"New" Veneziano amplitudes from "old" Fermat (hyper)surfaces

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Abstract

The history of the discovery of bosonic string theory is well documented. This theory evolved as an attempt to find a multidimensional analogue of Euler’s beta function to describe the multiparticle Veneziano amplitudes. Such an analogue had in fact been known in mathematics at least in 1922. Its mathematical meaning was studied subsequently from different angles by mathematicians such as Selberg, Weil and Deligne among others. The mathematical interpretation of this multidimensional beta function that was developed subsequently is markedly different from that described in physics literature. This work aims to bridge the gap between the mathematical and physical treatments. Using some results of recent publications (e.g. J.Geom.Phys.38 (2001) 81; ibid 43 (2002) 45) new topological, algebro-geometric, number-theoretic and combinatorial treatment of the multiparticle Veneziano amplitudes is developed. As a result, an entirely new physical meaning of these amplitudes is emerging: they are periods of differential forms associated with homology cycles on Fermat (hyper)surfaces. Such (hyper)surfaces are considered as complex projective varieties of Hodge type. Although the computational formalism developed in this work resembles that used in mirror symmetry calculations, many additional results from mathematics are used along with their suitable physical interpretation. For instance, the Hodge spectrum of the Fermat (hyper)surfaces is in one-to-one correspondence with the possible spectrum of particle masses. The formalism also allows us to obtain correlation functions of both conformal field theory and particle physics using the same type of the Picard-Fuchs equations whose solutions are being interpreted in terms of periods.

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1 Introduction

1.1 Some properties of Veneziano and Virasoro amplitudes

In 1968 Veneziano [1] had postulated 4-particle scattering amplitude $A(s, t, u)$ given (up to a common constant factor) by

$$A(s, t, u) = V(s, t) + V(s, u) + V(t, u)$$

(1.1)

where

$$V(s, t) = \int_0^1 x^{-\alpha(s)-1}(1 - x)^{-\alpha(t)-1}dx \equiv B(-\alpha(s), -\alpha(t))$$

(1.2)

is Euler’s beta function and $\alpha(x)$ is the Regge trajectory usually written as $\alpha(x) = \alpha(0) + \alpha'x$ with $\alpha(0)$ and $\alpha'$ being the Regge slope and the intercept, respectively. In case of Lorentzian metric with signature $\{+,-,-,-\}$ the Mandelstam variables $s$, $t$ and $u$ entering the Regge trajectory are given by [2]

$$s = (p_1 + q_1)^2,$$

$$t = (p_1 - p_2)^2,$$

$$u = (p_1 - q_2)^2.$$  

(1.3)

The 4-momenta $p_i$ and $q_i$ are constrained by the energy-momentum conservation $p_1 + q_1 = p_2 + q_2$ leading to relation between the Mandelstam variables:

$$s + t + u = \sum_{i=1}^4 m_i^2.$$  

(1.4)

Already Veneziano [1] had noticed\(^2\) that to fit the experimental data the Regge trajectories should obey the constraint

$$\alpha(s) + \alpha(t) + \alpha(u) = -1$$

(1.5)

consistent with Eq.(1.4). He also noticed that the amplitude $A(s, t, u)$ can be equivalently rewritten with help of this constraint as follows

$$A(s, t, u) = \Gamma(-\alpha(s))\Gamma(-\alpha(t))\Gamma(-\alpha(u))\sin \pi(-\alpha(s))\sin \pi(-\alpha(t))\sin \pi(-\alpha(u)).$$

(1.6)

This result looks strikingly similar to that suggested a bit later by Virasoro [3]. Up to a constant it is given by

$$\tilde{A}(s, t, u) = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b)\Gamma(b+c)\Gamma(c+a)}$$

(1.7)

\(^2\)To get our Eq.(1.5) from Eq.7 of Veneziano paper, it is sufficient to notice that his $1 - \alpha(s)$ corresponds to ours $-\alpha(s)$. 

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with \( a = -\frac{1}{2} \alpha(s) \), etc. also subjected to the constraint:

\[
\frac{1}{2} (\alpha(s) + \alpha(t) + \alpha(u)) = -1. \tag{1.8}
\]

Use of the formulas

\[
\Gamma(x)\Gamma(1 - x) = \frac{\pi}{\sin \pi x} \tag{1.9}
\]

and

\[
4 \sin x \sin y \sin z = \sin(x + y - z) + \sin(y + z - x) + \sin(z + x - y) - \sin(x + y + z) \tag{1.10}
\]

permits us to rewrite Eq.(1.7) in the alternative form (up to unimportant constant):

\[
\bar{A}(s,t,u) = \left[ \Gamma(-\frac{1}{2} \alpha(s))\Gamma(-\frac{1}{2} \alpha(t))\Gamma(-\frac{1}{2} \alpha(u)) \right]^2 \times
\]

\[
[\sin \pi(-\frac{1}{2} \alpha(s)) + \sin \pi(-\frac{1}{2} \alpha(t)) + \sin \pi(-\frac{1}{2} \alpha(u))].
\tag{1.11}
\]

Although these two amplitudes look deceptively similar, mathematically, they are markedly different. Indeed, by using Eq.(1.6) conveniently rewritten as

\[
A(a,b,c) = \Gamma(a)\Gamma(b)\Gamma(c)[\sin \pi a + \sin \pi b + \sin \pi c] \tag{1.12}
\]

and exploiting the identity

\[
\cos \frac{\pi z}{2} = \frac{\pi z}{2^{1-z}} \frac{1}{\Gamma(z)} \frac{\zeta(1-z)}{\zeta(z)} \tag{1.13}
\]

after some trigonometric calculations the following result is obtained:

\[
A(a,b,c) = \frac{\zeta(1-a)}{\zeta(a)} \frac{\zeta(1-b)}{\zeta(b)} \frac{\zeta(1-c)}{\zeta(c)}, \tag{1.14}
\]

provided that

\[
a + b + c = 1. \tag{1.15}
\]

For the Virasoro amplitude, apparently, no result like Eq.(1.14) can be obtained. The differences between the Veneziano and the Virasoro amplitudes are much more profound as the rest of this paper demonstrates.

The result, Eq.(1.14), is remarkable in the sense that it allows us to interpret the Veneziano amplitude from the point of view of number theory, the theory of dynamical systems, etc. Such interpretation is presented in some detail in Section 3.1 while Section 2 provides the necessary mathematical background.

Moreover, in our previous work, Ref. [4], the following partition function describing (train track) dynamics of 2+1 gravity was obtained

\[
Z(\beta) = \frac{\zeta(\beta - 1)}{\zeta(\beta)}. \tag{1.16a}
\]
This function is meant to describe a dynamical transition from the pseudo Anosov (Appendix B) to the Seifert fibered dynamical regime controlled by the temperature-like parameter $\beta$. The exact physical nature of the parameter $\beta$ was left unexplained. The train tracks "live" on the surface of the punctured torus—the simplest surface of negative Euler characteristic on which they can "live" [5]. Normally, torus without puncture(s) can be foliated by vector fields without singularities. The punctured torus (via Schottky double construction) can be converted into a double torus, that is into the Riemann surface of genus 2. If one tries to foliate such surface with some line field (not to be confused with the vector field [6]), one can easily discover that singularities of such field will occur inevitably. The number of singularities is controlled by the Poincare-Hopf index theorem\textsuperscript{3}. Foliations with singularities are known as pseudo Anosov (Appendix B) type. For the punctured torus only 2 singularities are allowed and their scattering on each other is very much like that which the Veneziano amplitude is describing. The Mandelstam variables $s,t,u$ can be used in the present case as well thus replacing our dynamical zeta function, Eq.(1.16a), with the product of zeta functions, i.e. with the Veneziano amplitude, Eq.(1.14). Such replacement is not complete however since our dynamical zeta function has "wrong" argument: $\beta - 1$, in the numerator of Eq.(1.16a), instead of $1 - \beta$. Fortunately, this deficiency is easy to correct if we recall the presentation of the Riemann zeta function as infinite product over primes. Then, instead of Eq.(1.16a), we obtain

$$Z(\beta) = \prod_p \frac{1 - p^{-\beta}}{1 - p^{1-\beta}},$$

(1.16b)

where the product is taken over all prime numbers. Such presentation invokes immediately connections with the p-adic string theory to be described below. For now, however, we still need to make several observations. To begin, let us introduce the p-adic analogue of zeta function, in particular,

$$\zeta_p(\beta - 1) = \frac{1}{1 - p^{1-\beta}},$$

to be reobtained in Section 2, Eq.(2.15). We can equivalently present it as

$$\zeta_p(\beta - 1) = \frac{-1}{p^{1-\beta}} \frac{1}{1 - p^{\beta-1}} = (-1)p^{\beta-1}\zeta_p(1 - \beta).$$

(1.17)

It should be clear from Eq.(1.17) that $\zeta_p(\beta - 1)$ has the same pole as $\zeta_p(1 - \beta)$ and, hence, the Veneziano amplitude, Eq.(1.14), is obtainable from dynamics of 2+1 gravity and, therefore, from the full Einsteinian gravity. Such arguments, although plausible, not quite rigorous yet. Indeed, in the case of the open string the world sheet is the Poincare upper half plane (or, equivalently, the open disc). The incoming/outgoing particles are represented as "insertions on the

\textsuperscript{3}It is possible to have different sets of singularities as long as their total index remains the same [5,6]
boundary” in terminology of Ref.[7], page 46, or as cusps in mathematical terminology. In the case of 4-particle Veneziano amplitude there are 4 ”insertions on the boundary”. But, as Fig.3 of our work on dynamics of train tracks, Ref.[4], indicates, in the case of the punctured torus there are in the simplest case also 4 insertions at the boundary. Moreover, the open disc Poincare model is not the model of the world sheet in the case of 2+1 gravity. It is used for the description of the dynamics of simplest train tracks. This dynamics is taking place in the Teichmüler space of the punctured torus which is just the open Poincare disc [8]. It can be shown [9], that the Teichmüller space of the punctured torus is the same as that of the 4 times punctured sphere (considered to be the world sheet for the closed string theory, Ref.[7], page 35). Hence, already at the tree level in string theory the description of physical processes actually takes place not in real but in the Teichmüller space. And, of course, mathematically rigorous study of the dynamics of train tracks indicates that, indeed, such dynamics is taking place in the Teichmüller space. The above picture, perhaps, can be generalized to surfaces of genus higher than one$^4$. In this paper, however, no attempts are made to do this in view of the fact that there is much more mathematically elegant and efficient way to accomplish the task. Before discussing details of this other way, we would like to make several comments regarding connections of the result, Eq.(1.14), with the p-adic string theory.

1.2 Connections with the p-adic string theory

Eq.(1.14) was used implicitly (but essentially) in the p-adic string theory as can be seen from the review by Bekke and Freund, Ref.[11], and is discussed below. Since Eq.(1.14) was also obtained in our earlier work, Ref.[4], in connection with dynamics of train tracks representing dynamics of 2+1 gravity, from the standpoint of an external observer, the evolution of the train track patterns takes place in normal 2+1 or even 3+1 physical space-time$^5$. Eq.(1.14), when it is obtained with help of the p-adic bosonic string formalism, ”lives” instead in 26 dimensional space-time (page 26 of Ref.[11]). Hence, it is worth discussing and comparing in some detail both cases now.

Let us begin with the case of the p-adic string theory. It is commonly believed that the only rationale of such theory lies in regularizing of the string world sheet by placing such sheet on the Bruhat-Tits (known in physics literature as Bethe lattice) tree (e.g see Appendix C) ”which can be embedded in ordinary real space-time” (page 29 of Ref.[11]). Consistency with known continuous results in bosonic string theory require, however, that such real space-time is 26

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$^4$And, indeed, in a series of papers by Takhtajan et al (e.g see [10] for the latest paper and references therein) a serious attempt has been made to accomplish this task. To our knowledge, the multiparticle Veneziano amplitudes have not been reobtained thus far in these papers.

$^5$Moreover, such picture is not just imaginary. It can be made entirely real if one looks at dynamics of disclinations in 2 dimensional liquid crystals. Such liquid crystal dynamics is in one to one correspondence with the dynamics of train tracks [5,6].
dimensional. Such conclusion is reached roughly speaking based on the following arguments. Consider the identity

\[ \pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma \left( \frac{1-s}{2} \right) \zeta(1-s). \]  

(1.18)

Following Ref. [11] we define the adelic zeta function \( \zeta_A(s) \) by

\[ \zeta_A(s) = \pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \zeta(s) \]  

(1.19)

then, Eq.(1.18) can be rewritten as follows

\[ \zeta_A(s) = \zeta_A(1-s). \]  

(1.20)

Using the same reference, an adelic gamma function \( \Gamma_A(s) \) is then defined by

\[ \Gamma_A(s) = \frac{\zeta_A(s)}{\zeta_A(1-s)}. \]  

(1.21)

In view of Eq.(1.20), clearly, \( \Gamma_A(s) = 1 \). This result is a particular example of the so called "product formula" well known in the algebraic number theory [12]. Following general rules of p-adic analysis beautifully explained in the classical book by Artin [12], one can rewrite the statement \( \Gamma_A(s) = 1 \) in a form of the product over both finite and infinite places, i.e.

\[ \Gamma_A(s) = \prod_p \Gamma_p(s) = 1, \]  

(1.22)

where \( p \) runs over all primes (just like in the case of Riemann zeta function) and, in addition, over the "prime at infinity": In our case

\[ \Gamma_p(s) = \frac{1 - p^{s-1}}{1 - p^{-s}} \equiv \frac{\zeta_p(s)}{\zeta_p(1-s)} \]  

(1.23)

but

\[ \Gamma_\infty(s) = \frac{\zeta_\infty(s)}{\zeta_\infty(1-s)} = \pi^{-s+\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) / \Gamma \left( \frac{1-s}{2} \right), \]  

(1.24)

where \( \zeta_\infty(s) = \pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \) in view of Eq.(1.18). Clearly, we get then

\[ \frac{\zeta_\infty(s)}{\zeta_\infty(1-s)} \prod_{p \neq \infty} \Gamma_p(s) = 1. \]  

(1.25)

This leads to

\[ \prod_{p \neq \infty} \frac{\zeta_p(s)}{\zeta_p(1-s)} = \prod_{p \neq \infty} \Gamma_p(s) = \frac{\zeta_\infty(1-s)}{\zeta_\infty(s)} = \frac{\zeta(s)}{\zeta(1-s)}. \]  

(1.26)
in view of Eq.(1.18). But the l.h.s. of Eq.(1.26) is obtainable from the p-adic string path integral [11,13]. Hence, the p-adic open string scattering amplitude is just an inverse of the standard Veneziano amplitude in view of Eq.(1.14). Since the standard amplitude is obtainable via Polyakov path integral in 26 dimensions, the p-adic string should also live in 26 dimensions in view of the product formula, Eq.(1.22). This is the conclusion reached in Ref.[11]. Fortunately (or unfortunately!), this conclusion is incorrect. Indeed, let us take another look at Eq-s (1.20)-(1.22). We have
\[ \prod_p \zeta_p(s) = \prod_p \zeta_p(1-s) \] (1.27)
or, equivalently,
\[ \zeta_\infty(s) \prod_{p \neq \infty} \zeta_p(s) = \zeta_\infty(1-s) \prod_{p \neq \infty} \zeta_p(1-s). \] (1.28)
Evidently, it is only natural to expect that "locally" we should also have
\[ \zeta_p(s) = \zeta_p(1-s) \] (1.29)
for all p's, including p=\infty. This is not the case, however, in view of definitions of local zeta functions, e.g. see Eq.(1.23). This negative observation cannot be simply repaired. Indeed, from p-adic analysis [14] and, even more advanced theory (which includes p-adic analysis as its part) of Drinfeld modules [15], it can be found that the existing in mathematics (e.g. see Eq.(3.45) below) expressions for the p-adic gamma functions (actually, there are several expressions [15]) differ markedly from that given in Eq.(1.23). Moreover, the Bruhat-Tits tree used for calculations of Veneziano and Virasoro amplitudes in Ref-s[11,13] is the discrete analogue of the upper Poincare' half plane [16, 17] which is the universal covering space for any hyperbolic Riemann surface while the conventional string theory formulations producing at the tree level Veneziano and Virasoro-Shapiro-like amplitudes involve flat metrics which is obviously nonhyperbolic. Hence, the whole chain of arguments leading to the conclusion that p-adic strings are "living" in 26 dimensions is apparently incorrect. Moreover, we argue in this work that use of the Bruhat-Tits tree in the p-adic string theory is motivated not by the necessity of regularization of the world sheet. It occurs rather naturally and is very closely associated with complex multiplication theory developed by Shimura and Taniyama [18]. It can be thought as some analogue of the Fourier component in the Fourier expansion\footnote{More exactly, it occurs as an analogue of the individual term in Euler's prime number product decomposition of zeta function.} of some continuous function. Just like in the linear Fourier analysis our knowledge of one Fourier component is practically sufficient for restoration of the whole function, the knowledge of one p-adic component is sufficient for restoration of the whole...
function as well. This statement is illustrated below on examples of calculation of the p-adic analogue of the Veneziano amplitude discussed in Sections 3.3., 4.3. and 5.3.1.

1.3 Organization of the rest of this paper

The history of development of the dual resonance models leading ultimately to models of relativistic bosonic and fermionic strings is well documented and can be found, for example, in the excellent collection of review articles edited by Jacob, Ref.[19]. This development one way or another stems from earlier efforts to develop the axiomatic S-matrix theory-the idea which can be traced back to two papers by Heisenberg [20] of 1943. The mathematical foundations of the S-matrix theory can be traced back to even much earlier works of Kramers and Krönig dating back to years 1926 and 1927. The usefulness of their results (related originally to the light scattering) to particle physics scattering is well documented in the classical monograph by Bjorken and Drell, Ref.[21]. The contribution of Kramers and Krönig to scattering problems lies in their observation of the usefulness of the Cauchy integral formula to such problems. Surely, from the modern perspective, one can talk equivalently about usefulness of the Riemann-Hilbert problem [22] to scattering problems. And this observation leads us ultimately to the Picard-Fuchs type equations considered in Section 5. If one reformulates the Kramers-Kronig results into the language of particle physics as it is done by Bjorken and Drell [21], then in few words the essence of the scattering problem can be formulated as the statement about the scattering amplitude

$$\text{Im} f(\omega) = \frac{\omega}{4\pi} \sigma(\omega)$$

connecting the imaginary part of the forward scattering amplitude \(\text{Im} f(\omega)\) with the total crosssection \(\sigma(\omega)\) for the "frequency" \(\omega > 0\). Suppose, we know the amplitude \(f(\omega)\), then, the familiar trick (which is just a corollary of the Cauchy theorem) tells us that

$$f(\omega + i\varepsilon) - f(\omega - i\varepsilon) = 2\pi i \text{Im} f(\omega), \varepsilon \to 0^+$$

thus allowing us to determine the total crosssection using Eq.(1.30). Alternatively, if such a crosssection can be obtained experimentally, one can restore the real part of the amplitude through Kramers-Kröning relations thus reconstructing the whole amplitude from the experimental data. Such simple minded picture becomes very complex when, for example, one is trying to analyze the analytic properties of the multiparticle vertex parts (contributing to scattering amplitude) based on analysis of the appropriate Feynman diagrams of quantum field theory. At the physical level of rigor this problem was studied by Landau [24]. Subsequently, much more sophisticated cohomological analysis of

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8The connection between the Picard-Fuchs equations and the Riemann-Hilbert problem is discussed, for example, in Ref. [23]
the whole problem was developed by Pham [25] whose results were brought to perfection by Milnor [26] and Brieskorn [27]. In physics literature the results of Pham were carefully analyzed in two fundamental papers by Ponzano, Regge, Speer and Westwater [28]. Unfortunately, subsequent development of particle physics went into different direction(s). Nevertheless, the major problem of reconstruction of the underlying particle physics model from scattering amplitudes with known analytic properties had ultimately created what is known today as string/membrane theory. As it was mentioned in Section 1.1., Veneziano had not derived his 4-particle amplitude theoretically. This amplitude was postulated and subsequently checked experimentally [7,19]. The attempts to extend this amplitude to multiparticle scattering had lead ultimately to formulation of string theory. Since mathematically, the simplest Veneziano amplitude is just Euler’s beta function, the extension of such amplitude to the multiparticle case mathematically is reduced to the correct multidimensional extension of Euler’s beta function integral. Such an extension can be found in the book by Edwards, e.g. see Ref [29], page 167, published in 1922. His result was rediscovered by many mathematicians. The latest in this list of people, perhaps, Varchenko whose two papers [30] of 1989-90 contain the result of Edwards and, surely, much more. Ironically, in 1967-a year earlier than Veneziano paper was published, the paper by Chowla and Selberg had appeared [31] which related Euler’s beta function to the periods of elliptic integrals. The usefulness of results of Chowla and Selberg is discussed in Section 4 within the context of conformal and particle physics theory. The result of Chowla and Selberg was generalized by Andre Weil whose two influential papers on the same subject [32, 33] had brought into picture periods of the Abelian varieties, Hodge rings, etc. thus inspiring Benedict Gross to rederive Edwards result in 1978 [34] without actually being aware of its existence. In the paper by Gross the beta function appears as period of the differential form on the Jacobian of the Fermat curve. His results as well as those of Rohlich (placed in the appendix to Gross paper) have been subsequently documented in the book by Lang [35]. These results are explained from physical standpoint in Section 3.2. where they are reinterpreted with help of Milnor’s fundamental results on singular points of complex hypersurfaces [26]. Milnor’s work to a large extent had shaped up the theory of singularities and, not surprisingly, all the results of this paper could be considered as some practical applications of the singularity theory as described, for example, in the monograph by Arnol’d, Varchenko and Gussein-Zade [36].

Although in the paper by Gross the multidimensional extension of beta function is considered, e.g. see page 207 of Gross paper [34], the details were not provided however. The details are provided (to our knowledge) only in this paper. They are based on ideas developed in lectures by Deligne [37] delivered in 1978-1979. Deligne was seemingly unaware of both results of Edwards and Gross. In developing of Hodge theory Deligne noticed that Hodge theory requires some essential changes (e.g. introduction of mixed Hodge structures, etc.) in the case if the Hodge-K"ahler manifolds possess singularities. We discuss these modifications briefly in Section 5 and in Appendix D within the context of standard singularity theory. Although this paper is an attempt to present
a balanced treatment of both the number-theoretic (including the p-adic) and standard analytic aspects of the problem of periods associated with Veneziano amplitudes, and in spite of the fact that the paper came out rather long, in reality, it presents just a gentle scratch on the surface of enormous amount of information. For instance, the number-theoretic aspects of the mixed Hodge structures involving theories of motives [37,38], of crystalline cohomology [37-39], of Tannakian categories [37-38], etc. are left completely outside the scope of this paper mainly because their physical meaning still remains to be uncovered. Also, the symplectic aspects of our scattering problem [40] needed for development of correct "string"-theoretic model associated with Veneziano amplitudes are left outside the scope of this paper as well. The emerging opportunity to "fill in the gaps" and to demonstrate that the correct formulation of "string" theory leads indeed to "the theory of everything", hopefully, should provide an inspiration for our readers. In Section 6 we provide some outline of what lies ahead. Very exciting connections between many branches of so far "pure" mathematics and "down to earth" and, hence, "dirty" physics are yet to be unveiled.

2 Number fields versus function fields

2.1 Valuations

The existing physical literature uses real, complex and p-adic numbers. Use of p-adic numbers is still rather exotic [11,13, 41] and to a large extent artificial and, hence, unpopular. It is strange that function fields have not been used in physics so far. In algebra [12,42] and number theory [12,43] it was recognized long ago that function fields have practically the same properties as number fields but in many cases are easier to handle. We would like to argue that the function fields are not only easier to handle than the number fields but also that these fields are physically more relevant. Not surprisingly, these fields found their widespread use in discrete mathematics, computer science and coding theory [44,45].

Elementary number theory begins with the ring of integers \( \mathbb{Z} \). This ring can be made into the field of fractions \( \mathbb{Q} \) if the division operation is included (provided that there is no divisions by zero). The field of rational numbers \( \mathbb{R} \) is obtained by the completion operation involving the Cauchy sequences and the usual absolute values [46]. The field of complex numbers \( \mathbb{C} \) is obtained by algebraic extension of \( \mathbb{Z}, \mathbb{Q} \) or \( \mathbb{R} \). E.g. solution of equation \( x^2 + 1 = 0 \) is the simplest basic example leading to complex numbers. Evidently every number \( n \in \mathbb{Z} \) can be decomposed into primes by the property of unique factorization, e.g.

\[
n = \pm p_1^{m_1} \cdots p_r^{m_r},
\]

(2.1)

eq etc. At the same time, every polynomial \( f(T) \) can also be uniquely decomposed over the algebraically closed field \( \mathbb{C} \) as follows:

\[
f(T) = c(T - \alpha_1)^{m_1} \cdots (T - \alpha_r)^{m_r}.
\]

(2.2)
Comparison between Eq.s (2.1) and (2.2) already suggests a close analogy. It is not our purpose here to reproduce long excerpts from algebra related to
polynomial rings $\mathbf{F}[T]$ over the field $\mathbf{F}$ where $\mathbf{F}$ can be either one of the fields above or finite field of $q$ elements. In the last case one usually writes $\mathbf{F}_q[T]$.
Evidently, the function ring $\mathbf{F}[T]$ is analogous to the number ring $\mathbb{Z}$ while the
quotients of polynomials taken from the ring $\mathbf{F}[T]$ form the field $\mathbf{F}(T)$ analogous
to $\mathbb{Q}$, etc. Computationally the function field has many advantages over the number field. We shall discuss only those which are related to the goals we have in mind. To this purpose, following Artin [12], we introduce the following

**Definition 2.1.** A valuation of a field $\mathbf{F}$ is a real-valued function $|x|$, defined for all $x \in \mathbf{F}$ satisfying the following requirements:

a) $|x| \geq 0$; $|x| = 0$ if and only if $x = 0$;
b) $|xy| = |x||y|$;
c) $|x + y| \leq |x| + |y|$.

There are many ways to introduce valuations explicitly as explained in Ref.[47] and we shall take advantage of this fact. Nevertheless, every valuation is equivalent to a valuation for which the triangular inequality holds. The triangular inequality can be sharpened for the non-archimedean valuations. In this case instead of c) one has the following inequality

$$|x + y| \leq \max(|x|, |y|).$$

The valuation is called trivial if $|x| = 1 \ \forall x \in \mathbf{F}^*$ ($\mathbf{F}^* = \mathbf{F} \setminus 0$). In the case of field $\mathbb{Q}$ it is convenient to choose some prime number $p$ so that any number $a \in \mathbb{Q}$, $a \neq 0$, can be presented in the form $a = p^\nu b$ with numerator and denominator of $b$ being prime to $p$.

**Definition 2.2.** The standard valuation at $p$ is defined according to the rule:

$$|a|_p = p^{−\nu}, \text{ where } \nu = \text{ord}_p(a) = \text{order of } a \text{ at } p.$$ 

(2.4)

Clearly, with such definition the whole field $\mathbb{Q}$ is divided into equivalence classes. Within such scheme the usual absolute value $|a| \equiv |a|_\infty$. This rule serves as definition of a fictitious "prime at infinity". The difference between the prime at infinity and the rest of primes is profound and represents the major difficulty in number theory as explained by Mazur, Ref. [48], page 15.

Fortunately, for the case of function fields this problem does not exist altogether since the prime at infinity is treated as any other prime. Let $\mathbf{F}[T]$ be a polynomial ring and $\mathbf{F}(T)$ its quotient field of rational functions. Let $p = p(T)$ be an irreducible polynomial (of degree $\geq 1$) with leading coefficient 1 in $\mathbf{F}[T]$.

For each rational function $\varphi(t) \in \mathbf{F}(T)$ one can write

$$\varphi(T) = (p(T))^\nu \frac{f(T)}{g(T)}$$

(2.5)

where $f(T)$ and $g(T) \in \mathbf{F}[T]$ are polynomials and $p \nmid fg$. By analogy with

Eq.(2.4) the standard valuation can be defined now as

$$|\varphi|_p = \exp\{-\nu \deg p\}.$$
In this case the analogue of the absolute value (prime at infinity) is just

\[ |\varphi|_\infty = \exp\{\deg \varphi\}. \]  (2.7)

These results can be made physically relevant if one considers the field of meromorphic functions on the Riemann sphere, i.e. on \( \mathbb{C}+\infty \). Let \( f(x) \) be such function defined on \( \mathbb{C} \). Then, near some point \( x_0 \) it can be presented as

\[ f(x) = (x - x_0)^{\text{ord}_{x_0} f(x)} g(x), \]  (2.8)

provided that \( g(x_0) \neq 0 \) or \( \infty \). Using these results one can write

\[ |f(x)|_{x_0} = \exp\{-\nu\} \text{ where } \nu = \text{ord}_{x_0} f(x). \]  (2.9)

If \( x_0=\infty \), one writes \( f(x) = \left(\frac{1}{x}\right)^{\text{ord}_\infty f(x)} g\left(\frac{1}{x}\right) \), where \( g(0) \neq 0 \) or \( \infty \), and using Eq.(2.7) one defines valuation at infinity as

\[ |f(x)|_\infty = \exp\{\deg f(x)\} \]  (2.10)

since in this case \( \text{ord}_\infty f(x) = \deg f(x) \). For the meromorphic functions

\[ \sum_{x_{x_0}} \text{ord}_{x_0} f(x) = 0. \]  (2.11)

This result can be interpreted as some sort of conservation law for Coulombic-like "charges". This analogy is not superficial. It will be discussed in a separate publication.

Eq.(2.11) is valid for both number and function fields and is known number-theoretically as the product formula

\[ \prod_p |a|_p = 1. \]  (2.12)

Taking log of both sides of Eq.(2.12) and using definitions of valuations one reobtains Eq.(2.11) back as required. From the discussion we had so far it is clear that the order and the degree have almost the same meaning. In particular, every element \( f(T) \in \mathbb{F}[T] \) has the form \( f(T)=a_0 T^n + a_1 T^{n-1} + \cdots + a_n \) and if \( a_0 \neq 0 \) it can be said that \( \deg(f) = n \). If \( f \) and \( g \) are non-zero polynomials, we have \([42,43]\)

\[ \deg(fg) = \deg(f) + \deg(g) \]

and

\[ \deg(f + g) \leq \max(\deg(f), \deg(g)). \]

Thus, in view of the definitions given above, for the function fields the valuations are always non-archimedean.

We would like to take advantage now of the fact that valuations can be introduced in many ways. In particular, the finite field \( \mathbb{F}_p \) with \( p \) elements is obtained as a quotient \( \mathbb{Z}/p\mathbb{Z} \) with \( p \) being a prime. If the coefficients \( \alpha_i \) of a polynomial \( f(T) \) of degree \( n \) belong to \( \mathbb{F}_p \), then the quotient \( \mathbb{F}_p[T]/f(T) \equiv \mathbb{F}_{p^n} \).
\( \text{GF}(p^n) \equiv \mathbb{F}_q \) forms the Galois field of \( q = p^n \) elements (congruence classes), e.g. see Ref.[49], page 422, and Appendix A of this work. The advantage of working with finite fields lies in the fact that for each \( n \) there is an irreducible polynomial in \( \mathbb{F}_p[T] \), e.g. see Ref.[49], page 472, and Appendix A of this work. This means that if we use the irreducible polynomials only, then the degree can be used instead of order. To facilitate reader’s understanding of these facts, a summary of relevant number theoretic results is provided in the Appendix A. In view of these results instead of Eq.(2.6) it is more advantageous to use \( |f| = q^{\deg(f)} \) as valuation and it can be checked [12,42,47] that such defined valuation obeys axioms a), b) and Eq.(2.3) and, hence, is non-archimedean.

