A Vector-Valued Almost Sure Invariance Principle for Random Expanding on Average Cocycles

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Abstract
We obtain a quenched vector-valued almost sure invariance principle (ASIP) for random expanding on average cocycles. This is achieved by combining the adapted version of Gouëzel’s approach for establishing ASIP (developed in Dragičević and Hafouta in Nonlinearity 34:6773–6798, 2021) and the recent construction of the so-called adapted norms (carried out in Dragičević and Sedro in Quenched limit theorems for expanding on average cocycles. arXiv:2105.00548, 2021), which in some sense eliminate the non-uniformity of the decay of correlations. For real-valued observables, we also show that the martingale approximation technique is applicable in our setup, and that it yields the ASIP with better error rates. Finally, we present an example showing the necessity of the scaling condition (12), answering a question of Dragičević and Sedro in Quenched limit theorems for expanding on average cocycles. arXiv:2105.00548, 2021.

Keywords Expanding on average cocycles · Almost sure invariance principle · Random dynamical systems

1 Introduction

The almost sure invariance principle (ASIP) is a powerful statistical tool that, given a sequence of vector-valued random variables $A_0, A_1, A_1, \ldots$, provides a coupling with an independent
sequence of Gaussian random vectors \( Z_0, Z_1, Z_2, \ldots \) such that

\[
\left| \sum_{j=0}^{n-1} A_j - \sum_{j=0}^{n-1} Z_j \right| = o(s_n),
\]

where \( s_n \) is called the rate of the ASIP, and the \( L^2 \)-norm of the sum \( \sum_{j=0}^{n-1} Z_j \) has the form \( s_n(1 + o(1)) \). Among several other limit theorems, ASIP implies the central and the functional central limit theorem (see [37] for details).

**ASIP for Deterministic Dynamics.** In the context of deterministic dynamics, one starts with a transformation \( T \) acting on a space \( X \) that admits a (physical) invariant measure \( \mu \). For sufficiently regular observables \( \psi : X \to \mathbb{R}^d \), we consider the process \( \psi, \psi \circ T, \psi \circ T^2, \ldots \) on the probability space \((X, \mu)\), and we are interested in formulating sufficient conditions under which it satisfies the ASIP. It is expected that this will occur when \( T \) is sufficiently chaotic, i.e. when \( T \) exhibits some form of hyperbolicity.

We emphasize that this problem has been thoroughly studied and that the literature is vast. Among many important contributions, we mention the works of Pollicott and Sharp [38], Field, Melbourne and Török [20] as well as Melbourne and Nicol [35, 36] (completed recently by Korepanov [30]), in which the authors obtained ASIP for large classes of (nonuniformly) hyperbolic maps. In addition, we mention the recent important papers by Cuny and Merlevede [10], Korepanov, Kosloff and Melbourne [32], Korepanov [31], as well as Cuny, Dedecker, Korepanov and Merlevede [8, 9] in which the authors further improved the error rates in ASIP for a wide class of (nonuniformly) hyperbolic deterministic dynamical systems.

Finally, and most relevant to the content of the present paper, we mention the seminal paper by Gouëzel [22], in which is developed the so-called spectral approach for establishing the ASIP, which is applicable whenever the transfer operator associated to \( T \) admits a spectral gap (on an appropriate Banach space).

**ASIP for Random Dynamics.** In the case of random dynamics, one starts with a base space which consists of a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) together with an invertible measure-preserving and ergodic transformation \( \sigma : \Omega \to \Omega \). Moreover, one considers a measurable family of transformations \((T_\omega)_{\omega \in \Omega}\) acting on some space \( X \). We study random compositions of the form

\[
T^n_\omega = T_{\sigma^{n-1} \omega} \circ \ldots \circ T_\omega, \quad \omega \in \Omega, \quad n \in \mathbb{N}.
\]

From this data, we can define the skew-product transformation \( \tau : \Omega \times X \to \Omega \times X \) by

\[
\tau(\omega, x) = (\sigma \omega, T_\omega(x)), \quad (\omega, x) \in \Omega \times X.
\]

(2)

For any \( \tau \)-invariant measure \( \mu \), there exists a family \((\mu_\omega)_{\omega \in \Omega}\) of measures on \( X \) such that

\[
\mu(A \times B) = \int_A \mu_\omega(B) \, d\mathbb{P}(\omega), \quad \text{for } \mu \in \mathcal{F} \text{ and } B \subset X \text{ measurable}.
\]

(3)

Given a sufficiently regular (random) observable \( \psi : \Omega \times X \to \mathbb{R}^d \), one can either study the process

\[
\psi, \psi \circ \tau, \ldots, \psi \circ \tau^n \quad \text{on } (\Omega \times X, \mu),
\]

(4)

or for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \), the process

\[
\psi(\omega, \cdot), \psi(\sigma \omega, \cdot) \circ T_\omega, \ldots, \psi(\sigma^n \omega, \cdot) \circ T^n_\omega \quad \text{on } (X, \mu_\omega).
\]

(5)
Then, limit laws related to (4) are called \textit{annealed}, whereas those concerned with (5) are called \textit{quenched}. We would like to stress that annealed limit theorems concern the stationary process $(\psi \circ T^n)_{n=0}^\infty$, while in the quenched case the process $(\psi(\sigma^n \omega, \cdot) \circ T^n)_{n=0}^\infty$ is not stationary, which to some extent makes the quenched limit theorems harder to prove. On the other hand, without some kind of mixing assumptions on the base map $\sigma$ annealed limit theorems cannot hold (in general). For i.i.d maps $T_{\sigma} \omega$ (as discussed in the next paragraph) limit theorems can be obtained by integration of the iterates of the random transfer operators (see [1]), while for some other classes of base maps such as Markov shifts or non-uniform Young towers a different type of integration argument yields several types of limit theorems (see [26]). However, it is still unclear which type of mixing conditions are necessary for annealed limit theorems to hold.

To the best of our knowledge, the quenched ASIP in the context of random dynamical systems was first discussed by Kifer [28], where it was briefly mentioned that the techniques developed there can be used to obtain the scalar-valued quenched ASIP for random expanding dynamics. More recently, both annealed and quenched ASIP were discussed for several classes of random dynamical systems [1, 40, 41]. The main idea in those papers is to show that the transfer operator associated to the skew-product transformation (see (2)) has a spectral gap on an appropriate space. Afterwards, one can simply apply Gouëzel’s results [22]. However, for this approach to work, one needs to impose strong (mixing) assumptions on the base space $(\Omega, \mathcal{F}, P)$. In fact, in all of those works, $(\Omega, \mathcal{F}, P)$ is a Bernoulli shift. By using martingale methods and relying on the techniques developed in [10, 27], in [12] a quenched scalar-valued ASIP for certain classes of random piecewise expanding dynamics was obtained, without any mixing assumptions on the base space. In addition, we mention two recent papers [42, 43] by Su devoted to the ASIP for certain classes of random expanding systems and maps which admit a random tower extension.

In order to apply directly the approach developed by Gouëzel [22] for establishing the ASIP in the context of random dynamics, it seems necessary to impose mixing assumptions on the base space $(\Omega, \mathcal{F}, P)$. To overcome this issue, the first two authors [17] have proposed a certain adaptation of Gouëzel’s method, by requiring a weaker control over the behavior of covariance matrices. This adaptation enabled them to extend the ASIP result from [12] to the case of vector-valued observables, still without any mixing assumptions on the base space. Moreover, it allowed to establish the first version of the vector-valued ASIP for broad classes of random (uniformly) hyperbolic dynamics as studied in [11, 14, 15].

**Contributions of the present paper.** Despite representing a significant progress, the abstract version of the ASIP for random dynamical systems formulated in [17, Theorem 4.18] is not entirely satisfactory as it is not directly applicable to random dynamical systems which exhibit nonuniform decay of correlations, such as expanding on average systems studied by Buzzi [6]. Indeed, condition [17, (4.6)] requires uniform decay of correlations: we refer to Remark 28 for more details. We note that the presence of the nonuniform decay of correlations essentially means that $D$ in (19) is allowed to depend on the random parameter $\omega$. This relaxation (with respect to previous works) is natural from the ergodic theory point of view. We refer to Remark 14 for details.

The purpose of the present paper is to fill the gap between the uniformly expanding case and the nonuniform one. More precisely, we combine techniques from [17] together with those developed by the first and the third author in [18], to obtain the quenched vector-valued ASIP for random expanding on average. We stress that in [18] several other limit theorems for random expanding on average cocycles have been discussed but not the ASIP. As mentioned above, in comparison with several existing result in literature (e.g. [12, 17, ...
we consider random dynamical systems which only expand on average. In another direction, in comparison with the expanding on average random maps considered in [1], our results do not require the maps $T_{\sigma_{j}}\omega$ to be independent, and the observable $\psi$ considered in this paper might depend on $\omega$. In fact, in Appendix A we show that, in general, when $\psi$ does not depend on $\omega$ then the usual asymptotic statistical behaviour might fail.

As in [18], the main idea is to construct suitable “adapted norms”, which enable us to absorb the nonuniformity in the decay of correlations. Unlike [18], this construction is not carried for an original cocycle of transfer operators but rather for an associated cocycle of normalized transfer operators. We highlight that, to the best of our understanding, one cannot simply rely on the construction developed in [18, Section 3.1]. In fact, we have to restrict to a slightly smaller class of cocycles from those considered in [18] (see Remark 4). Afterwards, it remains to verify that Theorem [17, Theorem 2.1] can be applied.

We also note that this approach is completely different from the techniques in [16, 28] which rely on inducing with respect to a region of the base space on which the random dynamics exhibits a uniform decay of correlations, and refer to [18, Section 1] for a detailed discussion.

In the second part of the paper, by using the martingale method together with techniques developed in [10] and [32], we establish the scalar-valued ASIP for a smaller class of observables but with better error rates (that is, with better estimates on the right hand side of (1)). We stress that even if we restrict to the setup from [12, 13] (namely assume uniform decay of correlations), we obtain slightly better rates than those given in [12, Theorem 1].

Finally, in an Appendix A, we present an example, essentially taken from [6, Appendix A], of a random system and an observable satisfying all of our assumptions, except for the scaling condition (12), for which the asymptotic variance fails to exist: this shows the sharpness of this scaling condition, answering in particular a question posed in (the original version of) [18].

**Comments and directions of future research.** We emphasize that our results (see for example Theorem 10) require a certain control over the size of an observable (condition (12)). This condition does not imply that the class of our observables is small: indeed, one can note that the observables satisfying (12) are in one-to-one correspondence with observables satisfying (16). However, condition (12) involves the so-called Oseledets-Lyapunov regularity function ($K$ in (12)) which is notoriously hard to compute explicitly (as it is given as a supremum of a certain quantity).

Beyond temperedness, the study of properties of Oseledets-Lyapunov regularity functions has been initiated only recently by Simić [39]. In particular, he shows that under appropriate regularity assumptions for a linear cocycle and mixing assumptions for a base space, one can achieve that these regularity functions belong to the $L^p$ space for sufficiently small values of $p > 0$. Other relevant contributions to this area of research were obtained more recently by Gouëzel and Stoyanov [23].

We would like to emphasize that conditions similar to (12) have appeared earlier in the study of random dynamical systems. Indeed, an analogous requirement is present in the study of invariant manifolds (see [2, (7.3.2)]) and linearization (see [2, Proposition 7.4.11]) of (nonlinear) random dynamical systems. We refer to [2, p.379] for a detailed discussion. Moreover, a similar condition ensures persistence of nonuniform behaviour under small perturbations (see [45, (2.2)]). Closer to the context of the present paper, the random variable

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1 One should notice that the conditions in [2] are stated in terms of the so-called Lyapunov norms. However, the random variable which measures the deviation of Lyapunov norms from the original norm is precisely Oseledets-Lyapunov regularity function.
measuring the speed of (exponential) decay of correlations for expanding on average cocycles (see \cite[Main Theorem]{6}) is also not given explicitly. This indicates that the complexity of (12) is not a consequence of our techniques but rather of intrinsic difficulties in treating nonuniformly hyperbolic dynamics.

Although this makes our results somewhat unsatisfactory, we still believe that the present approach offers a new insight on limit theorems for random systems which exhibit nonuniform decay of correlations. In particular, the above described connection with the Oseledets-Lyapunov regularity function sheds some light on the difficulty of providing explicit conditions for limit theorems in our setting. Moreover, our example from Appendix A, indicates that some condition similar to (12) needs to be imposed for limit theorems to hold.

Recently, the second author \cite{24} has made a significant breakthrough and obtained explicit conditions under which limit theorems hold for certain classes of random systems exhibiting nonuniform decay of correlations. However, these classes do not include random expanding of average cocycles. In particular, it is still an open problem to describe explicitly the set of observables for which limit theorems hold in the framework of the simple example presented in Appendix A.

In conclusion, if our results do not aim to say the final word on the topic of limit theorems for expanding on average cocycles, they certainly represent a step forward.

2 A Vector Valued ASIP via Gouëzel’s Approach

2.1 Preliminaries

We begin by recalling the setup from \cite{6} (and also from \cite{18}). Let \((X, \mathcal{G})\) be a measurable space endowed with a probability measure \(m\) and a notion of a variation \(\text{var}: L^1(X, m) \to [0, \infty)\) which satisfies the following conditions:

(V1) \(\text{var}(th) = |t| \text{var}(h)\);

(V2) \(\text{var}(g + h) \leq \text{var}(g) + \text{var}(h)\);

(V3) \(\|h\|_{L^\infty} \leq C_{\text{var}} (\|h\|_1 + \text{var}(h))\) for some constant \(1 \leq C_{\text{var}} < \infty\);

(V4) for any \(C > 0\), the set \(\{h: X \to \mathbb{R} : \|h\|_1 + \text{var}(h) \leq C\}\) is \(L^1(m)\)-compact;

(V5) \(\text{var}(1) = 0\), where \(1\) denotes the function equal to 1 on \(X\);

(V6) \(\{h: X \to \mathbb{R}_+ : \|h\|_1 = 1\text{ and } \text{var}(h) < \infty\}\) is \(L^1(m)\)-dense in \(\{h: X \to \mathbb{R}_+ : \|h\|_1 = 1\}\);

(V7) for any \(f \in L^1(X, m)\) such that \(\text{essinf} f > 0\), we have

\[
\text{var}(1/f) \leq \frac{\text{var}(f)}{(\text{essinf} f)^2}.
\]

(V8) \(\text{var}(fg) \leq \|f\|_{L^\infty} \cdot \text{var}(g) + \|g\|_{L^\infty} \cdot \text{var}(f)\);

(V9) for \(M > 0\), \(f: X \to [-M, M]\) measurable and every \(C^1\) function \(h: [-M, M] \to \mathbb{C}\),

we have \(\text{var}(h \circ f) \leq \|h\|_{L^\infty} \cdot \text{var}(f)\).

We define

\[
BV = BV(X, m) = \{g \in L^1(X, m) : \text{var}(g) < \infty\}.
\]

Then, \(BV\) is a Banach space with respect to the norm

\[
\|g\|_{BV} = \|g\|_1 + \text{var}(g).
\]
Remark 1 Observe that (V3) and (V8) imply that
\[ \|fg\|_{BV} \leq C_{\text{var}} \|f\|_{BV} \cdot \|g\|_{BV} \quad \text{for } f, g \in BV. \] (6)

Remark 2 We observe that in [6], assumption (V5) is replaced by the weaker \( \text{var} (\mathbb{I}) < +\infty \). However, for the examples we have in mind, our stronger version is satisfied. In particular, (V5) implies that \( \|\mathbb{I}\|_{BV} = 1 \).

