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Foreword

The first “Convegno Informale su Quantum Groups” was held in Florence from February 3 to 6, 1993. This Convegno was conceived as an informal meeting to bring together all the italian people working in the field of quantum groups and related topics. We are very happy indeed that about 30 theoretical physicists decided to take part presenting many aspects of this interesting and live subject of research. We thank all the participants for the stimulating and nice atmosphere that has characterized the meeting.

This paper has the intent to give a quick review in english of the contributions and related references.

We think useful to include the complete addresses and coordinate data of the participants. It is our intention to diffuse these proceedings by e-mail trough electronic data banks.

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Contents

A. Montorsi and M. Rasetti, \textit{Q-symmetries of the Hubbard model with phonons} \hspace{1cm} 3
A. Liguori and M. Mintchev, \textit{On Zamolodchikov’s equation} \hspace{1cm} 5
L. Castellani, \textit{Gauge theories of quantum unitary groups} \hspace{1cm} 7
E. Celeghini, M. Rasetti and G. Vitiello, \textit{Q-derivatives, coherent states and squeezing} \hspace{1cm} 10
L. Lusanna, \textit{Symplectic approach to relativistic localization: Dirac-Yukawa ultraviolet cutoff or classical basis of the Manin quantum plane?} \hspace{1cm} 11
G. Fiore, \textit{Realization of }\mathcal{U}_q(\mathfrak{so}(3))\textit{ within }\text{Diff}(\mathbb{R}^3)\textit{) } \hspace{1cm} 13
F. Bonechi, E. Celeghini, R. Giachetti, E. Sorace and M. Tarlini, \textit{Quantum groups and lattice physics} \hspace{1cm} 15
L. Bonora, \textit{Toda field theories and quantum groups} \hspace{1cm} 18
A. Sciarrino, \textit{“Embedding” of }q\text{-algebras} \hspace{1cm} 20
A. Lerda and S. Sciuto, \textit{Anyons and Quantum Groups} \hspace{1cm} 22
P. Truini, \textit{From classical to quantum: the problem of universality} \hspace{1cm} 23
L. Martina, O. Pashaev and G. Soliani, \textit{Anyons in planar ferromagnets} \hspace{1cm} 25
L. Dabrowski, \textit{Positive energy unitary representations of the conformal }q\text{-algebra} \hspace{1cm} 27
P. Cotta–Ramusino, L. Lambe and M. Rinaldi, \textit{Construction of quantum groups and the Y.B.E. with spectral parameter} \hspace{1cm} 29
E. Guadagnini, \textit{Topological field theory and invariants of three-manifolds} \hspace{1cm} 31
A. Nowicki, E. Sorace and M. Tarlini, \textit{The quantum Dirac equation associated to the }\kappa\text{-Poincaré} \hspace{1cm} 33
R. Floreanini and L. Vinet, \textit{Quantum algebras and basic hypergeometric functions} \hspace{1cm} 35
M. Carfora, M. Martellini and A. Marzuoli, \textit{4-Dimensional lattice gravity and 12j-symbols} \hspace{1cm} 37
D. Franco and C. Reina, \textit{The geometrical meaning of the quantum correction} \hspace{1cm} 39

Participants \hspace{1cm} 40
We show that the addition of a phonon field to the Hubbard model, although breaking the 'superconductive' $su(2)$ symmetry [1] of the Hubbard hamiltonian itself, restores it as a deformed (quantum group) $[su(2)]_q$ symmetry, where $q$ is related to the strength of electron-phonon coupling. Moreover, the chemical potential (and hence the filling) at which the symmetry is restored turns out to depend on the same interaction strength. This latter feature suggests the possibility of having 'superconducting' states with off diagonal long range order at a filling different from a half, with a non-vanishing projection over the ground state, thus reproducing the qualitative behavior of high-$T_c$ materials.

Here the phonons are identified with an ensemble of independent Einstein oscillators with frequency $\omega$ [2], and may be thought of as describing nothing but the ions oscillations around the lattice positions. When one switches on the phonon field, therefore, the grand-canonical Hubbard hamiltonian is changed into

$$H = H_{Hub}^{(loc)} + H_{ph} + H_{el-ph}^{(hop)} ,$$

where (in units in which the ion mass $M = 1$)

$$H_{Hub}^{(loc)} = \sum_j (-\mu (n_{j,\uparrow} + n_{j,\downarrow}) + U n_{j,\uparrow}n_{j,\downarrow}) , \quad H_{ph} = \frac{1}{2} \sum_j \left( p_j^2 + \omega^2 x_j^2 \right) ,$$

$n_{j,\uparrow}$ and $n_{j,\downarrow}$ being the number operators of electrons with up and down spin, $x_j$, $p_j$ the local ion displacement and momentum operators respectively ($[x_j, p_k] = i\delta_{j,k}$) commuting with the fermi operators, and [3,4]

$$H_{el-ph}^{(hop)} = \sum_{<j,k>} \sum_{\sigma} \left( t_{j,k} a_{k,\sigma}^\dagger a_{j,\sigma} + \text{h.c.} \right) = -\lambda \sum_j (n_{j,\uparrow} + n_{j,\downarrow}) x_j$$

$$+ t_0 \sum_{<j,k>} \sum_{\sigma} \exp \left\{ (-|j|\zeta (x_j - x_k)) \right\} \exp \{\kappa (p_j - p_k)\} a_{k,\sigma}^\dagger a_{j,\sigma} + \text{h.c.}$$

with $t_{j,k}$ giving the hopping amplitude due to the overlap of the electron orbitals centered at the displaced ion sites,

$$t_{j,k} = \int d\mathbf{r} \phi^*(\mathbf{r} - \mathbf{R}_j - \mathbf{x}_j) \left[ -\frac{1}{2} \Delta + V(\mathbf{r}) \right] \phi(\mathbf{r} - \mathbf{R}_1 - \mathbf{x}_1) ,$$
and $\lambda$ being the strength of the local electron-phonon coupling originated by $V(\mathbf{r})$ [3].

The q-deformed 'superconductive' $[su(2)]_q$ symmetry algebra of Hamiltonian (1) is generated by

$$
\hat{K}^{(z)} = \sum_{j} \mathbb{I} \otimes \cdots \mathbb{I} \otimes K^{(z)} \otimes \mathbb{I} \cdots \otimes \mathbb{I} = \sum_{j} K_{j}^{(z)}
$$

$$
\hat{K}^{(+)} = \sum_{j} e^{i\pi \cdot j} e^{-\alpha K^{(z)}} \otimes \cdots e^{-\alpha K^{(z)}} \otimes K^{(+)} \otimes e^{\alpha^* K^{(z)}} \otimes \cdots \otimes e^{\alpha^* K^{(z)}}
$$

$$
= \sum_{j} e^{i\pi \cdot j} \prod_{k<j} e^{-\alpha K^{(z)}_{k}} K_{j}^{(+)} \prod_{k>j} e^{\alpha^* K^{(z)}_{k}} ; \quad \hat{K}^{(-)} = [\hat{K}^{(+)}]^\dagger
$$

where

$$
K_{j}^{(+)} = e^{i\kappa p_{j}} a_{j,\uparrow}^\dagger a_{j,\downarrow}^\dagger , \quad K_{j}^{(-)} = K_{j}^{(+)}^\dagger , \quad K_{j}^{(z)} = \frac{1}{2} (n_{j,\uparrow} + n_{j,\downarrow} - 1)
$$

In order for the $[su(2)]_q$-symmetry to hold, it turns out that the parameters $\mu$, $\kappa$, and $\alpha$ have to be related to the physical parameters $U$, $\lambda$, $\omega$, and $\zeta$, by the following constraints:

$$
\mu = U - \frac{\lambda^2}{\omega} , \quad \zeta = \frac{\alpha}{f} , \quad \kappa = \frac{\lambda}{\omega} .
$$

