Baby-Skyrmions dressed by fermions, an analytic sector

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Abstract

We find an analytic solution of the backreacted coupled fermion-baby-Skyrmion system valid at all values of the coupling parameter. The solution, built on a finite cylinder, is generally given in terms of the Heun functions and satisfies the physical requirements of finite energy. For a special value of the coupling parameter, the solution becomes a periodic crystal of baby-Skyrmions and fermions defined on the plane $\mathbb{R}^2$. These solutions are trivially extended to multi-solitonic branches of higher Baryon number.

Contents

1 Introduction .................................................. 2

2 The System ..................................................... 3

3 Combined Skyrmion-Fermion ansatz ....................... 5

   3.1 The baby-Skyrmion ansatz ............................... 5

   3.2 The Fermion ansatz ..................................... 6

   3.3 The final set of equations ............................... 8

4 An analytic sector ........................................... 9

   4.1 Analytic solutions and Heun functions .................. 9

   4.2 Analytic periodic solution on $\mathbb{R}^2$ .................. 13

   4.3 Energy and currents .................................... 14

   4.4 Higher winding numbers ................................. 15

5 Large coupling limit ....................................... 16

6 Conclusions ................................................... 18
1 Introduction

The Skyrme model \cite{1} has long been the subject of intense study (for reviews see \cite{2} \cite{3} and references therein). The reason is its intimate relation to low energy Baryons in the limit of large number of colour charges \cite{4}. The model can be thought to arise as an effective field theory from the integration of high energy fermionic “quark” modes. In this sense the theory does not intrinsically possess any fermionic degrees of freedom. While phenomenologically appealing, the full Skyrme theory is not particularly easy to study. It is a highly non-linear theory in four dimensions and most theoretical progress has to be made numerically. While being a simpler model overall, the baby-Skyrmion model \cite{5} \cite{6} \cite{7} captures the essence and main features of its higher dimensional parent remarkably well. This lower dimensional model is particularly suited for the study of Skyrmion interactions in condensed matter systems \cite{8} \cite{9} \cite{10} and more recently for quantum computation purposes \cite{11} \cite{12} \cite{13}. This model also yields solitonic solutions with Baryon charge but, unlike the parent theory, it is not a-priori built from integrating fermionic degrees of freedom and therefore can be naturally coupled to them. The topic of fermion couplings to baby-Skyrmions is a very interesting one, neatly parametrized by the fermion-baby-Skyrmion coupling constant $g$. Most of the theoretical work regarding this coupled system is performed in the decoupling limit of $g \to 0$ \cite{14} \cite{15}, where one can safely assume the fermions do not modify the baby-Skyrmion profile. Supersymmetric extensions of this system were also previously considered \cite{16}. Recently however, back reaction of the fermions was included by raising the value of $g$ and solving the resulting coupled system numerically \cite{17} \cite{18}. While very promising, numerical calculations suffer from common computational problems, being generally valid over a subset of parameters for which one finds convergence.

In this paper we wish to show that by a simple redefinition of the topology of the underlying space, coupled with a particular choice of ansatz, the model reveals analytic solutions of the fermion-baby-Skyrmion coupled system, valid for all coupling parameters $g$. The main mathematical simplification is a particular choice of ansatz developed in \cite{19} \cite{20} \cite{21} \cite{22} \cite{23} \cite{24}. Within the context of the baby-Skyrme model, this method was used in \cite{25}. This ansatz decouples the fermions from the baby-Skyrmion profile equation of motion, while maintaining that of the baby-Skyrmion in the equation of motions of the fermions. In this sense, the system simplifies to solving for the fermion profiles in a background potential given by the baby-Skyrmion field, with the crucial difference that no limit for any coupling strength parameter has to ever be taken. The ansatz involves a particular locking between the time dependence of the baby-Skyrme field and the phases of the fermion components. In this analytic branch, valid for any $g$, the classical solitonic baby-Skyrmion profile is unaffected by the fermions that live with it even though their coupling can be arbitrarily strong. While the soliton profile remains unchanged, the coupling between the baby-Skyrme and the fermion profiles happens because of the time dependence of both fields through the frequency $\omega$. Remarkably, these solutions satisfy the physical requirement of having finite energy. Having this analytical power, we are able to make some statements regarding the large $g$ limit of this particular branch of solutions. Furthermore we will show that this solution is easily extendible to the case of multisolitonic configurations.

The paper is structured as follows: in section \ref{sec:system} we introduce the system we will work with throughout the paper, in section \ref{sec:ansatz} we provide details about the specific ansatz used in the construction of our solutions, specifying the physical interpretation of the corresponding fields. In section \ref{sec:solutions} we find our analytic solutions and explore their physical properties, including a brief discussion on the multi-soliton extension. Section \ref{sec:large_g} is devoted to the large fermion-baby-Skyrmion coupling limit, and finally in section \ref{sec:conclusions} we provide the conclusions of our research and some avenues of future investigation.
2 The System

The action we use describing a baby Skyrme system coupled to a Fermion is

\[ S = \int d^3x \left[ \frac{a_1}{2} (\nabla_\mu \bar{\Phi}) \cdot (\nabla^\mu \Phi) + \kappa_0 \left( \frac{1 - \bar{\Phi} \cdot \vec{n}}{2} \right) - \frac{a_2}{4} (\nabla_\mu \Phi \times \nabla_\nu \Phi) \cdot (\nabla^\mu \Phi \times \nabla^\nu \Phi) - \lambda (\bar{\Phi} \cdot \Phi - 1) + \mathcal{L}_F \right], \tag{1} \]

\[ \mathcal{L}_F = \bar{\Psi} \left( i\gamma_\mu \nabla_\mu + g \vec{\tau} \cdot \Phi - m \right) \Psi, \tag{2} \]

where \( \lambda \) is a Lagrange multiplier implementing the geometric constraint \( \Phi \cdot \Phi = 1 \) in isospin space, \( g \) is the coupling constant related to the baby-Skyrmion-Fermion interaction and the \( a_1, a_2 \) are constant coefficients. The isospin matrices are defined as \( \tau^i = I \otimes \sigma^i \), where \( \sigma^i \) are the Pauli matrices. The spin matrices are defined as \( \gamma_\mu = \tilde{\gamma}_\mu \otimes \mathbb{I} \). We also include a potential term for the baby-Skyrmion with its corresponding coupling constant \( \kappa_0 \).

