CURVATURE TENSOR UNDER THE COMPLETE NON-COMPACT RICCI FLOW

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Abstract. We prove that for a solution \((M^n, g(t)), t \in [0, T)\), where \(T < \infty\), to the Ricci flow on a complete non-compact Riemannian manifold with the Ricci curvature tensor uniformly bounded by some constant \(C\) on \(M^n \times [0, T)\), the curvature tensor stays uniformly bounded on \(M^n \times [0, T)\).

1. Introduction

In this paper, we continue our study of Ricci flow on complete non-compact Riemannian manifold \((M, g)\) (see [4] and [7]). For a compact Ricci flow, N.Sesum [9] shows an interesting result, which says that the bounded Ricci curvature tensor of the Ricci flow implies the boundedness of the full curvature operator. We can extend the result of N.Sesum [9] about compact Ricci flow to complete non-compact case. Our result is motivated by our recent works citeDM and [7]. We remark that in the beautiful book [2], B.Chow, P.Lu, L.Ni also give an improved version of Sesum’s result above for compact Ricci flows. For background about Ricci flow and Riemannian Geometry, we confer to the books of T.Aubin [1], B.Chow, P.Lu, L.Ni [2]. For more beautiful papers, one may see the works of R.Hamilton [8] and G.Perelman [10], [11] (see also the book of R.Schoen and S.T.Yau [14] for related topics).

The main theorem in this paper is the following

Theorem 1.1. Suppose \((M^n, g(t)), t \in [0, T)\), where \(T < \infty\), is a solution to the Ricci flow on a complete non-compact Riemannian manifold with the Ricci curvature uniformly bounded by some constant \(C\) on \(M^n \times [0, T)\). Then the curvature tensor stays uniformly bounded on \(M^n \times [0, T)\).

The difficulty in the study of the complete non-compact Ricci flow is that there is no corresponding non-local collapsing theorem in noncompact manifolds without injectivity radius bound conditions. Fortunately, to overcome this, we can use the precompactness theorem (due to D.Glickenstein [6]) of solutions to the Ricci flow in the absence of injectivity radius estimates. We

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make the convention that all Riemannian manifolds in this paper are complete non-compact Riemannian manifolds except explicitly stated.

2. Background

We now introduce some elementary parts about Ricci flow.

Lemma 2.1. Suppose \((M, g(t))\) is the solution to the Ricci flow for \(t \in [0, T)\) such that the Ricci curvatures are uniformly bounded by some constant \(C\). We have for all \(\epsilon > 0\) there exists a \(\delta(\epsilon, C) > 0\) such that if \(|t - t_0| < \delta\) then

\[
-\epsilon g(t_0) \leq g(t) - g(t_0) \leq \epsilon g(t_0),
\]

and

\[
|d_g(t)(x, y) - d_g(t_0)(x, y)| \leq \epsilon \frac{1}{2} d_g(t_0)(x, y),
\]

for all \(x, y \in M\).

We need the compactness properties of Ricci flow in the sense of Gromov-Hausdorff topology.

Definition 2.2. A sequence of pointed n-dimensional Riemannian manifolds \(\{(M^n_i, g_i, p_i)\}_{i=1}^\infty\) locally converges to a pointed metric space \((X, d, x)\) in the sense of \(C^\infty\)-local submersions at \(x\) if there is a Riemannian metric \(\overline{g}\) on an open neighborhood \(V \subset \mathbb{R}^n\) of \(o\), a pseudogroup \(\Gamma\) such that the quotient is well defined, an open set \(U \subset X\), and maps \(\phi_i : (V, o) \to (M_i, p_i)\) such that

1. \(\{(M^n_i, g_i, p_i)\}_{i=1}^\infty\) converges to \((X, d, x)\) in the pointed Gromov-Hausdorff topology,
2. the identity component of \(\Gamma\) is a Lie group germ,
3. \((V/\Gamma, \overline{d}_\overline{g})\) is isometric to \((U, d)\), where \(\overline{d}_\overline{g}\) is the induced distance in the quotient.
4. \((\phi_i)_*\) is nonsingular on \(V\) for all \(i \in \mathbb{N}\), and
5. \(\overline{g}\) is the \(C^\infty\) limit of \(\phi_i^*g_i\) (uniform convergence on compact sets together with all derivatives).

A slight strong convergence concept is the following.

