A NUMERICAL METHOD FOR COMPUTING RADially SYMMETRIC SOLUTIONS OF A DISSIPATIVE NONLINEAR MODIFIED KLEIN-GORDON EQUATION

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Abstract. In this paper we develop a finite-difference scheme to approximate radially symmetric solutions of the initial-value problem with smooth initial conditions

\[
\frac{\partial^2 w}{\partial t^2} - \nabla^2 w - \beta \frac{\partial}{\partial t} (\nabla^2 w) + \gamma \frac{\partial w}{\partial t} + m^2 w + G'(w) = 0
\]

subject to:

\[
\begin{align*}
\begin{cases}
w(x, 0) = \phi(x), & x \in D \\
\frac{\partial w}{\partial t}(x, 0) = \psi(x), & x \in D
\end{cases}
\end{align*}
\]

in an open sphere \(D\) around the origin, where the internal and external damping coefficients—\(\beta\) and \(\gamma\), respectively—are constant, and the nonlinear term has the form \(G'(w) = w^p\), with \(p > 1\) an odd number. The functions \(\phi\) and \(\psi\) are radially symmetric in \(D\), and \(\phi, \psi, r\phi\) and \(r\psi\) are assumed to be small at infinity. We prove that our scheme is consistent order \(O(\Delta t^2) + O(\Delta r^2)\) for \(G'\) identically equal to zero, and provide a necessary condition for it to be stable order \(n\). Part of our study will be devoted to compare the physical effects of \(\beta\) and \(\gamma\).

1. Introduction

Klein-Gordon-like equations appear in several branches of modern physics. A modified sine-Gordon equation appears for instance in the study of long Josephson junctions between superconductors when dissipative effects are taken into account [1]. A similar partial differential equation with different nonlinear term appears in the study of fluxons in Josephson transmission lines [2]. A modified Klein-Gordon equation appears in the statistical mechanics of nonlinear coherent structures such as solitary waves in the form of a Langevin equation (see [3] pp. 298–309); here no internal damping coefficient appears, though. Finally, our differential equation describes the motion of a damped string in a non-Hookean medium.

The classical (1 + 1)-dimensional linear Klein-Gordon equation has an exact soliton-like solution in the form of a traveling wave [4]. Some results concerning the analytic behavior of solutions of nonlinear Klein-Gordon equations have been established [5] [6] [7]; however, no exact method of solution is known for arbitrary
initial-value problems involving this equation. From that point of view it is important to investigate numerical techniques to describe the evolution of radially symmetric solutions of (1).

It is worth mentioning that some numerical research has been done in this direction. Strauss and Vázquez [8] developed a finite-difference scheme to approximate radially symmetric solutions of the nonlinear Klein-Gordon equation for the same nonlinear term we study in this paper; one of the most important features of their numerical method was that the discrete energy associated with the differential equation is conserved. The numerical study of the sine-Gordon model that describes the Josephson tunnel junctions has been undertaken by Lomdahl et al. [2]. Numerical simulations have also been performed to solve the (1 + 1)-dimensional Langevin equation [9].

In this paper we extend Strauss and Vázquez’s technique to include the effects of both internal and external damping, and validate our results against those in [8]. Section 2 is devoted to setting up the finite-difference scheme; the energy analysis of our problem is also carried out. Numerical results are presented in Section 3, followed by a brief discussion.

2. Analysis

Analytical results. The following is the major theoretic result we will use in our investigation. Here $M(t)$ represents the amplitude of a solution of (1) at time $t$, that is

$$M(t) = \max_x |w(x,t)|.$$  

**Theorem 1.** Let $\beta$ and $\gamma$ be both equal to zero, and let $G'(w) = |w|^{p-1}w$. Suppose that $\phi$ and $\psi$ are smooth and small at infinity. Then

1. If $p < 5$, a unique smooth solution of (1) exists with amplitude bounded at all time [6].
2. If $p \geq 5$, a weak solution exists for all time [11].
3. For $p > 8/3$ and for solutions of bounded amplitude, there is a scattering theory; in particular, they decay uniformly as fast as $M(t) \leq c(1 + |t|)^{-3/2}$.

Finite-difference scheme. Throughout this section we will assume that the functions $\phi(\bar{x})$ and $\psi(\bar{x})$ are smooth, of compact support, radially symmetric in the open sphere $D$ with center in the origin and radius $L$, and that $\phi, \psi, r\phi$ and $r\psi$ are small at infinity in $D$. Moreover, we will suppose that $w(\bar{x},t)$ is a radially symmetric solution of (1).

