Different degrees of non-compactness for optimal Sobolev embeddings

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Abstract. The structure of non-compactness of optimal Sobolev embeddings of $m$-th order into the class of Lebesgue spaces and into that of all rearrangement-invariant function spaces is quantitatively studied. Sharp two-sided estimates of Bernstein numbers of such embeddings are obtained. It is shown that, whereas the optimal Sobolev embedding within the class of Lebesgue spaces is finitely strictly singular, the optimal Sobolev embedding in the class of all rearrangement-invariant function spaces is not even strictly singular.

1. Introduction

Sobolev spaces and their embeddings into Lebesgue or Lorentz spaces (on an open set $\Omega \subseteq \mathbb{R}^d$) keep a prominent position in the theory of partial differential equations, and any information about structure of such embeddings is far-reaching.

There is a vast amount of literature devoted to study of conditions under which Sobolev embeddings are compact. Quality of compactness is often studied by the speed of decay of different $s$-numbers, which is connected to spectral theory of corresponding differential operators and provides estimates of the growth of their eigenvalues (see [11]). However, much less literature is devoted to study of the structure of non-compact Sobolev embeddings, which is related to the shape of essential spectrum (see [9]).

There are three common ways under which Sobolev embeddings can become non-compact:

(i) when the underlying domain is unbounded (see [1], cf. [12]);
(ii) when the boundary $\partial \Omega$ of $\Omega$ is too irregular (see [16, 17, 24, 25]);
(iii) when the target space is optimal or “almost optimal” (see [15, 20] and references therein).

In this paper we will focus on the third case. We will obtain new information about the structure of non-compactness of two optimal Sobolev

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embeddings—namely
\[ I : V_0^{m,p}(\Omega) \to L^{p^*}(\Omega) \]  
(1.1)
and
\[ I : V_0^{m,p}(\Omega) \to L^{p^*,p}(\Omega) \]
(1.2)
where \(1 \leq m < d\), \(p \in [1, d/m)\) and \(p^* = dp/(d - mp)\). Here \(\Omega\) is a bounded open set, and the subscript 0 means that the (ir)regularity of \(\Omega\) is immaterial (see Section 2 for precise definitions). The target spaces in both embeddings are in a sense optimal. The Lebesgue space \(L^p(\Omega)\) is well known to be the optimal target space in (1.1) among all Lebesgue spaces—that is, \(L^p(\Omega)\) is the smallest Lebesgue space \(L^q(\Omega)\) such that \(I : V_0^{m,p}(\Omega) \to L^q(\Omega)\) is valid. However, it is also well known ([26]) that (1.1) can be improved to (1.2) if one allows not only Lebesgue spaces but also Lorentz spaces, which form a richer class of function spaces. Since \(L^{p^*,p}(\Omega) \subsetneq L^{p^*}(\Omega)\), the latter is indeed an improvement. Furthermore, the Lorentz space \(L^{p^*,p}(\Omega)\) is actually the optimal target space in (1.2) among all rearrangement-invariant function spaces (see [14])—that is, if \(Y(\Omega)\) is a rearrangement-invariant function space (e.g., a Lebesgue space, a Lorentz space, or an Orlicz space, to name a few customary examples) such that \(I : V_0^{m,p}(\Omega) \to Y(\Omega)\) is valid, then \(L^{p^*,p}(\Omega) \subseteq Y(\Omega)\).

Not only are both embeddings (1.1) and (1.2) non-compact, but they are also in a sense “maximally non-compact” as their measures of non-compactness (in the sense of [9, Definition 2.7]) are equal to their norms. This was proved in [5, 13]. Moreover, even when \(L^{p^*}(\Omega)\) is enlarged to the weak Lebesgue space \(L^{p^*,\infty}(\Omega)\), which satisfies \(L^{p^*,p}(\Omega) \subsetneq L^{p^*}(\Omega) \subseteq L^{p^*,\infty}(\Omega)\), the resulting Sobolev embedding is still maximally non-compact. This was proved in [13]. These results may suggest that the “quality” and the structure of these non-compact embeddings should be the same.

However, there are other possible points of view on the quality of non-compactness. One of them is the question of whether a non-compact Sobolev embedding is strictly singular or even finitely strictly singular. Strictly singular operators and finitely strictly singular ones are important classes of operators as spectral properties of such operators are very close to those of compact ones. In this regard, it follows from [6] that the Sobolev embedding \(I : V_0^{1,1}(\Omega) \to L^{d/(d-1)}(\Omega)\), which is a particular case of (1.1) with \(m = p = 1\), is finitely strictly singular. Furthermore, it was also shown there that the almost optimal critical Sobolev embedding \(I : V_0^{d,1}((0, 1)^d) \to L^{\infty}((0, 1)^d)\) is finitely strictly singular, too. Finally, the same was proved in [18] for the optimal first-order Sobolev embedding into the space of continuous functions on a cube. These results suggest a hypothesis that non-compact Sobolev embeddings could be finitely strictly singular or at least strictly singular.

In this paper we will show that this hypothesis is correct for the “almost optimal” Sobolev embedding (1.1), but it is wrong for the “really optimal” Sobolev embedding (1.2). In other words, (1.2) is an example of a Sobolev
embedding whose target space is optimal among all rearrangement-invariant function spaces that is not a singular map (i.e., there exists an infinite dimensional subspace on which the embedding is invertible), but if the target space is slightly enlarged to an “almost optimal” one (i.e., the target space is optimal only in the smaller class of Lebesgue spaces), then the resulting Sobolev embedding (1.1) is finitely strictly singular (i.e., its Bernstein numbers are decaying to zero). In the case of (1.1), we prove a two-sided estimate of the Bernstein numbers corresponding to the embedding—the estimate is sharp up to multiplicative constants. In the case of (1.2), we show that all its Bernstein numbers coincide with the norm of the embedding.

The paper is structured as follows. In the next section, we recall definitions and notation used in this paper, as well as some background results. In Section 3, we start with a couple of auxiliary results, which may be of independent interest, then we focus on the “almost optimal” embedding (Theorem 3.3), and finally on the “really optimal” one (Theorem 3.4).

2. Preliminaries

Here we establish the notation used in this paper, and recall some basic definitions and auxiliary results.

Any rule \( s: T \rightarrow \{s_n(T)\}_{n=1}^{\infty} \) that assigns each bounded linear operator \( T \) from a Banach space \( X \) to a Banach space \( Y \) (we shall write \( T \in B(X,Y) \)) a sequence \( \{s_n(T)\}_{n=1}^{\infty} \) of nonnegative numbers having, for every \( n \in \mathbb{N} \), the following properties:

- (S1) \( ||T|| = s_1(T) \geq s_2(T) \geq \cdots \geq 0 \);
- (S2) \( s_n(S + T) \leq s_n(S) + ||T|| \) for every \( S \in B(X,Y) \);
- (S3) \( s_n(BTA) \leq \|B\|s_n(T)\|A\| \) for every \( A \in B(W,X) \) and \( B \in B(Y,Z) \), where \( W, Z \) are Banach spaces;
- (S4) \( s_n(I: E \rightarrow E) = 1 \) for every Banach space \( E \) with \( \dim E \geq n \);
- (S5) \( s_n(T) = 0 \) if \( \text{rank } T < n \);

is called a strict \( s \)-number. Notable examples of strict \( s \)-numbers are the approximation numbers \( a_n \), the Bernstein numbers \( b_n \), the Gelfand numbers \( c_n \), the Kolmogorov numbers \( d_n \), the isomorphism numbers \( i_n \), or the Mityagin numbers \( m_n \). For their definitions and the difference between strict \( s \)-numbers and ‘non-strict’ \( s \)-numbers, we refer the reader to [10, Chapter 5] and references therein. We say that a (strict) \( s \)-number is injective if the values of \( s_n(T) \) do not depend on the codomain of \( T \). More precisely, \( s_n(J_N^Y \circ T) = s_n(T) \) for every closed subspace \( N \subseteq Y \) and every \( T \in B(X,N) \), where \( J_N^Y: N \rightarrow Y \) is the canonical embedding operator.

In this paper, we will only need the definition of the Bernstein numbers. The \( n \)-th Bernstein number \( b_n(T) \) of \( T \in B(X,Y) \) is defined as

\[
b_n(T) = \sup_{X_n \subseteq X} \inf_{\|x\|_X = 1} \|Tx\|_Y,\]

where \( X_n \subseteq X \) is a finite-dimensional subspace of \( X \).
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where the supremum extends over all \(n\)-dimensional subspaces of \(X\). The Bernstein numbers are the smallest injective strict \(s\)-numbers ([28, Theorem 4.6]), that is,

\[
b_n(T) \leq s_n(T) \tag{2.1}
\]

for every injective strict \(s\)-number \(s\), for every \(T \in B(X,Y)\), and for every \(n \in \mathbb{N}\).

