Stochastic maximum principle for infinite dimensional control systems

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Abstract

The general maximum principle is proved for an infinite dimensional controlled stochastic evolution system. The control is allowed to take values in a nonconvex set and enter into both drift and diffusion terms. The operator-valued backward stochastic differential equation, which characterizes the second-order adjoint process, is understood via the concept of “generalized solution” proposed by Guatteri and Tessitore [SICON 44 (2006)].

Keywords. Stochastic maximum principle, stochastic evolution equation, optimal control, operator-valued backward stochastic differential equation, generalized solution.

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1 Introduction

1.1 Problem formulation and basic assumptions

In this paper we shall always indicate by $H$ a real separable Hilbert space and by $\langle \cdot, \cdot \rangle_H$ and $\| \cdot \|_H$ its inner scalar product and norm, respectively. Denote by $B(H)$ the Banach space of all bounded linear operators from $H$ to itself endowed with the norm $\| T \|_{B(H)} := \sup \{ \| Tx \|_H : \| x \|_H = 1 \}$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with the filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ generated by countable independent standard Wiener processes $\{W^i; i \in \mathbb{N}\}$ and augmented with all $\mathbb{P}$-null sets of $\mathcal{F}$. For simplicity, we write formally $f \cdot dW = \sum_{i \in \mathbb{N}} f_i \, dW^i$ with a sequence $f = (f_i; i \in \mathbb{N})$. We denote by $\mathbb{E}^{\mathcal{F}_t}$ the conditional expectation with respect to $\mathcal{F}_t$, and by $\mathcal{P}$ the predictable $\sigma$-field on $\Omega \times [0, 1]$.

In this paper, we study an infinite-dimensional optimal control problem governed by the following abstract semilinear stochastic evolution equation (SEE)

$$
\begin{align*}
&dx(t) = [Ax(t) + f(t, x(t), u(t))] \, dt + g(t, x(t), u(t)) \cdot dW_t, \\
&x(0) = x_0,
\end{align*}
$$

where $x(\cdot)$ is the state process and $u(\cdot)$ is the control. The control set $U$ is a nonempty Borel-measurable subset of a metric space whose metric is denoted by

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dist(·, ·). Fix an element (denoted by 0) in \( U \), and then define \( |u|_U = \text{dist}(u, 0) \).

An admissible control \( u(\cdot) \) is a \( U \)-valued predictable process such that

\[
\sup \{ \mathbb{E} |u(t)|_U^2 : t \in [0, 1] \} < \infty.
\]

Our optimal control problem is to find an admissible control \( u(\cdot) \) minimizing the cost functional

\[
J(u(\cdot)) = \mathbb{E} \int_0^1 l(t, x(t), u(t)) \, dt + \mathbb{E} h(x(1)).
\]

In the above statement, \( A \) is the infinitesimal generator of a \( C_0 \)-semigroup, and

\[
f : \Omega \times [0, 1] \times H \times U \to H, \quad g : \Omega \times [0, 1] \times H \times U \to l^2(H),
\]

\[
l : \Omega \times [0, 1] \times H \times U \to \mathbb{R}, \quad h : \Omega \times H \to \mathbb{R},
\]

where the Hilbert space

\[
l^2(H) := \left\{ z = (z_i; i \in \mathbb{N}) : \|z\|_{l^2(H)}^2 = \sum_{i \in \mathbb{N}} \|z_i\|_H^2 < \infty \right\}.
\]

Throughout this paper we make the following assumptions.

**Assumption 1.1** The operator \( A : D(A) \subset H \to H \) is the infinitesimal generator of a \( C_0 \)-semigroup \( \{e^{tA} : t \geq 0\} \). Set

\[
M_A := \sup \{ \|e^{tA}\|_{B(H)} : t \in [0, 1] \}.
\]

**Assumption 1.2** The functions \( f, g \) and \( l \) are all \( \mathcal{P} \times B(H) \times B(U) \)-measurable, and \( h \) is \( \mathcal{F}_t \times B(H) \)-measurable; for each \((t, u, \omega) \in [0, 1] \times U \times \Omega\), \( f, g, l \) and \( h \) are globally twice Fréchet differentiable with respect to \( x \); \( f_x, g_x, f_{xx}, g_{xx}, l_{xx} \) and \( h_{xx} \) are bounded by a constant \( M_0 \); \( f_x, g_x, l \) and \( h_x \) are bounded by \( M_0(1 + \|x\|_H + |u|_U) \); \( l \) and \( h \) is bounded by \( M_0(1 + \|x\|_H^2 + |u|_U^2) \).

In view of Assumption 1.1, the precise meaning of the equation (1.1) is

\[
x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A} \left[ f(s, x(s), u(s)) \, ds + g(s, x(s), u(s)) \cdot dW_s \right].
\]

A process \( x(\cdot) \) satisfying the above equality is usually called a mild solution to equation (1.1), cf. [3].

### 1.2 Developments of stochastic maximum principle and contributions of the paper

The aim of this work is to find a stochastic maximum principle (SMP for short) for the optimal control. As we know, SMP is one of the basic tools to study optimal stochastic control problems. Since [11], there have been a number of
results on such a subject. For finite dimensional systems, the problem in the general case was solved by Peng [13]. Hereafter, by the word “general” we mean the allowance of the control into the diffusion term and the nonconvexity of control domains. In contrast, most existing results on infinite dimensional systems are limited to the case in which the control domain is convex or the diffusion does not depend on the control (cf. [11, 9, 14]). Recently, several works [12, 7, 5] were devoted to the general SMP in infinite dimensions. Liu-Zhang [12] first addressed such a problem and formulated a general SMP in which they assumed the existence of the second-order adjoint process. Fuhrman et al. [7] focused on a concrete equation (which was a stochastic parabolic PDE with deterministic coefficients) and gave a complete formulation of SMP, while their approach depended on the special structure of the equation. In our previous work [5], a general SMP was obtained for abstract stochastic parabolic equations driven by finite Wiener processes. As far as we know, the general SMP for stochastic evolution equations as form (1.1) is not completely solved.

Our basic approach to derive the general SMP follows Peng’s idea of second-order expansion in calculating the variation of the cost functional caused by the spike variation. The key point is how to understand and solve a $B(H)$-valued backward stochastic differential equation (BSDE) which characterizes the second-order adjoint process in our SMP. To do this, we exploit a concept of solution to this equation called “generalized solution” which was first proposed by Guatteri-Tessitore [8] in the study of infinite dimensional LQ problems, and prove the existence-uniqueness result in our framework. However, the generalized solution only characterizes the first unknown component but says nothing about the second one. As a consequence, it seems difficult to derive our SMP via the traditional approach (i.e. applying Ito’s formula). To avoid this difficulty, we first derive a basic property of the generalized solution. Then our goal is achieved thanks to the Lebesgue differentiation theorem.

