Virtual signal in the heat equation and the enclosure method

Masaru Ikehata
Department of Mathematics, Graduate School of Engineering, Gunma University, Kiryu
376-8515, JAPAN
E-mail: ikehata@math.sci.gunma-u.ac.jp

Abstract. A relationship between travel time, heat equation and exponential solutions for backward heat equation are discussed. For the purpose an inverse source problem and inverse boundary value problems for the heat equations are considered. Those problems employ a single set of the temperature and heat flux on a known boundary and finite time interval as the observation data and are to extract an information about the shape and location of an unknown discontinuity in a material with known conductivity: (1) heat source; (2) boundary; (3) interface. For the problems (1) in two space dimensions and (2), (3) in one space dimension some direct extraction formulae of the information are given by employing the enclosure method. The results suggest a relationship between the travel time of a virtual signal and the observation data. Some conjectures for the corresponding problems to (2) in multi space dimensions are formulated and discussed.

1. Introduction
The enclosure method is a methodology in inverse problems for partial differential equations and was introduced by the author [7]. The method gave us how to use the exponential solution for extracting a partial information about the location of unknown discontinuity which appears as discontinuity of the coefficients of a partial differential equation or a part of the boundary of the domain of definition of the solution of the equation. This is completely different from the works [4, 19] in which they showed us how to use the exponential solution for the determination of an unknown coefficient in an elliptic equation from the associated Dirichlet-to-Neumann map. The original enclosure method makes use of infinitely many observation data. However, in [6] the author introduced a single measurement version of the enclosure method that can be divided into three parts.

1. Find a exponential solution of the formal adjoint of the governing equation for the background medium that is parameterized by a large parameter \( \tau \) and divides the whole space into two parts: in one part the absolute value of the solution decays as \( \tau \to \infty \); in another part the solution grows as \( \tau \to \infty \).

2. Construct an indicator function of independent variable \( \tau \) by multiplying the governing equation of the medium by the exponential solution, integrating over the domain of definition and extracting only the integral on the known boundary of the domain.

3. Study the asymptotic behaviour of the indicator function as \( \tau \to \infty \).

There are several existing applications of the single measurement version of the enclosure method to inverse problems for elliptic equations (see e.g., [9] and references therein) and quite recently...
this method has been applied to an inverse problem for the crack in an elastic body [13](an elliptic system of equations).

The aim of this paper is to report some new results of the application of a single measurement version of the enclosure method to the inverse problems for the heat equations (non elliptic equations) and give some conjectures. Those problems employ a single set of the temperature and heat flux on a known boundary and finite time interval as the observation data and are related to the thermal imaging of unknown discontinuity such as heat source, cavity, defect, interface or inclusion in a heat conductive material.

The outline of this paper is as follows. In Sections 2 and 3 we consider the problems of extracting an information about the shape and location of unknown heat source and boundary or interface in a material with known heat conductivity, respectively. We found that, for the heat equation the enclosure method yields some interesting and unexpected results. Those results suggest an introduction of a virtual signal with a finite propagation speed propagating in the heat conductive material. In Section 4 we describe some remarks and problems on the virtual signal appeared as an interpretation of the results in the previous sections; further applications and conjectures.

2. Inverse Source Problem

Let $\Omega$ be a bounded domain of $\mathbb{R}^2$ with smooth boundary. Let $T$ be an arbitrary positive number. Let $u = u(x, t)$ satisfy

$$u_t = \triangle u + f(x, t) \text{ in } \Omega \times [0, T],$$

$$u(x, 0) = 0 \text{ in } \Omega.$$

In this section we consider an inverse source problem for the heat equation. The problem is

**Problem 2.1.** Extract information about an unknown heat source $f(x, t)$ that occurred in a known domain from a single set of temperature $u(x, t)$ and flux distributions $\partial u / \partial \nu(x, t)$ for $(x, t) \in \partial \Omega \times [0, T]$.

Needless to say, one can not uniquely determine general $f(x, t)$ from a single set of temperature $u(x, t)$ and flux distributions $\partial u / \partial \nu(x, t)$ for $(x, t) \in \partial \Omega \times [0, T]$. However, under some a priori assumption on the form of the unknown source one can extract the full or partial information about the source. Here we give a list of the form of unknown sources in existing related results.

1. Yamatani-Ohnaka [24] assumed that the source takes the form

$$f(x, t) = \sum_{j=1}^{N} p_j \delta(x - x_j, t - t_j)$$

and that $0 < \max_j t_j < T$ and $p_j < 0$.

2. Yamamoto [23] assumed that the source takes the form

$$f(x, t) = \sigma(t)g(x), \sigma(0) \neq 0$$

and that $\sigma(t)$ is known.

3. EL Badia-Ha Duong [2] assumed that the source takes the form

$$f(x, t) = \sum_{j=1}^{N} c_j(t) \delta(x - x_j)$$
and that there exists a known $T_\star (< T)$ such that, for all $t \geq T_\star$, $c_j(t) = 0$ and for all $t < T_\star$, $c_j(t) < 0$.

In 2 and 3 the occurring time of the source is known; in 1 and 3 it is assumed that the source vanishes after a known time; in 2 the time varying part $\sigma(t)$ is assumed to be known.

