A Few Remarks on "Zero-Two" Law for Positive contractions in the Orlicz-Kantorovich spaces

Inomjon Ganiev¹,*, Farrukh Mukhamedov²,**, Dilmurod Bekbaev³

¹ Department Natural and Mathematical Sciences, Turin Polytechnic University Tashkent branch, 17, Kichik halqa yo'li str., 100095, Tashkent, Uzbekistan
² Department of Mathematical Sciences, College of Science, The United Arab Emirates University, Al Ain, Abu Dhabi, P.O. Box 15551, UAE
³ Department of Computational & Theoretical Sciences, Faculty of Science, International Islamic University Malaysia, 25200, Kuantan, Pahang, Malaysia

E-mail: *ganiev1@rambler.ru, **farrukh.m@uaeu.ac.ae

Abstract. In this paper we establish a vector version the "zero-two" law for positive contractions in the Orlicz-Kantorovich spaces.

1. Introduction

Let \((X, \mathcal{F}, \mu)\) be a measure space with a positive \(\sigma\)-additive measure \(\mu\). In what follows, for the sake of shortness, we denote by \(L^1(X, \mathcal{F}, \mu)\) the usual \(L^1\) space associated with \((X, \mathcal{F}, \mu)\). A linear operator \(T : L^1 \to L^1\) is called a positive contraction if \(Tf \geq 0\) whenever \(f \geq 0\) and \(\|T\|_1 \leq 1\). Ornstein and Sucheston [29] obtained an analytic proof of the Jamison-Orey result [20], and in their work they proved the following theorem [29, Theorem 1.1].

**Theorem 1.1.** Let \(T : L^1 \to L^1\) be a positive contraction. Then either

\[
\sup_{\|f\|_1 \leq 1} \lim_{n \to \infty} \|T^{n+1}f - T^nf\| = 2.
\]

or \(\|T^{n+1}f - T^nf\| \to 0\) for every \(f \in L^1\).

This result was later called a strong zero-two law. Consequently, [29, Theorem 1.3], if \(T\) is ergodic with \(T^n1 = 1\) (e.g. \(T\) is ergodic and conservative), then either (1.1) holds, or \(\|T^ng\|_1 \to 0\) for every \(g \in L^1\) with \(\int g \, d\mu = 0\). Some extensions of the strong zero-two law can be found in [31, 34].

Interchanging "sup" and "lim" in the strong zero-two law we have the following uniform zero-two law, proved by Foguel [8] using ideas of [29] and [7].

**Theorem 1.2.** Let \(T : L^1 \to L^1\) be a positive contraction. If for some \(m \in \mathbb{N} \cup \{0\}\) one has \(\|T^{m+1} - T^m\| < 2\), then

\[
\lim_{n \to \infty} \|T^{n+1} - T^n\| = 0.
\]

In [32] it was proved the following uniform strong zero-two law for positive contractions of \(L_p\)-spaces \((1 \leq p < +\infty)\):
Theorem 1.3. Let $1 \leq p < +\infty$ and $T$ be a positive contraction of $L_p$. If $\|T^{m+1} - T^m\| < 2$ for some $m \in \mathbb{N} \cup \{0\}$, then

$$\lim_{n \to \infty} \|T^{n+1} - T^n\| = 0.$$ 

A ”zero-two” law for Markov processes was proved in [3], which allowed to study random walks on locally compact groups. Other extensions and generalizations of the formulated law have been investigated by many authors [7, 9, 31, 24]. In all these investigations, the generalization was in direction replacement of the $L^1$-space by an abstract Banach lattice (see [25, 32]). In [28] we have proposed another kind of generalization of the uniform zero-two law in $L^1$-spaces.

Furthermore, in [13] we have established a vector-valued analog of uniform ”zero-two” law for the positive contractions of the Banach — Kantorovich $L_p$-lattices.

Theorem 1.4. Let $T : L_p(\nabla, \mu) \to L_p(\nabla, \mu)$, $p > 1$, $p \neq 2$ be a positive linear contraction such that $T1 \leq 1$. If one had $\|T^{m+1} - T^m\| < 2 \cdot 1$ for some $m \in \mathbb{N} \cup \{0\}$. Then

$$(o) - \lim_{n \to \infty} \|T^{n+1} - T^n\| = 0.$$ 

Later in [17] we have prove an vector-valued analog of Theorem 1.3.

