Moduli Structures, Separability of the Kinematic Hilbert Space and Frames in Loop Quantum Gravity

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Abstract

We reassess the problem of separability of the kinematic Hilbert space in loop quantum gravity under a new mathematical point of view. We use the formalism of frames, a tool used in signal analysis, in order to remove the redundancy of the moduli structures in high valence graphs, without resorting to set extension of diffeomorphism group. For this, we introduce a local redundancy which encodes the concentration of frame vectors on the tangent spaces $T_pM$ around points of intersections $p$ of smooth loops $\alpha$ in $\mathbb{R}^3$.

1 Introduction

Quantum gravity is expected to reconcile general relativity and quantum mechanics. Loop quantum gravity (LQG) \cite{1, 2} is a proposal for a non-perturbative and background independent quantization of general relativity. In LQG the quantization of space is described in terms of a basis of states known as spin-network states denoted by $s$-knots. They are labeled by quantum numbers which ensures a bijection with
the Natural numbers. Nevertheless, this bijection, which characterizes the separability of the set of states, can be violated. In fact, a few years ago, Fairbairn and Rovelli [4] studied the problem of the separability of the background-independent space of the quantum states of the gravitational field (the kinematical Hilbert space $H_{\text{diff}}$) in LQG. They have shown that $H_{\text{diff}}$ is nonseparable and redundant (physically meaningless – moduli) as a consequence of the fact that the knot space of the equivalence classes of graphs under diffeomorphisms are not countable. To circumvent this problem and get rid of the redundant parameters, they proposed a small extension of the group of homeomorphisms that are smooth (with their inverse) except possibly at a finite number of points. The space of the knot classes become countable and $H_{\text{diff}}$ is separable. In this contribution, we propose to (re)-examine the question of separability of $H_{\text{diff}}$ using another mathematical setting. We shall use the formalism of frames, a tool used in signal analysis, in order to remove the redundancy of the moduli structures in high valence graphs, without resorting to set extension of diffeomorphism.

2 Moduli structure of knots with intersections: the Grot-Rovelli case

Many of the most important problems in mathematics concern classification. One has a class of mathematical objects and a notion of when two objects should count as equivalent. It may well be that two equivalent objects look superficially very different, so one wishes to describe them in such a way that equivalent objects have the same description and inequivalent objects have different descriptions. In this section, we shall illustrate some of the key features of moduli structure of knots with intersections in LQG.

To begin with, we recall that in LQG the quantum states are labeled by knots with intersections [1]. These knots can be defined in two distinct ways: either as equivalence classes in $\mathbb{R}^3$ under a continuous deformations of the loop image – “c-knots” – or as equivalence classes under diffeomorphisms of $\mathbb{R}^3$ – “d-knots” (see Figure [1]). However, Grot-Rovelli showed in Ref. [3] that such knots equivalence classes of loops in $\mathbb{R}^3$ under diffeomorphisms are not countable; rather, they exhibit a moduli-space structure. So, the two definitions lose their equivalence when the knots have intersections, and d-knots are different from c-knots such that the space
$\mathcal{K}_d$ of $d$-knots is not countable. The continuous structure of the $\mathcal{K}_d$ comes from the differential structure of the underlying manifold, which means the presence of tangent spaces $T_pM$ at intersection points. The loops define lines in $T_pM$ and the diffeomorphisms act linearly on $T_pM$. Hence, diffeomorphism equivalence implies equivalence under linear transformations of $T_pM$. Therefore, for a large enough number of lines the linearity is lost, since we can not align all lines by a linear transformation if the set of parameters is greater than the dimension of the vector space considered.

