From $\mathcal{N} = 4$ gauge theory to $\mathcal{N} = 2$ conformal QCD: three-loop mixing of scalar composite operators

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Abstract. We derive the planar dilatation operator in the closed subsector of scalar composite operators of an $\mathcal{N} = 2$ superconformal quiver gauge theory to three loops. By tuning the ratio of its two gauge couplings we interpolate between a $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM theory and $\mathcal{N} = 2$ superconformal QCD. We find $\zeta(3)$ contributions at three loops that disappear when the theory is at the orbifold point. They are responsible for imaginary contributions to the dispersion relation of a single scalar excitation in the spin-chain picture. This points towards an interpretation of the individual scalar excitations as effective rather than as elementary magnons. We argue that the elementary excitations should be associated with certain fermions and covariant derivatives, and that integrability in the respective subsectors should persist at least to two loops.

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1 Introduction

The AdS/CFT correspondence [1-3] in its original formulation relates type II B string theory in $\text{AdS}_5 \times \mathcal{M}_5$ to $\mathcal{N} = 4$ SYM theory. Some closely related examples include $\mathbb{Z}_k$ orbifolds [4, 5] and the $\beta$-deformation [6] of the original correspondence. The resulting gravity backgrounds are of the form $\text{AdS}_5 \times M_5$ with a compact five-dimensional manifold $M_5$ that is given by $S^5/\mathbb{Z}_k$ in case of the $\mathbb{Z}_k$ orbifold or by a deformed $S^5$. The dual field theory preserves conformal symmetry and respectively is some quiver gauge theory or the $\beta$-deformed $\mathcal{N} = 4$ SYM theory with reduced supersymmetry. All these examples have in common that their dual string backgrounds are critical, i.e. ten-dimensional, and they share certain universal properties [7]. In the planar limit [8] the rank of the underlying gauge group is taken to infinity, while the number of matter fields has to be kept finite. This is the quenched approximation, where backreaction from matter fields is suppressed [9]. Furthermore, in the theories of the above type, the individual $a$ and $c$ anomaly coefficients [10] become equal $a = c$ in the limit. It is also important to note that planar Feynman diagrams in $\mathcal{N} = 4$ SYM theory and its $\mathbb{Z}_k$ orbifold theories are identical [11, 12]. The aforementioned four-dimensional superconformal theories should be considered as being members of the $\mathcal{N} = 4$ SYM universality class.

Lower-dimensional examples of the AdS/CFT correspondence include the ABJM duality [13] and its ABJ generalization [14] that involve $\mathcal{N} = 6$ supersymmetric Chern-Simons theory with levels $k, -k$ and respective product gauge groups $U(N) \times U(N)$ and $U(M) \times U(N)$ that are dual to M-theory on $\text{AdS}_4 \times S^7/\mathbb{Z}_k$ or to generalizations thereof involving flux. A gravity description in terms of type IIA string theory in the critical background $\text{AdS}_4 \times \mathbb{C}P^3$ is only possible in a certain regime of the parameters.

Recently, it was argued [7] that $\mathcal{N} = 2$ superconformal QCD (SCQCD) with gauge group $SU(N)$ and $N_f = 2N$ fundamental hypermultiplets should have a dual gravity description. $\mathcal{N} = 2$ SCQCD lies outside the $\mathcal{N} = 4$ SYM universality class, and its gravity dual is non-critical with $\text{AdS}_5$ and $S^1$ factors. It does not admit a quenched approximation, and its planar limit is the leading contribution in the Veneziano expansion [15]. The anomaly coefficients $a$ and $c$ differ even in the planar limit.

There is nevertheless a connection between $\mathcal{N} = 2$ SCQCD and the $\mathcal{N} = 4$ SYM universality class. The $\mathcal{N} = 2$ quiver gauge theory with product gauge group $SU(N) \times SU(N)$ and two coupling constants $g_{\text{YM}}$ and $\hat{g}_{\text{YM}}$ at its conformal point where $N = \hat{N}$, interpolates between $\mathcal{N} = 2$ SCQCD and the $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM theory [7,16]. We will refer to this superconformal theory as the interpolating theory. By considering this theory we can study how the transition from the $\mathcal{N} = 4$ SYM universality class to a theory outside this class is realized.

Of particular interest in a theory with conformal invariance is the spectrum of conformal dimensions of the gauge invariant composite operators. In perturbation theory where the coupling $\lambda$ is small, these operators mix under renormalization as

$$O_{a,\text{ren}} = \mathbb{Z}_a (\bar{\lambda}, \varepsilon) O_{a,\text{bare}},$$

where $a$ labels the composite operators in an appropriate basis and $\varepsilon$ is a UV regulator. The conformal dimensions of the composite operators are determined as a sum of their common bare scaling dimension and of individual anomalous dimensions. The latter
follow as eigenvalues of the dilatation operator that in terms of the renormalization constant $Z$ is defined as

$$D = \mu \frac{d}{d\mu} \ln Z(\bar{\lambda}^{2\varepsilon}, \varepsilon) = \lim_{\varepsilon \to 0} \left[ 2\varepsilon \frac{d}{d\bar{\lambda}} \ln Z(\bar{\lambda}, \varepsilon) \right], \quad (1.2)$$

where $\mu$ is the scale introduced by the regularization. In the planar limit, mixing occurs within the subset of single-trace operators in which the color contractions form a single cycle. Due to the equivalence of $\mathcal{N} = 4$ SYM theory and its orbifolds in the planar limit \[11,12\], the respective dilatation operators are identical. In these theories, in the $\beta$-deformed theory and also in the $\mathcal{N} = 6$ CS theory, the operator mixing problem appears to be integrable \[17-25\]. This can be seen by mapping the problem to an interacting spin chain \[17\]. The composite single-trace operators are interpreted as spin chain states, and the dilatation operator (1.2) is identified with the Hamiltonian acting on these chains. Integrability allows one to determine the eigenvalues from respective Bethe ansätze \[17,18,23,24,26-29\]. The latter rely on the factorization of multiple-particle scattering processes into products of two-particle scatterings captured in terms of a respective S-matrix with a dressing phase \[30,31\]. The particles that are fundamental excitations of the spin chain are called magnons. Further details can be found in the collection of reviews \[32\], see in particular \[33-37\].

In $\mathcal{N} = 4$ SYM theory, operator mixing can be studied in the closed $SU(2)$ subsector. This is the simplest setup since the respective operators contain only two different kinds of scalar field flavors. In the planar limit, mixing only occurs among operators with identical field content but different orderings within the single trace. The dilatation operator can hence not alter the length $L$, i.e. the total number of fields within the gauge trace, and it can be expressed in terms of permutations that act on the individual field flavors. Based on the assumption of integrability and some further input, the dilatation operator of $\mathcal{N} = 4$ SYM theory has been constructed to three loops \[38\] and then also to higher loops \[39,40\]. The three-loop result was then confirmed by only using the underlying symmetry, leaving undetermined some constants \[41\]. It was then also tested by field theory calculations that yield some of its eigenvalues \[42-44\]. At higher loops some particular terms in the dilatation operator were computed \[40,45-48\]. More details can be found in the collection of reviews \[32\], see in particular \[33,37,49,50\]. An explicit Feynman diagram calculation of the three-loop dilatation operator itself was accomplished recently \[51\]. In the employed $\mathcal{N} = 1$ superspace formalism the composite operators of the $SU(2)$ subsector appear as lowest components of superfields that are chiral. This chirality is crucial for the formulation of generalized finiteness conditions \[51\] that reduce significantly the calculational effort. The result is expressed in terms of the so-called chiral functions \[17,18\] that naturally emerge in the $\mathcal{N} = 1$ superspace formalism and are linear combinations of multiple permutations. They are generated by the chiral structures, i.e. by the configurations of the elementary chiral and anti-chiral propagators and vertices of the underlying Feynman diagrams. In the $\beta$-deformed theory the chiral functions have to be slightly modified \[52\] since the permutations contain phase factors \[20\]. However, the expression of the dilatation operator in terms of these chiral functions is not altered. In the aforementioned theories of the $\mathcal{N} = 4$ SYM universality class, the chiral functions are insensitive to potential further interactions.
involving gauge fields.

In the interpolating theory, operator mixing can be studied in a closed subsector that resembles the $SU(2)$ subsector of the $\mathcal{N} = 4$ SYM theory. Its composite operators consist only of certain scalar fields, and in an $\mathcal{N} = 1$ superfield formulation they are chiral. However, unlike in the $SU(2)$ subsector, they are built by choosing out of more than just two different types of fields. Moreover, their gauge traces involve the color indices from the two different $SU(N)$ factors in the product gauge group. Hence, the Feynman diagrams of operator renormalization are functions of the two gauge coupling constants $g_{YM}$ and $\hat{g}_{YM}$. They generate chiral functions that depend on the ratio of these couplings. Similarly to the case of the $\beta$-deformation \cite{20,52}, part of this dependence is associated with the chiral structure itself. It is thus included in the natural definition of the flavor operations as defined below in (3.10). But not all of the coupling dependence can be captured in this way. Gauge interactions that appear at higher loops introduce additional coupling dependence into the chiral functions (3.11). The relative coefficients that combine the flavor operations to the chiral functions become functions of the coupling ratio, yielding the deformed chiral functions (3.13). This kind of deformation should not occur in any theory of the $\mathcal{N} = 4$ SYM universality class. Yet, it should appear from six loop on in the $\mathcal{N} = 6$ CS theory of the ABJ generalization, involving the ratio of its two independent 't Hooft coupling constants. With the interpolating theory at hand we have a simpler setup than the CS theory that allows us to study the effects due to the presence of two gauge couplings.

The spin chain of the interpolating theory was constructed in \cite{16,53} with the purpose of investigating integrability of $\mathcal{N} = 2$ SCQCD. Already at one loop the scalar sector of the interpolating theory is not integrable, but the question is still open for its $\mathcal{N} = 2$ SCQCD limit. The purpose of this paper is to provide the dilatation operator to a loop order where it is sensitive to deformations that are generated by the gauge interactions. This requires a calculation at three loops. The insights obtained from our analysis allow us to predict that in other subsectors a first non-trivial test of integrability requires a three loop calculation. In particular, in the $SU(1|1)$ and $SL(2)$ subsectors integrability is inherited from the $\mathcal{N} = 4$ SYM theory to two loops, even for the interpolating theory. Similar arguments should hold for the respective subsectors in $\mathcal{N} = 1$ SQCD in the Veneziano limit with $N_f = 3N$ where the theory becomes conformal. So far, in this theory, only a one-loop study of the non-dynamical scalar excitations is available \cite{54}. A recent review on integrability in pure QCD and in its supersymmetric cousins can be found in \cite{55}.

The paper is organized as follows: Firstly, in section 2 we explain why in the interpolating theory the deformations of the chiral functions show up first at three loops. In section 3 we introduce the $\mathcal{N} = 1$ superfield formulation of the theory and describe the closed chiral subsector which is the subject of our investigation. Then, as a warm-up, in section 4 we present the calculation of the one- and two-loop dilatation operator. The three-loop calculation along the lines of \cite{51} is presented in section 5. In section 6 we extend the result beyond the asymptotic limit by computing the leading wrapping correction. Three-loop eigenvalues of the dilatation operator are summarized in section 7. This includes the calculations of the dispersion relation (7.5) for a single impurity and of the three-loop
anomalous dimensions of the composite single-trace operators that contain up to four elementary fields. Based on our calculation and its spin chain interpretation, in section 8 we discuss subsectors with elementary excitations in which integrability should persist to two loops. Finally, we conclude in section 9 and point towards interesting open problems that could be the subject of further studies. Details of our conventions, the Feynman rules, the derivation of required one- and two-loop subdiagrams, a list of the relevant integrals and a discussion of similarity transformations have been delegated to various appendices.

2 Why three loops?

As we have mentioned in the introduction, the presence of two gauge parameters in the interpolating theory leads to a dependence of the chiral functions on the ratio of these couplings. In particular, the chiral functions are deformed when additional gauge interactions are present in the underlying Feynman diagrams. We will now explain why in the closed chiral subsector these deformations first show up at three loops and lead to new effects.

In $\mathcal{N} = 1$ superspace the Feynman diagrams of the closed chiral subsector have the same form as the respective diagrams in $\mathcal{N} = 4$ SYM theory. As we conclude from the analysis in [51], in any diagram with an overall UV divergence at least one loop is generated by its chiral structure. Thus, one has to work at least at two loops in order to find a UV divergent diagram that also involves a gauge field generating a deformation of its chiral function. At two loops, the only diagram associated with such a deformation is depicted in figure 1(a). This diagram is finite and hence does not contribute to the dilatation operator of the theory. Deformations of the chiral functions can therefore appear in the dilatation operator only from three loops on. They originate from the diagrams with overall UV divergences that are displayed in figure 1(b), 1(c) and 1(d) and from the reflected diagrams where applicable. Note that the finite two-loop diagram of figure 1(a) appears as subdiagram within the second diagram of 1(b).

![Figure 1: Feynman diagrams to three loops that generate deformed chiral functions. The dotted lines denote all possible arrangements of bifundamental and adjoint field flavors. (a): The only two-loop diagram that generates a deformed $\chi(1)$ but is finite. (b): Three-loop diagrams that generate a deformed $\chi(1,2)$. The second diagram contains the diagram (a) as subdiagram. (c): Three-loop diagrams that generate a deformed $\chi(1)$. (d): Three-loop diagrams that generate a deformed $\chi(1,3)$ and only contribute to scattering.](image-url)
The above considerations imply that the one- and two-loop results for the dilatation operator depend only on chiral functions (3.11) that are in one-to-one correspondence to the chiral functions of $\mathcal{N} = 4$ SYM theory. This is not the case from three loops on. In the dilatation operator we find chiral functions that have their counterparts in the $\mathcal{N} = 4$ SYM case, but there are further contributions that depend on deformed chiral functions (3.13). They originate from the diagrams that are displayed in figures 1(b) and 1(c). These diagrams contribute with simple poles that have maximum transcendentality at this loop order, i.e. they are proportional to $\zeta(3)$. Some of these contributions add anti-Hermitean terms to the dilatation operator. Besides these contributions, we find additional transcendentality-three terms with undeformed chiral functions in the dilatation operator. All these terms disappear at the orbifold point, i.e. for equal couplings $g = \hat{g}$, such that one obtains the rational result of the $\mathcal{N} = 4$ SYM theory. Note that in [56] the circular Wilson loop of $\mathcal{N} = 2$ SCQCD was computed, and it deviates from its $\mathcal{N} = 4$ SYM counterpart first at three loops by $\zeta(3)$ terms.

