Canonical Gravity with Fermions

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Canonical gravity in real Ashtekar–Barbero variables is generalized to allow for fermionic matter. The resulting torsion changes several expressions in Holst’s original vacuum analysis, which are summarized here. This in turn requires adaptations to the known loop quantization of gravity coupled to fermions, which is discussed on the basis of the classical analysis. As a result, parity invariance is not manifestly realized in loop quantum gravity.

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I. INTRODUCTION

When matter is considered coupled to classical or quantum gravity, several important issues arise for fermions. This is, e.g., related to the chirality and possible parity violation of spinors or the fact that they contribute torsion to the space-time geometry. In loop quantum gravity, fermions have been treated occasionally but not yet, as detailed below, in a complete manner. They are therefore revisited here especially with canonical quantization in mind.

The canonical formulation of general relativity in complex Ashtekar variables [1] recasts gravity as a gauge theory similar to Yang-Mills theory, which offered a new way to a possible quantum theory of gravity. Although this reformulation of gravity, expressed in Ashtekar variables as a dynamical theory of complex-valued connections, has the advantage of obtaining algebraically simple constraints, rather complicated reality conditions have to be imposed on the basic canonical variables in order to recover real, Lorentzian general relativity. Moreover, since holonomies of the complex Ashtekar connections take values in the non-compact gauge group SL(2, C), this approach prevents one from taking advantage of much of the available mathematical arsenal of gauge theory built upon compact gauge groups. Therefore, real su(2) valued Ashtekar-Barbero connections, that is, $A^a_i = \Gamma^a_i + \gamma K^a_i$ (with the spin connection $\Gamma^a_i$, $K^a_i$ being a 1-form derived from extrinsic curvature and the Barbero-Immirzi parameter $\gamma$ [2, 3] taking any non-zero real value), have mainly been used for the passage to a quantum theory of gravity.

Real variables were initially introduced by Barbero in a purely canonical formalism [2] which left the relation of the real connection to possible pull-backs or projections of space-time objects unclear. Holst, motivated by this issue, carried out an analysis in [4] to re-derive Barbero’s canonical formulation from an action which generalizes the ordinary Hilbert-Palatini action. In this paper, we further generalize Holst’s analysis for pure gravity to allow for fermionic matter. In other words, we present the Hamiltonian formulation of the Einstein-Cartan action, which incorporates Holst’s action for the gravitational part. This issue has been considered in the literature several times, but the available discussions appear incomplete. In addition to filling this gap in the classical analysis, details of the canonical formulation whose results we summarize are crucial for a proper quantization of gravity in the presence of fermions.

In particular, non-zero torsion arising from the coupling of fermionic matter to gravity through the spin connection requires an analysis in terms of more general connections than used in Holst’s analysis, which inherit torsion contributions. Our results for the given Einstein-Cartan action, despite some resemblance to those in [4, 4, 5], differ in several details. Moreover, we generalize the canonical treatment to arbitrary non-minimal coupling of fermions without any inconsistencies as they occur in other approaches.

We summarize those derivations in a classical part in this paper, which are important to see the role of parity. These details will show us the crucial changes implied by torsion for the general form of dynamics as well as parity invariance, and thus also play a role for any quantization based on a formulation in Ashtekar variables. Consequences for a loop quantization and its dynamics, where several ingredients depend on the form of connections and the phase space structure, are thus described in the second and main part of this paper. Necessary adaptations to the loop quantization of gravity with fermions are explored and presented with the conclusion that previous constructions go through but require non-trivial changes. In particular, the form of basic variables used in the quantum representation makes it difficult to prove parity invariance of the quantum theory even if no parity violating classical interactions are used. This leaves open the potentially intriguing possibility that loop quantum gravity may provide small parity breaking effects due to the quantum space-time structure in the presence of fermions and torsion.
II. CANONICAL FORMULATION

For fermions, one can use a tetrad \( e^I_\mu \) rather than a space-time metric \( g_{\mu\nu} \), related by \( e^I_\mu e^J_\nu = g_{\mu\nu} \), in order to formulate an action with the appropriate covariant derivative of fermions. This naturally leads one to a first-order formalism of gravity in which the basic configuration variables are a connection 1-form and the tetrad. In vacuum the connection would, as a consequence of field equations, be the torsion-free connection compatible with the tetrad. In the presence of matter fields which couple directly to the connection, such as fermions, this is no longer the case and there is torsion \([8]\). For completeness and to introduce the notation, we start by demonstrating this well-known origin of torsion in the theory of gravity.

\[
S[e, \omega, \Psi] = S_G[e, \omega] + S_F[e, \omega, \Psi] = \frac{1}{16\pi G} \int_M d^4x \left| e \right| e^I_\mu e^J_\nu P^{IJ}_{KL} F^{\gamma}_{\mu\nu}(\omega) + \frac{1}{2} i \int_M d^4x \left| e \right| \left[ \nabla^I e^\mu_I e^\nu_J \right. \\
\left. \left( 1 - \frac{i}{\alpha \gamma_5} \right) \nabla_\mu \Psi - \nabla_\mu \Psi \left( 1 - \frac{i}{\alpha \gamma_5} \right) \gamma^\mu e^\mu \right],
\]

where \( \alpha \in \mathbb{R} \) is the parameter for non-minimal coupling. The \( \alpha \)-dependent terms of this form have been introduced in \([3]\) to generalize results of \([12, 10]\) (see also \([11]\)), where they played important (though indirect) roles in parity properties \([38]\). Here \( I, J, \ldots = 0, 1, 2, 3 \) denote the internal Lorentz indices and \( \mu, \nu, \ldots = 0, 1, 2, 3 \) the respective space-time indices. For simplicity, we ignore fermionic mass terms or potentials as they do not provide further complications.

The first term in \([11]\) is the Holst action \([4]\) of gravity \([39]\), \( e^I_\mu \) is the tetrad field, \( e \) is its determinant, and \( e^I_\mu \) its inverse. The Lorentz connection in this formulation is denoted by \( \omega^I_\mu \) and \( F^{IJ}_{\mu\nu}(\omega) = 2 \partial_\mu \omega_{IJ}^\nu + [\omega_\mu, \omega^\nu_{IJ}] \) is its curvature. In order to write the Holst action in a compact form, we have used the following tensor and its inverse

\[
P^{IJ}_{KL} = \delta[K]_I^J \delta[L]_K^J - \frac{1}{\gamma} \frac{e^{IJ}_{KL}}{2}, \\
P^{-1}_{IJ} = \frac{\gamma^2}{\gamma^2 + 1} \left( \delta[I]_J^J + \frac{e^{IJ}_{KL}}{\gamma} \right),
\]

where \( \gamma \) is again the Barbero–Immirzi parameter. Finally, the covariant derivative \( \nabla_\mu \) of Dirac spinors is defined by

\[
\nabla_\mu \equiv \partial_\mu + \frac{1}{4} \omega^J_\mu \gamma[IJ], \quad [\nabla_\mu, \nabla_\nu] \equiv \frac{1}{4} F^{IJ}_{\mu\nu} \gamma[IJ] (3)
\]

in terms of Dirac matrices \( \gamma_I \) (which will always carry an index such that no confusion with the Barbero–Immirzi parameter should arise). Note that we are ignoring the gauge connection required for describing an interaction between charged fermions in the definition of the covariant derivative \([40]\). However, this analysis can easily be generalized to incorporate such interactions.

Varying the action by \( \omega^I_\mu \) produces equations which can be solved for the torsion contribution in

\[
(\nabla_\mu - \nabla_\nu) V_I = C^I_{\mu J} V_J
\]

where \( \nabla_\mu \) is the covariant derivative compatible with the tetrad:

\[
e^I_\mu C_{JK} = 2\pi G \frac{\gamma}{\gamma^2 + 1} \left( \beta \epsilon_{IJKL} J^L - 2 \theta \eta_{IJJK} \right),
\]

where \( \beta := \gamma + 1/\alpha, \theta := 1 - \gamma/\alpha \). This contorsion tensor depends on the Immirzi parameter \( \gamma \) unless \( \alpha = \gamma \). This can then be inserted to produce an action of the form

\[
S[e, \omega, \Psi] = S_G[e, \omega] + S_F[e, \omega, \Psi] + S_{\text{int}}[e, \Psi] = \frac{1}{2\kappa} \int_M d^4x |e| e^I_\mu e^J_\nu \tilde{F}^{IJ}_{\mu\nu}(\tilde{\omega}) + \frac{1}{2} i \int_M d^4x |e| \left( \overline{\Psi} \gamma^I e^\mu_I \nabla_\mu \Psi - \nabla_\mu \Psi \gamma^\mu e^\mu \right)
\]

\[
+ \frac{3\kappa \gamma^2}{16 \gamma^2 + 1} \left( \frac{1}{\alpha^2} - \frac{2}{\alpha \gamma} - 1 \right) \int_M d^4x |e| \left( \overline{\Psi} \gamma_\mu \gamma_\nu \Psi \right) \left( \overline{\Psi} \gamma^\mu \gamma^\nu \Psi \right),
\]

A. Einstein–Cartan Action

The basic configuration variables in a Lagrangian formulation of fermionic field theory are the Dirac bi-spinor \( \Psi = (\psi, \eta)^T \) and its complex conjugate in \( \overline{\Psi} = (\psi^*, \eta^*)^T \) with \( \gamma^\alpha \) being the Minkowski signature Dirac matrices. We note that \( \psi \) and \( \eta \) transform with density weight zero and are spinors according to the fundamental representations of \( SL(2, \mathbb{C}) \). Then the non-minimal coupling of gravity to fermions can be expressed by the total action composed of the gravitational contribution \( S_G \) and the matter contribution \( S_F \) resulting from the fermion field:
with a simple interaction term in addition to gravity and fermion contributions of the torsion-free form.

Notice that the second term in the gravitational Holst action containing \( \gamma \) and the term involving non-minimal coupling \( \alpha \) in the Dirac action are dropped from the above effective action since both these terms can be expressed as boundary terms on-shell; see [4] for details concerning the second term in Holst action. The non-minimally coupled term in the Dirac action can be cast as a boundary term after using \( \tilde{\nabla}_\mu (\tilde{e}_\mu^\alpha) = 0 \) on solutions:

\[
\frac{1}{2\alpha} \int_M d^4x \, |e| \left( \tilde{\nabla}_\gamma \tilde{\nabla}_\gamma \Psi - \tilde{\nabla}_\mu \tilde{\nabla}_\mu \gamma \right) = \frac{1}{2\alpha} \int_M d^4x \, |e| \tilde{e}_I^\mu \left( \partial_\mu (\tilde{\nabla}_\gamma \gamma) - \frac{1}{4} \tilde{\omega}^{MN} \tilde{\nabla}_\gamma \gamma \right) \Psi \]

with the axial current \( J' = \tilde{\nabla}_\gamma \gamma \Psi \). In particular, as we will discuss in more detail in Sec. [11], the effective action in Eq. [3] is parity invariant for all real \( \alpha \). However, as noted in [3], there is an indirect effect of parity because not all torsion components transform as expected under parity unless \( \alpha = \gamma \). We will also see this in the canonical description in what follows, before discussing its significance in classical and quantum gravity. Moreover, for \( \alpha = \gamma \) the action becomes completely independent of \( \gamma \) as noted in [3]. In this case, we have equations for fermions minimally coupled to Einstein–Cartan gravity rather than gravity described by the Holst action. This case is also geometrically distinguished by topological invariance properties of boundary terms.