An operator \(T \in B(X,Y)\) is said to be strictly singular if there is no infinite dimensional closed subspace \(Z\) of \(X\) such that the restriction \(T|_Z\) of \(T\) to \(Z\) is an isomorphism of \(Z\) onto \(T(Z)\). Equivalently, for each infinite dimensional (closed) subspace \(Z\) of \(X\),

\[
\inf \{\|Tx\|_Y : \|x\|_X = 1, x \in Z\} = 0.
\]

An operator \(T \in B(X,Y)\) is said to be finitely strictly singular if it has the property that given any \(\varepsilon > 0\) there exists \(N(\varepsilon) \in \mathbb{N}\) such that if \(E\) is a subspace of \(X\) with \(\dim E \geq N(\varepsilon)\), then there exists \(x \in E, \|x\|_X = 1\), such that \(\|Tx\|_Y \leq \varepsilon\). This can be expressed in terms of the Bernstein numbers of \(T\). The operator \(T\) is finitely strictly singular if and only if

\[
\lim_{n \to \infty} b_n(T) = 0.
\]

The relations between these two notions and that of compactness of \(T\) are illustrated by the following diagram:

\(T\) is compact \(\implies\) \(T\) is finitely strictly singular \(\implies\) \(T\) is strictly singular; moreover, each reverse implication is false in general. For further details and general background information concerning these matters we refer the interested reader to [2], [21] and [29].

Throughout the rest of this section, \(X\) denotes a Banach space. The operator norm of the projection \(Q : L^2([0,1],X) \to L^2([0,1],X)\) defined as

\[
Qf(t) = \sum_{j=1}^{\infty} \left( \int_0^1 f(s) r_j(s) \, ds \right) r_j(t), \quad f \in L^2([0,1],X),
\]

is called the \(K\)-convexity constant of \(X\). Here \(\{r_j\}_{j=1}^{\infty}\) are the Rademacher functions. The \(K\)-convexity constant of \(X\) is denoted by \(K(X)\). If \(\dim X = n\), then (e.g., see [3, Theorem 6.2.4])

\[
K(X) \leq c \log(1 + d(X,\ell^n_2)). \tag{2.2}
\]

Here \(c\) is an absolute constant and \(d(X,\ell^n_2)\) is the Banach–Mazur distance, that is,

\[
d(X,\ell^n_2) = \inf \{\|T\|\|T^{-1}\| : T \text{ is a linear isomorphism of } X \text{ onto } \ell^n_2\}.
\]

We say that \(X\) is of cotype 2 if there is a constant \(\gamma\) such that

\[
\left( \sum_{j=1}^{m} \|x_j\|_X^2 \right)^{1/2} \leq \gamma \int_0^1 \left\| \sum_{j=1}^{m} x_j r_j(t) \right\|_X \, dt
\]

for every \(\{x_j\}_{j=1}^{m} \subseteq X, m \in \mathbb{N}\). We denote the least such a \(\gamma\) by \(C_2(X)\).
Let $Y$ be a Banach space such that $X \subseteq Y$. We say that the inclusion is \textit{2-absolutely summable} if there is a constant $\gamma$ such that

$$
\left( \sum_{j=1}^{m} \|x_j\|_{Y}^{2} \right)^{1/2} \leq \gamma \sup \left\{ \left( \sum_{j=1}^{m} |x^*(x_j)|^{2} \right)^{1/2} : \|x^*\|_{X^*} \leq 1 \right\}
$$

for every $\{x_j\}_{j=1}^{m} \subseteq X$, $m \in \mathbb{N}$. We denote the least such a $\gamma$ by $\pi_2(X \hookrightarrow Y)$.

Let $A, B$ be subsets of $X$. We denote the minimum number of points $x_1, \ldots, x_m \in X$ such that

$$
A \subseteq \bigcup_{j=1}^{m} (x_j + B)
$$

by $E(A, B)$. In general, it may happen that $E(A, B) = \infty$, but in our case it will always be a finite number. $E(A, B)$ denotes the minimum number of points $x_1, \ldots, x_m \in A$ such that (2.3) holds.

Let $(R, \mu)$ be a nonatomic measure space and $p \in [1, \infty)$. As usual, $L^p(R, \mu)$ denotes the \textit{Lebesgue space} endowed with the norm

$$
\|f\|_{L^p(R, \mu)} = \left( \int_{R} |f|^p \, d\mu \right)^{1/p}, \quad f \in L^p(R, \mu).
$$

Let $q \in [1, p]$. The \textit{Lorentz space} $L^{p,q}(R, \mu)$ is the Banach space of all $\mu$-measurable functions $f$ in $R$ for which the functional

$$
\|f\|_{L^{p,q}(R, \mu)} = \left( \int_{0}^{\infty} t^{\frac{q}{p}-1} f^*(t)^q \, dt \right)^{\frac{1}{q}}
$$

is finite—the norm on $L^{p,q}(R, \mu)$ is given by the functional. The function $f^* : (0, \infty) \to [0, \infty]$ is the (right-continuous) \textit{nonincreasing rearrangement} of $f$, that is,

$$
f^*(t) = \inf \{ \lambda > 0 : \mu(\{x \in R : |f(x)| > \lambda\}) \leq t \}, \quad t \in (0, \infty).
$$

Note that $f^*(t) = 0$ for every $t \in [\mu(R), \infty)$. Furthermore, we have (see [1, Chapter 2, Proposition 1.8])

$$
\|f\|_{L^{p,q}(R, \mu)} \leq \|f\|_{L^{p}(R, \mu)}.
$$

When $R \subseteq \mathbb{R}^d$ and $\mu$ is the $d$-dimensional Lebesgue measure, we write $L^p(R)$ and $L^{p,q}(R)$ instead of $L^p(R, \mu)$ and $L^{p,q}(R, \mu)$, respectively, and $|R|$ instead of $\mu(R)$ for short. We refer the interested reader to [27, Chapter 8] for more information on Lorentz spaces. Assume that $(R, \mu)$ is probabilistic. We denote by $L^{\psi_2}(R, \mu)$ the \textit{Orlicz space} generated by the Young function

$$
\psi_2(t) = \exp(t^2) - 1, \quad t \in [0, \infty).
$$

The norm on $L^{\psi_2}(R, \mu)$ is given by

$$
\|f\|_{L^{\psi_2}(R, \mu)} = \inf \left\{ \lambda > 0 : \int_{R} \psi_2 \left( \frac{|f(x)|}{\lambda} \right) \, d\mu(x) \leq 1 \right\}.
$$
We have (e.g., see [3, Lemma 3.5.5])

\[ c \sup_{p \in [1, \infty)} \frac{\|f\|_{L^p(R, \mu)}}{\sqrt{p}} \leq \|f\|_{L^{p_2}(R, \mu)} \leq \tilde{c} \sup_{p \in [1, \infty)} \frac{\|f\|_{L^p(R, \mu)}}{\sqrt{p}} \]  

(2.4)

for every \( f \in L^{p_2}(R, \mu) \). Here \( c \) and \( \tilde{c} \) are absolute constants. In particular, \( L^{p_2}(R, \mu) \) is continuously embedded in \( L^p(R, \mu) \) for every \( p \in [1, \infty) \).

Throughout the entire paper, we assume that \( d \in \mathbb{N}, d \geq 2 \). Let \( G \subseteq \mathbb{R}^d \) be a nonempty bounded open set. For \( m \in \mathbb{N} \) and \( p \in [1, \infty) \), \( V^{m,p}(G) \) denotes the vector space of all \( m \)-times weakly differentiable functions in \( G \) whose \( m \)-th order weak derivatives belong to \( L^p(G) \). By \( V_0^{m,p}(G) \) we denote the Banach space of all functions from \( V^{m,p}(G) \) whose continuation by \( 0 \) outside \( G \) is \( m \)-times weakly differentiable in \( \mathbb{R}^d \) equipped with the norm \( \|u\|_{V_0^{m,p}(G)} = \|\nabla^m u|_{\partial G}\|_{L^p(G)} \). By \( \nabla^m \) we denote the vector of all \( m \)-th order weak derivatives. When \( G \) is regular enough (for example, Lipschitz), \( V_0^{m,p}(G) \) coincides with the usual Sobolev space \( W_0^{m,p}(G) \), up to equivalent norms.

