Beyond the linear-order relativistic effect in galaxy clustering:
Second-order gauge-invariant formalism

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We present the second-order general relativistic description of the observed galaxy number density in a cosmological framework. The observed galaxy number density is affected by the volume and the source effects, both of which arise due to the mismatch between physical and observationally inferred quantities such as the redshift, the angular position, the volume, and the luminosity of the observed galaxies. These effects are computed to the second order in metric perturbations without choosing a gauge condition or adopting any restrictions on vector and tensor perturbations, extending the previous linear-order calculations. Paying particular attention to the second-order gauge transformation, we explicitly isolate unphysical gauge modes and construct second-order gauge-invariant variables. Moreover, by constructing second-order tetrads in the observer’s rest frame, we clarify the relation between the physical and the parametrized photon wavevectors. Our second-order relativistic description will provide an essential tool for going beyond the power spectrum in the era of precision measurements of galaxy clustering. We discuss potential applications and extensions of the second-order relativistic description of galaxy clustering.

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I. INTRODUCTION

Cosmology has seen its golden age, in particular due to the recent developments in the cosmic microwave background experiments such as the Wilkins Microwave Anisotropy Probe \cite{1} and Planck \cite{2} and in the large-scale galaxy surveys such as the Sloan Digital Sky Survey (SDSS; \cite{3}), the Two degree Field Galaxy Redshift Survey (2dFGRS; \cite{4}), and the Baryonic Oscillation Spectroscopic Survey (BOSS; \cite{5}). Furthermore, in order to exploit the enormous statistical power in three-dimensional volumes, a large number of galaxy surveys are planned to be operational in a near future such as Euclid, the Dark Energy Spectroscopic Instrument (DESI), the Large Synoptic Survey Telescope, and the Wide-Field InfraRed Survey Telescope (WFIRST), going progressively higher redshifts with larger sky coverage. These surveys will be able to deliver precise measurements of galaxy clustering on cosmological scales, in which alternative theories of modified gravity or dark energy models deviate from general relativity and in which the fingerprint of inflationary models remains intact. In particular, this is the regime, in which the standard Newtonian description of galaxy clustering breaks down, and therefore it is crucial to have a proper relativistic description to avoid misinterpretation of galaxy clustering measurements on large scales.

The standard Newtonian description of galaxy clustering is based on the assumption that the speed of light is infinite. However, the light we measure in galaxy surveys propagates throughout the Universe at a finite speed, and its path is affected, not only by the matter fluctuations, but also by the relativistic contributions such as the gravitational potential or the curvature of the Universe along its entire journey to reach us. Therefore, the relation between the physical quantities of source galaxies and the observable quantities in galaxy surveys is nontrivial, and it requires a proper relativistic treatment for solving the geodesic equation. Given the observed redshift and the observed galaxy position on the sky, the full relativistic formula of galaxy clustering can be derived \cite{6,7} by tracing the photon path backward in time and identifying the relation of these observable quantities to the physical quantities of source galaxies and the fluctuations that affect the photon path.

The relativistic formula provides the most accurate and complete description of galaxy clustering on large scales and clarifies the physical origin of all the effects in galaxy clustering \cite{3} such as the redshift-space distortion, the gravitational lensing, and the Sachs-Wolfe effect. It was shown \cite{3,9} that the relativistic effect in galaxy clustering is measurable in the current galaxy surveys and its detection significance can be greatly enhanced if the multi-tracer technique \cite{10,11} is employed, which altogether provides great opportunities to test general relativity and probe inflationary models on cosmological scales in upcoming galaxy surveys. Furthermore, the relativistic description of galaxy clustering has been independently verified in recent years \cite{12,13,14,15,16} and has received attention with various applications (e.g., see \cite{16,17,18}).
The relativistic description of galaxy clustering is computed so far to the linear order in metric perturbations. For Gaussian perturbations, the linear-order relativistic formula is all we need to describe galaxy clustering on large scales, where perturbations are linear. However, the Universe is far from being a complete Gaussian, even on large scales. For example, many inflationary models have extra degrees of freedom supplied by additional fields originating from the standard particle physics models or its supersymmetric extensions [25–27]. These new fields often couple to the inflaton field during the epoch of perturbation generation, and this nontrivial coupling leaves deviations from statistical isotropy in the two-point correlation function of curvature perturbation. Even in the absence of additional fields in the simplest inflationary model, tensor modes of gravitational waves can induce non-vanishing bispectrum in curvature perturbations [28].

Even on large scales the Universe deviates from perfect Gaussianity, and these inflationary models manifest themselves in the three-point correlation function or the bispectrum in Fourier space. Since these three-point statistics vanish at the linear order, the second-order relativistic effect is needed to compute the three-point statistics and thereby to extract additional information about non-standard inflationary models. Furthermore, this non-trivial coupling is intrinsically subtle second-order relativistic effect and affects not only the initial curvature power spectrum, but also the photon propagation, demanding consistent second-order relativistic treatments for its observable effects in galaxy clustering. In other words, while the second-order relativistic effect is subtle and small in comparison to the linear-order relativistic effect, it contains the distinct physical information about the early Universe. These arguments make a strong case for going beyond the linear-order relativistic effect in galaxy clustering. Here we develop the second-order general relativistic description of galaxy clustering, providing an essential tool in the era of precision cosmology.

Compared to the linear-order calculations, the second-order calculation is physically straightforward, albeit lengthy, but a few complications arise due to nonlinearity inherent in beyond-the-leading-order calculations. In particular, generation of vector and tensor perturbations is inevitable, and their contributions to the observable quantities need to be properly taken into consideration. The organization of this paper is as follows. The main result is presented in Sec. II where we construct the full second-order relativistic description of galaxy clustering. In Sec. II we provide the second-order gauge-invariant equations for those derived in Sec. II and we discuss the implications of our results in Sec. II The second-order gauge-invariant formalism is presented in Appendix A and the relation between the photon parametrization and the observed angle is clarified in Appendix B. Throughout the paper we use Greek indices for 3D spatial components and Latin indices for 4D spacetime components, respectively. Various symbols are summarized in Table 1.

Further in detail, the contents of Appendices and Sec. II are as follows. In Appendix A.1 we present the general metric representation of a FRW universe and its decomposition into scalar, vector, and tensor. In Appendix A.2 the second-order gauge transformation is derived and unphysical gauge mode is isolated. Finally, second-order gauge-invariant variables are constructed in Appendix A.3 In Appendix B.1 we construct the second-order tetrads, representing the observer’s rest frame in a FRW universe. In Appendix B.2 the photon wavevector is constructed by using local observable quantities, and the normalization of our photon parametrization is derived in Appendix B.3 In Appendix B.4 our choice for the normalization constant and its relation to the observed angle is clarified.

Section II further divides into ten subsections. The first subsection provides an overview of the calculation in Section II and explains the physical origin of all the contributions to the observed galaxy fluctuation. The parametrization of the photon wavevector and the geodesic equation is presented in Sec. II. The observed redshift and the distortion in the observed redshift are derived in Sec. II.C In Sec. II.D we briefly mention the relation of our photon parametrization to the observed angle. In Sec. II.E the position of the source galaxies is computed, and its deviation from the inferred position is derived. In terms of the distortion in the observed redshift and the inferred source position, we present the second-order deviations in the observed solid angle in Sec. II.F in the observed volume in Sec. II.G and in the luminosity distance in Sec. II.H Finally, the observed galaxy number density to the second-order is derived in Sec. II.I and its fluctuation is presented in Sec. II.J.

II. GALAXY CLUSTERING IN GENERAL RELATIVITY

A. Complete treatment of galaxy clustering: Overview of the calculations

Here we present a complete and unified treatment of galaxy clustering, providing an overview of the detailed calculations in Sec. II. This treatment unifies all the effects in galaxy clustering such as the redshift-space distortion, the gravitational lensing, the Sachs-Wolfe effect, and their relativistic effects into two physically distinct effects: the volume effect and the source effect 8. The volume effect describes the mismatch between the physical volume occupied by the observed source galaxies and the observationally inferred volume. The redshift-space distortion and the gravitational lensing convergence, for example, arise from the volume effect. The source effect describes the contributions associated with the physical properties of the source galaxy population. As the observationally inferred properties of the source galaxies are different from their physical properties, this mismatch gives rise to contributions to galaxy clustering. Magnification bias is one example of the source effect.

In general, statistics in galaxy clustering are derived based on the observed galaxy number density n_o^obs (or the observed galaxy fluctuation δ^obs), and the observed galaxy number density is further constructed using the basic observable quantities
in galaxy surveys: the observed redshift $z$, the observed angular position $\hat{n} = (\theta, \phi)$ of the source galaxy, and the number of galaxies $dN_{g}^{\text{obs}}$ counted within the observed redshift and solid angle. The observationally inferred volume $dV_{\text{obs}}$ occupied by the observed source galaxies is

$$dV_{\text{obs}} = \frac{\bar{r}^2(z)}{H(z)(1+z)^3} \sin \theta \, dz \, d\theta \, d\phi,$$

where $\bar{r}$ is the comoving line-of-sight distance and $H$ is the Hubble parameter, and the observed galaxy number density is then obtained as

$$n_{g}^{\text{obs}}(z, \hat{n}) = \frac{dN_{g}^{\text{obs}}(z, \hat{n})}{dV_{\text{obs}}}.$$

Considering that the observed galaxies have the physical galaxy number density $n_{g}$ and occupy the physical volume $dV_{\text{phy}}$, we can related the observed galaxy number density to those quantities as

$$dN_{g}^{\text{obs}} = n_{g}^{\text{obs}} dV_{\text{obs}} = n_{g} dV_{\text{phy}}, \quad n_{g}^{\text{obs}} = n_{g} (1 + \delta V), \quad \frac{dV_{\text{obs}}}{dV_{\text{phys}}} = 1 + \delta V,$$

where we defined the volume distortion $\delta V$. Regardless of which the source galaxy population $n_{g}$ is used, the volume distortion $\delta V$ will always contribute to galaxy clustering, and its contribution is collectively described as the volume effect. In the following subsections, we will compute the physical volume $dV_{\text{phy}}$ (and hence $\delta V$) and discuss the physical effects that contribute to the volume distortion.

The observed galaxies are grouped as galaxy samples based on various observable quantities such as the rest-frame luminosity, the spectral color, and so on. However, this classification is also based on the observationally inferred quantities, and they differ from the physical quantities of the source galaxy population. Therefore, when the source galaxy population is expressed in terms of observable quantities, this mismatch gives rise to contributions to the observed galaxy number density $n_{g}^{\text{phy}}$ as for the volume distortion $\delta V$. However, this source effect depends on which observable quantities are used to define the galaxy sample and how the physical galaxy number density $n_{g}$ representing the galaxy sample depends on the observable quantities. Therefore, we will consider only the rest-frame luminosity, the most frequently used quantity in galaxy surveys, to illustrate the source effect in Sec. II. In contrast, the time coordinate (or the distance from us) of the observed galaxy sample is computed based on the observed redshift, and hence the time evolution of the physical galaxy number density will always contribute to galaxy clustering as one of the source effect (see also Sec. II).

### B. Photon geodesic equation

The photon path is described by a null geodesic $x^{a}(v)$ with an affine parameter $v$, and its propagation direction is then $k^{a}(v) = dx^{a}/dv$ subject to the null condition $k^{a}k_{a} = 0$ and the geodesic equation $k^{a}a_{b}k^{b} = 0$. We choose the affine parameter $v$ such that the photon frequency measured in the rest frame of an observer with four velocity $u^{a}$ is

$$2\pi \nu = -g_{ab}k^{a}(v)u^{b}(v),$$

where the four velocity of the observer is normalized as $u^{a}u_{a} = -1$ (see Appendix A for our notation convention). Once the affine parameter is fixed in terms of physical quantities, the photon wavevector $k^{a}(v)$ is completely set without any further degrees of freedom. Since null geodesics are conformally invariant, we simplify the photon propagation equations by considering a conformal transformation $ds^2 = g_{ab}x^{a}x^{b} = a^{2}g_{ab}dx^{a}dx^{b}$, where the expansion factor $a$ is removed in a conformally transformed metric $\hat{g}_{ab}$. Under the conformal transformation, the null geodesic $x^{a}(v)$ remains unaffected, but its affine parameter is transformed to another affine parameter $\lambda$ [29]:

$$\frac{dv}{d\lambda} = C \alpha^{2}, \quad \hat{k}^{a} = \frac{dx^{a}}{d\lambda} = C \alpha^{2}k^{a}, \quad \hat{u}^{a} = au^{a},$$

where the unspecified proportionality constant $C$ represents arbitrariness or additional degree of freedom in the conformally transformed affine parameter.

The conformally transformed wavevector $\hat{k}^{a}$ still satisfies the same null condition $\hat{k}^{a}\hat{k}_{a} = 0$ and the geodesic equation $\delta_{ac}\hat{k}^{c} = 0$. While the physical wavevector $k^{a}$ and its affine parameter $v$ are completely fixed, we have additional freedom to choose its normalization and affine parameter $\lambda$ in the conformally transformed metric. With this freedom, we parametrize the photon wavevector as (see Appendix B)

$$\hat{k}^{a}(\lambda) = \left( \frac{d\tau}{d\lambda}, \frac{dx^{a}}{d\lambda} \right) = \left[ 1 + \delta \nu, - (e^{\alpha} + \delta e^{\alpha}) \right],$$
where $e^\alpha$ based on $\hat{g}_{\alpha\beta}$ is the photon propagation direction normalized as $e^\alpha e_\alpha = 1$ and the dimensionless quantities $\delta \nu$ and $\delta e^\alpha$ represent perturbations to the photon wavevector. These perturbation variables are defined in a non-perturbative way, such that they contain higher order perturbations, e.g., $\delta \nu = \delta \nu^{(1)} + \delta \nu^{(2)} + \cdots$ and by construction $\langle \delta \nu \rangle = 0$ to all orders in perturbation. Since the photon path is parametrized by the affine parameter $\lambda$, we have

$$
\frac{d}{d\lambda} = \frac{dx^\alpha}{d\lambda} \frac{\partial}{\partial x^\alpha} = \hat{k}_\alpha \partial_\alpha = (\partial_\tau - e^\alpha \partial_\alpha) + \delta \nu \partial_\tau - \delta e^\alpha \partial_\alpha,
$$

(7)

where the background relation in the round bracket simply represents that the photon propagation path is a straight line to the zeroth order. However, to the second order in perturbations, we need to consider the evolution of perturbations along the photon path deviating from the straight line.

Using the conformally transformed metric, the photon wavevector can be written to the second order in perturbations as

$$
\hat{k}_0 = \hat{g}_{0a} k^a = -(1 + \delta \nu + 2 A - B_\alpha e^\alpha + 2 A \delta \nu - B_\alpha \delta e^\alpha),
$$

$$
\hat{k}_\alpha = \hat{g}_{ab} k^b = -(e_\alpha + \delta e_\alpha + B_\alpha + 2 C_{\alpha \beta} e^\beta + \delta \nu B_\alpha + 2 C_{\alpha \beta} \delta e^\beta),
$$

(8)

and the null equation is then

$$
0 = \hat{k}_\alpha \hat{k}_\alpha = (e^\alpha e_\alpha - 1) + 2 \left(e^\alpha \delta e_\alpha - \delta \nu - A + B_\alpha + \alpha B_\alpha + 2 \mathcal{U}^\beta C_{\alpha \beta}\right) \delta \nu + 2 \left(A - B e^\alpha\right) \delta \nu + 2 \left(2 A \delta \nu - B_\alpha \delta e^\alpha\right) \delta \nu + \left(2 B_\alpha \delta \nu + 2 C_{\alpha \beta} \delta e^\beta\right) \delta e^\alpha,
$$

(9)

where the metric perturbations are defined in Appendix A. The background relation is trivially satisfied by the construction of the unit direction vector $e^\alpha$. Defining the perturbation to the observer four velocity, we derive the four velocity vector of the observer from the normalization condition $u^\alpha u_\alpha = -1$.

$$
u^\alpha \equiv \frac{\mathcal{U}^\alpha}{a}, \quad u^\alpha = \frac{1}{a} \left[1 + \frac{3}{2} \mathcal{U}^\alpha \mathcal{U}_\alpha - \mathcal{U}^\alpha B_\alpha\right] = 1 + \frac{\delta \nu^\alpha}{a},
$$

(10)

and by construction $\mathcal{U}^\alpha B_\alpha = 0$, the background relation for the temporal component of the geodesic equation is

$$
0 = \partial_\tau + 2 \partial_\alpha \mathcal{U}^\alpha = 0 \mathcal{U}^\tau + 2 \frac{\delta \nu}{a} \mathcal{U}^\alpha + \partial_\alpha \mathcal{U}^\alpha + \partial_\alpha \mathcal{U}^\alpha = 0, \quad \frac{d}{d\lambda} \delta \nu = -\delta \Gamma^\alpha_\alpha,
$$

(11)

where $\mathcal{U}^\alpha B_\alpha$ is the Christoffel symbol based on $\hat{g}_{ab}$, the background relation for the temporal component is already removed by the construction of the conformally transformed wavevector in Eq. (6), and we defined

$$
\delta \Gamma^\alpha_0 = \hat{g}^\alpha_0 \hat{k}^b \hat{k}^c - \mathcal{A}' = 2 A e^\alpha (B_{\alpha \beta} + C_{\alpha \beta}) e^\beta + 2 \delta \nu (A' - A e^\alpha) - 2 A A' - A e^\alpha + 2 \mathcal{U}^\beta C_{\alpha \beta} = (2 \mathcal{U}^\beta C_{\alpha \beta}, + \mathcal{B}^\gamma C_{\alpha \beta}) \delta e^\alpha,
$$

(12)

Similarly, the spatial component of the geodesic equation is

$$
0 = \partial_0 \mathcal{U}^\beta + \partial_\alpha \mathcal{U}^\alpha = 0 \mathcal{U}^\gamma + \hat{\Gamma}^\alpha_0 \hat{k}_0 \mathcal{U}^\alpha = 0 \mathcal{U}^\gamma + \hat{\Gamma}^\alpha_0 \hat{k}_0 \mathcal{U}^\alpha
$$

(13)

where we defined a perturbation quantity in a similar way as

$$
\delta \Gamma^\alpha_\beta = \hat{\Gamma}^\alpha_0 \hat{k}^c - \hat{\Gamma}^\alpha_0 \hat{k}^c = A e^\alpha - B \mathcal{U}^\alpha - \left(B_{\beta \gamma} - B_{\alpha \gamma} + 2 C_{\beta \gamma}\right) e^\beta + \left(2 C_{\beta \gamma} - C_{\beta \gamma}\right) e^\beta + \hat{\Gamma}^\alpha_0 \hat{k}_0 \mathcal{U}^\alpha = \delta \Gamma^\alpha_\beta,
$$

(14)

The spatial component of the geodesic equation indicates that the background photon propagation direction $e^\alpha$ is constant. Another useful quantity is

$$
\hat{k}_\alpha \hat{u}_\alpha = - \left[1 + \delta \nu + \mathcal{U}^\alpha (B_{\alpha \beta} - A e^\alpha) e^\alpha - \frac{1}{2} \mathcal{U}^2 \mathcal{U}_\alpha + \delta \nu \mathcal{U}^\alpha + (2 \mathcal{U}^\beta C_{\alpha \beta}) e^\alpha + (\mathcal{U}^\alpha - B_{\alpha \beta}) \delta e^\alpha\right],
$$

(15)

and for later reference we define the above quantity as $\hat{k}^\alpha \hat{u}_\alpha = -(1 + \Delta \nu)$. 


