CONVERGENCE OF A NUMERICAL SCHEME FOR THE
HAMILTON-JACOBI EQUATION:
A NEW APPROACH WITH THE ADJOINT METHOD

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Abstract. We consider a numerical scheme for the one dimensional time dependent Hamilton-Jacobi equation in the periodic setting. We present a new and simple proof of the rate of convergence of the approximations based on the adjoint method recently introduced by Evans.

1. Introduction

We consider here a numerical scheme and provide an error estimate for the numerical approximation of viscosity solutions of one dimensional time dependent Hamilton-Jacobi equation in the periodic setting:

\[
\begin{aligned}
  &u_t + H(u_x) = 0, & \text{in } T \times (0, \infty), \\
  &u = u_0, & \text{on } T \times \{ t = 0 \}.
\end{aligned}
\]

(1.1)

Here \( H : \mathbb{R} \to \mathbb{R} \) is a smooth, convex and coercive Hamiltonian, \( u_0 : T \to \mathbb{R} \) is a given smooth function, and \( T \) is the one dimensional torus identified, when convenient, with the interval \([0, 1]\).

Several authors investigated equation (1.1), and a number of results are available in literature (see [CL84, Son85, BS91, BCD97, DJ98, Kry00, BJ02, FF02, BJ05, Jak06, Obe06, JKLC08, CCDG08, Obe10], to name just a few).

The aim of this note is to take a first step on a new approach to this problem, using the adjoint method recently introduced by Evans (see [Eva10], and also [Tra, CG Tb, ES, CGTa]). Indeed, we will show how it is possible to recover some results, which are already well-known in literature, with a new and easy proof.

Though we consider only the one dimensional setting, most of the results can be extended without major changes to higher dimensions, with the exception of Section 4.2 where the argument we use is indeed one dimensional. Note however, that the main result, does not depend on the estimates of Section 4.2. To focus on the main ideas of our approach, we try to keep the formulation as simple as possible, while a more detailed study will be the subject of a forthcoming paper.

We consider a function \( F : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) with the following properties:

(F1) \( F \) is convex;

(F2) \( F(\cdot, q) \) is increasing for each \( q \in \mathbb{R} \) and \( F(p, \cdot) \) is increasing for each \( p \in \mathbb{R} \);
(F3) $F(-p, p) = H(p)$ for every $p \in \mathbb{R}$.

Such a function appears naturally. Indeed, if for instance $H(0) = 0 = \min_{p \in \mathbb{R}} H(p)$, then $F$ can be chosen as follows. Setting

$$F_1(p) := \begin{cases} 0 & p \leq 0, \\ H(-p) & p > 0, \end{cases}$$

$$F_2(q) := \begin{cases} 0 & q \leq 0, \\ H(q) & q > 0, \end{cases}$$

and $F(p, q) := F_1(p) + F_2(q)$ for $(p, q) \in \mathbb{R}^2$, properties (F1)–(F3) are satisfied.

At this point, for every $h > 0$ we introduce the solution $u^h : \mathbb{T} \times [0, \infty) \to \mathbb{R}$ to:

$$u^h_t + F(-\delta_h u^h, \delta_{-h} u^h) = 0, \quad \text{in } \mathbb{T} \times (0, \infty),$$

$$u = u_0, \quad \text{on } \mathbb{T} \times \{t = 0\},$$

where for every function $v : \mathbb{T} \to \mathbb{R}$ we set

$$\delta_h v(x) := \frac{v^h(x + h) - v^h(x)}{h}, \quad x \in \mathbb{T}.$$

Existence and uniqueness of $u^h$ can be easily proven (see the Appendix).

We point out that (1.2) is not a standard approximation for equation (1.1), for several reasons. First, we are not discretizing in the time variable. Also, note that the function $u^h$ is defined in all the torus $\mathbb{T}$, and not only in grid points. Finally, $h$ can take any value in $\mathbb{R}$. This gives us the advantage that we can differentiate $u^h$ with respect to the grid size without any technical problem.

We state now our main result.

**Theorem 1.1.** For every $T \in (0, \infty)$ there exists a positive constant $C = C(T)$, independent of $h$, such that

$$\sup_{t \in [0, T]} \|u(\cdot, t) - u^h(\cdot, t)\|_{L^\infty(\mathbb{T})} \leq C \sqrt{h}.$$  (1.3)

As already mentioned, inequality (1.3) is not new in literature and appeared, for instance, in the seminal paper [CL84], where Crandall and Lions studied Hamilton–Jacobi equation for coercive (not necessarily convex) Hamiltonians. The function $F$ they considered is given by

$$F(p, q) = H\left(\frac{q - p}{2}\right) + \gamma(p + q),$$

where $\gamma$ is a positive constant chosen in such a way that $|H'(p)| \leq 2\gamma$ for $|p| \leq R$, with $R > 0$ playing the role of an a priori bound on $|u_x|$. Note that, under this assumption, conditions (F2)–(F3) are satisfied, and (1.2) reads as

$$u^h_t + H\left(\frac{u^h(x + h, t) - u^h(x - h, t)}{2h}\right) = \gamma h \Delta_h u^h,$$  (1.5)
where for every function $v : T \rightarrow \mathbb{R}$ we set
\[ \Delta_h v(x) := \frac{v(x + h) - 2v(x) + v(x - h)}{h^2}, \quad x \in T. \]

Equation (1.5) is the analog to the usual regularized Hamilton–Jacobi equation $u_t + H(Du) = \varepsilon \Delta u$ (see also Crandall and Majda [CM80], and Souganidis [Sou85]), with the additional fact that the viscosity term vanishes as the grid size goes to zero.

Let us now briefly comment on the main ingredient of the present paper, that is how we prove Theorem 1.1. We start by linearizing (1.2), and then we consider the adjoint of the equation obtained, with final datum a Dirac delta (see (4.3)). It turns out that the solution of this last equation is a probability measure at every time (see Proposition 4.2), and satisfies a useful identity (see Proposition 4.3). Using these properties we are able to prove the necessary estimates.

We conclude by observing that, for technical reasons, at the moment we are not able to remove the convexity assumption on $H$ (see Remark 4.10).

The paper is organized as follows. Section 2 contains some preliminary observations, concerning finite difference quotients. In Section 3 we face the linear case, while a general convex Hamiltonian is the object of Section 4. The special case $H(p) = p^2/2$ is considered in Section 5. Finally, details about existence, uniqueness, and smoothness of the solution $u^h$ of (1.2) are given in the Appendix.

### 2. A few facts about finite difference quotients

For the convenience of the reader, we recall in this section a few facts about calculus with finite differences, whose proofs are elementary.

