EULER SEQUENCE AND KOSZUL COMPLEX OF A MODULE

BJÖRN ANDREAS, DARÍO SÁNCHEZ GÓMEZ, AND FERNANDO SANCHO DE SALAS

Abstract. We construct relative and global Euler sequences of a module. We apply it to prove some acyclicity results of the Koszul complex of a module and to compute the cohomology of the sheaves of (relative and absolute) differential p-forms of a projective bundle. In particular we generalize Bott’s formula for the projective space to a projective bundle over a scheme of characteristic zero.

Introduction

This paper deals with two related questions: the acyclicity of the Koszul complex of a module and the cohomology of the sheaves of (relative and absolute) differential p-forms of a projective bundle.

Let $M$ be a module over a commutative ring $A$. One has the Koszul complex $\text{Kos}(M) = \Lambda^\cdot M \otimes_A S^\cdot M$, where $\Lambda^\cdot M$ and $S^\cdot M$ stand for the exterior and symmetric algebras of $M$. It is a graded complex $\text{Kos}(M) = \bigoplus_{n \geq 0} \text{Kos}(M)_n$, whose $n$-th graded component $\text{Kos}(M)_n$ is the complex:

$0 \to \Lambda^n M \to \Lambda^{n-1} M \otimes M \to \Lambda^{n-2} M \otimes S^2 M \to \cdots \to M \otimes S^{n-1} M \to S^n M \to 0$

It has been known for many years that $\text{Kos}(M)_n$ is acyclic for $n > 0$, provided that $M$ is a flat $A$-module or $n$ is invertible in $A$ (see [3] or [10]). It was conjectured in [11] that $\text{Kos}(M)$ is always acyclic. A counterexample in characteristic 2 was given in [5], but it is also proved there that $H_\mu(\text{Kos}(M)_\mu) = 0$ for any $M$, where $\mu$ is the minimal number of generators of $M$. Leaving aside the case of characteristic 2 (whose pathology is clear for the exterior algebra), we prove two new evidences for the validity of the conjecture (for $A$ noetherian): firstly, we prove (Theorem 1.6) that, for any finitely generated $M$, $\text{Kos}(M)_n$ is acyclic for $n >> 0$; secondly, we prove (Theorem 1.7) that if $I$ is an ideal locally generated by a regular sequence, then $\text{Kos}(I)_n$ is acyclic for any $n > 0$. These two results are a consequence of relating the Koszul complex $\text{Kos}(M)$ with the geometry of the space $\mathbb{P} = \text{Proj} S^\cdot M$, as follows:

First of all, we shall reformulate the Koszul complex in terms of differential forms of $S^\cdot M$ over $A$: the canonical isomorphism $\Omega_{S^\cdot M/A} = M \otimes_A S^\cdot M$ allows us to interpret the Koszul complex $\text{Kos}(M)$ as the complex of differential forms $\Omega_{S^\cdot M/A}$ whose differential, $i_D: \Omega_{S^\cdot M/A}^p \to \Omega_{S^\cdot M/A}^{p-1}$, is the inner product with the $A$-derivation $D: S^\cdot M \to S^\cdot M$ consisting in multiplication by $n$ on $S^n M$. By homogeneous localization, one obtains a complex of $\mathcal{O}_\mathbb{P}$-modules $\tilde{\text{Kos}}(M)$ on $\mathbb{P}$.

Date: September 24, 2015.

This work was supported by the SFB 647 ‘Space-Time-Matter: Arithmetic and Geometric Structures’ of the DFG (German Research Foundation) and by the Spanish grants MTM2013-45935-P (MINECO) and FS/12-2014 (Samuel Solórzano Barruso Foundation).
Our first result (Theorem 1.4) is that the complex $\tilde{\text{Kos}}(M)$ is acyclic with factors (cycles or boundaries) the sheaves $\Omega^p_{\mathcal{P}/A}$. Moreover, one has a natural morphism

$$\text{Kos}(M)_n \to \pi_*[\tilde{\text{Kos}}(M) \otimes \mathcal{O}_\mathcal{P}(n)]$$

with $\pi: \mathcal{P} \to \text{Spec} A$ the canonical morphism. In Theorem 1.5 we give (cohomological) sufficient conditions for the acyclicity of the complexes $\text{Kos}(M)_n$ and $\pi_*[\tilde{\text{Kos}}(M) \otimes \mathcal{O}_\mathcal{P}(n)]$. These conditions, under noetherian hypothesis, are satisfied for $n \gg 0$, thus obtaining Theorem 1.6. The acyclicity of the Koszul complex of a locally regular ideal follows then from Theorem 1.5 and the theorem of formal functions.

The advantage of expressing the Koszul complex $\text{Kos}(M)$ as $(\Omega^*_S \otimes \mathcal{A}, i_D)$ is twofold. Firstly, it makes clear its relationship with the De Rham complex $(\Omega^*_S \otimes \mathcal{A}, A)$: The Koszul and De Rham differentials are related by Cartan’s formula: $i_D \circ d + d \circ i_D = \text{multiplication by } n$ on $\text{Kos}(M)_n$. This yields a splitting result (Proposition 1.10 or Corollary 1.11) which will be essential for some cohomological results in section 3 as we shall explain later on. Secondly, it allows a natural generalization (which is the subject of section 2): If $\mathcal{A}$ is a $k$-algebra, we define the complex $\text{Kos}(M/k)$ as the complex of differential forms (over $k$), $\Omega^*_S \otimes M/k$ whose differential is the inner product with the same $D$, i.e., $\partial_i \equiv D \cdot i$. Moreover, one has a natural morphism $\text{Kos}(M/k) = \bigoplus_{n \geq 0} \text{Kos}(M/k)_n$ and it induces, by homogeneous localization, a complex $\tilde{\text{Kos}}(M/k)$ of modules on $\mathcal{P}$ which is also acyclic and whose factors are the sheaves $\Omega^p_{\mathcal{P}/k}$ (Theorem 2.1). We can reproduce the aforementioned results about the complexes $\text{Kos}(M)_n$, $\tilde{\text{Kos}}(M)$, for the complexes $\text{Kos}(M/k)_n$, $\tilde{\text{Kos}}(M/k)$.

Section 3 deals with the second subject of the paper: let $\mathcal{E}$ be a locally free module of rank $r + 1$ on a $k$-scheme $X$ and let $\pi: \mathcal{P} \to X$ be the associated projective bundle, i.e., $\mathcal{P} = \text{Proj} S \mathcal{E}$. There are well known results about the (global and relative) cohomology of the sheaves $\Omega^p_{\mathcal{P}/X}(n)$ and $\Omega^p_{\mathcal{P}/k}(n)$ (we are using the standard abbreviated notation $\mathcal{N}(n) = \mathcal{N} \otimes \mathcal{O}_\mathcal{P}(n)$) due to Deligne, Verdier and Berthelot-Illusie (11, 12, 13) and about the cohomology of the sheaves $\Omega^p_{\mathcal{P}/k}(n)$ of the ordinary projective space due to Bott (the so called Bott’s formula, 12). We shall not use their results; instead, we refine the results and we obtain some new results, overall when $X$ is a $k$-scheme. Let us be more precise:

In Theorem 3.3 we compute the relative cohomology sheaves $R^* \pi_* \Omega^p_{\mathcal{P}/X}(n)$, obtaining Deligne’s result (see 11 and also 12) and a new (splitting) result, in the case of a $k$-scheme, concerning the sheaves $\pi_* \Omega^p_{\mathcal{P}/X}(n)$ and $R^* \pi_* \Omega^p_{\mathcal{P}/X}(-n)$ for $n > 0$. We obtain Bott formula for the projective space as a consequence. In Theorem 3.11 we compute the relative cohomology sheaves $R^* \pi_* \Omega^p_{\mathcal{P}/k}(n)$, obtaining Verdier’s results (see 12) and improving them in two ways: first, we give a more explicit description of $\pi_* \Omega^p_{\mathcal{P}/k}(n)$ and of $R^* \pi_* \Omega^p_{\mathcal{P}/k}(-n)$ for $n > 0$; secondly, we obtain a splitting result for these sheaves when $X$ is a $k$-scheme (as in the relative case).

Regarding Bott’s formula, we are able to generalize it for a projective bundle, computing the dimension of the cohomology vector spaces $H^q(\mathcal{P}, \Omega^p_{\mathcal{P}/X}(n))$ and $H^q(\mathcal{P}, \Omega^p_{\mathcal{P}/k}(n))$ when $X$ is a proper $k$-scheme of characteristic zero (Corollaries 3.7 and 3.14).

It should be mentioned that these results make use of the complexes $\tilde{\text{Kos}}(\mathcal{E})$ (as Deligne and Verdier) and $\tilde{\text{Kos}}(\mathcal{E}/k)$.