We will illustrate in more detail how this comes, by considering the example of Grot-Rovelli [3]. Consider a smooth loop $\alpha$ in $\mathbb{R}^3$, with a point of self-intersection $p \in \mathbb{R}^3$ and assume that $\alpha$ passes five times through $p$, such that, at this point we have five tangent vectors $v_1, \ldots, v_5$. Let us consider three of them linearly independent and denote by $\mathcal{K}_c[\alpha]$ the $c$ knot to which $\alpha$ belongs. Let now $\beta$ be an element in the same $\mathcal{K}_c[\alpha]$ and, as $\alpha$, it shall have five tangent vectors denoted by $w_1, \ldots, w_5$ in a point $q$. In order to guarantee that $\alpha$ and $\beta$ are in the same $d$-knot, there must be a diffeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ which transforms $\alpha$ into $\beta$. The tangent application $f^*$ maps vectors of the tangent space at $p$, $T_pM$, into tangent space at $q$, $T_qM$, and it should align the tangent vectors $v_i$ to the tangent vectors $w_i$ (see Figure 2). However, $f^*$ is a linear application between three-dimensional vector spaces.

![Diffeomorphism](image1.png)

Figure 1: Schematic illustration of a diffeomorphism equivalence class.

![Equivalence](image2.png)

Figure 2: Equivalence of diffeomorphisms and linear transformations between $T_pM$ and $T_qM$. 
tangent spaces, given by the Jacobian of the matrix of \( f \) at \( p \) which depends on nine parameters. Since the directions of the five vectors \( v_i \) depend on ten parameters, it is easily seen that no linear transformation exists that aligns such five given vectors \( v_i \) to five given vectors \( w_i \) (see Figure 3). Generally, \( \alpha \) and \( \beta \) shall not belong to the same \( d \) knot. Hence, there exists a continuous parameter which is function of the angle among the five tangents - which is invariant under diffeomorphism and distinguishes \( \alpha \) from \( \beta \). This is the root of moduli spaces. In other words, we see that our problem is related to the fact that there are linearly dependent vectors which makes the problem overcomplete, which means that the set becomes redundant. This redundancy is responsible for the lost of separability of \( \mathcal{H}_{\text{diff}} \). In order to restore the separability and to avoid singular points Fairbairn and Rovelli [4] proposed a small extension of the group of homeomorphisms that are smooth, with their inverse, except possibly at a finite number of points. In what follows, we propose an alternative way of treating the above moduli problem in order to remove the redundancy which causes the nonseparability of \( \mathcal{H}_{\text{diff}} \). For this, we introduce a local redundancy, which encodes the concentration of frame vectors (to be defined later on) on the tangent spaces \( T_p M \) around points of intersections \( p \) of smooth loops \( \alpha \) in \( \mathbb{R}^3 \).

Figure 3: Schematic illustration of the diffeomorphism problem in high valence graphs.

3 Frame theory in a nutshell

When working with vector spaces we are always attached to the concept of a basis. Indeed, through a basis we can rewrite any vector as a linear combination of such
vectors of basis. However, the conditions to a basis are very restrictive: one requires that the elements are linearly independent, and very often we even want them to be orthogonal with respect to an inner product. This makes it hard, or even impossible, to find bases satisfying extra conditions, and this is the reason that one might wish to look for a more flexible tool. Frames \[5, 6\] assume this role. In linear algebra, a frame in a Hilbert space can be seen as a generalization of the idea of a basis to sets which may be linearly dependent. A frame allows us to represent any vector as a set of “frame coefficients,” and to reconstruct a vector from its coefficients in a numerically stable way.

Below, following the Ref. \[5\], we briefly review the basic definitions and results related to finite frames. Let \( \mathcal{H} \) denote an \( \ell \)-dimensional real or complex Hilbert space. In this finite-dimensional situation, \( \{f_i\}_{i=1}^m \) is called a frame for \( \mathcal{H} \), if it is a – typically, but not necessarily linearly dependent – spanning set. This definition is equivalent to asking for the existence of constants \( 0 < A \leq B < \infty \) such that

\[
A \| f \|^2 \leq \| F(f_i) \|^2 \leq B \| f \|^2, \quad \forall f \in \mathcal{H},
\]

where

\[
\| F(f_i) \|^2 = \sum_{i \in I} | \langle f, f_i \rangle |^2.
\]