**Lemma 2.1.** Let $u, w : T \rightarrow \mathbb{R}$, and let $h \in \mathbb{R}$. Then, for every $x \in T$

\[ \delta_h w(x - h) = \delta_{-h} w(x); \] (2.1)

\[ \delta_{-h} [\delta_h w] (x) = \delta_h [\delta_{-h} w] (x) = \Delta_h w(x), \] (2.2)

\[ \delta_h^2 w(x) = \Delta_h w(x + h), \] (2.3)

\[ [\delta_h (vw)] (x) = v(x + h) \delta_h w(x) + w(x) \delta_h v(x) \] (2.4)

\[ \delta_h [w^2 (x)] = 2w(x) \delta_h w(x) + h [\delta_h w(x)]^2 \] (2.5)

\[ \Delta_h [w^2 (x)] = 2w(x) \Delta_h w(x) + (\delta_h w(x))^2 + (\delta_{-h} w(x))^2 \] (2.6)

The following lemma gives a discrete version of integration by parts.

**Lemma 2.2.** Let $v, w \in L^2(T)$ and let $h \in \mathbb{R}$. Then

\[ \int_T w \delta_h v \, dx = - \int_T v \delta_{-h} w \, dx. \]
We also recall the following formula

**Lemma 2.3.** Let \( v, w \in L^2(\mathbb{R}) \) and let \( h \in \mathbb{R} \). Then

\[
\int_{\mathbb{T}} \delta_h v \delta_h w \, dx = -\int_{\mathbb{T}} w \Delta_h v \, dx.
\]

3. **Toy model: linear case**

In this section, in order to motivate our approach, we consider the case of a linear Hamiltonian. To make the formulation easier, we focus on the stationary case, that is, we consider the equation

\[
u + u_x = f \quad \text{in} \quad \mathbb{T},
\]

where \( f \in C^2(\mathbb{T}) \), with \( |f_{xx}| \leq C \). We are going to show that, even in this simple case, one should be careful on the choice of the discretized equation, in order to obtain the desired properties for the solution of the adjoint equation. In particular, we will show that the correct discretized equation depends on the sign of \( h \). This phenomenon is known in literature as *upwinding effect*. By a straightforward calculation it is easy to see that

\[
u(x) = e^{-x}u(0) + \int_0^x f(s)e^{s-x} \, ds,
\]

where \( u(0) \) is uniquely determined by

\[
u(0) = \frac{\int_0^1 f(s)e^s \, ds}{e - 1}.
\]

3.1. **The case of \( h < 0 \).** We focus now on the case \( h < 0 \), and consider the following approximation of (3.1):

\[
u^h(x) + \delta_h \nu^h(x) = f(x).
\]

We observe that existence, uniqueness, and smoothness of \( \nu^h \) can be proved in a direct way. Indeed, straightforward calculations show that

\[
u^h(x) = \frac{-h}{1 - h} \sum_{n=0}^{\infty} \frac{1}{(1 - h)^n} f(x + nh), \quad \text{for} \ h < 0.
\]

This proves that \( \nu^h \in C^2(\mathbb{T}) \) and \( \frac{\partial \nu^h}{\partial h} \in C^1(\mathbb{T}) \).

**Remark 3.1.** One could consider equation (3.2) also for \( h > 0 \). Then, although formula (3.3) doesn’t hold anymore, due to convergence issues, a direct computation shows that for every \( N \in \mathbb{N} \)

\[
u^h(x) = \frac{-h}{1 - h} \sum_{n=0}^{N} \frac{1}{(1 - h)^n} f(x + nh) + \frac{1}{(1 - h)^{N+1}} u(x + (N + 1)h),
\]
provided a solution \( u^h \) exists. In particular, when \( h \) is rational, say \( h = p/q \) for some \( p, q \in \mathbb{N} \),
the formula above for \( N = q - 1 \) gives

\[
u^h(x) = \frac{-h(1-h)^{q-1}}{(1-h)^q - 1} \sum_{n=0}^{q-1} \frac{1}{(1-h)^n} f(x + nh).
\]

Since (3.2) is linear, we can directly pass to its adjoint equation. For every \( h < 0 \) and \( x_0 \in \mathbb{T} \)
we define \( \sigma^{h,x_0} \) as the solution of

\[
\sigma^{h,x_0} - \delta_{-h} \sigma^{h,x_0} = \delta_{x_0},
\]

(3.4)

where \( \delta_{x_0} \) denotes the Dirac delta measure concentrated in \( x_0 \), and \( x_0 \in \mathbb{T} \).

Let us show that \( \sigma^{h,x_0} \) is a probability measure, for every \( h < 0 \) and every \( x_0 \in \mathbb{T} \).

**Proposition 3.2.** For every choice of \( h < 0 \) and \( x_0 \in \mathbb{T} \), \( \sigma^{h,x_0} \) is a probability measure on \( \mathbb{T} \).

**Proof.** We denote with \( v^{h,F} \) the solution of the adjoint of equation (3.4):

\[
v^{h,F}(x) + \delta_h v^{h,F}(x) = F(x),
\]

(3.5)

with \( F \in C^\infty(\mathbb{T}) \).

First of all, observe that

\[
F \geq 0 \implies v^{h,F} \geq 0.
\]

(3.6)

Indeed, let \( \overline{x} \in \mathbb{T} \) be such that \( v^{h,F}(\overline{x}) = \min_{x \in \mathbb{T}} v^{h,F}(x) \). Then, evaluating equation (3.5) at \( x = \overline{x} \) we have

\[
\min_{x \in \mathbb{T}} v^{h,F}(x) = v^{h,F}(\overline{x}) = F(\overline{x}) - \frac{v^{h,F}(\overline{x} + h) - v^{h,F}(\overline{x})}{h} \geq 0,
\]

(3.7)

since \( h < 0 \). Now, multiplying equation (3.4) by the solution of (3.5), integrating over \( \mathbb{T} \) and using formula (3.8):

\[
v(x_0) = \int_{\mathbb{T}} v^{h,x_0} dx - \int_{\mathbb{T}} v^{h} \delta_h v^{h,x_0} dx = \int_{\mathbb{T}} (v + \delta_h v) \sigma^{h,x_0} dx = \int_{\mathbb{T}} F \sigma^{h,x_0} dx.
\]

In particular, thanks to (3.6), we have

\[
\int_{\mathbb{T}} F \sigma^{h,x_0} dx \geq 0, \quad \text{for every } F \geq 0,
\]

which implies that \( \sigma^{h,x_0} \) is nonnegative. To prove that \( \sigma^{h,x_0} \) has total mass 1, integrate (3.4) over \( \mathbb{T} \), to get

\[
1 = \int_{\mathbb{T}} \sigma^{h,x_0} dx - \int_{\mathbb{T}} \delta_h \sigma^{h,x_0} dx = \int_{\mathbb{T}} \sigma^{h,x_0} dx.
\]

\( \square \)

We prove now a useful formula.
Proposition 3.3. Let $h < 0$ and $x_0 \in \mathbb{T}$. Then, for every $g \in L^\infty(\mathbb{T})$

$$g(x_0) = \int_{\mathbb{T}} (g + \delta_h g) \sigma^{h,x_0} \, dx. \quad (3.8)$$

Proof. The proof simply follows by multiplying (3.4) by $g$, integrating over $\mathbb{T}$ and using (2.2). $\Box$

Proposition 3.2 motivates the choice of $h < 0$ in the remaining part of this section, since this was essential in the proof of inequality (3.7).