The mass dimensions of the parameters appearing in the Lagrangian are

\[ [g] = [m] = [a_1] = +1, \quad [a_2] = -1, \quad [\kappa_0] = [\lambda] = +3. \tag{3} \]

In order to analyze finite-density effects, it is convenient to introduce the following cylindrical flat metric

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - dx^2 - L^2 d\varphi^2, \tag{4} \]

where \( \varphi \) is a compactified coordinate with range \( 0 \leq \varphi \leq 2\pi \) and \( x \) belongs to a finite interval. In the following, we take \(-L/2 \leq x \leq L/2\) for the most of the cases but other intervals are also considered. With this choice of the metric, we choose the spin matrices \( \tilde{\gamma}_1 = -i\sigma_1, \tilde{\gamma}_2 = -iL\sigma_2 \) and \( \gamma_0 = \sigma_3 \) such that \( \gamma^\mu \) respect the Clifford algebra

\[ \{\gamma^\mu, \gamma^\nu\} = 2\mathbb{I} g^{\mu\nu}. \tag{5} \]

We will use the following common parametrization of the baby-Skyrmion field

\[ \Phi \cdot \Phi = 1 \iff \Phi = (\sin F \cos G, \sin F \sin G, \cos F), \tag{6} \]

\[ F = F(x^\mu), \quad G = G(x^\mu). \tag{7} \]

Without loss of generality, we also take the standard choice for the vacuum

\[ \vec{n} = (0,0,1). \tag{8} \]

Given the metric \([4]\), the positivity of the energy of the baby skyrmion sector is guaranteed requiring

\[ a_1 > 0, \quad a_2 \geq 0, \quad \kappa_0 \leq 0. \tag{9} \]

Finally, the topological charge of the baby Skyrmion is

\[ B = \frac{1}{4\pi} \int_\Sigma \rho_B, \quad \rho_B = \frac{1}{4\pi} (\sin F) dF \wedge dG. \tag{10} \]
The equations of motion for the baby skyrmion fields are

\[ 0 = -\Box F + \frac{\sin(2F)}{2} \nabla_\mu G \nabla^\mu G + c_1 \sin(2F) \left[ (\nabla_\mu F \nabla^\mu F)(\nabla_\nu G \nabla^\nu G) - (\nabla_\mu F \nabla^\mu G) \right]^2 \\
- c_1 \nabla_\mu \left[ \sin^2(F) \left[ (\nabla_\nu G \nabla^\nu G) \nabla^\mu F - (\nabla_\nu F \nabla^\nu G) \nabla^\mu G \right] \right] + \kappa_0 \frac{\partial}{\partial F} \left( \frac{1 - \cos F}{2} \right) - \frac{g}{a_1} \frac{\delta}{\delta F} \left( \bar{\Psi} \delta \cdot \Phi \right) \Psi \]

\[ (11) \]

\[ 0 = -\sin^2(F) \Box G - \sin(2F) \nabla_\mu F \nabla^\mu G - \frac{g}{a_1} \frac{\delta}{\delta G} \left( \bar{\Psi} \delta \cdot \Phi \right) \Psi + c_1 \nabla_\mu \left[ \sin^2(F) \left[ (\nabla_\nu F \nabla^\nu G) \nabla^\mu F(\nabla_\nu F \nabla^\nu F) \nabla^\mu G \right] \right]. \]

\[ (12) \]

In the above equations \( c_1 = \frac{a_2}{a_1}. \) Similarly, the Dirac equations for the Fermions read

\[ (i \gamma^\mu \nabla_\mu + g \bar{\tau} \cdot \bar{\Phi} - m) \Psi = 0. \]

\[ (13) \]

The solutions to this coupled system for the uncompactified case for which \( L d \varphi \equiv dy \) and \( y \) is extended to (plus or minus) infinity, have been studied numerically in much of the recent literature \[17\] \[18\]. In this paper we will show that the simple compactification of the \( y \) coordinate, effectively changing the topology of the underlying space from a flat plane to a cylinder, is sufficient to provide the first analytic solutions of this system with non vanishing topological charge. Moreover, in the specific case of a periodic solution, we are able to come back from the compact cylindrical space to the flat plane.

The study of analytic sector is of fundamental importance, as numerical control is quickly lost for extreme values of parameters or solutions which depends on more space time coordinates. Analytic solutions for topological soliton systems are rare and provide perfect toy models for the analysis of many fundamental physical effects.

We will consider for now a general Fermion ansatz of the form

\[ \Psi = 1/N_i \begin{pmatrix} v_1(x) \exp[i \Omega_1(\varphi,t)] \\
v_2(x) \exp[i \Omega_2(\varphi,t)] \\
v_1(x) \exp[i \Omega_3(\varphi,t)] \\
v_2(x) \exp[i \Omega_4(\varphi,t)] \end{pmatrix} \]

\[ (14) \]

where \( v_1, v_2, u_1 \) and \( u_2 \) are in general complex functions. The field \( \Psi \) is a spin-isospin spinor, meaning that it describes the wave function of a particle with spin one-half and iso-spin one-half. The coefficient \( N_i \) is a normalization coefficient which ensures that \( \int \Psi \Psi^\dagger = 1. \) In detail, this evaluates to

\[ N_i^2 = \int (u_1^* u_1 + u_2^* u_2 + v_1^* v_1 + v_2^* v_2) \, dx \, L d \varphi. \]

\[ (15) \]

The system admits at least two internal symmetries. The first symmetry of the action \( (\frac{1}{4}) \) is an \( SO(2) \sim U(1) \) rotation that acts only on the field components \( G, \Omega_2 \) and \( \Omega_4 \) as

\[ G \rightarrow G' = G + q \alpha \]
\[ \Omega_2 \rightarrow \Omega'_2 = \Omega_2 + q \alpha \]
\[ \Omega_4 \rightarrow \Omega'_4 = \Omega_4 + q \alpha \]

\[ (16) \quad (17) \quad (18) \]
where $\alpha$ is a constant phase and $q \in \mathbb{Z}$. Equivalently, this rotation acts on the fundamental field $\Phi$ and $\Psi$ as

$$\Phi \rightarrow \Phi' = O\Phi \quad \Psi \rightarrow \Psi' = e^{iQ\alpha}\Psi$$

(19)

where

$$O = \begin{pmatrix} \cos q\alpha & -\sin q\alpha & 0 \\ \sin q\alpha & \cos q\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad Q = \frac{q}{2}(\mathbb{I} - \tau_3)$$

(20)

and it can be interpreted as an iso-spin rotation. The corresponding conserved current $J^\mu_U(1) = J^\mu_s + J^\mu_f$ contains the scalar-field term $J^\mu_s$

$$J^\mu_s = qa_1 \sin^2 F \left(1 - \frac{a_2}{a_1} (F')^2 \right) \nabla^\mu G$$

(21)

and the fermion current component $J^\mu_f$

$$J^\mu_f = \bar{\Psi} \gamma^\mu Q \Psi.$$  

(22)

On the other hand, the system is also invariant under the phase transformation

$$\Omega_1 \rightarrow \Omega'_1 = \Omega_1 + \alpha$$
$$\Omega_2 \rightarrow \Omega'_2 = \Omega_2 + \alpha$$
$$\Omega_3 \rightarrow \Omega'_3 = \Omega_3 + \alpha$$
$$\Omega_4 \rightarrow \Omega'_4 = \Omega_4 + \alpha$$

(23) \quad (24) \quad (25) \quad (26)

that correspond to $\Psi \rightarrow \Psi' = e^{i\alpha}\Psi$. The associated Fermionic current $J^\mu_F$ is

$$J^\mu_F = \bar{\Psi} \gamma^\mu Q \Psi.$$  

(27)