Let \((\Omega, \mathcal{F}, \mathbb{P}, \sigma)\) be a probability space and \(\sigma : \Omega \to \Omega\) an invertible ergodic measure-preserving transformation. Let \(T_\omega : X \to X\), \(\omega \in \Omega\) be a collection of non-singular transformations (i.e. \(m \circ T_\omega^{-1} \ll m\) for each \(\omega\)) acting on \(X\). Each transformation \(T_\omega\) induces the corresponding transfer operator \(\mathcal{L}_{\omega}\) acting on \(L^1(X,m)\) and defined by the following duality relation
\[ \int_X (\mathcal{L}_{\omega} \phi) \psi \, dm = \int_X \phi(\psi \circ T_\omega) \, dm, \quad \phi \in L^1(X,m), \ \psi \in L^\infty(X,m). \]

Thus, we obtain a cocycle of transfer operators \((\Omega, \mathcal{F}, \mathbb{P}, \sigma, L^1(X,m), \mathcal{L})\) that we denote by \(\mathcal{L} = (\mathcal{L}_{\omega})_{\omega \in \Omega}\). For \(\omega \in \Omega\) and \(n \in \mathbb{N}\), set
\[ \mathcal{L}_{\omega}^n := \mathcal{L}_{\sigma^{-n} \omega} \circ \cdots \circ \mathcal{L}_{\sigma \omega} \circ \mathcal{L}_{\omega}. \]

Definition 3 A cocycle \(\mathcal{L} = (\mathcal{L}_{\omega})_{\omega \in \Omega}\) of transfer operators is said to be good if the following conditions hold:

- \(\Omega\) is a Borel subset of a separable, complete metric space and \(\sigma\) is a homeomorphism. Moreover, \(\mathcal{L}\) is \(\mathbb{P}\)-continuous, i.e. \(\Omega\) can be written as a countable union of measurable sets such that \(\omega \mapsto \mathcal{L}_{\omega}\) is continuous on each of those sets;
- there exists a random variable \(C : \Omega \to (0, +\infty)\) such that \(\log C \in L^1(\Omega, \mathbb{P})\) and \(\|\mathcal{L}_{\omega} h\|_{BV} \leq C(\omega) \|h\|_{BV}\) for \(\mathbb{P}\)-a.e. \(\omega \in \Omega\) and \(h \in BV\);
- there exist \(N \in \mathbb{N}\) and random variables \(\alpha^N, K^N : \Omega \to (0, +\infty)\) such that
\[ \int_\Omega \log \alpha^N \, d\mathbb{P} < 0, \quad \log K^N \in L^1(\Omega, \mathbb{P}) \]
and, for \(\mathbb{P}\)-a.e. \(\omega \in \Omega\) and \(h \in BV\),
\[ \text{var}(\mathcal{L}_{\omega}^N h) \leq \alpha^N(\omega) \text{var}(h) + K^N(\omega) \|h\|_1; \]
- for each \(a > 0\) and \(\mathbb{P}\)-a.e. \(\omega \in \Omega\), there exist random numbers \(n_\epsilon(\omega) < +\infty\) and \(\alpha_0(\omega), \alpha_1(\omega), \ldots\) such that for every \(h \in \mathcal{C}_a\),
\[ \text{essinf}_x (\mathcal{L}_{\omega}^N h)(x) \geq \alpha_n \|h\|_1 \quad \text{for } n \geq n_\epsilon, \] (7)
where
\[ \mathcal{C}_a := \{h \in L^\infty(X,m) : h \geq 0 \text{ and } \text{var}(h) \leq a \|h\|_1\}; \] (8)
- \(\log \left(\text{essinf}_{x \in X} (\mathcal{L}_{\omega}^N h)(x)\right) \in L^1(\Omega, \mathbb{P})\).

Remark 4 The first requirement of Definition 3, \(\mathbb{P}\)-continuity of the map \(\omega \in \Omega \mapsto \mathcal{L}_{\omega}\), may be seen as restrictive: this is the price to pay to apply the Multiplicative Ergodic Theorem in a non-separable Banach space such as \(BV\). We stress that the \(\mathbb{P}\)-continuity of the map \(\omega \mapsto \mathcal{L}_{\omega}\) holds whenever the map \(\omega \mapsto T_\omega\) has a countable range \(\{T_1, T_2, \ldots\}\) and for each \(j\), \(\{T_\omega = T_j\} \in \mathcal{F}\).
We note that when working with a separable Banach space $B$, as is the case in Example 6, we can replace this requirement with the looser strong measurability, i.e. measurability of $\omega \in \Omega \mapsto L_\omega h$ for $h \in B$.

- Definition 3 almost coincides with [18, Definition 13], the only difference being the addition of the last requirement in Definition 3.
- This log-integrability assumption may easily be checked on explicit examples (see e.g. the discussion in [3, Remark 2.12]).
- Furthermore, this assumption implies a certain version of the “random covering” similar to (7). More precisely, denoting by $C_+$ the cone of non-negative function in $L^\infty(X)$, and by $\theta_+$ the projective Hilbert metric on this cone, and assuming that the transfer operator cocycle $L$ satisfies

$$\log(\text{essinf } L_\omega 1) \in L^1(\Omega, \mathbb{P}),$$

we have, for any $h \in C_+$ such that $\theta_+(h, 1) \leq R$ for some $0 < R < +\infty$, the following: for $\mathbb{P}$-a.e $\omega \in \Omega$, there exists some random integer $n_c(\omega)$, non-random positive numbers $(\alpha_n)_{n \geq 0}$ such that for any $n \geq n_c(\omega)$,

$$\text{essinf}_{x \in X} L^n_\omega h \geq \alpha_n \|h\|_1.$$  

Recall (see e.g. [6, Sec. 1.3]) that $\theta_+(h, 1) = \log \left( \frac{\text{esssup} h}{\text{essinf} h} \right)$, so that

$$\theta_+(h, 1) \leq R \iff \text{essinf } h \geq e^{-R} \text{esssup } h.$$  

Since the sequence $\text{essinf } L^n_\omega 1$ is super-multiplicative, and by Birkhoff’s ergodic theorem, we have

$$\frac{1}{n} \log(\text{essinf } L^n_\omega 1) \geq \frac{1}{n} \sum_{k=0}^{n-1} \log(\text{essinf } L_{\alpha_k^k}\omega 1) \xrightarrow{n \to \infty} \int_{\Omega} \log(\text{essinf } L_\omega 1) d\mathbb{P}(\omega).$$  

In particular, there is some integer $n_c := n_c(\omega) < +\infty$, such that for $n \geq n_c$,

$$\text{essinf } L^n_\omega 1 \geq e^{n_1/2},$$

where

$$I := \int_{\Omega} \log(\text{essinf } L_\omega 1) d\mathbb{P}(\omega).$$

Hence, for any $h \in B_{\theta_+}(1, R)$, we have

$$\text{essinf } L^n_\omega h \geq (\text{essinf } h) \cdot (\text{essinf } L^n_\omega 1) \geq e^{-R} e^{n_1/2} \text{esssup } h \geq \alpha_n \|h\|_1,$$

with $\alpha_n := e^{-R+n_1/2} > 0$, which is non-random as announced.

Let us now give examples of systems satisfying our requirements: the following is essentially taken from [6].

**Example 5** (Lasota-Yorke cocycles) Consider $X = [0, 1]$, endowed with Lebesgue measure $m$ and the classical notion of variation var. We say that $T : X \to X$ is a piecewise monotonic non-singular map (p.m.n.s map for short) if the following conditions hold:

- $T$ is piecewise monotonic, i.e. there exists a subdivision $0 = a_0 < a_1 < \cdots < a_N = 1$ such that for each $i \in \{0, \ldots, N-1\}$, the restriction $T_i = T_{(a_i, a_{i+1})}$ is monotonic (in particular it is a homeomorphism on its image).

- $T$ is non-singular, i.e. there exists $|T'| : [0, 1] \to \mathbb{R}_+$ such that for any measurable $E \subset (a_i, a_{i+1})$, $m(T(E)) = \int_E |T'| dm$.

\[\text{we recall that a sequence of measurable functions } (f_n)n \text{ on } \Omega \text{ is said to be super-multiplicative if } f_{n+m}(\omega) \geq f_n(\sigma^n\omega) \cdot f_m(\omega) \text{ for } \mathbb{P}\text{-a.e. } \omega \in \Omega \text{ and } n, m \in \mathbb{N}.\]
The intervals \((a_i, a_{i+1})\) are called the intervals of \(T\). We also set \(N(T) := N\) and \(\lambda(T) := \text{ess inf}_{[0,1]} |T'|.\)

We consider a family \((T_\omega)_{\omega \in \Omega}\) of random p.m.n.s as above, and such that \(T : \Omega \times [0, 1] \to [0, 1]\), \((\omega, x) \mapsto T_\omega(x)\) is measurable. Denoting \(N_\omega = N(T_\omega)\) and \(\lambda_\omega = \lambda(T_\omega)\), we assume that

- The map \(\omega \mapsto \left(\text{var} \left(\frac{1}{T'_\omega}\right), N_\omega, \lambda_\omega, a_1, \ldots, a_{N_\omega-1}\right)\) is measurable.
- We have the following expanding-on-average property:

\[
\lim_{k \to \infty} \int_{\Omega} \log \min(\lambda_\omega, K) \, d\mathbb{P}(\omega) \in (0, +\infty)
\]

- The map \(\log \left(\frac{N_\omega}{\lambda_\omega}\right)\) is integrable.
- The map \(\log \left(\text{var} \left(\frac{1}{T'_\omega}\right)\right)\) is integrable.
- \(T_\omega\) is covering, i.e. for any interval \(I \subseteq [0, 1]\), there exists a random number \(n_c(\omega) > 0\) such that for any \(n \geq n_c\), one has

\[
\text{ess inf}_{[0,1]} \mathcal{L}^n_\omega (I) > 0. \quad (9)
\]
- \(\log (\text{ess inf}_{x \in X} (\mathcal{L}_\omega \mathbb{1}(x))) \in L^1(\Omega, \mathbb{P}).\)

We will call a cocycle satisfying the previous assumptions an expanding on average Lasota-Yorke cocycle. For a countably-valued measurable family \((T_\omega)_{\omega \in \Omega}\) of expanding on average Lasota-Yorke cocycle, the associated cocycle of transfer operators \((\mathcal{L}_\omega)_{\omega \in \Omega}\) is good (see [18]).

The following example can be fruitfully compared to a similar one by Kifer [29].

**Example 6** We consider \(X = \mathbb{S}^1\), endowed with the Lebesgue measure \(m\) and the notion of variation given by \(\text{var}(\phi) := \int_X |\phi'| \, dm = \|\phi'\|_{L^1}\). Notice that this notion of variation leads to define the \(W^{1,1} (\mathbb{S}^1)\) Sobolev space instead of the space of bounded variation observables \(BV\). We consider a measurable map \(T : \Omega \times X \to X\) such that \(T_\omega := T(\omega, \cdot)\) is \(C^r, r \geq 2\). In addition, we make the following assumptions:

- The map \(\omega \in \Omega \mapsto \left(\int_X \frac{|T'_\omega|}{(T'_\omega)^2} \, dm, \lambda_\omega\right)\) is measurable, where \(\lambda_\omega = \inf_{[0,1]} |T'_\omega|\).
- The following expanding on average property holds:

\[
\int_{\Omega} \log(\lambda_\omega) \, d\mathbb{P}(\omega) > 0. \quad (10)
\]

- The map \(\log \left(\int_X \frac{|T'_\omega|}{(T'_\omega)^2} \, dm\right)\) is \(\mathbb{P}\)-integrable.
- \(\log (\text{ess inf}_{x \in X} (\mathcal{L}_\omega \mathbb{1}(x))) \in L^1(\Omega, \mathbb{P}).\)

We call a family \((T_\omega)_{\omega \in \Omega}\) satisfying the previous assumptions a smooth expanding on average cocycle (see also [18, Example 16]).

We note that our expansion on average condition (10) implies that \(\mathbb{P}\)-a.s., \(T_\omega\) has non-vanishing derivative, hence is a local diffeomorphism and a monotonic map of the circle. Furthermore, smooth expanding on average cocycles satisfy a stronger version of the random covering property (it implies [6, Remark 0.1] the one formulated in (9)): for each non-trivial interval \(I \subseteq X\), for \(\mathbb{P}\)-a.e \(\omega \in \Omega\), there is a \(n_c := n_c(\omega, I) < \infty\) such that for all \(n \geq n_c\),

\[
T^n_\omega (I) = X.
\]
To see this, first remark that by smoothness of the maps $T_\omega$, one has, for any interval $I \subset X$, and any $n \in \mathbb{N}$, that
\[ m(T^n_\omega(I)) = \int_I |(T^n_\omega)'| dm \geq \lambda^n_\omega |I|, \]
where $\lambda^n_\omega := \lambda_{\sigma^{-n-1}\omega} \cdots \lambda_\omega$. Our expansion on average assumption (10) and Birkhoff’s ergodic theorem insures that
\[ \frac{1}{n} \log(\lambda^n_\omega) = \frac{1}{n} \sum_{j=0}^{n-1} \log(\lambda_{\sigma^{-j}\omega}) \xrightarrow{n \to \infty} \Lambda := \int_{\Omega} \log(\lambda_\omega) d\mathbb{P}(\omega) > 0. \]
Hence, for a.e $\omega \in \Omega$ we may choose measurably a $\tilde{n}_c(\omega) \in \mathbb{N}$, such that for $n \geq \tilde{n}_c(\omega)$ one has
\[ m(T^n_\omega(I)) \geq |I| e^{n\Lambda/2}. \]
Taking $n_c := 1 + \max(\tilde{n}_c, -\frac{2}{A} \log(|I|))$ gives the desired result.

Finally, for a measurable, possibly uncountably valued family $(T_\omega)_{\omega \in \Omega}$ of smooth expanding on average cocycle, the associated cocycle of transfer operators $(\mathcal{L}_\omega)_{\omega \in \Omega}$ is strongly measurable on $W^{1,1}$: this follows from [5, Prop. 4.11], by arguing as in [7, Proof of Prop. 5.2] (this fact was already noted in [19, Prop. 24]). Hence, the cocycle of transfer operators $(\mathcal{L}_\omega)_{\omega \in \Omega}$ is good in the sense of Remark 4, replacing the $P$-continuity requirement of Definition 3 by a strong measurability one. We emphasize that, since in the present setting, we work on the separable Banach space $W^{1,1}$, the range of the measurable map $\omega \in \Omega \mapsto T_\omega$ may be infinite uncountable, in contrast with the previous example.

Our abstract setup also covers multidimensional examples. The one we describe now is due to Buzzi [6, Appendix B].

**Example 7** (Multidimensional piecewise affine maps.) Recall that a polytope in $\mathbb{R}^d$ is defined as the intersection of half-spaces. If $X \subset \mathbb{R}^d$, let $P$ be a finite collection of pairwise disjoints, open polytopes $A$ of $\mathbb{R}^d$, such that $Y = \bigcup_{A \in P} A$ is dense in $X$. We now let $f : Y \to X$ be such that for any $A \in P$, $f : A \to f(A) \subset X$ is the restriction of an affine map $f_A$ of $\mathbb{R}^d$: we say that $(X, P, f)$ is a piecewise affine map. We will also assume that each $f_A$ is invertible.

We define the expansion rate of $f$,
\[ \lambda(f) := \inf_{x \in Y} \inf_{\|v\|=1} \|Df(x) \cdot v\|. \]

We also recall that, given a polytope $A \subset \mathbb{R}^d$, we can define the $\epsilon$-multiplicity of its boundary $\partial A$ at $x \in X$, $\text{mult}(\partial A, \epsilon, x)$, as the number of hyperplanes in $\partial A$ having non-empty intersection with $B(x, \epsilon)$ the ball of radius $\epsilon$ centered at $x$. We then set
\[ \text{mult}(\partial P, \epsilon) := \sup_{x \in X} \sum_{A \in P} \text{mult}(\partial A, \epsilon, x) \]
\[ \text{mult}(\partial P) := \lim_{\epsilon \to 0} \text{mult}(\partial A, \epsilon). \]

---

3 Here we abuse notations, identifying the Lebesgue measure on $\mathbb{S}^1$, the circle map $T^n_\omega$ and the small interval $I \subset \mathbb{S}^1$ with their lifted counterpart on $\mathbb{R}$. 