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On Zamolodchikov’s equation

Antonio Liguori and Mihail Mintchev

Recently Carter and Saito [1] discovered a simple but quite remarkable relationship between the quantum Yang-Baxter (YB) [2] and the Zamolodchikov tetrahedron (ZT) [3] equations. This relationship allows in particular to construct solutions of the ZT equation from solutions of the YB equation. Let $\mathcal{A}$ be an associative algebra (over $\mathbb{C}$) with unity $1$. Consider any three elements $\{A, M, B\}$ of $\mathcal{A} \otimes \mathcal{A}$, satisfying the YB

$$A_{12} A_{13} A_{23} = A_{23} A_{13} A_{12}, \quad B_{12} B_{13} B_{23} = B_{23} B_{13} B_{12},$$

and the mixed equations

$$M_{12} M_{13} A_{23} = A_{23} M_{13} M_{12}, \quad B_{12} M_{13} M_{23} = M_{23} M_{13} B_{12}.$$ 

We call $\{A, M, B\}$ a Carter-Saito (CS) triplet. It is not difficult to show that any CS triplet gives rise to a solution of the ZT equation. Indeed, using the decompositions

$$A \equiv \sum_{i \in I} a_i \otimes a_i', \quad B \equiv \sum_{j \in J} b_j \otimes b_j', \quad M \equiv \sum_{k \in K} m_k \otimes m_k',$$

one can verify by purely algebraic manipulations that

$$Z = \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} [a_i \otimes m_k] \otimes [a_i' \otimes b_j] \otimes [m_k' \otimes b_j']$$

satisfies the ZT equation

$$Z_{123} Z_{145} Z_{246} Z_{356} = Z_{356} Z_{246} Z_{145} Z_{123}$$

on $[\mathcal{A} \otimes \mathcal{A}]^\otimes 6$. Clearly, in order to implement effectively the above method for deriving solutions of the ZT equation, one should solve the preliminary problem of constructing CS triplets. This is precisely the problem we address in this talk (see also [4]).

Given a solution

$$R = \sum_{i \in I} c_i \otimes c_i' \in \mathcal{A} \otimes \mathcal{A}$$
of the YB equation, we have shown in [5] how to reconstruct a relative semigroup of spectral parameters \( S(R) \) belonging to \( \text{End}(\mathcal{A}) \). Let \( \mathcal{A}_\ell \) and \( \mathcal{A}_r \) be the subalgebras of \( \mathcal{A} \) generated by \( \{1, c_i : i \in I\} \) and \( \{1, c'_i : i \in I\} \) respectively. Define the subset \( S_\ell(R) \subset \text{End}(\mathcal{A}_\ell) \) as follows: \( \alpha \in S_\ell(R) \) if and only if there exists \( \beta \in \text{End}(\mathcal{A}_r) \) such that
\[
[\alpha \otimes \text{id}](R) = [\text{id} \otimes \beta](R) \ .
\]
It is easily seen that \( S_\ell(R) \) is actually a semigroup with respect to the composition of endomorphisms. Introducing
\[
R(\alpha) \equiv [\alpha \otimes \text{id}](R) \ ,
\]
one can also show that \( R(\alpha) \) satisfies the spectral YB equation
\[
R_{12}(\alpha_1)R_{13}(\alpha_1 \alpha_2)R_{23}(\alpha_2) = R_{23}(\alpha_2)R_{13}(\alpha_1 \alpha_2)R_{12}(\alpha_1) \ ,
\]
(1)
\( \alpha_1 \alpha_2 \) being the composition of the endomorphisms \( \alpha_1 \) and \( \alpha_2 \).

The idempotent elements of \( S_\ell(R) \)
\[
\mathcal{I}_\ell(R) \equiv \{\varepsilon \in S_\ell(R) : \varepsilon^2 = \varepsilon\}
\]
play a distinguished role in the above scheme. In fact, from eq.(1) it follows that \( R(\varepsilon) \) satisfies the YB equation for any \( \varepsilon \in \mathcal{I}_\ell(R) \). Furthermore, it is an immediate consequence of eq.(1) that
\[
\{R(\varepsilon_1), R(\varepsilon_2 \alpha \varepsilon_1), R(\varepsilon_2)\}
\]
is a SC triplet for any \( \alpha \in S_\ell(R) \) and \( \varepsilon_1, \varepsilon_2 \in \mathcal{I}_\ell(R) \). In this way one obtains a whole family of CS triplets, naturally generated by a solution of the quantum YB equation.

Some consequences of the above construction have been explored in [4]. We believe that further investigations in this framework will shed new light on the relationship between the YB and the ZT equations.

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Gauge theories of quantum unitary groups

Leonardo Castellani

In ref.s [1,2] we have proposed a geometric approach to the construction of q-gauge theories, based on the differential calculus on q-groups developed in [3-6]. These theories are continuously connected with ordinary Yang-Mills theories, just as quantum groups are continuously connected with ordinary Lie groups. Spacetime is taken to be ordinary (commutative) spacetime, but the whole discussion holds for a generic q-spacetime. The notations will be as in ref. [6].

The q-Lie algebra we obtain from a bicovariant differential calculus is given by the q-commutations between the quantum generators $T_i$:

$$T_i T_j - \Lambda_{ij}^{kl} T_k T_l = C_{ij}^k T_k$$

(1)

The braiding matrix $\Lambda$ and the q-structure constants $C_{ij}^k$ can be expressed in terms of the $R$-matrix of the corresponding $q$-group, as shown in [4] and further discussed in [5,6], and satisfy four conditions:

$$\Lambda_{ij}^{kl} \Lambda_{sp}^{lm} \Lambda_{qs}^{ks} = \Lambda_{ik}^{jm} \Lambda_{qs}^{sl} \Lambda_{up}^{ml}$$

(Yang – Baxter equation) (2a)

$$C_{mi}^r C_{nj}^s - \Lambda_{ij}^{kl} C_{mk}^r C_{rl}^n = C_{ij}^k C_{mk}^n$$

(q – Jacobi identities) (2b)

$$C_{is}^j \Lambda_{pq}^{sl} \Lambda_{rl}^{ir} A_{pq}^{pk} + C_{rl}^q \Lambda_{pr}^{ij} A_{pq}^{pk} = \Lambda_{ij}^{mq} \Lambda_{kp}^{si} \Lambda_{kl}^{ps} C_{ps}^r + \Lambda_{ij}^{mq} \Lambda_{kl}^{ps} C_{kl}^i$$

(2c)

$$\Lambda_{ij}^{irk} \Lambda_{kl}^{qsi} \Lambda_{mn}^{kl} C_{rs}^j = \Lambda_{ij}^{kl} C_{mn}^k$$

(2d)

The last two conditions are trivial in the limit $q \to 1$ ($\Lambda_{ij}^{kl} = \delta_i^j \delta_k^l$).

