RIEMANN EXTENSIONS IN THEORY OF THE FIRST ORDER SYSTEMS OF DIFFERENTIAL EQUATIONS

Valery Dryuma

Institute of Mathematics and Informatics AS Moldova, Kishinev

E-mail: valery@dryuma.com; cainar@mail.md

Abstract

The properties of the Riemann extensions of nonriemannian spaces defined by the first order systems of differential equations are considered.

1 Introduction

The first order polynomial systems of differential equations

\[
\frac{dx^i}{ds} = a^i_{jk} x^j x^k
\]

depending from the parameters \(a, b\) play an important role in various branches of modern mathematics and its applications.

However even in the case of the system of two equations

\[
\frac{dx}{ds} = k + ax + by + cz^2 + cxy + f y^2, \quad \frac{dy}{ds} = l + mx + ny + px^2 + qxy + ry^2
\]

there are many unsolved problems.

The most famous of them is the 16-th Hilbert problem about the quantity and position of the limit cycles in such type of the systems.

The spatial first order system of differential equations

\[
\frac{dx}{ds} = P(x, y, z), \quad \frac{dy}{ds} = Q(x, y, z), \quad \frac{dz}{ds} = R(x, y, z)
\]

with the functions \(P, Q, R\) polynomial on variables \(x, y, z\) are still more complicated object for the studying of their properties.

As example the studying of simplest spatial systems of equations such as the Lorenz equations

\[
\frac{dx}{ds} = \sigma(y - x), \quad \frac{dy}{ds} = r x - y - xz, \quad \frac{dz}{ds} = xy - bz
\]

or the Rössler system

\[
\frac{dx}{ds} = -y - z, \quad \frac{dy}{ds} = x + ay, \quad \frac{dz}{ds} = bx - cz + xz
\]

having chaotic behavior at some values of parameters represent the difficult task.

The systems of the first order differential equations are not suitable object of consideration from the usually point of Riemann geometry.
The systems of the second order differential equations in form
\[
\frac{d^2 x^i}{ds^2} + \Pi^i_{kj} (x) \frac{dx^k}{ds} \frac{dx^j}{ds} = 0
\] (6)
are best suited to do that.

They can be considered as geodesics of the affinely connected space \( M \) in local coordinates \( x^k \). The values \( \Pi^i_{kj} = \Pi^i_{jk} \) are the coefficients of affine connections on \( M \).

With the help of such coefficients can be constructed curvature tensor and others geometrical objects defined on variety \( M \).

There are many possibilities to present a given system of the first order of equations in the form of (6).

One of them is a following presentation.

For the system
\[
\frac{dx}{ds} = P(x, y), \quad \frac{dy}{ds} = Q(x, y)
\] (7)
after differentiation with respect to parameter \( s \) we get the second order system of differential equations of the form (6)
\[
\frac{d^2 x}{ds^2} = \frac{1}{P} (P_x \frac{dx}{ds} + P_y \frac{dy}{ds}),
\]
\[
\frac{d^2 y}{ds^2} = \frac{1}{Q} (Q_x \frac{dx}{ds} + Q_y \frac{dy}{ds}).
\] (8)

Such type of the system contains the integral curves of the system (7) as part of its solutions and can be considered as the equations of geodesics of two dimensional space \( M^2(x, y) \) equipped by affine connections with coefficients
\[
\Pi^1_{11} = -\frac{P_x}{P}, \quad \Pi^1_{12} = -\frac{P_y}{2P}, \quad \Pi^2_{12} = -\frac{Q_x}{2Q}, \quad \Pi^2_{22} = -\frac{Q_y}{Q}.
\] (9)

It is apparent that the properties of the system (7) have an influence on geometry of the variety \( M^2(x, y) \).

Remark that the system (8) is equivalent the second order differential equation
\[
\frac{d^2 y}{dx^2} = \ln(Q/P)_y \left( \frac{dy}{dx} \right)^2 + \ln(Q/P)_x \frac{dy}{dx}
\]
which has the solution in form
\[
\frac{dy}{dx} = \frac{Q}{P}.
\]

In this connection we can use the second order system of differential equation
\[
\frac{d^2 x}{ds^2} = -\ln(Q/P)_x \left( \frac{dx}{ds} \right)^2, \quad \frac{d^2 y}{ds^2} = \ln(Q/P)_y \left( \frac{dy}{ds} \right)^2
\] (10)
for the studying of the properties of the first order system of equations (7).

By analogy can be written the spatial system of the first order differential equations.

2 The Riemann extension of affinely connected space

For the studying of the geometry of the equations like (8) we apply the notion of the Riemann extension of nonriemannian space which was used earlier in [1, 2, 3].

Remind basic properties of this construction.
With help of the coefficients of affine connection of a given n-dimensional space can be introduced 2n-dimensional Riemann space \( D_{2n} \) in local coordinates \((x^i, \Psi_k)\) having the metric of form

\[
2^n ds^2 = -2\Pi^{ij}_k(x^l)\Psi_k dx^i dx^j + 2d\Psi_k dx^k
\]

(11)

where \( \Psi_k \) are the additional coordinates.

The important property of such type metric is that the geodesic equations of metric (11) decomposes into two parts

\[
\ddot{x}^k + \Pi^{k}_{ij}\dot{x}^i\dot{x}^j = 0,
\]

(12)

and

\[
\frac{\delta^2 \Psi_k}{ds^2} + R^l_{kji}\dot{x}^j\dot{x}^i\Psi_l = 0,
\]

(13)

where

\[
\frac{\delta \Psi_k}{ds} = \frac{d\Psi_k}{ds} - \Pi^{l}_{jk}\Psi_l \frac{dx^j}{ds}
\]

and \( R^l_{kji} \) are the curvature tensor of n-dimensional space with a given affine connection.

The first part (12) of the full system is the system of equations for geodesic of basic space with local coordinates \( x^i \) and it do not contains the supplementary coordinates \( \Psi_k \).

The second part (13) of the system has the form of linear \( N \times N \) matrix system of second order ODE’s for supplementary coordinates \( \Psi_k \)

\[
\frac{d^2 \bar{\Psi}}{ds^2} + A(s)\frac{d\bar{\Psi}}{ds} + B(s)\bar{\Psi} = 0.
\]

(14)

Remark that the full system of geodesics has the first integral

\[
-2\Pi^{ij}_k(x^l)\Psi_k \frac{dx^i}{ds} \frac{dx^j}{ds} + 2\frac{d\Psi_k}{ds} \frac{dx^k}{ds} = \nu
\]

(15)

which is equivalent to the relation

\[
2\Psi_k \frac{dx^k}{ds} = \nu s + \mu
\]

(16)

where \( \mu, \nu \) are parameters.

It is important to note that the geometry of extended space connects with geometry of basic space. For example the property of the space to be Ricci-flat \( R_{ij} = 0 \) or symmetrical \( R_{ijkl;m} = 0 \) keeps also for the extended space.

It is important to note that for extended space having the metric (11) all scalar curvature invariants are vanished.

As consequence the properties of linear system of equation (13-14) depending from the the invariants of \( N \times N \) matrix-function

\[
E = B - \frac{1}{2} \frac{dA}{ds} - \frac{1}{4} A^2
\]

under change of the coordinates \( \Psi_k \) can be of used for that.

The first applications the notion of extended spaces for the studying of nonlinear second order differential equations connected with nonlinear dynamical systems have been considered in the works of author [1, 2, 3].

3 The geometry of planar system of equations in form (8)

We shall consider from geometrical point the properties of planar systems of the equations (7).

To do this we use the second order system (8) having the solutions of the system (7) as the part of their own solutions.
Now the system (8) can be considered as the geodesics of two dimensional space with affine connections and we extend this space up to the four-dimensional space by the introducing two additional coordinates $\psi_1 = z, \psi_2 = t$.