**B. Canonical variables and second class constraints**

To set up a Hamiltonian formalism, one foliates space-time into spatial slices \( \Sigma_t \): \( t = \text{const} \) determined by a time function \( t \). Instead of working with space-time tensors, one uses spatial tensors which depend on \( t \), subject to evolution equations along a uniform evolution vector field \( t^\mu \) such that \( t^\mu \nabla_\mu t = 1 \). Since we are using the Lorentzian signature, the vector field \( t^\mu \) is required to be future directed. Let us decompose \( t^\mu \) into normal and tangential parts with respect to \( \Sigma_t \) by defining the lapse function \( N \) and the shift vector \( N^a \) as \( t^\mu = N_n^\mu + N^\mu \) with \( N^\mu n_\mu = 0 \), where \( n^\mu \) is the future directed unit normal vector field to the hypersurfaces \( \Sigma_t \). The space-time metric \( g_{\mu\nu} \) induces a spatial metric \( q_{\mu\nu} \) by the formula \( g_{\mu\nu} = q_{\mu\nu} - n_\mu n_\nu \). Since contractions of \( q_{\mu\nu} \) and \( N^a \) with the normal \( n^\mu \) vanish, they give rise to spatial tetrads \( q_{ab} \) and \( N^a \). Here, the lower case roman letters, \( a, b, c, \ldots \), are used to imply spatial tensorial indices.

Moreover, since we are using a tetrad formulation, in addition to the above foliation of the space-time manifold we perform a partial gauge fixing on the internal vector fields of the tetrad to decompose it into an internal unit time-like vector and a triad. Let us fix a constant internal vector field \( n_I = -\delta_{t,0} \) with \( n^J n_I = -1 \). Now we allow only those tetrads which are compatible with the fixed \( n^I \) in the sense that \( n^J = n^J e^I_j \) must be the unit normal to the given foliation. This implies that \( e^I_j = E^I_j - n^J n_I \) with \( E^I_j n_a = E^I_j n^a = 0 \) so that \( E^I_j \) is a triad.

Now using \( n^a = N^{-1} (t^a - N^a) \) to project fields normal and tangential to \( \Sigma_t \), one can decompose the Einstein–Cartan action [11] and extract the canonical fields as well as possible constraints. The only time derivative (along \( t^\mu \)) of gravitational variables in the action appears for the Ashtekar–Barbero connection

\[
A^I_b := -\gamma \omega_b^j \gamma^0 - \frac{1}{2} \epsilon_{jkl} \omega_b^{kl} = \gamma K^I_j + \Gamma^I_b \]

(7)

multiplied with \( P^I_a := \sqrt{q} \epsilon_{ijkl} / \gamma \kappa \) as the momentum conjugate to \( A^I_b \). The remaining components of the space-time connection,

\[
-A^j_b := \omega_b^j \gamma^0 - \frac{1}{2} \gamma \epsilon_{jkl} \omega_b^{kl} \]

(8)

and \( \epsilon_{jkl} \), as well as the lapse function \( N \) and shift vector \( N^a \) appearing in the metric are non-dynamical. As usually, variation by \( N \) and \( N^a \) gives the Hamiltonian and diffeomorphism constraints. The variation by the non-dynamical connection components, on the other hand, provides partially second class constraints which can be solved algebraically for

\[
\gamma \bar{A}_b^k = -A^k_b + 2 \Gamma^k_b \]

(9)

where

\[
\Gamma^k_b = \bar{\Gamma}_b^k + \frac{\gamma \kappa}{4(1 + \gamma^2)} \left( \theta \epsilon_{ijk} e^i_a J^j - \beta e^i_a J^0 \right) \]

(10)

is a combination of the torsion-free spin connection \( \bar{\Gamma}_b^k \) (see App. [A2]) and a torsion contribution

\[
C^i_a := \frac{\gamma \kappa}{4(1 + \gamma^2)} \left( \theta \epsilon^j_{kl} e^i_a J^j - \beta e^i_a J^0 \right) \]

(11)

Also \( \omega_b^{k0} \) is determined by the second class constraints, such that only \( \epsilon_{ijk} \omega_b^{jk} \) remain free as Lagrange multipliers of the Gauss constraint

\[
G_i = D_i P^b_i - \frac{1}{2} \sqrt{q} \bar{J}_i = \gamma [K_b, P^b_i] - \frac{\gamma \beta}{2(1 + \gamma^2)} \sqrt{q} J_i \]

(12)
where \( J^i = \psi^i \sigma^i \psi + \eta^i \sigma^i \eta \) (and \( J^0 = \psi^0 - \eta^0 \psi \) which will appear below).

Together with the diffeomorphism constraint

\[
C_a = P_j^b (F_{ab}^j - (\gamma^2 + 1) e_j^k K^k_a K^j_b) - i\sqrt{q} (\theta_L (\psi^j D_a \psi - \overline{D_a \eta} \eta) - c.c.) + \frac{\beta}{2} K^i_a \sqrt{q} J_i
\]

(13)

and the Hamiltonian constraint

\[
C = \frac{\gamma^2}{2\sqrt{q}} P^a_i P^b_j (e^{ij}_k F_{ab}^k - 2(\gamma^2 + 1) K^i_a K^j_b) + \frac{\gamma \kappa \beta}{2\sqrt{q}} P^a_i D_a (\sqrt{q} J^i) + (1 + \gamma^2) \kappa D_a \left( \frac{P^a_i G^i}{\sqrt{q}} \right)
\]

\[
+ i\gamma \kappa P^a_i \left( \theta_L (\psi^j \sigma^j D_a \psi + \overline{D_a \eta} \eta \sigma^0) - \theta_R (\eta^j \sigma^j D_a \eta + \overline{D_a \psi} \sigma^0 \psi) \right) - \frac{\kappa}{4} \left( 3 - \frac{\gamma}{\alpha} + 2\gamma^2 \right) \epsilon_{ijk} K^i_a P^k_a J^i
\]

(14)

(15)

(where \( \theta_{L/R} := \frac{1}{2} (1 \pm i/\alpha) \)) this provides a first class set of constraints.

At this point, we emphasize that we have not imposed any restriction on either the non-minimal coupling parameter \( \alpha \) or the Immirzi parameter \( \gamma \) (as long as they are both real). The formulation is thus consistent for all values, but as we will see the behavior under parity of the variables used appears different depending on whether \( \alpha = \gamma \) or not. We also emphasize that some of the terms in our constraints differ from those presented in [1] even for the case \( \alpha = \gamma \) considered there. In what follows, we will be led to consistency checks of our expressions, which confirm the presence of the terms listed here.

III. PARITY TRANSFORMATION OF THE CLASSICAL THEORY

In the presence of fermions, the parity behavior is not fully obvious even in the absence of explicitly parity violating interaction terms. A detailed analysis of transformation properties is then required.

A. The Torsion Contribution to Extrinsic Curvature

Torsion components play an indirect but important role in the behavior under parity. During the constraint analysis, second class constraints provide the torsion contribution to the connection as seen in [10]. However, although \( K^i_a \) is restricted by the Gauss constraint, constraints do not provide its complete torsion contribution. On the other hand, the transformation properties of the Ashtekar-Barbero connection \( A^i_a \) under parity cannot be determined without the knowledge of the torsion contribution to \( K^i_a \), or at least its parity behavior. Thus the splitting of extrinsic curvature into torsion-free and torsion parts is inevitable in order to arrive at a set of consistent parity transformations for gravity with fermions.

As in the case of [11], we have to solve partially equations of motion for the connection to derive the expression for the torsion part \( k^i_a \) of \( K^i_a \).

For the canonical pair \( (A^i_a, P^a_i) \) the equations of motion are \( \mathcal{L}_A A^i_a = \{ A^i_a, H \} = \delta H / \delta P^a_i \) and \( \mathcal{L}_P P^a_i = \{ P^a_i, H \} = - \delta H / \delta A^i_a \) where \( H[A^i_a, N^a] = \int d^3x (\Lambda^i_a G^i + N^a H) \) is the total constraint. While the first equation of motion entails all the dynamics of gravity coupled with matter, the second one yields the expression for the connection. After longer calculations, it takes the form

\[
\mathcal{L}_A P^a_j + \omega^0_i e_{ij}^k k^k P^a_j - P^a_j \partial_a N^a - N^a \partial_a P^a_j + P^a_j \partial_b N^b
\]

\[
+ N^a e_{jk}^b A^c_k + N^c G^j + \text{sgn det}(e^a_{\alpha\beta}) \left( \frac{\kappa}{\gamma^2} \partial_b (N e_{\alpha a}) \right)
\]

\[
+ \frac{\kappa}{\gamma^2} \left( e_{jk}^c A^k_b - e_{kj}^c A^k_b \right) \partial_j^c A^k_b = \frac{1}{\kappa} N e_{jk}^c J^n j^k + \frac{N \kappa}{2\alpha} P^c_j J^0
\]

where the sign of the determinant of the co-triad appears due to the use of \( \gamma \kappa P^a_i = \frac{1}{2} \text{sgn det}(e^a_{\alpha\beta}) e_{ij}^c e_{jk}^c e_{lk}^c. \)

In order to solve for \( k^i_a \) which appears via \( A^i_a \), we contract the equation with \( e^0_j \) and, as an internal tensor with indices \( l \) and \( j \), derive its trace and symmetric parts. Combined, this gives

\[
e^i_j \mathcal{L}_A P^c_j + e^i_j \mathcal{L}_A P^c_j - \delta^i_j e^c_i \mathcal{L}_A P^c + N^a (P^c_j \partial_a e^i_j + P^c_j \partial_b e^i_j) + e^i_c P^c_j \partial_b N^c + e^i_j P^c_j \partial_b N^c
\]

\[
- \text{sgn det}(e^a_{\alpha}) \left( \delta^i_j \gamma_k \kappa \partial_b (e^\alpha_{a\beta}) - N^a \gamma_k \kappa (e^\alpha_{a\beta} e^i_j + e^i_j e^\alpha_{a\beta}) \right) + \frac{N \sqrt{q}}{\gamma^2} (e^i_j A^k_b + e^i_j A^k_b) = - \frac{N \sqrt{q}}{2\alpha} \delta^i_j J^0.
\]
The extrinsic curvature contribution is contained in the term

\[(\sqrt{\gamma} e^{i} A_{ab} + e^{i} A_{b}^i) = (P_{a}^{b} \tilde{\Gamma}_{b}^{i} + P_{i}^{b} \tilde{\Gamma}_{b}^{i}) + (P_{a}^{i} C_{b}^{i} + P_{i}^{b} C_{b}^{i}) + \gamma (P_{b}^{i} \tilde{K}_{b}^{i} + P_{i}^{b} \tilde{K}_{b}^{i}) + \gamma (P_{b}^{i} k_b + P_{i}^{b} k_b), \]  

where we have used the decomposition \( A_{a}^{i} = \tilde{A}_{a}^{i} + \tilde{A}_{a}^{i} \) into the torsion-free part \( \tilde{A}_{a}^{i} = \tilde{\Gamma}_{a}^{i} + \gamma \tilde{K}_{a}^{i} \) and a torsion contribution \( \tilde{A}_{a}^{i} = C_{a}^{i} + \gamma k_{a}^{i} \).

