MINIMAX ESTIMATION OF NONREGULAR PARAMETERS AND DISCONTINUITY IN MINIMAX RISK

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Abstract. When a parameter of interest is nondifferentiable in the probability, the existing theory of semiparametric efficient estimation is not applicable, as it does not have an influence function. Song (2014) recently developed a local asymptotic minimax estimation theory for a parameter that is a nondifferentiable transform of a regular parameter, where the nondifferentiable transform is a composite map of a continuous piecewise linear map with a single kink point and a translation-scale equivariant map. The contribution of this paper is two fold. First, this paper extends the local asymptotic minimax theory to nondifferentiable transforms that are a composite map of a Lipschitz continuous map having a finite set of nondifferentiability points and a translation-scale equivariant map. Second, this paper investigates the discontinuity of the local asymptotic minimax risk in the true probability and shows that the proposed estimator remains to be optimal even when the risk is locally robustified not only over the scores at the true probability, but also over the true probability itself. However, the local robustification does not resolve the issue of discontinuity in the local asymptotic minimax risk.

Key words. Nonregular Parameters; Semiparametric Efficiency; Local Asymptotic Minimax Estimation; Translation-Scale Equivariant Maps

JEL Classification: C01, C13, C14, C44.

1. Introduction

Statistical inference on a parameter begins by choosing an appropriate estimator. For a finite dimensional parameter defined under local asymptotic normal experiments, it has become nearly a standard practice in statistics and econometrics to establish the optimality of an estimator through semiparametric efficiency, where optimality is expressed as a variance bound, and an estimator is taken to be optimal if it is asymptotically normal with its asymptotic variance achieving the bound. The literature along this approach is vast in statistics and econometrics.

A mathematical analysis of asymptotic optimal inference began with the famous paper by Wald (1943). While the approach of local asymptotic minimax estimation has
appeared in the previous literature (e.g. Le Cam (1953) and Chernoff (1956)), major breakthroughs were made by Hájek (1972) and Le Cam (1972). Koshevnik and Levit (1976), Pfanzagl and Wefelmeyer (1982), Begun, Hall, Huang and Wellner (1983) and Chamberlain (1986) extended asymptotic efficient estimation to nonparametric and semiparametric models. See also van der Vaart (1988, 1991) for further developments in this direction, and Newey (1990) for results that are relevant to econometrics. A general account of this approach is found in monographs such as Bickel, Klaassen, Ritov, and Wellner (1993), and in later chapters of van der Vaart and Wellner (1996) and van der Vaart (1998). While the references so far mostly focus on local asymptotic normal experiments (as this paper does), optimal estimation theory in local asymptotic mixed normal experiments has also received attention in the literature. See Jeganathan (1982) and Basawa and Scott (1983). In econometrics, Phillips (1991) developed optimal inference theory for cointegrating regression models using the framework of local asymptotic mixed normal experiments.

The existing notion of semiparametric efficiency is not directly applicable, when the parameter is not differentiable in the probability that identifies the parameter. Nondifferentiable parameters do not merely constitute a pathological case, for one can easily encounter such a parameter when the parameter (denoted by \( \theta \in \mathbb{R} \)) is defined through a nondifferentiable transform of another parameter vector, say, \( \beta \in \mathbb{R}^d \). For example, the parameter of interest might take the form of \( \theta = \max\{\beta_1, \ldots, \beta_d\} \) or \( \theta = \min\{\beta_1, \ldots, \beta_d\} \), where \( \beta_1, \ldots, \beta_d \) are average treatment effects from different treatment regimes or mean squared prediction errors from different predictive models, or boundaries of multiple intervals. (As for the last example, Chernozhukov, Lee and Rosen (2013) called the parameter an intersection bound. Examples of such bounds are found in Haile and Tamer (2003) and Manski and Tamer (2002) among many others.) The difficulty with estimation theory for such nondifferentiable parameters is emphasized by Doss and Sethuraman (1988). See Hirano and Porter (2010) for a general impossibility result for such parameters.

This paper focuses on the problem of optimal estimation when the parameter is nondifferentiable in a particular way. More specifically, this paper focuses on a parameter of interest in \( \mathbb{R} \) which takes the following form:

\[
(1.1) \quad \theta = (f \circ g)(\beta),
\]

where \( \beta \in \mathbb{R}^d \) is a regular parameter for which a semiparametric efficiency bound is well defined, \( g \) is a translation-scale equivariant map, and \( f \) is a continuous map that is potentially nondifferentiable. While the paper focuses on this particular way that
nondifferentiability arises, it accommodates various nondifferentiable parameters that are relevant in empirical researches (See Song (2014) for examples.)

A recent work by the author (Song (2014)) considers the case of \( f \) being a continuous piecewise linear map with a single kink point, and has demonstrated that the existing semiparametric efficient estimation can be extended to this case of nonregular parameter \( \theta \) through a local asymptotic minimax approach. While the result applies to various examples of nonregular parameters used in econometrics, the restriction on \( f \) excludes some interesting examples. For example, one might be interested in an optimal policy parameter that is censored on both upper and lower bounds, say, due to constraints on resources or in implementation.

This paper generalizes the theory to the case where \( f \) is Lipschitz continuous yet potentially nondifferentiable at a finite number of points. Similarly, as in Song (2014), it turns out that the local asymptotic minimax estimator takes the following form:

\[
\hat{\theta}_{mx} \equiv f\left(g(\tilde{\beta}) + \hat{c}^* \sqrt{\frac{1}{m}}\right),
\]

where \( \hat{c}^* \) is an optimal bias adjustment term, and \( \tilde{\beta} \) is a semiparametrically efficient estimator of \( \beta \). The optimal bias adjustment term can be determined by simulating the local asymptotic minimax risk.

Some researches in the literature have suggested various methods of bias adjustment and reported improved performances. (See for example Haile and Tamer (2003), and Chernozhukov, Lee, and Rosen (2013).) The approach of Song (2014) and this paper is distinct in the sense that it determines the optimal bias adjustment explicitly through theory of local asymptotic minimax estimation.

The resulting local asymptotic minimax risk for the kind of nonregular parameters considered in this paper is discontinuous in the underlying true probability in general. To appreciate the meaning of this discontinuity, it is worth recalling that the classical local asymptotic minimax risk approach imposes local uniformity over parametric submodels passing a fixed true probability. This local uniformity eliminates superefficient estimators such as Hodges estimator which is known to exhibit poor finite sample performance. (See Le Cam (1953) for a formal treatment of Hodges superefficient estimator. See also Weiss and Wolfowitz (1966)). In classical estimation theory for regular parameters, the local asymptotic minimax risk is continuous in the true probability. This feature stands in contrast with the local asymptotic minimax risk in this paper which is discontinuous in the true probability.

When the asymptotic distribution of a test statistic or an estimator exhibits discontinuity in the underlying true probability, it is a common practice to consider an alternative
asymptotic theory along a sequence of probabilities local around the true probability. Mostly, this alternative asymptotics involves a localization parameter which continuously "bridges" two distributions across the discontinuity point. The most common example of this approach is local asymptotic power analysis in hypothesis tests, where one adopts a sequence of probabilities that converge to a probability that belongs to the null hypothesis. A similar approach is found in local to unity models (Stock (1991)), weakly identified models (Staiger and Stock (1997)), and more recently, models of various moment inequality restrictions (Andrews and Guggenberger (2009)) among many others.

To deal with this issue of discontinuity in the local asymptotic minimax risk, this paper introduces a local robustification of the risk, where the risk is further robustified against a local perturbation of the true probability. Somewhat unexpectedly, the local asymptotic minimax risk remains unchanged after this local robustification of the risk. On the one hand, this means that the local asymptotic minimax estimator in (1.2) retains its optimality under this robustification, as long as the efficient estimator \( \tilde{\beta} \), after location-scale normalization, converges in distribution uniformly over the true probabilities. On the other hand, the discontinuity of the risk in the true probability is not resolved by the local robustification approach. Hence there may be a gap between the finite sample risk and its asymptotic version and the gap does not close uniformly over all the probabilities even in the limit. It remains an open question whether this renders the whole apparatus of the local asymptotic minimax theory dubious in practice, when the parameter is nondifferentiable.

