From Einstein’s 1905 Postulates 
to the Geometry of Flat Space-Time

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Minkowski diagrams in 1+1 dimensional flat space-time are given a strictly geometric derivation, directly from two gedanken experiments incorporating the principle of the constancy of the velocity of light and the principle of (special) relativity. Rectangles of photon trajectories play a central role in determining the simultaneity convention and in establishing the invariance of the interval.

For the hundredth anniversary of Zur Elektrodynamik bewegter Körper I would like to describe a derivation of the geometry of flat Minkowski space-time straight from Einstein’s two Voraussetzungen. No use will be made of either the Lorentz transformation or the invariant interval, which emerge as secondary features of the special-relativistic space-time geometry that follows directly from applying the two postulates to two gedanken experiments. I consider only one spatial dimension, commenting at the end on the higher dimensional generalization.

Assisting me in this approach to space-time geometry will be Alice and Bob, who have appeared in expositions of cryptography for many years. With the development of quantum cryptography and quantum information theory they have become well known to many physicists during the past decade. They can also be invaluable in making expositions of relativity more readable and concise.

1. Alice’s diagram: equilocs, equitemps, and scale factors

Alice, who uses the terminology appropriate to an inertial frame of reference, represents events as points in a plane diagram. In what follows I shall use “event” to signify the representative point as well as the event itself. Events that happen at the same place in Alice’s frame are all placed on the same straight line: an equiloc. (I would have preferred the term isotop but the similarity — not to mention the identity in German — to “isotope” makes this unacceptable.) Equilocs associated with different places must be parallel, for otherwise they would intersect and the point of intersection would represent a single event happening in two different places.

The distance between equilocs in Alice’s diagram is proportional to the distance between the places they represent. The proportionality constant $\lambda$ (or $\lambda_A$ if we wish to
emphasize its use by Alice) specifies, for example, the number of centimeters of diagram separating equilocs associated with events one foot apart in space. Her unit of distance, the *foot* (abbreviated f), is defined to be a light nanosecond: 1 f = 0.299792458 m. Although this term competes with the name still used in a few backward countries for a distance of 0.3048 m, Alice’s foot falls short of the English foot by a mere 1.6%. What makes the word irresistible is its phonetic resemblance to φωτός (light).

Alice positions events along her equilocs so that events she takes to be simultaneous also lie along a straight line: an *equitemp*. (The more beautiful *isochron* is linguistically incompatible with “equiloc” and, like “isotop”, is also used for other scientific purposes.) Equitemps associated with different times must also be parallel, since an event cannot happen at two distinct times. Alice’s equitemps cannot be parallel to her equilocs, since specifying both a place and a time specifies a unique event, and therefore a unique point in her diagram, but aside from that, she can choose the angle Θ (more fully, Θ_A) between her equitemps and equilocs as she wishes.

The distance between equitemps in Alice’s diagram is proportional to the time between the events they represent. It is convenient for her to use the same scale factor λ for equitemps associated with events 1 ns apart, as she uses for equilocs associated with events 1 f apart. It is also convenient to introduce a second scale factor µ which specifies the distance along a given equiloc of events whose times differ by 1 ns (or the distance along a given equitemp of events whose locations differ by 1 f). Evidently (see Figure 1) the two scale factors are related by

\[ \lambda = \mu \sin \Theta. \]

Because the speed of light is 1 f/ns and because of the relation Alice imposes on the scale factors for her equilocs and equitemps, it follows from Figure 1 that the space-time trajectories of two oppositely moving photons present at a given event, bisect the angles between the equiloc and equitemp on which that event lies, and that trajectories of oppositely moving photons are perpendicular.

