N) \) on \( \mathbb{R} \) defined by \( X_i : \text{Conf}^{\leq N}_\gamma(\mathbb{R}) \to \mathbb{R}; (x_1, \ldots, x_N) \mapsto x_i, i = 1, \ldots, N \). Let \( \alpha = (\alpha_1, \ldots, \alpha_N) \) be an \( N \)-tuple of real numbers. For given \( N \) points \( x = (x_1, \ldots, x_N) \in \text{Conf}^{\leq N}_\gamma(\mathbb{R}) \), we define a function on \( \mathbb{H} \)

\[
u^{\alpha}_H(z) = \sum_{i=1}^{N} \alpha_i \log |z - x_i|.
\]

Here \( z \) is the standard coordinate on \( \mathbb{H} \) embedded in \( \mathbb{C} \). Then the assignment

\[
(H^X_{\alpha}, X) : \Omega^F_\mathbb{H} \times \text{Conf}^{\leq N}_\gamma(\mathbb{R}) \ni (\omega, x) \mapsto (H^F_{\mathbb{H}}(\omega) + \nu^{\alpha}_H(x), x) \in \tilde{S}^{\text{Rot}_{\gamma,N+1}}(\mathbb{H})
\]
gives an $\widetilde{S}_{\gamma,N+1}^{\text{Rot}}(\mathbb{H})$-valued random field on $\Omega_{\text{Fr}}^N := \Omega_{\text{Fr}}^N \times \text{Conf}^\gamma_N(\mathbb{R})$ equipped with the product probability measure $\mathbb{P}_{\text{Fr}}^N \otimes \mathbb{P}_N$. We denote the probability measure on $\mathcal{S}_{\gamma,N+1}$ induced along the surjection

\begin{equation}
\pi_{\gamma,N+1}^\infty: S_{\gamma,N+1}^{\text{Rot}} \rightarrow \mathcal{S}_{\gamma,N+1}, \quad (h, x) \mapsto \left[\mathbb{H}, h, (x, \infty)\right]_\gamma
\end{equation}

by $\mathbb{P}_{\gamma,N+1}^{\text{X},\alpha}$.

**Definition 1.9** (Standard type). Let $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N$ and let $X = (X_1, \ldots, X_N)$ be a $\text{Conf}^\gamma_N(\mathbb{R})$-valued random variable. The associated probability measure $\mathbb{P}_{\gamma,N+1}^{X,\alpha}$ on $\mathcal{S}_{\gamma,N+1}$ constructed above is called a $\gamma$-quantum surface with $N+1$ marked boundary points of the $(X, \alpha)$-standard type. Equivalently, a $\gamma$-quantum surface with $N+1$ marked boundary points of the $(X, \alpha)$-standard type is a collection of the form $[\mathbb{H}, H_{\mathbb{H}}^{X,\alpha}, (X, \infty)]_\gamma$.

By definition, the random variable $X$ and the random field $H_{\mathbb{H}}^{\text{Fr}}$ are independent. They are combined when constructing the probability measure $\mathbb{P}_{\gamma,N+1}^{X,\alpha}$ giving rise to a $C^\infty(\mathbb{H})$-valued random field $H_{\mathbb{H}}^{X,\alpha} = H_{\mathbb{H}}^{\text{Fr}} + u_{\mathbb{H}}^{x,\alpha}$ depending on the random boundary points $X$ taking values in $\text{Conf}^\gamma_N(\mathbb{R})$.

Note that the function $u_{\mathbb{H}}^{x,\alpha}$ defined by Eq. (1.4) is harmonic with logarithmic singularities at the $(N+1)$-st point $\infty$ as well as at points $x_i \in \mathbb{R}$, $i = 1, \ldots, N$. Indeed, introducing a coordinate $w = 1/z$ vanishing at $\infty$, we see that $u_{\mathbb{H}}^{x,\alpha}(1/w) \sim -\sum_{i=1}^N \alpha_i \log |w|$ as $w \to 0$.

In the rest of the present paper, we will abbreviate a $\gamma$-(pre-)quantum surface with $N+1$ marked boundary points as a $\gamma$-(pre-)QS-$(N+1)$-MBPs.

1.2. The conformal welding problem. Now we propose the conformal welding problem for a $\gamma$-QS-$(N+1)$-MBPs of the $(X, \alpha)$-standard type with $N \in \mathbb{Z}_{\geq 1}$ as follows:

**Problem 1.** Let $\mathbb{P}_{\gamma,N+1}^{X,\alpha}, \gamma \in (0, 2]$, $N \in \mathbb{Z}_{\geq 1}$, be a $\gamma$-QS-$(N+1)$-MBPs of the $(X, \alpha)$-standard type. Then for each realization $(h, x) \in \widetilde{S}_{\gamma,N+1}^{\text{Rot}}(\mathbb{H})$ of $(H_{\mathbb{H}}^{X,\alpha}, X)$, find a member

$$
\left(\mathbb{H} \setminus \bigcup_{i=1}^N \eta^{(i)}(0, 1], \hat{h}, (\eta^{(1)}(1), \ldots, \eta^{(N)}(1), \infty)\right)
$$

in the equivalence class $[\mathbb{H}, h, (x, \infty)]_\gamma$ such that $\eta^{(i)} : (0, 1] \rightarrow \mathbb{H}$, $i = 1, \ldots, N$ are non-colliding and non-self-intersecting slits in $\mathbb{H}$ satisfying the following conditions:

1. The slits are anchored on the real axis: $\eta^{(i)}_0 := \lim_{t \to 0} \eta^{(i)}(t) \in \mathbb{R}$, $i = 1, \ldots, N$.
2. The slits are seams:

$$
\nu^\gamma_h(\eta^{(i)}(0, t]_{\text{L}}) = \nu^\gamma_h(\eta^{(i)}(0, t]_{\text{R}}), \quad t \in (0, 1], \quad i = 1, \ldots, N.
$$
Here $\eta^{(i)}(0,t)_{\text{L}}$ (resp. $\eta^{(i)}(0,t)_{\text{R}}$) is the boundary segment lying on the left (resp. right) of the slit $\eta^{(i)}(0,t)$.

To explain its geometric meaning, suppose that Problem 1 was solved. For each realization $H_{x,\alpha}^\omega$ of $H_{x,\alpha}^\omega$, let $\psi : \mathbb{H} \rightarrow \mathbb{H} \setminus \bigcup_{i=1}^N \eta^{(i)}(0,1]$ be the conformal equivalence. Write $z^-_i$ and $z^+_i$ for the points on the real axis such that $z^-_i < x_i < z^+_i$ and $\psi(z^+_i) = \eta^{(i)}_0$, $i = 1, \ldots, N$. Then the conformal mapping $\psi$ glues the intervals $[z^-_i, x_i]$ and $[x_i, z^+_i]$ by means of the boundary measure $\nu_{H_{x,\alpha}^\omega}^\gamma$ (see Fig. 1.1).

As a solution to Problem 1, a statistical ensemble of slits $\{\eta^{(i)}\}_{i=1}^N$ is obtained. Then, we would like to ask the probability law for the resulting curves:

**Problem 2.** Determine the probability law for the slits $\{\eta^{(i)}\}_{i=1}^N$.

Note that the anchor points of these curves on $\mathbb{R}$ are also random variables: $\eta^{(i)}_0 = \eta^{(i)}_0(\omega, x)$, $i = 1, \ldots, N$, while this seems to prevent us from capturing the ensemble of slits. Thus we also set the following problem as a sub-problem of Problem 2:

**Problem 3.** Find a $\gamma$-QS-$(N+1)$-MBPs of the $(X,\alpha)$-standard type such that the anchor points $\{\eta^{(i)}_0\}_{i=1}^N$ are deterministic.

In the case of $N = 1$, this problem reduces to the one addressed by Sheffield [She16]. The space $S_{\gamma,N+1}$ is fibered over $\text{Aut}(\mathbb{H}) \setminus \text{Conf}_{N+1}^\gamma(\partial \mathbb{H})$ by forgetting the distributions (see Appendix A). Owing to the fact that the space $\text{Aut}(\mathbb{H}) \setminus \text{Conf}_2^\gamma(\partial \mathbb{H}) = \{[0, \infty]\}$ consists of a single element, Problem 3 becomes trivial in this case. We could say that Sheffield [She16] addressed Problems 1 and 2 for a $\gamma$-QS-2-MBP of the $(X,\alpha)$-standard type with $\alpha_1 = \frac{2}{\gamma}$ and that he found a one-parameter family of solutions to Problem 1 namely, the reverse flow of the Schramm–Loewner evolution (SLE) [Sch00, RS05] gives the required conformal equivalences. Consequently, the resulting curve obeys the probability law of the one for the SLE of parameter $\kappa = \gamma^2$ as the solution to Problem 2.
1.3. Main result. We can find a hint to solve Problems [1] to [3] in the Loewner theory [Löw23, KSS68], which gives a characterization of a slit in a simply connected domain evolving from the boundary through the one-parameter family of uniformizing maps associated with it. We recall the result in [RS17] that dealt with the multiple-slit version of the Loewner theory. Let \( \eta_{(i)} : (0, \infty) \to \mathbb{H}, \ i = 1, \ldots, N \), be non-colliding and non-self-intersecting curves in \( \mathbb{H} \) anchored on \( \mathbb{R} \): \( \eta_{(i)}(0) \in \mathbb{R}, \ i = 1, \ldots, N \). We set \( H_{\eta} := \mathbb{H} \setminus \left( \bigcup_{i=1}^{N} \eta_{(i)}(0, t) \right) \). Then, at each time \( t \in (0, \infty) \), there is a unique conformal mapping \( g_t : H_{\eta(t)} \to \mathbb{H} \) under the hydrodynamic normalization (Fig. 1.2):

\[
g_t(z) = z + \frac{C(H_{\eta(t)})}{z} + O(|z|^{-2}), \quad z \to \infty.
\]

The constant \( C(H_{\eta(t)}) \) is called the half plane capacity of \( H_{\eta(t)} \). Notice that, by changing the parametrization of the curves, we can take \( C(H_{\eta(t)}) = 2Nt \). We call such a parametrization of curves a standard parametrization.

**Theorem 1.10** (Alternative expression of [RS17, Theorem 1.1]). Let \( \eta_{(i)} : (0, \infty) \to \mathbb{H}, \ i = 1, \ldots, N \), be non-colliding and non-self-intersecting curves in \( \mathbb{H} \) anchored on \( \mathbb{R} \) with a standard parametrization. There exists a unique set of continuous driving functions \( X_t = (X_{(1)^t}, \ldots, X_{(N)^t}) \in \mathbb{R}^N, \ t \in [0, \infty) \), such that the family of conformal mappings \( g_t : H_{\eta(t)} \to \mathbb{H} \) solves the multiple Loewner equation:

\[
(1.7) \quad \frac{d}{dt} g_t(z) = \sum_{i=1}^{N} \frac{2}{g_t(z) - X_{(i)^t}}, \quad t \geq 0, \quad g_0(z) = z \in \mathbb{H},
\]

i.e., \( \{g_t\}_{t \geq 0} \) is the Loewner chain driven by \( \{X_t : t \geq 0\} \). Moreover, the driving functions are determined by \( X_{(i)^t} = \lim_{t \to 0} g_t(\eta_{(i)}(t) + \epsilon), \ i = 1, \ldots, N \).

![Figure 1.2. Uniformization of the domain \( H_{\eta(t)} \) by the Loewner chain.](image)

By virtue of the above theorem, an \( N \)-tuple of random slits in \( \mathbb{H} \) anchored on \( \mathbb{R} \) is converted to a set of stochastic processes \( \{X_t \in \mathbb{R}^N : t \geq 0\} \). We assume that
it solves the system of stochastic differential equations (SDEs)
\begin{equation}
    dX^{(i)}_t = \sqrt{\kappa} dB^{(i)}_t + F^{(i)}(X_t) dt,
    \quad i = 1, \ldots, N, \quad t \geq 0,
\end{equation}
where \( \{B^{(i)}_t : t \geq 0\}_{i=1}^N \) are mutually independent one-dimensional standard Brownian motions and \( \{F^{(i)}(x)\}_{i=1}^N \) are suitable functions of \( x = (x_1, \ldots, x_N) \) not explicitly dependent on \( t \), i.e., \( \{X_t : t \geq 0\} \) are of the Markovian type (See [IW89, Eq. (2.11) in Chapter IV]). Then the solution \( \{g_t\}_{t \geq 0} \) is just the multiple SLE introduced in [BBK05]. While, in [BBK05], the set of driving processes was derived from a single auxiliary function in relation to conformal field theory (CFT) as is summarized in Appendix B, here we do not assume any CFT origin of the multiple SLE. We write SLE\( \kappa \) if we need to specify the parameter \( \kappa \).

