Phase structures emerging from holography with Einstein gravity - dilaton models at finite temperature

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Abstract: Asymptotic AdS Riemann space-times in five dimensions with a black brane (horizon) sourced by a fully back-reacted scalar field (dilaton) offer - via the holographic dictionary - various options for the thermodynamics of the flat four-dimensional boundary theory, uncovering Hawking-Page, first-order and second-order phase transitions up to a cross-over or featureless behavior. The relation of these phase structures to the dilaton potential is clarified and illustrating examples are presented. Having in mind applications to QCD we study vector mesons in the probe limit with the goal to figure out conditions for forming Regge type series of radial excitations and address the issue of meson melting.
1 Introduction

The advent of the insight into AdS/CFT correspondence [1–3] offered the option of having an alternative access to strongly coupled systems, e.g. to various facets of QCD in the non-pertubative regime, for instance. Phenomenologically interesting problems, e.g. the hadron spectrum or properties of the quark-gluon plasma, become treatable within a framework called holographic approaches. Most desirably would be to have a holographic QCD dual at our disposal, from which statements on QCD-related quantities can be derived in a unique manner. However, such a dual is presently not available [4–7]. Therefore, in practice, field-theory quantities in a five dimensional asymptotic Anti-de Sitter (AdS) are often related to observables (or expectation values of operators) in four dimensional Minkowski space-time in the spirit of the field-operator duality ([8], see also §5.3 in [9], §10.3 in [10], for instance). Top-down approaches attempt to use input from string theory constructions or elements thereof. These are to be contrasted with bottom-up approaches which aim at starting with an appropriate model on the field-theory side to mimic certain selected features of the boundary theory side, with the latter being connected to the quantum field
theory in Minkowski space, while the former includes the dynamics in the bulk. Besides early emphasis on accessing principal features of strongly coupled systems with many extensions to higher or lower dimensions than mentioned above, one can also take the attitude of adjusting sufficiently simple and thus transparent bottom-up models to a certain input and then employ them for predictions. Of course, the predictive power becomes a relevant issue here. Moreover, the foundations of the AdS/CFT correspondence, namely a very large number \( N_c \) of gauge degrees of freedom and a very large 't Hooft coupling, are often argued only to hold under special conditions, too. For instance, w.r.t. QCD one knows [11] that certain thermodynamic observables of the Yang-Mills gauge theory obey the proper scaling with \( N_c \) and hopes that the physics case of \( N_c = 3 \) is adequately captured.

Holographic modeling of QCD related problems became popular due to some particularly striking findings. Among them are the Regge type spectrum of hadronic and glue ball states, e.g. within the soft-wall model [7], the famous ratio of shear viscosity \( \eta \) to entropy density \( s \), \( \eta/s = 1/4\pi \) [12], and the phase diagram with a critical point [13, 14], to mention a few ones.

Besides gravity, the dilaton plays an important role as a breaker of the conformal symmetry since it introduces an energy scale. Obviously, the holographic models with gravity coupled to and sourced by a dilaton field – including the negative cosmological constant to ensure the asymptotic AdS geometry – represent some minimalistic set-up. To be specific, we restrict ourselves here to Einstein gravity. What remains is fixing the dilaton self-interaction. This may refer to roots in string theory, as recently put forward, e.g. in [15, 16] in a top-down approach, or to shape the dilaton self-interaction – encoded in the dilaton potential – by reproducing a certain set of wanted results within the dilaton engineering to reproduce Lattice QCD thermodynamics results of [17, 18].

The resulting set-up is called Einstein gravity - dilaton model. It continues numerous previous studies in cosmology, most notably in inflationary scenarios. Analogously, in holography such gravity-dilaton models enjoy some popularity due to their conceptual simplicity. There is an overwhelming number of studies, e.g. [15, 19–66], based on that type of model: [19, 20] represent an in-depth analysis and review of the model in detail and [21] touches the issue of consistency. References [15, 22–44] focus on thermodynamics, where [22] can be considered to be the prototype of modeling thermodynamics with a dual black hole. The authors of [23, 24] give a fundamental bound of the speed of sound and [25] derived a relation between the speed of sound and the single heavy quark free energy. Discussions about different phase structures can be found in [26–29], and for quantitative comparisons and parameter fits to results on Lattice QCD thermodynamics in case of vanishing and finite baryo-chemical potential we refer to [30–34]. There are investigations about the temperature dependence and the behavior during phase transitions of related quantities, e.g. [35] calculates string tension at finite temperature, [36] chooses an approach based on the beta function, [37–39] deal with electric and magnetic quantities, [15, 40] calculate the Debye screening mass; transport coefficients and bulk viscosities are the topics of [41, 42] and [43, 44], respectively. A holographic approach to the broad field of hydrodynamics and thermalization is given e.g. by [45–47] within the gravity - dilaton model class.
The soft-wall model [7] developed into a role model for computing particle spectra holographically. While in the original model the metric background is fixed by ad-hoc ansätze, the main idea of its generalizations [48–60] is to obtain the metric background as a solution of the Einstein equations. The particles are considered then as test particles. [48–52] give particular attention to Regge type spectra, [54] investigates the case of finite temperature and [55, 56] focus on chiral symmetry breaking. Due to the various applications of Einstein gravity - dilaton models (see as well [61] for a branless approach, [62] for fluctuating branes, [63] for a real-time formulation, [64] for cosmological discussions, [65] for scalar condensates or [66] for a generalization to higher dimensions) this list does not purport to be complete.