C. Observed redshift

The zero-th order photon path can be obtained by integrating the photon wave vector as a function of the affine parameter \( \lambda \) as

\[
\vec{x}^a(\lambda_s) - \vec{x}^a(\lambda_o) = \left[ \vec{r}_s - \vec{r}_o, \vec{x}^a_s \right] = \left[ \lambda_s - \lambda_o, (\lambda_o - \lambda_s)e^\alpha \right],
\]

where we set \( \vec{x}^a(\lambda_o) = \vec{x}^a_o = 0 \) and let \( \vec{x}^a_s = \vec{x}^a(\lambda_s) \). Without loss of generality (\( \lambda \rightarrow \lambda + \text{constant} \)), we set \( \lambda_o = 0 \) hereafter. Therefore, we have the defining relation between the affine parameter and the line-of-sight distance

\[
\lambda = \bar{\tau} - \bar{\tau}_o = -\int_0^z \frac{dz'}{H(z')} , \quad \bar{\tau}_o = \int_0^\infty \frac{dz}{H(z)} ,
\]

where \( H(z) = \dot{a}/a \) is the Hubble parameter and \( z \) is the redshift parameter corresponding to the conformal time \( \bar{\tau} \). Given a redshift parameter \( z \), we denote the affine parameter \( \lambda_z \), satisfying

\[
1 + z = \frac{1}{a(\bar{\tau}_z)} , \quad \lambda_z = \bar{\tau}_z - \bar{\tau}_o .
\]

In an inhomogeneous universe, the positions \( x^a_\lambda = x^a(\lambda) \) of the photon source and the observer given the affine parameters (or the redshift parameter) deviate from the positions \( \vec{x}^a_\lambda = \vec{x}^a(\lambda) \) in a homogeneous universe:

\[
x^a_\lambda = \vec{x}^a_\lambda + \delta x^a_\lambda , \quad \tau_\lambda = \bar{\tau}_\lambda + \delta \tau_\lambda .
\]

Photons emitted from \( x^a_o \) are received by the observer at \( x^a_o \), and the observed redshift \( z \) is the ratio of the photon wavelengths at source and observer

\[
1 + z = \frac{(k^a u^o_a)_s}{(k^a u^o_a)_o} = \frac{1 + \delta z}{a(\bar{\tau}_s)} , \quad (18)
\]

where we defined the distortion \( \delta z \) in the observed redshift. Compared to Eq. (18) in a homogeneous universe, the observed redshift in Eq. (20) is affected not only by the expansion of the Universe, but also by the perturbations such as peculiar velocities of the source and the observer. Using Eqs. (5) and (6), we derive

\[
\delta z = \delta a_o + \left[ \delta \nu + A + (U_\alpha - B_\alpha) e^\alpha + \delta \nu A + \delta e^\alpha (U_\alpha - B_\alpha) + (A B_\alpha + 2 C_{\alpha \beta} U^{\beta}) e^\alpha + \frac{1}{2} U^\alpha U_\alpha - \frac{1}{2} A^2 \right]_o ,
\]

where the brackets with superscript \( s \) and subscript \( o \) represents a difference of the quantities at the source position \( x^o(\lambda_s) \) and the observer position \( x^o(0) \) and the bracket with only subscript \( o \) represents the quantity is evaluated at the observer position. In deriving Eq. (21) we account for the fact that the observer position deviates from that in a homogeneous universe

\[
a(\tau_o) = a[\bar{\tau}(0) + \delta \tau(0)] = 1 + \mathcal{H}_o \delta \tau_o + \frac{1}{2} \left( \mathcal{H}_o^2 + \mathcal{H}'_o \right) \delta \tau_o^2 \equiv 1 + \delta a_o ,
\]

where the conformal Hubble parameter is \( \mathcal{H} = aH \), while the spatial position can be always set zero \( x^o_o = \bar{x}^o_o = 0 \) due to symmetry in a homogeneous universe.

In Eq. (21), perturbation quantities are evaluated at the source position \( x^o_s \), which is close but not exactly at the observed redshift \( z \), i.e., \( \bar{x}^a(\lambda_z) \). To facilitate further calculations, we define a perturbation \( \Delta \lambda_s \) in the affine parameter \( \lambda_s \) as \( \lambda_s = \lambda_z + \Delta \lambda_s \), where \( \lambda_z \) satisfies the relation in Eq. (18). To the second order in perturbations, the source position can be rephrased as

\[
\tau_s = \tau(\lambda_z + \Delta \lambda_s) = \bar{\tau}(\lambda_z + \Delta \lambda_s) + \delta \tau(\lambda_z + \Delta \lambda_s) = \bar{\tau}_z + \Delta \lambda_s + \delta \tau_z + \delta \nu \zeta \Delta \lambda_s = \bar{\tau}_z + \Delta \tau_z ,
\]

\[
x^o_s = x^o(\lambda_z + \Delta \lambda_s) = \bar{x}^o(\lambda_z + \Delta \lambda_s) + \delta x^o(\lambda_z + \Delta \lambda_s) = \bar{x}^o_z + e^\alpha \Delta \lambda_s + \delta \nu \zeta \Delta \lambda_s - \frac{1}{2} \Gamma^\alpha_{\beta \gamma} e^\beta \gamma \Delta \lambda_s^2 - \delta e^\alpha \zeta \Delta \lambda_s
\]

\[
\equiv \bar{x}^o_z + \Delta x^o_z ,
\]

where the subscript \( z \) indicates that quantities are evaluated at the affine parameter \( \lambda_z \) and \( \Gamma^\alpha_{\beta \gamma} \) is the Christoffel symbol based on \( g_{\alpha \beta} \). For the deviation of the source position in an inhomogeneous universe, we will need only the first order terms in
\[ \Delta x^a_z = (\Delta \tau_z, \Delta x^a_z), \] and we will compute it in detail in Sec. [III]. The distortion in the observed redshift is, therefore,

\[
\delta z^{(1)} = \mathcal{H}_o \tau_\alpha + \left[ \delta \nu + A + (\mathcal{U}_\alpha - B_\alpha) e^\alpha \right]
\]

\[
\delta z^{(2)} = \delta a_o + \left[ \delta \nu + A + (\mathcal{U}_\alpha - B_\alpha) e^\alpha + \delta \nu A + \delta e^\alpha (\mathcal{U}_\alpha - B_\alpha) + \left( AB_\alpha + 2C_{\alpha\beta} \mathcal{U}^\beta \right) e^\alpha + \frac{1}{2} \mathcal{U}^\alpha \mathcal{U}_\alpha - \frac{1}{2} A^2 \right]
\]

\[
+ \left[ \delta a - \delta \nu - A - (\mathcal{U}_\alpha - B_\alpha) e^\alpha \right] \left[ \delta \nu + A + (\mathcal{U}_\alpha - B_\alpha) e^\alpha \right] + \Delta x^a_z \left[ \delta \nu + A + (\mathcal{U}_\alpha - B_\alpha) e^\alpha \right]_b ,
\]  

where we omitted the superscripts of the perturbation orders for simplicity. Consistently to the second order in perturbations, first-order and second-order perturbation quantities in Eqs. (25) and (26) at the source position can be evaluated at the observed redshift, while the former results in additional second-order contributions due to the first-order deviation of the source position from the observed redshift. Similar calculations can be found in [30, 31].

So far, we left unspecified the perturbation \( \Delta \lambda_o \) in the affine parameter. Using the defining relation in Eq. (18), the observed redshift in Eq. (20) can be written as

\[
1 + z = \frac{1}{a(\tau_z)} = \frac{1 + \delta z}{a(\tau(\lambda_z + \Delta \lambda_o))},
\]

and note that the observed redshift is independent of how we label the source position using the affine parameter. Substituting Eq. (23) into Eq. (27) yields that the perturbation \( \Delta \lambda_o \) in the affine parameter satisfies

\[
\mathcal{H}_z \Delta \tau_z + \frac{1}{2} \left( \mathcal{H}_z^2 + \mathcal{H}_o' \right) (\Delta \lambda_o + \Delta \tau_z)^2 = \delta \tau_z ,
\]

and we derive

\[
\Delta \lambda^{(1)}_o = - \delta \tau^{(1)} - \frac{\delta \tau^{(1)}}{\mathcal{H}_z} , \quad \Delta \lambda^{(2)}_o = - \delta \tau^{(2)} - \delta \nu_z \left( - \delta \tau_z + \frac{\delta z}{\mathcal{H}_z} \right) - \frac{1}{2 \mathcal{H}_z} \left( \mathcal{H}_z^2 + \mathcal{H}_o' \right) \delta z^2 + \frac{\delta z^{(2)}}{\mathcal{H}_z} ,
\]

where the perturbation quantities in quadratic form are evaluated at the linear order and \( \mathcal{H}_z = \mathcal{H}(z) \).

**D. Observed angle of source galaxies**

The observed source position in the sky is described by the observed angle \( \hat{n} = (\theta, \phi) \) in the local observer frame. In a homogeneous universe, it is identical to the unit directional vector \( e^\alpha \). However, the observer frame is moving in an inhomogeneous universe, and these two unit directional vectors are different, simply because of the change of frame. Therefore, it is necessary to express the source galaxy position, not only in terms of the observed redshift \( z \), but also in terms of the observed angle \( (\theta, \phi) \).

In Appendix [B-4] we explicitly derive the photon wavevector \( k^\alpha \) in the FRW frame by transforming the observed photon wavevector in the observer’s rest frame. The photon wavevector is completely set by local observables quantities such as the photon frequency \( \nu \) and the angle \( (\theta, \phi) \). However, with additional degree of freedom \( C \) in the conformally transformed wavevector in Eq. (5), we can choose the normalization of the photon wavevector \( k^\alpha \) to simplify the calculations by aligning the two unit directional vectors \( n^\alpha = e^\alpha \). Though the choice has no impact on the description of observable quantities, other choice would make the calculation significantly complicated. The detailed calculations are presented in Appendix [B].

**E. Distortions in photon path**

Having computed the observed redshift and the observed angle, we now express the source position in terms of metric perturbations. In the presence of perturbations in an inhomogeneous universe, the photon path at the affine parameter \( \lambda \) is distorted as

\[
x^a_\lambda = x^a_o = [\tau_\lambda - \tau_o , x^a_\lambda] = \left[ \lambda + \int_0^\lambda d\lambda' \delta \nu , - \lambda e^\alpha - \int_0^\lambda d\lambda' \delta e^\alpha \right],
\]
and the deviation of the position from that in a homogeneous universe is

$$\delta x^\alpha = x^\alpha_\lambda - x^\alpha_\lambda = [\delta \tau_\lambda, \delta x^\alpha_\lambda] = \left[ \delta \tau_\nu, 0 \right] + \left[ \int_0^\lambda d\lambda' \delta \nu, -\int_0^\lambda d\lambda' \delta e^\alpha \right]$$,

(31)

where the integration over the affine parameter (d\lambda) represents the integration along the photon path x^\alpha(\lambda), not necessarily along the straight line x^\alpha(\lambda). Note that the affine parameter is defined as a parameter without resort to whether we consider homogeneous or inhomogeneous universes. Using the geodesic equations in (11) and (13), we derive the perturbations in the photon wavevector as

$$\delta \nu_\lambda - \delta \nu_o = - \int_0^\lambda d\lambda' \delta \Gamma^0$$,

$$\delta e^\alpha_\lambda - \delta e^\alpha_o = \int_0^\lambda d\lambda' \delta \Gamma^\alpha$$,

(32)

$$\delta \nu^{(1)}_o = -2 (A_z - A_o) - \int_{\bar{r}z}^{r_\lambda} d\bar{r} \left[ A' \left( - (B_{\alpha|\beta} + C'_{\alpha|\beta}) e^\alpha e^\beta \right) \right]$$,

(33)

$$\delta e^{\alpha(1)}_o = - \left[ B^\alpha + 2C^\beta_\alpha e^\beta \right] \int_{\bar{r}z}^{r_\lambda} d\bar{r} \left( A' \left( - (B_{\alpha|\beta} + C'_{\alpha|\beta}) e^\alpha e^\beta - 2 \delta \nu A_\alpha e^\alpha + 2 A A' \right) \right.$$,

$$+ A_{\alpha|\beta} B^\alpha - 2 \left( 2 A A_{\alpha|\beta} + B_{\beta|\alpha} B'_{\alpha|\beta} + B^\beta B_{[\alpha|\beta]} \right) e^\alpha + \delta e^\alpha \left( A_{\alpha|\beta} + C'_{\alpha|\beta} \right) e^\beta$$

$$\left. + \left[ A \left( 2B_{\alpha|\beta} + C'_{\alpha|\beta} \right) B + B_{\beta|\alpha} \left[ 2C^\gamma_{\alpha|\beta} \right] \right] e^\alpha e^\beta + \Delta x^\alpha \left[ A' \left( - (B_{\alpha|\beta} + C'_{\alpha|\beta}) e^\beta \right) \right. \right.$$

$$- 2 \left( A A_{\alpha|\beta} + C^\alpha_{\alpha|\beta} \right) e^\beta$$

$$\left. + \delta \nu B^\beta e^\beta \right) + 2 \left( C^\alpha_{\alpha|\beta} \right) e^\beta \delta \gamma + A' B^\alpha - 2 A_{\beta|\alpha} C_{\alpha|\beta} + 2 C^\beta_\beta \beta$$

$$- 2 B^\alpha A_{\beta|e^\beta} + 4 C^\alpha_{\beta|\gamma} (B_{[\beta|\gamma]} + C'_{[\beta|\gamma]}) e^\beta - \left[ 2 C^\beta_\beta \left( 2C^\gamma_{\beta|\gamma} - C_{\beta|\gamma} \right) \right] - B^\beta \left( B_{[\beta|\gamma]} + C'_{[\beta|\gamma]} \right) e^\beta e^\gamma$$

$$\left. + \Delta x^d \left( A_{\beta|e^\beta} - B_{\beta|e^\beta} - C_{\beta|\gamma|e^\gamma} \right) \right)$$,

(36)

where we used the total derivative with respect to the affine parameter along the photon path in Eq. (7) for simplification. The photon path in Eq. (30) can be further related to the integration over the metric perturbations defined in Eqs. (11) and (13) as

$$x^\alpha_\lambda = x^\alpha_\lambda = \left[ (1 + \delta \nu_o) \lambda - \int_0^\lambda d\lambda' (\lambda - \lambda') \delta \Gamma^0, - \lambda (e^\alpha + \delta e^\alpha_o) - \int_0^\lambda d\lambda' (\lambda - \lambda') \delta \Gamma^\alpha \right]$$.

(37)

Noting that the source position is parametrized by \lambda_s = \lambda + \Delta \lambda_s, we have the source position

$$\bar{x}^\alpha = \left[ \lambda + \tau_s, -\lambda e^\alpha \right] = [\bar{r}_z, \bar{r}_z e^\alpha]$$,

(38)

$$x^{(1)}_s = \left[ \delta \tau_0 + \Delta \lambda_s - \int_{\bar{r}z}^{r_\lambda} d\bar{r} \delta \nu, -\Delta \lambda_s e^\alpha + \int_{\bar{r}z}^{r_\lambda} d\bar{r} \delta e^\alpha \right]$$

$$= \left[ \delta \tau_0 + \lambda \delta \nu_0 + \Delta \lambda_s - \int_{\bar{r}z}^{r_\lambda} d\bar{r} \left( \bar{r}_z - \bar{r} \right) \delta \Gamma^0, \bar{r}_z \delta e^\alpha_o - \Delta \lambda_s e^\alpha - \int_{\bar{r}z}^{r_\lambda} d\bar{r} \left( \bar{r}_z - \bar{r} \right) \delta \Gamma^\alpha \right]$$,

(39)

$$\tau^{(2)}_s = \delta \tau_o + \Delta \lambda_s (1 + \delta \nu_o) - \int_{\bar{r}z}^{r_\lambda} d\bar{r} \left( \delta \nu + \Delta x^\alpha \delta \nu_o \right)$$

$$= \delta \tau_o + \lambda \delta \nu_o + \Delta \lambda_s (1 + \delta \nu_o) - \int_{\bar{r}z}^{r_\lambda} d\bar{r} \left( \bar{r}_z - \bar{r} \right) \left( \delta \Gamma^0 + \Delta x^\alpha \delta \Gamma^\alpha_o \right)$$,

(40)

$$x^{(2)}_s = -\Delta \lambda_s (e^\alpha + \delta e^\alpha_o) + \int_{\bar{r}z}^{r_\lambda} d\bar{r} \left( \delta e^\alpha + \Delta x^b \delta e^\alpha_o \right)$$

$$= \bar{r}_z \delta e^\alpha_o - \Delta \lambda_s (e^\alpha + \delta e^\alpha_o) - \int_{\bar{r}z}^{r_\lambda} d\bar{r} \left( \bar{r}_z - \bar{r} \right) \left( \delta \Gamma^\alpha + \Delta x^b \delta \Gamma^\alpha_o \right)$$,

(41)
where the line-of-sight integration here represents the integration over the unperturbed photon path $d\hat{r}$.

Since the observers identify the source position by measuring the observed redshift $z$ and the observed angle $(\theta, \phi)$, the inferred source position is in rectangular coordinates

$$\bar{x}^\alpha_s = \left[ \bar{r}_z, \bar{\hat{r}}_z \hat{n} \right] = \left[ \bar{r}_z, \bar{r}_z \sin \theta \cos \phi, \bar{r}_z \sin \theta \sin \phi, \bar{r}_z \cos \theta \right],$$

where $\bar{r}_z = \hat{r}(z) = \bar{r}_\alpha - \bar{r}_z$, $\hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ is a unit directional vector based on the observed angle $(\theta, \phi)$ of the source and $\bar{r}_z$ is the conformal time defined in Eq. [13]. Note that in general $\bar{x}^\alpha_s \neq \hat{x}^\alpha_s$ because of the difference between $e^\alpha$ and $n^\alpha$, but with our choice of normalization constant we have $\bar{x}^\alpha_s = \hat{x}^\alpha_s$. Given the source position in Eqs. (38)–(41), we define the distortion $(\delta T, \delta r, \delta \theta, \delta \phi)$ of the source position $x^\alpha_s$ with respect to the inferred source position $\bar{x}^\alpha_s$ by

$$x^\alpha_s \equiv \left[ \bar{r}_z + \delta T, (\bar{r}_z + \delta r) \sin(\theta + \delta \theta) \cos(\phi + \delta \phi), (\bar{r}_z + \delta r) \sin(\theta + \delta \theta) \sin(\phi + \delta \phi), (\bar{r}_z + \delta r) \cos(\theta + \delta \theta) \right],$$

where the deviation in the conformal time of the source position $\delta T \equiv \bar{r}_\alpha - \bar{r}_z = \Delta \bar{r}_z$ is different from $\delta r$. Since the source position $x^\alpha_s$ is unobservable, these deviations from the inferred source position $\bar{x}^\alpha_s$ are gauge-dependent [7]. While Eqs. (38)–(41) are valid in general coordinates, it is most convenient to evaluate the distortions in rectangular coordinates.

