Abstract. We introduce a solvable model of randomly growing systems consisted of many independent subunits. Scaling relations and growth rate distributions in the limit of infinite subunits are analyzed theoretically. Various types of scaling properties and distributions reported for growth rates of complex systems in wide fields can be derived from this basic physical model. Statistical data of growth rates for about 1 million business firms are analyzed as an example of randomly growing systems in the real-world. Not only scaling relations are consistent with the theoretical solution, the whole functional form of the growth rate distribution is fitted with a theoretical distribution having a power law tail.

Keywords Central limit theorem · Growth rates · Stable distribution · Power laws · Firm statistics · Gibrat’s laws

1 Introduction

Growth phenomena are generally highly irreversible dynamical processes far from thermal equilibrium [52]. From the viewpoint of statistical physics it is an important new topic that growth rates of complex systems often show non-trivial similar statistical behaviors across fields of sciences. Fat-tailed distributions of growth rates and a non-trivial shrink of variance as a function of the size are reported in various fields of sciences; business firms [17, 22], sales of pharmaceutical products [11], circulation numbers of newspapers [27], population of migratory birds [13], animals’ metabolic rate fluctuations [20], the amount of scientific funding [32], group size of religious activities [31], population size of cities [13], country’s whole economic activity observed by GDP [21] and the amount of exports and governmental debts [33]. Probability densities of logarithmic growth rates in most of these examples are typically approximated by double exponential (Laplace) distributions or by power law distributions, quite interestingly not by a Gaussian distribution.

Statistics on growth rates of business firms have a long history of study, and recently statistical physicists are involved in this topic. Gibrat postulated the “law of proportional effect” that the expected value of the growth rate of a business firm is proportional to the current size of the firm [17, 33]. The original Gibrat’s assumption states that the variance of growth rates is independent of the size, however, data analyses of business firm activities show various types of variance-size relations. There are papers
Fig. 1 Comparison of system size dependence of the probability density functions of growth rates for different values of $\alpha$. a; The case of $\alpha = 1.5$ ($<g_j(t)^{1.5}>=1$) and b; $\alpha = 0.5$ ($<g_j(t)^{0.5}>=1$) in semi-log plots. In both cases the growth rate distribution of individual subunits follows a uniform distribution. $N = 1$ (black thin line), $N = 2$ (red dash-dotted line), $N = 10$ (blue broken line), $N = 10^2$ (bold green line), $N = 10^3$ (purple dotted line), $N = 10^4$ (light blue dash-dotted line) and $N = 10^6$ (black broken line). Figures corresponding to the power law distribution, $P(<>g_i(t)) \propto g_i^{-1.6}$ are shown for $<g_i(t)^{1.5}>=1$ in the panel (c) and for $<g_i(t)^{0.5}>=1$ in the panel (d).

which support the original Gibrat’s assumption [12,41], on the other hand non-trivial fractional power laws are reported not only for business firms but also in many other phenomena [1] [9,11,13,18,20,21,27,32,43,52]. Also, the country-dependence [30] and the transition from Gibrat’s assumption to such power law decays are pointed out [48]. Various types of theoretical models of business firms have been introduced by physicists for better understanding of scaling properties of business firms from the standpoint of complex systems [2,3,4,8,28,34,35,42,50,53,55]. However, there has been no unified theory which can explain all these basic properties simultaneously. To understand these phenomena consistently, we introduce a simple random growth model of a complex system consisting of many independent subunits, and we consider the relationship between fluctuations in the growth rate of subunits with that of total system.

There are a number of pioneer studies on complex systems that focus on the growth rate statistics. Wyart et. al. introduced a company model that consisted of independent subunits characterized by a power law size distribution [54]. The growth rate of the overall system was shown to observe a symmetric stable distribution with its scale parameter, which corresponds to the standard deviation, either decaying in accordance with a power law or converging to a constant in the limit of infinite unit numbers. Schwarzkopf et. al. investigated a model that consisted of independent subunits whose number of summations changes with time, and they showed that the growth rate of this model also
observed a stable distribution \[37\]. Malevergne et. al introduced a theoretical model of firms based on the birth and death process and studied the contribution of the growth rate of the entire economy to the power law exponent of the distribution of sales \[24, 25\]. Solomon et. al. studied the statistical properties of growth rates in the framework of the nonlinear dynamics of a generalized Lotka-Volterra model \[6, 38, 16, 39\]. They demonstrated that the growth rate distribution for large time scale windows observes a stable distribution.

In the next section, we introduce a basic model of a complex system whose growth rate can be theoretically derived by a kind of renormalization of the many independent subunits of which the system consists. In the section 3, we show that in the limit of infinite number of subunits, the distribution of growth rates is shown to converge to a stable distribution with a power law tail and its scale indicator shrinks in a nontrivial manner. The stable distribution and the generalized central limit theorem were established in mathematics about 80 years ago \[10, 22\], however, these concepts were applied mostly for theoretical models assuming scaling properties for phenomena such as turbulence \[46, 47\]. Validity of the theoretical results for growth rates is confirmed in section 4 by analyzing a huge database about business firms. This example may be the first real world application of an asymmetric stable distribution fitted for the whole scale range. The final section is devoted for discussion.

2 The model

We consider a system consisting of \(N\) subunits characterized by non-negative scalar quantities, \(\{x_j(t)\}\). For each subunit we assume the following random multiplicative time evolution \[19\], which is known to be one of the basic process of producing power law fluctuations \[10, 19\]. (See Appendix A for brief review of random multiplicative process).

\[
x_j(t + \Delta t) = g_j(t)x_j(t) + f_j(t),
\]

where \(g_j(t)\) and \(f_j(t)\) for \(j = 1, 2, \cdots, N\), are growth rates and external forces, respectively, both are assumed to be independent identically distributed random variables taking only positive values. In the
case that the probability of occurrence of \( g_j(t) > 1 \) is not zero and if \(< \log(g_j(t)) < 0 \), where \(< \cdot \cdot \cdot \>\) denotes the average, it is known that there exists a statistically steady state in which the cumulative distribution follows a power law \([19] \),

\[
P(> x_j) \propto x_j^{-\alpha},
\]

for large value of \( x_j \) with positive exponent determined exactly only by the statistics of the growth rate by solving the following equation,

\[
< g_j(t)^\alpha > = 1.
\]

We note that Solomon et. al. elaborated a generalized version of this type of model, \( x(t + 1) = g(t)f_1(x(t)) + f_2(x(t)) \) where both \( f_1 \) and \( f_2 \) are nonlinear functions, in which a power law distribution holds generally in the steady state; although, the simple exact relation given in Eq. \([3] \), no longer holds \([39]\).

Based on a renormalization point of view we pay attention to the sum of all subunits, \( X(t; N) \equiv \sum_{j=1}^{N} x_j(t) \), which follows the same type of time evolution as that of the subunits,

\[
X(t + \Delta t; N) = G(t; N)X(t; N) + F(t; N),
\]

where \( F(t; N) \equiv \sum_{j=1}^{N} f_j(t) \), and the growth rate of the whole system is defined as

\[
G(t; N) = \frac{\sum_{j=1}^{N} g_j(t)x_j(t)}{\sum_{j=1}^{N} x_j(t)}. 
\]

It is easy to show that the mean value of growth rates is invariant, namely, \(< G(t; N) >= < g_j(t) > \equiv G \).

The system growth rate, \( G(t; N) \), in this study is closely related to the company growth model introduced elsewhere in which the probability density of the size change of a firm, assuming that a firm is a composite of independent \( K \) subunits, is given by the following \([54]\).

\[
Q(S, R) = \sum_{K=1}^{\infty} L(K) \int ds_1 \prod_{i=1}^{K} P_s(s_i) \delta(S - \sum_{i=1}^{K} s_i) \int dp \prod_{i=1}^{K} P_{\eta}(\eta_i) \delta(R - \sum_{i=1}^{K} s_i \eta_i),
\]

where \( S \) is the firm size for the previous term, \( R \) is that for the current term, \( P_s(s) \) is the PDF of the sizes of subunits supposing \( P_s(s) \propto s^{-\alpha-1} \), \( P_{\eta}(\eta) \) is the PDF of the growth rate of the subunits, \( L(K) \) is the number density of subunits and \( \delta(x) \) is the Dirac delta function for continuous variables and the Kronecker delta for discrete variables. Our model’s growth rate \( G(N, t) \) in the steady state corresponds to the ratio \( R/S \) for the case of fixed subunit number, \( L(K) = \delta(K - N) \). In contrast, authors in Ref \([54]\) mainly discussed the growth rate of the model on the condition \( L(K) = 1 \). In terms of this class of system growth models, our research newly clarifies the dependence of the distribution of the system growth rate both on the number of subunits and the distribution of subunit growth rates. In our study, we assume that \( \{g_i\} \) are identically distributed independent variables defined on a non-negative range, while in Ref. \([54]\) \( \{\eta_k\} \) are assumed to follow a normal distribution with zero mean to calculate the distribution of the system growth rate \( G \).

