(1, j)-set problem in graphs

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Abstract

A subset $D \subseteq V$ of a graph $G = (V, E)$ is a $(1, j)$-set if every vertex $v \in V \setminus D$ is adjacent to at least 1 but not more than $j$ vertices in $D$. The cardinality of a minimum $(1, j)$-set of $G$, denoted as $\gamma_{(1,j)}(G)$, is called the $(1, j)$-domination number of $G$. Given a graph $G = (V, E)$ and an integer $k$, the decision version of the $(1, j)$-set problem is to decide whether $G$ has a $(1, j)$-set of cardinality at most $k$. In this paper, we first obtain an upper bound on $\gamma_{(1,j)}(G)$ using probabilistic methods, for bounded minimum and maximum degree graphs. Our bound is constructive, by the randomized algorithm of Moser and Tardos. We also show that the $(1, j)$-set problem is NP-complete for chordal graphs. Finally, we design two algorithms for finding $\gamma_{(1,j)}(G)$ of a tree and a split graph, for any fixed $j$, which answers an open question posed in [CHHM13].

Keywords. Domination, $(1, j)$-set, NP-completeness, probabilistic methods, Chordal graphs

1 Introduction

The concept of domination and its variations is one of the most active area of research in graph theory because of its application in facility location problems, in problems involving finding a set of representatives, in monitoring communication or electrical networks, and in various other areas of practical applications (see [HHS98b, HHS98a]). Over the years, many different variants of domination have been introduced and studied in the literature. The concept of $(i, j)$-set is a very interesting and recent variant of domination [CHHM13, YW14].

1.1 Definitions

For a natural number $m$, let $[m]$ denote the set $\{1, 2, \ldots, m\}$. Let $G = (V, E)$ be a graph. For $v \in V$, let $N_G(v) = \{u | uv \in E\}$ denote the open neighborhood of $v$ and $N_G[v] = N_G(v) \cup \{v\}$ denote the closed neighborhood of $v$. The degree of a vertex $v \in V$, denoted by $d_G(v)$, is the number of neighbors of $v$. Let $\Delta_G$ and $\delta_G$ denote the maximum and minimum degree of $G$. (We will remove the subscript $G$ where it is obvious from the context). Let $G[S]$ denote the subgraph induced by the vertex set $S$ on $G$. A tree is a connected graph which has no cycle. A tree is called

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a rooted tree if one of its vertices, say $r$, has been designated as the root. The level of a vertex is the number of edges along the unique path between it and the root. A set $S \subseteq V$ of a graph $G = (V, E)$ is an independent set if no two vertices in $S$ are adjacent. If every pair of distinct vertices in $K \subseteq V$ are adjacent in $G$, then $K$ is called a clique. A graph $G$ is chordal if every cycle in $G$ of length at least four has a chord, that is, an edge between two non-consecutive vertices of the cycle. A graph $G = (V, E)$ is called a split graph if $V$ can be partitioned into two sets, say $S$ and $K$, such that $S$ is an independent set and $K$ is a clique of $G$. Note that trees and split graphs are chordal graphs. A claw is basically a $K_{1,3}$, a complete bipartite graph having one vertex in one partition and three vertices in the other partition. A vertex $u \in V$ is said to be dominated by a vertex $v \in V$ if $u \in N_G[v]$. A set $D \subseteq V$ is called a dominating set of $G$ if for every vertex $v \in V \setminus D$, $|N_G(v) \cap D| \geq 1$. The cardinality of a minimum dominating set of $G$ is called the domination number of $G$ and is denoted by $\gamma(G)$. Note that, a dominating set $D$ dominates each vertex of $V \setminus D$ at least once. If, for some positive integer $i$, a dominating set $D_i$ dominates each vertex of $V \setminus D_i$ at least $i$ times, then $D_i$ is called a $i$-dominating set. A restrained dominating set is a set $D_i \subseteq V$ where every vertex in $V \setminus S$ is adjacent to a vertex in $S$ as well as another vertex in $V \setminus S$. The cardinality of a minimum restrained dominating set of $G$ is called the restrained domination number of $G$.

1.2 Short review on $(i,j)$-set

A set $D \subseteq V$ of a graph $G = (V, E)$ is called a $(i,j)$-set if for every $v \in V \setminus D$, $i \leq |N_G(v) \cap D| \leq j$ for nonnegative integers $i$ and $j$, that is, every vertex $v \in V \setminus D$ is adjacent to at least $i$ but not more than $j$ vertices in $D$. The concept of $(i,j)$-set was introduced by Chellali et al. in [CHHM13]. Clearly, it is a generalization of the classical domination problem. Like domination problem, in this case, our goal is to find a $(i,j)$-set of minimum cardinality, which is called the $(i,j)$-domination number of $G$ and is denoted by $\gamma_{(i,j)}(G)$. The decision version of $(i,j)$-set problem is defined as follows.

$(i,j)$-Set problem ($(i,j)$-SET)

**Instance:** A graph $G = (V, E)$ and a positive integer $k \leq |V|$.

**Question:** Does there exist a $(i,j)$-set $D$ of $G$ such that $|D| \leq k$?

In domination, we are interested in finding a set $D$ which dominates all the vertices of $V \setminus D$ at least once. But in some situation, we need to dominate each vertex at least $i$ times and at the same time, dominating a vertex more than $j$ times, might cause a problem. Basically, we are interested in finding a $i$-dominating set with a bounded redundancy. In these type of situations, we need the concept of $(i,j)$-set. Also, $(i,j)$-set is a more general concept which involves nearly perfect set [DHJH+95], perfect dominating set [LS90] (also known as 1-fair dominating set [CHH12]) etc. as variants. There is a concept of set restricted domination which is defined as follows: for each vertex $v \in V$, we assign a set $S_v$. A set $DS$ is called a set restricted dominating set if for all $v \in V$, $|N_G[v] \cap D_S| \in S_v$. Note that if $S_v = \{j\}$ for all $v \in V$, we have a $(1,j)$-set. In that sense, $(1,j)$-set is a particular type of set restricted dominating set.

