Genuine hidden nonlocality without entanglement: from the perspective of local discrimination

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Quantum nonlocality without entanglement is a fantastic phenomenon in quantum theory. This kind of quantum nonlocality is based on the task of local discrimination of quantum states. Recently, Bandyopadhyay and Halder [Phys. Rev. A \textbf{104}, L050201 (2021)] studied the problem: is there any set of orthogonal states which can be locally distinguishable, but under some orthogonality preserving local measurement, each outcome will lead to a locally indistinguishable set. We say that the set with such property has hidden nonlocality. Moreover, if such phenomenon cannot arise from discarding subsystems which is termed as local irredundancy, we call it genuine hidden nonlocality. There, they presented several sets of entangled states with genuine hidden nonlocality. However, they doubted the existence of a set without entanglement but with genuine hidden nonlocality. In this paper, we eliminate this doubt by constructing a series of sets without entanglement but whose nonlocality can be genuinely activated. We derive a method to tackle with the local irredundancy problem which is a key tricky for the systems whose local dimensions are composite numbers. Unexpectedly, the constructions of genuine hidden nonlocal sets without entanglement seems to be easier than that with entanglement. Therefore, this kind of nonlocality is rather different from the Bell nonlocality.

I. INTRODUCTION

The predictions of quantum theory are incompatible with those based on the concept of locality [1, 2]. Such phenomenon is known as quantum nonlocality. Quantum nonlocality is usually revealed by violating Bell type inequalities [3–10] which can only arise from entangled states. The nonlocality arising from this way is termed as Bell nonlocality [1–3]. It is well-known that entanglement [11] cannot be generated from separable states under local operations. However, there exists some states which admit a local hidden variable model (hence is local) but violating Bell type inequalities after judicious local filters are applied. This kind of states is said to have hidden nonlocality [12–17].

Except the Bell nonlocality, there are other forms of nonlocality that have attracted our attention. In fact, the local indistinguishability of some set of orthogonal quantum states has been widely used to illustrate the phenomenon of quantum nonlocality. In this setting, two or more observers share the parts of a composite quantum system prepared in a state from a known orthogonal set. The task is to identify the state by performing local operations and classical communication (LOCC). Note that we can always identify the state correctly if global measurements are allowed as the set of states are assumed to be orthogonal. If such task can be accomplished perfectly under LOCC, we say that the set is locally distinguishable, otherwise, locally indistinguishable.

Bennett et al. [18] presented the first example of orthogonal product states that are locally indistinguishable and named such a phenomenon as quantum nonlocality without entanglement. The nonlocality here is in the sense that there exists some quantum information that could be inferred from global measurement but cannot be read from local correlations of the subsystems. Since then, this kind of nonlocality has been studied extensively (see Refs. [18–67] for an incomplete list). Most studies are focused on identifying sets of orthogonal product states or sets of the maximally entangled states. In addition, a stronger manifestation of nonlocality in multipartite systems has been studied which is based on the notion of local irreducibility in all bipartitions of subsystems [68–75]. An orthogonal set of pure states in multipartite quantum system is called locally irreducible if it is not possible to eliminate one or more states from the set by orthogonality-preserving local measurements.

The local indistinguishability of quantum states has been practically applied in quantum cryptography primitives such as data hiding [76, 77] and secret sharing [78–80]. Therefore, local indistinguishability of quantum states can be considered as a resource in quantum information processing. If there are only locally distinguishable sets at hand, how can we transfer them into resources that have applications in data hiding? This is what Bandyopadhyay and Halder [81] recently studied. In fact, they studied the problem: is there any set of orthogonal states which can be locally distinguishable, but under some orthogonality preserving local measurement, each outcome will lead to a locally indistinguishable set. As there are some trivial sets with this property, they introduced the concept of local irredundancy. An orthogonal set is said to be locally redundant if it remains orthogonal after discarding one or more subsystems. Otherwise, it is called to be locally irredundant. If a locally irredundant set satisfies the aforementioned property, then we said its nonlocality can be activated genuinely. There, they give several examples of such sets with entanglement. However, deeper research on this property remains to be explored. For example, is there any locally distinguishable set without entanglement whose nonlocality can be genuinely activated? Which state spaces have sets with this property? In this paper, we tend to solve the two problems partially.

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The rest of this article is organized as follows. In Sec. II, we review the concept of genuine activation of nonlocality (genuine hidden nonlocality). Then we study the constructions of sets with genuine hidden nonlocality with respect to two cases: this first is cardinality preserving while the second is cardinality decreasing. In Sec. III, we present examples for the first case and discuss its constructions in multipartite systems. In Sec. IV, we give an example for the second case and also and discuss its constructions in multipartite systems. Finally, we draw a conclusion in section V.

II. PRELIMINARY

It is well known that a set of quantum states is perfectly distinguished by global measurement if and only if the given set of states are mutually orthogonal. Therefore, every LOCC protocol that distinguishes a set of orthogonal states is a sequence of orthogonality-preserving-local-measurements (OPLM). In this paper, we consider this specific class of LOCC measurements, i.e., OPLM.

Now we give a brief review of the main problem studied in Ref. [81]. Suppose that \( \mathcal{S} \) is an orthogonal set of multipartite states which is locally distinguishable. The participants aim to find some OPLM \( \mathcal{M} \) such that for each outcome \( \mu \) of this measurement, they end up to a new orthogonal set \( \mathcal{S}'_\mu \) which is locally indistinguishable. If we could find such measurement, we call that the nonlocality of \( \mathcal{S} \) can be activated (Compared with setting in the Bell nonlocality, we also call that \( \mathcal{S} \) has hidden nonlocality here). However, the authors in Ref. [81] have pointed out that there are some trivial cases which arise from taking partial trace. Such cases would happen if the original set is local redundancy, i.e., the set remains orthogonal if we discard one or more subsystems. If the set \( \mathcal{S} \) is locally irredundant and satisfies the aforementioned property, we can call that the nonlocality of \( \mathcal{S} \) can be genuinely activated (And we also call that \( \mathcal{S} \) has genuine hidden nonlocality here). Note that the cardinality of \( \mathcal{S}'_\mu \) always satisfies \( |\mathcal{S}'_\mu| \leq |\mathcal{S}| \). According to the differences of the cardinality, we separate the genuine hidden nonlocality into two types.

(a) Genuine hidden nonlocality of type I: \( \mathcal{S} \) is locally distinguishable and locally irredundant. And there is some OPLM \( \mathcal{M} \) such that for each outcome \( \mu \) of this measurement, the set of post-measurement states \( \mathcal{S}'_\mu \) is locally indistinguishable and \( |\mathcal{S}'_\mu| = |\mathcal{S}| \). Moreover, if for each outcome \( \mu \) of this measurement, the set of post-measurement states \( \mathcal{S}'_\mu \) is locally irreducible, we call it has genuine hidden strong form nonlocality of type I.

