Penrose Limit and Enhançon Geometry

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Abstract

We study superstring theories on the Penrose limit of the enhançon geometry realized by the $D(p+4)$-branes wrapped on a $K3$ surface. We first examine the null geodesics with fixed radius in general brane backgrounds, which give solvable superstring theories with constant masses. In most cases, the superstring theories contain negative mass-squared. We clarify a condition that the world-sheet free fields have positive mass-squared. We then apply this condition to the enhançon geometry and find that the null geodesics with fixed radius exist only for $p = 0$ case. They define the superstring theories with positive mass-squared. For $p > 0$ case, we show that there is no null geodesic with fixed radius. We also discuss the decoupling limit which gives the dual geometry of super Yang-Mills theory with 8 supercharges. We discuss the $K3$-volume dependence of the superstring spectrum.

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1 Introduction

The work of Berenstein, Maldacena and Nastase (BMN) \cite{BMN} made a major step for understanding the holographic correspondence of string theory and gauge theories. Superstring theory on the maximally supersymmetric plane wave \cite{Zarembo:2000ei}, which is given by the Penrose limit \cite{Penrose:1974jy, Penrose:1974rq} of $\text{AdS}_5 \times S^5$, is exactly solvable \cite{Metsaev:1998it}. Based on this result and the AdS/CFT correspondence \cite{Maldacena:1997re}, BMN identified the string states on the plane wave background as certain sectors of the $D=4, \mathcal{N}=4$ super Yang-Mills theory. Supersymmetric gauge theories with conformal invariance but less supersymmetry has been also investigated extensively \cite{Gubser:2002tv, Witten:2002cs, Aharony:2008ug}.

It is an interesting problem to explore holographic correspondences for non-conformal gauge theories \cite{Gubser:2008px, Aharony:2010av, Fuji:2011wv} using string theories on brane backgrounds. In a previous paper \cite{Fuji:2008ee}, Fuji and the present authors have studied string theories on the Penrose limits of the brane solutions including the $Dp$-brane, $(p,q)$ fivebrane and $(p,q)$ string solutions. We have solved exactly the equations of motion of bosonic strings on certain backgrounds, which have time-dependent masses. Aspects of the quantization of strings on time-dependent plane wave backgrounds are discussed in refs. \cite{Fuji:2008ee, Fuji:2009wh}.

String theories on the Penrose limits of brane solutions and the possible connections to the gauge theories have been studied in the works such as ref. \cite{Gubser:2008yx, Aharony:2008an} for the $Dp$-branes, and refs. \cite{Aharony:2008an, Aharony:2008ej, Aharony:2008ug} for the fivebrane solutions. The Pilch-Warner solution \cite{Pilch:1993yv}, which provides the dual of the RG flow from the $\mathcal{N}=4$ super Yang-Mills theory to the superconformal $\mathcal{N}=1$ theory of Leigh and Strassler \cite{Leigh:1995ep}, have been studied in the Penrose limit \cite{Gubser:2008yx, Aharony:2008an}. String theories were analyzed near the region of the geometry corresponding to the IR fixed point. Recently, Gimon et al. \cite{Gimon:2009kd} studied the Maldacena-Núñez \cite{Maldacena:2001pb} and the Klebanov-Strassler \cite{Klebanov:2000hb} solutions, which are dual to $\mathcal{N}=1$ theories. They considered the Penrose limit along a special class of null geodesics which give solvable string theories. It was argued that the string spectrum represent hadrons in the IR limit of the gauge theories.

In this paper, we study geometries which are dual to super Yang-Mills theories with 8 supercharges. The brane system corresponds to the $D$-branes wrapped on $K3$, which is described by the ‘enhançon’ geometry \cite{Uranga:2008ne}. When $D(p+4)$-branes are wrapped on
$K^3$, the curvature of the $K^3$ induces negative $Dp$-brane charges. In the supergravity, they are effectively described by a geometry which is similar to the $D(p+4)$-$Dp$-brane system. This geometry is valid outside the ‘enhançon radius’. Inside the radius, we have flat space. It is proposed that this enhançon geometry is dual to the $(p+1)$-dimensional super Yang-Mills theory \cite{27}. As a result of the $K^3$ compactification, the gauge theory has half the maximal supersymmetry, and contain the massless fields which are the same as the $D=4, \mathcal{N}=2$ super Yang-Mills theory \cite{27}. We study superstring theory on the Penrose limit of enhançon geometry. We investigate the general null geodesics of the enhançon geometry for $D(p+4)$-branes wrapped on $K^3$. Our particular interest is the special null geodesics such that the radial transverse coordinate is fixed. The Penrose limit along such geodesics gives plane waves on which the light-cone string theory has constant masses.

In recent studies \cite{15,17}, it has been noted that many plane wave backgrounds which are obtained by the Penrose limits yield string theories which have negative mass-squared. Although stability analysis based on supergravity suggests the background is stable \cite{29}, it is difficult to make sense of those string theories which have instability on the world-sheet. The ground state of the string would be a stretched configuration due to the tidal force of the brane background \cite{17}. In this paper, we add an example of string theory where all the mass-squared are non-negative. In such a case, we expect that the string spectrum and their interpretation in terms of the gauge theory are clearly understood.

On the enhançon geometry, we find that the null geodesics with constant radius exist only for the $p=0$ case ($D4$-branes wrapped on $K^3$). Moreover, in this case, we obtain a superstring theory whose mass-squared are all non-negative. We will calculate the superstring spectrum based on the light-cone Hamiltonian. By taking the decoupling limit, we can discuss the correspondence with the gauge theory. In the present work, however, we do not discuss correspondence between string states and gauge-theory operators. We argue that our results of the string theory should give information on the gauge theory in the large $N$ limit with large effective ’t Hooft coupling. We also note a qualitative behavior of the string spectrum as the volume of $K^3$ varies. For $p=1,2$ cases, we show that null geodesic with constant radius does not exist.

This paper is organized as follows. We first review the enhançon geometry in sec-
In section 3, we discuss the null geodesic with constant radius in general brane backgrounds and the Penrose limit along such a geodesic. In section 4, we examine null geodesics in the enhançon geometry associated with D(p + 4)-branes wrapped on K3. We take the Penrose limit along the geodesic with constant radius for the $p = 0$ case. In section 5, we study the spectrum of the superstring theory in this Penrose limit. Discussion on the dual gauge theory is given in section 6. Conclusions and discussion are in section 7.

We include an Appendix, in which we apply the general discussion on the null geodesics with constant radius to other brane solutions. We analyze Dp-brane, (p, q) fivebrane, (p, q) string solutions, which have been treated in the previous paper [15]. The conditions for the existence of the null geodesics with constant radius on these backgrounds are carefully reexamined.

## 2 Enhancôn geometry

We begin with a brief review on the enhancôn geometry. When D(p + 4)-branes are wrapped on the whole K3, the curvature of the K3 induces one unit of negative Dp-brane charge per one D(p + 4)-brane. The supergravity solution which describes this system in the string frame is given as follows [27]

$$
\begin{align*}
    ds^2 &= Z_p^{-1/2}Z_{p+4}^{-1/2}(-dt^2 + dx_a^2) + Z_p^{1/2}Z_{p+4}^{1/2}(dr^2 + r^2d\Omega_{1-p}^2) \\
    &\quad + V^{1/2}Z_p^{1/2}Z_{p+4}^{-1/2}d\Omega_{K3}^2, \\
    A_{p+1} &= \frac{1}{g_sZ_p}dx^0 \wedge d\text{vol}(R^p), \\
    A_{p+5} &= \frac{V}{g_sZ_{p+4}}dx^0 \wedge d\text{vol}(R^p) \wedge d\text{vol}(K3), \\
    e^\phi &= g_sZ_p^{(3-p)/4}Z_{p+4}^{-(p+1)/4}, \\
    Z_p &= 1 - \frac{r^{3-p}}{r^{3-p}}, \quad Z_{p+4} = 1 + \frac{r^{3-p}}{r^{3-p}},
\end{align*}
$$

where $x^a$ ($a = 1, \ldots, p$) are the coordinates along the brane other than those wrapped on the K3. $r$ is the radial coordinate in the transverse $(5-p)$-dimensional space. $A_p$ denotes the RR $p$-form, and $\phi$ is the dilaton. The string coupling at the infinity is given by $g_s$. $ds^2_{K3}$ is the metric of the K3 with unit volume, $d\text{vol}(K3)$ is the corresponding volume.
form, and the constant $V$ is the volume of $K3$ measured at the infinity. The parameters $\tilde{r}_p$ and $r_{p+4}$, which are positive, are given by

$$r_{p+4}^{3-p} = (2\sqrt{\pi})^{1-p} \Gamma\left(\frac{3-p}{2}\right) g_s N \alpha' \tilde{r}_p^{3-p}, \quad \tilde{r}_p^{3-p} = \frac{V_s}{V} r_{p+4}^{3-p},$$

where $V_s \equiv (2\pi)^4 \alpha'^2$ and $N$ is the number of the D($p+4$)-branes. We will consider the cases of $p = 0, 1, 2$.