Now we state one of results in [12]. We assume that the unknown source takes the form

$$f(x, t) = \sum_{j=1}^{N} \chi_{P_j \times [T_j, T]}(x, t) \rho_j(x, t)$$

where

- Each $P_j \subset \Omega$ is given by the interior of a polygon
- If $j \neq j'$, then $\overline{P_j} \cap \overline{P_{j'}} = \emptyset$
- $T_j$ satisfies $0 \leq T_j < T$
- $\rho_j$ is Hölder continuous
- $\rho_j(p, T_j) \neq 0$ at all vertices of the convex hull of $P_j$

The exponential solution for this problem is given by the function

$$v(x, t) = e^{-z \cdot z_t} e^{x \cdot z}$$

where

$$z = c\tau \left( \omega + i \sqrt{1 - \frac{1}{c^2}} \omega^\perp \right), \quad \tau > c^{-2}.$$ 

The function $v$ satisfies the backward heat equation $\partial_t v + \Delta v = 0$ and the complex vector $z$ satisfies the equation $z \cdot z = \tau$.

Since for each fixed $s$ we have

$$|e^{\tau s} v(x, t)| = e^{\tau(s-t+c x \cdot \omega)},$$

we know that the asymptotic behaviour of $e^{\tau s} v(x, t)$ as $\tau \to \infty$ divides the space time into two parts $s - t + c x \cdot \omega > 0$ and $s - t + c x \cdot \omega < 0$: if $s - t + c x \cdot \omega > 0$, then $|e^{\tau s} v(x, t)|$ is growing; if $s - t + c x \cdot \omega < 0$, then $|e^{\tau s} v(x, t)|$ is decaying.

Using function $v$, we define the indicator function of independent variable $\tau(> c^{-2})$ by the formula

$$I_{\omega, c}(\tau; s) = e^{\tau s} \int_0^T \int_{\partial \Omega} \left( \frac{\partial v}{\partial \nu} u - \frac{\partial u}{\partial \nu} v \right) dSdt, \quad \tau > c^{-2}.$$ 

The indicator function can be calculated from a single set of temperature $u(x, t)$ and flux distributions $\partial u/\partial \nu(x, t)$ for $(x, t) \in \partial \Omega \times [0, T]$.

For the description of the asymptotic behaviour of the indicator function we introduce some notation.

We denote by $\omega(e)$ the unit vector in three dimensions

$$\omega(e) = \frac{1}{\sqrt{c^2 + 1}}(e \omega, -1)^T.$$ 

Set

$$D = \bigcup_{j=1}^{N} (P_j \times [T_j, T]).$$

and introduce the support function for $D$:

$$h_D(\vartheta) = \sup_{(x, t) \in D} (x, t)^T \cdot \vartheta, \quad \vartheta \in S^2.$$ 

3
The following theorem gives an extraction formula of the values of the support function for $D$.

**Theorem 2.1 ([12]).** Assume that

- $\omega(c)$ is regular with respect to $D$, i.e., the set $\partial D \cap \{(x, t) | (x, t)^T \cdot \omega(c) = h_D(\omega(c))\}$ consists of a single point
- Observation time $T$ and $c$ satisfy

$$
\sup_{x \in \Omega} (x, T)^T \cdot \omega(c) < h_D(\omega(c)).
$$

Then, the formula

$$
\lim_{\tau \to \infty} \log_{x=0} |I_{\omega, c}(\tau)| = \sqrt{c^2 + 1} h_D(\omega(c)),
$$

is valid. Moreover we have

- If $s \leq -\sqrt{c^2 + 1} h_D(\omega(c))$, then $\lim_{\tau \to \infty} |I_{\omega, c}(\tau; s)| = 0$
- If $s > -\sqrt{c^2 + 1} h_D(\omega(c))$, then $\lim_{\tau \to \infty} |I_{\omega, c}(\tau; s)| = \infty$

The main feature of our approach is

- it is based on a simple *one line* formula (2.2)
- we do not make use of the exact controllability of the heat equation nor the completeness of the eigenfunctions of the Dirichlet Laplacian in $\Omega$
- the assumptions on the unknown source is quite general
- the method provides us a *brief* information about the time and the location when and where the unknown source firstly appeared instead of the detailed information of the source
- it is possible to give a regularization of the formula (2.2) by employing the argument done in [8]

Here we give an *interpretation* of the result. The point is the interpretation of the condition (2.1). Assume that a *virtual* signal started at the point $x_0$ with $x_0 \cdot \omega(c) = h_D(\omega(c))$ with propagation speed $1/c$ and propagated on the plane spanned by two vectors $(c\omega, -1)^T$ and $(0, -1)^T$ to the exterior of $D$. Then condition (2.1) means that the arrival time of the signal at the boundary of $\Omega$ is less than $T$. From this point of view condition (2.1) is quite natural and understandable. However, it should be noted that we do not know what is the signal. However, one may say that the indicator function picks up a *virtual* signal with an arbitrary fixed propagation speed caused by the unknown source. One can apply the enclosure method to the corresponding inverse source problem for the wave equation. Then changing $c$ in the inverse source problem for the heat equation

$$
\partial_t u = \Delta u + f(x, t)
$$

$v = e^{-rt} e^{x \cdot z}$

$$
z = cr(\omega + i \sqrt{1 - \frac{1}{c^2 r^2}} \omega^\perp)
$$

$$
\tau > c^{-2}, c > 0
$$

has the same effect as changing *propagation speed* of a *real* signal in the inverse source problem
for the wave equation
\[ \partial_t^2 u = \frac{1}{c^2} \Delta u + f(x, t) \]
\[ v = e^{-\tau} e^{x \cdot z'} \]
\[ z' = c \tau \omega \]
\[ c > 0. \]

It should be noted that one can also extract the occurring time \( \min T_j \) directly by using another exponential solution. The construction for the variable coefficient case is closely related to the high frequency asymptotic solution of the corresponding wave equation. Those are given in [12].