The concept of $(i,j)$-set has been introduced recently in 2013. Unlike other variations of domination, it has not been well studied until now. As per our knowledge, only two papers have appeared on $(i,j)$-set [CHHM13, YW14]. The main focus of [CHHM13] is on a particular $(i,j)$-set, namely
(1, 2)-set. In [CHHM13], the authors have made a simple observation that for a simple graph $G$ with $n$ vertices, $\gamma(G) \leq \gamma(1, 2)(G) \leq n$. They have studied some graph classes for which these bounds are tight. They have shown that $\gamma(G) = \gamma(1, 2)(G)$ for claw-free graphs, $P_4$-free graphs, caterpillars etc. The authors have constructed a special type of split graph that achieves the upper bound. But there are some graph classes for which $\gamma(1, 2)(G)$ is strictly less than $n$. These graph classes involve graphs with maximum degree 4, graphs having a $k$-clique whose vertices have degree either $k$ or $k + 1$ etc [CHHM13]. In [CHHM13], the authors have studied the (1, 3)-set for grid graphs and showed that $\gamma(G) = \gamma(1, 3)(G)$. Using this result, they also showed that domination number is equal to restrained domination number. From complexity point of view, it is known that (1, 2)-SET is NP-complete for bipartite graphs [CHHM13]. A list of open problems were posed in [CHHM13] indicating some research directions in this field. In [YW14], the authors showed that some graphs with $\gamma(1, 2)(G) = n$ exist among some special families of graphs, such as planar graphs, bipartite graphs. These results answers some of the open problems posed in [CHHM13]. They also showed that for a tree $T$ with $k$ leaves, if $\deg_G(v) \geq 4$ for any non-leaf vertex $v$, then $\gamma(1, 2)(T) = n - k$.

Nordhaus-Gaddum-type inequalities are also established for (1, 2)-set in [YW14].

The main focus of [CHHM13] and [YW14] is (1, 2)-set. In this paper, we study the more general set, namely (1, $j$)-set. Apart from the open problems mentioned in [CHHM13], a bound on the (1, $j$)-domination number for general graphs is important. A general bound and its construction forms a major thrust of this paper, which is presented in Section 2. In Section 3, we tighten the hardness result by showing that (1, $j$)-set problem is NP-complete for chordal graphs. In Section 4, we propose two polynomial time algorithms that calculate minimum (1, $j$)-domination number for trees and split graphs, that solves an open problem mentioned in [CHHM13]. Finally, Section 5 concludes the paper.

2 Upper bounds

In this section, we shall prove an upper bound on the (1, $j$)-domination number, i.e. $\gamma(1, j)(G)$, of any graph $G = (V, E)$, having bounded minimum and maximum degree, for ‘sufficiently large’ $j$.

In [AS08], Alon and Spencer describe a similar upper bound on the domination number $\gamma(G)$, using probabilistic methods. Their strategy, (a classic example of the ‘alteration technique’ in probabilistic methods), was to select a random subset $X$ of vertices as a partial dominating set, and then to include the set $Y$ of vertices not dominated by $X$, to get the final dominating set. However, such a strategy is a priori not applicable for (1, $j$)-domination, because including or excluding vertices from the dominating set could change the number of dominating vertices adjacent to some vertex. Instead, we shall use a one-step process, and analyze it using the Lovász Local Lemma and Chernoff bounds to ensure that the conditions for (1, $j$)-set holds. Our proof also implies a randomized algorithm, using the Moser-Tardos constructive version of the Local Lemma [MT10], which would give a polynomial-time algorithm for obtaining a (1, $j$)-dominating set.

We first state two well-known results, Chernoff bound and Lovász Local Lemma, in a form suitable for our purposes. These results can be found in any standard text on probabilistic combinatorics, e.g. [AS08].

**Theorem 1** (Chernoff bound). *Suppose $X$ is the sum of $n$ independent variables, each equal to 1 with probability $p$ and 0 otherwise. Then for any $0 \leq \alpha$,

$$\Pr [X > (1 + \alpha)np] < \exp(-f(\alpha)np),$$
where \( f(\alpha) = (1 + \alpha) \ln(1 + \alpha) - \alpha \).

Lemma 2 (Lovász local lemma). Let \( \mathcal{A} = \{ E_1, E_2, ..., E_m \} \) be a collection of events over a probability space such that each \( E_i \) is totally independent of all but the events in \( \mathcal{D}_i \subseteq \mathcal{A} \setminus \{ E_i \} \). If there exists a real sequence \( \{ x_i \}_{i=1}^m \), \( x_i \in [0, 1) \), such that
\[
\forall i \in [m], \Pr[E_i] \leq x_i \prod_{j : E_j \in \mathcal{D}_i} (1 - x_j),
\]
then
\[
\Pr[\bigcap_{i=1}^m \overline{E_i}] > 0.
\]

In particular, if for all \( i \), \( |\mathcal{D}_i| = d \) and \( \Pr[E_i] \leq p \), then, if \( ep(d + 1) \leq 1 \), then
\[
\Pr[\bigcap_{i=1}^m \overline{E_i}] > \exp \left( -\frac{m}{d + 1} \right).
\]

Before stating the main theorem, we need some definitions: Given \( \alpha \in \mathbb{R}^+ \), let
\[
f(\alpha) \overset{\text{def}}{=} (1 + \alpha) \ln(1 + \alpha) - \alpha.
\]
Also let \( s(\alpha) \overset{\text{def}}{=} \min\{1, f(\alpha)\} \) and for \( \Delta \in \mathbb{Z}^+ \), let
\[
g(\Delta) \overset{\text{def}}{=} \ln(2e(\Delta^2 + 1)) = 1 + \ln 2 + 2 \ln \Delta + o(\Delta),
\]
where \( e \) is the base of the natural logarithm.

Theorem 3. Given \( j \in \mathbb{Z}^+ \), let \( \alpha > 0 \) be the maximum real number such that
\[
j + 1 \geq (1 + \alpha) \frac{\Gamma g(\Delta)}{s(\alpha)} \quad \text{where} \quad \Gamma = \frac{\Delta}{\delta}.
\]
Then, if such an \( \alpha \) exists,
\[
\gamma_{1,j}(G) \leq (1 + o_D(1)) \frac{g(\Delta)}{s(\alpha)\delta} n \leq (1 + o_D(1)) \left( \frac{1 + \ln 2 + 2 \ln \Delta}{s(\alpha)\delta} \right) n.
\]
Further, there is a randomized algorithm to obtain a \((1, j)\)-dominating set of size at most \( \frac{ng(\Delta)}{s(\alpha)\delta} \) that has expected runtime \( O(n) \).

Proof. Let \( D \subset V \) be a subset of vertices obtained by tossing a coin for each vertex \( v \in V \) independently and randomly with probability \( p = \frac{g(\Delta)}{s(\alpha)} \) and choosing \( v \) if the coin comes up Heads. We shall show using the Local Lemma, that the subset \( D \) is a \((1, j)\)-dominating set with non-zero probability.