The rest of this paper is organized as follows. Section 2 is devoted to some preliminary results on SEE and backward SEE (BSEE) in the Hilbert space. In Section 3, we study the well-posedness and basic properties of an operator-valued BSDE. With the previous preparations, we shall state and prove our main theorem, the stochastic maximum principle, in the final section.

We finish the introduction with some notations. Let $H$ be a separable Hilbert space and $B(H)$ be its Borel $\sigma$-field. The following classes of processes will be used in this article. Here $p, q \in [1, \infty]$.

- $L^p_p(\Omega \times [0, 1]; H)$ denotes the space of equivalence classes of processes $x(\cdot)$, admitting a predictable version such that $\mathbb{E} \int_0^1 \|x(t)\|_H^p \, dt < \infty$.
- $C_T([0, 1]; L^p(\Omega; H))$ denotes the space of $H$-valued processes $x(\cdot)$ such that $x(\cdot) : [0, 1] \rightarrow L^p(\Omega; H)$ is continuous and has a predictable modification.

Moreover, since the $\sigma$-field generated by the operator norm in $B(H)$ is too large, we shall define the following spaces with respect to $B(H)$-valued processes

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1 For more related studies, we refer to [14] and the references therein.
2 Normally, the solution of a BSDE consists of a pair of adapted processes of which the second one is the diffusion term. For more aspects on BSDE, we refer to [6].
and random variables.

- \( L^p_{\mathcal{P}, S}(\Omega \times [0, 1]; B(H)) \) denotes the space of equivalence classes of \( B(H) \)-valued processes \( T(\cdot) \) such that \( T(\cdot)x \in L^p_{\mathcal{P}}(\Omega \times [0, 1]; H) \) for each \( x \in H \). Here the subscript “S” stands for “strongly measurable”.

- \( L^p_{\mathcal{P}_1, S}(\Omega; B(H)) \) denotes the space of equivalence classes of \( B(H) \)-valued random variable \( T \) such that \( Tx \in L^p_{\mathcal{P}_1}(\Omega; H) \) for each \( x \in H \).

2 Preliminary results on SEEs and BSEEs

Let \( b : \Omega \times [0, T] \times H \to H \) and \( \sigma : \Omega \times [0, T] \times H \to l^2(H) \) be two \( \mathcal{P} \times B(H) \)-measurable mappings and \( F : \Omega \times [0, T] \times H \times l^2(H) \to H \) be a \( \mathcal{P} \times B(H) \times B(l^2(H)) \)-measurable mapping such that

\[
\|b(t, x) - b(t, \bar{x})\|_H + \|\sigma(t, x) - \sigma(t, \bar{x})\|_{\mathcal{I}(H)} \leq M_1 \|x - \bar{x}\|_H,
\]

\[
\|F(t, x, y) - F(t, \bar{x}, \bar{y})\|_H \leq M_1 \left( \|x - \bar{x}\|_H + \|y - \bar{y}\|_{\mathcal{I}(H)} \right) \quad (\text{a.s.})
\]

for some constant \( M_1 > 0 \) and any \( t \in [0, 1] \), \( x, \bar{x} \in H \) and \( y, \bar{y} \in l^2(H) \).

For given operator \( A \) satisfying Assumption 1.1 consider the following SEE

\[
x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A} [b(s, x(s)) \, ds + \sigma(s, x(s)) \cdot dW_s] \quad (2.1)
\]

and BSEE

\[
p(t) = e^{(1-t)A^*} \xi + \int_t^1 e^{(s-t)A^*} \left[ F(s, p(s), q(s)) \, ds - q(s) \cdot dW_s \right]. \quad (2.2)
\]

Now we present some preliminary results on SEE (2.1) and BSEE (2.2) which will be often used in this paper.

**Lemma 2.1** Under the above setting, we have the following assertions:

1. If \( p \in [2, \infty) \), \( b(\cdot, 0) \in L_{\mathcal{P}}^p(\Omega \times [0, 1]; H) \) and \( \sigma(\cdot, 0) \in L_{\mathcal{P}}^p(\Omega \times [0, 1]; l^2(H)) \), then SEE (2.1) has a unique solution \( x(\cdot) \) in the space \( \mathcal{C}_P([0, 1]; L^p(H)) \) for any given \( x_0 \in H \), with the \( L^p \)-estimate

\[
E \sup_{t \in [0, 1]} \|x(t)\|_H^p \leq K(M_A, M_1, p)\left[ \|x_0\|_H^p + \left( \int_0^1 \|b(t, 0)\|_H \, dt \right)^p + \left( \int_0^1 \|\sigma(t, 0)\|_{\mathcal{I}(H)}^2 \, dt \right)^{p/2} \right].
\]

2. If \( F(\cdot, 0, 0) \in L_{\mathcal{P}}^2(\Omega \times [0, 1]; H) \), then BSEE (2.2) has a unique solution \( (p(\cdot), q(\cdot)) \) in the space \( \mathcal{C}_P([0, 1]; L^2(H)) \times L_{\mathcal{P}}^2(\Omega \times [0, 1]; l^2(H)) \).
for any given $\xi \in L^2_{F_1}(\Omega; H)$, with the estimate
\[
\mathbb{E} \sup_{t \in [0,1]} \|p(t)\|_H^2 + \mathbb{E} \int_0^1 \|q(t)\|_{i_2(H)}^2 \, dt \\
\leq K(M_A, M_1) \mathbb{E} \left[ \|\xi\|_H^2 + \int_0^1 \|F(t,0,0)\|_H^2 \, dt \right].
\]
Hereafter $K(\cdot)$ is a positive constant depending only on the values in the brackets.

The above results can be found in, for example, [3, 10]. The following lemma, concerning the duality between SEE (2.1) and BSEE (2.2), can be easily derived by the Yoshida approximation (cf. [15]).

**Lemma 2.2** Under the conditions in Lemma 2.1, we have
\[
\mathbb{E} \langle x(t_2), p(t_1) \rangle_H + \mathbb{E} \int_{t_1}^{t_2} \left[ \langle b(s, x(s)), p(s) \rangle_H + \langle \sigma(s, x(s)), q(s) \rangle_{i_2(H)} \right] \, ds \\
= \mathbb{E} \langle x(t_2), p(t_2) \rangle_H + \mathbb{E} \int_{t_1}^{t_2} \langle x(s), F(s, p(s), q(s)) \rangle_H \, ds
\]
for any $0 \leq t_1 \leq t_2 \leq 1$.

