Discrete-Time Adaptive Control of a Class of Nonlinear Systems Using High-Order Tuners

Peter A. Fisher and Anuradha M. Annaswamy

Abstract—This paper concerns the adaptive control of a class of discrete-time nonlinear systems with all states accessible. Recently, a high-order tuner algorithm was developed for the minimization of convex loss functions with time-varying regressors in the context of an identification problem. Based on Nesterov’s algorithm, the high-order tuner was shown to guarantee bounded parameter estimation when regressors vary with time, and to lead to accelerated convergence of the tracking error when regressors are constant. In this paper, we apply the high-order tuner to the adaptive control of a particular class of discrete-time nonlinear dynamical systems. First, we show that for plants of this class, the underlying dynamical error model can be causally converted to an algebraic error model. Second, we show that using this algebraic error model, the high-order tuner can be applied to provably stabilize the class of dynamical systems around a reference trajectory.

I. INTRODUCTION

Adaptive control problems take the form of controlling a plant containing unknown parameters, which requires simultaneous online learning and control [1]–[4]. The field is rich with numerous applications and a theoretical history stretching back decades [2]. As autonomous systems become more and more pervasive, there is a growing need for faster learning and faster control. The many approaches to adaptive control that have been developed over the years can be roughly divided into two categories [4]: indirect adaptive control, in which the unknown plant parameters are learned and state feedback is calculated from the estimates; and direct adaptive control, in which the state feedback is directly learned.

Many recent approaches have taken an indirect approach. The approaches in [5], [6] are one illustration of indirect adaptive control of LTI systems, where the unknown parameters are first estimated using a least squares approach followed by a system-level synthesis method to determine the resulting LQR gain $K$. It should be noted that indirect adaptive control has a very rich history prior to [5], [6] as well [1]–[4].

Indirect approaches, however, have the requirement that the initial parameter estimate is sufficiently close, and they require a persistently exciting input so that the parameter estimates converge to their true values – [5], for example, calls for Gaussian noise as input. Direct adaptive control algorithms, on the other hand, determine a control structure wherein the parameters are directly adjusted based on a suitable performance error derived using a reference model.

Often the adaptive laws for adjusting these parameters are based on an error model [1] that leads to a stable adaptive law. The main advantage of this approach over the indirect one is that there is no requirement related to persistent excitation. As one cannot always guarantee that such an excitation is present, and as it is often counter to the system performance goals, this direct approach can be advantageous in many cases. For the most part, the adaptive laws for updating the parameter estimates are based on a gradient descent approach, both in continuous time [1], [2] and discrete-time [4].

We restrict our attention in this paper to the matched uncertainty setting. There has been previous literature on the unmatched uncertainty setting, especially the hybrid MRAC approach in [7], which relies on other methods such as concurrent learning [8] and composite learning [9] to avoid the need for persistent excitation. However, all of the above methods rely on perfect parameter learning, which requires at least a guarantee of finite excitation. Another hybrid MRAC approach is presented in [10] which does not place any assumption on excitation level, but does assume a bounded state and time derivative. Additionally, all of the papers above focus only on continuous time.

Within direct adaptive control, high-order tuners represent a more recent departure from gradient descent-based methods. High-order tuners for adaptive control were first studied in [11]. Within the past few years, a discrete-time high-order tuner was developed in [12] for parameter learning with time-varying regressors. Developed from a well-known theory of 2nd-order gradient algorithms for accelerated convergence [13]–[16], the high-order tuner algorithm in [12] was shown to lead to faster learning than gradient descent-based methods, as well as strong non-asymptotic convergence guarantees for constant regressors. Crucially, the discrete-time high-order tuner is provably stable when regressors vary with time. Its distinct advantage is accelerated convergence of the output error: it has been shown in [12] that when regressors are constant, the high-order tuner has convergence guarantees that are a log factor away from those of Nesterov’s algorithm [15] and that are significantly faster than those of gradient descent algorithms. Additionally, in [17], a continuous-time version of the high-order tuner algorithm was shown in simulation to result in an accelerated convergence of the output error to zero.

