Space with spinor structure and analytical properties of the solutions of Klein-Fock and Schrödinger equations in cylindrical parabolic coordinates

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Possible quantum mechanical corollaries of changing the vectorial geometrical model of the physical space, extending it twice, in order to describe its spinor structure (in other terminology and emphasis it is known as the Hopf’s bundle) are investigated. The extending procedure is realized in cylindrical parabolic coordinates: \( G(t, u, v, z) \Rightarrow G(t, u, v, z) \). It is done through expansion twice as much of the domain \( G \) so that instead of the half plane \( (u, v > 0) \) now the entire plane \((u, v)\) should be used accompanied with new identification rules over the boundary points. In the Cartesian picture this procedure there corresponds to taking the two-sheet surface \((x', y') \oplus (x'', y'')\) in place of the one-sheet surface \((x, y)\). Solutions of the Klein-Fock and Schrödinger equations \( \Psi_{e, p, a} = e^{itx}e^{iux}U_\xi(u)U_\nu(v) \) are constructed in terms of parabolic cylinder functions, \( a \) is a separating constant. Given quantum numbers \( e, p, a \) four types of solutions are possible: \( \Psi_{++}, \Psi_{--}; \Psi_{+-}, \Psi_{-+} \). The first two \( \Psi_{++, \Psi_{--} \text{ provide us with single-valued functions of the vectorial space points, whereas last two } \Psi_{+-}, \Psi_{-+} \text{ have discontinuities in the frame of vectorial space and therefore they must be rejected in this model. All four types of functions are continuous ones being regarded in the spinor space. Explicit form of a 2-order differential operator diagonalized on the constructed wave functions with the eigenvalue \( a \). \( \hat{A} \Psi_{e, p, a} = +a \Psi_{e, p, a} \) is found both in \((t, x, y, z)\) and \((u, v, z)\) representation. It is shown that solutions \( \Psi_{++, \Psi_{--}, \Psi_{+, \Psi_{--}}, \Psi_{+-}, \Psi_{-+} \) are the eigen-functions of two discrete spinor operators \( \delta \) and \( \hat{\pi} \). Both \( \hat{\pi}(u, v) = (u, -v) \). \( \hat{\pi}(u, v) = (u, -v) \). \( \delta \) \((x, y) = (x, y) \). \( \hat{\pi}(x, y) = (x, -y) \). Two other classifications of the wave functions over discrete quantum numbers are given. It is established that all solutions \( \Psi_{++, \Psi_{--}, \Psi_{+, \Psi_{--}}, \Psi_{+-}, \Psi_{-+} \) are orthogonal to each other provided that integration is done over extended domain parameterizing the spinor space. Simple selection rules for matric elements of the vector and spinor coordinates, \((x, y)\) and \((u, v)\), respectively, are derived. Selection rules for \((u, v)\) are substantially different in vector and spinor spaces. In the supplement some relationships describing primary geometric objects, spatial spinor \( \xi \) and \( \eta \), as functions of cylindrical parabolic coordinates, are given.

Key words: spinors, geometry, wave functions

I. INTRODUCTION

In the literature, there exist [1-31] three terminological different approaches though close in their intrinsic essence. There are a space-time spinor structure (see the book by by Penrose and Rindler [29] as a modern embodiment of the old idea [1-4] to use spinor groups instead of orthogonal); the Hooph bundle [5]; Kustaanheimo-Stiefel bundle [6,7]. Differences between three mentioned formalisms consist mainly in conceptual accents (see for more detail [32]). In the Hopf’s technique it is suggested to use in all parts only complex spinors \( \xi \) and conjugated \( \bar{\xi} \) instead of real-valued vector (tensor) quantities. In the Kustaanheimo-Stiefel approach we are to use four real-valued coordinates, form which Cartesian coordinates \((x, y, z)\) can be formed up by means of definite bilateral functions. These four variables by Kustaanheimo-Stiefel are real and imaginary parts of two spinor components. The known spinor invariant \((\xi^2 + \bar{\xi}^2)\) becomes the sum of four squared real quantities, so that we can associate spinor technique with geometry of the Riemann space \( S_4 \) of constant positive curvature.

In essence, the Kustaanheimo-Stiefel’s approach is other elaboration of the same Hopf’s technique based on complex spinors \( \xi \) and \( \bar{\xi} \), in terms of four real-valued variables. In so doing, we are able to hide in the formalism the presence of the non-analytical operation of complex conjugation. Spinor space structure, formalism developed in the present work, also exploits possibilities given by spinors to construct 3-vectors, however the emphasis is taken to doubling the set of spatial points so that we get an extended space model that is called a space with spinor structure [32-38]. In such an extended space, in place of \( 2\pi \)-rotation, only \( 4\pi \)-rotation transfers the space into itself.

The procedure itself of doubling the manifold can be realized easier when for parameterizing the space some curvilinear coordinate system is used instead of the Cartesian coordinates. In such context, spherical and parabolic coordinates were considered in [37]. In the present paper, the use of cylindrical parabolic coordinates is studied as applied for description of spinor space structure. Now we study analytical properties of Schrödinger and Klein-Fock the wave solutions depending on vector and spinor space models. It is demonstrated explicitly that transition to an extended space model (with spinor structure) lead us to augmenting the number of basis wave functions of a
quantum-mechanical scalar particle. Also, some possible manifestations of the extended space structure in matrix elements of physical quantities are discussed.

II. PARABOLIC CYLINDRICAL COORDINATES

These coordinates in the vector 3-space model are introduced by relations

\[ x = \frac{u^2 - v^2}{2}, \quad y = u v, \quad z = z. \]  

\[ v^2 = -x + \sqrt{x^2 + y^2}, \quad u^2 = +x + \sqrt{x^2 + y^2}. \]  

To cover all points of the vector space \((x, y, z)\) it suffices any one from the following four solutions:

\[ v = +\sqrt{-x + \sqrt{x^2 + y^2}}, \quad u = \pm\sqrt{+x + \sqrt{x^2 + y^2}}, \]  

\[ v = -\sqrt{-x + \sqrt{x^2 + y^2}}, \quad u = \pm\sqrt{+x + \sqrt{x^2 + y^2}}, \]  

\[ v = \pm\sqrt{-x + \sqrt{x^2 + y^2}}, \quad u = \pm\sqrt{+x + \sqrt{x^2 + y^2}}, \]  

\[ v = \pm\sqrt{-x + \sqrt{x^2 + y^2}}, \quad u = -\sqrt{+x + \sqrt{x^2 + y^2}}. \]

For definiteness, let us use the first variant from (3):

\[ v = +\sqrt{-x + \sqrt{x^2 + y^2}}, \quad u = \pm\sqrt{+x + \sqrt{x^2 + y^2}}. \]

Fig 1. The domain \(G(u, v)\) to parameterize the vector model

Correspondence between the points \((x, y)\) and \((u, v)\) can be illustrated by the formulas and schemes:

\[ u = k \cos \phi, \quad v = k \sin \phi, \quad \phi \in [0, \pi]; \]

\[ x = (k^2/2) \cos 2\phi, \quad y = (k^2/2) \sin 2\phi, \quad 2\phi \in [0, 2\pi] \]

