On First-Quantized Fermions in Compact Dimensions

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We discuss the path integral representation for the fermionic particles and strings along the lines of [1, 2] and concentrate at the problems arising when some target-space dimensions are compact. An example of partition function for fermionic particle at finite temperature or with one compact target-space dimension is considered in detail. It is demonstrated that the first-quantized path integral requires, in general, presence of nonvanishing "Wilson loops" and modulo some common problems for real fermions in Grassmannian formulation one can try to reinterpret them in terms of condensates of the world-line fermions. The properties of corresponding path integrals in string theory are also discussed.

1 Introduction

String theory is still believed to be the only possible candidate to the role of fundamental theory of all interactions though the exact sense of these words is rather far from being creative. One should better say that nowadays it is almost clear that certain problems which are in principle unresolvable in the context of conventional quantum field theory may be solved within this more fundamental theory in the future and, for example, the self-consistent formulation of quantum gravity is among them.

It is also clear that even terminologically string theory is not very exact name for this (not already quite) new fundamental theory, since strings are light excitations only in some part of the whole parameter or moduli space of nonperturbative string (or M-) theory and only there the theory can be formulated as perturbative expansion in terms of the Polyakov path integral [3]. Naively this situation is rather similar to quantum field theory when only at weak coupling it can be presented as (infinite) set of harmonic oscillators or free particles. However, the very fact that a theory is formulated as a sum over string world sheets implies completely different from conventional quantum field theory way of summing the states propagating along string loops, which basically means that string theory is not a quantum field theory. The problem of constructing string field theory or second-quantized theory of strings is a closely related issue.

This problem was among main interests of my teacher Vladimir Yakovlevich Fainberg when he has drawn my attention to this, very new at that time, branch of theoretical physics. We started to understand very soon that string theory is essentially first-quantized and that it is strongly related with the properties of two-dimensional geometry responsible for correct counting in string loops. The simplest analog of string path integral is the path integral over particle’s world lines and it can be used for computation of the Green functions of propagating particles, say, Dirac fermions, in external backgrounds [4], being a one-dimensional analog of the Fradkin-Tseytlin effective action [5] in string theory.

The one-dimensional analog of the Polyakov path integral simply results into the proper-time representation of propagators or partition functions of point particles, looking thus to be too complicated formulation of a very simple problem. However, it provides an important geometrical meaning and it is interesting to see methodologically how the well-known answers arise if one starts from one dimensional analog of string and integrate over co-ordinates of particle, extra variables living on a world-line (responsible for "internal" degrees of freedom) and one-dimensional "geometries". Introducing world-line supersymmetry [6] we immediately obtain space-time fermions, but already in this first nontrivial example the integral over one-dimensional "supergravity"
becomes ill-defined due to presence of real world-line fermions, this is a well-known problem, discussed, for example in \[3, 7, 7\]. The problem can be resolved, however, if one thinks of a fermionic particle as of a zero-length limit of fermionic Neveu-Schwarz-Ramond (NSR) string \[4\].

In this note we are going to discuss another side of the same problem which arises in the case when target-space is not just Minkowski or Euclidean space \(\mathbb{R}^D\) but has one or several compact dimensions. The simplest example of this situation – a partition function at finite temperature when one of dimensions is compactified and the target-space is \(\mathbb{R}^{D-1} \times S^1\). It will be shown below that this partition function may be defined "in spirit" of string theory, i.e. starting from one-dimensional (super)geometry though again we run into problems related to the existence of real fermions. Nevertheless these problems can be understood better if one thinks of a point particle as of a "field-theory" limit of string.

## 2 First Quantized Relativistic Bosons and Fermions

The path integral approach to the perturbative string theory \[3\] is quite nontrivial but very direct generalization of the first-quantized representation of quantum field theory as world-line integrals over the trajectories of relativistic particles (and anti-particles). The integral over embeddings of the world-sheets and/or world-lines should be supplemented by integration over two-dimensional and/or one-dimensional geometries (or super-geometries). In string theory the integral over two-dimensional geometries leads immediately to nontrivial consequences for the properties of the target space-time, but in the "model example" of point particles it can be simply reduced to the finite-dimensional integral over the lengths of world lines or what is called the integral over Feynman parameters in quantum field theory.