The discrete-time high-order tuner discussed above has only been studied in the context of system identification, and has employed algebraic error models for parameter learning. In this paper, we consider the adaptive control problem for
a class of feedback-linearizable dynamical systems whose states are accessible. For this class, we show that the high-order tuner can be applied, leading to global stability and convergence of the underlying tracking error in the state to zero. As in [18], a causal filtering approach based on [19] converts the underlying dynamical error model into an algebraic error model. Unlike in [20]–[24] where the underlying states are filtered as well, the approach used here only generates an augmented error signal, as in [19]. Using this error model and a high-order tuner, we show that the class of dynamic systems can be adaptively controlled in a stable manner (see [18] for a few preliminary results).

The main contribution of this paper is the application of the high-order tuner in [12] to direct adaptive control of a class of feedback-linearizable systems. We prove that this algorithm guarantees global boundedness of the closed-loop adaptive system and asymptotic tracking of a reference model regardless of the level of excitation in the input or the initial parameter estimate. Our proof technique is straightforward and generalizable to a broad class of laws for updating the parameter estimate.

To the authors’ knowledge, ours is the first paper to apply the high-order tuner in [12] to general discrete-time adaptive control. Our paper complements [18], which explores a simplified high-order tuner under noisy disturbances, and [25], which establishes parameter learning for identification problems in discrete-time dynamical systems with persistent excitation.

The paper proceeds as follows. Section II lays out the problem setting, describes the framework by which we convert the dynamical error model to an algebraic error model, and introduces useful notation for the subsequent proofs. Section III provides an illustrative example of our proof technique on a gradient descent-based adaptive law. Section IV presents the main result of our paper: a proof of stability using the high-order tuner in [12] as an adaptive law. Section V presents simulation results of the high-order tuner’s performance on a simple common dynamical system. Finally, Section VI provides concluding remarks. Full proofs are available in the arXiv preprint [26] – we provide only outlines here due to space constraints.

II. PROBLEM SETTING

The problem that we consider in this paper is the adaptive control of the states-accessible plant

\[ x_{p(k+1)} = A_p x_{pk} + B \left( \sum_{i=1}^{p} a_i f_i(x_{pk}) + u_k \right) \]  

(1)

where \( A_p \in \mathbb{R}^{n \times n} \) and all \( a_i \in \mathbb{R}^m \) are unknown, while \( B \in \mathbb{R}^{n \times m} \) and all \( f_i : \mathbb{R}^n \to \mathbb{R} \) are known, subject to the following assumptions:

**Assumption 1**: We assume that

1. the pair \((A_p, B)\) is controllable and all columns of \( B \) are linearly independent, and
2. each \( f_i(\cdot) \) is known and globally \( M_r \)-Lipschitz with \( f_i(0) = 0 \).

The goal is to determine the control input \( u_k \) in real time such that \( x_{pk} \) behaves in a desired manner.

A standard procedure in adaptive control is to choose a reference model

\[ x_{m(k+1)} = A_m x_{mk} + B r_k \]  

(2)

where \( A_m \) is chosen by the control designer to be Schur-stable with the desired closed-loop eigenvalues, and \( r_k \in \mathbb{R}^m \) is a reference input with \( \|r_k\| \leq r_{max} \) chosen such that \( x_{mk} \) follows the desired trajectory of the plant. For realizability, given that \( A_m \) is chosen without a priori knowledge of \( A_p \), the following matching condition is a standard assumption employed in adaptive control:

**Assumption 2**: We assume that there exists some \( K_* \in \mathbb{R}^{m \times n} \) such that

\[ A_m = A_p + BK_* \]  

(3)

If all parameters were known, choosing \( u_k \) of the form

\[ u_k = K_* x_{pk} + r_k - \sum_{i=1}^{p} a_i f_i(x_{pk}), \]  

(4)
a feedback linearizing controller, would ensure that the closed-loop plant response follows the same trajectory as the reference model.

It is well-known that an adaptive control input of the form

\[ u_k = \hat{K}_k x_{pk} + r_k - \sum_{i=1}^{p} \hat{a}_i f_i(x_{pk}), \]  

(5)

where \( \hat{K}_k \) and \( \hat{a}_i \) are estimates of the unknown parameters \( K_* \) and \( a_i \) in (4), can guarantee that the state error

\[ e_k := x_{pk} - x_{mk} \]  

(6)

corverges to zero if the estimates are suitably adjusted using an adaptive law [1], [4].

In the remainder of this section, we propose a new general algorithm for this adaptive control problem, based on results in [19] for system identification. In Section IV, we then propose the addition of the high-order tuner developed in [12], [25] as a particular adaptive law.