Fig 2. The mapping \(G(x, y) \implies G(u, v)\)
In the following, when turning to the case of spinor space, we will see the complete symmetry between coordinates \( u \) \( v \): namely, they are referred to Cartesian coordinates of the extended model \((x, y, z) \oplus (x, y, z)\) through the formulas (compare with the previous)

\[
v = \pm \sqrt{-x + \sqrt{x^2 + y^2}}, \quad u = \pm \sqrt{x + \sqrt{x^2 + y^2}}. \tag{6}
\]

the latter can be illustrated by the Fig 3:

\[
\text{Fig 3. } \tilde{G}(u, v) \text{ to cover spinor space}
\]

The metric of 3-space in parabolic cylindrical coordinates is

\[
dl^2 = dx^2 + dy^2 + dz^2 = (u^2 + v^2)(du^2 + dv^2) + dz^2;
\]

correspondingly, the Minkowsky metric looks as

\[
dS^2 = (dx^0)^2 - dl^2 = c^2 dt^2 - (u^2 + v^2)(du^2 + dv^2) - dz^2.
\]

III. SOLUTIONS OF THE KLEIN-FOCK EQUATION AND FUNCTIONS OF PARABOLIC CYLINDER

Let us consider the Klein-Fock equation specified for cylindric paraboholic coordinates:

\[
\left[ -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2} + \frac{1}{u^2 + v^2} \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) - \frac{m^2 c^2}{\hbar^2} \right] \Psi = 0,
\]

After separating the variables \((t, z)\) from \((u, v)\) by the substitution \(\Psi(t, u, v, \phi) = e^{-i\epsilon t/\hbar} e^{ipz/\hbar} U(u) V(v)\) one gets

\[
\left[ \frac{1}{U} \frac{d^2 U}{du^2} + \left( \frac{\epsilon^2}{\hbar^2 c^2} - \frac{m^2 c^2}{\hbar^2} \right) U \right] + \left[ \frac{1}{V} \frac{d^2 V}{dv^2} + \left( \frac{\epsilon^2}{\hbar^2 c^2} - \frac{m^2 c^2}{\hbar^2} \right) V \right] = 0. \tag{8}
\]

In the following, the notation is used

\[
\lambda^2 = \left( \frac{\epsilon^2}{\hbar^2 c^2} - \frac{m^2 c^2}{\hbar^2} - \frac{p^2}{\hbar^2} \right), \quad |\lambda| = \frac{1}{\text{meter}}.
\]

Introducing two separation constants \(a\) and \(b\) \((a + b = 0)\), from (8) we can derive two separate equations in variables \(u\) and \(v\) respectively:

\[
\frac{d^2 U}{du^2} + \left( \lambda^2 u^2 - a \right) U = 0, \quad \frac{d^2 V}{dv^2} + \left( \lambda^2 v^2 - b \right) V = 0. \tag{9}
\]

Canonical form of differential equation of parabolic cylinder (type 2, [39]) is

\[
\frac{d^2 F}{d\xi^2} + \left( \frac{\xi^2}{4} - \alpha \right) F = 0. \tag{10}
\]

Transition in equations (10) to the canonical form is reached through the use of dimensionless variables

\[
\sqrt{2\lambda} u \rightarrow u, \quad \frac{a}{2\lambda} \rightarrow a, \quad \sqrt{2\lambda} v \rightarrow v, \quad \frac{b}{2\lambda} \rightarrow b. \tag{11}
\]
So that equations (12) will take the form:

\[
\frac{d^2U}{du^2} + \left( \frac{u^2}{4} - a \right) U = 0 , \quad \frac{d^2V}{dv^2} + \left( \frac{v^2}{4} - b \right) V = 0 .
\]

(12)

As known, solutions of equation (10) can be found as a series:

\[
F(\xi) = c_0 + c_1 \xi + c_2 \xi^2 + \sum_{k=1,2,...} c_{2k+1} \xi^{2k+1} + \sum_{k=1,2,...} c_{2k+2} \xi^{2k+2} ;
\]

(13)

in (13) the terms of even and odd powers of \( \xi \) are distinguished. After substituting (13) into (10) we get:

\[
\left[ c_2^2 + \sum_{k=1,2,...} c_{2k+1}(2k + 1)(2k) \xi^{2k} - \sum_{k=1,2,...} c_{2k+2}(2k + 2)(2k + 1) \xi^{2k} \right] +
\]

\[
\frac{1}{4} \left[ c_0 \xi^2 + c_1 \xi^3 + c_2 \xi^4 + \sum_{k=1,2,...} c_{2k+1} \xi^{2k+3} + \sum_{k=1,2,...} c_{2k+2} \xi^{2k+4} \right] -
\]

\[-\alpha \left[ c_0 + c_1 \xi + c_2 \xi^2 + \sum_{k=1,2,...} c_{2k+1} \xi^{2k+1} + \sum_{k=1,2,...} c_{2k+2} \xi^{2k+2} \right] = 0 ,
\]

(14)

or separating terms of even and odd powers

\[
\left[ c_2^2 + \sum_{k=1,2,...} c_{2k+2}(2k + 2)(2k + 1) \xi^{2k} + \frac{1}{4} c_0 \xi^2 + \frac{1}{4} c_2 \xi^4 +
\]

\[
+ \frac{1}{4} \sum_{k=1,2,...} c_{2k+2} \xi^{2k+4} - \alpha c_0 - \alpha c_2 \xi^2 - \alpha \sum_{k=1,2,...} c_{2k+2} \xi^{2k+2} \right] \text{even} +
\]

\[
+ \left[ \sum_{k=1,2,...} c_{2k+1}(2k + 1)(2k) \xi^{2k-1} + \frac{1}{4} c_1 \xi^3 + \frac{1}{4} \sum_{k=1,2,...} c_{2k+1} \xi^{2k+3} -
\]

\[- \alpha c_1 \xi - \alpha \sum_{k=1,2,...} c_{2k+1} \xi^{2k+1} \right] \text{odd}
\]

(15)

and further

\[
\left[ \xi^0(2c_2 - \alpha c_0) + \xi^2(c_4 4 \times 3 + \frac{c_0}{4} - \alpha c_2) + \xi^4 (c_6 6 \times 5 + \frac{c_2}{4} - \alpha c_4) +
\]

\[
+ \sum_{k=3,4,...} c_{2k+2}(2k + 2)(2k + 1) \xi^{2k} + \frac{1}{4} \sum_{k=1,2,...} c_{2k+2} \xi^{2k+4} - \alpha \sum_{k=2,3,...} c_{2k+2} \xi^{2k+2} \right] \text{even} +
\]

\[
\left[ \xi(c_3 3 \times 2 - \alpha c_1) + \xi^3 (c_5 5 \times 4 + \frac{c_1}{4} - \alpha c_3) +
\]

\[
+ \sum_{k=3,4,...} c_{2k+1}(2k + 1)(2k) \xi^{2k-1} + \frac{1}{4} \sum_{k=1,2,...} c_{2k+1} \xi^{2k+3} - \alpha \sum_{k=2,3,...} c_{2k+1} \xi^{2k+1} \right] \text{odd}\]

