ON ESTIMATES FOR THE FU-YAU GENERALIZATION OF A STROMINGER SYSTEM

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Abstract

We study an equation proposed by Fu and Yau as a natural n-dimensional generalization of a Strominger system that they solved in dimension 2. It is a complex Hessian equation with right hand side depending on gradients. Building on the methods of Fu and Yau, we obtain \( C^0, C^2, \) and \( C^{2,\alpha} \) a priori estimates. We also identify difficulties in extending the Fu-Yau arguments for non-degeneracy from dimension 2 to higher dimensions.

1 Introduction

In 1985, Strominger [20] proposed a system of equations for compactifications of superstring theories which satisfy the key physical requirement of \( N = 1 \) supersymmetry. These equations are also remarkable from the mathematical standpoint, as they combine in a novel way features of Ricci-flat metrics on Calabi-Yau manifolds together with Hermitian-Einstein metrics on holomorphic vector bundles. Solutions of Strominger systems were indeed obtained perturbatively by Li and Yau [14] from Ricci-flat and Hermitian-Einstein metrics. However, non-perturbative solutions proved to be daunting, and it was a major breakthrough when Fu and Yau [7] obtained the first such solution, some twenty years after Strominger’s original proposal.

The particular Strominger solution obtained by Fu and Yau was a toric fibration over a \( K3 \) surface. For such manifolds, Fu and Yau succeeded in reducing the Strominger system to the special case in dimension \( n = 2 \) of the following equation, on a compact \( n \)-dimensional Kähler manifold \( (X, \omega) \),

\[
i\ddbar(e^u - \alpha fe^{-u}) \wedge \omega^{n-1} + n\alpha i\ddbar u \wedge i\ddbar u \wedge \omega^{n-2} + \mu \frac{\omega^n}{n!} = 0,
\]

where \( \alpha > 0 \) is a constant, \( f \geq 0 \) is a smooth function, \( \mu \) is a smooth function such that \( f_X \mu = 0 \), and the ellipticity condition described further below in (2.3) is imposed. When \( n = 2 \), it becomes a Monge-Ampère equation, and Fu and Yau [7] suggested the problem of studying the equation (1.1) for general dimension \( n \).

\[\text{1Work supported in part by the National Science Foundation under Grant DMS-12-66033 and DMS-1308136. Keywords: Hessian equations; symmetric functions of eigenvalues; Moser iteration, inequalities of Guan-Ren-Wang; maximum principles. AMS classification numbers: 32Q26 (32Q15, 32Q20, 32U05, 32W20), 35Kxx.}\]
In this paper, we provide a partial answer to the problem raised by Fu and Yau. More specifically, we express the equation (1.1) in a more standard complex Hessian type equation (see (2.7)), and we establish $C^0$, $C^2$, and $C^{2,\alpha}$ a priori estimates for the equation. An upper $C^1$ bound is automatic from the equation. But just as in the case $n = 2$ treated by Fu and Yau, the $C^2$ estimate is contingent upon a lower bound for the second symmetric function $\sigma_2(g')$ of the eigenvalues of the unknown Hermitian form $g'_{kj}$ given in (2.4). Indeed, this is equivalent to an improved gradient estimate. One of the key innovations of Fu and Yau was a proof of such a lower bound in dimension $n = 2$. However, while we were able to obtain a sharp generalization of their computations to arbitrary dimensions, it turned out that this was not strong enough to imply the desired lower bound (see §7), and it is at this time unclear whether such a lower bound does hold.

Our proof of the $C^0$ estimate is a close parallel of the proof by Moser iteration methods used in [7]. The $C^2$ estimate also builds in an essential way on the methods of [7], but we also exploit some new inequalities due to Guan, Ren, and Wang [10] in their work on real Hessian equations with gradient terms on the right hand side. Although the $C^{2,\alpha}$ estimate does not require much new work, it does not follow from the classical Evans-Krylov theory due to the dependence of the gradient on the right hand side. However, we can obtain the desired estimate by using the recent works of Wang [26] and Tosatti-Wang-Weinkove-Yang [24] which deal with the $C^{2,\alpha}$ regularity of complex Monge-Ampère type equations with Hölder regular right hand side. The estimate still open is the lower bound for $\sigma_2(g')$. We discuss in detail the difficulties in trying to extend to higher dimensions the Fu-Yau arguments for a lower bound for $\sigma_2(g')$. To handle higher dimensions, we work with general coordinate systems rather than the adapted ones with $\nabla u = (u_1, 0, \cdots, 0)$ used by Fu and Yau. This allows us a simplified and more transparent derivation of the Fu-Yau results for $n = 2$, and a clearer picture of why their arguments are not strong enough for higher dimensions. Because of the complexity of the calculations and possibly for future use, this is presented in detail in section §7.

2 The Fu-Yau Equation

We begin by writing equation (1.1) proposed by Fu-Yau [7] in a more explicit form. Let $\Lambda$ be a Hermitian (1, 1)-form, and let $\sigma_k(\Lambda)$ be the $k$-th symmetric function of its eigenvalues relative to the Kähler form $\omega$, that is,

$$\sigma_k(\Lambda) = \binom{n}{k} \Lambda^k \wedge \omega^{n-k} = \sum_{j_1 < \cdots < j_k} \Lambda_{j_1} \cdots \Lambda_{j_k}$$

(2.1)

where $\Lambda_j$ denotes the eigenvalues of $\Lambda$ relative to $\omega$. We shall also simplify equation (1.1) by writing $f$ instead of $\alpha f$ and $\mu$ instead of $\frac{\mu}{(n-2)!}$. Using this notation, equation (1.1) can then be rewritten as

$$(n - 1) \Delta (e^u - f e^{-u}) + 2n \alpha \sigma_2(i \partial \bar{\partial} u) + \mu = 0.$$  

(2.2)
The ellipticity condition for this equation is that the $(1,1)$-Hermitian form $\tilde{g}_{jk}$ defined below be strictly positive definite,

$$\tilde{g}_{jk} = (n-1)(e^u + fe^{-u})g_{jk} + 2n\alpha((\Delta u)g_{jk} - u_{jk}) > 0,$$

where $\omega = i \sum g_{jk}dz^k \wedge d\bar{z}^j$. It is convenient to introduce also the following Hermitian $(1,1)$-form,

$$g'_{jk} = (e^u + fe^{-u})g_{jk} + 2n\alpha u_{jk}. \quad (2.4)$$

If we denote by $\lambda_j$ the eigenvalues of $i\partial \bar{\partial} u$, by $\lambda'_j$ the eigenvalues of $g'_{jk}$, and by $\tilde{\lambda}_j$ the eigenvalues of $\tilde{g}_{jk}$, all with respect to $g_{jk}$, then it is easy to see that

$$\lambda'_j = (e^u + fe^{-u}) + 2n\alpha \lambda_j, \quad \tilde{\lambda}_j = \sum_{k \neq j} \lambda'_k, \quad (2.5)$$

and hence the following relations between the symmetric functions of $i\partial \bar{\partial} u$ and $g'_{jk}$,

$$\sigma_1(g') = n(e^u + fe^{-u}) + 2n\alpha \sigma_1(i\partial \bar{\partial} u)$$

$$\sigma_2(g') = 4n^2\alpha^2 \sigma_2(i\partial \bar{\partial} u) + 2n(n-1)\alpha(e^u + fe^{-u})\sigma_1(i\partial \bar{\partial} u) + \frac{n(n-1)}{2}(e^u + fe^{-u})^2$$

and between the symmetric functions of $g'_{jk}$ and $\tilde{g}_{jk}$,

$$\sigma_1(\tilde{g}) = (n-1)\sigma_1(g')$$

$$\sigma_2(\tilde{g}) = \frac{1}{2}(n-1)(n-2)\sigma_1(g')^2 + \sigma_2(g'). \quad (2.6)$$

Substituting equation (2.2) in the above expression for $\sigma_2(g')$, we can re-write the equation in terms of $g'_{jk}$ as

$$\sigma_2(g') = \frac{n(n-1)}{2}e^{2u}(1 - 4\alpha e^{-u}|Du|^2) + 2n\alpha(n-1)fe^{-u}|Du|^2$$

$$+ 2n\alpha(n-1)f + \frac{n(n-1)}{2}e^{-2u}f^2 - 2n\alpha \mu$$

$$+ 2n\alpha(n-1)e^{-u}(\Delta f - 2Re(g^{jk}f_j u_k)). \quad (2.7)$$

It follows that equations (2.2) and (2.7) are equivalent when $\alpha \neq 0$. Here as in the rest of the paper, we denote by $D$ the covariant derivative with respect to the given metric $g_{kj}$. Furthermore, as in [7], we impose a normalization condition on a solution $u$. Let $\beta = \frac{n}{n-1}$, and $\gamma = \frac{4}{\beta-1}$. For $A \ll 1$, we impose

$$\left(\int_X e^{-\gamma u}\right)^{\frac{1}{\gamma}} = A. \quad (2.8)$$
The ellipticity condition for equation (2.7) is that the eigenvalues of \( g'_{jk} \) with respect to the metric \( g_{jk} \) should be in the \( \Gamma_2 \) cone,

\[
\Gamma_2 = \{ \lambda' \in \mathbb{R}^n; \sigma_1(\lambda') > 0, \sigma_2(\lambda') > 0 \}.
\] (2.9)

Moreover, we remark that \( g' \in \Gamma_2 \) implies that \( \tilde{g}_{jk} > 0 \) by relation (2.5).

The equation (2.7) fits in the framework of complex Hessian equations on closed manifolds, which have been studied extensively by many authors in recent years, see for example, [1, 3, 4, 11, 12, 15, 16, 21, 22, 23, 28, 29]. However, in comparison with previous works, (2.7) has two new difficulties. The first difficulty is the dependence on the gradient of the right hand side of the equation. This causes some trouble when attempting to obtain a \( C^2 \) estimate. The second difficulty is the possible degeneracy of the equation. It is easy to see that even for the ideal case \( f = \mu = 0 \) in equation (2.7), the right hand side might be zero. Therefore, to get smooth solutions, one needs to show that it is not degenerate under certain conditions on \( A \). See §4 and §7 for more discussions of this particular difficulty.

Before moving to next subsection, we want to emphasize that these two difficulties occur when \( \alpha > 0 \) in equation (2.7). If \( \alpha < 0 \), the behavior of the equation is quite different and Fu-Yau studied the \( n = 2 \) case in [8]. We will investigate the higher dimensional case in other work.

2.1 The linearization \( F^{jk} \) of \( \sigma_2(g') \)

We can view the Fu-Yau equation (2.7) as a complex Hessian equation of \( \sigma_2 \) type, with a right hand side depending on \( Du \). In accordance with standard notation in partial differential equations, we also denote \( \sigma_2(g') \) by \( F \), viewed as a function of \( u \), \( Du \), and \( DD_u \). In particular, \( F^{jk} \equiv \partial F/\partial g'_{kj} \), and the linearization of \( \sigma_2(g') \) is given by

\[
\delta \sigma_2(g') = F^{jk} \delta g'_{kj}.
\] (2.10)

We shall need explicit formulas for \( F^{jk} \), and for the operator \( 2n\alpha F^{jk} D_j D_k \) acting on \( u \), the gradient \( Du \) of \( u \), the square \( |Du|^2 \) of the gradient, and the complex hessian \( D_p D_q u \).

We summarize briefly here our notations and conventions. The Hermitian form \( \omega \) defined by a Kähler metric \( g_{kj} \) is given by \( \omega = ig_{kj} dz^j \wedge d\bar{z}^k \). The Chern unitary connection with respect to the metric \( \omega \) is denoted by \( D_j = \partial_{\bar{z}^j} \), \( D_j V^p = g^{pq} \partial_j (g_{qm} V^m) \), and the curvature tensor is defined by

\[
[D_k, D_j] V^m = -R_{kj}^m g_{\ell p} V^p.
\]

The Ricci curvature \( R_{kj} \) is given by \( R_{kj} = R_{kj}^m g_{\ell m} \). Given a second Hermitian tensor \( g'_{km} \), the relative endomorphism \( h^{jk} \) from \( g'_{km} \) to \( g_{km} \) is defined by

\[
h^{jk} = g^{km} g'_{mj}.
\]
Writing $\sigma_2(g') = ((\text{Tr } h)^2 - \text{Tr } h^2)/2$, we readily find

$$F^{jk} = g^{jp}g^{kq}\tilde{g}_{pq},$$

(2.11)

where $\tilde{g}_{pq}$ is the metric introduced in (2.3). In particular $F^{jk}g_{kj} = (n-1)\text{Tr } h$, and hence

$$2n\alpha F^{jk}D_jD_ku = F^{jk}g'_{kj} - (e^u + fe^{-u})F^{jk}g_{kj} = 2F - (n-1)(e^u + fe^{-u})\text{Tr } h.$$  

(2.12)

Next, the variational formula for $\sigma_2(g')$ implies

$$\partial_p F = F^{jk}D_p g'_{kj}.$$  

(2.13)

Substituting in the definition of $g'_{kj}$, we obtain the following formula for $2n\alpha F^{jk}D_jD_k(D_pu)$,

$$2n\alpha F^{jk}D_jD_k(D_pu) = \partial_p F - (n-1)\text{Tr } h \partial_p(e^u + fe^{-u}).$$  

(2.14)

Similarly, we find

$$2n\alpha F^{jk}D_jD_k(D_pu) = \partial_p F - (n-1)\text{Tr } h \partial_p(e^u + fe^{-u}) + 2n\alpha \tilde{g}_{\ell m}R^{\ell m\bar{q}}D_q u$$  

(2.15)

where $R^{\bar{q}}_{\ell m\bar{q}}$ is the curvature of metric $\omega$. This additional curvature term resulted from the commutation of covariant derivatives $D_p D_j$ and $D_j D_p$ when acting on $D_k u$. It is now easy to deduce $2n\alpha F^{jk}D_jD_k|Du|^2$. Introduce the notation

$$|DU|^2_{Fg} = F^{jk}g^{\ell m}D_j\ell u D_k\bar{\ell}u, \quad |D\bar{D}u|^2_{F\bar{g}} = F^{jk}g^{\ell m}D_{j\bar{m}}u D_{k\bar{m}}u.$$  

(2.16)