The complex $\tilde{\text{Kos}}(\mathcal{E})$ is essentially equivalent
to the exact sequence

\[ 0 \to \Omega_{\mathcal{E}/X} \to (\pi^* \mathcal{E}) \otimes \mathcal{O}_X(-1) \to \mathcal{O}_X \to 0 \]

which is usually called Euler sequence. The complex \( \widetilde{\text{Kos}}(\mathcal{E}/k) \) is equivalent to the exact sequence

\[ 0 \to \Omega_{\mathcal{E}/k} \to \tilde{\Omega}_{\mathcal{E}/k} \to \mathcal{O}_X \to 0 \]

with \( B = S' \mathcal{E} \), which we have called global Euler sequence. These sequences still hold for any \( A \)-module \( M \) (which we have called relative and global Euler sequences of \( M \)). The aforementioned results about the acyclicity of the Koszul complexes of a module obtained in sections 1 and 2 are a consequence of this fact.

1. Relative Euler sequence of a module and Koszul complexes

Let \((X, \mathcal{O})\) be a scheme and let \( \mathcal{M} \) be quasi-coherent \( \mathcal{O} \)-module. Let \( B = S' \mathcal{M} \) be the symmetric algebra of \( \mathcal{M} \) (over \( \mathcal{O} \)), which is a graded \( \mathcal{O} \)-algebra: the homogeneous component of degree \( n \) is \( B_n = S^n \mathcal{M} \). The module \( \Omega_{\mathcal{B}/\mathcal{O}} \) of Kähler differentials is a graded \( B \)-module in a natural way: \( B \otimes \mathcal{O} B \) is a graded \( \mathcal{O} \)-algebra, with \( (B \otimes \mathcal{O} B)_n = \bigoplus_{p+q=n} B_p \otimes \mathcal{O} B_q \) and the natural morphism \( B \otimes \mathcal{O} B \to B \) is a degree 0 homogeneous morphism of graded algebras. Hence, the kernel \( \Delta \) is a homogeneous ideal and \( \Delta/\Delta^2 = \Omega_{\mathcal{B}/\mathcal{O}} \) is a graded \( B \)-module. If \( b_p, b_q \in B \) are homogeneous of degree \( p, q \), then \( b_p \partial b_q \) is an element of \( \Omega_{\mathcal{B}/\mathcal{O}} \) of degree \( p + q \). We shall denote by \( \Omega_{\mathcal{B}/\mathcal{O}}^p = \Lambda^p \mathcal{M} \otimes \mathcal{O} B \) the \( p \)-th exterior power of \( \Omega_{\mathcal{B}/\mathcal{O}} \), which is also a graded \( B \)-module in a natural way. For each \( \mathcal{O} \)-module \( N \), \( N \otimes \mathcal{O} B \) is a graded \( B \)-module with gradation: \( (N \otimes \mathcal{O} B)_n = N \otimes \mathcal{O} B_n \). Then one has the following basic result:

**Theorem 1.1.** The natural morphism of graded \( B \)-modules

\[
\mathcal{M} \otimes \mathcal{O} B[-1] \to \Omega_{\mathcal{B}/\mathcal{O}}^p
\]

\[
m \otimes b \mapsto b \partial m
\]

is an isomorphism. Hence \( \Omega_{\mathcal{B}/\mathcal{O}}^p \simeq \Lambda^p \mathcal{M} \otimes \mathcal{O} B[-p] \), where \( \Lambda^i \mathcal{M} = \Lambda^i_\mathcal{O} \mathcal{M} \).

The natural morphism \( \mathcal{M} \otimes \mathcal{O} S' \mathcal{M} \to S^{i+1} \mathcal{M} \) defines a degree zero homogeneous morphism of \( \mathcal{B} \)-modules \( \Omega_{\mathcal{B}/\mathcal{O}}^p = \mathcal{M} \otimes \mathcal{O} B[-1] \to \mathcal{B} \) which induces an \( \mathcal{O} \)-derivation (of degree 0) \( D: \mathcal{B} \to \mathcal{B} \), such that \( \Omega_{\mathcal{B}/\mathcal{O}} \to \mathcal{B} \) is the inner product with \( D \). This derivation consists in multiplication by \( n \) in degree \( n \). It induces homogeneous morphisms of degree zero:

\[
i_D: \Omega_{\mathcal{B}/\mathcal{O}}^p \to \Omega_{\mathcal{B}/\mathcal{O}}^{p-1}
\]

and we obtain:

**Definition 1.2.** The Koszul complex, denoted by \( \text{Kos}(\mathcal{M}) \), is the complex:

\[
\cdots \to \Omega_{\mathcal{B}/\mathcal{O}}^p \xrightarrow{i_D} \Omega_{\mathcal{B}/\mathcal{O}}^{p-1} \xrightarrow{i_D} \cdots \to \Omega_{\mathcal{B}/\mathcal{O}} \xrightarrow{i_D} \mathcal{B} \xrightarrow{i_D} 0
\]

Via Theorem 1.1, this complex is

\[
\cdots \xrightarrow{i_D} \Lambda^p \mathcal{M} \otimes \mathcal{O} B[-p] \xrightarrow{i_D} \cdots \xrightarrow{i_D} \mathcal{M} \otimes \mathcal{O} B[-1] \xrightarrow{i_D} \mathcal{B} \to 0
\]
Taking the homogeneous components of degree \( n \geq 0 \), we obtain a complex of \( O \)-modules, which we denote by \( \text{Kos}(M)_n \):

\[
0 \rightarrow \Lambda^n_M \rightarrow \Lambda^{n-1}_M \otimes_O M \rightarrow \cdots \rightarrow M \otimes_O S^{n-1}_M \rightarrow S^n_M \rightarrow 0
\]

such that \( \text{Kos}(M) = \bigoplus_{n \geq 0} \text{Kos}(M)_n \).

Now let \( P = \text{Proj} B \) and \( \pi : P \rightarrow X \) the natural morphism. We shall use the following standard notations: for each \( O_P \)-module \( N \), we shall denote \( N(n) = N \otimes_O O_P(n) \) and for each graded \( B \)-module \( N \) we shall denote by \( N \) the sheaf of \( O_P \)-modules obtained by homogeneous localization. We shall use without mention the following facts: homogeneous localization commutes with exterior powers and for any quasi-coherent module \( L \) on \( X \) one has \((L \otimes_O B[n]) = (\pi^* L)(n)\).

**Definition 1.3.** Taking homogeneous localization on the Koszul complex (1.1), we obtain a complex of \( O_P \)-modules, which we denote by \( \text{Kos}(M) \):

\[
\cdots \rightarrow \tilde{\Omega}_B^p \rightarrow \tilde{\Omega}_B^{p-1} \rightarrow \cdots \rightarrow \tilde{\Omega}_B^1 \rightarrow \tilde{\Omega}_B^0 \rightarrow O_P \rightarrow 0
\]

By Theorem [1.1] \( \tilde{\Omega}_B^d = (\pi^* \Lambda^d M)(-d) \), hence \( \text{Kos}(M) \) can be written as

\[
\cdots \rightarrow \tilde{\Omega}_B^d \rightarrow \tilde{\Omega}_B^{d-1} \rightarrow \cdots \rightarrow \tilde{\Omega}_B^1 \rightarrow \tilde{\Omega}_B^0 \rightarrow O_P \rightarrow 0
\]

**Theorem 1.4.** The complex \( \text{Kos}(M) \) is acyclic (that is, an exact sequence). Moreover,

\[
\Omega^p_{P/X} = \text{Ker} \left( \tilde{\Omega}_B^p \xrightarrow{id} \tilde{\Omega}_B^{p-1} \right).
\]

Hence one has exact sequences

\[
0 \rightarrow \Omega^p_{P/X} \rightarrow \tilde{\Omega}_B^p \rightarrow \tilde{\Omega}_B^{p-1} \rightarrow 0
\]

and right and left resolutions of \( \Omega^p_{P/X} \):

\[
0 \rightarrow \Omega^p_{P/X} \rightarrow \tilde{\Omega}_B^p \rightarrow \tilde{\Omega}_B^{p-1} \rightarrow \cdots \rightarrow \tilde{\Omega}_B^1 \rightarrow \tilde{\Omega}_B^0 \rightarrow O_P \rightarrow 0
\]

\[
\cdots \rightarrow \tilde{\Omega}_B^{r+1} \rightarrow \tilde{\Omega}_B^r \rightarrow \cdots \rightarrow \tilde{\Omega}_B^1 \rightarrow \tilde{\Omega}_B^0 \rightarrow \Omega^p_{P/X} \rightarrow 0
\]

In particular, for \( p = 1 \) the exact sequence

\[
0 \rightarrow \Omega^1_{P/X} \rightarrow \tilde{\Omega}_B^1 \rightarrow O_P \rightarrow 0
\]

is called the (relative) Euler sequence.

**Proof.** The morphism \( \tilde{\Omega}_B^d \rightarrow O_P \) is surjective, since \( M \otimes_O B[-1] \rightarrow B \) is surjective in positive degree. Let \( K \) be the kernel. We obtain an exact sequence

\[
0 \rightarrow K \rightarrow \Omega^1_{P/X} \rightarrow O_P \rightarrow 0
\]

Since \( O_P \) is free, this sequence splits locally; then, it induces exact sequences

\[
0 \rightarrow \Lambda^p K \rightarrow \tilde{\Omega}_B^p \rightarrow \Lambda^{p-1} K \rightarrow 0
\]

Joining these exact sequences one obtains the Koszul complex \( \text{Kos}(M) \). This proves the acyclicity of \( \text{Kos}(M) \). To conclude, it suffices to prove that \( K = \Omega^1_{P/X} \).