The simplest illustrative example of this technique is given by path integral representation for the propagator of bosonic spinless particle

\[
G(X_f, X_i) = \int D e^{\int_{X_i}^{X_f} DX e^{-\frac{1}{2}\int \frac{\dot{X}^2}{e} + m^2}} \tag{1}
\]

where integral over one-dimensional geometry is presented by the integral over (square root of) one dimensional metric \(e\) such that proper length of the world-line is \(T = \int_0^T e(t) dt\). Naive integration in \(\int\) first over \(De\)

\[
G(X_f, X_i) \sim \int_{X_i}^{X_f} DX e^{-m\sqrt{\int X^2}} \tag{2}
\]

gives rise to the non Gaussian integral over embeddings \(X = \{X_\mu(t)\}\) only \(\int\) so that the most effective way to deal with integrals like \(\int\) is the opposite – to integrate first over embeddings \(X\) (with boundary conditions \(X(0) = X_1\) and \(X(1) = X_f\)) in the Gaussian form \(\int\) and to use explicit cor-ordinates in the moduli space of one-dimensional geometries (metrics factorized over reparameterizations of world lines) to construct measure \(De\).

This is rather trivial procedure, which can be thought of as a variant of the Faddeev-Popov trick, or, even more simple – a generalization of the finite-dimensional formula of Gaussian integration, the details can be found, for example, in \(\int\) \(\int\) \(\int\). The pretty simple and well-known result is that one can choose a "gauge" \(\int\) \(e(t) = T = \text{const}\) \(T\) is the length of a path) and rewrite integration over \(De\) as an integral over reparameterizations and (one-dimensional) integral over the lengths \(T\). The first one, if measure \(De\) is normalized by dividing onto the volume of reparameterization group, is trivial (nothing depends on reparameterizations of world-lines!) and gives unity, so that, finally, for open trajectories one gets

\[
\int De^{\int_{X_i}^{X_f} DX e^{-\frac{1}{2}\int \frac{\dot{X}^2}{e} + \int T m^2}} = \int_0^\infty dT \int_{X_i}^{X_f} DX e^{-\frac{1}{2}\int \frac{\dot{X}^2}{e} + \int T m^2}} \tag{3}
\]

and performing (now gaussian!) integration over \(X\)’s we finally obtain a well-known formula \(\int\) \(\int\) \(\int\)

\[
\int_0^\infty dT e^{-\frac{1}{2}\int T m^2 - \int \frac{1}{2}(X_f - X_i)^2} \det \left( -\frac{1}{T^2} \frac{d^2}{dt^2} \right)^{-D/2} = \int \frac{d^D p}{(2\pi)^{D/2}} e^{ip(X_f - X_i)} \frac{1}{p^2 + m^2} \tag{4}
\]

\(\footnote{An alternative approach uses the Green-Schwarz target-space fermions from the very beginning and leads to the formalism of superparticle and Green-Schwarz superstring. In contrast to fermionic path integrals, describing the propagation of fermion only, the superparticle contains both boson and fermion, respecting space-time supersymmetry.}

\(\footnote{This is a naive infinite-dimensional analog of the exact formula \(\int\) to be used below.}

\(\footnote{The terminology "gauge" is not, in fact, very strict here since we are integrating over the whole space of metrics on world-line after choosing appropriate co-ordinates. However, we will keep this word through the paper since this is already commonly accepted terminology and since the integral over all metrics is equivalent to the integral over their "classes of equivalence" or lengths \(T\).}
where $D$ is dimension of space-time with (Euclidean) co-ordinates $X_\mu$, $\mu = 1, \ldots, D$.

This pretty simple logic can be generalized to the first-quantized fermions. Indeed, introducing supersymmetry on world line, one immediately comes to the space-time fermion [3], for example in the above sense – computation of the world-line integral with fixed ends gives the Dirac propagator.

The corresponding world-line action may be determined by invariance under world-line supersymmetry transformations

$$\delta X = \epsilon \Psi$$
$$\delta \Psi = -\epsilon \left( \dot{X} + \frac{1}{2} \chi \Psi \right) \Psi^{-1}$$\hspace{1cm} (5)

and contains Grassmann fermionic variables $\Psi_\mu$, which after quantization become gamma-matrices [5], so that the wave function carries now also a space-time fermionic index, since it is a vector in representation of the Clifford algebra.