Constructing two additional unit directional vectors $\hat{\theta} = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta)$ and $\hat{\phi} = (-\sin \phi, \cos \phi, 0)$ based on the observed angle, the distortions of the source position in spherical coordinates are to the linear order in perturbations

$$\delta r^{(1)} = (n_\alpha x^\alpha_s)^{(1)}(1) = -\Delta \lambda_s + \int_0^{\bar{r}_z} d\bar{r} \ e_\alpha \delta e^\alpha = \delta r_{\alpha} - \frac{\delta z}{\bar{H}_z} - \int_0^{\bar{r}_z} d\bar{r} \left( A - B_\alpha e^\alpha - C_{\alpha \beta} e^\beta e^\beta \right),$$

$$\bar{r}_z \delta \theta^{(1)} = (n_\alpha x^\alpha_s)^{(1)}(2) = \bar{r}_z e_{\theta \alpha} \delta e^\alpha - \int_0^{\bar{r}_z} d\bar{r} \left( \hat{r}_z - \bar{r} \right) e_{\theta \alpha} \delta \Gamma^\alpha,$n

$$\bar{r}_z \sin \theta \delta \phi^{(1)} = (n_\alpha x^\alpha_s)^{(1)}(3) = \bar{r}_z e_{\phi \alpha} \delta e^\alpha - \int_0^{\bar{r}_z} d\bar{r} \left( \hat{r}_z - \bar{r} \right) e_{\phi \alpha} \delta \Gamma^\alpha.$n

and to the second order in perturbations

$$\delta r^{(2)} = (n_\alpha x^\alpha_s)^{(2)} + \frac{1}{2 \bar{r}_z} \left[ (\theta_\alpha x^\alpha_s)^2 + (\phi_\alpha x^\alpha_s)^2 \right],$$

$$\bar{r}_z \delta \theta^{(2)} = (n_\alpha x^\alpha_s)^{(2)} - \frac{\left( n_\alpha x^\alpha_s \right)^2}{\bar{r}_z} + \cot \theta \frac{\theta_\alpha x^\alpha_s)^2}{\bar{r}_z},$$

$$\bar{r}_z \sin \theta \delta \phi^{(2)} = (n_\alpha x^\alpha_s)^{(2)} - \frac{\left( n_\alpha x^\alpha_s \right)^2}{\bar{r}_z} - \cot \theta \frac{\phi_\alpha x^\alpha_s)^2}{\bar{r}_z},$$

where the quadratic terms are at the first order and the remaining second-order pieces are

$$(n_\alpha x^\alpha_s)^{(2)} = -\Delta \lambda_s - \Delta \lambda_s e_\alpha \delta e^\alpha + \int_0^{\bar{r}_z} d\bar{r} \left[ e_\alpha \delta e^\alpha + e_\alpha \Delta x^b \delta e^\alpha_b \right],$$

$$= \delta r^{(2)} + \left( \delta r_s - \frac{\delta z}{\bar{H}_z} \right) \left( A - B_\alpha e^\alpha - C_{\alpha \beta} e^\beta e^\beta \right) + \frac{1}{2 \bar{H}_z^2} \left( \bar{H}_z^2 + \bar{H}_s^2 \right) \delta z^2 - \frac{\delta z^{(2)}}{\bar{H}_z},$$

$$+ \int_0^{\bar{r}_z} d\bar{r} \left[ \left( A - B_\alpha e^\alpha - C_{\alpha \beta} e^\beta e^\beta \right)^{(2)} + \frac{1}{2} \delta \nu \left( -2 A - B_\alpha e^\alpha \right) e_\alpha + \frac{1}{2} \delta e_\alpha + (B_\alpha + 2 C_{\alpha \beta} e^\beta) \right) \delta e^\alpha,$n

$$+ \Delta x^c \left( A - B_\alpha e^\alpha - C_{\alpha \beta} e^\beta e^\beta \right)_c,$n

$$(\theta_\alpha x^\alpha_s)^{(2)} = \bar{r}_z e_{\theta \alpha} \delta e^\alpha - \Delta \lambda_s e_{\theta \alpha} \delta e^\alpha - \int_0^{\bar{r}_z} d\bar{r} \left[ \bar{r}_z - \bar{r} \right] e_{\theta \alpha} \left( \delta \Gamma^\alpha + \Delta x^b \delta \Gamma^\alpha_b \right)$$

1 While the source position is on the past light cone, the separation of coordinates and metric perturbations is arbitrary and gauge-dependent. Furthermore, $\delta r$ only represents the radial displacement.
The distortions of the source position are decomposed as the radial and the angular displacements. Both of them arise due to the metric perturbations along the photon path, and the identification of the source at the observed redshift contributes to the radial displacement.

**F. Lensing magnification**

The distortion in the solid angle $d\Omega$ at the observed $(\theta, \phi)$ and the (unobserved) source $(\theta + \delta \theta, \phi + \delta \phi)$ is described by the deformation matrix $D$ (inverse of the magnification matrix), and it is conventionally decomposed as

$$D = \frac{\partial(\theta + \delta \theta, \phi + \delta \phi)}{\partial(\theta, \phi)} \equiv I - \begin{pmatrix} \kappa & 0 \\ 0 & \kappa \end{pmatrix} - \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} - \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix},$$

where $\kappa$ is the gravitational lensing convergence, $\omega$ is the rotation, and $(\gamma_1, \gamma_2)$ is the shear (e.g., see [32-34] for reviews). The ratio of the solid angles is the Jacobian of the angular transformation or the determinant of the deformation matrix:

$$\det D = 1 - 2\kappa + \kappa^2 - \gamma^2 + \omega^2 \equiv 1 - 2\kappa^{(1)} + \delta \Omega^{(2)},$$

where we defined the second-order part $\delta \Omega$ of the determinant. Note that the first-order term is simply the gravitational lensing convergence, and we only need the determinant term, not the individual components of shear and rotation. To the second order in perturbations, the determinant of the deformation matrix is

$$\det D = \frac{\sin(\theta + \delta \theta)}{\sin \theta} \left[ 1 + \frac{\partial}{\partial \theta} \delta \theta + \frac{\partial}{\partial \phi} \delta \phi + \frac{\partial}{\partial \phi} \delta \theta + \frac{\partial}{\partial \theta} \delta \phi - \frac{\partial}{\partial \theta} \delta \phi - \frac{\partial}{\partial \phi} \theta \right],$$

yielding the relation

$$\kappa^{(1)} = -\frac{1}{2} \left[ \left( \cot \theta + \frac{\partial}{\partial \phi} \right) \delta \theta + \frac{\partial}{\partial \phi} \delta \phi \right],$$

$$\delta \Omega^{(2)} = \left( \cot \theta + \frac{\partial}{\partial \phi} \right) \delta \theta + \frac{\partial}{\partial \phi} \delta \phi + \frac{\partial}{\partial \phi} \delta \theta + \frac{\partial}{\partial \phi} \delta \phi - \frac{\partial}{\partial \phi} \delta \phi - \frac{\partial}{\partial \phi} \theta - \frac{1}{2} \delta \theta^2 + \cot \theta \delta \theta \left( \frac{\partial}{\partial \phi} \delta \theta + \frac{\partial}{\partial \phi} \delta \phi \right).$$

The second-order calculations of the gravitational lensing and shear can be found in Bernardeau et al. [35, 36]. These expressions can be further related to the metric perturbations by using the distortions in the photon path computed in Sec. III but they provide a more physical transparent intuition as written in terms of the angular displacements $(\partial \theta, \partial \phi)$ of the source galaxy position.
G. Observed volume element

Having computed the distortion in photon path, we are now in a position to compute the physical volume occupied by the observed source galaxies over the small intervals $dz$ in observed redshift and $(d\theta, d\phi)$ in observed angle, and to express the volume in terms of the observed quantities. Since the real position $x^a_0$ of source galaxies is parametrized by using the observed quantities, the physical volume in the rest frame of the observed source galaxies can be written in a covariant way as

$$dV_{\text{phy}} = \sqrt{-g} \varepsilon_{abc} u^d_a \frac{\partial x^a_0}{\partial z} \frac{\partial x^b_0}{\partial \theta} \frac{\partial x^c_0}{\partial \phi} dz d\theta d\phi$$

where the subscript $s$ for the source position is omitted in the second line, and we simply expanded the summation of the Levi-Civita symbol $\varepsilon_{abcd}$ over the four velocity for further calculations. The distortion $\delta z$ in the observed redshift is given in Eqs. (55) and (56). Finally, the metric determinant is

$$\sqrt{-g} \equiv a^4(1 + \delta g) , \quad \delta g = A + C^\alpha - \frac{1}{2} A^2 + B^\alpha \theta_\alpha + A C^\alpha + \frac{1}{2} (C^\alpha)^2 - C^\beta C^\alpha .$$

To the second order in perturbations, we compute the individual terms in the square bracket in Eq. (58). First, the last term in the square bracket is

$$\varepsilon_{abc} U^a \frac{\partial x^a_0}{\partial z} \frac{\partial x^b_0}{\partial \theta} \frac{\partial x^c_0}{\partial \phi} = \frac{r^2}{H_z} \sin \theta \left\{ V_\parallel + V_\parallel \left[ 2 \frac{\delta r}{r_z} - 2 \kappa - H_z \frac{\partial}{\partial z} \delta T \right] - \frac{1}{r_z} \left( V_\parallel \frac{\partial}{\partial \theta} + V_\parallel \frac{\partial}{\sin \theta \partial \phi} \right) (\delta r + \delta T) + V_\theta \delta \theta + V_\phi \sin \theta \delta \phi \right\} ,$$

and the second term is

$$\varepsilon_{0\alpha\beta\gamma} \frac{\partial x^0_0}{\partial z} \frac{\partial x^\beta_0}{\partial \theta} \frac{\partial x^\gamma_0}{\partial \phi} = \frac{r^2}{H_z} \sin \theta \left\{ \delta u^0 + \delta u^0 \left[ 2 \frac{\delta r}{r_z} - 2 \kappa + H_z \frac{\partial}{\partial z} \delta r \right] \right\} ,$$

where the spatial component of the source four velocity is decomposed into the line-of-sight and the transverse velocities

$$U^a \equiv V_\parallel n^a + V_\theta \theta^a + V_\phi \phi^a .$$

These two terms in the square bracket in Eq. (58) vanish in a homogeneous universe. Finally, the first term in the square bracket is

$$\varepsilon_{0\alpha\beta\gamma} \frac{\partial x^0_0}{\partial z} \frac{\partial x^\beta_0}{\partial \theta} \frac{\partial x^\gamma_0}{\partial \phi} = \frac{r^2}{H_z} \sin \theta \left\{ 1 + 2 \frac{\delta r}{r_z} - 2 \kappa + H_z \frac{\partial}{\partial z} \delta r + \frac{\delta r^2}{r_z^2} + 2 \frac{\delta r}{r_z} \left( H_z \frac{\partial}{\partial z} \delta r - 2 \kappa \right) \right\}$$

$$- 2 H_z \frac{\partial}{\partial z} \delta \theta - H_z \frac{\partial}{\partial z} \delta \phi \frac{\partial}{\partial \phi} \delta r - H_z \frac{\partial}{\partial z} \delta \phi \frac{\partial}{\partial \phi} \delta r \right\} .$$

Summing up the individual contributions, we obtain the physical volume defined in Eq. (58) as

$$dV_{\text{phy}} = \sqrt{-g} \varepsilon_{abc} u^d_a \frac{\partial x^a_0}{\partial z} \frac{\partial x^b_0}{\partial \theta} \frac{\partial x^c_0}{\partial \phi} dz d\theta d\phi \equiv \frac{r^2}{H_z(1 + z)^3} dz d\theta d\phi (1 + \delta V) ,$$

and derive the volume distortion

$$\delta V^{(1)} = 3 \delta z + \delta g + 2 \frac{\delta r}{r_z} - 2 \kappa + H_z \frac{\partial}{\partial z} \delta r + \delta u^0 + V_\parallel ,$$

$$\delta V^{(2)} = 3 \delta z + \delta g + 2 \frac{\delta r}{r_z} + \delta \phi + H_z \frac{\partial}{\partial z} \delta r + \delta u^0 + V_\parallel + \delta u^0 \left[ 2 \frac{\delta r}{r_z} - 2 \kappa + H_z \frac{\partial}{\partial z} \delta r \right]$$

$$+ \frac{\delta r^2}{r_z^2} + 2 \frac{\delta r}{r_z} \left( H_z \frac{\partial}{\partial z} \delta r - 2 \kappa \right) - 2 H_z \frac{\partial}{\partial z} \delta \theta - H_z \frac{\partial}{\partial z} \delta \phi \frac{\partial}{\partial \phi} \delta r + V_\parallel \left[ 2 \frac{\delta r}{r_z} - 2 \kappa + H_z \frac{\partial}{\partial z} \delta r \right] + \frac{\delta r}{r_z} \left( V_\theta \frac{\partial}{\partial \theta} + V_\phi \frac{\partial}{\sin \theta \partial \phi} \right) (\delta r + \delta T) + V_\theta \delta \theta + V_\phi \sin \theta \delta \phi$$

$$+ (\delta g + 3 \delta z) \left[ 2 \frac{\delta r}{r_z} - 2 \kappa + H_z \frac{\partial}{\partial z} \delta r + \delta u^0 + V_\parallel \right] + 3 \delta z \delta g + 3 \delta z^2 + \Delta x^b \delta V^{(1)}_b ,$$

where the subscript $s$ for the source position is omitted in the second line, and we simply expanded the summation of the Levi-Civita symbol $\varepsilon_{abcd}$ over the four velocity for further calculations.
where the fluctuation quantities are now evaluated at the observed redshift and additional 2nd-order terms are added due to the 1st-order deviation of the photon path. It is noted that the partial derivatives with respect to the observed quantities \((z, \theta, \phi)\) are the partial derivatives with other observed quantities fixed; The derivative with respect to the observed redshift is the line-of-sight derivative along the past light cone, involving not only the spatial derivative, but also the time derivative, while the observed angular position \((\theta, \phi)\) is fixed.

The volume distortion to the linear order has a simple physical interpretation as the distortion compared to the volume element in a homogeneous universe in Eq. \((\ref{34})\) 

\[
\delta r/\bar{r} = \frac{3}{2} \delta z
\]

from the comoving factor \((1 + z)^3\), \(\delta g\), \(\delta u^0\), and \(V_\parallel\) from defining the source rest frame, \(2 \gamma \delta r/\bar{r}\) from the volume factor \(\Omega^2\), \(2 \kappa\) from the solid angle \(d\Omega\), and \(H_\parallel \partial_\tau \delta r\) from the change of the radial displacement at the observed redshift. To the second order in perturbations, these physical interpretations remain valid in the second-order volume distortion. However, additional physical effects need to be taken into account such as the contribution of the source tangential velocity and the tangential variation of the source position, similar to the transverse Doppler effect. There exist, of course, nonlinear coupling terms with the linear-order volume distortion.

\section{Fluctuation in luminosity distance}

Galaxy samples are often defined by its observed flux or the rest-frame luminosity inferred from the observed flux. The fluctuation in the luminosity distance at the observed redshift \(z\) is defined as

\[
D_L(z) \equiv \bar{D}_L(z)(1 + \delta D_L), \quad \bar{D}_L(z) = (1 + z)\bar{r}(z),
\]

where the fluctuation \(\delta D_L\) is dimensionless. Noting that the luminosity distance is related to the angular diameter distance

\[
D_A(z) = D_L(z)/\bar{r}^2.
\]

we can utilize the calculations of the photon path measured by the observer at origin to compute the angular diameter distance, and the fluctuation in the angular diameter distance is identical to the fluctuation in the luminosity distance. The fluctuation in the luminosity distance has been computed in \cite{4, 38–41}, and the second-order calculations are recently presented in \cite{31, 42}. Here we briefly present the calculation, but express it in terms of distortions in photon path we computed in \cite{11, 12} which clearly highlights the physical effects in play.

Let’s consider a unit area \(dA_{\text{phy}}\) in the source rest frame that appears subtended by the observed solid angle \(d\Omega = \sin \theta d\theta d\phi\). This unit area is related to the angular diameter distance as \(dA_{\text{phy}} = D_A^2(z) d\Omega\), and similar to the calculation in Sec. \cite{11, 12} it can be computed in a covariant way as

\[
dA_{\text{phy}} = \sqrt{-g} \varepsilon_{dabc} u^a_N a^b_N a^c_N \frac{\partial x^b}{\partial \theta} \frac{\partial x^c}{\partial \phi} d\theta d\phi,
\]

where the velocity four vector defines the source rest-frame and the observed photon direction defines the unit area in the source frame. The observed photon vector in Eq. \((\ref{28})\)

\[
N^a = \frac{k^a}{k^b u_b} + u^a
\]

is the observed photon direction expressed in a FRW frame and parallelly transported along the photon path. This is not to be confused with the observed photon direction \(n^\alpha = (\theta, \phi)\) measured in the observer rest frame. Therefore, the angular diameter distance is

\[
D_A^2(z) = \frac{-g}{\sin \theta} \varepsilon_{dabc} u^a_N a^b_N a^c_N \frac{\partial x^b}{\partial \theta} \frac{\partial x^c}{\partial \phi} = D_A^2(z)(1 + \delta g)(1 + \delta z)^2 \left[ \varepsilon_{dabc} \left( a u^d_N \right) \left( a N^a_N \right) \frac{\partial x^b_N}{\partial \theta} \frac{\partial x^c_N}{\partial \phi} \right]
\]

\[
= D_A^2(z) \left[ 1 + 2\delta D_L + \delta D_L^2 \right].
\]

To simplify the calculation, we compute the square bracket by splitting it into three components,

\[
\frac{\varepsilon_{dabc}}{r^2 \sin \theta} \left( a u^d_N \right) \left( a N^a_N \right) \frac{\partial x^b_N}{\partial \theta} \frac{\partial x^c_N}{\partial \phi} = -V_\theta \left( U^\alpha - B^\alpha \right) e_\alpha - \frac{1}{r^2} \left( V_\theta \frac{\partial}{\partial \theta} + V_\phi \frac{\partial}{\sin \theta \partial \phi} \right) \delta T,
\]

and the third component vanishes in the mean and the linear order in perturbations:

\[
\frac{\varepsilon_{dabc}}{r^2 \sin \theta} \left( a u^d_N \right) \left( a N^a_N \right) \frac{\partial x^b_N}{\partial \theta} \frac{\partial x^c_N}{\partial \phi} = -V_\theta \left( U^\alpha - B^\alpha \right) e_\alpha - \frac{1}{r^2} \left( V_\theta \frac{\partial}{\partial \theta} + V_\phi \frac{\partial}{\sin \theta \partial \phi} \right) \delta T.
\]
where the first term arises from the component \( \epsilon \delta_{0\beta\gamma} \) and the second term from \( \epsilon \delta_{abc} \). The second component in Eq. (72) is

\[
\frac{\epsilon_{0\alpha\beta\gamma}}{r_z^2 \sin \theta} \delta u^0 \left( a N^0_\alpha \right) \frac{\partial x^\beta}{\partial \theta} \frac{\partial x^\gamma}{\partial \phi} = \delta u^0 + \delta u^0 \left( -2 \kappa + 2 \frac{\delta r}{r_z} - C_{\alpha\beta} \epsilon^{\alpha\beta} \right),
\]

and finally the first component can be computed as

\[
\frac{\epsilon_{0\alpha\beta\gamma}}{r_z^2 \sin \theta} \left( a N^0_\alpha \right) \frac{\partial x^\beta}{\partial \theta} \frac{\partial x^\gamma}{\partial \phi} = 1 + V || + \delta e || - \Delta \nu + 2 \frac{\delta r}{r_z} + \left( \cot \theta + \frac{\partial}{\partial \theta} \right) \delta \theta + \frac{\partial}{\partial \phi} \delta \phi,
\]

\[
-2 \kappa \left( 2 \frac{\delta r}{r_z} - C_{\alpha\beta} \epsilon^{\alpha\beta} \right) - 2 \frac{\delta r}{r_z} C_{\alpha\beta} \epsilon^{\alpha\beta} + \left( \frac{\delta r}{r_z} \right)^2 - \delta \theta^2 - \frac{1}{2} \left( \sin \theta \delta \phi \right)^2 + \Delta \nu \left( V || + C_{\alpha\beta} \epsilon^{\alpha\beta} \right)
\]

\[
- \frac{1}{r_z} \left[ \left( V_\theta + \delta e_\theta \right) \frac{\partial}{\partial \theta} + \left( V_\phi + \delta e_\phi \right) \frac{\partial}{\partial \phi} \right] \delta r + \left( V_\theta + \delta e_\theta \right) \delta \theta + \left( V_\phi + \delta e_\phi \right) \sin \theta \delta \phi
\]

\[
- \frac{\partial}{\partial \phi} \delta \theta \delta \phi + \frac{\partial}{\partial \theta} \delta \phi \delta \theta + \cot \theta \delta \theta \left( \frac{\partial}{\partial \theta} \delta \theta + \frac{\partial}{\partial \phi} \delta \phi \right) + \left( \delta \theta \frac{\partial}{\partial \theta} + \delta \phi \frac{\partial}{\partial \phi} \right) \frac{\delta r}{r_z} + \Delta x^\alpha \delta D_{L,a}.
\]

Collecting terms altogether and using the null equation, the fluctuation in the luminosity distance is obtained as

\[
\delta D_L^{(1)} = \delta z - \kappa + \frac{\delta r}{r_z} + \frac{1}{2} \left( C_{\alpha\beta} - C_{\alpha\beta} \epsilon^{\alpha\beta} \right),
\]

\[
2 \delta D_L^{(2)} = 2 \delta z + \delta g + \delta u^0 + V || + e_{\alpha} \delta e_{\alpha} - \Delta \nu + 2 \frac{\delta r}{r_z} + \left( \cot \theta + \frac{\partial}{\partial \theta} \right) \delta \theta + \frac{\partial}{\partial \phi} \delta \phi.
\]

Given the distortion at the observed redshift, the fluctuation arises due to the distortion in the solid angle, the radial displacement, and the rest-frame of the source at the linear order in perturbations. To the second order, in addition to the nonlinear coupling of the linear order terms, there exist additional distortions along the tangential directions, as was the case in the nonline distortion.