The conservation of this current corresponds to the conservation of quantum probability since the time-component of the four-vector (27) represents the probability density ($\rho_F$ from now)

$$\rho_F = J^0_F = \Psi^\dagger \Psi.$$  

(28)

The divergence of the current (27) gives therefore the continuity equation

$$\frac{\partial \rho_F}{\partial t} + \nabla \cdot J_F = 0.$$  

(29)

3 Combined Skyrmion-Fermion ansatz

3.1 The baby-Skyrmion ansatz

For what concerns the baby skyrmion fields, the approach of [25] suggests the following ansatz

$$F = F(x), \quad G = p\varphi + \omega t, \quad p \in \mathbb{N}, \quad \text{with} \quad \frac{p^2}{L^2} - \omega^2 = 0.$$  

(30)

Note that this ansatz implies a non trivial periodic dependence on the time coordinate. This implies the field profiles have non vanishing topological windings around time-like surfaces ($\varphi = \text{const}$).
One of the fundamental aspects of this kind of ansatz is that even though the field profiles depend explicitly on time, the energy density does not, as we will show in section (4.3).

The topological charge density (10) simplifies to

\[ \rho_B = \frac{p}{4\pi} \sin F (\partial_x F) dx \wedge d\varphi. \] (31)

Choosing a finite interval for the \( x \)-coordinate as \( x \in [-L/2, L/2] \), we can build solutions with different topological charge depending on the boundary conditions of \( F \):

\[ F(L/2) - F(-L/2) = \pi (1 + 4n) \quad n \in \mathbb{N} \Rightarrow B = p \] (32)
\[ F(L/2) - F(-L/2) = \pi (3 + 4d) \quad d \in \mathbb{N} \Rightarrow B = -p \] (33)
\[ F(L/2) - F(-L/2) = 2s\pi \quad s \in \mathbb{N} \Rightarrow B = 0 \] (34)

(where \( B \) is the baby Baryon charge). The integers \( n, d \) or \( s \) appearing in Eqs. (32), (33), (34) do not appear in the final value of topological charge but, together with \( p \), they are relevant to classify the map. The meaning of these integers is that they classify the number of positive or negative “bumps” in the topological density charge. Thus, somehow, \( n, d \) and \( s \) could be interpreted as determining a second “charge” of the configuration [25].

Using the baby-Skyrmion ansatz, the Lagrangian for the Fermions can be reduced to

\[ \mathcal{L}_F = \Psi^\dagger D\Psi \] (35)

where the operator \( D \) has components

\[ D = \begin{bmatrix} -m + g \cos F + i\partial_t & g \sin F e^{i(p\varphi + \omega t)} & -\frac{L\partial_x - i\partial_y}{L} & 0 \\ g \sin F e^{i(p\varphi + \omega t)} & -m - g \cos F + i\partial_t & 0 & -\frac{L\partial_x - i\partial_y}{L} \\ L\partial_r + i\partial_\varphi & 0 & m - g \cos F + i\partial_t & -g \sin F e^{i(p\varphi + \omega t)} \\ 0 & \frac{L\partial_r + i\partial_\varphi}{L} & -g \sin F e^{i(p\varphi + \omega t)} & m + g \cos F + i\partial_t \end{bmatrix}. \] (36)

3.2 The Fermion ansatz

In [17][18], the fermion ansatz written in polar coordinates \( \{r, \theta\} \) on the plane \( \mathbb{R}^2 \) is of the form

\[ \begin{align*}
  v_1 &= v_1(r) \quad v_2 = v_2(r) \\
  u_1 &= u_1(r) \quad u_2 = u_2(r) \\
  \Omega_1 &= l \theta - \omega t \\
  \Omega_2 &= (p + l) \theta - \omega t \\
  \Omega_3 &= (l + 1) \theta - \omega t \\
  \Omega_4 &= (p + l + 1) \theta - \omega t
\end{align*} \] (37)

where \( \omega \) is the energy of the fermion eigenstates, \( p \) the topological charge and \( l \in \mathbb{Z} \) an additional integer that labels the solution. In these works, the full set of energy eigenstates for the fermions was found using for the baby skyrmions the axial symmetric ansatz. Furthermore, no assumption for the profile functions \( u_1, u_2, v_1, v_2 \) has been made \textit{a priori} and the whole system of coupled equations
was solved numerically with the proper boundary conditions. The aim of our work is instead to explore the existence of an analytic sector of solutions for the baby skyrmion-fermion system. These solutions must have finite energy to be physical but they should not be necessary eigenstates of the energy. Moreover, we are not using the usual axial symmetric baby skyrmions, but a cylindrical ansatz developed in [25].

For these reasons, we choose

$$v_2(x) = u_2(x) \quad u_1(x) = v_1(x)$$
$$\Omega_1 = l \varphi + \omega_2 t$$
$$\Omega_2 = (p + l) \varphi + (\omega + \omega_2) t$$
$$\Omega_3 = \Omega_1$$
$$\Omega_4 = \Omega_2$$

with $\omega = -p/L$ and $\omega_2 = -l/L$ with $p, l \in \mathbb{N}$. As we will show in the next section, this ansatz is able to reduce the whole system of coupled equations to:

- a single equation for the baby skyrmion profile
- a system of equations for the fermions in which the baby skyrmion is a background field.

Let us now spend some words on the interpretation of the ansatz (38). To this end, it is useful to identify the quantum numbers of the fermion wave-function $\Psi$ using the quantum operators

$$\mathcal{H} = \gamma_3 \left( i \gamma_i \partial_i - g \vec{\tau} \cdot \vec{\Phi} + m \right) \quad (39)$$
$$P_\varphi = -i \frac{\partial}{L \partial \varphi} \quad (40)$$
$$Q = \frac{q}{2} (\mathbb{1} - \tau_3) \quad (41)$$
$$S = \gamma_3 \quad (42)$$

where $\mathcal{H}$ is the Hamiltonian, $P_\varphi$ is the momentum along the $\varphi$-direction, $Q$ is a global $U(1)$ operator defined in (20) that is related to the iso-spin operator and $S$ is the spin operator.

