DLCQ Strings, Twist Fields  
and One-Loop Correlators on a Permutation Orbifold

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Abstract

We investigate some aspects of the relationship between matrix string theory and lightcone string field theory by analysing the correspondence between the two-loop thermal partition function of DLCQ strings in flat space and the integrated two-point correlator of twist fields in a symmetric product orbifold conformal field theory at one-loop order. This is carried out by deriving combinatorial expressions for generic twist field correlation functions in permutation orbifolds using the covering surface method, by deriving the one-loop modification of the twist field interaction vertex, and by relating the two-loop finite temperature DLCQ string theory to the theory of Prym varieties for genus two covers of an elliptic curve. The case of bosonic $\mathbb{Z}_2$ orbifolds is worked out explicitly and precise agreement between both amplitudes is found. We use these techniques to derive explicit expressions for $\mathbb{Z}_2$ orbifold spin twist field correlation functions in the Type II and heterotic string theories.
1 Introduction and Summary

Large $N$ matrix field theories obtained as dimensional reductions of maximally supersymmetric $U(N)$ Yang-Mills theory in ten spacetime dimensions provide nonperturbative descriptions of M-theory and string theory in various backgrounds, and associated superconformal field theories (see [1] for a review). The best understood example is matrix string theory [2]–[4] which takes the form of maximally supersymmetric $U(N)$ Yang-Mills theory in two dimensions. In this case the gauge coupling is inversely proportional to the string coupling, so that the free string limit corresponds to the infrared limit and the first order interaction term to the least irrelevant operator in the gauge theory. In this strong coupling limit the supersymmetric Yang-Mills theory approaches a superconformal fixed point which is conjectured to be the supersymmetric sigma model on the symmetric product orbifold $\mathbb{R}^8/N/S_N$. The spectrum of this orbifold superconformal field theory can be canonically identified with that of the free second quantized Type IIA string [4, 5].

This equivalence is demonstrated qualitatively in discrete light-cone quantization (DLCQ) by matching configurations obtained by gluing different copies of the strings winding around a light-like circle to twisted sectors of the symmetric product. The quantitative demonstration is given by matching the torus partition function of the superconformal field theory [5] to the thermodynamic free energy of the free Type IIA superstring [6]. It is conjectured [4] that the equivalence holds generally in the interacting string theory as well. Strings interact by means of splitting and joining, and the interaction points correspond to insertions of twist field operators in the orbifold superconformal field theory. It has been recently argued [7]–[10] that the structure of the contact interactions in Green-Schwarz light-cone superstring field theory simplifies within the twist field formulation of matrix string theory. Unlike the light-cone string field theory, however, the matrix model provides a full nonperturbative definition of the string dynamics in the large $N$ limit.

In this paper we will investigate this conjectural perturbative correspondence further by examining the relationship between the thermodynamic free energy of Type II superstring theory in DLCQ and correlation functions of the leading irrelevant twist field operators in the symmetric product orbifold conformal field theory. The Polyakov path integral for the former quantity is known [11] to truncate the sum over contributing string worldsheets to those which are branched covers of the spacetime torus arising from the null compactification at finite temperature. The free energy at the leading non-vanishing order in the string coupling constant is the two-loop string amplitude which has been calculated in [12]. In order to check the conjecture one needs to compute the corresponding amplitude in the orbifold conformal field theory, which is given by the one-loop two-point function of appropriate twist fields. These operators create twisted sectors out of the vacuum state, in that the local fields of the sigma model acquire non-trivial monodromy about the twist field insertion points. Computing their correlation functions is thus not straightforward, and a good portion of our analysis will centre around the technicalities involved in these calculations.

There are several strategies presented in the literature for computing twist field correlation functions. The stress tensor method was originally introduced in [13] and used to compute $Z_2$ orbifold [13]–[15], and more generally $Z_N$ orbifold [16, 17], correlation functions on worldsheets of arbitrary topology, and $S_N$ orbifold correlation functions on the sphere [18, 19]. In
this method one first determines the twisted Green’s function (the $n$-point function of the stress energy tensor in the twisted sector) by demanding the correct short distance behaviour and monodromy about the twist field insertion points. A closely related but more general technique is the covering space method. It makes direct use of the fact that a monodromy is associated to a covering surface. If a field is multi-valued when transported around a closed curve, then it is well-defined as a single-valued function on the appropriate cover of the worldsheet without any special points. In this way the twist field correlation functions can be expressed as vacuum amplitudes of the free conformal field theory on the covering surfaces. This method was exploited in [20]–[22]. It is also the main principle behind computing essentially all quantities in permutation orbifolds as shown in [23] where, in particular, the partition function was given for arbitrary orbifold twist group.

In this paper we use the covering space method for the definition and computation of twist field correlation functions in symmetric products defined on worldsheets of non-trivial topology. The vacuum amplitudes of these conformal field theories are known in complete generality, i.e., for worldsheets of arbitrary genus and arbitrary finite twist group [24]. We generalize these results to the $n$-point correlation functions of twist field operators. When the worldsheet has non-trivial fundamental group and the twist group is nonabelian, the definition of the corresponding twisted Green’s functions is problematic and the covering space technique is the only possible way to define the amplitudes. To make these formulæ completely explicit, one needs to determine the dependence of the complex structure of the covering space on that of the worldsheet and the location of the twist field operator insertions. This is a very difficult problem in the general case when the covering surface does not admit any conformal automorphisms. We have not been able to solve this problem in full generality and are not aware of any solution to it for any specific cases of such a cover. All known computations of twist field correlation functions are done with respect to covers with automorphisms (this is the case, in particular, for the $\mathbb{Z}_N$ orbifolds), or to worldsheets of trivial topology when the covering space can be parametrized explicitly in terms of the complex coordinate $z$ of the sphere. Nevertheless, using our technique we are able to determine the bosonic two-point twist field correlation function of the orbifold $\mathbb{R}^{2d} \rtimes \mathbb{Z}_2 := (\mathbb{R}^{24} \times \mathbb{R}^{24})/\mathbb{Z}_2$ and compare it to the appropriate power of the $\mathbb{Z}_2$ orbifold twist field correlation function of the one-dimensional free boson computed in [21], yielding a highly non-trivial check of our methods. Although throughout we deal only with orbifolds of flat space $\mathbb{R}^d$, most of our considerations and results apply to more general symmetric products as well.

When writing down generating functions of amplitudes in symmetric products, one has to sum over all covers of the worldsheet in such a way that only the connected covering surfaces contribute. This fact lies behind the conjecture that these amplitudes naturally arise in physical string theories. We generalize the resummation procedure which was done originally for the torus partition function in [5] and for the Klein bottle amplitude in [25] for the case of closed strings, and then for the annulus and Möbius diagrams in [26, 27] for the case of open strings. The generalization to the twist field $n$-point function is possible due to a general combinatorial formula [25] which is the crux of all of these calculations.

The main technical achievement of the two-loop calculation of [12] was a modification of the Weierstrass-Poincaré theory of reduction. Reduction may be described entirely in terms of the Riemann matrix of periods of a curve, and it has the effect of expressing theta functions at a given genus in terms of lower dimensional theta functions. This happens exactly when the curve
in question covers a surface of lower genus (but it may also occur without there being a covering map). The remarkable feature of this reduction is the simple universal form that the genus two DLCQ free energy takes in terms of Jacobi elliptic functions on the base torus. For the contributions from double covers of the torus to the two-loop free energy of the critical bosonic string, we find perfect agreement between the string free energy and the correlator of twist fields computed as the appropriate power of the $\mathbb{Z}_2$ orbifold twist field two-point function of [21]. We will find generally that the original genus zero interaction vertex proposed in [4] must be modified at one-loop order to ensure equivariance under the action of the non-trivial modular group in this instance. Since the structure of the result depends only on the orbifold twist group and not on the data of the specific string theory, we use this equivalence and the known formulae from [12] for the two-loop DLCQ free energy of the Type II and heterotic strings to derive the two-point functions of the appropriate spin twist fields in the corresponding $\mathbb{Z}_2$ orbifold superconformal field theories. To the best of our knowledge, these correlation functions have not been previously computed, and our explicit formulae should be useful for further clarifying the role of the twist field interaction vertex in light-cone string field theory.

The difficulty in establishing the correspondence is writing down the period matrix of the covering surface explicitly in terms of the modulus of the worldsheet torus and the branch point loci. This is achieved in part by elucidating the geometric meaning of the reduced genus two period matrix. In [12] it was shown that this period depends on two elliptic moduli, one of which lies in a modular orbit of an unramified (one-loop) cover of the base torus. Here we show that the second elliptic modulus determines the complex structure of a Prym variety. Prym varieties arise in special instances of covering surfaces. A theorem due to Mumford asserts that there are only three types of branched covers which give rise to Prym varieties, namely unramified double covers, ramified double covers with two branch points, and precisely our instance of genus two covers over an elliptic curve. When this is in addition a double cover of the torus, we use the canonical involution of the genus two surface to explicitly construct the dependence of the periods on the branch points (which are the images of the fixed points of the involution). This procedure unfortunately doesn’t generalize to higher degree covering surfaces (although the identification with a Prym variety always holds).

The organisation of the rest of this paper is as follows. In Section 2 we give a general introduction to the theory of bosonic permutation orbifolds, and use the one-loop sigma model to illustrate the typical combinatorial structure of amplitudes therein. We apply the combinatorial resummation formula for symmetric products to compute a large class of correlation functions which are invariant under the action of the twist group. We show how to generalize these formulae to correlation functions of twist field operators, and briefly review the structure of the DLCQ string partition function. In Section 3 we present detailed and explicit calculations for $\mathbb{Z}_2$ orbifolds. In the course of this analysis, we make the generic connection between DLCQ string theory and the theory of Prym varieties, and also derive the explicit modification of the twist field interaction vertex for toroidal worldsheets in the symmetric product sigma model. In Section 4 we discuss the technical issues surrounding the generalizations of these results to $S_N$ orbifolds with $N > 2$. We examine the uniformization construction, which is used to build vacuum amplitudes, in the context of a generic twist field $n$-point function, and the problem of determining the period of the genus two covering surface in terms of the branch point data. We also study the combinatorial expansion in more detail and indicate that, while computable in principle, the combinatorics become very non-trivial for $N > 2$. Finally, in Section 5 we
describe the modifications of permutation orbifolds required in the presence of fermionic degrees of freedom, and of twist field correlation functions therein. We then apply these and previous considerations to derive explicit formulae for the one-loop spin twist field correlation functions in the $\mathbb{Z}_2$ orbifold supersymmetric and heterotic string theories.

2 Correlation Functions on Permutation Orbifolds

In this section we will discuss some general aspects of permutation orbifolds of conformal field theories, and in particular the case of two-dimensional sigma models on symmetric product orbifolds of flat space. We first describe the general structure of the partition functions of these models, and then explain the construction of various classes of correlation functions including those of twist field operators. For the moment we treat only bosonic sigma models explicitly in order to highlight the essential details, deferring a more detailed analysis of the supersymmetric and heterotic cases to Section 5. We also explain how these orbifold theories can be interpreted as string field theories.

2.1 Permutation Orbifolds

When a two-dimensional conformal field theory has a discrete symmetry, one can consider the orbifold theory arising from quotienting with respect to the symmetry. The simplest example is the free boson on the circle $S^1$. Its action $\frac{1}{4\pi\alpha'} \int_{\Sigma} d^2z \|\partial X\|^2$ is invariant under the reflection $X \rightarrow -X$. The quotient of the target space is the well-known geometric orbifold $S^1/\mathbb{Z}_2$, and the coordinate field $X$ can have non-trivial monodromy when encircling a non-contractible cycle of the worldsheet $\Sigma$. If the radius of the circle is equal to the fundamental string length $\ell_s = \sqrt{\alpha'}$, then the resulting orbifold conformal field theory is the “square” of the critical Ising model [28].

Permutation orbifolds represent a large class of orbifolds where the parent conformal field theory (whose quotient is taken) has physical Hilbert space $\mathcal{H}$ with a discrete symmetry. This concept was first introduced in [29], and used for the construction of a $\mathbb{Z}_2$ orbifold of the $E_8 \times E_8$ heterotic string in [30]. One of their main applications is to the second quantization of string theory [5], and they have recently been argued [31] to describe new physical string theories at multiples of the critical dimension. A permutation orbifold of an arbitrary conformal field theory $\mathcal{C}$, by any finite symmetry group $G$ regarded as a subgroup of a symmetric group of some degree, is a consistent conformal field theory. All of its important quantities (central charge, conformal weights, genus one characters, modular $S$ and $T$ matrices, genus one partition function, etc.) were worked out originally for cyclic groups in [32], and then generalized to arbitrary finite groups in [23]. These formulae express a given quantity as a combinatorial expansion, depending on the twist group $G$, of the same quantity in the parent theory.

Highest weight states in a permutation orbifold $\mathcal{C} \lhd G := (\mathcal{C})^{\otimes N}/G$ correspond to orbits of a subgroup $G < S_N$ of the symmetric group of degree $N$ acting on the $N$-fold tensor product of states in the parent theory $\mathcal{C}$.\footnote{There is an additional label corresponding to the irreducible character of the double of the stabilizer of the orbit. See [23] for the precise definition.} In the case that $\mathcal{C}$ admits a sigma model description with embedding coordinate field $X \in M$, there is a corresponding sigma model description of $\mathcal{C} \lhd G$ on the geometric orbifold $M^N/G$ [26]. One introduces $N$ identical coordinate fields $X_a = X$, etc.
a = 1, . . . , N on the worldsheet Σ and allows for G-twisting of them along non-trivial cycles. For example, on the torus Σ = T with modulus τ, the boundary conditions of the N coordinate fields are labelled by two commuting permutations P, Q ∈ G < SN such that

\[ X^a(z + 1) = X^{P(a)}(z) \quad \text{and} \quad X^a(z + \tau) = X^{Q(a)}(z), \quad (2.1) \]

where in general g(a) denotes the image of the label a under the permutation g ∈ G. For a non-trivial pair (P, Q), these boundary conditions are called twisted sectors of the theory. Two pairs (P, Q) and (gPg−1, gQg−1) with g ∈ SN correspond to the same twisted sector, since we can get from one to the other by relabelling the coordinate fields \( a \rightarrow g(a) \).

In general, a twisted sector is given by an equivalence class of homomorphisms from the fundamental group \( \pi_1(Σ) \) of the worldsheet to the twist group G. (Since \( \pi_1(T) = \mathbb{Z}⊕\mathbb{Z} \), on the torus one specifies a homomorphism by choosing the image in G of the two commuting generators.) Two homomorphisms \( Φ, Φ' \) define the same twisted sector, and are said to be equivalent, if they are related by conjugation as \( Φ'(−) = gΦ(−)g^{-1} \) for some \( g ∈ SN \). The geometric interpretation is provided by the fact that every equivalence class \([Φ]\) of homomorphisms \( Φ : \pi_1(Σ) \rightarrow SN \) determines an unramified cover \( \hat{Σ} \) of degree N over the Riemann surface Σ. The coordinate label \( a \) corresponds to the label of a sheet and \( Φ \) is called the monodromy homomorphism of the covering. Conjugation of homomorphisms corresponds to relabelling of the sheets. In the case of the torus \( Σ = T \) with the boundary conditions (2.1), and with the subgroup generated by the pair of permutations P, Q acting transitively on the set of coordinate labels \( a = 1, . . . , N \), one can define a single new field \( X(z) \) which generates all of the fields \( X^a(z) \) through the identifications

\[ X(z + m + nτ) = X^{P^mQ^n(a)}(z) \quad (2.2) \]

with \( n, m ∈ \mathbb{Z} \) and a fixed choice of \( a \). This field is single-valued on a torus which is a cover of the original torus T, whose modular parameter can be determined from the doubly periodic function \( X(z) \) on T.

The modular invariant partition function of the permutation orbifold is determined entirely by the above data. It is given by [23] \(^2\)

\[ Z^G(τ) = \frac{1}{|G|} \sum_{Φ : \pi_1(Σ) \rightarrow G} \left( \prod_{ξ ∈ O(Φ)} Z(τξ) \right) \quad (2.3) \]

where the product runs over the orbits \( ξ \) of the image \( Φ(π_1(Σ)) \) in G and \( Z(τξ) \) is the modular invariant partition function of the parent conformal field theory on the connected component, corresponding to \( ξ \), of the cover of Σ given by the homomorphism \( Φ \). (The covering space \( \hat{Σ} \) is connected if and only if \( Φ(π_1(Σ)) \) acts transitively in \( SN \). The summation over \( Φ \) defines the projection onto \( G \)-invariant states and ensures modular invariance of the partition function. We will now explain how to determine (2.3) in practice.

The complex structure τ of the worldsheet \( Σ = Στ \) is encoded by a monomorphism \( u : π_1(Σ) \rightarrow I \), where I is the isometry group of the universal cover U of Σ. For genus \( g > 1 \) the latter space is a two-dimensional hyperbolic space, say the upper half plane \( U = \mathbb{U} \), and \( I = PSL(2, \mathbb{R}) \). The surface Σ equipped with a complex structure can be presented as

\(^2\)It is also expressible as a sesquilinear expansion in the Virasoro characters \( \text{Tr}_N(q^{L_0-c/24}) \), whose form is known in permutation orbifolds [23, 33].
the quotient $\Sigma_\tau = \mathbb{C}/u(\pi_1(\Sigma))$ and its complex structure inherited from $\mathbb{C}$ is encoded by the uniformizing group $u(\pi_1(\Sigma))$. Given a monodromy homomorphism $\Phi$, the fundamental group of the corresponding cover $\hat{\Sigma}$ is isomorphic to the stabilizer subgroup $H_a = \pi_1(\Sigma)_\xi := \{ \gamma \in \pi_1(\Sigma) | \Phi(\gamma)(a) = a \}$ with fixed $a \in \xi$ (represented by closed loops based at sheet $a$). Its index is equal to the length of the orbit $[\pi_1(\Sigma) : H_a] = |\xi|$, which is the number of sheets of the corresponding connected component of $\hat{\Sigma}$. Thus the monodromy homomorphism determines the topology of the covering space $\hat{\Sigma}$. We can now define the uniformizing group (and hence the complex structure $\tau$) of the cover $\Sigma$ to be given by $u(H_a)$ (i.e., $\Sigma_{\tau, u} = \mathbb{C}/u(H_a)$), which is a subgroup of $u(\pi_1(\Sigma))$ in accordance with the expected property $\pi_1(\Sigma) < \pi_1(\hat{\Sigma})$. Note that the representative of the orbit $a \in \xi$ can be arbitrarily chosen. This is because $H_a = \gamma H_a \gamma^{-1}$ with $\gamma \in \pi_1(\Sigma)$ for any $a, a' \in \xi$ and conjugate subgroups of $\pi_1(\Sigma)$ give rise to isometric quotients, hence determining equivalent surfaces. The homomorphism $u$ is not unique, as it can be composed with a modular transformation, but the partition function is modular invariant which makes the formula (2.3) well defined.

The expression (2.3) is an example of the typical structure of a quantity defined on a Riemann surface $\Sigma$ in a permutation orbifold. It is given by a combinatorial expansion (depending only on $G$) over the same quantity in the parent theory $\mathcal{C}$ defined on all of those surfaces which cover $\Sigma$ whose monodromy group is a subgroup of $G$. Its direct applicability is limited somewhat by the Riemann-Hurwitz formula for the genus $g$ of the unramified cover $\hat{\Sigma}$ given by

$$g = N(g - 1) + 1.$$ (2.4)

This implies that, unless $g = 1$, we would need to know the partition functions of the parent theory on surfaces of genera higher than $g$ in order to write down the genus $g$ partition function of the orbifold.

The case $g = 1$ is, however, much simpler. The universal cover of the torus $T$ is $U = \mathbb{C}$ and $I = \{T_c | c \in \mathbb{C} \}$ is the group of translations $T_c : z \mapsto z + c$ of the complex plane. We saw above that specifying a homomorphism $\Phi$ amounts to assigning commuting elements $P, Q \in G$ for the generators $(\alpha, \beta)$ of $\pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$. The stabilizer subgroup $H_a$ of any representative of an orbit $a \in \xi$ can be characterized by three positive integers $s, m, r$ such that $r$ is the smallest positive integer satisfying $P^r(a) = a$, $0 \leq s < r$ and $H_a$ is generated by $a^r, a^s a^m$. Then the index of this subgroup is given by $|\xi| = rm$. The image of $H_a$ under the isomorphism $u : (\alpha, \beta) \mapsto (T_1, T_\tau)$ determines a subgroup $< T_\tau, T_{s+m, \tau} >$ and the corresponding quotient of $\mathbb{C}$ is the torus with Teichmüller parameter given by

$$\tau^\xi = \frac{s + m \tau}{r}.$$ (2.5)

The fact that the finite index subgroups of the group $\mathbb{Z} \oplus \mathbb{Z}$ are all isomorphic to the group itself implies that all unramified covers of the torus are tori. In sigma model language, the path integral over the multi-valued fields $X^a$ on the torus $T$ is constructed by calculating the path integral over the single-valued field $X$ on the covering torus and summing over every possible $X$ constructed by different choices of the commuting pair $P, Q \in G$. For example, the genus one partition function of the $S_3$ orbifold is given by [34]

$$Z^S_3(\tau) = \frac{1}{6} Z(\tau)^3 + \frac{1}{2} Z(\tau) \left( Z(2\tau) + Z(\frac{\tau}{3}) + Z(\frac{\tau - 1}{3}) \right)$$

$$+ \frac{1}{3} \left( Z(3\tau) + Z(\frac{\tau}{3}) + Z(\frac{\tau + 1}{3}) + Z(\frac{\tau - 2}{3}) \right).$$ (2.6)

Note that the individual terms in (2.6) are not modular invariant, but their sum is.
2.2 Symmetric Products

Permutation orbifolds whose twist group $G$ is the full symmetric group $S_N$ are called symmetric products $\text{Sym}^N(C) := (C)^{S_N}/S_N$. In this case the formula (2.3) takes into account all $N$-sheeted coverings. Starting from a fixed parent theory $C$ and a given worldsheet genus $g$, the generating function of partition functions for all $N$ can be written in a closed form thanks to a combinatorial identity due to Bántay [25]. This identity translates the sum over homomorphisms in (2.3) to a sum over finite index subgroups of the group $\Gamma = \pi_1(\Sigma)$ and is given by

$$1 + \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{\Phi: \Gamma \rightarrow S_N} \left( \prod_{\xi \in O(\Phi)} \mathcal{Z}(\Gamma_\xi) \right) = \exp \left( \sum_{H \subseteq \Gamma} \frac{\mathcal{Z}(H)}{[\Gamma : H]} \right), \quad (2.7)$$

where $\Gamma_\xi$ is the stabilizer of the orbit $\xi$ and $[\Gamma : H]$ denotes the index of the subgroup $H$ in $\Gamma$. The formula (2.7) holds generally for any finitely generated group $\Gamma$ and any conjugation invariant function $\mathcal{Z}$ (i.e., $\mathcal{Z}(\gamma H \gamma^{-1}) = \mathcal{Z}(H)$ for all $\gamma \in \Gamma$) from the set of finite index subgroups of $\Gamma$ to a commutative ring $R$.