---

\[ \text{mult}(\partial P) := \lim_{\epsilon \to 0} \text{mult}(\partial A, \epsilon). \]
Finally we notice that there are some $\epsilon > 0$ for which $\text{mult}(\partial P, \epsilon) = \text{mult}(\partial P)$. We denote by $\epsilon(\partial P)$ the supremum of such $\epsilon$.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed as usual with an invertible, measure-preserving and ergodic self map $\sigma$, we consider countably-valued, measurable families of polytopes $(A_\omega)_{\omega \in \Omega}$ and affine maps $(f_{A_\omega})_{\omega \in \Omega}$ of $X \subset \mathbb{R}^d$: this data defines a cocycle of random piecewise affine map $(X, P_\omega, f_\omega)$, for which we assume:

1. For any $n \in \mathbb{N}$, the map $\omega \mapsto (\lambda(f^n_\omega), |P^n_\omega|, \text{mult}(\partial P^n_\omega))$ is measurable.
2. The map $\lambda$ is log-integrable.
3. The following expansion on average condition holds:
   \[ \lim_{n \to \infty} \frac{1}{n} \log \min \{(f^n_\omega(x), B_\epsilon(x)) : x \in X\} dx > 0. \]
4. The following random covering condition holds:
   For any ball $B \subset X$, $\mathbb{P}$-a.e $\omega \in \Omega$ there is a $n_c := n_c(\omega, B)$ such that $f^n_\omega(B) = X$ (modulo a null set for Lebesgue measure) for $n \geq n_c$.
5. $\log(\text{essinf}_{x \in X} C_\omega \mathbb{I}(x)) \in L^1(\Omega, \mathbb{P})$.

Under the previous assumptions, and for the notion of variation on $X$ given by
\[ \text{var}(f) = \sup_{0 < \epsilon \leq \epsilon_0} \frac{1}{\epsilon} \int_{\mathbb{R}^d} \text{osc}(f, B_\epsilon(x)) \, dm(x), \]

where \[ \text{osc}(f, B_\epsilon(x)) = \text{esssup}_{x_1, x_2 \in B_\epsilon(x)} |f(x_1) - f(x_2)|. \]

it is established in [6, Prop B.1] that a random piecewise affine map has a good random transfer operator, in the sense of [6, Def.1.1]. Together with the assumption that this map is countably valued, this shows that the associated transfer operator cocycle is good in the sense of Definition 3.

We recall the notion of a tempered random variable.

**Definition 8** We say that a measurable map $K : \Omega \to (0, +\infty)$ is tempered if
\[ \lim_{n \to \pm\infty} \frac{1}{n} \log K(\sigma^n_\omega) = 0, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \]

We will need the following classical result (see [2, Proposition 4.3.3.]).

**Proposition 9** Let $K : \Omega \to (0, +\infty)$ be a tempered random variable. For each $\epsilon > 0$, there exists a tempered random variable $K_\epsilon : \Omega \to (1, +\infty)$ such that
\[ \frac{1}{K_\epsilon(\omega)} \leq K(\omega) \leq K_\epsilon(\omega) \quad \text{and} \quad K_\epsilon(\omega)e^{-\epsilon|n|} \leq K_\epsilon(\sigma^n_\omega) \leq K_\epsilon(\omega)e^{\epsilon|n|}, \]

for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $n \in \mathbb{Z}$.

This notion of variation fulfills conditions (V1)-(V9): we refer to [13, Section 2.2.] for details.
2.2 Statement of the Main Result

We are now in a position to state the main result of our paper. By $x \cdot y$ we will denote the scalar product of $x, y \in \mathbb{C}^d$.

Theorem 10 Let $L = (L_\omega)_{\omega \in \Omega}$ be a good cocycle of transfer operators. Moreover, take $d \in \mathbb{N}$ and let $\psi = (\psi^1, \ldots, \psi^d) : \Omega \times X \to \mathbb{R}^d$ be a measurable map such that the following conditions hold:

- $\psi^i_\omega := \psi^i(\omega, \cdot) \in BV$ for $\omega \in \Omega$ and $1 \leq i \leq d$;
- for $1 \leq i \leq d$, we have that
  \[ \text{esssup}_{\omega \in \Omega} \left( K(\omega) \| \psi^i_\omega \|_{BV} \right) < +\infty, \]  
  (12)
- for $1 \leq i \leq d$ and $\mathbb{P}$-a.e. $\omega \in \Omega$,
  \[ \int_X \psi^i_\omega \, d\mu_\omega = 0, \]  
  (13)
where $\mu_\omega$ is a probability measure on $X$ as in the statement of Corollary 15 (namely $\{\mu_\omega\}$ is the unique family of absolutely continuous equivariant measures).

Then, we have the following:

1. there exists a positive semi-definite $d \times d$ matrix $\Sigma^2$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$ we have
  \[ \lim_{n \to \infty} \frac{1}{n} \int_X (S_n \psi(\omega, \cdot))^2 \, d\mu_\omega = \Sigma^2, \]
  where
  \[ S_n \psi(\omega, x) = \sum_{i=0}^{n-1} \psi(\sigma^i \omega, T^i_\omega(x)), \quad (\omega, x) \in \Omega \times X. \]  
  (14)
Moreover, $\Sigma^2$ is not positive definite if and only if there exist $v \in \mathbb{R}^d \setminus \{0\}$ and an $\mathbb{R}$-valued measurable function $r$ on $\Omega \times X$ such that $\mathbb{P}$-a.s. $r(\omega, \cdot) \in BV$, $\text{esssup}_{\omega \in \Omega} \| r(\omega, \cdot) \|_{BV} < \infty$ and
  \[ v \cdot \psi = r - r \circ \tau, \quad \mu - \text{a.e}, \]  
  (15)
where $\tau$ and $\mu$ are given by (2) and (3), respectively.

2. Suppose that $\Sigma^2$ is positive definite. Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and every $\delta > 0$, there exists a coupling between $\{\psi_{\sigma^i \omega} \circ T^n_\omega : n \geq 0\}$, considered as a sequence of random variables on $(X, \mathcal{B}, \mu_\omega)$, and a sequence $(Z_k)_k$ of independent centered (i.e. of zero mean) Gaussian random vectors such that
  \[ \left| \sum_{i=0}^{n-1} \psi(\sigma^i \omega, \cdot) \circ T^i_\omega - \sum_{i=1}^{n} Z_i \right| = O(n^{1/4+\delta}), \quad \text{almost-surely}. \]
Moreover, there exists a constant $C = C_\delta(\omega) > 0$ so that for every $n \geq 1$,
  \[ \left\| \sum_{i=0}^{n-1} \psi(\sigma^i \omega, \cdot) \circ T^i_\omega - \sum_{i=1}^{n} Z_i \right\|_{L^2} \leq C n^{1/4+\delta}. \]
Finally, there is a constant $C' = C_\delta'(\omega) > 0$ so that for every unit vector $v \in \mathbb{R}^d$,
\[
\left\| \sum_{i=1}^n Z_i \cdot v \right\|_{L^2}^2 - \left\| \sum_{i=0}^{n-1} \psi(\sigma^i \omega, \cdot) \circ T^i_\omega \cdot v \right\|_{L^2}^2 \leq C' n^{1/2+\delta}.
\]

**Remark 11** Let us comment on the statement of Theorem 10:

- Reasoning as in [18, Remark 34], it is easily seen that in the setting of [12], $K$ is constant. Hence, (12) is equivalent to
\[
\text{esssup}_{\omega \in \Omega} \| \psi^i_\omega \|_{BV} < +\infty, \quad 1 \leq i \leq d.
\] (16)
Therefore, in the setting of [12], Theorem 10 reduces to [17, Theorem 4.18].

- It is possible to construct observables satisfying assumption (12) by following, for each scalar map $\psi^i_\omega$, $i \in \{1, \ldots, d\}$, the procedure described in [18, Example 35].

**Remark 12** In [16] a version of Theorem 10 for the random expanding maps from [25, Ch. 5] (see also [34]) was established. While this was obtained only for Hölder continuous observables, we stress that these maps are not absolutely continuous with respect to a given reference measure, and so the setup of [16] is not included in the setup of the present paper. As discussed in Sect. 1, the results from [16] were obtained by passing to an induced system which exhibits uniform decay of correlations. Thus, a completely different type of assumptions on the observables were needed. However, we believe that the arguments in the present paper also yield Theorem 10 in the setup of [16]. The main obstacle is to establish (23) and (24) with a tempered random variable $\tilde{D}(\omega)$ (and not just with a one which is finite a.e.). Under appropriate log integrability conditions, when the maps $T_\omega$ in [16] act on the same space this can be achieved by an application of Oseledets theorem, while in the case of maps $T_\omega : \mathcal{E}_\omega \to \mathcal{E}_{\sigma \omega}$ between random measurable spaces $\{\mathcal{E}_\omega\}$, this can be achieved by using a recent version of Oseledets theorem for Banach fields established in [44].

### 2.3 Proof of Theorem 10: behaviour of the cocycle of normalized transfer operators

**Theorem 13** Let $\mathcal{L} = (\mathcal{L}_\omega)_{\omega \in \Omega}$ be a good cocycle of transfer operators. Then, the following holds:

- there exists an essentially unique measurable family $(v^0_\omega)_{\omega \in \Omega} \subset BV$ such that $v^0_\omega \geq 0, \int_X v^0_\omega dm = 1$ and
\[
\mathcal{L}_\omega v^0_\omega = v^0_{\sigma \omega}, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega;
\]

- there is a random variable $\ell : \Omega \to (0, +\infty)$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$,
\[
v^0_\omega \geq \ell(\omega) \text{ m.a.e.};
\] (17)

- for $\mathbb{P}$-a.e. $\omega \in \Omega$, $BV = \text{span}\{v^0_\omega\} \oplus BV^0$,
\[
BV^0 = \left\{ h \in BV : \int_X h dm = 0 \right\};
\] (18)

---

5 To apply the latter, it seems that we need to impose stronger integrability assumptions than [16]. More precisely, we believe that the random variable $Q_\omega$ defined in [34, (2.16)] needs to be integrable.
that

\[ \Omega_1 \]

We may now introduce the first hitting time of the positive measure set

\[ L_{\omega}^{n} \Pi(\omega) B_{V} \leq D(\omega) e^{-\lambda n} \]  

(19)

and

\[ L_{\omega}^{n}(Id - \Pi(\omega)) B_{V} \leq D(\omega) e^{\epsilon n}, \]  

(20)

where \( \Pi(\omega): B_{V} \rightarrow B_{V}^{0} \) is a projection associated to the splitting (18).

**Proof** The first assertion of the theorem is established in [6], while the third assertion is proved in [18, Proposition 24]. Moreover, the last two statements of the theorem follow from [18, Proposition 23] and [18, Proposition 28] respectively.

Thus, it only remains to establish the second assertion of the theorem: first, we remark that when \( \omega \) is good\(^6\) in the sense of [6, Definition 2.4], one has \( v_{\omega}^{0} \in C_{a} \) for some \( a > 0 \), where \( C_{a} \) is given by (8). Indeed, we may write, by [6, Lemma 2.1]

\[ \text{var}(L_{\sigma^{-n} \omega}^{n} \mathbb{I}) \leq C_{0}(\omega) \text{var}(\mathbb{I}) + C_{0}(\omega) \int_{X} \| dm = C_{0}(\omega), \]

where \( C_{0} \) is some a.e finite function. Moreover, by (19) and Proposition 9 we have that

\[ \| L_{\sigma^{-n} \omega}^{n} \mathbb{I} - v_{\omega}^{0} \| B_{V} \leq D(\sigma^{-n} \omega) e^{-\lambda n} \leq D_{\lambda/2}(\omega) e^{-\frac{\lambda}{2} n}. \]

Taking the limit as \( n \rightarrow \infty \), we obtain that

\[ \text{var}(v_{\omega}^{0}) \leq C_{0}(\omega) \leq B_{*}, \]

if \( \omega \) is good. Since \( a \geq 6B_{*} \), we obtain that \( v_{\omega}^{0} \in C_{a/6} \subset C_{a} \) for good \( \omega \). In particular, we get that

\[ \text{essinf} v_{\sigma^{R} \omega}^{0} = \text{essinf} L_{\omega}^{R} v_{\omega}^{0} \geq \alpha_{*}. \]

Hence, for every \( \omega \in \sigma^{-R}(\Omega_{*}) =: \Omega_{*}^{1} \), where \( \Omega_{*} \) is the set of good parameters, \( \text{essinf} v_{\omega}^{0} \geq \alpha_{*} \). We note that \( \mathbb{P}(\Omega_{*}^{1}) = 1 - \frac{1}{4} > 0 \) by [6, Lemma 2.6] and the measure preserving property of \( \sigma \).

Let us consider now the set \( \Omega_{*}^{2} := \{ \omega \in \Omega : \text{essinf} L_{\omega}^{R} \mathbb{I} > 0 \} \). Our log-integrability assumption on \( \text{essinf} L_{\omega}^{R} \mathbb{I} \) entails that this set has full measure, and up to replacing it by \( \bigcap_{k \in \mathbb{Z}} \sigma^{k}(\Omega_{*}^{2}) \), we can assume that it is \( \sigma \)-invariant. Hence, for a.e \( \omega \in \Omega \) and \( n \in \mathbb{N} \) we have that

\[ \text{essinf} L_{\sigma^{-n} \omega}^{n} \mathbb{I} \geq \text{essinf} L_{\sigma^{-1} \omega}^{1} \mathbb{I} \cdots \text{essinf} L_{\sigma^{-n} \omega}^{1} \mathbb{I} > 0. \]

We may now introduce the first hitting time of the positive measure set \( \Omega_{+} := \Omega_{*}^{1} \cap \Omega_{*}^{2} \), i.e. we set, for \( \omega \in \Omega \)

\[ \tilde{n}_{\omega} := \inf\{ n \in \mathbb{N} : \sigma^{-n} \omega \in \Omega_{*} \}. \]

Therefore, we have that

\[ \text{essinf} v_{\omega}^{0} = \text{essinf} L_{\sigma^{-\tilde{n}_{\omega}} \omega}^{\tilde{n}_{\omega}} v_{\sigma^{-\tilde{n}_{\omega}} \omega} \geq \alpha_{*} \cdot \text{essinf} L_{\sigma^{-\tilde{n}_{\omega}} \omega}^{\tilde{n}_{\omega}} \mathbb{I} > 0 \]

\(^{6}\) we note that several parameters \( a, R, B_{*}, \alpha_{*} \) associated with this notion will be used in the sequel (where \( \varepsilon \) from [6, Definition 2.4] is a sufficiently small fixed number)
Hence, (17) holds with
\[
\ell(\omega) := \alpha_\ast \cdot \text{essinf } L^n_{\sigma^{-n}\omega} \mathbb{1} > 0, \quad \omega \in \Omega.
\]
\[\square\]

**Remark 14** Using the language of the multiplicative ergodic theory (see [18, Section 2]), the estimates (19) and (20) mean that for the cocycle \( L = (L_\omega)_{\omega \in \Omega} \), the separation between the Oseledets subspace corresponding to the largest Lyapunov exponent (which is zero) and the sum of Oseledets subspaces corresponding to all other Lyapunov exponents is measured by a tempered random variable. We stress that this is a general fact that holds for arbitrary cocycles (of not necessarily transfer operators). We refer to [4, Proposition 3.2] for a precise formulation.