Properties of this model can be investigated by numerical simulation. Fig. \([1]\) shows an example of deformation of growth rate distributions for various values of \( N \) in the case that Eq. \([3]\) is fulfilled with \( \alpha = 1.5 \) observed in the statistically steady state realized for time steps larger than \( 10^6 \). Here the distribution of growth rates of subunits is given by an independent uniform distribution as shown by the case of \( N = 1 \). In this figure, the additive noise term, \( f_j(t) \), is set to be a positive constant for simplicity as we confirmed that the main results do not depend on the values of \( f_j(t) \) except the case that it is identically 0. As known from this figure the distribution of growth rates of the aggregated system tends to shrink slowly to a delta function as \( N \) goes to infinity. It is numerically confirmed that the same property of convergence to the delta-function holds for any distribution of growth rates of subunits if the growth rate distribution satisfies Eq. \([3]\) with \( \alpha > 1 \).

Fig. \([1]\) shows an example in the case of \( \alpha = 0.5 \). In this case the growth rate distribution stops shrinking for \( N \) larger than 10 and it converges to a non-trivial distribution in the limit of \( N \) goes to infinity. It is confirmed that this property is always observed if the value of \( \alpha \) in Eq. \([3]\) is between 0
and 1. The distribution of growth rate in the limit of \( N = \infty \) depends on the functional form of the growth rate distribution for the subunits.

Figs. 1a and 1d show results indicative of the case where \( g_i(t) \) observes a power law distribution. From these figures, we can also confirm that the distributions shrink for large \( N \), in the case of \( \alpha = 1.5 \). However, on the contrary, the distribution stops shrinking for \( N \) larger than 10 in the case of \( \alpha = 0.5 \).

### 3 Theoretical analyses

Here, we can theoretically evaluate the \( N \)-dependence of the width of the PDF of \( G(t; n) \) intuitively by introducing an approximation of the random variables \( \{ x_j(t) \} \) which are known to follow a power law in the steady state. A more precise derivation using the generalized central limit theorem is given in Appendix D. We introduce the following value as the measure of the width, \( \bar{G}(t; N)^2 \). Here, we define \( \bar{G}(t; N) \) as:

\[
\bar{G}(t; N) \equiv G(t; N) - G = \frac{\sum_{j=1}^{N}(g_j(t) - G) \cdot x_j(t)}{\sum_{j=1}^{N} x_j(t)}.
\]

\( \bar{G}(t; N)^2 \) takes zero in the case that the PDF of \( G(t; N) \) is a delta function. Let \( u \) be a random variable following a uniform distribution in the interval of \((0, 1]\). Then, the distribution of the new variable, \( u^{-1/\alpha} \), follows a power law with exponent \( \alpha \). For uniform random variables, \( \{ u \} \), we can approximate \( N \) samples by a set of deterministic values, \( \{ 1/N, 2/N, \ldots, 1 \} \), so that the set of power law variables \( \{ x_j(t) \} \), which follow Eq. (2), can be approximated by the deterministic set:

\[ \{ N^{1/\alpha}, (N/2)^{1/\alpha}, (N/3)^{1/\alpha}, \ldots, (N/j)^{1/\alpha}, \ldots, 1 \} \].

Therefore, the typical sample of \( \bar{G}(t; N)^2 \) is obtained,

\[
\bar{G}_r(t; N)^2 \equiv \frac{\left\{ \sum_{j=1}^{N} (g_j(t) - G) \cdot (j/N)^{-1/\alpha} \right\}^2}{\sum_{j=1}^{N} (j/N)^{-1/\alpha}}.
\]

Taking the average of \( \bar{G}_r(t; N)^2 \) with respect to \( g_j(t) \) for \( j = 1, 2, \ldots, N \) and applying the independence condition, \(< (g_n(t) - G)(g_m(t) - G) >= \sigma^2 \delta_{nm} \), we have

\[
< \bar{G}_r(t; N)^2 >= \sigma^2 \cdot \frac{\sum_{j=1}^{N} (j/N)^{-2/\alpha}}{\sum_{j=1}^{N} (j/N)^{-1/\alpha}}.
\]

where \( \sigma^2 \) is the variance of the growth rates for the subunits and \( \delta_{nm} \) is Kronecker’s delta. Then, we can calculate the summations in Eq. (8), \( N^{1/\alpha} \sum_{j=1}^{N} j^{-1/\alpha} \) and \( N^{2/\alpha} \sum_{j=1}^{N} j^{-2/\alpha} \), by applying an asymptotic expansion formula for the Riemann Zeta function,

| Value of \( \alpha \) | Width-Size(\( N \)) relation | Limit distribution of growth rates |
|---------------------|-----------------------------|---------------------------------|
| \( < g_i(t)^{\alpha} > = 1 \) | \( (\alpha-1)/(\alpha-2) N^{\alpha-2} \rightarrow 0 \) | Gaussian |
| \( 2 < \alpha < 1 \) | \( (\alpha-1)/(\alpha-2) N^{\alpha-2} \rightarrow 0 \) | Stable distribution with power law tails |
| \( \alpha = 1 \) | \( (\alpha-1)/(\alpha-2) N^{\alpha-2} \rightarrow 0 \) | Non-universal distribution |
| \( < g(t) > = 1 \) | \( (\alpha-1)/(\alpha-2) N^{\alpha-2} \rightarrow 0 \) | Depending on the subunit's property |
| \( 0 < \alpha < 1 \) | \( (\alpha-1)/(\alpha-2) N^{\alpha-2} \rightarrow 0 \) | Non-universal distribution |

Table 1 Summary of generalized central limit theorems for growth rates. The value of \( \alpha \) and the asymptotic functional form of the square root of Eq. (11) divided by \( \sigma \), which corresponds to the width of the growth rate, for large system size, \( N \), and the limit distributions. \( \gamma = 0.577... \) is the Euler constant.
Fig. 3 Convergence of the normalized growth rate distributions in the case of \( \alpha = 1.5 \). The growth rate for each subunit, \( g_j(t) \), distributes uniformly in the range \((0, (2.5)^{2/3})\) satisfying \( \langle g_j(t)^{1.5} \rangle = 1 \). The renormalized growth rate’s probability density functions for systems with \( N \) subunits are plotted for several values of \( N \). \( N = 1 \) (black thin line), \( N = 2 \) (red dash-dotted line), \( N = 10 \) (blue broken line), \( N = 10^2 \) (bold green line), \( N = 10^3 \) (purple dotted line), \( N = 10^4 \) (light blue dash-dotted line), \( N = 10^6 \) (black broken line), and the theoretical symmetric stable distribution, \( p(\bar{G}; 1.5, 0) \) (olive dotted line). The inserted figure shows the cumulative distribution function of the positive part in log-log scale confirming convergence to the power law.

