Observational Derivation of Einstein’s “Law of the Constancy of the Velocity of Light in Vacuo"

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Abstract. On the basis of Galilean invariance and the Doppler formula, combined with an observational condition, it is shown that the constancy of the velocity of light in vacuo can be derived, together with time-dilatation and Lorentz contraction. It is not necessary to take the constancy as a postulate.

INTRODUCTION

In this centenary of Einstein’s annus mirabilis, it is proper to reconsider one of his breakthroughs. We shall focus here on his 1905 paper on special relativity [1], in the language of the train-embankment setup he discussed in later works [2].

Motivated by the results of H. A. Lorentz on electrodynamics [3, 4], involving the invariance of the velocity of light, Einstein postulated that the speed of light is the same in all non-accelerated systems of reference. He deduced from this statement, combined with the principle of relativity, the kinematic base of the theory of special relativity.

In particular, Einstein drew attention to the issue of the non-absolutism of time, the effect of time dilatation: a material event lasting a certain time for an observer at rest with respect to it, lasts a longer time for an observer moving with constant speed with respect to it. This aspect explains why cosmic muons can be observed on the surface of the earth, since their rather short decay time is dilated for observers on the earth, so that part of them survive the travel through the earth’s atmosphere.

It is known that non-absolutism of time is behind the constancy of the velocity of the light. But, by itself, this is a difficult notion. In observational terms the constancy of the speed of light is also difficult to understand. We do not enter here any deep discussion about non-absolutism of time, except that we call attention to the fact that atomic clocks also emit light and, consequently, they are not so different from lamps.

It is the purpose of the present paper to offer an alternative, pedagogic derivation of the constancy of the speed of light. We do not make any assumption on the velocity of the light but on numbers of cycles and on frequencies.

In section II we present our setup, in section III we pose our conditions and in section IV we analyze the situation. We end up with concluding that our conditions imply a derivation of the constancy of the speed of light.

THE SETUP OF TRAIN, EMBANKMENT, LAMPS AND OBSERVERS

A train having speed \( v_e > 0 \) passes an embankment. A point in the train is called the origin \( O_{\text{train}} \) and at time \( t = 0 \) this passes the origin of the embankment called \( O_{\text{emb}} \). In the setup there are lamps. The first is at the origin of the embankment \( O_{\text{emb}} \), the second at the origin of the train \( O_{\text{train}} \). There are also two observers. One uses the point \( O_{\text{train}} \) as reference and is called observer-train. Another one takes the embankment as reference and is called observer-embankment. We also consider another point on the embankment at a distance \( X_e > 0 \) in the direction of the movement of the train, at which observer-embankment can measure light.

The lamp at the embankment has, according to observer-embankment, a certain frequency \( f_{\text{lamp}-e}^{\text{obs}} \). The lamp in the train has a different frequency, \( f_{\text{lamp}-t}^{\text{obs}} \), according to observer-train. This frequency is fixed such that, according to observer-train, it has the same frequency as the light he observes from the lamp at the embankment, \( f_{\text{lamp}-t}^{\text{obs}} = f_{\text{lamp}-e}^{\text{obs}} \).
Both observers have a clock.

**CONDITIONS**

1: Galilean invariance

The train moves with speed $v_e > 0$ according to the observer-embankment. We assume that the observer-train observes that the origin of the embankment $O_{emb}$ moves with respect to him with speed $v_t = -v_e$.

2: Applicability of Doppler formula

The lamp of the embankment produces in time-interval $(0, t_e)$ a number of cycles $N_{lamp-emb}^{obs-e}(t_e) = f_{lamp-emb}^{obs-e}$. The number $M_{lamp-emb}^{obs-e}(t_e)$ of cycles reaching the train in this interval is assumed, by observer-embankment, to be related by the classical Doppler formula to the number $N_{lamp-emb}^{obs-e}(t_e)$ emitted at the embankment,

$$M_{lamp-emb}^{obs-e}(t_e) = (1 - \beta_e)N_{lamp-emb}^{obs-e}(t_e),$$

where

$$\beta_e = \frac{v_e}{c_e},$$

with $c_e$ the speed of light according to observer-embankment.