The fact that three loops is the first loop order at which new contributions can arise can also be understood from the basis of loop integrals in four dimensions. Only from three loops on there appear several integrals with overall UV divergences that contain simple poles in $\varepsilon$. Among them one finds for the first time an integral with a simple pole that is proportional to $\zeta(3)$. The occurring integrals are listed in appendix D.

Three loops is also interesting for another reason. This is the lowest loop order at which the dilatation operator contains terms that do not contribute to the dispersion relation. We refer to them as pure scattering terms, since they only contribute to the scattering matrix. At three loops the only scattering term is generated by Feynman diagrams in figure 1(d), including their reflections where applicable. They contain a gauge field that deforms the chiral function and hence also modifies the scattering matrix compared to the $\mathcal{N} = 4$ SYM case.

3 The interpolating theory

The interpolating theory is given by an $\mathcal{N} = 2$ superconformal quiver gauge theory that depends on two couplings $g_{\text{YM}}$ and $\hat{g}_{\text{YM}}$, each of which is associated with one factor in the product gauge group $SU(N) \times SU(N)$. Since it has the same field content as a $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM theory, its action is roughly given by two copies of the $\mathcal{N} = 4$ SYM action, but with modified superpotentials that involve the bifundamental fields.
3.1 Action and parameters

In terms of \( \mathcal{N} = 1 \) superfields, with the conventions of \([57]\), the action reads

\[
S = \frac{1}{2} \int d^4x d^2\theta \left[ \frac{1}{g_{\text{YM}}^2} \text{tr} \left( W^\alpha W_\alpha \right) + \frac{1}{\hat{g}_{\text{YM}}^2} \text{tr} \left( \hat{W}^\alpha \hat{W}_\alpha \right) \right],
\]

\[
+ \int d^4x d^4\theta \left[ \text{tr} \left( e^{-g_{\text{YM}}V} \bar{\Phi} e^{g_{\text{YM}}V} \Phi \right) + \text{tr} \left( e^{-\hat{g}_{\text{YM}}\hat{V}} \bar{\hat{\Phi}} e^{\hat{g}_{\text{YM}}\hat{V}} \hat{\Phi} \right) \right]
\]

\[
+ \int d^4x d^4\theta \left[ \text{tr} \left( \tilde{Q}^I e^{g_{\text{YM}}V} Q_i e^{-g_{\text{YM}}V} \right) + \text{tr} \left( \tilde{\bar{Q}}_i e^{\hat{g}_{\text{YM}}\hat{V}} \bar{Q}^I e^{-\hat{g}_{\text{YM}}\hat{V}} \right) \right]
\]

\[
+ i \int d^4x d^2\theta \left[ g_{\text{YM}} \text{tr} \left( \tilde{Q}^I \Phi Q_I \right) - \hat{g}_{\text{YM}} \text{tr} \left( Q_I \hat{\Phi} \tilde{Q}^I \right) \right]
\]

\[
- i \int d^4x d^2\bar{\theta} \left[ \tilde{g}_{\text{YM}} \text{tr} \left( \tilde{Q}^I \Phi \tilde{Q}_I \right) - \hat{\tilde{g}}_{\text{YM}} \text{tr} \left( Q_I \hat{\Phi} \tilde{Q}^I \right) \right],
\]

where \( W_\alpha = i \tilde{D}^2 \left( e^{-g_{\text{YM}}V} D_\alpha e^{g_{\text{YM}}V} \right) \) and \( \hat{W}_\alpha = i \tilde{D}^2 \left( e^{-\hat{g}_{\text{YM}}\hat{V}} \hat{D}_\alpha e^{\hat{g}_{\text{YM}}\hat{V}} \right) \) are the chiral superfield strengths of the vector superfields \( V \) and \( \hat{V} \) that contain the gauge fields and transform in the adjoint representation of respectively the first and second copy of the gauge group. The field content of the theory and its transformation properties under the \( SU(N) \times SU(N) \) gauge and global \( SU(2)_L \) and \( SU(2)_R \) \( \times U(1) \) R-symmetry groups is shown in Table 1. We have grouped the fields according to their gauge group representations.

| field | \( SU(N) \times SU(N) \) | \( SU(2)_L \) | \( SU(2)_R \) | \( U(1) \) |
|-------|----------------|------------|------------|--------|
| \( V \) | (adj., 1) | 1 | 1 | 0 |
| \( \Phi \) | (adj., 1) | 1 | 1 | 1 |
| \( \bar{\Phi} \) | (adj., 1) | 1 | 1 | -1 |
| \( \tilde{V} \) | (1, adj.) | 1 | 1 | 0 |
| \( \tilde{\Phi} \) | (1, adj.) | 1 | 1 | 1 |
| \( \tilde{\bar{\Phi}} \) | (1, adj.) | 1 | 1 | -1 |
| \( Q_i \) | (\( \Box, \Box \)) | \( \Box \) | \( \Box \) | 0 |
| \( \bar{Q}_i \) | (\( \Box, \Box \)) | \( \Box \) | 0 |
| \( \tilde{Q}^I \) | (\( \Box, \Box \)) | \( \Box \) | 0 |
| \( \tilde{\bar{Q}}^I \) | (\( \Box, \Box \)) | \( \Box \) | 0 |

Table 1: The field content of the interpolating theory in terms of \( \mathcal{N} = 1 \) superfields. The fields are grouped according to their gauge group representations. \( \Box \) and \( \Box \) respectively denote fundamental and anti-fundamental representations. The global \( SU(2)_R \) symmetry that transforms chiral into anti-chiral superfields is not manifest in the \( \mathcal{N} = 1 \) superspace formulation.

representations. As in the \( \mathcal{N} = 4 \) SYM theory the superpotential is a cubic interaction of three different types of chiral fields, but here it contains a contraction of the \( SU(2)_L \) global symmetry index. The additional gauge fixing and ghost terms together with the Feynman rules required for a three-loop calculation in Fermi-Feynman gauge can be found in appendix \( A \). Moreover, in order to regulate the UV divergences of the Feynman diagrams, we will use dimensional reduction \([58]\) in \( D = 4 - 2\varepsilon \) dimensions.
In the following we will only consider the planar limit. The respective 't Hooft couplings are given by
\[ \lambda = g_{\text{YM}}^2 N, \quad \hat{\lambda} = \hat{g}_{\text{YM}}^2 N, \quad \bar{\lambda} = g_{\text{YM}} \hat{g}_{\text{YM}} N, \] (3.2)
where we have also introduced their geometric mean \( \bar{\lambda} \). For later convenience we also define the rescaled coupling constants
\[ g = \frac{\sqrt{\lambda}}{4\pi}, \quad \hat{g} = \frac{\sqrt{\hat{\lambda}}}{4\pi}, \quad \bar{g} = \frac{\sqrt{\bar{\lambda}}}{4\pi}, \] (3.3)
and introduce the following ratios
\[ \rho = \frac{\lambda}{\bar{\lambda}} = \frac{g}{\bar{g}}, \quad \hat{\rho} = \rho^{-1} = \frac{\hat{\lambda}}{\lambda} = \frac{\hat{g}}{g}. \] (3.4)
Even if \( \rho \hat{\rho} = 1 \), we will display the results in terms of both parameters for convenience.

The renormalization constant and dilatation operator are then given to three loops as
\[ Z = 1 + \bar{\lambda} Z_1 + \bar{\lambda}^2 Z_2 + \bar{\lambda}^3 Z_3 + \mathcal{O}(\bar{\lambda}^4) \]
\[ D = \bar{g}^2 D_1 + \bar{g}^4 D_2 + \bar{g}^6 D_3 + \mathcal{O}(\bar{g}^8), \] (3.5)
where the \( \ell \)-loop expansion coefficients \( Z_\ell \) and \( D_\ell \) are polynomial functions of degree \( \ell \) in the two coupling ratios \( \rho \) and \( \hat{\rho} \).

### 3.2 Closed chiral subsector

From the perspective of \( \mathcal{N} = 1 \) superfields the flavor \( SU(2) \) subsector of \( \mathcal{N} = 4 \) SYM is chiral, since its operators are composed only out of the elementary chiral superfields. Furthermore, it is closed, i.e. operator mixing due to renormalization only occurs among its members. In the interpolating theory there exists a closed chiral subsector that resembles the flavor \( SU(2) \) subsector of \( \mathcal{N} = 4 \) SYM theory. Its composite chiral single-trace operators have the form
\[ \mathcal{O} = \text{tr} \left( \Phi \ldots \Phi Q_I \hat{\Phi} \ldots \hat{\Phi} \hat{Q}_J \ldots \right), \] (3.6)
and they are symmetric traceless in their \( SU(2)_L \) indices \( I, J \). These composite operators are highest weight states w.r.t. the \( SU(2)_R \) symmetry, since they only include the chiral components of the \( SU(2)_R \) doublets of table 1. The different types of chiral superfields within the operators (3.6) we call field flavors, including also their different \( SU(2)_L \) components. This is not to be confused with the \( SU(N_f) \) flavor of \( \mathcal{N} = 2 \) SCQCD. The total number of elementary fields of the operator is denoted as its length \( L \). The fields \( Q_I \) and \( \hat{Q}_J \) that transform in the bifundamental and anti-bifundamental representation of the gauge group \( SU(N) \times SU(N) \) are regarded as impurities that have to appear pairwise in order to build a gauge invariant single-trace operator. Each impurity switches between

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1In the limit \( \hat{g}_{\text{YM}} \to 0 \), there is a symmetry enhancement. The respective gauge group becomes global and combines with the \( SU(2)_L \) to the \( SU(N_f) \) flavor group of \( \mathcal{N} = 2 \) SCQCD.
the two different types of fields \( \Phi \) and \( \hat{\Phi} \) that transform in the adjoint representation respectively of the first and second factor of the product gauge group. The fields \( \Phi \) and \( \hat{\Phi} \) are vacuum fields, since the operators

\[
\text{tr}(\Phi \ldots \Phi), \quad \text{tr}(\hat{\Phi} \ldots \hat{\Phi}) \tag{3.7}
\]

without impurities are protected from quantum corrections \([7, 16, 59]\) and hence yield two respective ground states.

The operators that contain exclusively impurities and are symmetric traceless representations of \( SU(2)_L \) read

\[
\text{tr} \left( Q_I \hat{Q}^I_\ldots Q_I \hat{Q}^I \right). \tag{3.8}
\]

They are also protected from quantum corrections \([7, 16, 59]\). This can be easily understood in terms of the generalized finiteness conditions formulated in \([51]\): for vanishing flavor subtraces the two fields cannot end up in a single anti-chiral vertex, i.e. the diagrams cannot contain the fundamental building block \((3.9)\). The remaining diagrams are all finite, since they either contain the finite self energies, or all their vertices appear in loops.

We close this section with a comparison of this closed chiral subsector with the \( SU(2) \) subsector of \( \mathcal{N} = 4 \) SYM theory. The operators \((3.6)\) of the former are free of flavor traces, but still contain all different types of chiral fields. This is a difference compared to the \( SU(2) \) subsector, where the operators are composed only out of two different kinds of chiral field flavors. In \( \mathcal{N} = 4 \) SYM theory, an inclusion of all three chiral flavors extends the operator mixing at least to the bigger subsector \( SU(2|3) \) \([41]\). Furthermore, the \( SU(2) \) subsector has only one type of vacuum and one maximally filled state. They are equivalent, and therefore it suffices to consider states in which the number of one type of fields does not exceed the other type. In contrast to this, the closed chiral subsector of the interpolating theory has two types of vacua \((3.6)\), and several maximally filled states \((3.8)\) that are built as alternating products of \( Q_I \) and \( Q^I \). It is not obvious whether the information obtained from the states above half-filling is redundant. Bearing the above comments in mind, the closed chiral subsector of the interpolating theory has a richer structure than its \( \mathcal{N} = 4 \) SYM counterpart.

### 3.3 Chiral functions

As mentioned in the introduction, the chiral functions capture the structure of the chiral and anti-chiral field lines in the Feynman diagrams of \( \mathcal{N} = 1 \) superfields. They are given as linear combinations of appropriately defined flavor operations that act on the different field flavors within the composite operators. The elementary building block of the chiral structure of the Feynman diagrams is given by a chiral and an anti-chiral vertex of the theory that are contracted with each other by a single chiral propagator. It includes the following combination of flavor operations

\[
\left( P - \Lambda_s - T \right)_{ij}, \tag{3.9}
\]
whether the first encountered impurity respectively is from left to right the legs involved in the interaction. The 'tilde' there by indicates where the sum runs over all possible configurations. The 'hat' as an involutive operator, i.e. \( \hat{\phi} = \phi \). Furthermore, we have introduced a parameter \( \alpha \) that drops out in the combination \( \Lambda_s \), and hence can be fixed to a convenient value. Note that the subscript \( s \) of \( \Lambda_s \) refers to the fact that from the perspective of the fields at positions \( (i, j) \) the chiral building block is an \( s \)-channel diagram.

In terms of the building block \( \mathbf{3.9} \), the chiral functions are defined as

\[
\chi(a_1, \ldots, a_n) = \sum_{r=0}^{L} \prod_{i=1}^{n} (P - \Lambda_s - T)_{r+a_r, r+a_r+1},
\]

where the summation considers the insertion of the interactions at all possible positions along the operators \( \mathbf{3.9} \) of length \( L \). The identity operation is given by \( \chi() \). When acting on single-trace operators the cyclic identification \( r + L \equiv r \) is understood.