The certainty equivalence input in (5) is equivalent to

\[ u_k = K_* x_{pk} + r_k - \sum_{i=1}^{p} a_i f_i(x_{pk}) + \hat{\Theta}_k \phi_k \]  

(7)

where

\[ \hat{\Theta}_k := [\hat{K}_k, \hat{a}_{1k}, \ldots, \hat{a}_{pk}] \]  

(8)

\[ \Theta_* := [K_*, a_{1}, \ldots, a_p] \]  

(9)

\[ \hat{\Theta}_k := \hat{\Theta}_k - \Theta_* \]  

(10)

\[ \phi_k^\top := [x_{pk}^\top, -f_1(x_{pk}), \ldots, -f_p(x_{pk})]. \]  

(11)

The closed-loop adaptive system for the plant in (1) with the controller in (7) is thus described by

\[ x_{p(k+1)} = A_m x_{pk} + B r_k + B \hat{\Theta}_k \phi_k. \]  

(12)

It is easy to see that (12), (2), and (6) yield the error model

\[ e_{k+1} = A_m e_k + B \hat{\Theta}_k \phi_k. \]  

(13)
A. An Equivalent Algebraic Error Model

Equation (13) is a dynamical error model, as it relates the two main errors, the state error \( e_k \) and the parameter error \( \Theta_k \), through a dynamical model. Our approach based on [19] transforms this problem into an algebraic error model of the form

\[
\varepsilon_{k+1} = \tilde{\Theta}_k \phi_k
\]

(14)

where [19]

\[
\varepsilon_{k+1} := (B^T B)^{-1} B^T (e_{k+1} - A_m e_k).
\]

(15)

It should be noted that the prediction error \( \varepsilon_{k+1} \) is another performance metric that depends on the state error through the relation

\[
e_{k+1} = A_m e_k + B \varepsilon_{k+1}.
\]

(16)

It should also be noted that a causal adaptive law can be derived for adjusting the parameter estimate \( \tilde{\Theta}_k \) defined in (8) by first measuring the state and the state errors on the right-hand side of (15) and then updating the parameter estimate. This overall algorithm is summarized in Algorithm 1 [18]. In line 11, ADAPT refers to any iterative algorithm for updating \( \tilde{\Theta}_k \).

As we shall show in the following sections, update laws based on the gradient of a loss function - in particular, normalized gradient descent and the high-order tuner - lead to global stability, regardless of the level of excitation in the input or the initial parameter estimate \( \tilde{\Theta}_0 \).

B. Preliminaries

In this section, we first review a well-known result pertaining to the Lyapunov stability of linear time-invariant systems:

**Proposition 1** ([27]): For any matrix \( A \in \mathbb{R}^{n \times n} \), the following conditions are equivalent:

1. \( A \) is Schur-stable, i.e. all eigenvalues of \( A \) are inside the unit circle.
2. For every SPD matrix \( Q \), there exists a unique SPD matrix \( P \) satisfying the discrete Lyapunov equation

\[
A^T P A - P = -Q.
\]

(17)

Finally, we provide a useful result pertaining to the relative growth rates of \( x_{pk} \) and \( \phi_k \):

**Lemma 1**: Let \( \phi_k \) be the regressor defined in (11). Then, under Assumption 1, there exists a known constant \( C > 0 \) such that \( \| \phi_k \| \leq C \| x_{pk} \| \).

We omit the proof, as it is fairly straightforward.

III. A FIRST-ORDER APPROACH TO ADAPTIVE CONTROL

The algebraic error model in (14) lends itself easily to a loss function given by [12]

\[
L_k(\tilde{\Theta}_k) = \frac{1}{2} \| \varepsilon_{k+1} \|^2 = \frac{1}{2} \| \tilde{\Theta}_k \phi_k \|^2.
\]

(18)

Using (14), the gradient of the loss function can then be calculated as

\[
\nabla L_k(\tilde{\Theta}_k) = \tilde{\Theta}_k \phi_k \phi_k^T = \varepsilon_{k+1} \phi_k^T.
\]

(19)

It is easy to see that \( L_k(\tilde{\Theta}_k) \) is non-strongly convex and has a time-varying smoothness parameter of \( \| \phi_k \|^2 \). We therefore use a normalized loss function given by [12]

\[
\tilde{J}_k(\tilde{\Theta}_k) = \frac{L_k(\tilde{\Theta}_k)}{N_k}
\]

