Capillary resonator as optical sensor for the measurement of the hydrostatic pressure. A theoretical study of its sensitivity

D Avila¹, S Horta¹, and C O Torres¹
¹ Laboratorio de Óptica e Informática, Universidad Popular del Cesar, Valledupar, Colombia

E-mail: duberavilap@unicesar.edu.co

Abstract. We report the study of an optical device for the measurement of hydrostatic pressure in fluids. The device studied is a sensor based on a dielectric optical resonator in the form of a capillary that confines the light in its interior through the phenomenon of total internal reflection. For the analytical study of sensitivity, we have considered the solution of the Helmholtz scalar equation for the case of a resonant cavity in form of cylinder composed of three layers with different refractive index. During the study, the excitation of the Whispering Gallery Modes WGMs on a transverse plane along the axial axis of the cavity is studied and the equation of eigenvalues that has information of the wavelengths of resonances for the TE and TM modes were obtained from the resonance conditions. To determine the sensitivity of the device, the shift of the wavelengths of resonances as a function of the internal pressure in the cavity were analyzed. The results show that it is possible to increase the sensitivity of the microresonator when the wall thickness of the cavity is thin, and the maximum diameter is increased.

1. Introduction

Optical microresonators are optical devices that are indispensable in circuits of integrated optics, which have applications in different areas of science and engineering. These optical microresonators have been manufactured with different morphologies in the form of capillaries [1-4], discs [5], rings [6-8], toroids [9], spheres [10], bubbles [11], bottles [12], etc, using different materials such as silicon, silica, polymers, with applications in areas such as biology [13], medicine [4], physics [14], chemistry [10] and specifically in the area of sensors for the measurement of temperature [15], humidity [12], refractive index [16] and some other physical variables of interest [6]. Specifically, the cylindrical optical microcavities that have been made using different types of materials, experience a series of resonances commonly known as Whispering Gallery modes WGMs, which are characteristic of the cavities that have rotational symmetry and can be explained through the phenomenon of total internal reflection occurring within these cavities when they are excited through an external source. Recently a series of studies of microcapillary resonators have been reported by its facility to interact simultaneously with two fluids of a different refractive index since they allow the flow of a substance in its interior with the facility. For the case of silica and PMMA polymer microcapillaries, a Q factor of the order of $1 \times 10^4$ has been reported [1,17]. In this research, we have analyzed the sensitivity of a capillary resonator for the measurement of hydrostatic pressure of microfluidics, analyzing the propagation of WGMs modes and the spectral shifts in the resonance wavelengths of the cavity in response to changes in hydrostatic pressure within the resonator.
2. Materials and methods
To perform the theoretical study of the sensitivity of a capillary microresonator for the measurement of hydrostatic pressure in microfluidics, the cavity is considered as a system composed of three layers with different refractive indexes and the WGMs modes are excited in the intermediate layer to the interior of the structure because in that region the light experiences the phenomenon of total internal reflection between the inner and outer interface of the microcavity. For the theoretical analysis, it is possible to determine the wave equation in the microcapillary and in this way it is possible to obtain the wavelengths of resonances and the field distribution of the corresponding WGMs modes with different radial and azimuthal orders. For this, it is necessary to choose a cylindrical symmetry as shown in Figure 1.

![Image](image-url)

**Figure 1.** Dielectric capillary scheme. Lateral image of the capillary and Cross-sectional view of the capillary. $n_1, n_2, n_3$ constitute respectively the indices of refraction in the media 1, 2 and 3, while $r$ and $z$ represent the external radius and $b$ is the internal radius. Outline of the elastic-optical effect inside the capillary by the action of the internal pressurization.

In the cavity, the confinement of the field is given in the material with an index of refraction $n_2$ which constitutes the wall of the capillary and to ensure the phenomenon of internal total reflection must be satisfied that $n_2 > n_3$ for the case of a cylindrical microcavity with thin wall thickness.