**Corollary 15** Let \( L = (L_\omega)_{\omega \in \Omega} \) be a good cocycle of transfer operators.

Then, the following holds:

- If \( (v^0_\omega)_{\omega \in \Omega} \subset BV \) is given by Theorem 13, then
  \[\omega \mapsto \|1/v^0_\omega\|_{BV} \text{ is tempered};\]
  \[(21)\]

- for \( \mathbb{P}\text{-a.e. } \omega \in \Omega, \)
  \[BV = \text{span}\{1\} \oplus BV^0_\omega,\]
  \[(22)\]
  where
  \[BV^0_\omega = \left\{ h \in BV : \int_X h \, d\mu_\omega = 0 \right\},\]
  and \( d\mu_\omega = v^0_\omega \, dm, \ \omega \in \Omega; \)
  \[\text{and there exist } \lambda' > 0 \text{ and a tempered random variable } \tilde{D} : \Omega \to (0, +\infty) \text{ such that for } \mathbb{P}\text{-a.e. } \omega \in \Omega \text{ and } n \in \mathbb{N}, \]
  \[\|L^n_\omega \tilde{\Pi}(\omega)\|_{BV} \leq \tilde{D}(\omega) e^{-\lambda'n} \]
  \[\|L^n_\omega (I - \tilde{\Pi}(\omega))\|_{BV} \leq \tilde{D}(\omega), \]
  \[(23)\]
  \[(24)\]
  where \( \tilde{\Pi}(\omega) : BV \to BV^0_\omega \) is a projection associated to the splitting (22), and
  \[L^n_\omega h = L^n_\omega (hv^0_\omega)/v^0_\sigma^n_\omega, \ h \in BV, \ n \in \mathbb{N}.\]

**Proof** We first establish (21). Given that
\[
\frac{\text{var}(v^0_\omega)}{\text{esssup}(v^0_\omega)^2} \leq \text{var}\left(\frac{1}{v^0_\omega}\right) \leq \frac{\text{var}(v^0_\omega)}{\text{essinf}(v^0_\omega)^2}, \quad (25)
\]
it is enough to show that \( \text{essinf}(v^0_\omega) \) is tempered. Indeed, \( \text{var}(v^0_\omega) \) is tempered by Theorem 13, which implies that \( \text{esssup} v^0_\omega \) is by (V3). We have, thanks to \( v^0_{\sigma^n_\omega} = L^n_\omega v^0_\omega \),
\[
\frac{1}{n} \log(\text{essinf } v^0_{\sigma^n_\omega}) \geq \frac{1}{n} \log(\text{essinf } v^0_\omega) + \frac{1}{n} \log(\text{essinf } L^n_\omega \mathbb{1}).
\]
By (17), the first term in the R.H.S. above goes to 0 as \( n \to \infty \), and for the second term, we notice that by the last item of Definition 3, super–multiplicativity of the sequence \( (\text{essinf } L^n_\omega \mathbb{1})_{n \geq 0} \) and Birkhoff’s ergodic theorem, one has
\[
\frac{1}{n} \log(\text{essinf } L^n_\omega \mathbb{1}) \geq \frac{1}{n} \sum_{k=0}^{n-1} \log \left( \text{essinf } L^n_{\sigma^k_\omega} \mathbb{1} \right) \to \infty \int_\Omega \log(\text{essinf } L_\omega \mathbb{1}) \, d\mathbb{P}(\omega),
\]
Theorem 13. Since
\[ \lim_{n \to \infty} \frac{1}{n} \log(\text{essinf } v_{\sigma^* \omega}) = 0, \]
i.e. that \( \text{essinf } (v^0) \) is tempered. Note that in the application of Tanny’s theorem we have used that \( \text{essinf } v^0_\omega \leq 1 \), which holds since \( \int_X v^0_\omega dm(\omega) = 1 \) and \( v^0_\omega \geq 0 \).

Next, we observe that for each \( h \in BV \) and \( \mathbb{P} \text{-a.e. } \omega \in \Omega \),
\[
\tilde{\Pi}(\omega)h = h - \left( \int_X h d\mu_\omega \right) \mathbb{1}, \quad \text{for } \mathbb{P} \text{-a.e. } \omega \in \Omega \text{ and } h \in BV. \tag{26}
\]

Since \( \omega \mapsto v^0_\omega \) is tempered, it follows that \( \omega \mapsto \|\tilde{\Pi}(\omega)\|_0 \) is tempered.

Take an arbitrary \( \epsilon > 0 \) and let \( \lambda > 0 \) and \( D = D_\epsilon : \Omega \to (0, +\infty) \) be given by Theorem 13. Since \( \omega \mapsto 1/v^0_\omega \) is tempered, by Proposition 9 there exists a tempered random variable \( K : \Omega \to (0, +\infty) \) such that
\[
\|1/v^0_\omega\|_0 \leq K(\omega) \quad \text{and} \quad K(\omega)e^{-\lambda/2|n|} \leq K(\sigma^\omega) \leq K(\omega)e^{\lambda/2|n|}, \tag{27}
\]
for \( \mathbb{P} \text{-a.e. } \omega \in \Omega \) and \( n \in \mathbb{Z} \). By (6), (19) and (27), it follows that
\[
\|L^n_\omega h\|_0 = \|L^n_\omega (hv^0_\omega)/v^0_\sigma\|_0 \
\leq C_{\text{var}} \|L^n_\omega (hv^0_\omega)/v^0_\sigma\|_0 \
\leq C_{\text{var}} D(\omega)e^{-\lambda n} \|h\|_0 \|v^0_\sigma\|_0 \|1/v^0_\omega\|_0 \|h\|_0 
\leq C_{\text{var}}^2 D(\omega)K(\sigma^n \omega)e^{-\lambda n} \|v^0_\sigma\|_0 \|h\|_0 \|h\|_0 
\leq C_{\text{var}}^2 D(\omega)K(\omega)e^{-\lambda n} \|v^0_\omega\|_0 \|h\|_0 \|h\|_0 ,
\]
for \( \mathbb{P} \text{-a.e. } \omega \in \Omega, h \in BV \) such that \( \int_X h d\mu_\omega = 0 \) and \( n \in \mathbb{N} \). Thus, (23) holds with \( \lambda' = \lambda/2 > 0 \) and
\[
\tilde{D}(\omega) = C_{\text{var}}^2 D(\omega)K(\omega)\|\tilde{\Pi}(\omega)\|_0 \|v^0_\omega\|_0 \|h\|_0 \quad \omega \in \Omega, \tag{28}
\]
which is a tempered random variable.

Note that we use a version of Tanny’s theorem for non-positive functions, whereas the “standard” version concerns non-negative ones.
On the other hand, (26) together with the simple observations that $L^n_\omega \mathbb{1} = \mathbb{1}$ and $\| \mathbb{1} \|_{BV} = 1$ implies that
\[
\| L^n_\omega (\text{Id} - \tilde{\Pi}(\omega)) h \|_{BV} = \left| \int_X h \, d\mu_\omega \right| \cdot \| L^n_\omega \mathbb{1} \|_{BV}
\leq \| h \|_\infty
\leq C_{\text{var}} \| h \|_{BV}
\]
for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $n \in \mathbb{N}$. Thus, (24) holds with
\[
\tilde{D}(\omega) = C_{\text{var}}
\]
which is constant and thus also tempered. Hence, (23) and (24) hold with $\tilde{D}$ being the maximum of the expressions in (28) and (29). This completes the proof. \qed

2.4 Proof of Theorem 10: Adapted norms

Lemma 16 Let $\mathcal{L} = (\mathcal{L}_\omega)_{\omega \in \Omega}$ be a good cocycle of transfer operators. Then, there is a family $\| \cdot \|_\omega$, $\omega \in \Omega$ of norms on $BV$ with the following properties:

1. There exists a tempered random variable $K : \Omega \to [1, +\infty)$ such that
\[
\| \phi \|_\omega \leq K(\omega) \| \phi \|_{BV} \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega \text{ and } \phi \in BV;
\]
   In particular, $\| \cdot \|_\omega$ is a complete norm.
2. For $\mathbb{P}$-a.e. $\omega \in \Omega$, $\phi \in BV$ and $n \in \mathbb{N}$,
\[
\| L^n_\omega \tilde{\Pi}(\omega) \phi \|_{\sigma^n_\omega} \leq e^{-\lambda' n} \| \phi \|_\omega.
\]
3. For $\mathbb{P}$-a.e. $\omega \in \Omega$ and $\phi \in BV$,
\[
\left| \int_X \phi \, d\mu_\omega \right| \leq \| \phi \|_\omega.
\]
4. For $\mathbb{P}$-a.e. $\omega \in \Omega$ and $\phi \in BV$,
\[
\| L_\omega \phi \|_{\sigma_\omega} \leq \| \phi \|_\omega.
\]
5. We have that
\[
\| \mathbb{1} \|_\omega = 1, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.
\]

Proof By using the same notation as in the statement of Corollary 15, we set
\[
\| \phi \|_\omega = \sup_{n \in \mathbb{N}}(\| L^n_\omega \tilde{\Pi}(\omega) \phi \|_{BV} e^{\lambda' n}) + \left| \int_X \phi \, d\mu_\omega \right|,
\]
for $\phi \in BV$ and $\mathbb{P}$-a.e. $\omega \in \Omega$.

We begin by observing that it follows from (23) and the simple observation $\| \cdot \|_{L^1(\mu_\omega)} \leq C_{\text{var}} \| \cdot \|_{BV}$ that
\[
\| \phi \|_\omega \leq (\tilde{D}(\omega) + C_{\text{var}}) \| \phi \|_{BV},
\]
for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $\phi \in BV$. On the other hand,
\[
\| \phi \|_\omega \geq \| \tilde{\Pi}(\omega) \phi \|_{BV} + \left| \int_X \phi \, d\mu_\omega \right|
\geq \| \tilde{\Pi}(\omega) \phi \|_{BV} + \| (\text{Id} - \tilde{\Pi}(\omega)) \phi \|_{BV}
\geq \| \phi \|_{BV},
\]
\[\square\]
for \( P \text{-a.e. } \omega \in \Omega \) and \( \phi \in BV \). Hence, (30) holds with
\[
K(\omega) = \tilde{D}(\omega) + C_{var}, \quad \omega \in \Omega.
\] (36)

Moreover, since \( \int_X \tilde{\Pi}(\omega) \phi d\mu_{\omega} = 0 \) we have that
\[
\| L^n_{\omega} \tilde{\Pi}(\omega) \phi \|_{\sigma^n\omega} = \sup_{m \in \mathbb{N}} \left( \| L^m_{\sigma^n\omega} \tilde{\Pi}(\sigma^n\omega) L^n_{\omega} \phi \|_{BV} e_{\lambda}^m \right)
\]
\[= \sup_{m \in \mathbb{N}} \left( \| L^{m+n}_{\omega} \tilde{\Pi}(\omega) \phi \|_{BV} e_{\lambda}^m \right)
\]
\[= e^{-\lambda n} \sup_{m \in \mathbb{N}} \left( \| L^{m+n}_{\omega} \tilde{\Pi}(\omega) \phi \|_{BV} e_{\lambda}^{(m+n)} \right)
\]
\[\leq e^{-\lambda n} \sup_{m \in \mathbb{N}} \left( \| L^m_{\omega} \tilde{\Pi}(\omega) \phi \|_{BV} e_{\lambda}^m \right)
\]
\[\leq e^{-\lambda n} \| \phi \|_{\omega},
\]
for \( P \text{-a.e. } \omega \in \Omega \) and \( n \in \mathbb{N} \) and \( \phi \in BV \). We conclude that (31) holds. Furthermore, (32) follows readily from (35). In addition, we have that
\[
\| L_{\omega} \phi \|_{\sigma^n\omega} = \sup_{n \in \mathbb{N}} \left( \| L^n_{\sigma^n\omega} \tilde{\Pi}(\sigma^n\omega) L_{\omega} \phi \|_{BV} e_{\lambda}^n \right) + \left| \int_X L_{\omega} \phi d\mu_{\sigma^n\omega} \right|
\]
\[= \sup_{n \in \mathbb{N}} \left( \| L^{n+1}_{\omega} \tilde{\Pi}(\omega) \phi \|_{BV} e_{\lambda}^n \right) + \left| \int_X \phi d\mu_{\omega} \right|
\]
\[= e^{-\lambda} \sup_{n \in \mathbb{N}} \left( \| L^{n+1}_{\omega} \tilde{\Pi}(\omega) \phi \|_{BV} e_{\lambda}^{n+1} \right) + \left| \int_X \phi d\mu_{\omega} \right|
\]
\[\leq \| \phi \|_{\omega},
\]
for \( P \text{-a.e. } \omega \in \Omega \) and \( h \in BV \). Thus, (33) holds. Finally, (34) follows directly from (35).

\[ \square \]

**Remark 17** In [18, Section 3.1], we introduced a similar class of norms, adapted to the original cocycle of transfer operators \( (L_{\omega})_{\omega \in \Omega} \). On the other hand, in Lemma 16 we construct norms adapted to the associated cocycle of normalized transfer operators \( (L_{\omega})_{\omega \in \Omega} \).

To the best of our understanding, in order to construct appropriate adapted norms for the cocycle \( (L_{\omega})_{\omega \in \Omega} \), one needs the additional requirement in Definition 3.

Finally, we note that adapted norms given by Lemma 16 have a simpler form than those constructed in the proof of [18, Proposition 30]. The reason is that the top Oseledets space (see [18, Section 2.1]) of the cocycle \( (L_{\omega})_{\omega \in \Omega} \) is spanned by \( 1 \) and \( L_{\omega} 1 = 1 \). Consequently, the cocycle \( (L_{\omega})_{\omega \in \Omega} \) does not exhibit any growth along the associated top Oseledets space.

We also describe the construction of dual adapted norms.

**Lemma 18** Let \( L = (L_{\omega})_{\omega \in \Omega} \) be a good cocycle of transfer operators. Then, there is a family \( \| \cdot \|_{\sigma^n\omega} \in \Omega \) of norms on \( BV^* \) with the following properties:

1. For \( P \text{-a.e. } \omega \in \Omega \) and \( \ell \in BV^* \),
\[
\frac{1}{K(\omega)} \| \ell \|_{BV^*} \leq \| \ell \|_{\sigma^n\omega} \leq \| \ell \|_{BV^*},
\]
(37)

where \( K : \Omega \to [1, +\infty) \) is as in (30).
2. For \( \mathbb{P} \text{-a.e } \omega \in \Omega \) and \( \ell \in BV^* \),
\[
\|L^* \omega \ell\|_{\sigma \omega}^* \leq \|\ell\|_{\sigma \omega}^*,
\]
(38)

3. For \( \mathbb{P} \text{-a.e } \omega \in \Omega \), \( \ell \in BV^* \) and \( n \in \mathbb{N} \),
\[
\|(L^n \omega)^* \Pi^*(\sigma^n \omega) \ell\|_{\sigma \omega}^* \leq e^{-\lambda' n} \|\ell\|_{\sigma \omega}^*,
\]
(39)
where \( \lambda' \) is as in (31) and \( \Pi^*(\omega) \ell := \ell - \ell(1) \mu_\omega \).

4. We have that
\[
\text{esssup}_{\omega \in \Omega} \|\mu_\omega\|_{\sigma \omega}^* < +\infty.
\]
(40)

5. For \( l \in BV^* \) and \( \mathbb{P} \text{-a.e. } \omega \in \Omega \),
\[
\|l(\mathbb{1})\|_{\sigma \omega}^* \leq \|l\|_{\sigma \omega}^*.
\]
(41)

**Proof** We follow closely the proof of [18, Proposition 33]. For \( \ell \in BV^* \) and \( \mathbb{P} \text{-a.e. } \omega \in \Omega \),
\[
\|\ell\|_{\sigma \omega}^* := \inf\{C > 0 : |\ell(\phi)| \leq C \|\phi\|_{\sigma \omega} \text{ for } \phi \in BV\},
\]
where \( \|\cdot\|_{\sigma \omega}, \omega \in \Omega \) is the family of norms given by Lemma 16. By (30), we have that
\[
|\ell(\phi)| \leq \|\ell\|_{BV^*} \cdot \|\phi\|_{BV} \leq \|\ell\|_{BV^*} \cdot \|\phi\|_{\sigma \omega},
\]
for \( \mathbb{P} \text{-a.e. } \omega \in \Omega \), \( \ell \in BV^* \) and \( \phi \in BV \). Hence, the second inequality in (37) holds. Moreover,
\[
|\ell(\phi)| \leq \|\ell\|_{\sigma \omega}^* \cdot \|\phi\|_{\sigma \omega} \leq K(\omega) \|\ell\|_{\sigma \omega}^* \cdot \|\phi\|_{BV},
\]
for \( \mathbb{P} \text{-a.e. } \omega \in \Omega \), \( \ell \in BV^* \) and \( \phi \in BV \), which yields the first inequality in (37).

Furthermore, using (33) we have that
\[
|L^* \omega \ell(\phi)| = |\ell(L_\omega \phi)| \leq \|\ell\|_{\sigma \omega}^* \cdot \|L_\omega \phi\|_{\sigma \omega} \leq \|\ell\|_{\sigma \omega}^* \cdot \|\phi\|_{\sigma \omega},
\]
for \( \mathbb{P} \text{-a.e. } \omega \in \Omega \), \( \ell \in BV^* \) and \( \phi \in BV \). Thus, (38) holds.

On the other hand, using (31) we have that
\[
|(L^n \omega)^* \Pi^*(\sigma^n \omega) \ell(\phi)| = |\ell(L^n \omega \tilde{\Pi}(\omega) \phi)|
\leq \|\ell\|_{\sigma \omega}^* \cdot \|L^n \omega \tilde{\Pi}(\omega) \phi\|_{\sigma \omega}
\leq e^{-\lambda' n} \|\ell\|_{\sigma \omega}^* \cdot \|\phi\|_{\sigma \omega},
\]
for \( \mathbb{P} \text{-a.e. } \omega \in \Omega \), \( n \in \mathbb{N} \) and \( \ell \in BV^* \). Therefore, (39) holds.

