Topology of Blow-ups and Enumerative Geometry

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Abstract
Let $\tilde{M}$ be the blow-up of a manifold $M$ along a submanifold $X$. We determine the integral cohomology ring and obtain a formula for the Chern classes of $\tilde{M}$.

As applications we determine the cohomology rings for the varieties of complete conics and complete quadrics in 3-space, and justify two enumerative results due to Schubert [S1, §20; §22].

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1 Introduction
All manifolds concerned in this paper are connected and are in the real and smooth category, but are not necessarily compact and orientable. The cohomologies are over integer coefficients, unless otherwise stated.

Let $X$ be a submanifold of a manifold $M$ whose normal bundle $\gamma_X$ has a complex structure. We construct the blow-up $\tilde{M}$ of $M$ along $X$ and present the integral cohomology ring $H^*(\tilde{M})$ in terms of the induced ring map $H^*(M) \rightarrow H^*(X)$ as well as the Chern classes of $\gamma_X$, see Theorem 1 in §3.

If $X \subset M$ is an embedding in the category of almost complex manifolds [MS, p.151], the blow-up $\tilde{M}$ then admits an almost complex structure. We obtain a formula expressing the Chern classes of $\tilde{M}$ by the Chern classes of $M$ and $\gamma_X$, together with the exceptional divisor, see Theorem 2 in §4.

As applications of Theorems 1 and 2 we determine the integral cohomology rings of the varieties of complete conics and quadrics on the 3-space $\mathbb{P}^3$, respectively in Theorem 3 in §5 and Theorem 4 in §6. They are used in §7 to justify two enumerative results due to Schubert [S1, §20; §22]:

Given 8 quadrics in the space $\mathbb{P}^3$ in general position, there are 4,407,296 conics tangent to all of them.

Given 9 quadrics in the space $\mathbb{P}^3$ in general position, there are 666,841,088 quadrics tangent to all of them.

For the historical importance and interests in these problems, see Kleiman [K1, K2], Fulton–Kleiman–MacPherson [FKM] and discussion in §7.

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2 Blow-up along a submanifold

For a submanifold $X \subset M$ let $\gamma_X$ be its normal bundle in $M$. Assume that $\dim \gamma_X = 2k$ and that $\gamma_X$ is equipped with a complex structure $J$. Let $\pi : E = \mathbb{P}(\gamma_X) \to X$ be the complex projective bundle associated with $\gamma_X$. The tautological line bundle on $E$ means disjoint union and where, as is canonical, the quotient space $\partial D(\lambda_X) \cup X/\sim$ is identified with $M$.

**Lemma 1.** There exists a smooth structure on $\widetilde{M}$ so that $\widetilde{f}$ is smooth.

Moreover, i) the diffeomorphism type of $\widetilde{M}$ is determined by the triple $\{M, \gamma_X, J\}$ and is independent on the choice of an Riemannian metric on $M$; ii) if $M$ and $X$ are furnished with almost complex structures [MS, p.151], then $\widetilde{M}$ has an almost complex structure compatible with that on $M$ and $X$.

**Proof.** With the quotient topology on $\widetilde{M}$ one has the decomposition

\[(2.2) \quad \widetilde{M} = (M \setminus D(\gamma_X)) \cup D(\lambda_E) \text{ (see [M; Definition 2.2])}\]

where $g$ is the natural identification $\partial D(\gamma_X) = S(\gamma_X) \to S(\lambda_E) = \partial D(\lambda_E)$. □

The map $f : \widetilde{M} \to M$ will be called the blow-up of $M$ along the submanifold $X$ (whose normal bundle has a complex structure) with exceptional divisor $E$ (resorting to terminology from algebraic geometry [GH, p.182, p.603]). Obvious but useful properties of the map $f$ are collected in Lemma 2 below.

**Lemma 2.** Let $i_E : E \to \widetilde{M}$ (resp. $i_X : X \to M$) be the embedding given by the zero section of $D(\lambda_E)$ in view of the decomposition (2.2). Then

i) the normal bundle of $E$ in $\widetilde{M}$ is $\lambda_E$;

ii) $f^{-1}(X) = E$ with $f \circ i_E = i_X \circ \pi$;

iii) $f$ restricts to a diffeomorphism: $\widetilde{M} \setminus E \to M \setminus X$. □

2
3 The Cohomology ring of a blow–up

We begin by developing some preliminary results. Given a ring $A$ and a finite set \{t_1, \cdots, t_n\}, write $A\{t_1, \cdots, t_n\}$ for the free $A$–module with basis \{t_1, \cdots, t_n\}. For an $m$–dimensional complex vector bundles $\alpha$ over a manifold $N$ with total Chern class $1 + c_1 + \cdots + c_m$ let $\pi_{\alpha} : \mathbb{P}(\alpha) \to N$ be the associated projective bundle of $\alpha$. The tautological line bundle on $\mathbb{P}(\alpha)$ is denoted by $\lambda_{\alpha}$, and write $\overline{\alpha}$ for the conjugation of $\lambda_{\alpha}$ [MS, p.167].

**Lemma 3.** The cohomology ring $H^*(\mathbb{P}(\alpha))$, as an $H^*(N)$–algebra, is generated by $t = c_1(\overline{\alpha}) \in H^2(\mathbb{P}(\alpha))$ subject to the single relation

\[
(3.1) \quad t^m + c_1 \cdot t^{m-1} + \cdots + c_m \cdot t = 0,
\]

In particular, $H^*(\mathbb{P}(\alpha)) = H^*(N)\{1, t, \cdots, t^{m-1}\}$.

Let $i_N : N \to W$ be a smooth embedding whose normal bundle $\gamma_N$ is oriented. Furnish $W$ with an Riemannian metric so that the total space of some $\epsilon$–disk bundle of $\gamma_N$ is identified with a tubular neighborhood $U_N$ of $N$ in $W$. Let $\mathcal{D}_N \in H^1(U_N; U_N \setminus N)$ be the Thom class of the oriented bundle $\gamma_N$, $r = \dim_{\mathbb{R}} \gamma_N$. Cup product with $\mathcal{D}_N$ yields the Thom isomorphism

\[
T_N : H^*(N) \xrightarrow{p_N^*} H^*(U_N) \xrightarrow{\mathcal{D}_N} H^*(U_N; U_N \setminus N), \quad T_N(\beta) = p_N^*(\beta) \cdot \mathcal{D}_N,
\]

where $p_N : U_N \to N$ is the obvious projection.

The inclusion $i_{U_N} : U_N \to W$ induces the Excision isomorphism

\[
i_{U_N}^* : H^*(W; W \setminus N) \xrightarrow{\cong} H^*(U_N, U_N \setminus N).
\]

and also satisfies $i_N = i_{U_N} \circ \sigma$, where $\sigma : N \to U_N$ is the zero section. Since $\sigma$ is a homotopy inverse of $p_N$ the latter implies that

\[
i_{U_N}^* = p_N^* \circ i_N^* : H^*(W) \to H^*(U_N).
\]

Write $\psi_N$ for the composed isomorphism

\[
\psi_N = i_{U_N}^{* -1} \circ T_N : H^*(N) \xrightarrow{\cong} H^*(W; W \setminus N)
\]

and let $j_N : (W, \emptyset) \to (W, W \setminus N)$ be the inclusion of topological pairs.

**Lemma 4.** For any $\alpha \in H^*(W)$, $\beta \in H^*(N)$ one has in $H^*(W)$ the relation

\[
\alpha \cdot j_N^* \psi_N(\beta) = j_N^* \psi_N(i_{U_N}^*(\alpha) \cdot \beta),
\]

where $\cdot$ means cup product in cohomology.

