Bethe ansatz equations for quantum chains combining different representations of $su(3)$

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Abstract
The general expression for the local matrix of a quantum chain $L(\theta)$ with the site space in any representation of $su(3)$ is obtained. This is made by generalizing $L(\theta)$ from the fundamental representation and imposing the fulfilment of the Yang-Baxter equation. With these operators and using a generalization of the nested Bethe ansatz, the Bethe equations for a multistate quantum chain combining two arbitrary representations of $su(3)$ are obtained.
In the study of integrable quantum systems, chains combining two kinds of spin have aroused great interest lately. The work was pioneered for $su(2)$ algebra by H. de Vega and F. Woynarovich [1]. In this paper a chain mixing sites with spin $1/2$ and $1$ and periodic boundary conditions was studied, and the generalization to a chain combining spin $1/2$ and any other $s$ was suggested. Several subsequent works have been published in which the thermodynamic properties of these systems are studied ([2]–[5]).

In this paper, we study an alternating chain the site states of which are a mixture of any two representations of $su(3)$. We made an initial approach to this problem in a previous paper [3], where we solved an alternating chain mixture of the two fundamental representations of $su(3)$ and presented a method, a modification of the nested Bethe ansatz (MNBA), needed to find the Bethe equation (BE) solutions of the problem. The process was as follows: in first place we sought the general form of the local operator $L(\theta)$ with its auxiliary space in the fundamental representation ([7]–[10]) and the site space in any representation of $su(3)$. This is done by departing from a general form inspired by the local operator $L(\theta)$ with the auxiliary and site space in the fundamental representation of $su(3)$ and by making that operator the YBE solution. The operator so obtained has several free parameters that are coming from the symmetries of the YBE. With this operator we can form integrable homogeneous chains and find the ansatz equations with usual nested Bethe ansatz (NBA) ([11],[12]). In a second step, alternating chains are formed by mixing any two representations of $su(3)$ and the solutions are formed by applying MNBA ref. [3]. From the results so obtained we can conjecture the BE for chains based on the algebra $su(n)$.

We denote a representation by the indices of its associated Dynkin diagram ($m_1, m_2$), where $m_1$ and $m_2$ correspond to the $\{3\}$ and $\{\bar{3}\}$ representations respectively. In the figures a continuous line was used for the fundamental representation $(1,0)$ and a wavy line for any other representation. Thus, the operators $L(\theta)$ are denoted as indicated in figure 1 and in order to simplify the writing of the formulae, we will adopt the following identifications: $L(\theta) \equiv L^{(1,0)(1,0)}(\theta)$ and $L'(\theta) \equiv L^{(1,0)(m_1,m_2)}(\theta)$.

\begin{align*}
L^{(1,0)(1,0)}(\theta) & = & L^{(1,0)(1,0)}(\theta) \\
L^{(1,0)(m_1,m_2)}(\theta) & = & L^{(1,0)(m_1,m_2)}(\theta)
\end{align*}

Figure. 1
The operator $L(\theta)$ can be written \[6\]

\[
L(\theta) = \begin{pmatrix}
\frac{1}{2}(\lambda^3 q^{-N^\alpha} - \lambda^{-3} q^{N^\alpha}) & \lambda \frac{(q^{-1} - q)}{2} f_1 & \lambda^{-1} \frac{(q^{-1} - q)}{2} [f_2, f_1] \\
\lambda^{-1} \frac{(q^{-1} - q)}{2} e_1 & \frac{1}{2}(\lambda^3 q^{-N^{\beta}} - \lambda^{-3} q^{N^{\beta}}) & \lambda \frac{(q^{-1} - q)}{2} f_2 \\
\lambda \frac{(q^{-1} - q)}{2} [e_1, e_2] & \lambda^{-1} \frac{(q^{-1} - q)}{2} e_2 & \frac{1}{2}(\lambda^3 q^{-N^{\gamma}} - \lambda^{-3} q^{N^{\gamma}})
\end{pmatrix},
\]