3. Inverse Initial Boundary Value Problem

3.1. Corrosion.

Let us describe the problem. Let \( a > 0 \) and \( \rho \geq 0 \). Let \( u = u(x, t) \) be an arbitrary solution of the problem:
\[ u_t = u_{xx} \text{ in } [0, a] \times [0, T], \]
\[ u_x(a, t) + \rho u(a, t) = 0 \text{ for } t \in ]0, T[ , \]
\[ u(x, 0) = 0 \text{ in } ]0, a[ . \]

**Problem 3.1.** Assume that both \( a \) and \( \rho \) are unknown. Extract \( a \) from \( u(0, t) \) and \( u_x(0, t) \) for \( 0 < t < T \).

There are extensive studies for the uniqueness and stability issue for the corresponding problem in multi space dimensions. See [3, 5, 21, 22] and references therein.

In one space dimensional case, in general, it is possible to calculate so-called the response operator for the heat equation from the data above. Therefore it may be possible to apply the method in [1] to the response operator and get the spectral data. Thus the problem may be reduced to the inverse spectral problem which has been well studied. However, this is due to the special situation of one space dimension and the procedure to get the spectral data is not an easy way.

In this section, without reducing to other inverse problems, we present an independent and simpler approach which is an application of the enclosure method to Problem 3.1.

Define
\[ v(x, t) = e^{-z^2 \tau} e^{xz} \]
where \( \tau > c^{-2} \) and \( z \) is given by
\[ z = -c \tau \left( 1 + i \sqrt{1 - \frac{1}{c^2 \tau}} \right). \]

The function \( v \) satisfies the backward heat equation \( \partial_t v + \Delta v = 0 \); \( z \) satisfies the equation
\[ z^2 = \tau + i 2 c^2 \tau^2 \sqrt{1 - \frac{1}{c^2 \tau}}. \]

For each \( s \) we have \( |e^{r \tau} v(x, t)| = e^{r(s-t-cx)} \). This yields that the asymptotic behaviour of \( e^{r \tau} v(x, t) \) as \( \tau \to \infty \) divides the space time into two parts \( s - t - cx > 0 \) and \( s - t - cx < 0 \): if \( s - t - cx > 0 \), then \( |e^{r \tau} v(x, t)| \) is growing; if \( s - t - cx < 0 \), then \( |e^{r \tau} v(x, t)| \) is decaying.
Define the indicator function of independent variable \( \tau(> c^{-2}) \) by the formula
\[
I_c(\tau; s) = e^{\tau s} \int_0^T (-v_x(0, t)u(0, t) + u_x(0, t)v(0, t)) \, dt.
\]

**Definition 3.1.** Condition \( c \) Let \( c > 0 \). We say that a function \( f \in L^2(0, T) \) satisfies the condition \( c \) if there exist positive constants \( C \) and real numbers \( \tau_0(\geq c^{-2}) \), \( \mu \) such that, for all \( \tau \geq \tau_0 \)
\[
C \tau^\mu \leq |\int_0^T f(t)e^{-z^2t} \, dt|
\]
where \( z \) is given by (3.2).

Two remarks are in order.
- if \( f(t) \) is given by a polynomial of \( t \) on \( [0, T] \) that is not identically zero, then for all \( c > 0 \) \( f(t) \) satisfies the condition \( c \).
- if \( f(t) \) is smooth on \( [0, T] \) and \( t = 0 \) is not a zero point of infinite order, then for all \( c > 0 \) \( f(t) \) satisfies the condition \( c \).

**Theorem 3.1 ([10]).** Assume that we know a positive number \( M \) such that \( M \geq 2a \). Let \( c \) be an arbitrary positive number satisfying \( Mc < T \). Let the \( u_x(0, t) \) satisfy the condition \( c \). Then the formula
\[
\lim_{\tau \to \infty} \frac{\log |I_c(\tau; 0)|}{\tau} = -2ca,
\]
is valid.

Once \( a \) is known, then one can explicitly extract also \( \rho \) from the data (Remark 4.1 in [10]).

Needless to say, from this theorem one automatically obtains the uniqueness theorem: \( u(0, t) \) and \( u_x(0, t) \) for \( 0 < t < T \) uniquely determine \( a \) (and \( \rho \)) provided \( u_x(0, t) \) satisfies the condition \( c \). This type uniqueness proof is completely different from existing one. See [3] for comparison.

Here we consider: what is \( 2ca \)?

