MANIFOLDS ASSOCIATED WITH \((\mathbb{Z}_2)^{n+1}\)-COLORED REGULAR GRAPHS

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Abstract. In this article we describe a canonical way to expand a certain kind of \((\mathbb{Z}_2)^{n+1}\)-colored regular graphs into closed \(n\)-manifolds by adding cells determined by the edge-colorings inductively. We show that every closed combinatorial \(n\)-manifold can be obtained in this way. When \(n \leq 3\), we give simple equivalent conditions for a colored graph to admit an expansion. In addition, we show that if a \((\mathbb{Z}_2)^{n+1}\)-colored regular graph admits an \(n\)-skeletal expansion, then it is realizable as the moment graph of an \((n+1)\)-dimensional closed \((\mathbb{Z}_2)^{n+1}\)-manifold.

1. Introduction

In 1998, Goresky, Kottwitz and MacPherson [GKM] established the GKM theory, which gives a direct link between equivariant topology and combinatorics (or more precisely, a link between GKM manifolds and GKM graphs). The most essential properties of a GKM manifold \(X\), such as the equivariant cohomology and Betti numbers of \(X\), can thus be explicitly expressed in terms of its corresponding GKM graph. A lot of work on this subject has been carried out since then ([BGH], [GH], [GZ1], [GZ2], [GZ3], [GZ4], [MMP], [MP]). For example, a series of papers by Guillemin and Zara showed that much geometrical and topological information of a GKM manifold can be read out from the corresponding GKM graph. In the GKM theory, the action group is a torus. Alternately, when the action group is chosen to be a mod 2-torus, there is an analogous theory, namely the mod 2 GKM theory (see [BGH] or [L]), which has successfully been applied to find the lower bound of the number of fixed points and to study the equivariant cobordism classification and the Smith problem in [L].

The work of this article is mainly motivated by the GKM theory. However, we shall carry out our work in the mod 2 category. Thus, let us first recall some basic facts about the mod 2 GKM theory, which inspired us to study colored regular graphs. Suppose that \(G = (\mathbb{Z}_2)^{n+1}\) (a mod 2-torus of rank \(n + 1\)), which is also an \((n + 1)\)-dimensional linear space over \(\mathbb{Z}_2\). Let \(EG \to BG\) be the universal principal \(G\)-bundle, where \(BG = EG/G\).
is the classifying space of $G$. It is well-known that $BG = (\mathbb{RP}^\infty)^{n+1}$ and 
\[ H^*(BG; \mathbb{Z}_2) = \mathbb{Z}_2[t_0, t_1, \ldots, t_n] \]
is a $\mathbb{Z}_2$-polynomial ring over $t_0, t_1, \ldots, t_n$, where the $t_i$’s are one-dimensional generators.

Now suppose that $X$ is a closed manifold with an effective $G$-action such that the fixed point set $X^G$ is finite, which implies $\dim X \geq n + 1$ (see [AP]). Then the Borel construction $EG \times_G X$ is defined as the orbit space of the induced diagonal action $\Delta_G$ on $EG \times X$, i.e.,
\[ EG \times_G X = (EG \times X)/\Delta_G, \]
so that the equivariant cohomology of $X$ is
\[ H^*_G(X; \mathbb{Z}_2) = H^*(EG \times_G X; \mathbb{Z}_2). \]
As shown in [L], to the $G$-manifold $X$, one can always assign a regular graph $\Gamma$ with a label $\alpha(e) \in \text{Hom}(G, \mathbb{Z}_2)$ for each edge $e$ in $\Gamma$. The GKM theory points out that when the $G$-manifold $X$ satisfies certain conditions, its equivariant cohomology is a free module over $H^*(BG; \mathbb{Z}_2)$ and can be explicitly read out from $(\Gamma, \alpha)$:
\[ H^*_G(X; \mathbb{Z}_2) = \{ f : V_\Gamma \rightarrow \mathbb{Z}_2[t_0, \ldots, t_n] \mid f(v) \equiv f(w) \mod \alpha(e) \text{ for } e \in E_v \cap E_w \} \]
where $V_\Gamma$ denotes the vertex set of $\Gamma$ and $E_v$ denotes the set of all edges in $\Gamma$ that are adjacent to $v$.

In the extreme case where $\dim X = n + 1$, the associated graph $\Gamma$ with $X^G$ as its vertex set can be derived as follows.

Every real irreducible representation of $G$ is one-dimensional, so it must be a homomorphism $\rho : G \rightarrow \text{GL}(1, \mathbb{R})$. Moreover, for each $g \in G$, $\rho(g) = (\pm 1)$ since $g$ is an involution. Therefore, by identifying the multiplicative group $\{(1), (-1)\}$ with $\mathbb{Z}_2$, one can identify the set of all real irreducible representations of $G$ with $\text{Hom}(G, \mathbb{Z}_2)$. Notice that $\text{Hom}(G, \mathbb{Z}_2) \cong G$ is also an $(n + 1)$-dimensional linear space over $\mathbb{Z}_2$.