Alice orients her diagram as in Figure 1, so the two families of photon trajectories make angles of π/4 with the vertical direction on the page. Her equilocs and equitemps are symmetrically disposed on either side of the diagonal directions of the photon trajectories, at angles \( \theta = \frac{1}{2} \Theta \) (or \( \theta' = \frac{1}{2} \Theta' \), \( \Theta' = \pi - \Theta \)) to the photon lines. Alice further orients her page so that the equitemps lie below the diagonals (i.e. they are more horizontal than vertical) and the equilocs lie above them (i.e. they are more vertical than horizontal). Finally, she orients the page so that equitemps higher on the page are associated with...
Figure 1. The parallel lines tilting slightly upward to the right are equitemps. Each represents events that happen at the same time in Alice’s frame. The times associated with the two equitemps are 1 ns apart. The parallel lines tilting steeply upward to the right are equilocs. Each represents events that happen in the same place. The positions associated with the two equilocs are 1 f apart. Alice’s scale factor $\lambda$ is the distance in the diagram between the equilocs or between the equitemps. Her scale factor $\mu$ is the length in the diagram of the (more heavily drawn) segments of the equitemps and equilocs between points (black circles) that represent events 1 f or 1 ns apart. Evidently $\lambda = \mu \sin \Theta$ where $\Theta = 2 \theta$ is the angle between equilocs and equitemps. The two equitemps and two equilocs bound a rhombus of area $\lambda \mu$. Both diagonals of the rhombus are photon space-time trajectories, since they connect events 1 f and 1 ns apart. Being diagonals of a rhombus they are perpendicular and bisect the angles at the vertices.

The structure and orientation of Alice’s diagrams are then uniquely determined except for two free parameters which we can take to be her scale factor $\lambda$ specifying how many centimeters of diagram separate two equitemps associated with events one ns apart, and her angle $\Theta$ between her families of equilocs and equitemps.
2. Bob’s use of Alice’s diagram with his own equilocs and equitemps.

Bob, who moves uniformly with velocity \( v \) in Alice’s frame, notes the same events that she does, and is free to record them in another diagram of his own making, appropriate to his proper frame. But since all events of interest are already present as points in Alice’s diagram, rather than putting events into a new diagram, he can try to impose on Alice’s his own set of equilocs and equitemps. His equilocs are entirely straightforward: they are just the space-time trajectories of objects that move with velocity \( v \) in Alice’s frame and are therefore also parallel straight lines, making the appropriate angle with Alice’s equilocs, and, like Alice’s, tilted by less than \( \pi/4 \) from the vertical, if \( v \) is less than 1 f/ns.

![Diagram](image_url)

**Figure 2.** The three equally spaced parallel lines are two of Bob’s equilocs and a third, associated with a position exactly halfway between the positions associated with the other two. Two photons leave a point on the middle equiloc and travel in opposite directions to the positions of the two outer equilocs, where they are reflected back to their starting equiloc. Because the photons start midway between the outer positions, their arrival at the outer positions, events marked by the two black circles, are simultaneous events in Bob’s frame and the line joining them is an equitemp for Bob. It is evident from the symmetry of the rectangle formed by the four photon trajectories that the angle \( \theta_B \) between Bob’s equitemp and a photon trajectory is the same as the angle \( \theta_B \) between his equilocs and that photon trajectory.
To establish the character of Bob’s equitemps in Alice’s diagram we must, for the first time, invoke Einstein’s postulates. The velocity of light is 1 f/ns, independent of the velocity of the source, in both Bob’s frame and Alice’s. Therefore two events determine an equitemp in Bob’s frame if and only if oppositely moving photons emitted at each event meet midway between their locations, or, equivalently, if it is possible for oppositely moving photons emitted together midway between their locations to arrive at the events just as they happen.

Two events satisfying these conditions are pictured as two black circles in Alice’s diagram in Figure 2. The three parallel lines are the two equilocs of Bob on which the events lie and his equiloc midway between them. The trajectories of four photons, demonstrating that two events on his outer two equilocs are simultaneous in Bob’s frame, form a rectangle. The more vertical diagonal of that rectangle is the middle equiloc of Bob. The more horizontal diagonal connects the points representing Bob’s two simultaneous events and is therefore an equitemp in his frame. It is evident from the symmetry of the rectangle that these two diagonals are symmetrically disposed about the photon lines: lines in Alice’s diagram connecting two events that are simultaneous in Bob’s frame, make the same angle $\theta_B$ with the photon lines as Bob’s equilocs do.