To complete the splitting, we use that the torsion-free extrinsic curvature from the usual expression \( \tilde{K}_{ab} = \sqrt{N} (\frac{1}{2} \gamma_{ab} - 2 \hat{D}_{(a} N_{b)}) \) satisfies

\[ P_{b}^{i} \tilde{K}_{b}^{i} + P_{i}^{b} \tilde{K}_{b}^{i} = -\frac{1}{N} \gamma ( (e^{i}_{a} \mathcal{L}_{i} P_{c}^{a} + e^{i}_{a} \mathcal{L}_{i} P_{c}^{a} - e^{b}_{a} \mathcal{L}_{i} P_{c}^{b} + (N^{a} (P_{c}^{i} \partial_{a} e_{c} + P_{b}^{i} \partial_{a} e_{b}) + e_{c} P_{b}^{i} \partial_{b} N^{c} + e_{c} P_{b}^{i} \partial_{b} N^{c})) ) \]

for \( \tilde{K}_{ab} = e^{i} \hat{K}_{ab} \). Combining this with (A10), (17), (11), we find \( e^{i}_{a} e^{b}_{i} + e^{b}_{i} k_{b} = \kappa \gamma J_{0}^i / 2(1 + \gamma^{2}) \). On the other hand, from the Gauss constraint it follows that \( k_{b}^{i} e^{b}_{i} - k_{b}^{i} e^{b}_{i} = \kappa \gamma \beta e^{b}_{i} J_{0}^i / 2(1 + \gamma^{2}) \). Thus,

\[ k_{a}^{m} = \frac{\gamma^{2}}{4(1 + \gamma^{2})} \left( \beta e_{b}^{a} J^{i} + \theta e_{b}^{a} J^{0} \right) \]  

is the contribution which provides the antisymmetric part of \( K_{ab} \), but also adds to the symmetric term. The expression for \( k_{a}^{m} \) can independently (but not fully canonically) be verified by computing it from (11) as \( k_{a}^{m} = -C_{a}^{m0} = -q_{a}^{m0} J^{0} \).

With (11) and (19), the Ashtekar–Barbero connection as split into its torsion and torsion-free parts is

\[ A_{a}^{i} = \tilde{\Gamma}_{a}^{i} + \gamma \tilde{K}_{a}^{i} + \frac{\kappa \gamma}{4} \epsilon_{k}^{i} e_{a}^{j} J^{j} - \frac{\kappa \gamma}{4 \alpha} e_{b}^{a} J^{0}, \]

where the first term is completely torsion-free and only the \( J \)-terms represent the torsion contribution.

## B. Parity transformation

We first define the parity transformation for both canonical gravitational variables and fermionic matter fields such that it respects the background independence of a theory of gravity non-minimally coupled with fermions. Parity conservation can then be determined by testing whether the effective action (5) in the Lagrangian formulation, or constraints as well as the symplectic structure of the Hamiltonian formulation are left invariant. As we will see, the torsion contributions to the connection play an important role in this, and we will be led to split all the constraints into their torsion-free and torsion parts to verify the parity behavior.

In a background-independent setting, we cannot refer to spatial coordinates changing their sign under parity reversal. Instead, as usually in formulations on curved manifolds we use the fact that triads change their orientations under parity reversal as one of the primary contributions to the parity transformation: \( e^{i} \rightarrow -e^{i} \).

For Dirac spinors, we use the conventional field theory definition \( \Psi \rightarrow \gamma_{0} \Psi \). These basic definitions imply

\[ J^{0} = \tilde{\Psi} \gamma_{0} \gamma^{5} \Psi \rightarrow -J^{0}, \quad J^{i} = \tilde{\Psi} \gamma^{i} \gamma^{5} \Psi \rightarrow J^{i} \]

\[ \Gamma_{a}^{i} = \tilde{\Gamma}_{a}^{i} + C_{a}^{i} \rightarrow \tilde{\Gamma}_{a}^{i} - \frac{\gamma \kappa}{4(1 + \gamma^{2})} \left( \beta \epsilon_{i}^{j} e_{b}^{b} J^{j} + \theta e_{b}^{b} J^{0} \right) \]

\[ \tilde{K}_{a}^{i} = K_{a}^{i} + k_{a}^{i} \rightarrow -\tilde{K}_{a}^{i} - \frac{\gamma \kappa}{4(1 + \gamma^{2})} \left( \beta \epsilon_{i}^{j} e_{b}^{b} J^{j} + \theta e_{b}^{b} J^{0} \right) \]

where we have used \( \tilde{K}_{ab}^{i} = \tilde{K}_{ab} e^{b j} \rightarrow -\tilde{K}_{ab}^{i} \). The \( \tilde{\gamma}_{a}^{i} \)-behavior follows from formulas in the Appendix, which keep track of factors of sgn det(\( e_{a}^{i} \)).

It is interesting to note that both \( \Gamma_{a}^{i} \) and \( K_{a}^{i} \) transform as their torsion-free counterparts \( \Gamma_{a}^{i} \) and \( \tilde{K}_{a}^{i} \), only for \( \alpha = \gamma \) (i.e. \( \theta = 0 \)), a result expected from (3). Also note that the Ashtekar–Barbero connection \( A_{a}^{i} \) does not have a simple transformation property because \( \Gamma_{a}^{i} \) and \( K_{a}^{i} \) transform differently. However, as seen in (20) the torsion contributions simplify when combined to \( C_{a}^{i} + \gamma k_{a}^{i} \). Regrouping the remaining terms by new combinations with \( \tilde{\Gamma}_{a}^{i} \) and \( \tilde{K}_{a}^{i} \) provides a transformation law

\[ A_{a}^{i} = \left( \tilde{\Gamma}_{a}^{i} + \frac{\gamma \kappa}{4 \alpha} e_{a}^{i} J^{0} \right) + \gamma \left( \tilde{K}_{a}^{i} + \frac{\kappa}{4} \epsilon_{j}^{i} e_{a}^{j} J^{k} \right) - \left( \tilde{\Gamma}_{a}^{i} - \frac{\gamma \kappa}{4 \alpha} e_{a}^{i} J^{0} \right) - \gamma \left( \tilde{K}_{a}^{i} + \frac{\kappa}{4} \epsilon_{j}^{i} e_{a}^{j} J^{k} \right) \]

just like the combination of torsion-free \( \tilde{\Gamma}_{a}^{i} \) and \( \tilde{K}_{a}^{i} \).

With these rules, the Liouville term in the action transforms as
\[ \int_{\Sigma} d^3x P_i^a \mathcal{L}_t A_i^a = \int_{\Sigma} d^3x P_i^a \mathcal{L}_t \left( \left( \tilde{\Gamma}^i_a - \frac{\gamma K}{4\alpha} \varepsilon_a^i J^0 \right) + \gamma \left( \tilde{K}^i_a + \frac{\kappa}{4} \varepsilon^i_{jk} \varepsilon^j_k J^k \right) \right) \]

and we have used the fact that \( \left\{ P_i^a, \tilde{\Gamma}^i_a - \frac{\gamma K}{4\alpha} \varepsilon_a^i J^0 \right\}_{PB} = 0 \) such that the \( \tilde{\Gamma} \)-term does not contribute to the symplectic structure. Therefore, the symplectic structure is invariant under the parity transformation.

Given that \( A_i^a \) consists of two terms transforming differently, it is useful for a parity analysis to rewrite all terms of the constraints by explicitly splitting off their torsion contributions. The torsion-free parts will then just have the vacuum parity behavior, which is parity invariant, while the torsion terms directly demonstrate the parity behavior in the presence of fermions through the currents. The Gauss constraint can easily be split in this way and formulated in torsion-free variables. The split Gauss constraint, \( \bar{G}_i = \gamma \varepsilon^i_{jk} \bar{K}^j_a P^a_k = 0 \), is independent of the fermion current and thus parity invariant. Splitting the diffeomorphism constraint into torsion and torsion-free components is more involved, and after a longer computation we obtain

\[ C_a = P_j^b \left( \bar{F}_{ab} + 2\partial_{[a} \bar{A}_{b]} + \varepsilon^i_{lm} \bar{A}_{a}^{lm} + \varepsilon^i_{lm} \bar{A}^{lm}_a + \varepsilon^i_{lm} \bar{A}_a^{lm} + \varepsilon^l_{im} \bar{A}^j_a \right) - \frac{1}{2} \gamma^2 K_a G_i \]

\[ + \frac{1}{2} i q \bar{D}_a \gamma^0 - \frac{1}{2} C_a^b \sqrt{q} J_i - \frac{\gamma}{2} K_a G_i \]

\[ = 2 \gamma P_j^b \bar{D}_a \bar{K}_b^j + \text{sgn det} (\varepsilon_a^i) \frac{\gamma K}{4} \varepsilon_{ca} a c \bar{P}_i^a \bar{D}_j \left( \sqrt{q} J_i + \frac{(1 + \gamma^2 J_k e_a - \gamma \varepsilon^i_{jk} J^0 - \frac{1 + \gamma^2 - \gamma^3}{\gamma} K_a^2) \bar{G}_j, \right) \]

where \( \Gamma^c_{ab} \) is the torsion-free Christoffel connection which can be expressed in terms of triads and co-triads as in (A2) and we have used (20) and \( \gamma \bar{K}_a \), \( P^b \), as \( \bar{G}_a \) to arrive at the final expression. Again, the splitting makes it obvious that the diffeomorphism constraint is invariant under parity transformations. Notice the importance of \( \text{sgn det}(\varepsilon_a^i) \)-factors which we carried through the calculation — see also the Appendix for some formulas.

Finally, the Hamiltonian constraint turns out to be

\[ C = \frac{\gamma^2}{2 \sqrt{q}} P_j^a P_j^b \left( \varepsilon^i_{jk} \bar{F}_{ab} + 2 \partial_{[a} \bar{A}_{b]} + \varepsilon^i_{lm} \bar{A}_{a}^{lm} + \varepsilon^i_{lm} \bar{A}_a^{lm} + \varepsilon^l_{im} \bar{A}_a^{lm} - 2 (\gamma^2 + 1) K_a^i K_b^j \right) \]

\[ \gamma^2 P_j^a \left( \sqrt{q} J_i + \frac{(1 + \gamma^2 J_k e_a - \gamma \varepsilon^i_{jk} J^0 - \frac{1 + \gamma^2 - \gamma^3}{\gamma} K_a^2) \bar{G}_j, \right) \]

where \( \bar{R}_{ab}^i \) is the curvature of \( \bar{\Gamma}_a^i \). Also this expression is parity invariant for all \( \alpha \).
Independently of the parity behavior, the last expression allows us to provide a cross-check of our constraints by comparing with the effective action \( \mathcal{L} \). From the Hamiltonian constraint (25) we read off the interaction term
\[
H_{\text{int}} = \frac{3\kappa}{16} \frac{\gamma^2}{1 + \gamma^2} \left( \frac{1}{\alpha^2} - \frac{2}{\alpha\gamma} - 1 \right) \sqrt{q}(J_0)^2 - \frac{3\kappa}{16} \frac{\gamma^2}{1 + \gamma^2} \left( \frac{1}{\alpha^2} - \frac{2}{\alpha\gamma} - 1 \right) \sqrt{q} J^i J^i, \tag{26}
\]

**IV. QUANTIZATION**

In Sec. III we have summarized all the necessary generalizations in the canonical formulation which are induced by the coupling between gravity and fermions through the Ashtekar–Barbero connection, and explicitly verified parity invariance in Sec. III which is not manifest in canonical variables \((A^a_i, P^b_j)\). In addition to checking the parity behavior, splitting the constraints into torsion-free/torsion parts thus provides a non-trivial cross-check by comparing our constraints with the interaction Hamiltonian of the effective action [41].