Let now $\mathfrak{T} \in \mathbb{T}$. Differentiating equation (3.2) w.r.t. $x$ we have

$$u^h_x(x) + \delta_h u^h_x(x) = f_x(x). \quad (3.9)$$

Then, thanks to (3.8)

$$u^h_x(\mathfrak{T}) = \int_{\mathbb{T}} (u^h_x + \delta_h u^h_x) \sigma^{h,\mathfrak{T}} \, dx = \int_{\mathbb{T}} f_x \sigma^{h,\mathfrak{T}} \, dx. \quad (3.10)$$

In the same way, applying operator $\delta_h$ to equation (3.2) and using once again (3.8)

$$\delta_h u^h(x) = \int_{\mathbb{T}} (\delta_h u^h + \delta^2_h u^h) \sigma^{h,\mathfrak{T}} \, dx = \int_{\mathbb{T}} \delta_h f \sigma^{h,\mathfrak{T}} \, dx. \quad (3.11)$$

Subtracting (3.10) from (3.11) we have

$$\delta_h u^h(\mathfrak{T}) - u^h_x(\mathfrak{T}) = \int_{\mathbb{T}} [\delta_h f - f_x] \sigma^{h,\mathfrak{T}} \, dx. \quad (3.12)$$

Also, applying operator $\delta_h$ to equation (3.9):

$$\delta_h u^h_x(x) + \delta^2_h u^h_x(x) = \delta_h f_x(x).$$

Thus, using relation (3.3) with $g = \delta_h u^h_x$:

$$\delta_h u^h_x(\mathfrak{T}) = \int_{\mathbb{T}} [\delta_h u^h_x(x) + \delta^2_h u^h_x(x)] \sigma^{h,\mathfrak{T}} \, dx = \int_{\mathbb{T}} \delta_h f_x(x) \sigma^{h,\mathfrak{T}} \, dx.$$

From last relation we infer that

$$u^h_x(\mathfrak{T} + h) = u^h_x(\mathfrak{T}) + h \int_{\mathbb{T}} \delta_h f_x(x) \sigma^{h,\mathfrak{T}} \, dx. \quad (3.13)$$

Let’s finally pass to the estimate of $\frac{\partial u^h}{\partial h}$.

Theorem 3.4. There exists $C > 0$, independent of $h$, such that

$$\left| \frac{\partial u^h}{\partial h}(x) \right| \leq C, \quad \text{for } x \in \mathbb{T}.$$  

In particular

$$\| u^h - u \|_{L^\infty(\mathbb{T})} \leq Ch.$$
Proof. Differentiating (3.2) w.r.t. \( h \) we have
\[
\frac{\partial u^h}{\partial h} + \delta_h \left( \frac{\partial u^h}{\partial h} \right) + \frac{1}{h} u^h_x |_{x+h} - \frac{1}{h} \delta_h u^h = 0. \tag{3.14}
\]
Let \( x_1 \in \mathbb{T} \) be such that
\[
\left| \frac{\partial u^h}{\partial h}(x_1) \right| = \max_{x \in \mathbb{T}} \left| \frac{\partial u^h}{\partial h}(x) \right|.
\]
Then, applying formula (3.8) with \( x_0 = x_1 \), thanks to (3.12), (3.13) and (3.13)
\[
\frac{\partial u^h}{\partial h}(x_1) = \int_\mathbb{T} \left[ \frac{\partial u^h}{\partial h} + \delta_h \left( \frac{\partial u^h}{\partial h} \right) \right] \sigma^{h,x_1} dx = \frac{1}{h} \int_\mathbb{T} [\delta_h u^h - u^h_x |_{x+h}] \sigma^{h,x_1} dx
\]
\[
= \frac{1}{h} \int_\mathbb{T} \left[ [\delta_h f(y) - f_x(y)] \sigma^{h,x} dy \right] \sigma^{h,x_1} dx - \int_\mathbb{T} \left[ \int_\mathbb{T} \delta_h f(y) \sigma^{h,x} dy \right] \sigma^{h,x_1} dx \leq C \frac{1}{h} \int_\mathbb{T} \left[ \int_\mathbb{T} \sigma^{h,x} dy \right] \sigma^{h,x_1} dx + C \int_\mathbb{T} \left[ \int_\mathbb{T} \sigma^{h,x} dy \right] \sigma^{h,x_1} dx \leq 2C,
\]
where we used the fact that
\[
\delta_h f(y) - f_x(y) \leq \sup_{x \in \mathbb{T}} |f_{xx}(z)| \leq C, \quad \delta_h f_x(y) \leq \sup_{x \in \mathbb{T}} |f_{xx}(z)| \leq C.
\]
Similarly, we have \( \frac{\partial u^h}{\partial h}(x_1) \geq -C \), which completes the proof. \( \square \)

3.2. The case \( h > 0 \). One can see that the correct discretized equation when \( h > 0 \) is given by
\[
u^h(x) + \delta_{-h} u^h(x) = f(x).
\]
Indeed, by repeating what was done in the previous subsection it is possible to show that also in this case the solution of the corresponding adjoint equation is a probability measure, and to obtain a result similar to the one in Theorem (3.4).

4. General Case

For every \( h > 0 \), we consider the following equation:
\[
\begin{cases}
\nu^h_t + F(-\delta_h u^h, \delta_{-h} u^h) = 0, & \text{in } \mathbb{T} \times (0, \infty),
\nu^h = u_0, & \text{on } \mathbb{T} \times \{t = 0\}.
\end{cases}
\tag{4.1}
\]
Next proposition, whose proof can be found in the Appendix, shows that existence and uniqueness of a smooth solution of the above equation are guaranteed.