Let $r = ||\bar{x}||$ be the Euclidean norm of $\bar{x}$ and let $G'(w) = w^p$, for $p > 1$ an odd number. Setting $v(r,t) = rw(r,t)$ for every $0 < r < L$ and $t \in \mathbb{R}$, it is evident that $v$ must satisfy the mixed-value problem

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial r^2} + \frac{\partial v}{\partial t} - \beta \frac{\partial^3 v}{\partial t \partial r^2} + m^2 v + r G'(v/r) = 0$$

subject to:

$$\begin{cases} v(r,0) = r\phi(r), & 0 < r < L \\ \frac{\partial v}{\partial t}(r,0) = r\psi(r), & 0 < r < L \\ v(0,t) = 0, & t \geq 0. \end{cases}$$


Proceeding now to discretize our problem, let \( a < L \) be a positive number with the property that \( \phi \) and \( \psi \) vanish outside of the sphere with center in the origin and radius \( a - \epsilon \), for some \( \epsilon > 0 \). Let \( 0 = r_0 < r_1 < \cdots < r_M = a \) and \( 0 = t_0 < t_1 < \cdots < t_N = T \) be partitions of \([0,a]\) and \([0,T]\), respectively, into \( M \) and \( N \) subintervals of lengths \( \Delta r = a/M \) and \( \Delta t = T/N \), respectively. Denote the approximate value of \( v(r_j, t_n) \) by \( v_j^n \). The finite-difference scheme associated with (2) is

\[
\begin{align*}
\frac{v_j^{n+1} - 2v_j^n + v_j^{n-1}}{(\Delta t)^2} - \frac{v_j^{n+1} - 2v_j^n + v_j^{n-1}}{(\Delta r)^2} + \gamma v_j^{n+1} - v_j^{n-1} - \frac{2\Delta t}{2}\beta \left( v_j^{n+1} - 2v_j^n + v_j^{n-1} \right) - \left( v_j^{n-1} - 2v_j^{n-1} + v_j^{n-1} \right) \\
\frac{m^2}{2} \left[ v_j^{n+1} + v_j^{n-1} \right] + \frac{1}{(j\Delta r)^p-1} G(v_j^{n+1}) - G(v_j^{n-1}) = 0,
\end{align*}
\]

where \( G(v) = v^{p+1}/(p+1) \).

Computationally, our method requires an application of Newton’s method for systems of nonlinear equations along with Crout’s reduction technique for tridiagonal linear systems.

An interesting property of this finite-difference scheme is that, for initial approximations \( \{w_0^n\} \) and \( \{w_1^n\} \) with zero centered-difference first spatial derivatives at the origin, the successive approximations provided by the method have likewise centered-difference first spatial derivative equal to zero at the origin. This claim follows by induction using the facts that

\[
v_j^0 + v_j^{-1} + \beta \left( v_j^{k+1} + v_j^{k-1} \right) - \left( v_j^{k-1} + v_j^{k-1} \right) = 0,
\]

that \( w_1^n = w_{n+1} \) iff \( v_1^n + v_n^n = 0 \), and the substitution \( v_j^n = j w_j^n \Delta r \). The induction hypothesis implies that \( v_1^n + v_{n-1}^n = 0 \) for every \( n \leq k \), whence the claim follows.

Our last statement implies that for a smooth initial profile at the origin, the subsequent approximations yielded by our method will be likewise smooth. As a test case, it is worthwhile mentioning that we have successfully obtained numerical results to verify this claim using a Gaussian initial profile centered at the origin.

**Stability analysis.** It is clear that (3) is consistent order \( O(\Delta t^2) + O(\Delta r^2) \) with (2) whenever \( G' \) is identically equal to zero. Moreover, in order for the finite-difference scheme to be stable order \( n \) it is necessary that

\[
\left( \frac{\Delta t}{\Delta r} \right)^2 < 1 + \gamma \frac{\Delta t}{4} + \beta \frac{\Delta t}{(\Delta r)^2} + m^2 \frac{(\Delta t)^2}{4}.
\]

To verify this claim, notice first that (3) can be rewritten as

\[
\frac{v_j^{n+1} - 2v_j^n + v_j^{n-1}}{(\Delta t)^2} - \frac{\delta v_j^n}{(\Delta r)^2} + \gamma v_j^{n+1} - v_j^{n-1} - \frac{2\Delta t}{2}\beta \frac{\delta v_j^{n+1} - \delta v_j^{n-1}}{2\Delta t (\Delta r)^2} + \frac{m^2}{2} \left[ v_j^{n+1} + v_j^{n-1} \right] = 0.
\]

