CONSTANT $Q$-CURVATURE METRICS ON CONIC 4-MANIFOLDS

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ABSTRACT. We consider the constant $Q$-curvature metric problem in the given conformal class on conic 4-manifolds and study related differential equations. We define subcritical, critical, and supercritical conic 4-manifolds. Following [Tr91] and [CY95], we prove the existence of constant $Q$-curvature metrics in subcritical cases. For conic 4-spheres with two singular points, we prove the uniqueness in critical cases and nonexistence in supercritical cases. As a byproduct, we also give the asymptotic expansion of the corresponding PDE near isolated singularities.

1. Introduction

In this paper, we study Branson’s $Q$-curvature on conic 4-manifolds. We first recall some definition. Let $(M^4, g)$ be a compact 4 dimensional smooth Riemannian manifold. Branson’s $Q$-curvature [BO91] is defined as

$$Q_g = \frac{-1}{6}\Delta R - \frac{1}{2}|Ric|^2 + \frac{1}{6}R^2,$$

where $R$ is the scalar curvature and $Ric$ is the Ricci curvature tensor of $g$ respectively. Similar to the role of Gaussian curvature plays in the theory of surfaces geometry, $Q$-curvature is related to the geometry of 4-manifolds by a Gauss-Bonnet-Chern formula:

$$\int_M QdV_g = 8\pi^2\chi(M) - \int_M \frac{1}{4}|W|^2dV_g,$$

where $W$ is the Weyl tensor. Let

$$k_g = \int_M QdV_g.$$ 

Since Weyl tensor is locally conformally invariant, identity (1.1) shows that $k_g$ is a global conformal invariant. Suppose that $g_w = e^{2w}g$, where $w \in C^\infty$. The corresponding $Q$-curvature for $g_w$ is given by

$$P_gw + Q_g = e^{4w}Q_{g_w}.$$ 

Here $P_g$ is the Paneitz operator

$$P_gw = \Delta^2_gw + \text{div} \left( \frac{2}{3}Rg - 2Ric \right) dw,$$

where $A_g = \frac{2}{3}Rg - 2Ric$. Paneitz operator is conformally covariant, namely $P_gu = e^{-4w}P_g$ for $g_w = e^{2w}g$.

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A natural question in conformal geometry is the existence of a constant $Q$-curvature metric in a given conformal class. This is the 4-dimensional extension of the classical uniformization theorem for surfaces. Finding a constant $Q$-curvature metric is equivalent to finding the solution of
\begin{equation}
\label{eq:1.2}
P_g w + Q_g = c \times e^{4w},
\end{equation}
where $c$ is a constant. This equation is in fact the Euler-Lagrange equation of $II_g$ functional which is first defined by Branson and Ørsted [BØ91]:
\begin{equation}
\label{eq:1.3}
II_g(u) = \int_M u P_g u dV_g + 2 \int_M Q u dV_g - \frac{k_g}{2} \log \int_M \exp(4u) dV_g.
\end{equation}
There are many works on this problem. A fundamental work by Chang and Yang [CY95] establishes the existence of constant $Q$-curvature metric if the conformal metric satisfies
a) nonnegative $P_g$ ,
b) $\text{Ker} P_g = \{\text{constants}\}$,
c) $k_g < 16\pi^2$.
The number $16\pi^2$ comes from a sharp Moser-Trudinger type inequality due to Adams [Ada88] which has been used by Chang-Yang [CY95] to obtain a minimizer of $II_g$. We should remark that this existence result covers many cases since condition (a), (b), and (c) are satisfied when the (i) Yamabe constant $Y_g \geq 0$, (ii) $k_g \geq 0$, and (iii) $M$ not conformal to $S^4$ by Gursky [Gur99]. If $k_g > 16\pi^2$, the $II_g$ might not have minimizers or even lower bound. However, Djadli and Malchiodi [DM08] are able to show that the existence holds for $k_g \neq 16l\pi^2, l = 1, 2, 3, \ldots$, by a delicate min-max argument. For $k_g = 16\pi^2$ with some additional assumptions, the existence results hold by J. Li, Y. Li and P. Liu [LLL12].

We now introduce conic 4-manifolds. Let $(M^4, g_0)$ be a compact smooth Riemannian 4-manifold. The conical singularities are represented by the conformal divisor
\begin{equation}
\label{eq:1.4}
D = \sum_{i=1}^k p_i \beta_i,
\end{equation}
where $p_i \in M$ and $0 > \beta_i > -1$. We assume that $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_k$. Let $\gamma(x)$ be a function in $C^\infty(M^4 - \{p_i\})$ with following form:
\begin{equation}
\label{eq:1.5}
\gamma(x) = \sum \beta_i \log(r_i)\eta_i(x) + f(x),
\end{equation}
where $r_i = \text{dist}_{g_0}(x, p_i)$ is the distance to $p_i$ and $f(x)$ is smooth in a neighborhood of $p_i$. $\eta_i(x)$ is a positive function that equals 1 in a neighborhood of $p_i$ and have support in a small ball around $p_i$ with radius smaller than the injective radius at $p_i$. Let
\begin{equation}
\label{eq:1.6}
g_D = e^{2\gamma} g_0.
\end{equation}
g_D is a metric conformal to $g_0$ on $M^4 - \{p_i\}$ which has a conical singularity at each $p_i$. Let $dV_D$ be the volume element of $g_D$. Let $H^2(dV_D) = W^{2,2}(dV_D)$ be corresponding Sobolev space. We define the conformal class of $g_D$
\begin{equation}
\label{eq:1.7}
[g_D] := \{g_w = e^{2w} g_D : w \in H^2(dV_D) \cap C^\infty(M - \{p_i\})\}.
\end{equation}
Note $[g_D]$ depends only on $(M^4, g_0)$ and $D$. Function $\log|x - p_i|$ is not in $H^2_{\text{loc}}(dV_D)$ and neither is $\gamma(x)$. Let $g_1 \in [g_D]$. We call a 4-tuple $(M^4, g_0, D, g_1)$ a conic manifold, $g_0$ the background metric and $g_1$ the conic metric. Just like conic
surfaces, for conic 4-manifolds, we have the corresponding Gauss-Bonnet-Chern formula states that if $g_1$ is a conic metric with divisor $D = \sum_{i=1}^{k} p_i \beta_i$, then
\[
k_{g_1} = \int_M Q_{g_1} dV_{g_1} = \int_M Q_{g_0} dV_{g_0} + \sum_{i=1}^{k} \beta_i.
\]
We will give a detailed proof of this Gauss-Bonnet-Chern formula in Section 3.

In this paper, we would like to study the constant $Q$-curvature metric for conic 4-manifolds. Our motivation are both analytical and geometrical. Analytically, Moser-Trudinger-Adams inequality gives the key estimate in works of [BCY92, CY95, DM08]. We would like to find a proper singular setting so that Moser-Trudinger-Adams inequality can be generalized and applied naturally. Geometrically, the study of stability condition and conic singularities plays a central role in recent developments in Kähler geometry [CDS14, CDS15, CDS13, Tia96, Tia15]. We would like to find a suitable conformal “stability” condition in conformal geometry.

We now describe our problem in detail. For conic 4-manifold $(M, g_0, D, g_1)$, our goal is to find a good metric $g_w \in [g_D]$ in the conformal class of $g_D$ such that the $Q$-curvature $Q_{g_w}$ is a constant. Finding such metrics is equivalent to finding (weak) solutions in $H^2(dV_D) = W^{2,2}(dV_D)$ of the equation
\[
P_{g_D} w + Q_D = c \times \exp(4w).
\]
In fact, $H^2(dV_D)$ is equivalent to $H^2(dV_0)$ space of $g_0$ and by Poincaré inequality, the $H^2(dV_0)$ norm of $w$ is given by
\[
\|u\|_{H^2(dV_0)}^2 = \int_M u P_{g_0} u dV_0 + \|u\|_{L^2(dV_0)}^2,
\]
when condition (a),(b) are satisfied, see Section 2.

Our approach to study conic 4-manifolds comes from the pioneering works of Troyanov for conic surfaces. He [Tro89, Tro91] systematically studied the prescribed curvature problem on conic surfaces. In particular, Troyanov [Tro91] classifies conformal metric on conic Riemann surfaces into three categories: subcritical, critical, and supercritical. He shows that in subcritical cases, there exists a unique constant Gaussian curvature metric. On sphere with 2 singularities, the existence happens only if the divisor $D = \beta_1 p + \beta_2 q$ for some $p, q$ [Tro89]. By a geometric construction, Luo and Tian [LT92] proved that with more than 2 conic points the solution exists if and only if in subcritical case. Chen and Li [CL95] gave the same results but from a PDE prospective. Chen and Li also proved that in critical case, the only solution is radial symmetric, like an American football. In a recent work, the first named author and Lai described the limiting process when a subcritical metric deforms continuously towards a critical one. They showed that the geometric picture converges to a “football” in Hausdorff-Gromov topology [FL16].

Motivated by Troyanov's classification, we define the following conic 4-manifolds:

i) **subcritical** $k_{g_0} + 8\pi^2 (\sum_i \beta_i) < 8\pi^2 (2 + 2\beta_1)$,

ii) **critical** $k_{g_0} + 8\pi^2 (\sum_i \beta_i) = 8\pi^2 (2 + 2\beta_1)$,

iii) **supercritical** $k_{g_0} + 8\pi^2 (\sum_i \beta_i) > 8\pi^2 (2 + 2\beta_1)$.

In this paper, we will primarily consider the subcritical case on general 4-manifolds and critical case on sphere.

We now state our main results. Our first result deals with the subcritical case:
Theorem 1.1. Let \((M^4,g_0,D,g_D)\) be a conic 4-manifold. Suppose that \(P_{g_0}\) is nonnegative and its kernel contains only constant functions on \((M^4,g_0)\). Let \(\beta_1 = \min\{\beta_i\}\). Suppose \(k_{g_0} + 8\pi^2(\sum_i \beta_i) < 8\pi^2(2 + 2\beta_1)\). Then there is a conformal metric \(g_w \in [g_D]\) represented by \(w \in C^\tau(M) \cap C^\infty(M - \{p_i\})\) for \(0 < \tau < \min\{2, 4(1 + \beta_1)\}\) such that \((M^4,g_w)\) has constant \(Q\)-curvature.

In fact, following [CY95] and [Tro91], we can prove that the functional \(I_g\) is coercive and hence we can show the existence of the minimizer of \(I_g\). The minimizer turns out to be a good solution of \((1.5)\) by elliptic regularity theory. The subcritical condition comes from a conic version of Moser-Trudinger-Adams inequality. The Euclidean case of this type of inequality is due to Lam and Lu [LL11]. Following [BCY92] and [CY95], with a little effort, we generalize this inequality to manifolds.

We also consider the solution with 2 singularities on conic 4 sphere. By stereographic projection, we only have to consider the PDE in \(\mathbb{R}^4\),

\[(1.6) \quad \Delta^2 u = e^{4u}.\]

By a conformal transformation, we can assume that the 2 singularities are at origin and infinity. We then discuss the radial symmetric solutions. Using the cylinder coordinate, we can reduce the PDE to a 4th order ODE,

\[(1.7) \quad v'''(t) - 4v''(t) = e^{4u}.\]

Theorem 1.2. There is a family of solutions \(v_\alpha\) of \((1.7)\), parametrized by \(\alpha = 1 + \beta > 0\) such that \(v_\alpha(t)\) goes to \(\pm \alpha\) as \(t\) goes to \(\pm \infty\). Differing by a constant and a translation in \(t\), these are the only solutions with linear growth at infinity.

Note that both singular points have the same \(\beta\) hence \(v_\alpha\) represents a 4 dimensional analogue of “football”. In fact, these are the only possible solutions with 2 conical singularities since we have the following uniqueness theorem:

Theorem 1.3. A constant \(Q\)-curvature 4-sphere with 2-singularities must be conformal to a radial symmetric conic sphere. Both singularities have the same index. Under the cylinder coordinate, the metric is given by Theorem 1.2.

In smooth cases, the symmetry of \((1.6)\) has been established by Lin [Lin98] using the moving plane method. Lin investigated the asymptotic behavior of the solution \(u\) and proved that \(\Delta u\) has an asymptotic harmonic expansion at infinity. This expansion implies certain monotonicity of \(u\) and allows one to initiate moving plane method near infinity. We should mention that Caffarelli, Gidas, and Spruck [CGS89] were the first to investigate such expansion and along with moving plane method to study semilinear elliptic equation \(-\Delta u = u^{\frac{n+2}{n-2}}\). Since we allow singularities at the origin and at the infinity, we do not expect to have an exact expansion like those in Lin [Lin98]. However, by careful analysis, we can establish a similar asymptotic expansion near each singularity. It generalizes our regularity theorem in Theorem 1.2. See details in section 7.

We intend to investigate the critical case and supercritical case further. We expect some nonexistence results for constant \(Q\)-curvature metric in supercritical cases and critical cases with more than 3 singular points. The supercritical cases are more elusive since we have the existence result [DM08]. We would like to mention that a recent work by the first named author and Wei [FW18] derives
a similar criterion for the existence of constant $\sigma_2$ curvature metric on conic 4-manifolds. In particular, they establish the nonexistence result for supercritical cases and uniqueness result for critical cases.

