Note About String with Deformed Dispersion Relation

J. Klusoň

Department of Theoretical Physics and Astrophysics
Faculty of Science, Masaryk University
Kotlářská 2, 611 37, Brno
Czech Republic
E-mail: klu@physics.muni.cz

ABSTRACT: We study string theory with global momentum living on de Sitter space. We also show that this presumption leads to the string with deformed dispersion relation.

KEYWORDS: Bosonic String, Deformed Dispersion Relation
1. Introduction and Summary

There was an intensive study of the particle in the curved momentum space going back to
60’s when these theories were studied as possible examples of divergence free theories [1].
Another very interesting situation was analysed in [2] and in [3] where the particle with
the four dimensional de Sitter space with curvature \( \kappa \) was investigated. Explicitly, in term
of five dimensional Minkowski space-time coordinates \( P_A \) it has the form
\[
-P_0^2 + P_1^2 + P_2^2 + P_3^2 + P_4^2 = \kappa^2.
\]
(1.1)

Further the Casimir for the first four coordinates forming \( SO(3,1) \) subgroup of \( SO(4,1) \)
that leaves the quadratic form given above invariant is
\[
\mathcal{C} = P_0^2 - P_1^2 - P_2^2 - P_3^2 - m^2.
\]
(1.2)

With these two main ingrediences the authors in [2] derived an action for massive particle
with de Sitter momentum space. It was shown that this action has deformed dispersion
relation and that it possesses non-trivial symplectic structure.

We would like to see whether similar idea can be applied in case of string theory
where the fundamental object is two dimensional string instead of one dimensional point
particle. Explicitly, we consider \( D \)--dimensional string theory and impose the condition
that momenta lives on \( D \)--dimensional de Sitter space. With this presumption we expect
that we get string theory with deformed dispersion relation whose existence was firstly
proposed in [4]. We studied this proposal from Hamiltonian point of view in [5] and we
showed that the formulation of string theory presented in [4] cannot lead to string theory
with deformed dispersion relation. We also suggested in [5] that the way how to define string
with deformed dispersion relation is to follow analysis performed in [3, 2]. As we argued
above we have to impose the condition that momenta lives on \( D \)-- dimensional de Sitter
space. However the question is whether this constraint should restrict momentum density
that depends on world-sheet spatial coordinate. We argue that such constraint cannot be
imposed due to the fact that its Poisson bracket with spatial diffeomorphism constraint
does not vanish on the constraint surface that would imply that they are not the first class
constraints breaking consistency of the string theory. For that reason we should rather impose the condition that the global momentum is restricted to live on $D$-dimensional de Sitter space. In other words we consider string with additional first class global constraint. Then, following \[2\], we fix this global constraint and introduce appropriate variables that parametrize reduced phase space. As a consequence of this procedure we find string theory with modified dispersion relation between global momenta. Note that since the inner part of the string is not affected by this modification clearly consistency and the spectrum of this string is the same as in case of undeformed one. In other words we can very easily implement the modification of dispersion relation into string theory. Then we can study properties of such a modified string theory in the same way as in point particle with all consequences.

This paper is organized as follows. In the next section (2) we introduce string theory with global momentum to be restricted on de Sitter manifold. Then in section (3) we study its canonical form and determine corresponding Dirac brackets.

2. String Theory with Deformed Dispersion Relation

Our starting point is the action for bosonic string in $D$-dimensions. In order to find its form with deformed dispersion relation it is convenient to write it in the canonical form together with an additional constraint whose explicit form will be specified below. Explicitly, we consider string theory action in the form

$$S = \int d\tau d\sigma (p_A \partial_\tau x^A - N^\tau \mathcal{H}_\tau - N^\sigma \mathcal{H}_\sigma - \Gamma S) ,$$  \hspace{1cm} (2.1)

where $\mathcal{H}_\tau \approx 0 , \mathcal{H}_\sigma \approx 0$ are standard Hamiltonian and diffeomorphism constraints of bosonic string in the form

$$\mathcal{H}_\tau = p_\mu \eta^{\mu\nu} p_\nu + T^2 \partial_\sigma x^\mu \eta_{\mu\nu} \partial_\sigma x^\nu \quad \mathcal{H}_\sigma = p_\mu \partial_\sigma x^\mu ,$$  \hspace{1cm} (2.2)

where $\mu, \nu = 0, 1, \ldots, D$ and $A,B = 0, 1, \ldots, D+1$. We further presume closed string with the length $L$. An important ingredient is the constraint that enforces momentum to be de Sitter

$$S = p_A p^A - \kappa^2 = p_\mu p^\mu + p_{D+1}^2 - \kappa^2 \approx 0 .$$  \hspace{1cm} (2.3)