Let us first define a morphism \( \Omega_{P/X} \rightarrow \tilde{\Omega}_B^p \). Assume for simplicity that \( X = \text{Spec} A \). For each \( b \in B \) of degree 1, let \( U_b \) the standard affine open subset of \( P \),
defined by $U_b = \text{Spec}(B(b))$, with $B(b)$ the 0-degree component of $B_b$. The natural inclusion $B(b) \to B_b$ induces a morphism $\Omega_{B(b)/A} \to \Omega_{B_b/A} = (\Omega_{B/A})_b$ which takes values in the 0-degree component, $(\Omega_{B/A})(b)$. Thus one has a morphism $\Omega_{B(b)/A} \to (\Omega_{B/A})(b)$, i.e., a morphism $\Gamma(U_b, \Omega_{P/X}) \to \Gamma(U_b, \Omega_{B/A})$. One checks that these morphisms glue to a morphism $f: \Omega_{P/X} \to \Omega_{B/A}$. This morphism is injective, because the inclusion $B(b) \to B_b$ has a retract, $c_n/b^k \mapsto c_n/b^n$, which induces a retract in the differentials. The composition $\Omega_{P/X} \to \Omega_{B/A} \to \mathcal{O}_P$ is null, as one checks in each $U_b$:

$$(i_D \circ f)(d(c_k/b^k)) = i_D \left( \frac{b^k d c_k - c_k d b^k}{b^{2k}} \right) = \frac{b^k i_D d c_k - c_k i_D d b^k}{b^{2k}} = 0$$

because $i_D d c_r = r c_r$ for any element $c_r$ of degree $r$. Thus, we have that $\Omega_{P/X}$ is contained in the kernel of $\Omega_{B/A} \to \mathcal{O}_P$. To conclude, it is enough to see that the image of $\Omega_{B/A}^2 \to \Omega_{B/A}$ is contained in $\Omega_{P/X}$. Again, this is a computation in each $U_b$; one checks the equality

$$i_D \left( \frac{d c_p \wedge d c_q}{b^{p+q}} \right) = \frac{p}{b^p} d \left( \frac{c_p}{b^q} \right) - \frac{q}{b^q} d \left( \frac{c_q}{b^p} \right)$$

and the right member belongs to $\Omega_{(B(b)/A)}^1$.

For each $n \in \mathbb{Z}$, we shall denote by $\widetilde{\text{Kos}}(\mathcal{M})(n)$ the complex $\text{Kos}(\mathcal{M})$ twisted by $\mathcal{O}_P(n)$ (notice that the differential of the Koszul complex is $\mathcal{O}_P$-linear). The differential of the complex $\text{Kos}(\mathcal{M})(n)$ is still denoted by $i_D$.

1.1. Acyclicity of the Koszul complex of a module.

Let us denote $\widetilde{\text{Kos}}(\mathcal{M})_n := \pi_*(\widetilde{\text{Kos}}(\mathcal{M})(n))$. The natural morphisms $[\Omega_{B/O}^n] \to \pi_*[\Omega_{B/O}^n]$ give a morphism of complexes

$$\text{Kos}(\mathcal{M})_n \to \widetilde{\text{Kos}}(\mathcal{M})_n$$

and one has:

**Theorem 1.5.** Let $\mathcal{M}$ be a finitely generated quasi-coherent module on a scheme $(X, \mathcal{O})$, $P = \text{Proj} \, \mathcal{S} \, \mathcal{M}$ and $\pi: \mathcal{P} \to X$ the natural morphism. Let $d$ be the minimal number of generators of $\mathcal{M}$ (i.e., it is the greatest integer such that $\Lambda^d \mathcal{M} \neq 0$) and $n > 0$. Then:

1. If $R^j \pi_*[\Omega_{B/O}^i(n)] = 0$ for any $j > 0$ and any $0 \leq i \leq d$, then $\widetilde{\text{Kos}}(\mathcal{M})_n$ is acyclic.

2. If (1) holds and the natural morphism $[\Omega_{B/O}^i] \to \pi_*[\Omega_{B/O}^i]$ is an isomorphism for any $0 \leq i \leq d$, then $\text{Kos}(\mathcal{M})_n$ is also acyclic.

**Proof.** (1) By Theorem 1.4, the complex $\text{Kos}(\mathcal{M})(n)$ is acyclic. Since the (non-zero) terms of this complex are $\Omega_{B/O}^i(n)$, the hypothesis tells us that $\pi_*(\text{Kos}(\mathcal{M}) \otimes \mathcal{O}_P(n))$ is acyclic, that is, $\text{Kos}(\mathcal{M})_n$ is acyclic.

(2) By hypothesis, $\text{Kos}(\mathcal{M})_n \to \text{Kos}(\mathcal{M})_n$ is an isomorphism and then $\text{Kos}(\mathcal{M})_n$ is also acyclic. \qed

**Theorem 1.6.** Let $X$ be a noetherian scheme and $\mathcal{M}$ a coherent module on $X$. The Koszul complexes $\text{Kos}(\mathcal{M})_n$ and $\text{Kos}(\mathcal{M})_n$ are acyclic for $n >> 0$. 

Proof. Indeed, the hypothesis (1) and (2) of Theorem 1.5 hold for \( n >> 0 \) (see Theorem 2.2.1 and [7, Section 3.3 and Section 3.4]).

**Theorem 1.7.** Let \( I \) be an ideal of a noetherian ring \( A \). If \( I \) is locally generated by a regular sequence, then \( \text{Kos}(I)_n \) and \( \overline{\text{Kos}}(I)_n \) are acyclic for any \( n > 0 \).

**Proof.** In this case \( \pi: \mathbb{P} \to X = \text{Spec} A \) is the blow-up with respect to \( I \), because \( S^n I = I^n \), since \( I \) is locally a regular ideal ([9]). Let \( d \) be the minimum number of generators of \( I \). By Theorem 1.5, it suffices to see that for any \( A \)-module \( M \) and any \( 0 \leq i \leq d \) one has:

\[
H^j(\mathbb{P}, (\pi^* M)(n - i)) = \begin{cases} 
0 & \text{if } j > 0 \\
M \otimes_A \mathcal{I}^{n-i} & \text{if } j = 0 
\end{cases}
\]

This is a consequence of the Theorem of formal functions (see [8, Corollary 4.1.7]). Indeed, let us denote \( Y_r = \text{Spec} A/I^r \), \( E_r = \pi^{-1}(Y_r) \) and \( \pi_r: E_r \to Y_r \). One has that \( E_r = \text{Proj} S_A/I^r (I/I^{r+1}) \) is a projective bundle over \( Y_r \), because \( I/I^{r+1} \) is a locally free \( A/I^r \)-module of rank \( d \), since \( I \) is locally regular. Hence, for any module \( N \) on \( Y \), and any \( m > -d \) one has

\[
H^j(E_r, (\pi_r^* N)(m)) = \begin{cases} 
0 & \text{if } j > 0 \\
N \otimes_{A/I^r} I^m/I^{m+r} & \text{if } j = 0 
\end{cases}
\]

Now, by the theorem of formal functions (let \( m = n - i \))

\[
H^j(\mathbb{P}, (\pi^* M)(m))^\wedge = \lim_r H^j(E_r, \pi_r^*(M/I^rM)(m)) = 0, \text{ for } j > 0.
\]

For \( j = 0 \), the natural morphism \( M \otimes_A I^m \to H^0(\mathbb{P}, (\pi^* M)(m)) \) is an isomorphism because it is an isomorphism after completion by \( I \):

\[
H^0(\mathbb{P}, (\pi^* M)(m))^\wedge = \lim_r H^0(E_r, \pi_r^*(M/I^rM)(m))
\]

\[
= \lim_r (M/I^rM) \otimes_{A/I^r} I^m/I^{m+r}
\]

\[
= \lim_r (M \otimes_A S^m I) \otimes_A A/I^r = (M \otimes_A I^m)^\wedge.
\]

\( \square \)