- In the case when \( \rho = 0 \), \( u(x, t) \) can be extended to the domain \( [0, 2a[\times]0, T[ \) as a solution of the heat equation by the reflection \( x \mapsto 2a - x \) at \( x = a \). In this case we think that the enclosure method yielded the information about the location of the set \( \{2a, \infty]\times]0, T[ \) and \( -2ca \) has the meaning
\[
-2ca = \sup \left\{ \left( \begin{array}{c} x \\ t \end{array} \right) : \left( \begin{array}{c} -c \\ -1 \end{array} \right) \cdot \left( \begin{array}{c} x \\ t \end{array} \right) \in [2a, \infty[ \times ]0, T[ \right\}.
\]

This is nothing but the information about the value of the support function for \( [2a, \infty[ \times ]0, T[ \) at the direction \( (-c, -1)^T \).

A more attractive interpretation is the following.
- \( 2ca \) is the travel time of a virtual signal with propagation speed \( 1/c \) that starts at the known boundary \( x = 0 \) and the initial time \( t = 0 \), reflects at another unknown boundary \( x = a \) and returns to the original boundary.

The idea behind this interpretation is the belief that, in an appropriate sense

Solution of **Heat Equation** \( v_t = v_{xx} \sim \sum_{\xi > 0} \) Solution of **Wave Equation** \( u_t = \xi^2 u_{xx} \).

It is well known that, at least, for the initial value problem for the heat equation there is a relationship with a corresponding initial value problem for the wave equation with arbitrary fixed propagation speed (see, for example [14, 15]). So if this is true in the present case, then it is reasonable to expect that the observation date coming from the heat equation should contain
some information coming from the wave equation with arbitrary propagation speed \( \xi = 1/c \).
This is the role of \( c \). This suggests that the indicator function for the heat equation is a mathematical instrument that picks up a signal coming from the corresponding wave equation with propagation speed \( 1/c \). Of course, to get \( a \) only from the observation data one can just use a small single \( c \) with \( Mc < T \).

Note that, in [11] we have already confirmed that this interpretation works also for the case when the heat conductivity of the material is given by a smooth function or piecewise constant function. See also 4.2 of Section 4.

### 3.2. Interface

Let \( 0 < b < a \). Define

\[
\gamma(x) = \begin{cases} 
\gamma_1, & \text{if } 0 < x < b, \\
\gamma_2, & \text{if } b < x < a,
\end{cases}
\]

where both \( \gamma_1 \) and \( \gamma_2 \) are positive constants and satisfies \( \gamma_2 \neq \gamma_1 \).

Let \( u \) be an arbitrary solution of the problem:

\[
\begin{align*}
u_t &= (\gamma u_x)_x & \text{in } ]0, a[ \times ]0, T[ , \\
u(x, 0) &= 0 & \text{in } ]0, a[.
\end{align*}
\]

#### Problem 3.2

Assume that \( \gamma_1 \) is known and that \( a, b \) and \( \gamma_2 \) are all unknown. Extract \( b \) from \( u(0, t) \) and \( \gamma_1 u_x(0, t) \) for \( 0 < t < T \).

Let \( c \) be an arbitrary positive number. Let

\[
v(x, t) = e^{-z^2t} e^{x z/\sqrt{\gamma_1}}
\]

where \( z \) is given by (3.2).

The function \( v \) satisfies the backward heat equation \( v_t + \gamma_1 v_{xx} = 0 \) in the whole space-time.

Define the indicator function of independent variable \( \tau(> c^{-2}) \) by the formula

\[
I_c(\tau) = \int_0^T (-\gamma_1 v_x(0, t)u(0, t) + \gamma_1 u_x(0, t)v(0, t)) dt.
\]

#### Theorem 3.2 ([11])

Assume that we know a positive number \( M \) such that \( M \geq 2b/\sqrt{\gamma_1} \). Let \( c \) satisfy \( Mc < T \). Assume that \( u_x(0, t) \) is stronger than \( u_x(a, t) \) in the sense

\[
\lim_{\tau \to \infty} \frac{\int_0^T e^{-z^2t}u_x(a, t)dt}{\int_0^T e^{-z^2t}u_x(0, t)dt} \exp \left( c \tau \left( \frac{b}{\sqrt{\gamma_1}} - \frac{(a - b)}{\sqrt{\gamma_2}} \right) \right) = 0
\]

and that \( u_x(0, t) \) satisfies the condition \( c \). Then the formula

\[
\lim_{\tau \to \infty} \frac{\log |I_c(\tau)|}{\tau} = -2 \frac{cb}{\sqrt{\gamma_1}}
\]

is valid.

Needless to say, this theorem also automatically yields a uniqueness theorem.

Note that \( 2cb/\sqrt{\gamma_1} \) is the travel time of a virtual signal with propagation speed \( \sqrt{\gamma_1}/c \) that starts at the known boundary \( x = 0 \) and the initial time \( t = 0 \), reflects at another unknown boundary \( x = b \) and returns to the original boundary. The condition (3.5) plays the role of killing another virtual signal with propagation speed \( \sqrt{\gamma_2}/c \) that starts at the unknown boundary \( x = a \) and the initial time \( t = 0 \), refracts at another unknown boundary \( x = b \) and goes to the boundary \( x = 0 \).
4. Questions, Remarks and Conjectures

4.1. What is the virtual signal?

- At the present time the signal is virtual. This means that assuming the existence of such a signal leads us to the better understanding of the results.
- Is there really such a signal?
- If it is so, how does the indicator function pick the signal up? Find a proof that really make use of the signal. Preferably fully time dependent proof.
- In the case $\rho = 0$ (Homogeneous Neumann) and $u_x(0,t)$ is given by a polynomial we have a time dependent proof [10] which is based on the eigenfunction expansions and the asymptotic behaviour of the special function studied by Olver [17]

$$S_1(w) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k^2 + w^2}, \quad \text{Re} \ w > 0.$$  

However, the proof does not indicate us the signal explicitly.

