De Sitter brane-world, localization of gravity, and the cosmological constant

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Cosmological models with a de Sitter 3-brane embedded in a 5-dimensional de Sitter spacetime (dS5) give rise to a finite 4D Planck mass similar to that in Randall-Sundrum (RS) brane-world models in anti–de Sitter 5-dimensional spacetime (AdS5). Yet, there arise a few important differences as compared to the results with a flat 3-brane or 4D Minkowski spacetime. For example, the mass reduction formula (MRF) \( M_{\text{Pl}}^{(4+n)} \) as well as the relationship \( M_{\text{Pl}}^{(4+n)} = M_{\text{Pl}}^{(5)} \ell_{\text{AdS}} \) are expected in models of product-space (or Kaluza-Klein) compactifications get modified in cosmological backgrounds. In an expanding universe, a physically relevant MRF encodes information upon the 4-dimensional Hubble expansion parameter, in addition to the length and mass parameters \( L, M_{\text{Pl}} \) and \( M_{\text{Pl}}^{(4+n)} \). If a bulk cosmological constant is present in the solution, then the reduction formula is further modified. With these new insights, we show that the localization of a massless 4D graviton as well as the mass hierarchy between \( M_{\text{Pl}} \) and \( M_{\text{Pl}}^{(4+n)} \) can be explained in cosmological brane-world models. A notable advantage of having a 5D de Sitter bulk is that in this case the zero-mass wave function is normalizable, which is not necessarily the case if the bulk spacetime is anti de Sitter. In spacetime dimensions \( D \geq 7 \), however, the bulk cosmological constant \( \Lambda_b \) can take either sign (\( \Lambda_b < 0, = 0, \) or \( > 0 \)). The \( D = 6 \) case is rather inconclusive, in which case \( \Lambda_b \) may be introduced together with 2-form gauge field (or flux). We obtain some interesting classical gravity solutions that compactify higher-dimensional spacetime to produce a Robertson-Walker universe with de Sitter-type expansion plus one extra noncompact direction. We also show that such models can admit both an effective 4-dimensional Newton constant that remains finite and a normalizable zero-mode graviton wave function.

I. INTRODUCTION

The Universe is endowed with a number of cosmological mysteries, but the one that most vexes physicists is the smallness of the observed vacuum energy density in the present Universe and its effects on an accelerated expansion of the Universe at a late epoch [1]. This cosmological enigma has so far defied an elegant and forthright explanation.

Brane-worlds are promising theories with extra spatial dimensions in which ordinary matter is localized on a (3+1)-dimensional subspace [2]. To this end, the Randall-Sundrum (RS) models in 5 dimensions [3, 4] could be viewed as the simplest brane-world configurations with 1 extra dimension of space. The RS models and their generalizations in higher spacetime dimensions are known to have interesting consequences for gravitational physics [5, 6] and cosmology, see, e.g., [7, 8]; we refer to [9] for review and further references.

In the simplest Randall-Sundrum brane-world models, one has a flat 3-brane (or a 4D Minkowski spacetime) embedded in a 5-dimensional anti-de Sitter spacetime, known as an AdS5 bulk. In this simple setting there exist a massless graviton (or zero-mode) and massive gravitons (or Kaluza-Klein modes) of metric tensor fluctuations. The massless graviton mode reproduces the standard Newtonian gravity on the 3-brane, while the Kaluza-Klein modes, which arise as the effect of graviton fluctuations in extra dimension(s), give corrections to the Newton’s force law [6]. The 5D bulk geometry is extremely warped in these models, as is reflected from a typical size of the 5D curvature radius, \( \ell_{\text{AdS}} < 0.1 \) mm. Consequently, the Newtonian gravity is recovered at distances larger than \( O(0.1) \) mm.

The requirement of an AdS5 bulk spacetime in the original RS brane-world models may not be something that is totally unexpected since certain versions of 10D string theory, particularly, type IIB string theory, are known to contain AdS5 space as a sub-background space of the full spacetime, which is AdS5 × S5, and string theory itself is viewed as the most promising candidate for the unified theory of everything. However, the original RS models also predict a zero cosmological constant on the brane or 4D spacetime. This result is not supported by cosmological observations, which favor a positive cosmological constant-like term in 4 dimensions.

To construct a natural theory of brane-world, we shall replace the flat 3-brane of the original RS setup by a dynamical brane or a physical \( 3 + 1 \)-dimensional hypersurface with a nonzero Hubble expansion parameter, for instance, by a Friedmann-Lamaître-Robertson-Walker metric. With such a simple modification of the original RS brane-world model, the zero-mode graviton fluctuation is not guaranteed to be localized on the brane, if the 5D bulk spacetime is anti–de Sitter. However, if the 5D bulk spacetime is de Sitter or positively curved, then there always exists a normalizable zero-mode graviton localized on a de Sitter brane. In such theories the smallness of the 4D cosmological constant term can be related to an infinitely large extension of the fifth dimension.

There is another motivation for considering a positively curved 5D background spacetime. When we consider compactifications of string/M-theory or classical supergravity theories with more than one extra dimension, then in a cosmological setting, and under the dimensional reduction from D dimensions to 5, we generally find that the 5D spacetime is de Sitter, if we also insist on the existence of a 4-dimensional de Sitter solution. Even though AdS5 is well–motivated from some aspects of type IIB supergravity, for its role in the AdS/CFT correspondence, it is difficult to realize an AdS5 background, while at the same time we also obtain a dS4 solution (or an inflating FRW universe in 4 dimensions) by solving the full D-dimensional Einstein equations.

In this paper we show how brane-world models with a positively curved bulk spacetime (dS5) can generate a 4-dimensional cosmological constant in the gravity sector of the...
effective 4D theory with a finite 4D Newton's constant and also help explain the localization of a normalizable zero-mode graviton in 4 dimensions. We also present some new insights on localization of gravity on a de Sitter brane embedded in a higher-dimensional bulk spacetime.

In Sec. II, we pay particular attention in the study of classical gravity solutions in a 5D de Sitter bulk spacetime (dS5) and derivation of the associated mass reduction formulæ. The idea of replacing an AdS5 bulk spacetime by a dS5 spacetime is not totally new; the behavior of classical gravity solutions on a dS3 brane embedded in dS5 was discussed before, e.g., in [14, 15]. Certain aspects of linearized gravity in a dS5 bulk spacetime were also discussed in these papers, mainly, the limits between which the value of the bulk cosmological constant has to lie in order to localize the graviton on a de Sitter 3-brane [14] and the correction to the static potential at short distances due to the massive KK modes living in a 5D bulk spacetime [15]. Part of our analysis in Sec. III finds a similarity with earlier works in the subject, but the details and some of our conclusions are different. For instance, for an embedding of dS3 brane into AdS5 bulk, we find that the massless graviton wave function is non-normalizable (despite being a bound state solution) once the Z2 symmetry is relaxed. This result is consistent with our observation in Sec. II that the 4D Newton’s constant is not finite in the absence of Z2 symmetry. A more detailed comparison between past and present results will be made in the context of the analysis below.

Different from the approaches in [12, 13], our analysis will be based on conformal coordinates (rather than on Gaussian normal coordinates) in terms of which the discussions of classical solutions and localization of gravity become simpler and more precise. For the sake of completeness, we shall also analyze a set of linearized bulk equations in a dS5 spacetime (the case of AdS5 bulk was considered before, for example, in [16, 17]). Furthermore, we provide new insights on the nature of mass gap in higher dimensions, by allowing more than one extra dimensions. Our focus in this paper will remain on the discussion of mass reduction formulæ and also on the behavior of classical gravity solutions in higher dimensions. These were not discussed before in the literature, including Refs. [14, 15], at least, at the level of clarity and details as have been done in this paper. As a canonical example in 5 dimensions, we also show that the effective 4D Newton’s constant can be finite for gravity theories coupled to a bulk scalar field, despite having a noncompact direction.