Re-writing the ansatz (38) as

$$\Psi = \frac{v_2(x)}{N_i} e^{i(p+l)\varphi + (\omega + \omega_2)t} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \frac{u_1(x)}{N_i} e^{i(\varphi + \omega_2 t)} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \equiv \Psi_1 + \Psi_2$$

we check that $\Psi$ is not an eigenstate of the energy since

$$\mathcal{H} \Psi = -(\omega + \omega_2) \Psi_1 - \omega_2 \Psi_2$$

where we already use the equations of motion that we show in detail in the next section. This result was expected from the very beginning since for the fermion field every eigenstate of the energy is written in the form $\Psi = e^{-i\omega t} \psi$, where $\omega$ is the energy eigenvalue and $\psi$ depends only on the space coordinates. A simple check shows that this was the case of (37) in [17] [18] but not of our ansatz (38). The energy of our fermion solution, we call it $E_f$ from now, simply represents the mean-value of $\mathcal{H}$ on the state $\Psi$

$$E_f = \int d^2x \Psi^* \mathcal{H} \Psi.$$
Since $E_f$ is finite, it can be possible to write $\Psi$ as a superposition of energy eigenstates. This is however beyond the aim of our paper. On the other hand, acting with the momentum $P_\phi$ we have that

$$P_\phi \Psi_1 = \frac{p + l}{L} \Psi_1 \quad Q\Psi_1 = q\Psi_1$$

$$P_\phi \Psi_2 = \frac{l}{L} \Psi_2 \quad Q\Psi_2 = 0.$$  \hspace{1cm} (46)

Furthermore, the states $\Psi_1$ and $\Psi_2$ are themselves a superposition of spin eigen-states

$$\Psi_1 = \frac{v_2(x)}{N_i} e^{i((p+l)\phi + (\omega + \omega_2)t)} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{v_2(x)}{N_i} e^{i((p+l)\phi + (\omega + \omega_2)t)} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \equiv \Psi_{1\uparrow} + \Psi_{1\downarrow}$$  \hspace{1cm} (48)

and similarly

$$\Psi_2 \equiv \Psi_{2\uparrow} + \Psi_{2\downarrow}$$  \hspace{1cm} (49)

where

$$S\Psi_{1\uparrow} = \frac{1}{2}\Psi_{1\uparrow} \quad S\Psi_{1\downarrow} = -\frac{1}{2}\Psi_{1\downarrow}$$

$$S\Psi_{2\uparrow} = \frac{1}{2}\Psi_{2\uparrow} \quad S\Psi_{2\downarrow} = -\frac{1}{2}\Psi_{2\downarrow}.$$  \hspace{1cm} (50)

In the end, using the ket notation $|P_\phi, Q\rangle$ for the eigenvector of the momentum and charge and $|\uparrow\downarrow\rangle$ for the spin, we can write $\Psi$ as a superposition of states

$$\Psi = \frac{v_2(x)}{N_i} e^{i(\omega + \omega_2)t} (| (p + l)/L, q \rangle (|\uparrow\rangle + |\downarrow\rangle)) + \frac{u_1(x)}{N_i} e^{i\omega_2 t} |l/L, 0 \rangle (|\uparrow\rangle + |\downarrow\rangle).$$  \hspace{1cm} (52)

in which $\omega = -p/L$ and $\omega_2 = -l/L$. If we consider this quantum solution at a given coordinate $x$ of the cylinder, the state (52) represents a superposition of two stationary waves propagating along $\phi$, each of which is characterized by spin up and down. The waves carry two different momenta $P_\phi$ and different charges $Q$ and they are weighed by the functions $v_2$ and $u_1$.

Without loss of generality, and unless explicitly stated for illustrative purposes, we will work in the following of the paper with the simplest choice of $l = \omega_2 = 0$.

3.3 The final set of equations

Finally, we arrive at our final form for the equations we wish to solve. Inserting ansatz (30) and (38) into (11)-(12) and (35) we arrive at the three following ODEs

$$F'' = -\frac{\kappa_0}{2a_1} \sin F,$$  \hspace{1cm} (53)

$$v_2' = g \sin F u_1 - (m + g \cos F) v_2,$$  \hspace{1cm} (54)

$$u_1' = g \sin F v_2 + (-m + g \cos F) u_1.$$  \hspace{1cm} (55)

8
As discussed in the previous section, the ansatz (30) and (38) reduce the full system of coupled field equations only to a single equation for the skyrmion profile and to a system for the fermion fields in which $F$ is a background field.

The baby skyrmion equation (53) can be integrated as

$$F'' = -\frac{\kappa_0}{2a_1} \sin F \Rightarrow F' = \sqrt{2 \left( E_0 + \frac{\kappa_0}{2a_1} \cos F \right)} ,$$

(56)

where $E_0$ is an integration constant which will be determined below in terms of the topological charge of the baby Skyrmion. The prime derivative notation refers here to derivatives w.r.t $x$.

On the other hand, we can separate the two fermion equations by taking the derivative of them with respect to $x$. Note that one can express one fermion component in terms of the other. In order to analyse the solutions of this coupled system analytically we further introduce the following change of variables

$$v_2 = I(x) K(x) ,$$

(57)

where

$$I(x) = I_0 e^{-mx} \sqrt{\sin(F(x))}$$

(58)

with $I_0$ an integration constant. The system then reduces to solving for the function $K(x)$, which in turn determines the fermion component $v_2$ and therefore also $u_1$ from equations (54) and (55). This remaining equation reduces to the Schrodinger-like form

$$-\frac{d^2K(x)}{dx^2} + V_{eff} (F) K(x) = 0 ,$$

(59)

where

$$V_{eff}(F) = g^2 - \csc F \left( -g \sqrt{2E_0 + \frac{\kappa_0}{a_1} \cos F - \frac{3\kappa_0}{4a_1} \cot F} - \frac{E_0}{2} (1 - 3 \csc^2 F) \right) .$$

(60)

While the potential in this Schrodinger-like equation is highly complicated we will show below that there exists a remarkably simple sector where the resulting solution (and therefore the full fermion and Skyrmion profiles) can be found analytically.

4 An analytic sector

4.1 Analytic solutions and Heun functions

Without doubt, the simplest non-trivial case to solve for the baby-Skyrmion profile corresponds to the choice $\kappa_0 = 0$ (namely, no explicit potential in the baby Skyrme action) and $m = 0$. In this case:

$$F = \sqrt{2E_0} x + \pi/2 ,$$

$$E_0 = E_0 (n) = \frac{\pi^2}{2} \left( \frac{1 + 4n}{L} \right)^2 ,$$

(61)

(62)

where the condition (62) arises from the requirement of having a positive non-vanishing topological charge in the interval $x = [-L/2, L/2]$ (see eq. (32)). From now, we will work with the simplest case of $n = 0$. 

9
With this choice of parameters and defining the dimensionless coordinate \( \tilde{x} = \sqrt{\frac{E_0}{2}} x \), the Schrödinger-like equation (59) reduces to
\[
-\frac{d^2 K(\tilde{x})}{d\tilde{x}^2} + \tilde{V}_{\text{eff}}(\tilde{x}) K(\tilde{x}) = 0,
\]
(63)
where
\[
\tilde{V}_{\text{eff}}(\tilde{x}) = \left[ -\tilde{E} + \frac{V_0}{1 - 2\cos^2(\tilde{x})} + \frac{V_1}{(1 - 2\cos^2(\tilde{x}))^2} \right],
\]
(64)
with \( V_0 = -2\sqrt{2/E_0}g \), \( V_1 = 3 \), \( \tilde{E} = 1 - 2g^2/E_0 \). Since \( x = [-L/2, L/2] \), we have \( \tilde{x} = [-\pi/4, \pi/4] \). Note also that with these boundary conditions the baryonic charge from (31) is \( B = 1 \). The profile of the topological charge density is shown in figure 1.