### 2.2 Zeta functions

**Definition 2.3**. The zeta function \( \zeta_{\mathbb{F}_q}(s) \) of the function field \( \mathbb{F}_q[T] \) is defined by

\[
\zeta_{\mathbb{F}_q}(s) = \sum_{f \in \mathbb{F}_q[T]} \frac{1}{|f|^s}, \text{ where } f \text{ is the monic polynomial.} \tag{2.13}
\]

Recall [46] that the monic polynomial \( f(T) \) is such for which \( \alpha_0 = 1 \). Using Appendix A we find that there are exactly \( q^d \) monic polynomials of degree \( d \) in \( \mathbb{F}_q[T] \). Hence, one obtains [43]

\[
\sum_{\deg(f)<d} \frac{1}{|f|^s} = 1 + \frac{q}{q^s} + \frac{q^2}{q^{2s}} + \cdots + \frac{q^d}{q^{ds}} \tag{2.14}
\]

and, accordingly,

\[
\zeta_{\mathbb{F}_q}(s) = \frac{1}{1 - q^{1-s}}. \tag{2.15}
\]

In view of Eq.(1.23) we obtain as well \( \zeta_{\mathbb{F}_q}(s) = \zeta_q(s-1) \). Next, we define zeta function over the rational function field \( K = \mathbb{F}(T) \). For this, we need a notion of the projective line \( \mathbb{P}^1(F) \) for the function fields. Since the usual projective line is just \( \mathbb{R} \cup \{\infty\} \), evidently for the field \( \mathbb{F}_q \) it is \( K = \mathbb{F}_q \cup \{\infty\} \). In our case we have two types of polynomials, e.g. see Eq.s(2.8) and (2.10). Clearly, \( \zeta_{\mathbb{F}_q}(s) \) is applicable for the first type. If we include the "prime at infinity", we obtain

\[
\zeta_K(s) = \frac{1}{1 - q^{1-s}} \frac{1}{1 - q^{-s}}. \tag{2.16}
\]

Let now \( h(f) \) be some function (to be discussed explicitly later) defined on the set of monic polynomials. With help of this function we obtain the following

**Definition 2.4**. The Dirichlet function \( D_h(s) \) associated with \( h(f) \) is given by

\[
D_h(s) = \sum_{f \text{ monic}} \frac{h(f)}{|f|^s} = \sum_{n=0}^{\infty} H(n)u^n \tag{2.17}
\]
where \( u = q^{-s} \) and
\[
H(n) = \sum_{\substack{f \text{ monic} \deg(f) \leq n}} h(f) . \tag{2.18}
\]

The function \( H(n) \) should possess the multiplicative property, that is \( H(m)H(n) = H(mn) \). The Dirichlet function \( D_h(s) \) sometimes is also called as zeta function \( Z_K(u) \) associated with the field \( K \) \cite{43}. One of the most fundamental achievements of number theory is summarized in the following theorem

**Theorem 2.5.** Let \( K \) be a function field in one variable \( T \) with a finite constant field \( F \) with \( q \) elements. Suppose that the genus of \( K \) is \( g \). Then there is a polynomial \( L_K(u) \in \mathbb{Z}[u] \) of degree \( 2g \) such that
\[
Z_K(u) = \frac{L_K(u)}{(1-qu)(1-u)} \tag{2.19}
\]
where \( L_K(u) = \prod_{i=1}^{2g} (1 - \alpha_i u) \) with \( \alpha_i \alpha_{g+i} = q, 1 \leq i \leq g \).

The proof of this theorem, attributed to A. Weil, can be found in many places, e.g. see Refs\[17,43-45\].

For the sake of space we refer our readers to the literature for details of relation between the genus \( g \) and the field \( K \). In short, we can embed \( f(T) \in F_q[T] \) into \( n \)-dimensional projective space \( \mathbb{P}^n(F_q) \) that is to convert it into the homogenous polynomial and to equate such polynomial to zero. The genus is an invariant of this homogenized polynomial with respect to some known transformations. All this is beautifully explained by Raoul Bott in Ref.\[50\].

Combining Eq.s\(2.17)-(2.19) \) produces
\[
Z_K(u) = \sum_{n=0}^{\infty} H(n)u^n. \tag{2.20}
\]

We shall need yet another interpretation of the same results. To facilitate matters, in view of Eq.\(2.15\), following Rosen, Ref. \[43\], we write formally
\[
\frac{1}{1-qu} = \prod_{l=1}^{\infty} (1 - q^{-l}u)^{-a_l}, \tag{2.21}
\]
where \( a_l \) is number of monic irreducible polynomials of degree \( l \). To obtain \( a_l \), we need to take the logarithmic irreducible derivatives with respect to \( u \) of both sides of Eq.\(2.21\). Then, multiplying the result by \( u \) produces
\[
\frac{qu}{1-qu} = \sum_{l=1}^{\infty} \frac{la_l u^l}{1-u^l} \tag{2.22}
\]
Finally, upon expansion comparing the coefficients of \( u^l \) produces the following result
\[
\sum_{l|n} la_l = q^n \tag{2.23}
\]
which provides the missing link between \( a_l \) and \( q \). Looking at Eqs (2.15), (2.17), (2.20) and (2.21) one can write as well for general case of genus \( g \) similar result:

\[
Z_K(u) = \prod_{l=1}^{\infty} (1 - u^l)^{-a_l}.
\] (2.24)

This time, however, the connection between \( a_l \) and \( q \) is more complicated. To obtain this connection we have to take the logarithm of both sides of Eq. (2.24). By expanding this logarithm in a power series in \( u \) we obtain,

\[
\ln Z_K(u) = \sum_{m=1}^{\infty} \frac{N_m}{m} u^m.
\] (2.25)

Here, by definition, \( N_m \) is the number of zeros of \( f(T) \) in \( \mathbb{P}^n(\mathbb{F}_q) \). For the projective curve of genus \( g \) associated with such polynomial, it can be shown [17, 43-45] that

\[
N_m = q^m + 1 - 2^g \sum_{i=1}^{\infty} \alpha_i^m.
\] (2.26)

For \( g = 0 \) substitution of this result into Eq. (2.25) reproduces back Eq. (2.16) as required. It should be noted, however, that the results just presented are only valid for the so-called non-singular curves, that is for the Riemann surfaces without nodes and holes. For singular surfaces (curves) the above formalism should be amended as discussed (albeit incompletely) in Ref. [51]. To bring all these results closer to those obtained in section 3 of our earlier work, Ref. [4], it is helpful to provide the alternative (dynamical) interpretation of just obtained results.

To this purpose, we begin with simple observations. Consider a map \( f : x \to y \) given by \( y = f(x) \). The number of the fixed points of this map can be easily obtained graphically by plotting together at the same plot the straight line \( y = x \) and \( y = f(x) \). Following Ref. [52] we define the global Lefshetz number \( L(f) \) as the intersection number \( I(\Delta, \text{graph}(f)) \) where \( \Delta \) is diagonal \((x, x)\) in the direct product \((x, y) \in X \times Y \). Next, we need a notion of the local Lefshetz number. It can be easily understood if we recall how the linearization of dynamical systems is done. For instance, for two-dimensional case we have \( f(x) = Ax + \) higher order terms where the dynamical matrix \( A \) can always be brought to the diagonal form and, hence, given by

\[
A = \begin{pmatrix}
\alpha_1 & 0 \\
0 & \alpha_2
\end{pmatrix}
\]

so that the local Lefshetz index \( L_0(f) \) is given by \( \text{sign } [(\alpha_1 - 1)(\alpha_2 - 1)] \). For a source or a sink both eigenvalues have the same sign while for a saddle they have the opposite sign. Clearly, the local Lefshetz number is equal to the index of the vector field, e.g. see our earlier work, Ref. [6], for illustrations and further details. Hence,

\[
L(f) = \sum_{f(x) = x} L_x(f)
\] (2.27)
and this is yet another way of writing the Poincare-Hopf index theorem extensively used in our work, Ref.[6]. In this reference the index $I(x) \equiv \mathcal{L}_x(f)$ of singularity at point $x$ was defined as

$$I(x) = \frac{1}{2\pi} \text{Im} \oint_C \frac{df}{f}$$

(2.28)

with $\mathcal{C}$ being a unit circle. This definition is valid for two dimensional case only. It is easy to extend this result to higher dimensions. Following Ref.[52] the local Lefshetz number can be identified as well with the degree (deg) of mapping $\varphi$:

$$z \rightarrow \frac{f(z) - z}{|f(z) - z|}$$

(2.29)

More accurately, suppose $z$ is an isolated fixed point of $f(x)$ in $\mathbb{R}^k$ (any manifold locally is $\mathbb{R}^k$). If $\mathcal{B}$ is a small closed ball centered at $z$ that contains no other fixed points, then the map $\varphi$ is a smooth map defined by $\varphi : \partial \mathcal{B} \rightarrow S^{k-1}$. For $k = 2$ this is just mapping of one circle onto another (or to itself). The number of times one circle winds around another (the winding number at the point $z$) is $\text{deg}_z \varphi$. In view of these remarks, such definition of the degree clearly coincides with Eq.(2.28) in two dimensions. Hence, $I(z) \equiv \mathcal{L}_z(f) = \text{deg}_z \varphi$.

These results can be easily generalized now as follows. Following Milnor[53], let us consider sequence of iterates $f_2(z) = f(f(z))$, $f_3(x) = f(f(f(z)))$, etc. and their fixed points. Clearly, we will obtain the associated Lefshetz numbers $\mathcal{L}(f_2), \mathcal{L}(f_3), \ldots$, and, accordingly, we can construct the zeta function

$$\ln Z_f(u) = \sum_{m=1}^{\infty} \frac{\mathcal{L}(f^m)}{m} u^m$$

(2.30)

whose meaning should be clear. Obtained results are to be used in the next section.

3 Variations on a theme by Fermat

3.1 Circle maps, function fields and the Veneziano amplitude

We would like to extend connections between the number theory and the theory of dynamical systems in order to reexamine results obtained earlier in Ref.[4]. Let us begin with the observation that every polynomial is completely defined by its coefficients $c_i$ (Appendix A).Suppose now that these coefficients belong to the field $\mathbb{F}_p$ and consider multiplication of such polynomials. This multiplication can be performed as follows. First, we remove the requirement that $p$ is the prime number. Next, we introduce the space $\Omega_N$ (where $N \geq 2$ replaces $p$)

$$\Omega_N := \{\omega = (c_0, c_1, c_2, \ldots) \mid c_i \in [0, 1, \ldots, N-1] \text{ for } i \in \mathbb{Z}\}$$

(3.1)
of one sided sequences of $N$ symbols which is in one-to-one correspondence with the set of polynomials. Multiplication of polynomials simply replaces $\omega$ by $\omega' = (c_0, c_1, c_2, \ldots)$ with coefficients $c_i \in [0, 1, \ldots, N - 1]$. That is polynomial multiplication corresponds to some permutation in the sequence of $c_i$'s i.e. to the automorphism $\Omega_N \to \Omega_N$. Such procedure can be computerized and serves as coding/decoding system (cryptography) for transmission of information [44,45]. A particular realization of such automorphism known as one sided $N$-shift can be defined as follows

$$\sigma^R_N : \Omega^R_N \to \Omega^R_N, \sigma^R_N \omega = \omega' = (c_1', c_2', c_3', \ldots). \quad (3.2)$$

The upper script $R$ stands for the "right". Surely, one can consider the iterates of such defined shift, and, accordingly, one can look for the fixed (periodic) points (orbits) for such iterates.

**Definition 3.1.** The periodic orbits for the one sided shift are the periodic sequences $(\sigma^R_N)^m \omega = \omega$ if and only if $c_{m+n} = c_n$. Every periodic sequence is uniquely determined by its basis $c_0, c_1, c_2, \ldots, c_{n-1}$.

Using results of Appendix A we conclude that there are $N^n$ periodic sequences. In particular, if $N$ is the prime number we get $p^n$ periodic sequences. This is the number of irreducible monic polynomials as discussed in Section 2. Hence, the dynamical zeta function for the one sided shifts coincides with that given by Eq.(2.15). This result is in accord with that obtained in Ref.[54] (e.g. see page 107, Eq.(3.1.5)) by slightly different set of arguments. Being armed with such result we are ready now to accomplish much more.

To this purpose we would like to remind our readers some results from the Nielsen-Thurston theory of surface homeomorphisms. As we have discussed in our previous works, Refs.[4-6], 2+1 gravity can be considered essentially as physically reformulated Nielsen-Thurston theory. For convenience of our readers the key points of this theory are summarized in the Appendix B. In this subsection we will develop our formalism without interruption referring to the Appendix B whenever it is necessary. In particular, using Proposition B.2., it should be clear that any surface homeomorphism can be lifted to the unit disk model of $H^2$. Then, the surface dynamics becomes dynamics of maps of the circle. Using Appendix B and Ref.[55] it follows that it is sufficient to restrict ourself by the periodic maps. If this is the case, then the circle $S^1$ can be identified with $F_m = \mathbb{Z}/m\mathbb{Z}$ and, if we are interested in the invertible transformations, then the number $m$ should be some prime, i.e. $m = p$. Hence, we arrive once again at the cyclotomic fixed point equation, Eq.(A3), written now as

$$f : x^p = x. \quad (3.3)$$

If we ignore the trivial solution $x = 0$, it can be rewritten as

$$x^{p-1} = 1 \quad (3.4)$$

and will possess $p - 1$ nontrivial fixed points. $n$ times iteration of Eq.(3.3) produces

$$f^n : x^{p^n} = x \quad (3.5)$$
and, again, it will have \( p^n - 1 \) non-trivial fixed points so that the Lefshetz number associated with the degree of such mapping is \( L_x(f^n) = p^n - 1 \) in view of Eq.(3.4). This result coincides with that obtained by different methods in Ref.[54], Proposition 8.2.4. Use of this number in Eq.(2.30) produces at once

\[
Z_f(t) = \exp\left(\sum_{n=1}^{\infty} \frac{p^n - 1}{n} t^n\right)
\]

Finally, by replacing \( t \) with \( p^{-s} \) in Eq.(3.6) we obtain the \( p \)-adic version of the Veneziano amplitude, Eq.(1.16b), discussed in Section 1.

### 3.2 From Brieskorn-Pham to Fermat and back

Previous result, as good as it is, by itself provides no clues about its extension to the multiparticle amplitudes. In this subsection we are going to make the first step towards this goal. The rest of the paper depends crucially on this first step.

To begin, consider a set of polynomials \( P_{B-P} \) of the type

\[
P_{B-P}(z) = z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n}
\]

defined in \( \mathbb{C}^{n+1} \) with \( a_0, \ldots, a_n \) being an \( n+1 \) tuple of positive integers greater than or equal to 2. The polynomial \( f(z) \) will be considered as the map \( f : \mathbb{C}^{n+1} \rightarrow \mathbb{C} \). The Brieskorn-Pham (B-P) variety \( V_B(f) \) is defined now as [25-27]

\[
V_{B-P}(f) = \{ z \in \mathbb{C}^{n+1} \mid f(z) = 0 \}.
\]

This variety has singularity only at one point: \( z = 0 \). The Fermat variety is a special case of the B-P variety for which \( a_0 = a_1 = \cdots = a_n = a \).

**Remark 3.2.** The Fermat variety is the Calabi-Yau variety when \( a = n + 1 \), Ref.[56].

**Corollary 3.3.** Because of this observation it is clear that if the Fermat variety has something to do with physics, then the methods used in mirror symmetry calculations [57] in physics can be applied directly to the Fermat varieties as well.

Fortunately this is the case as will be demonstrated in this subsection. Our goals are broader than just to establish this correspondence since we want to create an additional links with number and knot theories which, to our knowledge, are absent in the existing treatments [57].

Consider an intersection \( K = V_{B-P}(f) \cap S^{2n+1} \) (this intersection is actually a knot or link as it will become apparent from the discussed below) of the B-P variety with the boundary of \( 2n+2 \) dimensional ball \( B \) and define the mapping

\[
\phi(z) = \frac{f(z)}{|f(z)|}.
\]
It is clear that $\phi(z)$ is again the circle map of the type discussed in the previous subsection. This time, it is more than just this map since, actually, the mapping $\phi : S^{2n+1} \setminus K \to S^1$ defines a smooth fiber bundle Ref.[58], Lemma 19.10, with base a circle $S^1$ and a fiber the complement of a knot/link $K$ in $S^{2n+1}$. Earlier, in Ref.[4], Section 3, we discussed in detail the mapping torus construction of 3-manifolds fibering over the circle. We would like to argue here that for the Brieskorn variety associated with the polynomial $f(z) = z_0^{a_0} + z_1^{a_1}$ we can recover back results obtained earlier.

To this purpose let us recall the mapping torus construction first. Consider an orientation preserving surface homeomorphism $h : S \to S$. In the case of punctured (holed) torus this homeomorphism should respect the presence of a hole. The circumference of the hole is our base space $S_1$ (e.g. read Ref.[59]) which is our knot $K$, since the knot is just a circle embedded into $S^3$. The Seifert surface of the knot is a fiber and the 3-manifold is just a fiber bundle constructed in a following way. Begin with the product $S \times 0$ (the initial state) and $S_h \times 2\pi$ (the final state) so that for each point $x \in S$ we have $(x, 0)$ and $(h(x), 2\pi)$ respectively. The interval $I = (0, 2\pi)$ can be closed (to form a circle $S^1$) by identifying 0 and $2\pi$ causing the identification:

$$i : (x, 0) = (h(x), 2\pi).$$

(3.10)

The mapping torus $T_h$ fiber bundle is just the quotient

$$T_h = \frac{S \times I}{i}.$$  

(3.11)

It is 3-manifold which fibers over the circle and is complementary to the fibered knot (lying at the boundary of such (cusped) 3-manifold as explained in Section 5 of Ref.[4]).

In the present case let us consider the monodromy map $h(z)$ given by

$$h_t(z_0, ..., z_n) = (\exp\{it/a_0\}z_0, ..., \exp\{it/a_n\}z_n).$$

(3.12)

If the point $z_0, ..., z_n$ belongs to the variety $V_{B-P}(f)$ then, the identification analogous to that given by Eq.(3.10) takes place by construction. And, hence, the rest of the fiber bundle related arguments also goes through. Therefore, indeed, the inverse map $\phi^{-1}(z)$ produces the fiber bundle known as Milnor fibration. It is $n+2$ dimensional manifold embedded into $S^{2n+1}$. Let $n = 1$, then we are dealing with the usual knots embedded in $S^3$ and their complementary 3−manifolds. It is of interest for us to reobtain the Alexander polynomial $\Delta_K$ for such knots. This task can be accomplished if the monodromy matrix $M$ is known as it is explained in detail in our earlier work, Ref.[4]. Then, the polynomial $\Delta_K$ is given by

$$\Delta_K(t) = \det(tE - M)$$

(3.13)

with $E$ being the unit matrix. Although we had discussed in Ref.[4] how the matrix $M$ can be obtained, here we need yet another way of obtaining this
matrix. To this purpose following Pham [25] and further explained by Milnor [26], let us look at Eq.(3.9) and select such fiber for which

$$\phi(z) = \frac{f(z)}{|f(z)|} = 1.$$  

(3.14)

This can be achieved, for example, by requiring $f(z) = 1$ or

$$z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n} = 1.$$  

(3.15)

**Remark 3.4.** For $n = 1$ and $a_0 = a_1 = m$ this is just the standard Fermat equation written in the affine form.

Following Pham again, let us replace $z_i^{a_i}$ by $t_i$ so that we get

$$\sum_{i=0}^n t_i = 1.$$  

(3.16)

This homeomorphism can be represented as a composition of homeomorphisms acting on the homology basis: one generator for one basis element. The basis elements are just circles, more exactly, the quotients $\omega \in \{\mathbb{Z}/a_0\mathbb{Z}, \ldots, \mathbb{Z}/a_n\mathbb{Z}\}$. The generators $r_{a_i}$ are associated with rotations, i.e.

$$r_{a_i}(\omega) = \exp\left(\pm \frac{2\pi i \nu}{a_i}\right)\omega \equiv \zeta^{\nu}\omega$$  

(3.17a)

with $\nu$ being in the range $1 \leq \nu \leq a_i - 1$. Hence, $\zeta^{\nu}$ is just an eigenvalue of the rotation (homology) operator. The reader should consult Pham [25] and Milnor [26] for more details. For the case of torus we have just two elements in the homology basis so that we have

$$[r_{a_1} \otimes r_{a_2}] (\omega_1, \omega_2) = \xi^{\nu}\eta^{\mu}(\omega_1, \omega_2).$$  

(3.17b)

With help of above information the Alexander polynomial can be written now according to Milnor [26] (and Brieskorn [27]) as

$$\Delta_K(t) = \prod_{\omega_1^\nu=1, \omega_2^\mu=1 \atop \omega_1, \omega_2 \neq 1} (t - \omega_1\omega_2),$$  

(3.18)

where all Greek letters denote the corresponding roots of unity.

To get a feeling of this result, let us consider the simplest example of the trefoil knot $T$. It was discussed in our earlier work, Ref.[4], and it is a typical example of torus knot [59,61]. More specifically, it is 2,3-type of torus knot.
Therefore, for such knot we have two "homological" cyclotomic equations: $\omega^2 = 1$ and $\omega^3 = 1$. The first produces just one relevant root $\omega_1 = -1$, while the second has two: $\omega_2 = \rho = \frac{1}{2}(-1 \pm \sqrt{-3})$, and the complex conjugate $\bar{\rho}$. The Alexander polynomial, Eq. (3.18), can be calculated now momentarily,

$$\Delta_T = (t + \rho)(t + \bar{\rho}) = t^2 - t + 1.$$  

The result coincides with earlier obtained, Eq. (3.9) of Ref. [4], as required. Earlier, in Ref. [4], we argued, in accord with Ref. [59], that there are only two fibered knots for Seifert surfaces of genus one. The figure 8 knot is the only knot other than trefoil which has punctured torus as its Seifert surface. The Alexander polynomial for the figure 8 knot cannot be obtained from the polynomial given by Eq. (3.18) since the eigenvalues of the monodromy matrix always have modulus equal to one by construction [62]. We had explained in Ref. [4] that at least one of the eigenvalues of the monodromy matrix should be strictly larger than one in order to reproduce the figure 8 Alexander polynomial. The question arises: what happens if we use numbers other than 2 and 3 describing the trefoil? This issue was also addressed by Milnor [26] and Brieskorn [27]. The resolution of the apparent paradox lies in considering the Alexander polynomial for links instead of single knots. But this should be done with some care since links may require multicomponent Alexander polynomial for their description. This topic is discussed further in section 5. In principle, the beauty of this approach to knots and links lies in the fact that it is not restricted to knots and links embedded in $S^3$. Multidimensional analogs of knots and links can be readily considered, e.g. with help of the generalized Alexander polynomial

$$\Delta(t) = \prod_{\omega_i = 1; \omega_k \neq 1} (t - \omega_0 \omega_1 \cdots \omega_n),$$  

where $k = 0 - n$ and each $i_k$ runs between $0 < i_k < a_k$. This more general case is relevant for calculations associated with multiparticle Veneziano amplitudes. It is discussed in Section 5.

Being armed with these results we would like to reobtain them now in another way. This will bring us new physical interpretation of mathematically known results. Let us begin with the Fermat equation written in its standard form

$$x^N + y^N = z.$$  

For $N = 2$ this is just the Pythagorean equation. The problem of finding all integer solutions for this equation is known as problem about finding the Pythagorean triples. We had discussed this problem in Section 2 of Ref. [4]. This problem is completely solvable. For $N > 2$ the problem of proving that the above equation has only trivial solutions (the Fermat’s last theorem) was solved by Andrew Wiles only in 1995 [63]. Some physically relevant results dependent on his proof are discussed in Section 4.3. At the same time, for finite fields the above equation has many solutions. To realize that such Fermat equation
has actually many solutions it is sufficient to look at solutions of cyclotomic equations (using results on Appendix A)

\[ x^N = x, \quad y^N = y, \quad z^N = z. \] (3.22a)

With their help Eq.(3.21) can be replaced by the linear equation in the projective space \( \mathbb{CP}^2 \)

\[ x + y = z. \] (3.22b)

Eq.(3.22b) represents a hyperplane in such space. This hyperplane can be embedded into the Grassmanian associated with such projective space. This is done with help of Plücker embedding so that each hyperplane (that is the particular solution of the Fermat’s equation) produces a point in the Grassmanian. The situation in the present case becomes analogous to that we had discussed earlier in connection with the Witten-Kontsevich model, Ref.[64]. With each point in the Grassmanian associated particular Veneziano amplitude as we shall demonstrate below and in Section 5. Mathematically, such an amplitude is interpreted as one of the periods of the Fermat’s curve. Section 5 provides generalization of this result to Fermat hypersurfaces.

In his fundamental work, Ref.[65], Andre Weil had made estimates of number of solutions of Eq.(3.21) in cyclotomic fields and came up with his famous zeta function, Eq.(2.19). Below, in the next subsection, we shall demonstrate additional physical uses of such zeta function. For the time being, we would like to focus attention of our readers on issues of immediate relevance.

In particular, Eq.(3.21) is written in the so called projective form. In such form it represents the Fermat curve in the complex projective space \( \mathbb{PC}^2 \). It may be also convenient to write it in the affine form, i.e.

\[ x^N + y^N = 1, \] (3.23)

by dividing both sides of Eq.(3.21) by \( z^N \). For \( N > 2 \), following Gross [34], we denote such affine Fermat curve as \( F(N) \). Clearly, \( F(N) \) is just a special case of Eq.(3.15). This time, however, our treatment of Eq.(3.23) is going to be different. From the algebraic geometry it is known [66] that genus of \( F(N) \) is equal to \( \frac{1}{2}(N - 1)(N - 2) \). That is the Fermat curve is the Riemann surface of genus \( g \). We associate with such surface its Jacobian \( J(N) \) and subsequently we compute, following Gross and Rohrlich [34], the periods of differential 1-forms (to be constructed momentarily) on \( J(N) \). Let us call a double of integers \( (r, s) \) admissible if \( 1 \leq r, s \leq N - 1 \) (e.g. compare with Eq.(3.17a)). To any admissible double we associate the differential

\[ \eta_{r,s} = x^{r-1} y^{s-1} \frac{dx}{y^{N-1}} \] (3.24)

of the second kind "living" on \( F(N) \). If \( \zeta \) is a primitive \( N^{th} \) root of unity, let \( A \) and \( B \) be the automorphisms of the curve given by the formulas

\[ A(x, y) = (\zeta x, y), \quad B(x, y) = (x, \zeta y). \] (3.25)
The group generated by $A$ and $B$ acts naturally on $\eta_{r,s}$. The forms $\eta_{r,s}$ are the eigenforms for this action, with eigenvalues $\zeta^{r+j+sk}$. That is
\[
(A^j B^k) \eta_{r,s} = \zeta^{r+j+sk} \eta_{r,s}.
\] (3.26)

This result should be compared with Eq. (3.17b). Based on the result by Rohrlich (Appendix of Ref.[34]) it is possible to show that this similarity is not just coincidental. For the sake of space we only provide a sketch of an argument leaving details aside. These details are given in Refs[35, 67].

The affine version of the Fermat curve, Eq.(3.23), can be parametrized as follows
\[
x = t, \quad y = (1 - t^N)^{\frac{1}{N}}.
\] (3.27)

Such parametrization naturally limits the range of $t$ to the segment $[0, 1]$. Hence, we have effectively the mapping $\gamma_0 : [0, 1] \rightarrow F(N)$ of the segment $[0, 1]$ to the Fermat curve. By combining this result with Eq.(3.24) we obtain,
\[
I_{r,s} = \int_{\gamma_0} x^{r-1} y^{s-1} \frac{dx}{y^{N-1}} = \int_0^1 t^{r-1}(1 - t^N)^{\frac{1}{N}} \frac{dt}{(1 - t^N)^{\frac{N-1}{N}}} = \frac{1}{N} B\left(\frac{r}{N}, \frac{s}{N}\right).
\] (3.28)

Let $\frac{r}{N} = a, \frac{s}{N} = b$, then we obtain the nonsymmetrized Veneziano amplitude, Eq.(1.2),
\[
NI_{r,s} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.
\] (3.29)

Clearly, the symmetrized Veneziano amplitude can be obtained now without any problems.

Remark 3.5. The numbers $a, b$ and $c = 1 - a - b$, can be made negative in accord with Veneziano formula, Eq.(1.6). The details are explained in Section 5.

To obtain the symmetrized amplitude and also to make connection with knots and links (in view of Eqs(3.17b), (3.26)), we need to make few additional steps. Following Lang [35], let $\zeta = \exp(\frac{2\pi i}{N})$. Using the affine version of the Fermat curve, Eq.(3.23), it can be shown that the chain
\[
\kappa = \gamma_0 - (1,\zeta)\gamma_0 + (\zeta,\zeta)\gamma_0 - (\zeta, 1)\gamma_0,
\] (3.30)

where $(1, \zeta), etc.$ have the same meaning as in Eq.(3.25) and $\kappa$ is a closed path, i.e. a cycle on the Fermat curve. When lifted to the Jacobian $J(N)$ such a curve becomes one of the toral periods. Hence, using Eq.(3.30) and making trivial changes of variables one finds
\[
\int_{\kappa} \eta_{r,s} = (1 - \zeta^r)(1 - \zeta^s) \frac{1}{N} B\left(\frac{r}{N}, \frac{s}{N}\right).
\] (3.31)

Since
\[
B\left(\frac{r}{N}, \frac{s}{N}\right) = \frac{\Gamma\left(\frac{r}{N}\right)\Gamma\left(\frac{s}{N}\right)}{\Gamma\left(\frac{r}{N} + \frac{s}{N}\right)} = \frac{1}{\pi} \Gamma\left(\frac{s}{N}\right) \Gamma\left(\frac{t}{N}\right) \sin\left(\frac{\pi}{N} \frac{r + s}{N}\right),
\]

\[9\]
\[See\ Section\ 5.3.2.\ for\ additional\ details\]
where $t = N - r - s$, we can rewrite Eq.(3.31) (up to a constant) in a manifestly symmetric form:

$$\int_\kappa \eta_{r,s,t} = -\zeta^2 \zeta^r \Gamma(a) \Gamma(b) \Gamma(c) [\sin \pi a \sin \pi b \sin \pi c].$$

(3.32)

Finally, by multiplying both sides by $\zeta^2$ we obtain (up to constant again)

$$\zeta^2 \int_\kappa \eta_{r,s,t} = \Gamma(a) \Gamma(b) \Gamma(c) [\sin \pi a \sin \pi b \sin \pi c]$$

(3.33)

$$\Gamma(a) \Gamma(b) \Gamma(c) [\sin(2a + \sin 2b + \sin 2c)].$$

The r.h.s. of Eq.(3.33) looks almost the same as the Veneziano amplitude Eq.(1.12). It will have exactly the same particle spectrum. Surely, it can be made to look exactly the same if we use the symmetrized form of Eq.(3.28).