(b) Genuine hidden nonlocality of type II: \( \mathcal{S} \) is locally distinguishable and locally irredundant. And there is some OPLM \( \mathcal{M} \) such that for each outcome \( \mu \) of this measurement, the set of post-measurement states \( \mathcal{S}'_\mu \) is locally indistinguishable. In addition, there is some outcome \( \mu \) such that \( |\mathcal{S}'_\mu| < |\mathcal{S}| \).

In this paper, we will study how to construct sets without entanglement but with genuine hidden nonlocality of type I or II. Throughout this paper, we will use the following notations. Let \( d \geq 2 \) be an integer. Considering a quantum system \( \mathcal{H} \) of dimension \( d \), we usually denote \( \{ |0\rangle, |1\rangle, \ldots, |d-1\rangle \} \) as its a computational basis. We also use the notation \( |i+j\rangle \) (w.r.t. \( |i-j\rangle \)) which presents \( |i\rangle + |j\rangle \) (w.r.t. \( |i\rangle - |j\rangle \)). Moreover, we use \( |+\rangle = \sum_{i=0}^{d} |i\rangle \) and \( |m\rangle = \sum_{i=m}^{d} |i\rangle \) where \( m < n \). We also use \( I_d \) represent the identity operator on this system, i.e., \( I_d = \sum_{i=0}^{d-1} |i\rangle \langle i| \). And the states throughout this paper may be unnormalized.

III. GENUINE HIDDEN NONLOCALITY OF TYPE I

In this section, we study how to construct sets without entanglement but have genuine hidden nonlocality of type I. We will use the local indistinguishability of the sets without entanglement as follows.

**Theorem 1** ([42]). In \( \mathbb{C}^d \otimes \mathbb{C}^d \) system with \( d \geq 3 \), the following \( 2d-1 \) orthogonal pure product states are locally indistinguishable:

\[
\{|n\rangle |\delta_n\rangle\}_{n=1}^{d-1} \cup \{|\delta_n\rangle |n\rangle\}_{n=1}^{d-1} \cup \{|+(d-1)\rangle |+(d-1)\rangle\}
\]

where \( |\delta_n\rangle \equiv |0-n\rangle \) and \( n_+ = n+1 \) for \( 1 \leq n \leq d-2 \) while \( n_+ = 1 \) for \( n = d-1 \).

Now let’s start with the following example, the construction of whose states is inspired from the above Theorem for the cases \( \mathbb{C}^5 \otimes \mathbb{C}^5 \) and \( \mathbb{C}^6 \otimes \mathbb{C}^6 \) (see Fig. 1 for their structures).
Example 1. Let $\mathcal{S}$ be the set of product states in $\mathbb{C}^{11} \otimes \mathbb{C}^{11}$ whose elements are listed as below (see Fig. 2):

$$
\begin{align*}
|\psi_1\rangle &= |1\rangle|0 - 1 + 9 - 10\rangle, \\
|\psi_2\rangle &= |2\rangle|0 - 2 + 8 - 10\rangle, \\
|\psi_3\rangle &= |3\rangle|0 - 3 + 7 - 10\rangle, \\
|\psi_4\rangle &= |4\rangle|0 - 4 + 6 - 10\rangle, \\
|\phi_1\rangle &= |6\rangle|5 - 6 + 2 - 3\rangle, \\
|\phi_2\rangle &= |7\rangle|5 - 7 + 3 - 4\rangle, \\
|\phi_3\rangle &= |8\rangle|5 - 8 + 2 - 4\rangle, \\
|\phi_4\rangle &= |9\rangle|5 - 9 + 1 - 4\rangle, \\
|\phi_5\rangle &= |10\rangle|5 - 10 + 0 - 4\rangle.
\end{align*}
$$

Figure 1. This shows two examples of the structures of the product states constructed in Theorem 1 in $\mathbb{C}^5 \otimes \mathbb{C}^5$ and $\mathbb{C}^6 \otimes \mathbb{C}^6$. The squares indicated by the same label represent a unique state. For example, there are two squares (that is, $(2,0)$ and $(2,2)$) with label ‘$h_2$’ in the left hand side graph, they correspond to the state $|2\rangle|0 - 2\rangle \in \mathbb{C}^5 \otimes \mathbb{C}^5$ and there are two squares (that is, $(0,5)$ and $(4,5)$) with label ‘$V_3$’ in the right hand side graph, they correspond to the state $|0 - 4\rangle|5\rangle \in \mathbb{C}^6 \otimes \mathbb{C}^6$.

Figure 2. This shows the states structure of set of product states in Example 1. The squares indicated by the same label represent a unique state. For example, there are four squares (that is, $(1,0), (1,1), (1,9)$, and $(1,10)$) with label ‘$h_1$’, they correspond to the state $|\psi_1\rangle := |1\rangle|0 - 1 + 9 - 10\rangle$, and there are four squares (that is, $(5,2), (5,8), (7,2)$, and $(7,8)$) with label ‘$V_3$’, they correspond to the state $|\phi_5\rangle := |5 - 7\rangle|8 - 2\rangle$. The first five rows are axisymmetric about the middle column, and the first two columns and the last two columns of the last six rows are also axisymmetric about the middle column.
and $|\psi_9\rangle \equiv |+4\rangle|+10\rangle$, and $|\phi_{11}\rangle \equiv |5+10\rangle|+10\rangle$. Then the set $S$ has genuine hidden nonlocality of type I.

Now we show that $S$ is locally distinguishable. For each $|\Theta\rangle = |\theta_1\rangle|\theta_2\rangle \in S$, we denote

$$C(|\Theta\rangle) := \{(i,j) \in Z_{11} \times Z_{11} | \langle i|\theta_1\rangle \langle j|\theta_2\rangle \neq 0\}.$$  

We call it the coordinates of the state $|\Theta\rangle$. Note that except the states $|\psi_9\rangle$ and $|\phi_{11}\rangle$, the coordinates of other states in $S$ are all of cardinality 4 and these states can be formed into two types: their coordinates lie on the same line and are labeled with $h_i$ or $H_i$ in Fig. 2 (For examples, $C(|\psi_4\rangle) = \{(1,0),(1,1),(9,0),(10,0)\}$ and their coordinates lie on the four corners of a rectangle and are labeled with $v_j$ or $V_j$ in Fig. 2 (For example, $C(|\phi_5\rangle) = \{(5,3),(5,6),(10,3),(10,6)\}$). Moreover, for different states $|\Theta_1\rangle, |\Theta_2\rangle \in S \setminus \{|\psi_9\rangle, |\phi_{11}\rangle\}$, their coordinates are disjoint, i.e., $C(|\Theta_1\rangle) \cap C(|\Theta_2\rangle) = \emptyset$.