Now choose a point $v \in X^G$, and choose a linear identification $\phi_v : T_vX \rightarrow \mathbb{R}^{n+1}$. Then the induced tangent representation at $v$ can be expressed as an $(n + 1)$-dimensional real representation $\rho : G \rightarrow \text{GL}(n+1, \mathbb{R})$, which can be further split into $n + 1$ real irreducible representations
\[ \rho_0, \rho_1, \ldots, \rho_n \in \text{Hom}(G, \mathbb{Z}_2). \]
Clearly the set $\{\rho_0, \rho_1, \ldots, \rho_n\}$ is independent of the choice of $\phi_v$. Since $G$ acts effectively on $X$, $\rho_0, \rho_1, \ldots, \rho_n$ are linearly independent in $\text{Hom}(G, \mathbb{Z}_2)$, so they form a basis of $\text{Hom}(G, \mathbb{Z}_2)$.

For each $\rho_i$, the subgroup $g_i = \ker \rho_i$ is of codimension one in $G$, and the 1-dimensional component of the fixed set of $g_i$ acting on $X$ is a circle $S_i$ (c.f. [AP] or [CE]). Since $G/g_i \cong \mathbb{Z}_2$, the $G$-action on $X$ induces an involution on $S_i$, so $S_i$ contains only two fixed points $v$ and $w$ in $X^G$. Let us label $S_i$ by $\rho_i$ and choose an orientation on $S_i$. Then one gets two oriented edges with endpoints $v, w$ and with the same “label” $\rho_i$. Since the labeled oriented edges always appear in pairs with reversing orientations, one can merge
each pair into an unoriented edge, so that each $S_i$ determines one edge with two $G$-fixed points as its endpoints (i.e., a 1-valent graph). Then, our desired graph $\Gamma$ is the union of all these 1-valent graphs such that the vertex set of $\Gamma$ consists of all fixed points of $X^G$. $\Gamma$ is called the moment graph and the “labelling map” $\alpha : E_\Gamma \to \text{Hom}(G, \mathbb{Z}_2)$ is called the axial function, where $E_\Gamma$ denotes the set of all unoriented edges.

It is obvious that in this simple case, the moment graph $\Gamma$ is a regular $(n + 1)$-valent graph, and the axial function $\alpha : E_\Gamma \to \text{Hom}(G, \mathbb{Z}_2)$ satisfies the following two conditions:

(1) for each vertex $v$ in $\Gamma$, if $e_0, \ldots, e_n$ are all edges adjacent to $v$, then $\alpha(e_0), \ldots, \alpha(e_n)$ linearly span $\text{Hom}(G, \mathbb{Z}_2)$, so they are all nontrivial and distinct;

(2) for each edge $e$ in $\Gamma$, let $g_e = \ker \alpha(e)$ and $v, w$ denote the two endpoints of $e$. If $\{e_0, \ldots, e_n\}$ and $\{e'_0, \ldots, e'_n\}$ are the sets of edges adjacent to $v$ and $w$ respectively, then

$$\{\alpha(e_0)|_{g_e}, \ldots, \alpha(e_n)|_{g_e}\} = \{\alpha(e'_0)|_{g_e}, \ldots, \alpha(e'_n)|_{g_e}\}$$

(because they form irreducible decompositions of two equivalent tangent representations of $g_e$ at $v$ and $w$ respectively). It is not difficult to verify that this is equivalent to

$$\{\text{Span}(\alpha(e_0), \alpha(e)), \ldots, \text{Span}(\alpha(e_n), \alpha(e))\} = \{\text{Span}(\alpha(e'_0), \alpha(e)), \ldots, \text{Span}(\alpha(e'_n), \alpha(e))\}.$$

Conversely, given a finite connected regular graph $\Gamma$ together with some axial function $\alpha$ satisfying the previous two conditions, a natural question is: what information on the topology of manifolds can we extract from the pair $(\Gamma, \alpha)$? This seems to be an open-ended question. Recently, some interesting and beautiful work on this subject has been done by some mathematicians. For example, in [GZ2] Guillemin and Zara obtained some purely combinatorial analogues of the main results in the GKM theory. In addition, they also obtained a realization theorem for abstract GKM-graphs: for an abstract GKM graph $(\Gamma, \alpha)$, there exists a complex manifold $X$ and a GKM action of the torus $T$ on $X$ such that $(\Gamma, \alpha)$ is its GKM graph. Note that the constructed complex manifold $X$ is not compact, not equivariantly formal, and admits no canonical compactification.

This article will deal with the problem from a different viewpoint, in the mod 2 category. We shall introduce a “skeletal expansion technique” for any given abstract $G$-colored finite connected regular $(n + 1)$-valent graphs $(\Gamma, \alpha)$, where $\alpha$ satisfies the above two conditions. Specifically, we shall give a canonical way to inductively attach cells on $\Gamma$ to form a cell complex $K$ such that the 1-skeleton of $K$ is just $\Gamma$. We will call this technique the skeletal expansion, and we shall use it to carry out our work as follows.