In Sec. IV we consider classical gravity solutions in higher dimensions. In dimensions D ≥ 7, we find that the bulk cosmological term $\Lambda_b$ can take either sign, though a negative $\Lambda_b$ may be preferred over a positive $\Lambda_b$ for regularity of the metric. We will also make some general remarks about mass reduction formulæ (MRFs). In the standard Kaluza-Klein theories, the MRF is given by $M_{(\text{D})}^2 = M_{(4+n)}^{n+2} L^n$, which relates the 4-dimensional effective Planck mass $M_{(4)}$ with the $(4+n)$-dimensional Planck mass $M_{(4+n)}$ (with $L$ being the average size of the n extra dimensions). This result, also known as Gauss formula, gets naturally modified in the presence of a bulk cosmological term and also due a nonzero 4-dimensional Hubble expansion parameter. Finally, in Sec. V we will study a particular class of gravity solutions in the presence of a bulk scalar field. Concluding remarks are given in Sec. V.

The important contributions of this paper include (i) generalizations of the 5D results in higher dimensions, (ii) clear and more precise interpretations of the relationships between the 4D effective Planck mass and the D-dimensional fundamental mass scales, which take into account the effect of cosmic expansion or the Hubble expansion parameter plus the bulk cosmological constant term, and (iii) several new aspects of gravity localization both in 5D and higher dimensions.

II. DE SITTER BRANE-WORLDS

5-dimensional de Sitter brane-worlds characterized by a single extra dimension where the bulk spacetime is positively curved (instead of being flat or negatively curved) are among some highly plausible approaches to explaining the smallness of the observed cosmological vacuum energy density and localization gravity.

The basic idea behind the existence of a 4-dimensional de Sitter space solution (dS4) supported by warping of extra spaces can be illustrated by considering a curved 5-dimensional “warped metric”,

$$ds^2 = e^{2A(\phi)} (ds_4^2 + \rho^2 d\phi^2),$$

where $\rho$ is a free parameter with dimension of length and $e^{2A(\phi)}$ is the warp factor as a function of $\phi$. We look for solutions for which the 4-dimensional line–element takes the standard Friedmann-Lamaitre-Robertson-Walker (FLRW) form

$$ds^2_4 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega_2^2 \right],$$

where $\kappa$ is the 3D curvature constant with the dimension of inverse length squared. The 5D background Riemann tensor satisfies

$$(5) R_{ABCD} = \frac{\Lambda_5}{6} \left( (5) g_{AC} (5) g_{BD} - (5) g_{AD} (5) g_{BC} \right).$$

The 5D Einstein-Hilbert action takes the form

$$S_{\text{grav}} = M_5^3 \int d^5 x \sqrt{-g} (R - 2\Lambda_5),$$

where $M_5$ is the 5D Planck mass and $\Lambda_5 = 6/\ell^2$ and $\ell$ is the radius of curvature of the 5D bulk spacetime.

The gravitational action (2.4) may be supplemented with the following 3-brane action

$$S_{\text{brane}} = \int_{\partial M} \sqrt{-g_b} (-\tau),$$

where $\tau$ denotes the brane tension. The 5D Einstein field equations are given by

$$G_{AB} = -\frac{\tau}{2} \sqrt{-g} g^b_{\mu\nu} \delta_{AB} \delta(\phi - \phi_0) - \Lambda_5 g_{AB}.$$
The 3 independent equations of motion are

\[
\frac{6A'^2}{\rho^2} = 6 \left( \frac{\ddot{a}}{a^2} + \frac{\kappa}{a^2} \right) - \Lambda_5 e^{2A}, \tag{2.7}
\]
\[
\frac{6A''}{\rho^2} = -\Lambda_5 e^{2A} - \frac{\tau}{\rho M^3(5)} \delta(\phi - \phi_0) e^A, \tag{2.8}
\]
\[
\frac{\ddot{a}}{a} = \frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2}. \tag{2.9}
\]

Here we are interested in studying a theory without the orbifold boundary condition, that is a theory with infinite extent in both the positive and negative \(\phi\) direction. We allow both even and odd functions of \(\phi\) rather than the restriction to purely even functions demanded by the orbifold conditions in RS brane-world models.

### A. A spatially flat universe

First, we take \(\kappa = 0\) (spatially flat universe): The 5D Einstein equations are solved with the scale factor

\[
a(t) = a_0 e^{Ht} \tag{2.10}
\]

and the warp factor

\[
A(\phi) = \ln(2\ell_0 H) - \ln \left( \exp(\rho H\phi) + \frac{\ell_0^2}{\ell^2} \exp(-\rho H\phi) \right), \tag{2.11}
\]

where \(\ell_0\) and \(H\) are two integration constants. The standard results in AdS$_5$ space, see, for example \[16, 18, 19\], are obtained by replacing \(\ell^2\) with \(-\ell^2_{AdS}\) or \(6/\Lambda_5\).

From the explicit solution given above, we derive

\[
S^{(D=4)}_{\text{eff}} = M^2_{Pl} \int d^4x \sqrt{-g_4} (R_4 - \Lambda_4), \tag{2.12}
\]

where

\[
M^2_{Pl} = 8\rho M^3(5) \ell_0^3 H^3 \int_{-\infty}^{\infty} \left( e^{\rho H\phi} + \frac{\ell_0^2}{\ell^2} e^{-\rho H\phi} \right)^{-3} d\phi
= \frac{8\pi}{M^3(5)} \ell^3 H^3, \tag{2.13}
\]

\[
K = \frac{M^3(5)}{\rho} \int_{-\infty}^{+\infty} e^{3A} \left( 12A'^2 + 8A'' + \frac{12\rho^2}{\ell^2} e^{2A} \right) d\phi
= \frac{8\pi}{M^3(5)} \ell^3 H^4 \int_{-\infty}^{+\infty} 4 \left( 3 e^{2\varphi} + 3\lambda^2 e^{-2\varphi} - 2\lambda \right) (e^{\varphi} + \lambda e^{-\varphi})^5 d\varphi
= \frac{8\pi}{M^3(5)} \ell^3 H^4 \int_{-\infty}^{+\infty} \Lambda(\varphi), \tag{2.14}
\]

where we defined \(\varphi \equiv \rho H\phi\) and \(\lambda \equiv \ell_0^2/\ell^2\). This yields

\[
K = 8M^3(5) \ell^3 H^4 \frac{3\pi}{8\lambda^3/2} = 6H^2 M^2_{Pl}. \tag{2.15}
\]

A similar result was obtained in \[20\], taking \(\lambda = 1\) and \(\rho = 1\). We find interest only on smooth brane-world solutions, so we take \(\lambda \equiv \ell_0^2/\ell^2 > 0\).

### B. Inclusion of brane action

In the presence of a brane action, we have to consider the metric a step function in \(\phi\), while computing derivatives of \(A(\phi)\) (with respect to \(\phi\)). The solution valid for \(-\infty \leq \phi \leq +\infty\) then implies that

\[
A'' + \frac{4\lambda \rho^2 H^2}{\Phi^2} + \frac{\rho H \Phi_\pm}{\Phi_\pm} (2\delta(\phi - \phi_0)) = 0, \tag{2.20}
\]

where \(\tau' = \frac{d}{d\phi}\) and \(\Phi_\pm \equiv e^{\rho H\phi} \pm \lambda e^{-\rho H\phi}\). One could think of a de Sitter brane as the location \(\phi = \phi_0 (z = z_e)\) where the zero-mode graviton wave function is peaked.