Define \( R = \Delta t/\Delta r \). Let \( V_j^{n+1} = v_j^{n+1} \) and \( V_j^{n} = v_j^n \) for each \( j = 0, 1, \ldots, M \) and \( n = 0, 1, \ldots, N - 1 \). For every \( j = 0, 1, \ldots, M \) and \( n = 1, 2, \ldots, N \) let \( V_j^n \) be
the column vector whose components are $V_{n1}$ and $V_{n2}$. Our problem can be written then in matricial form as

$$
\begin{pmatrix}
k & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
V_{n+1}
\end{pmatrix}_j
= 
\begin{pmatrix}
2 + R^2 \delta \kappa^2 \\
1
\end{pmatrix}
\begin{pmatrix}
V
\end{pmatrix}_j
,$$

where

$$
k = 1 + \gamma \Delta t \frac{\beta}{2} - \frac{\beta \Delta t \delta^2}{2 (\Delta r)^2} + m^2 \frac{(\Delta t)^2}{2}
$$

and

$$
h = 1 - \gamma \Delta t \frac{\beta}{2} + \frac{\beta \Delta t \delta^2}{2 (\Delta r)^2} + m^2 \frac{(\Delta t)^2}{2}.
$$

Denoting the Fourier transform of each $\bar{V}_n^j$ by $\hat{V}_n^j$, we obtain that

$$
\hat{V}_{n+1}^j
= 
\begin{pmatrix}
\frac{2}{h(\xi)} \left( 1 - 2 R^2 \sin^2 \frac{\xi}{2} \right) - \frac{h(\xi) \hat{k}(\xi)}{k(\xi)}
\end{pmatrix}
\hat{V}_n^j,
$$

where

$$
\hat{k}(\xi) = 1 + \gamma \Delta t \frac{\beta}{2} + \frac{\beta \Delta t \delta^2}{2 (\Delta r)^2} \sin^2 \frac{\xi}{2} + m^2 \frac{(\Delta t)^2}{2}
$$

and

$$
\hat{h}(\xi) = 1 - \gamma \Delta t \frac{\beta}{2} - \frac{\beta \Delta t \delta^2}{2 (\Delta r)^2} \sin^2 \frac{\xi}{2} + m^2 \frac{(\Delta t)^2}{2}.
$$

The matrix $A(\xi)$ multiplying $\hat{V}_n^j$ in the last vector equation is the amplification matrix of the problem, which has eigenvalues given by

$$
\lambda_\pm = \frac{1 - 2 R^2 \sin^2 \frac{\xi}{2} \pm \sqrt{\left( 1 - 2 R^2 \sin^2 \frac{\xi}{2} \right) - h(\xi) \hat{k}(\xi)}}{k(\xi)}.
$$

In particular, for $\xi = \pi$ the eigenvalues of $A$ are

$$
\lambda_\pm = \frac{1 - 2 R^2 \pm \sqrt{\left( 1 - 2 R^2 \right)^2 - h(\pi) \hat{k}(\pi)}}{k(\pi)}.
$$

Suppose for a moment that $1 - 2 R^2 < -\hat{k}(\pi)$. If the radical in the expression for the eigenvalues of $A(\pi)$ is a pure real number then $|\lambda_-| > 1$. So for every $n \in \mathbb{N}$, $|A^n| \geq |\lambda_-|^n$ grows faster than $K_1 + n K_2$ for any constants $K_1$ and $K_2$. A similar situation happens when the radical is a pure imaginary number, except that in this case $| \cdot |$ represents the usual Euclidean norm in the field of complex numbers.

Summarizing, if $1 - 2 R^2 < -\hat{k}(\pi)$ then scheme 3 is unstable. Therefore in order for our numeric method to be stable order $n$ it is necessary that $1 - 2 R^2 > -\hat{k}(\pi)$, which is what we needed to establish.