So Bob’s equitemps are straight lines, making an angle $\Theta_B = 2\theta_B$ with his equilocs, symmetrically disposed with his equilocs about the photon lines, just as Alice’s are with angles $\Theta_A = 2\theta_A$. Given a diagram with both their sets of equitemps and equilocs there is therefore no way to tell which of them made the diagram first and which then added to it their own equitemps and equilocs: the frame independence of the velocity of light is explicitly consistent with the principle of relativity.

3. Relation between Alice’s and Bob’s scale factors: light rectangles.

It remains is to establish the relation between the scale factors $\lambda$ (or $\mu$) used by Alice and Bob. This has a simple geometric formulation in terms of rectangles of photon trajectories, like the one in Figure 2. One can associate with any two events a unique rectangle whose four sides are segments of the two pairs of photon lines passing through each of the events, as shown in Figure 3.

The relation between Alice’s and Bob’s scale factors is determined by the fact that the area of such a light rectangle for two events on an equiloc of Alice, a time $T$ apart in Alice’s frame, must be the same as the area of the light rectangle for two events on an equiloc of Bob, the same time $T$ apart in Bob’s frame. This equality of areas follows directly from the physical fact that if Alice and Bob each looks at a clock carried by the other, the rate
Figure 3. Two events $P$ and $Q$ determine a unique rectangle of photon line segments — a light rectangle — with the events at diagonally opposite vertices.

at which each sees the other’s clock running will differ from the rate of their own clock by the same factor $f$, as required by the principle of relativity and the source independence of the speed of light. That this reciprocity of Doppler shifts has this immediate geometric consequence is demonstrated in Figure 4.

Figure 4. The space-time trajectories of Alice’s clock and Bob’s intersect when both read 0. The heavier dashed line segments are photon trajectories demonstrating that when the clock of each reads $T$, each sees the clock of the other reading $t$. Two light rectangles, one with sides $A$ and $a$, the other with sides $B$ and $b$ are formed by the photon trajectories through the two clocks reading $T$ and the two (coincident) clocks reading 0.
When Alice and Bob are together they set their clocks to 0. Then they move uniformly apart. When their clocks read \( T \) each looks back at the other’s clock and sees it reading \( t \). Figure 4 reveals that the ratio \( t/T \) associated with Alice’s clock is equal to \( b/a \), while the ratio \( t/T \) associated with Bob’s is equal to \( A/B \). Therefore \( b/a = A/B \), so \( Aa = Bb \). But \( Aa \) is the area of the light rectangle with Alice’s clock reading \( T \) and 0 at opposite vertices, while \( Bb \) is the area of the corresponding light rectangle for Bob’s clock.

Figure 5 shows that the area of such a light rectangle is

\[
\Omega = \frac{1}{2} \lambda \mu T^2,
\]

so the simplest analytical expression of the geometric connection between Alice’s and Bob’s scale factors is just that the product \( \lambda \mu \) of the two scale factors is independent of frame:

\[
\lambda_A \mu_A = \lambda_B \mu_B.
\]

\[\theta\]

\(\mu T \sin \theta\)

\(\mu T \cos \theta\)

\(T\)

\(0\)

**Figure 5.** The length of an equiloc between two readings of the same clock a time \( T \) apart is \( \mu T \), where \( \mu \) is the scale factor for the equiloc. The area of the light rectangle with the events at opposite vertices is \( \Omega = (\mu T)^2 \sin \theta \cos \theta = \frac{1}{2} \mu^2 \sin \Theta T^2 = \frac{1}{2} \mu \lambda T^2 \).

It is useful to have an expression for this frame-independent product of scale factors, which we take in the form

\[
\Omega_0 = \frac{1}{2} \lambda \mu.
\]
If $\lambda$ and $\mu$ are given in cm of diagram per ns of time (or f of space), then $\Omega_0$ has dimensions of $\text{cm}^2/\text{ns}^2$ or $\text{cm}^2/f^2$. It is convenient to define for any frame — say Bob’s — a unit light rectangle as one with a diagonal that is an equiloc for Bob connecting events 1 ns apart. Figure 2 establishes that the other diagonal is an equitemp for Bob, and the relation between scales on equitemps and equilocs establishes that the vertices on the equitemp are events 1 f apart in his frame. Unit light rectangles in all frames have the same frame-independent area, $\Omega_0 = \frac{1}{2} \lambda \mu$.