(1)

where the parameters $\lambda$ and $q$ have be taken as the functions of $\theta$ and $\gamma$

\[
\lambda = e^{\frac{\theta}{2}}, \quad q = e^{-\gamma}
\]

(2)

and the $N$ matrices are

\[
N^{\alpha} = \frac{2}{3} h_1 + \frac{1}{3} h_2 + \frac{1}{3} I,
\]

(3a)

\[
N^{\beta} = -\frac{1}{3} h_1 + \frac{1}{3} h_2 + \frac{1}{3} I,
\]

(3b)

\[
N^{\gamma} = -\frac{1}{3} h_1 - \frac{2}{3} h_2 + \frac{1}{3} I,
\]

(3c)

where $\{e_i, f_i, q^{\pm h_i}\}, i = 1, 2$ are the Cartan generators of the deformed algebra $U_q(sl(3))$.

To obtain the operators $L'(\lambda)$ with the new parameters given in (2), we take (1) as a basis and write

\[
L'(\lambda) = \begin{pmatrix}
\frac{1}{2}(\lambda^3 q^{-N^\alpha} - \lambda^{-3} q^{N^\alpha}) & \lambda F_1 & \lambda^{-1} F_3 \\
\lambda^{-1} E_1 & \frac{1}{2}(\lambda^3 q^{-N^{\beta}} - \lambda^{-3} q^{N^{\beta}}) & \lambda F_2 \\
\lambda E_3 & \lambda^{-1} E_2 & \frac{1}{2}(\lambda^3 q^{-N^{\gamma}} - \lambda^{-3} q^{N^{\gamma}})
\end{pmatrix},
\]

(4)

where the operators $\{E_i, F_i\}, i = 1, 3$ are unknown and will be determined by imposing the YBE

\[
R(\lambda/\mu)[L'(\lambda) \otimes L'(\mu)] = [L'(\mu) \otimes L'(\lambda)]R(\lambda/\mu)
\]

(5)
as shown in figure 2. The $R_{c,d}^{b}(\theta) \equiv [L_{a,b}(\theta)]_{c,d}$ is given \[10\]

\[
R(\lambda, \mu) = \begin{pmatrix}
    a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & d & 0 & b & 0 & 0 & 0 & 0 \\
    0 & 0 & c & 0 & 0 & b & 0 & 0 \\
    0 & b & 0 & c & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & d & 0 & b & 0 \\
    0 & 0 & b & 0 & 0 & 0 & d & 0 \\
    0 & 0 & 0 & 0 & b & 0 & c & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & a
\end{pmatrix},
\]

with

\[
a(\lambda, \mu) = \frac{1}{2}(\lambda^3 \mu^{-3} q^{-1} - \lambda^{-3} \mu^3 q), \quad (7a)
\]

\[
b(\lambda, \mu) = \frac{1}{2}(\lambda^3 \mu^{-3} - \lambda^{-3} \mu^3), \quad (7b)
\]

\[
c(\lambda, \mu) = \frac{1}{2}(q^{-1} - q)\lambda \mu^{-1}, \quad (7c)
\]

\[
d(\lambda, \mu) = \frac{1}{2}(q^{-1} - q)\lambda^{-1} \mu. \quad (7d)
\]

The relations obtained are

\[
E_1 q^{N_\alpha} = q^{-1} q^{N_\alpha} E_1, \quad (8a)
\]

\[
E_1 q^{N_\beta} = q q^{N_\beta} E_1, \quad (8b)
\]

\[
F_1 q^{N_\alpha} = q q^{N_\alpha} F_1, \quad (8c)
\]

\[
F_1 q^{N_\beta} = q^{-1} q^{N_\beta} F_1, \quad (8d)
\]

\[
E_2 q^{N_\alpha} = q q^{N_\alpha} E_2, \quad (8e)
\]

\[
E_2 q^{N_\beta} = q^{-1} q^{N_\beta} E_2, \quad (8f)
\]

\[
F_2 q^{N_\alpha} = q^{-1} q^{N_\alpha} F_2, \quad (8g)
\]

\[
F_2 q^{N_\beta} = q q^{N_\beta} F_2, \quad (8h)
\]