## 1. Observed galaxy number density

In observation, the galaxy number density \( n^0_g \) is obtained by counting the number \( dN^0_g \) of galaxies observed within the volume defined by the observed direction \( (\theta, \phi) \) and the observed redshift \( z \): \( dN^0_g = n^0_g dV_{obs} \), where the volume element \( dV_{obs} \) over the small interval \( (dz, d\theta, d\phi) \) in observation is

\[
dV_{obs} = \frac{\bar{y}_z^2}{H_z (1 + z)^3} \sin \theta \ dz \ d\theta \ d\phi.
\]
correspond to an observer’s choice of gauge condition, uniform-redshift gauge.\(^2\)

However, since the Universe is far from being homogeneous, the constructed volume \(dV_{\text{obs}}\) in Eq. (72) differs from the physical volume \(dV_{\text{phy}}\) in Eq. (58) occupied by the observed galaxies on the sky. Using the conservation of the number of galaxies \(dN_g^{\text{obs}}\), the observed galaxy number density is related to the physical number density \(n_g\) of the observed source galaxies defined in their rest frame as

\[
n_g^{\text{obs}} = n_g (1 + \delta V) \,.
\]  

(80)

This relation highlights the contribution of the volume distortion \(\delta V\) in Eq. (64), and the volume effect is present in galaxy clustering, regardless of which galaxy sample is selected [8].

Furthermore, the physical number density \(n_g\) of source galaxies can be separated into the mean and the remaining fluctuation as

\[
n_g = \bar{n}_g(t_p) \left( 1 + \delta_{g}^{\text{int}} \right) ,
\]

(81)

where the mean is obtained by averaging the number density over a hypersurface defined by some time coordinate \(t_p\) and the intrinsic fluctuation around the mean vanishes when averaged:

\[
\bar{n}_g(t_p) \equiv \langle n_g \rangle_{t_p} , \quad \langle \delta_{g}^{\text{int}} \rangle_{t_p} = 0 .
\]

(82)

While the separation of the galaxy number density into the mean and the fluctuation is completely arbitrary and gauge-dependent in Eq. (81) as it relies on an unspecified choice of time \(t_p\), a physically meaningful choice of time coordinate (and hence gauge-invariant) can be made in relation to the biasing scheme, in which the galaxy fluctuation \(\delta_{g}^{\text{int}}\) can be further related to the underlying matter density fluctuation. To the linear order, a proper time (and hence the notation \(t_p\)) can be chosen to provide a physical biasing relation between the remaining fluctuation \(\delta_{g}^{\text{int}}\) and the underlying matter fluctuation \(\delta_m\) [9, 12–14, 17], as the local dynamics of galaxy formation can only be affected by the presence of long wavelength modes through the change in the local curvature and the local expansion rate [17]. Here we leave the second-order biasing to future work and proceed with an unspecified time coordinate (or unspecified gauge choice) for the intrinsic galaxy fluctuation \(\delta_{g}^{\text{int}}\).

In addition to the intrinsic fluctuation of the source galaxies, additional contribution to galaxy clustering arises from the source effect [8]: The source effect describes the contributions of the mean expressed in terms of observed quantities:

\[
n_g = \bar{n}_g(z) \left[ 1 - e_1 \delta z_{t_p} + \frac{1}{2} e_2 \delta z_{t_p}^2 \right] \left( 1 + \delta_{g}^{\text{int}} \right) ,
\]

(83)

where \(\delta z_{t_p}\) is the distortion in the observed redshift as in Eq. (21) but is evaluated at the time slicing specified by \(t_p\) and two additional coefficients

\[
e_1 = \frac{d \ln \bar{n}_g}{d \ln (1 + z)} , \quad e_2 = e_1 + e_1^2 + \frac{d e_1}{d \ln (1 + z)} ,
\]

(84)

are called the evolution biases. Since the mean number density \(\bar{n}_g\) here is a physical number density, even a sample with a constant comoving number density such as the matter density \(\bar{\rho}_m\) would have \(e_1 = 3\) and \(e_2 = 12\).

Furthermore, additional source effects will be present, if the source galaxy sample is defined by other observable quantities such as the rest-frame luminosity threshold inferred from the threshold in observed flux \(f_{\text{obs}}\). Similar to the observed volume \(dV_{\text{obs}}\), the luminosity distance \(D_L(z) = r_z (1 + z)\) in a homogeneous universe based on the observed redshift \(z\) is assigned to the source, and the inferred luminosity at a given observed flux \(f_{\text{obs}}\) is then

\[
\hat{L} = 4\pi D_L^2(z) f_{\text{obs}} .
\]

(85)

The physical luminosity \(L\) of the source galaxies is related to the inferred luminosity as

\[
L = 4\pi D_L^2(z) f_{\text{obs}} = \hat{L} (1 + \delta D_L)^2 ,
\]

(86)

where the physical luminosity distance \(D_L(z) \equiv D_L(z)(1 + \delta D_L)\). Therefore, the observed galaxy population defined its inferred luminosity above a given threshold is related to the galaxy population with corresponding physical luminosity above the same threshold as

\[
n_g = \bar{n}_g(\hat{L}) \left[ 1 - t_1 \delta D_L + \frac{1}{2} t_2 \delta D_L^2 \right] \left( 1 + \delta_{g}^{\text{int}} \right) ,
\]

(87)

\(^2\) The observed redshift is gauge-invariant, since its value (the observed redshift) remains unchanged, whatever coordinate system is used to describe it. However, it depends on the frame of an observer, i.e., the observed redshift is the spectral line ratio measured by an observer at rest. When an observer compares the real physical universe to a homogeneous universe, a choice of gauge condition needs to be made, and it is based on the observed redshift.
where two additional coefficients

\[ t_1 = -2 \frac{d \ln \bar{n}_g}{d \ln L}, \quad t_2 = t_1 + t_1^2 - 2 \frac{d t_1}{d \ln L}, \]  

describe the slope and running of the luminosity function. When the luminosity function \( d \bar{n}_g \propto L^{-s} \) is well approximated by a constant slope \( s \), these coefficients are

\[ t_1 = 2(s - 1) = 5p, \quad t_2 = 2(s - 1)(2s - 1) = 5p(5p + 1), \]  

where \( p = 0.4(s - 1) \) is the luminosity function slope in terms of magnitude \( M = \text{constant} - 2.5 \log_{10}(L/L_0) \).

### J. Observed galaxy fluctuation

Finally, by putting it altogether, the mean number density of the observed galaxies is expressed in terms of the observed redshift \( z \) and the observed flux \( f_{\text{obs}} \), and the observed galaxy number density is then decomposed of the mean and the remaining fluctuation as

\[ n_g^{\text{obs}}(z, \hat{n}) = \bar{n}_g(z) \left( 1 + \delta_g^{\text{int}} \right) \left( 1 + \delta V \right) \left( 1 - e_1 \delta z_{t_p} + \frac{1}{2} e_2 \delta z_{t_p}^2 \right) \left( 1 - t_1 \delta D_L + \frac{1}{2} t_2 \delta D_L^2 \right). \]  

This equation concisely summarizes the main result of the paper, in conjunction with the computation of all the perturbation quantities present in Eq. \[90\]. It is noted that only the metric perturbations are expanded to the second order, while the intrinsic fluctuation \( \delta_g^{\text{int}} \) that is likely to be highly nonlinear is left unexpanded. Furthermore, no gauge choice is made in the previous calculations.

In the absence of \textit{ab initio} knowledge of galaxy formation, the mean galaxy number density \( \bar{n}_g(z) \) cannot be computed \textit{a priori} — it has to be determined by the survey itself. Therefore, the observed mean at each redshift is obtained by averaging the observed number density \( n_g^{\text{obs}}(z, \hat{n}) \) over the survey area \( \Omega \):

\[ \bar{n}_g(z) = \frac{1}{\Omega} \int_{\Omega} d^2 \hat{n} n_g^{\text{obs}}(z, \hat{n}) \equiv \bar{n}_g(z) \delta n_g(z), \]  

where the residual fluctuation \( \delta n_g(z) \) in the mean number density arises if all the fluctuations in Eq. \[90\] may not average out over the survey area. Only in the limit of infinite volume survey, the residual fluctuation vanishes, and we have \( \bar{n}_g(z) = \bar{n}_g(z) \) at all redshifts. The detailed calculation of the observed mean involves the survey specifications and the spatial distribution of fluctuations in Eq. \[90\] that cannot be computed with generality. The observed galaxy fluctuation is defined in terms of the observed mean number density as

\[ \delta_g^{\text{obs}}(z, \hat{n}) \equiv \frac{n_g^{\text{obs}}(z, \hat{n})}{\bar{n}_g(z)} - 1, \]  

and it is noted that the residual fluctuation \( \delta n_g(z) \) contributes to the observed mean number density \( \bar{n}_g \) and the observed galaxy fluctuation \( \delta_g^{\text{obs}} \).

By assuming the infinite survey volume \( \bar{n}_g(z) = \bar{n}_g(z) \), we derive

\[ \delta_g^{\text{obs}(1)} = \delta_g^{\text{int}(1)} + \delta V - e_1 \delta z_{t_p} - t_1 \delta D_L, \]  

\[ \delta_g^{\text{obs}(2)} = \delta_g^{\text{int}(2)} + \delta V - e_1 \delta z_{t_p} + \frac{1}{2} e_2 \delta z_{t_p}^2 - t_1 \delta D_L + \frac{1}{2} t_2 \delta D_L^2 - e_1 \delta z_{t_p} \delta V \]  

\[ + \delta_g^{\text{int}} \left( \delta V - e_1 \delta z_{t_p} - t_1 \delta D_L \right) - t_1 \delta D_L \left( \delta V - e_1 \delta z_{t_p} \right). \]  

### III. GAUGE INvariant EQUATIONS

Having derived all the equations without choosing a gauge condition in Sec. \[\text{II}\] we construct the gauge-invariant equations for computing the observed galaxy fluctuation. Compared to the linear-order calculations, the second-order calculations regarding gauge transformations are more complicated, and they are further affected by the presence of unphysical gauge modes. However, once gauge modes are removed and gauge-invariant variables are constructed, it is straightforward to construct second-order
gauge-invariant equations, given the gauge-invariant equations at the linear order, although a complete verification of second-order gauge-invariance associated with those equations is more involved.

In Appendix A we explicitly derive the second-order gauge transformation to isolate and remove the gauge modes to the second order in perturbations. Using the gauge-transformation properties, second-order gauge-invariant variables are explicitly constructed in Appendix A2, and their structure takes a rather simple form. For example, the linear-order gauge-transformation \((\tilde{\tau} = \tau + T)\) at a given position yields the metric transformation

\[
\chi \equiv a(\beta + \gamma') , \quad \chi^{(1)} = \chi^{(1)} - aT^{(1)} , \quad \varphi^{(1)} = \varphi^{(1)} - \mathcal{H} T^{(1)} ,
\]

and we can construct a linear-order gauge-invariant variable

\[
\varphi^{(1)} \equiv \varphi^{(1)} - H\chi^{(1)} .
\]

However, as we explicitly show in Eq. (A33), this combination becomes gauge-dependent at the second order, and additional compensating terms are required to cancel the second-order corrections and guarantee its gauge-invariance. Therefore, the second-order gauge-invariant variable can be written in a form:

\[
\varphi^{(q)} = \varphi - H\chi + \varphi^{(q)} ,
\]

where the last term represents quadratic terms that compensate for the second-order gauge-transformation and its explicit expression is shown in Eq. (A37). As demonstrated in Appendix A2, a choice of gauge condition \(\chi = 0\) to the second order in perturbations completely removes unphysical gauge modes, and the remaining metric perturbations correspond to gauge-invariant variables associated with the choice of gauge condition \(\chi = 0\). While a variety of second-order gauge-invariant variables can be constructed satisfying Eq. (96) at the linear order, we constructed \(\varphi^{(q)}\) in Eq. (97), such that \(\varphi^{(q)}\) becomes \(\varphi\) when the gauge condition \(\chi = 0\) is adopted (hence the notation). Therefore, the quadratic terms \(\varphi^{(q)}\) in Eq. (97) or in Eq. (A37) satisfy the relation

\[
\varphi^{(q)} = 0 \quad \text{if} \quad \chi = 0 ,
\]

which greatly simplifies the way to construct second-order gauge-invariant equations.

Before we start constructing gauge-invariant equations for those we derived in Sec. II, we caution that not all equations can be made gauge-invariant, but this statement should not be confused with the fact that all equations with any proper choice of gauge condition are gauge-invariant. The gauge-invariance of equations itself does not guarantee that they describe observable or physical quantities, but it provides a necessary condition for those equations. We start constructing the second-order gauge-invariant equations in Sec. II.

To the linear order in perturbations, the null equation can be re-arranged as

\[
0 = 2 \left( e_\alpha e_\alpha - \delta \nu - A + B_\alpha e_\alpha + C_{\alpha\beta} e_\alpha e_\beta \right)^{(1)} = 2 \left( e_\alpha \delta e_\alpha - \delta \nu - \alpha_\chi + \varphi_\chi + \Psi_\alpha e_\alpha + C_{\alpha\beta} e_\alpha e_\beta \right)^{(1)} ,
\]

where the definition of the gauge-invariant variables are explicitly present in Appendix A2. However, the above expression is not gauge-invariant to the second order in perturbations, because of the quadratic terms in the gauge-invariant variables and the remaining quadratic terms in the null equations. To the second order, we re-arrange the null equation as

\[
0 = 2 \left( e_\alpha \delta e_\alpha - \delta \nu - \alpha_\chi + \varphi_\chi + \Psi_\alpha e_\alpha + C_{\alpha\beta} e_\alpha e_\beta \right) - 2 \left( e_\alpha \delta e_\alpha - \delta \nu - \alpha_\chi + \varphi_\chi + \Psi_\alpha e_\alpha + C_{\alpha\beta} e_\alpha e_\beta \right)^{(q)}
\]

\[
+ \delta e_\alpha \delta e_\alpha - \delta \nu^2 - 2 \delta \nu \left( 2A - B_\alpha e_\alpha \right) + 2 \left( B_\alpha + 2C_{\alpha\beta} e_\beta \right) e_\alpha ,
\]

and it is noted that a choice of scalar gauge condition is needed to construct second-order gauge-invariant vectors and tensors. Since we eliminate the unphysical gauge modes to the second order in perturbations by choosing the spatial C-gauge condition in Appendix A2, the metric tensor then corresponds to

\[
A = \alpha , \quad B_\alpha = \frac{1}{\alpha} \chi_\alpha + \Psi_\alpha , \quad C_{\alpha\beta} = \varphi_\alpha \beta + C_{\alpha\beta} .
\]

Therefore, the remaining quadratic terms in Eq. (100) can be readily re-arranged, and the gauge-invariant equation for the null condition becomes

\[
0 = \hat{k}^\alpha \hat{k}_\alpha = (\delta e_\alpha) - 2 \left( e_\alpha \delta e_\alpha - \delta \nu - \alpha_\chi + \varphi_\chi + \Psi_\alpha e_\alpha + C_{\alpha\beta} e_\alpha e_\beta \right) + \delta e_\alpha \delta e_\alpha - \delta \nu^2 - 2 \delta \nu \left( 2\alpha_\chi - \Psi_\alpha e_\alpha \right) + 2 \left( 2\varphi_\alpha + \Psi_\alpha + 2C_{\alpha\beta} e_\beta \right) e_\alpha .
\]
Construction of gauge-invariant expressions for geodesic equations can be made in a similar way by noting that the affine parameter integration deviates from the straight line at the second order. The temporal component of the geodesic equation is

$$-rac{d}{d\lambda} \delta \nu_\lambda = \alpha'_\lambda + \varphi'_\lambda - 2\alpha_{\lambda,\alpha} e^\alpha + \left(\Psi_{\lambda|\beta} + C'_{\lambda|\beta}\right)e^\alpha e^\beta + 2\delta \nu_\lambda \left(\alpha'_\lambda - \alpha_{\lambda,\alpha} e^\alpha\right) - 2\alpha_{\lambda,\alpha} \alpha'_\lambda - \alpha_{\lambda,\alpha} \Psi^\alpha + \Psi^\alpha \Psi'_\alpha$$

$$-2\alpha_{\lambda,\alpha} \varphi'_\lambda - 2\varphi_{\lambda,\alpha} e^\alpha \Psi_{\alpha} e^\beta + \varphi^\beta_{\lambda,\alpha} \Psi_{\alpha} - \left(\alpha_{\lambda, \lambda|\beta} + 2C_{\alpha|\beta}\right) + \Psi_{\lambda} \left(2C_{\gamma|\beta} - C_{\alpha|\beta}\right) e^\beta e^\gamma$$

$$+ 2\left(2\alpha_{\lambda,\lambda,\alpha} + \varphi'_\lambda e^\alpha + \Psi_{\beta} C_{\alpha|\beta} + \Psi_{\alpha|\beta} e^\beta - \delta e^\alpha \left[\varphi'_\lambda e^\alpha - e^\beta \left(\Psi_{\alpha|\beta} + C_{\alpha|\beta}\right)\right]\right), \quad (103)$$

and the spatial component is

$$\frac{d}{d\lambda} \delta e^\alpha_\lambda = \alpha^\alpha e^\alpha - \varphi^\beta_{\alpha|\beta} - 2 \left(\varphi^\beta_{\lambda,\beta|\alpha} e^\beta + \Psi_{\beta} C^\alpha_{\beta|\alpha} + \Psi_{\beta|\alpha} e^\alpha\right) e^\beta - \Psi_{\alpha|\beta} + 2\alpha_{\lambda,\alpha} e^\alpha + 2\alpha_{\lambda,\alpha} e^\alpha + \Psi_{\alpha|\beta} e^\alpha$$

$$-2\left(2\alpha_{\lambda,\alpha} + \varphi'_{\lambda} e^\alpha + \Psi_{\beta} C^\alpha_{\beta|\alpha} + \Psi_{\beta|\alpha} e^\alpha\right) e^\beta - 2\delta e^\alpha \left[\varphi'_{\lambda} e^\alpha + e^\beta \left(\Psi_{\alpha|\beta} + C_{\alpha|\beta}\right)\right]$$

$$+ \Delta e^\alpha \left[\alpha^\alpha e^\alpha - \varphi^\beta_{\alpha|\beta} - 2\alpha_{\lambda,\alpha} e^\alpha + \Psi_{\beta} C^\alpha_{\beta|\alpha} + \Psi_{\beta|\alpha} e^\alpha\right]. \quad (104)$$