![Fig. 1: (color online). The plot of the function \(\Lambda(\phi)\) with \(\lambda = 0.4, 0.6, \text{and} 0.8\) (from top to bottom) (black, blue, and pink underline line). Like the warp factor \(e^A\), \(\Lambda(\phi)\) is regular, has a peak at \(\phi = \phi_0\) and falls off rapidly away from the brane.](image)
The $\mu\nu$ components of the 5D Einstein equations yield

$$A'' + \frac{4A\rho^2 H}{\Phi^2} + \frac{2\rho H}{3M_5^2 \Phi^2} (\tau \delta(\phi - \phi_0)) = 0. \quad (2.21)$$

By comparing Eqs. (2.20) and (2.21), we get

$$\tau = \frac{3M_5^3}{\ell_0} \left( e^{\rho H(\phi_0)} - \lambda \rho e^{-\rho H(\phi_0)} \right). \quad (2.22)$$

Particularly, in the limit $\lambda \to 0$, the 5D spacetime becomes spatially flat and gravity is not localized in this case. Indeed, $\lambda > 0$ is required to keep the warp factor bounded from the below and above (cf. Figure 1).

For $\lambda > 0$, by writing

$$\phi = \frac{1}{\rho} \left( z + \ln \frac{\ell_0}{\ell} \right),$$

the solution for warp factor, Eq. (2.11), and the brane tension can be written in standard forms, i.e.,

$$e^{A(z)} = \frac{\ell H}{\cosh Hz}, \quad \tau = \frac{6M_5^3}{\ell_0} \sinh Hz_c, \quad (2.23)$$

where $z_c > 0$. The scale of warped compactification is

$$r_c \equiv \rho e^A = \frac{2\ell_0 \rho H}{(e^{\rho H(\phi_0)} + \lambda \rho e^{-\rho H(\phi_0)})} = \frac{\rho H \ell}{\cosh Hz}. \quad (2.24)$$

The $\rho \to \infty$ limit gives rise to a theory with a semi-infinite extra dimension.

In Sec. IIB of [14], the authors presented 5D solutions in terms of Gaussian normal coordinates, which admit a bulk singularity at $|y| = y_H$, especially, in the $\Lambda_5 > 0$ case, so there one required a cut-off in the bulk, which may be seen as the position of the horizon in the bulk. The bulk singularity at $|y| = y_H$, in terms of Gaussian normal coordinates, is not a physical singularity. Note that, in terms of the conformal coordinate $z$, the classical solution presented above is regular everywhere, particularly, when $\Lambda_5 > 0$. In the notation of [14], $\lambda \equiv H^2$, while in our notation $\lambda \equiv \ell_0^2/\ell^2 = \Lambda_5 \ell_0^2/6$.

### C. Nonflat universe

In a spatially nonflat universe ($\kappa \neq 0$), the 5D Einstein equations are explicitly solved when

$$a(t) = \frac{c_0^2 + \kappa \rho^2}{2c_0} \cosh \left( \frac{t}{\rho} \right) + \frac{c_0^2 - \kappa \rho^2}{2c_0} \sinh \left( \frac{t}{\rho} \right). \quad (2.25)$$

and

$$A(\phi) = \ln \left( \frac{2\ell_0}{\rho} \right) - \ln \left( \exp(\phi) + \frac{\ell_0^2}{\rho^2} \exp(-\phi) \right), \quad (2.26)$$

The Hubble-like parameter $H$ (which appeared in the $\kappa = 0$ case above) is no more arbitrary but it is fixed in terms of the length parameter $\rho$, i.e., $H \to 1/\rho$. The 4D effective action still takes the form of (2.12), but now

$$M_{Pl}^2 = \frac{\pi M_5^3}{2} \ell^3, \quad K = \frac{3\pi M_5^3}{\rho^4}. \quad (2.27)$$

Note that, unlike in the simplest RS brane-world models, we do not require the $Z_2$ symmetry in order to get a finite 4D Planck mass, as long as $\lambda > 0$ or $\ell_0^2/\ell^2 > 0$.

In the $\kappa \neq 0$ case, Eq. (2.22) is modified as

$$\tau = \frac{3M_5^3}{\ell_0} \left( e^{\phi_0} - \lambda e^{-\phi_0} \right). \quad (2.28)$$

The scale of warped compactification is now

$$r_c \equiv \rho e^A = \frac{2\ell_0}{(e^{\phi} + \lambda e^{-\phi})} = \frac{2\ell}{\cosh y}, \quad (2.29)$$

where $y \equiv \phi - \ln(\ell_0/\ell)$. This is exponentially suppressed as $y \to \pm \infty$. Clearly, there is no problem with taking the $\rho \to \infty$ limit of the background solution given above.

### III. LINEARIZED GRAVITY IN 5D

Brane-world models with one or more noncompact extra spaces are known to require the trapping of gravitational degrees of freedom on the brane [4, 8]. To determine whether the spectrum of linearized tensor fluctuations $\delta^{(5)}g_{AB}$ is consistent with 4D experimental gravity, we shall consider the perturbations around the background solution given above.

The $\kappa \neq 0$ solutions are slightly more restrictive than the $\kappa = 0$ solutions. So, henceforth, we focus our discussions on the $\kappa = 0$ case, for which $\rho$ is arbitrary. The perturbations of the 5D metric $\delta^{(5)}g_{AB} \equiv h_{AB}$ may be written as

$$\delta^{(5)}g_{AB} = \begin{bmatrix} -2e^{2A}\psi & e^{2A}a^2(\partial_iB - S_i) & e^{A}\xi \\ e^{2A}a^2(\partial_iB - S_i) & e^{2A}a^2(\partial_i\beta - \chi_i) & e^{2A}a^2(\partial_i\beta - \chi_i) \\ e^{A}\xi & e^{2A}a^2(\partial_i\beta - \chi_i) & 2\rho^2e^{2A}\zeta \end{bmatrix}, \quad (3.1)$$

where $\psi, \mathcal{R}, \mathcal{C}, \xi, \beta, \zeta$ are metric scalars, while $S_i, V_i, \chi_i$ are transverse 3D vector fields, and $h_{ij}$ represent transverse-traceless tensor modes. Here we focus on the analysis of tensor modes (we refer to [17] for the analysis of gauge-invariant scalar and
vector perturbations of maximally symmetric spacetimes), see also [21]. The transverse-traceless tensor modes $h_{ij} \equiv \delta g_{ij} - \delta^i_\mu \delta^j_\nu \gamma_{\mu\nu}(x, \phi)$ satisfy the following wave equation

$$e^{-2A} \left( \frac{1}{\rho^2} \frac{\partial}{\partial \phi^2} + \frac{3A'}{\rho^2} \partial_\phi - \partial_t^2 - 3 \frac{\dot{\alpha}}{a} \partial_t + \frac{\vec{\nabla}^2}{a^2} \right) h_{ij} + \frac{\tau}{2M_{(5)}} e^{-A} \delta(\phi - \phi_0) h_{ij} = 0. \quad (3.2)$$

The last term above has arisen from the first term on the right hand side in Eq. (2.6). By separating the variables as

$$h_{ij}(x^\mu, \phi) \equiv \sum \alpha_m(t) u_m(\phi) e^{i k \cdot x} \hat{e}_{ij}, \quad (3.3)$$

where $e_{ij}(x^\mu)$ is a transverse, tracefree harmonics on the spatially flat 3-space, $\vec{\nabla}^2 \hat{e}_{ij} = -k^2 \hat{e}_{ij}$, we get

$$\ddot{\alpha}_m + \frac{3}{a} \dot{\alpha}_m + \left( \frac{k^2}{a^2} + m^2 \right) \alpha_m = 0, \quad (3.4a)$$

$$\left( \frac{1}{\rho^2} \frac{d^2}{d\phi^2} + \frac{3A'}{\rho^2} \frac{d}{d\phi} + \frac{3H\Phi_+}{\rho\Phi_+} \delta(\phi - \phi_0) + m^2 \right) u_m = 0, \quad (3.4b)$$

where $m$ is a 4D mass parameter and $k$ is the comoving wavenumber along the 4D hypersurface.