\[
[E_1, F_1] = (q^{-1} - q) \left(q^{N_\beta} - q^{N_\alpha} - q^{N_\alpha} - q^{N_\beta}\right), \quad (8i)
\]

\[
[E_2, F_2] = (q^{-1} - q) \left(q^{N_\gamma} - q^{N_\beta} - q^{N_\beta} - q^{N_\gamma}\right), \quad (8j)
\]

\[
E_3 = \frac{1}{(q^{-1} - q)} q^{-N_\beta} [E_1, E_2], \quad (8k)
\]

\[
F_3 = \frac{1}{(q^{-1} - q)} q^{N_\beta} [F_2, F_1], \quad (8l)
\]
and besides, the modified Serre relations

\begin{align*}
q^{-1}E_1E_1E_2 - (q + q^{-1})E_1E_2E_1 + qE_2E_1E_1 &= 0, \\
qE_2E_2E_1 - (q + q^{-1})E_2E_1E_2 + q^{-1}E_1E_2E_2 &= 0, \\
q^{-1}F_1F_1F_2 - (q + q^{-1})F_1F_2F_1 + qF_2F_1F_1 &= 0, \\
qF_2F_2F_1 - (q + q^{-1})F_2F_1F_2 + q^{-1}F_1F_2F_2 &= 0,
\end{align*}

should be verified.

It must be noted that that the relations (8) are the usual ones for the quantum group \( U_q(sl(3)) \) while the relations (9) are not the usual ones for the said group and because of this the EYB is not verified if the generators \( e_i \) and \( f_i \), pertaining to deformed algebra, are taken as \( E_i \) and \( F_i \). This induces us to take

\begin{align*}
F_i &= \frac{1}{2}(q^{-1} - q)Z_if_i, \\
E_i &= \frac{1}{2}(q^{-1} - q)e_iZ_i^{-1}, \quad i = 1, 2
\end{align*}

(10a)

(10b)

where \( e_i \) and \( f_i \), \( i = 1, 2 \) are the generators of \( U_q(sl(3)) \) in the representation \((m_1, m_2)\) and \( Z_i \) are two diagonal operators that were obtained by imposing the verification of the relations (8) and (9). In this way, one obtains the general form of these operators given by

\begin{align*}
Z_1 &= q^{a_1h_1 - \frac{1}{3}h_2 + a_3I} \\
Z_2 &= q^{\frac{1}{3}h_1 + (a_1 + \frac{1}{3})h_2 + b_3I}
\end{align*}

(11a)

(11b)

where the operators \( h_i, i = 1, 2 \) are the diagonal elements of the algebra \( sl(3) \), and \( a_1, a_3 \) and \( b_3 \) are free parameters that are associated with the transformations that leave the EYB invariant.

The knowledge of the operator \( L' \) permits us to find the ansatz of any multistate chain that mixes various representations. For this purpose, the monodromy operator corresponding to the chain to be solved is built; as an example we will use the one which alternates the representations \((1, 0)\) and \((m_1, m_2)\)

\begin{align*}
T_{a,b}^{(alt)}(\theta) &= L_{a,a_1}^{(1)}(\theta)L_{a_1,a_2}^{(2)}(\theta) \ldots L_{a_{2N-2},a_{2N-1}}^{(2N-1)}(\theta)L_{a_{2N-1},b}^{(2N)}(\theta),
\end{align*}

(12)
that can be represented graphically as shown in figure 3.

Using the MNBA ([6], [13]) the ansatz for the chain can be found. To particularize to each case it is necessary to know the action of the diagonal operators $T_{i,i}^{\text{alt}}$ on the vacuum state if the chain is homogeneous or on the vacuum subspace if it is an alternating chain [8]. In both cases, it always characterized by the highest weight of the representation. Thus, for the representation $(m_1, m_2)$ it will be

$$\Lambda_h = \frac{2m_1 + m_2}{3} \alpha_1 + \frac{m_1 + 2m_2}{3} \alpha_2,$$

(13)

where $\alpha_1$ and $\alpha_2$ are the simple roots of $su(3)$.