As mentioned in section \( \mathbf{2.1} \) in the interpolating theory the chiral functions are subject to deformations caused by additional interactions involving the vector fields. We therefore have to decompose the chiral functions according to the different possible incoming and outgoing arrangements of the field flavors. The individual contributions are abbreviated as

\[
\chi^{(o_1, \ldots, o_l)}_{(i_1, \ldots, i_l)}(a_1, \ldots, a_n), \quad \tilde{\chi}^{(o_1, \ldots, o_l)}_{(i_1, \ldots, i_l)}(a_1, \ldots, a_n).
\]

The two lists \( (i_1, \ldots, i_l) \) and \( (o_1, \ldots, o_l) \) denote the positions of the total number of \( I \) impurities \( Q_I \) and \( \tilde{Q}^j \) that enter and respectively leave the interaction region, counting from left to right the legs involved in the interaction. The 'tilde' thereby indicates whether the first encountered impurity respectively is \( Q_I \) or \( \tilde{Q}^j \). Gauge invariance and planarity then fix uniquely the types of the other chiral fields. In terms of the individual contributions \( \mathbf{3.12} \) the deformed chiral functions have the following form

\[
\sum_{\tilde{i}, \sigma} (c_\tilde{i}^{(\rho)}(\rho) \chi^{(\sigma)}_{(\tilde{i})}(a_1, \ldots, a_n) + \tilde{c}_\tilde{i}^{(\tilde{\rho})}(\tilde{\rho}) \tilde{\chi}^{(\sigma)}_{(i)}(a_1, \ldots, a_n)) ,
\]

where the sum runs over all possible configurations \( \tilde{i}, \sigma \) of the ingoing and outgoing impurities. Note that for both terms in the sum the coefficients just differ by an exchange
$\rho \leftrightarrow \hat{\rho}$. The undeformed chiral functions (3.11) are recovered if all coefficients $c^A_i$ are set to one. Apart from the summation in (3.11) the elementary building block (3.9) is the simplest chiral function $\chi(1)$, and it is decomposed into the individual contributions as shown in figure 2. They read

$$
\begin{align*}
\chi^{(1)}_i(1) &= -\hat{\rho} O_{Q_j,\Phi \rightarrow Q_j,\Phi}, & \tilde{\chi}^{(1)}_i(1) &= -\rho O_{\tilde{Q}^j,\Phi \rightarrow \tilde{Q}^j,\Phi}, \\
\chi^{(2)}_i(1) &= O_{Q_j,\Phi \rightarrow \Phi Q_i}, & \tilde{\chi}^{(2)}_i(1) &= O_{\tilde{Q}^j,\Phi \rightarrow \tilde{Q}^j,\Phi}, \\
\chi^{(1)}_j(1) &= O_{\Phi Q_j \rightarrow \Phi Q_i}, & \tilde{\chi}^{(1)}_j(1) &= O_{\Phi \tilde{Q}^j \rightarrow \Phi \tilde{Q}^j}, \\
\chi^{(2)}_j(1) &= -\hat{\rho} O_{\Phi Q_j \rightarrow \Phi Q_i}, & \tilde{\chi}^{(2)}_j(1) &= -\rho O_{\Phi \tilde{Q}^j \rightarrow \Phi \tilde{Q}^j}, \\
\chi^{(1,2)}_{(1,2)}(1) &= -\hat{\rho} O_{Q_j \tilde{Q}^j \rightarrow \delta^j_i \tilde{Q}^{\kappa} Q_\kappa}, & \tilde{\chi}^{(1,2)}_{(1,2)}(1) &= -\rho O_{Q_j \tilde{Q}^j \rightarrow \delta^j_i \tilde{Q}^{\kappa} Q_\kappa},
\end{align*}
$$

(3.14)

where the operators $O_{A \rightarrow B}$ obey the Leibnitz rule when they replace the field configurations $A$ by the new configurations $B$. For completeness, in the last line, we have included the contributions that involve the trace operator $T$ of (3.10). They yield zero
when applied to the composite operators (3.6) of the closed chiral subsector.

We conclude this section by summarizing the action of Hermitean conjugation on the chiral functions. The relations read

\[
\chi(a_1, \ldots, a_n)\dagger = \chi(a_n, \ldots, a_1), \\
\chi^{(a_1, \ldots, a_2)}(a_1, \ldots, a_n)\dagger = \chi^{(a_1, \ldots, a_2)}(a_n, \ldots, a_1), \\
\tilde{\chi}^{(a_1, \ldots, a_2)}(a_1, \ldots, a_n)\dagger = \tilde{\chi}^{(a_1, \ldots, a_2)}(a_n, \ldots, a_1),
\]

(3.15)
i.e. under Hermitean conjugation the list of arguments is reversed and the lists of the incoming and outgoing impurities are interchanged.

4 One- and two-loop dilatation operator

In this section we calculate the one- and two-loop contribution to the dilatation operator of the closed chiral subsector introduced in section 3.2. At two loops the vanishing of flavor subtraces will allow us express the result exclusively in terms of (3.11), even if the contributions involving flavor traces undergo deformations. In the \(\mathcal{N}=1\) superfield formalism the one-loop self energies are identically zero, and the higher-loop chiral self energies are finite. Due to the finiteness conditions of [51], the appearing diagrams can only have overall UV divergences if they contain at least one chiral vertex that is not part of any loop. Therefore, all diagrams with only gauge-matter interactions or self-energy corrections are finite.

For the one-loop dilatation operator in the closed subsector, we have to evaluate the following diagram

\[
\begin{align*}
\frac{1}{\text{!}} = \hat{\lambda} I_1 \chi(1).
\end{align*}
\]

(4.1)
The one-loop contribution to the renormalization constant is given as the pole part of the above result. It reads

\[
Z_1 = -\mathcal{I}_1 \chi(1),
\]

(4.2)
where \(\mathcal{I}_1=\text{K}(I_1)\) is the pole part extracted by the operation K from the integral \(I_1\) in (D.3).

In order to determine the dilatation operator at two loops, we have to evaluate the diagrams that have interaction range \(R=2, 3\), i.e. in which two or three neighbouring field lines of the composite operator interact with each other. The maximum range \(R=3\) diagrams are determined as

\[
\begin{align*}
\frac{1}{\text{!}} = \hat{\lambda}^2 I_2 \chi(1, 2), \\
\frac{1}{\text{!}} = \hat{\lambda}^2 I_2 \chi(2, 1),
\end{align*}
\]

(4.3)
The \(R=2\) diagrams can be collectively written as a single diagram that contains as subdiagram the one-loop correction of the anti-chiral vertex given in (D.3). We introduce
generic relative couplings $\rho_i, i = 1, 2, 3$ for the individual faces of the diagram and obtain

\[ 2 \quad 3 = -\bar{\lambda}^2 I_2(\rho_2 + \rho_3)\chi^{(1)}(1). \tag{4.4} \]

Each possible ingoing and outgoing flavor combination is encoded in terms of the respective positions $\vec{i}, \vec{o}$ of the impurities and by $\tau$, the type of the first encountered impurity. Each such configuration determines the $\rho_i$ in terms of $\rho, \hat{\rho}$. With the explicit expressions for the chiral functions as given in (3.14) it is easy to conclude from figure 2 that in the closed chiral subsector the field separating faces 2 and 3 in the diagram has to be one of the bifundamentals. This means that the sum $\rho_2 + \rho_3$ yields $\rho + \hat{\rho}$ for all contributions in this subsector. This is not true for the diagrams in which two impurities interact, but as seen from (3.14) such an interaction is possible only for operators with non-vanishing flavor subtraces that are not members of the closed chiral subsector. We can therefore express the result in terms of $\chi(1)$, even if this does not hold for the trace terms. Summing up the contributions (4.3) and (4.4), we obtain for the two-loop renormalization constant

\[ Z_2 = -I_2 \left[ \chi(1, 2) + \chi(2, 1) - (\rho + \hat{\rho}) \chi(1) \right], \tag{4.5} \]

where $I_2 = KR(I_2)$ is the overall UV divergence that is extracted by $K$ after the subdivergence has been removed by the operation $R$. The expression of the two-loop integral is given in (D.3).

According to the definition (1.2), we multiply the $\frac{1}{\varepsilon}$ pole of (4.2) by 2, and the one of (4.5) by 4 and obtain for the one- and two-loop dilatation operator the following expressions

\[ D_1 = -2 \chi(1), \]
\[ D_2 = -2 \left( \chi(1, 2) + \chi(2, 1) \right) + 2(\rho + \hat{\rho}) \chi(1). \tag{4.6} \]

At $\rho = \hat{\rho} = 1$, where the interpolating theory becomes the $\mathbb{Z}_2$ orbifold of the $\mathcal{N} = 4$ SYM theory, this result is identical to the one in $\mathcal{N} = 4$ SYM theory, apart from a straightforward identification of the chiral functions in both theories.

## 5 Three-loop dilatation operator

In this section we calculate the three-loop contribution to the renormalization constant and the dilatation operator, following closely the analysis of [51]. We classify the underlying Feynman diagrams according to their range $R$, defined by the number of interacting elementary fields, and according to their chiral structure as captured in terms of the chiral functions (3.11) and (3.13).
5.1 Maximum range diagrams

Following [51], the chiral maximum-range diagrams are evaluated as

\[ \tilde{\lambda}^3 I_3 \chi(1, 2, 3), \quad \tilde{\lambda}^3 I_3 \chi(3, 2, 1), \]

\[ \tilde{\lambda}^3 I_{3bb} \chi(2, 1, 3), \quad \tilde{\lambda}^3 I_{3b} \chi(1, 3, 2). \]  

(5.1)

The non-chiral maximum range diagrams involve one vector field that leads to a deformation of the respective chiral function. For the contributions that have an overall UV divergence and that are free of flavor subtraces, one finds

\[ \tilde{\lambda}^3 I_3 \left[ \hat{\rho} \chi(1, 2, 3) + \rho \tilde{\chi}(1, 3) \right], \]

\[ -2\tilde{\lambda}^3 (I_3 + I_{32t}) \left[ \hat{\rho} \chi(1, 3) + \rho \tilde{\chi}(1, 3) \right], \]

\[ -(\rho + \hat{\rho}) \tilde{\lambda}^3 I_1 I_2 \chi(1, 3). \]  

(5.2)

The sum of the maximum range diagrams then contributes to the renormalization constant as

\[ Z_{3,R=4} = -I_3 \left( \chi(1, 2, 3) + \chi(3, 2, 1) \right) - I_{3bb} \chi(2, 1, 3) - I_{3b} \chi(1, 3, 2) + 2I_{32t} (\hat{\rho} \chi(1, 3) + \rho \tilde{\chi}(1, 3)) - 2I_1 I_2 (\rho + \hat{\rho}) \chi(1, 3), \]  

(5.3)

where as in [51] we have also added the respective contribution with chiral function \( \chi(1, 3) \) that only involves higher order poles in \( \varepsilon \) and hence does not contribute to the dilatation operator. It is required for the check that these poles indeed cancel in the logarithm of the renormalization constant. The first contribution in the second line contains a deformation of the chiral function \( \chi(1, 3) \) that has the form (3.13). It only contributes to the scattering of impurities.
5.2 Next-to-maximum range diagrams

The contributions from chiral next-to-maximum range diagrams are given by

\[ \lambda \chi(1, 2, 1) , \quad \lambda \chi(2, 1, 2). \]  

(5.4)

In the closed chiral subsector, i.e. for the terms not involving the flavor trace operator, the sum of their chiral functions can be replaced as

\[ \chi(1, 2, 1) + \chi(2, 1, 2) \rightarrow (\rho^2 + \hat{\rho}^2) \chi(1). \]  

(5.5)

The results for the only next-to-maximum range diagrams that involve a single vector field line are summarized by introducing generic coupling ratios \( \rho_i, i = 1, \ldots, 5 \), for the different faces of the diagrams. We find

\[ \lambda \chi(1) = \bar{\lambda} \left( - I_3 (\rho_1 + \rho_2 + \rho_5 + \rho_6) + I_{3t} (\rho_2 - \rho_3) \right) \chi^{(\bar{\rho})}(1), \]  

\[ \lambda \chi(2) = \bar{\lambda} \left( - I_3 (\rho_1 + \rho_3 + \rho_4 + \rho_6) + I_{3t} (\rho_3 - \rho_2) \right) \chi^{(\bar{\rho})}(2). \]  

(5.6)

where the contributions with the integral \( I_{3t} \) come from the diagrams displayed in figure 1(b). The individual diagrams and their expressions in the \( \mathcal{N} = 4 \) SYM theory can be found in [51]. We have to evaluate the above expression for all possible combinations of ingoing and outgoing field flavors that are encoded in terms of the positions \( \vec{i}, \vec{o} \) and type \( \tau \) of the respective ingoing and outgoing impurities. Thereby, configurations that lead to flavor traces are neglected. The gauge structure of a given combination fixes each \( \rho_i \) to either \( \rho \) or \( \hat{\rho} \).

There are further contributions from next-to-maximum range diagrams that involve two vector field lines. Following again closely the analysis in [51], the remaining contri-
The results from the individual diagrams contain different relative coefficients and hence seem to combine into a deformed chiral function. However, they in fact sum up to the undeformed chiral function $\chi(1)$ with a common global factor as long as contributions from flavor subtraces are of no concern.

From (5.4), (5.6) and (5.7) we determine the contribution of all range $R = 3$ diagrams to the renormalization constant as

$$Z_{3,R=3} = -\frac{1}{2} \lambda^3 I_3 (\rho^2 \chi_{(s)}(1) + \rho^2 \tilde{\chi}_{(s)}(1))$$

$$= -\frac{1}{2} \lambda^3 I_{3t} (\rho^2 \chi_{(s)}(1) + \rho^2 \tilde{\chi}_{(s)}(1))$$

$$= -\frac{1}{2} \lambda^3 I_3 (\rho^2 \chi_{(s)}(1) + \rho^2 \tilde{\chi}_{(s)}(1))$$

$$= -\frac{1}{2} \lambda^3 I_{3t} (\rho^2 \chi_{(s)}(1) + \rho^2 \tilde{\chi}_{(s)}(1)) .$$

(5.7)

The contributions with individual configurations of the impurities in the second and third line have the form (3.13) of a deformed chiral function. They are due to the second terms in (5.6) that originate from the diagrams displayed in figure 1(b).