The proof of (2.7) is instructive. A given term $\prod_{\xi} \mathcal{Z}(\Gamma_\xi)$ in the sum on the left-hand side of (2.7) depends only on the equivalence class of the homomorphism $\Phi$. An equivalence class can be written as

$$[\Phi] = \bigoplus_{k=1}^{N} n_k \phi_k, \quad (2.8)$$

where $\phi_k$ is a transitive equivalence class whose orbits all have length $k$ and $n_k \geq 0$ is its integer multiplicity with $\sum_k n_k = N$. One can then rewrite the product $\prod_{\xi} \mathcal{Z}(\Gamma_\xi) = \prod_k \mathcal{Z}(\Gamma_k)^{n_k}$, where $\Gamma_k$ is the stabilizer subgroup of an arbitrary representative of the image of $\phi_k$ in $S_N$. The cardinality of the equivalence class $[\Phi]$ can be determined as follows. The total number of possible elements to conjugate with is $|S_N| = N!$, but not all of these give inequivalent homomorphisms $\Phi$. The permutations which exchange the orbits that have the same $S_N$-action do not change $[\Phi]$, so we have to divide by their number which is $n_k!$. Finally, we have to divide out the number of cosets $\gamma \Gamma_k$ with $\gamma \Gamma_k \gamma^{-1} = \Gamma_k$, which is the index $\gamma_k = [N_{\Gamma}(\Gamma_k) : \Gamma_k]$ of the stabilizer $\Gamma_k$ in its normalizer subgroup $N_{\Gamma}(\Gamma_k)$. Thus $|[\Phi]| = N!/\prod_k n_k! \gamma_k^{n_k}$.

One can now rewrite the left-hand side of (2.7) as

$$1 + \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{\{n_k\}} \frac{N!}{\prod_{k=1}^{N} n_k! \gamma_k^{n_k}} \left( \prod_{k=1}^{N} \mathcal{Z}(\Gamma_k)^{n_k} \right) = \prod_{k=1}^{\infty} \left( \sum_{n_k=0}^{\infty} \frac{\mathcal{Z}(\Gamma_k)^{n_k}}{n_k! \gamma_k^{n_k}} \right) = \prod_{k=1}^{\infty} \exp \left( \frac{\mathcal{Z}(\Gamma_k)}{\gamma_k} \right). \quad (2.9)$$

Note that here a summation over conjugacy classes of index $k$ subgroups is implicitly assumed. The final step consists in rewriting the product of exponentials as the exponential of a sum over $k$, and then translating the latter summation into a sum over index $k$ subgroups. There are $[\Gamma : N_{\Gamma}(\Gamma_k)]$ distinct subgroups in the conjugacy class of $\Gamma_k$ (as $\gamma \Gamma_k \gamma^{-1} \neq \Gamma_k$ if $\gamma \notin N_{\Gamma}(\Gamma_k)$), so we need to divide by this number if we wish to sum over all index $k$ subgroups. Then the resulting factor in the denominator

$$\gamma_k [\Gamma : N_{\Gamma}(\Gamma_k)] = [N_{\Gamma}(\Gamma_k) : \Gamma_k] [\Gamma : N_{\Gamma}(\Gamma_k)] = [\Gamma : \Gamma_k] = k \quad (2.10)$$
is precisely the index of $\Gamma_k$ in $\Gamma$ and we have arrived at (2.7).

Let us now apply the identity (2.7) to the uniformizing group $\Gamma = u(\pi_1(\Sigma))$ of a compact Riemann surface $\Sigma = \Sigma_\tau$ with the definition

$$Z(H) := Z(\tau^H) \kappa^{[\Gamma:H]} \quad (2.11)$$

where $Z(\tau^H)$ is the modular invariant partition function of $\mathcal{C}$ defined on the surface $\Sigma_{\tau^H} = U/H$, with $U$ the universal cover of $\Sigma_{\tau}$, and $\kappa$ is a formal variable which is determined by physical constants in applications. The result is the grand canonical partition function

$$Z^{Sym}(\tau, \kappa) := 1 + \sum_{N=1}^{\infty} \kappa^N Z^{SN}(\tau) = \exp \left( \sum_{N=1}^{\infty} \kappa^N \mathcal{H}_N Z(\tau) \right), \quad (2.12)$$

where $Z^{SN}(\tau)$ is the partition function for the $S_N$ orbifold given by the formula (2.3) and the operator $\mathcal{H}_N$ is defined on modular invariant functions by

$$\mathcal{H}_N Z(\tau) = \frac{1}{N} \sum_{[\Gamma:H]=N} Z(\tau^H). \quad (2.13)$$

Note that the product over the orbits $\xi$ in (2.7) gives a sum for the power of $\kappa$ equal to $\sum_{\xi} [\Gamma : \Gamma_\xi] = \sum_{\xi} |\xi| = N$. This generating function is a sum over all possible (finite-sheeted) covers of the surface $\Sigma$ that the parent conformal field theory $\mathcal{C}$ is defined on, and its logarithm gives the restricted sum over connected covers. The operator defined by (2.13) yields a sum over subgroups $H < \pi_1(\Sigma)$ of index $N$, and in the case of the torus $\Sigma = T$ it coincides with the Hecke operator acting on the partition function of the parent theory by

$$\mathcal{H}_N Z(\tau) = \frac{1}{N} \sum_{r \cdot m \equiv N} \sum_{s \in \mathbb{Z}/r \mathbb{Z}} Z\left( \frac{s + m \tau}{r} \right). \quad (2.14)$$

### 2.3 Sigma Models at One-Loop

Our primary example of a permutation orbifold in this paper will be that of sigma models on symmetric products of flat space $\mathbb{R}^d$ at one-loop order in string perturbation theory. Let us describe this example explicitly in the case of a single boson $X$ in $\mathbb{R}$. The path integral of the sigma model conformal field theory on a symmetric product is gotten by considering the grand canonical partition function

$$Z^{Sym}(\tau, \kappa) = 1 + \sum_{N=1}^{\infty} \kappa^N \sum_{P,Q \in S_N, P \cdot Q = Q \cdot P} \frac{1}{N!} \int_{(P,Q)} \mathcal{D}X^1 \cdots \mathcal{D}X^N \exp \left( -\sum_{a=1}^{N} I(X^a) \right), \quad (2.15)$$

where

$$I(X) = \frac{1}{4\pi \alpha'} \int_{\mathbb{T}} d^2z \left( \frac{1}{2i\tau_2} \partial X(z) \overline{\partial X(z)} \right) \quad (2.16)$$

is the bosonic Polyakov action and $z = \sigma^1 - \tau \sigma^2$, $\sigma^1, \sigma^2 \in [0,1]$ are complex coordinates on the torus with respect to the complex structure $\tau = \tau_1 + i \tau_2$, $\tau_1 \in \mathbb{R}, \tau_2 > 0$. The sum over commuting pairs of permutations, specifying monodromy homomorphisms $\Phi : \pi_1(\mathbb{T}) \to S_N,$
is taken over worldsheet instantons of the field theory labelled by the boundary conditions (2.1).
Note that any metric on the torus can be written as
\[ ds^2 = e^{2\phi(z)} |dz|^2 \] (2.17)
where the scalar field \( \phi(z) \) on \( \mathbb{T} \) is an arbitrary conformal factor.

From the general formulas (2.12) and (2.14) above it follows that the partition function (2.15) is given by the combinatorial formula
\[ Z^{\text{Sym}}(\tau, \kappa) = \exp \left( \sum_{N=1}^{\infty} \kappa^N \sum_{r,m,N} \frac{1}{N} \zeta \left( \frac{s+m+r}{r} \right) \right), \] (2.18)
where
\[ \zeta(\tau) = \int \mathcal{D}X \ e^{-I(X)} \] (2.19)
is the sigma model partition function on the torus with target space \( \mathbb{R} \). This gives a sum of the partition function on a particular torus \( \mathbb{T} \) over the discrete set of covering tori. The Gaussian integral (2.19) can be evaluated in terms of a Quillen norm as
\[ \zeta(\tau) = \left( \frac{\text{vol}(\mathbb{T}) \det' \Delta}{4\pi^2 \alpha'} \right)^{-1/2} \] (2.20)
where \( \Delta \) is the scalar Laplacian operator on \( \mathbb{T} \) with respect to the torus metric (2.17), \( \text{vol}(\mathbb{T}) \) is the volume of the surface \( \mathbb{T} \) in (2.17), and \( \det' \Delta \) denotes the determinant of \( \Delta \) with zero modes excluded. At genus one, this determinant has a natural holomorphic splitting and \( \zeta(\tau) \) is a section of the determinant line bundle \( \det(\overline{\mathcal{O}})^{-1/2} \otimes \det(\overline{\mathcal{O}})^{-1/2} \) over the moduli space of complex structures on \( \mathbb{T} \). The determinant of the Dolbeault operator \( \overline{\mathcal{O}} \) is the automorphic form on Teichmüller space given by
\[ \det' \overline{\mathcal{O}} = e^{S_L(\phi)/24\pi} \eta(\tau)^2, \] (2.21)
where \( S_L(\phi) \) is the Liouville action and \( \eta(\tau) = e^{\pi i \tau/12} \prod_{n \in \mathbb{N}} (1 - e^{2\pi i n \tau}) \) is the Dedekind function. The partition function (2.19) is thus given explicitly by
\[ \zeta(\tau) = e^{-S_L(\phi)/24\pi} \left( \frac{1}{4\pi^2 \alpha'} \int_{\mathbb{T}} \eta^2 \ e^{\phi(z)} \right)^{-1/2} \left( \frac{1}{|\eta(\tau)|^2} \right). \] (2.22)

By replacing \( \zeta(\tau) \) with \( \zeta(\tau)^d \) in (2.18) we get the corresponding result for the parent conformal field theory of a free boson on the target space \( \mathbb{R}^d \). Moreover, the combinatorial formula (2.18) is completely generic and holds for any sigma model partition function on the torus. For example, we may simply replace \( \zeta(\tau) \) by the appropriate superstring or heterotic string partition functions at one-loop (with some modifications that we discuss in Section 5).

The formula (2.12) can also be used to compute any correlation function of fields which are unaffected by the orbifolding. These are the operators which are symmetric under permutations of the indices of the scalar field \( X \). Given any function \( f \), we use the notation \( \text{Tr} f(X) := \sum_a f(X^a) \) for such an operator referring to a diagonal matrix of the \( N \) independent fields \( X^a \). The (normalized) correlation function is defined by
\[ \langle \text{Tr} f(X) \rangle^{\text{Sym}}(\tau, \kappa) := \frac{1}{Z^{\text{Sym}}(\tau, \kappa)} \left( 1 + \sum_{N=1}^{\infty} \kappa^N \sum_{P,Q \in S_N} \sum_{P'Q'=Q \in P} \mathcal{D}X^1 \cdots \mathcal{D}X^N \text{Tr} f(X) \ e^{-\text{Tr} I(X)} \right). \] (2.23)
Rather than trying to determine the combinatorics of this amplitude directly, we will calculate instead the generating function

\[ Z_{\zeta}^{\text{Sym}}(\tau, \kappa) := \langle e^{\zeta \text{Tr} f(X)} \rangle_{\text{Sym}}^{\tau, \kappa} = \sum_{n=0}^{\infty} \langle \left( \text{Tr} f(X) \right)^n \rangle_{\text{Sym}}^{\tau, \kappa} \frac{\zeta^n}{n!}. \]  

(2.24)

Then we can get the correlation function (2.23) by differentiation as

\[ \langle \text{Tr} f(X) \rangle_{\text{Sym}}^{\tau, \kappa} = \frac{\partial Z_{\zeta}^{\text{Sym}}(\tau, \kappa)}{\partial \zeta} \bigg|_{\zeta=0}. \]  

(2.25)

The generating function (2.24) is just the symmetric product partition function of the sigma model conformal field theory with a shifted action

\[ I_{\zeta}(X) = I(X) - \zeta f(X) \]  

(2.26)

and the normalization \( Z_{\zeta=0}^{\text{Sym}}(\tau, \kappa) = 1 \). It can thus be calculated by using the combinatorial formulae (2.12) and (2.14) as above, with the result

\[ Z_{\zeta}^{\text{Sym}}(\tau, \kappa) = \frac{1}{Z^{\text{Sym}}(\tau, \kappa)} \exp \left( \sum_{N=1}^{\infty} \frac{\kappa^N}{N} \sum_{r=m=N} N \sum_{s \in \mathbb{Z}/r \mathbb{Z}} \delta_{\zeta} \left( \frac{s+m \tau}{r} \right) \right) \]  

(2.27)

where

\[ \delta_{\zeta}(\tau) = \int DX \ e^{-I_{\zeta}(X)} \]  

(2.28)

is the sigma model partition function on the torus with respect to the modified action (2.26). To carry out the differentiation in (2.25), we first calculate

\[ \frac{\partial \delta_{\zeta}(\tau)}{\partial \zeta} \bigg|_{\zeta=0} = \langle f(X) \rangle(\tau) \]  

(2.29)

where the (unnormalized) expectation values are calculated as Gaussian moments with respect to the original action (2.16). Combining these results along with the elementary identity \( \frac{d}{d\zeta} e^{F(\zeta)} = F'(\zeta) e^{F(\zeta)} \) gives finally

\[ \langle \text{Tr} f(X) \rangle_{\text{Sym}}^{\tau, \kappa} = \sum_{N=1}^{\infty} \frac{\kappa^N}{N} \sum_{r=m=N} N \sum_{s \in \mathbb{Z}/r \mathbb{Z}} \langle f(X) \rangle(\frac{s+m \tau}{r}). \]  

(2.30)

The correlation function of the symmetric operator \( \text{Tr} f(X) \) in the symmetric product is thus likewise expressed in terms of the correlation function of the operator \( f(X) \) on all unramified covering spaces over the base torus \( \mathbb{T} \). These formulae have natural extensions to higher loops, but in those instances they require knowledge of the correlation functions of \( f(X) \) on all higher genus Riemann surfaces.

### 2.4 Twist Fields

A twist field \( \sigma_P(w) \) in a generic permutation orbifold \( C \wr G \) is a primary field that creates the vacuum state of a twisted sector at a point \( w \in \Sigma \). In a sigma model conformal field theory, its insertion results in non-trivial local monodromy

\[ X^a((z-w) e^{2\pi i}) \sigma_P(w) = X^{P(a)}(z) \sigma_P(w) \]  

(2.31)
where the permutation $P$ is an element of the twist group $G < S_N$. Its effect is to thus make the local field $X$ multi-valued about the insertion point $w \in \Sigma$. If $P = (n)$ consists of a single cycle of length $n > 1$, then the corresponding twist field $\sigma_{(n)}(w)$ permutes $n$ copies of $C$ in a $\mathbb{Z}_n$-twisted sector and is a primary field with conformal weight [18]

$$\Delta_{(n)} = \frac{d}{24} \left( n - \frac{1}{n} \right)$$

(2.32)

for a $d$-dimensional boson. The corresponding fields $X^{a_i}(z)$, $i = 1, \ldots, n$ can then be glued together into one field $X(z)$ which is identified with a long string of length $n$.

In the general case, we have seen that twisted sectors are in one-to-one correspondence with conjugacy classes of $G$. The conjugacy class $[P]$ of an element $P \in S_N$ can be decomposed into combinations of cyclic permutations as $[P] = \prod_n (n)^{N_n}$ with $N_n \geq 0$ and $\sum_n n N_n = N$. For a bosonic sigma model in $d$ dimensions, the corresponding twist field has conformal dimension

$$\Delta_P = \sum_{n=1}^N N_n \Delta_{(n)} = \frac{d}{24} \left( N - \sum_{n=1}^N \frac{N_n}{n} \right).$$

(2.33)

An $S_N$-invariant twist field creating the twisted sector $[P]$ of the permutation orbifold is defined by averaging over all twist fields in the conjugacy class of $P$ to get

$$\sigma_{[P]}(w) = \frac{1}{N!} \sum_{g \in S_N} \sigma_{P g^{-1}}(w).$$

(2.34)

In this paper we will be primarily interested in correlation functions $\langle \sigma_{[P_1]}(w_1) \cdots \sigma_{[P_k]}(w_k) \rangle^G$ of twist field operators in the permutation orbifold $C/G$. These averages are difficult to calculate directly within a path integral formalism, because the twist fields are non-local operators. However, since these correlation functions are the vacuum functional with twisted boundary conditions due to (2.31), it is natural to extend the covering surface principle as in [20]–[22] and compute them via a generalization of the permutation orbifold partition function (2.3) on a Riemann surface $\Sigma$ of genus $g > 0$. Whenever we have twist fields inserted at $k$ distinct points $\underline{w} := \{w_1, \ldots, w_k\}$ of the worldsheet, a twisted sector is given by a conjugacy class of homomorphisms $\Phi : \pi_1(\Sigma_{\underline{w}}) \rightarrow G < S_N$ where $\Sigma_{\underline{w}} := \Sigma \setminus \underline{w}$ is the marked Riemann surface with the $k$ twist field insertion points deleted. It is restricted by admissibility criteria which require that the images of the generators $\gamma_i$ of $\pi_1(\Sigma_{\underline{w}})$ which are contractible to $w_i$ must be simple cycles of length $\nu_i > 1$ if a $\mathbb{Z}_{\nu_i}$ twist field $\sigma_{(\nu_i)}(w_i)$ is inserted at $w_i$. Each such homomorphism $\Phi$ determines a cover of the worldsheet $\Sigma$ on which a single new field $X(z)$, defined by a formula analogous to (2.2), is single-valued. Namely, after going around a curve $\gamma$ which is closed on the marked worldsheet $\Sigma_{\underline{w}}$, one sews the fields $X^{\Phi(\gamma)(a)}(z)$ into $X(z)$. Thus, the contribution to the correlation function from the worldsheet instanton sector determined by the homomorphism $\Phi$ is the free partition function on the cover of $\Sigma$ determined by $\Phi$.

While the sum arising in the orbifold partition function (2.3) is only over unramified covers $\hat{\Sigma}$ of $\Sigma$, the twist field correlation functions involve sums over branched covers $\hat{\Sigma}_{\underline{w}}$ where $\hat{\underline{w}} := f^{-1}(\underline{w})$ is the set of pre-images of the set $\underline{w}$ under the covering map $f : \hat{\Sigma} \rightarrow \Sigma$. The Riemann-Hurwitz formula for the genus $\hat{g}$ of the covering space with the given monodromy homomorphism is the general one for covers with ramification given by

$$\hat{g} = N (g - 1) + 1 + \frac{B}{2} \quad \text{with} \quad B = \sum_{i=1}^k (\nu_i - 1),$$

(2.35)
where \( \nu_i \) is the ramification index given by the length of the cycle of the \( i \)-th primary twist field. As before, we have to take into account those homomorphisms \( \Phi \) whose image does not act transitively on the coordinate labels \( a = 1, \ldots, N \). In this case the simple cycle condition for fixed length \( \nu_i \) has to hold for each orbit \( \xi \). This ensures that the genus of the connected component of the cover determined by the action of \( \Phi(\pi_1(\Sigma_w)) \) on each orbit \( \xi \) is equal to \( \hat{g} \).

We may now write down a formula analogous to (2.3) for the normalized \( k \)-point correlation function of twist field operators given by

\[
\left\langle \prod_{i=1}^{k} \sigma_{\pi_1}(w_i) \right\rangle^G = \frac{1}{|G|} \sum_{\Phi: \pi_1(\Sigma_w) \to G} \frac{1}{Z^G(\tau)} \left( \prod_{\xi \in \mathcal{O}(\Phi)} Z(\tau^{\xi}, w) \right),
\]

(2.36)

where \( \tau^{\xi}, w \) is the complex structure of the covering surface determined by the worldsheet modulus \( \tau \), the stabilizer \( \pi_1(\Sigma_w)\xi \), and the branch point loci \( w \).

There are three crucial differences between the formulæ (2.36) and (2.3). Firstly, the twist field correlation functions are not expressed in terms of correlation functions but instead in terms of partition functions. Secondly, there is a restriction on the admissible homomorphisms \( \Phi \) to ensure that they have the prescribed monodromy around the punctures, \( i.e., \Phi(\gamma_i) \) has to be a simple cycle of length \( \nu_i \) in each orbit. Thirdly, while the uniformization theorem provided us with a computational recipe for obtaining the Teichmüller coordinate \( \tau^* \) in terms of \( \tau \) via knowledge of \( \Phi \), it does not apply to the twist field \( k \)-point functions. The reason is that \( \tau \) parametrizes the uniformizing group of the compact Riemann surface \( \Sigma \), which is isomorphic to \( \pi_1(\Sigma) \), while the domain of the monodromy homomorphism \( \Phi \) is \( \pi_1(\Sigma_w) \) which differs from the domain of the isomorphism from the abstract group \( \pi_1(\Sigma) \) to the uniformizing group \( \pi_1(\Sigma_w) \). Therefore, the complex structure of the ramified cover \( \hat{\Sigma}_{w} \) is a function of that of the base space \( \Sigma \), the locations \( w \) of the branch points, and the monodromy homomorphism \( \Phi \).

We are also interested in twist field correlation functions on symmetric products. In order to apply a version of (2.7) we need to pass the constraint, which is imposed on the admissible homomorphisms \( \Phi \) in (2.36), to the definition of the function \( Z(H) \). Let us specialize the discussion to the torus \( \Sigma = \mathbb{T} \) for definiteness. In this case, the genus of the covering surface \( \hat{\Sigma} \) is \( \hat{g} \) whenever its branching number is \( B = 2(\hat{g} - 1) \). A standard presentation of the fundamental group of the marked torus is given by

\[
\Gamma := \pi_1(\mathbb{T}_w) = < \alpha, \beta, \gamma_1, \ldots, \gamma_k \mid [\alpha, \beta] \gamma_1 \cdots \gamma_k = 1 >.
\]

(2.37)

To each \( N \)-sheeted cover of \( \mathbb{T} \) there corresponds a conjugacy class of subgroups of \( \Gamma \) of index \( N \) \cite{35}, which is the stabilizer of the monodromy homomorphism \( \Phi \) acting in \( S_N \). Note that the group (2.37) is isomorphic to the free group on \( k + 1 \) generators \( \alpha, \beta, \gamma_1, \ldots, \gamma_k \), and any subgroup of a free group is also free. This is consistent with the fact \cite{35} that the stabilizer subgroup is isomorphic to \( \pi_1(\hat{\Sigma}_{w}) < \pi_1(\mathbb{T}_w) \). To decide when a given finite index subgroup \( H < \Gamma \) corresponds to a stabilizer subgroup of an admissible homomorphism \( \Phi \) in (2.36), we proceed as follows. Let \( i : \hat{\Sigma}_{w} \to \hat{\Sigma} \) be the natural inclusion of surfaces. The induced homomorphism \( i_* : \pi_1(\hat{\Sigma}_{w}) \to \pi_1(\hat{\Sigma}) \) is then the natural forgetful map. Since \( H \cong \pi_1(\hat{\Sigma}_{w}) \), a formal criterion for the admissibility of a finite index subgroup \( H < \pi_1(\mathbb{T}_w) \) is given by

\[
H/\ker(i_*) \cong \pi_1(\hat{\Sigma}).
\]

(2.38)

We can use (2.38) to check whether a given subgroup \( H \) is admissible. If the quotient is defined and it yields a group isomorphic to \( \pi_1(\hat{\Sigma}) \), then \( H \) is admissible. This property does not depend
on the conjugacy class of $H$ in $\pi_1(T_w)$. We can thus give an implicit definition for the function appearing in (2.7) as

$$Z(H) := \begin{cases} \frac{Z(\tau H, w)}{Z^{SN}(\tau)} \kappa^{[\Gamma; H]} & \text{if } H \text{ satisfies } (2.38), \\ 0 & \text{otherwise}. \end{cases}$$

We may then apply the formula (2.7) to get the generating function of twist field correlation functions.