Accordingly, we obtain the following

**Corollary 3.6.** With accuracy up to a root of unity, the Veneziano amplitude is just a period of the Jacobian variety for the Fermat curve.

The meaning of the phase factors $\zeta^r$, $\zeta^s$ will be explained from another point of view in Section 5.

Next, it can be shown [34,35] that

$$\int_{A^j B^k \kappa} \eta_{r,s} = \int_\kappa (A^j B^k) \eta_{r,s} = \zeta^{rj + sk} \int_\kappa \eta_{r,s}$$

(3.34)

and, therefore, the periods of the Fermat curve are eigenfunctions of the tensor product of the rotational (monodromy) operators, e.g. see Eq.s (3.17a),(3.17b), acting on the homology basis generated by closed $\kappa$ contours. From here we obtain yet another

**Corollary 3.7.** Because the parameters $r$ and $s$ in Eq.(3.31) have the same meaning as $p$ and $q$ in Eq.(3.18) for the Alexander polynomial, the systematics of particle mass spectrum is in one-to one correspondence with the systematics of torus knot/links described by the Alexander polynomial, Eq.(3.18). More general polynomial, Eq.(3.20), describing multidimensional knots/links is discussed further in Section 5.

Relevance and connections of these results with knots/links will be discussed further in Section 5 devoted to calculation of the multiparticle Veneziano amplitudes. In the meantime, we would like to discuss some additional links between the number theory and physics by reinterpreting mathematical work of Koblitz and Gross, Ref. [68], in physical terms.

### 3.3 Physical applications of Gross-Koblitz results

In Section 2 we introduced Weil’s zeta function, Eq.(2.19). From the associated discussion it should be clear that this function is effective measure of complexity

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\[\text{In Section 5 we argue that, actually, it is more advantageous to relate the particle mass spectrum to the Hodge spectrum (defined in Section 5 and Appendix D).}\]
of Riemann surfaces of genus $g \geq 1$ as compared to the "round sphere". This complexity is reflected in the polynomial $L_K(u) = \prod_{i=1}^{2g} (1 - \alpha_i u)$ whose inverse roots $\alpha_i$ carry all information about this complexity. The task of finding these roots was left unaccomplished in Section 2. Although the task of explicit finding of $\alpha_i$'s can be rather complicated for arbitrary algebraic curve, for the Fermat curves this task can be brought to completion as was demonstrated by Gross and Koblitz, Ref.[68], see also [69]. In order to connect their results with the results discussed earlier in this paper let us consider the simplest nontrivial case of the elliptic curve, i.e. the case when $g = 1$. Then Eq.(2.19) is reduced to

$$Z_K(u) = \frac{(1 - \alpha u)(1 - qu/\alpha)}{(1 - qu)(1 - u)},$$

(3.35)

It should be clear that $\alpha$ and $q/\alpha$ are associated with homology basis for torus. They might be thought as two periods of the elliptic curve, more exactly two $q$-adic periods (as explained below in Section 4.3). To bring all this "down to earth" consider, following Landau and Lifshitz, Ref.[70], a simple problem about dynamics of heavy pendulum. The period $T$ of such pendulum is determined essentially by the complete elliptic integral of the first kind: $T = 4K(k)$, where $k$ is known parameter and

$$K(k) = \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}.$$

(3.36)

Let, for example, $k = 1/\sqrt{2}$. Then, it can be shown [72] that

$$K(1/\sqrt{2}) = \frac{\sqrt{2}}{4} \frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)} = \frac{\sqrt{2}}{4} B(1/4, 1/2)$$

(3.37)

to be compared with Eq.(3.28).

Two questions immediately arise: a) is this a pure coincidence that mathematical expression for the period of the pendulum coincides with that obtained earlier for the Fermat curve? b) can heavy pendulum possess the second period and if it can what it means physically? To answer the first question we invoke the important result of Gerhard Frey [71] who discovered that the arithmetical properties of the elliptic curve $E_{A,B,C}$ defined by the Weierstrass equation

$$y^2 = x(x - A)(x + B)$$

(3.38)

are related to the arithmetic (diophantine) properties of the Fermat equation

$$a^p + b^p + c^p = 0$$

(3.39)

which is reduced to equation

$$A + B + C = 0$$

(3.40)
if $A = a^p$, $B = b^p$ and $C = c^p$. This observation paved the way for proving the Fermat last theorem by Wiles [63]. To answer the second question(s) we follow Mark Kac (as described in Ref. [72], page 77). The argument goes as follows. Suppose we change time in Newton’s equation for pendulum: $t \to \sqrt{-1}t$. Such substitution is equivalent to reversing of direction of the gravitational force and leads to a complementary periodic motion of the pendulum. That is the solution of the pendulum problem has not only its obvious real temporary period but also purely imaginary period as well. It is given explicitly by $\sqrt{-1}K(k')$ where $k' = \sqrt{1-k^2}$. For $k = 1/\sqrt{2}$ we get $k = k'$ so that the ratio of two periods is just $\sqrt{-1}$. But $\sqrt{-1}$ is the Gaussian prime (e.g. read our earlier work, Ref. [4]) that is it plays the same role as $q$ in Eq. (3.35) and, therefore, indeed, our conjecture about the meaning of $\alpha'$’s in Eq. (3.35) is apparently correct. Gross and Koblitz went much further in their analysis. We believe, that it is worthwhile to discuss their results in some detail since there is some physics associated with them.

To accomplish this task we need to introduce some definitions. For more details reader is urged to read pedagogically written book by Winnie Li [17]. Let $F_q$ be a finite field. For $a \in F_q$ define the additive character $\psi^a$ of the field $F_q$ as follows: $\psi^a(x) = \psi(ax) \forall x \in F_q$. The multiplicative character $\chi(x)$ is determined by the cyclotomic equation, Eq. (A.5). It is just one of the $q$-th nontrivial roots of unity and there are $q - 1$ of them. With help of these characters one can construct the Gaussian and the Jacobi sums. In particular, the Gauss sum is defined as

$$g(\chi, \psi) = - \sum_{x \in F_q^*} \chi(x)\psi(x)$$

(3.41)

where $F_q^* = F_q \setminus \{0\}$, while the Jacobi sum is defined as

$$J(\chi_1, \chi_2) = - \sum_{x \in F_q \atop x \neq 0, 1} \chi_1(x)\chi_2(1 - x).$$

(3.42)

The Gauss sum is analog in $F_q$ of the gamma function

$$\Gamma(s) = \int_0^\infty x^s e^{-x} \frac{dx}{x}$$

(3.43)

since this is ”sum” (i.e. integral with respect to its Haar measure $\frac{dx}{x}$) over the multiplicative character $x^s$ and the additive character $e^{-x}$. The Jacobi sum is analog of Euler’s beta function since

$$J(\chi_1, \chi_2) = \frac{g(\chi_1, \psi)g(\chi_2, \psi)}{g(\chi_1\chi_2, \psi)}.$$  

(3.44)

The $p$-adic gamma function (not to be confused with Eq. (1.21)) $\Gamma_p(n)$ for an integer $n \geq 2$ is defined by

$$\Gamma_p(n) = (-1)^n \prod_{j=1}^{n-1} \frac{1}{(p^j-1).}$$

(3.45)
It possesses almost standard properties, e.g.,
\[
\Gamma_p(x + 1) = \begin{cases} 
-x\Gamma_p(x) & \text{if } x \in \mathbb{Z}_p^* \\
-\Gamma_p(x) & \text{if } x \in p\mathbb{Z}_p
\end{cases}
\]

Koblitz had demonstrated in Ref. [69] that for the Fermat curve, Eq. (3.23), the zeta function of Weil is given by [see also [111], ch-r 6],
\[
Z(F(n); u | F_q) = \frac{1}{(1-qu)(1-u)} \prod_{1 \leq r,s < N-1 \atop r+s \neq N} (1 - J(\chi^r, \chi^s)u) \tag{3.46}
\]

In addition, he shows that
\[
J(\chi^r, \chi^s) = \frac{\Gamma_p(\frac{r}{N})\Gamma_p(\frac{s}{N})}{\Gamma_p(\frac{r+s}{N})} \tag{3.47}
\]
to be compared with Eq. (3.28). From here it follows that Eq. (3.47) is the p-adic analogue of the 4 particle Veneziano amplitude. In proving Theorem 4.13 of Ref. [68] Gross and Koblitz show that knowledge of the p-adic beta function allows to restore the full beta function, i.e., the full Veneziano amplitude and vice versa. This observation provides the strongest support to the conjecture of Raoul Bott who said, Ref. [50], page 159, that "I would not be too surprised if discrete mod p mathematics and the p-adic numbers would eventually be of use in the building of models for very small phenomena".

**Remark 3.8.** It is interesting to notice at this point that if instead of the complete elliptic integral, Eq. (3.36), we would consider an indefinite elliptic integral, e.g.,
\[
z = \int \frac{d\mathcal{P}}{4\mathcal{P}^3 - g_2\mathcal{P} - g_3}
\]
then, its inverse is the stationary solution of the KdV equation, i.e.,
\[
\left(\mathcal{P}'\right)^2 = 4\mathcal{P}^3 - g_2\mathcal{P} - g_3.
\]

Mulase [72] had shown how one can easily obtain the time-dependent solution from the time-independent one. Hence, the dual of KdV leading to the Virasoro algebra (as we had discussed in our earlier work, Ref. [73]) can be obtained as well.

Plausible as they are, these results can be considerably extended strengthened and generalized beyond the existing string theory formalism. Steps in these directions are presented in the following sections.

### 4 The Chowla-Selberg formula, the Fermat’s last theorem, the conformal field theory and the Veneziano amplitude

#### 4.1 Statement of the problem
Originally, the Chowla-Selberg formula was announced in 1949, Ref.[74], but was left practically unnoticed. This caused the authors to provide more details. The revised and extended paper received the widespread attention among mathematicians but was published only in 1967 [31]. Ironically, the Veneziano’s paper had appeared in 1968 [1] while the Annals of Mathematics paper by Ramachandra [75] published in 1964 containing detailed derivation of the Chowla-Selberg main results (to appear only in 1967!) also was left largely unnoticed.

In this work we would like to make connections between these historical papers. We begin with the description of the problem Chowla and Selberg wanted to solve. Looking at Eqs.(3.36) and (3.37) they wanted to know if the result, Eq.(3.37), is mere coincidence or if it is general rule. To make things more interesting, let us recall that the classical theory of elliptic functions provides us with the following results. Periods $K$ and $K'$ are given through the elliptic theta functions [76]

$$K = \frac{\pi}{2} \theta_3^2(\tau),$$

$$K' = \frac{\pi}{2} \theta_3^2(-\frac{1}{\tau}),$$

while $\tau$ is determined implicitly through

$$k^2 = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)},$$

with

$$\theta_2 = 2q^\frac{1}{4}H_0H_1^2 \quad \text{and} \quad \theta_3 = H_0H_2^2$$

provided that $q = \exp(\pi i \tau)$ and functions $H_0$, $H_1$ and $H_2$ are given by

$$H_0 = \prod_{k=1}^{\infty} (1-q^{2k}), \quad H_1 = \prod_{k=1}^{\infty} (1+q^{2k}), \quad H_2 = \prod_{k=1}^{\infty} (1+q^{2k-1}).$$

Based on these results, it seems like a hopeless task to calculate either of periods exactly. Needless to say the connection between Eqs.(3.36) and (3.37) looks totally mysterious. Nevertheless, Chowla and Selberg had demonstrated that this connection is not an accident. Under certain conditions to be discussed below, the representation of periods as products/ratios of gamma functions is the only possibility. In view of the Veneziano formula, Eq.(1.2) (or (1.6)), the conditions under which such representation is possible certainly is of physical interest.

Traditionally, calculation of the periods is done with help of the hypergeometric function as we had discussed earlier in Ref.[77]. It is appropriate to remind some aspects of this connection now. We have $K = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1; k^2\right)$, where the hypergeometric function is obtained as solution of the hypergeometric equation

$$z((1-z)F'' + [c - (a + b + 1)z]F' - abF = 0.$$
Since both $K$ and $K'$ can be obtained as solutions of Eq.(4.6) we can form their ratio
\[ \frac{y_1}{y_2} = w(\tau) = \frac{K'(\tau)}{K(\tau)}, \]
(4.7)
where $y_1$ and $y_2$ are solutions of the Fuchsian -type equation
\[ y'' + \frac{1}{2}\{w,\tau\}y = 0 \]
(4.8)
with the Schwarzian derivative $\{w,\tau\}$ known to be as
\[ \{w,\tau\} = \left( \frac{w''}{w'} \right)' - \frac{1}{2} \left( \frac{w''}{w'} \right)^2 \]
(4.9)
with $w = w(\tau), w' = \frac{dw}{d\tau}$, etc. Clearly, everybody who is familiar with string and conformal field theories will recognize at this moment connections between mathematics and physics. We are not going to develop such connections in this paper nevertheless. The only reason we have brought the hypergeometric equation to the attention of our readers is to emphasize that this is equation for the periods of the elliptic curve. As it is well known, the Knizhnik-Zamolodchikov (K-Z) equations are basic for all conformal statistical mechanical models [78-80] and are effectively reducible to the equations of hypergeometric type. Hence, all results of conformal field theories will remain unchanged in what follows. In particular, the expression for the nearest neighbor spin-spin correlator for the Ising model involves $F(\frac{1}{2}, \frac{1}{2}, 1; k^2)$ [81], page 69. But, much more can be actually accomplished thanks to results obtained by Chowla and Selberg.

To begin, we would like to remind our readers about the partition function for the free bosons on the torus described, for example, in Ref.[82], pages 340-344. The case of free fermions (Ising model) technically is almost the same. Indeed, in the first place the whole computation depends upon calculation of the sum
\[ G(s) = \sum_{\substack{m,n \neq n \pm n}} \frac{1}{|m + n\tau|^s}, \]
(4.10)
while in the second, of the sum
\[ G_{\mu,\nu}(s) = \sum_{\substack{m,n \neq n \pm n}} \frac{1}{|m + n\tau + (\mu + \nu\tau)|^{2s}}. \]
(4.11)
The purpose of the entire calculation in both cases lies in explicitly obtaining $G'(0)$ and $G'_{\mu,\nu}(0)$ where the prime means differentiation with respect to $s$ variable. It is clear that in both cases obtained expressions are $\tau-$ dependent. They also should be invariant with respect to modular transformations: $\tau \rightarrow \tau + 1$.

\[ ^{11}\text{Strong additional support of this claim is presented in Section 5.3.2 below.} \]
and \( \tau \to -\frac{1}{\tau} \). It is well known, e.g. see our earlier work, Ref.[4], that all transformations of the type
\[
\tau' = \frac{a\tau + b}{c\tau + d} \tag{4.12}
\]
with \( a, b, c, d \) being some integers subject to restriction \( ad - bc = 1 \) can be obtained by successive applications of the above two. In the same work we also explained what these transformations mean topologically. In view of generalizations which follow we need to present these results from the different perspective now.

### 4.2 Complex multiplication

To this purpose it is helpful to recall that, according to the theorem by Jacobi [76], the modular lattice \( L \),
\[
L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \text{ or, symbolically, } L = [\omega_1, \omega_2], \tag{4.13}
\]
has periods such that \( \tau = \text{Im} \frac{\omega_2}{\omega_1} > 0 \), that is one of the periods should be complex. Eq.(4.13) looks the same as Eq.(A.2) and, actually, has the same meaning. Moreover, from the algebraic point of view, \( L \) is just an ideal. In the Appendix B of our work, Ref.[4], we explained the notion of an ideal. For the sake of uninterrupted reading we would like to recall few things from that reference.

Suppose we have some set \( I \) and another set \( L \), then \( \forall \alpha, \beta \in I \) and \( \forall \xi \in L \), we have
\[
\alpha + \beta \in I \text{ (module property), } \alpha \xi \in L \text{ (ideal property).} \tag{4.14}
\]
In the present case, we have \( \alpha \) and \( \beta \in \mathbb{Z} \) and \( L = [\omega_1, \omega_2] \). Clearly, if \( \omega_1 \) and \( \omega_2 \) constitute the basic parallelogram of such lattice then, any point \( x = \alpha \omega_1 + \beta \omega_2 \) also belongs to the lattice. Surely, for a given lattice we can make many sublattices using equivalence relation \( x \equiv y \text{ (mod } m) \), that is \( x - y = mk \) for some integer \( k \). Specifically, if we initially had lattice \( L = [\omega_1, \omega_2] \) we can consider lattice \( L' = [\omega'_1, \omega'_2] \) where, in view of the results of the Appendix A, we have to demand
\[
\omega'_2 = a\omega_2 + b\omega_1 \tag{4.15} \\
\omega'_1 = c\omega_2 + d\omega_1.
\]
Eq.(4.12) is obtainable from these two equations by forming their ratio. We refrain from doing this for the moment since we want to consider the possibility \( \omega'_1 = \alpha \omega_1, \omega'_2 = \omega_2 \). That is we want to study the eigenvalue problem for the matrix \( A \) given by
\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4.16}
\]
According to Eq.(B.3) of Appendix B the problem of finding the eigenvalues of matrix $A$ is reduced to the problem of finding eigenvalues of the quadratic equation

$$\alpha^2 - \text{tr} A \alpha + \det A = 0. \quad (4.17)$$

This equation should be supplemented with some physical conditions in order to make sense of its solutions. In particular, let $a(L)$ denote the area of the period parallelogram associated with the lattice $L$. Evidently, [83],

$$a(L) = \frac{1}{2} |\omega_1 \bar{\omega}_2 - \omega_2 \bar{\omega}_1|, \quad (4.18)$$

where the overbar means the complex conjugation. In particular, for the lattice $L_\tau = [1, \tau]$ Eq.(4.18) produces $a(L) = |\text{Im}(\tau)|$. Now, if we rescale $\omega's$, we obtain $a(\alpha L) = |\alpha|^2 a(L)$. If we require that, upon such rescaling, the area $a(L)$ remains unchanged this leaves us with the only one option: $|\alpha|^2 = 1$. This result should be dealt with in connection with Eq.(4.17). This leaves us with not too many choices. One option is to have $\text{tr} A = 0$ thus producing rather trivial result: $\alpha = \pm 1$. The rest of options comes from the standard number-theoretic results about units in the quadratic fields as discussed for example by Hardy and Wright [84] (see also Appendix B of our earlier work, Ref.[4]). Hence, in addition to $\pm 1$, we obtain also $\alpha = \pm i$ and $\alpha = \pm \rho, \alpha = \pm \rho^2$ with $\rho = \frac{1}{2}(-1 + \sqrt{-3})$. The last result we had encountered earlier in connection with the Alexander polynomial for trefoil knot, Ref.[4], Section 3. These are the only options available for the quadratic fields. In the light of these results the following definition is useful:

**Definition 4.1.** Two complex tori $T = \mathbb{C}/L$ and $T' = \mathbb{C}/L'$ are isomorphic (respectively isogenous ) provided that there is an isomorphism (respectively isogeny) $\alpha : T \rightarrow T'$ induced by multiplication $\alpha : \mathbb{C} \rightarrow \mathbb{C}$ for $\alpha \in \mathbb{C}$ with $\alpha L = L'$ (respectively $\alpha L \subseteq L'$).

From this definition it follows that isogeny is an equivalence relation because $L'$ is homothetic to $L$. By establishing such relation we are effectively embedding our tori into the projective space. Homothetic lattices are associated with isomorphic elliptic curves [85]. This feature, known in literature as complex multiplication (CM) [85], admits generalization to Riemann surfaces of higher genus [18].

It is important to realize that in general an arbitrary torus $T$ may **not** possess CM. In the traditional (and, hence, more general) case there is no need to require $\omega'_1 = \alpha \omega_1, \omega'_2 = \alpha \omega_2$. In this case Eqs.(4.15) are reduced to Eq.(4.12), provided that $ad - bc = 1$. Eq.(4.12) might be interpreted as equation representing motion in the Teichmüller space of the punctured torus: Different initial points in the moduli space for $T$ will produce different orbits in the Teichmüller space of the torus. All this is explained in our earlier work, Ref.[4].

Let us now go back to Eqs (4.15). The second of these equations can be rewritten as $\alpha = c\tau + d$ and, since $\tau$ is a complex number, $\alpha$ should be a complex number as well.

**Remark 4.2.** At this point we would like to notice that if $\tau$ would be real, the multiplier $\alpha$ becomes real as well. Then, instead of CM theory one would
have real multiplication (RM) theory. Such theory would make no sense from the point of view of traditional algebraic geometry but would make sense from the point of view of non commutative geometry of Connes according to the recent papers by Manin [86,87]. In our work such possibility is not considered since CM is sufficient for description of physically relevant phenomena. Surely, we do not exclude the possibility that RM might have some physical significance and leave this option open for further study.

Going back to the subject of CM, we notice, that the first of Eq.s(4.15) can be written as \((ct + d)\tau = a\tau + b\). This result can also be written in the form of Eq.(4.12), i.e.

\[
\tau = \frac{a\tau + b}{ct + d}.
\]  

(4.19)

This is equation for the geodesic in \(H\) as we had discussed in Section 2 of Ref.[4]. Explicitly written, it looks like

\[
c\tau^2 + (d - a)\tau - b = 0.
\]

(4.20)

It can be equivalently rewritten as

\[
(ct)\tau = b + (d - a)\tau.
\]

(4.21)

If we denote \(ct\) as \(\beta\) then, the above equation can be presented as \(\beta\omega_2 = b\omega_1 + (d - a)\omega_2\) and, in addition, we have \(\beta\omega_1 = c\omega_2\). These equations are of the same kind as Eq.s(4.15) as required. Moreover, using the same equations and eliminating \(\tau\) we obtain the following equation for \(\alpha\):

\[
\alpha^2 - (a + d)\alpha + ad - bc = 0
\]

(4.22)

in agreement with Eq.(4.17) or, taking into account that \(ad - bc = 1\), we obtain

\[
\alpha^2 - (a + d)\alpha + 1 = 0
\]

(4.23)

This is equation for an integer in the quadratic field (Appendix B, Ref.[4]). But earlier we have obtained \(\alpha = ct + d\) and, hence, such integer must belong to the imaginary quadratic field. Therefore, \(\tau\) also belongs to the imaginary quadratic field but is not an integer in general since \(\tau = (\alpha - d)/c\). As we had discussed in Ref.[4], the imaginary quadratic fields are being characterized by their class numbers \(h(\sqrt{-d})\). These numbers directly associated with the unique factorization property for the ideals of these fields. Every number \(\kappa\) in imaginary quadratic field can be presented as \(\kappa = a + b\sqrt{-d}\), where \(a\) and \(b\in \mathbb{Q}\). Unique factorization producing class number \(h(\sqrt{-d}) = 1\) is possible if \(-d = 1, 2, 3, 7, 11, 19, 43, 67\) and 163. Sometimes, instead of 1 and 2 numbers 4 and 8 being used but, surely, these produce the same results. We mention this fact for the sake of comparison with the existing literature. If we would like to restrict ourself by these numbers, the parameter \(\tau\) in Eq.s(4.10) and (4.11) cannot be arbitrary anymore. It should be discrete. Nevertheless, because
we can vary integers $d$ and $c$ in the equation $\tau = (\alpha - d)/c$, this discreteness does not lead to just one value for $\tau$.

Now we would like to illustrate all these statements by simple but important examples. These examples serve to underscore the differences between what is known so far in the conformal field and string theories and new elements which, in our opinion, should be included. Inclusion of these new elements will enable us to use powerful number-theoretic methods in the conformal and string theories\(^{12}\).

Going back to Eq. (4.23) we obtain the following expressions for $\alpha'$'s :

$$\alpha_{1,2} = \frac{a + d}{2} \pm \frac{1}{2} \sqrt{(a + d)^2 - 4}.$$  \hspace{1cm} (4.24)

Since $\alpha$ should be an integer in the imaginary quadratic field we are left with the following set of options:

a) $a = d = 0$, thus producing $\alpha_{1,2} = \pm i$;

b) $a = \pm 1, b = 0$ (or $a = 0, b = \pm 1$), thus producing $\alpha_{1,2} = \frac{1}{2}(\pm 1 \pm \sqrt{-3})$

c) $a = d = \pm 1$, thus producing $\alpha_{1,2} = \pm 1$.

We have obtained all these results earlier and, surely, we would not reproduce them once again should these be the only options. Fortunately, there are other less trivial options leading to far reaching consequences.

Consider $a(L') = \frac{1}{2} \left| \omega_1' \bar{\omega}_2' - \omega_2' \bar{\omega}_1' \right|$ where $\omega_1'$ and $\omega_2'$ are given by the l.h.s. of Eq. (4.15). This expression can be rearranged with the help of Eq. (4.15) as follows:

$$a(L') = \frac{1}{2} \left| \omega_1' \bar{\omega}_2' - \omega_2' \bar{\omega}_1' \right|$$  \hspace{1cm} (4.25)

provided that $ad - bc = n$. We would like to argue now that such relation is legitimate. Indeed, the relation $ad - bc = 1$ comes as result of the requirement for the inverse of the matrix $A$, Eq. (4.16), should have only integer coefficients [88]. This is a legitimate requirement as long as we are dealing with the same lattice and just trying to change the basis. But if we are dealing with sublattices, relation $ad - bc = 1$ can be relaxed to $ad - bc = n$. This can be rigorously proven, e.g. see Ref. [89], ch-r1, and leads to a very deep results summarized in the Appendix C. The scaling result $a(\alpha L) = |\alpha|^2 a(L)$ discussed earlier can now be rewritten as $a(L') = a(\alpha L) = n a(L)$. This suggests that if $L$ is sublattice of $L'$, then these sublattices are homothetic and therefore are isogenic and, hence, are equivalent. Accordingly, Eq. (4.22) should be amended leading to

$$\alpha_{1,2} = \frac{a + d}{2} \pm \frac{1}{2} \sqrt{(a + d)^2 - 4n}.$$  \hspace{1cm} (4.26)

\(^{12}\)Usefulness of these "new elements" to conformal and "string" theories can be seen from both: the references provided at the end of Section 1.3 and from actual developments presented in Sections 5.2. and 5.3.
Thus, indeed, the complex multiplication is associated with integers in the imaginary quadratic field. From the above discussion it follows that there are only 9 imaginary quadratic fields whose ring of integers has class number \(h(\sqrt{-d}) = 1\). Actually, more can be said. Following Silverman, Ref.[85], we notice that the \(j(L)\) invariant of the elliptic curve (modular form of weight zero, e.g. see Eq.(5.8) below) is just some rational number \(\in \mathbb{Q}\) provided that \(j(L)\) characterizes the equivalence classes of elliptic curves with CM whose class number is 1. Moreover \(j(L) \in \mathbb{Q}\) only in 3 other cases so that there are exactly 13 classes of elliptic curves with complex multiplication. Such curves are known as Weil curves. They possess remarkable properties to be briefly discussed in the next subsection. It should be noted, however, that we are concerned about \(h = 1\) case only because it is technically much simpler as compared to other quadratic fields for which \(h > 1\). Altogether, quadratic imaginary fields are much simpler to handle than the real quadratic fields [90].

4.3 Fermat’s last theorem, periods of elliptic curves and Veneziano amplitudes

Zeta function of the elliptic curve \(Z_K(u)\) is given by Eq.(3.35). Now we would like to connect this function with modular forms of Appendix C. According to famous Taniyama-Shimura (T-S) conjecture (now proven, thanks to works of Wiles [63] and Taylor-Wiles [91]) every elliptic curve over \(\mathbb{Q}\) is modular. Naturally, we would like to explain what this notion means and how it is connected with \(Z_K(u)\). Since the results to be discussed depend crucially on (now proven) T – S conjecture and since the proof of this conjecture effectively implies, according to works by Frey [71] and Ribet [92], the Fermat’s last theorem, e.g. see Eqs.(3.38)-(3.40), it happens, that the mathematics related to Veneziano amplitudes is also related to the Fermat’s last theorem. Accordingly, the multiparticle generalization of these amplitudes, to be discussed in Section 5, is dependent to some extent also on the validity of the Fermat’s last theorem.

Let \(\mathcal{E}\) be an elliptic curve over \(\mathbb{Q}\). Analytically, it can be written in the so called (affine) Weierstrass form as

\[
y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6
\]

(4.27)

with coefficients in \(\mathbb{Q}\). Let \(\Delta(a_1,...,a_6)\) be the discriminant of the elliptic curve \(\mathcal{E}\) [83,85]. It carries information about the smoothness of \(\mathcal{E}\). If \(\Delta = 0\) the curve is singular, that is it may contain a node or a cusp. Suppose that our curve is nonsingular in the field \(\mathbb{Q}\). Suppose as well that we would like to know if such curve will still remain nonsingular if we reduce it, that is, if we would like to consider instead of Eq.(4.27) its \(\mathbb{F}_p\) analog (Appendix A)

\[
y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 \pmod{p}
\]

(4.28)

Remark 4.3. Such reduction allows us to use without change all results of Section 2.2.
Surely, to do so we need to get rid of denominators in Eq.(4.27) so that all coefficients belong to $\mathbb{Z}$. Upon reduction the curve $\mathcal{E}_p$ may become singular (even though it was not singular initially!). This happens when $\Delta(a_1,\ldots,a_6) \equiv 0(\text{mod } p)$ and represents the case of bad reduction. Otherwise the reduction is called good. Clearly, $\mathcal{E}$ has bad reduction only for a finite number of primes.

The bad reduction can be measured by means of the conductor. The conductor is a number $N$ such that

$$\text{Cond}(\mathcal{E}) \equiv N = \prod_p p^{f_p}.$$  \hspace{1cm} (4.29)

Here $f_p = 0$ if $p \nmid \Delta$ and $f_p \geq 1$ otherwise. Let $N_1$ be the number of solutions of Eq.(4.28) in the projective space (e.g.see Eq.(2.26)) then, the Hasse-Weil $L$ function of $\mathcal{E}$ can be defined as

$$L(\mathcal{E}, s) = \prod_p L_p(p^{-s}),$$  \hspace{1cm} (4.30)

where, in the case of good reduction, $[L_p(p^{-s})]^{-1} = (1-pu)(1-u)Z_p(u) |_{u=p^{-s}}$ with $Z_p(u)$ defined by Eq.(3.35). In general case we have as well

$$L_p(p^{-s}) = 1 - a_p p^{-s} + \psi(p) p^{1-2s}$$  \hspace{1cm} (4.31)

with $a_p = 1 + p - N_1(p)$ and $\psi(p) = 0$ or $1$ depending upon $p$ dividing or not $\text{Cond}(\mathcal{E})$. Hence, in general we can write

$$L(E, s) = \prod_{p|N} \frac{1}{1 - a_p p^{-s}} \prod_{p \nmid N} \frac{1}{1 - a_p p^{-s} + p^{1-2s}}.$$  \hspace{1cm} (4.32)

Now it is time to connect this result with the modular forms discussed in the Appendix C. More exactly, we need to make connections with the cusp forms of weight 2. The cusp form $f(\tau)$ is defined by its $q$ expansion: $f(\tau) = \sum_{n=1}^{\infty} c(n) q^n$, $q = \exp(2\pi i \tau)$. Using the properties of Hecke operators (Appendix C) and, in particular, Eq.s(C.9) and (C.16) (for $k=2$), we obtain (upon acting on $f(\tau)$) the following set of recursions for $c(n)'s$:

$$c(p^e)c(p) = c(p^{e+1}) + pc(p^{e-1})$$ for $p$ prime, $p \nmid N$, \hspace{1cm} (4.33a)

$$c(p^e) = [c(p)]^e,$$ for $p$ prime and $p \mid N$, and \hspace{1cm} (4.33b)

$$c(m)c(n) = c(mn).$$ \hspace{1cm} (4.33c)

The Mellin transform of the cusp form can be defined now as

$$F(s) = \int_0^\infty \frac{d\xi}{\xi} f(i\xi) \xi^s = \Lambda(f, s) := (2\pi)^{-s} \Gamma(s) \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$$  \hspace{1cm} (4.34)

$$\equiv (2\pi)^{-s} \Gamma(s)L(f, s).$$

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It can be easily demonstrated, e.g. see Ref. [93], that $L(f, s) = L(E, s)$ provided that the cusp form (of weight $k = 2$) is also an eigenform for the involution operator $W_N$ defined by

$$W_N f(\tau) = \pm N \tau^{-2} f\left(\frac{-1}{N \tau}\right).$$

(4.35)

More exactly, it is expected that the cusp form of weight 2 is an eigenform of two operators $T_2(n)$ and $W_N$:

$$T_2(n)f(\tau) = c(n)f(\tau),$$

(4.36a)

$$W_N f(\tau) = \eta f(\tau)$$

(4.36b)

with $\eta = \pm 1$ (fermionic property). If $f(\tau)$ lies in one of the two eigenspaces for $W_N$ and satisfies Eq.(4.36a) then, $L(f, s) = L(E, s)$.