The rest of the paper is structured as follows. In Section 2, the paper defines the scope of this paper by introducing assumptions about \( f, g, \beta \), and the set of underlying probabilities that identify \( \beta \). In Section 3, the paper gives a characterization of local asymptotic minimax risk, and proposes a general method to construct a local asymptotic minimax estimator. In Section 4, the paper considers a local robustification of the local asymptotic minimax risk, and shows that the results of Section 3 mostly remain unchanged. Section 5 concludes the paper. The mathematical proofs of the paper’s results appear in the Appendix.

A word of notation. Let \( \mathbf{1}_d \) be a \( d \times 1 \) vector of ones with \( d \geq 2 \). For a vector \( \mathbf{x} \in \mathbb{R}^d \) and a scalar \( c \), we simply write \( \mathbf{x} + c = \mathbf{x} + c \mathbf{1}_d \), or write \( \mathbf{x} = c \) instead of \( \mathbf{x} = c \mathbf{1}_d \). For \( \mathbf{x} \in \mathbb{R}^d \), the notation \( \max(\mathbf{x}) \) (or \( \min(\mathbf{x}) \)) means the maximum (or the minimum) over the entries of the vector \( \mathbf{x} \). We let \( \bar{\mathbb{R}} = [-\infty, \infty] \) and view it as a two-point compactification of \( \mathbb{R} \), and let \( \bar{\mathbb{R}}^d \) be the product of its \( d \) copies, so that \( \bar{\mathbb{R}}^d \) itself is a compactification of \( \mathbb{R}^d \). (e.g. Dudley (2002), p.74.) We follow the convention to set
\[ \infty \cdot 0 = 0 \text{ and } (\infty) \cdot 0 = 0. \] A supremum and an infimum of a nonnegative map over an empty set are set to be 0 and \( \infty \) respectively.

2. NONDIFFERENTIABLE TRANSFORMS OF A REGULAR PARAMETER

2.1. NONDIFFERENTIABLE TRANSFORMS. First, we begin with conditions for \( f \) and \( g \) in (1.1).

**Assumption 1:** (i) The map \( g : \mathbb{R}^d \to \mathbb{R} \) is Lipschitz continuous, and satisfies the following.

(a) (Translation Equivariance) For each \( c \in \mathbb{R} \) and \( x \in \mathbb{R}^d \), \( g(x + c) = g(x) + c \).

(b) (Scale Equivariance) For each \( u \geq 0 \) and \( x \in \mathbb{R}^d \), \( g(ux) = ug(x) \).

(c) (Directional Derivatives) For each \( z \in \mathbb{R}^d \) and \( x \in \mathbb{R}^d \),
\[
\tilde{g}(x; z) \equiv \lim_{t \downarrow 0} t^{-1} (g(x + tz) - g(x))
\]
exists.

(ii) The map \( f : \bar{\mathbb{R}} \to \bar{\mathbb{R}} \) is Lipschitz continuous and non-constant on \( \mathbb{R} \), and is continuously differentiable except at a finite number of points in \( \mathbb{R} \), with a Lipschitz continuous derivative.

Assumption 1(i) is the same as Assumption 1(i) of Song (2014) but the requirement for \( f \) is now substantially generalized by Assumption 1(ii). To give a sense of the map \( g \), consider the following examples.

**Examples 1:**

(a) \( g(x) = s'x \), where \( s \in S_1 \equiv \{ s \in \mathbb{R}^d : s'1_d = 1 \} \) and \( 1_d \) is the \( d \)-dimensional vector of ones.

(b) \( g(x) = \max(x) \) or \( g(x) = \min(x) \).

(c) \( g(x) = \max\{\min(x_1), x_2\} \), where \( x_1 \) and \( x_2 \) are (possibly overlapping) subvectors of \( x \).

(d) \( g(x) = \max(x_1) + \max(x_2), g(x) = \min(x_1) + \min(x_2), g(x) = \max(x_1) + \min(x_2), \) or \( g(x) = \max(x_1) + s'x_2 \) with \( s \in S_1 \).

Thus the examples of the parameters \( \theta \) in the form (1.1) are as follows.

**Examples 2:**

(a) \( \theta = |\max\{\beta_1, \beta_2\}| \)

(b) \( \theta = |\beta_1 - 1| \).

(c) \( \theta = \min\{\max\{\beta, 0\}, 1\} \).

(d) \( \theta = \min\{|\beta|, 1\} \).

(e) \( \theta = \max\{\min\{\max\{\beta_1, \beta_2\}, 1\}, 0\} \).
The framework of Song (2014) requires that \( f \) be a piecewise linear map with a single kink point, and hence excludes the examples of (c)-(e). In example (c), the parameter \( \theta \) is \( \beta \) censored at 0 and 1. In example (d), the parameter of interest is the absolute value of \( |\beta| \) censored at 1.

One might ask whether the representation of parameter \( \theta \) as a composition map \( f \circ g \) of \( \beta \) in (1.1) with \( f \) and \( g \) satisfying Assumption 1 is unique. Lemma 1 of Song (2014) gives an affirmative answer. As we shall see later, the asymptotic risk bound involves \( g \) and the optimal estimators involve the maps \( f \) and \( g \). This uniqueness result removes ambiguity that can potentially arise when \( \theta \) has multiple equivalent representations with different maps \( f \) and \( g \). When \( d \geq 2 \), the roles of \( f \) and \( g \) cannot be interchanged. When \( d = 1 \), Assumption 1(i) requires that 
\[
\alpha_h = 0 \quad \text{when} \quad h = 0,
\]
Let \( h \in L_2(P_{\alpha_0}) \),
\[
\int \left\{ \frac{1}{t} \left( dP_{\alpha_0}^{1/2} - dP_{\alpha_0}^{1/2} \right) - \frac{1}{2} hdP_{\alpha_0}^{1/2} \right\}^2 \to 0, \quad \text{as} \quad n \to \infty.
\]
When this convergence holds, we say that \( P_{\alpha_0} \) is differentiable in quadratic mean to \( P_{\alpha_0} \), call \( h \in L_2(P_{\alpha_0}) \) a score function associated with this convergence, and call the set of all such \( h \)'s a tangent set, denoting it by \( T(P_{\alpha_0}) \). We assume that the tangent set is a linear subspace of \( L_2(P_{\alpha_0}) \). Taking \( \langle \cdot, \cdot \rangle \) to be the usual inner product in \( L_2(P_{\alpha_0}) \), we view \((H, \langle \cdot, \cdot \rangle)\), with \( H \equiv T(P_{\alpha_0}) \), as a subspace of a separable Hilbert space. For each \( h \in H \), \( n \in \mathbb{N} \), and \( \alpha_h \in A \) such that \( \alpha_h = 0 \) when \( h = 0 \), let 
\[
P_{\alpha_0 + \alpha_h/\sqrt{n}} \text{ be probabilities converging to } P_{\alpha_0} \text{ (as in } (2.1) \text{) as } n \to \infty \text{ having } h \text{ as its score. We simply write } P_{n,h} = \frac{P_n^{\alpha_0 + \alpha_h/\sqrt{n}}} {P_{\alpha_0 + \alpha_h/\sqrt{n}}} \text{ and consider sequences of such probabilities } \{P_{n,h}\}_{n \geq 1} \text{ indexed by } h \in H.
\]
(See van der Vaart (1991) and van der Vaart and Wellner (1996), Section 3.11 for details.) Differentiability in quadratic mean and i.i.d. assumption imply local asymptotic normality (LAN): for any \( h \in H \),
\[
\log \frac{dP_{n,h}}{dP_{n,0}} = \zeta_n(h) - \frac{1}{2} \langle h, h \rangle,
\]
where for each \( h, h' \in H \),

\[
[\zeta_n(h), \zeta_n(h')] \xrightarrow{d} [\zeta(h), \zeta(h')], \text{ under } \{P_{n,0}\},
\]

and \( \zeta(\cdot) \) is a centered Gaussian process on \( H \) with covariance function \( \mathbb{E}[\zeta(h_1)\zeta(h_2)] = \langle h_1, h_2 \rangle \). (See the proof of Lemma 3.10.11 of van der Vaart and Wellner (1996).) Local asymptotic normality reduces the decision problem to one in which an optimal decision is sought under a single Gaussian shift experiment \( \mathcal{E} = (\mathcal{X}, \mathcal{G}, P_h; h \in H) \), where \( P_h \) is such that \( \log dP_h/dP_0 = \zeta(h) - \frac{1}{2} \langle h, h \rangle \).