**Proposition 4.1.** Let \( h > 0 \), and assume that \( F \in C^2(\mathbb{R}^2) \) and \( u_0 \in C^2(\mathbb{T}) \). Then, there exists a unique solution \( u^h \) to (4.1). Moreover, we have \( u^h, u^h_t, u^h_{xx} \in C(\mathbb{T} \times [0, \infty)) \) and
\[
u^h(x, \cdot), u^h_t(x, \cdot), u^h_{xx}(x, \cdot) \in C^1([0, \infty)) \quad \text{for every } x \in \mathbb{T}.
\]
We pass now to the Adjoint Method.

4.1. **Adjoint Method.** In order to apply the Adjoint Method, we consider the formal linearized operator $L^h$ corresponding to equation (4.1):

$$v \mapsto L^h v = v_t - D_p F(\delta_h v) + D_q F(\delta_-h v),$$

(4.2)

where $D_p F$ and $D_q F$ are evaluated at $(-\delta_h u^h, \delta_-h u^h)$. For each $h > 0$, $x_0 \in \mathbb{T}$ and $T \in (0, \infty)$ we denote by $\sigma^{h,x_0,T}$ the solution to

$$\begin{cases}
-\sigma_t^{h,x_0,T} + \delta_{-h}(\sigma^{h,x_0,T} D_p F) - \delta_h (\sigma^{h,x_0,T} D_q F) = 0, & \text{in } \mathbb{T} \times [0, T), \\
\sigma^{h,x_0,T} = \delta_{x_0}, & \text{in } \mathbb{T} \times \{t = T\},
\end{cases}$$

(4.3)

**Proposition 4.2 (Properties of $\sigma^{h,x_0,T}$).** Let $h > 0$, $x_0 \in \mathbb{T}$, and $T > 0$. For every $t \in [0, T]$ $\sigma^{h,x_0,T}(\cdot, t)$ is a probability measure on $\mathbb{T}$.

**Proof.** Let us fix $t_2 \in (0, T)$. We will proceed by steps.

**Step 1:** $\sigma^{h,x_0,T}(\cdot, t_2) \geq 0$.

In order to show that $\sigma^{h,x_0,T}(\cdot, t_2)$ is non-negative, for every $f \in C^\infty(\mathbb{T})$ let us denote by $v^{h,f,t_2}$ the solution of the adjoint of the equation (4.3):

$$\begin{cases}
v_t^{h,f,t_2} - D_p F(\delta_h v^{h,f,t_2}) + D_q F(\delta_-h v^{h,f,t_2}) = 0, & \text{in } \mathbb{T} \times (t_2, \infty), \\
v^{h,f,t_2} = f, & \text{on } \mathbb{T} \times \{t = t_2\}.
\end{cases}$$

(4.4)

First of all, observe that

$$f \geq 0 \implies v^{h,f,t_2} \geq 0, \quad \text{in } \mathbb{T} \times [t_2, \infty).$$

(4.5)

Indeed, let $f \geq 0$, and for every $\varepsilon > 0$ set $z^\varepsilon := v^{h,f,t_2} + \varepsilon t$. Using the same argument as in the previous proof, we can show that

$$\min_{(x,t) \in \mathbb{T} \times [t_2, T]} v^{h,f,t_2}(x,t) + \varepsilon T \geq \min_{(x,t) \in \mathbb{T} \times [t_2, T]} z^\varepsilon(x,t) = \min_{x \in \mathbb{T}} z^\varepsilon(x,t_2) = \min_{x \in \mathbb{T}} f(x) + \varepsilon t_2,$$

so that

$$\min_{(x,t) \in \mathbb{T} \times [t_2, T]} v^{h,f,t_2}(x,t) \geq \min_{x \in \mathbb{T}} f(x) - \varepsilon T.$$

Sending $\varepsilon \to 0^+$ claim (4.5) follows.

Let us now multiply equation (4.3) by $v^{h,f,t_2}$ and integrate, to get

$$- \int_{t_2}^T \int_{\mathbb{T}} v^{h,f,t_2} \sigma_t^{h,x_0,T} \, dx \, ds + \int_{t_2}^T \int_{\mathbb{T}} v^{h,f,t_2} \left[ \delta_{-h}(\sigma^{h,x_0,T} D_p F) - \delta_h (\sigma^{h,x_0,T} D_q F) \right] \, dx \, ds = 0.$$

Integrating by parts the first term becomes

$$- \int_{t_2}^T \int_{\mathbb{T}} v^{h,f,t_2} \sigma_t^{h,x_0,T} \, dx \, ds = -v^{h,f,t_2}(x_0,T) + \int_{\mathbb{T}} f(x) \sigma^{h,x_0,T}(x,t_2) \, dx + \int_{t_2}^T \int_{\mathbb{T}} v_t^{h,f,t_2} \sigma^{h,x_0,T} \, dx \, ds.$$
Thanks to (4.5), combining the last two equalities, integrating by parts, and using equation (4.4), we obtain
\[ \int_T \sigma^{h,x_0,T}(x,t_2) \sigma_{h,x_0}^{0,T}(x,t_2) \, dx \geq v_{h,f,t_2}^{h,x_0,T}(x_0,T) \geq 0, \quad \text{for each } f \geq 0, \]
from which we deduce that \( \sigma^{h,x_0,T}(\cdot,t_2) \geq 0. \)

**Step 2:** \( \sigma^{h,x_0,T}(\cdot,t_2) \) has total mass 1.

We integrate (4.3) from \( t_2 \) to \( T \) and over \( T \), to get
\[ 1 - \int_T \sigma^{h,x_0,T}(x,t_2) \, dx = \int_T \sigma_x^{h,x_0,T}(x,s) \, dx \, ds \]
\[ = \int_T \left[ \delta_h(\sigma^{h,x_0,T}D_pF) - \delta_h(\sigma^{h,x_0,T}D_qF) \right] \, dx \, ds = 0, \]
by periodicity. □

The following proposition establishes a useful formula.

**Proposition 4.3.** Let \( h > 0, x_0 \in T \), and \( T \in (0, +\infty) \). Then
\[ \int_0^T \int_T \sigma^{h,x_0,T} L^h \theta \, dx \, dt = \theta(x_0,T) - \int_T \theta(x,0) \sigma^{h,x_0,T}(x,0) \, dx, \]
whenever \( \theta \in C(T \times [0, \infty)) \) is such that \( \theta(x, \cdot) \in C^1([0, \infty)) \) for every \( x \in T \).

**Proof.** Multiplying equation (4.3) by \( \theta \) and integrating by parts, we have
\[ - \left[ \int_T \sigma^{h,x_0,T} \theta \, dx \right]_0^T + \int_0^T \int_T \sigma^{h,x_0,T} L^h \theta \, dx \, dt = 0, \]
and this shows the identity. □

In the next proposition we derive some useful equations.

