On the first-passage area of a Lévy process

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Abstract

Let be $X(t) = x - \mu t + \sigma B_t - N_t$ a Lévy process starting from $x > 0$, where $\mu \geq 0$, $\sigma \geq 0$, $B_t$ is a standard BM, and $N_t$ is a homogeneous Poisson process with intensity $\theta > 0$, starting from zero. We study the joint distribution of the first-passage time below zero, $\tau(x)$, and the first-passage area, $A(x)$, swept out by $X$ till the time $\tau(x)$. In particular, we establish differential-difference equations with outer conditions for the Laplace transforms of $\tau(x)$ and $A(x)$, and for their joint moments. In a special case ($\mu = \sigma = 0$), we show an algorithm to find recursively the moments $E[\tau(x)^m A(x)^n]$, for any integers $m$ and $n$; moreover, we obtain the expected value of the time average of $X$ till the time $\tau(x)$.

Keywords: First-passage time, first-passage area, jump-diffusion, Lévy process.

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1 Introduction

This is a continuation of the articles [1] and [2]; actually, in [1] we studied the distribution of the first-passage area (FPA) $A(x) = \int_0^{\tau(x)} X(t)\,dt$, swept out by a one-dimensional jump-diffusion process $X(t)$, starting from $x > 0$, till its first-passage time (FPT) $\tau(x)$ below zero, while in [2] we examined the special case (without jumps) when $X(t)$ is Brownian motion (BM) $B_t$ with negative drift $-\mu$, that is, $X(t) = x - \mu t + B_t$, studying in particular the joint distribution of $\tau(x)$ and $A(x)$. In the present paper, we aim to investigate the joint distributions of $\tau(x)$ and $A(x)$, in the case when $X(t)$ is a Lévy process of the form

$$X(t) = x - \mu t + \sigma B_t - N_t, \ x > 0,$$

(1.1)

where $\mu \geq 0$, $\sigma \geq 0$, $B_t$ is standard BM, and $N_t$ is a homogeneous Poisson process with intensity $\theta > 0$, starting from zero; thus, $X(t)$ turns out to be the superposition of drifted BM and Poisson process.

Referring to the Lévy process (1.1), we state and solve differential-difference equations for the Laplace transform of the two-dimensional random variable $(\tau(x), A(x))$. In particular, when $\mu = \sigma = 0$, we obtain the joint moments $E[\tau(x)^m A(x)^n]$ of the FPT and FPA, and we present an algorithm to find them recursively, for any $m$ and $n$; moreover, we find the expected value of the time average of $X(t)$ till its FPT below zero.

Studying the FPA of a process such as (1.1) is peculiar when modeling the evolution of certain random systems described by the superposition of a continuous stochastic process and...
a jump process (see references in [1]); these arise e.g. in solar physics studies, non-oriented animal movement patterns, and DNA breathing dynamics, as regards systems where the jump component can be absent (see e.g. [13], [14], [15] and references in [12]). Applications can be found in *Queueing Theory* (see e.g. [1]), and in *Finance*, in the framework of default-at-maturity model, which assumes the exchange rate follows a jump-diffusion process (see e.g. [5], [11]); for other examples from Economics and Biology, see e.g. [1].

The paper is organized as follows. Section 2 contains preliminary results on jump-diffusion processes. In section 3 we deal with the Laplace transform of \( \tau(x, A(x)) \) and the joint moments of \( \tau(x) \) and \( A(x) \), in the case when \( X(t) \) is a Poisson process (that is \( \mu = \sigma = 0 \)); precisely, we find explicitly \( E[\tau(x)A(x)] \), and we establish ODEs for the joint moments \( E[\tau(x)^m A(x)^n] \) of order \( n + m, \ n, m \geq 0 \), presenting also an algorithm to find recursively them; moreover, we find the expected value of the time average of \( X(t) \) till its FPT below zero. Sections 4 and 5 are respectively devoted to study the distributions of \( \tau(x) \), \( A(x) \) in the case of Poisson process with negative drift (\( \sigma = 0 \)), and drifted BM with Poisson jumps (\( \mu \neq 0, \ \sigma \neq 0 \)). Finally, Section 6 contains conclusions and final remarks.

## 2 Notations, formulation of the problem and preliminary results

We recall from [1] some definitions and results concerning FPT and FPA for a one-dimensional, time homogeneous jump-diffusion process \( X(t) \) driven by the SDE:

\[
dX(t) = b(X(t))dt + \sigma(X(t))dB_t + \int_{-\infty}^{+\infty} \gamma(X(t), u)\nu(dt, du)
\]

(2.1)

with assigned initial condition \( X(0) = x > 0 \); here \( \nu(\cdot, \cdot) \) is a temporally homogeneous Poisson random measure (see [7] for the definitions), and the functions \( b(\cdot), \sigma(\cdot), \gamma(\cdot, \cdot) \) satisfy suitable conditions for the existence and uniqueness of the solution (see [7], [10]). The random measure \( \nu \) is supposed to be homogeneous with respect to time translation, that is, its intensity measure \( E[\nu(dt, du)] \) is of the form \( E[\nu(dt, du)] = dt\pi(du) \) for some positive measure \( \pi \) defined on \( \mathcal{B}(\mathbb{R}) \), the Borel \( \sigma \)-field of subsets of \( \mathbb{R} \), and the jump intensity \( \Theta = \int_{-\infty}^{+\infty} \pi(du) \geq 0 \) is assumed to be finite.

If \( \gamma = 0, \) or \( \nu = 0 \), then the SDE (2.1) becomes the usual Itô's stochastic differential equation for a simple-diffusion (i.e. without jumps). If, for instance, the measure \( \pi \) is concentrated over the set \( \{u_1, u_2\} = \{-1, 1\} \) with \( \pi(u_i) = \theta_i \) and \( \gamma(u_i) = \epsilon_i \), the SDE (2.1) assumes the form \( dX(t) = b(X(t))dt + \sigma(X(t))dB_t + \epsilon_2 dN_2(t) + \epsilon_1 dN_1(t) \), where \( \epsilon_1 < 0, \ \epsilon_2 > 0 \) and \( N_i(t), \ t \geq 0 \) are independent homogeneous Poisson processes of amplitude 1 and rates \( \theta_1 \) and \( \theta_2 \), respectively governing downward \( (N_1) \) and upward \( (N_2) \) jumps.

Let \( D \) be the class of function \( f(x) \in C^2 \), for which the function \( f(x + \gamma(x, u)) - f(x) \) is \( \pi \)-integrable for any \( x \). The differential operator associated to the process \( X(t) \) is defined for \( f \in D \) by:

\[
Lf(x) = \frac{1}{2} \sigma^2(x) \frac{d^2}{dx^2} f(x) + b(x) \frac{d}{dx} f(x) + \int_{-\infty}^{+\infty} [f(x + \gamma(x, u)) - f(x)] \pi(du).
\]

(2.2)

Let us define, for \( x > 0 \):

\[
\tau(x) = \inf\{t > 0 : X(t) \leq 0|X(0) = x\},
\]

(2.3)
that is the first-passage time below zero of \(X(t)\), and assume that \(\tau(x)\) is finite with probability one. Really, it is possible to show (see [1], [16]) that the probability \(p_0(x)\) that \(X(t)\) ever leaves the interval \((0, +\infty)\) satisfies the partial differential-difference equation (PDDE) \(Lp_0 = 0\) with outer condition: \(p_0(y) = 1\) if \(y \leq 0\). The equality \(p_0(x) \equiv 1\) is equivalent to say that \(\tau(x)\) is finite with probability one. For diffusion processes without jumps (i.e. \(\gamma = 0\)) sufficient conditions are also available which ensure that \(\tau(x)\) is finite w.p. 1, and they concern the convergence of certain integral associated to the coefficients of \((2.1)\) (see e.g. [7], [9]).