Integrating the geodesic equation over the affine parameter, we derive temporal and spatial deviations

$$\delta \nu_{\chi}^2 = -2 \left(\alpha_{\chi z} - \alpha_{\chi,\alpha}\right) + \int_0^{\bar{r}_z} d\bar{r} \left\{ \left(\alpha - \phi - \psi_{\chi|\beta} e^\beta - C_{\chi|\beta} e^\beta\right)_{\alpha} + \delta \nu_{\chi} \left(2\alpha_{\chi,\alpha} - \Psi_{\chi}\right) - \Psi_{\chi|\beta} + 2\alpha_{\chi,\alpha} e^\alpha + 2\alpha_{\chi,\alpha} e^\alpha + \Psi_{\beta} C^\alpha_{\beta|\alpha} + \Psi_{\beta|\alpha} e^\alpha$$

$$-2\left(2\alpha_{\chi,\alpha} + \phi'_{\chi} e^\alpha + \Psi_{\beta} C^\alpha_{\beta|\alpha} + \Psi_{\beta|\alpha} e^\alpha\right) e^\beta - 2\delta e^\alpha \left[\phi'_{\chi} e^\alpha + e^\beta \left(\Psi_{\beta|\alpha} + C_{\beta|\alpha}\right)\right]$$

$$+ \Delta e^\alpha \left[\alpha_{\chi,\alpha} - \phi_{\chi|\beta} - \psi_{\chi|\beta} e^\beta - C_{\chi|\beta} e^\beta\right]. \quad (105)$$

Using the above expressions, the distortion in the observed redshift can be written in a gauge-invariant form as

$$\delta z_{(1)}^2 = \delta a_{o} + \left[\delta \nu_{\chi} + \alpha + (U_{\rho} - \Psi_{\rho}) \epsilon^\rho\right]^2 \delta a_{o} + \left[\left(U_{\rho} - \Psi_{\rho}\right) \epsilon^\rho - \alpha_{\rho}\right]^2 + \int_0^{\bar{r}_z} d\bar{r} \left(\alpha_{\rho} - \phi_{\rho}\right) - \left(\Psi_{\rho|\beta} + C'_{\rho|\beta}\right) e^\alpha e^\beta, \quad (106)$$

$$\delta z_{(2)}^2 = \delta a_{o} + \left[\delta \nu_{\chi} + \alpha + (U_{\rho} - \Psi_{\rho}) \epsilon^\rho + \delta \nu_{\alpha} + \alpha + (U_{\rho} - \Psi_{\rho}) \epsilon^\rho\right]^2 + \left[\delta a - \delta \nu_{\rho} - \alpha - (U_{\rho} - \Psi_{\rho}) \epsilon^\rho\right]^2 + \Delta_{\rho} \left[\delta \nu_{\chi} + \alpha + (U_{\rho} - \Psi_{\rho}) \epsilon^\rho\right]\right]. \quad (107)$$

Finally, the spatial deviations of the observed source position can be expressed as

$$\delta r_{(1)}^2 = \delta r_{o} + \delta z_{(2)} + \frac{\bar{r}_z}{H_{o}} + \int_0^{\bar{r}_z} d\bar{r} \left(\alpha_{\rho} - \phi_{\rho} - \Psi_{\rho} e^\rho - C_{\alpha|\beta} e^\rho e^\beta\right)\right), \quad (108)$$

$$\bar{r}_z \delta (\epsilon_{(1)}^\rho) = \bar{r}_z \epsilon_{o} \left(\epsilon_{\rho} + 2 C_{\beta|\rho} \right) - \int_0^{\bar{r}_z} d\bar{r} \left[\epsilon_{o} \left(\epsilon_{\rho} + 2 C_{\alpha|\beta} \right) + \left(\frac{\bar{r}_z - \bar{r}_{o}}{\bar{r}_{o}}\right) \frac{\partial}{\partial \theta} \left(\alpha_{\rho} - \phi_{\rho} - \Psi_{\rho} e^\rho - C_{\alpha|\beta} e^\rho e^\beta\right)\right], \quad (109)$$

$$\bar{r}_z \delta (\epsilon_{(2)}^\rho) = \bar{r}_z \epsilon_{o} \left(\epsilon_{\rho} + 2 C_{\beta|\rho} \right) - \int_0^{\bar{r}_z} d\bar{r} \left[\epsilon_{o} \left(\epsilon_{\rho} + 2 C_{\alpha|\beta} \right) + \left(\frac{\bar{r}_z - \bar{r}_{o}}{\bar{r}_{o}}\right) \frac{\partial}{\partial \theta} \left(\alpha_{\rho} - \phi_{\rho} - \Psi_{\rho} e^\rho - C_{\alpha|\beta} e^\rho e^\beta\right)\right], \quad (110)$$

$$\delta \nu_{\chi} = \alpha_{\chi,\alpha} + \frac{\varphi'_{\chi} + \varphi_{\chi,\alpha} e^\alpha + \Psi_{\beta} C_{\alpha|\beta} + \Psi_{\beta|\alpha} e^\alpha}{\alpha_{\chi,\alpha} - \alpha_{\chi,\alpha} e^\alpha - \alpha_{\chi,\alpha} e^\alpha + \Psi_{\beta} C_{\alpha|\beta} + \Psi_{\beta|\alpha} e^\alpha} - 2\delta \nu_{\chi} \left(2\alpha_{\chi,\alpha} + \varphi'_{\chi} e^\alpha + \Psi_{\beta} C_{\alpha|\beta} + \Psi_{\beta|\alpha} e^\alpha\right) e^\beta - 2\delta \nu_{\alpha} \left[\alpha_{\chi,\alpha} - \alpha_{\chi,\alpha} e^\alpha - \alpha_{\chi,\alpha} e^\alpha + \Psi_{\beta} C_{\alpha|\beta} + \Psi_{\beta|\alpha} e^\alpha\right] + \Delta_{\rho} \left[\delta \nu_{\chi} + \alpha_{\chi,\alpha} \epsilon^\rho\right]. \quad (111)$$
where the azimuthal deviation $\delta \phi^{(1)}$ can be readily inferred from $\delta \theta^{(1)}$. To the second order in perturbations, they are further related as

$$(n_{\alpha}x_{\alpha}^{(2)})_{X} = \delta r^{(2)} + \left(\delta z_{X} - \frac{\delta z}{H_{z}}\right)_{X} (\alpha_{X} - \varphi_{X} - \Psi_{\alpha}e^{\alpha} - C_{\alpha\beta}e^{\alpha}e^{\beta})_{z} + \frac{1}{2H_{z}^{3}} \left(H_{z}^{2} + H_{z}' \frac{\delta z_{X}^{(2)}}{H_{z}}\right)$$

\begin{equation}
+ \int_{0}^{r_{z}} d\tilde{r} \left\{ (\alpha_{X} - \varphi_{X} - \Psi_{\alpha}e^{\alpha} - C_{\alpha\beta}e^{\alpha}e^{\beta})^{(2)} + \delta \nu_{X} \left[ \frac{1}{2} \delta \nu + (2\alpha_{X} - \Psi_{\alpha}e^{\alpha})\right] \right\}
- \frac{1}{2} \delta \nu_{X} + 2\varphi_{X}e^{\alpha} + (\Psi_{\alpha} + 2C_{\alpha\beta}e^{\beta}) \delta e^{\alpha} + \Delta x^{c} (\alpha_{X} - \varphi_{X} - \Psi_{\alpha}e^{\alpha} - C_{\alpha\beta}e^{\alpha}e^{\beta})_{e},
\end{equation}

$$\left(\theta_{\alpha}x_{\alpha}^{(2)}\right)_{X} = \tilde{r}_{z}e_{\theta_{\alpha}} (\delta e^{\alpha} + \Psi_{\alpha}e^{\alpha} + 2C_{\alpha\beta}e^{\alpha}e^{\beta})^{(2)}_{o} - \Delta \lambda_{e_{\theta_{\alpha}}} \delta e^{\alpha}\right)$$

\begin{equation}
- \int_{0}^{r_{z}} d\tilde{r} \left\{ e_{\theta_{\alpha}} (1 + \Delta x_{\beta}^{b} \partial_{b}) (\Psi_{\alpha} + 2C_{\beta}e^{\beta}) + \left(\frac{\tilde{r}_{z} - \tilde{r}}{\tilde{r}}\right) \frac{\partial}{\partial \theta} \left(1 + \Delta x_{\beta}^{b} \partial_{b}\right) (\alpha_{X} - \varphi_{X} - \Psi_{\alpha}e^{\alpha} - C_{\alpha\beta}e^{\alpha}e^{\beta}) \right\}

+ (\tilde{r}_{z} - \tilde{r})e_{\theta_{\alpha}} \left[ \delta \nu_{X} (2\alpha_{X} - \Psi_{\alpha}e^{\alpha}) - \Psi_{\alpha} (\delta e^{\alpha} + \delta \nu_{X} e^{\beta}) - 2\varphi_{X} \delta e^{\alpha} - 2\varphi_{X} \alpha_{X}e^{\alpha}e^{\beta} - 2\alpha_{X}e^{\alpha} \right]

+ \delta \nu_{X} \chi_{\alpha\beta} e^{\beta} - 2C_{\beta} \delta e^{\beta} + 2 \left(C_{\beta\gamma}^{(o)} - C_{\beta\gamma}^{(1)}\right) e^{\gamma} \delta e^{\beta} + \chi_{\alpha\beta} \alpha_{X}e^{\alpha}e^{\beta} + 2C_{\beta} \Psi_{\beta} - 2\Psi_{\alpha} \alpha_{X}e^{\beta} e^{\beta}

+ 4(\varphi_{X} \tilde{g}^{\gamma} + C^{(o)}) \left(\varphi_{X} \tilde{g}^{\beta} + \Psi_{[\beta\gamma]} + C_{[\beta}^{\gamma]}\right) e^{\beta} - 2 (\varphi_{X} \delta e^{\alpha} + C_{\beta}^{(o)}) \left(2\varphi_{X} \delta e^{\alpha} + C_{\beta}^{(o)} - \varphi_{X} \delta e^{\beta} - C_{\beta}^{(o)}\right)

- \Psi_{\alpha} \left(\varphi_{X} \tilde{g}^{\beta} + \Psi_{[\beta\gamma]} + C_{[\beta}^{\gamma]}\right) e^{\beta} e^{\gamma}\right)\right\}.
\end{equation}

Subsequent calculations in Sec. IIII can be further expressed in terms of $\delta r_{\chi}$, $\delta \theta_{\chi}$, $\delta \phi_{\chi}$, $\delta z_{\chi}$ and other gauge-invariant variables in Appendix A.

IV. DISCUSSION

We have extended the calculation of the general relativistic description of galaxy clustering to the second order in metric perturbations without assuming any gauge conditions or adopting any restrictions on vector and tensor perturbations. On large scales, metric perturbations along the photon path affect the photon propagation, and these subtle relativistic effects need to be properly taken into account in considering the relation of the observable quantities such as the observed redshift and the angular position of source galaxies to the physical quantities of the source galaxies. In the past few years, linear-order relativistic effect in galaxy clustering has been computed, and it was shown that these subtle relativistic effects can be used to test general relativity and probe the early Universe in current and future galaxy surveys. Drawing on these previous works, we have computed the second-order relativistic effect in galaxy clustering, an essential tool for going beyond the power spectrum in the era of precision measurements of galaxy clustering.

Compared to the linear-order calculations, second-order calculations are more involved as the interchangeability between configuration and Fourier spaces is lost and the nonlinear coupling of the linear-order terms result in numerous additional terms (see, e.g., for reviews). Furthermore, scalar, vector, and tensor modes of perturbation variables become tangled as the nonlinear coupling of the linear-order terms source each component and affect their spatial transformation properties. To the second-order in perturbations, we have computed the transformation of the metric perturbations and removed the unphysical spatial gauge-modes. This procedure is necessary for the explicit construction of second-order gauge-invariant variables. As is often the case in many second-order calculations, one may assume no vector or tensor at the linear order and focus on scalar modes, because in this case no vector or tensor contributes to the scalar modes even at the second order, simplifying the situation. However, generation of vector and tensor is inevitable at the second order, and the observable quantities receive contributions from perturbations of all types, regardless of our calculational convenience. Hence we have constructed the second-order gauge-invariant variables with full generality on vector and tensor perturbations.

It is well-known that the observed redshift $z_{\text{obs}}$ is different from the redshift parameter $z_{h}$ in a homogeneous universe, because perturbations along the photon path such as the peculiar velocity and the gravitational potential contribute to the fluctuation $\delta z$ in the observed redshift: $1 + z_{\text{obs}} = (1 + z_{h})(1 + \delta z)$ in Eq. (21), where $1 + z_{h} = 1/a$. For exactly the same reason, the observed position $\eta^{a}$ of the source galaxy on the sky is different from the position $e^{a}$ in a homogeneous universe: $\eta^{a} = e^{a} + \delta \eta^{a}$ in Eq. (22), where two unit directional vectors can be obtained from the photon wavevector $k^{a}$ in each case. In Appendix B we have derived the relation between two unit directional vectors by explicitly constructing the physical photon wavevector in terms of the local observable quantities. This relation clarifies how the additional degree of freedom supplied by the conformal transformation in Eq. can be properly chosen to eliminate the distortion $\delta s^{a}$ in the observed position of the source galaxy.
With these issues resolved, it becomes rather straightforward, albeit lengthy, to extend the linear-order relativistic calculations to the second order. Compared to the inferred source position $\vec{r}_a = (\bar{r}_z, \bar{r}_T, \bar{n})$ in Eq. (42) based on the observed redshift $z$ and angle $(\theta, \phi)$, the physical source position can be parametrized in terms of the displacements $(\delta T, \delta r, \delta \theta, \delta \phi)$ in Eq. (43) to all orders in perturbations, and the volume effect in Eq. (64) can be readily computed to the desired orders in perturbations, although this separation of spatial and time components is gauge-dependent and these displacements need to be further related in terms of metric perturbations. Compared to the linear-order volume effect in Eq. (55), the notable difference in the second-order volume effect in Eq. (66) is the contribution of the tangential velocity $(V_\theta, V_\phi)$, and the displacement in the time coordinate $\delta T$ of the source galaxies, as in the transverse Doppler effect.

Finally, by using the second-order gauge-invariant variables, we have constructed the second-order gauge-invariant equations for the displacements. This step is necessary for numerically computing the displacements and the observed galaxy number density. To the second order, as quadratic terms are present in both the dynamical equations and the gauge-invariant variables, a proper choice of gauge-invariant variables is essential for simplifying the second-order gauge-invariant equations. An explicit construction of the second-order gauge-invariant equations was given in Sec. III.

The second-order relativistic description of galaxy clustering in this work provides the most accurate and complete description of galaxy clustering on large scales. While it is a step forward in the era of precision cosmology, proper applications of our second-order formalism to observations will require several steps beyond the scope of current investigation. First and foremost is galaxy bias to the second order in perturbations. Irrespective of nonlinear biasing schemes, computation of the second-order matter density fluctuation is necessary even for the simplest linear biasing. A physical choice of time slicing for the matter density $\delta n = (\bar{r}_z, \bar{r}_T, \bar{n})$ in Eq. (43) is the contribution of the tangential velocity $(V_\theta, V_\phi)$, and the displacement in the time coordinate $\delta T$ of the source galaxies, as in the transverse Doppler effect. Measurements of the three-point correlation function or the bispectrum in future surveys would not only complement the existing constraints from bolometric surveys but also provide new ways to test the general relativity and probe the early universe through the subtle relativistic effect in galaxy clustering.

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Appendix A: Second-Order Gauge-Invariant Formalism

Here we present our notation for second-order gauge-invariant formalism and discuss the gauge transformation properties in comparison to its linear-order counterparts. We then explicitly construct second-order gauge-invariant variables.

1. Spacetime metric perturbations

We describe the background for a spatially homogeneous and isotropic universe with a metric

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -a^2(\tau) d\tau^2 + a^2(\tau) \tilde{g}_{\alpha\beta} dx^\alpha dx^\beta,$$  

(A1)

where $a(\tau)$ is the scale factor and $\tilde{g}_{\alpha\beta}$ is the metric tensor for a three-space with a constant spatial curvature $K = -H^2_0 (1 - \Omega_{tot})$. To describe the real (inhomogeneous) universe, we parametrize the perturbations to the homogeneous background metric as

$$\delta g_{\alpha\beta} = -2 a^2 A, \quad \delta g_{\alpha\beta} = -a^2 B\alpha, \quad \delta g_{\alpha\beta} = 2 a^2 C_{\alpha\beta}.$$  

(A2)

These perturbation variables are defined in a non-perturbative way, such that they contain higher-order terms. To the second-order in perturbations, we may explicitly split the variables based on the perturbation order represented by the upper indices as

$$A = A^{(1)} + A^{(2)}, \quad B\alpha = B^{(1)}\alpha + B^{(2)}\alpha, \quad C_{\alpha\beta} = C^{(1)}_{\alpha\beta} + C^{(2)}_{\alpha\beta}.$$  

(A3)
Unless otherwise explicitly indicated, other perturbation variables should also be considered as those variables with higher-order terms.

It is customary in cosmological perturbation theory to decompose perturbation variables into scalar, vector, and tensor solutions of the generalized Helmholtz equation [52], according to their spatial-coordinate transformation properties. Therefore, the metric perturbations in Eq. (A3) are further decomposed as

\[
\begin{align*}
A &= \alpha, & B_\alpha &= \beta, & \alpha + B_\alpha, & C_{\alpha\beta} = \varphi + \gamma_{\alpha\beta} + C_{(\alpha\beta)} + C_{\alpha\beta},
\end{align*}
\]

where the vertical bar represents the covariant derivative with respect to the homogeneous spatial metric \( \tilde{g}_{\alpha\beta} \) and the round bracket is the symmetrization symbol. Separation of scalar, vector, and tensor can be readily made, based on the number of their spatial indices in the decomposed fields. The decomposed scalar perturbations can be obtained as

\[
\begin{align*}
\alpha &= \mathcal{A}, & \beta &= \Delta^{-1} \nabla^\alpha B_\alpha, & \gamma &= \frac{1}{2} \left( \Delta + \frac{1}{2} \bar{R} \right)^{-1} (3 \Delta^{-1} \nabla^\alpha \nabla_\beta C_{\alpha\beta} - C_\alpha^\alpha), & \\
\varphi &= \frac{1}{3} C_\alpha^\alpha - \frac{1}{6} \Delta \left( \Delta + \frac{1}{2} \bar{R} \right)^{-1} (3 \Delta^{-1} \nabla^\alpha \nabla_\beta C_{\alpha\beta} - C_\alpha^\alpha),
\end{align*}
\]

where \( \nabla_\alpha \) is the covariant derivative based on \( \tilde{g}_{\alpha\beta} \) (i.e., vertical bar) and \( \Delta = \nabla^\alpha \nabla_\alpha \) is the Laplacian operator. The presence of the Ricci scalar \( (\bar{R} = 6\bar{K}) \) for the three-space indicates that covariant derivatives are non-commutative. The decomposed vector and tensor components are computed in a similar manner as

\[
\begin{align*}
B_\alpha &= B_\alpha - \nabla_\alpha \Delta^{-1} \nabla^\beta B_\beta, & C_\alpha &= 2 \left( \Delta + \frac{1}{3} \bar{R} \right)^{-1} \left[ \nabla_\beta C_{\alpha\beta} - \nabla_\alpha \Delta^{-1} \nabla^\beta \nabla_\gamma C_{\alpha\beta} \right], \\
C_{\alpha\beta} &= C_{\alpha\beta} - \frac{1}{3} C_\gamma^\gamma \bar{g}_{\beta\gamma} - \frac{1}{2} \left( \nabla_\alpha \nabla_\beta - \frac{1}{3} \bar{g}_{\alpha\beta} \Delta \right) \left( \Delta + \frac{1}{2} \bar{R} \right)^{-1} \left[ \Delta^{-1} \nabla^\gamma \nabla_\delta C_{\gamma\delta} - C_\gamma^\gamma \right], \\
-2 \nabla_\alpha \left( \Delta + \frac{1}{3} \bar{R} \right)^{-1} \left[ \nabla_\delta C_{\beta\gamma} - \nabla_\beta \Delta^{-1} \nabla^\gamma \nabla_\delta C_{\gamma\delta} \right],
\end{align*}
\]

and they satisfy the transverse condition \( B_{\alpha} \mid_\alpha = C_{\alpha} \mid_\alpha = C_{\alpha\beta} \mid_\alpha = 0 \) and the traceless condition \( C_\alpha^\alpha = 0 \). Note that these variables \( (\alpha, \beta, \gamma, \varphi, B_\alpha, C_\alpha, C_{\alpha\beta}) \) are again non-linear perturbation variables, but Eqs. (A5) and (A6) show that the S-V-T decomposition is always possible in a non-perturbative way.