The Riemann metric of the four-dimensional extended space is defined by

$$4 ds^2 = 2z \frac{P_x}{P} dx^2 + (2z \frac{P_y}{P} + t \frac{Q_x}{Q}) dx dy + 2t \frac{Q_y}{Q} dy^2 + 2dz^2 + 2dy dt.$$  \hspace{1cm} (17)

Let us consider the basic geometric characteristics of a given metric. The nonzero components of the Ricci tensor are

$$R_{11} = \frac{1}{2} - 3 \left( \frac{\partial^2}{\partial x \partial y} P(x,y) \right)^2 Q(x,y) + 2 \left( \frac{\partial^2}{\partial x^2} P(x,y) \right) P(x,y) Q(x,y) + 2 \left( \frac{\partial}{\partial x} P(x,y) \right) Q(x,y) \frac{\partial}{\partial y} Q(x,y),$$

$$R_{12} = - \frac{1}{2} \left( \frac{\partial}{\partial x} P(x,y) \right) \left( \frac{\partial}{\partial y} P(x,y) \right) \left( \frac{\partial}{\partial y} Q(x,y) \right) - \frac{1}{2} \left( \frac{\partial}{\partial x} Q(x,y) \right) - 1/2 \frac{\partial}{\partial y} Q(x,y),$$

$$R_{22} = \frac{1}{2} - 3 \left( \frac{\partial}{\partial y} P(x,y) \right)^2 Q(x,y) + 2 \left( \frac{\partial^2}{\partial y^2} P(x,y) \right) P(x,y) Q(x,y) + 2 \left( \frac{\partial}{\partial y} Q(x,y) \right) P(x,y) \frac{\partial}{\partial y} P(x,y).$$

As was mentioned above all scalar invariants constructed from the curvature tensor and its covariant derivatives

$$p = R_{ij} R^{ij} = 0, \quad q = R_{ijkl} R^{ijkl} = 0...$$

are vanish.

The full system of geodesics of the metric (17) consists from the two groups of equations

$$\frac{d^2}{ds^2} x(s) - \left( \frac{\partial}{\partial s} P(x,y) \right) \left( \frac{d}{ds} x(s) \right)^2 \frac{P(x,y)}{P(x,y)} - \left( \frac{\partial}{\partial s} P(x,y) \right) \left( \frac{d}{ds} x(s) \right) \frac{d}{ds} y(s) = 0,$$

$$\frac{d^2}{ds^2} y(s) - \left( \frac{\partial}{\partial s} Q(x,y) \right) \left( \frac{d}{ds} y(s) \right)^2 \frac{Q(x,y)}{Q(x,y)} - \left( \frac{\partial}{\partial s} Q(x,y) \right) \left( \frac{d}{ds} x(s) \right) \frac{d}{ds} y(s) = 0,$$

and

$$\frac{d^2}{ds^2} z(s) + \left( \frac{\partial}{\partial s} (P(x,y))^3 \right) \left( \frac{d}{ds} Q(x,y) \right)^2 \frac{Q(x,y)}{Q(x,y)} = 0,$$

$$\frac{d^2}{ds^2} z(s) + \frac{\partial}{\partial s} \left( Q(x,y) \right)^2 \frac{P(x,y)}{P(x,y)} \left( \frac{\partial}{\partial s} Q(x,y) \right) \frac{\partial}{\partial y} P(x,y) + 2 \left( P(x,y) \right)^2 \left( Q(x,y) \right)^2 \frac{\partial^2}{\partial x \partial y} P(x,y) = 0,$$

$$\frac{d^2}{ds^2} z(s) + \frac{\partial}{\partial s} \left( Q(x,y) \right)^3 \frac{P(x,y)}{P(x,y)} \left( \frac{\partial}{\partial s} Q(x,y) \right)^3 \frac{P(x,y)}{P(x,y)} + \frac{\partial}{\partial s} \left( Q(x,y) \right)^2 \frac{P(x,y)}{P(x,y)} = 0.$$
3 THE GEOMETRY OF PLANAR SYSTEM OF EQUATIONS IN FORM (8)

Let us consider some examples.

Using the first integral of geodesics of the metric (16) we can get from the system (18) two independent linear second order differential equations for the each variable \( z(s) \) and \( t(s) \)

\[
\frac{d^2 z}{ds^2} + A_{11} \frac{dz}{ds} + A_{12} \frac{dt}{ds} + B_{11} z + B_{12} t = 0,
\]

\[
\frac{d^2 t}{ds^2} + A_{21} \frac{dz}{ds} + A_{22} \frac{dt}{ds} + B_{21} z + B_{22} t = 0.
\]  \hspace{1cm} (18)

Using the first integral of geodesics of the metric (16)

\[
\frac{dx}{ds} \frac{dz}{ds} + t \frac{dy}{ds} = \mu s/2 + \nu
\]

we can get from the system (18) two independent linear second order differential equations for the each variable \( z(s) \) and \( t(s) \)

\[
\frac{d^2 z}{ds^2} + A(s) \frac{dz}{ds} + B(s) z = 0,
\]

and

\[
\frac{d^2 t}{ds^2} + C(s) \frac{dt}{ds} + E(s) t = 0.
\]

Let us consider some examples.

1. The system of equations

\[
\frac{dx}{ds} = -y, \quad \frac{dy}{ds} = x
\]
corresponds the second order system of equations

\[
\frac{d^2}{ds^2} x(s) - \left( \frac{dx}{ds} (s) \right) \frac{d^2 y}{ds^2} (s) = 0, \\
\frac{d^2}{ds^2} y(s) - \left( \frac{dy}{ds} (s) \right) \frac{dx}{ds} (s) = 0.
\]

From the geodesics of extended space we get the equations for coordinate \( z \)

\[
\frac{d^2}{ds^2} z(s) - \frac{(x(s))^2 z(s)}{(y(s))^2} + 1/2 \frac{2 (y(s) (x(s))^2 + 2 (y(s))^3) \frac{dz}{ds}(s)}{x(s) (y(s))^2} + 1/2 \frac{\mu}{x(s)} = 0
\]

and

\[
\frac{d^2}{ds^2} t(s) - \frac{(y(s))^2 t(s)}{(x(s))^2} - 1/2 \frac{2 (x(s))^3 + 2 x(s) (y(s))^2) \frac{dt}{ds}(s)}{y(s) (x(s))^2} + 1/2 \frac{\mu}{y(s)} = 0
\]

for coordinate \( t \).

4 The geometry of planar system of equations in form (10)

The system (10) is more simple object for the studying of the properties of the system (7) from geometrical point of view.

Really, the metric of extended space defined by the system (10) can be written in form

\[
^4 ds^2 = -2z \frac{\partial(K(x,y))}{\partial x} dx^2 + 2t \frac{\partial(K(x,y))}{\partial x} dy^2 + 2dx dz + 2dy dt
\]

where \( z,t \) are the supplementary coordinates and the function \( K(x,y) = \ln(Q/P) \) is determined from the relations

\[
\frac{dy}{dx} = \exp(K(x,y))
\]
or

\[
\frac{dy}{ds} = \exp(K(x,y)) \frac{dx}{ds}.
\]

Contrary to the case (17) the Riemannian space with the metric (19) is a Ricci-flat \( R_{ij} = 0 \).

Its geodesics are defined by the system of equations

\[
\frac{d^2}{ds^2} x + \frac{\partial K(x,y)}{\partial x} \left( \frac{dx}{ds} \right)^2 = 0,
\]

\[
\frac{d^2}{ds^2} y - \frac{\partial K(x,y)}{\partial y} \left( \frac{dy}{ds} \right)^2 = 0.
\]

5 Rigorous approach at the geometry of the planar systems

We consider the system of paths of two-dimensional space \( S_2 \) in form

\[
\ddot{x} + \Pi_{11}(\dot{x})^2 + 2\Pi_{12} \dot{x} \dot{y} + \Pi_{22}(\dot{y})^2 = 0,
\]

\[
\ddot{y} + \Pi_{11}(\dot{x})^2 + 2\Pi_{12} \dot{x} \dot{y} + \Pi_{22}(\dot{y})^2 = 0,
\]

(20)
where coefficients $\Pi^{k}_{ij} = \Pi^{k}_{ji}$.

The Riemann extension of the space $S_2$ is determined by the metric

$$4ds^2 = -2z\Pi_{11}dx^2 - 2t\Pi_{12}dx^2 - 4s\Pi_{12}dxdy - 4t\Pi_{22}dxdy - 2s\Pi_{22}dy^2 - 2t\Pi_{22}dy^2. \quad (21)$$

A necessary condition for the equations (20) to admit a first integral

$$a_i(x, y) \frac{dx^i}{ds} = \text{const}$$

is

$$a_{i,j} + a_{j,i} = 0,$$

where

$$a_{i,j} = \frac{\partial a_i}{\partial x^j} - a_k \Gamma^k_{ij},$$

and $\Gamma^k_{ij}$ are the Christoffel symbols of the metric (21).