We assume that \( \beta \) is identified by \( P_{\alpha} \), \( \alpha \in \mathcal{A} \), and write \( \beta_n(h) = \beta(P_{\alpha_0 + \alpha h/\sqrt{n}}) \), regarding the parameter as an \( \mathbb{R}^d \)-valued map on \( H \). As a sequence of maps on \( H \), we assume that there exists a continuous linear \( \mathbb{R}^d \)-valued map, \( \dot{\beta} \), on \( H \) such that for any \( h \in H \),

(2.2) \[
\sqrt{n}(\beta_n(h) - \beta_n(0)) \to \dot{\beta}(h)
\]
as \( n \to \infty \). In other words, \( \beta_n(h) \) is regular in the sense of van der Vaart and Wellner (1996, Section 3.11).

As is well known the functional \( \dot{\beta} \) determines the efficiency bound for \( \beta \). Let \( e_m \) be a \( d \times 1 \) vector whose \( m \)-th entry is one and the other entries are zero. Certainly \( e'_m \dot{\beta}(\cdot) \) defines a continuous linear functional on \( H \), and hence there exists \( \dot{\beta}^*_m \in H \) such that \( e'_m \dot{\beta}(h) = \langle \dot{\beta}^*_m, h \rangle \), \( h \in H \). Then \( \|\dot{\beta}^*_m\|^2 = \langle \dot{\beta}^*_m, \dot{\beta}^*_m \rangle \) represents the asymptotic variance bound of the parameter \( \beta_m = e'_m \beta \). Let \( \Sigma \) be a \( d \times d \) matrix whose \( (m, k) \)-th entry is given by \( \langle \dot{\beta}^*_m, \dot{\beta}^*_k \rangle \). Throughout this paper, we assume that \( \Sigma \) is invertible. The inverse of matrix \( \Sigma \) is called the semiparametric efficiency bound for \( \beta \). (See Bickel, Klaassen, Ritov and Wellner (1993) for ways to compute \( \Sigma \).)

3. Local Asymptotic Minimax Estimation

3.1. Loss Functions. For a decision \( d \in \mathbb{R} \) and the object of interest \( \theta \in \mathbb{R} \), we consider the following form of a loss function:

(3.1) \[
L(d, \theta) \equiv \tau(|d - \theta|),
\]

where \( \tau: \mathbb{R} \to \mathbb{R} \) is a map that satisfies the following assumption.

Assumption 2: (i) \( \tau(\cdot) \) is increasing on \([0, \infty)\), \( \tau(0) = 0 \), and there exists \( \bar{\tau} \in (0, \infty) \) such that \( \tau^{-1}([0, y]) \) is bounded in \([0, \infty)\) for all \( 0 < y < \bar{\tau} \).

(ii) For each \( M > 0 \), there exists \( C_M > 0 \) such that for all \( x, y \in \mathbb{R} \),

(3.2) \[
|\tau_M(x) - \tau_M(y)| \leq C_M|x - y|,
\]
where \( \tau_M(\cdot) \equiv \min\{\tau(\cdot), M\} \).

The condition in (3.2) allows unbounded loss functions. The class of loss functions in this paper is mostly appropriate for the problem of optimal estimation, but excludes some other types of decision problems. For example, it excludes the hypothesis testing type loss function

\[
\tau(|d - \theta|) = 1\{|d - \theta| > c\}, \quad c \in \mathbb{R}.
\]

3.2. Pointwise Local Asymptotic Minimax Theory. First, we develop local asymptotic minimax theory for each fixed \( \alpha_0 \), and call it pointwise local asymptotic minimax theory, because the asymptotic approximation is pointwise at each \( \alpha_0 \). Let

\[
\theta_n(h) \equiv (f \circ g)(\beta_n(h))
\]

and, given any estimator \( \hat{\theta} \) which is a measurable function of \( X_n \), we define its local maximal risk: for each \( b \in [0, \infty) \),

(3.3)

\[
\mathcal{R}_{n,b}(\hat{\theta}) \equiv \sup_{h \in H_{n,b}} E_h \left[ \tau(\sqrt{n}\{\hat{\theta} - \theta_n(h)\}) \right],
\]

where \( H_{n,b} \equiv \{ h \in H : \|\beta_n(h) - \beta_n(0)\| \leq b/\sqrt{n} \} \), and \( E_h \) denotes expectation under \( P_{n,h} \).

Suppose that a Lipschitz continuous map \( f : \tilde{\mathbb{R}} \to \tilde{\mathbb{R}} \) that satisfies Assumption 1(ii) is given. Let \( \mathcal{Y} \subset \mathbb{R} \) be the set of differentiability points of \( f \) in \( \mathbb{R} \). By Assumption 1(ii), the set \( \mathcal{Y} \) is dense in \( \mathbb{R} \). We define for each \( x \in \mathbb{R} \)

\[
\tilde{f}'(x) \equiv \lim_{\varepsilon \searrow 0} \sup_{y \in [x - \varepsilon, x + \varepsilon] \cap \mathcal{Y}} |f'(y)|,
\]

where \( f'(y) \) denotes the first order derivative of \( f \) at \( y \). Certainly, the limit always exists by Assumption 1(ii), and hence \( \tilde{f}'(x) \) is well defined for both differentiable and nondifferentiable points. At a nondifferentiable point \( x \), it is equivalent to define \( \tilde{f}'(x) \) to be the maximum of the absolute left derivative and absolute right derivative. One may consider various alternative concepts of generalized derivatives (e.g. see Frank (1998)), but the definition \( \tilde{f}'(x) \) is simple enough for our purpose. The following result is a generalization of Theorem 1 in Song (2014).

**Theorem 1:** Suppose that Assumptions 1-2 hold. Then for any sequence of estimators \( \hat{\theta} \),

\[
\sup_{b \in [0, \infty)} \liminf_{n \to \infty} \mathcal{R}_{n,b}(\hat{\theta}) \geq \inf_{c \in \mathbb{R}} B(c),
\]

where

\[
B(c) \equiv \sup_{r \in \mathbb{R}^d} E \left[ \tau \left( \tilde{f}'(g(\beta_0)) \Big| \tilde{g}_0(Z + r) - \tilde{g}_0(r) + c \right) \right].
\]
The main feature of the local asymptotic risk bound in Theorem 1 is that it involves infimum over a line, instead of infimum over an infinite dimensional space. This convenient form is due to the same argument in Song (2014) based on the purification result of Dvoretsky, Wald, and Wolfowitz (1951) in zero sum games. This form is crucial for simulating the risk lower bound when we construct a local asymptotic minimax estimator, as explained below.

We consider an optimal estimator of $\theta$ that achieves the bound in Theorem 1. The procedure here is adapted from the proposal by Song (2014). Suppose that we are given a consistent estimator $\hat{\Sigma}$ of $\Sigma$ and a semiparametrically efficient estimator $\tilde{\beta}$ of $\beta$ which satisfy the following assumptions.

**Assumption 3:**

(i) For each $\varepsilon > 0$, there exists $a > 0$ such that

$$\limsup_{n \to \infty} \sup_{h \in H} P_{n,h} \left\{ \sqrt{n} \| \hat{\Sigma} - \Sigma \| > a \right\} < \varepsilon.$$  

(ii) For each $t \in \mathbb{R}^d$, $\sup_{h \in H} \left| P_{n,h} \left\{ \sqrt{n} (\hat{\beta} - \beta_n(h)) \leq t \right\} - P \left\{ Z \leq t \right\} \right| \to 0$ as $n \to \infty$.

Assumption 3 imposes $\sqrt{n}$-consistency of $\hat{\Sigma}$ and convergence in distribution of $\sqrt{n} (\tilde{\beta} - \beta_n(h))$, both uniform over $h \in H$. The uniform convergence can often be verified through the central limit theorem uniform in $h \in H$.

For a fixed large $M_1 > 0$, we define

$$\hat{\theta}_{mx} \equiv f \left( \tilde{g}(\hat{\beta}) + \hat{c}_{M_1} \sqrt{n} \right),$$

where $\hat{c}_{M_1}$ is a bias adjustment term constructed from the simulations of the risk lower bound in Theorem 1, as we explain now.

