Interpretation of Lorentz boosts in conformally deformed special relativity theory.

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Abstract

Conformally deformed special relativity is mathematically consistent example of a theory with two observer independent scales. As compare with recent DSR proposals, it is formulated starting from the position space. In this work we propose interpretation of Lorentz boosts of the model as transformations among accelerated observers. We point further that the model can be considered as relativistic version of MOND program and thus may be interesting in context of dark matter problem.

1 Introduction

In recent work [1] it has been proposed a model based on deformation of standard realization of the Lorentz group by means of special conformal transformation $\Lambda_{\lambda} \equiv (U_{\lambda})^{-1} \Lambda(U_{\lambda})$, $U_{\lambda} : x^\mu \rightarrow x^\mu + \delta^\mu_{\alpha} \lambda_{x^\alpha}^2$. The aim was to construct consistent example of doubly special relativity (DSR) model [2,3] (formulated in position space starting from the beginning), i.e. a theory with underlying symmetry group being the Lorentz group, but with kinematical predictions different from that of special relativity. Mathematical consistency of the model has been discussed, in particular, it resolve the problem of total momentum for multi-particle system presented in others DSR proposals. In some sense, our solution of the problem is an opposite as compare with others DSR proposals (see [2-5] and references therein). The known DSR proposals are based on various non-linear realizations of the Lorentz group in space of conserved momentum, i.e.

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are formulated as a list of kinematical rules of a theory. Ordinary energy-momentum relation \((P^\mu)^2 = -m^2\) is not invariant under the realization and is replaced by \([U(P^\mu)]^2 = -m^2\). The central problem of DSR kinematics is consistent definition of total momentum for many-particle system. Actually, due to non linear form of the transformations, ordinary sum of momenta does not transform as the constituents. Different covariant rules proposed in the literature lead to hardly acceptable features [4]. For our model, the Lorentz group is realized non linearly both on position and on conjugated momentum spaces, but conserved four-momentum turns out to be different from the conjugated one. On space of conserved momentum one has ordinary realization of the Lorentz transformations, so the composition rule is ordinary sum, see [1]. On other hand, energy and momentum have nonstandard relation with measurable quantities (coordinates and velocities). It suggests that kinematical predictions of our model differ from that of special relativity theory, see also below\(^1\).

In the present work we propose interpretation of Lorentz boosts in the conformally deformed special relativity (CDSR) model. In Sect. 2 we remind construction of deformed Lorentz realization as it was formulated in [1]. In Sect. 3 we demonstrate that the Lorentz boosts can be treated as transformations among mutually accelerated (in specific way) observers \(O\) and \(O'\). This interpretation is confirmed in Sect. 4, where we demonstrate that trajectory of \(O'\) is particular solution of geodesic-line equation. It suggests that the model may be relevant for description of expanding universe, we speculate on this possibility in the Conclusion.

2 Conformally deformed Lorentz group realization

In ordinary special relativity the requirement of invariance of the Minkowski interval: \(ds'^2 = ds^2\) immediately leads to the observer

\(^1\)The difference among the canonical momentum and the conserved one implies an interesting situation in canonically quantized version of the theory. While the conjugated variables \((x,p)\) have the standard brackets, commutators of the coordinates \(x^\mu\) with the energy and momentum \(P^\mu\) are deformed. Thus the phase space \((x,P)\) is endowed with the noncommutative geometry (with the commutators \([x,P]\) and \([P,P]\) being deformed). In particular, the energy-momentum subspace turns out to be noncommutative. The modified bracket \([x,P]\) suggests that the Planck’s constant has slight dependence on \(x\).
independent scale $|v^i| = c$. To construct a theory with one more scale, the invariance condition seems to be too restrictive. Actually, the most general transformations $x^\mu \longrightarrow x'^\mu(x')$ which preserve the interval are known to be Lorentz transformations in the standard realization [6] $x^\mu = \Lambda_{\mu \nu}x^\nu$, the latter does not admit one more invariant scale. So, one needs to relax the invariance condition keeping, as before, the speed of light invariant. It would be the case if $ds^2 = 0$ will imply $ds'^2 = 0$, which guarantees appearance of the invariant scale $c$ (in the case of linear relation $x^0 = ct$).

Thus, supposing existence of one more observer independent scale $R$, one assumes deformation of the invariance condition: $ds'^2 = A(x, R)ds^2$, where $A \xrightarrow{R \rightarrow \infty} 1$. By construction, the maximum velocity remains the invariant scale of the formulation. In the limit $R \rightarrow \infty$ one obtains ordinary special relativity theory.