(16)

From this it follows

\[
\left[ \xi^0(2c_2 - \alpha c_0) + \xi^2(c_4 4 \times 3 + \frac{c_0}{4} - \alpha c_2) + \xi^4 (c_6 6 \times 5 + \frac{c_2}{4} - \alpha c_4) +
\]

\[
+ \sum_{n=3,4,...} \left( c_{2n+2}(2n + 2)(2n + 1) + \frac{1}{4} c_{2n-2} - \alpha c_{2n} \right) \xi^{2n} \right] \text{even} +
\]

\[
\left[ \xi(c_3 3 \times 2 - \alpha c_1) + \xi^3 (c_5 5 \times 4 + \frac{c_1}{4} - \alpha c_3) +
\]

\[
+ \sum_{n=3,4,...} \left( c_{2n+1}(2n + 1)(2n) + \frac{1}{4} c_{2n-3} - \alpha c_{2n-1} \right) \xi^{2n-1} \right] \text{odd} .
\]

(17)
Setting each coefficient at a $\xi^k$ equal to zero one derives two independent groups of recurrent relations:

**even**

$$\xi^0 : \quad 2 c_2 - \alpha c_0 = 0 ,$$

$$\xi^2 : \quad c_4 4 \times 3 + \frac{c_2}{4^2} - \alpha c_2 = 0 ,$$

$$\xi^4 : \quad c_6 6 \times 5 + \frac{c_4}{4} - \alpha c_4 = 0 ,$$

$$n = 3 , 4 , \ldots , \quad \xi^{2n} : \quad c_{2n+2}(2n + 2)(2n + 1) + \frac{1}{4} c_{2n-2} - \alpha c_{2n} = 0 ;$$

**odd**

$$\xi^1 : \quad c_3 3 \times 2 - \alpha c_1 = 0 ,$$

$$\xi^3 : \quad c_5 5 \times 4 + \frac{c_3}{4} - \alpha c_3 = 0 ,$$

$$n = 3 , 4 , \ldots , \quad \xi^{2n-1} : \quad c_{2n+1}(2n + 1)(2n) + \frac{1}{4} c_{2n-3} - \alpha c_{2n-1} = 0 .$$

Taking into account the absence of any connection of equations (18) and (19) one can construct two linearly independent solutions (even and odd respectively): even

$$c_0 = 1 , c_1 = 0 , \quad F_1(\xi) = 1 + c_2 \xi^2 + c_4 \xi^4 + \ldots ,$$

$$c_2 = \frac{\alpha}{2} , \quad c_4 = \frac{1}{4 \times 3} (\alpha c_2 - \frac{1}{4}) = \frac{1}{4!} (\alpha^2 - \frac{1}{2}) ,$$

$$c_6 = \frac{1}{6 \times 5} (\alpha c_4 - \frac{c_2}{4}) = \frac{1}{6!} (\alpha^3 - \frac{7}{2}) ,$$

$$n = 3 , 4 , \ldots : \quad c_{2n+2} = \frac{1}{(2n + 2)(2n + 1)} (\alpha c_{2n} - \frac{1}{4} c_{2n-2})$$

odd

$$c_0 = 0 , c_1 = 1 , \quad F_2(\xi) = \xi + c_3 \xi^3 + c_5 \xi^5 + \ldots ,$$

$$c_3 = \frac{1}{3!} \alpha , \quad c_5 = \frac{1}{5 \times 4} (\alpha c_3 - \frac{c_1}{4}) = \frac{1}{5!} (\alpha^2 - \frac{3}{2}) ,$$

$$n = 3 , 4 , \ldots : \quad c_{2n+1} = \frac{1}{(2n + 1)(2n)} (\alpha c_{2n-1} - \frac{1}{4} c_{2n-3}) .$$

or differently 

**even**

$$F_1(\xi^2) = 1 + a_2 \frac{\xi^2}{2!} + a_4 \frac{\xi^4}{4!} + \ldots ,$$

$$a_2 = \alpha , \quad a_4 = \alpha^2 - \frac{1}{2} , \quad a_6 = \alpha^3 - \frac{7}{2} \alpha ,$$

$$n = 3 , 4 , \ldots : \quad a_{2n+2} = \alpha a_{2n} - \frac{(2n)(2n - 1)}{4} a_{2n-2} ;$$

**odd**

$$F_2(\xi) = \xi + a_3 \frac{\xi^3}{3!} + a_5 \frac{\xi^5}{5!} + \ldots ,$$

$$a_3 = \alpha , \quad a_5 = \alpha^2 - \frac{3}{2} \alpha ,$$

$$n = 3 , 4 , \ldots : \quad a_{2n+1} = \alpha a_{2n-1} - \frac{(2n - 1)(2n - 2)}{4} a_{2n-3} .$$

**IV. THE SET OF BASIS WAVE FUNCTIONS FOR KLEIN-FOCK PARTICLE, THE ROLE AND MANIFESTATION OF VECTOR AND SPINOR SPACE STRUCTURES RESPECTIVELY**

Having combined two previous solutions $F_1$ and $F_2$, we can obtain four types of the wave functions, solutions of the Klein-Fock equation in cylindrical parabolic coordinates (we will change the notation: $F_1 \Rightarrow E; F_1 \Rightarrow O$ :
(even \otimes \text{even}) : \quad \Phi_{++}(a, u^2) E(-a, v^2),
(odd \otimes \text{odd}) : \quad \Phi_{--}(a, u) O(-a, v^2),
(even \otimes \text{odd}) : \quad \Phi_{+-}(a, u^2) O(-a, v),
(odd \otimes \text{even}) : \quad \Phi_{-+}(a, u) E(-a, v^2). \tag{25}

Having in mind relation between \((u, v)\) and \((x, y)\), one readily notes behavior of the wave functions constructed at the point \(x = 0, y = 0\) (variable \(z\) is omitted):

\[
\begin{align*}
\text{(even \otimes \text{even})} : \quad & \Psi_{++}(x = 0, y = 0) \neq 0, \\
\text{(odd \otimes \text{odd})} : \quad & \Psi_{--}(x = 0, y = 0) = 0, \\
\text{(even \otimes \text{odd})} : \quad & \Psi_{+-}(x > 0, y = 0) = 0, \\
\text{(odd \otimes \text{even})} : \quad & \Psi_{-+}(x < 0, y = 0) = 0. \tag{26}
\end{align*}
\]

Now let us consider which restrictions for the wave functions \(\Psi\) are imposed by the requirement of single-valuedness. Two peculiarities in parameterizing are substantial:

\[
\begin{align*}
v = 0 : & \quad x = + \frac{u^2}{2} \geq 0, \quad y = 0; \\
u = 0 : & \quad x = - \frac{v^2}{2} \leq 0, \quad y = 0. \tag{27}
\end{align*}
\]

\[\text{Fig 4. The peculiarities in parametrization}\]