First, Alice perform the measurement $M_A^i := \{\pi^i = |i\rangle\langle i| | i \in Z_{11}\}$. If the outcome of $M_A^i$ is ‘i’, by the construction of the set $S$ and the above observation, the postmeasurement states of Bob’s part are mutually orthogonal and hence can be distinguished. For example, if the outcome of $M_A^2$ is ‘5’, the state must be one of $\{|\phi_7,8,9,10,11\rangle\}$ whose Bob’s part are the following five orthogonal states $|--4\rangle,|8-2\rangle,|9-1\rangle,|10-0\rangle,|+10\rangle$ respectively.

Now we prove that the set $S$ satisfies the second property. Suppose Bob perform the measurement $M_B^i := \{\pi^i = \sum_{i=0}^d |i\rangle\langle i|, \pi^i = \sum_{j=5}^{10} |j\rangle\langle j|\}$.

If the outcome of $M_B^i$ is ‘1’, the states are transferred to

$$|\psi_1\rangle = |1\rangle|0-1\rangle, \quad |\psi_2\rangle = |0\rangle|0-2\rangle, \quad |\psi_3\rangle = |3\rangle|0-3\rangle, \quad |\psi_4\rangle = |4\rangle|0-4\rangle, \quad |\psi_5\rangle = |0\rangle|0-4\rangle, \quad |\psi_7\rangle = |0\rangle|0-2\rangle, \quad |\phi_1\rangle = |0\rangle|0-3\rangle, \quad |\phi_2\rangle = |0\rangle|0-3\rangle, \quad |\phi_3\rangle = |8\rangle|2-4\rangle, \quad |\phi_4\rangle = |5\rangle|5-7\rangle, \quad |\phi_5\rangle = |10\rangle|0-4\rangle, \quad |\phi_6\rangle = |+4\rangle|+10\rangle, \quad |\phi_{11}\rangle = |+4\rangle|+10\rangle$$

which are mutually orthogonal and contain 9 states $\{|\phi_i\rangle\}_{i=1}^9$ that is known to be locally indistinguishable in $C^5 \otimes C^5$.

If the outcome of $M_B^i$ is ‘2’, the states are transferred to

$$|\psi_1\rangle = |1\rangle|9-10\rangle, \quad |\psi_2\rangle = |0\rangle|8-10\rangle, \quad |\psi_3\rangle = |3\rangle|7-10\rangle, \quad |\psi_4\rangle = |4\rangle|6-10\rangle, \quad |\psi_5\rangle = |5\rangle|5-6\rangle, \quad |\psi_6\rangle = |0\rangle|0-3\rangle, \quad |\phi_1\rangle = |6\rangle|5-6\rangle, \quad |\phi_2\rangle = |7\rangle|5-7\rangle, \quad |\phi_3\rangle = |8\rangle|5-8\rangle, \quad |\phi_4\rangle = |9\rangle|5-9\rangle, \quad |\phi_5\rangle = |10\rangle|5-10\rangle, \quad |\phi_6\rangle = |+4\rangle|5+10\rangle, \quad |\phi_{11}\rangle = |+4\rangle|5+10\rangle$$

which are mutually orthogonal and contain 11 states $\{|\phi_i\rangle\}_{i=1}^{11}$ that is known to be locally indistinguishable in $C^6 \otimes C^6$.

The above construction and argument can be easily extend to $C^d \otimes C^d$ for any prime number $d \geq 11$. Here we give one more example without proof.

**Example 2.** Let $S$ be the set of product states in $C^{13} \otimes C^{13}$ whose elements are listed as below (see Fig. 3):

$$|\psi_1\rangle \equiv |1\rangle|0-1+1+12\rangle, \quad |\psi_6\rangle \equiv |0\rangle|0-5\rangle|1-12\rangle, \quad |\psi_2\rangle \equiv |0\rangle|0-2+10\rangle, \quad |\psi_7\rangle \equiv |0\rangle|0-1\rangle|2-10\rangle, \quad |\psi_3\rangle \equiv |3\rangle|0-3+9\rangle, \quad |\psi_6\rangle \equiv |0\rangle|0-2\rangle|3-9\rangle, \quad |\psi_4\rangle \equiv |4\rangle|0-4+8\rangle, \quad |\psi_6\rangle \equiv |0\rangle|0-3\rangle|4-8\rangle, \quad |\psi_5\rangle \equiv |5\rangle|0-5+7\rangle, \quad |\psi_{10}\rangle \equiv |0\rangle|0-4\rangle|5-7\rangle, \quad |\phi_1\rangle \equiv |7\rangle|6-7+3\rangle, \quad |\phi_2\rangle \equiv |6\rangle|6-2\rangle|7-4\rangle, \quad |\phi_3\rangle \equiv |8\rangle|6-8+4\rangle, \quad |\phi_6\rangle \equiv |6\rangle|6-7\rangle|8-5\rangle, \quad |\phi_4\rangle \equiv |9\rangle|6-9+3\rangle, \quad |\phi_6\rangle \equiv |6\rangle|6-8\rangle|9-3\rangle, \quad |\phi_5\rangle \equiv |10\rangle|6-10+2\rangle, \quad |\phi_{10}\rangle \equiv |6\rangle|6-9\rangle|10-2\rangle, \quad |\phi_5\rangle \equiv |11\rangle|6-11+1\rangle, \quad |\phi_{11}\rangle \equiv |6\rangle|6-10\rangle|11-1\rangle, \quad |\phi_6\rangle \equiv |12\rangle|6-12+0\rangle, \quad |\phi_{12}\rangle \equiv |6\rangle|6-11\rangle|12-0\rangle.$$
and \(|\psi_{11}\rangle \equiv |+5\rangle |+12\rangle\), and \(|\phi_{13}\rangle \equiv |6+12\rangle |+12\rangle\). Then the set \(\mathcal{S}\) has genuine hidden nonlocality of type I.

For further discussion in the composite dimension, we need the following lemma.

**Lemma 1.** Let \(\mathcal{N}\) be a quantum channel from system \(\mathcal{H}_A\) to system \(\mathcal{H}_B\). Then \(\mathcal{N}\) preserves non-orthogonality. That is, for any density operators \(\rho, \sigma\) of systems \(\mathcal{H}_A\),

\[
\langle \rho, \sigma \rangle_A \neq 0 \Rightarrow \langle \mathcal{N}(\rho), \mathcal{N}(\sigma) \rangle_B \neq 0.
\]

Here, the inner product \(\langle \rho, \sigma \rangle_A := \text{Tr}[\rho^\dagger \sigma]\). Particularly, if we choose \(\mathcal{N}\) to be the partial trace operation, then we have that the partial trace operation preserves non-orthogonality.