4.2. Further applications

One can apply the method to the wave equation

$$u_{tt} = \frac{1}{c^2} u_{xx} \text{ in } [0, a[ \times ]0, T[,}$$

where $c$ is a fixed positive constant. In this case one can really extract the travel time by using the exponential solution of the wave equation:

$$v(x, t) = e^{-\tau (cx + t)}.$$  

The method covers also the heat equation with a variable coefficient ([11]):

$$u_t = (\gamma(x)u_x)_{x} \text{ in } [0, a[ \times ]0, T[,}$$

$$\gamma(a)u_x(a, t) + \rho u(a, t) = 0 \text{ for } t \in ]0, T[,}$$

$$u(x, 0) = 0 \text{ in } [0, a[.}$$

One can obtain an extraction formula of the quantity

$$2c \int_0^a \frac{dx}{\sqrt{\gamma(x)}}$$

from $u(0, t)$ and $u_x(0, t)$, $0 < t < T$ provided

- we know $M$ such that $M \geq a$
- the conductivity $\gamma$ can be given by a known positive function $\tilde{\gamma}$ of class $C^2$ on $[0, M]$
- $T$ and $c$ satisfies

$$T > 2c \int_0^M \frac{dx}{\sqrt{\tilde{\gamma}(x)}}$$

- $u_x(0, t)$, $0 < t < T$ satisfies the condition $c$

Note that: it is possible to make use of the real exponential solution of the backward heat equation instead of the complex one. Consider the simplest case $\gamma(x) \equiv 1$. Then the function $v(x, t) = e^{-\tau^2 t}e^{-\tau x}$ satisfies the backward heat equation. Using this function, we introduce another indicator function

$$J(\tau) = \int_0^T (-u_x(0,t)u(0,t) + u_x(0,t)v(0,t)) \, dt, \quad \tau > 0$$
where \( u \) satisfies (3.1). Then one obtains the formula

\[
\lim_{\tau \to \infty} \frac{\log |J(\tau)|}{\tau} = -2a
\]  

(4.1)

provided there exist positive constant \( C \) and a positive number \( \tau_0 \), real number \( \mu \) such that, for all \( \tau > \tau_0 \)

\[ C \tau^\mu \leq | \int_0^T e^{-\tau^2 t} u_x(0, t) dt | . \]

In this case the parameter \( c \) does not appear. See [11] for the variable coefficient case.

It would be interesting to apply our method to the linear equations of thermoelasticity with second sound in one space dimension where thermal disturbances are propagating as wave-like pules travelling at finite speed. See [18] and references therein for more information about the equations.

### 4.3. Extension to higher space dimensions. Four Conjectures

Let \( \Omega \) be a bounded domain of \( \mathbb{R}^n \) \((n = 2, 3)\) with smooth boundary. Let \( D \) an open subset of \( \Omega \) with smooth boundary and satisfy that: \( \overline{D} \subset \Omega; \Omega \setminus \overline{D} \) is connected. Let \( T \) be an arbitrary fixed positive number.

Let \( u = u(x, t) \) satisfy

\[
\begin{align*}
  u_t - \Delta u &= 0 \quad \text{in } \Omega \setminus \overline{D} \times [0, T], \\
  \frac{\partial u}{\partial \nu} + \rho u &= 0 \quad \text{on } \partial D \times [0, T], \\
  u(x, 0) &= 0 \quad \text{in } \Omega \setminus \overline{D}.
\end{align*}
\]

The main purpose of the previous section is to seek an approach that can be applied to the following inverse problem.

**Problem 4.1.** Extract information about the location and shape of \( D \) from the single set of the temperature \( u(x, t) \) and heat flux \( \partial u/\partial \nu(x, t) \) for \((x, t) \in \partial \Omega \times [0, T]\).

The proof of Theorems 3.1 and 3.2 is based on the asymptotic behaviour of the transform \( w(x, z) \) as \( \tau \to \infty \) for the solutions \( u(x, t) \) of (3.1), (3.4):

\[
w(x, z) = \int_0^T e^{-z^2 t} u(x, t) dt .
\]

This should be called a stationary approach. The extensions to the cases of two and three space dimensions belong to our next research projects. However, in this subsection, using this idea, we derive four conjectures.

Define

\[
w(x, \tau) = \int_0^T e^{-z^2 t} u(x, t) dt, \quad \tau > 0.
\]

Then we see that this \( w \) satisfies

\[
(\Delta - z \cdot z) w = u(x, T) e^{-z \cdot z T} \quad \text{in } \Omega \setminus \overline{D}
\]

\[
\frac{\partial w}{\partial \nu} + \rho w = 0 \quad \text{on } \partial D.
\]

(4.2)

Let \( v = v(x) \) satisfy

\[
(\Delta - z \cdot z) v = 0 \quad \text{in } \Omega.
\]

(4.3)
Integration by parts yields
\[
\int_{\partial \Omega} \left( \frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right) dS = \int_{\partial D} w \left( \rho v + \frac{\partial v}{\partial \nu} \right) dS - e^{-z \cdot z} T \int_{\Omega \setminus \mathcal{D}} u(x, T)v(x)dx. \tag{4.4}
\]
Note that the function \(e^{-z \cdot z} v(x)\) satisfies a backward heat equation.