Finite temperature effects are generated by plugging a black hole in the originally AdS and deform it accordingly. Thus, a Hawking temperature and a Bekenstein-Hawking entropy density link to thermodynamics. In the present paper, we also stay within such a framework: holographic gravity-dilaton model with the goal to elaborate the emerging thermodynamics w.r.t. conditions for the dilaton potential to catch certain phase structures with relevance to QCD.¹ The Columbia plot (cf. [68] for an updated version) provides several options for 2+1 flavor QCD: in dependence on the quark masses, first- and second-order phase transitions may show up as well as a cross-over and some others. We try to answer the question which properties the dilaton potential must have to enable these phase properties related to deconfinement and chiral restoration in QCD. On top of thermodynamic aspects we consider holographic vector mesons in the probe limit. That is, the gravity and dilaton background resulting from the field equations and equation of motion governs – besides the thermodynamic features – the behavior of vector mesons. We thus extend our previous studies [69] and investigate to which extent the disappearance of vector mesons as a possible indicator of deconfinement occurs at the QCD cross-over temperature. This is important for a holographic realization of the thermo-statistical interpretation of hadron multiplicities in ultra-relativistic heavy-ion collisions, e.g. at LHC [70]. The ultimate goal of such investigations, which however is beyond the scope of the present paper, is an extension to non-zero chemical potential, e.g. to address issues of the QCD phase diagram and the chemical freeze-out curve therein.

Our paper is organized as follows. In Section 2, we recall the holographic settings, that is the gravity-dilaton model, its field equations and equation of motion as well as the emerging thermodynamics and the access to phase structure. Also, the holographic description of vector mesons is recalled. Section 3 deals with thermodynamic scenarios by presenting a series of selected examples of transition types characterized by entropy density, sound velocity and pressure. This is supplemented by showing the Schrödinger equivalent potential which governs the existence or non-existence of vector mesons. After some general remarks on shaping the dilaton potential, we try to elucidate the conditions on the dilaton potential to enforce a first-order phase transition or a cross-over. In Section 4, we explain a relation between the Schrödinger equivalent potential at zero temperature and the thermodynamic

¹Such an investigation is timely, since a systematic study relating the dilaton potential and the emerging thermodynamics is currently lacking [67], see however [46], where selected cases are considered.
features. Both ones are linked by the field equations; details can be found in Appendices A and B. In the second part of Section 4, we reverse our view: instead of starting with a dilaton potential we model a certain shape of the Schrödinger equivalent potential which allows for a certain wanted hadron spectrum (ideally of a Regge type) and derive – again via field equations – the resulting dilaton potential.\(^2\) We conclude that part by considering the vector meson melting upon temperature increase. The summary and a discussion of possible extensions towards the goal of a consistent scenario of QCD thermodynamics with the chemical freeze-out model in the LHC energy regime can be found in Section 5.

2 Holographic settings

2.1 Thermodynamics from Einstein gravity - dilaton model

We consider the action

\[ S = \frac{1}{2\kappa} \int d^5x \sqrt{g} \left[ R - \frac{1}{2} (\partial\Phi)^2 - V(\Phi) \right] \] (2.1)

over the five-dimensional Riemann space-time with special ansatz of the metric given by the line element squared as

\[ ds^2 = e^{A(z)} \left[ f(z) dt^2 - d\vec{x}^2 - \frac{1}{f(z)} dz^2 \right] , \] (2.2)

where \(A\) denotes the warp factor with \(A(z) \to -2 \ln(z/L)\) as \(z \to 0\) to ensure an asymptotic Anti-de Sitter (AdS) and \(f\) denotes the blackness function with \(f(z) = 1 - O(z^4)\) which encodes the temperature via

\[ T(z_H) = -\frac{1}{4\pi} f'(z_H) \] (2.3)

with the horizon position \(z_H\) and the simple zero \(f(z_H) = 0\). The vacuum case, \(T = 0\), is equivalent to \(f = 1\). The dilaton \(\Phi\) in the action (2.1) is a dimensionless real-valued scalar bulk field. Its potential \(V(\Phi)\) has the asymptotic small-\(\Phi\) form

\[ L^2 V = -\frac{1}{12} \frac{m^2}{L^2} \Phi^2 + \cdots \]

where the first term refers to the negative cosmological constant and the second one has to obey the Breitenlohner-Freedman (BF) bound \(-4 \leq m^2 L^2 \leq 0\) \([71, 72]\); \(L\) sets a scale, as \(\kappa\) in (2.1), to make the action dimensionless in natural units. From (2.1) and (2.2) the field equations follow

\[ f'' + \frac{3}{2} A' f' = 0, \] (2.4)

\[ A'' - \frac{1}{2} A'^2 + \frac{1}{3} \Phi'^2 = 0, \] (2.5)

\[ (A'^2 - \frac{1}{6} \Phi'^2) f + \frac{1}{2} A' f' + \frac{1}{3} e^A V = 0, \] (2.6)

\(^2\)Alternatively, one could also start with an ansatz for the dilaton profile and derive all other quantities via field equations, cf. [53] and further references therein. Obviously, one could start equally well with other quantities or combinations thereof and derive the remaining functions from the field equations. [50] is an example for starting with the warp factor, defined below.
and the equation of motion

\[ \Phi'' + \left( \frac{3}{2} A' + \frac{f'}{f} \right) \Phi' - \frac{e^A}{f} \partial_\Phi V = 0 \]  

(2.7)

which is redundant since it follows from (2.6) with (2.4, 2.5). A prime means derivative w.r.t. the bulk coordinate \( z \). Equations (2.4-2.6) can be solved for a given \( V(\Phi) \) with the above side conditions. In such a way, the dilaton potential \( V(\Phi) \) determines the temperature via (2.3) and the speed of sound squared via

\[ c_s^2 = \frac{d \ln T}{d \ln s}, \]  

(2.8)

where \( s(z_H) = \frac{2}{\kappa} \exp\left\{-\frac{3}{2} A(z_H)\right\} \) stands for the entropy density. The quantities \( T, c_s^2 \) and \( s \) refer to the boundary theory according to the holographic dictionary. The pressure is calculated via \( p = \int dT s(T) \) with the side condition \( p(T = 0) = 0 \).