Let \(U\) be a functional of the process \(X\); for \(\lambda > 0\) denote by

\[ M_{U,\lambda}(x) = E\left[e^{-\lambda \int_0^{\tau(x)} U(X(s))ds}\right] \quad (2.4) \]

the Laplace transform of the integral \(\int_0^{\tau(x)} U(X(s))ds\). Then, it holds (see [1]):

**Theorem 2.1** Let \(X(t)\) be the solution of the SDE \((2.1)\), starting from \(X(0) = x > 0\); then, under the above assumptions, \(M_{U,\lambda}(x)\) is the solution of the problem with outer conditions:

\[
\begin{align*}
LM_{U,\lambda}(x) &= \lambda U(x)M_{U,\lambda}(x) \\
M_{U,\lambda}(y) &= 1, \text{ for } y \leq 0, \lim_{x \to +\infty} M_{U,\lambda}(x) = 0,
\end{align*}
\]

where \(L\) is the generator of \(X\), which is defined by \((2.2)\).

The \(n\)-th order moment of \(\int_0^{\tau(x)} U(X(s))ds\), if it exists finite, is given by \((n = 1, 2, \ldots)\):

\[ T_n(x) = E\left[\left(\int_0^{\tau(x)} U(X(s))ds\right)^n\right] = (-1)^n \left[\frac{\partial^n}{\partial \lambda^n} M_{U,\lambda}(x)\right]_{\lambda=0}. \]

Then, taking the \(n\)-th derivative with respect to \(\lambda\) in both members of the equation \((2.5)\), and calculating it for \(\lambda = 0\), one easily obtains that the \(n\)-th order moment \(T_n(x)\) \((n = 1, 2, \ldots)\) of \(\int_0^{\tau(x)} U(X(s))ds\), whenever it exists finite, is the solution of the PDDE:

\[ LT_n(x) = -nU(x)T_{n-1}(x), \quad x > 0, \quad (2.6) \]

which satisfies

\[ T_n(x) = 0, \text{ for } x \leq 0 \quad (2.7) \]

and an appropriate additional condition.

Indeed, the only condition \(T_n(x) = 0\) for \(x \leq 0\) is not sufficient to determinate uniquely the desired solution of the PDDE \((2.6)\), because it is a second order equation. Note that for a diffusion without jumps (\(\gamma = 0\)) and for \(U(x) \equiv 1\), \((2.6)\) is nothing but the celebrated Darling and Siegert’s equation \(([6])\) for the moments of the first-passage time, and \((2.7)\) becomes simply the boundary condition \(T_n(0) = 0\).

In the next section, we start considering the Poisson process \(X(t) = x - N_t\) \((x > 0)\), obtained from \((1.1)\) by taking \(\mu = \sigma = 0\), which is a special case of the jump-diffusion driven by \((2.1)\). As easily seen, the FPT of \(X(t)\) below zero, \(\tau(x)\), is finite with probability one. The successive sections regard Poisson process with negative drift \(X(t) = x - \mu t - N_t\) \((\sigma = 0)\) and drifted BM with Poisson jumps \(X(t) = x - \mu t + \sigma B_t - N_t\), \((\sigma > 0)\).

To illustrate the behavior of the process \(X(t)\) in \((2.1)\), in the figure below we show a sample path of the process \(X(t)\), respectively in the case of Poisson process, Poisson process with drift, and drifted BM with Poisson jumps.
3 The case of Poisson process \((\mu = \sigma = 0)\)

For \(x > 0\), let us consider the process \(X(t) = x - N_t\), where \(N_t\) \((N_0 = 0)\) is a homogeneous Poisson process with intensity \(\theta > 0\). The infinitesimal generator is:

\[
    Lf(x) = \theta[f(x-1) - f(x)], \quad f \in C^0(\mathbb{R}),
\]

and \(\tau(x) = \inf\{t > 0 : x - N_t \leq 0\}\), \(A(x) = \int_0^{\tau(x)} (x - N_t)dt\).

By Theorem 2.1 with \(U(x) = 1\), it follows that the Laplace transform \(M_{U,\lambda}(x)\) of \(\tau(x)\) is the solution of the equation \(LM_{U,\lambda}(x) = \lambda M_{U,\lambda}(x)\), with outer condition \(M_{U,\lambda}(y) = 1\) for \(y \leq 0\). By solving this equation, we get for any \(x > 0\) (see [1]):

\[
    M_{U,\lambda}(x) = \begin{cases} 
        \left(\frac{\theta}{\theta+\lambda}\right)^x & \text{if } x \in \mathbb{N} \\
        \left(\frac{\theta}{\theta+\lambda}\right)^{[x]+1} & \text{if } x \notin \mathbb{N},
    \end{cases}
\]  

(3.2)
where $[x]$ denotes the integer part of $x$. Recalling the expression of the Laplace transform of the Gamma density, we note that $\tau(x)$ has Gamma distribution with parameters $(x, \theta)$ if $x$ is a positive integer, while it has Gamma distribution with parameters $([x]+1, \theta)$ if $x$ is not an integer. In Fig. 2, we show the graph of the Laplace transform of $\tau(x)$, as a function of $\lambda > 0$, for some values of the parameters.

![Figure 2: Laplace Transform of $\tau(x)$](image)

The moments $T_n(x) = E[(\tau(x))^n]$ are soon obtained by the formula

$$T_n(x) = (-1)^n \left[\frac{\partial^n}{\partial x^n} M_{\lambda}(x)\right]_{\lambda=0}.$$  

In fact, we get (see [1]):

$$E(\tau(x)) = \begin{cases} \frac{x}{\theta} & \text{if } x \in \mathbb{N} \\ \frac{|x+1|}{\theta} & \text{if } x \notin \mathbb{N}, \end{cases}$$

and

$$E(\tau^2(x)) = \begin{cases} \frac{x^2 + x}{\theta^2} & \text{if } x \in \mathbb{N} \\ \frac{|x+1|^2}{\theta^2} + \frac{|x+1|}{\theta} & \text{if } x \notin \mathbb{N}. \end{cases}$$

Therefore:

$$Var(\tau(x)) = \begin{cases} \frac{x^2}{\theta^2} & \text{if } x \in \mathbb{N} \\ \frac{|x+1|}{\theta} & \text{if } x \notin \mathbb{N}. \end{cases}$$