The above four solutions \(25\) behave in peculiar regions as follows: (even \otimes \text{even}) :

\[
\begin{align*}
\Phi_{++}(a; u = 0, v) &= E(a, u^2 = 0) E(-a, v^2) = \\
&= E(-a, v^2) = + \Phi_{++}(a; u = 0, -v), \\
\Phi_{++}(a; +u, v = 0) &= E(a, u^2) E(-a, v^2 = 0) = \\
&= E(+a, u^2) = + \Phi_{++}(a; -u, v = 0),
\end{align*}
\]

(odd \otimes \text{odd}) :

\[
\begin{align*}
\Phi_{--}(a; u = 0, +v) &= O(a, u = 0) O(-a, v) = \\
&= + \Phi_{--}(a; u = 0, -v) = 0, \\
\Phi_{--}(a; u, v = 0) &= O(+a, u) O(-a, v = 0) = \\
&= + \Phi_{--}(a; -u, v = 0) = 0,
\end{align*}
\]

(even \otimes \text{odd}) :

\[
\begin{align*}
\Phi_{+-}(a; u = 0, +v) &= E(a, u^2 = 0) O(-a, v) = \\
&= O(-a, v) = - O(-a, v) = - \Phi_{+-}(a; u = 0, -v), \\
\Phi_{+-}(a; u, v = 0) &= E(+a, u^2) O(-a, v = 0) = \\
&= \Phi_{+-}(a; -u, v = 0) = 0,
\end{align*}
\]

(odd \otimes \text{even}) :

\[
\begin{align*}
\Phi_{-+}(a; u = 0, +v) &= O(+a, u = 0) E(-a, v^2) = \\
&= \Phi_{-+}(a; u = 0, -v) = 0, \\
\Phi_{-+}(a; +u, v = 0) &= O(+a, u) E(-a, v^2 = 0) = \\
&= O(+a, u) = - O(a, -u) = - \Phi_{-+}(a; -u, v = 0).
\end{align*}
\tag{28}\]
Taking in mind the Fig. 1 and the Fig. 3, one can immediately conclude: solutions $\Phi$ of the types $(++)$ and $(- -)$ are single-valued in the space with vector structure, whereas the solutions of the types $(+-)$ and $(- +)$ are not single-valued in space with vector structure, so these types $(+-)$ and $(- +)$ must be rejected. However, these solutions $(+-)$ and $(- +)$ must be retained in the space with spinor structure.

That dividing of the basis wave functions into two subsets may be formalized mathematically with the help of special discrete operator acting in spinor space:

$$\tilde{\delta} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \tilde{\delta} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -u \\ -v \end{bmatrix}. \quad (29)$$

It is easily verified that solutions single-valued in the vector space model are eigenfunctions of $\delta$ with eigenvalue $\delta = +1$:

$$\hat{\delta} \Phi_+(a; u, v) = + \Phi_+(a; u, v), \quad \hat{\delta} \Phi_-(a; u, v) = + \Phi_-(a; u, v), \quad (30)$$

and additional ones acceptable only in the spinor space model, are eigenfunction with the eigenvalue $\delta = -1$:

$$\hat{\delta} \Phi_+(a; u, v) = - \Phi_+(a; u, v), \quad \hat{\delta} \Phi_-(a; u, v) = - \Phi_-(a; u, v). \quad (31)$$

When using the spinor space model, two set $(u, v)$ and $(-u, -v)$ represent different geometrical points in the spinor space, so the requirement of single valuedness as applied in the case of spinor space does not presuppose that the values of the wave functions must be equal in the points $(u, v)$ and $(-u, -v)$:

$$\Phi(u, v) = \Phi((x, y)^{(1)}) \neq \Phi(-u, -v) = \Phi((x, y)^{(2)}). \quad (32)$$

Now let us add some details more. In general, the vector plane $(x, y)$ allows three inversion operations to which one can relate six discrete operations in spinor $^\circ$plane $\tilde{(u, v)}$:

$$(x, y) \Longrightarrow (x, -y), \quad \hat{\pi} = \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \hat{\pi}' = \hat{\delta} \hat{\pi} = -\hat{\pi},$$

$$(x, y) \Longrightarrow (-x, y), \quad \hat{\omega} = \begin{bmatrix} 0 & +1 \\ +1 & 0 \end{bmatrix}, \quad \hat{\omega}' = \hat{\delta} \hat{\omega} = -\hat{\omega},$$

$$(x, y) \Longrightarrow (-x, -y), \quad \hat{R} = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}, \quad \hat{R}' = \hat{\delta} \hat{R} = -\hat{R}. \quad (33)$$

One can easily construct eigenfunctions of these discrete operations, as well. For instance, let us consider the operator $\hat{R} = \hat{\omega} \hat{\pi}$. Noting two identities

$$\hat{R} \Phi_+(a; u, v) = \hat{R} E(a, u^2) E(-a, v^2) = E(a, v^2) E(-a, u^2) = \Phi_+(a; u, v), \quad (34)$$

$$\hat{R} \Phi_-(a; u, v) = \hat{R} O(a, u) O(-a, v) = O(a, -v) O(-a, u) = -\Phi_-(a; u, v). \quad (35)$$

one can easily construct the eigen-functions of the operator $\hat{R}$ (arguments are omitted):

$$\Phi^{(R=\pm 1)} = \Phi^{(+)}(a) \pm \Phi^{(+)}(-a), \quad \hat{R} \Phi^{(R=\pm 1)} = \pm \Phi^{(R=\pm 1)}; \quad (36)$$

and

$$\Phi^{(R=\pm 1)} = \Phi^{(-)}(a) \mp \Phi^{(-)}(-a), \quad \hat{R} \Phi^{(R=\pm 1)} = \pm \Phi^{(R=\pm 1)}. \quad (37)$$

In the same way, taking into account the identities

$$\hat{R} \Phi_-(a; u, v) = \hat{R} E(a, u^2) O(-a, v) = E(a, v^2) O(-a, u) = + \Phi_-(a; u, v); \quad (38)$$

and

$$\hat{R} \Phi_+(a; u, v) = \hat{R} O(a, u) E(-a, v^2) = O(a, -v) E(-a, u^2) = - \Phi_+(a; u, v). \quad (39)$$
therefore, the eigenfunctions may be given as:

\[ \varphi^{(R=\mp i)} = \Phi_{++}(a) \pm i \Phi_{+-}(a), \quad \hat{R} \varphi^{(\mp i)} = \pm i \varphi^{(\mp i)}; \]  

and

\[ \varphi^{(R=\mp i)}(-a) = \Phi_{--}(-a) \pm i \Phi_{-+}(+a), \quad \hat{R} \varphi^{(\mp i)}(-a) = \pm i \varphi^{(\mp i)}(-a). \]  