**Definition 4.4.** An elliptic curve over $\mathbb{Q}$ is said to be modular if there exist a cusp form of weight 2 on $S_N$ (Appendix C) for some $N$ such that $L(f, s) = L(E, s)$.

**Remark 4.5.** This identification means that $\alpha'_p s$ in Eq.(4.32) should be replaced by $c(p)'s$.

Shimura and Taniyama conjectured that every elliptic curve over $\mathbb{Q}$ is modular and Wiles [63] proved correctness this conjecture. Now it remains to explain what all these results have to do with the Veneziano amplitudes. We shall provide the answer based on results of the Eichler-Shimura theory (surely only those aspects that are relevant to our immediate needs) and keeping in mind further uses of the Chowla-Selberg formula (to be discussed later in Subsection 4.5.). In addition, the entire Section 5 is devoted to elaborations of these results from yet another perspective.

We begin with observation that the combination $f(\tau)d\tau$ is invariant with respect to action of modular transformations which belong to $S_N$. Define now a "path integral"

$$\Phi_f(\gamma) = \int_{\gamma(\tau_0)}^{\gamma(\tau_0)} f(\xi)d\xi.$$

(4.37)

This integral is actually independent of $\tau_0$ for $\gamma \in S_N$. If $\gamma \in S_N$ is elliptic ($|tr\gamma| < 2$) or parabolic ($|tr\gamma| = 2$) it can be shown [93] that $\Phi_f(\gamma) = 0$. This is consistent with Eq.(4.20) which is expected to have two solutions, that is to produce the geodesic generated by the hyperbolic transformations. Hence, only hyperbolic transformations should be taken into account when calculating $\Phi$. Since $\Phi_f(\gamma)$ is independent of $\tau_0$ this is possible only if $\Phi_f(\gamma)$ describes closed geodesic on the corresponding Riemann surface $R = \mathbb{H}/S_N$. But closed geodesic on the Riemann surface $R$ corresponds to the period (e.g. see Eq.(3.31)) when it is lifted to the Jacobian $J(R)$ of $R$. Surely, if $R$ has more than one cusp form, there will be more than one period. Since these cusp forms are independent of each other, the corresponding periods are also independent.
Remark 4.6. It is remarkable that such periods are defined on surfaces with cusps. In the case of dynamics of 2+1 gravity (equivalent to the dynamics of train tracks) the simplest non trivial Riemann surface on which the train track dynamics can be studied is represented by once punctured torus [4]. The simplest non trivial correlator for the Kontsevich-Witten model of 2 dimensional topological gravity also involves integration over the moduli space of once punctured torus [4].

The intersection pairing form \( \langle C, \omega \rangle \) can be defined now as
\[
\gamma(\tau_0) \int_{\tau_0} f(\xi) d\xi = \int C \omega = \langle C, \omega \rangle \equiv \Omega. \tag{4.38}
\]

This form can be easily defined for the Riemann surfaces of higher genus. Many interesting results related to Selberg trace formula, dynamics of hyperbolic flows, etc. can be found in Refs. [94,95] and references therein. In this work we are concerned with different issues however. In particular, we would like to explain why \( \alpha \) and \( \frac{q}{\alpha} \) defined in Eq.(3.35) can be called "periods", more exactly, \( p \)-adic periods. We also would like to connect these \( p \)-adic periods with \( \Omega \).

Ignoring for the moment bad reduction terms in Eq.(4.32) it is useful to rewrite Eq.(3.35) in the following equivalent form
\[
Z(p^{-s}) = \frac{1 + (N_p - p - 1)p^{-s} + p^{1-2s}}{(1 - p^{-s})(1 - p^{1-2s})} \equiv \frac{\zeta_p(s)\zeta_p(s - 1)}{L_p(s)} \tag{4.39}
\]
where, according to Eq.(4.31), \( N_p - p - 1 = -c(p) \) (see Remark 4.5). The global zeta function \( \hat{\zeta}(s) \) is obtained now as
\[
\hat{\zeta}(s) = \prod_p \frac{\zeta_p(s)\zeta_p(s - 1)}{L_p(s)} = \frac{\zeta(s)\zeta(s - 1)}{L(\mathcal{E}, s)}
\]
with
\[
L(\mathcal{E}, s) = \prod_p \left[ 1 + (N_p - p - 1)p^{-s} + p^{1-2s} \right]^{-1}. \tag{4.40}
\]

Let \( s = 1 \) in Eq.(4.40). Then, we obtain a very remarkable result, Ref.[96],
\[
L(\mathcal{E}, 1) = \prod_p \left( \frac{p}{N_p} \right). \tag{4.41}
\]

Moreover, using Eq.(4.34), we obtain as well
\[
L(\mathcal{E}, 1) = -2\pi i \int_0^{\infty} f(x) dx. \tag{4.42}
\]

Next, following Manin, Ref.[97], we would like to combine Eqs.(4.35), (4.36b) in order to get
\[
L(\mathcal{E}, 1) = (1 \pm 1) \sum_{n=1}^{\infty} \frac{c(n)}{n} \exp(-2\pi n/N) \tag{4.42}
\]
so that if $c(n)$ are known, $L(E, 1)$ can be calculated numerically very efficiently. Next, the function $L(f, s)$, Eq.(4.34), actually coincides with the Hecke zeta function $L(s, \chi)$ with Grössencharakter $\chi$. It is known, that such function has the following presentation and the associated with it Euler product

\[ L(s, \chi) = \sum_{a} \frac{\chi(a)}{N(a)^s} = \prod_{p} (1 - \chi(p)N(p)^{-s})^{-1}, \quad (4.43) \]

where the summation is taking place over all nonzero integral ideals with $N(a)$ being the norm of the ideal (e.g. see Ref.[98] for a quick introduction to these concepts) while the product is taken over all prime ideals. In Refs.[99], ch-r.18, and [100], ch-r.8, it is shown (using some examples) that, actually, $L(s, \chi) = L(f, s)$. Next, given this fact one needs to invoke the famous theorem by Weil [101] which identifies Hecke’s Grössencharakters with the Jacobi sums, e.g. see Eq.(3.42). Once this is done, Eqs.(3.35),(3.38), (3.46),(3.47) and (4.31) should be used along with Definition 4.4. and Remark 4.5. in order to claim that,

\[ a(p) = \alpha(r) + \bar{\alpha}(s) \approx J(\chi^r, \chi^s) = \frac{\Gamma_p(\frac{r}{N})\Gamma_p(\frac{s}{N})}{\Gamma_p(\frac{r+s}{N})} \equiv \omega_p. \quad (4.44) \]

Since according to Eq.(2.19) $\alpha \cdot \bar{\alpha} = p$, we obtain as well, $\bar{\alpha} = p/\alpha$. Because the r.h.s. of Eq.(4.44) is a $p$-adic period, the l.h.s., that is $\alpha(r)$ (or $a(p)$ ) is also a $p$-adic period. This proves the claim we made after Eq.(3.35) about the physical nature of $\alpha$. Moreover, this proves the claim made earlier that to reach the conclusion about $\alpha$ the last Fermat’s theorem should be used.

Remark 4.7. The connection between $\alpha$’s and the $p$-adic periods was first made by Dwork. Good and readable summary of this and related papers by Dwork is given in Ref.[102]. Relevant results are also summarized briefly in the book by Koblietz, Ref.[69]. The arguments presented above differ however from those of Dwork. They rely on the fundamental result of Andre Weil [101]. We would like now to connect these facts with Eq.(4.41) and also with Eqs.(1.23), (1.24). To do this let us consider, following Birch and Swinnerton-Dyer, Ref.[96], (and also Cassels, Ref.[103]) the product

\[ \prod_{p} \omega_p \sim \tau \quad (4.45) \]

where $\tau$ denotes the Tamagawa number. Calculation of Tamagawa numbers is described for example in the book by Weil [104]13. Nevertheless, the problem of finding $\tau$ in Eq.(4.45) is not completely solved to our knowledge. By analogy with Eq.(1.20), the above result remains unchanged if we replace $\omega_p$ by $|a|_p \omega_p$ since $\prod_{p} |a|_p = 1$. That is the product, Eq.(4.45), is invariant with respect to some scale changes. This is important in view of Eq.(4.44). The product in

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13More recent useful reference is the monograph by Maclachlan and Reid, Ref.[105], where the Tamagawa numbers are discussed along with many other relevant facts.
Eq.(4.45) is taken over all places, including $\infty$, therefore,

$$\tau \simeq \Omega \prod_{\text{finite } p} \omega_p \quad (4.46)$$

Surely, in view of Eq.(4.44) we expect $\Omega = B(r, s)$ with $B(r, s)$ being the usual beta function that is the Veneziano amplitude. Eq.(4.46) is the exact analog of Eq.(1.25) of the p-adic string theory [11]. To make all this connected with the number theory we have to look at Eq.(4.38) and to assume that

$$\int_C \omega \rightleftharpoons \int_{C_p} \hat{\omega}_p. \quad (4.47)$$

More accurately, this suggests that $\int_{C_p} \hat{\omega}_p \simeq c(p)$ and, since $N_p - p - 1 = -c(p)$ (according to Eq.s(4.31),(4.39)), one can divide both sides by $p$ in order to obtain

$$\frac{N_p}{p} = \frac{p + 1 - c(p)}{p}. \quad (4.47)$$

In view of Eq.s(4.41),(4.44), we finally obtain

$$L(\mathcal{E}, 1) = \prod_p \left[ \frac{p + 1 - \omega_p}{p} \right]^{-1} \quad (4.48)$$

with the product being taken again over all places, including $\infty$. The ramifications of this identity lead to the so called Birch-Swinnerton-Dyer conjecture (still unproven). It can be stated formally as

$$\lim_{s \to 1} (s - 1)^{-r} L(\mathcal{E}, s) = C \cdot \Omega \prod_{\text{finite } p} \omega_p \quad (4.49)$$

with $C$ and $r$ being some constants (details can be found in Ref.[85]). Evaluation of the constant $C$ is the major stumbling block in proving the conjecture. The constant $r$ is assumed to be known. Since this conjecture involves the Veneziano amplitude $\Omega$, Eq.(4.49) provides yet another link between the "string" and number theories. We would like to discuss now links between the number theory and the conformal field theories.

### 4.4 Number-theoretic formulation of statistical mechanical models of conformal field theory

The idea about deep links between the number theory and statistical mechanics is not new. It was discussed, for example, in the review paper by Tracy [106]. More recent works relating number theory and statistical mechanics can be found in Refs.[107-109]. Unlike these authors, we would like to address here different type of issues. In particular, we would like to understand better what
makes two dimensional models so special from the point of view of number theory. Once the answer to this question is found, there is an opportunity to think about higher dimensional exactly solvable models using number theory.

In Section 4.1, two functions $G(s)$ and $G_{\mu,\nu}(s)$ were introduced in Eqs (4.10) and (4.11). These are the basic functions to which one can relate many two dimensional exactly solvable statistical models at criticality. We would like to rewrite these functions in the number-theoretic form using simple example presented in detail below.

Consider a ring of complex numbers $A := \{a + bi \mid a, b, \in \mathbb{Z}\}$. This is the ring of Gaussian integers. Elements of $A$ correspond to points on the lattice in the complex plane. For the complex number $x = a + bi$ the norm $\|x\|$ is defined as usual by

$$\|x\| = a^2 + b^2 = x\bar{x}.$$  \hfill (4.50)

We define now the $\zeta$ function for $A$:

$$\zeta_A(s) := \sum_{a \in A, a \neq 0} \frac{1}{\|a\|^s}.$$ \hfill (4.51)

Since the ring of Gaussian integers has 4 units (that is there are 4 numbers for which $\|a\| = 1$) this should be taken into account when $\zeta_A(s)$ is presented as the Euler product

$$\zeta_A(s) = 4 \prod_q (1 - \|q\|^{-s})^{-1}$$ \hfill (4.52)

to be compared with the Hecke $L$ function, Eq.(4.43). The product in the r.h.s. of Eq.(4.52) is over prime ideals. To find these ideals, consider the following chain of arguments. Suppose $a + bi$ belong to the prime ideal, then $a - bi$ should also belong to the prime ideal because if it were possible to decompose $a - bi$ into two factors, the same would be possible for $a + bi$. Hence, $a^2 + b^2$ has the following decomposition into prime factors

$$a^2 + b^2 = (a + bi) (a - bi).$$ \hfill (4.53)

This suggests that either $a^2 + b^2 = p$ or $a^2 + b^2 = p^2$ where $p$ is the prime number. If $p \equiv 3 \text{ mod } 4$, then $a^2 + b^2 = p$ cannot be solved, that is $p$ is prime element of $A$. If, however, $p \equiv 1 \text{ mod } 4$, then $a^2 + b^2 = p$ is solvable so that $p = q_1 q_2$ with $q_1 = a + bi$ and $q_2 = a - bi$. These statements can be trivially checked for $p = 2$. This discussion provides a nice illustration to general results stated in Appendix B of our earlier work, Ref.[4]. Using these results we obtain the following result for $\zeta_A(s)$, Ref.[99]:

$$\zeta_A(s) = 4 \left( \frac{1}{1 - 2^{-s}} \right) \prod_{p \equiv 1 \text{ mod } 4} \left( \frac{1}{1 - p^{-s}} \right)^2 \prod_{p = 3 \text{ mod } 4} \left( \frac{1}{1 - p^{-2s}} \right).$$ \hfill (4.54)

Here the last term on the right comes from prime elements of $A$ with norm $p^2$, the middle comes from two factors both with norm $p$. Finally, the factor
containing 2 comes from the element $1 + i$ whose norm is 2. Eq.(4.54) can be rewritten now as

$$\zeta_A(s) = 4\zeta(s)L_{\chi}(s)$$

(4.55)

with $\zeta(s)$ being the usual Riemann zeta function and $L_{\chi}(s)$ being the Dirichlet $L$-function

$$L_{\chi}(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

(4.56)

where

$$\chi(n) = \begin{cases} 
0, & \text{if } n \text{ is even} \\
1, & \text{if } n \equiv 1 \mod 4 \\
-1, & \text{if } n \equiv -1 \mod 4
\end{cases}.$$

There is an alternative way of presenting the same thing, e.g. see Eq.(2.17). Namely, we can write as well

$$\zeta_A(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$$

(4.57)

where the factor $h(n)$ denotes the number of ways of representing $n$ as the sum of two squares $x^2 + y^2, x, y \in \mathbb{Z}$. Elementary exercise in number theory[99] produces the following result for $h(n)^{14}$ :

$$h(n) = 4 \sum_{m|n} \chi(n).$$

(4.58)

Clearly, if instead of the ring of Gaussian integers we would have a ring of integers of the imaginary quadratic field, that is $a + \sqrt{-d}b$, we could repeat almost word for word all the preceding arguments. This allows us to rewrite $G(s)$ of Eq.(4.10) as

$$G(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$$

(4.59)

where the function $h(n)$ counts the number of integer solutions to the equation $m^2 + \tau^2 l^2 = n$. Hence, it can be presented as well in the form of zeta function analogous to $\zeta_A(s)$. As for the function $G_{\mu,\nu}(s)$, this is nothing but the special case of the function $S_n(x, y, s)$ (for $y = a = 0$) introduced by Kronecker, e.g. read Chapter7(section 4) of the famous book by Weil, Ref.[110]. Already Kronecker knew how to connect this function with the theta functions and, since these functions can be made to satisfy the KdV equation [72] whose dual is related to the Virasoro algebra (e.g see Remark 3.8) it is clear that all results of the conformal field theory can be obtained now. One can do much better, however. Looking at Eq.s(4.50) and (4.51) one can easily realize that Eq.(4.51) is not restricted to numbers of quadratic imaginary field provided that one defines the norm for the number field. In the case of quadratic field the norm

---

$^{14}$Here for $m = 1 \mod 4$, $\chi(m) = 1$, for $m = 3 \mod 4$, $\chi(m) = -1$, etc., just like in Eq.(4.56).
∥x∥ = x̅. To generalize this result to more general number field it is sufficient to define the norm ∥x∥ as the product ∥x∥ = x₁ · x₂ · · · xₙ where x₂,...,xₙ are "conjugates" of x₁. To understand what all this means it is sufficient to realize that the algebraic equation
\[ c₀ + c₁x + \cdots + cₙxⁿ = 0 \] (4.60)
normally will have n roots. These roots are conjugates of each other. Let cᵢ belong to the field K while the roots to the field extension of K, say E. Let σ:E → E be an automorphism of E which leaves K (that is cᵢ) fixed. If x₁ is the root of Eq.(4.60), then σ(x₁) is also the root of the same equation. The group σ of automorphisms of E is known as Galois group, Gal(E/K). The Kronecker-Weber theorem states that every finite abelian extension of Q (that is finite extension of E/Q with Gal(E/Q) abelian) can be embedded into a cyclotomic extension Q(ω) with ω being some root of unity. This result underscores the importance of the cyclotomic fields discussed in the Appendix A.

Going back to Eq.(4.51) and replacing A with E we obtain the multidimensional generalization of G(s). The actual computations will depend crucially on the solvability of the Diophantine equation of the type
\[ ∥x∥_{E/K} = n. \] (4.61)
Hence, the multidimensional extension of the results of conformal field theories (CFT) depends on our ability to solve the Diophantine equations of the type shown above. The theory of Diophantine equations is sufficiently developed, e.g. see Ref.[99] and references therein. This fact provides hope that multidimensional extension of CFT can be developed gradually.

Before leaving this subsection, several comments are still in place. First, we would like to rewrite the Dirichlet L-function, Eq.(4.56), in a different form. To this purpose, we need to know that the Dirichlet character χ is actually some root of unity. If we identify Gal(Q(ζₙ)/Q) with (Z/nZ)* (where ζₙ is root of unity and * denotes the multiplicative group of residue classes (such group does not contain zero)), then the Dirichlet character mod n is the Galois character. Since all characters are of the form exp(2πiut/n) = χ(u)(t) it could happen that χₐ(t₁) = χₐ(t₂) if (t₁, n) = (t₂, n) = 1 and t₁ = t₂ mod f for some f smaller than n. Such f is called conductor of χ.³¹ It is important to realize that f divides n so that χ is periodic with period f. With this information in our hands, the Dirichlet L-function, Eq.(4.56), can be rewritten as [111]
\[ Lₕₙ(s) = \frac{1}{f^n} \sum_{a=1}^{f} \chi(a)H_{f,s} \] (4.62)
with the Hurwitz zeta function being defined as
\[ H(b, s) = \sum_{n=0}^{∞} \frac{1}{(b + n)^s}, \quad 0 < b \leq 1. \] (4.63)
³¹Not to be confused with that defined earlier in Eq.(4.29)!

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Using this result, Eq.(4.11) can be rewritten with help of Eq.(22) (Chapter 7, section 9 of Ref.[110]) in a similar form (which needs some minor modifications for higher dimensional lattices)

\[ G_{\mu,\nu}(s) = H(x, 2s) + H(1 - x, 2s) \]  

(4.64)

where \( x = \mu + \nu \tau \). Using results of Weil's book, Ref. [110], it is not difficult to figure out the higher dimensional analogue of Eq.(4.64). In fact, the reader may want to consult Refs.[112,113] where this topic is discussed. Their work is to be considered also in the next subsection.

Actually, physically interesting results are only those obtainable as limits \( G'(0) \) and \( G''_{\mu,\nu}(0) \) with prime being differentiation with respect to \( s \) [82]. Calculation of these limits leads us directly to the Chowla-Selberg formula for periods to be considered in the next subsection.

4.5 The Chowla-Selberg formula and the Veneziano amplitude

The Chowla-Selberg formula, Ref.[31], is essentially the same thing as the 1st Kronecker’s limit formula as discussed, for example, in the books by Weil, Ref.[110], and Lang, Ref.[114]. It deals with the unusual reinterpretation of this formula due to results of Lerch (Eq.(23) of Chapter 8, section 9 of Ref.[110]). Such reinterpretation does not come across easily upon reading Ref.[31]. More directly it was obtained in the paper by Ramachandra [75] and was subsequently reproduced in Weil’s book, Ref.[110].

Let \( K = \mathbb{Q}(\sqrt{-d}) \) be the quadratic imaginary field, i.e. imaginary quadratic extension of \( \mathbb{Q} \) of discriminant \( -d \). Let \( r_K \) be its ring of integers and suppose that \( r_K = \mathbb{Z} + \mathbb{Z}z \) with \( z \in K \). For such field the Dirichlet function can be written as

\[ L_\chi(s) = \frac{1}{d^s} \sum_{n=1}^{d} \chi(n)H(\frac{n}{d}, s) \]  

(4.65)

to be compared with Eq.(4.62). By analogy with Eq.(4.55) we consider the Dedekind zeta function for the field \( K \). Differentiating this function with respect to \( s \) we obtain:

\[ R(s) \equiv \frac{\partial}{\partial s} [\zeta(s)L_\chi(s)] = \zeta'(s)L_\chi(s) + \zeta(s)L'_\chi(s). \]  

(4.66)

Now we have to take the limit \( s \to 0 \) in this equation. Calculation of \( \zeta'(0) \) is easy and produces \( \ln 2\pi \). Calculation of \( L'_\chi(0) \) is somewhat more involved. Indeed, using Eq.(4.65) we get for \( L'_\chi \) the following result:

\[ L'_\chi(s) = [-\ln d] L_\chi(s) + \frac{1}{d^s} \sum_{n=1}^{d} \chi(n)H'(\frac{n}{d}, s). \]  

(4.67)
The Lerch formula obtained by Lerch in 1894 (Eq.(23) on page 60 of Weil’s book, Ref.[110]) comes to the rescue thus producing

$$H'(x, s = 0) = \ln \frac{\Gamma(x)}{\sqrt{2\pi}}.$$ (4.68)

The value of $L_\chi(0)$ was known already to Dirichlet:

$$L_\chi(0) = \frac{2h(\sqrt{-d})}{w_0},$$ (4.69)

where the class number $h(\sqrt{-d})$ was introduced in Subsection 4.2 and $w_0$ is the number of roots of unity in $K$. Finally, the value of $\zeta(0) = -1/2$ while that for $\zeta'(0) = -2 \ln 2 \pi$. Collecting all these results in Eq.(4.66) we obtain:

$$R(s = 0) = -\frac{4h}{w_0} \ln 2\pi + \left[ \frac{1}{2} \ln d \right] - \frac{1}{2} \sum_{n=1}^{d} \chi(n) \ln \frac{\Gamma(n/d)}{\sqrt{2\pi}}.$$ (4.70)

Specializing to those imaginary quadratic fields for which $h = 1$ the 1st limit formula of Kronecker (Eq.(17) on page 75 of Weil’s book) is given by

$$\zeta_K(s \to 0) = 1 - \frac{s}{12w_0} \ln(F(L))$$ (4.71)

with $F(L)$ denoting Siegel-Ramachandra invariant: $F(L) = (2\pi)^{-24} |N(L)|^{12} |\Delta(\tau)|^2$, Ref.[114], ch-rs19, 21. Here $\Delta(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^{2n})^{24} \equiv (2\pi)^{12} \eta(\tau)^{24}$ with $\eta(\tau)$ being the usual Dedekind eta-function while $N(L)$ being the norm of the principal ideal. This norm can be easily calculated for the quadratic fields [115].

Indeed, if the ideal $L = [\omega_1, \omega_2]$, then the different $D$ can be calculated as

$$D = \begin{vmatrix} \omega_1 & \omega_2 \\ \bar{\omega}_1 & \bar{\omega}_2 \end{vmatrix} = \omega_1 \bar{\omega}_2 - \omega_2 \bar{\omega}_1 \equiv \omega_1 \bar{\omega}_1 (\bar{\tau} - \tau)$$

while the same different for the basic lattice $\bar{L} = [1, (\bar{\tau} - \tau)]$ is given by $\bar{D} = (\bar{\tau} - \tau)$. The norm $N(L)$ is determined by the rule: $N(L) = D/\bar{D}$ so that $N(L) = \omega_1^2$. Keeping in mind that the Dedekind eta-function is the modular form of weight 12 and using Eq.(C.4) one obtains easily: $F(L) = (\text{Im}\tau)^{12} |\Delta(\tau)|^2$.

Differentiating Eq.(4.71) with respect to $s$ and equating this result with Eq.(4.70) (for $h = 1$), we obtain (omitting unimportant constants):

$$(\text{Im}\tau)^{12} |\Delta|^2 \approx \prod_{n=1}^{d} \Gamma\left(\frac{n}{d}\right)^{6w_0\chi(n)}.$$ (4.72)

We would like to compare this result with that known from the CFT. In particular, the free boson field theory on the torus is discussed in Ref.[82], ch-r...
10, section 10.2. If we let $w_0$ in Eq.(4.71) to be one then, using this formula we obtain at once

$$\frac{\partial}{\partial s} \zeta_K(s) \big|_{s=0} = -2 \ln \left[ \text{Im} \tau(\eta(\tau))^2 \right] = G'(0)$$

(4.73)

with $G'(0)$ being defined earlier by Eq.(4.10). This result coincides exactly with Eq.(10.29) obtained with help of the path integral for the boson field theory, Ref.[82]. The path integral for such theory is effectively the Euclidean version of the bosonic string in the conformal gauge [116]. More details on this topic are given in the next subsection.

In the meantime, use of Eq.(4.72) enables us to provide different interpretation of the results known in physics. First, using Eqs.(4.1)-(4.5) and following Chowla and Selberg, Ref.[31], we write $\Delta(\tau) = \eta K^{12}$ with $K$ being one of periods of the elliptic curve and $\eta$ being some (known in principle) algebraic number. Taking into account that $\text{Im} \tau$ is also an algebraic number (e.g. read Section 4.2) we notice that the combination $|\Delta|^2 \simeq (K)^{24}$. Therefore, using Eq.(4.72) we obtain

$$K = C \sqrt{\pi} \prod_{n=1}^{d} \Gamma \left( \frac{n}{d} \right)^{w_0 \chi(n)}$$

(4.74)

where $C$ is some constant (algebraic number). This is exactly the Chowla-Selberg formula (for $h = 1$), Eq.(4), page 110 of Ref.[31]. The factor of $\sqrt{\pi}$ comes in view of Eqs.(4.1) and (4.70).

**Remark.4.8.** At this point we would like to remind to our readers that exactly the same Dedekind zeta function $\zeta_K(s)$ used in obtaining the Chowla-Selberg formula was obtained earlier in our work, Ref.[4], and was interpreted as dynamical partition function of 2+1 gravity.

We would like to demonstrate now how the Veneziano-like amplitudes can be obtained with help of such partition function.

To do so, several things should be kept in mind. First, the number $w_0$ is equal to 4 for $d = 1$ (or $d=4$), it is equal 6 for $d = 3$ and it is equal to 2 in the rest of situations as it was explained earlier, e.g. see discussions after Eqs.(4.18) and (4.24). Next, the character $\chi(n)$ in Eq.(4.74) is equal to $\pm 1$ since it is the Kronecker symbol that is $\chi(n) = \left( \frac{d}{n} \right)$ according to p.109 of Ref.[31]. With this information, let us first look at the simplest case $d = 4$ having in mind to reproduce Eq.(3.37). In this case we need to evaluate the following Kronecker's symbols: $(\frac{1}{4}), (\frac{3}{2})$ and $(\frac{1}{2})$ (since $(\frac{3}{2}) = 0$ by definition). Using Ref.[115], pages 141-143, we find: $(\frac{1}{4}) = 1, (\frac{3}{2}) = 0, (\frac{1}{2}) = -1$. In arriving at these results we took into account that the discriminant of the field $d$ is not a square-free integer while $d/4$ is. Taking into account that $\sqrt{\pi} = \Gamma(\frac{1}{4})$ we obtain back the result given in Eq.(3.37). Following Chowla and Selberg, consider now another example. This time, let $d = 7$ so that $w_0 = 2$. Then, similar calculations produce

$$K = C \sqrt{\pi} \left[ \frac{\Gamma(\frac{1}{7}) \Gamma(\frac{3}{7}) \Gamma(\frac{4}{7})}{\Gamma(\frac{2}{7}) \Gamma(\frac{5}{7}) \Gamma(\frac{6}{7})} \right]^\frac{1}{2}$$

(4.75)
Using Eq.(1.9) it is possible to transform Eq.(4.75) into more comprehensible form:

\[ K = \left( \frac{\hat{C}}{\pi} \right) \Gamma \left( \frac{1}{7} \right) \Gamma \left( \frac{2}{7} \right) \Gamma \left( \frac{4}{7} \right). \] (4.76)

Looking at the Veneziano amplitude, Eq.(1.12), it is evident that the bracket containing sinus factors will not contribute to physically interesting poles. Hence, we may rewrite this amplitude as \( A(a, b, c) \simeq \Gamma(a)\Gamma(b)\Gamma(c) \) keeping in mind that \( a + b + c = 1 \) (or \(-1\)). In Ref.[117] Gross had shown that it is always possible to reexpress \( K \) as the product of gamma functions for other \( d's \) with \( h = 1 \). In this sense, the Chowla-Selberg formula, Eq.(4.74), could be considered as the Veneziano amplitude. In view of the Remark 4.8., it follows that the partition function of 2+1 gravity is capable of producing the Veneziano amplitudes and, since this function was not obtained in 26 dimensions, accordingly, our Veneziano-like amplitude(s) "live" in normal space-time dimensions. This conclusion is in accord with earlier obtained based on Eq.(3.6).

Obtained result, Eq.(4.74), is inconvenient, however, for use in particle physics since it is restricted to a very specific values of parameters, strongly depends upon the toroidal geometry and cannot be easily generalized to the Riemann surfaces of higher genus in spite of several attempts to do so [112,113]. We provided here all these derivations nevertheless because we believe that the number-theoretic methods discussed in this work could be potentially useful for development of CFT models either in dimensions higher than two or for models defined on Riemann surfaces of higher genus. This is especially important in view of the fact that such an extension for higher genus surfaces in mathematics literature is only possible for those Riemann surfaces (and, hence, the Abelian varieties, Jacobians, etc.) which admit CM. The theory of CM was developed by Taniyama and Shimura [18] and can be found also in the monograph by Lang, Ref.[118]. To our knowledge, the work of Shimura and Taniyama on CM had not been used in physics literature so far.

