The $SL(K + 3, \mathbb{C})$ Symmetry of the Bosonic String Theory

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Abstract

We discover that the exact string scattering amplitudes (SSA) of three tachyons and one arbitrary string state, or the Lauricella SSA (LSSA), in the 26$D$ open bosonic string theory can be expressed in terms of the basis functions in the infinite dimensional representation space of the $SL(K + 3, \mathbb{C})$ group. In addition, we find that the $K + 2$ recurrence relations among the LSSA discovered by the present authors previously can be used to reproduce the Cartan subalgebra and simple root system of the $SL(K + 3, \mathbb{C})$ group with rank $K + 2$. As a result, the $SL(K + 3, \mathbb{C})$ group can be used to solve all the LSSA and express them in terms of one amplitude. As an application in the hard scattering limit, the $SL(K + 3, \mathbb{C})$ group can be used to directly prove Gross conjecture [1–3], which was previously corrected and proved by the method of decoupling of zero norm states [4–10].

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I. INTRODUCTION

One of the most important issue of string theory is its spacetime symmetry structure. It has been widely believed that there exist huge spacetime symmetries of string theory. One way to study string symmetry is to calculate string scattering amplitudes (SSA). Indeed, it was conjectured by Gross [1–3] that there exist infinite number of linear relations among high energy, fixed angle or hard SSA of different string states. This conjecture was later corrected and explicitly proved in [4–9] by using the method of decoupling of zero-norm states [10]. Moreover, these infinite linear relations are so powerful that they can be used to reduce the number of independent hard SSA from $\infty$ down to 1. Other approaches of stringy symmetries can be found at [11–16]. For more details, see [17] for a recent review.

On the other hand, it was found that the high energy, fixed momentum transfer or Regge SSA of three tachyons and one arbitrary string states can be expressed in terms of a sum of Kummer functions $U$ [18, 20], which were then shown to be the first Appell function $F_1$  [20]. Regge stringy recurrence relations [19, 20] can then be constructed and used to reduce the number of independent Regge SSA from $\infty$ down to 1. Moreover, an interesting link between Regge SSA and hard SSA was pointed out in [18, 21], and for each mass level the ratios among hard SSA can be extracted from Regge SSA. It was then conjectured that the $SL(5; C)$ dynamical symmetry of the Appell function $F_1$ [22] is crucial to probe high energy
spacetime symmetry of string theory.

More recently, the Lauricella string scattering amplitudes (LSSA)\footnote{23} of three tachyons and one arbitrary string state in the 26D open bosonic string theory valid for arbitrary energies were calculated and expressed in terms of the D-type Lauricella functions $F^{(K)}_{D}$. Moreover, it was shown that\footnote{24} there exist $K + 2$ recurrence relations among $F^{(K)}_{D}$ which (together with a multiplication theorem of $F^{(1)}_{D}$) can be used to derive recurrence relations among LSSA and reduce the number of independent LSSA from $\infty$ down to 1.

In this paper, we will show the existence of the spacetime symmetry group structure of the LSSA. To be more specific, we will demonstrate that the LSSA can be expressed in terms of the basis functions in the infinite dimensional representation space of the $SL(K + 3, C)$ group\footnote{25, 26} which contains the $SO(2, 1)$ spacetime Lorentz group. In addition, we find that the $K + 2$ recurrence relations among the LSSA discovered by the present authors\footnote{24} previously can be used to reproduce the Cartan subalgebra and simple root system of the $SL(K + 3, C)$ group with rank $K + 2$.

We thus have demonstrated, for the first time, the existence of a spacetime symmetry group of the 26D open bosonic string theory. As a result, the $SL(K + 3, C)$ group can be used to solve all the LSSA and express them in terms of one amplitude. As an application in the hard scattering limit, the $SL(K + 3, C)$ group can be used to directly prove Gross conjecture\footnote{1–3}, which was previously corrected and proved by the method of decoupling of zero norm states\footnote{4–10}.

II. REVIEW OF THE LSSA

In this section, we first review the LSSA of three tachyons and one arbitrary string states of the 26D open bosonic string. The general states at mass level $M_2^2 = 2(N - 1)$, $N = \sum_{n,m,l>0} (nr_n^T + mr_m^P + lr_l^L)$ with polarizations on the scattering plane are of the form

$$\lvert r_n^T, r_m^P, r_l^L \rangle = \prod_{n>0} (\alpha_{-n}^T)^{r_n^T} \prod_{m>0} (\alpha_m^P)^{r_m^P} \prod_{l>0} (\alpha_{-l}^L)^{r_l^L} \lvert 0, k \rangle.$$  \hspace{1cm} (2.1)


In the CM frame, the kinematics are defined as

\[
k_1 = \left( \sqrt{M_1^2 + |k_1|^2}, -|k_1|, 0 \right), \quad (2.2)
\]

\[
k_2 = \left( \sqrt{M_2 + |k_1|^2}, |k_1|, 0 \right), \quad (2.3)
\]

\[
k_3 = \left( -\sqrt{M_3^2 + |k_3|^2}, -|k_3| \cos \phi, -|k_3| \sin \phi \right), \quad (2.4)
\]

\[
k_4 = \left( -\sqrt{M_4^2 + |k_3|^2}, |k_3| \cos \phi, +|k_3| \sin \phi \right) \quad (2.5)
\]

with \(M_1^2 = M_2^2 = M_3^2 = -2\) and \(\phi\) is the scattering angle. The Mandelstam variables are \(s = -(k_1 + k_2)^2\), \(t = -(k_2 + k_3)^2\) and \(u = -(k_1 + k_3)^2\). There are three polarizations on the scattering plane [4, 5].

\[
e^T = (0, 0, 1),
\]

\[
e^L = \frac{1}{M_2} \left( |k_1|, \sqrt{M_2 + |k_1|^2}, 0 \right),
\]

\[
e^P = \frac{1}{M_2} \left( \sqrt{M_2 + |k_1|^2}, |k_1|, 0 \right).
\]

For later use, we define

\[
k_i^X \equiv e^X \cdot k_i \quad \text{for} \quad X = (T, P, L).
\]

It is important to note that SSA of three tachyons and one arbitrary string state with polarizations orthogonal to the scattering plane vanish. Thus the Lorentz spacetime symmetry group is \(SO(2,1)\). The \((s, t)\) channel of the LSSA can be calculated to be [23]

\[
A_{st}^{(r_T^T, r_P^P, r_L^L)} = \prod_{n=1} \left[ -(n-1)! k_3^T \right]^{-r_T} \prod_{m=1} \left[ -(m-1)! k_3^P \right]^{-r_P^P} \prod_{l=1} \left[ -(l-1)! k_3^L \right]^{-r_L^L} \cdot B \left( \frac{t}{2} - \frac{1}{2}, \frac{s}{2} - 1 \right) F_D^{(K)} \left( \frac{-t}{2} - 1; R_n^T, R_m^P, R_l^L; \frac{u}{2} + 2 - N; \tilde{Z}_n^T, \tilde{Z}_m^P, \tilde{Z}_l^L \right)
\]

where we have defined \(R_k^X \equiv \{-r_1^X\}, \cdots, \{-r_k^X\}\) with \(\{a\}^n = a, a, \cdots, a\), \(Z_k^X \equiv [z_1^X], \cdots, [z_k^X]\), \(z_k^X = z_k^{X(k-1)}\) and \(z_k^X = \left( -\frac{k^X}{k_3^X} \right) \frac{1}{k} e^{\frac{2\pi ik^X}{k}}\), \(\tilde{z}_k^{X(k')} = \tilde{z}_k^X e^{\frac{2\pi ik^X}{k}}\), \(z_{kk'}^X \equiv 1 - z_{kk'}^X\) for \(k' = 0, \cdots, k - 1\).