3. Results and discussion
In the analysis of the polarization of TE modes, the electric field is axial to the axis of the capillary and in this case, we can separate variables for the field as a function of the radial and azimuthal component $E_z = (r, \phi) = E_z(r) \exp(\pm il\phi)$, where $r, \phi, z$ represent the radial, azimuthal and axial coordinates respectively and $l$ is an integer constant called the azimuth order number. According to Maxwell's equations, the resonant modes TE of the cylindrical dielectric cavity satisfy the differential equation in cylindrical coordinates given by Equation (1).

$$\nabla^2 E_z(r, \phi) + \frac{\partial}{\partial z} \left[ E_z(r, \phi) \frac{1}{\varepsilon} \frac{\partial E_z(r, \phi)}{\partial z} \right] + k^2 n^2 E_z(r, \phi) = 0$$

(1)

where $\varepsilon$ is the electrical permittivity of the material in which the field is propagating, $k_0$ is the wave number in the vacuum, $n$ is the refractive index and $E_z$ is the electric field polarized along the z-axis. In the case of a composite material with different layers of different refractive index, the refractive index $n$ is $n_1$ when $r \leq b$, $n_2$ when $b < r \leq a$ and $n_3$ when $r > a$, the wall thickness of the capillary is $g = a - b$. The solution for the field given by Equation (1) can be expressed as the Equation (2),

$$E_{z1} = A_j n_1 k_0 r$$
$$E_{z2} = B_j n_2 k_0 r + C_j n_1 k_0 r$$
$$E_{z3} = D_j n_3 k_0 r$$

(2)
In Equation (2), $A_i, B_i, C_i, D_i$ are complex constants, $J_0(r)$ and $Y_1(r)$ are respectively the cylindrical Bessel functions of first and second type with order 1, the function $H_1^{(1)}(r) = J_1(r) + iY_1(r)$ is the first type Hankel function. The solution of the wave equation is expressed in general as $E_z = E_{z1}\exp(il\theta)$ with $E_{z1} = E_{z2}, E_{z2}, E_{z3}$. For the study of TE modes, it is necessary to guarantee the continuity conditions of the field at the boundary at each dielectric interface, therefore some components of the electric and magnetic fields must be continuous. With the application of boundary conditions, it is possible to obtain the eigenvalue equation to determine the resonance wavelengths in the cylindrical cavity for TE modes as a function of the refractive indexes of the medium and the internal and external diameter of the microcavity, which was deduced, and its mathematical expression is observed in Equation (3),

$$
\frac{J_1(n_2k_br)Y_1(n_2k_ar) - Y_1(n_2k_ar)J_1(n_2k_br)}{J_1(n_2k_br)[J_1(n_2k_ar) - J_1(n_2k_ar)]} = \frac{Y_1(n_2k_ar)H_1^{(1)}(n_2k_ar) - H_1^{(1)}(n_2k_ar)Y_1(n_2k_ar)}{J_1(n_2k_br)[H_1^{(1)}(n_2k_ar) - J_1(n_2k_ar)]} (3)
$$

In general, the wave number $k_i$ is a complex number given by $k_i = 2\pi n_i/\lambda$ with $i = 1, 2, 3$. The eigenvalue equation for TE modes requires the use of numerical techniques for the determination of complex roots in order to determine the different wavelengths of resonances. We can consider that the optical resonances inside the microcapillary must satisfy a resonance condition $m\lambda_{eff} = n_{eff}(2\pi r)$, where $m$ is the modal order number, $n_{eff}$ is the effective refractive index of the mode and $r$ represents the average radius of the circle formed by the optical path length of the ray through the cross-section of the capillary. In the case of TM modes, the magnetic field is along the axial axis of the cylinder, in this case, it is assumed that $\epsilon$ does not present azimuthal dependence and so, the differential equation that solves the problem is observed in Equation (4),

$$
\nabla^2H_z(r, \theta) - \frac{1}{\varepsilon}\frac{\partial}{\partial r}\left(\frac{\partial H_z(r, \theta)}{\partial r}\right) + k_0^2 n^2 H_z(r, \theta) = 0 (4)
$$