**Proposition 4.4.** The following equations are satisfied in \( T \times (0, \infty) \):
\[
\begin{align*}
L^h u^h &= 0, \\
L^h u^h_{xx} + D_{pp} F(\delta_h u^h_x)^2 + D_{pq} F(\delta_h u^h_x)^2 + 2D_{pq} F(-\delta_h u^h_x)(\delta_h u^h_x) &= 0, \\
L^h w + \frac{h}{2} D_p F(\delta_h u^h_x)^2 + \frac{h}{2} D_q F(\delta_h u^h_x)^2 &= 0, \\
L^h u^h_x - \frac{1}{h} D_x F \left[ u^h_x |_{x+h} - \delta_h u^h \right] + \frac{1}{h} D_q F \left[ u^h_x |_{x-h} - \delta_h u^h \right] &= 0,
\end{align*}
\]
where \( w = (u^h)^2/2 \) and \( u^h = \partial u^h / \partial h \).
Proof. Equations (4.6) and (4.6) are obtained by differentiating (4.1) w.r.t. $x$ once or twice, respectively. Then, (4.6) follows multiplying (4.6) by $u^h$ and taking into account (2.5). Finally, differentiating (4.1) w.r.t. $h$ we have

$$(u^h)_{t} - D_p F \left[ \delta_h u^h + \frac{1}{h} (u^h_{x+h} - \delta_h u^h) \right] + D_q F \left[ \delta_{-h} u^h + \frac{1}{h} (u^h_{x-h} - \delta_{-h} u^h) \right] = 0,$$

which is (4.6).

□

We show now some a priori bounds which will be used in the proof of the main theorem.

Proposition 4.5. Let $h > 0$. Then, for every $t \in [0, \infty)$

$$\begin{align*}
\| u^h (\cdot, t) \|_{L^\infty(T)} &\leq \|(u_0)_x \|_{L^\infty(T)}, \\
\| u^h_{xx} (\cdot, t) \|_{L^\infty(T)} &\leq \|(u_0)_{xx} \|_{L^\infty(T)}, \\
\| \delta_{\pm h} u^h (\cdot, t) \|_{L^\infty(T)} &\leq \|(u_0)_x \|_{L^\infty(T)}.
\end{align*}$$

In particular,

$$\begin{align*}
(u^h_{x})_{x+h} - \delta_h u^h &\leq h \|(u_0)_{xx} \|_{L^\infty(T)}, \\
- (u^h_{x})_{x-h} - \delta_{-h} u^h &\leq h \|(u_0)_{xx} \|_{L^\infty(T)}, \\
u^h_{x} - \delta_h u^h &\geq -h \|(u_0)_{xx} \|_{L^\infty(T)}.
\end{align*}$$

Remark 4.6. We underline that in the proof of (4.7) and (4.8) we use the convexity assumption on $F$.

Proof. Let $t_1 \in (0, \infty)$, and choose $\hat{x} \in T$ such that

$$w(\hat{x}, t_1) = \max_{x \in T} w(x, t_1).$$

Multiplying (4.6) by $\sigma^h \tau^t_1$ and integrating, using Proposition 4.3

$$0 \geq \int_0^{t_1} \int_T \sigma^h \tau^t_1 L^h w \, dx \, dt \stackrel{u^h}{=} w(\hat{x}, t_1) - \int_T w(x, 0) \sigma^h \tau^t_1 (x, 0) \, dx$$

$$= w(\hat{x}, t_1) - \frac{1}{2} \int_T ((u_0)_x)^2 (x, 0) \sigma^h \tau^t_1 (x, 0) \, dx,$$

where the first inequality follows from the fact that $F$ is increasing in each variable. Since $\sigma^h \tau^t_1 (\cdot, 0)$ is a probability measure, (4.7) follows.

The second estimate is proven in a similar way. Let $t_1 \in (0, \infty)$, and choose $\tilde{x} \in T$ such that

$$u^h_{xx} (\tilde{x}, t_1) = \max_{x \in T} u^h_{xx} (x, t_1).$$
Multiplying equation (4.6) by \( \sigma^h,\bar{x},t_1 \), integrating, and using Proposition 4.3

\[
0 \geq \int_0^{t_1} \int_{\mathbb{T}} \sigma^h,\bar{x},t_1 L^h u^h_{\bar{x}} \, dx \, dt = u^h_{\bar{x}}(\bar{x},t_1) - \int_{\mathbb{T}} u^h_{\bar{x}}(x,0) \sigma^h,\bar{x},t_1(x,0) \, dx
\]

where the first inequality follows from the fact that \( F \) is convex. Last inequality implies (4.7)

Estimate (4.7) easily follows from (4.7).

Observe now that

\[
u^h|_{x+h} - \delta_h u^h = u^h(x+h) - \frac{u^h(x+h) - u^h(x)}{h} \]

for some \( \tau, \eta \in (0,1) \), and this gives (4.8). In a similar way one can prove (4.8) and (4.8). \( \square \)

The next proposition gives an upper bound for \( u^h_h \).

**Proposition 4.7.** There exists a positive constant \( C \) such that

\[
\max_{x \in \mathbb{T}} u^h_h(x,t_1) \leq Ct_1,
\]

for every \( h > 0 \) and \( t_1 \in (0,\infty) \).

**Proof.** Let \( t_1 \in (0,\infty) \) and choose \( \bar{x} \) such that

\[
u^h_h(\bar{x},t_1) = \max_{x \in \mathbb{T}} u^h_h(x,t_1).
\]

Then, multiplying equation (4.6) by \( \sigma^h,\bar{x},t_1 \), integrating, and using Proposition 4.3

\[
u^h_h(\bar{x},t_1) = \int_0^{t_1} \int_{\mathbb{T}} \left[ \frac{1}{h} D_p F \left[ u^h_h |_{x+h} - \delta_h u^h \right] - \frac{1}{h} D_q F \left[ u^h_h |_{x-h} - \delta_h u^h \right] \right] \sigma^h,\bar{x},t_1 \, dx \, dt,
\]

where we used the fact that \( u^h_h(\cdot,0) \equiv 0 \). The equality above, together with (4.7), (4.8) and (4.8), implies

\[
\frac{1}{h} D_p F \left[ u^h_h |_{x+h} - \delta_h u^h \right] - \frac{1}{h} D_q F \left[ u^h_h |_{x-h} - \delta_h u^h \right] \leq C,
\]

for some positive constant \( C \) independent of \( h \), so that the conclusion follows. \( \square \)

