Bounds for spiral and piecewise spiral splines

Alexey Kurnosenko

Abstract

This note is the updated outline of the article “Interpolational properties of planar spiral curves”, Fund. and Applied Math., 7(2001), N2, 441–463, published in Russian. The main result establishes boundary regions for spiral and piecewise spiral splines, matching given data. The width of such region can serve as the measure of fairness of the point set, subjected to interpolation. Application to tolerance control of curvilinear profiles is discussed.

1. Introduction

This note is intended to present in English the results of the article [1], which has aroused interest among researchers, working with spirals, i.e. planar curves with monotone curvature.

It is not a pure translation of [1]. Some auxiliary results of the article, such as statements (6), were originally developed for “very short” spiral arcs, namely, one-to-one projectable onto the chord. These statements are now essentially expanded, and presented to readers as the elements of the theory of spiral arcs (see [3, 4] and references herein). Here corresponding proofs are omitted.

Only final results of article [1] are reproduced in this note. They establish boundary regions for spiral and piecewise spiral splines, matching given planar point set. The width of such region is considered as the objective measure of fairness of the given data.

Section 2 and Appendix A, absent in the original article, consider the subject in the view of tolerances control of curvilinear profiles.

2. Demonstration of the bounding region

We consider problems, related to interpolation of planar point sets. Often one can more or less definitely decide: “this interpolant seems to be good, and this one is not” (e.g., contains unwanted oscillations). In brief, below we construct the region, enclosing all “good interpolants”.

To begin with, let us demonstrate such regions in the context of two problems, interpolation of function vs curve interpolation. Let points \((x_i, y_i)\) be defined by some function \(y = f(x)\) as in Fig. 1. For the point set a), among various interpolants, we definitely reject non-monotonous ones, because the sequence \(y_i\) is monotone. We assume that the designer of this point set takes care to present all existing extrema of the function to improve the quality of
future interpolation. The region, bounding all monotone interpolants, is shown as the union of rectangles.

Now consider the sequence of non-monotone data with maximum at point 5 (point set in Fig. 1(b,c)). If the point \((x_5, y(x_5))\) definitely corresponds to the maximum of \(y(x)\), bounding region can still be constructed (the case b)).

For function interpolation these regions are trivial and not interesting. Analogous regions for curve interpolation turned out to be much more interesting and useful. Drawing with curvilinear profile ABCD in Fig. 2 yields an example. The coordinates \((x_i, y_i)\) of the interpolation nodes could result from some known curve \([x(t), y(t)]\), which might have an exact representation in rather complicated expressions or, e.g., in terms of differential equations. In real practice such profile would be presented by 20–50 points. We do with only 4 points, just to be able to examine the bounding region without a microscope.

Data for curve interpolation include values of two functions \(x(t)\) and \(y(t)\), with no argument values \(t_i\). Monotonicity and extrema of these functions have no importance: these features are not even invariant under rotations. To obtain some invariant conclusions, one would like to analyze the behavior of curvature \(k(t)\). The analogues of “unwanted extrema” of \(f(x)\) might be extra extrema of curvature. Contrary to extrema of function, whose minimal number is directly visible from the plots like Fig. 1, a minimum of curvature extrema is far from being evident. Nevertheless, given data \(x_i, y_i\) (supplemented with tangents at the endpoints) allows us to detect if the original curve could be a spiral.

Fig. 2. Drawing with curvilinear profile ABCD; blue: bounding region, enclosing all possible spiral interpolants; green: bounds for the same profile with 7 interpolation nodes.
Below we prove that all possible spiral interpolants stay within the narrow region (shown in blue in the above example $ABCD$, of the width $\approx 1.1$). Conversely, any curve, going beyond these limits, is not a spiral.

3. Geometric preliminaries

First we explain the simplest construction of the bounding region. Denote

$$n(\varphi) = (\cos \varphi, \sin \varphi)$$

the unit vector, associated with the angle $\varphi$. A set of 12 points $P_1, \ldots, P_{12}$ and end tangents $n(\tau_1), n(\tau_{12})$ in Fig. 3(a) is the data to be interpolated. In Fig. 3(b) 12 circular arcs are traced as follows: arc $P_1P_2$ of the curvature $q_1$ passes through points $P_1$ and $P_2$, matching at $P_1$ given tangent $n(\tau_1)$; circular arcs $P_1P_2P_3$, $P_2P_3P_4$, $\ldots$, $P_10P_{11}P_{12}$, with curvatures $q_2$, $\ldots$, $q_{11}$ pass through triples of consecutive points; arc $P_{11}P_{12}$ of the curvature $q_{12}$ matches given tangent $n(\tau_{12})$ at the endpoint. Thus we obtain, on each chord $P_jP_{j+1}$, two circular arcs, forming the lens. Together these lenses form the sought for region.
Fig. 4. Definition of the angles $\alpha, \beta, \xi, \eta$ and illustration to the proof of Prop. 1. Nodes 1, 2, 3, 4 are those of interpolated spiral (dotted); dashed line are circles of curvature at nodes 2, 3.