As explained in ref. [27], this geometry is valid outside the ‘enhançon radius’ $r_e$. The D($p+4$)-branes form a shell at the enhançon radius, and the geometry inside is replaced with the flat space. A way to find $r_e$ is to consider a probe D($p+4$)-brane (also wrapped on $K3$). The enhançon radius is determined as the radius at which the tension of the probe brane vanishes. Following the arguments in refs. [27, 28], we have

$$r_e^{3-p} = \frac{2V}{V - V_s \tilde{r}_p^{3-p}}.$$  \hspace{1cm} (7)

The metric (11) has a naked singularity called the ‘repulson singularity’ at $r = \tilde{r}_p$, where repulsive force on a test particle diverges, but it is removed since the singularity is in the unphysical region $\tilde{r}_p < r_e$.

It is proposed that the decoupling limit of this geometry is dual to super Yang-Mills theory with 8 supercharges [27]. The decoupling limit is given by replacing $Z_p$ and $Z_{p+4}$ in (5) with

$$Z_p = 1 - \frac{\tilde{r}_p^{3-p}}{r_p^{3-p}}, \quad Z_{p+4} = \frac{r_{p+4}^{3-p}}{r_p^{3-p}}.$$ \hspace{1cm} (8)

This is obtained from the full geometry by taking $r \ll r_{p+4}$ with $r \sim \tilde{r}_p$. In other words, this corresponds to a ‘near-shell limit’ with $V/V_s$ taken large. Also note that the enhançon radius for the metric in the decoupling limit is given by $r_e^{3-p} = 2\tilde{r}_p^{3-p}$.

3 Null geodesics with constant radius in general brane backgrounds

The main purpose of this paper is to study enhançon geometry in the Penrose limit along null geodesics which stay at constant radius. Before beginning to study the enhançon geometry, we shall discuss some general properties of the null geodesics with constant radius. Assuming that the metric takes the form of generic brane solutions, we derive the
condition for the existence of the null geodesic with constant radius. We then obtain the formula for the Penrose limit along such a geodesic.

### 3.1 Geodesic equations

Let us consider generic brane solutions in $D$ spacetime dimensions, which are static, rotationally symmetric in the transverse space, and homogeneous in the spatial directions along the brane:

$$ds^2 = -A^2dt^2 + \tilde{A}^2dx^2 + B^2dr^2 + \tilde{B}^2r^2d\Omega^2_{D-p-2}. \quad (9)$$

where $x^a (a = 1, \ldots, p)$ are the coordinates along the branes. $A$, $\tilde{A}$, $B$ and $\tilde{B}$ are functions of the transverse radial distance $r$. Here, $d\Omega^2_{D-p-2}$ is the metric of the unit sphere $S^{D-p-2}$.

We will use the following parametrization of $S^{D-p-2}$:

$$d\Omega^2_{D-p-2} = \cos^2 \theta d\psi^2 + d\theta^2 + \sin^2 \theta d\Omega^2_{D-p-4}. \quad (10)$$

We consider null geodesics in the $(t, r, \psi)$ space. A null geodesic is given by the trajectory of a test particle with the action

$$S = \frac{1}{2} \int d\tau g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

$$= \frac{1}{2} \int d\tau \left( -A^2 \dot{t}^2 + B^2 \dot{r}^2 + \tilde{B}^2 r^2 \dot{\psi}^2 \right) \quad (11)$$

and the massless constraint

$$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -A^2 \dot{t}^2 + B^2 \dot{r}^2 + \tilde{B}^2 r^2 \dot{\psi}^2 = 0 \quad (12)$$

where $\cdot \equiv d/d\tau$.

From the equations of motion for $t$ and $\psi$, we see that the energy $E \equiv A^2 \dot{t}$ and the angular momentum $J \equiv \tilde{B}^2 r^2 \dot{\psi}$ are conserved. We rescale $\tau \rightarrow \tau/E$ and normalize the energy to one. Then, we have

$$\dot{t} = \frac{1}{A^2}, \quad \dot{\psi} = \frac{\ell}{\tilde{B}^2 r^2}, \quad (13)$$

where $\ell \equiv J/E$. The massless constraint becomes

$$-\frac{1}{A^2} + B^2 \dot{r}^2 + \frac{\ell^2}{\tilde{B}^2 r^2} = 0. \quad (14)$$
The velocity in the radial direction $\dot{r}$ is determined as

$$
\dot{r} = \pm \sqrt{\frac{1}{B^2 A^2} - \frac{\ell^2}{B^2 B^2 r^2}}. \tag{15}
$$

Since $\dot{r}^2 \geq 0$, (14) gives the region of $r$, which is determined by

$$
\frac{1}{B^2 A^2} \geq \frac{\ell^2}{B^2 B^2 r^2}. \tag{16}
$$

The points where the equality in (16) holds are generically the turning points of the geodesic. The direction to which the geodesic turns is determined by the second time-derivative $\ddot{r}$ at that point. From the equation of motion for $r$:

$$
-2B^2 \ddot{r} - (A^2)' \ell^2 + (B^2)' \dot{r}^2 + (B^2)' \dot{\psi}^2 = 0, \tag{17}
$$

we obtain

$$
\ddot{r} = \frac{1}{2B^2} \left( -\frac{(A^2)'}{A^4} + \frac{(B^2 r^2)'}{B^4 r^4} - (B^2)' \dot{r}^2 \right) \tag{18}
$$

where $' \equiv d/dr$.

We may consider a geodesic which stays at a fixed radius when $r$ and $\ell$ are adjusted such that $\dot{r} = \ddot{r} = 0$ are satisfied. From (15) and (18), we find that the fixed radius conditions are given by

$$
\frac{1}{B^2} \left( \frac{1}{A^2} - \frac{\ell^2}{B^2 r^2} \right) = 0, \quad \frac{1}{B^2} \left( \frac{(A^2)'}{A^4} - \frac{(B^2 r^2)'}{B^4 r^4} \right) = 0. \tag{19}
$$

It is useful to define the function

$$
F(r) \equiv -A^2 + \frac{\tilde{B}^2 r^2}{\ell^2}. \tag{20}
$$

The conditions (19) are equivalent to

$$
F(r) = 0, \quad F'(r) = 0, \tag{21}
$$

if $\ell^2/(A^2 B^2 \tilde{B}^2 r^2) \neq 0$. 

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3.2 Penrose limit along a null geodesic with constant radius

When a null geodesic stays at a constant radius, we can take the Penrose limit along it in the following way. The coordinates of the geodesic are parametrized by

\[
t = \frac{1}{A^2} \tau, \quad \psi = \frac{t}{B^2 r^2} \tau, \quad r = r_*,
\]

where \(r_*\) and \(\ell_*\) are the solutions of (21). \(A_*\) stands for \(A(r_*)\) etc.