For each vertex \( v \in V \), let \( E_v \) be the event that \( v \) is not \((1, j)\)-dominated by \( D \), i.e. that \( v \notin D \), and \( |N(v) \cap D| \not\in [j] \). We need to show that
\[
\Pr[\bigcap_{v \in V} \overline{E_v}] > 0.
\]
In order to use the Local Lemma, consider the dependency graph formed by having the set of events \( \{E_v\}_{v \in V} \) as vertices. Events \( E_u, E_v \) \((u, v \in V)\) are dependent if and only if their outcomes depend on at least one common coin toss. Then clearly, the events \( E_u, E_v \) will be dependent if and only if

\[ N[u] \cap N[v] \neq \emptyset. \]

This is possible only if the vertices \( u \) and \( v \) are at a distance at most 2 from each other in the graph \( G \). Hence, the degree of the dependency graph is at most \( \Delta^2 \). Now applying the symmetric form of the Local Lemma, we get that

\[
\Pr \left[ \bigcap_{v \in V} \overline{E_v} \right] > 0 \quad \text{if} \quad \Pr[E_v] \leq \frac{1}{e(\Delta^2 + 1)}.
\]

We also need to bound the size of the selected subset \(|D|\). However, this can easily be obtained by applying a Chernoff bound to the output of the Local Lemma. The proof of Theorem 3 is therefore completed with the following 2 claims: Let \( X \overset{\text{def}}{=} |D| \). With \( p, \alpha \) and \( E_v \) defined as above,

**Claim 1.** For all \( v \in V \),

\[
\Pr[E_v] \leq \frac{1}{e(\Delta^2 + 1)}.
\]

**Claim 2.** For any \( \varepsilon > \frac{1}{\sqrt{\Delta}} \), there exists \( \Delta_0 \in \mathbb{Z}^+ \), such that for all \( \Delta \geq \Delta_0 \), we have

\[
\Pr \left[ \left( X < (1 + \varepsilon)np \right) \cap \left( \bigcap_{v \in V} \overline{E_v} \right) \right] > 0.
\]

Thus the set \( D \) is of size at most

\[
(1 + o_\Delta(1))n \left( \frac{1 + \ln 2 + 2 \ln \Delta}{\delta} \right),
\]

and every vertex in \( V \setminus D \) has at least 1 and at most \( j \) neighbours in \( D \). The bound on \( \gamma_{(1,j)}(G) \) follows.

We will elaborate on the randomized algorithm, via Moser-Tardos’s local lemma implementation, for obtaining such a \((1,j)\)-dominating set in Remark 2 at the end of this section.

It only remains to prove the Claims 1 and 2.

**Proof of Claim 1.** Given any vertex \( v \in V \), define \( X_v = |N(v) \cap D| \), and let \( F_v \) denote the event that \( X_v \notin [1,j] \). Then we have that

\[
\Pr[E_v] = \Pr[E_v|\overline{F_v}] \cdot \Pr[F_v] + \Pr[E_v|F_v] \cdot \Pr[\overline{F_v}] = (1-p)\Pr[F_v] + 0
\]

We shall prove the stronger condition : \( \Pr[\bigcap_{v \in V} \overline{F_v}] > 0 \). Now,

\[
\Pr[F_v] = \Pr[X_v < 1] + \Pr[X_v > j].
\]

Observe that \( X_v \) has the binomial distribution \( \text{Bin}(d(v), p) \).
Note that, if \( j > d(v) \), then the event \( F_v \) occurs only when \( X_v = 0 \) and for \( j \leq d(v) \), the event \( F_v \) can occur when \( X_v = 0 \) or when \( X_v > j \). Therefore,

\[
\Pr[F_v] = \begin{cases} 
(1 - p)^{d(v)} & \text{if } j > d(v) \\
(1 - p)^{d(v)} + \Pr[X_v > j] & \text{if } j \leq d(v)
\end{cases}
\] (1)

By the premise of the Theorem, we get that

\[ j + 1 \geq (1 + \alpha) \frac{\Gamma g(\Delta)}{s(\alpha)} \geq (1 + \alpha) \frac{g(\Delta)}{s(\alpha)}, \]

and hence, substituting the value of \( p \), we get

\[ f(\alpha)d(v)p \geq f(\alpha) \frac{g(\Delta)}{s(\alpha)} \geq g(\Delta), \]

since \( f(\alpha) \geq s(\alpha) \). Substituting in the expression for \( f(v) \) gives

\[ (1 - p)^{d(v)} = e^{d(v)\ln(1 - p)} \leq e^{-d(v)p} \leq \frac{1}{2e(\Delta^2 + 1)}, \]

since \( d(v) \geq \delta \). To compute \( \Pr[X_v \geq j + 1] \), we use the Chernoff bound:

\[
\Pr[X_v \geq j + 1] \leq \Pr[\text{Bin}(d(v), p) \geq j + 1] \\
\leq \Pr[\text{Bin}(d(v), p) \geq (1 + \alpha)d(v)p] \\
\leq \exp(-f(\alpha)d(v)p) \\
\leq \frac{1}{2e(\Delta^2 + 1)}
\]

where the last inequality follows from the choice of \( p \).

Therefore, we get that

\[ \Pr[F_v] = \Pr[X_v < 1] + \Pr[X_v > 1] \leq \frac{2}{2e(\Delta^2 + 1)} \]

and hence that \( \Pr[F_v] \leq \frac{1}{e(\Delta^2 + 1)} \).

\[ \square \]

**Proof of Claim.** To show that \( \Pr[A \cap B] > 0 \) where \( A, B \) are events in a probability space, it suffices to show that

\[ \Pr[\bar{A} \cup \bar{B}] \leq \Pr[\bar{A}] + \Pr[\bar{B}] < 1, \text{ i.e., } \Pr[B] - \Pr[A] > 0. \]

Taking \( A \) to be the event \( (X < (1 + \varepsilon)np) \) and \( B \) to be \( \bigcap_{v \in V} \bar{E}_v \), we shall first upper bound \( \Pr[A] \), and then use the lower bound on \( \Pr[B] \) from the Local Lemma. Using Chernoff bound, we get that

\[
\Pr[X \geq (1 + \varepsilon)np] \leq \exp\left(-\frac{\varepsilon^2 np}{3}\right) \\
\leq \exp\left(-\frac{\varepsilon^2 ng(\Delta)}{3\delta}\right)
\]

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Now, from the Local Lemma, we get that
\[
\Pr \left[ \bigcap_{v \in V \setminus E_v} \right] > \left( 1 - \frac{1}{\Delta^2 + 1} \right)^n \\
\approx \exp \left( -\frac{n}{\Delta^2 + 1} \right).
\]

