Abstract. In the previous paper [17], the author defined equivariant Floer cohomology for a complete intersection in a toric variety and showed that it is isomorphic to the small quantum $D$-module after a mirror transformation when the first Chern class $c_1(M)$ of the tangent bundle is nef. In this paper, even when $c_1(M)$ is not nef, we show that the equivariant Floer cohomology reconstructs the big quantum $D$-module under certain conditions on the ambient toric variety. The proof is based on a mirror theorem by Coates and Givental [4]. The reconstruction procedure here gives a generalized mirror transformation first observed by Jinzenji in low degrees [20, 21].

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1. Introduction

The $S^1$-equivariant Floer cohomology proposed by Givental [9, 10] is a conjectural semi-infinite cohomology of the free loop space of a symplectic manifold $M$. The $S^1$ action here is the rotation of loops. Givental conjectured that this should have a natural $D$-module structure and is isomorphic to the small quantum $D$-module defined by quantum cohomology. Here, the small quantum $D$-module (henceforth small QDM) is a trivial $H^2(M, \mathbb{C}^*)$-bundle over $H^2(M, \mathbb{C}^*)$ with a flat connection $\nabla^h$, called the Dubrovin connection:

$$\nabla^h_a = \hbar Q_a \frac{\partial}{\partial Q_a} + p_a$$

where $Q^1, \ldots, Q^r$ is a co-ordinate system on $H^2(M, \mathbb{C}^*)$ and $p_1^*, \ldots, p_r^*$ denotes the quantum multiplication by a basis of $H^2(M, \mathbb{C}^*)$ dual to $Q^a$.

In [17], the author constructed the equivariant Floer cohomology $FH^*_{S^1}$ for toric complete intersections as an inductive limit of ordinary equivariant cohomology. He also introduced abstract quantum $D$-module (henceforth AQDM) which generalizes the small QDM. AQDM is a module over a Heisenberg algebra $D = \mathbb{C}[\hbar][Q_1, \ldots, Q_r]\langle P_1, \ldots, P_r \rangle$ where

$$[P_a, Q_b^\hbar] = \hbar \delta_a^b Q_b^\hbar, \quad 1 \leq a, b \leq r.$$ 

In case of the small QDM, $P_a$ is given by the Dubrovin connection $\nabla^h_a$. The author showed that the equivariant Floer cohomology $FH^*_{S^1}$ has the structure of an AQDM. Using a mirror theorem by Givental [11, 12], he also showed that $FH^*_{S^1}$ is isomorphic to the small QDM as an AQDM if the first Chern class $c_1(M)$ of the tangent bundle is nef.

This isomorphism of two AQDMs — small QDM and equivariant Floer cohomology — is the same as mirror transformation in the context of mirror symmetry. The small QDM is the A-model whereas the $FH^*_{S^1}$ plays the role of the B-model. More precisely, differential equations satisfied by a generator of $FH^*_{S^1}$ coincide with the Picard-Fuchs equations arising from the B-model. By the mirror transformation, we can compute the quantum cohomology i.e. the genus zero Gromov-Witten invariants.

When the first Chern class $c_1(M)$ is not nef, however, $FH^*_{S^1}$ is not isomorphic to the small QDM, and the ordinary mirror transformation does not work. A generalized mirror transformation is a prescription to remedy the situation. Here we need to consider not only the small QDM but also the big one. The big quantum $D$-module (henceforth big QDM) is a trivial $H^*(M)$-bundle on the total cohomology group $H^*(M)$ with the flat Dubrovin connection and its restriction to $H^2(M)$ can be identified with the small QDM. We introduce the notion of big AQDM which generalizes the big QDM. The AQDM in the previous paper [17] is called small AQDM in this paper. We show that the small AQDM reconstructs a big AQDM uniquely under certain conditions corresponding to that $H^*(M)$ is generated by $H^2(M)$. In particular, the small AQDM $FH^*_{S^1}$ reconstructs a certain big AQDM $\mathcal{E}^h$ such that $FH^*_{S^1}$ is isomorphic to the restriction of $\mathcal{E}^h$ to a subspace of its base space. Under Condition 5.6 on the ambient toric variety, we
will show that the reconstructed $\hat{E}^h$ is isomorphic to the big QDM of $M$ (Theorem 5.8). In this way, we can recover the big quantum cohomology from $FH^*_S$. Note that both $FH^*_S$ and the small QDM of $M$ are obtained as restrictions of $\hat{E}^h$ to certain subspaces, but the loci are different in general, i.e. the locus of $FH^*_S$, which can be considered to be the B-model locus, may not coincide with the linear subspace $H^2(M)$ (see Figure 1). The reconstruction from $FH^*_S$ to $\hat{E}^h$ can be viewed as a generalization of Kontsevich and Manin’s reconstruction theorem 23. Hertling and Manin 16 also proved a similar reconstruction theorem for (TE) structures.

In order to show that the reconstructed $\hat{E}^h$ is isomorphic to the big QDM of $M$, we use a mirror theorem by Coates and Givental 4. Coates and Givental introduced an infinite dimensional Lagrangian cone in the symplectic space $H^*(M) \otimes \mathbb{C}[\hbar, \hbar^{-1}][Q]$ encoding all the information of the genus zero Gromov-Witten theory. They described a relationship between the Gromov-Witten theory of $M$ itself and the twisted theory by a vector bundle $V$ on $M$ as a symplectic transformation of the corresponding Lagrangian cones. We interpret Coates–Givental’s symplectic transformation in terms of $D$-modules. In the framework of AQDMs, a symplectic transformation corresponds to a change of a frame at $Q = 0$ and a shift of the origin (Proposition 5.5).

In generalized mirror transformations, the use of (non-convergent) formal power series in the Novikov ring parameters is inevitable. In a forthcoming paper 19, we will discuss the convergence of generalized mirror transformations in some refined sense.

The paper is organized as follows. In section 2, we review the theory of quantum $D$-modules. In section 3, we review the equivariant Floer cohomology for toric complete intersections. In section 4, we formulate abstract big/small quantum $D$-modules and prove the reconstruction theorem. In section 5, we give a proof of the generalized mirror transformations. We also include the review of Coates–Givental’s theory. In section 6, we illustrate generalized mirror transformations by examples.

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2. Big and small QDMs

In 4, Coates and Givental introduced a twist of Gromov-Witten theory by a vector bundle and a multiplicative characteristic class. In this section, we review the genus zero twisted theory from a viewpoint of the $D$-module structure. We explain fundamental solutions, $J$-functions and Kontsevich and Manin’s reconstruction theorem.
2.1. **Twisted quantum cohomology.** Let $M$ be a smooth projective variety and $\mathcal{V}$ be a vector bundle over $M$. For simplicity, we assume that the total cohomology ring of $M$ consists only of the even degree part, $H^*(M) = H^{even}(M, \mathbb{C})$. Following [3], we introduce the following general multiplicative characteristic class $c$ for a vector bundle $F$:

$$c(F) := \exp \left( \sum_{k \geq 0} s_k \text{ch}_k(F) \right)$$

(2.1)

where $s_0, s_1, s_2, \ldots$ are arbitrary parameters. Let $\mathbb{C}[s]$ denote the completion of $\mathbb{C}[s_0, s_1, \ldots]$ with respect to the valuation $v: \mathbb{C}[s_0, s_1, \ldots] \rightarrow \mathbb{R} \cup \{\infty\}$ defined by

$$v(s_k) = k + 1, \quad k \geq 0.$$  

(2.2)

Then $c$ is defined over $\mathbb{C}[s]$. Let $\overline{M}_{0,n}(M, d)$ be the moduli space of genus zero, degree $d$ stable maps to $M$ with $n$ marked points. We have the following structure maps:

$$\overline{M}_{0,n+1}(M, d) \xrightarrow{\epsilon_i} \overline{M}_{0,n}(M, d)$$

where $\pi_i$ is the $i$-th forgetful map and $\epsilon_i$ is the evaluation map at the $i$-th marked point. The genus zero $(\mathcal{V}, c)$-twisted Gromov-Witten invariants are defined by

$$\langle \psi_1^{k_1} \alpha_1, \ldots, \psi_n^{k_n} \alpha_n \rangle_{\mathcal{V}} := \int_{[\overline{M}_{0,n}(M,d)]^{\text{virt}}} c(R^*\pi_{n+1}^*e_{n+1}\mathcal{V}) \cup \prod_{i=1}^{n} e_i^*(\alpha_i)\psi_i^{k_i}$$

(2.3)

where $\psi_i$ is the first Chern class of the $i$-th cotangent line, $\alpha_1, \ldots, \alpha_n \in H^*(M)$ and $[\overline{M}_{0,n}(M,d)]^{\text{virt}}$ is the virtual fundamental class. These correlators take values in $\mathbb{C}[s]$. We also introduce the twisted Poincaré pairing on $H^*(M)$:

$$\langle \alpha_1, \alpha_2 \rangle_{\mathcal{V}} := \int_M \alpha_1 \cup \alpha_2 \cup c(\mathcal{V}).$$

(2.4)

The twisted Gromov-Witten invariants satisfy almost all the formal properties of the ordinary Gromov-Witten invariants — the string equation, the dilaton equation, the topological recursion relations (TRR) and the divisor equation (see for instance [24, Section 1], [14]) — but the pairing appearing in the TRR should be replaced with the twisted Poincaré pairing. In [14], Givental explained these properties by geometry of a Lagrangian cone (see also Section 5.1).

Let $p_0$ be the unit class $1 \in H^0(M, \mathbb{C})$ and $\{p_1, \ldots, p_r\}$ be an integral basis of $H^2(M, \mathbb{Z})_{\text{free}} = H^2(M, \mathbb{Z})/H^2(M, \mathbb{Z})_{\text{tor}}$. We can choose a basis so that each $p_a$ ($1 \leq a \leq r$) is a nef class, i.e. $p_a$ intersects all effective curve classes non-negatively. Let $\{p_{r+1}, \ldots, p_s\}$ be a basis of $H^{2\times}(M, \mathbb{C})$. Let $t^0, t^1, \ldots, t^r, t^{r+1}, \ldots, t^s$ be linear co-ordinates of $H^*(M)$ dual to the basis $\{p_0, p_1, \ldots, p_r, p_{r+1}, \ldots, p_s\}$. Let $\tau = \sum_{i=0}^{s} t^i p_i$ be a general point on $H^*(M)$. Let $Q^a, 1 \leq a \leq r$ be the
Novikov variable dual to $p_a \in H^2(M, \mathbb{Z})_{\text{free}}$. This gives the following co-ordinate on $H^2(M, \mathbb{C}^*)$:

$$Q^a : H^2(M, \mathbb{C}^*) \cong H^2(M, \mathbb{C}/2\pi \sqrt{-1}\mathbb{Z}) \ni \left[ \sum_{a=1}^r t^a p_a \right] \mapsto \exp(t^a) \in \mathbb{C}^*. $$

For $d \in H_2(M, \mathbb{Z})$, we denote by $Q^d$ the monomial $(Q^1)^{[p_1,d]}(Q^2)^{[p_2,d]} \cdots (Q^r)^{[p_r,d]}$. The $(\mathcal{V}, \mathcal{E})$-twisted big and small quantum cohomology $QH^*_\mathcal{C}(M, \mathcal{V})$, $SQH^*_\mathcal{C}(M, \mathcal{V})$ are tensor products of the cohomology ring and the formal power series ring in co-ordinate variables:

$$QH^*_\mathcal{C}(M, \mathcal{V}) = H^*(M) \otimes \mathbb{C}[s][Q^1, \ldots, Q^r][t^0, t^1, \ldots, t^s], \quad \text{(big)}$$

$$SQH^*_\mathcal{C}(M, \mathcal{V}) = H^*(M) \otimes \mathbb{C}[s][Q^1, \ldots, Q^r]. \quad \text{(small)}$$

The quantum products $\ast$ on these modules are bilinear over the formal power series ring and deformations of the cup product $\cup$. The product of $SQH^*_\mathcal{C}(M, \mathcal{V})$ is obtained as the specialization $t^0 = t^1 = \cdots = t^s = 0$ of that of $QH^*_\mathcal{C}(M, \mathcal{V})$.

We define the big quantum product $\ast$ of $(M, \mathcal{V})$ by the formula

$$\langle p_i \ast p_j, p_k \rangle^\mathcal{V} = \sum_{d \in \Lambda} \sum_{n \geq 0} \frac{1}{n!} \left( \left\langle p_i, p_j, p_k, \underbrace{\tau, \ldots, \tau}_{n \text{ times}} \right\rangle_d^\mathcal{V} \right) Q^d \quad (2.5)$$

where $\tau = \sum_{i=0}^s n^i p_i$ and $\Lambda \subset H^2(M, \mathbb{Z})$ denotes the Mori cone, i.e. the semigroup generated by effective curves. Because $p_a$ is nef, $Q^d$ in the summation does not contain negative powers of $Q^a$, i.e. $Q^d \in \mathbb{C}[Q^1, \ldots, Q^r]$.

**Remark.** By applying the divisor equation, we get

$$\langle p_i \ast p_j, p_k \rangle^\mathcal{V} = \sum_{d \in \Lambda} \sum_{n \geq 0} \frac{1}{n!} \left( \left\langle p_i, p_j, p_k, \underbrace{\tau, \ldots, \tau}_{n \text{ times}} \right\rangle_d^\mathcal{V} \right) e^{(\tau, d)} Q^d,$$

where we write $\tau = t^0 p_0 + \sum_{a=1}^r t^a p_a + \tau_{\geq 4}$ with $\tau_{\geq 4} \in H^{\geq 4}(X, \mathbb{C})$. Thus for $\tau = \sum_{a=1}^r t^a p_a \in H^2(X, \mathbb{C})$, we have

$$\langle p_i \ast p_j, p_k \rangle^\mathcal{V} = \sum_{d \in \Lambda} \langle p_i, p_j, p_k \rangle_d^\mathcal{V} \prod_{a=1}^r (e^{t^a} Q^a)^{[p_a, d]}. $$

This shows that the small quantum product (restriction of the big one to $t^i = 0$ for $0 \leq i \leq s$) is equivalent to the big quantum product restricted to $\tau \in H^2(X, \mathbb{C})$. The big quantum product on $H^2(X, \mathbb{C})$ is obtained by substituting $Q^a$ in the small quantum product with $e^{t^a} Q^a$.

As an important example of the twists, we consider the twist by the $S^1$-equivariant Euler class $e$:

$$e(F) := \sum_{i \geq 0} \Lambda^{\text{rank}(F) - i} c_i(F) \quad (2.6)$$
where the $S^1$-action on $F$ is defined by the scalar multiplication on each fiber and $\lambda$ is a generator of $H^*_S(pt)$. This corresponds to the choice of parameters:

\[ s_0 = \log \lambda, \quad s_k = (-1)^{k-1}(k-1)!/\lambda^k. \]

In this case, the quantum product is defined on the following modules.

\[ QH^*_e(M, \mathcal{V}) = H^*(M) \otimes \mathbb{C}(\langle \lambda^{-1} \rangle)[Q^1, \ldots, Q^r][t^0, t^1, \ldots, t^s], \quad \text{(big)} \]
\[ SQH^*_e(M, \mathcal{V}) = H^*(M) \otimes \mathbb{C}(\langle \lambda^{-1} \rangle)[Q^1, \ldots, Q^r]. \quad \text{(small)} \]

Now assume that the bundle $\mathcal{V}$ is convex, i.e. for any holomorphic map $f: \mathbb{P}^1 \to M$, $H^1(\mathbb{P}^1, f^*(\mathcal{V})) = 0$ holds. In this case, the structure constants of $QH^*_e(M, \mathcal{V})$ take values in $\mathbb{C}[\lambda][Q^1, \ldots, Q^r][t^0, \ldots, t^s]$, i.e. we need not invert the variable $\lambda$.

We refer the reader to [24] for details. Thus we can define the quantum product on the smaller modules:

\[ QH^*_e(M, \mathcal{V}) = H^*(M) \otimes \mathbb{C}[\lambda][Q^1, \ldots, Q^r][t^0, \ldots, t^s] \quad \text{(big)} \]
\[ SQH^*_e(M, \mathcal{V}) = H^*(M) \otimes \mathbb{C}[\lambda][Q^1, \ldots, Q^r] \quad \text{(small)} \]

In particular, we can define the non-equivariant ($\lambda = 0$) Euler twisted quantum cohomology for a pair $(M, \mathcal{V})$. The non-equivariant version is closely related to the quantum cohomology of zero-locus $N \subset M$ of a transverse section of $\mathcal{V}$ as the following theorem shows.

**Theorem 2.1** ([22]). Let $\mathcal{V}$ be a convex vector bundle on $M$ and $\iota: N \hookrightarrow M$ be a smooth subvariety defined by a regular section of $\mathcal{V}$. Then

\[ \overline{M}_{0,n}(M, d)_{\text{virt}} \cap \text{Euler}(R^*(\pi_{n+1,*}e_{n+1}^*\mathcal{V})) = \sum_{d=\iota_*\beta} t_\beta \overline{M}_{0,n}([N, \beta])_{\text{virt}} \]

for $d \in H_2(M, \mathbb{Z})$. In particular, we have

\[ \lim_{\lambda \to 0} \langle \iota^*p_i \ast_{\mathcal{V}} p_j \ast_{\mathcal{N}} p_k \rangle^N(\tau) = \langle \iota^*p_i \ast_{\mathcal{N}} p_j \ast_{\mathcal{N}} p_k \rangle^N(\tau) \big|_{H_2(N) \to H_2(M)}. \]

Here, $\ast_{\mathcal{V}}$ and $\ast_{\mathcal{N}}$ are the products of $QH^*_e(M, \mathcal{V})$ and $QH^*(N)$ respectively, $\tau \in H^*(M)$ and $\langle \cdot, \cdot \rangle^N$ is the Poincaré pairing of $N$. The notation $|_{H_2(N) \to H_2(M)}$ means to replace $Q^d, d \in H_2(N)$ with $Q^{i_\mathcal{V}(d)}$.

**Notation.** As we have seen, we can define the quantum cohomology over various ground rings. In order to describe different versions in a unified way, we use the letter $K$ to denote the ground ring. In Section [2] $K$ means

\[ K = \begin{cases} 
\mathbb{C} & \text{untwisted,} \\
\mathbb{C}[\lambda] & c = e, \text{ convex } \mathcal{V}, \\
\mathbb{C}(\langle \lambda^{-1} \rangle) & c = e, \text{ general } \mathcal{V}, \\
\mathbb{C}[s] & \text{general } c.
\end{cases} \quad (2.7) \]

Also we use the following shorthand:

\[ K[Q, t] := K[Q^1, \ldots, Q^r][t^0, \ldots, t^s], \quad K[Q] := K[Q^1, \ldots, Q^r]. \]
2.2. QDM. Let $K$ be a topological ring $K$ endowed with a valuation $v: K \to \mathbb{R} \cup \{\infty\}$. The space $K\{h, h^{-1}\}$ of Laurent power series is defined to be

$$K\{h, h^{-1}\} := \left\{ \sum_{n \in \mathbb{Z}} a_n h^n \bigg| a_n \in K, \lim_{n \to \infty} v(a_n) = \infty, \inf_{n \in \mathbb{Z}} v(a_n) > -\infty \right\}.$$  

We define $K\{h\}$ (resp. $K\{\{h^{-1}\}\}$) to be the subspace of $K\{h, h^{-1}\}$ consisting of the positive (resp. negative) power series in $h$. These become rings when $K$ is complete. The valuation on $K = \mathbb{C}[s]$ was defined in (2.2) and that on $K = \mathbb{C}(\lambda^{-1})$ is given by $v(\sum_n a_n \lambda^{-n}) = \min\{n | a_n \neq 0\}$. These are complete. The valuations on $K = \mathbb{C}[\lambda]$ and $\mathbb{C}$ are induced from that on $\mathbb{C}(\lambda^{-1})$ and give them the discrete topology. For example, we have

$$\mathbb{C}(\lambda^{-1})\{h, h^{-1}\} = \mathbb{C}(h^{-1})((\lambda^{-1})),$$

$$\mathbb{C}[\lambda]\{h, h^{-1}\} = \mathbb{C}(h^{-1})[\lambda], \quad \mathbb{C}[\lambda]\{h\} = \mathbb{C}[h, \lambda].$$

**Definition 2.2.** The big and small QDMs are modules

$$QDM_e(M, V) = H^s(M) \otimes K\{h\}[Q, l], \quad \text{(big)}$$

$$SQDM_e(M, V) = H^s(M) \otimes K\{h\}[Q], \quad \text{(small)}$$

euendowed with the actions of the Dubrovin connection:

$$\nabla_i^h = h \frac{\partial}{\partial t_i} + p_i \ast \quad (0 \leq i \leq s, \text{ defined only for } QDM_e(M, V)),$$

$$\nabla_a^h = h Q^a \frac{\partial}{\partial Q^a} + p_a \ast \quad (1 \leq a \leq r)$$

where $\ast$ denotes the $S^1$-equivariant big/small quantum product of $(M, V)$ defined in (2.5). The Dubrovin connection is known to be flat.

The big QDM is considered as a flat bundle over a formal neighborhood of the origin in $\mathbb{C}^r \times H^s(M, \mathbb{C})$. Here, $\mathbb{C}^r$ is a partial compactification of $H^2(M, \mathbb{C}^*)$ given by a choice of co-ordinates $Q^1, \ldots, Q^r$. The small QDM is the restriction of the big one to $\mathbb{C}^r \times \{0\}$. The flat connection has a logarithmic singularity along $Q^1 \cdots Q^r = 0$. By the divisor equation, the big QDM restricted on the locus $\mathbb{C}^r \times H^s(M, \mathbb{C})$ can be recovered from the small one. We also consider the small QDM as a module over the Heisenberg algebra $\mathcal{D}$:

$$\mathcal{D} = K\{h\}[Q^1, \ldots, Q^r][P_1, \ldots, P_r]$$

whose generators satisfy the following commutation relations:

$$[P_a, Q^b] = h Q^b_a Q^b, \quad [P_a, P_b] = [Q^a, Q^b] = 0.$$  

Here, $\mathcal{D}$ acts on $SQDM_e(M, V)$ by

$$P_a \mapsto \nabla_a, \quad Q^a \mapsto \text{multiplication by } Q^a.$$
2.3. Fundamental solution and $J$-function. A fundamental solution to the big QDM is a formal section $L$ of the endomorphism bundle satisfying

\[ L \in \text{End}(H^*(M) \otimes K \{ h, h^{-1} \})[Q, t], \]

\[ \nabla^h_j L = L \circ h \frac{\partial}{\partial \psi^j}, \quad (0 \leq j \leq s) \]  \hspace{1cm} (2.8)

\[ \nabla^h_a L = L \circ (hQ^a \frac{\partial}{\partial Q^a} + p_a \cup), \quad (1 \leq a \leq r). \]  \hspace{1cm} (2.9)

An explicit form of a fundamental solution is given by the gravitational descendants (see [24, Section 1.3]. Note that our choice of the sign of $h$ is opposite to [24].)

\[ \langle Lp_i, p_j \rangle^V := \langle p_i, p_j \rangle^V + \sum_{d \in \Lambda, n \geq 0, (d,n) \neq (0,0)} \frac{1}{n!} \left( \frac{e^{-(t^{\psi_1} \cup \tau / h)} p_i}{-h - \psi_1}, p_j, \overbrace{\tau, \ldots, \tau}^{n \text{ times}} \right)^V Q^d. \]  \hspace{1cm} (2.10)

The fraction $\frac{1}{h - \psi_1}$ here should be expanded as a power series in $h^{-1}$. This $L$ does not contain positive powers in $h$ and belongs to $\text{End}(H^*(X) \otimes K \{ h^{-1} \})[Q, t]$. Using the TRR, one can show that $Lp_i, 0 \leq i \leq s$ form a basis of $\nabla^h$-parallel sections (see [24, Proposition 2]). Thus the equation (2.8) holds for this $L$. By using the divisor equation, we can rewrite the above $L$ as (see [24])

\[ \langle Lp_i, p_j \rangle^V = \langle e^{-\tau / h} p_i, p_j \rangle^V + \sum_{d \in \Lambda \setminus \{0\}} \sum_{n \geq 0} \frac{1}{n!} \left( \frac{e^{-(t^{\psi_1} \cup \tau / h)} p_i}{-h - \psi_1}, p_j, \overbrace{\tau, \ldots, \tau}^{n \text{ times}} \right)^V Q^d. \]

where $\tau = \sum_{i=0}^s t^i p_i = t^0 p_0 + \tau_2 + \tau_{2+}$ with $\tau_2 \in H^2(M)$ and $\tau_{2+} \in H^{2+}(M)$. In the expression like $e^{-\tau / h}$, $\tau$ is considered to be an operator acting on cohomology by the cup product. Therefore, we can decompose $L$ in the form $L = S \circ e^{-(t^{\psi_1} \cup \tau_2) / h}$, where $S$ is an element of $\text{End}(H^*(M)) \otimes K \{ h^{-1} \}[Q, t]$. It follows from the above expression that $S$ satisfies

\[ \frac{\partial}{\partial Q^a} S = Q^a \frac{\partial}{\partial Q^a} S. \]  \hspace{1cm} (2.11)

The equation (2.9) follows from (2.8) and (2.11).

**Proposition 2.3.** The above fundamental solution $L = L(Q, \tau, h)$ given by gravitational descendants is characterized by the following condition:

i) initial condition: $L(0, 0, h) = \text{id}$ and

ii) differential equations:

\[ hq^a \frac{\partial}{\partial q^a} L - L \circ (p_a \cup) + (p_a \ast) \circ L = 0, \quad (1 \leq a \leq r) \]

\[ h \frac{\partial}{\partial t} L + (p_j \ast) \circ L = 0, \quad (0 \leq j \leq s). \]

Moreover this satisfies
iii) unitarity:
\[ \langle L(Q, \tau, -\hbar)p_i, L(Q, \tau, \hbar)p_j \rangle^\nu = \langle p_i, p_j \rangle^\nu, \]

iv) divisor equation:
\[ (\frac{\partial}{\partial t^0} - Q^a \frac{\partial}{\partial Q^a})L + L \circ (\frac{p_n}{\hbar}) = 0. \quad (2.12) \]

Proof. The unitarity is stated in \[13\]. Set \( \mathcal{T} = L|_{\hbar \to -\hbar} \). By the Frobenius property \( \langle p_i * p_j, p_k \rangle^\nu = \langle p_i, p_j * p_k \rangle^\nu \) and the differential equation for \( L \), we have
\[ \hbar \frac{\partial}{\partial \ell^k}(\mathcal{T}p_i, Lp_j)^\nu = \langle p_k * \mathcal{T}p_i, Lp_j \rangle^\nu - \langle \mathcal{T}p_i, p_k * Lp_j \rangle^\nu = 0, \]
\[ \hbar Q^a \frac{\partial}{\partial Q^a}(\mathcal{T}p_i, Lp_j)^\nu = -\langle \mathcal{T}(p_n \cup)p_i, Lp_j \rangle^\nu + \langle \mathcal{T}p_i, L(p_n \cup)p_j \rangle^\nu. \]

Since the operation \( p_n \cup \) is nilpotent, \( (\hbar Q^a \frac{\partial}{\partial Q^a})^n(\mathcal{T}p_i, Lp_j)^\nu \) is zero for a sufficiently big \( n \). This shows that \( (\mathcal{T}p_i, Lp_j)^\nu \) is a constant and is equal to \( (\mathcal{T}p_i, Lp_j)|_{Q=\tau=0} = \langle p_i, p_j \rangle^\nu \). The divisor equation follows from \[2.11\].