**Energy analysis.** Assume that $G : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, and that $w(x, t)$ is a solution of (1) in a domain $D$ of $\mathbb{R}^3$. Moreover, we assume that $\nabla w \cdot \hat{n}$ is zero near the boundary of $D$ at all time, where $\hat{n}$ denotes the unit vector normal to the boundary of $D$. The Lagrangian associated with our nonlinear modified Klein-Gordon equation is given by

$$
\mathcal{L} = \frac{1}{2} \left\{ \left( \frac{\partial w}{\partial t} \right)^2 - |\nabla w|^2 - m^2 w^2 \right\} - G(w).
$$
It is easy to derive the following expression for the total energy associated with our nonlinear dissipative Klein-Gordon-like equation:

\[
E(t) = \iint_D \left\{ \frac{1}{2} \left( \frac{\partial w}{\partial t} \right)^2 + \frac{1}{2} |\nabla w|^2 + \frac{m^2}{2} w^2 + G(w) \right\} d\bar{x}.
\]
$G'(u) = 0$  

$G'(u) = u^3$

$G'(u) = u^5$  

$G'(u) = u^7$

$G'(u) = u^9$  

$G'(u) = \sin(5u) - 5u$

**Figure 2.** Approximate radial solutions of (1) with $G'(u)$ at $t = 0.2$, for initial data $\phi(r) = 0$ and $\psi(r) = 100h(r)$, $\beta = 0$ and $\gamma = 0$ (solid), $\gamma = 5$ (dashed) and $\gamma = 10$ (dotted).

**Proposition 2.** The instantaneous rate of change with respect to time of the total energy associated with the PDE in (1) is given by

$$E'(t) = -\iiint_D \left\{ \beta \left\| \nabla \left( \frac{\partial w}{\partial t} \right) \right\|^2 + \gamma \left( \frac{\partial w}{\partial t} \right)^2 \right\} \, d\vec{x}. $$
Proof. Taking derivative on both sides of Equation (4), we obtain that

$$\frac{dE}{dt} = \iiint_D \frac{\partial w}{\partial t} \left( \frac{\partial^2 w}{\partial t^2} + m^2 w + G'(u) \right) \, d\bar{x} + \frac{1}{2} \iiint_D \frac{\partial}{\partial t} |\nabla w|^2 \, d\bar{x}$$

$$= \iiint_D \frac{\partial w}{\partial t} \left( \frac{\partial^2 w}{\partial t^2} - \nabla^2 w + m^2 w + G'(w) \right) \, d\bar{x} + \iint_{\partial D} \frac{\partial w}{\partial t} \nabla w \cdot \hat{n} \, d\sigma$$

$$= \beta \iiint_D \frac{\partial w}{\partial t} \left( \frac{\partial w}{\partial t} \right)^2 \, d\bar{x} - \gamma \iiint_D \left( \frac{\partial w}{\partial t} \right)^2 \, d\bar{x} + \iint_{\partial D} \frac{\partial w}{\partial t} \nabla w \cdot \hat{n} \, d\sigma.$$
On the other hand, from Green’s first identity we see that

$$
\int \int \int_D \frac{\partial w}{\partial t} \left( \nabla^2 \left( \frac{\partial w}{\partial t} \right) \right) \, d\vec{x} = \int \int_{\partial D} \frac{\partial w}{\partial t} \frac{\partial}{\partial t} (\nabla w \cdot \vec{n}) \, d\sigma - \int \int \int_D \left\| \nabla \left( \frac{\partial w}{\partial t} \right) \right\|^2 \, d\vec{x}.
$$

The surface integrals in these last two equations are equal to zero, whence the result follows. □
It is worthwhile noticing that if $\beta$ and $\gamma$ are positive then the total energy is decreasing in time. Also, if $\beta$ and $\gamma$ are both equal to zero then the energy is conserved. Finally, if $\beta$ is zero then the expression of $E'(t)$ coincides with the one derived in [12].
Let us assume now that $G$ is nonnegative. The total energy in this case is likewise nonnegative and the integral of every term in (4) is bounded by $\sqrt{2E(t)/m}$. In particular, this last statement implies that the integral of $w^2$ at time $t$ is bounded by $E(t)$. For those times $t$ for which $E(t)$ is finite (and particularly for the case when $\beta$ and $\gamma$ are both equal to zero), this means that $w$ is a square-integrable function in the first variable at time $t$. 