Plots corresponding to the case of a power law distribution of subunits, \( P(> g_i(t)) \propto g_i^{-1.6} \) are shown for \( \langle g_i(t)^{1.5} \rangle = 1 \) in panel (b) and the theoretical asymmetric stable distribution, \( p(\bar{G}; 1.5, 0.85) \), is also plotted by the olive dotted line for comparison.

\[
\zeta(\lambda) \equiv \sum_{j=1}^{\infty} \frac{1}{j^\lambda} = \sum_{j=1}^{N} \frac{1}{j^\lambda} + \frac{1}{(\lambda - 1)N^{\lambda-1}} - \frac{1}{2N^\lambda} + \cdots .
\]
Neglecting the third term and higher order terms in the right hand side of Eq. (10), we have the following approximation of Eq. (9) for $0 < \alpha$:

$$
\langle G_r(t, N)^2 \rangle = \sigma^2 \cdot \left\{ \frac{\zeta^{(\frac{1}{\alpha})}}{\zeta^{(\frac{1}{\alpha})}} N^{1 - \frac{2}{\alpha}} \right\}^2 -\frac{\alpha N^{1 - \frac{2}{\alpha}}}{\frac{\alpha N^{1 - \frac{2}{\alpha}}}} \cdot \frac{\sigma^2}{t} \cdot \frac{\zeta^{(\frac{1}{\alpha})}}{\zeta^{(\frac{1}{\alpha})}} N^{1 - \frac{2}{\alpha}} \right\}^2.
$$

(11)

These theoretical evaluations are checked numerically in Fig. 2 in which the widths of growth rate distributions for the aggregated system are plotted as functions of the number of subunits, $N$. The theoretical asymptotic functional forms in Table 1 fit quite well asymptotically for all cases. It should be noted that the standard deviation (or the variance) observes a slightly different scaling from that given Eq. (11) and is not suitable for the scale parameters of our model, except for extremely large $N$ (Appendix E). As an alternative of the standard deviation, we introduce the median of $G(t; N)^2$ and the interquartile range (IQR) as the measure of the width of the distribution in Figs. 2(a) and (b), respectively. From Fig. 2(a) we can confirm that the medians of $G(t; N)^2$ almost correspond to the width given by Eq. (11) and from Fig. 2(b) IQRs are proportional to asymptotic behavior of this equation given by Table 1. In addition, from Table 1 and Appendix D, we can also confirm that Eq. (11) is proportional to the scale parameter of the stable distribution given by Eq. (12) for $N >> 1$.

From Eq. (11) we find that the width of PDF of $G(t; N)$ converges to 0 following a power law of $N$ in the case of $0 < \alpha < 1$, on the other hand the width takes a finite value even in the limit of $N \to \infty$ in the case of $0 < \alpha < 1$. At the marginal case of $\alpha = 1$ the width decays to 0 very slowly for increasing $N$. For $2 < \alpha$ the width decays inversely proportional to $N$, that is, the typical $N$ dependence in the case of ordinary limit theorem. These functional forms of $N$ dependence are summarized in Table 1 in the second column. As known from this result Gibrat’s assumption of constant variance is fulfilled in the case of $0 < \alpha < 1$, and the non-trivial power law decays of width-size relations observed in many complex systems are realized in the case of $1 < \alpha < 2$. As shown in the first column of this table the range of $\alpha$ is characterized by the form of equality or inequality for the moment function of the growth rates, $M(s) \equiv \langle g_j(t)^s \rangle$. Derivation of these relations is summarized in Appendix B with the basic properties of the moment function. From this table we find that Gibrat’s assumption holds for systems in which the mean growth rate is larger than 1, while the non-trivial power law decay of width-size relation for growth rate is expected in the situation that the mean growth rate of subunits is less than 1.

Next, we theoretically derive functional forms of the distribution of growth rates normalized by the width in the limit of $N \to \infty$. We consider the following 3 cases according to the value of $\alpha$. The details of the derivation are given in Appendix D.

I. The case of $0 < \alpha < 1$: The width of growth rate does not shrink to 0 but it converges to a finite value even in the limit of $N \to \infty$ as known from Eq. (11). This reason can be understood by a general property of power law distribution with exponent less than 1. In such a case the mean value $\langle x_j(t) \rangle$ diverges implying that some samples in $\{x_j(t)\}$ take extraordinarily large values compared with others. So, both the denominator and numerator of Eq. (10) can be approximated by only finite numbers of extraordinarily large contributors, therefore, the value of Eq. (11) is finite even in the limit of large $N$. The distribution of growth rate shows the same property, namely, even in the limit of $N \to \infty$, the limit distribution is determined only by a small number of large contributors, therefore, we cannot expect a universal functional form in this case.

II. The case of $1 < \alpha < 2$: As indicated in Fig. 2 the width of distribution shrinks to 0 in the limit of $N \to \infty$ we can expect existence of a universal limit distribution independent of the initial condition. In this case the average, $\langle x_j(t) \rangle$, takes a finite value in the steady state, the denominator in Eq. (10) can be roughly estimated as $N < x_j(t)$ for very large values of $N$. On the other hand, the numerator in Eq. (9) is given by the sum of $(g_j(t) - G)x_j(t)$, in which $g_j(t) - G$ gives a coefficient taking either positive or negative sign randomly with respect to $j$, and $x_j(t)$ follows a power law with the exponent $\alpha$. Namely, the numerator becomes a summation of $N$ independent identically distributed random variables that have both positive and negative power law tails with the exponent $\alpha$. As the generalized central limit theorem can be applied to such a sum of random variables, the limit distribution of growth rate $G$, which is normalized by the width of the distribution around the mean value, is expected to
converge to a stable distribution, which has the form of an inverse Fourier transform \[^{10}\], (see also Appendix C for a brief summary of the central limit theorem and the stable distributions),

\[
p(\tilde{G}; \alpha, \beta) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left\{ -i\rho \tilde{G} - |\rho|^\alpha (1 - i\beta \psi(\rho, \alpha)) \right\} d\rho, \tag{12}
\]

where the asymmetry parameter \( \beta \), which takes a value in the interval \([-1, 1]\), and the function \( \psi(\rho, \alpha) \) are given as follows,

\[
\beta = \frac{\langle (g_j - \bar{G})|g_j - \bar{G}|^\alpha >}{\langle |g_j - \bar{G}|^\alpha >}, \quad \phi(\rho, \alpha) \equiv \frac{\rho}{|\rho|} \tan \frac{\pi \alpha}{2}. \tag{13}
\]

It is well known that the limit probability density, Eq. (12), has a power law tail with the exponent \( \alpha \) just like the distribution of \( \{x_j(t)\} \).

In Fig. 3(a), we confirm the validity of this theoretical result by a numerical simulation for the case of \( \alpha = 1.5 \) and \( \beta = 0 \). The normalized growth rates for the system consisted of \( N \)-subunits, \( \tilde{G} \), are calculated by subtracting the mean value and normalized by the width of the distribution. As known from this figure the distribution of normalized growth rates changes its functional form for different values of \( N \). The probability density functions are converging clearly to the theoretical function, \( p(\tilde{G}, 1.5, 0.0) \). In addition, Fig. 3(b) indicates the results for the case of \( \alpha = 1.5 \) and \( \beta = 0.85 \). From this figure, we can also confirm convergence to the asymmetric stable distribution \( p(\tilde{G}, 1.5, 0.85) \). Such skew distribution does not appear in the case that \( g_i(t) \) observes the normal distribution with zero mean as discussed in Ref. [54].

**III. The case of \( 2 \leq \alpha \):** Similar estimation for denominator of Eq. (4) is valid and the ordinary central limit theorem can be applied to the numerator of Eq. (7) as the variances for \( \{x_j(t)\} \) are finite. The expression of Eq. (12) is also valid in this case, however, the parameters are limited to \( \alpha = 2 \) and \( \beta = 0 \), namely, the limit distribution of normalized growth rate is always \( p(\tilde{G}, 2, 0) \), which is the well-known Gaussian distribution with no long tail.

It is interesting to consider a special situation that the growth rates \( \{g_j(t)\} \) are distributed symmetrically around 1. Then, we can derive \( \alpha = 1 \) from the basic relation \( < g_j(t) >= 1 \), and \( \beta \) in Eq. (13) is 0 by symmetry. In such a case we can expect that the limit distribution of normalized growth rate converges to \( p(\tilde{G}; 1, 0) = 1/\left\{2\pi (1 + \tilde{G}^2) \right\} \) from Eq. (12).

Results for all these cases are summarized in the third column of Table 1. The limit distribution of growth rate is determined by the value of moment function for the growth rates of subunits. The important point is that the ordinary central limit theorem can be applied only in the limited cases of relatively small growth rates, the mean value of growth rate is less than 1 and the second moment of growth rate is less or equal to 1. For systems in the real world it is expected that the systems are nearly in the statistically steady state and the mean values of growth rates of subunits may take a value around 1. Then, as known from Table 1, the limit distributions of growth rates belong to either power laws or non-universal functional forms.