### 2.2 Vector mesons in probe limit

Additionally, we study the behavior of vector mesons in the probe limit, i.e. they are not back-reacted. To capture their features, we use the standard action [7]

\[ S_V \propto \int d^5x \sqrt{g} F^2, \]  

(2.9)

where \( F^2 \) is the squared field strength tensor of a \( U(1) \) vector field. The equation of motion follows, after some manipulations [73], as one-dimensional Schrödinger type equation [69]

\[ \left( \partial^2_\xi - (U_T - m_n^2) \right) \psi = 0, \]  

(2.10)

where \( \partial_\xi \equiv (1/f)\partial_z \) and

\[ U_T = U_0 f^2 + \frac{1}{2} S f f' \]  

(2.11)

with the Schrödinger equivalent potential

\[ U_0 = \frac{1}{2} S' + \frac{1}{4} S^2, \quad S \equiv \frac{1}{2} A - \frac{2}{3} \Phi. \]  

(2.12)

For \( T = 0 \), the tortoise coordinate \( \xi \) becomes \( z \) and only \( U_0(z) \) is relevant. \( m_n \) denotes the masses of normalizable modes as solutions of (2.10) with \( n = 0, 1, 2, \cdots \) as the quantum number of radial excitations. In general, \( A \) and \( \Phi \) depend on both, \( z \) and \( z_H \). To distinguish the vacuum case \( (T = 0; A_0(z), \Phi_0(z), f(z) = 1) \) from the non-zero temperature case \( (T > 0; A(z, z_H), \Phi(z, z_H), f(z, z_H) \leq 1) \) we add the index 0 to \( A \) and \( \Phi \), as above for \( U_0 \).

By employing the field equations, we find the following relation for \( U_0(z) \):

\[ U_0 = \frac{17}{48} A_0' + \frac{1}{3} A_0' \Phi_0 + \frac{1}{3} e^{A_0(\partial_\Phi V - \frac{1}{6} V)}. \]  

(2.13)

In Section 3.1 we study the vector meson spectrum over the background determined by solutions of (2.4-2.7) within \( U_0 \) from (2.13), while in Section 4.2 an ansatz for \( U_0(z) \), which facilitates a certain mass spectrum, is used as an input for (2.4-2.7) to figure out the related thermodynamics and phase structure.
| Example | $\gamma$ | $a$  | $b$  | Transition       |
|---------|---------|------|------|------------------|
| a)      | 0.56    | -0.077 | 0    | none             |
| b)      | 0       | 1.155 | 0.18 | cross-over      |
| c)      | 0       | 1.155 | 0.20 | second-order     |
| d)      | 0       | 1.155 | 0.25 | first-order      |
| e)      | 0.83    | -2.069 | 0    | Hawking-Page    |

Table 1. Parameter selection yielding the examples in Fig. 1 and the characterization of the thermodynamic features.

3 Thermodynamic scenarios

3.1 Selected examples

To illustrate the systematics of the thermodynamics related to the dilaton potential $V$ we choose the three-parameter ansatz [22, 74]

$$-L^2V(\Phi) = 12\cosh(\gamma\Phi) + a\Phi^2 + b\Phi^4$$

(3.1)because we can go through several thermodynamic scenarios (see Fig. 1) by changing the values of parameters $\gamma$, $a$ and $b$, which are related by $-L^2 m^2_\Phi = 12\gamma^2 + 2a$ to the dilaton mass parameter $m^2_\Phi$.

As already remarked in [22], quite featureless dilaton potentials $V(\Phi)$ can lead to fairly different thermodynamic features. Since the field equations (2.4-2.6) can be rearranged to display only a sensitivity to $\partial_\Phi V/V$ as a function of $\Phi$, we plot this key quantity in the left column of Fig. 1. Example a) does not exhibit any features: $T(z_H)$ is monotonously decreasing, $c^2_s(T)$ is increasing, and $U_0$ has no minimum, meaning that normalisable modes as vector mesons do not exist at all. Examples b) - d) exhibit $\partial_\Phi V/V$ with a pronounced maximum which becomes gradually higher. In case b), $T(z_H)$ is monotonously dropping, albeit with a shallow near-flat section; it causes a pronounced minimum of the sound velocity squared; $U_0$ does not allow for any normalizable states due to the lacking minimum. That case is classified as cross-over. Case c) features a lifted maximum of $\partial_\Phi V/V$, resulting in a flat section of $T(z_H)$, and the sound velocity drops at a certain temperature to zero, thus representing an example of a second-order phase transition. The Schrödinger potential $U_0(z)$ has here a very shallow minimum, i.e. vector mesons as normalizable modes cannot show up. Lifting the maximum of $\partial_\Phi V/V$ further (case d)), $T(z_H)$ shows a local minimum that is connected to an inflection point, implying metastable states, unstable states and spinodales as well. That becomes most evident by the sound velocity squared $c^2_s(T)$ and the pressure $p(T)$ (see Fig. 2): metastable states are depicted by thin solid curve sections and the unstable ones by dotted sections. Clearly, states with $c^2_s < 0$ cannot be realized in nature. All these features classify a first-order phase transition, for which $U_0(z)$ exhibits a local minimum; it is - for the given parameters - still too shallow to accommodate vector mesons. Only if $\partial_\Phi V/V$ exceeds $\sqrt{2/3}$ (case e)), the global minimum of $U_0(z)$ allows for meson states, depicted by horizontal lines. At the same time, $T(z_H)$ has a global minimum.
Figure 1. Selected examples of parameter choices (see Tab. 1) for the dilaton potential (3.1) (left column, for $\partial_0 V/V$), the resulting temperature $LT$ as a function of horizon position $z_H/L$ (second column), velocity of sound squared $c_s^2$ as a function of temperature $LT$ (third column) and the Schrödinger potential $L^2 U_0$ as a function of $z/L$ (right column). The dilaton potential (3.1) serves as an input to solve the field equations (2.4-2.6) and obtain $T(z_H)$ from (2.3), $c_s^2(T)$ from (2.8) combined with (2.3), and $U_0(z)$ from (2.13).