Before closing this section, we would like to discuss yet another reason for discussing the Chowla-Selberg results.

### 4.6 Heights of elliptic curves, the Chowla-Selberg formula and string theory

In this subsection we would like to discuss still another connection between the Veneziano amplitudes and the number theory. It is based on realization that the l.h.s. of Chowla-Selberg formula, Eq.(4.72), can be written in a different way. This is possible because the function \( \Delta(\tau) \) is the modular form and, hence, following ideas of Eichler-Shimura theory discussed in Section 4.3., we can construct the invariant differential form \( \alpha = (\Delta(\tau))^{1/2} dz \) and, with help of
such form, the intersection pairing [119]:

\[
\frac{i}{2} \int_{C/L} \alpha \wedge \bar{\alpha} = \frac{i}{2} \int_{C/L} |\Delta(\tau)|^{\frac{1}{2}} \, dz \wedge d\bar{z} \quad (4.77)
\]

\[
= |\Delta(\tau)|^{\frac{1}{2}} \int_{A} dx \wedge dy
\]

\[
= |\Delta(\tau)|^{\frac{1}{2}} \text{Im} \tau.
\]

Following Silverman [89,119], denote the height \( h(E/Q) \) of the elliptic curve \( E \) as

\[
h(E/Q) = -\frac{1}{2} \ln \frac{i}{2} \int_{C/L} \alpha \wedge \bar{\alpha}. \quad (4.78)
\]

The above definition is not the most complete. Pedagogically clear explanation of the concept of height and its usefulness in arithmetic algebraic geometry is given in the review paper by Mazur [120]. Rigorous connection between the height and the Chowla-Selberg formula is presented in relatively recent Annals of Mathematics paper by Colmez [121]. We deliberately avoid all intricacies of this concept\(^{16}\) since our goal is more modest: we want to connect Eq.(4.78) with string theory.

Following Ref.[8], the Weil-Petersson fundamental \((1,1)\) Kähler form \( \omega_{W-P} \) on the Teichmüller space of the torus (and/or the punctured torus[9]) is given by

\[
\omega_{W-P} = \frac{i}{4(\text{Im} \tau)^2} d\tau \wedge d\bar{\tau}. \quad (4.79)
\]

At the same time, the genus one partition function for the bosonic string in 26 space-time dimensions is given by [123,124]:

\[
Z_1 = \text{const} \int_{M_{1,1}} \frac{\omega_{W-P}}{(\text{Im} \tau)^{12} |\Delta|^2} \quad (4.80)
\]

\[
= \text{const} \int_{M_{1,1}} \omega_{W-P} \exp(24h(E/Q))
\]

with integration domain taken over the moduli space \( M_{1,1} \) of once punctured torus. In our previous work, Ref.[64], we had discussed extensively the above integral for the case when the height function is equal to zero. This leads immediately to the Kontsevich-Witten matrix models, etc. Since the height function is closely connected with the Arakelov theory [119,121], naturally, extension of Eq.(4.80) to the Riemann surfaces of higher genus involves elements of this theory. This indeed was accomplished by Manin [125] and Bost and Jolicoeur [126] (see also paper by Smit [127]). The above formula and its higher genus

\(^{16}\)These can be found easily in the literature just cited, e.g. see Ref.[122] in addition.
generalizations contains several important drawbacks. First, the modular form \( \Delta(\tau) \) is a cusp form. That is, it can exist only if the torus is cusped, i.e. it has at least one puncture. (The dimension (and, hence, the number) of cusp forms surely depends upon the actual number of cusps). This topology is the simplest possible for the case of 2+1 gravity as it was extensively discussed in our earlier work, Ref.[4]. But, in string theory the puncture is normally associated with some particle via vertex operator insertion. **By definition**, the string partition function **without external sources (or sinks)** should **not** contain punctures. Second, the expression for the height makes sense in the Diophantine geometry dealing with finite number of points on the algebraic curves. The Chowla-Selberg formula, Eq.(4.72), is in perfect agreement with such statement since it involves **discrete** \( \tau \)'s coming from the imaginary quadratic fields. Use of Eq.(4.80) destroys the complex multiplication option and, hence, disconnects the l.h.s. of Eq.(4.72) from the r.h.s.\(^{17}\) If, however, the complex multiplication is destroyed (not used, ignored) then, according to Weil, page 40, paragraph 7, Ref.[110], the very basic transformation law, Eq.(C.7), of Appendix C for the modular functions is violated. Third, the genus zero partition function seemingly **does not** require integration over the moduli space [7]. This, however, is suspicious. Indeed, the open string world sheet for genus zero case is represented by the upper Poincare halfplane \( \mathcal{H} \) which is a model of hyperbolic (that is non-Euclidean) space. The minimal number of open external states is four and they appear as insertions at the boundary of \( \mathcal{H} \) (e.g. see Fig.1.15(c) of Ref.[7]). Technically speaking, these are just the cusp points. The four times punctured sphere is a **hyperbolic** surface whose cover is \( \mathcal{H} \) with four cusps. The locations of cusps is not fixed. The motion of the cusp points is described by the motion in the parameter, i.e. in the Teichmüller, space. Nag had demonstrated [9] that the Teichmüller space of the four times punctured sphere is **the same** as of once punctured torus. Notice, that in view of these arguments there is no need to discuss separately open and closed strings. This explains why the Veneziano amplitude makes more sense than the Shapiro-Virasoro. If one accepts the above arguments, then, 2+1 gravity and string theory are in agreement with each other so that formally Eq.(4.80) is a partition function for "time-reduced" 2+1 gravity. This was noticed already in our earlier work, Ref.[64] and further developed by Krasnov [131]\(^{18}\). Fortunately, there is much more efficient alternative route of obtaining the multiparticle Veneziano amplitudes. It is presented in the next section.

\(^{17}\)This can be repaired if the traditional Teichmüller/moduli space is replaced by its \( p \)-adic analogue [128]. Even then, as the recent developments indicate [129, 130], one still will end up with the \( p \)-adic version of the results presented in Section 5 below.

\(^{18}\)For more recent references, please, see the footnote \# 3.
5 Fermat’s hypersurfaces and Veneziano amplitudes

5.1 The Picard-Fuchs equations

In Section 3 we had shown that the standard Veneziano four particle amplitude is just one of the periods of the Jacobian variety for the Fermat curve. For physical applications it is of interest to obtain the multiparticle amplitudes. This can be done in two ways: either by explicit calculation based on generalization of the results presented in Section 3 or by considering some sort of equations whose solutions produce the desired periods. The situation in the present case is very similar to that encountered in mirror symmetry calculations [57]. Because of this similarity we will be brief in discussing the second option. But, we feel, that this option should be mentioned explicitly in the text because it provides some additional insight into physical and mathematical meaning of periods and their calculation.

We begin with the simplest example borrowed from the pedagogically written paper by Griffiths [132]. He considers calculation of the period of the following integral along the closed contour $\Gamma$ in the complex $z$-plane:

$$\pi(\lambda) = \oint_{\Gamma} \frac{dz}{z(z - \lambda)}. \tag{5.1}$$

Since this integral depends upon parameter $\lambda$ the period $\pi(\lambda)$ is some function of $\lambda$ which can be determined as follows. Differentiate $\pi(\lambda)$ in Eq.(5.1) with respect to $\lambda$.

This produces:

$$\pi'(\lambda) = \frac{1}{\pi(z - \lambda)}. \tag{5.2}$$

The combination

$$\lambda \pi'(\lambda) + \pi(\lambda) = 0 \tag{5.2}$$

produces the desired differential equation which enables us to calculate $\pi(\lambda)$. This simple result can be vastly generalized to cover the case of periods of integrals of the type

$$\Omega(\lambda) = \oint_{\Gamma} \frac{P(z_1, ..., z_n)}{Q(z_1, ..., z_n)} dz_1 \wedge dz_2 \wedge ... \wedge dz_n \tag{5.3}$$

The equation $Q(z_1, ..., z_n) = 0$ determines the variety. It may contain parameter (or parameters) $\lambda$ so that the polar locus of values of $z'$s satisfying equation $Q = 0$ depends upon this parameter. By analogy with Eq.(5.2) one obtains

$$\sum_{n=0}^{k} P_n(\lambda) \frac{d^n}{d\lambda^n} \Omega(\lambda) = 0. \tag{5.4}$$

This is the Picard-Fuchs equation for periods. To solve this equation one needs to know the explicit form of polynomials $P_n(\lambda)$. In general this is not an easy task as it was demonstrated by Manin [133] many years ago. And, because this is not an easy task, we believe, that calculation of periods using generalization
of section 3 is more efficient. Nevertheless, method of differential equations teaches us things which are not immediately apparent when direct calculations of periods is made.

In Section 4.1, we noticed that for the elliptic curves the periods can be calculated as solutions of the hypergeometric equation, Eq.(4.6). Surely, this equation is of Picard-Fuchs type. However, in Eq.(4.6) there is no explicit differentiation with respect to parameter \(\lambda\) and, hence, now we would like to correct this deficiency. Fortunately, this task was accomplished by Manin [133]. We follow Ref.[134], however, which is more elementary. The Legendre form, Ref.[83], page 179, of the elliptic curve \(E\) is given by

\[
y^2 = x(x-1)(x-\lambda).
\]

The invariant differential form \(\omega\) on \(E\) is given by

\[
(5.5)
\]

Differentiation with respect to \(\lambda\) produces

\[
\lambda(\lambda - 1) \frac{\partial^2}{\partial \lambda^2} \omega(\lambda) + (2\lambda - 1) \frac{\partial}{\partial \lambda} \omega(\lambda) + \frac{1}{4} \omega(\lambda) = d\left( -\sqrt{x(x-1)(x-\lambda)} \right)
\]

Integrating this relation along the contours \(2\int_0^1 = \oint_\alpha\) and/or \(2\int_0^1 = \oint_\beta\) we obtain the equation for periods

\[
\lambda(\lambda - 1) \frac{\partial^2}{\partial \lambda^2} \Omega(\lambda) + (2\lambda - 1) \frac{\partial}{\partial \lambda} \Omega(\lambda) + \frac{1}{4} \Omega(\lambda) = 0 (5.7)
\]

in accord with Manin, Ref.[133]. Let us recall [83] that for the elliptic curve in the Legendre form the \(j\)–invariant is known to be

\[
(5.8)
\]

Use of this invariant allows us to compare elliptic curves between each other. In particular, if two curves are isogenous they have the same \(j\)–invariant. This means that \(j\)–invariant (or parameter \(\lambda\)) describes the family of all elliptic curves and, hence, \(\lambda\) can be identified with the point in the moduli space of elliptic curves. This means that the differential operator in Eq.(5.7) is defined on the moduli space of elliptic curves and it effectively describes the deformation of complex structure for such curves. Since the moduli space is non Euclidean, the differential operator in Eq.(5.7) is actually defined in some "curved" space characterized by some connection (Gauss-Manin connection). The commutator of covariant derivatives on such space produces zero curvature (so that connection is flat but not trivial nevertheless). Surely all this machinery can be extended to more sophisticated algebraic varieties. A good summary of results applicable to general case can be found in Refs.[36,134,135]. In this paper we are not going to use these results however and, hence, the above discussion is only meant
to indicate the alternative route to reach the same destination. The necessity to mention about such a route comes from the fact that it actually provides the unified description of both conformal and "string" theories as mentioned briefly already in Section 4.1. Although the Picard-Fuchs equation for the B-P type singularities can be easily derived (as discussed in the Appendix D), this route for obtaining the multiparticle Veneziano amplitudes is not unique. In the next subsection we discuss another route closer in spirit to that envisioned by Landau [24] and further developed by Pham [25] and others.

**Conjecture:** Based on the results of Pham [25], Lefshetz [136] and Griffiths [132], and those presented in this work, it is natural to expect that any scattering amplitude (correlation function) in particle physics (in conformal field theories) should come out as solution of some kind of the Picard-Fuchs equation and, hence, should be interpreted as one of the periods of differential forms "living" on the moduli space appropriate for the particular scattering problem. In the case of conformal field theories this conjecture could be considered as actually proven in view of the work of Schehtman and Varchenko [137] (to be discussed briefly below in connection with hyperplane arrangements) while in the case of particle physics this work along with those on mirror symmetry [57] (and references therein) could be considered as first steps towards its proof.

### 5.2 Veneziano amplitudes from Fermat hypersurfaces

To obtain the multiparticle Veneziano amplitude we have to generalize results of Section 3.2. It is appropriate to mention at this point that development of the multiparticle Veneziano amplitudes leading to string theories (as discussed in detail in Ref.[19]) differs markedly from presentation which follows. Our presentation is based on works of Deligne [37] and Milnor [26]. Additional mathematical details and references can be found in Ref.[38].

In section 3.2 the main object of interest was the differential 1-form, Eq.(3.24), leading to the Veneziano amplitude, Eq.(3.33). Now we need to find the analogous form for the Fermat hypersurface, Eq.(3.15), with \(a_0 = \cdots = a_n = N\). To do so, we would like to remind our readers about some facts from the theory of Riemann surfaces to be generalized to more complicated case of Fermat hypersurfaces. In the case of Riemann surface \(R\) there are three types of differential forms \(\omega\). For the first and second type the period integrals \(\int_\gamma \omega\) are non-zero only for non-null homologous cycles \(\gamma\) on \(R\) whereas for the third type one obtains the so called residual periods associated with poles of differential forms \(\omega\). They can occur for null homologous cycles as well and, hence, are of no interest to us [138]. Suppose now that we have a \(p\)-form \(\omega\) on a manifold \(X\). If we integrate such a form over \(p\)-cycle, then one obtains a period (usually a complex number) depending only on a homology class of the cycle provided that \(\omega\) is closed (that is \(d\omega = 0\)). The homology classes form \(H_p\) homology group of \(X\) with complex coefficients, \(H_\omega(p, C)\). The dual to it is \(p\)-th cohomology group \(H^p(p, C)\). A closed \(p\)-form \(\omega\) yields a cohomology class \([\omega] \in H^p(X, C)\). The Stokes’ theorem shows that this class remains unchanged if we add to it
an exact form, that is differential \( d\psi \) of the \( p-1 \) form \( \psi \). This circumstance allows us to introduce the equivalence relation and, hence, to study the quotients, e.g. the de Rahm \( p \)th cohomology group \( H^p_{DR}(X) = C^p(X)/E^p(X) \), with \( C^p \) and \( E^p \) being respectively the vector spaces of closed and exact \( p \)-forms with complex coefficients. According to de Rham, Ref.[138,139], there is a canonical homomorphism: \( H^p_{DR}(X) \rightarrow H^p(X,\mathbb{C}) \).

Consider first how all this applies to the Riemann surface \( X \) which is complex one-dimensional manifold. For such manifold \( H^1_{DR}(X) \) is isomorphic to the vector space of holomorphic 1-forms. Each holomorphic 1-form \( \omega \) is closed. This is so because \( d\omega \) is a holomorphic 2-form which is equal to zero automatically since \( X \) is one dimensional. From topology it is known that for the Riemann surface of genus \( g \) the cohomology group \( H^1(X,\mathbb{C}) \) has dimension \( 2g \). Meanwhile the dimension of holomorphic 1-forms is \( g \) so that the holomorphic 1-forms yield only \( 1/2 \) of cohomology.

To repair this situation one needs to consider not only the cohomology classes coming from the holomorphic forms but also coming from the anti holomorphic forms. For a general manifold \( X \), following Hodge, one introduces the differential \((p,q)\)-form \( \omega \) (in local complex coordinates \( z_1,\ldots,z_n \)) such that

\[
\omega = \sum a_{i_1,\ldots,i_p,j_1,\ldots,j_q} \, dz_{i_1} \wedge \cdots \wedge d\bar{z}_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q} \quad (5.9)
\]

Each \( m \)-form on \( X \) can be expressed uniquely as a sum of \((p,q)\)-forms with \( p + q = m \).

The following theorem plays a very important role.

**Theorem 5.1.** If \( X \) is compact complex manifold, then it is Hodge manifold (or Hodge-type manifold) if and only if it is a projective algebraic manifold at the same time.

This statement is known as the Kodaira’s embedding theorem, Ref.[139], ch-r 6. Evidently, the projective algebraic manifold must be complex.

**Definition 5.2.** A complex projective manifold \( X \) is a nonsingular subvariety in \( \mathbb{CP}^n \) consisting of the common zeroes of a system of homogenous polynomial equations in coordinates \( (z_0,\ldots,z_n) \) of \( \mathbb{C}^{n+1} \).

On \( \mathbb{C}^{n+1} \) one can introduce the Hermitean scalar product (which is just the finite dimensional analog of what is being used for the Hilbert space of quantum mechanics):

\[
<z,z> = \sum_{i=0}^{n} z_i \bar{z}_i. \quad (5.10)
\]

With help of such defined scalar product it is possible to recover the metric and the curvature for such spaces. Moreover, with little efforts this can be extended.
to $\mathbb{CP}^n$ [139]. If the Hermitian metric $h$ of the projective manifold $X$ embedded in $\mathbb{CP}^n$ is given by

$$ h = \frac{i}{2} \sum h_{\mu\nu}(z) dz_{\mu} \otimes d\bar{z}_{\mu}. \tag{5.11} $$

then, the fundamental $(1,1)$ form $\Omega$ is given by

$$ \Omega = \frac{i}{2} \sum h_{\mu\nu}(z) dz_{\mu} \wedge d\bar{z}_{\mu}. \tag{5.12} $$

**Definition 5.3.** The Hermitian metric $h$ on $X$ is of Kähler type if $\Omega$ is closed, i.e. $d\Omega = 0$.

Not all complex manifolds admit the Kähler metric$^{19}$. Fortunately, those which admit have direct physical relevance. In particular, all known physically meaningful symplectic manifolds of exactly integrable finite dimensional classical dynamical system admit Kähler type metric and $\Omega$ form.[40,141,142]. Using Kodaira’s embedding theorem we conclude that all physically interesting projective manifolds of Kähler type are also manifolds of Hodge type$^{20}$.

In particular, let us consider the case of Riemann surface of genus $g$. It can be mapped into $g$ dimensional complex torus (the Jacobian) under appropriate conditions (e.g. it must be a polarized Abelian variety) [118,134,139,140]. And such torus surely admits Kähler metric and, hence it is a Hodge-type manifold with Hodge decomposition:

$$ H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X). \tag{5.13} $$

**Remark 5.4.** The condition that an abelian variety is of complex multiplication type can be interpreted as a choice of basis for the above decomposition, Eq.(5.13),[143].

**Remark 5.5** The situation becomes more complicated if complex hypersurface possess some kind of singularities, e.g B–P type discussed in Section 3.2. In this case the mixed Hodge structures should be considered instead of pure Hodge structures as shown by Deligne [37]. Systematic up to date exposition of this topic can be found in Ref.[135]. Mathematically rigorous and physically accessible treatment of these structures can be found in excellent review paper by Varchenko [144] (Appendix D).

Complex multiplication (discussed in Section 4.2) is closely associated with manifolds of Hodge type. Fortunately, this is indeed the case for both the Fermat curves considered in Section 3 [34] and for the Fermat hypersurfaces [37] to be considered below in this section. Construction of Hodge type manifolds admitting complex multiplication is concisely described in the paper by Weil [33] and also in his book, Ref.[140]. More recent and much more advanced

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$^{19}$Comprehensive and readable account of Kähler manifolds is given in Andre Weil’s book, Ref.[140]. The book by Wells, Ref.[139], is also an excellent source.

$^{20}$For additional details, please read Section 6.
presentation of this topic involving theories of motives, Tannakian categories, crystalline cohomology, etc. can be found in Refs. [38,145].

Consider now the projective form of Eq.(3.15) adopted to the Fermat case, i.e.

\[ z_0^N + \cdots + z_n^N - z_{n+1}^N = 0. \]  

(5.14)

Let us consider it as a variety over the cyclotomic number field \( K = \mathbb{Q}(\exp\{2\pi i/N\}) \).

Then, using results of Appendix A we can rewrite it as a hyperplane in \( \mathbb{P}^n \), i.e.

\[ z_0 + \cdots + z_n - z_{n+1} = 0. \]  

(5.15)

Such a hyperplane can be presented as a point in the Grassmanian as we had discussed in our earlier work, Ref.[64]. Each hyperplane, Eq.(5.15), is just a result of a particular solution of the associated system of cyclotomic equations (e.g. see Eq.(A.3)). Naturally, there are many other solutions and each of them will be represented by its own point in the Grassmanian. One can think about the transformations which connect different hyperplanes. Such transformations are presented by the matrices \( G \) belonging to the rotation group. Among these transformations might be those, call them \( N \), which have the fixed point. If this is the case, then, one should consider a quotient \( G/N \). This quotient is essentially what is known in the literature as the Griffiths domain [139,146,147].

**Griffiths domain is the moduli space for periods.** We shall demonstrate momentarily that each hyperplane is associated with its period (that is with the multiparticle Veneziano amplitude) so that the totality of hyperplanes are in one to one correspondence with the totality of periods. This fact is discussed further in Section 5.3.2. and Appendix D. In principle, if one can construct the matrix of periods then, one can determine the Griffiths domain as well. To keep focus of attention of our readers on physics, mathematically relevant issues of physical importance are discussed in the Appendix D. This allows us to come to the main purpose of this work- calculation of multiparticle Veneziano amplitudes.

To this purpose, we need to use the Corollary 2.11. from Griffiths work, Ref.[132]. It is related to the integral, Eq.(5.3). To explain things better, we adopt the explanation of this Corollary using discussion from the Brieskorn’s book, Ref.[138], page 647.

Let \((x_0, \ldots, x_n)\) be homogenous coordinates of the point in the projective space and \((z_1, \ldots, z_n)\) be the associated with them coordinates in the affine space with \( z_i = x_i/x_0 \), then the rational \( n \)-form is given by

\[
\frac{p(z_1, \ldots, z_n)}{q(z_1, \ldots, z_n)} dz_1 \wedge dz_2 \cdots \wedge dz_n
\]  

(5.16)

with rational function \( p(z)/q(z) \) being a quotient of two homogenous polynomials of the same degree. Under substitution \( z_i = x_i/x_0 \), the form \( dz_1 \wedge dz_2 \cdots \wedge dz_n \) changes to

\[
dz_1 \wedge dz_2 \cdots \wedge dz_n = x_0^{-(n+1)} \sum_{i=0}^{n} (-1)^i x_i dx_0 \wedge \cdots \wedge d\hat{x}_i \wedge \cdots dx_n
\]  

(5.17)
where the hat on top of \( x_i \) means that it is excluded from the product. Define now the form

\[
\omega_0 := \sum_{i=0}^{n} (-1)^i x_i dx_0 \wedge \ldots \wedge d\hat{x}_i \wedge \ldots dx_n. \tag{5.18}
\]

In order to rewrite the form, Eq.(5.16), in terms of \( x \) coordinates it is sufficient to multiply the denominator of \( p(z)/q(z) \) by \( x_0^{n+1} \) and to replace \( z \) by \( x \) in the quotient. Thus one obtains a homogenous expression

\[
\omega = \frac{P(x)}{Q(x)} \omega_0 \tag{5.19}
\]

in which the (degree of \( Q \)) = (degree of \( P \)) + (n+1). This is Griffiths result, Corollary 2.11.of Ref.[132]. Conversely, each homogenous form, Eq.(5.19), yields an affine rational form, Eq.(5.16), by substitution: \( x_0 = 1, x_i = z_i \).

Next, consider the set \( X(S^1) \) of numbers which we call the spectrum of the critical point (Appendix D) of the B-P singularity, Eq. (3.7). It is given by

\[
X(S^1) = \{ a \bar{\in} (\mathbb{Z}/N\mathbb{Z})^{n+2} \equiv (\mathbb{Z}/N\mathbb{Z}) \times \cdots \times (\mathbb{Z}/N\mathbb{Z}) | a = (a_0, \ldots, a_{n+1}), \sum_i a_i = 0 \mod N \} \tag{5.20a}
\]

Let \( < a_i > \) denote the representative of \( a_i \) in \( \mathbb{Z} \) such that \( 1 \leq < a_i > \leq N-1 \)(e.g. see Eq.(3.17a)). Also, define the average

\[
< a > = \frac{1}{N} \sum_i < a_i > \tag{5.20b}
\]

then, for the Fermat hypersurface the differential form \( \omega \), Eq.(5.19), can be written as

\[
\omega = \frac{x_0^{<a_0>-1} \cdots x_n^{<a_{n+1}>-1}}{(x_0^{N} + \cdots + x_n^{N})^{<a>} \omega_0} \tag{5.20c}
\]

with \( \omega_0 \) defined by Eq.(5.18) (with \( n \) replaced by \( n+1 \)). With such form at our disposal it will be advantageous for us to use the affine representation of this form using results just described. To this purpose, by analogy with developments in Section 3.2, e.g. see the discussion after Eq.(3.15), we make change of variables:

\[ z_i = t_i^N \exp(\frac{2\pi i a_i}{N}) \equiv t_i^N \zeta, \]

provided that \( \sum_i t_i = 1, t_i \geq 0 \), thus forming the \( n+1 \) simplex \( \Delta \). Such change of variables allows us to consider a map \( f(\Delta) : \Delta \rightarrow V(F) \) from the simplex \( \Delta \) to the affine variety \( V(F) \)

\[
Y_0^N + \cdots + Y_n^N = 1, \ Y_i = x_i / x_{n+1} \equiv z_i. \tag{5.21}
\]

This map can be made in many ways: for example we can suppress all \( \zeta \)'s, then, suppress all, except one, then, all, except two, etc. and still will get the desired mapping. To avoid guessings, we need to reobtain back the result, Eq.(3.31), for the simplest four particle Veneziano amplitude. Clearly, such amplitude should be obtainable from general result, Eq.(5.20), i.e. it should come up as period of the integral \( \int_\gamma \omega \). Such an integral contains a pole. There should be a procedure
generalizing that known in the standard complex analysis of one variable of calculating residues of multidimensional complex integrals. Fortunately such procedure exist and was developed by Leray, Pham and others [136, 148] in connection with Landau’s work on analytical properties of scattering amplitudes of Feynman’s diagrams [24]. With such information at our disposal, we would like to reobtain the simplest Veneziano amplitude now. Although results of Deligne [37] are of great help in accomplishing this task, unfortunately, we cannot borrow them entirely for (physical) reasons which will become clear upon comparing of our derivations (below) with those by Deligne.

We begin with Eq.(3.31) written (up to constant factor of $N$) as follows:

$$I = \frac{(-1)}{2\pi i} \zeta \frac{d}{d\zeta} (1 - \zeta^t)(1 - \zeta^s)(1 - \zeta^r) \Gamma(a)\Gamma(b)\Gamma(1 - a - b)$$

(5.22)

This form clearly calls for elimination of the factor $\zeta^{-t}$ for reasons of symmetry. This was actually done in Eq.(3.33). This time, we would like to do such elimination more systematically. Hence, for the time being we shall keep this factor dropping it at the end. Eq.(5.22) can be presented equivalently in the following form:

$$\Gamma(a + b)I = (1 - \zeta^s)(1 - \zeta^r) \int_0^\infty \int_0^\infty dx_1 dx_2 x_1^{a-1} x_2^{b-1} \exp(-x_1 - x_2)$$

(5.23)

$$= (1 - \zeta^s)(1 - \zeta^r) \int_0^\infty \int_0^\infty d\hat{x}_1 d\hat{x}_2^{a-1} \exp(-\hat{x}_1 - \hat{x}_2)$$

In going from the first line to the second we have introduced new variables: $x_1 = \hat{x}_1 t, x_2 = \hat{x}_2 t, x_1 + x_2 = t$ implying that $\hat{x}_1 + \hat{x}_2 = 1$ and the Jacobian of transformation equal to one. The above can be rewritten therefore as

$$I = (1 - \zeta^s)(1 - \zeta^r) \int_0^1 d\hat{x}_1 \hat{x}_1^{a-1}(1 - \hat{x}_1)^{b-1}$$

(5.24)

Looking at Eqs (5.17) and (5.20) and comparing with Eq.(5.24) we can formally write: $I = (1 - \zeta^s)(1 - \zeta^r) \int_0^\infty \omega$ . And, indeed, this expression coincides with Eq.(3.31) in view of Eq.(3.28). These results can be vastly generalized now. First, we would like to notice that if we would make a change: $r \rightarrow -r, s \rightarrow -s$, the arguments leading to Eq.(5.24) will remain unchanged. Therefore, this means that this result can be used for calculation of physical Veneziano amplitude, e.g. see Eqs.(1.6) and (3.33). Second, based on this observation, we can also change $\zeta^s$ into $\zeta^{-s}$ and $\zeta^r$ into $\zeta^{-r}$ without changing $a$ and $b$ in the integral of Eq.(5.24). Vice versa : we can change $a$ and $b$ without change of phase factors. To illustrate why this is possible consider as before change of variables: $z_i = t_i^\frac{1}{N} \exp(\frac{2\pi i}{N})$. In the integral, Eq.(5.24), we choose $z_1 = \hat{x}_1 = t_1^\frac{1}{N} \zeta$ (where $\zeta$ means that the $\zeta$ factor may or may not be present. It may be present if we

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bring the external phase factors inside the integral, Eq.(5.24)) and, accordingly, 
\[ z_2 = \hat{x}_2 = t_2^\frac{1}{N} \hat{\xi} \] so that for the Fermat curve in the affine form: 
\[ z_1^N + z_2^N = 1, \]
we obtain back equation for the simplex: 
\[ t_1 + t_2 = 1. \]
Hence, we may consider the most general case of the integral of the Veneziano-type:

\[
I_{V(F)} = \int_\Delta Y_1^{a_1} \cdots Y_{n+1}^{a_{n+1}} \frac{dY_1}{Y_1} \wedge \cdots \wedge \frac{dY_n}{Y_n}. \tag{5.25}
\]

Naturally, it is determined up to a constant. The constant is easily determined according to the following rules:

a) if the number of phase factors is even the overall sign of the integral is 
"+", otherwise, it is "-";

b) the total phase factor is just \( \zeta^{\hat{a}_1 + \hat{a}_2 + \cdots + \hat{a}_{n+1}} \), where \( \hat{a}_i \) means that this factor may be either zero or some rational fraction;

c) the sum \( \sum_{i=0}^{n+1} a_i = 0 \mod N \);

d) upon change of variables: \( Y_i = \zeta t_i^\frac{1}{N} \), the overall c-factor \( c = (\frac{1}{N})^n \) emerges (and can be dropped);

f) the resulting integral

\[
I_{V(F)} = \int_\Delta t_1^{b_1} \cdots t_{n+1}^{b_{n+1}} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}
\]

with \( b_i = a_i/N \) can be calculated in the spirit of ingenious trick by Deligne [37] which, of course, can be as well inferred directly from Eq.(5.23). To this purpose, we multiply both sides of the last equation by \( \Gamma(\sum_{i=1}^{n+1} b_i)I_{V(F)} = \Gamma(\sum_{i=1}^{n+1} b_i) \int_\Delta t_1^{b_1} \cdots t_{n+1}^{b_{n+1}} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n} \). For the r.h.s. we get:

\[
\Gamma(\sum_{i=1}^{n+1} b_i) \int_\Delta t_1^{b_1} \cdots t_{n+1}^{b_{n+1}} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}
\]

\[
= \int_\Delta \int_0^\infty \frac{dt}{t} \exp(-t) t_1^{b_1} \cdots t_{n+1}^{b_{n+1}} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}
\]

Let now \( s_i = tt_i \) so that, as before, \( t = \sum_{i=1}^{n+1} s_i, s_i \geq 0 \). This then produces the following integral

\[
\int_0^\infty \cdots \int_0^\infty \exp(-\sum_{i=1}^{n+1} s_i) \frac{s_1^{b_1}}{s_1} \cdots \frac{s_{n+1}^{b_{n+1}}}{s_{n+1}} \wedge \frac{ds_1}{s_1} \wedge \cdots \wedge \frac{ds_{n+1}}{s_{n+1}}
\]

\[
= \Gamma(b_1) \cdots \Gamma(b_{n+1})
\]

so that finally we obtain the contribution to the Veneziano amplitude

\[
I_{V(F)} = \frac{\prod_{i=1}^{n+1} \Gamma(b_i)}{\Gamma(\sum_{i=1}^{n+1} b_i)}, \tag{5.26}
\]
The total amplitude is the sum of the above with the appropriate phase factors.