Figure 1: Topological charge density \( \rho_B \) in the finite interval \( x \in [-L/2, L/2] \) for the skyrmion solution with \( n = 0, p = 1 \) and \( g = L = 1 \).

The Schrödinger equation with the potential of equation (64) has standard solutions in terms of Heun\( G \) functions [26]. Interestingly, similar solutions in terms of Heun functions for a different solitonic system, which separates its fields in a similar way, were also recently found in [27]. In our notation, the solution takes the form
\[
K(\tilde{x}) = (a - \sin^2(\tilde{x}))^\lambda \phi(\sin^2(\tilde{x}))
\]
(65)
where
\[
\phi(\sin^2(\tilde{x})) = c_1 \mathcal{H}_G[a, q, \alpha, \beta, \gamma, \delta, \sin^2(\tilde{x})] + c_2 \sin(\tilde{x}) \mathcal{H}_G[a, q - (-1 + \gamma)d, 1 + \beta - \gamma, 1 + \alpha - \gamma, 2 - \gamma, \delta, \sin^2(\tilde{x})].
\]
(66)
with \( c_1 \) and \( c_2 \) arbitrary (complex) integration constants, \( d = (1 + \alpha + \beta - \gamma + (-1 + a)\delta) \) and
\[
\lambda = \frac{1}{2} \left( 1 + \sqrt{1 + V_1} \right), \quad a = \frac{1}{2}, \quad \gamma = \delta = \frac{1}{2}, \quad \alpha = \lambda - \frac{\sqrt{\tilde{E}}}{2}, \quad \beta = \lambda + \frac{\sqrt{\tilde{E}}}{2},
\]
(67)
\[
\mu = \lambda^2 - \tilde{E}/4, \quad \nu = -\lambda(\lambda - 1) - \lambda(a + 1/2) + V_0/8 + \tilde{E}a/2, \quad q = -\mu a - \nu.
\]
(68)
Our full analytic solution for the fermion field therefore reads
\[
v_2(x) = \sqrt{\cos(2\tilde{x})} \left( a - \sin^2(\tilde{x}) \right)^\lambda \phi(\sin^2(\tilde{x}))
\]
(69)
\[ u_1(x) = -\frac{\sec 2\tilde{x}}{g} \left( g \sin(2\tilde{x}) v_2(\tilde{x}) - \sqrt{\frac{E_0}{2}} \partial_{\tilde{x}} v_2(\tilde{x}) \right), \] (70)

with \( E_0 \) given in terms of \( L \) from equation (61) with \( n = 0 \). We present some sample solutions in figures 2 and 3.

An important remark must be made here: the \( H_G \) functions are analytical in the range

\[ \sin^2(\tilde{x}) < \min\{1,|a|\} \] (71)

that in our case corresponds to \(-\pi/4 < \tilde{x} < \pi/4\). This therefore might indicate a problem for the validity of the analytical form of the solutions at the boundary \( \tilde{x} = \pm \pi/4 \) (or \( x = \pm L/2 \)). However, remarkably, the equations (54), (55) are well-defined on these points and then the corresponding values of the solutions can be added by analytic continuation or by numerical interpolation. Moreover, it is possible to extend our analytical solution for a general interval \( \tilde{x} \in \pi/2 (-1/2+j,1/2+j) \) with \( j \in \mathbb{Z} \) (that corresponds to \( x \in (-L/2+jL,L/2+jL) \)). In this interval, the solution of the Schrodinger-like equation (59) with potential (64) reads

\[ K_j(\tilde{x}) = (a - \sin^2(\tilde{x}))^\lambda \phi_j(\tilde{x}) \] (72)

where

\[
\begin{align*}
\phi_j(\tilde{x}) &= c_1^{(j)} H_G[a,q^{(j)},\alpha,\beta,\gamma,\delta,\sin^2(\tilde{x} - j\pi/2)] \\
&+ c_2^{(j)} \sin(\tilde{x} - j\pi/2) \times \\
&\times H_G[a,q^{(j)} - (-1 + \gamma)d,1 + \beta - \gamma,1 + \alpha - \gamma,2 - \gamma,\delta,\sin^2(\tilde{x} - j\pi/2)].
\end{align*}
\] (73)

and \( q^{(j)} \) is re-defined as

\[
\begin{align*}
q^{(j)} &= -\mu a - \nu^{(j)} \\
\nu^{(j)} &= -\lambda(\lambda - 1) - \lambda(a + 1/2) + (-1)^j V_0/8 + \tilde{E}a/2.
\end{align*}
\] (74)

The final solution for \( v_2 \) and \( u_1 \) in this generic interval is again of the form

\[
v_2^{(j)}(x) = \sqrt{\cos(2\tilde{x})} \left( a - \sin^2(\tilde{x}) \right)^\lambda \phi_j(\tilde{x})
\] (76)

\[
u_1^{(j)}(x) = -\frac{\sec 2\tilde{x}}{g} \left( g \sin(2\tilde{x}) v_2^{(j)}(\tilde{x}) - \sqrt{\frac{E_0}{2}} \partial_{\tilde{x}} v_2^{(j)}(\tilde{x}) \right).
\] (77)

In order to build a solution in a larger interval, it is necessary to glue different solutions labelled by consecutive integers \( j \). The parameters \( c_{1}^{(j)}, c_{2}^{(j)} \) must be chosen to respect the continuity of the first derivative and the function. In figure (4), we show an example of a solution defined in the interval \( x \in (-3L/2,3L/2) \) or equivalently \( \tilde{x} \in (-3\pi/4,3\pi/4) \).
Figure 2: Solutions for fermion profiles $v_2$ and $u_1$ in the finite interval $x \in (-L/2, L/2)$ with parameters $c_1 = 1$, $c_2 = 0$, $p = 1$, $n = 0$, $g = L = 1$.

Figure 3: Solutions for fermion profiles $v_2$ and $u_1$ in the finite interval $x \in (-L/2, L/2)$ with parameters $c_1 = 0$, $c_2 = 1$, $p = 1$, $n = 0$, $g = L = 1$.