Let us first consider Eq. (3.4a). If we write $\alpha_m = \varphi_m/a(\eta)$ and use conformal time $\eta = -\int (dt/a)$, then the wave equation on the brane reads as

$$\frac{d^2 \varphi_m}{d\eta^2} + \left[ - \frac{d^2 a}{a d\eta^2} + a^2 m^2 + k^2 \right] \varphi_m = 0. \quad (3.5)$$

With $a \propto e^{H}$ (and hence $\eta = -1/(aH)$), we get

$$\frac{d^2 \varphi_m}{d\eta^2} + \left[ - \frac{2}{\eta^2} + \frac{m^2}{\eta^2 H^2} + k^2 \right] \varphi_m = 0. \quad (3.6)$$

The general solution is

$$\varphi_m(\eta, k) = \sqrt{\eta k} Z(\lambda, \eta k), \quad \nu \equiv \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}, \quad (3.7)$$

where $Z(\nu, \eta k)$ is a linear combination of Bessel functions of order $\nu$. One recovers the RS solution in the limit $a(\eta) \to \text{const} \equiv 1$, in which case $\varphi_m = \text{exp}(\pm i \omega t)$, with $\omega^2 = k^2 + m^2$. The perturbations are over-damped for all light modes with $0 < m < 3H/2$, while all heavy modes with $m > 3H/2$ oscillate and decay more rapidly and the modes with $m^2 < 0$ do not exist. In a 4D de Sitter space, the eigen modes satisfying $m^2 > 0$ are not localized on an inflating (de Sitter) brane, see below.

Defining $u_m \equiv e^{-3A/2} \psi_m$, it is possible to rewrite Eq. (3.4b) in a Schrödinger-like form

$$\frac{d^2 \psi_m}{d\phi^2} - V \psi_m + m^2 \rho^2 \psi_m = 0 \quad (3.8)$$

with the scalar potential

$$V = \frac{9}{4} A'' + \frac{3}{2} A'' - \frac{3H\rho\Phi_-}{\Phi_+} \delta(\phi - \phi_0)$$

$$= \frac{9\rho^2 H^2}{4} - \frac{15\lambda\rho^2 H^2}{\Phi_+} - \frac{6H\rho\Phi_+}{\Phi_+} \delta(\phi - \phi_0), \quad (3.9)$$

where we used the solution (2.11) and $\Phi_\pm = e^{\rho H}\pm \lambda e^{-\rho H}$.

Introducing a new coordinate variable

$$z \equiv \rho \phi - \ln \sqrt{\lambda},$$

we get

$$\frac{d^2 \psi_m}{dz^2} - V \psi_m = -m^2 \psi_m, \quad (3.10)$$

where

$$V = \frac{9H^2}{4} - \frac{15H^2}{4 \cosh^2(Hz)} - 6H \tanh(Hz) \delta(z - z_c). \quad (3.11)$$

The brane is now located at $z = z_c$. The zero-mode solution ($m^2 = 0$) is given by

$$\psi_0(z) = \frac{b_0}{(\cosh(Hz))^{3/2}}, \quad (3.12)$$

which is clearly normalizable since

$$\int_{-\infty}^{\infty} |\psi_0(z)|^2 dz = \frac{\pi b_0^2}{2H}.$$

There is one more bound state solution, i.e.,

$$\psi_1(z) = b_1 \frac{\sqrt{\cosh^2(Hz) - 1}}{(\cosh(Hz))^{3/2}}, \quad (3.13)$$

which is obtained by taking $m^2 = 2H^2$. This solution is also normalizable. However, only the zero-mode solution ($m^2 = 0$) is localized on the de Sitter brane. This can be seen by substituting (3.12) into Eq. (3.10) and comparing the delta-function terms (cf Figs. 2, 3).

![FIG. 2: (color online). The plot of the function $H \int |\psi_0|^2$ with $b_0 = 1$.](image-url)
The zero-mode graviton is localized on the brane. The first excited state with mass \( m^2 = 2H^2 \) is normalizable but this mode is not localized on the brane.

The general solution to Eq. (3.10) is given by

\[
\psi(z) = c_1 X^{5/2} \, _2F_1 \left( \frac{5 + 2i\mu}{4}, \frac{5 - 2i\mu}{4}; \frac{1}{2}; 1 - X^2 \right) + c_2 \sqrt{X^2 - 1} X^{5/2} \, _2F_1 \left( \frac{7 + 2i\mu}{4}, \frac{7 - 2i\mu}{4}; \frac{3}{2}; 1 - X^2 \right),
\]

(3.15)

where \( X = \cosh(Hz) \) and \( \mu \equiv \sqrt{\frac{m^2}{4} - \frac{1}{4}} = \pm i\nu \). The allowed values of \( m \) are quantized in units of \( H \) or the index \( \gamma \equiv m^2/H^2 \). Around the brane’s position at \( z \equiv z_c \), satisfying \( Hz \ll 1 \), the solution looks like

\[
\psi(z) = c_1 P_\mu + c_2 Q_\mu,
\]

(3.16)

where

\[
P_\mu = 1 - \frac{3 + 2\gamma}{4} (Hz)^2 + \frac{39 + 12\gamma + 4\gamma^2}{96} (Hz)^4 + \cdots ,
\]

(3.17a)

\[
Q_\mu = Hz - \frac{3 + 2\gamma}{12} (Hz)^3 + \frac{99 + 12\gamma + 4\gamma^2}{480} (Hz)^5 + \cdots .
\]

(3.17b)

Note that the condition \( Hz \ll 1 \) signifies a cosmological scale for which \( H^{-1} \gg z \), i.e. the Hubble radius is much larger than the radial extension of the fifth dimension.

Further, using the following property:

\[
_2F_1(a, b; c; z) = (1 - z)^{-b} \, _2F_1(c - a, b; c; \frac{z}{z - 1}),
\]

the eigenfunctions for the massive continuous modes with \( m^2 \geq 2H^2 \) can be expressed as

\[
\psi_m(z) = c_1 X^{i\mu} \times \, _2F_1 \left( \frac{3 + 2\gamma}{4}, \frac{5 - 2\mu}{4}; \frac{1}{2}; \frac{X^2 - 1}{X} \right) + c_2 \sqrt{X^2 - 1} X^{i\mu} \times \, _2F_1 \left( \frac{1 + 2\gamma}{4}, \frac{7 - 2\mu}{4}; \frac{3}{2}; \frac{X^2 - 1}{X^2} \right),
\]

(3.18)

where \( X = \cosh(Hz) \). This result may be brought to the form given in [14] [cf. Eq. (49)], applying Euler’s hypergeometric transformations. A direct comparison with the result in [15], i.e., Eq. (14), is less straightforward as the result there was presented using Gaussian normal coordinates. In the large \( zH \) limit, we get

\[
\psi_{Hz \to \infty} = c_1 e^{izH} + c_2 e^{-izH},
\]

(3.19)

where \( \mu \equiv \sqrt{\frac{m^2}{4} - \frac{1}{4}} \). With \( c_1 = 0 \), all heavy modes with \( \mu > 0 \) become oscillating plane waves, which represent the de-localized KK modes (cf. Figures 4[5]). The time-evolution of the mode functions of these heavy modes [cf. Eq. (3.7)] shows that they remain underdamped at late times \( |\eta| \to 0 \).
A couple of remarks are in order. In the $\Lambda_5 < 0$ case, though the analysis closely follows the one given above, the opposite sign of $\Lambda_5$ modifies some of the results. For example, when $\Lambda_5 < 0$, the scalar potential $V$ in (3.10) is given by

$$V = \frac{9H^2}{4} + \frac{15H^2}{4\sinh^2(Hz)} - 6H \coth(Hz)\delta(z-z_c).$$

The zero-mode solution ($m^2 = 0$) is given by

$$\psi_0(z) = \frac{c_0}{\sinh(Hz)^{3/2}}, \hspace{0.5cm} (3.21)$$

which is normalizable only when one allows a cutoff scale satisfying $0 < z_c \leq z$ or $z \leq z_c < 0$.

