THE FUNCTIONAL EQUATION OF THE JACQUET-SHALIKA
INTEGRAL REPRESENTATION OF THE LOCAL EXTERIOR-SQUARE
L-FUNCTION

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1. Introduction

An integral representation for the exterior square $L$-function for $GL_n$ was given by Jacquet and Shalika in 1990 [9]. In the mid 1990’s the first author, with Piatetski-Shapiro, embarked on the local analysis of the exterior square $L$-function via this integral representation in conjunction with their project to establish functoriality from $SO_{2n+1}$ to $GL_{2n}$ via the converse theorem and integral representations [4]. The approach there was by the Bernstein-Zelevinsky theory of derivatives as in [3]. This was set aside and never published, other than [4].

Recently there has been renewed interest in the local and global theory of the exterior square $L$-function via this integral representation [11, 10, 11]. In particular, in [14] the second author used his results on the connection between linear periods [13] and Shalika periods to analyze the local exterior square $L$-functions via Bernstein-Zelevinsky derivatives and prove the local functional equation in the case of $GL_{2m}$. This approach seems simpler than the approach used in [4].

In this paper we complete the work in [14] and derive the local functional equation for the exterior square $L$-function for $GL_{2m+1}$. In their original paper, Jacquet and Shalika considered the odd case only briefly in their last section, Section 9. We deduce the shape of the local functional equation from the global one in [9] and then prove it as in [14] using Bernstein-Zelevinsky derivatives and the theory of linear periods [7] extended to the odd case. (We should point out that our version of the global and local functional equation in the odd case is different from that given by Kewat and Raghunathan in [11].)

The local functional equation of the exterior square $L$-function is now available for irreducible representations of $G_n$, for any $n$. We will use these local functional equations in the future to prove the inductivity, or multiplicativity, of the local exterior square $L$-function and $\gamma$-factor, and then complete the local nonarchimedean theory of the exterior square $L$-function at the ramified places.

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We should point out that the exterior square \(L\)-function is available from the Langlands-Shahidi method \([16]\) and the main result of \([11]\) is that for discrete series representations the \(L\)-functions from the Langlands-Shahidi method and the integral representation of Jacquet and Shalika agree.

2. Preliminaries

Let \(F\) be a nonarchimedean local field, with ring of integers \(\mathfrak{O}\), prime ideal \(\mathfrak{P}\), and fix a uniformizer \(\varpi\) so that \(\mathfrak{P} = (\varpi)\). Let \(q = |\mathfrak{O}/\mathfrak{P}|\) denote the cardinality of the residue class field. We let \(\text{val} : F^\times \to \mathbb{Z}\) be the associated discrete valuation with \(\text{val}(\varpi) = 1\) and normalize the absolute value so that \(|a| = q^{-\text{val}(a)}\).

Let \(\mathcal{M}_k\) denote the algebra of \(k \times k\) square matrices with entries in \(F\) and \(\mathcal{M}_{a,b}\) the \(a \times b\) matrices with entries in \(F\).

We denote \(GL_n(F)\) by \(G_n\) for \(n \geq 1\). We will denote \(|\det(g)|\) by \(|g|\) for a matrix in \(G_n\). The group \(N_n\) will be the unipotent radical of the standard Borel subgroup \(B_n\) of \(G_n\) given by upper triangular matrices. For \(n \geq 2\) we denote by \(U_n\) the group of matrices 
\[
u(x) = \begin{pmatrix} I_{n-1} & x \\ 1 & \end{pmatrix}
\]
for \(x\) in \(F^{n-1}\).

For \(n > 1\), the map \(g \mapsto \begin{pmatrix} g \\ 1 \end{pmatrix}\) is an embedding of the group \(G_{n-1}\) in \(G_n\). We denote by \(P_n\) the subgroup \(G_{n-1}U_n\) of \(G_n\). This is the mirabolic subgroup of \(G_n\). We fix a nontrivial additive character \(\theta\) of \(F\), and denote by \(\theta\) again the character 
\[

\nu(n) \mapsto \theta \left( \sum_{i=1}^{n-1} n_{i,i+1} \right)
\]
of \(N_n\). The normalizer of \(\theta|_{U_n}\) in \(G_{n-1}\) is then \(P_{n-1}\).

Suppose \(n = 2m\) is even. Let \(\sigma_n \in G_n\) be the permutation matrix for the permutation given by
\[
\sigma_n = \begin{pmatrix} 1 & 2 & \cdots & m \\ 1 & 3 & \cdots & 2m-1 \end{pmatrix} \begin{pmatrix} m+1 & m+2 & \cdots & 2m \\ 2 & 4 & \cdots & 2m \end{pmatrix}.
\]
In this case we denote by \(M_n\) the standard Levi of \(G_n\) associated to the partition \((m, m)\) of \(n\). Let \(w_n = \sigma_n\) and then let \(H_n = w_n M_n w_n^{-1}\).

Suppose \(n = 2m + 1\) is odd. In this case we let \(\sigma_n\) be the permutation matrix in \(G_n\) associated to the permutation
\[
\sigma_{2m+1} = \begin{pmatrix} 1 & 2 & \cdots & m \\ 1 & 3 & \cdots & 2m-1 \end{pmatrix} \begin{pmatrix} m+1 & m+2 & \cdots & 2m & 2m+1 \\ 2 & 4 & \cdots & 2m & 2m+1 \end{pmatrix}.
\]
so that \(\sigma_{2m} = \sigma_{2m+1}|_{G_{2m}}\) and let \(w_{2m+1} = w_{2m+2}|_{GL_{2m+1}}\) so that
\[
w_{2m+1} = \begin{pmatrix} 1 & 2 & \cdots & m+1 \\ 1 & 3 & \cdots & 2m+1 \end{pmatrix} \begin{pmatrix} m+3 & m+4 & \cdots & 2m+1 \\ 2 & 4 & \cdots & 2m-2 \end{pmatrix}.
\]
In the odd case, \( \sigma_{2m+1} \neq w_{2m+1} \). We let \( M_n \) denote the standard parabolic associated to the partition \((m + 1, m)\) of \( n \) and set \( H_n = w_n M_n w_n^{-1} \) as in the even case.

Note that the \( H_n \) are compatible in the sense that \( H_n \cap G_{n-1} = H_{n-1} \).