By Theorem 2.1 with $U(x) = x$, we get the Laplace transform $M_{U,\lambda}(x)$ of $A(x)$ as the solution of the equation $LM_{U,\lambda}(x) = \lambda x M_{U,\lambda}(x)$, with outer condition $M_{U,\lambda}(y) = 1$ for $y \leq 0$. By solving this equation, we obtain for any $x > 0$ (see [1]):

$$M_{U,\lambda}(x) = \begin{cases} \theta^x \cdot ((\theta + \lambda)(\theta + 2\lambda) \cdots (\theta + x\lambda))^{-1} & \text{if } x \in \mathbb{N} \\ \theta^{|x|+1} \cdot ((\theta + \lambda x)(\theta + \lambda(x-1)) \cdots (\theta + \lambda(x-[x])))^{-1} & \text{if } x \notin \mathbb{N} \end{cases}$$

that can be written in the unique form, valid for any $x > 0$:

$$M_{U,\lambda}(x) = \frac{\theta^{|x|+1}}{(\theta + \lambda x)(\theta + \lambda(x-1)) \cdots (\theta + \lambda(x-[x]))}.$$ 

Notice that $M_{U,\lambda}(x)$ turns out to be the Laplace transform of a linear combination of $[x] + 1$ independent exponential random variables with parameter $\theta$, with coefficients $x, x-1, \ldots, x-[x]$. In the Figure 3, we show the graph of the Laplace transform of $A(x)$, as a function of $\lambda > 0$, for some values of the parameters.

![Figure 3.2: Laplace Transform of $x$ for $\lambda = 1$ and $\lambda = 1$ (blue), $\lambda = 2$ (red), and $\lambda = 3$ (cyan).](image)

The $n$–th order moment of $A(x)$ is given by $(-1)^n \left[\frac{\partial^n}{\partial x^n} M_{U,\lambda}(x)\right]_{\lambda=0}$; calculating the first and second derivative, we obtain (see [1]):

$$E(A(x)) = \frac{(2x - [x])([x] + 1)}{2\theta}$$

(3.5)
Figure 3: Laplace Transform of $A(x)$, as a function of $\lambda > 0$, for $\theta = 1$; on the left, from top to bottom: $x = 1$ (blue), $x = 2$ (red), and $x = 3$ (cyan); on the right, from top to bottom: $x = 1.5$ (blue), $x = 2.5$ (red), and $x = 3.5$ (cyan).

and

$$E(A^2(x)) = \frac{[x] + 1}{12\theta^2} \left(12x(x-[x])([x]+2)+[x](3[x]^2+7[x]+2)\right). \quad (3.6)$$

In the Figure 4, we show the densities of $\tau(x)$ and $A(x)$, estimated by Monte Carlo simulation, for some values of the parameters.

Figure 4: For $\theta = 1$ and $x = 2$, estimated density of $\tau(x)$ (left) and of $A(x)$ (right).

### 3.1 Joint moments of $\tau(x)$ and $A(x)$.

For $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}_+^2$, the joint Laplace transform of $(\tau(x), A(x))$ is

$$M_\lambda(x) = M_{(\lambda_1, \lambda_2)}(x) = E \left( e^{-\lambda_1 \tau(x) - \lambda_2 A(x)} \right)$$

$$= E \left( e^{-\lambda_1 \tau(x) - \lambda_2 \int_0^x X(t) dt} \right) = E \left( e^{-\int_0^x \lambda_1(\lambda_1 + \lambda_2 X(t)) dt} \right). \quad (3.7)$$

By using (2.5) with $\lambda = 1$ and $U(x) = \lambda_1 + \lambda_2 x$, we obtain that the function $M_{(\lambda_1, \lambda_2)}(x)$ solves the problem:

$$\begin{cases}
\theta[M_\lambda(x-1) - M_\lambda(x)] = (\lambda_1 + \lambda_2 x)M_\lambda(x), \quad x > 0; \\
M_\lambda(y) = 1, \quad y \leq 0; \quad \lim_{x \to +\infty} M_\lambda(x) = 0.
\end{cases} \quad (3.8)$$

The joint moments $E[\tau(x)^m A(x)^n]$, if they exist finite, are given by:

$$E[\tau(x)^m A(x)^n] = (-1)^{m+n} \frac{\partial^m}{\partial \lambda_1^m} \frac{\partial^n}{\partial \lambda_2^n} M_\lambda(x) \bigg|_{\lambda_1 = \lambda_2 = 0}. \quad (3.9)$$

By taking the derivatives $\frac{\partial^2}{\partial \lambda_1 \partial \lambda_2}|_{\lambda_1 = \lambda_2 = 0}$ in both sides of (3.8), and equaling them, we obtain:
Theorem 3.1 \( V(x) = E[\tau(x)A(x)] \) is the solution of the problem
\[
\begin{align*}
\theta[V(x-1) - V(x)] &= -xE[\tau(x)] - E[A(x)], \; x > 0; \\
V(y) &= 0, \; y \leq 0; \; \lim_{\theta \to +\infty} V(x) = 0.
\end{align*}
\] (3.10)

Now we solve (3.10), separately in the case when \( x \in \mathbb{N} \), and when \( x \notin \mathbb{N} \).

1) \( x \in \mathbb{N} \): from (3.3) and (3.5), we obtain \(-xE[\tau(x)] - E[A(x)] = -\frac{x^2}{\theta} - \frac{x(x+1)}{2\theta} = -\frac{x(3x+1)}{2\theta}\).

Then, from the first equation of (3.10) we get
\[
V(x) = V(x-1) + \frac{x(3x+1)}{2\theta^2}.
\]

It follows that
\[
V(x) = \frac{x(x+1)^2}{2\theta^2}, \quad \text{where we used that } \sum_{i=1}^{n} i = n(n+1)/2 \text{ and } \sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6.
\]

The covariance of \( \tau(x) \) and \( A(x) \) is
\[
Cov(\tau(x), A(x)) = E[A(x)\tau(x)] - E[A(x)]E[\tau(x)]
\]
\[
= \frac{x(x+1)^2}{2\theta^2} - \frac{x}{\theta} \cdot \frac{x(x+1)}{2\theta} = \frac{x(x+1)}{2\theta^2}.
\] (3.12)

The correlation coefficient is obtained by (3.3) and (3.6):
\[
\rho(x) = \frac{Cov(\tau(x), A(x))}{\sqrt{Var(A(x))Var(\tau(x))}} = \frac{3(x+1)}{2(2x+1)}.
\] (3.13)

2) \( x \notin \mathbb{N} \): from (3.3) and (3.5) we have \(-xE[\tau(x)] - E[A(x)] = -\frac{[x]+1)(4x-[x])}{2\theta}\); therefore, from the first of (3.10) we obtain that \( V(x) = V(x-1) + \frac{([x]+1)(4x-[x])}{2\theta^2} \). As in the previous case, it follows that \( V(x) = \sum_{n=0}^{[x]} \frac{([x-n]+1)(4x-n-[x-n])}{2\theta^2} \). Since \( \forall k \in \mathbb{Z} \) and \( x \in \mathbb{R}, \; [x+k] = k + [x] \), after calculation, we obtain:
\[
V(x) = \frac{([x]+1)([x]+2)(2x-[x])}{2\theta^2}.
\] (3.14)