![Cutting operation by multiple SLE](image)

**Figure 1.3.** Cutting operation by multiple SLE.

To solve Problems 1 to 3 we define the cutting operation on \( S^{\text{Rot}}_{\gamma,N+1}(\mathbb{H}) \)-valued random fields associated with the multiple SLE. Let \( (H, X) \) be an \( S^{\text{Rot}}_{\gamma,N+1}(\mathbb{H}) \)-valued random field (see the right picture in Fig. 1.3). Suppose that we have an \( N \)-tuple of interacting stochastic processes \( \{X_t = (X^{(1)}_t, \ldots, X^{(N)}_t) \in \mathbb{R}^N : t \geq 0\} \) with the initial conditions \( X^{(i)}_0 = X_i \), \( i = 1, \ldots, N \), which is conditionally independent of \( (H, X) \). We assume that it determines random slits \( \{\eta^{(i)}\}_{i=1}^N \) through the correspondence in Theorem 1.10. Then these slits are anchored at the initial marked boundary points \( X_i \), i.e., \( \lim_{t \to 0} \eta^{(i)}(t) = X_i \), \( i = 1, \ldots, N \), a.s. The configuration space for \( \{X_t : t \geq 0\} \) is identified with \( \text{Conf}_{\gamma,N}(\mathbb{R}) \times [0, \infty) \). We denote the probability law on the space \( \text{Conf}_{\gamma,N}(\mathbb{R}) \times [0, \infty) \) which governs \( \{X_t : t \geq 0\} \) by \( \mathbb{P}^{\text{SLE}} \), since it also governs the multiple SLE given in the form (1.7). Then at each time \( t \in [0, \infty) \), \( \mathbb{P}^{\text{SLE}} \) induces a probability measure on \( \text{Conf}_{\gamma,N}(\mathbb{R}) \) by
\[
    \mathbb{P}^{\text{SLE}}(dx_1, \ldots, dx_N) = \mathbb{P}^{\text{SLE}}(X^{(1)}_t \in dx_1, \ldots, X^{(N)}_t \in dx_N).
\]

We fix a time \( T \in [0, \infty) \) and restrict the random distribution \( H \) on the domain \( \mathbb{H}^T \subseteq \mathbb{H} \) to obtain a new \( \gamma \)-QS-(\( N + 1 \))-MBPs (see the middle picture in Fig. 1.3)
\begin{equation}
    \left[ \mathbb{H}^T, (\eta^{(1)}(T), \ldots, \eta^{(N)}(T), \infty), H|_{\mathbb{H}^T} \right]_{\gamma}.
\end{equation}

It is manifest from this construction that we have
\[
    \nu_{H|_{\mathbb{H}^T}}^{\gamma}(\eta^{(i)}(0,t)|_L) = \nu_{H|_{\mathbb{H}^T}}^{\gamma}(\eta^{(i)}(0,t)|_R), \quad t \in [0,T], \quad i = 1, \ldots, N, \quad \text{a.s.}
\]
We define an $\mathcal{S}_{N+1}^{\text{Rot}}(H)$-valued random field (see the left picture in Fig. 1.3)

\begin{equation}
\mathfrak{q}(X_{t_0 \leq t \leq T})(H, X) := \left( g_T^{-1}\left| H_{\mathbb{H}^2}, (X_T^{(1)}, \ldots, X_T^{(N)}) \right. \right),
\end{equation}

where we wrote $g_T^{-1}\left| H_{\mathbb{H}^2} = H_{\mathbb{H}^2} \circ g_T^{-1} + Q \log |g_T^{-1}|$. Then, uniformizing the domain $\mathbb{H}^2_T$ to $\mathbb{H}$ by the conformal mapping $g_T$, we can find that the $\gamma\text{-QS-}(N+1)$-MBPs (1.9) coincides with $\pi_{\gamma,N+1}^{\infty}(\mathfrak{q}(X_{t_0 \leq t \leq T})(H, X))$. We call this assignment $\mathfrak{q}(X_{t_0 \leq t \leq T})$ of the $\mathcal{S}_{N+1}^{\text{Rot}}(H)$-valued random field (1.10) to a given $\mathcal{S}_{N+1}^{\text{Rot}}(H)$-valued random field $(H, X)$ the cutting operation associated with the multiple SLE driven by $S\text{LE}$ driven by $\gamma\text{-QS-}$.

Notice that for a quantum surface $\pi_{\gamma,N+1}^{\infty}(\mathfrak{q}(X_{t_0 \leq t \leq T})(H, X))$ obtained by the cutting operation associated with the multiple SLE, the versions of Problems 1 to 3 can also be answered: Since the Loewner theory ensures that the slits $\{\eta(t)\}_{t \geq 0}$ are deterministic if and only if the initial configuration $X$ is deterministic. In this case, the probability law of the resulting slits $\{\eta(t)\}_{t \geq 0}$ shall be completely determined by the multiple SLE, answering Problem 2.

Therefore, if a $\gamma\text{-QS-}(N+1)$-MBPs obtained by the composition of the cutting operation and the surjection $\pi_{\gamma,N+1}^{\infty}$ is of the $(X, \alpha)$-standard type for some $\text{Conf}_{\kappa}(\mathbb{R})$-valued random variable $X$, the conformal welding problem for it is solved in the above arguments. The following proposition gives such examples.

**Proposition 1.11.** Let $(H_{\mathbb{H}}^{X,\alpha}, X)$ be an $\mathcal{S}_{N+1}^{\text{Rot}}(H)$-valued random field introduced in (1.3) with $(\alpha_1, \ldots, \alpha_N) = (\frac{\gamma}{2}, \ldots, \frac{\gamma}{2})$ and assume that $\{X_t = (X_t^{(1)}, \ldots, X_t^{(N)}) : t \geq 0\}$ solves the set of SDEs (1.8) with $\kappa = \gamma^2$ and

\begin{equation}
F^{(i)}(x) = \sum_{j=1}^{N} \frac{4}{x_i - x_j}, \quad i = 1, \ldots, N,
\end{equation}

starting at $X$. Then, for an arbitrary $T \in [0, \infty)$, we have

\begin{equation}
\mathfrak{q}(X_{t_0 \leq t \leq T})(H_{\mathbb{H}}^{X,\alpha}, X) \overset{\text{(law)}}{=} \left( H_{\mathbb{H}}^{X_T,\alpha}, X_T \right)
\end{equation}

as $\mathcal{S}_{N+1}^{\text{Rot}}(H)$-valued random fields. In particular, $\pi_{\gamma,N+1}^{\infty}(\mathfrak{q}(X_{t_0 \leq t \leq T})(H, X))$ is a $\gamma\text{-QS-}(N+1)$-MBPs of the $(X_T, \alpha)$-standard type.

For the choice of the functions $\{F^{(i)}(x)\}_{i=1}^{N}$ as (1.11), the corresponding set of driving processes $\{X_t : t \geq 0\}$ solves a time change of the Dyson model [Dys62]. In the usual convention, the Dyson model of parameter $\beta$ is the system of SDEs
on \( \{ X_t^{D_0} = (X_t^{D_0(1)}, \ldots, X_t^{D_0(N)}) \in \mathbb{R}^N : t \geq 0 \} \) \cite{Dys62,Kat13}:

\[
(1.13) \quad dX_t^{D_0(i)} = dB_t^{(i)} + \frac{\beta}{2} \sum_{j=1}^{N} \frac{1}{X_t^{D_0(i)} - X_t^{D_0(j)}} dt, \quad t \geq 0, \quad i = 1, \ldots, N.
\]

Then the set of driving processes \( \{ X_t : t \geq 0 \} \) above are identified with \( \{ X_t^{D_0/\kappa} : t \geq 0 \} \). It is known that when \( \beta \geq 1 \), the Dyson model with an arbitrary finite number of particles \( N \in \mathbb{Z}_{\geq 2} \) has a strong and pathwise unique non-colliding solution for general initial conditions \cite{RS93,CL97,GM13,GM15}. The non-colliding condition \( \beta \geq 1 \) for the Dyson model corresponds to \( \kappa \in (0, 8] \) through the relation \( \beta = 8/\kappa \). See Section 1.1 for more detail on the relations among parameters.

As a consequence of Proposition 1.11, we obtain a solution to Problems 1 to 3 for \( \gamma \)-QSs-(\( N + 1 \))-MBPs due to the above arguments.

**Theorem 1.12.** Let \( \gamma \in (0, 2] \) and \( N \in \mathbb{Z}_{\geq 1} \). Suppose that \( \{ X_t = (X_t^{(1)}, \ldots, X_t^{(N)}) : t \geq 0 \} \) is the time change of the Dyson model \( \{ X_t^{D_0/\kappa} : t \geq 0 \} \) starting at a deterministic initial state \( X_0 = x \in \text{Conf}^e_N(\mathbb{R}) \). Then, at each time \( T \in (0, \infty) \), the conformal welding problem for \( \left[ \mathbb{H}, H_{\mathbb{H}}^{X_{T}, \alpha}, (X_T, \infty) \right] \) with \( (\alpha_1, \ldots, \alpha_N) = \left( \frac{2}{\gamma}, \ldots, \frac{2}{\gamma} \right) \) is solved as follows:

1. The solution of the Loewner equation (1.7) driven by the time change of the Dyson model \( \{ X_t^{D_0/\kappa} : 0 \leq t \leq T \} \) gives a solution to Problem 1. Namely, \( g_T^{-1} : \mathbb{H} \to \mathbb{H}^T_T \) is the desired conformal equivalence.
2. The probability law for resulting slits \( \{ \eta^{(i)} \}_{i=1}^N \) is the one for the multiple \( \text{SLE}_8 \). This gives a solution to Problem 2.
3. Problem 3 is answered positively with \( \eta_0 = x \) a.s.

### 1.4. The multiple flow line problem.

Another topic in random geometry that stems from the GFF is the imaginary geometry \cite{MS16a,MS16b,MS16c,MS17}, which sees the flow line of the vector field \( e^{\sqrt{-1}H/\chi} \), where \( H \) is a \( C_0^\infty(\mathbb{D}) \)'-valued random field and \( \chi \) is a real parameter. Let us temporarily suppose that \( h \) was a smooth function on \( D \subset \mathbb{C} \). Then \( e^{\sqrt{-1}h/\chi} \) is a smooth vector field and its flow line \( \eta : [0, \infty) \to D \) starting at \( x_0 \in D \) is defined as the solution of the ordinary differential equation

\[
\frac{d\eta(t)}{dt} = e^{\sqrt{-1}h(\eta(t))/\chi}, \quad t \geq 0, \quad \eta(0) = x_0 \in D.
\]

For another simply connected domain \( \tilde{D} \) and a conformal equivalence \( \psi : \tilde{D} \to D \), we can consider the pull-back \( \tilde{\eta} = \psi^{-1} \circ \eta \) of the flow line \( \eta \) by \( \psi \). Then \( \tilde{\eta} \) satisfies the following differential equation:

\[
\frac{d\tilde{\eta}(t)}{dt} = \frac{1}{|\psi'(\tilde{\eta}(t))|} e^{\sqrt{-1}(h_0 \psi - \chi \arg \psi')(\tilde{\eta}(t))/\chi}, \quad t \geq 0.
\]
When we adopt a time change \( \tilde{\eta}(f(t)) = \tilde{\eta}(f(t)) \) with \( f(t) = \int_0^t |\psi'(\tilde{\eta}(s))|ds \), we see that
\[
\frac{d\tilde{\eta}(f(t))}{dt} = e^{\sqrt{\frac{-1}{h}}/h / (\psi - \chi \arg \psi')}(\tilde{\eta}(f(t))), \quad t \geq 0.
\]
Since a time reparametrization does not change the whole curve, we can say that the domains \( D \) with a smooth function \( h \) and \( \tilde{D} \) with \( h \circ \psi - \chi \arg \psi' \) are equivalent as long as flow lines of vector fields \( e^{\sqrt{\frac{-1}{h}}/h / (\psi - \chi \arg \psi')}/h \) are concerned.

Interestingly, this equivalence relation also makes sense even when we work with a \( C_\infty^\alpha(D)' \)-valued random field \( H \) instead of a smooth function [MS16a, MS16b, MS16c, MS17].

Definition 1.13. A \( \chi \)-imaginary surface (\( \chi \)-IS) is an equivalence class of pairs \((D, H)\) of simply connected domains \( \tilde{D} \subset \mathbb{C} \) and \( C_\infty^\alpha(D)' \)-valued random fields \( H \) under the equivalence relation
\[
(D, H) \sim (\tilde{D}, \tilde{H}) := (\psi^{-1}(D), H \circ \psi - \chi \arg \psi'),
\]
where \( \tilde{D} \subset \mathbb{C} \) is another simply connected domain and \( \psi : \tilde{D} \to D \) is a conformal equivalence.

It has been shown in [MS16a, She16] that for a \( \chi \)-IS (\( \mathbb{H} \), \( H_{\mathbb{H}}^{\text{Dir}} - \frac{2}{\sqrt{\kappa}} \arg(\cdot) \)) with \( \chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2} \), the flow line starting at the origin can be identified with the curve for the SLE\( _{\kappa} \). Note that the arg function is a harmonic function on \( \mathbb{H} \) with the boundary value changes at the origin and infinity by \( \pi \). In this sense, above imaginary surface is said to have boundary condition changing points (BCCPs) at the origin and infinity.

