Anomalous Scaling and Fractal Dimensions

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The relation between critical exponents, characterizing a continuous phase transition, and the fractal structure of physical lines, proliferating at the critical point, is established by considering the two-dimensional O(N) spin model for which many exact results are available.

A field theory describes a grand canonical ensemble of fluctuation loops of arbitrary shape and length, a so-called loop gas. In the context of quantum field theories, the fluctuating loops physically represent the worldlines traced out by the quantum particles, while in a classical context they represent, for example, high-temperature (HT) graphs (in the case of spin models), current loops (in the case of the Ginzburg-Landau theory) or closed stream lines (in the case of the complex $|\phi|^4$ theory). In a recent comment, Prokof’ev and Svistunov\textsuperscript{1} pointed out that a relation proposed by Hove, Mo, and Sudbø\textsuperscript{2} between the anomalous scaling dimension of a field and the fractal or Hausdorff dimension of critical loops is incorrect. They concluded furthermore that the anomalous scaling dimension cannot be deduced from simulations of closed loops only. They supported their criticism by Monte Carlo simulations of the HT representation of the three-dimensional complex $|\phi|^4$ theory and reported the value $D = 1.7655(20)$ as fractal dimension of HT graphs at the critical point.

As was first pointed out by Helfrich and Müller\textsuperscript{3}, the HT expansion of an $N$-vector spin or O(N) lattice model, which can be visualized by closed graphs along the bonds on the underlying lattice\textsuperscript{4}, describes at the same time a loop gas of sterically interacting physical lines. Specifically, the HT expansion of the three-dimensional XY model ($N = 2$), which belongs to the same universality class as the $|\phi|^4$ theory, simultaneously represents the vortex loop gas of a bulk superconductor\textsuperscript{5}. As the magnetic interaction is screened, vortices in superconductors have short-range interactions that can be accurately described by a steric repulsion. The equivalence implies in particular that the fractal dimensions of HT graphs of the three-dimensional XY model and vortex loops in bulk superconductors, studied in Ref.\textsuperscript{2}, coincide. Prokof’ev and Svistunov\textsuperscript{1} therefore rightly compare their results with those of Ref.\textsuperscript{2}, where the value $D = 1.92(4)$ was reported for magnetic vortex loops at the critical point.

In their reply to the comment\textsuperscript{1}, Hove and Sudbø\textsuperscript{6} questioned the validity of the criticism leveled at their earlier work. By considering the HT representation of the two-dimensional O(N) spin model for $-2 \leq N \leq 2$, for which many exact results are available, we give in this short note analytic arguments supporting the findings of Prokof’ev and Svistunov and, hopefully, settle the issue. The HT graphs of the two-dimensional Ising model ($N = 1$) physically represent Peierls domain walls, separating spin clusters of opposite orientation on the dual lattice. These closed lines, which we recently investigated numerically in Ref.\textsuperscript{7}, form the two-dimensional counterpart of magnetic vortex loops in three dimensions.

A loop gas can be conveniently characterized by the loop distribution $\ell_n$, giving the average number of loops of $n$ steps per unit volume. Close to the critical point $K_c$, $\ell_n$ asymptotically takes a form similar to the cluster distribution in percolation theory\textsuperscript{8},

$$\ell_n \sim n^{-\tau} e^{-\theta n}, \quad \theta \propto (K - K_c)^{1/\sigma}. \quad (1)$$

Here, $\theta$ is the line tension, $\tau$ and $\sigma$ are two exponents characterizing the distribution, and $K$ is the tuning parameter. When the line tension is finite, the Boltzmann factor in the distribution\textsuperscript{10} exponentially suppresses large loops. Upon approaching the critical point, $\tau$ vanishes at a rate determined by the exponent $\sigma$. At $K_c$, the loops proliferate as they can now become arbitrary long without energy penalty. The remaining factor in the loop distribution is an entropy factor, giving a measure of the number of ways a loop of $n$ steps can be embedded in the lattice. The entropy exponent $\tau$ is related to the fractal dimension $D$ of the loops at the critical point via

$$\tau = d/D + 1, \quad (2)$$

where $d$ denotes the dimension of space (in the case of classical theories) or spacetime (in the case of quantum theories). The entropy factor decreases with increasing $n$ because it becomes increasingly more difficult for paths to close the longer they are.