We introduce the light-cone coordinates

\[
x^\pm \equiv \frac{1}{2} \left( A^2 t \pm \frac{B^2 r^2}{\ell_*} \psi \right),
\]

and rewrite the metric (21) of the brane solution as

\[
ds^2 = -A^2 A^4 (dx^+ + dx^-)^2 + \frac{B^2 r^2 \cos^2 \theta \ell_*^2}{B_4^2 r_*^4} (dx^+ - dx^-)^2 \\
+ \tilde{A}^2 dx_a^2 + B^2 dr^2 + \tilde{B}^2 r^2 (d\theta^2 + \sin^2 \theta d\Omega_6^2). \tag{24}
\]

The Penrose limit is given by the following combination of the coordinate transformation and the rescaling of the metric,

\[
x^+ \rightarrow x^+, \quad x^- \rightarrow \Omega^2 x^-, \quad z \rightarrow \Omega z, \quad \theta \rightarrow \Omega \theta, \quad x^a \rightarrow \Omega x^a, \\
ds^2 \rightarrow \Omega^{-2} ds^2, \tag{25}
\]

in the limit \(\Omega \rightarrow 0\). Here \(z\) is a shifted radial coordinate \(z \equiv r - r_*\).

The divergent terms in (24) cancel due to (19). The parts which remain finite take the plane wave form. After rescaling the coordinates by suitable constants, we obtain\(^3\)

\[
ds^2 = 2 dx^+ dx^- - \left( m^2_x \sum_{a=1}^p x_a^2 + m^2_y \sum_{i=1}^{D-p-3} y_i^2 + m^2_z z^2 \right) (dx^+)^2 \\
+ \sum_{a=1}^p dx_a^2 + \sum_{i=1}^{D-p-3} dy_i^2 + dz^2. \tag{26}
\]

The coefficients \(m_i\) are constant and are given by

\[
m^2_x = 0, \quad m^2_y = \frac{\ell_*^2}{B^2 r_*^4}, \quad m^2_z = -\frac{1}{2} \frac{1}{B^2 A_*^4} F''(r), \quad \tag{27}
\]

\(^3\)In this paper we have changed the sign of the definition of \(m^2_i\) in (21) from the one in the previous paper [15].
where $F''(r)_* \equiv F''(r)|_{r=r_*}$, and $F(r)$ is defined in (20). The coordinates $y_l \ (l = 1, \ldots, D - p - 3)$ are defined by

$$
\sum_{\ell=1}^{D-p-3} dy_l^2 = d\theta^2 + \theta^2 d\Omega_{D-p-4}^2. \tag{28}
$$

As we will see in section 5, $m_i^2$ are the mass-squared (times a positive constant) for the bosonic modes $X^i$ of the string in the light-cone gauge on this background. Note that the sign of $m_y^2$ is always positive, but the sign of $m_z^2$ is given by the sign of $-F''(r)_*$. In the present work, we will study string theory on the plane wave where $m_i^2$ are all non-negative. When some of the $m_i^2$ become negative, there will be an instability on the world-sheet. Note that this does not necessarily mean an instability of the spacetime (26). In fact, the analyses of the linearized fluctuations around some plane waves with negative $m_i^2$ suggests that the spacetime is stable [29]. This instability on the world-sheet suggests that the ground state of the string is not point-like, but is rather a stretched configuration, due to the tidal force, as mentioned in ref. [17]. Semi-classical analysis of strings around expanded configurations were performed in refs. [30] for the $AdS_5 \times S^5$ background. Study of string theories on various brane backgrounds along this line would be interesting, but it is beyond the scope of the present paper.

In Appendix, we apply the general discussion in this section to various brane backgrounds. We examine the conditions for the fixed radius null geodesics for the D$p$-brane, $(p, q)$ fivebrane, $(p, q)$ string solutions. We will find several examples of the null geodesics with fixed radius. We will also see that the resulting string theories have non-negative mass-squared only for the near-horizon limits of the fivebrane solutions.

### 4 Penrose limit of enhançon geometry

We now study the Penrose limit of enhançon geometry. In the first subsection, we will discuss possible null geodesics in the enhançon geometries associated with D$(p+4)$-branes. In particular, we look for the null geodesics which stay at constant radius, by using the formulas in the previous section. We will show that such geodesics exist for the $p = 0$ case. In the second subsection, we will obtain the Penrose limit of the $p = 0$ enhançon geometry along the geodesic.
4.1 Possible types of the null geodesics

We consider null geodesics in the \((t, r, \psi)\) space in the enhançon geometry, where \(r\) is the radial coordinate and \(\psi\) is one of the angular coordinates on the \(S^{4-p}\) in the transverse space. We assume that the geodesic is fixed in the \(K3\) or in the directions \(x^a\) along the \(p\)-brane. We may use the formulas in the last section by setting \(D = 6\), \(A^2 = Z_{p-1/2}^{-1/2}Z_{p+4}^{1/2}\) and \(B^2 = \tilde{B}^2 = Z_p^{1/2}Z_{p+4}^{1/2}\). For our background, it is convenient to define

\[
 f(r) \equiv r^{4-2p}\ell^2 Z_p^{1/2}Z_{p+4}^{1/2} F(r)
 = r^{6-2p} + (r_{p+4}^3 - \tilde{r}_p^3) r^{3-p} - \ell^2 r^{4-2p} - \tilde{r}_p^3 r_{p+4}^{3-p},
\]

(29)

The condition (16) for the allowed region of a geodesic is equivalent to \(f(r) \geq 0\). Moreover, the condition for the fixed radius \(\dot{r} = \ddot{r} = 0\) is equivalent to \(f(r) = f'(r) = 0\), and the sign of \(-f''(r)\) evaluated at the fixed radius is the same as the sign of \(m_z^2\).

Firstly we examine what kind of null geodesics are possible for the cases of \(p = 1\) and \(p = 2\). We can see from (29), that we have \(f(r) > 0\) for sufficiently large \(r\), and that \(f(r) = 0\) has one solution \(r = r_T\). If the parameter \(\ell\) is chosen such that \(r_T\) is in the physically sensible region \((r_T \geq r_e)\), \(r_T\) is a turning point where a geodesic which comes from \(r = \infty\) turns outward. On the other hand, if \(r_T < r_e\), the geodesic from the infinity reaches the enhançon radius, and it would pass through the flat region \((r < r_e)\) and then goes back to infinity. We can easily see that \(f(r) = f'(r) = 0\) do not have a solution, and there is not a geodesic which stay at constant radius for \(p = 1, 2\).

For the \(p = 0\) case, on the other hand, null geodesics with fixed radius is possible. The conditions for the fixed radius for \(p = 0\) become

\[
 f(r) = r^6 - \ell^2 r^4 + (r_4^3 - \tilde{r}_0^3) r^3 - \tilde{r}_0^3 r_4^3 = 0,
\]

(30)

\[
 f'(r) = 6r^5 - 4\ell^2 r^3 + 3(r_4^3 - \tilde{r}_0^3) r^2 = 0.
\]

(31)

By eliminating \(\ell^2\) from (30) and (31), we obtain the equation which determines the fixed radius

\[
 2r^6 - (r_4^3 - \tilde{r}_0^3) r^3 + 4\tilde{r}_0^3 r_4^3 = 0,
\]

which have two solutions

\[
 r_{e,}^3 = \frac{1}{4} \left( x - 1 \pm \sqrt{(x-1)^2 - 32x} \right) \tilde{r}_0^3.
\]

(32)
Here we have set $x \equiv V/V_*$, and used the relation $r_4 = x\tilde{r}_0$. The formal solutions in (32) are both positive when $x - 1 > 0$ and $(x - 1)^2 - 32x > 0$, which is equivalent to

$$x \geq 17 + 12\sqrt{2}. \quad (33)$$

The two positive solutions $r_{*\pm}$ in this case are both in the physical region, i.e. outside the enhançon radius $r_e$. In fact, the difference $r_{*+}^3 - r_{*-}^3$ is given by

$$r_{*+}^3 - r_{*-}^3 = \frac{1}{4} \left\{ x - 1 - \frac{8x}{x - 1} - \sqrt{(x - 1)^2 - 32x} \right\} \tilde{r}_0^3.$$

When (33) holds, the r.h.s. is positive since

$$\{(x - 1)^2 - 8x\}^2 - (x - 1)^2\{(x - 1)^2 - 32x\} = 16x(x + 1)^2 > 0.$$

Thus, $r_{*-}$ and $r_{*+}$ are greater than $r_e$. If (33) is not satisfied, (30) and (31) do not have positive solution for any $\ell$.