Figure 4: Solutions for fermion profiles $v_2$ and $u_1$ in the finite interval $x \in (-3L/2, 3L/2)$ with parameters $c_1^{(0)} = 1$, $c_2^{(0)} = 0$, $c_1^{(1)} = c_1^{(-1)} = 1.9823$, $c_2^{(1)} = c_2^{(-1)} = 0.2229$, $p = 1$, $n = 0$, $g = L = 1$. 
4.2 Analytic periodic solution on $\mathbb{R}^2$

For the special choice of coupling constant

$$g = \sqrt{\frac{E_0}{2}},$$

the system of equations (54), (55) and (56) with $\kappa_0 = m = 0$ can be explicitly solved for any choice of $E_0$ with solutions

$$F(x) = \sqrt{2E_0} x + \pi/2 = 2\tilde{x} + \pi/2,$$  \hspace{1cm} (79)

$$v_2(x) = A \sin\left(\sqrt{\frac{E_0}{2}} x\right) = A \sin(\tilde{x}),$$  \hspace{1cm} (80)

$$u_1(x) = A \cos\left(\sqrt{\frac{E_0}{2}} x\right) = A \cos(\tilde{x}),$$  \hspace{1cm} (81)

where $A$ is a complex constant. These remarkably simple solutions describe a periodic field configuration along the $x-$direction. The period $\lambda$ of the whole skyrmion-fermion system along this direction can be found looking at the least common multiple between the period of the scalar and fermion field. For the scalar field, we have

$$\left\{ \begin{array}{l}
\Phi(0, \varphi) = \Phi(\lambda_s, \varphi) \\
\partial_x \Phi(0, \varphi) = \partial_x \Phi(\lambda_s, \varphi)
\end{array} \right. \quad \text{with} \quad \lambda_s = \frac{2\pi}{\sqrt{2E_0}} = \frac{\pi}{g},$$ \hspace{1cm} (82)

while for the fermion field

$$\left\{ \begin{array}{l}
\Psi(0, \varphi) = \Psi(\lambda_f, \varphi) \\
\partial_x \Psi(0, \varphi) = \partial_x \Psi(\lambda_f, \varphi)
\end{array} \right. \quad \text{with} \quad \lambda_f = \frac{4\pi}{\sqrt{2E_0}} = \frac{2\pi}{g}.$$ \hspace{1cm} (83)

Since $\lambda_f = 2\lambda_s$, the period of the whole system along $x$ becomes $\lambda = 4\pi/\sqrt{2E_0}$.

The periodicity of this solution along $x$ allows us to build it not only on a finite cylinder but also on the whole $\mathbb{R}^2$ plane. This is possible following the procedure below. Firstly, after a simple re-definition of the variable $dy^2 = L^2 dz^2$ in the metric (4), the cylindrical space becomes equivalent to a flat rectangular space with periodic boundary condition along $y$. If now we have periodic conditions even along $x$, we can replicate such a periodic rectangular flat cell to build the whole $\mathbb{R}^2$ space. In our special case in which $g$ is given by (78), we can then build a baby skyrmion-fermion periodic crystal on $\mathbb{R}^2$ with the solutions

$$\Phi = \begin{pmatrix}
\sin (2\tilde{x} + \pi/2) \cos (p(y - t)/L) \\
\sin (2\tilde{x} + \pi/2) \sin (p(y - t)/L) \\
\cos (2\tilde{x} + \pi/2)
\end{pmatrix} \quad \Psi = A \begin{pmatrix}
\cos(\tilde{x}) e^{i(l(y-t)/L)} \\
\sin(\tilde{x}) e^{i((p+l)(y-t)/L)} \\
\cos(\tilde{x}) e^{i((p+l)(y-t)/L)}
\end{pmatrix}$$

replicating the rectangular $\{x, y\}$ cell of size $[0, \lambda] \times [0, 2\pi L]$.

If instead we consider again the case of a finite cylinder with $x \in [-L/2, L/2]$, a periodic solution in this interval needs

$$L = k\lambda \quad \Rightarrow \quad E_0 = \frac{8\pi^2 k^2}{L^2}$$ \hspace{1cm} (85)
where \( k \in \mathbb{N} \). Therefore, at finite density we have only a discrete set of possible values for \( E_0 \) as well as for the coupling constant \( g = 2\pi k/L \). Note that the condition \((85)\) implies

\[
F(L/2) - F(-L/2) = 4\pi k
\]

and then, due to \((34)\), the total topological charge of this periodic configuration in this finite interval is always \( B = 0 \). This is expected, since the periodic configuration built with boundaries \((34)\) represents an array of baby-Skyrmion and anti-baby-Skyrmion as discussed in \([25]\).

As a further check for the solutions \((80)\), \((81)\), we verified explicitly that they can be expressed in terms of the HeunG functions presented above upon setting \( g = \sqrt{E_0/2} \). The special choice of the coupling constant reduces \((69), (70)\) to this surprisingly simple solution.

### 4.3 Energy and currents

The energy density \( \rho_s \) for the baby skyrmion is given by the skyrmionic component of energy tensor \( T^{\mu\nu} \)

\[
\rho_s = T^{00} = T_{b0} = a_1 \left( \nabla_0 F \nabla_0 F + \sin^2 F \left( \nabla_0 G \right) \left( \nabla_0 G \right) \right) - a_2 \left\{ \sin^2 F \left( \nabla_0 F \nabla_0 F \right) \left( \nabla_0 G \nabla_0 G \right) + \left( \nabla_0 F \nabla_0 F \right) \left( \nabla_0 G \nabla_0 G \right) \right\} - 2(\nabla_0 F \nabla_0 G)(\nabla_0 F \nabla_0 G) \right\} \\
g_{00} \left\{ \frac{a_1}{2} \left[ \nabla_0 G \nabla_0 G + \sin^2 F \left( \nabla_0 G \nabla_0 G \right) \right] - \frac{a_2}{2} \left[ \sin^2 F \left( \nabla_0 F \nabla_0 F \right) \left( \nabla_0 G \nabla_0 G \right) - \left( \nabla_0 F \nabla_0 G \right)^2 \right] \right\} \\
+ \kappa_0 \left( \frac{1 - \Phi \cdot \vec{n}}{2} \right) \right\}.
\]

With the ansatz given by \((61), (62)\), the baby skyrmion energy density reads

\[
\rho_s = a_1 \left( p^2 \sin^2 F \frac{L}{2} \left( 1 + \frac{2a_2 E_0}{a_1} \right) + 2E_0 \right).
\]

The fermion energy \( E_f \) defined in \((45)\) with the ansatz \((38)\) reads instead

\[
E_f = \int \left[ \Psi^* H \Psi \right] dx L d\varphi = \int \left[ \frac{2(p + l)}{L} v_2 v_2 + \frac{2l}{L} u_1 u_1 \right] dx L d\varphi
\]

For what concerns the conserved current, we can write the explicit form of the skyrmion component of the global \( J_{U(1)}^\mu \) current defined in \((21)\) that results

\[
J_s^0 = \frac{qa_1 p}{L} \sin^2 F \left( 1 + \frac{2a_2 E_0}{a_1} \right) \\
J_s^1 = J_s^x = 0 \\
J_s^2 = J_s^\varphi = \frac{qa_1 p}{L^2} \sin^2 F \left( 1 + \frac{2a_2 E_0}{a_1} \right).
\]

On the other hand, given the decomposition \((52)\), the probability density \( \rho_q \) of finding a fermion with global charge \( Q = q \) with spin up or down is

\[
\rho_q = \frac{2v_2 v_2}{N_i^2}.
\]
Finally, we write explicitly the form of the fermion probability density and the probability density current defined in (27), (28) and (29)

\[ J^0_F = \rho_F = 2 (u_1 u_1^* + v_2 v_2^*) \] (95)
\[ J^x_F = J^\varphi_F = 0 \] (96)
\[ J^2_F = J^\varphi_F = \frac{2}{L} (u_1 u_1^* + v_2 v_2^*) \] (97)

As it is clear from above, the fermionic probability density current wraps the cylinder around the \( \varphi \)-direction, much like a solenoid.