We apply this conditions for determination of the coefficients of equations $\Gamma^k_{ij}$ using the vector $a_i$ in form

$$a_i = [Q(x, y), -P(x, y), 0, 0].$$

This means that the first order of equation

$$\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)}$$

or

$$Q(x, y)dx - P(x, y)dy = 0$$

is an integral of the paths equations.

Using such condition it is possible to find only three coefficients of affine connections $\Gamma^k_{ij}$.

For determination of the rest coefficients we use yet another the first order equation

$$\frac{dy}{dx} = -\frac{y(y-1)}{x(x-1)}, \quad (22)$$

with the first integral

$$y(x) = \frac{C(x-1)}{x-C}.$$

The equation (22) plays an important role in theory of of the planar first order system of equations (2) ([4]).

In result the coefficients $\Pi^{k}_{ij}$ of the paths equation are defined by

$$\Pi^{1}_{11} = \frac{\left(\frac{\partial}{\partial x}Q(x, y)\right) x(x-1)}{y^2P(x, y) - yP(x, y) + Q(x, y)x^2 - Q(x, y)x},$$

$$\Pi^{1}_{12} = \frac{2yP(x, y) - 2P(x, y) + 2xP(x, y) + \left(\frac{\partial}{\partial y}Q(x, y)\right)x^2 - \left(\frac{\partial}{\partial y}Q(x, y)\right)x - \left(\frac{\partial}{\partial x}P(x, y)\right)x^2 + \left(\frac{\partial}{\partial x}P(x, y)\right)x}{y^2P(x, y) - yP(x, y) + Q(x, y)x^2 - Q(x, y)x},$$

$$\Pi^{1}_{22} = \frac{\left(\frac{\partial}{\partial y}P(x, y)\right)x(x-1)}{y^2P(x, y) - yP(x, y) + Q(x, y)x^2 - Q(x, y)x},$$

and corresponding expressions for the coefficients $\Pi^{2}_{11}, \Pi^{2}_{12}, \Pi^{2}_{22}$. 

In result the geodesics take the form

\[ \frac{d^2}{ds^2} x(s) + \frac{\left( \frac{\partial}{\partial y} P(x, y) \right) x (x-1) \left( \frac{d}{dx} x(s) \right)^2}{y^2 P(x, y) - y P(x, y) + Q(x, y) x^2 - Q(x, y) x} + \]
\[ \frac{\left( 2 y P(x, y) - 2 P(x, y) + 2 x P(x, y) + \left( \frac{\partial}{\partial y} Q \right) x^2 - \left( \frac{\partial}{\partial y} P \right) x - \left( \frac{\partial}{\partial y} P \right) \left( \frac{d}{dx} x(s) \right)^2 \right) \frac{d}{dx} y(s)}{y^2 P - y P + Q x^2 - Q x} = 0, \]
\[ \frac{d^2}{ds^2} y(s) - \frac{\left( \frac{\partial}{\partial y} Q(x, y) \right) y (y-1) \left( \frac{d}{dx} y(s) \right)^2}{y^2 P(x, y) - y P(x, y) + Q(x, y) x^2 - Q(x, y) x} + \]
\[ \frac{\left( 2 y Q - 2 Q + 2 y Q + y^2 \frac{\partial}{\partial y} P - y \frac{\partial}{\partial y} P \right) \left( \frac{d}{dx} x(s) \right)^2 \frac{d}{dx} y(s)}{y^2 P - y P + Q x^2 - Q x} + \]
\[ \frac{\left( \frac{\partial}{\partial y} P(x, y) \right) y (y-1) \left( \frac{d}{dx} y(s) \right)^2}{y^2 P(x, y) - y P(x, y) + Q(x, y) x^2 - Q(x, y) x} = 0, \] (23)

and

\[ \frac{d^2 z}{ds^2} + A(s) \frac{dz}{ds} + B(s) \frac{dt}{ds} + C(s) \frac{dz}{ds} + E(s) t(s) = 0, \]
\[ \frac{d^2 t}{ds^2} + F(s) \frac{dz}{ds} + H(s) \frac{dt}{ds} + K(s) \frac{dz}{ds} + L(s) t(s) = 0 \]

with the coefficients depending from the parameter \( s \) and the functions \( Q(x, y), P(x, y) \).

Remark that last two equations are reduced at the independent equations

\[ \frac{d^2 z}{ds^2} + M(s) \frac{dz}{ds} + N(s) z(s) = 0 \]

and

\[ \frac{d^2 t}{ds^2} + U(s) \frac{dz}{ds} + V(s) t(s) = 0 \]

with the help of the first integral of geodesics

\[ z(s) \frac{dz}{ds} + t(s) \frac{dy}{ds} - \frac{s}{2} - \beta = 0 \]

of the metric (21).

6 The second order ODE’s cubic on the first derivative in theory of the planar systems

The first two equations of geodesic of the metric (23) are equivalent to the one second order differential equation

\[ \frac{d^2}{dx^2} y(x) + \left( \frac{\partial}{\partial y} P(x, y) \right) x^2 - \left( \frac{\partial}{\partial y} P(x, y) \right) x \left( \frac{d}{dx} y(x) \right)^3 + \]
\[ \frac{\left( \frac{\partial}{\partial x} P - \frac{\partial}{\partial y} Q \right) x^2 + \left( \frac{\partial}{\partial y} Q - \frac{\partial}{\partial x} P - 2 P \right) x + \left( \frac{\partial}{\partial y} P \right) y^2 + \left( 2 P - \frac{\partial}{\partial y} P \right) y + 2 P \left( \frac{d}{dx} y(x) \right)^2}{y^2 P - y P + Q x^2 - Q(x, y) x} = 0. \]
6  THE SECOND ORDER ODE’S CUBIC ON THE FIRST DERIVATIVE IN THEORY OF THE PLANAR SYSTEMS

\[ \left( -\frac{\partial}{\partial x} Q \right) x^2 + \left( \frac{\partial}{\partial y} Q + 2Q \right) x + \left( \frac{\partial}{\partial x} P - \frac{\partial}{\partial y} Q \right) y^2 + \left( 2Q - \frac{\partial}{\partial x} P + \frac{\partial}{\partial y} Q \right) y - 2Q \right) \frac{dy}{dx} y(x) + \\
\frac{y^2 P - yP + Qx^2 - Qx}{y^2 P(x,y) - yP(x,y) + Q(x,y)x^2 - Q(x,y)x} = 0. \tag{24} \]

The equation (24) has the equation

\[ \frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)} \]

as particular integral and the function

\[ y(x) = \frac{C(x - 1)}{x - C} \tag{25} \]

as the first integral.

In result we get two-parametric solutions of the second order ODE from the one-parametric solutions of a given first order equation.

This fact allow us to describe the properties of the first order system (2) with respect the values of parameter \( C \) of the solution (25).

Let us consider some example.