The path integral representation for the Dirac particle propagator [4] has a similar form to (1)

$$G(X_f, X_i) = \int D\psi D\chi \int_{X_i}^{X_f} DX \int D\psi e^{-\frac{i}{\hbar} \int dt \frac{\delta S}{\delta \chi} + \frac{\delta S}{\delta \psi} + \frac{1}{2} \psi \dot{\chi} + m^2(e + 1/4 \chi d^{-1} \chi)}$$\hspace{1cm} (6)

where one adds the integration over the Grassmann variables $\Psi_\mu$ as well as the integral over one-dimensional geometries $\int D\epsilon$ is replaced by the integral over one-dimensional super-geometries $\int D\epsilon D\chi$. The path integral (6) has the following properties (investigated in detail in [1]):

- The bosonic part is the same as in (1) giving rise to a contribution to the integral identical to l.h.s. of (1).
- The fermionic integral is ill-defined since we deal with real fermions $\Psi_\mu$ and $\chi$.
- Real first-order fermion $\Psi_\mu$ coincides with its own momentum $\delta S / \delta \Psi_\mu$. That leads to appearance of the Clifford algebra after quantization. In the path integral (6) it means that we integrate around the classical trajectory $\Psi_\mu = \gamma_\mu$ with $\gamma_\mu$ to be replace by Dirac gamma-matrices $[\gamma_\mu, \gamma_\nu]_+ = \delta_{\mu\nu}$. More strictly the path integral (6) defines the symbol of the propagator (it is especially clear if one considers sample computations in external fields, see [6] for details).
- The Gaussian integral over the fluctuations $(\Psi_\mu - \gamma_\mu)$ gives rise to the determinant of the first-order differential operator $\det \left( \frac{d\tau}{\tau} \right)^D/2$ which does not depends on metric $e$ on world line or its length $T$. This can be established in several ways, for example, one may compute it via ”discretization”. Another method of computation the same determinant is to assign to the fluctuations the antiperiodic boundary conditions at the end of the world line.
- Like ”metric” $e$ the ”gravitino” $\chi$ by local transformations can be ”gauged” to the form $\chi(t) = X = \text{const}$, where $X$ is now a Grassmannian constant. The Grassmann integral over $X$ gives rise to the most essential contribution into Dirac propagator

$$\int_0^\infty \frac{dT}{T^{1+D/2}} e^{-\frac{1}{4} T m^2 \frac{(X_f - X_i)^2}{2 T}} \int dX e \frac{\delta S}{\delta \chi} = \int_0^\infty \frac{dT}{T^{1+D/2}} e^{-\frac{1}{4} T m^2 \frac{(X_f - X_i)^2}{2 T}} \frac{1}{2 T} (\gamma \cdot (X_f - X_i) + m \gamma_5) = \int \frac{d^D p}{(2\pi)^D/2} e^{\frac{\gamma \cdot p + m \gamma_5}{2T}} \frac{1}{p^2 + m^2}$$\hspace{1cm} (7)

where $\gamma_5$ (to be replaced by product of all Dirac matrices $\gamma_\mu$) can be thought of $[3]$ as a classical value of auxiliary Grassmann variable $\Psi_5$ appearing when one gets rid of nonlocal term in the action in (6).

- The ”illness” of real fermions is seen when one derives the integration measure over $d\chi$ from the ”first principles” of world-line theory. To do this, one needs, for example, to complexify the field $\chi [3][4]$. The only reason for doing this, known to the author, is if one says that the propagator for Dirac particle is a limiting case for the propagator of the open Ramond string $[3][4]$. In string theory worldsheet supergeometry has two gravitino fields $\chi$ and $\bar{\chi}$ (see sect. [3] below) and the integration measure can be defined rigorously.
Apart for the problems with real fermions listed above, the first-quantized technique works quite effective both for bosons and fermions. One can illustrate this, for example, considering generating functionals for particles in external fields (see details and references in [8]) – the one-dimensional analogs of the Fradkin-Tseytlin string effective actions. One adds, for example, exponent of the interaction term with the gauge field

$$\int dt \left( \dot{X}_\mu(t) A_\mu(X(t)) + \frac{e(t)}{2} F_{\mu\nu}(X(t)) \Psi_\mu(t) \Psi_\nu(t) \right)$$

(ordered $P$-exponent in the non-Abelian case) and this generating functional gives rise to all desired results.

In what follows we will see that background terms and especially those exactly given by (8) are extremely important for the theory with compact dimensions. The interaction of string with gauge field is given by the same integral (8) which is taken, in this case, over the one-dimensional boundary of two-dimensional world-sheet. In the first-quantized ideology of string theory one starts with some "internal" degrees of freedom, living on world-lines and world-sheets. In the first-quantized ideology of string theory one starts with some string effective actions. One adds, for example, exponent of the interaction term with the gauge field, which from the point of view of string theory which is then "tuned" by propagating string. If we think of a particle being the zero-length limit of string, the same ideology may be applied directly in the case of integration over world-lines.