In the following we will apply this formalism to study the perturbation of the sigma model conformal field theory, on the symmetric product of $\mathbb{R}^d$, by an irrelevant operator of conformal dimension $\frac{3}{2}$. For this, we introduce the bosonic Dijkgraaf-Verlinde-Verlinde (DVV) interaction vertex \cite{4, 36} which is defined with respect to the $\mathbb{Z}_2$ twist field $\sigma_{ab}(w)$ corresponding to the transposition in $S_N$ that interchanges the fields $X^a$ and $X^b$ while leaving all others invariant. These twist fields generate the elementary joining and splitting of strings in the symmetric product, and they can be built out of standard $\mathbb{Z}_2$ orbifold twist operators \cite{13, 20}. Then the translationally invariant vertex operator is defined by

$$V_{bos} = -\frac{\lambda N}{\text{vol}(T)} \int_T d\mu(z) \sum_{1 \leq a < b \leq N} \sigma_{ab}(z),$$

where $\lambda$ is a coupling constant proportional to the string coupling $g_s$. In contrast to the originally proposed genus zero case \cite{4, 18, 36}, we will find that the DVV vertex operator at genus one needs to be defined using a non-constant measure $d\mu(z) = d^2z/\mu(z)$ on the torus $T$. It will be determined explicitly in the ensuing sections (as will the coupling constant $\lambda$) by modular invariance requirements. When $d = 24$, the twist field $\sigma_{ab}(w)$ is a primary field of conformal weight $\frac{3}{2}$. Starting from the one-loop action (2.16), the interacting symmetric product sigma model is defined by the action

$$I_{int}^{SN}(X) = \text{Tr} I(X) + V_{bos}$$

with $\text{Tr} I(X) = \sum_a I(X^a)$.

In this paper we will compute the leading order effect of this perturbation. Using translational invariance of the sigma model path integral to move one of the branch points to the origin $z = 0$, we are thus interested in computing the translationally invariant correlator

$$\langle \circ V_{bos} \circ V_{bos} \circ \rangle^{SN} = \frac{\lambda^2 N^2}{\text{vol}(T) \mu(0)} \sum_{a_i < b_i} \int_T d\mu(z) \langle \sigma_{a_1 b_1}(z) \sigma_{a_2 b_2}(0) \rangle^{SN}.$$  

The computation of the two-point functions in (2.42) specializes the above discussion to the case $g = 1$, $\hat{g} = 2$, and $k = 2$. There are two simple branch points with ramification indices $\nu_1 = \nu_2 = 2$ and $\Gamma = \pi_1(T \setminus \{z, 0\})$. Then the logarithm of the generating function (2.7) with the definition (2.39) is given by a sum over the modular invariant vacuum amplitudes on all connected $N$-sheeted genus two covers $\tilde{\Sigma}$ with two fixed simple branch points. In this case the first quantized modular invariant partition function for the parent theory is the two-loop version of (2.20) on $\tilde{\Sigma}$ (with vanishing Liouville field $\phi = 0$ for simplicity) given by \cite{37, 38}

$$\text{vol}(\tau) = \frac{(\text{det(Im}(\tau)))^{3-d/2}}{(4\pi^2 \alpha')^{-d/2} |\Psi_{10}(\tau)|^2},$$

(2.43)
where \( d \) is the spacetime dimension (\( d = 26 \) for the critical bosonic string). Here \( \Psi_{10}(\tau) \) is the genus two parabolic modular form of weight ten with no zeroes or singularities (the Igusa cusp form), defined on the Siegel half-space \( \mathbb{U}^2 = \{ \tau \mid \text{Im}(\tau_{11}) > 0 \text{, } \text{Im}(\tau_{22}) > 0 \text{, } \text{det(Im } \tau) > 0 \} \) of \( 2 \times 2 \) Riemann period matrices \( \tau \) with the boundary component \( \mathbb{U} \times \mathbb{U} \) consisting of diagonal matrices removed. It can be expressed in terms of the ten genus two theta-constants \( \Theta(\mathbf{a} \mathbf{b})(\tau) \) as

\[
\Psi_{10}(\tau) = 2^{-12} \prod_{\mathbf{a} \cdot \mathbf{b} \equiv 0 \text{ mod } 2} \Theta(\mathbf{a} \mathbf{b})(\tau)^2. \tag{2.44}
\]

2.5 Thermodynamics of DLCQ Strings

In the genus one case \( \Sigma = \mathbb{T} \), the logarithm of the right-hand side of (2.12) coincides with the free energy of second quantized string theory on the target space \( M \times \mathbb{S}^1 \times \mathbb{R} \) when the parent theory is the corresponding conformal field theory on the spacetime \( M \) in the free string limit \( g_s \to 0 \) [5, 6]. The matching is provided by identifying the modulus of the worldsheet and that of the spacetime torus, where the second compact direction is timelike and is generated by the trace taken in computing the free energy amplitude. Its radius is identified with the inverse temperature \( \beta \). In discrete light cone quantization (DLCQ), the light cone Hamiltonian and momentum are given by

\[
H = P^+ \quad \text{and} \quad P^- = N/R \tag{2.45}
\]

where \( R \) is the radius of the compactified light-like direction \( x^+ \in \mathbb{S}^1 \) and \( N \in \mathbb{N}_0 \). The thermodynamic free energy \( F_{\text{DLCQ}}^{(1)} \) is then defined by

\[
e^{-\beta F_{\text{DLCQ}}^{(1)}} = \text{Tr} \ e^{-\frac{\beta}{\sqrt{2}} (P^+ + P^-)} = \sum_{N=0}^{\infty} e^{-\beta N/\sqrt{2} R} \text{Tr} \mathcal{H}_N e^{-\beta P^+/\sqrt{2}}, \tag{2.46}
\]

where \( \mathcal{H}_N \) denotes the sector of the physical Hilbert space with definite total light cone momentum \( P^- = N/R \). The trace over this subspace can be computed by using the mass-shell relation \( P^+ = H^\perp / P^- \), where \( H^\perp \) is the Hamiltonian for the transverse degrees of freedom along \( M \).

In this way one arrives at the expression (2.12) with the definition (2.14) and \( \kappa := e^{-\beta/\sqrt{2} R} \). The Teichmüller parameter of the base torus \( \mathbb{T} \) on which the string bits live is

\[
\tau^* := \frac{4 \pi i \alpha'}{\sqrt{2} \beta R}. \tag{2.47}
\]

The qualitative reason for the equivalence is that the second quantized vacuum amplitude is given by the integral of the conformal field theory partition function over the moduli space of complex structures, but the only contributing surfaces at one-loop order are those which arise by winding the string around the compact directions. In other words, only the discretized moduli space of unramified covers of the torus \( \mathbb{T} \) is summed over and taking the logarithm eliminates the disconnected covers.

When \( M = \mathbb{R}^{24} \) one finds that the DLCQ partition function for bosonic string theory coincides exactly with the partition function of the symmetric product in the limit \( N \to \infty \), with the length \( n_i \) of a long string identified with the light cone momentum \( P_i^- = n_i / R \) for \( i = 1, \ldots, 24 \).

Checking the equivalence of perturbative bosonic string dynamics and the corresponding interacting symmetric product of \( \mathbb{R}^{24} \) beyond the free string limit \( g_s \to 0 \) requires computing the...
thermal free energy in DLCQ at higher genus and the appropriate amplitudes in the permutation orbifold perturbed by the DVV interaction vertex (2.40). The former amplitudes truncate to sums over branched covers of the spacetime torus $\mathbb{T}$ arising in the null compactification at finite temperature [11], while the local structure of the operator $V_{\text{bos}}$ matches nicely with the cubic string interaction vertices in light cone Green-Schwarz string field theory [7]–[10]. In this setting the string interactions are generated by sewing together torus worldsheets along branch cuts.

On the DLCQ side, the next-to-leading order contribution is the two-loop free energy which was computed in [12] with the result

$$F_{\text{DLCQ}}^{(2)}(\tau^*, \kappa) = -g_s^2 \left| \frac{\tau^*}{32\pi^2 \alpha'} \right|^{12} \sum_{N=2}^{\infty} \frac{\kappa^N}{N^2} \sum_{rm=N} \left( \frac{r}{m} \right)^{10} \times \sum_{s,t \in \mathbb{Z}/r \mathbb{Z}} \int_{\triangle} \frac{d^2\tau^\#}{(\tau^\#)^{12}} \left| \Psi_{10}(\tau_{r,m,s,t}(\tau^*, \tau^\#)) \right|^{-2}.$$  

This thermal string amplitude is just the weighted integral over a fundamental modular domain of the genus two bosonic string partition function (2.43) with respect to the modular invariant integration measure on the space of $2 \times 2$ Riemann period matrices with diagonal matrices excluded, but with integration domain restricted to the partially discretized moduli space of genus two simple branched covers $\hat{\Sigma}$ of the torus $\mathbb{T}$ with modulus $\tau^*$. The integers appearing in (2.48) can be assembled into the $2 \times 4$ matrix

$$M = \begin{pmatrix} 0 & 0 & -m & 0 \\ r & 0 & -s & -t \end{pmatrix}$$

which determines a homology basis for the cover in which the push-forward $f_* : H_1(\hat{\Sigma}, \mathbb{Z}) \to H_1(\mathbb{T}, \mathbb{Z})$, induced by the holomorphic covering map $f : \hat{\Sigma} \to \mathbb{T}$, is given on a basis of canonical homology cycles $\hat{\alpha}_i, \hat{\beta}_i, i = 1, 2$ for $\hat{\Sigma}$ by

$$f_*(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2) = (\alpha, \beta) M$$

with respect to a canonical homology basis $(\alpha, \beta)$ of the base torus. It specifies the way in which the cycles of the cover $\hat{\Sigma}$ wind around the cycles of $\mathbb{T}$. The period matrix $\tau \in \mathbb{U}^2 \setminus (\mathbb{U} \times \mathbb{U})$ of the cover in this basis is given by the normal form

$$\tau_{r,m,s,t}(\tau^*, \tau^\#) = \begin{pmatrix} -\frac{s+m/\tau^*}{r} & -\frac{l}{\tau} \\ -\frac{l}{\tau} & -\frac{t}{\tau^\#} \end{pmatrix}$$

with $\tau^\# \in \mathbb{U}$, and the integration in (2.48) is taken over the standard fundamental domain $\triangle \subset \mathbb{U}$ for the action of the genus one modular group $SL(2, \mathbb{Z})$ on $\tau^\#$.

The diagonal elements of the period matrix (2.51) naturally capture the modulus of the degree $N = rm$ unramified cover of the base torus $\mathbb{T}$ of modulus $(\tau^*)^{-1}$, along with a second torus of modulus $\tau^\#$. The key feature of the homology basis in which we have expressed the genus two amplitude (2.48) is that the genus two theta functions appearing in (2.44) admit reduction to genus one theta functions on these two tori, due to the rational-valued off-diagonal entries of (2.51). Hence the $\tau^*$-dependence of the two-loop free energy is expressible in terms of
elliptic functions, analogously to the one-loop case. Recall that the elliptic Jacobi theta function with characteristics \( a, b \in \mathbb{Z}/2\mathbb{Z} \) is defined by

\[
\theta\left(\begin{smallmatrix} a \\ b \\
\end{smallmatrix}\right)(z|\tau) = \sum_{n \in \mathbb{Z}} \exp\left(\pi i \tau (n + \frac{a}{2})^2 + 2\pi i (n + \frac{a}{2}) (z + \frac{b}{2})\right)
\]

(2.52)

along with the Erdélyi notation

\[
\theta_1(z|\tau) = \theta\left(\begin{smallmatrix} 0 \\
\end{smallmatrix}\right)(z|\tau) \quad \text{and} \quad \theta_2(z|\tau) = \theta\left(\begin{smallmatrix} 0 \\ 1 \\
\end{smallmatrix}\right)(z|\tau),
\]

\[
\theta_3(z|\tau) = \theta\left(\begin{smallmatrix} 1 \\
\end{smallmatrix}\right)(z|\tau) \quad \text{and} \quad \theta_4(z|\tau) = \theta\left(\begin{smallmatrix} 1 \\ 0 \\
\end{smallmatrix}\right)(z|\tau).
\]

(2.53)

Then one has the decompositions [12]

\[
\Theta\left(\begin{smallmatrix} a \\
\end{smallmatrix}\right)(\tau_{r,m,s,t}(\tau^*, \tau^\#)) = \frac{e^{\pi i a_2 b_2/2}}{N \sqrt{-\tau^*}} \sum_{n=0}^{N-1} (-1)^{b_2 n} \theta\left(\begin{smallmatrix} a_1 \\
\end{smallmatrix}\right)(n + \frac{a_1}{2}, \frac{mt}{N}, \frac{m+2}{N} \tau^*) \times \theta_j\left(\frac{n+2a_2/2}{N}, \frac{1}{N^2 \tau^\#}\right)
\]

(2.54)

where \( j = 2 \) (resp. \( j = 3 \)) when the integer \( a_1 m t + b_2 N \) is odd (resp. even). For notational ease, this formula is written after performing a projective rotation \( \tau_{r,m,s,t}(\tau^*, \tau^\#) \rightarrow -\tau_{r,m,s,t}(\tau^*, \tau^\#) \) along with a reflection in the modulus \( \tau^\# \).

In this paper we shall present a detailed comparison between the free energy (2.48) and the integrated (with respect to the branch point loci) two-point correlation function (2.42) of twist fields corresponding to transpositions, which requires the generalization of the combinatorial identity (2.12) to coverings with two simple branch points as explained in Section 2.4 above. While the auxiliary genus one surface of modulus \( \tau^\# \) above is anticipated \textit{a posteriori} on general grounds from the Weierstrass-Poincaré reduction theory for branched covers [12], its geometrical significance has been hitherto unclear. In the following we will identify this torus explicitly, which among other things will provide the transformation from the branch point loci to the modulus \( \tau^\# \) required to match the expressions (2.42) and (2.48), as well as the measure \( d\mu(z) \) and coupling constant \( \lambda \) required to define the DVV vertex operator (2.40) on an elliptic curve.

3 \( \mathbb{Z}_2 \) Orbifolds

The purpose of this section is to establish the equivalence of the two-point function for the DVV vertex operator in the symmetric product \( \mathbb{R}^{24} \wr \mathbb{Z}_2 \) with the \( N = 2 \) contribution to the genus two free energy (2.48) of the bosonic DLCQ string. For the former calculation we will exploit the known formulae [21] for the multi-loop partition functions and twist field correlation functions on the geometric orbifold \( S^1/\mathbb{Z}_2 \). For the latter computation we connect the form of the total reduced free energy (2.48) to the theory of Prym varieties for generic genus two covers of the torus \( T \) of modulus \( \tau^* \). By a theorem due to Mumford [39], the only coverings that generate Prym varieties are double covers with at most two branch points, and our case of genus two covers over an elliptic curve. Our proof puts the covering surface principle sketched in Section 2.4 on more solid ground, and provides a non-trivial explicit check for the computation of twist field correlation functions through two rather distinct methods.
3.1 Target Space vs. Permutation Orbifold

For later use, we begin by elucidating the correspondence between the sigma model conformal field theories on the geometric orbifold $\mathbb{R}^{24}/\mathbb{Z}_2$ and on the permutation orbifold $\mathbb{R}^{24} \wr \mathbb{Z}_2$. For this, let us consider the $S^1/\mathbb{Z}_2$ target space orbifold of a free boson $X$ compactified on a circle $S^1$ of radius $R$, where the group action is the reflection involution $X \mapsto -X$. On the other hand, the permutation orbifold $S^1 \wr \mathbb{Z}_2$ is defined on the tensor product of the $S^1$ conformal field theory with itself. Labelling the two copies of the boson $X$ by $X^a$, $a = 1, 2$, the group action of the permutation orbifold is given by $X^1 \mapsto X^2$, $X^2 \mapsto X^1$. This can be compared to the geometric orbifold group action by introducing new coordinate fields $X^\pm = X^1 \pm X^2$, so that the $\mathbb{Z}_2$ permutation group now acts as $X^\pm \mapsto \pm X^\pm$. It follows that the permutation orbifold is equivalent to the target space orbifold plus an independent free boson $X^+$ on $S^1$. The partition functions of the two theories are thus related by

$$Z^{\mathbb{Z}_2}(\tau, R) = \mathfrak{z}(\tau, R) \ Z_{\text{orb}}(\tau, R),$$

where $\mathfrak{z}(\tau, R)$ denotes the partition function of the compactified scalar field $X^+$ and $Z_{\text{orb}}(\tau, R)$ that of the $S^1/\mathbb{Z}_2$ theory.

It is instructive to check the identity (3.1) explicitly at one-loop order in the decompactified circle theory. The amplitude for the boson $X^+$ on $S^1$ is given by the worldsheet instanton sum

$$\mathfrak{z}(\tau, R) = \mathfrak{z}(\tau) \, \mathfrak{z}_{\text{cl}}(\tau, R) := \frac{\sqrt{4\pi^2 \alpha'}}{\sqrt{\eta(\tau)}} \sum_{m, m' \in \mathbb{Z}} \frac{R}{\sqrt{\alpha'}} \exp \left( - \frac{\pi R^2 |m \tau - m'|^2}{\alpha' \tau_2} \right),$$

where $\mathfrak{z}(\tau)$ is the modular invariant amplitude (2.22) for the free boson on the real line (so that $\mathfrak{z}_{\text{cl}}(\tau, R = \infty) = 1$) and henceforth we set the Liouville field $\phi = 0$. The sum in (3.2) runs over classical solutions with the given winding numbers around the generating cycles of a canonical homology basis. For the partition function of the target space orbifold, we note that the oscillator part $\mathfrak{z}(\tau)$ of the partition function (3.2) is independent of the radius $R$. A monodromy homomorphism $\Phi$ for an unramified double cover of a genus one surface is characterized by a binary pair $(\varepsilon, \delta) \in (\mathbb{Z}/2\mathbb{Z})^2$, where 0 (resp. 1) labels periodic (resp. antiperiodic) global monodromy around the canonical homology cycles $(\alpha, \beta)$ of the base. In the twisted sectors, the $\mathbb{Z}_2$ action $X \mapsto -X$ kills non-trivial instantons at one-loop (as a consequence of the Riemann-Roch theorem), while the quantum parts may be computed by equating the $\mathbb{Z}_2$-twisted partition function at $R = \sqrt{\alpha'}$ with that of the untwisted $S^1$ theory at the self-dual radius $R = 1/\sqrt{\alpha'}$ which coincides with the multi-critical Ashkin-Teller model. The result is [21]

$$Z_{\text{orb}}(\tau, R) = \frac{1}{2} \mathfrak{z}(\tau, R) + \frac{\eta(\tau)}{\theta_2(\tau)} + \frac{\eta(\tau)}{\theta_3(\tau)} + \frac{\eta(\tau)}{\theta_4(\tau)},$$

where we have denoted the Jacobi-Erdélyi theta constants by $\theta_i(\tau) := \theta_i(0|\tau)$. Finally, the vacuum amplitude of the $\mathbb{Z}_2$ permutation orbifold can be determined from the formula (2.3) as

$$Z^{\mathbb{Z}_2}(\tau, R) = \frac{1}{2} \left( \mathfrak{z}(\tau, R)^2 + 3(2\tau, R) + 3(\tau, R) + 3\left(\frac{\tau}{2}, R\right) \right).$$

Clearly the contributions to both sides of the formula (3.1) from the untwisted sector match. For the contributions from the twisted sectors, we use the identities $\theta_3(\tau + 1) = \theta_4(\tau)$ and $\theta_2(\tau) \, \theta_3(\tau) \, \theta_4(\tau) = 2\eta(\tau)^3$ (3.5)
to derive the elliptic function relation
\[
\frac{1}{|\theta_2(\tau) \eta(\tau)|} + \frac{1}{|\theta_3(\tau) \eta(\tau)|} + \frac{1}{|\theta_4(\tau) \eta(\tau)|} = \frac{\theta_3(\tau) \theta_3(\tau + 1)}{2\eta(\tau)^4} + \frac{1}{|\theta_3(\tau) \eta(\tau)|} + \frac{1}{|\theta_3(\tau + 1) \eta(\tau)|}.
\]
(3.6)
where in the last line we substituted the identity \( \theta_3(\tau) = \eta(\frac{\tau + 1}{2})^2/\eta(\tau + 1) \) and used \(|\eta(\tau + 1)| = |\eta(\tau)|\). This equation establishes the \( R \to \infty \) limit of the formula (3.1), for each twisted sector, which easily generalizes to \( \mathbb{Z}_2 \) orbifolds of \( \mathbb{R}^d \) by taking appropriate powers.

### 3.2 DLCQ Strings on Double Covers

We now turn to the explicit form of the \( N = 2 \) part of the genus two bosonic DLCQ free energy (2.48) which is given explicitly by
\[
\mathcal{F}_2^2(\tau^*) = -\frac{g_2^2}{16} \left( \frac{\tau^*}{16\pi^2 \alpha'} \right)^{12} \sum_{s=0,1} \int_{\triangle} \frac{d^2 \tau^\#}{(\tau^\#_2)^{12}} |\Psi_{10}(\tau^*(\tau^*),\tau^\#)|^{-2},
\]
(3.7)
where the corresponding period matrices read
\[
\tau_s(\tau^*, \tau^\#) := \tau_{r=2,m=1,s,t=1}(\tau^*, \tau^\#) = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.
\]
(3.8)
By modular invariance it suffices to restrict to the \( s = 0 \) contribution. To see this, we define the \( SL(2,\mathbb{Z}) \) modular transformation \( \tilde{\tau}^\# = \tau^\#/\left(2\tau^\# + 1\right) \). Then the period matrices \( \tau_1(\tau^*, \tau^\#) \) and \( \tau_0(\tau^*, \tilde{\tau}^\#) \) are related by the \( Sp(4,\mathbb{Z}) \) modular transformation
\[
\tau_0(\tau^*, \tilde{\tau}^\#) = (A \tau_1(\tau^*, \tau^\#) + B)(C \tau_1(\tau^*, \tau^\#) + D)^{-1}
\]
(3.9)
given by the matrix
\[
g = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 \end{pmatrix} =: \begin{pmatrix} A & B \\ C & D \end{pmatrix}.
\]
(3.10)
Since the integration over \( \tau^\# \) in (3.7) runs over a fundamental domain \( \triangle \) for \( SL(2,\mathbb{Z}) \), we can compensate the omission of the \( s = 1 \) term by simply doubling the \( s = 0 \) contribution.

Let us now simplify the integrand of (3.7) by working out explicitly the product of theta constants appearing in the genus two modular form (2.44). Starting from the reduction (2.54) with \( N = 2 \), one has \( j = 2 \) when \( a_1 = 1 \) and \( j = 3 \) when \( a_1 = 0 \), and hence
\[
\Theta(a_0)\left(\tau_0(\tau^*, \tilde{\tau}^\#)\right) = \frac{e^{\pi \lambda_2 b_2/2}}{2 \sqrt{-i \tilde{\tau}^\#}} \left( \theta_{(a_0)}\left(\frac{\tilde{\tau}^\#}{4} \left\lfloor \frac{1}{2} \right\rfloor \right)^2 \theta_{(a_1)}\left(\frac{\tilde{\tau}^\#}{4} \left\lfloor \frac{1}{2} \right\rfloor \right)^{-1} \right)^{1/4}
\]
(3.11)
Using the property
\[ \theta_i(nz + \frac{1}{2} | \tau) = (-1)^{ab} \theta_i(a+1)(nz | \tau) \]  
where \( b + 1 \) is understood modulo 2, one can now write down the product of the even genus two theta constants in (2.44). To simplify the formulae somewhat, in the ensuing calculations we will use the shorthand notations \( \tilde{\theta}_1 := \theta_i(0|\tau), \tilde{\theta}_1^* := \theta_i(\frac{1}{2}|\tau), \theta_i := \theta_i(0|\frac{1}{4\tau}) \) and \( \tilde{\theta}_1^* := \theta_i(\frac{1}{4} - \frac{1}{4\tau}) \).