**Remark 1.8.** Let \( d \) be the minimum number of generators of \( \mathcal{M} \). Since \( \overline{\text{Kos}}(\mathcal{M}) \) is acyclic and \( \pi_* \) is left exact, one has that \( \text{H}_d(\text{Kos}(\mathcal{M}), n) = 0 \) for any \( n \). One the other hand, it is proved in [5] that \( \text{H}_d(\text{Kos}(\mathcal{M}), n) = 0 \) for any \( n \). One cannot expect \( \text{Kos}(\mathcal{M}) \to \overline{\text{Kos}}(\mathcal{M}) \) to be an isomorphism in general. For instance, consider \( X = \text{Spec} A \) with \( A = k[u, v, s_1, s_2, t_1, t_2]/I \) where \( k \) is a field and \( I = (-us_1 + vt_1 + ut_2, vs_1 + us_2 - vt_2, v^2_2, ut_1) \). Let \( M = (Ax \oplus Ay)/A(\bar{u} x + \bar{v} y) \), where \( \bar{u} \) (resp. \( \bar{v} \)) is the class of \( u \) (resp. \( v \)) in \( A \). Then one can prove that the map \( M \to \pi_* O_\mathbb{P}(1) \) is not injective (for details we refer to section 26.21 of The Stacks project). So that the question which arises here is whether \( \text{Kos}(\mathcal{M}) \) is a quasi-isomorphism. We do not know the answer, besides the acyclicity theorems for both complexes mentioned above.
1.2. **Koszul versus De Rham.** The exterior differential defines morphisms
\[ d: \Omega^p_{B/O} \to \Omega^{p+1}_{B/O} \]
which are \( \mathcal{O} \)-linear, but not \( \mathcal{B} \)-linear. One has then the De Rham complex:
\[
\text{DeRham(} \mathcal{M} \text{)} \equiv 0 \to \mathcal{B} \xrightarrow{d} \Omega^0_{B/O} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^p_{B/O} \xrightarrow{d} \Omega^{p+1}_{B/O} \to \cdots
\]
which can be reformulated as
\[
0 \to \mathcal{B} \xrightarrow{d} \mathcal{M} \otimes_\mathcal{O} \mathcal{B}[-1] \to \cdots \xrightarrow{d} \Lambda^p \mathcal{M} \otimes_\mathcal{O} \mathcal{B}[-p] \to \Lambda^{p+1} \mathcal{M} \otimes_\mathcal{O} \mathcal{B}[-p-1] \to \cdots
\]
Taking into account that \( d \) is homogeneous of degree 0, one has for each \( n \geq 0 \) a complex of \( \mathcal{O} \)-modules
\[
\text{DeRham(} \mathcal{M} \text{)}_n \equiv 0 \to S^n \mathcal{M} \to \mathcal{M} \otimes_\mathcal{O} S^{n-1} \to \cdots \to \Lambda^{n-1} \mathcal{M} \otimes_\mathcal{O} \mathcal{M} \to \Lambda^n \mathcal{M} \to 0
\]

The differentials of the Koszul and De Rham complexes are related by Cartan’s formula: \( i_D \circ d + d \circ i_D = \text{multiplication by } n \) on \( \Lambda^p \mathcal{M} \otimes_\mathcal{O} S^{n-p} \mathcal{M} \). This immediately implies the following result:

**Proposition 1.9.** *If \( X \) is a scheme over \( \mathbb{Q} \), then Kos(\( \mathcal{M} \))_n and DeRham(\( \mathcal{M} \))_n are homotopically trivial for any \( n > 0 \). In particular, they are acyclic.*

Now we pass to homogeneous localizations. The differential \( d: \Omega^p_{B/O} \to \Omega^{p+1}_{B/O} \) is compatible with homogeneous localization, since for any \( \omega_{k+n} \in \Omega^p_{B/O} \) of degree \( k + n \) and any \( b \in \mathcal{B} \) of degree 1, one has:
\[
d\left( \frac{\omega_{k+n}}{b^n} \right) = \frac{b^n d \omega_{k+n} - (d b^n) \wedge \omega_{k+n}}{b^{2n}}.
\]
Thus, for any \( n \in \mathbb{Z} \), one has (\( \mathcal{O} \)-linear) morphisms of sheaves
\[
d: \tilde{\Omega}^p_{B/O}(n) \to \tilde{\Omega}^{p+1}_{B/O}(n)
\]
and we obtain, for each \( n \), a complex of sheaves on \( \mathbb{P} \):
\[
\tilde{\text{DeRham}}(\mathcal{M}, n) = 0 \to \mathcal{O}_p(n) \xrightarrow{d} \tilde{\Omega}^0_{B/O}(n) \xrightarrow{d} \cdots \xrightarrow{d} \tilde{\Omega}^p_{B/O}(n) \to \cdots
\]
which can be reformulated as
\[
0 \to \mathcal{O}_p(n) \xrightarrow{d} (\pi^* \mathcal{M})(n-1) \to \cdots \to (\pi^* \Lambda^p \mathcal{M})(n-p) \to \cdots
\]
It should be noticed that \( \tilde{\text{DeRham}}(\mathcal{M}, n) \) is not the complex obtained for \( n = 0 \) twisted by \( \mathcal{O}_p(n) \), because the differential is not \( \mathcal{O}_p \)-linear.

Again, one has that \( i_D \circ d + d \circ i_D = \text{multiplication by } n \), on \( \tilde{\Omega}^p_{B/O}(n) \). Hence, one has:

**Proposition 1.10.** *If \( X \) is a scheme over \( \mathbb{Q} \), then the complexes \( \tilde{\text{Kos}}(\mathcal{M})(n) \) and \( \tilde{\text{DeRham}}(\mathcal{M}, n) \) are homotopically trivial for any \( n \neq 0 \).*

**Corollary 1.11.** *Let \( X \) be a scheme over \( \mathbb{Q} \). For any \( n \neq 0 \), the exact sequences
\[
0 \to \Omega^p_{\mathbb{P}/X}(n) \to \tilde{\Omega}^p_{B/O}(n) \to \Omega^{p-1}_{\mathbb{P}/X}(n) \to 0
\]
split as sheaves of \( \mathcal{O} \)-modules (but not as \( \mathcal{O}_p \)-modules).*
2. Global Euler sequence of a module and Koszul complexes

Assume that \((X, \mathcal{O})\) is a \(k\)-scheme, where \(k\) is a ring (just for simplicity, one could assume that \(k\) is another scheme). Let \(\mathcal{M}\) be an \(\mathcal{O}\)-module and \(\mathcal{B} = S' \mathcal{M}\) the symmetric algebra over \(\mathcal{O}\). Instead of considering the module of Kähler differentials of \(\mathcal{B}\) over \(\mathcal{O}\), we shall now consider the module of Kähler differentials over \(k\), that is, \(\Omega_{\mathcal{B}/k}\). As it happened with \(\Omega_{\mathcal{B}/\mathcal{O}}\) (section 1), the module \(\Omega_{\mathcal{B}/k}\) is a graded \(\mathcal{B}\)-module in a natural way. The \(\mathcal{O}\)-derivation \(D: \mathcal{B} \to \mathcal{B}\) is in particular a \(k\)-derivation, hence it defines a morphism \(i_D: \Omega_{\mathcal{B}/k} \to \mathcal{B}\), which is nothing but the composition of the natural morphism \(\Omega_{\mathcal{B}/k} \to \Omega_{\mathcal{B}/\mathcal{O}}\) with the inner product \(\pi: \Omega_{\mathcal{B}/\mathcal{O}} \to \mathcal{B}\) defined in section 1. Again we obtain a complex of \(\mathcal{B}\)-modules \((\Omega_{\mathcal{B}/k}, i_D)\) which we denote by \(\text{Kos}(\mathcal{M}/k)\):

\[
\cdots \to \Omega_{\mathcal{B}/k}^p i_D \to \Omega_{\mathcal{B}/k}^{p-1} i_D \to \cdots \to \Omega_{\mathcal{B}/k} i_D \Omega_{\mathcal{B}/k} \to \mathcal{B} \to 0
\]  

(2.1)

and for each \(n \geq 0\) a complex of \(\mathcal{O}\)-modules

\[
\text{Kos}(\mathcal{M}/k)_n = \cdots \to [\Omega_{\mathcal{B}/k}^p]_n i_D \to \cdots \to [\Omega_{\mathcal{B}/k}]_n i_D \to S^n \mathcal{M} \to 0
\]

By homogeneous localization one has a complex of \(\mathcal{O}_p\)-modules, denoted by \(\tilde{\text{Kos}}(\mathcal{M}/k)\):

\[
\cdots \to \tilde{\Omega}_{\mathcal{B}/k}^p i_D \to \tilde{\Omega}_{\mathcal{B}/k}^{p-1} i_D \to \cdots \to \tilde{\Omega}_{\mathcal{B}/k} i_D \tilde{\Omega}_{\mathcal{B}/k} \to \mathcal{O}_p \to 0
\]

Theorem 2.1. The complex \(\tilde{\text{Kos}}(\mathcal{M}/k)\) is acyclic (that is, an exact sequence). Moreover,

\[
\Omega_{\mathcal{B}/k}^p = \text{Ker} \left( \Omega_{\mathcal{B}/k}^p \to \tilde{\Omega}_{\mathcal{B}/k}^p \right).
\]

Hence one has exact sequences

\[
0 \to \Omega_{\mathcal{B}/k}^p \to \tilde{\Omega}_{\mathcal{B}/k}^p \to \Omega_{\mathcal{B}/k}^{p-1} \to 0
\]

and right and left resolutions of \(\Omega_{\mathcal{B}/k}^p\):

\[
0 \to \Omega_{\mathcal{B}/k}^p \to \tilde{\Omega}_{\mathcal{B}/k}^p \to \tilde{\Omega}_{\mathcal{B}/k}^{p-1} \to \cdots \to \tilde{\Omega}_{\mathcal{B}/k} \to \mathcal{O}_p \to 0
\]

\[
\cdots \to \tilde{\Omega}_{\mathcal{B}/k} \to \tilde{\Omega}_{\mathcal{B}/k}^{p-1} \to \cdots \to \tilde{\Omega}_{\mathcal{B}/k}^{p+1} \to \Omega_{\mathcal{B}/k}^p \to 0
\]

In particular, for \(p = 1\) the exact sequence

\[
0 \to \Omega_{\mathcal{B}/k} \to \tilde{\Omega}_{\mathcal{B}/k} \to \mathcal{O}_p \to 0
\]

(2.2)

is called the (global) Euler sequence.