Definition 2.3. A sequence of pointed n-dimensional Riemannian manifolds \(\{(M^n_i, g_i, p_i)\}_{i=1}^\infty\) converges to a pointed metric space \((X, d, x)\) in the sense of \(C^\infty\)-local submersions if for every \(y \in X\) there exist \(q_i \in M\) such that \(\{(M^n_i, g_i, q_i)\}_{i=1}^\infty\) locally converges to \((X, d, y)\) in the sense of \(C^\infty\)-local submersions at \(y\).

We now recall the precompactness result.

Theorem 2.4. Let \(C_k > 0\) be constants for \(k \in \mathbb{N}\) and \(\{(M^n_i, g_i(t), p_i)\}_{i=1}^\infty\), where \(t \in [0, T]\), be a sequence of pointed solutions to the Ricci flow on complete manifolds such that

\[
|Rm(g_i(t))| g_i(t) \leq C,
\]
and
\[ |D_{g_{i}(t)} \operatorname{Rm}(g_{i}(t))|_{g_{i}(t)} \leq C_{k}, \]
for all \( i, k \in \mathbb{N} \) and \( t \in [0, T] \).

Then there is a subsequence \( \{ M_{i_{k}}, g_{i_{k}}(t), p_{i_{k}} \}_{k=1}^{\infty} \) and a one parameter family of complete pointed metric spaces \( (X, d(t), x) \) such that for each \( t \in [0, T] \), \( (M_{i_{k}}, d_{g_{i_{k}}}(t), p_{i_{k}}) \) converges to \( (X, d(t), x) \) in the sense of \( C^{\infty} \)-local submersions and the metrics \( g(t) \) in definition 2.3 are solutions to the Ricci flow.

3. Preparation

We begin with simple results about Ricci flow on complete non-compact Riemannian manifolds.

Lemma 3.1. Suppose \((M, g(t))\) is the solution to the Ricci flow for \( t \in [0, T) \) such that the Ricci curvatures are uniformly bounded by some constant \( C \). We have for all \( \epsilon > 0 \) there exists a \( \delta(\epsilon, C) > 0 \) such that if \( t \in [t_{0}, t_{0} + \delta] \) then

\[
\operatorname{Vol}_{g(t)}(B_{g(t)}(x, r)) \geq (1 - \epsilon)^{\frac{n}{2}} \operatorname{Vol}_{g(t_{0})}(B_{g(t_{0})}(x, \frac{r}{1 + \epsilon})),
\]

for all \( x \in M \), where \( \operatorname{Vol}_{g(t)}(B_{g(t)}(x, r)) \) is the volume of ball centered at point \( x \) with respect to metric \( g(t) \).

Proof. For any \( x \in M^{n}, r > 0 \), and \( t \in [t_{0}, t_{0} + \delta] \), by lemma 2.1 we have

\[
B_{g(t_{0})}(x, \frac{r}{1 + \epsilon}) \subset B_{g(t)}(x, r),
\]

and

\[
\operatorname{Vol}_{g(t)}(B_{g(t)}(x, \frac{r}{1 + \epsilon})) \geq (1 - \epsilon)^{\frac{n}{2}} \operatorname{Vol}_{g(t_{0})}(B_{g(t_{0})}(x, \frac{r}{1 + \epsilon})).
\]

Hence we have

\[
\operatorname{Vol}_{g(t)}(B_{g(t)}(x, r)) \geq (1 - \epsilon)^{\frac{n}{2}} \operatorname{Vol}_{g(t_{0})}(B_{g(t_{0})}(x, \frac{r}{1 + \epsilon})).
\]

The following is about the volume growth of geometric balls under a scalar curvature condition.

Lemma 3.2. Let \( M \) be the \( n \)-dimensional complete manifold with bounded scalar curvature. Then the volume growth ratio \( \frac{\operatorname{Vol}(B(p, r))}{\omega_{n} r^{n}} \) uniformly converges to \( \omega_{n} \) as \( r \) converges to zero, where \( \operatorname{Vol}(B(p, r)) \) is the volume of ball centered at point \( p \) with radius \( r \) and \( \omega_{n} \) is the volume of Euclidean unit \( n \)-ball. Here the uniform limit is with respect to a Ricci flow with bounded curvature.
Proof. Let \( \{ x^i \} \) be the geodesic coordinates at the point \( p \in M \). Within the injectivity radius at \( p \), we have the expansion of metric

\[
g_{ij} = \delta_{ij} - \frac{1}{3} R^m_{ijn}(q)x^m x^n + o(r^2),
\]

where \( q \) is a point in \( B(p, r) \) (see lemma 3.4 on p.210 of [14]). Then

\[
det(g_{ij}) = 1 - \frac{1}{3} R^q_{ij}(q)x^i x^j + o(r^2).
\]