Then the modular form (2.44) can be expressed as
\[ \Psi_{10}(\tau(z, \tau^2)) = \frac{A^2 B^2}{2^{32} \tau^4} \]  
where
\[ A = (\theta_2^* \theta_3^# + \theta_3^* \theta_2^#) (\theta_2^* \theta_2^# + \theta_4^* \theta_4^#) (\theta_3^* \theta_3^# + \theta_4^* \theta_2^#) \times (\theta_3^* \theta_3^# - \theta_4^* \theta_4^#) (\theta_2^* \theta_2^# - \theta_1^* \theta_1^#), \]
\[ B = (\tilde{\theta}_2^* \tilde{\theta}_3^# + \tilde{\theta}_3^* \tilde{\theta}_2^#) (\tilde{\theta}_2^* \tilde{\theta}_2^# + \tilde{\theta}_4^* \tilde{\theta}_4^#) (\tilde{\theta}_3^* \tilde{\theta}_3^# + \tilde{\theta}_4^* \tilde{\theta}_2^#) \]
\[ \times (\tilde{\theta}_3^* \tilde{\theta}_3^# - \tilde{\theta}_4^* \tilde{\theta}_4^#) (\tilde{\theta}_2^* \tilde{\theta}_2^# - \tilde{\theta}_1^* \tilde{\theta}_1^#), \]
\[ \theta_2^* \theta_3^# + \theta_3^* \theta_2^# = 0 \]  
and \( \tilde{\theta}_2^* \tilde{\theta}_3^# + \tilde{\theta}_3^* \tilde{\theta}_2^# = 0 \). One finds
\[ \theta_3^*(z | \tau) \theta_2(0 | \tau) \theta_3(0 | \tau) \theta_4(0 | \tau) = 2 \theta_1(z | \tau) \theta_2(z | \tau) \theta_3(z | \tau) \theta_4(z | \tau), \]
from which the second equality is a consequence of the identity for products of theta functions with identical modulus given by
\[ \theta_3(0 | \tau) \theta_3(0 | \tau) = 2 \theta_1(z | \tau) \theta_2(z | \tau) \theta_3(z | \tau) \theta_4(z | \tau), \]
with \( z = \frac{1}{4} \).

The next step consists in using the modulus doubling identities
\[ \theta_2(0 | 2\tau) = 2 \theta_2(0 | \tau), \]
\[ \theta_3(0 | 2\tau) = \theta_3(0 | \tau), \]
\[ \theta_3(0 | \tau) \theta_4(0 | \tau) = \theta_4(0 | \tau), \]
\[ \theta_3(0 | \tau)^2 + \theta_4(0 | \tau)^2 = 2 \theta_3(0 | 2\tau)^2, \]
along with the Jacobi abstruse identity
\[ \theta_3(0 | \tau)^4 - \theta_4(0 | \tau)^4 = \theta_2(0 | \tau)^4 \]
on both \( \theta_1^{*} \) and \( \theta_1^{#} \). After introducing the notations \( \tilde{\theta}_i := \theta_i(0|\tau) \) and \( \tilde{\theta}_i := \theta_i(0|\frac{1}{4\tau}) \) we find
\[ \mathcal{A} B = -128 \tilde{\theta}_2^* \tilde{\theta}_3^# \tilde{\theta}_2^* \tilde{\theta}_2^# \tilde{\theta}_3^# \tilde{\theta}_4^# \left( \tilde{\theta}_1^*(\tilde{\theta}_2^# + \tilde{\theta}_3^#) - \tilde{\theta}_1^#(\tilde{\theta}_2^# + \tilde{\theta}_3^#) \right). \]
We now undo the projective rotation $\tau_0 \to -\tau_0$ and the reflection $\tau^\# \to -\tau^\#$ that were used to write (2.54), in order to use theta functions which are convergent on the standard domain of genus one moduli $\tau_2 > 0$. This affects only $\theta_i^*$, because its modulus changes as $\theta_i(0\,|\frac{1}{\tau}) \to \theta_i(0\,|\frac{1}{\tau})$. The reflection of the off-diagonal elements of the period matrix (2.51) which flips the sign of the argument of $\theta_i$ via (2.54) is easily checked to have no effect on the product (3.22).

The final transformation we perform on the product (3.22) is a modular $S$ transformation on both $\theta_i^*$ and $\theta_i^\#$ given by

$$
\begin{align*}
\theta_2(0\,| -\frac{1}{\tau}) &= \sqrt{-17} \theta_4(0\,| \tau), \\
\theta_3(0\,| -\frac{1}{\tau}) &= \sqrt{-17} \theta_3(0\,| \tau), \\
\theta_4(0\,| -\frac{1}{\tau}) &= \sqrt{-17} \theta_2(0\,| \tau).
\end{align*}
$$

Then we can write the modular form (3.13) as

$$
\Psi_{10}(\tau_0(\tau^*, \tau^\#)) = (\tau^*)^{10} \eta(\tau^*)^{12} \eta(2\tau^\#)^{12} \times \left( \theta_2(2\tau^\#)^4 (\theta_4(\tau^*)^4 + \theta_3(\tau^*)^4) - \theta_2(\tau^*)^4 (\theta_4(2\tau^\#)^4 + \theta_3(2\tau^\#)^4) \right)^2,
$$

where we have used (3.5). Substituting into (3.7) and using (3.21) we arrive at our final form for the two-loop DLCQ free energy given by

$$
\mathcal{F}_2(\tau^*) = -\frac{g_s^2}{8(16\pi^2 \alpha^s)^{12}} \frac{|\eta(\tau^*)|^{-24}}{\tau^*^{8}} \int_{\Delta} \frac{d^2\tau^\#}{(\tau_2^\#)^{12}} \left| \theta_3(\tau^*)^4 \theta_4(2\tau^\#)^4 - \theta_2(\tau^*)^4 \theta_3(2\tau^\#)^4 \right|^4.
$$

### 3.3 Prym Varieties

Our next goal is to determine the genus one modulus $\tau^\#$ explicitly in terms of the branch point loci on the base torus $T$. This modulus arose generically from the algebraic Weierstrass-Poincaré reduction of the period matrix $\tau$ of the covering surface $\hat{\Sigma}$ to the normal form (2.51), which is a consequence of the fact that the genus two Riemann period matrix in this instance satisfies a Hopf condition [12]. We will now elucidate the geometrical significance of this modulus for a generic genus two cover over $T$ of degree $N = rm$, and then show how in the case of double covers this geometrical realization determines it explicitly as a function of branch points on the worldsheets $T$.

Let $f: \hat{\Sigma} \to T$ be a holomorphic map. Let $\omega_i$, $i = 1, 2$ be the canonical, normalized abelian holomorphic differentials on $\hat{\Sigma}$ with the periods

$$
\oint_{\delta_i} \omega_j = \delta_{ij} \quad \text{and} \quad \oint_{\beta_i} \omega_j = \tau_{ij}.
$$

On the base elliptic curve $T$ the holomorphic one-form is $dz$ with the periods $\oint_{\alpha} dz = 1$ and $\oint_{\beta} dz = \tau^*$. The two sets of differentials are related by the pull-back homomorphism $f^* : H^{1,0}(T, \mathbb{C}) \to H^{1,0}(\hat{\Sigma}, \mathbb{C})$ through

$$
f^*(dz) = h_1 \omega_1 + h_2 \omega_2.
$$
for some complex numbers $h_i$. These numbers can be determined by integrating the relation (3.27) over a canonical homology basis of $H_1(\hat{\Sigma}, \mathbb{Z})$ using (2.50), and with respect to the basis specified by (2.49) they are given by

$$h_1 = r \tau^* \quad \text{and} \quad h_2 = 0 .$$

(3.28)

Let $\text{Jac}(\hat{\Sigma}) := H^{1,0}(\hat{\Sigma}, \mathbb{C})/H^{1,0}(\hat{\Sigma}, \mathbb{Z})$ be the principally polarized Jacobian variety of $\hat{\Sigma}$, where $\Lambda_\tau = \mathbb{Z}^2 \oplus \tau \mathbb{Z}^2$ is the lattice of rank four induced by the period matrix $\tau$ of $\hat{\Sigma}$. It can be identified with the Picard group $\text{Pic}^0(\hat{\Sigma})$ of isomorphism classes of flat line bundles over $\hat{\Sigma}$, in correspondence with degree zero divisors, and it is isomorphic to the complex two-dimensional torus $\mathbb{C}^2/\Lambda_\tau$. There is an embedding of $\hat{\Sigma}$ into $\text{Jac}(\hat{\Sigma})$ provided by the Abel map $A: \hat{z} \mapsto \int_{\hat{z}}(\omega_1, \omega_2)$, which also provides the mapping from divisors to the Jacobian variety. The theta divisor is the analytic subvariety of the Jacobian defined by the equation $\Theta(0, z_1, z_2|\tau) = 0$. On the base, the Jacobian torus can instead be identified with the elliptic curve $T$ itself and one has $\text{Jac}(T) \cong T$.

It follows from a general property of finite morphisms between smooth projective curves [39] that the holomorphic map $f: \hat{T} \to T$ can be factorized by means of a commutative triangle

$$
\begin{array}{ccc}
\hat{T} & \xrightarrow{g} & \Sigma_1 \\
\downarrow f & & \downarrow f_1 \\
T & & T
\end{array}
$$

(3.29)

where $f_1: \Sigma_1 \to T$ is an unramified cover. The induced pullback morphisms on the Jacobian tori have the properties that $\ker(f^*) \cong \ker(f_1^*)$ and $g^*: \Sigma_1 \to \text{Jac}(\hat{\Sigma})$ is injective. This accounts for the first diagonal entry in the period matrix (2.51). The complimentary subvariety to $\text{im}(f^*) \cong T$ in the Jacobian torus $\mathbb{C}^2/\Lambda_\tau$ is gotten from the norm morphism

$$\Omega_f : \text{Jac}(\hat{\Sigma}) \to T \quad \text{with} \quad \Omega_f(z_1, z_2) := h_1 z_1 + h_2 z_2$$

(3.30)

which takes the divisor class $D$ of degree zero by applying $f$ to each point of the divisor. The kernel of this morphism is a principally polarized subvariety of $\text{Jac}(\hat{\Sigma})$ called the Prym variety of the cover and in the present case it is a complex one-dimensional torus $\mathbb{C}/(\mathbb{Z} \oplus \Pi \mathbb{Z})$ whose period $\Pi$ is called the Prym modulus. In the basis defined by (2.49), from (3.28) it follows that the kernel of (3.30) in $\mathbb{C}^2$ consists of all points of the form $(z_1, z_2) = (m, z)$ with $m \in \mathbb{Z}$ and $z \in \mathbb{C}$. Passing to the quotient $\mathbb{C}^2/\Lambda_\tau$ using (2.51) truncates to points $(0, z)$ with the identifications $z \sim z + m_1 \tau^* + m_2 \tau#$ for any $m_1, m_2 \in \mathbb{Z}$. It follows that the Prym modulus in this basis is given by

$$\Pi = r \tau^#$$

(3.31)

and we have explicitly identified the second elliptic modulus in (2.51). Using the factorization (3.29) one shows [39] that the induced theta divisor on $\ker(\Omega_f)$ is $r$ times the theta divisor defining its principal polarization, and hence that $\ker(\Omega_f)$ is a Prym-Tyurin variety.

So far everything we have said holds generally for any $N$-sheeted genus two cover of the torus $T$. When $N = 2$, wherein only the $r = 2$ term contributes in (2.48), the Prym variety possesses a special characterization [40] which enables one to make this construction much more explicit.
Consider the element of the symplectic group \( Sp(4, \mathbb{Z}) \) given by
\[
g = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{pmatrix} =: \begin{pmatrix} A & B \\ C & D \end{pmatrix}.
\] (3.32)

It induces the change in basis of \( H_1(\hat{\Sigma}, \mathbb{Z}) \) represented by
\[
M = M' \begin{pmatrix} D^T & B^T \\ C^T & A^T \end{pmatrix}
\text{ with } M' = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},
\] (3.33)

and the genus two modular transformation
\[
\tau_0^r(\tau^\bullet, \tau^\#) = (A \tau_0^r(\tau^\bullet, \tau^\#) + B) (C \tau_0^r(\tau^\bullet, \tau^\#) + D)^{-1}
\] (3.34)

with
\[
\tau_0^r(\tau^\bullet, \tau^\#) = \frac{1}{2} \begin{pmatrix} \Pi + \tau^\bullet & \Pi - \tau^\bullet \\ \Pi - \tau^\bullet & \Pi + \tau^\bullet \end{pmatrix}
\] (3.35)

where we have used (3.31) with \( r = 2 \). From (2.50) it follows that
\[
f_*(\hat{\alpha}_1) = -f_*(\hat{\alpha}_2) = \alpha \quad \text{and} \quad f_*(\hat{\beta}_1) = -f_*(\hat{\beta}_2) = \beta.
\] (3.36)

Integrating both sides of (3.27) in this basis thus gives \( h_1' = -h_2' = 1 \), and hence
\[
f^*(dz) = \omega_1 - \omega_2.
\] (3.37)

What makes the instance of a double cover \( f : \hat{\Sigma} \rightarrow \mathbb{T} \) special is that it has a canonical conformal automorphism \( \iota : \hat{\Sigma} \rightarrow \hat{\Sigma} \), satisfying \( f \circ \iota = f \), which is the involution permuting the sheets of the cover. It uniquely determines the covering with \( T = \hat{\Sigma}/\iota \). From (3.36) it follows that
\[
\iota(\hat{\alpha}_1) = -\iota(\hat{\alpha}_2) \quad \text{and} \quad \iota(\hat{\beta}_1) = -\iota(\hat{\beta}_2),
\] (3.38)

and hence that
\[
\iota^* (\omega_1) = -\omega_2.
\] (3.39)

The holomorphic one-form
\[
\nu = \omega_1 + \omega_2
\] (3.40)

is called the Prym differential and it is the unique holomorphic differential on the two-sheeted cover \( \hat{\Sigma} \) which is odd under the defining involution with \( \iota^*(\nu) = -\nu \). It follows from (3.37)–(3.40) and the form (3.35) of the period matrix in this basis that the Prym period is determined by
\[
\Pi = \oint_{\hat{\beta}_1} \nu.
\] (3.41)

The Prym differential \( \nu \) is normalized with respect to the \( \hat{\alpha}_1 \) cycle, while it has vanishing periods around \( \hat{\alpha}_1 - \hat{\alpha}_2 \) and \( \hat{\beta}_1 - \hat{\beta}_2 \). At the level of Jacobian varieties, the Prym variety \( \ker(\Omega_f) \) is isomorphic to the subvariety of \( \text{Jac}(\hat{\Sigma}) \) consisting of degree zero divisor classes which are odd under the involution \( \iota \). Note that from (3.37) it follows that the embedding \( f^* : \mathbb{T} \hookrightarrow \text{Jac}(\hat{\Sigma}) \) is isomorphic to the subvariety invariant under \( \iota \).
Similarly to the even holomorphic one-form (3.37), the Prym differential (3.40) may be given explicitly as the pull-back \( \nu = f^*(pr(w_1, w_2)) \) of a multiplicative differential \( pr(z; w_1, w_2) \) on the base elliptic curve \( \mathcal{T} \) with modulus \( \tau^* \). It is required to have a square root cut singularity about each of the branch points \( w_1, w_2 \in \mathcal{T} \) of the cover and to have global periodicity under \( z \to z + m + n \tau^* \) for any \( m, n \in \mathbb{Z} \). This uniquely determines the multiplicative differential on \( \mathcal{T} \) in terms of Jacobi-Erdélyi elliptic functions as

\[
pr(z; w_1, w_2) = \frac{\theta_1(z - \frac{w_1 + w_2}{2} | \tau^*)}{\sqrt{\theta_1(z - w_1 | \tau^*) \theta_1(z - w_2 | \tau^*)}}.
\]

The Prym modulus (3.41) may then be written as

\[
\tau^\# = \frac{1}{2} \Pi = \frac{1}{2} \oint_{\beta} pr(w_1, w_2) \oint_{\alpha} pr(w_1, w_2),
\]

thereby determining the desired explicit dependence of the elliptic modulus \( \tau^\# \) on the branch point loci. As expected, \( \Pi \to \tau^* \) in the unramified limit \( w_1 \to w_2 \) wherein the branch cut on \( \mathcal{T} \) closes up. It follows from (3.35) that this limit corresponds to approaching a separating boundary component of moduli space, wherein the genus two Riemann surface \( \hat{\Sigma} \) degenerates into two copies of the base torus \( \mathbb{T} \).

Thus far we have not accounted for global monodromy \( \Phi \) of the covering map \( f : \hat{\Sigma} \to \mathcal{T} \), i.e., the above formulas are written in the untwisted sector \( (\varepsilon, \delta) = (0, 0) \). For each twisted sector \( (\varepsilon, \delta) \in (\mathbb{Z}/2\mathbb{Z})^2 \) there is a holomorphic Prym form \( \nu_{\varepsilon, \delta} \) which is odd under the involution \( \iota \) and which has non-vanishing periods only around the \( (\hat{a}_1, \hat{b}_1) \) cycles of the homology group \( H_1(\hat{\Sigma}, \mathbb{Z}) \). They project onto multiplicative differentials \( pr_{\varepsilon, \delta}(w_1, w_2) \) on \( \mathcal{T} \) which have square root cut singularities about the branch points \( w_1, w_2 \in \mathcal{T} \). The Prym form corresponding to the characteristic \( (\varepsilon, \delta) \) can be gotten from the untwisted one via a crossing transformation of the branch points

\[
w_1 \to w_1 + \delta + \varepsilon \tau^* \quad \text{and} \quad w_2 \to w_2
\]

with \( pr_{0,0}(w_1, w_2) = pr(w_1, w_2) \). The corresponding Prym modulus is defined by

\[
\Pi_{\varepsilon, \delta} = \frac{\oint_{\beta} pr_{\varepsilon, \delta}(w_1, w_2)}{\oint_{\alpha} pr_{\varepsilon, \delta}(w_1, w_2)},
\]

with \( \Pi_{0,0} = \Pi \).

These constructions of Prym varieties and Prym differentials have natural generalizations to double covers \( \hat{\Sigma} \) of a genus \( g \) surface \( \Sigma \) with \( k = 2n \) branch points \( (n = 0, 1) \), with genus \( g = 2g + n - 1 \) determined by the Riemann-Hurwitz formula (2.35). In this case the Prym variety is a complex torus of dimension \( g + n - 1 \). By the Riemann-Roch theorem, there are exactly \( g + n - 1 \) independent holomorphic one-forms which are odd under the automorphism \( \iota \) and which form a basis for the Prym differentials. The remaining \( g \) even ones on \( \hat{\Sigma} \) are preimages of the
holomorphic differentials on the base space \(\Sigma\). A further generalization exists to more general abelian automorphism groups of a cover. The action of the group on \(H^{1,0}(\hat{\Sigma}, \mathbb{C})\) is then always diagonal on a suitable basis of holomorphic differentials and the subspace corresponding to a non-trivial set of eigenvalues are pull-backs of multiplicative elliptic differentials, whose multiplicative factors are given by these eigenvalues. This is exploited implicitly in the computation of \(\mathbb{Z}_N\) orbifold twist field amplitudes in [16].

### 3.4 Correlation Functions of Twist Field Operators

We now come to the computation of the two-point function \(\langle \sigma(z) \sigma(0) \rangle_{\mathbb{Z}_2}\) of \(\mathbb{Z}_2\) twist fields \(\sigma(z) = \sigma_{12}(z)\) in the \(\mathbb{R}^{24}\)\(\mathbb{Z}_2\) permutation orbifold. We begin by discussing some general aspects concerning global monodromy in the covering surface construction of Section 2.4. Recall that the sum appearing in the correlation function (2.36) of interest (computed with the amplitude (2.43)) is restricted to the set of admissible monodromy homomorphisms \(\Phi\) such that each connected component of the corresponding cover \(\hat{\Sigma}\) of the base torus \(\Sigma\) is isomorphic to \(\pi_1(\hat{\Sigma}) \cong \mathbb{Z}_N\) and since there are \(2\) squares of the generators of \(\pi_1(\hat{\Sigma})\), the problem is reduced to determining \(\mathbb{Z}_N\)-isomorphic to \(\pi_1(\hat{\Sigma})\) for the case at hand.

It is ensured by the requirement that the monodromy of the generators of \(\pi_1(\hat{\Sigma})\) encircling the punctures be a simple transposition in each orbit \(\xi \in \mathcal{O}(\Phi)\). The period matrix \(\tau_{\mathbf{w}}\) depends on the monodromy only via its stabilizer subgroups, which are the finite index subgroups \(H < \pi_1(\hat{\Sigma})\) obeying the admissibility criterion (2.38). Consider the stabilizer subgroup \(H = H_a\) of a given sheet \(a\) corresponding to a transitive homomorphism \(\Phi : \pi_1(\hat{\Sigma}) \rightarrow S_N\). Since it is isomorphic to \(\pi_1(\hat{\Sigma})\) and since there are \(2N - 2\) preimages of the two branch points of \(\mathcal{T}_\mathbf{w}\), it is a group freely generated by \(2N + 1\) elements. The kernel of the forgetful homomorphism \(\hat{i}_*: \pi_1(\hat{\Sigma}) \rightarrow \pi_1(\Sigma)\) is given by the normal closure

\[
\hat{N}_H(\hat{\gamma}_1, \ldots, \hat{\gamma}_{2N-2}) = \langle h \hat{\gamma}_1 h^{-1}, \ldots, h \hat{\gamma}_{2N-2} h^{-1} \mid h \in H \rangle
\]

of the generators \(\hat{\gamma}_i\) encircling the ramification points.

When \(N = 2\) the generators \(\hat{\gamma}_i\) are easily determined. Let us use the presentation \(\pi_1(\hat{\Sigma}) = \langle \alpha, \beta, \gamma \rangle\). The generators of \(\pi_1(\hat{\Sigma})\) encircling the ramification points are the (pullbacks of the) squares of the generators of \(\pi_1(\hat{\Sigma})\) which encircle the punctures. For \(N = 2\), the preimages of the punctures are precisely the ramification points, and hence one has

\[
\ker(\hat{i}_*) = \hat{N}_H(\gamma^2, ([\alpha, \beta] \gamma)^2).
\]

There are four homomorphisms with the prescribed monodromy representing the four twisted sectors \((\varepsilon, \delta) \in (\mathbb{Z}/2\mathbb{Z})^2\), and all of them are transitive. There are correspondingly exactly four admissible subgroups \(H\) of index two. Since \(\mathbb{Z}_2\) is an abelian group, conjugacy classes of homomorphisms contain only one element. Their precise forms and the corresponding stabilizers can be determined explicitly.

The simplest example is provided by the admissible homomorphism \(\Phi_1\) which sends \(\gamma\) to the transposition \((1\ 2)\) and \(\alpha, \beta\) both to the identity. Its stabilizer \(H_1\) is freely generated by the words \(\alpha, \beta, \alpha \gamma \alpha^{-1}, \beta \gamma \beta^{-1}, \gamma^2\). We then seek a presentation of the generators \(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2, \hat{\gamma}\) of \(\pi_1(\hat{\Sigma})\) such that the quotient by the relations \(\gamma^2 = ([\alpha, \beta] \gamma)^2 = 1\) yields the group \(\pi_1(\hat{\Sigma})\) with \([\hat{\alpha}_1, \hat{\beta}_1][\alpha_2, \hat{\beta}_2] \in \hat{N}_{H_1}(\gamma^2, ([\alpha, \beta] \gamma)^2)\). For the case at hand, one sees that the assignments \(\hat{\alpha}_1 = \alpha, \hat{\beta}_1 = \beta, \hat{\alpha}_2 = \alpha \gamma \alpha^{-1}, \hat{\beta}_2 = \beta \gamma \beta^{-1}, \hat{\gamma} = \gamma^2\) suffice. This determines the homomorphism of fundamental groups \(i_* \circ f_*^{-1}\), where \(f\) is the restriction of the covering map to the marked
surfaces. Since the abelianization of $\pi_1(T^2)$ factors through this map, the powers of $\alpha, \beta$ in the canonical homology generators $\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2$ gives the map (2.50). This yields the covering homology matrix

$$M_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

which obeys the Hopf condition. Reduction of this matrix via an $Sp(4, \mathbb{Z})$ modular transformation as in Section 3.3 above yields the normal form (2.49) with $r = 2, m = t = 1, s = 0$. The other three admissible homomorphisms are similarly treated.