**4 An application to business firm activities**

Now, we apply these theoretical results for data analysis. Among various types of dynamical complex systems in the real world, business firms are attracting the attention of scientists because there are precise observation data in the form of financial reports \[^{17-30,44}\]. In order to check the validity of our theory, we analyze comprehensive business firm data of Japan provided by the governmental research institute, RIETI (Research Institute for Economy, Trade and Industry). The data contains financial reports of 961,318 business firms, which practically covers all active firms in Japan in 2004 and 2005. It is already confirmed that basic quantities of these business firms, such as annual sales, incomes and number of employees, follow power law distributions \[^{29}\].

There are several quantities that characterize the size of a business firm, such as assets, number of employees, sales, and incomes. Among these quantities, we focus on sales because this quantity reflects the present scale of activity of a firm most directly. Also, we simply assume that the whole activity of a
business firm is given by the sum of the activities of individual employees; namely, we regard \( X(t;N) \) in Eq. (4) as the annual sales of a business firm with \( N \) employees in the \( t \)-th year. Neglecting the additive term in Eq. (4) as well as the change of the number of employees in a year, we estimate the growth rate \( G(t;N) \) by the ratio of the \( t+1 \)-th year’s annual sales over that of the \( t \)-th year’s for a business firm with a number of employees \( N \). In application of our mathematical model to real data, we need to specify the minimal independently acting subunits for actual firms. However, this is tends to be quite problematic because real firms may consist of various divisions of different sizes. To make a rough estimation, here, we simply assume that \( N \) is given by the number of employees. It is already confirmed from the data that the autocorrelation of the growth rate averaged over all business firms is very close to 0, implying that the growth of a business firm can be roughly viewed as an independent random growth process approximated by Eq. (4) with a negligibly small external force term.

Categorizing the business firms by the number of employees, \( N \), we can measure the width of growth rate distribution for each category. Fig. 4 shows \( N \)-dependence of the width of growth rate distribution in log-log scale. It is confirmed that Gibrat’s assumption of size-independence does not hold in this case, and the width of growth rate decays clearly for large \( N \). Here, the theoretical line is given by a power law, \( N^{-0.046} \), which is derived from Eq. (11) in the case of \( \alpha = 1.06 \). In the inserted figure the cumulative distribution of annual sales of all firms, corresponding to a superposition of distribution of \( X(t;N) \) for all \( N \), is plotted in log-log scale. We can confirm that the tail of the distribution is approximated by a power law with the exponent \( \alpha = 1.06 \) as expected.

A theoretical limit distribution of growth rate, \( p(G;1.06,0.50) \) in Eq. (12), is plotted together with that for real data estimated for \( N \) larger than 300 in Fig. 5. The inserted figure shows the cumulative distribution of annual sales of all firms, corresponding to a superposition of distribution of \( X(t;N) \) for all \( N \), is plotted in log-log scale. We can confirm that the tail of the distribution is approximated by a power law with the exponent \( \alpha = 1.06 \) as expected.

The exponent takes \( \alpha_s = 1.07 \) [1.063,1.068] for the CDF of sales, \( \alpha_s = 1.06 \) [1.055,1.065] for the CDF of the growth rate, \( \alpha_w = 0.0053 = 1 - 1/1.06 \) [1-1/1.056,1-1/1.069] for the power law of the

![Fig. 4 Width of growth rate fluctuation of Japanese business firms as a function of number of workers, \( N \), in log-log plot. The dashed line shows a theoretical curve given by the inverse power law, \( N^{-0.057} \), which is derived from Eq. (11) with \( \alpha = 1.06 \). The inserted figure shows cumulative distribution function of annual sales of about 1 million Japanese firms in 2005 with the dotted line showing a power law distribution with the same exponent \( \alpha = 1.06 \).](image)
width of growth rates, where $[\ldots]$ is the 95% confidence interval and the domains of the power laws are assumed as $s \geq 4.22 \cdot 10^{10}$ (yen) for sales, $g \geq 1.11$ for the growth rate, and employees $\geq 10$ for the width. Here, to estimate the exponents, we employed a robust linear regression after taking the logarithmic transformation for power law regions [26]. The reason why we apply this method is to reduce the effect of outliers, that cannot be clearly distinguished from the data. However, it must be noted that this estimation depends on the choice of the domain of the power law, such that, for example, $\alpha_s = 1.09$ [1.086,1.089] for $s > 1.01 \cdot 10^{11}$ (yen), $\alpha_g = 1.07$ [1.066,1.078] for $g > 1.31$, and $\alpha_{w} = 1 - 1/1.06$ [1 - 1/1.037, 1 - 1/1.079] for employees $\geq 100$. Because of the limitation of data accuracy as well as the ambiguity of the correspondence between real business firm activity and that of the simple mathematical model, we cannot form any definite conclusions. It may be fair to conclude that the conjecture that a business firm’s growth rate distribution is approximated by an asymmetric stable distribution does not contradict the data.

5 Discussions

In this paper we introduced a theoretically solvable model of sum of randomly growing independent subunits. As summarized in Table 1 we found generalized central limit theorems applicable for a composite of randomly growing subunits, in which we can find various types in both width-size relations and the limit distributions of growth rates. As an example of a real world application, we analyzed a huge database of business firm growth rates of Japan and confirmed consistency with the theory.

The crucial study regarding the sum of randomly growing independent subunits was investigated by Wyart and Bouchaud, and our results regarding the size dependence on the width of the PDF of growth rates are consistent therewith [54]. Our research newly clarifies the dependence of the distribution of the growth rate of the system, $G(t; N)$, on the functional form of the growth rate of the subunits. In our analysis, we consider an arbitrary functional form for the growth rate of subunits, while, in the case of Wyart and Bouchaud’s study, only the normal distribution with zero mean is assumed for the growth rate of subunits. In particular, we clarify the categorization of the limit behaviors with
a clear condition on the growth rate distribution of subunits, as summarized in Table 1. This is a very general result which is expected to be applied widely to randomly growing phenomena in various fields involving power laws. In addition, our model can derive the expression of the system growth rate \( G(t; N) \) from subunit growth rate \( g_i \), more simply than previous models without assuming power law distributions.

Application study of this theory for growth rate distribution of complex systems is highly encouraged. It is expected that the shrinking of growth rate width and the functional form of limit distributions can be directly compared with real data of various phenomena to check the universality of this aggregated system of randomly growing subunits. For Fig. 1 the normalized growth rate distribution looks quite similar to a double-exponential (Laplace) distribution in the case of intermediate numbers of subunits. This type of finite size effect should be treated carefully in real-world data analysis. There is the possibility that the varieties of empirically known properties of the growth rate of complex systems introduced in the beginning of this paper can be understood using our approach as a framework. Real-world systems may not be in a steady-state, so it is important for future work to investigate transient behaviors of this independent subunit system before it reaches the statistically steady state. Extension of this novel renormalization view of growth rates to cases of interacting subunits may also be an attractive new research topic. It is expected that a variation of generalized central limit theorem for the growth rates might be found for the wider category of growing complex systems like the case of non-standard statistical physics for long-range interaction systems [14].

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Appendix A: An introduction to random multiplicative process

As random multiplicative process is not widely known, here we introduce a simple exactly solvable case of random multiplicative process and show intuitively how the process realizes a power law distribution in the statistically steady state. Also, a continuum limit version of this multiplicative process is discussed.