We settle the flow line problem for a \( \chi \)-IS with more BCCPs than two as follows: Let \( x = (x_1, \ldots, x_N) \in \text{Conf}^{<}_{N}(\mathbb{R}) \) and \( \beta_1, \ldots, \beta_N \) be real numbers. The \( C_\infty^\alpha(\mathbb{H})' \)-valued random field
\[
H_{\mathbb{H}}^{x, \beta, \mathcal{I}} = H_{\mathbb{H}}^{\text{Dir}} - \sum_{i=1}^{N} \beta_i \arg(\cdot - x_i)
\]
on \( \mathbb{H} \) defines a \( \chi \)-IS with BCCPs \((x_1, \ldots, x_N, \infty)\), whose boundary value has discontinuity at \( x_i \) by \( \pi \beta_i \), \( i = 1, \ldots, N \) and at \( \infty \) by \( -\pi \sum_{i=1}^{N} \beta_i \). Here \( \mathcal{I} \) stands for imaginary.

Problem 4. Determine the probability law for multiple flow lines of a \( \chi \)-IS with \( N \) BCCPs (\( \chi \)-IS-(\( N \))-BCCPs) \( H_{\mathbb{H}}^{x, \beta, \mathcal{I}} \) starting at boundary points \( x \in \text{Conf}^{<}_{N}(\mathbb{R}) \).

The flow line problem is addressed in Sect. 3 in the case when \( (\beta_1, \ldots, \beta_N) = \left( \frac{2}{\sqrt{\kappa}}, \ldots, \frac{2}{\sqrt{\kappa}} \right) \) with \( 0 < \kappa \leq 4 \). An answer to Problem 4 is given as Theorem 3.4.

The present paper is organized as follows: In Sect. 2 we prove Proposition 1.11 owing to the coupling of the reverse flow of the multiple SLE and the free boundary GFF. In Sect. 3 we address the multiple flow line problem in an analogous
manner for the conformal welding problem and solve it using the forward flow of the multiple SLE. In the final Sect. 4, we discuss related topics and future direction. In Appendix A, we give a construction of the spaces $S_\gamma$ and $S_{\gamma,N+1}$ as orbifolds and investigate their structures in detail. In Appendix B, we summarize the approach in [BBK05] that formulated the multiple SLE in relation to CFT and the analogous way of defining the reverse flow of the multiple SLE.

Acknowledgements. The authors would like to thank Takuya Murayama, Ikkei Hotta and Daiya Yamauchi for useful discussion. The present work was partially supported by the Research Institute for Mathematical Sciences (RIMS), a Joint Usage/Research Center located in Kyoto University. The authors thank Naotaka Kajino, Takashi Kumagai, and Daisuke Shiraishi for organizing the very fruitful workshop, “RIMS Research Project: Gaussian Free Fields and Related Topics”, held in 18-21 September 2018 at RIMS. MK was supported by the Grant-in-Aid for Scientific Research (C) (No.26400405), (B) (No.18H01124), and (S) (No.16H06338) of Japan Society for the Promotion of Science (JSPS). SK was supported by the Grant-in-Aid for JSPS Fellows (No. 17J09658).

2. Proof of Proposition 1.11

In this section, we prove Proposition 1.11. It suffices to prove that $H_{X,T,\alpha}$ and $g_{T}^{-1}H_{|\mathbb{H}_{\eta}^{\alpha}}$ obey the same probability law. The central idea is to interpolate these two $C^\infty(\mathbb{H})$-valued random fields by a single stochastic process and show its stationarity. The cutting operation $A(X_t:0\leq t\leq T)$ indeed defines a candidate of such an interpolating stochastic process, but it turns out that the reverse flow behaves much better. Before proceeding, let us introduce the reverse flow of the multiple Loewner chain needed in our proof.

2.1. Reverse flow of the multiple Loewner chain. Let $\{X_t = (X^{(1)}_t, \ldots, X^{(N)}_t) : t \geq 0\}$ be a set of continuous functions of $t$ that drives the Loewner equation (1.7). We assume that given $\{X_t : t \geq 0\}$, the Loewner equation (1.7) has a unique solution and determines an $N$-tuple of slits $\{\eta^{(i)}\}_{i=1}^N$. We fix a time $T \in (0, \infty)$ and set $Y^{(i)}_{T,t} := X^{(i)}_{T-t}$, $t \in [0,T]$, $i = 1, \ldots, N$. The reverse flow of the Loewner chain is defined as the solution of

\[ \frac{d}{dt} f^T_t(z) = - \sum_{i=1}^N \frac{2}{f^T_t(z) - Y^{(i)}_{T,t}}, \quad t \in [0,T], \quad f^T_0(z) = z \in \mathbb{H}. \]

Lemma 2.1. The identity $f^T_t = g^{-1}_{T,t}$ holds, where $g^{-1}_T : \mathbb{H} \to \mathbb{H}^\eta_T$ is the inverse map of the uniformizing map $g_T : \mathbb{H}^\eta_T \to \mathbb{H}$ that solves the Loewner equation (1.7).

Proof. Set $u^T_t := f^T_{T-t}(z)$, $z \in \mathbb{H}$, $t \in [0,T]$. Then it satisfies

\[ \frac{d}{dt} u^T_t(z) = \sum_{i=1}^N \frac{2}{u^T_t(z) - X^{(i)}_{T-t}}, \quad t \in [0,T], \quad u^T_0(z) = f^T_T(z), \quad z \in \mathbb{H}. \]
Since we have assumed that the multiple Loewner equation (1.7) has a unique solution, this implies that \( u_t^T(z) = g_t(f_t^T(z)) \). Indeed, both sides satisfy the same differential equation with the same initial condition. In particular, at time \( t = T \),
\[
u_T^T(z) = f_0^T(z) = z = g_T(f_T^T(z)), \quad z \in \mathbb{H}
\]
implies that \( f_T^T = g_T^{-1} \). \( \square \)

The reverse flow of the multiple SLE can also be formulated in connection to CFT as described in Appendix \([3]\).

2.2. Interpolation of random fields. We assume that the set of driving processes \( \{X_t = (X_t^{(1)}, \ldots, X_t^{(N)}) : t \geq 0\} \) is given by the system of SDEs (1.8).

For a fixed \( T \in (0, \infty) \), we set the time reversed process \( Y^T_{t,i} = X^T_{T-t}, \; t \in [0, T], \; i = 1, \ldots, N \), and let \( \{f^T_t(\cdot) : t \in [0, T]\} \) be the reverse flow defined in Eq. (2.2) driven by \( \{Y_{T,t}; (Y_{T,t}^{(1)}, \ldots, Y_{T,t}^{(N)}): t \in [0, T]\} \). From Lemma 2.1, we can conclude that \( g_T^{-1}H_{\mathbb{H}^T} = f^T_tH|_{\mathbb{H}^T} \) a.s.

Let us define a stochastic process
\[
n_t(z) = u_{\mathbb{H}^T}(2/\gamma, \ldots, 2/\gamma) = 2 \sum_{i=1}^{N} \gamma \log |z - Y^T_{t,i}|, \quad t \in [0, T], \quad z \in \mathbb{H}.
\]

We also consider \( H_{\mathbb{H}^T} \) which is independent of \( \{B^T_t: t \geq 0\} \). Then we see that, at each time \( t \in [0, T], \; \mathbb{H}_t = H_{\mathbb{H}^T} + n_t, (Y_{T,t}; \infty) \) is a \( \gamma \)-QS-(\( N + 1 \))-MBPs of the \( (Y_{T,t}, \alpha) \)-standard type with \( (\alpha_1, \ldots, \alpha_N) = (\frac{2}{\gamma}, \ldots, \frac{2}{\gamma}) \). We set
\[
h_t(z) := n_t(f^T_t(z)) + Q \log |f^T_t(z)|, \quad z \in \mathbb{H}, \quad t \in [0, T]
\]
with \( Q = \frac{2}{\gamma} + \frac{3}{2} \) and set
\[
p_t = h_t + H_{\mathbb{H}^T} \circ f^T_t, \quad t \in [0, T].
\]

Then we can see that the stochastic process \( p_t, t \in [0, T] \) interpolates two \( C^\infty(\mathbb{H}) \)-valued random fields so that \( p_0 = H_{\mathbb{H}^T} X_{t, \alpha} \) and \( p_T = f^T_tH|_{\mathbb{H}^T} \).

2.3. Stationarity of the stochastic process. We claim that \( p_0 \) and \( p_T \) obey the same probability law. The proof relies on the following key lemmas:

**Lemma 2.2.** The stochastic process \( h_t(z), z \in \mathbb{H}, t \in [0, T] \) is a local martingale with increment
\[
dh_t(z) = 2 \sum_{i=1}^{N} \text{Re} \frac{-2}{f^T_t(z) - Y^T_{t,i}} dB^T_t, \quad z \in \mathbb{H}, \quad t \in [0, T],
\]
if \( \kappa = \gamma^2 \) and the functions \( \{F^{(i)}(x)\}_{i=1}^N \) are chosen as (1.7).

**Proof.** Note that \( h_t(z) \) is the real part of
\[
h^*_t(z) = \frac{2}{\gamma} \sum_{i=1}^{N} \log(f^T_t(z) - Y^T_{t,i}) + Q \log f^T_t(z), \quad z \in \mathbb{H}, \quad t \in [0, T].
\]
We will show that \( h_1^*(z), z \in \mathbb{H}, t \in [0, T] \) is a local martingale if \( \kappa = \gamma^2 \) and the functions \( \{F^{(i)}(x)\}_{i=1}^N \) are chosen as \([1.11]\). Owing to the time reversibility of the Brownian motions, the set of time reversed driving processes \( \{Y_{T,t} : 0 \leq t \leq T\} \) solves the following system of SDEs:

\[
(2.3) \quad dY_{T,t}^{(i)} = \sqrt{\kappa} dB_t^{(i)} - F^{(i)}(Y_{T,t}) dt, \quad t \in [0, T], \quad i = 1, \ldots, N.
\]

By Eqs. (2.1) and (2.3), Itô’s formula gives

\[
d \log(f_t^{(i)}(z) - Y_{T,t}^{(i)}) = \frac{1}{f_t^{(i)}(z) - Y_{T,t}^{(i)}} \left( \sum_{j=1}^N \frac{-2 dt}{f_t^{(j)}(z) - Y_{T,t}^{(j)}} - \sqrt{\kappa} dB_t^{(i)} + F^{(i)}(Y_{T,t}) dt \right)
\]

\[
- \frac{1}{2} \left( f_t^{(i)}(z) - Y_{T,t}^{(i)} \right)^2 dt.
\]

They are assembled to give the increment of \( h_1^*(z), z \in \mathbb{H}, \)

\[
dh_t^*(z) = \frac{-4}{\sqrt{\kappa}} \left( \sum_{i=1}^N \frac{1}{f_t^{(i)}(z) - Y_{T,t}^{(i)}} \right)^2 dt - \sum_{i=1}^N \frac{2 dB_t^{(i)}}{f_t^{(i)}(z) - Y_{T,t}^{(i)}} + \frac{2}{\sqrt{\kappa}} \sum_{i=1}^N F^{(i)}(Y_{T,t}) dt
\]

\[
- \sum_{i=1}^N \left( f_t^{(i)}(z) - Y_{T,t}^{(i)} \right)^2 dt + 2Q \sum_{i=1}^N \frac{dt}{f_t^{(i)}(z) - Y_{T,t}^{(i)}}, \quad t \in [0, T],
\]

where we have used the relation \( \kappa = \gamma^2 \). It can be verified that

\[
\sum_{i,j=1}^N \frac{1}{f_t^{(i)}(z) - Y_{T,t}^{(i)}} \frac{1}{f_t^{(j)}(z) - Y_{T,t}^{(j)}} = \frac{1}{2} \sum_{i,j=1}^N \frac{1}{(f_t^{(i)}(z) - Y_{T,t}^{(i)})(f_t^{(j)}(z) - Y_{T,t}^{(j)})}.
\]

Using this, we see that the increment of \( h_1^*(z), z \in \mathbb{H} \) becomes

\[
dh_t^*(z) = \frac{2}{\sqrt{\kappa}} \sum_{i=1}^N \frac{1}{f_t^{(i)}(z) - Y_{T,t}^{(i)}} \left( F^{(i)}(Y_{T,t}) - \sum_{j=1}^N \frac{4}{Y_{T,t}^{(i)} - Y_{T,t}^{(j)}} \right) dt
\]

\[
- \sum_{i=1}^N \frac{2 dB_t^{(i)}}{f_t^{(i)}(z) - Y_{T,t}^{(i)}}, \quad t \in [0, T]
\]

and conclude that the stochastic process \( h_1^*(z) \) is a local martingale if the functions \( \{F^{(i)}(x)\}_{i=1}^N \) are chosen as \([1.11]\). Moreover, under such a choice of the functions \( \{F^{(i)}(x)\}_{i=1}^N \), Eq. (2.2) is obtained.