To simulate the risk lower bound in Theorem 1, we first draw $\{\xi_i\}_{i=1}^L$ i.i.d. from $N(0, I_d)$. Since $\tilde{g}_0(\cdot)$ depends on $\beta_0$ that is unknown to the researcher, we first construct a consistent estimator of $\tilde{g}_0(\cdot)$. Take a sequence $\varepsilon_n \to 0$ such that $\sqrt{n} \varepsilon_n \to \infty$ as $n \to \infty$. Examples of $\varepsilon_n$ are $\varepsilon_n = n^{-1/3}$ or $\varepsilon_n = n^{-1/2} \log n$. Let

$$\hat{g}_n(z) \equiv g \left( z + \varepsilon_n^{-1} (\hat{\beta} - g(\hat{\beta})) \right).$$

Then it is not hard to see that $\hat{g}_n(z)$ is consistent for $\tilde{g}_0(z)$. Define

$$\hat{a}_n \equiv \sup_{x \in [g(\hat{\beta}) - \varepsilon_n, g(\hat{\beta}) + \varepsilon_n] \cap \mathcal{Y}} |f'(x)|.$$

Let

$$\hat{B}_{M_1}(c) \equiv \sup_{r \in [-M_1, M_1]^d} \frac{1}{L} \sum_{i=1}^L \tau_{M_1} \left( \hat{a}_n \left| \hat{g}_n (\hat{\Sigma}^{1/2} \xi_i + r) - \hat{g}_n(r) + c \right| \right).$$
Then we define
\begin{equation}
\hat{c}_{M_1} \equiv \frac{1}{2} \left\{ \sup \hat{E}_{M_1} + \inf \hat{E}_{M_1} \right\},
\end{equation}
where, with \( \eta_{n,L} \to 0 \), \( \eta_{n,L} (\sqrt{L} + \varepsilon_n \sqrt{n} + \varepsilon_n^{-1}) \to \infty \) as \( n, L \to \infty \),
\[ \hat{E}_{M_1} \equiv \left\{ c \in [-M_1, M_1] : \hat{B}_{M_1}(c) \leq \inf_{c_1 \in [-M_1, M_1]} \hat{B}_{M_1}(c_1) + \eta_{n,L} \right\}. \]

The following theorem affirms that \( \hat{\theta}_{mx} \) is local asymptotic minimax for \( \theta = g(\beta) \).
(For technical facility, we follow a suggestion by Strasser (1985) (p.440) and consider a truncated loss: \( \tau_M(\cdot) = \min\{\tau(\cdot), M\} \) for large \( M \).)

**Theorem 2:** Suppose that Assumptions 1-3 hold. Then, for any \( M > 0 \) and any \( M_1 \geq M \) that constitutes constant \( \hat{c}_{M_1} \),
\[ \sup_{b \in [0, \infty)} \limsup_{n \to \infty} R_{n,b,M}(\hat{\theta}_{mx}) \leq \inf_{c \in \mathbb{R}} B(c), \]
where \( R_{n,b,M}(\hat{\theta}_{mx}) \) coincides with \( R_{n,b}(\hat{\theta}_{mx}) \) with \( \tau(\cdot) \) replaced by \( \min\{\tau(\cdot), M\} \).

Therefore, the risk lower bound
\[ \inf_{c \in \mathbb{R}} B(c) \]
in Theorem 1 is sharp. We call it the *local asymptotic minimax risk* in this paper.

When \( \tau(x) = |x|^p \), for some \( p \geq 1 \), the minimizer of \( B(c) \) does not depend on the shape of \( f \). Hence, in constructing \( \hat{B}_{M_1}(c) \) in (3.4), it suffices to take \( \hat{a}_n = 1 \).

When \( \theta = g(\beta) \) is a regular parameter, taking the form of \( g(\beta) = s'\beta \) with \( s \in S_1 \), the local asymptotic minimax risk bound becomes
\[ \inf_{c \in \mathbb{R}} \mathbb{E} \left[ \tau \left( \hat{f}'(s'\beta_0)|s'Z + c| \right) \right] = \mathbb{E} \left[ \tau \left( \hat{f}'(s'\beta_0)|s'Z| \right) \right] , \]
where the equality above follows by Anderson’s Lemma. In this case, it suffices to set \( \hat{c}^{*}_{M_1} = 0 \), for the infimum over \( c \in \mathbb{R} \) is achieved at \( c = 0 \). This is true regardless of whether \( f \) is symmetric around zero or not. Hence the minimax decision becomes simply
\begin{equation}
\hat{\theta}_{mx} = f(\tilde{\beta}'s).
\end{equation}
This has the following consequences.

**Example 5:** (a) When \( \theta = \beta's \) for a known vector \( s \in S_1 \), \( \hat{\theta}_{mx} = \tilde{\beta}'s \). Therefore, the decision in (3.6) reduces to the well-known semiparametric efficient estimator of \( \beta's \).
(b) When \( \theta = \max\{a \cdot \beta's + b, 0\} \) for a known vector \( s \in S_1 \) and known constants \( a, b \in \mathbb{R} \), \( \hat{\theta}_{mx} = \max\{a \cdot \tilde{\beta}'s + b, 0\} \).
(c) When $\theta = |\beta|$ for a scalar parameter $\beta$, $\hat{\theta}_{mx} = |\hat{\beta}|$. This decision is analogous to Blumenthal and Cohen (1968).
(d) When $\theta = \max\{\beta_1 + \beta_2 - 1, 0\}$, $\hat{\theta}_{mx} = \max\{\hat{\beta}_1 + \hat{\beta}_2 - 1, 0\}$.

The examples of (b)-(d) involve nondifferentiable transform $f$, and hence $\hat{\theta}_{mx} = f(\hat{\beta}'s)$ as an estimator of $\theta$ is asymptotically biased in these examples. Nevertheless, the plug-in estimator $\hat{\theta}_{mx}$ that does not involve any bias-reduction is local asymptotic minimax.

4. Discontinuity in the Local Asymptotic Minimax Risk

4.1. Local Robustification. The local asymptotic minimax risk $\inf_{c \in \mathbb{R}} B(c)$ depends on $\beta_0$ discontinuously in general. This is easily seen from the form of $B(c)$, for we may have $\overline{f}'(x)$ and $\overline{f}'(y)$ stay apart, even as $x$ and $y$ get closer to each other. This discontinuity may imply that the local asymptotic minimax approach may serve as a poor approximation of a finite sample risk bound.

We consider an alternative approach of optimality that is robustified against a local perturbation of $\beta_0 = \beta(P_{\alpha_0})$ (i.e. of $\alpha_0 \in \mathcal{A}$). Note that Ibragimov and Khas’minski (1986) pointed out the desirability of local robustification with respect to such an initial parameter $\beta_0$.

For each $y \in \mathbb{R}^d$ and a positive sequence $\varepsilon_n \downarrow 0$, define

$$\mathcal{A}(\alpha_0; \varepsilon_n) \equiv \{\alpha \in \mathcal{A} : ||\beta(P_{\alpha}) - \beta(P_{\alpha_0})|| \leq \varepsilon_n\}.$$  

The set $\mathcal{A}(\alpha_0; \varepsilon_n)$ is the collection of $\alpha$’s such that the regular parameter vectors $\beta(P_{\alpha})$ and $\beta(P_{\alpha_0})$ are close to each other. Then define the local maximal risk under local robustification: for each $b \in [0, \infty)$, and a positive sequence $\varepsilon_n \downarrow 0$,

$$\mathcal{R}_{n,b}(\hat{\theta}; \varepsilon_n) \equiv \sup_{\alpha_1 \in \mathcal{A}(\alpha_0; \varepsilon_n)} \sup_{h \in H_{n,b}(\alpha_1)} \mathbb{E}_{h,\alpha_1} \left[ \tau(|\sqrt{n}(\hat{\theta} - \theta_n(h))|) \right],$$

where $\mathbb{E}_{h,\alpha_1}$ denotes the expectation under $P_{n,h,\alpha_1} \equiv P_{\alpha_1 + \alpha_h/\sqrt{n}}$ and

$$H_{n,b}(\alpha_1) \equiv \{h \in H : ||\beta_n(h) - \beta(P_{\alpha_1})|| \leq b/\sqrt{n}\}.$$  

Then certainly by Theorem 1, we have for any sequence of estimators $\hat{\theta}$,

$$\sup_{b \in [0, \infty)} \liminf_{n \to \infty} \mathcal{R}_{n,b}(\hat{\theta}; \varepsilon_n) \geq \inf_{c \in \mathbb{R}} B(c).$$

\footnote{While nondifferentiability of $f$ yields discontinuity in the minimax risk in most cases, there are counterexamples. For example, when $f(a) = |a|$, we have $f'(a) = 1$ for all $a \in \mathbb{R}$, and if further $g(\beta) = s'\beta$, the minimax risk is continuous in $\alpha_0$, although $f$ is nondifferentiable at 0.}
The main question is whether this lower bound is sharp. For this, we show that the optimal estimator $\hat{\theta}_{mx}$ continues to achieve this lower bound, when Assumption 3 is strengthened as follows.