We consider a positive random variable, \( x(t) \), which follows the following stochastic equation,

\[
x(t + \Delta t) = g(t)x(t) + 1,
\]

where \( g(t) \) is a stochastic noise term which takes either a positive constant, \( g \), or 0 with probability 1/2, respectively. Starting with the initial condition, \( x(0) = 1 \), the time evolution is given as

\[
x(\Delta t) = \begin{cases} 
1 \\
1/2
\end{cases} (\text{prob. } 1/2)
\]

\[
x(2\Delta t) = \begin{cases} 
9 + 1 (\text{prob. } 1/4) \\
g + 1 (\text{prob. } 1/4)
\end{cases}
\]

The general solution at time step \( \tau \) is obtained as

\[
x(\tau \Delta t) = \begin{cases} 
(g^{\tau - 1} - 1)/(g - 1) (\text{prob. } 1/2^2) \\
(g^{\tau - 2} - 1)/(g - 1) (\text{prob. } 1/2^2) \\
\vdots \\
(g - 1)/(g - 1) (\text{prob. } 1/2^k) \\
1 (\text{prob. } 1/2)
\end{cases}
\]

where \( k \) is an integer from 1 to \( \tau \). From this solution we can calculate the cumulative distribution of \( x(t) \) in the limit of \( t \to \infty \) as

\[
P(\geq x) = 2 \{1 + (g - 1)x\} = \frac{\text{Prob}(x)}{\text{Prob}(\infty)},
\]

where \( P(\geq x) \) denotes the probability that \( x(\infty) \) takes a value larger than or equal to \( x \). In the case that \( g > 1 \) we have the asymptotic power law distribution for very large value of \( x \),

\[
P(> x) \propto x^{-\alpha},
\]

where the exponent, \( \alpha = \log(2)/\log(g) \), fulfills Eq. (3) in the range, \( 0 < \alpha < \infty \), namely,

\[
< g(t)^\alpha > = \frac{1}{2} \cdot g^\alpha + \frac{1}{2} \cdot 0 = 1
\]
With this special example we confirm the validity of Eq. (1) to Eq. (3). Note that in this example, the stationary condition, \(< \log(g(t)) > < 0\), is automatically satisfied because the condition that \(g(t) = 0\) with probability \(1/2\) gives the value, \(< \log(g(t)) > = -\infty\).

The key point of realizing the power law in this multiplicative random process is understood intuitively by neglecting the additive term. The probability of repeating \(g(t) = g\) for \(k\) time steps is given as \(p(k) \equiv 1/2^k = e^{-k \log(2)}\) and the corresponding value of \(x(t)\) is approximated as \(x(t) \approx g^k = e^{k \log(g)}\), then by deleting \(k\) from these relations we have Eq. (A.5). Namely, successive exponential growth with an exponential distribution of duration time gives the power law distribution.

This type of derivation of power law can be generalized in the following way. Let us consider the following general form of random multiplicative process,
\[ x(t + \Delta t) = g(t)x(t) + f(t) \]  
(A.7)
where \(g(t)\) and \(f(t)\) are independent random variables taking positive values, and we assume the situation that \(\log(g(t))\) fluctuates around 0 [11]. Taking logarithm for both sides and introducing variables \(y(t) \equiv \log(x(t))\) and \(r(t) = \log(g(t))\), Eq. (A.7) can be transformed as
\[ y(t + \Delta t) = \log \{ g(t)x(t) + f(t) \} = y(t) + r(t) + \frac{f(t)}{y(t)x(t)} + \cdots \]  
(A.8)
Neglecting the terms including \(f(t)\) as higher order terms, time evolution of the probability density of \(y(t)\), \(p(y, t)\), is approximated by a Fokker-Planck equation,
\[ p(y + \Delta y, t + \Delta t) \approx \int_{-\infty}^{\infty} \omega(r)p(y - r, t)dr = p(y, t) - < r > \frac{\partial p(r, t)}{\partial y} + \frac{< r^2 >}{2} \frac{\partial^2 p(r, t)}{\partial y^2} + \cdots, \]  
(A.9)
where \(\omega(r)\) denotes the probability density of \(r\). Assuming existence of a statistically steady state we have the following exponential distribution.
\[ p(y) \propto \exp \frac{\Delta}{\Delta + y} \]  
(A.10)
In the situation, \(< r >= < \log(g) > < 0\), which is equivalent to the condition of existence of steady state for random multiplicative process [10]. Eq. (A.10) is shown to be equivalent to the power law, Eq. (A.5), with its exponent given as
\[ \alpha \approx \frac{2}{< r^2 >} \]  
(A.11)
This value is derived from the exact relation, Eq. (3). By expanding the left hand side of the equation, \(< g(t)^\alpha > = 1\), in terms \(\alpha\) as follows, and by equating the second and third terms in right hand side as an approximation.
\[ < g(t)^\alpha > = < e^{\alpha \log(g(t))} > \]
\[ = 1 + \alpha < \log(g(t)) > + \alpha^2 < \{ \log(g(t))^2 \} > + \cdots \]  
(A.12)
The key equation for determining the value of exponent, Eq. (3), can be derived roughly by the following way. Neglecting the additive term in the right hand side of Eq. (A.7) and taking an average over realizations after taking the \(s\)-th power of both sides, we have the following relation.
\[ < x(t + \Delta t)^s > \approx < g(t)^s > < x(t)^s >, \]  
(A.13)
In the case that \(< g(t)^s > \gg 1\) it is clear that \(< x(t)^s > \) diverges in the limit of \(t \to \infty\). On the other hand in the case that \(< g(t)^s > < 1\) the value of \(< x(t)^s > \) is always finite. Therefore, we have the following relations,
\[ < x(\infty)^s > = \int_{\text{finite}}^{\infty} f(s > \alpha) \]  
(A.14)
where \(\alpha\) satisfies Eq. (3). The property of Eq. (A.14) is one of typical characteristics of the power law distribution, Eq. (A.5). So, we can find that the power law exponent Eq. (A.5) is consistent with Eq. (3). Note that Solomon et. al. elaborated on a generalized version of this type of model, \(x(t + 1) = g(t)f_1(x(t)) + f_2(x(t))\), where both \(f_1\) and \(f_2\) are nonlinear functions, and they derived the relationship between the exponents and these functions [39].

A rigorous mathematical derivation of this relation was done by Kesten in 1973 in more general form of real-valued matrix considering also the case that the distribution of the additive noise follows a power law [19]. In his proof the value of \(\alpha\) is limited in the range of \(0 < \alpha \leq 2\) as he applies the theory of stable distribution, however, our numerical analysis and the above intuitive theoretical analysis suggests that the value of \(\alpha\) can be extended to the whole range \(0 < \alpha\).

It should be noted that the existence of the additive term in Eq. (1) or Eq. (A.5) is essential to realize the statistically steady state. As known from Eq. (A.5) the stochastic process is a random walk with a negative trend in view of \(\log(x(t))\), therefore, without any additive noise term the random walker tends to shrink to
shown in Fig. B-1. So, that Eq. (3) holds with $0 < \alpha < 1$ for a positive value of basic properties of the moment function for growth rate of a subunit, resistivity in the electric circuit and in the situation that fluctuation of voltage is amplified as a whole, and it is concave with respect to $s$. For any positive value of viscosity state by introducing an amplifier in the circuit. Namely, the value of $\mu(t)$ corresponds to resistivity in the electric circuit and in the situation that fluctuation of voltage is amplified as a whole, and the effective resistivity takes a negative value. By introducing an electric circuit in which an amplifier works at random timing we have a physical situation which is described by Eq. (A.15) and power law distributions of voltage fluctuation are confirmed experimentally.\[36\]

As known from this equation the case of $g(t)$ > 1 corresponds to a negative value of viscosity, $\mu(t) < 0$. In the case of a colloidal particle’s diffusion in water such a negative viscosity cannot be realized, however, in the case of voltage fluctuation in an electric circuit, which is approximated also by a Langevin equation, we can consider a negative viscosity state by introducing an amplifier in the circuit. Namely, the value of $\mu(t)$ corresponds to resistivity in the electric circuit and in the situation that fluctuation of voltage is amplified as a whole, and the effective resistivity takes a negative value. By introducing an electric circuit in which an amplifier works at random timing we have a physical situation which is described by Eq. (A.15) and power law distributions of voltage fluctuation are confirmed experimentally.\[36\]

Appendix B: Basic properties of the moment function

As Eq. (3) is the key of determining the exponent of the power law of Eq. (2), here, we summarize the basic properties of the moment function for growth rate of a subunit, $M(s) \equiv < g_j(t)^s >$. This is a continuous function and it is concave with respect to $s$ for any distribution of $g_j(t)$ because the second derivative of this function is always positive, $M(s)^{''} \equiv \langle \log(g_j(t))^2 g_j(t)^s \rangle > 0$. As $M(0) = 1$ is an identity, if $M(\alpha) = 1$ holds for a positive value of $\alpha$, then we know that $M(s) < 1$ for $0 < s < \alpha$ and $M(s) > 1$ for $\alpha < s$ as schematically shown in Fig. (2c). So, $M(2) < 1$ corresponds to $2 < \alpha$ as shown in the first column of Table 1. In the situation that Eq. (3) holds with $0 < \alpha < 1$, then we have $M(1) = < g_j(t) > 1$, while in the situation, $1 < \alpha$, we have $M(1) = < g_j(t) > 1$. The stationary condition, $< \log(g(t)) > 0$, means that the slope at the origin, $M(0)'$, is negative, so if this condition is not fulfilled $M(s) > 1$ for any positive $s$, implying that the stochastic process of Eq. (1) is not stationary. On the other hand in the case that the probability of occurrence of $g_j(t) > 1$ is 0, it is trivial that $M(s) < 1$ for any positive $s$. Ref. [23] provides more details on discussions and applications to financial bubbles.