Thus, at each \( z \in \mathbb{H} \), the stochastic process \( \{h_t(z) : t \in [0, T]\} \) can be regarded as a Brownian motion after an appropriate time change. In the following, we
assume that the functions \( \{F^{(i)}(x)\}_{i=1}^{N} \) are as \( \text{(1.11)} \). By Eq. \( (2.2) \), the cross variation between \( h_t(z) \) and \( h_t(w) \), \( z, w \in \mathbb{H} \) is given by

\[
d \langle h_t(z), h_t(w) \rangle = \sum_{i=1}^{N} \left( \text{Re} \frac{2}{f_t^i(z) - Y_{T,t}^{i}(z)} \right) \left( \text{Re} \frac{2}{f_t^i(w) - Y_{T,t}^{i}(w)} \right) dt, \quad z, w \in \mathbb{H}.
\]

**Lemma 2.3.** Define \( G_{\mathbb{H}^{T}}^{F_t}(z, w) := G_{\mathbb{H}}^{F_t}(f_t^T(z), f_t^T(w)), t \in [0, T], z, w \in \mathbb{H} \). Then

\[
d \langle h_t(z), h_t(w) \rangle = -dG_{\mathbb{H}^{T}}^{F_t}(z, w), \quad t \in [0, T], \quad z, w \in \mathbb{H}.
\]

**Proof.** This can be verified by direct computation. By definition, we have

\[
G_{\mathbb{H}^{T}}^{F_t}(z, w) = -\log |f_t^T(z) - f_t^T(w)| - \log |f_t^T(z) - \overline{f_t^T(w)}|.
\]

Thus its increment is computed as

\[
dG_{\mathbb{H}^{T}}^{F_t}(z, w) = -2 \text{Re} \frac{df_t^T(z) - df_t^T(w)}{f_t^T(z) - f_t^T(w)} - 2 \text{Re} \frac{df_t^T(z) - df_t^T(w)}{f_t^T(z) - f_t^T(w)}
\]

\[
= \sum_{i=1}^{N} \text{Re} \frac{-2d}{(f_t^i(z) - Y_{T,t}^{i}(z))(f_t^i(w) - Y_{T,t}^{i}(w))}
\]

\[
+ \sum_{i=1}^{N} \text{Re} \frac{-2d}{(f_t^i(z) - Y_{T,t}^{i}(z))(f_t^i(w) - Y_{T,t}^{i}(w))}
\]

\[
= -\sum_{i=1}^{N} \left( \text{Re} \frac{2}{f_t^i(z) - Y_{T,t}^{i}(z)} \right) \left( \text{Re} \frac{2}{f_t^i(w) - Y_{T,t}^{i}(w)} \right) dt
\]

which is the same as \(-d \langle h_t(z), h_t(w) \rangle, t \in [0, T], z, w \in \mathbb{H} \). \( \square \)

For a test function \( \rho \in C^\infty(\mathbb{H}) \) of zero-mass \( \int_{\mathbb{H}} \rho(z)d\mu(z) = 0 \), we have

\[
d \langle (h_t, \rho), (h_t, \rho) \rangle = -dE_t^{F_t}(\rho),
\]

where

\[
E_t^{F_t}(\rho) = \int_{\mathbb{H} \times \mathbb{H}} \rho(z)G_{\mathbb{H}^{T}}^{F_t}(z, w)\rho(w)d(\mu \otimes \mu)(z, w)
\]

is non-increasing in the time variable \( t \in [0, T] \). This implies that \( (h_t, \rho), t \in [0, T] \), is a Brownian motion such that we can regard \(-E_t^{F_t}(\rho)\) as time variable. Thus \( (h_T, \rho) \) is normally distributed with mean \( (h_0, \rho) \) and variance \(-E_T^{F_t}(\rho)\). The random variable \( (H_T^{F_t} \circ f_T^T, \rho) \) is also normally distributed with mean zero and variance \( E_T^{F_t}(\rho) \) by the conformal invariance of the GFF. Since the random variable \( (H_T^{F_t} \circ f_T^T, \rho) \) is conditionally independent of \( (h_T, \rho) \), their sum \( (p_T, \rho) \) is a normal random variable with mean \( (h_0, \rho) \) and variance \( E_T^{F_t}(\rho) \) coinciding with \((h_0 + H_T^{F_t}, \rho) = (p_0, \rho)\) in probability law. This implies \( p_T \stackrel{\text{law}}{=} p_0 \) as \( C^\infty(\mathbb{H}) \)-valued random fields. The proof of Proposition \( \text{(1.11)} \) is complete.
3. Solution to the flow line problem

This section is devoted to presenting and proving another main result in the present paper, which we did not explain in any detail in Introduction. This concerns the flow line problem for a $\chi$-IS-(N + 1)-BCCPs. Again, we let $\{X_t = (X_t^{(1)}, \ldots, X_t^{(N)}) : t \geq 0\}$ be a set of driving processes satisfying \(^{(1.8)}\) associated with parameter $\kappa > 0$ and functions $\{F^{(i)}(x)\}_{i=1}^{N}$ with initial conditions $(X_0^{(1)}, \ldots, X_0^{(N)}) = (x_1, \ldots, x_N) \in \text{Conf}_{\kappa}(\mathbb{R})$. We also assume that the Loewner equation \(^{(1.7)}\) driven by $\{X_t; t \geq 0\}$ has the unique solution $g_t$, $t \geq 0$ and determines $N$ non-noncolliding and non-selfintersecting slits $\{\eta^{(i)}\}_{i=1}^{N}$ in $\mathbb{H}$ anchored on $\mathbb{R}$.

3.1. Key statement. At each $z \in \mathbb{H}$, consider a stochastic process

$$n^2_t(z) = \frac{-2}{\sqrt{\kappa}} \sum_{i=1}^{N} \arg(z - X_t^{(i)}), \quad t \geq 0.$$  

At each time $t \in [0, \infty)$, the random field $n^2_t$ is harmonic on $\mathbb{H}$, in which its boundary value changes at $N$ points $X_t^{(i)} \in \mathbb{R}$, $i = 1, \ldots, N$. Let $H_{\text{Dir}}$ be independent of $\{B_t^{(i)} : t \geq 0\}_{i=1}^{N}$. Put $H_{\text{Dir}}^{X_t, \beta} = n^2_t + H_{\text{Dir}}^{\beta}$, where $(\beta_1, \ldots, \beta_N) = (\frac{2}{\sqrt{\kappa}}, \ldots, \frac{2}{\sqrt{\kappa}})$ are fixed here and in the sequel. We also define

$$h_t^\kappa(z) := n^2_t(g_t(z)) - \chi \arg g_t'(z)$$

with $\chi \in \mathbb{R}$ and

$$p_t^\kappa := H_{\text{Dir}}^{X_t, \beta} \circ g_t - \chi \arg g_t' = h_t^\kappa + H_{\text{Dir}}^{\beta} \circ g_t.$$  

Note that $(\mathbb{H}, H_{\text{Dir}}^{X_t, \beta}) \sim (\mathbb{H}, H_{\text{Dir}}^{\beta})$. Due to the initial condition $g_0(z) = z \in \mathbb{H}$, we can see that $p_0^\kappa = H_{\text{Dir}}^{\beta}$, where $x = (x_1, \ldots, x_N) \in \text{Conf}_{\kappa}(\mathbb{R})$ are the initial values of the driving processes.

Proposition 3.1. Let $\kappa \in (0, 4]$, $N \in \mathbb{Z}_{\geq 1}$ and $\chi \in \mathbb{R}$. Suppose that $\{X_t = (X_t^{(1)}, \ldots, X_t^{(N)}) : t \geq 0\}$ is the solution of the system of SDEs \(^{(1.8)}\) starting at $x = (x_1, \ldots, x_N) \in \text{Conf}_{\kappa}(\mathbb{R})$. If $\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$ and the functions $\{F^{(i)}(x)\}_{i=1}^{N}$ are given by \(^{(1.7)}\), then at each time $T \in [0, \infty)$, two $C_0^\infty(\mathbb{H})$-valued random fields $p_0^\kappa$ and $p_T^\kappa$ obey the same probability law.

3.2. Proof of Proposition 3.1. The proof is very similar to that of Proposition \(^{(1.14)}\). Thus we omit the computational details. The following lemmas play key roles:

Lemma 3.2. At each $z \in \mathbb{H}$, the stochastic process $h_t^\kappa(z)$, $t \geq 0$ is a local martingale with increment

$$dh_t^\kappa(z) = \sum_{i=1}^{N} \text{Im} \frac{2}{g_t(z) - X_t^{(i)}} dB_t^{(i)}, \quad t \geq 0,$$

if the functions $\{F^{(i)}(x)\}_{i=1}^{N}$ are chosen as \(^{(1.7)}\).
Thus, at each \( t \in [0, \infty) \), the cross variation between \( h_t^3(z) \) and \( h_t^3(w) \) is given by

\[
d \langle h_t^3(z), h_t^3(w) \rangle = \sum_{i=1}^{N} \left( \text{Im} \frac{2}{g_t(z) - X_t^{(i)}} \right) \left( \text{Im} \frac{2}{g_t(w) - X_t^{(i)}} \right) dt, \quad z, w \in \mathbb{H},
\]

which turns out to be expressed using the Green function.

**Lemma 3.3.** Let \( G_{\mathbb{H}_t^q}^{\text{Dir}}(z, w) := G_{\mathbb{H}_t^q}^{\text{Dir}}(g_t(z), g_t(w)), \quad t \geq 0, \quad z, w \in \mathbb{H}_t^q \). Then we have

\[
d \langle h_t^3(z), h_t^3(w) \rangle = -dG_{\mathbb{H}_t^q}^{\text{Dir}}(z, w), \quad t \geq 0, \quad z, w \in \mathbb{H}_t^q.
\]

For \( \rho \in C_0^\infty(\mathbb{H}) \), the quadratic variation of \( (h_t^3, \rho) \) becomes

\[
d \langle (h_t^3, \rho), (h_t^3, \rho) \rangle = -dE_t^{\text{Dir}}(\rho),
\]

where

\[
E_t^{\text{Dir}}(\rho) = \int_{\mathbb{H}_t^q \times \mathbb{H}_t^q} \rho(z) G_{\mathbb{H}_t^q}(z, w) \rho(w) d(\mu \otimes \mu)(z, w)
\]

is the Dirichlet energy of \( \rho \) in the domain \( \mathbb{H}_t^q \), \( t \geq 0 \). Here \( \rho \) is supposed to be a function on \( \mathbb{H}_t^q \) by restriction. Since the process \( \mathbb{H}_t^q \), \( t \geq 0 \) is non-increasing, the minus of the Dirichlet energy \( -E_t^{\text{Dir}}(\rho) \) is non-decreasing. Thus regarding \( -E_t^{\text{Dir}}(\rho) \) as new time, the stochastic process \( (h_t^3, \rho) \) is a Brownian motion. This implies that at any fixed time \( T \in [0, \infty) \), the random variable \( (h_T^3, \rho) \) is normal with mean \( (h_0^3, \rho) \) and variance \( -E_T^{\text{Dir}}(\rho) - (-E_0^{\text{Dir}}(\rho)) \). The random variable \( (H_{\mathbb{H}_t}^{\text{Dir}} \circ g_T, \rho) \) is also a mean-zero normal variable with variance \( E_T^{\text{Dir}}(\rho) \). Since \( (h_T^3, \rho) \) and \( (H_{\mathbb{H}_t}^{\text{Dir}} \circ g_T, \rho) \) are conditionally independent, their sum \( (p_T^3, \rho) \) is normally distributed with mean \( (h_0^3, \rho) \) and variance \( E_0^{\text{Dir}}(\rho) \), thus it coincides with \( (p_0^3, \rho) \) in probability law.

### 3.3 Arguments

We here present a geometric interpretation of Proposition 3.1. That is, Proposition 3.1 provides two distinct samplings of \( C_0^\infty(\mathbb{H})' \)-valued random fields which obey the same probability law. The first one is to directly sample the random field \( p_0^3 = H_{\mathbb{H}_t}^{\text{Dir},\beta,3} \) (see the upper-left picture in Fig. 3.1), and the other one is to sample multiple SLE paths \( \{\eta^{(i)}\}_{i=1}^N \) up to time \( T \in (0, \infty) \), sample the random field \( p_T^3 = h_T^3 + H_{\mathbb{H}_t}^{\text{Dir}} \circ g_T \) on the domain \( \mathbb{H}_T^q \), and then extend it to \( \mathbb{H} \) (see the upper-right picture of Fig. 3.1). Coincidence in probability law between these two samplings roughly means that there is a one-to-one correspondence among instances of two samplings with the same weights. In particular, with each instance \( h \) of \( p_0^3 \), one can associate an \( N \)-tuple of multiple SLE paths \( \{\eta^{(i)}\}_{i=1}^N \). We describe this correspondence more concretely in the sequel.

Notice that \( n_t^3, t \geq 0 \), is the unique harmonic function with boundary conditions

\[
n_t^3(x) = -\frac{2\pi}{\sqrt{\kappa}}(N - i), \quad \text{if} \; x \in (X_t^{(i)}, X_t^{(i+1)}), \quad i = 0, 1, \ldots, N.
\]


Figure 3.1. Geometric interpretation of Proposition 3.1: The boundary values on the real axis are defined by $\lambda_i = -\frac{2\pi (N - i - 1)}{\sqrt{\kappa}}, i = 1, \ldots, N + 1$.