In general, exponent of (8) corresponds to nontrivial external gauge field. For vanishing field strength $F_{\mu\nu} = 0$, $A_\mu$ is a pure gauge and the first term in (8) can be integrated out. This is not the case, however, if one allows compact dimensions, then vanishing $F_{\mu\nu}$ still allows nonvanishing "Wilson loops" and the path integral is defined, up to exponent of some phase

$$\int dt \dot{X}_\mu(t) A_\mu(X(t)) = \oint A_\mu dX_\mu$$

For the flat target space one may always choose $A_\mu = const$ and the nonzero phase can be "generated" only for the compact direction, say in the case of $R^{D−1} \times S^1$ only $A_0$ may be nonvanishing (index "0" here and below would correspond to the compact part of $R^{D−1} \times S^1$). Possible arising of nontrivial phase for the first-quantized path integral will be essentially used below and one may even try to interpret this phase as coming from "internal" degrees of freedom, living on world-lines and world-sheets.

3 Free Energy of Point Particles

Let us now turn more closely to the problem we are going to discuss below in detail. In contrast to sect. [8] consider closed trajectories, corresponding to the 1-loop partition functions, which from the point of view of quantum field theory are equal to

$$\pm \frac{1}{2} \log \det(p^2 + m^2)$$

where two different signs correspond either to fermions or to bosons. These determinants have well-known proper-time representations, for example, in bosonic case one can write

$$-\log \det(p^2 + m^2) = -\text{Tr} \log(p^2 + m^2) = - \int \frac{dDp}{(2\pi)^D} \log(p^2 + m^2) =$$

$$= \int_0^\infty \frac{dT}{T} \int \frac{dDp}{(2\pi)^D} e^{-\frac{T}{2m^2}(p^2 + m^2)} = \frac{1}{(2\pi)^{D/2}} \int_0^\infty \frac{dT}{T^{1+\frac{D}{2}}} e^{-\frac{T}{2m^2}}$$

The last expression can be considered as a similar to (8) (first-quantized) path integral representation for a particle,

$$\int DDe X e^{-\frac{T}{2} \int \frac{X^2}{m^2} + e m^2}$$

but now on periodic trajectories i.e. with boundary conditions $X(0) = X(1)$ [8]. The computation can be performed in a standard way, again choosing the "gauge" $e(t) = T = \int_0^1 e(t) dt$, when it is reduced to

$$\int_0^\infty \frac{dT}{T} e^{-\frac{T}{2} m^2} \int DX e^{-\int \frac{X^2}{m^2}}$$

since the integration measure over the metric $De$ on world-line, induced by $\|\delta e\|^2 = \int_0^\infty \left(\frac{\delta e}{e}\right)^2$, gives rise on periodic trajectories to $\int_0^\infty \frac{dT}{T}$ (in contrast to $\int_0^\infty dT$ on open world lines). The integral over $DX$ separates into $X_f = X_i$ with extra integration over this point if compare to the integral over co-ordinates in [8].
the zero-mode part \(dX^{(0)}\) which gives the contribution \(\left(\frac{4\pi}{T}\right)\frac{d^2}{dt^2}\) coming from the reparameterization invariant definition of the integration measure over zero modes, times the volume of target space \(\text{Vol}(D)\), while the rest is again \(\text{det} \left(-\frac{1}{T^2} \frac{d^2}{dt^2}\right)\), to be computed in a standard way. For periodic boundary conditions

\[
\log \text{det} \left(-\frac{1}{T^2} \frac{d^2}{dt^2}\right) = 4 \sum_{n=1}^{\infty} \log \frac{2\pi n}{T} = 2 \log T + \text{const}
\]

(14)
since we have double degenerated eigenvalues \((\frac{2\pi n}{T})^2\) with \(n = 1, 2, \ldots\) Altogether, after normalization to the volume of target space \(\text{Vol}(D)\), it gives the last expression in (11).