**Proposition 4.8.** There exists a positive constant \( C \) such that

\[
\min_{x \in \mathbb{T}} u^h_h(x,t_1) \geq -\frac{1}{\sqrt{h}} C(1+t_1),
\]

for every \( h > 0 \) and \( t_1 \in (0,\infty) \).
Proof. Let $t_1 \in (0, \infty)$ and choose $\overline{x}$ such that

$$u_h^b(\overline{x}, t_1) = \min_{x \in \mathbb{T}} u_h^b(x, t_1).$$

As in the previous proof, we have

$$u_h^b(x, t_1) = \int_0^{t_1} \int_{\mathbb{T}} \left[ \frac{1}{h} D_p F \left[ u^h_x \left| x+h - \delta_h u^h \right| \right] - \frac{1}{h} D_q F \left[ u^h_x \left| x-h - \delta_h u^h \right| \right] \right] \sigma^h, x, t_1 \, dx \, dt.$$

Using Young’s inequality and (4.8)

$$\frac{1}{h} D_p F \left[ u^h_x \left| x+h - \delta_h u^h \right| \right] \geq \frac{1}{2} \frac{D_p F}{\sqrt{h}} - \frac{\sqrt{h}}{2} (D_p F)(\delta_h u^h)^2 - C,$$

(4.9)

In a similar way we obtain

$$- \frac{1}{h} D_q F \left[ u^h_x \left| x-h - \delta_h u^h \right| \right] \geq - \frac{1}{2} \frac{D_q F}{\sqrt{h}} - \frac{\sqrt{h}}{2} (D_q F)(\delta_h u^h)^2 - C.$$

(4.10)

Thus, adding relations (4.9) and (4.10)

$$u_h^b(\overline{x}, t_1) \geq - \frac{1}{2} \frac{D_p F}{\sqrt{h}} - \frac{\sqrt{h}}{2} (D_p F)(\delta_h u^h)^2 - C.$$

(4.11)

The next result is a direct consequence of the previous two propositions and implies Theorem 1.1.

**Proposition 4.9.** There exists a positive constant $C$ such that

$$\|u_h^b(\cdot, t)\|_{L^\infty(\mathbb{T})} \leq \frac{1}{\sqrt{h}} C(1 + t),$$

for every $h > 0$ and $t \in (0, \infty)$.

**Remark 4.10.** To prove (4.11) we used the new inequality

$$h \int_0^{t_1} \int_{\mathbb{T}} \left[ D_p F(\delta_h u^h)^2 + D_q F(\delta_{-h} u^h)^2 \right] \sigma^h, x, t_1 \, dx \, dt \leq C,$$

(4.12)

which can be easily derived by multiplying (4.6) by $\sigma^h, x, t_1$ and integrating by parts. If we choose $F$ as in (1.4), then (4.12) reads as

$$h \int_0^{t_1} \int_{\mathbb{T}} \left[ (\delta_h u^h)^2 + (\delta_{-h} u^h)^2 \right] \sigma^h, x, t_1 \, dx \, dt \leq C,$$

which is the analog of the new and important inequality

$$\varepsilon \int_0^{t_1} \int_{\mathbb{T}} |D^2 u|^2 \sigma \, dx \, dt \leq C,$$
which Evans derived in [Eva10]. Notice that (4.12) and (4.13) hold for general (non convex) coercive Hamiltonians. However, we do not know whether (4.13) is still correct if we replace \( \delta_h u_x^h \) by \( u_{xx}^h \) or by \( \frac{u_x^h - \delta_h u_x^h}{h} \). That is one of the reasons why we have to require the convexity assumption on \( F \) in order to have (4.8), which we use, for instance, in proving (4.9) and (4.10).

**Remark 4.11.** If \( F \) is as in (1.4), and we assume further that \( H \) is uniformly convex, we can improve (4.13). Indeed, let \( \sigma^{h,v,t_1}_t \) be a solution of the adjoint equation

\[
\begin{cases}
-s_t^{h,v,t_1} + \delta_h (\sigma^{h,v,t_1}_t D_y F) - \delta_h (\sigma^{h,v,t_1}_t D_q F) = 0, & \text{in } \mathbb{T} \times [0, t_1], \\
\sigma^{h,v,t_1}_t = v, & \text{on } \mathbb{T} \times \{ t = t_1 \},
\end{cases}
\]

where \( v \) is a probability measure on \( \mathbb{T} \) with a smooth density. Then, multiplying (4.6) by \( \sigma^{h,v,t_1}_t \) and integrating by parts we have

\[
\int_0^{t_1} \int_{\mathbb{T}} [(\delta_h u_x^h)^2 + (\delta_h u_x^h)^2] |\sigma^{h,v,t_1}_t| \, dx \, dt \leq C
\]

(4.14)

for some \( C = C(t_1, v) \). See [Eva10 CGTb] for more applications of inequalities (4.12), (4.13) and (4.14).

4.2. An additional estimate. Let us now choose \( F \) as in (1.4); then equation (1.2) becomes

\[
u_t^h + H \left( \frac{\delta_h u^h + \delta_{-h} u^h}{2} \right) = \gamma h \Delta_h u^h .
\]

We are able to get the following estimate

**Lemma 4.12.** There exists \( C > 0 \), independent of \( h \) and \( T \), such that

\[
h \int_0^T \int_{\mathbb{T}} \Delta_h u^h (\delta_h u_x^h + \delta_{-h} u_x^h) \, dx \, dt \leq C, \quad \text{for every } h, T > 0.
\]

(4.16)

**Proof.** Differentiate (4.15) w.r.t. \( x \), and then multiply by \( \delta_h u^h + \delta_{-h} u^h \), to get

\[
(\delta_h u^h + \delta_{-h} u^h) u_{x^2_1} + \frac{1}{2} H' \left( \frac{\delta_h u^h + \delta_{-h} u^h}{2} \right) (\delta_h u^h + \delta_{-h} u^h) (\delta_h u_x^h + \delta_{-h} u_x^h) = \gamma h \Delta_h u_x^h (\delta_h u^h + \delta_{-h} u^h).
\]

(4.17)

Choose \( G \) such that \( G'(s) = 2H'(s)s \) for \( s \in \mathbb{R} \) then

\[
\int_0^T \int_{\mathbb{T}} \frac{1}{2} H' \left( \frac{\delta_h u^h + \delta_{-h} u^h}{2} \right) (\delta_h u^h + \delta_{-h} u^h) (\delta_h u_x^h + \delta_{-h} u_x^h) \, dx \, dt
\]

\[
= \int_0^T \int_{\mathbb{T}} G' \left( \frac{\delta_h u^h + \delta_{-h} u^h}{2} \right) \left( \frac{\delta_h u_x^h + \delta_{-h} u_x^h}{2} \right) \, dx \, dt = 0.
\]