The unitarity means that \( L(Q, \tau, \hbar^{-1}) \) is the adjoint of \( L(Q, \tau, -\hbar) \). Put \( g_{ij} = \int_M p_i \cup p_j \) and let \( (g^{ij})_{0 \leq i, j \leq s} \) be the matrix inverse to \( (g_{ij})_{0 \leq i, j \leq s} \). Using the unitarity, we calculate \( L^{-1} \) as
\[ L^{-1}p_i = p_i + \sum_{d \in \Lambda, n \geq 0, (d,n) \neq (0,0)} \sum_{j,k} \frac{1}{n!} \frac{p_k}{c(V)} g^{kj} \left\langle \frac{p_j}{\hbar - \psi}, p_i, \tau_1, \ldots, \tau \right\rangle^\nu_d \]
\[ = e^{\tau/\hbar} p_i + e^{(\ell^i + \tau_2)/\hbar} \sum_{d \in \Lambda \setminus \{0\}} \sum_{n \geq 0} \frac{1}{n!} \frac{p_k}{c(V)} g^{kj} \left\langle \frac{p_j}{\hbar - \psi_1}, p_i, \tau_{4, \ldots, 4} \right\rangle^\nu_d e^{(\tau, d)} Q^d. \quad (2.13) \]

**Definition 2.4.** The big \( J \)-function of \((M, V)\) is a cohomology-valued formal function defined as \( J := L^{-1}p_0 \), where \( L \) is the fundamental solution in \[2.10\] and \( p_0 \) is the unit. This is an element of \( H^*(M) \otimes K\{[h^{-1}]\}[Q, \ell] \). The small \( J \)-function is the restriction of the big one to \( \tau = 0 \in H^*(M) \).

**Remark 2.5.** The small \( J \)-function is slightly different from what the author used in the previous paper \[17\]. The \( J \)-function \( J_{prev}(Q, \hbar) \) in \[17\] is related to the above small \( J \)-function \( J_{small}(Q, \hbar) \) by
\[ J_{prev}(Q, \hbar) = e^{\sum_{a=1}^r p_a \log Q_a/\hbar} J_{small}(Q, \hbar), \]
where \( J_{prev} \) lives in \( H^*(X) \otimes K\{[h^{-1}]\}[Q][\log Q^1, \ldots, \log Q^r] \). It follows from \[2.13\] that the big \( J \)-function restricted to \( \tau_2 \in H^2(X) \) is related to the small \( J \)-function by
\[ J_{big}(Q, \tau_2, \hbar) = e^{\tau_2/\hbar} J_{small}(e^{\ell_1} Q^1, \ldots, e^{\ell_r} Q^r, \hbar), \quad \tau_2 = \sum_{a=1}^r \ell^a p_a \in H^2(X). \]
The big QDM is generated by the unit section $p_0$ as a $D$-module. In fact, $QDM_c(M, V)$ is generated by $p_0$ and its derivatives by the Dubrovin connection:

$$\nabla^h_1 p_0 = p_1, \ldots, \nabla^h_s p_0 = p_s,$$

as a $K\{\hbar\}[Q, t]$-module. Because the big $J$-function is defined to be the inverse image of $p_0$ under $L$, it plays the role of a generator of the big QDM. In fact, if a differential operator $P$ annihilates $J$:

$$P(Q, t, h Q^a \frac{\partial}{\partial Q^a} + p_a, h \frac{\partial}{\partial t^i}, \hbar) J(Q, \tau, \hbar) = 0,$$

then we have a relation in $QDM_c(M, V)$:

$$P(Q, t, \nabla^h_a, \nabla^h_i, \hbar)p_0 = 0$$

and vice versa. On the other hand, the small QDM is generated by the unit section under the $H^2$-generation condition; in this case, the $J$-function plays the role of a generator of the small QDM.

**Proposition 2.6** ([17, Theorem 2.4]). Let $M$ be a smooth projective variety and $V$ be a vector bundle on $M$. Assume that the total cohomology ring of $M$ is generated by the second cohomology group. Then the small QDM is generated by the unit section $p_0$ as a $D$-module. Moreover, we have an isomorphism of $D$-modules $D/I \cong SQDM_c(M, V)$ which sends $1$ to $p_0$, where $I$ is the left ideal of $D$ consisting of elements $f(Q, P, \hbar) \in D$ satisfying

$$f(Q^1, \ldots, Q^r, Q^i \frac{\partial}{\partial Q^1} + p_1, \ldots, Q^r \frac{\partial}{\partial Q^r} + p_r, \hbar) J_{small}(Q, \hbar) = 0.$$

**Proof.** This theorem was shown in [17, Theorem 2.4] for convex $V$. The proof applies without change to this more general case. \qed

In this paper, we are mainly interested in the quantum cohomology of toric varieties and their twists. In this case, the $H^2$-generation of the total cohomology always holds. When the total cohomology ring is generated by $H^2(M)$, we have the following reconstruction theorem by Kontsevich and Manin [23], whose generalization is the main theme of this paper.

**Theorem 2.7** (Kontsevich and Manin). If the total cohomology ring $H^*(M)$ is generated by the second cohomology group $H^2(M)$ as a ring, the big quantum cohomology $QH^*(M)$ can be reconstructed from the small quantum cohomology $SQH^*(M)$. In other words, if we know all the three point functions of the form $\langle p_a, p_i, p_j \rangle_{0,3,d}$ with $1 \leq a \leq r$, $1 \leq i, j \leq s$, we can reconstruct all genus 0, $n$-point functions for all $n$.

3. **Equivariant Floer Theory**

In this section, we review the equivariant Floer theory for toric complete intersections [9, 10, 17]. The exposition here slightly differs from what was given in [17] so that it contains a little more general case where $V$ is not necessarily nef. See also [1] for background materials on toric varieties.
3.1. Toric varieties. A projective toric variety $X$ can be defined by the following data:

1. algebraic torus $\mathbb{T}_C \cong (\mathbb{C}^*)^r$ and its maximal compact subgroup $\mathbb{T} \cong (S^1)^r$;
2. $N$-tuple of integral vectors $u_1, \ldots, u_N$ in the weight lattice $\text{Hom}(\mathbb{T}_C, \mathbb{C}^*)$;
3. a vector $\eta$ in $\text{Lie}(\mathbb{T})^\vee := \text{Hom}(\mathbb{T}_C, \mathbb{C}^*) \otimes \mathbb{R}$.

The integral vectors $u_1, \ldots, u_N$ define a homomorphism $\mathbb{T}_C \to (\mathbb{C}^*)^N$ ($\mathbb{T} \to (S^1)^N$) and an action of $\mathbb{T}_C$ (resp. $\mathbb{T}$) on $\mathbb{C}^N$. Define

$$A_\eta := \left\{ I \subset \{1, 2, \ldots, N\} \mid \sum_{i \in I} \mathbb{R}_{>0} u_i \ni \eta \right\}.$$ 

This is a subset of the power set $\mathcal{P}(\{1, 2, \ldots, N\})$. Then $X$ is defined as a GIT quotient of $\mathbb{C}^N$ by $\mathbb{T}_C$.

$$X := \mathbb{C}^N//_\eta \mathbb{T}_C = \mathcal{U}_\eta/\mathbb{T}_C, \quad \mathcal{U}_\eta := \mathbb{C}^N \setminus \bigcup_{I \notin A_\eta} \mathbb{C}^I,$$

where $\mathbb{C}^I \subset \mathbb{C}^N$ denote the co-ordinate subspace $\{(z_1, \ldots, z_N) ; z_i = 0 \text{ for } i \notin I\}$. The toric variety $X$ is non-empty, smooth and compact under the following conditions.

(A) $\{1, \ldots, N\} \setminus \{i\} \in A_\eta$ for all $1 \leq i \leq N$.
(B) For any $I \in A_\eta$, $\{u_i\}_{i \in I}$ generate $\text{Hom}(\mathbb{T}_C, \mathbb{C}^*)$ as a $\mathbb{Z}$-module.
(C) If $\sum_{i=1}^{N} c_i u_i = 0$ with $c_i \geq 0$, then $c_j = 0$ for all $j$.

Henceforth we assume these. A representation $\rho \in \text{Hom}(\mathbb{T}_C, \mathbb{C}^*)$ of $\mathbb{T}_C$ gives rise to a line bundle $\mathcal{L}_\rho$:

$$\mathcal{L}_\rho := \mathbb{C} \times_\rho \mathcal{U}_\eta \longrightarrow X,$$

where $\mathbb{T}_C$ acts on $\mathbb{C} \times \mathbb{C}^N$ by the representation $\rho$ on the $\mathbb{C}$ factor and in the given way on the $\mathbb{C}^N$ factor. By this, we have a natural identification:

$$\text{Hom}(\mathbb{T}_C, \mathbb{C}^*) \cong \text{Pic}(X) \cong \mathcal{H}^2(X, \mathbb{Z}).$$

Let $\mu : \mathbb{C}^N \to \text{Lie}(\mathbb{T})^\vee$ be the moment map for the $\mathbb{T}$ action on $\mathbb{C}^N$. This is given by $\mu(z_1, \ldots, z_N) = \sum_{i=1}^{N} \frac{1}{2} |z_i|^2 u_i$. We can give $X$ also as a symplectic form:

$$X \cong \mu^{-1}(0)/\mathbb{T}.$$

The vector $\eta \in \text{Lie}(\mathbb{T})^\vee$ is identified with the class of the reduced symplectic form. The Kähler cone $C_X$ of $X$ i.e. the cone of Kähler classes are given by

$$C_X = \bigcap_{I \in A_\eta} \sum_{i \in I} \mathbb{R}_{>0} u_i \subset \text{Hom}(\mathbb{T}_C, \mathbb{C}^*) \otimes \mathbb{R} \cong \mathcal{H}^2(X, \mathbb{R}).$$

We have $\eta \in C_X$. Take a nef integral basis $p_1, \ldots, p_r$ of $\mathcal{H}^2(X, \mathbb{Z}) \cong \text{Hom}(\mathbb{T}_C, \mathbb{C}^*)$ as we did in Section 2. Here $p_a \in C_X$ since $p_a$ is nef. We write

$$u_i = \sum_{a=1}^{r} m_a^i p_a$$

(3.3)

The vector $u_i \in \text{Hom}(\mathbb{T}_C, \mathbb{C}^*)$ represents the Poincaré dual of the torus invariant divisor $\{z_i = 0\}$, where $z_i$ is the standard $i$-th co-ordinate on $\mathbb{C}^N$. 
We will also consider the equivariant Floer cohomology of $X$ twisted by a vector bundle $\mathcal{V}$. We take $\mathcal{V}$ as a sum $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_r$ of line bundles $\mathcal{V}_i$. We put

$$v_i := c_1(\mathcal{V}_i) = \sum_{a=1}^{r} l_a^i p_a \in H^2(X, \mathbb{Z}).$$

### 3.2. Algebraic model for the universal cover of free loop spaces.

Following Givental [10] and Vlassopoulos [25], we introduce the algebraic model $L_X$ for the universal cover $\tilde{X}^{S^1}$ of the free loop space $X^{S^1} = \text{Map}(S^1, X)$. The universal cover $\tilde{X}^{S^1}$ is given by

$$\tilde{X}^{S^1} = \text{Map}(S^1, \mathcal{U}_\eta)/\text{Map}_0(S^1, \mathcal{T}_C),$$

where $\text{Map}_0(S^1, \mathcal{T}_C)$ denotes the set of contracting loops in $\mathcal{T}_C$. We will replace $\text{Map}(S^1, \mathcal{U}_\eta)$ with the space of Laurent polynomial loops and $\text{Map}_0(S^1, \mathcal{T}_C)$ with $\mathcal{T}_C$ to construct the algebraic model $L_X$. Define

$$L_X := C[\zeta, \zeta^{-1}]^N//\eta \mathcal{T}_C = L\mathcal{U}_\eta/\mathcal{T}_C, \quad L\mathcal{U}_\eta := C[\zeta, \zeta^{-1}]^N \bigcup_{I \notin A_\eta} C[\zeta, \zeta^{-1}]^\ell.$$

Here, $\zeta$ is considered to be a parameter of loops. The $\mathcal{T}_C$ action on $C[\zeta, \zeta^{-1}]^N \cong C^N \otimes C[\zeta, \zeta^{-1}]$ is induced from the given action on $C^N$ and the trivial action on $C[\zeta, \zeta^{-1}]$. We write a general point on $L_X$ as

$$[z_1(\zeta), \ldots, z_N(\zeta)] \in L_X, \quad \text{where } z_i(\zeta) = \sum_{\nu \in \mathbb{Z}} a_{\mu \nu} \zeta^\nu \in C[\zeta, \zeta^{-1}].$$

The variables $\{a_{\mu \nu}\}$ play the role of homogeneous co-ordinates on $L_X$. We can write $L_X$ also as a symplectic quotient. The moment map $\mu_\infty: C[\zeta, \zeta^{-1}]^N \to \text{Lie}(\mathbb{T})^\mathbb{N}$ of $T$ action is given by $\mu_\infty[z_1(\zeta), \ldots, z_N(\zeta)] = \sum_{1 \leq i \leq N, r \in \mathbb{Z}} \frac{1}{2} |a_{\mu \nu}|^2 u_i$ and we have

$$L_X \cong \mu_\infty^{-1}(\eta)/T.$$ 

Note that we use the same $\eta$ as we used to define $X$. The space $L_X$ here is infinite dimensional, but by cutting off Laurent series at a finite exponent, we can write $L_X$ as an inductive limit of smooth projective toric varieties (or compact symplectic toric manifolds). We endow $L_X$ with the inductive limit topology.

The algebraic model $L_X$ has several properties analogous to the universal cover $\tilde{X}^{S^1}$ of free loop space. First, this has an $S^1$-action of the loop rotation:

$$[z_1(\zeta), \ldots, z_N(\zeta)] \mapsto [z_1(e^{-\sqrt{-1} \theta} \zeta), \ldots, z_N(e^{-\sqrt{-1} \theta} \zeta)], \quad e^{\sqrt{-1} \theta} \in S^1.$$ 

Second, this $S^1$ action is Hamiltonian with respect to the reduced symplectic form on $L_X$. The Hamiltonian $H$ is given by

$$H[z_1(\zeta), \ldots, z_N(\zeta)] = \frac{1}{2} \sum_{i=1}^{N} \sum_{\nu \in \mathbb{Z}} \nu |a_{\mu \nu}|^2 \quad \text{on the level set } \mu_\infty^{-1}(\eta).$$
This is an analogue of the classical action functional \( \int pdq \) on \( \tilde{X}^{S^1} \). Third, the group \( H_2(X, \mathbb{Z}) \cong \pi_1(X^{S^1}) \) of covering transformations on \( \tilde{X}^{S^1} \) acts on the model \( L_X \). For \( d \in H_2(X, \mathbb{Z}) \), we define a covering transformation \( Q^d \) as
\[
Q^d: [z_1(\zeta), \ldots, z_N(\zeta)] \mapsto [\zeta^{-u_1, d}z_1(\zeta), \ldots, \zeta^{-(u_N, d)}z_N(\zeta)]
\]
Take a basis of \( H_2(X, \mathbb{Z}) \) dual to \( p_1, \ldots, p_r \) and let \( Q^1, \ldots, Q^r \) be the corresponding covering transformations. Then we have \( Q^d = (Q^1)^{p_1, d} \cdots (Q^r)^{p_r, d} \).

We introduce \( S^1 \)-equivariant cohomology classes \( p_1, \ldots, p_r \in H^2_S(L_X) \) corresponding to the basis \( p_1, \ldots, p_r \) of \( H^2(X, \mathbb{Z}) \). By the natural isomorphism \( H^2(X, \mathbb{Z}) \cong \text{Hom}(\mathbb{T}_C, \mathbb{C}^*) \), \( p_a \) gives a representation of \( \mathbb{T}_C \) and gives a line bundle \( L_{pa} \) on \( L_X \):
\[
L_{pa} = \mathbb{C} \times_{p_a} LU_\eta \longrightarrow L_X.
\]
This \( L_{pa} \) admits an \( S^1 \)-action such that \( S^1 \) acts on Laurent polynomials by the loop rotation \( \zeta \mapsto e^{-\sqrt{-1} \theta} \zeta \) and trivially on the \( \mathbb{C} \) factor. We define
\[
P_a := e^{\sum p_a (L_{pa})} \in H^2_S(L_X).
\]
It is easy to see that the pull-back by the covering transformation \( Q^b \) and the multiplication by \( P_a \) satisfies the commutation relation
\[
[P_a, (Q^b)^*] = h \delta^b_a (Q^b)^*
\]
as operators acting on \( H^*_S(L_X) \). Here, \( h \) is a generator of the \( S^1 \)-equivariant cohomology of a point. This commutation relation, discovered by Givental [9, 10], yields a \( D \)-module structure on the equivariant Floer cohomology.

### 3.3. Equivariant Floer cohomology

The equivariant Floer theory here is considered to be a Morse-Bott theory on \( L_X \) with respect to the Hamiltonian function \( H \). The critical set of \( H \) is equal to the \( S^1 \)-fixed point set in \( L_X \) and is given by the union of copies \( X_d \) of \( X \) over \( d \in H_2(X, \mathbb{Z}) \), where
\[
X_d := \{[a_1 \zeta^{(u_1, d)}, \ldots, a_N \zeta^{(u_N, d)}] \in L_X \mid a_i \in \mathbb{C} \}.
\]
The gradient vector field of \( H \) generates a flow \( \phi_t \) on \( L_X \) given by
\[
\phi_t[z_1(\zeta), \ldots, z_N(\zeta)] = [z_1(e^{-t} \zeta), \ldots, z_N(e^{-t} \zeta)], \quad t \in \mathbb{R}.
\]
Let \( L^\infty_d \) be the closure (with respect to the inductive limit topology) of the stable manifold of \( X_d \):
\[
L^\infty_d := \{ z(\zeta) \in L_X \mid \lim_{t \to \infty} \phi_t(z) \in X_d \}
\]
\[
= \left\{ [z_1(\zeta), \ldots, z_N(\zeta)] \in L_X \mid z_i(\zeta) = \sum_{\nu \leq (u_i, d)} a_{i\nu} \zeta^\nu \right\}.
\]
Similarly, let \( L^{-\infty}_d \) be the closure of the unstable manifold of \( X_d \):
\[
L^{-\infty}_d := \left\{ [z_1(\zeta), \ldots, z_N(\zeta)] \in L_X \mid z_i(\zeta) = \sum_{\nu \geq (u_i, d)} a_{i\nu} \zeta^\nu \right\}.
\]
We set $L_{d_1} = L_{d_1}^\infty \cap L_{d_2}^{\infty}$. This is a finite dimensional smooth toric variety and considered to be a compactification of the union of gradient flowlines connecting $X_{d_1}$ and $X_{d_2}$. Note that the infinite dimensional spaces $L_X$ and $L_{d_2}^{\infty}$ here are quotients of open subsets of $\mathbb{C}^\infty$ by the torus $\mathbb{T}_C$; the open subset here is the complement of union of infinite codimensional subspaces in $\mathbb{C}^\infty$, so in particular contractible. Therefore, $L_X$ and $L_{d_2}^{\infty}$ are homotopy equivalent to the classifying space $BT_C$ and we have

$$H^*(L_X) \cong H^*(L_{d_2}^{\infty}) \cong \mathbb{C}[c_1(L_{p_1}), \ldots, c_1(L_{p_r})].$$

Repeating the same argument for the Borel construction $L_X \times_{S^1} ES^1$, $L_{d_2}^{\infty} \times_{S^1} ES^1$, we obtain

$$H^*_S(L_X) \cong H^*_S(L_{d_2}^{\infty}) \cong \mathbb{C}[P_1, \ldots, P_r, h].$$

First we explain the construction of equivariant Floer cohomology in the case of toric variety itself ($\forall = 0$). Introduce a partial order ($\leq$) on $H_2(X, \mathbb{Z})$ as

$$d_1 \leq d_2 \iff L_{d_1}^{\infty} \subset L_{d_2}^{\infty}, \quad d_1, d_2 \in H_2(X, \mathbb{Z}).$$

Then $H_2(X, \mathbb{Z})$ is a directed set. When $d_1 \leq d_2$, there exists a push-forward $H^*_S(L_{d_1}^{\infty}) \to H^*_S(L_{d_2}^{\infty})$ of $S^1$-equivariant cohomology. This push-forward is defined by the multiplication by the Euler class of the (finite rank) normal bundle $N$ of $L_{d_1}^{\infty}$ in $L_{d_2}^{\infty}$.

$$\cup \text{Euler}_S(N) : H^*_S(L_{d_1}^{\infty}) \cong \mathbb{C}[P_1, \ldots, P_r, h] \to \mathbb{C}[P_1, \ldots, P_r, h] \cong H^*_S(L_{d_2}^{\infty}),$$

where the Euler class is given by

$$\text{Euler}_S(N) = \prod_{i=1}^N \prod_{\nu=(u_1, u_2)} \left( \sum_{a=1}^r m_a^\nu P_a - \nu h \right).$$

By this push-forward, $\{H^*_S(L_{d_1}^{\infty})\}_{d \in H_2(X, \mathbb{Z})}$ forms an inductive system. Define the semi-infinite $S^1$-equivariant cohomology $H^*_S(L_X^{\infty})$ as the inductive limit:

$$H^*_S(L_X^{\infty}) := \text{inj lim}_d H^*_S(L_{d_2}^{\infty}).$$

The covering transformation $Q^d$ acts on the inductive system by pull-backs

$$(Q^d)^* : H^*_S(L_{d_1}^{\infty}) \cong H^*_S(L_{d_2}^{\infty})$$

so induces a map $(Q^d)^* : H^*_S(L_X^{\infty}) \to H^*_S(L_X^{\infty})$. The equivariant class $P_a$ also acts on the semi-infinite cohomology by the cup product. These actions satisfy the commutation relation

$$[P_a, (Q^b)^*] = h\delta_a^b(Q^b)^* \quad \text{on } H^*_S(L_X^{\infty}).$$

Hence $H^*_S(L_X^{\infty})$ becomes a module over the polynomial Heisenberg algebra. The semi-infinite cohomology defined here contains classes of “semi-infinite” cycles. For example, the class

$$\Delta := \text{the image of } 1 \in H^*_S(L_0^{\infty}) \text{ in } H^*_S(L_X^{\infty})$$
can be viewed as the Poincaré dual of the fundamental class of the space \( L_\infty^\infty \).
Define a filtration of \( H^{{\infty}/2}_D(L_X) \) as

\[
F_n^\infty(H^{{\infty}/2}_D(L_X)) := \sum_{d \in \square_n} H^D_i(L_d^\infty) = \sum_{d \in \square_n} Q^d \mathbb{C}[P_1, \ldots, P_r, \hbar] \Delta,
\]
where \( \square_n := \{ d \in H_2(X, \mathbb{Z}) \mid \sum_{a=1}^r \langle p_a, d \rangle \geq n, \langle p_a, d \rangle \geq 0, \forall a \} \).

**Definition 3.1.** The \( S^1 \)-equivariant Floer cohomology \( FH^*_D(L_X) \) is defined to be the completion of \( F^\infty(H^{{\infty}/2}_D(L_X)) \) with respect to the above filtration. Then \( FH^*_D(L_X) \) becomes a module over \( \mathbb{C}[\hbar][Q^1, \ldots, Q^r, P_1, \ldots, P_r] \).

Second, we explain the case where we have a vector bundle \( V \) on \( X \). We introduce another fiberwise \( S^1 \)-action on \( V \) and work \( T^2 \)-equivariantly. We use \( \hbar \) and \( \lambda \) for generators of \( T^2 \)-equivariant cohomology of a point, where \( \hbar \) corresponds to the \( S^1 \)-action rotating loops and \( \lambda \) corresponds to the additional fiberwise \( S^1 \)-action. Recall that \( V \) is a sum of line bundles \( V_1 \oplus \cdots \oplus V_l \) with the first Chern class \( v_i = c_1(V_i) \). Let \( V_{i, \nu} \) be a line bundle on \( L_X \), given by \( v_i \in H^2(X, \mathbb{Z}) \cong \text{Hom}(\mathbb{T}_C, \mathbb{C}^*) \):

\[
V_{i, \nu} := \mathbb{C} \times v_i \mathcal{U}_n \rightarrow L_X
\]
endowed with the \( S^1 \)-action (corresponding to the loop rotation):

\[
[v, (z_1(\zeta), \ldots, z_N(\zeta))] \mapsto [e^{-\sqrt{-1} \theta} v, (z_1(e^{-\sqrt{-1} \theta} \zeta), \ldots, z_N(e^{-\sqrt{-1} \theta} \zeta))], \quad e^{\sqrt{-1} \theta} \in S^1.
\]

The additional \( S^1 \)-action acts on \( V_{i, \nu} \) by scalar multiplication on each fiber and trivially on the base. For a function \( f : \{1, \ldots, l\} \rightarrow \mathbb{Z} \), let \( V_f^\infty \) be the infinite dimensional \( T^2 \)-equivariant vector bundle on \( L_X \):

\[
V_f^\infty := \bigoplus_{i=1}^l \bigoplus_{\nu \geq f(i)} V_{i, \nu}.
\]

For \( d \in H_2(X, \mathbb{Z}) \), we define \( V_d^\infty \) by regarding \( d \) as a function \( i \mapsto \langle v_i, d \rangle \) on \( \{1, \ldots, l\} \). Define a partial order on \( H_2(X, \mathbb{Z}) \times \mathbb{Z}^l \) as

\[
(d_1, f_1) \leq (d_2, f_2) \iff L_{d_1}^\infty \subset L_{d_2}^\infty \quad \text{and} \quad f_1(i) \leq f_2(i) \quad \forall i \in \{1, \ldots, l\}.
\]

We define for \( (d, f) \in H_2(X, \mathbb{Z}) \times \mathbb{Z}^l \),

\[
H^T_2(L_d^\infty / V_f^\infty) := H^T_2(L_d^\infty) \cong \mathbb{C}[P_1, \ldots, P_r, \lambda, \hbar].
\]

We could imagine \( H^T_2(L_d^\infty / V_f^\infty) \) as the cohomology of the zero-locus \( s^{-1}(0) \subset L_d^\infty \) if we have a transverse section \( s \) of \( V_f^\infty \) over \( L_d^\infty \). In this case, an element of \( H^T_2(L_d^\infty / V_f^\infty) \) could be viewed as the restriction to \( s^{-1}(0) \) of a cohomology class of the ambient \( L_d^\infty \). When \( (d_1, f_1) \leq (d_2, f_2) \), we define the push-forward \( H^T_2(L_{d_1}^\infty / V_{f_1}^\infty) \rightarrow H^T_2(L_{d_2}^\infty / V_{f_2}^\infty) \) by multiplying the Euler class of the normal bundle \( N \) of \( L_{d_1}^\infty \) in \( L_{d_2}^\infty \) and the quotient \( V_{f_1}^\infty / V_{f_2}^\infty \):

\[
\cup \text{Euler}_{S^1}(N) \text{ Euler}_{T^2}(V_{f_1}^\infty / V_{f_2}^\infty) : H^T_2(L_{d_1}^\infty / V_{f_1}^\infty) \rightarrow H^T_2(L_{d_2}^\infty / V_{f_2}^\infty).
\]
Note that we have a natural inclusion $V^\infty_f \subset V^\infty_{f_1}$ with finite rank quotient. The $S^1$-equivariant characteristic class of the quotient is given by

$$\text{Euler}_{T^2}(V^\infty_{f_2} / V^\infty_{f_1}) = \prod_{i=1}^{l} \prod_{\nu = \nu_1(i)}^r \left( \sum_{a=1}^l t^a \nu - \nu h + \lambda \right).$$

As before, $\{H_{T^2}(L^\infty_d / V^\infty_f)\}_{d,f}$ forms an inductive system. Let $H^\infty_{T^2}(L_X/V)$ denote its inductive limit. This time, we define $H^\infty_{T^2}(L_X/V)$ as a submodule generated by the images of $H^*_T(L^\infty_d / V^\infty_d)$ in the inductive limit $H^\infty_{T^2}(L_X/V)$:

$$H^\infty_{T^2}(L_X/V) := \sum_{d \in H_2(X, \mathbb{Z})} H^*_T(L^\infty_d / V^\infty_d) \subset H^\infty_{T^2}(L_X/V).$$

This semi-infinite cohomology is considered to contain semi-infinite cycles like $\left( [L^\infty_d] \cap \text{Euler}_{T^2}(V^\infty_d) \right)$. The covering transformation $Q^d$ acts on the inductive system by the pull-backs:

$$(Q^d)^*: H^*_T(L^\infty_d / V^\infty_d) \cong H^*_T(L^\infty_{d+d'} / V^\infty_{d+d'}),$$

so acts on the inductive limit. This action together with the multiplication by $P^a$ preserves the submodule $H^\infty_{T^2}(L_X/V)$ and makes it a module over the polynomial Heisenberg algebra. Let $\Delta$ be the image of $1 \in H^0_{T^2}(L^\infty_0 / V^\infty_0)$ in $H^\infty_{T^2}(L_X/V)$. As in Section 2, we set

$$K := \begin{cases} \mathbb{C}[\lambda] & \text{if } V \text{ is convex } i.e. v_1 \text{ is nef for all } i, \\ \mathbb{C}(\langle \lambda^{-1} \rangle) & \text{otherwise.} \end{cases}$$

We will consider the semi-infinite cohomology over $K\{h\}$ (see Section 2.2 for this notation). Define a filtration on $H^\infty_{T^2}(L_X/V)$ by:

$$F^n(H^\infty_{T^2}(L_X/V)) := \sum_{d \in \mathbb{C}_n} H^*_T(L^\infty_d / V^\infty_d) = \sum_{d \in \mathbb{C}_n} Q^d \mathbb{C}[\lambda, h, P_1, \ldots, P_r] \Delta. \quad (3.5)$$

**Definition 3.2.** The $T^2$-equivariant Floer cohomology $FH^\infty_{T^2}(L_X/V)$ for a pair $(X, V)$ is defined to be the completion of $F^0(H^\infty_{T^2}(L_X/V)) \otimes_{H^*_T(pt)} K\{h\}$ with respect to the above filtration. This is a module over $\mathcal{D} \cong K\{h\}[Q^1, \ldots, Q^r](P_1, \ldots, P_r)$.

