Two-Loop Calculations in $\phi^4$ Light-Front Field Theory

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Abstract

We perform a two-loop calculation in light-front $\phi^4$ theory to determine the effective mass renormalization of the light-front Hamiltonian. The renormalization scheme adopted here is manifestly boost invariant, and yields results that are in perfect agreement with the explicitly covariant Feynman diagram approach.
1 Introduction

It has been known for some time that the renormalization of light-front field theories is an extremely challenging task, although systematic procedures have been developed over the years to handle these problems consistently [1, 2, 3, 4, 5, 6, 7], including boost-invariant regularization schemes [5, 6, 7].

Relatively recently, it was observed that a different boost-invariant regularization scheme may be employed in a treatment of light-front matrix field theories with $\phi^3$ interactions [8], following earlier work on the small-$x$ behavior of light-cone wave functions [9, 10].

In this work, we follow the approach suggested in [8] to determine the effective two-loop mass renormalization of the $\phi^4$ light-front Hamiltonian. In Section 2, we begin by presenting the light-front formulation of scalar $\phi^4$ field theory in $D \geq 2$ dimensions, and its subsequent quantization via commutation relations. In Section 3, we employ a manifestly boost-invariant procedure to extract the effective two-loop mass renormalization of the light-front Hamiltonian. This result will be compared to a Feynman diagram calculation that can be performed straightforwardly in two dimensions. A brief discussion of our results will appear in Section 4.

2 Scalar $\phi^4$ Theory in Light-Cone Coordinates

Consider the $D$-dimensional field theory described by the action

$$S = \int d^Dx \left[ \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right], \quad (1)$$

where $\phi(x)$ is a scalar field defined on the $D$-dimensional Minkowski space $x \equiv (x^0, x^1, \ldots, x^{D-1})$.

Working in the light-cone coordinate frame

$$x^+ = \frac{1}{\sqrt{2}} (x^0 + x^{D-1}), \quad \text{“time coordinate”} \quad (2)$$

$$x^- = \frac{1}{\sqrt{2}} (x^0 - x^{D-1}), \quad \text{“longitudinal space coordinate”} \quad (3)$$

$$x^\perp = (x^1, \ldots, x^{D-2}), \quad \text{“transverse coordinates”} \quad (4)$$

one may derive from the light-cone energy-momentum tensor expressions for the light-cone Hamiltonian $P^-$ and conserved total momenta $(P^+, P^\perp)$:

$$P^+ = \int dx^- dx^\perp (\partial_\perp \phi)^2, \quad \text{“longitudinal momentum”} \quad (5)$$
The light-cone Hamiltonian propagates a given field configuration in light-cone time \( x^+ = 0 \), say:

\[
[\phi(x^-, x^\perp), \partial_- \phi(y^-, y^\perp)] = \frac{i}{2} \delta(x^- - y^-) \delta(x^\perp - y^\perp).
\]

The light-cone Hamiltonian \( P^- \) is performed in the usual way – namely, we impose commutation relations at some fixed light-cone time (\( x^+ = 0 \)) while preserving this quantization condition. At fixed \( x^+ = 0 \), the Fourier representation

\[
\phi(x^-, x^\perp) = \frac{1}{(\sqrt{2\pi})^{D-1}} \int_0^\infty \frac{dk^+}{\sqrt{2k^+}} \int dk^\perp \times
\]

\[
\left[ a(k^+, k^\perp) e^{-i(k^+ x^+ - k^\perp \cdot x^\perp)} + a^\dagger(k^+, k^\perp) e^{i(k^+ x^+ - k^\perp \cdot x^\perp)} \right],
\]

together with the quantization condition (8), imply the relation

\[
[a(k^+, k^\perp), a^\dagger(\tilde{k}^+, \tilde{k}^\perp)] = \delta(k^+ - \tilde{k}^+) \delta(k^\perp - \tilde{k}^\perp).
\]