**Proof.** Consider the commutative diagram induced by $i_{U_N}$

\[
\begin{array}{ccc}
H^*(W) \times H^*(W; W \setminus N) & \xrightarrow{i_{U_N}^* \times i_N^*} & H^*(U_N) \times H^*(U_N, U_N \setminus N) \\
\downarrow & & \downarrow \\
H^*(W; W \setminus N) & \xrightarrow{i_N^*} & H^*(U_N, U_N \setminus N)
\end{array}
\]
Applying Lemma 5. It is shown in [MS; Theorem 11.3] that the cycle class \( \omega \) where the vertical maps are given by cup products. It follows that for any \( \alpha \in H^*(W) \), \( \beta \in H^*(N) \) one has in \( H^*(U_N, U_N \setminus N) \) that
\[
\begin{align*}
i_{U_N}^*(\alpha \cdot \psi_N(\beta)) &= i_{U_N}^*(\alpha) \cdot i_{U_N}^*(\psi_N(\beta)) \\
&= (i_{U_N}^*(\alpha) \cdot p_N^*(\beta)) \cdot D_N \quad \text{(by the definition of \( \psi_N \))}
\end{align*}
\]

\( = p_N^*(i_{U_N}^*(\alpha) \cdot \beta) \cdot D_N \quad \text{(since \( i_{U_N}^* = p_N^* \circ i_N^* \))}
\]
\( = T_N(i_N^*(\alpha) \cdot \beta) \quad \text{(by the definition of \( T_N \)).}
\]

Applying \( j_N^* \circ i_N^{*-1} \) to both ends of the equalities verifies the relation. \( \square \)

Let \( e(\gamma_N) \in H^*(N) \) be the Euler class of the oriented bundle \( \gamma_N \) and set \( \omega_N = j_N^* \psi_N(1) \in H^*(W) \) with \( 1 \in H^0(N) \) the multiplicative unit. It is essentially shown in [MS; Theorem 11.3] that

**Lemma 5.** Suppose that \( i_N \) embeds \( N \) as a closed subset of \( W \). Then
i) \( i_N^* \omega_N = e(\gamma_N) \);
ii) if \( W \) is compact and oriented, \( \omega_N \) is the Poincare dual of the oriented cycle class \( i_N^*[N] \in H_*(W) \). \( \square \)

Let \( \alpha \) be an oriented subbundle of an oriented vector bundle \( \beta \) over a space \( Y \) and let \( D_\alpha \in H^*(E_\alpha, E_\alpha^0) \) be the Thom class of \( \alpha \), where \( E_\alpha \) is the total space of \( \alpha \) and \( E_\alpha^0 \) is the complement of the zero section \( \sigma_\alpha : Y \rightarrow E_\alpha \). Write \( j : (E_\alpha, E_\alpha^0) \rightarrow (E_\beta, E_\beta^0) \) for the inclusion and let \( \gamma \) be the quotient of \( \alpha \) in \( \beta \).

**Lemma 6.** The induced map \( j^* \) carries \( D_\beta \) to \( D_\alpha \cdot p_\alpha^* e_\gamma \), where \( e_\gamma \in H^*(Y) \) is the Euler class of \( \gamma \) and \( p_\alpha \) is the projection of \( \alpha \).

**Proof.** The inclusion \( j \) factors through the pair \((E_\alpha, E_\alpha^0) \times Y\) in the fashion
\[
\begin{align*}
(E_\alpha, E_\alpha^0) \times Y & \xrightarrow{id \times p_\alpha} (E_\alpha, E_\alpha^0) \\
& \xrightarrow{id \times \sigma_\gamma} (E_\alpha, E_\alpha^0) \times E_\gamma \quad \xrightarrow{(E_\alpha, E_\alpha^0) \times E_\gamma} (E_\alpha, E_\alpha^0) \times (E_\gamma, E_\gamma^0)
\end{align*}
\]
The proof is done by \( D_\beta = D_\alpha \times D_\gamma \) and \( \sigma_\gamma^* D_\gamma = e_\gamma \). \( \square \)

We shall also need the following standard fact from homological algebra.

**Lemma 7.** Consider the exact ladder of abelian groups
\[
\begin{align*}
\cdots & \rightarrow A_1 & \rightarrow & A_2 & \rightarrow & A_3 & \rightarrow & A_4 & \rightarrow & \cdots \\
i_1 \downarrow \cong & \quad & i_2 \downarrow & \quad & i_3 \downarrow & \quad & i_4 \downarrow \cong & & \\
\cdots & \rightarrow B_1 & \rightarrow & B_2 & \xrightarrow{\beta} & B_3 & \rightarrow & B_4 & \rightarrow & \cdots
\end{align*}
\]
If \( i_1 \) and \( i_4 \) are isomorphic and \( i_2 \) is monomorphic, then \( i_3 \) is monomorphic.

Moreover, if the short exact sequence \( 0 \rightarrow A_2 \rightarrow B_2 \rightarrow B_2 / \Im i_2 \rightarrow 0 \) is split, then
i) the sequence \( 0 \rightarrow A_2 \rightarrow B_2 \rightarrow B_2 / \Im i_2 \rightarrow 0 \) is split,
ii) the map \( \beta \) induces an isomorphism \( B_2 / \Im i_2 \rightarrow B_3 / \Im i_3 \). \( \square \)

Carrying on discussion from the previous section assume that the embedding \( i_X : X \rightarrow M \) maps \( X \) onto a closed subset of \( M \), and let \( f : M \rightarrow M \) be the blow-up of \( M \) along \( X \) with exceptional divisor \( E \). Write \( C(\gamma_X) = 1 + c_1 + \cdots + c_k \in H^*(X) \) for the total Chern class of the normal bundle \((\gamma_X, J)\), \( 2k = \dim_R \gamma_X \). By i) of Lemma 5 we have
The problem we are about to solve is to describe the ring $H^* (\tilde{M})$ in terms of the induced ring map $i_X^* : H^* (M) \to H^* (X)$, as well as the total Chern class of $\gamma_X$. We shall make use of the composition

$$H^* (X) \xrightarrow{\iota_X^*} H^* (E) \xrightarrow{\cong \psi_E} H^* (\tilde{M}; \tilde{M} \setminus E) \xrightarrow{j_E^*} H^* (\tilde{M}),$$

which will be shown to be monomorphic into the ideal in $H^* (\tilde{M})$ generated by $\omega_E$ in the proof of the coming result.

**Theorem 1.** The induced ring map $f^* : H^* (M) \to H^* (\tilde{M})$ embeds $H^* (M)$ as a direct summand of $H^* (\tilde{M})$.

Moreover, one has the direct sum decomposition

$$H^* (\tilde{M}) = f^* (H^* (M)) \oplus H^* (X) \{ \omega_E, \cdots, \omega_{E}^{k-1} \}, 2k = \dim_{\mathbb{R}} \gamma_X$$

that is subject to the relations:

i) $f^* (\omega_X) = \sum_{1 \leq r \leq k} (-1)^{r-1} c_{k-r} \cdot \omega_E$;

ii) for any $y \in H^r (M)$, $f^* (y) \cdot \omega_E = i_X^* (y) \cdot \omega_E$.

The proof will be organized in view of the exact ladder induced by the map $f : (M; M \setminus X) \to (\tilde{M}; \tilde{M} \setminus E)$ of topological pairs

\[
\begin{array}{ccccccc}
& & H^{r-2k} (X) & & & & \\
& \psi_X \downarrow \cong & \cdots & \rightarrow & H^{r-1} (M \setminus X) & \rightarrow & H^r (M; M \setminus X) \\
\| & & f^* \downarrow & & j_X^* & & H^r (M) \\
\| & & & & f^* \downarrow & & f^* \downarrow \\
\cdots & \rightarrow & H^{r-1} (\tilde{M} \setminus E) & \rightarrow & H^r (\tilde{M}, \tilde{M} \setminus E) & \rightarrow & H^r (\tilde{M}) \\
& \psi_E \uparrow \cong & & & & j_E^* & & H^r (\tilde{M} \setminus E) \\
& & & & \| & & \| \\
& & & & H^{r-2} (E) & & H^r (\tilde{M} \setminus E) \\
\end{array}
\]

where the identification $H^* (M \setminus X) = H^* (\tilde{M} \setminus E)$ comes from iii) of Lemma 2.

We specify useful information about certain homomorphisms in this diagram.