We organize the paper as follows. In Section 2, we discuss some function spaces and embedding theorems with conical singularities. In Section 3, we derive a Gauss-Bonnet-Chern formula resembling the one for Riemann surfaces. In Section 4, we establish a Moser-Trudinger-Adams type inequality for conic manifolds. In Section 5, we give the proof of Theorem 1.1. In Section 6, we study the radial symmetric solutions and prove Theorem 1.2. In Section 7, we study the asymptotic behavior of the solutions near a conical singularity. In Section 8, we use the asymptotic expansion from Section 7 to prove Theorem 1.3.

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2. Function spaces

In this section, we describe several function spaces that we will use later to study the $\mathcal{H}$ functional. Note this time we consider the general dimension $n$ instead of 4-dimension.

Let $(M^n, g_0, D, g_D)$ be an $n$ dimensional conic manifold according to the definitions similar to (1.3) and (1.4) where $D = \sum_{i=1}^{\infty} \beta_i p_i$. Likewise, we assume that $g_D = e^{2\gamma(x)} g_0$.

$\gamma$ is given by:

$$
\gamma(x) = \sum \beta_i \log(r_i) \eta_i(x) + f(x),
$$

where $f(x)$ is smooth and $\eta_i(x)$ is a smooth cutoff function equal to 1 in a neighborhood of $p_i$ and $\text{supp}(\eta_i) \subset B_{\delta_i}(p_i)$. Here $\delta_i$ is the injective radius at each $p_i$ with metric $g_0$. Let $g_i = g_D$ and $dV_i$ be the volume elements of $g_i$ for $i = 0, 1$ respectively. We can define $H^2(dV_i)$ norm of a function $u$ in $C^\infty(M)$ by

$$
\|u\|_{H^2(dV_i)}^2 = \int_M |u|^2 dV_i + \int_M |\nabla_{g_i} u|^2 dV_i + \int_M |\nabla^2_{g_i} u|^2 dV_i,
$$

for $i = 0, 1$. Let $H^2(dV_i)$ be the closure of $C^\infty(M)$ under the $H^2(dV_i)$ norm. Although $g_i$ is not smooth, the related $L^p$ spaces have certain comparison theorem and $H^2(dV_i)$ are actually the same. This is in fact a crucial point in Troyanov’s proof [Tro91]. Besides, we still have Poincaré inequality and compact embedding theorem for Sobolev space just like the classic one. The following results are similar to [Tro91] which was originally applied to 2 dimensional surfaces and $H^1$ space.

**Proposition 2.1.** (Weighted Sobolev inequality) Let $\Omega$ be an open domain in $\mathbb{R}^n$ and $\beta > -1$. Then there is some constant $C(\Omega)$, independent of $p$ such that for any $u \in C^2_c(\Omega)$,

$$
\left( \int_\Omega |u|^p \cdot |x|^{n\beta} \, dx \right)^{\frac{1}{p}} \leq C(\Omega) p^{\frac{n-2}{2}} \cdot \|\Delta u\|_{L^\frac{n}{n-2}(\Omega)}.
$$

**Proof.** See the Appendix in [Tro91].

By partition of unity and Proposition 2.1 we have
Proposition 2.2. (Sobolev’s embedding) There is a constant $C$ such that for all $u \in H^2(dV_i)$ and $p \in [1, \infty)$, we have $\|u\|_{L^p(dV_i)} \leq C\|u\|_{H^2(dV_i)}$.

Proposition 2.3. ($L^p$ comparison). Let $\alpha = \min\{\beta_i + 1\}$ and $\omega = \max\{\beta_i + 1\}$. If $p > \frac{2}{\alpha}$, then $L^p(dV_0) \subset L^2(dV_i)$. If $p > q\omega$, then $L^p(dV_i) \subset L^q(dV_0)$.

Proof. See [Tro89]. □

Proposition 2.4. $H^2(dV_0) = H^2(dV_i)$.

Proof. By Poincaré’s inequality, $H^2(dV_i)$ norm is given by $\|u\|_{L^2(dV_i)} + \|\Delta u\|_{L^2(dV_i)}$. We claim that $\|\Delta g_0 u\|_{L^2(dV_0)}$ and $\|\Delta g_i u\|_{L^2(dV_i)}$ are equivalent. In fact, we have that

$$e^{2\gamma(x)} \Delta g_i u(x) = \Delta g_0 u(x) + 2\nabla g_0 u(x) \cdot \nabla g_0 \gamma(x).$$

Observe that, $|\nabla g_0 \gamma(x)| \sim |x - p_i|^{-1}$ at a neighborhood of $p_i$ and smooth elsewhere. By the well known Hardy’s inequality,

$$\|\nabla g_0 u(x) \cdot \nabla g_0 \gamma(x)\|_{L^2(dV_0)} \leq C\|\nabla g_0^2 u\|_{L^2(dV_0)}.$$  

Hence, $\|\Delta g_i u\|_{L^2(dV_i)} < C\|\Delta g_0 u\|_{L^2(dV_0)}$. The other direction is the same. We only have to show

$$u\|_{L^2(dV_0)} \leq C\|u\|_{H^2(dV_i)} \text{ and } \|u\|_{L^2(dV_i)} \leq C\|u\|_{H^2(dV_0)}.$$ 

Let $\alpha = \min\{\beta_i + 1\}$ and $\omega = \max\{\beta_i + 1\}$. By Proposition 2.3, we can choose $p > 2\omega$ to get $\|u\|_{L^2(dV_0)} \leq \|u\|_{L^p(dV_i)}$. Then we use Proposition 2.2 to get $\|u\|_{L^2(dV_0)} \leq C\|u\|_{H^2(dV_i)}$. The second inequality in (2.1) can be proved by choosing $p > \frac{2}{\alpha}$. □

Since we have established Proposition 2.4, from now on, we do not distinguish $H^2(dV_i)$.

The following two propositions are quite standard. See [Tro89] for details.

Proposition 2.5. (compact embedding) The embedding $H^2(dV_i) \hookrightarrow L^p(dV_i)$ for $i = 0, 1$ is compact for $1 < p < \infty$.

Proposition 2.6. (Poincaré’s inequality) If $\int_M v dV_i = 0$, $i = 0, 1$, then $\|v\|_{H^2} \leq C\|\Delta v\|_2$ for some constant $C = C(M)$.

3. Gauss-Bonnet-Chern Formula

On $(M^4, g)$, recall that we have Gauss-Bonnet-Chern formula

$$\int_M Q(x)dV_g(x) + \frac{1}{4}\int_M |W(x)|^2dV_g(x) = 8\pi^2\chi(M).$$

Here $W(x)$ is the Weyl tensor. In this section, we will describe the contribution of conical singularities to this Gauss-Bonnet-Chern formula.

Let $(M^4, g_0, D, g_D)$ be a conic 4-manifold and $g_1 = g_D$. A good choice of base metric $g_0$ will simplify the discussion. Here we use the conformal normal coordinates [LP87]. That is we can replace $g_0$ by a smooth metric in $[g_0]$ such that around each point $p_i$, the normal coordinates satisfies

$$\det(g_0(x)) = 1 + O(|x|^N)$$

and we can pick $N \in \mathbb{N}$ large enough.
Lemma 3.1. Suppose that $B_\epsilon(x)$ is a ball with radius $\epsilon$ around some point $x \in M$. We assume the injective radius at $x$ is $\delta$. Let $h(x,y) = \log(|x-y|)f(y)$ where $f(y)$ has support in $B_\delta(x)$ and equals 1 in $B_\epsilon(x)$. Then
\[
\lim_{\epsilon \to 0} \int_{M \setminus B_\epsilon(x)} P_y(h(x,y))dV_y(y) = 8 \pi^2 = 4|S^3|,
\]
where $P_y$ is the Paneitz operator with respect to y and $|S^3|$ is the volume of a unit 3-sphere.

Proof. Let $(r,\theta^i)$ be the normal coordinates at $x$. Let $f \in C^2(M \setminus \{x\})$. If $f = f(r)$ then
\[
\Delta f(r) = f'' + \frac{n-1}{r} f' + f' \log \sqrt{\det g},
\]
where $f'$ denotes the derivative with respect to $r$. In particular, $\Delta \log r = \frac{4}{r} + \frac{1}{r^2} \log \det g$. Suppose that $Pu = \Delta^2 u + \text{div}(A_g du)$ where $A_g = \frac{2}{3} Rg - 2 \text{Ric}$. Divergence theorem then gives
\[
\int_{M \setminus B_\epsilon(x)} P_y(h(x,y))dV_y(y) = - \int_{\partial B_\epsilon(x)} \left( \frac{\partial}{\partial r} \Delta \log r + O(r^{-2}) \right) d\Omega_g
\]
(3.2)
\[
= \int_{\partial B_\epsilon} 4 r^2 d\Omega_g + o(1),
\]
since $\partial_r \Delta \log r = -\frac{4}{r} + O(r^{N-3})$ by our assumption (3.1). Take $\epsilon \to 0$ then the right hand side of (3.2) approaches $8\pi^2$ since the unit sphere $S^3$ has volume 2$\pi^2$. □

Proposition 3.2. (Gauss-Bonnet-Chern) Suppose that $g_1 = e^{2\gamma}g_0$ is the metric with $k$ singular points given by divisor $D = \sum_{i=1}^k p_i \beta_i$, where $p_i \in M$, $\beta_i > -1$. Suppose that $\gamma(x) = \beta_i \log r + f(x)$ in a neighborhood of $p_i$, $r = \text{dist}(p_i, x)$ and $f(x) \in H^2(M) \cap C^\infty(M \setminus \{p_i\})$. Then we have the formula
\[
\int_M Q_{g_1} dV_{g_1} = \int_M Q_{g_0} dV_{g_0} + 8 \pi^2 \left( \sum_i \beta_i \right)
\]

Proof. First, we assume that $f$ is bounded. Observe that
\[
\int_M Q_{g_1} dV_{g_1} = \int_M e^{-4\gamma}(P_{g_0} \gamma + Q_{g_0}) dV_{g_1}
\]
\[
= \int_M (P_{g_0} \gamma + Q_{g_0}) dV_{g_0}.
\]
We need to compute $\int_M P_{g_0} \gamma dV_{g_0}$. Suppose that $\epsilon$ is a positive real number smaller than the injective radius at each $p_i$. Let $B_\epsilon = B_\epsilon(p_i)$ be the ball with radius $\epsilon$ at $p_i$. We consider the integral $\int_{M \setminus \cup B_\epsilon} P_{g_0} \gamma dV_{g_0}$. By divergence theorem, we see that this integral is just $\sum_{i=1}^k \int_{\partial B_\epsilon} P_{g_0} \gamma d\Omega_{g_0}$, $d\Omega_{g_0}$ is the area element on the geodesic sphere. Note $\gamma(x) = f(x) + \beta_i \log(r)$ near $p_i$ and $f(x)$ is bounded in $B_\epsilon$. Thus, by Lemma 3.1 we see that
\[
\lim_{\epsilon \to 0} \int_{M \setminus \cup B_\epsilon(p_i)} P_{g_0} \gamma = \sum_{i=1}^k 8 \pi^2 \beta_i.
\]
For more general $f \in H^2(M) \cap C^\infty(M \setminus \{p_i\})$, let $\rho_\epsilon$ be a smooth function such that $\text{supp}(1 - \rho_\epsilon) \subset \cup B_{2\epsilon}(p_i)$ and $\rho_\epsilon = 0$ on $B_\epsilon(p_i)$. We can assume that $|D^k \rho_\epsilon| < C\epsilon^{-k}$
for \( k = 1, 2, 3, 4 \). Now it is suffice to prove

\[
\lim_{\epsilon \to 0} \int_M \rho \epsilon \| \nabla \phi \|_H^2 \, dV_{g_0} = 0.
\]

Let \( B_\epsilon = \bigcup_i B_i(p_i) \). Note that the highest order term of \( P_{g_0} \) can be estimated by H"older’s inequality

\[
| \int_M \Delta f \Delta \rho \epsilon \, dV_{g_0} | \leq | \int_{B_\epsilon} C \epsilon^{-2} \Delta f \, dV_{g_0} | \\
\leq C | \int_{B_\epsilon} \epsilon^{-4} | \partial x |^2 | dV_{g_0} | \int_{B_\epsilon} (\Delta f)^2 | \partial x |^2 \, dV_{g_0} \\
(3.4)
\leq C \| f \|_{H^2(B_\epsilon)} \to 0 \quad \text{as} \ \epsilon \to 0.
\]

The lower order term can be estimated by

\[
\int_M \rho \epsilon \text{div}(A_{g_0} \epsilon \nabla f) \, dV_{g_0} = \int_M A_{g_0} \nabla \rho \epsilon \nabla f \, dV_{g_0} = \int_{B_\epsilon} A_{g_0} \nabla \rho \epsilon \nabla f \, dV_{g_0}.
\]

By H"older’s inequality, the lower order term goes to 0 as \( \epsilon \) goes to 0. \( (3.4) \) and \( (3.3) \) proves \( (3.3) \) which concludes the whole proof. \( \square \)

For later use, we state the following corollary.