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Figure 6. Approximate value of solutions to (1) near the origin vs. time for different nonlinear terms, and initial conditions $\phi(r) = 0$ and $\psi(r) = 100h(r)$. Left column: $\beta = 0$ and $\gamma = 0$ (solid), $\gamma = 10$ (dashed) and $\gamma = 20$ (dotted); right column: $\gamma = 0$ and $\beta = 0$ (solid), $\beta = 0.001$ (dashed), $\beta = 0.0025$ (dashed-dotted) and $\beta = 0.005$ (dotted).
Let $G'(w) = w^p$ with $p > 1$. Assuming that $w$ is a radially symmetric solution of the damped nonlinear Klein-Gordon equation in a sphere $D$ with center in the origin and radius $L$, and using the transformation $v(r,t) = rw(r,t)$ the energy expression adopts the form $E(t) = 4\pi E_0(t)$, with

\begin{equation}
E_0(t) = \int_0^L \left\{ \frac{1}{2} \left( \frac{\partial v}{\partial t} \right)^2 + \frac{1}{2} \left( \frac{\partial v}{\partial r} \right)^2 + \frac{m^2}{2} v^2 + r^{1-p}G(v) \right\} dr.
\end{equation}
Table 1. Relative differences of externally damped solutions to (1) with respect to the corresponding undamped solution at different time steps.

| Time step | Relative differences |  
|-----------|----------------------|
|           | $\gamma = 0.1$ | $\gamma = 0.5$ | $\gamma = 1$ | $\gamma = 5$ | $\gamma = 10$ |
| 0         | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 20        | 0.0028 | 0.0142 | 0.0283 | 0.1395 | 0.2693 |
| 40        | 0.0103 | 0.0509 | 0.1006 | 0.4491 | 0.7706 |
| 60        | 0.0167 | 0.0821 | 0.1611 | 0.6579 | 0.9573 |
| 80        | 0.0192 | 0.0942 | 0.1836 | 0.6954 | 0.9387 |
| 100       | 0.0200 | 0.0977 | 0.1896 | 0.6994 | 0.9308 |

The instantaneous rate of change of energy is given by $E'(t) = 4\pi E_0'(t)$, where

$$E_0'(t) = -\int_0^L \left\{ \beta \left( \frac{\partial^2 v}{\partial r \partial t} - \frac{1}{r} \frac{\partial v}{\partial t} \right)^2 + \gamma \left( \frac{\partial v}{\partial t} \right)^2 \right\} dr.$$

It is possible to reproduce now the argument in [8] to show that for every $t$ and nonzero $r$, $|w(r,t)| \leq \sqrt{2E_0(t)/r}$. This means in particular that if a solution were unbounded, it would have to be unbounded at the origin.

The discrete energy is given by

$$\frac{E_n^0}{\Delta r} = \frac{1}{2} \sum_{j=0}^{m-1} \left( \frac{v_j^{n+1} - v_j^n}{\Delta t} \right)^2 + \frac{1}{2} \sum_{j=0}^{m-1} \left( \frac{v_j^{n+1} - v_j^n}{\Delta r} \right)^2 + \frac{1}{2} \sum_{j=0}^{m-1} \left( v_j^n \right)^2 + \sum_{j=1}^{m-1} G(v_j^{n+1}) + G(v_j^n)$$

This expression is obviously consistent with (5). Moreover, taking the difference between $E_0^m/\Delta r$ and $E_0^{m-1}/\Delta r$ and simplifying after using (3), it can be shown that

$$\frac{E_n^0 - E_0^{n-1}}{\Delta t} = -\beta \sum_{j=1}^{m-1} \left( \frac{v_j^{n+1} - v_j^n}{2\Delta t} \right) \left( \frac{(v_j^{n+1} - v_j^n) - (v_j^{n+1} - v_j^{n-1})}{\Delta t(\Delta r)^2} \right) \Delta r$$

For $\beta = 0$ this expression provides us with a consistent approximation to the instantaneous rate of change of energy. Numerical results demonstrate that energy decreases as a function of time for $\beta > 0$, which is in agreement with the corresponding instantaneous change of energy as a function of time.

3. Numerical results

The numerical results presented in this section correspond to approximate solutions of the dissipative, nonlinear, modified Klein-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \nabla^2 u - \beta \frac{\partial}{\partial t} (\nabla^2 u) + \gamma \frac{\partial u}{\partial t} + u + G'(u) = 0,$$
Table 2. Table of relative differences of externally damped solutions of \( u \) with respect to the corresponding undamped solution at \( t = 0.2 \).