Moreover, by (32) we have that
\[
|\mu_\omega(\phi)| = \left| \int_X \phi \, d\mu_\omega \right| \leq \|\phi\|_{\sigma \omega},
\]
for \( \mathbb{P} \text{-a.e. } \omega \in \Omega \) and \( \phi \in BV \). Hence, (40) holds. Finally, (41) follows readily from (34). \( \Box \)

\[8 \] We identify \( \mu_\omega \) with the functional \( \phi \mapsto \int_X \phi \, d\mu_\omega \) on \( BV \).
2.5 Proof of Theorem 10: Perturbation Results and Consequences

Throughout this section, we consider a good cocycle \((L_\omega)_{\omega \in \Omega}\), and an observable \(\psi : \Omega \times X \to \mathbb{R}^d\), as in the statement of Theorem 10. By \(|x|\) we will denote the Euclidean norm of \(x \in \mathbb{R}^d\). Moreover, we write \(\psi_\omega\) instead of \(\psi(\omega, \cdot)\).

For \(\theta \in \mathbb{C}^d\), \(\omega \in \Omega\) and \(\phi \in BV\), we (formally) set

\[
L_\theta^\omega \phi := L_\omega(e^{\theta \cdot \psi_\omega} \phi).
\]

The proof of the following lemma is inspired by the proof of [18, Lemma 36].

**Lemma 19** There exists \(C' > 0\) such that

\[
\|L_\theta^\omega \phi\|_{\sigma_\omega} \leq C' \|\phi\|_{\omega} \quad \text{for} \quad \mathbb{P}\text{-a.e. } \omega \in \Omega, \ |\theta| \leq 1 \quad \text{and } \phi \in BV,
\]

where \(\|\cdot\|_{\omega}\), \(\omega \in \Omega\) is the family of norms given by Lemma 16.

**Proof** Since \(K(\omega) \geq 1\), it follows from (12) that

\[
\text{esssup}_{\omega \in \Omega} \|\psi_i^j\|_{BV} < +\infty,
\]

for \(1 \leq i \leq d\). Take \(\theta \in \mathbb{C}^d\) such that \(|\theta| \leq 1\). By (30) and (33), we have that

\[
\|L_\theta^\omega \phi - L_\omega \phi\|_{\sigma_\omega} = \|L_\omega((e^{\theta \cdot \psi_\omega} - 1)\phi)\|_{\sigma_\omega}
\leq \|(e^{\theta \cdot \psi_\omega} - 1)\phi\|_{\omega}
\leq K(\omega) \|(e^{\theta \cdot \psi_\omega} - 1)\phi\|_{BV}
\leq C_{\text{var}} K(\omega) \|e^{\theta \cdot \psi_\omega} - 1\|_{BV} \cdot \|\phi\|_{\omega},
\]

for \(\mathbb{P}\text{-a.e. } \omega \in \Omega\) and \(\phi \in BV\).

On the other hand, we have that

\[
\|e^{\theta \cdot \psi_\omega} - 1\|_{BV} = \left\| \prod_{i=1}^d e^{\theta_i \psi_i^j} - 1 \right\|_{BV}
\leq C_{\text{var}}^d \prod_{i=1}^d \prod_{j=1}^{i-1} \|e^{\theta_j \psi_i^j}\|_{BV} \cdot \|e^{\theta_i \psi_i^j} - 1\|_{BV}.
\]

Moreover, for \(1 \leq i \leq d\) we have (see the proof of [18, Lemma 36]) that

\[
\|e^{\theta_i \psi_i^j} - 1\|_{BV} \leq (1 + C_{\text{var}}) e \|\psi_i^j\|_{\infty} \|\psi_i^j\|_{BV},
\]

and consequently

\[
\|e^{\theta_i \psi_i^j}\|_{BV} \leq 1 + (1 + C_{\text{var}}) e \|\psi_i^j\|_{\infty} \|\psi_i^j\|_{BV}.
\]

It follows from (43), (45), (46) and (47) that there exists a constant \(D > 0\) such that

\[
\|e^{\theta \cdot \psi_\omega} - 1\|_{BV} \leq D \sum_{i=1}^d \|\psi_i^j\|_{BV}, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.
\]

By (12), (44) and (48), we find that there exists another constant \(D' > 0\) such that

\[
\|L_\theta^\omega \phi - L_\omega \phi\|_{\sigma_\omega} \leq D' \|\phi\|_{\omega}, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega \text{ and } \phi \in BV.
\]
Finally, we observe that (42) follows readily from (33) together with the above estimate. The proof of the lemma is completed. ⊓⊔

Let $S$ denote the space consisting of all measurable $V : \Omega \times X \to \mathbb{R}$ such that $V_\omega := V(\omega, \cdot) \in BV$ for $\mathbb{P}$-a.e. $\omega \in \Omega$ and

$$
\|V\|_S := \text{esssup}_{\omega \in \Omega} \|V_\omega\|_\omega < +\infty,
$$

where $\|\cdot\|_\omega$, $\omega \in \Omega$ is the family of norms given by Lemma 16. Then, $(S, \|\cdot\|_S)$ is a Banach space.

Furthermore, let $S_0$ denote the set of all $V \in S$ such that

$$
\int_X V_\omega d\mu_\omega = 0, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.
$$

Using (32), it is easy to verify that $S_0$ is a closed subspace of $S$.

For $(\theta, W) \in \mathbb{C}^d \times S_0$, we formally define

$$
F(\theta, W)(\omega, \cdot) = \frac{L_\theta W_{\sigma^{-1}\omega} (1 + W_{\sigma^{-1}\omega})}{\int_X L_\theta W_{\sigma^{-1}\omega} (1 + W_{\sigma^{-1}\omega}) d\mu_\omega} - 1 - W_\omega, \quad \omega \in \Omega.
$$

By arguing as in the proof of [18, Lemma 41], one can establish the following result.

**Lemma 20** There exists a neighborhood $U$ of $(0, 0) \in \mathbb{C}^d \times S_0$ such that $F : U \to S_0$ is well-defined and analytic. Furthermore, its differential w.r.t $W$ at $(0, 0)$, $D_2 F(0, 0) : S_0 \to S_0$ is invertible.

The following result follows from Lemma 20 and the implicit function theorem (exactly as in the proof of [18, Theorem 42]).

**Lemma 21** There exists a neighborhood $U$ of $0 \in \mathbb{C}^d$, such that for any $\theta \in U$, there exist $v^\theta \in S$, $\lambda^\theta \in L^\infty(\Omega)$, satisfying:

1. The maps $U \ni \theta \mapsto v^\theta \in S$ and $U \ni \theta \mapsto \lambda^\theta \in L^\infty(\Omega)$ are analytic.

2. For any $\theta \in U$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, $v^\theta_\omega$, $\lambda^\theta_\omega$ satisfy:

$$
L_\theta^{\cdot} v^\theta_\omega = \lambda^\theta_\omega v^\theta_{\sigma \omega},
$$

$$
\lambda^\theta_\omega = \int_X L_\theta^{\cdot} v^\theta_\omega d\mu_{\sigma \omega},
$$

$$
1 = \int_X v^\theta_\omega d\mu_\omega.
$$

**Remark 22** As noted, Lemma 21 is close in spirit to [18, Theorem 42]. However, there are some important differences. Indeed, in [18, Theorem 42] we considered the case when $d = 1$ and our perturbation result was stated for our original cocycle of transfer operators $(L_\omega)_{\omega \in \Omega}$, while here we deal with the associated cocycle of normalized transfer operators $(L_\omega)_{\omega \in \Omega}$.

Let $N$ consist of all measurable $\Phi : \Omega \to BV^*$ such that

$$
\|\Phi\|_N := \text{esssup}_{\omega \in \Omega} \|\Phi_\omega\|^*_\omega < +\infty,
$$

where $\|\cdot\|^*_\omega$, $\omega \in \Omega$ is the family of norms given by Lemma 18. By $N_0$ we denote the subspace of $N$ consisting of all $\Phi \in N$ such that

$$
\Phi_\omega(1) = 0, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.
$$
Then, it follows easily from (41) that $N_0$ is a closed subspace of $N$.

For $(\theta, W) \in \mathbb{C}^d \times S_0$, we formally define

$$F^*(\theta, \Phi)_{\omega} = \frac{(L^\theta_{\omega})^*(\Phi_{\sigma\omega} + \mu_{\sigma\omega})}{(L^\theta_{\omega})^*(\Phi_{\sigma\omega} + \mu_{\sigma\omega})(\mathbb{I})} - \Phi_{\omega} - \mu_{\omega}, \ \omega \in \Omega.$$  

One can show that $F^*$ is well-defined and analytic on a neighborhood of $(0, 0) \in \mathbb{C}^d \times S_0$. Moreover, by arguing as in the proof of [18, Proposition 44] (see also [18, Remark 45]), one has:

**Lemma 23** There exists a neighborhood $U$ of $0$ in $\mathbb{C}^d$ and an analytic map $U \ni \theta \mapsto \phi_{\theta} \in N$ such that

$$(L^\theta_{\omega})^* \phi^\theta_{\sigma_{\omega} \omega} = \lambda^\theta_{\omega} \phi^\theta_{\omega}, \ \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega \text{ and } \theta \in U'.$$

By arguing as in the proof of [18, Lemma 59], one can also establish the following result.

**Lemma 24** There exists $r \in (0, 1)$ such that for $\theta \in \mathbb{C}^d$ sufficiently close to $0$, $\mathbb{P}$-a.e. $\omega \in \Omega$, $h \in BV$ and $n \in \mathbb{N}$,

$$\|L^\theta_{\omega} (h - \phi^\theta_{\omega}(h)v^\theta_{\omega})\|_{\sigma^\omega} \leq r^n \|h\|_{\omega}.$$  

The following auxiliary result plays an important role in the proof of Theorem 10.

**Lemma 25** There exist constants $K, \rho > 0$ such that

$$\|L^\theta_{\omega} h\|_{\sigma^\omega} \leq K \|h\|_{\omega},$$

for $t \in \mathbb{R}^d, |t| \leq \rho$, $\mathbb{P}$-a.e. $\omega \in \Omega$, $h \in BV$ and $n \in \mathbb{N}$.

**Proof** We have that

$$\int_X e^{it \cdot S_n \psi(\omega, \cdot)} d\mu_\omega = \int_X L^{it, n}_{\omega} \mathbb{I} d\mu_{\sigma^\omega}$$

$$= \int_X L^{it, n}_{\omega} (\mathbb{I} - v^{it \omega}_{\omega}) d\mu_{\sigma^\omega} + \int_X L^{it, n}_{\omega} v^{it \omega}_{\omega} d\mu_{\sigma^\omega}$$

$$= \int_X \left[ \sum_{j=0}^{n-1} \lambda^{it \omega}_{\sigma^\omega j} \int_X e^{it \cdot S_n \psi(\omega, \cdot)} d\mu_\omega \right]$$

where $S_n \psi$ is given by (14). Thus,

$$\left| \sum_{j=0}^{n-1} \lambda^{it \omega}_{\sigma^\omega j} \right| \leq \int_X L^{it, n}_{\omega} (\mathbb{I} - v^{it \omega}_{\omega}) d\mu_{\sigma^\omega} + \int_X e^{it \cdot S_n \psi(\omega, \cdot)} d\mu_\omega$$

$$\leq 1 + \int_X \left| e^{it \cdot S_n \psi(\omega, \cdot)} (\mathbb{I} - v^{it \omega}_{\omega}) \right| d\mu_\omega$$

$$\leq 1 + \| \mathbb{I} - v^{it \omega}_{\omega} \|_{L^1(\mu_\omega)}$$

$$\leq 1 + C_{\text{var}} \| \mathbb{I} - v^{it \omega}_{\omega} \|_{BV}$$

$$\leq 1 + C_{\text{var}} \| \mathbb{I} - v^{it \omega}_{\omega} \|_{\omega}$$

$$\leq 1 + C_{\text{var}} \| \mathbb{I} - v^{it \omega}_{\omega} \|_{S},$$
from which it follows that there exists $\rho > 0$ such that

$$\left| \prod_{j=0}^{n-1} \lambda_{\sigma_j \omega}^{i_t} \right| \leq 2, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega, n \in \mathbb{N} \text{ and } t \in \mathbb{R}^d, \ |t| \leq \rho. \quad (49)$$

On the other hand, we have that

$$L_{\omega}^{i_t, n} h = L_{\omega}^{i_t, n} (h - \phi_{\omega}^{i_t}(h)v_{\omega}^{i_t}) + \phi_{\omega}^{i_t}(h)L_{\omega}^{i_t, n} v_{\omega}^{i_t},$$

and thus

$$\|L_{\omega}^{i_t, n} h\|_{\sigma^{n \omega}} \leq \|L_{\omega}^{i_t, n} (h - \phi_{\omega}^{i_t}(h)v_{\omega}^{i_t})\|_{\sigma^{n \omega}} + \|\phi_{\omega}^{i_t}\|_{\mathcal{N}} \cdot \left| \prod_{j=0}^{n-1} \lambda_{\sigma_j \omega}^{i_t} \right| \cdot \|v_{\omega}^{i_t}\|_{S} \cdot \|h\|_{\omega}, \quad (50)$$

for $\mathbb{P}$-a.e. $\omega \in \Omega, n \in \mathbb{N}, h \in BV$ and $t \in \mathbb{R}^d$ sufficiently close to 0. The desired conclusion follows directly from (49) and (50) together with Lemma 24 and the continuity of $t \rightarrow \|\phi_{\omega}^{i_t}\|_{\mathcal{N}}$. \hfill \Box

2.6 Completing the Proof of Theorem 10

In order to establish the first assertion of Theorem 10, we will rely on the arguments in [12, 17, 18]. By $E_\omega(\phi)$ we will denote $\int_X \phi d\mu_\omega$. In addition, let $\tau$ and $\mu$ be as in (2) and (3), respectively.

Firstly, as in [17, Proposition 4.17], by replacing $\psi$ with $\psi \cdot \cdot v$ for an arbitrary unit vector $v$, it is enough to consider scalar valued functions $\psi$. In the scalar case, by using (43) and arguing as in the proof of [12, Lemma 12], we find that

$$E_\omega \left( \sum_{k=0}^{n-1} \psi_{\sigma_k \omega} \circ T_{\omega}^{k} \right)^2 = \sum_{k=0}^{n-1} E_\omega((\psi_{\sigma_k \omega})^2 \circ T_{\omega}^{k}) + 2 \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} E_\sigma(\psi_{\sigma_i \omega}(\psi_{\sigma_j \omega} \circ T_{\sigma_j \omega}^{i-j}))$$

and

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} E_\omega((\psi_{\sigma_k \omega})^2 \circ T_{\omega}^{k}) = \int_{\Omega \times X} \psi(\omega, x)^2 d\mu(\omega, x),$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$. Set

