Application of the geometric immersion method based on the Castigliano variational principle for the axisymmetric problems of elasticity theory

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Abstract. The fundamentals of the geometric immersion stress method for axisymmetric problems of elasticity theory have been stated. The geometric immersion method involves reduction of the initial problem for Clapeyron's free form elastic body to an iteration sequence of elasticity theory problems on some canonical domain. The iteration procedure for the variational equation of the geometric immersion method has been stated, as well as its discrete analog construction technique suggested with the finite-element stress method for the axisymmetric problem of the elasticity theory in the cylindrical coordinates. A practical application of the method has been demonstrated by a test problem. A reasonably good fit between the stress fields definition and the numerical solution by the traditional finite-element displacement method has been obtained.

1. Introduction
The boundary problem of the elasticity theory can be traditionally stated as displacement or stress, which under variational formulation implies the principle of minimum potential energy of the system (Lagrange principle) and the principle of the least work (Castigliano principle), respectively. Approximative and numerical methods of the elasticity theory such as the finite-element method (FEM) and the variational difference method, which implement extreme principles, in most cases, base on the displacement formulation as it is more convenient for choosing basis functions that should satisfy rather simple requirements for kinematic acceptability, that is fulfillment of boundary displacement conditions and existence of space variable first derivatives. Well-known shortcomings of numerical displacement solutions consist in low precision in strain and stress fields definition, weakly compressible material analysis problems, and impossibility to calculate incompressible bodies. Therefore, different mixed formulations based on Reissner, Hu-Washizu, etc. variational principles, which are more difficult to realize, but, to a certain extent, free from the given shortcomings, have been developed sufficiently widely.

Approximate and numerical stress solutions based on the Castigliano variational principle have found a fairly limited application, primarily, due to the construction problems with basis functions which, in this formulation, must be statically acceptable, i.e. satisfy the equilibrium equations and static boundary conditions in the domain [1-5]. Generally, solutions for such elasticity theory problems have been obtained in the canonical form domains.

There are some methods to reduce the elasticity theory boundary problem for an arbitrary shape
domain to the canonical domain: the fictitious domain method, perturbation technique, etc. The paper offers theoretical justification and practical examples of the geometrical immersion method (GIM) for the elasticity theory stress problem. The appeal of this approach is connected with the possibility in principle to obtain stress fields in the numerical realization of the GIM with higher precision and to solve incompressible and weakly compressible material problems. Moreover, displacement and stress solutions enable construction of lower and upper variational boundaries for the guaranteed exact solution.

2. Problem statement

Let us consider the functional of complementary work of an elastic body [6]:

\[ V(\hat{\sigma}) = \frac{1}{2} \int_\Omega \sigma_{ij} e_{ij} (\hat{\sigma}) dD - \int_{S_u} t_i (\hat{\sigma}) U_i dS_u, \]  

where \( \hat{\sigma} \) is symmetrical stress tensor with components \( \sigma_{ij} \); \( \hat{\epsilon} \) – symmetrical strain tensor with components \( e_{ij} \); \( U_i \) – displacement vector components specified on boundary \( S_u \), \( t_i = \sigma_{ij} n_j \) – forces on surface \( S_u \), \( n_i \) – components of the boundary \( S \) unit outward the normal vector. Functional (1) is defined in domain \( D \) on the set of statically acceptable stress fields \( \sigma_{ij} \) which satisfy equilibrium equations and static boundary conditions on boundary \( S_u \); \( S = S_u \cup S_o \).

From the functional (1) stationarity condition, given that \( \delta \sigma_{ij} \) variations are also statically acceptable stress fields, we have:

\[ \delta V(\hat{\sigma}) = \frac{1}{2} \int_\Omega \delta \sigma_{ij} e_{ij} (\hat{\sigma}) dD - \int_{S_u} t_i (\delta \hat{\sigma}) U_i dS_u = 0. \]  

Hence, the realization of compatibility equations in the Beltrami-Michell form in the \( D \) domain and boundary conditions in displacements on \( S_u \) follow:

\[ \text{Figure 1. Initial domain } D, \text{ complement domain } D_\Delta \text{ and boundary delineation.} \]