2. Second-order gauge transformation

A coordinate transformation in general relativity accompanies a transformation of the metric tensor \( g_{ab} \) and affects its correspondence of a coordinate position to the homogeneous background universe, called a gauge transformation. Thus, it is necessary to separate the physical degree-of-freedom from fictitious gauge freedoms due to coordinate transformation. Here we consider the most general coordinate transformation to the second order,

\[
\tilde{x}^a = x^a + \xi^a,
\]

and decompose the infinitesimal transformation \( \xi^a \) into scalar parts \( T, L \) and a vector part \( L^a \) based on \( \tilde{g}_{\alpha\beta} \) as

\[
\xi^a = (T, L^a) = (T, L^a + L^a).
\]

While the gauge-transformation of general tensors can be derived in terms of the Lie derivatives [53], we simply use the tensor transformation properties induced by the coordinate transformation

\[
\tilde{g}_{ab}(x^e) = \frac{\partial x^e}{\partial \tilde{x}^a} \frac{\partial x^d}{\partial \tilde{x}^b} g_{cd}(x^e),
\]

where they are evaluated at the same spacetime position, represented by two different values of coordinate components. Evaluating \( g_{ab} \) in Eq. (A9) at \( x^e \) and relating to \( g_{ab}(x^e) \), we derive the transformation of the metric perturbations in Eq. (A2) as

\[
\Delta = \mathcal{A} - (T' + \mathcal{H}T) - \mathcal{A}'T - 2\mathcal{A}'(T' + \mathcal{H}T) + \frac{3}{2} T'T' + t T'T' + 3\mathcal{H} \mathcal{H}T + \frac{1}{2} (2\mathcal{H}^2 + \mathcal{H}') T^2
\]

\[
- \mathcal{A}_{\alpha} \mathcal{L}^\alpha - \mathcal{B}_{\alpha} \mathcal{L}^{\alpha} + \mathcal{T}_{\alpha} \mathcal{L}^{\alpha} + \mathcal{L}^\alpha (\mathcal{T}_{\alpha} + \mathcal{H} \mathcal{T}_{\alpha}) - \frac{1}{2} \mathcal{L}^{\alpha} \mathcal{L}_{\alpha},
\]
\[
\tilde{B}_\alpha = B_\alpha - T_{,\alpha} - 2AT_{,\alpha} - (B'_\alpha + 2\mathcal{H}B_\alpha) T - B_\alpha T' + 2T'T_{,\alpha} + T (T'_{,\alpha} + 2\mathcal{H}T_{,\alpha})
\]
\[
+ L_\alpha - B_{\alpha,\beta} L^\beta - B_\beta L^\alpha_{,\alpha} + 2C_{\alpha,\beta} L^\beta_{,\alpha} - L^\beta_{,\alpha} T' + T_\beta L^\delta_{,\alpha} + L^\gamma T_{,\alpha} - T (L^\alpha_{,\alpha} + 2\mathcal{H}L^\alpha)
\]
\[
- L^\beta_{,\alpha} L^\delta_{,\alpha} - g_{\alpha\beta} L^\delta_{,\gamma} L^\gamma_{,\alpha} - L^\gamma (g_{\alpha\beta,\gamma} L^\delta_{,\gamma} + g_{\alpha\beta} L^\delta_{,\gamma}'),
\]
\[
\tilde{C}_{\alpha\beta} = C_{\alpha\beta} - \mathcal{H}T g_{\alpha\beta} + B_{\alpha,\beta} T - (C'_{\alpha\beta} + 2\mathcal{H}C_{\alpha\beta}) T - \frac{1}{2} T_\alpha T_{,\beta} + \mathcal{H} g_{\alpha\beta} T'T + \frac{\mathcal{H}^2 + \mathcal{H}'}{2} g_{\alpha\beta} T^2
\]
\[
- \frac{1}{2} g_{\alpha\beta,\gamma} L^\gamma - g_{\alpha,\beta} L^\gamma_{,\beta} - C_{\alpha,\beta} L^\gamma - 2C_{\alpha,\gamma} L^\gamma_{,\beta} + g_{\alpha,\beta} L^\gamma_{,\alpha} + \mathcal{H} g_{\alpha,\beta} L^\gamma_{,\alpha} + \frac{1}{2} g_{\alpha,\beta} L^\gamma_{,\alpha} L^\delta_{,\beta}
\]
\[
+ L^\delta (\frac{1}{2} g_{\alpha,\beta,\gamma} L^\gamma_{,\delta} + g_{\alpha,\beta,\gamma} L^\gamma_{,\delta} + \frac{1}{4} g_{\alpha,\beta,\gamma} L^\gamma_{,\delta} + \frac{1}{2} \mathcal{H} g_{\alpha,\beta} L^\gamma_{,\alpha} L^\delta_{,\beta})
\].

It is possible to further decompose these metric perturbations into scalar, vector, and tensor and to derive the transformation of the decomposed metric perturbations by using Eqs. (A5) and (A6). However, a few words in regard to spatial gauge transformation are in order. The spatial homogeneity of the background universe keeps the spatial diffeomorphism intact to all orders in perturbations, and the physics is invariant under spatial gauge-transformation. However, it is well known [52, 54, 55] that the perturbation variables \((\beta, \gamma, B_\alpha, C_\alpha)\) change with the spatial transformation \(L\) or \(L_\alpha\) at the linear order, carrying unphysical gauge modes,

\[
\tilde{\beta} = \beta - T + L', \quad \tilde{\gamma} = \gamma - L, \quad \tilde{B}_\alpha = B_\alpha + L'_\alpha, \quad \tilde{C}_\alpha = C_\alpha - L_\alpha.
\]

As physical quantities are invariant under spatial gauge transformation, they can depend only on two combinations \(\chi = a(\beta + \gamma')\) and \(\Psi_\alpha = B_\alpha + C'_\alpha\) that are invariant under spatial gauge transformations [56]. Writing the metric perturbations in terms of these spatially invariant variables is readily achieved by choosing a spatial gauge that leaves no unphysical gauge freedom \(L_0 = 0\) (i.e., \(L = L_0 = 0\)). We choose the \(C\)-gauge [45] as our spatial gauge choice

\[
\tilde{\gamma} = \gamma = 0, \quad \tilde{C}_\alpha = C_\alpha = 0
\]

which completely sets \(L = 0\) and \(L_\alpha = 0\) to the linear order, while a choice of temporal gauge is left free. Combining the \(C\)-gauge choice with any choice of a temporal gauge \((T = 0)\) at the linear order, the second-order gauge transformation of the spatial metric perturbation in Eqs. (A12) can be simplified as

\[
\tilde{C}_{\alpha\beta} = C_{\alpha\beta} - \mathcal{H}T (2) g_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta,\gamma} L (2)\gamma - \frac{1}{2} (g_{\gamma\alpha} L (2)\gamma + g_{\gamma\beta} L (2)\gamma_{,\alpha}) = C_{\alpha\beta} - \mathcal{H}T (2) g_{\alpha\beta} - L (2)_{,\alpha\beta},
\]

and using Eqs. (A5) and (A6) its decomposed perturbations transform as

\[
\tilde{\gamma} = \gamma - L (2), \quad \tilde{C}_\alpha = C_\alpha - L_\alpha (2).
\]

Therefore, the \(C\)-gauge condition in Eq. (A14) to the second order in perturbations completely removes the unphysical gauge freedom \(L_0 = 0\) to the same order in perturbations, and we take the \(C\)-gauge as our spatial gauge choice throughout the paper. As opposed to choosing a (physical) temporal gauge, the spatial gauge choice affects no physical quantities or the Einstein equations, as the spatial diffeomorphism is unbroken symmetry.

With the spatial \(C\)-gauge choice but without any temporal gauge choice, the metric perturbations transform as

\[
\tilde{\alpha} = \alpha - T' - \mathcal{H}T - \alpha T' - 2\alpha \mathcal{H}T + \frac{3}{2} T'' T' + 3\mathcal{H} T' + \mathcal{H}^2 T^2 + \frac{\mathcal{H}^3}{2} T^2,
\]

\[
\tilde{B}_\alpha = B_\alpha - T_{,\alpha} - 2\alpha T_{,\alpha} - (B'_\alpha + 2\mathcal{H}B_\alpha) T - B_\alpha T' + 2T'T_{,\alpha} + T (T'_{,\alpha} + 2\mathcal{H}T_{,\alpha}),
\]

\[
\tilde{C}_{\alpha\beta} = C_{\alpha\beta} - \mathcal{H}T g_{\alpha\beta} + B_{\alpha,\beta} T - (C'_{\alpha\beta} + 2\mathcal{H}C_{\alpha\beta}) T - \frac{1}{2} T_\alpha T_{,\beta} + T (H g_{\alpha\beta} T' + \frac{1}{2} (\mathcal{H}^2 + \mathcal{H}') g_{\alpha\beta} T'),
\]

and in terms of scalar, vector, and tensor components the metric is

\[
A = \alpha, \quad B_\alpha = \frac{1}{a} \chi_{,\alpha} + \Psi_\alpha, \quad C_{\alpha\beta} = \varphi g_{\alpha\beta} + C_{\alpha\beta},
\]

where we used \(\chi = a(\beta + \gamma')\) and \(\Psi_\alpha = B_\alpha + C'_\alpha\). The above equations are fully general to the second order — No temporal (physical) gauge choice is made, while unphysical spatial gauge freedom is eliminated. The gauge-transformation equations of
the decomposed variables can be readily derived by using Eqs. (A33) and (A36). In order to simplify the situation, we assume that the three-space is flat ($K = 0$). Therefore, the scalar perturbations transform as

$$\ddot{\chi} = \chi - aT + aT\dot{T} + a\Delta T^2 + a\Delta^{-1} \nabla^\alpha \left[ -2aT_{,\alpha} - \frac{1}{a} \left( \chi' + \nabla H \right) T - \left( \Psi_{,\alpha} + 2\nabla T^\alpha T' - T\nabla T_{,\alpha} \right) \right]$$

$$\frac{1}{2} \Delta^{-1} \left[ \left( \frac{1}{a} \chi_{,\alpha} + \Psi^\alpha \right) T_{,\alpha} - \frac{1}{2} T_{,\alpha} T_{,\alpha} - 3 \Delta^{-1} \nabla^\alpha \nabla^\beta \left( \frac{1}{a} \chi_{,\alpha} T_{,\beta} - \frac{1}{2} T_{,\alpha} T_{,\beta} + \Psi_{,\alpha} T_{,\beta} \right) - (C'_{,\alpha} + 2\nabla C_{,\alpha}) T \right], \quad (A21)$$

$$\ddot{\phi} = \phi' - \nabla T' - 2\nabla \phi T' + aT' T + \left( \frac{1}{2} \nabla^2 + \nabla' \right) T^2 + \frac{1}{2} \left( \frac{1}{a} \chi_{,\alpha} T_{,\alpha} + \Psi_{,\alpha} T_{,\alpha} - \frac{1}{2} T_{,\alpha} T_{,\alpha} \right)$$

$$\frac{1}{2} \Delta^{-1} \nabla^\alpha \nabla^\beta \left[ \frac{1}{a} \chi_{,\alpha} T_{,\beta} + \Psi_{,\alpha} T_{,\beta} - \left( C'_{,\alpha} + 2\nabla C_{,\alpha} \right) T - \frac{1}{2} T_{,\alpha} T_{,\beta} \right], \quad (A22)$$

and the vector and tensor perturbations transform as

$$\ddot{\Psi}_{,\alpha} = \Psi_{,\alpha} - 2aT_{,\alpha} - \frac{1}{a} \left( \chi_{,\alpha} + \nabla H \right) T - \left( \Psi_{,\alpha} + 2\nabla T^\alpha T' - T\nabla T_{,\alpha} \right)$$

$$\nabla \alpha \Delta^{-1} \nabla^\beta \left[ -2aT_{,\beta} - \frac{1}{a} \left( \chi_{,\beta} + \nabla H \right) T - \left( \Psi_{,\beta} + 2\nabla T^\beta T' - T\nabla T_{,\beta} \right) \right]$$

$$\nabla \alpha \Delta^{-1} \nabla^\beta \nabla^\gamma \left[ \frac{1}{a} \chi_{,\alpha} T_{,\beta} + \Psi_{,\alpha} T_{,\beta} - \left( C'_{,\alpha} + 2\nabla C_{,\alpha} \right) T - \frac{1}{2} T_{,\alpha} T_{,\beta} \right]' \right]$$

$$\tilde{C}_{,\alpha\beta} = C_{,\alpha\beta} + \frac{1}{a} \chi_{,\alpha} T_{,\beta} + \Psi_{,\alpha} T_{,\beta} - (C'_{,\alpha} + 2\nabla C_{,\alpha}) T - \frac{1}{2} T_{,\alpha} T_{,\beta} - \frac{1}{3} \left( \frac{1}{a} \chi_{,\gamma} + \Psi^\gamma \right) T_{,\gamma} T_{,\beta}$$

$$- \frac{1}{2} \left( \nabla_{,\gamma} \nabla_{,\beta} - \frac{1}{3} g_{,\alpha\beta} \Delta \right) \Delta^{-1} \left[ 3 \Delta^{-1} \nabla^\gamma \nabla^\delta \left( \frac{1}{a} \chi_{,\alpha} T_{,\beta} + \Psi_{,\alpha} T_{,\beta} - \left( C'_{,\alpha} + 2\nabla C_{,\alpha} \right) T - \frac{1}{2} T_{,\alpha} T_{,\beta} \right) \right]$$

$$- \left( \frac{1}{a} \chi_{,\gamma} + \Psi^\gamma \right) T_{,\gamma} + \frac{1}{2} T_{,\gamma} T_{,\gamma} - 2 \nabla_{,\gamma} \Delta^{-1} \nabla^\gamma \left[ \frac{1}{2a} \left( \chi_{,\beta} + \nabla H \right) T_{,\gamma} + \Psi_{,\beta} T_{,\gamma} - \left( C'_{,\beta} + 2\nabla C_{,\beta} \right) \right] T - \frac{1}{T_{,\beta} T_{,\gamma} T_{,\gamma}}$$

It is evident that scalar, vector, and tensor components mix together due to the nonlinear quadratic terms, present in the second-order gauge transformation. Furthermore, even with no vector or tensor at the linear order, the second-order scalar perturbations generate the second-order vector and tensor perturbations. Nevertheless, the equations greatly simplify in the absence of linear order vector and tensor perturbations, especially for scalar perturbations.

Equations (A21)–(A24) are completely general to the second order in perturbations, and no physical gauge choice is made yet. Furthermore, it is apparent that combined with the spatial $C$-gauge, a proper temporal gauge choice at the linear order provides a valid gauge choice at the second order, e.g., if we choose a gauge condition at the linear order

$$\ddot{\chi}^{(1)} = \chi^{(1)} = 0, \quad T^{(1)} = 0, \quad \ochrome$$

the transformation equation in Eq. (A21) at the second order in perturbations takes the form

$$\ddot{\dot{\chi}}^{(2)} = \chi^{(2)} - aT^{(2)}, \quad \ochrome$$

identical to its linear order transformation equation. Therefore, by choosing a gauge condition to the second order, $\chi^{(1,2)} = 0$ in this example, we completely fix the gauge condition to the second order in perturbations, leaving no gauge ambiguities.

Similarly to the transformation in metric tensor, the coordinate transformation in Eq. (A7) induces the vector transformation

$$\dddot{V}^a(x^e) = \frac{\partial \phi^a}{\partial x^b} V^b(x^e), \quad \ochrome$$

and by evaluating the transformed vector at the same coordinate $x^e$, we can derive the vector gauge transformation relation as

$$\dddot{\tilde{U}}^a = U^a + \nabla T U^a - T U^{\alpha}, \quad \ochrome$$
and the perturbations in photon wavevector transform as
\[
\delta \nu = \delta \nu + \frac{d}{d\lambda} T + 2H T (1 + \delta \nu) - \delta \nu T - T'' - (\mathcal{H}' - 2\mathcal{H}'') T^2 + TT_\alpha e^\alpha - 2\mathcal{H} T_\alpha e^\alpha, \tag{A29}
\]
\[
\delta \epsilon^\alpha = \delta \epsilon^\alpha + 2H T (\epsilon^\alpha + \delta \epsilon^\alpha) - T \delta \epsilon^\alpha - 2\mathcal{H} T T^\alpha e^\alpha - (\mathcal{H}' - 2\mathcal{H}'') T^2 e^\alpha. \tag{A30}
\]

3. Second-order gauge-invariant variables

Here we construct second-order gauge-invariant variables. First, comparing to the linear-order calculations, we discuss the difference in the second-order calculation by explicitly constructing a second-order gauge-invariant variable. We then give expressions for other second-order gauge-invariant variables used in the text. Our construction of second-order gauge-invariant variables follows the work in [45], but without the restriction that there is no vector or tensor at the linear order.

As an example, we construct the second-order gauge-invariant variable \(\varphi_\chi\). The linear-order gauge-transformation equations for \(\chi\) and \(\varphi\) in Eqs. (A21) and (A22) are
\[
\dot{\chi}^{(1)} = \chi^{(1)} - aT^{(1)}, \quad \dot{\varphi}^{(1)} = \varphi^{(1)} - H T^{(1)}, \tag{A31}
\]
and we can easily construct a linear-order gauge-invariant variable
\[
\varphi_\chi^{(1)} \equiv \varphi^{(1)} - H\chi^{(1)}. \tag{A32}
\]

The notation of the gauge-invariant variable is set up such that \(\varphi_\chi\) becomes \(\varphi\) when the gauge condition \(\chi = 0\) is adopted (similarly, we can also construct a gauge-invariant variable \(\chi_\varphi \equiv -\varphi_\chi / H = \chi - \varphi / H\), such that \(\chi_\varphi\) becomes \(\chi\) when the gauge condition \(\varphi = 0\) is adopted). Therefore, it is desirable to construct such gauge-invariant variables at the second order. The simplest guess is to extend the definition of \(\varphi_\chi\) at the linear order to the second order. Using Eqs. (A21) and (A22), we verify that the simplest choice transforms as
\[
\dot{\varphi} - H\dot{\chi} = \varphi - H\chi - \varphi T - 2H\varphi T + \frac{1}{2} H'T^2 + \frac{1}{2} \left(\frac{1}{a} \chi'^\alpha T_\alpha + \Psi'^\alpha T_\alpha - \frac{1}{2} T'^\alpha T_\alpha\right)
\]
\[
- \frac{1}{2} \Delta^{-1} \nabla^\alpha \nabla^\beta \left(\frac{1}{a} \chi + \Psi \right) T_{\alpha \beta} - (C'_{\alpha \beta} + 2\mathcal{H} C_{\alpha \beta}) T - \frac{1}{2} T_\alpha T_{\alpha \beta}
\]
\[
- \mathcal{H} \Delta^{-1} \nabla^\alpha - 2a \Delta^{-1} \nabla^\alpha \nabla^\beta \left(\frac{1}{a} \chi + \Psi \right) T_{\alpha \beta} - (C'_{\alpha \beta} + 2\mathcal{H} C_{\alpha \beta}) T - \frac{1}{2} T_\alpha T_{\alpha \beta} + \Psi T_{\alpha \beta} - (C'_{\alpha \beta} + 2\mathcal{H} C_{\alpha \beta}) T\right)'.
\]

It is evident that this particular combination is not gauge-invariant to the second-order. However, the transformation in Eq. (A33) suggests that a quadratic correction to the simplest combination be needed to construct a second-order gauge-invariant variable and the correction should vanish if we choose the gauge condition \(\chi = 0\). Since the quadratic correction only involves the linear-order transformation, we use
\[
T^{(1)} = \frac{1}{a} \left(\chi^{(1)} - \tilde{\chi}^{(1)}\right), \tag{A34}
\]
to find the quadratic correction for the gauge-invariant variable that vanishes if we choose the gauge-condition \(\chi = 0\).