In the limit $N \to 0$, the HT graphs of the O(N) model reduce to self-avoiding random walks (SAWs) which physically represent polymer rings in good solvents\textsuperscript{9}. Using this equivalence, de Gennes determined the fractal structure of SAWs in terms of the critical exponents of the O(N $\to 0$) model. Generalizing his results to arbitrary $-2 \leq N \leq 2$, we showed in a recent Letter\textsuperscript{11} that the fractal dimensions $D$ of the HT graphs and the exponents $\sigma$ of the loop distribution\textsuperscript{10} are related to the correlation length exponent $\nu$ of the O(N) model in the following way:

$$\nu = 1/\sigma D. \quad (3)$$
For SAWs \( (N \to 0) \), \( \sigma = 1 \), but in general \( \sigma \) takes a different value.

In the context of SAWs it is well established that loops alone do not yield all the critical exponents of the universality class defined by the O(\( N \to 0 \)) model [2]. To that end, also the total number \( z_n = \sum_{x'} z_n(x, x') \) of SAWs of \( n \) steps starting at \( x \) and ending at an arbitrary site \( x' \) is needed. Because of translation symmetry \( z_n(x, x') = z_n(|x - x'|) \), and \( z_n \) does not depend on \( x \). The ratio of \( z_n(x, x') \) and \( z_n \) defines the probability \( P_n(x, x') \) of finding a path connecting \( x \) and \( x' \) in \( n \) steps. As \( n \to \infty \), it scales as [11]

\[
P_n(x, x') = z_n(x, x')/z_n \sim n^{-d/D} P\left(|x - x'|/n^{1/D}\right),
\]

(4)

with \( P \) a scaling function, while the number \( z_n \) scales as

\[
z_n \sim n^{-d/D} e^{-\vartheta n},
\]

(5)

with \( \vartheta \) an exponent which is to be identified. Since the number of possible rooted open paths with no constraint on their endpoint increases with the number \( n \) of steps, \( \vartheta \) is expected to be positive.

The loop distribution [11] is related to the number of paths \( z_n(x, x + ai) = z_n(a) \) of \( n \) steps returning to a site \( x + ai \) adjacent to the starting point \( x \) (see Fig. 1 and Ref. [12]) through

\[
\ell_n = \frac{1}{V} \sum_{x} \frac{1}{n} z_n(a),
\]

(6)

with \( i \) any of the (positive or negative) directions on the lattice, \( V \) the lattice volume, and \( a \) the lattice spacing. The latter serves as a microscopic cutoff, whence \( z_n(a) \) rather than \( z_n(0) \) appears when closing open paths. The factor \( 1/n \) in Eq. (6) accounts for the fact that a loop can be traced out starting at any lattice point along the loop. As first shown by McKenzie and Moore for self-avoiding random walks [13], consistency of Eqs. (6) with \( z_n(a) = z_n P_n(a) \) and [11] requires that the scaling function \( P(t) \) must vanish for \( t \to 0 \) and behave for small argument \( t \) as \( P(t) \approx t^\vartheta \) with an exponent determined by the asymptotic behavior [10] of the number \( z_n \) of open paths at the critical point \( (\vartheta = 0) \).

The fractal dimension \( D \) and the exponents \( \sigma \) and \( \vartheta \) determine the critical exponents of the theory. As for self-avoiding random walks, the relations can be derived by writing the correlation function \( G(x, x') \) in terms of \( z_n(x, x') \approx \sum_{x''} z_n(x, x'') \), where because of translational invariance, \( G(x, x') = G(|x - x'|) \). When evaluated at the critical point, where \( G(x, x') \sim 1/|x - x'|^{d-2+\eta} \), this gives the relation proposed by Prokof'ev and Svistunov [1]

\[
\eta = 2 - D - \vartheta.
\]

(7)

Using high-precision data for \( \eta \), together with their estimate for \( D \), they arrived at the estimate \( \vartheta = 0.1965(20) \) for the three-dimensional \(|\varphi|^4 \) theory. The relation proposed by Hove, Mo, and Sudbø [2] corresponds to setting \( \vartheta \) to zero in Eq. (7), which in general is not allowed. Given the exact values for \( \eta \) [14] and the fractal dimensions \( D \) of the HT graphs [15], \( \vartheta \) can be determined exactly for the two-dimensional O(\( N \)) model (see Table I). Through the exact enumeration and analysis of the number \( z_n \) of self-avoiding walks on a square lattice up to length 71, the expected value \( \vartheta/D = 11/32 \) for \( N = 0 \) has been established to high precision [16], while our Monte Carlo simulation [7] of the high-temperature representation of the two-dimensional Ising model unambiguously shows that \( \vartheta \) is nonzero. In their numerical study of the three-dimensional \(|\varphi|^4 \) theory [1], Prokof'ev and Svistunov considered not the asymptotic behavior of the number \( z_n \) of open graphs to verify the value \( \vartheta = 0.1965(20) \), but the behavior of the probability [10] for small arguments. Both should give the same results, according to our analysis.