3. Different degrees of noncompactness

An important property of both optimal Sobolev embeddings (1.1) and (1.2), which we will exploit in both cases, is that their norms are homothetic invariant.

**Proposition 3.1.** Let \( \Omega \subseteq \mathbb{R}^d \) be a nonempty bounded open set, \( m \in \mathbb{N}, 1 \leq m < d, \) and \( p \in [1, d/m) \). Let \( p^* = dp/(d - mp) \) and \( q \in [p, p^*] \). Denote by \( I \) the identity operator \( I: V_0^{m,p}(\Omega) \to L^{p^*,q}(\Omega) \). For every \( 0 < \lambda < \|I\| \) and every \( \varepsilon > 0 \), there exist a system of functions \( \{u_j\}_{j=1}^\infty \subseteq V_0^{m,p}(\Omega) \) and a system of open balls \( \{B_{r_j}(x_j)\}_{j=1}^\infty \subseteq \Omega \) with the following properties.

(i) The balls \( \{B_{r_j}(x_j)\}_{j=1}^\infty \) are pairwise disjoint.

(ii) \( \|u_j\|_{V_0^{m,p}(\Omega)} = 1 \) and \( \|u_j\|_{L^{p^*,q}(\Omega)} = \lambda \) for every \( j \in \mathbb{N} \).

(iii) \( \text{supp} u_j \subseteq B_{r_j}(x_j) \) for every \( j \in \mathbb{N} \).

(iv) For every sequence \( \{\alpha_j\}_{j=1}^\infty \subseteq \mathbb{R} \), we have

\[
\left\| \sum_{j=1}^\infty \alpha_j u_j \right\|_{L^{p^*,q}(\Omega)} \geq \frac{\lambda}{(1 + \varepsilon)^\frac{1}{q}} \left( \sum_{j=1}^\infty |\alpha_j|^q \right)^{\frac{1}{q}}.
\]  

(3.1)

**Proof.** It is known that

\[
\|I\| = \|I_G: V_0^{m,p}(G) \to L^{p^*,q}(G)\| \quad \text{for every open set } \emptyset \neq G \subseteq \Omega. \quad (3.2)
\]

Indeed, arguing as in the proof of [13, Proposition 3.1], we observe that the proof of (3.2) amounts to showing that, if \( u \in V_0^{m,p}(B_r(0)) \) and \( 0 < s < r \), then

\[
\frac{\|u(\kappa \cdot)\|_{L^{p^*,q}(B_s(0))}}{\|u(\kappa \cdot)\|_{V_0^{m,p}(B_s(0))}} = \frac{\|u\|_{L^{p^*,q}(B_s(0))}}{\|u\|_{V_0^{m,p}(B_s(0))}}.
\]  

(3.3)
where $\kappa = r/s$. It is a matter of simple straightforward computations to show that
\[
\|u(\kappa \cdot)\|_{L^p(\mathbb{S}(B_r(0)))} = \kappa^{\frac{d}{p}} \|u\|_{L^p(\mathbb{S}(B_r(0)))}
\]
and
\[
\|\nabla (u(\kappa \cdot))\|_{L^p(\mathbb{S}(B_r(0)))} = \kappa^{\frac{d}{p}} \|\nabla u\|_{L^p(\mathbb{S}(B_r(0)))},
\]
whence (3.3) immediately follows.

We now start with the desired systems. We will use induction. First, using (3.2), we find a ball $B_{\tau_1}(x_1) \subseteq \overline{B}_{\tau_1}(x_1) \subseteq \Omega$ and a function $u_1 \in V_0^{m,p}(\Omega)$ such that $\text{supp} \ u_1 \subseteq B_{\tau_1}(x_1)$, $\|u_1\|_{L^{p^*}(\Omega)} = \lambda$ and $\|u_1\|_{V_0^{m,p}(\Omega)} = 1$. Set $\delta_0 = |B_{\tau_1}(x_1)|$. By the monotone convergence theorem, there is $0 < \delta_1 < \delta_0$ such that
\[
(1 + \varepsilon) \int_{\delta_1}^{\delta_0} \left( t^{\frac{1}{p^*} - \frac{1}{q}} u_1^*(t) \right)^q \, dt \geq \int_0^{\delta_0} \left( t^{\frac{1}{p^*} - \frac{1}{q}} u_1^*(t) \right)^q \, dt = \|u_1\|^q_{L^{p^*}(\Omega)}.
\]
Next, assume that we have already found functions $u_j \in V_0^{m,p}(\Omega)$, pairwise disjoint balls $B_{\tau_j}(x_j) \subseteq \overline{B}_{\tau_j}(x_j) \subseteq \Omega$, and $0 < \delta_k < \cdots < \delta_1 < \delta_0$, $j = 1, \ldots, k$, where $k \in \mathbb{N}$, such that $\|u_j\|_{V_0^{m,p}(\Omega)} = 1$ and $\|u_j\|_{L^{p^*}(\Omega)} = \lambda$, $\text{supp} \ u_j \subseteq B_{\tau_j}(x_j)$, and
\[
(1 + \varepsilon) \int_{\delta_{j-1}}^{\delta_j} \left( t^{\frac{1}{p^*} - \frac{1}{q}} u_j^*(t) \right)^q \, dt \geq \int_{\delta_{j-1}}^{\delta_j} \left( t^{\frac{1}{p^*} - \frac{1}{q}} u_j^*(t) \right)^q \, dt = \|u_j\|^q_{L^{p^*}(\Omega)}.
\]
\[\tag{3.4}\]
Take any ball $B_{\tau_{k+1}}(x_{k+1})$ such that $B_{\tau_{k+1}}(x_{k+1}) \subseteq \overline{B}_{\tau_{k+1}}(x_{k+1}) \subseteq \Omega \setminus \bigcup_{j=1}^k \overline{B}_{\tau_j}(x_j)$ and $|B_{\tau_{k+1}}(x_{k+1})| < \delta_k$. Thanks to (3.2), we find a function $u_{k+1} \in V_0^{m,p}(\Omega)$ such that $\text{supp} \ u_{k+1} \subseteq B_{\tau_{k+1}}(x_{k+1})$, $\|u_{k+1}\|_{L^{p^*}(\Omega)} = \lambda$ and $\|u_{k+1}\|_{V_0^{m,p}(\Omega)} = 1$. By the monotone convergence theorem again, there is $0 < \delta_{k+1} < \delta_k$ such that
\[
(1 + \varepsilon) \int_{\delta_{k+1}}^{\delta_k} \left( t^{\frac{1}{p^*} - \frac{1}{q}} u_{k+1}^*(t) \right)^q \, dt \geq \int_{\delta_{k+1}}^{\delta_k} \left( t^{\frac{1}{p^*} - \frac{1}{q}} u_{k+1}^*(t) \right)^q \, dt = \|u_{k+1}\|^q_{L^{p^*}(\Omega)}.
\]
This finishes the inductive step.

Clearly, the constructed systems $\{u_j\}_{j=1}^\infty \subseteq V_0^{m,p}(\Omega)$ and $\{B_{\tau_j}(x_j)\}_{j=1}^\infty \subseteq \Omega$ have the properties (i)–(iii), and (3.4) is valid for every $j \in \mathbb{N}$. It remains to verify that (iv) is also valid. Let $\{\alpha_j\}_{j=1}^\infty \subseteq \mathbb{R}$. Since the functions $\{\alpha_j u_j\}_{j=1}^\infty$ have pairwise disjoint supports, we have
\[
\left\{ x \in \Omega : \sum_{j=1}^\infty |\alpha_j u_j(x)| > \gamma \right\} = \sum_{j=1}^\infty \left\{ x \in \Omega : |\alpha_j u_j(x)| > \gamma \right\}
\]
for every $\gamma > 0$. It follows that
\[
\left( \sum_{j=1}^\infty |\alpha_j u_j| \right)^* \geq \sum_{j=1}^\infty |\alpha_j| u_j^*(\delta_j, \delta_{j-1}). \tag{3.5}
\]
Indeed, suppose that there is \( t \in (0, |\Omega|) \) such that
\[
\left( \sum_{j=1}^{\infty} \alpha_j u_j \right)^*(t) < \sum_{j=1}^{\infty} |\alpha_j| u_j^*(t) \chi_{(\delta_j, \delta_{j-1})}(t).
\]

Plainly, there is a unique index \( k \) such that \( t \in (\delta_k, \delta_{k-1}) \). By the definition of the nonincreasing rearrangement, there is \( \gamma > 0 \) such that
\[
\left\{ x \in \Omega : \left| \sum_{j=1}^{\infty} \alpha_j u_j(x) \right| > \gamma \right\} \leq t \quad \text{and} \quad \gamma < |\alpha_k| u_k^*(t).
\]

Consequently, using the definition again, we have
\[
\{|x \in \Omega : |\alpha_k u_k(x)| > \gamma\} > t,
\]
however. Thus we have reached a contradiction, and so \((3.5)\) is proved.