If the $K3$ volume $V$ is large enough to satisfy (33), we may consider the null geodesic which is fixed at $r = r_{*+}$ or at $r = r_{*-}$. The parameter $\ell$ is fixed to $\ell_{*+}$ or to $\ell_{*-}$, correspondingly, by the condition $f'(r) = 0$:

$$\ell_{*\pm}^2 = \frac{3}{2r_{*\pm}^2} + \frac{3}{4r_{*\pm}}(r_4^3 - \tilde{r}_0^3). \quad (34)$$

Evaluating $f''(r)$ with $r = r_{*\pm}$, $\ell = \ell_{*\pm}$, we obtain

$$f''(r) = \pm 3r_{*\pm}\sqrt{(r_4^3 - \tilde{r}_0^3)^2 - 32\tilde{r}_0^3 r_4^3}.$$ 

That is, we have $f''(r) < 0$ at $r = r_{*-}$, but $f''(r) > 0$ at $r = r_{*+}$.

We may analyze the null geodesics in the enhançon geometries in the decoupling limit (8) by replacing the function $f(r)$ in (29) with

$$f(r) = -\ell^2 r^{4-2p} + r_{p+4}^{3-p} r^{3-p} - \tilde{r}_0^{3-p} r_{p+4}^{3-p}. \quad (35)$$

For $p = 0$, the condition for the constant radius $f(r) = f'(r) = 0$ has one solution:

$$r_{*d}^3 = 4\tilde{r}_0^3, \quad \ell_{*d}^2 = \frac{3}{4} \frac{r_4^3}{r_{*d}}. \quad (36)$$

When evaluated with this solution, we have $f''(r) = -3r_4^3 r_{*d}^3 < 0$. This radius $r_{*d}$ is greater than the enhançon radius $r_e^3 = 2\tilde{r}_0^3$. Note that this $r_{*d}$ corresponds to $r_{*-}$ in the full geometry. Indeed, we may obtain (36) by taking the $x \to \infty$ limit for $r_{*-}$ in (32). The larger solution $r_{*+}$ is scaled out in the decoupling limit. For the $p = 1, 2$, we do not have a solution of $f(r) = f'(r) = 0$ with $f(r)$ in (35).
4.2 Penrose limit of the $p = 0$ enhançon along the null geodesic with constant radius

We take the Penrose limit along the null geodesic with constant radius which we have found for the $p = 0$ enhançon. We can readily obtain the plane wave using the formula in section 3.

From (26) and (27), we get

$$ds^2 = 2dx^+ dx^- - (m_y^2 y^2 + m_z^2 z^2)(dx^+)^2 + dy^2 + dz^2 + dw_s^2$$  \hspace{1cm} (37)

where coordinates $w_s$ ($s = 5, 6, 7, 8$) are the ones along the $K_3$. In our case where the geodesic is fixed in the $K_3$, the $w_s$-part of the metric becomes flat. Note that we do not have $x^a$ (spatial directions along the $p$-brane), since $p = 0$. The constant coefficients $m_y^2$ and $m_z^2$ are given by

$$m_y^2 = \frac{\ell_s^2}{Z_{ps} Z_{4s} r_s^4}, \quad m_z^2 = -\frac{1}{2} \frac{1}{\ell_s^2 r_s^4} f''(r)_s.$$ \hspace{1cm} (38)

where $Z_{ps}, Z_{p+4s}$ mean $Z_p(r_s), Z_{p+4}(r_s)$, and $f''(r)_s \equiv f''(r)|_{r=r_s}$. We denote the solutions of $f(r) = f'(r) = 0$, which are given in \textbf{32} and \textbf{34} or in \textbf{36}, by $r_s$ and $\ell_s$.

If the Penrose limit is taken along $r_{s-}$ or $r_{s+}$, $m_z^2$ is positive, but if the limit is taken along $r_{s+}$, $m_z^2$ is negative, as we see from the signs of $-f''(r)_s$. We will study string theory in the former cases, for the reason mentioned at the end of section 3. Substituting $r_s = r_{s-}, \ell_s = \ell_{s-}$ into (38), we obtain the explicit forms of $m_i^2$ as follows:

$$m_y^2 = \frac{3}{2} \frac{(x - 1 - h(x))(3x - 3 - h(x))}{(x - 5 - h(x))(5x - 1 - h(x))} \frac{1}{r_{s-}^2},$$

$$m_z^2 = \frac{4h(x)}{(3x - 3 - h(x))} \frac{1}{r_{s-}^2},$$ \hspace{1cm} (39)

where

$$h(x) \equiv \sqrt{(x - 1)^2 - 32x},$$

and $r_{s-}$ is given in \textbf{32}. The plane wave for the geometry in the decoupling limit, obtained by substituting $r_s = r_{sd}, \ell_s = \ell_{sd}$ in \textbf{38}, is given by

$$m_y^2 = \frac{1}{r_{sd}^2}, \quad m_z^2 = \frac{2}{r_{sd}^2},$$ \hspace{1cm} (40)
where $r_{sd}$ is in (36). Note that (40) can be obtained from (39) by taking the $x \to \infty$ limit.

The RR fields in the Penrose limit are obtained by applying the coordinate transformations and the rescalings of the coordinates mentioned in section 3.2, to (2), (3):

$$F_2 = dA_1 = \frac{\ell_s}{g_{st_s} Z_{0s}^{1/4} Z_{4s}^{1/4}} \left( \frac{1}{Z_0} \right)_*^' dx^+ \wedge dz,$$

$$F_6 = dA_5 = \frac{\ell_s}{g_{st_s} Z_{4s}^{3/4}} \left( \frac{1}{Z_4} \right)_*^' dx^+ \wedge dw^5 \wedge dw^6 \wedge dw^7 \wedge dw^8 \wedge dz.$$ (41)

Here, we have used the relation

$$V Z_0 Z_4^{-1} \text{dvol}(K3) \to dw^5 \wedge dw^6 \wedge dw^7 \wedge dw^8,$$

which follows from the fact that the $w$-part of the metric becomes flat in the Penrose limit. In the following, we will use the field strength $F_4$, dual to $F_6$:

$$F_4 = \frac{\ell_s}{g_{st_s} Z_{4s}^{3/4}} \left( \frac{1}{Z_4} \right)_*^' dx^+ \wedge dy^1 \wedge dy^2 \wedge dy^3.$$ (42)

The explicit forms of the RR-field strengths (times a dilaton factor) for $r_s = r_{s-}$ are given by

$$e^\phi F_{+z} = -2^{3/2} \cdot 3^{3/2} x (3x - 3 - h(x))^{1/2} (x - 1 - h(x))^{1/2} \cdot \frac{1}{(5x - 1 - h(x))^{1/2} (x - 5 - h(x))^{3/2} r_{s-}},$$

$$e^\phi F_{+123} = 2^{3/2} \cdot 3^{3/2} x (3x - 3 - h(x))^{1/2} (x - 1 - h(x))^{1/2} \cdot \frac{1}{(5x - 1 - h(x))^{3/2} (x - 5 - h(x))^{1/2} r_{s-}}.$$ (43)

For $r_s = r_{sd}$, we have

$$e^\phi F_{+z} = -\frac{1}{r_{sd}}, \quad e^\phi F_{+123} = \frac{3}{r_{sd}}.$$ (44)