**Assumption 3’**: (i) There exists $M > 0$ such that

$$\limsup_{n \to \infty} \sup_{\alpha \in A} \sup_{h \in H} P_{n,h,\alpha} \{ \sqrt{n} \| \hat{\Sigma} - \Sigma \| > M \} < \varepsilon.$$  

(ii) For each $t \in \mathbb{R}^d$,

$$\sup_{\alpha \in A} \sup_{h \in H} \left| P_{n,h,\alpha} \{ \sqrt{n}(\hat{\beta} - \beta_n(h)) \leq t \} - P\{ Z \leq t \} \right| \to 0,$$

as $n \to \infty$.

Assumption 3’ strengthens the uniformity in convergence in Assumption 3 to that over $\alpha \in A$. In many cases, it is not hard to verify this condition.

**Theorem 3**: Suppose that Assumptions 1-2 and 3’ hold. Then for each $\varepsilon_n \downarrow 0$ such that $\varepsilon_n \sqrt{n} \to \infty$, as $n \to \infty$, for any sequence of estimators $\hat{\theta}$, and for any $M_1 > M$ such that $M_1$ constitutes $\hat{c}_{M_1}$,

$$\lim_{M \uparrow \infty} \sup_{b \in (0,\infty)} \limsup_{n \to \infty} R_{n,b,M}(\hat{\theta}_{mx}; \varepsilon_n) \leq \inf_{c \in \mathbb{R}} B(c).$$

The result of Theorem 3 shows that the local asymptotic minimax risk

$$\inf_{c \in \mathbb{R}} B(c)$$

remains unchanged, even when we locally robustify the maximal risk, and that the estimator $\hat{\theta}_{mx}$ continues to satisfy the local asymptotic minimaxity after local robustification. At the same time, local robustification does not resolve the issue of discontinuity in the local asymptotic minimax risk.

Given Theorems 1-3, we find that fixing $\delta > 0$, and considering $A(\alpha_0; \delta/\sqrt{n})$ instead of $A(\alpha_0; \varepsilon_n)$ will not change the result. The local asymptotic minimax risk does not depend on this choice of $\delta$. This is because the local asymptotic minimax risk remains the same either we take $R_{n,b}(\hat{\theta}; 0)$ (Theorems 1-2) or we take $R_{n,b}(\hat{\theta}; \varepsilon_n)$ as our local maximal risk.

4.2. **Discussion.** To understand the result of Theorem 3, let us consider localization with a fixed Pitman direction. First, let $\alpha_0 \in A$ be as before such that $\beta(P_{\alpha_0}) = \beta_0$, and consider

$$\alpha_n(\delta) = \alpha_0 + \frac{\delta}{\sqrt{n}},$$
where $\delta \in \mathcal{A}$ is a Pitman direction. We assume that $\{P_{\alpha_n(\delta)}\}_{n=1}^{\infty}$ is quadratic mean differentiable at $P_{\alpha_0}$ with a score $h_\delta \in H$. Then, we have

$$\beta(P_{\alpha_n(\delta)}) = \beta(P_{\alpha_0}) + \beta(P_{\alpha_0 + \delta/\sqrt{n}}) - \beta(P_{\alpha_0})$$

$$= \beta(P_{\alpha_0}) + \frac{\dot{\beta}(h_\delta)}{\sqrt{n}} + o(n^{-1/2}),$$

by the regularity of $\beta$. In other words, the Pitman direction $\delta$ for $\alpha_n(\delta)$ is now translated into the Pitman direction $\dot{\beta}(h_\delta)$ for $\beta(P_{\alpha_n(\delta)})$.

Recall that the local asymptotic minimax risk arises as a consequence of robustification against all the scores $h \in H$ at $P_{\alpha_0}$. Since $\Sigma$ is invertible, the range of $\dot{\beta}$ (when $\dot{\beta}$ is extended to a completion $\tilde{H}$ of $H$) is equal to $\mathbb{R}^d$, i.e., for any $r \in \mathbb{R}^d$, there exists $h \in \tilde{H}$ such that $\dot{\beta}(h) = r$. Therefore, robustification against all the Pitman directions such that $\{P_{\alpha_n(\delta)}\}_{n=1}^{\infty}$ is quadratic mean differentiable at $P_{\alpha_0}$ is equivalent to robustification against all the $\sqrt{n}$-converging Pitman deviations from $\beta_0$. Thus the local robustification against Pitman deviations from $\beta$ is already incorporated in the results of local asymptotic minimax risk in Theorems 1 and 2. This is why local robustification around $\beta_0$ does not alter the results.

One might suggest considering a single Pitman direction $\delta$ and focusing on a sequence of probabilities $\{P_{\alpha_n(\delta)}\}_{n=1}^{\infty}$, derive the local asymptotic minimax risk in a way that depends on $\delta$, and see if the risk continuously depends on $\delta$. This approach is analogous to many other approaches used to deal with discontinuity of asymptotic distributions such as Pitman local asymptotic power analysis, local-to-unity models, and weak identification. However, such an approach in this context counters to the basic motivation of the local asymptotic minimax approach, because restricting attention to a single sequence of probabilities fails to robustify the decision problem properly against local perturbations of the underlying probability and hence fails to exclude superefficient estimators.

5. Conclusion

This paper focuses on the problem of optimal estimation for a parameter that is a nondifferentiable transform of a regular parameter. First, this paper extends the results of Song (2014) allowing for a more general class of nondifferentiable transforms. Second,

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2In fact, Theorem 3 is stronger than this, because for each $\bar{\delta} \in (0, \infty)$,

$$\mathcal{A}(\alpha_0; \bar{\delta}/\sqrt{n}) \subset \mathcal{A}(\alpha_0; \varepsilon_n),$$

from some large $n$ on.

3A still alternative way is an approach of global robustification, where one robustifies against all $\alpha_0$’s in $\mathcal{A}$. The problem with this approach is that the minimax decision problem often becomes trivial, with the minimax risk being infinity. Such a trivial case arises, for example, when $\sup_{x \in \mathbb{R}} f'(x) = \infty$. This is the case when $f(x) = x^2$ for example.
this paper investigates the issue of discontinuity in local asymptotic minimax risk, and considers the approach of local robustification of the true probability. As it turns out, the local robustification does not alter the local asymptotic minimax risk. This means that the optimal estimator remains optimal under this additional dimension of local robustification. On the other hand, it also means that the discontinuity in the minimax risk is not resolved by the local robustification. Hence, there still remains the question of whether local asymptotic minimax theory gives a good approximation of a finite sample decision problem when the parameter is nondifferentiable. A full investigation of this issue is relegated to a future research.