Then

$$2n\alpha F^{jk}D_jD_k|Du|^2 = 2n\alpha g^{\ell m}F^{jk}(D_jD_kD_\ell u D_{\bar{m}}u + D_{\ell u}D_jD_kD_{\bar{m}}u) + 2n\alpha (|DU|_{Fg}^2 + |D\bar{D}u|_{F\bar{g}}^2)$$  

(2.17)

and hence, in view of the formulas (2.14) and (2.15),

$$2n\alpha F^{jk}D_jD_k|Du|^2 = g^{\ell m}(\partial_\ell F\partial_{\bar{m}}u + \partial_{\bar{m}}F\partial_\ell u) + 2n\alpha \tilde{g}_{\ell m}\partial_\ell \partial_{\bar{m}}u R^{\ell m\bar{q}}\partial_\bar{q} u$$

$$- (n-1)\text{Tr } h g^{\ell m}(\partial_\ell (e^u + fe^{-u})\partial_{\bar{m}}u + e^u + fe^{-u})\partial_{\bar{m}}u$$

$$+ 2n\alpha (|DU|_{Fg}^2 + |D\bar{D}u|_{F\bar{g}}^2).$$  

(2.18)

Finally, the operator $2n\alpha F^{jk}D_jD_k$ acting on the Hessian $D_pD_q u$ can be obtained in a similar way from differentiating the equation (2.13) again, giving

$$F^{jk}D_pD_qg'_{kj} = \partial_p\partial_q F - D_p(\text{Tr } h)D_q(\text{Tr } h) + D_jh^jD_kh^k.$$  

(2.19)
We can extract the term \( D_p D_q D_j D_k u \) from the left-hand side. Permuting the order of differentiation, we find

\[
2n\alpha F^{jk} D_p D_q g_{kj} u
\]

\[
= 2n\alpha F^{jk} D_p D_q D_j D_k u + 2n\alpha (F^{jk} R_{qjk} a_{ap} - F^{jk} R_{qpk} a_{uj})
\]

\[
= F^{jk} D_p D_q g_{k}^{j} - (e^u - f e^{-u}) u_p g_{qk} F^{jk} g_{kj} - (e^u + f e^{-u}) u_p g_{qk} F^{jk} g_{kj}
\]

\[
+ 2e^{-u} \text{Re}(u_p f_q) F^{jk} g_{kj} - f_q e^{-u} F^{jk} g_{kj} + 2n\alpha (F^{jk} R_{qjk} a_{ap} - F^{jk} R_{qpk} a_{uj})
\]

\[
= \partial_p \partial_q F - D_p (\text{Tr} h) D_q (\text{Tr} h) + D_j h^p D_k h^q + 2n\alpha (F^{jk} R_{qjk} a_{ap} - F^{jk} R_{qpk} a_{uj})
\]

\[
+ \left\{ -(e^u - f e^{-u}) D_p D_q u - (e^u + f e^{-u}) D_p u D_q u + 2e^{-u} \text{Re}(u_p f_q) - e^{-u} D_p u D_q f \right\} (n - 1) \text{Tr} h.
\]

All these formulas are quite general. For the specific Fu-Yau equation, we can substitute the right hand side of equation (2.7) for \( F = \sigma_2 (g') \), as we shall do in sections $\S 4$ and $\S 7$.

### 3 The \( C^0 \) Estimate

The following \( C^0 \) estimate holds:

**Theorem 1** Let \((X, \omega)\) be a compact Kähler manifold of dimension \(n\) with \(\text{Vol}(X, \omega) = 1\). Let \(u\) be a solution of (2.7) under ellipticity condition (2.3) and normalization condition (2.8). Then, for \(A < 1\), there exists a constant \(C_0\) depending only on \((X, \omega)\), \(\alpha\), \(\|f\|_{C^2}\), and \(\|\mu\|_{L^\infty}\) such that

\[
e^{-\inf u} \leq C_0 A.
\]

Furthermore, if \(A\) is chosen small enough such that \(C_0 A < 1\), then there is a constant \(C_1\) depending also only on \((X, \omega)\), \(\alpha\), \(\|f\|_{C^2}\), and \(\|\mu\|_{L^\infty}\) such that

\[
e^{\sup u} \leq C_1 A^{-1}.
\]

**Proof.** We proceed by Moser iteration. First, we define the Hermitian form corresponding to \(\tilde{g}_{ji}\):

\[
\tilde{\omega} = (n - 1)(e^u + f e^{-u}) \omega + 2n\alpha((\Delta u) \omega - i \partial \overline{\partial} u) > 0.
\]

Let \(k \geq 2\). The starting point is to compute the quantity

\[
\int_X i \partial \overline{\partial} (e^{-ku}) \wedge \tilde{\omega} \wedge \omega^{n-2}
\]

in two different ways. On one hand, by the definition of \(\tilde{\omega}\) and Stokes’ theorem, we have

\[
\int_X i \partial \overline{\partial} (e^{-ku}) \wedge \tilde{\omega} \wedge \omega^{n-2} = \int_X (n - 1) (e^{u} + f e^{-u}) i \partial \overline{\partial} (e^{-ku}) \wedge \omega^{n-1} + 2n\alpha \int_X (\Delta u) i \partial \overline{\partial} (e^{-ku}) \wedge \omega^{n-1}.
\]
Using the volume form \( \omega^n \), we compute

\[
\frac{1}{(n-1)!} \int_X i\bar{\partial}(e^{-ku}) \wedge \bar{\omega} \wedge \omega^{n-2} \tag{3.6}
\]

\[
= \int_X (n-1)(e^u + f e^{-u}) \Delta(e^{-ku}) + 2n\alpha \int_X (\Delta u)\Delta(e^{-ku})
\]

\[
= k^2(n-1) \int_X (e^u + f e^{-u})e^{-ku}|Du|^2 - k(n-1) \int_X (e^u + f e^{-u})e^{-ku} \Delta u
\]

\[
+ 2k^2n\alpha \int_X e^{-ku} \Delta u|Du|^2 - 2k\alpha \int_X e^{-ku}(\Delta u)^2.
\]

On the other hand, using equation (2.2), we obtain

\[
\int_X i\bar{\partial}(e^{-ku}) \wedge \bar{\omega} \wedge \omega^{n-2} \tag{3.7}
\]

\[
= k^2 \int_X e^{-ku}i\bar{\partial}u \wedge \bar{\partial}u \wedge \bar{\omega} \wedge \omega^{n-2} - k \int_X (n-1)e^{-ku}(e^u + f e^{-u})i\bar{\partial}u \wedge \omega^{n-1}
\]

\[
-2k\alpha \int_X e^{-ku}i\bar{\partial}u \wedge i\bar{\partial}u \wedge \omega^{n-2} + 2k\alpha \int_X e^{-ku}i\bar{\partial}u \wedge i\bar{\partial}u \wedge \omega^{n-2}
\]

\[
= k^2 \int_X e^{-ku}i\bar{\partial}u \wedge \bar{\partial}u \wedge \bar{\omega} \wedge \omega^{n-2} - k \int_X (n-1)e^{-ku}(e^u + f e^{-u})i\bar{\partial}u \wedge \omega^{n-1}
\]

\[
-2k\alpha \int_X e^{-ku}i\bar{\partial}u \wedge \bar{\partial}u \wedge \omega^{n-2} - 2k(n-2)! \int_X e^{-ku}u \frac{\omega^n}{n!} - 2k \int_X e^{-ku}i\bar{\partial}(e^u - f e^{-u}) \wedge \omega^{n-1}.
\]

Expanding out terms and using the definition of \( \bar{\omega} \) yields

\[
\frac{1}{(n-1)!} \int_X i\bar{\partial}(e^{-ku}) \wedge \bar{\omega} \wedge \omega^{n-2} \tag{3.8}
\]

\[
= k^2(n-1) \int_X e^{-ku}(e^u + f e^{-u})|Du|^2 + k^2(2n\alpha) \int_X e^{-ku} \Delta u|Du|^2
\]

\[
- \frac{k^2(2n\alpha)}{(n-1)!} \int_X e^{-ku}i\bar{\partial}u \wedge i\bar{\partial}u \wedge \omega^{n-2} - k(n-1) \int_X e^{-ku}(e^u + f e^{-u}) \Delta u
\]

\[
- 2k\alpha \int_X e^{-ku}(\Delta u)^2 - \frac{2k}{n-1} \int_X e^{-ku} \mu - 2k \int_X e^{-(k-1)u}(|Du|^2 + \Delta u) - 2k \int_X e^{-(k+1)u} f \Delta u
\]

\[
+ 2k \int_X e^{-(k+1)u} \Delta f + 2k \int_X e^{-(k+1)u} f |Du|^2 - 4k \int_X e^{-(k+1)u} Re(g^{ik} f_j u_k).
\]

We now equate (3.6) and (3.8) and cancel repeating terms.

\[
0 = -\frac{k\alpha}{(n-1)!} \int_X e^{-ku}i\bar{\partial}u \wedge \bar{\partial}u \wedge \bar{\omega} \wedge \omega^{n-2} - \int_X e^{-(k-1)u}|Du|^2 - \int_X e^{-(k-1)u} \Delta u + \int_X e^{-(k+1)u} \Delta f + \int_X e^{-(k+1)u} f |Du|^2
\]

\[
- 2 \int_X e^{-(k+1)u} Re(g^{ik} f_j u_k) - \int_X e^{-(k+1)u} f \Delta u.
\]
Integration by parts gives
\[
0 = -\frac{k\alpha}{(n-1)!} \int_X e^{-ku} i\partial u \wedge \partial u \wedge i\partial \bar{u} \wedge \omega^{n-2} - \frac{1}{n-1} \int_X e^{-ku} \mu - k \int_X e^{-(k-1)u} |Du|^2 \\
+ \int_X e^{-(k+1)u} \Delta f - k \int_X e^{-(k+1)u} f |Du|^2 - \int_X e^{-(k+1)u} g^{ij} f_k u_k. \tag{3.10}
\]

One more integration by parts yields the following identity:
\[
k \int_X e^{-ku} |Du|^2 (e^u + fe^{-u}) = -\frac{k\alpha}{(n-1)!} \int_X e^{-ku} i\partial u \wedge \partial u \wedge i\partial \bar{u} \wedge \omega^{n-2} - \frac{1}{n-1} \int_X e^{-ku} \mu + \left(1 - \frac{1}{k+1}\right) \int_X e^{-(k+1)u} \Delta f. \tag{3.11}
\]

We now estimate the first term on the right hand side. At a point \( p \in X \), choose coordinates such that \( \bar{g}_{ij} = \delta_{ij} \) and \( u_{kj} \) is diagonal. From the condition \( \bar{g} > 0 \), we see that \( \bar{g}_{kk} = (n-1)(e^u + fe^{-u}) + 2n\alpha(\Delta u - u_{kk}) > 0 \) at \( p \). We compute
\[
i\partial u \wedge \partial u \wedge i\partial \bar{u} \wedge \omega^{n-2} = (n-2)! \sum_i |u_i|^2 (\Delta u - u_{ii}) \omega^n_{n!} > \frac{(n-1)!}{2n\alpha} |Du|^2 (e^u + fe^{-u}) \omega^n_{n!}. \tag{3.12}
\]

Using this inequality in (3.11), we obtain
\[
\frac{k}{2} \int_X e^{-ku} |Du|^2 (e^u + fe^{-u}) \leq -\frac{1}{n-1} \int_X e^{-ku} \mu + \left(1 - \frac{1}{k+1}\right) \int_X e^{-(k+1)u} \Delta f. \tag{3.13}
\]

Since \( f \geq 0 \), we can deduce the following estimate:
\[
k \int_X e^{-(k-1)u} |Du|^2 \leq C \left( \int_X e^{-ku} + \int_X e^{-(k+1)u} \right). \tag{3.14}
\]

Therefore, for \( k \geq 1 \), we have
\[
\int_X |De^{-\frac{k}{2}u}|^2 \leq Ck \left( \int_X e^{-(k+1)u} + \int_X e^{-(k+2)u} \right). \tag{3.15}
\]

To obtain a \( C^0 \) estimate, we use the method of Moser iteration as done in [7]. We set \( \beta = \frac{n}{(n-1)} \). The Sobolev inequality gives us
\[
\left( \int_X |e^{-\frac{1}{2}u}|^{2\beta} \right)^{\frac{1}{\beta}} \leq C \left( \int_X |e^{-\frac{1}{2}u}|^2 + \int_X |De^{-\frac{1}{2}u}|^2 \right). \tag{3.16}
\]

Combining the Sobolev inequality with (3.15) yields
\[
\left( \int_X e^{-k\beta u} \right)^{\frac{1}{\beta}} \leq Ck \left( \int_X e^{-ku} + \int_X e^{-(k+1)u} + \int_X e^{-(k+2)u} \right). \tag{3.17}
\]
Applying Hölder’s inequality, we get
\[
\left( \int_X e^{-k\beta u} \right)^{\frac{1}{\beta}} \leq Ck\left\{ \left( \int_X e^{-(k+2)u} \right)^{\frac{1}{k+2}} + \left( \int_X e^{-(k+2)u} \right)^{1 + \frac{1}{k+2}} \right\}.
\] (3.18)

For this inequality to be useful, we need to take \( k \) large enough so that \( k\beta \geq k + 2 \). In order to proceed with the iteration, we consider two cases.