Before starting the quantization, the first question concerns the choice of basic variables. We have two sets, given by the canonical variables \((A^a_i, P^b_j)\) in the presence of torsion as well as the torsion-free components \((\bar{A}^a_i, \bar{P}^b_j)\) with explicit expressions for torsion in terms of the fermion current in (20). However, as we have seen, equations of motion are required to find the torsion contribution to extrinsic curvature in explicit form. The use of classical equations of motion is not suitable for a quantization, and there is thus no choice but to use the canonical variables with implicit torsion terms.

**A. Half-densitized fermions**

In addition to torsion terms, there will be a further contribution to the connection once we formulate the fermions in terms of half-densities as required for consistency 12. For fermions, we have the canonical pair \((\psi, \pi)\) with \(\pi = -i\sqrt{q}\bar{\psi}^i\). These canonical variables cannot be promoted to operators on a Hilbert space with a suitable inner product in a way incorporating the reality condition \(\pi^\dagger = i\sqrt{q}\bar{\psi}\) by satisfying \(\hat{\pi}^\dagger = i\sqrt{q}\bar{\psi}\). First, if \(f(A)\) is a non-trivial real valued function of the connection \(A\), then the inconsistent relation
\[
0 = 0^\dagger = (\hat{\pi}, f(A))^\dagger = i\sqrt{q} f(A)\bar{\psi} \neq 0 \tag{27}
\]
ensues. Here the first commutator is expected to vanish since the corresponding classical Poisson bracket vanishes. On the contrary, the classical Poisson bracket corresponding to the second commutator is non-zero; hence the inconsistency arises. A second problem can be seen to arise from the symplectic structure obtained from the fermion Liouville form
\[
\Theta = -i \int_{\Sigma_t} d^3x \sqrt{\gamma} \left( \theta_L \bar{\psi}^i \hat{\psi} - \theta_R \bar{\psi}^i \psi \right) = \int_{\Sigma_t} d^3x \left( \pi \bar{\psi} - i \frac{\gamma}{2} \theta_R \bar{\psi}^i \psi e^i_\alpha \hat{P}^\alpha \right) - \int_{\Sigma_t} d^3x \sqrt{\gamma} \mathcal{L}_i(\pi \psi). \tag{28}
\]

Here, it follows from the second term of the first integral that the connection \(A^a_i\) acquires an imaginary correction term \(\frac{i}{2} \theta_R \bar{\psi}^i \psi e^i_\alpha \hat{P}^\alpha\), which endows the theory with a complex connection. This, in turn, would require the use of a complexification of SU(2) in holonomies, for which, due to the non-compactness, none of the loop quantization
techniques relying on the existence of a normalized Haar measure would be available (see e.g. [18]).

Both problems were solved by Thiemann who observed in [12] that, in order to obtain a well-defined canonical loop quantization with a real Ashtekar-Barbero connection also in the presence of fermions, one should cast fermion fields into Grassmann-valued half-densities. Thus $\xi := \sqrt{q} \psi$ instead of $\psi$ (and $\chi := \sqrt{q} \eta$ instead of $\eta$)

is considered to be the classical canonical variable, and $\pi_\xi = -i\xi^\dagger$ is the conjugate momentum for $\xi$. The inconsistencies in [27] are naturally removed as the new canonical variables imply the reality condition $\pi^\dagger_\xi = i\xi$ without any appearance of $\sqrt{q}$.

In half-densities, the symplectic structure becomes

$$
\Theta = -i \int_{\Sigma_i} d^3x \sqrt{q} \left( \theta_L (\psi^\dagger \psi - \dot{\eta}^\dagger \eta) - \theta_R (\dot{\psi}^\dagger \psi - \eta^\dagger \dot{\eta}) \right) = \int_{\Sigma_i} d^3x \left( \pi_\xi \dot{\xi} + \pi_\chi \dot{\chi} \right) + \int_{\Sigma_i} d^3x \frac{\gamma K}{4\alpha} L(e_i, j^0),
$$

(29)

where we have ignored total time derivatives which would drop out of the action for appropriate boundary conditions. The classical anti-Poisson brackets for Grassmann-valued fields are $\{\xi_A(x), \pi_{\xi B}(y)\} = \delta_{AB}\delta(x,y)$. Moreover, as the extra term shows, $\sqrt{q}$ can be absorbed in spinors without changing the symplectic structure of the gravitational variables only when $\alpha \rightarrow \infty$, i.e. for minimal coupling. Combining the last term in (29) with the gravitational Liouville term $\int d^3x P^i_\xi L_i A^j_a$ a real-valued correction term $\frac{\gamma K}{4\alpha} e^i_a j^0$ must be added to the Ashtekar-Barbero connection $A^i_a$. This is a new feature that is present in the non-minimally coupled theory if the fermion fields are expressed in terms of half-densities. Therefore, the new canonical connection can be written as

$$
A^i_a := A^i_a + \frac{\gamma K}{4\alpha} e^i_a j^0 = \bar{A}^i_a + C^i_a + \gamma K^i_a,
$$

(30)

where

$$
C^i_a := \frac{\theta \gamma^2 \kappa}{4(1 + \gamma^2)} \left( \frac{1}{\gamma} e^i_k e^k_a J^0 - e^i_a J^0 \right).
$$

(31)

Absorbing the correction term into the torsion contribution to the spatial spin connection allows one to keep $K^i_a$ unchanged in the course of expressing all the constraints in terms of the corrected connection. Note that the corrected torsion contribution, $C^i_a$, to the spin connection vanishes for $\alpha = \gamma$. (If one would use the fully split connection [20] based on partial solutions of the equations of motion, the new contribution in the presence of half-densities would cancel the $J^0$-dependence of $A^i_a$ completely.)

In terms of the corrected connection and half-densities, the total Dirac Hamiltonian constraint (modulo the Gauss constraint) in (15) takes the smeared form

$$
H_{\text{total}} = \int_{\Sigma_i} d^3 x \left( \frac{\gamma^2 \kappa}{2\sqrt{q}} P^a_i \Pi^b_j \left( e^i_{jk} \mathcal{F}^j_a - 2(\gamma^2 + 1) K^i_{[a} K^j_{b]} \right) - \frac{\gamma K \beta P^a_i}{\sqrt{q}} D_a \left( \pi_\xi \tau^i \chi + \pi_\chi \tau^i \xi \right) - \frac{2\gamma \kappa P^a_i}{\sqrt{q}} \left( \theta_L \pi_\xi \tau^i D_a \xi - \theta_R \pi_\chi \tau^i D_a \chi - c.c. \right) \right) + \frac{\gamma K \beta}{2\sqrt{q}(1 + \gamma^2)} \left( 3 - \frac{\gamma}{\alpha} + 2\gamma^2 \right) \left( \pi_\xi \tau_\eta \xi + \pi_\chi \tau_\eta \chi \right) \left( \pi_\xi \tau^i \xi + \pi_\chi \tau^i \chi \right)
$$

+ \frac{i \gamma^2 \kappa}{4\alpha q} e^i_j P^a_i \left( \pi_\xi - \pi_\chi \chi \right) D_a P^b_j + \frac{3\gamma K \theta}{8\alpha \sqrt{q}} \left( \pi_\xi - \pi_\chi \chi \right) \left( \pi_\xi - \pi_\chi \chi \right) \right)
$$

(32)

where $\mathcal{F}^j_{ab}$ is the curvature and $D_i$ now and in the rest of the paper, is the covariant derivative related to the corrected connection $A$.

**B. Quantum representation**

The ordinary kinematical constructions of loop quantum gravity do not refer to torsion or torsion-freedom and thus go through unchanged. We thus present only the bare concepts relevant for the construction of con-
1. Fermion fields

The space of all Grassmann-valued half-densitized 2-component spinors $\xi(x)$ and $\chi(x)$ constitutes the classical configuration space $\mathcal{F}$ for fermion fields. The loop quantization presents smeared objects

$$\Xi_A(x) := \int_{\Sigma_t} \frac{d^3 y \delta(x,y) \xi_A}{\epsilon^3} \quad \chi_A(x,y) := \lim_{\epsilon \to 0} \int_{\Sigma_t} \frac{d^3 y \chi_A(x,y)}{\epsilon^3}$$

to operators, where $\chi_A(x,y)$ is the characteristic function of a box of Lebesgue measure $\epsilon^3$ centered at $x$. Note that $\Xi_A$ are scalar Grassmann valued functions since the $\delta$ distribution is a density of weight one. It is also easy to see that $\Xi$ and their adjoint satisfy anti-Poisson brackets similar to those presented above for $\xi$. Upon quantization, the anti-Poisson bracket is replaced by the anti-commutator $[\Xi_A(x), \bar{\pi}_B(y)]_+ = i\hbar \delta_{AB} \delta_x y$ with $\delta_{x,y}$ being the Kronecker symbol (rather than a $\delta$-distribution thanks to the smearing involved in $\Xi_A$).

This algebra can be represented on a non-separable Hilbert space $\mathcal{H}_F = L^2(\mathcal{S}_t, d\mu_F)$ where each copy $\mathcal{H}_\alpha$ for any point $v$ in space is an ordinary Grassmann-valued Hilbert space of multi-linear functions of $\Xi_A(v)$ and $\Xi_A(v)$ of two-component spinors in their Grassmann space $S_v$, with integration measure $d\mu_v = d\mathcal{S}_v d\Xi_v d\Xi^\dagger_v$. The full space of the fields can then be written as $\mathcal{S} := \bigotimes_v S_v$ with measure $d\mu_F(\Xi, \Xi) = \prod_v d\mu_v$. On this space, $\Xi_A$ acts as a multiplication operator, and its momentum $\bar{\pi}_B = -i\hbar \partial / \partial \Xi_B$ by a derivative. In addition, we have a second copy of these point-wise Hilbert spaces for $\chi$ smeared to $X$.

A dense subset of functions in this Hilbert space is formed by cylindrical functions which are superpositions only of products of finitely many vertex-wise Grassmann-factors. These functions can be seen to arise if one starts with a cyclic state independent of $\Xi$ and $X$ and uses the $\Xi_v$ and $X_v$ as “creation” operators. Since all the constraints depend on the fermion only via currents, which are polynomials in $\Xi_A$ and $X_A$, they can easily be represented on this subspace of cylindrical functions.