It is now a matter of substituting the Fourier representation (8) for the quantized matrix field \( \phi \) into definitions (7), (6) and (5), to obtain the following quantized expressions for the light-cone Hamiltonian and conserved total momenta:

\[
: P^+ : = \int_0^\infty \frac{dk^+}{2k^+} \int dk^\perp k^\perp \cdot a^\dagger(k^+, k^\perp)a(k^+, k^\perp),
\]

\[
: P^\perp : = \int_0^\infty \frac{dk^+}{2k^+} \int dk^\perp k^\perp \cdot a^\dagger(k^+, k^\perp)a(k^+, k^\perp),
\]

\[
: P^- : = \int_0^\infty \frac{dk^+}{2k^+} \int dk^\perp \left( \frac{m^2 + |k^\perp|^2}{2k^+} \right) a^\dagger(k^+, k^\perp)a(k^+, k^\perp)
\]

\[
+ \frac{\lambda}{4! \cdot 4(2\pi)^D} \int_0^\infty \frac{dk^+_1 dk^+_2 dk^+_3 dk^+_4}{\sqrt{k^+_1^2 + k^+_2^2 + k^+_3^2 + k^+_4^2}} \int dk^\perp_1 dk^\perp_2 dk^\perp_3 dk^\perp_4 \times
\]

\[
\left\{ 4 \cdot \delta(k_1 + k_2 + k_3 - k_4) \cdot a^\dagger(k_1)a^\dagger(k_2)a^\dagger(k_3)a(k_4)
\right. \\
+ 6 \cdot \delta(k_1 + k_2 - k_3 - k_4) \cdot a^\dagger(k_1)a^\dagger(k_3)a(k_4)a(k_2)
\right. \\
+ 4 \cdot \delta(k_1 + k_2 + k_3 - k_4) \cdot a^\dagger(k_4)a(k_1)a(k_2)a(k_3) \right\},
\]

where \( \mathbf{k} \equiv (k^+, k^\perp) \).
3 Two-Loop Mass Renormalization

We are now interested in calculating the effective two-loop contribution to the mass term in the light-cone Hamiltonian, which is represented diagrammatically in Fig 1. One advantage of working in light-cone coordinates is the absence of tadpole diagrams, i.e. the only two-loop contribution to the mass is given by the ‘setting sun’ diagram below. A convenient strategy is to compute this diagram for vanishing (external) longitudinal momentum \( k^+ \to 0 \). The details of this method appear in an earlier article [8], so we will mention only the key ideas here.

First, note that the two-loop process can be obtained by two applications of the light-cone Hamiltonian; one application corresponds to the creation of three partons from a single parton via the interaction \( a^\dagger a^\dagger a^\dagger a \), while the subsequent interaction \( a^\dagger a a a \) will absorb these same three partons to leave one parton left over. During the intermediate step, where there are three partons in flight, the sum of the light-cone momenta must be equal to the initial parton momentum, which we write as \((k^+, k^\perp)\).

We now consider what happens in Fig 1, when the incoming longitudinal momentum \( k^+ \) is made vanishingly small by taking the limit \( k^+ \to 0 \). It is important to stress that in this limit, we never set \( k^+ \) identically to zero; we simply investigate the limiting contribution of the diagram as \( k^+ \) – which is always positive – is made arbitrarily small.

A detailed calculation paralleling the one carried out in [8] – in which one studies the bound state integral equations governing the light-cone wave functions in the limit of vanishing \( k^+ \) – may be used to calculate the limiting contribution of this diagram to the mass renormalization. The result is

\[
\Gamma(k^+) = -\frac{\lambda^2}{4!(2\pi)^{2D-2}} \int_0^1 d\alpha d\beta d\gamma \delta(\alpha + \beta + \gamma - 1) \times \\
\int dk_1^+ dk_2^+ dk_3^+ \frac{\delta(k_1^+ + k_2^+ + k_3^+ - k^+)}{\beta\gamma(m^2 + |k_1^+|^2) + \alpha\gamma(m^2 + |k_2^+|^2) + \alpha\beta(m^2 + |k_3^+|^2)}.
\]

(14)
This quantity depends on $k^\perp$ only, since any dependence on $k^+$ was eliminated after taking the limit $k^+ \to 0$. In order to find the renormalized mass, we set $k^\perp = 0$.

Some remarks are in order. Firstly, note that a remnant of the longitudinal integration survives through the variables $\alpha, \beta$ and $\gamma$. These quantities represent the fractions of the external longitudinal momentum $k^+$, and are thus integrated over the interval $(0,1)$. They are not eliminated in the limit $k^+ \to 0$. To further illuminate the significance of these variables, we consider the special case of $1 + 1$ dimensions (i.e. $D = 2$). In this case, the mass renormalization is

$$- \frac{\lambda^2}{4!(2\pi)^2} \int_0^1 d\alpha d\beta d\gamma \frac{1}{m^2(\beta\gamma + \alpha\gamma + \alpha\beta)}.$$  \hspace{1cm} (15)

Now consider the Feynman integral for the ‘setting sun’ diagram of scalar $\phi^4$ theory in $1 + 1$ dimensions, with zero external momenta \[11\]. In this case, the mass renormalization turns out to be

$$- \frac{\lambda^2}{96\pi^4} \int \frac{d^2p_1 d^2p_2}{(p_1^2 + m^2)(p_2^2 + m^2)[(p_1 + p_2)^2 + m^2]}.$$ \hspace{1cm} (16)

If we use Feynman parameters to rewrite this last integral \[12\], and perform the momentum integration, we end up with expression \[13\]. i.e. the integrals (14) and (16) are equal in $1 + 1$ dimensions. It would be interesting to test for equality in $D \geq 3$ dimensions, and we leave this for future work.