The main finding in [14] was that the choices $\Lambda_5 > 0$ and $\Lambda_5 < 0$ both may lead to a normalizable zero-mode graviton wave function, provided that one satisfies

$$\Lambda_5 \geq -\frac{\tau^2}{6M^6_5} \hspace{0.5cm} \text{and} \hspace{0.5cm} \Lambda_5 \leq \frac{\tau^2}{4M^6_5},$$

respectively, in the AdS$_5$ and dS$_5$ cases, where $\tau$ is the 3-brane tension. It is easy to understand the latter case, which follows from the classical solution [cf. Eq. (2.23)] plus the positivity of the off-brane potential [cf. Eq. (3.11)]. However, in the former case ($\Lambda_5 < 0$), it was also necessary to have $Z_2$ symmetry, otherwise the zero-mode solution is non-normalizable. Such constraints become much weaker (or even do not exist), especially in dimensions, $D \geq 7$. Similar remarks apply to mass gap or the mass of Kaluza-Klein states in higher dimensions, which are determined in terms of a free parameter $M$ (associated with the size of fifth dimension) rather than in terms of the 4D Hubble expansion parameter $H$.

### A. Linearized bulk equation

Next we consider the bulk perturbations of the 5D metric, which can be analyzed by considering a wave equation for a master variable $\Omega \equiv \Omega(x^\mu; \phi)$ introduced in [17]. In a 5D de Sitter background defined by (2.1) we find that

$$\frac{1}{\rho^2} \left( e^{-3A} \Omega' \right)' - \left( e^{-3A} \frac{a^3}{a^3} \Omega' \right) + \left( \frac{\tilde{\nu}^2}{a^2} - e^{2A} \frac{a^2}{\ell^2} \right) e^{-3A} \frac{a^3}{a^3} \Omega = 0. \hspace{0.5cm} (3.22)$$

By using the following change of variable

$$\Omega \equiv a(t)^3 e^{3A(\phi)} \tilde{\Omega}, \hspace{0.5cm} (3.23)$$

Eq. (3.22) can be written as

$$\frac{1}{\rho^2} \left( \tilde{\Omega}'' + 3A' \tilde{\Omega}' + 3A'' \tilde{\Omega} \right) - \left( \frac{\tilde{\nu}^2}{a^2} - e^{2A} \frac{a^2}{\ell^2} \right) \tilde{\Omega} = 0. \hspace{0.5cm} (3.24)$$

By separating the newly defined master variable as

$$\tilde{\Omega}(x^\mu, \phi) = \sum \alpha_m(t) u_m(\phi) e^{ik \cdot x},$$

we get

$$\ddot{\alpha}_m + \frac{3}{\alpha} \dot{\alpha}_m + \left( \frac{k^2}{a^2} + m^2 \right) \alpha_m = 0, \hspace{0.5cm} (3.25a)$$

$$\frac{d^2 u_m}{d\phi^2} + 3A' u_m' + \left( 3A'' + m^2 \rho^2 - \frac{\rho^2}{\ell^2} e^{2A} \right) u_m = 0. \hspace{0.5cm} (3.25b)$$

The first equation above is the same as (3.4a), so we only have to consider the second equation. Defining $u_m \equiv e^{-3A/2} \Phi_m$, we get

$$\frac{d^2 \Phi_m}{d\phi^2} - V \Phi_m + m^2 \rho^2 \Phi_m = 0, \hspace{0.5cm} (3.26)$$

with the off-brane potential $V$ of the form

$$V = \frac{9}{4} A'^2 - \frac{3}{2} A'' + \frac{\rho^2}{\ell^2} e^{2A} \Phi_m = \frac{9}{4} \rho^2 H^2 \Phi_m + \lambda \rho^2 H^2 \Phi_m, \hspace{0.5cm} (3.27)$$

where we used the solution (2.11) and $\Phi_m \equiv e^{\rho H \phi} \pm \lambda e^{-\rho H \phi}$. Defining $z \equiv \rho \phi - \ln \sqrt{\chi}$, as before, we get

$$- \frac{d^2 \Phi_m}{dz^2} + V \Phi_m = m^2 \Phi_m, \hspace{0.5cm} (3.28)$$

with the scalar potential $V$ of the form

$$V = \frac{9H^2}{4} + \frac{H^2}{4 \sinh^2(Hz)}. \hspace{0.5cm} (3.29)$$

This equation can also be obtained directly from Eq. (3.22) but using $\Omega \equiv \sum \alpha_m(t) u_m(\phi)$ and $u_m \equiv e^{-3A/2} \Phi_m$, see, e.g., [17]. The main difference, as compared to the result in AdS$_5$ spacetimes, is the sign of the second term above. The general solution to (3.28) is given by

$$\Phi_m(z) = c_1 \left( \cosh(Hz) \right)^{1/2} \times 2F_1 \left( \frac{1 + 2\nu}{4}, \frac{1 - 2\nu}{4}; \frac{1}{2}; -\sinh^2(Hz) \right) + c_2 \left| \sinh(Hz) \right| \left( \cosh(Hz) \right)^{1/2} \times 2F_1 \left( \frac{1 + 2\nu}{4}, \frac{1 - 2\nu}{4}; \frac{3}{2}; -\sinh^2(Hz) \right), \hspace{0.5cm} (3.30)$$

where $\nu \equiv \sqrt{\frac{9}{4} - m^2 \rho^2}$. Again, there are 2 bound state solutions: (i) $\nu = 3/2 (m^2 = 0)$ and

$$\Phi_0(z) = c_1 \sqrt{X} \sqrt{X^2 - 1} + c_2 \sqrt{X} \left( 1 - \sqrt{X^2 - 1} \tan^{-1} \frac{1}{\sqrt{X^2 - 1}} \right), \hspace{0.5cm} (3.31)$$

(ii) $\nu = 1/2 (m^2 = 2H^2)$ and

$$\Phi_1(z) = \sqrt{X} \left( c_1 + c_2 \tan^{-1} \frac{1}{\sqrt{X^2 - 1}} \right), \hspace{0.5cm} (3.32)$$

where $X \equiv \cosh(Hz)$. Both these solutions are non-normalizable. This result is desirable as it implies that a massless bulk scalar mode may not be localized on a de Sitter brane. Note that there are no any tachyonic or growing modes localized on the brane either.
B. Projected Weyl tensor

The nonexistence of arbitrarily light KK excitations can be seen also by considering the wave equation for a projected 5D Weyl tensor, which is given by \[ e^A \frac{\partial}{\partial z} e^{-A} \frac{\partial}{\partial z} + \Box_4 - 4H^2 \] \( E_{\mu\nu} = 0 \), \( \mu, \nu = C^{\alpha\beta\gamma\delta}, n^A, n^B \) is the projected 5D Weyl tensor, \( n^A \) is the vector unit normal to the brane, \( \Box_4 \equiv g^{\mu\nu} D_\mu D_\nu \) is the 4-dimensional d’Alembertian with respect to the metric \( g_{\mu\nu} \), and \( D_\mu \) is the covariant derivative. With a separation of variable \( E_{\mu\nu} = \Psi(\phi) Y_{\mu\nu}^{(m)}(x^\mu) \), Eq. (3.33) yields

\[ e^A \frac{\partial}{\partial z} e^{-A} \frac{\partial}{\partial z} - 2H^2 \Psi_m = -m^2 \Psi_m, \] \( \Box_4 - m^2 - 2H^2 \) \( Y_{\mu\nu}^{(m)} = 0 \), \( \mu, \nu = C^{\alpha\beta\gamma\delta}, n^A, n^B \)

where \( m \) is a 4D mass parameter, which has been introduced here as a separation constant. In this formalism, we clearly see that there is a mode \( \Psi = \text{constant} \) with \( m^2 = 2H^2 \) that trivially satisfies the equation. Defining \( \Psi_m \equiv e^{A/2} \Phi_m \), we can rewrite the off-brane wave equation (3.34a) as \[ -\frac{d^2 \Phi_m}{dx^2} + V \Phi_m = m^2 \Phi_m, \] \( V \equiv \frac{A^2}{4} - \frac{A''}{2} + 2H^2 = \frac{9H^2}{4} + \frac{H^2}{4 \cosh^2(Hz)} \)

This is the same potential as in (3.28). The mode \( m^2 = 2H^2 \) translates to \( \Phi_m \propto e^{-A/2} \propto \cosh(Hz) \), and it is the first eigenmode and is obtained from (3.32) with \( c_2 = 0 \).