5.3 Nearest-neighbour interactions

The nearest-neighbour interactions either involve the finite two-loop corrections of the anti-chiral vertex or the finite two-loop self energies of the chiral fields. Introducing again generic coupling ratios $\rho_i, i = 1, 2, 3$ for the different faces in the diagram, we
obtain from (C.10) the result

\[
\begin{align*}
2 \lambda^3 &= \frac{1}{2} I_{3t} (2 \rho_1 (\rho_2 + \rho_3) + \rho_2^2 + \rho_3^2) \chi_0^{(r)}(1). 
\end{align*}
\] (5.9)

Like the respective two-loop expression (4.4) also this result has to be evaluated for all possible field flavor arrangements, i.e. positions \( \vec{i}, \vec{o} \) and type \( \tau \) of the respective in- and outgoing impurities in the closed chiral subsector. The second type of diagram involves the finite two-loop self energies of the different chiral field flavors. Since according to (C.7) they are the same for all flavors, we find immediately

\[
2 \lambda^3 I_{3t} \chi(1),
\] (5.10)

The sum of all \( R = 2 \) diagrams yields the following contribution to the renormalization constant

\[
\begin{align*}
Z_{3,R=2} &= \left( -I_3 (\rho + \hat{\rho})^2 + \frac{1}{2} I_{3t} (\rho^2 + \hat{\rho}^2 - 4) \right) \chi(1) \\
&\quad + I_{3t} \left( (\rho^2 - 1) \left( \chi_1^{(c)}(1) + \chi_2^{(c)}(1) \right) + (\rho^2 - 1) \left( \chi_1^{(c)}(1) + \chi_3^{(c)}(1) \right) \right). 
\end{align*}
\] (5.11)

By examining the individual diagrams of the two-loop chiral vertex correction in (5.9), we find that the diagrams displayed in figure 1(c) generate the contributions with deformed chiral functions (3.13) in (5.11).

### 5.4 Result

The renormalization constant is a sum of (5.3), (5.8) and (5.11), and it is given by

\[
Z_3 = -I_3 \left( \chi_1(1, 2, 3) + \chi_3(3, 2, 1) - 2(\rho + \hat{\rho})(\chi(1, 2) + \chi(2, 1)) \right) \\
+ \chi(1, 2, 1) + \chi(2, 1, 2) + (\rho + \hat{\rho})^2 \chi(1) \\
\quad - I_{3ab} \chi(2, 1, 3) - I_{3ab} \chi(1, 3, 2) \\
+ 2I_{3a} (\rho \chi_1^{(c)}(1, 3) + \rho \tilde{\chi}_1^{(c)}(1, 3)) - 2I_{3a} (\rho + \hat{\rho}) \chi(1, 3) \\
+ I_{3a} (\rho - \hat{\rho} \chi_2^{(c)}(1) - \rho \tilde{\chi}_2^{(c)}(1)) + \rho \left( \chi_2^{(c)}(1) + \tilde{\chi}_2^{(c)}(1) \right) \\
\quad - \chi_2^{(c)}(1, 2) + \chi_3^{(c)}(1, 2) + \chi_2^{(c)}(2, 1) + \chi_3^{(c)}(2, 1) \\
+ \tilde{\chi}_2^{(c)}(1, 2) - \tilde{\chi}_3^{(c)}(1, 2) - \tilde{\chi}_2^{(c)}(2, 1) - \tilde{\chi}_3^{(c)}(2, 1)).
\] (5.12)
According to the definition \[ (12) \], we have to extract the \( \frac{1}{z} \) pole and multiply the result by 6. The three-loop contribution to the dilatation operator then reads

\[ D_3 = -4(\chi(1, 2, 3) + \chi(3, 2, 1) - 2(\rho + \tilde{\rho})(\chi(1, 2) + \chi(2, 1)) \]
\[ + \chi(1, 2, 1) + \chi(2, 1, 2) + (\rho + \tilde{\rho})^2 \chi(1)) \]
\[ + 2(\chi(2, 1, 3) - \chi(1, 3, 2)) - 4(\tilde{\rho}\chi^{(3)}_{(1, 3)} + \rho\tilde{\chi}^{(3)}_{(1, 3)}) \]
\[ + 2\zeta(3)(\rho - \tilde{\rho})(\rho - \tilde{\rho}) \chi(1) - \rho^2(\chi^{(1)}_{(1)}(1) + \tilde{\chi}^{(1)}_{(1)}(1)) \]
\[ - \chi^{(2)}_{(2)}(1, 2) + \chi^{(s, 3)}_{(2, 2)}(1, 2) + \chi^{(s)}_{(2, 1)}(2, 1) + \chi^{(1, s)}_{(1, 1)}(2, 1) \]
\[ + \tilde{\chi}^{(s)}_{(1, 2)}(1, 2) - \tilde{\chi}^{(s, 3)}_{(1, 2)}(1, 2) - \tilde{\chi}^{(s)}_{(2, 2)}(2, 1) - \tilde{\chi}^{(1, s)}_{(2, 3)}(2, 1)) \].

At the orbifold point, where \( \rho = \tilde{\rho} = 1 \), the result is identical to the one of the \( N = 4 \) SYM theory after a straightforward identification of the chiral functions that is required due to the differing chiral field content. Note that apart from the deformed pure scattering term in the third line, all the other deformations are homogeneous maximal transcendental due to \( \zeta(3) \).

Using the relations \[ (3.15) \] for the conjugation of the chiral functions, we see that the above result \[ (5.13) \] of the Feynman diagram calculation is not Hermitian. A similar phenomenon has already been observed in the context of QCD \[ 60 \]. While there the mixing matrix is non-Hermitian at leading order, here the one- and two-loop results \[ (4.6) \] are Hermitian. As in \[ 60 \] also in our case the eigenvalues of gauge invariant operator are real. Therefore, there should exist a non-unitary similarity transformation that transforms the dilatation operator to an Hermitian form in a new basis. In appendix \[ E \] we construct general similarity transformations and determine the one that casts \[ (5.13) \] into the simplest Hermitian form given by

\[ D_3 = -4(\chi(1, 2, 3) + \chi(3, 2, 1) - 2(\rho + \tilde{\rho})(\chi(1, 2) + \chi(2, 1)) \]
\[ + \chi(1, 2, 1) + \chi(2, 1, 2) + (\rho + \tilde{\rho})^2 \chi(1)) \]
\[ - 4(\tilde{\rho}\chi^{(3)}_{(1, 3)} + \rho\tilde{\chi}^{(3)}_{(1, 3)}) \]
\[ + 2\zeta(3)(\rho - \tilde{\rho})((\rho - \tilde{\rho}) \chi(1) - (\rho + \tilde{\rho})((\chi^{(1)}_{(1)}(1) - \chi^{(3)}_{(2)}(1) - \tilde{\chi}^{(1)}_{(1)}(1) + \tilde{\chi}^{(3)}_{(2)}(1))) \].

### 6 Wrapping interactions

The mixing of operators of length \( L \leq 3 \) cannot be studied by using the dilatation operator \[ (5.13) \], since it contains contributions from Feynman diagrams with interaction range \( R \geq L \). In case of shorter operators, these contributions have to be replaced \[ (17, 48) \] by the so-called wrapping interactions \[ (26, 61) \], that arise due to the truncation of the genus expansion beyond the planar contributions \[ 62 \]. For operators of length \( L = 3 \) in the closed chiral subsector, the respective analysis is very similar to the one in the \( \beta \)-deformed case \[ (52, 63) \]. Finite size corrections in the \( SL(2) \) subsectors of \( Z_\alpha \) orbifolds have recently been studied by means of the TBA and \( Y \)-system in \[ (64, 65) \].
Following [52][63], the chiral wrapping diagrams

\[ \lambda^3 I_3 \chi(1, 2, 3), \quad = \bar{\lambda}^3 I_3 \chi(3, 2, 1) \]  

replace the chiral maximum range diagrams in (5.1). Thereby, a cyclic identification \((P - \Lambda_s - T)_{34} \simeq (P - \Lambda_s - T)_{31}\) in the definition of the chiral functions (3.11) is understood. The above diagrams yield the same expressions as the diagrams in the first line of (5.1). Therefore, these terms persist in the dilatation operator. However, the contributions from the diagrams in the second line of (5.1) and the ones from (5.2) are removed by the subtraction procedure.

It turns out that there are no further contributions from wrapping diagrams. At three loops their overall UV divergences cancel with each other as in the case of the \(\beta\)-deformed theory [63]. The respective cancellations read

\[ \begin{align*}
\chi^{(1)} (1, 2, 3) & + \chi^{(1)} (3, 2, 1) = 0, \\
\chi^{(2)} (1, 2) & + \chi^{(2)} (2, 1) = 0,
\end{align*} \]

and they also hold for the respective reflected diagrams.

With the above described modifications, the dilatation operator that considers the leading wrapping correction for length \(L = 3\) operators then reads

\[
D_{3,w} = -4\left( \chi(1, 2, 3) + \chi(3, 2, 1) - 2(\rho + \hat{\rho})(\chi(1, 2) + \chi(2, 1)) \\
+ \chi(1, 2, 1) + \chi(2, 1, 2) + (\rho + \hat{\rho})^2 \chi(1) \right) \\
+ 2\zeta(3)(\rho - \hat{\rho})\left( (\rho - \hat{\rho}) \chi(1) - \hat{\rho}(\chi^{(1)} (1) + \bar{\chi}^{(2)} (1)) + \rho(\chi^{(2)} (1) + \bar{\chi}^{(1)} (1)) \right) \\
- \chi^{(2)} (1, 2) + \chi^{(2)} (2) (1, 2) + \chi^{(3)} (2, 1) + \chi^{(4)} (2) (2, 1) \\
+ \bar{\chi}^{(2)} (1, 2) - \bar{\chi}^{(3)} (1, 2) - \bar{\chi}^{(4)} (2, 1) - \bar{\chi}^{(4)} (2, 1). \]

Since all \(L = 2\) operators are protected (they correspond to the types of states given in (3.7), (3.8)), there is no need to calculate the next wrapping correction explicitly.

## 7 Eigenvalues

In this section we calculate some eigenvalues of the three-loop dilatation operator. First, we determine the dispersion relations of the scalar impurities. Then, we derive the anomalous dimensions for the shortest non-protected operators of length \(L = 3\) and \(L = 4\).
7.1 Dispersion relation

The momentum eigenstates of a single impurity are given by

\[ \psi(p) = \sum_m e^{imp_}\Phi_m \Phi Q_1 \Phi \ldots \Phi, \quad \tilde{\psi}(p) = \sum_m e^{imp_}\Phi_m \Phi \tilde{Q}_1 \Phi \ldots \Phi. \quad (7.1) \]

Note that these states are bifundamental in contrast to the one s in \( N = 4 \) SYM theory that are in the adjoint representation. Due to this difference, already at one-loop the local action of the dilatation operator depends on the gauge fixing parameter [53, 66].

Since we only work in Fermi-Feynman gauge, we can make no prediction of how a different gauge choice affects the result. Of course, gauge independence is guaranteed whenever the dilatation operator acts on a gauge invariant composite operator.

When the chiral functions (3.12) with a specific position of the impurity within the ingoing and outgoing interacting field lines are applied to the states in (7.1), they yield the phase shifts

\[
\begin{align*}
\chi_n^{(s)} (1, \ldots, n) \psi &= e^{i(n-1)p} \left( e^{-ip} - \hat{\rho} \right) \psi, \\
\tilde{\chi}_n^{(s)} (1, \ldots, n) \tilde{\psi} &= e^{i(n-1)p} \left( e^{-ip} - \hat{\rho} \right) \tilde{\psi}, \\
\chi_{n+1}^{(s)} (1, \ldots, n) \psi &= e^{i(n-1)p} \left( e^{-ip} - \hat{\rho} \right) \psi, \\
\tilde{\chi}_{n+1}^{(s)} (1, \ldots, n) \tilde{\psi} &= e^{i(n-1)p} \left( e^{-ip} - \hat{\rho} \right) \tilde{\psi}, \\
\chi_{n}^{(s)} (n, \ldots, 1) \psi &= e^{-i(n-1)p} \left( e^{-ip} - \hat{\rho} \right) \psi, \\
\tilde{\chi}_{n}^{(s)} (n, \ldots, 1) \tilde{\psi} &= e^{-i(n-1)p} \left( e^{-ip} - \hat{\rho} \right) \tilde{\psi}, \\
\chi_{n}^{(s)} (n, \ldots, 1) \psi &= e^{-i(n-1)p} \left( e^{-ip} - \hat{\rho} \right) \psi, \\
\tilde{\chi}_{n}^{(s)} (n, \ldots, 1) \tilde{\psi} &= e^{-i(n-1)p} \left( e^{-ip} - \hat{\rho} \right) \tilde{\psi}. \\
\end{align*}
\]

(7.2)

The above results for either \( \psi \) or \( \tilde{\psi} \) at fixed \( n \) combine to the phase shift for the respective impurity eigenstate generated by the sum

\[
\frac{1}{2} \left[ \chi(1, 2, \ldots, n) + \chi(n, \ldots, 2, 1) \right] \to - \cos(n-1)p \left[ 4 \sin^2 \frac{\rho}{2} + (\rho^2 - \hat{\rho}^2)^2 \right]. \quad (7.3)
\]

We then find that the phase shifts generated by an application of the dilatation operator (4.6), (5.13) as obtained from Feynman diagrams to the states in (7.1) are given by

\[
E(p) = 2\bar{g}^2 \left[ 4 \sin^2 \frac{\rho}{2} + (\rho^2 - \hat{\rho}^2)^2 \right] - 2\bar{g}^4 \left[ 4 \sin^2 \frac{\rho}{2} + (\rho^2 - \hat{\rho}^2)^2 \right]^2 \\
+ \bar{g}^6 \left[ 4 \left( 4 \sin^2 \frac{\rho}{2} + (\rho^2 - \hat{\rho}^2)^2 \right)^3 - 2\zeta(3)(\rho - \hat{\rho})^2 \left[ 4 \sin^2 \frac{\rho}{2} + (\rho^2 - \hat{\rho}^2)^2 \right] \\
- 2\zeta(3)(\rho^2 - \hat{\rho}^2)(\rho - \hat{\rho} + 2i \sin p) \right] + O(\bar{g}^8) ,
\]

(7.4)

where the upper and lower sign is assumed respectively for the momentum eigenstate \( \psi(p) \) and \( \tilde{\psi}(p) \) in (7.1). The above expression determines the first three orders in the ansatz of the dispersion relation

\[ E(p) = \sqrt{1 + h^2(g, \hat{g}) \left[ 4 \sin^2 \frac{\rho}{2} + (\rho^2 - \hat{\rho}^2)^2 \right] - 1 + f(g, \hat{g})(\rho - \hat{\rho} + 2i \sin p), \quad (7.5) \]

where the functions \( h^2(g, \hat{g}) \) and \( f(g, \hat{g}) \) are given by

\[ h^2(g, \hat{g}) = 4\bar{g}^2 - 4\bar{g}^6(\rho - \hat{\rho})^2 \zeta(3) + O(\bar{g}^8), \quad f(g, \hat{g}) = -2\bar{g}^6(\rho - \hat{\rho}^2)\zeta(3) + O(\bar{g}^8). \quad (7.6) \]

The function \( h^2(g, \hat{g}) \) begins with a one-loop contribution and is corrected at three loops, where \( f(g, \hat{g}) \) appears for the first time. Both three-loop contributions are proportional
to $\zeta(3)$, i.e. they are homogeneous maximal transcendental at that loop order and they vanish at the orbifold point where $g = \hat{g}$. Note also that the function $h^2(g, \hat{g})$ is strikingly similar to its counterpart in the ABJM and ABJ setups. There, its four-loop contribution is proportional to $\zeta(2)$ and hence homogeneous maximal transcendental in three dimensions. The three-loop term in $h^2(g, \hat{g})$ is due to an undeformed chiral function and thus fits into the all-order expression conjectured in [66]. This is not the case for the complex contribution with coefficient $f(g, \hat{g})$ that is generated by deformed chiral functions. Its imaginary part is generated by an anti-Hermitian combination in the dilatation operator.