### 3 Operator-valued BSDEs: well-posedness and properties

In this section, we study the following operator-valued BSDE (OBSDE)
\[
dP(t) = - \{ A^* P(t) + P(t) A + A^*_1(t) P(t) + P(t) A^*_2(t) \} \, dt + \{ Q(t) \cdot dW_t \}
\]
\[
P(1) = P_1
\]
with the unknown processes $P(\cdot)$ and $Q(\cdot)$, where $A$ satisfies Assumption 1.1. $A^*_1(\cdot)$ and $C(\cdot)$ are given coefficients, and
\[
\text{Tr}[C^* PC + QC + C^* Q] = \sum_{i \in \mathbb{N}} [C^*_i PC_i + Q_i C_i + C^*_i Q_i](t).
\]
We call the pair $(G, P_1)$ the **input** of OBSDE (3.1). Such a equation will be used to characterize the second order adjoint process of the controlled system in the next section. We make the following assumption.

**Assumption 3.1** $A_2(\cdot) \in L^\infty_{FS}(\Omega \times [0,1]; B(H))$. $C(\cdot) = (C_i(\cdot) : i \in \mathbb{N})$ with $C_i(\cdot) \in L^\infty_{FS}(\Omega \times [0,1]; B(H))$. Assume
\[
M_2 := \text{ess sup}_{t, \omega} \left\{ \|A_2(t, \omega)\|_{i_2(B(H))}^2, \sum_{i \in \mathbb{N}} \|C_i(t, \omega)\|_{i_2(B(H))}^2 \right\} < \infty.
\]
3.1 The well-posedness

The solvability theory of $B(H)$-valued BSDEs is still far from complete. A first remarkable work on such a subject was found in Guatteri-Tessitore [8] where, inspired by the notion of “strong solution” for PDEs (cf. [2]), the authors proposed for an OBSDE the concept of generalized solution which only involved the first unknown $P(\cdot)$, and obtained the corresponding existence-uniqueness result. Their approach based on the solvability of (3.1) in the Hilbert space $B_2(H)$ of all Hilbert-Schmidt operators from $H$ to itself, see Theorem 5.4 in [8].

Following the spirit of Guatteri-Tessitore [8], we give the following definition.

Definition 3.2 (generalized solution) $P(\cdot)$ is called a generalized solution to OBSDE (3.1) in $L^2_{F_1}S(\Omega \times [0, 1]; B(H))$ if there exists a sequence $(P^n, Q^n, G^n)$ such that

1) $P^n(1) \in L^2_{F_1}(\Omega; B_2(H))$, $G^n \in L^2_{F_1}(\Omega \times [0, 1]; B_2(H))$, and there exists a constant $\lambda \geq 1$ such that

$$\|P^n(1)\|_{B(H)} \leq \lambda \|P_1\|_{B(H)} \quad \text{and} \quad \|G^n(t)\|_{B(H)} \leq \lambda \|G(t)\|_{B(H)} \quad (a.s.)$$

for any $n \in \mathbb{N}$ and $t \in [0, 1]$.

2) $(P^n, Q^n)$ is a mild solution to OBSDE (3.1) with the input $(G^n, P^n(1))$, that is, for all $t \in [0, 1],

$$P^n(t) = e^{(1-t)A^*}P^n(1)e^{(1-t)A} + \int_t^1 e^{(s-t)A^*}[A_1^*P^n + P^nA_2 + G^n](s)e^{(s-t)A} \, ds$$

$$+ \int_t^1 e^{(s-t)A^*}Tr[C^*P^nC + Q^nC + C^*Q^n](s)e^{(s-t)A} \, ds$$

$$+ \int_t^1 e^{(s-t)A^*}Q^n(s)e^{(s-t)A} \cdot dW_s, \quad (a.s.) \quad (3.2)$$

3) for any $x, y \in H$ and $t \in [0, 1],

$$\langle x, G^n(t)y \rangle_H \to \langle x, G(t)y \rangle_H \quad \text{and} \quad \langle x, P^n(t)y \rangle_H \to \langle x, P(t)y \rangle_H \quad (a.s.)$$

In the above definition only the process $P(\cdot)$ is characterized. Nevertheless, this is sufficient and even natural for the optimal control theory since $Q(\cdot)$ is not involved in the formulation of the SMP (see Theorem 4.1). For more detailed account we refer to Remark 6.3 in [8]. Now we give the following well-posedness result on the generalized solution to OBSDE (3.1).

Theorem 3.3 Let Assumptions [3.1] and [3.7] be satisfied. Suppose $P_1 \in L^2_{F_1}(\Omega; B(H))$ and $G \in L^2_{F_1}(\Omega \times [0, 1]; B(H))$. Then

i) there exists a unique generalized solution $P(\cdot)$ to OBSDE (3.1).
ii) for each $\tau \in [0,1]$ and any $\xi, \zeta \in L^2_H(\Omega; H)$,
\[ \langle \xi, P(\tau)\zeta \rangle_H = E^{F_\tau} \langle y^{\tau, \xi}(1), P_1 y^{\tau, \zeta}(1) \rangle_H \]
\[ + E^{F_\tau} \int_\tau^1 \langle y^{\tau, \xi}(t), G(t) y^{\tau, \zeta}(t) \rangle_H dt \quad (a.s.) \tag{3.3} \]
with $y^{\tau, \xi}$ (similarly for $y^{\tau, \zeta}$) being the solution to equation
\[ y^{\tau, \xi}(t) = e^{(t-\tau)A} \xi + \int_\tau^t e^{(t-s)A} A_4(s)y^{\tau, \xi}(s)ds \]
\[ + \int_\tau^t e^{(t-s)A} C(s)y^{\tau, \xi}(s)\cdot dW_s, \quad t \in [\tau, 1]; \tag{3.4} \]
iii) for each $\tau \in [0,1]$, it holds almost surely that
\[ \|P(\tau)\|_{B(\mathcal{H})}^2 \leq K(M_A, M_2) E^{F_\tau} \left[ \|P_1\|_{B(\mathcal{H})}^2 + \int_0^1 \|G(t)\|_{B(\mathcal{H})}^2 dt \right] =: \Lambda_\tau; \tag{3.5} \]
iv) for each $\tau \in [0,1]$ and any $\xi, \zeta \in L^4_{\mathcal{F}_\tau}(\Omega; H)$,
\[ \lim_{|s-t| \to 0} E\langle (P(s)-P(t))\zeta \rangle_H = 0 \quad \text{with} \ s, t \in [\tau, 1]. \]

The proof of the above theorem depends on the following result in which the ODBSE (6.1) is considered in the Hilbert space $B_2(\mathcal{H})$. Such a lemma is similar to Theorems 5.4 and 5.5 in Guatteri-Tessitore [3].