Appendix C: A brief review of the generalized central limit theorem

The central limit theorem is one of the most powerful mathematical tools; however, it is too often used to approximate the sum of random variables, of the form, $Y(N) \equiv y_1 + y_2 + \cdots + y_N$, by normal distributions. In fact there are three required conditions on random variables for their sums to obey the central limit theorem [19].

1. All random variables follow the identical distribution.
2. The variables are independent.
3. The variance of the variables is finite.
If one of these conditions is violated, then the central limit theorem does not apply.

Violation of the first condition has recently been attracting attention as super-statistics, that is, superposition of stochastic variables having different statistics. As an old example, a fat-tailed velocity distribution observed in randomly stirred granular particles can be explained by superposition of normal distributions having different variances due to clustering caused by inelastic collisions.

Giving a general discussion of correlated variables violating the second condition is rather difficult, as the details depend on the details of correlation. It is known that universal properties independent of the details of the system can be expected at the critical point of phase transition at which power law distributions and power law scaling relations play important roles. Also, theoretical approaches based on the concept of “non-extensive entropy” can provide general solutions for strongly correlated systems having scale-free interactions such as charged particles. Here, we review the domains of attraction of the GCLT more precisely. We consider the scaled sum of random variables, having different variances due to clustering caused by inelastic collisions.

C.I The domain of attraction of the Generalized Central Limit Theorem (GCLT)

Here, we review the domains of attraction of the GCLT more precisely. We consider the scaled sum of random variables,

\[ Z_N = \frac{1}{B_N} \left( \sum_{i=1}^{N} z_i - A_N \right). \]  

(C.1)

where \( A_N \) and \( B_N \) are sequences, as discussed later and we assume that \( z_i \) has the PDF satisfying the following conditions:

\[ P_c(x) \approx c^+ \cdot \lambda \cdot x^{-\lambda - 1}, \quad (x \to \infty) \quad (j = 1, 2, \cdots, N) \]  

(C.2)

\[ P_c(x) \approx c^- \cdot \lambda \cdot |x|^{-\lambda - 1}, \quad (x \to -\infty) \quad (j = 1, 2, \cdots, N), \]  

(C.3)

where \( c^+ \), \( c^- \) and \( \lambda \) are positive constants and \( P_c(x) \) is the PDFs of \( z_j \) \( (j = 1, 2, \cdots, N) \).

According to the GCLT, \( Z_N \) observes the stable distribution with the parameters:

\[ \alpha = \begin{cases} \lambda & (\lambda \leq 2) \\ 2 & (\lambda > 2) \end{cases} \]  

(C.4)

and

\[ \beta = \frac{c^+ - c^-}{c^+ + c^-}. \]  

(C.5)

Here, the characteristic function of the stable distribution is defined as

\[ \phi(z; \alpha, \beta, \gamma, \delta) = \exp \left[ i \delta z - (\gamma |z|)^{\alpha} \left\{ 1 + i \beta \frac{z}{\alpha} \omega(z, \alpha) \right\} \right]. \]  

(C.6)

\[ \omega(z, \alpha) = \begin{cases} \tan(\frac{z}{\beta}) & (\alpha \neq 1) \\ \frac{1}{\Delta \log(z)} & (\alpha = 1) \end{cases}, \]  

(C.7)

and the coefficients \( A_N \) and \( B_N \) were defined as:

\[ A_N = \begin{cases} 0 & (0 < \lambda < 1) \\ \beta \cdot (c^+ + c^-) \cdot N \cdot \log(N) & (\lambda = 1) \\ N \cdot |z| > & (\lambda > 1) \end{cases} \]  

(C.8)

\[ B_N = \begin{cases} \left[ \frac{(c^+ + c^-)^{1/\alpha}}{\pi} \cdot \left( 2 \Gamma(\alpha) \sin(\alpha \pi/2) \right)^{1/\alpha} \cdot N^{1/\alpha} \right] & (0 < \lambda < 1) \\ \left( c^+ + c^- \right)^{1/\alpha} \cdot N & (\lambda = 1) \\ \left( c^+ + c^- \right)^{1/\alpha} \cdot [N \log(N)]^{1/\alpha} & (1 < \lambda < 2) \end{cases} \]  

(C.9)

\[ \left( 1/2 \right) < (z - \alpha < \alpha) >^{1/2} \cdot N^{1/2} & (\lambda > 2) \]
Appendix D: Growth rate of the system for $\alpha \geq 1$

We calculate the growth rate $G(t;N)$ using the generalized central limit theorem (GCLT) in the case that the distribution converges to the stable distribution. Here, we assume that

$$x_j(t+1) = g_j(t)x_j(t) + f_j(t) \quad (j = 1, 2, \ldots, N)$$

and for the steady state

$$P_{x_j}(x) \approx d_g \cdot \alpha_g \cdot x^{-\alpha_g - 1}, \quad (x \to \infty) \quad (j = 1, 2, \ldots, N)$$

where $d_g$ is the constant which is determined by the PDF of $g_j$, $P_g(g)$, and $\alpha_g$ satisfies $< g_j(t)^{\alpha_g} > = 1$.

D.I: $\alpha_g > 1$

First, we investigate the case of $\alpha_g > 1$, namely, where the mean of $g_j$ takes a value less than 1.

Considering the domain of attraction of the GCLT, as given in Appendix C, we transform $G(t;N)$ for the steady state as follows:

$$G_N = \frac{\sum_{j=1}^{N} g_j x_j}{\sum_{j=1}^{N} x_j} = < g > + \frac{1}{N} \frac{B_N^{(1)} J_1}{(B_N^{(2)}/N) J_2 + < x >},$$

where $G_N$ is the growth rate for the steady state, $G_N \equiv G(t;N)$, $(t \to \infty)$,

$$< g > \equiv < g_j > \quad (i = 1, 2, \ldots, N)$$

$$g_j \equiv g_j - < g >$$

$$\zeta_j \equiv g_j \cdot x_j$$

$$J_1 \equiv \sum_{j=1}^{N} \zeta_j$$

$$J_2 \equiv \sum_{j=1}^{N} x_j - N < x >$$

$$B_N^{(1)} \equiv \begin{cases} N^{1/\alpha_g} \cdot \frac{(c_1^+ + c_1^-)^{1/\alpha_g} \cdot F(c_1^+) \cdot (1 < \alpha_g < 2)}{(N \log(N))^{1/2} \cdot (c_1^+ + c_1^-)^{1/2} \cdot (N^2 \cdot (1/2 < (gx - < gx >)^2))^{1/2} \cdot (\alpha_g > 2)} \\ N^{1/2} \cdot (1/2 < \alpha_g > 2) \end{cases}$$

$$B_N^{(2)} \equiv \begin{cases} N^{1/\alpha_g} \cdot \frac{(c_2^+ + c_2^-)^{1/\alpha_g} \cdot F(c_2^+) \cdot (1 < \alpha_g < 2)}{(N \log(N))^{1/2} \cdot (c_2^+ + c_2^-)^{1/2} \cdot (N^2 \cdot (1/2 < (x - < x >)^2))^{1/2} \cdot (\alpha_g > 2)} \\ N^{1/2} \cdot (1/2 < \alpha_g > 2) \end{cases}$$

$$F(\alpha) \equiv \frac{1}{\pi} \cdot 2\Gamma(\alpha) \sin(\alpha\pi/2)$$

$$= 1/\alpha_g$$

and

$$c_1^+ \equiv \lim_{y \to -\infty} (1/\alpha_g) y^{\alpha_g + 1} P_{x_j}(y)$$

$$c_1^- \equiv \lim_{y \to -\infty} (1/\alpha_g) y^{\alpha_g + 1} P_{x_j}(y)$$

$$c_2^+ \equiv \lim_{y \to -\infty} (1/\alpha_g) y^{\alpha_g + 1} P_{x_j}(y)$$

$$c_2^- \equiv \lim_{y \to -\infty} (1/\alpha_g) y^{\alpha_g + 1} P_{x_j}(y).$$

Because $B_N^{(2)}/N \to 0 \quad (N \to \infty)$, we can neglect the term $B_N^{(2)}/N \cdot J_2$ in the denominator of Eq. (D.13) for $N \gg 1$, and we obtain the approximation,

$$G_N \approx < g > + \frac{B_N^{(1)}}{N} \cdot \frac{J_1}{< x >},$$

From the GCLT, $J_1$ observes the stable distribution with the parameters $\alpha = \alpha_g$, $\beta = (c_1^+ - c_1^-)/(c_1^+ + c_1^-)$, $\gamma = 1$ and $\delta = 0$, where the characteristic function of the stable distribution is defined as in Eq. (D.13).