First, we determine under what conditions this procedure of cell-gluing can be performed to the end until one obtains a closed manifold (see Theorem 2.4). We further show that every closed combinatorial $n$-manifold can be obtained in this way (see Theorem 2.5).
Next, we consider the realization problem: can we always construct a $G$-action on an $(n+1)$-dimensional closed manifold such that its moment graph is exactly the given $G$-colored regular $(n+1)$-valent graph $(\Gamma, \alpha)$? To do this, our strategy is first to try to construct the orbit space of an action from $(\Gamma, \alpha)$. Then by the work of Davis and Januszkiewicz [DJ] on the reconstruction of small covers (which is stated briefly below), we shall use the obtained orbit space to construct the desired $G$-action. We shall show that under certain conditions, any abstract $G$-colored finite connected regular $(n+1)$-valent graph $(\Gamma, \alpha)$ is realizable as the moment graph of an $(n+1)$-dimensional closed $G$-manifold (see Theorem 2.6).

An $(n+1)$-dimensional convex polytope $P \subset \mathbb{R}^{n+1}$ is just an $(n+1)$-dimensional compact manifold which is the intersection of a finite set of half-spaces in $\mathbb{R}^{n+1}$. It naturally becomes a cell decomposition of $B^{n+1}$ and has a minimum set of defining half-spaces, each of which corresponds to an $n$-dimensional face of $P$. Denote the set of all these codimension one faces by $F(P)$. If every vertex is surrounded by exactly $n+1$ faces in $F(P)$, then $P$ is called a simple convex polytope. Clearly, the 1-skeleton of each $(n+1)$-dimensional simple convex polytope is a regular $(n+1)$-valent graph $\Gamma(P)$.

If $P$ is an $(n+1)$-dimensional simple convex polytope, a characteristic function is a map $\lambda : F(P) \to G = (\mathbb{Z}_2)^{n+1}$ such that the $n+1$ faces in $F(P)$ adjacent to each vertex are sent to $n+1$ linearly independent vectors in $G$. Each such $\lambda : F(P) \to G$ is dual to an axial function $\alpha_\lambda : E_{\Gamma(P)} \to \text{Hom}(G, \mathbb{Z}_2)$, such that for each edge $e \in E_{\Gamma(P)}$ and those faces $F \in F(P)$ containing $e$, $\alpha_\lambda(e)(\lambda(F)) = 0$. It is easy to see that both $\lambda$ and $\alpha_\lambda$ are determined by each other.

An $(n+1)$-dimensional closed manifold $X$ is called a small cover over $P$ if it admits an effective and locally standard $G$-action such that the orbit space $X/G = P$. In [DJ], Davis and Januszkiewicz observed that for each simple convex polytope $P$ and each characteristic function $\lambda : F(P) \to G$, there is a canonical way to construct a small cover $X(\lambda)$ over $P$, and every small cover over $P$ can be reconstructed in this way. Moreover, $\Gamma(P)$ is exactly the moment graph for $X(\lambda)$ and $\alpha_\lambda$ is the corresponding axial function.

In the case of small covers, reconstructing the orbit spaces and $G$-manifolds from moment graphs is simple. In fact, the orbit space $X/G$ of a small cover $X$ is bounded by $S^n$ with a very nice regular cell decomposition corresponding to $\partial P$, and this cell complex is topologically dual to a triangulation of $S^n$. The skeletal expansion of the moment graph $(\Gamma, \alpha)$ of $X$ will exactly reproduce the pre-defined cell decomposition on $\partial P$, so that this can lead us to recover the orbit space $P$ and its characteristic function $\lambda$, and thus $X$ can be reconstructed. More generally, if $X$ is not a small cover, the problem of reconstructing the $G$-manifold $X$ and its orbit space compatible with the moment graph becomes much more complicated. We remark that if the orbit space $X/G$ is a compact $(n+1)$-manifold with boundary such that the pre-image of each component of the boundary of $X/G$ contains a fixed point, then the skeletal expansion of the moment graph $(\Gamma, \alpha)$ will produce a cell decomposition of the boundary of $X/G$.
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2. Notation and main results

In this section, we will first define some standard notation and state our main results. Suppose \( G = (\mathbb{Z}_2)^{n+1} \) and \( \Gamma \) is a finite connected regular \((n+1)\)-valent graph. A \( G \)-coloring is a map

\[
\alpha : E_\Gamma \rightarrow \text{Hom}(G, \mathbb{Z}_2),
\]

such that at each vertex, the colors of the \( n+1 \) edges adjacent to it are linearly independent vectors (here the color of an edge \( e \) is simply the vector \( \alpha(e) \)). Notice that for some technical reason, the edge colors are in \( \text{Hom}(G, \mathbb{Z}_2) \cong G \), but not in \( G \). If in addition the total image of \( \alpha \) contains only \( n+1 \) vectors \( x_0, \ldots, x_n \), then we call \( \alpha \) a "pure" \( G \)-coloring, which is just the ordinary edge coloring (see Fig. 1).