(20)

with the normalization term

\[
N_k = \max \{ \mu, \| \phi_k \|^2 \}
\]

(21)

for some \( \mu > 0 \). It is thus apparent that \( \tilde{J}_k(\tilde{\Theta}_k) \) is convex and 1-smooth, with a gradient that can be calculated as

\[
\nabla \tilde{J}_k(\tilde{\Theta}_k) = \frac{1}{N_k} \varepsilon_{k+1} \phi_k^T.
\]

(22)

The problem of minimizing the loss function in (20) leads naturally to the multivariable form of the well-known normalized gradient descent adaptive law given by [4]

\[
\tilde{\Theta}_{k+1} = \tilde{\Theta}_k - \gamma \nabla \tilde{J}_k(\tilde{\Theta}_k).
\]

(23)

A. Stability of the Gradient Descent Adaptive Law

We now show that Algorithm 1 with (21)-(23) in place of ADAPT on line 11 is a globally stable adaptive controller. The first step is to quantify the evolution of the parameter error, which is addressed in Proposition 2.

**Proposition 2**: The adaptive law in (21)-(23) results in a bounded parameter error \( \tilde{\Theta}_k \) for all \( k \) if \( \mu > 0 \) and \( 0 < \gamma < 2 \) with

\[
V_k = \| \tilde{\Theta}_k \|^2
\]

(24)

as a Lyapunov function.

**Proof**: Define \( \Delta V_k = V_{k+1} - V_k \). Using (22)-(23) and the fact that \( N_k \geq \| \phi_k \|^2 \), one can show by manipulating the Lyapunov increment that

\[
\Delta V_k \leq -\gamma(2 - \gamma) \| \varepsilon_{k+1} \|^2 \leq 0
\]

(25)

if \( 0 < \gamma < 2 \). See Appendix A in [26] for details.

We now prove the convergence of the overall closed loop adaptive system described by the error model in (13):
Theorem 1: The closed-loop adaptive system defined by (1), (5), (13), (15), and (21)-(23) with \( \mu > 0 \) and \( 0 < \gamma < 2 \) results in \( \lim_{k \to \infty} \| e_k \| = 0 \).

Proof: Because \( V_k \geq 0 \) and \( \Delta V_k \leq 0 \), we know that \( \Delta V_k \to 0 \) as \( k \to \infty \). Therefore, we have \( \lim_{k \to \infty} \frac{\| x_{k+1} \|_N^2}{N} = 0 \), which implies that either (1) \( e_k \to 0 \) or (2) \( \| x_{k+1} \| = 0 \).

Remark 1: Proposition 2 and Theorem 1 are both well-known in the existing literature – see e.g. [28]. We include both proofs in order to make apparent the similarity with the high-order tuner algorithm in the next section.

Remark 2: Theorem 1 together with Proposition 2 extends the results of [19] to show that the approach based on an algebraic error model can be used to guarantee closed-loop stability for discrete-time adaptive control.

Another interesting point to note is the generality of the proof. As will become evident from our discussions in the next section, the same method of proof can be employed for any adaptive law that can guarantee the property \( \lim_{k \to \infty} \frac{\| x_{k+1} \|_N^2}{N} = 0 \). In this case, this property followed from the structure of the Lyapunov increment (see (25)).

IV. ADAPTIVE CONTROL WITH A HIGH-ORDER TUNER

We now state the main result of this paper. For the plant given in (1), we propose the adaptive controller given in Algorithm 1, with Algorithm 2 in place of ADAPT in line 11. Algorithm 2 summarizes the high-order tuner adaptive law [12].

The specific updates that constitute the high-order tuner are given by

\[
\begin{align*}
\hat{x}_{k+1} &= \hat{x}_k - \gamma \nabla f_k (\hat{x}_{k+1}) \\
\tilde{\Theta}_{k+1} &= \Theta_k - \beta (\tilde{\Theta}_k - \hat{\Theta}_k) \\
\Theta_{k+1} &= \Theta_k - \beta (\Theta_k - \hat{x}_k) \\
\hat{\Theta}_{k+1} &= \hat{x}_k - \gamma \nabla f_k (\hat{x}_{k+1})
\end{align*}
\]

where \( \hat{x}_{k+1} \) is an auxiliary parameter estimate, \( \nabla f_k (\hat{x}_{k+1}) \) and \( \nabla f_k (\hat{x}_{k+1}) = \nabla f_k (\hat{x}_{k+1}) / N_k \), and the gradients of \( L_k \) are given by

\[
\begin{align*}
\nabla L_k (\hat{x}_k) &= \Theta_k \phi_k \phi_k^T = \hat{x}_{k+1} \phi_k^T \\
\nabla L_k (\hat{x}_{k+1}) &= \Theta_k \phi_k \phi_k^T = \hat{x}_{k+1} \phi_k^T \\
\nabla L_k (\hat{x}_{k+1}) &= \Theta_k \phi_k \phi_k^T = \hat{x}_{k+1} \phi_k^T
\end{align*}
\]