Thus, there exist quite a definite classification of the Klein-Fock solutions in cylindrical parabolic coordinates in terms of quantum numbers, eigenvalues of the following operator (an explicit form \( \hat{A} \) will be given below)

\[ i \frac{\partial}{\partial t} \implies \epsilon, \quad -i \frac{\partial}{\partial z} \implies p, \quad \hat{A} \implies a, \quad (\hat{\delta}, \hat{\tilde{R}}) \implies (\delta = \pm 1, R = \pm 1). \]  

As a base to classify solutions of the Klein-Fock equation, instead of \( (\hat{\delta}, \hat{\tilde{R}}) \) one might have taken other two operator: for instance, \( \hat{\omega} \) and \( \hat{\omega} \). Then, allowing for the identities

\[ \hat{\omega} \Phi_{++}(a; u, v) = \hat{\omega} \quad E(a, u^2) E(-a, v^2) = \]

\[ = E(a, v^2) E(-a, u^2) = \Phi_{++}(-a; u, v), \]

\[ \hat{\omega} \Phi_{--}(a; u, v) = \hat{\omega} \quad O(a, u) O(-a, v) = \]

\[ = O(a, v) O(-a, u) = \Phi_{--}(-a; u, v). \]  

We can construct eigenfunctions of the operator \( \hat{\omega} \):

\[ \Phi^{(\omega=\pm 1)}_{++} = \Phi_{++}(a) \pm \Phi_{++}(-a), \quad \hat{\omega} \quad \Phi^{(\omega=\pm 1)}_{++} = \pm \Phi^{(\omega=\pm 1)}_{++}; \]  

and

\[ \Phi^{(\omega=\pm 1)}_{--} = \Phi_{--}(a) \pm \Phi_{--}(-a), \quad \hat{\omega} \quad \Phi^{(\omega=\pm 1)}_{--} = \pm \Phi^{(\omega=\pm 1)}_{--}. \]  

In the same manner, for additional solutions we have

\[ \hat{\omega} \Phi_{+-}(a; u, v) = \hat{\omega} \quad E(a, u^2) O(-a, v) = \]

\[ = E(a, v^2) O(-a, u) = + \Phi_{+-}(-a; u, v) \]  

and

\[ \hat{\omega} \Phi_{-+}(a; u, v) = \hat{\omega} \quad O(a, u) E(-a, v^2) = \]

\[ = O(a, v) E(-a, u^2) = \Phi_{-+}(-a; u, v), \]  

therefore, the eigenfunctions may be given as

\[ \varphi^{(\omega=\pm 1)} = \Phi_{++}(a) \pm \Phi_{++}(-a), \quad \hat{\omega} \quad \varphi^{(\omega=\pm 1)} = \pm \varphi^{(\omega=\pm 1)}; \]  

\[ \varphi^{(\omega=\pm 1)} = \Phi_{--}(a) \pm \Phi_{--}(-a), \quad \hat{\omega} \quad \varphi^{(\omega=\pm 1)} = \pm \varphi^{(\omega=\pm 1)}; \]  

It is easy to obtain some classifications with the help of \( (\hat{\delta}, \hat{\pi}) \). Indeed,

\[ \hat{\pi} \Psi_{++}(a; u, v) = \hat{\pi} F_1(a, u^2) F_1(-a, v^2) = F_1(a, u^2) F_1(-a, v^2) = + \Psi_{++}(a; u, v), \]

\[ \hat{\pi} \Psi_{--}(a; u, v) = \hat{\pi} F_2(a, u^2) F_2(-a, v^2) = F_2(a, u^2) F_2(-a, v^2) = - \Psi_{--}(a; u, v), \]

\[ \hat{\pi} \Psi_{-+}(a; u, v) = \hat{\pi} F_1(a, u^2) F_2(-a, v) = F_1(a, u^2) F_2(-a, v) = + \Psi_{-+}(a; u, v), \]

\[ \hat{\pi} \Psi_{+-}(a; u, v) = \hat{\pi} F_2(a, u^2) F_1(-a, v^2) = F_2(a, u^2) F_1(-a, v^2) = - \Psi_{+-}(a; u, v). \]  

Remembering eqs. \( (29) \sim (32) \), one can conclude that the basic solutions are eigenfunctions of two discrete operators \( \hat{\delta} \) and \( \hat{\pi} \):

\[ (\Psi_{++}, \Psi_{--}, \Psi_{-+}, \Psi_{+-}) \iff (\hat{\delta}, \hat{\pi}) \]

All three ways to classify solutions with the help of discrete operators \( (\hat{\delta}, \hat{\pi}), (\hat{\delta}, \hat{\tilde{R}}), (\hat{\delta}, \hat{\omega}) \) are equally acceptable. It is understandable that an operator \( \hat{A} \) related to the quantum number \( a \), must commute with \( \hat{\delta} \) and \( \hat{\pi} \), and it is not commute with \( \hat{R} \) and \( \hat{\omega} \).
Boundary properties of the wave functions constructed can be illustrated by the schemes:

\[ \psi^+ + \psi^- \]

Fig 7. Boundary behavior of the wave functions in \((x, y)\)-plane

V. EXPLICIT FORM OF A DIAGONALIZED OPERATOR \( \hat{A} \)

Let us find an explicit, form of the operator \( \hat{A} \) introduced above by the equation \( \hat{A} \psi = a \psi \). To this end, remembering eq. (9)

\[
\frac{d^2 U}{du^2} + \lambda^2 u^2 U = a U, \quad \frac{d^2 V}{dv^2} + \lambda^2 v^2 V = -a V, \tag{52}
\]

one derives

\[
\frac{1}{2} \left[ \left( \frac{\partial^2}{\partial u^2} + \lambda^2 u^2 \right) - \left( \frac{\partial^2}{\partial v^2} + \lambda^2 v^2 \right) \right] U(u)V(v) = a U(u)V(v). \tag{53}
\]

From where one gets an explicit form of \( \hat{A} \):

\[
\hat{A} \psi_{\epsilon,p,a}(t,u,v,z) = a \psi_{\epsilon,p,a}(t,u,v,z), \tag{54}
\]

and

\[
\hat{A} = \frac{1}{2} \left\{ \left[ \frac{\partial^2}{\partial u^2} + \left( -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2} - m^2 \right) u^2 \right] - \left[ \frac{\partial^2}{\partial v^2} + \left( -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2} - m^2 \right) v^2 \right] \right\}. \]