Finally, using \((3.4)\) and \((3.5)\), we observe that
\[
\left\| \sum_{j=1}^{\infty} \alpha_j u_j \right\|_{L^{p^*, q}(\Omega)}^q \leq \int_0^\infty \left( t^{p^* - \frac{1}{q}} \left( \sum_{j=1}^{\infty} |\alpha_j| u_j^*(t) \chi_{(\delta_j, \delta_{j-1})}(t) \right) \right)^q dt
\]
\[
\geq \sum_{j=1}^{\infty} |\alpha_j|^q \int_{\delta_j}^{\delta_{j-1}} \left( t^{p^* - \frac{1}{q}} u_j^*(t) \right)^q dt
\]
\[
\geq \frac{1}{1 + \varepsilon} \sum_{j=1}^{\infty} |\alpha_j|^q \left\| u_j \right\|_{L^{p^*, q}(\Omega)}^q
\]
\[
= \frac{\lambda^q}{1 + \varepsilon} \sum_{j=1}^{\infty} |\alpha_j|^q.
\]

We start with the Lebesgue case. The following lemma of independent interest is a key ingredient for the proof of the fact that the embedding \((1.1)\) is finitely strictly singular. Its proof is inspired by that of [6, Lemma 2.9].

**Lemma 3.2.** Let \((R, \mu)\) be a probability measure space and \(p \in [1, \infty)\). Let \(X_n\) be a \(n\)-dimensional subspace of \(L^p(R, \mu)\). There is a positive \(\mu\)-measurable function \(g\) on \(R\) and a linear isometry \(L : L^p(R, \mu) \to L^p(R, \nu)\) defined as \(Lf = g^{-1/p}f\), where \(d\nu = g \, d\mu\), with the following properties. The measure \(\nu\) is probabilistic, and in every subspace \(Y \subseteq X_n\) with \(\dim Y \geq n/2\), there exists a function \(h \in Y\) such that
\[
\|h\|_{L^p(R, \mu)} = 1
\]
and
\[
\sup_{q \in [1, \infty)} \frac{\|Lh\|_{L^q(R, \nu)}}{\sqrt{q}} \leq C.
\]
Here $C$ is an absolute constant depending only on $\min\{p, 2\}$.

**Proof.** Let $X_n$ be a $n$-dimensional subspace of $L^p(R, \mu)$. Thanks to \cite{22} (cf. \cite{31}, Theorem 2.1), there exists a positive \(\mu\)-measurable function $g$ on $R$ such that $\|g\|_{L^1(R, \mu)} = 1$ and the following is true: Upon setting $d\nu = g \, d\mu$ and defining $Lf = g^{-1/p} f$, $f \in L^p(R, \mu)$, the subspace $\tilde{X}_n = LX_n$ of $L^p(R, \nu)$ has a basis $\{\psi_1, \ldots, \psi_n\}$ that is orthonormal in $L^2(R, \nu)$ and satisfies

\[
\sum_{j=1}^n |\psi_j|^2 \equiv n \quad \mu\text{-a.e. on } R. \tag{3.6}
\]

Note that, since $\tilde{X}_n$ has a basis consisting of functions from $L^2(R, \nu)$, we have $\tilde{X}_n \subseteq L^2(R, \nu)$ even for $p \in [1, 2)$.

Let $Y$ be a subspace of $X_n$ with $\dim Y \geq n/2$. Set

\[ B_p(Z) = \{ f \in Z : \|f\|_{L^p(R, \nu)} \leq 1 \} \]

and

\[ B_{\exp}(Z) = \{ f \in Z : \|f\|_\mathcal{L}(\nu) \leq 1 \}, \]

in which $Z$ is $\tilde{X}_n$ or $\tilde{Y} = L\tilde{Y}$. By \cite{7}, Lemma 9.2, we have

\[
\log E(B_2(\tilde{X}_n), tB_{\exp}(\tilde{X}_n)) \leq c_1 \frac{1}{t^2} n \quad \text{for every } t \geq 1; \tag{3.7}
\]

where $c_1$ is an absolute constant, which is independent of $n$ and $t$. Since $\dim \tilde{Y} \geq n/2$, we have

\[
\log E(B_p(\tilde{Y}), \frac{1}{4} B_p(\tilde{Y})) \geq n \log 2 \tag{3.8}
\]

by a standard volumetric argument (e.g., see \cite{3}, (1.1.10)).

We start with the case $p \in [2, \infty)$, which is simpler. Since $B_p(\tilde{Y}) \subseteq B_2(\tilde{Y}) \subseteq B_2(\tilde{X}_n)$, it follows from (3.7) that

\[
\log E(B_p(\tilde{Y}), tB_{\exp}(\tilde{X}_n)) \leq c_1 \frac{1}{t^2} n \quad \text{for every } t \geq 1. \tag{3.9}
\]

Moreover, since $E(B_p(\tilde{Y}), 2tB_{\exp}(\tilde{X}_n)) \leq E(B_p(\tilde{Y}), tB_{\exp}(\tilde{X}_n))$ (e.g., see \cite{3}, Fact 4.1.9), we actually have

\[
\log E(B_p(\tilde{Y}), 2tB_{\exp}(\tilde{Y})) \leq c_1 \frac{1}{t^2} n \quad \text{for every } t \geq 1.
\]

Therefore, we can find $t_0 \geq 1$, not depending on $n$, so large that

\[
\log E(B_p(\tilde{Y}), 2t_0B_{\exp}(\tilde{Y})) \leq n \log \frac{2}{2}.
\]

It follows that $2t_0B_{\exp}(\tilde{Y}) \not\subset \frac{1}{4} B_p(\tilde{Y})$. Indeed, if $2t_0B_{\exp}(\tilde{Y}) \subset \frac{1}{4} B_p(\tilde{Y})$, then

\[
\log E(B_p(\tilde{Y}), \frac{1}{4} B_p(\tilde{Y})) \leq \log E(B_p(\tilde{Y}), 2t_0B_{\exp}(\tilde{Y})) \leq n \log \frac{2}{2},
\]
which would contradict (3.9). Hence there is a function $h_0 \in \tilde{Y}$ such that
\[ \|h_0\|_{L^2(R,\nu)} \leq 2t_0 \quad \text{and} \quad \|h_0\|_{L^p(R,\nu)} > \frac{1}{4}. \]
Then $h = L^{-1}h_0/\|h_0\|_{L^p(R,\nu)}$ is the desired function thanks to (3.4).

We now turn our attention to the case $p \in [1, 2)$. Assume for the moment that we know that
\[ \log E(B_p(\tilde{X}_n), tB_2(\tilde{X}_n)) \leq c_2 \frac{\log^2(1 + t)}{t^2} n \quad \text{for every } t \geq 1. \tag{3.10} \]
Here $c_2$ is a constant depending only on $p$. Clearly
\[ E(B_p(\tilde{Y}), tB_{exp}(\tilde{X}_n)) \leq E(B_p(\tilde{X}_n), tB_{exp}(\tilde{X}_n)) \]
\[ \leq E(B_p(\tilde{X}_n), sB_2(\tilde{X}_n)) \cdot E(sB_2(\tilde{X}_n), tB_{exp}(\tilde{X}_n)) \]
\[ = E(B_p(\tilde{X}_n), sB_2(\tilde{X}_n)) \cdot E(B_2(\tilde{X}_n), \frac{t}{s}B_{exp}(\tilde{X}_n)), \]
and so
\[ \log E(B_p(\tilde{Y}), tB_{exp}(\tilde{X}_n)) \leq c_3 \left( \frac{\log^2(1 + s)}{s^2} + \frac{s^2}{t^2} \right) n \]
for every $1 \leq s \leq t$ thanks to (3.7) and (3.10). Here $c_3$ is a constant depending only on $p$. Plugging $s = \sqrt{t}$ into this inequality, we arrive at
\[ \log E(B_p(\tilde{Y}), tB_{exp}(\tilde{X}_n)) \leq c_3 \frac{1 + \log^2(1 + \sqrt{t})}{t} n \quad \text{for every } t \geq 1. \]
Since $\lim_{t \to \infty} \frac{1 + \log^2(1 + \sqrt{t})}{t} = 0$, we can now proceed in the same way as in the case $p \in [2, \infty)$, using this inequality instead of (3.9). Therefore, the proof will be complete once we prove (3.10)—to that end, we adapt the argument of [3, Proposition 9.6]. The proof of (3.10) is divided into several steps.