**Case 1:** For all \( k \geq \gamma = \frac{4}{\beta-1} \), we have \( \int_X e^{-ku} \leq 1 \). In this case, for each \( k \geq \gamma \), (3.18) gives us
\[
\left( \int_X e^{-k\beta u} \right)^{\frac{1}{\beta}} \leq Ck \left( \int_X e^{-(k+2)u} \right)^{\frac{1}{k+2}}.
\] (3.19)

Using Hölder’s inequality, we also have
\[
\int_X e^{-(k+2)u} = \int_X (e^{-u}) \left( e^{-u} \right)^{2k-\gamma} \leq \left( \int_X e^{-k\beta u} \right)^{\frac{1}{k+2}} \left( \int_X e^{-ku} \right)^{1 - \frac{1}{k+2}}.
\] (3.20)

Therefore
\[
\left( \int_X e^{-k\beta u} \right)^{\frac{1}{\beta}} \leq Ck \left( \int_X e^{-k\beta u} \right)^{\frac{1}{k+2}} \left( \int_X e^{-ku} \right)^{2k-\gamma}. \] (3.21)

By regrouping and using the identity \( \gamma\beta = 4 + \gamma \), we obtain
\[
\left( \int_X e^{-k\beta u} \right)^{\frac{1}{\beta}} \leq \left( Ck \right)^{\frac{2(k+2)}{2k-\gamma}} \int_X e^{-ku}.
\] (3.22)

Since \( k \geq \gamma \geq 4 \), we have \( \frac{2(k+2)}{2k-\gamma} \leq \frac{2(k+2)}{k} \leq 3 \). Thus
\[
\|e^{-u}\|_{L^{k\beta}} \leq \left( Ck \right)^{3/k} \|e^{-u}\|_{L^k}, \] (3.23)

for \( k \geq \gamma \). We iterate this estimate and conclude
\[
e^{-\inf u} = \|e^{-u}\|_{L^\infty} \leq C\|e^{-u}\|_{L^\gamma} = CA. \] (3.24)

**Case 2:** There exists a \( k_0 > \gamma \) such that \( \int_X e^{-k_0u} > 1 \). In this case, using \( \text{Vol}(X, \omega) = 1 \) and Hölder’s inequality, we have \( \int_X e^{-ku} > 1 \) for all \( k \geq k_0 \). After possibly increasing \( k_0 \), we take \( k \geq k_0 \geq \gamma\beta > \gamma \). From (3.18) and (3.20), we have
\[
\left( \int_X e^{-k\beta u} \right)^{\frac{1}{\beta}} \leq Ck \int_X e^{-(k+2)u} \leq Ck \left( \int_X e^{-k\beta u} \right)^{\frac{1}{k+2}} \left( \int_X e^{-ku} \right)^{1 - \frac{1}{k+2}}.
\] (3.25)

After rearranging, we obtain
\[
\|e^{-u}\|_{L^{k\beta}} \leq \left( Ck \right)^{\frac{2}{2k-\gamma}} \|e^{-u}\|_{L^k}^{2k-\gamma}. \] (3.26)
Since we assume $k \geq \gamma \beta$, we conclude
\[
\|e^{-u}\|_{L^{k,\beta}} \leq (Ck)^{\frac{2}{\beta}} \|e^{-u}\|_{L^{k,\beta}}^{2k-\gamma}. \tag{3.27}
\]
We set
\[
\Theta(\nu) = \frac{1}{\beta} \left( \frac{2k_0 \beta^\nu - \gamma}{2k_0 \beta^\nu - 1 - \gamma} \right). \tag{3.28}
\]
For $i, j \in \mathbb{N}$ with $j \geq 0$ and $i \geq j$, we have
\[
\prod_{\nu=j}^i \Theta(\nu) = \frac{2k_0 - \frac{\gamma}{\beta^j}}{2k_0 - \frac{\gamma}{\beta^{i-j+1}} - \gamma} \leq \frac{2k_0}{2k_0 - \gamma \beta} \leq 2. \tag{3.29}
\]
Therefore we can iterate our estimate in the following way:
\[
\|e^{-u}\|_{L^{k_0,\beta^{i+1}}} \leq (Ck_0 \beta^i)^{\frac{2}{\beta} \|e^{-u}\|_{L^{k_0,\beta}}} \leq \left( \prod_{j=0}^{i} (Ck_0 \beta^j)^{\frac{2}{\beta} \prod_{\nu=j+1}^i \Theta(\nu)} \right) \|e^{-u}\|_{L^{k_0}}^{\prod_{\nu=0}^i \Theta(\nu)} \leq \left( \prod_{j=0}^\infty (Ck_0 \beta^j)^{\frac{4}{\beta}} \right) \|e^{-u}\|_{L^{k_0}} \leq C \|e^{-u}\|_{L^{k_0}}^2. \tag{3.30}
\]
As we let $i \to \infty$, we have
\[
\|e^{-u}\|_{L^\infty} \leq C \|e^{-u}\|_{L^{k_0}}^2. \tag{3.31}
\]
We would like to estimate $\|e^{-u}\|_{L^{k_0}}$ in terms of $\|e^{-u}\|_{L^\gamma}$. Starting from (3.18), we can follow either case 1 or case 2, depending on the size of $\int e^{-(k+2)u}$. We then arrive at estimate (3.23) or (3.26):
\[
\|e^{-u}\|_{L^{k_0}} \leq C \|e^{-u}\|_{L^{k_0}}, \text{ or } \|e^{-u}\|_{L^{k_0}} \leq C \|e^{-u}\|_{L^{k_0}}^r, \text{ for some } r > 1. \tag{3.32}
\]
By repeating this process finitely many times, we can control $\|e^{-u}\|_{L^{k_0}} \leq C \|e^{-u}\|_{L^\gamma}$ for some $a \geq 1$. Since $\|e^{-u}\|_{L^\gamma} = A < 1$, we have
\[
e^{-\inf u} = \|e^{-u}\|_{L^\infty} \leq C \|e^{-u}\|_{L^\gamma}^2 \leq CA. \tag{3.33}
\]
To control the supremum of $u$, we replace $k$ with $-k$ in (3.11). Then, for $k \neq 1$,
\[
k \int_X e^{ku} |D^2 u|^2 (e^u + f e^{-u}) \tag{3.34}
\]
\[
= -\frac{k \alpha}{(n-1)!} \int_X e^{ku} i \partial u \wedge \partial u \wedge i \partial \overline{\partial} u \wedge \omega^{n-2} + \frac{1}{n-1} \int_X e^{ku} \mu - \frac{1}{1-k} \int_X e^{(k-1)u} \Delta f.
\]
Proceeding as before in the case of the infimum estimate, we can use (3.12) to derive the following estimate for any $k$ greater than a fixed number greater than 1
\[
k \int_X e^{(k+1)u} |D^2 u|^2 \leq C \left( \int_X e^{ku} + \int_X e^{(k-1)u} \right). \tag{3.35}
\]
Thus for $k \geq 2\beta$, we can estimate

$$\int_X |De^{\frac{k}{2}u}|^2 \leq Ck \left( \int_X e^{(k-1)u} + \int_X e^{(k-2)u} \right). \quad (3.36)$$

Since $e^{-\inf u} = CA \ll 1$, we can conclude

$$\int_X |De^{\frac{k}{2}u}|^2 \leq Ck \int_X e^{ku} , \quad (3.37)$$

for $k \geq 2\beta$. The Sobolev inequality yields

$$\left( \int_X e^{k\beta u} \right)^{1/\beta} \leq Ck \int_X e^{ku}. \quad (3.38)$$

By iterating this estimate, we have

$$e^{\sup u} \leq C\|e^{u}\|_{L^{2\beta}}. \quad (3.39)$$

To complete the supremum estimate, we need another inequality. Setting $k = -1$ in (3.10), we have

$$\int_X e^{2u} |Du|^2 = -\frac{n\alpha}{(n-1)!} \int_X e^u i\bar{\partial}u \wedge \bar{i\bar{\partial}u} \wedge \omega^{n-2} + \frac{1}{n-1} \int_X e^u \mu - \int_X \Delta f - \int_X f |Du|^2 + \int_X g^{jk}f_{j\bar{k}}. \quad (3.40)$$

We estimate the first term on the RHS by using (3.12), and since $e^u \geq 1$ we obtain

$$\int_X e^{2u} |Du|^2 \leq C \left( \int_X e^u + \int_X |u| + 1 \right) \leq C \left( \int_X e^u \right) . \quad (3.41)$$

Therefore

$$\int_X |De^u|^2 \leq C \left( \int_X e^u \right). \quad (3.42)$$

Either by using this estimate, or using a scaling argument, one can obtain from (3.39) that

$$e^{\sup u} \leq C\|e^{u}\|_{L^2}, \quad (3.43)$$

so the objective now is to control $\|e^{u}\|_{L^2}$. Consider the set $U = \{ x : e^u \leq \frac{2}{A} \}$. We have

$$A^\gamma = \int_U e^{-\gamma u} + \int_{X \setminus U} e^{-\gamma u} \leq e^{-\gamma \inf U} |U| + \frac{A^\gamma}{2^\gamma} (1 - |U|) \leq \left( CA^\gamma - \frac{A^\gamma}{2^\gamma} \right) |U| + \frac{A^\gamma}{2^\gamma}.$$

Therefore,

$$|U| \geq \frac{1 - \frac{1}{2^\gamma}}{C - \frac{1}{2^\gamma}} := \delta > 0. \quad (3.44)$$
To estimate the $L^2$ norm of $e^u$, we follow the argument from Tosatti-Weinkove [25]. Let $\psi = e^u$, and let $\overline{\psi} := \int_X \psi$. By the Poincaré inequality and (3.42),
\[ \|\psi - \overline{\psi}\|_{L^2} \leq \|D\psi\|_{L^2} \leq C\|\psi\|_{L^1}^{1/2}. \]  
(3.45)

We compute
\[ \delta \psi \leq \int_U |\psi| \leq \int_U |\psi - \overline{\psi}| + |\psi| \leq \int_X |\psi - \overline{\psi}| + \frac{2}{A}|U| \leq \frac{C}{A}(1 + \|\psi - \overline{\psi}\|_{L^1}). \]  
(3.46)

Using the previous estimate and (3.45), it is now easy to obtain an $L^1$ estimate:
\[ \|\psi\|_{L^1} \leq \|\psi - \overline{\psi}\|_{L^1} + \|\overline{\psi}\|_{L^1} \leq CA^{-1}(1 + \|\psi - \overline{\psi}\|_{L^1}) \]
\[ \leq CA^{-1}(1 + \|\psi\|_{L^2}^{1/2}) \leq CA^{-1}(1 + \|\psi\|_{L^1}^{1/2}). \]  
(3.47)

Therefore $\|\psi\|_{L^1}$ is under control, and by (3.45), we can deduce that $\|\psi\|_{L^2} \leq CA^{-1}$. By (3.43), we have
\[ e^{\sup u} \leq CA^{-1}. \]  
(3.48)

4 The $C^1$ Estimate

As we mentioned previously, the $a priori$ gradient estimate is easy due to the special structure of the right hand side of equation (2.7). Define the constant $\kappa_c$ by
\[ \kappa_c = \frac{n(n - 1)}{2}. \]  
(4.1)

Our equation (2.7) is
\[ e^{-2u}F = \kappa_c - 4\alpha\kappa_c\left\{ e^{-u}|Du|^2 - fe^{-3u}|Du|^2 + e^{-3u}(g^{ik}f_{ik} - g^{ik}f_{ik}) \right\} + \kappa_c e^{-2u}\left\{ 2f + f^2 e^{-2u} + 4\alpha e^{-u}\Delta f \right\} - 2n\alpha e^{-2u}\mu. \]  
(4.2)

We first estimate
\[ 0 < e^{-2u}F \leq \kappa_c - 4\alpha\kappa_c e^{-u}|Du|^2 \left\{ 1 - (\|f\|_{L^\infty} + 1)e^{-2u} \right\} + O(e^{-2u}). \]  
(4.3)

For a choice of $A$ small enough, we can make $e^{-u} \leq CA \ll 1$. It follows that
\[ e^{-u}|Du|^2 \leq C. \]  
(4.4)

**Theorem 2** Let $u$ be a solution of (2.7) under ellipticity condition (2.3) and normalization condition (2.8). If $A$ is small enough, then there exists a positive constant $C$ depending on $(X, g)$, $\alpha$, $\|f\|_{C^2}$, and $\|\mu\|_{L^\infty}$ such that
\[ e^{-u}|Du|^2 \leq C. \]  
(4.5)
We observe that the present situation is different from the situation for the standard complex Hessian equation
\[ \sigma_k(g_{ji} + u_{ji}) = f(z) \] (4.6)
on a compact Kähler manifold \((X, g)\) with \(0 < f(x) \in C^\infty(X)\), see [3, 11, 29]. For the standard equation (4.6) with non-degenerate right hand side \(f(z)\), one needs to work very hard to get the gradient estimate since the upper bound of the \(C^2\) estimate depends on the \(C^1\) estimate. Once the gradient estimate is obtained, the non-degeneracy of \(f(z)\) together with the \(C^2\) upper bound imply the uniform ellipticity of the equation. In our current situation, the structure of equation (1.1) is better in the sense that it automatically gives a \(C^1\) upper bound. However, in this case, the \(C^1\) estimate is not good enough to give uniform ellipticity. For that purpose, we need to get a uniform positive lower bound for \(e^{-2u}F\), which turns out to be equivalent to a sharper \(C^1\) upper bound. From this viewpoint, the desired gradient estimate here is much more involved than in the standard case. We will continue to discuss this in §7.

5 The \(C^2\) Estimate

In this section, we derive the \textit{a priori} \(C^2\) estimate of equation (2.7) under the assumption of a \textit{sharp gradient estimate}. As previously mentioned, the presence of the gradient of \(u\) on the right hand side brings substantial difficulties. For real Hessian equations, this problem was recently addressed by Guan-Ren-Wang [10] under some assumptions. Here, we adapt some of their ideas to the complex setting. However, there are still some troublesome terms such as \(|DDu|^2\) which cannot be handled as in the real case. This is the reason for the \textit{sharp gradient estimate} assumption in our estimate. Our theorem is the following.

\textbf{Theorem 3} Let \(u\) be a solution of (2.7) under ellipticity condition (2.3) and normalization condition (2.8). Suppose that for every \(0 < \delta < 1\), there exists an \(0 < A_\delta \ll 1\) such that for all \(0 < A \leq A_\delta\), the following bound holds:
\[ e^{-u}|Du|^2 \leq \delta. \] (5.1)

Then there exists \(0 < A_0 \ll 1\) such that for all \(0 < A \leq A_0\), there holds
\[ \frac{1}{C}g \leq \tilde{g} \leq Cg, \] (5.2)
where \(C\) is a constant depending on \(\|u\|_{L^\infty}, \|Du\|_{L^\infty}, (X, \omega), \|f\|_{C^2}, \|\mu\|_{C^2}, \alpha, A\).