Proof. It is completely analogous to the proof of Theorem 1.4 \(\square\)

Let us denote \(\tilde{\text{Kos}}(\mathcal{M}/k)_n := \pi_*(\tilde{\text{Kos}}(\mathcal{M}/k)(n))\). The natural morphisms

\[
[\Omega_{\mathcal{B}/k}^p]_n \to \pi_*(\tilde{\Omega}_{\mathcal{B}/k}^p(n))
\]

give a morphism of complexes

\[
\text{Kos}(\mathcal{M}/k)_n \to \tilde{\text{Kos}}(\mathcal{M}/k)_n.
\]

In complete analogy to the relative setting we have the following:
**Theorem 2.2.** Let $\mathcal{M}$ be a finitely generated quasi-coherent module on a scheme $(X, \mathcal{O})$, $B = \mathcal{M}$, $P = \text{Proj} B$ and $\pi : P \to X$ the natural morphism. Let $d'$ be the greatest integer such that $\Omega_{B/k}^{d'} \neq 0$ and $n > 0$. Then:

1. If $R^j \pi_*(\Omega_{B/k}^i(n)) = 0$ for any $j > 0$ and any $0 \leq i \leq d'$, then $\tilde{\text{Kos}}(\mathcal{M}/k)_n$ is acyclic.
2. If (1) holds and the natural morphism $[\Omega_{B/k}^i]_n \to \pi_*(\tilde{\Omega}_{B/k}^i(n))$ is an isomorphism for any $0 \leq i \leq d'$, then $\text{Kos}(\mathcal{M}/k)_n$ is also acyclic.

**Theorem 2.3.** Let $X$ be a noetherian scheme and $\mathcal{M}$ a coherent module on $X$. The Koszul complexes $\text{Kos}(\mathcal{M}/k)_n$ and $\tilde{\text{Kos}}(\mathcal{M}/k)_n$ are acyclic for $n \gg 0$.

2.1. **Koszul versus De Rham (Global case).** Now we pass to the De Rham complex (over $k$). The $k$-linear differentials

$$d : \Omega_{B/k}^p \to \Omega_{B/k}^{p+1}$$

give a (global) De Rham complex

$$\text{DeRham}(\mathcal{M}/k) \equiv 0 \to B \xrightarrow{d} \Omega_{B/k} \xrightarrow{d} \cdots \xrightarrow{d} \Omega_{B/k}^{p-1} \xrightarrow{d} \Omega_{B/k}^p \to \cdots$$

which is bounded if $X$ is of finite type over $k$. Since $d$ is homogeneous of degree 0, one has for each $n \geq 0$ a complex of $\mathcal{O}$-modules (with $k$-linear differential)

$$\text{DeRham}(\mathcal{M}/k)_n \equiv 0 \to S^n \mathcal{M} \xrightarrow{d} [\Omega_{B/k}]_n \xrightarrow{d} \cdots \xrightarrow{d} [\Omega_{B/k}^p]_n \to \cdots.$$  

One has again Cartan’s formula: $i_D \circ d + d \circ i_D = \text{multiplication by } n$, on $[\Omega_{B/k}^p]_n$ and then:

**Proposition 2.4.** If $X$ is a scheme over $\mathbb{Q}$, then $\text{Kos}(\mathcal{M}/k)_n$ and $\text{DeRham}(\mathcal{M}/k)_n$ are homotopically trivial (in particular, acyclic) for any $n > 0$.

As in section 1.2, we can take homogeneous localizations: for each $n \in \mathbb{Z}$, the differentials $\Omega_{B/k}^p \to \Omega_{B/k}^{p+1}$ induce $k$-linear morphisms

$$d : \tilde{\Omega}_{B/k}^p(n) \to \tilde{\Omega}_{B/k}^{p+1}(n)$$

and one obtains a complex of $\mathcal{O}_P$-modules (with $k$-linear differential)

$$\tilde{\text{DeRham}}(\mathcal{M}/k, n) = 0 \to \mathcal{O}_P(n) \xrightarrow{d} \tilde{\Omega}_{B/k}(n) \xrightarrow{d} \cdots \xrightarrow{d} \tilde{\Omega}_{B/k}^p(n) \to \cdots$$

Again, the differentials of Koszul and De Rham complexes are related by Cartan’s formula: $i_D \circ d + d \circ i_D = \text{multiplication by } n$, on $\tilde{\Omega}_{B/k}^p(n)$, so one has:

**Proposition 2.5.** Let $X$ be a scheme over $\mathbb{Q}$. The complexes $\tilde{\text{Kos}}(\mathcal{M}/k)(n)$ and $\tilde{\text{DeRham}}(\mathcal{M}/k, n)$ are homotopically trivial (in particular, acyclic) for any $n \neq 0$.

**Corollary 2.6.** If $X$ is a scheme over $\mathbb{Q}$, then for any $n \neq 0$, the exact sequences

$$0 \to \Omega_{\mathbb{Q}/k}^p(n) \to \tilde{\Omega}_{B/k}^p(n) \to \Omega_{\mathbb{Q}/k}^{p-1}(n) \to 0$$

split as sheaves of $k$-modules (but not as $\mathcal{O}_P$-modules).
3. Cohomology of projective bundles

In this section we assume that $E$ is a locally free sheaf of rank $r+1$ on a $k$-scheme $(X, \mathcal{O})$. Let $B = S E$ be its symmetric algebra over $\mathcal{O}$ and $P = \text{Proj} B \to X$ the corresponding projective bundle. Our aim is to determine the cohomology of the sheaves $\Omega^p_{P/X}(n)$ and $\Omega^p_{P/k}(n)$.

3.1. Cohomology of $\Omega^p_{P/X}(n)$.

**Notations:** In order to simplify some statements, we shall use the following conventions:

1. $S^p E = 0$ whenever $p < 0$, and analogously for exterior powers.
2. For any integer $p$, we shall denote $\bar{p} = r + 1 - p$.
3. For any $\mathcal{O}$-module $M$, we shall denote by $M^*$ its dual: $M^* = \text{Hom}(M, \mathcal{O})$.

We shall use the following well known result on the cohomology of a projective bundle:

**Proposition 3.1.** Let $n$ be a non negative integer. Then

$$R^i \pi_* \mathcal{O}_P(n) = \begin{cases} 0 & \text{for } i \neq 0 \\ S^n \mathcal{E} & \text{for } i = 0 \end{cases}$$

If $n$ is a positive integer, then

$$R^i \pi_* \Omega^p_{B/\mathcal{O}}(−n) = \begin{cases} 0 & \text{for } i \neq r \\ S^{n−r} \mathcal{E}^* \otimes \Lambda^{r+1} \mathcal{E} & \text{for } i = r \end{cases}$$

We shall also use without further explanation a particular case of projection formula: for any quasi-coherent module $N$ on $X$ and any locally free module $L$ on $P$ such that $R^j \pi_* L$ is locally free (for any $j$), one has

$$R^i \pi_*(\pi^* N \otimes L) = N \otimes R^i \pi_* L$$

**Proposition 3.2.** Let $n$ be a non negative integer. Then

$$R^i \pi_* \tilde{\Omega}^p_{B/\mathcal{O}}(n) = \begin{cases} 0 & \text{for } i \neq 0 \\ \Lambda^p \mathcal{E} \otimes S^{n−p} \mathcal{E} & \text{for } i = 0 \end{cases}$$

For any positive integer $n$, one has

$$R^i \pi_* \tilde{\Omega}^p_{B/\mathcal{O}}(−n) = \begin{cases} 0 & \text{for } i \neq r \\ \Lambda^p \mathcal{E}^* \otimes S^{n−p} \mathcal{E}^* & \text{for } i = r \text{ with } \bar{p} = r + 1 - p \end{cases}$$

**Proof.** Since $\tilde{\Omega}^p_{B/\mathcal{O}} = (\pi^* \Lambda^p \mathcal{E})(−p)$, the results follows from Proposition 3.1. For the second formula we have also used the natural isomorphism $\Lambda^p \mathcal{E} = \Lambda^p \mathcal{E}^* \otimes \Lambda^{r+1} \mathcal{E}$.

**Remark 3.3.** Notice that $\Lambda^p \mathcal{E} \otimes S^{n−p} \mathcal{E} = [\Omega^p_{B/\mathcal{O}}]_n$. Thus, Proposition 3.2 and Theorem 1.5 tell us that $\text{Kos}(\mathcal{E})_n \to \tilde{\text{Kos}}(\mathcal{E})_n$ is an isomorphism for any $n \geq 0$ and the Koszul complexes $\tilde{\text{Kos}}(\mathcal{E})_n$ and $\text{Kos}(\mathcal{E})_n$ are acyclic for any $n > 0$ (thus we obtain the well known fact of the acyclicity of the Koszul complex of a locally free module).

Let us denote by $K_{p,n}$ the kernels of the morphisms $i_\mathcal{O}$ in $\text{Kos}(\mathcal{E})_n$, that is,

$$K_{p,n} := \text{Ker}(\Lambda^p \mathcal{E} \otimes S^{n−p} \mathcal{E} \to \Lambda^{p−1} \mathcal{E} \otimes S^{n−p+1} \mathcal{E})$$

One has the following result (see [12] or [4, Exposè XI] for different approaches).
Theorem 3.4. Let $E$ be a locally free sheaf of rank $r + 1$ on a $k$-scheme $(X, \mathcal{O})$ and $P = \text{Proj} S \cong X$ the corresponding projective bundle.