Relation [10] can, incidentally, be derived by using the second-moment definition of the correlation length \( \xi \),

\[
\xi^2 = \sum_{x} x^2 G(x)/\sum_{x} G(x),
\]

(8)

with \( x \equiv |x| \). Finally, using the definition of the sus-

| Model   | \( N \) | \( \gamma \) | \( \eta \) | \( \nu \) | \( D \) | \( \sigma \) | \( \vartheta \) |
|---------|--------|----------|--------|------|-----|--------|--------|
| Gaussian| -2     | 0        | 1/2    | 5/4  | 5/4 | 3/4    | 3/4    |
| SAW     | 1/4    | 3/4      | 1/4    | 1/4  | 1/4 | 1/4    | 1/4    |
| Ising   | 1/4    | 1/4      | 1/4    | 1/4  | 1/4 | 1/4    | 1/4    |
| XY      | 2/∞    | 1/∞      | 1/2    | 0    | 1/2 | 1/2    | 1/2    |
| Spherical| ∞      | 0        | 0      | 2    | 0   | 0      | 0      |

TABLE I: Critical exponents of the two-dimensional critical O(\( N \)) spin models, with \( N = -2, 0, 1, 2, \infty \), respectively, together with the fractal dimensions \( D \) of the HT graphs as well as the two exponents \( \sigma \) and \( \vartheta \).
ceptibility $\chi$, $\chi = \sum_x G(x,x')$, which diverges as $\chi \sim |K - K_c|^{-\gamma}$, we obtain

$$\gamma = (D + \vartheta)/\sigma D.$$  \hfill (9)

This relation generalizes one due to Cloizeaux [17] for SAWs for which $\sigma = 1$. The explicit expressions for $\nu, \eta, \gamma$ satisfy Fisher’s scaling relation, $\gamma/\nu = 2 - \eta$. The critical exponents are a function of the two independent variables $z_1 \equiv D + \vartheta$ and $z_2 \equiv \sigma D$ which determine the anomalous scaling dimension of the $\phi$ and $\phi^2$ fields:

$$d_\phi = \frac{1}{2}(d - 2 - \eta) = \frac{1}{2}(d - z_1),$$  
$$d_{\phi^2} = d - \frac{1}{\nu} = d - z_2,$$ \hfill (10)

respectively.

With $\vartheta$ set to zero, Eq. (9) reduces to the relation $\gamma = 1/\sigma$ proposed by Nguyen and Sudbø in Ref. [18]. Given that $\vartheta$ is in general positive, that relation, too, is generally incorrect. It is, however, not impossible for $\vartheta$ to be zero. A class of models with the special value $\vartheta = 0$ is provided by a noninteracting Bose gas in $d$ space dimensions with modified energy spectrum $\epsilon(k) \propto k^{D-2}$ with $D < d \leq 2D$ at fixed particle number density [19]. This class belongs to the universality class of the spherical model, obtained by taking $N \to \infty$ in the O($N$) model, with long-range interactions [20]. The critical exponents of this universality class are exactly known [21]. Those of the two-dimensional spherical model without long-range interactions, corresponding to $D = 2$ and zero critical temperature, are included in Table I.

The presence of the exponent $\vartheta$ is specific to having geometrical objects, viz. lines, that can be open or closed as it provides consistency when closing open lines. For geometrical objects where the notion of open and closed does not apply, such as for spin clusters, the analog of the exponent $\vartheta$ is absent. This is, for example, the case in the Fortuin-Kasteleyn formulation of the $Q$-state Potts model [22], and for the improved estimators in the cluster update Monte Carlo algorithms [23].

In conclusion, using exact results for the two-dimensional O($N$) model with $-2 \leq N \leq 2$, we provided analytic support to the findings of Ref. [1] that the full set of critical exponents for a given theory cannot be determined from the loop distribution alone. For the full set also the asymptotic behavior of the number of open paths is needed. Generalizing results known for SAWs, we showed that the exponent $\vartheta$ governing this behavior is connected to that of the probability $P_n(x,x')$ of finding a path connecting $x$ and $x'$ in $n$ steps for $x' \to x$, i.e., in the limit of closing open paths. Finally, we showed that this connection, together with a relation recently proposed by us [24], determines the full set of critical exponents in terms of the fractal structure of open and closed physical lines.

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