![Figure 1. Examples of \( G \)-colored graphs](image)

For each subgraph \( \Delta \subseteq \Gamma \), let \( \text{Span} \alpha(\Delta) \) denote the linear space spanned by all colors of edges in \( \Delta \), and call it the color of \( \Delta \). Let

\[
\dim_\alpha \Delta = \dim \text{Span} \alpha(\Delta).
\]

If \( K \) is a triangulation of a manifold \( M \) and the graph \( \Gamma \) is its 1-skeleton, then each \( k \)-dimensional simplex \( \Delta \in K \) can be characterized by the 1-skeleton of \( \partial \Delta \), which is a subgraph of \( \Gamma \). Motivated by this idea, we introduce the following concept of \( k \)-nest for colored graph.

**Definition 2.1.** We say that a proper subgraph \( \Delta \subseteq \Gamma \) is a \( k \)-dimensional colored nest (or just a \( k \)-nest) associated with \( (\Gamma, \alpha) \) if

1. \( \Delta \) is connected;
2. \( \dim_\alpha \Delta = k \);
3. for every connected subgraph \( \Delta' \supseteq \Delta \), \( \dim_\alpha \Delta' > k \).
If $\Delta' \supseteq \Delta$, and they are both colored nests, then we say that $\Delta$ is a face of $\Delta'$. Let $K(\Gamma, \alpha)$ denote the set of all colored nests of $(\Gamma, \alpha)$. We call

$$K_k(\Gamma, \alpha) = \{ \Delta \in K(\Gamma, \alpha) \mid \dim \Delta \leq k \}$$

the $k$-dimensional colored nest-skeleton (or just the $k$-skeleton) of $K(\Gamma, \alpha)$.

**Remark.** Suppose $\text{Hom}(G, \mathbb{Z}_2)$ is spanned by $n + 1$ vectors $x_0, \ldots, x_n$. Let $S(G, \mathbb{Z}_2)$ denote the $\mathbb{Z}_2$-exterior algebra generated by these vectors, which consists of all square-free (i.e., we require $x_0^2 = 0, \ldots, x_n^2 = 0$ in $S(G, \mathbb{Z}_2)$) $\mathbb{Z}_2$-polynomials of $x_0, \ldots, x_n$, and let $S^k(G, \mathbb{Z}_2)$ denote its linear subspace spanned by all degree $k$ monomials, namely

$$x_0 \cdots x_{k-1}, x_0 \cdots x_{k-2}x_k, \ldots, x_{n-k+1} \cdots x_n.$$

Then $S^1(G, \mathbb{Z}_2) = \text{Hom}(G, \mathbb{Z}_2)$, so the color of each edge can also be considered as an element in $S(G, \mathbb{Z}_2)$. Furthermore, for any $m$-dimensional linear subspace $L \subseteq S^1(G, \mathbb{Z}_2)$, choose a basis $y_1, \ldots, y_m$ for $L$, one can easily verify that $\eta_L = y_1 \cdots y_m \in S^m(G, \mathbb{Z}_2)$ is independent of the choice of $y_1, \ldots, y_m$. Therefore, for each connected subgraph $\Delta \subseteq \Gamma$, $L = \text{Span} \alpha(\Delta)$ determines a unique homogeneous $\mathbb{Z}_2$-polynomial $\eta_L \in S^{\text{dim} \Delta}(G, \mathbb{Z}_2)$. In particular, one can extend the coloring $\alpha$ to a “coloring”

$$\tilde{\alpha} : K(\Gamma, \alpha) \to S(G, \mathbb{Z}_2).$$

Clearly a 0-nest $v$ is just a vertex of $\Gamma$. A 1-nest $e$ is an edge in $E_\Gamma$ together with its two ends. A 2-nest $\gamma$ is a maximal arc or maximal circle such that $\dim_G \gamma = 2$.

**Definition 2.2.** If for each edge $e_1$ with endpoints $v, w$ and each edge $e_0$ adjacent to $v$, there exists one and only one edge $e_2$ adjacent to $w$ such that

$$\text{Span}(\alpha(e_0), \alpha(e_1)) = \text{Span}(\alpha(e_1), \alpha(e_2)),$$

then we call $\alpha$ a “good” coloring. Clearly “pure” colorings are always “good” colorings.