Here we follow the convention that $X^0_t = -\infty$ and $X^{N+1}_t = +\infty$. In particular, it has discontinuity at $X^{(i)}_t$ by $2\pi/\sqrt{\kappa}$ along $\mathbb{R}$, $i = 1, \ldots, N$. Note that $H_{X^{(i)}_t}$, $t \geq 0$ is regarded as the GFF with the same boundary condition as $n^2_t$, $t \geq 0$.

Let us investigate the behavior of $p^3_t$, $T \in [0, \infty)$ near the boundary of $H_{X^{(i)}_t}$. Near a point $x \in (x_i, x_{i+1})$ on the real axis, we have $\lim_{z \to x^+} \arg g^3_T(z) = 0$. Thus $p^3_t$, $T \in [0, \infty)$ has the same boundary value as $p^3_0$ on the real axis. We next suppose that $x$ lies on the $i$-th strand of the $N$-tuple of SLE paths, i.e., $x \in \eta^{(i)}(0, T)$. Although the strand $\eta^{(i)}(0, T)$ is not a smooth curve, we temporarily let $\theta(x)$ be the angle of the tangent line of $\eta^{(i)}(0, T)$ at $x$ (see Fig. 3.2). Then we have $\lim_{z \to x^+} \arg g^3_T(z) = \pi - \theta(x)$ where $z$ approaches $x$ from the right, while $\lim_{z \to x^-} \arg g^3_T(z) = -\theta(x)$ where $z$ approaches $x$ from the left. Thus

$$\lim_{z \to x^+} p^3_T(z) = -\frac{2\pi}{\sqrt{\kappa}}(N - i) + \chi\theta(x) - \chi\pi,$$

$$\lim_{z \to x^-} p^3_T(z) = -\frac{2\pi}{\sqrt{\kappa}}(N - i + 1) + \chi\theta(x)$$
Figure 3.2. The angle $\theta(x)$

(see the upper- and lower-right pictures of Fig. 3.1). Although these two values themselves are not well-defined because $\theta(x)$ is not, their difference does not depend on $\theta(x)$ yielding

$$\lim_{z \to x^+} p_T^3(z) - \lim_{z \to x^-} p_T^3(z) = \frac{\sqrt{\kappa \pi}}{2},$$

where we have used the relation $\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\pi}}{2}$. Thus we will see that $p_T^3$ has discontinuity by $\sqrt{\kappa \pi}/2$ from the left side to the right side of a strand. Conversely, from Proposition 3.1 one can find in an instance $h$ of $p_0^3 = H_{\chi}^{\kappa,\beta,3}$ strands evolving from $x_i$, $i = 1, \ldots, N$ so that the value of $h$ has discontinuity by $\sqrt{\kappa \pi}/2$ from the left side to the right side of a strand. Following the argument in [SS13, MS16a, Shel16], these strands can be regarded as the flow lines of the vector field $e^{\sqrt{-1}H_{\chi}^{\kappa,\beta,3}/\chi}$ starting from $x_i$, $i = 1, \ldots, N$. Moreover, the law of these strands agrees with the one of the slits $\{\eta^{(i)}\}_{i=1}^N$ determined by the multiple SLE$_{\kappa}$. The above argument will prove the following:

**Theorem 3.4.** Let $\kappa \in (0,4]$ and $N \in \mathbb{Z}_{\geq 1}$. The flow line problem for the $\chi$-IS-$(N+1)$-BCCPs $(\mathbb{H}, H_{\chi}^{\kappa,\beta,3})$ with $\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\pi}}{2}$ is solved for any boundary points $x = (x_1, \ldots, x_N) \in \text{Conf}_N^c(\mathbb{R})$ if $\beta = (\beta_1, \ldots, \beta_N)$ is given by $\beta_i = \frac{2}{\sqrt{\kappa}}$, $i = 1, \ldots, N$. The probability law of the flow lines is given by the multiple SLE$_{\kappa}$ driven by the time change of the Dyson model (1.8) associated with the functions (1.11).

4. CONCLUDING REMARKS

As conclusion, we make some discussions and remarks on related topics and future direction.

4.1. Other driving processes. The conformal welding problem introduced in Section 1.2 requires precise description of the statistical behavior of slits $\{\eta^{(i)}\}_{i=1}^N$
in \( \mathbb{H} \), which are non-colliding and non-self-intersecting. Our strategy to solve this problem is to identify \( \{ \eta^{(i)} \}_{i=1}^{N} \) in \( \mathbb{H} \) with “\( N \)-tuple of SLE curves.” Based on Theorem 1.10 from [RS17], we have reduced the problem for \( \{ \eta^{(i)} \}_{i=1}^{N} \) in \( \mathbb{H} \) to the problem to find a stochastic process \( \mathbf{X}_t = (X_t^{(1)}, \ldots, X_t^{(N)}) \in \mathbb{R}^N, t \geq 0, \) of \( N \) particles on \( \mathbb{R} \). Then we have assumed that \( \mathbf{X}_t, t \geq 0 \) satisfies a system of SDEs in a general form (1.8). There, the drift terms \( \int_0^t F^{(i)}(X_s)ds, i = 1, \ldots, N, t \geq 0, \) which determine the interaction among \( N \) particles, are arbitrary.

In the deterministic Loewner equation (1.7), it was proved that [Sch12], if driving functions \( X_t^{(i)}, i = 1, \ldots, N, \) are \( \frac{1}{2} \)-Hölder continuous and \( X_t \in \text{Conf}^{< N}(\mathbb{R}) \) for all \( t \in [0, \infty) \), then the Loewner chain driven by them determines multiple slits in \( \mathbb{H} \). Then we have expected that the stochastic process \( \mathbf{X}_t, t \geq 0 \) solving our conformal welding problem will be a non-colliding system of \( N \) particles on \( \mathbb{R} \), for which the drift terms should give sufficiently strong repulsive interactions. The main result in the present paper given by Theorem 1.12 states that the solution can be given by a proper time change of the Dyson model, which may be the most studied process in non-colliding particle systems in probability theory and random matrix theory (see, for instance [For10, AGZ10, Kat15]). As shown in (1.13), in the Dyson model, the repulsive force acts between any pair of particles, whose strength is proportional to the inverse of distances between the particles and the proportionality coefficient is given by \( \beta/2, \beta \in (0, \infty) \). We note that the \( \gamma \)-QS-\((N+1)\)-MBPs of the \(( \mathbf{X}, \alpha \) )-standard type has the set of parameters \( \gamma \in (0, 2) \) and \( \alpha_i, i = 1, \ldots, N, \) and the multiple SLE does one parameter \( \kappa \in (0, \infty) \). Theorem 1.12 determines the relations among them as

\[
\kappa = \gamma^2, \quad \alpha_i = \frac{2}{\gamma} \quad (i = 1, \ldots, N), \quad \beta = \frac{8}{\kappa}.
\]

The equality \( \kappa = \gamma^2 \) is the same as that given by Sheffield [She16] and \( \alpha_i = 2/\gamma, i = 1, \ldots, N \) are a simple \( N \)-variable extension of his result \( \alpha_1 = 2/\gamma \) for the original conformal welding problem with two marked boundary points. The equality \( \beta = 8/\kappa \) is found in the literatures [Car03a, Car03b, Car04, BBK05], but its derivations heavily depended on CFT and the so-called group theoretical formulation of SLE [BB03, BB04, Kos18] (see Appendix B). Our derivation given in the proof of Lemma 2.2 is purely probability theoretical and simple. This relation, however, gives another puzzle, since the non-colliding condition \( \beta \geq 1 \) for the Dyson model [RS93, CL97, GM13, GM14] corresponds to \( \kappa \in (0, 8] \), while when \( \kappa > 4 \) the multiple SLE curves will collide with each other and become self-intersecting in \( \mathbb{H} \).

As interacting particle systems related with random matrix theory, a variety of non-colliding particle systems have been studied (see, for instance, [KT04]). We hope that we can address the conformal welding problems in other situations and, in solving them, interesting relations between non-colliding particle systems and multiple SLEs will be discovered. We will depict, in the following subsections,
examples of such other situations for which the conformal welding problem is solvable and produces an another type of driving processes.

4.2. Inhomogeneous systems. The setting (2.1) and (2.3) for the conformal welding problem can be generalized as follows:

\[ \frac{d}{dt}f^T_t(z) = -\sum_{i=1}^{N} \frac{2\lambda_i}{f^T_t(z) - Y^{(i)}_{T,t}}, \quad t \in [0, T], \quad f^T_0(z) = z \in \mathbb{H}, \]

\[ dY^{(i)}_{T,t} = \sqrt{\kappa_i} dB^{(i)}_t - F^{(i)}(Y_{T,t}) dt, \quad t \in [0, T], \quad i = 1, \ldots, N, \]

where \( \lambda_i > 0, \kappa_i > 0, i = 1, \ldots, N \) with \( \sum_{i=1}^{N} \lambda_i = N \) \text{[Sch12, RS17, dMHS18]}. Let

\[ h^*_t(z) = \sum_{i=1}^{N} \alpha_i \log(f^T_t(z) - Y^{(i)}_{T,t}) + Q \log f^T_t(z), \quad z \in \mathbb{H}, \quad t \in [0, T], \]

where \( \alpha_i, i = 1, \ldots, N \) are indeterminate real numbers and \( Q = \frac{2}{7} + \frac{2}{7} \). By the similar calculation to that given in the proof of Lemma \text{[2.2]} we can show that

\[ dh^*_t(z) = -\sum_{i=1}^{N} \frac{\alpha_i \kappa_i}{f^T_t(z) - Y^{(i)}_{T,t}} dB^{(i)}_t + \sum_{i=1}^{N} \frac{2C(\alpha_i, \kappa_i, \lambda_i, \gamma)}{(f^T_t(z) - Y^{(i)}_{T,t})^2} dt \]

\[ + \sum_{i=1}^{N} \frac{1}{f^T_t(z) - Y^{(i)}_{T,t}} \left( \alpha_i F^{(i)}(Y_{T,t}) - 2 \sum_{j=1}^{N} \alpha_i \lambda_j + \alpha_i \lambda_i \right) dt, \quad t \in [0, T], \]

with

\[ C(\alpha_i, \kappa_i, \lambda_i, \gamma) = -\left( \lambda_i + \frac{\kappa_i}{4} \right) \alpha_i + Q \lambda_i, \quad i = 1, \ldots, N. \]

Hence if

\[ C(\alpha_i, \kappa_i, \lambda_i, \gamma) = 0, \quad i = 1, \ldots, N \]

and

\[ F^{(i)}(x) = \frac{2}{\alpha_i} \sum_{j=1}^{N} \frac{\alpha_i \lambda_j + \alpha_i \lambda_i}{x_i - x_j} \]

\[ = 2 \sum_{j=1}^{N} \frac{\lambda_j}{x_i - x_j} + \frac{2\lambda_i}{\alpha_i} \sum_{j=1}^{N} \frac{\alpha_j}{x_i - x_j}, \quad i = 1, \ldots, N, \]

then \( h^*_t(z), z \in \mathbb{H}, t \in [0, T] \) becomes a local martingale. When \( \lambda_i = 1, \kappa_i = \kappa, \]

\( i = 1, \ldots, N, \) \text{[4.4]} with \text{[4.3]} gives

\[ \alpha_i = \frac{4Q}{4 + \kappa} = \frac{2Q}{\sqrt{\kappa(\frac{2}{\sqrt{\kappa}} + \frac{2}{\sqrt{\kappa}})}}, \quad i = 1, \ldots, N. \]
That is, the weights to the marked boundary points are homogeneous. When we further assume the relation $\kappa = \gamma^2$, then $\alpha_i = \frac{2}{\gamma}$ as we have seen in one of our main theorems (Theorem 1.12).

Inhomogeneous setting of the conformal welding problem with $\alpha_i \neq \alpha_j$, $i \neq j$ in general (as well as the inhomogeneous flow line problem with $\beta_i \neq \beta_j$, $i \neq j$ in general) will be studied in which inhomogeneous multiple SLE (4.1) driven by inhomogeneous interacting particles on $\mathbb{R}$ (4.2) shall be analyzed under the conditions (4.3), and (4.5) to solve the problems.

4.3. Multiple quadrant SLE and the Wishart process. In this paper, we have formulated the conformal welding problem for a $\gamma$-QS-(N+1)-MBPs $[\mathbb{H}, H^X_{\alpha}, (X, \infty)]_\gamma$ of the $(X, \alpha)$-standard type. In solving this, we have adopted the form of multiple Loewner equation (1.7) and assumed that the set of driving processes is determined by the system of SDEs (1.8) with drift functions $\{F(i)(x)\}_{i=1}^N$ motivated by preceding works [BBK05, RS17]. As a result, we have found that the parameters $\kappa = \gamma^2$ and $\alpha_i = \frac{2}{\gamma}$ are determined and the functions are chosen as (1.11) to obtain a one-parameter family $\{[\mathbb{H}, H^X_{\alpha}, (X_T, \infty)]_\gamma : T \in (0, \infty)\}$ of $\gamma$-QSs-(N+1)-MBPs, for each of which the conformal welding problem is solvable.