At the end of this section, we finally show the plots of some relevant physical quantities for different examples of solutions. In particular, in figures (5) and (6) we consider the case of a finite cylindrical box and in figure (7) the periodic cell of the periodic solution defined in section 4.2. In figure (7) it becomes evident the difference between the solution studied in this work and the cases considered in [17]. In [17], the use of axial symmetric baby skyrmions leads to an axial localized fermion density on the plane. In our case instead, the periodic crystal-like form of the baby skyrmion field developed in [25] leads to a constant fermion probability density on the plane and a constant fermion probability current along \( y \).

\[ (a) \rho_B(x) \quad (b) \rho_s(x) \quad (c) J^\varphi_s(x) \]
\[ (d) \rho_F(x) \quad (e) J^\varphi_F(x) \quad (f) \rho_q(x) \]

Figure 5: Plots for the baby skyrmion (blue) and fermion (red) solution with parameters \( g = L = 1, a_1 = 1, a_2 = 1, p = 1, l = 0, q = 1, n = 0, c_1 = 1, c_2 = 0 \) in the finite interval \( x \in (-L/2, L/2) \). In order: (a) the topological density charge \( \rho_B \), (b) the baby skyrmion energy density \( \rho_s \), (c) the \( \varphi \)-component of the skyrmion global \( U(1) \) current \( J^\varphi_s \), (d) the fermion probability density \( \rho_F = J^0_F \), (e) the \( \varphi \)-component of the fermion probability density current \( J^\varphi_F \), (f) the probability density \( \rho_q \).

4.4 Higher winding numbers

We conclude this section by reporting on some cases where \( p > 1 \) and \( n \neq 0 \). Note that the Baryon charge is completely contained in the winding along the \( \varphi \)-direction, this means that the profiles in the \( x \) direction do not depend on the value of this winding explicitly. Therefore, all the profile functions remain unchanged for the case of higher Baryon number while the energies and the currents depend
on it explicitly as shown in section 4.3. We show some illustrative plots for higher Baryon charge $p$ in figure 8. We further have freedom in changing the additional "winding" denoted by the integer $n$ of eq. (62). In this case, the interval of analyticity of our solution due to eq. (71) changes. This interval is defined as

\[-\frac{\pi}{4} < \tilde{x} < \frac{\pi}{4} \iff -\frac{\sqrt{2}\pi}{4\sqrt{E_0}} < x < \frac{\sqrt{2}\pi}{4\sqrt{E_0}} \text{ with } E_0 = \frac{\pi^2}{2} \left( \frac{1 + 4n}{L} \right)^2 \]  

(98)

and then it depends on $n$. In figure 9 we show the plots of the fermion profiles and other physical quantities along $x$ for the case of $n = 1$ in the finite interval $x \in (-L/10, L/10)$. Otherwise, in figure 10, we show the fermion profiles for different values of $n$ in the fixed range $\tilde{x} \in (-\pi/4, \pi/4)$.

5 Large coupling limit

Having analytical solutions we are in a position of being able to extend these to large values of the coupling parameter $g$. Theoretically, quantum perturbations of the baby-Skyrme profile are suppressed by $1/M_S$, where $M_S$ is the mass scale of the soliton solution (or just the soliton energy). For our case $M_S \propto a_1$ and therefore we can safely take large $g$ assuming that $g/a_1$ remains small. This result ties well with the general physical intuition that for large background soliton solutions (with large values of $M_S$) quantum perturbations can be safely neglected. Numerically, taking large $g$ values is a hard task as extreme values of the parameters usually involve loss of convergence or the requirement of extreme precision and hence large hardware requirements or computing times. Moreover, we want to emphasize that the peculiar ansatz developed in this work leads to a system in which only the fermion profiles are modified by the choice of $g$. As shown in eqs. (53), (54) and (55), due to the special relation among $v_1, v_2, u_1$ and $u_2$, the equation for the skyrme profile is decoupled from the
fermions and then for every choice of $g$ the baby skyrmion represents a rigid background field. This solution differs from the case described in [17] in which the increase of the coupling $g$ leads to a strong deformation of the baby skyrmion solution. In figures 11 and 12, we show the plots for the normalized profile functions for the fermions and for other physical quantities for different values of large $g$ in the finite interval $-L/2 < x < L/2$. From these plots, we see that independently from the choice of the coefficients $c_1, c_2$ in the solutions (69) and (70) at large $g$ the fermionic probability density is concentrated at the border of the box where the topological charge density is low. We can gain some analytic understanding of this behaviour studying the equations in the large $g$ limit. In this limit, we have that the potential $V_{\text{eff}}$ in (60) (with $m = 0$ and $\kappa_0 = 0$) tends to a potential well in which $V_{\text{eff}} \sim g^2$ inside. A plot of $V_{\text{eff}}$ is shown in figure 13. Then, the Schrodinger-like equation for $K(x)$, given in equation (59), near the origin gives simply

$$K(x) \sim \hat{c}_1(e^{gx} + e^{-gx}) + \hat{c}_2(e^{gx} - e^{-gx}),$$

where we separate the odd and even components with the arbitrary constants $\hat{c}_1, \hat{c}_2$ in the same way of (66). We conjecture that these constants are respectively related with $c_1$ and $c_2$. Therefore, since $F \sim \pi/2$ close to $x = 0$, we obtain from (69), (70) in the large $g$ limit

$$v_2^{\text{approx}}(x) \sim \hat{c}_1(e^{gx} + e^{-gx}) + \hat{c}_2(e^{gx} - e^{-gx})$$

$$u_1^{\text{approx}}(x) \sim (e^{gx} + e^{-gx})(\hat{c}_2 - \hat{c}_1 \sin(\sqrt{2E_0}x)) + (e^{gx} - e^{-gx})(\hat{c}_1 - \hat{c}_2 \sin(\sqrt{2E_0}x)).$$

Comparing qualitatively the approximate solutions with the exact ones close to the $x = 0$ origin, we confirm the quality of the approximation. In figures 14, 15 we show the plots of this comparison for
different values of the coupling $g$ in the small interval $x \in [-L/10, L/10]$. The form of the approximates solutions indicates that the magnitude of the fermion profile increases significantly at large $g$ going from the center to the borders of the finite interval (it goes like $\exp\{gx\}$). Therefore, the fermion probability density ($\rho_F \sim v_F^2 + u_F^2$) has the same behaviour as shown in figures 11 and 12.