We start by defining the field strength as

$$F_{\mu \nu} \equiv \frac{1}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu A_\nu - A_\nu A_\mu) = \partial_{[\mu} A_{\nu]} - A_{[\mu} A_{\nu]}$$

(3)

where $A_\mu \equiv A_\mu^i T_i$. The gauge potentials $A_\mu^i$ are taken to satisfy the q-commutations:

$$A_{[\mu} A_{\nu]}^i = - \frac{1}{q^2 + q^{-2}} (\Lambda + \Lambda^{-1})_{kl}^{ij} A_{[\mu}^k A_{\nu]}^l$$

(4)
and simply commute with the quantum generators $T_i$. The square parentheses around the $\mu, \nu$ indices stand for ordinary antisymmetrization. The inverse $\Lambda^{-1}$ of the braiding matrix always exists and is defined by $(\Lambda^{-1})^{ij}_{kl} \Lambda^{kl}_{mn} = \delta^i_m \delta^j_n$. The rule (4) is inspired by the $q$-commutations of exterior products of left-invariant one-forms on the quantum groups $U_q(N)$ deduced in ref. [6]. As shown in [2], the field strength (3) can be rewritten as

$$F^i_{\mu\nu} = \partial_{[\mu} A_{\nu]}^i + P_A^{k\ell} C_{k\ell}^{ij} A_{[\mu}^i A_{\nu]}^n$$

where $P_A$ is a projector $q$-generalizing the antisymmetrizer, and $C_{k\ell}^{ij} \equiv C_{k\ell}^{ij} - \Lambda^{ij}_{kl} C_{ij}^{kl}$. We next define the gauge variations:

$$\delta A_{\mu} = -\partial_{\mu} \varepsilon - A_{\mu} \varepsilon + \varepsilon A_{\mu}$$

with $\varepsilon \equiv \varepsilon^i T_i$ and postulate the commutations $\varepsilon^i A^j_{\mu} = \Lambda^i_{kl} A^k_{\mu} \varepsilon^l$. Under the variations (6) the field strength transforms as $\delta F_{\mu\nu} = \varepsilon F_{\mu\nu} - F_{\mu\nu} \varepsilon$. Indeed the calculation is identical to the usual one, since both in the definition (3) for $F_{\mu\nu}$ and in the variations (6) we have ordinary commutators. The $q$-commutativity enters the game only when we want to factorize the $q$-Lie algebra generators $T_i$, or to reorder terms containing $q$-commuting objects like $A, \varepsilon$. The ordinary commutator in (6) leads to a composition law $(\delta_1 \delta_2 - \delta_2 \delta_1) A_{\mu} = -\partial_{\mu} [\varepsilon_2, \varepsilon_1] - [A_{\mu}, [\varepsilon_2, \varepsilon_1]]$ formally identical to the classical one.

Using the $A, \varepsilon$ commutations and eq. (1), the gauge variations of $A_{\mu}^i$ take the familiar form $\delta A_{\mu}^i = -\partial_{\mu} \varepsilon^i - A_{\mu}^i \varepsilon + \varepsilon A_{\mu}^i$.

By using the bicovariance conditions (2a) and (2d) we can prove that $\varepsilon^i F^j = \Lambda^i_{kl} F^k_{\mu} \varepsilon^l$ so that $\delta F^i_{\mu\nu} = -F^i_{\mu\nu} \varepsilon^k C_{jk}^i$. Condition (2c) ensures that the commutation relations (4) are preserved under the $q$-gauge transformations (6), cf. [2].

There is a simple way to obtain the commutations between $F^i$ and $A^j$, and between $F^i$ and $F^j$, see [2].

Finally, we construct the $q$-lagrangian invariant under the $U_q(N)$ quantum Lie algebra. We set $L = F^i_{\mu\nu} F^j_{\mu\nu} g_{ij}$ where $g_{ij}$, the $q$-analogue of the Killing metric, is determined by requiring the invariance of $L$ under the $q$-gauge transformations (6). Under these transformations, the variation of $L$ is given by

$$\delta L = -C_{mn}^{i} F^m_{\mu\nu} \varepsilon^n F^j_{\mu\nu} g_{ij} - C_{mn}^{j} F^i_{\mu\nu} F^m_{\mu\nu} \varepsilon^n g_{ij}$$

After reordering the terms as $FF\varepsilon$ we find that $\delta L$ vanishes when

$$C_{mn}^{i} A^{nj}_{rs} g_{ij} + C_{rs}^{j} g_{mj} = 0$$

This eq. is not difficult to solve in particular cases. For example, we have given in ref.[2] the most general $q$-metric satisfying (8) for the case of $U_q(2) = [SU(2) \otimes U(1)]_q$. For
conventions, and a detailed study of $U_q(2)$, we refer to [6].

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Q-derivatives, coherent states and squeezing

Enrico Celeghini, Mario Rasetti and Giuseppe Vitiello

The q-commutator is discussed in the framework of the Fock-Bargmann representation and is functionally realized in terms of the commutator of the z-multiplication operator with the q-derivative (z complex number). We obtain a weak relation between the q-deformation of the W-H algebra and the generator of the Glauber coherent states and of the squeezed states. This study may turn out to be fruitful in relating the quantum algebraic structures with the theory of the entire analytical functions and of the theta functions[1].

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Symplectic approach to relativistic localization: Dirac-Yukawa ultraviolet cutoff or classical basis of the Manin quantum plane?

Luca Lusanna

The center-of-mass (c.o.m.) and relative variable symplectic basis for extended relativistic systems (particles, Nambu string, classical fields) described by first class constraints is described. The action of the Poincaré algebra on the constraint set is such that this set is the disjoint union of strata, one for each kind of Poincaré orbit; in each stratum the relative variables have the associated Wigner covariance with the degrees of freedom associated with the relative times playing the role of gauge variables conjugated to some first class constraint. Instead the canonical c.o.m. variable is not a four-vector: if one draws in a given reference frame all the trajectories of this canonical variable (associated with all possible frames), one obtains a world-tube around the covariant (but not canonical) Fokker c.o.m. variable. The invariant radius of this world-tube is an intrinsic classical length determined by the Poincaré Casimirs: it could the classical basis of a deformation parameter for the definition of the quantum Manin plane; since this length is Casimir dependent it is not clear which role, if any, it could play in the developments of the quantum Poincaré groups. To make frame independent statements at the classical level about an extended relativistic system, one cannot localize its symplectic c.o.m. inside the world-tube. The standard canonical quantization with these localization restrictions allows to define an intrinsic invariant ultraviolet cutoff, when the configuration of the extended system corresponds to an irreducible Poincaré representation with $P^2 > 0$ and $W^2 \neq 0$; it is the Compton wave length multiplied the value of the rest frame spin. The theorems of Hegerfeldt about the violation of Einstein causality, when one studies the spreading of localized wave packets of the Newton-Wigner position operator, imply that the canonical c.o.m. position operator, and therefore the absolute positions of the individual components of the extended system, cannot be self-adjoint operators.
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Realization of $U_q(so(3))$ within $Diff(\mathbb{R}^3_q)$

Gaetano Fiore

As known, the $N$-dim real quantum euclidean space $[1]$ ($\mathbb{R}^N_q$ in our notation) and its differential calculus $[2]$ are covariant w.r.t. the action of the quantum group of rotations $SO_q(N)$. Here we restrict to the case $N = 3$ and show that when $q \in \mathbb{R}$ it is possible to realize $U_q(so(3))$ (which is the dual Hopf Algebra of $Fun(SO_q(3))$) in terms of $q$-antisymmetric differential operators on $\mathbb{R}^3_q$, i.e. in terms of the $q$-analog of the angular momentum operators in $\mathbb{R}^3$. The explicit analogous result for any $N \geq 3$ will be given in [3] in detail.