The solution of the differential Equation (4) is again a set of Bessel functions for the magnetic field in each of the regions with different refractive index given by Equation (5),

$$
H_z(mk_br) = \begin{cases} 
a_i J_1(n_2k_ar) & r \leq b 
b_i J_1(n_2k_ar) + c_i Y_1(n_2k_ar) & b < r \leq a 
d_1 H_1^{(1)}(n_2k_ar) & r > a 
\end{cases} (5)
$$

In the same way that the equation of eigenvalues for the TE modes was obtained in Equation (3), the boundary conditions were applied in each of the interfaces, guaranteeing the continuity of the fields between the different media obtaining the expression of eigenvalues in the Equation (6),

$$
\frac{n_2^2 J_1(n_2k_ar)Y_1(n_2k_ar) - n_2^2 J_1(n_2k_ar)Y_1(n_2k_ar)}{n_2^2 J_1(n_2k_ar)Y_1(n_2k_ar) - n_2^2 J_1(n_2k_ar)Y_1(n_2k_ar)} = \frac{n_2^2 H_1^{(1)}(n_2k_ar)Y_1(n_2k_ar) - H_1^{(1)}(n_2k_ar)Y_1(n_2k_ar)}{n_2^2 H_1^{(1)}(n_2k_ar)H_1^{(1)}(n_2k_ar) - H_1^{(1)}(n_2k_ar)H_1^{(1)}(n_2k_ar)} (6)
$$

In Figure 1, a diagram of a capillary is observed which is pressurized in its interior with a fluid at a constant pressure $p$. This pressurization creates radial forces that are uniformly distributed to the interior of the microcapillary, so that when the capillary is pressurized, a uniform positive circumferential displacement will occur at the inner and outer radius. It is possible to determine the circumferential displacement of the radius in terms of the characteristics of the material by knowing the elastic-optical phenomenon experienced by the capillary. In Figure 1, $u(b, p)$ is the circumferential displacement of the internal radius shown in the discontinuous curve and $u(a, p)$ is defined as the circumferential
displacement of the external radius, also observed in the external discontinuous curve, \( g \) represents the thickness of the capillary. In order to solve the equation of eigenvalues Equation (3) and Equation (6) of TE and TM modes which are excited to the interior of the cavity, it is necessary to determine the circumferential displacements of the internal and external radius in the microcapillary as a consequence of the pressurization. In this way, the internal and external radius of the microcapillary is given by the expression \( b(p) = b_0 + u(b, p) \) and \( a(p) = a_0 + u(a, p) \) respectively, where \( b_0 \) and \( a_0 \) are the internal and external radios without pressurizing.

During the computational simulation to determine the roots of the eigenvalue equations of the WGMs modes, it is necessary to set some capillary parameters, such as the material that compos the microcavity and for this we have chosen the material PMMA (Polymethyl-methacrylate) of which some experimental results have been reported [17]. For the PMMA, the circumferential displacement when the capillary is pressurized can be determined from the theory of elasticity assuming a linear behavior according to Hooke's law given by Equation (7) and Equation (8).

\[
\begin{align*}
    u(b, p) &= u(r, p)_{r=b} = \frac{pb}{E} \left( \frac{a^2 + b^2}{a^2 - b^2} + v \right) \\
    u(a, p) &= u(r, p)_{r=a} = \frac{2pb^2}{E(a^2 - b^2)}
\end{align*}
\]

where, \( r \) is the radial distance measured from the axial, \( E \) is the Young module and \( v \) is the Poisson coefficient. In the same way, the theoretical sensitivity can be determined for different azimuth and radial order numbers for the different TE and TM modes. In the analysis of the results, was possible to numerically determine the sensitivity for microcapillaries with different geometric parameters and were compared with some experimental results obtained by some authors [17]. In Table 1, the theoretical sensitivity determined numerically for three different microcapillaries with different geometric parameters for TE and TM modes is observed and different azimuth order numbers. In the results of the Table 1, the geometrical parameters of the PMMA capillary optical micro-resonators that were simulated, have been chosen according to the geometric characteristics of some capillary tubes that have been manufactured by some authors and whose results have been reported in the literature [17]. The numerical simulation for the calculation of sensitivity, the following parameters were taken into account: \( n_1=1.000299, n_2=1.49, n_3=1.0002924, m=2000, \nu=0.37 \) and \( E=2.2\times10^8 \text{ Pa} \).