Lemma 1 can be understood as a corollary of the well known result: a set of quantum states is perfect distinguishable under global measurement if and only if the given set of states are mutually orthogonal. For the sake of completeness, we give a direct proof in the APPENDIX.

**Method for checking local irredundancy:** Now we return to discuss the genuine activation of nonlocality in general dimension. Suppose \(d = 2k + 1 \geq 11\), we can construct the set \(\mathcal{S}\) to be the union of \(|\psi_i\rangle\}_{i=1}^{2k-1}\) and \(|\phi_i\rangle\}_{i=1}^{2k-1}\) similarly to Example 1 and Example 2. In the following, we show that even for the case of composite number \(d\), the above set of states is locally irredundant whenever \(d \geq 11\). Let \(d = p_1 p_2 \cdots p_n\), where \(p_i\) is some prime divisor of \(d\) (here we assume that \(3 \leq p_1 \leq p_2 \leq \cdots \leq p_n\) and \(n \geq 2\)). Suppose that \(\mathcal{H}_X = \mathbb{C}^d\), it can be factored into an \(n\)-particles systems \(\otimes_{i=1}^n \mathcal{H}_{X_i}\) where \(\mathcal{H}_{X_i} = \mathbb{C}^{p_i}\) and \(X \in \{A, B\}\). In fact, we will show that Alice and Bob can not preserve the orthogonality of the given states by discarding one or more of their subsystems. Written \(|\psi_i\rangle = |\psi_{iA}\rangle |\psi_{iB}\rangle\) for \(i = 1, 2, \cdots, 2(k-1)\), one observes that

\[
\langle \psi_i | \psi_j \rangle_B = 0, \langle \psi_{k-i+1} | \psi_{k-i+j} \rangle_A \neq 0
\]

for different \(i, j \in \{0, 1, \cdots, k-1\}\). If the set is local redundancy, there must exist not all empty sets \(\mathcal{S}_A \subseteq \{A_1, A_2, \cdots, A_n\}\), \(\mathcal{S}_B \subseteq \{B_1, B_2, \cdots, B_n\}\) and some unitaries \(U_A, U_B \in U(d)\), such that

\[
\text{Tr}_{\mathcal{S}_A \cup \mathcal{S}_B}[(U_A \otimes U_B) |\psi_i\rangle \langle \psi_i | (U_A^\dagger \otimes U_B^\dagger)]_{i=1}^{2k-2}
\]

are mutually orthogonal. And this is equivalent to the mutual orthogonality of the following set

\[
\text{Tr}_{\mathcal{S}_A} [U_A |\psi_i\rangle_A \langle \psi_i | U_A^\dagger] \otimes \text{Tr}_{\mathcal{S}_B} [U_B |\psi_i\rangle_B \langle \psi_i | U_B^\dagger)]_{i=1}^{2k-2}.
\]
To ensure the satisfaction of these orthogonality relations, by Eq. (1) and Lemma 1, both \( \{ \text{Tr}_{\mathcal{A}}[U_A|\psi_i\rangle_A\langle\psi_i|U_A^\dagger]\}_i \) and \( \{ \text{Tr}_{\mathcal{B}}[U_B|\psi_{k-1+i}\rangle_B\langle\psi_{k-1+i}|U_B^\dagger]\}_i \) are mutually orthogonal. By assumption, at least one of \( \mathcal{A} \) and \( \mathcal{B} \) is nonempty. Without loss of generality, we assume that \( \mathcal{A} \neq \emptyset \). Therefore, the resulting states \( \{ \text{Tr}_{\mathcal{A}}[U_A|\psi_i\rangle_A\langle\psi_i|U_A^\dagger]\}_i \) are \( (k - 1) \) orthogonal states in the systems corresponding to \( \{ A_1, \ldots, A_N \} \setminus \mathcal{A} \) whose dimension is at most \( d/p_1 \leq d/3 < k - 1 \). This deduces a contradiction as there are at most \( N \) orthogonal positive semidefinite matrices in system with dimensional \( N \). Therefore, our constructing set is locally irredundant.

**Theorem 2.** Let \( d \geq 11 \) be an odd integer. Then there exists some orthogonal set without entanglement in \( \mathbb{C}^d \otimes \mathbb{C}^d \) with genuine hidden nonlocality of type I.

One finds that the above results can be also used to construct sets with this property in multipartite systems. For example, let \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D \otimes \mathcal{H}_E \otimes \mathcal{H}_F \) where

\[
\dim_{\mathcal{C}}(\mathcal{H}_A) = \dim_{\mathcal{C}}(\mathcal{H}_B) = \dim_{\mathcal{C}}(\mathcal{H}_C) = \dim_{\mathcal{C}}(\mathcal{H}_D) = \dim_{\mathcal{C}}(\mathcal{H}_F) = 11
\]

and \( \dim_{\mathcal{C}}(\mathcal{H}_E) = 13 \). Set \( \mathcal{M} := \{ |i\rangle \langle i|, j\rangle \langle j| \} \). Denote \( \{ |\Phi\rangle_{AB}\}_{i=1}^{10} \) be the set constructed in Example 1. Finding four unfilled squares (with labels \( h_i, h_j, v_i, v_j \)) in Fig. 2 such that they come from exactly four different colored blocks and form a rectangle. For example, \( \{ (3, 2), (3, 8), (9, 2), (9, 8) \} \). Using the four cubics, we can define a state \( |\Phi\rangle_{AB} := |3 - 9\rangle |2 - 8\rangle \) with the following properties: (1). The set \( \{ |\Phi\rangle_{AB}\}_{i=1}^{10} \) is an orthogonal set which is still locally distinguishable (whose proof is similar to the proof showed in Example 1); (2). \( \pi^B |\Phi\rangle_{AB} \) is nonzero for each \( k = 1, 2, \ldots, 10 \); (3). \( \pi^B |\Phi\rangle_{AB} \) is orthogonal to each of \( \{ \pi^B |\Phi\rangle_{AB}\}_{i=1}^{20} \) for \( k = 1, 2 \). Denote \( \{ |\Psi\rangle_{EF}\}_{i=1}^{24} \) be the set constructed in Example 2. And we choose \( |\Psi\rangle_{EF} \) to be \( |4 - 10\rangle |3 - 9\rangle \) which has similar properties with \( |\Phi\rangle_{AB} \). One can check that the union of the following set