First, observe that
\[ E(B_p(\tilde{X}_n), tB_2(\tilde{X}_n)) \tag{3.11} \]
\[ \leq E(B_p(\tilde{X}_n), 2B_p(\tilde{X}_n) \cap 2tB_2(\tilde{X}_n)) \cdot E(B_p(\tilde{X}_n) \cap tB_2(\tilde{X}_n), \frac{t}{2}B_2(\tilde{X}_n)). \]
Since $(\tilde{X}_n, \|\cdot\|_{L^2(R,\nu)})$ is a Hilbert space, $2tB_2(\tilde{X}_n)$ is a multiple of its unit ball and $B_p(\tilde{X}_n)$ is a (nonempty) closed convex subset of $(\tilde{X}_n, \|\cdot\|_{L^2(R,\nu)})$, we have $E(B_p(\tilde{X}_n), 2tB_2(\tilde{X}_n)) = \tilde{E}(B_p(\tilde{X}_n), 2tB_2(\tilde{X}_n))$ (e.g., see [3, Fact 4.1.4]). Consequently, if $B_p(\tilde{X}_n) \subseteq \bigcup_{k=1}^m (u_k + 2tB_2(\tilde{X}_n))$, where $u_k \in B_p(\tilde{X}_n)$, then $B_p(\tilde{X}_n) \subseteq \bigcup_{k=1}^m (u_k + 2B_p(\tilde{X}_n) \cap 2tB_2(\tilde{X}_n))$. Hence
\[ E(B_p(\tilde{X}_n), 2B_p(\tilde{X}_n) \cap 2tB_2(\tilde{X}_n)) \leq E(B_p(\tilde{X}_n), 2tB_2(\tilde{X}_n)). \tag{3.12} \]
Combining (3.11) and (3.12), we obtain
\[
\log E(B_p(\tilde{X}_n), tB_2(\tilde{X}_n)) \\
\leq \log E(B_p(\tilde{X}_n), 2tB_2(\tilde{X}_n)) \\
+ \log E(B_p(\tilde{X}_n) \cap tB_2(\tilde{X}_n), \frac{t}{2}B_2(\tilde{X}_n)).
\]
(3.13)

Now, thanks to (3.6) and the fact that \(p < 2\), for every \(k\)

Note that
\[
B_{2k}(\tilde{X}_n) \\
\text{together with Lemma 9.1.3} \quad \text{combined with [7, Lemma 4.4], we have}
\]

Combining (3.11) and (3.12), we obtain
\[
\log E(B_p(\tilde{X}_n), tB_2(\tilde{X}_n)) \\
\leq \log E(B_p(\tilde{X}_n), 2tB_2(\tilde{X}_n)) \\
+ \log E(B_p(\tilde{X}_n) \cap tB_2(\tilde{X}_n), \frac{t}{2}B_2(\tilde{X}_n)).
\]
(3.13)

for every \(u = \sum_{j=1}^{n} \alpha_j \psi_j \in \tilde{X}_n\), whence it follows that
\[
\|u\|_{L^2(R, \nu)}^2 \\
\leq n \frac{2-p}{2} \|u\|_{L^p(R, \nu)}^p \\
\text{for every } u \in \tilde{X}_n.
\]

Clearly, this implies that \(B_p(\tilde{X}_n) \subseteq rB_2(\tilde{X}_n)\) for every \(r \geq n \frac{2-p}{2p}\); hence
\[
\log E(B_p(\tilde{X}_n), 2^k tB_2(\tilde{X}_n)) = 0
\]
for every \(k \in \mathbb{N}\) such that \(2^k t \geq n \frac{2-p}{2p}\). Therefore, iterating (3.13) with \(t\) replaced by \(2^k t\), \(k \in \mathbb{N}\), we arrive at
\[
\log E(B_p(\tilde{X}_n), tB_2(\tilde{X}_n)) \leq \sum_{k=0}^{\infty} \log E(B_p(\tilde{X}_n) \cap 2^k tB_2(\tilde{X}_n), 2^{k-1} tB_2(\tilde{X}_n)).
\]
(3.14)

Second, we claim that
\[
\log E(B_p(\tilde{X}_n) \cap sB_2(\tilde{X}_n), \frac{s}{2}B_2(\tilde{X}_n)) \leq c_4 \frac{\log^2(1 + s)}{s^2} n
\]
(3.15)
for every \(s \geq 1\). Here \(c_4\) is a constant depending only on \(p\). Fix \(s \geq 1\). Let \(Z\) denote \(X_n\) endowed with the norm
\[
\|u\|_Z = \max \left\{ \|u\|_{L^p(R, \nu)}, \frac{1}{s} \|u\|_{L^2(R, \nu)} \right\}.
\]

Note that \(B_p(\tilde{X}_n) \cap sB_2(\tilde{X}_n)\) is the unit ball of \(Z\). Owing to (3.9.17) together with Lemma 9.1.3 combined with [7, Lemma 4.4], we have
\[
\log E(B_p(\tilde{X}_n) \cap sB_2(\tilde{X}_n), \frac{s}{2}B_2(\tilde{X}_n)) \\
\leq c_5 K(Z)^2 C_2(Z)^2 \pi_2^2 (Z \rightarrow (\tilde{X}_n, \| \cdot \|_{L^2(R, \nu)})) \frac{1}{s^2}.
\]
(3.16)

Here \(c_5\) is an absolute constant.

As for \(K(Z)\), we have
\[
K(Z) \leq c_6 \log(1 + d(Z, \ell_2^n))
\]
(3.17)
by \[22\]. Here \(c_6\) is an absolute constant.

We claim that \(d(Z, \ell^n_2) \leq s\). To this end, consider the linear isomorphism \(T: Z \to \ell^n_2\) defined as

\[ Tf = \{\alpha_j\}_{j=1}^n, \ f = \sum_{j=1}^n \alpha_j \psi_j \in Z. \]

Clearly, \(T\) is onto \(\ell^n_2\), and we have

\[ \|Tf\|_{\ell^p_2} = \|f\|_{L^2(R,\nu)} \leq s\|f\|_Z \quad (3.18) \]

for every \(f \in Z\). On the other hand, using (3.6) and the fact that \(p < 2\), we obtain

\[
\|T^{-1}(\{\alpha_j\}_{j=1}^n)\|_{L^p(R,\nu)}^p = \int_R \left( \sum_{j=1}^n \alpha_j \psi_j(t) \right)^p \, d\nu(t) \\
= \int_R \left( \sum_{j=1}^n \alpha_j \psi_j(t) \right)^{p-2} \left( \sum_{j=1}^n \alpha_j \psi_j(t) \right)^2 \, d\nu(t) \\
\leq \int_R \left( \|\{\alpha_j\}_{j=1}^n\|_{\ell^n_2} \sqrt{n} \right)^{p-2} \left( \sum_{j=1}^n \alpha_j \psi_j(t) \right)^2 \, d\nu(t) \\
\leq \|\{\alpha_j\}_{j=1}^n\|_{\ell^n_2}^{p-2} \| \sum_{j=1}^n \alpha_j \psi_j \|_{L^2(R,\nu)}^2 \\
= \|\{\alpha_j\}_{j=1}^n\|_{\ell^n_2}^p 
\]

for every \(\{\alpha_j\}_{j=1}^n\). Furthermore, we plainly have

\[
\frac{1}{s} \|T^{-1}(\{\alpha_j\}_{j=1}^n)\|_{L^2(R,\nu)} \leq \|T^{-1}(\{\alpha_j\}_{j=1}^n)\|_{L^2(R,\nu)} = \|\{\alpha_j\}_{j=1}^n\|_{\ell^n_2}. 
\]

Therefore \(\|T^{-1}\| \leq 1\). By combining this with (3.18), it follows that

\[ d(Z, \ell^n_2) \leq s. \]

Plugging this into (3.17), we obtain

\[ K(Z) \leq c_6 \log(1 + s). \quad (3.19) \]

As for \(C_2(Z)\), we claim that

\[ C_2(Z) \leq c_7, \quad (3.20) \]

where \(c_7\) is a constant depending only on \(p\). To this end, recall that

\[ \max\{C_2(L^p(R,\nu)), C_2(L^2(R,\nu))\} < \infty, \]
and $C_2(L^p(R, \nu))$ depends only $p$ (e.g., see [2, Theorem 6.2.14]). Now, since 

$$\left( \sum_{j=1}^{m} \|f_j\|_Z^2 \right)^{1/2} \leq \left( \sum_{j=1}^{m} \|f_j\|_{L^p(R, \nu)}^2 \right)^{1/2} + \frac{1}{s} \left( \sum_{j=1}^{m} \|f_j\|_{L^2(R, \nu)}^2 \right)^{1/2} \leq 2 \max \{C_2(L^p(R, \nu)), C_2(L^2(R, \nu))\} \int_0^1 \left\| \sum_{j=1}^{m} f_j \rho_j(t) \right\|_Z \, dt$$

for every $\{f_j\}_{j=1}^m \subseteq Z$, $m \in \mathbb{N}$, the claim immediately follows.