4. Light rectangles and the invariant interval.

Abstracting from Alice and Bob, we can say that the area (in units of $\Omega_0$) of the light rectangle determined by two time-like separated events is the square of the time between them in the frame in which they happen in the same place. This is precisely the definition of the squared interval $I^2$ between the events, so

$$I^2 = \frac{\Omega}{\Omega_0}. \quad (5)$$

Because the same scale factor $\lambda$ is associated with equilocs and equitemps this also holds for the interval between two space-like separated events: $I^2 = \frac{\Omega}{\Omega_0}$, where $I^2$ is the square of the distance between the events in the frame in which they happen at the same time, and $\Omega$ is the area of their light rectangle.

The light rectangle for two light-like separated events degenerates to a line, which has zero area, again agreeing with the definition of the interval.

5. Interval in terms of coordinates.

Figure 6 relates the geometric representation of the interval as a light rectangle to its expression in terms of coordinates in a particular frame, showing that the squared interval between two time-like separated events is $I^2 = T^2 - D^2$ where $T$ and $D$ are the time and distance between the events in any frame.

The two large black circles are the events $E_1$ and $E_2$. The solid lines are Alice’s equitemp and equiloc, which intersect at an event $E_3$, where they make the same angle $\theta$ with a photon trajectory. As a result, the right triangle with sides $c$ and $d$ is similar to the right triangle with sides $a$ and $b$, so $a/b = c/d$ and therefore

$$ad = bc. \quad (6)$$

The squared interval between $E_1$ and $E_2$ is proportional to the area $(a - c)(b + d)$ of their light rectangle:

$$I^2 = \frac{(a - c)(b + d)}{\Omega_0}, \quad (7)$$
Figure 6. The two larger black circles are two time-like separated events $E_1$ and $E_2$. The dashed lines are photon trajectories. The two solid lines are Alice’s equitemp and equiloc, which intersect at an event $E_3$. The squared distance between $E_1$ and $E_2$ in Alice’s frame is proportional to the area of the rectangle with $E_3$ and $E_2$ at opposite vertices. Her squared time between $E_1$ and $E_2$ is proportional to the area of the rectangle with $E_1$ and $E_3$ at opposite vertices. The squared interval between $E_2$ and $E_1$ is proportional to the area of the rectangle with $E_1$ and $E_2$ at opposite vertices.

which (6) simplifies to

$$I^2 = \frac{(ab - cd)}{\Omega_0}. \quad (8)$$

But $ab/\Omega_0$ is the squared interval between $E_1$ and $E_3$ while $cd/\Omega_0$ is the squared interval between $E_2$ and $E_3$. Since $E_1$ and $E_3$ are on an equiloc in Alice’s frame, the squared interval between them is $T^2$, the square of Alice’s time between them; since $E_2$ and $E_3$ are on an equitemp in Alice’s frame the squared interval between them is $D^2$, the square of Alice’s distance between them. But since $E_3$ happens at the same place as $E_1$ and the same time as $E_2$ in Alice’s frame, $T$ and $D$ are also Alice’s time and distance between $E_1$ and $E_2$. So

$$I^2 = T^2 - D^2. \quad (9)$$
The analogous argument for space-like separated events follows from reflecting Figure 6 in any of the photon trajectories.

![Diagram](image1)

**Figure 7.** A proof of $I^2 = T^2 - D^2$ more purely geometric than that of Figure 6. Each of the three light rectangles has been replaced in part (a) by a rhombus of twice the area. The quadrilaterals bounding parts (b) and (c) are identical, leading to the relation between areas shown symbolically in part (d).