If now one considers a particle in space-time with compact directions (for example the so called theories at finite temperature when \(X_0 \sim X_0 + \beta\), one should make a substitution \(\int \frac{dp}{2\pi} \to \frac{1}{\beta} \sum_{n \in \mathbb{Z}}\) when computing the trace in (11), so that

\[
\frac{1}{2} \int \frac{d^D p}{(2\pi)^D} \log(p^2 + m^2) \to \frac{1}{2\beta} \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \log(p^2 + m^2 + \frac{4\pi^2 n^2}{\beta^2}) = \frac{1}{\beta} \int_{\mathbb{R}} \log(1 - e^{-\beta \omega_p})
\]

(15)
up to a divergent constant, including the vacuum energy of the infinite system of oscillators \(\int_{\mathbb{R}} \frac{d\omega}{\omega}\), where \(\omega_p = \sqrt{p^2 + m^2}\) and \(d\omega = \frac{d^{D-1} p}{(2\pi)^{D-1}}\) stays for the integration only over the "space" momenta. In (15) we have used that

\[
\sinh \pi x = \pi x \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right)
\]

(16)
Now the path integral representation (12) gives for this case

\[
\int_0^\infty \frac{dT}{T} e^{-\frac{1}{2} T m^2} \int DX e^{-\int \frac{1}{2} \frac{d^D X}{(2\pi)^D}} \cdot \frac{\beta}{\sqrt{2\pi}} \int_0^\infty \frac{dT}{T^{1+\frac{D}{2}}} e^{-\frac{1}{2} T m^2} \sum_{n \in \mathbb{Z}} e^{-\frac{\beta^2 n^2}{2T}}
\]

(17)
which includes the \(\theta\)-function sum over "wrappings" along the compact \(X_0\)-direction

\[
\theta\left(0 \left| \frac{i\beta^2}{2\pi T}\right.\right) = \sum_{n \in \mathbb{Z}} e^{-\frac{\beta^2 n^2}{2T}}
\]

(18)
and the extra factor \(\beta \sim \int dX^{(0)}\) comes from the integration over the zero mode in compact direction \((\text{Vol}(D) \to \beta \cdot \text{Vol}(D^{-1}))\). This sum can be interpreted as a sum over ensemble of particle (each term corresponds to \(n\) particles, propagating in the loop), thus giving rise to statistic quantity in terms of formally one-particle, first-quantized integral. The result, normalized now onto the volume of "space-dimensional" part \(\text{Vol}(D^{-1})\) only, is free energy \(\beta F\), since

\[
\frac{\beta}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \int_0^\infty \frac{dT}{T^{1+\frac{D}{2}}} e^{-\frac{\beta^2 n^2}{2T}} = \frac{\beta}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \int_{\mathbb{R}} e^{-\frac{\beta^2 \omega_p^2}{2T}} = \frac{\beta}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{1}{\beta n} \left(1 - e^{-\beta \omega_p}\right) = \beta F = \log Z
\]

(19)
In order to present the last computation one should use the integration formula

\[
\int_0^\infty \frac{dT}{\sqrt{T}} e^{-a^2 T - \frac{b^2}{2T}} = \frac{\sqrt{2\pi}}{|a|} e^{-2|ab|}
\]

(20)
which is literally used in (15) after a "modular" transformation \(T \to \frac{1}{T}\). Thus, we have checked that in bosonic case the first-quantized formalism immediately, without any subtleties, like introducing nontrivial backgrounds, leads to correct well-known result.
4 The path integral for fermions

However, the situation is not as simple in the fermionic case, when one should get instead of (19) [19]

$$- \frac{1}{\beta} \int dp \log (1 + e^{-\beta \omega \mu}) = \int dp \sum_{n=1}^{\infty} \frac{e^{-\beta \omega \mu}}{\beta n} e^{i \pi n} \sim \int_0^{\infty} \frac{dT}{T} e^{-\frac{\beta}{2} T m^2} \sum_{n \in \mathbb{Z}} e^{-\frac{\beta\omega^2}{4T} + i \pi n}$$

where the last factor

$$\theta \left( \frac{1}{2} \left| \frac{i \beta^2}{2\pi T} \right| \sum_{n \in \mathbb{Z}} e^{-\frac{\beta\omega^2}{4T} + i \pi n}$$

is the only essential contribution of the world-sheet fermions. In conventional terms of quantum field theory, or better space-time fermions, this effect comes from well-known shift to the Matsubara half-integer frequencies. But let us stress here that this has nothing in common with our discussion based on world-line fermions and below in this section we are going to look at this problem from the perspective of first-quantized formalism.