As for $\pi_2(Z \hookrightarrow (\bar{X}_n, \| \cdot \|_{L^2(R, \nu)})$, note that $\|f\|_{L^1(R, \nu)} \leq \|f\|_{L^p(R, \nu)} \leq \|f\|_Z$ for every $f \in Z$, and so the unit ball of $(\bar{X}_n, \| \cdot \|_{L^1(R, \nu)})^*$ is contained in the unit ball of $Z^*$. It follows that

$$\pi_2(Z \hookrightarrow (\bar{X}_n, \| \cdot \|_{L^2(R, \nu)}) \leq \pi_2((\bar{X}_n, \| \cdot \|_{L^1(R, \nu)}) \hookrightarrow (\bar{X}_n, \| \cdot \|_{L^2(R, \nu)})),$$

By [2, Proof of Lemma 4.5], we have

$$\pi_2((\bar{X}_n, \| \cdot \|_{L^1(R, \nu)}) \hookrightarrow (\bar{X}_n, \| \cdot \|_{L^2(R, \nu)}) \leq c_8 \sqrt{n}.$$

Here $c_8$ is an absolute constant. Hence

$$\pi_2(Z \hookrightarrow (\bar{X}_n, \| \cdot \|_{L^2(R, \nu)}) \leq c_8 \sqrt{n}. \quad (3.21)$$

The desired estimate (3.15) now follows by combining (3.19), (3.20) and (3.21) with (3.10).

Finally, now that we have (3.15) at our disposal, the rest is simple. Combining (3.15) with (3.14), we obtain

$$\log E(B_p(\bar{X}_n), tB_2(\bar{X}_n)) \leq c_4 \sum_{k=0}^{\infty} \log^2(1 + 2^k t) \frac{1}{4^k k^2} n \leq 2c_4 \left( \sum_{k=0}^{\infty} \frac{k^2 \log^2 2 + \log^2(1 + t)}{4^k} \right) \frac{1}{t^2} n \leq 2c_4 \left( \sum_{k=0}^{\infty} \frac{k^2 + 1}{4^k} \right) \frac{\log^2(1 + t)}{t^2} n.$$

This finishes the proof of (3.10). \hfill \Box

We are now in a position to prove the main result concerning the embedding (1.1).

**Theorem 3.3.** Let $\Omega \subseteq \mathbb{R}^d$ be a nonempty bounded open set, $m \in \mathbb{N}$, $1 \leq m < d$, and $p \in [1, d/m)$. Denote by $I$ the identity operator $I: V_0^{m,p}(\Omega) \to L^p(\Omega)$, where $p^* = dp/(d - mp)$. There exists $n_0 \in \mathbb{N}$, depending only on $d$ and $m$, such that

$$C_1 n^{-\frac{m}{d}} \leq b_n(I) \leq C_2 n^{-\frac{m}{d}} \quad \text{for every } n \geq n_0. \quad (3.22)$$

Here $C_1$ and $C_2$ are constants depending only on $d$, $m$ and $p$.

In particular, $I$ is finitely strictly singular.
Proof. First, we prove the upper bound on \( b_n(I) \). Set \( l = d^m \). We may without loss of generality assume that \(|\Omega| = 1/l\); otherwise we replace \( dx/(l|\Omega|) \). We start with a few definitions. By \( G : V_{0}^{m,p}(\Omega) \to \bigoplus_{j=1}^{l} L^p(\Omega) \) we denote the linear isometric operator defined as
\[
Gu = \nabla^m u, \; u \in V_{0}^{m,p}(\Omega).
\]
Here \( \bigoplus_{j=1}^{l} \) stands for the \( \ell^p \)-direct sum, and the way in which the vector \( \nabla^m u \) is ordered is completely immaterial—we fix arbitrary order. Furthermore, let \( R = \bigoplus_{j=1}^{l} \Omega^{(j)} \) consist of \( l \) disjoint copies of \( \Omega \), each endowed with the Lebesgue measure. We denote the corresponding probabilistic measure space by \( (R, \mu) \). Finally, \( S : \bigoplus_{j=1}^{l} L^p(\Omega) \to L^p(R, \mu) \) denotes the linear isometry defined as
\[
S(f_1, \ldots, f_l) = \sum_{j=1}^{l} f_j \chi_{\Omega^{(j)}}, \; (f_1, \ldots, f_l) \in \bigoplus_{j=1}^{l} L^p(\Omega).
\]
Let \( c_1 \) be the Besicovitch constant in \( \mathbb{R}^d \). Recall that \( c_1 \) depends only on \( d \). Set \( c_2 = \binom{d+m-1}{m-1} \). Note that \( c_2 \) is the dimension of the vector space of polynomials in \( \mathbb{R}^d \) of degree at most \( m-1 \), which we will denote by \( P_{m-1}(\mathbb{R}^d) \). Assume that \( n \geq 2c_1c_2 \). Let \( X_n \) be a \( n \)-dimensional subspace of \( V_{0}^{m,p}(\Omega) \), and \( \overline{X}_n \subseteq L^p(R, \mu) \) its image under the linear isometric operator \( S \circ G \). Clearly, \( \dim \overline{X}_n = n \). Let \( L, g \) and \( \nu \) be those from Lemma 3.2 applied to \( \overline{X}_n \). Since \( \Omega \) is bounded and \( \|g\|_{L^1(R, \mu)} = 1 \), for each \( x \in \Omega \) we can find \( r_x \in (0, \text{diam } \Omega) \) such that
\[
\int_{\bigoplus_{j=1}^{l} B_{r_x}^{(j)}(x)} g \, d\mu = \frac{2c_1c_2}{n}.
\]
Here \( B_{r_x}^{(j)}(x) \) are disjoint copies of \( B_{r_x}(x) \) in \( \Omega^{(j)} \). Using the Besicovitch covering lemma, we find a countable subcollection \( \{B_{r_k}(x_k)\}_{k=1}^{M} \) such that
\[
\Omega \subseteq \bigcup_{k=1}^{M} B_{r_k}(x_k)
\]
and
\[
\sum_{k=1}^{M} \chi_{B_{r_k}(x_k)} \leq c_1.
\]
We claim that
\[
M \leq \frac{n}{2c_2}.
\]
Indeed, we have

\[ M \frac{2c_1c_2}{n} = \sum_{k=1}^{M} \int \bigoplus_{j=1}^{l} B_{r_k}^{(j)}(x_k) g \, d\mu \]

\[ \leq \left\| \sum_{k=1}^{M} \chi \bigoplus_{j=1}^{l} B_{r_k}^{(j)}(x_k) \right\|_{L^\infty(R,\mu)} \left\| g \right\|_{L^1(R,\mu)} \]

\[ \leq c_1. \]

Recall that, for every \( u \in V_{m,p}^n(B_{r_k}(x_k)) \), \( k = 1, \ldots, M \), there is a polynomial \( P_{u,k} \in \mathcal{P}_{m-1}(\mathbb{R}^d) \), depending on \( u \) and \( B_{r_k}(x_k) \), such that

\[ \| u - P_{u,k} \|_{L^p(B_{r_k}(x_k))} \leq c_3 \| \nabla^m u \|_{L^p(B_{r_k}(x_k))}; \quad (3.27) \]

moreover, the dependence of \( P_{u,k} \) on \( u \) is linear. Here \( c_3 \) depends only on \( d, m \) and \( p \). This follows easily by iterating the classical Sobolev–Poincaré inequality on balls (e.g., see [23, Corollary 1.64]).