First choice of \(v\) satisfying (4.3).

Given \(\omega \in S^2\) choose \(\omega_\perp \in S^2\) with \(\omega \cdot \omega_\perp = 0\).

Define
\[
v(x) = e^{x \cdot z} \tag{4.5}
\]
where \(\tau\) satisfies \(\tau > c^{-2}\) and
\[
z = c\tau \left( \omega + i \sqrt{1 - \frac{1}{c^2}} \omega_\perp \right).
\]
This \(v\) satisfies (4.3). By virtue of multi space dimension we could choose \(z\) in such a way that \(z \cdot z\) is positive (compare with (3.3)). This \(z\) is exactly same as in Section 2.

Now define the first indicator function by the left hand side of (4.4):
\[
I_{\omega, \omega_\perp}(\tau; c) = \int_{\partial \Omega} \left( \frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right) dS.
\]
Of course studying the behaviour of \(I_{\omega, \omega_\perp}(\tau; c)\) is equivalent to that of the right hand side of (4.4). Since \(z \cdot z = \tau\), first we know that
\[
I_{\omega, \omega_\perp}(\tau; c) = \int_{\partial D} w \left( \rho v + \frac{\partial v}{\partial \nu} \right) dS + O \left( e^{-\tau (T - c h_\Omega(\omega))} \right). \tag{4.6}
\]
Here for simplicity we assume that \(D\) is strictly convex (say a ball). Then one can find a unique point \(q = q(\omega)\) on \(\partial D\) such that \(q \cdot \omega = h_D(\omega)\). Then using a stationary phase method, we may have
\[
\int_{\partial D} w \left( \rho v + \frac{\partial v}{\partial \nu} \right) dS \sim \left( \rho(q)v(q) + \frac{\partial v}{\partial \nu}(q) \right) \int_{\text{near } q \cap \partial D} w dS \tag{4.7}
\]
\[
\sim e^{\tau c h_D(\omega)} e^{i c \gamma q \omega_\perp \sqrt{1 - \frac{1}{c^2}}} \int_{\text{near } q \cap \partial D} w dS
\]
The main point is the analysis of \(w\) near \(q\) on \(\partial D\). Let \(\tilde{w} = \tilde{w}(x, \tau)\) solve
\[
(\Delta - \tau) \tilde{w} = 0 \text{ in } \Omega \setminus \mathcal{D},
\]
\[
\frac{\partial \tilde{w}}{\partial \nu} + \rho \tilde{w} = 0 \text{ on } \partial D, \tag{4.8}
\]
\[
\frac{\partial \tilde{w}}{\partial \nu} = \frac{\partial w}{\partial \nu} \text{ on } \partial \Omega.
\]
The well posedness yields
\[
w = \tilde{w} + O \left( e^{-\tau T} \right).
\]
From this and (4.7) we will obtain

$$\int_{\partial D} w \left( \rho v + \frac{\partial v}{\partial \nu} \right) dS \sim e^{\tau c h_D(\omega)} e^{i\tau q \cdot \omega} \sqrt{1 - \frac{1}{c^2 \tau}} \int_{\text{near } q \cap \partial D} \tilde{w} dS.$$  \hspace{1cm} (4.9)

Therefore the main analytical part of this approach can be reduced to the analysis of the solution (4.8) as $\tau \rightarrow \infty$. We think that the method of geometrical optics may work for this case and yield, as $\tau \rightarrow \infty$

$$\tilde{w} \sim e^{-\sqrt{\tau} \text{dist}(q, \partial \Omega)}$$  \hspace{1cm} (4.10)

under suitable condition on $\partial w/\partial \nu$ or $\partial u/\partial \nu(x, t)$ for $(x, t) \in \partial \Omega \times ]0, T[$. Note that, in a simpler situation ($D = \emptyset$ and $\tilde{w} \equiv 1$ on $\partial \Omega$) a similar asymptotic formula has been established by Varadhan [20]. This is related to the short time asymptotics of the heat kernel through the Laplace transform. See also Kannai [15], Norris [16] and references therein.

Now let $T > c(h_\Omega(\omega) - h_D(\omega))$. From (4.6), (4.9) and (4.10) we may see that: there exist real number $\mu$ and a positive constant $A$ such that, as $\tau \rightarrow \infty$

$$\lim_{\tau \to \infty} \tau^\mu e^{-\tau c h_D(\omega) + \sqrt{\tau} \text{dist}(q, \partial \Omega)} |I_{\omega, \omega^\perp}(\tau; c)| = A.$$  

This yields

$$\frac{\log |I_{\omega, \omega^\perp}(\tau; c)|}{\tau} = c h_D(\omega) - \frac{1}{\sqrt{\tau}} \text{dist}(q, \partial \Omega) + O \left( \frac{\log \tau}{\tau} \right).$$

Summing up, we may propose

**Conjecture A.** Let $c$ satisfy $T > c(h_\Omega(\omega) - h_D(\omega))$. The formula

$$\lim_{\tau \to \infty} \frac{\log |I_{\omega, \omega^\perp}(\tau; c)|}{\tau} = c h_D(\omega),$$

is valid.