## 5 Superstring spectrum on the plane wave

Having obtained the Penrose limit of the $p = 0$ enhançon, we now study superstring theory on the plane wave background. Our convention for the fermionic part of the action follows that of Cvetič et.al. [31]. The covariant action of type IIA superstring up to the quadratic order in fermions\(^4\) is given by

$$S = \frac{1}{2\pi \alpha'} \int d\tau \int_0^{2\pi} d\sigma (\mathcal{L}_b + \mathcal{L}_f)$$

\(^4\)It has been shown that the superstring action in the light-cone gauge is quadratic in fermions for fairly general pp-wave backgrounds [32].
\[ L_b = - \frac{1}{2} \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu} + \frac{1}{2} \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu} \] (45)

\[ L_f = -i \partial_\alpha X^\mu \overline{\Theta} (\sqrt{-h} h^{\alpha\beta} - \epsilon^{\alpha\beta} \Gamma_{11}) \Gamma_\mu \tilde{D}_\beta \Theta. \] (46)

Indices \( \alpha, \beta \) denote the world-sheet directions \( \tau \) and \( \sigma \). \( h_{\alpha\beta} \) is the world-sheet metric and \( \epsilon^{\tau\sigma} = +1 \). Spacetime indices are \( \mu, \nu = 0, 1, \ldots, 9 \), and the indices for the transverse directions are \( i, j = 1, \ldots, 8 \). We will use \( m, n = 0, 1, \ldots, 9 \) for the tangent frame. Fermion \( \Theta \) is the 32-component Majorana spinor, \( g_{\mu\nu} \) and \( B_{\mu\nu} \) are the background metric and the NS-NS 2-form, respectively. The \( \Gamma \)-matrices \( \Gamma_m \) in 10 dimensions satisfy \( \{ \Gamma_m, \Gamma_n \} = 2 \eta_{mn} \).

Chirality matrix \( \Gamma_{11} \) is given by \( \Gamma_{11} = \Gamma_0 \Gamma_1 \cdots \Gamma_9 \). Also note \( \Gamma_\pm = (\Gamma_9 \pm \Gamma_0)/\sqrt{2} \) and \( \Gamma_{\pm} = \Gamma_{\mp} \). The conjugate spinor \( \overline{\Theta} \) is defined by \( \overline{\Theta} = \Theta^\dagger \Gamma^0 \). Curved-space \( \Gamma \)-matrices \( \Gamma_\mu \) are given by applying the vielbein on \( \Gamma_m \).

The derivative \( \tilde{D}_\alpha \) is the ‘pull-back of the supercovariant derivative’ defined by

\[ \tilde{D}_\alpha \Theta \equiv \partial_\alpha \Theta + \partial_\alpha X^\mu \left( \frac{1}{4} \omega^m_{\mu} \Gamma_{mn} + \Omega_\mu \right) \Theta, \] (47)

where \( \omega^m_{\mu} \) is the spin connection, and

\[ \Omega_\mu = \frac{1}{8} \Gamma_{11} \Gamma^{\rho\sigma} F_{\mu\rho\sigma} - \frac{1}{16} \epsilon^\phi (\Gamma_{11} \Gamma^{\rho\sigma} F_{\rho\sigma} - \frac{1}{12} \Gamma^{\rho\sigma\lambda\tau} F_{\rho\sigma\lambda\tau}) \Gamma_\mu. \] (48)

Here, \( F_{\mu\rho\sigma} \) is the NS-NS 3-form field strength, \( F_{\mu\nu} \) and \( F_{\mu\nu\rho\sigma} \) are the RR 2-form and 4-form field strengths, respectively, and \( \phi \) is the dilaton background. The antisymmetrized product of \( \Gamma \)-matrices \( \Gamma^{m_1 \ldots m_n} \) is defined with weight 1.

On the pp-wave background, the only non-vanishing contribution from the spin connection is \( \omega^m_{\mu} \Gamma_{mn} = -\partial_i g_{++} \Gamma_i \Gamma_- \). Since the NS-NS flux is absent and the non-vanishing components of the RR fluxes are \( F_{+i} \) and \( F_{+ijk} \) in our case, \( \Omega_\mu \) is written as

\[ \Omega_\pm = \bar{\Omega} \Gamma_\pm \Gamma_\pm, \quad \Omega_- = 0, \quad \Omega_i = \bar{\Omega} \Gamma_\pm \Gamma_i, \]

where

\[ \bar{\Omega} \equiv \frac{1}{8} \epsilon^\phi \left( \Gamma_{11} \Gamma^i F_{+i} - \frac{1}{6} \Gamma^{ijk} F_{+ijk} \right). \]

We substitute the plane wave background \((37)\) for the \( p = 0 \) enhançon into the action, and impose the light-cone gauge condition

\[ \sqrt{-h} h^{\alpha\beta} = \eta^{\alpha\beta}, \quad X^+ = \alpha' p^+ \tau, \quad \Gamma_- \Theta = 0. \] (49)
The equations of motion for the transverse bosonic fields $X^i$ become

$$\partial^2_\tau X^i - \partial^2_\sigma X^i + \tilde{m}_i^2 X^i = 0,$$  \hspace{1cm} (50)

where $\tilde{m}_i^2 \equiv (\alpha' p^+)^2 m_i^2$. $m_i^2$ are obtained in section 4.2. There are four massless fields ($m_w = 0$), three massive fields with mass $m_y$ and one massive field with mass $m_z$.

The fermionic part of the Lagrangian in the light-cone gauge reads

$$L_f = i(\alpha' p^+) \left\{ \bar{\Theta} \Gamma_{\pm} \partial_\tau \Theta + \bar{\Theta} \Gamma_{11} \partial_\sigma \Theta + (\alpha' p^+) \bar{\Theta} \Gamma_{\pm} \tilde{\Omega} \Gamma_{\pm} \Theta \right\}. \hspace{1cm} (51)$$

We now decompose $\Gamma_m$ into $16 \times 16$ blocks following ref. [31]:

$$\Gamma_+ = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \hspace{0.5cm} \Gamma_- = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}, \hspace{0.5cm} \Gamma_i = \begin{pmatrix} \gamma_i & 0 \\ 0 & -\gamma_i \end{pmatrix}, \hspace{0.5cm} \Gamma_{11} = \begin{pmatrix} \gamma_9 & 0 \\ 0 & -\gamma_9 \end{pmatrix}$$

where $\gamma_i$ are the SO(8) $\Gamma$-matrices which satisfy $\{\gamma_i, \gamma_j\} = \delta_{ij}$, and $\gamma_9 \equiv \gamma_1 \cdots \gamma_8$. In this representation, spinors which satisfy $\Gamma_- \Theta = 0$ are of the form

$$\Theta = \begin{pmatrix} 0 \\ \theta \end{pmatrix}.$$  \hspace{1cm} (52)

We further decompose $\theta$ into a pair of 8 component spinors $\theta_{\pm}$ according to their SO(8) chiralities:

$$\theta = \theta_+ + \theta_-, \hspace{0.5cm} \gamma_9 \theta_{\pm} = \pm \theta_{\pm}. \hspace{1cm} (52)$$

We can write (51) in the form

$$L_f = \sqrt{2} i(\alpha' p^+) \left\{ \bar{\theta}_+ (\partial_\tau + \partial_\sigma) \theta_+ + \bar{\theta}_- (\partial_\tau - \partial_\sigma) \theta_- + 2 \bar{\theta}_+ \tilde{\Omega}_- \theta_- + 2 \bar{\theta}_- \tilde{\Omega}_+ \theta_+ \right\} \hspace{1cm} (53)$$

where

$$\tilde{\Omega}_{\pm} \equiv \frac{\alpha' p^+}{8} e^\phi \left( \mp \gamma_i F_{+i} + \frac{1}{6} \gamma_{ijk} F_{+ijk} \right).$$