Notice that there is a room for changing the model of uniformization maps. As a generalized multiple Loewner equation for $N$ slits, we consider the following form

\begin{equation}
\frac{d}{dt}g_t(z) = \Psi(g_t(z), X_t), \quad t \geq 0, \quad g_0(z) = z,
\end{equation}

where $\Psi(z, x)$ is a suitable functions of $z$ and $x = (x_1, \ldots, x_N)$, and $\{X_t = (X_t^{(1)}, \ldots, X_t^{(N)}) : t \geq 0\}$ is a set of driving processes. We do not specify the domain of definition for the function $\Psi(z, x)$ since it will depend on models. When we take

\begin{equation}
\Psi(z, x) = \sum_{i=1}^N \frac{2}{z - x_i}, \quad z \in \mathbb{H}, \quad x \in \mathbb{R}^N,
\end{equation}

the associated Loewner equation (4.6) reduces to (1.7).

Let us see the case for another choice of $\Psi(z, x)$. Let $\mathbb{D} := \{z \in \mathbb{C} | \text{Re}z > 0, \text{Im}z > 0\}$ be an orthant in $\mathbb{C}$. We adopt

\begin{equation}
\Psi(z, x) = \Psi_{\mathbb{D}}(z, x) := \sum_{i=1}^N \left( \frac{2}{z - x_i} + \frac{2}{z + x_i} \right) + \frac{\delta}{z}, \quad z \in \mathbb{D}, \quad x \in (\mathbb{R}_{>0})^N.
\end{equation}

Here $\delta \in \mathbb{R}$ is a parameter and $\mathbb{R}_{>0} = \{x \in \mathbb{R} | x > 0\}$ is the set of positive real numbers. The associated Loewner equation (4.6) becomes

\begin{equation}
\frac{d}{dt}g_t(z) = \sum_{i=1}^N \left( \frac{2}{g_t(z) - X_t^{(i)}} + \frac{2}{g_t(z) + X_t^{(i)}} \right) + \frac{\delta}{g_t(z)}, \quad t \geq 0,
\end{equation}

$g_0(z) = z \in \mathbb{D}$. 

\end{document}
We again assume that the set of driving processes \( \{ X_t = (X_t^{(1)}, \ldots, X_t^{(N)}) \in (\mathbb{R}_>0)^N : t \geq 0 \} \) solves the system of SDEs

\[
dX_t^{(i)} = \sqrt{\kappa}dB_t^{(i)} + F^{(i)}(X_t)dt, \quad t \geq 0, \quad i = 1, \ldots, N,
\]

where \( \kappa > 0 \) is a parameter, \( \{ B_t^{(i)} : t \geq 0 \}_{i=1}^N \) are mutually independent standard Brownian motions and \( \{ F^{(i)}(X) \}_{i=1}^N \) are suitable functions of \( X = (x_1, \ldots, x_N) \) so that \( X_t \) lies in \( (\mathbb{R}_>0)^N \) for all \( t \geq 0 \). The equation (4.7) is the multiple version of the quadrant Loewner equation considered in [Tak14]. We assume that, if the initial value of \( X_t \) satisfies \( 0 < X_0^{(1)} < X_0^{(2)} < \cdots < X_0^{(N)} \), each realization determines \( N \) non-colliding and non-intersecting slits \( \{ \eta^{(i)} : (0, \infty) \to \mathbb{O} \}_{i=1}^N \) anchored on \( \mathbb{R}_>0 \): \( \eta_0^{(i)} = X_0^{(i)}, i = 1, \ldots, N \). Namely \( g_t(\cdot), t \geq 0, \) becomes a uniformization map

\[
g_t : \mathbb{O}_t^N := \mathbb{O} \backslash \bigcup_{i=1}^N \eta_t^{(i)}(0, t) \to \mathbb{O}.
\]

Let \( \gamma \in (0, 2] \). For \( N \) points \( x = (x_1, \ldots, x_N) \), where \( 0 < x_1 < x_2 < \cdots < x_N \) and an \( N \)-tuple of real numbers \( \alpha = (\alpha_1, \ldots, \alpha_N) \), we define the following function on \( \mathbb{O} \):

\[
w^{x, \alpha}_\mathbb{O}(z) = \sum_{i=1}^N \alpha_i (\log|z - x_i| + \log|z + x_i|) + Q \log|z|,
\]

where \( Q = \frac{2}{\gamma} + \frac{\gamma}{2} \), and a \( C^\infty(\mathbb{O})' \)-valued random field \( H^x, \alpha_\mathbb{O} := H^\mathbb{O} + w^{x, \alpha}_\mathbb{O} \). For a random \( N \) point configuration \( X = (X_1, \ldots, X_N) \) valued in \( \text{Conf}_N(\mathbb{R}_>0) \), we can see that

\[
[\mathbb{O}, H^X, \alpha, (X, \infty)]_\gamma = [\mathbb{I}, H^X, \alpha, (X^2, \infty)]_\gamma,
\]

where \( X^2 = ((X_1)^2, \ldots, (X_N)^2) \), is of the \( (X^2, \alpha) \)-standard type.

We define the Wishart process on parameters \( \beta > 0 \) and \( \nu > -1 \) as a solution of the system of SDEs on \( X_t^{W_{\beta, \nu}(i)}, t \geq 0, i = 1, \ldots, N \) such that [Bru91][KT04]

\[
dX_t^{W_{\beta, \nu}(i)} = dB_t^{(i)} + \left[ \frac{2\nu + 1}{2} \frac{1}{X_t^{W_{\beta, \nu}(i)}} \right] dt,
\]

\[
+ \frac{\beta}{2} \sum_{j \neq i}^N \left( \frac{1}{X_t^{W_{\beta, \nu}(i)} - X_t^{W_{\beta, \nu}(j)}} + \frac{1}{X_t^{W_{\beta, \nu}(j)} + X_t^{W_{\beta, \nu}(i)}} \right) dt,
\]

\( t \geq 0, \quad i = 1, \ldots, N. \)

Using the function \( \Psi_\mathbb{O}(z, x) \) as a model of uniformization maps, we can obtain solutions to the conformal welding problem.

**Theorem 4.1.** Let \( \gamma \in (0, 2] \) and \( N \in \mathbb{Z}_{>1} \). Suppose that \( \{ X_t = (X_t^{(1)}, \ldots, X_t^{(N)}) : t \geq 0 \} \) is the time change of the Wishart process \( \{ X_t^{W_{\beta, \nu}(i)} : t \geq 0 \} \) starting at a deterministic initial state \( X_0 = x \in \text{Conf}_N(\mathbb{R}_>0) \). Then, at each time \( T \in (0, \infty) \),

the conformal welding problem for \([\mathbb{D}, H^{\mathbb{D}, \alpha}_t, (X_T, \infty)]\) with \((\alpha_1, \ldots, \alpha_N) = (\frac{2}{\gamma}, \ldots, \frac{2}{\gamma})\) is solved as follows:

1. The solution of the Loewner equation (4.7) driven by the time change of the Wishart process \(\{X^W_{\kappa r, \kappa + (\kappa + \delta)/\kappa}, 0 \leq t \leq T\}\) gives a solution to Problem 1. Namely, \(g^{-1}_T : \mathbb{D} \to \mathbb{D}_T^\gamma\) is the desired conformal equivalence.

2. The probability law for resulting slits \(\{\eta^{(i)}\}_{i=1}^N\) is the one for the quadrant multiple SLE\(_\kappa\). This gives a solution to Problem 2.

3. Problem 3 is answered positively with \(\eta_0 = x\) a.s.

From the above observation, we could say that interacting particle systems such as the Dyson model and the Wishart process are associated with models \(\Psi(z, x)\) of uniformization maps. Along this line, we could expect a new classification of interacting particle systems from a perspective of the Loewner theory and coupling with GFF.

A detail of this subject including the associated flow line problem will be published elsewhere.

4.4. Other variants of SLE. There are variants of SLE other than the chordal SLE including radial and dipole. It was shown in [SW05] that these three versions of SLE are transformed one another by conformal mappings in the framework of the SLE\(_\kappa,\rho\), while the force points are allowed to be interior of the domain. For example, the radial SLE\(_\kappa\) is transformed to the chordal SLE\(_\kappa,\kappa-\delta\) with an interior force point by a Möbius transformation. In [MS16, She16], the coupling with the SLE and the GFF is formulated for the SLE\(_\kappa,\rho\), and our result in the present paper is also expected to be extended to the case of the multiple SLE\(_\kappa,\rho\). Then, it would be interesting to study how the transformation of these SLEs can be compatible to the coordinate transformation of quantum surfaces in Eq.(1.2) under the connection between the SLE and the LQG.

4.5. The limit \(N \to \infty\). It would be interesting to consider the conformal welding problem and the flow line problem in the case with infinitely many boundary points. In the present paper, the multiple SLE driven by the \(N\)-particle Dyson model arose as the solution to the conformal welding problem for a \(\gamma\)-QS-(\(N+1\))-MBPs of the \((X, \alpha)\)-standard type. If the method in the present paper is applicable at the limit \(N \to \infty\), it can be expected that the multiple SLE driven by the infinite dimensional Dyson model [KT10, Osa12, Osa13, Tsa16, OT16, KO18] would appear in such systems. Although the multiple SLE driven by infinitely many driving processes is not well-posed so far, we hope that it is captured when the coupling with GFF is considered.

Another limit of \(N \to \infty\) is the hydrodynamic limit of the multiple SLE [dMS16, HK18]. In this case, the ensemble of slits gets deterministic as \(N \to \infty\). Accordingly, a quantum surface must be subject to a boundary condition. An interesting question is, then, how multiple slits condition the GFF on the domain and finally impose a boundary condition.
4.6. **Discrete models converging to the present systems.** It has been reported that random planar maps converge to SLE-decorated LQG in several topology (see [GMS17] and references therein). As the chordal SLE describes the scaling limit of a single interface in various critical lattice models, the multiple SLEs describe scaling limits of collections of interfaces in critical lattice models with alternating boundary conditions (see [BPW18] and references therein). In the present paper we introduced new kinds of continuous systems, the \( \gamma \)-quantum surface with \( N+1 \) marked boundary points (\( \gamma \)-QS-(\( N+1 \))-MBPs) and the \( \chi \)-imaginary surface with \( N+1 \) boundary condition changing points (\( \chi \)-IS-(\( N+1 \))-BCCPs) for \( N \in \mathbb{Z}_{\geq 1} \). Both have been related with the multiple SLE driven by the Dyson model in solving the conformal welding problem and the flow line problem. Discrete counterparts of these random systems and corresponding problems will be studied.

**Appendix A. Construction of \( \mathcal{S}_\gamma \) and \( \mathcal{S}_{\gamma,N+1} \)**

In this appendix, we construct the spaces \( \mathcal{S}_\gamma \) and \( \mathcal{S}_{\gamma,N+1} \) as orbifolds and study their structures.

**A.1. Without marked boundary points.** Let us begin with \( \mathcal{S}_\gamma \), \( \gamma \in (0,2] \). Consider the following lift of \( \mathcal{S}_\gamma \):

\[
\mathcal{S}_\text{univ} := \bigcup_{D \subseteq \mathbb{C}: \text{1-conn.}} C^\infty(\overline{D})',
\]

where \( D \subseteq \mathbb{C} \) runs over all simply connected domains. We consider the canonical surjection \( \mathcal{S}_\text{univ} \rightarrow \mathcal{S}_\gamma \). Notice that each component \( C^\infty(\overline{D})' \) carries a right action \( a_{D,\gamma} : C^\infty(\overline{D})' \times \text{Aut}(D) \rightarrow C^\infty(\overline{D})' \text{ of Aut}(D) \) depending on the parameter \( \gamma \) defined by

\[
a_{D,\gamma}(h, \psi) := h \circ \psi + Q \log |\psi'|, \quad h \in C^\infty(\overline{D}), \quad \psi \in \text{Aut}(D),
\]

where we set \( Q = \frac{\gamma}{2} + \frac{3}{2} \). The chain rule ensures that \( a_{D,\gamma} \) defines a right action of the group. We write the quotient space \( C^\infty(\overline{D})' / a_{D,\gamma} \) as \( C^\infty(\overline{D})'_{\gamma\text{-red}} \) (we read “\( \gamma \)-red” as “\( \gamma \)-reduced”), and set

\[
\widehat{\mathcal{S}}_\gamma := \bigcup_{D \subseteq \mathbb{C}: \text{1-conn.}} C^\infty(\overline{D})'_{\gamma\text{-red}}.
\]

Let us introduce a groupoid \( \mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1) \) whose objects are simply connected domains in \( \mathbb{C} \): \( \mathcal{G}_0 = \{D \subseteq \mathbb{C} : \text{simply connected}\} \) and the set of morphisms of which is given by \( \mathcal{G}_1(D_1, D_2) = \text{Aut}(D_2) \backslash \text{Iso}(D_1, D_2) / \text{Aut}(D_1) \), \( D_1, D_2 \in \mathcal{G}_0 \). It is obvious that each set \( \mathcal{G}_1(D_1, D_2) \) consists of a single element, which we denote by the symbol \( (D_1 \rightarrow D_2) \). Then the anti-action of \( \mathcal{G} \) on \( \widehat{\mathcal{S}}_\gamma \) is given as follows. We consider a mapping \( c_\gamma : \mathcal{G}_1 \times \widehat{\mathcal{S}}_\gamma \rightarrow \widehat{\mathcal{S}}_\gamma \) which is defined for pairs \( ((D_1 \rightarrow D_2), h \in \mathcal{G}_1) \).
$C^\infty(D_2')_{\gamma - \text{red}}, D_1, D_2 \in \mathcal{G}_0$ as
\[
c_\gamma((D_1 \to D_2), h) := h \circ \psi + Q \log |\psi'| \in C^\infty(D_1')_{\gamma - \text{red}},
\]
where $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$ and $\psi : D_1 \to D_2$ is a conformal equivalence. It can be verified that the above definition does not depend on the choice of a conformal equivalence $\psi$. Then the quotient $\tilde{S}_\gamma/c_\gamma$ is just the totality of $\gamma$-pre-quantum surfaces $S_\gamma$.