$$\Psi(\omega) = \sum_{n=1}^{\infty} \int_X \psi(\omega, x) \psi(\tau^n(\omega, x)) d\mu_\omega(x) = \sum_{n=1}^{\infty} \int_X L_{\omega}^{n}(\psi_{\omega}) \psi_{\sigma^n \omega} d\mu_{\sigma^n \omega}.$$
By (30), (31) and (32), we have that
\[ \left| \int_X L_n^\infty(\psi_\omega) \psi_{\sigma_n \omega} d\mu_{\sigma_n \omega} \right| \leq \| L_n^\infty(\psi_\omega) \cdot \psi_{\sigma_n \omega} \|_{\sigma_n \omega} \]
\[ \leq K(\sigma_n \omega) \| L_n^\infty(\psi_\omega) \cdot \psi_{\sigma_n \omega} \|_{BV} \]
\[ \leq C_{var} K(\sigma_n \omega) \| L_n^\infty(\psi_\omega) \|_{BV} \cdot \| \psi_{\sigma_n \omega} \|_{BV} \]
\[ \leq C_{var} K(\sigma_n \omega) e^{-\lambda_n} \| \psi_\omega \|_{\omega} \cdot \| \psi_{\sigma_n \omega} \|_{BV} \]
\[ \leq C_{var} e^{-\lambda_n} K(\omega) \| \psi_\omega \|_{BV} K(\sigma_n \omega) \| \psi_{\sigma_n \omega} \|_{BV}, \tag{51} \]
for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \) and \( n \in \mathbb{N} \). Therefore by (12) we get that \( \text{esssup}_{\omega \in \Omega} |\Psi(\omega)| < +\infty \). Thus, \( \Psi \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \). Hence, it follows from Birkhoff’s ergodic theorem that, for \( \mathbb{P} \) a.e. \( \omega \in \Omega \),
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \Psi(\sigma^i \omega) = \int_{\Omega} \Psi(\omega) d\mathbb{P}(\omega) \]
\[ = \sum_{n=1}^{\infty} \int_{\Omega \times X} \psi(\omega, x) \psi(\tau^n(\omega, x)) d\mu(\omega, x). \tag{52} \]
Moreover, we have that
\[ \left| \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} E_{\sigma^i \omega}(\psi_{\sigma^i \omega}(\psi_{\sigma_j \omega} \circ T_{\sigma^i \omega}^{j-i})) - \sum_{i=0}^{n-1} \Psi(\sigma^i \omega) \right| \]
\[ \leq \sum_{i=0}^{n-1} \sum_{k=n-i}^{\infty} \int_X L_k(\sigma^i \omega) \psi_{\sigma^{k+i} \omega} d\mu_{\sigma^{k+i} \omega} \]
\[ \leq C_{var} \left( \text{esssup}_{\omega \in \Omega} K(\omega) \| \psi_\omega \|_{BV} \right)^2 \sum_{i=0}^{n-1} \sum_{k=n-i}^{\infty} e^{-\lambda k}. \]
We thus derive that
\[ \lim_{n \to \infty} \frac{1}{n} \left( \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} E_{\sigma^i \omega}(\psi_{\sigma^i \omega}(\psi_{\sigma_j \omega} \circ T_{\sigma^i \omega}^{j-i})) - \sum_{i=0}^{n-1} \Psi(\sigma^i \omega) \right) = 0, \tag{53} \]
It follows from (52) and (53) that
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} E_{\sigma^i \omega}(\psi_{\sigma^i \omega}(\psi_{\sigma_j \omega} \circ T_{\sigma^i \omega}^{j-i})) = \sum_{n=1}^{\infty} \int_{\Omega \times X} \psi(\omega, x) \psi(\tau^n(\omega, x)) d\mu(\omega, x), \]
for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \). Hence, we conclude that
\[ \lim_{n \to \infty} \frac{1}{n} E_{\omega}(\sum_{k=0}^{n-1} \psi_{\tau^k \omega} \circ T_{\omega}^k)^2 = \int_{\Omega \times X} \psi(\omega, x)^2 d\mu(\omega, x) \]
\[ + 2 \sum_{n=1}^{\infty} \int_{\Omega \times X} \psi(\omega, x) \psi(\tau^n(\omega, x)) d\mu(\omega, x) \]
\[ =: \Sigma^2 \geq 0, \]
We have that
\[ \mathbb{P}-a.e. \, \omega \in \Omega. \]
In order to show that \( \Sigma^2 = 0 \) if and only if \( \psi \) is a coboundary with \( r \in L^2(\Omega \times X, \mu) \) (see (15)), one can argue as in [12]. The proof that we can get (15) with a function \( r \) so that \( \text{esssup}_{\omega \in \Omega} \|r(\omega, \cdot)\|_{BV} < \infty \) is postponed to the next section, see Lemma 34 (note that it is enough to consider real valued functions \( \psi \)).

We now establish the second assertion of Theorem 10. In the following lemma, we verify [17, condition (2.1)].

**Lemma 26** There exist constants \( C, c, \rho > 0 \) such that for \( \mathbb{P}-a.e. \, \omega \in \Omega, \) any \( n, m > 0, \) \( b_1 < b_2 < \ldots < b_{n+m+1}, k > 0 \) and \( t_1, \ldots, t_{n+m} \in \mathbb{R}^d \) with \( |t_j| \leq \rho \), we have that

\[
\left| \mathbb{E}_\omega \left( i^\sum_{j=1}^n t_j \left( \sum_{\ell=0}^{b_j} A_\ell \right) + i^\sum_{j=n+1}^{n+m} t_j \left( \sum_{\ell=0}^{b_j+b_\ell} A_\ell \right) \right) \right|
\leq C^{n+m} e^{-ck},
\]

where
\[ A_\ell := \psi_\sigma^\ell \omega \circ T^\ell_\omega, \quad \ell \in \mathbb{N}. \]

**Proof** Set
\[ Q_wh = \left( \int_X h \, d\mu_\omega \right) \mathbb{I}, \quad \text{for } \omega \in \Omega \text{ and } h \in BV. \]

We have that
\[
\mathbb{E}_\omega \left( i^\sum_{j=1}^n t_j \left( \sum_{\ell=0}^{b_j} A_\ell \right) + i^\sum_{j=n+1}^{n+m} t_j \left( \sum_{\ell=0}^{b_j+b_\ell} A_\ell \right) \right)
= \mathbb{E}_{\sigma^{b_{n+m+1}+k}} \left( \prod_{j=n+1}^{n+m} L_{\sigma^{b_{j+1}+b_j}} \left( L_{\sigma^{b_{n+1}}} \prod_{j=1}^{n} L_{\sigma^{b_{j-1}+b_j}} (\mathbb{I}) \right) \right)
= \mathbb{E}_{\sigma^{b_{n+m+1}+k}} \left( \prod_{j=n+1}^{n+m} L_{\sigma^{b_{j+1}+b_j}} \left( L_{\sigma^{b_{n+1}}} - Q_{\sigma^{b_{n+1}}} \right) \prod_{j=1}^{n} L_{\sigma^{b_{j-1}+b_j}} (\mathbb{I}) \right)
+ \mathbb{E}_{\sigma^{b_{n+m+1}+k}} \left( \prod_{j=n+1}^{n+m} L_{\sigma^{b_{j+1}+b_j}} Q_{\sigma^{b_{n+1}}} \prod_{j=1}^{n} L_{\sigma^{b_{j-1}+b_j}} (\mathbb{I}) \right).
\]

It follows from (31), (32), (34) and Lemma 25 that
\[
\left| \mathbb{E}_{\sigma^{b_{n+m+1}+k}} \left( \prod_{j=n+1}^{n+m} L_{\sigma^{b_{j+1}+b_j}} \left( L_{\sigma^{b_{n+1}}} - Q_{\sigma^{b_{n+1}}} \right) \prod_{j=1}^{n} L_{\sigma^{b_{j-1}+b_j}} (\mathbb{I}) \right) \right|
\leq K^{n+m} e^{-\lambda^2 k} \text{esssup}_{\omega \in \Omega} \|\mathbb{I}\|_{\omega}
= K^{n+m} e^{-\lambda^2 k}.
\]

\[ \square \]
The conclusion of the lemma follows from an observation that
\[
\mathbb{E}_{\sigma^{b_{n+m+1}+k}\omega} \left( \prod_{j=n+1}^{n+m} L_{\sigma^{b_{j}+k}\omega}^{i_{t_{j},b_{j+1}-b_{j}}} Q_{\sigma^{b_{j+1}}\omega} \prod_{j=1}^{n} L_{\sigma^{b_{j}}\omega}^{i_{t_{j},b_{j+1}-b_{j}}} (1) \right) \\
= \mathbb{E}_{\omega} \left( e^{i \sum_{j=1}^{n} t_{j} \left( \sum_{l=b_{j}}^{b_{j+1}-1} \mathcal{A}_{c} \right)} \right) \cdot \mathbb{E}_{\omega} \left( e^{i \sum_{j=n+1}^{n+m} t_{j} \left( \sum_{l=b_{j}+k}^{b_{j+1}-1} \mathcal{A}_{c} \right)} \right).
\]

Next we verify [17, condition (2.5)].

**Lemma 27** There exist constants \(C_0 > 0\) and \(r \in (0, 1)\) such that
\[
| \text{Cov}(A_n \cdot v, A_{n+k} \cdot v) | \leq C_0 r^k,
\]
for \(\mathbb{P}\)-a.e. \(\omega \in \Omega\), \(n, k \in \mathbb{N}\) and \(v \in \mathbb{R}^d\) such that \(|v| = 1\), where \(A_n = \psi_{\sigma^n \omega} \circ T^n_\omega\).

**Proof** We have that
\[
\text{Cov}(A_n \cdot v, A_{n+k} \cdot v) = \sum_{1 \leq i,j \leq d} \int_X v_i v_j \psi_{\sigma^n \omega}^{i} \circ T^n_\omega \cdot \psi_{\sigma^{n+k} \omega}^{j} \circ T^{n+k}_\omega \, d\mu_\omega \\
= \sum_{1 \leq i,j \leq d} v_i v_j \int_X \psi_{\sigma^n \omega}^{i} \cdot \psi_{\sigma^{n+k} \omega}^{j} \circ T^{n+k}_\omega \, d\mu_\omega \\
= \sum_{1 \leq i,j \leq d} v_i v_j \int_X L_{\sigma^{n} \omega}^{k} (\psi^{i}_{\sigma^{n} \omega}) \psi^{j}_{\sigma^{n+k} \omega} \, d\mu_{\sigma^{n+k} \omega}. \tag{55}
\]
The same computation as (51) now yields
\[
\left| \int_X L_{\sigma^{n} \omega}^{k} (\psi^{i}_{\sigma^{n} \omega}) \psi^{j}_{\sigma^{n+k} \omega} \, d\mu_{\sigma^{n+k} \omega} \right| \leq C_{\text{var}} e^{-\lambda^k K (\sigma^n \omega)} \| \psi^{i}_{\sigma^{n} \omega} \|_{BV} K (\sigma^{n+k} \omega) \| \psi^{j}_{\sigma^{n+k} \omega} \|_{BV}, \tag{56}
\]
which, together with (12) and (55) imply the conclusion of the lemma. \(\square\)

The conclusion of Theorem 10 now follows directly from the previous two lemmas by applying the abstract version of ASIP given in [17, Theorem 2.1].

**Remark 28** We can now explain the reason for introducing adapted norms. We first observe that it follows from (30) and Lemma 25 that
\[
\| L_{t_{j}}^{i_{t_{j}}n} \|_{BV} \leq M(\omega) \quad \text{for } t \in \mathbb{R}^d, |t| \leq \rho, n \in \mathbb{N} \text{ and } \mathbb{P}\text{-a.e. } \omega \in \Omega, \tag{57}
\]
where \(M: \Omega \to (0, +\infty)\) is a tempered random variable. By relying solely on (23) and (57), we observe that the L.H.S in (54) can be bounded by
\[
\left( \prod_{j=n+1}^{n+m} M(\sigma^{b_{j}+k} \omega) \right) \tilde{D}(\sigma^{b_{n+1} \omega}) e^{-\lambda^k K (\sigma^{n} \omega)} \left( \prod_{j=1}^{n} M(\sigma^{b_{j} \omega}) \right).
\]
Note that even in the case when \(M\) is a constant, the above expression depends on \(\omega\) and thus [17, Theorem 2.1] is not directly applicable. Our construction of adapted norms is precisely tailored to overcome this difficulty.
3 A Scalar-Valued Almost Sure Invariance Principle

In this section we will improve the rates obtained in the previous section for a class of real valued observables. In order to achieve that, we need to impose an additional requirement. More precisely, we suppose that there exists a tempered random variable $N : \Omega \rightarrow (0, +\infty)$ such that

$$\|g \circ T_\omega\|_{BV} \leq N(\omega)\|g\|_{BV}, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega \text{ and } g \in BV.$$  \hfill (58)

**Remark 29** In the setting of Example 5, the above condition is satisfied whenever the number $N_\omega$ of monotonicity intervals is tempered.

**Theorem 30** Let $L = (L_\omega)_{\omega \in \Omega}$ be a good cocycle of transfer operators. Then, there exists a tempered random variable $\tilde{K} : \Omega \rightarrow [1, +\infty)$ with the property that for each measurable observable $\psi : \Omega \times X \rightarrow \mathbb{R}$ satisfying the following properties:

$$\text{esssup}_{\omega \in \Omega} \left( \tilde{K}(\omega)\|\psi_\omega\|_{BV} \right) < +\infty,$$  \hfill (59)

and for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\int_X \psi_\omega \, d\mu_\omega = 0,$$

the following holds:

1. there exists $\Sigma^2 \geq 0$ such that

$$\Sigma^2 = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}_\omega \left( \sum_{k=0}^{n-1} \psi_{\sigma^k\omega} \circ T_\omega^k \right)^2, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega;$$

Moreover, $\Sigma^2 = 0$ if and only if $\psi = r - r \circ \tau$ for some measurable $r : \Omega \times X \rightarrow \mathbb{R}$ so that $\text{esssup}_{\omega \in \Omega} \|r(\omega, \cdot)\|_{BV} < \infty$.

2. Assume $\Sigma^2 > 0$. Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $\forall \epsilon > \frac{3}{4}$, by enlarging the probability space $(X, \mathcal{B}, \mu_\omega)$ if necessary, it is possible to find a sequence $(Z_k)_k$ of independent and centered (i.e. of zero mean) Gaussian random variables such that $\mu_\omega$ almost surely,

$$\sup_{1 \leq k \leq n} \left| \sum_{i=0}^{k-1} (\psi_{\sigma^i\omega} \circ T_\omega^i) - \sum_{i=1}^{k} Z_i \right| = O(n^{1/4} \log^e(n)).$$

Moreover, the difference between the $L^2$-norms of $\sum_{i=0}^{k-1} (\psi_{\sigma^i\omega} \circ T_\omega^i)$ and $\sum_{i=1}^{k} Z_i$ is bounded in $k$, and the variance of $Z_i$ equals $\int_X m_{\sigma^i\omega}^2 \, d\mu_{\sigma^i\omega}$, with $m_\omega$ given by (63).

**Remark 31** We note that $\tilde{K}$ will be constructed so that $K \leq \tilde{K}$, where $K$ is as in the statement of Theorem 10. Consequently, the first part of Theorem 30 will be a consequence of Theorem 10.

Let $K_1 = \max\{K, N, \tilde{D}\}$, where $K$ is as in the statement of Theorem 10, $\tilde{D}$ is as in the statement of Corollary 15 and $N$ is from (58). By taking into account Proposition 9, we can assume that

$$K_1(\sigma^n\omega) \leq K_1(\omega)e^{\delta|n|} \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega \text{ and } n \in \mathbb{Z},$$  \hfill (60)

where $\delta \in (0, 2\lambda'/5)$ is fixed but arbitrary and $\lambda' > 0$ is given by Corollary 15. Finally, set $\tilde{K} = K_1^{7/2}$. Then, $\tilde{K}$ is tempered.
Set
\[ \chi_\omega := \sum_{n=1}^{\infty} L_n^{\sigma^{-n}_\omega}(\psi_{\sigma^{-n}_\omega}), \quad \omega \in \Omega. \quad (61) \]

**Lemma 32** We have that
\[ \text{esssup}_{\omega \in \Omega} (K_1(\omega)^{5/2} \| \chi_\omega \|_{BV}) < +\infty. \quad (62) \]

**Proof** By (23) and (60), it follows that
\[
\| \chi_\omega \|_{BV} \leq \sum_{n=1}^{\infty} K_1(\sigma^{-n}_\omega)e^{-\lambda'n} \| \psi_{\sigma^{-n}_\omega} \|_{BV}
\]
\[
\leq \text{esssup}_{\omega \in \Omega} \left( \tilde{K}(\omega) \| \psi_{\omega} \|_{BV} \right) \sum_{n=1}^{\infty} \frac{1}{K_1(\sigma^{-n}_\omega)^{5/2}} e^{-\lambda'n}
\]
\[
\leq \text{esssup}_{\omega \in \Omega} \left( \tilde{K}(\omega) \| \psi_{\omega} \|_{BV} \right) K_1(\omega)^{-5/2} \sum_{n=1}^{\infty} e^{-(\lambda' - \delta/2)n},
\]
for \( \mathbb{P}\)-a.e. \( \omega \in \Omega \). Since \( \delta \in (0, 2\lambda'/5) \), we conclude that the statement of the lemma holds. \( \square \)

Next, set
\[ m_\omega = \psi_\omega + \chi_\omega - \chi_{\sigma \omega} \circ T_\omega, \quad \omega \in \Omega. \quad (63) \]