For what concerns the periodic solution defined on the plane $\mathbb{R}^2$ in section 4.2, the relation (78) locks the coupling constant $g$ with the parameter $E_0$ that defines the baby skyrmion profile (79). In the periodic case, therefore, the increase of $g$ modifies both the skyrmion and fermion fields meaning that the period of the solution decreases (82), (83). In figure 16 we show the plots for the periodic solution with $g = 2$ in the same $(x, y)$-range of figure 7 (where $g = 1$). As we see, the period of this new solution is a half of the first one.

6 Conclusions

In this paper we have shown that the baby-Skyrmion system coupled to fermions has a branch of analytic solutions with non-vanishing Baryon number valid at all values of the coupling parameter $g$. We first built these analytic solutions on a finite cylinder of arbitrary length and, in the special case of a particular choice of coupling, we found a periodic solution on the infinite plane $\mathbb{R}^2$. The fermion solutions are not eigenvalues of the Hamiltonian but are still normalizable solutions with fi-
Figure 9: Plots for the baby skyrmion (blue) and fermion (red) solution with \( n = 1 \) and \( g = L = 1 \), \( a_1 = 1 \), \( a_2 = 1 \), \( p = 1 \), \( l = 0 \), \( q = 1 \), \( c_1 = 1 \), \( c_2 = 0 \) in the finite interval \( x \in (-L/10, L/10) \). In order: (a) the fermionic solution \( v_2(x) \), (b) the fermionic solution \( u_1(x) \), (c) the topological density charge \( \rho_B(x) \), (d) the baby skyrmion energy density \( \rho_s(x) \), (e) the \( \varphi \)-component of the skyrmion global \( U(1) \) current \( J_\mu^s \), (f) the fermion probability density \( \rho_F(x) = J_0^F \), (g) the \( \varphi \)-component of the fermion probability density current \( J_\mu^F \), (h) the probability density \( \rho_q \).

Finite mean-value of the energy and therefore physical. They describe the superposition of particular fermion states, as described in detail in section 3. We showed that these solutions reveal interesting insights into the limit of strong coupling. A natural line of further investigation is the coupling of these solutions to an electromagnetic field where one expects interesting novel physics arising from the coupling of the fermions to the background electromagnetic fields. Furthermore, without the necessity of adding additional fields, more solutions of the system we presented can be investigated by considering \( \kappa \neq 0 \) in the potential (60). This system however should be analysed numerically.

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Figure 10: Fermion profiles varying $n$, for $g = L = 1$, values of $n$ shown in legend.
Figure 11: Plots for the baby skyrmion and fermion solutions with parameters $g = \{10, 20, 30, 50\}$, $L = 1$, $a_1 = 100$, $a_2 = 1$, $p = 1$, $l = 0$, $q = 1$, $n = 0$, $c_1 = 1$, $c_2 = 0$ in the finite interval $x \in (-L/2, L/2)$. In order: (a) the normalized fermionic solution $v_2/N_i$, (b) the normalized fermionic solution $u_1/N_i$, (c) the topological density charge $\rho_B$, (d) the baby skyrmion energy density $\rho_s$, (e) the $\varphi$–component of the skyrmion global $U(1)$ current $J_\mu^s$, (f) the fermion probability density $\rho_F = J_\mu^0$, (g) the $\varphi$–component of the fermion probability density current $J_\mu^\varphi_F$, (h) the probability density $\rho_q$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure11}
\caption{Plots for the baby skyrmion and fermion solutions with parameters $g = \{10, 20, 30, 50\}$, $L = 1$, $a_1 = 100$, $a_2 = 1$, $p = 1$, $l = 0$, $q = 1$, $n = 0$, $c_1 = 1$, $c_2 = 0$ in the finite interval $x \in (-L/2, L/2)$. In order: (a) the normalized fermionic solution $v_2/N_i$, (b) the normalized fermionic solution $u_1/N_i$, (c) the topological density charge $\rho_B$, (d) the baby skyrmion energy density $\rho_s$, (e) the $\varphi$–component of the skyrmion global $U(1)$ current $J_\mu^s$, (f) the fermion probability density $\rho_F = J_\mu^0$, (g) the $\varphi$–component of the fermion probability density current $J_\mu^\varphi_F$, (h) the probability density $\rho_q$.}\end{figure}
Figure 12: Plots for the baby skyrmion and fermion solutions with parameters $g = \{10, 20, 30, 50\}$, $L = 1$, $a_1 = 100$, $a_2 = 1$, $l = 0$, $q = 1$, $n = 0$, $c_1 = 0$, $c_2 = 1$ in the finite interval $x \in (-L/2, L/2)$. In order: (a) the normalized fermionic solution $v_2/N_i$, (b) the normalized fermionic solution $u_1/N_i$, (c) the topological density charge $\rho_B$, (d) the baby skyrmion energy density $\rho_s$, (e) the $\varphi$–component of the skyrmion global $U(1)$ current $J_\mu^s$, (f) the fermion probability density $\rho_F = J_\mu^F$, (g) the $\varphi$–component of the fermion probability density current $J_\mu^F$, (h) the probability density $\rho_q$.

Figure 13: Plot of $V_{eff}$ for different values of coupling constants $g = \{10, 20, 30, 50\}$ and $L = 1$, $a_1 = 1$, $n = 0$ in the finite interval $x \in [-L/2, L/2]$. 
Figure 14: Plots of the approximate solutions \( v_2^{\text{approx}} \), \( u_1^{\text{approx}} \) compared with the exact solutions \( v_2 \) and \( u_1 \) for different coupling constants \( g = \{50, 60, 70\} \) and \( \hat{c}_1 = 0.2, \hat{c}_2 = 0, c_1 = 1, c_2 = 0, L = 1, a_1 = 1, a_2 = 1, p = 1, l = 0, q = 1, n = 0 \) in the finite interval near the origin \( x \in \left[-L/10, L/10\right] \).

Figure 15: Plots of the approximate solutions \( v_2^{\text{approx}} \), \( u_1^{\text{approx}} \) compared with the exact solutions \( v_2 \) and \( u_1 \) for different coupling constants \( g = \{50, 60, 70\} \) and \( \hat{c}_1 = 0, \hat{c}_2 = 0 \) and \( c_1 = 0, c_2 = 1, L = 1, a_1 = 1, a_2 = 1, p = 1, l = 0, q = 1, n = 0 \) in the finite interval near the origin \( x \in \left[-L/10, L/10\right] \).
Figure 16: Plots for the periodic baby skyrmion (blue) and fermion (red) solution for $g = 2$ ($E_0 = 8$) in the cell of dimensions $[0, 2\pi] \times [0, 2\pi L]$ with parameters $A = 1$, $a_1 = 1$, $a_2 = 1$, $p = 1$, $l = 0$, $q = 1$ and $L = 1$. In order: (a) the topological density charge $\rho_B$, (b) the baby skyrmion energy density $\rho_s$, (c) the $y$–component of the skyrmion global $U(1)$ current $J^y$, (d) the fermion probability density $\rho_F = J^y_F$, (e) the $y$–component of the fermion probability density current $J^y_F$, (f) the probability density $\rho_q$. 
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