**Lemma 3.4** In addition to the conditions in Theorem 3.3 suppose $P_1 \in L^2_{\mathcal{F}_\tau}(\Omega; B_2(\mathcal{H}))$ and $G \in L^2_{\mathcal{F}_\tau}(\Omega \times [0,1]; B_2(\mathcal{H}))$. Then there exists a unique mild solution $(P(\cdot), Q(\cdot))$ in the space
\[ C_P([0,1]; L^2(\Omega; B_2(\mathcal{H}))) \times L^2_{\mathcal{F}_\tau}(\Omega \times [0,1]; l^2(B_2(\mathcal{H}))), \]

that is, $(P, Q)$ satisfies relation (3.2) instead of $(P^n, Q^n)$. Moreover, the assertions (ii) and (iii) in Theorem 3.3 hold true with respect to such $P(\cdot)$.

**Proof**. The existence and uniqueness of the (mild) solution $(P(\cdot), Q(\cdot))$ were given by Theorem 5.4 in [3]. Next, we indicate that Lemma 2.1 implies
\[ \|y^{\tau, \xi}(\tau)\|^4_H \leq K(M_A, M_2) \|\xi\|^4_H \quad (a.s.) \tag{3.6} \]
for any $\tau \in [0,1]$ and $\xi \in L^2_{\mathcal{F}_\tau}(\Omega; H)$. In the case of $A \in B(\mathcal{H})$, one can show the relation (3.6) by the generalized Itô formula (see e.g. [3] Theorem 4.17). Then the general case can be obtained by a standard argument of the Yoshida approximation. The inequality (3.5) follows from (3.3) and (3.6). □

**Proof of Theorem 3.3**. For the sake of convenience, we write the right-hand side of equality (3.3) as $T_\tau(\xi, \zeta; G, P_1)$. Then it follows from (3.6) and Young’s inequality that for any $\xi, \zeta \in L^2_{\mathcal{F}_\tau}(\Omega; H)$,
\[ |T_\tau(\xi, \zeta; G, P_1)| \leq \sqrt{\Lambda_\tau} \|\xi\|_H \|\zeta\|_H \quad (a.s.). \]
We shall always select an RCLL version of the martingale \((\Lambda_\tau; \tau \in [0,1])\). Fix arbitrary \(\tau \in [0,1]\) and take a standard complete orthonormal basis \(\{e^H_i\}\) in \(H\). Then there is a set of full probability \(\Omega_1 \subset \Omega\) such that for each \(\omega \in \Omega_1\),

\[
\left| [T_\tau(e^H_i, e^H_j; G, P_1)](\omega) \right| \leq \sqrt{\Lambda_\tau(\omega)}, \quad \forall i, j \in \mathbb{N}.
\]

Hence, from the Riesz representation theorem, there is a unique \(P(\tau, \omega) \in B(H)\) for each \(\omega \in \Omega_1\) such that

\[
\langle e^H_i, P(\tau, \omega) e^H_j \rangle_H = [T_\tau(e^H_i, e^H_j; G, P_1)](\omega), \quad \forall i, j \in \mathbb{N},
\]

and

\[
\|P(\tau, \omega)\|_{B(H)} \leq \sqrt{\Lambda_\tau(\omega)}, \quad \forall \omega \in \Omega_1.
\]

It is easy to check that \(\langle x, P(\tau) y \rangle_H = [T_\tau(x, y; G, P_1)](a.s.)\) for any \(x, y \in H\); and furthermore, for any simple \(H\)-valued \(F_\tau\)-measurable random variables \(\xi, \zeta\), we have

\[
\langle \xi, P(\tau) \zeta \rangle_H = [T_\tau(\xi, \zeta; G, P_1)], \quad (a.s.)
\]

Then by a standard argument of approximation, the above relation holds true for any \(\xi, \zeta \in L^4_{F_\tau}(\Omega; H)\).

We claim: the operator-valued process \(P(\cdot)\) defined in (3.7) is the desired generalized solution. Indeed, for each \(n \in \mathbb{N}\), we introduce the finite dimensional projection \(\Pi_n: H \rightarrow H: x \rightarrow \sum_{i=1}^n \langle x, e^H_i \rangle_H e^H_i\), define

\[
G^n(t, \omega) := \Pi_n G(t, \omega) \Pi_n \quad \text{and} \quad P^n_1(\omega) := \Pi_n P_1(\omega) \Pi_n
\]

and find from Lemma 3.4 the (unique) mild solution \((P^n, Q^n)\) to OBSDE (3.1) with the input \((G^n, P^n_1)\). Obviously, the conditions (1) and (2) in Definition 3.2 are satisfied. Noticing the construction of \(G^n\), it remains to show \(\langle x, P^n(t) y \rangle_H \rightarrow \langle x, P(t) y \rangle_H (a.s.)\) for any \(x, y \in H\) and \(t \in [0,1]\). It follows from Lemma 3.4 that \(\langle x, P^n(\tau) y \rangle_H = T_\tau(x, y; G^n, P^n_1) (a.s.)\). By the Lebesgue dominated convergence theorem, we have

\[
T_\tau(x, y; G^n, P^n_1) \rightarrow T_\tau(x, y; G, P_1) \quad (a.s.)
\]

This implies \(\langle x, P^n(\tau) y \rangle_H \rightarrow \langle x, P(\tau) y \rangle_H\). The claim is proved.

On the other hand, from Lemma 3.4 Definition 3.2 and the Lebesgue dominated convergence theorem, the relations (3.3) and (3.5) hold true for every generalized solution to OBSDE (3.1), which yields assertions (ii) and (iii). Moreover, the relation (3.3) implies the uniqueness of the generalized solution. Thus the assertion (i) is proved.