The equations of motion become

$$(\partial_\tau + \partial_\sigma) \theta_+ + 2 \tilde{\Omega}_- \theta_- = 0, \hspace{0.5cm} (\partial_\tau - \partial_\sigma) \theta_- + 2 \tilde{\Omega}_+ \theta_+ = 0. \hspace{1cm} (54)$$

Bringing these equations into diagonalized second-order forms, we find that the masses of the fermionic modes are given by the eigenvalues of $-4\tilde{\Omega}_+ \tilde{\Omega}_-$. Evaluating it with the
plane wave fluxes \[ (43) \] for the fixed radius \( r^*_-, \) we obtain

\[
-4\tilde{\Omega}_+ \tilde{\Omega}_- = -\frac{(\alpha' p^+)^2}{16} \left\{ -(e^\phi F_{1+23})^2 - (e^\phi F_{4+})^2 + 2e^{2\phi} F_{1+23} F_{4+} \gamma_{1234} \right\}
\]

\[
= \frac{27 (\alpha' p^+)^2}{2} \left( \frac{3x - 3 - h(x)}{r^*_-} \right) \left( x - 5 - h(x) \right) (5x - 1 - h(x))^3 \times \left\{ 5x - 1 - h(x) \pm x (x - 5 - h(x)) \right\}^2.
\]

(55)

Here, indices 1, 2, 3 denote the \( y \)-directions, and 4 denotes the \( z \)-direction. Signs \( \pm \) in (55) denote the eigenvalues of \( \gamma_{1234} \). In the decoupling limit, where the fluxes are given by (44), this reduces to

\[
-4\tilde{\Omega}_+ \tilde{\Omega}_- = \frac{3(\pm 1)^2 (\alpha' p^+)^2}{16} \frac{r^*_-}{r^*_d}.
\]

(56)

For the world-sheet fermions, we have four fields of the same mass \( \tilde{m}^2_{f_1} \) given by the plus signs of (55) or (56), and four fields of the same mass \( \tilde{m}^2_{f_2} \) given by the minus signs of (55) or (56).

The frequency of the oscillator at the \( n \)-th level is given by \( \omega_n = \sqrt{n^2 + \tilde{m}^2} \), where \( \tilde{m}^2 \) is either the mass for the bosonic modes \( \tilde{m}^2_i \) or for the fermionic modes \( \tilde{m}^2_{f_1}, \tilde{m}^2_{f_2} \). We note here that the sum of \( \omega^2_n \) for the bosonic oscillators is equal to that for the fermionic oscillators, at each \( n \). Indeed, the following equality (relation for the sum of \( \omega^2_0 \)'s) holds

\[
4 \times \tilde{m}^2_w + 3 \times \tilde{m}^2_y + \tilde{m}^2_z = 4 \times \tilde{m}^2_{f_1} + 4 \times \tilde{m}^2_{f_2},
\]

as we see from (59) and (55), or from (44) and (56). This property, which was also observed for the plane waves for the Maldacena-Núñez and the Klebanov-Strassler solutions in ref. [24], guarantees that the string theory is finite.

Mode expansion can be performed in the standard way. (See e.g. refs. [5, 1].) We obtain the light-cone Hamiltonian as follows:

\[
H = H_{b0} + H_b + H_{f0} + H_f + E_0,
\]

(57)

\[
H_{b0} = \frac{1}{2p^+} \sum_{i=1}^{4} p_i^2 + \frac{1}{2\alpha' p^+} \left( \sum_{i=5}^{7} \tilde{m}_y a_i^{0\dagger} a_0^i + \tilde{m}_z a_i^{0\dagger} a_0^i \right),
\]

(58)

\[
H_b = \frac{1}{2\alpha' p^+} \sum_{n=1}^{\infty} \left( \sum_{i=5}^{7} \sqrt{n^2 + \tilde{m}_y^2} (a_i^{n\dagger} a_n^i + \tilde{a}_i^{n\dagger} \tilde{a}_n^i) + \sqrt{n^2 + \tilde{m}_z^2} (a_z^{n\dagger} a_n^z + \tilde{a}_z^{n\dagger} \tilde{a}_n^z) \right. \\
\left. + \sum_{i=1}^{4} n (a_i^{n\dagger} a_n^i + \tilde{a}_i^{n\dagger} \tilde{a}_n^i) \right),
\]

(59)
Here $P_i$ ($i = 1, \ldots, 4$) is the center of mass momentum in the $w$-directions. $a_n^i$ ($n \geq 0$; $i = 1, \ldots, 8$) and $\tilde{a}_n^i$ ($n \geq 1$; $i = 1, \ldots, 8$) are the annihilation operators for the bosonic harmonic oscillators. $b_{n,\rho}^A$, $\tilde{b}_{n,\rho}^A$ ($A = 1, 2$) are the fermionic harmonic oscillators, where we have written the spinor indices $\rho = 1, \ldots, 4$ explicitly. These oscillators obey the standard commutation relations: $[a_n^i, a_m^{j\dagger}] = \delta^{ij} \delta_{nm}$, $[\tilde{a}_n^i, \tilde{a}_m^{j\dagger}] = \delta^{ij} \delta_{nm}$, $\{b_{n,\rho}^A, b_{m,\sigma}^{A\dagger}\} = \delta^{AB} \delta_{nm} \delta_{\rho\sigma}$, $\{\tilde{b}_{n,\rho}^A, \tilde{b}_{m,\sigma}^{A\dagger}\} = \delta^{AB} \delta_{nm} \delta_{\rho\sigma}$. There should also be a vacuum energy $E_0$ in (60). A physical state $|\psi\rangle$ is subject to the level matching condition:

$$\left(\sum_{n=1}^{\infty} \sum_{i=1}^{8} n a_n^i a_n^i + \sum_{n=1}^{\infty} \sum_{A=1}^{4} \sum_{\rho=1}^{4} n b_{n,\rho}^A b_{n,\rho}^A \right) |\psi\rangle = \left(\sum_{n=1}^{\infty} \sum_{i=1}^{8} n \tilde{a}_n^i \tilde{a}_n^i + \sum_{n=1}^{\infty} \sum_{A=1}^{4} \sum_{\rho=1}^{4} n \tilde{b}_{n,\rho}^A \tilde{b}_{n,\rho}^A \right) |\psi\rangle.$$  

### 6 Comment on the dual gauge theory

The gauge theory which is dual to the string theory on the decoupling limit of the $p = 0$ enhaçon geometry is the five dimensional super Yang-Mills theory compactified on $K3$ with the volume $V$. Firstly, we recall that the gauge coupling of the five dimensional theory and that of the $K3$-compactified one dimensional theory are given by

$$g_{YM,5}^2 = (2\pi)^2 g_s \alpha'^{1/2}, \quad g_{YM,1}^2 = (2\pi)^2 g_s \alpha'^{1/2} V^{-1},$$  

respectively. For the energy scale larger than the inverse radius of $K3$ ($E > V^{-1/4}$), the five dimensional gauge theory provides a good description, and for the small energy scale ($E < V^{-1/4}$), the one dimensional gauge theory will take over. The effective coupling at energy scale $E$ for the gauge theory in the large $N$ limit will be given by the following dimensionless 't Hooft couplings [27]:

$$g_{YM,5}^2 N E^{1/2} \quad (\text{for } E > V^{-1/4}), \quad g_{YM,1}^2 N E^{-3/2} \quad (\text{for } E < V^{-1/4}).$$  

Let us examine what is the parameter region of the gauge theory to which our string-theory results correspond. The string theory on the supergravity background gives a good
description when the effective string coupling (determined by the dilaton v.e.v.) and the curvature of the background measured in the string unit are both small. Let us examine these conditions at the constant radius\(^5\) \(r_{sd} = 4^{1/3} \tilde{r}_0\). We have

\[
e^{\phi}|_{r_{sd}} = g_s Z_0^{3/4} Z_{-1/4}^{4/4}|_{r_{sd}} = 2^{-1} \cdot 3^{3/4} g_s \left( \frac{V_*}{V} \right)^{1/4} = 2^{-1} \cdot 3^{3/4} \frac{\mu}{\tilde{N}},
\]

\[
R|_{r_{sd}} = Z_{0}^{-1/2} Z_{-1/2}^{-1/2} |_{r_{sd}} = 4^{1/3} 3^{-1/2} \pi^{-2/3} \frac{\mu^{-2/3}}{\alpha'} \mu, \tag{65}
\]

where \(\mu \equiv g_s N (V_*/V)^{1/4}\). Thus, we need to set \(\mu \gg 1\) fixed but large, and take the \(N \to \infty\) limit, in order to have the string coupling and the curvature both small.