### 3.1. Spline data

For the given set of $N$ points denote $M = N - 1$ the number of chords; denote also the length of $j$-th chord as $2c_j = |P_j P_{j+1}|$, and its angular direction as $\mu_j$:

$$1 \leq j \leq M: \quad c_j = \frac{1}{2} \sqrt{(x_{j+1} - x_j)^2 + (y_{j+1} - y_j)^2}, \quad \cos \mu_j = \frac{x_{j+1} - x_j}{2c_j}, \sin \mu_j = \frac{y_{j+1} - y_j}{2c_j};$$

$$j = 0, j = N: \quad c_0 = c_N = 0, \mu_0 = \tau_1, \mu_N = \tau_N. \quad (1)$$

Definitions (1) imitate boundary tangents, as if two additional points, $P_0$ and $P_{N+1}$, were added, forming two infinitesimal pseudo-chords, $P_0 P_1$ and $P_N P_{N+1}$, keeping specified directions. Two-point arcs $P_1 P_2$ and $P_{N-1} P_N$ can now be described by three-point formulas (2),(3) as $P_0 P_1 P_2$ and $P_{N-1} P_N P_{N+1}$. Turning angles $\rho_j$ of the chord in every $j$-th node, $1 \leq j \leq N$, and signed curvatures $q_j$ of directed circles $P_{j-1} P_j P_{j+1}$ can be defined by

$$\rho_j = \mu_j - \mu_{j-1}, \quad q_j = \frac{\sin \rho_j}{d_j}, \quad \text{where} \quad d_j = \frac{1}{2} |P_{j-1} P_{j+1}| = \sqrt{c_{j-1}^2 + 2c_{j-1} c_j \cos \rho_j + c_j^2}. \quad (2)$$

Curvatures $q_j$ are expected to be close to unknown curvatures $k_j$ in the nodes of the original curve.

Fig. 4(a) illustrates unknown tangents $n(\alpha_j)$, $\alpha_j = \tau_j - \mu_j$, and $n(\beta_j)$, $\beta_j = \tau_{j+1} - \mu_j$, at the endpoints of the $j$-th segment of the interpolated spiral, measured with respect to the direction $n(\mu_j)$ of the $j$-th chord. To estimate $\alpha_j$, $\beta_j$, the angles $\xi_j$ and $\eta_j$ are used (Fig. 4(b)). Vector $n(\xi_j)$ is the tangent to the circular arc $P_{j-1} P_j P_{j+1}$ at $P_j$; vector $n(\eta_j)$ is the tangent to the arc $P_j P_{j+1} P_{j+2}$ at $P_{j+1}$; both $\xi_j$ and $\eta_j$ are also measured with respect to $n(\mu_j)$:

$$\sin \xi_j = \frac{-c_j \sin \rho_j}{d_j} = -c_j q_j, \quad \sin \eta_j = \frac{c_j \sin \rho_{j+1}}{d_{j+1}} = c_j q_{j+1}, \quad \cos \xi_j = \frac{c_{j-1} + c_j \cos \rho_j}{d_j}, \quad \cos \eta_j = \frac{c_{j+1} + c_j \cos \rho_{j+1}}{d_{j+1}}, \quad 1 \leq j \leq M. \quad (3)$$
Let \( n(\vartheta_j) \) denote the tangent to the circular arc \( P_{j-1}P_jP_{j+1} \) at \( P_j \), measured in global coordinates. Note that
\[
\begin{align*}
\tau_j &= \mu_{j-1} + \beta_{j-1} = \mu_j + \alpha_j \\
\vartheta_j &= \mu_{j-1} + \eta_{j-1} = \mu_j + \xi_j
\end{align*}
\]  
\( \implies \beta_{j-1} = \rho_j + \alpha_j, \quad \eta_{j-1} = \rho_j + \xi_j. \) \( \tag{4} \)

### 3.2. Spiral arc in normalized position

We consider spiral and circular arcs between two neighbouring nodes as curves, one-to-one projectable onto the subtending chord. Therefore they can be represented as functions \( y(x) = f_j(x) \) in the local coordinate system such that the \( j \)-th chord becomes the segment \([ -c_j, c_j ]\) of the local \( x \)-axis:
\[
y(\pm c_j) = 0; \quad y'(c_j) = \tan \alpha_j, \quad y'(c_j) = \tan \beta_j, \quad \kappa_j(x) = \frac{y''}{(1 + y'^2)^{3/2}} \quad \tag{5}
\] is monotone curvature function.