The fermionic analog of the path integral (12) has the form

$$\int D\chi D\Psi e^{-\frac{i}{\hbar} \int dt \chi \dot{X} + \frac{\Psi}{2} \chi X + m^2 (\epsilon + \frac{1}{4}\chi d t^{-1} \chi)) = \int D\chi D\Psi e^{-\frac{i}{\hbar} \int dt \chi \dot{X} + \frac{\Psi}{2} \chi X + m^2 (\epsilon + \frac{1}{4}\chi d t^{-1} \chi)$$

where integral over $DX$ is again considered over closed trajectories $X(1) = X(0)$. As in the case of propagators considered in sect. [3] the integration is almost identical to the bosonic case and gives

$$\int_0^{\infty} \frac{dT}{T} e^{-\frac{\beta}{2} T m^2} \int DX e^{-\frac{i}{\hbar} \int dt \chi \dot{X}} \int D\Psi e^{-\frac{i}{\hbar} \int dt \chi \dot{\chi}} \int D\chi e^{-\int dt \chi d t^{-1} \chi} = \frac{\text{Vol}(\mathbb{R}^{D-1})}{(2\pi)^{\frac{D-1}{2}}} \cdot \frac{\beta}{\sqrt{2\pi}} \int_0^{\infty} \frac{dT}{T} e^{-\frac{\beta}{2} T m^2} \frac{1}{T} \left( \text{det} d t^{-1} \chi \right) \left( \text{det} d t^{-1} \chi \right) e^{\frac{1}{\hbar} A_{\mu} d X^\mu}$$

The only essential factor, which distinguishes (24) from the bosonic case is the last factor of the type (12) and which can be even thought of as coming from fermionic degrees of freedom $\frac{1}{\hbar} A_{\mu} d X^\mu = \int dt \chi \dot{\Psi} X$, having exactly the meaning of a "theta-term" or interaction with gauge field (12) $A_{\mu} = -\frac{1}{\hbar} \chi \Psi$. But let us stress again that such factor can be the only source for "fermionic anomaly" since the fermionic determinant $\text{det} d t_{\mu}$ does not depends on metric and its extra power in (24) is absolutely inessential. Such gauge potential in the example with target-space $\mathbb{R}^{D-1} \times \mathbb{S}^1$ corresponds to a nontrivial loop in compact direction $X_0 \sim X_0 + \beta$, and it means that in our case one may consider only $A_0 = -\frac{1}{\hbar} \Psi_0$ being nonzero. The bosonic part of the computation is identical to that one considered above while two extra determinants in (24) – the degrees of $d t_{\mu}$ – correspond to the integration over $D\Psi$ and $D\chi$ and, thus, are independent on metric $\epsilon(t) = T$ and/or should be computed with antiperiodic boundary conditions.

Let us now turn directly to the only possible classical and "fermion-dependent" contribution

$$\int A_{\mu} d X^\mu = -\int dt \frac{\chi \Psi_0}{2T} \dot{X}_0$$

coming from the loop in "temperature" $X_0$-direction. First, already from general physical arguments it is clear that the only self-consistent value for this "background loop" is $A_0 = \frac{2\pi i}{\hbar} (K + \frac{1}{2})$ with any integer $K \in \mathbb{Z}$. The corresponding "gauge" symmetry is a (discrete) $\mathbb{Z}_2$-symmetry of changing the fermionic sign and thus one may interpret $A_{\mu} = -\frac{1}{\hbar} \Psi_\mu$ just as a "Wilson-line" background (12). In particular, it is supersymmetric, since

$$\delta \left( \frac{\chi \Psi_\mu}{e} \right) = - \frac{d}{dt} \left( \frac{2\epsilon \Psi_\mu}{e} \right) + \frac{e}{e} \left( 2\dot{\Psi}_\mu - \frac{\chi}{e} \dot{X}_\mu \right)$$

where the first term is total derivative and vanishes on periodic trajectories (even if $\Psi$ is antiperiodic, $\epsilon$ is also antiperiodic and still $\epsilon \Psi$ is periodic) while the second is proportional to the equations of motion and vanishes on classical trajectories.

So, in short, there are few things to be fixed for the computation (24) of the fermionic path integral (23):

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[1] Notice, that the first-quantized path integral gives the logarithm of the partition function (free energy) so that in both bosonic (19) and fermionic (21) cases one gets the sums of infinitely many terms, corresponding to the expansion of logarithms. If the result of the first-quantized computation were the partition functions themselves (without logarithms), then in bosonic case one would still see the sum of infinitely many terms (geometric progression) while in the fermionic case there would be only two contributions of two-level system.
The integration over fluctuations of world-line fermionic fields $\Psi_\mu$ gives no nontrivial contribution to the answer for the temperature partition function, unlike the "Matsubara" target-space fermions. In more conventional string theory terms this is an extra difference between the world-sheet Neveu-Schwarz-Ramond and target-space Green-Schwarz fermions.