So

\[
Vol(B(p, r)) = \omega_n r^n (1 - \frac{R(q)}{6(n+2)} r^2 + o(r^2)).
\]

Since \( M \) has the bounded scalar curvature, we have

\[
|Vol(B(p, r)) - \omega_n r^n| = |\frac{R(q)}{6(n+2)} r^2 + o(r^2)| \leq C r^2,
\]

where \( C \) is a constant only depends on dimension \( n \) and the bound of scalar curvature. So the lemma follows immediately. \( \square \)

4. Proof of Theorem 1.1 by blowing up argument

In this section, we prove Theorem 1.1.

Proof. The argument is by contradiction. Assume that the curvature blows up at a finite time \( T \) under the assumptions of the theorem. Then there exist sequences \( t_i \to T^- \) and \( p_i \in M \), such that

\[
Q_i := |Rm|(p_i, t_i) \geq C^{-1} \sup_{M \times [0, t_i]} |Rm|(x, t) \to \infty,
\]

as \( i \to \infty \), where \( C \) is any constant bigger than 1. We define the pointed rescaled solutions \( (M^i, g_i(t), p_i) \), \( t \in [-t_i Q_i, 0] \), by \( g_i(t) = Q_i^{-1} g(t_i + Q_i t) \).

Then we have

\[
\sup_{M \times (-t_i Q_i, 0]} |Rm| g_i(x, t) \leq C
\]

and

\[
|Rm g_i(p_i, 0)| = 1.
\]

Hence by Shi’s derivative estimates of Ricci flow (see [13]), we have

\[
|D^k g_i(t) Rm g_i(t)|_{g_i(t)} \leq C_k.
\]

By theorem 2.4 we have \( (M_t, g_i(t), p_i) \) subconverges to metric space \( (X, d(t), x) \) in the sense of \( C^\infty \)-local submersions. Hence we have a neighborhood \( V \subset \mathbb{R}^n \) of \( o \) with metric \( \overline{f}(t) \) is the ancient solution to Ricci flow, and \( (V, \overline{f}(t)) \) modulo a isometric pseudogroup action is isometric to a neighborhood of \( x \) in the limit metric space \( (X, d(t)) \). Moreover, there are maps \( \phi_i : (V, o) \to (M_i, p_i) \) such that \( \overline{g} \) is the \( C^\infty \) limit of \( \phi_i^* g_i \). Hence \( |Rm(\overline{g})(o, 0)| = 1 \). Since the Ricci curvatures of our original flow are uniformly bounded, we have that \( |R(t)| \leq C \), where \( R(t) \) is a scalar curvature with respect to \( g(t) \). Since \( R_i(t) = \frac{R(t)}{Q_i} \) for all \( t \in (-t_i Q_i, 0] \), we get that
\[ R_i(t) \to 0 \text{ as } i \to \infty, \text{ i.e. } \overline{R}(t) = 0 \text{ for } t \in (-\infty, 0]. \] By the evolution equation for a scalar curvature \( \frac{d}{dt}\overline{R} = \Delta \overline{R} + 2|\overline{Ric}|^2, \) we have \( \overline{\text{Ric}}(t) = 0 \) for all \( t \in (-\infty, 0], \) i.e. our solution \( \overline{g} \) is stationary, where \( \overline{R}(t) \) is the scalar curvature with respect to metric \( \overline{g}(t). \) Therefore it can be extended for all times \( t \in (-\infty, +\infty) \) to an eternal solution, where \( \overline{g}(t) = \overline{g}. \) Take sufficient small \( r > 0 \) such that \( B(o, r) \subset V. \) Since \( \overline{g} \) is the \( C^\infty \) limit of \( \phi_i^*g_i, \) we have

\[
\frac{\text{Vol}\phi_i^*g_i(0)}{r^n} B_{\phi_i^*g_i(0)}(o, r) = \lim_{i \to \infty} \frac{\text{Vol}\phi_i^*g_i(t_i)B_{\phi_i^*g_i(t_i)}(o, rQ_i^{-\frac{1}{2}})}{(rQ_i^{-\frac{1}{2}})^n},
\]