A purely geometric proof of (9) that avoids even the tiny bit of analysis in (6)-(8) is given in Figure 7. Part (a) of Figure 7 reproduces the content of Figure 6, except that the three light rectangles associated with the three pairs of events have been replaced by three rhombi, each with twice the area of the rectangle it replaces. In part (b) the two smaller rhombi have been combined with two copies of the triangle formed by the three events to form a quadrilateral. In part (c) the largest rhombus is combined with two copies of that triangle to form that same quadrilateral, thereby demonstrating that the area of the largest rhombus is the sum of the areas of the two smaller ones, leading to the relation between areas shown in part (d).

6. Measuring the interval with a single clock; an application of light rectangles.

There is an elegant way to measure the interval using only light signals and a single clock [1]. Figure 8 illustrates the method for two space-like separated events and demon-
Figure 8. The solid line is Alice’s equiloc. Three readings of her clock are shown. It is seen at event $E_2$ to read $-T_1$; it reads 0 at event $E_1$; and it reads $T_2$ when $E_2$ is seen at the location of the clock. The area of the light rectangle with $E_1$ and $E_2$ at opposite vertices is $(\mu T_1 \sin \theta)(\mu T_2 \cos \theta) = \Omega_0 T_1 T_2$.

strates why it works. The events are $E_1$ and $E_2$. A uniformly moving clock (stationary in Alice’s frame) is present at $E_1$. Call the reading of the clock when $E_1$ takes place 0. When $E_2$ takes place the clock is seen at $E_2$ to read $-T_1$. And when $E_2$ is seen to take place at the clock, the clock reads $T_2$. The segments of Alice’s equiloc between her clock reading 0 and reading $T_1$ or $T_2$ have length $\mu T_1$ or $\mu T_2$ where $\mu$ is Alice’s scale factor. Consequently the area of the light rectangle with $E_1$ and $E_2$ at opposite vertices is

$$
\Omega = (\mu T_1 \sin \theta)(\mu T_2 \cos \theta) = \frac{1}{2}\mu^2 \sin \Theta T_1 T_2 = \frac{1}{2}\mu \lambda T_1 T_2 = \Omega_0 T_1 T_2.
$$

(10)

The squared interval $I^2$ between $E_1$ and $E_2$ is $\Omega/\Omega_0$ and therefore

$$
I^2 = T_1 T_2.
$$

(11)

In much the same way, Figure 9 establishes for time-like separated events the relation
Figure 9. The solid line is Alice’s equiloc. Three readings of her clock are shown. It reads 0 at event $E_1$; it is seen at event $E_2$ to read $T_1$; and it reads $T_2$ when $E_2$ is seen at the location of the clock. The area of the light rectangle with $E_1$ and $E_2$ at opposite vertices is $(\mu T_1 \sin \theta)(\mu T_2 \cos \theta) = \Omega_0 T_1 T_2$.

(11) between the interval and the three readings on Alice’s clock.

7. Shapes of light rectangles and relative velocities of frames. What is the relation between the shapes of light rectangles whose diagonals are equilocs in different frames? Since the shape for any given frame is independent of the size, it suffices to consider two light rectangles in Figure 10. One has as its diagonal Bob’s equiloc between events $P$ and $R$, and the other, Alice’s equiloc between events $P$ and $Q$. The event $Q$ has been chosen so that the lighter solid line between $Q$ and $R$ is an equitemp in Alice’s frame. The aspect ratios of Bob’s and Alice’s light rectangles are $b/B$ and $a/A$.

We can relate these two aspect ratios to Bob’s velocity $v$ in Alice’s frame, by noting that $v$ is the ratio of the length of Alice’s equitemp, from $Q$ to $R$, to the length of her equiloc, from $P$ to $Q$. Each of these two lines is the hypothenuse of a right triangle whose other sides are photon trajectories, and because the lines are an equitemp and equiloc for Alice, the two right triangles are similar. Consequently the ratio $v$ of the hypothenuse
Figure 10. The line from $P$ to $R$ is an equiloc in Bob’s frame. The lines from $P$ to $Q$ and from $Q$ to $R$, making the same angle $\theta$ with the photon line through $Q$, are an equiloc and equitemp in Alice’s frame. Bob’s velocity $v$ in Alice’s frame is the ratio of the lengths of these two lines.