Consequently, the canonical quotient map $S_{\text{univ}} \to S_\gamma$ is the composition
\[
S_{\text{univ}} \xrightarrow{\bigcup_D a_D, \gamma} \tilde{S}_\gamma \xrightarrow{c_\gamma} S_\gamma.
\]

By uniformizing any domain $D$ to the upper half plane, the collection $S_\gamma$ of all $\gamma$-pre-quantum surfaces is identified with the space of $\gamma$-reduced distributions $C^\infty(H)'_{\gamma - \text{red}}$ on $\mathbb{H}$.

A.2. With marked boundary points. Let us move on to $S_{\gamma,N+1}, \gamma \in (0, 2]$, $N \in \mathbb{Z}_{\geq 0}$. We consider the following space
\[
S_{\text{univ}}^{N+1} = \bigcup_{D \subset \subset \mathbb{C} \text{ 1-conn.}} C^\infty(D)' \times \text{Conf}^{\leq}_{N+1}(\partial D),
\]
where $D \subset \subset \mathbb{C}$ runs over all simply connected domains. As we have seen, for each $D \subset \subset \mathbb{C}$, the component $C^\infty(D)'$ has a right action $a_{\gamma,D}$ of $\text{Aut}(D)$ depending on the parameter $\gamma$. The same group also acts on $\text{Conf}^{\leq}_{N+1}(\partial D)$ from the left. We write the diagonal action of $\text{Aut}(D)$ on $C^\infty(D)' \times \text{Conf}^{\leq}_{N+1}(\partial D)$ as $a_{D,\gamma,N+1}$ and set
\[
\tilde{S}_{\gamma,N+1} := \bigcup_{D \subset \subset \mathbb{C} \text{ 1-conn.}} C^\infty(D)' \times a_{D,\gamma,N+1} \text{Conf}^{\leq}_{N+1}(\partial D).
\]

It can be verified that the groupoid $\mathcal{G}$ again acts on $\tilde{S}_{\gamma,N+1}$. Then the space $\tilde{S}_{\gamma,N+1}$ is constructed as
\[
S_{\text{univ}}^{N+1} \xrightarrow{\bigcup_D a_{D,\gamma,N+1}} \tilde{S}_{\gamma,N+1} \xrightarrow{\mathcal{G}} S_{\gamma,N+1}.
\]

Let us write each component of $\tilde{S}_{\gamma,N+1}$ as
\[
\tilde{S}_{\gamma,N+1}(D) := C^\infty(D)' \times a_{D,\gamma,N+1} \text{Conf}^{\leq}_{N+1}(\partial D).
\]

Because the action of the groupoid $\mathcal{G}$ is simply transitive, the space $\tilde{S}_{\gamma,N+1}$ is noncanonically isomorphic to $\tilde{S}_{\gamma,N+1}(D)$ for every $D$.

For simplicity, let us identify $S_{\gamma,N+1}$ with $\tilde{S}_{\gamma,N+1}(\mathbb{H})$ and consider the following commutative diagram:
\[
\begin{array}{ccc}
C^\infty(\mathbb{H})' \times \text{Conf}^{\leq}_{N+1}(\partial \mathbb{H}) & \xrightarrow{\phi} & \text{Conf}^{\leq}_{N+1}(\partial \mathbb{H}) \\
\downarrow & & \downarrow \\
\tilde{S}_{\gamma,N+1}(\mathbb{H}) & \xrightarrow{\pi} & \text{Aut}(\mathbb{H}) \setminus \text{Conf}^{\leq}_{N+1}(\partial \mathbb{H}).
\end{array}
\]
1. If $N \leq 2$, the space $\text{Aut}(\mathbb{H}) \backslash \text{Conf}_{N+1}^<(\partial \mathbb{H}) = \{\ast\}$ consists of a single element. Thus the space $S_{\gamma,N+1}$ is isomorphic to the fiber $\pi^{-1}(\ast)$.
   (a) If $N = 0$, the point $\infty \in \text{Conf}_1^<(\partial \mathbb{H})$ is fixed by the subgroup $\text{Aff}(\mathbb{H}) = \{ z \mapsto az + b | a > 0, b \in \mathbb{R}\}$ of affine transformations in $\text{Aut}(\mathbb{H})$. Thus, the fiber over $[\infty] \in \text{Aut}(\mathbb{H}) \backslash \text{Conf}_1^<(\partial \mathbb{H})$ becomes
   \[\pi^{-1}[\infty] \simeq C^\infty(\mathbb{H})' / a_{\mathbb{H},\gamma}(\text{Aff}(\mathbb{H})).\]
   (b) If $N = 1$, the point $(0, \infty) \in \text{Conf}_2^<(\partial \mathbb{H})$ is fixed by the subgroup $\text{Scl}(\mathbb{H}) = \{ z \mapsto az | a > 0\}$ of scale transformations in $\text{Aut}(\mathbb{H})$. Thus, the fiber over $[0, \infty] \in \text{Aut}(\mathbb{H}) \backslash \text{Conf}_2^<(\partial \mathbb{H})$ becomes
   \[\pi^{-1}[0, \infty] \simeq C^\infty(\mathbb{H})' / a_{\mathbb{H},\gamma}(\text{Scl}(\mathbb{H})).\]
   (c) If $N = 2$, the action of $\text{Aut}(\mathbb{H})$ on $\text{Conf}_2^<(\partial \mathbb{H})$ is simply transitive. Thus, the fiber over $[0, 1, \infty] \in \text{Aut}(\mathbb{H}) \backslash \text{Conf}_2^<(\partial \mathbb{H})$ becomes
   \[\pi^{-1}[0, 1, \infty] \simeq C^\infty(\mathbb{H})'.\]

2. If $N \geq 3$, the space $\text{Aut}(\mathbb{H}) \backslash \text{Conf}_{N+1}^<(\partial \mathbb{H})$ is $(N-2)$-dimensional over $\mathbb{R}$ and each fiber becomes
   \[\pi^{-1}[x_1, \ldots, x_{N+1}] \simeq C^\infty(\mathbb{H})', \quad (x_1, \ldots, x_{N+1}) \in \text{Conf}_{N+1}^<(\partial \mathbb{H}).\]

We show an alternative construction of $\widetilde{S}_{\gamma,N+1}^\text{Rot}(\mathbb{H})$ used to define a $\gamma$-QS-$(N+1)$-MBPs of the $(X, \alpha)$-standard type in Introduction. Let $\text{Rot}(\mathbb{H}) \subset \text{Aut}(\mathbb{H})$ be the subgroup consisting of rotations of $\mathbb{H}$. We consider the following object:
   \[\widetilde{S}_{\gamma,N+1}^\text{Rot}(\mathbb{H}) := C^\infty(\mathbb{H})' \times_{a_{\mathbb{H},\gamma,N+1}(\text{Rot}(\mathbb{H}))} \text{Conf}_{N+1}^<(\partial \mathbb{H}),\]
   which is a fiber bundle over $\text{Rot}(\mathbb{H}) \backslash \text{Conf}_{N+1}^<(\partial \mathbb{H})$. By sending the $(N+1)$-st point to $\infty$, we have
   \[\text{Rot}(\mathbb{H}) \backslash \text{Conf}_{N+1}^<(\partial \mathbb{H}) \simeq \text{Conf}_{N}^<(\mathbb{R}),\]
   and the fiber over $x = (x_1, \ldots, x_N) \in \text{Conf}_{N}^<(\mathbb{R})$ is isomorphic to $C^\infty(\mathbb{H})'$. Since $\text{Conf}_{N}^<(\mathbb{R})$ is contractible, the fiber bundle $\widetilde{S}_{\gamma,N+1}^\text{Rot}(\mathbb{H}) \to \text{Conf}_{N}^<(\mathbb{R})$ is trivial:
   \[\widetilde{S}_{\gamma,N+1}^\text{Rot}(\mathbb{H}) \simeq C^\infty(\mathbb{H})' \times \text{Conf}_{N}^<(\mathbb{R}),\]
   reducing to Eq. (1.3). In this construction, it becomes clear that the surjection $\pi_{\gamma,N+1}^\infty$ defined in (1.6) is just the quotient map.

**APPENDIX B. DRIVING PROCESSES OF MULTIPLE SLEs FROM AN AUXILIARY FUNCTION**

A time change of the Dyson model also appeared in [BBK05] as a particular example of a set of driving processes. In their work, in connection to CFT, a set
of driving processes $X_t = (X_t^{(1)}, \ldots, X_t^{(N)})$, $t \geq 0$, $N \in \mathbb{Z}_{\geq 1}$, was derived from an auxiliary function $Z(x_1, \ldots, x_N)$ annihilated by operators

$$\mathcal{D}_i = \frac{\kappa}{2} \partial_{x_i}^2 + 2 \sum_{j,j\neq i} \left( \frac{1}{x_i - x_j} \partial_{x_j} \frac{h_{\kappa}}{(x_i - x_j)^2} \right), \quad i = 1, \ldots, N,$$

where $h_{\kappa} = \frac{\kappa - \frac{2}{\kappa}}{2\kappa}$ so that \{\(X_t = (X_t^{(1)}, \ldots, X_t^{(N)}) : t \geq 0\)\} satisfies the system of SDEs

$$dX_t^{(i)} = \sqrt{\kappa}dB_t^{(i)} + \kappa(\partial_{x_i} \log Z)(X_t)dt + \sum_{j,j\neq i} 2dt \frac{X_t^{(i)} - X_t^{(j)}}{X_t^{(i)} - X_t^{(j)}}, \quad t \geq 0, \quad i = 1, \ldots, N.$$

Our set of driving processes \([1.8]\) associated with functions \([1.11]\) can be obtained by taking the following auxiliary function:

$$Z(x_1, \ldots, x_N) = \prod_{i<j} (x_i - x_j)^2 \kappa.$$ 

This auxiliary function is a correlation function of the Coulomb gas and was argued in [BBK05] to be related to interfaces according to the criterion by Dubédat [Dub07].

It is also possible to directly derive the reverse flow of the multiple SLE in an analogous way as the one used in [BBK05]. The central idea is to require correlation functions of CFT to be local martingales. Let us begin with the reverse flow of single SLE [Law09, VL12]:

$$\frac{d}{dt}g_{t}^{R}(z) = -\frac{2}{g_{t}^{R}(z) - \sqrt{\kappa}B_{t}}, \quad t \geq 0, \quad g_{0}^{R}(z) = z \in \mathbb{H},$$

where \{\(B_{t} : t \geq 0\)\} is a standard Brownian motion and $\kappa > 0$ is a parameter. If we define $f_{t}^{R}(z) := g_{t}^{R}(z) - \sqrt{\kappa}B_{t}$, then it satisfies

$$df_{t}^{R}(z) = -\frac{2dt}{f_{t}^{R}(z)} - \sqrt{\kappa}dB_{t}, \quad t \geq 0, \quad z \in \mathbb{H}.$$ 

Now let us recall the group theoretical formulation of SLE [BB03, BB04] (see also [Kos18, Sect. II]), in which this evolution can be enhanced to an operator-valued stochastic process acting on the representation space of the Virasoro algebra. The resulting stochastic process denoted as $R(f_{t}^{R})$ satisfies

$$R(f_{t}^{R})^{-1}dR(f_{t}^{R}) = \left(2L_{-2} + \frac{\kappa}{2}L_{-1}^2 \right)dt + \sqrt{\kappa}L_{-1}dB_{t}, \quad t \geq 0, \quad R(f_{0}) = \text{Id},$$

where $L_n, n \in \mathbb{Z}$ are the standard Virasoro generators. By the standard argument (see also [Enk17]), for a highest weight vector $|c, h\rangle$ for the Virasoro algebra, the stochastic process $R(f_{t}^{R})|c, h\rangle, t \geq 0, \quad$ is a local martingale if the highest weight is chosen as

$$c = c_{\kappa}^{R} = 1 + \frac{3(\kappa + 4)^2}{2\kappa}, \quad h = h_{\kappa}^{R} = -\frac{\kappa + 6}{2\kappa}. $$

Note that the same local martingale is also expressed as

$$R(f_{t}^{R})|c_{\kappa}^{R}, h_{\kappa}^{R}\rangle = R(g_{t}^{R})\Psi_{h_{\kappa}^{R}}(\sqrt{\kappa}B_{t})|0\rangle,$$
where $\Psi_{hR}$ is the primary field of conformal weight $h_R$.