The main result is the following. We define

\[
\begin{align*}
    L_m &:= q[x^3\partial^1 - x^1\partial^3 + (q^{-\frac{1}{2}} - q^\frac{1}{2})x^2\partial^2]B^{-1} \\
    L_+ &:= (q^\frac{1}{2} + q^{-\frac{1}{2}})q^\frac{3}{4}(x^2\partial^3 - qx^3\partial^2)B^{-1} \\
    L_- &:= (q^\frac{1}{2} + q^{-\frac{1}{2}})q^{-\frac{3}{4}}(x^1\partial^2 - qx^2\partial^1)B^{-1},
\end{align*}
\]

(1)

where for the $\mathbb{R}^3_q$ coordinates $x^i$ and the corresponding partial derivatives $\partial^i$ we follow the conventions of [1],[2], and $B := 1 + q(q - 1)x^i\partial_i$. Then the generators $L_m, L_+, L_-$ make up a closed algebra with commutation rules

\[
\begin{align*}
    q^{\frac{1}{2}}L_mL_+ - q^{-\frac{1}{2}}L_+L_m &= L_+ \\
    q^{\frac{1}{2}}L_mL_- - q^{-\frac{1}{2}}L_-L_m &= -L_- \\
    [L_+, L_-] &= (q^\frac{1}{2} + q^{-\frac{1}{2}})L_m[1 + (q^\frac{1}{2} - q^{-\frac{1}{2}})L_m];
\end{align*}
\]

(2)

moreover $(L_+)^* = L_-$, $L^*_m = L_m$, and the Casimir operator $L^2$ (the square angular momentum) is the following quadratic expression in $L_+, L_-, L_m$

\[L^2 = L_m^2 + (q^\frac{1}{2} + q^{-\frac{1}{2}})^{-1}[q^\frac{1}{2}L_+L_- + q^{-\frac{1}{2}}L_-L_+].\]

(3)

In the classical limit $q = 1$ we recover the usual $so(3)$ generators $L_3, L_+, L_-$ satisfying the relations $[L_3, L_\pm] = \pm L_\pm$, $[L_+, L_-] = 2L_3$ after introducing suitable real coordinates.

The algebra generated by $L_m, L_+, L_-$ coincides with the Hopf algebra $U_q(su(2)) = U_q(so(3))$. In fact the transformation

\[
\begin{align*}
    L_\pm &= (q^\frac{1}{2} + q^{-\frac{1}{2}})[2 + (q^\frac{1}{2} - q^{-\frac{1}{2}})^2C]^{-1}X_{\pm}q^{\frac{H_{\pm}^2}{2}} \\
    L_z &= (q^\frac{1}{2} - q^{-\frac{1}{2}})^{-1}\left\{ -1 + \frac{q^\frac{1}{2}(q^\frac{1}{2} + q^{-\frac{1}{2}})}{2 + (q^\frac{1}{2} - q^{-\frac{1}{2}})^2C} \right\};
\end{align*}
\]

(4)
maps the set of generators $H, X_+, X_- \text{ of } U_q(su(2))$ satisfying the standard \[1\] commutation relations (with $q \to q^{\frac{1}{2}}$) into generators $L_m, L_+, L_-$ satisfying relations (2).

The unitary representation of integral spin $k$ of $U_q(su(2))$ can be realized in terms of the differential operators (1) acting on the $(2k + 1)$-dimensional vector space $W_k$ made out of the q-deformed “symmetric” polynomials of degree $k$ in $x$. Then the scalar product between polynomials can be defined in terms of the integration on $\mathbb{R}^3_q$ defined in \[4\],[\[5\]] by imposing Stoke’s theorem. $W_k$ can be generated through the iterated application of $L_+$ to the lowest weight eigenvector $u_{k,-k} := (x^1)^k$ of $L_m$. In fact the vectors $u_{k,h} := (L_+)^{k+h}u_{k,-k}, h = -k, -k + 1, ..., k (L_+u_{k,k} = 0)$ make up a basis of $W_k$ consisting of eigenvectors of $L_2, L_3$ with eigenvalues $l_k^2, \lambda_{k,h}$ respectively given by

$$l_k^2 = [k]_q[k + 1]_q \left( \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{q^{k - \frac{1}{2}} + q^{\frac{1}{2} - k}} \right)$$

$$\lambda_{k,-k} = -\frac{1 + q^{-1}}{q^k + q^{-k-1}}q^{\frac{1}{2}}[k]_q \quad \lambda_{k,h+1} = q\lambda_{k,h} + q^{\frac{1}{2}}. \quad (5)$$

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Quantum groups and lattice physics

F. Bonechi, E. Celeghini, R. Giachetti, E. Sorace and M. Tarlini

0. A number of inhomogeneous quantum algebras of large physical interest has recently been determined by extending the contraction procedure from semisimple Lie algebras to their quantized version [1-3]. In order to preserve the Hopf algebra structure, it may occur that sometimes the quantum parameter itself must undergo a rescaling, so that, after the contraction, the parameter can acquire a physical dimension. It emerges then naturally a picture in which the quantum algebra represents the symmetry of a dynamical system with a fundamental length or time scale, $a$, and in which the coproduct $\Delta$ establishes the rules for combining the elementary excitations [4-6].

Here we shall review from this point of view the properties of phonons, which is the physical system associated with the two dimensional pseudo euclidean (or 1+1 – Poincaré) group, $E_\ell(1,1)$, [5], and then we describe the quantization of the linear modes of magnetic chains, the magnons, by using a quantum analogue of the Galilei group [6].

1. The quantum algebra $E_\ell(1,1)$ is generated by $k^{\pm 1} = e^{\pm iaP}$, $P_0$, $J$, where $P$ represents the momentum, $P_0$ and $J$ the energy and boost respectively, satisfying

$$kP_0k^{-1} = P_0, \quad kJk^{-1} = J + aP_0, \quad JP_0 - P_0J = (k - k^{-1})/(2a),$$

$$\Delta(k) = k \otimes k, \quad \Delta(P_0) = k^{-1/2} \otimes P_0 + P_0 \otimes k^{1/2},$$

$$\Delta(J) = k^{-1/2} \otimes J + J \otimes k^{1/2}. $$

The Casimir of $E_\ell(1,1)$ is $C = P_0^2 - (2/a)^2 \sin^2(aP/2)$ and the differential realization $P_0 = (i/v) \partial_t$, $k = \exp(a \partial_x)$, $J = i(x/v)\partial_t - (vt/a)\sin(-ia\partial_x)$, turns the eigenvalue equation $C z(x,t) = m^2 v^2 z(x,t)$ for the Casimir into the PDE

$$(\partial_t^2 + (2v/a)^2 \sin^2(-ia\partial_x/2) + m^2 v^4) z(x,t) = 0,$$

which, for $m = 0$, describes the phonons on a lattice with spacing $a$. The single particle properties are well described by $E_\ell(1,1)$. For instance the position operator $X = (1/2) \{ P_0^{-1}, J \}_+$ has a time derivative $v_g = \dot{X} = iv [P_0, X] = v \cos(aP/2)$, reproducing the well known expression for the group velocity of the phonons.

The coproduct gives the two-phonon global variables. For energy and boost we get

$$P_0 = e^{-iaP^{(1)}/2} P_0^{(2)} + P_0^{(1)} e^{iaP^{(2)}/2}, \quad J = e^{-iaP^{(1)}/2} J^{(2)} + J^{(1)} e^{iaP^{(2)}/2}. $$

15
Moreover $k = k^{(1)}k^{(2)}$, from which $P = P^{(1)} + P^{(2)} + 2\pi n/a$, whith $n$ chosen so that $P$ is kept in the fixed Brillouin zone: quantum symmetry implies Umklapp process.

Explicitly, for two possibly differently polarized phonons with velocity parameters $v^{(1)}, v^{(2)}$ and dispersion relations $\Omega^{(r)} = v^{(r)}P_0^{(r)} = (2v^{(r)}/a)\sin(aP^{(r)}/2)$, $(r = 1, 2)$, the total energy $P_0$ reads $P_0 = (2/a)\sin(a(P^{(1)} + P^{(2)})/2)$. Therefore, by the energy conservation, the physical process actually occurs if there exists a velocity $v$ such that $\Omega = \Omega^{(1)} + \Omega^{(2)}$, where $\Omega = (2v/a)\sin(aP/2)$ is the dispersion relation of the composite system.