Theorem 1.5. Let $T : L_p(\nabla, \mu) \to L_p(\nabla, \mu)$ be a positive linear contraction such that $T1 \leq 1$. If one has $\|T^{m+1} - T^m\| < 2 \cdot 1$ for some $m \in \mathbb{N} \cup \{0\}$. Then

$$(o) - \lim_{n \to \infty} \|T^{n+1} - T^n\| = 0.$$ 

In [21] it was proved the following ”zero-two” law for positive contractions in the Banach lattices having a weak unit and uniformly monotone norm.

Theorem 1.6. Let $E$ be a Banach lattice having a weak unit and uniformly monotone norm. Let $T$ be a positive contraction of $E$. If for some $m \in \mathbb{N} \cup \{0\}$ one has $\|T^{m+1} - T^m\| < 2$, then

$$\lim_{n \to \infty} \|T^{n+1} - T^n\| = 0.$$ 

Uniform monotone properties of Orlicz spaces considered in [1] and [2]. The monotonicity properties of the Luxemburg norm in Musielak — Orlicz spaces are characterized in [5]. In particular, if the Orlicz function $M$ has $\Delta_2$ - condition, then Orlicz space is vector lattice with the uniform monotone norm and having a weak unit. Hence, if the Orlicz function $M$ has $\Delta_2$ - condition, the uniform ”zero-two” law is valid for positive contractions in the Orlicz spaces.

The main aim of this paper is to prove the uniform ”zero-two” law for the positive contractions of the Orlicz — Kantorovich lattices $L_M(\nabla, \mu)$. We notice that in [6, 10, 11],[14]-[16] several ergodic theorems have been obtained for positive contractions of $L_p(\nabla, \mu)$-spaces.

2. Preliminaries

Let $(\Omega, \Sigma, \lambda)$ be a measurable space with finite measure $\lambda$, and $L_0(\Omega)$ be the algebra of all measurable functions on $\Omega$ (here the functions equal a.e. are identified) and let $\nabla(\Omega)$ be the Boolean algebra of all idempotents in $L_0(\Omega)$. By $\nabla$ we denote an arbitrary complete Boolean subalgebra of $\nabla(\Omega)$.

Let $E$ be a linear space over the real field $\mathbb{R}$. By $\| \cdot \|$ we denote a $L_0(\Omega)$-valued norm on $E$. Then the pair $(E, \| \cdot \|)$ is called a lattice-normed space (LNS) over $L_0(\Omega)$. An LNS $E$ is said to be $d$-decomposable if for every $x \in E$ and the decomposition $\|x\| = f + g$ with $f$ and $g$ disjoint positive elements in $L_0(\Omega)$ there exist $y, z \in E$ such that $x = y + z$ with $\|y\| = f$, $\|z\| = g$. 


Suppose that \((E, \| \cdot \|)\) is an LNS over \(L_0(\Omega)\). A net \(\{x_\alpha\}\) of elements of \(E\) is said to be \((bo)\)-converging to \(x \in E\) (in this case we write \(x = (bo)\)-lim \(x_\alpha\)) if the net \(\{\|x_\alpha - x\|\}\) \((o)\)-converges to zero (here \((o)\)-convergence means the order convergence) in \(L_0(\Omega)\) (written as \((o)\)-lim \(\|x_\alpha - x\| = 0\)). A net \(\{x_\alpha\}_{\alpha \in A}\) is called \((bo)\)-fundamental if \((x_\alpha - x_\beta)_{(\alpha, \beta) \in A \times A}\) \((bo)\)-converges to zero.

An LNS in which every \((bo)\)-fundamental net \((bo)\)-converges is called \((bo)\)-complete. A Banach–Kantorovich space (BKS) over \(L_0(\Omega)\) is a \((bo)\)-complete \(d\)-decomposable LNS over \(L_0(\Omega)\). It is well known [26],[27] that every BKS \(E\) over \(L_0(\Omega)\) admits an \(L_0(\Omega)\)-module structure such that \(\|fx\| = |f| \cdot \|x\|\) for every \(x \in E\), \(f \in L_0(\Omega)\), where \(|f|\) is the modulus of a function \(f \in L_0(\Omega)\). A BKS \((\mathcal{U}, \| \cdot \|)\) is called a Banach–Kantorovich lattice if \(\mathcal{U}\) is a vector lattice and the norm \(\| \cdot \|\) is monotone, i.e., \(|u_1| \leq |u_2|\) implies \(|u_1| \leq |u_2|\). It is known [26] that the cone \(\mathcal{U}_+\) of positive elements is \((bo)\)-closed.