Therefore,
\[
Cov(\tau(x), A(x)) = \frac{([x]+1)([x]+2)(2x-[x])}{2\theta^2} - \frac{([x]+1)(2x-[x])([x]+1)}{\theta} \cdot \frac{2\theta}{2\theta} = \frac{([x]+1)(2x-[x])}{2\theta^2}.
\] (3.15)

The correlation coefficient is:
\[
\rho(x) = \frac{Cov(\tau(x), A(x))}{\sqrt{Var(A(x))Var(\tau(x))}}
\]
\[
= \frac{([x]+1)(2x-[x])}{2\theta^2} \cdot \frac{([x]+1)(2x-[x])}{12\theta^2} \cdot \{12x (x - [x]) + 2x ([x] + 1)\}
\]
\[
= \sqrt{\frac{3(2x-[x])^2}{12x (x - [x]) + 2x ([x] + 1)}}.
\] (3.16)

Notice that, both in the case when \( x \) is an integer, and when it is not, \( \rho(x) \) depends only on \( x \) and, \( \rho(x) \to \sqrt{\frac{3}{4}} \approx 0.8666 \), as \( x \to +\infty \).
3.3.1 Joint Laplace Transform of \((\tau(x), A(x))\)

Let us consider the problem (3.17), given by

\[
\mathbb{E}[\tau(x) | A(y)] = A(y), \quad y \in \mathbb{N},
\]

\[
\lim_{y \to +\infty} A(y) = 0.
\]

(3.26)

We can find an explicit solution to this problem, analyzing separately the case when \(x \in \mathbb{N}\) and the case when \(x \not\in \mathbb{N}\).

1) \(x \in \mathbb{N}\): as easily seen, one has

\[
M_\lambda(x) = \theta^x \cdot \prod_{k=1}^{x} \frac{1}{\lambda_1 + \lambda_2 k + \theta},
\]

(3.17)

2) \(x \not\in \mathbb{N}\): if \(x \in (0, 1)\), then \(M_\lambda(x) = \theta/[(\lambda_1 + \lambda_2 x + \theta)]\), because \(M_\lambda(x-1) = 1\); if \(x \in (1, 2)\), then \(M_\lambda(x) = \theta^2/[(\lambda_1 + \lambda_2 x + \theta)(\lambda_1 + \lambda_2 (x-1) + \theta)]\).

Thus, iterating the procedure, one gets:

\[
M_\lambda(x) = \theta^{|x|+1} \prod_{k=0}^{|x|} \frac{1}{\lambda_1 + \lambda_2 (x-k) + \theta}.
\]

(3.18)

It is easy to check that \(\frac{\partial^2}{\partial \lambda_1 \partial \lambda_2} M_\lambda(x) \big|_{\lambda_1=\lambda_2=0} = V(x)\) where \(V(x)\) is given by (3.11) and (3.14).

In the Figure 6, we show, in the case of Poisson process, the graph of the joint Laplace transform of \((\tau(x), A(x))\), as a function of \((\lambda_1, \lambda_2)\), for \(x = 1\).

3.3 Moments of any order

The aim of this subsection is to find the quantity

\[
V_{m,n}(x) := \mathbb{E}[\tau(x)^m A(x)^n],
\]

with \(m, n \in \mathbb{N}\), following the same arguments used in [2] in the corresponding case. Taking the \(m\)-th partial derivative with respect to \(\lambda_1\) and the \(n\)-th partial derivative with respect to \(\lambda_2\) in the first equation of (3.8) and evaluating at \(\lambda_1 = \lambda_2 = 0\), we obtain the following:

\[
\begin{align*}
\theta [V_{m,n}(x-1) - V_{m,n}(x)] &= -nxV_{m,n-1}(x) - mV_{m-1,n}(x) \\
V_{m,n}(y) &= 0, \quad y \leq 0, \quad \lim_{\theta \to +\infty} V_{m,n}(x) = 0.
\end{align*}
\]

(3.19)
Figure 6: Joint Laplace Transform of \((\tau(x), A(x))\), as a function of \((\lambda_1, \lambda_2)\), in the case of Poisson process, for \(x = 1\).

\[\square\]

**Remark 3.3** For \(m = n = 1\), we have \(V_{1,1}(x) = V(x) = E[\tau(x)A(x)]\), \(V_{0,1}(x) = E[A(x)]\) and \(V_{1,0}(x) = E[\tau(x)]\), so (3.19) becomes (3.10).

By solving (3.19) for \(m = 2, n = 1\), one finds:

\[
V_{2,1}(x) = \begin{cases} 
\frac{x(x+1)^2(x+2)}{2^6}, & x \in \mathbb{N} \\
\frac{(x+1)(x+2)(x+3)(2x-1)}{2^6}, & x \notin \mathbb{N}
\end{cases}
\] (3.20)

that is, if \(x \in \mathbb{N}\), \(V_{2,1}(x)\) is a polynomial of degree 4, otherwise it has polynomial growth of degree 4. Similar polynomial expressions can be obtained for any \(m\) and \(n\), implying that the moments \(E[\tau(x)^mA(x)^n]\) are finite, for all \(m\) and \(n\). As far as the form of the solution \(V_{m,n}(x)\) of (3.19) is concerned, proceeding by induction, as done in [2], one gets:

**Theorem 3.4** For integers \(m, n \geq 0\), the solution of (3.19) vanishes at zero, and it is a polynomial of degree \(m + 2n\) if \(x \in \mathbb{N}\), otherwise it has a polynomial growth of degree \(m + 2n\).

\[\square\]

When \(x\) is an integer, by proceeding as done in [2] for the analogous situation, it is possible to obtain a compact closed form of \(V_{m,n}(x)\): in fact, by using Theorem 3.4 for \(x, m, n \in \mathbb{N}\), we obtain that there exist \(a_{1}^{(m,n)}, \ldots a_{m+2n}^{(m,n)} \in \mathbb{R}\) such that \(V_{m,n}(x) = \sum_{k=1}^{m+2n} a_{k}^{(m,n)} x^{k}\), moreover, there exist real numbers \(a_{k+1}^{(m-1,n)}\) for \(k = 1, \ldots, m + 2n - 1\) and \(a_{k}^{(m,n-1)}\) for \(k = 1, \ldots, m + 2n - 2\) such that:

\[
V_{m-1,n}(x) = \sum_{k=1}^{m+2n-1} a_{k}^{(m-1,n)} x^{k} = a_{1}^{(m-1,n)} x + \sum_{k=1}^{m+2n-2} a_{k+1}^{(m-1,n)} x^{k+1},
\] (3.21)

and

\[
V_{m,n-1}(x) = \sum_{k=1}^{m+2n-2} a_{k}^{(m,n-1)} x^{k}.
\] (3.22)
Now, we introduce three matrices \( A^{(m,n)} \), \( B^{(m-1,n)} \) and \( C^{(m,n-1)} \in \mathbb{R}^{(m+2n)\times(m+2n)} \), such that

\[
A^{(m,n)}_{i,j} = \begin{cases} 
(j-i) \theta, & j \geq i = 1, \ldots, m + 2n \\
0, & \text{otherwise},
\end{cases} \tag{3.23}
\]

\[
B^{(m-1,n)} = \begin{pmatrix} 
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & -mI_{m+2n-1}
\end{pmatrix}
\]

and

\[
C^{(m,n-1)} = \begin{pmatrix} 
0 & \cdots & 0 \\
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & -nI_{m+2n-2}
\end{pmatrix} \tag{3.24}
\]

where \( I_k \in \mathbb{R}^{k\times k} \) is the identity matrix. Finally, we denote by \( \mathbf{a}_{m,n} \in \mathbb{R}^{m+2n} \) the vector of the coefficients of \( V_{m,n}(x) \), i.e. \( \mathbf{a}_{m,n} = \left( a^{(m,n)}_1, \ldots, a^{(m,n)}_{m+2n} \right)^T \).