\[
\{ |\Phi\rangle_{AB}|\Phi\rangle_{CD}|\Psi\rangle_{EF}\}_{i=1}^{20},
\{ |\Phi\rangle_{AB}|\Phi\rangle_{CD}|\Psi\rangle_{EF}\}_{i=1}^{20},
\{ |\Phi\rangle_{AB}|\Phi\rangle_{CD}|\Psi\rangle_{EF}\}_{i=1}^{24},
\]

has genuine hidden nonlocality of type I in \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D \otimes \mathcal{H}_E \otimes \mathcal{H}_F \). Firstly, the set is locally distinguishable as \( \{ |\Phi\rangle_{AB}\}_{i=1}^{20} \cup \{ |\Phi\rangle_{AB}\}_{i=1}^{20} \cup \{ |\Phi\rangle_{CD}\}_{i=1}^{20} \cup \{ |\Phi\rangle_{CD}\}_{i=1}^{24} \) and \( \{ |\Psi\rangle_{EF}\}_{i=1}^{24} \) all are locally distinguishable. Now if the \( B, D, F \) parts make the following local measurements

\[
\mathcal{M} := \{ |i\rangle \langle i|, j\rangle \langle j| \},
\]

there are eight outcomes. One can check that the post-measurement states of each outcome form an orthogonal set with 64 elements which is locally indistinguishable. Generally, this result can be extended to \( \otimes_{i=1}^{N} (\mathcal{H}_A \otimes \mathcal{H}_B) \) where each \( d_i = \dim_{\mathcal{C}}(\mathcal{H}_A) = \dim_{\mathcal{C}}(\mathcal{H}_B) \) is an odd integer greater than 10 for \( i \in \{ 1, 2, \ldots, N \} \).

Note that locally irreducible (see [68]) is a stronger form of nonlocality than locally indistinguishable. Therefore, it is interesting to find that whether we can genuinely activate this stronger form of nonlocality under OPLM without decreasing the cardinality of the given set. The examples with three elements provided by [81] do satisfy this property. In the following, we present an example of such set without entanglement.

**Example 3.** Let \( \mathcal{S} \) be the set of product states in \( \mathbb{C}^{11} \otimes \mathbb{C}^{11} \) whose elements are listed as below (see Fig. 4):

\[
|\psi_1\rangle \equiv |1\rangle |3 - 4 + 5 - 6\rangle,
|\psi_2\rangle \equiv |2\rangle |2 - 4 + 5 - 7\rangle,
|\psi_3\rangle \equiv |3\rangle |1 - 4 + 5 - 8\rangle,
|\psi_4\rangle \equiv |4\rangle |0 - 4 + 5 - 9\rangle,
|\phi_1\rangle \equiv |6\rangle |5 - 6 + 2 - 3\rangle,
|\phi_2\rangle \equiv |7\rangle |5 - 7 + 3 - 4\rangle,
|\phi_3\rangle \equiv |8\rangle |5 - 8 + 2 - 4\rangle,
|\phi_4\rangle \equiv |9\rangle |5 - 9 + 1 - 4\rangle,
|\phi_5\rangle \equiv |10\rangle |5 - 10 + 0 - 4\rangle,
\]

and \( |S\rangle \equiv |+10\rangle |+10\rangle \), and \( |\mathcal{S}\rangle \equiv |1 - 6\rangle |0 - 9\rangle \). Then the set \( \mathcal{S} \) has genuine hidden strong form nonlocality of type I.

Similar with Example 1, we can show that \( \mathcal{S} \) is locally distinguishable. Suppose Bob perform the measurement \( \mathcal{M} := \{ |i\rangle \langle i|, j\rangle \langle j| \} \). Note that, for both outcomes of the measurement, the cardinality of the possible states are unchanged. In appendix, we show that for both outcomes of the measurement the postmeasurement states are locally irreducible.
Bob perform the measurement $M$ if the outcome of $\psi$ is locally indistinguishable. First, Alice perform the measurement $M$ if the outcome of $\psi$ is locally indistinguishable. Now we show that $S$ is locally distinguishable. In this section, we study how to construct sets without entanglement but have genuine hidden nonlocality of type II. We will use the local indistinguishability of the sets without entanglement as follows.

**Theorem 3 ([65]).** In $\mathbb{C}^m \otimes \mathbb{C}^n$ system with $m, n \geq 3$, there exists a set with $2(m + n) - 4$ orthogonal pure product states which is locally indistinguishable.

According to the parity of $m$ and $n$, the structure of the set of states can be divided into four types (see Fig. 5). For more details on the sets of quantum states, the authors may see the reference [65]. Now let’s start with the following example.

**Example 4.** Let $\mathcal{S}$ be the set of product states in $\mathbb{C}^7 \otimes \mathbb{C}^8$ whose elements are listed as below:

$$
\begin{align*}
|\psi_{1,2}\rangle & \equiv |0\rangle |0 \pm 1\rangle, \\
|\psi_{6,7}\rangle & \equiv |0\rangle |2 \pm 3\rangle, \\
|\psi_{5,6,7}\rangle & \equiv |0 + w + w^2 2\rangle |4\rangle, \\
|\psi_{12,13,14}\rangle & \equiv |1 + w + w^2 3\rangle |0\rangle, \\
|\phi_{1,2}\rangle & \equiv |4 \pm 6 + 3 \pm 4\rangle, \\
|\phi_{3,4}\rangle & \equiv |4 \pm 5\rangle |7\rangle, \\
|\phi_{3,4}\rangle & \equiv |4 \pm 5\rangle |7\rangle,
\end{align*}
$$

where $w \in \{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\}$ which is any cubic root of unit. Then the set $\mathcal{S}$ has genuine hidden nonlocality of type II.

Now we show that $\mathcal{S}$ is locally distinguishable. First, Alice perform the measurement $M_1^A := \{\pi_i^A = \sum_{j=0}^{1} |i\rangle \langle j|, \pi_2^A = \sum_{j=4}^{9} |j\rangle \langle j|\}.$

If the outcome of $M_1^A$ is ‘1’, the state must be one of $\{|\psi_i\rangle\}_{i=1}^{14}$. And the states do not change under this measurement. Now Bob perform the measurement:

$$M_2^B := \{\pi_1^B = |4\rangle |4\rangle + |5\rangle |5\rangle, \pi_2^B = I_8 - \pi_1^B\}.$$

If the outcome of $M_1^B$ is ‘1’, the state must be one of $\{|\psi_{5,6,7,8,9}\rangle\}$ which may change to be

$$|\psi_{5,6,7}\rangle = |0 + w + w^2 2\rangle |4\rangle, \quad |\psi_{6,9}\rangle = |3\rangle |3 + 6 + 4\rangle.$$

It is easy to derive a LOCC protocol to distinguish the states of above set. If the outcome of $M_1^B$ is ‘2’, the state must be one of $\{|\psi_i\rangle\}_{i=1}^{14} \setminus \{|\psi_{5,6,7}\rangle\}$ (without the states corresponding to the label $v_1$ in Fig. 6) which may change to

$$|\psi_{1,2}\rangle = |0\rangle |0 \pm 1\rangle, \quad |\psi_{8,9}\rangle = |3\rangle |3 + 6\rangle, \quad |\psi_{10,11}\rangle = |3\rangle |1 + 2\rangle, \quad |\psi_{12,13,14}\rangle = |1 + w + w^2 3\rangle |0\rangle.$$
It is also easy to derive a LOCC protocol to distinguish the states of above set.