Remarkably, the dispersion relation (7.5) is not real for real momentum $p$, but its three-loop term contains an imaginary contribution. This is due to anti-Hermitian terms in the three-loop dilatation operator (5.13). As explained in appendix E, the dilatation operator and the basis of operators can be altered by similarity transformations without affecting the eigenvalues of gauge invariant operators. We find that it can be transformed into the Hermitian expression (5.14) that leads to a dispersion relation without an imaginary term which read

$$E'(p') = \sqrt{1 + h^2(g, \hat{g})[4\sin^2\frac{p}{2} + (\rho^2 - \hat{\rho}^2)^2]} - 1 + f(g, \hat{g})(\rho - \hat{\rho}). \tag{7.7}$$

The transformed momentum eigenstates $\psi'(p')$ and $\tilde{\psi}'(p')$ given in (E.17) have the same form as $\psi(p)$ and $\tilde{\psi}(p)$ in (7.1) apart from a normalization factor and an imaginary shift of the original momentum $p$. It is given by

$$p' = p \pm i\bar{g}^4(\rho^2 - \hat{\rho}^2)\zeta(3) + \mathcal{O}(\bar{g}^6), \tag{7.8}$$

such that the energies obey $E(p) = E'(p')$.

We conclude this section with a discussion of the limiting cases. When the interpolating theory becomes the orbifold theory at $\rho = \hat{\rho} = 1$ or $\mathcal{N} = 2$ SCQCD, the imaginary part in (7.4) drops out and the dispersion relations become identical for both states $\psi(p)$ and $\tilde{\psi}(p)$. The respective limits read to three-loop order

$$E(p) = \begin{cases} 8g^2\sin^2\frac{p}{2} - 32g^4\sin^4\frac{p}{2} + 256g^6\sin^6\frac{p}{2} + \mathcal{O}(g^8) & \text{orbifold} \\ 2g^2 - 2g^4 + 4g^6(1 - \zeta(3)) + \mathcal{O}(g^8) & \mathcal{N} = 2 \text{ SCQCD} \end{cases}, \tag{7.9}$$

where at the orbifold point one recovers the dispersion relation of $\mathcal{N} = 4$ SYM theory [26], while in $\mathcal{N} = 2$ SCQCD it is independent of $p$ as observed before at one loop [16].

### 7.2 Eigenvalues of some short operators

In this section we explicitly give the results for the anomalous dimensions of $L = 3, 4$ operators. For later convenience we define the following parameters

$$G = \frac{g + \hat{g}}{2}, \quad \sigma = \frac{g - \hat{g}}{g + \hat{g}}. \tag{7.10}$$

The chiral composite operators of length $L = 2$ are all protected, since they correspond to the types of states given in (3.7) and (3.8).
The chiral composite operators of length \( L = 3 \) are the two groundstates of the form given in (3.7) and the two states
\[
O_0 = \hat{\rho} \, \text{tr} \left( Q_I \hat{Q}^I \Phi \right) + \text{tr} \left( Q_I \hat{Q}^I \Phi \right), \quad O_1 = -\rho \, \text{tr} \left( Q_I \hat{Q}^I \Phi \right) + \text{tr} \left( Q_I \hat{Q}^I \Phi \right),
\]
that are eigenstates of the dilatation operator given by (4.6) and by the wrapping corrected three-loop contribution (6.3). Their eigenvalues read
\[
\gamma_0 = 0, \quad \gamma_1 = 8G^2(1 + \sigma^2) - 32G^4(1 + \sigma^2)^2 + 256G^6(1 + \sigma^2)(1 + \sigma^2)^2 - \sigma^2 \zeta(3) \, . \quad (7.12)
\]

The chiral composite operators of length \( L = 4 \) are the three protected states of the form given in (3.7) and (3.8) and the three states
\[
O_1 = \text{tr} \left( Q_I \hat{Q}^I \Phi \Phi \right), \quad O_2 = \text{tr} \left( Q_I \hat{Q}^I \Phi \Phi \right), \quad O_3 = \text{tr} \left( Q_I \hat{Q}^I \Phi \Phi \right),
\]
that mix under renormalization. At the orbifold point, where the couplings are equal and hence \( G = g = \hat{g}, \sigma = 0 \), the eigenvectors do not depend on the coupling, since all chiral functions become proportional to a single mixing matrix. This simplification does not hold in the generic case. In terms of the parameters (7.10) the three eigenvalues are given by
\[
\gamma_0 = 0, \quad \gamma_{\pm} = 8G^2(1 + \sigma^2) - 4G^4(7(1 + \sigma^4) + 10\sigma^2) + 8G^6(1 + \sigma^2)(23(1 + \sigma^4) + 2\sigma^2(13 - 16\zeta(3)))
\]
\[
\pm 4G^2\sqrt{X}, \quad X = (1 - \sigma^2)^2(1 - 10G^2(1 + \sigma^2))
\]
\[
+ G^4(101(1 + \sigma^8) + 12\sigma^2(1 + \sigma^4) - 162\sigma^4 - 32\sigma^2(1 - \sigma^2)^2 \zeta(3))
\]
\[
- 4G^6((1 + \sigma^2)(95(1 + \sigma^8) + 84\sigma^2(1 + \sigma^4) + 90\sigma^4) - 48\sigma^2(1 - \sigma^2)^2(1 + \sigma^2) \zeta(3))
\]
\[
+ 4G^8(361(1 + \sigma^{12}) + 1298\sigma^2(1 + \sigma^8) + 2759\sigma^4(1 + \sigma^4) + 3708\sigma^6)
\]
\[
- 16\sigma^2(1 - \sigma^2)^2(27(1 + \sigma^4) + 58\sigma^2) \zeta(3)). \quad (7.14)
\]
At the orbifold point \( G = g = \hat{g}, \sigma = 0 \), the anomalous dimensions simplify to
\[
\gamma_0 = 0, \quad \gamma_{\pm} = 8g^2 - 28g^4 + 184g^6 \pm 4g^2(1 - 5g^2 + 38g^4) \, . \quad (7.15)
\]
For \( \mathcal{N} = 2 \) supersymmetric QCD, where \( G = \frac{2}{3}, g = \hat{g} = 0, \sigma = 1 \), the anomalous dimensions simplify to
\[
\gamma_0 = 0, \quad \gamma_{\pm} = 4g^2 - 6g^4 + g^6(18 - 8\zeta(3)) \pm 2g^4(1 - 7g^2) \, . \quad (7.16)
\]
In both cases, the square-root could be taken exactly without employing further series expansions.

8 Where are the elementary excitations?

The calculation presented in this paper determines the three-loop mixing in the closed chiral subsector that contains the operators (3.6). The excitations in this sector are
the bifundamental fields $Q_I$ and $\tilde{Q}^J$. As we have found they have complex dispersion relations (7.3), and this makes it difficult to interpret them as elementary excitations. In this section, we discuss the excitations of other closed subsectors that might be regarded as fundamental ones.

The fields of the theory have a definite scaling dimension $\Delta$ and $U(1)_R$ charge $J$. The excitations above the vacuum fields are classified according to the difference of $\Delta - J$. While the vacuum fields have $\Delta = 1$, $J = 1$ and hence $\Delta - J = 0$, the excitations have $\Delta - J \geq 1$. In $\mathcal{N} = 4$ SYM theory, all excitations with $\Delta - J = 1$ are elementary, i.e. they are magnons, while the ones with $\Delta - J \geq 2$ are their bound states. In contrast to this, in the interpolating theory the $\Delta - J = 1$ excitations $Q_I$ and $\tilde{Q}^J$ of the closed chiral subsector should not be thought of as being elementary. Therefore, we should search in other closed subsectors of the theory for $\Delta - J = 1$ excitations with real dispersion relations.

From our analysis we conclude that the complex terms in the dispersion relations of $Q_I$ and $\tilde{Q}^J$ appear due to the bifundamental nature of these excitations, resulting in a dependence on two gauge couplings. In order to avoid such terms we should focus on sectors that involve one gauge group only. We should look for $\Delta - J = 1$ excitations that transform in the same adjoint representation as the vacuum fields. They are either given by two of the adjoint fermions or by a lightcone component of covariant spacetime derivatives involving the respective gauge field. In a vacuum given by the first of the groundstates in (3.7), the respective fermions are the ones included in the superfields $\Phi$ and $V$. The composite operators that contain only one type of the fermionic or derivative excitations respectively correspond to the closed $SU(1|1)$ or $SL(2)$ subsector of $\mathcal{N} = 4$ SYM theory.

We will now argue why in these subsectors non-trivial deviations from the case of $\mathcal{N} = 4$ SYM theory should only show up from three loops on. Firstly, since the vacuum fields and excitations in these sectors transform in the adjoint representation of only one factor of the gauge group, a dependence on the coupling constant of the other gauge group factor can only arise when bifundamental matter fields form a loop in which at least one further interaction occurs. This requires at least two loops. For example, a diagram of this type is given by the first one in (C.6). It contributes to the two-loop self energies of the adjoint chiral matter fields. Due to its presence, the final result (C.8) for the self energy is the same for all chiral matter fields, and it depends on the product of both gauge couplings. This dependence then occurs in the overall UV divergence of diagrams at three and higher loops, in which the finite two-loop chiral self energy appears as subdiagram. Such diagrams contribute in the orbifold theory, where the two couplings are equal, while they are absent in $\mathcal{N} = 2$ SCQCD, where one of the couplings is zero. If one does not find other diagrams that compensate this behavior, the dilatation operator of $\mathcal{N} = 2$ SCQCD would start to deviate from the $\mathcal{N} = 4$ SYM result in the respective subsector at three loops. The one- and two-loop dilatation operators of the $SU(1|1)$ and $SL(2)$ subsectors is hence identical to their counterparts in $\mathcal{N} = 4$ SYM theory up to a trivial identification of the coupling constants. Deviations can first appear at three loops, where integrability can then be non-trivially tested for the first time.
9 Conclusions

The central result of this paper is the expression for the dilatation operator as given in \((5.13)\) at three loops. For the discovery of new effects in the interpolating theory and their investigation it was necessary to work at least at this order. Three loops is the first order at which the chiral functions of the theory are deformed by gauge interactions. There occur homogeneous transcendental contributions that involve \(\zeta(3)\). They vanish at the point of equal couplings, where the theory becomes a \(\mathbb{Z}_2\) orbifold of \(\mathcal{N} = 4\) SYM theory, and the dilatation operator reduces to the \(\mathcal{N} = 4\) SYM result.

We have found that the dispersion relation \((7.5)\) contains a function \(h^2(g, \hat{g})\) with non-vanishing three-loop contribution. Furthermore, for real momentum \(p\) it develops an imaginary part away from the orbifold and \(\mathcal{N} = 2\) SCQCD points. Both of these deviations from the \(\mathcal{N} = 4\) result are homogeneous maximal transcendental, i.e. proportional to \(\zeta(3)\) at three loops. The imaginary contribution is due to anti-Hermitean terms in the three-loop dilatation operator. They can be removed by a non-unitary similarity transformation that also transforms the basis of eigenstates. In particular, the momenta of eigenstates of single excitations acquire imaginary two-loop shifts. The dispersion relation \((7.5)\) contains extra terms involving \(f(g, \hat{g})\) that are missing in a symmetry-based all-order conjecture in \([66]\). Concerning a spin chain interpretation of operator mixing in the closed chiral subsector of the theory, the excitations \(Q_i\) and \(\tilde{Q}_i\) should be regarded as effective rather than as elementary magnons. In addition, compared to \(\mathcal{N} = 4\) SYM theory, the two-body S-matrix is further deformed by pure scattering terms that first show up at three loops and come with a deformed chiral function.

Based on our findings we have argued that one should search for integrability in other subsectors of the interpolating theory that contain elementary excitations. They should correspond to the \(SU(1|1)\) and \(SL(2)\) subsectors of \(\mathcal{N} = 4\) SYM theory and involve the adjoint fields associated with only one factor of the product gauge group. The respective excitations should have simpler dispersion relations than \((7.5)\). On the basis of the finite two-loop self energy corrections we have predicted that non-trivial deviations from the respective sectors of \(\mathcal{N} = 4\) SYM theory can only show up from three loops on.