The integer \(K\) in Eq. (2.10) is defined to be

\[
K = \sum_{\{\text{for all } r_j^T \neq 0\}} j + \sum_{\{\text{for all } r_j^P \neq 0\}} j + \sum_{\{\text{for all } r_j^L \neq 0\}} j.
\]
The D-type Lauricella function \( F_D^{(K)} \) is one of the four extensions of the Gauss hypergeometric function to \( K \) variables and is defined as

\[
F_D^{(K)}(\alpha; \beta_1, \ldots, \beta_K; \gamma; x_1, \ldots, x_K) = \sum_{n_1, \ldots, n_K} \frac{(\alpha)_{n_1+\ldots+n_K} (\beta_1)_{n_1} \cdots (\beta_K)_{n_K} x_1^{n_1} \cdots x_K^{n_K}}{n_1! \cdots n_K!} \tag{2.12}
\]

where \((\alpha)_n = \alpha \cdot (\alpha + 1) \cdots (\alpha + n - 1)\) is the Pochhammer symbol. There was a integral representation of the Lauricella function \( F_D^{(K)} \) discovered by Appell and Kampe de Feriet (1926) \[27\]

\[
F_D^{(K)}(\alpha; \beta_1, \ldots, \beta_K; \gamma; x_1, \ldots, x_K) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 dt \, t^{\alpha-1}(1-t)^{\gamma-\alpha-1} \cdot (1-x_1 t)^{-\beta_1} \cdot (1-x_2 t)^{-\beta_2} \cdots (1-x_K t)^{-\beta_K}, \tag{2.13}
\]

which was used to calculate Eq. (2.10).

As an application of Eq. (2.10), it can be shown that in the hard scattering limit \( e^P = e^L \) \[4, 5\], the leading order LSSA corresponds to \( r_1^T = N - 2m - 2q, r_1^L = 2m \) and \( r_2^L = q \), and the LSSA in the hard scattering limit can be calculated to be \[23\]

\[
A^{(N-2m-2q,2m,q)}_{st} \simeq B \left( -\frac{t}{2} - 1, -\frac{s}{2} - 1 \right) (E \sin \phi)^N \left( \frac{(2m)!}{m!} \right) \left( -\frac{1}{2M_2} \right)^{2m+q} \left( \frac{1}{2} \right)^{m+q} A^{(N,0,0)}_{st}, \tag{2.14}
\]

which gives the ratios \[17\]

\[
\frac{A^{(N-2m-2q,2m,q)}_{st}}{A^{(N,0,0)}_{st}} = (2m - 1)!! \left( -\frac{1}{M_2} \right)^{2m+q} \left( \frac{1}{2} \right)^{m+q}, \tag{2.15}
\]

and is consistent with the previous result \[4, 9\]. The first example calculated was the ratios at mass level \( M^2 = 4 \) \[4, 5\]

\[
\mathcal{T}_{TTT} : \mathcal{T}_{LLT} : \mathcal{T}_{(LT)} : \mathcal{T}_{[LT]} = 8 : 1 : -1 : -1. \tag{2.16}
\]

The ratios among SSA in Eq. (2.15) and Eq. (2.16) are generalization of ratios among field theory scattering amplitudes. Let’s consider a simple analogy from particle physics. The ratios of the nucleon-nucleon scattering processes

\[
(a) \ p + p \rightarrow d + \pi^+,
(b) \ p + n \rightarrow d + \pi^0,
(c) \ n + n \rightarrow d + \pi^-
\]

\[2.17\]
can be calculated to be (ignore the tiny mass difference between proton and neutron)

\[ T_a : T_b : T_c = 1 : \frac{1}{ \sqrt{2}} : 1 \]  

(2.18)

from \( SU(2) \) isospin symmetry. Is there any symmetry group structure which can be used to calculate SSA ratios in Eq. (2.15) and Eq. (2.16)? This is the main issue we want to address in this paper and it turns out that the relevant group is the noncompact \( SL(K+3, C) \) group as we will discuss in the rest of the paper. Since the spacetime symmetry group of the LSSA needs to include the noncompact Lorentz group \( SO(2,1) \), the noncompact \( SL(K + 3, C) \) group seems to be a reasonable one.

III. THE \( SL(4,C) \) SYMMETRY

In this section, for illustration we first consider the simplest \( K = 1 \) case with \( SL(4,C) \) symmetry. For a given \( K \), there can be LSSA with different mass level \( N \). For illustration, for \( K = 1 \) as an example there are three types of LSSA

\[
\begin{align*}
(\alpha^{p}_{-1})^{p_1}, F_{D}^{(1)} \left( -\frac{t}{2} - 1, -p_1, \frac{u}{2} + 2 - p_1, 1 \right), N = p_1, \\
(\alpha^{q}_{-1})^{q_1}, F_{D}^{(1)} \left( -\frac{t}{2} - 1, -q_1, \frac{u}{2} + 2 - q_1, [\tilde{Z}^1_{P}] \right), N = q_1, \\
(\alpha^{r}_{-1})^{r_1}, F_{D}^{(1)} \left( -\frac{t}{2} - 1, -r_1, \frac{u}{2} + 2 - r_1, [\tilde{Z}^1_{L}] \right), N = r_1.
\end{align*}
\]  

(3.1)

To calculate the group representation of the LSSA for \( K = 1 \), we first define 

\[
f_{abc}^{b}(\alpha; \beta; \gamma; x) = B(\gamma - \alpha, \alpha) F_{D}^{(1)}(\alpha; \beta; \gamma; x) a^{\alpha} b^{\beta} c^{\gamma}. \]  

(3.2)

Note that the LSSA in Eq. (2.10) for \( K = 1 \) corresponds to the case \( a = 1 = c \), and can be written as

\[
A_{st}^{R_X} = f_{11}^{-k_3^X} \left( -\frac{t}{2} - 1; R_X^X, \frac{u}{2} + 2 - N; \tilde{Z}^X \right). \]  