2. A deformation $\Gamma_\ell(1)$ of the one-dimensional Galilei algebra is obtained by the four generators $B$, $M$, $K$, and $T$ representing the Galilean boost, the mass, the momentum and the energy which satisfy

$$[B,K] = iM, \quad [B,T] = (i/\ell) \sin(\ell K), \quad [M,\cdot] = [T,K] = 0$$

$$\Delta B = e^{-iaK} \otimes B + B \otimes e^{iaK}, \quad \Delta M = e^{-iaK} \otimes M + M \otimes e^{iaK},$$

$$\Delta K = 1 \otimes K + K \otimes 1, \quad \Delta T = 1 \otimes T + T \otimes 1,$$

The Casimir of $\Gamma_\ell(1)$ reads $C = MT - (1/a^2)\left((1 - \cos(aK))\right)$ and this algebra admits the differential realization $B = mx$, $M = m$, $P = -i\partial_x$, $T = (ma^2)^{-1}\left(1 - \cos(-ia\partial_x)\right) + c/m$, where $c$ is the constant value of the Casimir.

The physical system that can be studied by means of the quantum symmetry $\Gamma_q(1)$ is a spin 1/2 system referred to as XXZ model and known to be integrable by the Bethe Ansatz method. Its Hamiltonian is $H = 2J \sum_{i=1}^N \left((1-\alpha)(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + S_i^z S_{i+1}^z\right)$, with $S_{N+1}^n = S_1^n$. Firstly, from the Galileian position operator $X = B/M$, we find the well known magnon velocity $\dot{X} = i[T,X] = J\ell \sin(\ell K)$. Then, following the standard method for analyzing such models, the eigenvalue equation for $\mathcal{H}$, in terms of the states $\psi = \sum_i f_i S_i^+ |0\rangle$ with a single spin deviate, becomes an algebraic system in $f_i$ which can be embedded into the PDE for the continuous amplitude $f(x)$

$$-4J\sigma \left(1 - (1-\alpha)\cos(-ia\partial_x)\right)f(x) = (\epsilon - \epsilon_0)f(x).$$

The operator on the left hand side coincides with the differential realization of $T$, by identifying $(m\ell^2)^{-1} = -2J(1-\alpha)$ and $c/m = -2J\alpha$.

Let us now discuss the two magnon states $\psi = \sum_{ij} f_{ij} S_i^+ S_j^+ |0\rangle$, where $f_{ij} = f_{ji}$, $i \neq j$, while $f_{ii}$ are physically meaningless and have no part in the theory. The algebraic system for the coefficients $f_{ij}$ is easily found and can again be analyzed by embedding it into a PDE for continuous amplitudes $f(x_1, x_2)$ and using the $\Gamma_q(1)$ symmetry. From $\Delta T$ we find the total energy $T_{12} = (M_1\ell^2)^{-1}\left(1 - \cos(\ell K_1)\right) + (M_2\ell^2)^{-1}\left(1 - \cos(\ell K_2)\right) + (c_1/M_1) + (c_2/M_2)$.

For $M_1 = M_2 = M$, using the previous identifications and the differential realization $K_1 = -i\partial_{x_1}$, $K_2 = -i\partial_{x_2}$, the eigenvalue equation $T_{12} f(x_1, x_2) = (\epsilon - \epsilon_0)f(x_1, x_2)$ for the two magnon amplitude $f(x_1, x_2)$ is equivalent to the free system of the Bethe ansatz. The
two magnon bound states are obtained by requiring that the energy has a homogeneous
dependence of degree $-1$ upon the total mass, exactly as in the single magnon cases. As
a result we obtain a total mass $M_{12} = 2M/(1 - \alpha)$ and the energy of bound states is
$T_{12} = -2J\left(1 - (1 - \alpha)^2 \cos^2(\ell K/2)\right)$.

The procedure can be extended to any number of magnons by using the coproduct
and its associativity: we find two recurrence relations for the $n$-magnon mass and energy
that can be solved, yielding

$$M_{12...k} = -\left(2J(1 - \alpha)\ell^2\right)^{-1} \mathcal{U}_{k-1}(1/(1 - \alpha)),$$

$$k = 2, \ldots n,$$

$$T_{12...n} = \frac{-2J(1 - \alpha)}{\mathcal{U}_{n-1}(1/(1 - \alpha))} \left(\mathcal{T}_n(1/(1 - \alpha)) - \cos(\ell K_{12...n})\right),$$

so that the bound state energy of the $n$ magnon bound states has a closed form in terms
of the Tchebisheff polynomials $\mathcal{U}_k$ and $\mathcal{T}_k$.

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Toda field theories and quantum groups

Loriano Bonora

The simplest Toda theory in 1+1 dimensions is the Liouville theory. The Liouville equation
\[ \partial_{x_+} \partial_{x_-} \phi = e^{2\phi}, \quad x_\pm = x \pm t, \quad x \in S^1, \ t \in \mathbb{R} \]
is conformal invariant. It can be solved through the following procedure. Write the associated Drinfeld-Sokolov linear systems
\[ \partial_{x_+} Q(x_+) = \left( p(x_+) H - E_+ \right) Q(x_+), \quad \partial_{x_-} \bar{Q}(x_-) = -\bar{Q}(x_-) \left( \bar{p}(x_-) H - E_- \right) \]
where \( H, E_+, E_- \) are the generators of the Lie algebra \( sl_2 \). For any solutions \( Q_+ \) and \( Q_- \) of these systems and any highest weight vector \( |\Lambda> \) of \( sl_2 \) corresponding to the weight \( \lambda \)
\[ e^{\lambda(\phi(x_+,x_-) H)} = <\Lambda| Q(x_+) M \bar{Q}(x_-)|\Lambda> \]
is a solution of the Liouville equation for any constant matrix \( M \). The matrix \( M \) is chosen in such a way as to guarantee periodicity and locality of the solutions. Vice versa one can prove that to any local periodic solutions of the Liouville equation there corresponds a couple of free periodic chiral boson fields \( p \) and \( \bar{p} \). This allows us to describe the classical phase space of the Liouville theory by means of free bosonic oscillators [1].

This analysis can be extended to any Toda field theory based on a finite dimensional Lie algebra [2], to Toda field theories based on affine algebras [3] and to Toda field theories defined on Riemann surfaces [4].

The particularly simple parametrization of the classical phase space (free bosonic oscillators) makes the canonical quantization procedure very effective. An intermediate quantum formula is, for example, the exchange algebra
\[ \psi(x) \psi(y) = \psi(y) \psi(x) R_{12}^\pm(p_0), \quad +(-) \text{ if } x > (<) y \]
where \( \psi = <\Lambda|g\rho, \text{ and } g,\rho \text{ are suitable matrices that guarantee periodicity and locality. } R_{12} \text{ is the quantum } R \text{ matrix in the Block wave basis [5,6].} \]

Quantization of the Liouville theory eventually leads to calculating the correlation functions in particular of the conformal minimal models. Quantization of the other Toda
theories based on finite dimensional Lie algebras leads to W minimal models, and finally quantization of Toda theories based on affine Lie algebras should lead to a thorough understanding of theories like the sine–Gordon or sinh–Gordon models and their generalizations.