**Corollary 3.3.** \( Q_1 \rho \in L^p(dV_0) \) for \( 1 < p < 2 \) and \( Q_1 \in L^2(dV_1) \)

**Proof.** Recall \( P_{g} v = \Delta^2_g v + \text{div}(A_{g} \epsilon \nabla v) \). In a neighborhood of \( p_i, \gamma(x) = \beta_i \log(r) + f(x) \) for some smooth \( f(x) \). By calculation in Lemma 3.1 we see that, \( \Delta^2_{g_0} \gamma(x) = O(r^{-N-4}) \). Thus \( |P_{g_0} \gamma| \leq C r^{-2} \). Since \( Q_1 \rho = (P_{g_0} \gamma + Q_0) \sim O(r^{-2}) \). We see that, \( Q_1 \rho \in L^p(dV_0) \) for \( 1 < p < 2 \). Besides, \( Q_1 \sim O(r^{-2-4\beta_i}) \). Hence, \( Q_1^2 \rho \sim O(r^{-4-4\beta_i}) \). Since \(-4-4\beta_i > -4\), we have \( Q_1 \in L^2(dV_1) \). \( \square \)

4. A Modified Moser-Trudinger-Adams’ Inequality

The singular version of Moser-Trudinger-Adams’ inequality on bounded domains in Euclidean spaces has been proved by Lam and Lu [LL11]. In this section, we will give a quick proof based on a comparison principle of Talenti [Tal76] and a lemma by Tarsi [Tar12]. Then we will generalize this inequality to conic 4-manifolds.

We first introduce Talenti’s comparison principle. Let \( f \) be a measurable function with support in a bounded domain \( \Omega \subset \mathbb{R}^n \). Let \( \lambda(s) = m(\{x : |f(x)| > s\}) \) and \( f^*(t) = \sup\{s > 0 : \lambda(s) > t\} \). The spherical rearrangement \( f^*(x) \) of \( f \) is defined to be

\[
u^*(x) = u^*(\omega_n |x|^n), \ x \in \Omega^#.
\]

Here \( \Omega^# \) is a open ball in \( \mathbb{R}^n \) with the same measure as \( \Omega \), \( \omega_n \) is the volume of a unit \( n \) dimensional ball.

**Lemma 4.1** (Talenti’s principle). If \( u, v \) are solutions of the following equations,

\[
\begin{align*}
\Delta u(x) &= f(x) \quad x \in \Omega, \\
u(x) &= 0 \quad x \in \partial\Omega, \\
\Delta v(x) &= f^*(x) \quad x \in \Omega^#, \\
v(x) &= 0 \quad x \in \partial\Omega^#,
\end{align*}
\]

then we have that

\[v \geq u^#.
\]
Let
\[ b_{n,2} = \frac{1}{\omega_n} \left[ \frac{4\pi^2}{\Gamma\left( \frac{n}{2} - 1 \right)} \right]^{\frac{2}{n-2}}, \]
following [Ada88]. Note that \( \omega_n = \pi^{n/2}/\Gamma\left( \frac{n}{2} + 1 \right) \). We have \( b_{n,2} = [\omega_n^2 n(n-2)]^{n/2} \).

By definition, \( u^# \) is non-increasing. Hence, we obtain that
\[
\int_\Omega e^{bu^{n/2}} |x|^{n\beta} \, dx \leq \int_\Omega e^{b(u^#)^{n/2}} |x|^{n\beta} \, dx + C(\Omega)
\]
\[
\leq \int_\Omega e^{bu^{n/2}} |x|^{n\beta} \, dx + C(\Omega),
\]
where \( b > 0 \) and \( -1 < \beta < 0 \).

Next, we state a lemma of Tarsi [Tar12].

**Lemma 4.2.** [Tar12] Let \( p > 1 \). For any \( r > 0 \) there is a constant \( C = C(p,r) \) such that for any positive measurable function \( f(s) \) on \((1, +\infty)\), satisfying
\[
\int_1^\infty f^p s^{2p-1} \, ds \leq 1
\]
then
\[
\int_1^\infty e^{r F(t)} \frac{dt}{t^{r+1}} \leq C
\]
where \( \frac{1}{p} + \frac{1}{q} = 1 \) and
\[ F(t) = \int_t^1 \int_1^\infty f(s) \, ds \, dr. \]

We now state the modified Moser-Trudinger-Adams inequality in bounded Euclidean domains.

**Theorem 4.3.** Suppose that \( \Omega \subset \mathbb{R}^4 \) is a bounded domain and \( 0 \in \Omega \). Suppose \( u \in C^2(\Omega) \) and \( -1 < \beta < 0 \). There is a \( C \) such that if \( \|\Delta u\|_2 \leq 1 \) then
\[
\int_\Omega \exp(b_{n,2}(1 + \beta)u^2) |x|^{n\beta} \, dx \leq C.
\]
The coefficient \( b_{n,2}(1 + \beta) \) here is sharp in the sense that the inequality fails for bigger constant. In particular, \( b_{4,2} = 32\pi^2 \).

**Proof.** Suppose that \( v(x) \) is a \( C^2 \) radical decreasing function with support in ball \( B_R \). Let \( w(t) = n\pi^2/\omega_n^2 (n-2)^2 - \pi^2 v(Rt^{-1/(n-2)}) \). We see that
\[
\int_{B_R} \exp(b_{n,2}\alpha v^{n/2}) |x|^{n\beta} \, dx = \frac{n\omega_n R^{n\alpha}}{n-2} \int_1^\infty \exp\left( \frac{n\alpha}{n-2} \frac{w^{n/2}}{w} \right) \frac{dt}{t^{1+\frac{1}{n-2}}},
\]
where \( \alpha = \beta + 1 < 1 \). The condition that \( \int_{B_R} |\Delta v|^2 \, dx \leq 1 \) is equivalent to \( \int_1^\infty |w''(s)| \tilde{s}^{n-1} \, ds \leq 1 \) where \( w(t) = \int_t^1 \int_{s=1}^\infty w''(s) \, ds \, dz \). Then, by the Lemma 4.2, we see that
\[
\int_1^\infty \exp\left( rw^{n/2}(t) \right) \frac{dt}{t^{r+1}} \leq C_0,
\]
where \( C_0 = C_0(n,r) \). If we pick \( r = n\alpha/n-2 \), we have
\[
\int_{B_R} \exp(b_{n,2}\alpha v^{n/2}) |x|^{n\beta} \, dx \leq C_0.
\]
We now show that the constant $b_{n,2\alpha}$ is sharp. Let

$$B = \left\{ b : \exists C_0, \int_B e^{b|\nabla \psi|^2} \geq C_0, \forall \psi \in C^\infty_c(B) \text{ and } \|\Delta \psi\|_2 = 1 \right\}.$$  

Let $u$ be a positive function with support in unit ball $B_1$ and equals 1 in $B_r$ with $0 < r < 1$. Let $b \in B$, then

$$C_0 \geq r^{4\alpha} \exp(b/\|\Delta u\|_2^{\frac{n}{n-2}}).$$

This leads to

$$b \leq \alpha n \lim_{r \to 0} C_2 \beta_1(B_r; B_1)^{-\frac{n}{n-2}} \left( \log \frac{1}{r} \right),$$

where $C_2 \beta_1(B_r; B_1) = \inf \|\Delta u\|_2^2$ and the infimum is taken over all $u \in C^\infty_c(B)$ such that $u = 1$ on $B_r$. By [Ada88], the right hand side of (4.2) is just $b_{n,2\alpha}$. □

We apply Theorem 4.3 to obtain an estimate on conic 4-manifolds. We first make a variation to Theorem 4.3 since Talenti’s principle works for Newton potential.

**Corollary 4.4.** Suppose that $\Omega \subset \mathbb{R}^4$ is a bounded domain, $f \in C_c(\Omega)$ and $\|f\|_{L^2(\Omega)} \leq 1$. Let $u$ be the Newton potential of $f$, i.e.

$$u = \int_\Omega \Gamma(x-y)f(y)dy,$$

where $\Gamma(x) = \frac{1}{4\pi|x|^2}$. Then we have that

$$\int_\Omega \exp(32\pi^2(1+\beta)u^2)|x|^{4\beta}dx \leq C.$$

We have the following results.

**Corollary 4.5.** Suppose that $(M, g_0, D, g_D)$ is a conic 4-manifolds with $D = \sum_{i=1}^k \rho_i \beta_i$ and $\beta_1 = \min\{\beta_i\}$. Let $g_i = g_D$ and $dV_i$ be the volume elements of $g_i$, $i = 0, 1$. Let $\bar{u} = \int_M u dV_1$. Then for any $u \in H^2(dV_1)$

$$\log \int_M \exp(4|u - \bar{u}|)dV_1 \leq C + \frac{1}{8\pi^2(1+\beta_1)}\|\Delta u\|^2,$$

where $C = C(M)$ is a constant.

**Proof.** Since the Green function $G(x,y)$ of $\Delta$ exists on smooth manifold $(M, g_0)$, we have $u(x) - \bar{u} = \int_M \Delta u(y)G(x,y)dV_0(y)$ where $\bar{u} = \frac{1}{V_0(M)} \int_M u(y)dV_0(y)$. Note

$$G(x,y) = r^{-2}g(r) + H(x,y)$$

for $H$ a bounded function on $M \times M$, $r = d(x,y)$ and $g(r)$ a function with support in $r < \text{injection radius}$. First, let $f(y) = \Delta u(y)$ and suppose that

$$\|f\|_2 \leq 1.$$

Suppose that $dV_1 = \rho(x)dV_0 = e^{4\gamma(x)}dV_0$ is the volume element for the singular metric. Consider the following PDE on $M$

$$\Delta \psi(x) = \frac{1}{V_1(M)}(\rho(x) - V_1(M)).$$
This PDE has a weak solution \( \psi(x) \in W^{2,p} \) for \( 1 < p < -\frac{1}{\beta_1} \). By Sobolev embedding, we have that \( \psi(x) \in L^{\frac{4p}{2p-1}}(M) \subset L^2(M) \). Let and \( \bar{u} = \int_M u(y)dV(y) \). Then we have that
\[
u - \bar{u} = \int_M f(y)(G(x,y) - \psi(y))dV(y).
\]
By Hölder’s inequality, we have
\[
\int_M f(y)\psi(y) \leq \|f\|_2\|\psi\|_2.
\]
Combining (4.3) and (4.4), we have
\[
|u(x) - \bar{u}| \leq \left|\int_{B_{\delta}(x)} f(y)r^{-2}dV_0(y)\right| + C\|f\|_2,
\]
where \( \delta \) is the injective radius. Pick a normal coordinates around \( x \). The metric \( g_{ij}(y) = \delta_{ij} + O(|y|^2) \). Then we see that
\[
\left|\int_{B_{\delta}(x)} f(y)r^{-2}dV_0(y)\right| = \left|\int_{B_{\delta}(x)} f(y)r^{-2}(1 + O(r^2))dy\right|
\leq \left|\int_{B_{\delta}(x)} f(y)|x - y|^{-2}dy\right| + C\|f\|_2.
\]
We can assume that \( f \) has compact support in \( B_{\delta}(x) \), because the integral over the rest part can be controlled by its \( L^2 \) norm. Let \( u_1(x) = \int_{\mathbb{R}^4} f(y)|x - y|^{-2}dy \). By Corollary 4.4 we have that
\[
\int_{B_{\delta}(x)} \exp(32\pi^2\alpha_i(u_1 - \bar{u})^2(z))|z|^{4\beta_i}dz \leq c_0
\]
where \( \alpha_i = (1 + \beta_i) \). Let \( \alpha = 1 + \beta_1 \). Suppose that \( v(x) = ku_1(x) \) for some positive \( k \), then by mean value inequality we have
\[
\int_{B_{\delta}(x)} \exp(4|v(z) - \bar{v}|)|z|^{4\beta}dz \leq \int_{B_{\delta}(x)} \exp\left(32\pi^2\alpha\left(\frac{v(z) - \bar{v}}{k}\right)^2 + \frac{1}{8\pi^2\alpha}k^2\right)|z|^{4\beta}dz
\]
\[
\leq c_0 \exp\left(\frac{k^2}{8\pi^2\alpha}\right).
\]
Hence, on \( B_{\delta}(p_i) \), we will have \( \int_{B_{\delta}(p_i)} \exp(4|v(z) - \bar{v}|)|x|dV_0(z) \leq C \exp(\frac{1}{8\pi^2\alpha}\|\Delta v\|^2) \).
On \( M - \cup_i B_{\delta/2}(p_i) \), we can assume that \( v(z) - \bar{v} \) vanishes in \( B_{\delta/2}(p_i) \). Then we can apply Adams’ inequality in the form of [BCY92] which gives
\[
\int_{M - \cup B_{\delta/2}(p_i)} \exp(4|v(z) - \bar{v}|)|x|dV(z) \leq c_0 \exp\left(\frac{\|\Delta v\|^2}{8\pi^2}\right)
\]
\[
\leq c_0 \exp\left(\frac{\|\Delta v\|^2}{8\pi^2\alpha}\right).
\]
Thus
\[
\int_{M} \exp(4|v(z) - \bar{v}|)|x|dV(z) \leq C \exp\left(\frac{\|\Delta v\|^2}{8\pi^2\alpha}\right).
\]
This concludes the proof.
If the Paneitz operator $P$ is nonnegative with $\text{Ker}(P) = \{\text{constants}\}$, we can define the pseudo differential operator $\sqrt{P}$ and the Green function of $\sqrt{P}$ has the same leading term as $-\Delta$. See Lemma 1.6 in [BCH92] for details. Then we can follow the proof in Corollary 4.5 to obtain the lower bound of $H$ on conic manifolds.