It remains to prove the assertion (iv). Fix arbitrary \(\tau \in [0,1]\). Without loss of generality, we assume \(t < s\). Then for any \(\xi, \zeta \in L^4_{F_\tau}(\Omega; H)\) we have (recall
First, it follows from (3.6) and Young’s inequality that
\[ |I_3|^2 \leq K(M_A, M_2) \mathbb{E} \int_t^s \|G(r)\|_H^2 \, dr \cdot \sqrt{\mathbb{E} \|\xi\|_H^4} \cdot \sqrt{\mathbb{E} \|\zeta - y(t,\xi)\|_H^4} \to 0, \] as \( |s - t| \to 0. \) On the other hand, the trajectory of \( y(t,\xi)(\cdot) \) is continuous in \( H \), which along with (3.6) and the Lebesgue dominated convergence theorem yields
\[ |I_2|^2 \leq K(M_A, M_2, G, P_1) \left( \frac{1}{2} \sqrt{\mathbb{E} \|\xi\|_H^4} \sqrt{\mathbb{E} \|\zeta - y(t,\xi)\|_H^4} \right) \to 0, \] as \( |s - t| \to 0. \)

Similarly, we can show
\[ |I_1| \to 0, \] as \( |s - t| \to 0. \)

Therefore, we have for any \( \xi, \zeta \in L_2^\beta(\Omega; H), \)
\[ \lim_{|s - t| \to 0} \mathbb{E} \langle \xi, (P(s) - P(t))\zeta \rangle_H = 0, \] with \( s, t \in [\tau, 1]. \)

The assertion (iv) is proved. This concludes the theorem. \( \square \)

3.2 A basic property of the generalized solution

The absence of \( Q(\cdot) \) in the definition of generalized solution brings a new difficulty in our derivation of the stochastic maximum principle compared with the traditional duality approach (cf. [13]). The following result will play a key role to overcome the difficulty.

Proposition 3.5 Let the conditions in Theorem 3.3 be satisfied. For any \( \tau \in [0, 1], \vartheta, \theta \in L_2^\beta(\Omega; l^2(H)) \) and \( \varepsilon \in (0, 1 - \tau), \) let \( y_{\varepsilon,\vartheta}^{\tau,\theta}(\cdot) \) be the mild solution to equation
\[
y_{\varepsilon,\vartheta}^{\tau,\theta}(t) = \int_\tau^t e^{(t-s)A} A_2(s) y_{\varepsilon,\vartheta}^{\tau,\theta}(s) \, ds
+ \int_\tau^t e^{(t-s)A} \left[ C(s)y_{\varepsilon,\vartheta}^{\tau,\theta}(s) + \varepsilon^{-\frac{1}{2}} 1_{[\tau,\tau+\varepsilon]} \right] \cdot dW_s.
\]
Then we have
\[
E(\vartheta, P(\tau)\vartheta)_{\mathcal{F}_1(H)} = \lim_{\varepsilon \downarrow 0} E\left[ \left\langle y_{\varepsilon}^{\tau, \vartheta}(1), P_{1} y_{\varepsilon}^{\tau, \vartheta}(1) \right\rangle_H + \int_\tau^1 \left\langle y_{\varepsilon}^{\tau, \vartheta}(t), G(t) y_{\varepsilon}^{\tau, \vartheta}(t) \right\rangle_H \, dt \right].
\]

Remark 3.6 In the $B_2(H)$-framework, the above result can be easily proved by the Itô formula (recalling Lemma 3.7); however, this approach fails in the general $B(H)$-framework due to the absence of $Q(\cdot)$ in the generalized solution. Besides, an approximation argument by using the sequence $(P^n, Q^n, G^n)$ seems also difficult to prove the above result.

The previous proposition follows from several lemmas. For the sake of convenience, we denote
\[
T_{\varepsilon}^\tau(\vartheta, \theta) = E\left[ \left\langle y_{\varepsilon}^{\tau, \vartheta}(1), P_{1} y_{\varepsilon}^{\tau, \vartheta}(1) \right\rangle_H + \int_\tau^1 \left\langle y_{\varepsilon}^{\tau, \vartheta}(t), G(t) y_{\varepsilon}^{\tau, \vartheta}(t) \right\rangle_H \, dt \right].
\]
Define
\[
B_A(\tau) := \left\{ \xi \in L^2_{\mathcal{F}_\tau}(\Omega; H) : \xi(\omega) \in D(A) \quad \text{and} \quad \|\xi\|_A := \sup_{\omega} (\|A\xi\|_H + \|\xi\|_H) < \infty \right\}
\]
which is dense in $L^2_{\mathcal{F}_\tau}(\Omega; H)$. Set $e_i = (0, \ldots, 0, 1, 0, \ldots)$ with only the $i$-th element nonzero. Then $\xi e_i \in L^2_{\mathcal{F}_\tau}(\Omega; l^2(H))$ for any $\xi \in L^2_{\mathcal{F}_\tau}(\Omega; H)$ and $i \in \mathbb{N}$. Moreover, we define
\[
\xi^i(t) := \varepsilon^{-\frac{1}{2}}(W^1_t - W^\varepsilon_t)\xi \quad \text{and} \quad \zeta^i(t) := \varepsilon^{-\frac{1}{2}}(W^1_t - W^\varepsilon_t)\zeta.
\]

Lemma 3.7 For any $\tau \in [0, 1]$, $\xi \in B_A(\tau)$ and $i \in \mathbb{N}$, we have
\[
E\|y_{\varepsilon}^{\tau, \xi^i}(\tau + \varepsilon) - \xi^i(\tau + \varepsilon)\|_H^4 \leq K\varepsilon^2 \|\xi\|_A^4.
\]
Proof. For simplicity, we set $y^i(t) := y_{\varepsilon}^{\tau, \xi^i}(t)$. Then for $t \in [\tau, \tau + \varepsilon]$,
\[
(y^i - \xi^i)(t) = \int_\tau^t e^{(t-s)A} \left[ A_1(s)(y^i - \xi^i)(s) + (A + A_2(s))\xi^i(s) \right] \, ds + \int_\tau^t e^{(t-s)A} [C(s)(y^i - \xi^i)(s) + C(s)\xi^i(s)] \cdot dW_t.
\]
Then by Lemma 2.4, we have
\[
E\|y^i - \xi^i(\tau + \varepsilon)\|_H^4 \leq K E\left( \int_\tau^{\tau + \varepsilon}\|\xi^i(t)\|_A^2 \, dt \right)^2 \leq K\varepsilon^2 \|\xi\|_A^4.
\]
The lemma is proved.  \blacksquare
Notice the fact that for any \( \xi, \zeta \in B_A(\tau) \),
\[
T^{\varepsilon}_i(\xi_{i_1}, \xi_{i_2}) = \mathbb{E} \int_{\tau}^{\tau+\varepsilon} \langle y^{\varepsilon, \xi_{i_1}}(t), G(t)y^{\varepsilon, \xi_{i_2}}(t) \rangle_H \, dt \\
+ \mathbb{E} \langle y^{\varepsilon, \xi_{i_1}}(\tau+\varepsilon), P(\tau+\varepsilon)y^{\varepsilon, \xi_{i_2}}(\tau+\varepsilon) \rangle_H \\
=: J_1 + J_2.
\]

Now we let \( \varepsilon \) tend to 0. On the one hand, one can show that the term \( J_1 \) tends to 0 similarly as in (3.8); on the other hand, by means of Lemma 3.7, the term \( J_2 \) should tend to the same limit as \( \mathbb{E} \langle \xi^i(\tau+\varepsilon), P(\tau+\varepsilon)\zeta^j(\tau+\varepsilon) \rangle_H \). Indeed, we have