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“Embedding” of q-algebras

Antonio Sciarrino

The underlying idea in some applications of q-algebras is to use a q-deformed algebra instead of a Lie algebra to realize a dynamical symmetry. The dynamical symmetry in many physical models is displayed through embedding chains of algebras of the type

$$G_0 \supset G_1 \supset \ldots \supset SO(3) \supset SO(2)$$

An essential step to carry forward the program of application of q-algebras as dynamical symmetry is to dispose on a formalism which allows to build up chains analogous to eq.(1) replacing the Lie algebras by the deformed ones. The existence of 3-dim principal q-subalgebra for $Gl_q(3)$, has been shown [1] in the symmetric basis, but the the coproduct of $Gl_q(3)$ does not induce a coproduct in the 3-dim principal subalgebra. In ref.[2] we tried to solve the problem the other way around: to define $SO_q(3)$ and to build up a deformed structure of the type $Gl_q(3)$. Indeed a “deformed $Gl(3)$” can be obtained but it is not clear how to impose on it the Hopf structure. The origin of the difficulties is on the Chevalley-Cartan basis which is not suitable to discuss embedding of subalgebras except the trivial ones. We present here an alternative deformation scheme [3]. Let us immediately emphasize that really the word “embedding” is used in some loose sense: the algebra $G_q$ deformed according the following deformation scheme is not the same as the $G_q$ defined in the Chevalley basis. Let us sketch what the underlying idea is. Consider a semisimple Lie algebra $G$ and a not regular maximal subalgebra $L \subset G$: $ad_G \rightarrow ad_L \oplus R_L$ where $R_L$ is a representation of $L$. Let $\{E_i^\pm\}$ be the generators of $L$ in the Chevalley basis and $\{X_k^\pm\}$ some elements of $R_L$ with suitable properties. Then we define a deformation scheme in which the Cartan subalgebra of $G$, which is partly in the Cartan subalgebra of $L$ and partly in $R_L$, is left invariant; the set of $\{E_i^\pm\}$ is deformed in the standard way. In the simplest case where the rank of $L$ is one unit less the rank of $G$ there is one element $K_0$ in $R_L$ which commutes with the Cartan subalgebra of $L$ such that (for fixed $j, i \neq j$):

$$[K_0, E_j^\pm] = \pm X_j^\pm \quad [K_0, E_i^+] = 0 \quad [X_j^+, X_j^-] = [H_j]_{q_j}$$

20
Then we impose the Hopf structure on $K_0$ as an element of the Cartan subalgebra. This scheme defines $G_q \supset L_q$.

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Anyons and Quantum Groups

Alberto Lerda and Stefano Sciuto

Anyonic oscillators with fractional statistics are built on a two-dimensional square lattice by means of a generalized Jordan-Wigner construction, and their deformed commutation relations are thoroughly discussed. Such anyonic oscillators, which are non-local objects that must not be confused with $q$-oscillators, are then combined à la Schwinger to construct the generators of the quantum group $SU(2)_q$ with $q = \exp(i\pi\nu)$, where $\nu$ is the anyonic statistical parameter [1].

The construction can be generalized to $SU(n)_q$ [2] and more generally to the deformations of the Lie algebras $A_n$, $B_n$ and $C_n$, which can be built by a fermionic Schwinger construction [3].

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From classical to quantum: the problem of universality

Piero Truini

A basic question for a theory is that of universality: one should be able to build within a certain class of objects a general structure which incorporates every model of that class as a particular case. We have addressed this question within the theory of Quantum Groups, in particular, for deformations of the universal enveloping algebras of reductive Lie algebras (a Lie algebra is reductive if it is the direct sum of a semisimple Lie algebra and an abelian one, as in the case of \( GL(n) \)).

We have solved the question, \([1,2]\) starting from very simple concepts and an Ansatz. The main results of our work can be summarized as follows. Given an algebra or a Hopf algebra \( A \) over \( \mathbb{C} \) we define a deformation of it as an algebra or a Hopf algebra over \( R = \mathbb{C}[u_1, u_2, \ldots, u_m] \), the ring of formal power series in the deformation parameters \( u_1, u_2, \ldots, u_m \), which is torsion free as an \( R \)-module, that reduces to \( A \) when \( u_1 = u_2 = \ldots = u_m = 0 \). The number of deformation parameters is arbitrary to start with.

The first important step towards our goal is the construction of what we call the extended enveloping algebra, which is an enlargement of the classical enveloping algebra (but where the algebra structure is deformed) so as to include power series in the Cartan generators. This leads to a result that we regard as fundamental in the characterization of quantum groups. The deformations of universal enveloping algebras involve some infinite series in the generators of the Cartan subalgebras, but we find the characterization of such series, given in the literature, quite unsatisfactory and sometimes not fully justified. We show that these series are strongly constrained. The coalgebra structure forces them to satisfy harmonic, constant coefficient differential equations; the algebra structure implies invariance under translations. Both constraints together leave as possibilities only certain combinations of polynomials and exponentials.

Having established this we prove, under the Ansatz that the two Borel subalgebras deform as Hopf algebras, that all deformations of a reductive Lie algebra can be obtained as specializations of a universal multiparameter deformation. The number of parameters in the universal deformation is \( \frac{1}{2}[N(N-1) - C(C-1)] + M \), where \( N \) is the rank of the reductive algebra, and \( C = N - N_1 \) is the dimension of its center, \( N_1 \) being the rank of the semisimple part, while \( M \) is the number of simple components.
We also prove that the algebra structure can always be reduced (on the simple components) to that of the standard one-parameter quantization that can be found in the literature. The extra parameters then appear only in the comultiplication. If we restrict our result to the case of a simple algebra, we find precisely the twisted quasi triangular Hopf algebras, the twisting being a particular case of *gauge* transformation in the category of quasi triangular quasi-Hopf algebras.

The example of $A_1 \oplus A_1$, that we had treated in [3], having in mind there non-standard deformations of the Lorentz group ($A_1 \oplus A_1$ is the complexification of its Lie algebra), serves as an illustration of the nature of the extra parameters. This example suggests that part of them is closely related to quantizations of a semisimple algebra in which the simple components remain classical throughout the deformation.

In a recent paper, [4], we found the dual structure of the universal deformations found in [1,2], in the case of semisimple A-type Lie algebras. In particular we can define, through a pairing, a natural action of the universal deformation of the Lorentz algebra on the deformed Minkowski space that suggests a new way of deforming the Poincaré Lie algebra.

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Anyons in planar ferromagnets

Luigi Martina, Oktay Pashaev and Giulio Soliani

As is well known, the Gauss law constraint in the Chern-Simons (CS) theory can be related to the comultiplication of a quantum group [1]. This emerges as a hidden symmetry of the quantized CS theory [2]. Correspondingly, the soliton (vortex) excitations at the quantum level satisfy an arbitrary statistics and can be regarded as anyons, identified with Laughlin’s quasiparticles in the Quantum Hall Effect [3].

We show that, at least at the classical level, a planar continuum Heisenberg ferromagnet, given by the Landau-Lifshitz equation

$$
S_t = S \times (\partial^2_1 + \partial^2_2) S,
$$

where $S = S(x_1, x_2, t), S^2 = 1$, can be described in terms of the CS theory.

By resorting to the tangent space representation [4], in the static case Eq. (1) can be reduced to the (anti) self-dual CS model

$$
D_\pm \psi_\pm = 0,
[D_1, D_2] = 4i\kappa^2(|\psi_+|^2 - |\psi_-|^2).
$$

If, for instance, $\psi_-$ is vanishing, then $|\psi_+|^2$ satisfies the Liouville equation

$$
(\partial^2_1 + \partial^2_2) \ln |\psi_+|^2 = -8\kappa^2 |\psi_+|^2.
$$

Thus, the magnetic vortices of model (1) correspond to the CS and Liouville solitons, while the topological charge of the former corresponds to the electric charge of the latter.