By using (3.19), and proceeding as in [2], one gets the following matrix equation:

\[
A^{(m,n)} \mathbf{a}_{m,n} = B^{(m-1,n)} \begin{pmatrix} 0 \\ \vdots \\ \mathbf{a}_{m-1,n} \end{pmatrix} + C^{(m,n-1)} \begin{pmatrix} 0 \\ \vdots \\ \mathbf{a}_{m,n-1} \end{pmatrix} \tag{3.25}
\]

Being \( A^{(m,n)} \) invertible, it results

\[
\mathbf{a}_{m,n} = \left(A^{(m,n)}\right)^{-1} B^{(m-1,n)} \begin{pmatrix} 0 \\ \vdots \\ \mathbf{a}_{m-1,n} \end{pmatrix} + \left(A^{(m,n)}\right)^{-1} C^{(m,n-1)} \begin{pmatrix} 0 \\ \vdots \\ \mathbf{a}_{m,n-1} \end{pmatrix} \tag{3.26}
\]

Equation (3.26) provides a recursive formula to find the coefficient \( a^{(m,n)}_k \), for \( x \in \mathbb{N} \). Thanks to the fact that the involved matrices are triangular, for \( m \) and \( n \) not too large, this formula represents a faster way to obtain the coefficients of the polynomial \( V_{m,n}(x) \) than solving directly the equation (3.19).

### 3.4 Expected value of the time average till the FPT

We will find a closed form for

\[
E\left( \frac{A(x)}{\tau(x)} \right) = E\left( \frac{1}{\tau(x)} \int_0^{\tau(x)} X(t) dt \right),
\]

that is the expected value of the time average of Poisson process \( X(t) = x - N_t \) till its FPT below zero.

If \( M_\lambda(x) = M_{\lambda_1,\lambda_2}(x) \) is the joint Laplace transform of \( (\tau(x), A(x)) \), defined by (3.7), we note that

\[
E \left( A(x)e^{-\lambda_1 \tau(x)} \right) = -\lim_{\lambda_2 \to 0^+} \frac{\partial}{\partial \lambda_2} M_{\lambda_1,\lambda_2}(x),
\]

and

\[
E \left( \frac{A(x)}{\tau(x)} \right) = \int_0^{+\infty} E \left( A(x)e^{-\lambda_1 \tau(x)} \right) d\lambda_1. \tag{3.27}
\]
Thus, the calculation of $E(A(x)/\tau(x))$ is reduced to integrate $-\lim_{\lambda_2 \to 0^+} \frac{\partial}{\partial \lambda_2} M_{\lambda_1, \lambda_2}(x)$ with respect to $\lambda_1$.

By taking the partial derivative with respect to $\lambda_2$ in (3.17) and (3.18), and evaluating it at $\lambda_2 = 0$, we easily obtain:

$$E(A(x)e^{-\lambda_1 \tau(x)}) = \begin{cases} \frac{x(x+1)}{2} \frac{\theta^x}{(\lambda_1 + \theta)^{x+1}} & \text{if } x \in \mathbb{N}, \\ \left(\frac{\theta^x}{(\lambda_1 + \theta)^{x+1}}\right) \frac{\theta^x}{(\lambda_1 + \theta)^{x+1}} & \text{if } x \notin \mathbb{N}. \end{cases} \quad (3.28)$$

In the calculations, we have used the formula $(\prod_{i=1}^n f_i)' = (\prod_{i=1}^n f_i)(\sum_{i=1}^n f_i'/f_i)$, giving the derivative of the product of $n$ functions.

Finally, by calculating the integral, we get:

$$E\left(\frac{A(x)}{\tau(x)}\right) = \int_0^\infty E(A(x)e^{-\lambda_1 \tau(x)})d\lambda_1 = \begin{cases} \frac{x+1}{2} & \text{if } x \in \mathbb{N}, \\ x - \frac{\lfloor x \rfloor}{2} & \text{if } x \notin \mathbb{N}. \end{cases} \quad (3.29)$$

Notice that, for any $x > 0$, $E[A(x)/\tau(x)]$ turns out to be $\geq x/2$.

### 4 The case of Poisson Process with drift

For $x > 0$, let us consider the process

$$X(t) = x - \mu t - N_t \quad (4.1)$$

where $\mu > 0$. As before, we denote by $\tau(x)$ the FPT below zero and by $A(x) = \int_0^{\tau(x)} X(t)dt$ the FPA. The infinitesimal generator is

$$Lf(x) = -\mu \frac{\partial f}{\partial x} + \theta[f(x-1) - f(x)], \quad f \in C_b^1(\mathbb{R}). \quad (4.2)$$

**Remark 4.1** Set $Y(t) = x - N_t$; the FPT $\tau_Y(x) = \inf\{t > 0 : Y_t \leq 0\}$ and the FPA $A_Y(x) = \int_0^{\tau_Y(x)} Y(t)dt$ have already studied in the previous section. Since $X(t) \leq Y(t)$, one gets $\tau(x) \leq \tau_Y(x)$, and also $A(x) = \int_0^{\tau(x)} X(t)dt \leq \int_0^{\tau_Y(x)} Y(t)dt = A_Y(x)$.

Let $U$ be a functional of the process $X(t)$; as before, for $\lambda > 0$ we denote by $M_\lambda(x) = \mathbb{E}\left[e^{-\lambda \int_0^{\tau(x)} U(X_s)ds}\right]$ the Laplace transform of the integral $\int_0^{\tau(x)} U(X_s)ds$. Then, from (2.5) of Theorem 2.1, with the generator $L$ given by (4.2), it follows that $M_\lambda(x)$ satisfies the problem with outer conditions:

$$\begin{cases} -\mu M_\lambda'(x) + \theta[M_\lambda(x-1) - M_\lambda(x)] = \lambda U(x)M_\lambda(x), \quad x > 0; \\ M_\lambda(y) = 1, \quad y \leq 0; \quad \lim_{x \to +\infty} M_\lambda(x) = 0. \end{cases} \quad (4.3)$$