We claim that there is a subspace \( Y \) of \( X_n \) with \( \dim Y \geq n/2 \) such that

\[ \| u \|_{L^p(B_{r_k}(x_k))} \leq c_3 \| \nabla^m u \|_{L^p(B_{r_k}(x_k))} \quad \text{for every } u \in Y \text{ and } k. \quad (3.28) \]

Indeed, set \( Y_0 = X_n \), and let \( Y_1 \) be the kernel of the linear operator \( Y_0 \ni u \mapsto P_{u,1} \in \mathcal{P}_{m-1}(\mathbb{R}^d) \). By the rank-nullity theorem, we have

\[ \dim Y_1 \geq n - \dim \mathcal{P}_{m-1}(\mathbb{R}^d) = n - c_2. \]

Now, let \( Y_2 \) be the kernel of the linear operator \( Y_1 \ni u \mapsto P_{u,2} \). It follows that \( \dim Y_2 \geq n - 2c_2 \). Proceeding in the obvious way, we find a subspace \( Y = Y_M \) of \( X_n \) with \( \dim Y \geq n - Mc_2 \) such that \( P_{u,k} \equiv 0 \) for every \( u \in Y \) and \( k = 1, \ldots, M \). The claim now immediately follows from (3.26) and (3.27).

Let \( \tilde{Y} \) be the image of \( Y \) under the linear isometric operator \( S \circ G \). Thanks to [Lemma 3.2], there is \( \tilde{u} \in \tilde{Y} \subseteq \tilde{X}_n \) such that

\[ \| u \|_{V_0^{m,p}(\Omega)} = \| \tilde{u} \|_{L^p(R,\mu)} = 1 \quad (3.29) \]

and

\[ \sup_{q \in [1,\infty)} \frac{\| L\tilde{u} \|_{L^q(R,\nu)}}{\sqrt{q}} \leq c_4, \quad (3.30) \]
where \( u = (SG)^{-1}\tilde{u} \in V^{m,p}_0(\Omega) \) and \( d\nu = g\,d\mu \). Here \( c_4 \) is a constant depending only on \( p \). By (3.21) and (3.28), we have

\[
\|u\|_{L^p(\Omega)} \leq \sum_{k=1}^M \|u\|_{L^p(B_r(x_k))} \leq c_3^{\|\nabla u\|_{L^p(B_r(x_k))}}
\]

Finally, we turn our attention to the lower bound in (3.22), whose proof is simpler. To that end, recall that we have (e.g., see [30, Remark 7])

\[
b_n(\ell_p \to \ell_{p^*}) = n^{\frac{p^*-p}{p^*}} = n^{-\frac{mp}{d-mp}} \quad \text{for every } n \in \mathbb{N}.
\]
Here we used the fact that $1 \leq p < p^*$. Let $0 < \lambda < \|I\|$ and $\varepsilon > 0$. By \(\text{(3.34)}\), there is a subspace $E_n$ of $\ell_p$ with $\dim E_n = n$ such that

$$\inf_{\{\alpha_j\}_{j=1}^n \in E_n} \|\{\alpha_j\}_{j=1}^n\|_{\ell_{p^*}} \geq n^{-\frac{m}{d}} - \varepsilon. \quad (3.35)$$

Furthermore, let $\{u_j\}_{j=1}^\infty$ and $\{B_j\}_{j=1}^\infty$ be systems whose existence is guaranteed by Proposition 3.1 with $q = p^*$. Note that the linear operator $T: \ell_p \to V_0^{m,p}(\Omega)$ defined as

$$T(\{\alpha_j\}_{j=1}^\infty) = \sum_{j=1}^\infty \alpha_j u_j, \quad \{\alpha_j\}_{j=1}^\infty \in \ell_p,$$

is well defined and isometric. Indeed, since the functions $u_j$ have mutually disjoint supports and $\|u_j\|_{V_0^{m,p}(\Omega)} = 1$, we have

$$\left\| \sum_{j=1}^\infty \alpha_j u_j \right\|_{V_0^{m,p}(\Omega)}^p = \sum_{j=1}^\infty |\alpha_j|^p \|\nabla u_j\|_{L^p(\Omega)}^p = \sum_{j=1}^\infty |\alpha_j|^p.$$

In particular, $T$ is injective. Furthermore, we also have

$$\left\| \sum_{j=1}^\infty \alpha_j u_j \right\|_{L^{p^*}(\Omega)} = \lambda \|\{\alpha_j\}_{j=1}^\infty\|_{\ell_{p^*}} \quad \text{for every } \{\alpha_j\}_{j=1}^\infty \in \ell_{p^*} \quad (3.36)$$

since $\|u_j\|_{L^{p^*}(\Omega)} = \lambda$ for every $j \in \mathbb{N}$. Set $X_n = TE_n$. We have $\dim X_n = \dim E_n = n$. Combining \(\text{(3.35)}\) and \(\text{(3.36)}\) with the fact that $T$ is isometric, we arrive at

$$b_n(I) \geq \inf_{u \in X_n} \|u\|_{L^{p^*}(\Omega)} = \inf_{\{\alpha_j\}_{j=1}^\infty \in E_n} \left\| \sum_{j=1}^\infty \alpha_j u_j \right\|_{L^{p^*}(\Omega)} = \lambda \inf_{\{\alpha_j\}_{j=1}^\infty \in E_n} \|\{\alpha_j\}_{j=1}^\infty\|_{\ell_{p^*}} \geq \lambda (n^{-\frac{m}{d}} - \varepsilon).$$

Letting $\varepsilon \to 0^+$ and $\lambda \to \|I\|^{-}$, we obtain

$$b_n(I) \geq \|I\|n^{-\frac{m}{d}}.$$

Note that this is actually the desired lower bound in \(\text{(3.22)}\) because we can take $C_1 = \|I\|$ and indeed, the norm of the embedding $V_0^{m,p}(\Omega) \to L^{p^*}(\Omega)$ depends only on $d$, $m$ and $p$ but not on $\Omega$. This follows from the simple observation that

$$\|I: V_0^{m,p}(B) \to L^{p^*}(B)\| \leq \|I: V_0^{m,p}(\Omega) \to L^{p^*}(\Omega)\| \leq \|I: V_0^{m,p}(\tilde{B}) \to L^{p^*}(\tilde{B})\|,$$

where $B$ and $\tilde{B}$ are (any) open balls in $\mathbb{R}^d$ such that $B \subseteq \Omega \subseteq \tilde{B}$, and from the fact that $\|I: V_0^{m,p}(B) \to L^{p^*}(B)\|$ is constant for every open ball $B \subseteq \mathbb{R}^d$ and depends only on $d$, $m$ and $p$—to that end, recall \(\text{(3.2)}\).
We conclude with the Lorentz case. The following theorem tells us that the “really optimal” Sobolev embedding (1.2) is not strictly singular (let alone finitely strictly singular); moreover, all its Bernstein numbers coincide with its norm.

**Theorem 3.4.** Let $\Omega \subseteq \mathbb{R}^d$ be a nonempty bounded open set, $m \in \mathbb{N}$, $1 \leq m < d$, and $p \in [1, d/m)$. Denote by $I$ the identity operator $I : V_0^{m,p}(\Omega) \to L^{p^*,p}(\Omega)$, where $p^* = dp/(d - mp)$. We have

$$b_n(I) = \|I\| \quad \text{for every } n \in \mathbb{N},$$

(3.37)

where $\|I\|$ denotes the operator norm.

Furthermore, $I$ is not strictly singular.

**Proof.** Let $\varepsilon > 0$ and $0 < \lambda < \|I\|$, and $\{u_j\}_{j=1}^\infty \subseteq V_0^{m,p}(\Omega)$ be a system of functions from Proposition 3.1 with $q = p$.