Let $n$ be a positive integer number.

(1) \[ R^i \pi_* \Omega_{P/X}^p = \begin{cases} \mathcal{O} & \text{if } 0 \leq i = p \leq r \\ 0 & \text{otherwise} \end{cases} \]

(2) \[ R^i \pi_* \Omega_{P/X}^p(n) = \begin{cases} 0 & \text{if } i \neq 0 \\ \mathcal{K}_{p,n} & \text{if } i = 0 \end{cases} \]

and, if $X$ is a $\mathbb{Q}$-scheme, then
\[ \mathcal{K}_{p,n} \oplus \mathcal{K}_{p-1,n} = \Lambda^p E \otimes S^{n-p} E. \]

(3) \[ R^i \pi_* \Omega_{P/X}^p(-n) = \begin{cases} 0 & \text{if } i \neq r \\ \mathcal{K}_{r-p,n}^* & \text{if } i = r \end{cases} \]

and, if $X$ is a $\mathbb{Q}$-scheme, then
\[ \mathcal{K}_{r-p,n}^* \oplus \mathcal{K}_{r-p+1,n} = \Lambda^p E^* \otimes S^{n-p} E^*. \]

Proof. Let $n \geq 0$. By Theorem 1.4
\[ 0 \to \Omega_{P/X}^p(n) \to \Omega_{B/O}^p(n) \to \cdots \to \Omega_{B/O}(n) \to \mathcal{O}_P(n) \to 0 \]
is a resolution of $\Omega_{P/X}^p(n)$ by $\pi_*$-acyclic sheaves (by Proposition 3.2). One concludes then by Proposition 3.2 and Remark 3.3.

(3) follows from (2) and (relative) Grothendieck duality: one has an isomorphism
\[ \Omega_{P/X}^p = \mathcal{H}om (\Omega_{P/X}^{r-p}, \Omega_{P/X}^r) \]
and then
\[ \mathbb{R} \pi_* \Omega_{P/X}(n) \cong \mathbb{R} \pi_* \mathcal{H}om (\Omega_{P/X}^{r-p}(n), \Omega_{P/X}^r) \cong \mathbb{R} \mathcal{H}om (\mathbb{R} \pi_* \Omega_{P/X}^{r-p}(n)[r], \mathcal{O}) \]
and one concludes by (2).

Finally, the statements of (2) and (3) regarding the case that $X$ is a $\mathbb{Q}$-scheme follow from Corollary 1.11.

\[ \square \]

Corollary 3.5. (Bott’s formula) Let $P_r$ be the projective space of dimension $r$ over a field $k$. Let $n$ be a positive integer number.

(1) \[ \dim_k H^q(P_r, \Omega_{P_r}^p) = \begin{cases} 1 & \text{if } 0 \leq q = p \leq r \\ 0 & \text{otherwise} \end{cases} \]

(2) \[ \dim_k H^q(P_r, \Omega_{P_r}^p(n)) = \begin{cases} 0 & \text{if } q \neq 0 \\ \binom{n+r-p}{n} \binom{n-1}{p} & \text{if } q = 0 \end{cases} \]

(3) \[ \dim_k H^q(P_r, \Omega_{P_r}^p(-n)) = \begin{cases} 0 & \text{if } q \neq r \\ \binom{n+p}{n} \binom{n-1}{r-p} & \text{if } q = r \end{cases} \]
Proof. It follows from Theorem 3.4 once one proves that \( \dim_k K_{p,n} = \binom{n+r-p}{p}(n-1) \).

From the exact sequence \( 0 \to K_{p,n} \to \Lambda^r \mathcal{E} \otimes S^{n-r} \mathcal{E} \to K_{p-1,n} \to 0 \) it follows that \( \dim_k K_{p,n} + \dim_k K_{p-1,n} = \binom{r+1}{r}(n-r) \); hence it suffices to prove that

\[
\binom{n+r-p}{n} \binom{n-1}{p} + \binom{n+r-p+1}{n} \binom{n-1}{p-1} = \binom{r+1}{p} \binom{n-r+p}{r}
\]

which is an easy computation if one writes \( \binom{a}{b} = \frac{a!}{b!(a-b)!} \).

\[\square\]

Remark 3.6. (1) We can give an interpretation of \( H^0(\mathbb{P}_r, \Omega^p_{\mathbb{P}_r}(n)) \) in terms of differentials forms of the polynomial ring \( k[x_0, \ldots, x_r] \): one has the exact sequence

\[
0 \to H^0(\mathbb{P}_r, \Omega^p_{\mathbb{P}_r}(n)) \to [\Omega^p_{k[x_0, \ldots, x_r]/k}]_n \xrightarrow{i_D} [\Omega^p_{k[x_0, \ldots, x_r]/k}]_n
\]

that is, \( H^0(\mathbb{P}_r, \Omega^p_{\mathbb{P}_r}(n)) \) are those \( p \)-forms \( \omega \in \Omega^p_{k[x_0, \ldots, x_r]/k} \) which are homogeneous of degree \( n \) and such that \( i_D \omega = 0 \), where \( D = \sum_{i=0}^r x_i \frac{\partial}{\partial x_i} \).

(2) From the exact sequence

\[
0 \to H^0(\mathbb{P}_r, \Omega^p_{\mathbb{P}_r}(n)) \to \Lambda^r \mathcal{E} \otimes S^{n-r} \mathcal{E} \to \cdots \to \mathcal{E} \otimes S^{n-1} \mathcal{E} \to S^n \mathcal{E} \to 0
\]

we can give a different combinatorial expression of \( \dim_k H^0(\mathbb{P}_r, \Omega^p_{\mathbb{P}_r}(n)) \) (as Verdier does):

\[
\dim_k H^0(\mathbb{P}_r, \Omega^p_{\mathbb{P}_r}(n)) = \sum_{i=0}^p (-1)^i \binom{r+1}{p-i} \binom{n+r-p+i}{r}
\]

It follows from Theorem 3.4 that \( H^q(\mathbb{P}, \Omega^p_{\mathbb{P}}) = H^q(X, \mathcal{O}) \). For the twisted case we have the following:

Corollary 3.7. Let \( X \) be a proper scheme over a field \( k \) of characteristic zero. Let \( \mathcal{E} \) be a locally free module on \( X \) of rank \( r+1 \) and \( \mathbb{P} = \text{Proj} \mathcal{E} \) the associated projective bundle. Then, for any positive integer \( n \), one has:

(1) \( \dim_k H^q(\mathbb{P}, \Omega^p_{\mathbb{P}}(n)) = \sum_{i=0}^p (-1)^i \dim H^q(X, \Lambda^{r-i} \mathcal{E} \otimes S^{n-p+i} \mathcal{E}) \).

(2) \( \dim_k H^q(\mathbb{P}, \Omega^p_{\mathbb{P}}(n)) = \sum_{i=0}^p (-1)^i \dim H^{q-i}(X, \Lambda^{r-i} \mathcal{E}^* \otimes S^{n-p-i} \mathcal{E}^*) \).

with \( \tilde{p} = r+1-p \).

Proof. (1) By Corollary 1.11 one has

\[
H^q(\mathbb{P}, \Omega^p_{\mathbb{P}}(n)) \oplus H^q(\mathbb{P}, \Omega^{p-1}_{\mathbb{P}}(n)) = H^q(\mathbb{P}, \tilde{\Omega}^p_{\mathbb{P}/\mathcal{O}}(n))
\]

and \( H^q(\mathbb{P}, \tilde{\Omega}^p_{\mathbb{P}/\mathcal{O}}(n)) = H^q(X, \Lambda^p \mathcal{E} \otimes S^{n-p} \mathcal{E}) \) by Proposition 3.2. Conclusion follows.

(2) is completely analogous.

\[\square\]

3.2. Cohomology of \( \Omega^p_{\mathbb{P}/k}(n) \).

Let us consider the exact sequence of differentials

\[
0 \to \Omega_{X/k} \otimes \mathcal{O} \to \Omega_{B/k} \to \Omega_{B/\mathcal{O}} \to 0
\]

This sequence locally splits: indeed, if \( \mathcal{E} \) is trivial, then \( \mathcal{E} = E \otimes_k \mathcal{O} \) and \( B = B \otimes_k \mathcal{O} \), with \( B = S' \); hence, \( \Omega_{B/\mathcal{O}} = \Omega_{B/k} \otimes_k \mathcal{O} \) and there is a natural morphism \( \Omega_{B/k} \otimes_k \mathcal{O} \to \Omega_{B/k} \) which is a section of \( \Omega_{B/k} \to \Omega_{B/\mathcal{O}} \).
Remark 3.8. The exact sequence is a sequence of graded \( \mathcal{B} \)-modules, hence it gives an exact sequence of \( \mathcal{O} \)-modules in each degree. In particular, in degree 0 one obtains an isomorphism \( \Omega_{X/k} = [\Omega_{\mathcal{B}/k}]_0 \), and an exact sequence in degree 1:

\[
0 \to \Omega_{X/k} \otimes_{\mathcal{O}} \mathcal{E} \to [\Omega_{\mathcal{B}/k}]_1 \to \mathcal{E} \to 0
\]

which is nothing but the Atiyah extension.