By substituting Eq. (A34) into Eqs. (A21) and (A22), we have
\[
\tilde{\chi} = \chi - aT + H(\chi - \chi^2) - (\chi - \tilde{\chi})\tilde{\chi} - \Delta^{-1} \nabla^\alpha \left[2\alpha_\chi (\chi - \tilde{\chi})_\alpha + (\chi - H\chi) \chi_\alpha - (\chi - H\tilde{\chi}) \tilde{\chi}_\alpha\right]
\]
\[
+ (\Psi' + 2\mathcal{H}\Psi_\alpha)(\chi - \tilde{\chi}) + \Psi_\alpha (\chi - \tilde{\chi})' - a^2 \Delta^{-1} \left[\frac{1}{4a^2} (\chi'^\alpha \chi_\alpha - \chi'^\alpha \chi_\alpha) + \frac{1}{2a} \Psi'^\alpha (\chi - \tilde{\chi})_{\alpha}\right]
\]
\[
- \frac{3}{2a} \Delta^{-1} \nabla^\alpha \nabla^\beta \left(\frac{1}{a} \chi - \tilde{\chi} \chi_\beta + \Psi_\alpha (\chi - \tilde{\chi})_{\beta} - (C'_{\alpha \beta} + 2\mathcal{H} C_{\alpha \beta})(\chi - \tilde{\chi})\right), \tag{A35}
\]
\[
\tilde{\varphi} = \varphi - HT - (\varphi_\chi + 2H\varphi_\chi)(\chi - \tilde{\chi}) - \frac{1}{2} (H^2 + \tilde{H})(\chi^2 - \tilde{\chi}^2) + H^2 (\chi - \tilde{\chi})^2 - H(\chi - \tilde{\chi})\tilde{\chi} + \frac{1}{4a^2} (\chi'^\alpha \chi_\alpha - \chi'^\alpha \chi_\alpha)
\]
\[
+ \frac{1}{2a} \Psi'^\alpha (\chi - \tilde{\chi})_{\alpha} - \frac{1}{2a} \Delta^{-1} \nabla^\alpha \nabla^\beta \left(\frac{1}{a} \chi_{\alpha \beta} - \chi_{\alpha \beta} \tilde{\chi}_{\alpha} + \Psi_\alpha (\chi - \tilde{\chi})_{\beta} - (C'_{\alpha \beta} + 2\mathcal{H} C_{\alpha \beta})(\chi - \tilde{\chi})\right). \tag{A36}
\]
Collecting terms with tilde on the left-hand side, we obtain the second-order gauge-invariant variable

$$
\varphi_\chi \equiv \varphi - H \chi - (\dot{\varphi}_\chi + 2H \varphi_\chi) \chi - \frac{1}{2} (\dot{H} + H^2) \chi^2 + H \Delta^{-1} \nabla^\alpha \left[ 2\alpha_{\chi,\alpha} + (\chi - H \chi) \chi_{\alpha} + (\Psi'_\alpha + 2H \Psi_\alpha) \chi + \Psi_\alpha \chi \right] \\
+ \frac{1}{4a^2} \chi^\alpha \chi_\alpha + \frac{1}{2a} \Psi_\alpha \chi_\alpha - \frac{1}{2a} \Delta^{-1} \nabla^\alpha \nabla^\beta \left[ \frac{1}{2a} \chi_{\alpha,\alpha} + \Psi_\alpha \chi_\beta - (C'_{\alpha\beta} + 2HC_{\alpha\beta}) \chi \right] \\
+ a^2 H \Delta^{-1} \left[ \frac{1}{4a^2} \chi^\alpha \chi_\alpha + \frac{1}{2a} \Psi_\alpha \chi_\alpha - \frac{3}{2a} \Delta^{-1} \nabla^\alpha \nabla^\beta \left( \frac{1}{2a} \chi_{\alpha,\alpha} + \Psi_\alpha \chi_\beta - (C'_{\alpha\beta} + 2HC_{\alpha\beta}) \chi \right) \right] \equiv \varphi - \chi + \chi^{(q)} \chi, \quad (A37)
$$

and we check its gauge-invariance by explicitly performing transformation. The gauge-invariant variable \( \varphi_\chi \) has the property that \( \varphi_\chi \rightarrow \varphi \) under the gauge condition \( \chi = 0 \). Similar calculations can be performed to construct the second-order gauge-invariant variable

$$
\alpha_\chi \equiv \alpha - \dot{\alpha}_\chi - 2\alpha_{\chi} + \frac{1}{2} \dot{H} \chi^2 + H \Delta^{-1} \nabla^\alpha \left[ 2\alpha_{\chi,\alpha} + (\chi - H \chi) \chi_{\alpha} + (\Psi'_\alpha + 2H \Psi_\alpha) \chi + \Psi_\alpha \chi \right] \\
+ \frac{1}{4a^2} \chi^\alpha \chi_\alpha + \frac{1}{2a} \Psi_\alpha \chi_\alpha - \frac{3}{2a} \Delta^{-1} \nabla^\alpha \nabla^\beta \left( \frac{1}{2a} \chi_{\alpha,\alpha} + \Psi_\alpha \chi_\beta - (C'_{\alpha\beta} + 2HC_{\alpha\beta}) \chi \right) \equiv \alpha - \chi + \alpha^{(q)} \chi, \quad (A38)
$$

where we defined the quadratic correction terms that vanish under the gauge condition indicated in the subscript (similarly, we also define \( \varphi_\chi \equiv \varphi - H \chi + \varphi^{(q)} \chi \) in Eq. [A37]).

We continue to repeat the exercise to construct gauge-invariant variables for four vectors. Perturbations to the photon wavevector and four velocity can be rearranged to define second-order gauge-invariant variables as

$$
\delta e^\alpha_\chi \equiv \delta e^\alpha + 2H (e^\alpha + \delta e^\alpha) - \chi \delta e^\alpha + (\dot{H} - 3H^2) \chi^2 e^\alpha
$$

$$
-2H e^\alpha \Delta^{-1} \nabla^\beta \left[ 2\alpha_{\chi,\beta} + (\chi - H \chi) \chi_{\alpha} + (\Psi'_\alpha + 2H \Psi_\alpha) \chi + \Psi_\alpha \chi \right] \\
-2a H e^\alpha \Delta^{-1} \left[ \frac{1}{4a^2} \chi^\beta \chi_{\beta} + \frac{1}{2a} \Psi^\beta \chi_{\beta} - \frac{3}{2a} \Delta^{-1} \nabla^\beta \nabla^\gamma \left( \frac{\chi_{\beta \gamma}}{2a} + \Psi_\beta \chi_{\gamma} - (C'_{\beta \gamma} + 2HC_{\beta \gamma}) \chi \right) \right] \equiv \delta e^\alpha + 2H \delta e^\alpha - \chi \delta e^\alpha + (\dot{H} - 3H^2) \chi^2 e^\alpha \quad (A39)
$$

$$
\delta \nu_\chi \equiv \delta \nu + H \chi + \dot{\chi} - \frac{1}{a} \chi_{\alpha} e^\alpha + 2H \delta \nu - \chi \delta e^\alpha + \delta \nu (\chi - H \chi) - \frac{1}{a} \chi_{\alpha} e^\alpha - \chi - H \chi^2 - 2H \chi
$$

$$
+ \frac{3H}{a} \chi_{\alpha} e^\alpha + \frac{1}{a} \chi_{\alpha} e^\alpha - H \Delta^{-1} \nabla^\alpha \left[ 2\alpha_{\chi,\alpha} + (\chi - H \chi) \chi_{\alpha} + (\Psi'_\alpha + 2H \Psi_\alpha) \chi + \Psi_\alpha \chi \right] \\
+ a^2 H \Delta^{-1} \left[ \frac{1}{4a^2} \chi^\alpha \chi_\alpha + \frac{1}{2a} \Psi_\alpha \chi_\alpha - \frac{3}{2a} \Delta^{-1} \nabla^\alpha \nabla^\beta \left( \frac{\chi_{\alpha \beta}}{2a} + \Psi_\alpha \chi_{\beta} - (C'_{\alpha \beta} + 2HC_{\alpha \beta}) \chi \right) \right] \\
- \frac{1}{a} \frac{d}{d\chi} \left[ \Delta^{-1} \nabla^\alpha \left( 2\alpha_{\chi,\alpha} + (\chi - H \chi) \chi_{\alpha} + (\Psi'_\alpha + 2H \Psi_\alpha) \chi + \Psi_\alpha \chi \right) \right] \\
- \frac{1}{a} \frac{d}{d\chi} \left[ a^2 \Delta^{-1} \left[ \frac{1}{4a^2} \chi^\alpha \chi_\alpha + \frac{1}{2a} \Psi_\alpha \chi_\alpha - \frac{3}{2a} \Delta^{-1} \nabla^\alpha \nabla^\beta \left( \frac{\chi_{\alpha \beta}}{2a} + \Psi_\alpha \chi_{\beta} - (C'_{\alpha \beta} + 2HC_{\alpha \beta}) \chi \right) \right] \right] \quad (A40)
$$

$$
U_\chi^\alpha \equiv U^\alpha + H \chi U^\alpha - \chi U^\alpha \quad (A41)
$$

where the spatial part of the four velocity itself is gauge-invariant at the linear order. Finally, gauge-invariant variables for vector and tensor in metric perturbations are

$$
\Psi_{\chi \alpha} \equiv \Psi_\alpha - \frac{2}{a} \alpha_{\chi \alpha} + \frac{1}{a} (\chi_{\chi \alpha} - H \chi \chi_{\alpha}) - \left( \Psi_\alpha + 2H \Psi_\alpha \right) \chi - \Psi_\alpha (\chi - H \chi)
$$

$$
- \nabla^\alpha \Delta^{-1} \nabla^\beta \left[ \frac{2}{a} \alpha_{\chi \beta} + \frac{1}{a} (\chi_{\chi \beta} - H \chi \chi_{\beta}) - \left( \Psi_\beta + 2H \Psi_\beta \right) \chi - \Psi_\beta (\chi - H \chi) \right] \\
+ 2a \Delta^{-1} \nabla^\alpha \left[ \frac{1}{2a^2} \chi_{\alpha \beta} + \frac{1}{a} \Psi_{\alpha \beta} - \left( \delta_{\alpha \beta} + 2HC_{\alpha \beta} \right) \chi \right] \\
- 2a \Delta^{-1} \nabla^\alpha \Delta^{-1} \nabla^\beta \nabla^\gamma \left[ \frac{1}{2a^2} \chi_{\beta \gamma} + \frac{1}{a} \Psi_{\beta \gamma} - \left( \delta_{\beta \gamma} + 2HC_{\beta \gamma} \right) \chi \right] \quad (A42)
$$

$$
C_{\chi \alpha \beta} \equiv C_{\alpha \beta} + \frac{1}{2a^2} \chi_{\alpha \beta} + \frac{1}{a} \Psi_{\alpha \beta} - \left( \delta_{\alpha \beta} + 2HC_{\alpha \beta} \right) \chi - \frac{1}{3a} g_{\alpha \beta} \left( \frac{1}{2a} \chi^\gamma + \Psi^\gamma \right) \chi_{\gamma}
$$
inhomogeneous universe: we add perturbations to describe the deviation from homogeneity, defining the observer four velocity in an hypersurface orthogonal to (and also vector of scalar, vector, and tensor contributions. Exactly for this reason, the second-order gauge-invariant variable for tensor \( C_{\alpha\beta} \) (and also vector \( \Psi_{\beta} \)) requires a choice of scalar gauge condition. It is noted that other choices of scalar gauge condition can be made to construct these second-order gauge-invariant variables, and even with no vector or tensor at the linear order one needs a choice of scalar gauge condition for second-order vector and tensor gauge-invariant variables, as scalar contributions generate vector and tensor to the second order in perturbations.

\[ \text{Appendix B: Photon Wavevector} \]

In this section, we explicitly construct second-order tetrads to derive the photon wavevector \( k^a \) in the FRW frame in terms of local observable quantities. We then clarify the relation of the photon wavevector to the observed angles \((\theta, \phi)\) in the presence of additional extra degree of freedom supplied by the conformal transformation of the metric. The basic description of the geometric optics can be found in [57-69].

1. Second-order tetrads in the observer rest frame

Here we construct a second-order orthonormal basis in the observer rest frame defined by the observer’s four velocity \( u^a \). The time-like velocity \((-1 = u_a u^a)\) of the observer defines the (proper) time-direction \([e_1]^a = u^a\) in the local Lorentz frame and its hypersurface orthogonal to \(u^a\). Three spacelike four vectors \([e_i]^a\) \((i = 1, 2, 3)\) can be further defined to serve as spatial directions in the observer rest frame. These four orthonormal vectors are called tetrads. In the local Lorentz frame, the metric is Minkowsky \((g_{\mu\nu} = \eta_{\mu\nu} = g_{ab}[e_i]^a[e_i]^b, \mu, \nu = t, x, y, z\) and the tetrads are simply unit vectors: \([e_1]^a = (1, 0, 0, 0)\) and \([e_i]^a = (0, \delta_i^a)\). However, we are interested in tetrad expressions in an inhomogeneous expanding (FRW) universe with the metric described by Eqs. (A1) and (A2).

Accounting for the expansion, the tetrads in a homogeneous universe are

\[
[e_i]^a = u^a = \frac{1}{a}(1, 0, 0, 0), \quad [e_x]^a = \frac{1}{a}(0, 1, 0, 0), \quad [e_y]^a = \frac{1}{a}(0, 0, 1, 0), \quad [e_z]^a = \frac{1}{a}(0, 0, 0, 1),
\]

with the tetrad index \( \mu \) of \([e_i]^a\) raised or lowered by \( \eta_{\mu\nu} \), while the FRW index \( a \) is raised or lowered by \( g_{ab}\),\(^3\) In an inhomogeneous universe, we add perturbations to describe the deviation from homogeneity, defining the observer four velocity in an inhomogeneous universe:

\[
[e_i]^a = u^a = \frac{1}{a} (1 + \delta u^0, \mathcal{U}^a).
\]

With the normalization condition \(-1 = g_{ab}[e_i]^a[e_i]^b\), we have the observer four velocity to the second order in perturbations

\[
[e_i]^a = \frac{1}{a} \left[ 1 - A + \frac{3}{2} A^2 + \frac{1}{2} \mathcal{U}^a \mathcal{U}_a - \mathcal{U}^a \mathcal{U}_a, \mathcal{U}^a \right], \quad [e_1]^a = a \left[ -1 - A + \frac{1}{2} A^2 - \frac{1}{2} \mathcal{U}^a \mathcal{U}_a, \mathcal{U}_a - A \mathcal{B}_a + A \mathcal{B}_a + 2 \mathcal{U}^a \mathcal{C}_{\alpha\beta} \right],
\]

consistent with Eq. (10). The remaining spatial directions are also derived by using the orthonormality condition \( \delta_{ij} = g_{ab}[e_i]^a[e_j]^b \) and \( \mathcal{U}^a = g_{ab}[e_i]^a[e_i]^b \) as

\[
[e_i]^a = \frac{1}{a} \left[ \mathcal{U}_a - A \mathcal{B}_a + 2 A \mathcal{B}_a - A \mathcal{C}_i^a + A \mathcal{C}_i^a - A \mathcal{C}_i^a + \frac{1}{2} \left( \mathcal{U} \mathcal{U}^a - A \mathcal{B}^a \right) + \frac{3}{2} \mathcal{C}_{\alpha\beta} \mathcal{C}_{\alpha\beta} \right],
\]

\(^3\) In the local Lorentz frame of the observer, the time coordinate \(x_T^a\) is equivalent to the proper time of the observer. Therefore, the path \(x_T^a\) of the observer in a FRW coordinate parametrized by the proper time yields the observer four velocity \(u^a = \partial x_T^a / \partial s_p = [e_i]^a\). While our convention for tetrads is set consistent with this notion \((|e_i|^a \equiv u^a = [-e_i]^a)\), it can be set with different sign, \([e_i]^a \equiv u^a = [-e_i]^a\), which simply changes the direction of proper time backward, leaving the construction of other spatial directions \([e_i]^a\) unaffected.
\[ [e_i] \alpha = a \left[ -U_i - A U_i - C_{i \beta} U^\beta, \delta_{i \alpha} + C_{i \alpha} + \frac{1}{2} (B_i B_\alpha - C_{i \beta} C_\beta + U_i U_\alpha) - U_\alpha B_i \right]. \]  

(B5)

To ensure that our second-order construction of tetrads is correct, we reconstruct the FRW metric \( g_{ab} \) by transforming the local coordinate \( x^\mu_L \) to the FRW coordinate \( x^\mu_F \):

\[ g_{ab}(x_F) = \frac{\partial x^\mu_L}{\partial x^\mu_F} \frac{\partial x^\nu_F}{\partial x^\nu_L} \eta_{\mu \nu} = [e^a]_\alpha [e^b]_\beta \eta_{\alpha \beta} = -\sum_i [e_i]_\alpha [e_i]_\beta , \]

(B6)

and check if each component of the reconstructed metric is identical to the metric defined in Eqs. (A1) and (A2).

2. Photon wavevector in FRW coordinates

The photon propagation direction is the direction orthogonal to the hypersurface defined by the same phase \( \theta = k \cdot x_L - \omega t \). The components of the photon wavevector in the local Lorentz frame are

\[ k^\alpha_L = \eta^{\mu \nu} \theta_{\mu \nu} = (\omega, k) = 2\pi \nu \left( 1, -\hat{n} \right), \]

(B7)

where the angular frequency is \( \omega = 2\pi \nu \), \( k_L = |k| = 2\pi / \lambda \), the speed of light \( c = \lambda \nu = 1 \), and we put the subscript \( L \) to emphasize that the components are written in the local Lorentz frame (as opposed to the FRW frame). We defined a unit directional vector \( \hat{n} \) for photon propagation measured by the observer, \( \hat{n} \propto -k \). The photon frequency measured by the observer is then

\[ -\eta_{\mu \nu} u^\mu_L k^\nu_L = \omega = 2\pi \nu , \]

(B8)

where \( u^\mu_L = [e_\mu] = (1, 0, 0, 0) \) in the local Lorentz frame. Since the photon wavevector is expressed in terms of physical quantities (the observed frequency and angle), there are no additional degrees of freedom associated with the photon wavevector in Eq. (B7).

Now we compute the photon wavevector in a FRW coordinate by transforming the photon wavevector in Eq. (B7) as

\[ k^a = \frac{\partial x^a}{\partial x^L} k^\mu_L = [e_\mu]^a k^\mu_L , \]

(B9)

\[ k^0 = \frac{2\pi \nu}{a} \left\{ 1 - A - n^i (U_i - B_i) + \frac{3}{2} \theta^2 + \frac{1}{2} U^\beta U_\beta - U^\beta B_\beta - n^i \left[ 2AB_i - A U_i + C_i \beta (U^\beta + B^\beta) \right] \right\} , \]

(B10)

\[ k^\alpha = \frac{2\pi \nu}{a} \left\{ -n^\alpha + U^\alpha + n^i C_i - n^i \left[ \frac{1}{2} (U_i U_\alpha - B_i B_\alpha) + \frac{3}{2} C_i \beta C_\beta \right] \right\} , \]

(B11)

where \( n^\alpha \) is the spatial component of the unit directional vector \( \hat{n} \) in a local Lorentz frame, other perturbation quantities are those in a FRW frame, and the repeated indices indicate the summation over the spatial components.4 Because of the observer velocity \( U^\alpha \) and the gravitational potential \( A \), the components of the photon wavevector appear different in a FRW coordinate, but its physical interpretation depends on gauge choice. The spatial photon direction \( k^\alpha \) in a FRW coordinate is different from that \( n^\alpha \) in the observer rest frame, because the observer is not at rest in the FRW frame. However, the photon frequency measured by the observer is a Lorentz scalar, independent of frame:

\[ -g_{ab} u^a k^b = -\eta_{\mu \nu} u^\mu_L k^\nu_L = \omega = 2\pi \nu , \]

(B12)

which sets the affine parameter \( \nu \) as in Eq. (4) — Given the spacetime metric \( g_{ab} \) and locally measured observables \( (\nu, \hat{n}) \), the wavevector \( k^a \) in Eq. (B9) is therefore completely set in terms of physical quantities.