If the outcome of $M^B_1$ is ‘2’, the state must be one of $\{|\phi_i\rangle\}_{i=1}^8$. And the states do not change under this measurement. Now Bob perform the measurement:

$M^B_1 \equiv \{\pi^B = |4\rangle \langle 4| + |5\rangle \langle 5|, \pi^B_2 = 4 - \pi^B_1\}.$

If the outcome of $M^B_1$ is ‘1’, the state must be one of $\{|\phi_i\rangle\}_{i=1}^8$ which may change to be

$|\tilde{\phi}_{1,2}\rangle = |4\rangle|5\rangle, |\tilde{\phi}_{7,8}\rangle = |5\rangle|6\rangle.$

It is easy to derive a LOCC protocol to distinguish these four states. If the outcome of $M^B_1$ is ‘2’, the state must be one of $\{|\phi_i\rangle\}_{i=1}^8 \setminus \{|\phi_{7,8}\rangle\}$ (without the states corresponding to the label $V_2$ in Fig. 6) which may change to be

$|\phi_{1,2}\rangle = |4\rangle|3\rangle, |\phi_{3,4}\rangle = |6\rangle|6\rangle.$

Figure 5. This shows the examples of the four types of states structure in Theorem 3 in $\mathbb{C}^3 \otimes \mathbb{C}^3$, $\mathbb{C}^4 \otimes \mathbb{C}^2$, $\mathbb{C}^5 \otimes \mathbb{C}^5$, and $\mathbb{C}^6 \otimes \mathbb{C}^4$ respectively. The squares indicated by the same label of cardinality 2,3 represent 2,3 states respectively. For example, there are three squares (that is, (0,3), (1,3), and (2,3)) with label ‘$v_1$’ at the right down corner figure, they correspond to $|0 + w1 + w^22\rangle|3\rangle \in \mathbb{C}^6 \otimes \mathbb{C}^4$ where $w \in \{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\}$. It is easy to derive a LOCC protocol to distinguish these four states. If the outcome of $M^B_1$ is ‘1’, the state must be one of $\{|\phi_i\rangle\}_{i=1}^8 \setminus \{|\phi_{7,8}\rangle\}$ (without the states corresponding to the label $V_2$ in Fig. 6) which may change to be

$|\phi_{1,2}\rangle = |4\rangle|3\rangle, |\phi_{3,4}\rangle = |6\rangle|6\rangle.$
It is also easy to derive a LOCC protocol to distinguish the states of above set. 

Now we prove the second property. Suppose Bob perform the measurement \( \mathcal{M}_B := \{ \pi_1^B = \sum_{i=0}^4 |i\rangle \langle i|, \pi_2^B = \sum_{j=3}^7 |j\rangle \langle j| \} \).

If the outcome of \( \mathcal{M}_B \) is ‘1’, the states are transferred to
\[
|\psi_{1,2}\rangle = |0\rangle |0\pm 1\rangle, \\
|\psi_{3,4}\rangle = |0\rangle |2\pm 3\rangle, \\
|\psi_{5,6,7}\rangle = |0+w+1+w^2\rangle |4\rangle, \\
|\phi_{1,2}\rangle = |4\rangle |3\pm 4\rangle,
\]
which are mutually orthogonal and contains 14 states that is known to be locally indistinguishable in \( \mathbb{C}^4 \otimes \mathbb{C}^5 \).

If the outcome of \( \mathcal{M}_B \) is ‘2’, the states are transferred to
\[
|\phi_{1,2}\rangle \equiv |4\rangle |5\pm 6\rangle, \\
|\phi_{3,4}\rangle \equiv |4\pm 5\rangle |7\rangle, \\
|\phi_{7,8}\rangle \equiv |5\pm 6\rangle |5\rangle,
\]
which are mutually orthogonal and contains 8 states that is known to be locally indistinguishable in \( \mathbb{C}^3 \otimes \mathbb{C}^3 \).

Now we prove that the set is locally irredundant. Written \(|\psi_i\rangle = |\psi_i\rangle_A |\psi_i\rangle_B\) for \( i = 1, 2, \ldots, 14 \). One may observe that
\[
\langle \psi_i | \psi_j \rangle_A \neq 0, \text{ and } (|\psi_i\rangle |\psi_i\rangle)_B \neq 0
\]
for different \( i, j \in \{1, 2, 3, 4, 5\} \) and different \( k, l \in \{1, 12, 13, 14\} \). If the set \( \mathcal{S} \) is local irredundancy, there must exist \( \mathcal{S}_B \subseteq \{B_1, B_2, B_3\} \) and some unitary \( U_B \in U(d) \), such that \( \text{Tr}_{\mathcal{S}_B}[U_B |\psi_i\rangle_B \langle |\psi_i| B\rangle_B] \rangle_{i=1}^5 \) are mutually orthogonal. However, the dimension of the system corresponding to \( \{B_1, B_2, B_3\} \setminus \mathcal{S}_B \) is at most 4 which is less than 5.

The above argument can be easily extend to \( \mathbb{C}^m \otimes \mathbb{C}^n \) provided the conditions \( m, n \geq 7 \). Therefore, we arrive at the following theorem.

**Theorem 4.** Let \( m, n \geq 7 \) be an integer. Then there exists some orthogonal set without entanglement in \( \mathbb{C}^m \otimes \mathbb{C}^n \) which has genuine hidden nonlocality of type II.

For an integer \( m \geq 7 \) define \( f_m = \lceil \frac{m-1}{2} \rceil \) and \( c_m = m - f_m \). Similarly, one finds that \( c_m \) is the smallest integer which is larger than \( m/2 \). Similarly, one can construct a set with \( 2(m+n) - 8 \) states which are inspired by Xu et al.’s results [65] in \( \mathbb{C}^{20} \otimes \mathbb{C}^{20} \) and \( \mathbb{C}^{100} \otimes \mathbb{C}^{100} \). The local irredundancy can be obtained similarly as the above argument since there are at least \( c_n > \frac{n}{2} \) states for which Alice’s part are mutually non-orthogonal and \( c_m > \frac{m}{2} \) states for which Bob’s part are mutually non-orthogonal. Similar with the discussion in the end of Sec. III, we can extend the construction to multipartite systems whose local dimensions are greater than 6.