**Theorem 4.6.** Suppose that $(M, g_0, D, g_D)$ is a conic 4-manifolds with $D = \sum_{i=1}^k p_i \beta_i$ and $\beta_1 = \min\{\beta_i\}$. Let $g_0 = g_D$ and $dV_i$ be the volume elements of $g_i$, $i = 0, 1$. Let $\bar{u} = \int_M u(y)dV_1(y)$. Let $P$ be the Paneitz operator of $g_0$ on $M$. Suppose that $P$ is nonnegative and $\text{Ker}(P) = \{\text{constants}\}$. Then for any $u \in H^2(dV_1)$

$$\log \int_M \exp(4|u(x) - \bar{u}|)dV_1(x) \leq C + \frac{1}{8\pi^2(1 + \beta_1)} \int_M u(x)Pu(x)dV_0(x),$$

where $C = C(\beta_1, M)$ is a constant.

5. **Proof of Theorem 1.**

The equation of constant $Q$-curvature is the Euler-Lagrange equation of $H$ functional, which is studied in [BCH92] and [CY95] for smooth metrics.

$$\text{II}_3(u) = \langle Pu, u \rangle_g + 2 \int_M Qu dV_g - \frac{k_g}{2} \log \int_M \exp(4u)dV_g,$$

where $k_g = \int_M QdV_g$ and $\langle Pu, u \rangle_g = \int_M uPu_dV_g$. If we integrate by part, we can see that $\langle Pu, u \rangle$ is well defined for $u \in H^2(dV_g)$. It is obvious that $\langle Pu, u \rangle$ is conformal invariant. So we will use the notation $\langle Pu, u \rangle$.

Let $(M^4, g_0, D, g_D)$ be a conic manifold with $g_D(x) = e^{2\gamma(x)}g_0(x)$. Let $g_1 = g_D$ and $dV_i$ be the volume elements of $g_i$, $i = 0, 1$. The key estimate of [BCH92] [CY95] is to employ Adams’ inequality [Ada88] to derive a low bound of $H$ functional. By Adams’ inequality and its modified form [BCH92] [CY95], if $u \in H^2(dV_0)$, $\langle Pu, u \rangle \leq 1$, we have that

$$\int_M \exp(32\pi^2|u(x) - \bar{u}|^2) dV_g(x) \leq c_0 V_g(M),$$

where the mean value $\bar{u} = \int_M u dV_0$. In conic 4-manifolds, we use the modified Adams’ inequality [1.6] to obtain the estimate for $H$ functional.

**Proof of Theorem 1.** First assume that $\langle Pu, u \rangle \leq 1$. By Theorem 4.6

$$\log \left( \int_M \exp(4|u - \bar{u}|)dV_1 \right) \leq \frac{1}{8\pi^2\alpha} \langle Pu, u \rangle + C(\alpha, |M|),$$

where $\alpha = (1 + \beta_1)$ and $\bar{u} = \int_M u dV_1$. We apply mean value inequality

$$\int_M Q_1 dV_1 = \int_M Q_1(u - \bar{u})^2 dV_1 \leq \frac{1}{4\epsilon} \int_M Q_1^2dV_1 + \epsilon \int_M (u - \bar{u})^2 dV_1 + \bar{u}k_g.$$ 

Note that $\|u - \bar{u}\|_{L^2(dV_1)} \leq \|u - \bar{u}\|_{H^2} \leq \langle Pu, u \rangle$, by Propositions [2.2] and [2.6]. Then this gives us the desired estimate (as in [CY95]) that if $\bar{u} = 0$, then

$$\langle Pu, u \rangle + 2 \int_M Q_1 dV_1 - \frac{k_{g_0}}{2} \log \left( \int_M \exp(4u)dV_1 \right) \geq \left(1 - \frac{k_{g_0}}{16\pi^2\alpha} - \epsilon \right) \langle Pu, u \rangle + C(\alpha, \epsilon).$$

(5.2)
Here \( k_{g_1} = k_{g_0} + 8\pi^2 \sum \beta_i \). If \( k_{g_1} < 16\pi^2 \alpha \) we can always choose an \( \epsilon \) small enough such that \( \frac{k_{g_1}}{16\pi^2 \alpha} + \epsilon < 1 \). Then (3.2) shows

\[
II(u) \geq C' \langle Pu, u \rangle + C \geq C''.
\]

Let

\[
\Lambda = \inf \{ II(u) : u \in H^2(dV_1), \int_{S^4} u dV_1 = 0 \}.
\]

Take a minimizing sequence of \( II \), namely \( \{ u_i \}_{i=1}^\infty \) such that \( II(u_i) \to \Lambda \) and \( \int_{S^4} u_i dV_1 = 0 \). By (3.3) we see that \( \| \Delta u_i \|_2^2 \) is bounded. Hence, \( u_i \) is bounded in \( H^2(dV_1) \) by Poincaré’s inequality. Replaced by a subsequence, we may assume that \( u_i \) converges weakly to some \( w \) in \( H^2(dV_1) \) and strongly to the same \( w \) in \( L^2(dV_0) \).

**Proposition 5.1.** \( II(w) = \Lambda \)

**Proof.** Since \( u_i \) converges to \( w \) weakly in \( H^2(dV_1) \), we see that

\[
\langle Pw, w \rangle \leq \lim \inf \langle Pu_i, u_i \rangle
\]

and

\[
\int Qw dV_1 = \lim_{i \to 1} \int Qu_i dV_1.
\]

Now we want to control the last term of \( II \). First we note that

\[
| \exp(4u_i) - \exp(4w) | = \left| \int_{u_i}^w 4 \exp(4s) ds \right| \\
\leq 4 \exp(4|w| + 4|u_i|)|u - u_i|.
\]

This leads to

\[
\int | \exp(4u_i) - \exp(4w) | dV_1 \leq 4 \int | \exp(4|w| + 4|u_i|) | w - u_i | dV_1 \\
\leq 4 \left( \int | \exp(4|w|) | \right)^{1/4} \left( \int | \exp(4|u_i|) | \right)^{1/4} \\
\times \left( \int |w - u_i|^2 \right)^{1/2}.
\]

By Adams’ Inequality, \( \left( \int | \exp(4|w|) | \right)^{1/4} \) and \( \left( \int | \exp(4|u_i|) | \right)^{1/4} \) are bounded. This shows that \( \int | \exp(4u_i) - \exp(4w) | dV_1 \to 0 \) as \( i \to \infty \). Hence

\[
\lim_{i \to \infty} \log \left( \int | \exp(4u_i) | dV_1 \right) = \log \left( \int | \exp(4w) | dV_1 \right).
\]

Therefore, we have that \( II(w) \leq \lim \inf II(u_i) = \Lambda \). By the definition of \( \Lambda \), we have \( II(w) = \Lambda \).  

Since we have the minimizer of functional \( II \), the Euler-Lagrange equation for critical points of \( II \) reads

\[
P_{g_1} w + Q_1 = \frac{k_{g_1}}{2 \int_M e^{4w} dV_1} e^{4w}.
\]

It is equivalent to \( Q_w = c \) where \( Q_w \) is the \( Q \)-curvature of metric \( g_w = e^{2w} g_1 \). So our minimizer \( w \) is a weak solution of

\[
(5.4) \quad P_{g_1} w + Q_1 = c \cdot e^{4w}.
\]
Proof of the regularity. Suppose we have a $H^2$ solution $w$ of $P_g w + Q_1 = c \cdot e^{4w}$. By a dilation, we assume that $c = \pm 1$ or 0. We may assume that $c = 1$ and $w$ is a weak solution of $P_g w + Q_1 = e^{4w}$ since the proofs for other cases are similar. To be precise, for any $v \in H^2(dV_0)$,

$$0 = \int_M v P_g w dV + \int_M (Q_1 - e^{4w}) v dV_1$$

$$= \int_M v P_{g_0} w \rho dV_0 + \int_M (Q_1 - e^{4w}) v \rho dV_0$$

Thus, $w$ is a weak solution of

$$P_{g_0} w = e^{4w} \rho - Q_1 \rho$$

in $H^2(dV_0)$. Let

$$h(x) = e^{4w(x)} \rho(x) - Q_1(x) \rho(x) - \text{div} (A_{g_0} \rho w).$$

By Corollary 3.3, $Q_1 \rho \in L^p(dV_0)$ for $1 < p < 2$. $\rho \in L^{q_1}$ for $1 < q_1 < \frac{1}{\alpha - 1}$ and by Adams’ inequality, $e^{4w} \in L^p(dV_0)$ for any $p > 1$. This implies that $e^{4w} \rho \in L^{q_1}$ for $1 < q_1 < \frac{1}{\alpha - 1}$. Hence $h \in L^q(dV_0)$ for some $1 < q < \min\{2, \frac{1}{1 - \alpha}\}$. Let $z(x) = \Delta w$. Then $\Delta z = h(x)$ in weak sense, i.e.

$$\int_M z \Delta v dV_0 = \int_M h v dV_0,$$

for any $v(x) \in H^2(dV_0)$. Now let $\Gamma(x, y)$ be the Green function for $\Delta$. Let $H(x) = \int_M h(y) \Gamma(x, y) dV_0$. Then by regularity theory of elliptic equations $[GT98]$, we see that $H(x) \in W^{2,q}(dV_0)$ and $\Delta H(x) = h(x)$ a.e.. Thus $z(x) = H(x) + \bar{z}$ a.e., where $\bar{z} = \int_M z dV_0$. We apply regularity theory of elliptic equation again to obtain $w \in W^{4,3}(dV_0)$. Note $4q > 4$, so we may embed the solution $w$ into Hölder spaces which implies

$$w \in C^\tau, \tau < \min\{2, 4(1 + \beta)\}.$$

For any $x_0 \in M$ and $x_0 \neq p_i$, $i = 1, 2, ..., k$, we can find a small neighborhood $B_{x_0}(x_0)$ of $x_0$ which does not intersect with a neighborhood of $p_i$. By the discussion in $[CY95, Mal06]$, the Green function of $P_{g_0}$ with respect to $g_0$, denoted as $G(x, y)$, exists and is smooth on $M \times M \setminus \{(x, x)\}$. Furthermore, $G(x, y)$ has following asymptotic properties $[Mal06]$,

$$|G(x, y) - \frac{1}{8\pi^2} \log \frac{1}{|x - y|}| \leq C, \ x \neq y,$$

and for derivatives,

$$|\nabla^i G(x, y)| \leq C_i \frac{1}{|x - y|}, \ i = 1, 2, 3.$$ 

Here $C, C_i, i = 1, 2, 3$ are some constants depend on $(M, g_0)$. This gives the representation of $w$,

$$w(x) - \bar{w} = \int_M G(x, y) P_{g_0} w(y) dV_0(y).$$

Let $f(x) = e^{4w(x)} \rho(x) - Q_1(x) \rho(x)$. $f(x)$ is clearly integrable and bounded in $B_{x_0}(x_0)$. This implies that $w(x) \in C^3(B_{x_0}(x_0))$. Then we can apply the regularity theory to $\Delta z(x) = h(x)$. Since $h(x) \in C^1(M \setminus \{p_i\})$, the regularity theory shows
Then $w \in C^{4,\tau}_{\text{loc}}(M\setminus\{p_i\})$. Iterate this procedure, we see that $w \in C^\infty(M\setminus\{p_i\})$.

**Remark 5.2.** From the regularity argument, we can see the number 2 in (5.6) is introduced by the $L^2$-integrability of $Q_1 \rho$. This term disappears when the original metric is conformal flat. In other words, the solution is in $C^{\tau}(M)$ for any $\tau < 4(1 + \beta_1)$ if the metric is conformal flat.

### 6. Radial Symmetric Solutions

In this section, we consider the radial symmetric solution on 4-sphere with standard background metric. We assume that there are two conic points at south and north pole such that $D = \beta_0 x_S + \beta_1 x_N$. Let $\eta : \mathbb{R}^4 \to S$ be the inverse of the stereographic projection from north pole. Then

$$\eta^*(g_0) = e^{2z(x)} \ ds^2$$

where $g_0$ is the standard metric on sphere, $ds^2$ is the Euclidean metric, and

$$z(x) = \log \frac{2}{|x|^2 + 1}.$$ 

Note $\eta^{-1}$ maps $x_N$ to infinity and $x_S$ to 0. Let $T = S^3 \times \mathbb{R}$. This is like a tube or a cylinder. Suppose $G : S^3 \times \mathbb{R} \to \mathbb{R}^4, G(w, t) = e^t w$.