6. Appendix: Mathematical Proofs

Proof of Theorem 1: As in the proof of Lemma 3 of Song (2014), we begin by fixing $r \in \mathbb{R}^d$ so that for some $h' \in \overline{H}$, $r = \dot{\beta}(h')$. (Existence of such $h' \in \overline{H}$ for each $r \in \mathbb{R}^d$ follows from the condition that $\Sigma$ is invertible.) Also fix $h \in H$ such that $\langle h, h' \rangle = 0$. We write

$$\sqrt{n}\{\hat{\theta} - f(g(\beta_n(h + h')))\} \equiv \sqrt{n}\{\hat{\theta} - f(g(\beta_n(h')))\} - \sqrt{n}\{f(g(\beta_n(h + h'))) - f(g(\beta_n(h'))))\}.$$

Suppose first that $f$ is continuously differentiable at $g(\beta_0)$. Then by Assumption 1(ii), we have $x_1 < x_2$ such that $x_1 < g(\beta_0) < x_2$, where $f$ is continuously differentiable on $[x_1, x_2]$. Furthermore, by regularity of $\beta$ and Lipschitz continuity of $g$, we have

$$g(\beta_n(h + h')) = g(\beta_n(h + h') - \beta_0 + \beta_0) \to g(\beta_0),$$

as $n \to \infty$. Therefore, we note that from some large $n$ on, by the mean value theorem,

$$\sqrt{n}\{f(g(\beta_n(h + h'))) - f(g(\beta_0)))\} \equiv \sqrt{n}f'(a_n(h, h'))\{g(\beta_n(h + h')) - g(\beta_0))\},$$

where $a_n(h, h') \equiv t_n\{g(\beta_n(h + h') - g(\beta_0))\} + g(\beta_0)$ for some $t_n \in [0, 1]$. From (6.1), we have

$$a_n(h, h') \to g(\beta_0), \text{ as } n \to \infty.$$

From (A.10) of Song (2014) on page 149, we also find that

$$\sqrt{n}\{g(\beta_n(h + h')) - g(\beta_n(h')))\} = \tilde{g}_0(\tilde{\beta}(h) + r) - \tilde{g}_0(r) + o(1).$$
Since \( f' \) is continuous at \( g(\beta_0) \) (Assumption 1(ii)) and \( \hat{\theta} \) is bounded, we combine these results to deduce that

\[
\sqrt{n} \left\{ f(g(\beta_n(h + h'))) - f(g(\beta_n(h'))) \right\} 
\rightarrow f'(g(\beta_0)) \left( \tilde{g}_0(\hat{\beta}(h) + r) - \tilde{g}_0(r) \right) ,
\]

for all \( h \in H \), as \( n \rightarrow \infty \).

For any sequence of estimators \( \hat{\theta} \), the sequence \( \{ \hat{\theta} \}_{n \geq 1} \) is uniformly tight in \( \bar{\mathbb{R}} \), and hence by using the LAN property and (6.3), applying Prohorov’s Theorem, we find that for each subsequence of \( \{ n \} \), there exists a further subsequence \( \{ n' \} \) such that under \( \{ P_{n',h'} \} \),

\[
\sqrt{n'} \left\{ \hat{\theta} - g(\beta_{n'}(h + h')) \right\} 
\rightarrow \left[ V - f'(g(\beta_0)) \left( \tilde{g}_0(\hat{\beta}(h) + r) - \tilde{g}_0(r) \right) \right],
\]

where \( V \in \bar{\mathbb{R}} \) is a random variable having a potentially deficient distribution. The rest of the proof can be proceeded precisely as in the proofs of Lemma 3 and Theorem 1 of Song (2014).

Second, suppose that \( f \) is not continuously differentiable at \( g(\beta_0) \). Since there is a finite number of nondifferentiability points for \( f \), we have \( x_1, x_2 \in \mathbb{R} \) such that \( x_1 < g(\beta_0) < x_2 \), and \( f \) is continuously differentiable on \([x_1, g(\beta_0))\) and \((g(\beta_0), x_2]\) by Assumption 1(ii).

As previously, we choose arbitrary \( r \in \mathbb{R}^d \) so that for some \( h' \in H, r = \hat{\beta}(h') \), and fix \( b/2 \geq ||h'|| \cdot ||\hat{\beta}'|| \). Now, define

\[
H_{n,b,1}^* \equiv \{ h \in H_{n,b}^* : R_n(h) \geq 0 \}, \quad \text{and}
H_{n,b,2}^* \equiv \{ h \in H_{n,b}^* : R_n(h) \leq 0 \},
\]

where \( H_{n,b}^* \equiv \{ h \in H_{n,b} : \langle h, h' \rangle = 0 \} \) and \( R_n(h) \equiv g(\beta_n(h + h')) - g(\beta_0) \), and observe that for all \( h \in H \),

\[
|R_n(h)| \rightarrow 0,
\]

as \( n \rightarrow \infty \). (This is (A.31) of Song (2014). See the arguments for details.) Thus we have for each \( h \in H \), as \( n \rightarrow \infty \),

\[
1 \{ h \in H_{n,b,1}^* \} \rightarrow 1 \{ h \in H_b \} \quad \text{and}
1 \{ h \in H_{n,b,2}^* \} \rightarrow 1 \{ h \in H_b \},
\]

where \( H_b \equiv \{ h \in H : ||\hat{\beta}(h)|| \leq b \} \).
Note that
\[
\lim_{b \to \infty} \liminf_{n \to \infty} \sup_{h \in H_{n,b}} \mathbf{E}_h \left[ \tau_M \left( |\sqrt{n}(\hat{\theta} - f(g(\beta_n(h))))| \right) \right]
\geq \max_{l=1,2} \lim_{b \to \infty} \liminf_{n \to \infty} \sup_{h \in H_{n,b/2,l}} \mathbf{E}_h \left[ \tau_M \left( |\sqrt{n}(\hat{\theta} - f(g(\beta_n(h))))| \right) \right].
\]
Due to the liminf and supremum over \( h \in H_{n,b/2,l} \) (where the supremum over an empty set of a nonnegative function is taken to be zero), it suffices to focus on \( h \in H \) such that \( h \in H_{n,b/2,1} \) or \( h \in H_{n,b/2,2} \), eventually from some large \( n \) on.

For each \( h \in H_{n,b,1}^* \), we have
\[
g(\beta_0) \leq g(\beta_n(h + h')), \]
and by the mean-value theorem,
\[
\sqrt{n} \left\{ f(g(\beta_n(h + h'))) - f(g(\beta_0)) \right\} = f'_+(g(\beta_0)) + t_n \sqrt{n} (g(\beta_n(h + h')) - g(\beta_0)),
\]
where \( t_n \geq 0 \) and \( t_n \leq g(\beta_n(h + h')) - g(\beta_0) \) and \( f'_+(x) \) denotes the right derivative of \( f \) at \( x \). By using (6.2) and the Lipschitz continuity of \( f' \) on \( (g(\beta_0), x_2] \), for any \( h \in H \), such that \( h \in H_{n,b/2,1}^* \) eventually, we have
\[
\sqrt{n} \left\{ f(g(\beta_n(h + h'))) - f(g(\beta_n(h'))) \right\} \to f'_+(g(\beta_0)) \left( \tilde{g}_0(\hat{\beta}(h) + r) - \tilde{g}_0(r) \right),
\]
as \( n \to \infty \).

Therefore, for any \( h \in H \), such that \( h \in H_{n,b/2,1}^* \) eventually, and for each subsequence of \( \{n\} \), there exists a further subsequence \( \{n'\} \) such that under \( \{P_{n',h'}\} \),
\[
\begin{bmatrix}
\sqrt{n'} \left\{ \hat{\theta} - f(g(\beta_n(h + h'))) \right\} \\
\log dP_{n',h+h'}/dP_{n',h'}
\end{bmatrix} \xrightarrow{d} \begin{bmatrix}
V_+ - f'_+(g(\beta_0)) \left( \tilde{g}_0(\hat{\beta}(h) + r) - \tilde{g}_0(r) \right) \\
\zeta(h) - \frac{1}{2} \langle h, h \rangle
\end{bmatrix},
\]
where \( V_+ \in \mathbb{R} \) is a random variable having a potentially deficient distribution. Using (6.3) and following the same arguments as in the proofs of Lemma 3 and Theorem 1 of Song (2014), we deduce that
\[
\lim_{b \to \infty} \liminf_{n \to \infty} \sup_{h \in H_{n,b/2,1}^*} \mathbf{E}_h \left[ \tau_M \left( |\sqrt{n}(\hat{\theta} - f(g(\beta_n(h))))| \right) \right]
\geq \inf_{c \in \mathbb{R}} \sup_{r \in \mathbb{R}^d} \mathbb{E} \left[ \tau \left( |f'_+(g(\beta_0))| \left| \tilde{g}_0(Z + r) - \tilde{g}_0(r) + c \right| \right) \right].
\]
Similarly, for any \( h \in H \), such that \( h \in H_{n,b/2,2}^* \) eventually, we have
\[
\sqrt{n} \left\{ f(g(\beta_n(h_n + h'))) - f(g(\beta_n(h'))) \right\} \to f'_-(g(\beta_0)) \left( \tilde{g}_0(\hat{\beta}(h) + r) - \tilde{g}_0(r) \right),
\]
as \( n \to \infty \), where \( f'_-(x) \) denotes the left derivative of \( f \) at \( x \). Hence similarly as before, for any \( h \in H \), such that \( h \in H_{n,b/2}^* \) eventually, and for each subsequence of \( \{n\} \), there exists a further subsequence \( \{n'\} \) such that under \( \{P_{n',h'}\} \),