After substitution of the relation (25) into the equation (24) one gets the expression

\[ \alpha(x,y)C^3 + \beta(x,y)C^4 + \gamma(x,y)C^3 + \delta(x,y)C^2 + \epsilon(x,y)C + \mu(x,y) = 0 \tag{26} \]

, where

\[ \alpha(x,y) = (a_{12} + b_{12})y^2 + \\
+ (b_1 + b_2 + (2a_{22} + 2a_{11} + 2b_{22} + 2b_{11})x + a_2 + a_1)y + (a_{12} + b_{12})x^2 + (a_1 + b_2 + b_1 + a_2)x + \\
+ 2a_0 + 2b_0, \]

\[ \beta(x,y) = 2b_{22}y^3 + ((-5a_{12} - b_{12} - 2b_{22})x + 2b_2 - b_{12})y^2 + \\
+ ((-4a_{22} - 3b_2 - 5a_1 - 3a_2 - 2b_{11} - 6b_{22} - b_{11})x - a_2) + \\
+ ((-4b_{11} + a_{12} - 10a_{11} - b_{12} - 4a_{22})x^2 + 2b_0 - 2b_2 - b_{11})y + \\
+ (-2a_{12} + 2a_{11})x^3 + (-2a_{12} - 2b_1 - 2a_1 - 3b_{12} - 2a_2) x^2 - 4b_0 + (-6a_0 - 2b_1 - 4b_0 - 3b_2 - a_1 - 2a_2)x - \\
- 2a_0, \]

\[ \gamma(x,y) = (-4xb_{22} - 2b_{22})y^3 + ((-b_{12} + 2b_{22} + 10a_{12})x^2 - 2b_2 + (b_{12} + 4b_{22} - 4b_2)x)y^2 + \\
+ (b_2 + (-3a_{12} + 2a_{22} + 2a_{11} + b_{12} + 2a_{11})x^3 + (6b_{22} + 2a_{22} + b_1 + 3a_2 - 4b_0 + 6b_2)x)y + \\
+ ((-b_1 + 10a_1 + 3a_2 + 2b_2 + 4b_{11} + 2b_{12} - a_{12} - 8a_{22})x^2 - 2b_0)y + \\
+ (a_{12} - 6a_{11})x^4 + (4a_{12} + a_2 - 2a_{11} + b_1)x^3 + (b_1 + 6a_0 + 3b_2 + a_2 + 8b_0)x + \\
+ (a_{12} + 6a_0 + 4b_1 + 2a_1 + 2b_0 + 3b_{12} + 4a_{22})x^2 + 2b_0, \]

\[ \delta(x,y) = (2x^2b_{22} + 4xb_{22})y^3 + ((b_{12} - 10a_{12})x^3 + (2b_2 - 4b_{22} + b_{12})x^2 + (4b_2 - 2b_{22})x)y^2 + \\
+ ((b_1 - 4a_{22} + 3a_{12} - 2b_{11} - a_2 - 10a_1 - 2b_{12})x^3 + (-b_{12} - 4b_2 - 4a_{22} - 3a_2 + b_1 + 2b_0)x^2)y + \\
+ ((b_1 - 4a_{22} + 3a_{12} - 2b_{11} - a_2 - 10a_1 - 2b_{12})x^3 + (-b_{12} - 4b_2 - 4a_{22} - 3a_2 + b_1 + 2b_0)x^2)y + \]
where
\[+ (3 \, a_{12} - 20 \, a_{11}) \, x^4 + \left( -3 \, b_2 + 4 \, b_0 - 2 \, b_{22} \right) \, x \, y +
+ 6 \, x^5 \, a_{11} + (6 \, a_{11} + 2 \, a_1 - 2 \, a_{12}) \, x^4 + (-4 \, b_0 - b_{12} - 2 \, b_{1} - 2 \, a_2 - 6 \, a_0) \, x^2 +
+ (-2 \, a_{12} - 2 \, a_0 - 2 \, a_2 - 2 \, b_1) \, x^3 + (-4 \, b_0 - b_2) \, x],
\]

\[\epsilon(x, y) = -2 \, y^3 \, x^2 \, b_{22} + \left( -x^3 b_{12} + 5 \, x^4 \, a_{12} + (-2 \, b_2 + 2 \, b_{22}) \, x^2 \right) \, y^2 +
+ \left( (5 \, a_1 - 3 \, a_{12}) \, x^4 + (10 \, a_{11} - a_{12}) \, x^3 + (-b_1 + b_{12} + a_2 + 2 \, a_{22}) \, x^3 + (2 \, b_2 - 2 \, b_0) \, x^2 \right) \, y - 2 \, x^6 \, a_{11} +
+ (-a_1 - 6 \, a_{11}) \, x^5 + (b_1 + 2 \, a_0 + a_2) \, x^3 + (a_{12} - 2 \, a_1) \, x^4 + 2 \, x^2 \, b_0,
\]

\[\mu(x, y) = -y^2 \, x^5 \, a_{12} + (-2 \, x^6 \, a_{11} + (a_{12} - a_1) \, x^5) \, y + 2 \, x^6 \, a_{11} + x^5 \, a_1.
\]

The substitution of the expression (25) into the relation (26) give us the expression
\[A(x)C^6 + B(x)C^5 + E(x)C^4 + F(x)C^3 + H(x)C^2 + K(x)C + L(x) = 0,\]

where
\[A(x) = (2 \, b_{12} - 2 \, b_{22} - 2 \, b_{11} + 2 \, a_{12} - 2 \, a_2 - 2 \, a_{11}) \, x^2 +
+ (2 \, b_{22} - 2 \, b_{12} - 2 \, a_{12} + 2 \, a_{11} + 2 \, a_{22} + 2 \, b_{11}) \, x +
+ a_2 + a_{12} + b_1 + b_{12} + b_2 + 2 \, a_0 + a_1 + 2 \, b_0,
\]

\[B(x) = (-8 \, a_{12} + 12 \, a_{11} + 4 \, b_{11} - 4 \, b_{22} + 4 \, a_{22}) \, x^3 +
+ (-2 \, b_1 + 7 \, a_{12} + 16 \, b_{22} + 2 \, a_1 + 4 \, b_2 - 2 \, b_{11} - 10 \, a_{11} - 5 \, b_{12}) \, x^2 +
+ (-2 \, b_1 - 8 \, b_2 - 8 \, a_0 - 4 \, a_{22} - 4 \, a_2 - 2 \, b_{12} - 4 \, a_{12} - 14 \, b_{22} - 8 \, b_0 - 6 \, a_1 - 2 \, b_{11}) \, x -
- 2 \, b_0 - 2 \, a_0 - b_1 - a_2 - b_{12} + 2 \, b_{22},
\]

\[E(x) = (-28 \, a_{11} - 2 \, b_{11} - 2 \, a_{22} - 2 \, b_{12} + 12 \, a_{12}) \, x^4 +
+ (-8 \, a_1 + 18 \, a_{11} + 6 \, b_{22} + 4 \, b_1 - 2 \, b_{11} + 4 \, b_{12} - 6 \, a_{22} - 8 \, a_{12} - 4 \, b_2) \, x^3 +
+ (4 \, b_1 + 12 \, a_0 + 6 \, a_2 + 13 \, a_1 + b_2 + 6 \, a_{12} + 6 \, a_{22} + 4 \, b_{11} + 2 \, a_{22} - 20 \, b_{22} + 10 \, b_0) \, x^2 +
+ (4 \, a_2 + 2 \, b_1 + 10 \, b_0 + 10 \, b_2 + 2 \, a_{22} + 16 \, b_{22} + 8 \, a_0) \, x - b_2 - 2 \, b_{22},
\]

\[F(x) = (32 \, a_{11} - 8 \, a_{12}) \, x^5 + (-2 \, b_1 + 4 \, a_{22} + 12 \, a_1 + 3 \, b_{12} + 2 \, b_{11} + 2 \, a_{12} - 12 \, a_{11}) \, x^4 +
+ (-4 \, a_2 - 6 \, b_1 - 12 \, a_1 - 4 \, a_{12} - 6 \, b_{12} - 8 \, a_0 - 2 \, b_{22} - 2 \, b_{11} - 4 \, b_0 + 6 \, b_2) \, x^3 +
+ (-2 \, b_1 - 8 \, b_2 - 14 \, b_0 - 12 \, a_0 - b_{12} - 6 \, a_2 - 4 \, a_{22} + 6 \, b_{22}) \, x^2 + (-4 \, b_{22} - 2 \, b_0 - 2 \, b_2) \, x,
\]

\[H(x) = (2 \, a_{12} - 18 \, a_{11}) \, x^6 + (2 \, a_{12} - 2 \, a_{11} - 8 \, a_1) \, x^5 +
+ (-2 \, a_{22} + 2 \, a_0 - b_{12} + a_2 + 3 \, b_1 + 3 \, a_1 + a_{12}) \, x^4 +
+ (2 \, b_1 + 2 \, b_{12} + 6 \, b_0 + 4 \, a_2 - 2 \, b_2 + 8 \, a_0 + 2 \, a_{22}) \, x^3 + (3 \, b_2 + 4 \, b_0) \, x^2,
\]
\[ K(x) = 4x^7a_{11} + (6a_{11} + 2a_1 - a_{12})x^6 + 2x^5a_1 + (-2a_0 - a_2 - b_1)x^4 - 2x^3b_0, \]

\[ L(x) = -2x^7a_{11} - x^6a_1. \]