lengths is equal to the ratio of the lengths of either pair of corresponding sides, and we have

$$v = \frac{a - b}{a} = \frac{B - A}{A}. \quad (12)$$

These relations tell us that

$$b/a = 1 - v \quad \text{and} \quad B/A = 1 + v. \quad (13)$$

Consequently the aspect ratio $b/B$ of Bob’s light rectangle is related to the aspect ratio $a/A$ of Alice’s by

$$\frac{b/B}{a/A} = \frac{1 - v}{1 + v}. \quad (14)$$

With this information we can extract the quantitative expression for the relativistic Doppler shift from Figure 4. Applied to the two photon rectangles in that Figure, (14)
tells us that
\[
\frac{1 - v}{1 + v} = (A/B)(b/a),
\]  
where \( v \) is the velocity of Bob in Alice’s frame of reference. But the equality of the rates \( f \) at which Alice or Bob sees time passing on the other’s clock, as measured by their own clock, requires that
\[
A/B = f = b/a,
\]  
and therefore
\[
f = \sqrt{\frac{1 - v}{1 + v}}.
\]

8. Doppler shifted lengths; another application of light rectangles.

The relation (15) also provides a diagrammatic demonstration of the less frequently noted fact that a train moving toward one with speed \( v \) is seen, compared with stationary objects, to be longer than its proper length by the Doppler factor \( \sqrt{\frac{1 + v}{1 - v}} \). The white circle in the upper right of Figure 11 is the event in which the front of the train reaches Bob. The second white circle in the lower left, connected to the first circle by the heavy photon line is the event at the rear of the train that Bob sees when the front of the train reaches him.

The distance between these two events in Bob’s frame (which is the length he sees the train as having) is the distance between events represented by the lower white circle and the black circle on Bob’s equiloc at its intersection with his equitemp through the lower white circle. That distance squared is the interval between the events, and is therefore given by the area (in units of \( \Omega_0 \)) of the larger light rectangle, reproduced in the inset on the lower right. The squared proper length of the train, on the other hand, is given by the area of the smaller light rectangle, reproduced in the inset on the upper left.

Since the two rectangles have a side in common, the ratio of their areas is just the ratio, \( \frac{1 + v}{1 - v} \), of their aspect ratios, so the length of the train as seen by Bob exceeds its proper length by \( \sqrt{\frac{1 + v}{1 - v}} \). A similar construction establishes that when the train is moving away, it is seen to be shorter than its proper length by the factor \( \sqrt{\frac{1 - v}{1 + v}} \).

9. More spatial dimensions.

The simplicity of these constructions diminishes as one introduces additional spatial dimensions, but one feature of the higher dimensional diagrams should be translated into the
Figure 11. Demonstration that Bob sees a train moving toward him with speed $v$ as having a length that exceeds its proper length by the relativistic Doppler factor $\sqrt{\frac{1+v}{1-v}}$. 

terminology developed above. Suppose Alice represents events in two spatial dimensions in a three-dimensional space-time diagram, of which her two-dimensional diagram is now one of a family of parallel two-dimensional slices. She applies the rules enunciated above to every such two-dimensional slice. She then aligns the slices so that the one-dimensional equitemps in each slice associated with the same time all lie in an equitemporal plane perpendicular to the slices, and so that the one-dimensional equilocs associated with the same position in the first dimension all lie in a plane perpendicular to the plane of the slices. Each such plane of equilocs can be further resolved into linear equilocs according to the positions they represent in the direction orthogonal to the first spatial dimension.