C. Correction to Newton’s law

To estimate the correction to Newton’s force law generated by a discrete tower of Kaluza-Klein modes, one may go to the thin brane limit, i.e. \( H^{-1} \to 0 \), but keeping the ratio \( z_c/H^{-1} \) finite. One also assumes that the matter fields in the 4-dimensional theory is smeared over the width of the brane and the brane thickness is smaller compared with the bulk curvature, \( H^{-1} < \ell \), so \( H\ell > 1 \). Under this approximation, the gravitational potential between 2 point-like sources of masses \( M_1 \) and \( M_2 \) located on the brane is modified via exchange of gravitons living in 5 dimensions as

\[ U(r) = G_4 \frac{M_1 M_2}{r} + G_4 \int_0^{\infty} \frac{d\ell}{H^4} m e^{-m r} (1 + O(1)), \] \( m \geq \sqrt{2H}, r \) is the distance between the two pointlike sources, and \( \alpha \) is a constant of order unity. This result qualitatively agrees with that given in [20, 22]. In [13] a concern was raised, especially, in the dS case, that the correction to the gravitational potential due to the massive KK states may dominate for \( r < H^{-1} < \ell \), leading to a 5-dimensional behaviour which is not Newtonian. But one should also note that the corrections to the gravitational potential are suppressed by a factor of \( \sum e^{-m r} \). As a result, there is perhaps no restriction in taking \( \ell \gg r \), provided that the KK modes are sufficiently heavy. For instance, with \( m_i \geq \text{TeV} \sim 10^{-15} \text{cm}, \alpha \sim O(1) \), the correction term may not show up unless we probe a sufficiently small distance scale, like \( r \sim 10^{-12} \text{cm} \).

In the presence of matter or gauge fields, one may require a more complete analysis, involving the effects of the overlap of the gravitational modes with the matter modes, but we will not consider this case here.

IV. GENERALIZATION TO HIGHER DIMENSIONS

For a consistent description of gravity plus gauge field theories, one may require models with more than one extra dimensions, and the world around us could have up to \( n = 6 \) (or \( n = 7 \)) extra spatial dimensions, if our universe is described by string theory (or M-theory). In the following, as some canonical examples, we will consider the \( n = 2 \) and \( n = 3 \) cases, but it is straightforward to generalize the present discussion to \( D = 10 \) or \( D = 11 \) dimensions.

First, we allow 2 extra dimensions \( (n = 2) \) and write the 6D action as

\[ S_{\text{grav}} = M_4^{(6)} \int d^6 x \sqrt{-g} R, \]

where \( M_4^{(6)} \) is the 6D Planck mass. It is not difficult to check that the metric ansatz

\[ ds^2 = \frac{1}{K^{2p}} \left( -dt^2 + \alpha(t)^2 dx_{3,k}^2 + \frac{p^2 K}{H^2 K^2} dz^2 + L^2 K^{2p} dt^2 \right) \]

where \( K' = dK/dz \) and \( 0 \leq \theta \leq 2\pi \), solves the 6D Einstein equations with the 4D scale factor

\[ \alpha(t) = \frac{c_2}{2} e^{H t} + \frac{k}{2c_0 H^2} e^{-H t}. \]

In the above, \( K(z) \) is an arbitrary function of \( z \).

As a simple example, we take \( K(z) = \cosh(Mz) \), where \( M \) has a mass dimension of one. Then, under the dimensional reduction, from \( D = 6 \) to \( D = 4 \), we get

\[ M_4^{(6)} \int d^6 x \sqrt{-g} R = M_{4}^{(4)} \int d^4 x \sqrt{-g_4} \left( R_{(4)} - \Lambda_4 \right), \]

where \( \Lambda_4 = 6H^2 \) and

\[ M_{4}^{(4)} \int d^4 x \sqrt{-g_4} \left( R_{(4)} - \Lambda_4 \right) = \frac{4\pi L M_4^{(6)}}{3H}. \]
introduced into the Einstein action without considering other source terms.

Next we note that the following 7-dimensional metric,
\[ ds_7^2 = \frac{1}{F(z)} (-dt^2 + a(t)^2 d\bar{x}_{\alpha,b}^2 + G(z) dz^2 + Ed\Omega_2^2), \tag{4.6} \]
solves the Einstein field equations following from
\[ S = M_7^5 \int d^7 x \sqrt{-g} \left( R - 2\Lambda_b \right), \tag{4.7} \]
when
\[ G(z) = \frac{15 F'(z)^2}{36 H^2 F(z)^2 - 4\Lambda_b F(z)}, \quad E = \frac{1}{3H^2}. \tag{4.8} \]
where \( F' = dF(z)/dz \). There can exist a large class of 4D de Sitter solutions with different choices of \( F(z) \). Generalization of this solution to higher dimensions is straightforward.

Here we consider 2 physically interesting examples.

**Example 1**

Take
\[ F(z) \equiv F_0^2 \cosh^2(Mz). \tag{4.9} \]
The dimensionally reduced action reads as
\[ M_7^5 \int d^7 x \sqrt{-g} \left( R - 2\Lambda_b \right) = M_{Pl}^2 \int d^4 x \sqrt{-g_4} \left( R_{(4)} - \Lambda_4 \right), \tag{4.10} \]
where \( \Lambda_4 = 6H^2 \),
\[ M_{Pl}^2 = M_7^5 \frac{2\pi M \sqrt{15}}{9F_0^3H^3} I(z) \tag{4.11} \]
and
\[ I(z) \equiv \int_{-\infty}^{\infty} \frac{\cosh^2(Mz) - 1}{\cosh^3(Mz) \sqrt{\cosh^2(Mz) - \beta}} dz, \tag{4.12} \]
with
\[ \beta = \frac{\Lambda_b}{9H^2F_0^2}. \]
With \( \Lambda_b = 0 \), so \( \beta = 0 \), one can easily evaluate the above integral and find that \( I(z) = 2/(5M) \). In the \( \Lambda_b > 0 \) case, we require \( 0 < \beta < 1 \). With \( \beta = 1 \), we get \( I(z) = 3\pi/(8M) \).
That is, provided that \( 0 < \beta < 1 \), the integral converges and its value ranges between \( 2/(5M) \) and \( 3\pi/(8M) \).