Firstly, with respect to the presentation by Lemma 3

$$H^* (E) = H^* (X) \{ 1, t, \cdots, t^{k-1} \}, t = c_1 (\lambda_E) \in H^2 (E)$$

we get from (3.2) that

$$(3.4) \quad \text{the endomorphism } H^* (E) \xrightarrow{\psi_E} H^* (\tilde{M}; \tilde{M} \setminus E) \xrightarrow{j_E^*} H^* (\tilde{M}) \xrightarrow{i_E^*} H^* (E) \text{ is given by } e \mapsto -e \cdot t \text{ for all } e \in H^* (E).$$

Next, let $\lambda_E^\perp$ be the complement of $\lambda_E$ in $\pi^* \gamma_X$ with $\delta \in H^* (E)$ its Euler class. Applying Lemma 6 to the decomposition $\pi^* \gamma_X = \lambda_E \oplus \lambda_E^\perp$ on $E$ yields that

$$(3.5) \quad \delta = \psi_E^{-1} \circ f^* \circ \psi_X (1) \text{ and the composition } \psi_E^{-1} \circ f^* \circ \psi_X : H^* (X) \to H^{*+2(k-1)} (E) \text{ is given by } x \mapsto \delta \cdot x, x \in H^* (X).$$
In addition, one deduces from \( \pi^* \gamma_X = \lambda_E \oplus \lambda_E^\perp \) with \( C(\gamma_X) = 1 + c_1 + \cdots + c_k \) and \( C(\lambda_E) = 1 - t \) the formual in \( H^*(E) \)

\[(3.6) \quad \delta = \pi^* \delta_1 + \pi^* \delta_2 \cdot t + \pi^* \delta_3 \cdot t^2 + \cdots + t^{k-1}; \quad \delta t = -c_k.\]

**Proof of Theorem 1.** By (3.5) and (3.6) we have the short exact sequence

\[0 \to H^*(X) \xrightarrow{\psi^{-1} \circ f^* \circ \psi_X} H^*(E) \to H^*(E)/\text{Im} \psi^{-1} \circ f^* \circ \psi_X \to 0\]

in which the quotient \( H^*(E)/\text{Im} \psi^{-1} \circ f^* \circ \psi_X \) can be identified with the direct summand \( H^*(X)\{1, t, \cdots, t^{k-2}\} \) of \( H^*(E) \) by (3.6), hence is split. It follows from Lemma 7 that the ring map \( f^* : H^*(M) \to H^*(\tilde{M}) \) embeds \( H^*(M) \) as a direct summand of \( H^*(\tilde{M}) \). Moreover, one obtains the additive decomposition \( (3.3) \) from ii) of Lemma 7 and \( j_E^* \psi_E(t^n) = (-1)^n \omega_E^n \) by (3.4).

Proceeding to the relation i) it follows from the commutativity of the second diagram in the ladder that in \( H^*(\tilde{M}) \)

\[(3.7) \quad f^*(j_E^* \circ \psi_X(x)) = j_E^* \circ \psi_E(x \delta) \text{ for all } x \in H^*(X).\]

Granted with the formula for \( \delta \) in (3.6) one obtains i) by setting \( x = 1 \) in (3.7).

Turning to the relation ii) one has by Lemma 4 that for all \( y \in H^*(M) \)

\[f^*(y) \cdot j_E^* \circ \psi_E(1) = j_E^* \circ \psi_E(i_E^* \circ f^*(y) \cdot 1)\]

where \( 1 \in H^0(E) = H^0(X) \) is the unit. One obtains ii) from \( i_E^* \circ f^*(y) = \pi^* \circ i_X^*(y) \) by ii) of Lemma 2. □

**Remark 1.** In [GH, p.605] Griffiths and Harris obtained the decomposition (3.3) in the category of complex manifolds, while the relations i) and ii) were absent. Partial information on (3.3) was also obtained by McDuff [M; Proposition 2.4].

It is clear that the ring \( H^*(\tilde{M}) \) is uniquely characterized by the additive decomposition (3.3) together with the relations i) and ii) in Theorem 1. Indeed, granted with the fact that \( f^*(H^*(M)) \subset H^*(\tilde{M}) \) is a subring, i) and ii) expand, respectively, the products of elements in the second summand and the products between elements in the first and second summand as elements in the decomposition (3.3). □

### 4 The Chern classes of a blow-up

For a CW-complex \( X \) let \( K(X) \) be the topological \( K \)-theory of complex vector bundle on \( X \). The **Chern character** \( Ch : K(X) \to H^*(X; \mathbb{Q}) \) is the transformation defined by [A; F, p.56]

\[Ch(\xi) = \dim \xi + \sum_{k \geq 1} \frac{s_k(c_1(\xi), c_2(\xi), \cdots, c_n(\xi))}{k!},\]

where \( s_k \) is the \( k^{th} \) Newton polynomial expressing the power sum \( t^k_1 + \cdots + t^k_n \) by the elementary symmetric polynomials \( e_i(t_1, \cdots, t_n), n = \dim \xi, 1 \leq i \leq n \).

Assume in this section that \( i : X \to M \) is a smooth embedding in the category of almost complex manifolds [MS, p.151], and let the blow–up \( \tilde{M} \) be furnished with the induced almost complex structure (Lemma 1) so that the
Chern classes of $\widetilde{M}$ is defined. In this section we express the Chern class of $\widetilde{M}$ in terms the Chern classes of $M$ and $\gamma_X$ (the normal bundle), together with the exceptional divisor.

For an almost complex manifold $N$ write $\tau_N$ for its tangent bundle and denote the total Chern class of $\tau_N$ by $C(\tau) =: C(\tau)$. Let $\alpha$ be an $m$–dimensional complex vector bundles over $N$ with total Chern class $1 + c_1 + \cdots + c_m$. We shall put $t = c_1(\bar{\lambda}) \in H^2(\mathbb{P}(\alpha))$.

**Lemma 8.** In the ring $K(\mathbb{P}(\alpha))$ one has

$$\tau(\alpha) \oplus \epsilon^1 = \pi^*_\alpha \tau_N \oplus \bar{\lambda}_\alpha \otimes \pi_\alpha^* \alpha,$$

where $\epsilon^1$ is the trivial line bundle on $\mathbb{P}(\alpha)$. In particular,

i) $Ch(\tau(\alpha)) = \pi^*_\alpha Ch(\tau_N) + \epsilon^1 \cdot \pi^*_\alpha Ch(\alpha) - 1$;

ii) the total Chern class of $\mathbb{P}(\alpha)$ is $C(\mathbb{P}(\alpha)) = \pi^*_\alpha (C(N)) \cdot G$ with

$$G = (1 + t)^m + (1 + t)^{m-1} \pi^*_\alpha c_1 + (1 + t)^{m-2} \pi^*_\alpha c_2 + \cdots + \pi^*_\alpha c_m.$$

**Proof.** Let $\lambda^\perp_\alpha$ be the quotient of $\pi^*_\alpha \alpha$ by the subbundle $\lambda_\alpha$. From

$$\tau(\alpha) = \pi^*_\alpha \tau_N \oplus Hom(\lambda_\alpha, \lambda^\perp_\alpha),$$

and $Hom(\lambda_\alpha, \lambda_\alpha) = \epsilon^1$, $Hom(\xi, \eta) = \bar{\xi} \otimes \eta$, $\pi^*_\alpha \alpha = \lambda_\alpha \oplus \lambda^\perp_\alpha$ one gets (4.1).

Item i) comes directly from (4.1). It follows also from (4.1) that

$$C(\mathbb{P}(\alpha)) = \pi^*_\alpha (C(N)) \cdot G \text{ with } G = C(\bar{\lambda}_\alpha \otimes \pi^*_\alpha \alpha).$$

To verify the expression for $G$ assume that the Chern roots of $\alpha$ is $s_1, \cdots, s_m$. That is $C(\alpha) = \prod_{1 \leq i \leq m} (1 + s_i)$ with $c_r(\alpha) = c_r(s_1, \cdots, s_m)$ (the $r^{th}$ elementary symmetric function). The proof is done by the calculation

$$C(\bar{\lambda}_\alpha \otimes \pi^*_\alpha \alpha) = \prod_{1 \leq i \leq m} (1 + t + \pi^*_\alpha s_i) = (1 + t)^m \prod_{1 \leq i \leq m} (1 + \pi^*_\alpha s_i)$$

$$= (1 + t)^m [1 + \frac{\pi^*_\alpha c_1}{1+t} + \frac{\pi^*_\alpha c_2}{(1+t)^2} + \cdots + \frac{\pi^*_\alpha c_m(\alpha)}{(1+t)^m}] \square$$

**Theorem 2.** Let $f : \widetilde{M} \to M$ be the blow–up of $M$ along $X$, and let $\lambda_E \in K(M)$ be the line bundle on $M$ with $c_1(\lambda_E) = \omega_E$. In the ring $K(M)$ one has

$$\tau_M = f^* \tau_M + (\bar{\lambda}_E - 1) \otimes (\pi^* \gamma_X - \lambda_E).$$