We will need the work of Bernstein and Zelevinsky concerning the classification of irreducible representations of \( G_n \). We first define the following functors following [2]:

- The functor \( \Phi^+ \) from \( \text{Alg}(P_{k-1}) \) to \( \text{Alg}(P_k) \) such that, for \( \pi \) in \( \text{Alg}(P_{k-1}) \), one has \( \Phi^+ \pi = \text{ind}_{P_{k-1}}^{P_k} (\delta^{1/2}_{k} \pi \otimes \theta) \).

- The functor \( \Psi^+ \) from \( \text{Alg}(G_{k-1}) \) to \( \text{Alg}(P_k) \), such that for \( \pi \) in \( \text{Alg}(G_{k-1}) \), one has \( \Psi^+ \pi = \text{ind}_{G_{k-1}}^{P_k} (\delta^{1/2}_{k} \pi \otimes 1) = \delta^{1/2}_{k} \pi \otimes 1 \).

We recall the following proposition which follows from Propositions 2.1 and 2.2 of [13] (in which one has injections instead of isomorphisms):

**Proposition 2.1.** Let \( \sigma \) belong to \( \text{Alg}(P_{n-1}) \), and \( \chi \) be a character of \( P_{n-1} \cap H_{n-1} \). Then there is a positive character \( \chi' \) of \( P_{n-1} \cap H_{n-1} \), independent of \( \sigma \), such that
\[
\text{Hom}_{P_{n-1} \cap H_{n-1}} (\Phi^+ \sigma, \chi) \simeq \text{Hom}_{P_{n-1} \cap H_{n-1}} (\sigma, \chi \chi')
\]

Let \( \sigma' \) belong to \( \text{Alg}(P_{n-2}) \), and \( \chi' \) be a character of \( P_{n-2} \cap H_{n-2} \). Then there is a positive character \( \chi'' \) of \( P_{n-2} \cap H_{n-2} \), independent of \( \sigma' \), such that
\[
\text{Hom}_{P_{n-2} \cap H_{n-2}} (\Phi^+ \sigma', \chi') \simeq \text{Hom}_{P_{n-2} \cap H_{n-2}} (\sigma', \chi' \chi'').
\]

As a corollary we have the following.

**Corollary 2.1.** Let \( n = 2m + 1 \) be an odd integer. Let \( \rho \) be an irreducible representation of \( G_k \) for \( k \leq n - 1 \), and \( \chi \) be a character of \( H_n \cap P_n \). Then
\[
\text{Hom}_{H_n \cap P_n} (\Phi^+ (\Psi^+)^{n-k-1} \Psi^+ (\rho), \chi) \simeq \text{Hom}_{H_k} (\rho, \chi \mu_n^k)
\]
for a character \( \mu_n^k \) of \( H_k \) independent of \( \rho \).

**Proof.** By the previous propositions we have
\[
\text{Hom}_{H_n \cap P_n} (\Phi^+ (\Psi^+)^{n-k-1} \Psi^+ (\rho), \chi) \simeq \text{Hom}_{P_{k+1} \cap H_{k+1}} (\Psi^+ (\rho), \chi \mu')
\]
for an appropriate character \( \mu' \). \( \Psi^+ \) is just twisting by \( |\det(.)|^{1/2} \), and then extending a representation of \( H_k \) to \( P_{k+1} \) by \( 1 \) on \( U_{k+1} \), so it is quite straightforward that a linear form on \( \Psi^+ (\rho) \) quasi-invariant under \( P_{k+1} \cap H_{k+1} \) is just a linear form on a twist of \( \rho \) by a positive character, quasi-invariant under \( H_k \).

3. **The functional equation of the local exterior square \( L \)-function, when \( n \) is odd**

In this section, \( n = 2m + 1 \) is odd.
3.1. **An action of the Shalika subgroup on** \(C_c^\infty(F^m)\). We consider the Shalika subgroup \(S_n\) of \(G_n\):

\[
S_n = \left\{ \begin{pmatrix} g & z & y \\ g & x & 1 \end{pmatrix} \middle| g \in G_m, \ x \in M_{1,m}, \ y \in M_{m,1}, \ z \in M_m \right\}
\]

We recall that

\[
\Theta \left( \begin{pmatrix} I_m & z \\ I_m & 1 \end{pmatrix} \begin{pmatrix} g & y \\ I_m & 1 \end{pmatrix} \right) = \theta(\text{Tr}(z))
\]

defines a character of \(P_n \cap S_n\). We claim that \(S_n\) admits a certain linear representation on the space \(C_c^\infty(F^m)\).

**Proposition 3.1.** There is a linear representation \(R_\theta\) of \(S_n\) on the space \(C_c^\infty(F^m)\), such that:

- \(R_\theta \begin{pmatrix} g \\ g \\ 1 \end{pmatrix} \phi(x) = \phi(x.g).
- \(R_\theta \begin{pmatrix} I_m & z_0 \\ I_m & 1 \end{pmatrix} \phi(x) = \theta(\text{Tr}(-z_0))\phi(x).
- \(R_\theta \begin{pmatrix} I_m & y_0 \\ I_m & 1 \end{pmatrix} \phi(x) = \theta(\langle x, y_0 \rangle)\phi(x).
- \(R_\theta \begin{pmatrix} I_m & x_0 \\ x & 1 \end{pmatrix} \phi(x) = \phi(x + x_0).

In fact, \(R_\theta\) is simply the model of \(\text{ind}^{S_n}_{P_n \cap S_n}(\Theta^{-1})\), given by the restriction

\[
f \in \text{ind}^{S_n}_{P_n \cap S_n}(\Theta^{-1}) \mapsto \phi \in C_c^\infty(F^m),
\]

where \(\phi(x) = f \begin{pmatrix} I_m \\ I_m \\ x \end{pmatrix} \).

**Proof.** Just check that this is indeed the model of \(\text{ind}^{S_n}_{P_n \cap S_n}(\Theta^{-1})\) given by the restriction map defined in the statement above. \(\square\)

Let \(\tau = \tau_n\) be the matrix \(\begin{pmatrix} I_m & I_m \\ I_m & 1 \end{pmatrix}\). For \(h \in G_n\), we denote by \(h^\tau\) the matrix \(\tau h \tau^{-1}\).

One checks the following lemma.

**Lemma 3.1.** The map \(s \mapsto {}^t(s^{-1})^\tau = {}^t s^\tau\) defines an automorphism of the group \(S_n\).
Proof. If $s = \begin{pmatrix} g & g \\ 1 & 1 \end{pmatrix}$, then $t' s^{-\tau} = \begin{pmatrix} t g^{-1} & t g^{-1} \\ 1 & 1 \end{pmatrix}$.