The substitutions \( x = 0 \) and \( x = 1 \) into the (27) lead to the conditions on the value \( C \)

\[
(a_2 + a_{12} + b_1 + b_{12} + b_2 + 2a_0 + a_1 + 2b_0)C^2 +
-2b_0 - 2a_0 - b_1 - a_2 - b_{12} + 2b_{22})C - b_2 - 2b_{22} = 0,
\]

(28)

\[
(a_2 + a_{12} + b_1 + b_{12} + b_2 + 2a_0 + a_1 + 2b_0)C^2 +
(2a_{11} - 2b_0 - b_1 - 2a_0 - a_{12} - a_2)C - 2a_{11} - a_1 = 0.
\]

(29)

The substitution \((x = C, y = 1 - C)\) into the (26) lead to the conditions on the value \( C \)

\[
(b_{12} - 2b_{22})C + 2b_{22} + b_2 = 0.
\]

(30)

After substitution \((y = 1 - x, x = 1, C = 1/C_1)\) we get

\[
(a_1 + 2a_{11})C_1^2 + (a_{12} - 2a_{11} + b_1 + a_2 + 2a_0 + 2b_0)C_1 - a_1 - a_2 - a_{12} - 2b_0 -
-b_{12} - b_1 - b_2 - 2a_0 = 0,
\]

(31)

and the substitution \((y = 1 - x, x = 0, C = 1/C_1)\) lead to the condition

\[
(-2b_{22} - b_2)C_1^2 + (-2b_0 - b_{12} - b_1 - a_2 - 2a_0 + 2b_{22})C_1 + a_1 + a_2 + a_{12} + 2b_0 +
+b_{12} + b_1 + b_2 + 2a_0 = 0.
\]

(32)

Remark that for the comparison with results of the article \([4]\) in all above formulas the equation (2) was presented in the form

\[
\frac{dy}{dx} = \frac{a_0 + a_1x + a_2y + a_{11}x^2 + a_{12}xy + a_{22}y^2}{b_0 + b_1x + b_2y + b_{11}x^2 + b_{12}xy + b_{22}y^2}.
\]

Remark 1 In the famous article \([4]\) was developed the approach to the studying of the problem of the limit cycles of the equations (2).

Let us remind the basic facts of the Petrovsky-Landis theory.

For every closed curve of the system (2) the solution

\[
y(x) = \frac{C(x - 1)}{(x - C)}
\]

of equation

\[
\frac{dy}{dx} = -\frac{y(y - 1)}{x(x - 1)}
\]

(33)

is corresponded.

At the same time the value \( C \) satisfies the algebraic equations

\[
\sum a_n(\mu_1)C^n = 0,
\]
where the coefficients $a_n(\mu_i)$ are dependent from the parameters of equation (2).

This equation arises from the condition

$$\int_c (x-C)^2 [x(x-1)Q(x,y) + y(y-1)P(x,y)] \frac{dx}{x^3(x-1)^3} = 0.$$

The substitution of the function $y$ from the (33) into this expression and calculation of the residues with regard the points $0, 1, C$ lead to the equations on the parameter $C$.

According the (4) general quantity of the values $C$ defined by such equations is equal 14 and this number coincide with the quantity of closed solutions determined by the equations (2).

As it was shown in (4) 11 curves from 14 can be transformed into the vicinity of the particular points of the equation (30).

As result only three closed curves do not be transformed into the vicinity of the particular points and the quantity of the limit cycles defined by the equation (2) was found equal three.

In spite of the fact that the statement of the article (4) about the quantity of the limit cycles defined by the system (2) was fallacious its approach is useful from point of our consideration.

In fact the conditions on the value $C$ given by the (28)- (32) are the same with those which was used in the article (4) for estimation of the quantity of the limit cycles in quadratic polynomial system.

It can be shown that all conditions on residues of functions considered in the article (4) are followed from the (28)- (30).

### 7 The examples

We apply the second order ODE (24) for the studying of the properties of the first order ODE’s

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}.$$

1. Let us consider the system

$$\dot{y} = 8 - 3a - 14ax - 2axy - 8y^2,$$

$$\dot{x} = 2 + 4x - 4ax^2 + 12xy.$$  \tag{34}

Corresponding equation (24) for this system becomes

$$(-12y(x)^3x + (-2 + 4ax^2 + 8x^2)y(x)^2 + (2ax^3 + 2 - 6ax^2 + 4x)y(x) - 3ax - 8x^2 + 8x - 11ax^2 + 14ax^3) \frac{d^2}{dx^2} y(x) +$$

$$(-12 x^3 + 12 x^2) \left( \frac{d}{dx} y(x) \right)^3 +$$

$$+ (-4y(x)x^2 - 4 + 4x^2 - 2ax^2 + 24y(x)x - 8ax^2y(x) - 2ax^3 + 12 (y(x))^2 x + 4y(x)) \left( \frac{d}{dx} y(x) \right)^2 +$$

$$+ (-12y(x)^3 + (10ax + 16 + 8)y(x)^2 + (6a + 20ax + 2ax^2 - 12)y(x) + 16 - 8ax + 14ax^2 - 6a - 16x) \frac{dy(x)}{dx} y(x) +$$

$$+ 14ay(x) - 12a (y(x))^3 - 2a (y(x))^3 = 0$$  \tag{35}

and the function

$$\frac{dy(x)}{dx} = \frac{8 - 3a - 14ax - 2axy - 8y^2}{2 + 4x - 4ax^2 + 12xy}.$$  \tag{36}

is solution of this equation.

Moreover the function

$$\frac{dy(x)}{dx} = -\frac{y(y-1)}{x(x-1)}$$
also is solution of the equation (35).

Now we shall get another particular solutions of our equation (35).

The substitution
\[
\frac{dy}{dx} = \frac{M(x, y)}{N(x, y)}
\]
into the equation (35) give us the relation between the function \(M(x, y), N(x, y)\).

In spite of the fact that such relation is very complicated one can get with help of the MAPLE-6 some solutions of them.

For example, the presentation of the functions \(M(x, y), N(x, y)\) in form
\[
M(x, y) = A(x) + B(x)y + C(x)y^2, \quad N(x, y) = E(x) + F(x)y + H(x)y^2,
\]
where
\[
A(x) = 2x - 3ax^2 - 1, \quad B(x) = -1, \quad C(x) = -2x,
E(x) = x, \quad F(x) = 2x^2, \quad H(x) = 0,
\]
corresponds the system
\[
\frac{dy(x)}{dx} = \frac{2x - 3ax^2 - 1 - y - 2xy^2}{x + 2x^2y},
\]
compatible with the equation
\[
\frac{d}{dx} y(x) = \frac{8 - 3a - 14ax - 2axy(x) - 8y(x)^2}{2 + 4x + 4ax^2 + 12y(x)x}.
\]

In result we get the solution
\[
\frac{1}{4} + x - x^2 + ax^3 + xy(x) + x^2y(x)^2 = 0
\]
which present the limit cycle at the some value of the parameter \(a\).