**Lemma 3.8** For any \( \tau \in [0, 1] \), \( \xi, \zeta \in B_A(\tau) \) and \( i, j \in \mathbb{N} \), we have
\[
\lim_{\varepsilon \downarrow 0} \{ \mathbb{E} \langle \xi^i(\tau+\varepsilon), P(\tau+\varepsilon)\zeta^j(\tau+\varepsilon) \rangle_H - T^{\varepsilon}_i(\xi_{i_1}, \xi_{i_2}) \} = 0.
\]

**Proof.** It is sufficient to show
\[
\lim_{\varepsilon \downarrow 0} \{ \mathbb{E} \langle \xi^i(\tau+\varepsilon), P(\tau+\varepsilon)\zeta^j(\tau+\varepsilon) \rangle_H - J_2 \} = 0.
\]
Indeed, from Theorem 3.8, Lemmas 2.1 and 3.7, we have
\[
|\mathbb{E} \langle \xi^i(\tau+\varepsilon), P(\tau+\varepsilon)\zeta^j(\tau+\varepsilon) \rangle_H - J_2|
\leq
K \left( \mathbb{E} \|\xi\|_H^4 \right)^{\frac{1}{4}} \left( \mathbb{E} \|y^{\varepsilon, \xi_{i_1}}(\tau+\varepsilon) - \zeta^j(\tau+\varepsilon)\|_H^4 \right)^{\frac{1}{4}}
+ K \left( \mathbb{E} \|\xi\|_H^4 \right)^{\frac{1}{4}} \left( \mathbb{E} \|y^{\varepsilon, \xi_{i_1}}(\tau+\varepsilon) - \xi^i(\tau+\varepsilon)\|_H^4 \right)^{\frac{1}{4}}
\rightarrow 0, \quad \text{as } \varepsilon \downarrow 0.
\]
This concludes this lemma. ■

On the other hand, from the continuity of \( P(\cdot) \) we have the following

**Lemma 3.9** For any \( \tau \in [0, 1] \), \( \xi, \zeta \in B_A(\tau) \) and \( i, j \in \mathbb{N} \), we have
\[
\lim_{\varepsilon \downarrow 0} \mathbb{E} \langle \xi^i(\tau+\varepsilon), P(\tau+\varepsilon)\zeta^j(\tau+\varepsilon) \rangle_H = \mathbb{E} \langle \xi, P(\tau)\zeta \rangle_H.
\]

**Proof.** It is easy to see
\[
\mathbb{E} \langle \xi^i(\tau+\varepsilon), P(\tau)\zeta^j(\tau+\varepsilon) \rangle_H = \mathbb{E} \langle \xi, P(\tau)\zeta \rangle_H.
\]
Thus we need prove
\[
\lim_{\varepsilon \downarrow 0} \mathbb{E} \langle \xi^i(\tau+\varepsilon), \{P(\tau+\varepsilon) - P(\tau)\}\zeta^j(\tau+\varepsilon) \rangle_H = 0.
\]
It follows from (3.9), the boundedness of \( \xi, \zeta \), and Doob’s martingale inequality (cf. [4]) that (recall (3.3))
\[
\|\xi, \{P(\tau+\varepsilon) - P(\tau)\}\zeta\|_H^2 \leq 4 \left( \max_{\varepsilon \in [0, 1]} A_1 \right) \|\xi\|_H^2 \|\zeta\|_H^2 \in L^1(\Omega),
\]

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Note that
\[
\langle \xi^i(\tau + \varepsilon), [P(\tau + \varepsilon) - P(\tau)]\zeta^j(\tau + \varepsilon) \rangle_H
= \varepsilon^{-1}(W^i_{\tau+\varepsilon} - W^i_\tau)(W^j_{\tau+\varepsilon} - W^j_\tau) \langle \xi, [P(\tau + \varepsilon) - P(\tau)]\zeta \rangle_H
\]
Then from Theorem 3.3(iv) and the Lebesgue dominated convergence theorem, we have
\[
\begin{align*}
|E \langle \xi^i(\tau + \varepsilon), [P(\tau + \varepsilon) - P(\tau)]\zeta^j(\tau + \varepsilon) \rangle_H |^2 \\
\leq \varepsilon^{-2} \cdot E((W^i_{\tau+\varepsilon} - W^i_\tau)^2(W^j_{\tau+\varepsilon} - W^j_\tau)^2) \\
\cdot E|\langle \xi, [P(\tau + \varepsilon) - P(\tau)]\zeta \rangle_H |^2
\end{align*}
\]
\[
\leq 3E|\langle \xi, [P(\tau + \varepsilon) - P(\tau)]\zeta \rangle_H |^2 \to 0, \quad \text{as } \varepsilon \downarrow 0.
\]

The lemma is proved. ■

Now we are in a position to complete the proof of Proposition 3.5.

**Proof of Proposition 3.5.** Fix any \( \vartheta, \theta \in L^2_{\mathcal{F}_T}(\Omega; L^2(H)) \). For arbitrary \( \delta > 0 \), we can find (from the density) a large \( m_\delta \) and \( \{\xi_i, \zeta_i : i = 1, \ldots, m_\delta\} \subset B_A(\tau) \) such that
\[
\begin{align*}
\vartheta_{m_\delta} := (\xi_1, \ldots, \xi_{m_\delta}), \\
\theta_{m_\delta} := (\zeta_1, \ldots, \zeta_{m_\delta}), \\
E\|\vartheta - \vartheta_{m_\delta}\|_{L^2(H)}^4 + E\|\theta - \theta_{m_\delta}\|_{L^2(H)}^4 < \delta^4.
\end{align*}
\]
Then we have
\[
\begin{align*}
E \langle \vartheta, P(\tau)\theta \rangle_{L^2(H)} - E \langle \vartheta_{m_\delta}, P(\tau)\theta_{m_\delta} \rangle_{L^2(H)} < K(\vartheta, \theta, P) \delta, \\
|T^\varepsilon_\tau(\vartheta, \theta) - T^\varepsilon_\tau(\vartheta_{m_\delta}, \theta_{m_\delta})| < K(\vartheta, \theta, P) \delta.
\end{align*}
\]
On the other hand, from Lemmas 3.8 and 3.9 one can easily check that
\[
E \langle \vartheta_{m_\delta}, P(\tau)\theta_{m_\delta} \rangle_{L^2(H)} = \lim_{\varepsilon \downarrow 0} T^\varepsilon_\tau(\vartheta_{m_\delta}, \theta_{m_\delta}).
\]
Thus we have
\[
\limsup_{\varepsilon \downarrow 0} \left| E \langle \vartheta, P(\tau)\theta \rangle_{L^2(H)} - T^\varepsilon_\tau(\vartheta, \theta) \right| < K(\vartheta, \theta, P) \delta.
\]
From the arbitrariness of \( \delta \), we conclude the proposition. ■