If $s = \begin{pmatrix} I_m & z \\ I_m & 1 \end{pmatrix}$, then $t' s^{-\tau} = \begin{pmatrix} I_m & -t z \\ I_m & 1 \end{pmatrix}$.

If $s = \begin{pmatrix} I_m & y \\ I_m & 1 \end{pmatrix}$, then $t' s^{-\tau} = \begin{pmatrix} I_m & -t y \\ I_m & 1 \end{pmatrix}$.

If $s = \begin{pmatrix} I_m & x \\ x & 1 \end{pmatrix}$, then $t' s^{-\tau} = \begin{pmatrix} I_m & -t x \\ I_m & 1 \end{pmatrix}$. □

We now investigate the effect of the Fourier transform on the representation $R_{\theta}$.

**Proposition 3.2.** For $\phi$ in $C^\infty_c(F^m)$, we denote by

$$
\hat{\phi}(y) = \int_{u \in F^m} \phi(u) \theta^{-1}(\langle u, y \rangle) du
$$

for $du$ such that the Fourier inversion formula holds. We denote by $\mathcal{F}$ the operator $\phi \mapsto \hat{\phi}$ on $C^\infty_c(F^m)$. Then it satisfies

$$
\mathcal{F}(R_{\theta}(s) \phi) = |s|^{-1/2} R_{\theta^{-1}}(t' s^{-\tau}) \mathcal{F}(\phi).
$$

**Proof.** One checks this on the generators, using the proof of Lemma 3.1. □

### 3.2. The integral representation for the exterior square $L$-function

Let $\pi$ be an irreducible admissible representation of $G_n$. If $\pi$ is generic, we let $\mathcal{W}(\pi, \theta)$ denote its Whittaker model; if not, then $\pi$ is an irreducible quotient of an induced representation $\Xi$ of Langlands type which has a Whittaker model and we set $\mathcal{W}(\pi, \theta) = \mathcal{W}(\Xi, \theta)$ [8]. Following Section 9 of [9] we now define two families of integrals, for $W$ in $\mathcal{W}(\pi, \theta)$, $\phi$ in $C^\infty_c(F^m)$, and $s$ in $\mathbb{C}$:

$$
J_{\theta}(s, W) = \int W \left( \sigma_n \begin{pmatrix} I_m & z \\ I_m & 1 \end{pmatrix} \begin{pmatrix} g & g \\ 1 & 1 \end{pmatrix} \right) \theta(Tr(-z)) |g|^{s-1} dz dg,
$$

where $g$ is integrated over $N_m \backslash G_m$, and $z$ over $N_n \backslash M_n$, where $N_n$ is the space of upper triangular matrices, and

$$
J_{\theta}(s, W, \phi) = J_{\theta}(s, \rho(\phi) W),
$$

where

$$
\rho(\phi) W(g) = \int_{x \in F^m} W \left( g \begin{pmatrix} I_m & I_m \\ x & 1 \end{pmatrix} \right) \phi(x) dx.
$$

Notice that in fact, $\rho(\phi) W$ is a finite sum of right translates of $W$. 

It is proved in \cite{9} that there exists $r_\pi$ in $\mathbb{R}$, such that the integrals $J_\theta(s,W)$ converge for $\text{Re}(s) > r_\pi$, and that they are in fact elements of $\mathbb{C}(q^{-s})$. This implies the same property for the integrals $J_\theta(s,W,\hat{\phi})$. It is moreover proved in \cite{15} that the integrals $J_\theta(s,W)$ span a fractional ideal $J_\pi$ of $\mathbb{C}[q^{\pm s}]$, generated by an Euler factor $L(s,\pi,\wedge^2)$.

**Lemma 3.2.** The integrals $J_\theta(s,W,\hat{\phi})$ also span $J_\pi = L(s,\pi,\wedge^2)\mathbb{C}[q^{\pm s}]$.

**Proof.** One has $\langle J_\theta(s,W,\hat{\phi}) \rangle \subset \langle J_\theta(s,W) \rangle$, as $\rho(\phi)W$ is a finite sum of right translates of $W$. Conversely, for $\phi$ the characteristic function of a small enough subgroup of $F^m$, the integral $J_\theta(s,W,\hat{\phi})$ becomes equal to a positive multiple of $J_\theta(s,W)$ by smoothness of $\mathcal{W}(\pi,\theta)$, hence $\langle J_\theta(s,W,\hat{\phi}) \rangle \supset \langle J_\theta(s,W) \rangle$. \hfill $\square$

We now check that the integrals $J_\theta(s,W,\hat{\phi})$ define invariant bilinear forms under the group $S_n$.

**Lemma 3.3.** The map $B_{s,\pi,\theta} : (W,\phi) \mapsto J_\theta(s,W,\hat{\phi})/L(s,\pi,\wedge^2)$ defines a bilinear form on $\mathcal{W}(\pi,\theta) \times \mathcal{C}_c^\infty(F^m)$, which satisfies the relation $B_{s,\pi,\theta}(\rho(h)W,R_\theta(h)\phi) = |h|^{-s/2}B_{s,\pi,\theta}(W,\phi)$.

**Proof.** We recall that, for $\sigma = \sigma_n$,

$$
J_\theta(s,W,\hat{\phi}) = \int W \left( \sigma \begin{pmatrix} I_m & z \\ I_m & 1 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} I_m & z \\ I_m & 1 \end{pmatrix} \right) \phi(x)\theta(Tr(-z))[g]^{|s-1|}dx dz dg
$$

which is absolutely convergent for $s$ large enough. One just needs to check the invariance of $B_{s,\pi,\theta}$ under the generators of $S_n$ given in Lemma 3.1. This follows from simple change of variables. \hfill $\square$

Let $w$ be long Weyl element of of $G_n$, represented by the antidiagonal matrix with ones on the second diagonal. We denote by $\widetilde{W}$ the map on $G_n$ defined by $\widetilde{W}(g) = W(\check{w}^tg^{-1})$. Then $W \mapsto \widetilde{W}$ is a vector space isomorphism between $\mathcal{W}(\pi,\theta)$ and $\mathcal{W}(\pi^\vee,\theta^{-1})$, where $\pi^\vee$ denotes the (admissible) contragredient of $\pi$, which satisfies $\rho(h)\widetilde{W} = \rho(h^{-1})\widetilde{W}$. Now, Proposition 3.2 has the following consequence.