V. CONCLUSIONS AND DISCUSSIONS

We studied genuine hidden nonlocality without entanglement in the settings of local discrimination of quantum states. There are two cases: the first one is cardinality preserving under OPLM and the second one is cardinality decreasing under OPLM. For both cases, we provided examples to demonstrate how to construct set of product states with genuine hidden nonlocality. For the first case, the construction can even be extended to \( 2N \)-parties systems \( \otimes_{i=1}^N (\mathcal{H}_A \otimes \mathcal{H}_B) \) whenever each \( d_i = \dim(\mathcal{H}_A) = \dim(\mathcal{H}_B) \) is an odd integer greater than 10 for \( i \in \{1, 2, \ldots, N\} \). And for the second case, we can extend the idea to any \( N \)-parties systems whose local dimensions are greater than 6. Moreover, we provided a method to deal with the local irredundancy problem which might be helpful to tackling this problem. Compared with results with those in [81], it is a bit surprise as sets without entanglement seems can be more easier to be genuinely activated than sets with entanglement. Therefore, it is interesting find more sets with entanglement (but with more complicated structure) and genuine hidden nonlocality. In this case, finding a way to check the local irredundanity is a tricky problem.

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APPENDIX

Proof of Lemma 1. Suppose not, we have \( \langle \mathcal{N}(\rho) | \mathcal{N}(\sigma) \rangle_B = 0 \). Let \( \rho = \sum_{i=1}^m \lambda_i |x_i\rangle \langle x_i|, \sigma = \sum_{j=1}^n \mu_j |y_j\rangle \langle y_j| \) be the spectral decomposition of \( \rho \) and \( \sigma \) respectively. By the linearity of \( \mathcal{N} \) and the inner product,
\[
\langle \mathcal{N}(\rho), \mathcal{N}(\sigma) \rangle_B = \sum_{i=1}^m \sum_{j=1}^n \lambda_i \mu_j \langle \mathcal{N}(x_i) | \mathcal{N}(y_j) \rangle_B.
\]  

(2)
It is well known that $\text{Tr}[MN] \geq 0$ if $M$ and $N$ are positive semidefinite matrices. Therefore, the zero of $\langle \mathcal{N} (\rho), \mathcal{N} (\sigma) \rangle_B$ implies all the terms on the right hand side of Eq. (2) are zeros, i.e.,

$$
\langle \mathcal{N} (|x_i\rangle \langle x_i|), \mathcal{N} (|y_j\rangle \langle y_j|) \rangle_B = 0
$$

for all $i, j$. For each $i, j$, we compute the $\langle \mathcal{N} (|x_i\rangle \langle x_i|), \mathcal{N} (|y_j\rangle \langle y_j|) \rangle_B$ by using the Kraus representation of $\mathcal{N}$ which is given by

$$
\mathcal{N} (\tau) = \sum_{k=1}^{K} A_k \tau A_k^\dagger,
$$

where $\sum_{k=1}^{K} A_k^\dagger A_k = I_A$.

Therefore, we have

$$
\langle \mathcal{N} (|x_i\rangle \langle x_i|), \mathcal{N} (|y_j\rangle \langle y_j|) \rangle_B = \sum_{k=1}^{K} \sum_{j=1}^{K} \text{Tr}[A_k|x_i\rangle \langle x_i|A_j^\dagger |y_j\rangle \langle y_j|A_k^\dagger]
$$

$$
= \sum_{k=1}^{K} \sum_{j=1}^{K} \text{Tr}[|x_i\rangle \langle x_i|A_k^\dagger |y_j\rangle \langle y_j|A_j^\dagger]
$$

$$
= \sum_{k=1}^{K} \sum_{j=1}^{K} (|x_i\rangle \langle x_i|A_k^\dagger A_j^\dagger)^2.
$$

As $\langle \mathcal{N} (|x_i\rangle \langle x_i|), \mathcal{N} (|y_j\rangle \langle y_j|) \rangle_B = 0$, we have $|\langle x_i|A_k^\dagger A_j^\dagger \rangle|^2 = 0$ for all $1 \leq k, l \leq K$. Particularly, $\langle x_i|A_k^\dagger A_k |y_j\rangle = 0$ for $1 \leq k \leq K$ which imply that $|x_i\rangle = 0$ as $\sum_{k=1}^{K} A_k = I_A$. Using these results and the spectral decomposition of $\rho$ and $\sigma$, we obtain that $\rho \sigma = 0$ which is a zero matrix. This is contradicted with the non-orthogonality of $\rho$ and $\sigma$. \hfill \Box

**Proof of the local irreducibility of postmeasurement states in Example 3.** If the outcome of $M^B$ is ‘1’, the states are transferred to $\mathcal{H}_1$

$$
|\psi_1\rangle = |1\rangle |3 - 4\rangle, \quad |\psi_2\rangle = |2\rangle |2 - 4\rangle, \quad |\psi_3\rangle = |3\rangle |1 - 4\rangle, \quad |\psi_4\rangle = |4\rangle |0 - 4\rangle,
$$

$$
|\phi_1\rangle = |6\rangle |2 - 3\rangle, \quad |\phi_2\rangle = |7\rangle |3 - 4\rangle, \quad |\phi_3\rangle = |8\rangle |2 - 4\rangle, \quad |\phi_4\rangle = |9\rangle |1 - 4\rangle,
$$

$$
|\phi_5\rangle = |10\rangle |0 - 4\rangle, \quad |\phi_6\rangle = |11\rangle |+10\rangle + |+4\rangle, \quad |\phi_7\rangle = |12\rangle |-1\rangle - |5\rangle
$$

which can be seen as states in $\mathcal{H}_A \otimes \mathcal{H}_B_1 := \mathbb{C}^{11} \otimes \mathbb{C}^{5}$ with computation basis $\{|i\rangle |j\rangle \mid i \in \{1, \ldots, 11\}, j \in \{1, \ldots, 5\}\}$. We only need to show that $A$ and $B_1$ can only start with trivial OPLMs. In fact, to preserve the orthogonality of $\{|\psi_i\rangle\}_{i=1}^{8} \cup \{|\phi_i\rangle\}_{i=1}^{5}$, $B_1$ can only start with trivial measurement. Now it is sufficient to prove that if $E = (a_{i,j})_{i,j \in \{1, \ldots, 11\}}$ is an $11 \times 11$ Hermitian matrix and

$$
\langle \Theta_1 | E \otimes I_{B_1} | \Theta_2 \rangle = 0, \quad \langle \Theta_1 \rangle \neq \langle \Theta_2 \rangle \in \mathcal{H}_1,
$$

then $E \propto I_A$. As the $B_1$ part of the states $\{|\phi_i\rangle\}_{i=1}^{5} \cup \{|\psi_i\rangle\}_{i=1}^{6}$, applying Eq. (4) to this subset, we obtain

$$
a_{i,k} = 0, \quad \forall i, k \in \{1, \ldots, 11\}, \quad k \neq i.
$$

Substitute $\langle \Theta_1 \rangle$ and $\langle \Theta_2 \rangle$ in Eq. (3) by one of $\{|\psi_i\rangle\}_{i=1}^{6} \cup \{|\phi_i\rangle\}_{i=1}^{5}$ and $\langle \Theta \rangle$ respectively, we obtain $a_{x,5} = a_{x,6} = 0$ for $x \in \{1, 2, 3, 4, 7, 8, 9, 10\}$.