According to the geometric immersion procedure, we will consider canonical domain \( D_0 \) (\( D_\Delta = D_0 \setminus D \), Figure 1) and redefine our functional (1) on the canonical domain:

\[ V(\hat{\sigma}) = \frac{1}{2} \int_\Omega \sigma_{ij} e_{ij} (\hat{\sigma}) dD - \int_{S_u} t_i (\hat{\sigma}) U_i dS_u = \frac{1}{2} \int_\Omega \sigma_{ij} e_{ij} (\hat{\sigma}) dD - \int_{S_u} t_i (\hat{\sigma}) U_i dS_u = \frac{1}{2} \int_\Omega \sigma_{ij} e_{ij} (\hat{\sigma}) dD - \int_{S_o} t_i (\hat{\sigma}) U_i dS_u - \frac{1}{2} \int_{D_\Delta} \sigma_{ij} e_{ij} (\hat{\sigma}) dD_\Delta = \frac{1}{2} \int_\Omega \sigma_{ij} e_{ij} (\hat{\sigma}) dD - \int_{S_u} t_i (\hat{\sigma}) U_i dS_u - \frac{1}{2} \int_{D_\Delta} \sigma_{ij} e_{ij} (\hat{\sigma}) dD_\Delta = \frac{1}{2} \int_\Omega \sigma_{ij} e_{ij} (\hat{\sigma}) dD - \int_{S_u} t_i (\hat{\sigma}) U_i dS_u. \]  

By varying the stress fields, we come to the following GIM variational equation which is defined in canonical domain \( D_0 \):
\[ \delta V(\hat{\sigma}) = \int_{D} \delta \sigma_{ij} \epsilon_{ij}(\hat{\sigma}) dD - \int_{\delta D} \delta \sigma_{ij} \epsilon_{ij}(\hat{\sigma}) d\delta D - \int_{S} t_{i}(\hat{\sigma}) U_{i} dS_{u} = 0. \] (4)

Functional analysis methods have determined requirements for functions according to which functional (3) is defined in domain \( D_{\Delta} \) that complements initial domain \( D \) up to canonical \( D_{0} \); established a correspondence between the stress fields that minimize functionals (1) and (3); shown the correspondence of variational equations solutions (2) and (4) in the \( D \) domain; and obtained differential formulation of the elasticity theory boundary stress problem in domain \( D_{0} \), which corresponds to variational formulation (4). Necessary conditions for the functional (4) stationarity are realization of compatibility equations in the Beltrami-Michelle form in domain \( D_{0} \) and kinematic boundary conditions on boundaries \( S_{u}^{1} \) and \( S_{u}^{2} \). It has been theoretically obtained that on transition via boundaries \( S_{u}^{2} \) and \( S_{u}^{1} \), a jump of the displacement vector is possible. Immersion of the initial domain into the canonical one evidently leads to the appearance of new boundary \( S_{u}^{1} \) on which the boundary conditions should be formulated. Three possible types of such boundary conditions, which enable the selection of the most acceptable boundary condition type for the current problem, have been ascertained.

It is suggested that equation (4) be solved with the following iteration procedure:

\[ \int_{D} \delta \sigma_{ij}^{(k+1)} \epsilon_{ij}(\hat{\sigma}^{(k+1)}) dD_{0} = \int_{D} \delta \sigma_{ij}^{(k)} \epsilon_{ij}(\hat{\sigma}^{(k)}) dD_{0} + \int_{S} t_{i}(\delta \hat{\sigma}^{(k+1)}) U_{i} dS_{u}, \]

\[ \sigma_{ij}^{(0)} = 0, \quad k = 1, 2, 3, \ldots \] (5)

The iterative process convergence theorem in the geometric immersion method (5), irrespective of the initial \( D \) domain difference from the canonical \( D_{0} \) one, has been proved.

In a number of cases, the GIM allows analytical realization, but in most cases, one has to resort to numerical methods. Numerical realization presupposes that variational difference or finite element methods are applied, which enable the construction of a discrete analog of the GIM variational iterative procedure [7-10]. The present paper expands on the finite element realization of the method by the example of axisymmetrical elasticity problems.

3. Exemplifications

As an example of the described GIM procedure, let us consider an axisymmetrical elasticity problem: determination of the stress state of the finite hollow cylinder with a rigid rectangle in the cross-section and a circular inclusion under the action of uniformly distributed internal pressure \( P \) (Figure 2). On the inclusion contour, both displacement vector components are zero. A cylinder without inclusion, completely covering the real domain, was chosen as the canonical domain.