(4.18)
Integrating the first term in the left hand side of (4.17), we have

\[ L_1 = \int_0^T \int_{\mathbb{T}} \left( (\delta_h u^h + \delta_{-h} u^h) u_x^h \right) dx dt \]

\[ = \left[ \int_\mathbb{T} (\delta_h u^h + \delta_{-h} u^h) u_x^h dx \right]_{t=0}^{t=T} + \int_0^T \int_{\mathbb{T}} (\delta_h u_x^h + \delta_{-h} u_x^h) u_x^h dx dt \]

\[ = \left[ \int_\mathbb{T} (\delta_h u^h + \delta_{-h} u^h) u_x^h dx \right]_{t=0}^{t=T} - \int_0^T \int_{\mathbb{T}} (\delta_h u^h + \delta_{-h} u^h) u_x^h dx dt \]

\[ = \left[ \int_\mathbb{T} (\delta_h u^h + \delta_{-h} u^h) u_x^h dx \right]_{t=0}^{t=T} - L_1, \]

and therefore, using (4.11) and (4.13),

\[ L_1 = \frac{1}{2} \left[ \int_\mathbb{T} (\delta_h u^h + \delta_{-h} u^h) u_x^h dx \right]_{t=0}^{t=T} \geq -C. \]  

(4.19)

Integrating (4.17) and taking into account (4.18) and (4.19),

\[ -C \leq \int_0^T \int_{\mathbb{T}} \gamma h \Delta_h u_x^h (\delta_h u^h + \delta_{-h} u^h) dx dt = - \int_0^T \int_{\mathbb{T}} \gamma h \Delta_h u^h (\delta_h u_x^h + \delta_{-h} u_x^h) dx dt, \]

from which (4.16) follows. \(\square\)

**Remark 4.13.** Inequality (4.16) is the analog of the following one

\[ \varepsilon \int_0^T \int_{\mathbb{T}} |u_x^\varepsilon|^2 dx dt \leq C \]

if we consider the usual regularized equation

\[ u_t^\varepsilon + H(u_x^\varepsilon) = \varepsilon u_{xx}^\varepsilon \]

and the space dimension is 1.

5. A Special Case: \( H(p) = p^2/2 \)

We consider in this section the special case

\[ H(p) = \frac{p^2}{2}. \]

Hence, we will study the Hamilton-Jacobi equation

\[ \begin{cases} 
  u_t + \frac{u_x^2}{2} = 0, & \text{in } \mathbb{T} \times (0, \infty), \\
  u = u_0, & \text{in } \mathbb{T} \times \{t = 0\}. 
\end{cases} \]  

(5.1)

We choose \( F : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) defined as

\[ F(p, q) := \frac{(p^+)^2}{2} + \frac{(q^+)^2}{2}, \]
where we used the notation
\[ a^+ := \max\{a, 0\}, \quad a^- := \min\{a, 0\}, \quad a \in \mathbb{R}. \]

Notice that in this case properties (F1)–(F3) are satisfied. In particular
\[ F(p,q) = ((-p)^+)^2 + ((\delta_h - p)^+)^2 = p^2 + (\delta_h - p)^+^2 = H(p), \]
so that (F3) holds. For every \( h > 0 \), we are then lead to study the following approximation of equation (5.1):
\[
\begin{aligned}
&u^h_t + \frac{[(-\delta_h u^h)^+]^2}{2} + \frac{[(\delta_{-h} u^h)^+]^2}{2} = 0, \quad \text{in } T \times (0, \infty), \\
&u^h = u_0, \quad \text{in } T \times \{t = 0\},
\end{aligned}
\]
or equivalently,
\[
\begin{aligned}
&u^h_t + \frac{[(\delta_h u^h)^+]^2}{2} + \frac{[(\delta_{-h} u^h)^+]^2}{2} = 0, \quad \text{in } T \times (0, \infty), \\
&u^h = u_0, \quad \text{in } T \times \{t = 0\},
\end{aligned}
\]
where we used the fact that \((-\delta_h u^h)^+ = - (\delta_{-h} u^h)^+\). The linear operator correspondent to (5.2) is given by
\[ v \mapsto L^h v := v_t + (\delta_h u^h)^- (\delta_h v) + (\delta_{-h} u^h)^+ (\delta_{-h} v). \]

Observe that, although the function \( F \) just defined is not of class \( C^2 \), we have \( F \in C^{1,1} \). Then, we can approximate \( F \) with a sequence of smooth functions satisfying (F1)–(F3) with equibounded Hessian (for instance by convolution). Thus, since all the constants appearing in the previous section just depended on the bounds on \( DF \), we can pass to the limit and still obtain Theorem 1.1.

6. Appendix

In this section we study the properties of the solution \( u^h \) of equation
\[
\begin{aligned}
&u^h_t + F(-\delta_h u^h, \delta_{-h} u^h) = 0, \quad \text{in } T \times (0, \infty), \\
&u^h = u_0, \quad \text{on } T \times \{t = 0\},
\end{aligned}
\]
(6.1)

Proof of Proposition 4.1

Step 1: Local Existence and Uniqueness. Consider the following ODE in the Banach space \( C(\mathbb{T}) \):
\[
\begin{aligned}
&z^h(t) = G^h(z^h(t)) \quad t \in (0, \infty) \\
&z^h(0) = u_0
\end{aligned}
\]
(6.2)
where \( G^h : C(\mathbb{T}) \to C(\mathbb{T}) \) is given by
\[ G^h(z) := - F(-\delta_h z, \delta_{-h} z). \]
(6.3)
Here with the dot we denoted the derivative of the function \([0, \infty) \ni t \mapsto z^h(t) \in C(\mathbb{T})\). Since \(G^h\) is locally Lipschitz continuous, there exists \(\delta > 0\) and a unique function \(z^h \in C^1([0, \delta); C(\mathbb{T}))\) satisfying (6.3) for \(t \in [0, \delta)\). In particular, from the fact that \(z^h \in C^1([0, \delta); C(\mathbb{T}))\) it follows that \((x, t) \mapsto z^h(x, t) \in C(\mathbb{T} \times [0, \delta))\) and \(z^h(x, \cdot) \in C^1([0, \delta))\) for every \(x \in \mathbb{T}\). Thus, \(z^h\) is a solution to (6.1). On the other hand, every solution of (6.1) has to satisfy (6.3) as well. This shows local existence and uniqueness of \(u^h\).