**Remark.** In [10], in case of hypersurfaces in the projective space, the semi-infinite class $\Delta$ was given in the form of the infinite product

$$\Delta = \prod_{i=1}^N \prod_{\nu < 0}^r \left( \sum_{a=1}^l m^a_i \nu - \nu h \right) \cup \prod_{i=1}^l \prod_{\nu \geq 0}^r \left( \sum_{a=1}^l t^a_i \nu - \nu h + \lambda \right).$$

This is here interpreted as an element in the inductive limit.
3.4. Localization map, pairing and the freeness of Floer cohomology.
A solution to the D-module $FH^{\ast}_T(L_{X/V})$, i.e., a D-module homomorphism from $FH^{\ast}_T(L_{X/V})$ to the trivial $D$-module $K\{h,h^{-1}\}[Q^1,\ldots,Q^r]$, is given by the localization of cohomology on the $S^1$-fixed set of the algebraic model $L_X$. For $\alpha \in H^{\ast}_T(L_{X/V})$ and $d \in H_2(X,\mathbb{Z})$, we define a localization at $X_d \cong X$ by

$$\text{Loc}_d(\alpha) := \frac{\alpha(d,f)}{\text{Euler}_\ast(N) \text{Euler}_T(V^\infty_d/V^\infty_f)} |_{X_d} \in H^\ast(X) \otimes \mathbb{C}[h,h^{-1}][\lambda^{-1}],$$

where $\alpha(d,f) \in H^{\ast}_T(L^\infty_d/V^\infty_f)$ is a representative of $\alpha$ such that $(d',f) \geq (d,d)$ and $N$ is the normal bundle of $L^\infty_d$ in $L^\infty_f$. Here, we expand the inverse Euler classes around $\lambda = \infty$ so that $\text{Loc}_d(\alpha)$ is defined over $\mathbb{C}[h,h^{-1}][\lambda^{-1}]$. It was shown in [17] Lemma 4.4 that on the submodule $H^{\ast}_T(L_{X/V})$, $\text{Loc}_d$ takes values in $H^\ast(X) \otimes \mathbb{C}[h,h^{-1}][\lambda]$ if $V$ is convex. Thus, in any case, $\text{Loc}_d(\alpha) \in K\{h,h^{-1}\}$ for $\alpha \in H^{\ast}_T(L_{X/V})$. Moreover, we can show that the support $\{d \in H_2(X,\mathbb{Z}) \mid \text{Loc}_d(\alpha) \neq 0\}$ is contained in a cone of the form $d_0 + \Lambda$, where $\Lambda$ is the Mori cone of $X$ and $d_0 \in H_2(X,\mathbb{Z})$ depends on $\alpha$. Therefore, we can define the map $\text{Loc}$ as

$$\text{Loc}: H^{\ast}_T(L_{X/V}) \to H^\ast(X) \otimes K\{h,h^{-1}\}[Q^1,\ldots,Q^r][(Q^1)^{-1},\ldots,(Q^r)^{-1}],$$

$$\alpha \mapsto \sum_{d \in H_2(X,\mathbb{Z})} \text{Loc}_d(\alpha)Q^d.$$ 

This was denoted by $\Xi$ in [17]. The following proposition was shown for convex $V$ in [17], but the proof there applies to a general $V$.

**Proposition 3.3 ([17] Proposition 4.5.4.6).** The map $\text{Loc}$ is a homomorphism of $H^{\ast}_T(\text{pt}) \otimes \mathbb{C}[(Q^1)^\pm,\ldots,(Q^r)^\pm]$-modules and injective. Moreover, it satisfies the differential equation:

$$\text{Loc}(P_\alpha) = (hQ^a \frac{\partial}{\partial Q^a} + p_a \cup) \text{Loc}(\alpha).$$

Here, $Q^a$ acts on the $H^{\ast}_T(L_{X/V})$ by the pull-back $(Q^a)^\ast$. It follows from the injectivity of $\text{Loc}$ that the filtration defined in (3.5) is Hausdorff. This $\text{Loc}$ induces an embedding of $K\{h\}[Q^1,\ldots,Q^r]$-modules:

$$\text{Loc}: FH^{\ast}_T(L_{X/V}) \to H^\ast(X) \otimes K\{h,h^{-1}\}[Q^1,\ldots,Q^r].$$

By the localization map, we can define the $I$-function [12] for $(X,V)$ as $I_{X,V}(Q,h) := \text{Loc}(\Delta)$. The $I$-function is given explicitly as

$$I_{X,V}(Q,h) := \sum_{d \in \Lambda} \prod_{i=1}^N \prod_{u_i = -\infty}^0 (u_i + \nu h) \prod_{\nu_i = -\infty}^\infty (u_i + \nu h) \prod_{j=1}^l \prod_{\nu_j = -\infty}^\infty (v_j + \nu h + \lambda) Q^d. \quad (3.6)$$

In Section 5 we will show that the $I$-function determines the big QDM of $(X,V)$ by a (generalized) mirror transformation.
We can also introduce a pairing on our Floer cohomology. We have an anti-$S^1$-equivariant automorphism $\overline{\cdot}: L_X \to L_X$ defined by $z_i(\zeta) \mapsto z_i(\zeta^{-1})$. This induces a map $\overline{\cdot}$:

$$\overline{\cdot}: H^*_F(L^\infty_d) \to H^*_F(L^{-\infty}_d),$$

$$f(P_1, \ldots, P_r, \lambda, \hbar) \mapsto f(P_1, \ldots, P_r, \lambda, -\hbar).$$

For $\alpha, \beta$ in $H^\infty_2(L_X/V)$, define $\int_{L_X/V} \overline{\alpha} \cup \beta \in \mathbb{C}[\hbar][((\lambda)^{-1})]$ by

$$\int_{L_X/V} \overline{\alpha} \cup \beta = \int_{L^d_2} \alpha(-d, -f')_{(d, f)} \beta_{(d, f)} \prod_{i=1}^{r} \prod_{i=\nu}^{f(i)} \text{Euler}_{T_2}(V_{i, \nu}) \prod_{i=\nu}^{f(i)} \text{Euler}_{T_2}(V_{i, \nu}).$$

Here, $\alpha(-d, -f')$ and $\beta_{(d, f)}$ are representatives of $\alpha$ and $\beta$ such that $L^\infty_d \subset L^\infty_{\infty}$. The Euler class in the denominator is expanded around $\lambda = \infty$. It is easy to check that this is independent of the choice of representatives. When $V$ is convex, the above pairing restricted to the submodule $H^\infty_2(L_X/V)$ is the same as what is given in [17], so takes values in $\mathbb{C}[[\lambda, \hbar]]$. For $\alpha, \beta$ in $H^\infty_2(L_X/V)$, we define another pairing $\langle \overline{\alpha}, \beta \rangle$ as

$$\langle \overline{\alpha}, \beta \rangle := \sum_{d \in H_2(X, \mathbb{Z})} Q^{-d} \int_{L_X/V} \overline{\alpha} \cup (Q^d)^{*}(\beta).$$

\textbf{Proposition 3.4} ([17] Proposition 4.8). The pairing $\langle \cdot, \cdot \rangle$ on $H^\infty_2(L_X/V)$ takes values in $K\{\hbar\}[Q^1, \ldots, Q^r][(Q^1)^{-1}, \ldots, (Q^r)^{-1}]$. We have

$$\langle \overline{\alpha}, \beta \rangle = \int_X \text{Loc}(\overline{\alpha}) \cup \text{Loc}(\beta) \cup \text{EulerS}_1(V),$$

where $\text{Loc}(\alpha) = \text{Loc}(\alpha)|_{\hbar \to -\hbar}$ is defined by flipping the sign of $\hbar$. The pairing $\langle \cdot, \cdot \rangle$ can be extended on $FH^*_F(L_X/V)$ via the right-hand side and takes values in $K\{\hbar\}[Q^1, \ldots, Q^r]$ on $FH^*_F(L_X/V)$.

\textbf{Remark.} In [17], the author introduced the Floer homology $FH_*$ and defined a pairing between $FH_*$ and $FH^*$. The pairing above is equivalent to that in [17] when we identify Floer homology with cohomology via the Poincaré duality map $\overline{\cdot}: FH^* \cong FH_*$ given in [17].

\textbf{Theorem 3.5} ([17] Theorem 4.10]). The $T^2$-equivariant Floer cohomology $FH^*_F(L_X/V)$ is a free module of rank $\dim H^*(X)$ over $K\{\hbar\}[Q^1, \ldots, Q^r]$. Moreover, the localization $\text{Loc}_0$ at $X_0 \cong X$ gives a canonical isomorphism:

$$\text{Loc}_0: FH^*_F(L_X/V) \big/ \sum_{a=1}^{r} Q^a FH^*_F(L_X/V) \cong H^*(X) \otimes K\{\hbar\}.$$

satisfying $\text{Loc}_0(P_0, \alpha) = p_a \text{Loc}_0(\alpha)$, $\text{Loc}_0(\Delta) = 1$. In particular, for a set of polynomials $T_1, \ldots, T_s \in \mathbb{C}[x_1, \ldots, x_r]$, if $\{T_i(p_1, \ldots, p_r)\}_{i=0}^{r}$ forms a basis of $H^*(X)$, then $\{T_i(P_1, \ldots, P_r)\}_{i=0}^{r}$ gives a free basis of $FH^*_F(L_X/V)$ over $K\{\hbar\}[Q^1, \ldots, Q^r]$. 

We can prove this theorem by using the localization map and the pairing defined above. The proof for convex $\mathcal{V}$ in [17] again applies to this general case. In Section 4.1, we will see that $FH_{\tau_2}^*(L_{X/\mathcal{V}})$ is the small AQDM from this theorem.

4. Reconstruction of Big AQDM

In this section, we formulate the big and small abstract quantum $D$-modules (AQDM) and prove a reconstruction theorem from the small AQDM to the big one. We start from their definitions and explain canonical frames, affine structures on the base and reconstruction theorem.

4.1. Definitions. Let $K$ be an integral domain with a valuation $v_K: K \to \mathbb{R} \cup \{\infty\}$. We assume $K$ is complete with respect to the topology defined by $v_K$. See Section 2.2 for notation used here. We assume that $K$ contains the field $\mathbb{Q}$ of rational numbers and $v_K|_\mathbb{Q} = 0$. Basic examples of $K$ include $\mathbb{Q}, \mathbb{C}, \mathbb{C}[\lambda], \mathbb{C}(\lambda^{-1}), \mathbb{C}[s]$, $H^1_t(pt)$ and $H^1_t(pt)((\lambda^{-1}))$, where the last two rings appear when we consider the $T$-equivariant quantum cohomology and its twist. We set the ideal $m_K = \{ x \in K \mid v_K(x) > 0 \}$ if $v_K(K) \subset \mathbb{R}_{\geq 0}$ and $m_K = \{0\}$ otherwise.

Define

$$O := K[Q^1, \ldots, Q^n], \quad O^h := K\{\hbar\}[Q^1, \ldots, Q^n].$$

Let $B$ be the formal neighborhood of zero in the affine space $A^{n+1}_K$ over $O$ endowed with the structure sheaf $O_B$ and the sheaf $O^h_B$ of algebra:

$$O_B := O[t^0, \ldots, t^n], \quad O^h_B := O^h[\hbar^0, \ldots, t^n].$$

The rings $O, O_B$ have a natural topology. In general, the topology on the formal power series ring $K[x_1, \ldots, x_n]$ over $K$ is given by the following fundamental neighborhood system of zero:

$$U_n := \left\{ \sum_{i_1, \ldots, i_n \geq 0} a_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n} \bigg| v_K(a_{i_1, \ldots, i_n}) \geq n \text{ if } i_1 + \cdots + i_n \leq n \right\}.$$

A set of elements $\hat{t}^0, \ldots, \hat{t}^s$ of $O_B$ is called a co-ordinate system on $B$ if $\hat{t}^i|_{t^i = 0}$ belongs to $m_K$ and if $O_B$ is topologically generated by $\hat{t}^s$ as an $O$-algebra, i.e. $O_B = O[\hat{t}^0, \ldots, \hat{t}^s]$. By an $O$-valued point on $B$, we mean a continuous $O$-algebra homomorphism $\phi: O_B \to O$. For example, any co-ordinate system $\hat{t}^0, \ldots, \hat{t}^s$ on $B$ defines an $O$-valued point $\hat{t}^0 = \cdots = \hat{t}^s = 0$. Conversely, any $O$-valued point $\hat{b}$ on $B$ is given by $\hat{t}^0 = \cdots = \hat{t}^s = 0$ for some co-ordinate system. In this case, $\hat{t}^0, \ldots, \hat{t}^s$ are called co-ordinates centered at $\hat{b}$. Let $D_B$ denote the Heisenberg algebra defined by

$$D_B := O_B^h(P_1, \ldots, P_r, \varphi_0, \ldots, \varphi_s)/I_B.$$

Here, $I_B$ is the two-sided ideal generated by

$$[P_a, f(Q, t, \hbar)] - \hbar Q^a \frac{\partial}{\partial Q^a} f(Q, t, \hbar), \quad [\varphi_i, f(Q, t, \hbar)] - \hbar \frac{\partial}{\partial t^i} f(Q, t, \hbar),$$

$$[P_a, P_b], \quad [\varphi_i, \varphi_j], \quad [P_a, \varphi_i].$$
where \( f(Q, t, h) \in \mathcal{O}_B^B \). When \( B \) is zero-dimensional over \( \mathcal{O} \), i.e. \( \mathcal{O}_B = \mathcal{O} \), \( \mathcal{D}_B \) coincides with the Heisenberg algebra \( \mathcal{D} \) in Section \ref{2.2}. The generators \( P_a, \varphi_i \) of \( \mathcal{D}_B \) depend on the choice of co-ordinates on \( B \). Another co-ordinate system \( \hat{t}^1, \ldots, \hat{t}^s \) on \( B \) yields the following generators \( \tilde{P}_a, \tilde{\varphi}_i \) of \( \mathcal{D}_B \):

\[
\tilde{P}_a = P_a + \sum_{j=0}^s Q^j \frac{\partial \hat{t}^j(Q, \hat{t})}{\partial \hat{t}^j} \varphi_j, \quad \tilde{\varphi}_i = \sum_{j=0}^s \frac{\partial \hat{t}^j(Q, \hat{t})}{\partial \hat{t}^i} \varphi_j. \tag{4.1}
\]

For a \( \mathcal{D}_B \)-module \( \mathcal{E}^h \) and an \( \mathcal{O} \)-valued point \( b \) on \( B \), we set

\[
\mathcal{E}^h_b := \mathcal{E}^h / \sum_{i=0}^s t^i \mathcal{E}^h, \quad \mathcal{E}^h_{b,0} := \mathcal{E}^h / \sum_{a=1}^r Q^a \mathcal{E}^h_b, \quad \mathcal{E}_{b,0} := \mathcal{E}_{b,0}^h / h\mathcal{E}_{b,0}^h,
\]

where \( t^0, \ldots, t^s \) are co-ordinates centered at \( b \). When \( B \) is zero-dimensional over \( \mathcal{O} \), we will use the notation \( \mathcal{E}^h_0 \), \( \mathcal{E}_0 \) instead of \( \mathcal{E}^h_{b,0}, \mathcal{E}^h_{b,0} \). Let \( p_a \) be the operator acting on \( \mathcal{E}^h_{b,0} \) and \( \varphi_i \) induced from \( P_a \) and \( \varphi_i \) be the operator acting on \( \mathcal{E}_{b,0} \) induced from \( \varphi_i \). Note that \( p_a \) does not depend on the choice of co-ordinates \( t^i \) on \( B \) whereas \( \varphi_i \) does. Then \( \mathcal{E}^h_b \) has the structure of a \( \mathcal{D} \)-module, \( \mathcal{E}^h_{b,0} \) becomes a \( K[H[p_1, \ldots, p_r]] \)-module and \( \mathcal{E}_{b,0} \) is a \( K[p_1, \ldots, p_r, \varphi_0, \ldots, \varphi_s] \)-module.

**Definition 4.1.** An abstract quantum \( D \)-module or AQDM on the base \( B \) is a \( \mathcal{D}_B \)-module \( \mathcal{E}^h \) satisfying the following conditions at an \( \mathcal{O} \)-valued point \( b \) on \( B \):

1. \( \mathcal{E}^h \) is a finitely generated free \( \mathcal{O}^B \)-module. In particular, \( \mathcal{E}^h_{b,0} \) (resp. \( \mathcal{E}_{b,0} \)) is a free \( K[H^0] \)-module (resp. free \( K \)-module) of finite rank.
2. There exists a splitting \( \Phi_0 : \mathcal{E}_{b,0} \to \mathcal{E}^h_{b,0} \) such that the induced map \( \Phi_0 : \mathcal{E}_{b,0} \otimes_K K[H^0] \to \mathcal{E}^h_{b,0} \) is an isomorphism of \( K[H^0] \)-modules.
3. Matrix elements of \( p_a^\alpha \in \text{End}_K(\mathcal{E}_{b,0}) \) (with respect to a certain \( K \)-basis) have valuations bounded from below uniformly in \( n \geq 0 \).
4. There exists an element \( e_0 \in \mathcal{E}_{b,0} \) such that \( \{ \varphi_i e_0 \}_{i=0}^s \) is a \( K \)-basis of \( \mathcal{E}_{b,0} \).

A small AQDM is an AQDM over the zero-dimensional base \( B \), i.e. \( \mathcal{O}_B = \mathcal{O} \). A big AQDM is an AQDM satisfying the additional condition

The condition (2) above is equivalent to that \( p_a \in \text{End}(\mathcal{E}^h_{b,0}) \) is represented by an \( h \)-independent matrix in a suitable \( K[H^0] \)-basis of \( \mathcal{E}^h_{b,0} \). The condition (3) is technically necessary; we will use this to construct a fundamental solution below. This is always satisfied when \( v_K(K) \subset \mathbb{R}_{\geq 0} \). The condition (4) implies that the rank of \( \mathcal{E}^h \) is same as the dimension \( s+1 \) of the base space \( B \) over \( \mathcal{O} \).

**Remark.** In \cite{17}, we assumed that \( \{ e_0, p_1 e_0, \ldots, p_r e_0 \} \) is part of a \( K \)-basis of \( \mathcal{E}_0 \) for some \( e_0 \in \mathcal{E}_0 \) for a small AQDM \( \mathcal{E}^h \). We will see this condition in Proposition \ref{1.8} below. We will see in Theorem \ref{1.6} below that the conditions (2), (3), (4) do not depend on the choice of an \( \mathcal{O} \)-valued point \( b \), i.e. if they hold at one \( b \), then they hold at any other point.

**Example 4.2.** (1) The small and big QDM for a pair \( (M, \mathcal{V}) \) in Section \ref{2} are the small and big AQDM respectively. The actions of \( P_a \) and \( \varphi_i \) are given respectively by the Dubrovin connections \( \nabla^h_a \) and \( \nabla^h_i \). We have a natural identification
A free basis (trivialization) of $E^h$ over $O_B^h$ is not a priori given. By the conditions (1) and (2), we can choose a splitting $\Phi: E_{b,0} \to E^h$ inducing an isomorphism of $O_B^h$-modules $\hat{\Phi} = \Phi^0$ and $\Phi^1$ of $E_{b,0} \otimes_K O_B^h \to E^h$ such that

$$\Phi: E_{b,0} \otimes_K K\{h\} \to E_{b,0}$$

is an isomorphism of $K\{p_1, \ldots, p_r\}$-modules.

We call such a splitting $\Phi$ (resp. $\Phi^0$) a frame of $E^h$ (resp. $E_{b,0}^h$). Two frames $\Phi$ and $\Phi^1$ of $E^h$ are related by a gauge transformation $G$ in $\text{Aut}_{O_B^h}(E_{b,0}^h \otimes_K O_B^h)$ as $\Phi \mapsto \hat{\Phi} := \Phi \circ G$. For a gauge transformation $G$, we need to assume that $G_{b,0} := G|_{q=t=0} \in \text{Aut}_{O_B^h}(E_{b,0}^h \otimes_K K\{h\})$ is a homomorphism of $K\{h\}[p_1, \ldots, p_r]$-modules and $G|_{q=t=h=0} = \text{id}_{E_{b,0}^h}$ so that $\hat{\Phi} = \Phi \circ G$ gives a new frame. Using a frame $\Phi$, we can write down the $D_B$-module structure of $E^h$ as follows. Define operators $\nabla_a^h, 1 \leq a \leq r$ and $\nabla_i^h, 0 \leq i \leq s$ acting on $E_{b,0}^h \otimes_K O_B^h$ by

$$\Phi(\nabla_a^h v) = P_a \Phi(v), \quad \Phi(\nabla_i^h v) = \varphi_i \Phi(v), \quad v \in E_{b,0}^h \otimes_K O_B^h.$$

These operators commute and take the form

$$\nabla_a^h = hQ^a \frac{\partial}{\partial Q^a} + A_a(Q, t, h), \quad \nabla_i^h = \hbar t_i \frac{\partial}{\partial t_i} + \Omega_i(Q, t, h)$$

where $A_a, \Omega_i \in \text{End}(E_{b,0}^h) \otimes_K O_B^h$. They correspond to the Dubrovin connection of QDMs and are considered to be a flat connection on the trivial bundle $E_{b,0}^h \times B \to B$. Note that by the condition (2) and our choice of $\Phi$, $p_a = A_a(0, 0, h)$ is independent of $\hbar$. Under a gauge transformation $G$, $A_a$ and $\Omega_i$ are transformed into

$$G^{-1}A_a G + hG^{-1}Q^a \frac{\partial}{\partial Q^a}, \quad G^{-1}\Omega_i G + hG^{-1} \frac{\partial G}{\partial t_i}. \quad (4.3)$$

Under the generator change (4.1) of $D_B$ induced from a co-ordinate change on $B$, $A_a$ and $\Omega_i$ are transformed into

$$\hat{A}_a = A_a + \sum_{j=0}^s Q^a \frac{\partial v_j(Q, \hbar)}{\partial Q^a} \hat{\Omega}_j, \quad \hat{\Omega}_i = \sum_{j=0}^s \frac{\partial v_j(Q, \hbar)}{\partial \hbar} \hat{\Omega}_j. \quad (4.4)$$

4.2. Canonical frames. A frame $\Phi$ of an AQDM $E^h$ is said to be canonical if the associated connection operators $A_a(Q, t, \hbar), 1 \leq a \leq r$ and $\Omega_i(Q, t, \hbar), 0 \leq i \leq s$ are independent of $\hbar$. In case of QDMs, $A_a$ and $\Omega_i$ are given as the quantum multiplications by $p_a$ and $p_i$, so they are by definition $\hbar$-independent.
Therefore the QDM is a priori endowed with a canonical frame. In general, if the connection operators $A_a$ and $\Omega_i$ are $\hbar$-independent, we have

$$[A_a, A_b] = [A_a, \Omega_i] = [\Omega_i, \Omega_j] = 0, \quad (4.5)$$
$$Q^b \partial A_a \over \partial Q^b = Q^b \partial A_a \over \partial Q^b, \quad \partial A_a \over \partial t^i = Q^a \partial \Omega_i \over \partial Q^a, \quad \partial \Omega_i \over \partial t^j = \partial \Omega_i \over \partial t^j. \quad (4.6)$$

These follow from the flatness: $[\nabla^h_a, \nabla^h_i] = [\nabla^h_a, \nabla^h_j] = [\nabla^h_i, \nabla^h_j] = 0$. In particular, $\mathcal{E}_{b,0} \otimes O_B$ has the structure of an $O_B[A_a, \Omega_i]$-module; if moreover it is generated by $e_0$ as an $O_B[A_a, \Omega_i]$-module (this holds when $\mathcal{E}^h$ is a big AQDM), $\mathcal{E}_{b,0} \otimes O_B$ has the structure of an $O_B[A_a, \Omega_i]$-algebra such that $e_0$ is the unit. This algebra is an analogue of the quantum cohomology.

Let $t^0, \ldots, t^r$ be a co-ordinate system centered at the base point $b$. Take a frame $\Phi$ of $\mathcal{E}^h$. A fundamental solution is an element $L = L(Q, t, \hbar)$ of $\text{End}(\mathcal{E}_{b,0}) \otimes K\{\hbar, \hbar^{-1}\}[Q, t]$ satisfying

$$\nabla^h_i L = \hbar \partial \over \partial t^i L + \Omega_i L = 0, \quad (4.7)$$
$$\nabla^h_a L - L p_a = \hbar Q^a \partial \over \partial Q^a L + A_a L - L p_a = 0, \quad (4.8)$$

where $0 \leq i \leq s, 1 \leq a \leq r$. Here, $p_a = A_a|_{Q=t=0} \in \text{End}(\mathcal{E}_{b,0})$. These equations correspond to (2.8), (2.9) in case of QDM. We normalize a fundamental solution $L$ by the condition:

$$L(0, 0, \hbar) = \text{id}_{\mathcal{E}_{b,0}} \quad (4.9)$$

**Proposition 4.3.** There exists a unique fundamental solution $L$ normalized by (4.9). This $L$ depends on the base point $b$ and the frame $\Phi$. If $\Phi$ is a canonical frame, then $L = \text{id}_{\mathcal{E}_{b,0}} + O(\hbar^{-1})$.