The composition map $\eta G : T \to S^4$ gives a cylinder coordinate. It is easy to see that

$$P_T = (\partial_t^2 + \Delta_{S^3})^2 - 4 \partial_t^2$$

is the Paneitz operator on $T$ with product metric $g_T$. A radial symmetric function on $\mathbb{R}^4$ depends only on $t$. Thus we can write down the constant $Q$-curvature equation to

$$v^{(m)}(t) - 4v''(t) = c \cdot e^{4v}.$$ 

In our case, we only consider the positive $Q$-curvature since $k_{g} = 8\pi^2(2 + \beta_0 + \beta_1) > 0$. By adding a constant, we can normalize the equation

$$v^{(m)}(t) - v''(t) = e^{4v(t)}.$$ 

The corresponding solution for standard sphere is given by

$$v(t) = -\log \cosh t.$$ 

If we have a symmetric conic metric with singularities given by $\beta_0 x_S + \beta_1 x_N$, the corresponding $v(t)$ must have linear growth at $\pm \infty$. Thus, we have the following boundary conditions:

$$\lim_{t \to -\infty} v'(t) = 1 + \beta_0, \lim_{t \to +\infty} v'(t) = -\beta_1 - 1.$$ 

We only have to classify all solutions with bounded first derivative.

Define $x_1(t) = v'(t)$, $x_2(t) = x_1'(t)$, $x_3(t) = x_2'(t) - 4x_1(t)$ and $x_4(t) = x_3'(t)$. Then we derive the following system

$$\begin{align*}
x_1' &= x_2 \\
x_2' &= 4x_1 + x_3 \\
x_3' &= x_4 \\
x_4' &= 4x_1 x_3
\end{align*}$$

(6.3)
The standard metric associates to the solution given by \( X(t) = (x_1, x_2, x_3, x_4)^T \) with
\[
X(t) = (-\tanh t, -\text{sech}^2 t, 2 \tanh t (\text{sech}^2 t + 2), 6\text{sech}^4 t)^T.
\]
Note that \( Q \)-curvature being positive means that \( x_4 > 0 \) for all \( t \).

We can find a first integral as the following.

**Proposition 6.1.** We have the following first integral
\[
2x_2^2 - 8x_1^2 - 4x_1x_3 + x_4 = c,
\]
or equivalently,
\[
2x_2^2 + \frac{1}{2}x_3^2 - \frac{1}{2}(x_2')^2 + x_4 = c.
\]
where \( c \) is a constant.

**Remark 6.2.** This formula indicates that if the system has a bounded solution then
\[
\lim_{t \to \infty} |x_1(t)|^2 = \lim_{t \to -\infty} |x_1(t)|^2.
\]
In other words, \( \beta_0 = \beta_1 \) is a necessary condition for the existence of a solution for (6.1) with (6.2).

**Proof.** Two formulae are equivalent. Multiply \( v'(t) \) on both sides of (6.1) and integrate by parts. Then we get the first identity. To get the second formula, simply plug \( 4x_1(t) = x_2'(t) - x_3(t) \) into the first identity. \( \square \)

Since the system the system (6.3) is invariant if we change the variable \( t \to t + c \), we need to fix the gauge of (6.3). First, we consider a special case with the following initial data:
\[
x_1(0) = x_3(0) = 0, \ x_2(0) = p, \ x_4(0) = q.
\]
Here \( p < 0 \) and \( q > 0 \). Such solution is symmetric with respect to \( t = 0 \), i.e. \( x_1(t) \) and \( x_3(t) \) are odd functions while \( x_2(t) \) and \( x_4(t) \) are even functions. The constant \( c \) in (6.4) and (6.5) is given by \( c = 2p^2 + q \). The standard solution of 4 sphere coincides with the solution starting with \( p = -1 \) and \( q = 6 \).

Fix \( p \) and define
\[
Q = \{ q > 0 : \forall t > 0, \ x_2(t) < 0 with x_2(0) = p, \ x_4(0) = q \}.
\]
We state some lemmas to show that \( Q \) is connected, nonempty, and bounded from above.

**Lemma 6.3.** (Monotonicity lemma) If \( x_i(t) \) and \( y_i(t) \) are two solutions for the system (6.3) and \( x_i(0) \geq y_i(0), i = 1, 2, 3, 4 \), then \( x_i(t) \geq y_i(t) \) for all \( t > 0 \). The equality holds if and only if \( x_i(0) = y_i(0), i = 1, 2, 3, 4 \).

**Proof.** The uniqueness of ODE tells that the equality holds if \( x_i(0) = y_i(0), i = 1, 2, 3, 4 \). So suppose that \( x_j(0) > y_j(0) \), for some \( j \in \{1, 2, 3, 4\} \). Then there will be at least a small interval \( (0, \epsilon) \) on which \( x_i(t) > y_i(t) \) for \( i \leq j \). Since \( (\log x_4)' - (\log y_4)' = x_1 - y_1 > 0 \) on this interval, we have \( x_4(t) > y_4(t) \) on \( (0, \epsilon) \). We must have \( x_3(t) = y_3(t) \) on \( (0, \epsilon) \). Let \( J = \{ t > 0 : x_i(t) > y_i(t) \} \) and \( t_0 = \inf\{ t > 0 : t \notin J \} \). If \( t_0 < \infty \), \( x_i(t_0) > y_i(t_0) \) for \( i = 1, 2, 3 \) and \( x_4(t_0) = y_4(t_0) \). However,
\[
(\log x_4)'(t_0) - (\log y_4)'(t_0) = x_1(t_0) - y_1(t_0) > 0.
\]
It is impossible since \( t_0 \) is the first point. This concludes that \( J = (0, \infty) \). \( \square \)
Lemma 6.4. If $4p + q \leq 0$, then $x_2(t) \to -\infty$ as $t \to \infty$.

Proof. If $4p + q < 0$, then $x_2''(0) = 0$, $x_2'''(0) = 4p + q < 0$. Hence $x_2''(t) < 0$ on a small interval $(0, \epsilon)$. Clearly,

$$x_2'''(t) = 4x_2'(t) + x_2''(t),$$

$x_2'''(t)$ is negative as long as $x_2'(t) < 0$. Therefore $x_2'(t)$ must go to $-\infty$ and $x_2(t)$ must go to $-\infty$ as well.

If $4p + q = 0$, $x_2^{(4)}(0) = x_4''(0) = 4pq < 0$. Then $x_2'(t) < 0$ on a small interval $(0, \epsilon)$. The above argument yields the same result. □

Lemma 6.5. For each $p < 0$, there is some $q > 0$ such that $\exists T > 0$, $x_2(t) > 0$ for $t > T$.

Proof. Suppose that on the contrary, for any $q > 0$, $x_2(t) < 0$ for all $t > 0$. This implies $x_1(t) < 0$, $x_4'(t) < 0$, and

$$x_4''(t) = 4(x_2(t) + 4x_1'(t))x_1(t) \Rightarrow x_4''(t) \geq 4x_2(t)x_1(t).$$

We have $x_2'(t) = x_3'(t) + 4x_1(t)$. By 6.4 we can assume that $4p + q > 0$. Thus $x_2(t) > 0$ on a interval $(0, \epsilon)$. Let’s assume that $(0, T]$ is the biggest interval on which $x_2(t) \geq 0$. We have $x_4(t) \leq q$ and $x_2(t) \geq p$. Hence, $x_4'(t) = 4x_2(t)x_4(t) \geq 4pq$, $x_4'(t) \geq 4pq t$ and $x_4(t) \geq q + 2pq t^2$. Since

$$x_2''(t) = 4x_2 + x_4 \geq 4p + q + 2pq t^2,$$

we have

$$x_2'(t) \geq (4p + q)t + \frac{2}{3}pq t^3.$$ 

These are true for all $t \in (0, T]$ especially for $t = T$. By (6.6), we see that

$$T \geq \sqrt{\frac{4p + q}{-\frac{2}{3}pq}} = \sqrt{\frac{\frac{4p + q}{2}}{-\frac{2}{3}p}}.$$ 

The estimate for $x_2'(t)$ leads to

$$x_2(t) \geq p + \frac{t^2}{2}(4p + q) + \frac{1}{6}pq t^4.$$ 

The right hand side has to positive roots $t_1 < t_2$. If we can pick a $t$ such that

$$t_1^2 = \frac{-\sqrt{(4p + q)^2 - \frac{8pq}{3} + (4p + q)}}{-\frac{2}{3}pq} < t^2 < \sqrt{(4p + q)^2 - \frac{8pq}{3} + (4p + q)} = t_2^2,$$

then $x_2(t) > 0$ and we get the contradiction. Thus, it is sufficient to show the interval $(t_1, t_2) \cap (0, T)$ is not empty for large $q$. Let $z = \frac{4}{q}$. We see that

$$t_1^2 = \frac{-\sqrt{(4z + 1)^2 - \frac{8z^2}{3} + (4z + 1)}}{-\frac{2}{3}p} \to 0, \quad \text{as } q \to \infty,$$

$$\sqrt{\frac{4p + q}{-\frac{2}{3}pq}} = \sqrt{\frac{4z + 1}{-\frac{2}{3}p}} \to \sqrt{\frac{3}{2p} > 0, \quad \text{as } q \to \infty}.$$ 

Hence, for large $q$,

$$t_1 < \sqrt{\frac{4p + q}{-\frac{2}{3}pq}} \leq \min\{T, t_2\}.$$
Theorem 6.6. For any fixed \( p < 0 \), there is a unique \( q > 0 \) such that \( 4p+q > 0 \) and the solution of system (6.3) is bounded for all \( t \) with initial data \( x_1(0) = x_3(0) = 0 \), \( x_2(0) = p \) and \( x_4(0) = q \).

Proof. Let

\[ Q = \{ q > 0 : \forall t > 0, \ x_2(t) < 0 \text{ with } x_2(0) = p, \ x_4(0) = q \} \]

By Lemma 6.4 if \( 4p + q \leq 0 \), \( q \in Q \). By monotonicity lemma 6.3, \( Q \) is a connected set and by Lemma 6.5, \( Q \) does not contain the whole half line. This implies that \( q_0 = \sup \{ q \in Q \} < \infty \). We want to prove that \( q_0 \) is the unique choice such that the solution is bounded. Let \( \{ y_i(t), i = 1, 2, 3, 4 \} \) be a solution of (6.3) with initial value:

\[ (y_1, y_2, y_3, y_4)(0) = (0, p, 0, q). \]

We will prove that \( y_2(t) \to 0 \) by excluding several plausible cases for \( y(t) \).

Case 1. \( \exists \tau > 0 \) such that \( y_2(t) < 0 \) for \( 0 \leq t < \tau \), \( y_2(t_0) = 0 \), and \( y_2'(t_0) > 0 \).

It can be ruled out easily because the our solutions are continuously dependent on the initial value. In fact, if \( y_2'(t_0) > 0 \), \( y_2(t_1) > 0 \) for some \( t_1 > 0 \). We can find a \( q' < q_0 \) such that the solution \( z(t) \) with \( z_i(0) = y_i(0), i = 1, 2, 3 \) and \( z_4(0) = q' \) will be close enough to \( y_2 \) at \( t_1 \). Precisely, we only need \( |z_2(t_1) - y_2(t_1)| < y_2(t_1)/2 \).

Then \( z(t_1) > 0 \) and this contradicts the definition of \( q_0 \).

Case 2. \( \exists \tau > 0 \) such that \( y_2(t) < 0 \) for \( 0 \leq t < \tau \), \( y_2(t_0) = 0 \), and \( y_2'(t_0) = 0 \).

This can be ruled out since

\[ y_2''(t_0) = 4y_2(t_0) + y_4(t_0) > 0, \]

and \( y_2(t_0) \) is a local minimum. It contradicts the assumption of \( t_0 \).

Case 3. \( \liminf_{t \to +\infty} y_2(t) = -\infty \). Then, there is an increasing sequence of \( t_k \to \infty \) such that \( y_2(t_k) \to -\infty \) as \( k \to +\infty \). For each \( k \), we can assume that \( y_2(t) < -\varepsilon_k \) for some \( \varepsilon_k \) on \( (0, t_k) \). By the definition of \( q_0 \), there is a sequence \( q_k \to q_0 \) such that \( q_k > q_0 \) and there is a sequence of solutions \( \{ x_i(t) \}_{i=1}^\infty \) with initial value \( (0, p, 0, q_i) \) such that \( ||x_j(t) - y_j(t)||_\infty \to 0, j = 1, 2, 3, 4 \) as \( i \to \infty \) in any compact subset of \( \mathbb{R} \). For \( t \in (0, t_k) \), we can find \( i_k \) such that \( ||x_j(t) - y_j(t)||_\infty < \varepsilon_k \). However, since \( q_k > q_0 \), \( x_j(t) \) must be close to the line \( x_2 = 0 \). By mean value theorem there is a \( \tau_{i_k} > 0 \) such that \( (x_j(t) - y_j(t))(\tau_{i_k}) < 0 \) and \( x_j(t)(\tau_{i_k}) \leq y_2(t_k) \). The first integral (6.3) at \( \tau_{i_k} \) gives

\[ (y_2(t_k))^2 \leq 2(x_2(t_k))^2 - (x_4(t_k))^2 + (x_3(t_k))^2 + x_4(t_k)^2 = 2p^2 + q_{i_k} < 2p^2 + 2q_0. \]

Now we reach a contradiction since the right hand side of the above equation is bounded.