Complete symmetry group for the case is the conformal group. It involves, in particular, special conformal transformations with the parameter $b^\mu$

$$U_b : x^\mu \longrightarrow \frac{1}{\Omega}(x^\mu + b^\mu x^2),$$
$$\Omega(x, b) \equiv 1 + 2bx + b^2x^2. \tag{1}$$

Similarly to momentum DSR proposals [2, 3], let us deform the Lorentz group realization in accordance with the rule

$$\Lambda_b \equiv (U_b)^{-1}\Lambda(U_b),$$
$$\Lambda_b : x^\mu \longrightarrow \frac{1}{G}[(\Lambda x)^\mu + [(1 - \Lambda)b\mu x^2],$$
$$G(x, b, \Lambda) \equiv 1 - 2b(1 - \Lambda)x + 2b(1 - \Lambda)bx^2. \tag{2}$$

The above mentioned proportionality factor for the case is $A = G^{-2}$. The parameters $b^\mu$ can be further specified by the requirement that space rotations $\Lambda^{\mu}_\nu = (\Lambda^0_0 = 1, \Lambda^0_i = \Lambda^i_0 = 0, \Lambda^i_j \equiv R^i_j, R^T = R^{-1})$ are not deformed by $b^\mu$. Then the only choice is $b^\mu = (\lambda, 0, 0, 0)$, which gives final form of the deformed Lorentz group realization

$$\Lambda_\lambda : x^\mu \longrightarrow \frac{1}{G}
[(\Lambda x)^\mu + (\delta^\mu_0 - \Lambda^\mu_0)x^2],$$
$$G(x, \lambda, \Lambda) \equiv 1 + 2\lambda(x^0 - \Lambda^0_\mu x^\mu) - 2\lambda^2(1 - \Lambda^0_0)x^2. \tag{3}$$

Our convention for the Minkowski metric is $\eta_{\mu\nu} = (-, +, +, +)$. One confirms now emergence of one more observer independent scale:
there is exist unique vector $x^\mu$ with zero component unaltered by the transformations (3). Namely, from the condition $x^0 = x^0$ one has the only solution $x^\mu = (R \equiv -\frac{1}{\lambda}, 0, 0, 0)$ (the latter turns out to be the fixed vector). Thus all observers should agree to identify $R$ as the invariant scale. Let us point that the transformations (3) are not equivalent to either the Fock-Lorentz realization [7], or to recent DSR proposals (the realizations lead to varying speed of light).

Inspection of transformation properties of quantities in our disposal allows one [1] to find invariant interval under the transformations (3)

$$ds^2 = \frac{\eta_{\mu\nu}dx^\mu dx^\nu}{(1 + 2\lambda x^0 - \lambda^2 x^2)^2} \equiv g_{\mu\nu}(x)dx^\mu dx^\nu,$$

(5)

On the domain where the metric is non degenerated, the corresponding four dimensional scalar curvature is zero, while three-dimensional space-like slice $x^0 = 0$ is curved space with constant curvature $R_{(3)} = -\frac{24\lambda}{R^2}$.

### 3 Interpretation of Lorentz boosts in CDSR

As it was mentioned above, space part of the deformed Lorentz transformations (3) represents usual rotations, the latter are not deformed by the scale $\lambda$. Let us discuss the remaining part, corresponding to generators $M_0$, i.e. Lorentz boosts of the model: $\Lambda^\mu_\nu = \exp(2\omega M_0)^\mu_\nu$. Keeping $\omega^{01} \equiv \alpha \neq 0$ only, non zero matrix elements of $\Lambda^\mu_\nu$ are $\Lambda^0_0 = \sinh \alpha, \Lambda^0_1 = \Lambda^1_0 = \cosh \alpha, \Lambda^2_2 = \Lambda^3_3 = 1$. Supposing further the standard relation $x^0 = ct$, one obtains the following form of Eq.(3),

$$t' = \frac{1}{c} \left[ (t + \lambda ct^2 - \frac{\lambda}{c} x^2) \cosh \alpha + \frac{x}{c} \sinh \alpha + \frac{1}{c} (c^2t^2 + x^2) \right],$$

$$x' = \frac{1}{c} \left[ c \left( t + \lambda ct^2 - \frac{\lambda}{c} x^2 \right) \sinh \alpha + x \cosh \alpha \right],$$

$$x'^2 = 0,$$

$$x'^3 = 0,$$

$$G \equiv 1 + 2\lambda \left[ c \left( t + \lambda ct^2 - \frac{\lambda}{c} x^2 \right) (1 - \cosh \alpha) - x \sinh \alpha \right],$$