pointing to a Hawking-Page (HP) phase transition. Let be $T_{\text{min}} = \min T$ at $z_H^{\text{min}}$. Then, the branch for $z_H < z_H^{\text{min}}$ is stable (for $p > 0$) and metastable (for $p < 0$), while the branch for $z_H > z_H^{\text{min}}$ is unstable, and its free energy is above the thermal gas solution (see [26] for the related construction) which applies for $0 < T < T_{\text{min}} < T_c < \infty$, where $T_c$ (slightly above $T_{\text{min}}$) is the first-order phase transition temperature. (While $T_{\text{min}}$ follows as minimizer of (2.3) over $z_H$, $T_c$ is obtained by the loop construction sketched in the rightmost graphs of Fig. 2.) The velocity of sound drops to zero at $T_c$.

Figures 3 and 4 exhibit that region in the $\gamma$-$a$-$b$ parameter space, where a first-order or HP phase transition occurs; and in Fig. 4, curves on which a second-order phase transition happens are depicted too. Thanks to the three-parameter potential (3.1), the visualization
of the parameter space structure is quite straightforward, while multi-parameter ansätze may lead to intricate structures. In the present case, the HP transition happens for \( \gamma \geq 0.8 \) for all BF permitted values of \( a \) and \( b \) as well, as exhibited in the left panel of Fig. 4. For \( \gamma < 0.8 \), the blue areas depict the first-order phase transition, which are limited by the second-order transition (red curves). Further left (gray parts of the panels with constant \( \gamma \) or \( a \) or \( b \) cross sections), a cross-over occurs, which turns smoothly into a featureless behavior for smaller values of \( b \).

### 3.2 Shaping the dilaton potential

We refrain here from a quantitative comparison with QCD thermodynamics and refer the interested reader to [15, 22, 26, 32, 33], for example, where the thermodynamics of Yang-Mills and 2+1 flavor QCD with physical quark masses is considered. The essence is to employ multi-parameter ansätze of the dilaton potential with the aim to reproduce the Lattice QCD thermodynamics results as good as possible.\(^3\) Information of QCD is thus imported and mapped in a cumulative manner on \( V(\Phi) \) without explicit reference to quarks and gluons and their masses, colors, flavors, couplings etc.\(^4\) Having successfully accomplished the shaping of \( V(\Phi) \), one can proceed to derive further quantities, such as viscosities [43, 44], diffusion constants [41, 42] etc. as predictions. By extending the model (2.1) by further fields, e.g. a Maxwell type \( U(1) \) gauge field [13, 33, 81], one may address non-zero chemical potential effects to access the phase diagram and issues of a critical point. Here, susceptibilities serve as crucial further information to be imported from QCD. However, our present goal here is to answer the question whether one can describe - within the modeling (2.1, 2.9) - at the same time QCD thermodynamic features, i.e. a cross-over at about 155 MeV [17, 18], and a proper in-medium behavior of hadrons, i.e. vector mesons in probe limit as representatives thereof. By a proper “in-medium behavior” we mean that

\(^3\)For a more generic study of the potential and the related RG flow, cf. [75].

\(^4\)In contrast, the IHQCD model [19, 20, 26–28] aims at anchoring fundamental QCD features in the chosen ansatz from the beginning; the approaches in [15, 16], following the 1-R charge black hole (1RCBH) model [76–80], in turn are string theory driven.
Figure 3. Plot of the region in the $\gamma$-a-b parameter space, where a HP or first-order phase transition in the BF allowed range occurs for the dilaton potential (3.1). Cross sections of constant values of $\gamma$, $a$ and $b$ are exhibited in Fig. 4.

Figure 4. Plot of parameter ranges of the dilaton potential (3.1) for first-order phase transition (blue areas) or HP transition (yellow areas) for a series of selected constant values of $\gamma = 0, \cdots, 1$ (left), $a = -6, \cdots, 2$ (middle) and $b = 0, \cdots, 0.4$ (right). The red curves mark the second-order phase transition ranges; beyond, the transition turns into a cross-over, followed by a featureless behavior (dark gray areas). The BF bound restricts the values of $a$ to the strip $-6\gamma^2 < a < 2 - 6\gamma^2$ (depicted by the two black solid curves in left and middle panels). The panels continue to larger values of $b$ and $\gamma$ without changes.

at chemical freeze-out temperature of about 155 MeV [70], hadrons do exist with hardly medium-modified properties. Otherwise, the famous thermo-statistical interpretation of hadron multiplicities in ultra-relativistic heavy-ion collisions [70] would be invalidated.