This is natural generalization of the constraint used in \[2\]. However an important point is that this local constraint $S$ does not have weakly vanishing Poisson bracket with $\mathcal{H}_\sigma \approx 0$ on the constraint surface. Such a situation would imply that $\mathcal{H}_\tau, \mathcal{H}_\sigma$ were not the first class constraints with the breaking of the consistency of string theory. For that reason we would rather presume that the total momentum is forced to live on de Sitter space and that $p_{D+1}$ depends on $\tau$ only. Explicitly we consider following constraint

$$S = (\int d\sigma p_\mu)(\int d\sigma p^\mu) + p_{D+1}^2 - \kappa^2 \approx 0 .$$  \hspace{1cm} (2.4)

In order to demonstrate consistency of the theory let us introduce smeared form of spatial diffeomorphism and Hamiltonian constraints

$$T_\sigma (N^\sigma) = \int d\sigma N^\sigma \mathcal{H}_\sigma , \quad T_\tau (N) = \int d\sigma N \mathcal{H}_\tau .$$  \hspace{1cm} (2.5)
Then using canonical Poisson brackets \( \{ x^\mu(\sigma), p_\nu(\sigma') \} = \delta_\nu^\mu \delta(\sigma - \sigma') \) we obtain following Poisson brackets between all constraints

\[
\{ T_\sigma(N^\tau), S \} = 0 , \quad \{ T_\tau(N), S \} = 0 , \\
\{ T_\sigma(N^\tau), T_\sigma(M^\sigma) \} = T_\sigma(N^\sigma \partial_\sigma M^\sigma - M^\sigma \partial_\sigma N^\sigma) , \\
\{ T_\sigma(N^\tau), T_\tau(M) \} = T_\tau(-\partial_\tau N^\sigma M + N^\sigma \partial_\tau M) , \\
\{ T_\tau(N), T_\tau(M) \} = 4T_\sigma(N\partial_\sigma M - M\partial_\sigma N) .
\]

(2.6)

The formulas given in (2.6) show that all Poisson brackets between constraints vanish on the constraint surface and hence they are the first class constraints. Hence the action has the form

\[
S = \int d\tau d\sigma(p_A \partial_\tau x^A - N^\tau \mathcal{H}_\tau - N^\sigma \mathcal{H}_\sigma) - \int d\tau \Gamma S .
\]

(2.7)

Since \( S \) is the first class constraint it is appropriate to fix it. In fact, since \( S \approx 0 \) is global constraint that does not depend on \( \sigma \) we can presume that its gauge fixing eliminates global variables only. For that reason we split \( p_\mu \) and \( x^\mu \) in the following way

\[
p_\mu(\tau, \sigma) = \tilde{p}_\mu(\tau, \sigma) + \frac{1}{L} P_\mu(\tau) , \quad x^\mu(\tau, \sigma) = \tilde{x}^\mu(\tau, \sigma) + X^\mu(\tau) ,
\]

(2.8)

where

\[
P_\mu = \int d\sigma p_\mu , \quad X^\mu = \frac{1}{L} \int d\sigma x^\mu ,
\]

(2.9)

where \( L \) is the length of the string and we have chosen the pre factor \( L^{-1} \) in the expression for \( X^\mu \) so that it has correct dimension \([\text{length}]\). Note that (2.8) implies

\[
\int d\sigma \tilde{p}_\mu = 0 , \quad \int d\sigma \tilde{x}^\mu = 0 .
\]

(2.10)