\[
\begin{bmatrix}
\sqrt{n'} \left\{ \hat{\theta} - f (g(\beta_n h + h')) \right\} \\
\log dP_{n',h'+h'/dP_{n',h'}}
\end{bmatrix} \xrightarrow{d} \begin{bmatrix}
V_- - f'_-(g(\bar{\beta}(h)) \left( \bar{g}_0(\hat{\beta}(h) + r) - \bar{g}_0(r) \right) \\
\zeta(h) - \frac{1}{2} (h, h)
\end{bmatrix}
\]

where \( V_- \in \hat{R} \) is a random variable having a potentially deficient distribution. Using (6.5) and following the same arguments as in the proofs of Lemma 3 and Theorem 1 of Song (2014), we deduce that

\[
\lim \inf \lim \sup_{n \to \infty} \sup_{h \in H_{n,b/2}} E_h \left[ \tau_M \left( \left| \sqrt{n} \{ \hat{\theta} - f (g(\beta_n h)) \} \right| \right) \right] 
\geq \inf_{c \in R} \sup \mathbb{E} \left[ \tau \left( \left| f'_-(g(\beta_0)) \right| \left| \bar{g}_0(Z + r) - \bar{g}_0(r) + c \right| \right) \right].
\]

Combining the bounds in (6.7) and (6.8) into (6.6), we conclude that

\[
\lim \inf \sup_{n \to \infty} \sup_{h \in H_{n,b}} E_h \left[ \tau_M \left( \left| \sqrt{n} \{ \hat{\theta} - f (g(\beta_n h)) \} \right| \right) \right] \geq \max_{l=1,2} \Psi(a_l),
\]

where

\[
\Psi(a) \equiv \inf_{c \in R} \sup \mathbb{E} \left[ \tau \left( a \left| \bar{g}_0(Z + r) - \bar{g}_0(r) + c \right| \right) \right],
\]

and

\[
a_+ \equiv \left| f'_+(g(\beta_0)) \right| \quad \text{and} \quad a_- \equiv \left| f'_-(g(\beta_0)) \right|.
\]

Note that \( \Psi(a) \) is an increasing function of \( a \) on \([0, \infty)\). Hence the last bound is equal to

\[
\Psi(\max \{a_+, a_-\}).
\]

Since \( f' \) is Lipschitz continuous on \([x_1, g(\beta_0)]\) and \((g(\beta_0), x_2] \) by Assumption 1(ii), we have

\[
a_+ = \lim_{y \downarrow 0} |f'(g(\beta_0) + y)| = \lim_{\varepsilon \downarrow 0} \sup_{0 < y \leq \varepsilon} |f'(g(\beta_0) + y)| \quad \text{and}
\]

\[
a_- = \lim_{y \uparrow 0} |f'(g(\beta_0) - y)| = \lim_{\varepsilon \downarrow 0} \sup_{0 < y \leq \varepsilon} |f'(g(\beta_0) - y)|.
\]

Since max function is continuous,

\[
\max \{a_+, a_-\} = \lim_{\varepsilon \downarrow 0} \max \left\{ \sup_{0 < y \leq \varepsilon} |f'(g(\beta_0) + y)|, \sup_{0 < y \leq \varepsilon} |f'(g(\beta_0) - y)| \right\}
\]

\[
= \lim_{\varepsilon \downarrow 0} \sup_{y \in [-\varepsilon, \varepsilon] \setminus \{0\}} |f'(g(\beta_0) + y)| = \hat{f}'(g(\beta_0)).
\]

Thus we have a desired lower bound. \( \blacksquare \)
For a given $M_1 > 0$, define
\[
B_{M_1}(c) \equiv \sup_{r \in \mathbb{R}^d} \mathbb{E} \left[ \tau_{M_1} \left( a_0 \mid g_0(Z + r) - \bar{g}_n(r) + c \right) \right],
\]
where $a_0 \equiv \hat{f}'(g(\beta_0))$, and let
\[
E_{M_1} \equiv \left\{ c \in [-M_1, M_1] : B_{M_1}(c) \leq \inf_{c_1 \in [-M_1, M_1]} B_{M_1}(c_1) \right\}.
\]
Define $c_{M_1} \equiv 0.5 \max E_{M_1} + \min E_{M_1}$. We also define
\[
\bar{g}_n(z) \equiv g \left( z + \varepsilon_n^{-1}(\beta_0 - g(\beta_0)) \right),
\]
for $z \in \mathbb{R}^d$, and
\[
\bar{B}_{M_1}(c) \equiv \sup_{r \in [-M_1, M_1]^d} \frac{1}{L} \sum_{i=1}^{L} \tau_{M_1} \left( a_0 \mid \bar{g}_n(\Sigma^{1/2} \xi_i + r) - \bar{g}_n(r) + c \right),
\]
\[
\hat{B}_{M_1}(c) \equiv \sup_{r \in [-M_1, M_1]^d} \frac{1}{L} \sum_{i=1}^{L} \tau_{M_1} \left( a_0 \mid \hat{g}_n(\Sigma^{1/2} \xi_i + r) - \hat{g}_n(r) + c \right),
\]
and
\[
B^*_{M_1}(c) \equiv \sup_{r \in [-M_1, M_1]^d} \mathbb{E} \left[ \tau_{M_1} \left( a_0 \mid \bar{g}_n(\Sigma^{1/2} \xi_i + r) - \bar{g}_n(r) + c \right) \right].
\]
We also define
\[
E^*_{M_1} \equiv \left\{ c \in [-M_1, M_1] : B^*_{M_1}(c) \leq \inf_{c_1 \in [-M_1, M_1]} B^*_{M_1}(c_1) \right\}.
\]

**Lemma A1:** Suppose that Assumptions 1(i), 2, and 3 hold.

(i) As $K \to \infty$,
\[
\limsup_{n \to \infty, h \in H} P_{n,h} \left\{ \sup_{c \in [-M_1, M_1]} \left| B^*_{M_1}(c) - \hat{B}_{M_1}(c) \right| > K(L^{-1/2} + n^{-1/2} \varepsilon_n^{-1} + \varepsilon_n) \right\} \to 0.
\]

(ii) As $n \to \infty$,
\[
\sup_{c \in [-M_1, M_1]} \left| B^*_{M_1}(c) - B_{M_1}(c) \right| \to 0.
\]

**Proof:** (i) As shown in the proof of Lemma A5 of Song (2014), we find that as $K \to \infty$,
\[
\limsup_{n \to \infty, h \in H} P_{n,h} \left\{ \sup_{z \in \mathbb{R}^d} \left| \bar{g}_n(z) - \hat{g}_n(z) \right| > Kn^{-1/2} \varepsilon_n^{-1} \right\} \to 0.
\]
Therefore, as $K \to \infty$,
\[
\limsup_{n \to \infty, h \in H} P_{n,h} \left\{ \sup_{c \in [-M_1, M_1]} \left| B_{M_1}(c) - \hat{B}_{M_1}(c) \right| > Kn^{-1/2} \varepsilon_n^{-1} \right\} \to 0.
\]
Also, for any $\tilde{\varepsilon}_n \downarrow 0$ such that $\tilde{\varepsilon}_n / \varepsilon_n \to 0$ and $\tilde{\varepsilon}_n \sqrt{n} \to \infty$ as $n \to \infty$, we have that
\[
\inf_{h \in H} P_{n,h} \left\{ \left| g(\hat{\beta}) - g(\beta_0) \right| \leq C \tilde{\varepsilon}_n \right\} \to 1,
\]
for some $C > 0$. Therefore, with probability approaching one (uniformly over $h \in H$),
\[
|\hat{a}_n - a_0(\tilde{\varepsilon}_n)| \leq \sup_{x \in [g(\tilde{\beta}) - \varepsilon_n, g(\tilde{\beta}) + \varepsilon_n] \cap \mathcal{Y}} |f'(x)| \leq C \varepsilon_n \to 0,
\]
as $n \to \infty$, for some constant $C > 0$. The last bound $C \varepsilon_n$ follows from the assumption that the derivative $f'(x)$ is Lipschitz continuous on $\mathcal{Y}$. Following the proof of Lemma A5 of Song (2014), we conclude that
\[
\lim_{n \to \infty} \sup_{h \in H} P_{n,h} \left\{ \sup_{c \in [-M_1, M_1]} \left| \hat{B}_{M_1}(c) - \bar{B}_{M_1}(c) \right| > K \{n^{-1/2} + \varepsilon_n\} \right\} \to 0 \quad \text{and}
\]
\[
\lim_{n \to \infty} P \left\{ \sup_{c \in [-M_1, M_1]} \left| B^*_M(c) - \hat{B}_{M_1}(c) \right| > K (L^{-1/2} + n^{-1/2}) \right\} \to 0,
\]
as $K \to \infty$. Combining these results, we obtain the desired result.