Next, we specify $c_1^+$, $c_1^-$, $c_2^+$ and $c_2^-$. Applying the formula of the transformation of random variables to $\zeta_j = b_j x_j$,

$$P_{\zeta_j}(z) = \int_{\zeta_{min}}^{\zeta_{max}} \frac{1}{|y|} P_y(g) \cdot P_z(z/\bar{y}) dg,$$
where \( \zeta_{\text{min}} = \max\{g_{\text{min}} - <g>, \min\{z/x_{\text{min}}, 0\}\}, \zeta_{\text{max}} = \min\{g_{\text{max}} - <g>, \max\{z/x_{\text{min}}, 0\}\}, \) the support of PDF of \( g \) is \([g_{\text{min}}, g_{\text{max}}]\) and the support of the PDF of \( x \) is \([x_{\text{min}}, \infty]\).

Taking the limit of \( z \),

\[
P_{\zeta}(z) = \begin{cases} 
|z|^{-\alpha_{g}^{-1}} \cdot \int_{g_{\text{min}} - <g>}^{0} p_{g}(g) \cdot |\bar{g}|_{g}^{\alpha_{g}} \, dg & (z \to -\infty) \\
|z|^{-\alpha_{g}^{-1}} \cdot \int_{g_{\text{max}} - <g>}^{0} p_{g}(g) \cdot |\bar{g}|_{g}^\alpha \, dg & (z \to \infty)
\end{cases}
\] (D.18)

Thus,

\[
c_{1}^{+} = d_{g} \cdot \int_{g_{\text{min}} - <g>}^{0} p_{g}(g) \cdot |\bar{g}|_{g}^\alpha \, dg
\] (D.19)

\[
c_{1}^{-} = d_{g} \cdot \int_{g_{\text{max}} - <g>}^{0} p_{g}(g) \cdot |\bar{g}|_{g}^\alpha \, dg
\] (D.20)

In addition, from Eq. (D.2)

\[
c_{2}^{+} = d_{g}
\] (D.22)

\[
c_{2}^{-} = 0.
\] (D.23)

As a result, \( G_{N} \) observes the stable distribution with the parameters:

\[
\alpha = \begin{cases} 
\alpha_{g} & (1 < \alpha_{g} < 2) \\
2 & (\alpha_{g} \geq 2)
\end{cases}
\] (D.24)

\[
\beta = \frac{c_{1}^{+} - c_{1}^{-}}{c_{1}^{+} + c_{1}^{-}} = \frac{\langle (g - <g>) |(g - <g>) \rangle^{\alpha - 1}}{\langle (g - <g>) \rangle^{\alpha}}
\] (D.25)

\[
\gamma = \frac{B_{N}}{N < x >} = \begin{cases} 
N^{1/\alpha - 1} \cdot \langle (g - <g>) \rangle^{\alpha} \cdot F(\alpha) \cdot d_{g} / \mu_{x} & (1 < \alpha_{g} < 2) \\
N^{-1/2} \cdot \log(N)^{1/2} \cdot \langle (g - <g>) \rangle^{1/2} \cdot d_{g} / \mu_{x} & (\alpha_{g} = 2) \\
N^{1/2} \cdot 1/\sqrt{2} \cdot \sigma_{\zeta} / \mu_{x} & (\alpha_{g} > 2)
\end{cases}
\] (D.26)

where \( \mu_{x} \equiv x \Rightarrow f > / (1 - <g>) \) and \( \sigma_{\zeta} \equiv \langle (g - <g>)x \rangle^{2} > 1/2 = \langle <g - <g>)^{2} \rangle \cdot (\langle (f - <f >)^{2} > + \mu_{x}^{2}) / (1 - <g^{2} > + \mu_{x}^{2}) \rangle^{1/2} \).

D.II: \( \alpha_{g} = 1 \)

In the same manner as was done for the case of \( \alpha_{g} > 1 \), we calculate the case of \( \alpha_{g} = 1 \), namely, where \( <g_{j} > = 1 \). Considering the domain of attraction of the GCLT given in Appendix C, we transform the expression of the growth rate of the system for the steady state, \( G_{N} \),

\[
G_{N} = \frac{\sum_{j=1}^{N} g_{j} \cdot x_{j}}{\sum_{j=1}^{N} x_{j}}
\] (D.27)

\[
= <g > + \frac{1}{N \log(N)} \cdot \frac{B_{N}^{(1)} \cdot J_{1} + N \log(N) \cdot \beta_{1}(c_{1}^{+} + c_{1}^{-})}{B_{N}^{(2)} / (N \log(N)) \cdot J_{1} + \beta_{2}(c_{2}^{+} + c_{2}^{-})}.
\] (D.28)
where
\[< g > \equiv < g_j > \quad (i = 1, 2, \cdots, N) \]  
\[\zeta_i \equiv (g_j - < g >) \cdot x_j \]  
\[J_1 \equiv \frac{\sum_{j=1}^{N} \zeta_j - N \cdot \log(N) \cdot \beta_1 \cdot (c_1^+ + c_1^-)}{B_N^{(1)}} \]  
\[J_2 \equiv \frac{\sum_{j=1}^{N} x_j - N \cdot \log(N) \cdot \beta_1 \cdot (c_1^+ + c_1^-)}{B_N^{(2)}} \]  
\[B_N^{(1)} \equiv (\pi/2)(c_1^+ + c_1^-) N = (\pi/2) d g < g - < g > \mid > \cdot N \]  
\[B_N^{(2)} \equiv (\pi/2)(c_2^+ + c_2^-) N = (\pi/2) d g \cdot N \]  
\[\beta_1 = \frac{e_1^+ - e_1^-}{e_1^+ + e_1^-} = \frac{<(g - < g >) >}{< |g - < g >| >} = 0 \]  
\[\beta_2 = \frac{e_2^+ - e_2^-}{e_2^+ + e_2^-} = \frac{d - 0}{d - 0} = 1 \]  
\[\text{Because } B_N^{(2)}/(N \log(N)) \propto 1/\log(N) \to 0 \quad (N \to \infty), \text{ roughly speaking, we can neglect the } B_N^{(2)}/(N \log(N)) \cdot J_2 \text{ in the denominator for } N >> 1. \text{ Then we obtain the following approximation:} \]
\[G_N \approx < g > + \frac{B_N^{(1)}}{N \cdot \log(N)} \cdot \frac{J_1}{d g} \]  
From the GCLT and the properties of the stable distribution, \(G_N\) observes the stable distribution with the parameters:
\[\alpha = 1, \]  
\[\beta = \beta_1 = \frac{<(g - < g >)(|g - < g >|)^{\alpha - 1}>}{<(|g - < g >|)^{\alpha}>} = 0, \]  
\[\gamma = \frac{1}{\log(N)} \cdot \frac{\pi}{2} \cdot \frac{d g}{d g} \cdot \frac{<|g - < g >|>}{<g - 1>}, \]  
\[\delta = < g >> = 1. \]
This distribution is equivalent to the Cauchy distribution with the following parameters:
\[\mu = 1, \]  
\[\sigma_N = \frac{\pi}{2} \cdot \frac{1}{\log(N)} \cdot <g - 1>, \]  
where the Cauchy distribution is defined as
\[f_c(G_N; \mu, \sigma_N) = \frac{1}{\pi} \cdot \frac{\sigma_N}{(G_N - \mu)^2 + \sigma_N^2}. \]

Appendix E: Scaling of the variance of the model

In Ref. [53], it is reported that the scaling of the variance (or the standard deviation) is different from that of the PDF of the growth rates. In an analogous way, we investigate the scaling of the variance of our model.