Further, using (2.8) and the canonical Poisson brackets \( \{ x^\mu(\sigma), p_\nu(\sigma') \} = \delta_\nu^\mu \delta(\sigma - \sigma') \) we obtain

\[
\{ X^\mu, P_\nu \} = \delta_\nu^\mu , \quad \{ \tilde{x}^\mu(\sigma), P_\nu \} = 0 , \\
\{ X^\mu, \tilde{p}_\nu(\sigma) \} = 0 , \quad \{ \tilde{x}^\mu(\sigma), \tilde{p}_\nu(\sigma') \} = \delta(\sigma - \sigma') \delta^\mu_\nu + \frac{1}{L} \delta^\mu_\nu .
\]

(2.11)

Finally, using (2.8) we can rewrite the Hamiltonian constraint and spatial diffeomorphism constraint in the form

\[
\mathcal{H}_\tau = \tilde{p}_\mu \eta^{\mu\nu} \tilde{p}_\nu + T^2 \partial_\sigma \tilde{x}^\mu \eta_{\mu\nu} \partial_\sigma \tilde{x}^\nu + \frac{1}{L} P_\mu \eta^{\mu\nu} P_\nu = \frac{1}{L} P_{\mu\nu} P_\nu + \tilde{\mathcal{H}}_\tau ,
\]

\[
\mathcal{H}_\sigma = \tilde{p}_\mu \partial_\sigma \tilde{x}^\mu + \frac{1}{L} P_\mu \partial_\sigma \tilde{x}^\mu \equiv \tilde{\mathcal{H}}_\sigma + \frac{1}{L} P_\mu \partial_\sigma \tilde{x}^\mu .
\]

(2.12)

As we argued above \( S \) Poisson commutes with all constraints so it is appropriate to gauge fix it. It is important to stress that the inner variables \( \tilde{x}^\mu, \tilde{p}_\mu \) are not affected by this.
procedure. Following [2] we introduce gauge fixing function $G$ that does not depend on $\sigma$ and that obeys
\begin{equation}
\{S, G\} \equiv \triangle \neq 0 \quad (2.13)
\end{equation}
so that $S$ and $G$ are two second class constraints and hence we reduce global degrees of freedom from $2(D+1)$ into $2D$. Following [2] we parametrize reduced phase space with variables $y^\mu, k_\mu$ that have vanishing Poisson brackets with constraints
\begin{align}
\{k_\mu, S\} &\approx 0, \quad \{k_\mu, G\} \approx 0, \\
\{y^\mu, S\} &\approx 0, \quad \{y^\mu, G\} \approx 0.
\end{align}
(2.14)

Now we find that Dirac brackets of $k_\mu$ and $y^\mu$ with arbitrary functions are equal to Poisson brackets since
\begin{align}
\{y^\mu, F\}_D &= \{y^\mu, F\} - \{y^\mu, S\} \Delta^{-1} \{G, F\} + \{y^\mu, G\} \Delta^{-1} \{S, F\} = \{y^\mu, F\}, \\
\{k_\mu, F\}_D &= \{k_\mu, F\} - \{k_\mu, S\} \Delta^{-1} \{G, F\} + \{k_\mu, G\} \Delta^{-1} \{S, F\} = \{k_\mu, F\}.
\end{align}
(2.15)

As the next step we introduce gauge fixing function $G$ that, following [2], has the form
\begin{equation}
G = \frac{1}{2}(X^{D+1} - X^0) - T, \quad (2.16)
\end{equation}
where $T$ is real number. Then we have
\begin{equation}
\{S, G\} = -(P_0 + P_{D+1}). \quad (2.17)
\end{equation}

In what follows we will consider $AN(D)$—group submanifold defined by two conditions $S = 0, P_0 + P_{D+1} > 0$. Let us choose $k_\mu$ coordinates in the following way
\begin{align}
P_0(k_0, k_i) &= \kappa \sinh \frac{k_0}{\kappa} + \frac{k_i k^i}{2\kappa} e^{\frac{k_0}{\kappa}}, \\
P_i(k_0, k_i) &= k_i e^{\frac{k_0}{\kappa}}, \\
P_{D+1}(k_0, k_i) &= \kappa \cosh \frac{k_0}{\kappa} - \frac{k_i k^i}{2\kappa} e^{\frac{k_0}{\kappa}},
\end{align}
(2.18)

where $i = 1, \ldots, D$. Then using the first and the third equation in (2.18) we can express $k_0$ as
\begin{equation}
k_0 = \kappa \ln \frac{P_0 + P_{D+1}}{\kappa}. \quad (2.19)
\end{equation}