2. Gravitational variables

Classical configuration variables for gravity are SU(2)-connections on a principal fiber bundle over the spatial manifold $\Sigma$, represented by smooth su(2)-valued local 1-forms $A^a_i$ from (10); the space $\mathcal{A}$ of all such 1-forms is the classical configuration space. The phase space is coordinatized by the pair $(A^a_i, P^b_i)$, where $P^b_i$ is the conjugate momentum, an su(2)-valued vector density on $\Sigma$ proportional to the densitized triad. Then the only non-vanishing Poisson bracket is

$$\{A^a_i(x), P^b_j(y)\} = \delta^a_j \delta^b_0 \delta(x,y).$$

No well-defined quantum analogs for these canonical variables exist in a direct form without smearing. The elementary classical variables that have well-defined quantum analogs are rather given by (complex valued) matrix elements of holonomies $h_i(\mathcal{A}) = P \exp(\int_a A^a_i r_i^a d\mathcal{S}) \in SU(2)$ along paths $e$ in $\Sigma$ and fluxes $F_S^f(\mathcal{A}) := \int_S f_i n_a P^a_i dS^2 y$, where $f$ are su(2)-valued functions across 2-surfaces $S$ in $\Sigma$ and $n_a$ is the (metric-independent) non-normal to the surface.

This provides the appropriate smearing for gravitational variables. The resulting holonomy-flux algebra is represented on a Hilbert space $\mathcal{H} = L^2(\mathcal{A}, d\mu_{\mathcal{A}})$ constructed as follows (18): We first introduce cylindrical functions whose space will eventually be completed to a Hilbert space. Cylindrical functions are functions on $\mathcal{A}$ which depend on $A^a_i$ only through holonomies $h_i(\mathcal{A})$ along edges $e$ of a graph $\alpha$ (a finite set of edges) in $\Sigma$. If a graph $\alpha$ has $n$ edges, then, given a $C^\infty$ complex-valued function $\psi$ on $SU(2)^n$, a cylindrical function $\Psi_{\alpha}$ on $\mathcal{A}$ with respect to the graph $\alpha$ can be written as

$$\Psi_{\alpha}(\mathcal{A}) := \psi(h_{e_1}(\mathcal{A}), \ldots, h_{e_n}(\mathcal{A})).$$

Let $\text{Cyl}_\alpha$ denote the space of such functions with respect to the graph $\alpha$, and $\text{Cyl} = \cup_{\alpha} \text{Cyl}_\alpha$ the space of all cylindrical functions. A natural inner product on $\text{Cyl}_\alpha$ can be introduced by defining the measure $d\mu_{\alpha}$ by

$$\langle \Psi_{\alpha}, \Phi_{\alpha} \rangle = \int d\mu_{\alpha} \overline{\Psi}_{\alpha} \Phi_{\alpha} := \int_{SU(2)^n} d\mu_{\alpha}^n \overline{\psi}(h_{e_1}, \ldots, h_{e_n}) \phi(h_{e_1}, \ldots, h_{e_n})$$

with the Haar measure $d\mu_{SU(2)}$ on SU(2). The Cauchy completion of $\text{Cyl}_\alpha$ with respect to this inner product gives rise to a Hilbert space $\mathcal{H}_\alpha := L^2(\mathcal{A}_\alpha, d\mu_{\alpha})$, where $\mathcal{A}_\alpha := \mathcal{A}_\alpha / G^0_{\alpha}$ is the space of smooth connections restricted to the graph $\alpha$ modulo all local gauge transformations $g_\alpha \in G^0_{\alpha}$ which are the identity on the vertices.
full Hilbert space $\mathcal{H} := L^2(\mathcal{A}, d\mu_{\text{AL}})$ where $d\mu_{\text{AL}}$ is the Ashtekar–Lewandowski measure constructed in this way and $\mathcal{A}$ the space of generalized connections. The latter space represents the quantum configuration space as an enlargement from the classical configuration space $\mathcal{A}$ of connections by distributions. On $\mathcal{H}$, holonomies act as multiplication operators which change the graph when acting on a cylindrical state whose graph does not contain the edge used in the holonomy. Flux operators are represented by invariant vector field operators on SU(2)-copies corresponding to the edges intersected by the surface of the flux. As operators on function spaces over SU(2), invariant vector fields have discrete spectra, and so do flux operators. From fluxes, one can construct further operators of spatial geometry such as area and volume [19, 20, 21] which also have discrete spectra.

All this remains unchanged in the presence of torsion. By construction, the Ashtekar–Barbero connection inherits the total torsion contribution and thus the effect of torsion on the system is concealed in holonomies which acquire the tensor product of the basic representation $\rho$ of their edges by irreducible SU(2)-representations where for all vertex labels $v$ states take the form $n$ networks [22, 23]: graphs together with a labeling it is useful to use special cylindrical states based on spin enlargement from the classical configuration space $H \otimes H$ torsion connection. At each level, which thus has no choice but to refer to the unsplit part of $\mathcal{G}$, this is not available at the kinematical level, with all contributions from torsion in quantum kinematics. A complete split of torsion-free and torsion components is possible only once equations of motion are partially used. This is not available at the kinematical level, which thus has no choice but to refer to the unsplit torsion connection.

3. Combined Hilbert space of gravity and fermions

For the combined system, we simply take the tensor product $\mathcal{H} \otimes \mathcal{H}_{\text{F}}$ as the Hilbert space, which acquires the tensor product of the basic representations. All cylindrical states can be written in the form $\psi(h_{c_1}, \ldots, h_{c_m}, \Xi_{e_1}, \ldots, \Xi_{e_l}, X_{e_1}, \ldots, X_{e_l})$ with integer $n$, $m$, and $l$. Especially for the gravitational dependence it is useful to use special cylindrical states based on spin networks [22, 23]: graphs together with a labeling $j_e$ of their edges by irreducible SU(2)-representations $\rho^{(j_e)}$, and of vertices with spinor representations $\sigma_v$ of SU(2) (obtained from tensor products of the fundamental representation given by the basic 2-spinors) as well as contractions $C_v$ in vertices to contract the matrix-represented holonomies of edges incoming and outgoing at $v$. Such states take the form

$$\prod_{v,e} \Gamma_{v,e}^{(i_1)} \ldots \Gamma_{v,e}^{(i_k)} \mu_1 \ldots \mu_m \nu_v^{(k)} (h_v(A))^{\mu_1} \ldots^{\mu_m} \sigma_v(\Xi_v, X_v) \rho^{(j_e)}$$

(36)

where for all vertex labels $v_e$ are to be contracted with indices $j_e$ on represented matrices $\rho^{(j_e)}(h_v(A))^{\mu_1} \ldots^{\mu_m}$ of all $n_v$ outgoing edges as well as the spinor index $\nu_v^{(k)}$, and $\mu_e^k$ with indices $\mu_e$ of all $m_v$ incoming edges as well as the spinor index $\nu_v$. C. Constraints

General relativity is a background independent theory and is fully constrained in the canonical formulation. Thus the quantization of the constraints is necessary to obtain physical states. Having identified elementary operators and their quantum representation, this kinematical structure is now used to construct a set of quantum operators corresponding to constraints relevant for the system. Subsequently, these quantum constraints have to be solved to obtain physical states. The existence of torsion may change the form of each of the quantum constraint operators and consequently influence their solutions. Here, we will show that extra terms can be quantized consistently.

1. Kinematical constraints

We first express the Gauss constraint in terms of half-densities and the new canonical connection $A_i^\ell$

$$G_i := \partial_\mu P^\mu_i - \frac{1}{2} \sqrt{\gamma} J_i = \partial_\mu P^\mu_i + \pi_\ell \tau_\ell \xi + \pi_\chi \tau_\chi \lambda \cdot \xi$$

(37)

Upon smearing the constraint with an su(2)-valued function $A^\ell$ on $\Sigma$, it is easy to see that $G[A^\ell] = \int_\Sigma d^3x A^\ell G_i$ generates internal SU(2) rotations on the phase space of general relativity:

$$\{ A^\ell_i, G[A] \} = -\partial_\ell A^i$$

and $$\{ P^a_i, G[A] \} = \epsilon_{ij}^k \lambda^j P^a_k$$

together with a spinor transformation in the fundamental representation of SU(2). Thus, the quantization of the Gauss constraint is carried out in a similar fashion as it is done in the torsion-free case, restricting gauge invariant states to be supported on $A/\mathcal{G}$. For our configuration variables, we have the transformations $h_v \mapsto g_v(h_v) h_v^{-1}$, $\Xi_v \mapsto g_v \Xi_v$ and $X_v \mapsto g_v X_v$ under a gauge transformation $g_v \in SU(2)$. A spin network state, when gauge transformed, acquires at each vertex $v$ factors of $\rho^{(j_e)}(g_v^{-1})$ from all incoming edges, $\rho^{(j_e)}(g_v)$ from outgoing edges and $f_e(g_v)$ from spinor factors in the state. For a gauge invariant state, these factors must cancel each other when contracted with the $C_v$ in (36), which implies that representation matrices (including the spinor) must be multiplied by contraction with an intertwiner of all relevant representations to the trivial one. The resulting gauge invariant states satisfy the quantum constraint equation $G[A^\ell] \Psi_\alpha = 0$ for all $A^\ell$. Similarly, one can use the action of the spatial diffeomorphism group on the phase space by computing infinitesimal canonical transformations generated by $D[N^\alpha] = \int_\Sigma d^3x N^\alpha C_\alpha$. In terms of half-densities and the corrected connection, the constraint turns out to be
up to contributions from the Gauss constraint. This constraint generates transformations

$$\{A^a_i, D[N^a]\} = N^b D_{bc} + D_a (N^c A^a_c) = \mathcal{L}_N A^a_i$$

and

$$\{P^a_i, D[N^a]\} = N^b \partial_b P^a_i - P^b_i \partial_b N^a + P^a_i \partial_b N^b = \mathcal{L}_N P^a_i$$

simply by moving the graph (which presents a unitary transformation with respect to the Ashtekar–Lewandowski measure). Thus, invariant states can be determined by constructing a new, diffeomorphism invariant Hilbert space via group averaging.

2. **Hamiltonian constraint**

While the Gauss and diffeomorphism constraints generate the canonical transformations that represent the well-known kinematical gauge symmetries in the classical phase space independently of torsion, the scalar constraint entails the essence of dynamics of the theory. Hence the scalar quantum operator describes quantum dynamics of the physical states which must be in accordance with the presence of torsion. Unfortunately, a complete quantization of this scalar constraint is yet to be satisfactorily realized. Therefore, we present only the necessary adaptations to the existing quantization attempts. In this approach, it is essential to re-express the classical expression of the scalar constraint in terms of those phase space functions which can be promoted to well-defined operators.