The numbers \( A \) and \( B \) are called frame bounds. They are not unique. The optimal upper frame bound is the infimum over all upper frame bounds, and the optimal lower frame bound is the supremum over all lower frame bounds. Note that the optimal frame bounds actually are frame bounds. In fact,

\[
A \| f \|^2 \leq \| F(f_i) \|^2,
\]

guarantees the uniqueness of frame coefficients, since

\[
\| F(f_i) - F(f_j) \| = 0 \implies \| f_i - f_j \| = 0,
\]

while

\[
\| F(f_i) \|^2 \leq B \| f \|^2,
\]

guarantees the numerical stability in computing frame coefficients, as long as \( B \) is reasonable, since

\[
\| F(f_i + \delta f_i) - F(f_i) \| = \| F(\delta f_i) \| \leq \sqrt{B} \| \delta f_i \|.
\]
Proposition 3.1 (Christensen [5, Proposition 1.1.2]). Let \( \{f_i\}_{i=1}^m \) be a sequence in \( \mathcal{H} \). Then \( \{f_i\}_{i=1}^m \) is a frame for a vector space \( W := \text{span}\{f_i\}_{i=1}^m \), where \( W \subset \mathcal{H} \) is a proper subset of \( \mathcal{H} \).

From this proposition follows the

Corollary 3.2 (Christensen [5, Corollary 1.1.3]). A family of elements \( \{f_i\}_{i=1}^m \) in \( \mathcal{H} \) is a frame for \( \mathcal{H} \) if and only if \( \text{span}\{f_i\}_{i=1}^m = \mathcal{H} \).

According to Corollary 3.2, a frame could support more elements than is demand for a basis, since the only requirement is that the elements generate all the space. In particular, given a frame \( \{f_i\}_{i=1}^m \) for \( \mathcal{H} \) and \( \{g_i\}_{i=1}^n \) an arbitrary collection of vectors in \( \mathcal{H} \), then, \( \{f_i\}_{i=1}^m \cup \{g_i\}_{i=1}^n \) also is a frame for \( \mathcal{H} \). A frame that is not a basis is called overcompleted, or redundant.

Let us consider a Hilbert space \( \mathcal{H} \), \( \ell \)-dimensional, equipped with a frame \( \{f_i\}_{i=1}^m \), with \( m \geq \ell \), and define the following application:

\[
T : \mathbb{C}^m \rightarrow \mathcal{H}, \quad T\{c_k\}_{k=1}^m = \sum_{k=1}^m c_k f_k, \tag{3.2}
\]

where \( T \) is called pre-frame operator and its adjoint is given by

\[
T^* : \mathcal{H} \rightarrow \mathbb{C}^m, \quad T^* f = \{\langle f, f_k \rangle\}_{k=1}^m. \tag{3.3}
\]

Notice that in terms of the frame operator, we have

\[
\langle Ff, f \rangle = \sum_{k=1}^m |\langle f, f_k \rangle|^2, \quad \forall f \in \mathcal{H}. \tag{3.4}
\]

Thus, the lower frame bound might be sought as a kind of the lower bound of the frame operator. A frame \( \{f_k\}_{k=1}^m \) is called tight if we choose \( A = B \) in the frame definition, that is,

\[
\sum_{k=1}^m |\langle f, f_k \rangle|^2 = A\|f\|^2. \tag{3.5}
\]

Notice that besides the tight requirement, we are assuming the optimal frame bounds given by the infimum and the supremum. We should also notice that the upper bound condition is always assured by the Cauchy-Schwarz-Bunjakowski inequality. Hence, our task is to determine the lower frame bound. Thus, we consider the following
Proposition 3.3 (Christensen [3, Proposition 1.1.4]). Let \( \{f_k\}_{k=1}^m \) be a tight frame for \( \mathcal{H} \) with frame bound \( A \). Let \( F = A \mathbb{I} \), where \( \mathbb{I} \) is the identity operator in \( \mathcal{H} \), then

\[
f = \frac{1}{A} \sum_{k=1}^m \langle f, f_k \rangle f_k, \quad \forall f \in \mathcal{H}.
\] (3.6)

We see that the above representation is quite similar to the representation of an orthonormal basis with the factor \( 1/A \) which makes the difference.