Now substitute $\langle \Theta_1 \rangle$ and $\langle \Theta_2 \rangle$ in Eq. (3) by $\langle \phi_1 \rangle$ and $\langle \phi_2 \rangle$, we obtain that $a_{x,5} = a_{x,6} = 0$. Now applying Eq. (3) to the following pairs of states $\{|\psi_i\rangle\}_{i=1}^{6} \cup \{|\phi_i\rangle\}_{i=1}^{5}$, we get $a_{x,k} = 0$ for $x \in \{1, \ldots, 11\} \setminus \{0\}$. Therefore, $E$ is a diagonal matrix. At last, applying Eq. (3) to one of $\{|\psi_i\rangle\}_{i=1}^{6} \cup \{|\phi_i\rangle\}_{i=1}^{5}$ and $\langle \Theta \rangle$, we have $E \propto I_A$.

If the outcome of $M^B$ is ‘2’, the states are transferred to $\mathcal{H}_2$

$$
|\psi_1\rangle = |1\rangle |5 - 6\rangle, \quad |\psi_2\rangle = |2\rangle |5 - 7\rangle, \quad |\psi_3\rangle = |3\rangle |5 - 8\rangle, \quad |\psi_4\rangle = |4\rangle |5 - 9\rangle,
$$

$$
|\psi_5\rangle = |6\rangle |5 - 6\rangle, \quad |\psi_6\rangle = |7\rangle |5 - 7\rangle, \quad |\psi_7\rangle = |8\rangle |5 - 8\rangle, \quad |\psi_8\rangle = |9\rangle |5 - 9\rangle,
$$

$$
|\psi_9\rangle = |10\rangle |5 - 10\rangle, \quad |\phi_10\rangle = |5 - 9\rangle, \quad |\phi_11\rangle = |1 - 6\rangle |9\rangle
$$

which can be seen as states in $\mathcal{H}_A \otimes \mathcal{H}_B_2 := \mathbb{C}^{11} \otimes \mathbb{C}^{6}$ with computation basis $\{|i\rangle |j\rangle \mid i \in \{1, \ldots, 11\}, j \in \{5, 6, \ldots, 10\}\}$. We only need to show that $A$ and $B_2$ can only start with trivial OPLMs. In fact, to preserve the orthogonality of $\{|\phi_i\rangle\}_{i=1}^{10} \cup \{|\psi_i\rangle\}_{i=1}^{6}$, $B_2$ can only start with trivial measurement. Now it is sufficient to prove that if $E = (a_{i,j})_{i,j \in \{1, \ldots, 11\}}$ is an $11 \times 11$ Hermitian matrix and

$$
\langle \Theta_1 | E \otimes I_{B_2} | \Theta_2 \rangle = 0, \quad \langle \Theta_1 \rangle \neq \langle \Theta_2 \rangle \in \mathcal{H}_2,
$$

(4)
then $E \propto I_A$. Written $E = \left( \begin{array}{cc} E_{11} & E_{12} \\ E_{21} & E_{22} \end{array} \right)$ where $E_{11}, E_{12} = E_{21}^\dagger, E_{22}$ are $5 \times 5, 5 \times 6$, and $6 \times 6$ matrices respectively. Following similar proof with that in Ref. [42], we can prove that $E_{11} = \alpha_1 \sum_{i=0}^{8} |i\rangle \langle i|$ and $E_{22} = \alpha_2 \sum_{i=5}^{10} |j\rangle \langle j|$ for some $\alpha_1, \alpha_2 \geq 0$ by using the orthogonality relations of Eq. (4) from the sets $\{ \{ \tilde{\psi}_i \}_{i=1}^{8} \cup \{ \tilde{\mathbf{S}} \}$ and $\{ \{ \tilde{\phi}_j \}_{j=1}^{10} \cup \{ \tilde{\mathbf{S}} \}$ respectively. In the following, we show that $E_{12}$ is a zero matrix. Note that the $B_2$ part of $|\tilde{\psi}_i\rangle$ and $|\tilde{\phi}_j\rangle$ are pairwise nonorthogonal for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5$. Substitute $|\Theta_1\rangle$ and $|\Theta_2\rangle$ in Eq. (4) by $|\tilde{\psi}_i\rangle$ and $|\tilde{\phi}_j\rangle$ respectively, we obtain

$$a_{i,j+5} = \langle i| E |j+5 \rangle = 0$$

for $i = 1, 2, 3, 4$ and $j = 1, 2, 3, 4, 5$. Now substitute $|\Theta_1\rangle$ and $|\Theta_2\rangle$ in Eq. (4) by $|\tilde{\psi}_{i+4}\rangle$ and $|\tilde{\phi}_{i+1}\rangle$ ($i = 5, 6$), we obtain $a_{0,6} = a_{6,6} = 0$, $a_{0,7} = a_{1,7} = 0$, $a_{0,8} = a_{2,8} = 0$, $a_{0,9} = a_{3,9} = 0$.

Now substitute $|\Theta_1\rangle$ and $|\Theta_2\rangle$ in Eq. (4) by $|\tilde{\psi}_5\rangle$ and $|\tilde{\phi}_1\rangle$ ($i = 5, 6$), we obtain $a_{0,5} = a_{0,10} = a_{4,5} = a_{4,10} = 0$, and $a_{0,5} + a_{0,6} + a_{1,5} + a_{1,6} = 0$.

By the previous result, we get $a_{0,5} = a_{0,6} = 0$. Therefore, the matrix $E_{12}$ is a zero matrix. Hence $E$ is a diagonal matrix. At last, consider that $\langle \mathbf{M} | E \otimes I_{B_2} | \tilde{\mathbf{S}} \rangle = 0$, we have $a_{1,1} = a_{6,6}$. That is, $\alpha_1 = \alpha_2$. So we conclude that $E \propto I_A$.

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