Case 4. \( \lim_{t \to +\infty} y_2(t) = -c < 0 \). Then, \( y_1(t) < \frac{-c}{2}t + b \) for some constant \( b \). Hence \( y_1(t) < (b - \frac{c}{2}t)y_4 \) and \( y_4(t) \leq C \exp(bt - \frac{c}{2}t^2) \). This estimate shows \( y_3(t) \) is bounded. Since \( y_3(t) = 4y_1 + y_3 \), we see \( y_2(t) < -\frac{c}{2}t + b_1 \) for some constant \( b_1 \). It’s clear that \( y_2 \) is not bounded and cannot have negative limit. It is still a contradiction.
Case 5. $\liminf_{t \to +\infty} y_2(t) = -c < 0$ while $\limsup_{t \to +\infty} y_2(t) > -c$. We must have a sequence $t_n \to +\infty$ such that $y_2(t_n) \to -c$, $y_2''(t_n) = 0$, $y_2''(t_n) = 4y_2(t_n) + y_4(t_n) \geq 0$. This means $\lim_{n \to \infty} y_4(t_n) \geq 4c$. But $y_4'(t) = 4y_4(t)y_1(t) < 0$ implies that $y_4$ is monotone and for $t$ big enough $y_4(t) \geq 4c$. However, $y_3(t)$ is then unbounded. Use the first integral at $t_n$ again we see that is a contradiction.

By ruling out the above 5 cases, we see that the only possibility for $y_2(t)$ is that it goes to 0 as $t \to \infty$. Now if $y_2(t)$ oscillates as $t \to \infty$, $y_3$ is monotone hence the first integral implies that $y_3$ is bounded and the rest terms in the first integral must be bounded too. So $y_2'(t)$ is bounded. If $y_2(t)$ is monotone for big $t$, then $y_2(t)$ must be bounded as well as $y_3(t)$. Either case shows $y_2(t)$ and $y_3(t)$ are bounded. Then, $y_2'(t) = 4y_1(t) + y_3(t)$ yields that $y_1(t)$ is bounded.

Finally, we discuss the uniqueness result as follows:

**Theorem 6.7.** Fix a constant in the right hand side of the first integral [6.4], the bounded solution to the system (6.3) is unique up to a translation (dilation) in $t$.

**Proof of the uniqueness.** By a translation, we may assume that $x_1(0) = 0$. If the initial data is give by

$$(x_1, x_2, x_3, x_4)(0) = (0, a, b, c),$$

where $c > 0$ and it also yields a bounded solution $z_i(t)$, then $z_i(-t)$ with initial value $(0, a, -b, c)$ is also a solution. We can assume $b > 0$ and $2a^2 + c = 2p^2 + q$. Suppose $y_i(t)$ is a bounded solution such that

$$(y_1, y_2, y_3, y_4)(0) = (0, p, 0, q).$$

If $p \leq a < 0$ then $q \leq c$. By monotonicity lemma, we see that $z_i(t) > y_i(t)$ for $t > 0$. Hence $z_2(t) - y_2(t) > 0$ and $(z_2(t) - y_2(t))' > 0$. With a positive difference, if $y_2'(t)$ goes to 0, $z_1(t)$ must go to $\infty$. If $p > a$ then $q > c$, let $z_i(t) = z_i(-t)$. Then still $z_i(t) < y_i(t)$ for $t > 0$ and $z_i(t)$ must go to $-\infty$. Thus, $z(t)$ can not bounded either. We have proved the uniqueness.

### 7. Asymptotic Behavior

In this section, we establish a local asymptotic expansion for the solutions of (1.1) in $\mathbb{R}^4$. First, we need to solve the Laplace equation on polynomial space. Let $P_m$ be the space of homogeneous polynomials with degree $m$. Then, eigenvalues and eigenfunctions of Laplacian $\Delta$ are described as follows.

**Lemma 7.1.** (e.g. [LP87]) Suppose that $x \in \mathbb{R}^4 r = |x|$. The eigenvalues of $r^2\Delta$ on $P_m$ are

$$\{\lambda_j = 2j(2 + 2m - 2j) : j = 0, 1..., \lfloor m/2 \rfloor\}.$$ 

The eigenfunctions corresponding to $\lambda_j$ are the functions of the form $r^{2j}u$, where $u \in P_{m-2j}$ is harmonic.

The above lemma shows: if $a$ is not an eigenvalue $(r^2\Delta - a)$ is invertible. Next, we state a technical lemma.

**Lemma 7.2.** Let $\beta \in \mathbb{R}$, $-1 < \beta < 0$.

1. $\beta \neq -1/2$. For any polynomial $f(x)$, there is a polynomial $q(x)$ such that

$$\Delta^2(q(x)r^{4\beta + 4}) = f(x)r^{4\beta}.$$
There is a collection of polynomials \( \{q_i\}_{i=0}^k \) such that
\[
\Delta^2 \left( \sum_{l=0}^k q_l(x) r^{4\beta + 4 (\log r)^l} \right) = f(x) r^{4\beta} (\log r)^k.
\]
\[\tag{7.1}\]

2. \( \beta = -1/2 \). For any polynomial \( f \) with degree \( \leq 2 \) with \( f(x) = \sum a_{ij} x_i x_j + \sum b_i x_i + c \), we have a function
\[
q(x) = a_0 r^2 + \tilde{a}_{ij} x_i x_j \log r + \tilde{b}_i x_i \log r + \tilde{c} \log r,
\]
such that \( \Delta^2 (q(x) r^2) = f(x) r^{-2} \). In particular, \( a_0 \) vanishes if \( \tilde{a}_{ii} = 0 \), \( i = 1, 2, 3, 4 \), and \( \tilde{a}_{ij} = 0 \) if the degree is less than 2. Besides, we have a collection of polynomials \( q_i \) such that
\[
\Delta^2 \left( r^2 \sum_{l=0}^{k+1} q_l(x) (\log r)^l \right) = f(x) r^{-2} (\log r)^k.
\]
\[\tag{7.3}\]

Proof. See the appendix.

Suppose \( u \) is a desired weak solution of (1.6). We consider in a unit ball
\[
\Delta^2 u(x) = e^{4u(x)} \text{ in } B_1(0).
\]
Then by regularity theory \( u - \beta \log r \in C^\infty (B_1(0) - \{0\}) \cap C^{4\beta + 4 - \epsilon} (B_1(0)), \forall \epsilon > 0 \). Here \( \beta = \check{\beta}_0 \). Let \( w = u(x) - \beta \log r \). Then \( w \) satisfies
\[
\Delta^2 w = e^{4w} |x|^{4\beta}.
\]
\[\tag{7.4}\]

Theorem 7.3. Suppose that \( w \) is a solution of (7.4) in \( B_1 \) and \( w \in C^\infty (B_1(0) - \{0\}) \cap C^{4\beta + 4 - \epsilon} (B_1(0)), \forall \epsilon > 0 \). Then
1. \( -\frac{k+1}{k+2} < \beta < -\frac{k}{k+1} \), for some \( k = 0, 1, 2, \cdots \). Then
\[
w = \sum_{l=1}^{k+1} q_l(x) r^{4l(\beta + 1)} + \psi(x),
\]
where \( \psi(x) \in C^{4, \gamma} \) and \( q_l \) are polynomials.
2. \( \beta = -\frac{2k}{2k+1} \), for some \( k = 0, 1, 2, \cdots \).
\[
w = \sum_{l=1}^{2k+2} q_l(x) r^{4l(\beta + 1)} + \psi(x),
\]
where \( \psi(x) \in C^{4, \gamma} \) and \( q_l \) are polynomials.
3. \( \beta = -\frac{2k-1}{2k} \), for some \( k = 0, 1, 2, \cdots \). Then
\[
w = \sum_{l=1}^{2k} q_l(x) r^{4l(\beta + 1)} P_l(\log r) + \psi(x).
\]
where \( \psi(x) \in C^{4, \gamma} \), \( q_l \) and \( P_l \) are polynomials.

We break the proof into 3 cases according to the value of \( \beta \).

proof of Case 1, \( -\frac{k+1}{k+2} < \beta < -\frac{k}{k+1} \). Since \( w \in C^{4\beta + 4 - \epsilon} \), we can find a polynomial \( g_0 \) with degree not exceeding \( 4\beta + 4 \) such that \( \check{w} := w - g_0 \) and \( \check{w}(x) = o(r^{4\beta + 4 - \epsilon}) \). Then \( \check{w} \) satisfies
\[
\Delta^2 \check{w} = e^{4\check{w}} (e^{4\check{w}} - 1) r^{4\beta} + e^{4\check{w}} r^{4\beta}.
\]
Since $e^{4g_0}$ is smooth enough, we can pick a polynomial $\phi_0(x)$ such that $(e^{4g_0(x)} - \phi_0(x))r^{4\beta} = O(r^{-\gamma})$ where $\gamma > 0$. In fact, since $\beta \neq -\frac{1}{2}$, we can pick $\gamma = 4\beta + 4$.

By Lemma 7.2 we can find a polynomial $q_0(x)$ such that
\[
\Delta^2 (q_0(x)r^{4\beta+4}) = \phi_0(x)r^{4\beta}.
\]

Now, we can see that
\[
\Delta^2 (\bar{w} - q_0(x)r^{4\beta+4}) = e^{4\bar{w}}(e^{4\bar{w}} - 1)r^{4\beta+4} + o(r^{8\beta+4-\epsilon}),
\]

If $\beta > -\frac{1}{2}$, then the right hand side of (7.10) is H"older continuous and by elliptic regularity theory, we are done. If $\beta < -\frac{1}{2}$, let $w_1 = \bar{w} - q_0(x)r^{4\beta+4}$. Then we see that
\[
\Delta^2 w_1 \in L^p, 1 < p < \frac{4}{4 + 8\beta}.
\]

By elliptic regularity theory, we have that $w_1 \in W^{4,p} \hookrightarrow C^{8\beta+8-\epsilon}$. We now use induction. Suppose that by above argument $w_l \in C^{4(l+1)(\beta+1)-\epsilon}$, for $1 \leq j \leq k$. In addition, $w_j$ assumes the following expansion
\[
w_j = \bar{w}_{j-1} - \sum_{l=0}^k \phi^l_j(x) r^{4[l+1](\beta+1)}.
\]

(7.7)\[
\Delta^2 \bar{w}_j = e^{4s_{j-1}}(e^{4\bar{w}_j} + e^{4s_{j-1}}r^{4[l+1](\beta+1)} - 1) + \xi_j(x)
\]

(7.8)\[
= e^{4s_{j-1}}(e^{4\bar{w}_j} - 1) + e^{4s_{j-1}}[e^{4g_j + \sum_{l=0}^k \phi^{l+1}_j(x)r^{4(l+1)(\beta+1)}} - 1] + \xi_j(x)
\]

(7.9)\[
= o(r^{4[j+2](\beta+j+1)-\epsilon}) + R_j(x) + \xi_j(x).
\]

We see that
\[
R_j(x) = \sum_{l=0}^k \phi^l_j(x)r^{4[l+1](\beta+1)} + O(r^{4(k+2)(\beta+k+1)}),
\]

where $\phi^l_j(x)$ are polynomials. $\xi_j(x)$ are H"older continuous functions.

We can find polynomials $q^l_j(x)$ such that
\[
\Delta^2 (q^l_j(x)r^{4[l+1](\beta+1)}) = \phi^l_j(x)r^{4[l+1](\beta+1)}.
\]

Let $Q_j(x) = \sum_l q^l_j(x)r^{4[l+1](\beta+1)}$ and $w_{j+1} = \bar{w}_j(x) - Q_j(x)$. Then
\[
\Delta^2 w_{j+1} = o(r^{4[j+2](\beta+j+1)-\epsilon}) + \xi_{j+1}(x),
\]

where $\xi_{j+1}(x)$ is a H"older continuous function. If $j < k$, the right hand side of (7.10) is in $L^p$ for $1 < p < \frac{1}{(j+2)(\beta+j+1)-\epsilon}$. Thus $w_{j+1} \in W^{4,p}$ by elliptic regularity theory and hence in $C^{4(j+2)(\beta+1)-\epsilon}$. We can keep using induction until $j = k$ when the right hand side of (7.10) is H"older and $w_{k+1} \in C^{4,\gamma}$ for some $\gamma > 0$. This clearly gives the asymptotic expansion.

\[\square\]

\textit{proof of Case 2, $\beta = \frac{1}{2k+1} - 1$.} This case is just like Case 1 with minor changes. We can assume that $k \geq 1$. The beginning steps are the same as in the first case.
We use induction. In this case, we assume that \( w_1 \in C^{8\beta+8} \). If \( \beta = \frac{1}{2k} - 1 \), the remaining term like \((7.9)\) has the following form
\[
R_1(x) = \phi_0 r^{4\beta} + \phi_1 r^{8\beta+4} + \cdots + \phi_{2k-1} r^{8k\beta+8k-4} + \phi_{2k} + O(r^\gamma)
\]
where \( \phi_l \) are polynomials.

\[
4(l+1)(\beta+1) - 4 \neq -2, \quad l = 0, 1, \cdots, 2k.
\]

So by lemma \((7.2)\) we can find polynomials \( q_1(x) \) and \( Q_1(x) = \sum q_1^l(x) r^{4(l+1)(\beta+1)} \) such that
\[
\Delta^2 Q_1(x) = \sum_{l=0}^{2k} \phi_l(x) r^{4(\beta+1)4l}.
\]

We can still use the induction but this time, the induction will continue until \( j \) hits \( 2k - 1 \). Then
\[
\Delta^2 w_{2k} = o(r^{-1}) + \xi_{2k}(x),
\]

This however shows that \( w_{2k} \) is in \( C^{4-\epsilon} \). Note that
\[
\Delta^2 w_{2k} = e^{4x_{2k-1}} (\exp (4w_{2k} + 2q_{2k-1} - 1) r^{4\beta} + \xi_{2k-1})
\]

(7.11)

\[
= e^{4x_{2k-1}} (e^{4q_{2k-1}} - 1) r^{4\beta} + r^{4x} e^{4x} (e^{2q_{2k-1}} - 1) + \xi_{2k-1}.
\]

Hence, the right hand side of \((7.11)\) is Hölder continuous. By regularity theory of elliptic equations, we conclude the proof.

proof of Case 3, \( \beta = \frac{1}{2k} - 1 \). Finally, suppose that \( \beta = -\frac{2k-1}{2k} \). The first few steps are the same. We replace \( w \) by \( \tilde{w} \) such that \( \tilde{w} = o(r^{4\beta+4-\epsilon}) \).