Inserting this relation into the second equation in (2.18) we can express $k_i$ as
\begin{equation}
k_i = \kappa \frac{P_i}{P_0 + P_{D+1}}. \quad (2.20)
\end{equation}
In other words we have $k_{\mu} = k_{\mu}(P_A)$. Then using these relations we immediately get
\begin{equation}
\{k_{\mu}, S\} = 0, \quad \{k_0, G\} = 0.
\end{equation}

In the same way we can show that $\{k_i, G\} = 0$ since $k_i$ depend on the linear combination $P_0 + P_{D+1}$ only. We can further argue that
\begin{equation}
\{k_{\mu}, \tilde{x}^\nu\} = 0
\end{equation}
due to the fact that $k_{\mu}$ are functions of $P_{\mu}$ and $P_{\mu}$ Poisson commute with $\tilde{x}^\mu$ by definition.

To proceed further we introduce generator of rotations that for the full string has the form
\begin{equation}
J_{AB} = \int d\sigma (x_{AB} - x_{BA}) = (X_A P_B - X_B P_A) + \int d\sigma (\tilde{x}_A \tilde{p}_B - \tilde{x}_B \tilde{p}_A) = J_{AB} + \tilde{J}_{AB}.
\end{equation}

Then it is easy to see that
\begin{equation}
\{J_{AB}, S\} = 0.
\end{equation}

Following we define $y^\mu$ using components of $J_{AB}$ as
\begin{equation}
y^0 = \frac{1}{\kappa} J_{0(D+1)} , \quad y^i = \frac{1}{\kappa} (J_{(D+1)i} + J_{(D+1)i})
\end{equation}
and calculate (recall that $X_0 = -X^0, X_{D+1} = X^{D+1}$)
\begin{equation}
\{y^0, G\} = \frac{1}{2} (X^0 - X^{D+1}) = -(G + T) \approx -T
\end{equation}
and we see that this Poisson bracket is zero for $T = 0$. Let us further calculate Poisson bracket between $y^i$ and $G$ and we get
\begin{equation}
\{y_i, G\} = 0,
\end{equation}

where we used the fact that we have
\begin{equation}
\{J_{AB}, J_{CD}\} = (\eta_{AD} J_{CB} + \eta_{BC} J_{DA} + \eta_{AC} J_{BD} + \eta_{BD} J_{AC}).
\end{equation}

Then with the help of definition and the Poisson bracket we get
\begin{align*}
\{y^i, y^j\} &= \frac{1}{\kappa^2} \left\{ \{J_{0i}, J_{0j}\} + \{J_{0i}, J_{(D+1)j}\} + \{J_{(D+1)i}, J_{0j}\} + \{J_{(D+1)i}, J_{(D+1)j}\} \right\} = 0, \\
\{y^0, y^i\} &= \frac{1}{\kappa^2} \{J_{0(D+1)}, J_{0i} + J_{(D+1)i}\} = -\frac{1}{\kappa^2} (J_{0i} + J_{(D+1)i}) = -\frac{1}{\kappa} y^i.
\end{align*}
If we finally take into account the gauge fixing function $G = 0$ we can find inverse relation between $X^A$ and $y^\mu$ in the form

$$X^{D+1} = -\frac{\kappa y^0}{P_{D+1} + P_0} = -y^0 e^{-\frac{k_0}{\kappa}},$$

$$X^0 = -\frac{\kappa y^0}{P_{D+1} + P_0} = -y^0 e^{-\frac{k_0}{\kappa}},$$

$$X^i = -\frac{\kappa y^i}{P_0 + P_{D+1}} = -x^i e^{-\frac{k_0}{\kappa}}.
$$

Then inserting these relations into string theory action we obtain

$$S = \int d\tau (\dot{y}^\mu k_\mu - \frac{k_0}{\kappa} y^i k_i) + \int d\tau d\sigma (\tilde{p}_\mu \partial_\tau \tilde{x}^\mu - N (P_\mu (k) P^\mu (k) + \tilde{H}_\tau) - N^\sigma \mathcal{H}_\sigma (k)),$$

where

$$P_\mu (k) P^\mu (k) = -4\kappa^2 \sinh^2 \left(\frac{k_0}{2\kappa}\right) + k_i k^j e^{\frac{k_0}{\kappa}}.
$$

The action (2.31) is final form of the string theory action with modified dispersion relation. We see that the inner part has the same form as in non-deformed case. In the next section we perform canonical analysis of the action (2.31).