Let us transform this operator \( \hat{A} \) to Cartesian coordinates. To this end, taking into account the formulas

\[
\frac{\partial}{\partial u} = u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial v} = -v \frac{\partial}{\partial x} + u \frac{\partial}{\partial y},
\]

\[
\frac{\partial^2}{\partial u^2} = u \frac{\partial}{\partial x} \frac{\partial}{\partial x} + v \frac{\partial}{\partial x} \frac{\partial}{\partial y} + v \frac{\partial}{\partial y} \frac{\partial}{\partial x} + u \frac{\partial}{\partial y} \frac{\partial}{\partial y},
\]

\[
\frac{\partial^2}{\partial v^2} = v \frac{\partial}{\partial x} \frac{\partial}{\partial x} - v \frac{\partial}{\partial x} \frac{\partial}{\partial y} - u \frac{\partial}{\partial y} \frac{\partial}{\partial x} + u \frac{\partial}{\partial y} \frac{\partial}{\partial y}.
\]
and also

\[ \frac{\partial u}{\partial x} = \frac{u}{u^2 + v^2}, \quad \frac{\partial u}{\partial y} = \frac{v}{u^2 + v^2}, \quad \frac{\partial v}{\partial x} = \frac{-v}{u^2 + v^2}, \quad \frac{\partial v}{\partial y} = \frac{u}{u^2 + v^2}, \]

one finds

\[ \frac{\partial^2}{\partial u^2} = u^2 \frac{\partial^2}{\partial x^2} + v^2 \frac{\partial^2}{\partial y^2} + 2uv \frac{\partial^2}{\partial x \partial y} + \frac{\partial}{\partial x}, \]
\[ \frac{\partial^2}{\partial v^2} = v^2 \frac{\partial^2}{\partial x^2} + u^2 \frac{\partial^2}{\partial y^2} - 2uv \frac{\partial^2}{\partial x \partial y} - \frac{\partial}{\partial x}, \]

that is

\[ \frac{1}{2} \left( \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) = \frac{u^2 - v^2}{2} \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) + 2uv \frac{\partial^2}{\partial x \partial y} + \frac{\partial}{\partial x} = x \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) + 2y \frac{\partial^2}{\partial x \partial y} + \frac{\partial}{\partial x}. \]

Therefore, for \( \hat{A} \) in Cartesian coordinates one has the following representation

\[ \hat{A} = x \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) + 2y \frac{\partial^2}{\partial x \partial y} + \frac{\partial}{\partial x} + x \left( -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2} - m^2 \right); \]

whereas in \((u, \vartheta, z)\)-coordinates it looks as

\[ \hat{A} = \frac{1}{2} \left[ \left( \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) - \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2} + m^2 \right) (u^2 - v^2) \right]. \]

The solutions constructed above, \( \Psi_{++}, \Psi_{--}, \Psi_{+-}, \Psi_{-+} \), behave themselves in exact correspondence with the following commutation relations:

\[ \hat{\delta} \hat{A} = + \hat{A} \hat{\delta}, \quad \hat{\pi} \hat{A} = + \hat{A} \hat{\pi}, \]
\[ \hat{\omega} \hat{A} = - \hat{A} \hat{\omega}, \quad \hat{R} \hat{A} = - \hat{A} \hat{R}, \]
\[ \hat{\pi} \hat{\omega} = - \hat{\omega} \hat{\pi}, \quad \hat{\pi} \hat{R} = - \hat{R} \hat{\pi}. \]

VI. ORTHOGONALITY AND COMPLETENESS OF THE BASES FOR VECTOR AND SPINOR SPACE MODELS

Now let us consider the scalar multiplication

\[ \int \Psi^*_\mu \Psi_\mu \sqrt{-g} \, dtdzdudv. \]

of the basic wave functions constructed:

\[ \Psi_{++}(\epsilon, p, a) = e^{i\epsilon t} e^{ipz} \Phi_{++}(a; u, v) = e^{i\epsilon t} e^{ipz} E(+a, u^2) E(-a, v^2), \]
\[ \Psi_{--}(\epsilon, p, a) = e^{i\epsilon t} e^{ipz} \Phi_{--}(a; u, v) = e^{i\epsilon t} e^{ipz} O(+a, u) O(-a, v), \]
\[ \Psi_{+-}(\epsilon, p, a) = e^{i\epsilon t} e^{ipz} \Phi_{+-}(a; u, v) = e^{i\epsilon t} e^{ipz} E(+a, u^2) O(-a, v), \]
\[ \Psi_{-+}(\epsilon, p, a) = e^{i\epsilon t} e^{ipz} \Phi_{-+}(a; u, v) = e^{i\epsilon t} e^{ipz} O(+a, u) E(-a, v^2). \]

\( \mu \) and \( \mu' \) stand for generalized quantum numbers. In the first place, interesting integrals are (arguments \((a;u,v)\) are omitted):

in vector space

\[ I_0 = \int_0^{+\infty} dv \int_{-\infty}^{+\infty} du \Phi^*_++ \Phi_{--} (u^2 + v^2), \]
in spinor space

\[
I_1 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du \Phi^{+}_{++} \Phi^{--} (u^2 + v^2),
\]

\[
I_2 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du \Phi^{+}_{+-} \Phi^{--} (u^2 + v^2),
\]

\[
I_3 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du \Phi^{+}_{++} \Phi^{--} (u^2 + v^2),
\]

\[
I_4 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du \Phi^{+}_{+-} \Phi^{--} (u^2 + v^2),
\]

\[
I_5 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du \Phi^{+}_{--} \Phi^{+-} (u^2 + v^2),
\]

\[
I_6 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du \Phi^{+}_{--} \Phi^{+-} (u^2 + v^2).
\]  \hspace{1cm} (62)

Integral \(I_6\) in vector space vanishes identically

\[
I = \int_{0}^{+\infty} dv \int_{-\infty}^{+\infty} du \ E(\pm a, u^2) \ E(-a, v^2) \ O(a, v) \ (u^2 + v^2) =
\]

\[
= \int_{0}^{+\infty} dv \int_{-\infty}^{+\infty} du \ E(\pm a, u^2) \ O(a, v) \times E(-a, v^2); O(-a, v) \ (u^2 + v^2) \equiv 0,
\]

because integration in variable \(u \in (-\infty, +\infty)\) is done for an odd function of \(u\) in symmetrical region \(u \in (-\infty, +\infty)\).

By the same reasons, integral \(I_1\) in spinor space vanishes as well.

The integral \(I_2\) vanishes

\[
I_2 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du \ E(\pm a, u^2) \ O(a, v) \ E(-a, v^2) \ (u^2 + v^2) \equiv 0,
\]

because integration is done for an odd function in \(v, u\)-variables, in symmetrical regions \(v \in (-\infty, +\infty)\) and \(u \in (-\infty, +\infty)\).

Integral \(I_3\) vanishes

\[
I_3 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du \ E(\pm a, u^2) \ E(-a, v^2) \ E(a, u^2) \ O(a, v) \ u^2 + v^2 \equiv 0,
\]

because integration is done for odd function of \(v\) variable, in the symmetrical region \(v \in (-\infty, +\infty)\).

Integral \(I_4\) vanishes

\[
I_4 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du \ E(\pm a, u^2) \ E(-a, v^2) \ O(a, u) \ E(-a, v^2) \ (u^2 + v^2) \equiv 0,
\]

because integration is done for an odd function of \(U\) in symmetrical region \(u \in (-\infty, +\infty)\).