(6)

where it was taken $x^1 \equiv x$, $x^2 = x^3 = 0$. Suppose that this expression corresponds to transformation low for observers $O$ and $O'$ in some state of motion in $x$-direction among themselves. Consistency of the picture requires that the transformation parameter $\alpha$ does
not depend on $x, t$. We demonstrate that it can be achieved by the following choice of the state of motion

$$t + \lambda ct^2 - \frac{\lambda}{c} (x(t))^2 = \frac{1}{V} x(t), \quad V = \text{const},$$

(7)

or, equivalently

$$x(t) = \frac{c}{2\lambda} \left[ -\frac{1}{V} + \sqrt{\frac{1}{V^2} + \frac{4\lambda}{c} (t + \lambda ct^2)} \right] =$$

$$V t + \lambda c V \left( 1 - \frac{V^2}{c^2} \right) t^2 + O(\lambda^2),$$

(8)

$$x(0) = 0, \quad \dot{x}(0) = V.$$

Choice of sign for the square root is dictated by the limit $x \to V t$ as $\lambda \to 0$. Similarly to the special relativity case, the parameter $\alpha$ can be determined now as follows: suppose the systems $O, O'$ coincide at $t = 0$, with instantaneous relative velocity being $V$. Let $(t, x)$ be coordinates of some event, which happens at the origin of $O'$ at the moment $t$: $(t', x' = 0)$. From second equation of the system (6) one immediately obtains

$$0 = \frac{1}{G} \left[ \frac{c}{V} x(t) \sinh \alpha + x(t) \cosh \alpha \right], \quad \Longrightarrow \quad \tanh \alpha = -\frac{V}{c},$$

(9)

i.e. the standard SR relation. Substitution of this result into Eq.(6) gives expression for the Lorentz boost in terms of (instantaneous) relative velocity among the observers

$$x' = G^{-1} \left( 1 - \frac{V^2}{c^2} \right)^{-\frac{1}{2}} \left( x - V \left( t + \lambda ct^2 - \frac{1}{c} x^2 \right) \right),$$

$$t' = G^{-1} \left( 1 - \frac{V^2}{c^2} \right)^{-\frac{1}{2}} \left( t - \frac{V^2}{c^2} x + \lambda ct^2 - \frac{1}{c} x^2 \right) + \frac{1}{G} \left( -c^2 t^2 + x^2 \right),$$

$$G \equiv 1 + 2\lambda \left( c \left( t + \lambda ct^2 - \frac{1}{c} x^2 \right) \left( 1 - \left( 1 - \frac{V^2}{c^2} \right)^{-\frac{1}{2}} \right) + \right.$$

$$\left. \left( 1 - \frac{V^2}{c^2} \right)^{-\frac{1}{2}} \frac{V}{c} x \right).$$

(10)

One concludes that Lorentz boosts of the model describe transformation among accelerated (according to Eq.(8)) observers. In the next section we demonstrate that it is reasonable interpretation. Namely, analysis of geodesic motion of a particle in the model shows that inertial motion is not the geodesic one, i.e. mutually inertial observers are not free. In contrast, Eq.(8) turns out to be solution of geodesic equations of motion.
As it should be, the transformations obtained coincide with the Lorentz boosts in the limit $\lambda \to 0$. Note also that Eq. (8) is consistent with the maximum signal velocity. Actually

$$v = \frac{dx}{dt} = \frac{1 + 2\lambda ct}{\sqrt{\frac{1}{v^2} + \frac{4\lambda}{c}(t + \lambda ct^2)}},$$ (11)

from which it follows $v \to c$ as $t \to \infty$. Then an observer with initial velocity $V < c$ will have velocity $v$ less than $c$ in the future, as it should be. Let us point out that this picture suggests interpretation of singularity presented in Eqs. (5), (6) in terms of event horizon of an observer.