If \( k = 1 \), in \((7.3)\) we can find \( q_0(x) \) in the form of \((7.2)\). Then \( w_1 = \tilde{w} - q_0(x)r^2 \in C^{4-\epsilon} \). Suppose that \( \tilde{w}_1 = w_1 - g_1(x) \) such that \( g_1 \) is a polynomial and \( \tilde{w}_1 = o(r^{4-\epsilon}) \). Then \((e^{4q_1+4q_0r^2} - 1)r^{-2} = c \log r + o(r^\gamma) \) where \( 0 < \gamma_1 < 1 \). So \( \Delta^2 \tilde{w}_1 = O(r^{\gamma_1}) + c \log r \). Then we pick \( c_1 \) such that
\[
\Delta^2 (\tilde{w}_1 - c_1 r^4 \log r) = c_2 + O(r^\gamma).
\]

This shows that
\[
w = (a_{ij} x_i x_j + b_i x_i + c) r^2 \log r + c_1 r^4 \log r + \psi(x),
\]

where \( \psi(x) \in C^{4, \gamma_1} \). This concludes the case where \( \beta = -1/2 \).

Otherwise, we can find \( q_0(x) \) a polynomial satisfying \((7.3)\). Then \( w_1 = \tilde{w} - q_0(x)r^{4\beta+4} \in C^{8\beta+4-\epsilon} \). Subtracting a polynomial \( g_1(x) \) from \( w_1 \), we assume \( \tilde{w}_1 = w_1 - g_1 = o(r^{8\beta+8-\epsilon}) \). We expand \( \Delta^2 \tilde{w}_1 \) in the form of \((7.8)\). Note that the remaining term \( R_1(x) \) in \((7.9)\) has the following form:
\[
R_1(x) = \sum_{l=1}^{2k} \phi_l r^{4\beta+4} + O(r^\gamma).
\]

(7.12)

In \((7.12)\), there is a term \( \phi_l(x) r^{4\beta+4} \). This term will introduce \( \log r \).

We can assume that \( \deg \phi_k \leq 2 \), since higher degree terms times \( r^{-2} \) are Hölder continuous. We can find polynomials \( q_l^l(x) \) for \( l \neq k \), such that \( \Delta^2 (q_l^l(x) r^{4\beta+4}) =
\( \phi_t(x) r^{4l(\beta+1)-4} \) and a function \( q^2_t(x) \) in the form of \( \binom{7}{2} \) such that \( \Delta^2(q^2_t(x)r^2) = \phi_k r^{-2} \). Let
\[
Q_1(x) = \sum_{l=1}^{2k-1} q^l_1(x) r^{4l(\beta+1)}.
\]

Let \( w_2 = \tilde{w}_1 - Q_1(x) \).
\[
\Delta^2 w_2 = o(r^{16(\beta+1)-\epsilon}) + \xi_2(x).
\]

We then use induction. Suppose that \( w_l \in C^{4l+1}(\beta+1) \), \( l \leq j \leq 2k \). Suppose that \( w_j = \tilde{w}_{j-1} - Q_{j-1} \).
\[
Q_j-1 = \sum_{l=1}^{2k-1} q^l_1 - 1 r^{4l(\beta+1)} P^l_j(\log r),
\]

where \( q^l_1 \) and \( P^l_j \) are polynomials. Then let \( \tilde{w}_j = w_j - g_j \) such that \( \deg g_j \leq 3 \) and \( \tilde{w}_j = o(r^{4l+1}(\beta+1)-\epsilon) \). We have
\[
\Delta^2 \tilde{w}_j = o(r^{4l+1}(\beta+1)-\epsilon) + R_j(x) + \xi_{j-1}(x).
\]

Here \( R_j(x) \) can be written as
\[
R_j(x) = \sum_{l \neq k, 1 \leq l \leq 2k} \phi^l_j \bar{P}^l_j(\log r) r^{2l+4} + \phi^\beta_r \bar{P}^\beta(r) + O(r^{j}).
\]

We can assume that \( \deg \phi^\beta_r \leq 2 \) because the higher degree terms by \( r^{-2} \bar{P}^\beta(\log r) \) are Hölder continuous. By Lemma \( \binom{7}{2} \) we can find
\[
Q_j(x) = \sum_{l=1}^{2k} q^l_1(x) P^l_j(\log r) r^{4l(\beta+1)}.
\]

such that \( \Delta^2(Q_j - R_j) \) is Hölder continuous. Then
\[
\Delta^2 (\tilde{w}_j - Q_j) = o(r^{4l+2}(\beta+1)-4-\epsilon) + \xi_j(x).
\]

The rest of proof is the same to the previous cases. \( \square \)

8. Uniqueness Result with 2 Singularities

In this section, we give a proof of Theorem \( \binom{13}{3} \). Let \( M = S^4, g_0 \) be the standard metric on 4-sphere. Let \( (M, g_0, D, g_1) \) be the conic sphere with divisor \( D = \beta_0 p_0 + \beta_1 p_1 \), \( \int_M Q_{g_1} d\nu_{g_1} = 8\pi^2 (2 + \beta_0 + \beta_1) \). Note if \( w \) is a solution on the sphere with divisor \( D \), we have
\[
P_{g_1} w + 6 = 3(2 + \beta_0 + \beta_1) e^{4w}.
\]

Here we have normalized the equation such that the conic sphere has the same volume as a standard 4-sphere. Let \( k_d = 3(2 + \beta_0 + \beta_1) \). We restrict the solution such that
\[
w - \sum_{i=0,1} \eta_i(x) \beta_i \log |x - p_i| \in H^2(d\nu_0),
\]

at where \( p_i \) are two points on the sphere and \( \eta_i(x) \) are cut off functions in the neighborhood of \( p_i \) like those in \( \binom{13}{3} \). By a conformal transform on the sphere, we can assume that two points are south and north poles. By stereographic projection from north pole, we obtain the equation on \( \mathbb{R}^4 \)
\[
\Delta^2 u = k_d e^{4w}.
\]

We state two lemmas that describe the asymptotic behavior.
Lemma 8.1. $\Delta u - 2\beta_0 \frac{1}{|x|^2} = -\frac{k}{2\pi^2} \int_{\mathbb{R}^4} \frac{4u(y)}{|x-y|^4} dy - C_1$ where $C_1 \geq 0$ is a constant.

Lemma 8.2. $u - \beta_0 \log |x| = -\frac{k}{2\pi^2} \int_{\mathbb{R}^4} \log \frac{|x-y|}{|y|} e^{4u(y)} dy + C_0$ where $C_0$ is a constant.

Besides, for any $\epsilon > 0$ there is an $R_\epsilon$ such that

$$(-2 - \beta_1) \log |x| \leq u \leq (-2 - \beta_1 + \epsilon) \log |x|,$$

for $|x| \geq R_\epsilon$.

For the proof of Lemma 8.1 and Lemma 8.2, see Lemma 2.1 - 2.5 in [Lin98]. We should mention that since we always assume that the solution $u$ comes from a $H^2$ function on $S^4$, $u$ satisfies assumptions in Lin’s paper for both lemmas.

Now we can derive an asymptotic expansion at infinity.

Lemma 8.3. $u(x) = -(2 + \beta_1) \log |x| + c + O(|x|^{-1})$ and

\begin{equation}
\begin{aligned}
\Delta u &= |x|^{-2} (a_0 + \sum_{i=1}^4 a_{i,1} |x|^{-4(\beta_1+1)} P_{i,1}(-\log |x|)) \\
&+ \sum_{i=1}^4 a_{i,1} |x|^{-2} + \sum_{i=1}^4 a_{i,1} |x|^{-4(\beta_1+1)-2} P_{i,1}(-\log |x|)) + O(|x|^{-4}), \\
-\frac{\partial}{\partial x_i} \Delta u &= a_{0,i} |x|^{-4} + O(|x|^{-6+\beta_1}), \\
-\frac{\partial^2}{\partial x_i \partial x_j} \Delta u &= O(|x|^{-6}).
\end{aligned}
\end{equation}

for large $|x|$, where $c, 0 < \delta < 4(\beta_1+1), a_{i,1}, 1 \leq i \leq 4$ are constants and $P_{i,1}$ are polynomials. Note that $a_0 = 2(2 + \beta_1)$ is positive.

Proof. Let $w(x) = u \left( \frac{x}{|x|^4} \right) - (2 + \beta_1) \log |x|$. By Lemma 8.2 we see that $w(x)$ satisfies

\begin{equation}
\begin{aligned}
\Delta^2 w(x) &= \bar{k}_g e^{4w(x)} |x|^{4\beta_1} \quad \text{in } \mathbb{R}^4 - \{0\}, \\
|w(x)| &= o(|\log |x||) \quad \text{as } |x| \to 0, \\
|\Delta w| &= o(|x|^{-2}) \quad \text{as } |x| \to 0.
\end{aligned}
\end{equation}

Let $h(x)$ be a weak solution of

\begin{equation}
\begin{aligned}
\Delta^2 h(x) &= \bar{k}_g e^{4w(x)} |x|^{4\beta_1} \quad \text{in } B_1, \\
h(x) &= w(x) \quad \text{on } \partial B_1, \\
\Delta h(x) &= \Delta w(x) \quad \text{on } \partial B_1.
\end{aligned}
\end{equation}

By Lemma 8.2 $e^{4w(x)} |x|^{4\beta_1}$ is in $L^p(B_1)$ for $(-\beta_1)^{-1} > p > 1$. By regularity theorem of elliptic equations, $\Delta h(x) \in W^{2,p}(B_1)$ and $h(x) \in W^{4,p}(B_1)$ and hence $h(x) \in C^{4\tau}$ for $0 < \tau < (1 + \beta_1)$. Now, let $q(x) = w(x) - h(x)$. Then it satisfies that

\begin{equation}
\begin{aligned}
\Delta^2 q &= 0 \quad \text{in } B_1 - \{0\}, \\
q &= \Delta q = 0 \quad \text{on } \partial B_1, \\
|q(x)| &= o(|\log |x||), |\Delta q| = o(|x|^{-2}) \quad \text{as } |x| \to 0.
\end{aligned}
\end{equation}

Thanks to the asymptotic property, we can still apply maximum principle to $\Delta q$ with boundary $\partial B_1$. Thus $\Delta q \equiv 0$ and similarly, $q \equiv 0$. Therefore, $w(x) = h(x)$.

Now by the asymptotic expansion in Theorem 7.3 the lemma follows immediately.

We exploit the moving plane methods to prove the symmetry of the solution with two conical singularities. Following the convention in the literature, see for example [GNN79, CGS89, Lin98], let $\lambda \in \mathbb{R}$, $T_\lambda = \{(x_1, x_2, x_3, x_4) : x_1 = \lambda\}$,
\[ \Sigma_\lambda = \{ x : x_1 > \lambda \} \text{, and } x^\lambda = (2\lambda - x_1, x_2, x_3, x_4). \] In order to initiate the moving plane along \( x_1 \) direction, we need the following two lemmas.

**Lemma 8.4.** Let \( v \) be a positive function defined in a neighborhood of infinity satisfying the asymptotic expansion \( (8.1) \). Then there exists \( \lambda \) and \( R > 0 \) such that
\[ v(x) > v(x^\lambda) \]
holds for \( \lambda < \bar{\lambda} \), \( |x| \geq R \) and \( x \in \Sigma_\lambda \).

**Lemma 8.5.** Suppose \( v \) satisfies the assumption of 8.4 and \( v(x) > v(x^{\lambda_0}) \) for \( x \in \Sigma_{\lambda_0} \). Assume \( v(x) - v(x^{\lambda_0}) \) is superharmonic in \( \Sigma_{\lambda_0} \). Then there exist \( \epsilon > 0 \), \( S > 0 \) such that the followings hold.