**Table 1.** Theoretical sensitivity determined numerically for three different microcapillaries with different geometric parameters for TE and TM modes.

| Capillaries | Experimental sensitivity (nm/Bar) [17] | Internal radius (µm) | Thickness (µm) | Theoretical sensitivity (nm/Bar). |
|-------------|--------------------------------------|----------------------|---------------|---------------------------------|
|             |                                      |                      |               | TE Mode                        |
|             |                                      |                      |               | 0.165 (l=1)                    |
| No 1        | 0.147                                | 296.70               | 58.88         | 0.159 (l=1)                    |
|             |                                      |                      |               | 0.154 (l=3)                    |
|             |                                      |                      |               | 0.163 (l=3)                    |
|             |                                      |                      |               | 0.203 (l=10)                   |
|             |                                      |                      |               | 0.158 (l=10)                   |
|             |                                      |                      |               | 0.156 (l=50)                   |
|             |                                      |                      |               | 0.158 (l=50)                   |
| No 2        | 0.024                                | 71.30                | 82.82         | 0.020 (l=1)                    |
|             |                                      |                      |               | 0.019 (l=1)                    |
|             |                                      |                      |               | 0.019 (l=3)                    |
|             |                                      |                      |               | 0.019 (l=10)                   |
|             |                                      |                      |               | 0.019 (l=10)                   |
|             |                                      |                      |               | 0.018 (l=50)                   |
|             |                                      |                      |               | 0.019 (l=50)                   |
| No 3        | 0.351                                | 832.96               | 79.93         | 0.3466 (l=1)                   |
|             |                                      |                      |               | 0.3207 (l=3)                   |
|             |                                      |                      |               | 0.3659 (l=3)                   |
|             |                                      |                      |               | 0.3601                         |
|             |                                      |                      |               | 0.3584 (l=10)                  |
|             |                                      |                      |               | 0.3417 (l=50)                  |
|             |                                      |                      |               | 0.3427 (l=50)                  |
In Figure 2, shows the numerical and experimental results of the sensitivity for capillary No 1. During the simulation, it was possible to study capillaries with different geometric parameters.

![Figure 2](image2.png)

**Figure 2.** Numerically obtained sensitivity of capillary No. 1 with a wall thickness of 58.88x10^{-6} m for different modal orders. TE and TM Modes.

These results show that TE modes in this type of cavities may experience better sensitivity levels than TM modes, therefore it is necessary to use experimental techniques that allow to control the states of polarization for the modes TM. In the same way, although the theoretical model does not allow to determine the spectrum obtained at the exit of the capillary resonator, it is possible to verify that the sensitivity obtained experimentally presents a good approximation with the numerical results for the modes TE and TM. On the other hand, during the theoretical analysis, it was determined that the Q factor and the losses associated to the dispersion phenomena and the light attenuation in the cavity are not important factors in the measurement applications because the sensitivity of the device is determined from the spectral shifts of resonance wavelengths. During the theoretical development, it was possible to develop a computational simulation using the finite element method which allows observing the coupling of the field to the interior of the cylindrical optical cavity using the program COMSOL Multiphysics 5.1, as seen in the 3D image of Figure 3.

![Figure 3](image3.png)

**Figure 3.** Optical field coupled to the interior to the cavity cylindrical of 16x10^{-6} m of external diameter through a cylindrical waveguide of 16x10^{-6} m of external diameter. Light enters from the bottom in the graph.

During the simulation, a cylindrical waveguide with an external diameter of the order of 2x10^{-6} m couples the field into a cylindrical microcavity with an outer diameter of 16x10^{-6} m, the refractive index of the core of the cylindrical waveguide is 1.52, while the refractive index of the cladding is 1.5.
4. Conclusions
The measurement of the sensitivity of an optical sensor is an important parameter because it determines the ability of the device to respond to the variation of some external parameter, in this sense, of the results obtained in the development of the present investigation. It is shown that micro-resonators manufactured in the form of capillary tubes from certain polymers, can experience a good sensitivity for the measurement of hydrostatic pressure changes inside the resonant cavity and can improve the sensitivity reported by some authors in systems where it is required to measure the hydrostatic pressure of small amounts of a fluid. The sensitivity of the device depends on the geometrical parameters of the structure, such as the external, internal diameter and the wall thickness of the cavity.