When the regressors are constants (i.e. \( \phi_k = \phi = \text{constant} \)), (26)-(28) reduce to Nesterov’s algorithm [12],

\[
\Theta_{k+1} = \Theta_k + \beta (\Theta_k - \hat{x}_{k+1}) - \gamma \nabla L_k (\Theta_k + \beta (\Theta_k - \hat{x}_{k+1}))
\]

where

\[
\beta = 1 - \beta, \quad \gamma = \gamma / \beta.
\]

(26)-(28) are therefore a high-order counterpart to the adaptive law in (23), and include momentum components (second term in (31)) and acceleration components (third term in (31)).

A. Stability of the High-Order Tuner Adaptive Law

As before, we first quantify the evolution of the parameter error \( \hat{x}_k \) and the auxiliary parameter estimate \( \hat{x}_k \) in the following proposition.

Proposition 3: The adaptive law in (21) and (26)-(30) results in a bounded parameter error \( \hat{x}_k \) and a bounded auxiliary parameter estimate \( \hat{x}_k \) for all \( k \leq N_k \) as \( \mu > 0 \), \( 0 < \beta < 2 \), \( 0 < \gamma < \sqrt{\frac{2-\beta}{\beta}} \), and \( \alpha > 0 \) as defined in (35) with

\[
V_k = \| \hat{x}_{k+1} - \Theta_k \|_F^2 + \| \hat{x}_k - \hat{x}_{k+1} \|_F^2
\]

as a Lyapunov function.

Proof: Define \( \Delta V_k = V_{k+1} - V_k \). Using (26)-(30) and the fact that \( N_k \geq \| \phi_k \|_F^2 \), one can show by manipulating the Lyapunov increment that

\[
\Delta V_k \leq -\gamma \alpha (1 - \gamma / 2) (1 - \frac{\alpha}{2 + \alpha}) \| \hat{x}_{k+1} \|_F^2 / N_k
\]

where

\[
\alpha = 2(1 - \gamma) - \frac{\gamma (2 - 3 \beta)^2}{\beta (2 - (1 + \gamma^2) \beta)}
\]

if \( 0 < \beta < 2 \) and \( 0 < \gamma < \sqrt{\frac{2-\beta}{\beta}} \). It follows that \( \Delta V_k \leq 0 \) if \( \alpha > 0 \). See Appendix C in [26] for details.

The main result of this paper, that the high-order tuner algorithm accomplishes the control objective of \( \| e_k \| \to 0 \) as \( k \to \infty \), is now given in the following theorem.

Theorem 2: For the plant given in (1), Algorithm 1 with Algorithm 2 as ADAPT and \( \mu > 0 \), \( 0 < \beta < 2 \), \( 0 < \gamma < \sqrt{\frac{2-\beta}{\beta}} \), and \( \alpha > 0 \) as defined in (35) results in \( \lim_{k \to \infty} \| e_k \| = 0 \).
Proof: The proof proceeds identically to the proof of Theorem 1. The crucial ingredient is the fact that the Lyapunov increment takes the form in (34), and thus the high-order tuner also guarantees \( \lim_{k \to \infty} \frac{\|e_{k+1}\|^2}{N_k} = 0 \). See Appendix D in [26] for details.

Remark 3: In Proposition 3 and Theorem 2 (as in Proposition 2 and Theorem 1), we make no assumptions on the level of excitation in the input or on the initial parameter estimate \( \hat{\theta}_0 \). Therefore, this adaptive law can be applied with any bounded input \( \{r_k\}_{k \geq 0} \) and any initial parameter estimate.

V. SIMULATION RESULTS

To show that the high-order tuner achieves state tracking performance that is comparable to or better than that of gradient descent, we conducted simulations of a simple plant as in (1) using Algorithm 1. We compare the results using (21)-(23) and Algorithm 2 as ADAPT in line 11.