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A Gauge fixed action and Feynman rules in $\mathcal{N} = 1$ superspace

The interpolating theory in an $\mathcal{N} = 1$ superspace formulation is presented in section 3. It contains two real vector superfields $V, \tilde{V}$ and two chiral superfields $\Phi, \tilde{\Phi}$ that all transform in the adjoint representation of respectively the first and second factor of the product gauge group $SU(N) \times SU(N)$. Furthermore, it contains chiral superfields $Q, \tilde{Q}$ that respectively transform in the bifundamental and anti-bifundamental representation of $SU(N) \times SU(N)$. The gauge fixing proceeds independently for each gauge field $V, \tilde{V}$ as in the case of $\mathcal{N} = 4$ SYM theory \cite{57}, and we obtain the gauge fixed action

$$S_{gf} = S - \frac{1}{2} \int d^4x \, d^4\theta \left[ \frac{1}{\alpha} \text{tr} \left( (D^2 V)(\bar{D}^2 V) \right) + \frac{1}{\alpha} \text{tr} \left( (D^2 \tilde{V})(\bar{D}^2 \tilde{V}) \right) \right]$$

$$+ \int d^4x \, d^4\theta \left[ \text{tr} \left( (c' + \bar{c}) \text{L}_{\text{YM}}(c + \bar{c} + \coth L_{\text{YM}}(c - \bar{c})) \right) + \text{tr} \left( (\bar{c}' + c) \text{L}_{\text{YM}}(\bar{c} + c + \coth L_{\text{YM}}(\bar{c} - c)) \right) \right], \quad (A.1)$$

where $S$ is the action given in (3.1), and $L_{\text{YM}} X = [V,X]$. The fields are decomposed as

$$V = V_a T^a, \quad \tilde{V} = \tilde{V}_{\tilde{a}} \tilde{T}^\tilde{a},$$

$$c = c_a T^a, \quad \tilde{c} = \tilde{c}_{\tilde{a}} \tilde{T}^\tilde{a}, \quad Q_i = Q_i^a B_a, \quad Q^i = \tilde{Q}_{\tilde{i}}^a B_a, \quad (A.2)$$

$$\Phi = \Phi_a T^a, \quad \tilde{\Phi} = \tilde{\Phi}_{\tilde{a}} \tilde{T}^\tilde{a},$$

in terms of representation matrices for the product gauge group $SU(N) \times SU(N)$

$$(T^a)^i_j, \quad (\tilde{T}^\tilde{a})^i_{\tilde{j}}, \quad (B_a)^i_j, \quad (B^\tilde{a})^i_{\tilde{j}}, \quad (A.3)$$

that transform the fundamental indices $i$ and $\tilde{i}$ of the respective $SU(N)$ factor into adjoint $a$ and $\tilde{a}$, (anti-)bifundamental $\underline{a}$ indices. The matrices of the adjoint representations fulfill the commutation relations

$$[T^a, T^b] = i f_{abc} T^c, \quad [\tilde{T}^\tilde{a}, \tilde{T}^\tilde{b}] = i f_{\underline{a}bc} \tilde{T}^\underline{c} \quad (A.4)$$

of the respective Lie-algebra. Furthermore, the matrices obey

$$(T^a)^i_j (T^a)^k_l = \delta^i_j \delta^k_l - \frac{1}{N} \delta^i_j \delta^k_l, \quad (\tilde{T}^\tilde{a})^i_{\tilde{j}} (\tilde{T}^\tilde{a})^k_{\tilde{l}} = \delta^i_{\tilde{j}} \delta^k_{\tilde{l}} - \frac{1}{N} \delta^i_{\tilde{j}} \delta^k_{\tilde{l}}, \quad (B_a)^i_j (B_a)^k_l = \delta^i_j \delta^k_l, \quad (B_{\underline{a}})^i_{\underline{j}} (B_{\underline{a}})^k_{\underline{l}} = \delta^i_{\underline{j}} \delta^k_{\underline{l}}, \quad (A.5)$$

where summations over $a, \tilde{a}, \underline{a}$ are understood.

B Feynman rules

In this appendix we present the Feynman rules that are required for a three-loop calculation. We use the Wick rotated rules, i.e. we have transformed $e^{-iS} \to e^{S}$ in the path
integral. In supersymmetric Fermi-Feynman gauge where \( \alpha = 1 + \mathcal{O}(g_{YM}^2) \), the vector, chiral and ghost propagators are given by

\[
\langle V_a V_b \rangle = \frac{-\delta_{ab}}{p^2} \delta^4(\theta_1 - \theta_2),
\]

\[
\langle \Phi_a \bar{\Phi}_b \rangle = \frac{-\delta_{ab}}{p^2} \delta^4(\theta_1 - \theta_2),
\]

\[
\langle Q^i_a \bar{Q}^k_j \rangle = \frac{-\delta^i_j \delta^k_a}{p^2} \delta^4(\theta_1 - \theta_2), \quad (B.1)
\]

\[
\langle \tilde{Q}^i_a \tilde{Q}^k_j \rangle = \frac{-\delta^i_j \delta^k_a}{p^2} \delta^4(\theta_1 - \theta_2),
\]

\[
\langle \bar{c}'_a c_b \rangle = -\langle c'_a \bar{c}_b \rangle = \frac{-\delta_{ab}}{p^2} \delta^4(\theta_1 - \theta_2).
\]

The expressions for the propagators of \( \hat{V} \), \( \hat{\Phi} \) and \( \hat{c} \), \( \hat{\bar{c}}' \) are identical up to the replacement \( \delta_{ab} \rightarrow \delta_{\hat{a}\hat{b}} \).

The cubic gauge vertex for the vector field \( V \) is given by

\[
V^{V3} = \left( \left( \frac{2}{g_{YM}^2} \text{tr} \left( \frac{T^a}{2} \left[ T^b, T^c \right] \right) \right) \right) \left( \left( \begin{array}{ccc}
D^a & D^b & D^c \\
D^a & D^b & D^c \\
D^a & D^b & D^c
\end{array} \right) \right)
\]

\[
(B.2)
\]

where the color indices are labeled \( (a, b, c) \) clockwise, starting with the leg to the left. The respective vertex for the field \( \hat{V} \) is identical up to a replacement of the gauge coupling and the color trace. The D-algebra has to be performed for all six permutations of the structure of the covariant derivatives at its legs. The only purpose of the vertices that appear on the r.h.s. of the equation is to display this structure. They do not contain any other non-trivial factors.

The corrections from the gauge parameter \( \alpha \) do not appear in the diagrams explicitly considered in this paper.
The cubic gauge-matter vertices are given by

\[ V_{\Phi \Phi} = \frac{1}{2} g_{YM} \text{tr} (T^a [T^b, T^c]) , \quad V_{\Phi \Phi} = \frac{1}{2} \hat{g}_{YM} \text{tr} (\hat{T}^a [\hat{T}^b, \hat{T}^c]) , \]

\[ V_{Q \hat{Q} i} = \frac{1}{2} g_{YM} \delta^i_j \text{tr} (T^a B^a B^b) , \quad V_{Q \hat{Q} i} = \frac{1}{2} (\hat{g}_{YM}) \delta^i_j \text{tr} (\hat{T}^a B^a B^b) , \]

\[ V_{\Phi \Phi} = \frac{1}{2} i g_{YM} \delta^j_i \text{tr} (T^a B^a B^b) , \quad V_{\Phi \Phi} = \frac{1}{2} (\hat{i} \hat{g}_{YM}) \delta^j_i \text{tr} (\hat{T}^a B^a B^b) , \]

The color indices are labeled \((a, b, c)\) clockwise, starting with the leg to the left.

For the three-loop calculation we only need some of the quartic gauge-matter vertices.

(B.3)
They read

\[
V_{V^2\phi} = \left( -\frac{g_{YM}^2}{2} \right) \left( \text{tr} \left([T^a, T^b][T^c, T^d]\right) + \text{tr} \left([T^a, T^c][T^b, T^d]\right) \right),
\]

\[
V_{V^2Q_i Q_j} = \frac{g_{YM}^2}{2} \delta^i_j \text{tr} \left( \{T^a, T^b\}B_{2a}B_{2d}\right),
\]

\[
V_{V^2Q_i \hat{Q}^j} = \frac{\hat{g}_{YM}^2}{2} \delta^i_j \text{tr} \left( \{\hat{T}^a, \hat{T}^b\}B_{2a}B_{2d}\right),
\]

\[
V_{VQ_i \hat{Q}^j} = \frac{g_{YM}^2}{2} \delta^i_j \text{tr} \left( T^a B_{2a} \hat{T}^{b} B_{2d}\right),
\]

\[
V_{\hat{V}^2Q_i \hat{Q}^j} = \frac{\hat{g}_{YM}^2}{2} \delta^i_j \text{tr} \left( \{\hat{T}^a, \hat{T}^b\}B_{2a}B_{2d}\right),
\]

\[
V_{\hat{V}Q_i \hat{Q}^j} = \frac{g_{YM}^2}{2} \delta^i_j \text{tr} \left( T^a B_{2a} \hat{T}^{b} B_{2d}\right),
\]  

where the color indices are labeled \((a, b, c, d)\) clockwise starting with the leg in the upper left corner.

### C One- and two-loop subdiagrams

In this appendix we present the non-vanishing one- and two-loop subdiagrams that appear in our calculation. In order to collectively represent the subdiagrams with all possible combinations of adjoint and bifundamental fields, we introduce generic \('t\) Hooft couplings \(\lambda_1, \lambda_2, \lambda_3\) and dotted lines as chiral field lines that denote the adjoint and (anti)-bifundamental fields. Each of the couplings \(\lambda_i\) is then built from the Yang-Mills gauge coupling and rank that is associated with a respective face of the subdiagram that carries label \(i\). In order to obtain the subdiagram for a specific field configuration, one just has to replace the dotted field lines by the matter fields of the theory and then determine the gauge factor that is associated with each face.
C.1 One-loop chiral vertex correction

Omitting all factors from the respective tree-level vertex of (B.3) the one-loop corrections to the chiral vertices are easily summarized as

\[
\bar{D}^2 \bar{D}^2 \lambda_1 + \ldots = \left( \begin{array}{c}
\bar{D}^2 \bar{D}^2 \\
\bar{D}^2 \bar{D}^2
\end{array} \right),
\]

where the ellipsis denote the remaining two diagrams obtained by clockwise cyclic permutations of the interactions of one sector to the next sector. Under each such permutation the coupling constants associated with the individual sectors also have to be replaced as \( \lambda_1 \rightarrow \lambda_2, \lambda_2 \rightarrow \lambda_3, \lambda_3 \rightarrow \lambda_1 \). The d’Alembertian \( \Box \) cancels the respective propagator and thereby produces a minus.

The one-loop correction to the cubic gauge-matter vertex is given by

\[
\bar{D}^2 \bar{D}^2 \\
\bar{D}^2 \bar{D}^2
\]

where we have omitted the covariant derivatives. The result for the first term containing the cubic gauge vertex (B.2) can be found e.g. in [51]. We just have to generalize it including the different 't Hooft couplings, and then we obtain

\[
\frac{\lambda_2}{2}.
\]

The covariant derivatives and also momenta are read-off when leaving the vertices. The other two contributions involving only cubic vertices are also easily adopted from the
expressions in [51]. They read

\[
\begin{pmatrix}
\bar{D}^2 & D^2 \\
\bar{D}^2 & D^2 \\
\bar{D}^2 & D^2 \\
\bar{D}^2 & D^2 \\
\end{pmatrix}
\begin{pmatrix}
\bar{D}^2 & D^2 \\
\bar{D}^2 & D^2 \\
\bar{D}^2 & D^2 \\
\bar{D}^2 & D^2 \\
\end{pmatrix}
= \begin{pmatrix}
(D^2)_{\beta}^\alpha - (p_3 - t)^{\alpha \delta} \frac{D_\delta D_\alpha}{(p^2)^2}

(\bar{D}^2)_{\beta}^\alpha - (l + p_2)^{\alpha \delta} \frac{D_\delta D_\alpha}{(p^2)^2}

(D^2)_{\beta}^\alpha - (l + p_2)^{\alpha \delta} \frac{D_\delta D_\alpha}{(p^2)^2}

(\bar{D}^2)_{\beta}^\alpha - (l + p_2)^{\alpha \delta} \frac{D_\delta D_\alpha}{(p^2)^2}
\end{pmatrix}
\begin{pmatrix}
\lambda_1 \\
\lambda_1 \\
\lambda_1 \\
\lambda_1 \\
\end{pmatrix},
\]

(C.4)

where we have inserted \(-\frac{\bar{D}}{p^2} = 1\) in order to obtain triangle integrals. We sum up the above expressions and simplify the result as explained in [51]. The expression for the one-loop corrections of the cubic gauge-matter vertices is then given by

\[
\begin{pmatrix}
\bar{D}^2 & D^2 \\
\bar{D}^2 & D^2 \\
\end{pmatrix}
\begin{pmatrix}
\bar{D}^2 & D^2 \\
\bar{D}^2 & D^2 \\
\end{pmatrix}
= \begin{pmatrix}
\frac{\lambda_2}{2}

\frac{\lambda_2}{2}
\end{pmatrix},
\]

(C.5)

(C.2) Two-loop chiral self energies

All one-loop self energies are identically zero at the conformal point. The two-loop chiral self energies are finite and do not contribute to the two-loop \(\beta\)-function. However, at higher loops overall UV divergences may be generated by diagrams that contain them as subdiagrams, and we therefore have to calculate them. The two-loop self energies of the chiral adjoint and bifundamental matter fields are determined by the following

\[
\begin{pmatrix}
\bar{D}^2 & D^2 \\
\bar{D}^2 & D^2 \\
\end{pmatrix}
\begin{pmatrix}
\bar{D}^2 & D^2 \\
\bar{D}^2 & D^2 \\
\end{pmatrix}
= \lambda_1 - \frac{1}{4}
\begin{pmatrix}
[\bar{D}_\alpha D_\beta]

[p_3 D_\delta]
\end{pmatrix}
\begin{pmatrix}
(\bar{D}^2)_{\beta}^\alpha - (l + p_2)^{\alpha \delta} \frac{D_\delta D_\alpha}{(p^2)^2}

(D^2)_{\beta}^\alpha - (l + p_2)^{\alpha \delta} \frac{D_\delta D_\alpha}{(p^2)^2}
\end{pmatrix}
\lambda_2.
\]

(C.5)
Feynman diagrams

\[ S_\Phi = -2\lambda \hat{\lambda} I_{2t}, \quad S_{Q, \tilde{Q}} = -\left( \lambda^2 + \hat{\lambda}^2 \right) I_{2t}, \]
\[ S_{2a} = \ldots \ldots + \ldots \ldots + \ldots \ldots = \left( \lambda_1^2 + \lambda_2^2 \right) I_2, \]
\[ S_{2b} = \ldots \ldots + \ldots \ldots + \ldots \ldots = 4\lambda_1 \lambda_2 I_2, \]
\[ S_{3a} = \ldots \ldots + \ldots \ldots = \frac{\lambda_1^2 + \lambda_2^2}{2} I_1^2, \]
\[ S_{3b} = \ldots \ldots + \ldots \ldots = 2\lambda_1 \lambda_2 I_1^2, \]
\[ S_4 = \ldots \ldots + \ldots \ldots = -2\lambda_1 \lambda_2 (I_1^2 + I_{2t}), \]
\[ S_5 = \ldots \ldots + \ldots \ldots + \ldots \ldots = -2(\lambda_1 + \lambda_2)^2 I_2, \]
\[ S_6 = \ldots \ldots + \ldots \ldots = \frac{\lambda_1^2 + \lambda_2^2}{2} (-I_1^2 + 2I_2 + 2I_{2t}), \]

where \( \lambda_1 = \lambda_2 = \lambda, \hat{\lambda} \) for \( \Phi, \hat{\Phi} \) and \( (\lambda_1, \lambda_2) = (\lambda, \hat{\lambda}), (\hat{\lambda}, \lambda) \) respectively for \( Q, \tilde{Q} \).