Let \( \varepsilon = \frac{\sqrt{c\delta}}{\Delta} \), where \( c > 0 \) is any constant. Then we get that for sufficiently large \( \Delta \),
\[
\Pr [B] - \Pr [\bar{A}] \geq \exp \left( -\frac{n}{\Delta^2 + 1} \right) - \exp \left( -\varepsilon^2 ng(\Delta) \right) > 0
\]
since for \( \varepsilon = \frac{\sqrt{c\delta}}{\Delta} \), we get that
\[
\frac{\varepsilon^2 ng(\Delta)}{3\delta} \geq \frac{c n \ln \Delta}{3\Delta^2} > \frac{n}{\Delta^2 + 1}.
\]

In particular, taking \( \alpha = e - 1 \) and \( G \) \( d \)-regular, we get:

**Corollary 4.** If \( G \) is a \( d \)-regular graph, and \( j > eg(d) \) then
\[
\gamma(1,j)(G) \leq (2 + o_d(1)) \frac{n \ln d}{d}.
\]

**Remark 1.** We remark that from the known lower bounds on the domination number of random graph, our results can be seen to be tight up to constant multiplicative factors, since \( \gamma(1,j)(G) \geq \gamma(G) \). For instance, the result of Glebov, Liebenau and Szabó [GLS13], implies that there exist graphs \( G \) on \( n \) vertices such that their domination number \( \gamma(G) \geq \frac{n \log d}{d} \).

**Remark 2.** Elaboration on the Moser-Tardos’s (MT) implementation: We set up a SAT formula for domination, where each vertex \( v_i \in V \) corresponds to a variable \( x_i \) and there is a clause \( C(v) \) corresponding to each neighbourhood \( N(v) \). \( x_i = \text{“true”} \) means the vertex \( v_i \) is selected in the dominating set. Clause \( C(v) \) is said to have failed if it is not satisfied in the given assignment. In addition, for each vertex \( v \in V \) having degree \( d(v) > j \), there is a unique clause for every subset of \( N(v) \) of size \( j + 1 \), which fails only if all the vertices in the corresponding subset of \( N(v) \) are selected in the dominating set. Now we run the MT algorithm on this formula, (i.e. take a random assignment where each variable is set to true independently with the probability \( p \) used in the proof; choose an arbitrary failing clause and randomly reset all variables inside the clause; repeat until all clauses are satisfied). The LLL condition guarantees that the MT algorithm will terminate in expected time linear in the number of clauses, i.e. \( O(n\Delta^{j+1}) = O(n^{j+2}) \), and when this happens, the Chernoff bound guarantees that with high probability, not more than \( (1 + o_d(1)) \frac{\Delta n}{\alpha \delta} \) many variables will be set to “true”.

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3 NP-complete for chordal graphs

In this section, we show that $(1, j)$-SET is NP-complete when restricted to chordal graphs. Note that, for $j = 1$, the problem is basically perfect domination problem, which is known to be NP-complete for chordal graphs [YL90]. For $j \geq 2$, we prove the NP-completeness by using a reduction from Exact 3-Cover problem (EX3C), which is known to be NP-complete [GJ79].

**Exact 3-Cover problem (EX3C)**

**Instance:** A finite set $X$ with $|X| = 3q$, where $q$ is a positive integer and a collection $C$ of 3-element subsets of $X$.

**Question:** Is there a subcollection $C'$ of $C$ such that every element of $X$ appears in exactly one element of $C'$?

**Theorem 5.** $(1, j)$-SET is NP-complete for chordal graphs.

Clearly $(1, j)$-SET for chordal graph is in NP. We describe a polynomial reduction from EX3C to $(1, j)$-SET for chordal graphs. Given any instance $(X, C)$ of EX3C, we obtain a chordal graph $G = (V, E)$ and an integer $k$ such that EX3C has a solution if and only if $G$ has a $(1, j)$-set of cardinality at most $k$.

Let $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{C_1, C_2, \ldots, C_t\}$ be an arbitrary instance of EX3C. The vertex set of the newly formed graph $G = (V, E)$ is formed as a disjoint union of $V_1, V_2$ and $V_3$, that is,

$$V = V_1 \sqcup V_2 \sqcup V_3.$$  

For each $C_p \in C$, $p \in [t]$, we have a claw centered at a vertex $u_p$ and let $v_p, y_p$ and $z_p$ be the pendant vertices of that claw. The set $V_1$ is given by

$$V_1 = \bigcup_{p=1}^{t} \{u_p, v_p, y_p, z_p\}.$$  

Also, we have a set $V_2$ of $3q$ vertices $x_1, x_2, \ldots, x_{3q}$, each corresponding to an element of $X$. Furthermore, for each $i \in \{1, 2, \ldots, 3q\}$, we add a gadget $G_i$, as shown in Figure 1. The gadget $G_i$ is basically a forest of $q$ number of rooted trees of depth 2 rooted at the vertices $w_1^i, w_2^i, \ldots, w_q^i$ as shown in Figure 2. In the gadget $G_i$, each $w_1^i, w_2^i, \ldots, w_q^i$ has $j$ children and each of these $j$ children has 2 more children. The set $V_3$ is given by

$$V_3 = \bigcup_{i=1}^{3q} V(G_i),$$

where $V(G_i)$ is the vertex set of $G_i$. Now we add the edges between $x_i$ and $v_p$ if the element corresponding to $x_i$ is in $C_p$. Note that degree of each $v_p$ is 4 for all $p \in \{1, 2, \ldots, t\}$. Also, we add edges between every pair of distinct vertices of $\{x_1, x_2, \ldots, x_{3q}\}$, making it a clique. Finally, for each $i \in \{1, 2, \ldots, 3q\}$, we add the edges $x_iw_1^i, x_iw_2^i, \ldots, x_iw_q^i$. The construction of $G$ from the instance $(X, C)$ of EX3C is illustrated in Figure 1. Clearly, the graph $G$ is a chordal graph. Let $k = t + q + 3jq^2$.

Theorem 5 directly follows from the following result.
Lemma 6. EX3C has a solution if and only if $G$ has a $(1,j)$-set of cardinality at most $k = t + q + 3jq^2$.