#### 4.1 Laplace transform of $\tau(x)$

Taking $U(x) = 1$ in (4.3), we get that the Laplace transform of $\tau(x)$ satisfies:

$$\begin{cases} -\mu M_\lambda'(x) + \theta[M_\lambda(x-1) - M_\lambda(x)] = \lambda M_\lambda(x), \quad x > 0; \\ M_\lambda(y) = 1, \quad y \leq 0; \quad \lim_{x \to +\infty} M_\lambda(x) = 0. \end{cases} \quad (4.4)$$
1) if \( x \in (0, 1] \), since \( M_\lambda(x - 1) = 1 \), the first equation of (4.4) becomes \( \mu M'_\lambda(x) + (\theta + \lambda) M_\lambda(x) = \theta \). By solving and taking into account the conditions of (4.4), we find that the Laplace transform of \( \tau(x) \) is \( M_\lambda(x) = \frac{\lambda}{\theta + \lambda} e^{-\frac{\theta x}{\mu}} + \frac{\theta}{\theta + \lambda} e^{-\frac{\theta x}{\mu}} \). Therefore, \( E[\tau(x)] = -\frac{\partial}{\partial \lambda} M_\lambda(x) |_{\lambda=0} = \frac{1}{\theta} - \frac{1}{\theta} e^{-\frac{\theta x}{\mu}}. \)

2) if \( x \in (1, 2] \), one has \( M_\lambda(x - 1) = \frac{\lambda}{\theta + \lambda} e^{-\frac{\theta x}{\mu}} + \frac{\theta}{\theta + \lambda} \). Then, the first equation of (4.4) becomes \( \mu M'_\lambda(x) + (\theta + \lambda) M_\lambda(x) = \frac{\lambda}{\theta + \lambda} e^{-\frac{\theta x}{\mu}} + \frac{\theta}{\theta + \lambda} \). By solving and taking into account the conditions, we find that the Laplace transform of \( \tau(x) \) is

\[
M_\lambda(x) = \frac{\lambda(\lambda + 2\theta)}{(\theta + \lambda)^2} e^{-\frac{\theta x}{\mu}} + \frac{\theta}{\mu(\theta + \lambda)} e^{-\frac{\theta x}{\mu}} (x-1) e^{-\frac{\theta x}{\mu}} + \frac{\theta^2}{(\theta + \lambda)^2}.
\]

Therefore:

\[
E[\tau(x)] = -\frac{\partial}{\partial \lambda} M_\lambda(x) |_{\lambda=0} = \frac{2}{\theta} - \frac{2}{\theta} e^{-\frac{\theta x}{\mu}} + \frac{x}{\mu} e^{-\frac{\theta x}{\mu}} (x-1).
\]

3) if \( x \in (2, 3] \), one has \( x - 1 \in (1, 2] \) and so \( M_\lambda(x - 1) = \frac{\lambda(\lambda + 2\theta)}{(\theta + \lambda)^2} e^{-\frac{\theta x}{\mu}} + \frac{\theta}{\mu(\theta + \lambda)} (x-1) e^{-\frac{\theta x}{\mu}} + \frac{\theta^3}{(\theta + \lambda)^3}. \)

By solving and taking into account the conditions of (4.4), we find that the Laplace transform of \( \tau(x) \) is

\[
M_\lambda(x) = \left[ 1 - \frac{\theta^3}{(\theta + \lambda)^3} \right] e^{-\frac{\theta x}{\mu}} + \frac{\theta}{\mu(\theta + \lambda)} e^{-\frac{\theta x}{\mu}} + \frac{\theta^2}{2\mu^2(\theta + \lambda)} e^{-\frac{\theta x}{\mu}} (x-2) + \frac{\theta^3}{(\theta + \lambda)^3}.
\]

In conclusion, for \( x \in [0, 3] \):

\[
M_\lambda(x) = \begin{cases} 
\frac{\lambda}{\theta + \lambda} e^{-\frac{\theta x}{\mu}} + \frac{\theta}{\theta + \lambda}, & x \in [0, 1) \\
\frac{\lambda(\lambda + 2\theta)}{(\theta + \lambda)^2} e^{-\frac{\theta x}{\mu}} + \frac{\theta}{\mu(\theta + \lambda)} e^{-\frac{\theta x}{\mu}} (x-1) + \frac{\theta^2}{(\theta + \lambda)^2}, & x \in [1, 2) \\
\left[ 1 - \frac{\theta^3}{(\theta + \lambda)^3} \right] e^{-\frac{\theta x}{\mu}} + \frac{\theta}{\mu(\theta + \lambda)} e^{-\frac{\theta x}{\mu}} (x-1) + \frac{\theta^2}{2\mu^2(\theta + \lambda)} (x-2), & x \in [2, 3]
\end{cases}
\]

Notice that, letting \( \mu \) go to 0, (4.6) becomes (3.2), which holds in the case of Poisson process.

In the Figure 7 we report the shapes of the Laplace transform of \( \tau(x) \) for \( \mu = 1, \theta = 1 \) and various values of the starting point \( x \in [0, 3] \).

Owing to the complexity of the form, unlike the case of Poisson process, we are not able to find, neither a closed form for the Laplace transform of \( \tau(x) \) for any \( x > 0 \), nor a closed form for \( E[\tau(x)] \). However, the upper and lower bounds hold:

\[
\frac{x}{\mu + \theta} \leq E[\tau(x)] < \frac{x + 1}{\mu + \theta}
\]
As before, we look for a solution, taking \( x > x \in U \).

Remark 4.2 \( \tau(x) \leq \tau_Y(x) \), where \( \tau_Y(x) = \inf\{t > 0 : Y(t) \leq 0\} \), \( Y(t) = x - N_t \).

4.2 Laplace transform of \( A(x) \)

Taking \( U(x) = x \) in (4.3), we get that the Laplace transform of \( A(x) \) satisfies:

\[
\begin{align*}
-\mu M'_\lambda(x) + \theta [M_\lambda(x) - M_\lambda(x-1)] &= \lambda x M_\lambda(x), \quad x > 0; \\
M_\lambda(y) &= 1, \quad y \leq 0; \quad \lim_{x \to +\infty} M_\lambda(x) = 0.
\end{align*}
\]

As before, we look for a solution, taking \( x > 0 \) in successive intervals. We start with \( x \in (0, 1] \): it results \( x - 1 \leq 0 \), and so \( M_\lambda(x-1) = 1 \). Then, the first equation of (4.8) becomes \( \mu M'_\lambda(x) + (\theta + \lambda x) M_\lambda(x) = \theta \), whose solution is:

\[
M_\lambda(x) = \exp \left\{ -\left( \frac{\theta}{\mu} x + \frac{\lambda x^2}{2\mu} \right) \right\} \left[ c + \frac{\theta}{\mu} \sqrt{\lambda} e^{-\frac{\theta^2}{2\lambda \mu}} \int_0^x e^{\frac{t^2}{2\mu}} \, dt \right],
\]

where \( c \) is a constant. Since the last integral cannot be written in terms of elementary functions, we cannot iterate this procedure, to find the solution of (4.8) for every \( x > 0 \). However, it is possible to integrate numerically (4.8) by the Euler method; for \( x = 1 \), the shape of \( M_\lambda(x) \), so obtained, as a function of \( \lambda > 0 \), is reported in Figure 8.

We have found \( E[A(x)] \) as the value, changed in sign, of the slope of the tangent line at zero of the Laplace transform of \( A(x) \). Then, we have estimated \( E[A(x)] \) also by Monte Carlo simulation. In Figure 8, the slope of the tangent line at zero of the Laplace transform is \( m = -0.404 \) and so, for \( x = 1 \), \( \theta = 1 \) and \( \mu = 1 \), we obtain the estimate \( E[A(x)] = 0.404 \), while Monte Carlo simulation provides the value 0.399.