3. Hamiltonian Analysis

In this section we perform canonical analysis of the action (2.31), following [2]. To do this we treat $y^\mu$ and $k_\mu$ as independent variables. In other words we can interpret the action (2.31) as Lagrangian form of the action and determine conjugate momenta from it. We define $l_\mu$ as momentum conjugate to $y^\mu$ and $\Pi^\mu$ as momentum conjugate to $k_\mu$

$$l_\mu = \frac{\partial L}{\partial \partial_\tau y^\mu} = k_\mu, \quad \Pi^i = \frac{\partial L}{\partial \partial_\tau k_i} = 0, \quad \Pi^0 = \frac{\partial L}{\partial \partial_\tau k_0} = -\frac{1}{\kappa} y^i k_i,$$

where we have canonical Poisson brackets

$$\{y^\mu, l_\nu\} = \delta^\mu_\nu, \quad \{k_\mu, \Pi^\nu\} = \delta^\nu_\mu .
$$

Further, from (3.1) we get following primary constraints

$$\phi_\mu \equiv l_\mu - k_\mu \approx 0, \quad \Psi^i \equiv \Pi^i \approx 0, \quad \Psi^0 = \Pi^0 + \frac{1}{\kappa} y^i k_i .
$$

Then with the help of (3.2) we obtain Poisson brackets between constraints in the form

$$\{\phi_0, \Psi^0\} = -1, \quad \{\phi_0, \Psi^i\} = 0, \quad \{\phi_i, \Psi^0\} = -\frac{1}{\kappa} k_i,$$

$$\{\phi_i, \Psi^j\} = -\delta^j_i, \quad \{\Psi^i, \Psi^0\} = -\frac{1}{\kappa} y^i .
$$

(3.4)
It is convenient to write these Poisson brackets into matrix form introducing common notation for all constraints $\Phi_A \equiv (\phi_0, \phi_i, \Psi^0, \Psi^i)$ and hence we can rewrite (3.4) into more symmetric form

$$\{ \Phi_A, \Phi_B \} = M_{AB} \, , \quad M_{AB} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -\frac{1}{\kappa} k_i & -\delta^j_i \\ 1 \frac{1}{\kappa} k_j & 0 & \frac{1}{\kappa} y^j & 0 \\ 0 & \delta^j_i & -\frac{1}{\kappa} y^j & 0 \end{pmatrix} . \quad (3.5)$$

Then it is easy to determine inverse matrix $M^{AB}$ from (3.5)

$$M^{AB} = \begin{pmatrix} 0 & \frac{1}{\kappa} y^j & 1 \frac{1}{\kappa} k_j \\ -\frac{1}{\kappa} y^j & 0 & 0 & \delta^j_i \\ -1 & 0 & 0 & 0 \\ -\frac{1}{\kappa} k_i & -\delta^j_i & 0 & 0 \end{pmatrix} . \quad (3.6)$$

Now we are ready to determine Dirac brackets between canonical variables (Note that due to the fact that $\Phi_A$ are second class constraints we can solve them for $\Pi^\mu$ and $l_\mu$ so that independent variables will be $y^\mu$ and $k_\mu$). First of all we have

$$\{ y^0, \Phi_A \} = (1, 0, 0, 0) \, , \quad \{ y^i, \Phi_A \} = (0, \delta^j_i, 0, 0) \, ,$$

$$\{ k_0, \Phi_A \} = (0, 0, 1, 0) \, , \quad \{ k_i, \Phi_A \} = (0, 0, 0, \delta^j_i) \, ,$$

so that we have

$$\{ y^i, y^j \}_D = \{ y^i, y^j \} - \{ y^i, \Phi_A \} M^{AB} \{ \Phi_B, y^j \} = 0 \, ,$$