Through (4), (3a) and (13), together with the commutation rules of $su(3)$ it was possible to know the action of $L'_{i,i}(\theta)$ on the highest weight, obtaining

$$L'_{1,1}(\theta)\Lambda_h = \sinh\left(\frac{3}{2} \theta + \left(\frac{2}{3} m_1 + \frac{1}{3} m_2 + \frac{1}{3}\right) \gamma\right) \Lambda_h,$$

(14a)

$$L'_{2,2}(\theta)\Lambda_h = \sinh\left(\frac{3}{2} \theta + \left(-\frac{1}{3} m_1 + \frac{1}{3} m_2 + \frac{1}{3}\right) \gamma\right) \Lambda_h,$$

(14b)

$$L'_{3,3}(\theta)\Lambda_h = \sinh\left(\frac{3}{2} \theta + \left(-\frac{1}{3} m_1 - \frac{2}{3} m_2 + \frac{1}{3}\right) \gamma\right) \Lambda_h.$$

(14c)

It is also applicable for obtaining the action of the operators $L_{i,i}(\theta)$ on the corresponding highest weight state taking $m_1 = 1$ and $m_2 = 0$. In this way, in the alternate chain that mixes $N$ representations $(1, 0)$ with $N$ representations $(m_1, m_2)$, the BE are given by

$$[g(\mu_k)]^N [\tilde{g}_1(\mu_k)]^N = \prod_{j \neq k}^{r} \frac{g(\mu_k - \mu_j)}{g(\mu_j - \mu_k)} \prod_{i=1}^{s} g(\lambda_i - \mu_k),$$

(15a)

$$[\tilde{g}_2(\lambda_k)]^N = \prod_{j=1}^{r} g(\lambda_k - \mu_j) \prod_{i \neq k}^{s} \frac{g(\lambda_i - \lambda_k)}{g(\lambda_k - \lambda_i)},$$

(15b)
where $\mu_i, i = 1, \ldots, r$ and $\lambda_j, j = 1, \ldots, s$ the roots of the ansatz, the function $g$ is,

$$g(\theta) = \frac{\sinh\left(\frac{3}{2} \theta + \gamma\right)}{\sinh\left(\frac{3}{2} \theta\right)}$$  \hfill (16)

and $\tilde{g}_1(\theta)$ and $\tilde{g}_2(\theta)$ are obtained from (14) giving

$$\tilde{g}_1(\theta) = \frac{\sinh\left(\frac{3}{2} \theta + \left(\frac{2}{3} m_1 + \frac{1}{3} m_2 + \frac{1}{3}\right)\gamma\right)}{\sinh\left(\frac{3}{2} \theta + \left(-\frac{1}{3} m_1 + \frac{2}{3} m_2 + \frac{1}{3}\right)\gamma\right)},$$  \hfill (17a)

$$\tilde{g}_2(\theta) = \frac{\sinh\left(\frac{3}{2} \theta + \left(-\frac{1}{3} m_1 - \frac{2}{3} m_2 + \frac{1}{3}\right)\gamma\right)}{\sinh\left(\frac{3}{2} \theta + \left(-\frac{1}{3} m_1 + \frac{2}{3} m_2 + \frac{1}{3}\right)\gamma\right)}.$$  \hfill (17b)

The procedure can be generalized to chains that mix non-fundamental representations, irrespective of the number of sites and their distribution in the representations. For this purpose, it is necessary to build the monodromy matrix following an analogous process to use in (12). If we use a dashed line for the representation $(m'_1, m'_2)$, the monodromy matrix $T^{(\text{gen})}(\theta)$ can be represented graphically as shown in figure 4.

$$T^{(\text{gen})}_{a,b}(\theta) = \begin{array}{cccc}
\theta & \theta & \theta & \theta \\
\theta & \theta & \theta & \theta \\
\theta & \theta & \theta & \theta \\
\theta & \theta & \theta & \theta \\
\end{array}$$