This is the extraction formula of the convex hull of $D$. However, it has better rewrite the formula as

$$\lim_{\tau \to \infty} \frac{\log e^{-\tau c h_\Omega(\omega)} |I_{\omega, \omega^\perp}(\tau; c)|}{\tau} = -c \{ (h_\Omega(\omega) - h_D(\omega)) \}.$$  

Note that the function $e^{-\tau c h_\Omega(\omega)} e^{x \cdot z}$ is exponentially growing in $x \cdot \omega > h_\Omega(\omega)$ and decaying in $x \cdot \omega < h_\Omega(\omega)$ as $\tau \to \infty$. Clearly this conjecture should not be considered as the generalization of Theorem 3.1. In the right hand side a contribution by a reflection of the virtual signal caused by cavity does not appear.

However, there is another possibility of the choice of complex $z$.

**Second choice of $v$ satisfying (4.3).**

It is given by (4.5) with

$$z = c\tau \left( 1 + i \sqrt{1 - \frac{1}{c^2 \tau}} \right) \omega, \quad \tau > c^{-2}.$$  

In this case we have

$$z \cdot z = \{ c\tau \left( 1 + i \sqrt{1 - \frac{1}{c^2 \tau}} \right) \}^2$$

and, however, the real part of $z \cdot z$ is still just $\tau$. For this $z$ define

$$I_\omega(\tau; c) = \int_{\partial \Omega} \left( \frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right) dS$$
where \( w \) is same as that of the first indicator function.

Using a similar heuristic argument as above we may propose

**Conjecture B.** Given \( \omega \) assume that there exists a unique \( q \in \partial D \) such that \( q \cdot \omega = h_D(\omega) \). Let \( c \) satisfy \( T > c (h_\Omega(\omega) - h_D(\omega)) \). The formula

\[
\lim_{\tau \to \infty} \log \frac{e^{-\tau c h_\Omega(\omega)} I_\omega(\tau; c)}{\tau} = -c \{ (h_\Omega(\omega) - h_D(\omega)) + \text{dist}(q, \partial \Omega) \},
\]

is valid.

The quantity \( c (h_\Omega(\omega) - h_D(\omega)) \) can be interpreted as the first arrival time of a *virtual* signal at \( \partial D \) which started from the point \( x \cdot \omega = h_\Omega(\omega) \) with propagation speed \( 1/c \). The time for returning to the boundary \( \partial \Omega \) from \( \partial D \) should be \( c \text{dist}(q, \partial \Omega) \). Therefore Conjecture B can be considered as the extension of Theorem 3.1 to the higher space dimensional case.

**Third choice of \( v \) satisfying (4.3).**

It is given by (4.5) with

\[
z = \sqrt{\tau} \omega.
\]

Define the third indicator function by the formula

\[
I_\omega(\tau) = \int_{\partial \Omega} \left( \frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right) dS
\]

where \( w \) is same as that of the first indicator function.

**Conjecture C.** Given \( \omega \) assume that there exists a unique \( q \in \partial D \) such that \( q \cdot \omega = h_D(\omega) \). The formula

\[
\lim_{\tau \to \infty} \log \frac{e^{-\sqrt{\tau} h_\Omega(\omega)} I_\omega(\tau)}{\sqrt{\tau}} = -\{ (h_\Omega(\omega) - h_D(\omega)) + \text{dist}(q, \partial \Omega) \},
\]

is valid.

Note that the function \( e^{-\sqrt{\tau} h_\Omega(\omega)} e^{\sqrt{\tau} x \cdot \omega} \) is exponentially growing in \( x \cdot \omega > h_\Omega(\omega) \) and decaying in \( x \cdot \omega < h_\Omega(\omega) \) as \( \tau \to \infty \). So replacing \( \tau \) in (4.11) with \( \tau^2 \), one can consider (4.11) the multi space dimensional version of (4.1).

Finally we introduce the fourth indicator function.

Let \( p \) be an *arbitrary* point outside \( \Omega \). Then the function

\[
v(x, \tau, p) = \frac{e^{\pm \sqrt{\tau} |x - p|}}{|x - p|}
\]

satisfies \( (\Delta - \tau) v = 0 \) in a neighbourhood of \( \overline{\Omega} \). Now define the fourth indicator function by the formula

\[
I_{\pm}(\tau; p) = \int_{\partial \Omega} \left( \frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right) dS
\]

where

\[
w(x, t) = \int_0^T e^{-\tau t} u(x, t) dt.
\]

Using a similar argument and (4.10), one may propose:

**Conjecture D.**
(1) Given \( q \) outside \( \Omega \) assume that there exists a unique \( q_- \in \partial D \) such that \( |q_- - p| = \inf_{y \in \partial D} |y - p| \). The formula
\[
\lim_{\tau \to \infty} \frac{\log |I_-(\tau; p)|}{\sqrt{\tau}} = -\{ \text{dist}(q_-, \partial \Omega) + |q_- - p| \},
\]
is valid.

(2) Given \( q \) outside \( \Omega \) assume that there exists a unique \( q_+ \in \partial D \) such that \( |q_+ - p| = \sup_{y \in \partial D} |y - p| \). The formula
\[
\lim_{\tau \to \infty} \frac{\log |I_+(\tau; p)|}{\sqrt{\tau}} = -\{ \text{dist}(q_+, \partial \Omega) - |q_+ - p| \},
\]
is valid.