However, the above formalism is sensitive only to the induced homomorphism $f_*$ between homology groups rather than homotopy groups, and it is difficult to proceed further with the explicit construction of the modular invariant amplitude (2.36). We will return to this issue in some more detail in the next section. Here we shall compute the twist field correlation function using results of [21] where the correlation functions are computed for a free boson $X$ in the geometric orbifold $S^1/\mathbb{Z}_2$ using the covering space method explained in Section 2.4. The two-point correlation function on the torus $T$ with twist field insertions may be computed from the path integral over field configurations $\hat{X}$ on the double cover $\hat{\Sigma}$ which are odd under the canonical involution with $\hat{X} \circ \iota = -\hat{X} \mod 2\pi R$. As in Section 3.1 above, in each twisted sector $(\varepsilon, \delta)$ the amplitude is a product of a radius independent quantum piece and a classical piece. The instanton configurations on the worldsheet $\hat{\Sigma}$ that contribute to the classical part of the correlation function are analogous to the untwisted ones used in Section 3.1 above. In the homology basis specified by (3.33), the boundary conditions of the boson $\hat{X}$ in the given twisted sector are characterized by the Prym differential $\nu_{\varepsilon, \delta}$. The classical contribution is then completely analogous to that in (3.2) with the period $\tau$ equal to the Prym modulus $\Pi_{\varepsilon, \delta}$.

The quantum contributions may be computed by equating the two-loop orbifold amplitude with that of the circle theory at the self-dual radius as before, with the additional observation that the twist fields in this correspondence are equivalent to magnetic vertex operators [21]. At this radius the momentum lattices appearing in the classical partition sums can be built up from a finite number of square sublattices. A term by term comparison of the chiral blocks gives an expression for the ratio of a twisted determinant to the untwisted determinant $z^{2\tau}$ as the modulus squared of a holomorphic function of the positions of the branch points on $T$. In this way the normalized twist field two-point function on $T$ with the twist characteristic $(\varepsilon, \delta)$ in the $\mathbb{Z}_2$ target space orbifold of the compactified boson $X$ can be written as [21]

$$\langle \sigma(z) \sigma(0) \rangle^{\varepsilon, \delta}_{\text{orb}} = \Phi^{(\tau^*)} |c(\xi)\rangle^{-2} \Phi^{cl(\Pi_{\varepsilon, \delta}, R)} ,$$

where

$$c(\xi) = E(z)^{1/8} \frac{\theta(\frac{\xi}{2}) (0 | \Pi_{\varepsilon, \delta})}{\sqrt{\theta(\frac{a+b+\delta}{2} | \tau^*) \theta(\frac{b}{2} | \tau^*)}} .$$

Here we have used translation invariance to fix one of the twist field insertion points at the origin, and $(a,b) \neq (1,1)$ is a fixed arbitrary characteristic. The quantity $E(z)$ is the prime form of the elliptic curve $\mathbb{T}$ given by

$$E(z) = \frac{\theta_1(z | \tau^*)}{\theta_1'(0 | \tau^*)}$$

with $\theta_1'(z | \tau) := \frac{\partial}{\partial z} \theta_1(z | \tau)$, and it is the doubly periodic elementary solution of the Laplace
equation on the torus. The independence of the expression (3.51) on the choice of characteristic \((a, b)\) is the mathematical statement of the Schottky relations [40] (see Section 3.5 below).

We can now write down the desired amplitude in the permutation orbifold \(\mathbb{R}^{24} \wr \mathbb{Z}_2\). For this, we redefine the independent bosons \(X_i^a, i = 1, \ldots, 24, a = 1, 2\) to \(X_i^\pm = X_i^1 \pm X_i^2\) as in Section 3.1 above. Since the \(\mathbb{Z}_2\) permutation group acts on the 24 bosons simultaneously, both the global and local monodromy of the fields \(X_i^+\) are trivial, and the twist operators act as the identity on these fields. The path integral over \(X_i^-\) thus leads simply to an overall factor \(\mathcal{Z}(\tau^\bullet, R)^{24}\). On the other hand, the twist operators act as a \(\mathbb{Z}_2\) twist field simultaneously on all sigma model fields \(X_i^-\). It follows that the correct prescription is to raise the geometric \(\mathbb{Z}_2\) orbifold twist field correlation function in each sector to the power 24, and then sum over the twisted sectors.

The \(X_i^+\) contribution is cancelled in the suitably normalized correlation function by the same factors coming from the partition function (3.1). One should then take the decompactification limit \(R \to \infty\), wherein \(\mathcal{Z}(\tau^\bullet, R = \infty) = 1\) as before. This gives the two-point function

\[
\langle \sigma(z) \sigma(0) \rangle_{\mathbb{Z}_2}^{Z_2} = \lim_{R \to \infty} \frac{1}{2} \sum_{(\varepsilon, \delta) \in (\mathbb{Z}/2\mathbb{Z})^2} \left( \langle \sigma(z) \sigma(0) \rangle_{\text{orb}}^{\varepsilon, \delta} \right)^{24}.
\]

Substituting (2.22) and (3.50)–(3.52), and using the identity

\[
\theta'_1(0 | \tau) = -2\pi \eta(\tau)^3,
\]

then leads to the explicit formula

\[
\langle \sigma(z) \sigma(0) \rangle_{\mathbb{Z}_2}^{Z_2} = \frac{1}{2} \left( \frac{2 \sqrt{2} \pi^{5/2}}{\tau_2^*} \right)^{12} \frac{\theta(\nu^*) \theta(\nu^*)^4}{\theta_1(z | \nu^*) \eta(\nu^*)^5} \sum_{(\varepsilon, \delta) \in (\mathbb{Z}/2\mathbb{Z})^2} \frac{\theta(z^{\varepsilon, \delta}) \theta(z^{\varepsilon, \delta})}{\theta(0^{\varepsilon, \delta}) \eta(\nu^*)^2} ^{24}.
\]

\[
\text{(3.55)}
\]

### 3.5 DLCQ Free Energy = DVV Correlator

We will now prove the main result of this section, establishing the equivalence

\[
\mathcal{F}_2(\tau^\bullet) = \frac{4\lambda^2}{\tau_2^* \mu(0)} \int_{\mathbb{T}} d\mu(z) \langle \sigma(z) \sigma(0) \rangle_{\mathbb{Z}_2}^{Z_2}
\]

between the DLCQ free energy on the double cover \(\tilde{\Sigma} \to \mathbb{T}\) given by (3.25) and the translationally invariant correlator (2.42) of the DVV vertex operator determined by the twist field two-point function (3.55) on \(\mathbb{R}^{24} \wr \mathbb{Z}_2\). We begin by observing that the right-hand side of the formula (3.56) is independent of the twist characteristic \((\varepsilon, \delta)\) in (3.55). This follows from the fact that one can get any twisted sector from the untwisted one \((\varepsilon, \delta) = (0, 0)\) by a crossing transformation (3.44). Crossing symmetry of the orbifold theory, along with modular invariance at genus one, is the remnant of genus two modular invariance on the covering space [21]. One can check this invariance explicitly by showing that the \(z\)-dependent part of the correlation function (3.50) transforms under the crossing transformation (3.44) precisely by changing \((0, 0) \to (\varepsilon, \delta)\), just like the Prym modulus according to (3.45).

Next we examine the change of integration variables from the modulus \(\tau^\#\) in (3.25) to the branch point location in (3.56). For this, we require the Jacobian \(|d\tau^\#/dz|^2\). The explicit dependence of the Prym modulus \(\Pi\) on the branch point loci is given by the formula (3.43)
with \( w_1 = z, w_2 = 0 \), but this is not convenient for computing the requisite derivative \( d\Pi/dz \). Instead, it is more useful to use the implicit dependence of the Prym modulus on the branch point \( z \) dictated by the Schottky relations. For zero characteristics \((\varepsilon, \delta) = (0, 0)\), they are given by

\[
\sqrt{\theta_i(\frac{z}{2} | \tau^*)} \theta_i(0 | \tau^*) = \sqrt{\theta_j(\frac{z}{2} | \tau^*)} \theta_j(0 | \tau^*) \quad (3.57)
\]

By separating the explicit \( z \) and \( \Pi \) dependences for \( i = 4 \) and \( j = 2 \), we can write (3.57) as

\[
\frac{\theta_2(0 | \Pi)}{\theta_4(0 | \Pi)} = \sqrt{\frac{\theta_2(0 | \tau^*) \theta_2(\frac{z}{2} | \tau^*)}{\theta_4(0 | \tau^*) \theta_4(\frac{z}{2} | \tau^*)}}. \quad (3.58)
\]

Taking the total derivative of the relation (3.58) with respect to \( z \) yields

\[
\frac{\partial}{\partial \Pi} \left( \frac{\theta_2(0 | \Pi)}{\theta_4(0 | \Pi)} \right) \frac{d\Pi}{dz} = \frac{d}{dz} \left( \frac{\theta_2(0 | \tau^*) \theta_2(\frac{z}{2} | \tau^*)}{\theta_4(0 | \tau^*) \theta_4(\frac{z}{2} | \tau^*)} \right). \quad (3.59)
\]

We can transform the \( \Pi \) derivative by using the heat equation

\[
\partial \frac{\partial \theta_i(z | \Pi)}{\partial \Pi} + \frac{i}{4\pi} \partial^2 \frac{\partial \theta_i(z | \Pi)}{\partial z^2} = 0 \quad (3.60)
\]

to get the form

\[
\frac{\partial}{\partial \Pi} \left( \frac{\theta_2(0 | \Pi)}{\theta_4(0 | \Pi)} \right) = -\frac{\partial}{\partial w} \left( \frac{\theta_3(0 | \Pi)^2 \theta_2(w | \Pi) \theta_3(w | \Pi)}{\theta_4(w | \Pi)^2} \right) \bigg|_{w=0}. \quad (3.61)
\]

We may then use the identity for the derivative of a ratio of theta functions given by

\[
\frac{\partial}{\partial w} \left( \frac{\theta_2(w | \Pi)}{\theta_4(w | \Pi)} \right) = -\pi \theta_3(0 | \Pi)^2 \frac{\theta_2(w | \Pi) \theta_3(w | \Pi)}{\theta_4(w | \Pi)^2} \quad (3.62)
\]

to arrive at

\[
\frac{\partial}{\partial \Pi} \left( \frac{\theta_2(0 | \Pi)}{\theta_4(0 | \Pi)} \right) = \frac{i}{4} \frac{\theta_3(0 | \Pi)^3 \theta_1'(0 | \Pi)}{\theta_4(0 | \Pi)^2}. \quad (3.63)
\]

The differentiation on the right-hand side of (3.59) is an easy exercise. This calculation can be repeated starting from the Schottky relation (3.57) with \( i = 4 \) and \( j = 3 \). The final result is identical to that above with the replacements \( \theta_2 \leftrightarrow \theta_3 \) of theta functions everywhere. In this way we can finally write

\[
\left| \frac{d\Pi}{dz} \right|^2 = \pi^2 \left| \frac{\theta_2(0 | \tau^*) \theta_2(\frac{z}{2} | \tau^*)}{\theta_2(0 | \Pi)} \right|^2 \left| \frac{\theta_3(0 | \tau^*) \theta_3(\frac{z}{2} | \tau^*)}{\theta_3(0 | \Pi)} \right|^2 \left| \frac{\theta_4(0 | \tau^*) \theta_4(\frac{z}{2} | \tau^*)}{\theta_4(0 | \Pi)} \right|^2 \left| \frac{\theta_1(0 | \Pi)^4}{\theta_1(0 | \Pi)} \right|^2 \left| \frac{\theta_1'(0 | \Pi)}{\theta_1'(0 | \Pi)} \right|^2 \left| \frac{\theta_3(0 | \Pi)^2 \theta_3(w | \Pi) \theta_3(w | \Pi)}{\theta_4(w | \Pi)^2} \right|^2 \left| \frac{\theta_3(0 | \Pi)^3 \theta_1'(0 | \Pi)}{\theta_4(0 | \Pi)^2} \right|^2 . \quad (3.64)
\]

To compare (3.64) with the elliptic functions appearing in the expressions (3.25) and (3.55) for \((\varepsilon, \delta) = (0, 0)\), we exploit the identity (3.19) and the Schottky relations (3.57) again to write

\[
\left| \frac{d\Pi}{dz} \right|^2 = \pi^2 \left| \frac{\theta_1(\frac{z}{2} | \tau^*)^3}{\theta_1(z | \tau^*) \theta_1'(0 | \Pi)^2} \prod_{i=1,2} \frac{\theta(a_i | \Pi)}{\theta(a_i | 0 | \Pi)} \right|^2. \quad (3.65)
\]
where \((a_i, b_i) \in \{(0, 0), (0, 1), (1, 0)\}\) are arbitrary characteristics which we will choose conveniently. We can now use the identities (3.5), (3.19) and (3.54) along with

\[
\theta_3\left(\frac{\tau}{2} \mid \tau^\ast\right)^2 \theta_4(0 \mid \tau^\ast)^2 - \theta_4\left(\frac{\tau}{2} \mid \tau^\ast\right)^2 \theta_3(0 \mid \tau^\ast)^2 = -\theta_1\left(\frac{\tau}{2} \mid \tau^\ast\right)^2 \theta_2(0 \mid \tau^\ast)^2
\]

(3.66)
to expand the expression (3.65) into

\[
\frac{|d\Pi|}{dz}^2 = \frac{1}{2^{18}} \left| \frac{\eta(\tau^\ast)^{-42}}{\theta_1(z \mid \tau^\ast)^9} \prod_{i=1}^8 \sqrt{\theta(a_i^0)\left(\frac{\tau}{2} \mid \tau^\ast\right) \theta(b_i^0)\left(0 \mid \tau^\ast\right)} \right| \times \left[ \theta_2\left(\frac{\tau}{2} \mid \tau^\ast\right) \theta_3\left(\frac{\tau}{2} \mid \tau^\ast\right) \theta_4(0 \mid \tau^\ast)^2 \theta_3(0 \mid \tau^\ast)^2 \right]^2 \times \left[ \theta_3\left(\frac{\tau}{2} \mid \tau^\ast\right)^2 \theta_4(0 \mid \tau^\ast)^2 - \theta_4\left(\frac{\tau}{2} \mid \tau^\ast\right)^2 \theta_3(0 \mid \tau^\ast)^2 \right]^4.
\]

(3.67)

We have again used (3.57) to infer that every term of the product in (3.67) is independent of the chosen characteristic \((a_i, b_i)\).

Let us now substitute (3.67) into the integral (3.25), recalling that \(\Pi = 2\tau^\#\). We can again exploit the freedom in choice of characteristics \((a_i, b_i)\) to combine the theta functions in (3.67) with the ones \(\theta_i(0 \Pi) = \theta(a_i^0)(0 \Pi)\) and \(\theta_i(0 \tau^\ast) = \theta(b_i^0)(0 \tau^\ast)\) appearing in (3.25) by re-expressing Dedekind functions as theta functions using (3.5). The simplification effectively amounts to replacing each factor \(\theta_i(0 \Pi)\) with \(\sqrt{\theta_i(\frac{\tau}{2} \mid \tau^\ast)} \theta_i(0 \tau^\ast)\). We can use this trick to cancel the difference of theta functions appearing in the integrand of (3.25) by simply doing this replacement for every term, and remembering that there are in total 40 factors of \(\theta_i(0 \Pi)\) in each term of the expansion of the fourth power of the difference.

In this way, it is straightforward to see after some inspection that the free energy (3.25) may be written in terms of an integral over the branch point location on the torus \(T\) as

\[
\mathcal{F}_2(\tau^\ast) = \frac{g_s^2}{(2\pi^2 \alpha')^2} \left| \frac{\eta(\tau^\ast)}{4 \mid \tau^\ast}^{30} \right| \frac{d^2 z}{(\text{Im } \Pi(z))^{12}} \frac{1}{\theta_1(z \mid \tau^\ast)^6} \prod_{i=1}^{48} \sqrt{\theta(a_i^0)\left(\frac{\tau}{2} \mid \tau^\ast\right) \theta(b_i^0)\left(0 \mid \tau^\ast\right)}.
\]

(3.68)

It is now clear that with (3.55) the DLCQ free energy function (3.68) can be expressed in the form (3.56) if we choose the measure

\[
d\mu(z) = \frac{d^2 z}{\mu(z)} \quad \text{with} \quad \mu(z) = \left(\frac{2\pi^2 \alpha'}{\tau^\ast 2} \text{Im } \Pi(z)\right)^{d/2}
\]

(3.69)

where \(d = 24\) is the spacetime dimension of the permutation orbifold. Using \(\Pi(0) = \tau^\ast\), the coupling constant \(\lambda\) is then given by

\[
\lambda = \frac{4g_s}{\pi^3 |512 \tau^\ast|^4} \sqrt{\tau^\ast}.
\]

(3.70)

Note that the coupling (3.70) has the correct infrared behaviour \(\lambda \to 0\) as \(\tau^\ast \to \infty\) to ensure that the interacting sigma model approaches a conformal fixed point in the infrared limit.

From the genus two perspective the origin of the measure (3.69) is clear. It arises from the \(Sp(4, \mathbb{Z})\) modular invariant integration over the moduli space of genus two branched covering maps \(f : \Sigma \to T\). From the genus one perspective it is a consequence of the conformal anomaly, implying that the local twist field correlation functions depend on the coordinatization chosen.
on the Riemann surface $\mathbb{T}$. For the twist field operators the natural choice is the coordinate $z$ of $\mathbb{T}$, but to induce the modular invariant interactions of strings in the symmetric product a non-trivial integration measure (3.69) must be adapted. We will see this explicitly in the next section when we study the action of the mapping class group of the punctured torus $\mathbb{T}_w$.

4 Nonabelian Orbifolds

In this section we address some issues surrounding the extensions of the results of the previous section to $S_N$ orbifolds with $N > 2$. At this stage, however, we have not succeeded in making the construction as explicit as for the $\mathbb{Z}_2$ orbifold. The main technical obstruction is the combined noncommutativity of the twist group $S_N$ and the fundamental group $\pi_1(\mathbb{T}_w)$ of the punctured torus. For twist group $\mathbb{Z}_2$ the image of the latter group under a given monodromy homomorphism $\Phi$ is of course an abelian group, enabling explicit constructions. But these constructions become ambiguous and inconsistent in the nonabelian case, as one must deal with the full nonabelian homotopy group and not just its abelianization to the homology group. We are not aware of any direct computation of the twist field correlation functions in these specific instances. In the following we will highlight some of the main technical issues surrounding these calculations in the higher degree permutation orbifolds, and in particular to what extent the DLCQ free energy (2.48) can be used to provide an explicit representative for the DVV correlator (2.42) using the combinatorial formula (2.36). One of the outcomes of this analysis will be a more precise, general description of the measure $d\mu(z)$ required in the definition of the vertex operator (2.40).

4.1 Uniformization Construction

Let us recall the general construction of Section 2.4. A correlation function involving twist fields alone in any permutation orbifold is defined through the generalized partition function (2.36). It gives a twist field correlation function on a worldsheet $\Sigma$ as a sum over twisted sectors, each characterized by a conjugacy class of monodromy homomorphisms. One term is given by the partition function of the covering space $\hat{\Sigma}$ determined by Hurwitz data, comprising the monodromy, the complex structure of the worldsheet $\Sigma$ and the insertion points of the twist field operators. The issue is how to determine the covering space and its complex structure in terms of the Hurwitz data. The monodromy in the case of $k$ distinct insertion points on the worldsheet is a homomorphism $\Phi : \pi_1(\Sigma_w) \to G < S_N$, and the general Riemann-Hurwitz formula (2.35) for ramified coverings gives the genus $\hat{g}$ of the covering space. Determining the topological type of the cover is analogous to the unramified case. The fundamental group of the marked cover $\hat{\Sigma}_w$ is given by a stabilizer subgroup $H_a < \pi_1(\Sigma_w)$. The index $a$ is the label of a sheet, which is permuted by the twist group $G < S_N$, and different choices of $a$ result in conjugate subgroups of $\pi_1(\Sigma_w)$ corresponding to different choices of pre-image of the base point of $\pi_1(\Sigma_w)$ as the base point of $\pi_1(\hat{\Sigma}_w)$.

However, it is much more difficult to determine the complex structure of the cover. Recall that the prescription for the unramified case was to choose a uniformizing homomorphism $u : \pi_1(\Sigma) \to U$ such that $\Sigma_r = U/u(\pi_1(\Sigma))$. Then one needs to restrict $u$ to the stabilizer subgroup of $\pi_1(\Sigma)$ corresponding to the monodromy homomorphism $\Phi$. But the domain of the monodromy is $\pi_1(\Sigma_w)$ for the ramified case, which is a group distinct from $\pi_1(\Sigma)$. Hence it
is not straightforward to extend this uniformization method to the case of branched coverings. Consider the commutative diagram

\[
\begin{array}{c}
\hat{\Sigma}^\hat{w} \xrightarrow{i} \hat{\Sigma} \\
\downarrow \hat{f} \quad \downarrow f \\
\Sigma^\Sigma \xrightarrow{i} \Sigma
\end{array}
\]

(4.1)

where the maps \(i\) and \(\hat{i}\) are the canonical inclusions (filling in the deleted points), and \(\hat{f}\) is the restriction of the covering map \(f\) to the punctured surfaces. Passing to the corresponding pushforwards, this diagram induces a commutative diagram of fundamental groups given by

\[
\begin{array}{c}
\pi_1(\hat{\Sigma}^\hat{w}) \xrightarrow{i_*} \pi_1(\hat{\Sigma}) \\
\downarrow \hat{f}_* \quad \downarrow f_* \\
\pi_1(\Sigma^\Sigma) \xrightarrow{i_*} \pi_1(\Sigma)
\end{array}
\]

(4.2)

Let \(T(k, g)\) denote the Teichmüller space of genus \(g\) Riemann surfaces with \(k\) punctures. Let \(\mathcal{M}(k, g)\) be the mapping class group of the (marked) Riemann surface \(\Sigma^\Sigma\) acting on \(T(k, g)\). One seeks maps which fit into the commutative diagram

\[
\begin{array}{c}
T(\hat{k}, \hat{g}) \longrightarrow T(0, \hat{g}) \\
\downarrow \quad \downarrow \\
T(k, g) \longrightarrow T(0, g)
\end{array}
\]

(4.3)

associated to the covering and the inclusions such that the vertical arrow on the left is given by the surjective map \(U/u(\pi_1(\Sigma^\Sigma)) \cong U/u(H_a) \rightarrow U/u(\pi_1(\Sigma^\Sigma))\), where \(u\) is a uniformizing map of punctured surfaces. In this way one can incorporate the information from the monodromy contained in the admissible finite index subgroup \(H_a\). Note that the corresponding complex dimensions of the spaces involved in (4.3) map as

\[
\begin{array}{c}
3\hat{g} - 3 + \hat{k} \quad \longrightarrow \quad 3\hat{g} - 3 \\
\downarrow \quad \downarrow \\
3g - 3 + k \quad \longrightarrow \quad 3g - 3
\end{array}
\]

(4.4)

for \(g > 0\) (except for \(\dim \mathcal{T}(0, 1) = 1\)).