Taking homogeneous localizations we obtain an exact sequence of \( \mathcal{O}_{\mathcal{P}} \)-modules

\[
0 \to \pi^* \Omega_{X/k} \to \tilde{\Omega}_{\mathcal{B}/k} \to \tilde{\Omega}_{\mathcal{B}/\mathcal{O}} \to 0
\]

which splits locally (on \( X \)).

Proposition 3.9. Let \( n \) be a positive integer. Then:

\[
R^i \pi_* \tilde{\Omega}_{\mathcal{B}/k}^p = \begin{cases} 
0 & \text{for } i \neq 0, r \\
\Omega_{X/k}^p & \text{for } i = 0 \\
\tilde{\Omega}_{X/k}^{p-r-1} & \text{for } i = r 
\end{cases}
\]

\[
R^i \pi_* \tilde{\Omega}_{\mathcal{B}/k}^p(n) = \begin{cases} 
0 & \text{for } i \neq 0 \\
[\tilde{\Omega}_{\mathcal{B}/k}^p]_n & \text{for } i = 0 
\end{cases}
\]

\( 3 \) \( R^i \pi_* \tilde{\Omega}_{\mathcal{B}/k}^p(-n) = 0 \) for \( i \neq r \) and \( R^i \pi_* \tilde{\Omega}_{\mathcal{B}/k}^p(-n) \) is locally isomorphic to \( \bigoplus_{q=0}^n (\Omega_{X/k}^{p-q} \otimes \Lambda^q \mathcal{E}^* \otimes S^{r-q} \mathcal{E}^*) \), with \( q = r + 1 - n \).

\( 4 \) Furthermore, if \( X \) is a smooth \( k \)-scheme (of relative dimension \( d \)), then

\[
R^r \pi_* \tilde{\Omega}_{\mathcal{B}/k}^p(-n) = [\tilde{\Omega}_{\mathcal{B}/k}^{d+p}]_n \otimes \Omega_{X/k}^d
\]

Proof. If \( \mathcal{E} \) is trivial, then \( \tilde{\Omega}_{\mathcal{B}/k} = \pi^* \Omega_{X/k} \oplus \tilde{\Omega}_{\mathcal{B}/\mathcal{O}} \), so \( \tilde{\Omega}_{\mathcal{B}/k}^p = \bigoplus_{q=0}^p \pi^* \Omega_{X/k}^{p-q} \otimes \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^q \) and (1)-(3) follow from Proposition 3.2 in this case. Since \( \mathcal{E} \) is locally trivial, we obtain the vanishing statements of (1)-(3).

1 The natural morphism \( \Omega_{X/k}^p \to \pi_* \tilde{\Omega}_{\mathcal{B}/k}^p \) is an isomorphism because it is locally so. The natural morphism \( \Omega_{\mathcal{B}/k}^{r+1} \to \Omega_{\mathcal{B}/\mathcal{O}}^{r+1} \) gives a morphism \( R^r \pi_* \tilde{\Omega}_{\mathcal{B}/k}^{r+1} \to R^r \pi_* \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^{r+1} = \mathcal{O} \), which is an isomorphism because it is locally so. Finally, for any \( p \geq 0 \), the natural morphism \( \tilde{\Omega}_{\mathcal{B}/k}^p \otimes \tilde{\Omega}_{\mathcal{B}/k}^{r+1} \to \tilde{\Omega}_{\mathcal{B}/k}^{p+r+1} \) induces a morphism \( \pi_* (\tilde{\Omega}_{\mathcal{B}/k}^p \otimes \tilde{\Omega}_{\mathcal{B}/k}^{r+1}) \to R^r \pi_* \tilde{\Omega}_{\mathcal{B}/k}^{p+r+1} \), i.e. a morphism \( \Omega_{X/k}^p \to R^r \pi_* \tilde{\Omega}_{\mathcal{B}/k}^{p+r+1} \), which is an isomorphism because it is locally so.

2 The natural morphism \( [\Omega_{\mathcal{B}/k}^p]_n \to \pi_* \tilde{\Omega}_{\mathcal{B}/k}^p(n) \) is an isomorphism because it is locally so.

It only remains to prove (4), which is a consequence of (relative) Grothendieck duality. Indeed, notice that, under the smoothness hypothesis, \( R^r \pi_* \tilde{\Omega}_{\mathcal{B}/k}^p(-n) \) is locally free, by (3). Hence, if suffices to compute its dual. This is given by duality: the relative dualizing sheaf is \( \Omega_{\mathcal{F}/X}^{r+1} = \tilde{\Omega}_{\mathcal{B}/\mathcal{O}}^{r+1} \) and one has isomorphisms
\[ \Omega_{B/k}^{d+r+1} = \Omega_{B/k}^{r+1} \otimes \pi^* \Omega_{X/k}^d \] and \( \text{Hom}(\Omega_{B/k}^p, \Omega_{B/k}^{d+r+1}) = \Omega_{B/k}^{d+r+p}. \) Then:

\[
\left[ R^i \pi_* \Omega_{B/k}^p (-n) \right] = \pi_* \text{Hom}_F(\Omega_{B/k}^p(-n), \Omega_{B/k}^{r+1}) = \pi_* \left[ \text{Hom}_F(\Omega_{B/k}^p(-n), \Omega_{B/k}^{d+r+1}) \otimes \pi^*(\Omega_{X/k}^d)^* \right] = (\pi_* \Omega_{B/k}^{d+p+n}(n)) \otimes (\Omega_{X/k}^d)^* \overset{(2)}{=} [\Omega_{B/k}^{d+p}]_n \otimes (\Omega_{X/k}^d)^*.
\]

\[ \square \]

**Corollary 3.10.** The Koszul complexes \( \text{Kos}(E/k)_n \) and \( \text{Kos}(E/k)_{n} \) are acyclic for \( n > 0 \) and \( \text{Kos}(E/k)_n \to \text{Kos}(E/k)_{n} \) is an isomorphism for any \( n \geq 0 \).

Let us denote by \( \mathcal{T}_{p,n} \) the kernels of the morphisms \( i_D \) in the Koszul complex \( \text{Kos}(E/k)_n \); that is,

\[ \mathcal{T}_{p,n} := \text{Ker} \left( [\Omega_{B/k}^p]_n \to [\Omega_{B/k}^{p-1}]_n \right) \]

**Theorem 3.11.** Let \( E \) be a locally free sheaf of rank \( r + 1 \) on a \( k \)-scheme \( (X, \mathcal{O}) \) and \( \mathbb{P} = \text{Proj}^r E \overset{i_D}{\to} X \) the corresponding projective bundle.

Let \( n \) be a positive integer. One has:

1. \( R^i \pi_* \Omega_{\mathbb{P}/k}^p = \Omega_{X/k}^d \)
2. \( R^i \pi_* \Omega_{\mathbb{P}/k}^p(n) = \begin{cases} 0 & \text{for } i \neq 0 \\ \mathcal{T}_{p,n} & \text{for } i = 0 \end{cases} \)

and, if \( X \) is a \( \mathbb{Q} \)-scheme, then one has an isomorphism (of \( k \)-modules, not of \( \mathcal{O} \)-modules)

\[ \mathcal{T}_{p,n} \oplus \mathcal{T}_{p-1,n} = [\Omega_{B/k}^p]_n. \]

3. \( R^i \pi_* \Omega_{\mathbb{P}/k}^p(-n) = 0 \) for \( i \neq r \) and \( R^r \pi_* \Omega_{\mathbb{P}/k}^p(-n) \) is locally isomorphic to \( \bigoplus_{p=0}^\infty \Omega_{\mathbb{P}/k}^p(-r+1) \otimes K_{r-q,n}. \) Moreover, if \( X \) is a \( \mathbb{Q} \)-scheme, then one has an isomorphism (of \( k \)-modules, not of \( \mathcal{O} \)-modules)

\[ R^r \pi_* \Omega_{\mathbb{P}/k}^p(-n) \oplus R^r \pi_* \Omega_{\mathbb{P}/k}^{p-1}(-n) = R^r \pi_* \Omega_{\mathbb{P}/k}^p(-n) \]

4. If \( X \) is a smooth \( k \)-scheme (of relative dimension \( d \)), then

\[ R^r \pi_* \Omega_{\mathbb{P}/k}^p(-n) = \mathcal{T}_{d+r-p,n} \otimes \Omega_{X/k}^d \]

and, if \( X \) is a \( \mathbb{Q} \)-scheme, then one has an isomorphism (of \( k \)-modules, not of \( \mathcal{O} \)-modules)

\[ R^r \pi_* \Omega_{\mathbb{P}/k}^p(-n) \oplus R^r \pi_* \Omega_{\mathbb{P}/k}^{p-1}(-n) = [\Omega_{B/k}^{d+p}]_n \otimes \Omega_{X/k}^d. \]

**Proof.** If \( E \) is trivial, then \( \Omega_{\mathbb{P}/k} = \pi^* \Omega_{X/k} \oplus \mathcal{O}_{\mathbb{P}/X} \), so \( \Omega_{\mathbb{P}/k} = \bigoplus_{q=0}^\infty \pi^* \Omega_{X/k}^q \otimes \mathcal{O}_{\mathbb{P}/X}^{q-r} \) and (1)-(3) follow from Theorems 3.4. In this case. Since \( E \) is locally trivial, we obtain the vanishing statements of (1)-(3).