| Nonlinear Term | Relative differences |
|----------------|----------------------|
| \( G'(u) \)    | \( \gamma = 0.1 \)  | \( \gamma = 0.5 \)  | \( \gamma = 1 \)  | \( \gamma = 5 \)  | \( \gamma = 10 \) |
| 0              | 0.0098               | 0.0478               | 0.0923               | 0.3642               | 0.5631               |
| \( u^3 \)      | 0.0097               | 0.0477               | 0.0929               | 0.3528               | 0.5554               |
| \( u^5 \)      | 0.0137               | 0.0665               | 0.1287               | 0.4024               | 0.6418               |
| \( u^7 \)      | 0.0171               | 0.0833               | 0.1618               | 0.5068               | 0.7819               |
| \( u^9 \)      | 0.0204               | 0.0999               | 0.1728               | 0.5736               | 0.8488               |
| \( \sinh(5u) - 5u \) | 0.0263               | 0.1377               | 0.2518               | 0.6284               | 0.8813               |

Table 3. Relative differences of internally damped solutions to \( u \) with respect to the corresponding undamped solution at different time steps.

| Time step | Relative differences |
|-----------|----------------------|
| \( n \)  | \( \beta = 10^{-6} \)  | \( \beta = 10^{-5} \)  | \( \beta = 10^{-4} \)  | \( \beta = 0.0005 \)  | \( \beta = 0.001 \)  |
| 0         | 0.0000               | 0.0000               | 0.0000               | 0.0000               | 0.0000               |
| 20        | 0.0005               | 0.0054               | 0.0517               | 0.2188               | 0.3682               |
| 40        | 0.0030               | 0.0300               | 0.2640               | 0.8281               | 1.1457               |
| 60        | 0.0156               | 0.0996               | 0.1493               | 1.6536               | 1.1460               |
| 80        | 0.0102               | 0.0970               | 0.7242               | 1.1530               | 1.2138               |
| 100       | 0.0080               | 0.0772               | 0.5751               | 1.0406               | 1.1435               |

Table 4. Table of relative differences of internally damped solutions of \( u \) with respect to the corresponding undamped solution at \( t = 0.2 \).

| Nonlinear Term | Relative differences |
|----------------|----------------------|
| \( G'(u) \)    | \( \beta = 10^{-6} \)  | \( \beta = 10^{-5} \)  | \( \beta = 10^{-4} \)  | \( \beta = 0.0005 \)  | \( \beta = 0.001 \)  |
| 0              | 0.0003               | 0.0027               | 0.0242               | 0.0859               | 0.1326               |
| \( u^3 \)      | 0.0003               | 0.0032               | 0.0289               | 0.1040               | 0.1621               |
| \( u^5 \)      | 0.0011               | 0.0105               | 0.0948               | 0.3374               | 0.5043               |
| \( u^7 \)      | 0.0023               | 0.0224               | 0.1825               | 0.5663               | 0.7327               |
| \( u^9 \)      | 0.0041               | 0.0397               | 0.3133               | 0.7318               | 0.9256               |
| \( \sinh(5u) - 5u \) | 0.0063               | 0.0577               | 0.4717               | 0.9403               | 1.1007               |

obtained using a tolerance of \( 10^{-5} \) and a maximum number of 20 iterations on every application of Newton’s method. The space and time steps are always fixed as \( \Delta r = \Delta t = 0.002 \).

**External damping.** Throughout this section we fix \( \beta = 0 \).

Let us start considering the problem of approximating radially symmetric solutions of \( u \) with \( G'(u) = u^7 \), and initial data \( \phi(r) = h(r) \) and \( \psi(r) = h'(r)+h(r)/r \), where

\[
h(r) = \begin{cases} 
5 \exp \left\{ \frac{1}{100} \left[ 1 - \frac{1}{1 - (10r - 1)^2} \right] \right\}, & \text{if } 0 \leq r < 0.2, \\
0, & \text{if } 0.2 \leq r \leq 0.4.
\end{cases}
\]

We have plotted numerical solutions of this problem for several values of \( \gamma \). The graphical results are presented in Figure 1 for \( \gamma = 0, 5, 10 \). We observe first of all
that the solutions of the damped nonlinear Klein-Gordon-like equation corresponding to small values of $\gamma$ are consistently similar to those of the undamped case. To verify this claim quantitatively, we consider the approximations $\bar{v}^n_\gamma$ and $\bar{v}^n_0$ to the undamped and damped cases, respectively, and compute the relative difference in the $\ell_2,\Delta_x$-norm via

$$\delta(\bar{v}^n_\gamma, \bar{v}^n_0) = \frac{||\bar{v}^n_\gamma - \bar{v}^n_0||_{2,\Delta x}}{||\bar{v}^n_0||_{2,\Delta x}}$$

(here we follow [13]). The relative differences for several values of $\gamma$ at consecutive time steps are shown in Table 1. We observe that the difference between the solutions of the nonlinear Klein-Gordon-like equation with damping coefficient $\gamma$ and the corresponding undamped equation can be made arbitrarily small by taking $\gamma$ sufficiently close to 0.