**Remark.** Notice that the $\alpha(e_i)$’s can be considered as elements in $S^1(G, \mathbb{Z}_2)$. Assume that the colors of edges adjacent to each vertex are linearly independent, then the above equation is equivalent to $\alpha(e_0)\alpha(e_1) = \alpha(e_1)\alpha(e_2)$, and is also equivalent to

$$\prod_{e \in E_v \setminus \{e_1\}} \alpha(e) \equiv \prod_{e \in E_w \setminus \{e_1\}} \alpha(e) \mod \alpha(e_1),$$

where $E_p$ denotes the set of all edges in $\Gamma$ adjacent to a vertex $p$. This also implies that there must be a unique bijection $\theta_{e_1} : E_v \to E_w$ such that for any $e \in E_v \setminus \{e_1\}$,

$$\alpha(e) \equiv \alpha(\theta_{e_1}(e)) \mod \alpha(e_1).$$

The collection $\theta = \{\theta_e \mid e \in E_\Gamma\}$ is called a connection of $(\Gamma, \alpha)$. Notice also that if $X$ is an $(n + 1)$-dimensional closed manifold with an effective and locally standard $G$-action such that the fixed point set $X^G$ is finite, then the orbit space $X/G$ is a nice manifold with corners (see, [D] and [LM]) and each axial function on its moment graph is in fact a “good” coloring.
Lemma 2.1. Suppose $n \geq 2$ and $(\Gamma, \alpha)$ is a finite connected $G$-colored regular graph. Then $\alpha$ is “good” if and only if each $k$-nest of $K(\Gamma, \alpha)$ is a connected regular $k$-valent subgraph.

Proof. Suppose $\alpha$ is “good”. Fix a $k$-nest $\Delta$. For any vertex $p$, let $E(p)$ denote the set of all edges inside $\Delta$ adjacent to $p$, and let $L_p$ denote the linear space spanned by $\{\alpha(e) \mid e \in E(p)\}$. Now let us show that for any two vertices $u, v$ connected by an edge $e_1 \in \Delta$, $L_u = L_v$.

If this is not true, suppose without loss of generality that there is an edge $e_0 \in E(u)$ such that $\alpha(e_0) \notin L_v$. Then the connection of $\alpha$ points out an edge $e_2 \in E_{\Gamma}$ adjacent to $v$ such that

$$\text{Span}(\alpha(e_0), \alpha(e_1)) = \text{Span}(\alpha(e_1), \alpha(e_2)).$$

Since $\alpha(e_2) \in \text{Span}(\alpha(e_0), \alpha(e_1)) \subseteq \text{Span} \alpha(\Delta)$ and $\Delta$ is maximal, $e_2$ must be an edge inside $\Delta$, so $e_2 \in E(v)$ and $\alpha(e_2) \in L_v$, which implies

$$\alpha(e_0) \in \text{Span}(\alpha(e_1), \alpha(e_2)) \subseteq L_v,$$

a contradiction.

Since $\Delta$ is connected, this implies that $L_p$ is the same for all vertices $p$ in $\Delta$. Therefore for any vertex $p$ in $\Delta$, $L_p = \text{Span } \alpha(\Delta)$. However, this is a $k$-dimensional linear space, thus the valence of $p$ in $\Delta$ must be always $k$. Notice that for $0 < l \leq n + 1$, the colors on $l$ edges emerging from the same vertex are always linearly independent so they span a $l$-dimensional linear space.

On the other hand, suppose every 2-nest is a regular 2-valent graph. For each edge $e_1$ with endpoints $v, w$ and each edge $e_0$ adjacent to $v$, let $\tilde{\gamma}$ be the graph formed by edges with colors inside $\text{Span}(\alpha(e_0), \alpha(e_1))$ together with their vertices, and let $\gamma$ be its connected component containing $e_0$ and $e_1$. Then $\gamma$ is a 2-nest, so the valence of $\gamma$ at $w$ equals two. This implies that there is exactly one edge $e_2 \neq e_1$ adjacent to $w$, such that $\alpha(e_2) \in \text{Span}(\alpha(e_0), \alpha(e_1))$. Hence $\alpha$ is “good”. \hfill $\square$

Remark. The proof of Lemma 2.1 actually shows that if $\alpha$ is “good”, then for any linear subspace $L$, the edges in $\alpha^{-1}(L) = \{e \in E_{\Gamma} \mid \alpha(e) \in L\}$ together with their vertices also forms a disjoint union of some regular graphs.

Recall that a regular cell decomposition of a topological space $M$ is a cell decomposition $K$ such that

1. for each cell $\Delta \in K$, let $B$ be the unit open ball with dimension equal to dim $\Delta$, and let $S$ be its boundary sphere, then the characteristic map $\phi : B \to \Delta \subseteq M$ extends to a homeomorphism $\phi : (\overline{B}, S) \to (\overline{\Delta}, \partial \Delta)$ where $\partial \Delta = \overline{\Delta} \setminus \Delta \subseteq M$;

2. for each cell $\Delta \in K$, its boundary sphere $\partial \Delta$ is the union of some cells in $K$.