**Lemma 3.4.** The bilinear form $C_{s,\pi,\theta} : (W,\phi) \mapsto B_{1-s,\pi^\vee,\theta^{-1}}(\rho(\tau)\widetilde{W},\phi)$ on $\mathcal{W}(\pi,\theta) \times \mathcal{C}_c^\infty(F^m)$ also belongs to the space $\text{Hom}_{S_n}(W(\pi,\theta) \otimes \mathcal{C}_c^\infty(F^m),|.|^{-s/2})$.

**Proof.** By definition, for $h \in S_n$ we have

$$
C_{s,\pi,\theta}(\rho(h)W,R_\theta(h)\phi) = B_{1-s,\pi^\vee,\theta^{-1}}(\rho(\tau)\widetilde{W},\phi,\mathcal{F}(R_\theta(h)\phi)).
$$

If we now use Proposition 3.2 and Lemma 3.3 to compute the right hand side we have

$$
B_{1-s,\pi^\vee,\theta^{-1}}(\rho(\tau)\widetilde{W},\mathcal{F}(R_\theta(h)\phi)) = B_{1-s,\pi^\vee,\theta^{-1}}(\rho(\tau)\rho(h^{-1})\widetilde{W},|h|^{-1/2}R_{\theta^{-1}}(\tau^{h^{-1}})\phi)
$$

$$
= |h|^{-1/2}B_{1-s,\pi^\vee,\theta^{-1}}(\rho(h^{-1})\rho(\tau)\widetilde{W},R_{\theta^{-1}}(\tau^{h^{-1}})\phi)
$$

$$
= |h|^{-s/2}C_{s,\pi,\theta}(W,\phi).
$$

\hfill $\square$
The functional equation will then follow if we can prove that for almost all \( s \), the space \( \text{Hom}_{S_n}(\mathcal{W}(\pi, \theta) \otimes C^\infty(F^m), |.|^{-s/2}) \) is of dimension at most 1. That is what we do in the next section.

3.3. The local functional equation. We denote by \( L_n \) the maximal (non-standard) Levi subgroup of \( G_n \) of type \((m + 1, m)\), given by

\[
L_n = \left\{ \begin{pmatrix} g_1 & u \\ g_2 & \lambda \\ v & \end{pmatrix} \in G_n | u \in M_{m,1}, v \in M_{1,m}, \lambda \in F, g_1, g_2 \in G_m \right\}.
\]

We first show that if \( \pi \) is an irreducible representation of \( G_n \), then there is an injection of the vector space \( \text{Hom}_{P_n nS_n}(\mathcal{W}(\pi, \theta, \Theta)) \) into \( \text{Hom}_{P_n nL_n}(\mathcal{W}(\pi, \theta, \chi)) \) for some character \( \chi \) of \( L_n \). This will be a consequence of the technique in Paragraph 6.2 in [7]. This will then give us a multiplicity one result which we can apply to the functionals \( B_{s, \pi, \theta} \) and \( C_{s, \pi, \theta} \) above to obtain the functional equation.

Let \( \Pi = \mathcal{W}(\pi, \theta) \). Recall that if \( \pi \) is generic then \( \mathcal{W}(\pi, \theta) \simeq \pi \) while if \( \pi \) is not generic then \( \mathcal{W}(\pi, \theta) \simeq \Xi \) where \( \Xi \) is the induced representations of Langlands type having \( \pi \) as its unique irreducible quotient. If \( L \) is an element of the space \( \text{Hom}_{P_n nG}(\Pi, \Theta) \) and \( v \) belongs to \( \Pi \), we denote by \( S_{L, v} \) the function on \( G_n \) defined as \( S_{L, v}(g) = L(\Pi(g)v) \). If we formally set

\[
I(S_{L, v}, s) = \int_{G_n} S_{L, v}(\text{diag}(g, I_{m+1})) |g|^s dg
\]

and

\[
\Gamma_L(v) = I(S_{L, v}, s)
\]

then a simple change of variables in the integral gives that \( \Gamma_L \in \text{Hom}_{P_n nL_n}(\Pi, \chi_s) \) where

\[
\chi_s \begin{pmatrix} g_1 & u \\ g_2 & \lambda \\ v & \end{pmatrix} = \left( \frac{|g_1|}{|g_2|} \right)^{-s}. 
\]

To actually implement this we need to first understand the convergence of \( I(S_{L, v}, s) \) and then in the realm of convergence show that the map \( L \mapsto \Gamma_L \) is indeed injective.

We begin with convergence. We write \( U'_1 \) for the unipotent radical of the standard parabolic of type \((i, n - i) = (i, 2m + 1 - i)\). In what follows all that is important is that \( \Pi \) has finite length.

**Proposition 3.3.** For \( a \) in \((F^\times)^m\), we denote by \( m(a) \) the matrix \( \text{diag}(b_1, \ldots, b_m, I_{m+1}) \), with \( b_i = a_i \ldots a_m \). For \( 1 \leq i \leq m \), there is a finite set \( X_{\Pi, i} \) of characters of \( F^\times \) (namely the central characters of the irreducible sub-quotients of the Jacquet modules \( \Pi_{U'_1} \) of \( \Pi \)), such that if \( S_{L, v} \) is as above, and \( |a_i| \leq 1 \) when \( i \) is between 1 and \( m - 1 \), then \( S_{L, v}(m(a)) \) is a sum of functions of the form

\[
\prod_{i=1}^m \chi_i(a_i) \text{val}(a_i)^{m_i} \varphi(a)
\]
with \( \chi_i \in X_{\Pi,i} \), integers \( m_i \geq 0 \), and \( \varphi \) a Schwartz function on \( F^m \). This implies that there is a real number \( r_{\Pi} \), such that the integral
\[
I(S_{L,v}, s) = \int_{G_m} S_{L,v}(\text{diag}(g, I_{m+1})) |g|^s \, dg
\]
is absolutely convergent for \( \text{Re}(s) > r_{\Pi} \).

Proof. Let \( V \) be the space of \( \Pi \). As in p.118 of [7], we see that there is \( c = c_{L,v} > 0 \), such that \( |a_m| \geq c \), and \( |a_i| \leq 1 \) for \( i \in \{1, \ldots, m-1\} \) implies \( S_{L,v}(m(a)) = 0 \), thanks to the relation \( L(\pi(a)\pi(u)v) = \Theta(aua^{-1})L(\pi(a)v) \) for \( u \in U_m \subset S_n \).