Figure 4

The eigenvalues for the local operators on the highest weight states, in straightforward notation are

$$\tilde{l}_{1,1}(\theta) = \sinh\left(\frac{3}{2} \theta + \left(\frac{2}{3} m_1 + \frac{1}{3} m_2 + \frac{1}{3}\right)\gamma\right)$$  \hfill (18a)

$$\tilde{l}_{2,2}(\theta) = \sinh\left(\frac{3}{2} \theta + \left(-\frac{1}{3} m_1 + \frac{1}{3} m_2 + \frac{1}{3}\right)\gamma\right)$$  \hfill (18b)

$$\tilde{l}_{3,3}(\theta) = \sinh\left(\frac{3}{2} \theta + \left(-\frac{1}{3} m_1 - \frac{2}{3} m_2 + \frac{1}{3}\right)\gamma\right)$$  \hfill (18c)

$$\tilde{l}_{1,1}(\theta) = \sinh\left(\frac{3}{2} \theta + \left(\frac{2}{3} m'_1 + \frac{1}{3} m'_2 + \frac{1}{3}\right)\gamma\right)$$  \hfill (18d)

$$\tilde{l}_{2,2}(\theta) = \sinh\left(\frac{3}{2} \theta + \left(-\frac{1}{3} m'_1 + \frac{1}{3} m'_2 + \frac{1}{3}\right)\gamma\right)$$  \hfill (18e)

$$\tilde{l}_{3,3}(\theta) = \sinh\left(\frac{3}{2} \theta + \left(-\frac{1}{3} m'_1 - \frac{2}{3} m'_2 + \frac{1}{3}\right)\gamma\right)$$  \hfill (18f)
By calling the number of sites in the representations \((m_1, m_2)\) and \((m'_1, m'_2)\) \(N_1\) and \(N_2\) respectively, we found the eigenvalue of the transfer matrix for this general chain

\[
\Delta(\theta) = [\bar{t}_{1,1}(\theta)]^{N_1}[\tilde{t}_{1,1}(\theta)]^{N_2} \prod_{j=1}^{r} g(\mu_j - \theta) + \prod_{j=1}^{r} g(\theta - \mu_j) \left[ [\bar{t}_{2,2}(\theta)]^{N_1}[\tilde{t}_{2,2}(\theta)]^{N_2} \prod_{i=1}^{s} g(\lambda_i - \theta) + \right. \\
\left. [\bar{t}_{3,3}(\theta)]^{N_1}[\tilde{t}_{3,3}(\theta)]^{N_2} \prod_{l=1}^{r} \frac{1}{g(\theta - \mu_l)} \prod_{i=1}^{s} g(\theta - \lambda_i) \right] 
\]

(19)

and the BE are

\[
[\bar{g}_1(\mu_k)]^{N_1}[\tilde{g}_1(\mu_k)]^{N_2} = \prod_{j=1}^{r} \frac{g(\mu_k - \mu_j)}{g(\mu_j - \mu_k)} \prod_{i=1}^{s} g(\lambda_i - \mu_k) 
\]

(20a)

\[
[\bar{g}_2(\lambda_k)]^{N_1}[\tilde{g}_2(\lambda_k)]^{N_2} = \prod_{j=1}^{r} g(\lambda_k - \mu_j) \prod_{i=1}^{s} \frac{g(\lambda_i - \lambda_k)}{g(\lambda_k - \lambda_i)} 
\]

(20b)

where \(\bar{g}_1\) and \(\bar{g}_2\) are given in (15a, b) and \(\tilde{g}_1\) and \(\tilde{g}_2\) are the same as the previous ones but \((m_1, m_2)\) is replaced by \((m'_1, m'_2)\).

In the light of this, the generalization for the case of mixed chains with more than two different representations seems simple, although the physical models that they represent will be less local and the interaction more complex.

In a non-homogeneous chain combining different representations of \(su(n)\), each representation introduces \((n - 1)\) functions (that we call source functions). Each solution will have \((n - 1)\) sets of equations (the same number of dots in its Dynkin diagram). The first member of the equations will be a product of the respective source functions powered to the number of sites of each representation and the second a product of source functions similar to (20).

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