The problem rests in the construction of the horizontal arrows of (4.3). Since the pushforward \(\iota_*\) is a group homomorphism, the image of an element of a uniformizing group \(u(\pi_1(\Sigma^\Sigma)) < PSL(2, \mathbb{R})\), which we identify with the complex structure given by \(U/u(\pi_1(\Sigma^\Sigma)) \in \mathcal{T}(k, g)\), is a coset and thus not an element in \(PSL(2, \mathbb{R})\). Thus even though the quotient of the uniformizing group of the marked surface by the normal closure of the parabolic generators is isomorphic to \(\pi_1(\Sigma)\) (by the admissibility constraint), it is not a subgroup of \(PSL(2, \mathbb{R})\). The same remarks apply to the map \(\hat{i}\) inducing the top horizontal arrow in (4.3). Therefore it is not possible to apply the method of uniformization which worked for the unramified case, and the forgetful maps (i.e., the horizontal arrows in (4.3)) need to be constructed by hand.

Let us specialize to our main problem of interest, where the base space is the torus \(\Sigma = \mathbb{T}\) with \(k = 2\) simple branch points. For an \(N\)-sheeted cover of genus \(\hat{g} = 2\) there are \(\hat{k} = 2N - 2\).
preimages of these branch points, so that two of the \( N \) preimages of a generic point of the base coincide for a branch point. The main obstacle in constructing the map \( \nu_T : T(2,1) \to T(0,1) \) rests in the fact that a flat torus admits a complete euclidean metric, whereas a punctured torus admits a complete hyperbolic metric. Thus in order to apply uniformization one needs to construct a map between the space of flat tori and the space of hyperbolic tori. Let us assume that the branch points are distinguished points of the flat metric on \( \mathbb{T} \). Using the automorphism group of the torus we may fix the location of one of the branch points at the origin. Then one requires a bijection \( T(2,1) \to T(0,1) \times \mathbb{U} \), where the second branch point \( z \) varies in the complex upper half plane \( \mathbb{U} \). This must be done in such a way that a lift of the mapping class group \( M(0,1) = SL(2, \mathbb{Z}) \) to \( M(2,1) \) acts equivariantly on \( T(2,1) \) with respect to this bijection.

An element of \( T(2,1) \) is a twice punctured hyperbolic torus. Using the uniformizing homomorphism \( u : \pi_1(\mathbb{T}_w) \to PSL(2,\mathbb{R}) \), it can be characterized as a discrete Fuchsian group

\[
u(\pi_1(\mathbb{T}_w)) = \langle \alpha, \beta, \gamma \in PSL(2,\mathbb{R}) \mid |\text{tr} \alpha| > 2, |\text{tr} \beta| > 2, |\text{tr} \gamma| = |\text{tr} [\alpha, \beta] \gamma| = 2 > . \tag{4.5}\]

The hyperbolic generators \( \alpha, \beta \) correspond to translation along a canonical homology basis of the unmarked torus, while \( \gamma \) and \([\alpha, \beta] \gamma\) are the parabolic generators corresponding to the punctures.\(^3\) Then the complex structure is given by \( \mathbb{U}/u(\pi_1(\mathbb{T})) \). The subgroup (4.5) contains three real parameters for each generator, two trace relations for parabolicity and a conjugation symmetry which eliminates three parameters, hence the real dimension of \( T(2,1) \) is \( 3 \cdot 3 - 2 - 3 = 4 \), as anticipated.

The space \( T(0,1) \times \mathbb{U} \) is coordinatized by ordered pairs \((\tau, z)\), where \( \tau \) is a genus one modulus and \( z \) is a distinguished point on \( \mathbb{T} \). The mapping class group \( M(0,1) \cong SL(2,\mathbb{Z}) \) of the flat torus acts on these pairs through the generators

\[
\begin{align*}
T : (\tau, z) &\longmapsto (\tau + 1, z) \quad \text{and} \quad S : (\tau, z) &\longmapsto \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) .
\end{align*} \tag{4.6}
\]

obeying \( S^4 = (TS)^3 S^2 = 1 \). The modular \( S \)-transformation here is defined via analytic continuation along a clockwise oriented path around the origin in the complex \( z \)-plane. A lift of these generators to the mapping class group \( M(2,1) \) of the twice punctured hyperbolic torus is presented in [41] as an action on the generators of (4.5) by

\[
\tilde{T} : \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \longmapsto \begin{pmatrix} \alpha \\ \beta \alpha \\ \gamma \end{pmatrix} \quad \text{and} \quad \tilde{S} : \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \longmapsto \begin{pmatrix} \beta^{-1} \\ \alpha \\ \beta^{-1} \gamma \beta \end{pmatrix} . \tag{4.7}
\]

This lift of \( SL(2,\mathbb{Z}) \) is not unique. In fact, the modular group \( M(2,1) \) is an extension of \( SL(2,\mathbb{Z}) \) by \( B(2,1)/\Gamma_{M(2,1)} \), where \( \Gamma_{M(2,1)} \) is the center of \( M(2,1) \) and \( B(2,1) \) denotes the two-stranded braid group of the torus [41]. Equivariance of the bijection \( \nu_T : T(2,1) \to T(0,1) \) with respect to these actions is then the statement

\[
\nu_T \circ \tilde{T} = T \circ \nu_T \quad \text{and} \quad \nu_T \circ \tilde{S} = S \circ \nu_T . \tag{4.8}
\]

We have not succeeded in constructing explicitly the required modular equivariant bijections, and it is not possible to write an algebraic formula [42]. One could try to surpass this problem by working directly with the hyperbolic presentation of the tori, and the known bijection between the Fenchel-Nielsen coordinates of Teichmüller space and the Fuchsian coordinates

\(^3\)We could have equivalently used an independent parabolic generator \( \gamma' \) with the relation \([\alpha, \beta] \gamma \gamma' = 1\).
parametrizing the uniformizing group \([42]\). But there is a great deal of ambiguity in this procedure which prevents an explicit construction, and there is no canonical way to identify the modular parameters of the torus itself and those corresponding to the branch points.

4.2 Homology Construction

Given the technical difficulties encountered above, we now turn to an alternative approach to determining the complex structure of the cover via the push-forward induced on homology groups \(f_* : H_1(\hat{\Sigma}, \mathbb{Z}) \to H_1(\Sigma, \mathbb{Z})\), which is provided by the abelianization of the diagram (4.2) for the fundamental groups. If a canonical basis is fixed both in the homology group of the base and that of the cover, then this map is given by a \(2g \times 2\hat{g}\) matrix \(M^T\). This matrix can then be used to determine the period matrix of the cover in terms of the period matrix of the base and some additional parameters \([12]\). For sufficiently low genus, the period matrix \(\tau\) uniquely characterizes the complex structure. We will go through this construction in detail for the relevant case of the genus two cover \(f : \hat{\Sigma} \to \Sigma\) for the two point function of twist fields corresponding to simple branch points. In this case the complex structure on \(\hat{\Sigma}\) is determined by a canonical map \(H^{1,0}(\hat{\Sigma}, \mathbb{C}) \otimes H_1(\hat{\Sigma}, \mathbb{Z}) \to \mathbb{C}\).

Let us see first how the matrix representation \(M\) of \(f_*\) can be determined and compared to the construction of Section 2.5. The main difference from the \(N = 2\) case studied at the beginning of Section 3.4 is that for \(N > 2\) the preimages of the punctures are no longer just the ramification points, since there are \(2N - 2 > 2\) preimages of the branch points. Let \(\pi_1(\mathbb{T}_w) = <\alpha, \beta, \gamma>\), the free group on three generators such that \(\ker(\hat{\iota}_a)\) is the normalizer \(N_{\pi_1(\mathbb{T}_w)}(\gamma, [\alpha, \beta] \gamma)\). In other words, the image of \(\alpha\) and \(\beta\) are the standard generators of \(\pi_1(\mathbb{T})\), whereas \(\gamma\) and \([\alpha, \beta] \gamma\) correspond to simple closed curves which are contractible to the branch points. The stabilizer \(H_a < \pi_1(\mathbb{T}_w)\) corresponding to a monodromy homomorphism \(\Phi\) is a subgroup of index \(N\) in the case of an \(N\)-sheeted cover. It can be presented in terms of \(4 + (2N - 2) - 1\) words from \(\pi_1(\mathbb{T}_w)\) which freely generate the group \(H_a\). By identifying \(H_a\) with \(\pi_1(\hat{\Sigma}_w)\), this presentation gives the homomorphism \(\hat{f}_*\) explicitly. There are \(N - 2\) independent elements from \(H_a\) which are conjugate to \(\gamma\) in \(\pi_1(\mathbb{T}_w)\), and another \(N - 2\) elements which are conjugate to \([\alpha, \beta] \gamma\). There is one further element conjugate to \(\gamma^2\) and another one conjugate to \(([\alpha, \beta] \gamma)^2\). This is because the \(N - 2\) generators of \(\pi_1(\hat{\Sigma}_w)\) corresponding to simple closed curves contractible to \(N - 2\) preimages of a branch point project to the simple closed curve contractible to the branch point, whereas the other two generators project to curves with winding number two about each of the branch points.

The normalizer of these \(2N - 2\) generators in \(H_a\) is the subgroup \(\ker(\hat{\iota}_a)\). One then seeks \(4 + 2N - 3\) generating elements such that \(2N - 3\) are in \(\ker(\hat{\iota}_a)\) and also the commutator product \([\hat{\alpha}_1, \hat{\beta}_1] [\hat{\alpha}_2, \hat{\beta}_2]\) of a suitably chosen remaining four. In other words, \(\hat{\alpha}_i, \hat{\beta}_i\) are representatives of the cosets that project to a canonical homology basis of \(\pi_1(\hat{\Sigma})\) under the map \(\hat{\iota}_a\). Due to the commutativity of the diagram (4.2) and the abelianization, the entry \(M_{ij}\) of the \(2 \times 4\) homology covering matrix is the sum of powers of the \(i\)-th generator of \(H_1(\mathbb{T}, \mathbb{Z})\) \((\alpha\) or \(\beta)\) appearing in the expression of the \(j\)-th generator of \(H_1(\hat{\Sigma}, \mathbb{Z})\) \((\hat{\alpha}_1, \hat{\beta}_1, \hat{\alpha}_2\) or \(\hat{\beta}_2)\). In this way, the two-point function may be computed by summing over admissible finite index subgroups \(H_a < \Gamma = \pi_1(\mathbb{T}_w)\).

Let us look at an explicit example of how this works. For \(N = 3\), there are 16 conjugacy classes of transitive monodromy homomorphisms, each class containing 6 homomorphisms. Ac-
Accordingly, there are 16 conjugacy classes of admissible index three subgroups of $\Gamma$, each class having $[\Gamma : N_{\Gamma}(\Gamma_k)] = 3$ representatives. Consider the admissible monodromy homomorphism $\Phi_1$ given by

$$\Phi_1 : \alpha \mapsto (2 \, 3), \quad \beta \mapsto (1 \, 2) \quad \text{and} \quad \gamma \mapsto (2 \, 3). \quad (4.9)$$

The corresponding three sheeted cover may be depicted schematically as

```
   3  
  / \  
 /   \ 
/     \f
 |     |
 |     |
 |     |
 /     \ 
 2  1  4
   |
   |
   |
   [\alpha,\beta]\gamma
```

(4.10)

with the parallelogram representing the base torus $T$. The sheets 2 and 3 are ramified over the branch point corresponding to $\gamma$, while the sheets 1 and 2 are ramified over the other branch point corresponding to $[\alpha,\beta]\gamma$ (since $\Phi_1 : [\alpha,\beta]\gamma \mapsto (1 \, 2)$).

The stabilizer subgroup of $\Gamma = \pi_1(T_w) = \langle \alpha, \beta, \gamma \rangle$ can be presented by

$$H_1 = \langle g_1, \ldots, g_7 \rangle := \langle \alpha, \beta^2, \gamma, \beta \alpha^2 \beta^{-1}, \beta \gamma \alpha^{-1} \beta^{-1}, \beta \alpha \gamma \beta^{-1}, \beta \alpha \beta \alpha^{-1} \beta^{-1} \rangle. \quad (4.11)$$

The elements $g_1, \ldots, g_7$ generate the the group $H_1$ freely. One can then determine the generators of $\ker(\hat{i}_*)$ as

$$\hat{g}_1 = g_3 = \gamma, \quad \hat{g}_2 = g_6 g_5 = \beta \alpha^{-2} \alpha^{-1} \beta^{-1}, \quad \hat{g}_3 = g_4 g_2 g_1^{-1} g_2^{-1} g_5 = \beta \alpha [\alpha, \beta] \gamma \alpha^{-1} \beta^{-1}, \quad \hat{g}_4 = g_1 g_4^{-1} g_7^{-1} g_6 g_7 g_3 = ([\alpha, \beta] \gamma)^2, \quad (4.12)$$

where we have underlined the curves on the base that they are conjugate to. Finally, it is possible to write down the generators

$$\hat{\alpha}_1 = g_7, \quad \hat{\beta}_1 = g_1 g_4^{-1}, \quad \hat{\alpha}_2 = g_1 \quad \text{and} \quad \hat{\beta}_2 = g_2 \quad (4.13)$$

such that

$$H_1/N_{H_1}(\hat{g}_1, \hat{g}_2, \hat{g}_3, \hat{g}_4) = \langle \hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2 | [\hat{\alpha}_1, \hat{\beta}_1] [\hat{\alpha}_2, \hat{\beta}_2] = 1 \rangle = \pi_1(\Sigma). \quad (4.14)$$

We can now count the powers of $\alpha, \beta$ appearing in (4.13) to determine the matrix representation $M = M_1$ of $f_*$ in (2.50) with

$$M_1 = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}. \quad (4.15)$$

---

4In practice it is easier to determine the monodromy homomorphism corresponding to a given presentation of a finite index subgroup.

5One can check that the elements (4.13) are independent representatives of the generators of the quotient modulo $\hat{g}_1, \hat{g}_2$ and $\hat{g}_4$, except for $\hat{g}_3 = [\hat{\alpha}_1, \hat{\beta}_1][\hat{\alpha}_2, \hat{\beta}_2]$. 

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The remaining 15 admissible finite index subgroups are similarly treated. All instances provide a matrix representation $M$ which satisfies the Hopf condition and which leads to the normal form (2.49) after reduction using the symplectic group $Sp(4, \mathbb{Z})$. However, the map from the set of admissible finite index subgroups to the set of normal forms (2.49) obeying the Hopf condition is not unique, and there is a large degree of arbitrariness in this procedure. The reason is that the partial reduction leading to (2.49) involves only $Sp(4, \mathbb{Z})$ transformations, but not modular transformations of the base. It may happen that an admissible finite index subgroup $H_a$ is invariant under a $SL(2, \mathbb{Z})$ transformation of the base (e.g., $\alpha \leftrightarrow \beta$), in which case one may get matrices $M$ leading to period matrices which are not related by a modular transformation on the cover $\hat{\Sigma}$. Thus it is only onto the set of fully reduced Poincaré normal forms of $M$, which incorporates a sum over all such $SL(2, \mathbb{Z})$ transformations of the base $T$, that this reduction map is unique. However, the reduced moduli space for the Poincaré normal form is very complicated and depends sensitively on number theoretic properties of the degree $N$ [12].

4.3 Equivariance of the DVV Correlator

The construction of Section 4.2 above determines the dependence of the $2 \times 2$ period matrix $\tau_H$ on a given admissible monodromy homomorphism, or equivalently a given admissible finite index subgroup $H < \Gamma$, with $\tau_H = \tau_{r,m,s,t}$ in (2.51). At this stage we are faced with the problem of finding the dependence (either explicit or implicit) of the Prym modulus $\Pi = r \tau^#$ on the branch point location $z \in T$. The construction of Prym differentials in Section 3.3 does not carry through to the higher degree branched covers, because for any genus two cover $f: \hat{\Sigma} \to T$ of degree $N \geq 3$ there are no non-trivial automorphisms $\iota: \hat{\Sigma} \to \hat{\Sigma}$ such that $f \circ \iota = f$ [43]. As any genus two Riemann surface is a hyperelliptic curve, the cover $\hat{\Sigma}$ does have a canonical hyperelliptic involution $\iota_{\hat{\Sigma}}$ and its hyperelliptic divisor which is the effective divisor of degree six consisting of the fixed points of $\iota_{\hat{\Sigma}}$. Then there is a unique involution $\iota_T: T \to T$ of the base such that $f \circ \iota_{\hat{\Sigma}} = \iota_T \circ f$ [43]. However, given that the above construction is not invariant under $SL(2, \mathbb{Z})$ transformations of the base, it is not clear how to exploit the hyperelliptic representation of $\hat{\Sigma}$, and the corresponding Schottky relations, to determine the branch point dependence as before. This is further reflected in the fact that the standard constructions of cut abelian differentials (such as (3.42)) for cyclic orbifolds [16] become ambiguous for nonabelian monodromy. We are not aware of any constructions of Prym differentials or Prym moduli for higher degree genus two covers $f: \hat{\Sigma} \to T$ in terms of branch point loci.

On general grounds it follows that the complex structure on the covering surface $\hat{\Sigma}$ is uniquely determined by the holomorphic map $f: \hat{\Sigma} \to T$ in terms of the moduli $\tau^*$ and $z$, but not necessarily in an explicit parametrization. We can use results of [44] to ascertain that the desired explicit branch point dependence does exist and can be used to give some insight into the modular behaviour of the DVV correlator. One of the advantages of the formalism of Section 4.2 over that of Section 4.1 above is that one can study equivariance properties in the genus two modular group $Sp(4, \mathbb{Z})$, rather than in the more complicated mapping class group $\mathcal{M}(2,1)$. For fixed monodromy given by an admissible finite index subgroup $H < \Gamma$, there is a holomorphic map

$$
\tau_H : T(0,1) \times U \longrightarrow U^2 , \quad (\tau^*, z) \longmapsto \tau_H(\tau^*, z) \quad (4.16)
$$

which is determined generically in [44] via a sewing construction on twice-punctured tori in
terms of Jacobi-Erdélyi theta functions, Weierstrass functions and Eisenstein series on the base torus $T$. The primary difference in our specific case is that the modulus $\tau^\#$ has a square root cut singularity at each of the branch points $w_1 = z$ and $w_2 = 0$, rather than the logarithmic cut singularity which arises in [44].

Consider the monomorphism $SL(2, \mathbb{Z}) \hookrightarrow Sp(4, \mathbb{Z})$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (4.17)$$

This lift of $SL(2, \mathbb{Z})$ acts in the expected way on the domain of the map (4.16) as

$$\begin{pmatrix} \tau^\bullet, z \end{pmatrix} \mapsto \begin{pmatrix} \frac{a \tau^\bullet + b}{c \tau^\bullet + d} \\ \frac{c \tau^\bullet + d}{c \tau^\bullet + d} \end{pmatrix}. \quad (4.18)$$

For each choice of branch for $\tau^\#$, the map $\tau_H$ is equivariant with respect to this action of $SL(2, \mathbb{Z}) < Sp(4, \mathbb{Z})$ [44] and there is a commutative diagram

$$\begin{array}{ccc}
T(0, 1) \times \mathbb{U} & \xrightarrow{\tau_H} & \mathbb{U}^2 \\
\downarrow & & \downarrow \\
SL(2, \mathbb{Z}) & \xrightarrow{\tau_H} & SL(2, \mathbb{Z})
\end{array} \quad (4.19)$$

This property determines the equivariance of the DVV correlator (2.42), represented by the genus two DLCQ free energy (2.48) at a fixed value of the degree $N$. Since under (4.18) the flat area form on the torus transforms as $d^2z \mapsto d^2z / |c \tau^\bullet + d|^2$, and since the local twist field correlation functions $\langle \sigma_{a_1 b_1}(z) \sigma_{a_2 b_2}(0) \rangle_{S_N}^{S_N}$ have total scaling dimension 6, the scaling properties of the measure $\mu(z)$ under (4.18) can be explicitly determined.

Given the remarkable agreement of the $N = 2$ free energy with the twist field two-point function in the $Z_2$ orbifold, it is natural to extrapolate this correspondence and to take the fixed $N$ DLCQ free energy integrand in (2.48) as the definition of the local twist field correlation function $\langle \sigma_{a_1 b_1}(z) \sigma_{a_2 b_2}(0) \rangle_{S_N}^{S_N}$ on the $\mathbb{R}^{24} \wr S_N$ permutation orbifold, according to the covering surface principle of Section 2.4. However, the explicit form of the mapping (4.16) displayed in [44, Proposition 6.2] is far too complicated for an explicit determination of the required Jacobian $|d\tau^\#/dz|^2$ (and furthermore one needs an $Sp(4, \mathbb{Z})$ transformation relating their period matrix to ours). Moreover, it is difficult to arrive at explicit formulas which are illuminating, as the products (2.44) of theta functions (2.54) are rather involved for $N \geq 3$.

### 5 Fermionic Orbifolds

In this final section we will study fermionic extensions of the permutation orbifolds considered thus far, in particular those orbifold sigma models arising in discrete light-cone quantization of superstrings and heterotic strings in ten spacetime dimensions. We will describe the modifications of the covering surface principle and twist field operators of Section 2 required in these cases. The genus two DLCQ free energy amplitudes in these instances are derived in [12]. Given the success of the bosonic $Z_2$ orbifold model of Section 3, we will use the appropriately modified
Consider the superconformal sigma model on the torus with target space \( \mathbb{R}^8 \) defined by the action
\[
I(X, \psi) = \frac{1}{4\pi \alpha'} \int \frac{d^2 z}{2i\tau_2} \left( \partial X_i(z) \bar{\partial} X_i(z) + \psi_i(z) \bar{\partial} \psi_i(z) + \bar{\psi}_i(z) \partial \bar{\psi}_i(z) \right),
\]
where the real bosonic fields \( X_i, i = 1, \ldots, 8 \) transform in the eight-dimensional vector representation \( 8_e \) of the R-symmetry group \( SO(8) \), while the components \( \psi_i, \bar{\psi}_i, i = 1, \ldots, 8 \) of the 16-component Majorana-Weyl spinor field \( \psi \) transform in the spinor \( 8_e \) and conjugate spinor \( 8_e \) representations of \( SO(8) \), respectively. The spinor fields are sections of the twisted spin line bundle \( S_T \otimes L_\delta \) over the torus, where \( L_\delta \) is a real line bundle over \( T \) with flat connection determined by one of the four spin structures \( \delta = (\delta_\alpha) \in H^1(T, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \) and \( [S_T] \in \text{Pic}^0(T) \) is chosen to correspond to the theta divisor in the given homology basis \((\alpha, \beta)\). The \( \mathcal{N} = 8 \) worldsheet supersymmetry of the sigma-model is generated by the fermionic supercurrents
\[
G^i(z) = -\frac{i}{2} \gamma^i \left( \psi' \left( \partial X_i(z) + \bar{\psi}' \left( \bar{\partial} X_i(z) \right) \right) \right)
\]
where \( \gamma^i \) are the \( Spin(8) \) Dirac matrices.

In the corresponding permutation orbifold, the monodromy conditions on the bosonic fields \( X \) in a given twisted sector \((P, Q)\) are as in (2.1), while the fermion monodromy is given by
\[
\psi^a(z + 1) = (-1)^{\delta_\alpha} \psi^{P(a)}(z) \quad \text{and} \quad \psi^a(z + \tau) = (-1)^{\delta_\beta} \psi^{Q(a)}(z),
\]
where for simplicity we have omitted a potential extra sign depending on the reference spin structure \([S_T]\). This symmetry is compatible with \( \mathcal{N} = 8 \) worldsheet superconformal invariance [45], and it means that on the fermionic fields the twist group \( G \) is extended to \( G \times (\mathbb{Z}/2\mathbb{Z})^N \). The consistency condition \( PQ = QP \) implies [27] that the spin structure phases in (5.3) are independent of the coordinate label \( a \) in the permutation orbifold, and hence that only the diagonal subgroup of \((\mathbb{Z}/2\mathbb{Z})^N \) acts nontrivially on the fermions. The asymmetry between the twistorings of bosons and fermions implies that the modular invariant sum over monodromy homomorphisms breaks spacetime supersymmetry of the orbifold sigma model.