In particular, the Chern character of $\tau_M$ is given by

$$Ch(\tau_M) = f^* Ch(\tau_M) + (e^{-\omega_E} - 1)(Ch(\gamma_X) - e^{\omega_E}).$$

**Proof.** Let $U_E \in \bar{K}(\widetilde{M}, \widetilde{M} \setminus E)$ be a Thom class and denote by $\varphi_E : K(E) \to \bar{K}(\widetilde{M}, \widetilde{M} \setminus E)$ for the Thom isomorphism

$$\varphi_E(\beta) = U_E \otimes \beta, \beta \in K(E) [A, p.111],$$

where $\bar{K}$ is the reduced $K$–theory of topological pairs. In view of the exact seque
we can assume that $j^*U_E = (X - 1)$. Moreover, since $\tau_{\tilde{M}} = f^*\tau_M$ on $\tilde{M}\setminus E$ by iii) of Lemma 2, the difference $\tau_{\tilde{M}} - f^*\tau_M \in K(\tilde{M})$ satisfies $\tau_{\tilde{M}} - f^*\tau_M = (X - 1) \otimes \beta$ for some $\beta \in K(E)$ by the exactness of the sequence at $K(\tilde{M})$. To find the expression of $\beta$ we examine the restriction

\[
\alpha = (\tau_{\tilde{M}} - f^*\tau_M) | E = (\lambda_E + \tau_E) - (\pi^*\tau_X + \pi^*\gamma_X)
\]

where $\pi : E = \mathbb{P}(\gamma_X) \to X$ is the projection. From $\pi^*\gamma_X = \lambda_E \oplus \lambda_E^\perp$ and $\tau_E = \pi^*\tau_X \oplus \text{Hom}(\lambda_E, \lambda_E^\perp)$ by (4.2) we get in $K(E)$ that

\[
\alpha = \text{Hom}(\lambda_E, \lambda_E^\perp) - \lambda_E^\perp = (\lambda_E - 1) \otimes \lambda_E^\perp = (\lambda_E - 1) \otimes (\pi^*\gamma_X - \lambda_E).
\]

That is $\beta = \pi^*\gamma_X - \lambda_E. \square$

**Example 1.** By the definition of $Ch(\xi)$ there is an algorithm to fashion $C(\xi)$ from $Ch(\xi)$. As examples we get from (4.4) that:

\[
c_1(\tilde{M}) = c_1(M) - (n - k - 1)\omega_E;
\]

\[
c_2(\tilde{M}) = c_2(M) + (c_1(\gamma_X) - (n - k - 1)c_1(M))\omega_E + \frac{(n-k)(n-k-3)}{2}\omega_E^2;
\]

\[
c_3(\tilde{M}) = c_3(M) + (-c_1^2(\gamma_X) + 2c_2(\gamma_X) + c_1(M)c_1(\gamma_X))
\]

\[
-(n - k - 1)c_2(M))\omega_E + \frac{(n-k)(n-k-3)}{2}c_1(M)
\]

\[
-(n - k - 2)c_1(\gamma_X))\omega_E^2 - \frac{(n-k-5)(n-k-1)(n-k)}{6}\omega_E^3;
\]

\[
\ldots
\]

where for simplicity, $c_i(M)$ is used in the place of $f^*c_i(M)$, and where $k = \dim X$, $n = \dim M. \square$

**Remark.** In [GH, p.609] Griffith and Harris showed their interests in finding a general formula for the Chern classes of a blow–up. Applying the formulæ for $c_1(\tilde{M})$ and $c_2(\tilde{M})$ above one can recover the calculations in [GH, p.608–611].

In [F, Theorem 15.4] Fulton obtained a formula for the Chern classes of a blow–up in the context of intersection theory.

In [G] Gromov introduced blow–ups for symplectic manifolds. Theorem 2 is applicable, in particular, to compute the Chern classes of blow–ups in the category of symplectic manifolds [M].

A companion example that assures us that Theorem 2 is not trivial in the category of almost complex manifolds reads as follows. Recall that the 6–dimensional sphere $S^6$ has a canonical almost complex structure. The blow–up of $S^6$ at a point $X \in S^6$ is diffeomorphic to the projective 3–space $\mathbb{P}^3$, together with an almost complex structure $J$. By Theorem 2 we have $C(\mathbb{P}^3, J) = 1 - 2x - 4x^3$, where $x \in H^2(\mathbb{P}^3)$ is the exceptional divisor. This computation indicates that $J$ is different with the canonical complex structure on $\mathbb{P}^3. \square$
5 The variety of complete conics

Applying Theorem 1 we determine the cohomology ring of the variety of complete conics on \( \mathbb{P}^3 \). We start with some calculation based on Lemmas 3 and 4. Let \( \mathbb{P}^n \) be the projective space of lines through the origin in \( \mathbb{C}^{n+1} \). The canonical Hopf complex line bundle on \( \mathbb{P}^3 \) is denoted by \( \lambda_n \).

**Example 2.** Let \( x =: c_1(\lambda_3) \in H^*(\mathbb{P}^3) \) be the class of hyperplane, and let \( \alpha \) be the complement of \( \lambda_3 \) in the trivial bundle \( \mathbb{P}^3 \times \mathbb{C}^4 \). Then
\[
C(\alpha) = 1 + x + x^2 + x^3 \in H^*(\mathbb{P}^3).
\]

We get from Lemma 3 that
\[
(5.1) \quad H^*(\mathbb{P}(\alpha)) = \mathbb{Z}[x, \rho]/\langle x^4, \rho^3 + \rho^2x + \rho x^2 + x^3 \rangle,
\]
where \( \rho = c_1(\lambda_3) \in H^*(\mathbb{P}(\alpha)) \), and from Lemma 8 that
\[
(5.2) \quad C(\mathbb{P}(\alpha)) = (1 + x)^4((1 + \rho)^3 + (1 + \rho)^2x + (1 + \rho)x^2 + x^3). \quad \Box
\]

**Example 3.** Let \( \text{Sym}^2(\alpha) \subset \alpha \otimes \alpha \) be the symmetric product with \( \alpha \) as in Example 1, and let \( M = \mathbb{P}(\text{Sym}^2(\alpha)) \). Since
\[
C(\text{Sym}^2(\alpha)) = 1 + 4x + 10x^2 + 20x^3
\]
we get from Lemma 3 that
\[
(5.3) \quad H^*(M) = \mathbb{Z}[x, y]/\langle x^4, y^6 + 4xy^5 + 10x^2y^4 + 20x^3y^3 \rangle
\]
where \( y = c_1(\lambda_3) \in H^*(M) \). By Lemma 8 we have
\[
(5.4) \quad C(M) = (1 + x)^4[(1 + y)^6 + 4x(1 + y)^5 + 10x^2(1 + y)^4 + 20x^3(1 + y)^3]. \quad \Box
\]

The manifold \( M = \mathbb{P}(\text{Sym}^2(\alpha)) \) in Example 3 is a \( \mathbb{P}^5 \)-bundle on \( \mathbb{P}^3 \), called the variety of conics on \( \mathbb{P}^3 \). The bundle map \( s : \alpha \rightarrow \text{Sym}^2(\alpha) \) by \( v \rightarrow v \otimes v \) over the identity of \( \mathbb{P}^3 \) satisfies \( s(\lambda v) = \lambda^2 s(v) \) for \( \lambda \in \mathbb{C} \), hence induces a smooth embedding of the associated projective bundles
\[
(5.5) \quad i : \mathbb{P}(\alpha) \rightarrow M = \mathbb{P}(\text{Sym}^2(\alpha))
\]
whose image is the degenerate locus of all double lines, called the \( \mathbb{P}^3 \)-parameterized Veronese surface in \( M \). The blow-up \( \bar{M} \) of \( M \) along \( \mathbb{P}(\alpha) \) is the variety of complete conics on \( \mathbb{P}^3 \).