The circular arc, resting on the segment \([ -c, c ]\) of the \( x \)-axis with the tangent vector \( n(\varphi) \) at the start point, is denoted as
\[A(x; c, \varphi) = \frac{\cos \varphi}{c} \sin \varphi + \frac{\cos \varphi}{c} \cos \varphi + x, \quad |x| \leq c, \quad |\varphi| \leq \frac{\pi}{2}.\]

The spiral segment has tangent \( n(\alpha) \) and curvature \( a \) at the startpoint \( A = (-c, 0) \), and tangent \( n(\beta) \) and curvature \( b \) at the endpoint \( B = (c, 0) \). Below two biarcs \( A J B \) illustrate this data. Assuming that the curvature (5) is increasing, \( a = \kappa(-c) < \kappa(c) = b \), the following relations are valid:
\[
\begin{align*}
\alpha + \beta &> 0 \quad \text{[or \( \text{sgn}(b-a) = \text{sgn}(\alpha+\beta) \);]} \quad \tag{6a}\\
ac &< -\sin \alpha, \quad \sin \beta < bc; \quad \tag{6b}\\
A(x; c, -\beta) < f(x) &< A(x; c, \alpha), \quad |x| < c. \quad \tag{6c}
\end{align*}
\]

Eq. (6a) is modified Vogt’s theorem: (see [3], St. 3). Eqs. (6b) are commented in [3], St. 6. Eq. (6c) is lens theorem (see [3], St. 5). These relations turn to equalities if the curvature is constant.

### 4. Bound for a spiral spline

Interpolation of a curve usually assumes that the neighbouring nodes are located rather closely. The corresponding restrictions could be expressed in terms of sufficiently small turning angles \( \rho_j \), e.g., \( |\rho_j| \leq \pi/2 \). We apply even more weak constraints,
\[
c_j-1 + c_j \cos \rho_j \geq 0 \quad \text{and} \quad c_j + c_j-1 \cos \rho_j \geq 0. \quad \tag{7}
\]
This requires each of two arcs, \( P_{j-1}P_j \) and \( P_jP_{j+1} \), of the circle \( P_{j-1}P_jP_{j+1} \) not to exceed 180°. Three last examples below (red) violate these conditions.

Propositions 1 and 2 constitute Theorem 5 in [1].
Proposition 1. The union of lenses, formed by arcs \( A(x; c_j, -\eta_j) \) and \( A(x; c_j, \xi_j) \) on every chord \( P_jP_{j+1}, j = 1, \ldots, M \), covers all possible spirals, matching given data \( \{P_1, \ldots, P_{M+1}; \tau_1, \tau_{M+1}\} \).

**Proof.** In the below proofs only the case of increasing curvature is considered. Denote \( f_j(x) \) the \( j \)-th segment of original (or interpolated) spiral. Inequalities near \( f_j(x) \) in the chain
\[
A(x; c_j, -\eta_j) \leq A(x; c_j, -\beta_j) \leq f_j(x) \leq A(x; c_j, \alpha_j) \leq A(x; c_j, \xi_j)
\]
constitute the lens theorem for a spiral arc (Eq. (6c) and Fig. 4(a)); non-strict form “\( \leq \)” takes into account the possibility of constant curvature. Outer inequalities express the statement of Prop. 1 for every \( j \)-th chord. To show that the lens, defined by \((\alpha_j, -\beta_j)\), is enclosed by the lens \((\xi_j, -\eta_j)\), we have to prove
\[
\xi_j \geq \alpha_j \geq -\beta_j \geq -\eta_j.
\]
The inner inequality, \( \alpha + \beta \geq 0 \), is Vogt’s theorem (6a) for increasing (\( > \)) or constant (\( = \)) curvature of the spiral arc \( f_j(x) \). Note also that, by construction,
\[
\alpha_1 = \xi_1, \quad \beta_M = \eta_M.
\]
To illustrate proof of (9), the circle of curvature of the spiral at node \( j \) \( (j = 2) \) is traced by dashed line in Fig. 4(c); tangent \( n(\alpha_2) \) to the spiral at \( P_2 \) is also tangent to this circle. It is known that a spiral arc with increasing curvature intersects the osculating circle from right to left. The point \( P_1 \) is located to the right of the circle, and the point \( P_3 \) to the left. Therefore the circular arc, passing through points \( P_{1,2,3} \), intersects the osculating circle from right to left, and \( \xi_2 > \alpha_2 \). The case \( \xi_2 = \alpha_2 \) occurs if the segment \( P_1P_2P_3 \) of the original spiral is coincident with the osculating circle. Similarly, from mutual position of the circular arc through \( P_{2,3,4} \), and the osculating circle at \( P_3 \) (Fig. 4(d)) we deduce \( \eta_2 \geq \beta_2 \). □

The width \( \varnothing \) of the united region is
\[
\varnothing = \max_{1 \leq j \leq M} \varnothing_j, \quad \varnothing_j = c_j \left| \tan \frac{\xi_j}{2} + \tan \frac{\eta_j}{2} \right| \approx \frac{1}{2} c_j^2 |q_j - q_{j+1}|.
\]
Increasing \( N \) decreases \( c_j \) and \( |q_j - q_{j+1}| \approx |k_j - k_{j+1}| \), which means cubic convergence of the width \( \varnothing \) with \( N \). Fig. 3(c), showing the region for 16 points, illustrates this effect.

Proposition 2. If turning angles \( \rho_j \) obey constraints (7), the sequence \( q_j, j = 1, \ldots, N \), of 3-point curvatures is monotone.