### 4 The stochastic maximum principle and its proof

#### 4.1 The statement of the main theorem

Now we are in a position to formulate the stochastic maximum principle for optimal controls. Define the *Hamiltonian*
\[
\mathcal{H} : \Omega \times [0,1] \times H \times U \times H \times L^2(H) \to \mathbb{R},
\]
as the form
\[
\mathcal{H}(t,x,u,p,q) := l(t,x,u) \langle p, f(t,x,u) \rangle_H + \langle q, g(t,x,u) \rangle_{L^2(H)}, \quad (4.1)
\]
then our main result can be stated as follows.
Theorem 4.1 (stochastic maximum principle) Let Assumptions [14] and [15] be satisfied, \( \bar{x}(\cdot) \) be the state process with respect to an optimal control \( \bar{u}(\cdot) \). Then for each \( u \in U \), the variational inequality

\[
0 \leq \mathcal{H}(\tau, \bar{x}(\tau), u, p(\tau), q(\tau)) - \mathcal{H}(\tau, \bar{x}(\tau), \bar{u}(\tau), p(\tau), q(\tau)) + \frac{1}{2} \langle g(\tau, \bar{x}(\tau), u) - g(\tau, \bar{x}(\tau), \bar{u}(\tau)), P(\tau)[g(\tau, \bar{x}(t), u) - g(\tau, \bar{x}(\tau), \bar{u}(\tau))] \rangle_{\mathcal{P}(\mathcal{H})}
\]

holds for a.e. \((\tau, \omega) \in [0, 1] \times \Omega\), where \((p(\cdot), q(\cdot))\) is the solution to BSEE (2.2) with

\[
F(t, p, q) = \mathcal{H}_x(t, \bar{x}(t), \bar{u}(t), p, q), \quad \xi = h_x(\bar{x}(1)),
\]

and \( P(\cdot) \) is the generalized solution to OBSDE (3.1) with

\[
A_t(t) = f_x(t, \bar{x}(t), \bar{u}(t)), \quad C(t) = g_x(t, \bar{x}(t), \bar{u}(t)), \quad G(t) = \mathcal{H}_{xx}(t, \bar{x}(t), \bar{u}(t), p(t), q(t)), \quad P_1 = h_{xx}(\bar{x}(1)).
\]

4.2 Proof of the main theorem

The proof is divided into the following three steps.

Step 1. The spike variation and second-order expansion.

The approach in this step is quite standard (cf. [13]). Recall that \( \bar{x}(\cdot) \) is the state process with respect to an optimal control \( \bar{u}(\cdot) \). We construct a perturbed admissible control in the following way

\[
u^\varepsilon(t) := \begin{cases} u, & \text{if } t \in [\tau, \tau + \varepsilon], \\ \bar{u}(t), & \text{otherwise}, \end{cases}
\]

with fixed \( \tau \in [0, 1] \), sufficiently small positive \( \varepsilon \), and an arbitrary \( U \)-valued \( \mathcal{F}_\tau \)-measurable random variable \( u \) satisfying \( \mathbb{E}[|u|_U^4] < \infty \).

Let \( x^\varepsilon(\cdot) \) be the state process with respect to control \( u^\varepsilon(\cdot) \). For the sake of convenience, we denote for \( \varphi = f, g, l, f_x, g_x, l_x, f_{xx}, g_{xx}, l_{xx} \),

\[
\tilde{\varphi}(t) := \varphi(t, \bar{x}(t), \bar{u}(t)),
\]

\[
\varphi^\Delta(t) := \varphi(t, \chi(t), u^\varepsilon(t)) - \tilde{\varphi}(t, \bar{x}(t), \bar{u}(t)),
\]

Let \( x_1(\cdot) \) and \( x_2(\cdot) \) be the solutions respectively to

\[
x_1(t) = \int_0^t e^{(t-s)A} \tilde{f}_x(s)x_1(s) \, ds + \int_0^t e^{(t-s)A} \tilde{g}_x(s)x_1(s) + g(\Delta(s)) \cdot dW_s,
\]

\[
x_2(t) = \int_0^t e^{(t-s)A} \left[ \tilde{f}_x(s)x_2(s) + \frac{1}{2} \tilde{f}_{xx}(s)(x_1 \otimes x_1)(s) + f(\Delta(s)) \right] \, ds
\]