If both the functions $\psi_\pm$ are nonvanishing [5], then we can introduce a holomorphic function $U = \bar{\psi}_+ \psi_-$ and rewrite Eq. (2) as the conformally invariant Sinh - Gordon equation

$$
(\partial^2_1 + \partial^2_2) \ln |\psi_+|^2 = -8\kappa^2(|\psi_+|^2 - |U|^2 / |\psi_+|^2),
(\partial^2_1 + \partial^2_2) \ln |U|^2 = 0,
$$

which can be extended to the affine Liouville model [6].
In the *non-static case*, following the same procedure, we find a *CS* gauged Nonlinear Schrödinger Equation for the charged matter fields $\psi_{\pm}$ coupled to a ”statistical gauge field” $A_\mu$ [5], namely

\[
\begin{align*}
i D_0 \psi_{\pm} + (D_1^2 + D_2^2) \psi_{\pm} + 8\kappa^2 |\psi_{\pm}|^2 \psi_{\pm} &= 0, \\
\partial_1 A_2 - \partial_2 A_1 &= -8\kappa^2 (|\psi_+|^2 - |\psi_-|^2), \\
\partial_0 A_i - \partial_i A_0 &= 8\kappa^2 i\epsilon_{ij} (\overline{\psi_+} D_j \psi_{+} - \overline{D_j \psi_{+}} \psi_{+}) - (\overline{\psi_-} D_j \psi_- - \overline{D_j \psi_-} \psi_-)
\end{align*}
\]

and connected by the relation

\[
D_+ \psi_- = D_- \psi_+,
\]

where $D_\mu = \partial_\mu - i/2A_\mu$, $D_\pm \equiv D_1 \pm iD_2$, $A_0 = V_0 - 8\kappa^2 (|\psi_+|^2 + |\psi_-|^2)$, $A_i \equiv V_i$ ($i = 1, 2$), and $V_\mu$ is the $U(1)$ gauge field associated with the tangent space.

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Positive energy unitary representations of the conformal q-algebra

Ludwik Dabrowski

We report the work [1] on positive-energy unitary irreducible representations of a q-deformation $U_q(su(2, 2))$ of the conformal algebra, which has the q-deformed Lorentz algebra $U_q(so(3, 1))$ as a Hopf subalgebra [2]. For generic $q$, the positive energy UIRs of $U_q(su(2, 2))$ are deformations of the respective representations of $su(2, 2)$ labelled by $[j_1, j_2, d]$, where $2j_1, 2j_2$ are non-negative integers fixing finite dimensional irreducible representations of the Lorentz subalgebra $so(3, 1)$, and $d$ is the conformal dimension [3]. We extend for $|q| = 1$, the hermitian conjugation used in [3] to an anti-linear anti-involution $\omega$ in a *-Hopf algebra $U_q(sl(4, C))$, which is a complexification of $U_q(su(2, 2))$. With the help of $\omega$ we introduce the scalar product of the Poincaré-Birkhof-Witt basis in the Verma module $V_q$. By using some new results in the theory of Verma modules [4], we find the singular (null) vectors and after quotienting them and their descendents in $V_q$, we obtain irreducible positive-energy unitary representations. When $q$ is $N$-th root of unity, all these unitary representations become finite-dimensional. We discuss in some detail the massless representations, which are also irreducible representations of the $q$-deformed Poincaré subalgebra. Generically, their dimensions are smaller than the corresponding finite-dimensional non-unitary representation of $su(2, 2)$, except when $N = 2|h| + 1$, where $h$ is the helicity (this includes the fundamental representations). As examples we give explicitly a $4 \times 4$ representation for $N = 2 \ (q = -1), j_1 = 1/2, j_2 = 0$, and list the orthonormal basis for the cases $N = 3 \ (q = e^{2\pi i/3}), j_2 = 0, j_1 = 1/2$ and $j_1 = 1$.

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Construction of quantum groups and the Y.B.E. with spectral parameter

P. Cotta Ramusino, L. Lambe and M. Rinaldi

Faddeev, Reshetikhin and Takhtadzhyan showed, some time ago [1] how to construct a bialgebra $A_R$ out of a given matrix $R \in \text{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ satisfying the Yang-Baxter equation. This bialgebra may have an antipode, i.e. may be Hopf algebras, when some extra conditions are satisfied.

All known Quantum Groups arise as Hopf algebras which are duals to the above bialgebras. In this framework, the Universal R-Matrix can be thought as a bilinear form over $A_R$ [2].

In our paper we analyze some aspects of the construction of [1] and discuss some applications to the case of Yang-Baxter equations with spectral parameter.

We first start with a matrix $R \in \text{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$, which is only supposed to verify the Yang-Baxter equation (without spectral parameter) and then we construct canonically a pair of bilinear forms on $A_R$ satisfying the quasi-commutativity and quasi-triangularity properties, as well as the Yang-Baxter equation.

These bilinear forms (Generalized Universal R-Matrices) allow the construction of other solutions of the Yang-Baxter equations which can be seen as "tensor products" of the given solution.

When $R$, $R^{t_2}$ and $(R^{-1})^{t_2}$ are all invertible then we show explicitly that the dual of $A_R$ is a ribbon Hopf-Algebras (see [3,4]). Above, given $R = \sum_i \alpha_i \otimes \beta_i$, we have set $R^{t_2} \equiv \sum_i \alpha_i \otimes \beta_i^T$.

The construction of Generalized Universal R-Matrices, can be easily adapted to the case of Yang-baxter equations with spectral parameter. Moreover the spectral parameter can take values in a generic non commutative group.

If we are given a solution of such of an equation, then we can construct in a canonical way a bialgebra, very similar to the bialgebra $A_R$ considered in the constant case and a pair of bilinear forms over such bialgebra. These bilinear forms have all the corresponding properties of the constant case (quasi-commutativity, quasi-triangularity) and they satisfy the Yang-Baxter equation without the spectral parameter.

This allows us to construct tensor products of solutions, exactly as in the constant case. Moreover we discuss the search for solutions of the Yang-Baxter equation with spectral
parameter, starting from the solutions of the corresponding constant equation.

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Topological field theory and invariants of three-manifolds

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The quantum Chern-Simons model \[1,2\] is a topological gauge field theory which is defined in a generic three-manifold \( \mathcal{M} \) which is orientable. The gauge invariant observables of this system are the vacuum expectation values of the Wilson line operators associated with framed and oriented links in \( \mathcal{M} \); each link component has a colour which is given by the labels of the inequivalent irreducible representations of the gauge group.

Quite remarkably, for compact simple Lie groups, the Chern-Simons theory is soluble \[3\] in any closed and connected three-manifold. This means that, for any given link in \( \mathcal{M} \), one can easily compute the exact expression of the associated expectation value of the Wilson line operator. These expectation values represent invariants of ambient isotopy for framed links in \( \mathcal{M} \).

In addition to the expectation values of the Wilson line operators, one can also compute the value \( \mathcal{I}(\mathcal{M}) \) of the improved partition function \[3\] of the theory in the three-manifold \( \mathcal{M} \). In order to give a precise definition of \( \mathcal{I}(\mathcal{M}) \), one must connect topologically inequivalent manifolds. This problem cannot be solved by means of the standard methods of field theory. For this reason, in ref.\[3\] the operator surgery method has been introduced. The basic idea is to find an appropriate generalization of the familiar Casimir effect. Any three-manifold \( \mathcal{M} \) is cobordant with the sphere \( S^3 \); this means that \( \mathcal{M} \) can be obtained by means of Den surgery on \( S^3 \). The operator surgery method consists in finding the field theory rules which correspond to the surgery instructions. The crucial property, which permits us to solve this problem, is the fact that a generic surgery can be obtained by combining twist homeomorphisms of solid tori.