Acknowledgements
The authors thank the contribution of the LaFe research group of the University of Campinas and Professor Cristiano M. B. Cordeiro for the cooperation provided for the development of the experimental results.

References
[1] Zamora V, Diez A, Andries M V and Gimeno B 2009 Chemical sensor applications of whispering-gallery modes resonances of thin capillaries with submicrometric wall Proc. SPIE Optical Sensors (Prague) vol 7356 (United States: SPIE Digital Library) pp 73560Z 1-8
[2] Zamora V, Diez A, Andries MV and Gimeno B 2011 Cylindrical optical microcavities: Basic properties and sensor applications Photonics and Nanostructures-Fundamentals and Applications 9(2) 149
[3] Avila D A 2014 Polymeric Capillary Optical Resonator Sensors Latin America Optics and Photonics Conference LAOP (Cancun, Mexico) (Washington: Optical Society of America) p LTu4A 38
[4] Ling T, Majd S, Mayer M and Guo L 2008 Detection and quantification of lipid membrane binding on silica microtube resonator sensor Proc. SPIE Single Molecule Spectroscopy and Imaging (San Jose, California) vol 6862 (United States: SPIE Digital Library) pp 68620B 1-8
[5] Grist S M, Schmidt S A, Flueckiger J, Donzella V, Shi W, Talebi F, Kirk J T, Ratner D M, Cheung K C and Chrostowski L 2013 Silicon photonic micro-disk resonators for label-free biosensing Optics Express 21(7) 7994
[6] Orghici R, Lützow P, Burgmeier J, Koch J, Heidrich H, Schade W, Welschhoff N and Waldvogel S 2010 A microring resonator sensor for sensitive detection of 1,3,5-trinitrotoluene (TNT) Sensors 10(7) 6788
[7] Gouveia M A, Pellegrini P, Dos Santos J, Raimundo I and Cordeiro C 2010 Analysis of immersed silica optical microfiber knot resonator and its application as a moisture sensor Applied Optics 53(31) 7454
[8] Kiyat I, Aydilini A and Dagli N 2005 High-Q silicon-on-insulator optical rib waveguide racetrack resonators Optics Express 13(6) 1900
[9] Zhang X, Choi H S and Armani A M 2010 Ultimate quality factor of silica microtoroid resonant cavities Applied Physics Letters 96(15) 153304
[10] Ren H, Vollmer F, Arnold S and Libchaber A 2007 High-Q microsphere biosensor - analysis for adsorption of rodlike bacteria Optics Express 15(25) 17410
[11] Chen Z, Li M, Wu X, Liu L and Xu L 2015 2-D optical/opto-mechanical microfluidic sensing with micro-bubble resonators Optics Express 23(14) 17659
[12] Avila D A, Horta S D and Torres C O 2017 PMMA Solid bottle optical microresonator for measure relative humidity Journal of Physics: Conference Series 792 012059 1-6
[13] Zhu H, White I, Suter J D, Dale P and Fan X 2007 Analysis of biomolecule detection with optofluidic ring resonator sensors Optics Express 15(15) 9139
[14] Avila D A, Torres C O and Cordeiro M C 2017 Sensitivity of a PMMA polymer capillary microresonator for measuring relative humidity Journal of Physics: Conference Series 792 012050 1-7
[15] Rahman A 2011 Temperature sensor based on dielectric optical microresonator Optical Fiber Technology 17(6) 536-540
[16] Calixto S, Aguilar R M, Monzon H D and Minkovich V P 2008 Capillary refractometer integrated in a microfluidic configuration Applied Optics 47(6) 843-848
[17] Gouveia M A, Avila D, Marques T, Torres M and Cordeiro M 2015 Morphology dependent polymeric capillary optical resonator hydrostatic pressure sensor Optics Express 23(8) 10643-10652