Lemma 6.2. of [7], which asserts that if \( i \) is a positive integer \( \leq m \) and if \( v \in V(U'_1) = \{ \pi(u)v' - v' \mid v' \in V, u \in U'_1 \} \), then \( S_{L,v}(\text{diag}(m(a))) \) vanishes if \( |a_i| \) is small enough, and \( |a_j| \leq 1 \) for \( 1 \leq j \leq m \), is also valid in our case. This lemma only uses the quasi-invariance of \( S_{L,v} \) under the Shalika subgroup \( S_n \), and its right smoothness. We indicate the notational changes to be made in Lemma 6.2 of [7] for our situation: \( a = m(a) := \text{diag}(b_1, \ldots, b_m, I_{m+1}) \) instead of \( \text{diag}(b_1, \ldots, b_m, I_m) \), \( u_1 := \left( \begin{pmatrix} I_m & Z \\ I_{m+1} \end{pmatrix} \right) \), \( u_2 := \left( \begin{pmatrix} u' \\ I_{m+1} \end{pmatrix} \right) \), \( u' \) is the same, and \( \left( \begin{pmatrix} I_m & bu'^{-1}b^{-1} \\ 1 \end{pmatrix} \right) \) replaced by \( \left( \begin{pmatrix} I_m \\ bu'^{-1}b^{-1} \end{pmatrix} \right) \). Notice that there is a typo in [7], equality at the top of p.120, where the second \( \pi(a) \) should stand just before \( \nu_0 \). This shows that the lemma applies in our situation.

Now, let \( H_i \) be the group \( \{ \text{diag}(tI_{i}, I_{m+1-i}), t \in F^X \} \), \( H_i^1 \) is the subgroup \( \{ \text{diag}(tI_{i}, I_{m+1-i}), t \in O - 0 \} \), \( H = \prod_{i=1}^m H_i \), and \( H^1 = \prod_{i=1}^m H_i^1 \). For \( i \leq m \), the Jacquet module \( V_{U'_i} = V/V(U'_i) \) has finite length and \( H_i \) acts by a character on each irreducible subquotient. Fix \( L \in H\text{om}_{F, \text{cusp}}(\Pi, \Theta) \), and call \( \mathcal{V} \) the space of maps \( \phi_{L,v} : a \in H \mapsto S_{L,v}(m(a)) \) for \( v \in V \). \( \mathcal{V} \) is certainly a smooth \( H \)-module. Let \( \mathcal{V}_i \) denotes the \( H_i \)-submodule of functions \( \phi \) in \( \mathcal{V} \), such that there is \( c_\phi > 0 \), which satisfies that \( \rho(h_i)\phi \) vanishes on \( H_i^1 \) when \( |h_i| \leq c_\phi \) (with \( h_i \in H_i \simeq F^X \)). Then, \( \mathcal{V}/\mathcal{V}_i \) is a quotient of \( V_{U'_i} \) (thanks to our version of Lemma 6.2), and we can apply Lemma 3.5 below, which tells us that for any \( v \in V \), \( \phi_{L,v} \) restricts to \( H^1 \) as we expect. Now, let \( (z_{\beta})_\beta \) be a finite set of representatives of \( \{ a_m \in H_m \mid 1 \leq |a_m| \leq c_{L,v} \}/U \) for a \( U \) compact open subgroup of \( H_m \) fixing \( \phi_{L,v} \), we can then write \( 1_{\{1 \leq |a_m| \leq c_{L,v}\}} \phi_{L,v}(a_1, \ldots, a_{m-1}, a_m) \) as
\[
\sum_{\beta} \phi_{L,v}(a_1, \ldots, a_{m-1}, z_{\beta}) 1_{z_{\beta}U}(a_m) = \sum_{\beta} \phi_{L,v}(z_{\beta}a_1, \ldots, a_{m-1}, 1) 1_{z_{\beta}U}(a_m).
\]
We now conclude (as \( 1 \in H_{m}^1 \)), thanks to the relation
\[
\phi_{L,v}(a_1, \ldots, a_{m-1}, a_m) = 1_{\{a_m \leq 1\}} \phi_{L,v}(a_1, \ldots, a_{m-1}, a_m) + 1_{\{1 \leq |a_m| \leq c_{L,v}\}} \phi_{L,v}(a_1, \ldots, a_{m-1}, a_m)
\]
for \( |a_i| \leq 1 \) when \( i \leq m - 1 \).

The asymptotic expansion implies the convergence of the integral as on the top of p.119 of [7], as we can here as well write \( L(\Pi(h)v) = L(\Pi(m(a)k)v \) (see [7]), because \( \{\text{diag}(g,g,1) \in G_m \} \) fixes \( L \). (This part was a problem in the even case [14].) \( \square \)
We are left with proving Lemma 3.5 below. This lemma is very similar to Lemma 2.2.1 of [6]. We will give a slightly different proof, based on [12], where the exponents of the representation appear.

**Lemma 3.5.** Let $\mathcal{V}$ be a subspace of the smooth complex functions on the torus $H = \prod_{i=1}^{m} H_i$ with $H_i = F^x$, which is a smooth $H$-module. Write $H^1_1$ for $\mathcal{O} - \{0\}$. Let $\mathcal{V}_i$ be the $H_i$-submodule of functions $\phi$ in $\mathcal{V}$, such that there is $c_\phi > 0$, which satisfies that $\rho(h_i)\phi$ vanishes on $H^1 = \prod_{i=1}^{m} H^1_i$ when $|h_i| \leq c_\phi$ (with $h_i \in H_i$). Suppose moreover that for every $i \in \{1, \ldots, m\}$, the $H_i$-module $Q_i = \mathcal{V}/\mathcal{V}_i$ has a finite filtration $0 \subset Q_{1,i} \subset \cdots \subset Q_{n_i,i} = Q_i$, such that $H_i$ acts by a character on each successive subquotient $Q_{t+1,i}/Q_{t,i}$, and call $X_i$ the finite family of such characters. Then any $\phi$ in $\mathcal{V}$ restricts to $H^1$ as a sum of maps of the form $\phi(a) = \prod_{i=1}^{m} \chi_i(a_i)\text{val}(a_i)^{m}\varphi(a)$, for $\chi_i \in X_i$, $m_i \in \mathbb{N}$, and $\varphi$ a Schwartz function on $\mathcal{O}^m$.