**Proof.** Comparison of (7) and (3) yields \( |\xi_j| \leq \frac{\pi}{2} \), and \( |\eta_j| \leq \frac{\pi}{2} \); so,
\[
-\eta_j \overset{(9)}{\leq} \xi_j \quad \Rightarrow \quad -\sin \eta_j \leq \sin \xi_j \quad \overset{(3)}{\Rightarrow} \quad -q_{j+1}c_j \leq -q_jc_j \quad \Rightarrow \quad q_{j+1} \geq q_j. \quad \square
\]

5. Bound for spline with distinguished vertices (piecewise spiral spline)

Consider the case when the sequence of 3-point curvatures is not monotone, although conditions (7) are satisfied. This means that the original curve is not a spiral. Assume that
\[
\text{nodes with minimal/maximal curvature } q_j \text{ are exactly the vertices of the curve; } \quad (11a)
\]
\[
\text{there is at least one node between two neighbouring vertices. } \quad (11b)
\]
Under these assumptions there is also a simple way to construct the bounding region. In Fig. 5 the ellipse is subdivided by 13 points. Analyzing the sequence of 3-point curvatures
and assuming (11), we detect vertices at $P_1, P_3, P_8, P_{10}$. The 3-point arcs $P_{j-1}P_jP_{j+1}$ are traced only for the triples, wherein the midpoint $P_j$ is not a vertex (Fig. 5(a)). As before, we obtain the pair of boundary arcs on chords $2–3, 3–4, 6–7, 11–12, 12–13$. We have only one circular arc on each chord, adjacent to a vertex. Lacking arcs (12) are shown in Fig. 5(b) by bold (red) lines. The angles in (12) are such that, e.g., bold arc 1–2 shares tangent at $P_2$ with the arc 2–3–4; tangent to the arc 2–3–4 at $P_4$ is matched by the 2-point arc 4–5, and so on.

The circle of curvature of the ellipse at the point $P_8$ (Fig. 5(a)) shows that the distances $|P_7P_8|, |P_8P_9|$ are too big, compared to the local radius of curvature. As consequence, the estimated bounding regions 7–8 and 8–9 are too wide.

**Proposition 3.** Under assumptions (11) the pair of boundary arcs for the point set with distinguished vertices is defined as follows:

| vertex at $P_j$: | vertex at $P_{j+1}$: | boundary arcs for the chord $P_jP_{j+1}$: |
|------------------|------------------|----------------------------------|
| yes              | no               | $A(x; c_j, -\eta_j)$, $A(x; c_j, -\xi_{j-1} - \rho_j)$; |
| no               | yes              | $A(x; c_j, \eta_{j+1} - \rho_{j+1})$, $A(x; c_j, \xi_j)$; |
| no               | no               | $A(x; c_j, -\eta_j)$, $A_j(x; c_j, \xi_j)$.

**Proof.** The case no/no is copied from (8). Condition (11b) excludes the case yes/yes.

First, we note that for the closed curve there is no need to specify the boundary tangents $\tau_1, \tau_N$: to define the turning angles $\rho_1, \rho_N$ definitions (1) should be replaced by $P_0 = P_N$ and $P_{N+1} = P_1$, assuming $M = N$ (the number of chords is equal to the number of nodes).

Consider the segment (3-4-5-6-7) with curvature decreasing in (3-4-5) to the vertex at node 5 (minimum), and then increasing in (5-6-7). Inequalities (9) look like

$$P_4P_5: \xi_4 < \alpha_4 < -\beta_4 < ?, \quad P_5P_6: -\eta_5 < -\beta_5 < \alpha_5 < \hat{\iota},$$

and yield $-\alpha_5 < \eta_5$, and $\beta_4 < -\xi_4$. Missed right-most restrictions can now be obtained as follows:

$$-\beta_4 \overset{(4)}{=} -\alpha_5 - \rho_5 < \eta_5 - \rho_5; \quad \alpha_5 = \beta_4 - \rho_5 < -\xi_4 - \rho_5.$$
Similar reasoning yields similar bounds for the vertex of maximal curvature; e.g., for vertex 8
\[ P_7P_8: \xi_7 > -\beta_7 > \eta_8 - \rho_8; \quad P_8P_9: -\eta_8 < -\beta_8 < \alpha_8 < -\xi_7 - \rho_8. \]
Together they look like (with upper signs for a vertex-minimum)
\[
\begin{align*}
\xi_j \lesssim & \alpha_j \lesssim -\beta_j \lesssim \eta_{j+1} - \rho_{j+1}; \\
-\eta_j \lesssim & -\beta_j \lesssim \alpha_j \lesssim -\xi_{j-1} - \rho_j.
\end{align*}
\]
This yields bounds (12) for yes/no and no/yes cases.

Note that the discrete curvature plot may look like \( \overbrace{\text{ }} \), including the segments of constant curvature, which can be named “extended vertices”. Any node of the extended vertex can be chosen as the vertex to apply the above described algorithm.

6. Narrowing the bounding region for spiral splines

Fig. 6 illustrates the strengthened version of Prop. 1: some of the bounding arcs are replaced by biarcs. As the consequence, the bound for the whole region turned into a pair of smooth curves, intersecting at the nodes.