(i) \( v_{x_1} > 0 \) in \( |x_1 - \lambda_0| < \epsilon \) and \( |x| > S \).
(ii) \( v(x) > v(x^\lambda) \) in \( x_1 \geq \lambda_0 + \frac{\epsilon}{2} > \) \( \lambda \) and \( |x| > S \)
for all \( x \in \Sigma_\lambda, \lambda \leq \lambda_1 \) with \( |\lambda_1 - \lambda_0| < \epsilon \), where \( c = c(\lambda_0, v) \) is a small positive number.

Both lemmas are contained in the celebrated paper by Caffarelli-Gidas-Spruck [CGS89]. For their proofs, please see Lemma 2.3 and 2.4 in [CGS89]. We should remind the readers that although our asymptotic expansion is not the exact form in the above paper, the leading terms are the same. Hence the argument in [CGS89] works here.

**Proof of Theorem 1.5.** For any \( \lambda \neq 0 \), let \( w_\lambda(x) = u(x) - u(x^\lambda) \) in \( \Sigma_\lambda \). Then \( w_\lambda(x) \) satisfies

\[
\begin{align*}
\Delta^2w_\lambda &= b_\lambda(x)w_\lambda & \text{if } x \in \Sigma_\lambda, \\
w_\lambda &= \Delta w_\lambda = 0 & \text{if } x \in T_\lambda,
\end{align*}
\]
where \( b_\lambda(x) = \bar{k}g^{\lambda}(x) - e^{4u(x^\lambda)} > 0 \). By Lemma 8.4, \( \Delta w_\lambda < 0 \) for \( x \in \Sigma_\lambda, \lambda \leq \bar{\lambda} < 0 \), \( |x| > R \). Since \( v(x) > 0 \), there is \( \bar{\lambda}_1 \leq \bar{\lambda} \) such that \( v(x^\lambda) < v(x) \) for \( |x| < R \) and \( \lambda < \lambda_1 \). Hence
\[ \Delta w_\lambda(x) < 0. \]
in \( \Sigma_\lambda \) for \( \lambda \leq \bar{\lambda}_1 \). By Lemma 8.3, \( \lim_{|x| \to \infty} w_\lambda(x) = 0 \). By maximum principle, we have \( w_\lambda(x) > 0 \) in \( \Sigma_\lambda \) for \( \lambda \leq \bar{\lambda}_1 \). Move \( T_\lambda \) to the right. Let \( \lambda_0 = \sup \{ \lambda < 0 : v(x^\mu) \leq v(x), \ x \in \Sigma_\mu \text{ for } \mu \leq \lambda \} \). If \( \lambda_0 = 0 \) then we are finished. Otherwise, we claim that \( u(x) \equiv u(x^{\lambda_0}) \) for \( x \in \Sigma_{\lambda_0} \). This also implies that \( \lambda_0 = 0 \). We argue by contradiction. Suppose that \( \lambda_0 < 0 \) and \( w_{\lambda_0} \neq 0 \) in \( \Sigma_{\lambda_0} \). By continuity, \( \Delta w_{\lambda_0} \leq 0 \) in \( \Sigma_{\lambda_0} \). Since \( w_{\lambda_0}(x) \to 0 \) as \( |x| \to \infty \), by strong maximum principle \( w_{\lambda_0} > 0 \) in \( \Sigma_{\lambda_0} \). Then we have
\[ \Delta^2w_{\lambda_0} = \bar{k}g(e^{4u}(x) - e^{4u(x^{\lambda_0})}) > 0. \]
Hence \( \Delta w_{\lambda_0} \) is subharmonic. By strong maximum principle, we have \( \Delta w_{\lambda_0} < 0 \) in \( \Sigma_{\lambda_0} \). By the definition of \( \lambda_0 \) there is a sequence \( \lambda_n \downarrow \lambda_0 \) and \( \lambda_n < 0 \) such that \( \sup_{x \in \Sigma_{\lambda_n}} \Delta w_{\lambda_n} > 0 \). Since \( \lim_{|x| \to \infty} \Delta w_{\lambda_n}(x) = 0 \), there exists \( z_n \in \Sigma_{\lambda_n} \) such that
\[ \Delta w_{\lambda_n}(z_n) = \sup_{x \in \Sigma_{\lambda_n}} \Delta w_{\lambda_n}(x) > 0. \]
Note that at \( z_n \),
\[ \nabla \Delta w_{\lambda_n}(z_n) = 0. \]
By Lemma 8.3 we see that $z_\alpha$ are bounded. Suppose that $z_0$ is a limit point of $z_n$. If $z_0 \in \Sigma_{\lambda_0}$, by continuity, we have $\Delta w_{\lambda_0}(z_0) = 0$. This contradicts that $\Delta w_{\lambda_0} < 0$ in $\Sigma_{\lambda_0}$. If $z_0$ is on $T_{\lambda_0}$, then $\nabla(\Delta w_{\lambda_0}(z_0)) = 0$, which yields a contradiction to Hopf’s lemma. Hence the claim is proved. The symmetry with respect to $x_1 = 0$ clearly follows from $\lambda_0 = 0$. By a rotation, the solution is clearly symmetric with respect to any hyperplane through the origin and hence must be radial symmetric. \[\square\]

**Appendix**

**Proof of Lemma 7.2** For each degree $m$, we only have to consider homogeneous polynomials in $P_m$. 
1. Suppose that $\beta \neq -\frac{1}{2}$. Let $p(x) \in P_m$. By Euler formula, $x_i D_i p(x) = mp(x)$. 

$$\Delta (p(x)r^{4\beta+2}) = r^{4\beta} \left( r^2 \Delta p(x) + (4\beta + 2)(4\beta + 4 + 2m)p(x) \right)$$

So $\Delta (p(x)r^{4\beta+2}) = f(x)r^{4\beta}$ if 

$$r^2 \Delta p(x) + (4\beta + 2)(4\beta + 4 + 2m)p(x) = f(x).$$

Then by Lemma 7.1 we see that there exists a $p(x)$ such that 

$$\Delta (p(x)r^{4\beta+2}) = f(x)r^{4\beta},$$

if $\beta \neq -1/2$. Now let $q(x)$ be a homogeneous polynomial with degree $m$. Then 

$$\Delta (q(x)r^{4\beta+4}) = r^{4\beta+2} \left( r^2 \Delta q(x) + (4\beta + 4)(4\beta + 6 + 2m)q(x) \right)$$

Apply Lemma 7.1 again. There is a polynomial such that $\Delta (q(x)r^{4\beta+4}) = p(x)r^{4\beta+2}$. If $\log r$ is involved let $p(x)$ be a polynomial in $P_m$. If $\beta \neq -\frac{1}{2}$, 

$$\Delta [p(x)r^{4\beta+2} \log r] = r^{4\beta} \log r \left( r^2 \Delta p(x) + (4\beta + 2)(2m + 4\beta + 4)p(x) \right) + r^{4\beta} \phi(x),$$

where $\phi(x)$ is in $P_{m-2}$. First, we can solve 

$$r^2 \Delta p_1(x) + (4\beta + 2)(2m + 4\beta + 4)p_1(x) = f(x).$$

We can also find a polynomial $p_2(x)$ such that $\Delta (p_2(x)r^{4\beta+2}) = \phi(x)r^{4\beta}$. Thus, 

$$\Delta [r^{4\beta+2}(p_1 \log r - p_2)] = f(x)r^{4\beta} \log r.$$

With a similar argument we can find $q_1$, $q_2$ such that 

$$\Delta [r^{4\beta+4}(q_1(x) \log r + q_2(x))] = (p_1(x)|x|^2 \log r - p_2(x)|x|^2)r^{4\beta},$$

hence 

$$\Delta^2 \left[ (q_1(x) \log r + q_2(x))r^{4\beta+4} \right] = f(x)r^{4\beta} \log r.$$ 

Then we use induction. Suppose that we can find solutions of (7.1) for $0 \leq l \leq k - 1$ 

$$\Delta \left( p(x)r^{4\beta+2} \log r \right)^{k} = r^{4\beta} \left( \log r \right)^{k} \left( r^2 \Delta p(x) + (4\beta + 2)(4\beta + 4 + 2m)p(x) \right) + \sum_{j=0}^{k-1} r^{4\beta} \phi_j(x) \left( \log r \right)^{j},$$

where $\phi_j(x)$ are polynomials. Then we can find $p(x)$ such that 

$$r^2 \Delta p(x) + (4\beta + 2)(4\beta + 4 + 2m)p(x) = f(x).$$
The remaining terms can be solved by induction. Therefore, we can find \( p_1(x) \)
such that \( \sum_{i} p_i r^{4i\beta}(\log r)^i = f(x)r^{4\beta}(\log r)^k \). Then repeat argument for each
\( \tilde{f}(x) = p_i(x)|x|^2 \). We can find \( q_i \).

1. \( \beta = -\frac{1}{2} \). If homogeneous degree \( m = 0 \), we see that \( \Delta^2 (\frac{c}{16} r^2 \log r) = cr^{-2} \).
   So this is true for degree 0 polynomial. If \( m = 1 \), direct computation shows that
   \[
   \Delta^2 \left( \frac{c_i r^2}{48} \log r \right) = a_i r^i.
   \]
   If \( m = 2 \), we have for \( i \neq j \)
   \[
   \Delta^2 (x_i x_j r^2 \log r) = 96 x_i x_j r^{-2}.
   \]
   For \( i = j \), we have
   \[
   \Delta^2 (x_i^2 r^2 \log r) = 32 + 96 x_i^2 r^{-2} + 48 \log r.
   \]
   Note that \( \Delta^2 (r^4 \log r) = 7 \times 64 + 3 \times 128 \log r \). Since \( \Delta^2 r^4 = 192 \), we can still find
   a solution for \( \Delta^2 (g(x)r^4) = x_i^4 r^2 \) in the form of \( (7.2) \).

   For functions in the form of \( (7.3) \), we argue by induction with respect to \( k \). Note
   the above argument is for \( l = 0 \). Suppose that for \( 0 \leq l \leq k - 2 \), we have a solution
   for \( (7.3) \). We prove for \( l = k - 1 \). First, if \( f(x) = 1 \)
   \[
   \Delta^2 [c r^2 (\log r)^k] = cr^{-2}(c_1(\log r)^{k-4} + c_2(\log r)^{k-3} + c_3(\log r)^{k-2} + 24(\log r)^{k-1}),
   \]
   where \( c_i \) are polynomials of \( k \) and \( c_i = 0 \) for \( k = 1, 2, \ldots, 4 - i \). Let \( c = 2^{-4} \) and the
   first term can be cancelled. The remaining terms are lower order terms and can be
   solved by induction. For \( f(x) = x_i \), we have that
   \[
   \Delta^2 (c x_i r^2 (\log r)^k) = c k \frac{x_i^2 r^2}{r^2} \log^{-k-1} (r) \left( \sum_{i=0}^{2} c_i (\log r)^i + 48 \log^3 (r) \right).
   \]
   Likewise, take \( c = (48k)^{-1} \) and we can solve the remaining terms by induction.

   For \( f(x) = x_i x_j \), \( i \neq j \), take the test function in the form of \( c x_i x_j r^2 (\log r)^k \). The
   argument is similar. The last case is when \( f(x) = x_i^2 \). We compute directly,
   \[
   \Delta^2 (x_i^2 r^2 (\log r)^k) = x_i^2 r^2 (\log r)^{k-1} r \times 96 + x_i^2 r^{-2} \log P (\log r) + Q (\log r),
   \]
   where \( P \) and \( Q \) are polynomials and \( \text{deg} P(x) \leq k - 2 \), \( \text{deg} Q(x) \leq k \). Note that
   the degree of \( P(x) \) is less than \( k - 1 \). We can solve by induction so that we have
   a function \( P(x) \) in the form of \( (7.3) \) such that \( \Delta^2 (P(x)r^2) = x_i^2 r^{-2} P(\log r) \). For
   \( Q(x) \), if it has degree \( k \), we pick a function in the form of \( cr^4 (\log r)^k \).

   \[
   \Delta^2 (r^4 (\log r)^k) = k \log^{-k-4} (r) ((k - 1) (k - 2)(k - 3)
   \]
   + \begin{align*}
   b_1 k (k - 1) (k - 2) \log (r) \\
   b_2 (k - 1) k \log^2 (r) + b_3 k \log^3 (r) \end{align*} + 12 \times 16 \log^5 (r),
   \]
   where \( b_i \) are constants independent of \( k \). Suppose \( a_k \neq 0 \) is the top coefficient of
   \( Q(x) \). If we subtract \( Q(\log r) \) by \( \Delta^2 \left( \frac{a_k}{12 \times 10} r^4 (\log r)^k \right) \), what left is a polynomial of
   \( \log r \) with degree less than \( k \). Then we can apply the induction. As a result, we can
   find a polynomial \( \tilde{Q}(x) \) with \( \text{deg} \tilde{Q}(x) \leq k \) and \( \Delta^2 (r^4 \tilde{Q}(\log r)) = Q(\log r) \). Now let
   \[\tilde{f}(x) = \frac{1}{96 k} (x_i^2 r^2 (\log r)^k - P(x)r^2 - r^4 \tilde{Q}(\log r)) \].
   Clearly, \( \Delta^2 \tilde{f}(x) = x_i^2 r^{-2} (\log r)^{k-1} \). This completes the last case and the whole proof.
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