**Proof.** We do an induction on $m$. We start with $m = 1$, so $H = H_1 = F^x$ and $H^1 = H^1_1 = \mathcal{O} - 0$, this case being almost as significant as the general case. Let $\phi$ belong to $\mathcal{V}$, and $\overline{\phi}$ its image in $Q_1$. Then by Lemma 2.1 of [12], $\overline{\phi}$ generates a finite dimensional $H_1$-submodule of $Q_1$. Let $\overline{B} = \{\overline{e}_1, \ldots, \overline{e}_r\}$ be a basis of the span $\langle \rho(h)\phi \mid h \in H^1 \rangle$ and for each $i$ let $e_i$ be a lift of $\overline{e}_i$ in $\mathcal{V}$. By Proposition 2.8 of [12], the matrix $M(h) = \text{Mat}_\mathbb{F}(\rho(h))$ (for $h \in H_1$) has coefficients of the form $\chi(h)P(\text{val}(h))$, for $\chi$ in $X_1$, and $P$ a polynomial. Let $e = (e_1, \ldots, e_r) \in \mathcal{V}^r$, so that if $\overline{\phi} = \sum_{i=1}^{r} x_i \overline{e}_i$, the map $d(h) = \phi(h) - (x_1, \ldots, x_r)e(h)$ vanishes for $|h| \leq q^{-t}$ for some $t \geq 0$. Notice that for any $a \in H_1$, there is $n_a \in \mathbb{N}$, such that for any $l \in \{1, \ldots, r\}$, the map $\rho(a)e_l - \sum_k M(a)_{k,l}e_k$ vanishes on the set $|h| \leq q^{-n_a}$. We claim the following. (We will state and prove a more general version below.)

**Claim.** There is $z \in H^1_1$ such that $e(zh) = t(M(h)e(z)$ for all $h \in H^1_1$.

From this claim, we deduce that $e(h) = t(M(z^{-1})M(h)e(z)$ for $|h| \leq q^{-n'}$, hence if we set $N = \max(n', t)$, we obtain $\phi(h) = (x_1, \ldots, x_r)tM(z^{-1}h)e(z)$ for $|h| \leq q^{-N}$, hence

$$
\phi(h) = 1_{\{|h| \leq q^{-N}\}}(x_1, \ldots, x_r)tM(z^{-1})M(h)e(z) + 1_{\{|h| \leq q^{-N} \leq |h| \leq 1\}}\phi(h)
$$

for $h \in H^1_1$, which is of the desired form.

Now we do the induction step from $m-1$ to $m$. So let $\phi$ belong to $\mathcal{V}$, and $\overline{\phi}$ its image in $Q_m$. Again $\langle \rho(h)\phi \mid h \in H_m \rangle$ is finite dimensional with basis $\overline{B} = \{\overline{e}_1, \ldots, \overline{e}_r\}$, and we use the same notations $e_i$ and $e$ as above, so that $\overline{\phi} = \sum_{i=1}^{r} x_i \overline{e}_i$. The difference $d(h_1, \ldots, h_{m-1}, h) = \phi(h_1, \ldots, h_{m-1}, h) - (x_1, \ldots, x_r)e(h_1, \ldots, h_{m-1}, h)$ vanishes for all $|h_i| \leq 1$ and $|h| \leq q^{-t}$ for some $t \geq 0$. As before, for any $a \in H_m$, there is $n_a \in \mathbb{N}$, such that for any $l \in \{1, \ldots, r\}$, the map $\rho(a)e_l - \sum_k M(a)_{k,l}e_k$ vanishes on the set $\{|h| \leq 1, |h| \leq q^{-n_a}\}$.

To proceed we need the general form of the previous claim.

**Claim.** There is $z \in H^1_1$ such that

$$
e(h_1, \ldots, h_{m-1}, zh) = t(M(h)e(h_1, \ldots, h_{m-1}, z)
$$

for all $h_i \in H^1_i$ and $h \in H^1_m$.  

Proof of the Claim. Let $\varpi$ be a uniformizer of $F^\times$, and $U$ a compact open subgroup of $\mathfrak{O}^\times$ such that $(1, \ldots, 1, U)$ fixes $e$, as well as the representation $h \mapsto M(h)$ of $H$. Choose a set $u_1, \ldots, u_l$ of representatives of $\mathfrak{O}^\times/U$, and let $n' = \max(n_u, n_\varpi)$ and fix $z$ with $|z| = q^{-n'}$. If $|h| \leq 1$, we can write $h = \varpi^r u_i u$ for some $r \geq 0$, $i \in \{1, \ldots, l\}$, and $u \in U$. We then have, for $|h_i| \leq 1$, the equalities
\[
e(h_1, \ldots, h_{m-1}, zh) = e(h_1, \ldots, h_{m-1}, z\varpi^r u_i) = e(h_1, \ldots, h_{m-1}, z\varpi^r)
\]
because $|z\varpi^r| \leq |z| \leq q^{-n_u}$. If $r \geq 1$, we then have
\[
e(h_1, \ldots, h_{m-1}, z\varpi^r) = e(h_1, \ldots, h_{m-1}, z\varpi^{r-1})
\]
because $|z\varpi^{r-1}| \leq |z| \leq q^{-n_u}$. Repeating this last step as needed, we obtain the desired relation
\[
e(h_1, \ldots, h_{m-1}, zh) = e(h_1, \ldots, h_{m-1}, z) = e(h_1, \ldots, h_{m-1}, z).
\]

We now return to the proof of the lemma. Set $N = \max(n', t)$, so that for $|h| \leq q^{-N}$ and $h_i \leq 1$, we have
\[
e(h_1, \ldots, h_{m-1}, h) = e(h_1, \ldots, h_{m-1}, h).
\]
Writing $f(h_1, \ldots, h_{m-1}, h)$ for $(x_1, \ldots, x_r)M(z) M(h)$, we obtain for $h_i \in H^1_i$, an $h \in H^m$, $\phi(h_1, \ldots, h_{m-1}, h) = 1_{\{|h| \leq q^{-N}\}} f(h_1, \ldots, h_{m-1}, h) + 1_{\{|q^{-N}| \leq h| \leq 1\}} \phi(h_1, \ldots, h_{m-1}, h)$. Now, if we fix $y$ in $H^1_m$, and denote by $\mathcal{V}_y$ the space of functions on $H' = \prod_{i=1}^{m-1} H_i$ of the form $h' \mapsto \phi(h', y)$ for $\phi \in \mathcal{V}$, we can apply our induction hypothesis to this space, so any function $\phi_y$ in $\mathcal{V}_y$ is a sum of functions of the form $a \mapsto \prod_{i=1}^{m-1} \chi_i(a_i) val(a_i)^m \varphi(a)$, for $\chi_i \in X_i$, $m_i \in \mathbb{N}$, and $\varphi$ a Schwartz function on $\mathfrak{O}^{m-1}$. As $z$ belongs to $H^1_m$, and $h' \mapsto e_i(h', z)$ belongs to $\mathcal{V}_z$ for $i \in \{1, \ldots, r\}$, we deduce that the map $1_{\{|h| \leq q^{-N}\}} f(h_1, \ldots, h_{m-1}, h)$ is of the desired form on $H^1$. It remains to show that the same is true for the map $1_{\{|q^{-N}| \leq h| \leq 1\}} \phi(h_1, \ldots, h_{m-1}, h)$ on $H^1$. However, taking $U$ an open subgroup of $\mathfrak{O}^\times$ such that $(1, \ldots, 1, U)$ fixes $\phi$, and representatives $(z_\alpha)$ of $\{q^{-N} \leq |h| \leq 1\}/U$, we can write
\[
1_{\{|q^{-N}| \leq h| \leq 1\}} \phi(h_1, \ldots, h_{m-1}, h) = \sum_{\alpha} 1_{z_\alpha U} (h) \phi(h_1, \ldots, h_{m-1}, z_\alpha)
\]
and we conclude by the induction hypothesis again, that $1_{\{|q^{-N}| \leq h| \leq 1\}} (\phi_1, \ldots, h_{m-1}, h)$ is of the desired form as well, which concludes the proof. \hfill \Box