4 Inertial observers are replaced by accelerated ones in CDSR

The invariant interval (5) suggests the following action for a particle motion [1]

$$S = \frac{1}{2} \int d\tau \left[ \frac{\eta_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}{e(1 + 2\lambda x^0 - \lambda^2 x^2)^2} - em^2 \right].$$ (12)

It is invariant under the global symmetry (3), under the ”translations”: $x'\mu = (SC\lambda)^{-1}e^a_\mu \partial (SC\lambda)x^\mu$ with the parameters $a^\mu$, as well as under the reparametrizations $\tau \longrightarrow \tau'(\tau)$, $x'(\tau)' = x^\mu(\tau)$, $e'(\tau') = \partial \tau$ $\partial \tau'/e(\tau)$. Hamiltonian formulation of the theory has been described in some details in [1]. Dynamics is governed by the following equations of motion and constraint:

$$\dot{x}^\mu = \tilde{\Omega}^2 p^\mu, \quad \dot{p}^\mu = -\frac{2m^2}{\tilde{\Omega}}(\delta^\mu_0 \lambda + \lambda^2 x^\mu),$$ (13)

$$p^2 = -\tilde{\Omega}^{-2} m^2.$$(14)

where $\tilde{\Omega} \equiv (1 + 2\lambda x^0 - \lambda^2 x^2)$, $p$ represents conjugated momentum for $x$, and the standard gauge $e = 1$ for the constraint $p_e = 0$ has been chosen.

Let us find equations of motion for the physical variables $x^i(t)$, where $t = x^0$. To this end one needs to fix a gauge for the first class constraint (14). Note that the standard gauge $x^0 = p^0 \tau$ is not
covariant, since \( x^0 \) and \( p^0 \) have different transformation laws. The covariant gauge turns out to be (see [1] for interpretation of this expression)

\[
x^0 = \tilde{\Omega} \left[ \Omega \left( p^0 - 2\lambda(xp) \right) - 2 \left( x^0 - \lambda x^2 \right) \left( \lambda p^0 - \lambda^2(xp) \right) \right] \tau + \lambda(x x) = p^0 \tau + O(\lambda),
\]

(15)

It can be shown that the gauge is consistent with Eq. (13). In the gauge chosen equations of motion for \( x^i(x^0) \), \( p^i(x^0) \) can be written in the form

\[
\dot{x}^i = \tilde{\Omega}^{ij} \left( 1 + \left( \frac{\dot{\Omega}^{ij}}{m} \right)^2 \right)^{-\frac{1}{2}},
\]

\[
\dot{p}^i = -2m\lambda^2 \tilde{\Omega}^{-2} x^i \left( 1 + \left( \frac{\dot{\Omega}^{ij}}{m} \right)^2 \right)^{-\frac{1}{2}},
\]

(16)

or, equivalently

\[
m\tilde{\Omega}^{-1} (1 - (\dot{x}^j)^2)^{-\frac{1}{2}} \dot{x}^i = p^i,
\]

\[
\dot{p}^i = -2m\lambda^2 \tilde{\Omega}^{-2} (1 - (\dot{x}^j)^2)^{\frac{1}{2}} x^i.
\]

(17)

It implies the following equations for \( x^i(t) \)

\[
\frac{dx^i}{dt} = 2\lambda c \tilde{\Omega}^{-1} \left( 1 - \left( \frac{\dot{x}^j}{c} \right)^2 \right) \left( (1 + \lambda c t) \dot{x}^i - \lambda c x^i \right).
\]

(18)

In the first order on \( \lambda \) one has

\[
\ddot{x}^i = 2\lambda c \left( 1 - \left( \frac{\dot{x}^j}{c} \right)^2 \right) \dot{x}^i + O(\lambda^2).
\]

(19)

One notes that \( x^i = dt + b^i \) is not a solution of the equations. In contrast, it can be verified by direct computation that the trajectory of observer \( O' \) obeys the equation and represents example of geodesic line of the model. Thus the Lorentz boost describe transformation among geodesically moving observers, the latter replace inertially moving observers of special relativity.

5 Conclusion

As it was discussed, CDSR model can be considered as special relativity theory of accelerated according to Eq. (8) observers. This
expression turns out to be solution of geodesic-line equation of the theory (18), the parameter $\lambda$ then has interpretation as a rate of expansion, and can be identified$^2$ with the Hubble constant $\lambda c \sim H_0$. It suggests modification of ordem $\lambda$ of Newtonian dynamics in average-velocities region in accordance with Eq. (8): $x(t) = Vt + H_0V \left(1 - \frac{V^2}{c^2}\right)t^2 = Vt + a_0t^2$. Departure from Newton laws occurs in the limit of small accelerations $a \sim a_0$ of test particles. Let us point that such a kind modification of non relativistic dynamics (MOND program) has been quite successfull in explaining of rotational curves of galaxies [8] without introducing of dark matter. So, our suggestion is that CDSR model may represent relativistic basis for the MOND program.

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$^2$It is not in contradiction with discussion in the end of Sect. 2, since our analysis is the local one (metric is singular).
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