Proof. Suppose the instance $(X, C)$ has a solution $C'$. Since each element of $X$ is covered by exactly one element of $C'$, $|C'| = q$. For each gadget $G_i$, $i \in [3q]$, let $S_i$ be the set of all children of $w_i^1, w_i^2, \ldots, w_i^q$. Clearly, each $S_i$ contains $jq$ vertices. We form a set $D$ as follows:

$$D = \{u_i | 1 \leq i \leq t\} \cup \{v_p | C_p \in C'\} \cup \left(\bigcup_{i=1}^{3q} S_i\right).$$

Since $|C'| = q$, $D$ contains $t + q + 3jq^2$ vertices. One can easily check that $D$ forms a $(1,j)$-set of $G$.

Conversely, suppose that $G$ has a $(1,j)$-set $D$ of cardinality at most $k = t + q + 3jq^2$. First
observe that since \( D \) is a dominating set, \( D \) must contain at least \( t \) vertices from the set

\[
V_4 \overset{\text{def}}{=} \{y_1, \ldots, y_t\} \cup \{z_1, \ldots, z_t\} \cup \{u_1, \ldots, u_t\}
\]

to dominate the pendant vertices

\[
\{y_1, \ldots, y_t\} \cup \{z_1, \ldots, z_t\}.
\]

Similarly, for a fixed \( i \) and \( r \), consider the tree \( T^i_r \). To dominate the pendant vertices of the tree \( T^i_r = (V^i_r, E^i_r) \) we need to select at least \( j \) vertices from the set \( V^i_r \setminus \{w^i_r\} \). Summing up over all \( i \) and \( r \), we get that \( D \) contains more than \( 3jq^2 \) vertices from the set

\[
V_5 \overset{\text{def}}{=} \bigcup_{1 \leq i \leq 3q, 1 \leq r \leq q} (V^i_r \setminus \{w^i_r\}).
\]

Observe now that the cardinality of \( D \) is at least \( t + 3jq^2 = t + 3jq^2 \).

Now to complete the proof we will only have to show that \( V_2 \cap D = \emptyset \). Since if this is the case then each \( x_1 \) has to be dominated by either some \( w^i_r \in G_i \) or a \( v_s \in V_1, s \in [t] \). We have to dominate the \( 3q \) vertices of \( V_2 \) using at most \( q \) vertices, since we have used up the other \( t + 3jq^2 \) vertices. Since each \( w^i_r \) dominates only one \( x_i \), while each \( v_i \in V_1 \) dominates 3 \( x_i \)’s, this is possible only if there exist \( q \) vertices \( v_{i_1}, \ldots, v_{i_q} \), which can dominate the \( 3q \) vertices \( x_i \in V_2 \). Now define \( C' \) to be the sets corresponding to these vertices, i.e. \( C' = \{C_{i_1}, \ldots, C_{i_q}\} \). Clearly \( C' \) is an exact cover of \( X \), and has only \( q \) sets.

Till now we have only used the fact that \( D \) is a dominating set but for showing \( D \cap V_2 = \emptyset \) we will be crucially using the fact that \( D \) is a \((1, j)\)-set. To reach a contradiction let us suppose some \( x_i \in D \). Then each \( w^i_r, r \in [q] \) is 1-dominated by \( x_i \), and either has to be in \( D \) or can have at most \( j - 1 \) other neighbours that are in \( D \). In either case, we get that for each tree \( T^i_r \in G_i, |T^i_r \cap D| \geq j + 1 \). Hence, \( |G_i \cap D| \geq jq + q \). This implies that \( |D| \geq t + 1 + (3q - 1)(jq) + (j + 1)q = t + 1 + 3jq^2 + q \), which contradicts the assumption that \( |D| \leq k \). Therefore \( D \cap V_2 = \emptyset \).

\[\text{Remark 3.} \] Following observations directly follow from the NP hardness reduction:

1. The only possibility of dominating \( V_4 \) by \( t \) vertices is to take \( \{u_1, u_2, \ldots, u_t\} \) and this set also dominates the set \( \{v_1, v_2, \ldots, v_t\} \).

2. The only possibility of dominating \( V_5 \) by \( 3jq^2 \) vertices is to take \( \bigcup_{i=1}^{3q} S_i \) and this set dominates each \( w^i_q \) exactly \( j \) times. Note that \( S_i \) is the set of all children of \( w^i_1, w^i_2, \ldots, w^i_q \) and each \( S_i \) contains \( jq \) vertices.

4 Polynomial time algorithms

4.1 Tree

To design an efficient algorithm for finding \((1, j)\)-domination number of a given tree \( T \), we need the concept of a more generalized set, namely \( M \)-set of an \( M \)-labeled tree. In fact, we design a dynamic programming algorithm for finding the minimum cardinality of an \( M \)-set of an \( M \)-labeled tree. First let us define an \( M \)-labeled tree and an \( M \)-set.
**Definition 7.** A tree $T$ is called an $M$-labeled tree if each vertex $v$ is associated with two nonnegative integers $M_a(v)$ and $M_b(v)$ such that $M_a(v) \leq M_b(v)$. A subset $S \subseteq V$ of an $M$-labeled tree $T = (V,E)$ is called an $M$-set if $M_a(v) \leq |N_T(v) \cap S| \leq M_b(v)$ for every $v \in V \setminus S$. The minimum cardinality of an $M$-set of an $M$-labeled tree $T$ is called the $M$-domination number of $T$ and is denoted by $\gamma_M(T)$.

Note that if all the vertices of an $M$-labeled tree $T$ can be labeled as $M_a(v) = 1$ and $M_b(v) = j$, then an $M$-set of $T$ is nothing but a $(1,j)$-set of the underlying tree.

The main idea of the dynamic programming algorithm is to choose a specific vertex $u$ from $T$. Any minimum $M$-set of $T$ should either contain $u$ or does not contain $u$. So the problem of finding the minimum cardinality of an $M$-set of $T$ boils down to finding two parameters: (i) $\gamma_M(T,u)$, the minimum cardinality of an $M$-set of $T$ that contains the specific vertex $u$ and (ii) $\gamma_M(T,\bar{u})$, the minimum cardinality of an $M$-set of $T$ that does not contain the specific vertex $u$.

Suppose $uv$ is an edge of the $M$-labeled tree $T$. Let $H_1$ and $H_2$ be the subtrees of $T$ rooted at $u$ and $v$ respectively. Note that $H_1$ and $H_2$ are $M$-labeled trees and the labels of the vertices of $H_1$ and $H_2$ remain the same as they are in $T$. Our aim is to use the parameters $\gamma_M(H_1, u)$, $\gamma_M(H_1, \bar{u})$, $\gamma_M(H_2, v)$, and $\gamma_M(H_2, \bar{v})$ (with suitable labeling $M$) to find $\gamma_M(T, u)$ and $\gamma_M(T, \bar{u})$. The following lemma shows how the values of $\gamma_M(T, u)$ and $\gamma_M(T, \bar{u})$ are obtained.