**Theoretical approximation** Here, we assume that the distribution of the unit growth rates \(g_i\) have the support \([0, g_{\text{max}}]\). It is trivial that \(G_N \leq g_{\text{max}}\). The variance of \(G_N\) is written as
\[< (G_N - < G_N >)^2 > = \int_0^{g_{\text{max}}} (G_N - < G_N >)^2 \cdot P_{G_N}(G_N) \cdot dG_N. \]  
For large \(N\), the PDF of the system growth rate \(P_{G_N}(G_N)\) is approximated by the stable distribution with the parameter given by Eqs. (D.24)(D.25)(D.26) and (D.24). On this condition, the asymptotic tail behavior of the PDF is approximated as
\[P_{G_N}(G_N) \approx \frac{1}{2} \cdot \alpha \cdot \left(1 + \beta \cdot \frac{G_N - < g_i >}{|G_N - < g_i >|} \right) \cdot |G_N - < g_i >|^{-\alpha - 1} \cdot N^{1 - \alpha}. \]
Since the contribution of the central part of the PDF to the variance is much smaller than that of the tail part of the PDF (i.e., the power law part) for large \( N \), we neglect the central part. Then we obtain

\[
< (G_N - < G_N >)^2 > \approx \int_{g_{\text{max}}}^{g_{\text{max}}^+} (\bar{G}_N)^2 \cdot \frac{1}{2} \alpha \cdot (1 + \beta) \cdot \frac{\bar{G}_N}{|\bar{G}_N|} \cdot N^{1-\alpha} \cdot |\bar{G}_N|^{-\alpha-1} \cdot d\bar{G}_N \quad \text{(E.3)}
\]

\[
= \int_{< g_i >}^{g_{\text{max}}^+} (\bar{G}_N)^2 \cdot \frac{1}{2} \alpha \cdot (1 + \beta) \cdot \frac{\bar{G}_N}{|\bar{G}_N|} \cdot N^{1-\alpha} \cdot |\bar{G}_N|^{-\alpha-1} \cdot d\bar{G}_N \quad \text{(E.4)}
\]

\[
\propto N^{1-\alpha} \quad \text{(E.5)}
\]

where \( \bar{G}_N \equiv G_N - < g_i > \). Therefore, the scaling of the standard deviation of the system growth rate is \( \propto N^{(1-\alpha)/2} \). This scaling is different from the scaling of the scale parameter of the stable distribution \( N^{1-\alpha} \).

**Numerical confirmation** We confirm the abovementioned result numerically. We calculate \( G(t; N) \) on the following condition:

\[
g_i(t) = \begin{cases} 0 & \text{prob 0.5} \\ g_0 & \text{prob 0.5} \end{cases}
\]

where \( < g_i^2 > \equiv 1 \), namely, \( g_0 = 2^{1/\alpha} \).

\[
f_i(t) = \begin{cases} 1 & x_i(t) < 1 \\ 0 & x_i(t) \geq 1 \end{cases}
\]

For very large \( N \), we transform \( G_N \) as follows:

\[
G_N = \frac{\sum_{i=1}^{N} g_i \cdot x_i}{\sum_{i=1}^{N} x_i} = \frac{g_0 \cdot \sum_{i \in \{i|x_i=g_0\}} x_i}{\sum_{i \in \{i|x_i=g_0\}} x_i + \sum_{i \in \{i|x_i=0\}} x_i} = \frac{g_0 \cdot S_a}{S_a + S_b} \quad \text{(E.6)}
\]

where

\[
S_a \equiv \sum_{i \in \{i|x_i=g_0\}} x_i \quad \text{(E.7)}
\]

\[
S_b \equiv \sum_{i \in \{i|x_i=0\}} x_i \quad \text{(E.8)}
\]

For \( N >> 1 \), \( S_a \) and \( S_b \) can be approximated by the stable distribution with the parameters \( \alpha = \alpha_g = \beta = 1, \gamma = (N/2)^{1/\alpha_g}, F(\alpha_g), \delta = N/2 < x_i > \) because \( \sum_{i \in \{i|x_i=g_0\}} 1 \sim \sum_{i \in \{i|x_i=0\}} 1 \sim N/2 \) and \( x_i \) observes the following power law distribution:

\[
p_{x_i}(x_i) = \alpha_g \cdot x_i^{-\alpha_g - 1} \quad \text{(E.9)}
\]

We apply this approximation for calculations of \( G_N \) in this section.

Figs. 12(a) and (b) illustrate the standard deviation and the IQR of the system growth rate \( G_N \). From the black triangle in the figure, we confirm that the standard deviation is in accordance with the theoretical result \( N^{(1-\alpha)/2} \) given by Eq. (E.5) in the case of \( \alpha_g = 1.5 \), namely \( g^{1.5} = 1 \) for \( N < 10^{1.5} \) and Eq. (E.5) holds for all \( N \) in the case of \( \alpha_g = 1.06 \).

Note that in the case of \( \alpha_g = 1.5 \) shown in Fig. 12(a), for \( N > 10^{1.5} \), the standard deviation is proportional to \( N^{1-\alpha_g} \), whose exponent is closer to that of the IQR. The cause of this transition of the scaling exponent from \( 1-\alpha_g/2 \) to \( 1/\alpha_g - 1 \) is the finite size effect for the sample number. In a finite sample number the maximum values of the asymptotic stable distribution, \( g_{\text{stable}} \), are estimated at \( N^{1-\alpha_g} \cdot m^{1/\alpha_g} \) for \( \alpha_g < 1 \) and \( 1/\log(N) \cdot m^{1/\alpha_g} \) for \( \alpha_g = 1 \) using the extreme value theory \([15]\), where we denote the sample number as \( m \) (we apply \( m = 10^5 \) in the simulation.). We can neglect the cut-off of \( g_{\text{max}} \) on the condition \( g_{\text{stable}} < g_{\text{max}} \), namely, \( N^{1-\alpha_g} \cdot m^{1/\alpha_g} < g_{\text{max}} \) for \( 1 < \alpha_g < 2 \) or \( 1/\log(N) \cdot m^{1/\alpha_g} < g_{\text{max}} \) for \( \alpha_g = 1 \). Therefore, the standard deviation observes the scaling \( N^{1-\alpha_g} \) for \( N >> m \). In fact, from Figs. 12(b) and (c), we can confirm that the maximum value of the samples of \( G_N \) approximately equals to \( g_{\text{max}} \) in the case that the standard deviation follows the scaling, \( N^{(1-\alpha_g)/2} \), namely, the case that \( N < 10^{1.5} \) for \( \alpha_g = 1.5 \) and the case of any \( N \) for \( \alpha_g = 1.06 \). We should also note that the above-mentioned conditions that the scalings of the standard
deviation are in consonant with the scalings of the IQR (i.e., \( N^{1-1/\alpha} \) for \( \alpha > 1 \) or \( \propto 1 / \log(N) \) for \( \alpha > 1 \)) are associated with the condition that the denominator of Eq. (D.3) can be approximated by \( \langle x \rangle \) even considering fluctuations, as \( \langle x \rangle \) is sufficiently larger than the maximum value of \( B_N^{(2)} / N \cdot J_2 \) for a given sample number \( m \). In particular, in the case of \( 1 < \alpha_g < 2 \), the condition that the standard deviation observes \( N^{1-1/\alpha_g} \) is given by the condition \( g_{\text{stable}} \approx N^{1-1/\alpha_g} \cdot m^{1/\alpha_g} << g_{\max} = \text{const.} \) as already discussed above. Conversely, the condition that the denominator of Eq. (D.3) can be approximated by \( \langle x \rangle \) is given by the condition, \( N^{1-1/\alpha_g} \cdot m^{1/\alpha} << \langle x \rangle = \text{const.} \). To approximate the maximum value of \( J_2 \), we use the approximation of \( m \) samples for power law random variables with the exponent \( \alpha \), as \( \{m^{1/\alpha}, (m/2)^{1/\alpha}, (m/3)^{1/\alpha}, \ldots \} \). From these calculations, we can confirm that both conditions are in accordance with the functional form of \( N \). The same discussion is applicable also for the case \( \langle \alpha \rangle \approx 1 \).

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