2. Let us consider the system
\[
\dot{y} = x + 2y + 4xy + (2 + 3a)y^2,
\]
\[
\dot{x} = 5x + 6x^2 + 4(1 + a)xy + ay^2. \tag{37}
\]

Corresponding equation (24) for this system becomes
\[
\frac{(2x^2ay - 4x^2a - 2xy - 4x^3a + 4x^3 - 4x^2)}{\Delta(x, y)} \left( \frac{d}{dx} y(x) \right)^3 +
\]
\[
\frac{(-6xy - 20y^2a - 3x^2 - 10xy - 2y^2x - 4x^3 + 7x + 6xy)}{\Delta(x, y)} \left( \frac{d}{dx} y(x) \right)^2 +
\]
\[
\frac{(6xy^2a - 4ay^2 - 7y + 3y^2 + 20y^2x - x + x^2 + 4y^3 - 6xy + 4yx^2 + 4ay^3)}{\Delta(x, y)} \frac{d}{dx} y(x) +
\]
\[
\frac{d^2}{dx^2} y(x) + \frac{y + 3y^2 - 4y^3}{\Delta(x, y)} = 0, \tag{38}
\]
where
\[
\Delta(x, y) = y^4a + (-a + 4x + 4xa)y^3 + (-7xa - x + 3x^2a + 8x^2)y^2 + (-8x^2 - 7x + 4x^3)y - x^2 + x^3.
\]
The first order equation
\[
\frac{dy(x)}{dx} = \frac{x + 2y + 4xy + (2 + 3a)y^2}{5x + 6x^2 + 4(1 + a)xy + ay^2}.
\]
is particular integral of the second order equation (38).
The function \(y(x)\) defined by the relation
\[
x^2 + x^3 + x^2y(x) + 2axy(x)^2 + 2axy(x)^3 + a^2y(x)^4 = 0
\]
satisfies both equations (38) and (39).
Beyond this point the first order equation
\[
\frac{dy(x)}{dx} = \frac{2x + 3x^2 + 2xy + 2ay^2 + 2ay^3}{x^2 + 4axy + 6axy^2 + 4a^2y^3}
\]
resulting from the (40) by differentiation also satisfies the equation (38).
The presence of the two particular first order integrals for a given second order differential equation can be used for finding one them in evident form.
In concerned case we use the integral (39) as known and seek for the expression
\[
\frac{dy(x)}{dx} = \frac{M(x, y)}{N(x, y)}
\]
as the second integral with some functions \(M(x, y)\) and \(N(x, y)\) (for example as polynomial on variable \(y\) with the coefficients depending from the variable \(x\)).
After substitution both particular integrals into a given second order ODE (38) the functions \(M(x, y)\) and \(N(x, y)\) can be find in evident form from the corresponding condition of compatibility.

3. The next example is the system ([6])
\[
\dot{y} = 15(1 + a)y + 3a(1 + a)x^2 - 2(9 + 5a)xy + 16y^2;
\]
\[
\dot{x} = 6(1 + a)x + 2y - 6(2 + a)x^2 + 12xy.
\]
Corresponding equation (24) for this system becomes
\[
\frac{(-2x + 12x^3 - 10x^2)(\frac{d}{dx}y(x))^3}{\Delta(x, y)} + \\
\frac{(16xy - 20yx^2 + 18x^3 - 39x^2 - 2y^2 - 12y^2x + 21x - 12yxa + 10x^3a - 31ax^2a + 12yxa^2a + 21xa + 2y)(\frac{d}{dx}y(x))^2}{\Delta(x, y)} + \\
\frac{(6yx^2a^2 - 18yx^2 - 21y + 21y^2a - 10y^2x + 12y^3 - 21ya - 22y^2xa + 9y^2 - 4yx^2a + 54xy + 42yxa)(\frac{d}{dx}y(x))}{\Delta(x, y)} + \\
\frac{\frac{d^2}{dx^2}y(x) + 18y^3 - 18y^2 + 6xa^2y - 10yx^2 - 6xa^2y^2 + 10y^3a - 6y^2xa + 6yxa}{\Delta(x, y)} = 0,
\]
where
\[
\Delta(x, y) = (2 + 12x) y^3 + (-22x + 4x^2 - 2 - 6ax^2 + 6xa)y^2 + \\
+ (-10x^3a - 18x^3 + 31ax^2 - 21xa + 45x^2 - 21x)y - 3x^3a - 3x^3a^2 + 3x^4a + 3x^4a^2.
\]

8 The spatial first order system of equations

Generalization of the Petrovsky-Landis approach to the spatial first order systems of equations may be realized by the following way.
8.1

Instead of the spatial first order system of equations

$$\frac{dx}{ds} = P(x, y, z), \quad \frac{dy}{ds} = Q(x, y, z), \quad \frac{dz}{ds} = R(x, y, z)$$ (43)

we consider the Pfaff equation

$$P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = 0$$ (44)

for the orthogonal trajectories of the system (43).

The relation (44) can be considered as the linear first integral of geodesics of the six-dimensional space in local coordinates \((x, y, z, U, V, W)\) with the Riemann metric of the form

$$\frac{6}{ds^2} = -2(\Gamma^1_{33}U + \Gamma^2_{33}V + \Gamma^3_{33}W)dx^2 - 2(\Gamma^1_{22}U + \Gamma^2_{22}V + \Gamma^3_{22}W)dy^2 -$$
$$-4(\Gamma^1_{12}U + \Gamma^2_{12}V + \Gamma^3_{12}W)dxdy -$$
$$-4(\Gamma^1_{13}U + \Gamma^2_{13}V + \Gamma^3_{13}W)dxdz +$$
$$+2dxdu + 2dydV + 2dzdW,$$ (45)

where \(\Gamma^k_{ij} = \Gamma^k_{ij}(x, y, z)\).

One part of geodesics of the metric (45) has the form of deodesics of the three-dimensional space with the affine connection \(\Gamma^k_{ij} = \Gamma^k_{ij}(x, y, z)\)

$$\frac{d^2x}{ds^2} + \Gamma^1_{11} \left( \frac{dx}{ds} \right)^2 + \Gamma^1_{12} \left( \frac{dy}{ds} \right)^2 + \Gamma^1_{13} \left( \frac{dz}{ds} \right)^2 + 2\Gamma^1_{12} \frac{dx}{ds} \frac{dy}{ds} + 2\Gamma^1_{13} \frac{dx}{ds} \frac{dz}{ds} + 2\Gamma^1_{23} \frac{dy}{ds} \frac{dz}{ds} = 0,$$

$$\frac{d^2y}{ds^2} + \Gamma^2_{11} \left( \frac{dx}{ds} \right)^2 + \Gamma^2_{22} \left( \frac{dy}{ds} \right)^2 + \Gamma^2_{23} \left( \frac{dz}{ds} \right)^2 + 2\Gamma^2_{12} \frac{dx}{ds} \frac{dy}{ds} + 2\Gamma^2_{13} \frac{dx}{ds} \frac{dz}{ds} + 2\Gamma^2_{23} \frac{dy}{ds} \frac{dz}{ds} = 0,$$

$$\frac{d^2z}{ds^2} + \Gamma^3_{11} \left( \frac{dx}{ds} \right)^2 + \Gamma^3_{22} \left( \frac{dy}{ds} \right)^2 + \Gamma^3_{33} \left( \frac{dz}{ds} \right)^2 + 2\Gamma^3_{12} \frac{dx}{ds} \frac{dy}{ds} + 2\Gamma^3_{13} \frac{dx}{ds} \frac{dz}{ds} + 2\Gamma^3_{23} \frac{dy}{ds} \frac{dz}{ds} = 0$$

and another part of geodesics is the linear system of the second order differential equations for the coordinates \(U, V, W\).

Remind that if the relation

$$a_i \frac{dx^i}{ds} = 0$$

is the linear integral of geodesics

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0$$

then the conditions

$$a_{i,j} + a_{j,i} = 0$$

must be satisfy.

In the space with the metric (45) these conditions allow us to determine only six coefficients \(\Gamma^k_{ij}(x, y, z)\) from the eighteen one.