The case in which the gauge group is \( G = SU(2) \) has been studied in detail in ref.\[3\]; the operator surgery method can also be used for more complicated groups. So, the non-Abelian Chern-Simons field theory (with a compact gauge group) has been solved in any three-manifold \( \mathcal{M} \). As a result, one can define and compute a set of topological invariants of three-manifolds. These invariants are equivalent to those defined by Reshetikhin and Turaev \[4\] by means of quantum groups. This equivalence is not a coincidence because any function, which is invariant under the diagonal action of the standard quantum groups, really represents an invariant function of ordinary Lie groups.
Given a set of topological invariants in three dimensions, one can naturally define a set of associated invariants in two dimensions. Thus, by means of the invariants of the Chern-Simons field theory, one can define [5] a set of topological invariants for the punctured Riemann surfaces of arbitrary genus. These new invariants are integer numbers and represent the dimensions of the physical state spaces associated with the punctured Riemann surfaces.

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The quantum Dirac equation associated to the $\kappa$–Poincaré

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In these last years the method of "$q$–contracting" known semisimple quantum groups to nonsemisimple ones [1] has been successfully applied to get the deformed counterpart of the most physically relevant kinematical algebras [2,3]. The construction of 4–dimensional quantum Poincaré (with 10 parameters and an hermitian involutive coproduct) [3,4] which has been obtained by $q$–contracting the $q$–antiDeSitter $SO_q(3, 2)$ Hopf algebra rewritten in terms of $q$–commutators of "physical generators". Since only one direction can be deformed it is obvious to choose the time axis so that the quantum relativistic mass square operator becomes

$$ C_1 = (2\kappa \sinh(P_0/(2\kappa)))^2 - P_i P_i \quad \text{(as proposed in [2]) and the square}$$

$$ C_2 = (\cosh(P_0/\kappa) - P_i P_i/(4\kappa^2)) W_0^2 - W_i W_i, $$

with $W_0 = P_i M_i$ and $W_i = \kappa \sinh(P_0/\kappa) M_i + \epsilon_{ijk} P_j L_k$.

A $q$–deformed Dirac equation was then proposed in [5,6] as the square root of $C_1$. Our criterion [7] is that the $\kappa$–deformed Dirac equation must be invariant under the spinorial representations of the $\kappa$–Poincaré and our method makes use of the contraction applied to representations. So we $q$–contract the sum of two $q$–antiDeSitter representations one of them 4–dimensional, rephrasing the procedure that in the classical case gives rise to the Dirac equation (see [8]). In this way we derive the corresponding representation of the $\kappa$–Poincaré generators in which the orbital spinless operators are q–composed with the $\gamma$ representation of the Lorentz algebra (which is a zero–momentum representation also of the $\kappa$–Poincaré Hopf algebra). For these representations the relationship between the two Casimirs is $C_2 = -(3/4)C_1(1 + C_1/(4\kappa^2))$.

The quantum Dirac operator is then found by imposing the invariance under the global $\kappa$–spinorial representations:

$$ D = -\exp(-\frac{P_0}{2\kappa}) \gamma_i P_i + \gamma_4 \kappa \sinh(\frac{P_0}{\kappa}) - \gamma_4 \frac{1}{2\kappa} P_i P_i. $$

It is noteworthy that it differs from [5,6] as $D^2 = C_1(1 + C_1/(4\kappa^2)) = -(4/3) C_2$ and that $D$ gives rise to a 4–spinorial wave equation with finite–difference operators in the time with a delay $i/(2\kappa)$, the classical limit $\kappa \to \infty$ gives the standard Dirac operator. The
phenomenological implications concerning the coupling to electromagnetic field are now under investigations [9,10].

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Quantum algebras and basic hypergeometric functions

Roberto Floreanini and Luc Vinet

The representation theory of quantum algebras is deeply rooted in the theory of \( q \)-special functions. These functions are extensions to a base \( q \) of the standard special functions. One of the best known example is the basic hypergeometric series \( (|q|<1) \) \[1\]

\[
\begin{align*}
2\phi_1(a,b;c,q,z) &= \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} z^n, \quad |z| < 1, \\
(a;q)_n &= (1-a)(1-aq) \cdots (1-aq^{n-1}),
\end{align*}
\]

which is the \( q \)-generalization of the usual hypergeometric series \( 2F_1 \) of Gauss. The following two \( q \)-exponentials are also very important:

\[
\begin{align*}
e_q(z) &= \sum_{n=0}^{\infty} \frac{1}{(q;q)_n} z^n, \quad |z| < 1, \\
E_q(z) &= \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n} z^n.
\end{align*}
\]

The basic idea underlying the quantum algebra interpretation of \( q \)-special functions is to replace the exponential mappings from the Lie algebra \( \mathcal{G} \) into the corresponding group by \( q \)-exponentials from the quantum algebra \( \mathcal{U}_q(\mathcal{G}) \) into the completion of it. The matrix elements of products of \( q \)-exponentials in quantum algebra generators, in specific representations, are then found to be expressible in terms of \( q \)-special functions.

The hypergeometric function \( 2\phi_1 \) is in this way connected with the quantum algebra \( \mathcal{U}_q(sl(2)) \), \([5,6,7]\) while known \( q \)-generalizations of classical orthogonal polynomials (Hermite, Laguerre, Gegenbauer) appear in the representation theory of \( q \)-oscillators and of \( \mathcal{U}_q(su(1,1)) \). \([2,3,8]\) The basic version of Bessel functions are instead connected with the euclidean algebra in two dimensions. \([4,9]\)

This quantum algebra interpretation of \( q \)-special functions is very useful since it allows deriving new properties for these functions (orthogonality relations, generating functions, addition formulas), hard to obtain otherwise.
As in the case of ordinary special functions, the basic special functions enter the solution of many physical problems: they arise whenever quantum algebras are relevant to the description of physical models (e.g. infinite spin chains and massive minimal models). The study of the properties of these functions is thus of great importance and the approach presented here seems to be simple and promising.

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4–Dimensional lattice gravity and 12j-symbols

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In 1968 Ponzano and Regge [1] discovered a deep connection between the asymptotic expansion of a Racah-Wigner 6j- symbol and the partition function for 3D-euclidean quantum gravity with an action discretized according to the prescription of Regge Calculus [2]. During the last three years there have been important developments of this original idea, which provide a new insight in the search for the semiclassical limit of discretized 3D-gravity models and which allow to discuss their connection with topological quantum field theories (see [3]-[10]).

I present here a model (see [11],[12]) which provides a 4-dimensional version of Ponzano and Regge’s result [1]. In particular, I show that the exponential of the euclidean Einstein-Regge action for a 4D- discretized block is given, in the semiclassical limit, by a suitable gaussian integral of a 12j-symbol. I also discuss a model of 4D-topological lattice gravity which involves a 15j-symbol [13]. Differences between the two models are stressed.

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The geometrical meaning of the quantum correction

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The ring (Frobenius algebra) of local observables for topological \( \sigma \)-models on the Riemann sphere \( P^1 \) with values in the grassmanian \( G(s;n) \) is known to be the quotient of the cohomology ring of the target space by the (inhomogeneous) ideal generated by the so-called quantum correction. While the need for a quantum correction comes from algebraic motivations in field theory, the aim of our paper [1] is to understand its geometrical meaning. The simple examples of \( P^1 \to P^n \) models tell us that the quantum correction is a form of Poincaré duality which allows to compute intersections on moduli spaces of lower degrees. We will check this point of view for the case of \( P^1 \to G(s;n) \) models, yielding a proof of the algebraic result from physics [2] in terms of the geometry of the \( \sigma \)-model itself. Finally we generalize our geometrical reasoning to flag-manifold valued models, getting a new result which needs some physical interpretation.

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42