It should be clear by now why it is permissible to multiply both sides of Eq. (3.33) by factor of $\zeta^2$ (and, of course, to do the same in the most general case).

**Remark 5.6.** Eq. (5.27) can be found in Gross paper, Ref. [34], where it is given without derivation. The role of complex multiplication in obtaining this result is strongly emphasized in his paper. The same equation was formally obtained by Edwards [29] already in 1922.

**Remark 5.7** In accord with Section 3.2, different combinations of $a_i's$ correspond to different periods of Fermat hypersurface. Each such period corresponds to a point in the Grassmannian so that different periods correspond to different points. This makes calculation of the Veneziano amplitudes similar to our earlier calculations of the Witten-Kontsevich averages, Ref. [64].

**Remark 5.8.** There is yet another method of obtaining the results of this subsection. It is based on the asymptotic analysis of complex oscillatory integrals as discussed in the book by Arnol’d, Gusein-Zade and Varchenko, Ref. [36]. Fortunately, the results of this reference not only useful for reobtaining the multiparticle Veneziano amplitudes (as discussed in the Appendix D) but also they provide a very economical way of getting all the results associated with the mixed Hodge structures. These results will be utilized in Section 5.3.2, in connection with the Hodge spectrum which is in one-to-one correspondence with the particle mass spectrum.

### 5.3 Dynamics of Seifert fibered phase and Veneziano amplitudes

#### 5.3.1 Connections between zeta function and Alexander polynomial

In our previous work, Ref. [4], we had considered dynamics of the Seifert fibered phase of 2+1 gravity (e.g. see Appendix 3 for definitions) associated with solution of the Witten-Kontsevich model. In this work we have reobtained the partition function (the p-adic version of this function) of 2+1 gravity in Section 3.1., Eq. (3.6), working with dynamics of gravity in this phase. Earlier, we argued that such partition function is associated also with the p-adic analogue of the four particle Veneziano amplitude, Eq.s (1.15), (1.16b), (1.17). In Section 3.2, we introduced the generalized Alexander polynomial, Eq. (3.20). According to Milnor [26] and Brieskorn [27], such polynomial describes the multidimensional fibered knots and links. Since according to the Corollary 3.7, the spectrum of particle masses emerging as poles in four particle scattering amplitude is in one-to-one correspondence with the appropriate Alexander polynomial describing fibered knots and/or links in 3 dimensions, it is natural to expect that the same will be true for the multiparticle scattering^21.

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21 As the development presented below indicates, this option is less convenient than that emerging from the correspondence of mass spectrum with the Hodge spectrum.
Remark 5.9. It should be noted at this point that the Alexander polynomial may describe more than one knot/link. This happens in the case of cable knots/links, Ref.[149], pages 121-122. In such cases one might think of one-to-one correspondence between the mass spectrum and knot/link cable families. This option is not going to be explored in this paper in view of the remarks made in the footnote below.

To proceed, we need to explain why two apparently different equations, Eq.(3.13) and (3.18), describe the same Alexander polynomial for the Seifert-fibered knots/links. The connection between the above equations could be thought as a corollary to a very deep result of Grothendieck [62] which can be formulated in the form of the following

Theorem 5.10. The Alexander polynomial for algebraic knots/links embedded into $S^{2n+1}$ is equal to the product of cyclotomic polynomials.

Remark 5.11. For the purposes of this work it is sufficient to keep in mind that the link $K = V_{B-P}(f) \cap S^{2n+1}$ introduced in Section 3.2. is algebraic [149].

Remark 5.12. The Alexander polynomial $\Delta_K(t)$, Eq.(3.18), can be obtained from the ratio of cyclotomic polynomials given below

$$\Delta_K(t) = \frac{(t^{pq} - 1)(t - 1)}{(t^p - 1)(t^q - 1)}.$$ (5.27)

Indeed, using equations (A.6)-(A.8) of Appendix A we obtain for instance,

$$t^q - 1 = (t - 1)(t^{q-1} + \cdots) = \prod_{i} (t - \xi_i),$$

with $\xi_i$ being one of the roots of unity (including one). Use of such type of products in Eq.(5.27) produces, indeed, the desired result, Eq.(3.18). For 3-dimensional Seifert-fibered knots/links the result, Eq.(5.27), had been obtained by Le Trang [150] and, independently, by Sumners and Woods [151]. The result, Eq.(5.27), leads to the following additional observation originally made by Milnor [26]. It appears, that with help of Eq.s (2.30),(3.16), (3.17) one can connect the Alexander polynomial and the Weil-type zeta function. Earlier, in Eq.(3.6) we had obtained such zeta function for the p-adic version of Veneziano amplitude. Moreover, for the torus the Weil-type zeta function, Eq.(3.35), was obtained earlier by Dwork [102], e.g. see Remark.4.7. Looking at Eq.(3.35) it is educational to rewrite it in the following form:

$$\frac{1}{Z_K(u)} = \frac{(u - 1)(qu - 1)}{(u - 1)(qu/\alpha - 1)}.$$ (5.28)

Assume now that $q = t^{mn}, \alpha = t^m$ so that $q/\alpha = t^n$. In addition, assume that $u = u^n$ (or $u = u^m$, etc.). Then, using cyclotomic-type expansion (displayed above) and letting $u = 1$ at the end of calculations produces back the Alexander polynomial, Eq.(3.18), for the torus knots. This then implies that, at least for
torus knots, $Z_K(t)\Delta_K(t) = 1$. This result is not totally surprising if one recalls [152] that for a well behaving matrix $A$ one has the following identity

$$[\det(I - tA)]^{-1} = \exp\left\{\sum_{n=1}^{\infty} \frac{t^n}{n} (tr A^n)\right\}$$

(5.29)

Looking at Eq.(3.13) and recalling that $\det M = \pm 1$ (e.g. read Section 3 of Ref.[4]) and comparing the r.h.s. of the above identity with Eq.(2.25) one obtains the desired relationship between the zeta function and the Alexander polynomial. In Section 3 of Ref.[4] we have demonstrated that the monodromy matrix $M$ is indeed involved in surface dynamics.

**Remark 5.13.** The cyclotomicity (Theorem 5.9) implies that such dynamics is characteristic only for the Seifert fibered phase (regime). This is in accord with results of Milnor’s book [26], e.g read chapter 10.

The question remains: will such relationship hold for surfaces more complicated than torus? Although the answer to this question can be found in Milnor’s book [26], we prefer to approach it differently. To this purpose let us notice that the arguments used in obtaining of Eq.(5.28) can be also applied to zeta function, Eq.(3.6), the p-adic Veneziano amplitude. In this case we obtain

$$\frac{1}{Z_f(t)} = \frac{1 - pt}{1 - t} = \prod_{\xi^n = 1, \xi \neq 1} (t - \xi) = \Delta_{K_0}(t),$$

(5.30)

where $\Delta_{K_0}(t)$ is the Alexander polynomial for the "empty" knot [58].This result follows from Milnor’s general theory discussed in Section 3.2. Indeed, let in Eq.(3.7) there is only one term, i.e. $f(z) = z^n$.We can still construct a circle mapping, Eq.(3.9), and after applying the same type of arguments leading to the Alexander polynomial, Eq.(3.20), we will get $\Delta_{K_0}(t)$. Just made observation can be vastly generalized with help of the fundamental theorem of Thom and Sebastiani [153]. The arguments by presented in Ref.[154] are based on this theorem and summarized in the following additional theorem (Oka, Ref.[154]):

**Theorem 5.14.** Let $f$ be a polynomial in $\mathbb{C}^n \times \mathbb{C}^m$ such that $f(z,w) = g(z) + h(w)$ for each $(z,w) \in \mathbb{C}^n \times \mathbb{C}^m$ where $g(z)$ and $h(z)$ are weighted homogenous polynomials in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively. Let $X = f^{-1}(1) \subset \mathbb{C}^n \times \mathbb{C}^m$, $y = g^{-1}(1) \subset \mathbb{C}^n$ and $Z = h^{-1}(1) \subset \mathbb{C}^m$. Then there is a natural homotopy equivalence $\alpha$ between $X$ and $Y \star Z$ where $Y \star Z$ is the join of $Y$ and $z$ with strong topology.

As a corollary, the same author obtains the following result:

**Theorem 5.15.** Let $g$, $h$ and $f$ be the same as in Theorem 5.14, and assume that $g$ and $h$ have isolated critical points at the origin. Then the characteristic (Alexander) polynomials of the associated fiberings $\Delta_g(t)$, $\Delta_h(t)$ and $\Delta_f(t)$ satisfy the equation

$$\Delta_f(t) = \Delta_g(t) \star \Delta_h(t).$$

(5.31)
That is letting $\Delta_g(t) = \prod_i (t - \lambda_i)$ and $\Delta_h(t) = \prod_i (t - \mu_i)$ we should obtain $\Delta_f(t) = \prod_{i,j}^{} (t - \lambda_i \mu_j)$.

**Remark 5.16.** The concept of a join could be found in any book on topology, e.g. see Ref.[60]. We actually have effectively used it already when we had considered the map $\Delta \longrightarrow V(F)$ as discussed before Eq.(5.21). Roughly speaking, the concept of a join could be understood based on the following example taken from Ref.[155] (see also Pham[25] and Milnor[26]). Let $\mathbb{Z}/p$ and $\mathbb{Z}/q$ be finite cyclic groups consisting of all p-th, respectively q-th roots of unity. The join $J = \mathbb{Z}/p \ast \mathbb{Z}/q$ could be thought as totality of all vectors $(s\xi,t\eta) \in \mathbb{C}^2$ with $s,t \geq 0$, $s + t = 1$, and $\xi \in \mathbb{Z}/p$, $\eta \in \mathbb{Z}/q$. For the circle map of the type

$$
\psi(z,w) = \frac{z^p + w^q}{|z^p + w^q|}
$$

it could be shown [26], that $J$ is the deformation retract of the fiber $\psi^{-1}(1)$.

**Remark 5.17.** From Theorems 5.14. and 5.15 it follows that the Alexander polynomial of the multidimensional fibered knot/ link can be constructed with help of the Alexander polynomials for empty knots, e.g. $\Delta_g(t) = \prod_i (t - \lambda_i)$, Eq.(5.30), associated with zeta functions of circle maps related to periodic surface automorphisms characteristic for the Seifert-fibered phase (Appendix B)\textsuperscript{22}. With all useful information just discussed we still have not provided an answer to the question we had posed earlier about the relationship between the Alexander polynomial and zeta function for multicomponent links. Fortunately, such problem was discussed by A’Campo [156] so that it remains only to illustrate his findings. In particular, for the B-P type singularity, e.g. see Eqs.(3.7),(3.8), the Alexander polynomial $\Delta(t)$ should be

$$
\Delta(t) = \left[ (t-1)^{-1}(t^N - 1)\chi(S_N) \right]^{(-1)^n}
$$

with $\chi(S_N)$ being the Euler characteristic of the complement of the Fermat hypersurface in $\mathbb{C}P^n$. A’Campo provides the following result for the characteristic $\chi(S_N) = \frac{1-(1-N)^{n+1}}{N}$. Next, if the Milnor number (that is the multiplicity of the critical point) is given by

$$
\mu = (-1)^n[-1 + N\chi(S_N)] = (N - 1)^{n+1},
$$

then the zeta function $Z(t)$ and the Alexander polynomial $\Delta(t)$ are connected through

$$
\Delta(t) = t^\mu \left[ \frac{t-1}{t} - Z(\frac{1}{t}) \right]^{(-1)^{n+1}}.
$$

Consider now a special case : $N = 3$, $n = 2$, treated by A’Campo. It is relevant for 4-particle Veneziano amplitude discussed in Section 3.2. The treatment of general case will become obvious upon completion of discussion for this special case.\textsuperscript{22}Essentially the same conclusions can be reached using the laguage of singularity theory (Appendix D), e.g. see the Fubini's theorem, Eq.(D.7), and the discussion which follows.
case. The Alexander polynomial for this special case can be written at once as follows:

\[
\Delta(t) = (1 - t)^{-1}(t^3 - 1)^3 \\
= (1 - t)^2 [(t - \rho)(t - \bar{\rho})]^3 \\
= (1 - t)^2 (t^2 + t + 1)^3
\]  

(5.35)

here \( \rho \) is the same as in Eq.(3.19). The reader familiar with knot theory might recognize immediately in the above expression the Hosokawa-type Alexander polynomial for 2 component link. To prove that the above expression is, indeed, valid for a link, it is sufficient to invoke the theorem by Murasugi, Ref.[157], page 117:

**Theorem 5.18.** Let \( \mathcal{L} \) be a link with number of components \( \mu \geq 2 \), then the Alexander polynomial \( \Delta_{\mathcal{L}}(1) = 0 \).

To prove that the above polynomial is of Hosokawa type (denote it by \( \hat{\Delta}_{\mathcal{L}}(t) \)) we use the following facts from knot theory [158]: if the Alexander polynomial is of Hosokawa type then, a) the multiplicity of the factor \((t - 1)\) is even, b) \( \hat{\Delta}_{\mathcal{L}}(t) \) is a Laurent polynomial of even degree (up to multiplication by factors \( t^m \) with \( m = 0, \pm 1, \pm 2, ... \)), c) the number of components \( \mu \) of the link should be greater or equal to two but **otherwise is arbitrary** Ref.[157], page 121. In the present case Milnor [26] (and also A’Campo [156]) had shown that the number \( \mu \) of components in the link is given by Eq.(5.33)).; d) the Hosokawa type Alexander polynomial possess the property: \( \Delta_{\mathcal{L}}(t) \hat{=} \Delta_{\mathcal{L}}(\frac{1}{t}) \). Looking at the Alexander polynomial given by Eq.(5.35) it is straightforward to demonstrate that, indeed, the equivalence \( \Delta_{\mathcal{L}}(t) \hat{=} \Delta_{\mathcal{L}}(\frac{1}{t}) \) holds.

**Remark 5.19.** The above arguments apply, strictly speaking, only to 3d links embedded in \( S^3 \). The Alexander polynomial obtained by A’Campo [156] describes, however, knots/links embedded not into \( S^{2n+1} \) but into \( \text{CP}^n \). This circumstance creates no additional problems locally (since locally \( \text{CP}^n \supset \text{C}^{n+1} \supset \text{R}^{2n+2} \supset S^{2n+1} \)). Nevertheless, the results related to the Alexander polynomial derived by A’Campo should be treated with some caution. We hope, that the Theorems 5.14. and 5.15. provide the necessary assurance that, indeed, it is possible to extend results for links in 3d to links of dimensions higher than 3.

**Remark 5.20.** The Alexander polynomial, Eq.(5.32), formally looks different from that given by Eq.(3.20). This difference is superficial however since Eq.(5.32) is written for a special case of the Brieskorn-Pham singularity: when all \( a'_i s = N \). In this case all roots of unity factors in Eq.(3.20) originate from the same Eq.(A.3) (with fixed \( n = N \)).

**Remark 5.21.** The Corollary 3.7. should be amended now in view of Eq.(5.32). The mass spectrum is determined actually by two parameters: \( n \) and \( N \). This option, is only plausible but, in fact, is less convenient than that discussed below.
5.3.2 Newton’s polyhedra and the Hodge spectrum

Based on the results of Appendix D, we conclude that the above two parameters control the Hodge spectrum of the B-P singularity. As results of Ref.[36], page 328, indicate, the total number of differential n+1 forms \( \omega_b \) of the type

\[
\omega_b = s_1^{b_1} \cdots s_{n+1}^{b_{n+1}} \frac{ds_1}{s_1} \wedge \cdots \wedge \frac{ds_n}{s_n} \wedge \frac{ds_{n+1}}{s_{n+1}}
\]  

(5.36)

is given by the Milnor number, Eq.(5.33), (e.g., please, read in addition the Remark 5.26 below). Hence, this number provides as well the total number of possible Veneziano amplitudes. This number has a very interesting geometrical and topological interpretation which we would like to discuss now. We begin by introducing some definitions.

**Definition 5.22.** The function \( x^k = x_1^{k_1} \cdots x_n^{k_n} \) is called *monomial* with the (vector) exponent \( k = (k_1, \ldots, k_n) \), \( k_i \in \mathbb{Z}^+ \). The monomial is called *Laurent* if \( k_i \in \mathbb{Z} \) [158].

Let \( \mathcal{A} \subset \mathbb{Z}^k \) be a finite set of integer vectors. Then, this set can be identified with the set of corresponding monomials. Let \( \mathbb{C}^A \) be the space of Laurent polynomials made out of monomials from the set \( \mathcal{A} \), i.e.

\[
f(x) = f(x_1, ..., x_n) = \sum_{k \in \mathcal{A}} a_k x^k.
\]  

(5.37)

Evidently, the B-P variety, Eq.(3.8), is just a special case of the variety defined by

\[
V(f) = \{ z \in (\mathbb{C}^*)^k \mid f(z) = 0 \}
\]  

(5.38)

where \( \mathbb{C}^* = \mathbb{C} \setminus 0 \). Such variety is known in literature as the *algebraic toric variety* [158]. The name “toric” will be explained shortly.

**Definition 5.23.** The Newton polytope (polyhedron) \( N(f) \) associated with function \( f(x) \) defined by Eq.(5.37) is the convex hull in \( \mathbb{R}^n \) of the set \{ \( k : a_k \neq 0 \) \}.

**Remark 5.24.** Use of the above definition of \( N(f) \) in actual calculations may or may not be convenient. There are several algorithmically different (but combinatorially equivalent) ways to define a convex polytope in \( \mathbb{R}^n \)[159]. According to Ref.[159], a convex polytope \( P \) is either

a) a convex hull of finite set of points in \( \mathbb{R}^n \) (which is essentially just the Definition 5.23) or

b) is the result of an intersection of finitely many half spaces in \( \mathbb{R}^n \). In the last case \( P \) is defined by the set if inequalities:

\[
P = \{ x \in \mathbb{R}^n : (l_i, x) \geq -a_i, \ i = 1, ..., m \}
\]  

(5.39)

The vectors \( l_i \in (\mathbb{R}^n)^* \) are from the *dual* space to be defined shortly below and \( a_i \in \mathbb{R}_i, i = 1, ..., m \). Let \( H_i \) be an affine hyperplane in \( \mathbb{R}^n \) then the intersection
\( P \cap H_i \) is called face of the polytope. The boundary \( \partial P \) of \( P \) is the union of faces.

Let us discuss now the rationale for introducing the dual space. To this purpose let us recall that the function \( f \) is considered to be quasi homogenous of degree \( d \) with exponents \( l_1, ..., l_n \) if
\[
 f(\lambda^{l_1}x_1, ..., \lambda^{l_n}x_n) = \lambda^d f(x_1, ..., x_n) \tag{5.40}
\]
provided that \( \lambda \in \mathbb{C}^* \). The exponents \( l_i \) associated with variables \( x_i \) are called weights.

In the light of the above definition, it is clear that each monomial in Eq.(5.37) is associated with the scalar product of the type:
\[
 \sum_j (l_j) i k_j = d_i \tag{5.41}
\]
with index \( i \) numbering the monomials. Hence, the vector \( l = (l_1, ..., l_n) \) is the dual of the vector \( k \). The name ”toric” comes from the observation that \( \lambda \) can be chosen in such a way that \( |\lambda| = 1 \). Hence, the usual topological torus \( T^n \) (which is a product of \( n \) circles) defined by
\[
 T^n = \{(e^{2\pi i \varphi_1}, ..., e^{2\pi i \varphi_n}) \in (\mathbb{C}^*)^n \},
\]
with \( \varphi = (\varphi_1, ..., \varphi_n) \) running through \( \mathbb{R}^n \), is just a subgroup of the algebraic torus. The algebraic torus \( (\mathbb{C}^*)^n \) is the semigroup with respect to component-wise multiplication [158], i.e. if \( t \in (\mathbb{C}^*)^n \) then, the action of \( t \) on \( x^k \) is given by
\[
 t^l \cdot x^k = t^{l_k} x^k
\]
which is just a special case of Eq.(5.40). Also, Eq.(3.26) is just a special case of this equation. The scalar product \( l \cdot k = d \) defines a set \( L \) of hyperplanes. Following Ref.[160] we call the set of all exponents of monomials that appear in \( f(x) \), Eq. (5.37), the support. It is clear that the members of the support lie either inside of the \( N(f) \) or at its faces. The hyperplane set \( L \) plays a major role in what follows.

**Remark 5.25.** It can be shown [161] that the hyperplane set \( L \) determines convex polyhedron \( P \) completely and vice versa (i.e. they are dual to each other). Since the polyhedron \( N(f) \) defines (determines) toric variety, Eq.(5.38), this means that the hyperplane set \( L \) determines the toric variety as well so that all results of this work can be deduced from (encoded in) some properties of the hyperplane set \( L \). The theory of these sets is known in the literature as the theory of arrangements. Condensed, physically relevant and pedagogically clear exposition of this theory can be found in the book by Orlik and Terrao [162]. This book among many other things discusses work of Schehtman and Varchenko [137] connecting theory of hyperplane arrangements with the theory of K-Z equations\(^{23}\).
With such background we are ready now to go back to the differential n+1 form, Eq.(5.36), in order to rewrite it in the language of toric varieties. It should be clear that $\omega_b$ in Eq.(5.36) can be presented as follows

$$\omega_b = x^b \omega_0$$

(5.42a)

where $b = (b_1, ..., b_{n+1})$ and $\omega_0$ is given by

$$\omega_0 = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_{n+1}}{x_{n+1}}.$$  

(5.42b)

Taking into account Eq.s(3.26) and (5.41) and using Eq.(5.42a) we obtain the hyperplane equation which, in view of Eq.(5.20b), can be written as

$$-\alpha = \frac{1}{N} \sum_i b_i = < b >$$

(5.43)

and should be compared with Eq.(5.39).

In writing this equation we have taken into account that the eigenvalue $\lambda$ of the monodromy operator (Section 3.2) is related to $\alpha$ as

$$\alpha = -\frac{1}{2\pi i} \ln \lambda$$

(5.44)

in accord with notations of the Appendix D.

**Remark 5.26.** From Eq.(5.20a) it should be clear that the vector $b$ is just one of many possible allowed vectors. The totality of these vectors is given by the Milnor number $\mu$, Eq.(5.33). The set of these vectors is in one-to-one correspondence with the set of eigenvalues $\{\lambda\}$ of the monodromy operator, Eq.(3.17).

For the B-P polynomial, Eq.(3.7), it is easy to give useful geometric description of these vectors. Indeed, consider a hypercube $(0, N)^{n+1} = (0, N) \times \cdots \times (0, N)$. Then a vector $b$ is represented by one of the points strictly inside of the hypercube. If the volume of the unit cube cell is chosen to be unity, then the number of points inside the cube coincides with the volume $(N - 1)^{n+1}$ which is just the Milnor number $\mu$, Eq.(5.33).

In view of the results of Appendix D, the totality of hyperplanes described by Eq.(5.43) is in one-to-one correspondence with the totality of the Hodge numbers associated with the differential forms, Eq.(5.36). We would like to describe this connection in some detail following the work of Arnol’d [163].

From Eq.(D.6) it follows, that $-\alpha$ in Eq.(5.44) should be a positive rational number. Mathematically, both integer and non integer numbers are allowed. In the case of integers the corresponding monodromy eigenvalue $\lambda = 1$, while in the non integer case $\lambda \neq 1$. In view of Eq.(3.17a), only non integers should be of physical interest. Physical relevance of the integer case requires further study. So, let $-\alpha$ be some integer + a fraction (in Appendix D it is denoted as $\hat{\alpha}$). The following number-theoretic problem can be posed now: For the set of integers $b$
lying inside the hypercube \((0,N)^{n+1}\) and for positive rational number \(\hat{\alpha}\) lying in the range
\[
    n - p < \hat{\alpha} \leq n - p + 1
\]
(with \(p\) being defined either by Eq.(5.9) or in the Appendix D) find how many vectors \(b\) produce \(-\alpha\) in Eq.(5.44) provided that \(\hat{\alpha}\) lies in the above domain.

The Hodge number \(h_{pq}^{\alpha}\) can be found now if in addition to the number-theoretic problem just described we would find the number of solutions of Eq.(5.43) for case when
\[
    -\alpha = \hat{\alpha} = n - p + 1.
\]

Denote \(h_{pq}^{\alpha}\) the number of solutions for the first problem and \(h_{1pq}\) the number of solutions for the second problem then, the Hodge number \(h_{pq}\) is given by
\[
    h_{pq} = h_{pq}^{\alpha} + h_{1pq},
\]
e.g. see Eq.(D.10) of Appendix D.

Geometrically, \(h_{pq}^{\alpha}\) is just the number of integer points in the hypercube lying between the hyperplanes defined by the left and right sides of the inequality, Eq.(5.45), while \(h_{1pq}\) is the number of integer points lying in the hyperplane defined by Eq.(5.46). According to the Ref.s[159] and [163], \(N(f)\) for the B-P polynomial is combinatorially equivalent to the hypercube \((0, N)^{n+1} = \Delta\) whose vertices belong to the integral lattice \(\mathbb{Z}^{n+1} \subset \mathbb{R}^{n+1}\). For any positive integer \(k\), let \(l(k\Delta)\) be the number of integral points in \(k\Delta \cap \mathbb{Z}^{n+1}\). By definition, \(l(0\Delta) = 1\). In addition, let \(l^*(k\Delta)\) be the number of integral points in the interior of \(k\Delta\). Based on these definitions, two generating functions (Poincaré polynomials) can be constructed now:

\[
P_{\Delta}(t) = \sum_{k \geq 0} l(k\Delta) t^k
\]
and
\[
Q_{\Delta}(t) = \sum_{k \geq 0} l^*(k\Delta) t^k.
\]

Batyrev [164] proved that, if additionally one defines two related functions:
\[
\Psi_{\Delta}(t) = (1 - t)^{n+2} P_{\Delta}(t) = \sum_{i \geq 0} \psi_i(\Delta) t^i
\]
and
\[
\Phi_{\Delta}(t) = (1 - t)^{n+2} Q_{\Delta}(t) = \sum_{i \geq 0} \varphi_i(\Delta) t^i,
\]
then,
\[
t^{n+2} \Psi_{\Delta}(t^{-1}) = \Phi_{\Delta}(t).
\]
According to Arnol’d, Ref.[163], for the B-P variety one obtains
\[
Q_{\Delta}(t) = \left( \frac{t^{\frac{1}{n+1}} - t}{1 - t^{\frac{1}{n+1}}} \right)^{n+1}.
\]
Consider now the limit: \( t \to 1 + \varepsilon, \varepsilon \to 0^+ \) of the above generating function. An easy calculation produces:

\[
Q_\Delta(1) = \mu = (N - 1)^{n+1}, \tag{5.52}
\]

where, again, \( \mu \) is Milnor number, Eq.(5.33). According to Eq.(5.33) this result also allows calculation of the Euler characteristic \( \chi(S_N) \) of the complement of the Fermat hypersurface in \( \mathbb{CP}^n \) and, hence, the Euler characteristic of the Fermat hypersurface as well [156].

**Remark 5.27.** The result, Eq.(5.52), is known in the literature as Kouchnirenko theorem [165].

**Remark 5.28.** Based on the observation that [166] for any Kähler manifold \( X \)

\[
\chi(X) = \sum_{p,q=0}^{n+1} (-1)^{p+q} h^{p,q}(X), \tag{5.53}
\]

Danilov and Khovanskii [167] had proposed to look for properties of the generating function

\[
e(X; x, \bar{x}) = \sum_{p,q} e^{pq}(X)x^p\bar{x}^q, \tag{5.54}
\]

where \( e^{pq}(X) = (-1)^{p+q} h^{p,q}(X) \). Because of the connection between \( N(f) \) and \( \chi \) exhibited in Eqs.(5.33) and (5.52), these authors had demonstrated how to recover the Hodge numbers \( h^{p,q} \) from generating functions, Eq.(5.48), of the Newton polyhedron thereby connecting the generating function \( e(X; x, \bar{x}) \) with that for the Newton polyhedron. Useful condensed summary of results of Ref.[167] can be found in the review paper by Cox [168].

**Remark 5.29.** The formula of Arnol’d for \( Q_\Delta(t) \), Eq. (5.51), is a special case of a more general formula obtained by Steenbrink [169], Brieskorn (unpublished) and Saito [170]. Their results are alternative to that obtained by Danilov and Khovanskii and, in our opinion, more efficient computationally. In the case if \( f(x) \), Eq.(5.37), is linear combination of monomials \( x^k \) such that Eq.(5.41) acquires the form \( \mathbf{k} \cdot \mathbf{l} = 1 \) for all monomials entering \( f(x) \), it is possible to demonstrate that

\[
Q_\Delta(t) = \prod_{i=1}^{n+1} \left( \frac{t^l_i - t}{1 - t^l_i} \right) \tag{5.55}
\]

which for the case of B-P polynomial is reduced back to Eq.(5.51). However, the interpretation of the polynomial \( Q_\Delta(t) \) by Saito [170] is different. According to Saito

\[
Q_\Delta(t) = \sum_i t^{\alpha_i}
\]

with \( \alpha_i = -\frac{1}{2\pi i} \ln \lambda_i \) (e.g. see the Appendix D and Refs. [135,170]). Hence, \( Q_\Delta(t) \) is also a generating function for the Hodge spectrum. Since earlier (and in the Appendix D) we had mentioned that the spectrum is degenerate, the above
generating function acquires the following form:

$$Q_\Delta(t) = \sum_{\{\alpha\}} n_\alpha t^\alpha$$

(5.56)

with degeneracies $$n_\alpha = \sum_l n_{\alpha,l}$$. The numbers $$n_{\alpha,l}$$ participate in yet another generating function

$$F = \sum_{\alpha,l} n_{\alpha,l}(\alpha,l)$$

(5.57)

with the spectral pair $$(\alpha,l)$$ being defined in the Appendix D. Since both $$\alpha$$ and $$l$$ in this pair are related to $$p$$ and $$q$$, one can identify $$n_{\alpha,l}$$ with $$h_{p,q}^{\alpha,l}$$ [135]. Hence, the generating function $$Q_\Delta(t)$$ allows, in principle, determination of the Hodge numbers $$h_{p,q}^{\alpha,l}$$. Examples are provided in Ref.[135]. The same results (with much less details) can be found in a short paper by Varchenko and Khovanskii [171].