Let $s_0$ be a real number strictly greater than $r_\Pi$. Given $L \in Hom_{P_u \cap S_u}(\Pi, \Theta)$ and $v \in V_\Pi$, set
\[
\Gamma_L(v) = I(S_{L,v}, s_0).
\]
This now converges and, as noted above, a simple change of variables in the integral defining $I(S_{L,v}, s_0)$ gives that $\Gamma_L \in Hom_{P_u \cap L_u}(\Pi, \chi_{s_0})$ where $\chi_{s_0} \left(\begin{array}{cc} g_1 & u \\ g_2 & 1 \end{array}\right) = \left(\frac{|g_1|}{|g_2|}\right)^{-s_0}$. 

**Proposition 3.4.** Suppose \( s_0 \) is a real number greater than \( r_1 \). The map \( L \mapsto \Gamma_L \) gives a linear injection of \( \text{Hom}_{P_n \cap L_n}(\Pi, \Theta) \) into the space \( \text{Hom}_{P_n \cap L_n}(\Pi, \chi_{s_0}) \).

**Proof.** We only need to check that if \( \Gamma_L \) is zero, then so is \( L \). So we suppose that \( \Gamma_L \) is zero. Consider \( \Phi(y) \) the function on \( M_m \), equal to \( \mathcal{S}_{L,v}(\text{diag}(., I_{m+1}))|.|^{s_0-m} \) on \( G_m \), and to zero outside \( G_m \). Then \( \Phi(y) \) is \( L^1 \) for a Haar measure on \( M_m \), because \( \mathcal{S}_{L,v}(\text{diag}(g, I_{m+1}))|g|^{s_0} \) is \( L^1 \) for a Haar measure on \( G_m \) and \( M_m - G_m \) is of measure zero in \( M_m \), and we have the equality

\[
\int_{M_m} \Phi(y)dy = \int_{G_m} \mathcal{S}_{L,v}(\text{diag}(g, I_{m+1}))|g|^{s_0} dg.
\]

More generally, for any \( x \) in \( M_m \), we have the equalities of absolutely convergent integrals:

\[
\int_{G_m} S_{L,v} \left( \begin{array}{ccc} I_m & x & 1 \\ \cdot & I_m & \cdot \\ \cdot & \cdot & \cdot \end{array} \right)^v \mathcal{S}_{L,v}(\text{diag}(g, I_{m+1}))|g|^{s_0} dg = \int_{G_m} \theta(Tr(gx)) \mathcal{S}_{L,v}(\text{diag}(g, I_{m+1}))|g|^{s_0} dg
\]

\[
= \int_{M_m} \theta(Tr(xy))\Phi(y)dy = \Phi(x).
\]

But \( \Gamma_L \) being zero implies that the first integral in this series of equality is zero, hence \( \Phi \)'s Fourier transform on \( M_m \) is zero. In particular, \( \Phi \) is zero almost everywhere on \( M_m \), but as it is continuous on \( G_m \), it must be zero on \( G_m \). This implies that \( \mathcal{S}_{L,v}(I_n) = L(v) \) is zero for every \( v \), i.e. that \( L \) is zero.

From here, we get the following multiplicity one result that is the key to proving the local functional equation.

**Proposition 3.5.** For almost all \( s \), the space \( \text{Hom}_{S_n}(\mathcal{W}(\pi, \theta) \otimes C_c(\mathbb{F}^m), \cdot, |.|^{-s/2}) \) is of dimension at most 1.

**Proof.** We again let \( \Pi \) denote \( \mathcal{W}(\pi, \theta) \). Set \( \chi = \chi_{s_0} \) as in Proposition 3.4. We first prove that for all values of \( q^{-s} \), except possibly a finite number, we have \( \dim(\text{Hom}_{P_n \cap L_n}(\Pi, \chi, \cdot, \cdot^{s})) \leq 1 \). We can replace \( P_n \cap L_n \) by \( P_n \cap H_n \) in the statement we wish to prove, as both are conjugate in \( G_n \) (and actually in \( P_n \)). Then according to Section 3.5. of [2], the restriction of \( \Pi \) to \( P_n \) has a filtration by derivatives with each successive quotient of the form \((\Phi^+)^{n-k-1}\Psi^+(\tau)\) for \( k \leq n - 1 \) and \( \tau = \pi^{(n-k)} \) a representation of \( G_k \), the \((n-k)\)th derivative of \( \pi \). Since the functors \( \Phi^+ \) and \( \Psi^+ \) are exact, we can replace each \( \tau \) with its composition series (it is of finite length) and assume a filtration with successive quotients of the form \((\Phi^+)^{n-k-1}\Psi^+(\tau)\) with \( \tau \) irreducible. For every irreducible representation \( \tau \) of \( G_k \), for \( k \geq 1 \), from Corollary 2.1 we deduce that

\[
\text{Hom}_{P_n \cap H_n}(\cdot, \cdot, \cdot, \cdot^{s}) = \text{Hom}_{H_n}(\tau, \chi_{s_0}^{k}, |.|^{s})
\]

and this last space is zero except for a finite number of \( q^{-s} \) as \( \tau \) has a central character. For all other values of \( q^{-s} \), we deduce that the functional must be non-zero on the bottom piece of the Bernstein-Zelevinsky filtration which is \((\Phi^+)^{n-1}\Psi^+(1)\). Thus for all but finitely many values of \( q^{-s} \) we have

\[
\dim(\text{Hom}_{P_n \cap H_n}(\Pi, \chi, \cdot, \cdot^{s})) \leq \dim(\text{Hom}_{P_n \cap H_n}(\cdot, \cdot, \cdot, \cdot^{s})) + \dim(\text{Hom}_{P_n \cap H_n}(\cdot, \cdot, \cdot, \cdot^{s}))
\]
Again by Corollary 2.1 we have $Hom_{P_n \cap H_n}((\Phi^+)^{n-1} \Psi^+(1), \chi|.|^s) \simeq Hom_{H_n}(1, \chi \mu_n'|.|^s)$ which has dimension one. Hence this proves our assertion about $\dim(Hom_{P_n \cap L_n}(\Pi, \chi|.|^s))$.