**Lemma 8.** Let $uv$ be an edge of an $M$-labeled tree $T$ and $H_1$ and $H_2$ be the subtrees of $T$ rooted at $u$ and $v$ respectively. Then the following statements hold.

(a) $\gamma_M(T,u) = \gamma_M(H_1,u) + \gamma_M'(H_2)$, where the label $M'$ is same as $M$ except $M'_a(v) = \max\{M_a(v) - 1, 0\}$ and $M'_b(v) = \max\{M_b(v) - 1, 0\}$

(b) $\gamma_M(T,\bar{u}) = \min\{\gamma_M(H_1,\bar{u}) + \gamma_M(H_2,\bar{v}), \gamma_M'(H_1,\bar{u}) + \gamma_M(H_2,v)\}$, where the label $M'$ is same as $M$ except $M'_a(u) = \max\{M_a(u) - 1, 0\}$ and $M'_b(u) = \max\{M_b(u) - 1, 0\}$.

**Proof.** (a) Let $D$ be a minimum cardinality $M$-set of $T$ containing $u$. Let $D_1 = V(H_1) \cap D$ and $D_2 = V(H_2) \cap D$. Clearly $D_1$ is an $M$-set of $H_1$ containing $u$. Now, $D_2$ may or may not contain the vertex $v$.

**Case** $v \in D_2$: In this case, $D_2$ is an $M'$-set of $H_2$ containing $v$.

**Case** $v \notin D_2$: In this case, $D_2$ is an $M'$-set of $H_2$ not containing $v$.

Since $\gamma_M'(H_2) = \min\{\gamma_M'(H_2,v), \gamma_M'(H_2,\bar{v})\}$, we have $\gamma_M(H_1, u) + \gamma_M'(H_2) \leq \gamma_M(T, u)$.

On the other hand, let $D_1$ be a minimum cardinality $M$-set of $H_1$ containing $u$ and $D_2$ be a minimum cardinality $M'$-set of $H_2$. Let $D = D_1 \cup D_2$. Clearly, whatever be the case ($v \in D_2$ or $v \notin D_2$), we can verify that $D$ is a $M$-set of $T$ and $u \in D$. Hence, $\gamma_M(T, u) \leq \gamma_M(H_1, u) + \gamma_M'(H_2)$.

Thus we have, $\gamma_M(T, u) = \gamma_M(H_1, u) + \gamma_M'(H_2)$.

(b) Let $D$ be a minimum cardinality $M$-set of $T$ not containing $u$. Let $D_1 = V(H_1) \cap D$ and $D_2 = V(H_2) \cap D$. Now, $D_2$ may or may not contain the vertex $v$.

**Case** $v \notin D$: In this case, $D_1$ is an $M$-set of $H_1$ not containing $u$ and $D_2$ is an $M$-set of $H_2$ not containing $v$. Hence, $\gamma_M(H_1, \bar{u}) + \gamma_M(H_2, v) \leq \gamma_M(T, \bar{u})$.

**Case** $v \in D$: In this case, $D_1$ is an $M'$-set of $H_1$ not containing $u$ and $D_2$ is an $M$-set of $H_2$ containing $v$. Hence, $\gamma_M'(H_1, \bar{u}) + \gamma_M(H_2, v) \leq \gamma_M(T, \bar{u})$. 

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So we have, \( \min\{\gamma_M(H_1, \bar{u}) + \gamma_M(H_2, \bar{v}), \gamma_M'(H_1, \bar{u}) + \gamma_M(H_2, v)\} \leq \gamma_M(T, \bar{u}) \).

On the other hand, for showing \( \gamma_M(T, \bar{u}) \leq \min\{\gamma_M(H_1, \bar{u}) + \gamma_M(H_2, \bar{v}), \gamma_M'(H_1, \bar{u}) + \gamma_M(H_2, v)\} \), we have the following two cases:

**Case** \( \min\{\gamma_M(H_1, \bar{u}) + \gamma_M(H_2, \bar{v}), \gamma_M'(H_1, \bar{u}) + \gamma_M(H_2, v)\} = \gamma_M(H_1, \bar{u}) + \gamma_M(H_2, \bar{v}) \): Let \( D_1 \) be a minimum cardinality \( M \)-set of \( H_1 \) not containing \( u \) and \( D_2 \) be a minimum cardinality \( M \)-set of \( H_2 \) not containing \( v \). Let \( D = D_1 \cup D_2 \). We can easily verify that \( D \) a minimum cardinality \( M \)-set of \( T \) not containing \( u \). So, \( \gamma_M(T, \bar{u}) \leq \gamma_M(H_1, \bar{u}) + \gamma_M(H_2, \bar{v}) \).

**Case** \( \min\{\gamma_M(H_1, \bar{u}) + \gamma_M(H_2, \bar{v}), \gamma_M'(H_1, \bar{u}) + \gamma_M(H_2, v)\} = \gamma_M'(H_1, \bar{u}) + \gamma_M(H_2, v) \): In this case, similarly we can show that \( \gamma_M(T, \bar{u}) \leq \gamma_M'(H_1, \bar{u}) + \gamma_M(H_2, v) \).

Hence in both the cases, \( \gamma_M(T, \bar{u}) \leq \min\{\gamma_M(H_1, \bar{u}) + \gamma_M(H_2, \bar{v}), \gamma_M'(H_1, \bar{u}) + \gamma_M(H_2, v)\} \).

Thus we have, \( \gamma_M(T, \bar{u}) = \min\{\gamma_M(H_1, \bar{u}) + \gamma_M(H_2, \bar{v}), \gamma_M'(H_1, \bar{u}) + \gamma_M(H_2, v)\} \).

Based on the above lemma, we have the following dynamic programming algorithm for finding \( \gamma_M(T) \) for an \( M \)-labeled tree \( T \). Note that, a tree with a single vertex forms the base case at which \( \gamma_M(T) \) can be easily computed depending upon the \( M \) label.