Let \(B_0, B_1\) be constants depending on \((X, \omega), \|f\|_{C^2}, \|\mu\|_{C^2}, \alpha\). Recall that we have the following \(C^0\) estimates
\[ e^{-u} \leq B_0A \ll 1, \quad e^u \leq B_1A^{-1}. \] (5.3)
The estimate in the assumption (5.1) was obtained by Fu-Yau in [7] when \( X \) has dimension \( n = 2 \). The Fu-Yau estimate is rederived in §7 and can be found in (7.35). Whether a Fu-Yau type gradient estimate holds for dimension \( n > 2 \) is still unknown. From equation (4.2), one can see that such an estimate implies a lower bound for \( e^{-2u} F \). For the purpose of the \( C^2 \) estimate, we shall take

\[
e^{-2u} F \geq \frac{1}{2}.
\]  

(5.4)

To prove the theorem, it suffices to obtain an upper bound on the maximal eigenvalue of \( g' \). The upper and lower bounds of \( \tilde{g} \) will then follow from the relations between \( g' \) and \( \tilde{g} \) as discussed in §2.

Before proceeding with the \( C^2 \) estimate, we state a lemma due to Guan-Ren-Wang [10].

**Lemma 1** Suppose \( v^i_j \) is an endomorphism such that \( v \in \Gamma_2 \). Then for any tensor \( A_{ij} \),

\[
- \sum_{i \neq j} A_{ii} A_{jj} \geq - \frac{|\sigma_2 (v) A_{ii}|^2}{|\sigma_2(v)|}.
\]  

(5.5)

In particular,

\[
- \sum_{i \neq j} D_k g^i_i D_k g^j_j \geq - \frac{|D_k F|^2}{|F|}.
\]  

(5.6)

**Proof.** We reproduce the proof of the Guan-Ren-Wang inequality for completeness. Let \( H = \frac{\sigma_2(v)}{\sigma_1(v)} \). Differentiate \( \log H \) with respect to the \((p,p)\) entry to obtain

\[
\frac{H^p_p}{H} = \frac{\sigma_2^{pp}}{\sigma_2} - \frac{\sigma_1^{pp}}{\sigma_1}.
\]  

(5.7)

Differentiate again

\[
\frac{H^{pp,qq}}{H} = \frac{H^{pp} H^{qq}}{H^2} + \sigma_2^{pp,qq} - \frac{\sigma_2^{pp} A_{qq} A_{qq}}{(\sigma_2)^2} + \frac{\sigma_1^{pp} \sigma_1^{qq}}{(\sigma_1)^2}.
\]  

(5.8)

Since \( H \) is concave, we have

\[
0 \geq \frac{|H^{pp} A_{pp}|^2}{H^2} + \sigma_2^{pp,qq} A_{pp} A_{qq} - \frac{|\sigma_2^{pp} A_{pp}|^2}{(\sigma_2)^2} + \frac{|\sigma_1^{pp} A_{pp}|^2}{(\sigma_1)^2}
\]

\[
\geq \frac{\sigma_2^{pp,qq} A_{pp} A_{qq}}{\sigma_2} - \left| \frac{\sigma_2^{pp} A_{pp}}{\sigma_2} \right|^2.
\]  

(5.9)

This completes the proof of the Guan-Ren-Wang inequality.
We now proceed to the proof of the $C^2$ estimate. We shall apply the maximum principle to a function similar to the one used by Hou-Ma-Wu in [11]. Let $M > 0$ be a large constant to be determined later. Let $\sup_X |u| \leq L$. Define

$$
\psi(t) = \frac{M}{2n\alpha} \log \left( 1 + \frac{t}{L} \right),
$$

(5.10)

It follows that

$$
\frac{M}{2n\alpha L} > \psi' > \frac{M}{4n\alpha L}, \quad 2n\alpha \psi'' = -\frac{|2n\alpha \psi'|^2}{M}.
$$

(5.11)

For small $\delta > 0$ to be chosen later, we define

$$
\phi(t) = -\log (M_2 - t), \quad M_2 = \frac{17\delta B_1}{A}.
$$

(5.12)

Note that $\phi(|Du|^2)$ is well-defined by the assumption on gradient estimate (5.1). Indeed, we may choose $A_0 \ll 1$ depending on $\delta$ such that, for any $0 < A \leq A_0$,

$$
|Du|^2 \leq \delta e^u \leq \frac{\delta B_1}{A},
$$

(5.13)

and hence

$$
\phi'(|Du|^2) \leq \frac{A}{16\delta B_1}.
$$

(5.14)

Furthermore, we have the lower bound

$$
\phi'(|Du|^2) \geq \frac{A}{17\delta B_1} \geq \frac{e^{-u}}{17\delta B_0 B_1},
$$

(5.15)

and the relationship

$$
\phi'' = (\phi')^2.
$$

(5.16)

First, consider

$$
G_0(z, \xi) = \log (g_{jk}' \xi^k \xi^j) - 2n\alpha \psi(u) + \phi(|Du|^2),
$$

(5.17)

for $z \in X$ and $\xi \in T_{z}^{1,0}(X)$ a unit vector. $G_0$ is not defined everywhere, but we may restrict to the compact set where $g_{jk}' \xi^k \xi^j \geq 0$ and obtain an upper semicontinuous function. Let $(p, \xi_0)$ be the maximum of $G_0$. Choose coordinates centered at $p$ such that $g_{jk} = \delta_{jk}$ and $g_{jk}'$ is diagonal. Suppose $g_{11}'$ is the largest eigenvalue of $g'$. Then $\xi_0(p) = \partial_1$, and we extend this to a local unit vector field $\xi_0 = g_{11}^{-1/2} \frac{\partial}{\partial z^1}$. Define the local function

$$
G(z) = \log (g_{11}^{-1} g_{11}') - 2n\alpha \psi(u) + \phi(|Du|^2).
$$

(5.18)

This function $G$ also attains a maximum at $p \in X$. We will compute at the point $p$. We shall be assuming that $g_{11}'(p) \gg 1$, otherwise we would already have an upper bound on the maximal eigenvalue of $g'$ and the $C^2$ estimate would be complete.
Covariantly differentiating $G$ gives

$$G_{\bar{j}} = \frac{(e^u + f e^{-u})\bar{\bar{j}} + 2n\alpha D_1 D_1 u}{g'_{11}} + \phi'|D\bar{u}|^2_F - 2n\alpha \psi' u_{\bar{j}}. \quad (5.19)$$

Differentiating $G$ a second time and contracting with $F^{\bar{i}\bar{j}}$ yields

$$F^{\bar{i}\bar{j}} G_{\bar{j}i} = \frac{2n\alpha}{g'_{11}} F^{\bar{i}\bar{j}} D_1 D_1 D_1 D_1 u + \frac{(e^u - f e^{-u})}{g'_{11}} F^{\bar{i}\bar{j}} u_{\bar{\bar{j}}} + \frac{(e^u + f e^{-u})}{g'_{11}} |D\bar{u}|^2_F$$

$$- \frac{2e^{-u}}{g'_{11}} \text{Re}(F^{\bar{i}\bar{j}} u_i f_j) + \frac{e^{-u}}{g'_{11}} F^{\bar{i}\bar{j}} f_{\bar{\bar{j}}} - \frac{|Dg'_{11}|^2_F}{(g'_{11})^2} + \phi' F^{\bar{i}\bar{j}} |D\bar{u}|^2_{\bar{j}i} + \phi'' |D| D\bar{u}|^2_{\bar{j}i}$$

$$- 2n\alpha \psi' F^{\bar{i}\bar{j}} u_{\bar{\bar{j}}} - 2n\alpha \psi'' |D\bar{u}|^2_{\bar{j}i}. \quad (5.20)$$

Here we introduced the notation

$$|D\chi|^2_F = F^{\bar{i}\bar{k}} D_\bar{i} \chi D_\bar{k} \chi. \quad (5.21)$$

We will get an estimate for $D_1 D_2 D_3 D_4 u$ using our formula (2.20). First, notice

$$g'_{11} \leq \text{Tr} h \leq ng'_{11}. \quad (5.22)$$

Furthermore, since $g' \in \Gamma_2$, we can estimate for each $k$,

$$2n\alpha |u_{kk}| \leq |g'_{kk}| + |e^{u} + f e^{-u}| \leq C(g'_{11} + 1). \quad (5.23)$$

Using these inequalities, we may estimate (2.20) in the following way

$$2n\alpha F^{\bar{i}\bar{j}} D_1 D_2 D_3 D_4 u \geq D_1 D_1 F - D_1(\text{Tr} h) D_1(\text{Tr} h) + |D_1 g'|^2 - C(g'_{11})^2 - C. \quad (5.24)$$

We now substitute this inequality into (5.20) to obtain

$$F^{\bar{i}\bar{j}} G_{\bar{j}i} \geq \frac{1}{g'_{11}} \left( |D_1 g'|^2 - |D_1 \text{Tr} h|^2 + D_1 D_1 F \right) + \frac{(e^u - f e^{-u})}{g'_{11}} F^{\bar{i}\bar{j}} u_{\bar{\bar{j}}} - \frac{|Dg'_{11}|^2_F}{(g'_{11})^2} + \phi' F^{\bar{i}\bar{j}} |D\bar{u}|^2_{\bar{j}i}$$

$$+ \phi'' |D| D\bar{u}|^2_{\bar{j}i} - 2n\alpha \psi' F^{\bar{i}\bar{j}} u_{\bar{\bar{j}}} - 2n\alpha \psi'' |D\bar{u}|^2_{\bar{j}i} - C g'_{11} - C. \quad (5.25)$$

We have the identity

$$2n\alpha F^{\bar{i}\bar{j}} u_{\bar{\bar{j}}} = F^{\bar{i}\bar{j}} g'_{\bar{\bar{j}}} - (e^u + f e^{-u}) F^{\bar{i}\bar{j}} g'_{\bar{\bar{j}}} = 2F - (e^u + f e^{-u})(n - 1) \text{Tr} h. \quad (5.26)$$

Note that by estimates (5.3), we have that $(n - 1)e^u \geq 1$ for small enough choice of $A$. Using this fact with (5.22), we obtain

$$0 \geq \frac{1}{g'_{11}} \left( |D_1 g'|^2 - D_1(\text{Tr} h) D_1(\text{Tr} h) + D_1 D_1 F \right) - \frac{|Dg'_{11}|^2_F}{(g'_{11})^2} + \phi' F^{\bar{i}\bar{j}} |D\bar{u}|^2_{\bar{j}i}$$

$$+ \phi'' |D| D\bar{u}|^2_{\bar{j}i} + (\psi' - C) g'_{11} - 2n\alpha \psi'' |D\bar{u}|^2_{\bar{j}i} - C. \quad (5.27)$$
We now compute the term involving $\phi'$. By (2.18), we have

$$2n\alpha F |D\bar{u}|_F^2 \geq -2|D\bar{u}| |DF| - Cg_{11} + 2n\alpha(D\bar{D}u|_{F_g}^2 + |D\bar{D}u|_{F_g}^2) - C. \quad (5.28)$$

Therefore

$$0 \geq \frac{1}{g_{11}} \left\{ |D_1 g'|^2 - D_1 \text{Tr} h D_1 \text{Tr} h + D_1 D_1 F \right\} - \frac{|Dg'_{11}|_F^2}{(g'_{11})^2} + \phi' D\bar{D}u|_{F_g}^2 + \phi'|D\bar{D}u|_{F_g}^2$$

$$+ \phi'' |D\bar{D}u|_F^2 \left\{ - \frac{n\alpha}{\bar{g}^2} |D\bar{u}| |DF| + (\psi' - C\phi' - C)g'_{11} - 2n\alpha\psi'' |D\bar{u}|_F^2 - C. \right\} \quad (5.29)$$

Define

$$\tau = \frac{1}{1 + M}. \quad (5.30)$$

Using (5.16), $DG(p) = 0$, and (5.11),

$$\phi'' |D\bar{D}u|_F^2 = \sum_i F_i^2 |\phi'| |D\bar{u}|_i^2 = \sum_i F_i^2 \left| -\frac{D_1 g'_{11}}{g'_{11}} + 2n\alpha \psi' u_i \right|^2$$

$$\geq \tau \sum_i F_i^2 \left| \frac{D_1 g'_{11}}{g'_{11}} \right|^2 - \frac{\tau}{1 - \tau} \sum_i F_i^2 |2n\alpha \psi' u_i|^2$$

$$= \tau \sum_i F_i^2 \left| \frac{D_1 g'_{11}}{g'_{11}} \right|^2 + \frac{\tau M}{1 - \tau} 2n\alpha \psi'' \sum_i F_i^2 |u_i|^2$$

$$= \tau \sum_i F_i^2 \left| \frac{D_1 g'_{11}}{g'_{11}} \right|^2 + 2n\alpha \psi'' \sum_i F_i^2 |u_i|^2. \quad (5.31)$$

Thus

$$0 \geq \frac{\phi'}{2} |D\bar{D}u|_{F_g}^2 + \frac{\phi'}{2} |D\bar{D}u|_{F_g}^2 + \frac{1}{g_{11}} D_1 D_1 F - \frac{\phi'}{n\alpha} |DF||D\bar{u}|$$

$$+ \frac{\phi'}{2} \left\{ |D_1 g'|^2 - D_1 \text{Tr} h D_1 \text{Tr} h \right\} - (1 - \tau) \frac{|Dg'_{11}|_F^2}{(g'_{11})^2}$$

$$+ \frac{\phi'}{2} |D\bar{D}u|_{F_g}^2 + \frac{\phi'}{2} |D\bar{D}u|_{F_g}^2 + \left\{ \frac{M}{4n\alpha L} - C\phi' - C \right\} g'_{11} - C. \quad (5.32)$$

Computing in coordinates and applying the Guan-Ren-Wang inequality (Lemma 1) yields

\begin{align*}
\left( |D_1 g'|^2 - D_1 \text{Tr} h D_1 \text{Tr} h \right) &= \sum_{i \neq j} |D_1 g'_{ij}|^2 - \sum_{i \neq j} D_1 g'_{ij} D_1 g'_{ij} \geq \sum_{i \neq j} |D_1 g'_{ij}|^2 - \frac{|D_1 F|^2}{F}.
\end{align*}

Using the definition of $g'$, we obtain

\begin{align*}
\sum_{i \neq j} |D_1 g'_{ij}|^2 &\geq \sum_{j>1} |D_1 g'_{ij}|^2 = \sum_{j>1} |D_j g'_{11} - (e^u + f e^{-u})_j|^2
\geq \left( 1 - \frac{\tau}{2} \right) \sum_{j>1} |D_j g'_{11}|^2 - C_M
\end{align*}

(5.33)
where the last constant $C_M$ depends on $\tau$, and hence on $M$. We therefore arrive at

$$0 \geq \frac{\phi'}{2} |D\overline{D}u|^2_{F_g} + \frac{\phi'}{2} |DDu|^2_{F_g} + \frac{1}{g'_{11}} D_1 D_1 F - \frac{\phi'}{n\alpha} |DF||Du|$$

$$+ \left(1 - \frac{\tau}{2}\right) \frac{1}{g'_{11}} \sum_{j>1} |D_j g'_{11}|^2 - (1 - \tau) \frac{|Dg'_{11}|^2_F}{(g'_{11})^2} + \frac{\phi'}{2} |D\overline{D}u|^2_{F_g} + \frac{\phi'}{2} |DDu|^2_{F_g}$$

$$- \frac{|D_1 F|^2}{g'_{11} F} + \left\{ \frac{M}{4n\alpha L} - C|\phi' - C\right\} g'_{11} - C_M. \quad (5.34)$$

At this point, it will be important to distinguish constants which depend on $A$ from those that do not. Let $B$ denote a constant depending on $(X, g), \|f\|_{C^2}, \|\mu\|_{C^2}, \alpha$. As before, we use $C$ to denote a constant depending on $(X, g), \|u\|_{\infty}, \|Du\|_{\infty}, \|f\|_{C^2}, \|\mu\|_{C^2}, \alpha$ and use $C_M$ to denote the constants which may also depend on $M$. We now state two lemmas.