The answer to the posed problem seems to be negative. Hints come, for example, from [73, 82–85], where the melting (disappearance) of hadrons was found to happen at temperatures significantly below 155 MeV. While [69] offers an avenue to remedy such an insanity, the given framework of (2.1, 2.9) seems to be too restricted and calls for extensions. Leaving the latter ones for separate work, we try to find a loophole to join the cross-over
The authors of [22] derived the relation (henceforth called Gubser’s adiabatic criterion)
\[ c_s^2 \approx \frac{1}{3} - \frac{1}{2} \left( \frac{\partial_b V}{V} \right)^2, \]  
(3.2)
where ‘\( \approx \)’ indicates the validity in adiabatic approximation. The formula implies that for \( \partial_b V/V > \sqrt{2/3} \) the sound velocity becomes imaginary, thus pointing to a first-order phase transition, either as a standard construction à la example d) or the HP transition à la example e) in Fig. 2. To systematize the various thermodynamic scenarios, we plot \( \partial_b V/V \) of the above examples a) - e) in one diagram, see Fig. 5. Based on such a comparison, the impact of \( \partial_b V/V \) on the thermodynamics can be summarized qualitatively as follows: If \( \partial_b V/V \) reaches the value \( \sqrt{2/3} \) (or somewhat below, depending on the concrete \( V(\Phi) \)), \( T(z_H) \) forms a local extremum, where the nearest one to the boundary becomes a minimum. So if \( \partial_b V/V \) intersects the \( \sqrt{2/3} \) line once, we have a global minimum of \( T(z_H) \) and a HP phase transition. Otherwise, if \( \partial_b V/V \) intersects twice, \( T(z_H) \) forms a local minimum followed by a local maximum and we have a first-order phase transition. A second-order phase transition arises if \( \partial_b V/V \) touches the \( \sqrt{2/3} \) line. Additionally, each extreme point of \( \partial_b V/V \) implies an inflection point of \( T(z_H) \), i.e. a cross-over is generated by a maximum of \( \partial_b V/V \) whose altitude stays below \( \sqrt{2/3} \).
To formalize these findings we derive in Appendix A the relation

\[
\frac{1}{T} \frac{dT}{dz_H} = \frac{1}{2} \frac{V}{\dot{\Phi}} \left( \frac{\partial_b V}{V} \right)^2 \frac{2}{3} \Phi' + \frac{\partial A}{\partial z_H} + \left( \frac{3}{2} \frac{\partial A'}{\partial z_H} + \Phi' \frac{\partial \Phi}{\partial z_H} \right) \frac{\partial_b V}{V},
\]

(3.3)

where \( \Phi' \equiv (\partial_z \Phi(z, z_H)) |_{z=z_H}, A' \equiv (\partial_z A(z, z_H)) |_{z=z_H}. \)

Given the facts that (i) \( T(z_H \to 0) \to \frac{1}{\pi z_H} \) [23, 24], (ii) the monotonous behavior of \( \Phi(z, z_H) \) as a function of \( z \) with \( \Phi' > 0 \), and (iii) the above quoted asymptotic behavior of \( V(\Phi) \) at small \( \Phi \) (implying \( \partial_b V/V = m_b^2 L^2 \Phi/12 \)), one recognizes from the first line of (3.3) that the slope \( dT/dz_H \), which is negative at small \( z_H \), can turn into a positive one, once \( \partial_b V/V > \sqrt{2/3} \) is reached, indicating a local or global minimum of \( T(z_H) \).

Understanding “adiabatic approximation” as a situation where \( \partial A/\partial z_H, \partial A'/\partial z_H \) and \( \partial \Phi/\partial z_H \) are small, one thus recovers Gubser’s adiabatic criterion. Otherwise, the second line of (3.3) provides corrections. In fact, in example d), \( \partial_b V/V \) stays below the \( \sqrt{2/3} \) line but facilitates a first-order phase transition. (The relation of \( T(z_H) \) to transitions is discussed in [19, 20]: in essence, a minimum of \( T(z_H) \) points to a first-order phase transition, since \( s(z_H) \) is a monotonously increasing function.) The corrections give eventually a border line (called \( T_{\text{min}} \) curve) for each type of dilaton potential which is determined by calculating the minimal value of \( \partial_b V/V \) (as a function of \( \Phi \) and depending on all parameters denoted shortly by \( \vec{p} \)) such that \( T \) forms a minimum. Systematic numerical analyses with the dilaton potential (3.1) show that this line is shifted down if \( \partial_b V/V \) as a function of \( \Phi \) becomes steeper when varying the parameters \( \vec{p} \) in \( V(\Phi; \vec{p}) \). The right part of Figure 5 shows an example of such a line and the dependence of the difference between Gubser’s criterion and the \( T_{\text{min}} \) curve as a function of \( \Phi \). This is further visualized in Fig. 6 for two special parameters, \( b = 0 \) and 2, in the projections on the \( \gamma-a \) plane: the offset of the regions determined by (3.2) and the true onset of a first-order phase transition increases with \( b \); in addition, the region where the Schrödinger equivalent potential \( U_0 \) displays a minimum is shown by the red curves.

4 Schrödinger potential

4.1 Adiabatic approximation to Schrödinger potential

The relationship between the situation of \( T = 0 \) and features at \( T > 0 \) has been stressed in [19, 20]. Here, we envisage a relation of \( U_0(z) \) and \( T(z_H) \) in adiabatic approximation. In Appendix B we derive the relation

\[
9U_0 \cong \frac{17}{12} \frac{T^2}{T^2} + \frac{28}{3} \pi T' + \frac{44}{3} \pi^2 T^2 + 2 \left( 2 \pi T - \frac{T'}{T} \right) \sqrt{\frac{8}{3} \left( \frac{T'^2}{T^2} + 5 \pi T' + 4 \pi^2 T^2 \right)}. \tag{4.1}
\]