**Algorithm 1:** Min\(_M\)-set\(_T\)re

**Input:** A \( M \)-labeled tree \( T = (V, E) \).

**Output:** A minimum cardinality of an \( M \)-set of \( T \), i.e., \( \gamma_M(T) \).

\[
\text{begin} \\
1 \quad \text{Select a vertex } u \text{ from } V; \\
2 \quad \text{Select an edge } uv \text{ from } E; \\
3 \quad \text{Calculate } \gamma_M(T, u) \text{ and } \gamma_M(T, \bar{u}) \text{ according to Lemma 8}; \\
4 \quad \gamma_M(T) = \min\{\gamma_M(T, u), \gamma_M(T, \bar{u})\}; \\
5 \quad \text{return } \gamma_M(T); \\
\text{end}
\]

The correctness of Algorithm 1 is based on Lemma 8. Since the dynamic programming runs over the edges of the given tree, Algorithm 1 take linear time. Also, as noted earlier, if we initialize the \( M \)-label as \( M_a(v) = 1 \) and \( M_b(v) = j \) for all \( v \in V \), then \( \gamma_{(1,j)}(T) = \gamma_M(T) \). Hence we have the following theorem.

**Theorem 9.** The \((1, j)\)-domination number of a given tree can be computed in linear time.

### 4.2 Split graph

In this subsection, we design an algorithm which finds \((1, j)\)-domination number for a given split graph \( G \) in polynomial time. This algorithm is important because most of the domination type problems like domination [Ber84], total domination [LP83], \( k \)-tuple domination [LC03] etc. are NP-complete for split graphs.

Let the vertex set \( V \) of a split graph \( G = (V, E) \) is partitioned into a clique \( K \) and an independent set \( S \), i.e., \( V = K \cup S \). Also assume that \( |K| = n_1 \) and \( |S| = n_2 \). Note that in finding a minimum \((1,j)\)-set, \( j \) can be considered as a constant. Now if \( n_1 \leq j \), then we are done. Because, in that case, we consider all possible subsets of \( K \) and based on the neighborhood set of these subsets we
can find a minimum cardinality \((1, j)\)-set. Since \(j\) is a constant, the number of subsets of \(K\) is bounded by a constant (this constant is huge, \(2^j\)). This implies that, in this case, we can find a minimum \((1, j)\)-set in polynomial time. Hence we assume that \(j < n_1\). The idea of the algorithm is based on a simple fact that if a \((1, j)\)-set, say \(D\), contains more than \(j\) but less than \(n_1\) vertices from \(K\), then there exists a vertex in \(K \setminus D\) which is dominated by more than \(j\) vertices, which is a contradiction to the definition of \((1, j)\)-set. Hence we have the following observation.

**Observation 10.** Every \((1, j)\)-set of a given split graph \(G\) contains only \(i\) vertices from \(K\) where \(i \in \{0, 1, 2, \ldots, j, n_1\}\).

Now, for each \(i = 0, 1, 2, \ldots, j\) and \(n_1\), we find a minimum cardinality \((1, j)\)-set \(D\) of \(G\) such that \(|K \cap D| = i\). Finally we pick the minimum cardinality \((1, j)\)-set among these \(j + 2\) types of \((1, j)\)-sets of \(G\). Hence the main task in this algorithm is to find a minimum cardinality \((1, j)\)-set \(D\) of \(G\) such that \(|K \cap D| = i\) for each \(i = 0, 1, 2, \ldots, j\) and \(n_1\). The following lemma gives a complete characterization of these \(j + 2\) types of \((1, j)\)-sets.

**Lemma 11.** Let the vertex set \(V\) of a connected split graph \(G = (V, E)\) is partitioned into a clique \(K\) and an independent set \(S\), i.e., \(V = K \cup S\) and \(|K| = n_1\) and \(|S| = n_2\). Let \(D\) be a minimum \((1, j)\)-set of \(G\). Then the following statements are true.

(a) If \(K \cap D = \emptyset\), then \(d_G(v) \in \{n_1, n_1 + 1, \ldots, n_1 + j - 1\}\) for all \(v \in K\). In this case, \(D = S\) is the only \((1, j)\)-set of \(G\).

(b) For all \(i \in [j - 1]\), if \(K \cap D = \{v_1, \ldots, v_i\} = K_i\), then \(d_G[K \cup S_i](v) \in \{n_1 - 1, n_1, \ldots, n_1 + (j - i) - 1\}\) for all \(v \in K \setminus K_i\), where \(S_i = S \setminus N_G(K_i)\). In this case, \(D = K_i \cup S_i\) is a minimum cardinality \((1, j)\)-set of \(G\) containing \(K_i\).

(c) If \(K \cap D = \{v_1, v_2, \ldots, v_j\} = K_j\), then \(S \subseteq N_G(K_j)\). In this case, \(D = \{v_1, v_2, \ldots, v_j\}\) is a \((1, j)\)-set of \(G\) of minimum cardinality.

(d) If \(K \subseteq D\), then \(S_2 \subseteq D\), where \(S_2 = \{v \in S| d_G(v) \geq (j + 1)\}\). In this case, \(D = K \cup S_2\) is a \((1, j)\)-set of \(G\) of minimum cardinality.

**Proof.** (a) In this case, since \(S\) is an independent set, \(D = S\). Again, since \(K\) is a clique, \(d_G(v) \geq n_1 - 1\) for all \(v \in K\). If \(d_G(v_i) = n_1 - 1\) for some vertex \(v_i \in K\), then

\[N_G(v_i) \cap D = \emptyset.\]

This is a contradiction to the definition of \((1, j)\)-set. Again if \(d_G(v_i) \geq n_1 + j\) for some vertex \(v_i \in K\), then

\[|N_G(v_i) \cap D| \geq (j + 1).\]

This will force \(v_i\) to be in \(D\), which is a contradiction to \(K \cap D = \emptyset\). Thus, we have

\[d_G(v) \in \{n_1, n_1 + 1, \ldots, n_1 + j - 1\}\quad \forall v \in K.\]

(b) In this case, clearly \((K_i \cup S_i) \subseteq D\). Again, since \(K\) is a clique, \(d_G[K \cup S_i](v) \geq n_1 - 1\) for all \(v \in K\). If \(d_G[K \cup S_i](v_i) \geq n_1 + (j - i)\) for some vertex \(v_i \in (K \setminus K_i)\), then \(|N_G(v_i) \cap D| \geq j\). This will force \(v_i\) to be in \(D\), which is a contradiction to \(|K \cap D| = i\). Thus

\[d_G[K \cup S_i](v) \in \{n_1 - 1, n_1, \ldots, n_1 + (j - i) - 1\}\quad \forall v \in K \setminus \{v_i\}.\]
In this case, clearly \( D = K_i \cup S_i \) is a minimum cardinality \( (1,j) \)-set of \( G \) containing \( K_i \).