In case the Hubble expansion parameter \( H \) is large, or equivalently, when \( |\beta| < 1 \), the mass reduction formula is well approximated by
\[ M_{Pl}^2 \sim \frac{M_7^5}{H^3F_0^2}, \tag{4.13} \]
However, when \( H \) becomes small, or one considers a large cosmological distance scale, then one should allow a negative bulk cosmological term. The integral converges for any negative value of \( \beta \). In the limit \( |\beta| \gg 1 \), the mass reduction formula is well approximated by the relationship
\[ M_{Pl}^2 \sim \frac{M_7^5}{H^2 F_0^4} \frac{1}{(-\Lambda_b)^{1/2}}. \tag{4.14} \]
This result may be analyzed further by taking \( H \sim 10^{-60} \) \( M_{Pl} \), especially, if one wants to tune \( \Lambda_b \) to the present value of 4D cosmological constant. To satisfy phenomenological constraints, such as, \( M_7 \gtrsim 1 \) TeV and \( (-\Lambda_b)^{1/2} \ll M_7 \), one then has to allow \( F_0 \) to take a reasonably large value, \( F_0 \gtrsim 10^{14} \gg 1 \). A similar constraint could arise in five dimensions as well; see, for example, [22], where a realistic construction required \( H^2/\Lambda_b \sim 10^{-15} \) (in the \( D = 7 \) case, the ratio \( H^2/\Lambda_b \) equals to \( 1/9F_0 \)). A constraint like this becomes much weaker when one applies the model to explain the early universe inflation. For instance, with \( H \sim 10^{-5} \) \( M_{Pl} \), we get \( M_{Pl} \sim 10^{-3} M_7/F_0 \), in which case \( F_0 \) may be taken to be small, say, \( F_0 \sim O(1) \).

To say anything further in a concrete way, we need to have some physical information about the constant \( F_0 \), which may be related to the D-dimensional dilaton coupling.

**Example 2**

Take
\[ F(z) \equiv F_0^2 \exp \left( 2 \arctan \left( e^{Mz} \right) \right). \tag{4.15} \]
The warp factor \( F(z) \) and the function \( G(z) \) are regular everywhere. The integral (4.12) is now modified as
\[ I(z) \equiv \int \frac{(1 - \beta(z))^{-1/2}}{4 \cosh(Mz) \exp \left( 5 \arctan \left( e^{Mz} \right) \right)} dz, \tag{4.16} \]
where
\[ \beta(z) \equiv \frac{\Lambda_b}{9H^2F_0^2} \exp \left( -2 \arctan \left( e^{Mz} \right) \right). \tag{4.17} \]
Since \( \arctan(x) \to \pm \frac{\pi}{2} \) as \( x \to \pm \infty \), the above integral gives a finite result provided that \( \Lambda_b < 9H^2F_0^2 e^{\pi} \), leading to a dynamical mechanism of compactification. The choice \( \Lambda_b < 0 \) is fairly safe from the viewpoint of metric regularity, or the smoothness of \( G(z) \).

The classical solutions given above are regular everywhere, \( -\infty < z < \infty \). However, at the linearized level, one may be required to introduce a cutoff along \( z \)-direction. In the \( \Lambda_b = 0 \) case, the off-brane Schrödinger-type potential and the zero-mode solution \( (m^2 = 0) \) are given, respectively, by
\[ V(z) = \frac{21}{16} \left( \frac{F'}{F} \right)^2 + \frac{3}{4} \left( \frac{F''}{F'} \right)^2 - \frac{1}{2} \left( \frac{F'''}{F'} \right) \tag{4.18} \]
and
\[ \psi_0(z) \propto \left( \frac{dF(z)}{dz} \right)^{-1/2} F(z)^{-3/4}. \tag{4.19} \]
For instance, with $F(z) \propto \cosh^2(Mz)$, we obtain
\[ \psi_0(z) = \frac{c_0}{\cosh^2(Mz)|\sinh(Mz)|^{1/2}}. \] (4.20)

Here one is required to take $0 < z_c \leq z$ or $z \leq z_c < 0$ so as to ensure normalizability of the zero-mode wavefunction. For a choice like $F(z) \propto \exp(\alpha \arctan \exp(Mz))$ (where $\alpha$ is arbitrary), however, there arises no such restriction.

In the $\Lambda_b \neq 0$ case, the off-brane wave equation for tensor perturbations satisfies
\[ \frac{d^2 u_m}{dz^2} - \left( \frac{5F'}{2F} + \frac{G'}{2G} \right) \frac{du_m}{dz} + m^2 u_m(z) = 0. \] (4.21)

$G(z)$ was defined in (4.8). By defining
\[ u_m(z) \propto \left( \frac{F'F^2}{\sqrt{9H^2F^2 - 9b^2}} \right)^{1/2} \psi(z), \] (4.22)

we can bring Eq. (4.21) into the standard Schrodinger equation; the explicit expression of the potential $V$, which is rather lengthy, is not important for our discussion here. We only mention that the zero-mode solution is given by
\[ \psi_0(z) \propto \frac{(9H^2F(z) - \Lambda_b)^{1/4}}{F(z)\sqrt{dF/dz}}. \] (4.23)

As for classical solutions, there exists a constraint that $\Lambda_b < 9H^2F^2(0)$. As another rather simple example, we may take $F(z) \propto e^{Mz}$ and restrict the radial coordinate in the range $0 \leq z < \infty$. Indeed, the zero-mode graviton wave function is normalizable with all of the above choices of $F(z)$.

All the results above can easily be generalised to higher dimensions, including the 10- and 11-dimensional models inspired by string/M-theory [23, 24]. We refer to [24] for further discussions on localization of gravity. The method can also be extended to a class of thick domain walls (or de Sitter brane solutions) in gravity coupled to a bulk scalar field [24, 29]. In the next section, we only consider the $D = 5$ case. The discussions in general D dimensions will be given elsewhere.

V. A THICK DOMAIN-WALL SOLUTION IN D=5

The 5D gravitational action is modified as
\[ S_5 = \frac{1}{2} \int d^5x \sqrt{-g} \left[ R_{\kappa_5^2} - g^{AB}\partial_A \phi \partial_B \phi - 2V(\phi) \right], \] (5.1)

where $\kappa_5^2 \equiv 1/M_5^3$. The full 5D warped metric is
\[ ds^2 = e^{2A(z)} (dz^2 + L^2). \] (5.2)

As above we search for a class of 4D FLRW cosmologies with de Sitter expansion, characterized by the metric
\[ ds_4^2 = L^2 \left[ -dt^2 + a(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega_2^2 \right) \right]. \] (5.3)

where $L$ is a scale associated with the size of the physical 3+1 spacetime. For simplicity, we begin with the case $k = 0$ and $a(t) \propto e^{Ht}$. This simplification is also justified in a realistic cosmological scenario, at least, for late time cosmologies.

Next, consider that $\phi$ depends only on the extra dimension, $\phi \equiv \phi(z)$. The 5D field equations then take the form
\[ \phi'' = \frac{3}{\kappa_5^2} \left( A'' - A' - H^2 \right), \] (5.4)
\[ V(\phi) = \frac{3e^{-2A}}{2\kappa_5^2} \left( 3H^2 - 3A'' - A' \right), \] (5.5)
\[ \frac{dV}{d\phi} = e^{-2A} (\phi'' + 3A' \phi'), \] (5.6)

where $t \equiv d/dz$. These equations admit a series of solutions with different choice of $\phi(z)$ or $V(\phi(z))$. As an illustrative case, we look for the standard domain-wall type solution:
\[ \kappa_5 \phi = \phi_0 - \phi_1 \arcsin \tanh \left( \frac{Hz}{L\delta} \right), \] (5.7)

where $\phi_1$ and $\delta$ are dimensionless constants. The solution for the warp factor is given by
\[ A(z) = A_0 - \delta \ln \cosh \left( \frac{Hz}{L\delta} \right). \] (5.8)

The coefficient $\phi_1$ in (5.7) is fixed as $\phi_1 = \sqrt{3\delta(1 - \delta)}$. Note that $\phi$ behaves as a canonical scalar field only when $0 < \delta < 1$. This will remain our choice in the following discussion. The scalar potential is
\[ V(\phi) = \frac{3(1 + 3\delta)H^2}{2\kappa_5^2 L^2} \left[ \cos^2 \left( \frac{\phi - \kappa_5 \phi_0}{\phi_1} \right) \right]^{(1-\delta)}. \] (5.9)

The case $\delta = 1$ is special for which $\phi$ and hence $V(\phi)$ are constants. This potential has a maximum at $z = 0$. This is seen by noting that
\[ \cos^2 \left( \frac{\phi - \kappa_5 \phi}{\phi_1} \right) = \cosh^2 \left( \frac{Hz}{L\delta} \right), \] (5.10)

where we used the solution (5.7). The height of the potential decreases with the expansion of the universe since $L$ is required to be larger ($H$ becomes smaller) at late epochs. These solutions are available with a nonzero 3D curvature, i.e. with the scale factor $a(t) = (a_0/2) e^{Ht} + (k/2H^2a_0) e^{-Ht}$.