Figure 7: Laplace Transform of \( \tau(x) \), as a function of \( \lambda > 0 \), for \( \mu = 1, \theta = 1 \); from top to bottom: \( x = 0.5 \) (blue), \( x = 1.8 \) (red) and \( x = 2.5 \) (cyan).
CHAPTER 4. THE CASE OF POISSON PROCESS WITH DRIFT

Figure 4.3: Laplace transform of $A(x)$ obtained by the Euler method with $x = 1$, $\theta = 1$ and $\mu = 1$ and a discretization step $h = 0.1$.

Remark 4.0.1 Indeed, the drift carries the process $\mu_t$ toward zero faster. We have tested this numerically, by Monte Carlo method and we have obtained for $\lambda = 10$, $\mu = 2$, $\theta = 1$, the estimate:

$A(\lambda) = 17,2015$ $A(\lambda) = 54,9731$

as expected.

Figure 8: Laplace transform of $A(x)$, as a function of $\lambda > 0$, obtained numerically by the Euler method with $x = 1$, $\theta = 1$ and $\mu = 1$ and a discretization step $h = 0.1$.

Remark 4.3 Of course, the FPA of Poisson process with drift is less than the one regarding Poisson process.

Notice that, neither the joint moments of $\tau(x)$ and $A(x)$, nor their joint Laplace transform, can be obtained in closed form, because the corresponding PDEs cannot be solved explicitly.

5 The case of Drifted Brownian motion with Poisson jumps

For $x > 0$, $\mu > 0$, and $\sigma > 0$, let be $X(t) = x - \mu t + \sigma B_t - N_t$. The infinitesimal generator is

$$Lf(x) = -\mu \frac{\partial f}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial x^2} + \theta[f(x-1) - f(x)], \quad f \in C^2_b(\mathbb{R})$$

(5.1)

As before, let $U$ be a functional of the process $X$ and let $M_\lambda(x) = E\left[e^{-\lambda \int_0^{\tau(x)} U(X_s) \, ds}\right]$ the Laplace transform of the integral $\int_0^{\tau(x)} U(X_s) \, ds$. Then, from (2.5) of Theorem 2.1, with generator $L$ given by (5.1), one gets that $M_\lambda(x)$ satisfies the problem with outer conditions:

$$\begin{cases} \frac{\sigma^2}{2} M''_\lambda(x) - \mu M'_\lambda(x) + \theta[M_\lambda(x-1) - M_\lambda(x)] = \lambda U(x)M_\lambda(x), \quad x > 0; \\ M_\lambda(y) = 1, \quad y \leq 0; \quad \lim_{x \to +\infty} M_\lambda(x) = 0. \end{cases}$$

(5.2)

5.1 Laplace transform of $\tau(x)$

Taking $U(x) = 1$ in (5.2), we get that the Laplace transform of $\tau(x)$ is the solution of the PDDE:

$$\begin{cases} \frac{\sigma^2}{2} M''_\lambda(x) - \mu M'_\lambda(x) + \theta[M_\lambda(x-1) - M_\lambda(x)] = \lambda M_\lambda(x), \quad x > 0; \\ M_\lambda(y) = 1, \quad y \leq 0; \quad \lim_{x \to +\infty} M_\lambda(x) = 0; \end{cases}$$

(5.3)

1) if $x \in (0, 1]$, then $M_\lambda(x-1) = 1$; the first equation of (5.3) becomes $\frac{\sigma^2}{2} M''_\lambda(x) - \mu M'_\lambda(x) + \theta[1 - M_\lambda(x)] = \lambda M_\lambda(x)$. By solving and taking into account the conditions of (5.3), we find
that the Laplace transform of $\tau(x)$ is

$$M_\lambda(x) = \frac{\lambda}{\lambda + \theta} e^{(\mu/\sigma^2-(1/\sigma^2)\sqrt{\mu^2+2(\lambda+\theta)})x} + \frac{\theta}{\lambda + \theta}. \quad (5.4)$$

2) if $x \in (1, 2]$, one has $M_\lambda(x-1) = \frac{\lambda}{\lambda + \theta} e^{(\mu/\sigma^2-(1/\sigma^2)\sqrt{\mu^2+2(\lambda+\theta)})x} + \frac{\theta}{\lambda + \theta}$. Then, the first equation of (5.3) becomes:

$$\frac{\sigma^2}{2} M''_\lambda(x) - \mu M'_\lambda(x) - (\lambda + \theta) M_\lambda(x) = -\frac{\lambda \theta}{\sigma^2(\lambda + \theta)} e^{(\mu/\sigma^2-(1/\sigma^2)\sqrt{\mu^2+2\sigma^2(\lambda+\theta)})x} \frac{\theta^2}{\sigma^2(\lambda + \theta)}. \quad (5.5)$$

By solving and taking into account the conditions, we find that the Laplace transform of $\tau(x)$ is

$$M_\lambda(x) = \frac{\lambda(\lambda + 2\theta)}{\sigma^2(\lambda + \theta)} e^{(\mu/\sigma^2-(1/\sigma^2)\sqrt{\mu^2+2\sigma^2(\lambda+\theta)})x} + \frac{\theta}{\lambda + \theta}. \quad (5.6)$$

3) if $x \in (2, 3]$, one has $M_\lambda(x-1) = \frac{\lambda(\lambda+2\theta)}{\sigma^2(\lambda+\theta)} e^{(\mu/\sigma^2-(1/\sigma^2)\sqrt{\mu^2+2\sigma^2(\lambda+\theta)})x} + \frac{\theta}{\lambda + \theta}$. Then, the first equation of (5.3) becomes:

$$\frac{\sigma^2}{2} M''_\lambda(x) - \mu M'_\lambda(x) - (\lambda + \theta) M_\lambda(x) = -\frac{\lambda \theta^2}{\sigma^4(\lambda + \theta)\sqrt{\mu^2+2\sigma^2(\lambda+\theta)}} e^{(\mu/\sigma^2-(1/\sigma^2)\sqrt{\mu^2+2\sigma^2(\lambda+\theta)})x} \frac{\theta^3}{\sigma^2(\lambda + \theta)^2}. \quad (5.6)$$

By solving and taking into account the conditions, we find that the Laplace transform of $\tau(x)$ is

$$M_\lambda(x) = (1 - \theta/(\lambda + \theta))^3 \frac{\lambda \theta^2}{\sigma^4(\lambda + \theta)^2\sqrt{\mu^2+2\sigma^2(\lambda+\theta)}} e^{(\mu/\sigma^2-(1/\sigma^2)\sqrt{\mu^2+2\sigma^2(\lambda+\theta)})x} \frac{\theta^2}{\lambda + \theta}$$

$$+ \frac{\theta^2 \lambda}{2(\lambda+2)(\mu+2\sigma^2(\lambda+\theta))} \left[ x^2 + 2x(\sigma^2 \sqrt{\mu^2+2(\lambda+\theta)}) - 1 \right] e^{(\mu/\sigma^2-(1/\sigma^2)\sqrt{\mu^2+2\sigma^2(\lambda+\theta)})x} \frac{\theta^3}{\lambda + \theta} \cdot \quad (5.6)$$

In line of principle, it is possible to iterate this procedure to obtain the Laplace transform of $\tau(x)$, also for $x > 3$, but the calculations are very complicated. Therefore, for $x > 3$ it is more convenient to estimate numerically the Laplace transform of $\tau(x)$, and $E[\tau(x)]$, by Monte Carlo simulations.