Note that \(\mu\) is the effective coupling \((63)\) at the energy scale \(E = V^{-1/4}\). The effective coupling grows as we increase the energy above \(V^{-1/4}\), and also grows as we decrease the energy below \(V^{-1/4}\). Thus, the condition \(\mu \gg 1\) implies that the gauge theory to which our string results should correspond is strongly coupled at every energy region\(^6\).

We note a qualitative behavior of the string spectrum as we vary the volume \(V\) of the \(K^3\). The world-sheet mass is written as

\[
m^2 \sim \frac{1}{r_{sd}^2} \sim \left( \frac{V_*}{V} g_s N \right)^{-2/3} \frac{1}{\alpha'} = \left( \frac{V_*}{V} \right)^{3/4} g_s N \mu^{-2/3} \frac{1}{\alpha'}. \tag{66}
\]

From this, we see that in the five dimensional limit (when \(V/V_* \gg 1\)), we have \(m^2 \alpha' \gg 1\), and in the one dimensional limit (when \(V/V_* \ll 1\)), \(m^2 \alpha' \ll 1\). This suggests that in the five dimensional limit, the energy of the low lying states are degenerate, since \(\omega_n = \sqrt{n^2 + (p^+ \alpha' m)^2} \sim (p^+ \alpha' m) + n^2/(p^+ \alpha' m)\). On the other hand, in the one dimensional limit, \(m^2\) becomes small, and the spectrum approaches that of the string in the flat space.

We note that there are natural candidates for the gauge-theory operators which correspond to the string states in the Penrose limit. The bosonic fields in the five dimensional super Yang-Mills theory on the D4-branes are \(A_0, A_a, \phi_i\), where the indices \(a(= 1, \ldots, 4)\) denote the spatial directions along the branes, and the indices \(i(= 5, \ldots, 9)\) denote the directions transverse to the branes. Upon the K3 compactification, the one dimensional

---

\(^5\)It may be more appropriate to require that the above two conditions are satisfied throughout the ‘near-shell’ region \(r_e \leq r \leq r_4\), as assumed in the case of the holography for the D0-branes \[13\]. If we take this viewpoint, the condition for the smallness of the dilaton is replaced by \(g_s \ll 1\).

\(^6\)In ref. \[27\], identification of weakly coupled descriptions at various energy scales was attempted. It was assumed that \(\mu \ll 1\), which is opposite to our requirement here.
gauge field $A_0$ and five scalars $\phi_i$ remain massless. Following the idea of BMN \[^1\], the ground state of the string, which have angular momentum in the $S^4$ in the transverse space, would be identified as the trace of the product of $Z \equiv \phi_8 + i\phi_9$, $\text{Tr}(Z \cdots Z)$. Creation of the state by the oscillators in the ‘$y$-directions’ $a^i_n$ ($i = 5, 6, 7$) will correspond to the $\phi_i$ insertions in the trace, oscillators in the ‘$z$-direction’ will be represented by replacing one $Z$ with the covariant derivative $D_0 Z$, and oscillators in the ‘$w$-directions’ $a^a_n$ ($a = 1, \ldots, 4$) will be given by replacing one $Z$ with $D_a Z$. There should be the ordering-dependent phase factors, to represent each string mode $n$, as in ref. \[^1\].

We leave the explicit prediction of the correlation functions of the gauge theory for future study. In the case of $\mathcal{N} = 4$ super Yang-Mills theory, only a subset of the Feynman diagrams contribute to the scaling dimensions of the operators studied by BMN \[^1\]. Whether similar non-renormalization properties exist for the theory with half the maximal supersymmetry is a non-trivial question. Indeed, the superstring spectrum for our case is not supersymmetric. It would be necessary to add SUSY breaking term to the Lagrangian. It is an interesting problem to reproduce the behavior of the spectrum from the gauge theory calculation.

7 Conclusions

In this paper, we have studied the Penrose limit along the null geodesics with fixed radius both in general brane background and in enhançon geometry. In general background we have obtained the explicit formula for the plane wave metric and discussed the condition that resulting string theories have tachyonic mass. In enhançon geometry associated with $D(p+4)$ ($p = 0, 1, 2$) branes wrapped on the $K^3$ surface, we examined possible types of null geodesics with fixed radius and obtained solvable superstring theories from the Penrose limit of the $p = 0$ enhançon geometry. This geometry is dual to the five dimensional super Yang-Mills theory wrapped on $K^3$ with eight supercharges.

In order to formulate the holography for this background, and to extract the information on the gauge theory from the string theory on the null geodesic with constant radius, further analysis is necessary. It would be important to study the superstring theory on the Penrose limit along generic null geodesic, which has time-dependent masses, and regard
the present result as a special limit. In particular, we note that the null geodesic with constant radius found in this paper can be regarded as a limit of a family of the ‘tunneling null geodesics’ introduced in ref. [33]. These are the null geodesics in the space-time in which $t$ and $\psi$ are simultaneously Wick rotated. Such a geodesic passes the region where $f(r) \leq 0$, in the notation of section 4.1. Authors of ref. [33] formulated the holography for $AdS_5 \times S^5$, based on the string theory along tunneling null geodesic which connect two points of the base space of the gauge theory living on the boundary of the background in the decoupling limit. The null geodesic with constant radius found for our background corresponds to the limiting case where the two points on the boundary are infinitely separated. This suggests that the string on this special geodesic represents the gauge theory in the IR limit. By clarifying the interpretation of the null geodesic with constant radius, it might be possible to have an explanation for the fact that such a geodesic exist only for $p = 0$ case from the gauge-theory point of view.

Another direction for future study would be to apply the analysis of the present paper on the null geodesics to other geometries. A possibly interesting background is the Pilch-Warner solution [34] corresponding to the flow from $\mathcal{N} = 4$ to $\mathcal{N} = 2$ supersymmetric Yang-Mills theory. The massless field content of the above $\mathcal{N} = 2$ theory is the one for the pure $\mathcal{N} = 2$ super Yang-Mills theory, which is the same as the dual theory for the enhançon geometry given by the D7-brane wrapped on $K3$. In refs. [35], it was shown that the enhançon mechanism also takes place for this $\mathcal{N} = 2$ Pilch-Warner solution. It might be the case that the structure of the holographic RG flow has some similarities with the ones for our enhançon geometries.

Recently, the authors of ref. [21] proposed an interpretation of the string spectrum at fixed radius on the Maldacena-Núñez and the Klebanov-Strassler solutions. It is argued that the string states correspond to stable hadrons which are described by composite operators made from massive fields. It is an interesting question whether a similar interpretation can be applied to our case.

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Appendix: Null geodesics with constant radius for various brane solutions

As an application of the general procedure described in section 3, we analyze the $D_p$-brane, $(p, q)$ fivebrane $(p, q)$ string solutions, which have been studied in the previous paper [15]. We examine the conditions for the existence of the null geodesic with fixed radius for each brane solution. When such a geodesic exist, we take the Penrose limit. We note here again that we have changed the sign in the definition of the $m_i^2$ from ref. [15].

We investigate the $D_p$-branes in appendix A.1, the $(p, q)$ fivebranes including the NS5-branes in appendix A.2, the $(p, q)$ strings including the fundamental strings in appendix A.3. We will see that a stable geodesic for which all $m_i^2$ are non-negative, is possible only for the fivebrane solutions in the near-horizon limit.