If 4 points \( P_{j-1}, \ldots, P_{j+2} \) are cocircular (e.g., collinear), we obtain \( q_j = q_{j+1} \), and zero width \( \varnothing_j = 0 \). Any spiral, matching such data, includes arc \( P_{j-1}P_jP_{j+1}P_{j+2} \) of constant curvature \( q_j \). “Simple” construction, defined by (8), yields the region of zero width on the segment \( P_jP_{j+1} \). Strengthened version (18) returns zero width on the whole segment \( P_{j-1}P_jP_{j+1}P_{j+2} \). Example in Fig. 7 illustrates this situation.

To describe the narrowed region (Prop. 5) we first introduce notation for biarc curves. Because all involved biarcs are one-to-one projectable onto the chord, they can be considered as functions \( B(x; \ldots) \) in the local coordinate system (6). The first circular arc \( AJ \) of a biarc in (6) has tangent \( \mathbf{n}(\alpha) \) and curvature \( a \) at the startpoint \( A = (-c, 0) \), and is smoothly continued at the joint point \( J \) by the second arc \( JB \) of curvature \( b \) to the endpoint \( (c, 0) \) with end tangent \( \mathbf{n}(\beta) \). The condition of tangency of arcs \( AJ \) and \( JB \) looks like
\[
(ac + \sin \alpha)(bc - \sin \beta) + \sin^2 \omega = 0, \quad \text{where} \quad \omega = \frac{\alpha + \beta}{2}
\]
Narrowed region (nodes 2-3-4-5 are not exactly collinear)

Narrowed region (nodes 2-3-4-5 are collinear)

Simple bounding region for the above data

Fig. 7. Regions for data with 4 cocircular (collinear) points

Fig. 8. Defining "degenerate" biarcs (15a) and (15b); lens boundaries are shown dashed

is the angular half-width of the associated lens. Details and reference formulae for a family of biarcs are given in [4].

It is well known that, fitting two-point $G^1$ data (tangents at the endpoints) with a biarc, we have one degree of freedom: one curvature, either $a$ or $b$, can be selected, the other should be defined from condition of tangency (13). We use the following notation for biarc curves:

$B_1(x; c, \alpha, \beta, a)$ for biarc, matching end tangents $n(\alpha), n(\beta)$, and the start curvature $a$;

$B_2(x; c, \alpha, \beta, b)$ for a biarc, matching end tangents $n(\alpha), n(\beta)$, and the end curvature $b$;

$B_0(x; c, \alpha, \beta, p), p \in [0; \infty]$, for a family of short biarcs [4] with curvatures

$$a = -\frac{1}{c} \left( \sin \alpha - \frac{1}{p} \sin \omega \right), \quad b = \frac{1}{c} \left( \sin \beta + p \sin \omega \right) \quad \left[ p = -\frac{\sin \omega}{ac + \sin \alpha} = \frac{bc - \sin \beta}{\sin \omega} \right].$$

(14)

Now define degenerate biarcs as follows. For $\alpha + \beta \geq 0$, and either $p = 0$ or $p = \infty$

$B_0(x; c, \alpha, \beta, 0) = \lim_{p \to 0} B_0(x; c, \alpha, \beta, p) = B_1(x; c, \alpha, \beta, +\infty) = A(x; c, -\beta)$;

$B_0(x; c, \alpha, \beta, \infty) = \lim_{p \to \infty} B_0(x; c, \alpha, \beta, p) = B_2(x; c, \alpha, \beta, \pm \infty) = A(x; c, \alpha)$.  

(15a)
For $\alpha + \beta = 0$:
\[
B_0(x; c, \gamma, -\gamma, p) = \lim_{\omega \to 0} B_0(x; c, \omega + \gamma, -\gamma, p) = A(x; c, \gamma).
\] (15b)

Fig. 8(a) illustrates definitions (15a): when, e.g., the join point tends to startpoint $A$, the first curvature $\alpha$ tends to infinite impulse of curvature at $A$, the first subarc of a biarc vanishes; simultaneously the second arc tends to the lens boundary. In Fig. 8(b) the case $\alpha + \beta = 2\omega \to 0$ (15b) is illustrated with lenses, narrowing down in width.

**Proposition 4.** Consider a spiral arc $y = f(x)$ with increasing curvature $\kappa(x)$. If boundary angles $\alpha, \beta$ and curvature fall in ranges
\[
\alpha' \leq \alpha \leq \alpha'', \quad \beta' \leq \beta \leq \beta'', \quad -\infty \leq a \leq \kappa(x) \leq b \leq +\infty,
\] (16)

than the curve is bounded by
\[
\begin{align*}
&\text{if } \alpha' + \beta'' \geq 0: \quad B_1(x; c, \alpha', \beta'', a) \leq f(x); \\
&\text{if } \alpha'' + \beta' \geq 0: \quad f(x) \leq B_2(x; c, \alpha'', \beta', b).
\end{align*}
\] (17)

The proof ([1], Th. 4) includes inequalities (labels correspond to Fig. 9):
\[
\begin{align*}
&\overbrace{AJ_1B}^{A_{J_1}B} \leq \overbrace{AJ_2B}^{A_{J_2}B} \leq \overbrace{AJ_3B}^{A_{J_3}B} \leq f(x); \\
&f(x) \leq \overbrace{AJ_4B}^{A_{J_4}B} \leq \overbrace{AJ_5B}^{A_{J_5}B} \leq \overbrace{AJ_6B}^{A_{J_6}B} \leq B_2(x; c, \alpha'', \beta', b).
\end{align*}
\]

Inequalities near $f(x)$ result from bilens theorem ([1, Th. 3], [4, Th. 1]). The rest can be obtained by inspecting the involved circular arcs.