Integral \(I_5\) vanishes

\[
I_5 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du \ O(a, u) \ O(a, v) \ E(a, u^2) \ O(-a, v) \ (u^2 + v^2) \equiv 0,
\]

because one integrates an odd function of \(u\) in symmetrical region \(u \in (-\infty, +\infty)\).

Integral \(I_6\) vanishes

\[
I_6 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du \ O(a, u) \ O(a, v) \ E(-a, u^2) \ O(-a, v) \ (u^2 + v^2) \equiv 0,
\]

because one integrates an odd function of \(v\) in symmetrical region \(v \in (-\infty, +\infty)\).

Thus, vanishing integrates \(I_6, I_1...I_6\) from (61), (62) shows that the formulas (61) provide us with orthogonal basis for Hilbert space \(\Psi(u a v, v)\), where \((u, v, z)\) belong to an extended (spinor) space model.
VII. ON MATRIX ELEMENTS OF PHYSICAL OBSERVABLES, IN VECTOR AND SPINOR SPACE MODELS

The question of principle is how transition from vector to spinor space model can influence result of calculation of matrix elements for physical quantities. As an example, let us consider matrix elements for operator of coordinates: One may calculate matrix elements of basic initial coordinates $u$, $v$ or there 2-order derivative coordinates $x$, $y$:

$$(u, v) \quad \text{or} \quad x = \frac{u^2 - v^2}{2}, \quad y = uv.$$  \hfill (63)

With the use of the above rules – integral for an odd function in symmetrical region vanishes identically – one can derive simple section rules for matrix elements (for simplicity we restrict ourselves only to the degeneracy in discrete quantum number $++, --, +- , --$ taking $c, p, a$ fixed):

**in vector space**

\[
\begin{array}{cccc}
  x_{\mu', \mu} & ++ & - - & y_{\mu', \mu} \\
  ++ & \neq 0 & 0 & 0 \\
  - - & 0 & \neq 0 & 0 \\
\end{array}
\]

**in spinor space**

\[
\begin{array}{cccc}
  x_{\mu', \mu} & ++ & - - & - - \\
  ++ & \neq 0 & 0 & 0 \\
  - - & 0 & \neq 0 & 0 \\
\end{array}
\]

The same for coordinates $u$ and $v$ looks: **in vector space**

\[
\begin{array}{cccc}
  u_{\mu', \mu} & ++ & - - & v_{\mu', \mu} \\
  ++ & 0 & \neq 0 & 0 \\
  - - & \neq 0 & 0 & 0 \\
\end{array}
\]

**in spinor space**

\[
\begin{array}{cccc}
  u_{\mu', \mu} & ++ & - - & - - \\
  ++ & 0 & \neq 0 & 0 \\
  - - & 0 & \neq 0 & 0 \\
\end{array}
\]

Let us give some detail of calculation needed. For example,

**In vector space**

\[
x_{a'++, a--} =
\]

\[
= \int_{0}^{+\infty} \int_{-\infty}^{+\infty} dv E(+a', u^2) E(-a', v^2) \frac{u^2 - v^2}{2} O(+a, u) O(-a, v) (u^2 + v^2) \equiv 0,
\]

**in spinor space**

\[
x_{a'++, a--} =
\]

\[
= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dv E(+a', u^2) E(-a', v^2) \frac{u^2 - v^2}{2} O(+a, u) O(-a, v) (u^2 + v^2) \equiv 0.
\]

**In vector space**

\[
u_{a'++, a--} =
\]

\[
= \int_{0}^{+\infty} \int_{-\infty}^{+\infty} dv E(+a', u^2) E(-a', v^2) u O(+a, u) O(-a, v) (u^2 + v^2) \neq 0,
\]
in spinor space

\[ u_{a'++, a--} = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du \ E_{a'}(u) \ E(a', v) \ u \ O(a, u) \ O(-a, v) \ (u^2 + v^2) \equiv 0. \]

In vector space:

\[ v_{a'++, a++} = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du \ E_{a'}(u) \ E(a', v) \ v \ E(a, u) \ O(-a, v) \ (u^2 + v^2) \neq 0. \]

in spinor space:

\[ v_{a'++, a--} = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du \ E_{a'}(u) \ O(-a', v) \ E(a, u) \ O(-a, v) \ (u^2 + v^2) \equiv 0. \]

VIII. SCHRODINGER EQUATION

Analysis given on analytical properties of Klein-Fock wave solutions in vector and spinor space models still retains its applicability with slight changes for the non-relativistic Schrödinger equation as well:

\[ i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \left[ \frac{\partial^2}{\partial z^2} + \frac{1}{u^2 + v^2} \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \right] \Psi, \]  

(64)

substitution for wave functions is the same

\[ \Psi(t, u, v, z) = e^{-i\epsilon t/\hbar} e^{ipz/\hbar} U(u) V(v), \]

equation for \( U(u)V(v) \) is

\[ \left[ \frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + (\epsilon - \frac{p^2}{2m}(u^2 + v^2)) \right] U(u)V(v) = 0. \]  

(65)

explicit form of \( \hat{A} \) in \((u, v)\)-representation is

\[ \hat{A} = \frac{1}{2} \left[ \frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) + (i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} ) (u^2 - v^2) \right], \]  

(66)

in \((x, y)\) form it looks

\[ \hat{A} = \frac{\hbar^2}{2m} \left( x \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) + 2y \frac{\partial^2}{\partial x \partial y} + \frac{\partial}{\partial x} \right) + x \left( i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} \right). \]  

(67)

IX. PARAMETRIZATION OF SPACIAL SPINORS BY PARABOLIC CYLINDRAL COORDINATES

In [35,37] concepts of two sorts of spatial spinors, depending on P-orientation of primary vector space model, vector \( E_3 \) and pseudovector \( \Pi_3 \), correspond \( \eta \)-spinor and \( \xi \)-spinor respectively. Procedure of extending the space model is realized simpler on the base of curvilinear coordinate systems. Here let us consider this procedure in cylindrical parabolic coordinates. In vector model they are introduced by relations:

\[ x = \frac{u^2 - v^2}{2}, \quad y = u \ v, \quad z = z, \]

\[ v \in [0, +\infty), \quad u, z \in (-\infty, +\infty); \]  