(ii) The proof is precisely the same as the proof of Lemma A6 of Song (2014). 

The following lemma deals with the discrepancy between $\hat{c}_{M_1}$ and $c_{M_1}$.

**Lemma A2:** Suppose that Assumptions 1(i), 2, and 3 hold. Then there exists $M_0$ such that for any $M_1 > M_0$, and any $\varepsilon > 0$,
\[
\sup_{h \in H} P_{n,h} \left\{ |\hat{c}_{M_1} - c_{M_1}| > \varepsilon \right\} \to 0,
\]
as $n, L \to \infty$ jointly.

**Proof of Lemma A2:** The proof essentially modifies that of Lemma A7 of Song (2014). From the latter proof, it suffice to show (a) and (b) in the proof of Lemma A7 of Song (2014) for our context. We can derive these using Lemma A1 precisely in the same way. 

■
Proof of Theorem 2: Take $\bar{e}_n \downarrow 0$ such that $\bar{e}_n/e_n \to 0$ but $\bar{e}_n/\sqrt{n} \to \infty$. Then, observe that

$$\inf_{h \in H} P_n, h \left\{ \left| g(\tilde{\beta}) - g(\beta_0) + \frac{\hat{c}_{M_1}}{\sqrt{n}} \right| \leq C \bar{e}_n \right\} \to 1,$$

for some constant $C > 0$ that does not depend on $h \in H$ by Assumption 3(i) and Lipschitz continuity of $g$ and the fact that

$$\frac{|\hat{c}_{M_1}|}{\sqrt{n}} \leq \frac{M_1}{\sqrt{n}} \to 0,$$

as $n \to \infty$ for each fixed $M_1 > 0$. Then, with probability approaching one,

$$\limsup_{n \to \infty} \sup_{h \in H_n} \left| \sqrt{n} \{ \tilde{\theta}_{mx} - f(g(\beta_n(h))) \} \right| \leq a_n \left| g(\tilde{\beta}) - g(\beta_0) + \frac{\hat{c}_{M_1}}{\sqrt{n}} \right| = a_n \left| g(\tilde{\beta}) - g(\beta_0) + \frac{c_{M_1}}{\sqrt{n}} \right| + o_P(n^{-1/2}), \tag{6.10}$$

where

$$a_n \equiv \sup_{x \in [g(\beta_0) - \bar{e}_n/2, g(\beta_0) + \bar{e}_n/2]} |f'(x)|, \tag{6.11}$$

and $o_P(1)$ is uniform over $h \in H$. The last equality follows by Assumption 3(i), Lipschitz continuity of $f$, and Lemma A1. Note that the supremum in (6.11) is monotone decreasing in $n$, so that

$$a_n \to \bar{f}'(g(\beta_0)), \tag{6.12}$$

as $n \to \infty$.

Therefore, using (6.10) and following precisely the same proof as that of Theorem 2 in Song (2014), we have

$$\limsup_{n \to \infty} \sup_{h \in H_n} \mathbb{E}_n \left[ \tau_M \left( |\sqrt{n} \{ \tilde{\theta}_{mx} - f(g(\beta_n(h))) \}| \right) \right] \leq \limsup_{n \to \infty} \sup_{r \in \mathbb{R}^d} \mathbb{E}_n \left[ \tau_M \left( a_n \left\{ \tilde{g}_0 \left( Z + r \right) - \bar{g}_0 \left( r \right) + c_{M_1} \right\} \right) \right].$$

By (6.12), the last term is bounded by

$$\sup_{r \in \mathbb{R}^d} \mathbb{E} \left[ \tau_M \left( \tilde{f}'(g(\beta_0)) \left\{ \tilde{g}_0 \left( Z + r \right) - \bar{g}_0 \left( r \right) + c_{M_1} \right\} \right) \right] = \inf_{c \in [-M_1, M_1]} \sup_{r \in \mathbb{R}^d} \mathbb{E} \left[ \tau_M \left( \tilde{f}'(g(\beta_0)) \left\{ \tilde{g}_0 \left( Z + r \right) - \bar{g}_0 \left( r \right) + c \right\} \right) \right].$$

The last equality follows by the definition of $c_{M_1}$. Finally, we increase $M_1 \uparrow \infty$ to obtain the desired result. ☐
Proof of Theorem 3: First, due to Assumption 3', the convergences in Lemmas A1 and A2 are uniform over \( \alpha_0 \in A \). Take \( \tilde{\varepsilon}_n \downarrow 0 \) such that \( \tilde{\varepsilon}_n / \varepsilon_n \to 0 \) but \( \tilde{\varepsilon}_n / \sqrt{n} \to \infty \). Then, observe that by Assumption 3'(i), Lipschitz continuity of \( g \),

\[
\inf_{\alpha \in A_n(\alpha_0; \varepsilon_n)} \inf_{h \in H} P_{n,h} \left\{ \left| g(\hat{\beta}) - g(\beta(\alpha)) + \frac{\tilde{\varepsilon}_n}{\sqrt{n}} \right| \leq C \tilde{\varepsilon}_n \right\} \to 1,
\]

for some constant \( C > 0 \) that does not depend on \( h \in H \) or \( \alpha \in A \). Then, we have

\[
\left| \sqrt{n} \{ \hat{\theta}_{mx} - f(g(\beta_n(h))) \} \right| \leq \bar{a}_n \left| g(\hat{\beta}) - g(\beta_n(h)) + \frac{cM_1}{\sqrt{n}} \right| + o_P(n^{-1/2})
\]

where \( o_P(1) \) is uniform over \( h \in H \) and over \( \alpha \in A \), and

\[
\bar{a}_n \equiv \sup_{\alpha \in A_n(\alpha_0; \varepsilon_n)} \sup_{x \in [g(\hat{\beta}) - \tilde{\varepsilon}_n, g(\beta) + \tilde{\varepsilon}_n]} |f'(x)|.
\]

(Recall that \( \beta_0 = \beta(P_{\alpha_0}) \) and hence it depends on \( \alpha_0 \in A \).) Similarly as in the proof of Theorem 2,

\[
\limsup_{n \to \infty} \sup_{\alpha \in A_n(\alpha_0; \varepsilon_n)} \sup_{h \in H, h} E_h \left[ \tau_M(\{ \sqrt{n} \{ \hat{\theta}_{mx} - f(g(\beta_n(h))) \} \}) \right]
\]

\[
\leq \limsup_{n \to \infty} \sup_{r \in \mathbb{R}^d} E \left[ \tau_M(\bar{a}_n | \hat{g}_0(Z + r) - g_0(r) + cM_1|) \right]
\]

\[
\leq \sup_{r \in \mathbb{R}^d} E \left[ \tau_M(\bar{a}_n | \hat{g}_0(Z + r) - g_0(r) + cM_1|) \right];
\]

for any fixed \( n_0 \geq 1 \). The last inequality follows because \( \varepsilon_n \downarrow 0 \) and \( \tilde{\varepsilon}_n \downarrow 0 \) as \( n \to \infty \), and \( \bar{a}_n \) and \( A_n(\alpha_0; \varepsilon_n) \) are decreasing in \( n \). We send \( n_0 \to \infty \) and apply the monotone convergence theorem to obtain the last bound as

\[
\sup_{r \in \mathbb{R}^d} E \left[ \tau_M(\hat{f}'(\beta_0) | \hat{g}_0(Z + r) - g_0(r) + cM_1|) \right]
\]

\[
= \inf_{c \in [-M_1, M_1]} \sup_{r \in \mathbb{R}^d} E \left[ \tau_M(\hat{f}'(\beta_0) | \hat{g}_0(Z + r) - g_0(r) + c|) \right].
\]

Finally, we increase \( M_1 \uparrow \infty \) to obtain the desired result. \( \blacksquare \)

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