With respect to the presentations for \( H^*(\mathbb{P}(\alpha)) \) and \( H^*(M) \) in (5.1) and (5.3), the induced ring map \( i^* : H^*(M) \rightarrow H^*(\mathbb{P}(\alpha)) \) is clearly given by
\[
i^*(x) = x, \quad i^*(y) = 2\rho.
\]
It follows then from (5.2) and (5.4) that the total Chern class of the normal bundle \( \gamma_{\mathbb{P}(\alpha)} \) is
\[
i) \quad C(\gamma_{\mathbb{P}(\alpha)}) = 1 + (3x + 9\rho) + (30\rho^2 + 20x\rho + 6x^2) + (32\rho^3 + 32x\rho^2 + 16x^2\rho).
\]
Proceeding to determine the Poincare dual $\omega_{\mathcal{P}(\alpha)} \in H^*(M)$ of the cycle $i_\ast[\mathcal{P}(\alpha)]$ we may assume (with respect to the additive basis $\{x^3, xy^2, x^2y, x^3\}$ of $H^6(M)$ by (5.3)) that

$$\omega_{\mathcal{P}(\alpha)} = ay^3 + bxy^2 + cxy + dx^3, \quad a, b, c, d \in \mathbb{Z}.$$  

Since the restriction of $\omega_{\mathcal{P}(\alpha)}$ to the fiber $\mathbb{P}^5$ is equal to $\omega_{\mathcal{P}^2} = 4\omega^3 \in H^6(\mathbb{P}^5)$ (the Poincare dual of the Veronese surface on $\mathbb{P}^5$) we have $a = 4$. It follows then from $i^\ast(\omega_{\mathcal{P}(\alpha)}) = c_4(\gamma_{\mathcal{P}(\alpha)})$ by (3.2) that

$$i^\ast(\omega_{\mathcal{P}(\alpha)}) = 32\rho^3 + 4bx^2\rho + 2bx^2\rho + dx^3$$

$$= (32 - d)\rho^3 + (4b - d)x^2\rho + (2c - d)x^2\rho = 32\rho^3 + 32x^2\rho + 16x^2\rho,$$

where the second equation is obtained from the relation $x^3 = -(\rho^2 + x^2\rho^2 + x^2\rho)$ on $\mathbb{P}(\alpha)$ by (5.1). Coefficients comparison in the third equality in $H^6(\mathbb{P}(\alpha))$ then yields that $d = 0$, $b = c = 8$. Consequently

ii) $\omega_{\mathcal{P}(\alpha)} = 4y^3 + 8xy^2 + 8x^2y.$

Let $E \subset \widetilde{M}$ be the exceptional divisor and set $z = \omega_E \in H^2(\widetilde{M})$. According to Theorem 1 we get from i) and ii) that

**Theorem 3.** The integral cohomology of the variety $\widetilde{M}$ of complete conics on $\mathbb{P}^3$ is given by

$$H^*(\widetilde{M}) = \frac{\langle 2[x, y] \rangle}{(z, y^3 + 4xy^2 + 10x^2y + 20x^3y)} \oplus \frac{\langle 2[x, \rho] \rangle}{(z, \rho^3 + \rho^2x + \rho x^2 + x^3)} \{z, z^2\}$$

that is subject to the relations

i) $4y^3 + 8xy^2 + 8x^2y = (30\rho^2 + 20\rho x + 6x^2)z - (3x + 9\rho)z^2 + z^3.$

ii) $yz = 2\rho z.$ \[\square\]

Over rationals the monomials $\rho^r z$ in the second summand of (5.6) agrees $\frac{1}{2}y^r z$ by the relation ii). Therefore, we get from Theorem 3 that

**Corollary 1.** Let $\widetilde{M}$ be the variety of complete conics on $\mathbb{P}^3$. Then

$$H^*(\widetilde{M}, \mathbb{Q}) = \mathbb{Q}[x, y, z]/\langle x^4, y^3, g_3, g_4, g_6 \rangle$$

where

$$g_3 = 2z^3 - (6x + 9y)z^2 + (15y^2 + 20xy + 12x^2)z - 8y^3 - 16xy^2 - 16x^2y;$$

$$g_4 = (y^3 + 2xy^2 + 4x^2y + 8x^3)z;$$

$$g_6 = y^6 + 4xy^5 + 10x^2y^4 + 20x^3y^3.\square$$

6 The variety of complete quadrics

Consider the map $s : \mathbb{C}^4 \times \mathbb{C}^4 \to Sym^2(\mathbb{C}^4) \subset \mathbb{C}^4 \otimes \mathbb{C}^4$ defined by $s(u, v) = u \otimes v$. Since $s(\lambda u, v) = s(u, \lambda v) = \lambda s(u, v)$ it induces a smooth map on the quotients

$$\varphi : \mathbb{P}^3 \times \mathbb{P}^3 \to \mathbb{P}^9.$$
which restricts to an embedding \( \varphi | \Delta : \mathbb{P}^3 \to \mathbb{P}^9 \) along the diagonal \( \Delta = \mathbb{P}^3 \subset \mathbb{P}^3 \times \mathbb{P}^3 \), and is 2 to 1 on the complement \( \mathbb{P}^3 \times \mathbb{P}^3 \setminus \Delta \). We set \( X_1 = \text{Im} \varphi | \Delta, \ X_2 = \text{Im} \varphi \subset \mathbb{P}^9 \).

Geometrically, \( \mathbb{P}^9 \) is the space of quadrics on \( \mathbb{P}^3 \), \( \varphi(l_1,l_2) = L_1 \cup L_2 \) with \( L_i \) the hyperplane perpendicular to \( l_i \), and \( X_r \) is the degenerate locus of the quadrics of rank \( \leq r \), \( r = 1,2 \). Let \( \mathbb{P}^9 \) be the blow-up of \( \mathbb{P}^9 \) along \( X_1 \) with exceptional divisor \( E \), and let \( X \subset \mathbb{P}^9 \) be the strict transformation of \( X_2 \) in \( \mathbb{P}^9 \). The blow-up \( \tilde{N} \) of \( \mathbb{P}^9 \) along \( X \) is the variety of complete quadrics on \( \mathbb{P}^3 \) \([V]\).

In this section we determine the cohomology ring \( H^*(\tilde{N}) \). Firstly, it comes directly from Theorem 1 and 2 that

**Example 4.** Let \( u = c_1(\lambda_9) \in H^*(\mathbb{P}^9) \), \( y = c_1(\lambda_3) \in H^*(\mathbb{P}^3) \), and put \( v = -\omega_E \). Then

\[
H^*(\mathbb{P}^9) = \mathbb{Z}[u]/\langle u^{10} \rangle \oplus \mathbb{Z}[y]/\langle y^4 \rangle \{v, v^2, \cdots, v^5\} \text{ subject to three relations:} \\
v^6 + 16v^5y + 110v^4y^2 + 420v^3y^3 + 8u^6 = 0; \\
wv = 2yu; \quad y^4v = 0.
\]

\[
(6.2) \quad Ch(\tau_{\mathbb{P}^9}) = 9 + 10u + 5v + \frac{1}{2}(10u^2 + 16uv + 7v^2) \\
+ \frac{3}{2}(10u^3 + 27u^2v + 24uv^2 + 5v^3) + \cdots \square
\]

Without resorting to the map \( \varphi \) the variety \( X \) admits an interesting description (implicitly indicated by Vainsencher \([V]\) and Concini–Procesi \([CP]\)). Let \( G_{4,2} \) be the Grassmannian of 2–dimensional linear subspaces on \( \mathbb{C}^4 \), and let \( \eta \) be the canonical vector bundle on \( G_{4,2} \). Then

\[
(6.3) \quad X = \mathbb{P}(\text{Sym}^2(\eta)),
\]

To compute the cohomology ring of \( \tilde{N} \) we need to describe the induced ring map \( j^* : H^*(\mathbb{P}^9) \to H^*(X) \) and find the expression of \( C(\gamma_X) \in H^*(X) \). These are done in the next result.