Let \(\nabla : \Omega \to L_0(\Omega)\) be a strictly positive \(L_0\)-valued measure on \(\nabla\) that is modular, i.e., \(\mu(ge) = g\mu(e)\) for all \(e \in \nabla, g \in \nabla(\Omega)\) (see [27]). By \(L_0(\nabla)\) we denote the order complete vector lattice \(C_\infty(\Omega, \nabla)\), where \(Q(\nabla)\) is the Stonian compact associated with complete Boolean algebra \(\nabla\).

Denote by
\[
L_p(\nabla, \mu) = \left\{ f \in L_0(\nabla) : \int |f|^p d\mu - \text{exist} \right\}, p \geq 1.
\]
It is known [26] that \(L_p(\nabla, \mu)\) is a BKS over \(L_0(\Omega)\) with respect to the \(L_0(\Omega)\)-valued norm \(\|f\|_{L_p(\nabla, \mu)} = \left(\int |f|^p d\mu\right)^{\frac{1}{p}}\). Moreover, \(L_p(\nabla, \mu)\) is a Banach–Kantorovich lattice (see [12]).

An even continuous convex function \(M : R \to [0, \infty)\) is called an \(N\)-function, if \(\lim_{t \to 0} \frac{M(t)}{t} = 0\) and \(\lim_{t \to \infty} \frac{M(t)}{t} = \infty\). An \(N\)-function \(M\) is said to satisfy \(\Delta_2\)-condition on \([s_0, \infty)\), \(s_0 \geq 0\), if there exists a constant \(k\) such that \(M(2s) \leq kM(s)\) for every \(s \geq s_0\) (see [22]). Every \(N\)-function \(M\) has the form \(M(t) = \int_0^t \! p(s)ds\), where \(p(t)\) is a nondecreasing function that is positive for \(t > 0\), right-continuous for \(t \geq 0\), and such that \(p(0) = 0\) and \(\lim_{t \to \infty} p(t) = \infty\). Put \(q(s) := \sup\{t : p(t) \leq s\}, s \geq 0\). The function \(N(t) := \int_0^t \! q(s)ds\) is an \(N\)-function which is called the complementary \(N\)-function to \(M\) (see [22]).

The set
\[
L^0_M := L^0_M(\nabla, \mu) := \{ x \in L_0(\nabla) : M(x) \in L_1(\nabla, \mu) \}
\]
is called the Orlicz \(L_0\)-class, and the vector space
\[
L_M := L_M(\nabla, \mu) := \{ x \in L_0(\nabla, \mu) : xy \in L_1(\nabla, \mu) \text{ for all } y \in L^0_M \}
\]
is called the \(L_M\) \(L_0\)-space, where \(N\) is the complementary \(N\)-function to \(M\).

One can see that \(L_M(\nabla, \mu) \subset L_1(\nabla, \mu)\). Define the \(L_0\)-valued Orlicz norm on \(L_M(\nabla, \mu)\) as follows
\[
\|x\|_M := \sup\left\{ | \int xyd\mu| : y \in A(N) \right\}, x \in L_M(\nabla, \mu),
\]
where \(A(N) = \{ y \in L^0_M : \int N(y)d\mu \leq 1 \}\). The pair \((L_M(\nabla, \mu), \| \cdot \|_M)\) is a Banach–Kantorovich lattice which is called the Orlicz–Kantorovich lattice associated with the \(L_0\)-valued measure [23].

As in the classical setting, the Orlicz spaces, along with the Orlicz norm \(\| \cdot \|_M\) on \(L_M(\nabla, \mu)\), one can consider the \(L_0\)-valued Luxemburg norm
\[
\|x\|_{(M)} := \inf\left\{ \lambda \in L_0 : \int \! M(\lambda^{-1}x)d\mu \leq 1, \lambda \text{ is an invertible positive element} \right\}.
\]
Moreover, the pair \((L_M(\nabla, \mu), \| \cdot \|_{(M)})\) is also a Banach—Kantorovich lattice [24].

Now we mention necessary facts from the theory of measurable bundles of Boolean algebras and Banach spaces (see [18] for more details).