**Proof.** The corresponding statement in the small case is proven in Proposition 3.5 in [17]. For a function $f = f(Q^1, \ldots, Q^r, t^0, \ldots, t^r, \hbar)$, we write

$$f^{k,l} = f(Q^1, \ldots, Q^k, 0, \ldots, 0, t^0, \ldots, t^{l-1}, 0, \ldots, 0, \hbar).$$

Suppose by induction that we have a solution $L^{k,l}$ satisfying (4.7) with $0 \leq i \leq l - 1$ and (4.8) with $1 \leq a \leq k$ and

$$A^{k,l} L^k = L^k p_a \quad (4.10)$$

for $k + 1 \leq a \leq r$. When $k = l = 0$, these are satisfied for $L^{0,0} = \text{id}_{\mathcal{E}_{b,0}}$. We expand $L^{k+1,l}$ and $A^{k+1,l}$ as

$$L^{k+1,l} = L^{k,l} \sum_{n=0}^{\infty} T_n(Q^{k+1})^n, \quad A^{k+1,l} = \sum_{n=0}^{\infty} B_n(Q^{k+1})^n,$$

where $T_n$ and $B_n$ are $\text{End}(\mathcal{E}_{b,0})$-valued functions in $Q^1, \ldots, Q^k, t^0, \ldots, t^{l-1}, \hbar$ and $T_0 = \text{id}_{\mathcal{E}_{b,0}}$ and $B_0 = A^{k+1}_{k+1}$. From (4.3) with $a = k + 1$, we have

$$n \hbar L^{k,l} T_n + \sum_{i=0}^{n} B_{n-i} L^{k,l} T_i - L^{k,l} T_n p_{k+1} = 0.$$
Since \(B_0 L^{k,l} = A^{k,l}_{k+1} L^{k,l} = L^{k,l} p_{k+1}\) by (4.10), we have
\[
\left(1 + \frac{1}{n\hbar} \text{ad}(p_a)\right) T_n = \frac{1}{n\hbar} \sum_{i=0}^{n-1} (L^{k,l})^{-1} B_{n-i} L^{k,l} T_i.
\]
This determines \(T_n\) in terms of \(T_0, \ldots, T_{n-1}\) because the operator \(1 + \text{ad}(p_a)/(n\hbar)\) is invertible in \(\text{End}(\mathcal{E}_{b,0}) \otimes_K K\{\hbar, \hbar^{-1}\}\) by the condition (4). In fact,
\[
\left(1 + \frac{\text{ad}(p_a)}{n\hbar}\right)^{-1} = \sum_{m=0}^{\infty} \frac{1}{(n\hbar)^m} \sum_{i=0}^{m} (-1)^i \binom{m}{i} p_i B p_{m-i}, \quad B \in \text{End}(\mathcal{E}_{b,0}).
\]
Note that the right-hand side is well-defined in \(\text{End}(\mathcal{E}_{b,0}) \otimes_K K\{\hbar, \hbar^{-1}\}\) by the condition (4). Thus we obtain \(L^{k+1,l}\). Now we need to check that (4.7), \(0 \leq k \leq L\), holds for \(L^{k+1,l}\). We have for \(1 \leq a \leq k\)
\[
\nabla_{k+1}^h (\nabla_{a}^h L^{k+1,l} - L^{k+1,l} p_a) = \nabla_{a}^h \nabla_{k+1}^h L^{k+1,l} - L^{k+1,l} p_{k+1} p_a
\]
and \((\nabla_{a}^h L^{k+1,l} - L^{k+1,l} p_a)|_{Q^{k+1}=0} = \nabla_{a}^h L^{k,l} - L^{k,l} p_a = 0\). Thus \(T := \nabla_{a}^h L^{k+1,l} - L^{k+1,l} p_a\) satisfies \(\nabla_{k+1}^h T = T p_{k+1} = 0\) and \(T|_{Q^{k+1}=0} = 0\). This differential equation for \(T\) turns out to have a unique solution, thus we have \(T = 0\). We also need to check
\[
A^{k+1,l} L^{k+1,l} - L^{k+1,l} p_a = 0
\]
for \(k+2 \leq a \leq r\). But the left-hand side equals \(\nabla_{a}^h L^{k+1,l} - L^{k+1,l} p_a\) restricted to \(Q^{k+2} = \cdots = Q^{r} = t^l = \cdots = t^{r} = 0\), thus this follows from the same argument.

In the same way, we can also check that (4.7), \(0 \leq i \leq l - 1\) hold for \(L^{k,l+1}\).

Next we solve for \(L^{k,l+1}\) under the same induction hypothesis. We expand \(L^{k,l+1}\) and \(\Omega^{k,l+1}_l\) as
\[
L^{k,l+1} = \sum_{n=0}^{\infty} U_n (t^l)^n, \quad \Omega^{k,l+1}_l = \sum_{n=0}^{\infty} C_n (t^l)^n,
\]
where \(U_n\) and \(C_n\) are \(\text{End}(\mathcal{E}_{b,0})\)-valued functions in \(Q^1, \ldots, Q^k, t^0, \ldots, t^{l-1}\) and \(U_0 = L^{k,l}\) and \(C_0 = \Omega^{k,l}_l\). By (4.7) with \(i = l\), we have
\[
(n+1)\hbar U_{n+1} + \sum_{i=0}^{n} C_{n-i} U_i = 0.
\]
Therefore, \(U_{n+1}\) is recursively determined by \(U_0, \ldots, U_n\) and we obtain \(L^{k,l+1}\). By a routine argument, we can check that (4.7), \(0 \leq i \leq l\) and (4.8), \(1 \leq a \leq k\) hold for \(L^{k,l+1}\) and that \(A^{k+1,l} L^{k,l+1} = L^{k,l+1} p_a\) holds for \(k+1 \leq a \leq r\). This completes the induction step. The last statement follows from the above procedure. \(\square\)

The fundamental solution \(L\) above depends on a choice of frame. For another frame \(\Phi = \Phi \circ G\), the corresponding fundamental solution \(\hat{L}\) is given by
\[
\hat{L} = G^{-1} \circ L \circ G_{b,0}
\]
where \(G_{b,0} = G|_{Q=t=0}\). This easily follows from that \(\hat{L}|_{Q=t=0} = \text{id}_{\mathcal{E}_{b,0}}\) and that \(G_{b,0}\) commutes with the action of \(p_a\).
Definition 4.4. Take a frame $\Phi$ and the fundamental solution $L$ associated to $\Phi$. Also choose an element $e_0 \in \mathcal{E}_{b,0}$. We assume $e_0$ satisfies the condition (4) in case of the big AQDM. The $J$-function of the AQDM $\mathcal{E}^h$ is defined to be $J(Q, t, h) := L^{-1}e_0 \in \mathcal{E}_{b,0} \otimes_K K \{h, h^{-1}\}\{Q, t\}$. This depends on the choice of $b$, $\Phi$ and $e_0$.

In case of QDM, this definition coincides with the original $J$-function (Definition 2.4). In equivariant Floer theory, we choose a frame $\Phi$ of $FH^*_T(L_X/\nu)$ such that $\Phi_0$ coincides with $\text{Loc}_0^{-1}$. Then by Proposition 3.3, the inverse $L^{-1}$ of the corresponding fundamental solution is given by

$$L^{-1}: H^*(X) \otimes K\{h\}[Q] \xrightarrow{\Phi} FH^*_T(L_X/\nu) \xrightarrow{\text{Loc}} H^*(X) \otimes K\{h, h^{-1}\}[Q].$$

We take $e_0$ to be the image of $\Delta$. and choose $\Phi$ so that it satisfies $\Phi(e_0) = \Delta$. Then the $J$-function of $FH^*_T(L_X/\nu)$ as an AQDM is given by the $I$-function because $I_{X,\nu} = \text{Loc}(\Delta) = L^{-1}(e_0)$.

Because the $J$-function is the inverse image of $e_0$ under $L$, we have a relation in $\mathcal{E}^h$

$$f(Q, t, P_a, \nu_i, h) \Phi(e_0) = 0, \quad f \in \mathcal{D}_B,$$

if and only if the $J$-function satisfies the differential equation

$$f(Q, t, hQ^a \frac{\partial}{\partial Q^a} + p_a, h\frac{\partial}{\partial h}, h)J(Q, t, h) = 0.$$

Because the big AQDM is generated by $e_0$ as a $\mathcal{D}_B$-module by the condition (4), it is reconstructed by the $J$-function as a $\mathcal{D}_B$-module. When we view $\mathcal{E}^h$ as a module over $\mathcal{D}'_B := \mathcal{O}^h_B(P_1, \ldots, P_r) \subset \mathcal{D}_B$, we have the following analogue of Proposition 2.6

Proposition 4.5. Let $\mathcal{E}^h$ be an AQDM and $\Phi$ be a frame of $\mathcal{E}^h$. Let $J(Q, t, h)$ be the $J$-function associated with $\Phi$ and $e_0 \in \mathcal{E}_{b,0}$. Assume that $\mathcal{E}_{b,0}$ is generated by $e_0$ as a $K\{p_1, \ldots, p_r\}$-module. Then $\mathcal{E}^h$ is generated by $\Phi(e_0)$ as a $\mathcal{D}_B := \mathcal{O}^h_B(P_1, \ldots, P_r)$-module. In other words, $\mathcal{E}^h$ is generated by $J(Q, t, h)$ as a $\mathcal{D}_B$-module, i.e. we have an isomorphism of $\mathcal{D}_B$-module $\mathcal{D}_B/\mathcal{I} \cong \mathcal{E}^h$ which sends 1 to $\Phi(e_0)$, where $\mathcal{I}$ is the left ideal of $\mathcal{D}_B$ consisting of elements $f(Q, t, P, h) \in \mathcal{D}_B$ satisfying

$$f(Q, t, hQ^a \frac{\partial}{\partial Q^a} + p_a, h)J(Q, t, h) = 0.$$

In particular, the equivariant Floer cohomology in Section 2 is generated by the $I$-function $\{3.6\}$ as a $\mathcal{D}$-module.

Proof. This was stated in Theorem 3.17 in [17] for small AQDMs. We omit the proof since it is elementary and is similar to that of Theorem 2.4 in [17].

Theorem 4.6. For a given frame $\Phi_0$ of $\mathcal{E}^h_{b,0}$ satisfying (4.2), there exists a unique canonical frame $\Phi_{\text{can}}$ which induces $\Phi_0$. The existence of a canonical frame implies that the conditions (2), (3) (and (4) in case of big AQDM) in Definition 4.7 hold at any $\mathcal{O}$-valued point $b'$. 

Proof. The proof here is similar to that in [17, Theorem 3.9]. We owe the idea of the Birkhoff factorization here to Guest [15]. This method also appeared in [4]. We begin with an arbitrary frame $\Phi$ which induces $\Phi_0$. Let $L$ be the fundamental solution associated with $\Phi$. We claim that there exist $L_+ \in \text{End}(E_{b,0}) \otimes K \{h\} [Q, t]$ and $L_- \in \text{End}(E_{b,0}) \otimes K \{h^{-1}\} [Q, t]$ such that

$$L = L_+ L_-, \quad L_-|_{h = \infty} = \text{id}_{E_{b,0}}, \quad L_+|_{Q = t = 0} = \text{id}_{E_{b,0}}. \quad (4.12)$$

By expanding $L$, $L_+$ and $L_-$ in power series in $Q$ and $t$, we can solve for each coefficient of $L_+$ and $L_-$ recursively. We refer the reader to [17, Theorem 3.9] for details. In Equation (4.13) below, we will also give a formula of $L_+$ in terms of $L^{-1}$. By substituting $L_+ L_-$ in the differential equations (4.7), (4.8), we have

$$L_+^{-1} \frac{\partial L_+}{\partial t} + L_+^{-1} \Omega_i L_+ = -h \frac{\partial L_-}{\partial t} L_-^{-1},$$

$$L_-^{-1} \partial L_+ \partial Q^a + L_+^{-1} A_a L_+ = -h Q^a \partial L_- \partial Q^a L_-^{-1} + L_- p_a L_-^{-1}.$$

The left-hand sides belong to $\text{End}(E_{b,0}) \otimes K \{h\} [Q, t]$ and the right-hand sides belong to $\text{End}(E_{b,0}) \otimes K \{h^{-1}\} [Q, t]$. Therefore, the both hand sides are $h$-independent. Hence, the gauge transformation by $G := L_+$ makes the connection operators $h$-independent and $\Phi_0 = \Phi \circ G$ is a canonical frame. Note that $G$ does not change $\Phi$ since $G|_{Q = t = 0} = \text{id}_{E_{b,0}}$.

Next we show the uniqueness of a canonical frame. Let $\Phi$ and $\Phi'$ be two canonical frames inducing the same frame $\Phi_0$ of $E_{b,0}$. Then there exists a gauge transformation $G$ such that $\Phi' = \Phi \circ G$ and $G|_{Q = t = 0} = \text{id}_{E_{b,0}}$. Let $L$ and $L'$ be the fundamental solutions associated to $\Phi$ and $\Phi'$ respectively. Then by (4.11), we have $G = L L'^{-1}$. Because $\Phi$ and $\Phi'$ are canonical, we have $L L'^{-1} = \text{id}_{E_{b,0}} + O(h^{-1})$ by Proposition 4.3. On the other hand, since $G$ does not contain negative powers of $h$, we have $G = L L'^{-1}|_{h = \infty} = \text{id}_{E_{b,0}}$, so $\Phi = \Phi'$.

Finally, we see that the conditions (2), (3), (4) hold at any $O$-valued point $b'$. Let $A_a(0, Q, t)$ and $\Omega_i(0, Q, t)$ be the connection operators associated to a canonical frame. The frame $\Phi$ induces an identification $E_{b,0} \otimes K \{h\} \cong E_{b',0}$. Under this isomorphism, $p_a$ action on $E_{b',0}$ corresponds to an $h$-independent operator $A_a(0, t(b')|_{Q = 0}) \in \text{End}(E_{b,0}) \otimes K \{h\}$. Thus the condition (2) holds at $b'$. The condition (4) implies that $\{\Omega_i(0, t)|_{Q = 0}\}_{i=0}^a$ is an $O_B$-basis of $E_{b,0} \otimes O_B$. By specialization, we know (4) holds also at $b$.

Remark. The positive part $L_+$ of the Birkhoff factorization, which serves as a gauge transformation to a canonical frame, may be calculated by the following formula:

$$L_+ v = \sum_{k=0}^{\infty} (\text{id} - \pi_+ \circ L^{-1})^k v \quad \text{for } v \in E_{b,0}. \quad (4.13)$$
Here, $\pi_+$ is the map $\pi_+: E_{b,0} \otimes K \{ h, h^{-1} \}[[ Q, t ]] \to E_{b,0} \otimes K \{ h \}[[ Q, t ]]$ induced from the projection $K \{ h, h^{-1} \} = K \{ h \} \oplus h^{-1} K \{ h^{-1} \} \to K \{ h \}$. To show this, first note that the right-hand side converges in $( Q, t )$-adic topology since $\pi_+ \circ L^{-1} = id + O( Q, t )$ on $E_{b,0} \otimes K \{ h \}$. From the Birkhoff factorization, we have $v = \pi_+ ( L^{-1} v ) = \pi_+ L^{-1} ( L_+ v )$ for $v \in E_{b,0}$. Therefore $\sum_{k=0}^{\infty} (id - \pi_+ L^{-1})^k v = \sum_{k=0}^{\infty} (id - \pi_+ L^{-1})^k \pi_+ L^{-1} ( L_+ v ) = \sum_{k=0}^{\infty} (id - \pi_+ L^{-1})^k - (id - \pi_+ L^{-1})^{k+1} ) ( L_+ v ) = L_+ v$. Note also that the right-hand side of (4.13) is not $K \{ h \}$-linear but only $K$-linear in $v$.

4.3. Affine structures. We describe an affine structure on the base space of the big AQDM. The base space of the big QDM has a natural linear structure and the covariant derivative along a linear vector field $\partial / \partial t$ gives

$$\nabla^h_1 = p_i * 1 = p_i \cup 1.$$ 

where $\nabla^h_i$ is the Dubrovin connection in Section 2. This motivates the following definition:

**Theorem-Definition 4.7.** Let $E^h$ be a big AQDM endowed with a canonical frame $\Phi$. Let $e_0$ be an element of $E_{b,0}$ satisfying the condition (4) in Definition 4.1. There exists a co-ordinate system $t^0, \ldots, t^s$ on $B$ such that

$$\Phi( \hat{\varphi}_i e_0 ) = \hat{\varphi}_i \Phi(e_0), \quad \Phi( p_a e_0 ) = \hat{P}_a \Phi(e_0)$$

(4.14)

for $0 \leq i \leq s$ and $1 \leq a \leq r$. Here, $\hat{P}_a$ and $\hat{\varphi}_i$ are generators of $\mathcal{D}_B$ corresponding to the co-ordinates $t^i$ and $\hat{\varphi}_i \in \text{End}(E_{b,0})$ is induced from $\hat{\varphi}_i$. The co-ordinate system having this property is unique up to affine transformations over $K$, thus defines an affine structure on $B$ over $K$. This affine structure depends on the choice of $\Phi$ and $e_0$. We call such $t^0, \ldots, t^s$ a flat co-ordinate system associated with $\Phi$ and $e_0$.

(ii) The $J$-function associated with the canonical frame $\Phi$ and $e_0$ has the $h^{-1}$-expansion of the form

$$J = e_0 + \frac{1}{h} \sum_{\ell=0}^{s} \ell^i ( \hat{\varphi}_i e_0 ) + O( h^{-2} ).$$

(4.15)

**Proof.** Let $t^0, \ldots, t^s$ be an arbitrary co-ordinate system centered at $b$. Let $P_a$, $\varphi_i$ be the corresponding generators of $\mathcal{D}_B$. Let $A_a(Q, t)$ and $\Omega_i(Q, t)$ be the connection operators associated with the canonical frame $\Phi$. By Proposition 4.3 and $J = L^{-1} e_0$, the $J$-function associated with the canonical frame is of the form $J = e_0 + O( h^{-1} )$. Therefore, we can define an element $\bar{t}^i \in O_B$ by the expansion (note that $\{ \varphi_i e_0 \}$ is a $K$-basis of $E_{b,0}$):

$$J = e_0 + \frac{1}{h} \sum_{\ell=0}^{s} \ell^i ( \varphi_i e_0 ) + O( h^{-2} ).$$
First we check that $\hat{t}^0, \ldots, \hat{t}^s$ form a co-ordinate system on $B$. We have
\[
\hbar \frac{\partial}{\partial \hat{t}^i} J = L^{-1}(\Omega_i e_0) = \Omega_j e_0 + O(\hbar^{-1}),
\]
\[
(\hbar Q^a \frac{\partial}{\partial Q^a} + p_a) J = L^{-1}(A_a e_0) = A_a e_0 + O(\hbar^{-1}).
\]
Hence, by comparing the leading terms in the $\hbar^{-1}$-expansions,
\[
\sum_{i=0}^s \frac{\partial \hat{t}^i}{\partial \hat{t}^j}(\varsigma_i e_0) = \Omega_j e_0 = \varsigma_j e_0 + O(Q, t), \quad p_a e_0 + \sum_{i=0}^s Q^a \frac{\partial \hat{t}^i}{\partial Q^a}(\varsigma_i e_0) = A_a e_0.
\]
From the first equation, the Jacobi matrix $(\partial \hat{t}^i / \partial \hat{t}^j) = \delta^i_j + O(Q, t)$ turns out to be invertible. Therefore $\hat{t}^0, \ldots, \hat{t}^s$ gives a co-ordinate system. On the other hand, it follows from the above formulas that
\[
\varsigma_i e_0 = \sum_{j=0}^s \frac{\partial \hat{t}^j(Q, \hat{t})}{\partial \hat{t}^i} \Omega_j e_0, \quad p_a e_0 = A_a e_0 - \sum_{a=1}^r Q^a \frac{\partial \hat{t}^i(Q, t)}{\partial Q^a}(\varsigma_i e_0).
\]
By the chain rule
\[
0 = \sum_{k=0}^s \frac{\partial \hat{t}^i(Q, t)}{\partial t^k} Q^a \frac{\partial t^k(Q, \hat{t})}{\partial Q^a} + Q^a \frac{\partial \hat{t}^i(Q, t)}{\partial Q^a}
\]
and (4.16), we find
\[
p_a e_0 = A_a e_0 + \sum_{a=1}^r \sum_{k=0}^s \frac{\partial \hat{t}^i(Q, t)}{\partial t^k} Q^a \frac{\partial t^k(Q, \hat{t})}{\partial Q^a}(\varsigma_i e_0) = (A_a + \sum_{a=1}^r Q^a \frac{\partial \hat{t}^i(Q, \hat{t})}{\partial Q^a} \Omega_j) e_0.
\]
This and the first equation of (4.16) show that
\[
\varsigma_i e_0 = \Omega_i e_0, \quad p_a e_0 = \hat{A}_a e_0
\]
where $\Omega_i$ and $\hat{A}_a$ are connection operators corresponding to the new co-ordinates $\hat{t}^i$ (see (4.4)). Because $\varsigma_i = \hat{\varsigma}_i$ in this case, this is equivalent to (4.14). Thus we proved the existence of flat co-ordinates and (ii).

Let $t^{l'}, \ldots, t^{r'}$ be another flat co-ordinate system. Let $\hat{\varsigma}_i, P^a_{l'} \in D_B$ and $\hat{\varsigma}_i' \in \text{End}(\mathcal{E}_{b,0})$ be the corresponding operators. Since two bases $\{\hat{\varsigma}_i e_0\}_{i=0}^s$ and $\{\hat{\varsigma}_i' e_0\}_{i=0}^s$ of $\mathcal{E}_{b,0}$ are related by an element in $GL(s + 1, K)$, it follows from (4.4) and (4.14) that the matrix $\partial t^{l'}(Q, \hat{t}) / \partial \hat{t}^j$ is in $GL(s + 1, K)$. Similarly, we know that $Q^a(\partial t^{l'}(Q, \hat{t}) / \partial Q^a) = 0$. Therefore, $t^{l'} = c^j + \sum_{j=0}^s b^j_{l'} \hat{t}^j$ for some $c^j, b^j_{l'} \in K$. \hfill \Box

In case of the big QDM, linear co-ordinates on cohomology give a flat co-ordinate system in the above sense. We also have an analogue of the string and divisor equations in the context of big AQDM.

**Proposition 4.8.** Let $\mathcal{E}^h$ be a big AQDM. Let $\Phi$ be a canonical frame of $\mathcal{E}^h$ and $e_0$ be an element of $\mathcal{E}_{b,0}$ satisfying the condition (4) in Definition 4.1. Assume
that \(\{e_0,p_1 e_0, \ldots, p_r e_0\}\) is part of a \(K\)-basis of \(E_{b,0}\). There exists a flat co-ordinate system \(t^0, \ldots, t^s\) on \(B\) associated with \(\Phi\) and \(e_0\) such that
\[
\varphi_0 e_0 = e_0, \quad \varphi_a e_0 = p_a e_0, \quad 1 \leq a \leq r. \tag{4.17}
\]
Then the fundamental solution \(L\) associated with \(\Phi\) satisfies
\[
\left(\frac{\partial}{\partial t^a} - Q^a \frac{\partial}{\partial Q^a}\right)L + L \circ \frac{p_a}{\hbar} = 0, \quad \hbar \frac{\partial}{\partial t^0} L = L
\]
and the corresponding connection operators \(A_a\) and \(\Omega_i\) are independent of \(t^0\) and satisfy
\[
A_a = \Omega_a, \quad \left(\frac{\partial}{\partial t^a} - Q^a \frac{\partial}{\partial Q^a}\right)\Omega_i = 0, \quad \Omega_0 = \text{id},
\]
where \(1 \leq a \leq r\) and \(0 \leq i \leq s\).

\textbf{Proof.} By assumption, we can make flat co-ordinates satisfy (4.17) by a \(K\)-linear co-ordinate change. Then by the differential equations for \(L\), we have
\[
\left(\hbar \frac{\partial}{\partial t^a} - (\hbar Q^a \frac{\partial}{\partial Q^a} + p_a)\right) L^{-1} \varphi_j e_0 = L^{-1}(\Omega^a_{\varphi_j} - \Delta^a_{\varphi_j}) \varphi_j e_0 = 0.
\]
where we used \(\varphi_j e_0 = \Omega_j e_0\) in the second line and used \((4.15)\) and \(A_a e_0 = p_a e_0 = \varphi_a e_0 = \Omega_a e_0\) in the third line. Thus we have \((\hbar \frac{\partial}{\partial t^0} - (\hbar Q^a \frac{\partial}{\partial Q^a} + p_a)) L^{-1} = 0\). Similarly,
\[
\hbar \frac{\partial}{\partial t^0} L^{-1} \varphi_j e_0 = L^{-1} \Omega_0 \varphi_j e_0 = L^{-1} \Omega_0 \Omega_j e_0 = L^{-1} \Omega_j \varphi_0 e_0 = L^{-1} \Omega_j e_0 = L^{-1} \varphi_j e_0.
\]
Thus \(\hbar \frac{\partial}{\partial t^0} L^{-1} = L^{-1}\). The differential equations for \(L\) follow from these. The statement on the connection operators easily follows from the differential equations for \(L\). The details are left to the reader. \qed

\textbf{Remark.} This proposition shows that the connection \(A_a\), \(\Omega_i\) are functions in \(Q^1 e^t, \ldots, Q^r e^t, \ell^{r+1}, \ldots, \ell^s\) in suitable flat co-ordinates. The variable \(q^a\) used in the previous paper [17] corresponds to the combination \(q^a = Q^a e^t\). There, the small AQDM was a module over the Heisenberg algebra generated by \(p_a\), \(q^b\) with commutation relation \([p_a, q^b] = \delta^c_a \hbar q^c\). In [17], we allow a co-ordinate change in \(q\)-variables to solve for flat co-ordinates \(\hat{q}^a\). This method works for a small AQDM satisfying a certain “nef” assumption (Assumption 3.15 in [17]) but does not work beyond nef case. In this paper, introducing redundant co-ordinates \(Q^a\) and \(t^i\), we solve for flat co-ordinates \(\hat{t}^i\) for the big AQDM (which will be reconstructed from the small one in the next section). Note that we keep a co-ordinate \(Q^a\) fixed and allow a co-ordinate change only in \(t\)-variables.
4.4. **Reconstruction.** For an AQDM $\mathcal{E}^h$ over $B$ and an $\mathcal{O}$-valued point $b$ on $B$, a fiber $\mathcal{E}_b^h$ at $b$ becomes a small AQDM. Here we consider the reconstruction of a big AQDM from its fiber. The reconstruction theorem below is considered as a generalization of Kontsevich and Manin’s reconstruction theorem (Theorem 4.9).

Let $\mathcal{E}^h$ be an AQDM over $B$. Let $t^0, \ldots, t^s$ be a co-ordinate system on $B$. Let $B'$ be the formal neighborhood of zero in the affine space $K_{t^0+1}$ over $\mathcal{O}$ endowed with the structure sheaf $\mathcal{O}_{B'} = \mathcal{O}[e^0, \ldots, e^s]$ and the sheaf $\mathcal{O}^h_{B'} = \mathcal{O}^h[e^0, \ldots, e^s]$ of algebras. By a map $f : B' \to B$, we mean a continuous $\mathcal{O}$-algebra homomorphism $f^* : \mathcal{O}_B \to \mathcal{O}_{B'}$. More concretely, this is given by $s + 1$ elements $t^0(Q, \epsilon), \ldots, t^s(Q, \epsilon)$ in $\mathcal{O}_{B'}$ such that $t^i(0, 0) \in m_K$, where $t^i(Q, \epsilon) = f^* t^i$. We define a pulled back $\mathcal{D}_{B'}$-module $f^* \mathcal{E}^h$ as

$$f^* \mathcal{E}^h = \mathcal{O}^h_{B'} \otimes_{\mathcal{O}^h_B} \mathcal{E}^h$$

where the actions of $P^a'_\alpha, \phi^i'_\alpha \in \mathcal{D}_{B'}$ (corresponding to co-ordinates $\epsilon^i$) are given by

$$P^a'_\alpha = h Q^a \frac{\partial}{\partial Q^a} \otimes 1 + 1 \otimes (P^a + \sum_{j=0}^s Q^a \frac{\partial^j (Q, \epsilon)}{\partial Q^a} \phi_j),$$

$$\phi^i'_\alpha = h \frac{\partial}{\partial \epsilon^i} \otimes 1 + 1 \otimes \sum_{j=0}^s \frac{\partial^j (Q, \epsilon)}{\partial \epsilon^i} \phi_j.$$

It is easy to check that $f^* \mathcal{E}^h$ becomes an AQDM over $B'$.

**Theorem 4.9.** Let $\mathcal{E}^h$ be a small AQDM. Assume that $\mathcal{E}_0$ is generated by a single element $e_0$ as a $K[p_0, \ldots, p_s]$-module. Then there exist a big AQDM $\hat{\mathcal{E}}^h$ over a base $B$, an $\mathcal{O}$-valued point $b$ on $B$ and an isomorphism $\varphi : \mathcal{E}^h \cong \hat{\mathcal{E}}^h_b$ of $\mathcal{D}$-modules satisfying the following universal property: For any AQDM $\mathcal{F}^h$ over $B'$ with an isomorphism $\varphi' : \mathcal{E}^h \cong \mathcal{F}^h_{b'}$ of $\mathcal{D}$-modules at $b'$ on $B'$, there exist a unique map $f : B' \to B$ and a unique isomorphism $\theta : \mathcal{F}^h \cong f^* \mathcal{E}^h$ of $\mathcal{D}$-modules such that $f(b') = b$ and $\theta_{b'} \circ \varphi' = \varphi$. Here, $\theta : \mathcal{F}^h_{b'} \cong f^* \mathcal{E}^h_b$ is an isomorphism of $\mathcal{D}$-modules induced from $\theta$. In particular, $\hat{\mathcal{E}}^h$ is unique up to isomorphisms.