The only essential fermionic factor, which can arise, has the form of (9)

$$e^{\int A_\mu dX_\mu}$$

with very special imaginary

$$A_\mu = \delta_{\mu,0} \frac{2\pi i}{\beta} (K + \frac{1}{2}) \quad K \in \mathbb{Z}$$

as usual taking values in the "dual torus" – a circumference with radius proportional to the inverse radius of the compact dimension.

This "vacuum expectation value" can be thought of as coming from world line fermions $A_\mu = -\chi_{2} e^{\Psi_\mu}$. This is a very subtle point, since even on classical trajectories if $\langle \chi \rangle = X \neq 0$ one gets $\psi_\mu \sim X$ and due to Grassmannian nature of fermionic variables their product vanishes, but $\langle \chi \Psi_\mu \rangle \neq 0$, with $A_\mu$ defined by (28). The naive inconsistency of this setting is related, again, to the problem of real fermions (the Grassmann algebra is one-dimensional) we are coming back all the time. Below, in sect. 5, we consider this problem in "regularized" setup of string theory, looking at the fermionic particle integral as at the limit of path integral for the fermionic string.

On more rough level, one may even forget about all complications with the fermionic variables. For example, just say, that any first-quantized path integral in space with compact directions is defined up to the factors like (9) with $F_{\mu\nu} = 0$ but without any other restrictions to $A_\mu$. It means, in particular, that the partition function will get a contribution

$$\theta \left( \alpha \frac{i\beta^2}{2\pi T} \right) = \sum_{n \in \mathbb{Z}} e^{-\frac{\beta^2 n^2}{2T} + 2\pi i n \alpha}$$

of which (18) and (22) are particular cases. The generic factor (29) would correspond to anyon, a particle with fractional statistics determined by arbitrary phase $\alpha \in \mathbb{R}/\mathbb{Z}$. If we consider, however, a simply-connected target-space (like $\mathbb{R}^{D-1}$), and wave-function depends only on position in this target-space, or co-ordinate $X$, and does not "remember" the path – by standard "Landau argument" the only choices for the phase are $\alpha = 0, \frac{1}{2}$, so that from (29) we come back either to (18) or to (22).

So, we have tried to argue in this section that the careful treatment of the fermionic world-line integral gives rise to expected result in the case of compact target-space as well as in the case of usual Minkowski/Euclidean space. However, we again run into a problem with the (integration over) world-line real fermions which has no rigorous resolution without complexification or "raising" the problem to the string level.

5 The fermionic string path integral

Let us, finally, consider the fermionic string path integral (3)

$$\int Dg \ D\chi \ DX \ D\Psi \ \exp \left( -\frac{1}{2\pi \alpha'} \int_{\Sigma} \tilde{\partial}X \partial X + \Psi \tilde{\partial}\Psi + \tilde{\Psi} \partial \Psi + \chi \Psi \tilde{\partial}X + \tilde{\chi} \tilde{\Psi} \partial X + \frac{1}{2} \tilde{\chi} \tilde{\Psi} \Psi \right)$$

where action now is invariant under local two-dimensional supersymmetry (10). In contrast to the particle case (23) action in (30) contains now two cubic "complex conjugated" terms and quartic term $\chi \Psi_\mu \bar{\Psi}_\mu$, containing both fermions and both gravitinos. In the case of open fermionic string one should impose boundary conditions $\Psi = \pm \bar{\Psi}$, $\chi = \pm \bar{\chi}$ on the free boundaries of the world-sheet, and for the Ramond sector (the same sign at all components of the boundary) one gets in the limit $\alpha' \to 0$ the Dirac particle. Due to existence of both $\Psi$ and $\bar{\Psi}$ and two gravitinos there is no problem any more with constructing measure for the fermionic string.

This is a kind of ”two-dimensional regularization” of one-dimensional Grassmann algebra considered in sect. 4. Only at the boundaries of the world-sheet two independent generators become connected by the boundary conditions, but now requirement that $\langle \chi \Psi \rangle \neq 0$ (and $\langle \bar{\chi} \bar{\Psi} \rangle \neq 0$) is absolutely consistent in the
"interior" of $\Sigma$. Reparameterization invariance requires only for these quantities to be (linear combinations of) holomorphic or antiholomorphic differentials, i.e. existence of "vacuum averages"

$$\langle \chi \Psi \rangle = \sum_k a^{(k)}_\mu d\omega_k$$ (31)

and

$$\langle \bar{\chi} \bar{\Psi} \rangle = \sum_k \bar{a}^{(k)}_\mu d\bar{\omega}_k$$ (32)

where we have chosen $\{d\omega_k\}$ to be the set of canonical differentials on $\Sigma$, $\oint_a \delta j, j, k = 1, \ldots, g =$ genus($\Sigma$) and $\{d\bar{\omega}_k\}$ - their complex conjugated.