**Lemma 33** We have
\[ \text{esssup}_{\omega \in \Omega} K_1(\omega)^{3/2} \| m_\omega \|_{BV} < +\infty. \]

**Proof** We have that
\[
\| m_\omega \|_{BV} \leq \| \psi_\omega \|_{BV} + \| \chi_\omega \|_{BV} + N(\omega) \| \chi_{\sigma \omega} \|_{BV}
\]
\[
\leq \| \psi_\omega \|_{BV} + \| \chi_\omega \|_{BV} + K_1(\omega) \| \chi_{\sigma \omega} \|_{BV},
\]
for \( \mathbb{P}\)-a.e. \( \omega \in \Omega \), which together with \( K_1(\omega) \geq 1 \), Lemma 32 and (60) easily implies the lemma. \( \square \)

Before proceeding with the proof of the ASIP, we complete the proof about the existence of a coboundary representation with a \( BV \) function:

**Lemma 34** Suppose that there exists a measurable map \( c : \Omega \times X \to \mathbb{R} \) such that
\[ \psi = c \circ \tau - c \quad \text{and} \quad \int_{\Omega \times X} |c(\omega, x)|^2 d\mu(\omega, x) < \infty. \quad (64) \]

Then for \( \mathbb{P}\)-a.e \( \omega \in \Omega \), \( c_\omega := c(\omega, \cdot) \in BV \) and \emph{esssup}_{\omega \in \Omega} \| c(\omega, \cdot) \|_{BV} < \infty. \]

**Proof** First, notice that the function \( c \) in (64) satisfies \( c_\omega = c(\omega, \cdot) \in L^2(\mu_\omega) \) for \( \mathbb{P}\)-a.e. \( \omega \in \Omega \). Next, by Lemma 32 we have
\[ \text{esssup}_{\omega \in \Omega} \| \chi_\omega \|_{BV} < \infty. \quad (65) \]

\( \square \)
The rest of the proof will only rely on (65) (and not the stronger condition (62)) and thus remains valid in the circumstances of Theorem 10, since the arguments in the proof of Lemma 32 give (65) when $K(\omega)\|\psi_\omega\|_{BV}$ is bounded. A straightforward computation yields that

$$\chi_\omega - L_{\sigma^{-1}\omega}\chi_{\sigma^{-1}\omega} = L_{\sigma^{-1}\omega}\psi_{\sigma^{-1}\omega}, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$ \hfill (66)

On the other hand, (64) together with the fact that $L_{\sigma^{-1}\omega}(c_\omega \circ T_{\sigma^{-1}\omega}) = c_\omega$ (see [12, Lemma 7]) imply that

$$c_\omega - L_{\sigma^{-1}\omega}c_{\sigma^{-1}\omega} = L_{\sigma^{-1}\omega}\psi_{\sigma^{-1}\omega}, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$ \hfill (67)

Setting $d_\omega := c_\omega - \chi_\omega$, it follows from the last two identities that

$$d_{\sigma \omega} = L_{\omega}d_\omega \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega,$$ \hfill (68)

Since $\sigma$ is ergodic, we conclude that the integral $d_0 := \int_X d_\omega d\mu_\omega$ does not depend on $\omega$. Moreover, since $|L_\omega d_\omega| \leq L_\omega |d_\omega|$ we have that the norm $\|d_\omega\|_{L^1(\mu_\omega)}$ does not depend on $\omega$. Next, by iterating (66) we obtain that

$$d_{\sigma^n \omega} = L^n_\omega d_\omega, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega \text{ and } n \in \mathbb{N}.$$ \hfill (69)

Observe now that for all $g_1, g_2 \in L^1(\mu_\omega)$ we have

$$\|L^n_\omega g_1 - L^n_\omega g_2\|_{L^1(\mu_{\sigma^n \omega})} \leq \int_X (|g_1 - g_2|)d\mu_{\sigma^n \omega} = \|g_1 - g_2\|_{L^1(\mu_\omega)}.$$ \hfill (70)

Now, since functions with bounded variation are dense in $L^1(\mu_\omega)$, by approximating $d_\omega$ by a function with bounded variation and then using (23) we obtain that for $\mathbb{P}$ a.e. $\omega \in \Omega$, we have

$$\lim_{n \to \infty} \left\| L^n_\omega d_\omega - \left( \int_X d_\omega d\mu_\omega \right) \right\|_{L^1(\mu_{\sigma^n \omega})} = 0.$$ \hfill (71)

By (67) and (68), we get that

$$\lim_{n \to \infty} \|d_{\sigma^n \omega} - d_0\|_{L^1(\mu_{\sigma^n \omega})} = 0, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$ \hfill (72)

Since $\sigma$ is ergodic we conclude that $d_\omega = d_0$ in $L^1(\mu_\omega)$ for $\mathbb{P}$ a.e. $\omega \in \Omega$. Indeed,

$$\int_{\Omega \times X} |d(\omega, x) - d_0|d\mu_\omega(x) = \int_{\Omega} \left( \int_X |d \circ \tau^n(\omega, x) - d_0|d\mu_\omega(x) \right) d\mathbb{P}(\omega)$$

$$= \int_{\Omega} \left( \int_X |d_{\sigma^n \omega} - d_0|d\mu_{\sigma^n \omega}(x) \right) d\mathbb{P}(\omega)$$

$$= \int_{\Omega} \|d_{\sigma^n \omega} - d_0\|_{L^1(\mu_{\sigma^n \omega})} d\mathbb{P}(\omega),$$

where $d(\omega, x) := d_\omega(x)$. Next, observe that the random variables $G_n(\omega) = \|d_{\sigma^n \omega} - d_0\|_{L^1(\mu_{\sigma^n \omega})}$ are uniformly bounded since $\|d_\omega\|_{L^1(\mu_\omega)}$ does not depend on $\omega$. By using the dominated convergence theorem we conclude that the above right hand side converges to 0. Therefore, we get that for $\mathbb{P}$-a.e. $\omega \in \Omega$, $d_\omega = d_0$. Hence,

$$c_\omega = d_\omega + \chi_\omega = d_0 + \chi_\omega \in BV,$$

with $\text{esssup}_{\omega \in \Omega} \|c_\omega\|_{BV} < +\infty$, for $c$ satisfying (64). \hfill \square
Going back to 33, one has via a straightforward computation:

$$L_\omega(m_\omega) = 0$$ for \(\mathbb{P}\text{-a.e. } \omega \in \Omega\).

In particular, it follows (see [12, Proposition 2.]) that

$$\mathbb{E}_\omega[m_\sigma^n \circ T^n_\omega | (T^n_\omega)^{-1}(B)] = L_\sigma^n(m_\sigma^n \circ T^n_\omega) = 0.$$ 

In other words, \((m_\sigma^n \circ T^n_\omega)_{n\geq 0}\) is a so-called reversed martingale difference with respect to the sequence of \(\sigma\)-algebras \(((T^n_\omega)^{-1}(B))_{n\geq 0}\). In view of the above lemmas together with (63), we have that

$$\sup_n \text{esssup}_{\omega \in \Omega} \left\| \sum_{k=0}^{n-1} \psi_{\sigma^k \omega} \circ T^k_\omega - \sum_{k=0}^{n-1} m_{\sigma^k \omega} \circ T^k_\omega \right\|_\infty < \infty. \quad (69)$$

Next, we define a new observable \(\hat{\psi} : \Omega \times X \to \mathbb{R}\) by

$$\hat{\psi}_\omega = L_\omega(m_\omega^2) \circ T_\omega - \int_X m_\omega^2 d\mu_\omega, \quad \omega \in \Omega.$$ 

Clearly,

$$\int_X \hat{\psi}_\omega d\mu_\omega = 0, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$ 

Moreover, we have that

$$\text{esssup}_{\omega \in \Omega} K_1(\omega) \| \hat{\psi}_\omega \|_{BV} < \infty. \quad (70)$$

Indeed, it follows from (23), (24), (58) and Lemma 33 that there exists \(C > 0\) such that

$$\| L_\omega(m_\omega^2) \circ T_\omega \|_{BV} \leq 2 \tilde{D}(\omega) N(\omega) \| m_\omega^2 \|_{BV} \leq 2 K_1(\omega)^2 \| m_\omega^2 \|_{BV} \leq C K_1(\omega)^{-1},$$

for \(\mathbb{P}\text{-a.e. } \omega \in \Omega\). The above estimate together with Lemma 33 implies that (70) holds. We will need the following lemma.

**Lemma 35** For \(\mathbb{P}\text{-a.e. } \omega \in \Omega\), we have that

$$\sum_{k=0}^{n-1} \hat{\psi}_{\sigma^k \omega} \circ T^k_\omega = O(n^{1/2} \sqrt{\log \log n}).$$

**Proof** There are two possibilities. Either the variance associated to \(\hat{\psi}\) is nonzero or zero. If it is nonzero, then it follows from Theorem 10 together with (70) (recall also that \(K_1 \geq K\)) that the process \((\hat{\psi}_{\sigma^k \omega} \circ T^k_\omega)\) satisfies the ASIP. In particular the law of iterated logarithm holds true, which implies the desired conclusion.

In the case the variance vanishes, as in the proof of Theorem 10 there exists a bounded measurable function \(c \in L^2(\Omega \times X, \mu)\) so that \(\hat{\psi} = c \circ \tau - c\), \(\mu\) almost everywhere. Thus,

$$\sum_{k=0}^{n-1} \hat{\psi}_{\sigma^k \omega} \circ T^k_\omega = -c_\omega + c_{\sigma^n \omega} \circ T^n_\omega, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$ 

We remark that we may construct for \(\hat{\psi}\), via (61), an observable \(\hat{\chi}\) satisfying Lemma 32 and thus (65), as we did for \(\psi\). Hence, Lemma 34 applies, and we actually have

\[\mathbb{E}_\omega[\phi|G]\] denotes the conditional expectation of \(\phi\) with respect to the \(\sigma\)-algebra \(G\) and \(\mu_\omega\).
\[
\text{esssup}_{\omega \in \Omega} \| c(\omega, \cdot) \|_{BV} < \infty. \quad \text{Using (V3) we then get that}
\]
\[
\sum_{k=0}^{n-1} \hat{\psi}_{\sigma^k \omega} \circ T_{\omega}^k = O(1).
\]

Alternatively, since \( \mu \) is ergodic and \( c^2 \) integrable we have that
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} c^2_{\sigma^j \omega} \circ T_{\omega}^j = \int_{\Omega \times X} c^2 \, d\mu
\]
and so for \( \mathbb{P} \text{-a.e} \, \omega \in \Omega \) and for \( \mu_{\omega \text{-a.e}} \, x \) we have
\[
c^2_{\sigma^n \omega} \circ T_{\omega}^n = o(n),
\]
which implies that
\[
\sum_{k=0}^{n-1} \hat{\psi}_{n^k \omega} \circ T_{\omega}^k = o(\sqrt{n}),
\]
giving us the announced result. \( \square \)

Finally, we recall the following result.

**Lemma 36** ([10]) Let \((X_n)_n\) be a sequence of square integrable random variables adapted to a non-increasing filtration \((G_n)_n\). Assume that \(\mathbb{E}(X_n | G_{n+1}) = 0 \) almost surely, that
\[
v^2_n := \sum_{k=1}^{n} \mathbb{E}(X_k^2) \quad \longrightarrow \quad \infty \quad \text{(71)}
\]
and that \(\sup_n \mathbb{E}(X_n^2) < \infty\). Moreover, let \((a_n)_n\) be a non-decreasing sequence of positive numbers such that \((a_n/v_n^2)_n\) is non-increasing, \((a_n/v_n)_n\) is non-decreasing and:
\[
\sum_{k=1}^{n} (\mathbb{E}(X_k^2 | G_{k+1}) - \mathbb{E}(X_k^2)) = o(a_n) \quad \text{a.s.;} \quad \text{(72)}
\]
\[
\sum_{n \geq 1} a_n^{-v} \mathbb{E}(|X_n|^{2v}) < \infty \quad \text{for some } 1 \leq v \leq 2. \quad \text{(73)}
\]
Then, up to enlarging our probability space, it is possible to find a sequence \((Z_k)_k\) of independent and centered Gaussian variables with \(\mathbb{E}(X_k^2) = \mathbb{E}(Z_k^2)\) such that, almost surely:
\[
\sup_{1 \leq k \leq n} \left| \sum_{i=1}^{k} X_i - \sum_{i=1}^{k} Z_i \right| = o \left( (a_n (|\log(v_n^2/a_n)| + \log \log a_n))^{1/2} \right)
\]
We are now in a position to complete the proof of Theorem 30.
Proof of the Theorem 30  Observe that
\begin{align*}
\sum_{k=0}^{n-1} \left( \mathbb{E}_\omega [m^2_{\sigma^k \omega} \circ T^k_\omega \mid (T^{k+1}_\omega)^{-1} (B)] - \mathbb{E}_\omega (m^2_{\sigma^k \omega} \circ T^k_\omega) \right) &= \sum_{k=0}^{n-1} \left( L_{\sigma^k \omega} (m^2_{\sigma^k \omega}) \circ T^k_\omega - \int_X m^2_{\sigma^k \omega} \, d\mu_{\sigma^k \omega} \right) \\
&= \sum_{k=0}^{n-1} \hat{\psi}_{\sigma^k \omega} \circ T^k_\omega,
\end{align*}
and thus it follows from Lemma 35 that for \( P \)-a.e. \( \omega \in \Omega \) we have that
\begin{align*}
\sum_{k=0}^{n-1} \left( \mathbb{E}_\omega [m^2_{\sigma^k \omega} \circ T^k_\omega \mid (T^{k+1}_\omega)^{-1} (B)] - \mathbb{E}_\omega (m^2_{\sigma^k \omega} \circ T^k_\omega) \right) &= O(n^{1/2} \sqrt{\log \log n}).
\end{align*}
This, in particular, shows that (72) holds true with \( X_n = m_{\sigma^n \omega} \circ T^n_\omega \) and \( a_n = n^{1/2} (\log n)^\epsilon \), for any \( \epsilon > \frac{1}{2} \). We next show that (73) holds true with \( v = 2 \). Indeed, it follows from Lemma 33 that there exists \( C > 0 \) such that
\begin{align*}
\sum_{n \geq 1} a_n^{-2} \mathbb{E}_\omega [m^4_{\sigma^n \omega} \circ T^n_\omega] &\leq C \sum_{n \geq 1} a_n^{-2} < \infty.
\end{align*}
By applying Lemma 36 for \( X_n = m_{\sigma^n \omega} \circ T^n_\omega, n \in \mathbb{N}, \) using (69) and that \( \Sigma^2 > 0 \) (which insures that \( v_n^2 \) grows linearly fast in \( n \)), we complete the proof of the Theorem 30. \( \Box \)

Remark 37  Using (69), Lemma 63 and the Chernoff bounding scheme we can obtain an exponential concentration inequality in the sense of [15, Proposition 4.5].

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Data availability  Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Appendix A: A Counter Example to the Existence of the Asymptotic Variance

In this section, we present an explicit example of a good cocycle of transfer operators and an real-valued observable such that the conclusion of Theorem 10 fails. Our observable will satisfy all assumptions of Theorem 10 except for (12). We stress that the example is essentially taken from [6, Appendix A].

Consider \((\hat{\Omega}, \hat{\mathcal{B}}, \hat{Q}, S)\) the full-shift over \( \{1, 2, \ldots\} \), with probability vector \( (Z, Z/Z^{2+\delta}, \ldots, Z/n^{2+\delta}, \ldots) \), for some \( 0 \leq \delta \leq 1 \), \( Z \) being the normalization constant.

Let \( h : \hat{\Omega} \rightarrow \mathbb{R} \) be the (positive) observable defined by \( h(\omega) = \omega_0 \) if \( \omega := (\omega_n)_{n \in \mathbb{Z}} \in \hat{\Omega} \). Note that
\begin{align*}
\int_{\hat{\Omega}} h \, d\hat{Q} &= \sum_{i \geq 1} i \cdot \frac{Z}{i^{2+\delta}} = Z \sum_{i=1}^{\infty} \frac{1}{i^{1+\delta}} < +\infty,
\end{align*}
when $0 < \delta \leq 1$.

Define $(\Omega, \mathcal{F}, \mathbb{P}, \sigma)$ to be the suspension over $S$ with roof function $h$, i.e. $\Omega := \{(\omega, i) \in \tilde{\Omega} \times \mathbb{N}, 0 \leq i < h(\omega)\}$, $\sigma : \Omega \circlearrowleft$ is given by

$$
\sigma(\omega, i) := \begin{cases} 
(\omega, i + 1) & \text{if } i < h(\omega) - 1 \\
(S\omega, 0) & \text{if } i = h(\omega) - 1
\end{cases}
$$

and $\mathbb{P}(A) := \left(\int_{\tilde{\Omega}} h \, dQ\right)^{-1} \sum_{i \geq 0} Q \left(A \cap (\tilde{\Omega} \times \{i\})\right)$.