**Lemma 2** Under the non-degeneracy assumption (5.4) and $C^0$ estimate (5.3), there holds

$$|DF| \leq B(e^u |Du| + e^{-u}) \left(|DDu| + |D\overline{D}u|\right) + C. \quad (5.35)$$

$$\frac{|DF|^2}{|F|} \leq Be^u \left(|DDu|^2 + |D\overline{D}u|^2\right) + C. \quad (5.36)$$

$$|D_1 D_1 F| \leq Be^u \left(|DDu|^2 + |D\overline{D}u|^2 + e^u |Du_{11}|\right) + C. \quad (5.37)$$

**Lemma 3** Let $p \in X$ be a point where $G$ attains a maximum. Assuming $g'_{11}(p) \gg 1$ is large enough, then at $p$ we have

$$\frac{1 - \tau}{2} \sum_{j>1} |D_j g'_{11}|^2 - (1 - \tau) \frac{|Dg'_{11}|^2_F}{(g'_{11})^2} + \frac{\phi'}{2} |D\overline{D}u|^2_{F_g} + \frac{\phi'}{2} |DDu|^2_{F_g} \geq 0. \quad (5.38)$$

Assuming these lemmas, we shall now prove the $C^2$ estimate. We may assume $g'_{11} \gg 1$ is large at the point $p \in X$, otherwise we already have the desired estimate. Applying both lemmas to (5.34), we have

$$0 \geq \frac{\phi'}{2} |D\overline{D}u|^2_{F_g} + \frac{\phi'}{2} |DDu|^2_{F_g} - \frac{Be^u}{g'_{11}} \left\{ |DDu|^2 + |D\overline{D}u|^2 + e^u |Du_{11}|\right\} - \frac{\phi'}{n\alpha} |DF||Du|$$

$$+ \left\{ \frac{M}{4n\alpha L} - C(1 + \phi')\right\} g'_{11} - C_M. \quad (5.39)$$

Using $DG(p) = 0$ (5.19), we may estimate

$$|Du_{11}| \leq C + |Du|g'_{11} \left\{ \psi' + \frac{\phi'}{2n\alpha} (|DDu| + |D\overline{D}u|)\right\}$$

$$\leq C + \left(e^{-\frac{\tau}{2}} (|DDu| + |D\overline{D}u|)\right) \left(e^{\frac{\tau}{2}} |Du|g'_{11} \frac{\phi'}{2n\alpha}\right) + |Du|\psi' g'_{11} \quad (5.40)$$

$$\leq C + e^{-u} \left(|DDu|^2 + |D\overline{D}u|^2\right) + \frac{(\phi')^2}{(2n\alpha)^2} \frac{e^u |Du|^2}{2} (g'_{11})^2 + |Du|\psi' g'_{11}. \quad (5.40)$$

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Using the estimate (5.40), the estimate (5.11) for $\psi'$, and $F^{ii} \geq F^{11}$, (5.39) becomes

$$0 \geq \frac{1}{g_{11}'} \left\{ \phi' g_{11}' F^{11} - B e^u \right\} \left( |DDu|^2 + |D\bar{D}u|^2 \right) - \frac{\phi'}{n\lambda} |DF| |Du|$$

$$+ \left\{ \frac{M}{4n\alpha L} - C(1 + \phi' + (\phi')^2) \right\} g_{11}' - C_M. \quad (5.41)$$

We shall show that for small enough $A$, we can ensure

$$\frac{\phi'}{2} g_{11}' F^{11} - B e^u \geq 1. \quad (5.42)$$

Indeed, this follows from the basic fact that $g_{11}' F^{11} \geq c(n)F$. Note that $g_{11}'$ is the largest eigenvalue and hence $g_{11}' \geq \frac{1}{n-1} \sigma_1(\lambda'|1)$. We use the notation $\sigma_k(\lambda'|j)$ for the $k$-th symmetric function of $(\lambda'|j) = (\lambda'_1, \ldots, \lambda'_j, \ldots, \lambda'_n) \in \mathbb{R}^{n-1}$. For example, $\sigma_1(\lambda'|1) = \sum_{i \neq 1} \lambda'_i = F^{11}$. This implies

$$\sigma_2(\lambda') = \sigma_1(\lambda'|1) \lambda_1 + \sigma_2(\lambda'|1) \leq \sigma_1(\lambda'|1) \lambda_1 + \frac{n-2}{2(n-1)} \sigma_1^2(\lambda'|1)$$

$$\leq \sigma_1(\lambda'|1) \lambda_1 + \frac{n-2}{2} \sigma_1(\lambda'|1) \lambda_1'.$$

which gives the desired estimate $g_{11}' F^{11} \geq \frac{2}{n} F$. Therefore, using (5.4) and (5.15)

$$g_{11}' \frac{\phi'}{2} F^{11} - B e^u \geq \frac{\phi' e^{2u}}{n} (F e^{-2u}) - B e^u \geq e^u \left( \frac{1}{2n(17\delta B_0 B_1)} - B \right) \geq 1, \quad (5.43)$$

when our parameter $A_0$ is chosen such that $\delta$ is sufficiently small and $e^u$ sufficiently large. The estimate is possible because the $B_0, B_1, B$ are independent of $A$. Now that our normalization $A \leq A_0$ has been chosen, we recall the bounds (5.14) and (5.15) for $\phi'$, and set $M_1 = \frac{M}{4n\alpha L} - C(1 + \frac{A}{168B_1} + (\frac{A}{168B_1})^2)$, which is positive for $M$ large enough. The inequality (5.41) implies

$$0 \geq \frac{1}{g_{11}'} (|DDu|^2 + |D\bar{D}u|^2) + M_1 g_{11}' - \frac{\phi'}{n\lambda} |DF| |Du| - C_M$$

$$\geq M_1^\frac{1}{2} (|DDu|^2 + |D\bar{D}u|^2)^{\frac{1}{2}} + \frac{3}{4} M_1 g_{11}' - \frac{\phi'}{n\lambda} |DF| |Du| - C_M$$

$$\geq \frac{3}{4} M_1 g_{11}' - C_M$$

In the last inequality, we made use of the estimate (5.35) for $|DF|$, and chose $M$ large enough. Thus we have established

$$g_{11}' \leq C. \quad (5.44)$$

This completes the proof of Theorem 3.
Proof of Lemma 2. Using the definition (2.7) of $F = \sigma_2(g')$, we shall estimate $DF$ and $D_1 D_1 F$. The expression (2.7) shows that $F$ is a linear combination of the expressions

$$e^{au}, \quad e^{\pm u}|Du|^2, \quad e^{-u}Du, \quad e^{-u}\bar{D}u,$$

with coefficients given by smooth functions whose derivatives of any fixed order can be bounded by constants $C$. The constant $a$ can take the values $0, \pm 1, 2$. Thus $DF$ can be bounded by a linear combination of the above expressions and their derivatives. The expressions can themselves be bounded by constants $C$, while $D(e^{au})$ can be bounded by constants $C$, and

$$D(e^{\pm u}|Du|^2) = e^{\pm u}(DDu \cdot \bar{D}u + Du \cdot D\bar{D}u) \pm e^{\pm u}(Du)|Du|^2,$$

$$D(e^{-u}Du) = e^{-u}DDu - e^{-u}Du Du, \quad D(e^{-u}\bar{D}u) = e^{-u}D\bar{D}u - e^{-u}Du \bar{D}u.$$

All the last terms on the right hand side of each of the above three equations can be bounded by $C$. The estimate (5.35) for $|DF|$ follows.

Next, we turn to $D_1 D_1 F$. For this, we view $D_1 F$ as a linear combination of the above expressions and their derivatives, and apply $D_1$. In this process, we can ignore all terms bounded by expressions of the form

$$e^{pu}|Du|^q(|DDu| + |D\bar{D}u|) + e^{ru} + |Du|^s$$

for some $p, q, r, s \geq 0$ since they can all be absorbed into $e^u(|DDu|^2 + |D\bar{D}u|^2) + C$ (recall that $e^u > 1$ by the assumption (5.1)). Examples of such terms are the bounds obtained in (5.35) for $|DF|$. Thus, when the derivative $D_1$ lands on the coefficients of the linear combination giving $D_1 F$, we obtain only expressions that can be bounded by the right hand side of (5.35) and can be ignored. This means that, to establish the bound (5.37), it suffices to consider the expressions $D_1 D_1(e^{au})$, $D_1 D_1(e^{\pm u}|Du|^2)$, $D_1 D_1(e^{-u}Du)$, and $D_1 D_1(e^{-u}\bar{D}u)$. Modulo $O(e^{pu}|Du|^q(|DDu| + |D\bar{D}u|) + e^{ru} + |Du|^s)$, we can write

$$D_1 D_1(e^{au}) = 0,$$

and

$$D_1 D_1(e^{-u}Du) = e^{-u}D_1 D_1 Du, \quad D_1 D_1(e^{-u}\bar{D}u) = e^{-u}D_1 D_1 \bar{D}u,$$

which can clearly be bounded by the right hand side of (5.37). Similarly,

$$D_1 D_1(e^{\pm u}|Du|^2) = e^{\pm u}(D_1 Du \cdot \bar{D}u + Du \cdot D_1 \bar{D}u) = e^{\pm u}(D_1 D_1 Du \cdot \bar{D}u + Du \cdot D_1 D_1 \bar{D}u) + e^{\pm u}(D_1 D_1 Du \cdot D_1 Du + D_1 Du \cdot D_1 D_1 Du).$$

It follows that

$$|D_1 D_1(e^{\pm u}|Du|^2)| \leq e^u(|D\bar{D}u|^2 + |DDu|^2) + e^u|Du| |Du_{11}|. \quad (5.45)$$
We note that by $|Du|^2 \leq e^u$ (5.1), we may estimate $|Du|$ by $e^u$ since $e^u \geq 1$. Thus all the terms in $D_1 D_1 F$ can be bounded by the right hand side of (5.37), completing the proof of (5.37).

Using the lower bound for $F$ in (5.4) and the fact that $(e^u |Du| + e^{-u})^2 \leq 2(e^{2u}|Du|^2 + e^{-2u}) \leq Be^{3u}$, in view of the assumption $e^{-u} \leq 1$ and $|Du|^2 \leq e^u (5.1)$, we have

$$\frac{|DF|^2}{F} \leq \frac{Be^{3u}}{F} (|DDu|^2 + |D\tilde{D}u|^2) + C \leq Be^u (|DDu|^2 + |D\tilde{D}u|^2) + C.$$

Q.E.D.