Interestingly, the parameters $\alpha, \beta, \gamma$ above – which correspond to fractions of longitudinal light-cone momentum in the light-cone approach – represent the familiar Feynman parameters in the usual covariant Feynman diagram approach. We have therefore discovered a physical basis for the Feynman parameters.

The actual value of the integral may be computed \[11, 13\], and is given by

$$\frac{\lambda^2}{144\pi^2 m^2} \left[ \frac{2\pi^2}{3} - \psi'(\frac{1}{3}) \right],$$ \hspace{1cm} (17)

where $\psi(x)$ is Euler’s psi function. The terms in the parenthesis has the numerical value $-3.51586 \ldots$

We leave an analysis of the self-energy contribution $\Gamma(k^\perp)$ in $D > 2$ dimensions for future work.

4 Discussion

We have calculated the two-loop mass renormalization for the effective light-cone Hamiltonian of scalar $\phi^4$ field theory. An explicit evaluation of the corresponding integral in
two dimensions was given, and shown to be precisely equal to the corresponding two-loop ‘setting sun’ Feynman diagram. Moreover, we were able to attribute physical significance to the familiar Feynman parameters; namely, as fractions of vanishingly small external light-cone momenta.

Evidently, it would be interesting to study the integral (14) in more detail for higher space-time dimensions. In particular, ultraviolet divergences are expected in dimensions $D \geq 3$, and so we may invoke dimensional regularization to regulate the integration over transverse momentum space. This would facilitate a straightforward comparison with existing Feynman diagram calculations in $\phi^4$ theory [12].

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References

[1] S.Glazek and K.G.Wilson, Phys.Rev D48 (1993) 5863; ibid D49 (1994) 4214.
[2] F.Wegner, Ann. der Physik 3, (1994) 77.
[3] K.G.Wilson, T.S.Walhout, A.Harindrinath, W.M.Zhang, R,J.Perry, and S.D.Glazek, Phys.Rev D49 (1994) 6720.
[4] S.Glazek, A.Harindrinath, S.Pinsky, J.Shigemitsu, and K.G.Wilson, Phys.Rev D47 (1993) 1599.
[5] R.Perry, “Light-Front Quantum Chromodynamics,” to be published in Proceedings of the APCTP-RCNP Joint International School on Physics of Hadrons and QCD, Osaka, Japan. (nucl-th/9901080);
[6] Brent H. Allen, Robert J. Perry, “Systematic Renormalization in Hamiltonian Light-Front Field Theory,” Phys.Rev. D58 (1998) 125017; Roger D. Kylin, Brent H. Allen, Robert J. Perry, “Systematic Renormalization in Hamiltonian Light-Front Field Theory: The Massive Generalization”, hep-th/9812080.
[7] Stanislaw D. Glazek, “Boost-Invariant Running Couplings in Effective Hamiltonians,” hep-th/9904029.

[8] F. Antonuccio, “Light-Cone Quantization and Renormalization of Large-N Scalar Matrix Models,” hep-th/9705043.

[9] F. Antonuccio, S.J. Brodsky, and S. Dalley, Phys. Lett. B412 (1997) 104-110.

[10] S. Dalley, “Collinear QCD Models”, hep-th/9812080. Invited talk at ‘New Non-Perturbative Methods and Quantisation on the Light-Cone’, Les Houches 24 Feb - 4 Mar 1997. (To appear in proceedings), (hep-th/9704211); S. Dalley, Phys. Rev. D58 (1998) 087705.

[11] Jens O. Andersen, “Two-Loop Calculations for $\phi^4$ Theory in 1 + 1 Dimensions”, (unpublished notes, 1993).

[12] J.C. Collins, Phys. Rev. D10, 1213 (1974).

[13] Gradshteyn, I.S. and Ryzhik, I.M, Table of Integrals, Series and Products, Academic Press, London (1980).