1. The exact sequences \( 0 \to \Omega_{\mathbb{P}/k}^p \to \Omega_{\mathbb{P}/k}^p \to \Omega_{\mathbb{P}/k}^{p-1} \to 0 \) induce morphisms

\[ \pi_* \Omega_{\mathbb{P}/k}^{p-i} \to R^1 \pi_* \Omega_{\mathbb{P}/k}^{p-i+1} \to \cdots \to R^i \pi_* \Omega_{\mathbb{P}/k}^p \]

whose composition with the natural morphism \( \Omega_{X/k}^{p-i} \to \pi_* \Omega_{\mathbb{P}/k}^{p-i} \) gives a morphism \( \Omega_{X/k}^{p-i} \to R^i \pi_* \Omega_{\mathbb{P}/k}^{p} \). This morphism is an isomorphism because it is locally so.
(2) The exact sequence $0 \to \Omega_p^p(n) \to \Omega^p_{\mathcal{B}/k}(n) \to \Omega^p_{\mathcal{B}/k}(n)$ induces, taking direct image, the isomorphism $\pi_* \Omega^p_{\mathcal{B}/k}(n) = \mathcal{K}_{p,n}$.

(4) follows from (2) and (relative) Grothendieck duality. Indeed, notice that, under the smoothness hypothesis, $R^r \pi_* \Omega^p_{\mathcal{B}/k}(n)$ is locally free, by (3). Hence, if suffices to compute its dual. This is given by duality: the relative dualizing sheaf is $\mathcal{K}^r_{\mathcal{B}/X}$ and one has isomorphisms $\Omega_{\mathcal{B}/X}^{d+r} = \Omega_{\mathcal{B}/X}^r$ and $\text{Hom}(\Omega_{\mathcal{B}/k}^p, \mathcal{K}^{d+r}_{\mathcal{B}/k}) = \mathcal{K}^{d+r}_{\mathcal{B}/k}$; then:

$$
\left[ R^r \pi_* \Omega^p_{\mathcal{B}/k}(-n) \right]^* = \pi_* \text{Hom}_{\mathcal{B}}(\Omega^p_{\mathcal{B}/k}(-n), \Omega^r_{\mathcal{B}/k}) = \pi_* [\text{Hom}_{\mathcal{B}}(\Omega^p_{\mathcal{B}/k}(-n), \Omega^d_{\mathcal{B}/k}) \otimes \pi^*(\Omega^d_{\mathcal{B}/k})^*] = (\pi_* \Omega^d_{\mathcal{B}/k}(n))^* \otimes (\Omega^d_{\mathcal{B}/k})^*.
$$

Finally, the statements of (2)-(4) regarding the case of a $\mathbb{Q}$-scheme follow from Corollary 2.6.

**Remark 3.12.** For $n = 1$ a little more can be said (as Verdier does): The natural morphism $\Omega^p_{\mathcal{B}/k} \otimes \mathcal{E} \to \pi_* \Omega^p_{\mathcal{B}/k}(1)$ is an isomorphism. Indeed, the exact sequence

$$0 \to \Omega^p_{\mathcal{B}/k} \otimes \mathcal{B} \to \Omega^p_{\mathcal{B}/k} \to \Omega^p_{\mathcal{B}/O} \to 0$$

induces for each $p$ an exact sequence

$$0 \to \Omega^p_{\mathcal{B}/k} \otimes \mathcal{B} \to \Omega^p_{\mathcal{B}/k} \to \Omega^p_{\mathcal{B}/k} \otimes \Omega^p_{\mathcal{B}/O} \to \Omega^p_{\mathcal{B}/k} \otimes S^2 \Omega^p_{\mathcal{B}/O} \to \cdots$$

and taking degree 1, an exact sequence

$$0 \to \Omega^p_{\mathcal{B}/k} \otimes \mathcal{E} \to \Omega^p_{\mathcal{B}/k} \otimes \mathcal{E} \to 0$$

On the other hand, taking $\pi_*$ in the exact sequence

$$0 \to \Omega^p_{\mathcal{B}/k}(1) \to \Omega^p_{\mathcal{B}/k}(1) \to \Omega^p_{\mathcal{B}/k}(1) \to 0$$

gives the exact sequence

$$0 \to \pi_* \Omega^p_{\mathcal{B}/k}(1) \to \pi_* \Omega^p_{\mathcal{B}/k}(1) \to \pi_* \Omega^p_{\mathcal{B}/k}(1) \to 0.$$
The decomposability of $\mathbb{R}_\pi \Omega_{\mathcal{P}/k}^p$ implies an isomorphism

$$H^q(\mathcal{P}, \Omega_{\mathcal{P}/k}^p) = \bigoplus_{i=0}^r H^{q-i}(X, \Omega_{X/k}^{p-i})$$

For the twisted case we have the following:

**Corollary 3.14.** Let $X$ be a proper scheme over a field $k$ of characteristic zero. Let $E$ be a locally free module on $X$ of rank $r+1$ and $\mathcal{P} = \text{Proj} S \cdot E$ the associated projective bundle. Then, for any positive integer $n$, one has:

1. $\dim_k H^q(\mathcal{P}, \Omega_{\mathcal{P}/k}^p(n)) = \sum_{i=0}^p (-1)^i \dim_k H^{q-i}(X, \Omega_{X/k}^{p-i}n)$.
2. If $X$ is smooth over $k$ of dimension $d$, then $\dim_k H^q(\mathcal{P}, \Omega_{\mathcal{P}/k}^p(-n)) = \sum_{i=0}^{d+r-p} (-1)^i \dim_k H^{d+r-q}(X, \Omega_{X/k}^{d+r-p-i}n)$.

**Proof.** (1) By Corollary 2.6,

$$H^q(\mathcal{P}, \Omega_{\mathcal{P}/k}^p(n)) \oplus H^q(\mathcal{P}, \Omega_{\mathcal{P}/k}^{p-1}(n)) = H^q(\mathcal{P}, \widetilde{\Omega}_{\mathcal{P}/k}^p(n))$$

and $H^q(\mathcal{P}, \widetilde{\Omega}_{\mathcal{P}/k}^p(n)) = H^q(X, \Omega_{X/k}^p)$. Conclusion follows.

(2) follows from (1) and duality.

References

[1] P. Berthelot and L. Illusie, *Classes de Chern en cohomologie cristalline*, C. R. Acad. Sci. Paris Sér. A-B 270 (1970), A1695-A1697; ibid., 270 (1970), pp. A1750–A1752.
[2] R. Bott, *Homogeneous vector bundles*, Ann. of Math. (2), 66 (1957), pp. 203–248.
[3] N. Bourbaki, *Algèbre, Chapitre 10*, Masson, Paris, 1980.
[4] P. Deligne, *Cohomologie des intersections complètes*, Lecture Notes in Mathematics, Vol. 340, Springer-Verlag, Berlin-New York, 1973. Groupes de monodromie en géométrie algébrique. II. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II), Dirigé par P. Deligne et N. Katz.
[5] J. M. Giral and F. Planas-Vilanova, *A note on the acyclicity of the Koszul complex of a module*, Ark. Mat., 45 (2007), pp. 273–278.
[6] M. Gros, *Classes de Chern et classes de cycles en cohomologie de Hodge-Witt logarithmique*, Mém. Soc. Math. France (N.S.), (1985), p. 87.
[7] A. Grothendieck, *Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes*, Inst. Hautes Études Sci. Publ. Math., (1961), p. 222.
[8] ———, *Éléments de géométrie algébrique. III. Étude cohéologique des faisceaux cohérents*, I, Inst. Hautes Études Sci. Publ. Math., (1961), p. 167.
[9] A. Micali, *Sur les algèbres universelles*, Ann. Inst. Fourier (Grenoble), 14 (1964), pp. 33–87.
[10] D. Quillen, *Homology of commutative rings*, MIT mimeographed notes, Massachusetts Institute of Technology, Cambridge, MA, (1967).
[11] F. Sancho de Salas, *Residues of a Pfaff system relative to an invariant subscheme*, Trans. Amer. Math. Soc., 352 (2000), pp. 4019–4035.
[12] J. L. Verdier, *Le théorème de Le Potier*, in Différents aspects de la positivité (Sém. Géom. Anal., École Norm. Sup., Paris, 1972–1973), Soc. Math. France, Paris, 1974, pp. 68–78. Astérisque, No. 17.
EULER SEQUENCE AND KOSZUL COMPLEX OF A MODULE

DEPARTAMENTO DE MATEMÁTICAS AND INSTITUTO UNIVERSITARIO DE FÍSICA FUNDAMENTAL Y MATEMÁTICAS (IUFFyM), UNIVERSIDAD DE SALAMANCA, PLAZA DE LA MERCED 1-4, 37008 SALAMANCA, SPAIN.

E-mail address: dario@usal.es
E-mail address: fsancho@usal.es