**Lemma 9.** Let \( 1 + c_1 + c_2 \in H^*(G_{4,2}) \) be the total Chern class of \( \eta \). One has the ring presentation

\[
(6.4) \quad H^*(X) = \frac{\mathbb{Z}[c_1,c_2,t]}{(2c_1c_2 - c_1^2 + c_2^2 + 4c_1^2 + t(2c_1^2 + 4c_2) + 2c_1^2)}
\]

in which \( t = c_1(\lambda_{\text{Sym}^2(\eta)}) \). Moreover, with respect to the presentations (6.1) and (6.4) we have

i) the induced ring map \( j^* : H^*(\mathbb{P}^9) \to H^*(X) \) is given by

\[
 j^*(u) = t; \quad j^*(v) = -2(c_1 + t);
\]

ii) the total Chern class of the normal bundle \( \gamma_X \) is

\[
 C(\gamma_X) = 1 - (9c_1 + 3t) + (30c_1^2 + 18c_1t + 3t^2 - 4c_2) \\
- (32c_1^3 + 32c_1^2t + 12c_1t^2 + 2t^3);
\]

iii) the Poincare dual \( \omega_X \) of the cycle class \( j_*[X] \) on \( \mathbb{P}^9 \) is

11
\[ \omega_X = 10u^3 + 22u^2v + 16uv^2 + 4v^3 \]

**Proof.** The presentation (6.4) comes directly from Lemma 8, and

\[ H^*(G_{4,2}) = \frac{2 c_1 c_2}{(2c_1 c_2 - c_1^2 - c_2^2)}; \]
\[ C(Sym^2(\eta)) = 1 + 3c_1 + (2c_1^2 + 4c_2) + 2c_1^3 \in H^6(G_{4,2}). \]

The proof for i) is somewhat more delicate. We note that the embedding \( j : X \rightarrow E \) fits in the commutative diagram (e.g. [V])

\[
\begin{array}{ccc}
\mathbb{P}(\alpha) & \xrightarrow{i} & \mathbb{P}(Sym^2(\alpha)) \\
\downarrow g_X & & \downarrow \quad \downarrow g_E \\
X & \xrightarrow{j} & \tilde{E}^0
\end{array}
\]

where \( i \) is as that in (5.2), \( g_E \) is the diffeomorphism onto the exceptional divisor \( E \subset \tilde{E}^0 \), and \( g_X \) is the embedding corresponding to the identification \( \text{Im} g_E \circ i = \text{Im} j \cap E \) in \( \tilde{E}^0 \). Item i) will be deduced in view of the diagram (6.5).

Firstly, with respect to the explicit presentations of the rings \( H^*(\tilde{E}^0) \) and \( H^*(\mathbb{P}(Sym^2(\alpha))) \) in (6.1) and (5.3) by explicit generators and relations we apply the method illustrated in [FD] to get

\[ (6.6) \quad g_E^*(u) = 2x; g_E^*(v) = -2x + y. \]

Recall next that with respect to the presentations of \( H^*(\mathbb{P}(\alpha)) \) and \( H^*(M) \) in (5.1) and (5.3), the ring map \( i^* \) satisfies

\[ (6.7) \quad i^*(x) = x, \quad i^*(y) = 2 \rho \]

Now, by ii) of Lemma 2 and (6.6) the normal bundle \( \beta \) of the embedding \( g_E \) is characterized by \( c_1(\beta) = 2x - y. \) It follows then from (6.5) and (6.7) that

\[ (6.8) \quad \text{the normal bundle of } \mathbb{P}(\alpha) \subset X \text{ is } \zeta = i^* \beta \text{ with } c_1(\zeta) = 2x - 2 \rho. \]

It implies that with respect to the presentations of \( H^*(X) \) in (6.2) and \( H^*(\mathbb{P}(\alpha)) \) in (6.1) the induced ring map \( g_X^* \) is given by

\[ (6.9) \quad g_X^*(c_1) = -(x + \rho); \quad g_X^*(c_2) = x \rho; \quad g_X^*(t) = 2x. \]

We obtain i) from (6.6), (6.7), (6.9), and \( g_E^* \circ i^* = j^* \circ g_X^* \) by (6.5).

For ii) we have by Lemma 8 that

\[
\begin{align*}
\text{Ch}(\tau_X) &= \pi_0^* \text{Ch}(\tau_{G_{4,2}}) + e^t \cdot \pi_0^* \text{Ch}(\text{Sym}^2 \eta) - 1 \\
&= 6 + (-c_1 + 3t) + \frac{1}{2}(7c_1^2 + 3t^2 - 8c_2 + 6c_1t) \\
&\quad + \frac{1}{3}(c_1^3 + 15c_1^2t - 24c_2t + 9c_1t^2 + 3t^3) + \cdots.
\end{align*}
\]

It follows then from the relation \( \gamma_X = j^* \tau_{\tilde{E}^0} - \tau_X \) that

\[
\begin{align*}
\text{Ch}(\gamma_X) &= j^* \text{Ch}(\tau_{\tilde{E}^0}) - \text{Ch}(\tau_X) \\
&= 3 - (9c_1 + 3t) + \frac{1}{2}(3t^2 + 18c_1t + 21c_1^2 + 8c_2) \\
&\quad + \frac{1}{3}(9t^3 + 9c_1t^2 - 39c_1^2t - 39c_1^3 + 24c_2t) + \cdots.
\end{align*}
\]
This implies that (e.g. Example 1)
\begin{align*}
c_1(\gamma_X) &= -(9c_1 + 3t), \\
c_2(\gamma_X) &= 30c_1^2 + 18ct + 3t^2 - 4c_2, \\
c_3(\gamma_X) &= -(32c_1^3 + 32c_1^2t + 12ct^2 + 2t^3).
\end{align*}

Finally consider iii). Assume with respect to the basis \{w^3, uy^2, yv^2, v^3\} of 
\(H^6(\overline{\mathbb{P}^9})\) by (6.1) that \(\omega_X = aw^3 + by^3 + cy^2v + dv^3, a, b, c, d \in \mathbb{Z}\). From
\[
\omega_X = c_3(\gamma_X) = -(32c_1^3 + 32c_1^2t + 12ct^2 + 2t^3)
\]
we find that \(a = 10, b = 88, c = 32, d = 4\). Item iii) is verified by notifying further the relations 
\[4y^2v = u^2v, 2yv = uv\] on \(H^*(\overline{\mathbb{P}^9})\) by (6.1).□

Let \(\overline{E} \subset \overline{N}\) be the exceptional divisor corresponding to \(X \subset \overline{\mathbb{P}^9}, w = \omega_\overline{E} \in H^2(\overline{N})\). According to Theorem 1 we get from (6.1) and Lemma 9 that

**Theorem 4.** The integral cohomology of the variety \(\overline{N}\) of complete quadrics on \(\mathbb{P}^3\) is given by
\[
(6.10) \ H^*(\overline{N}) = \mathbb{Z}[u]/\langle u^{10} \rangle \oplus \mathbb{Z}[y]/\langle y^4 \rangle \{v, v^2, \ldots, v^5\} \\
\quad \quad \quad \oplus \mathbb{Z}[c_1, c_2, t]/\langle (2c_1c_2 - c_1^2 - c_2^2 + 3c_1t + 18c_2 + 2c_1t^2 + 24c_2t + 2t^3) w, w^2 \rangle
\]
that is subject to the relations
\begin{enumerate}
  \item \(v^6 + 16v^5y + 110v^4y^2 + 420v^3y^3 + 8u^6 = 0;\)
  \item \(uv = 2yv;\)
  \item \(uw = tw, vw = -2(c_1 + t)w;\)
  \item \(10u^3 + 22u^2v + 16uv^2 + 4v^3 = (30c_1^2 + 18c_1t + 3t^2 - 4c_2)w + (9c_1 + 3t)w^2 + w^3. □\)
\end{enumerate}

**Corollary 2.** Let \(\overline{N}\) be the variety of complete quadrics on \(\mathbb{P}^3\). Then
\[
(6.11) \ H^*(\overline{N}; \mathbb{Q}) = \mathbb{Q}[u, v, w]/\langle g_{4,1}, g_{4,2}, g_{5,1}, g_{5,2}, g_6 \rangle,
\]
where, setting \(h = 3u + 2v - w,\)
\begin{align*}
g_{4,1} &= -8u^4 - 14u^3v - 9u^2v^2 - 2uv^3 + 2(2u + v)^3h - 3(2u + v)^2h^2 \\
&\quad + 2(2u + v)h^3 \\
g_{4,2} &= 8u^4 + 4u^3v - 6u^2v^2 - 7uv^3 - 2v^4 - (16u^3 + 14u^2v - v^3 + 2uv^2)h \\
&\quad + 6(2u^2 + uv)h^2 - 4uh^3 \\
g_{5,1} &= (2w - 4u - 3v)h^4; \quad g_{5,2} = u^4v; \\
g_6 &= 16u^6 + 105u^5v^3 + 55u^4v^4 + 16uv^5 + 2v^6. □
\end{align*}