Definition 2.3. Suppose $k \leq n$ and $M$ is a topological space which has a $k$-dimensional finite regular cell decomposition $K$. If there is a one-to-one correspondence $\kappa : K \to K_k(\Gamma, \alpha)$ preserving dimensions and face relations, then $(M, K)$ is called a $k$-skeletal expansion of $(\Gamma, \alpha)$, and $K_k(\Gamma, \alpha)$ is called a $G$-colored cell decomposition of $M$. If $k = n$ and $M$ is a closed manifold, then $M$ is called a $(\mathbb{Z}_2)^{n+1}$-colorable manifold.
Remark. If \( \Gamma \) is the 1-skeleton of an \((n+1)\)-dimensional simple convex polytope \( P \), and \( \alpha : E_\Gamma \to \text{Hom}(G, \mathbb{Z}_2) \) is dual to a characteristic function of \( P \) (c.f. section 1), then \( \partial P \) with the induced cell decomposition is an \( n \)-skeletal expansion of \((\Gamma, \alpha)\).

Obviously, 1-skeletal expansions always exist. One can simply choose the vertices of \( \Gamma \) as the 0-cells and choose the edges in \( E_\Gamma \) as the 1-cells.

Lemma 2.2. Suppose \( n \geq 2 \) and the coloring \( \alpha \) is “good”. Then \((\Gamma, \alpha)\) always has a 2-skeletal expansion.

Proof. By Lemma 2.1, each 2-nest is a connected regular 2-valent graph. Therefore all 2-nests must be embedded circles. Furthermore, one can glue discs to each of these circles in the graph to get a 2-skeletal expansion. \( \square \)

Remark. If \( n \geq 2 \) and \( \alpha : E_\Gamma \to \text{Hom}(G, \mathbb{Z}_2) \) is “pure”, then \((\Gamma, \alpha)\) always has a 2-skeletal expansion. Notice that a 2-nest in this case must be a simple closed curve in \( \Gamma \) consisting of an even number of edges with alternating edge-coloring. We call them bi-colored circles.

Higher dimensional skeletal expansions can be built up inductively using the following lemma, which can be easily proved similarly.

Lemma 2.3. Suppose that \( \alpha \) is “good”, \((\Gamma, \alpha)\) has a \( k \)-skeletal expansion \((M, K)\), and \( \kappa : K \to K_k(\Gamma, \alpha) \) is a one-to-one correspondence. For each \( k+1 \) edges \( e_0, \ldots, e_k \in E_\Gamma \) sharing one common vertex \( v \), let \( \Delta \) be the unique \((k+1)\)-nest containing \( e_0, \ldots, e_k \), and let
\[
F^k(e_0, \ldots, e_k) = \bigcup_{\kappa(\sigma) \subseteq \Delta} \sigma.
\]
If each such \( F^k(e_0, \ldots, e_k) \) is a \( k \)-sphere, then one can glue a \((k+1)\)-cell to each \( k \)-sphere and obtain a \((k+1)\)-skeletal expansion of \((\Gamma, \alpha)\). \( \square \)

In general, \( M \) is only a topological space but not a manifold, so it would be quite natural to ask the following three questions:

(Q1) When will a finite connected regular \((n+1)\)-valent graph \( \Gamma \) with a “good” \((\mathbb{Z}_2)^{n+1}\)-coloring admit an \( n \)-skeletal expansion?
(Q2) If \((\Gamma, \alpha)\) admits an \( n \)-skeletal expansion \((M, K)\), when will \( M \) be a closed manifold?
(Q3) What kind of \( n \)-dimensional manifold can be \((\mathbb{Z}_2)^{n+1}\)-colorable?

In the next two sections, we will give a complete answer to these questions in dimensions 2 and 3. For dimensions greater than 3, question (Q1) is still open, due to the difficulties in recognizing regular cell decompositions of \( S^3 \). However, questions (Q2) and (Q3) are solved completely by the following two theorems.

Theorem 2.4. Suppose \( G = (\mathbb{Z}_2)^{n+1}, (\Gamma, \alpha) \) is a \( G \)-colored finite connected regular \((n+1)\)-valent graph in which \( \alpha \) is a “good” coloring, and \((M, K)\) is an \( n \)-skeletal expansion of \((\Gamma, \alpha)\). Then \( M \) is an \( n \)-dimensional closed manifold.
An $n$-dimensional closed combinatorial manifold is a connected topological space with a finite $n$-dimensional regular cell decomposition, such that the link complex of each $k$-cell is an embedded $(n - k - 1)$-sphere. Notice that every closed differentiable manifold is a closed combinatorial manifold (see e.g. [W]), and every closed combinatorial manifold is a closed (topological) manifold.

**Theorem 2.5.** Every $n$-dimensional closed combinatorial manifold is $(\mathbb{Z}_2)^{n+1}$-colorable. Moreover, one can choose “pure” colorings for doing this.

**Remark.** It is not difficult to see that this is also a necessary condition for being colorable, namely an $n$-dimensional closed manifold is $(\mathbb{Z}_2)^{n+1}$-colorable if and only if it is a closed combinatorial manifold.