**Step 2: Global Existence and Uniqueness.** We claim that

\[
||u^h(\cdot, t)||_{L^\infty(\mathbb{T})} \leq ||u_0||_{L^\infty(\mathbb{T})} + |F(0, 0)| t, \quad \text{for every } t \in (0, \infty). \tag{6.4}
\]

To prove the claim fix \(t_1 > 0\), choose any constant \(c_1 < F(0, 0)\), and set \(v^h := u^h + c_1 t\). Let \((\overline{x}, \overline{t}) \in \mathbb{T} \times [0, t_1]\) be such that

\[
v^h(\overline{x}, \overline{t}) = \max_{(x, t) \in \mathbb{T} \times [0, t_1]} u^h(x, t). \tag{6.5}
\]

Assume that \(\overline{t} \in (0, t_1]\). Then,

\[
\begin{align*}
v^h_{\xi}(\overline{x}, \overline{t}) &= u^h_{\xi}(\overline{x}, \overline{t}) + c_1 = -F(-\delta_h u^h(\overline{x}, \overline{t}), \delta_{-h} u^h(\overline{x}, \overline{t})) + c_1 \\
&= -F(-\delta_h v^h(\overline{x}, \overline{t}), \delta_{-h} v^h(\overline{x}, \overline{t})) + c_1 \leq -F(0, 0) + c_1 < 0,
\end{align*}
\]

which is not possible by (6.5). This implies \(\overline{t} = 0\). Thus, we conclude by (6.3) that

\[
\max_{x \in \mathbb{T}} u^h(x, t) \leq F(0, 0) t \leq \max_{x \in \mathbb{T}} u_0(x), \quad \text{for every } t \in [0, t_1],
\]

so that

\[
\max_{x \in \mathbb{T}} u^h(x, t) \leq \max_{x \in \mathbb{T}} u_0(x) + |F(0, 0)| t, \quad \text{for every } t \in [0, t_1].
\]

In the same way we can show that

\[
\min_{x \in \mathbb{T}} u^h(x, t) \geq \min_{x \in \mathbb{T}} u_0(x) - |F(0, 0)| t, \quad \text{for every } t \in [0, t_1].
\]

This shows (6.4) and, in turn, global existence and uniqueness.

**Step 3: Smoothness.** Consider the following equation

\[
\begin{align*}
v^h_{\xi}(t) &= P^h(t, v^h(t)) \quad t \in (0, \infty), \\
v^h(0) &= (u_0)_x,
\end{align*} \tag{6.6}
\]

where \(P^h : (0, \infty) \times C(\mathbb{T}) \to C(\mathbb{T})\) is defined as the formal linearization of \(G^h:\)

\[
P^h(t, w) = D_g F |_{(-\delta_h u^h, \delta_{-h} u^h)} \delta_h w - D_q F |_{(-\delta_h u^h, \delta_{-h} u^h)} \delta_{-h} w.
\]
Since $DF$ is continuous, $P^h$ is continuous and $P^h(t, \cdot)$ is linear. Then, there exists a unique global solution to (6.6). By repeating what was done in the previous step, we have that $(x, t) \mapsto v^h(x, t) \in C(\mathbb{T} \times [0, \infty))$ and $v^h(x, \cdot) \in C^1([0, \infty))$ for every $x \in \mathbb{T}$. We claim that $v^h = u^h_x$.

To show this observe that, for every $y \in \mathbb{R} \setminus \{0\}$, $\delta_y u^h \in C^1((0, \infty); C(\mathbb{T}))$ is the unique solution of the equation

$$
\begin{cases}
  \dot{w}(t) = R^h(t, w(t)) & t \in (0, \infty), \\
  w(0) = \delta_y u_0,
\end{cases}
$$

where $R^h$ is given by

$$
R^h(t, z) := D_y F \big|_x \delta_h z - D_q F \big|_\xi \delta_{-h} z,
$$

with

$$
\xi := (-\theta \delta_h u^h(\cdot) + (1 - \theta) \delta_h u^h(\cdot + y), \theta \delta_{-h} u^h(\cdot) + (1 - \theta) \delta_{-h} u^h(\cdot + y)),
$$

for some $\theta = \theta(t, y) \in (0, 1)$. Also, we have

$$
\|P^h(t, w_2) - P^h(t, w_1)\|_{C(\mathbb{T})} \leq C_1 \|w_2 - w_1\|_{C(\mathbb{T})}, \quad C_1 = C_1(t, h),
$$

and

$$
\|P^h(t, v^h(t)) - R^h(t, v^h(t))\|_{C(\mathbb{T})} \leq \varphi^{h, y}(t),
$$

where

$$
\varphi^{h, y}(t) := \|D_y F \big|_x - D_q F \big|_{(-\delta_h u^h, \delta_{-h} u^h)}\| \delta_h v^h(t)\|_{C(\mathbb{T})}
$$

$$
+ \|D_q F \big|_\xi - D_q F \big|_{(-\delta_h u^h, \delta_{-h} u^h)}\| \delta_{-h} v^h(t)\|_{C(\mathbb{T})}
$$

satisfies

$$
\lim_{y \to 0} \sup_{t \in [0, T]} \varphi^{h, y}(t) = 0, \quad \text{for every } T > 0 \text{ and } h > 0.
$$

Using the version of Gronwall’s Inequality stated at the end of the section we have

$$
\|\delta_y u^h(t) - v^h(t)\|_{C(\mathbb{T})} \leq e^{C_1 t} \|\delta_y u_0 - (u_0)_x\|_{C(\mathbb{T})} + e^{C_1 t} \int_0^t e^{-C_1 s} \varphi^{h, y}(s) ds,
$$

for every $t \in (0, \infty)$. From this, we conclude that $(u^h)_x(\cdot, t) = v^h(\cdot, t)$ for every $t \in [0, \infty)$ and thus $u^h(\cdot, t) \in C^1(\mathbb{T})$.

In a similar way, one can show the part of the statement concerning $u^h_x$ and $u^h_{xx}$. \hfill \Box

We conclude by stating the version of Gronwall’s inequality which was used in the previous proof.
Lemma 6.1 (Gronwall’s inequality). Let $X$ be a Banach space and $U \subset X$ an open set in $X$. Let $f, g : [a, b] \times X \to X$ be continuous functions and let $y, z : [a, b] \to U$ satisfy the initial value problems

\[
\begin{aligned}
\dot{y}(t) &= f(t, y(t)) & & t \in (a, b), \\
y(a) &= y_0, \\
\dot{z}(t) &= g(t, z(t)) & & t \in (a, b), \\
z(a) &= z_0.
\end{aligned}
\]

Also assume there is a constant $C \geq 0$ so that

\[
\|g(t, x_2) - g(t, x_1)\| \leq C\|x_2 - x_1\|
\]

and a continuous function $\varphi : [a, b] \to [0, \infty)$ so that

\[
\|f(t, y(t)) - g(t, y(t))\| \leq \varphi(t).
\]

Then for $t \in [a, b]$

\[
\|y(t) - z(t)\| \leq e^{C|t-a|}\|y_0 - z_0\| + e^{C|t-a|}\int_a^t e^{-C(s-a)}\varphi(s) \, ds.
\]

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