3: Identification of times

Both observers count $t = 0$ for the event where $O_{train}$ passes $O_{emb}$. Let us denote the time of observer-embankment as $t_e$ and the time of observer-train as $t_t$. Later we shall see that these times are indeed different. We relate their times by equating the number of cycles observed by observer-train with the number of cycles presumed by observer-embankment:

$$M_{lamp-emb}^{obs-t}(t_t) = M_{lamp-emb}^{obs-e}(t_e).$$

4: Conservation of information

This is a physical condition: Observer-embankment measures the same frequency from lamp-embankment and lamp-train:

$$f_{lamp-emb}^{obs-e} = f_{lamp-emb}^{obs-t}.$$  

If it is not satisfied, the speed of light will not be constant.

**ANALYSIS OF FREQUENCIES**

The frequency for observer-train

The observer-train counts the number $M_{lamp-emb}^{obs-t}$ in a period he denotes as $(0, t_t)$. Observer-embankment knows about this counting and calls the period $(0, t_e)$. We shall see that it is necessary to assume that for observer-embankment this
counting time is dilated to be
\[ t_e = \gamma t_t, \tag{5} \]
where \( \gamma \) is for now some unknown factor. If it would appear to be equal to unity, there would be no time-dilatation. But we shall show below that it is equal the Lorentz factor \( 1/\sqrt{1 - \beta^2} \).

The frequency of the lamp at the embankment is for observer-train
\[ f_{\text{lamp} - e} = \frac{M_{\text{lamp} - e}(t_t)}{t_t}. \tag{6} \]
This becomes equal to
\[ f_{\text{lamp} - e} = \gamma t_e = \gamma (1 - \beta_t) \frac{N_{\text{lamp} - e}(t_e)}{t_e} = \gamma (1 - \beta_t) f_{\text{lamp} - e}. \tag{7} \]
From the expression for \( \gamma \) that we shall deduce, this will appear to be red-shifted.

In the setup outlined above, the lamp in the train emits light with exactly this frequency, according to observer-train. In accordance with Eq. (5), the time to reach the position \( X_e \) of observer-embankment is according to observer-train
\[ t_{x,e} = \frac{1}{\gamma} t_{x,e}. \tag{8} \]

**Frequency of the lamp in the train according to observer-embankment**

Now we repeat this argument for the lamp in the train, which is observed by observer-embankment. This means that there is an interchange in the role of lamps and clocks. Whereas the lamp in the embankment was going away from this observer, there is a time interval, according to observer-train \((0, t_{x,e})\), in which the train approaches position \( X_e \). Observer-embankment calls this the interval \((0, t_{x,e})\), with \( t_{x,e} = X_e/v_e \).

Consider observer-train’s time interval \((t'_t, t_{x,e})\). In this interval the number of cycles produced by the lamp in the train is
\[ N_{\text{lamp} - e}(\Delta t'_t) = f_{\text{lamp} - e}(\Delta t'_t) = \gamma (1 - \beta_t) \frac{N_{\text{lamp} - e}(\Delta t'_t)}{t'_t}, \tag{9} \]
where the primes indicates that now lamp-train is discussed. According to the Doppler formula, observer-train assumes that the number of cycles received by observer-embankment in the related time-interval is
\[ M_{\text{lamp} - e}(\Delta t'_t) = (1 - \beta_t) N_{\text{lamp} - e}(\Delta t'_t), \tag{10} \]
where
\[ \beta_t = -\frac{v_t}{c_t} \tag{11} \]
involve the speed of light as observed by observer-train. The sign changes with respect to Eq. (1) because now the lamp, located in the train, is approaching \( X_e \).

Observer-embankment counts these pulses in a time interval \( \Delta t'_e = t_{x,e} - t'_t \). In analogy with previous case, we have to assume that observer-train considers this period to last a time
\[ \Delta t'_e = \gamma \Delta t'_e, \tag{12} \]
where also the factor \( \gamma \) is to be determined.

We now have
\[ f_{\text{lamp} - t} = \frac{M_{\text{lamp} - e}(\Delta t'_t)}{\Delta t'_e} = \gamma (1 - \beta_t) \frac{N_{\text{lamp} - e}(\Delta t'_t)}{\Delta t'_t} = \gamma (1 - \beta_t) f_{\text{lamp} - e}. \tag{13} \]
Because in our setup is designed such that $f_{\text{lamp} \rightarrow t}^{\text{obs} \rightarrow t} = f_{\text{lamp} \rightarrow e}^{\text{obs} \rightarrow e}$, it follows from Eq. (7) that

$$f_{\text{lamp} \rightarrow t} = \gamma (1 - \beta_t) \gamma_e (1 - \beta_e) f_{\text{lamp} \rightarrow e}^{\text{obs} \rightarrow e}. \tag{14}$$