We wish to corroborate this pattern for different nonlinear terms and a different set of initial conditions. With this objective in mind, Figure 2 depicts numerical solutions of (1) with $\gamma = 0, 5, 10$, nonlinear terms $G'(u) = u^3, u^5, u^7, u^9$, and sinh$(5u) - 5u$, initial conditions $\phi(r) = 0$ and $\psi(r) = 100h(r)$, and values of $\gamma = 0, 5, 10$. More accurately, Table 2 provides relative differences of these solutions at $t = 0.2$ (for the nonlinear functions listed above and varying values of $\gamma$) with respect to the corresponding undamped solution, for a wider selection of values of the parameter $\gamma$.

It must be mentioned that, as it was expected, the total energy was invariably decreasing for positive values of $\gamma$, and increasing for negative values. For the value $\gamma = 0$, the rate of change of energy is equal to zero and our numeric results agree with [8]. Experimental results show that small values of $\gamma$ correspond with small values of the discrete rate of change of the energy. This last observation corroborates stability of our method.

We also observe that the amplitude of solutions corresponding to positive values of $\gamma$ tend to decrease as time or $\gamma$ increases. Figure 2 partially corroborates that behavior. We have computed solutions corresponding to negative values of $\gamma$ (graphs not included in this paper) and have verified that the amplitude of solutions increases with time and with $|\gamma|$.

Finally, we have obtained graphs of the energy $E_0$ vs. time for $G'(u) = u^3, u^5, u^7$, initial data $\phi(r) = 0$ and $\psi(r) = 100h(r)$, and values of $\gamma = 1, 5, 10$. The results (depicted in the left column of Figure 7) show a loss in the total energy as a function of time.

**Internal damping.** Consider first the case when $\gamma$ equals zero. Figure 3 shows numerical solutions of (1) at consecutive times, for initial data $\phi(r) = h(r)$ and $\psi(r) = h'(r) + h(r)/r$, with nonlinear term $G'(u) = u^7$, and values of $\beta = 0, 0.001, 0.003$. We observe that small values of $\beta$ produce results similar to those of the corresponding undamped case. To corroborate this claim, we appeal once more to the relative differences in the $\ell_2,\Delta_x$-norm of dissipative solutions with respect to the non-dissipative one. The results are shown in Table 3. The results evidence the continuity of solutions with respect to the parameter $\beta$ for this particular choice of nonlinearity, providing thus numerical support in favor of the stability of our method.

We want to establish now the continuity of our method for several nonlinear terms at a fixed large time. In order to do it, Figure 4 shows the numerical solutions
of \( \Psi \) at time \( t = 0.2 \), for the nonlinear terms \( G'(u) = u^3, u^5, u^7 \), for two different sets of initial conditions: \( \phi(r) = 0 \) and \( \psi(r) = 100h(r) \), and \( \phi(r) = h(r) \) and \( \psi(r) = 0 \), and values of \( \beta = 0, 0.0001, 0.0002 \). The graphs in this figure, together with the analysis of relative differences in the \( \ell_2, \Delta t \)-norm supplied in Table 4 for the first set of initial conditions, evidence the continuity of the numerical solution given by our method with respect to the parameter \( \beta \) for different nonlinearities.

We now consider the case when \( \gamma \) is nonzero. We use \( G'(u) = 0, u^3, u^5, u^7, u^9 \), and \( \sinh(5u) - 5u \), initial data \( \phi(r) = 0 \) and \( \psi(r) = 100h(r) \), and time \( t = 0.2 \). Figure 3 shows numerical solutions of (1) for values of \( \beta \) for different nonlinearities. The solutions for smaller nonzero values of \( \beta \) are indeed closer to the corresponding internally undamped solution, while the larger values of \( \beta \) spread out the internally undamped solution at the same time that the maximum amplitude is decreased.

In order to study the time behavior of the solutions near the origin we have included Figure 5 using initial data \( \phi(r) = 0 \) and \( \psi(r) = 100h(r) \), the nonlinear terms \( G'(u) = u^3, u^5 \) and \( u^7 \), different choices of values for \( \beta \) and \( \gamma \), and \( \Delta r = \Delta t = 0.002 \). The left column shows the time-dependence of solutions at the origin for \( \beta = 0 \) and three positive values of \( \gamma \), whereas the right column shows similar results for \( \gamma = 0 \) and three positive values of \( \beta \). We observe that the value of solutions at the origin for large times is always approximately equal to zero for \( \beta = 0 \), which is in agreement with our experience of the \((1+1)\)-dimensional case.