It is valid if, in a decomposition \( A(z, z_H) = A_0(z) + a(z, z_H) \) and \( \Phi(z, z_H) = \Phi_0(z) + \varphi(z, z_H) \), the terms \( a(z, z_H) \) and \( \varphi(z, z_H) \) are sub-leading and can be neglected. The Chamblin-Reall solution [86] with \( V(\Phi) \propto \exp{\{\gamma \Phi} \) is an example, where \( a = \varphi = 0 \) can
Figure 6. Projection of border lines (for $b = 0$ and $b = 0.2$) of parameter regions on $\gamma$ vs. $a$ plane, where $T(z_H)$ has a minimum (blue hatched regions, left and above to solid blue curves), $U_0$ has a minimum (left and above to red curves), and $\partial_k V/V$ is greater than $\sqrt{2/3}$ (left and above to dashed blue curves) for the dilaton potential (3.1); if $b$ and therefore the slope of $\partial_k V/V$ increase, the difference between Gubser’s adiabatic criterion (blue dashed) and the $T_{\text{min}}$ curve (solid blue) becomes larger. In the blue double-hatched region, $\gamma$ is great enough to have a first-order phase transition for all $a$ and $b$. The region of a HP transition is in yellow. The green hatched regions are excluded due to BF.

be chosen, despite of $f(z, z_H) \leq 1$.

Equation (4.1) is exact for the AdS-BH metric, i.e. $A = -2 \ln(z/L)$ and $f(z, z_H) = 1 - (z/z_H)^4$ which is generated by $V(\Phi) = -12/L^2$. Thus, the left side converges against the right side, if $z \to 0$ and also if $z \to \infty$ because then one has $A \to A_0$ and $\Phi \to \Phi_0$. If we assume that $T(z_H)$ has a minimum at $z_H = z_{H}^{\text{min}}$, (4.1) implies $U_0'' = \frac{2}{3}(43 - 8\sqrt{6})T'' > 0$ meaning $U_0$ decreases at the minimum position of $T$. Due to the AdS asymptotics of the Schrödinger potential, there has to be a minimum as well, at a position nearer to the boundary.

4.2 Requiring a minimum of $U_0(z)$

The above examples demonstrate that for many parameter choices of the dilaton potential $V(\Phi)$ the Schrödinger potential $U_0(z)$ does not exhibit a minimum and thus does not allow for modes which can be interpreted as vector mesons. Instead of deriving the background (warp factor and dilaton profile at $T = 0$) from given dilaton potential, we start now with an ansatz for $U_0(z)$ such as to have a minimum. Assuming the latter one is sufficiently deep, normalizable modes would then be expected. Our ansatz for demonstrative purposes is

$$U_0(z) = \frac{3}{4z^2} + \left(\frac{z}{L}\right)^p \frac{1}{L^2}, \quad (4.2)$$
where the first term comes from the asymptotic warp factor at \( z \to 0 \); the second term facilitates the required minimum at \( \hat{z}_{\text{min}}/L = (3/2p)^{1/p} (1+p/2) \).

\[ (2.12) \] is solved at \( f = 1 \) by

\[
S = 2 \frac{d}{dz} \ln \left( c_1 \hat{z}^{-\frac{1}{2}} F_1 \left( \frac{p}{p+2}, \frac{\hat{z}^{p+2}}{(p+2)^2} \right) + c_2 \hat{z}^{\frac{3}{2}} F_1 \left( \frac{p+4}{p+2}, \frac{\hat{z}^{p+2}}{(p+2)^2} \right) \right)
\]

with \( \hat{z} \equiv z/L \) and \( c_1 = 1 \) (due to AdS behavior at boundary \( z \to 0 \)) and \( c_2 = -\frac{1}{2}(p+2) \left( \frac{p+2}{2p} \right) \Gamma \left( \frac{2}{p+2} \right) / \Gamma \left( \frac{2}{p+2} \right) \) (due to the assumption that \( (4.2) \) is globally valid). The field equations \((2.4-2.6)\) must be solved numerically to get \( A(z), \Phi(z) \) and \( V(\Phi) \). The same \( V(\Phi) \), which is supposed to be independent of \( T \), is then used to derive \( U_T(z,z_H) \) and \( T(z_H) \). Figure 7 exhibits such solutions for \( p = 0.5, 1 \) and 2, where the latter value reproduces the soft-wall model [7] with a strictly linear Regge type spectrum \( L^2 m_n^2 = 4(n+1) \) for \( n = 0, 1, 2 \cdots \). The left panel is for \( U_0(z) \) according to \( (4.2) \), while the middle panel shows \( T(z_H) \); the right panel displays \( \partial_\Phi V/V \) as a function of \( \Phi \). There is a striking similarity of the curves \( T(z_H) \) and \( U_0(z) \) (isotopy referring to the monotony behavior) which we interpret as follows: a minimum of \( U_0(z) \) is related to a minimum of \( T(z_H) \), i.e. a first-order phase transition - here a HP transition, since it is a global minimum. The behavior of \( \partial_\Phi V/V \) as a function of \( \Phi \) is in agreement with our assessments in Sections 3.1 and 3.3.

To get an idea on the related scale, the left-hand plot of Fig. 8 shows the first three states, that is \( L^2 m_n^2 \) as a function of \( p \) for \( n = 0 \) (ground state) and \( n = 1, 2 \) (first two radial excitations). For comparison, the middle plot of Fig. 8 displays \( LT_{\text{min}} \) also as a function of \( p \). Both figures can be combined to \( T_{\text{min}}/m_n \) as a function of \( p \) (right-hand plot of Fig. 8). Having in mind applications to QCD and identifying \( T_{\text{min}} \) with 155 MeV (see above) and \( m_0 \) with the \( \rho \) meson ground state mass of 770 MeV, one arrives at \( T_{\text{min}}/m_0 \approx 0.20 \), i.e. a value not too far from the range of values shown in the right-hand plot of Fig. 8. However, 2+1 flavor QCD with physical quark masses does not provide a first-order phase transition. Insofar, the present set-up is more appropriate for 2+1 flavor QCD in the chiral limit, which in fact enjoys a first-order phase transition [68], but detailed information on
Figure 8. Scaled vector meson masses squared $L^2m_n^2$ as a function of $p$ (left panel), scaled minimum temperature $LT_{\text{min}}$ as a function of $p$ (middle panel), and ratio $T_{\text{min}}/m_n$ as a function of $p$ for the ansatz (4.2). Blue solid/green dashed/red dotted for ground state/first excitation/second radial excitation (there is an infinite tower of excitations for all $p$).