**Proof. Construction of $\hat{\mathcal{E}}^h$:** Take a canonical frame $\Phi$ of $\mathcal{E}^h$. Let $A_a(Q)$, $1 \leq a \leq r$ be the associated ($h$-independent) connection operators. Let $\{e_0, \ldots, e_s\}$ be a basis of $\mathcal{E}_0$ over $K$. Let $B$ be a space with co-ordinates $t^0, \ldots, t^s$ and $b$ be the point defined by $t^0 = \cdots = t^s = 0$. We set $\mathcal{O}_B = \mathcal{O}[t^0, \ldots, t^s]$ and $\mathcal{O}_B^h = \mathcal{O}^h[t^0, \ldots, t^s]$ as usual. We claim that there exist unique flat connections $\nabla^h_a$, $1 \leq a \leq r$ and $\nabla^h_i$, $0 \leq i \leq s$ acting on $\mathcal{E}_0 \otimes_K \mathcal{O}^h_B$ of the form:

$$\nabla^h_a = h Q^a \frac{\partial}{\partial Q^a} + A_a(Q, t), \quad \nabla^h_i = h \frac{\partial}{\partial e^i} + \Omega_i(Q, t)$$

such that $A_a(Q, 0) = A_a(Q)$ and $\Omega_i(Q, t)e_0 = e_i$. Then $\hat{\mathcal{E}}^h := \mathcal{E}_0 \otimes_K \mathcal{O}_B^h$ has the structure of a big AQDM endowed with a canonical frame such that $\hat{\mathcal{E}}^h_b = \mathcal{E}_0 \otimes \mathcal{O}^h_B$. 


is isomorphic to $\mathcal{E}^h$ via $\Phi$ (i.e. $\varphi = \Phi^{-1}$). We expand $A_{a}(Q, t)$ and $\Omega_{i}(Q, t)$ as

$$A_{a}(Q, t) = \sum_{n \geq 0} A_{a}^{(n)}(Q, t), \quad \Omega_{i}(Q, t) = \sum_{n \geq 0} \Omega_{i}^{(n)}(Q, t),$$

where $A_{a}^{(n)}(Q, t)$ (resp. $\Omega_{i}^{(n)}(Q, t)$) is the degree $n$ part of $A_{a}(Q, t)$ (resp. $\Omega_{i}(q, t)$) with respect to the variables $t^{0}, \ldots, t^{n}$. We also write $A_{a}^{\leq n} := \sum_{k=0}^{n} A_{a}^{(k)}$ and $\Omega_{i}^{\leq n} = \sum_{k=0}^{n} \Omega_{i}^{(k)}$. Then we must have

$$\Omega_{i}^{(n)}(Q, t) e_{0} = \delta_{n, 0} e_{i} \quad (4.18)$$

The flatness of the connection implies (see (4.15), (4.16))

$$\sum_{k+l=m} [A_{a}^{(k)}, A_{b}^{(l)}] = 0, \quad \sum_{k+l=m} [A_{a}^{(k)}, \Omega_{j}^{(l)}] = 0, \quad \sum_{k+l=m} [A_{a}^{(k)}, \Omega_{j}^{(l)}] = 0, \quad (4.19)$$

$$\partial_{b}A_{a}^{(n)} = \partial_{a}A_{b}^{(n)}, \quad \partial_{j}\Omega_{i}^{(n)} = \partial_{i}\Omega_{j}^{(n)}, \quad \partial_{j}A_{a}^{(n)} = \partial_{a}\Omega_{j}^{(n-1)}, \quad (4.20)$$

where $\partial_{a} = Q^{a}\partial / \partial Q^{a}$ and $\partial_{i} = \partial / \partial t^{i}$. Since $\mathcal{E}_{0}$ is generated by $e_{0}$ as a $K[p_{1}, \ldots, p_{r}]$-module and $p_{a} = A_{a}^{(0)}|_{Q=0}$, we know that $\mathcal{E}_{0} \otimes_{K} \mathcal{O}$ is generated by $e_{0}$ as an $\mathcal{O}[A_{a}^{(0)}, \ldots, A_{a}^{(0)}]$-module. Thus, $\mathcal{E}_{0} \otimes_{K} \mathcal{O}$ has a unique $\mathcal{O}[A_{a}^{(0)}, \ldots, A_{a}^{(0)}]$-algebra structure such that $e_{0}$ is a unit. Since $\Omega_{i}^{(0)}$ commutes with $A_{a}^{(0)}$ and $\Omega_{i}^{(0)} e_{0} = e_{i}$, $\Omega_{i}^{(0)}$ is identified with the multiplication by $e_{i}$ in this algebra $\mathcal{E}_{0} \otimes_{K} \mathcal{O}$.

Suppose by induction that we have $A_{a}^{(k)}$ for $0 \leq k \leq m - 1$ and $\Omega_{i}^{(k)}$ for $0 \leq k \leq m - 1$ satisfying all the conditions (4.18), (4.19), (4.20) up to $n = m - 1$. First, we can solve for $A_{a}^{(m)}$ uniquely using the third equation of (4.20) for $n = m$. This is possible since the integrability condition is satisfied: $\partial_{i}(\partial_{a}O_{j}^{m-1}) = \partial_{a}\partial_{j}O_{i}^{m-1} = \partial_{a}\partial_{j}O_{m-1}^{i} = \partial_{j}(\partial_{a}O_{i}^{m-1})$. We need to check the first equation of (4.20) with $n = m$. This follows from $\partial_{i}(\partial_{a}A_{m}^{(m)} - \partial_{a}A_{b}^{(m)}) = \partial_{b}\partial_{a}O_{i}^{m-1} = \partial_{a}\partial_{j}O_{i}^{m-1} = 0$. Also we need to check the first equation of (4.19) with $n = m$. This follows from

$$\partial_{i} \sum_{k+l=m} [A_{a}^{(k)}, A_{b}^{(l)}] = \sum_{k+l=m} [\partial_{a}O_{i}^{(k)}, A_{b}^{(l)}] + \sum_{k+l=m} [A_{a}^{(k)}, \partial_{a}O_{i}^{(l)}] = \sum_{k+l=m-1} \left\{ \partial_{a}[O_{i}^{(k)}, A_{b}^{(l)}] - [O_{i}^{(k)}, \partial_{a}A_{b}^{(l)}] + \partial_{b}[A_{a}^{(k)}, O_{i}^{(l)}] - [\partial_{b}A_{a}^{(k)}, O_{i}^{(l)}] \right\} = \sum_{k+l=m-1} [O_{i}^{(k)}, \partial_{b}A_{a}^{(l)}] - \partial_{a}A_{b}^{(l)} = 0. \quad (4.21)$$

Secondly, we solve for $\Omega_{j}^{(m)}$. Let $m \subset \mathcal{O}_{B}$ be the ideal generated by $t^{0}, \ldots, t^{m}$. The equation (4.19) with $n \leq m$ imply that $\mathcal{E}_{0} \otimes (\mathcal{O}_{B}/m^{m+1})$ has the structure of an $(\mathcal{O}_{B}/m^{m+1})[A_{a}^{\leq m}]$-module. Since this is again generated by $e_{0}$, this has a unique $(\mathcal{O}_{B}/m^{m+1})[A_{a}^{\leq m}]$-algebra structure such that $e_{0}$ is a unit. By (4.18) and (4.19), $\Omega_{i}^{\leq m}$ commutes with $A_{a}^{\leq m}$ on $\mathcal{E}_{0} \otimes (\mathcal{O}_{B}/m^{m+1})$ and $\Omega_{i}^{\leq m} e_{0} = e_{i}$, $\Omega_{i}^{\leq m}$ is uniquely determined as the multiplication by $e_{i}$ in this algebra. We need to
check $\partial_j \Omega_i^{\leq m} = \partial_i \Omega_j^{\leq m}$. We have
\[
\begin{align*}
[\partial_j \Omega_i^{\leq m} - \partial_j \Omega_i^{\leq m}, A_a^{\leq m}] &= \partial_j [\Omega_i^{\leq m}, A_a^{\leq m}] - \partial_i [\Omega_j^{\leq m}, A_a^{\leq m}] \\
&+ [\Omega_i^{\leq m}, \partial_a \Omega_j^{\leq m-1}] - [\Omega_j^{\leq m}, \partial_a \Omega_i^{\leq m-1}] \\
&= \partial_a [\Omega_i^{\leq m}, \Omega_j^{\leq m-1}] - [\partial_a \Omega_i^{\leq m}, \Omega_j^{\leq m-1}] - [\Omega_j^{\leq m}, \partial_a \Omega_i^{\leq m-1}] \equiv 0 \mod m^m
\end{align*}
\]
Thus in particular, the action of $\partial_j \Omega_i^{\leq m} - \partial_j \Omega_i^{\leq m}$ on $E_0 \otimes (O_{B^\prime}/m^m)$ commutes with $A_a^{\leq m-1}$. On the other hand, $(\partial_j \Omega_i^{\leq m} - \partial_i \Omega_j^{\leq m})e_0 = \partial_j e_i - \partial_i e_j = 0$. Since $E_0 \otimes (O_{B^\prime}/m^m)$ is generated by $e_0$ as an $(O_{B^\prime}/m^m)[A_a^{\leq m-1}]$-module, we must have $\partial_j \Omega_i^{\leq m} - \partial_i \Omega_j^{\leq m} = 0$. This completes the induction step.

**Universal property:** Let $\mathcal{F}^\hbar$ be an AQDM over $B'$ with an isomorphism $\varphi^\prime: \mathcal{E}^\hbar \cong \mathcal{F}^\hbar_{b'}$ of $\mathcal{D}$-modules. Let $e^0, \ldots, e^l$ be co-ordinates on $B'$ centered at $b'$. First we construct a map $f: B' \to B$ and an isomorphism $\theta: \mathcal{F}^\hbar \cong f^* \mathcal{E}^\hbar$ such that $f(b') = b$ and $\theta \circ \varphi^\prime = \varphi$. The canonical frame $\Phi$ of $\mathcal{E}^\hbar$ used in the first half of the proof induces a frame $\Phi^\prime_{b'}$ of $\mathcal{F}^\hbar_{b'}$ via the isomorphism $\varphi^\prime$.

\[
\begin{align*}
\mathcal{E}_0 \otimes O^\hbar & \xrightarrow{\Phi} \mathcal{E}^\hbar \\
\varphi^\prime_0 \circ id & \quad \varphi^\prime_0: \mathcal{E}_0 \to \mathcal{F}^\hbar_{b',0} \quad \text{is the map induced from } \varphi^\prime.
\end{align*}
\]

By Theorem 4.6, $\Phi^\prime_{b'}$ uniquely extends to a canonical frame $\Phi^\prime: \mathcal{F}^\hbar_{b',0} \otimes O^\hbar_{b'} \to \mathcal{F}^\hbar$. Let $J'(Q, e, h)$ be the $J$-function of $\mathcal{F}^\hbar$ associated with $\Phi^\prime$ and $\varphi^\prime_0(e_0) \in \mathcal{F}^\hbar_{b',0}$. Define $\tilde{t}'(Q, e) \in O_{B'}$ by the $h^{-1}$-expansion of $J'$:

\[
J' = \varphi^\prime_0(e_0) + \frac{1}{h} \sum_{i=0}^s \tilde{t}'(Q, e) \varphi^\prime_0(e_i) + O(h^{-2}).
\]

Note that $\{ \varphi^\prime_0(e_i) \}_{i=0}^l$ is a basis of $\mathcal{F}_b, 0$. The functions $t'(Q, e) := \tilde{t}'(Q, e) - \tilde{t}'(Q, 0)$ determine a map $f: B' \to B$ such that $f(b') = b$. We define an $O_{B'}$-module isomorphism $\theta: \mathcal{F}^\hbar \to f^* \mathcal{E}^\hbar$ by

\[
\theta: \mathcal{F}^\hbar \xrightarrow{\Phi^{-1}} \mathcal{F}_{b',0} \otimes_K O^\hbar_{B'} \xrightarrow{\varphi^\prime_0^{-1} \otimes id} \mathcal{E}_0 \otimes_K O^\hbar_{B'} = f^* \mathcal{E}^\hbar.
\]

Then $\theta \circ \varphi^\prime$ coincides with $\Phi^{-1} = \varphi$ by the above commutative diagram. We show that $\theta$ is an isomorphism of $\mathcal{D}_{B'}$-modules. Let $A^\prime_a(Q, e)$, $1 \leq a \leq r$ and $\Omega^\prime_i(Q, e)$, $0 \leq i \leq l$ be the connection operators of $\mathcal{F}^\hbar$ associated with the canonical frame $\Phi^\prime$. Since $\varphi^\prime$ gives an isomorphism of $\mathcal{D}$-modules,

\[
A^\prime_a(Q, 0) = \varphi^\prime_0 A_a(Q) \varphi^\prime_0^{-1}.
\]

Let $L'$ be the fundamental solution of $\mathcal{F}^\hbar$ associated with $\Phi$. Then one has

\[
h \frac{\partial}{\partial e^i} J' = h \frac{\partial}{\partial e^i} L'^{-1} \varphi'(e_0) = L'^{-1}(\Omega^\prime_i(Q, e) \varphi'(e_0)) = \Omega^\prime_i(Q, e) \varphi'(e_0) + O(h^{-1}).
\]
Hence, by comparing the leading term of the $\hbar^{-1}$-expansion
\[
\Omega'_i(Q,\epsilon)\varphi'_0(e_0) = \sum_{j=0}^{s} \frac{\partial \hat{\psi}(Q,\epsilon)}{\partial e^j} \varphi'_0(e_j) = \varphi'_0 \left( \sum_{j=0}^{s} \frac{\partial \hat{\psi}(Q,\epsilon)}{\partial e^j} e_j \right).
\] 
(4.22)

On the other hand, the connection operators $f^*A_a(Q,\epsilon)$, $1 \leq a \leq r$ and $f^*\Omega_i(Q,\epsilon)$, $0 \leq i \leq l$ of $f^*\hat{\mathcal{E}}^\hbar$ satisfy
\[
f^*A_a(Q,0) = A_a(Q), \quad f^*\Omega_i(Q,\epsilon)e_0 = \sum_{j=0}^{s} \frac{\partial \hat{\psi}(Q,\epsilon)}{\partial e^j} e_j.
\]

But the flat connections $\hbar Q^a \frac{\partial}{\partial x^a} + A'_a$ and $\hbar \frac{\partial}{\partial x^a} + \Omega'_i$ satisfying (4.21) and (4.22) are unique; this follows from the same argument as the first half. Therefore, we conclude
\[
A'_a(Q,\epsilon) = \varphi'_0 A_a(Q,\epsilon) \varphi'^{-1}_0, \quad \Omega'_i(Q,\epsilon) = \varphi'_0 \Omega_i(Q,\epsilon) \varphi'^{-1}_0.
\]

This shows that $\theta$ is a homomorphism of $\mathcal{D}_{B^r}$-modules.

Finally, we show that such $f$ and $\theta$ are unique. Let $f: B' \to B$ and $\theta: \mathcal{F}^\hbar \cong f^*\hat{\mathcal{E}}^\hbar$ be an arbitrary map and an isomorphism of $\mathcal{D}_{B^r}$-modules satisfying $f(b') = b$ and $\theta_B \circ \varphi' = \varphi$. Let $\hat{J}(Q,t,\hbar)$ be the $J$-function of $\hat{\mathcal{E}}^\hbar$ associated with the given trivialization and $e_0 \in \mathcal{E}_0 = \hat{\mathcal{E}}_{b,0}$. Let $\hat{\psi}(Q,t) \in \mathcal{O}_B$ be the flat co-ordinate given by the expansion (see Theorem-Definition 1.7)
\[
\hat{J} = e_0 + \frac{1}{\hbar} \sum_{i=0}^{s} \hat{\psi}(Q,t) e_i + O(\hbar^{-2}).
\]

By applying the discussion preceding (4.22) to this $\hat{J}$, we have $e_i = \Omega_i(Q,t)e_0 = \sum_{j=0}^{s} (\partial \hat{\psi}(Q,t) / \partial t^i) e_j$. Therefore, $\partial \hat{\psi}(Q,t) / \partial t^i = \delta^i_j$ and
\[
\hat{\psi}(Q,t) = t^i + g^i(Q), \quad \text{or} \quad \hat{t}^i(Q,t) = \hat{\psi}(Q,t) - \hat{\psi}(Q,0).
\] 
(4.23)

for some $g^i \in \mathcal{O}$. The given canonical frame of $\hat{\mathcal{E}}^\hbar$ induces a canonical frame $\Phi'$ of $\mathcal{F}^\hbar$ via $\theta$. From $\theta_B \circ \varphi' = \varphi$ and Theorem 4.3 it follows that this canonical frame $\Phi'$ coincides with the $\Phi'$ in the previous paragraph. The $J$-function $J'$ of $\mathcal{F}^\hbar$ with respect to $\Phi'$ and $\varphi'_0(\epsilon_0) \in \mathcal{F}_{\Phi',0}$ is given by
\[
J' = \varphi'_0(f^*\hat{J}) = \varphi'_0(e_0) + \frac{1}{\hbar} \sum_{i=0}^{s} (f^*\hat{\psi}) \varphi'_0(e_i) + O(\hbar^{-2}).
\]

From this, (4.23) and $f(b') = b$, it follows that the map $f$ coincides with what was given above. The uniqueness of $\theta$ follows from that $\theta$ intertwines the canonical frame $\Phi'$ of $\mathcal{F}^\hbar$ with the given canonical frame of $f^*\hat{\mathcal{E}}^\hbar$.

Remark 4.10. The uniqueness of the reconstruction of a big AQDM holds in a more general situation. For a small AQDM $\mathcal{E}^\hbar$, $\mathcal{E} := \mathcal{E}^\hbar / \hbar \mathcal{E}^\hbar$ has the structure of an $\mathcal{O}[P_1,\ldots,P_r]$-module. Through a canonical frame, $\mathcal{E}$ is identified with the $\mathcal{O}[A_1(Q),\ldots,A_r(Q)]$-module $\mathcal{E}_0 \otimes \mathcal{O}$, where $A_a(Q)$ is the connection operator associated with the canonical frame. In the above proof, we used the fact that
modules at an O with the canonical frame Φ.

A (Theorem 4.6). Let \( H \) since the Euler vector field (in quantum cohomology having a tame semisimple point satisfies this condition in the above theorem. In particular, \( \hat{\mathcal{E}}^h \) is unique if it exists. For small quantum cohomology SQH, this condition (4.24) corresponds to that \( SQH^* \) is generated by \( H^2 \) at a generic value of the quantum parameter \( Q \). For example, the small quantum cohomology having a tame semisimple point satisfies this condition since the Euler vector field (in \( H^2 \)) generates the quantum cohomology algebra in a neighborhood of a tame semisimple point. In this case, the uniqueness of the reconstruction agrees with the Dubrovin’s reconstruction theorem [8].

Remark 4.11. Hertling-Manin [16 Theorem 2.5] proved a similar reconstruction theorem for (TE)-structures. The (TE) structure is endowed with a family of flat connections \( d + \frac{1}{\hbar} \Omega(t, \hbar) \) parametrized by \( \hbar \) which has poles at \( \hbar = 0 \). It also has a flat connection in the \( \hbar \)-direction which corresponds to the grading in the quantum cohomology. Our reconstruction theorem differs from theirs in that (i) we do not consider differential equations in the \( \hbar \)-direction, that (ii) the AQDM here is regular singular along \( Q^1 \cdots Q^r = 0 \) and that (iii) they reconstructed (TE) structures in analytic category.

We can now formulate generalized mirror transformations.

Definition 4.12 (Generalized mirror transformation). (i) We begin with a small AQDM \( \mathcal{E}^h \) such that \( \mathcal{E}_0 \) is generated by a single element \( e_0 \) as a \( K[p_1, \ldots, p_r] \)-module. The equivariant Floer cohomology \( \mathcal{F}H^*_\mathcal{L}(L_X/V) \) in Section 3 gives an example of such small AQDMs.

(ii) Choose a frame \( \Phi_0 \) of \( \mathcal{E}_0^h \) satisfying (4.2) and calculate the canonical frame \( \Phi \) of \( \mathcal{E}^h \) inducing \( \Phi_0 \) by the Birkhoff factorization of the fundamental solution \( L \) (Theorem 4.7). Let \( A_a(Q) \), 1 \( \leq a \leq r \) be the connection operators associated with the canonical frame \( \Phi \).

(iii) Reconstruct a big AQDM \( \hat{\mathcal{E}}^h \) on the base \( B \) such that \( \hat{\mathcal{E}}_b^h \cong \mathcal{E}^h \) as \( \mathcal{D} \)-modules at an \( \mathcal{O} \)-valued point \( b \) on \( B \) (Theorem 4.9). More concretely, we take \( \hat{\mathcal{E}}^h \) to be \( \mathcal{E}_0^h \otimes \mathcal{O}^h_b \) endowed with flat connections \( hQ^a \frac{\partial}{\partial Q^a} + A_a(Q, t) \), 1 \( \leq a \leq r \) and \( h \frac{\partial}{\partial Q^a} + \Omega_i(Q, t), 0 \leq i \leq s \). Here, \( t^0, \ldots, t^r \) are co-ordinates on \( B \) centered at \( b \) and \( A_a, \Omega_i \) are \( \hbar \)-independent. Given a \( K \)-basis \( \{e_i\}_{i=0}^s \) of \( \mathcal{E}_0^h \), \( A_a, \Omega_i \) are uniquely determined by (4.3), (4.6), \( A_a(Q, 0) = A_a(Q) \) and \( \Omega_i(Q, t)e_0 = e_i \).

(iv) Take a flat co-ordinate system \( t^0, \ldots, t^r \) on \( B \) (Theorem-Definition 4.7) of the form \( \hat{t} = \hat{t}(Q, t) = t^i + g_i(Q), g_i(Q) \in \mathcal{O}, g_i(0) = 0 \) (see (4.23)). This new co-ordinate system gives the new connection operators \( \hat{A}_a(Q, t), \hat{\Omega}_i(Q, t) \) (by (4.4)) which satisfy \( \hat{A}_a e_0 = p_a e_0, \hat{\Omega}_i e_0 = \hat{\phi}_i e_0 \), where \( p_a = \hat{A}_a|_{Q=t=0} \) and \( \hat{\phi}_i = \hat{\Omega}_i|_{Q=t=0} \).

By a generalized mirror transformation, the original small AQDM \( \mathcal{E}^h \) becomes isomorphic to the restriction of \( \hat{\mathcal{E}}^h \) to the locus \( \hat{t} = \hat{t}(Q, 0) \), where \( \hat{t}(Q, t) \) are
flat co-ordinates taken in (iv). We call this subspace \( \{ \hat{\mathfrak{i}} = \hat{\mathfrak{i}}(Q, 0) \} \) of the \((Q, \hat{\mathfrak{i}})\)-space a locus of \( \mathcal{E}^h \). When \( \{ \{a \in \hat{\mathfrak{i}} \} \} \) is part of a \( K \)-basis of \( \mathcal{E}_0 \), by Proposition 4.8 we can assume that the connection operators \( \hat{A}_a, \Omega_0 \) in (iv) moreover satisfy \( \hat{A}_a = \Omega_a, \Omega_0 = \text{id} \) and depend only on \( Q^1 e^{\hat{\mathfrak{i}}}, \ldots, Q^r e^{\hat{\mathfrak{i}}}, \hat{\mathfrak{i}}^r, \ldots, \hat{\mathfrak{i}}^s \). Then we can take

\[
\ell^0, \; \hat{q}^a := Q^a e^{\hat{\mathfrak{i}}^a} \quad 1 \leq a \leq r, \quad \hat{\mathfrak{i}}^r, \quad r + 1 \leq j \leq s
\]

to be effective parameters of \( \hat{\mathcal{E}}^h \). In these effective parameters, the locus of the original small AQDM \( \mathcal{E}^h \) is given by (see Figure 1)

\[
\hat{\mathfrak{i}}^0 = \hat{\mathfrak{i}}^0(Q, 0), \; \hat{q}^a = Q^a e^{\hat{\mathfrak{i}}^a(Q, 0)} \quad 1 \leq a \leq r, \quad \hat{\mathfrak{i}}^r = \hat{\mathfrak{i}}^r(Q, 0) \quad r + 1 \leq j \leq s.
\]

In ordinary mirror transformations, the locus of \( \mathcal{E}^h \) is contained in the small quantum cohomology locus defined by \( \hat{\mathfrak{i}}^r = \cdots = \hat{\mathfrak{i}}^s = 0 \) and \( \hat{q}^a = Q^a e^{\hat{\mathfrak{i}}^a(Q, 0)} \) can be interpreted as a co-ordinate change on the \( q \)-space. The specific characteristic in generalized case is that the locus of \( \mathcal{E}^h \) is not necessarily contained in the small quantum cohomology locus in the effective parameter space.

![Figure 1. The effective parameter space of \( \hat{\mathcal{E}}^h \)](image)

We now discuss a relationship among the \( J \)-functions of \( \mathcal{E}^h \) and \( \hat{\mathcal{E}}^h \) and the locus of \( \mathcal{E}^h \) in \( B \). Let \( \Phi_{\text{ini}} \) be an arbitrary frame of \( \mathcal{E}^h \). Let \( L_{\text{ini}} \) be the fundamental solution of \( \mathcal{E}^h \) associated with \( \Phi_{\text{ini}} \) and \( I(Q, h) := L_{\text{ini}}^{-1} e_0 \) be the \( J \)-function of \( \mathcal{E}^h \) associated with \( e_0 \in \mathcal{E}_0 \). The canonical frame \( \Phi \) calculated in step (ii) is given by \( \Phi = \Phi_{\text{ini}} \circ G \), where \( G = L_{\text{ini}} \) is the positive part of the Birkhoff factorization (4.12) of \( L_{\text{ini}} \). (Here, \( \Phi \) and \( \Phi_{\text{ini}} \) induce the same frame of \( \mathcal{E}^h \).) By Proposition 4.5, \( \mathcal{E}^h \) is generated by \( \Phi_{\text{ini}}(e_0) \) as a \( \mathcal{D} \)-module. Hence there exists \( V(Q, P, h) \in \mathcal{D} \) such that \( V(Q, P, h) \Phi_{\text{ini}}(e_0) = \Phi(e_0) = \Phi_{\text{ini}}(G e_0) \), i.e. \( V(Q, \nabla_a^h, h) e_0 = G e_0 \). Note that we can assume \( V(0, P, h) = 1 \) since \( G_{|Q=0} = L_{\text{ini}, +}|_{Q=0} = \text{id} \). Then by (4.11), the \( J \)-function \( J \) of \( \mathcal{E}^h \) associated with the new frame \( \Phi \) and \( e_0 \) is given by

\[
J(Q, h) = L_{\text{ini}}^{-1}(G e_0) = V(Q, hQ^a \frac{\partial}{\partial Q^a} + p_a, h) I(Q, h).
\] (4.25) (4.26)

The big AQDM \( \hat{\mathcal{E}}^h \) reconstructed in step (iii) is naturally equipped with a canonical frame (compatible with \( \Phi \)). Let \( J(Q, \hat{\mathfrak{i}}, h) \) be the \( J \)-function of \( \hat{\mathcal{E}}^h \) with respect
to the given canonical frame, \( e_0 \in \mathcal{E}_0 = \mathcal{E}_{\nu,0} \) and flat co-ordinates \( \tilde{t}^i = \tilde{t}^i(Q,t) \) in (iv). (Recall that \( b \) is given by \( t = 0 \).) Then it is easy to see that
\[
\hat{J}(Q,\tilde{t}(Q,0),\hbar) = J(Q,\hbar).
\]
(4.27)

The equations (4.25), (4.26) and (4.27) give a relationship between the two \( J \)-functions \( I \) and \( J \). Moreover, since \( \tilde{t}^i \) can be read off from the \( \hbar^{-1} \)-expansion of \( \hat{J} \) (Theorem–Definition 4.7), we have by (4.27), (4.25), \( G = L_{\text{ini},+} \) and (4.13),
\[
\sum_{j=0}^{s} \hat{t}^j(Q,0)(\hat{\varphi}_j e_0) = \text{Res}_{\hbar=0} d\hbar \left\{ \sum_{k=0}^{\infty} (id - L_{\text{ini}}^{-1} \circ \pi_+)^k I(Q,\hbar) \right\}.
\]

This formula calculates the locus of \( \mathcal{E}^\hbar \) perturbatively. For a nef complete intersection in toric varieties, the \( I \)-function takes of the form \( I(Q,\hbar) = f(Q)e_0 + O(\hbar^{-1}) \). In this case, the left-hand side is simplified to the form \( \text{Res}_{\hbar=0} d\hbar(I(Q,\hbar)/f(Q)) \). This recovers the original mirror transformation in [11, 12].