**Proof.** With rational coefficients we have
\[
y^3v = \frac{1}{2} u' v
\]
by the relation ii);
\[ t^r w = u^r w, \ c_1^r w = (-1)^r (\frac{1}{r} v + u)^r w \]
by the relations in iii), and
\[
\begin{align*}
    c_2 w &= \frac{1}{4} ((30c_1^2 + 18c_1 t + 3t^2) w + (9c_1 + 3t) w^2 + w^3) \\
    &\quad - (10u^3 + 22u^2 v + 16uv^2 + 4v^3)) \\
    c_2^2 w &= 9twc_1 c_2 + w^3 c_2 + 3tw^2 c_2 + 2t^2 wc_1 c_2 + 4wc_1^2 c_2 + 9w^2 c_1 c_2
\end{align*}
\]
by the relation iv). These implies that the ring \( H^*(\overline{N}; \mathbb{Q}) \) is generated multiplicatively by \( u, v, w \). The relations \( g_{4,1}, g_{4,2}, g_{5,1}, g_{5,2}, g_6 \) in (6.11) correspond in order of the relations on \( H^*(\overline{N}) \) by Theorem 4
\[
\begin{align*}
    (2c_1 c_2 - c_1^3) w &= 0; \\
    (t^3 + 3t^2 c_1 + t(2c_1^2 + 4c_2)) w &= 0; \\
    (c_2^2 - c_1^2 c_2) w &= 0; \\
    y^4 v &= 0; \\
    v^6 + 16v^5 y + 110v^4 y^2 + 420v^3 y^3 + 8u^6 &= 0. \square
\end{align*}
\]
From diagram (6.5) and the proof of Theorem 4 we obtain:

**Corollary 3.** The variety \( \overline{M} \) of complete conics on \( \mathbb{P}^3 \) is the strict transformation of the subvariety \( \mathbb{P}(\text{Sym}^2(\alpha)) \subset \mathbb{P}^9 \) in \( \overline{N} \). In particular, the induced cohomology ring map of the embedding \( i^*_N : \overline{M} \to \overline{N} \) is given by
\[
(6.12) \quad i^*_N(u) = 2x, \quad i^*_N(v) = -2x + y, \quad i^*_N(w) = z \text{ (see Corollary 1). } \square
\]

### 7 Application to enumerative geometry

Let \( M \) be an oriented manifold of dimension \( n \) whose rational cohomology ring is presented as a quotient of a polynomial ring
\[
(7.1) \quad H^*(M; \mathbb{Q}) = \mathbb{Q}[x_1, \ldots, x_k]/(g_1, \ldots, g_m), \ g_i \in \mathbb{Q}[x_1, \ldots, x_k],
\]
and let \( S \) be the set of monomials in \( x_1, \ldots, x_k \) with degree \( n \). The function \( f_M : S \to \mathbb{Q} \) defined by \( f_M x = \langle x, [M] \rangle \), \( x \in S \), is called the characteristic of \( M \) with respect to the generators \( x_1, \ldots, x_k \), where \( [M] \in H_n(M) \) is the orientation class and \( \langle \cdot, \cdot \rangle \) is the Kronecker pairing. As being indicated by the notation \( f_M \) can be interpreted as “integration along \( M \)” in De Rham theory. The enumerative problems on \( M \) asks an effective algorithm to evaluate \( f_M \) ([S1, FKM, D]).

Our approach to the enumerative problems posed in section 1 is strongly influenced by the work of Griffiths and Harris [GH] on the five-conic problem. It suggests that the relations \( g_1, \ldots, g_m \) in (5.1) may be used to evaluate \( f_M \) systematically, as illustrated by the forthcoming discussion.

**7.1. The five-conic problem** (see [GH, 749–753]). Recall that the variety \( \overline{M} \) of complete conics admits a fibration \( \overline{M} \to \mathbb{P}^3 \) whose fiber over a point \( l \in \mathbb{P}^3 \)
is the variety $W$ of complete conics on $\mathbb{P}^2$ (the hyperplane perpendicular to $l$). Therefore, setting $x = 0$ in (5.7) yields

**Corollary 4.** Let $W$ be the variety of complete conics on $\mathbb{P}^2$. Then

$$(7.2) \quad H^*(W; \mathbb{Q}) = \mathbb{Q}[y, z]/\langle g_3, g_4, g_6 \rangle$$

where $g_3 = 8y^3 - 15y^2z + 9yz^2 - 2z^3; \quad g_4 = y^3z; \quad g_6 = y^6$. □

Using the relations $g_3, g_4$ in (7.2) one can evaluate all the characteristics \( \int_W y^s z^t \) with $s + t = 5$ in $H^{10}(N; \mathbb{Q}) = \mathbb{Q}$. Starting from the obvious fact \( \int_W y^5 = 1 \) one gets from the relation $g_4$ that

$$\int_W y^4 z = \int_W y^3 z^2 = 0.$$  

Combining these with the relation $g_3$ one obtains:

$$\int_W y^2 z^3 = 4, \quad \int_W y^3 z^2 = 18, \quad \int_W z^5 = 51.$$  

On the other hand for a generic conic $C \subset \mathbb{P}^2$ let $\tilde{V}_C \subset W$ be the strict transformation of the variety $V_C \subset \mathbb{P}^5$ of conics tangent to $C$. Then

$$\tilde{V}_C \sim 6y - 2z$$  

([GH, p.751]).

By the characteristics computed above: \( \int_W (6y - 2z)^5 = 3264 \). This implies that

**Remark 3.** The five-conic problem is of historical importance for being one of the first enumerative problems whose rigorous verification requires nontrivial intersection theory ([GH], [FM]). Interesting and inspiring accounts for its history from Charsles on were given by Kleiman in [K1, p.469–472; K2]. □

The above calculation can be mechanized. Using certain build-in functions of *Mathematica* an algorithm evaluating \( \int_M x, x \in S \), is given below. Let $G$ be a Gröbner basis of the ideal generated by the set $\{g_1, \cdots, g_m\}$ of polynomials (7.1). Take a monomial $x_0 \in S$ with \( \int_M x_0 = 1 \) as a reference.

**Algorithm: Characteristics**

1. Call GroebnerBasis[ , ] to compute $G$;

2. For $x \in S$ call PolynomialReduce[ , ] to compute the residue $h(a)$ of the difference $x - ax_0$ module $G$ with $a$ an indeterminacy;

3. Set $\int_M x =: a_0$ with $a_0$ the solution to equation $h(a) = 0$. □

We may emphasis at this point that in step 2 the residue $h(a)$ obtained is always a linear equation in $a$.

**7.2. The eight-quadric problem.** Consider the variety $\tilde{M}$ of complete conics on $\mathbb{P}^3$ (§5). With respect to the presentation of the ring $H^*(\tilde{M}; \mathbb{Q})$ in Corollary 1, all the numbers $\int_{\tilde{M}} x^r y^s z^t$ with $r + s + t = 8$ generated by Characteristics are tabulated below (with the symbol $\int_{\tilde{M}}$ omitted), where apparently the monomial $x^3 y^5$ can be taken as a reference:
For a line \( l \subset \mathbb{P}^3 \) and a plane \( L \subset \mathbb{P}^3 \) let \( V_l \) and \( V_L \subset M = \mathbb{P}(\text{Sym}^2(\alpha)) \) be, respectively, the variety of conics meeting the line \( l \), and tangent to the plane \( L \). Then we have

\[
V_l \sim 2x + y; \quad V_L \sim 2x + 2y \quad \text{in} \quad H^2(M; \mathbb{Q}).
\]

Moreover, let \( \overline{V}_l \) and \( \overline{V}_L \) be their strict transformations in the blow-up \( \overline{M} \) of \( M \) along \( \mathbb{P}(\alpha) \). Then the calculation in [GH, p.754] implies that

\[
(7.3) \quad \overline{V}_l \sim 2x + y; \quad \overline{V}_L \sim 2x + 2y - z \quad \text{in} \quad H^2(\overline{M}; \mathbb{Q}).
\]

On the other hand for a generic quadric \( S \subset \mathbb{P}^3 \) let \( \overline{V}_S \subset \overline{M} \) be the strict transformation of the variety \( V_S \subset M \) of conics tangent to \( S \). It is well known that (see [F, p.192])

\[
\overline{V}_S \sim 2\overline{V}_l + 2\overline{V}_L = 8x + 6y - 2z.
\]

From characteristics in the table we get

\[
\overline{V}_S^8 = (8x + 6y - 2z)^8 = 4,407,296.
\]

This implies (see discussion in Kleiman [K1, p.474–475]) that

\[\text{Given 8 quadrics in the space } \mathbb{P}^3 \text{ in general position, there are } 4,407,296 \text{ conics tangent to all of them}.\]