According to our condition 4, these frequencies are the same, so we conclude that

$$\gamma (1 - \beta_t) \gamma_e (1 - \beta_e) = 1. \tag{15}$$

### A mirror of the setup

In our setup, the origin of the train $O_{\text{train}}$ had just passed the origin of the embankment $O_{\text{emb}}$ and moved on. In an alternative setup, we might have considered the approach of the train, while observer-embankment performs a measurement at the point $-X_e < 0$. For observer-train, approaching the lamp of the embankment, its frequency would be blue-shifted rather than red-shifted, and the lamp in the train is supposed to be adjusted to this new value.

Repeating all steps one-by-one in the two related time-intervals, we would deduce that now Eq. (15) holds with the signs of $\beta_e$ and $\beta_t$ altered, because of the interchange of approaching and separating. Keeping our previous definitions of $\beta_e$ and $\beta_t$, this means that

$$\gamma_e (1 + \beta_e) \gamma_t (1 + \beta_t) = 1. \tag{16}$$

### Derivation of the constancy of the speed of light

When dividing Eq. (15) and Eq. (16) we obtain

$$\frac{(1 - \beta_t)(1 - \beta_e)}{(1 + \beta_e)(1 + \beta_t)} = 1. \tag{17}$$

This has the solution

$$\beta_e = -\beta_t \quad \text{or} \quad \frac{v_e}{c_e} = -\frac{v_t}{c_t}. \tag{18}$$

Since by our Galilean condition we have assumed that $v_t = -v_e$, we deduce that the speed of light is the same in both systems of reference, even though they move with respect to each other,

$$c_e = c_t. \tag{19}$$

Now going back to (15), we are left with

$$\gamma_e \gamma_t = \frac{1}{1 - \beta_e^2}. \tag{20}$$

This reveals that the classical choice $\gamma_e = \gamma_t = 1$ is excluded, as is the choice $\gamma = 1/\gamma_e$, that one might naively have guessed when comparing (5) and (12). Instead, the solution is to assume that both observers are completely equivalent, so that $\gamma_e = \gamma_t$, which yields

$$\gamma_e = \frac{1}{\sqrt{1 - \beta_e^2}}. \tag{21}$$

This is indeed the Lorentz factor. So given our Condition 4, we did have to assume time-dilatation in order to solve the problem.

The distance between the origin $O_{\text{emb}}$ and the position $X_e$ of observer-embankment is for observer-train

$$X_t = -v_t t_{X,t} = v_e t_{X,e} = \frac{1}{\gamma_e} X_e \tag{22}$$

where we used (8). So also the Lorentz contraction is derived in this analysis.
DISCUSSION

We have demanded a few observational conditions. The most important one is the condition on conservation of information: the frequency directly observed from a lamp is the same as when this light is absorbed and re-emitted in a moving reference frame. From this we were able to derive the constancy of the speed of light in vacuo, and, with it, the time dilatation and the Lorentz contraction. This provides conditions under which the theory of special relativity is valid.

It is interesting to see that in this approach the observers are to a large extent classical: they adopt the Galilean principle and the Doppler formula applied to the number of cycles of the light. It is only when their times are compared, that a time-dilatation has to be taken into account. Indeed, without it our conditions have been shown to lead to contradictions.

The time-dilatation needs not be known to the observers themselves, they need not communicate with each other about their findings. It is only known to “us”, as external observers, trying to unify the observations of these two individuals.

The physical presence of observers can as usual be replaced by e.g. automated photo camera’s, as is often done in practice. But somewhere down the line someone is needed to read and reconcile these observations. This reconciliation is accomplished with Einstein’s theory of special relativity, which holds provided the conditions for our analysis are valid.

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REFERENCES

1. A. Einstein, On the electrodynamics of moving bodies, Ann. der Phys. 17, 891, (1905).
2. A. Einstein, Relativity, (Three Rivers Press, New York, 1961).
3. H. A. Lorentz, Versuch einer Theorie der elektrischen und optischen Erscheinungen in bewegten Körpeln (Brill, Leiden 1895).
4. H. Lorentz, A. Einstein, H. Minkowski, and H. Weyl, The Principle of Relativity (Methuen, London 1923).