These configuration yield a common result for the finite two-loop self energies of the chiral matter fields that is given by

\[ \Sigma_{\Phi, Q, \tilde{Q}} = S_{\Phi, Q, \tilde{Q}} + S_{2a} + S_{2b} + S_{3a} + S_{3b} + S_4 + S_5 + S_6 = -2\bar{\lambda}^2 I_{2t}. \]  

Restoring the dependence on the spinor derivative and the correct proportionality to the external momentum \( p \), the two-loop chiral self energy can be written as

\[ \ldots \ldots = -2\bar{\lambda}^2 p^{2(D-3)} D^2 \].  

The gray scaled part of the graph is identified as the integral \( I_{2t} \) given in (D.4).
C.3 Two-loop chiral vertex corrections

The two-loop correction of the chiral vertex is given as a sum of the following non-vanishing contributions

\[ 2 = 1 + 1 + 1 + 1 + \ldots, \]  

where the ellipsis denote again the omitted diagrams that are obtained by cyclic rotations of the interactions of the displayed diagrams to the other sectors. After D-algebra, the final result can be cast into the form

\[
\begin{align*}
\frac{3}{2} \cdot \frac{1}{2} = & \quad \left( 2\lambda_1^2 \raisebox{-0.5em}{\begin{array}{c} \lambda_2 \\ \end{array}} - (2\lambda_2\lambda_3 - (\lambda_2 - \lambda_3)^2) \right) \\
& + (\lambda_2^2 - \lambda_3^2) \\
& - (\lambda_2^2 + (\lambda_1 - \lambda_2)^2) \\
& + \lambda_2^2 \\
& + \lambda_3^2 \ldots,
\end{align*}
\]

where the ellipsis denotes the clockwise cyclic permutations of the above structures, thereby replacing \( \lambda_1 \rightarrow \lambda_2, \lambda_2 \rightarrow \lambda_3, \lambda_3 \rightarrow \lambda_1 \). The coupling constants are thereby associated with the three sectors, counting clockwise and starting with the sector to the right.


\section*{D Integrals}

In this appendix we collect the integral that are relevant for our three-loop calculation.

In \( D \)-dimensional Euclidean space the simple loop integral that involves two propagators of massless fields with respective weights \( \alpha \) and \( \beta \) and external momentum \( p^2 = 1 \) can be expressed in terms of a \( G \)-function. It is given by

\[
G(\alpha, \beta) = \frac{1}{(2\pi)^D} \int \frac{d^Dk}{k^{2\alpha}(k - p)^{2\beta}} \bigg|_{p^2=1} = \frac{\Gamma(\frac{D}{2} - \alpha)\Gamma(\frac{D}{2} - \beta)\Gamma(\alpha + \beta - \frac{D}{2})}{(4\pi)^{\frac{D}{2}}\Gamma(\alpha)\Gamma(\beta)\Gamma(D - \alpha - \beta)} .
\]  

(D.1)

Likewise, the \( G \)-functions for integrals with one momentum or two momenta in the numerators can respectively be written as

\[
G_1(\alpha, \beta) = \frac{1}{2}(-G(\alpha, \beta - 1) + G(\alpha - 1, \beta) + G(\alpha, \beta)) ,
\]

\[
G_2(\alpha, \beta) = \frac{1}{2}(-G(\alpha, \beta - 1) - G(\alpha - 1, \beta) + G(\alpha, \beta)) .
\]  

(D.2)

To three loops, we need the following integrals and their overall UV divergences

\[
I_1 = G(1,1) ,
\]

\[
I_2 = G(1,1)G(3 - \frac{D}{2}, 1) ,
\]

\[
I_3 = G(1,1)G(3 - \frac{D}{2}, 1)G(5 - D, 1) ,
\]

\[
I_{3t} = I_{2t}G(5 - D, 1) ,
\]

\[
I_{3b} = ,
\]

\[
I_{3bb} = G(1,1)^2G(3 - \frac{D}{2}, 3 - \frac{D}{2}) ,
\]

\[
I_{32t} = G_1(2,1)G_1(4 - \frac{D}{2}, 1)G_2(6 - \frac{D}{2}, 1) ,
\]

\[
I_{32t} = \frac{2}{D-4}G(1,1)(G(1,2) + G(3 - \frac{D}{2}, 2)) = \frac{1}{(4\pi)^4}6\zeta(3) + \mathcal{O}(\epsilon) .
\]  

(D.3)

where \( I = KR(I) \) denotes the pole part of the respective integral \( I \). It is extracted by \( K \) after the subdivergences have been removed by the operation \( R \). The integral \( I_{3t} \) that appears as substructure in \( I_{3t} \) and in the final expression for the two-loop chiral self energy \((C.8)\) is finite and given by

\[
I_{3t} = \frac{2}{D-4}G(1,1)(G(1,2) + G(3 - \frac{D}{2}, 2)) = \frac{1}{(4\pi)^4}6\zeta(3) + \mathcal{O}(\epsilon) .
\]  

(D.4)
E  Similarity transformations

The representation of the dilatation operator is not unique, but it may be transformed by a change of the basis of operators that does not alter its eigenvalues. In this appendix, we work out the most general transformation that preserves the structural constraints coming from the underlying Feynman graphs. We include non-unitary transformations that allow us to remove the anti-Hermitian contributions in the three-loop dilatation operator (5.13).

The similarity transformations can be realized as

\[ D' = e^{-\chi} D e^{\chi} = D + \delta D , \]  

where \( \chi \) is a linear combination of flavor operations. We demand that the transformation preserves the structural constraints coming from the underlying Feynman diagrams, i.e. the transformation must not increase the maximum range and the maximum power in \( \rho \) and \( \hat{\rho} \) found in the dilatation operator at a given order in \( \bar{g} \). This is guaranteed if the weak coupling expansion of \( \chi \) only contains those flavor operations that can be associated with Feynman diagrams at the considered order. It does not matter whether these Feynman diagrams have an overall UV divergence or are finite. In addition to the chiral functions we therefore also have to consider operators in flavor space that are generated by the finite Feynman diagrams involving gauge interactions only. Their elementary building block is given by

\[ i_j = (\Lambda_t)^{ij}, \quad (\Lambda_t)^{ij} = \left\{ \begin{array}{l} \rho \mathbb{1} \cup (\Phi^*, \Phi) \cup (\hat{Q}^i, Q_j) \\ \hat{\rho} \mathbb{1} \cup (\hat{\Phi}^*, \hat{\Phi}) \cup (\hat{Q}^i, \hat{Q}^j) \end{array} \right. , \]  

where the subscript \( t \) of \( \Lambda_t \) is there to remind us that the underlying diagram is a \( t \)-channel when regarding the fields at positions \( (i, j) \) as incoming. In analogy to the chiral functions (3.11), we introduce the operators

\[ \Lambda_t(a_1, \ldots, a_n) = \sum_{r=0}^{L-1} \prod_{i=1}^{n} (\Lambda_t)_{r+a_i, r+a_i+1} , \quad a_1 \leq \cdots \leq a_n \]  

that are generated by Feynman diagrams involving \( n \) gauge fields. The identity operation is given by \( \Lambda_t() = \chi() \). Note that we can impose the order \( a_1 \leq \cdots \leq a_n \) on the list of arguments, since the operators \( (\Lambda_t)^{ij}, (\Lambda_t)^{kl} \) commute for any \( i, j, k, l \). If \( \hat{\rho} = \rho \) these operators reduce to the identity and cause no effect in the transformation (E.1). This is the reason why in \( \mathcal{N} = 4 \) SYM theory they need not be considered when constructing the similarity transformations.

Based on the aforementioned considerations, the most general ansatz for \( \chi \) that leads to similarity transformations up to three loops is given by

\[ \chi = \bar{g}^2 \left( \delta_1 \chi(1) + \delta_{11} \Lambda_t(1) \right) + \bar{g}^4 \left( \delta_{21} \chi^{(1)}(1) + \delta_{22} \chi^{(2)}(1) + \delta_{23} \chi^{(1)}(1) + \delta_{24} \chi^{(2)}(1) + \delta_{21} \chi^{(1)}(1) + \delta_{22} \chi^{(2)}(1) + \delta_{23} \chi^{(1)}(1) + \delta_{24} \chi^{(2)}(1) + \delta_{25} \chi(1, 2) + \delta_{26} \chi(2, 1) + \delta_{27} \Lambda_t(1) \right) . \]  

(E.4)
Chiral functions that involve the flavor trace operator are not included, since they do not contribute in the closed chiral subsector we are interested in. Furthermore, it is not necessary to consider at two loops the operators $\Lambda_i(1, 2)$ and combinations of $(\Lambda_i)_{ij}$ with the flavor operations of the chiral functions. They act identical to the operators already present in the above ansatz apart from additional dependences on $\rho$ and $\tilde{\rho}$ that appear as prefactors. Their contributions can be absorbed into the coefficients $\delta_{2i}$, $\delta_{2i}$, $i = 1, \ldots, 4$ and $\delta_{2i}$ by allowing them to be complex linear functions of $\rho$, and $\tilde{\rho}$. They should then fulfill

$$
\delta_{2i}(\rho, \tilde{\rho}) = \delta_{2i}(\rho, \rho), \quad i = 1, \ldots, 4,
$$

while the remaining coefficients $\delta_1, \delta_{11}, \delta_{25}, \delta_{26}$ are complex constants.

The one-loop contribution in Eq. (6) coming with $\chi(1)$ is essentially the one-loop dilatation operator $D_1$ itself as given in Eq. (16). It therefore commutes with $D_1$ and does not generate a transformation of the two-loop dilatation operator $D_2$. However, the one-loop term involving $\Lambda_i(1)$ leads to a transformation of $D_2$. We do not want this to happen, since $D_1$ and also $D_2$ are already Hermitean and in a minimal form. Hence, from now on we set $\delta_{11} = 0$. The three-loop dilatation operator $D_3$ is then modified by the following terms

$$
\delta D_3 = -[\chi_1, D_2] - [\chi_2, D_1]
= 2(\epsilon_2[\chi(1), \chi(1, 2)] + \epsilon_2[\chi(1), \chi(2, 1)])
+ \epsilon_2(\chi(2)^{(1)}_1, \chi(2)_1) + (\epsilon_2 + \epsilon_2)[\chi(2)^{(1)}_1, \chi(2)_1] + \epsilon_2[\chi(2)^{(1)}_1, \chi(2)_1]
+ \epsilon_2[\chi(2)^{(1)}_1, \chi(2)_1] + (\epsilon_2 - \epsilon_2)[\chi(2)^{(1)}_1, \chi(2)_1] + (\epsilon_2 - \epsilon_2)[\chi(2)^{(1)}_1, \chi(2)_1]
+ \epsilon_2[\chi(2)^{(1)}_1, \chi(2)_1] + (\epsilon_2 + \epsilon_2)[\chi(2)^{(1)}_1, \chi(2)_1] + (\epsilon_2 - \epsilon_2)[\chi(2)^{(1)}_1, \chi(2)_1]
+ \epsilon_2[\chi(2)^{(1)}_1, \chi(2)_1] + (\epsilon_2 - \epsilon_2)[\chi(2)^{(1)}_1, \chi(2)_1] + (\epsilon_2 - \epsilon_2)[\chi(2)^{(1)}_1, \chi(2)_1]
+ \mu[\chi(1)_1, \chi(1)_1] + (\mu - \epsilon_2)[\chi(1)_1, \chi(1)_1] + (\mu - \epsilon_2)[\chi(1)_1, \chi(1)_1]
+ (\mu - \epsilon_2)[\chi(1)_1, \chi(1)_1] + (\mu - \epsilon_2)[\chi(1)_1, \chi(1)_1] + (\mu - \epsilon_2)[\chi(1)_1, \chi(1)_1]
+ (\mu - \epsilon_2)[\chi(1)_1, \chi(1)_1] + (\mu - \epsilon_2)[\chi(1)_1, \chi(1)_1] + (\mu - \epsilon_2)[\chi(1)_1, \chi(1)_1]
+ (\mu - \epsilon_2)[\chi(1)_1, \chi(1)_1] + (\mu - \epsilon_2)[\chi(1)_1, \chi(1)_1] + (\mu - \epsilon_2)[\chi(1)_1, \chi(1)_1]
$$

where $\chi_1, \chi_2$ are the respective one- and two-loop contributions in Eq. (16), and we have introduced the new parameters

$$
\epsilon_2 = \delta_1 - \delta_{25}, \quad \epsilon_2 = \delta_1 - \delta_{26}, \quad \epsilon_2 = \delta_21 - \delta_{24}, \quad \epsilon_2 = \delta_21 - \delta_{24}, \quad \epsilon_2 = \delta_{22} - \delta_{23}, \quad \epsilon_2 = \delta_{22} - \delta_{23}, \quad \mu = \delta_{21} - \delta_{21}, \quad \nu = \delta_{27}.
$$

The coefficient $\delta_1$ is absorbed into the redefinitions of $\delta_{25}$ and $\delta_{26}$. We can hence set $\delta_1 = 0$ such that $\delta D_3$ is entirely generated by the commutator of $\chi_2$ with $D_1$. The
individual commutators of chiral functions that appear in (E.6) are evaluated to
\[
[\chi(1), \chi(1, 2)] = \chi(2, 1, 2) + \chi(1, 3, 2) - \chi(2, 1, 3) - \chi(1, 2, 1),
\]
\[
[\chi(1), \chi(2, 1)] = \chi(1, 3, 2) + \chi(1, 2, 1) - \chi(2, 1, 2) - \chi(2, 1, 3).
\]
(E.8)