4.5. Reconstruction via generators. In order to obtain the reconstruction, in view of its uniqueness, we only need to find at least one big AQDM which is an extension of the given small AQDM. Here, we consider the extension of a \( D \)-module via generators or \( J \)-functions.

Let \( \mathcal{O}^\hbar \) be a subring of \( \mathcal{O}^\hbar \) and set \( \mathcal{D} := \mathcal{O}^\hbar \langle P_1, \ldots, P_r \rangle \subset \mathcal{D} \). Let \( \mathcal{J}(Q,\hbar) \) be an element of a suitable function space \( \mathcal{F} \) of functions in \( Q^1, \ldots, Q^r \) and \( \hbar \). We assume that \( \mathcal{D} \) acts on \( \mathcal{F} \) and that \( \mathcal{J}(Q,\hbar) \) generates a \( \mathcal{D} \)-module \( \mathcal{E}^\hbar = \mathcal{D}\mathcal{J}(Q,\hbar) \subset \mathcal{F} \) which is finitely generated as an \( \mathcal{O}^\hbar \)-module. Here, \( \mathcal{E}^\hbar \) is isomorphic to \( \mathcal{D}/I \) where \( I \) is the left ideal consisting of an element \( f(Q,P,\hbar) \in \mathcal{D} \) which annihilates \( \mathcal{J}(Q,\hbar) \) (c.f. Proposition 4.13). We also set \( \mathcal{O}_B^\hbar := \mathcal{O}^\hbar \langle t^0, \ldots, t^s \rangle \) and \( \mathcal{D}_B := \mathcal{O}_B^\hbar \langle P_1, \ldots, P_r, \varphi_0, \ldots, \varphi_s \rangle \subset \mathcal{D}_B \). Let \( \mathcal{D}_B \) act on \( \mathcal{F}[t] := \mathcal{F}[t^0, \ldots, t^s] \) by \( \varphi_i \mapsto \hbar \partial/\partial t^i \).

**Proposition 4.13.** For a \( \mathcal{D}_B \)-module \( \hat{\mathcal{E}}^\hbar \subset \mathcal{F}[t] \) generated by \( \hat{\mathcal{J}} \in \mathcal{F}[t] \) such that \( \hat{\mathcal{J}}(Q,t,\hbar) = \mathcal{J}(Q,\hbar) + O(t) \), there exists an isomorphism \( \hat{\mathcal{E}}^\hbar / \sum_{i=0}^{s} t^i \mathcal{E}^\hbar \cong \mathcal{E}^\hbar \) of \( \mathcal{D} \)-modules which sends \( [\hat{\mathcal{J}}] \) to \( \mathcal{J} \) if and only if \( \hat{\mathcal{J}}(Q,t,\hbar) \) satisfies
\[
\hbar \frac{\partial}{\partial t^i} \hat{\mathcal{J}}(Q,t,\hbar) \in \mathcal{O}_B^\hbar \langle P_1, \ldots, P_r, t^0 \varphi_0, \ldots, t^s \varphi_s \rangle \hat{\mathcal{J}}(Q,t,\hbar).
\]
(4.28)

In this case, \( \hat{\mathcal{E}}^\hbar \) is a finitely generated \( \mathcal{O}_B^\hbar \)-module. If moreover \( \mathcal{E}^\hbar \) is free over \( \mathcal{O}^\hbar \), then \( \hat{\mathcal{E}}^\hbar \) is free over \( \mathcal{O}_B^\hbar \).

**Proof.** First we prove the “if part”. Set \( \mathcal{D}'_B := \mathcal{O}_B^\hbar \langle P_1, \ldots, P_r, t^0 \varphi_0, \ldots, t^s \varphi_s \rangle \subset \mathcal{D}_B \). It follows from (4.28) that \( \hat{\mathcal{E}}^\hbar \) is generated by \( \hat{\mathcal{J}} \) as a \( \mathcal{D}_B \)-module. Therefore one can define a surjective homomorphism of \( \mathcal{D} \)-modules:
\[
\hat{\mathcal{E}}^\hbar \ni f(Q,t,P,t \varphi,\hbar) \hat{\mathcal{J}} \mapsto f(Q,0,P,0,\hbar) \mathcal{J} \in \mathcal{E}^\hbar, \quad f(Q,t,P,t \varphi,\hbar) \in \mathcal{D}'_B.
\]

We need to show that the kernel equals \( \sum_{i=0}^{s} t^i \hat{\mathcal{E}}^\hbar \). To see this, it suffices to show that for \( f \in \mathcal{D} \) satisfying \( f \mathcal{J} = 0 \), there exists \( g \in \mathcal{D}'_B \) such that \( g|_{t=0} = f \) and \( g \hat{\mathcal{J}} = 0 \). We will prove this by induction on \( s \). As an inductive hypothesis, we can assume that there exists \( g \in \mathcal{D}'_B \) such that \( g|_{t=0} = f \) and \( (g \hat{\mathcal{J}})|_{t=0} = 0 \). Let
$T_i\mathcal{J} \in \mathbb{E}^h$, $1 \leq i \leq N$ be generators of $\mathbb{E}^h$ as an $\mathcal{O}^h$-module, where $T_i \in \mathbb{D}_h$. Let $n_0$ be the maximum of the degrees of $T_i$, $1 \leq i \leq N$ with respect to $\mathbb{P}_1, \ldots, \mathbb{P}_r$. Let $\mathbb{D}'_B$ be the subspace of $\mathcal{O}^h_B(\mathbb{P}_1, \ldots, \mathbb{P}_r) \subset \mathbb{D}_B$ consisting of all elements which are of degree less than or equal to $n_0$ in $\mathbb{P}_1, \ldots, \mathbb{P}_r$. We show by induction on $m$ that if $g\hat{\mathcal{J}} = O((t^*)^m)$ for $g \in \mathbb{D}'_B$, there exists $g_m \in (t^*)^m \mathbb{D}'_B$ such that $(g + g_m)\hat{\mathcal{J}} = O((t^*)^{m+1})$. Let $m$ be zero. There exists $c_i \in \mathcal{O}^h$ such that $(g\hat{\mathcal{J}})_{t=0} = \sum_{i=1}^{N} c_i T_i\mathcal{J}$. Thus we can take $g_0$ to be $-\sum_{i=1}^{N} c_i T_i \in \mathbb{D}'_B$. Let $m$ be positive. Set $\partial_s := \partial/\partial t^s$. First note that we have for a function $c(t^s)$ in $t^s$,

$$[\partial_s, c(t^s)(ht^s\partial_s)^n] = c'(t^s)(ht^s\partial_s)^n + c(t^s) \sum_{k=0}^{n-1} \binom{n}{k}(ht^s\partial_s)^k h^{n-k-1}(h\partial_s).$$

This means $[\partial_s, \mathbb{D}'_B] \subset \mathbb{D}'_B + \mathbb{D}'_B h\partial_s$. Therefore, we can write $[\partial_s, g] = g' + g''h\partial_s$ for some $g', g'' \in \mathbb{D}'_B$. By \cite{41,28}, there exists $F \in \mathbb{D}'_B$ such that $(h\partial_s)\hat{\mathcal{J}} = F\hat{\mathcal{J}}$. We calculate

$$h\partial_s(g\hat{\mathcal{J}}) = h(g' + g''h\partial_s)\hat{\mathcal{J}} + g'h\partial_s\hat{\mathcal{J}} = (h(g' + g''F) + gF)\hat{\mathcal{J}} = (h(g' + g''F) + [g, F])\hat{\mathcal{J}} + Fg\hat{\mathcal{J}} = h\phi\hat{\mathcal{J}} + O((t^*)^m)$$

where we set $\phi = g' + g''F + [g, F]/h$. Note that $\phi \in \mathbb{D}'_B$ because the commutator $[g, F]$ is divisible by $h$. Since $\phi\hat{\mathcal{J}} = O((t^*)^{m-1})$, by the induction hypothesis, there exists $\phi_{m-1} \in (t^*)^{m-1} \mathbb{D}'_B$ such that $(\phi + \phi_{m-1})\hat{\mathcal{J}} = O((t^*)^m)$. We can take $g_m \in (t^*)^m \mathbb{D}'_B$ such that $\phi_{m-1} = \partial g_m/\partial t^s$. Then we have

$$h\partial_s((g + g_m)\hat{\mathcal{J}}) = h(\phi + \phi_{m-1})\hat{\mathcal{J}} + g_m h\partial_s\hat{\mathcal{J}} + O((t^*)^m) = O((t^*)^{m+1}).$$

Thus $(g + g_m)\hat{\mathcal{J}} = O((t^*)^{m+1})$. This completes the proof of the “if part”.

Next we show that $\bar{\mathbb{E}}^h$ is finitely generated over $\bar{\mathcal{O}}^h_B$ when $\bar{\mathbb{E}}^h/m\bar{\mathbb{E}}^h \cong \bar{\mathbb{E}}^h$, where $m = \sum_{i=1}^{N} t^i\bar{\mathbb{O}}^h_B$. It follows by an elementary argument that $\{T_i\mathcal{J}\}_{i=1}^{N}$ generates $\mathbb{E}^h$ over $\bar{\mathcal{O}}^h_B$ when $\{T_i\mathcal{J}\}_{i=1}^{N}$ generates $\mathbb{E}^h$ over $\mathcal{O}^h_B$. Here, we need the fact that $\mathcal{O}^h_B$ is complete and $\bar{\mathbb{E}}^h$ is Hausdorff for their $m$-adic topology ($\bigcap_{n \geq 0} m^n\bar{\mathbb{E}}^h \subset \bigcap_{n \geq 0} m^n\mathcal{O}^h_{B} = \{0\}$). See for instance \cite{26, Corollary 2, §3, Ch.VIII}.

Next we show that $\bar{\mathbb{E}}^h$ is free over $\bar{\mathbb{O}}^h_B$ if $\mathbb{E}^h$ is free over $\mathcal{O}^h_B$. We take $\{T_i\mathcal{J}\}$ to be a free basis over $\mathcal{O}^h_B$. We claim that $\{T_i\mathcal{J}\}$ is a free basis over $\bar{\mathbb{O}}^h_B$. Since we know that $T_i\mathcal{J}$ generate $\mathbb{E}^h$ over $\mathcal{O}^h_B$, we have $(h\partial/\partial t^s)(T_i\mathcal{J}) = \sum_{k=1}^{N} \Omega_{j_k}^h(T_k\mathcal{J})$ for some $\Omega_{j_k}^h \in \mathcal{O}^h_B$. For $c \in \mathcal{O}^h_B$, let $\operatorname{ord}(c) := \sup\{n; c \in m^n\}$ denote the order of zero at $t = 0$. Suppose we have $\sum_{i=1}^{N} c_i(T_i\mathcal{J}) = 0$ and set

$$\nu := \min_{1 \leq i \leq N} (\operatorname{ord}(c_i)) = \operatorname{ord}(c_{i_0}).$$

Because $T_i\mathcal{J}$ is a basis of $\mathbb{E}^h$ and $\sum_{i=1}^{N} c_i|_{t=0} T_i\mathcal{J} = 0$, we have $c_i|_{t=0} = 0$. Thus $\nu > 0$. If $0 < \nu < \infty$, we calculate

$$0 = h \frac{\partial}{\partial t^s} \sum_{i=1}^{N} c_i(T_i\mathcal{J}) = \sum_{k=1}^{N} (h\partial \frac{\partial c_k}{\partial t^s} + \sum_{i=1}^{N} c_i \Omega_{j_k}^h)(T_k\mathcal{J}).$$
Let \( P \)

Example 4.14. Let \( E^h \) be the small AQDM such that \( E_0 \) is generated by \( e_0 \in E_0 \) as a \( K[p_1, \ldots, p_r] \)-module. Let \( J(Q, h) \) be the \( J \)-function associated with some frame and \( e_0 \). We take \( \mathfrak{F} \) to be \( E^h_0 \otimes K\{h, h^{-1}\}\{Q\} \) and let \( \mathcal{D} \) act on \( \mathfrak{F} \) by \( P_a \mapsto hQ^n(\partial/\partial Q^n) + p_a \). By Proposition 4.13, \( J(Q, h) \in \mathfrak{F} \) generates \( E^h \). Let \( \{T_i(p_1, \ldots, p_r)e_0\}_{i=0}^s \) be a \( K \)-basis of \( E_0 \). Then, the following function \( J \) satisfies (4.23) and formally reconstructs a big AQDM:

\[
J(Q, t, h) = \exp \left( \frac{1}{h} \sum_{i=0}^s t_i T_i(hQ^n \frac{\partial}{\partial Q^n} + p_a) \right) J(Q, h) \in \mathfrak{F}[t]
\]

Note that \( t_i \) here is not necessarily a flat co-ordinate.

Example 4.15. We construct a formal mirror corresponding to the big quantum cohomology of \( \mathbb{P}^n \). Following Givental [12], we take a mirror of the small quantum cohomology of \( \mathbb{P}^n \) to be the oscillatory integral

\[
J(Q, h) = \int_{C(\mathbb{C}^*)^n} \exp \left( \frac{1}{h} (x_1 + \cdots + x_n + \frac{Q}{x_1x_2\cdots x_n}) \right) dx
\]

where \( \Gamma \) is a non-compact Morse cycle of the Morse function \( (x_1, \ldots, x_n) \mapsto \Re(x_1 + \cdots + x_n + \frac{Q}{x_1x_2\cdots x_n}) \) and \( dx = \prod_{i=1}^n(dx_i/x_i) \). This \( J \) is a multi-valued function in \( Q \) and \( h \). We take \( \mathfrak{F} \) to be the space of multi-valued analytic functions on \( (Q, h) \in \mathbb{C}^* \times \mathbb{C}^* \), i.e. single-valued functions on the universal cover of \( \mathbb{C}^* \times \mathbb{C}^* \). Then \( J \in \mathfrak{F} \). We also take \( O^h \) to be the subspace of \( O^h = \mathbb{C}[h][[Q]] \) (here \( K = \mathbb{C} \)) consisting of convergent functions for all \( Q, h \in \mathbb{C} \). Let \( \mathcal{D} := O^h(\mathcal{P}) \) act on \( \mathfrak{F} \) by \( P \mapsto hQ(\partial/\partial Q) \). The oscillatory integral (4.29) satisfies the differential equation

\[
(h\partial)^{n+1} - QJ(Q, h) = 0, \quad \partial = Q(\partial/\partial Q)
\]

and generates the small QDM of \( \mathbb{P}^n \): \( SQDM(\mathbb{P}^n) \cong O^h \otimes_{O^h} (\mathcal{D}/(h\partial)^{n+1} - Q) \).

Let \( t^0, t^1, \ldots, t^n \) be co-ordinates on \( B \) and consider \( \tilde{J} \) in \( \mathfrak{F}[t^0, \ldots, t^n] \):

\[
\tilde{J}(Q, t, h) = \int_{C(\mathbb{C}^*)^2} \exp \left( \frac{1}{h} (t^0 + \sum_{i=1}^n t^i x_i + \frac{Q}{x_1\cdots x_n} + \sum_{i=1}^n t^i x_1 \cdots x_i) \right) dx
\]

Here, the integrand should be expanded as a power series in \( t \). We can easily show that \( \tilde{J} \) satisfies

\[
(R_1R_2\cdots R_n h\partial - Q)\tilde{J} = 0,
\]

where \( R_i := h\partial - h^2\partial_i - \cdots - h^n\partial_i \) and \( \partial_i = \partial/\partial t^i \). The equation (4.31) is a deformation of the relation (4.30) and (4.32) shows that \( \tilde{J} \) satisfies the condition
Therefore, the $D_B$-module $\hat{E}^h$ generated by $\hat{f}$ is a free $O_B^h$-module of rank $n + 1$ and an extension of the $D$-module generated by $\hat{f}$. By the uniqueness of the reconstruction, $O_B^h \otimes_{O_B^h} \hat{E}^h$ is canonically isomorphic to the big QDM of $\mathbb{P}^n$.

On the other hand, the Jacobi ring of the phase function $t^0 + \sum_{i=1}^n x_i + Q/(x_1 \cdots x_n) + \sum_{i=1}^n t_i x^i \cdots x^i$ is of dimension $2n > \dim H^*(\mathbb{P}^n)$ for non-zero generic $t$. Thus, it seems to be important to treat $\hat{f}$ as a formal power series in $t^i$. This formal mirror construction for big quantum cohomology will be generalized to toric varieties in general. We hope to discuss this problem elsewhere.

In the literature, starting from a Laurent polynomial, Barannikov [3] (for toric varieties) and Douai and Sabbah [6, 7] (for weighted projective spaces or for a general tame function) constructed a Frobenius manifold of big quantum cohomology. It would be interesting to study a relationship between their methods and ours.

5. The Proof of Generalized Mirror Transformations

In this section, we prove that the big AQDM reconstructed from the equivariant Floer cohomology $FH_{T^2}(L_X, \nu)$ by the generalized mirror transformation (Definition 4.12) is isomorphic to the big QDM for a pair $(X, \nu)$.

5.1. Review of quantum Lefschetz theorem. We review the quantum Lefschetz theorem by Coates and Givental [4]. We use the notation in Section 2. Let $M$ be a smooth projective variety and $\nu$ be a vector bundle on $M$. Let $c$ be a general multiplicative characteristic class $2,1$. We take $K = \mathbb{C}[s]$ to be the ground ring and consider the Gromov-Witten theory twisted by $\nu$ and $c$. The genus zero twisted Gromov-Witten potential $F_s$ is defined to be

$$ F_s(t_0, t_1, t_2, \ldots) = \sum_{n=0}^\infty \sum_{d \in \Lambda} \frac{Q^d}{n!} (t(\psi_1), \ldots, t(\psi_n))^\nu, \quad t(\psi_i) = \sum_{k=0}^\infty t_k \psi_i^k, $$

where $t_i$ is a $H^*(M)$-valued co-ordinate. Introduce an infinite dimensional space $\mathcal{H}$ over $K[Q]$ with a symplectic form $\Omega_s$:

$$ \mathcal{H} := H^*(M) \otimes K\{\hbar, h^{-1}\}[Q], $$

$$ \Omega_s(f(h), g(h)) := \text{Res}_{h=0} \langle f(-h), g(h) \rangle^\nu dh, \quad f, g \in \mathcal{H}, $$

where $\langle \cdot, \cdot \rangle^\nu$ is the twisted Poincaré pairing $2,4$. The symplectic space $\mathcal{H}$ is decomposed into two isotropic subspaces $\mathcal{H}_+, \mathcal{H}_-$ as $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-:

$$ \mathcal{H}_+ = H^*(M) \otimes K(\hbar)[Q], \quad \mathcal{H}_- = h^{-1}H^*(M) \otimes K\{h^{-1}\}[Q]. $$

We call this a polarization of $\mathcal{H}$. By the polarization, we can regard $\mathcal{H}$ as the cotangent bundle $T^*\mathcal{H}_+$ of $\mathcal{H}_+$ with $\mathcal{H}_-$ being identified with a fiber of $T^*\mathcal{H}_+$.

We write a general element in $\mathcal{H}_+$ as

$$ q(h) := q_0 + q_1 \hbar + q_2 \hbar^2 + \cdots \in \mathcal{H}_+, $$

where $q_i$ is a $H^*(M)$-valued co-ordinate. We relate the co-ordinates $t_n$ to the co-ordinate $q_n$ by the following dilaton shift:

$$ q(h) = t(h) - \delta h, \quad i.e. \quad q_n = t_n - \delta_n h, $$

where $\delta_n$ are constants.
More explicitly, its graph defines a formal germ \( L \) in \( g \) and (5.1), the potential \( F \) \( \{ s \} \) \( \in \) where 1 \( \in \) \( H^s(M) \) is a basis of \( H^s(M) \) and we write \( t_n = \sum \delta_{ij} t^i \), \( J \) is a basis of \( H^s(M) \) \( \otimes \) \( K \{ h \} \{ Q \} \). The J-function in Definition 2.4 is considered to be a family of elements on the cone. In fact, by (2.13) and (5.1), the J-function \( J_s \) of the twisted theory satisfies

\[
-\hbar J_s(Q, \tau, -\hbar) = d_{-\hbar, +}F_s = L_0 \cap \{-\hbar + \tau + H_0\}.
\]

Thus, the derivatives of the J-function (recall that \( \tau = \sum_{i=0}^{s} t_i p_i \))

\[
-\hbar \frac{\partial}{\partial t^i} J_s(Q, \tau, -\hbar) = L(Q, \tau, -\hbar)^{-1} p_j
\]

are tangent vectors to \( L_0 \). Coates and Givental [4] observed that

- The tangent space \( \mathbb{L}_\tau \) of \( L_0 \) at \( d_{-\hbar, +}F_s \) a free module over \( K\{h\}\{Q\} \) generated by the derivatives (5.2) of the J-function:

\[
\mathbb{L}_\tau = L(Q, \tau, -\hbar)^{-1}(H^s(M) \otimes K\{h\}\{Q\}) \subset \mathcal{H}
\]

- We have \( \hbar \mathbb{L}_\tau \subset L_0 \) and the tangent space to \( L_0 \) at any point on \( \hbar \mathbb{L}_\tau \) equals \( \mathbb{L}_\tau \). The Lagrangian \( L_0 \) is ruled by these tangent spaces:

\[
L_0 = \bigcup_{\tau \in H^s(M)} \hbar \mathbb{L}_\tau
\]

(5.3)

In particular, \( L_0 \) is a Lagrangian cone. The family \( \{ \mathbb{L}_\tau ; \tau \in H^s(M) \} \) of subspaces in \( \mathcal{H} \) forms a semi-infinite variation of Hodge structures in the sense of Barannikov [2]. In [14], Givental explained that the above geometric properties of \( L_0 \) correspond to the string equation, the dilaton equation and the topological recursion relations for the genus zero Gromov-Witten invariants.

**Theorem 5.1** (Coates-Givental [4]). The Lagrangian cone \( L_0 \subset (\mathcal{H}, \Omega_0) \) of the twisted Gromov-Witten theory and that \( L_0 \subset (\mathcal{H}, \Omega_0) \) of the untwisted theory are related by a symplectic transformation \( \Delta : (\mathcal{H}, \Omega_0) \rightarrow (\mathcal{H}, \Omega_0) \):

\[
L_0 = \Delta L_0, \quad \Delta = c(V)^{-1/2} \exp \left( \sum_{m \geq 0} s_{2m-1} \frac{B_{2m}}{(2m)!} \text{ch}_1(V) \hbar^{2m-1} \right),
\]

where we set \( s_{-1} = 0 \) and Bernoulli numbers \( B_{2m} \) are defined by \( x/(1 - e^{-x}) = x/2 + \sum_{m=0}^{\infty} B_{2m} x^{2m}/(2m)! \).

The J-function is obtained from the cone \( L_0 \) as a slice \( L_0 \cap \{-\hbar + \tau + H_0\} \) and its derivatives (5.2) recover the fundamental solution \( L \) of the big QDM. The Dubrovin connection and the quantum product can be obtained from the
differential equations (2.8), (2.9) for $L$, so the big QDM with its canonical frame is determined by the cone $L_s$.

Let $V$ be the sum $V_1 ⊕ ⋯ ⊕ V_n$ of line bundles and $c$ be the $S^1$-equivariant Euler class $e$ (2.6). In this case, Coates and Givental found another family of elements lying on the cone $L_s$. Recall that the ground ring $K$ is $\mathbb{C}(\lambda^{-1})$ when $c = e$. Set $v_i := c_i(V_i)$ and let $J(Q, \tau, h) = \sum_{d \in A} J_d(\tau, h)Q^d$ be the untwisted $J$-function of $M$. The hypergeometric modification $I^{tw}(Q, \tau, h)$ of the $J$-function is defined to be

$$I^{tw}(Q, \tau, h) := \sum_{d \in A} \prod_{i=1}^l \frac{\Gamma((v_i)_d + (v_i + \nu h + \lambda))}{\Gamma((v_i)_d)} \cup J_d(\tau, h)Q^d. \quad (5.4)$$

**Theorem 5.2** (Coates-Givental [3]). The family $\tau \mapsto -hI^{tw}(Q, \tau, -h)$ is lying on the Lagrangian cone of the $(V, e)$-twisted Gromov-Witten theory of $M$.

As we see below, the hypergeometric modification $I^{tw}$ recovers the cone $L_s$ as $J_s$ does. The above theorem and (5.3) imply that

$$-hI^{tw}(Q, \tau, -h) = -hL(Q, \hat{\tau}, -h)^{-1}v(Q, \tau, -h) \in h\mathbb{L}_S$$

for some $v \in H^*(M) \otimes K\{\hat{h}\}[Q, \hat{t}]$ and $\hat{\tau} = \hat{\tau}(Q, \tau) \in H^*(M) \otimes K\{\hat{Q}, \hat{t}\}$. Because $L(0, \hat{\tau}, h) = e^{\hat{\tau}/h}$ and $I^{tw}(0, \tau, h) = e^{\tau/h}$, we have $v(0, \tau, h) = e^{(\tau - \hat{\tau}(h, \tau))/h}$. But $v$ does not contain negative powers in $h$, so it follows that $\hat{\tau}(0, \tau) = \tau$ and $v(0, \tau, h) = 1$. Hence, $\tau \mapsto \hat{\tau} = \hat{\tau}(Q, \tau)$ is an invertible co-ordinate change on $H^*(M)$. The derivatives of $I^{tw}$

$$-h\frac{\partial}{\partial \hat{t}}I^{tw}(Q, \tau, -h) = L(Q, \hat{\tau}, -h)^{-1}\nabla^{-h}_j v(Q, \tau, -h),$$

where $\nabla^{-h}_j = -h(\partial/\partial \hat{t}^j) + (\partial \hat{\tau}/\partial \hat{t}^j)^*$ is the Dubrovin connection, generate $\mathbb{L}_S$ over $K\{\hat{h}\}[Q, \hat{t}]$. Therefore the derivatives of $I^{tw}$ determines the cone $L_s$ by (5.3). These imply the following corollary (c.f. Proposition 2.6, Proposition 1.5):

**Corollary 5.3.** (1) Let $B$ and $\mathcal{D}_B$ be as in Section 4. Let $\mathcal{D}_B$ act on $H^*(M) \otimes K\{\hat{h}, \hat{h}^{-1}\}[Q, \hat{t}]$ by $\mathcal{P}_a \mapsto hQ^a \partial/\partial Q^a + p_a$ and $\varphi \mapsto h\partial/\partial \hat{t}$. The $\mathcal{D}_B$-module generated by $I^{tw}(Q, \tau, h)$ (where $\tau = \sum_{i=0}^s \hat{t}^i p_i$) is isomorphic to QDM$_e(M, V)$ under some co-ordinate change $\hat{\tau} = \hat{\tau}(Q, \tau)$. Here, $\hat{\tau}$ is the natural $H^*(M)$-valued co-ordinate on the base of the QDM. We have $\hat{\tau}(0, \tau) = \tau$ and the isomorphism $\mathcal{D}_B I^{tw} \cong QDM_e(M, V)$ sends $I^{tw}$ to a section $v(Q, \hat{\tau}, h)$ such that $v(0, \hat{\tau}, h) = 1$.