Although a manifold constructed as above does not naturally carry any group actions, in some sense, it is associated with the boundary of the orbit space of a manifold with group actions. Thus we can pose the following realization problem:

(Q4) Let $(\Gamma, \alpha)$ be a $(\mathbb{Z}_2)^{n+1}$-colored finite connected regular $(n + 1)$-valent graph such that $\alpha$ is a “good” coloring. Under what conditions will $(\Gamma, \alpha)$ be realizable as the moment graph of a $(\mathbb{Z}_2)^{n+1}$-action on an $(n + 1)$-dimensional closed manifold?

We know by Theorem 2.4 that if $(\Gamma, \alpha)$ admits an $n$-skeletal expansion $(M, K)$, then $M$ is an $n$-dimensional closed manifold. By a simple trick, we can modify $M$ into the boundary of another manifold $N$ and turn $N$ into the orbit space of a $G$-manifold. This leads us to the following result.

**Theorem 2.6.** Suppose $(\Gamma, \alpha)$ is a $(\mathbb{Z}_2)^{n+1}$-colored finite connected regular $(n + 1)$-valent graph such that $\alpha$ is a “good” coloring. If $(\Gamma, \alpha)$ admits an $n$-skeletal expansion $(M, K)$, then it is realizable as the moment graph of some $(\mathbb{Z}_2)^{n+1}$-action on an $(n + 1)$-dimensional closed manifold.

Although we have not achieved a complete answer for the realization problem, it is very tempting to make the following conjecture:

**Conjecture.** For each $(\mathbb{Z}_2)^{n+1}$-colored finite connected regular $(n + 1)$-valent graph $(\Gamma, \alpha)$ with $\alpha$ being a “good” coloring, $(\Gamma, \alpha)$ can be realized as the moment graph of some $(\mathbb{Z}_2)^{n+1}$-action on an $(n + 1)$-dimensional closed manifold.

3. $(\mathbb{Z}_2)^3$-COLORABLE CLOSED SURFACES

Let us first examine the case $n = 2$, which is surprisingly simple.

**Theorem 3.1.** Suppose $G = (\mathbb{Z}_2)^3$, $\Gamma$ is a finite connected regular 3-valent graph, and $\alpha : E_{\Gamma} \to \text{Hom}(G, \mathbb{Z}_2)$ is a “good” coloring. Then $(\Gamma, \alpha)$ has a 2-skeletal expansion $(M, K)$ in which $M$ is a closed surface.

**Proof.** By Lemma 2.2 $(\Gamma, \alpha)$ must have a 2-skeletal expansion $(M, K)$. Moreover, since each edge is used exactly twice by 2-cells, this gives a pairing of arcs on the boundaries of those 2-cells. Therefore, gluing these cells together along their boundaries will give a closed surface. \qed
As an example of application, the two graphs in Fig. 1 both satisfy the above conditions. It is not difficult to see that the surface corresponding to Fig. 1 (a) is a 2-sphere, while the surface for Fig. 1 (b) is a real projective plane $P^2$.

**Theorem 3.2.** Every closed surface $M$ is $(\mathbb{Z}_2)^3$-colorable. Moreover, one can choose planar graphs and “pure” colorings for doing this.

**Remark.** The first sentence is in fact a corollary of Theorem 2.5. Notice that the graph used in Fig. 2 for a real projective plane $P^2$ is exactly the same as Fig. 1 (b).

**Proof.** The graphs for building up closed surfaces are shown in Fig. 2, where $\{x_0, x_1, x_2\}$ is a chosen basis for $(\mathbb{Z}_2)^3$. By Theorem 3.1, these graphs all have 2-skeletal expansions, and the corresponding topological space will automatically become closed surfaces. Now let us verify the topological type for these surfaces.

The graph for $S^2$ is just a simple exercise. Let us examine the graph for $gT^2$ (the connected sum of $g$ copies of $T^2$). The graph has $8g$ vertices, $12g$ edges and $2g + 2$ bi-colored circles, which are listed below:

| color | notation | 2-nest |
|-------|----------|--------|
| $\beta$ | $\Span(x_0, x_1)$ | $(A_{11}A_{12}A_{15}A_{16}\cdots A_{g_1}A_{g_2}A_{g_5}A_{g_6})$ |
| $\gamma_1$ | $\Span(x_0, x_1)$ | $(A_{13}A_{14}A_{17}A_{18})$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $\gamma_g$ | $\Span(x_0, x_1)$ | $(A_{g_3}A_{g_4}A_{g_7}A_{g_8})$ |

| $\xi$ | $\Span(x_0, x_2)$ | $(A_{11}A_{18}A_{13}A_{12}A_{15}A_{14}A_{17}A_{16}\cdots A_{g_1}A_{g_8}A_{g_3}A_{g_2}A_{g_5}A_{g_4}A_{g_7}A_{g_6})$ |

| $\eta_1$ | $\Span(x_1, x_2)$ | $(A_{11}A_{12}A_{13}A_{14}A_{15}A_{16}A_{17}A_{18})$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $\eta_g$ | $\Span(x_1, x_2)$ | $(A_{g_1}A_{g_2}A_{g_3}A_{g_4}A_{g_5}A_{g_6}A_{g_7}A_{g_8})$ |

Suppose $(M, K)$ is its 2-skeletal expansion. Then one can choose an orientation on each of these bi-colored circles, such that when two bi-colored circles share a common edge, they induce opposite orientations on that edge. This implies that $M$ is an orientable surface. Moreover, its Euler characteristic is $8g - 12g + (2g + 2) = 2 - 2g$, so $M$ is indeed an orientable surface with genus $g$.