6 Discussion

To our knowledge, so far the results from geometry of toric varieties had been used in the so called mirror symmetry calculations [57]. The results presented above indicate that actually all results of conformal field theories and high energy physics could be described from the standpoint of geometry of toric varieties. This discipline itself is not new however. Actually, it can be considered as branch of the theory of the linear algebraic groups to which some results of the theory of Lie groups and Lie algebras had been adopted. For instance, the result, Eq.(5.56), has its analogue in the theory of characters in the theory of Lie groups/algebras and, hence, one can derive the Weil formula for characters, etc. Under such interpretation, the Milnor number $$\mu$$, Eq.(5.33), plays a role of the dimension of representation of the Weil/Coxeter reflection group. With little extra effort one can reobtain Kac-Weil character formula central for developments of conformal field theories. In addition, obtained results allow the symplectic interpretation enabling us to restore the underlying physical model producing, for instance, the Veneziano amplitudes. The results of Atiyah documented in Ref. [40] indicate that this model is also going to be a string. The combinatorial aspects of the Lie group theory representations will allow us to restore the Verlinde fusion rules as a special case of Schubert calculus considered in our earlier work on Kontsevich-Witten model [64]. All results of this earlier work and the work just presented can be reobtained with help of the symplectic formalism based on the Duistermaat-Heckman formula [40]. If one is interested in development of p-adic aspects of string theory, the concept of motives and its latest development culminated in the works by the latest Fields medalist, Vladimir Voevodsky, can be found in the form accessible to physicists in Refs.[186,187]. Details of the results outlined above are to be presented in subsequent publications.
Acknowledgements. The author owes this work to Tatyana Zhebentyayeva (Clemson) whose late night phone call had triggered the chain of thoughts leading to this work. In addition, the author would like to thank Anatoly Libgober (UIC), Winnie Li (Penn State U), Michael Rosen (Brown U), Ulrich Geckeler (U Saarbrucken) for helpful correspondence.
Appendix A. Some results from cyclotomic field theory

Suppose we have the number field $F_p$ of characteristic $p$, that is the coset $\mathbb{Z}/p\mathbb{Z}$. Let $c_i \in F_p$. It can take $p$ values. Suppose now that we would like to extend the field $F_p$. This can be accomplished very easy using concepts of standard linear algebra. To construct the vector space of dimension $n$ we need to have a set of $n$ linearly independent (basis) vectors $e_i$ so that any vector $A$ can be decomposed as usual:

$$A = \sum_{i=1}^{n} a_i e_i. \quad (A.1)$$

Let now the role of $a_i$ is to be played by $c_i$ and the role of $e_i$ -by some basis elements $\alpha_i$ to be determined momentarily. Then, by construction, a number

$$\beta = \sum_{i=1}^{n} c_i \alpha_i \quad (A.2)$$

belongs to the extension of the field $F_p$. Since each $c_i$ can have $n$ values, the field $F_p[\alpha]$ contains $p^n$ elements and is known in literature as the Galois field $\text{GF}(p^n)$. The task remains to find the basis set explicitly. The familiar example of complex numbers helps a lot in this respect. Indeed, the complex number $i = \sqrt{-1}$ is just solution of the equation $x^2 + 1 = 0$. Clearly, all complex numbers $z$ are given by $z = a_1 + ib$ with $a$ and $b$ belonging to $\mathbb{R}, \mathbb{Q}$ or $\mathbb{Z}$. The basis elements are 1 and $i$. To go beyond this elementary case it is sufficient to consider all solutions of the equation

$$x^n - x = 0. \quad (A.3)$$

It can be shown [99] that the above equation can be also rewritten as

$$x^n - x = \prod_{i=1}^{n} (x - \alpha_i) \quad (A.4)$$

with $\alpha_i$ being a root of Eq.(A.3). In addition, it can be shown that all roots are distinct and, hence, there are $n$ of them. Naturally, they can form the basis which we are looking for. Moreover, Eq.(A.3) can be equivalently rewritten as

$$x^{n-1} = 1 \quad (A.5)$$

and, hence, the cyclotomic (that is circular) nature of this equation becomes clear. Indeed, consider $\zeta_k = \exp\{i\alpha\}$ with $\alpha = k \frac{2\pi}{n}$. If for $k$ we assign values $0, 1, 2, ..., n - 2$, we obtain $n - 1$ equidistant points on the circle. It is clear that this construction can be performed for any $n$ and, hence, if we consider instead of Eq.(A.5) the equivalent equation

$$x^n = 1 \quad (A.5a)$$
we shall obtain exactly \( n \) values for \( \zeta_k \). These are independent of each other and can be used instead of \( \alpha_i \)'s in Eq.(A.2). Moreover, there actually no need of substituting all \( \zeta_i \)'s into Eq.(A.2). It is sufficient to use just one of the so called \textit{primitive} roots of unity in Eq.(A.2). By definition \textbf{all} roots of Eq.(A.5a) can be obtained as powers of those which are considered to be primitive. To obtain primitive roots the following steps should be taken. For a given \( n \) one should look for all \( k \)'s such that \( k \nmid n \) and \( k < n \). The number of such possibilities is given by the Euler’s function \( \varphi(n) \). Recall that the coset \( \mathbb{Z}/n\mathbb{Z} \) is a field (and is denoted \( \mathbb{F}_n \)) only if \( n \) is some prime number \( p \). An element \( c_i \) is a \textit{unit} in \( \mathbb{F}_n \) if \((c_i,n) = 1\) (that is \( c_i \nmid n \)). Hence, there is one-to-one correspondence between the units in \( \mathbb{F}_n \) and the primitive roots in cyclotomic fields. So if \( n = p \) we have \( \varphi(p) = p - 1 \) primitive roots. If we take one of them, say \( \zeta \), then the desired basis is given by: 
\[
(x - \zeta_1) \cdots (x - \zeta_{\varphi(p)}) \equiv \Phi_{r_i}(x). \quad (A.6)
\]
It is monic irreducible polynomial of degree \( \varphi(r_i) \) by construction. Using multiplicative property of Euler’s function it is clear that the following identity should hold
\[
x^n - 1 = \prod_{d|n} \Phi_d(x) \quad (A.7)
\]
with \( d \) running over all divisors of \( n \). For \( n = p \) the result can be considerably simplified by noticing that \( x = 1 \) is the root of Eq.(A.7). This means, in view of Eq.(A.6), that Eq.(A.7) can be simplified to
\[
x^p - 1 = (x - 1)(x^{p-1} + x^{p-2} + \cdots + x + 1). \quad (A.8)
\]
By construction, the second multiplier in the r.h.s of Eq.(A.8) is monic irreducible (in \( \mathbb{R}, \mathbb{Q} \) or \( \mathbb{Z} \)) polynomial. It should be clear that the roots of the equation
\[
x^{p-1} + x^{p-2} + \cdots + x + 1 = 0 \quad (A.9)
\]
are primitive roots of the cyclotomic Eq.(A.5a). Thus, the connection between the cyclotomic fields and polynomials is clear. This connection can be extended further with help of Eq.(A.2). Indeed, let us write
\[
\beta = \sum_{i=0}^{n-1} c_{1i} \alpha^i
\]
and also
\[
\beta^2 = \sum_{i=0}^{n-1} c_{2i} \alpha^i
\]
etc. and, finally,
\[ \beta^n = \sum_{i=0}^{n-1} c_n \alpha^i \]
for some \( \alpha_{i,j} \) in \( \mathbb{F}_p \). Now we want to look at equation
\[ \sum_{i=0}^{n} x_i \beta^i = 0 \tag{A.10} \]
with \( x_i \in \mathbb{F}_p \). Upon substitution of equations for powers of \( \beta' \)s in Eq. (A.10) the representation of \( x'_i \)s through \( \alpha'_{i,j} \)s can be found and, hence, to construct equation of order \( n \) with \( \beta \) being a root. The polynomial associated with such an equation is irreducible and always can be made monic \cite{49}, page 430. Since for a given prime \( p \) there are \( p^n \beta' \)s according to Eq. (A.2), hence, there are \( p^n \) monic irreducible polynomials of degree \( n \). If we exclude the trivial case \( \beta = 0 \), then there are \( p^n - 1 \) nontrivial irreducible polynomials.

**Remark A.1.** Let \( S_m \) be symmetric group acting on complex space \( \mathbb{C}^m \) by exchanging coordinates. Then
\[ Y_m := \{ (z_1, ..., z_m) \in \mathbb{C}^m \mid z_i \neq z_j \text{ for } i \neq j \}/S_m \]
is the set of all unordered \( m \)-tuples of distinct complex numbers. It can be shown (Brieskorn, Malle & Matzat) that the set \( Y_m \) is in one-to-one correspondence with the set of monic irreducible polynomials with roots \( z_1, ..., z_m \). Define a braid on \( m \) strings (and initial point \( z_1^0, ..., z_m^0 \)) as homotopy class of a closed paths in \( Y_m \) with initial and final point at \( z_1^0, ..., z_m^0 \). Such constructed braid is in one-to-one correspondence with the set of monic irreducible polynomials\footnote{Braids are connected directly with the hyperplane arrangements mentioned in Section 5.3.2. in connection with the K-Z equations.}.

**Appendix B. Some results from Nielsen-Thurston theory of surface homeomorphisms**

Let \( \mathcal{S} \) be closed orientable Riemann surface of genus \( g \). The first homotopy group, the fundamental group \( \pi_1(\mathcal{S}) \) of surface \( \mathcal{S} \) is made of \( 2g \) generators \( \{x_i, y_i\}, i = 1 - g \) and a single relation so that its presentation is known to be
\[ \pi_1(\mathcal{S}) = \langle x_1, y_1, ..., x_g, y_g \mid [x_1, y_1] \cdots [x_g, y_g] \rangle. \tag{B.1} \]
Nielsen has noted that there is one to one correspondence between automorphisms of \( \pi_1(\mathcal{S}) \) and surface self-homeomorphisms. This is summarized in the following proposition

**Proposition B.1.** (Nielsen \cite{172}) If \( g > 1 \), then every element of \( Out(\pi_1(\mathcal{S})) \) is represented by a unique isotopy class of self-homeomorphisms of \( \mathcal{S} \).
An important subgroup of $\text{Out}(\pi_1(S))$ is the mapping class group $\mathcal{M}_g$. Geometrically, this group is finitely generated by the Dehn twists in simple closed curves (lamination set) on $S$ whose physical significance was discussed extensively in our previous work, Refs[4-6]. A simple closed curve $C$ on the orientable surface $S$ has a neighborhood $E$ homeomorphic to an annulus which is convenient to parametrize by $\{(r,\theta)|1 \leq r < 2\}$. The Dehn twist in $C$ can be imagined as an automorphism $T_C: S \to S$. It is given by the identity off $E$ and by $[r,\theta] \to [r,\theta + 2\pi r]$ on $E$. We would like to illustrate these concepts on the simplest example of the punctured torus $T^2$ using results of our previous work, Ref. [4]. In this case $\text{Out}(\pi_1(T^2)) = \text{GL}_2(\mathbb{Z})$ and $\mathcal{M}_{1,1} = \text{PSL}(2, \mathbb{Z})$. Since any transformation from $\text{PSL}(2, \mathbb{Z})$ is obtainable by projectivisation of $\text{SL}(2, \mathbb{Z})$ we discuss everything in terms of $\text{SL}(2, \mathbb{Z})$ with projectivisation at the end. Any transformation which belongs to $\text{SL}(2, \mathbb{Z})$ is expressible in terms of $2 \times 2$ matrix $A$ given by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(B.2)

with integer coefficients subject to condition: $\det A = ab - cd = 1$. The characteristic polynomial for this matrix is given by

$$t^2 - trA \cdot t + \det A = 0.$$  (B.3)

This implies that the eigenvalues of $A$ are either:

- a) both complex (when $\text{tr} A = 0, 1, -1$),
- b) both equal to $\pm 1$ (when $\text{tr} A = \pm 2$),
- c) distinct and real (when $|\text{tr} A| > 2$).

If $\Phi_A$ is toral automorphism then, transformation a) is called periodic since $(\Phi_A)^n = 1$ for some $n$ (actually, $n = 12$ by the Hamilton-Cayley theorem), transformation b) is called reducible since it leaves a simple closed curve $C$ invariant, transformation c) is called (pseudo) Anosov if the (line) vector field on surface (does not contain singularities. Physical significance of this difference is explained and illustrated in our previous work, Refs.[5,6].

The largest of two eigenvalues is associated with the topological entropy of the (line)vector flow and is related to the amount of stretching of surface $S$ and, hence, with the dilatation parameter of the Teichmüller theory. The Nielsen-Thurston theory generalizes the above classification of surface automorphisms to all surfaces of genus $g > 1$. Already Nielsen had realized [172] that for $g > 1$ it is more convenient to study homeomorphisms of surface $S$ by considering their image on the universal cover of $S$ which we choose as Poincare disc model of $H^2$, i.e. $\text{int} \ D \cup S^1_{\infty} = H^2$. According to Nielsen [172],

**Proposition B.2.** Any lift $\tilde{h}$ of the surface self-homeomorphism $h: S \to S$ to the universal cover of $S$ extends to a unique self-homeomorphism of the unit disc $D$, i.e. to $\text{int} \ D \cup S^1_{\infty}$.

Surface self-homeomorphisms $h$ are associated with the Dehn twists connected with a set of simple closed nonintersecting curves homotopic to geodesics.
(such set is called geodesic lamination \( \mathcal{L} \)). Their lifts \( \hat{\mathcal{L}} \) are associated with some maps of the circle \( S^1_\infty \) extendable (quasi conformally) to the interior of the disc \( D \) e.g. see Ref.[73]. An image of the closed geodesic on \( S \), when lifted to \( H^2 \), is just a segment of a circle whose both ends lie on \( S^1_\infty \). Since geodesics are nonintersecting, circle segments on \( S^1_\infty \) are also nonintersecting. In this work we are interested only in the periodic maps of the circle since non periodic maps can be always systematically approximated by periodic ones as it is explained in terms of continuous fractions and associated with them Dehn twists in our earlier work, Ref.[4] or, alternatively, in Milnor’s lecture notes [173] and Ref.[55].

In connection with such circle maps the following remark is of importance.

**Remark B.3.** (A variant of Sarkovskii theorem, Ref.[174], page 88). Let \( f: S^1 \rightarrow S^1 \) be a continuous map of the circle with a periodic orbit of period 3. If the lift \( \hat{f}: \mathbb{R} \rightarrow \mathbb{R} \) has also a periodic orbit of period 3 then, \( f \) has periodic orbits of every period. The condition on the lift map \( \hat{f} \) cannot be dropped. The continuous maps of the circle can be replaced by the piecewise linear maps while 3 still remains as minimal period.

**Remark B.4.** As noted by Kontsevich [175], the moduli space problem makes sense only for Riemann surfaces obeying the following set of inequalities

\[
    g \geq 0, \ n > 0, \ 2 - 2g - n < 0
\]

with \( n \) being the number of distinct marked points (effectively, distinct boundary components). Boundary components can be eliminated by the Schottky double construction. This construction can be performed as follows. If \( M \) is a complex manifold with \( C_1, ..., C_n \) boundary components, one can consider an exact duplicate of it, say \( \hat{M} \), with the same number of boundary components, say, \( \hat{C}_1, ..., \hat{C}_n \). Evidently, for each point \( x \in M \) there is a "symmetric" point \( \hat{x} \in \hat{M} \). The Schottky double \( 2M \) is formed as a disjoint union \( M \cup \hat{M} \) and identifying each point \( x \in C_i \) with point \( \hat{x} \in \hat{C}_i \) for \( 1 \leq i \leq n \). In the simplest case we have initially either punctured torus, i.e. \( g = 1, n = 1 \), or the trice punctured sphere, i.e. \( g = 0, n = 3 \). In both cases the Schottky double is a double torus. A double torus has 3 geodesics which belong to the geodesic lamination \( \mathcal{L} \). The image of these geodesics lifted to \( H^2 \) produces 3 circular arcs whose ends lie on \( S^1_\infty \). This is the minimal number of arcs required for the moduli space problem to make sense. According to Remark B.3. this is also the minimal period for the periodic homeomorphisms of the circle in view of the Sarkovskii theorem.

In Ref.[176] it is argued that the total number of geodesics on the Schottky double is \( 6g - 6 + 3n \). This is the dimension of space of holomorphic quadratic differentials (real on each of the boundary components). Hence, in accordance with Teichmüller theory [8], it is the dimension of the Teichmüller and, accordingly, the moduli space of such Schottky doubled surface.

**Appendix C. Complex multiplication and Hecke operators**

In Section 4b we have introduced both lattice \( L' \) and sublattice \( L \). This caused us to introduce the relationship: \( ad - bc = n, n \geq 1 \). We would like now
to study physical implications of this relationship. To this purpose, following Ref.[89], we introduce 3 types of matrices:

\[ \mathcal{D}_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) ; \ ad - bc = n \right\}, \]

\[ \mathcal{S}_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(\mathbb{Z}) ; \ ad = n, a, d > 0, 0 \leq b < d \right\}. \]

\[ \Xi = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z}) ; \ \alpha, \beta, \gamma, \delta \in \mathbb{Z}, \alpha \delta - \beta \gamma = 1 \right\}. \]

Naturally, \( \mathcal{S}_n \leftrightarrow SL_2(\mathbb{Z}) \setminus \mathcal{D}_n \). Introduce now the homothety operator \( R_\lambda \) as follows

\[ R_\lambda L = \lambda L \quad \text{(C.1)} \]

where \( \lambda \in \mathbb{C} \) and the action of \( n \)th Hecke operator \( T(n) \) on the lattice \( L \) is defined by

\[ T(n)L = \sum_{L' \subseteq L'} (L'). \quad \text{(C.2)} \]

In order to use this equation, it should be rewritten in more explicit form as

\[ T(n)L = \sum_{\alpha \in \delta_n} \alpha(L). \quad \text{(C.3)} \]

It can be demonstrated [89] that

a) \( R_\lambda R_\mu = R_{\lambda \mu} \ \forall \lambda, \mu \in \mathbb{C} \);

b) \( R_\lambda T(n) = T(n)R_\lambda \ \forall \lambda \in \mathbb{C}, n \geq 1 \);

c) \( T(mn) = T(m)T(n) \ \forall m, n \geq 1 \) with \( \gcd(m, n) = 1 \);

d) \( T(p^e)T(p) = T(p^{e+1}) + pT(p^{e-1})R_p \) for \( p \) prime, \( e \geq 1 \)

and \( p \nmid n \), while for \( p \mid n \) we get

\[ d')T(p^e) = [T(p)]^{d'}. \]

Since any number \( n \) can be decomposed into primes the above rules are sufficient for any \( n \), prime or not. From the definition of the Hecke operator it is clear that it is acting on the lattices, more exactly, it acts on equivalence classes between different lattices. Let now \( \tilde{f}(L) \) be some function defined on such lattices. We shall assume that this function is homogenous of degree \( -k \).

That is

\[ \tilde{f}(Z\omega_1 + Z\omega_2) = \omega_2^{-k} f(\omega_2/\omega_1). \quad \text{(C.4)} \]

The requirement of homogeneity makes this function invariant with respect to changes of the basis. Indeed, if we have change of the basis given by

\[ \begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} \]

then,

\[ \tilde{f}(Z\omega_1' + Z\omega_2') = \omega_1'^{-k} f(\omega_2'/\omega_1') = (c\omega_1 + d\omega_2)^{-k}(d + c(\omega_2/\omega_1))^{k} f(\omega_2/\omega_1) = \omega_2'^{-k} f(\omega_2/\omega_1), \quad \text{(C.5)} \]
provided that \( f \) is modular function of weight \( k \), that is
\[
f\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \tau = (c \tau + d)^k f(\tau) \quad \text{for} \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}(2, \mathbb{Z}). \tag{C.6}
\]
As consequence of Eqs.(C.4) and (C.5), function \( \tilde{f}(L) \) possess the homogeneity property
\[
\tilde{f}(\alpha L) = \alpha^{-k} \tilde{f}(L). \tag{C.7}
\]
Consider now action of the Hecke operator on function \( \tilde{f}(L) \). In the case if such a function is a modular form of weight \( k \) we obtain
\[
(T_k(n) f)(\tau) = n^{k-1} \sum_{L \subset L' \atop [L:L'] = n} \tilde{f}(L') = n^{k-1} \sum_{ad=n, a \geq 1} \sum_{0 \leq b < d} d^{-k} f\left( \frac{a \tau + b}{d} \right), \tag{C.8}
\]
where, following Ref.[89], we defined \( (T_k(n) f)(\tau) = \left( n^{k-1} T(n) \tilde{f} \right)(L) \). The result, Eq.(C.8), was obtained with help of Eqs.(C.3)and.(C6). In actual calculations one should take into account that the weight \( k \) should be only even number. The modular forms with odd weight are zero as can be easily shown. The factor \( n^{k-1} \) looks somewhat artificial at this point. Hence, we would like to explain its origin now.

The modular function \( f(\tau) \) admits the Fourier decomposition
\[
f(\tau) = \sum_{m} c(m) q^m,
\]
where \( q = \exp(2\pi i \tau) \). The eigenvalue problem
\[
(T_k(n) f)(\tau) = \lambda(n) f(\tau) \tag{C.9}
\]
in view of Eq.(C.8) can be written as follows
\[
\lambda(n) f(\tau) = n^{k-1} \sum_{ad=n, a \geq 1} \sum_{0 \leq b < d} d^{-k} \sum_{m \in \mathbb{Z}} c(m) \exp(2\pi im(a \tau + b)/d) \tag{C.10}
\]
Since
\[
\sum_{0 \leq b < d} e^{2\pi imb/d} = \begin{cases} d & \text{if } d \mid m \\ 0 & \text{if } d \nmid m \end{cases}
\]
we have to replace \( m \) by \( md = mn/a \) thus producing \( n/d = a \) resulting in cancellation of the prefactor \( n^{k-1} \). After this, Eq.(C.10) acquires the following form
\[
\lambda(n) f(\tau) = \sum_{m \in \mathbb{Z}} \sum_{ad=n, a \geq 1} a^{k-1} c\left( \frac{mn}{a} \right) e^{2\pi ima \tau}. \tag{C.11}
\]
The last result can be further rearranged thus producing
\[
\lambda(n) c(l) = \sum_{a \mid \gcd(l, n)} a^{k-1} c\left( \frac{nl}{a^2} \right). \tag{C.12}
\]
Suppose \( \gcd(l,n) = 1 \), then, surely, \( a = 1 \) which leads us to \( \lambda(n)c(l) = c(nl) \). Let further, \( l = 1 \). This produces \( \lambda(n)c(1) = c(n) \). From here it follows that \( c(1) \neq 0 \) and it can actually be taken as 1, Ref. [89], page 78 and Ref. [100], page 101. If this is so, then we are left with a very nice result: \( \lambda(n) = c(n) \), to be used in the main text. Moreover, we will also need the following result:

\[
\left( R_{\lambda} \tilde{f} \right)(L) = \tilde{f}(\lambda L) = \lambda^{-k} \tilde{f}(L).
\]

Using this result in equation

\[
\left( T(p^{e})T(p)\tilde{f} \right)(L) = \left( T(p^{e+1})\tilde{f} \right)(L) + \left( pT(p^{e-1})R_{p}\tilde{f} \right)(L)
\]

we obtain

\[
\left( T(p^{e})T(p)\tilde{f} \right)(L) = \left( T(p^{e+1})\tilde{f} \right)(L) + \left( p^{1-k}T(p^{e-1})\tilde{f} \right)(L).
\]

Using the definition \( (T_{k}(n)f)(\tau) = \left( n^{k-1}T(n)\tilde{f} \right)(L) \) in Eq. (C.15) and multiplying both sides by factor \( p^{(e+1)(k-1)} \) produces our final result

\[
T(p^{e})T(p)f = T(p^{e+1})f + p^{k-1}T(p^{e-1})f
\]

to be discussed from yet another (physical) point of view immediately below.

To this purpose we would like to remind our readers some excerpts from the graph theory. A graph \( G \) consists of finite set of vertices \( V \) and edges \( E \) along with the following two maps:

\[
E \rightarrow V \times V \quad \text{and} \quad e \rightarrow (o(e), t(e))
\]

so that \( \forall e \in E \) we may have \( e \rightarrow \hat{e} \) which means that the edge can change the orientation into opposite so that \( o(e) = t(\hat{e}) \). The element \( o(e) \) is the origin of \( e \) and \( t(e) \) is the terminus of \( e \). Two vertices \( v_{0} \) and \( v_{1} \) are adjacent if there is an edge with \( o(e) = v_{0} \) and \( t(e) = v_{1} \). The order of \( G \), denoted as \( |G| \), is determined by the number of vertices in \( G \). A walk \( W \) in \( G \) is a sequence of oriented edges \( e_{1}, \ldots, e_{r} \) such that \( t(e_{i}) = o(e_{i+1}) \) for \( 1 \leq i \leq r-1 \). If walk is from \( v_{0} = o(e_{1}) \) to \( v_{r} = t(e_{r}) \) it means that the walk is from \( v_{0} \) to \( v_{r} \) of length \( r \). We shall be interested only in walks without backtracking. For such walks, let \( |G| = n \) be the order of the graph then, the matrix \( A_{r} = \left( a_{i,j}^{(r)} \right) \) is a symmetric \( n \times n \) matrix with nonnegative integer entries counting the number of walks of length \( r \) from \( v_{i} \) to \( v_{j} \). By definition, \( A_{1} \) is the adjacency matrix [177]. The degree \( \deg v_{i} \) of the vertex \( v_{i} \) is the number of edges emanating at \( v_{i} \). Let \( D \) be the diagonal matrix with entries \( \deg v_{1}, \ldots, \deg v_{n} \) along the diagonal then, matrices \( A_{r} \) can be determined recursively from the following set of equations

\[
A_{1}A_{1} = A_{2} + D,
\]

\[
A_{r}A_{1} = A_{r+1} + A_{r-1}(D-I) \quad \text{for} \quad r \geq 2.
\]
The proof of this result can be found in Ref.[178]. In order to use these recursions we would like to make a few simplifications. First, we would like to consider only $k$-regular graphs (i.e. such for which degree of all vertices is the same and equal $k$). Next, let $k = p + 1$ for some prime $p$. And, finally, let $A_r = B_r - B_{r-2}$ for all $r \geq 1$. Then, the system of recursions given by Eq.(C.18) is reduced to

$$B_rB_1 = B_{r+1} + pB_{r-1} \text{ for } r \geq 1,$$  \hspace{1cm} (C.19)

provided that $B_{-1} = 0$, $B_0 = I$, $B_1 = A_1$. This result should be compared with Eq.(C.16) (for $k = 2$). Evidently, the total agreement is reached. It should be clear that the edges of $G$ represent equivalence classes between different lattices represented by vertices of $G$. The universal covering of such graph is $p + 1$-valent tree known in the literature as Bruhat-Tits tree [179]. Should our lattices come from the lattices whose base is made of more than 2 elements then, instead of trees, we would have buildings [179, 180]. Such necessity indeed occurs for higher dimensional complex tori (e.g. read Section 5) and had been discussed in connection with geometry of toric varieties (Section 5) by Mumford [181] already in 1970.

Because of the agreement between Eqs.(C.19) and (C.16), it is sufficient, in principle, to do all relevant physics ”locally”, that is for the specific prime $p$, in order to restore the ”global” picture for all primes. Although this idea is not new, as discussed in the Introduction, its implementation in the present context may still require considerable efforts. E.g. to rewrite this paper using the terminology of motives [38, 145] would require entirely new (and much more advanced) level of presentation. This task is left for the future.

For now we only would like to notice that since $A_1$ is the adjacency matrix, this means that the Hecke operators can be identified with the adjacency matrix of some graph $G$. This observation makes them already physical especially because the discrete (combinatorial) Laplacian $\Delta$ for $k$-regular graph $G$ is known to be

$$\Delta = kI - A_1.$$  \hspace{1cm} (C.20)

Harmonic analysis for such type of Laplacians is discussed in detail in Refs.[16, 17, 182-184]. Higher dimensional analogs of Hecke operators, Hecke algebras, etc. are discussed in Ref.[185].

**Appendix D. Veneziano amplitudes from the oscillatory integrals.**

In this appendix we would like to provide the condensed summary of known results for oscillatory integrals [36, 144]. Fortunately, these results are sufficient for reobtain the Veneziano amplitudes discussed in the main text.

Following Ref.[36], Ch-r 11, we are interested in study of integrals of the type

$$\int_{[\Gamma]} e^{\tau f(z)} \omega$$  \hspace{1cm} (D.1)

for

$$f(z) = \sum_{n=1}^{\infty} \frac{(\alpha n^2)}{n^2} \left(1 - \frac{1}{n} \right) z^n.$$
where the function \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) is holomorphic in a neighborhood of its only one critical point located at the origin, \([\Gamma]\) is \( n\)–dimensional homology chain associated with the cohomological differential \( n\)–form \( \omega \). The parameter \( \tau \) is expected to be positive. At the boundary of the chain \([\Gamma]\) the real part of the function \( f(z) \) is negative. The chain \([\Gamma]\) with such property is called admissible. We shall need the following identity (Lemma 11.2. of Ref.[36]) valid for large \( \tau \)'s:

\[
\int_{[\Gamma]} e^{\tau f(z)} \omega \approx \int_0^{t_0} e^{-\tau t} \left( \int_{\partial_{-\tau}[\Gamma]} \omega / df \right) dt
\]

(D.2)

with \( t_0 \) being a some positive number which eventually can be taken to the infinity. This result can be easily understood in physical terms. Indeed, let us consider an auxiliary integral

\[
I(t) = \int_{[\Gamma]} \delta(t - f(z)) \omega
\]

(D.3)

with \( \delta \) being the usual Dirac’s delta function. Such type of integrals formally occur in many calculations, e.g. in many-body quantum mechanical problems involving the density of states calculations, especially for the phonon spectra, etc. As it is shown in Ref.[148], Ch-r 4,

\[
I(t) = \int_{\partial_{-\tau}[\Gamma]} \omega / df
\]

(D.4)

so that for \( t_0 \to \infty \) the r.h.s. of the Eq.(D.2) is just the Laplace-like transform of the density of states.