(3.3)
We are now ready to introduce the 15 generators of $SL(4, C)$ group \[25, 26\]

$$E_{\alpha} = a \left( x \partial_x + a \partial_a \right),$$
$$E_{-\alpha} = \frac{1}{a} \left[ x (1 - x) \partial_x + c \partial_c - a \partial_a - xb \partial_b \right],$$
$$E_{\beta} = b \left( x \partial_x + b \partial_b \right),$$
$$E_{-\beta} = \frac{1}{b} \left[ x (1 - x) \partial_x + c \partial_c - b \partial_b - xa \partial_a \right],$$
$$E_{\gamma} = c \left[ (1 - x) \partial_x + c \partial_c - a \partial_a - b \partial_b \right],$$
$$E_{-\gamma} = -\frac{1}{c} \left( x \partial_x + c \partial_c - 1 \right),$$
$$E_{\beta\gamma} = bc \left[ (x - 1) \partial_x + b \partial_b \right],$$
$$E_{-\beta,-\gamma} = \frac{1}{bc} \left[ x (x - 1) \partial_x + xa \partial_a - c \partial_c + 1 \right],$$
$$E_{\alpha\gamma} = ac \left[ (1 - x) \partial_x - a \partial_a \right],$$
$$E_{-\alpha,-\gamma} = \frac{1}{ac} \left[ x (1 - x) \partial_x - xb \partial_b + c \partial_c - 1 \right],$$
$$E_{\alpha\beta\gamma} = abc \partial_x,$$
$$E_{-\alpha,-\beta,-\gamma} = \frac{1}{abc} \left[ x (x - 1) \partial_x - c \partial_c + xb \partial_b + xa \partial_a - x + 1 \right],$$
$$J_{\alpha} = a \partial_a,$$
$$J_{\beta} = b \partial_b,$$
$$J_{\gamma} = c \partial_c,$$

and calculate their operations on the basis functions \[25, 26\]
which suggest the Cartan subalgebra

Here the representation is infinite dimensional. On the other hand, a simple calculation terminated as in the case of the finite dimensional representation of a compact Lie group.

Note, for example, that since \( \beta \) is a nonpositive integer, the operation by \( E_{-\beta} \) will not be terminated as in the case of the finite dimensional representation of a compact Lie group. Here the representation is infinite dimensional. On the other hand, a simple calculation gives

\[
E_{\alpha} f^b_{ac} (\alpha; \beta; \gamma; x) = (\gamma - \alpha - 1) f^b_{ac} (\alpha + 1; \beta; \gamma; x),
\]
\[
E_{\beta} f^b_{ac} (\alpha; \beta; \gamma; x) = \beta f^b_{ac} (\alpha; \beta + 1; \gamma; x),
\]
\[
E_{\gamma} f^b_{ac} (\alpha; \beta; \gamma; x) = (\gamma - \beta) f^b_{ac} (\alpha; \beta; \gamma + 1; x),
\]
\[
E_{\beta\gamma} f^b_{ac} (\alpha; \beta; \gamma; x) = \beta f^b_{ac} (\alpha; \beta + 1; \gamma + 1; x),
\]
\[
E_{\alpha\beta\gamma} f^b_{ac} (\alpha; \beta; \gamma; x) = \beta f^b_{ac} (\alpha + 1; \beta + 1; \gamma + 1; x),
\]
\[
E_{-\alpha} f^b_{ac} (\alpha; \beta; \gamma; x) = (\alpha - 1) f^b_{ac} (\alpha - 1; \beta; \gamma; x),
\]
\[
E_{-\beta} f^b_{ac} (\alpha; \beta; \gamma; x) = (\gamma - \beta) f^b_{ac} (\alpha; \beta - 1; \gamma; x),
\]
\[
E_{-\gamma} f^b_{ac} (\alpha; \beta; \gamma; x) = (\alpha + 1 - \gamma) f^b_{ac} (\alpha; \beta; \gamma - 1; x),
\]
\[
E_{-\beta\gamma} f^b_{ac} (\alpha; \beta; \gamma; x) = (\alpha - \gamma + 1) f^b_{ac} (\alpha; \beta - 1; \gamma - 1; x),
\]
\[
E_{-\alpha\beta\gamma} f^b_{ac} (\alpha; \beta; \gamma; x) = (\alpha - 1 - \gamma) f^b_{ac} (\alpha - 1; \beta - 1; \gamma - 1; x),
\]
\[
J_{\alpha} f^b_{ac} (\alpha; \beta; \gamma; x) = \alpha f^b_{ac} (\alpha; \beta; \gamma; x),
\]
\[
J_{\beta} f^b_{ac} (\alpha; \beta; \gamma; x) = \beta f^b_{ac} (\alpha; \beta; \gamma; x),
\]
\[
J_{\gamma} f^b_{ac} (\alpha; \beta; \gamma; x) = \gamma f^b_{ac} (\alpha; \beta; \gamma; x).
\] (3.5)

Note, for example, that since \( \beta \) is a nonpositive integer, the operation by \( E_{-\beta} \) will not be terminated as in the case of the finite dimensional representation of a compact Lie group.

Here the representation is infinite dimensional. On the other hand, a simple calculation gives

\[
[E_{\alpha}, E_{-\alpha}] = 2J_{\alpha} - J_{\gamma},
\]
\[
[E_{\beta}, E_{-\beta}] = 2J_{\beta} - J_{\gamma},
\]
\[
[E_{\gamma}, E_{-\gamma}] = 2J_{\gamma} - (J_{\alpha} + J_{\beta} + 1),
\]

which suggest the Cartan subalgebra

\[
[J_{\alpha}, J_{\beta}] = 0, [J_{\beta}, J_{\gamma}] = 0, [J_{\alpha}, J_{\gamma}] = 0.
\] (3.6)
Indeed, if we redefine
\[ J'_\alpha = J_\alpha - \frac{1}{2} J_\gamma, \]
\[ J'_\beta = J_\beta - \frac{1}{2} J_\gamma, \]
\[ J'_\gamma = J_\gamma - \frac{1}{2} (J_\alpha + J_\beta + 1), \]
we find out that each of the triplets \( \{J^+, J^-, J^0\} \) \( \equiv \{E_\alpha, E_{-\alpha}, J'_\alpha\} \), \( \{E_\beta, E_{-\beta}, J'_\beta\} \), \( \{E_\gamma, E_{-\gamma}, J'_\gamma\} \), \( \{E_{\alpha\gamma}, E_{-\alpha}, J'_\alpha + J'_\gamma\} \), \( \{E_{\alpha\beta}, E_{-\beta}, J'_\alpha + J'_\beta\} \)
constitutes the well known commutation relations
\[ [J^0, J^\pm] = \pm J^\pm, \quad [J^+, J^-] = 2J^0. \tag{3.7} \]

In the following, we want to further relate the \( SL(4, \mathbb{C}) \) group to the recurrence relations of \( F^{(1)}_D(\alpha; \beta; \gamma; x) \) or of the LSSA in Eq.(3.1). For our purpose, there are \( K + 2 = 1 + 2 = 3 \) recurrence relations among \( F^{(1)}_D(\alpha; \beta; \gamma; x) \) or Gauss hypergeometry functions

\[ (\alpha - \beta) F^{(1)}_D(\alpha; \beta; \gamma; x) - \alpha F^{(1)}_D(\alpha + 1; \beta; \gamma; x) + \beta F^{(1)}_D(\alpha; \beta + 1; \gamma; x) = 0, \tag{3.8} \]
\[ \gamma F^{(1)}_D(\alpha; \beta; \gamma; x) - (\gamma - \alpha) F^{(1)}_D(\alpha; \beta; \gamma + 1; x) - \alpha F^{(1)}_D(\alpha + 1; \beta; \gamma + 1; x) = 0, \tag{3.9} \]
\[ \gamma F^{(1)}_D(\alpha; \beta; \gamma; x) + \gamma (x - 1) F^{(1)}_D(\alpha; \beta + 1; \gamma; x) - (\gamma - \alpha) x F^{(1)}_D(\alpha; \beta + 1; \gamma + 1; x) = 0. \tag{3.10} \]

The three recurrence relations can be used to derive recurrence relations among LSSA in Eq.(3.1).