For determination of others coefficients \(\Gamma^k_{ij}(x, y, z)\) we use the auxiliary equations same with the Petrovsky-Landis theory

$$y(y - 1)dx + x(x - 1)dy = 0, \quad z(z - 1)dx + x(x - 1)dz = 0.$$

Every of these relations can be considered as the linear first integral of geodesics of the metric (45) and this allow us to determined all coefficients of connection \(\Gamma^k_{ij}(x, y, z)\).
Some of them are

\[
\Gamma_{11}^{1}(x, y, z) = \frac{(\partial_{x} P(x, y, z)) x(x - 1)}{P(x, y, z) x^2 - P(x, y, z)x - Q(x, y, z)y^2 + Q(x, y, z)y - R(x, y, z) z^2 + R(x, y, z) z},
\]

\[
\Gamma_{11}^{3}(x, y, z) = \frac{y(y - 1) \frac{\partial}{\partial y} P(x, y, z)}{P(x, y, z) x^2 - P(x, y, z)x - Q(x, y, z)y^2 + Q(x, y, z)y - R(x, y, z) z^2 + R(x, y, z) z},
\]

\[
\Gamma_{13}^{1}(x, y, z) = \frac{y(y - 1) (\frac{\partial}{\partial y} R(x, y, z)(x - x^2) + 2R(x, y, z)(x - 1 + z) - \frac{\partial}{\partial x} P(x, y, z)(x^2 - x))}{P(x, y, z) x^2 - P(x, y, z)x - Q(x, y, z)y^2 + Q(x, y, z)y - R(x, y, z) z^2 + R(x, y, z) z},
\]

\[
\Gamma_{22}^{1}(x, y, z) = \frac{y(y - 1) \frac{\partial}{\partial y} Q(x, y, z)}{P(x, y, z) x^2 - P(x, y, z)x - Q(x, y, z)y^2 + Q(x, y, z)y - R(x, y, z) z^2 + R(x, y, z) z},
\]

\[
\Gamma_{22}^{2}(x, y, z) = - \frac{z(z - 1) \frac{\partial}{\partial y} Q(x, y, z)}{P(x, y, z) x^2 - P(x, y, z)x - Q(x, y, z)y^2 + Q(x, y, z)y - R(x, y, z) z^2 + R(x, y, z) z},
\]

\[
\Gamma_{23}^{1}(x, y, z) = \frac{x(x - 1) (\frac{\partial}{\partial y} R(x, y, z) + \frac{\partial}{\partial z} Q(x, y, z))}{P(x, y, z) x^2 - P(x, y, z)x - Q(x, y, z)y^2 + Q(x, y, z)y - R(x, y, z) z^2 + R(x, y, z) z},
\]

\[
\Gamma_{23}^{3}(x, y, z) = \frac{1}{y(y - 1)} \frac{\partial}{\partial y} R(x, y, z) + \frac{\partial}{\partial z} Q(x, y, z)
\]

So the six-dimensional space with the metric (45) is suitable geometric object for the studying of the properties of the spatial first order system of equations (43).
Another approach for the studying of the spatial system of equations is connected with consideration of the system (43)

\[ Q(x, y, z)dx - P(x, y, z)dy = 0, \quad R(x, y, z)dx - P(x, y, z)dz = 0 \] (46)

together with the relation

\[ xdx + ydy + zdz = 0, \quad x^2 + y^2 + z^2 = C \] (47)

(or another ones) as the linear integrals of geodesics of the six-dimensional space.

From these conditions the eighteen coefficients of connections \( \Gamma^k_{ij} \) can be uniquely determined. In fact, after the continuation of the system (46) on the projective plane we get the Pfaff equation

\[
\frac{d^2}{ds^2}x(s) + \left( \left( \frac{\partial}{\partial x}R(x, y, z) \right) z + P(x, y, z) + y \frac{\partial}{\partial x}Q(x, y, z) \right) \left( \frac{d}{ds}x(s) \right)^2
\]

\[ + \left( - \left( \frac{\partial}{\partial y}P(x, y, z) \right) y + \left( \frac{\partial}{\partial y}Q(x, y, z) \right) y + \left( \frac{\partial}{\partial z}R(x, y, z) \right) z \right) \left( \frac{d}{ds}x(s) \right) \frac{d}{ds}y(s) - \right.

\[ \left. \left( \frac{\partial}{\partial y}P(x, y, z) \right) y - P(x, y, z) \right) \left( \frac{d}{ds}y(s) \right)^2
\]

\[ - \left( \frac{\partial}{\partial z}R(x, y, z) \right) z + P(x, y, z) \right) \left( \frac{d}{ds}z(s) \right) \frac{d}{ds}z(s)
\]

\[ yQ(x, y, z) + R(x, y, z)z + P(x, y, z)x \]

\[ \left( \frac{\partial}{\partial y}P(x, y, z) \right) y - P(x, y, z) \right) \left( \frac{d}{ds}y(s) \right)^2
\]

\[ yQ(x, y, z) + R(x, y, z)z + P(x, y, z)x \]

\[ \left( \frac{\partial}{\partial z}R(x, y, z) \right) z + P(x, y, z) \right) \left( \frac{d}{ds}z(s) \right) \frac{d}{ds}z(s)
\]

\[ yQ(x, y, z) + R(x, y, z)z + P(x, y, z)x \]

\[ = 0, \]

and analogous equations for the

\[ \frac{d^2}{ds^2}y(s) + ... = 0 \]

and

\[ \frac{d^2}{ds^2}z(s) + ... = 0 \]

In result the six-dimensional space with the geodesics, having the linear integral in form (46)-(47) has been constructed.

## 9 Projectivization of the planar system

We apply the geometric approach at the studying of the Pfaff equations (44) connected with a planar systems of equations

\[
\frac{dx}{ds} = p(x, y), \quad \frac{dy}{ds} = q(x, y)
\] (48)

after their projectivization.

In fact after the continuation of the system (46) on the projective plane we get the Pfaff equation

\[-zQ(x, y, z)dx + zP(x, y, z)dy + (xQ(x, y, z) - yP(x, y, z))dz = 0\]

where \( P(x, y, z), Q(x, y, z) \) are the homogeneous functions constructed with the help of given functions \( p(x, y), q(x, y) \).
9 PROJECTIVIZATION OF THE PLANAR SYSTEM

As example we consider the equation
\[
\frac{dx}{ds} = \lambda x - y - 10x^2 + (5 + \delta)xy + y^2, \quad \frac{dy}{ds} = x + x^2 + (\epsilon - 25)xy
\]
having at least a four limit cycles at some restriction on the parameters \(\lambda, \delta, \epsilon\).

After its projectivization we get the Pfaff equation
\[
(-y^2x\delta - y\lambda xz + zy^2 - 5y^2x + zz^2 + x^3 + x^2y\epsilon - 15x^2y - y^3)\,dz + (z^2\lambda x - z^2y + 5zxy\delta + zy^2 - 10z\,x^2)\,dy + (-zxy\epsilon + 25zxy - z^2x - zz^2)\,dx = 0.
\]

For determination of the connections coefficients of corresponding six-dimensional space we add at this equation the relations
\[
y(y - 1)dx + x(x - 1)dy = 0, \quad z(z - 1)dx + x(x - 1)dz = 0
\]
and consider all of them as the linear integrals of geodesics.

In result we get a six-dimensional space whose geodesics have at least a four limit cycles.

9.1 The Lorenz system

The next example is the system of equations
\[
\frac{dx}{ds} = P(x, y, z), \quad \frac{dy}{ds} = Q(x, y, z), \quad \frac{dz}{ds} = R(x, y, z)
\]
with some functions \(P, Q, R\).

We consider the relations
\[
Q(x, y, z)dx - P(x, y, z)dy = 0, \quad R(x, y, z)dx - P(x, y, z)dz = 0
\]
as the linear first integrals of geodesics of corresponding six-dimensional space.

For determination of the connection coefficients we use the invariant conditions
\[
\Gamma_{11} + \Gamma_{12} + \Gamma_{13} = 0, \quad \Gamma_{12} + \Gamma_{22} + \Gamma_{23} = 0, \quad \Gamma_{13} + \Gamma_{23} + \Gamma_{33} = 0,
\]
and the freedom in the choice of coordinate systems.