$$\{ y^0, y^j \}_D = \{ y^0, y^j \} - \{ y^0, \Phi_A \} M^{AB} \{ \Phi_B, y^j \} = \frac{1}{\kappa} y^j \, ,$$

$$\{ y^0, k_0 \}_D = \{ y^0, k_0 \} - \{ y^0, \Phi_A \} M^{AB} \{ \Phi_B, k_0 \} = 1 \, ,$$

$$\{ y^0, k_i \}_D = \{ y^0, k_i \} - \{ y^0, \Phi_A \} M^{AB} \{ \Phi_B, k_i \} = \frac{1}{\kappa} k_i \, ,$$

$$\{ y^i, k_0 \}_D = \{ y^i, k_0 \} - \{ y^i, \Phi_A \} M^{AB} \{ \Phi_B, k_0 \} = 0 \, ,$$

$$\{ y^i, k_j \}_D = \{ y^i, k_j \} - \{ y^i, \Phi_A \} M^{AB} \{ \Phi_B, k_j \} = \delta^j_i \, ,$$

$$\{ k_0, k_i \}_D = \{ k_0, k_i \} - \{ k_0, \Phi_A \} M^{AB} \{ \Phi_B, k_i \} = 0 \, ,$$

$$\{ k_i, k_j \}_D = \{ k_i, k_j \} - \{ k_i, \Phi_A \} M^{AB} \{ \Phi_B, k_j \} = 0 \, .$$

(3.8)

These Dirac brackets are the same as in the point particle case derived in [3] and that are counterparts of $\kappa$–deformed phase space [6]. Once again we should stress that $k_\mu, y^\mu$ have zero Dirac brackets with $\tilde{x}^\mu, \tilde{p}_\mu$. In fact, there are still Hamiltonian and diffeomorphism constraints

$$\mathcal{H}_\tau = P_\mu(k) y^{\mu\nu} P_{\nu}(k) + \tilde{\mathcal{H}}_\tau \, , \quad \mathcal{H}_\sigma = \tilde{\mathcal{H}}_\sigma + P_\mu(k) \partial_\sigma \tilde{x}^\mu$$

(3.9)
that are still first class constraints. Finally, since inner part of the string represented by
variables \( \tilde{z}^\mu, \tilde{p}_\mu \) decouple from \( k_\mu, y^\mu \) it is clear that the quantum consistency of the string
is the same as in undeformed case.

Let us outline main result that was derived in this paper. The goal was to find string
theory with deformed dispersion relation in an alternative way to the proposal presented
in [4]. We showed that it can be done when we generalized procedure suggested in [2, 3]
from one dimensional object to two dimensional object-string.

Acknowledgement:
This work is supported by the grant “Integrable Deformations” (GA20-04800S) from the
Czech Science Foundation (GACR).

References

[1] Y. A. Golfand, “On the introduction of an "elementary length" in the relativistic theory of
elementary particles,” Zh. Eksp. Teor. Fiz. 37 (1959) no.2, 504-509

[2] M. Arzano and J. Kowalski-Glikman, “Kinematics of a relativistic particle with de Sitter
momentum space,” Class. Quant. Grav. 28 (2011), 105009
doi:10.1088/0264-9381/28/10/105009 [arXiv:1008.2962 [hep-th]].

[3] F. Girelli, T. Konopka, J. Kowalski-Glikman and E. R. Livine, “The Free particle in
deformed special relativity,” Phys. Rev. D 73 (2006), 045009
doi:10.1103/PhysRevD.73.045009 [arXiv:hep-th/0512107 [hep-th]].

[4] J. Magueijo and L. Smolin, “String theories with deformed energy momentum relations, and
a possible nontachyonic bosonic string,” Phys. Rev. D 71 (2005), 026010
doi:10.1103/PhysRevD.71.026010 [arXiv:hep-th/0401087 [hep-th]].

[5] J. Kluson, “Note About String Theory with Deformed Dispersion Relations,”
[arXiv:2106.15870 [hep-th]].

[6] J. Lukierski, H. Ruegg and W. J. Zakrzewski, “Classical quantum mechanics of free kappa
relativistic systems,” Annals Phys. 243 (1995), 90-116 doi:10.1006/aphy.1995.1092
[arXiv:hep-th/9312153 [hep-th]].