To find an effective 4D potential, we shall consider a dimensionally reduced action. From the solution given above, we derive
\[ S_{\text{eff}} = \int e^{3A(z)} dz \int d^4x \sqrt{-g_4} \times \left( \frac{R_4}{2\kappa_5^2} - \left( \frac{6A'' + 4A'}{\kappa_5^2} \right) - \frac{1}{2} \phi'^2 - V(\phi)e^{2A} \right) \]
\[ = \frac{1}{2\kappa_5^2} \int e^{3A(z)} dz \int d^4x \sqrt{-g_4} \times \left( \frac{R_4}{2} - (3A'' + A') - 3H^2 \right) \]
\[ = \frac{1}{2\kappa_5^2} \int d^4x \sqrt{-g_4} R_4 - \int d^4x \sqrt{-g_4} \Lambda_4, \] (5.11)
where

\[ \frac{1}{ \kappa^2_4} = \frac{\epsilon^{3A_0}}{\kappa^2_5} \int_{-\infty}^{\infty} \frac{dz}{\cosh \left( \frac{z \delta}{L} \right)^{3\delta}} \]

\[ = \frac{\epsilon^{3A_0} \delta^2}{\kappa^2_5 H} \int_{\varphi_1}^{\varphi_2} \frac{d\varphi}{(\cosh \varphi)^{3\delta}}, \quad (5.12) \]

and

\[ \Lambda_4 \equiv \frac{e^{3A_0} H}{\kappa^2_5} \int_{\varphi_1}^{\varphi_2} \frac{6\delta \cosh^2 \varphi - 3\delta - 1}{(\cosh \varphi)^{2+3\delta}} d\varphi \]

\[ = \frac{e^{3A_0} H}{\kappa^2_5} \int \Lambda(\varphi) d\varphi, \quad (5.13) \]

where we introduced a new variable \( \varphi \equiv \frac{H}{L} \delta \). For \( 0 < \delta < 1 \) the potential is Mexican-hat type (cf. Figure 6).

![Figure 6: (color online). The plot of the function \( \Lambda(\varphi) \) with \( \delta = 1, 0.5, \) and 0.2 (top to bottom). In the \( \delta = 0 \) case, the potential is unbounded, leading to a diverging \( \Lambda_4 \).](image)

Note that the integral

\[ I_\delta \equiv \int_{-\infty}^{\infty} \frac{d\varphi}{(\cosh \varphi)^{3\delta}} = \frac{\sqrt{\pi}}{\Gamma(3\delta/2)} \Gamma((3\delta + 1)/2) \quad (5.14) \]

is finite as long as \( \delta > 0 \). For instance \( I_{\delta=0.5} = 2.3958 \) and \( I_{\delta=1} = \pi/2 \). This all shows that

\[ M_{Pl}^2 \sim \frac{L M_{Pl}^3 \epsilon^{3A_0}}{H} = \mathcal{R} M_{Pl}^3, \quad (5.15) \]

where \( \mathcal{R} \equiv (L/H) \epsilon^{3A_0} \) is the effective size of the fifth dimension. In our notation above, \( H^{-1} \) measures the Hubble radius, but \( L \) is a dimensionless constant. As is clear, to get a small \( \mathcal{R} \) we shall take an exponentially small \( \epsilon^{A_0} \). For \( \delta > 0 \) the effective 4D potential is positive. We rule out the case \( \delta = 0 \) because in this case the 4D Planck mass is not finite. Moreover, \( \int \Lambda(\varphi) = -2 \), which implies that \( \Lambda_4 \) is negative in 4-dimensions.

We have shown that, as in the model without a scalar Lagrangian, the effective 4D Newton’s constant is finite despite having a noncompact direction.

VI. CONCLUSION

Finally, we summarize the main results in the paper. Simple five-dimensional brane-world models defined in an AdS_5 spacetime have been known to provide a rich phenomenology for exploring some of the intriguing ideas that are emerging from string/M-theory, such as, AdS/CFT correspondence, AdS holography and mass hierarchies.

The replacement of AdS_5 spacetime by dS_5 spacetime, along with replacement of a flat 3-brane by a physical 4D universe, is found to give us new problems and new possibilities. The problem is that the embedding of dS_4 in dS_5 may be viewed as a O(4) symmetric bubble described by a Coleman-De Luccia instanton [30], in which case the size of the bubble may not exceed the radius of dS_5. As a result, a constraint like \( cH^{-1} \leq \ell_{AdS} \) could bring the 5D Planck mass down to TeV scale or even much lower. This problem can easily be overcome by introducing two or more extra dimensions, along with a higher-dimensional bulk cosmological term and background fluxes.

Another issue could be that dS_5 allows a foliation by a flat space but that is a spacelike hypersurface. There is no way to cut dS_5 by Minkowski spacetime. That is, in a cosmological setting, a flat 4D Minkowski spacetime is not a solution to 5D Einstein equations, if the bulk spacetime is de Sitter. This is in contrast to the results in Randall-Sundrum brane-world models in AdS_5 spacetimes. But this is anyway not a real problem since the Universe has probably never gone through a phase of being close to a static universe or a flat 3-brane.

There is perhaps no necessity of having a Minkowski spacetime embedded in a dS_5 spacetime as long as the massless graviton wave function becomes normalizable on a 4D de Sitter spacetime, which is indeed the case within the model considered in this paper. We have shown that the effective 4-dimensional Planck mass derived from the fundamental D-dimensional Planck mass can be finite because of the large but finite warped volume and also the large world-volume of a de Sitter brane (i.e., the physical Universe), implying that a Z_2 symmetry available to 5D brane-world models in an AdS background can be simply relaxed.

We have followed the most general approach for obtaining cosmologies with de Sitter expansion from the higher-dimensional Einstein equations, which could yield characteristic linear 4-dimensional spacetime sizes of many orders of magnitude bigger than linear sizes in extra coordinates. We have obtained some interesting classical gravity solutions that compactify higher-dimensional spacetime \((D \geq 5)\) to produce a Robertson-Walker universe with de Sitter type expansion plus 1 extra noncompact direction. We have also shown that such models can admit both an effective 4-dimensional Newton constant that remains finite and a normalizable graviton wave function.

Compared to some known results in five dimensions, for instance, in Refs. [14, 15, 27, 29], we find that both the classical and linearized solutions are less constrained in higher space-time dimensions. De Sitter brane-world models in higher dimensions appear to be less restrictive (more viable) also on a phenomenological ground. For instance, in dimensions \( D \geq 7 \), the mass gap or the mass of KK modes may be determined in terms of a free parameter \( M \) (associated with
the size of fifth dimension) rather than in terms of the 4D Hubble expansion parameter $H$. The latter is the situation in five dimensions, where the lightest KK mode will have mass $m = \sqrt{2H}$, which is not sufficiently massive in low energy or in the present Universe with $H \sim 10^{-46} M_{Pl}$. Moreover, in spacetime dimensions $D \geq 7$, the classical solutions (with or without a bulk cosmological constant) could lead to a spontaneous compactification of the extra dimensions.

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