In the Figure 9, we show the shapes of the Laplace transform of $\tau(x)$ for $\mu = 1$, $\sigma = 1$, $\theta = 1$ and various values of the starting point $x \in [0, 3]$, obtained by using (5.4), (5.5) and (5.6). By calculating the derivative of $M_\lambda(x)$ with respect to $\lambda$, at $\lambda = 0$, one obtains $-E(\tau(x))$; for instance, for $x \in (0, 1]$ one gets:

$$E(\tau(x)) = \frac{1}{\theta} \left[ 1 - e^{\frac{\sigma^2}{\theta}(\mu-\sqrt{\mu^2+2\theta})x} \right]. \quad (5.7)$$
5.2 Laplace transform of $A(x)$

Taking $U(x) = x$ in (5.2), we get that the Laplace transform of $A(x)$ is the solution of the PDDE:

$$\frac{1}{2}M'(x) - \mu M(x) + \theta[M(x-1) - M(x)] = \lambda x M(x), \quad x > 0;$$

$$M(0) = 1, \quad y \leq 0; \lim_{x \to +\infty} M(x) = 0.$$  \hfill (5.8)

Unfortunately, even for $x \in (0, 1]$, the first equation of (5.8) is a second order ODE with non-constant coefficients, so its explicit solution cannot be found; neither the solution can be calculated numerically by the Euler method, since the initial condition $M(0)$ is unknown. As far as the expectation of $A(x)$ is concerned, one has to estimate it by Monte Carlo simulation by using

$$A(x) = \int_0^{\tau(x)} X(t) \, dt = x \tau(x) - \mu \frac{\tau(x)^2}{2} + \int_0^{\tau(x)} B_t \, dt - \int_0^{\tau(x)} N_t \, dt.$$  

For instance, for $x = 10, \theta = 1, \mu = 2,$ and $\sigma = 1,$ Monte Carlo simulation provides the estimate $\hat{E}[A(x)] = 17.782.$ Notice, however, this estimate is not the more accurate possible, but it has only an illustrative purpose; in fact, we have not considered any procedure to avoid the overestimate of the FPT time $\tau(x)$ (see e.g. [8]).

6 Conclusions and final remarks

In this paper, we have carried on the study, already undertook in [1] for general jump-diffusions, and in [2] for drifted Brownian motion, of the first-passage area (FPA) $A(x)$, swept out by a one-dimensional jump-diffusion process $X(t)$, starting from $x > 0,$ till its first-passage time (FPT) $\tau(x)$ below zero. Here, we have investigated the joint distribution of $\tau(x)$ and $A(x)$, in the special case when $X(t)$ is a Lévy process of the form $X(t) = x - \mu t + \sigma B_t - N_t$, where $\mu \geq 0$, $\sigma \geq 0$, $B_t$ is standard Brownian motion, and $N_t$ is a homogeneous Poisson process with intensity $\theta$, starting at zero. We have established partial differential-difference
equations (PDDE’s) with outer conditions for the Laplace transform of the random vector 
\((\tau(x), A(x))\), and for the joint moments \(E[\tau(x)^m A(x)^n]\) of the FPT and FPA. In the special 
case of Poisson process \((\mu = \sigma = 0)\), that is \(X(t) = x - N_t\), we have presented an algorithm to 
find recursively \(E[\tau(x)^m A(x)^n]\), for any \(m\) and \(n\); moreover, we have calculated the expected 
value of the time average of \(X(t)\) till its FPT below zero. In the other cases, we have not been 
always able to carry on explicit calculations, whenever the corresponding PDDEs equations 
cannot be solved in closed form.

We observe that, for a given barrier \(S\), one can extend the results of this paper to the FPT 
and FPA through \(S\) of a one-dimensional Lévy process \(X(t) = x + \mu t + \sigma B_t + N_t\) \((\mu, \sigma \geq 0)\), 
starting from \(x < S\); in this article, we have considered the case of crossing zero, for the sake 
of simplicity. Really, for \(x < S\), one can consider the FPT \(\tau_S(x) = \inf\{t > 0 : X(t) \geq S\}\) and 
the FPA \(A_S(x) = \int_0^{\tau_S(x)} X(t)\,dt\) determined by \(X(t)\) till its FPT \(\tau_S(x)\) through \(S\); note that 
\(A_S(x)\) is improperly called the FPA of \(X(t)\) through \(S\), since the area of the plane region 
determined by the trajectory of \(X(t)\) and the time axis in the first-passage period \([0, \tau_S(x)]\) is 
\(\int_0^{\tau_S(x)} |X(t)|\,dt\), which coincides with \(A_S(x)\) only if \(X(t)\) is non-negative in the entire interval 
\([0, \tau_S(x)]\). However, also in this case one obtains that the Laplace transforms and moments 
of \(\tau_S(x)\) and \(A_S(x)\) are solutions to certain PDDE’s with outer conditions, which are similar 
to those analyzed in the present paper. In fact, for \(X(t) = x + \mu t + \sigma B_t + N_t\), with \(x < S\), 
one has

\[
\tau_S(x) = \inf\{t > 0 : x + \mu t + \sigma B_t + N_t \geq S\} = \inf\{t > 0 : -x - \mu t - \sigma B_t - N_t \leq -S\};
\]

Since \(-B_t \sim B_t\), then \(\tau_S(x)\) has the same distribution as 

\[
\inf\{t > 0 : S - x - \mu t + \sigma B_t - N_t \leq 0\} \equiv \tilde{\tau}(S - x),
\]

where \(\tilde{\tau}(z)\) is the FPT of Lévy process \(Z(t) = z - \mu t + \sigma B_t - N_t\) below zero, which was 
studied in this paper. Moreover:

\[
A_S(x) = \int_0^{\tau_S(x)} X(t)\,dt = S\tau(x) - \tilde{A}(S - x) = S\tilde{\tau}(S - x) - \tilde{A}(S - x),
\]

where

\[
\tilde{A}(z) = \int_0^{\tilde{\tau}(z)} (z - \mu t + \sigma B_t - N_t)\,dt
\]

is the FPA below zero of drifted BM with Poisson jumps, already studied. This implies 
that the Laplace transforms and moments of \(\tau_S(x)\) and \(A_S(x)\) are again solutions to suitable 
PDDE’s with outer conditions (see e.g. \([3]\)).

In line of principle, all the results contained in this paper can be extended to a jump-
diffusion process \(X(t)\) in which the drifted BM is replaced by a one-dimensional, time homo-
genous diffusion, and also to a more general Lévy process, in which the jump sizes are not 
constant, but they are independent random variables with assigned distribution, implying 
that \(N_t\) becomes a compound Poisson process; of course, explicit calculations become more 
complicated.

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