In the following we will show that the three recurrence relations can be used to reproduce the Cartan subalgebra and simple root system of the \( SL(4, \mathbb{C}) \) group with rank 3. With the identification in Eq.(3.2), the first recurrence relation in Eq.(3.8) can be rewritten as
\[ \frac{(\alpha - \beta) f^b_{\alpha c}(\alpha; \beta; \gamma; x)}{B(\gamma - \alpha, \alpha) a^\alpha b^{\beta - 1} c^\gamma} - \frac{\alpha f^b_{\alpha c}(\alpha + 1; \beta; \gamma; x)}{B(\gamma - \alpha - 1, \alpha + 1) a^{\alpha + 1} b^{\beta - 1} c^\gamma} + \frac{\beta f^b_{\alpha c}(\alpha; \beta + 1; \gamma; x)}{B(\gamma - \alpha, \alpha) a^\alpha b^{\beta + 1} c^\gamma} = 0. \tag{3.11} \]
By using the identity
\[ B(\gamma - \alpha - 1, \alpha + 1) = \frac{\Gamma(\gamma - \alpha - 1) \Gamma(\alpha + 1)}{\Gamma(\gamma)} = \frac{\alpha}{\gamma - \alpha - 1} \frac{\Gamma(\gamma - \alpha) \Gamma(\alpha)}{\Gamma(\gamma)}, \tag{3.12} \]
the recurrence relation then becomes
\[ (\alpha - \beta) f^b_{ac}(\alpha; \beta; \gamma; x) - \frac{\alpha f^b_{ac}(\alpha + 1; \beta; \gamma; x)}{\frac{\beta f^b_{ac}(\alpha + 1; \gamma; x)}{b}} = 0, \tag{3.13} \]
or
\[ \left( \alpha - \beta - \frac{E_\alpha}{a} + \frac{E_\beta}{b} \right) f^b_{ac}(\alpha; \beta; \gamma; x) = 0, \tag{3.14} \]
which means
\[ [\alpha - \beta -(x \partial_x + a \partial_a) + (x \partial_x + b \partial_b)] f^b_{ac}(\alpha; \beta; \gamma; x) = 0, \tag{3.15} \]
or
\[ [(\alpha - J_\alpha) - (\beta - J_\beta)] f^b_{ac}(\alpha; \beta; \gamma; x) = 0. \tag{3.16} \]

Similarly for the second recurrence relation in Eq. (3.9), we obtain
\[ f^b_{ac}(\alpha; \beta; \gamma; x) - \frac{E_\gamma f^b_{ac}(\alpha; \beta; \gamma; x)}{c (\gamma - \beta)} - \frac{E_\alpha E_\gamma f^b_{ac}(\alpha; \beta; \gamma; x)}{ac (\beta - \gamma)} = 0. \tag{3.17} \]

After some calculations, we end up with
\[ [(\gamma - c \partial_c) - (\beta - b \partial_b)] f^b_{ac}(\alpha; \beta; \gamma; x) = 0, \tag{3.18} \]
or
\[ [(\gamma - J_\gamma) - (\beta - J_\beta)] f^b_{ac}(\alpha; \beta; \gamma; x) = 0. \tag{3.19} \]

Finally the third recurrence relation in Eq. (3.10) can be rewritten as
\[ \gamma f^b_{ac}(\alpha; \beta; \gamma; x) + \frac{\gamma (x - 1) E_\beta f^b_{ac}(\alpha; \beta; \gamma; x)}{b \beta} - \frac{(\gamma - \alpha) x E_\beta \gamma f^b_{ac}(\alpha; \beta; \gamma; x)}{\gamma - \alpha} b \beta c = 0, \tag{3.20} \]
which gives after some computation
\[ (\beta - J_\beta) f^b_{ac}(\alpha; \beta; \gamma; x) = 0. \tag{3.21} \]

It is easy to see that Eq. (3.16), Eq. (3.19) and Eq. (3.21) imply the last three equations of Eq. (3.5) or the Cartan subalgebra in Eq. (3.6) as expected.

In addition to the Cartan subalgebra, we need to derive the operations of the \( \{E_\alpha, E_\beta, E_\gamma\} \) from the recurrence relations. With the operations of Cartan subalgebra and \( \{E_\alpha, E_\beta, E_\gamma\} \),
one can reproduce the whole $SL(4, \mathbb{C})$ algebra. Note that the first recurrence relation in Eq. (3.8) can be rewritten as

$$(\alpha - \beta) f^b_{ac} (\alpha; \beta; \gamma; x) - \frac{\alpha f^b_{ac} (\alpha + 1; \beta; \gamma; x)}{\gamma - \alpha - 1} a + \frac{\beta f^b_{ac} (\alpha; \beta + 1; \gamma; x)}{b} = 0, \quad (3.22)$$

which means

$$(\alpha - \beta) f^b_{ac} (\alpha; \beta; \gamma; x) - \frac{E_a f^b_{ac} (\alpha; \beta; \gamma; x)}{a} + \frac{\beta f^b_{ac} (\alpha; \beta + 1; \gamma; x)}{b} = 0 \quad (3.23)$$

where we have used the operation of $E_a$ in Eq. (3.15). The next step is to use the definition of $E_a$ in Eq. (3.4) to obtain

$$\left(\alpha - \beta - \frac{a (x \partial_x + a \partial_a)}{a}\right) f^b_{ac} (\alpha; \beta; \gamma; x) = - \frac{\beta f^b_{ac} (\alpha; \beta + 1; \gamma; x)}{b}, \quad (3.24)$$

which implies

$$[b (b \partial_b + x \partial_x)] f^b_{ac} (\alpha; \beta; \gamma; x) = E_{\beta} f^b_{ac} (\alpha; \beta; \gamma; x) = \beta f^b_{ac} (\alpha; \beta + 1; \gamma; x) \quad (3.25)$$

where we have used the definition of $E_{\beta}$ in Eq. (3.4). Eq. (3.25) is consistent with the operation of $E_{\beta}$ in Eq. (3.5).