\[
+ \int_0^t e^{(t-s)A} \left[ \tilde{g}_x(s)x_2(s) + \frac{1}{2} \tilde{g}_{xx}(s)(x_1 \otimes x_1)(s) + g(\Delta(s))x_1(s) \right] \cdot dW_s.
\]
It follows from Lemma 2.1 that for all $t \in [0, 1]$,
\[
\begin{align*}
\epsilon^{-2}E \|x_1(t)\|^4_H + \epsilon^{-1}E \|x_1(t)\|^2_H + \epsilon^{-2}E \|x_2(t)\|^2_H \leq K, \\
\epsilon^{-2}E \|\bar{x}(t) - \bar{x}(t)\|^2_H + \epsilon^{-1}E \|\bar{x}(t) - \bar{x}(t)\|^2_H \leq K, \\
\epsilon^{-2}E \|\bar{x}(t) - \bar{x}(t) - x_1(t)\|^2_H \leq K, \\
\epsilon^{-2}E \|\bar{x}(t) - \bar{x}(t) - x_1(t) - x_2(t)\|^2_H = o(1).
\end{align*}
\] (4.2)
This along with the fact
\[
J(u^\varepsilon(\cdot)) - J(\bar{u}(\cdot)) \geq 0
\]
yields (for details, we refer to [16] or [5])
\[
o(\varepsilon) \leq \mathbb{E} \int_0^1 \left[ \int \left( t_1(t) + \langle \bar{\bar{H}}(t), x_1(t) + x_2(t) \rangle_H + \frac{1}{2} \langle x_1(t), \bar{\bar{H}}x(t)x_1(t) \rangle_H \right) dt \\
+ \mathbb{E} \langle \bar{h}_x(\bar{x}(1)), x_1(1) + x_2(1) \rangle_H + \frac{1}{2} \langle x_1(1), \bar{h}_xx(\bar{x}(1))x_1(1) \rangle_H \right] dt.
\]
**Step 2. First-order duality analysis.**
It follows from Lemma 2.1(2) that BSEE (2.2.2) has a unique solution $(p(\cdot), q(\cdot))$ in this situation. Recalling the Hamiltonian (4.1), and from Lemma 2.2 we have
\[
\mathbb{E} \int_0^1 \left[ \int \left( t_1(t) + \langle \bar{\bar{H}}(t), x_1(t) + x_2(t) \rangle_H + \frac{1}{2} \langle x_1(t), \bar{\bar{H}}x(t)x_1(t) \rangle_H \right) dt \\
+ \mathbb{E} \langle \bar{h}_x(\bar{x}(1)), x_1(1) + x_2(1) \rangle_H + \frac{1}{2} \langle x_1(1), \bar{h}_xx(\bar{x}(1))x_1(1) \rangle_H \right] dt.
\]
Hence, we get
\[
o(1) \leq \epsilon^{-1}E \int_\tau^{\tau+\varepsilon} [\mathcal{H}(t, \bar{x}(t), u, p(t), q(t)) - \mathcal{H}(t, \bar{x}(t), \bar{u}(t), p(t), q(t))] dt \\
+ \frac{1}{2} \epsilon^{-2}E \int_0^1 \langle x_1(t), \mathcal{H}_x(t, \bar{x}(t), \bar{u}(t), p(t), q(t))x_1(t) \rangle_H dt \\
+ \frac{1}{2} \epsilon^{-1}E \langle x_1(1), \mathcal{H}_{xx}(\bar{x}(1))x_1(1) \rangle_H.
\] (4.3)
**Step 3. Second-order duality analysis and completion of the proof.**
This is the key step in the proof. From Theorem 3.3 it is easy to check that OBSDE (3.1) has a unique generalized solution $P(\cdot)$ in this situation. Now we introduce the following equation
\[
y^\varepsilon(t) = \int_\tau^t e^{(t-s)A} A_s g^\varepsilon(s) ds \\
+ \int_\tau^t e^{(t-s)A} |C(s)| g^\varepsilon(s) + \epsilon^{-\frac{1}{2}}1_[\tau, \tau+\varepsilon] g^A(\tau) \cdot dW_s, \ t \in [\tau, 1].
\]
Then we have
Lemma 4.2 For a.e. $\tau \in [0,1]$, we have

$$\lim_{\varepsilon \downarrow 0} \sup_{t \in [\tau,1]} \mathbb{E}\|e^{-\frac{\varepsilon}{2}} x_1(t) - y^\varepsilon(t)\|_H^4 = 0.$$ 

Proof. By Lemma 2.1, we have for each $\tau \in [0,1]$,

$$\sup_{t \in [\tau,1]} \mathbb{E}\|e^{-\frac{\varepsilon}{2}} x_1(t) - y^\varepsilon(t)\|_H^4 \leq K \cdot \frac{1}{\varepsilon} \int_{\tau}^{\tau + \varepsilon} \mathbb{E}\|g^\Delta(t) - g^\Delta(\tau)\|_{\mathcal{P}(H)}^4 \, dt.$$ 

From the Lebesgue differentiation theorem, we have for each $X \in L^4_{\mathcal{P}}(\Omega; l^2(H))$,

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{\tau}^{\tau + \varepsilon} \mathbb{E}\|g^\Delta(t) - X\|_{\mathcal{P}(H)}^4 \, dt = \mathbb{E}\|g^\Delta(\tau) - X\|_{\mathcal{P}(H)}^4, \text{ for a.e. } \tau \in [0,1].$$ 

Since $L^4_{\mathcal{P}}(\Omega; l^2(H))$ is separable, let $X$ run through a countable density subset $Q$ in $L^4_{\mathcal{P}}(\Omega; l^2(H))$, and denote

$$E := \bigcup E_X := \bigcup \{ \tau : \text{the above relation does not hold for } X \}.$$ 

Then we have $\text{meas}(E) = 0$. For arbitrary positive $\eta$, take an $X \in Q$ such that

$$\mathbb{E}\|g^\Delta(\tau) - X\|_{\mathcal{P}(H)}^4 < \eta.$$ 

then for each $\tau \in [0,1) \setminus E$,

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{\tau}^{\tau + \varepsilon} \mathbb{E}\|g^\Delta(t) - g^\Delta(\tau)\|_{\mathcal{P}(H)}^4 \, dt$$

$$\leq \lim_{\varepsilon \downarrow 0} \frac{8}{\varepsilon} \int_{\tau}^{\tau + \varepsilon} \mathbb{E}\|g^\Delta(t) - X\|_{\mathcal{P}(H)}^4 \, dt + 8\mathbb{E}\|g^\Delta(\tau) - X\|_{\mathcal{P}(H)}^4$$

$$\leq 16\mathbb{E}\|g^\Delta(\tau) - X\|_{\mathcal{P}(H)}^4 < 16\eta.$$ 

From the arbitrariness of $\eta$, we conclude this lemma. ■

From the the above lemma, we have

$$\varepsilon^{-1} \mathbb{E} \int_0^1 \langle x_1(t), G(t) x_1(t) \rangle_H \, dt + \varepsilon^{-1} \mathbb{E} \langle x_1(1), P_1 x_1(1) \rangle_H$$

$$= o(1) + \mathbb{E} \int_\tau^1 \langle y^\varepsilon(t), G(t) y^\varepsilon(t) \rangle_H \, dt + \mathbb{E} \langle y^\varepsilon(1), P_1 y^\varepsilon(1) \rangle_H$$

$$, \forall \tau \in [0,1) \setminus E.$$ 

Keeping in mind the above relation, and applying Proposition 3.3, we conclude for each $\tau \in [0,1) \setminus E$,

$$\mathbb{E} \langle g^\Delta(\tau), P(\tau) g^\Delta(\tau) \rangle_{\mathcal{P}(H)}$$

$$= \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \left\{ \mathbb{E} \int_0^1 \langle x_1(t), G(t) x_1(t) \rangle_H \, dt + \mathbb{E} \langle x_1(1), P_1 x_1(1) \rangle_H \right\}.$$ 

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This along with (4.3) and the Lebesgue differentiation theorem yields for each \( u \in U \),

\[
0 \leq \mathbb{E} [\mathcal{H}(\tau, \bar{x}(\tau), u, p(\tau), q(\tau)) - \mathcal{H}(\tau, \bar{x}(\tau), \bar{u}(\tau), p(\tau), q(\tau))] \\
+ \frac{1}{2} \mathbb{E} \left< g^\Delta(\tau), P(\tau)g^\Delta(\tau) \right>_{L^2(H)}, \quad \text{a.e. } \tau \in [0, 1).
\]

Therefore, the desired variational inequality follows from a standard argument (cf. [11]). This completes the proof of the stochastic maximum principle.

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