![Figure 2. The design diagram.](image)

The problem solution by the geometric immersion method is reduced to the variational-iterative procedure (5). To ensure static admissibility of the stress fields, the following boundary conditions should be satisfied:
\[
\sigma_r(2,z) = -P, \quad \tau_r(2,z) = 0, \quad z \in S^1_\sigma,
\]
\[
\sigma_r(6,z) = 0, \quad \tau_r(6,z) = 0, \quad z \in S^2_\sigma,
\]
\[
\sigma_z(r,2) = \tau_z(r,2) = 0, \quad r \in S^2_\sigma,
\]
\[
\tau_r(r,0) = 0, \quad r \in S^2_{u_1} \cup S^2_{u_2}.
\]

Kinematic boundary conditions are:
\[
u_i(r,z) = u_i(r,z) = 0, \quad (r,z) \in S^2_\nu,
\]
\[
u_i(r,0) = 0, \quad (r,z) \in S^2_{u_1} \cup S^2_{u_2}.
\]

During immersion, new boundary \( S^1_{\nu_i} \) is also formed, for which a boundary condition should be stated; in this case, it is convenient to extend symmetry conditions, such as in \( S^1_{\nu_i} \) and \( S^1_{u_1} \):
\[
u_i(r,0) = u_i(r,0) = 0, \quad (r,z) \in S^1_{\nu_i}.
\]

Material parameters are: \( E = 2.3E11 \) Pa, \( \nu = 0.3 \). Pressure - \( P = 10 \) Pa.

In order to establish the GIM iterative convergence on the discretization of the canonical domain, a series of calculations of a sequence of uniformly condensing meshes was made (Figure 3). Results obtained by the finite-element displacement method (discretization degree is 6088) were taken as the reference solution.

![Figure 3](image-url)

**Figure 3.** Distribution \( \sigma_r, Pa \) depending on the degrees of freedom on the cylinder section at \( r = 4m \).

It was established that the 3267 nodal unknown mesh will be sufficient for the problem. Further condensing of the mesh does not result in a significant change of the numerical solution.

In order to determine the iterative convergence of the number of iterations, let us consider a change in the relative residuals at each solution step in Figure 4.
Figure 4. Change in the relative residual on the iteration number.

Figure 4 shows that we need about 40 iterations to obtain a quality solution. If we increase the iteration number further, relative residual $\left\| \delta^{(k+1)} - \delta^{(k)} \right\| / \left\| \delta^{(k)} \right\|$ will be less than 0.0001, which will change results insignificantly.

Since an undisputable advantage of the numerical approaches based on the Castigliano variational principle is calculation of structures from incompressible materials, let us consider practical application of GIM in the investigation problem of the stressed state of the rubber-metal shock absorber in the form of shaped bushing designed to compensate for the longitudinal impact (Figure 5(a)). Steel elements are supposed to be absolutely rigid and continuously bonded with rubber. Figure 5(b) shows a simplified design model and indicates complement domain $D_\Delta$ and new boundary $S_{1r}^l$, which appeared as a result of geometric immersion.

![Figure 5. The rubber-metal shock absorber in the form of shaped bushing.](image)

The given problem was also explored for convergence; the results are presented as a stress intensity pattern in the whole shock absorber in Figure 6.
4. Conclusion

Thus, the paper has demonstrated practical application of the geometric immersion method based on the Castigliano variational principle for the solution of axisymmetrical elastic problems, including problems for incompressible elastic materials. The iteration procedure for solving the variational equation by the geometric immersion method has been formulated, as well as the procedure for its discrete analog construction by the finite-element stress method for the axisymmetric elasticity problem in the cylindrical coordinates has been suggested. The considered model axisymmetric problems have demonstrated the GIM efficiency. A sufficiently good agreement of the stress fields determination in comparison with the numerical solution by the traditional finite-element displacement method has been obtained.

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References

[1] Girija Vallabhan C V, Muluneh Azene 1982 *Int. J. Numer. Meth. Eng.* **18**(2) 291-309
[2] Sarigul N, Gallagher R H 1989 *Int. J. Numer. Meth. Eng.* **28**(7) 1577-98
[3] Watwood V B Jr, Hartz B J 1968 *Int. J. Solids Struct.* **4**(9) 857-73
[4] Fraeijs de Veubeke B 2001 *Int. J. Numer. Meth. Eng.* **52**(3) 287-342
[5] Pin Tong, Pian Theodore H H 1969 *Int. J. Solids Struct.* **5**(5) 463-72
[6] Lurie A I 2005 *Theory of elasticity* (Berlin: Springer)
[7] Gallager R 1975 *Finite element method. Fundamentals* (Englewood Cliffs, N.J.: Prentice Hall)
[8] Norrie D, de Vries J 1978 *An introduction to finite element analysis* (New York: Academic Press)
[9] Zienkiewicz O C, Taylor R L, Zhu J Z 2005 *The Finite Element Method: Its Basis and Fundamentals* (Oxford: Elsevier Butterworth-Heinemann)
[10] Spilker R L, Pian T H H 1978 *Computers and Structures* **9** 273-9