The multiple analogue of the reverse flow of SLE would require a CFT datum as an input. Let us consider the following correlation function:

$$Z(x_1, \cdots, x_N) = \langle \psi_{hR}(x_1) \cdots \psi_{hR}(x_N) \rangle_0,$$

the central charge for which is $c_R$. Due to the existence of a singular vector

$$(2L_{-2} + \frac{\kappa}{2}L_{-1}^2) |c_R, h_R\rangle,$$

the correlation function $Z(x_1, \cdots, x_N)$ solves the following system of differential equations:

$$(B.1) \quad D_i^R Z = 0, \quad i = 1, \ldots, N,$$

where

$$D_i^R = \frac{\kappa}{2} \partial^2_{x_i} - 2 \sum_{j \neq i} \left( \frac{1}{x_j - x_i} \partial_{x_j} - \frac{h_R}{(x_j - x_i)^2} \right), \quad i = 1, \ldots, N.$$

**Example B.1.** The following function gives a solution to the system of differential equations (B.1):

$$(B.2) \quad Z(x_1, \cdots, x_N) = \prod_{i<j} (x_i - x_j)^{-\frac{2}{\kappa}}.$$

As in the case of the forward flow of multiple SLE [BBK05], we define driving processes as follows:

**Definition B.2.** Let $\{B^{(i)}_t : t \geq 0\}_{i=1}^N$ be mutually independent standard Brownian motions, and $Z(x_1, \cdots, x_N)$ be a solution to the system of differential equations (B.1). The associated set of driving processes $\{Y^{(i)}_t = (Y^{(i)(1)}, \ldots, Y^{(i)(N)}) : t \geq 0\}$ is defined as a solution to the system of SDEs:

$$dY^{(i)}_t = \sqrt{\kappa} dB^{(i)}_t + \kappa(\partial_{x_i} \log Z)(Y_t) dt - \sum_{j \neq i} \frac{2 dt}{Y^{(i)}_t - Y^{(j)}_t}, \quad i = 1, \ldots, N, \quad t \geq 0.$$

**Example B.3.** For the function given in (B.2), the associated set of driving processes satisfies (2.3). Thus the reverse flow of multiple SLE considered in Sect. 2 is a particular example of ones defined below.

Associated with these data, we define the reverse flow of multiple SLE as follows:

**Definition B.4.** Let $Z(x_1, \cdots, x_N)$ be a solution to the system of differential equations (B.1) and $\{Y_t : t \geq 0\}$ be the associated driving processes. The reverse flow of the multiple SLE associated with these data is a stochastic process $\{g^R_t(\cdot)\}_{t \geq 0}$ solving the following multiple version of the reverse flow:

$$\frac{d}{dt} g^R_t(z) = - \sum_{i=1}^N \frac{2}{g^R_t(z) - Y^{(i)}_t}, \quad t \geq 0, \quad g^R_0(z) = z.$$

The reverse flow defined above is connected to CFT in the following sense:
Theorem B.5. Let $Z(x_1, \cdots, x_N)$ be a solution to the system of differential equations (B.1), \{Y_t : t \geq 0\} be the associated set of driving processes, and let \{g^R_t(\cdot)\}_{t \geq 0}$ be the corresponding reverse flow of multiple SLE. Then the stochastic process
\[
M_t := \frac{1}{Z(Y_t)} R(g^R_t(Y^{(1)}_t)) \Psi_{h^R_t}(Y^{(1)}_t) \cdots \Psi_{h^R_t}(Y^{(N)}_t) |0\rangle, \quad t \geq 0,
\]
is a local martingale on a representation space of the Virasoro algebra.

REFERENCES

[AGZ10] G. W. Anderson, A. Guionnet, and O. Zeitouni. An Introduction to Random Matrices. Cambridge University Press, Cambridge, 2010.

[BB03] M. Bauer and D. Bernard. Conformal field theories of stochastic Loewner evolutions. Commun. Math. Phys., 239:493–521, 2003.

[BB04] M. Bauer and D. Bernard. Conformal transformations and the SLE partition function martingale. Ann. Henri Poincaré, 5:289–326, 2004.

[BBK05] M. Bauer, D. Bernard, and K. Kytölä. Multiple Schramm–Loewner evolutions and statistical mechanics martingales. J. Stat. Phys., 120:1125–1163, 2005.

[Ber16] N. Berestycki. Introduction to the Gaussian free field and Liouville quantum gravity, 2016. available at https://homepage.univie.ac.at/nathanael.berestycki/articles.html.

[BPW18] V. Beffara, E. Peltola, and H. Wu. On the uniqueness of global multiple SLE, 2018. arXiv:1801.07699.

[Bru91] M. F. Bru. Wishart process. J. Theor. Probab., 4:725–751, 1991.

[Car03a] J. Cardy. Stochastic Loewner evolution and Dyson’s circular ensembles. J. Phys. A: Math. Gen., 36:L379–L386, 2003.

[Car03b] J. Cardy. Corrigendum: Stochastic Loewner evolution and Dyson’s circular ensembles. J. Phys. A: Math. Gen., 36:12343, 2003.

[Car04] J. Cardy. Calogero–Sutherland model and bulk-boundary correlations in conformal field theory. Phys. Lett. B, 582:121–126, 2004.

[CL97] E. Cépa and D. Lépingle. Diffusing particles with electrostatic repulsion. Probab. Theory Relat. Fields, 107:429–449, 1997.

[dMHS18] A. del Monaco, I. Hotta, and S. Schleissinger. Tightness results for infinite-slit limits of the chordal Loewner equation. Comput. Methods Funct. Theory, 18:9–33, 2018.

[dMS16] A. del Monaco and S. Schleissinger. Multiple SLE and the complex Burgers equation. Math. Nachr., 289:2007–2018, 2016.

[DS11] B. Duplantier and S. Sheffield. Liouville quantum gravity and KPZ. Invent. Math., 185:333–393, 2011.

[Dub07] J. Dubédat. Commutation relations for Schramm-Loewner evolutions. Commun. Pure and Appl. Math., LX:1792–1847, 2007.

[Dys62] F. J. Dyson. A Brownian-motion model for the eigenvalues of a random matrix. J. Math. Phys., 3:1191–1198, 1962.

[For10] P. J. Forrester. Log-gases and Random Matrices. London Math. Soc. Monographs. Princeton University Press, Princeton, 2010.

[Fuk17] Y. Fukusumi. Time reversing procedure of SLE and 2d gravity, 2017. arXiv:1710.08670.

[GM13] P. Graczyk and J. Małecki. Multidimensional Yamada-Watanabe theorem and its applications to particle systems. J. Math. Phys., 54:021503, 2013.

[GM14] P. Graczyk and J. Małecki. Strong solutions of non-colliding particle systems. Electron. J. Probab., 19:1–21, 2014.
[GMS17] E. Gwynne, J. Miller, and S. Sheffield. The Tutte embedding of the mated-CRT map converges to Liouville quantum gravity, 2017. arXiv:1705.11161.

[Hid80] T. Hida. Brownian Motion, volume 11 of Applications of Mathematics. Springer-Verlag New York Heidelberg Berlin, 1980.

[HK18] I. Hotta and M. Katori. Hydrodynamic limit of multiple SLE. *J. Stat. Phys.*, 171:166–188, 2018.

[IW89] N. Ikeda and S. Watanabe. Stochastic Differential Equations and Diffusion Processes. North-Holland/Kodansha, Tokyo, 2nd edition, 1989.

[Kat15] M. Katori. Bessel Processes, Schramm–Loewner Evolution, and the Dyson Model, volume 11 of SpringerBriefs in Mathematical Physics. Springer, 2015.

[KO18] Y. Kawamoto and H. Osada. Finite-particle approximations for interacting Brownian particles with logarithmic potential. *J. Math. Soc. Japan*, 70:921–952, 2018.

[Kos18] S. Koshida. Local martingales associated with Schramm-Loewner evolutions with internal symmetry. *J. Math. Phys.*, 59:101703, 2018.

[KSS68] P. P. Kufarev, V. V. Sobolev, and L. V. Sporyseva. A certain method of investigation of extremal problems for functions that are univalent in the half-plane. *Trudy Tomsk. Gos. Univ. Ser. Meh.-Mat.*, 200:142–164, 1968.

[KT04] M. Katori and H. Tanemura. Symmetry of matrix-valued stochastic processes and noncolliding diffusion particle systems. *J. Math. Phys.*, 45:3058–3085, 2004.

[KT10] M. Katori and H. Tanemura. Non-equilibrium dynamics of Dyson’s model with an infinite number of particles. *Commun. Math. Phys.*, 293:469–497, 2010.

[Law09] G. F. Lawler. Multifractal analysis of the reverse flow for the Schramm-Loewner evolution. *Progress in Probability*, 61:73–107, 2009.

[Löw23] K. Löwner. Untersuchungen über schlichte konforme Abbildungen des Einheitskreises. I. *Mathematische Annalen*, 89:103–121, 1923.

[MS16a] J. Miller and S. Sheffield. Imaginary geometry I: Interacting SLEs. *Probab. Theory Relat. Fields*, 164:553–705, 2016.

[MS16b] J. Miller and S. Sheffield. Imaginary geometry II: Reversibility of $\text{SLE}_\kappa(p_1,p_2)$ for $\kappa \in (0, 4)$. *Ann. Prob.*, 44:1647–1722, 2016.

[MS16c] J. Miller and S. Sheffield. Imaginary geometry III: reversibility of $\text{SLE}_\kappa$ for $\kappa \in (4, 8)$. *Ann. Math.*, 184:455–486, 2016.

[MS17] J. Miller and S. Sheffield. Imaginary geometry IV: interior rays, whole-plane reversibility, and space-filling trees. *Probab. Theory Relat. Fields*, 169:729–869, 2017.

[Osa12] H. Osada. Infinite-dimensional stochastic differential equations related to random matrices. *Probab. Theory Relat. Fields*, 153:471–509, 2012.

[Osa13] H. Osada. Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials. *Ann. of Probab.*, 41:1–49, 2013.

[OT16] H. Osada and H. Tanemura. Strong Markov property of determinantal processes with extended kernels. *Stoch. Proc. Appl.*, 126:186–208, 2016.

[Pol81a] A. M. Polyakov. Quantum geometry of bosonic strings. *Phys. Lett. B*, 103:207–210, 1981.

[Pol81b] A. M. Polyakov. Quantum geometry of fermionic strings. *Phys. Lett. B*, 103:211–213, 1981.

[QW18] W. Qian and W. Werner. Coupling the Gaussian free fields with free and with zero boundary conditions via common level lines. *Commun. Math. Phys.*, 361:53–80, 2018.

[RS93] L. C. G. Rogers and Z. Shi. Interacting Brownian particles and the Wigner law. *Probab. Theory Relat. Fields*, 95:555–570, 1993.

[RS05] S. Rohde and O. Schramm. Basic properties of SLE. *Ann. Math.*, 161:883–924, 2005.

[RS17] D. Roth and S. Schleissinger. The Schramm-Loewner equation for multiple slits. *J. Anal. Math.*, 131:73–99, 2017.
CONFORMAL WELDING PROBLEM

[Sch00] O. Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, 118:221–288, 2000.

[Sch12] S. Schleissinger. The multiple-slit version of Loewner’s differential equation and pointwise Hölder continuity of driving functions. *Ann. Acad. sci. Fenn., Math.*, 37:191–201, 2012.

[She07] S. Sheffield. Gaussian free fields for mathematicians. *Probab. Theory Relat. Fields*, 139:521–541, 2007.

[She16] S. Sheffield. Conformal weldings of random surfaces: SLE and the quantum gravity zipper. *Ann. Prob.*, 44:3474–3545, 2016.

[SS13] O. Schramm and S. Sheffield. A contour line of the continuum Gaussian free field. *Probab. Theory Relat. Fields*, 157:47–80, 2013.

[SW05] O. Schramm and D. B. Wilson. SLE coordinate changes. *New York J. Math.*, 11:659–669, 2005.

[Tak14] T. Takebe. Dispersionless BKP hierarchy and quadrant Löwner equation. *SIGMA*, 10:023, 2014.

[Tsa16] L.-C. Tsai. Infinite dimensional stochastic differential equations for Dyson’s model. *Probab. Theory Relat. Fields*, 166:801–850, 2016.

[VL12] F. J. Viklund and G. F. Lawler. Almost sure multifractal spectrum for the tip of an SLE curve. *Acta Math.*, 209:265–322, 2012.

(Makoto KATORI) DEPARTMENT OF PHYSICS, FACULTY OF SCIENCE AND ENGINEERING, CHUO UNIVERSITY, KASUGA, BUNKYO-ku, TOKYO 112-8551, JAPAN
E-mail address: katori@phys.chuo-u.ac.jp

(Shinji KOSHIDA) DEPARTMENT OF BASIC SCIENCE, THE UNIVERSITY OF TOKYO, KOMABA, MEGURO, TOKYO 153-8902, JAPAN
E-mail address: koshida@vortex.c.u-tokyo.ac.jp