Generally, the sum over \((\mathbb{Z}/2\mathbb{Z})^N \) monodromy in the fermionic sector is weighted by a consistent set of GSO phases \( \zeta[\delta; \Phi] \), generically dependent upon the twisted sector \( \Phi : \pi_1(\Sigma) \to G \), which are constrained by modular covariance requirements. In the untwisted sector \( \Phi(-) = e \), the phase corresponding to a spin structure \( \delta = (\delta_\alpha) \in \mathbb{Z}/2\mathbb{Z} \) is the mod 2 index of the Dirac operator on \( \Sigma \) twisted by the flat line bundle \( L_\delta \to \Sigma \) given by [46]
\[
\zeta[\delta; e] = (-1)^{\dim H^0(\Sigma, S_8 \otimes L_\delta)} = (-1)^{\delta_\alpha \delta_\beta},
\]
where \( \dim H^0(\Sigma, S_\Sigma \otimes L_\delta) \) is the number of linearly independent holomorphic sections of the spin bundle \( S_\Sigma \otimes L_\delta \). Schematically then, the modification of the formula (2.3) for the partition function of the supersymmetric permutation orbifold is given by

\[
Z^{G \times (\mathbb{Z}_2)^N}(\tau) = \frac{1}{2^{N|G|}} \sum_{\Phi : \pi_1(\Sigma) \rightarrow G} \sum_{\delta \in H^1(\Sigma, \mathbb{Z}/2\mathbb{Z})} \zeta[\delta; \Phi] \left( \prod_{\xi \in \mathcal{O}(\Phi)} Z_\delta(\tau^\xi) \right) \tag{5.5}
\]

where \( Z_\delta(\tau^\xi) \) is the partition function of the parent superconformal field theory computed with the global fermionic monodromy determined by the spin structure \( \delta \).

For example, the partition function of the supersymmetric \( \mathbb{R}^8 \times (S_N \times (\mathbb{Z}_2)^N) \) permutation orbifold on \( \Sigma = \mathbb{T} \) can be determined by first calculating the contribution from a given spin structure (say the Ramond-Ramond sector) to the path integral over the complex fermionic fields, and then summing over the modular orbits using either of the two GSO projections of Type II string theory. Then the parent partition function appearing in the formula (2.3) is given by [27]

\[
Z(\tau) = \frac{1}{2} \left| \hat{3}(0) (\tau)^4 - \hat{3}(1) (\tau)^4 - \hat{3}(1) (\tau)^4 \pm \hat{3}(1) (\tau)^4 \right|^2, \tag{5.6}
\]

where the +/- sign corresponds to the Type II A/B string amplitude and

\[
\hat{3}(\delta)(\tau) = \left( \frac{4\pi^2 \alpha'}{\tau_2} \right)^4 e^{\frac{\pi}{4\tau_2} (2\delta_0 - 1) \tau} e^{\pi i \delta_0 \delta_\beta/4} \times \prod_{n=1}^\infty \left( 1 - (-1)^{\delta_\beta} e^{\pi i \tau (2n-1+\delta_0)} \right) \left( 1 - (-1)^{\delta_\beta} e^{\pi i \tau (2n-1-\delta_0)} \right). \tag{5.7}
\]

The corresponding grand canonical partition function (2.12) matches the Type II DLCQ free energy at finite temperature, with (5.6) producing the action of the (restricted) Hecke operator on the partition function of the first quantized Green-Schwarz superstring [6, 11, 12]. A completely analogous correspondence holds for the thermal partition function of Type IIB DLCQ superstrings on the maximally supersymmetric plane wave background in ten dimensions [47].

The operators which create local monodromy in the superconformal sigma model with respect to the action of \( S_N \) are products \( \sigma_P(z) S_P(z) \) of bosonic and fermionic twist fields. Let us work in the sector of trivial global \( \mathbb{Z}_2 \) monodromy for the spinor fields, i.e., with the Ramond-Ramond spin structure \( \delta = (0) \). The other sectors are treated similarly as in [7]. In a \( \mathbb{Z}_n \)-twisted sector corresponding to a cyclic permutation \( P = (n) \), the vacuum state then carries an irreducible representation of the Clifford algebra for \( Spin(8) \). By using an \( SO(8) \) triality isomorphism, the representation space can be taken to be the direct sum \( 8_+ \oplus 8_- \). The corresponding components of the 16-dimensional ground state vector are created respectively by the primary spin fields \( S^{(n)}_i(z) \) and \( \tilde{S}^{(n)}_i(z) \), \( i = 1, \ldots, 8 \). They each have conformal dimension [19]

\[
\Delta^{RR}_{(n)} = \frac{n}{6} + \frac{1}{3n}. \tag{5.8}
\]

To describe the supersymmetric version of the DVV interaction vertex [4], we need another kind of spin twist field to ensure that the operators generating the basic joining and splitting of superstrings yield an irrelevant deformation of the superconformal sigma model. The bosonic twist field \( \sigma_{ab}(z) \) transposing the fields \( X^a \) and \( X^b \) has conformal dimension \( \frac{1}{2} \) when \( d = 8 \) (see (2.32)), as does the fermionic twist field \( S_{ab}(z) \) interchanging \( \psi^a \) and \( \psi^b \). To increase the
scaling dimension by $\frac{1}{2}$ in a supersymmetric fashion, we use the supersymmetric descendent of the primary twist field operators $\sigma(z) \tilde{S}(z)$ given by

$$[Q^\ell, \sigma(z) \tilde{S}^{\ell'}(z)] + [\sigma(z) \tilde{S}^{\ell}(z), Q^{\ell'}] = g^\ell(z) S^{\ell'}(z) \delta^{\ell\ell'} =: \Lambda(z) \delta^{\ell\ell'}$$

(5.9)

where

$$Q^\ell = \oint \frac{dz}{2\pi i} G^\ell(z)$$

(5.10)

are the $\mathcal{N} = 8$ supercharges and the contour integral is taken around the origin $z = 0$. The descendent bosonic twist fields $g^\ell_P(z)$ create the first excited states in the twisted sector $[P]$. Since the combination $\psi^a - \psi^b$ has Ramond boundary conditions under transposition in $S_N$, the corresponding spin field carries a representation of the Clifford algebra. The twist field $S^\ell_{ab}(z)$ transforms as a vector of $SO(8)$, and it coincides with the standard spin field of the supersymmetric $\mathbb{R}^8 \wr \mathbb{Z}_2$ permutation orbifold which can be constructed explicitly via bosonization of the fermion fields $\psi_i$ [48, 49].

The fermionic DVV vertex operator is now defined by

$$V_{\text{ferm}} = -\lambda N \frac{\text{vol}(\mathcal{T})}{\text{vol}(\mathcal{T})} \int_{\mathcal{T}} d\mu(z) \sum_{1 \leq a < b \leq N} \Lambda_{ab}(z).$$

(5.11)

The descendent twist field $\Lambda_{ab}(z)$ is a primary field of conformal weight $\frac{3}{2}$. The interaction vertex (5.11) is spacetime supersymmetric, $SO(8)$ invariant and describes elementary string interactions [4].

The computation of the local twist field correlations functions $\langle \Lambda_{a_1 b_1}(z) \Lambda_{a_2 b_2}(0) \rangle_{S_N \times (\mathbb{Z}_2)^N}$ requires a modification of the covering surface principle of Section 2.4. This is because one should no longer simply close the punctures on the covering space $\hat{\Sigma}$ corresponding to the branch points to get the identity state at those points. Rather, one must insert the operator that creates a Ramond vacuum at the insertion points in order to give the fermions the correct local monodromy. Thus in the supersymmetric orbifold theory one uses the same covering spaces $\hat{\Sigma}$ as in the case of the bosonic orbifold, but instead of computing the partition function on $\hat{\Sigma}$ one computes a correlation function of spin fields on $\hat{\Sigma}$. Schematically, the modification of a generic, normalized bosonic twist field correlation function (2.36) is given by

$$\left\langle \prod_{i=1}^k \Lambda_{P_i}(w_i) \right\rangle_{G \times (\mathbb{Z}_2)^N}^{G \times (\mathbb{Z}_2)^N} = \frac{1}{2^N|G|} \sum_{\Phi: \pi_1(\Sigma) \rightarrow G \times (\mathbb{Z}_2)^N} \frac{1}{Z^{G \times (\mathbb{Z}_2)^N}(\tau)} \prod_{\xi \in O(\Phi)} \left\langle \prod_{i=1}^k \hat{S}_{P_i}(\hat{w}_i) \right\rangle_{(\tau, \xi, w)},$$

(5.12)

where the global $(\mathbb{Z}_2)^N$ monodromy acts trivially in the bosonic sector and diagonally in the fermionic sector as in (5.5). A similar prescription for $\mathcal{N} = 4$ supersymmetric orbifold sigma models is used in [50]. When $\Sigma = \mathbb{T}$, this will be provided by the corresponding DLCQ free energy through the required modification of the GSO projection at finite temperature which breaks supersymmetry by making spacetime fermions antiperiodic around the thermal cycle [12].

\footnote{A similar statement is also true in the NS–NS sector. In the mixed R–NS and NS–R sectors, there are no combinations of $\psi^a$ which possess zero modes, so that these sectors have trivial local spin monodromy and the prescription instead follows that of Section 2.4.}
Similar considerations also apply to the heterotic sigma model on the torus with target space \( \mathbb{R}^8 \), which is defined by the action

\[
I(X, \psi, \chi) = \frac{1}{4\pi \alpha'} \int_T d^2z \frac{1}{2i \tau_2} \left( \partial X_i(z) \overline{\partial X_i(z)} + \psi_i(z) \overline{\partial \psi_i(z)} + \chi_A(z) \partial \chi_A(z) \right)
\]

(5.13)

where the Majorana-Weyl fermion fields \( \chi_A, A = 1, \ldots, 32 \) are \( SO(8) \) singlets. The holomorphic sector of this worldsheet field theory coincides with that of the supersymmetric sigma model (5.1), while after bosonization of \( \chi_A \) the antiholomorphic sector coincides with the bosonic sigma model (2.16) in \( d = 24 \) with 16 of the bosons compactified on the Cartan torus of the heterotic gauge group \( G = SO(16) \times SO(16) \). The heterotic sigma model (5.13) is a superconformal field theory with chiral \((8,0)\) worldsheet supersymmetry.

The corresponding \((8,0)\) supersymmetric permutation orbifold \([36, 51]\) is \( (\mathbb{R}^8 \times G) \wr (G \times (\mathbb{Z}_2)^N) \). The twist subgroup \((\mathbb{Z}_2)^N\) acts on the holomorphic sector exactly as in the supersymmetric case. In the antiholomorphic sector, the gauge fermions \( \chi_A \) are sections of flat real line bundles \( L_\delta \to \mathbb{T} \) like \( \psi_i \), and so have global fermionic monodromy conditions as in (5.3). In contrast to the fields \( \psi_i \), however, the spin structure phases for \( \chi_A \) in the permutation orbifold generally depend on the coordinate label \( a \). Perturbative string interactions are now generated by the heterotic version of the DVV vertex operator \([36, 51]\). For this, we must explicitly write worldsheet fields as products of holomorphic and antiholomorphic fields (which was implicitly understood in all previous formulae). As the holomorphic sector consists of the usual supersymmetric orbifold theory in eight dimensions, the holomorphic part of the vertex is constructed using the dimension \( 2 \) spin twist operators \( \Lambda_{ab}(z) \) defined in (5.9). On the other hand, the antiholomorphic sector consists of \( d = 24 \) bosons, and since the local monodromies about branch points are insensitive to the compactness of the 16 bosons on the Cartan torus, the antiholomorphic part of the vertex is built from the dimension \( 2 \) bosonic twist fields \( \overline{\sigma}_{ab}(z) \) of Section 2.4.

It follows that the heterotic DVV vertex operator is defined by

\[
V_{\text{het}} = -\frac{\lambda N}{\text{vol}(\mathbb{T})} \int_T d\mu(z) \sum_{1 \leq a < b \leq N} \left( \Lambda \otimes \overline{\sigma} \right)_{ab}(z).
\]

(5.14)

The computation of local twist field two-point functions proceeds by using a formula analogous to (5.12).

### 5.2 Supersymmetric DLCQ Strings

The genus two DLCQ free energy \( F^{(2)}_{\text{form}}(\tau^*, \kappa) \) for Type IIA superstrings at finite temperature is computed in [12]. To write the result, we require some preliminary definitions. The ten even reduced, genus two integer characteristics \( \left( \begin{array}{c} a \\ b \end{array} \right) = \left( \begin{array}{c} a_1 \\ a_2 \\ b_1 \\ b_2 \end{array} \right) \in H^1(\Sigma, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}^4/2\mathbb{Z}^4 \) obey \( a \cdot b \equiv 0 \mod 2 \) and are denoted by

\[
\delta_1 = \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right), \quad \delta_2 = \left( \begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \end{array} \right), \quad \delta_3 = \left( \begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \end{array} \right), \quad \delta_4 = \left( \begin{array}{c} 0 \\ 1 \\ 1 \\ 0 \end{array} \right), \quad \delta_5 = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array} \right), \quad \delta_6 = \left( \begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \end{array} \right), \quad \delta_7 = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right), \quad \delta_8 = \left( \begin{array}{c} 1 \\ 0 \\ 1 \\ 1 \end{array} \right), \quad \delta_9 = \left( \begin{array}{c} 1 \\ 1 \\ 0 \\ 1 \end{array} \right) \quad \text{and} \quad \delta_{10} = \left( \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array} \right).
\]

(5.15)

We use the shorthand notation \( \delta_i := \Theta(\delta_i)(\tau)^4 \), where the genus two period matrix \( \tau = \tau_{r,m,s,t}(\tau^*, \tau^F) \) is given by (2.51). On the last four characteristics in (5.15) we define genus
two functions \( \Xi_6(\delta_i)(\tau) \) of modular weight six by the formulae

\[
\begin{align*}
\Xi_6(\delta_7) &= \vartheta_2 \vartheta_3 \vartheta_5 + \vartheta_8 \vartheta_9 \vartheta_0 - \vartheta_1 \vartheta_4 \vartheta_6 , \\
\Xi_6(\delta_8) &= \vartheta_7 \vartheta_9 \vartheta_0 - \vartheta_1 \vartheta_4 \vartheta_5 + \vartheta_2 \vartheta_3 \vartheta_6 , \\
\Xi_6(\delta_9) &= \vartheta_7 \vartheta_8 \vartheta_0 - \vartheta_1 \vartheta_2 \vartheta_5 + \vartheta_3 \vartheta_4 \vartheta_6 , \\
\Xi_6(\delta_0) &= \vartheta_7 \vartheta_8 \vartheta_9 + \vartheta_3 \vartheta_4 \vartheta_6 - \vartheta_1 \vartheta_2 \vartheta_5 .
\end{align*}
\] (5.16)

Then one has

\[
F_{\text{ferm}}^{(2)}(\tau^\ast, \kappa) = -\frac{g_2^2}{4} \left[ \frac{\tau^\ast}{64\pi^2 \alpha'} \right]^4 \sum_{N=2}^{\infty} \frac{k_N}{N} \sum_{\substack{m,N \text{ odd} \ 1 \neq 0 \ s,t \in \mathbb{Z}/r \mathbb{Z}}} \int_{\Delta} \left| \frac{d^2\tau^\#}{(\tau^\#_2)^4} \right| |\Psi_{10}(\tau)|^{-2}
\]

\[
\times \left| \Xi_6(\delta_7)(\tau) \Theta(\delta_7)(\tau)^4 + \Xi_6(\delta_8)(\tau) \Theta(\delta_8)(\tau)^4
\]

\[
+ \Xi_6(\delta_9)(\tau) \Theta(\delta_9)(\tau)^4 + \Xi_6(\delta_0)(\tau) \Theta(\delta_0)(\tau)^4 \right|^2 .
\] (5.17)

Note that the fermionic contribution to (5.17) consists of a sum of four terms in the Weierstrass-Poincaré reduction. We may identify these terms as resulting from the modular invariant sum over genus one spin structures, as in (5.6). The free energy (5.17) should now be equated to the translationally invariant correlator \( \langle V_{\text{form}} V_{\text{form}} V^\ast \rangle^{S_N \times (\mathbb{Z}/2)^N} \). As in the bosonic case, one is then faced with the problem of equating the two continuous parametrizations of the partially discretized genus two moduli space, one in terms of the elliptic Prym modulus \( \Pi = r \tau^\# \) and the other in terms of the branch point location \( z \in \mathbb{T} \). This can again be done explicitly for the degree two contribution to (5.17), corresponding to double covers of the torus \( \mathbb{T} \), and used to compute local spin twist field correlation functions explicitly in each twisted sector of the \( \mathbb{R}^8 \) \( (\mathbb{Z}/2)^3 \) permutation orbifold.

The \( N = 2 \) contribution to (5.17) is given by

\[
F_{\text{ferm}}^{(2)}(\tau^\ast) = -\frac{g_2^2}{4} \left[ \frac{\tau^\ast}{64\pi^2 \alpha'} \right]^4 \int_{\Delta} \left| \frac{d^2\tau^\#}{(\tau^\#_2)^4} \right| |\mathcal{C}(\tau_0(\tau^\ast, \tau^\#_2))/\Psi_{10}(\tau_0(\tau^\ast, \tau^\#_2))|^2
\] (5.18)

where

\[
C = \Xi_6(\delta_7) \vartheta_7 + \Xi_6(\delta_8) \vartheta_8 + \Xi_6(\delta_9) \vartheta_9 + \Xi_6(\delta_0) \vartheta_0 .
\] (5.19)

We will begin by simplifying the elliptic function (5.19) using the decomposition (2.54) for \( N = 2 \). We use the notation of Section 3.2 throughout. To simplify the formulae somewhat, we momentarily omit the overall factor of \( 1/2\sqrt{-1r^\#} \) in (3.11) and reinstate it at the end of the calculation. For reference, let us tabulate the ten reduced even genus two theta constants according to the spin structures (5.15) as

\[
\begin{array}{cccccccc}
\delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 \\
\theta_3^\# \theta_4^\# + \theta_4^\# \theta_3^\# & \theta_3^\# \theta_4^\# - \theta_4^\# \theta_3^\# & \theta_3^\# \theta_4^\# + \theta_5^\# \theta_1^\# & \theta_3^\# \theta_4^\# - \theta_5^\# \theta_1^\# & 2 \theta_3^\# \theta_3^\# \\
\delta_6 & \delta_7 & \delta_8 & \delta_9 & \delta_0 \\
\theta_3^\# \theta_3^\# & \theta_4^\# \theta_4^\# & \theta_2^\# \theta_2^\ast & 2 \theta_1^\# \theta_1^\# & -2i \theta_1^\# \theta_1^\ast \\
2 \theta_3^\# \theta_3^\# & \theta_2^\# \theta_2^\# & \theta_2^\# \theta_2^\ast & 2 \theta_1^\# \theta_1^\# & -2i \theta_1^\# \theta_1^\ast
\end{array}
\] .

(5.20)

Since one has the equalities \( \vartheta_3 = \vartheta_6, \vartheta_7 = \vartheta_8 \) and \( \vartheta_9 = \vartheta_0 \) for the given reduction, we immediately find that \( \Xi_6(\delta_7) = \Xi_6(\delta_8) \) and \( \Xi_6(\delta_9) = \Xi_6(\delta_0) \).
After some elementary manipulations we can bring (5.19) into the form

\[
C = 2^8 \theta_2^4 \theta_2^4 \theta_3^4 \theta_4^4 \theta_3^4 \theta_3^2 (\theta_3^2 - \theta_4^2) \left( \theta_3^2 - \theta_4^2 \right) \theta_3^4 \theta_3^4 \theta_3^4 \\
\times \left[ \theta_3^2 \theta_4^2 \left( \theta_3^2 - \theta_4^2 \right)^2 + \theta_3^2 \theta_4^2 \left( \theta_3^2 - \theta_4^2 \right)^2 \right] \\
+ 2^{10} \theta_2^8 \theta_2^8 \theta_1^8 \theta_1^8 + 2^{11} \theta_2^4 \theta_4^2 \theta_2^4 \theta_4^2 \theta_3^4 \theta_3^4 - 2^9 \theta_2^4 \theta_3^4 \theta_1^4 \theta_1^4 \theta_2^4 \theta_2^4 \left( \theta_3^4 - \theta_4^4 \right) \left( \theta_3^4 - \theta_4^4 \right). 
\]

(5.21)

We can now proceed as in Section 3.2 by doubling the modulus of the theta functions. In addition to the identities displayed in (3.20), we will also require the doubling identities

\[
\theta_1(z|\tau) \theta_2(z|\tau) = \theta_1(2z|2\tau) \theta_4(0|2\tau), \\
\theta_3(z|\tau) \theta_4(z|\tau) = \theta_4(2z|2\tau) \theta_4(0|2\tau)
\]

(5.22)

with \( z = \frac{1}{4} \). We may then take into account that the theta functions with argument \( z = \frac{1}{4} \) satisfy \( \theta_3^* = \theta_4^* \) and \( \theta_1^* = -\theta_2^* \), and analogously for \( \theta_3^\# \). The calculation is neither difficult nor illuminating, and the result is

\[
C = \frac{2}{(\tau^\#)^8} \theta_2^8 \theta_3^4 \theta_4^4 \theta_3^4 \theta_3^4 \theta_4^4 \theta_4^4,
\]

(5.23)

where we have inserted back the factor \( (1/2 \sqrt{-i \tau^\#})^{16} \) and the bar stands for doubled modulus as in Section 3.2.

Let us now perform a modular \( S \) transformation (3.23) on the modulus of both types of theta functions in (5.23). Then the final result for the numerator of the integrand in (5.35) reads

\[
\left| C(\tau_0|\tau^\#) \right|^2 = 2^{34} \left| (\tau^\#)^{16} \eta(2\tau^\#) \eta(\tau^\#) \theta_4(2\tau^\#) \theta_4(\tau^\#) \right|^4.
\]

(5.24)

Substituting (5.24) along with (3.24) into (5.35), and using the abstruse identity (3.21), we arrive at the final form for the supersymmetric two-loop DLCQ free energy given by

\[
\mathcal{F}_2^{\text{form}}(\tau^\#) = -\frac{16 g_s^2}{(\pi^2 \alpha')^4} \left| \theta_4(\tau^\#) \right|^8 \int \frac{d^2 \tau^\#}{(\tau_2^\#)^4} \left| \frac{\theta_4(2\tau^\#)}{\theta_3(\tau^\#) \theta_4(2\tau^\#)} \right|^4.
\]

(5.25)

Analogously to the bosonic case of Section 3.5, this integral should be matched to the worldsheet averaged two-point correlation function of spin twist field operators \( \Lambda(z) = \Lambda_{12}(z) \) in the \( \mathbb{R}^8 \times (\mathbb{Z}_2)^3 \) permutation orbifold given by

\[
\mathcal{F}_2^{\text{form}}(\tau^\#) = \frac{4 \lambda^2}{\tau_2^8 \mu(0)} \int \frac{d\mu(z)}{\pi} \left( \Lambda(z) \Lambda(0) \right)^{(\mathbb{Z}_2)^3}.
\]

(5.26)

We recall that, by modular invariance at genus two, the branch point integration in (5.26) projects all contributions to the correlation function onto the trivial twist sector \( (\varepsilon, \delta) = (0, 0) \), so that the local integrand that we can read off from (5.26) is \( 4 - \frac{1}{2\tau} \left( \Lambda(z) \Lambda(0) \right)^{(\mathbb{Z}_2)^3} \). We substitute (3.69) with \( d = 8 \) and (3.70), and recall that the Prym modulus is given by \( \Pi = \Pi_{0,0} = 2\tau^\# \). The crucial observation is that the bosonic contribution to (5.25) involving the difference of theta functions is identical to that of the purely bosonic case (3.25), due to the universal dimension independent contribution of the modular form \( \Psi_{10}(\tau) \) to the bosonic genus two partition function.
(2.43). We can therefore use the same calculation of the Jacobian $|d\tau^#/dz|^2$ that was carried out in Section 3.5, wherein it was shown that

$$\left|\frac{d\tau^#}{dz}\right|^2 = \left|\theta_3(\tau^\bullet)\theta_4(2\tau^#)^4 - \theta_4(\tau^\bullet)\theta_3(2\tau^#)^4\right|^{-4}$$

$$= \frac{1}{2^{24}} \left|\eta(\Pi)^4\theta_1(z | \tau^\bullet)\theta(\eta)\right|^6 \left|\frac{\theta(q)^{(\xi)}(\tau^\bullet)}{\theta(q)^{(0)}(0 | \Pi)}\right|^{24}$$

for an arbitrary fixed characteristic $(a, b) \neq (1, 1)$. Using the identity (3.54) and recalling the definition of the prime form (3.52), after a little algebra we can use (5.25)–(5.27) to compute

$$\langle \Lambda(z) \Lambda(0) \rangle_{0,0}^{(Z_2)^3} = \hat{\lambda}(\tau^\bullet)^8 |E(z)|^{-6} 64\tau^\bullet \eta(\Pi)^3 \theta_4(\Pi)^6 \left|\frac{\theta(q)^{(\xi)}(\tau^\bullet)^3 \theta(q)^{(0)}(0 | \tau^\bullet)^3}{\theta(q)^{(0)}(0 | \Pi)}\right|^6$$

where

$$\hat{\lambda}^\bullet = \sqrt{4\pi^2 \alpha'} \frac{1}{\eta(\tau)} \left|\frac{\theta_4(\tau)}{\eta(\tau)}\right|$$

is the one-loop, first quantized partition function of the Green-Schwarz superstring in $\mathbb{R}$ evaluated with the genus one spin structure $(\tilde{\Pi})$.