### A.1 D$p$-branes

The $D_p$-brane solution in the string frame is given by

$$ds^2 = H^{-1/2}(-dt^2 + dx_a^2) + H^{1/2}(dr^2 + d\Omega^2_{8-p}),$$

$$H = 1 + \frac{Q_p}{r^{7-p}}.$$  \hspace{1cm} (A.1)

We can apply the formulas of section 3 by substituting $A^2 = H^{-1/2}$, $B^2 = H^{1/2}$, and $D = 10$. We study the $D_p$-branes with $0 \leq p \leq 6$. To study the behavior of the null geodesics on this background, it is convenient to define $f(r)$ which is related to $F(r)$ in (20) by

$$f(r) \equiv r^{5-p}\ell^2 H^{1/2} F(r).$$ \hspace{1cm} (A.2)

The conditions $\dot{r} = \ddot{r} = 0$ for the null geodesic with fixed radius are equivalent to

$$f(r) = r^{7-p} - \ell^2 r^{5-p} + Q_p = 0,$$ \hspace{1cm} (A.3)

$$f'(r) = (7 - p) r^{6-p} - (5 - p) \ell^2 r^{5-p} = 0.$$ \hspace{1cm} (A.4)

Note that the sign of $m_i^2$ is the same as that of $f(r)$. 

In the near-horizon limit, the metric is given by replacing $H$ in (A.1) with

$$H = \frac{Q_p}{r^{5-p}}. \quad (A.5)$$

In this case, the conditions for $\dot{r} = \ddot{r} = 0$ become

$$f(r) = r^{5-p} Q_p = 0, \quad (A.6)$$

$$f'(r) = - (5-p) r^{5-p} Q_p = 0. \quad (A.7)$$

We shall examine whether there are solutions to (A.3) and (A.4), or (A.6) and (A.7) for each brane background.

For the $D_p$-branes with $p \leq 4$, we have the solution $r = r_*$, $\ell = \ell_*$ of (A.3) and (A.4)

$$r_* = \left(\frac{5-p}{2} Q_p\right)^{\frac{1}{1-p}},$$

$$\ell_*^2 = (7-p)(5-p) \left(\frac{5-p}{2}\right)^2 Q_p^{\frac{2}{1-p}}. \quad (A.8)$$

Penrose limit along the null geodesic fixed at the radius $r_*$ is obtained from the formula in section 3. We get the plane wave metric (26) with

$$m_x = 0, \quad m_y = \frac{1}{r_*^2}, \quad m_z = -\frac{5-p}{r_*^2}. \quad (A.9)$$

Since we have $m_z^2 < 0$, string theory has tachyonic mass on this plane wave.

In the near-horizon limit, there is no solution to (A.6) and (A.7). Indeed, this is clear from the fact that the fixed radius $r_*$ is of the same order as the boundary of the near-horizon region $r \sim Q_5^{1-p}$.

For $p = 5$, (A.3) and (A.4) is satisfied by $r_* = 0$ and $\ell_*^2 = Q_5$. Taking the Penrose limit along this geodesic, we get $m_y^2, m_z^2$ which are divergent. On the other hand, when we consider the near-horizon geometry, (A.6) and (A.7) are satisfied by an arbitrary $r_*$ if $\ell_*^2 = Q_5$ is satisfied. The Penrose limit is given by

$$m_x^2 = m_z^2 = 0, \quad (A.10)$$

$$m_y^2 = \frac{1}{r_*^2}. \quad (A.11)$$

In this case, we have $m_z^2 = 0$, and there is no tachyonic mass on the world-sheet of the string on this background. The Penrose limit of fivebrane solutions along the null geodesic with constant radius was obtained by Oz and Sakai [20].
For $p = 6$, the conditions for the constant radius (A.3) and (A.4), or (A.6) and (A.7) do not have solutions.

### A.2 $(p, q)$ fivebranes

The $(p, q)$ fivebrane metric in the string frame is

$$ds^2 = h^{-1/2}(-dt^2 + dx_a^2 + \tilde{H}(dr^2 + r^2d\Omega_3^2)),$$

$$\tilde{H} = 1 + \frac{\bar{Q}_5}{r^2}, \quad h^{-1} = \sin^2 \gamma \tilde{H}^{-1} + \cos^2 \gamma.$$ (A.12)

where $\tan \gamma = q/p$. This includes the NS5-branes ($\cos \gamma = 1$) and the D5-branes ($\cos \gamma = 0$) discussed above. The conditions for the fixed radius $\dot{r} = \ddot{r} = 0$ are equivalent to $f(r) = f'(r) = 0$, where

$$f(r) \equiv h^{1/2}\ell^2 F(r) = r^2 - \ell^2 + \bar{Q}_5.$$ (A.13)

The conditions are of the similar form as for the D5-branes, i.e. the $p = 5$ case of (A.3). (A.13) can be satisfied by $r_* = 0$ and $\ell^2_* = \bar{Q}_5$, but $m_y^2$, $m_z^2$ are divergent.

The $(p, q)$ fivebrane metric in the near-horizon limit is given by replacing $\tilde{H}$ in (A.12) with

$$\tilde{H} = \frac{\bar{Q}_5}{r^2}.$$ (A.14)

The conditions for the constant radius is satisfied by arbitrary $r_*$, if we have $\ell^2_* = \bar{Q}_5$. In the Penrose limit, we have

$$m_x^2 = m_z^2 = 0,$$ (A.15)

$$m_y^2 = \frac{1}{\bar{Q}_5 \cos^2 \gamma + \bar{r}_s^2 \sin^2 \gamma},$$ (A.16)

which is consistent with the result given in ref. [20]. As in the case of the D5-branes, we have $m_z^2 = 0$ which is non-negative, for the $(p, q)$ fivebranes in the near-horizon limit.

### A.3 $(p, q)$ strings

The $(p, q)$ string metric in the string frame is given by

$$ds^2 = h^{-1/2}(\tilde{H}^{-1}(-dt^2 + dx^2) + dr^2 + r^2d\Omega_3^2),$$

$$\tilde{H} = 1 + \frac{\bar{Q}_4}{r^o}, \quad h^{-1} = \sin^2 \gamma \tilde{H} + \cos^2 \gamma.$$ (A.17)
where \( \tan \gamma = q/p \). We obtain the fundamental strings by setting \( \cos \gamma = 1 \), and the D1-branes discussed above by setting \( \cos \gamma = 0 \). The conditions \( \dot{r} = \ddot{r} = 0 \) are equivalent to \( f(r) = f'(r) = 0 \), where

\[
f(r) \equiv h^{1/2} \ell^2 r^4 \tilde{H} F(r) = r^6 - \ell^2 r^4 + \tilde{Q}_1.
\]  (A.18)

Since (A.18) is of the same form as the \( p = 1 \) case of (A.3), the conditions for the constant radius are the same as the ones for the D1-branes:

\[
\begin{align*}
r_* &= (2\tilde{Q}_1)^{\frac{1}{5}} \\
\ell_*^2 &= 3 \cdot 2^{-\frac{2}{5}} \tilde{Q}_1^{\frac{1}{5}}.
\end{align*}
\]  (A.19)

Taking the Penrose limit, we obtain

\[
\begin{align*}
m_x^2 &= 0, \\
m_y^2 &= \frac{3}{3 - \cos^2 \gamma r_*^2}, \\
m_z^2 &= -\frac{12}{3 - \cos^2 \gamma r_*^2}.
\end{align*}
\]  (A.20) (A.21) (A.22)

We have negative mass-squared \( m_z^2 < 0 \).

The near-horizon limit of the \((p, q)\) string metric is given by replacing \( \tilde{H} \) in (A.17) with

\[
\tilde{H} = \frac{Q_1}{r^6}.
\]  (A.23)

In this case, there is no null geodesic with constant radius.

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