Comparing these conjectures, one knows that Conjecture A (of course, if it is correct) indicates the advantage of making use of complex plane wave solutions. However, the restriction on \( c \) that \( T > c(h_{\Omega}(\omega) - h_D(\Omega)) \) and the result are different from the one space dimensional case. So the conjecture A does not fit the interpretation in one space dimensional case. Maybe Conjecture A corresponds to the original formula for the Laplace equation in [6]. For this point Conjectures B, C and (1) of D may fit with the interpretation. In a forthcoming paper we will study conjectures A to D.

It would be interested to consider also the unknown interface instead of cavity or the corresponding problems for the thermoelasticity, etc. There are a lot of inverse problems to be approached by using the idea of the enclosure method.

Needless to say, it may be possible to generalize the enclosure method for the heat equation by using other solutions for the backward heat equation instead of exponential solutions. This also belongs to our future plan.

Throughout the study we found that the information contained in observation data on a finite time interval is quite rich. The idea of the enclosure method makes a direct link between the study of inverse initial boundary value problems for the heat equations on a finite time interval and that of the high frequency asymptotics of the solutions of elliptic equations.

Acknowledgments
This research was partially supported by Grant-in-Aid for Scientific Research (C)(No. 18540160) of Japan Society for the Promotion of Science.

The author would like to thank Mikhail Belishev for remarks about the BC method for the heat equation in one space dimension and comments on the results in [10, 11]. He would like to thank Shigeru Sakaguchi for bringing reference [20] to his attention and having related discussion.

References
[1] Avdonin S, Belishev M and Rozhkov Yu 1997 The BC-method in the inverse problem for the heat equation J. Inv. Ill-Posed Probl. 5 309-22
[2] EL Badia A and Ha Duong T 2002 On an inverse source problem for the heat equation. Application to a pollution detection problem J. Inv. Ill-Posed Probl. 10 585-99
[3] Bryan K and Caudill F Jr 1997 Uniqueness for a boundary identification problem in thermal imaging Differential Equations and Computational Simulations III (Electric Journal of Differential Equations Conference 01) 23-39
[4] Calderón A P 1980 On an inverse boundary value problem in Seminar on Numerical Analysis and its Applications to Continuum Physics ed Meyer W H and Raupp M A (Rio de Janeiro: Brazilian Math. Soc.) p 65-73

13
[5] Canuto B, Rosset E and Vessella S 2002 Quantitative estimates of unique continuation for parabolic equations and inverse initial-boundary value problems with unknown boundaries Trans. Amer. Math. Soc. 354 491-535
[6] Ikehata M 1999 Enclosing a polygonal cavity in a two-dimensional bounded domain from Cauchy data Inverse Problems 15 1231-41
[7] Ikehata M 2000 Reconstruction of the support function for inclusion from boundary measurements J. Inv. Ill-Posed Probl. 8 367-78
[8] Ikehata M 2002 A regularized extraction formula in the enclosure method Inverse Problems 18 435-40
[9] Ikehata M 2005 The Herglotz wave function, the Vekua transform and the enclosure method Hiroshima Math. J. 35 485-506
[10] Ikehata M 2006 Travel time and heat equation. One space dimensional case Preprint arXiv:math.AP/0611447
[11] Ikehata M 2006 Travel time and heat equation. One space dimensional case II Preprint arXiv:math.AP/0611449
[12] Ikehata M 2007 An inverse source problem for the heat equation and the enclosure method Inverse Problems 23 183-202
[13] Ikehata M and Itou H 2007 Reconstruction of a linear crack in an isotropic elastic body from a single set of measured data Inverse Problems 23 589-607
[14] Jost J 2002 Partial differential equations (Graduate texts in mathematics 214)(New York: Springer)
[15] Kannai Y 1977 Off diagonal short time asymptotics for fundamental solutions of diffusion equations Comm. PDE 2 781-830
[16] Norris J R 1997 Heat kernel asymptotics and the distance function in Lipschitz Riemannian manifolds Acta Math. 179 79-103
[17] Olver F W 1974 Asymptotics and special functions (New York and London: Academic Press)
[18] Racke R 2002 Thermoelasticity with second sound-exponential stability in linear and non-linear 1-d Math. Meth. Appl. Sci. 25 409-41
[19] Sylvestre J and Uhlmann G 1987 A global uniqueness theorem for an inverse boundary value problems Ann. Math. 125 153-69
[20] Varadhan S R S 1967 On the behaviour of the fundamental solution of the heat equation with variable coefficients Comm. Pure Appl. Math. 20 431-55
[21] Vessella S 1995 A domain identification problem for the heat equation-Hölder stability estimate J. Inv. Ill-Posed Probl. 3 495-503
[22] Vessella S 1997 Stability estimates in an inverse problem for a three-dimensional heat equation SIAM J. Math. Anal. 28 1354-70
[23] Yamamoto M 1995 Stability, reconstruction formula and regularization for an inverse source hyperbolic problem by a control method Inverse Problems 11 481-96
[24] Yamatani K and Ohnaka K 1997 An estimation method for point sources of multidimensional diffusion equation Appl. Math. Modelling 21 77-84