The only remaining freedom lies in a new scale factor $\sigma$ relating separation of equilocs in the same plane perpendicular to the slices, to separation in space of the events they represent, in the direction orthogonal to the first spatial dimension. This is determined
by a requirement of isotropy: the two orthogonal photon trajectories passing through any event in the 1+1 dimensional diagrams should expand into a complete right circular cone of photon trajectories in the 2+1 dimensional diagram. Figure 12 demonstrates that this determines the scale factor $\sigma$ in the perpendicular direction to be the frame-independent geometric mean of the scale factors used in the two-dimensional slices:

$$\sigma = \sqrt{\lambda \mu}.$$ 

(18)

**Figure 12.** A two dimensional slice of Alice’s (2+1)-dimensional space-time diagram. Alice’s equitemps are now planes perpendicular to the slice that intersect it in the two parallel heavy lines. The third heavy line is an equiloc of Alice, lying in the slice. The two photon trajectories are the intersection with the slice of a right circular cone of photon trajectories with the lower black circle for its vertex, and the line connecting that circle to the white circle for its axis.

The figure shows a 2-dimensional diagram of Alice, now to be viewed as a 2-dimensional slice of her 3-dimensional diagram. Two of her equitemps in the 3-dimensional diagram are parallel planes, perpendicular to the 2-dimensional slice, that intersect it in the two parallel heavy lines. The two equitemps represent events 1 ns apart. The third heavy line, connecting the two black circles is an equiloc of Alice (that remains just a line in the higher dimensional diagram). The black circles represent events 1 ns apart in Alice’s frame. The dashed photon trajectories are the intersection of a right circular cone of photon trajectories with the 2-dimensional diagram. The axis of the cone is the light vertical line connecting the lower black circle with the upper white circle.
Consider a photon that connects the event represented by the black circle on the lower equitemporal plane with an event on the upper equitemporal plane that is displaced from the upper black circle in the direction perpendicular to the 2-dimensional slice. Since the projection of the photon’s trajectory into the plane of the slice coincides with an equiloc in that plane, the motion of the photon out of the plane represents its entire change in position in the 3-dimensional diagram. Since the two equitemporal planes represent events 1 ns apart, the photon must cover a distance of 1 f, and it must therefore finish displaced by the distance $\sigma$ from the upper black circle in the direction perpendicular to the slice.

This final position must lie on the cone of photon trajectories through the lower black circle, and must therefore be a distance $r$ from the white circle in the 3-dimensional diagram. It is clear from the diagram that $r$ is related to $\mu$ and $\theta$ by

$$ r = \mu \sin(\pi/4 + \theta). \quad (19) $$

On the other hand the final position of the photon is displaced from the upper black circle by a distance $\sigma$ perpendicular to the slice, and displaced from the white circle in the plane of the slice by a distance $\mu \cos(\pi/4 + \theta)$, so its distance in the 3-dimensional diagram from the white circle must be

$$ r = \sqrt{\sigma^2 + \mu^2 \cos^2(\pi/4 + \theta)}. \quad (20) $$

This is equal to the distance in (19) provided

$$ \sigma^2 = \mu^2 \sin^2(\pi/4 + \theta) - \mu^2 \cos^2(\pi/4 + \theta) = \mu^2 (-\cos(\frac{1}{2} \pi + 2\theta)) = \mu^2 \sin 2\theta = \mu \lambda. \quad (21) $$

So the distance perpendicular to the 2-dimensional slice between equilocs containing events 1 f apart is just the invariant quantity $\sqrt{\mu \lambda}$. All observers with equilocs in the plane of Alice’s 2-dimensional slice use this same scale factor for spatial separations perpendicular to that slice. (This may well be the most difficult derivation of $y' = y, z' = z$ ever to appear in the literature.)

10. Related work.

This way of developing 1+1 dimensional flat space-time diagrams refines and extends an approach I recommended several years ago for teaching special relativity to nonscientists who know a little elementary algebra and plane geometry [2,3]. An expanded exposition can be found in my forthcoming book [4] on special relativity for nonscientists. Dieter Brill and Ted Jacobson have recently given a similar geometric treatment of the interval [5]. I learned from Brill and Jacobson that features of this point of view go back at least to
1913 [6], and that some beautiful film clips illustrating some of these geometric relations are at the website of Dierck-Ekkehard Liebscher [7]. Closely related material can be found in Liebscher’s book [8].

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