We use a following normalization of coordinate system
\[
\Gamma_{11} = 0, \quad \Gamma_{12} = -\Gamma_{23}, \quad \Gamma_{22} = 0, \quad \Gamma_{12} = -\Gamma_{13}, \quad \Gamma_{33} = 0 \quad \Gamma_{13} = -\Gamma_{23}.
\]

In result all connection coefficients of the space can be determined uniquely and we get the expressions for the connection coefficients
\[
\Gamma_{23}^3(x, y, z) = -1/4 \frac{-Q\frac{\partial}{\partial y}P + R\frac{\partial}{\partial z}P + P\frac{\partial}{\partial x}R - 2P\frac{\partial}{\partial y}P + P\frac{\partial}{\partial y}Q}{PQ},
\]
\[
\Gamma_{33}^3(x, y, z) = - \frac{Q(x, y, z)\frac{\partial}{\partial z}P(x, y, z)}{P(x, y, z)R(x, y, z)},
\]
\[
\Gamma_{12}^3(x, y, z) = 1/4 \frac{-RQ\frac{\partial}{\partial y}P + (R)^2\frac{\partial}{\partial z}P + PR\frac{\partial}{\partial z}R - 2QR\frac{\partial}{\partial y}P - 2PR\frac{\partial}{\partial z}P + PR\frac{\partial}{\partial y}Q}{Q(P)^2}
\]
\[
\Gamma_{13}^2(x, y, z) = -1/4 \frac{2QP\frac{\partial}{\partial z}P - QP\frac{\partial}{\partial y}Q - (Q)^2\frac{\partial}{\partial y}P + QR\frac{\partial}{\partial z}P - QP\frac{\partial}{\partial z}R}{P(R)^2},
\]
\[
\Gamma_{23}^2(x, y, z) = \frac{2PR\frac{\partial}{\partial z}P - QR\frac{\partial}{\partial z}R + 2\frac{\partial}{\partial y}Q}{P(R)^2}.
\]
\[ \Gamma_{22}^1(x, y, z) = -\frac{\partial^2 P(x, y, z)}{Q(x, y, z)}, \]

\[ \Gamma_{22}^3(x, y, z) = -\frac{R(x, y, z) \frac{\partial^2 P(x, y, z)}{P(x, y, z)} - \frac{\partial^2 Q(x, y, z)}{P(x, y, z)}}{Q(x, y, z)}, \]

\[ \Gamma_{23}^1(x, y, z) = -\frac{1}{2} \frac{2 P \frac{\partial}{\partial x} P - P \frac{\partial}{\partial y} Q - Q \frac{\partial}{\partial y} P + R \frac{\partial}{\partial z} P + P \frac{\partial}{\partial z} R}{R Q}. \]

The corresponding metric is

\[ 6 ds^2 = -2 \left( \Gamma_{11}^2(x, y, z) V + \Gamma_{11}^3(x, y, z) W \right) dx^2 - 
-4 \left( -\Gamma_{23}^3(x, y, z) U - \Gamma_{13}^3(x, y, z) V + \Gamma_{12}^3(x, y, z) W \right) dydz - 
-4 \left( -\Gamma_{23}^3(x, y, z) U + \Gamma_{13}^3(x, y, z) V + \Gamma_{12}^3(x, y, z) W \right) dydz - 
-4 \left( -\Gamma_{23}^3(x, y, z) U + \Gamma_{13}^3(x, y, z) V + \Gamma_{12}^3(x, y, z) W \right) dydz - 
-2 \left( \Gamma_{13}^3(x, y, z) V + \Gamma_{33}^3(x, y, z) V \right) dz^2 + 2 dx dy + 2 dy dz + 2 dz dw. \tag{49} \]

Remark that the conditions

\[ a_{i;j;k} + R_{kij}^m a_m = 0 \]

for the vectors

\[ a_i = [P(x, y, z), -Q(x, y, z), 0, 0, 0, 0] \]

and

\[ a_i = [P(x, y, z), 0, -R(x, y, z), 0, 0, 0] \]

are obeyed for the metric (49).

In particular case of the Lorenz system of equations

\[ \frac{dx}{ds} = \sigma(y - x), \quad \frac{dy}{ds} = rx - y - xz, \quad \frac{dz}{ds} = xy - bz, \]

we have

\[ P(x, y, z) = \sigma(y - x), \quad Q(x, y, z) = rx - y - xz, \quad R(x, y, z) = xy - bz, \]

where \( \sigma, r, b \) are the parameters and geodesics of the metric (49) are equivalent to the expressions

\[ \frac{d^2 y(x)}{dx^2} = \frac{\sigma \left( \frac{d^2 y(x)}{dx^2} \right)^3}{-rx + y + xz} + 1/2 \left( \frac{x + xb - 2 \sigma x + rx - xz + 2 \sigma y - yb - 2 yz}{rx^2 y - yzx^2 - y^2 x + xzb^2 - xzrb + ybz} \right) \frac{d^2 z(x)}{dx^2} - 
\frac{(x + xb - 2 \sigma x + rx - xz + 2 \sigma y - yb - 2 y) \left( \frac{d^2 y(x)}{dx^2} \right) \frac{d^2 z(x)}{dx^2}}{bzx - ybz - yx^2 + y^2 x}. \]
-1/2 \left( -\frac{yb + xb + zz - 2 x \sigma + x - rx + 2 \sigma y}{rx - y^2 - yzx - rx^2 + xy + zx^2} \right) \left( \frac{dy}{dx} \right)^2 
+ \frac{d^2}{dx^2} z(x) \left( \frac{dx}{dy} \right)^2 
+ \frac{1}{2} \left( -\frac{yb + xb + zz - x - rx + 2 y}{y^2 - 2xy + x^2} \right) \frac{dz}{dx} \left( \frac{dx}{dy} \right) 
\frac{\sigma}{\sigma - y + x} 
\frac{\frac{d^2}{dx^2} z(x)}{\frac{d}{dx} y(x)} = 0,
and the second order linear system differential equations for the coordinates $U(s), V(s), W(s)$.

These equations have the solutions in form of the first order differential equations

$$
\frac{dy}{dx} = \frac{rx - y - xz}{\sigma(y - x)}, \quad \frac{dz}{dx} = \frac{xy - bz}{\sigma(y - x)}.
$$

Geometrical properties of the metric (49) are dependent from the parameters $\sigma, b, r$.
For example the component $R_{zz}$ the Ricci tensor $R_{ij}$ look as

$$
R_{zz} = -\frac{1}{2} \frac{x}{(xy + bz)(-y + x)} \left( \frac{dx}{dy} \right)^2 
$$
and a more complicated expressions for another components.

Finally we present the expression for the density of the Chern-Simons invariant of affine connection for the space defined by the Lorenz system of equation.

In case of three dimensional space it is defined as ([6])

$$
CS(\Gamma) = e^{ijk}(\Gamma^p_{iq} \Gamma^q_{kp,j} + \frac{2}{3} \Gamma^p_{iq} \Gamma^q_{jr} \Gamma^r_{kp})
$$

(50)
where $e^{ijk}$ is a Levi-Civita symbols.

For the six-dimensional metric we get the expression

$$
4 \sigma \left( -y + x \right)^4 \left( -rx + y + zx \right)^2 \left( -xy + bz \right)^2 CS(\Gamma) = 
-9 z^5 x^6 b + \left( (10 x^5 + (11 b^2 + 48 \sigma b - 40 b) x^3) y + (11 b^2 + 4 b + 36 rb - 25 \sigma b) x^4 - 23 \sigma y^2 x^2 b - x^6 \right) z^4 + 
+ M3(x,y) z^3 + M4(x,y) z^2 + M5(x,y) z + M6(x,y),
$$

(51)
where

$$
M3(x,y) = (4 r - 2 \sigma) x^6 + (18 b - 40 r - 8 + 32 \sigma) y x^5 + 
+ \left( (18 b + 44 - 56 \sigma) y^2 - 14 \sigma b^2 - 54 b x^2 - 12 b + 18 \sigma b^2 - 4 \sigma b + 75 \sigma b r + 5 b^3 - 12 rb + 26 rb^2 \right) x^4 + 
+ (26 \sigma y^3 + (-144 \sigma rb - 2 b + 80 \sigma^2 b + 120 rb - 66 \sigma b - 10 b^3 - 50 b^2 - 26 rb^2 + 14 \sigma b^2) y) x^3 + 
$$
The functions $M4(x, y) - M6(x, y)$ are dependent from the parameters and are too cumbersome. Remark that in the case $x = y$ for the right part of the (51) we get the expression

$$-9y^4(z + 1 - r)^4(bz - y^2).$$

which is equal to zero on the stationary points of the Lorenz model

$$z = r - 1, \quad y = \sqrt{b(r - 1)}.$$

More detail information on the properties of the six-dimensional Riemann space connected with the Lorenz system of equations can be obtained with the help of the studying of the linear system of equations for the coordinates $U, V, W$.

The studying of the equation

$$g^{ij}\nabla_i \nabla_j A_k - R^l_k A_l + \lambda A_k = 0$$

for the eigenvalues $\lambda$ of the de Rham operator, defined on the 1-forms

$$A(x, y, z) = A_i(x, y, z)dx^i$$

of manifold can be also useful for that.

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