The commutators of chiral functions with fixed incoming and outgoing flavor arrangements are given by
\[
[\chi_{(1)}^{(1)}(1), \chi_{(1)}^{(2)}(1)] = \chi_{(1)}^{(1)}(2, 1) + \rho \chi_{(1)}^{(2)}(1),
\]
\[
[\chi_{(1)}^{(1)}(1), \chi_{(1)}^{(2)}(1)] = -\rho \chi_{(2)}^{(1)}(1) - \chi_{(2)}^{(1)}(1, 2),
\]
\[
[\chi_{(1)}^{(1)}(1), \chi_{(2)}^{(2)}(1)] = \chi_{(2)}^{(2)}(2, 1) - \chi_{(2)}^{(2)}(1, 2),
\]
\[
[\chi_{(1)}^{(2)}(1), \chi_{(1)}^{(1)}(1)] = \rho \chi_{(2)}^{(1)}(1) - \rho \chi_{(2)}^{(1)}(1),
\]
\[
[\chi_{(1)}^{(2)}(1), \chi_{(2)}^{(2)}(1)] = \chi_{(2)}^{(2)}(2, 1) + \rho \chi_{(2)}^{(2)}(1),
\]
\[
[\chi_{(1)}^{(2)}(1), \chi_{(2)}^{(2)}(1)] = -\rho \chi_{(2)}^{(1)}(1) - \chi_{(2)}^{(1)}(1, 2),
\]
and by
\[
[\chi_{(1)}^{(1)}(1), \tilde{\chi}_{(1)}^{(1)}(1)] = 0,
\]
\[
[\chi_{(1)}^{(1)}(1), \tilde{\chi}_{(1)}^{(2)}(1)] = \chi_{(1,2)}^{(3)}(1, 2),
\]
\[
[\chi_{(1)}^{(1)}(1), \tilde{\chi}_{(2)}^{(2)}(1)] = -\chi_{(1,2)}^{(3)}(2, 1),
\]
\[
[\chi_{(1)}^{(1)}(1), \tilde{\chi}_{(2)}^{(2)}(1)] = \chi_{(1,3)}^{(3)}(1, 2) - \chi_{(1,3)}^{(1,2)}(2, 1),
\]
\[
[\chi_{(2)}^{(1)}(1), \tilde{\chi}_{(1)}^{(1)}(1)] = -\tilde{\chi}_{(1,2)}^{(3)}(1, 2),
\]
\[
[\chi_{(2)}^{(1)}(1), \tilde{\chi}_{(2)}^{(2)}(1)] = \chi_{(2,2)}^{(3)}(2, 1) - \chi_{(2,2)}^{(3)}(2, 1),
\]
\[
[\chi_{(2)}^{(2)}(1), \tilde{\chi}_{(1)}^{(1)}(1)] = \chi_{(2,1)}^{(3)}(1, 2) - \chi_{(2,1)}^{(3)}(1, 2),
\]
\[
[\chi_{(2)}^{(2)}(1), \tilde{\chi}_{(2)}^{(2)}(1)] = 0,
\]
\[
[\chi_{(2)}^{(2)}(1), \tilde{\chi}_{(2)}^{(2)}(1)] = 0.
\]
(E.9)

Furthermore, \(\Lambda_{r}(1)\) does not alter the positions of the impurities but only produces a factor of either \(\rho\) or \(\tilde{\rho}\). Thus, its commutators with the chiral functions with fixed positions of the incoming and outgoing impurities read
\[
[\Lambda_{r}(1), \chi_{(i_{1}, \ldots, i_{l})}^{(o_{1}, \ldots, o_{l})} (a_{1}, \ldots, a_{n})] = (\rho - \tilde{\rho}) \sum_{k=1}^{l} (-1)^{k} (i_{k} - o_{k}) \chi_{(i_{1}, \ldots, i_{l})}^{(o_{1}, \ldots, o_{l})} (a_{1}, \ldots, a_{n}),
\]
\[
[\Lambda_{r}(1), \tilde{\chi}_{(i_{1}, \ldots, i_{l})}^{(o_{1}, \ldots, o_{l})} (a_{1}, \ldots, a_{n})] = - (\rho - \tilde{\rho}) \sum_{k=1}^{l} (-1)^{k} (i_{k} - o_{k}) \tilde{\chi}_{(i_{1}, \ldots, i_{l})}^{(o_{1}, \ldots, o_{l})} (a_{1}, \ldots, a_{n}).
\]
(E.11)

Inserting the above expressions for the commutators into (E.6) and adding the result to the three-loop dilatation operator \(\mathcal{D}_{3}\) given in (5.13), we obtain the transformed
expression

\[
\begin{align*}
D'_3 &= -4(\chi(1,2,3) + \chi(3,2,1)) + 8(\rho + \tilde{\rho})(\chi(1,2) + \chi(2,1)) \\
&\quad - 2(\tilde{\rho} + \epsilon_2 - \epsilon_2)\chi(1,2,1) - 2(2 - \epsilon_2 + \epsilon_2)\chi(2,1,2) - 2(2(\rho + \tilde{\rho})^2 - (\rho - \tilde{\rho})^2\zeta(3))\chi(1) \\
&\quad - 2(1 - \epsilon_2 - \epsilon_2)(\chi(1,3,2) - \chi(2,1,3)) - 4(\rho\chi_{(\star,\star)}^{(\star,\star)}((1,1,3) + \rho\chi_{(\star,\star)}^{(\star,\star)}((1,3,3)) \\
&\quad + 2(\rho\tilde{\epsilon}_2b - (1 - \rho^2)\zeta(3))\chi_{(1,2)}^{(1)}(1) + 2(\rho\tilde{\epsilon}_2b - (1 - \rho^2)\zeta(3))\tilde{\chi}_{(1,2)}^{(1)}(1) \\
&\quad + 2(\rho\tilde{\epsilon}_2a - (\rho - \tilde{\rho})(\epsilon_2 - \nu) - (1 - \rho^2)\zeta(3))\chi_{(2,1)}^{(2)}(1) \\
&\quad + 2(\rho\tilde{\epsilon}_2a + (\rho - \tilde{\rho})(\epsilon_2 + \nu) - (1 - \rho^2)\zeta(3))\tilde{\chi}_{(2,1)}^{(2)}(1) \\
&\quad - 2(\rho\tilde{\epsilon}_2a - (\rho - \tilde{\rho})(\epsilon_2b + \epsilon_2c) - (1 - \rho^2)\zeta(3))\chi_{(1,2)}^{(2)}(1) \\
&\quad - 2(\rho\tilde{\epsilon}_2a + (\rho - \tilde{\rho})(\epsilon_2b + \epsilon_2c) - (1 - \rho^2)\zeta(3))\tilde{\chi}_{(1,2)}^{(2)}(1) \\
&\quad - 2(\rho\tilde{\epsilon}_2b + (1 - \rho^2)\zeta(3))\chi_{(2,1)}^{(2)}(1) - 2(\rho\tilde{\epsilon}_2b + (1 - \rho^2)\zeta(3))\tilde{\chi}_{(2,1)}^{(2)}(1) \\
&\quad - 2(\epsilon_2b + \epsilon_2c + (\rho - \tilde{\rho})\chi_{(2,1)}^{(1)}(1,2) - 2(\tilde{\epsilon}_2b + \tilde{\epsilon}_2c - (\rho - \tilde{\rho})\tilde{\chi}_{(2,1)}^{(1)}(1,2) \\
&\quad + 2\epsilon_2c\chi_{(1,2)}^{(1)}(2,1) + 2\epsilon_2c\tilde{\chi}_{(1,2)}^{(1)}(2,1) \\
&\quad - 2(\epsilon_2a + (\rho - \tilde{\rho})(\chi_{(2)}^{(2)}(1,2) - \chi_{(2)}^{(2)}(2,1)) - 2(\epsilon_2a - (\rho - \tilde{\rho}))\chi_{(2,1)}^{(1)}(2,2) - \tilde{\chi}_{(2,1)}^{(2)}(2,1)) \\
&\quad - 2(\epsilon_2a - \epsilon_2b - \epsilon_2c)\chi_{(1,2)}^{(2)}(1,2) - 2(\epsilon_2a - \epsilon_2b - \tilde{\epsilon}_2c)\tilde{\chi}_{(1,2)}^{(2)}(1,2) \\
&\quad + 2(\epsilon_2a - \epsilon_2b + (\rho - \tilde{\rho})\zeta(3))\chi_{(2,1)}^{(2)}(1,2) + 2(\epsilon_2a - \epsilon_2c - (\rho - \tilde{\rho})\zeta(3))\tilde{\chi}_{(2,1)}^{(2)}(1,2) \\
&\quad + 2(\mu + \epsilon_2b + \epsilon_2c)\chi_{(1,3)}^{(2)}(1,2,1) + 2(\mu - \epsilon_2b - \epsilon_2c)\tilde{\chi}_{(1,3)}^{(1,2)}(2,1) \\
&\quad + 2(\mu - \epsilon_2c + \tilde{\epsilon}_2b + (\rho - \tilde{\rho})\zeta(3))\chi_{(1,2)}^{(3)}(1,2,1) - \tilde{\chi}_{(1,2,1)}^{(2)}(1,2)) \\
&\quad - 2(\mu - \epsilon_2b - \epsilon_2c + \tilde{\epsilon}_2b + \tilde{\epsilon}_2c + (\rho - \tilde{\rho})\zeta(3))(\chi_{(2,1,3)}^{(1,2)}(2,1) - \tilde{\chi}_{(2,1,3)}^{(1,2)}(2,1)) \\
&\quad + 2(\mu + \tilde{\epsilon}_2a)(\chi_{(1,3)}^{(3)}(1,2) - \chi_{(1,3)}^{(1,3)}(2,1)) - 2(\mu - \epsilon_2b)(\chi_{(1,3,1)}^{(1,3)}(1,2) - \tilde{\chi}_{(1,3,1)}^{(1,3)}(2,1)) \\
&\quad + 2(\mu - \epsilon_2c + \tilde{\epsilon}_2a)\chi_{(1,3)}^{(2,3)}(1,2) - 2(\mu - \epsilon_2a + \tilde{\epsilon}_2c)\tilde{\chi}_{(1,3)}^{(2,3)}(1,2) \\
&\quad - 2(\mu - \epsilon_2b - \epsilon_2c - \tilde{\epsilon}_2a - (\rho - \tilde{\rho})\zeta(3))\chi_{(2,3)}^{(1,3)}(2,1) \\
&\quad + 2(\mu - \epsilon_2a + \tilde{\epsilon}_2b + \tilde{\epsilon}_2c - (\rho - \tilde{\rho})\zeta(3))\tilde{\chi}_{(2,3)}^{(1,3)}(2,1)
\end{align*}
\]  
(E.12)

A particular choice of the parameters of the transformation then allows us to simplify the expression of \(D_3\) as obtained from Feynman diagrams and given in (5.13). The anti-Hermitian terms can be completely removed with the following choice of the parameters

\[
\text{Re}\, \epsilon_2 = \text{Re}\, \epsilon_2 = 1
\]  
(E.13)

in analogy to the case of \(N = 4\) SYM theory and moreover with

\[
\begin{align*}
\text{Re}\, \epsilon_{2a} &= \text{Re}\, \mu \quad \Rightarrow \quad \text{Re}\, \tilde{\epsilon}_{2a} = - \text{Re}\, \mu \\
\text{Re}\, \epsilon_{2b} &= -2\text{Re}\, \epsilon_{2c} + \text{Re}\, \mu \quad \Rightarrow \quad \text{Re}\, \tilde{\epsilon}_{2b} = -2\text{Re}\, \epsilon_{2c} - \text{Re}\, \mu \\
\text{Im}\, \epsilon_{2b} &= 0 \quad \Rightarrow \quad \text{Im}\, \tilde{\epsilon}_{2b} = 0
\end{align*}
\]

(E.14)

Since \(\tilde{\mu} = -\mu, \tilde{\nu} = \nu\) according to (E.7), the above choices respect the constraints (E.5).

38
With these choices and furthermore by setting
\[ \text{Im} \epsilon_2 = \text{Im} \epsilon_2 = 0, \quad \text{Re} \epsilon_2 = 0, \quad \text{Re} \epsilon_2 = 0, \quad \text{Im} \mu = 0, \quad \text{Im} \nu = 0, \]
(E.15)
one obtains the simplified and Hermitean result (5.14) that is presented in section 5.4.

The identified transformation explicitly reads
\[ \chi = \bar{g}^4 (\rho - \bar{\rho}) \zeta (3) (\chi_1^{(1)}(1) + \chi_2^{(1)}(1) + \bar{\chi}_1^{(1)}(1) + \bar{\chi}_2^{(1)}(1)) + \delta_{21} \chi(1) + \chi(1,2) + \chi(2,1) \]
\[ + (\rho + \bar{\rho}) \zeta (3) \Lambda_1(1), \]
(E.16)
where the parameter $\delta_{21}$ can be chosen at will, since in (E.6) the respective term drops out when taking the commutator with $D_1$.

Applying the transformation (E.16) to the momentum eigenstates (7.1) and using (7.2), (7.3), we obtain for the terms relevant to three loops
\[ \psi'(p') = e^{-\chi} \psi(p) = [1 + \tilde{g}_4 N_{2,L,p}(\rho, \bar{\rho})] \psi(p + i \tilde{g}_4 (\rho^2 - \bar{\rho}^2) \zeta(3)) , \]
\[ \tilde{\psi}'(p') = e^{-\chi} \tilde{\psi}(p) = [1 + \tilde{g}_4 N_{2,L,p}(\rho, \bar{\rho})] \tilde{\psi}(p - i \tilde{g}_4 (\rho^2 - \bar{\rho}^2) \zeta(3)) . \]
(E.17)
The momentum acquires a constant imaginary two-loop shift, and the normalization is corrected by
\[ N_{2,L,p}(\rho, \bar{\rho}) = (\delta_{21} + \cos p) [4 \sin^2 \varphi + (\rho^2 - \bar{\rho}^2)^2] - (\rho - \bar{\rho}) \zeta (3) [1 - e^{i\varphi} \rho - (\rho - L \bar{\rho})] . \]
(E.18)
Note that the rational term already appears in the $N = 4$ case, where it removes the anti-Hermitean contribution $2(\chi(2,1,3) - \chi(1,3,2))$ in the third line of (5.13).

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