Similarly, we can check the operation of $E_{\alpha}$. Note that the first recurrence relation in Eq. (3.8) can be rewritten as

$$(\alpha - \beta) f^b_{ac} (\alpha; \beta; \gamma; x) - \frac{(\gamma - \alpha - 1) f^b_{ac} (\alpha + 1; \beta; \gamma; x)}{a} + \frac{E_{\alpha} f^b_{ac} (\alpha; \beta; \gamma; x)}{b} = 0 \quad (3.26)$$

where we have used the operation of $E_{\beta}$ in Eq. (3.5). The next step is to use the definition of $E_{\beta}$ in Eq. (3.4) to obtain

$$\left(\alpha - \beta + \frac{b (x \partial_x + a \partial_a)}{b}\right) f^b_{ac} (\alpha; \beta; \gamma; x) = \frac{(\gamma - \alpha - 1) f^b_{ac} (\alpha + 1; \beta; \gamma; x)}{a}, \quad (3.27)$$

which implies

$$[a (a \partial_a + x \partial_x)] f^b_{ac} (\alpha; \beta; \gamma; x) = E_{\alpha} f^b_{ac} (\alpha; \beta; \gamma; x) = (\gamma - \alpha - 1) f^b_{ac} (\alpha + 1; \beta; \gamma; x). \quad (3.28)$$

where we have used the definition of $E_{\alpha}$ in Eq. (3.4). Eq. (3.28) is consistent with the operation of $E_{\alpha}$ in Eq. (3.5).

Finally we check the operation of $E_{\gamma}$. Note that Eq. (3.9) can be written as

$$\frac{\gamma f^b_{ac} (\alpha; \beta; \gamma; x)}{B (\gamma - \alpha, \alpha) a^{\alpha} b^{\beta} c^{\gamma}} - \frac{(\gamma - \alpha) f^b_{ac} (\alpha; \beta; \gamma + 1; x)}{B (\gamma - \alpha, \alpha) a^{\alpha} b^{\beta} c^{\gamma + 1}} - \frac{\alpha f^b_{ac} (\alpha + 1; \beta; \gamma + 1; x)}{B (\gamma - \alpha, \alpha) a^{\alpha + 1} b^{\beta} c^{\gamma + 1}} = 0, \quad (3.29)$$
which gives
\[ f_{ac}^b (\alpha; \beta; \gamma; x) - \frac{f_{ac}^b (\alpha; \beta; \gamma + 1; x)}{c} - \frac{f_{ac}^b (\alpha + 1; \beta; \gamma + 1; x)}{ac} = 0. \] (3.30)

The next step is to use the definition and operation of \( E_{\alpha\gamma} \) to obtain
\[ f_{ac}^b (\alpha; \beta; \gamma; x) - \frac{f_{ac}^b (\alpha; \beta; \gamma + 1; x)}{c} - \frac{E_{\alpha\gamma} f_{ac}^b (\alpha; \beta; \gamma; x)}{ac (\beta - \gamma)} = 0, \]
which gives
\[ f_{ac}^b (\alpha; \beta; \gamma; x) - \frac{ac [(1 - x) \partial_x - a \partial_a] f_{ac}^b (\alpha; \beta; \gamma; x)}{ac (\beta - \gamma)} \]
\[ = \frac{f_{ac}^b (\alpha; \beta; \gamma + 1; x)}{c}. \] (3.31)

After some simple computation, we get
\[ -c [b \partial_b - c \partial_c - (1 - x) \partial_x + a \partial_a] f_{ac}^b (\alpha; \beta; \gamma; x) \]
\[ = E_{\gamma} f_{ac}^b (\alpha; \beta; \gamma; x) = (\gamma - \beta) f_{ac}^b (\alpha; \beta; \gamma + 1; x). \] (3.32)

Eq. (3.32) is consistent with the operation of \( E_{\gamma} \) in Eq. (3.5).

We thus have shown that the extended LSSA \( f_{ac}^b (\alpha; \beta; \gamma; x) \) in Eq. (3.2) with arbitrary \( a \) and \( c \) form an infinite dimensional representation of the \( SL(4, C) \) group. Moreover, the 3 recurrence relations among the LSSA can be used to reproduce the Cartan subalgebra and simple root system of the \( SL(4, C) \) group with rank 3. The recurrence relations are thus equivalent to the representation of the \( SL(4, C) \) symmetry group.

**IV. THE GENERAL SL(K + 3, C) SYMMETRY**

To calculate the group representation of the LSSA for general \( K \), we first define
\[ f_{ac}^{b_1 \cdots b_K} (\alpha; \beta_1, \cdots, \beta_K; \gamma; x_1, \cdots, x_K) \]
\[ = B (\gamma - \alpha, \alpha) F_{D}^{(K)} (\alpha; \beta_1, \cdots, \beta_K; \gamma; x_1, \cdots, x_K) a^{\alpha} b_1^{\beta_1} \cdots b_K^{\beta_K} c^\gamma. \] (4.1)

Note that the LSSA in Eq. (2.10) corresponds to the case \( a = 1 = c \), and can be written as
\[ A_{st} (r^T_n, r^P_m, r^L_i) = f_{11}^{-(n-1)k^T_n, -(m-1)k^P_m, -(l-1)k^L_i} \left( -\frac{t}{2} - 1; R^T_n, R^P_m, R^L_i; \frac{u}{2} + 2 - N; \tilde{Z}_n^T, \tilde{Z}_m^P, \tilde{Z}_i^L \right). \] (4.2)

It is possible to generalize the \( SL(4, C) \) symmetry group for the \( K = 1 \) case discussed in the previous section to the general \( SL(K + 3, C) \) group. We first introduce the \((K + 3)^2 - 1\)
generators of $SL(K + 3, C)$ group ($k = 1, 2, \ldots K$) \cite{25, 26}

\begin{align*}
E^\alpha &= a \left( \sum_j x_j \partial_j + a \partial_a \right), \\
E^\beta_k &= b_k (x_k \partial_k + b_k \partial b_k), \\
E^\gamma &= c \left( \sum_j (1 - x_j) \partial x_j + c \partial c - a \partial a - \sum_j b_j \partial b_j \right), \\
E^{\alpha\gamma} &= ac \left( \sum_j (1 - x_j) \partial x_j - a \partial a \right), \\
E^{\beta_k\gamma} &= b_k c [(x_k - 1) \partial x_k + b_k \partial b_k], \\
E^{\alpha\beta_k\gamma} &= ab_k c \partial x_k,
\end{align*}