Figure 9. Vector meson masses squared $L^2m_n^2$ of the first three states (color code as in Fig. 8) as a function of the temperature for the parameter $p = 2$ (solid) and $p = 1$ (dashed). The masses are constant during the thermal gas phase and all excitations disappear at $T_c$, where the black hole solution begins to apply.

thermodynamic quantities as well as the vector meson spectrum is lacking (cf. [87] for a search for the delineation curve in the Columbia plot where cross-over and first-order phase transitions touch each other; very preliminary first estimates [88] point to a ratio of $T_c/m_0$ in the same order of magnitude as for the case of physical quark masses).

The temperature dependence of $U_T$ (not shown) is such to cause the instantaneous disappearance (melting) of vector meson states at $T_c \approx T_{\text{min}}$ (see Fig. 9). Denoting the disappearance temperature by $T_{\text{dis}}$ and choosing $m_0 = m_\rho$ as scale, Fig. 9 translates into $T_{\text{dis}} = m_\rho \frac{LT_{\text{dis}}}{Lm_0}$, meaning $T_{\text{dis}} = 116$ MeV for $LT_{\text{dis}} = 0.27$ (0.3) and $Tm_0 = 1.7$ (2). That value of $T_{\text{dis}}$ does not apply to 2+1 flavour QCD with physical quark masses, since, as stressed above, the present scenario is more suitable for the chiral limit where a first-order phase transition occurs.
5 Summary and discussion

We focus here on the QCD relevant first-order, second-order and cross-over transitions. Obviously, more complicated structures are possible, e.g. a sequence or nested first-order transitions for functions $T(z_H)$ with multiple local minima (see [26] for the case of a double transition). These require further shaping of the dilaton potential. This can be easily done by combining the elements of our systematics presented in this paper: an extreme point of $\partial_\Phi V/V$ generates an inflection point of $T(z_H)$ which points to a cross-over or a second-order phase transition (if it is a horizontal turning point) or to a first-order phase transition if the temperature exhibits additional extreme points which can be controlled by the altitude of $\partial_\Phi V/V$.

We did not touch such issues as good and bad curvature singularities [89], adding further (e.g. charged scalar) fields which can bridge to order parameters and/or condensation [90], larger classes of dilaton potentials (e.g. Liouville potentials or linear combination thereof [91]) and fluctuations. Unfortunately, the Einstein gravity - dilaton model seems to be not flexible enough to allow simultaneously for a cross-over and meson states in the probe limit because the existence of the latter ones requires a minimum in $T(z_H)$. The obvious idea to construct a dilaton potential such that $T(z_H)$ has a minimum at a horizon $z_H^*$ with $T(z_H^*)$ being small (to ensure the existence of the mesons) and a cross-over at the QCD critical temperature of about 155 MeV does not work very well, since all mesons states disappear already at the minimum temperature $T(z_H^*)$.

A first step further on the road to a fully consistent approach could be the consideration of a $U(1)$ Maxwell type gauge field. Such a field has been used to address the question of the behavior of vector mesons in relation to the thermodynamics: $\rho$ mesons are described through the $U(1)$ gauge field (see (2.9)) and putting together the actions (2.1) and (2.9) would yield a model with full back reaction from the mesons to the gravity background. Since QCD thermodynamics is not driven by vector mesons alone, another step is adding flavor by including the pseudo-scalar and scalar sectors via the bulk fundamental fields and its vacuum expectation values. Some works point directly in this direction: the authors of [92] introduce a second scalar field (glue ball field) and solve the field equations for the case $T = 0$; many other investigate the behavior of hadron species in a given background without back reaction (see e.g. [73, 82, 93–99]). [100, 101] give a study of phase transitions in relation to a flavor containing model with given metric background. Bringing the characteristic features of the mentioned works together would be an improvement. The Holy Grail would be a model with parameters steering quark masses and condensates for different flavors separately with proper hadron spectra in all (scalar, pseudo-scalar, vector, axial-vector, tensor and axial-tensor) sectors.

All mentioned extensions point to the leading question, which framework is needed to have a QCD consistent thermodynamics and proper in-medium modifications of the hadron species. However, increasing the variety of a model means increasing its complexity and requires much follow-up work.
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A Derivation of (3.3)

We use \( f'(z_H, z_H) = -4\pi T(z_H) \) and \( f(z_H, z_H) = 0 \) to evaluate (2.6) and (2.7) at \( z = z_H \) which imply

\[
T(z_H) = \frac{1}{6\pi} \frac{e^{A(z_H, z_H)}}{A'(z_H, z_H)} V(\Phi(z_H, z_H)),
\]

and

\[
T(z_H) = -\frac{1}{4\pi} \frac{e^{A(z_H, z_H)}}{\Phi'(z_H, z_H)} \partial_\phi V(\Phi(z_H, z_H)),
\]

respectively. Differentiating (A.1) w.r.t. \( z_H \) yields

\[
\frac{1}{T} \frac{dT}{dz_H} = \frac{1}{T} \frac{d}{dz_H} \frac{\partial A}{\partial z_H} \left( A'' - \frac{\partial A'}{\partial z_H} \right) + \frac{\partial_\phi V}{V} \frac{d\Phi}{dz_H},
\]

where all functions are to be taken at \((z_H, z_H)\). Equating (A.1) and (A.2) yields \( A' \) at \( z = z_H \) as

\[
A' \bigg|_{z=z_H} = -\frac{2}{3} \frac{\Phi'}{\Phi} V \bigg|_{z=z_H}.
\]