Theorem 3.4 (Christensen [3, Theorem 1.1.5]). Let \( \{f_k\}_{k=1}^m \) be a frame for \( \mathcal{H} \) with a frame operator \( F \). Then the following properties are satisfied:

(i) \( F \) is invertible and self-adjoint.

(ii) Every \( f \in \mathcal{H} \) can be represented as

\[
f = \sum_{k=1}^m \langle f, F^{-1} f_k \rangle f_k = \sum_{k=1}^m \langle f, f_k \rangle F^{-1} f_k.
\] (3.7)

Every frame in a finite space contain a subset which is a basis. Hence, if \( \{f_k\}_{k=1}^m \) is a frame but not a basis, there exists a sequence of nonzero elements \( \{d_k\}_{k=1}^m \) such that \( \sum_{k=1}^m d_k f_k = 0 \). So, for all element \( f \in \mathcal{H} \) we can write it as

\[
f = \sum_{k=1}^m \langle f, F^{-1} f_k \rangle + \sum_{k=1}^m d_k f_k
\]

\[
= \sum_{k=1}^m \left( \langle f, F^{-1} f_k \rangle + d_k \right) f_k.
\]

This shows us that \( f \) has many representations as superposition of elements of the frame. We shall verify in some examples, quite simple, how to use these frames in order to “eliminate” the redundancy of basis overcompleted and see how these can be applied in the case of high valence graphs.

4 The Grot-Rovelli case revisited

To begin with, let us consider a three-dimensional space with a basis \( \{v_k\}_{k=1}^3 \) and let \( \{v_k, \sum_{k=1}^3 \alpha_k v_k, \sum_{k=1}^3 \beta_k v_k\} \) be a frame. From the frame condition we obtain
that

\[ A\|f\|^2 \leq \sum_{k=1}^{5} |\langle f, f_k \rangle|^2 \]

\[ A\|f\|^2 \leq \|f\|^2 + \sum_{k=1}^{3} \left( |\alpha_k|^2 + |\beta_k|^2 \right) |\langle f, v_k \rangle|^2 \]

\[ A \leq 1 + \frac{\sum_{k=1}^{3} \left( |\alpha_k|^2 + |\beta_k|^2 \right) |\langle f, v_k \rangle|^2}{\|f\|^2} . \]

We can see that all redundant information is constrained in the frame bounds, and since these bounds are not unique we can take it as optimal ones. In this case we should take the infimum of the values

\[ \inf \left\{ 1 + \frac{\sum_{k=1}^{3} \left( |\alpha_k|^2 + |\beta_k|^2 \right) |\langle f, v_k \rangle|^2}{\|f\|^2} \right\} , \quad (4.1) \]

where we can eliminate the “extra” information because in this case \( A = 1 \). This process can be done for \( n \) redundant elements and it is interesting that we can restore from the frame bounds only the necessary “information”. We can also notice that we had, initially, a overcomplete set from the perspective of basis. Nonetheless, when looking at it as a frame and applying the definitions we recover the expansion only in terms of the basis elements. Moreover, we could add to a basis any number of elements, linearly dependent, which in the frame point of view we recover only what is really necessary, in other words (as in signal analysis) we can filter all redundant information that makes the set overcomplete.

Therefore, if we consider a \( Q \)-valence graph in a three-dimensional hypersurface, we could look at the tangent vectors at the knot as forming a frame and as shown above, filter the information which generates the moduli structure and the nonseparability of \( \mathcal{H}_{\text{diff}} \), without need to extend the diffeomorphism group.