\begin{align*}
E_\alpha &= \frac{1}{a} \left[ \sum_j x_j (1 - x_j) \partial x_j + c \partial c - a \partial a - \sum_j x_j b_j \partial b_j \right], \\
E_\beta_k &= \frac{1}{b_k} \left[ x_k (1 - x_k) \partial x_k + x_k \sum_{j \neq k} (1 - x_j) x_j \partial x_j + c \partial c - x_k a \partial a - \sum_j b_j \partial u_j \right], \\
E_\gamma &= -\frac{1}{c} \left( \sum_j x_j \partial x_j + c \partial c - 1 \right), \\
E_{\alpha\gamma} &= \frac{1}{ac} \left[ \sum_j x_j (1 - x_j) \partial x_j - \sum_j x_j b_j \partial b_j + c \partial c - 1 \right], \\
E_{\beta_k\gamma} &= \frac{1}{b_k c} \left[ x_k (x_k - 1) \partial x_k + \sum_{j \neq k} (x_j - 1) x_j \partial x_j + x_k a \partial a - c \partial c + 1 \right], \\
E_{\alpha\beta_k\gamma} &= \frac{1}{ab_k c} \left[ \sum_j x_j (x_j - 1) \partial x_j - c \partial c + x_k a \partial a + \sum_j x_j b_j \partial b_j - x_k + 1 \right], \\
E_{\beta_kp}^\beta &= \frac{b_k}{b_p} [(x_k - x_p) \partial z_k + b_k \partial b_k], (k \neq p), \\
J_\alpha &= a \partial a, \\
J_\beta_k &= b_k \partial b_k, \\
J_\gamma &= c \partial c.
\end{align*}

(4.3)

Note that we have used the upper indices to denote the "raising operators" and the lower indices to denote the "lowering operators". The number of generators can be counted by the following way. There are $1 E^\alpha, K E^\beta_k, 1 E^\gamma, 1 E^{\alpha\gamma}, K E^{\beta_k\gamma}$ and $K E^{\alpha\beta_k\gamma}$ which sum up to $3K + 3$ raising generators. There are also $3K + 3$ lowering operators. In addition,
there are $K(K - 1) E^{\alpha}_{b_p} + K + 2 J$, the Cartan subalgebra. In sum, the total number of generators are $2(3K + 3) + K(K - 1) + K + 2 = (K + 3)^2 - 1$. It is straightforward to calculate the operation of these generators on the basis functions $(k = 1, 2, \ldots K)$.

\[
E^{\alpha} f^{b_1 \cdots b_K}_{ac} (\alpha) = (\gamma - \alpha - 1) f^{b_1 \cdots b_K}_{ac} (\alpha + 1),
\]
\[
E^{\beta} f^{b_1 \cdots b_K}_{ac} (\beta) = \beta f^{b_1 \cdots b_K}_{ac} (\beta + 1),
\]
\[
E^{\gamma} f^{b_1 \cdots b_K}_{ac} (\gamma) = \left( \gamma - \sum_{j} \beta_j \right) f^{b_1 \cdots b_K}_{ac} (\gamma + 1),
\]
\[
E^{\alpha \gamma} f^{b_1 \cdots b_K}_{ac} (\alpha; \gamma) = \left( \sum_{j} \beta_j - \gamma \right) f^{b_1 \cdots b_K}_{ac} (\alpha + 1; \gamma + 1),
\]
\[
E^{\beta \gamma} f^{b_1 \cdots b_K}_{ac} (\beta; \gamma) = \beta f^{b_1 \cdots b_K}_{ac} (\beta + 1; \gamma + 1),
\]
\[
E^{\alpha \beta} f^{b_1 \cdots b_K}_{ac} (\alpha; \beta; \gamma) = \beta f^{b_1 \cdots b_K}_{ac} (\alpha + 1; \beta + 1; \gamma + 1),
\]
\[
E^{\alpha} f^{b_1 \cdots b_K}_{ac} (\alpha) = (\alpha - 1) f^{b_1 \cdots b_K}_{ac} (\alpha - 1),
\]
\[
E^{\beta} f^{b_1 \cdots b_K}_{ac} (\beta) = \left( \gamma - \sum_{j} \beta_j \right) f^{b_1 \cdots b_K}_{ac} (\beta - 1),
\]
\[
E^{\gamma} f^{b_1 \cdots b_K}_{ac} (\gamma) = (\alpha - \gamma + 1) f^{b_1 \cdots b_K}_{ac} (\gamma - 1),
\]
\[
E^{\alpha \gamma} f^{b_1 \cdots b_K}_{ac} (\alpha; \gamma) = (\alpha - 1) f^{b_1 \cdots b_K}_{ac} (\alpha - 1; \gamma - 1),
\]
\[
E^{\beta \gamma} f^{b_1 \cdots b_K}_{ac} (\beta; \gamma) = (\alpha - \gamma + 1) f^{b_1 \cdots b_K}_{ac} (\beta - 1; \gamma - 1),
\]
\[
E^{\alpha \beta} f^{b_1 \cdots b_K}_{ac} (\alpha; \beta; \gamma) = (1 - \alpha) f^{b_1 \cdots b_K}_{ac} (\alpha - 1; \beta - 1; \gamma - 1),
\]
\[
E^{\beta \gamma} f^{b_1 \cdots b_K}_{ac} (\beta; \beta) = \beta f^{b_1 \cdots b_K}_{ac} (\beta + 1; \beta - 1),
\]
\[
J^{\alpha} f^{b_1 \cdots b_K}_{ac} (\alpha; \beta; \gamma) = \alpha f^{b_1 \cdots b_K}_{ac} (\alpha; \beta; \gamma),
\]
\[
J^{\beta} f^{b_1 \cdots b_K}_{ac} (\alpha; \beta; \gamma) = \beta f^{b_1 \cdots b_K}_{ac} (\alpha; \beta; \gamma),
\]
\[
J^{\gamma} f^{b_1 \cdots b_K}_{ac} (\alpha; \beta; \gamma) = \gamma f^{b_1 \cdots b_K}_{ac} (\alpha; \beta; \gamma)
\]

where, for simplicity, we have omitted those arguments in $f^{b_1 \cdots b_K}_{ac}$ which remain the same after the operation. The commutation relations of the $SL(K + 3)$ Lie algebra can be calculated in the following way. In addition to the Cartan subalgebra for the $K + 2$ generators $\{J_{\alpha}, J_{\beta}, J_{\gamma}\}$,
let’s redefine
\[
J'_\alpha = J_\alpha - \frac{1}{2} J_\gamma,
\]
\[
J'_{\beta_k} = J_{\beta_k} - \frac{1}{2} J_\gamma + \sum_{j \neq k} J_{\beta_j},
\]
\[
J'_\gamma = J_\gamma - \frac{1}{2} \left( J_\alpha + \sum_j J_{\beta_j} + 1 \right).
\] (4.5)

One can show that each of the following triplets \[26\]
\[
\{ J^+, J^-, J^0 \} \equiv \{ E_\alpha, E_\alpha', J'_\alpha \}, \{ E_{\beta_k}, E_{\beta_k'}, J'_{\beta_k} \},
\]
\[
\{ E_{\gamma}, E_{\gamma'}, J'_\gamma \}, \{ E_{\alpha\beta\gamma}, J'_\alpha + J'_{\beta_k} + J'_\gamma \},
\]
\[
\{ E_{\alpha\gamma}, E_{\alpha\gamma'}, J'_\gamma \}, \{ E_{\alpha\beta}, E_{\beta_k}, J'_\alpha + J'_{\beta_k} \},
\]
\[
\{ E_{\beta_l}, E_{\beta_l'}, J'_{\beta_l} - J'_{\beta_l} \}
\] (4.6)
satisfies the commutation relations in Eq.(3.7).