By inserting (A.4) in (A.3) and eliminating \( A'' \) via (2.5) we find

\[
\frac{1}{\Phi' T} \frac{dT}{dz_H} = \frac{1}{2} \frac{\partial_\phi V}{V} - \frac{1}{3} \frac{\partial A}{\partial z_H} V + \frac{1}{\Phi'} \left( 2 \frac{\partial A'}{\partial z_H} + \frac{\partial_\phi V}{\partial z_H} \right) \partial_\phi V
\]

at \( z = z_H \). This leads directly to (3.3). The next step is to solve the field equation (2.4):

\[
f(z, z_H) = 1 - \frac{h(z, z_H)}{h(z_H, z_H)},
\]

where \( h(z, z_H) := \int_0^z \exp(-3/2A(\tilde{z}, z_H)) \, d\tilde{z} \). This solution is well defined and can be employed to compute the temperature at a for third time:

\[
T(z_H) = \frac{1}{4\pi} \frac{e^{-\frac{3}{2}A(z_H, z_H)}}{h(z_H, z_H)}.
\]

After differentiating (A.1) w.r.t. \( z_H \) and some manipulations we end at

\[
\frac{1}{T} \frac{dT}{dz_H} + 4\pi T = -\frac{3}{2} A' \bigg|_{z=z_H} + \frac{3}{2h} \int_0^{z_H} \frac{\partial A}{\partial z_H}(\tilde{z}, z_H) e^{-\frac{3}{2}A(\tilde{z}, z_H)} \, d\tilde{z}
\]

which we will use in Appendix B.
Figure 10. Left: $T_{\text{min}}$ curve (solid black) for the potential (C.1) with $p_{i \geq 3} = 0$, analogue to Fig. 5. By varying a second parameter (here $p_3 = -0.01 \cdots 0.01$), the $T_{\text{min}}$ curve becomes a strip (red area), where a negative (positive) value of $p_3$ belongs to the lower (upper) part, since increasing $p_3$ means increasing the slope of $\partial_b V/V$ and therefore increasing also the difference between Gubser’s adiabatic criterion and the $T_{\text{min}}$ curve. If $\partial_b V/V$ as a function of $\Phi$ with given parameter set $\vec{p}$ has a section with the corresponding $T_{\text{min}}$ area (here only displayed of a $p_3$ interval) then a first-order or HP phase transition is facilitated. Right: Border lines of parameter regions in $-L^2m^2_\Phi = p_2/24$ vs. $p_4$ space, where either $T(z_H)$ has a minimum (right to blue curve), or $U_0(z)$ has a minimum (right to red curve), or $\partial_b V/V$ is greater than $\sqrt{2/3}$ (right to green curve) for the dilaton potential (C.1) with $p_3 = 0$ and $p_{i \geq 5} = 0$; if $m_\Phi$ and therefore the slope of $\partial_b V/V$ increase, the difference between Gubser’s adiabatic criterion and the $T_{\text{min}}$ curve becomes larger.

B Derivation of (4.1)

We start with (2.13). To relate $U_0$ with $T$, we evaluate (A.1, A.2, A.5, A.8) in leading order, i.e. neglecting $a$ and $\varphi$:

\[
T = \frac{1}{6\pi} e^{A_0} V(\Phi_0), \tag{B.1}
\]

\[
T = -\frac{1}{4\pi} e^{A_0} \Phi_0' \partial_b V(\Phi_0), \tag{B.2}
\]

\[
\frac{1}{\Phi_0' T \ dz_H} = \frac{1}{2} \frac{\partial_b V}{V} - \frac{1}{3} \frac{V}{\partial_b V}, \tag{B.3}
\]

\[
\frac{1}{T} \ dz_H + 4\pi T = -\frac{3}{2} A_0'. \tag{B.4}
\]

These four equations allow for eliminating all $A$’s and $\Phi$’s in (2.13). Finally we arrive at (4.1).

C Towards a systematic dilaton potential expansion analysis

Another useful form of the dilaton potential is

\[
-L^2 V(\Phi) = 12 \exp \left( \sum_{i=2}^5 p_i \Phi^i \right), \tag{C.1}
\]
since \( \partial \Phi V/V \) runs over all BF permitted and AdS conform polynomials if \( \vec{p} = (p_2, p_3, \cdots) \) runs over all vectors. The case \( p_{i \geq 3} = 0 \) characterizes the leading order (straight lines) in the spirit of an expansion of \( \partial \Phi V/V \) in powers of \( \Phi \). The left panel in Fig. 10 shows the influence of the varying \( p_3 \)-depending term (red strip) on the \( T_{\text{min}} \) curve (black curve) which is generated with running \( p_2 \) analogously to Fig. 5. The parameter regions in the \(-L^2m_\Phi^2\) vs. \( p_4 \) plane, where \( T(z_H) \) (blue curve) or \( U_0(z) \) (red curve) exhibits a minimum, as well as the area, where \( \partial \Phi V/V \) is greater than \( \sqrt{2/3} \) (green curve), are also shown (see right panel and compare with Fig. 6). In such a manner one can study, piece by piece, the impact of the individual terms in (C.1) on the issue of phase structure and capabilities to permit vector meson modes in the probe limit.

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