Finally, Figure 6 shows the graphs of the energy \( E_0 \) vs. time for \( G'(u) = u^3, u^5, u^7 \), initial data \( \phi(r) = 0 \) and \( \psi(r) = 100h(r) \), and values of \( \beta = 0.0005, 0.001, 0.005 \). The results (depicted in the right column) show a loss in the total energy as a function of time. It is clear that the rate at which the total energy is lost due to internal damping is greater than the corresponding rate due to external damping.

4. Discussion

A numerical method has been designed to approximate radially symmetric solutions of some dissipative, nonlinear, modified Klein-Gordon equations with constant internal and external damping coefficients \( \beta \) and \( \gamma \), respectively. Our finite-difference scheme is in general agreement with the non-dissipative results presented in [S]. The method is consistent \( O(\Delta t^2) + O(\Delta r^2) \), conditionally stable, and continuous with respect to the parameters \( \beta \) and \( \gamma \); as expected, the total energy decays in time for positive choices of the parameters. The corresponding scheme to approximate the total energy of the system is consistent and has the property that the discrete rate of change of the discrete energy with respect to time approximates the corresponding continuous rate of change for \( \beta = 0 \).

Several conclusions can be drawn from our numerical computations. First of all, we have seen that both internal and external damping tend to decrease the magnitude of solutions, as it was expected. Our results clearly exhibit the dispersive effects of the parameter \( \beta \) and the dissipative effects of \( \gamma \). Our energy computations evidence the fact that the rate at which the energy is dissipated by the internal damping is faster than the corresponding rate of external damping. Finally, we observe that the effect of the nonlinear term in the temporal behavior near the origin is to increase the number of oscillations as the degree of the nonlinearity is increased. Invariably, the solutions of the dissipative modified Klein-Gordon equation converge in time to the trivial solution.
References

[1] Remoissenet, M. Waves Called Solitons. Springer-Verlag, New York, third edition, 1999.
[2] Lomdahl, P. S.; Soerensen, O. H.; Christiansen, P. L. Soliton Excitations in Josephson Tunnel Junctions. Phys. Rev. B, 25(9):5737–5748, 1982.
[3] Makhankov, V. G.; Bishop, A. R.; Holm, D. D., editor. Nonlinear Evolution Equations and Dynamical Systems Needs ’94: Los Alamos, NM, USA 11-18 September ’94: 10th International Workshop. World Scientific Pub. Co. Inc., Singapore, first edition, 1995.
[4] Polyanin, A. D.; Zaitsev, V. F. Handbook of Nonlinear Partial Differential Equations. Chapman & Hall CRC Press, Boca Raton, Fla., first edition, 2004.
[5] Glassey, R. M. Blow-up theorems for nonlinear wave equations. Math. Zeit., 132:183–203, 1973.
[6] Jörgens, K. Das Anfangswertproblem im Grossen für eine Klasse nichtlinearer Wellengleichungen. Math. Zeit., 77:295–308, 1961.
[7] Barone, A.; Esposito, F.; Magee, C. J.; Scott, A. C. Theory and Applications of the Sine-Gordon Equation. Riv. Nuovo Cim., 1:227–267, 1971.
[8] Strauss, W. A.; Vázquez, L. Numerical Solution of a Nonlinear Klein-Gordon Equation. J. Comput. Phys., 28:271–278, 1978.
[9] Alexander, F. J.; Habib, S. Statistical Mechanics of Kinks in 1 + 1 Dimensions. Phys. Rev. Lett., 71:955–958, 1993.
[10] Segal, I. E. The Global Cauchy problem for a relativistic scalar field with power interaction. Bull. Soc. Math. Fr., 91:129–135, 1963.
[11] Morawetz, C. S.; Strauss, W. A. Decay and scattering of solutions of a nonlinear relativistic wave equation. Comm. Pure and Appl. Math., 25:1–31, 1972.
[12] Strauss, W. A. Partial Differential Equations: An Introduction. John Wiley & Sons, Inc., New York, second edition, 1992.
[13] Thomas, J. W. Numerical Partial Differential Equations. Springer-Verlag, New York, first edition, 1995.