(2) Via the isomorphism in (1), $\mathcal{E}^h := \mathcal{D}_B I^{tw}$ becomes a big AQDM having a canonical frame $\Phi$ induced from the standard trivialization of QDM$_e(M, V)$. The frame $\Phi_0 : \mathcal{E}_{b, 0} \otimes K\{\hat{h}\} \rightarrow \mathcal{E}_{b, 0}$ induced by $\Phi$ satisfies $\Phi_0[I^{tw}] = [I^{tw}]$, where $b = \{\tau = 0\} \in B$. The co-ordinate $\hat{\tau} = \hat{\tau}(Q, \tau)$ are obtained as flat co-ordinates associated with $\Phi$ and $e_0 = [I^{tw}] \in \mathcal{E}_{0, b}$.

**Proof.** (1) is clear from the preceding discussion. (2) follows from that the classes $[I^{tw}]$ of $I^{tw}$ in both $\mathcal{E}_{b, 0}$ and $\mathcal{E}_{0, b}$ correspond to the same element $v(0, \hat{\tau}, h) = 1 = v(0, \hat{\tau}, 0)$ on the QDM side. \qed
5.2. Cones of big AQDMs. Here we study a relationship between cones and general big AQDMs. We use the notation in Section 4. Let $\mathcal{E}^h$ be a big AQDM on the base $B$ with a base point $b$. Let $t^i, 0 \leq i \leq s$ be co-ordinates centered at $b$. Here we take the infinite dimensional space $\mathcal{H}$ to be

$$\mathcal{H} = \mathcal{E}_{b,0} \otimes K\{h, h^{-1}\}[Q]$$

**Definition 5.4.** Let $\Phi$ be a frame of $\mathcal{E}^h$ and $L = L(Q, t, h)$ be the fundamental solution (Proposition 4.3) associated with an $O$-valued point $b$ on $B$ and $\Phi$. A cone $\mathcal{L} \subset \mathcal{H}$ of the big AQDM $\mathcal{E}^h$ is defined to be the following subset of $\mathcal{H}$:

$$\mathcal{L} := \bigcup_{b' \in B} h \mathcal{L}_{Q,0}(b') \quad \text{where} \quad \mathcal{L}_{Q,0}(b') := L(Q, t(b'), -h)^{-1}(\mathcal{E}_{b,0} \otimes K\{h\}[Q]) \subset \mathcal{H}.$$ 

Here, $b'$ ranges over the set of $O$-valued points in $B$. This depends on the choice of a base point $b$ and a frame $\Phi$.

**Proposition 5.5.** Let $\mathcal{E}^h$ be a big AQDM endowed with a frame $\Phi: \mathcal{E}_{b,0} \otimes \mathcal{O}^h_B \to \mathcal{E}^h$. Let $t^i, 0 \leq i \leq s$ be co-ordinates centered at $b$. Assume that the connection operators $A_a(Q, t, h), \Omega_i(Q, t, h) \in \text{End}(\mathcal{E}_{b,0} \otimes \mathcal{O}^h_B)$ associated with $\Phi$ satisfy

$$A_a(0, t, h) = p_a, \quad \Omega_i(0, t, h) = \varphi_i.$$ 

Let $b'$ be an $O$-valued point and $\Phi': \mathcal{E}_{b',0} \otimes \mathcal{O}^h_B \to \mathcal{E}^h$ be a frame such that for $G := \Phi^{-1} \circ \Phi' \in \text{Hom}(\mathcal{E}_{b',0}, \mathcal{E}_{b,0}) \otimes \mathcal{O}^h_B$, $G_0 := G|_{Q=0}$ is independent of $t$. Then the cone $\mathcal{L}'$ associated with $b'$ and $\Phi'$ is related to the cone $\mathcal{L}$ associated with $b$ and $\Phi$ as

$$\mathcal{L}' = G_0(-h)^{-1} \exp \left( -\frac{1}{h} \sum_{i=0}^{s} c^i \varphi_i \right) \mathcal{L},$$

where $c^i = t^i(b')|_{Q=0}$.

**Proof.** Let $L(Q, t, h)$ be the fundamental solution associated with $b$ and $\Phi$. By the assumption, we have $L(0, t, h) = \exp(\sum_{i=0}^{s} t^i \varphi_i/h)$. We claim that the fundamental solution $L'$ associated with $\Phi'$ and $b'$ is given by (c.f. (4.11))

$$L'(Q, t, h) = G(Q, t, h)^{-1}L(Q, t, h) \exp(-\sum_{i=0}^{s} c^i \varphi_i/h)G_0(h)$$

This $L'$ satisfies the initial condition $L'(Q, t(b'), h)|_{Q=0} = \text{id}$. Let $A'_a, \Omega'_i$ be the connection operators associated with $\Phi'$. Then by (4.3), $A'_a|_{Q=0} = G^{-1}A_a|_{Q=0}G_0 = G_0^{-1}p_aG_0$ is independent of $t$. This implies $p_a = A'_a|_{Q=0} = G^{-1}p_aG_0$, i.e., $G_0$ commutes with $p_a$. By this and (4.3), it easily follows that $(h \partial/\partial t^i + \Omega'_i)L' = 0$ and $(h(Q^a \partial/\partial Q^a) + A'_a)L' + L'p_a = 0$. Now the conclusion follows from the definition of $\mathcal{L}$ and $\mathcal{L}'$. 

The symplectic transformation $\triangle$ in Theorem 5.1 can be interpreted as the combination of a gauge transformation $G_0$ at $Q = 0$ and a shift of the $K$-valued origin $b|_{Q=0}$. Note that the symplectic transformation does not change the $D_B$-module itself. By Theorem 5.1 and Proposition 5.3, we see that the twist of the
quantum cohomology corresponds to the shift of the origin by \(-\sum_{l \geq 1} s_{l-1} \text{ch}_l(V)\) and the gauge transformation at \(Q = 0:\)

\[
G_0(\hbar) = c(V)^{1/2} \exp \left( \sum_{m \geq 1, l \geq 0} s_{2m-1+l} \frac{B_{2m}}{(2m)!} \text{ch}_l(V) \hbar^{2m-1} \right).
\]

5.3. Main Theorem. We consider the following condition on a smooth projective toric variety \(X\).

**Condition 5.6.** There exists a smooth projective toric variety \(Y\) such that \(c_1(Y)\) is nef and \(X\) is a complete intersection of nef toric divisors \(D_1, \ldots, D_k\) in \(Y\).

We give a sufficient condition for Condition 5.6 to hold.

**Proposition 5.7.** Assume that there exist nef toric divisors \(D'_1, \ldots, D'_l\) in \(X\) and integers \(n_1, \ldots, n_l \geq 0\) such that \(c_1(X) + \sum_{i=1}^l n_i [D'_i]\) is nef. Then \(X\) satisfies Condition 5.6.

**Proof.** Recall the construction of a toric variety \(X\) in Section 3.1. We use the same notation for \(X\) as there. The first Chern class of \(X\) is given by the sum \(c_1(X) = \sum_{i=1}^N u_i\) of toric divisor classes. Without loss of generality, we can assume \(u_i = [D'_i]\) for \(1 \leq i \leq l\). By the assumption, \(c_1(X) + \sum_{i=1}^l n_i u_i\) are in the closure \(\overline{C}_X\) of the Kähler cone \(C_X(5.2)\) and so are \(u_i, 1 \leq i \leq l\). We define a toric variety \(Y\) by the following data (see Section 3.1):

1. The same algebraic torus \(\mathbb{T}_C \cong (\mathbb{C}^*)^r\).
2. \((N + \sum_{i=1}^l n_i)\)-tuple of integral vectors \((u_1, \ldots, u_i, \ldots, u_l, u_{l+1}, \ldots, u_N)\) in \(\text{Hom}(\mathbb{T}_C, \mathbb{C}^*)\) where each \(u_i\) is repeated \(n_i + 1\) times for \(1 \leq i \leq l\).
3. The same vector \(\eta \in \text{Hom}(\mathbb{T}_C, \mathbb{C}^*) \otimes \mathbb{R}\).

It is easy to check that these data satisfy the conditions (A)–(C) in Section 3.1 so \(Y\) is a smooth projective toric variety. We have \(H^2(Y, \mathbb{Z}) \cong H^2(X, \mathbb{Z})\) and \(C_Y \cong C_X\). By the construction, \(c_1(Y)\) is nef and \(X\) is a complete intersection of \((\sum_{i=1}^l n_i)\) nef toric divisors in \(Y\).

By using the same method as above, we can show that every smooth projective toric variety \(X\) can be realized as a complete intersection of not necessarily nef toric divisors in a smooth projective Fano toric variety \(Y\).

**Theorem 5.8** (Main theorem). Let \(X\) be a smooth projective toric variety satisfying Condition 5.6 and \(V\) be a sum of line bundles on \(X\). By the generalized mirror transformation in Definition 4.12, the equivariant Floer cohomology \(\mathcal{E}_h = FH_{T_2}(L_{X/V})\) together with the choice of a frame \(\Phi_0 = \text{Loc}_{e_0}^{-1}\) of \(\mathcal{E}_h\) and \(e_0 = [\Delta] \in \mathcal{E}_0\) reconstructs the big QDM \(QDM_e(X, V)\) twisted by \(V\) and the \(S^1\)-equivariant Euler class \(e\).

**Remark.** In the previous preprint (version 4) [18], the main theorem was stated for an arbitrary smooth projective toric variety. The proof there contained technical mistakes when \(X\) is a complete intersection of not nef toric divisors in a Fano toric variety \(Y\). Here, we impose Condition 5.6 on \(X\) instead. The author, however, hopes that this condition will be removed in the future.
The rest of Section 5.3 is devoted to the proof of Theorem 5.8. Let \( \mathcal{V} \) be a sum \( \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_t \) of line bundles over \( X \) and \( v_i = c_1(\mathcal{V}_i) \in H^2(X) \). See Section 4.4 for the notation on toric varieties and Section 2 for that on QDMs.

5.3.1. When \( X \) is Fano. By Givental’s mirror theorem \([12]\), we know that for a Fano toric variety \( X \), the small \( J \)-function \( J_X(Q,h) \) coincides with the \( I \)-function \( I_X(Q,h) \) of \( X \) itself, where \( I_X \) is (3.6) with \( \mathcal{V} = 0 \). Let \( I^w_\mathcal{V}(Q,\tau,h) \) denote the hypergeometric modification (5.4) of the big \( J \)-function of \( X \) with respect to \( \mathcal{V} \). Then it is immediate to see that

\[
I^w_\mathcal{V}(Q,0,h) = I_{X,\mathcal{V}}(Q,h)
\]

where the right-hand side is the \( I \)-function in (3.6). Let \( B \) be the formal neighborhood of zero in \( H(x) \otimes \mathcal{O} \) and take the ground ring \( K \) to be \( \mathbb{C}((\lambda^{-1})) \). The right-hand side generates \( \mathcal{E}^h := FH_{T^2}(L_X/\mathcal{V}) \) over the Heisenberg algebra \( \mathcal{D} \) by Proposition 4.3. On the other hand, the \( \mathcal{D}_B \)-module \( \mathcal{F}^h \) generated by \( I^w_\mathcal{V} \) is isomorphic to \( QDM_\mathcal{F}_e(X,\mathcal{V}) \) by Corollary 5.3. By the same corollary, the class \( [I^w_\mathcal{V}] \) in \( \mathcal{F}_{0,b} \) corresponds to \( 1 \in H^*(X) \), so generates \( \mathcal{F}_{0,b} \) as a \( K[p_1,\ldots,p_r] \)-module. Then by Proposition 4.5, \( \mathcal{F}^h \) is generated by \( I^w_\mathcal{V} \) as an \( \mathcal{O}^h_B(\{P_1,\ldots,P_r\}) \)-module and the restriction of \( \mathcal{F}^h \) to \( \tau = 0 \) is generated by \( I^w_\mathcal{V}(Q,0) \) as a \( \mathcal{D} \)-module. Therefore, \( \mathcal{F}^h|_{\tau=0} \cong \mathcal{E}^h \) as \( \mathcal{D} \)-modules. By the uniqueness of the reconstruction in Theorem 4.9, \( \mathcal{F}^h \) must be isomorphic to the big AQDM \( \mathcal{E}^h \) reconstructed from \( \mathcal{E}^h \). Finally we check that the frame \( Loc^{-1}_0 \) of \( \mathcal{E}^h_0 \) and \( e_0 = [\Delta] \in \mathcal{E}_0 \) correspond to the standard trivialization of the QDM at \( Q = 0 \) and the unit \( 1 \in H^*(X) \). The class \( [\Delta] \in \mathcal{E}_0 \) corresponds to \( [I^w_\mathcal{V}] \in \mathcal{F}_{b,0} \), so corresponds to \( 1 \in H^*(X) \) by Corollary 5.3. By the same corollary, the frame \( \Phi_0 \) of \( \mathcal{E}^h \) induced from the standard trivialization of \( QDM_e(X,\mathcal{V}) \) satisfies \( \Phi_0[\Delta] = [\Delta] \). Since \( \Phi_0 \) is a homomorphism of \( K\{\hbar\}[p_1,\ldots,p_r] \)-modules \([12]\), this completely determines \( \Phi_0 \). By Theorem 5.5, \( Loc^{-1}_0 \) satisfies the same properties as \( \Phi_0 \), thus \( \Phi_0 = Loc^{-1}_0 \).

5.3.2. From \( \mathcal{V} = 0 \) to \( \mathcal{V} \neq 0 \). We will show that Theorem 5.8 holds for a pair \((X,\mathcal{V})\) when it holds for \( X \) itself. We will assume that \( \mathcal{V} \) is a line bundle and set \( v = c_1(\mathcal{V}) \). The proof in the case where \( \mathcal{V} \) is a sum of line bundles requires only notational changes. By assumption, \( FH^*_{T^2}(L_X) \) is isomorphic to the restriction of \( QDM(X) \) to some \( c(Q) \in H^*(X) \otimes \mathbb{C}[Q] \) such that \( c(0) = 0 \). By an argument similar to that showing (4.26), we have

\[
I_X(Q,h) = f(Q,hQ\frac{\partial}{\partial Q} + p,h)J_X(Q,c(Q),h)
\]

for some \( f(Q,P,h) \in \mathbb{C}[\hbar][Q][P] \) such that \( f(0,P,h) = 1 \). Set

\[
M_d(x) = \prod_{\nu = -\infty}^{\nu = \infty} \frac{(x + \lambda + \nu \hbar)}{(x + \lambda + \nu \hbar)} \in \lambda^{(\nu,d)} \mathbb{C}[\hbar,x][\lambda^{-1}] .
\]

Put \( v = \sum_{a=1}^v v^a p_a \) and \( Q^a_\lambda := Q^a \lambda^{-a} \). We define the modification of a general cohomology-valued \( Q \)-series \( \sum_d Q^d H_d(t,h) \in H^*(X) \otimes \mathbb{C}(\hbar^{-1})[Q,t] \) to be

\[
\text{modif}(H) := \sum_d Q^d M_d(v) H_d(t,h) \in H^*(X) \otimes \mathbb{C}(\hbar^{-1})[\lambda^{-1},Q_\lambda,t].
\]
We consider a submodule $W$ of $\mathbb{C}(\mathcal{H})[\lambda^{-1}, Q, \lambda]$. For example, we have $I_X \mathcal{V} = \text{modif}(I_X)$ and $I^{tw}_\mathcal{V} = \text{modif}(J_X)$. Set $\partial_{\lambda} = \partial / \partial \lambda^s$ and $\partial_t = \partial / \partial t^s$. The modification of a differential operator $g = \sum_d Q^d g_d(t, h \partial + p, h \partial, h)$ in $\mathbb{C}[h, h \partial + p][t][\partial][Q]$ is defined to be

$$\text{modif}(g) := \sum_d Q^d M_d(h \partial_e + v) g_d(t, h \partial, h \partial, h) \in \mathbb{C}[h, h \partial + p][\lambda^{-1}, t][\partial][Q_L]$$

where $\partial_v = \sum_{a=1}^r v^a \partial_a$.

**Lemma 5.9.** Let $g_1, g_2$ be differential operators in $\mathbb{C}[h, h \partial + p][t][\partial][Q]$ and $H$ be a cohomology-valued power series in $H^i(X) \otimes \mathbb{C}(\mathcal{H})[Q, t]$. We have

$$\text{modif}(g_1 g_2) = \text{modif}(g_1) \text{modif}(g_2), \quad \text{modif}(g_1 H) = \text{modif}(g_1) \text{modif}(H).$$

**Proof.** We expand $g_1 = \sum_d Q^d g_1,d$ and $g_2 = \sum_d Q^d g_2,d$, where $g_i,d \in \mathbb{C}[h, h \partial + p][t][\partial][Q]$. By using $M_{d_1+d_2}(x) = M_{d_1}(h(v, d_2) + x) M_{d_2}(x)$,

$$\text{modif}(g_1 g_2) = \sum_d Q^d M_d(h \partial_e + v) \sum_{d_1+d_2} g_1,d_1 g_2,d_2$$

$$= \sum_{d_1, d_2} Q^d_1 Q^d_2 M_{d_1}(h(d_2, v) + h \partial_v + v) g_1,d_1 M_{d_2}(h \partial_e + v) g_2,d_2$$

$$= \text{modif}(g_1) \text{modif}(g_2).$$

This proves the first equation. The proof of the second equation is similar. \(\square\)

Put $c(Q) = \sum_{i=0}^s c^i(Q)p_i$. By using (5.5) and Lemma 5.9 we have

$$I_{X, \mathcal{V}}(Q, h) = \text{modif}(I_X(Q, h)) = \text{modif}(f) \text{modif}(J_X(Q, c(Q), h)) = \text{modif}(f) e^{\sum_{i=0}^s \text{modif}(c^i(Q) \partial)} I^{tw}_{\mathcal{V}}(Q, \tau, h) \bigg|_{\tau=0}$$

where we used $J_X(Q, c(Q), h) = \exp(\sum_{i=0}^s c^i(Q) \partial_i) J_X(Q, \tau, h) |_{\tau=0}$ in the third line. By Corollary 5.3 it follows that the $D_B$-module generated by $I^{tw}_{\mathcal{V}}$ has the following free basis over $\mathcal{O}^h_B = \mathbb{C}[h][\lambda^{-1}] [Q, t]$: $h \partial_0 I^{tw}_{\mathcal{V}} = I^{tw}_{\mathcal{V}}, h \partial_1 I^{tw}_{\mathcal{V}}, \ldots, h \partial_s I^{tw}_{\mathcal{V}}$.

We consider a submodule $\mathcal{F}^h$ generated by these elements over $\mathcal{O}^h_B := \mathbb{C}[h][\lambda^{-1}, Q, \lambda, t]$. This is invariant under the action of $h \partial_a + p_a$ and $h \partial_t$, so has the structure of a $D_B := \mathcal{O}^h_B(h \partial + p, h \partial_t)$-module. In fact, the connection matrices $A_a = (A^j_{ak})$, $\Omega_i = (\Omega^j_{ik})$ corresponding to $h \partial_a + p_a$ and $h \partial_t$ are given by

$$(h \partial_a + p_a) h \partial_k I^{tw}_{\mathcal{V}} = \sum_{j=0}^s (h \partial_j I^{tw}_{\mathcal{V}}) A^j_{ak}, \quad h \partial h (h \partial_k I^{tw}_{\mathcal{V}}) = \sum_{j=0}^s (h \partial_j I^{tw}_{\mathcal{V}}) \Omega^j_{ik}$$

It is obvious that the matrix entries $A^j_{ak}, \Omega^j_{ik}$ belong to $\mathcal{O}^h_B$, but from $I^{tw}_{\mathcal{V}} \in H^i(X) \otimes \mathbb{C}(\mathcal{H})[\lambda^{-1}, Q, \lambda, t]$ and $I^{tw}_{\mathcal{V}} |_{\lambda=0} = \exp(\tau / h)$, it follows that they belong
also to $O^h_B$. Put $\nabla^h_a = h\partial_a + A_a$ and $\nabla^h_i = h\partial_i + \Omega_i$. In the basis $(5.7)$, the element $e^{\sum_{i=0}^s \text{modif}(c^i(Q)\partial_i)} I^w_{\nu}(Q,\tau,h)$ is represented by

$$\exp(g/h)[1,0,\ldots,0]^T, \quad \text{where} \quad g = \sum_{i=0}^s \sum_d Q^d M_d(\nabla^h_v) c^i_d \nabla^h_i. \quad (5.8)$$

Here $c^i(Q) = \sum_d Q^d c^i_d$ and $[1,0,\ldots,0]^T$ represents $I^w_{\nu}$.

**Lemma 5.10.** Let $g$ be a differential operator in $\mathbb{C}[h,\nabla^h][\lambda^{-1},t][\nabla^h][Q_\lambda]$ such that $g|_{Q_\lambda=0} = 0$. Then there exist a differential operator $h$ in the same ring and $c^i(Q,\tau) \in O_B := \mathbb{C}[\lambda^{-1},Q_\lambda,t]$ such that $h|_{Q_\lambda=0} = 0$, $c^i(0,\tau) = 0$,

$$\exp(g/h) = \exp\left(\frac{1}{h} \sum_{i=0}^s c^i(Q,\tau) \nabla^h_i\right) \exp(h/h)$$

and that the action of $h/h$ preserves the module $\mathcal{F}^h \cong \mathbb{C}^{s+1} \otimes O^h_B$.

**Proof.** We assume by induction that there exist $h_n \in \mathbb{C}[h,\nabla^h][\lambda^{-1},t][\nabla^h][Q_\lambda]$ and $c^i_n(Q,\tau) \in O_B$ such that

$$\exp(g/h) = \exp\left(\frac{1}{h} \sum_{i=0}^s c^i_n(Q,\tau) \nabla^h_i\right) \exp(h_n/h) \quad (5.9)$$

and that

$$\frac{h_n}{h} \mathcal{F}^h \subset \mathcal{F}^h + \frac{1}{h} \sum_{|d|>n} Q^d \mathcal{F}^h. \quad (5.10)$$

Here, we identify $\mathcal{F}^h$ with $\mathbb{C}^{s+1} \otimes O^h_B$ and $|d| = \sum_{a=1}^s (p_a,d)$. When $n = 0$, we can take $c^0_0 = 0$. We have for $h_n = h_n(Q,t,\nabla^h,\nabla^h,h)$,

$$\text{Res}_{h=0} dh((h_n/h)v) = h_n(Q,t,A_a|_{h=0},\Omega_i|_{h=0},0)v(Q,t,0),$$

where any $v = v(Q,t,h) \in \mathbb{C}^{s+1} \otimes O^h_B$. By the induction hypothesis,

$$[w_1(Q,t),\ldots,w_s(Q,t)]^T := h_n(Q,t,A_a|_{h=0},\Omega_i|_{h=0},0)[1,0,\ldots,0]^T$$

is in $\sum_{|d|>n} Q^d \mathbb{C}^{s+1} \otimes O_B$. Since $\Omega_i[1,0,\ldots,0]^T = [0,\ldots,0,1,0,\ldots,0]^T$, where $1$ is in the $i$-th position, we know that the application of

$$h_n(Q,t,A_a|_{h=0},\Omega_i|_{h=0},0) - \sum_{i=0}^s w^i(Q,t) \Omega_i|_{h=0}$$

to the vector $[1,0,\ldots,0]^T$ is zero. Because $A_a|_{h=0}$ and $\Omega_i|_{h=0}$ commute with each other (by the flatness of $\nabla^h$ and $\nabla^h$) and because $\mathbb{C}^{s+1} \otimes O_B$ is generated by $[1,0,\ldots,0]^T$ as a $\mathbb{C}[A_a|_{h=0}][\lambda^{-1},t][\Omega_i|_{h=0}][Q_\lambda]$-module, we know that the above endomorphism is identically zero. Put $c^i_{n+1}(Q,t) = c^i_n(Q,t) + w^i(Q,t)$. We define $\nabla_{c_n} := (1/h)\nabla^h_{c_n} := (1/h)\sum_{i=0}^s c^i_n \nabla^h_i$. We define $\nabla_{w_n}, \nabla_w$ and $\nabla_{c_{n+1}}$ similarly.
By Baker-Campbell-Hausdorff formula, we have
\[
\exp(-\nabla_{c_{n+1}}) \exp\left(\frac{g}{\hbar}\right) = \exp(-\nabla_{c_n} + \frac{g}{\hbar} - \frac{1}{2}(\nabla_{c_n}^2, \frac{g}{\hbar}) + \cdots)
\]
\[
= \exp((-\nabla_{c_n} + \frac{g}{\hbar} - \frac{1}{2}(\nabla_{c_n}, \frac{g}{\hbar}) + \cdots) - (\nabla_w + \frac{1}{2} \nabla_w, \frac{g}{\hbar}) + \cdots))
\]
\[
= \exp((h_n - \nabla_w^h - R_n)/\hbar).
\]
In the third line, we used the Baker-Campbell-Hausdorff formula for \( e^{h_n/\hbar} = e^{-\nabla c_n e^{g/\hbar}} \) and \( R_n/\hbar = \frac{1}{2} \nabla_w, \frac{g}{\hbar} + \frac{1}{12} \nabla_w, \frac{g}{\hbar} + \cdots \) is the remainder. Because the space \( \frac{1}{\pi} \mathbb{C}[\hbar, \nabla^h]\{\lambda^{-1}, t\} \nabla^h_\lambda \mathbb{C}[\hbar, \nabla^h]\) is closed under the commutator \([\cdot, \cdot]\), \( R_n/\hbar \) belongs to this space. Since \( e^v \in \mathcal{O}_B \) and \( g|_{Q, \tau = 0} = 0 \), it follows that \( R_n \in \sum_{d > n+1} Q^d \mathbb{C}[\hbar, \nabla^h]\{\lambda^{-1}, t\} \nabla^h_\lambda \mathbb{C}[\hbar, \nabla^h]\). Now \( \mathcal{c}_n^\tau = h_n - \nabla^h_w - R_n \) satisfy (5.9) and (5.10). By construction, \( \mathcal{c}_n \) and \( h_n \) converges in \( \mathcal{Q}^{-}\)-adic topology and the conclusion follows. □

By (5.8) and Lemma 5.10 there exist \( \mathcal{c}'(Q, \tau) \in \mathcal{O}_B \) and \( v^i(Q, \tau) \in \mathcal{O}_B \) such that \( \mathcal{c}'(0, \tau) = 0, v^i(0, \tau) = \delta_{i0}, \)
\[
\mathcal{c}^e \sum_{i=0}^s \text{mod}(c^e(Q) \partial_i) I^w_{\tau^*}(Q, \tau, h) = \mathcal{c}^e \sum_{i=0}^s \mathcal{c}(Q, \tau) \partial_i \left( \sum_{i=0}^s v^i(Q, \tau, h) \partial_i I^w_{\tau^*}(Q, \tau, h) \right)
\]
\[
= \sum_{i=0}^s v^i(Q, \tau^*)(\partial_i I^w_{\tau^*})(Q, \tau^*, h)
\]
where \( \tau^* = \tau^e(Q, \tau) = e^\sum_{i=0}^s \mathcal{c}^e(Q, \tau) \partial_i \tau \). From what we have discussed, the \( \mathcal{D}_B^\mathcal{Q} \)-module generated by \( I^w_{\tau^*} \) is considered to be a big AQDM over the ground ring \( \mathcal{K} = \mathbb{C}\{\lambda^{-1}\} \) (with \( Q \) replaced with \( Q_\lambda \)) and is isomorphic to a \( \mathcal{D}_B \)-submodule of \( QDM_{\mathcal{Q}}(X, \mathcal{V}) \). Under this isomorphism, \( I^w_{\tau^*}(\tau^e(Q, 0), \hbar) \) corresponds to a section \( v^e(Q, \tau) \) of the QDM on a locus \( \tau^e = \tau^e(Q) \in H^e(X) \otimes \mathbb{C}[\lambda^{-1}, Q_\lambda] \) such that \( v^e(0, \hbar) = 1 \) and \( \tau^e(0) = 0 \). Thus by Proposition 4.5, there exists a differential operator \( f^e(Q, \hbar \partial p + p, \hbar) \in \mathbb{C}[\hbar][\lambda^{-1}, Q_\lambda][h \partial + p] \) such that \( f^e(0, \hbar \partial p + p, \hbar) = 1 \) and
\[
\sum_{i=0}^s v^i(Q, \tau^*)(h \partial_i I^w_{\tau^*})(Q, \tau^*, \hbar)|_{\tau^e} = f^e(Q, \hbar \partial p + p, \hbar) I^w_{\tau^*}(Q, \tau^*(Q, 0), \hbar).
\]
Then by (5.10), for \( f'' = \text{mod}(f)^e f', \)
\[
I_{X, \mathcal{V}}(Q, h) = f''(Q, \hbar \partial p + p, \hbar) I^w_{\tau^*}(Q, \tau^*(Q, 0), \hbar)
\]
(5.11)
The differential operator \( f'' \) belongs to \( \mathcal{D} = \mathbb{C}[\hbar, \hbar \partial p + p][\lambda^{-1}, Q_\lambda] \) which is considered to be a certain completion of \( \mathcal{D} = \mathbb{C}[\hbar][\lambda^{-1}, Q_\lambda](\hbar \partial p + p) \). It is easy to show that a small AQDM (over \( \mathcal{K} = \mathbb{C}\{\lambda^{-1}\} \)) has the structure of a \( \mathcal{D} \)-module. Thus, by Corollary 5.3 the above formula means that \( I_{X, \mathcal{V}} \) corresponds to a section \( v'' \) of \( QDM_{\mathcal{Q}}(M, \mathcal{V}) \) on the locus \( \tau^e(Q) \) such that \( v''|_{Q, \tau = 0} = 1 \). Thus by Proposition 4.5 the \( \mathcal{D} \)-module generated by \( I_{X, \mathcal{V}}(Q, h) \) is isomorphic to the restriction of \( QDM_{\mathcal{Q}}(X, \mathcal{V}) \) to the subspace \( \{ \tau' = \tau^e(Q) \} \). Now the conclusion follows from that \( F H_{I_{X, \mathcal{V}}}(L_{X, \mathcal{V}}) \) is generated by \( I_{X, \mathcal{V}} \) and the uniqueness of the reconstruction
in Theorem 4.9. A routine argument shows that Loc\(^{-1}\) and \([\Delta]\) corresponds to the standard trivialization of the QDM and the unit \(1 \in H^*(X)\).