**Proof of Lemma 3.** This argument will adapt the proof of Proposition 9 in Guan-Ren-Wang [10] to the complex setting. Recall that we are working at a point $p$ with $DG = 0$, $g_{kj} = \delta_{kj}$ and $u_{kj}$, $g'_{11} \geq g'_{22} \geq \ldots \geq g'_{nn}$. The first step is the following computation, for $a \in \{1,2,\ldots,n\}$ fixed.

$$-(1 - \tau)F^{a\bar{a}} \left| \frac{D_a g'_{11}}{g'_{11}} \right|^2 + \frac{\phi'}{2} F^{a\bar{a}} \left( |u_{a\bar{a}}|^2 + \sum_k |u_{ka}|^2 \right)$$

$$= -(1 - \tau)F^{a\bar{a}} \left| \phi' |Du|^2 - 2n\alpha \psi' u_a \right|^2 + \frac{\phi'}{2} F^{a\bar{a}} \left( |u_{a\bar{a}}|^2 + \sum_k |u_{ka}|^2 \right)$$

$$\geq -2(1 - \tau)F^{a\bar{a}} \left( |\phi' |Du|^2 | + 2n\alpha \psi' u_a|^2 \right) + \frac{\phi'}{2} F^{a\bar{a}} \left( |u_{a\bar{a}}|^2 + \sum_k |u_{ka}|^2 \right)$$

$$\geq F^{a\bar{a}} \left\{ \phi' \cdot \left( \frac{1}{2} - 4(1 - \tau)\phi' |Du|^2 \right) \left( |u_{a\bar{a}}|^2 + \sum_k |u_{ka}|^2 \right) - C \right\}. \quad (5.46)$$

By our choice of $\phi$, we have (5.13), (5.14) and (5.15), hence

$$\phi' |Du|^2 \leq \frac{1}{16}, \quad \phi' > \frac{1}{C} > 0. \quad (5.47)$$

Thus we have for $a \in \{1,2,\ldots,n\}$,

$$-(1 - \tau)F^{a\bar{a}} \left| \frac{D_a g'_{11}}{g'_{11}} \right|^2 + \frac{\phi'}{2} F^{a\bar{a}} \left( |u_{a\bar{a}}|^2 + \sum_k |u_{ka}|^2 \right) \geq F^{a\bar{a}} \left( \frac{1}{4} |u_{a\bar{a}}|^2 - C \right) \quad (5.48)$$

If $g'_{11} \gg 1$, then

$$u_{11} = \frac{1}{2n\alpha} \left( g'_{11} - (e^u + f e^{-u}) \right) \gg 1, \quad (5.49)$$

hence by letting $a = 1$ in (5.48)

$$-(1 - \tau)F^{1\bar{1}} \left| \frac{D_1 g'_{11}}{g'_{11}} \right|^2 + \frac{\phi'}{2} F^{1\bar{1}} \left( |u_{1\bar{1}}|^2 + \sum_k |u_{k1}|^2 \right) \geq 0 \quad (5.50)$$
for $g'_{11}$ sufficiently large. Thus to prove the lemma, it needs to be shown that

$$\frac{1 - \frac{\varepsilon}{2}}{g'_{11}} \sum_{i > 1} |D_i g'_{11}|^2 - (1 - \tau) \sum_{i > 1} F^{ij} \left| \frac{D_{ij} g'_{11}}{g'_{11}} \right|^2 + \sum_{i > 1} \frac{\phi'}{2} F^{ij} \left( |u_{ij}|^2 + \sum_k |u_{ki}|^2 \right) \geq 0. \quad (5.51)$$

To prove this estimate, we proceed by cases. Let

$$0 < \varepsilon < \frac{\tau}{2(1 - \tau)}. \quad (5.52)$$

Case (A): $\sum_{i=2}^{n-1} g'_{ii} < \varepsilon g'_{11}$. In this case, we have $F^{nn} = g'_{11} + \sum_{i=2}^{n-1} g'_{ii} < (1 + \varepsilon)g'_{11}$. Thus

$$\frac{1 - \frac{\varepsilon}{2}}{g'_{11}} \sum_{i > 1} |D_i g'_{11}|^2 - (1 - \tau) \sum_{i > 1} F^{ij} \left| \frac{D_{ij} g'_{11}}{g'_{11}} \right|^2 \geq \frac{1 - \frac{\tau}{2}}{g'_{11}} \sum_{i > 1} |D_i g'_{11}|^2 - (1 - \tau) \sum_{i > 1} F^{nn} \frac{D_{ij} g'_{11}}{g'_{11}}^2 \geq \left( 1 - \frac{\tau}{2} - (1 - \tau)(1 + \varepsilon) \right) \frac{1}{g'_{11}} \sum_{i > 1} |D_i g'_{11}|^2,$$

which is nonnegative by the choice of $\varepsilon$. This proves (5.51).

Case (B): $\sum_{i=2}^{n-1} g'_{ii} \geq \varepsilon g'_{11}$. In this case, we have $g'_{22} \geq \frac{\varepsilon}{n-2} g'_{11}$. For $g'_{11}$ large enough,

$$u_{22} = \frac{1}{2n\alpha} \left( g'_{22} - (e^u + f e^{-u}) \right) \geq \frac{1}{2n\alpha} \frac{\varepsilon}{n-2} g'_{11} - C \geq \frac{\varepsilon}{4n(n-2)\alpha} g'_{11}. \quad (5.53)$$

We divide case (B) into subcases.

Case (B1): $F^{22} \geq 1$. By (5.48)

$$-(1 - \tau) \sum_{i > 1} F^{ij} \left| \frac{D_{ij} g'_{11}}{g'_{11}} \right|^2 + \sum_{i > 1} \frac{\phi'}{2} F^{ij} \left( |u_{ij}|^2 + \sum_k |u_{ki}|^2 \right) \geq F^{22} \left( \frac{1}{4} |u_{22}|^2 - C \right) + \sum_{i > 2} F^{ij} \left( \frac{1}{4} |u_{ij}|^2 - C \right) \geq \left( \frac{1}{4} |u_{22}|^2 - C \right) - C \sum_{i > 2} F^{ij} \geq \frac{\varepsilon^2}{4^3(n(n-2)\alpha)^2} (g'_{11})^2 - C g'_{11} - C \geq 0. \quad (5.54)$$

Case (B2): $F^{22} < 1$. In this case,

$$-g'_{nn} \geq -1 + (g'_{11} - g'_{22}) + \sum_{i=2}^{n-1} g'_{ii} \geq -1 + \varepsilon g'_{11} \geq \frac{\varepsilon}{2} g'_{11}. \quad (5.55)$$

$$-u_{nn} = \frac{1}{2n\alpha} \left( -g'_{nn} + (e^u + f e^{-u}) \right) \geq \frac{\varepsilon}{4n\alpha} g'_{11}. \quad (5.56)$$
Note that the assumption of case (B) implies \( F^{\bar{n}} \geq (1 + \varepsilon)g_{11}' \). Another computation using (5.48) yields

\[
-(1 - \tau) \sum_{i > 1} F^{\bar{i}} \frac{D_{i}g'_{11}}{g_{11}'} + \sum_{i > 1} \frac{g'_{11}}{2} F^{\bar{i}} \left( |u_{i}|^2 + \sum_{k} |u_{ki}|^2 \right) \\
\geq F^{\bar{n}} \left( \frac{1}{4} |u_{nn}|^2 - C \right) + \sum_{i=2}^{n-1} F^{\bar{i}} \left( \frac{1}{4} |u_{i}|^2 - C \right) \\
\geq F^{\bar{n}} \left( \frac{\varepsilon^2}{4^3(n\alpha)^2} |g'_{11}|^2 - C \right) - Cg'_{11} \\
\geq (1 + \varepsilon)g'_{11} \left( \frac{\varepsilon^2}{4^3(n\alpha)^2} |g'_{11}|^2 - C \right) - Cg'_{11} \geq 0. \tag{5.57}
\]

This establishes (5.51), and thus proves Lemma 3. Q.E.D.

6 The \( C^{2,\eta} \) Estimate

At this point, we have shown the \textit{a priori} \( C^2 \) estimates (5.2) for equation (2.7), under the assumption of a sharp \( C^1 \) upper bound (5.1). This \( C^2 \) estimate implies that the equation is uniformly elliptic and that it is also a concave operator. We would like to apply the Evans-Krylov theorem [6, 13, 19] to show the \( C^{2,\eta} \) bound. However, we cannot apply the standard theorem directly.

In fact, equation (2.7) is of the following form

\[
\sigma_2 \left( \chi_{jk}(z, u) + u_{jk} \right) = \varphi(z, u, Du). 
\]

By the \textit{a priori} \( C^2 \) estimate, we have uniform bounds for the complex Hessian \( \partial \bar{\partial} u \) and hence for \( \Delta u \). This implies that \( u \in C^{1,\theta} \) for some \( \theta \in (0, 1) \). Therefore, the function \( \varphi(z, u, Du) \) is right hand side of equation (2.7) is only \( C^\theta \) even if \( f \) and \( \mu \) are smooth on \( X \). Thus, the standard Evans-Krylov theorem is not directly applicable as it requires a \( C^{1,1} \) bound for \( \varphi \), which depends on the \( C^3 \) norm of \( u \) in our case.

The \( C^{2,\eta} \) regularity for the complex Monge-Ampère equations with only Hölder continuous right hand side was obtained by Dinew-Zhang-Zhang [5] for \( u \in C^{1,1} \). The assumption on \( u \) was weaken to be \( \Delta u \in L^\infty \) by Wang [26] and it was later extended to more general settings by Tosatti-Wang-Weinkove-Yang [24]. Indeed, our setup here fits well into the general picture in [24] (Theorem 1.1 for equation (1.4) in [24]). We note that our \( \chi_{jk} = (e^u + f e^{-u}) g_{jk} \in C^{1,\theta} \) and \( \varphi \in C^\theta \). And thus we can apply their main result to conclude the following \( C^{2,\eta} \) bound for \( u \). We refer the reader to [24] for details.

\textbf{Theorem 4} Let \( u \in C^2(X) \) be a solution to (2.7) with normalization condition (2.8). Then, there exist positive constants \( 0 < \eta < 1 \) and \( C \) depending on \( n, (X, g), \|u\|_{L^\infty}, \|Du\|_{L^\infty}, \|\Delta u\|_{L^\infty}, \|f\|_{C^3}, \|\mu\|_{C^1} \) and \( \alpha \) such that

\[
\|u\|_{C^{2,\eta}(X)} \leq C. \tag{6.1}
\]
7 Non-Degeneracy and Sharp Gradient Bounds

In order to solve equation (2.7) subject to normalization condition (2.8), one can use the method of continuity. This can be done by introducing the parameter $t$, and replacing $f$ by $tf$ and $\mu$ by $t\mu$.

$$e^{-2u}F = \kappa_e \{ 1 - 4\alpha e^{-u}|Du|^2 \} + 4 \alpha \kappa_e \left( t f e^{-3u}|Du|^2 - te^{-3u}(g^{ik} f_{uk} + g^{ik} f_k u_i) \right)$$
$$+ \kappa_e t e^{-2u} \left( 2f + f^2 e^{-2u} + 4\alpha e^{-u}\Delta f \right) - 2n\alpha t e^{-2u} \mu. \quad (7.1)$$

We see that when $t = 0$, the equation admits the trivial solution $u = -\log A$, and the right hand side is equal to $\kappa_e$. The issue addressed in this section is whether the right-hand side can degenerate to zero as $t$ tends to $t = 1$. For simplicity, we shall suppress the parameter $t$ in our computations and write $f$ instead of $tf$ and $\mu$ instead of $t\mu$. The theorem of Fu-Yau [7] is the following.

**Theorem 5 (Fu-Yau [7])** Let the dimension of $X$ be equal to $n = 2$. For any $\delta > 0$, there exists $A_0 > 0$ depending on $(X, \omega)$, $f$, $\alpha$, $\mu$, such that if $A < A_0$, then for any solution $u$ of the Fu-Yau equation (2.7) with normalized condition (2.8), there holds

$$e^{-2u}F \geq \kappa_e - \delta. \quad (7.2)$$

In the rest of this section, we investigate the non-degeneracy estimate for the higher dimensional case. As mentioned in the Introduction, we follow the idea of Fu-Yau closely, but we work with general coordinate systems rather than the adapted ones with $\nabla u = (u_1, 0, \cdots, 0)$ used by Fu and Yau. This allows us a simplified and more transparent derivation of the Fu-Yau results for $n = 2$, and a clearer picture of why their arguments are not strong enough for higher dimensions. Following Fu-Yau, we apply the maximum principle to the following function

$$G = 1 - 4\alpha e^{-u}|Du|^2 + 4\alpha e^{-\varepsilon u} - 4\alpha e^{-\varepsilon \inf u}. \quad (7.3)$$

7.1 First computation of $F^{jk}D_jD_k G$

We begin by computing $F^{jk}D_jD_k(-4\alpha e^{-u}|Du|^2)$. Because we shall ultimately evaluate this expression as a critical point of $G$, where

$$D(e^{-u}|Du|^2) = D(e^{-\varepsilon u}) \quad (7.4)$$

it is advantageous to express $D_jD_k(-4\alpha e^{-u}|Du|^2)$ in terms of $D(e^{-u}|Du|^2)$ as much as possible. Thus we write

$$F^{jk}D_jD_k(-4\alpha e^{-u}|Du|^2) = 4\alpha F^{jk}D_k u D_j(e^{-u}|Du|^2) + 4\alpha F^{jk}D_j u D_k(e^{-u}|Du|^2)$$
$$+ 4\alpha e^{-u}|Du|^2 F^{jk}D_j u D_k - 4\alpha e^{-u} F^{jk}D_j D_k|Du|^2$$
$$+ 4\alpha (F^{jk}D_j D_k u) e^{-u}|Du|^2. \quad (7.5)$$

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On the other hand, a straightforward computation gives

\[
F^{jk} D_j D_k(4\alpha e^{-\varepsilon u}) = 4\alpha \varepsilon^2 |Du|^2 e^{-\varepsilon u} - 4\alpha \varepsilon (F^{jk} D_j D_k u) e^{-\varepsilon u}. \tag{7.6}
\]

and thus

\[
F^{jk} D_j D_k G = 4\alpha F^{jk} D_k u D_j (e^{-u} |Du|^2) + 4\alpha F^{jk} D_j u D_k (e^{-u} |Du|^2)
+ 4\alpha (e^{-u} |Du|^2 + \varepsilon^2 e^{-\varepsilon u}) |Du|^2 - 4\alpha e^{-u} F^{jk} D_j D_k |Du|^2
+ 4\alpha (F^{jk} D_j D_k u) (e^{-u} |Du|^2 - \varepsilon e^{-\varepsilon u}) \tag{7.7}
\]

where we have introduced the notation \( |Du|^2_F = F^{jk} D_j D_k u \).