There are \(K+2\) fundamental recurrence relations among \(F_D^{(K)}(\alpha; \beta; \gamma; x)\) or the Lauricella functions. The three different kinds of recurrence relations are \[24\]
\[
\left( \alpha - \sum_j \beta_j \right) F^{(K)}_D(\alpha) - \alpha F^{(K)}_D(\alpha + 1) + \sum_j \beta_j F^{(K)}_D(\beta_j + 1) = 0,
\] (4.7)
\[
\gamma F^{(K)}_D(\gamma) - (\gamma - \alpha) F^{(K)}_D(\gamma + 1) - \alpha F^{(K)}_D(\alpha + 1; \gamma + 1) = 0,
\] (4.8)
and
\[
\gamma F^{(K)}_D + \gamma (x_m - 1) F^{(K)}_D(\beta_m + 1) + (\alpha - \gamma) x_m F^{(K)}_D(\beta_m + 1; \gamma + 1) = 0
\] (4.9)
where \(m = 1, 2, ...K\). The three types of recurrence relations can be used to derive recurrence relations among LSSA and reduce the number of independent LSSA from \(\infty\) down to \(1\) \[24\].

In the following we will show that the three types of recurrence relations above imply the Cartan subalgebra of the \(SL(K + 3, \mathbb{C})\) group with rank \(K + 2\). With the identification in Eq.(4.1), the first type of recurrence relation in Eq.(4.7) can be rewritten as
\[
\left( \alpha - \sum_j \beta_j \right) f_{ac}^{b_1 ... b_K} - \frac{E_a f_{ac}^{b_1 ... b_K}}{a} (\alpha) + \sum_j \frac{E_{\beta_j} f_{ac}^{b_1 ... b_K}}{b_j} (\beta_j) = 0,
\] (4.10)
which gives
\[
\left( \alpha - \sum_j \beta_j \right) f_{ac}^{b_1 ... b_K} - \left( \sum_j x_j \partial_j + a \partial_a \right) f_{ac}^{b_1 ... b_K} + \sum_j \left( x_j \partial_j + b_j \partial_{b_j} \right) f_{ac}^{b_1 ... b_K} = 0
\] (4.11)
or

\[
\left[ (\alpha - a \partial_a) + \sum_j (\beta_j - b_j \partial_{b_j}) \right] f_{ac}^{b_1 \cdots b_K} = 0, \tag{4.12}
\]

which means

\[
\left[ (\alpha - J_{\alpha}) + \sum_j (\beta_j - J_{\beta_j}) \right] f_{ac}^{b_1 \cdots b_K} = 0. \tag{4.13}
\]

The second type of recurrence relation in Eq.(4.8) can be rewritten as

\[
f_{ac}^{b_1 \cdots b_K} - \frac{E^\gamma f_{ac}^{b_1 \cdots b_K} (\gamma)}{c \left( \gamma - \sum_j \beta_j \right)} - \frac{E^{\alpha \gamma} f_{ac}^{b_1 \cdots b_K} (\alpha; \gamma)}{ac \left( \sum_j \beta_j - \gamma \right)} = 0, \tag{4.14}
\]

which gives

\[
\left[ \gamma - \sum_j \beta_j - \left( \sum_j (1 - x_j) \partial_{x_j} + c \partial_c - a \partial_a - \sum_j b_j \partial_{b_j} \right) \right] f_{ac}^{b_1 \cdots b_K} = 0 \tag{4.15}
\]

or

\[
\left[ (\gamma - c \partial_c) - \sum_j (\beta_j - b_j \partial_{b_j}) \right] f_{ac}^{b_1 \cdots b_K} = 0. \tag{4.16}
\]

Eq.(4.16) can be written as

\[
\left[ (\gamma - J_{\gamma}) - \sum_j (\beta_j - J_{\beta_j}) \right] f_{ac}^{b_1 \cdots b_K} = 0. \tag{4.17}
\]

The third type of recurrence relation in Eq.(4.9) can be rewritten as \((m = 1, 2, \ldots K)\)

\[
f_{ac}^{b_1 \cdots b_K} + \frac{(m - 1)E^\beta_m f_{ac}^{b_1 \cdots b_K}}{b_m \beta_m} - \frac{x_m E^{\beta_m \gamma} f_{ac}^{b_1 \cdots b_K}}{b_m c \beta_m} = 0, \tag{4.18}
\]

which gives

\[
\beta_m f_{ac}^{b_1 \cdots b_K} + (m - 1) (x_m \partial_m + b_m \partial_{b_m}) f_{ac}^{b_1 \cdots b_K} - x_m [ (m - 1) \partial_{x_m} + b_m \partial_{b_m} ] f_{ac}^{b_1 \cdots b_K} = 0 \tag{4.19}
\]

or

\[
(\beta_m - b_m \partial_{b_m}) f_{ac}^{b_1 \cdots b_K} = 0. \tag{4.20}
\]

In the above calculation, we have used the definition and operation of \(E^{\beta_m \gamma}\) in Eq.(4.13) and Eq.(4.4), respectively.
Eq. (4.20) can be written as
\[(\beta_m - J_{\beta_m}) f_{ac}^{b_1 \cdots b_K} = 0, m = 1, 2, \ldots K.\] (4.21)

It is important to see that Eq. (4.13), Eq. (4.17) and Eq. (4.21) imply the last three equations of Eq. (4.4) or the Cartan subalgebra of \(SL(K+3, \mathbb{C})\) as expected.

In addition to the Cartan subalgebra, we need to derive the operations of \(\{E_\alpha, E_{\beta_k}, E_\gamma\}\) from the recurrence relations. With the operations of Cartan subalgebra and \(\{E_\alpha, E_{\beta_k}, E_\gamma\}\), one can reproduce the whole \(SL(K+3, \mathbb{C})\) algebra. The calculations of \(E_\alpha\) and \(E_\gamma\) are straightforward and are similar to the case of \(SL(4, \mathbb{C})\) in the previous section. Here we present only the calculation of \(E_{\beta_k}\). The recurrence relation in Eq. (4.17) can be rewritten as
\[
\left(\alpha - \sum_j \beta_j\right) f_{ac}^{b_1 \cdots b_K} - \frac{E_\alpha f_{ac}^{b_1 \cdots b_K}(\alpha)}{a} + \sum_{j \neq k} \frac{E_{\beta_j} f_{ac}^{b_1 \cdots b_K}(\beta_j)}{b_j} + \frac{\beta_k f_{ac}^{b_1 \cdots b_K}(\beta_k + 1)}{b_k} = 0. \tag{4.22}
\]