Proposition 3.4 then implies that

$$\dim(Hom_{P_n \cap S_n}(\Pi, |.|^s \Theta)) \leq 1$$

for all values of $q^{-s}$ except a finite number. Now, we have the following series of isomorphisms:

$$Hom_{S_n}(\Pi \otimes C_c^\infty(F^m), |.|^{-s/2}) \simeq Hom_{S_n}(\Pi \otimes ind_{P_n \cap S_n}^G(\Theta^{-1}), |.|^{-s/2})$$

$$\simeq Hom_{S_n}(\Pi, Ind_{S_n \cap P_n}(|.|^{-s+1/2} \Theta))$$

$$\simeq Hom_{S_n}(\Pi, |.|^{-s+1/2} \Theta),$$

the last isomorphism by Frobenius reciprocity. Hence for all but finitely many values of $q^{-s}$ the space $Hom_{S_n}(\Pi \otimes C_c^\infty(F^m), |.|^{-s/2})$ is of dimension at most one. □

This has as a consequence the functional equation of the exterior-square $L$-function in the odd case.

Theorem 3.1. Let $\pi$ be an irreducible representation of $G_n$. There exists an invertible element $\epsilon(s, \pi, \lambda^2, \theta)$ of $\mathbb{C}[q^{\pm s}]$, such that for every $W$ in $\mathcal{W}(\pi, \theta)$, one has the following functional equation:

$$\epsilon(s, \pi, \lambda^2, \theta) \frac{J_\theta(s, W, \phi)}{L(s, \pi, \lambda^2)} = \frac{J_{\theta-1}(1 - s, \rho(\tau)\hat{W}, \hat{\phi})}{L(1 - s, \pi^\vee, \lambda^2)}.$$

Proof. As the bilinear forms $C_{s, \pi, \theta}$ and $B_{s, \pi, \theta}$ defined in Lemmas 3.3 and 3.4 belong to $Hom_{S_n}(\mathcal{W}(\pi, \theta) \otimes C_c^\infty(F^m), |.|^{-s/2})$, Proposition 3.5 gives the existence of $\epsilon(s, \pi, \lambda^2, \theta)$ in $\mathbb{C}(q^{-s})$ as in the statement.

As the integrals $J_{\theta-1}(1 - s, \hat{W}, \hat{\phi})$ span the fractional ideal $L(1 - s, \pi^\vee, \lambda^2)\mathbb{C}[q^{\pm s}]$, one can always find a finite set of Whittaker functions $W_i$, and Schwartz functions $\phi_i$ satisfying

$$\sum_i J_{\theta-1}(1 - s, \rho(\tau)\hat{W}_i, \hat{\phi}_i) = L(1 - s, \pi^\vee, \lambda^2) \in \mathbb{C}[q^{-s}].$$

Therefore, for this choice of $\{(W_i, \phi_i)\}$ we have

$$\epsilon(s, \pi, \lambda^2, \theta) \frac{\sum_i J_\theta(s, W_i, \phi_i)}{L(s, \pi, \lambda^2)} = 1,$$

and the factor $\epsilon(s, \pi, \lambda^2, \theta)$ is nonzero in $\mathbb{C}(q^{-s})$, with $\epsilon(s, \pi, \lambda^2, \theta)^{-1} \in \mathbb{C}[q^{\pm s}]$. Now, there is also a choice of a finite set of Whittaker functions $W_j$, and Schwartz functions $\phi_j$ satisfying

$$\sum_j J_\theta(s, W_j, \phi_j) = L(s, \pi, \lambda^2),$$

hence

$$\epsilon(s, \pi, \lambda^2, \theta) = \frac{\sum_j J_{\theta-1}(1 - s, \rho(\tau)\hat{W}_j, \hat{\phi}_j)}{L(1 - s, \pi^\vee, \lambda^2)} \in \mathbb{C}[q^{\pm s}],$$

thus it $\epsilon(s, \pi, \lambda^2, \theta)$ is a unit in $\mathbb{C}[q^{\pm s}]$. □
3.4. Remarks on the local functional equation in the even case. In Theorem 4.1 of [14], the second author proves the functional equation for generic irreducible representations of $G_n$, with $n = 2m$ even. To generalize to any irreducible admissible representation, one must argue with the Whittaker model $W(\pi, \theta)$ as we do above. The proof uses Proposition 4.3 of the same paper, hence one needs to check that the irreducible representation $\pi$ of the statement of this Proposition can safely be replaced by the Whittaker model $W(\pi, \theta)$. Looking at the proof of this Proposition, we see that we need to extend Proposition 4.2 of [14], and its immediate Corollary 4.1, to $W(\pi, \theta)$. Proposition 4.2 is itself a consequence of Proposition 4.1, so that the only point is to extend Proposition 4.1 of [14] from an irreducible generic representation to a representation of the form $\Pi = W(\pi, \theta) \simeq \Delta_1 \times \cdots \times \Delta_r$, which is parabolically induced from irreducible generic representations $\Delta_i$ of smaller linear groups. Setting $L(s, \Pi) = \prod_i L(s, \Delta_i)$, the statement of Proposition 4.1 (and hence of Propositions 4.2 and 4.3) is still true for $\Pi$ thanks to Section 3 of Godement-Jacquet’s first chapter [5]. All the other arguments in the proof of Theorem 4.1 are valid for any irreducible representation, just as in the proof of Theorem 3.1 above. There is however one point, namely that the $\epsilon$ factor is a unit, the proof of which is not correct in Theorem 4.1 of [14]. One just needs to modify this bit as in the proof of Theorem 3.1 above.

The conclusion of this discussion is that the local functional equation of the exterior square $L$-function is now available for irreducible representations of $G_n$, for any $n$.

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