Let now \( f(z) = z^{N+1} \) and \( \omega = z^{l-1} dz \) with \( 1 \leq l \leq N \). Then, \( I(t) \) is given by (equation (7) on page 305 of Ref.[36])

\[
I(t) = \text{const} |t|^{1/N+1}
\]

(D.5)

with \( \text{const} = \left( \frac{\zeta_j^l - \zeta_j^{l+1}}{N+1} \right) \). The phase factor \( \zeta_j^l \) plays the same role here as in Sections 3.2. and 5.2. Using this result in Eq.(D.2) and letting \( t_0 \) to become infinity produces

\[
\int_{[\Gamma]} e^{\tau f(z)} \omega \approx \text{const} \Gamma(-\alpha) \tau^{\alpha}
\]

(D.6)

with \( \alpha = \frac{l}{N+1} \). The result just obtained can be broadly generalized with help of the Fubini’s theorem which can be stated as follows:

\[
\int_{[\Gamma_1] \times [\Gamma_2]} e^{\tau (f+g)} \omega \wedge \eta = \int_{[\Gamma_1]} e^{\tau f} \omega \int_{[\Gamma_2]} e^{\tau g} \eta.
\]

(D.7)
If we combine this formula with the asymptotic result, Eq.(D.2), then, the Fubini’s theorem becomes a statement from the theory of Laplace transforms: the Laplace transform of the convolution is equal to the product of the Laplace transforms. Looking at Eq.(D.5) it is easy to write a convolution. With accuracy up to a constant we obtain

\[ I(\hat{\alpha} + \hat{\beta}) = \int_{0}^{t} ds \, |t - s|^{\hat{\alpha}} |s|^{\hat{\beta}} = t^{\hat{\alpha} + \hat{\beta} + 1} B(\hat{\alpha} + 1, \hat{\beta} + 1) \quad (D.8) \]

where \( \hat{\alpha} = \frac{l}{N+1} - 1, \hat{\beta} = \frac{m}{N+1} - 1, 1 \leq l, m \leq N \). Evidently, use of this result in the l.h.s. of Fubini’s relation superimposed with Eq.s(D.2-D.5) produces back Eq.(5.24a) of the main text. Since the results just described and many others can be found in the comprehensive paper by Varchenko, Ref. [144], there is no need to repeat here his arguments related to Hodge and mixed Hodge structures associated with such type of oscillatory integrals. It also should be clear from reading of the same reference that the result, Eq.(5.32), for the Alexander polynomials is connected with Fubini’s theorem just discussed.

Following Refs.[160,163] we shall call the exponent \( \alpha \) in Eq.(D.6) an order of the differential form \( \omega \). It is also called the spectrum of the critical point of the germ \( f(z) = z^{N+1} \). Let now we have two germs \( f \) and \( g \) with respective spectrum \( \{\alpha_i\} \) and \( \{\beta_j\} \) so that \( i = 1, ..., \mu \) and \( j = 1, ..., \eta \). Then the spectrum of the critical point of the germ \( f + g \) is obtained \( \{\alpha_i + \beta_j + 1\} \). For the composition of germs leading to the B-P variety, e.g.see Eq.s (3.7)-(3.8), the spectrum is made essentially of numbers which belong to the set \( X(S^1) \) (actually we have to divide these numbers by \( N \)) defined by Eq.(5.22a). In view of the Eq.s(3.17a) and (3.26) each number \( \alpha \) is also associated with the eigenvalue \( \lambda \) of the monodromy operator of the critical point, i.e. \( \lambda = \exp(\pm 2\pi i \alpha) \).

The dimensions of spaces of \((p,q)\) differential forms \( \omega \) defined by Eq.(5.9) are given by the Hodge numbers \( h^{p,q} \). These numbers had been related to the spectrum by Steenbink [169]. We follow, nevertheless, more recent and more clear exposition of this subject presented in Ref.[135] (Ref.[144] is also very helpful). Let \( p + q = m, m \leq n, \) and choose a pair of numbers \((\hat{\alpha}, l)\) where \( \hat{\alpha} \) is a spectral number \( \alpha = -\frac{1}{2\pi} \ln \lambda \) normalized by the level \( p, i.e. \) \( n-p-1 < \hat{\alpha} \leq n-p \), and \( l \) is the weight number given by

\[
 l = \begin{cases} 
 p + q & \text{if } \lambda \neq 1 \\
 p + q - 1 & \text{if } \lambda = 1 
\end{cases} \quad (D.9)
\]

Let \( h^p_q \) be the dimension (in physics terminology, the degeneracy) of the eigenspace related to the eigenvalue \( \lambda \). Then, the Hodge numbers \( h^{p,q} \) are determined simply by

\[ h^{p,q} = \sum_{\lambda} h^p_q(\lambda). \quad (D.10) \]

Taking into account the symmetry of the Hodge numbers (and also the numbers \( h^p_q(\lambda) \)):

\[ h^{p,q} = h^{q,p}, \quad h^p_q = h^{n-1-q,n-1-p} \quad \text{for } \lambda \neq 1 \quad (D.11a) \]
and

\[ h_1^{p,q} = h_1^{n-q,n-p} \text{ for } \lambda = 1 \]  

(D.11b)

and using this symmetry the following "sum rule" is obtained:

\[ \sum_{p,q} h_1^{p,q}(p + q + 1) + \sum_{p,q} h_1^{p,q}(p + q) = n\mu. \]  

(D.12a)

where the Milnor number \( \mu \) is defined in the main text (for the special case of B-P variety only!) by Eq.(5.54). In addition, there is yet another sum rule

\[ \sum_i \alpha_i = \mu\left(\frac{N}{2} - 1\right). \]  

(D.12b)

Consider now the simplest example based on analysis of the germ \( f(z) = z^{N+1} \). According to Eq.(D.6) we have \( \alpha = -\frac{l}{N+1} \) with \( 1 \leq l \leq N \). Clearly, \( |\alpha| < 1 \) so that \( \lambda \neq 1, \hat{\alpha} = \frac{l}{N+1} \). In view of the results presented above, we conclude that \( p = 0 \) and, due to symmetry, also \( q = 0 \). This produces \( h_0^{0,0} = 1 \) and \( h^{0,0} = N = \mu \). Let us consider less trivial example of germ of the Fermat-like curve. It is relevant to the discussion in Section 3.2. In particular, we had obtained the set of differential forms, Eq.(3.24), distinguished by the numbers \( r \) and \( s \) (or \( a = rN \) and \( b = sN \)). Replacing now \( N \) by \( N+1 \) and using Eq.(D.8) and the discussion which follows this equation, we obtain the spectrum as \( a + b - 1 \). Clearly, the factor of \(-1\) can be discarded since it does not carry any additional useful physical information. Hence, we are left with the spectral set \( \frac{m+n}{N+1} \) with \( 1 \leq m, n \leq N \). Using the arguments presented above the Hodge spectrum can be determined rather easily on a case by case basis. **Hence, the particle spectrum is in one to one correspondence with the Hodge numbers of the spectral set.** The alternative way of computation of Hodge numbers is presented in Section 5.3.2.

We would like to conclude this appendix by writing down the set of the Picard -Fuchs equations (e.g. see page 336 of Ref.[36]) for the B-P type of singularity (Sections 3 and 5). They are given below:

\[ \frac{dI^k}{dt} = (r_k - 1)\frac{I^k}{t}, \]  

(D.13)

where the subscript \( k = (k_0, ..., k_n) \) numbers different integrals of the type given by Eq.(D.3) and \( r_k \) is given by

\[ r_k = \frac{k_0 + ... + k_n}{N + 1}, \]

with \( 1 \leq k_0, ..., k_n \leq N \). 81
References

[1] G. Veneziano, Construction of a crossing symmetric, Regge-behaved amplitude for linearly rising trajectories, Il Nuovo Cimento 57A (1968) 190-197.
[2] V. De Alfaro, S. Fubini, G. Furlan, C. Rossetti, Currents in Hadron Physics, Elsevier, Amsterdam, 1973.
[3] M. Virasoro, Alternative construction of crossing-symmetric amplitudes with Regge behavior, Phys.Rev. 177 (1969) 2309-2314.
[4] A. Kholodenko, Statistical mechanics of 2+1 gravity from Riemann zeta function and Alexander polynomial: exact results, J.Geom.Phys. 38 (2001) 81-139.
[5] A. Kholodenko, Use of quadratic differentials for description of defects and textures in liquid crystals and 2+1 gravity, J.Geom.Phys. 33 (2000) 59-102.
[6] A. Kholodenko, Use of meanders and train tracks for description of defects and textures in liquid crystals and 2+1 gravity, J.Geom.Phys. 33 (2000) 23-58.
[7] M. Green, J. Schwarz, E. Witten, Superstring Theory, Vol.1, Cambridge University Press, Cambridge, 1987.
[8] Y. Imayoshi, M. Taniguchi, An Introduction to Teichmüller Spaces, Springer, Berlin, 1992.
[9] S. Nag, The Complex Analytic Theory of Teichmüller Spaces, Wiley Interscience, New York, 1998.
[10] L. Takhtajan, L-P. Teo, Liouville action and Weil-Petersson metric on deformation spaces, global Kleinian reciprocity and holography, math.CV/0204318.
[11] L. Bekke, P. Freund, p-adic numbers in physics, Phys.Reports 233 (1993) 1-66.
[12] E. Artin, Algebraic Numbers and Algebraic Functions, Gordon and Breach, London, 1967.
[13] A. Zabrodin, Non-Archimedean strings and Bruhat-Tits trees, Comm.Math.Phys.123 (1989) 463-483.
[14] S. Lang, Cyclotomic Fields, Springer-Verlag, Berlin, 1990.
[15] D. Thakur, On Gamma function for function fields, in: D. Goss, D. Hayes, M. Rosen (Eds.), The Arithmetic of Function Fields, de Gruyter, Berlin, 1992.
[16] A. Terras, Fourier Analysis on Finite Groups and Applications, Cambridge University press, Cambridge, 1999.
[17] W. Li, Number Theory With Applications, World Scientific, Singapore, 1996.
[18] G. Shimura and Y. Taniyama, Complex Multiplication of Abelian Varieties and its Applications to Number Theory, Publ.Math.Soc.Japan, Tokyo, 1961.
[19] M. Jacob (Editor), Dual Theory, Elsevier, Amsterdam, 1974.
[20] W. Heisenberg, Die beobachtbaren Grössen in der theorie der
Elementarteilchen, I, Z. Phys. 120 (1943) 513-538; ibid, II, 673-702.

[21] J. Bjorken, S. Drell, Relativistic Quantum Fields, Vol. 2, McGraw-Hill, New York, 1965.

[22] S. Novikov, S. Manakov, L. Pitaevskii, V. Zakharov, Theory of Solitons, Consultants Bureau, New York, 1984.

[23] D. Anosov, S. Aranson, V. Arnol’d, I. Bronstein, V. Grines, Yu. Il’yaschenko, Ordinary Differential Equations and Smooth Dynamical Systems, Springer-Verlag, Berlin, 1997.

[24] L. Landau, On analytic properties of vertex parts in quantum field theory, Nucl. Phys. 13 (1959) 181-192.

[25] F. Pham, Formules de Picard-Lefschetz generalisees et ramification des integrales, Bull. Soc. Math. France 93 (1965) 333-367.

[26] J. Milnor, Singular Points of Complex Hypersurfaces, Princeton University Press, Princeton, N.J., 1968.

[27] E. Brieskorn, Beispiele zur Differentialtopologie von Singularit"aten, Inv. Math. 2 (1966) 1-14.

[28] G. Ponzano, T. Regge, E. Speer, M. Westwater, The monodromy rings of a class of self-energy graphs, Comm. Math. Phys. 15 (1969) 83-132; ibid. 18 (1970) 1-64.

[29] J. Edwards, A Treatise on the Integral Calculus, Vol. 2, Macmillan, London, 1922.

[30] A. Varchenko, Euler’s Beta-function, Vandermonde determinant, Legendre equation and critical values of linear functions on hyperplane configurations, part I, Sov. Math. Doklady 53 (1989) 1206-1235; ibid, part II, 54 (1990) 146-155.

[31] S. Chowla, A. Selberg, On Epstein’s Zeta-function, J. Für die Reine und Angew. Math. 227 (1967) 86-110.

[32] A. Weil, Sur les periods des integrales Abeliennes, Comm. Pure and Appl. Math. 29 (1976) 813-819.

[33] A. Weil, Abelian varieties and the Hodge ring, in: Collected Papers, Vol. 3, Springer-Verlag, Berlin, 1979.

[34] B. Gross, On the periods of Abelian integrals and formula of Chowla and Selberg, Inv. Math. 45 (1978) 193-211.

[35] S. Lang, Introduction to Algebraic and Abelian Functions, Springer-Verlag, Berlin, 1982.

[36] V. Arnol’d, S. Gusein-Zade, A. Varchenko, Singularities of Differentiable Maps, Vol. 2, Birkhäuser, Boston, 1988.

[37] P. Deligne, Hodge cycles on Abelian Varieties, in: Lecture Notes in Math., Vol. 900, Springer-Verlag, Berlin, 1982.

[38] N. N. Schappacher, Periods of Hecke Characters, Lecture Notes in Math. Vol. 1301, Springer-Verlag, Belin, 1988.

[39] P. Bertolot, A. Ogus, Notes on Crystalline Cohomology, Mathematical Notes, Vol. 21, Princeton University Press, Princeton, N.J., 1978.

[40] D. McDuff, D. Salamon, Introduction to Symplectic Topology, Clarendon Press, Oxford, 1998.
[41] V.Vladimirov, I.Volovich, E.Zelenov, p-adic Analysis and Mathematical Physics, World Scientific, Singapore, 1994.
[42] D.Goldschmidt, Algebraic Functions and Projective Curves, Springer-Verlag, Berlin, 2003.
[43] M.Rosen, Number Theory in Function Fields, Springer-Verlag, Berlin, 2002.
[44] W.Gilbert, Modern Algebra With Applications, John Wiley&Sons, New York, 1976.
[45] H.Stichtenoth, Algebraic Function Fields and Codes, Springer-Verlag, Berlin, 1993.
[46] S.Lang, Undergraduate Algebra, Springer-Verlag, Berlin, 1990.
[47] B.Van der Waerden, Algebra, Springer-Verlag, Berlin, 1967.
[48] B.Mazur, On the passage from local to global in number theory, AMS Bulletin 29 (1993) 14-50.
[49] L.Childs, A Concrete Introduction to Higher Algebra, Springer-Verlag, Berlin, 1995.
[50] R.Bott, On the Shape of a Curve, Adv.Math.16 (1975)144-159.
[51] Y.Aurby, M.Perret, A Weil theorem for singular curves, in : R.Pellikaan, M.Perret, S.Vladut (Eds), Arithmetic,Geometry and Coding Theory, Walter de Gruyter, Berlin, 1996.
[52] V.Guillemin, A.Pollack, Differential Topology, Prentice Hall, New York, 1974.
[53] J.Milnor, Infinite cyclic coverings, in: Collected Papers, Vol.2, Publish or Perish, Inc., Houston, TX, 1995.
[54] A.Katok, B.Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, Cambridge University Press, Cambridge, 1995.
[55] J.Birman, M.Kidwell, Fixed points of pseudo-Anosov diffeomorphisms of surfaces, Adv.Math. 46 (1982) 217-220.
[56] N.Yui, Arithmetic of certain Calabi-Yau varieties and mirror symmetry, in : B.Conrad, K.Rubin (Eds), Arithmetic Algebraic Geometry, AMS Publications, Providence, RI, 2001.
[57] D.Cox, S.Katz, Mirror Symmetry and Algebraic Geometry, AMS Publications, Providence, RI, 1999.
[58] L.Kauffman, On Knots, Princeton University Press, Princeton, 1987.
[59] G.Brue, H.Zieschang, Knots, Walter de Gryter, Berlin, 1985.
[60] M.Armstrong, Basic Topology, Springer-Verlag, 1983.
[61] D.Rolfsen, Knots and Links, Publish or Perish, Houston, 1990.
[62] A.Grothendieck, Modeles de Neron et monodromie, in: LNM Vol.288, Springer-Verlag, Berlin, 1972.
[63] A.Wiles, Modular elliptic curves and Fermat’s last theorem, Ann.Math. 141 (1995) 443-551.
[64] A.Kholodenko, Kontsevich-Witten model rom 2+1 gravity: new exact combinatorial solution, J.Geom.Phys 43 (2002) 45-91.
[65] A.Weil. Number of solutions of equations in finite fields,
AMS Bulletin 55 (1949) 497-508.

[66] R.Miranda, Algebraic Curves and Riemann Surfaces, AMS Publications, Providence, RI, 1997.

[67] D.Rorlich, Points at infinity on the Fermat curves, Inv. Math. 39 (1977) 95-127.

[68] B.Gross, N.Koblitz, Gauss sums and the p-adic Γ-function, Ann.Math. 109 (1979) 569-581.

[69] N.Koblitz, p-adic Analysis: A short Course on Recent Work, Cambridge University Press, Cambridge, 1980.

[70] L.Landau, E.Lifshitz, Mechanics, Nauka, Moscow, 1965.

[71] G.Frey, Links between stable elliptic curves and certain Diophantine equations, Ann.Univ.Saraviensis, Ser.Math.1 (1986) 1-40.

[72] H.McKean, V.Moll, Elliptic Curves, Cambridge University Press, Cambridge, 1999.

[73] A.Kholodenko, Boundary conformal field theories, limit sets of Kleinian groups and holography, J.Geom.Phys. 35 (2000) 193-238.

[74] S.Chowla, A.Selberg, On Epstein’s zeta function, PNAS 35 (1949) 371-374.

[75] K.Ramachandra, Some applications of Kronecker’s limit formulas, Ann.Math. 80 (1964) 104-148.

[76] N.Akhieser, Elements of the Theory of Elliptic Functions, Nauka, Moscow, 1970.

[77] A.Kholodenko, Some thoughts about random walks on figure eight, Phys.A 289 (2001) 377-408.

[78] P.Etingof, I.Frenkel, A.Kirillov, Jr., Lectures on Representation Theory and Knizhnik-Zamolodchikov Equations, AMS Publications, Providence, RI, 1998.

[79] B.Bakalov, A.Kirillov, Jr., Lectures on Tensor Categories and Modular Functors, AMS Publications, Providence, R.I. 2001.

[80] A.Varchenko, Special Functions, KZ Type Equations and Representation Theory, Lecture Notes given at MIT in spring of 2002, Math.QA/0205313.

[81] C.Itzykson, J-M. Drouffe, Statistical Field Theory, Vol.1., Cambridge University Press, Cambridge, 1989.

[82] P.Di Francesco, P.Mathieu, D.Senechal, Conformal Field Theory, Springer-Verlag, Berlin, 1997.

[83] D.Husemöller, Elliptic Curves, Springer-Verlag, Berlin, 1987.

[84] G.Hardy, E.Wright, An Introduction to the Theory of Numbers, Clarendon, Oxford, 1962.

[85] J.Silverman, The Arithmetic of Elliptic Curves, Springer-Verlag, Berlin, 1986.

[86] Y.Manin, Real multiplication and noncommutative geometry,
[87] Y. Manin, M. Marcolli, Holography principle and arithmetic of algebraic curves, hep-th/0201036
[88] G. Jones, D. Singerman, Complex Functions, Cambridge University Press, Cambridge, 1987.
[89] J. Silverman, Advanced Topics in the Arithmetic of Elliptic Curves, Springer-Verlag, Berlin, 1994.
[90] H. Stark, L-functions at s=1. III. Totally real fields and Hilbert’s 12th problem, Adv. Math. 22 (1976) 64-84.
[91] R. Taylor, A. Wiles, Ring theoretic properties of some Hecke algebras, Ann. Math. 141 (1995) 553-572.
[92] K. Ribet, On modular representations of Gal (\overline{\mathbb{Q}}/\mathbb{Q}) arising from modular forms, Inv. Math. 100 (1990) 431-476.
[93] A. Knapp, Elliptic Curves, Princeton University Press, Princeton, 1992.
[94] S. Katok, J. Millson, Eichler-Shimura homology, intersection numbers and rational structures on spaces of modular forms, AMS Transactions 300 (1987) 737-757.
[95] R. Sharp, Closed geodesics and periods of automorphic forms, Adv. Math. 160 (2001) 205-216.
[96] B. Birch, H. Swinnerton-Dyer, Notes on elliptic curves. II. Jour. für Reine und Angew. Math. 218 (1965) 79-108.
[97] Y. Manin, Cyclotomic fields and modular curves, Russ. Math. Surv. 26 (1971) 7-71.
[98] E. Hecke, Lectures on the Theory of Algebraic Numbers, Springer-Verlag, Berlin, 1981.
[99] K. Ireland, M. Rosen, A Classical Introduction to Modern Number Theory, Springer-Verlag, Berlin, 1990.
[100] H. Iwaniec, Topics in Classical Automorphic Forms, AMS Publications, Providence, RI, 1997.
[101] A. Weil, Jacobi sums as ”Grossencharaktere”, AMS Transactions, 73 (1952) 487-495.
[102] N. Katz, Travaux de Dwork, Seminaire BOURBAKI, # 409, 1972.
[103] J. Cassels, Diophantine equations with special reference to elliptic curves, Jour. Lond. Math. Soc. 41 (1966) 193-291.
[104] A. Weil, Adeles and Algebraic Groups, Birkhauser, Boston, 1982.
[105] C. Maclachlan, A. Reid, The Arithmetic of Hyperbolic 3-Manifolds, Springer-Verlag, Berlin, 2003.
[106] C. Tracy, The emerging role of number theory in exactly solvable models in lattice statistical mechanics, Physica D 25 (1987) 1-19.
[107] Y. Zinoviev, The Ising model and the L-function, Theor. Math. Phys. 126 (2001) 66-80.
[108] P. Cohen, A C* dynamical system with Dedekind zeta partition function and spontaneous symmetry breaking,
109 J-B. Bost, A. Connes, Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking, Sel. Math. 1 (1995) 411-457.
110 A. Weil, Elliptic Functions According to Eisenstein and Kronecker, Springer-Verlag, Berlin, 1976.
111 L. Washington, Introduction to Cyclotomic Fields, Springer-Verlag, Berlin, 1997.
112 C. Moreno, The Chowla-Selberg formula, J. Number Theory 17 (1983) 226-245.
113 T. Asai, On a certain function analogous to $\ln|\eta(z)|$, Nagoya Math. J. 40 (1970) 193-211.
114 S. Lang, Elliptic Functions, Springer-Verlag, Berlin, 1987.
115 H. Cohn, Advanced Number Theory, Dover Publications, Inc., NY, 1962.
116 H. Tsukada, String Path Integral Realization of Vertex Operator Algebras, AMS Memoirs Vol.444, AMS Publications, Providence, RI, 1991.
117 B. Gross, Arithmetic on Elliptic Curves with Complex Multiplication, LNM 776, Springer-Verlag, Berlin, 1980.
118 S. Lang, Complex Multiplication, Springer-Verlag, Berlin, 1983.
119 J. Silverman, The theory of height functions, in: G. Cornell, J. Silverman (Eds), Arithmetic Geometry, Springer-Verlag, Berlin, 1986.
120 B. Mazur, Arithmetic on curves, AMS Bulletin 14 (1986) 207-259.
121 P. Colmez, Periodes des varietes abeliennes a multiplication complexe, Ann. Math. 138 (1993) 625-683.
122 S. Zhang, Heights of Heegner points on Shimura curves, Ann. Math. 153 (2001) 27-147.
123 J. Polchinski, String Theory, Vol.1, Cambridge University Press, Cambridge, 1998.
124 A. Beilinson, Y. Manin, The Mumford form and the Polyakov measure in string theory, Comm. Math. Phys. 107 (1986) 359-376.
125 Y. Manin, The partition function of the string can be expressed in terms of theta functions, Phys. Lett. B 172 (1986) 184-188.
126 J. Bost, T. Jolicoeur, A holomorphy property and the critical dimension in string theory from an index theorem, Phys. Lett. B 174 (1986) 273-276.
127 J-D. Smit, String theory and algebraic geometry of moduli spaces, Comm. Math. Phys. 114 (1988) 645-685.
128 S. Mochizuki, Foundations of $p$-adic Teichmüller Theory, AMS Publications, Providence, RI, 1999.
129 A. Ogus, Frobenius and Hodge spectral sequence, Adv. in Math. 162 (2001) 141-172.
130 A. Ogus, Elliptic crystals and modular motives, Adv. in Math. 162 (2001) 173-216.
[131] K.Krasnov, Holography and Riemann surfaces, Adv. in Theor.Math.Phys. 4 (2000) 929-979.
[132] P.Griffiths, On periods of certain rational integrals: I, Ann.Math. 90 (1969) 460-495; ibid II, 496-541.
[133] Y.Manin, Algebraic curves over fields with differentiation, AMS Translations 37 (1964) 59-78.
[134] Y.Shimizu, K.Ueno, Advances in Moduli Theory, AMS Translations 206 (2002) 1-175.
[135] V.Kulikov, Mixed Hodge Structures and Singularities, Cambridge University Press, Cambridge, 1998.
[136] S.Lefshetz, Applications of Algebraic Topology: Graphs and Networks, The Picard-Lefshetz Theory and Feynman Integrals, Springer-Verlag, Berlin, 1975.
[137] V.Schechtman, A.Varchenko, Arrangement of hyperplanes and Lie Algebra homology, Inv.Math. 106 (1991) 139-194.
[138] E.Brieskorn, H.Knorrer, Plane Algebraic Curves, Birkhäuser, Boston, 1986.
[139] R.Wells, Differential Analysis on Complex Manifolds, Prentice Hall, Inc., NJ, 1973.
[140] A.Weil, Introduction a L’étude des Varietes Kähleriennes, Hermann, Paris, 1958.
[141] V.Arnol’d, Mathematical Methods of Classical Mechanics, Nauka, Moscow, 1974.
[142] V.Arnol’d, A.Givental, Symplectic Geometry, in: V.Arnol’d, S.Novikov (Eds), Dynamical Systems IV, Springer-Verlag, Berlin, 1990.
[143] H.Pohlmann, Algebraic Cycles on Abelian varieties of complex multiplication type, Ann.Math. 88 (1968) 161-180.
[144] A.Varchenko, Asymptotic Hodge structure in the vanishing cohomology, Math.USSR, Izvestija 18 (1982) 469-512.
[145] U.Jannsen, S.Kleiman, J-P Serre (Eds), Motives, Parts 1 and 2, AMS Publications, Providence, RI, 1994.
[146] P.Griffiths, Periods of integrals on Algebraic manifolds, I. (Construction and properties of modular varieties), Am.J.of Math. 90 (1968) 568-626.
[147] P.Griffiths, Periods of integrals on algebraic manifolds, II. (Local study of the period mapping), Am.J.of Math. 90 (1968) 801-865.
[148] R.Hwa, V.Teplitz, Homology and Feynman Integrals, W.A.Benjamin, Inc., NY, 1966.
[149] D.Eisenbud, W.Neumann, Three-Dimensional Link Theory and Invariants of Plane Curve Singularities, Princeton University Press, Princeton, 1985.
[150] LeTrang , Sur les noeuds algebriques, Compositio Math.25 (1972) 281-321.
[151] D.Sumners, J.Woods, The monodromy of reducible plane
curves, Inv.Math.40 (1977) 107-141.
[152] J.Franks, Homology and Dynamical Systems, Regional Conf.
Series in Math.49, AMS Publications, Providence, RI, 1982.
[153] M.Sebastiani, R.Thom, Un resultat sur la monodromie,
Inv.Math.13 (1971) 90-96.
[154] M.Oka, On the homotopy types of hypersurfaces defined
by weighted homogenenous polynomials, Topology 12 (1973)19-32.
[155] N.Saveliev, Lectures on the Topology of 3-Manifolds,
Walter de Gruyter, Belin, 1999.
[156] N.A’Campo, La fonction zeta d’une monodromie,
Comm.Math.Helv. 50 (1975) 233-248.
[157] K.Murasugi, Knot Theory and its Applications,
Birkhäuser, Boston, 1996.
[158] A.Kwanchi, A Survey of Knot Theory, Birkhäuser, Boston, 1996.
[159] I.Gelfand, M.Kapranov, A.Zelevinski, Discriminants, Resultants and
Multidimensional Determinants, Birkhäuser, Boston, 1994.
[160] V.Buchstaber, T.Panov, Torus Actions and Their Applications
in Topology and Combinatorics, Univ.Lect.Series Vol.24,
AMS Publications, Providence, RI, 2002.
[161] V.Arnol’d, V.Vaasil’ev, V.Goryunov, O.Lyashko,
Singularities, local and global theory, in V.Arnold (Editor),
Dynamical Systems VI, Springer-Verlag, Berlin, 1993.
[162] T.Zaslavsky, Facing up to Arrangements: Face-count
Formulas for Partitions of Space by Hyperplanes, AMS
Memoirs, Vol 154, AMS Publications, Providence, RI, 1975.
[163] P.Orlik, H.Terao, Arrangements and Hypergeometric Integrals,
Math.Sci.Japan Memoirs, Vol.9, Tokyo, 2001.
[164] V.Batyrev, Variations of the mixed Hodge structure of affine
hypersurfaces in algebraic tori, Duke Math.journ. 69 (1993) 349-409.
[165] A.Koushirenko, Polyedres de Newton et nombres de Milnor,
Inv.Math.32 (1976) 1-31.
[166] S.Salamon, On the cohomology of Kähler and hyper-Kähler
manifolds, Topology 35 (1996) 137-155.
[167] V.Danilov, A.Khovanskii, Newton polyhedra and an algorithm
for computing Hodge-Deligne numbers,
Math.USSR-Izv. 29 (1987) 279-298.
[168] D.Cox, Recent developments in toric geometry, math.AG/9606016.
[169] J.Steenbrink, Mixed Hodge structure on vanishing cohomology,
Proc.Nordic Summer Scool, Oslo, 1976.
[170] M.Saito, On Steenbrink’s conjecture, Math.Ann.289 (1991) 703-716.
[171] A.Varchenko, A.Khovanskii, Sov.Math.Dokl. 32 (1985) 122-127.
[172] J.Nielsen, Untersh hungenzur topologie der geslossen zweisichtig en
flachen I, Acta Math.50 (1927) 189-358.
[173] J.Milnor, Dynamics in One Complex Variable, Vieweg, Wiesbaden, 1999.
[174] W de Mello, S.van Strien, One-Dimensional Dynamics, Springer-Verlag, Berlin, 1993.
[175] M.Kontsevich, Intersection theory on the moduli space of curves and the matrix Airy function, Comm.Math.Phys. 147 (1992) 1-23.
[176] F.Tomi, A.Tromba, The Index Theorem for Minimal Surfaces of Higher Genus, AMS Memoirs 117 (1995) 1-78.
[177] C.Godsil, G.Royle, Algebraic Graph Theory, Springer-Verlag, Berlin, 2001.
[178] A.Pitzer, Ramanujan Graphs, Studies in Adv.Math. 7 (1998) 159-178.
[179] J-P. Serre. Trees, Springer-Verlag, Berlin, 1980.
[180] K.Brown, Buildings, Springer-Verlag, Berlin, 1991.
[181] D.Mumford, Further applications, LNM 339 (1973) 165-190.
[182] A.Figa-Talamanca, C.Nebbia, Harmonic Analysis and Representation Theory for Groups Acting on Homogenous Trees, Cambridge University Press, Cambridge, 1991.
[183] A.Lubotsky, Discrete Groups, Expanding Graphs and Invariant Measures, Birkhäuser, Boston, 1994.
[184] P.Diaconis, Group Representations in Probability and Statistics, Inst.of Math.Statistics Lecture Notes Series, Vol.11, Hayward, CA, 1988.
[185] A.Kreig, Hecke Algebras, AMS Memoirs, Vol.435, AMS Publications, Providence, RI, 1990.
[186] M.Kontsevich, D.Zagier, Periods, in: B.Engquist, W.Schid (Eds), Mathematics Unlimited, Springer-Verlag, Berlin, 2001.
[187] E.Frielander, A.Suslin, The work of Vladimir Voevodsky, AMS Notices 50 (2003) 214-217.