Our starting point is expression (32) of the Hamiltonian constraint in half-densitized fermions. The fermion terms in the Dirac Hamiltonian coupled with gravity, can be quantized using the strategy developed by Thiemann in [13]. Note that this Dirac Hamiltonian is different from the one presented in [13] (which took a second order viewpoint) in two aspects: the covariant derivative \(D\) now contains the Ashtekar–Barbero connection with torsion and the interaction term is new. Also the gravitational term has torsion contributions which have to be taken into account when applying the standard quantization strategy of [6].

\[
D[N^a] = \int_\Sigma d^3x \ N^a \left( 2P_b^b \partial_{[a} A^a_{b]} - A^a_i \partial_b P^b_i + \frac{1}{2} \left( \pi_\xi \partial_a \xi - (\partial_a \pi_\xi) \xi + \pi_\chi \partial_a \chi - (\partial_a \pi_\chi) \chi \right) \right) \tag{38}
\]

as well as the correct Lie derivative \(\delta \xi = N^a \partial_a \xi + \frac{1}{2} \xi \partial_a N^a\) of half-densitized fermions. Hence, this constraint can be quantized as in the torsion-free case via the finite action of the diffeomorphism group. A finite diffeomorphism \(\varphi\) is represented on cylindrical states by

\[
\bar{D}_\varphi \psi(h_{e_1}, \ldots, h_{e_m}, \Xi_{v_1}, \ldots, \Xi_{v_n}, X_{w_1}, \ldots, X_{w_l}) = \psi(h_{\varphi(e_1)}, \ldots, h_{\varphi(e_m)}, \Xi_{\varphi(v_1)}, \ldots, \Xi_{\varphi(v_n)}, X_{\varphi(w_1)}, \ldots, X_{\varphi(w_l)}) \tag{39}
\]

As usually, the expression involving extrinsic curvature \(K^a_i\) would vanish for \(\gamma = 1\) in Euclidean signature which in turn implies that the first term in the gravitational constraint reduces to the scalar constraint \(H^E[N]\) of Euclidean general relativity. Then let us write the scalar constraint for gravity alone as

\[
H[N] = \sqrt{\gamma} H^E[N] - 2(1 + \gamma^2) T[N], \tag{40}
\]

where

\[
T[N] := \sqrt{\frac{\gamma}{4\kappa}} \int_\Sigma d^3x N \frac{P^a_i P^b_j}{\sqrt{\det P}} K^a_i K^b_j. \tag{41}
\]

In order to quantize the scalar constraint for gravity, it is first necessary to express it in terms of classical phase space functions which have well-defined quantum analogs. In this regard, the following classical objects and relationships are crucial as building blocks: The total volume \(V = (\gamma \kappa)^{3/2} \int_\Sigma d^3x \sqrt{\det P} \) of \(\Sigma\), the co-triad

\[
e^i_a(x) := \frac{\sqrt{\kappa}}{2} \text{sgn det}(e^i_a) \epsilon^{ijk} P^b_k P^c_b \sqrt{\det P} = \frac{2}{\gamma \kappa} \{A^i_a(x), V\}, \tag{42}
\]

the integrated trace of extrinsic curvature

\[
K := \gamma \kappa \int_\Sigma d^3x K^i_a P^a_i. \tag{43}
\]

as well as expansions

\[
h_{e, A}(A) = 1 + \delta s^a \tau_a A^i_a + O(\delta^2) \tag{44}
\]

\[
h_{\alpha, j}^a(A) = 1 + \delta s^a \sqrt{\gamma} \gamma^i_j \partial_a \gamma^i_k + O(\delta^3) \tag{45}
\]

of holonomies along small open edges \(e\) in direction \(s^a\) of coordinate length \(\delta\) or small square loops \(\delta_{ij}^a\) of coordinate area \(\delta^2\) with sides in the directions \(s^a_i\).
The first step in a regularization of a spatial integral is to introduce a triangulation of $\Sigma$ as the union of tetrahedra with edges of coordinate length $\delta$ and $\epsilon$ at a given vertex pointing in directions $s_i^d$, $i = 1, 2, 3$. To use this for a construction of operators, the positions and directions of tetrahedra are usually adapted to vertices and edges of the graph underlying a state to be acted on. The coordinate volumes of tetrahedra then replace the integration measure: $e^{abc}d^4x \rightarrow \delta^4e^{ijk}s_i^d s_j^e s_k^f$. Moreover, internal tensors can be written in terms of Pauli matrices, such as $\epsilon^{ijk}s_i^d s_j^e s_k^f$. The tangents $s_i^d$, factors of $\delta$ and Pauli matrices can then be combined with Poisson brackets to obtain

$$\tau_h \delta s_i^d \{A^k_a, O\} \rightarrow -\frac{1}{i\hbar}h_{st}[h_{si}^{-1}, \hat{O}] \quad (46)$$

in terms of holonomies with their well-defined quantization, where $O$ could be the volume if (42) is used, or the integrated trace of extrinsic curvature $K$. For fine triangulations, $\delta \ll 1$, the error in replacing connection components by holonomies is small, and it goes to zero in the limit where all edge lengths of tetrahedra vanish. Similarly, covariant derivatives can be combined to $\delta s_i^d D^a$ and then regularized to a difference of values at the endpoints of a small edge in direction $s_i^d$. If there are always three factors where $\delta$ can be absorbed and the quantized contributions vanish only when acting on vertices of a graph, a well-defined operator results even in the limit when the regulator is removed because for finite graphs finitely many terms remain in the triangulation sum.

We first turn to the matter terms which arise in (32). Some of them agree with the Dirac Hamiltonian used in (13), and can thus be quantized along the same lines. However, our analysis has provided extra terms which must be ensured to have well-defined quantum expressions, too. The current interaction terms can directly be quantized with fermion operators and using

$$\frac{\text{sgn} \det(e^d)}{\sqrt{q}} = \frac{1}{6\delta} e^{abc} \epsilon_{ijk} e_i^a e_j^b e_k^c$$

$$= \frac{36e^{abc} \delta_{ijk}}{\gamma^3_{\kappa} \kappa^3} \{A^k_a, V^{1/3}\} \{A^0_a, V^{1/3}\} \{A^k_a, V^{1/3}\}$$

for a quantizable expression in terms of commutators of holonomies and the volume operator. Edge tangents of the holonomies for the three Poisson brackets provide the elementary coordinate volumes of the triangulation, while half-densitized fermions in the current products will simply be vertex-wise operators.

Terms of the form $q^{-1/2}P_a^k D_a^b O$ where $O$ is an expression of fermions can be reformulated using $\gamma \kappa P_a^k = \frac{\text{sgn} \det(e^a)}{\sqrt{q}} \epsilon^{abc} \epsilon_{ijk} e_i^a e_j^b e_k^c$ in which we can again absorb the inverse $\sqrt{q}$ after expressing the co-triads as Poisson brackets. Here, we will have two holonomies requiring an edge tangent vector as well as the covariant derivative which will become a directional derivative once the triangulation volumes are expressed via edge vectors: we use the expansion $h e(\delta)O(e(\delta)) - O(e(0)) \approx \delta e^a D_a^b O$ where $h e(\delta)$ is a holonomy along an edge $e$ of coordinate length $\delta$. Also these terms can thus be quantized by standard techniques, which involves a discretization of the derivative.

Finally, we have to turn $q^{-1}\epsilon^{ijk} P^a_b P^b_a D_a^b P^b_a$ into an expression which can be quantized. We first rewrite this as

$$\frac{\gamma \kappa^2}{q} \epsilon^{ijk} P^a_b \epsilon^c_de_i^b \partial_a^b P^b_j = -\frac{\gamma \kappa^2}{q} e^{ijk} P^a_b P^b_j \partial_a^c e^b_k$$

$$= -\frac{\text{sgn} \det(e^d)}{\sqrt{q}} \epsilon^{abc} e^k_b \partial_a^c e^b_k = -\text{sgn} \det(e^d) \epsilon^{abc} e^b_k \partial_a^c e^b_k$$

which provides two factors of co-triads and one partial derivative. Each of them will be combined with a tangent vector to provide either holonomies or a discretized derivative. The inverse powers of $q^{1/4}$ can be absorbed by choosing appropriate positive powers of volume in Poisson brackets expressing the co-triads. (Note that this is the reason why we had to move one $q^{-1/4}$ past the partial derivative, because absorbing a single $q^{-1/2}$ would require the ill-defined logarithm of volume.)

For the gravitational part of the constraint, the curvature components $F^a_{ab}$ appear in a term which can be expressed as $\int d^3x e^{abc} F^k_{ab} \epsilon^{ijk} \epsilon^{ijkl} P^d \epsilon^{ijkl} / \sqrt{\text{det} P}$. After triangulation, this takes the form $e^{ijk} \epsilon^a_b \epsilon^b_c \epsilon^c_k \epsilon^k_j \epsilon^j_i \text{tr}(e_{ab} e_{cd} \epsilon^{ijk} S^k_{sk_j} \gamma_i \{A^k_a, V\})$ which can be written in terms of holonomies via $e^{ijk} \epsilon^j_i \gamma_{s_k} \gamma^k \{A^k_a, V\}$.

It remains to quantize the extrinsic curvature terms, where our goal is to express $K^i_a$ in terms of Poisson brackets such as $\{A^0_a, K\}$ and $\{A^i_a, V\}$ which can be promoted to commutators of well-defined operators. In the torsion-free case the integrated extrinsic curvature is used in the expression $K^i_a = \frac{1}{\kappa \gamma} \{A^i_a, K\}$ for extrinsic curvature components. This relation, proven e.g. in (24), turns out to be one of the main places where torsion changes the quantization procedure of the Hamiltonian constraint. Viewing (13) as a functional of the canonical pair $(A^i_a, P^d_j)$, i.e. expressing $K^i_a$ in terms of $A^i_a$ and $P^d_j$, yields

$$\{A^i_a(y), K\} = \kappa(A^i_a(y) - \Gamma^i_a(y)) - \kappa \int_{\Sigma} d^3x P^d_j(x) \frac{\delta \Gamma^i_a(x)}{\delta P^d_j(y)} = \kappa \gamma K^i_a(y) + \kappa \frac{\gamma^2 q^2 \theta}{4(1 + \gamma^2)} \left( \frac{1}{\gamma} \epsilon^i_{kl} e^k_b(y) J^l(y) + \frac{1}{2} \epsilon^i_a(y) J^0(y) \right).$$

Here, we have used $\Gamma^i_a = \bar{\Gamma}^i_a + C^i_a$ (which only re-
second step together with (11) and the fact that 
\[ \kappa \int_{\Sigma} d^3x \ F^a_i(x) \delta F^i_1(x) = 0, \]
which can be proven by a direct calculation or using the fact that \( \tilde{F} := \kappa \gamma \int_{\Sigma} d^3x \ P^a_i(x) \tilde{\Gamma}^a_i(x) \) is the generating functional of \( \tilde{\Gamma} \).
(Due to the presence of torsion, unless \( \theta = 0 \) the functional \( F := \kappa \gamma \int_{\Sigma} d^3x \ P^a_i(x) \Gamma^a_i(x) \) no longer generates
a canonical transformation to \( (K^i_a, P^b_i) \) since \( \{ A^a_i, F \} \neq \Gamma^a_i \). Many of the differences between torsion and torsion-
free canonical gravity are reflected in this property of the canonical structure.)