**Proposition 5.** The narrowed bounding region for spiral data $\{(x_j, y_j), j = 1, \ldots, M+1, \tau_1, \tau_{M+1}\}$ with increasing curvature is defined by
\[
B_0(x; c_j, \alpha'_j, \beta''_j, p'_j) \leq f_j(x) \leq B_0(x; c_j, \alpha''_j, \beta'_j, p''_j).
\] (18a)

Biars’ parameters are
\[
\begin{array}{|c|c|c|c|c|}
\hline
& j = 1 & 1 < j \leq M & 1 \leq j < M & j = M \\
\hline
\alpha'_j & \xi_1 & \max(-\rho_j - \xi_{j-1}, -\eta_j) & \max(-\xi_j, \rho_{j+1} - \eta_{j+1}) & \eta_M \\
\hline
\alpha''_j & \xi_1 & \xi_j & \eta_j & \eta_M \\
\hline
a_j & -\infty & \frac{1}{c_{j-1}} \sin \beta'_{j-1} & -\frac{1}{c_{j+1}} \sin \alpha'_{j+1} & +\infty \\
\hline
p'_j & -\frac{\sin \omega}{a_j c_j + \sin \alpha'_j}, \quad \omega = \frac{\alpha'_j + \beta''_j}{2} & b_j = \frac{b_j c_j - \sin \beta'_{j+1}}{\sin \omega}, \quad \omega = \frac{\alpha''_j + \beta'_j}{2} & & \\
\hline
\end{array}
\] (18b)
Proof. Transform (9), following the scheme:

\[- \xi_1 \leq \beta_1 \leq \eta_1 \quad \Rightarrow \quad - \xi_1 - \rho_2 \leq \beta_1 - \rho_2 \leq \eta_1 - \rho_2 \quad \Rightarrow \quad - \xi_1 - \rho_2 \leq \alpha_2 \leq \xi_2,\]

\[- \eta_2 \leq \alpha_2 \leq \xi_2 \quad \Rightarrow \quad - \eta_2 + \rho_2 \leq \alpha_2 + \rho_2 \leq \xi_2 + \rho_2 \quad \Rightarrow \quad - \eta_2 + \rho_2 \leq \beta_1 \leq \eta_1.\]

Combining the first and the last column yields ranges (16) for \( \beta_1 \) and \( \alpha_2 \):

\[
\begin{align*}
\max(-\xi_1, \rho_2-\eta_2) & \leq \beta_1 \leq \eta_1, & \max(-\rho_2-\xi_1, -\eta_2) & \leq \alpha_2 \leq \xi_2. \\
\beta_1' & = \beta_1, & \beta_1'' & = \alpha_2, & \alpha_2' & = \beta_1, & \alpha_2'' & = \alpha_2.
\end{align*}
\]

Similarly, we obtain ranges for pairs \((\beta_2, \alpha_3), (\beta_3, \alpha_4), \ldots, (\beta_{M-1}, \alpha_M)\). Missing ranges for \( \alpha_1 \) and \( \beta_M \) are known from (10): \( \alpha_1' = \alpha_1'' = \xi_1, \beta_M' = \beta_M'' = \eta_M \). Two rows of the table (18b) are thus filled.

Now check conditions in (17). E. g., for \( \alpha_j' + \beta_j'' \geq 0 \) we have

\[
\begin{align*}
& j = 1: \quad \alpha_1' + \beta_1'' = \xi_1 + \eta_1 \geq 0; \\
& j > 1: \quad \alpha_j' + \beta_j'' = \max(-\eta_j, -\rho_j - \xi_{j-1}) + \eta_j = \max(0, \eta_j - \rho_j - \xi_{j-1}) \geq 0.
\end{align*}
\]

Similarly, conditions \( \alpha_j'' + \beta_j' \geq 0 \) hold.

To estimate ranges for unknown curvatures \( k_j \) in nodes \( j = 1, \ldots, M+1 \), we make use of inequalities (6b); non-strict form accounts for the possible case of constant curvature:

\[
\text{on the chord} \quad \begin{array}{ll}
P_1P_2: & k_1c_1 \leq -\sin \alpha_1, \quad k_2c_1 \geq \sin \beta_1; \\
P_2P_3: & k_2c_2 \leq -\sin \alpha_2, \quad k_3c_2 \geq \sin \beta_2; \\
\cdots & \cdots \\
P_{M-1}P_M: & k_{M-1}c_{M-1} \leq -\sin \alpha_{M-1}, \quad k_Mc_{M-1} \geq \sin \beta_{M-1}; \\
\text{last chord} & \begin{array}{ll}
P_MP_{M+1}: & k_Mc_M \leq -\sin \alpha_M, \quad k_{M+1}c_M \geq \sin \beta_M.
\end{array}
\end{array}
\]