where \( \text{Vol}B(o, r) \) is the volume of ball \( B(o, r) \) on \( V \) with respect to metric \( \overline{g}. \) For any \( \epsilon > 0, \) let \( \delta(\epsilon, 1) \) be the constant defined in lemma \( \text{3.1}. \) Since \( t_i \to T \) and \( T \) is finite, there there exists sufficient large \( i_0 \) such that \( |t_i - t_{i_0}| < \delta(\epsilon, 1) \) and

\[
\frac{\text{Vol}B(o, r)}{r^n} \geq \frac{\text{Vol}\phi_i^*g(t_i)B_{\phi_i^*g(t_i)}(o, rQ_i^{-\frac{1}{2}})}{(rQ_i^{-\frac{1}{2}})^n} - \epsilon,
\]

when \( i > i_0. \) Hence by lemma \( \text{3.1} \) we have

\[
\frac{\text{Vol}B(o, r)}{r^n} \geq (1 - \epsilon)^\frac{n}{2} \frac{\text{Vol}\phi_i^*g(t_{i_0})B_{\phi_i^*g(t_{i_0})}(o, \frac{1}{1+\epsilon}rQ_i^{-\frac{1}{2}})}{(rQ_i^{-\frac{1}{2}})^n} - \epsilon.
\]

Since curvature is bounded at time \( t_{i_0}, \) we have

\[
\frac{\text{Vol}B(o, r)}{r^n} \geq (1 - \epsilon)^\frac{n}{2} \frac{\text{Vol}\phi_i^*g(t_{i_0})B_{\phi_i^*g(t_{i_0})}(o, \frac{1}{1+\epsilon}rQ_i^{-\frac{1}{2}})}{(rQ_i^{-\frac{1}{2}})^n} - \epsilon.
\]

Let \( \epsilon \to 0, \) we conclude that

\[
\frac{\text{Vol}B(o, r)}{r^n} \geq \omega_n.
\]

Finally by \( \overline{\text{Ric}}(t) = 0 \) and Bishop-Gromov volume comparison, we have metric \( \overline{g} \) is flat, which contradicts to the fact \( |Rm(\overline{g})|(o, 0) = 1. \]

\( \square \)

**Remark 4.1.** Note that in the proof of Theorem \( \text{1.1} \) we only used the facts that \( |R| \leq C \) for \( t \in [0, T] \) and that \( \int_0^T |\text{Ric}|ds \leq C. \)

We have immediately the following simple result of Theorem \( \text{1.1} \)

**Corollary 4.2.** Suppose \( (M^n, g(t)), t \in [0, T], \) where \( T < \infty, \) is a solution to the Ricci flow on a complete manifold with the Ricci curvature uniformly bounded by some constant \( C \) on \( M^n \times [0, T), \) then the solution can be extended past time \( T. \)

**Proof.** This is a simple consequence of theorem \( \text{1.1} \) and Shi’s derivative estimates of Ricci flow (see \( \text{13} \)). \( \square \)
5. Applications

We point out some applications of our Theorem 1.1.

Lemma 5.1. Suppose \((M^3, g(t)), t \in [0, T)\), where \(T < \infty\), is a solution to the Ricci flow on a complete Riemannian 3-manifold with bounded curvature tensor at initial time \(t = 0\). There exists \(C = C(g(0), T)\) so that

\[
\frac{dv(g(t))}{dv(g(t_0))} = \sqrt{\frac{A(t)}{A(t_0)}} \leq C
\]

for all \(t \in [0, T)\) and all \(x \in M\), where

\[
dv(g(t)) := \sqrt{A(t)}dx := \sqrt{\det(g_{ij}(t))}dx
\]

and \(t_0\) is any fixed time in \([0, T)\).

Proof. The evolution equation for \(\ln A(t)\) is

\[
\frac{d}{dt} \ln A(t) = -R. \tag{5.1}
\]

The evolution equation for the scalar curvature \(R(t)\)

\[
\frac{d}{dt} R = \Delta R + 2|Ric|^2,
\]

implies by a straightforward maximum principle argument that at any time \(t \in [0, T)\)

\[
R(t) \geq \frac{1}{(\min R(0))^{-1} - \frac{2t}{3}} \geq C_0,
\]

for some constant \(C_0\). If we integrate the equation (9) over \(s \in [t_0, t]\) for any \(t \in [t_0, T)\) we will get

\[
\ln A(t) - \ln A(t_0) = - \int_{t_0}^{t} R(s)ds \leq C_1. \tag{5.2}
\]

Now we get the results immediately. \(\square\)

With this result, we have the following result, which is an interesting consequence of Theorem 1.1.