Thanks to the property (S1) of (strict) $s$-numbers, it is sufficient to show that

$$b_n(I) \geq \|I\| \quad \text{for every } n \in \mathbb{N}.$$

Let $X_n$ be the subspace of $V_0^{m,p}(\Omega)$ spanned by the functions $u_1, \ldots, u_n$. Since the functions $u_j$ have mutually disjoint supports, it follows that $\dim X_n = n$. Since the functions $u_j$ have mutually disjoint supports, we have, for every $u = \sum_{j=1}^n \alpha_j u_j \in X_n$,

$$\|u\|_{V_0^{m,p}(\Omega)}^p = \sum_{j=1}^n |\alpha_j|^p.$$

Furthermore, thanks to (3.1),

$$\|u\|_{L^{p^*,p}(\Omega)}^p = \left\| \sum_{j=1}^n \alpha_j u_j \right\|_{L^{p}(\Omega)}^p \geq \frac{\lambda^p}{1 + \varepsilon} \sum_{j=1}^n |\alpha_j|^p.$$

Hence

$$b_n(I) \geq \inf_{u \in X_n \setminus \{0\}} \frac{\|u\|_{L^{p^*,p}(\Omega)}}{\|u\|_{V_0^{m,p}(\Omega)}} \geq \frac{\lambda}{(1 + \varepsilon)^{\frac{p}{p^*}}}.$$

Since this holds for every $\varepsilon > 0$ and $0 < \lambda < \|I\|$, it follows that $b_n(I) \geq \|I\|$.

Finally, to show that $I$ is not strictly singular, it is sufficient to take any $\varepsilon > 0$ and $0 < \lambda < \|I\|$ and consider the infinite dimensional subspace of $V_0^{m,p}(\Omega)$ spanned by the functions $u_1, u_2, \ldots$ Arguing as above, we immediately see that $I$ is bounded from below on this infinite dimensional subspace. Therefore, $I$ is not strictly singular. \hfill $\Box$

**Remark 3.5.** In light of (2.1), (3.37) actually tells us that, in the case of the “really optimal” Sobolev embedding (1.2), we have

$$s_n(I) = \|I\|$$

for every $n \in \mathbb{N}$ and every injective strict $s$-number $s$. 

References

[1] R. A. Adams and J. J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003. ISBN 0-12-044143-8.

[2] F. Albiac and N. J. Kalton. *Topics in Banach space theory*, volume 233 of *Graduate Texts in Mathematics*. Springer, [Cham], second edition, 2016. doi: 10.1007/978-3-319-31557-7.

[3] S. Artstein-Avidan, A. Giannopoulos, and V. Milman. *Asymptotic geometric analysis. Part I*, volume 202 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015. doi: 10.1090/surv/202.

[4] C. Bennett and R. Sharpley. *Interpolation of operators*, volume 129 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988. ISBN 0-12-088730-4.

[5] O. Bouchala. Measures of non-compactness and Sobolev-Lorentz spaces. *Z. Anal. Anwend.*, 39(1):27–40, 2020. ISSN 0232-2064. doi: 10.4171/zaa/1649.

[6] J. Bourgain and M. Gromov. Estimates of Bernstein widths of Sobolev spaces. In *Geometric aspects of functional analysis (1987–88)*, volume 1376 of *Lecture Notes in Math.*, pages 176–185. Springer, Berlin, 1989. doi: 10.1007/BFb0090054.

[7] J. Bourgain, J. Lindenstrauss, and V. Milman. Approximation of zonoids by zonotopes. *Acta Math.*, 162(1-2):73–141, 1989. doi: 10.1007/BF02392835.

[8] B. Carl and I. Stephani. *Entropy, Compactness and the Approximation of Operators*. Cambridge Tracts in Mathematics. Cambridge University Press, 1990. doi: 10.1017/CBO9780511897467.

[9] D. E. Edmunds and W. D. Evans. *Spectral theory and differential operators*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1987. ISBN 0-19-853542-2. Oxford Science Publications.

[10] D. E. Edmunds and J. Lang. *Eigenvalues, embeddings and generalised trigonometric functions*, volume 2016 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011. ISBN 978-3-642-18267-9. doi: 10.1007/978-3-642-18429-1.

[11] D. E. Edmunds and H. Triebel. *Function spaces, entropy numbers, differential operators*, volume 120 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996. ISBN 0-521-56036-5. doi: 10.1017/CBO9780511662201.

[12] D. E. Edmunds, J. Lang, and Z. Mihula. Measure of noncompactness of Sobolev embeddings on strip-like domains. *J. Approx. Theory*, 269: Paper No. 105608, 13, 2021. ISSN 0021-9045. doi: 10.1016/j.jat.2021.105608.
[13] S. Hencl. Measures of non-compactness of classical embeddings of Sobolev spaces. Math. Nachr., 258:28–43, 2003. ISSN 0025-584X. doi: 10.1002/mana.200310085.

[14] R. Kerman and L. Pick. Optimal Sobolev imbeddings. Forum Math., 18(4):535–570, 2006. ISSN 0933-7741. doi: 10.1515/Forum.2006.028.

[15] R. Kerman and L. Pick. Compactness of Sobolev imbeddings involving rearrangement-invariant norms. Studia Math., 186(2):127–160, 2008. ISSN 0039-3223. doi: 10.4064/sm186-2-2.

[16] V. A. Kozlov, V. G. Maz’ya, and J. Rossmann. Spectral problems associated with corner singularities of solutions to elliptic equations, volume 85 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2001. ISBN 0-8218-2727-8. doi: 10.1090/surv/085.

[17] J. Lang and V. G. Maz’ya. Essential norms and localization moduli of Sobolev embeddings for general domains. J. Lond. Math. Soc. (2), 78(2):373–391, 2008. ISSN 0024-6107. doi: 10.1112/jlms/jdn035.

[18] J. Lang and V. Musil. Strict s-numbers of non-compact Sobolev embeddings into continuous functions. Constr. Approx., 50(2):271–291, 2019. ISSN 0176-4276. doi: 10.1007/s00365-018-9448-0.

[19] J. Lang, V. Musil, M. Olšák, and L. Pick. Maximal non-compactness of Sobolev embeddings. J. Geom. Anal., 31(9):9406–9431, 2021. doi: 10.1007/s12220-020-00522-y.

[20] J. Lang, Z. Mihula, and L. Pick. Compactness of Sobolev embeddings and decay of norms. Studia Math., 265(1):1–35, 2022. ISSN 0039-3223. doi: 10.4064/sm201119-29-9.

[21] P. Lefèvre and L. Rodríguez-Piazza. Finitely strictly singular operators in harmonic analysis and function theory. Adv. Math., 255:119–152, 2014. ISSN 0001-8708. doi: 10.1016/j.aim.2013.12.034.

[22] D. R. Lewis. Finite dimensional subspaces of $L_p$. Studia Math., 63(2):207–212, 1978. doi: 10.4064/sm-63-2-207-212.

[23] J. Malý and W. P. Ziemer. Fine regularity of solutions of elliptic partial differential equations, volume 51 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1997. doi: 10.1090/surv/051.

[24] V. G. Maz’ya. Sobolev spaces with applications to elliptic partial differential equations, volume 342 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Heidelberg, augmented edition, 2011. ISBN 978-3-642-15563-5. doi: 10.1007/978-3-642-15564-2.

[25] V. G. Maz’ya and S. V. Poborchi. Differentiable functions on bad domains. World Scientific Publishing Co., Inc., River Edge, NJ, 1997. ISBN 981-02-2767-1.

[26] J. Peetre. Espaces d’interpolation et théorème de Soboleff. Ann. Inst. Fourier (Grenoble), 16(1):279–317, 1966. ISSN 0373-0956. doi: 10.5802/aif.232.
[27] L. Pick, A. Kufner, O. John, and S. Fučík. *Function spaces. Vol. 1*, volume 14 of *De Gruyter Series in Nonlinear Analysis and Applications*. Walter de Gruyter & Co., Berlin, extended edition, 2013. ISBN 978-3-11-025041-1; 978-3-11-025042-8. doi: 10.1515/9783110250428.

[28] A. Pietsch. *s*-numbers of operators in Banach spaces. *Studia Math.*, 51: 201–223, 1974. doi: 10.4064/sm-51-3-201-223.

[29] A. Pietsch. *History of Banach spaces and linear operators*. Birkhäuser Boston, Inc., Boston, MA, 2007. ISBN 978-0-8176-4367-6; 0-8176-4367-2.

[30] A. Plichko. Superstrictly singular and superstrictly cosingular operators. In *Functional analysis and its applications*, volume 197 of *North-Holland Math. Stud.*., pages 239–255. Elsevier Sci. B. V., Amsterdam, 2004. doi: 10.1016/S0304-0208(04)80172-5.

[31] G. Schechtman and A. Zvavitch. Embedding subspaces of $L_p$ into $l_p^N$, $0 < p < 1$. *Math. Nachr.*, 227:133–142, 2001. doi: 10.1002/1522-2616(200107)227:1<133::AID-MANA133>3.0.CO;2-8.

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