5.3.3. When \(V = 0\). By Condition 5.6 we embed \(X\) into \(Y\) as a complete intersection of nef toric divisors \(D_1, \ldots, D_k\). Without loss of generality, we can assume that \(X\) and \(Y\) are given as the GIT quotients \(\mathbb{C}^N / \mathbb{T}_\mathbb{C}\) and \(\mathbb{C}^{N+k} / \mathbb{T}_\mathbb{C}\) respectively and that the embedding \(i: X \hookrightarrow Y\) is induced from the inclusion of the co-ordinate subspace \(\mathbb{C}^N \subset \mathbb{C}^{N+k}\) consisting of first \(N\) factors. Let \(z_1, \ldots, z_{N+k}\) be the standard co-ordinates on \(\mathbb{C}^{N+k}\) and \(u_i \in H^2(Y)\) be the class of the toric divisor \(\{z_i = 0\}\). Then we have \(u_{N+i} = [D_i]\) for \(1 \leq i \leq k\). In particular, \(u_{N+1}, \ldots, u_{N+k}\) are nef. Because the cohomology rings of \(X\) and \(Y\) are generated by toric divisor classes, the restriction map \(i^*: H^*(X) \to H^*(Y)\) is surjective. Thus, the push-forward \(i_*: H_*(X) \to H_*(Y)\) is injective. Let \(p_1, \ldots, p_{r+l}\) be a nef integral basis of \(H^2(Y)\) such that \(i^*p_{r+1} = \cdots = i^*p_{r+l} = 0\) and that \(i^*p_1, \ldots, i^*p_r\) form a nef integral basis of \(H^2(X)\). Introduce Novikov variables \(Q^1, \ldots, Q^{r+l}\) dual to \(p_1, \ldots, p_{r+l}\). We use

\[
Q^d = (Q^1)^{(p_1,d)} \cdots (Q^{r+l})^{(p_{r+l},d)}
\]

for \(d \in H_2(Y)\),

\[
Q^d = (Q^1)^{(i^*p_1,d)} \cdots (Q^{r+l})^{(i^*p_{r+l},d)}
\]

for \(d \in H_2(X)\).

Let \(W\) denote the sum of nef line bundles on \(Y\):

\[
W = O(D_1) \oplus \cdots \oplus O(D_k).
\]

**Lemma 5.11.** Let \(J_Y;W(Q, \tau, h)\) be the \(J\)-function of the \((W, e)\)-twisted quantum cohomology of \(Y\). Let \(J_X;W(Q, \tau, h)\) be the \(J\)-function of the quantum cohomology of \(X\). Then we have

\[
\lim_{\lambda \to 0} e(W) \cup J_Y;W(Q, \tau, h) = i_*J_X(Q, i^*\tau, h).
\]

**Proof.** This follows from Theorem 2.1. \(\square\)

Let \(J_Y(Q, \tau, h)\) be the \(J\)-function of \(Y\) and \(I_Y;W(Q, \tau, h)\) be the hypergeometric modification (5.4) of \(J_Y\) with respect to \(W\). Let \(I_Y;W(Q, h)\) be the \(I\)-function (3.6) of \((Y, W)\). Since \(c_1(Y)\) is nef, by Givental’s mirror theorem [12], there exist \(f_i(Q) \in \mathbb{C}[Q]\), \(1 \leq i \leq r + l\) such that \(f_i(0) = 0\) and

\[
J_Y(Q, c(Q), h) = I_Y(Q, h), \quad c(Q) = \sum_{a=1}^{r+l} f_a(Q)p_a.
\]

Since \(D_1, \ldots, D_k\) are nef, the modification factor with respect to \(W\) is polynomial in \(\lambda\). We repeat the same argument as in Section 5.3.3 by replacing the rings

\[
\mathbb{C}[h][\lambda^{-1}, Q_\lambda, t], \quad \mathbb{C}((h^{-1}))[\lambda^{-1}, Q_\lambda, t], \quad \mathbb{C}[h, h^\partial + p][\lambda^{-1}, t][h^\partial][Q_\lambda]
\]

appearing there with

\[
\mathbb{C}[h, \lambda][Q, t], \quad \mathbb{C}((h^{-1})[\lambda][Q, t], \quad \mathbb{C}[h, h^\partial + p, \lambda][t][h^\partial][Q_\lambda].
\]

Then we can show that the relation (5.11) holds when \(I_X;W\) and \(I_W;W\) replaced with \(I_Y;W\) and \(I_W;W\) and for some \(f'' \in D := \mathbb{C}[h, h^\partial + p, \lambda][Q]\) and \(\tau''(Q, 0) \in \mathbb{C}[\lambda][Q]\).
This again shows (by Proposition 4.5) that \( D := \mathbb{C}[\lambda][Q]/(\hbar \partial + p) \)-module generated by \( I_{Y,W}(Q, \hbar) \) equals the \( D \)-module generated by \( I_{W}^{tw}(Q, \tau^*(Q, 0), \hbar) \) as a submodule of \( H^*(Y) \otimes \mathbb{C}(\hbar^{-1})[\lambda][Q] \). In particular, there exists a differential operator \( f_1(Q, \hbar \partial + p, \hbar) \in D \) such that \( f_1(0, \hbar \partial + p, \hbar) = 1 \) and

\[
I_{W}^{tw}(Q, \tau^*(Q, 0), \hbar) = f_1(Q, \hbar \partial + p, \hbar) I_{Y,W}(Q, \hbar).
\]  

(5.12)

A routine argument using Corollary 5.3 and Proposition 4.5 shows that the \( D \)-module generated by \( I_{W}^{tw}(Q, \tau^*(Q, 0), \hbar) \) is isomorphic to the restriction of the QDM\(_e\)(\( Y, W \)) to some \( \tilde{\tau}(Q) \in H^*(Y) \otimes \mathbb{C}[\lambda][Q] \). It is easy to see that \( I_{W}^{tw}(Q, \tau^*(Q, 0), \hbar) \) is a \( J \)-function of the small AQDM generated by \( I_{W}^{tw}(Q, \tau^*(Q, 0), \hbar) \) for some frame. By the relationships \((\ref{5.26}), (\ref{5.27})\) among \( J \)-functions, there exists a differential operator \( f_2(Q, \hbar \partial + p, \hbar) \in D \) such that \( f_2(0, \hbar \partial + p, \hbar) = 1 \) and

\[
J_{Y,W}(Q, \tilde{\tau}(Q), \hbar) = f_2(Q, \hbar \partial + p, \hbar) I_{W}^{tw}(Q, \tau^*(Q, 0), \hbar)
= V(Q, \hbar \partial + p, \hbar) I_{Y,W}(Q, \hbar).
\]  

(5.13)

where in the second line we used \((\ref{5.12})\) and put \( V = f_2 f_1 \). On the other hand, using

\[
I_{Y,W}(Q, \hbar) = \sum_{d \in \Lambda_Y} Q^d \prod_{\nu = 1}^N (u_i + \nu \hbar) \cup \prod_{j = N+1}^{N+k} \prod_{\nu = 1}^{(u_j, d)} (u_i + \nu \hbar) (u_j + \nu \hbar)
\]  

(5.14)

and \( i_* i^*(\alpha) = \alpha \cup \prod_{j = N+1}^{N+k} u_j \) for \( \alpha \in H^*(Y) \), we can easily check that

\[
\lim_{\lambda \to 0} e(W) \cup J_{Y,W}(Q, \hbar) = i_* I_X(Q, \hbar)
\]  

(5.15)

By Lemma 5.11 and Equations \((\ref{5.13}), (\ref{5.15})\), we have

\[
i_* J_X(Q, \lim_{\lambda \to 0} i^* \tilde{\tau}(Q), \hbar) = \left( \lim_{\lambda \to 0} V(Q, \hbar \partial + p, \hbar) \right) i_* I_X(Q, \hbar).
\]

Note that the limit \( \lambda \to 0 \) exists since everything is defined over \( K = \mathbb{C}[\lambda] \). Set \( V_0(Q, P, \hbar) = \lim_{\lambda \to 0} V(Q, P, \hbar) \) and \( \tilde{\tau}_0(Q) = \lim_{\lambda \to 0} i^* \tilde{\tau}(Q) \). By the injectivity of \( i_* \),

\[
J_X(Q, \tilde{\tau}_0(Q), \hbar) = V_0(Q, \hbar \partial + i^* p, \hbar) I_X(Q, \hbar).
\]  

(5.16)

Since \( I_X \) and \( J_X \) does not depend on \( Q^{r+1}, \ldots, Q^{r+l} \), by putting \( Q^{r+1} = \cdots = Q^{r+l} = 0 \) if necessary, we can assume \( \tilde{\tau}_0(Q) \) and \( V_0(Q, P, \hbar) \) depend only on \( Q^1, \ldots, Q^r \) and \( P_1, \ldots, P_r \) and \( \hbar \). Then \((\ref{5.10})\) implies that \( J_X(Q, \tilde{\tau}_0(Q), \hbar) \) belongs to the \( D \)-module \( FH^{*_{\lambda}}_Z(L_X) \) generated by \( I_X(Q, \hbar) \). Because \( V_0|_{Q=0} = 1 \), by Proposition 4.5, \( J_X(Q, \tilde{\tau}_0(Q), \hbar) \) also generates \( FH^{*_{\lambda}}_Z(L_X) \). Therefore, the restriction of \( QDM(X) \) to \( \tau = \tilde{\tau}_0(Q) \) (which is generated by \( J_X(Q, \tilde{\tau}_0(Q), \hbar) \)) by Proposition 4.5 is isomorphic to \( FH^{*_{\lambda}}_Z(L_X) \). Now the conclusion follows from the uniqueness of the reconstruction in Theorem 4.9.
6. Example

Let $M^N_k$ be a degree $k$ hypersurface of $\mathbb{P}^{N-1}$. This variety is Fano if $k < N$, Calabi-Yau if $k = N$ and of general type if $k > N$. We compute the big quantum cohomology of a general type hypersurface $M^N_k$. More precisely, we will compute the Euler-twisted quantum cohomology $\text{QH}_E^*(\mathbb{P}^7, \mathcal{O}(9))$. We follow the procedure of the generalized mirror transformation in Definition 4.12.

Since $\mathcal{O}(9)$ is ample, the $T^2$-equivariant Floer cohomology $\text{FH}^*_{T^2}(L_{\mathbb{P}^7/\mathcal{O}(9)})$ is defined over $\mathbb{C}[\lambda]$. We will denote the specialization of $\text{FH}^*_{T^2}(L_{\mathbb{P}^7/\mathcal{O}(9)})$ to $\lambda = 0$ by $\text{FH}^*_S(L_{\mathbb{P}^7/\mathcal{O}(9)})$. This is generated by the Floer fundamental class $\Delta$ over the Heisenberg algebra $\mathcal{D}$ with the following relation:

$$P^8\Delta = Q(9P + 9h)(9P + 8h) \cdots (9P + 2h)(9P + h)\Delta,$$

where $[P, Q] = hQ$. By Theorem 3.5, a frame $\Phi$ of the Floer cohomology is given by

$$\Phi(1) = \Delta, \quad \Phi(p) = P\Delta, \quad \ldots, \quad \Phi(p^7) = P^7\Delta,$$

where we choose $p$ to be a positive generator of $H^2(\mathbb{P}^7, \mathbb{Z})$. The connection matrix $A = (A_{kj})$ associated with $\Phi$ is defined by

$$P\Phi(p^j) = \sum_{k=0}^7 \Phi(p^k)A_{kj}.$$ 

The matrix $A$ is of the form

$$A(Q, h) = \begin{bmatrix}
0 & 0 & 0 & C_0(Q, h) \\
1 & 0 & 0 & C_1(Q, h) \\
0 & 1 & 0 & C_2(Q, h) \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 1 & C_7(Q, h)
\end{bmatrix},$$

where $P^8\Delta = \sum_{k=0}^7 C_k(Q, h)P^k\Delta$. The last column $P^8\Delta$ is calculated in the following way. By expanding (6.1) in $P$, we get

$$P^8\Delta = Q(9^9 P + 9^8 \cdot 45h)P^8\Delta + O(P^7)\Delta.$$ 

The term $P^8\Delta$ in the right-hand side is again replaced with the right-hand side of (6.1). Repeating the substitution for infinitely many times, we will arrive at $P^8\Delta = \sum_{k=0}^7 C_k(Q, h)P^k\Delta$. Note that the remainder in each step will go to zero in the $Q$-adic topology. First few terms of the (in fact non-convergent) power
series \( C_k(Q, h) \) are given by

\[
\begin{align*}
C_0(Q, h) &= 362880h^8Q + 84352288228920h^{10}Q^2 + 287295183309735497205760h^{11}Q^3 + \cdots, \\
C_1(Q, h) &= 92918h^8Q + 2161722824649176h^2Q^2 + 7384638765710370538901268h^{10}Q^3 + \cdots, \\
C_2(Q, h) &= 9498870h^7Q + 224382860804086767h^8Q^2 + 770022503217483472097175312h^3Q^3 + \cdots, \\
C_3(Q, h) &= 527562720h^3Q + 1263132210366894780h^7Q^2 + 43629721017495550435312780h^{10}Q^3 + \cdots, \\
C_4(Q, h) &= 1767041325h^6Q^2 + 4311916692248817630h^8Q^2 + 1503173349971730690260067660h^9Q^3 + \cdots, \\
C_5(Q, h) &= 373620737h^5Q + 936948774231192043h^5Q^2 + 3310328844753977849031223849h^6Q^3 + \cdots, \\
C_6(Q, h) &= 502217450h^2Q + 1312151047876934563h^4Q^2 + 47311019540125905135150100740h^7Q^3 + \cdots, \\
C_7(Q, h) &= 416183030h^2Q + 116184365841040307h^3Q^2 + 33004425486383221173017327h^3Q^3 + \cdots.
\end{align*}
\]

Next we perform the Birkhoff factorization of the fundamental solution \( L^{-1} \) of the flat connection \( \nabla^h = hQ(\partial/\partial Q) + A(Q, h) \). By \( L^{-1} = \text{Loc} \circ \Phi, \ L^{-1}(1) \) is the I-function \((\ref{eq:2})\) for \((\mathbb{P}^7, \mathcal{O}(9))\) with \( \lambda = 0 \) and \( L^{-1}(p^h) = \text{Loc}(P^h\Delta) \) is given by the derivatives of the I-function.

\[
L^{-1} = \begin{bmatrix} I_0 & \cdots & I_7 \end{bmatrix}, \quad I_k(Q, h) = (hQ\partial/\partial Q + p)^k \sum_{d=0}^{k} Q^d \prod_{m=1}^{bd} (9p + mh) \prod_{m=1}^{pd} (p + mh)^8.
\]

Here, we regard \( I_k \) as a column vector by expanding it in a basis \( \{1, p, \ldots, p^7\} \). Let \( L = L_+L_- \) be the Birkhoff factorization \((\ref{eq:1})\) of \( L \). Let \( \pi_+ \) denote the projection \( \mathbb{C}(h^{-1}) = \mathbb{C}[h] \oplus h^{-1}\mathbb{C}[h^{-1}] \to \mathbb{C}[h] \). By \( \pi_+(L^{-1}L_+) = \pi_+(L_-^{-1}) = \text{id} \), we have the following recursive formula for \( L_{+,d} \), where \( L_+ = \sum_{d=0}^{\infty} L_{+,d}Q^d \).

\[
L_{+,0} = \text{id}, \quad L_{+,d} = - \sum_{k=1}^{d} \pi_+(T_kL_{+,d-k}), \quad \text{where} \ L^{-1} = \text{id} + \sum_{d=1}^{\infty} T_dQ^d.
\]

On the other hand, for the reason of the grading \( \deg Q = \deg h^{-1} = -2 \), \( L_- \) becomes a finite series in \( Q \) and \( h^{-1} \). For example, the first column of \( L_-^{-1} \) (\( J \)-function for a canonical frame) is given by

\[
L_-^{-1}(1) = 1 + (34138908Q/h)p^2 + (56718144Q/h^2 + 840493443598718Q^2/h)p^3
+ \left(-22818915Q/h^3 + 649236650349369Q^2/h^2 + 3815935305370046256215462Q^3/h^3ight) p^4
+ \left(-4479543Q/h^4 - 4116161741786333Q^2/16h^3 + 1568163327547517306411844Q^3/h^3ight.
+ 219544798390763529724114822821260793Q^4/128h^5
+ \left.(8995906Q/h^5 - 2387486769247188Q^2/h^4 - 1841141718101141933423191Q^3/2h^3
+ 16559324859503519472166239017258Q^4/h^2
+ 7727223622317492116815015841479602342513631Q^2/12500h^6
+ (83567124Q/h^6 + 128193071705368551Q^2/32h^2 + 2536603825689258986824613Q^3/12h^4
- 1982932699581533560131149922355550Q^4/1024h^5
- 13718052707679235154906002198915935664875845527Q^5/500000h^7
+ 354241938428523717528257996282946380767628355279112051Q^6/20000h^8)\right)p^7.
\]
The connection matrix $A$ associated with a canonical frame $\Phi_{can} = \Phi \circ L_+$ is independent of $\hbar$ and is given by

$$A = L_+^{-1} A L_+ + \hbar L_+^{-1} Q \frac{\partial L_+}{\partial Q} = (L_+|_{\hbar=0})^{-1} A|_{\hbar=0} L_+|_{\hbar=0}$$

where $\alpha, \beta, \gamma, \ldots$ are the following constants:

$$\alpha = 34138908, \quad \beta = 16809868887197436, \quad \gamma = 90857052, \quad \delta = 11447799161101387518646386,$$

$$\epsilon = 81506931029963973/2, \quad \phi = 124756281, \quad \rho = 21954479839076352972414822821260793/32,$$

$$\xi = 1889246459939144055742585, \quad \eta = 77272362317492411681510195184170620342513631/2500,$$

$$\omega = 10627258152857115284755398884883914230185065837361713/10000,$$

$$\nu = 241135263679641337470680547162167530786173/1250, \quad \lambda = 8186578061602904275032886226470995/32,$$

$$\pi = 272776447102990732569280, \quad \mu = 2985296281746390, \quad \sigma = 5973264.$$

Now we perform the reconstruction. Let $t^0, \ldots, t^7$ be a co-ordinate system on $H^*(\mathbb{P}^7)$ dual to a basis $1, p, \ldots, p^7$. We solve for $\hbar$-independent connection matrices $A(Q, t), \Omega_0(Q, t), \Omega_1(Q, t), \ldots, \Omega_7(Q, t)$ such that they satisfy the conditions (4.5), (4.6) of flatness and

$$A(Q, 0) = A(Q), \quad \Omega_i(Q, t)e_0 = e_i,$$

Here, $\{e_0, \ldots, e_7\}$ denotes the standard basis of $\mathbb{C}^8 \cong H^*(\mathbb{P}^7)$ corresponding to $\{1, p, \ldots, p^7\}$. In view of the string and the divisor equation, it suffices to compute the deformation in parameters $t^2, \ldots, t^7$. Let $A^{(n)}$ and $\Omega_k^{(n)}$ be the degree $n$ part of $A(Q, t)|_{e^0=t^1=0}$ and $\Omega(Q, t)|_{e^0=t^1=0}$ with respect to the variables $t^2, \ldots, t^7$. Put $A^{\leq n} = \sum_{j=0}^{n} A^{(j)}$ and $\Omega^{\leq n} = \sum_{j=0}^{n} \Omega^{(j)}$. Because $t^2, t^3, \ldots, t^7$ have negative degrees $-2, -4, \ldots, -12$, it turns out that $A(Q, t)|_{e^0=t^1=0}$ and $\Omega(Q, t)|_{e^0=t^1=0}$ are polynomial in $t^2, \ldots, t^7$ and $A(Q, t)|_{e^0=t^1=0} = A^{\leq 5}, \Omega_k(Q, t)|_{e^0=t^1=0} = \Omega_k^{\leq 5}$. Assume inductively that we know $A^{\leq n}$. Because $\mathbb{C}^8 \otimes \mathbb{C}[Q, t]$ is generated by $e_0$ as a $\mathbb{C}[Q, t][A^{\leq n}]$-module, this admits a unique $\mathbb{C}[Q, t][A^{\leq n}]$-algebra structure such that $e_0$ is a unit. Define $\Omega_k^{\leq n}$ to be the multiplication matrix by $e_k$ in this ring. This is calculated in the following way.

$$\Omega_k^{\leq n} = \sum_{j=0}^{7} B_{kj} (A^{\leq n})^j, \quad \text{where } B_{kj} \text{ is determined by } e_k = \sum_{j=0}^{7} B_{kj} (A^{\leq n})^j,$$

By the relation $(Q \partial / \partial Q) \Omega_k^{\leq n} = (\partial / \partial t^k) A^{\leq n+1}$, we compute $A^{\leq n+1}$ as

$$A^{\leq n+1} = A^{\leq n} + \int_0^{(t^2, \ldots, t^7)} \sum_{k=2}^{7} Q \frac{\partial \Omega_k^{\leq n}}{\partial Q} dt^k.$$
Suppose that we have obtained $A(Q,t)|_{\hat{e}_1=\hat{t}^1=0}$ and $\Omega(Q,t)|_{\hat{e}_1=\hat{t}^1=0}$ in the above way. We will take flat co-ordinates $\hat{t}^0, \hat{t}^1, \ldots, \hat{t}^7$ of the form $\hat{t}^k = t^k + g^k(Q)$, $g^k(0) = 0$. By \cite{[14]}, the connection matrices $\hat{A}, \hat{\Omega}_k$ associated with $\hat{t}^k$ are given by

$$
\hat{A}(Q,t) = A(Q,t) - \sum_{k=0}^{7} Q \frac{\partial g^k(Q)}{\partial Q} \Omega_k(Q,t), \quad \hat{\Omega}_k(Q,t) = \Omega_k(Q,t).
$$

The functions $g^k$ are determined by the condition $e_1 = \hat{A}(Q,0)e_0 = A(Q)e_0 - \sum_{k=0}^{7} Q(\partial g^k/\partial Q)e_k$. We find (only from $A(Q)$)

$$
\begin{align*}
\hat{t}^0 &= t^0, \quad \hat{t}^1 = t^1, \quad \hat{t}^2 = t^2 + \alpha Q, \quad \hat{t}^3 = t^3 + \frac{1}{2} \beta Q^2, \quad \hat{t}^4 = t^4 + \frac{1}{3} \delta Q^3, \\
\hat{t}^5 &= t^5 + \frac{1}{4} \rho Q^4, \quad \hat{t}^6 = t^6 + \frac{1}{5} \eta Q^5, \quad \hat{t}^7 = t^7 + \frac{1}{6} \omega Q^6.
\end{align*}
$$

By the string and the divisor equation in Proposition 4.8, we have $\hat{A} = \Omega_1$ and

$$
\begin{align*}
\Omega_k(Q,t(Q,\hat{t})) &= \Omega_k(Qe^{\hat{t}^1}, t(Q,0,0,\hat{t}^2,\ldots,\hat{t}^7)) \\
&= \Omega_k(Qe^{\hat{t}^1}, 0,0,\hat{t}^2 - g^2(Q),\ldots,\hat{t}^7 - g^7(Q)).
\end{align*}
$$

The matrix-valued function $(Q,\hat{t}) \mapsto \Omega_k(Q,t(Q,\hat{t}))$ represents the multiplication by $p^k$ in the big quantum cohomology $\mathcal{Q}H_{\text{Euler}}^*(\mathbb{P}^7, \mathcal{O}(9))$ with respect to the basis $\{1, p, \ldots, p^7\}$. For simplicity, we present a $(7 \times 7)$ submatrix $\Omega_{1,j,k}$ of $\Omega_1 = (\Omega_{1,j,k})_{0 \leq j, k \leq 7}$.

$$
(7 \times 7)\text{-submatrix of } \Omega_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & A & 1 & 0 & 0 & 0 & 0 \\
0 & B & D & 1 & 0 & 0 & 0 \\
0 & C & B & A & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix},
$$
where the functions \( A, B, C, D \) are given by

\[
A = (\gamma - \alpha)Qe^t_1 = 56718144Qe^t_1, \\
D = (\phi - \alpha)Qe^t_1 = 90617373Qe^t_1, \\
B = (\epsilon + 2\alpha(\alpha - \phi) - \beta)(Qe^t_1)^2 + (\phi - \alpha)t^2Qe^t_1 \\
= \frac{35512880615374365}{2}(Qe^t_1)^2 + 90617373t^2Qe^t_1, \\
C = \left(\frac{9}{2}\alpha^2(\phi - \alpha) + \frac{3}{2}\beta(3\alpha - \gamma) - 3\epsilon\alpha - \delta + \xi\right)(Qe^t_1)^3 \\
+ (4\alpha(\alpha - \phi) + 2(\epsilon - \beta))t^2(Qe^t_1)^2 + \left(\frac{\phi - \alpha}{2}(t^2)\right)^2 + (\gamma - \alpha)t^3\right)Qe^t_1 \\
= 4037555975532386945225553(Qe^t_1)^3 + 35512880615374365t^2(Qe^t_1)^2 \\
+ \left(\frac{90617373}{2}(t^2)^2 + 56718144t^3\right)Qe^t_1.
\]

By Theorem 2.1 this submatrix is related to the product by \( p \) in \( QH^*(M_8^9) \) as

\[
\int \frac{1}{9} \int_{M_8^9}(p* p^j) \cup p^{6-j} = \Omega_{1,j}, \quad 0 \leq i, j \leq 6.
\]

They agree with the Jinzenji’s calculations [20, Section 6]. In particular, the genus 0 Gromov-Witten potential \( F_{M_8^9} \) of \( M_8^9 \) restricted to the image of \( H^*(\mathbb{P}^7) \to H^*(M_8^9) \) is determined by \( (\partial/\partial t^1)^3F_{M_8^9} = 9C \) and the classical part.

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