We can now substitute in the critical point equation (7.4) of \( G \), and obtain

\[
F^{jk} D_j D_k G = 4\alpha (e^{-u} |Du|^2 - 2\varepsilon e^{-\varepsilon u} + \varepsilon^2 e^{-\varepsilon u}) |Du|^2_F
+ 4\alpha F^{jk} D_j D_k u (e^{-u} |Du|^2 - \varepsilon e^{-\varepsilon u}) - 4\alpha e^{-u} F^{jk} D_j D_k |Du|^2 \tag{7.8}
\]

Both expressions \( F^{jk} D_j D_k u \) and \( F^{jk} D_j D_k |Du|^2 \) have been computed in section \( \S 2 \) and are found in equations (2.12) and (2.18). Substituting in the formulas derived there, we obtain

\[
F^{jk} D_j D_k G = 4\alpha (e^{-u} |Du|^2 - 2\varepsilon e^{-\varepsilon u} + \varepsilon^2 e^{-\varepsilon u}) |Du|^2_F
+ 4\alpha (\frac{4}{n} F - \frac{2(n-1)}{n} (e^{u} + f e^{-u}) Tr \bar{h}) (e^{-u} |Du|^2 - \varepsilon e^{-\varepsilon u})
+ \frac{2(n-1)}{n} Tr \bar{h} e^{-u} g^\ell \rho m \{ \partial_\ell (e^{u} + f e^{-u}) \partial_m u + \partial_m (e^{u} + f e^{-u}) \partial_\ell u \}
- \frac{2}{n} e^{-u} g^\ell \rho m \{ \partial_\ell F \partial_m u + \partial_m F \partial_\ell u \} + 4\alpha e^{-u} g_{\rho m} R^{\mu \rho \nu \sigma} \partial_\mu F \partial_\nu u \partial_\sigma u. \tag{7.9}
\]

We now make use of a key partial cancellation, observed by Bobcki in his proof of \( C^1 \) estimates for the Monge-Ampère equation [2] (see also [9, 29], and [17, 18] for other applications of this partial cancellation), between \( |DDu|^2_{F_g} \) and \( |Du|^2 |Du|^2_{F_g} \), which is the following. At a critical point of \( G \), the relation (7.4) implies

\[
g^{ij} D_i D_j u D_j u = -g^{ij} D_j D_p u D_p u + |Du|^2 - \varepsilon e^{(1-\varepsilon)u} D_p u. \tag{7.10}
\]

We can now estimate \( |DDu|^2_{F_g} \) from below by

\[
|DDu|^2_{F_g} \geq \frac{1}{|Du|^2} |g^{ij} D_p D_i u D_j u|^2 = \frac{1}{|Du|^2} g^{ij} D_j D_p u D_i u - (|Du|^2 - \varepsilon e^{(1-\varepsilon)u}) D_p u \tag{7.11}
\]

\[
= |Du|^2 |Du|^2_{F} + \varepsilon^2 e^{2(1-\varepsilon)u} |Du|^2_{F} - 2\varepsilon |Du|^2_{F} e^{(1-\varepsilon)u}
+ \frac{1}{|Du|^2} |g^{ij} D_i u D_j D_p u|^2_{F} - \frac{2}{|Du|^2} (|Du|^2 - \varepsilon e^{(1-\varepsilon)u}) Re(F^{pq} g^{ij} D_q u D_i u D_j D_p u). \tag{7.12}
\]

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The terms $|Du|^2|D_{ij}u|^2_F$ and $-2\varepsilon e^{(1-\varepsilon)u}|Du|^2_F$ will cancel out similar terms in $F^{jk}D_jD_kG$. The expression $\text{Re}(F^{pq}g^{ij}D_qD_iuD_jD_pu)$ can be rewritten as

$$\text{Re}(F^{pq}g^{ij}D_qD_iuD_jD_pu) = \frac{1}{2n\alpha}\text{Re}(F^{pq}g^{ij}D_qD_iuD_jD_pu) - \frac{e^u + fe^{-u}}{2n\alpha}|Du|^2_F$$

by going to coordinates where $g_{kj} = \delta_{kj}$, and $u_{kj}$ is diagonal at the point $p$ where the function $G$ attains its minimum. Here $\sigma_k(\lambda'|j)$ denotes the $k$-th symmetric function of the $(n-1) \times (n-1)$ diagonal matrix with eigenvalues $\lambda'_m$, $m \neq j$.

We also make use of the other term $|D\tilde{u}|^2_{F,g}$, which we rewrite as

$$|D\tilde{u}|^2_{F,g} = \frac{F^{pq}g^{ij}}{(2n\alpha)^2}(g'_{jp} - (e^u + fe^{-u})g_{jp})(g'_{qi} - (e^u + fe^{-u})g_{qi})$$

Again using the above coordinates, we can work out a more explicit expression for $|g'|^2_{F,g}$,

$$-\frac{e^{-u}}{n^2\alpha}|g'|^2_{F,g} = -\frac{e^{-u}}{n^2\alpha}\sum_{j=1}^{n}\hat{\lambda}_j(\hat{\lambda}'_j)^2 = -\frac{e^{-u}}{n^2\alpha}(F\sum_{j=1}^{n}\lambda'_j - \sum_{j=1}^{n}\sigma_2(\lambda'|j)\lambda'_j)$$

$$= -\frac{e^{-u}}{n^2\alpha}(F\text{Tr}h - \sum_{j=1}^{n}(\sigma_3(g') - \sigma_3(\lambda'|j))) = -\frac{e^{-u}}{n^2\alpha}F\text{Tr}h + \frac{e^{-u}}{n^2\alpha}3\sigma_3(g').$$

Thus we find, at a critical point of the test function $G$,

$$F^{jk}D_jD_kG \leq \left\{ -\frac{4}{n}e^{-2\alpha}\varepsilon(e^u + fe^{-u})|Du|^2 + \frac{4}{n}\varepsilon e^{(1-\varepsilon)u}(e^u + fe^{-u}) - 4\alpha\varepsilon^2e^{-2\varepsilon u}$$

$$+4\alpha\varepsilon^2e^{-\varepsilon u}e^{-u}|Du|^2 + 4\alpha\varepsilon^2\varepsilon^2e^{-2\varepsilon u}e^{-u}|Du|^2 - \frac{4\alpha\varepsilon^2e^{-\varepsilon u}e^{-u}|Du|^2}{2n}\right\}$$

$$+\varepsilon e^{-u}\left\{ \frac{8}{n}F - 4\sum_{j=1}^{n}\sigma_2(\lambda'|j)|u_j|^2 - \frac{2(n-1)}{n}(e^u + fe^{-u})\text{Tr}h \right\}$$

$$-\frac{e^{-u}}{n^2\alpha}F\text{Tr}h + \frac{e^{-u}}{n^2\alpha}3\sigma_3(g') + \frac{4}{n^2\alpha}e^{-u}(e^u + fe^{-u})F - \frac{n-1}{n^2\alpha}e^{-u}(e^u + fe^{-u})^2\text{Tr}h$$

$$+\frac{2(n-1)}{n}\text{Tr}h e^{-u}2\Re\langle D(e^u + fe^{-u}), Du \rangle - \frac{2}{n}e^{-u}2\Re\langle DF, Du \rangle$$

$$+4\alpha\varepsilon^2\varepsilon^2e^{-2\varepsilon u}e^{-u}|Du|^2\right\}$$

\text{(7.11)}

\section{7.2 Using the equation}

So far, we have not used the equation (2.7). We shall now use it to evaluate and simplify the preceding estimate for $F^{jk}D_jD_kG$. It is convenient to rewrite the equation (2.7) in the
At a critical point (7.4) for \( G_\alpha e \) and the expression 4 details, it is convenient to introduce the following groups of expressions:

\[
\text{The preceding expression is unwieldy if we write it down in full. To avoid unnecessary}
\]

\[
\text{applying the critical point equation (7.4), we see that this term is of the form}
\]

\[
\text{following form}
\]

\[
F = \frac{n(n-1)}{2} e^{2u} (1 - 4\alpha e^{-u}|Du|^2) - 2n\alpha \nu
\]

where the function \( \nu \) is defined to be

\[
\nu = \mu - (n-1)f e^{-u}|Du|^2 - \frac{n-1}{2\alpha} f - \frac{n-1}{4\alpha} e^{-2u} f^2
\]

\[-(n-1)e^{-u}(\Delta f - g^I(D_j f D_k u + D_k u D_j f)).\]

(7.13)

At a critical point (7.4) for \( G \), we have

\[
\partial_F F = n(n-1)e^{2u} \partial_F u (1 - 4\alpha e^{-u}|Du|^2) + \frac{n(n-1)}{2} e^{2u} \partial_F (-4\alpha e^{-u}|Du|^2) - 2n\alpha \nu
\]

\[= 2F \partial_F u - 4n\alpha \partial_E \nu + 2\alpha(n-1)\varepsilon \partial_E u^{(2-\varepsilon)u} - 2n\alpha \partial_E \nu.\]

(7.14)

The preceding expression is unwieldy if we write it down in full. To avoid unnecessary details, it is convenient to introduce the following groups of expressions:

- The group \( \mathcal{E}_0 \) consists of the following expressions

\[
e^{-u} \text{Tr} h, \quad e^{-2u} F, \quad e^{-3u} \sigma_3(g'), \quad e^{-u} \frac{|Du|^2}{|F|^2}, \quad e^{-2u} \sum_j \sigma_2(\lambda'\langle j\rangle) \frac{|u_j|^2}{|Du|^2}, \quad 1
\]

(7.15)

where \( \sigma_3 \) and \( \sigma_2(\lambda'\langle j\rangle) \) denote the symmetric functions of the eigenvalues of the matrix \( g'_k \).

- The group \( \mathcal{E}_1 \) consists of expressions of the form

\[
\varepsilon e^{-\varepsilon u} \Phi, \quad \Phi \in \mathcal{E}_0.
\]

(7.16)

- The group \( \mathcal{E}_2 \) consists of expressions of the form

\[
c \Phi, \quad \text{with } \Phi \in \mathcal{E}_0 \text{ and } c << \varepsilon e^{-\varepsilon u}
\]

(7.17)

where the inequality indicated on the coefficient \( c \) should hold for \( \varepsilon << 1 \) and \( A << 1 \).

For example, any function of the form \( e^{-2u} v \) with \( v \) a bounded function can be classified into the group \( \mathcal{E}_2 \). Another example is the expression \( 4\alpha e^2 e^{-\varepsilon u}|Du|^2 \), which can be viewed as belonging to \( e^{2u} \mathcal{E}_2 \), since

\[
4\alpha e^2 e^{-\varepsilon u}|Du|^2 = e^{2u} \left\{ \varepsilon^2 e^{-\varepsilon u}(4\alpha e^{-u}|Du|^2) e^{-\varepsilon u} \frac{|Du|^2}{|Du|^2} \right\}
\]

(7.18)

and the expression \( 4\alpha e^{-u}|Du|^2 \) is bounded as we vary \( A \) by the \( C^1 \) estimate (4.5).

We now claim that

\[
-\frac{2}{n} e^{-u} 2\text{Re}<DF, Du> = -\frac{8}{n} F e^{-u}|Du|^2 - 8\alpha(n-1)\varepsilon e^{(2-\varepsilon)u}(e^{-u}|Du|^2)
\]

(7.19)

modulo terms of the form \( e^{2u} \mathcal{E}_2 \). Indeed, absorbing all terms \( e^{-u}|Du|^2 \) into \( O(1) \) yields

\[
-\frac{2}{n} e^{-u}(-2\alpha) 2\text{Re}<Dv, Du> = -4\alpha(n-1)f e^{-u} 2\text{Re}<Du, D(e^{-u}|Du|^2)>
\]

\[+ 4\alpha(n-1)e^{-u} 2\text{Re}<Df, D(e^{-u}|Du|^2)> + O(1).
\]

Applying the critical point equation (7.4), we see that this term is of the form \( e^{2u} \mathcal{E}_2 \).
7.2.1 The expression for \( F^{jk}D_jD_kG \) up to \( \mathcal{E}_2 \) terms

It is now easy to clean up considerably the expression for \( F^{jk}D_jD_kG \). Up to \( e^{2u}\mathcal{E}_1 \) and \( e^{2u}\mathcal{E}_2 \) terms, (7.11) is

\[
F^{jk}D_jD_kG \leq -\left\{ \frac{4}{n} e^{-u} |Du|^2 \right\} e^{u} \left| \frac{Du}{|Du|^2} \right|^2 + e^{-u} |Du|^2 \left\{ \frac{8}{n} F - \frac{4}{n} \sum_j \sigma_2(\lambda'|j) \left| \frac{u_j}{|Du|^2} \right|^2 - \frac{2(n-1)}{n} e^{u} \text{Tr} h \right\} \\
- \frac{4\alpha e^{-u}}{|Du|^2} g^{ij} D_i u D_j D_p u |F|^2 - \frac{e^{-u}}{n^2 \alpha} \text{Tr} h \left( \frac{4}{n^2 \alpha} - \frac{e^{-u}}{n^2 \alpha} \right) + \frac{4}{n^2 \alpha} F - \left\{ \frac{4}{n} e^{-u} |Du|^2 \right\} e^{u} \left| \frac{Du}{|Du|^2} \right|^2 \\
- \frac{4\alpha e^{-u}}{|Du|^2} \sum_j \sigma_2(\lambda'|j) \left| \frac{u_j}{|Du|^2} \right|^2 + \frac{e^{-u}}{n^2 \alpha} 3\sigma_3 - \frac{4\alpha e^{-u}}{|Du|^2} g^{ij} D_i u D_j D_p u |F|^2. \tag{7.20}
\]

We can now make use of the formula (7.19) for \( \langle DF, Du \rangle \) modulo \( e^{2u}\mathcal{E}_1 \) and \( e^{2u}\mathcal{E}_2 \) obtained in the previous section. The expression \( 8F/n \) in the top line cancels out. Regrouping terms in terms of \( |Du|^2/|Du|^2 \), \( \text{Tr} h \) and \( F \), we have

\[
F^{jk}D_jD_kG \leq e^{u} \text{Tr} h \left\{ \frac{2}{n} \frac{n-1}{n^2 \alpha} e^{-u} |Du|^2 - \frac{n-1}{n^2 \alpha} - \frac{e^{-2u} F}{n^2 \alpha} \right\} + \frac{4}{n^2 \alpha} F - \left\{ \frac{4}{n} e^{-u} |Du|^2 \right\} e^{u} \left| \frac{Du}{|Du|^2} \right|^2 \\
- \frac{4\alpha e^{-u}}{|Du|^2} \sum_j \sigma_2(\lambda'|j) \left| \frac{u_j}{|Du|^2} \right|^2 + \frac{e^{-u}}{n^2 \alpha} 3\sigma_3 - \frac{4\alpha e^{-u}}{|Du|^2} g^{ij} D_i u D_j D_p u |F|^2. \tag{7.20}
\]

We can now eliminate systematically \( 4\alpha e^{-u} |Du|^2 \) using the equation

\[
4\alpha e^{-u} |Du|^2 = 1 - \frac{e^{-2u} F}{\kappa_c} \quad \text{modulo } \mathcal{E}_2. \tag{7.21}
\]

where it is convenient to introduce the critical value \( \kappa_c \) as in (4.1). The coefficient of \( e^{u} \text{Tr} h \) above becomes

\[
2\frac{n-1}{n} e^{-u} |Du|^2 - \frac{n-1}{n^2 \alpha} - \frac{e^{-2u} F}{n^2 \alpha} = \frac{n-1}{n\alpha} \left\{ \frac{1}{2} - \frac{1}{n} e^{-2u} \frac{F}{\kappa_c} \right\}. \tag{7.22}
\]