(68)

with graphical illustration in Fig. A1.
It suffices to use the semi-plane shown in Fig. 1 to cover the whole plane \((x, y)\) Spacial spinor \(\xi\) is given in \((u, v, z)\)-coordinates as
\[
\xi(u, v, z) = \begin{vmatrix} \sqrt{z^2 + (u^2 + v^2)^2/4 + z} & e^{-i\gamma/2} \\ \sqrt{z^2 + (u^2 + v^2)^2/4 - z} & e^{i\gamma/2} \end{vmatrix}, \quad e^{i\gamma/2} = \frac{u + iv}{\sqrt{u^2 + v^2}}; \quad (69)
\]
here the factor \(e^{i\gamma/2}\) belongs to upper complex half-plan (in the case of spinor model, it will cover the whole complex plane). At the plane \(z = 0\) (designated by \(\Pi^{+\cap-}\)), spinor \(\xi\) is given by
\[
\xi^{+\cap-}(u, v, z = 0) = \frac{1}{\sqrt{2}} \begin{vmatrix} u - i v \\ u + i v \end{vmatrix}. \quad (70)
\]
Spatial spinor of the type \(\eta\) is given by
\[
\eta^\sigma(u, v, z) = \begin{vmatrix} \sqrt{z^2 + (u^2 + v^2)^2/4 - (u^2 + v^2)/2} & (\sigma e^{-i\gamma/2}) \\ \sqrt{z^2 + (u^2 + v^2)^2/4 + (u^2 + v^2)/2} & (e^{-i\gamma/2}) \end{vmatrix}, \quad \sigma = +1 \text{ corresponds to upper semi-space } (z > 0), \quad \sigma = -1 \text{ corresponds to lower semi-space } (z < 0). \quad (71)
\]
To the plane \((z = 0)\) corresponds the simpler spinor
\[
\eta^{+\cap-}(u, v, z = 0) = \begin{vmatrix} 0 \\ u + i v \end{vmatrix}. \quad (72)
\]
The way to parameterize the vector \((x, y)\)-plane by \((u, v)\)-coordinates prescribes the following identification rules for the domain \(G(u, v)\) :

\[
v \in [0, +\infty) \implies v \in (-\infty, +\infty); \quad (73)
\]
at this the factor \(e^{+i\gamma/2}\) in \(\xi\) will belong to the whole circle in the complex plane.
Fig. A3. Transition to a spinor model

One may specially note that the identification rule in $\tilde{G}(u, v)$ covering spinor models $\tilde{\Pi}_3$ and $\tilde{E}_3$ seems simpler than that in $G(u, v)$ for a vector models $\Pi_3$ and $E_3$:

$$\tilde{G}(u, v) = \begin{cases} 
  u & \in (-\infty, +\infty), \\
  v & \in (-\infty, +\infty), \\
  z & \in (-\infty, +\infty).
\end{cases}$$

The domain $\tilde{G}(u, v, z)$ does not require any special identification rules on its (infinite) boundary, in addition to Euclidean structure of the $(u, v, z)$-space.

The domain $\tilde{G}(u, v, z)$ looks the same both for $\Pi_3$ and $E_3$ spinor spaces. This means that the domain $\tilde{G}(u, v, z)$ with Euclidean topology does not determine in full the properties of spinor models, $\Pi_3$ or $E_3$. Same specific distinction between $\tilde{\Pi}_3$ and $\tilde{E}_3$ models can be seen if one follows how the orientation of cylindrical surfaces $\xi^i, \eta^i$ changes when passing from upper to lower half-space.

Indeed, accordingly (69) and (71), $(\xi^1, \xi^2)$-components are oriented as follows

$$\begin{align*}
\xi^1 & : & & v & & z > 0 & & \delta = 2 & & \delta = 1 \\
\xi^2 & : & & v & & z > 0 & & \delta = 2 & & \delta = 1 \\
\xi^1 & : & & v & & z < 0 & & \delta = 1 & & \delta = 2 \\
\xi^2 & : & & v & & z > 0 & & \delta = 1 & & \delta = 2
\end{align*}$$

Fig. A4. Spacial spinor $\xi^i(u, v, z)$

Instead $\eta^1, \eta^2$ for $\tilde{\Pi}_3$ model are characterized by the schemes
Thus, we can conclude that the concept of spinor space should be defined by giving
1) the form of extended region $\tilde{G}$;
2) the way to identify its boundary points
(see applying spherical or parabolic coordinates in the same contest [37]);
3) substantial element of the concept of spinor space consists in indication of orientation in $\tilde{G}$ region – the latter is determined by explicit functions $\eta^i$ and $\xi^i$ of $(u, v, z)$.

One other aspect of the spinor space models can be clarified with the help of the the derivatives of $\xi^i(u, v, z)$ and $\eta^i(u, v, z)$ with respect to $(u, v)$

$$\frac{\partial \xi^1}{\partial u} = \frac{\xi^1}{2} \left( \frac{\rho}{r(r+z)} u + \frac{i}{\rho} v \right), \quad \frac{\partial \xi^1}{\partial v} = \frac{\xi^1}{2} \left( \frac{\rho}{r(r+z)} v - \frac{i}{\rho} u \right),$$

$$\frac{\partial \xi^2}{\partial u} = \frac{\xi^2}{2} \left( \frac{\rho}{r(r-z)} u - \frac{i}{\rho} v \right), \quad \frac{\partial \xi^2}{\partial v} = \frac{\xi^2}{2} \left( \frac{\rho}{r(r-z)} v + \frac{i}{\rho} u \right),$$

(74)

$$\frac{\partial \eta^1}{\partial u} = \frac{\eta^1}{2} \left( -\frac{u}{r} + \frac{i}{\rho} v \right), \quad \frac{\partial \eta^1}{\partial v} = \frac{\eta^1}{2} \left( -\frac{v}{r} - \frac{i}{\rho} u \right),$$

$$\frac{\partial \eta^2}{\partial u} = \frac{\eta^2}{2} \left( +\frac{u}{r} - \frac{i}{\rho} v \right), \quad \frac{\partial \eta^2}{\partial v} = \frac{\eta^2}{2} \left( +\frac{v}{r} + \frac{i}{\rho} u \right).$$

(75)

One should note that relations (74) and (75) are singular only on the axis $z$. With the help of (74), (75) one can readily find derivatives along directions

$$\vec{w} = (u, v), \quad \vec{\nu} = (a, b),$$

$$(\vec{\nu} \times \vec{w}) = (a \, v - b \, u), \quad (\vec{\nu} \times \vec{w}) = (a \, v - b \, u),$$

$$\nabla_{\vec{\nu}} = \vec{\nu} \hat{\nabla}, \quad \hat{\nabla} = \left( \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right).$$

For $\nabla_{\vec{\nu}} \xi$:

$$\nabla_{\vec{\nu}} \xi^1 = \frac{\xi^1}{2} \left[ \frac{\rho}{r(r+z)} (\vec{\nu} \times \vec{w}) + \frac{i}{\rho} (\vec{\nu} \times \vec{w}) \right],$$

$$\nabla_{\vec{\nu}} \xi^2 = \frac{\xi^2}{2} \left[ \frac{\rho}{r(r-z)} (\vec{\nu} \times \vec{w}) - \frac{i}{\rho} (\vec{\nu} \times \vec{w}) \right].$$

(76)
For $\nabla_\varphi \eta$:

$$\nabla_\varphi \eta^1 = \frac{\eta^1}{2} \left[ -\frac{\nabla^i \omega^j}{r} + \frac{i}{\rho} (\nabla \times \omega) \right], \quad \nabla_\varphi \eta^2 = \frac{\eta^2}{2} \left[ -\frac{\omega^i}{r} - \frac{i}{\rho} (\nabla \times \omega) \right].$$

These equations can be considered as fundamental equations underlying spatial spinors, because solutions of these equations provide us with spatial spinors.

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