Together with (12) it is then straightforward to show that

\[
K^i_a = \frac{1}{\gamma \kappa} \left\{ A^i_a, K \right\} - \frac{\theta}{2(1 + \gamma^2)^2} \epsilon^i_{kl} \left\{ A^k_b, V \right\} \sqrt{\mathcal{F}^l} - \frac{\theta}{4(1 + \gamma^2)^2} \left\{ A^i_a, V \right\} \sqrt{\mathcal{F}^0}.
\]

With these classical identities, the contributions \( H^E[N] \) and \( T[N] \) to the Hamiltonian constraint become

\[
H^E[N] = \frac{1}{\kappa^3 \gamma^2} \int_{\Sigma} d^3x \ N(x) e^{abc} \mathcal{F}^{k}_{ab}(x) \left\{ A^k_b(x), V \right\} \text{sgn} \text{det}(e^l_d),
\]

and

\[
T[N] = \frac{1}{2\kappa^2 \gamma^2} \int_{\Sigma} d^3x \ N(x) e^{abc} \epsilon_{kmn} \left\{ A^k_b(x), V \right\} K^m_c K^n_c \text{sgn} \text{det}(e^l_d)
\]

\[
- \frac{2 \theta}{\gamma^3 \kappa^2 (1 + \gamma^2)} \int_{\Sigma} d^3x \ N(x) e^{abc} \epsilon_{kmn} e^{ij} \left\{ A^k_b(x), V \right\} \left\{ A^m_c(x), V \right\} \sqrt{q} J^i \text{sgn} \text{det}(e^l_d)
\]

\[
- \frac{\theta}{\gamma^3 \kappa^2 (1 + \gamma^2)} \int_{\Sigma} d^3x \ N(x) e^{abc} \epsilon_{kmn} e^{ij} \left\{ A^k_b(x), V \right\} \left\{ A^m_c(x), V \right\} \left\{ A^n_l(x), V \right\} \sqrt{q} J^i J^j \text{sgn} \text{det}(e^l_d)
\]

\[
+ \frac{2 \theta^2}{8 \gamma^3 \kappa^2 (1 + \gamma^2)} \int_{\Sigma} d^3x \ N(x) e^{abc} \epsilon_{kmn} e^{ij} \left\{ A^k_b(x), V \right\} \left\{ A^m_c(x), V \right\} \left\{ A^n_l(x), V \right\} \sqrt{q} J^i J^j \text{sgn} \text{det}(e^l_d)
\]

\[
+ \frac{2 \theta^2}{8 \gamma^3 \kappa^2 (1 + \gamma^2)} \int_{\Sigma} d^3x \ N(x) e^{abc} \epsilon_{kmn} \left\{ A^k_b(x), V \right\} \left\{ A^m_c(x), V \right\} \left\{ A^n_l(x), V \right\} \sqrt{q} J^i J^j \text{sgn} \text{det}(e^l_d).
\]

Here, we have already absorbed inverse powers of \( \sqrt{\mathcal{F}} \) in
the Poisson brackets, while keeping one factor of \( \sqrt{\mathcal{F}} \) with
each current component to make the product quadratic
in half-densities of fermions without other metric
components.

It is thus clear that the presence of torsion introduces
non-trivial additional terms in the gravitational Hamiltonian
constraint when it is written in a form suitable for
quantization.

While no changes to the torsion-free construction of
the Hamiltonian constraint are required for expressing \( \mathcal{F}^{ab} \) and \( A^a_i \) in terms of holonomies, there is a further
difference to the treatment of \( K \) in [6]. This quantity
is not directly related to a basic variable, but can be
obtained from a Poisson bracket \( \{ H^E[1], V \} \) where
both ingredients are already written as quantizable functions
of basic quantities. With \( \Gamma^a_i \) having contributions from
torsion, we obtain, using \( \{ A^a_i \} \) and the trace of (11),

\[
\{ H^E[1], V \} = \sqrt{\gamma \kappa} \frac{1}{2} \int_{\Sigma} d^3x \ \left( \epsilon^{ij} P^a_i P^b_j + 2 P^b_n (\Gamma^a_n + \gamma K^a_n) \right)
\]

\[
= \gamma^{3/2} \kappa \int_{\Sigma} d^3x \ \left( P^b_n \epsilon^{mn} + \gamma P^b_n \Gamma^a_n \right) = \gamma^{3/2} K - \frac{3 \theta}{4} \frac{\gamma^2}{1 + \gamma^2} \int_{\Sigma} d^3x \sqrt{q} J^0,
\]

which implies

\[
K = \gamma^{3/2} \{ H^E[1], V \} - i \frac{6 \alpha^2 \gamma \kappa \theta}{(1 + \gamma^2)(1 + \alpha^2)} \int_{\Sigma} d^3x \ (\theta_R \pi_\xi \xi - \theta_L \pi_\chi \chi).
\]
Again, the presence of torsion implies that $K$ can no longer be expressed just as the Poisson bracket of $H^E[1]$ and $V$; the extra term involving the fermion charge density in (31) is necessary if the torsion is included in the connection. This result is consistent since splitting the torsion contribution from $K^i_a$ and taking the trace of (19) reduces $K$ to the Poisson bracket $\gamma^{-2} \{ H^E[1], V \}$ without any extra terms. The additional term in (31), however, does not have much effect since it only depends on the canonical fermion half-densities, and thus drops out of the Poisson bracket with $A^i_j$ in (19) which is the only form in which $K$ appears.

It is interesting to note that, for $\alpha = \gamma$, the equations (17), (19), (20), and (21) take the standard forms of the torsion-free case (without any extra terms) since $\theta$ vanishes. This results since the torsion contribution to the spatial spin connection, $C^i_a$, vanishes for $\alpha = \gamma$ when the fermion fields are expressed in half-densities as shown in (11). Therefore, except for the extra terms in (22), the strategy for a loop quantization of the gravitational sector of gravity non-minimally coupled to fermions is exactly the same as that in vacuum for $\alpha = \gamma$. Although this is the case which was also addressed in [5], we emphasize that the complete canonical derivation for real variables has to be done to recognize the roles of all possible contributions to the variables and constraints. In particular, there are extra terms in (22) whose correct form must be used to quantize the Hamiltonian constraint.

For $\alpha \neq \gamma$, the quantization of the scalar constraint of gravity with fermions demands the quantization of the non-trivial extra terms in (19) in addition to the terms appearing in (22). This can be carried out using the standard strategy: All extra terms have the structure $\int d^3x N e^{abc} e_{kmn} \{ A^i_a, O_i \} \{ A^m_a, O_2 \} \{ A^c_a, O_3 \} O^p$ where $O_1$, $O_2$, and $O_3$ are either powers of $V$ or $K$, and $O^p$ is $e^i_n \sqrt{q} J^i$, $\delta^i_n \sqrt{q} J^0$, $e^c_n q J^0 J^i$, $\delta^c_n q(J^0)^2$ and $q J^0 J_i$, respectively, in all the required terms. The operators $\tilde{O}_i$ are obtained either as the volume operator or its commutator with the Euclidean part of the Hamiltonian constraint. The current terms also provide vertex operators directly in terms of the smeared fermion operators $\dot{\Xi}_a$ and $\dot{X}_a$. For $J_0$, this can directly be multiplied with the commutators, while $J^c$ can be inserted into the trace through $\tau_3 J^c$. We do not list the long expressions for complete operators here, but it is clear now that well-defined quantizations exist for all the extra terms. This provides quantizations of all terms in (19), completing the quantization of the gravitational constraint in the presence of torsion.

### D. Parity

In loop quantum gravity, the parity behavior is not manifest because the Ashtekar connection transforms as $\Gamma^i_a + \gamma K^i_a \rightarrow \Gamma^i_a - \gamma K^i_a$ under parity, which does not result in a straightforward transformation of its holonomies. For states in the connection representation, there is thus no simple parity transformation on the Hilbert space for which one could check invariance of the theory. Sometimes the relation between $K^i_a$ and extrinsic curvature is changed in the definition of basic variables, making use of $\text{sgn} \det(e^a_i) K^i_a e^b_i$ with a sign factor which would make the redefined $K^i_a$ and thus the whole Ashtekar connection invariant under a reversal of the triad orientation. However, the symplectic structure would be invariant under this transformation only if a corresponding sign factor is included in the momentum, now being $\text{det}(e^a_i) P^a_i$ instead of $P^a_i$. This momentum would also be invariant under orientation reversal. With all the basic gravitational variables being invariant under orientation reversal, one would simply lose any possibility to implement non-trivial parity transformations at all. Thus, the only possibility is to work with a theory whose parity behavior is rather concealed.

While this may appear only as a technical problem in vacuum or with non-fermionic matter, it becomes acute in the presence of fermions and torsion. (Note that a second order formalism, where fermions would not imply torsion contributions to the connection and thus allow a parity behavior as in the vacuum theory, is unnatural for the connection variables of the Ashtekar formulation as it underlies the loop quantization.) As our classical discussion in Sec. III showed, the precise behavior of the variables and constraints under parity transformations is no longer obvious in the presence of torsion. Even classically, the behavior is fully determined only on-shell, making use not only of the constraints but also of some equations of motion. While the classical solution space turns out to be parity invariant for any $\alpha$, specific torsion contributions to $\Gamma^i_a$ and $K^i_a$ acquire a behavior different from the torsion-free parity behavior unless $\alpha = \gamma$. This observation, consistent with [2], indicates that the situation of parity after quantization, where information about solutions of equations of motion cannot be used, may be much more involved.

In fact, now the non-trivial parity behavior is hidden in holonomies used as basic operators. At the kinematical level, there is no way of knowing what unitary transformation could possibly represent a change in parity, given that even classically one would have to make use of constraints and equations of motion to determine that. In the classical case, the behavior of the theory under parity became obvious only after explicitly splitting off the torsion contributions from the basic variables—a procedure which we are denied in the quantum theory. Triads have a much simpler (and obvious) behavior under parity, but this, too, is difficult to implement at the quantum level because no triad representation exists in the full theory [25]. Thus, the triad transformation cannot simply be implemented at the state level.

It is thus quite likely that loop quantum gravity provides for parity violating effects especially once fermions are included, even if the classical fermion interactions are used preserve parity. With the hidden nature of torsion contributions and parity in the quantum formulation, the precise form and magnitude of those parity violating ef-
fects is not easy to discern. But some implications can be explored either with effective equations (in their canonical form as described in [24,27]) which would allow one to perform some of the steps required in the classical analysis of parity, or with symmetry reduced models. An advantage of the latter would be that some models exist (such as those introduced in [28,29,30,31,32]) which do allow a triad representation and thus a more direct implementation of parity transformations.

V. CONCLUSIONS

We have summarized results of a complete canonical formulation of gravity non-minimally coupled to fermions in Ashtekar variables. This includes generalizations of basic results in the recent and some older literature, such as the torsion-mediated four-fermion interaction, and puts them on a firm canonical basis. We have used this for a demonstration of parity invariance of classical solutions, which required us to derive all contributions to the Ashtekar connection explicitly and to write several new versions of the canonical constraints, with explicit or implicit torsion contributions. The different forms of the constraints are needed to understand the parity behavior, and they also facilitate comparisons with earlier derivations and allow crucial cross-checks of the results. Here, we have noticed that our analysis fills in several gaps of previously available derivations and generalizes them to arbitrary non-minimal coupling.

The main purpose of the paper, however, is to provide a better and more complete foundation for the loop quantization of gravity coupled to fermions than can be found in the existing literature. Also this requires knowledge of the details given in the derivation of the canonical formalism to appreciate which of the established quantization steps of the torsion-free case go through in the presence of torsion, and where adaptations may be necessary. Overall, we find that the quantization of fermion fields and their dynamics given by Thiemann and others goes through in a well-defined manner. In details, however, we have clarified several steps where previously gaps existed, although they were not always realized. For all values of the non-minimal coupling parameter $\alpha$ there are new terms in the constraints due to torsion which are derived here in complete form. We have shown that torsion contributions and terms which arise from using half-densitized spinors cancel in the connection for the case where the non-minimal coupling parameter $\alpha$ equals the Barbero–Immirzi parameter $\gamma$. As a consequence, the presence of fermions does not change the quantization procedure much in this case, although there are still additional terms. For $\alpha \neq \gamma$, on the other hand, several additional adaptations to the usual construction steps of the Hamiltonian constraint operator are necessary.