We can now define our random system: take $T_0 : [0, 1] \circlearrowleft$ to be the doubling map and let $T_1(x) = \frac{1}{2} (E(2x) + \{4x\})$, where $E(x)$ denotes the integer part of $x$ and $\{x\}$ its fractional part. We consider the random interval map $T_{(\omega, i)}$, $(\omega, i) \in \Omega$ defined by

$$
T_{(\omega, i)} := \begin{cases} 
T_1 & \text{if } i < h(\omega) - 1 \\
T_0 & \text{if } i = h(\omega) - 1.
\end{cases}
$$

It is verified in [6] that the associated cocyle of transfer operators $(L_{(\omega, i)})_{(\omega, i) \in \Omega}$ is good in the sense of [18, Definition 13]. Moreover, we have that $\mu_{(\omega, i)} = m$ for $(\omega, i) \in \Omega$, where $m$ denotes the Lebesgue measure on $[0, 1]$. Hence $L_{(\omega, i)} \mathbb{1} = \mathbb{1}$, which implies that the log-integrability condition is trivially satisfied. Let

$$
n_c(\omega, i) := \min\{k \in \mathbb{N} : T_{(\omega, i)}^k([0, 1/2]) = [0, 1]\}, \quad (\omega, i) \in \Omega.
$$

Then, we have (see [6, p.47]) that $n_c(\omega, i) = h(\omega) - i$. Observe that $n_c$ is not integrable. Indeed, for each $N \in \mathbb{N}$ we have that

$$
\mathbb{P}(n_c(\omega, i) = N) = \left(\int_{\tilde{\Omega}} h \, dQ\right)^{-1} \sum_{i \geq 0} Q(h(\omega) - i = N)
= \left(\int_{\tilde{\Omega}} h \, dQ\right)^{-1} \sum_{i \geq N} Q(\omega_0 = i)
= \left(\int_{\tilde{\Omega}} h \, dQ\right)^{-1} \sum_{i \geq N} Z \frac{1}{2^{i+\delta}} \sim \frac{C}{N^{1+\delta}},
$$

for some constant $C > 0$, which easily implies that $n_c$ is not integrable.

We can finally introduce the observables of interest: consider $\phi = 2 \cdot \mathbb{1}_{[0, 1/2]}$, and $\psi = \phi - \int_{[0, 1]} \phi \, dm$. We have (see [6, p.47]) that

$$
\int_{[0, 1]} \phi \cdot \phi \circ T_{(\omega, i)}^n \, dm = \begin{cases} 
2 & \text{if } n < n_c(\omega, i)
1 & \text{otherwise}.
\end{cases}
$$

Therefore,

$$
\int_{[0, 1]} \psi \cdot \psi \circ T_{(\omega, i)}^n \, dm = \begin{cases} 
1 & \text{if } n < n_c(\omega, i)
0 & \text{otherwise}.
\end{cases}
$$

Now, observe that

$$
\mathbb{E} \left( \sum_{n=0}^{N-1} \psi \circ T_{(\omega, i)}^n \right)^2 = \sum_{n=0}^{N-1} \mathbb{E}(\psi^2 \circ T_{(\omega, i)}^n) + 2 \sum_{n=0}^{N-1} \sum_{m=n+1}^{N-1} \mathbb{E} \left( \psi \cdot \psi \circ T_{(\omega, i)}^{m-n} \right),
$$
where $E$ denotes the expectation w.r.t. $m$. Since $\mu(\omega,i) = m$ for $(\omega,i) \in \Omega$, we have that
\[
E(\psi^2 \circ T^{n}_{(\omega,i)}) = \int_{[0,1]} \psi^2 dm = 2.
\]
For the other term, remark that by (74)
\[
\sum_{m=n+1}^{N-1} E(\psi \circ \sigma^n(\omega,i)) = \min(n_c(\sigma^n(\omega,i)), N-n) - 1
\]
so that we get
\[
\lim_{N \to \infty} \frac{1}{N} \int \left( \sum_{n=0}^{N-1} \psi \circ T^{n}_{(\omega,i)} \right)^2 = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \min(n_c(\sigma^n(\omega,i)), N-n),
\]
for $P$-a.e $(\omega,i) \in \Omega$. On the other hand, since $n_c$ is a measurable, positive and non-integrable function, it follows from Lemma 40 below (applied to $f_m(\omega,i) = \min(n_c(\sigma^n(\omega,i)), m)$ and $f(\omega,i) = n_c(\omega,i)$) that
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \min(n_c(\sigma^n(\omega,i)), N-n) = +\infty, \quad \text{for } P\text{-a.e. } (\omega,i) \in \Omega.
\]
In particular, we see that the first assertion of Theorem 10 does not hold.

**Remark 38** Observe that the observable $\psi$ constructed above is deterministic, i.e. it does not depend on $(\omega,i) \in \Omega$. In particular, it satisfies (43). Thus, the example we discussed shows that the condition such as (12) is needed for Theorem 10 to hold. This provides an affirmative answer to the question posed in the first version of [18].

**Remark 39** We note that our example can be further simplified: we can replace $T_1$ with the identity map on $[0,1]$.

**Lemma 40** (Maker’s theorem for positive non-integrable functions) Let $(\mathcal{X}, \mathcal{B}, \nu)$ be a probability space and $T : \mathcal{X} \to \mathcal{X}$ an ergodic probability preserving transformation. Let $(f_m)_{m \in \mathbb{N}}$ be a sequences of measurable real-valued and nonnegative functions on $\mathcal{X}$ so that $\lim_{m \to \infty} f_m = f$ exists $\nu$-a.e. and $\int_{\mathcal{X}} f(x) d\nu(x) = \infty$. Then
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f_{N-n} \circ T^n = +\infty, \quad \text{for } \nu\text{-a.e. } x \in \mathcal{X}.
\]

**Proof** Firstly, by replacing $f_m$ with $g_m = \inf\{f_n : n \geq m\}$, we can assume without any loss of generality that $f_m \leq f_{m+1}$ for all $m \in \mathbb{N}$. Indeed, it suffices to observe that $g_m$ also converges to $f$ and that
\[
\frac{1}{N} \sum_{n=0}^{N-1} f_{N-n} \circ T^n \geq \frac{1}{N} \sum_{n=0}^{N-1} g_{N-n} \circ T^n, \quad N \in \mathbb{N}.
\]

Let us fix some $M > 0$ and set $A_M = \{x \in \mathcal{X} : f(x) \leq M\}$. Set $f_m^{(M)} = f_m \cdot 1_{A_M}$ and $f^{(M)} = f \cdot 1_{A_M}$, where $1_A$ denotes the indicator function of a set $A$. Then, since $f_m$ is increasing in $m$, we have that
\[
f_m^{(M)} \leq f^{(M)} \leq M.
\]
Thus, by applying the classical Maker’s theorem \cite{33}, we obtain that
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f_{N-n}^{(M)} \circ T^n = \int_{\Lambda^M} f^{(M)} d\nu = \int_{\Lambda^M} f d\nu, \quad \nu\text{-a.e.}
\]
On the other hand, since \( f_m \geq f_m^{(M)} \) we have that
\[
\lim \inf_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f_{N-n} \circ T^n \geq \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f_{N-n}^{(M)} \circ T^n = \int_{\Lambda^M} f d\nu.
\]
By taking the limit as \( M \to \infty \), we conclude that
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f_{N-n} \circ T^n = \int_{\Lambda^M} f d\nu = +\infty, \quad \nu\text{-a.e.}
\]
The proof of the lemma is completed. \( \Box \)

5 Appendix B: Estimation of \( K \)

In this section, we explore possible strategies to estimate the tempered random variable \( K \) that appears in the statement of Theorem 10. More precisely, under additional assumptions
\[
\text{esssup}_{\omega \in \Omega_1} \| L_\omega \| < +\infty,
\]
we show that \( K \in L^p \) for small enough \( p > 0 \).

5.1 General Strategy

We begin by observing that \( \tilde{D}(\omega) \) given by (26) satisfies
\[
\| \tilde{D}(\omega) h \|_{BV} \leq \| h \|_{BV} + \left| \int_X h d\mu_\omega \right| \leq (1 + C_{var}) \| h \|_{BV},
\]
and
\[
\| (\text{Id} - \tilde{D}(\omega)) h \|_{BV} \leq C_{var} \| h \|_{BV}, \quad h \in BV.
\]
Hence, we observe that \( \tilde{D} \) appearing in the statement of Corollary 15 (see (23) and (24)) can be constructed as:
\[
\tilde{D}(\omega) := (1 + C_{var}) \sup_n (\| L^n_\omega \|_{BV} \| e^{\lambda n} ).
\]
We now recall that \( K \) is given as a sum of \( \tilde{D} \) and a number \( C_{var} \) (see (36)). Therefore, from now on we will concentrate on a possible strategy to estimate \( \tilde{D} \), also ignoring the constant factor \( 1 + C_{var} \). We suppose that
\[
B := \text{esssup}_{\omega \in \Omega_1} \| L_\omega \| < +\infty.
\]
We begin by noting (see the proof of \cite[Proposition 28]{18}) that \( \lambda' \) can be chosen as \( \lambda' = -\lambda_2 - \epsilon \) for any \( \epsilon > 0 \), where \( \lambda_2 < 0 \) is the second largest Lyapunov exponent of the
cocycle \((L_\omega)_{\omega \in \Omega}\). By Kingman’s subadditive ergodic theorem, we have that
\[
\lambda_2 = \inf_{n \in \mathbb{N}} \frac{1}{n} \int_{\Omega} \log \|L^n_\omega|_{BV_\omega}\| \, d\mathbb{P}(\omega).
\]
In particular, there exists \(n_0 \in \mathbb{N}\) such that
\[
\frac{1}{n_0} \int_{\Omega} \log \|L^n_\omega|_{BV_\omega}\| \, d\mathbb{P}(\omega) < \lambda_2 + \frac{\epsilon}{2}.
\]
Set
\[
A_n = \{ \omega \in \Omega : \|L^n_\omega|_{BV_\omega}\| > e^{(\lambda_2 + \epsilon)n} \} = \left\{ \omega \in \Omega : \frac{1}{n} \log \|L^n_\omega|_{BV_\omega}\| > \lambda_2 + \epsilon \right\}.
\]
Set
\[
g(\omega) := \frac{1}{n_0} \log \|L^n_\omega|_{BV_\omega}\|, \quad \omega \in \Omega.
\]
By \(S_n g\) we will denote the \(n\)-th Birkhoff sum of \(g\) with respect to \(\sigma\). It follows from (76) and [23, Lemma 3.1] that there exists \(M > 0\) such that
\[
\log \|L^n_\omega|_{BV_\omega}\| \leq S_n g(\omega) + M,
\]
for \(\mathbb{P}\)-a.e. \(\omega \in \Omega\) and \(n \in \mathbb{N}\). Hence, if \(\omega \in A_n\)
\[
\frac{1}{n} S_n g(\omega) + \frac{M}{n} > \frac{1}{n} \int_{\Omega} g \, d\mathbb{P} + \frac{\epsilon}{2},
\]
so that
\[
\frac{1}{n} S_n g(\omega) - \frac{1}{n} \int_{\Omega} g \, d\mathbb{P} > \epsilon - \frac{M}{n} > \frac{\epsilon}{4}
\]
for \(n\) sufficiently large. In particular,
\[
A_n \subset \left\{ \omega \in \Omega : \frac{1}{n} S_n g(\omega) - \frac{1}{n} \int_{\Omega} g \, d\mathbb{P} > \frac{\epsilon}{4} \right\}.
\]
Provided that \(g\) satisfies the large deviation property (as stated in [13, Theorem A]), there exists \(\alpha \in (0, 1)\) such that \(\mathbb{P}(A_n) \leq \alpha^n\) for \(n\) sufficiently large. In particular, \(\sum_{n=1}^{\infty} \mathbb{P}(A_n) < +\infty\). By the Borel-Cantelli lemma, we can conclude that for \(\mathbb{P}\)-a.e. \(\omega \in \Omega\), there exists a smallest \(N_\omega \in \mathbb{N}\) such that
\[
\|L^n_\omega|_{BV_\omega}\| \leq e^{-\lambda n}, \quad n \geq N_\omega.
\]
Taking into account (76) we see that
\[
\tilde{D}(\omega) \leq e^{(\lambda + \log B)N_\omega}, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.
\]
Moreover, observe that \(\{N_\omega = k\} \subset A_{k-1}\), and thus \(\mathbb{P}((N_\omega = k)) \leq \alpha^{k-1}\) for large \(k \in \mathbb{N}\). In particular, for each \(q > 0\), we have that there exists \(C > 0\) such that
\[
\int_{\Omega} \tilde{D}(\omega)^q \, d\mathbb{P}(\omega) \leq C \sum_{k=0}^{\infty} e^{q(\lambda + \log B)k} \alpha^{k-1} < +\infty,
\]
provided that
\[
\log \alpha + q(\lambda + \log B) < 0,
\]
which is satisfied whenever \(q > 0\) is sufficiently small.
5.2 A Concrete Example

Finally, let us demonstrate the above strategy in a concrete example where the idea can be considerably simplified. In particular, there is no need to compute the exact value of $\lambda_2$.

Let us assume that $X = [0, 1]$ is unit interval and that $\mathbb{P}$ a.s. we have $T_{\omega} \in \{ T_1, T_2 \}$, where $T_1 x = 2x \mod 1$ and $T_2$ is the identity map on $X$. We assume also that $\mathbb{P}(\{ \omega : T_{\omega} = T_1 \}) > 0$ for $i = 1, 2$. In this case we have that $\mu_{\omega} = m$, where $m$ is the Lebesgue measure on $X$. In particular, $L_{\omega} = L_{\omega}$ for $\omega \in \Omega$. Since $T_1$ is expanding, there are constants $C, \lambda_1 > 0$ such that for every $k \in \mathbb{N}$ and $h \in BV^0$, we have that

$$\| L_{i}^k h \|_{BV} \leq Ce^{-\lambda_1 k} \| h \|_{BV},$$

(77)

where $L_i$ denotes the transfer operator associated to $T_i$. Let $N_n(\omega)$ be the number of $0 \leq j < n$ such that $T^j_{\sigma^j \omega} = T_1$, i.e. $N_n(\omega) = \sum_{j=0}^{n-1} \mathbb{I}_A(\sigma^j \omega)$ where $A = \{ \omega : T_{\omega} = T_1 \}$ and $\mathbb{I}_A$ denotes the indicator function of $A$. By (77) and using that the transfer operator of $T_2$ is the identity operator, we have that

$$\| L_{n}^n h \|_{BV} \leq Ce^{-\lambda_1 N_n(\omega)} \| h \|_{BV},$$

(78)

for $\omega \in \Omega$, $h \in BV^0$ and $n \in \mathbb{N}$. Let $a := \mathbb{P}(A) > 0$ and set

$$N_{\omega} = \inf \left \{ N : N_n(\omega) \geq \frac{1}{2} an, \ \forall n \geq N \right \}.$$

Then, (78) implies that for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $n \in \mathbb{N}$,

$$\| L_{n}^n \|_{BV^0} \leq K(\omega)e^{-\lambda n},$$

where $K(\omega) = Ce^{\lambda_1 N_{\omega}}$ and $\lambda = a \lambda_1 / 2$. Observe that for $k \geq 1$,

$$N_{\omega} = k + 1 \subset \left \{ \left | \frac{N_k(\omega)}{k} - a \right | > \frac{1}{2} a \right \}.$$

Thus, if the stationary process $(\mathbb{I}_A \circ \sigma^n)$ satisfies an appropriate large deviations principle (e.g. choosing maps $T_{\omega} \in \{ T_1, T_2 \}$ in an i.i.d. fashion), we can conclude that $N_{\omega}$ is integrable. Hence, log $K$ is integrable and consequently also tempered. This provides an explicit formula for $K(\omega)$, and our scaling condition (12) means that $\| \psi_{\omega} \|_{BV}$ is small when it takes a lot of time for the Birkhoff average to get close enough to its mean.

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