(c) If possible, let \( u \in S \setminus N_G(K_j) \). Clearly \( u \in D \). Since \( G \) is connected, there exists at least one vertex \( v \) in \( N_G(u) \setminus K_j \). Now for that vertex \( v \),

\[
|N_G(v) \cap D| \geq j.
\]

This will force \( v \) to be in \( D \), which is a contradiction to \( |K \cap D| = j \). Thus \( S \subset N_G(K_j) \). Clearly in this case, \( D = K_j \) is a \( (1,j) \)-set of \( G \) of minimum cardinality.

(d) The proof is trivial and hence omitted. \( \square \)

Based on the above lemma, we have Algorithm 2 that finds a minimum cardinality \( (1,j) \)-set of a given split graph \( G \).

**Algorithm 2: Min\_(1,j\)-set Split**

---

**Input:** A split graph \( G = (V,E) \) with \( V = K \cup S \).

**Output:** A minimum cardinality \( (1,j) \)-set \( D \) of \( G \).

1. **begin**
2.  
3.  **if** \( d_G(v) \in \{n_1, n_1 + 1, \ldots, n_1 + j - 1\} \) for all \( v \in K \) **then**
4.      \( D_0 \leftarrow S; \)
5.  **else**
6.      \( D_0 \leftarrow \emptyset; \)
7.  **foreach** \( i \in [j - 1] \) **do**
8.      **foreach** \( i \) element subsets \( K_i^t \) of \( K \) **do**
9.          **if** \( d_G[K \cup S_i^t](v) \in \{n_1 - 1, n_1, \ldots, n_1 + (j - i) - 1\} \) for all \( v \in K \setminus K_i^t \), where \( S_i^t = S \setminus N_G(K_i^t) \) **then**
10.             \( D_i^t \leftarrow K_i^t \cup S_i^t; \)
11.          **else**
12.             \( D_i^t \leftarrow \emptyset; \)
13.      **foreach** \( j \) element subsets \( K_j^t \) of \( K \) **do**
14.          **if** \( S \subset N_G(K_j^t) \) **then**
15.             \( D_j \leftarrow K_j^t; \)
16.          **else**
17.             \( D_j \leftarrow \emptyset \)
18.      \( D_{j+1} \leftarrow K \cup S', \) where \( S' = \{u \in S | d_G(u) \geq (j + 1)\}; \)
19.  \( D \leftarrow D_{j+1}; \) where \( D_{j+1} \) is the minimum cardinality nonempty set among all \( D_{i+1} \), \( 0 \leq i \leq (j + 1) \).
20. **return** \( D \);

---

The correctness of Algorithm 2 is based on Observation 10 and Lemma 11. Next we analyze the complexity of Algorithm 2. Note that we can compute \( D_0 \) and \( D_{j+1} \) in \( O(n) \) time. For each \( 1 \leq i \leq (j - 1) \), the set \( D_i \) can be computed in \( O(n^i + in^i \log n) \) time because in each case, we have to check all possible \( i \) element subsets of \( K_i \), i.e., \( O(n^i) \) subsets and after that we have to assign the minimum cardinality subset to \( D_i \), which takes \( O(in^i \log n) \) time. Hence computing all the sets \( D_i \) for \( 0 \leq i \leq (j + 1) \) can be done in polynomial time. For \( D_j \), we have to check all \( j \) element subsets of \( K_j \), i.e., \( O(n^j) \), and since \( j \) is a constant, it takes polynomial time to compute \( D_j \). Also finding the minimum cardinality nonempty set in line 19 takes polynomial time. Hence, Algorithm 2 can be done polynomial time. Thus, we have the main theorem in this subsection as follows.
Theorem 12. For any fixed \( j \), the cardinality of a minimum \((1,j)\)-set of a given split graph can be computed in polynomial time.

5 Concluding remarks

In this paper, we have obtained an upper bound on \((1,j)\)-domination number. We have shown that \((1,j)\)-SET is NP-complete for chordal graphs. We have also designed two algorithms for finding \( \gamma_{(1,j)}(G) \) of a tree and a split graph. In [CHHM13], the authors constructed a special type of split graph \( G \) with \( n \) vertices for which \( \gamma_{(1,2)}(G) = n \). Lemma 11 gives a more general type of split graph for which \( \gamma_{(1,j)}(G) = n \). It actually characterizes the split graphs with \( n \) vertices having \( \gamma_{(1,j)}(G) = n \). The characterization is as follows:

**Corollary 13.** Let \( G \) be a split graph with \( V = K \cup S \) and \( |K| = n_1 \), \( |S| = n_2 \). Then \( \gamma_{(1,j)}(G) = n \) if and only if the following conditions hold.

(i) There exists at least one vertex \( v \) in \( K \) such that \( d_G(v) \geq n_1 + j \).

(ii) For all \( i \in [j-1] \) and for each \( i \) element subset \( K_i = \{v_1, \ldots, v_i\} \) of \( K \), there exists some vertex \( v_t \in K \setminus K_i \) such that \( d_G[K_i \cup S_i](v_t) \geq n_1 + (j - i) \), where \( S_i = S \setminus N_G(K_i) \).

(iii) For each \( j \) element subset \( K_j \) of \( K \), \( S \not\subset N_G(K_j) \).

(iv) For every \( u \in S \), \( d_G(u) \geq (j + 1) \).

Condition (i), (ii) and (iii) actually force all the vertices of \( K \) in a minimum cardinality \((1,j)\)-set \( D \) and condition (iv) forces all vertices of \( S \) in \( D \). Using this type of split graphs, we can construct a graph \( G \) (not a split graph) having \( \gamma_{(1,j)}(G) = n \). The construction is as follows: Let

\[
G_1 = (V_1, E_1), G_2 = (V_2, E_2), \ldots, G_p = (V_p, E_p)
\]

be \( p \) split graphs having partitions

\[
V_1 = K_1 \cup S_1, V_2 = K_2 \cup S_2, \ldots, V_p = K_p \cup S_p.
\]

The vertex set of the constructed graph \( G = (V, E) \) is given by

\[
V = \bigcup_{i=1}^{p} V_i
\]

and the edge set is given by

\[
E = \left( \bigcup_{i=1}^{p} E_i \right) \bigcup E',
\]

where \( E' \) is an arbitrary edge set between the vertices of

\[
K_1, K_2, \ldots, K_p.
\]

We can easily verify that \( \gamma_{(1,j)}(G) = |V| \). But characterizing the graphs with \( n \) vertices having \( \gamma_{(1,j)}(G) = n \) seems to be an interesting but difficult question. Also, it would be interesting to study the open problems mentioned in [CHHM13].
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