### 4.1 Redundancy

The redundancy of a frame, at least in finite dimensions, is a precise quantitative notion. In fact, the following theorem shows that the redundancy of a frame can be measured through the frame bounds \( A \) and \( B \).
Teorema 4.1 (Mallat [6, Theorem 5.2]). In a space of finite dimension $N$, a frame of $M \geq N$ normalized vectors has frame bounds $A$ and $B$, which satisfy

$$A \leq \frac{M}{N} \leq B.$$  

For a tight frame $A = B = M/N$. As this theorem is valid for any finite $M$, we can apply it in the situation treated in Ref. [4] in order to avoid moduli structures, without requiring the extension of the diffeomorphism group at a quantum level.

### 4.2 Q-Valence Graphs in LQG

Now consider a $Q = 4$ valence graph in LQG where three of them are linearly independent, for example, $v_1, v_2, v_3$. We shall denote them as $\{\varphi_i\} = \{v_1, v_2, v_3, v_4 = \sum_{i=1}^{3} \alpha_i v_i\}$. Then, for any vector $f$ on $H$ we have

$$\sum_{i=1}^{4} |\langle f, \varphi_i \rangle|^2 = \sum_{i=1}^{3} |\langle f, v_i \rangle|^2 + |\langle f, \sum_{i=1}^{3} \alpha_i v_i \rangle|^2$$

$$= \|f\|^2 + \sum_{i=1}^{3} |\langle f, \alpha_i v_i \rangle|^2$$

$$\leq \|f\|^2 + \sum_{i=1}^{3} \alpha_i^2 |\langle f, v_i \rangle|^2$$

$$\leq \sum_{i=1}^{3} (1 + \alpha_i^2) |\langle f, v_i \rangle|^2.$$  

On the other hand, by the Theorem 4.1, if we are dealing with a tight frame, then

$$\sum_{i=1}^{4} |\langle f, \varphi_i \rangle|^2 = \frac{4}{3} \sum_{i=1}^{3} |\langle f, v_i \rangle|^2.$$  

(4.3)

Note that the expansion coefficients of the redundant vector yield the following value for $\alpha$:

$$\alpha_i^2 = \frac{1}{3}.$$  

Finally, let us consider again the simple case considered by Grot-Rovelli [3] in which a moduli space appear, which is a five valence graph. Hence, we have
an intersection point \( p \) crossed by the loop \( \alpha \) five times. Let us fix an arbitrary coordinate chart in the neighborhood of \( p \), and let \( v_i \) for \( i = 1, 2, 3, 4, 5 \) be the components of the five tangents of the loop \( \alpha \) at \( p \). Assume any three of the five vectors are linearly independent, again \( v_1, v_2, v_3 \) (these three vectors define a basis in the tangent space at \( p \)), then the set \( \{ \varphi_i \} = \{ v_1, v_2, v_3, v_4 = \sum_{i=1}^{3} \alpha_i v_i, v_5 = \sum_{i=1}^{3} \beta_i v_i \} \) is used to build the frame. Thus, we obtain a constrained redundancy

\[
\alpha_i^2 + \beta_i^2 = \frac{2}{3}.
\]

Naturally, for an \( Q \)-valence graph in LQG (with a finite set of redundant vectors) we obtain that

\[
\sum_{j=1}^{M-N} \alpha_{i,j}^2 = \frac{(M - N)}{N},
\]

where \( \alpha_{i,j} \) is the \( i \)-th coefficient of the \( j \)-th redundant vector, \( M \) is the total number of elements in the frame (basis + redundancy) and \( N \) is the dimension of the space. Thus, we see that through the perspective of a frame we expand a set, \textit{a priori}, larger than a basis in a stable way and convert the redundancy into a \textit{multiplicative coefficient} that measures and controls the problem of high valence graphs in LQG. In turn, since the redundant vectors can be removed in a stable way, we do not have more the problem of the mapping between tangent spaces that requires linearity. In fact, the redundancy \textit{does not change} once an invertible operator is applied to a frame [7]. Thus, the tangent map \( f^*: T_p M \to T_q M \) is allowed and we do not need to extend the diffeomorphism group. In conclusion, by using the setting of frames, the moduli structures are not anymore present, and the kinematical Hilbert space of the diffeomorphism invariant states of LQG is separable.

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