---

4 Greek indices \( \alpha, \beta \) are used to represent the spatial components of four vectors in a FRW coordinate, and Latin indices \( i, j \) are used to represent those in a Local Lorentz frame. However, as far as the summation is concerned, there is no distinction, as the three vectors in a FRW coordinate are based on the mean three-metric \( \bar{g}_{i \alpha} \) in a flat universe \( (K = 0) \). Nevertheless, it is noted that those three vectors have different values of their components depending on frames.
3. Normalization constant

With the conformal transformation relation in Eq. (5) and the wavevector in Eqs. (110) and (111), we have one degree of freedom in the overall amplitude of the photon wavevector, given the metric tensor \( g_{ab} \):

\[
\bar{k}^\alpha = C a^2 k^\alpha = C \nu a .
\] (B13)

While the normalization coefficient \( C \) is constant, the photon frequency measured by local observers changes at each spacetime due to redshift and perturbations, and so does the product \( C \nu a \). The conformally transformed photon wavevector can be explicitly written as

\[
\begin{align*}
\bar{k}^0 & = 2 \pi C \nu a \left( 1 - A - n^i (U_i - B_i) + \frac{3}{2} A^2 - \frac{1}{2} U^\beta U_\beta - n^i \left[ 2 A B_i - A U_i + C_{i\beta} \left( U^\beta + B^\beta \right) \right] \right) , \\
\bar{k}^\alpha & = 2 \pi C \nu a \left( - n^\alpha + U^\alpha + n^i C_{i\alpha} - n^i \left[ \frac{1}{2} \left( U_i U^\alpha - B_i B^\alpha \right) + \frac{3}{2} C_{i\beta} C_{\beta\alpha} \right] \right) .
\end{align*}
\] (B14)

(B15)

Noting that due to expansion of the universe the photon frequency redshifts as \( \bar{\nu} \propto 1 / a \) in a homogeneous universe, we have the background relation and the normalization constant as

\[
\bar{k}^\alpha = \bar{C} a^2 k^\alpha = 2 \pi \bar{C} \nu a (1 - n^\alpha) \equiv (1 - e^\alpha) , \quad \bar{C} = \frac{1}{2 \pi \bar{\nu} a} = \frac{1}{2 \pi \bar{\nu}_0} , \quad e^\alpha = n^\alpha ,
\] (B16)

where \( \bar{\nu} \equiv \bar{\nu}_0 / a \) and \( \bar{\nu}_0 \) is a constant. Equation (B116) indicates that all the observers along the photon path measures the same direction \( n^\alpha = e^\alpha \) (constant) and infers the same frequency \( \nu \) based on the observed redshift in a homogeneous universe.

Therefore, our parametrization of the photon wavevector in Eq. (6) is completely general with perturbations described by \((\delta \nu, \delta e^\alpha)\), subject to the null condition in Eq. (9) and the geodesic equations (11) and (13). However, these perturbations are related to the physical quantities \((\nu, \hat{n})\) measured by local observers at each spacetime point with only one degree of freedom in the overall amplitude \( C \), arising from the conformal transformation. This can be further understood as follows. Splitting the photon frequency and the normalization constant into the mean and its fluctuation as

\[
\nu \equiv \bar{\nu} a (1 + \Delta \nu) , \quad C \equiv \bar{C} (1 + \delta C) .
\] (B17)

The perturbations of the photon wavevector are related to the normalization constant as

\[
\begin{align*}
\delta \nu^{(1)} & = - A - n^\alpha (U_\alpha - B_\alpha) + \delta C + \Delta \nu , \\
\delta e^{\alpha(1)} & = - U^\alpha - n^i C_{i\beta} + n^i (\delta C + \Delta \nu) , \\
\delta \nu^{(2)} & = - A - n^\beta (U_\beta - B_\beta) + \frac{3}{2} A^2 - \frac{1}{2} U^\beta U_\beta - n^\beta \left[ 2 A B_\beta - A U_\beta + C_{\beta\gamma} \left( U^\gamma + B^\gamma \right) \right] \\
& \quad + (\delta C + \Delta \nu) \left( 1 - A - n^\beta (U_\beta - B_\beta) \right) + \delta C \Delta \nu , \\
\delta e^{\alpha(2)} & = - U^\alpha - n^\beta C_{\beta\gamma} + n^\gamma \left[ \frac{1}{2} \left( U_i U^\alpha - B_i B^\alpha \right) + \frac{3}{2} C_{i\beta} C_{\beta\alpha} \right] \\
& \quad + (\delta C + \Delta \nu) (n^\alpha - U^\alpha - C_{i\beta} n^i) + \delta C \Delta \nu n^\alpha ,
\end{align*}
\] (B18)
(B19)
(B20)
(B21)

where the observed angle \( n^\alpha \) is defined in a nonperturbative way and the perturbation order in each quantity can be straightforwardly understood in conjunction with those in the left-hand side. This relation explicitly shows that there is only one degree of freedom \( C \) and the wavevector is completely set once the normalization constant is chosen. By removing the normalization constant at each order, we derive

\[
\begin{align*}
n^\alpha & = e^\alpha + \delta e^\alpha + U^\alpha + C_{i\beta} e^\beta + \left\{ \delta e^\beta + U^\beta - \frac{1}{2} e^\gamma C_{\gamma\beta} + [\delta \nu + A + e^\gamma (U_i - B_i)] e^\beta \right\} C_{\beta\alpha} - \frac{1}{2} \left( U_i U^\alpha - B_i B^\alpha \right) e^\gamma \\
& \quad + \left[ - \delta \nu - A - (U_\beta - B_\beta) e^\beta + \frac{3}{2} A^2 - \frac{1}{2} U^\beta U_\beta - 3 A B_\beta e^\beta + 2 A U_\beta e^\beta - 2 C_{\beta\gamma} U_\gamma e^\beta - \delta e^\beta (U_\beta - B_\beta) \right] \\
& \quad + \delta \nu \left[ \delta \nu + A + 2 e^\beta (U_\beta - B_\beta) \right] + (U_\beta - B_\beta) (U_i - B_i) e^\gamma e^\gamma \right\} e^\alpha - \left[ \delta \nu + A + e^\beta (U_\beta - B_\beta) \right] \delta e^\alpha \\
& \equiv e^\alpha + \delta n^\alpha ,
\end{align*}
\] (B22)
where we defined the perturbation \( \delta n^\alpha \) in the observed angle \( n^\alpha \) with respect to \( e^\alpha \). Note that the expression is independent of \( \mathbb{C} \), because it is an observable quantity, while individual components \( \delta e^\alpha \) and \( \delta e^\alpha \) are affected by the choice of the normalization constant and so is \( e^\alpha \), because we split one observable quantity \( n^\alpha \) into the mean \( e^\alpha \) and the perturbation \( \delta n^\alpha \) around it. Furthermore, the perturbation is subject to the unit normalization condition: \( n_\alpha n^\alpha = 1 \), which implies

\[
e^\alpha \delta n^{\alpha(1)} = 0, \quad 2e^\alpha \delta n^{\alpha(2)} + \delta n^{\alpha(1)} \delta n^{\gamma(1)} = 0,
\]

and the orthogonality condition for another unit directional vector \( \hat{\theta} \) (and similarly for \( \hat{\phi} \))

\[
e^\alpha \delta n^{\alpha(1)} + \delta n^{\alpha(1)} e^\alpha = 0, \quad e^\alpha \delta n^{\alpha(2)} + \delta n^{\alpha(1)} e^\alpha + \delta n^{\alpha(1)} \delta n^{\alpha(1)} = 0,
\]

where the two unit directional vectors constructed from the observed angle \( \hat{n} \) are

\[
\hat{\theta} = \frac{\partial}{\partial \theta} \hat{n} \equiv e^\alpha + \delta n^\alpha, \quad \hat{\phi} = \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \hat{n} \equiv e^\phi + \delta n^\phi,
\]

and a similar set for the unit directional vector \( e^\alpha \) is defined as \( (e^\alpha, e^\alpha) \).

4. Observed angle

The photon wavevector is measured by the observer, and the observed direction of the source galaxies is independent of our choice of the normalization constant (or the parametrization of the photon wavevector). However, the observed direction is characterized by the observed angle \( (\theta, \phi) \) of unit directional vector \( \hat{n} \) in the local Lorentz frame as

\[
\hat{n} = n^\alpha_L = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \tag{B26}
\]

and these components depend on the choice of frame. For example, suppose the observer frame is moving with velocity \( v \) relative to the rest frame (say, the rest frame of CMB). The observed angle in the rest frame is \([60]\)

\[
\hat{n}' = \left( \frac{\hat{n} \cdot \hat{v} - v}{1 - \hat{n} \cdot \hat{v}} \right) \hat{v} + \frac{\hat{n} - (\hat{n} \cdot \hat{v}) \hat{v}}{\gamma (1 - \hat{n} \cdot \hat{v})} = (\sin \theta' \cos \phi', \sin \theta' \sin \phi', \cos \theta'), \tag{B27}
\]

where \( v = |v| \) and \( \gamma = (1 - v^2)^{-1/2} \). The aberration due to the relative velocity affects the observed angle. The generalization of this relation to the general relativistic case is the four vector of the photon direction \([38]\)

\[
N^\alpha = -\frac{k^\alpha}{2\pi \nu} + u^\alpha, \tag{B28}
\]

which satisfies the spacelike condition \( N^\alpha N_\alpha = 1 \) and the orthogonality condition \( N^\alpha u_\alpha = 0 \). Equation (B28) can be readily derived from Eq. (B28) by Lorentz boosting the observer velocity. However, since we are interested in expressing quantities in the FRW frame in terms of local observable \( (\theta, \phi) \), we have to use the relation in Eq. (B27) between the local and the FRW components of the photon direction expressed in each frame. We use the observed photon direction in Eq. (B28) in the FRW frame for computing the fluctuation in the luminosity distance.

Though Eq. (B27) is completely general, a dramatic simplification can be made by a choice of the normalization constant \( \mathbb{C} \): While for illustration the normalization constant was split into the mean \( \mathbb{C} \) and its fluctuation \( \delta \mathbb{C} \) in Eq. (B17), the normalization constant represents only one degree-of-freedom, and it needs to be specified independent of whether the universe is homogeneous or inhomogeneous. Our choice of the mean part in Eq. (B16) is automatically related to our choice of the perturbation part:

\[
2\pi \nu \equiv -\tilde{\eta}_{ab} \hat{k}^a \hat{k}^b = 2\pi \mathbb{C} \nu \bigg|_{\lambda_0} \equiv 1, \quad (1 + \delta \mathbb{C})(1 + \Delta \nu) \bigg|_{\lambda_0} \equiv 1. \tag{B29}
\]

This condition constrains the perturbations of the photon wavevectors in Eqs. (B18) – (B21) as

\[
\delta \nu^{(1)} = -A - n^\alpha (U_\alpha - B_\alpha), \quad \delta e^{\alpha(1)} = -U^\alpha - n^\beta C^\beta_{\alpha}, \tag{B30}
\]

\[
\delta \nu^{(2)} = -A - n^\beta (U_\beta - B_\beta) + \frac{3}{2} A^2 + \frac{1}{2} U^\beta U_\beta - U^\beta B_\beta - n^\beta [2AB_\beta - AU_\beta + C_{\beta\gamma}(U^\gamma + B^\gamma)], \tag{B31}
\]

\[
\delta e^{\alpha(2)} = -U^\alpha - n^\beta C^\alpha_{\beta} + n^\gamma \left[ \frac{1}{2} (U_\gamma U^\alpha - B_\gamma B^\alpha) + \frac{3}{2} C^\beta_{\gamma\delta} C^\delta_{\alpha} \right], \tag{B32}
\]
where metric perturbations are all evaluated at the observer position $x^a(\lambda_0)$. Using Eq. (B22), we derive

$$e^\alpha = n^\alpha, \quad \delta n^\alpha = 0,$$

(B33)

the two unit directional vectors of which are described by the same observed angle $(\theta, \phi)$. Consequently, the other two orthonormal directional vectors coincide with each other, given this choice of normalization condition:

$$\hat{n} \equiv e^\alpha, \quad \hat{\theta} = \frac{\partial}{\partial \theta} \hat{n} \equiv e_\theta^\alpha, \quad \hat{\phi} = \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \hat{n} \equiv e_\phi^\alpha, \quad \delta n_\theta^\alpha = \delta n_\phi^\alpha = 0.$$

(B34)

Equation (B15) (also Eq. [B11]) shows that metric perturbations are well separated from the observable quantities (the observed angle $n^\alpha$ and the photon frequency $\nu$), such that when ensemble averaged given the observable quantities it yields the desired relations in Eqs. (B30)–(B34). Physically, the normalization condition is a mathematical choice, and our choice yields $n^\alpha = e^\alpha$ in a homogeneous universe. Therefore, this relation should remain valid even in an inhomogeneous universe. This choice for $\delta \nu$ and $\delta e^\alpha$ is found [14, 19] to the linear order in perturbations. However, we explicitly provide physical justification as to the presence of additional degree of freedom and its relation to the photon wavevector $k^a$. It is noted that there is only one degree-of-freedom in the normalization constant (i.e., $\bar{C}$ and $\delta \bar{C}$ are not independent), and other choice of the normalization constant $C$ is equally valid, while other choice would significantly complicate the relation between $e^\alpha$ and $n^\alpha$. 

| Symbol | Definition | Equation |
|--------|------------|----------|
| \( z \) | observed redshift of source galaxy | (20) |
| \( \theta, \phi \) | observed angular position of source galaxy \( \hat{n} = (\theta, \phi) \) | |
| \( n^\alpha \) | unit vector in FRW coordinate for observed angle \( \hat{n} \) | (B22) |
| \( n_g, n_{g,\text{obs}} \) | physical and observed galaxy number densities | (2) and (3) |
| \( dV_{\text{phy}}, dV_{\text{obs}} \) | physical and observationally inferred volumes occupied by source galaxies | (64) and (79) |
| \( \delta V \) | distortion in volume between \( dV_{\text{phy}} \) and \( dV_{\text{obs}} \) | (64) – (66) |
| \( e_1, e_2 \) | coefficients describing the time evolution of galaxy sample | (84) |
| \( t_1, t_2 \) | coefficients describing the luminosity function of galaxy sample | (85) |
| \( \delta_{\text{int}}, \delta_{\text{obs}} \) | intrinsic fluctuation of galaxy number density, observationally constructed galaxy fluctuation | (81) and (92) – (94) |
| \( \hat{x}_a \) | observationally inferred position of source galaxies | (42) |
| \( x_a = (\tau_s, x_{\alpha}^s) \) | real source galaxy position | (39) and (41) – (43) |
| \( \hat{x}_a = (\tau_s, \hat{x}_{\alpha}^s) \) | source galaxy position in a homogeneous universe | (38) |
| \( \delta z \) | distortion in observed redshift | (20) |
| \( \delta \tau, \delta \phi \) | distortion in comoving distance from the inferred distance based on observed redshift | (43) |
| \( \delta \tau_s, \delta \phi \) | distortion in angle from observed angle | (43) |
| \( \Delta \tau_s (= \delta T), \Delta x^A_a \) | distortion in comoving coordinate from the inferred | (23) and (24) |
| \( k^a \) | photon wavevector | (B9) |
| \( \hat{k}^a \) | conformally transformed photon wavevector | (5) and (B13) – (B15) |
| \( k_0^a \) | photon wavevector in the observer rest frame | (B7) |
| \( \nu, \omega \) | frequency and angular frequency of photon | (B7) |
| \( [e_{\mu}]^a \) | orthnormal tetrads vectors | (B1) – (B5) |
| \( \bar{\nu} \) | photon frequency in a homogeneous universe | (B16) |
| \( \bar{\Delta \nu} \) | perturbation part in \( k^a \bar{u}_a \) | (15) and (B29) |
| \( \nu \) | physical affine parameter | (4) |
| \( \lambda \) | conformally transformed affine parameter | (5) and (17) |
| \( \Delta \lambda_s \) | perturbation in conformally transformed affine parameter | (23) |
| \( \mathcal{C} \) | normalization constant due to conformal transformation | (5) and (B29) |
| \( \delta \nu, \delta e^a \) | perturbations in temporal and spatial components of the photon wavevector | (6) |
| \( \delta e_{\mu}, \delta e_\theta, \delta e_\phi \) | spatial decomposition of \( \delta e^a \) along \( (\hat{n}, \hat{\theta}, \hat{\phi}) \) | (76) |
| \( N^a \) | observed photon four vector in FRW coordinate | (B28) |
| \( u^a \) | four velocity | (10) |
| \( \delta u^0 \) | perturbation in temporal component of conformally transformed four velocity | (10) |
| \( U^a \) | spatial component of conformally transformed four velocity | (10) |
| \( V_{\mu}, V_{\theta}, V_{\phi} \) | decomposition of spatial velocity along \( (\hat{n}, \hat{\theta}, \hat{\phi}) \) | (62) |
| \( \kappa \) | gravitational lensing convergence | (53) |
| \( D_L, D_A \) | physical luminosity and angular distances | (67) and (68) |
| \( \mathcal{D} \) | deformation matrix in angle | (53) |
| \( \delta \mathcal{D} \) | second-order distortion in solid angle | (53) |
| \( \delta D_L \) | fluctuation in luminosity distance | (67) |
| \( \chi \) | spatially gauge-invariant metric perturbation variable | (93) |
| \( \varphi_\chi \) | scalar gauge-invariant variable in metric tensor | (A37) |
| \( \alpha_\chi \) | scalar gauge-invariant variable in metric tensor | (A38) |
| \( \delta \varphi_\chi \) | temporal gauge-invariant variable for photon wavevector | (A39) |
| \( \delta e_\chi^a \) | spatial gauge-invariant variable for photon wavevector | (A39) |
| \( \Psi_\chi \) | vector gauge-invariant variable in metric tensor | (A42) |
| \( C_{\chi \alpha \beta} \) | tensor gauge-invariant variable in metric tensor | (A44) |
| \( \delta r_s, \delta \theta_s \) | gauge-invariant radial and angular distortions of the source galaxy position | (108) – (110) |
| Symbol   | Definition                                                                 | Equation |
|----------|-----------------------------------------------------------------------------|----------|
| $a$      | comoving scale factor                                                       | (A1)     |
| $H, \dot{H}$ | Hubble and conformal Hubble parameters                                      |          |
| $\bar{r}$ | comoving line-of-sight distance                                             | (17)     |
| $g_{\alpha\beta}, \tilde{g}_{\alpha\beta}$ | metric and conformally transformed metric tensors                          | (A2) and (5) |
| $g, \delta g$ | metric determinant and perturbation part                                    | (59)     |
| $\eta_{\mu\nu}$ | Minkowsky metric in a local frame                                           | (A2)     |
| $A$      | temporal perturbation in metric tensor                                       | (A2)     |
| $B_{\alpha}$ | off-diagonal perturbation in metric tensor                                  | (A2)     |
| $C_{\alpha\beta}$ | spatial perturbation in metric tensor                                       | (A2)     |
| $\alpha$ | temporal perturbation in metric tensor                                       | (A2)     |
| $\beta$  | off-diagonal scalar perturbation in metric tensor                           | (10)     |
| $B_{\alpha}$ | off-diagonal vector perturbation in metric tensor                           | (10)     |
| $\varphi$ | scalar spatial perturbation in metric tensor                                | (10)     |
| $C_{\alpha}$ | spatial vector perturbation in metric tensor                                | (10)     |
| $C_{\alpha\beta}$ | spatial tensor perturbation in metric tensor                               | (10)     |
| $\xi^a = (T, L^a)$ | coordinate transformation                                                   | (A7) and (A8) |
| $L, L^a$  | scalar and vector decomposition of $L^a$                                    | (A8)     |
| $\delta T^0, \delta T^a$ | source perturbations in photon geodesic equation                           | (12) and (14) |