This results in

\[
\begin{align*}
\sin \beta_{j-1} \quad \sin \beta_{j-1} \quad \sin \beta_{j-1} \quad \sin \beta_{j-1} \quad \sin \beta_{j-1}
\end{align*}
\]

i.e. \( -\infty \leq \kappa_1(x) \leq b_1, \quad a_2 \leq \kappa_2(x) \leq b_2, \quad \ldots, \quad a_M \leq \kappa_M(x) \leq +\infty \). Prop. 4 can now be applied to define boundary biarcs:

\[
B_1(x; c_j, \alpha_j', \beta_j'', a_j) \leq f_j(x) \leq B_2(x; c_j, \alpha_j'', \beta_j', b_j).
\]

It is easy to verify that degenerate cases, arising in (20) with \( a_1 = -\infty \), or \( b_M = \infty \), or \( \alpha + \beta = 0 \), result in “simple” bounds (8). E. g., if \( \alpha_j' + \beta_j'' = 0 \) is the case, \( \alpha_j' = -\beta_j'' = -\eta_j \), and

\[
B_1(x; c_j, \alpha_j', \beta_j'', a_j) = B_0(x; c_j, -\eta_j, \eta_j, p) \quad \text{(15b)} = A(x; c_j, -\eta_j).
\]

Eq. (20) is rewritten in (18) in terms of “universal” function \( B_0(x; \alpha, \beta, p) \) (14).

Note also that possible continuation of (19) is

\[
q_{j-1} = -\frac{\sin \xi_{j-1}}{c_{j-1}} \leq \sin \beta_{j-1} \quad \sin \beta_{j-1} \quad \sin \beta_{j-1} \quad \sin \beta_{j-1} \quad \sin \beta_{j-1}
\]

unknown curvatures \( k_j \) are limited by known 3-point neighbouring curvatures \( q_{j \pm 1} \). The above chosen ranges \([a_j; b_j] \) yield more narrow region.

It may also happen that limits \( \pm \infty \) for end curvatures (18b) can be specified from some additional considerations. E. g., in the 3-point data \((M = 2)\), shown below, one can assume

\[
\text{In (20) “hidden” degenerate biarcs may occur. E. g., the biarc } B_2(x; c, \alpha'', \beta', b), \text{ regular at first sight, becomes degenerate if } bc - \sin \beta' = 0, \text{ and equal to } B_1(x; c, \alpha'', \beta', \infty) = B_0(x; c, \alpha'', \beta', 0). \text{ This happens in the data like shown in Fig. 7, chord 2.}
\]
positivity of curvature, and replace \( a_1 = -\infty \) by \( a_1 = 0 \). This replaces the lower boundary arc \( A(x; c_1, -\eta_1) \) (dashed) by the regular biarc \( B_1(x; c_1, \xi_1, \eta_1, 0) \):

\[
\begin{array}{c}
1 \quad \tau_1 \quad 2 \\
1 \quad \tau_1
\end{array}
\]

\(* * *

The specific occasion of the research [1] was eventuated in about 1995\(^2\), in the workshop of the Institute for High Energy Physics (IHEP, Protvino). Research originated from a practical problem of inspection of a certain cam profile: there was, on a small part of the 360°-profile, a big discrepancy between nominal and measured curves. The reason, expressed in present-day terms, was that the design of the profile included very unfair point subsets.

While exploring the situation, the author has become interested to find some quantitative measure of such drawbacks in drawings, possibly, a tool for an expert-metrologist. Purely geometrically the problem was shaped as follows: to describe a set of curves with as few vertices as possible, matching given interpolation data. The simplest case (spiral, zero vertices) gave a well restricted solution, justifying such problem setting. Proposition 3 is the step towards non-spiral curves.

A. Application to computer-aided tolerancing

A.1. Bounding regions in the view of tolerance control

Bounding region is not quite a new concept for industrial design. Returning to Fig. 2, consider the hole, dimensioned as \( \varnothing 26 \pm 0.2 \). This can be treated as the bounding region (in the form of ring) for the circular profile. The given tolerance admits inaccuracy of order 0.2 in coordinates of the center of the hole. If declared as the datum element, the hole defines the origin of the coordinate system; consequently, any other dimension of the type “coordinate” on this drawing should be agreed with this inaccuracy. E.g., setting tolerance of order 0.1 for the dimension 100 would be an evident error in the drawing.

Normally such error will be corrected by an expert-metrologist before starting the manufacturing process. Possible correction could be specifying the roundness tolerance \( \varepsilon \ll 0.1 \) for the hole. This allows the value of diameter still to be in the range 25.8 \( \leq \varnothing \leq 26.2 \), but reduces the width of the ring-shaped bounding region to much smaller acceptable value.