A. Simulation Details

As in [18], the numerical experiments were conducted using the linearized short-period dynamics of a transport aircraft flying at a low altitude at 250 ft/s, taken from Exercise 1.2 in [29]. We add an integral error state \( \dot{q}_e = q - r \) so that the pitch rate tracks a command signal \( r \) with zero steady-state error, assume a discrete-time controller with a 100 Hz sampling rate, and discretize the resulting dynamics using a zero-order hold to obtain the nominal discrete-time dynamics

\[
x_{p(k+1)} = Ax_{pk} + bu_k + b_r r_k
\]

where \( x_{pk} = [\alpha, q, q_e]^\top \). Further details can be found in [18].

We considered the reference model

\[
x_{m(k+1)} = A_m x_{mk} + b_r r_k = (A + b b_{LQR}^\top) x_{mk} + b_r r_k
\]

where \( \theta_{LQR} \) is the gain matrix obtained from LQR on the nominal discrete-time dynamics with cost matrices \( Q = \text{diag}([0, 0, 1]) \) and \( R = 1 \). We then assumed a parametric uncertainty such that \( A_p = A_m - b \theta_{LQR}^\top \) for some unknown \( k \), and applied the certainty equivalence control input

\[
u_k = \tilde{\theta}_k^\top x_{pk}
\]

with \( \tilde{\theta}_0 = \theta_{LQR} \). The resulting error model was given by

\[
e_{k+1} = A_m e_k + b \tilde{\theta}_k^\top \phi_k = A_m e_k + b e_{k+1}
\]

with \( \phi_k = x_{pk} \).

B. Results and Discussion

A Monte Carlo simulation was conducted with 2000 trials. In each trial, \( \theta_0 \) was obtained by multiplying each element of \( \theta_{LQR} \) by an i.i.d random value uniformly distributed over \([-0.5, 2]\). The adaptive control task was to track the reference model with \( r_k = 5 \) \( \forall k \geq 0 \) starting from \( x_{p0} = x_{m0} = 0 \). For simplicity, we chose \( \mu = 1 \). Hyperparameter tuning was carried out to ensure the fastest possible convergence of \( \|e_k\| \) to zero. For gradient descent we chose \( \gamma = 1 \), corresponding to the well-known projection algorithm [4]. For the high-order tuner, we found that choosing \( \gamma \) as large as possible for any given \( \beta \) (see Appendix E in [26]) led to fastest reduction of both \( \|e_k\| \) and \( |e_k| \).

Figures 1 and 2 show the results of our simulations. Solid lines are mean values over all trials, and the darker and lighter windows around them are 50% and 90% confidence intervals, respectively. We find that under both performance metrics, the high-order tuner performs comparably to gradient descent. Intriguingly, however, the high-order tuner tends to produce slightly larger values of \( |e_k| \) and slightly smaller values of \( \|e_k\| \). This result runs counter to the intuition provided by (16).

Further research is needed to understand this result, as well as to understand why it appears best to choose \( \gamma \) as large as possible for any given \( \beta \). One possible source of intuition
may be the reduction to Nesterov’s algorithm in (31)-(32). It is possible that having an extra hyperparameter allows the adaptive law to be somewhat tuned to the particular dynamical system. It is also possible that there exists another Lyapunov function besides the one in (33), which could provide more clarity.

VI. CONCLUSIONS AND FUTURE WORKS
In this paper, we present a novel algorithm for model-reference adaptive control of the class of nonlinear systems given in (1). This algorithm uses a causal filtering method to convert the resulting dynamical error model into an algebraic error model and applies the discrete-time high-order tuner presented in [12] as the adaptive law. Crucially, the algorithm is shown to guarantee that $\|e_k\| \to 0$ as $k \to \infty$ using a simple and general proof method for systems with all states accessible. In this proof, we make no assumptions on the initial parameter estimate or the amount of excitation in the input.

We also provide simulation results showing that the high-order tuner achieves comparable or slightly better performance than the standard gradient descent-based adaptive law. Simulations in [17] have shown accelerated convergence of the state error $e_k$ to zero using a continuous-time equivalent of the high-order tuner algorithm. Further research is needed to understand the influence of the choice of gains on the performance of the high-order tuner, and to explore how this accelerated convergence might be realized in discrete time.

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