Theorem 5.2. Suppose \((M^3, g(t)), t \in [0, T)\), where \(T < \infty\), is a solution to the Ricci flow on a complete Riemannian 3-manifold with bounded curvature tensor at initial time \(t = 0\). If \(g(t)\) is singular at time \(T\), then \(\lim_{t \to T} \dot{V}(t) = 0\), where

\[
\dot{V}(t)^2 := \dot{A}(t) := \min_{x \in M} \frac{A(t)}{A(t_0)},
\]

\(A(t) = \det(g_{ij}(t))\) and \(t_0\) is any fixed time in \([0, T)\).
Proof. We will prove theorem 5.2 by contradiction. Assume that there exist a sequence of times \( t_i \to T \) and \( \delta > 0 \) so that \( \hat{A}_i > \delta \) for all \( i \) and all \( x \in M \).

From the equation (5.1) we get

\[
\ln A(t_i) - \ln A(t_0) = - \int_{t_0}^{t_i} R(s) ds.
\]

Since \( \ln \delta < \frac{\ln A(t_i)}{\ln A(t_0)} \leq \ln C \), where \( C \) is a constant from the lemma 5.1, we get

\[
\left| \int_{t_0}^{T} R(s) ds \right| \leq C_2,
\]

where \( C_2 \) is a constant depending on \( g(t_0) \) and \( T \). In dimension 3 we have a pinching estimate, i.e. there exists constants \( C_3 \) and \( C_4 \) so that

\[
|Ric| \leq C_3(R + C_4),
\]

for all times \( t \in [0, T) \). This gives that \( \int_{t_0}^{T} |Ric| ds \leq C_3(\int_{t_0}^{T} (R(s) + C_4) ds \leq C_5 \) and the estimate does not depend on \( x \in M \).

If \( g(t) \) has a singularity at time \( T \), by theorem 1.1 we have there exist a sequence of times \( s_i \to T \) and \( x_i \in M \) so that

\[
Q_i = |Ric|(p_i, s_i) \geq C^{-1} \sup_{M \times [0, s_i]} |Ric|(x, t) \to \infty,
\]

where \( C \) is some constant bigger than 1. In dimension 3 the curvature tensor is controlled by the Ricci curvature. We define the pointed rescaled solutions \((M^n, g_i(t), p_i), t \in [-s_iQ_i, 0]\), by \( g_i(t) = Q_i g(s_i + Q_i^{-1}t) \). Then we have

\[
\sup_{M \times (-s_iQ_i, 0]} |Ric|_{g_i}(x, t) \leq C,
\]

and

\[
|Ric_{g_i}(p_i, 0)| = 1.
\]

By theorem 2.4 we have \((M_i, g_i(t), p_i)\) sub-converges to metric space \((X, d(t), x)\) in the sense of \( C^\infty\)-local submersions. Hence we have a neighborhood \( V \subset \mathbb{R}^n \) of \( o \) with metric \( \overline{g}(t) \) is the ancient solution to Ricci flow, and \((V, \overline{g}(t))\) modulo a isometric pseudogroup action is isometric to a neighborhood of \( x \) in the limit metric space \((X, d(t))\). Moreover, there are maps \( \phi_i : (V, o) \to (M_i, p_i) \) such that \( \overline{g} \) is the \( C^\infty \) limit of \( \phi_i^* g_i \). Hence \( |\overline{Ric}(o, 0)| = 1 \), where \( \overline{Ric} \) is the Ricci tensor with respect to \( \overline{g}(t) \). Because of the Hamilton-Ivey’s pinching estimate, \((V, \overline{g}(t))\) has non-negative sectional curvature. This allows us to apply the relative volume comparison theorem to \((V, \overline{g}(t))\) to conclude that for sufficient small \( r > 0 \)

\[
\frac{\text{Vol}_{\overline{g}(0)} B(o, r)}{r^n} \leq w_n,
\]
where $w_n$ is the volume of a unit euclidean ball. Fix $\epsilon > 0$, we can show that
\[
\liminf_{r \to 0} \frac{\text{Vol}_{\bar{g}(0)} B(o, r)}{r^n} \geq w_n - \epsilon,
\]
in the same way as in theorem [1.1] since we used only integral bound on $|Ric|$ in order to control changes in volumes and distances for $t$ close to $T$. Since this estimate holds for every $\epsilon > 0$, we can conclude that $(V, g(0))$ would have to be the euclidean space, which is not possible since $|\text{Ric}|(o, 0) = 1$.

\[\square\]

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