We multiply by \( e^{-2u} \) and summarize the previous calculations in the following inequality,

\[
(F^{jk}D_jD_kG)e^{-2u} \leq \left\{ \frac{1}{2} - \frac{1}{n} e^{-2u} \frac{F}{\kappa_c} \right\} \frac{n-1}{n\alpha} e^{-u} \text{Tr} h + \frac{3}{n^2 \alpha} e^{-3u} \sigma_3 + \frac{4}{n^2 \alpha} e^{-2u} F \\
- \left\{ \frac{1}{n} e^{-2u} \frac{F}{\kappa_c} \right\} \frac{1}{n\alpha} e^{-u} \left| \frac{Du}{|Du|^2} \right|^2 - \frac{1}{n\alpha} \left\{ \frac{e^{-2u} F}{\kappa_c} \right\} e^{-2u} \sum_{j=1}^n \sigma_2(\lambda'|j) \left| \frac{u_j}{|Du|^2} \right|^2 \\
- \frac{4\alpha e^{-3u}}{|Du|^2} g^{ij} D_i u D_j D_p u |F|^2. \tag{7.23}
\]

modulo terms in groups \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \). The terms in group \( \mathcal{E}_1 \) in the expression for \( (F^{jk}D_jD_kG)e^{-2u} \) come from (7.11) and (7.19) and can be worked out to be

\[
\varepsilon e^{-2u} e^{-u} \left\{ \frac{2}{n} \frac{n-1}{n} e^{-u} \text{Tr} h - \frac{8}{n} F + \frac{4}{n} e^{-u} \left| \frac{Du}{|Du|^2} \right|^2 + \frac{4}{n} \sum_j \sigma_2(\lambda'|j) \left| \frac{u_j}{|Du|^2} \right|^2 - 2(n-1) e^{2u} \left( 1 - \frac{e^{-2u} F}{\kappa_c} \right) \right\} \\
= \varepsilon e^{-2u} e^{-u} \left\{ \frac{2}{n} \frac{n-1}{n} e^{-u} \text{Tr} h - \frac{4}{n} F - 2(n-1) e^{2u} + \frac{4}{n} e^{-u} \left| \frac{Du}{|Du|^2} \right|^2 + \frac{4}{n} \sum_j \sigma_2(\lambda'|j) \left| \frac{u_j}{|Du|^2} \right|^2 \right\}. \tag{7.24}
\]
An explicit expression for the expression $|g^{ij}u_{ij}|^2_F$ occurring above is

$$- \frac{4\alpha e^{-3u}}{|Du|^2} |g^{ij}u_{ij}|^2_F = - \frac{e^{-3u}}{n^2|Du|^2} |g^{ij}u_{ij}|^2_F - \frac{e^{-3u}(e^u + fe^{-u})|Du|^2}{|Du|^2 n^2\alpha}
$$

$$+ \frac{2e^{-3u}(e^u + fe^{-u})}{|Du|^2 n^2\alpha} Fqg^{ij}u_{ij}g'_{ij}
$$

$$= - \frac{e^{-3u}}{n^2\alpha} F \cdot h + \frac{2e^{-2u}}{n^2\alpha} F + \left( \frac{e^{-2u}F}{n^2\alpha} - \frac{1}{n^2\alpha} \right) e^{-u}|Du|^2_F
$$

$$+ \frac{e^{-3u}}{n^2\alpha} \sigma_3 - \frac{e^{-3u}}{n^2\alpha} \sum_{j=1}^n \sigma_3(\lambda^i|j) |u_j|^2 |Du|^2 - \frac{2e^{-2u}}{n^2\alpha} \sum_{j=1}^n \sigma_2(\lambda^i|j) |u_j|^2 |Du|^2
$$

modulo terms in group $E_2$. Here we used

$$|g^{ij}u_{ij}|^2_F = \sum_{j=1}^n \bar{\lambda}_j e^{-u} |u_j|^2 = \sum_{j=1}^n (F - \sigma_2(\lambda^i|j)) |u_j|^2 (Tr h - \bar{\lambda}_j).
$$

Combining this expression with the previous two expressions, we obtain the following

**Theorem 6** Let $p \in X$ be a point where the function $G$ achieves its minimum. Set

$$\kappa_p = (e^{-2u}F)(p), \quad \theta = 2\alpha e^{-\varepsilon u}(p). \quad (7.25)
$$

Then we have

$$0 \leq \left\{ \frac{1}{2} - \frac{1}{n} - \frac{3\kappa_p}{2\kappa_c} + \theta \right\} \frac{n-1}{n} e^{-u} Tr h
$$

$$+ \left\{ \frac{n+1}{n} \left( \kappa_p \frac{1}{n-1} - 1 \right) + 2\theta \right\} \frac{1}{n} e^{-u}|Du|^2_F
$$

$$+ \left\{ \frac{6}{n^2} - \frac{2}{n} \theta \right\} \kappa_p - (n-1)\theta - \frac{e^{-3u}}{n^2} \sum_{j=1}^n \sigma_3(\lambda^i|j) |u_j|^2 |Du|^2
$$

$$+ \frac{4}{n^2} e^{-2u} \sigma_3 - \frac{e^{-2u}}{n} \left\{ \frac{n+2}{n} - \frac{\kappa_p}{\kappa_c} - 2\theta \right\} \sum_{j=1}^n \sigma_2(\lambda^i|j) |u_j|^2 |Du|^2 \quad (7.26)
$$

up to terms in group $E_2$.

### 7.3 A simplified Fu-Yau argument in dimension $n = 2$

We can now rederive the following key estimate of Fu-Yau [7] when $n = 2$ (and hence $\kappa_c = 1$): for any $\delta > 0$, there exists $A_\delta > 0$ so that, if $A < A_\delta$, then the minimum $\kappa = \min_X(e^{-2u}F)$ at any time $t$ satisfies the lower bound

$$\kappa > 1 - 2\delta. \quad (7.27)
$$
Indeed, fix $\delta > 0$, with $\delta << 1$. Recall that the test function $G(z)$ assumes its minimum at a point $p$, and set $\kappa_p = (e^{-2u}F)(p)$. In view of the $C^0$ estimate,

$$e^{-2u}F = \kappa_c G + O(A^\varepsilon),$$

and hence $\kappa \geq \kappa_p + O(A^\varepsilon)$. Thus it suffices to show that (7.27) holds with $\kappa$ replaced by $\kappa_p$. It also suffices to show that if $\kappa_p > 1/4$, then $\kappa_p > 1 - \delta$ for $A_\delta$ small enough. This is because $\kappa_p = \kappa = 1$ when $t = 0$, as discussed in (7.1). As $t$ varies, $\kappa$ cannot reach 1/2, since the first time it does so, we would have then $\kappa_p > 1/4$ (for $A_0$ small enough), and hence $\kappa > 1 - 3\delta$, which is a contradiction. But then $\kappa > 1/2$ for all time, and hence $\kappa > 1 - 2\delta$ for all time, as desired.

We now argue by contradiction. Assume that $\kappa_p > 1/4$. If $\kappa_p > 1 - \delta/2$, we are done, so we assume that $\kappa_p \leq 1 - \delta/2$. In dimension $n = 2$, $\sigma_3$ and $\sigma_2(\lambda'_j)$ all vanish. Incorporating the error terms in $E_1$ and $E_2$, the inequality (7.26) implies, for $A$ small enough,

$$c_1 e^{-u} \text{Tr} h(p) + c_2 e^{-u} \frac{|Du|^2_\kappa(p)}{|Du|^2}(p) \leq c_3 \kappa_p + c_4$$  \hspace{1cm} (7.28)

where $c_1, c_2, c_3, c_4$ are strictly positive constants, depending only on $\delta$. Since $\kappa_p$ is bounded by an absolute constant, it follows that

$$e^{-u} \left( \text{Tr} h(p) + \frac{|Du|^2_\kappa(p)}{|Du|^2}(p) \right) \leq c_5,$$

where $c_5$ is a constant depending only on $\delta$. This implies that all terms in $\theta^{-1}E_2$ can be bounded by $c$, where $c$ is a constant that can be made arbitrarily small by taking $\varepsilon$ and $A$ to be small.

Going back again to the inequality (7.26), we can bound the term $|Du|^2_\kappa$ as follows,

$$\left\{ \frac{n+1}{n} \left( \kappa_p \frac{1}{n-1} - 1 \right) + 2\theta \right\} \frac{1}{n\alpha} e^u \frac{|Du|^2_\kappa(p)}{|Du|^2} \leq \left\{ \frac{n+1}{n} \left( \kappa_p \frac{1}{n-1} - 1 \right) + 2\theta \right\} \frac{1}{n\alpha} e^u \lambda'_1$$  \hspace{1cm} (7.29)

where $\lambda'_1$ is either the largest or the lowest eigenvalue of $g'_{pq}$, depending on the sign of the coefficient. In dimension $n = 2$, $\kappa_c = 1$, and Theorem 6 implies, modulo additive terms of order $E_2$,

$$0 \leq \left( \frac{3}{4} \kappa_p + \frac{1}{2}\theta \right)(\lambda'_1 + \lambda'_2) e^{-u} + \frac{3}{4}(\kappa_p - 1 + \theta) e^{-u} \lambda'_1 + \left( \frac{3}{2} - \theta \right) \kappa_p - \theta$$

$$= -\left( \frac{3}{4} - \frac{3\theta}{2} \right) e^{-u} \lambda'_1 - \left( \frac{3}{4} \kappa_p - \frac{\theta}{2} \right) e^{-u} \lambda'_2 + \left( \frac{3}{2} - \theta \right) \kappa_p - \theta.$$  \hspace{1cm} (7.30)

Since $\lambda'_1 + \lambda'_2 \geq 2\sqrt{a_1a_2} \sqrt{\lambda'_1 \lambda'_2}$ for any $a_1, a_2 \geq 0$, and since $\lambda'_1 \lambda'_2 = e^{2u} \kappa_p$, we obtain

$$\left( \frac{3}{4} - \frac{3\theta}{2} \right)^{\frac{1}{2}} \left( \frac{3}{4} \kappa_p - \frac{\theta}{2} \right)^{\frac{1}{2}} \kappa_p \leq \frac{3}{4} \kappa_p - \frac{\theta}{2}(\kappa_p + 1).$$  \hspace{1cm} (7.31)
The leading term \( \kappa_p^2 \) cancels upon squaring both sides. Since we have assumed that \( \kappa \) is bounded away from 0, we can also divide by \( \kappa_p \) and the error terms of type \( \mathcal{E}_2 \) will remain of type \( \mathcal{E}_2 \). We obtain, discarding terms of order \( \theta^2 \) and dividing through by \( \theta \kappa_p \),

\[-\kappa_p - \frac{1}{3} \leq -\frac{2}{3}(\kappa_p + 1), \]

or equivalently,

\[\kappa_p \geq 1 \quad (7.32)\]

modulo additive constants which can be made arbitrarily small by taking \( A \) small. This establishes the desired lower bound for \( \kappa \).

To finish the discussion on dimension \( n = 2 \), we note that Theorem 5 implies the sharp gradient estimate assumption (5.1) in the \( C^2 \) estimate. From the previous analysis, we may choose \( A_\delta \) such that

\[\kappa = \min_X (e^{-2u}F) \geq 1 - \alpha \delta. \quad (7.33)\]

From (4.3), we have

\[1 - \alpha \delta \leq e^{-2u}F \leq 1 - 4\alpha e^{-u} |Du|^2 \left\{ 1 - (\|f\|_\infty + 1)e^{-2u} \right\} + O(e^{-2u}). \quad (7.34)\]

After choosing to be \( A_\delta \) smaller if necessary, we see that the previous inequality implies

\[e^{-u} |Du|^2 \leq \delta. \quad (7.35)\]

7.4 The case of higher dimension \( n \)

In higher dimensions, it is not difficult to see that the inequality (7.26) obtained in Theorem 6 is not powerful enough to provide a lower bound for \( \kappa_p \). In fact, even if we restrict ourselves only to the leading terms by setting formally \( \theta = 0 \), the computation and examples indicate that the case \( n = 2 \) case is quite special. In the \( n = 2 \) case, as shown in (7.31) with \( \theta = 0 \), it is easy to see that the leading terms about \( \kappa_p \) cancel perfectly between both sides. However, this is not the case for higher dimensions.

We illustrate the problem in the case \( n = 3 \). Suppose that at the point \( p \in X \) where \( G \) achieves its minimum, \( Du \) happens to be in the direction of \( \lambda_1' \). Substituting \( \kappa_c = 3 \) and \( n = 3 \), the inequality (7.26) with \( \theta = 0 \) obtained in Theorem 6 becomes

\[0 \leq \left( \frac{1}{9} - \frac{\kappa_p}{3} \right) (\lambda_1' + \lambda_2' + \lambda_3') e^{-u} + \left( \frac{2}{9} \kappa_p - \frac{4}{9} \right) e^{-u} (\lambda_1' + \lambda_3') + \frac{2}{3} \kappa_p \]

\[+ \frac{4}{9} e^{-3u} \lambda_1' \lambda_2' \lambda_3' + \left( \frac{\kappa_p}{9} - \frac{5}{9} \right) e^{-2u} \lambda_2' \lambda_3'. \quad (7.36)\]

This inequality cannot prevent \( \kappa_p = e^{-2u}\sigma_2(\lambda') \) from starting at \( \kappa_c = 3 \) and then going to zero along the method of continuity. Indeed, the path \( e^{-u}\lambda' = (1, s, s) \) gives \( \kappa_p = 2s + s^2 \) and the previous inequality reduces to

\[0 \leq \frac{1}{9} (s^4 - 2s^2 + 1). \quad (7.37)\]
Thus it is unclear whether the non-degeneracy estimate holds in higher dimensions, and it would certainly require a different method.

Acknowledgements: The authors would like to thank Pengfei Guan for stimulating conversations and for his notes on Fu-Yau’s equation. They would also like to thank Valentino Tosatti for his lectures and notes on Strominger systems. The authors are also very grateful to the referee for a particularly careful reading of the paper, and for numerous suggestions which helped clarify the paper a great deal.

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