**Remark 4.** In the notation \( \mu, \nu, \varrho \) of Schubert [S1, \S20] one has \( \mu = x, \quad \nu = 2x + y, \quad \varrho = 2x + 2y - z \). This transformation allows one to recover all the characteristic numbers \( \mu^r \nu^s \varrho^t, s + s + t = 8 \) given in Schubert’s book [S1; p.95] from the numbers tabulated above. \( \square \)

### 7.3. The nine-quadric problem.

Consider the variety \( \overline{N} \) of complete quadrics (§6). With respect to the presentation of \( H^*(\overline{N}; \mathbb{Q}) \) in Corollary 2, all the numbers \( \int_{\overline{N}} u^r v^s w^t \) with \( r + s + t = 9 \) generated by Characteristics are tabulated below, where apparently \( u^9 \) can be taken as a reference.
For a point \( p \in \mathbb{P}^3 \) (resp. a line \( l \subset \mathbb{P}^3 \); a plane \( L \subset \mathbb{P}^3 \)) let \( W_p \subset \widetilde{N} \) (resp. \( W_l; W_L \subset \widetilde{N} \)) be the strict transformation the variety \( \subset \mathbb{P}^9 \) of the quadrics containing \( p \) (resp. tangent to the line \( l \); tangent to the plane \( L \)). Then \( W_p \sim u \). Moreover, since with respect to the embedding \( i_S : \tilde{M} \rightarrow \tilde{N} \) one has \( W_l \cap \tilde{M} = \tilde{V}_l, W_L \cap \tilde{M} = \tilde{V}_L \), we get from (7.3) and Corollary 3 that

\[
W_l \sim 2u + v; \ W_L \sim 3u + 2v - w \text{ in } H^2(\tilde{N}; \mathbb{Q}).
\]

On the other hand for a generic quadric \( S \subset \mathbb{P}^3 \) let \( W_S \subset \widetilde{N} \) be the strict transformation of the variety \( \subset \mathbb{P}^9 \) of the quadrics tangent to \( S \). It is well known that (see [F, p.192])

\[
W_S \sim 2W_p + 2W_l + 2W_L = 12u + 6v - 2w.
\]

From characteristics in the table we get

\[
W_S^9 = (12u + 6v - 2w)^9 = 666,841,088.
\]

This implies (see discussion in Kleiman [K1, p.474–475]) that

\[
\text{Given 9 quadrics in the space } \mathbb{P}^3 \text{ in general position, there are 666,841,088 quadrics tangent to all of them.}
\]

Remark 5. In the notation \( \mu, \nu, \varphi \) of Schubert [S1, §22] one has \( \mu =: u, \nu =: 2u + v, \varphi =: 3u + 2v - w \). This transformation allows one to recover all the characteristic numbers \( \mu^s \nu^t \varphi^r \), \( s + t + r = 9 \) given in Schubert’s book [S1; p.105] from the numbers tabulated above.

According to Fulton–Kleiman–MacPherson [FKM], the verification of the nine–quadric problem has been obtained independently by van der Waerden (pvt. ms., 1981), De Concini–Procesi [CP] (1982), Vainsencher [V] (1982), and by Laksov (pvt. ms., 1982).

7.4. Endnotes. The two classical varieties concerned in §5 and §6 admit a natural generalization as the variety \( V_{n,r} \) of complete quadric \( r \)–folds in \( n \)–space, whose geometric aspects has been investigated extensively from various point of views during the history (see [F, GH, CP, L, T, V] as well as its references).
particular, a very nice brief survey of the theory of complete quadrics is given by Laksov in [L]. However, concrete presentations for their cohomologies (or Chow rings) remains unknown. The proofs of Theorems 3 and 4 indicate that Theorems 1 and 2 may serve as the general principles ([V]) required to build up the cohomology of $V_{n,r}$ from its geometric formulation.

One of the main difficulties in the classical approaches to an enumerative problem on a variety is to compute the characteristic numbers with respect to a family of subvarieties involved [FKM]. Traditionally, for the variety $V_{n,r}$ of complete quadric $r$-folds in $n$-space $\mathbb{P}^n$, they were evaluated one by one either by some geometric algorithms [CP, V, FM, GH, p.754–756], or by some combinatorial recurrence formulas [T], both of them are inspired by Schubert’s original approaches [S1, S2]. Alternatively, one may draw the following idea from Griffiths and Harris’s approach to the five-conic problem (i.e. 7.1): once the cohomology of $M$ is presented in terms of generators and relations as in (7.1), the evaluation of all characteristic numbers becomes a routine matter.

If $M$ is a flag variety general method to find a presentation of the ring $H^\ast(M)$ in the form of (7.1) can be found in [DZ1, DZ2]. If $M$ is a flag variety associated to the classical groups $G = U(n), SO(n), Sp(n)$, formulae to evaluate the characteristic numbers (with respect to canonical generators on $H^\ast(M)$) can be found in [D].

References

[A] M. Atiyah, K-theory, New York-Amsterdam: W.A. Benjamin, Inc. (1967).

[CP] C. De Concini, C. Procesi, Complete symmetric varieties, Invariant theory, Proc. 1st 1982 Sess. C.I.M.E., Lect. Notes Math. 996, 1-44 (1983).

[D] H. Duan, Some enumerative formulas for flag varieties, Commun. Algebra 29, No. 10, 4395-4419 (2001).

[DZ1] H. Duan, Xuezhi Zhao, The Chow rings of generalized Grassmannians, arXiv: math.AG/0511332 (to appear in Foundations of Computational Mathematics).

[DZ1] H. Duan, Xuezhi Zhao, The integral cohomology of complete flag manifolds, arXiv: math.AT/0801.2444.

[F] W. Fulton, Intersection theory (second edition), Springer 1998.

[FM] W. Fulton, and R. MacPherson, Defining algebraic intersections, Algebr. Geom., Proc., Tromso Symp. 1977, Lect. Notes Math. 687, 1-30 (1978).

[FKM] W. Fulton; S. Kleiman; R. MacPherson, About the enumeration of contacts, Algebraic geometry–open problems, Proc. Conf., Ravello/Italy 1982, Lect. Notes Math. 997, 156-196 (1983).

[G] M. Gromov, Partial Differential Relations, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3, A Series of Modern Surveys in Mathematics, Vol. 9, 1986.

[GH] P. Griffith and J. Harris, Principles of algebraic geometry, Wiley, New York 1978.
[K1] S. Kleiman, Problem 15. Rigorous foundation of the Schubert’s enumerative calculus, Proceedings of Symposia in Pure Math., 28 (1976), 445-482.

[K2] S. Kleiman, Chasles’s enumerative theory of conics: A historical introduction, Studies in algebraic Geometry, MAA Stud. Math. 20, 117-138, 142-143 (1980).

[K3] S. Kleiman, Intersection theory and enumerative geometry: A decade in review. (With the collaboration of A. Thorup on §3), Algebraic geometry, Proc. Summer Res. Inst., Brunswick/Maine 1985, part 2, Proc. Symp. Pure Math. 46, 321-370 (1987).

[L] D. Laksov, Completed quadrics and linear maps, Algebraic geometry, Proc. Summer Res. Inst., Brunswick/Maine 1985, part 2, Proc. Symp. Pure Math. 46, 371-387 (1987).

[LD] F. Li and H. Duan, Homology rigidity of Grassmannians, Vol.29, no.3, (2009), 697-704.

[M] D. McDuff, Examples of simply connected symplectic non-kahler manifolds, Journal of Differential Geometry, 20(1984), 267–277.

[MS] J. Milnor and J. Stasheff, Characteristic classes, Ann. of Math. Studies 76, Princeton Univ. Press, 1975.

[S1] H. Schubert, Kalkül der abzählenden Geometrie, Berlin, Heidelberg, New York: Springer-Verlag (1979).

[S2] H. Schubert, Allgemeine Anzahlfunctionen für Kegelschnitte, Flächen und Räume zweiten Grades in n Dimensionen, Math. Ann. XLV. 153-206 (1894).

[T] A. Thorup, Parameter spaces for quadrics, Pragacz, Piotr (ed.), Parameter spaces: enumerative geometry, algebra and combinatorics. Proceedings of the Banach Center conference, Banach Cent. Publ. 36, 199-216 (1996).

[V] I. Vainsencher, Schubert calculus for complete quadrics, Enumerative geometry and classical algebraic geometry, Prog. Math. 24, 199-235 (1982).