After operation of \(E_{\beta_j}\), we obtain
\[
\left(\alpha - \sum_j \beta_j\right) f_{ac}^{b_1 \cdots b_K} - \left(\sum_j x_j \partial_j + a \partial_a\right) f_{ac}^{b_1 \cdots b_K} + \sum_{j \neq k} \left( x_j \partial_j + b_j \partial_{b_j}\right) f_{ac}^{b_1 \cdots b_K} = \frac{-\beta_k f_{ac}^{b_1 \cdots b_K}(\beta_k + 1)}{b_k}, \tag{4.23}
\]
which gives the consistent result
\[
b_k \left( b_k \partial_{b_k} + x_k \partial_k\right) f_{ac}^{b_1 \cdots b_K}(\beta_k) = E_{\beta_k} f_{ac}^{b_1 \cdots b_K} = \beta_k f_{ac}^{b_1 \cdots b_K}(\beta_k + 1), k = 1, 2, \ldots K. \tag{4.24}
\]

In the above calculation, we have used the definitions and operations of \(E_{\beta_k}\) and \(E_\alpha\) in Eq. (4.3) and Eq. (4.4), respectively.

The \(K+2\) equations in Eq. (4.13), Eq. (4.17) and Eq. (4.21) together with \(K+2\) equations for the operations \(\{E_\alpha, E_{\beta_k}, E_\gamma\}\) are equivalent to the Cartan subalgebra and the simple root system of \(SL(K+3, \mathbb{C})\) with rank \(K+2\). With the Cartan subalgebra and the simple roots, one can easily write down the whole Lie algebra of the \(SL(K+3, \mathbb{C})\) group. So one can construct the Lie algebra from the recurrence relations and vice versa.

In the previous publication, it was shown that \([24]\) the \(K+2\) recurrence relations among \(F_D^{(K)}\) can be used to derive recurrence relations among LSSA and reduce the number of independent LSSA from \(\infty\) down to 1. We conclude that the \(SL(K+3, \mathbb{C})\) group can be
used to derive infinite number of recurrence relations among LSSA, and one can solve all
the LSSA and express them in terms of one amplitude.

Finally, in addition to Eq. (4.6), there is a simple way to write down the Lie algebra
commutation relations of $SL(K+3, C)$, namely

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj} \quad (4.25)$$

with the identifications

$$E^\alpha = E_{12}, E_\alpha = E_{21}, E^{\beta_k} = E_{k+3,3}, E_\beta = E_{3,k+3},$$

$$E^\gamma = E_{31}, E_\gamma = E_{13}, E^{\alpha\gamma} = E_{32}, E_{\alpha\gamma} = E_{23},$$

$$E^{3\kappa\gamma} = -E_{k+3,1}, E_{3\kappa\gamma} = -E_{1,k+3}, E_{3\alpha\beta\kappa\gamma} = -E_{k+3,2},$$

$$E_{\alpha\beta\kappa\gamma} = -E_{2,k+3}, J'_{\alpha} = \frac{1}{2} (E_{11} - E_{22}), J'_{\beta_k} = \frac{1}{2} (E_{k+3,k+3} - E_{33}), J'_{\gamma} = \frac{1}{2} (E_{33} - E_{11}). \quad (4.26)$$

V. CONCLUSION AND DISCUSSION

In this paper, we point out that the exact LSSA in the 26D open bosonic string theory
can be expressed in terms of the basis functions in the infinite dimensional representation
space $V$ of the $SL(K + 3, \mathbb{C})$ group which contains the $SO(2, 1)$ spacetime Lorentz group.
In addition, we find that the $K + 2$ recurrence relations among the LSSA can be used to
reproduce the Cartan subalgebra and simple root system of the $SL(K+3, \mathbb{C})$ group with rank
$K + 2$. Thus the recurrence relations are equivalent to the representation of $SL(K + 3, \mathbb{C})$
group of the LSSA. As a result, the $SL(K+3, \mathbb{C})$ group can be used to solve all the LSSA
and express them in terms of one amplitude \[24\]. As an application in the hard scattering
limit, the $SL(K + 3, \mathbb{C})$ group can be used to directly prove Gross conjecture \[1–3\], which
was previously corrected and proved by the method of decoupling of zero norm states \[4–10\].

There are some special properties in the $SL(K + 3, \mathbb{C})$ group representation of the LSSA,
which make it different from the usual symmetry group representation of a physical system.
First, the set of LSSA does not fill up the whole representation space $V$. For example, for
states $f_{ac}^{\beta_1...\beta_K} (\alpha; \beta_1, \cdots, \beta_K; \gamma; x_1, \cdots, x_K)$ in $V$ with $a \neq 1$ or $c \neq 1$, they are not LSSA.

Indeed, there are more states in $V$ with $K \geq 2$ which are not LSSA either. We give one
example in the following. For $K = 2$ there are six type of LSSA ($\omega = -1$)

\begin{align}
(\alpha_{-1}^T)^{p_1}(\alpha_{-1}^P)^{q_1}, F_D^{(2)}(a, -p_1, -q_1, c - p_1 - q_1, 1, \left[\tilde{z}_1^P\right]), N = p_1 + q_1, & \quad (5.1) \\
(\alpha_{-1}^T)^{p_1}(\alpha_{-1}^L)^{r_1}, F_D^{(2)}(a, -p_1, -r_1, c - p_1 - r_1, 1, \left[\tilde{z}_1^L\right]), N = p_1 + r_1, \quad (5.2) \\
(\alpha_{-1}^P)^{q_1}(\alpha_{-1}^L)^{r_1}, F_D^{(2)}(a, -q_1, -r_1, c - q_1 - r_1, \left[\tilde{z}_1^P\right], \left[\tilde{z}_1^L\right]), N = q_1 + r_1, & \quad (5.3) \\
(\alpha_{-2}^T)^{p_2}, F_D^{(2)}(a, -p_2, -p_2, c - 2p_2, 1, 1), N = 2p_2, & \quad (5.4) \\
(\alpha_{-2}^P)^{q_2}, F_D^{(2)}(a, -q_2, -q_2, c - 2q_2, 1 - z_2^P, 1 - \omega z_2^P), N = 2q_2, & \quad (5.5) \\
(\alpha_{-2}^L)^{r_2}, F_D^{(2)}(a, -r_2, -r_2, c - 2r_2, 1 - z_2^L, 1 - \omega z_2^L), N = 2r_2. & \quad (5.6)
\end{align}

One can show that the states obtained from operation by $E_\beta$ on either states in Eq.\ref{eq:5.4} to Eq.\ref{eq:5.6} are not LSSA. However, all states in $V$ including those "auxiliary states" which are not LSSA can be exactly solved by recurrence relations or the $SL(K + 3, \mathbb{C})$ group and express them in terms of one amplitude. These "auxiliary states" and states with $a \neq 1$ or $c \neq 1$ in $V$ may represent other SSA, e.g. SSA of two tachyon and two arbitrary string states etc. Work in this direction is in progress.

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