We can generate from (5.28) the contribution of a generic twisted sector $(\varepsilon, \delta) \in (\mathbb{Z}/2\mathbb{Z})^2$ to the spin twist field correlation function by using a crossing transformation $z \mapsto z + \delta + \varepsilon \tau^\bullet$ and the corresponding twisted Prym modulus (3.46), along with the transformation formula for Jacobi elliptic functions given by

$$\theta(q)^{(\xi)}(z + \delta + \varepsilon \tau^\bullet | \tau^\bullet) = \exp\left(-\frac{\pi i}{2} \varepsilon^2 \tau^\bullet - \pi i \varepsilon z - \frac{\pi}{2} (b + \varepsilon) \varepsilon\right) \theta(q)^{(\xi + \delta)}(z | \tau^\bullet)$$

which is valid for arbitrary $a, b \in \mathbb{Q}$ and $\varepsilon, \delta \in \mathbb{Q}$. In fact, the $z$-dependence of the correlation function (5.28) is identical to that of Section 3.4 (up to an overall power), and hence the twisted sector two-point function is an appropriate supersymmetric completion of the bosonic correlation function (3.50) (with $R = \infty$ and $d = 8$). The final result is

$$\langle \Lambda(z) \Lambda(0) \rangle_{\varepsilon,\delta}^{(Z_2)^3} = \hat{\lambda}(\tau^\bullet)^8 |\hat{\lambda}(\xi)^3|^{-16}$$

where

$$\hat{\lambda}(\xi)^3 = \frac{\theta(q)^{(\xi)}( \Pi_{\varepsilon,\delta})^3 \theta_4(\Pi_{\varepsilon,\delta})}{8 \sqrt{\tau^\bullet \eta(\Pi_{\varepsilon,\delta})^3 \theta_4(\Pi_{\varepsilon,\delta})}}$$

and the twisted bosonic determinant $\hat{\lambda}(\xi)^3$ is given by (3.51). The cubic power in the supersymmetric twisted determinant (5.32) reflects the fact that the effective twist group of the supersymmetric permutation orbifold is $(Z_2)^3$.

### 5.3 Heterotic DLCQ Strings

Finally, we come to the thermodynamic, genus two DLCQ free energy $F_{\text{het}}^{(2)}(\tau^\bullet, \kappa)$ for heterotic strings with heterotic gauge group $\mathcal{G} = \text{Spin}(32)/\mathbb{Z}_2$ or $\mathcal{G} = E_8 \times E_8$. The holomorphic sector consists of the usual chiral superstring contribution at genus two. In the antiholomorphic sector, the non-compact bosons produce the usual antichiral bosonic contribution, while the compactified bosonic fields produce an instanton sum over the root lattice of $\mathcal{G}$. The latter contribution
yields a theta function of the root lattice which is the unique genus two modular form of weight eight given by

$$\Psi_8(\tau) = \sum_{i=0}^{9} \Theta(\delta_i)(\tau)^{16}. \quad (5.33)$$

In the notation of Section 5.2 above, one then has [12]

$$F_{\text{het}}^{(2)}(\tau^*, \kappa) = \frac{g_2^2}{8} \left| \frac{\tau^*}{2048\pi^4 (\alpha')^2} \right|^4 \sum_{N=2}^{\infty} \frac{\kappa N}{N} \sum_{m=1}^{N \text{ even}} \frac{1}{m^4} \sum_{s,t \in \mathbb{Z}/r \mathbb{Z}, t \neq 0} \int_{\Delta} d^2 \tau^\# \frac{\Psi_8(\tau)}{|\Psi_{10}(\tau)|^2}$$

$$\times \left( \Xi_6(\delta_7)(\tau) \Theta(\delta_7)(\tau)^4 + \Xi_6(\delta_8)(\tau) \Theta(\delta_8)(\tau)^4 \right.$$

$$+ \Xi_6(\delta_9)(\tau) \Theta(\delta_9)(\tau)^4 + \Xi_6(\delta_9)(\tau) \Theta(\delta_9)(\tau)^4 \right). \quad (5.34)$$

Again we deal explicitly only with the contribution of double covers to the formula (5.34), which is given by

$$F_{\text{het}}^{(2)}(\tau^*) = \frac{g_2^2}{8} \left| \frac{\tau^*}{2048\pi^4 (\alpha')^2} \right|^4 \int_{\Delta} d^2 \tau^\# \frac{\Psi_8(\tau_0(\tau^*, \tau^\#))}{|\Psi_{10}(\tau_0(\tau^*, \tau^\#))|^2} C(\tau_0(\tau^*, \tau^\#)) \quad (5.35)$$

where from (5.24) one has

$$C(\tau_0(\tau^*, \tau^\#)) = 2^{17} (\tau^*)^8 \eta(2\tau^\#)^{12} \eta(\tau^*)^{12} \theta_4(2\tau^\#)^4 \theta_4(\tau^*)^4. \quad (5.36)$$

To simplify the combination of elliptic functions arising in the genus two modular form (5.33), we follow the same steps as in the bosonic and supersymmetric calculations. Namely, we expand the terms in the sum over even genus two spin structures in (5.33) using the table (5.20), transform it to a form that is suitable for doubling the moduli of the Jacobi theta functions, write the doubling identities, and then make an elliptic $S$ transformation. The final result is again conveniently written in terms of theta functions of moduli $2\tau^\#$ and $\tau^*$ as

$$\Psi_8(\tau_0(\tau^*, \tau^\#)) = 2^{10} (\tau^*)^8 \theta_4(\tau^*)^{16} \theta_4(2\tau^\#)^{16} P_{\hat{G}} \left( \frac{\theta_4(\tau^*)^4}{\theta_4(\tau^*)^4}, \frac{\theta_4(2\tau^\#)^4}{\theta_4(2\tau^\#)^4} \right), \quad (5.37)$$

where $P_{\hat{G}}(x, y)$ is the symmetric polynomial defined by

$$P_{\hat{G}}(x, y) = 256 \left( x^4 y^4 + 1 \right) - 512 \left( x^4 y^3 + x^3 y^4 + x + y \right) + 1984 \left( x^3 y^3 + x y \right)$$

$$+ 288 \left( x^2 y^2 + x^2 y^3 + x^2 + x y \right) - 2016 \left( x^3 y^2 + x^2 y^3 + x^2 y + x y \right)$$

$$+ x^4 + 604 \left( x^3 y + x y^3 \right) + 3654 x^2 y^2. \quad (5.38)$$

Substituting (5.36) and (5.37), along with (3.24) and the abstruse identity (3.21), we find that the heterotic DLCQ free energy is given by

$$F_{\text{het}}^{(2)}(\tau^*) = \frac{g_2^2}{64} \left( \frac{\theta_4(\tau^*)}{\eta(\tau^*)} \right)^8 \theta_4(-\tau^*)^{12}$$

$$\times \left( \frac{\theta_4(2\tau^\#)}{\eta(-2\tau^\#)} \right)^2 \theta_3(\tau^*)^4 \theta_4(2\tau^\#)^4 - \theta_4(\tau^*)^4 \theta_3(2\tau^\#)^4$$

$$\times \left( \frac{\theta_4(-2\tau^\#)}{\eta(-2\tau^\#)} \right)^4 P_{\hat{G}} \left( \frac{\theta_4(-\tau^*)^4}{\theta_4(-\tau^*)^4}, \frac{\theta_4(-2\tau^\#)^4}{\theta_4(-2\tau^\#)^4} \right). \quad (5.39)$$
where we have used the complex conjugation properties $\overline{\theta_i(\tau)}^4 = \theta_i(-\tau)^4$ and $\overline{\eta(\tau)}^{12} = \eta(-\tau)^{12}$.

We equate (5.39) to the integrated two-point correlation function in the heterotic ($\mathbb{R}^8 \times \mathcal{G}) \times (\mathbb{Z}_2 \ltimes (\mathbb{Z}_2)^2$) permutation orbifold given by

$$\mathcal{F}_2^{\text{het}}(\tau^*) = \frac{4\lambda^2}{\bar{\tau}_2} \mu(0) \int d\mu(z) \left( \langle (\Lambda \otimes \overline{\sigma})(z) (\Lambda \otimes \overline{\sigma})(0) \rangle_{\mathbb{Z}_2 \ltimes (\mathbb{Z}_2)^2} \right).$$

Using the identities (3.52), (3.54) and (5.27) we then arrive at the two-point function of twist fields in the untwisted sector given by

$$\left( \langle (\Lambda \otimes \overline{\sigma})(z) (\Lambda \otimes \overline{\sigma})(0) \rangle_{\mathbb{Z}_2 \ltimes (\mathbb{Z}_2)^2} \right)_{\Lambda, \overline{\sigma}} = \frac{32}{(4\pi^2 \alpha')^8} \left( \frac{\theta_4(-\tau)\eta(\tau)}{\eta(-\tau)} \right)^{12} \left| E(z) \right|^{-6} \left| 8\tau^* \eta(\Pi)^3 \theta_4(\Pi) \right|^8$$

with $(a, b) \neq (1, 1)$, where $\hat{j}(\tau)$ is the supersymmetric partition function.

The structure of the formula (5.41) can be understood as follows. Generally, the separating degeneration limit $\tau_{12} \to 0$ of the genus two modular form (5.33) factorizes into the unique elliptic modular form of weight eight under $SL(2, \mathbb{Z})$ as

$$\Psi_5(\tau) = (\theta_2(\tau_{11})^{16} + \theta_3(\tau_{11})^{16} + \theta_4(\tau_{11})^{16}) (\theta_2(\tau_{22})^{16} + \theta_3(\tau_{22})^{16} + \theta_4(\tau_{22})^{16}) + \mathcal{O}(\tau_{12}^2).$$

For the covering surface $\hat{\Sigma}$, in the homology basis wherein the period matrix is given by (3.35) this degeneration limit corresponds to $\Pi \to \tau^*$, or equivalently $z \to 0$. Since the $x \to y$ limit of the symmetric polynomial (5.38) factorizes as

$$P_0^\theta(x, y) = 64 \left( (x - 1)^4 + x^4 + 1 \right)^2,$$

we see that the $z \to 0$ limit of the two-point function (5.41) factors into the one-loop heterotic string partition function on $\mathbb{R}^8$ evaluated with the spin structure (9) which is given by

$$\hat{\Theta}_0(\tau) = \left( \frac{4\pi^2 \alpha'}{\tau_2} \right)^4 \frac{1}{|\eta(\tau)|^{16}} \left( \frac{\theta_4(\tau)}{\eta(\tau)} \right)^4 \left( \frac{\theta_2(-\tau)^{16} + \theta_3(-\tau)^{16} + \theta_4(-\tau)^{16}}{2\eta(-\tau)^{16}} \right).$$

However, in contrast to the bosonic and supersymmetric twist field correlation functions, for distinct branch points the two-point function (5.41) does not neatly factor out a component corresponding to the untwisted fluctuation determinant of the heterotic orbifold sigma model. The reason generally is that the effective twist group is now a semi-direct product $S_N \ltimes (\mathbb{Z}_2)^N$ acting on the gauge fermions $\chi^a$. This means that the discrete $(\mathbb{Z}_2)^N$ gauge symmetry acts in the gauge sector together with the monodromy conditions of the permutation orbifold, and a disentanglement of the twisted and untwisted determinants arising from integration over the fermion fields $\chi$ in terms of branch point data as previously is not possible.

For example, by applying a crossing transformation to (5.41) as before one arrives at the twisted sector two-point functions

$$\left( \langle (\Lambda \otimes \overline{\sigma})(z) (\Lambda \otimes \overline{\sigma})(0) \rangle_{\mathbb{Z}_2 \ltimes (\mathbb{Z}_2)^2} \right)_{\epsilon, \delta} = 32 \left( \frac{\hat{j}(\tau^*)}{(4\pi^2 \alpha')^8} \right)^8 \left| \hat{\Theta}_0(\tau) \right|^{-16} \left( \frac{\theta_4(-\tau)^{16}}{2\eta(-\tau)^{16}} \right)^{12}$$

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$$P_0^\theta(x, y) = 64 \left( (x - 1)^4 + x^4 + 1 \right)^2,$$
with the supersymmetric twisted determinant $\hat{c}(\hat{x})$ given by (5.32). The extra gauge symmetry is implemented by $O(N)$ vector reflections of $\chi^a$ and holonomies of the corresponding flat real line bundles $L_\delta \to \mathbb{T}$. The latter phases correspond to $\mathbb{Z}_2$-valued Wilson lines which break the spacetime heterotic gauge group $\hat{G}$ to $G = SO(16) \times SO(16)$. They yield the extra GSO projection required to match to the spectrum of the free $E_8 \times E_8$ heterotic string [36, 51, 52] and to light-cone heterotic string field theory.

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References

[1] W. Taylor, “Matrix Theory: Matrix Quantum Mechanics as a Fundamental Theory”, Rev. Mod. Phys. 73 (2001) 419–462 [hep-th/0101126].

[2] L. Motl, “Proposals on Nonperturbative Superstring Interactions”, hep-th/9701025.

[3] T. Banks and N. Seiberg, “Strings from Matrices”, Nucl. Phys. B497 (1997) 41–55 [hep-th/9702187].

[4] R. Dijkgraaf, E.P. Verlinde and H.L. Verlinde, “Matrix String Theory”, Nucl. Phys. B500 (1997) 43–61 [hep-th/9703030].

[5] R. Dijkgraaf, G.W. Moore, E.P. Verlinde and H.L. Verlinde, “Elliptic Genera of Symmetric Products and Second Quantized Strings”, Commun. Math. Phys. 185 (1997) 197–209 [hep-th/9608096].

[6] G. Grignani and G.W. Semenoff, “Thermodynamic Partition Function of Matrix Superstrings”, Nucl. Phys. B561 (1999) 243–272 [hep-th/9903246].

[7] R. Dijkgraaf and L. Motl, “Matrix String Theory, Contact Terms and Superstring Field Theory”, hep-th/0309238.

[8] S. Moriyama, “Comments on Supersymmetry Algebra and Contact Term in Matrix String Theory”, J. High Energy Phys. 0409 (2004) 013 [hep-th/0405091].

[9] I. Kishimoto, S. Moriyama and S. Teraguchi, “Twist Field as Three String Interaction Vertex in Light Cone String Field Theory”, Nucl. Phys. B744 (2006) 221–237 [hep-th/0603068].

[10] I. Kishimoto and S. Moriyama, “On LCSFT/MST Correspondence”, hep-th/0611113.

[11] G. Grignani, P. Orland, L.D. Paniak and G.W. Semenoff, “Matrix Theory Interpretation of DLCQ String Worldsheets”, Phys. Rev. Lett. 85 (2000) 3343–3346 [hep-th/0004194].

[12] H.C.D. Cove and R.J. Szabo, “Two-Loop String Theory on Null Compactifications”, Nucl. Phys. B741 (2006) 313–352 [hep-th/0601220].
[13] L.J. Dixon, D. Friedan, E.J. Martinec and S.H. Shenker, “The Conformal Field Theory of Orbifolds”, Nucl. Phys. B282 (1987) 13–73.

[14] D. Bernard, “$Z_2$-Twisted Fields and Bosonization on Riemann Surfaces”, Nucl. Phys. B302 (1988) 251–279.

[15] H. Saleur, “Correlation Functions of the Critical Ashkin-Teller Model on a Torus”, J. Stat. Phys. 50 (1988) 475–508.

[16] J.J. Atick, L.J. Dixon, P.A. Griffin and D. Nemeschansky, “Multi-Loop Twist Field Correlation Functions for $Z_N$ Orbifolds”, Nucl. Phys. B298 (1988) 1–35.

[17] A. Lawrence and A. Sever, “Scattering of Twist Fields from D-Branes and Orientifolds”, arXiv:0706.3199 [hep-th].

[18] G.E. Arutyunov and S.A. Frolov, “Virasoro Amplitude from the Sym$^N(\mathbb{R}^{24})$ Orbifold Sigma Model”, Theor. Math. Phys. 114 (1998) 43–66 [hep-th/9708129].

[19] G.E. Arutyunov and S.A. Frolov, “Four Graviton Scattering Amplitude from Sym$^N(\mathbb{R}^8)$ Supersymmetric Orbifold Sigma Model”, Nucl. Phys. B524 (1998) 159–206 [hep-th/9712061].

[20] S. Hamidi and C. Vafa, “Interactions on Orbifolds”, Nucl. Phys. B279 (1987) 465–513.

[21] R. Dijkgraaf, E.P. Verlinde and H.L. Verlinde, “$c = 1$ Conformal Field Theories on Riemann Surfaces”, Commun. Math. Phys. 115 (1988) 649–690.

[22] O. Lunin and S.D. Mathur, “Correlation Functions for $M^N/S_N$ Orbifolds”, Commun. Math. Phys. 219 (2001) 399–442 [hep-th/0006196].

[23] P. Bántay, “Characters and Modular Properties of Permutation Orbifolds”, Phys. Lett. B419 (1998) 175–178 [hep-th/9708120].

[24] P. Bántay, “Orbifoldization, Covering Surfaces and Uniformization Theory”, Lett. Math. Phys. 57 (2001) 1–5 [hep-th/9808023].

[25] P. Bántay, “Symmetric Products, Permutation Orbifolds and Discrete Torsion”, Lett. Math. Phys. 63 (2003) 209–218 [hep-th/0004025].

[26] H. Fuji and Y. Matsuo, “Open String on Symmetric Product”, Int. J. Mod. Phys. A16 (2001) 557–608 [hep-th/0005111].

[27] H. Fuji, “Open Superstring on Symmetric Product”, hep-th/0112116.

[28] P.H. Ginsparg, “Applied Conformal Field Theory”, in: Fields, Strings, Critical Phenomena, eds. E. Brézin and J. Zinn-Justin (North-Holland, 1990), pp. 1–168 [hep-th/9108028].

[29] A. Klemm and M.G. Schmidt, “Orbifolds by Cyclic Permutations of Tensor Product Conformal Field Theories”, Phys. Lett. B245 (1990) 53–58.

[30] P. Forgács, Z. Horváth, L. Palla and P. Vecsey, “Higher Level Kac-Moody Representations and Rank Reduction in String Models”, Nucl. Phys. B308 (1988) 477–508.

[31] M.B. Halpern, “The Orbifolds of Permutation-Type as Physical String Systems at Multiples of $c = 26. I$: Extended Actions and New Twisted Worldsheet Gravities”, J. High Energy Phys. 0706 (2007) 068 [hep-th/0703044].

[32] L. Borisov, M.B. Halpern and C. Schweigert, “Systematic Approach to Cyclic Orbifolds”, Int. J. Mod. Phys. A13 (1998) 125–168 [hep-th/9701061].

[33] Z. Kádár, “The Torus and the Klein Bottle Amplitude of Permutation Orbifolds”, Phys. Lett. B484 (2000) 289–294 [hep-th/0004122].
[34] P. Bántay, “Permutation Orbifolds”, Nucl. Phys. B633 (2002) 365–378 [hep-th/9910079].

[35] C.L. Ezell, “Branch Point Structure of Covering Maps onto Nonorientable Surfaces”, Trans. Amer. Math. Soc. 243 (1978) 123–133.

[36] S.-J. Rey, “Heterotic Matrix Strings and their Interactions”, Nucl. Phys. B502 (1997) 170–190 [hep-th/9704158].

[37] A.A. Belavin, V.G. Knizhnik, A. Morozov and A.M. Perelomov, “Two and Three Loop Amplitudes in the Bosonic String Theory”, Phys. Lett. B177 (1986) 324–328.

[38] G.W. Moore, “Modular Forms and Two-Loop String Physics”, Phys. Lett. B176 (1986) 369–379.

[39] C. Birkenhake and H. Lange, Complex Abelian Varieties (Springer, 2004).

[40] J.D. Fay, Theta Functions on Riemann Surfaces (Springer, 1973).

[41] J.M. Smyrnakis, “Representations of the Mapping Class Group of the Two-Punctured Torus on the Space of sl(2, C) Spin 1/2 – Spin 1/2 Kac-Moody Blocks”, Nucl. Phys. B496 (1997) 630–642 [hep-th/9611225].

[42] H.M. Farkas and I. Kra, Theta Constants, Riemann Surfaces and the Modular Group (American Mathematical Society, 2001).

[43] E. Kani, “Hurwitz Spaces of Genus 2 Covers of Elliptic Curves”, Collect. Math. 54 (2003) 1–51.

[44] G. Mason and M.P. Tuite, “On Genus Two Riemann Surfaces Formed from Sewn Tori”, Commun. Math. Phys. 270 (2007) 587–634 [math.QA/0603088].

[45] L.J. Dixon, P.H. Ginsparg and J.A. Harvey, “c = 1 Superconformal Field Theory”, Nucl. Phys. B306 (1988) 470–496.

[46] D. Mumford, Tata Lectures on Theta (Birkhäuser, 1983).

[47] Y. Sugawara, “Thermal Amplitudes in DLCQ Superstrings on pp-Waves”, Nucl. Phys. B650 (2003) 75–113 [hep-th/0209145].

[48] D. Friedan, Z. Qiu and S.H. Shenker, “Superconformal Invariance in Two Dimensions and the Tricritical Ising Model”, Phys. Lett. B151 (1985) 37–43.

[49] D. Friedan, E.J. Martinec and S.H. Shenker, “Conformal Invariance, Supersymmetry and String Theory”, Nucl. Phys. B271 (1986) 93–165.

[50] O. Lunin and S.D. Mathur, “Three-Point Functions for MN/SN Orbifolds with N = 4 Supersymmetry”, Commun. Math. Phys. 227 (2002) 385–419 [hep-th/0103169].

[51] D.A. Lowe, “Heterotic Matrix String Theory”, Phys. Lett. B403 (1997) 243–249 [hep-th/9704041].

[52] T. Banks and L. Motl, “Heterotic Strings from Matrices”, J. High Energy Phys. 9712 (1997) 004 [hep-th/9703218].