In the Ramond sector, which directly "regularizes" the fermionic particle case, one can choose $\langle \chi \rangle = \chi'$ and $\langle \bar{\chi} \rangle = \bar{\chi}'$, $\chi' = \pm \chi'$ on the boundaries of the world-sheet, so that quartic term vanishes and the normalization conditions come only from the cubic terms in (31), giving

$$\int_\Sigma \chi \Psi \tilde{\partial} X = \sum k a^{(k)}_\mu R_\mu^{(j)} (\text{Im} T)_{ij}^{-1} \int_\sigma d\omega_k \wedge d\bar{\omega}_i = \sum k a^{(k)}_\mu R_\mu^{(j)} = \text{(half) integer}$$ (33)

together with complex conjugated. In (33) we have introduced $T_{ij} = \oint_i d\omega_j$ – the period matrix of $\Sigma$.

Relation (33) is a direct two-dimensional generalization of (25) giving rise to the same conclusions – the "background loop" takes values into the dual torus to the compactified part of the target-space. This is not much more than we had in sect. 4, and it is enough to compute the one-loop temperature partition function of a superstring in NSR formalism [11]. We see again, that the "vacuum expectation values" $a^{(k)}_\mu$ take values in dual torus and, in complete analogy with the particle case, they should be integer for the bosons, half-integer for the fermions and generic point of a dual torus corresponds to anyons. So, in order to compute the partition function one could "insert by hands" the extra "$Z_2$ factor" for each fermionic sector propagating along the loop in "temperature direction".

This is still not quite satisfactory prescription. On the other hand, setting both $\langle \chi \Psi \rangle$ and $\langle \bar{\chi} \bar{\Psi} \rangle$ nonvanishing on the world-sheet one sees that the quartic term in (30) becomes also nonzero. Moreover, normalization (33) requires that the contribution of the quartic terms to the action is proportional to the inverse powers $\beta^{-1}$ of the radius $\beta$. It means that the string path integral sum over windings respects duality $\beta \leftrightarrow \beta^{-1}$ already before integrating over two-dimensional geometries. Thus, we see that the properties of NSR string path integral in presence of compact dimensions deserve further careful analysis, in particular the questions of fermionic condensates and background Wilson loops are not yet fully understood.

6 Conclusion

In this note we have considered the first-quantized formulation of the temperature partition functions for the bosonic and fermionic particles and a generalization of this path-integral representation to the case of fermionic string. We have seen that the ensemble of particles is generated by wrappings of the world-lines (world-sheets) in "temperature direction".

The first-quantized integrals of this kind may be also applied for the computations of string path integrals in the presence of D-branes (see, for example, [12] and references therein). In particular, it is easy to see that the partition function of strings in the background of D-branes does not differ too much from the old propagator computations [3, 2] and leads, say, for N-branes lead to an integral with $N$ boundaries of the typical form

$$\int \prod_{i=1}^N dX_i \ e^{-\sum_{i<j} (X_i - X_j)^2} \ldots$$ (34)

as an "eigenvalue" matrix integral. It would be interesting to study fermionic string path integrals of this sort, where the main problems are expected in application of the NSR formalism (before GSO projection!) for the computations with D-branes. One may expect all problems considered in this note together with imposing of boundary conditions on the NSR fermionic fields on the surfaces of D-branes.

Finally, let us point out that we have demonstrated in this note that the "thermodynamic" computations in the first-quantized formalism for the fermions, involving nontrivial effects with closed trajectories, do not differ
too much from the old computations in spirit of \cite{1,9}. It reminds the similarity between the two interesting physical outcomes of the first-quantized string computations \cite{2,11}, related to the exponential growth of the density of massive string states. In the "field theory" part this is a modification of causality behavior of the string theory Green functions \cite{2} while in "thermodynamic part" it is related with the famous Hagedorn phase transition (see \cite{11} and references therein). Both effects are different sides of the same coin and our arguments from sect. \cite{11} may serve as an extra sign in favour of appearance of all "thermodynamic" effects directly from the NSR string path integral.

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