While our results do not challenge the previous claims that all fields necessary for the standard model of particle physics can be quantized by loop techniques, some of the details of a specific quantization have to be corrected. As such Hamiltonians may become relevant for phenomenological considerations, e.g. in cosmology [33,34], a precise understanding of the quantum states and dynamical operators is not only necessary for a complete quantization but even for potential physical applications. In particular, we have highlighted the fact that current constructions of loop quantum gravity do not suffice to show that it exactly preserves parity.

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APPENDIX A: THE SU(2) SPIN CONNECTION

$\Gamma_a^b$ ON $\Sigma$

1. Torsion-free spin connection

In the torsion-free case, an explicit expression for the su(2) valued spin connection $\bar{\Gamma}_a^b$ can be derived from the fact that the covariant derivative of a co-triad vanishes: $D_a e^b_i = \partial_a e^b_i - \Gamma^b_{ab} e^a_i + \Gamma^b_{al} e^l_i = 0$. Thus, $\bar{\Gamma}_{ak}^b = -e^b_k (\partial_a e^b_l - \Gamma^c_{ab} e^c_l)$ and

$$\bar{\Gamma}_a^k = \frac{1}{2} \varepsilon^{ij}_k \bar{\Gamma}_a^j = -\frac{1}{2} \varepsilon^{ij}_k (\partial_a e^b_i - \Gamma^c_{ab} e^c_i)$$

(A1)

where $\Gamma^c_{ab}$ is the usual torsion-free Levi-Civita connection for $\Gamma_{aj}^k := \bar{\Gamma}_a^j e^j_k$ is used. With the definition of the Levi-Civita connection and $q_{ab} := e^a_i e^b_i$ we obtain

$$e^j_i \Gamma^c_{ab} = \frac{1}{2} \left( e^j_d e^k_b \partial_a e^d_k + 2 \partial_l (e^j_b) + e^d_j e^k_b \partial_a e^d_k - e^d_j e^k_b \partial_a e^d_b - e^d_j e^k_b \partial_d e^a_k \right).$$

(A2)
Inserting (A2) into (A1), we finally obtain the desired expression for the spin connection

\[ \Gamma^k_a = -\frac{1}{2} \epsilon_{ij} \Gamma_a^k = \frac{1}{2} \epsilon_{ij} \epsilon_{kl} \partial_a \epsilon^l_k - \Gamma^k_a = \frac{1}{2} \epsilon_{ij} \epsilon_{kl} \partial_a \epsilon^l_k + \epsilon_{ij} \epsilon_{kl} \partial_a \epsilon^l_k. \]  

(A3)

The following expressions are useful for computing \( \Gamma^k_a \) with torsion from the variational equations in the presence of fermions:

\[ e^a_i \tilde{\Gamma}^i_a = -\frac{1}{2} \epsilon_{ijk} \epsilon^b_k \partial_a \epsilon^l_i + \frac{1}{2 \sqrt{q}} \epsilon_{abc} \partial_a \epsilon^l_i, \]

(A4)

and

\[ \delta^k_i \epsilon^{bcd} e^a_i \partial_b \epsilon_c d_n + 2 \epsilon^{bcd} \epsilon^k_e \partial_e \epsilon^l_i = \text{sgn det}(e^a_i) \left( \frac{\sqrt{q}}{2} \epsilon_{ijk} \epsilon_{ijm} e^a_m \epsilon_{eap} + 2 \sqrt{q} \epsilon_{ijk} \epsilon_{ijm} \epsilon_{eal} \right) \]

\[ = \text{sgn det}(e^a_i) \sqrt{q} \epsilon_{ijk} \epsilon_{ijm} \epsilon_{eal}. \]  

(A5)

Finally, the Gauss constraint \( D_b P^{bm} = \partial_b P_m^m + \epsilon_{ij} m \Gamma^i_b P^{bj} \) implies

\[ \Gamma^k_a \Gamma^l_a - \Gamma^k_b \Gamma^l_b = \epsilon_{kl} \partial_a P^b_m + \frac{1}{2 (1 + \gamma^2)} \epsilon_{m}^{kl} \sqrt{q} J_m \]

\[ \text{sgn det}(e^a_i) \left( -\epsilon^{bcd} \epsilon^l_i \epsilon^a_c + \epsilon^{bcd} \epsilon^k_e \partial_b \epsilon^l_i \right) + \frac{1}{2 (1 + \gamma^2)} \epsilon_{m}^{kl} \sqrt{q} J_m. \]  

(A6)

2. Connection with torsion

Varying the action by connection components, we obtain

\[ \frac{\delta L}{\delta (\mathcal{A}^l)} = 1 + \frac{\gamma^2}{2} \epsilon_{lk} P^c_j \omega_i k^0 + \frac{1 + \gamma^2}{2} \epsilon_{lj} P^c_i \epsilon^a N \alpha^a + \frac{1 + \gamma^2}{2 \gamma^2} \epsilon^{abc} \partial_a (\epsilon_{dl} N) \]

\[ + \frac{\gamma^2 (1 + \gamma^2) N}{2 \sqrt{q}} \frac{P^l_k P^k_j \epsilon^a N \alpha^{a} - \epsilon^a N \alpha^{a}}{N} \frac{N_c}{4} \sqrt{q} \left( \gamma + \frac{1}{\alpha} \right) J_l - \frac{\gamma N}{4} \frac{P^l_i}{N} \left( \gamma + \frac{1}{\alpha} \right) J^0 - \frac{\gamma N}{4} \epsilon_{jk} \frac{1}{N} \left( 1 - \frac{1}{\alpha} \right) J^k = 0. \]  

(A7)

which in the canonical formulation serves as one of the second class constraints. After expressing (A7) in terms of \( \Gamma^a_i \) and \( K^a_i \) first and then contracting with \( e^e_c \), we obtain

\[ \frac{1 + \gamma^2}{2 \gamma^2} \epsilon_{e}^{m l k} \sqrt{q} \omega_l k^0 - \frac{1 + \gamma^2}{2 \gamma^2} \epsilon_{l k} \sqrt{q} N \alpha^k + \frac{1 + \gamma^2}{2 \gamma^2} \sqrt{q} \epsilon^a e^e c \epsilon^k l k N \alpha^c K^a + \text{sgn det}(e^a_i) \frac{1 + \gamma^2}{2 \gamma^2} \epsilon^{bcd} e^l_i c \epsilon^e d \partial_b N \]

\[ + \text{sgn det}(e^a_i) \frac{1 + \gamma^2}{2 \gamma^2} \epsilon^{bcd} e^e c \epsilon^l_i \partial_b d l - \frac{(1 + \gamma^2)}{2 \gamma^2} \sqrt{q} \left( N \epsilon^a e^e c - e^a e^e c \right) \Gamma^i_a = -\beta N \epsilon^{e} e^{l m} \frac{1}{4} \sqrt{q} J^0 - \gamma N \frac{1}{4} e^{e} e^{l m} \frac{1}{4} \sqrt{q} J^k + \frac{\beta N}{4} \delta^m_i \sqrt{q} J^0. \]  

(A8)

Contracting it with \( \delta^l_i \) and using the Gauss constraint, this equation simplifies considerably to

\[ \text{sgn det}(e^a_i) \frac{1 + \gamma^2}{2 \gamma^2} N \epsilon^{e} e^{l m} \partial_b d l - (1 + \gamma^2) N P^m l \Gamma^l_a = \frac{3}{4} \beta N \sqrt{q} J^0. \]  

(A9)

Symmetrizing the indices \( m \) and \( l \) in (A8) and using (A9) for \( e^e_i \Gamma^i_a \), we obtain the following symmetric combination of \( P^l_i \) and \( P^m l \Gamma^l_a \)

\[ \gamma_K (P^l_i \Gamma^m a + P^m l \Gamma^l a) = \text{sgn det}(e^a_i) \frac{1}{2 \gamma^2} \epsilon^{bcd} e^e c \epsilon^l_i \partial_b d l - \epsilon^{bcd} e^e c \partial_b d l - \epsilon^{bcd} e^e c \partial_e d l - \frac{\beta N}{2 \gamma} \delta^m_l \sqrt{q} J^0. \]  

(A10)

On the other hand, the second class constraints can be seen to appear an equation \( 2 \partial_b P^{bm} + 2 \epsilon_{ij} m \Gamma^i_B \) which \( \theta \sqrt{q} J^m / (1 + \gamma^2) \), or

\[ \gamma_K (P^l_i \Gamma^m a - P^m l \Gamma^l a) = \text{sgn det}(e^a_i) \left( \epsilon^{bcd} e^l_i \partial_b d l + \epsilon^{bcd} e^c l c \right) + \frac{\theta N}{2 \gamma} \epsilon^{m l} \sqrt{q} J^0. \]  

(A11)
Combining (A10) and (A11) yields

\[ 2\gamma_K P^a_l \Gamma^k_a = \text{sgn}(e^i_c)(\delta^{kl} e^a_c \partial_l e_{da} + 2\epsilon^{bcd} e^k_d \partial_b e_{c}) + \frac{\gamma_K}{2(1 + \gamma^2)} \left( \theta \epsilon^i_j \sqrt{\gamma} J^j - \beta \delta^{kl} \sqrt{\gamma} J^0 \right). \]  

(A12)

Next, inserting (A5) into (A12), we find

\[ e^i_{l} \Gamma^k_c = \frac{1}{2} \epsilon^{ijk} e_i (2\epsilon^b_d \partial_a e^j_d + \epsilon^b_c \partial_a e^j_d) + \frac{\gamma_K}{4(1 + \gamma^2)} \left( \theta \epsilon^i_j \sqrt{\gamma} J^j - \beta \delta^{kl} \sqrt{\gamma} J^0 \right), \]  

(A13)

and finally (14).

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[38] In the gravitational part, however, we will not follow exactly the notation of [4] but rather that of [41].
[39] At this point, it is noteworthy that we intend to use the signature (−+++), instead of (−−−) which is common in quantum field theory for both gravity and fermions since this is the signature most prevalent in the literature for canonical gravity. This demands certain modifications in the representations of the Clifford algebra, where it turns out that changing the signature from (−−−) to (−+++), only requires all the Dirac matrices to be multiplied by η.
[40] More generally, the triads can be allowed to transform as \( e^a \rightarrow A^a \), where \( A^a \) is an orthogonal transformation matrix with determinant −1. Also, the gamma matrices transform like \( \gamma^0 \rightarrow \gamma^0 + \gamma^0 - A^a \gamma^a \). It is easy to check that the torsion-free spin connection and the extrinsic curvature transform as \( \Gamma^a \rightarrow -A^b_j \Gamma^a_j \) and \( K^a \rightarrow A^a \). Finally, the transformation of \( A^a \) can be obtained from these two one-forms. Our arguments about parity invariance remain unchanged if this more general transformation is used.
[41] This demonstrates that terms presented above, and which differ from those in [3] (for \( a = \gamma \)), must be contained in the constraints. Ignoring the interaction term in [20], on the other hand, provides the Hamiltonian constraint of a second-order formalism which can be compared directly with the Appendix of [12] (for \( \gamma = 1 \)). Notice that the derivation sketched in [12] does not work purely in real variables and assumes properties of the projection from complex variables. As the comparison with our results shows, the calculations of [12] leave some extra terms in the constraint which are absent in a complete derivation based only on real variables.