Let \((\Omega, \Sigma, \lambda)\) be the same as above and \(X\) be a mapping assigning an \(L_p\)-space constructed by a real-valued measure \(\mu_{\omega}\), i.e. \(L_p(\nabla_{\omega}, \mu_{\omega})\) to each point \(\omega \in \Omega\) and let

\[
L = \left\{ \sum_{i=1}^{n} \alpha_i e_i : \alpha_i \in \mathbb{R}, \ e_i(\omega) \in \nabla_{\omega}, \ i = \overline{1,n}, \ n \in \mathbb{N} \right\}
\]

be a set of sections. In [12] it has been established that the pair \((X, L)\) is a measurable bundle of Banach lattices and \(L_0(\Omega, X)\) is modulo ordered isomorphic to \(L_p(\nabla, \mu)\).

3. The ”zero-two” law for positive contractions in the Orlicz-Kantorovich spaces

In this section we provide the uniform ”zero-two” law for the positive contractions of the Orlicz—Kantorovich lattices \(L_M(\nabla, \mu)\). Using methods of measurable bundles of Orlicz—Kantorovich lattices.

**Lemma 3.1.** Let \(\hat{f} \in L_1(\nabla, \mu)\) and the \(N\)-function \(M\) meets \(\Delta_2\) - condition. Then \(\hat{f} \in L_M(\nabla, \mu)\) if and only if \(f(\omega) \in L_M(\nabla_{\omega}, \mu_{\omega})\) for almost all \(\omega \in \Omega\), moreover \(\|\hat{f}\|_{L_M(\nabla, \mu)} = \|\hat{f}(\omega)\|_{L_M(\nabla_{\omega}, \mu_{\omega})}\) for almost all \(\omega \in \Omega\).

**Proposition 3.2.** If \(N\)-function \(M\) meets \(\Delta_2\)-condition then the Orlicz - Kantorovich lattice \((L_M(\nabla, \mu), \| \cdot \|_{(M)})\) order and isometrically isomorphic to \(L_0(\Omega, X)\), where \((X, \mathcal{E})\) is measurable bundle of Banach lattices on \(\Omega\) such that \(X(\omega) = (L_M(\nabla_{\omega}, \mu_{\omega}), \| \cdot \|_{L_M(\nabla_{\omega}, \mu_{\omega})})\) and \(\mathcal{E} = \{ \sum_{i=1}^{n} \alpha_i c_i : \alpha_i \in \mathbb{R}, c_i \in M(\Omega, \nabla) \} \).

**Definition 3.3.** A positive \(L_0\) - bounded, \(L_0\) - linear mapping \(T : L_M(\nabla, \mu) \to L_M(\nabla, \mu)\) is called an \(L_1 - L_\infty\) - contraction if \(\|T \hat{f}\|_{L_1(\nabla, \mu)} \leq \|\hat{f}\|_{L_1(\nabla, \mu)}\) and \(T 1 \leq 1\).

Using the methods of [13] one can establish the following facts.

**Lemma 3.4.** Let \(T : L_1(\nabla, \mu) \to L_1(\nabla, \mu)\) is \(L_1 - L_\infty\) - contraction, then for each \(\omega \in \Omega\) there exists \(L_1 - L_\infty\) - contraction \(T_\omega : L_M(\nabla_{\omega}, \mu_{\omega}) \to L_M(\nabla_{\omega}, \mu_{\omega})\) such that \(T_\omega f(\omega) = (T \hat{f})(\omega)\) for almost all \(\omega \in \Omega\).

**Lemma 3.5.** Let \(M\) has \(\Delta_2\) - condition and \(T^{(i)} : L_M(\nabla, \mu) \to L_M(\nabla, \mu)\) be positive \(L_1 - L_\infty\) - contraction. \(i = 1, 2\). Then

\[
\|T^{(1)} - T^{(2)}\|_{(M)}(\omega) = \|T^{(1)}_\omega - T^{(2)}_\omega\|_{(M)}(\omega)
\]

for almost all \(\omega \in \Omega\).

**Lemma 3.6.** Let \(M\) has \(\Delta_2\) - condition and \(T^{(i)} : L_M(\nabla, \mu) \to L_M(\nabla, \mu), i = 1, 2\) be positive \(L_1 - L_\infty\) - contraction. Then

\[
\|T^{(1)}_\omega - T^{(2)}_\omega\|_{(M)}(\omega) \leq \|T^{(1)} - T^{(2)}\|_{(M)}(\omega)
\]

where \(| \cdot |\) is module of an operator.