Note that, given all dimensions and tolerances, one can usually construct the exact model of the ideal part, corresponding, for the case of the hole, to \( \varnothing 26 \pm 0. \). The position of any point of the circular and straight line elements will be perfectly determined. But this is not so for the elements like profile \( ABCD \). Since it is not presented by exact equations, and the interpolation method is not exactly specified, there occurs certain nondeterminancy in constructing the nominal profile itself.

Assuming a coordinate inspection machine as a device for tolerance control, one should assure precision of coordinate measurements about 10% of tolerances under inspection. The

\(^2\)Article [1] was first published in 1998 as Preprint IHEP, 98-9, Protvino, 1998.
more accurate is the device, the more precisely the distance from the measured point to a circle/line can be defined. Again, this is not the case for the curvilinear profile: one could define exact distance, say, to the currently chosen interpolant. The latter could be different from one, previously used in NC-machining process.

So, the nondeterminacy of the profile behaves as inevitable addition to the measurement error. Constructing the bounding region estimates this nondeterminacy, and allows us to set its acceptable limits, called by the accuracy requirements. The width of the region is the objective measure of fairness of the given point set, and may serve as the criterion to demand a designer for more detailed profile description.

For traditional interpolation methods nothing ensures that some interpolant passes within the prescribed bounding region. But, on the other hand, one can verify whether it does or not. An example is discussed below.
A.2. Test with cubic spline interpolation

Algorithms of interpolation with spirals are not widely distributed with CAD software. On the other hand, interpolating spiral data by, e.g., a cubic spline, will usually not return a spiral. However, as soon as the region is known, wherein the sought for curve definitely lies, acceptance test for any interpolation method can be performed. We add to this note an example of such test. In Fig. 10 given data points and end tangents were interpolated by a cubic spline curve. The nodes in the example are spaced out such as one could visually estimate if the spline fits the prescribed region (three last chords are shown magnified).

The spline can be parametrized by accumulated chord length, which approximates arc length of the interpolated curve. Assuming that the 3-rd order Bezier polynomials \( x(t), y(t) \) are close to the Taylor series \( x(s), y(s) \) of the naturally parametrized unknown curve gives rise to set boundary conditions at the start point of the cubic spline as \( (x'(0), y'(0)) \approx n(\tau_1) \), and similarly at the endpoint.

Points setting of 5 first nodes, compared to expected curvatures, is evidently unacceptable for good interpolation. It is exhibited in both location of the spline out of prescribed region, and large beatings in the curvature plot.

Situation on the right side looks much better, although points are set still too rare, compared with typical industrial drawings. Recall that the bounding region encloses some unknown, but well defined curve (e.g., solution of a differential equation, or some other implementation of designer’s concept). With such test it is possible:

- to estimate whether the bounding region is sufficiently narrow, i.e. the unknown curve is well/ill-defined by the given point set;
- to estimate closeness of the unknown curve and the interpolant, which, even for non-spiral interpolation method, may fall inside the bounding region.

A.3. Is the interpolant in Fig. 10 “unfair”?

Appearance of the curvature plot in Fig. 10, with curvature extrema in every node and between them, is rather similar to the examples in [2, Figures 9.8–9.13]. The question arises if the non-monotonicity of the curvature plot is essential, and should be somehow corrected.

The general answer depends on the values of curvatures, and the particular reason to avoid the beatings. Analyzing the problem, one should not forget that the objects of computer-aided design become the objects of further manufacturing, which is not geometrically precise. We can compare the model and the real part (cam profile, highway path, etc.), and find that the latter fits well prescribed tolerances. But small difference between two curves could result in rather big distortions of curvature plot:

In the above illustration the data is chosen on the circular arc of radius 10 with \( P_1 = (0, 0) \), and \( \tau_1 = 0 \). Ideal discrete curvature plot is the line \( q(i) = 0.10 = \text{const} \). Manufacturing/measurement errors were imitated by rounding the coordinates to two decimal digits:
$P_2 \approx (0.5233, 0.0137)$ was replaced by $(0.52, 0.01)$, $P_3 \approx (1.0453, 0.0548)$ by $(1.05, 0.05)$, \ldots, and $P_{21} \approx (8.6603, 5.0)$ by $(8.66, 5.0)$. This rounding caused well visible beatings in discrete curvature plot. The geometric reason of such behavior is evident: for a small circular arc $ABC$ a slight shift of the midpoint $B$ effects a dramatic change in radius or curvature. The smaller is the angular measure of the arc, the greater is this effect.

The above considerations seem to agree with the note in [2, Sec. 23.1]: the definition of fairness, based on minimum of curvature extrama, “is certainly subjective; however, it has proved to be a practical concept”.

References

[1] Kurnosenko A.I. Interpolational properties of planar spiral curves. Fund. and Applied Math. 7(2001), N2, 441–463 (in Russian).

[2] Farin, G. Curves and Surfaces for CAGD: A Practical Guide, 5-th edition, Acad. Press, 2002.

[3] Kurnosenko A.I. Applying inversion to construct planar, rational, spiral curves that satisfy two-point $G^2$ Hermite data. Comp. Aided Geom. Design 27(2010), 262–280.

[4] Kurnosenko A.I. Biarcs and bilens. Computer Aided Geometric Design, 30 (2013), 310–330.