By means of the argument of [17, Theorem 4.3] we can prove the following result.

**Theorem 3.7.** Let the Orlicz function \(M\) with \(\Delta_2\) - condition. If \(T : L_M(\nabla, \mu) \to L_M(\nabla, \mu)\) is positive \(L_1 - L_\infty\) - contraction and for some \(m \in \mathbb{N} \cup \{0\}\), \(\|T^{m+1} - T^m\| < 2 \cdot 1\), then

\[
(o) - \lim_{n \to \infty} \|T^{n+1} - T^n\| = 0
\]
[1] Akcoglu M A and Sucheston L 1985 *Rend. Circ. Mat. Palermo* **8** 325–335
[2] Bru B and Heinich H 1985 *C. R. Acad. Sci. Paris.* **301** 893–894
[3] Derriennic Y 1976 *Ann. Inst. H. Poincaré Sec. B* **12** (1976) 111–129
[4] Hudzik H 1983 *Comment. Math.* **23** 21–32
[5] Kurc W 1992 *J. Approx. Theory* 173–187.
[6] Chilin V I and Ganiev I G 2000 Russian Math. (Iz. VUZ) **44** 77–79
[7] Foguel S R 1971 *Israel J. Math.* **10** 275–280.
[8] Foguel S R 1976 *Proc. Amer. Math. Soc.* **61** (1976) 262–264
[9] Foguel S R 1983 *Israel J. Math.* **45** (1983) 219–224
[10] Ganiev I G. 1998 *Uzbek Math. Zh.* (1998), N.5, 14–21 (Russian).
[11] Ganiev I G. 2000 *Uzb. Math. Jour.* 2000, N.1, 18–26.
[12] Ganiev I G. In book : *Studies on Functional Analysis and its Applications*, pp. 9–49. Nauka, Moscow (2006) (Russian).
[13] Ganiev I G and Mukhamedov F 2006 *Comment. Math. Univ. Carolinae* **47** 427–436
[14] Ganiev I G and Mukhamedov F 2013 *Lobachevskii Jour. Math.* **34** 1–10
[15] Ganiev I and Mukhamedov F 2014 *Positivity* **18** 687–702
[16] Ganiev I and Mukhamedov F 2015 *Bull. Malays. Math. Sci. Soc.* **38**(2015) 387–397.
[17] Ganiev I G, Mukhamedov F and Bekbaev D 2015 *Turkish J Math* **39** 583–594
[18] Gutman A E. 1993 *Siber. Adv. Math.* **3** 8–40
[19] Gutman A E. 1995 *Trudy Inst. Mat.* **29** 63–211 (Russian)
[20] Jamison B and Orey S 1967 *Z. Wahrsch. Verw. Geb.* **8** (1967) 4148
[21] Katznelson Y and Tzafriri L 1986 *J. Funct. Anal.* **68** 313–328.
[22] Krasnosel’skii M A and Rutickii J A 1961 *Convex functions and Orlicz spaces*, Groningen
[23] Zakirov B S and Chilin V I 2009 *Sib. Math. J.* **50** 1027–1037.
[24] Zakirov B S 2007 *Uzbek. Mat. Zh.* No 2, 3244
[25] Kusraev A G 1985 *Vector duality and its applications* (Novosibirsk, Nauka) 1985 (Russian).
[26] Kusraev A G 2000 *Dominated operators*, (DKluwer Academic Publishers, Dordrecht)
[27] Mukhamedov F. On dominant contractions and a generalization of the zero-two law. *Positivity.* **15** (2011) 497–508.
[28] Ornstein D and Sucheston L 1970 *Ann. Math. Statis.* **41** 1631–1639.
[29] Vulikh B Z. 1967 *Introduction to theory of partially ordered spaces*.
[30] Wittmann R 1987 *Israel J. Math.* **59**(1987) 828
[31] Wittmann R 1988, *Math. Z.* **197**(1988) 223–229
[32] Zaharopol R. 1986 *J. Funct. Anal.* **68** 300–312.
[33] Zaharopol R 2000 *Turkish J. Math.* **24** (2000) 109120.