As for the graph for $kP^2$ (i.e., a connected sum of $k$ copies of $P^2$), it has $4k$ vertices, $6k$ edges and $k + 2$ bi-colored circles, which are listed below:

| color | notation | 2-nest |
|-------|----------|--------|
| $\beta$ | $\Span(x_0, x_1)$ | $(A_{11}A_{14}A_{12}A_{13}\cdots A_{k_1}A_{k_4}A_{k_2}A_{k_3})$ |
| $\xi$ | $\Span(x_0, x_2)$ | $(A_{11}A_{12}A_{14}A_{13}\cdots A_{k_1}A_{k_2}A_{k_4}A_{k_3})$ |
| $\eta_1$ | $\Span(x_1, x_2)$ | $(A_{11}A_{12}A_{13}A_{14})$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $\eta_k$ | $\Span(x_1, x_2)$ | $(A_{k_1}A_{k_2}A_{k_3}A_{k_4})$ |
Suppose \((M, K)\) is its 2-skeletal expansion. The 2-cells bounded by \(\beta\) and \(\xi\) cannot have opposite orientations on all their common boundary arcs, so \(M\) must be a non-orientable surface, while its Euler characteristic is equal to \(4k - 6k + (k + 2) = 2 - k\). Thus \(M\) is indeed a non-orientable surface with genus \(k\). □

**Remark.** The graphs for \(gT^2\) and \(kP^2\) are actually constructed from the graphs for \(1T^2\) (torus) and \(1P^2\) (real projective plane) by doing connected sums. Suppose \(\Gamma_i\) is a regular 3-valent graph with “pure” coloring \(\alpha_i : E_{\Gamma_i} \to \{x_0, x_1, x_2\}\) and \((M_i, K_i)\) is its 2-skeletal expansion \((i = 1 \text{ or } 2)\). If \(e_1, e_2\) are two edges in \(\Gamma_1, \Gamma_2\) respectively with the same color, then one can cut them open, reconnect the four ends in the other way and form a new regular 3-valent graph \(\Gamma\) with “pure” coloring \(\alpha : E_{\Gamma} \to \{x_0, x_1, x_2\}\). Suppose \((M, K)\) is
its 2-skeletal expansion, then it is not difficult to see that $M$ is exactly the connected sum of $M_1$ and $M_2$.

4. $(\mathbb{Z}_2)^4$-COLORABLE CLOSED 3-MANIFOLDS

Now let us consider the case $n = 3$.

**Theorem 4.1.** Suppose $G = (\mathbb{Z}_2)^4$, $\Gamma$ is a connected regular 4-valent graph and $\alpha : E_\Gamma \to \text{Hom}(G, \mathbb{Z}_2)$ is a "good" $G$-coloring. Then $(\Gamma, \alpha)$ has a 3-skeletal expansion $(M, K)$ (by Theorem 2.4 $M$ must be a closed 3-manifold) if and only if:

$$\# \{3\text{-nests}\} = \# \{2\text{-nests}\} - \# \{\text{vertices}\}.$$

**Proof.** First let us use $\nu_k$ to denote (only in this proof) the number of $k$-nests in $\Gamma$. Now suppose $(\Gamma, \alpha)$ has a 3-skeletal expansion $(M, K)$. Then by Theorem 2.4, $M$ is a closed 3-manifold. So it is not difficult to see that $\nu_0 - \nu_1 + \nu_2 - \nu_3$ is equal to the Euler characteristic of $M$, which must be 0. Since $\Gamma$ is a regular 4-valent graph, $\nu_1 = 2\nu_0$ so we get $\nu_3 = \nu_2 - \nu_0$.

On the other hand, suppose that $\nu_3 = \nu_2 - \nu_0$. First choose a 2-skeletal expansion $(M_2, K_2)$ for $(\Gamma, \alpha)$ according to Lemma 2.3. For each three edges $e_0, e_1, e_2$ sharing a common vertex, suppose $\Delta$ is the unique 3-nest containing them and suppose $F = F^2(e_0, e_1, e_2)$. Let $\nu_k(\Delta)$ denote the number of $k$-nests in $\Delta$. Then the Euler characteristic of $F$ is equal to $\chi(\Delta) = \nu_0(\Delta) - \nu_1(\Delta) + \nu_2(\Delta)$, since $F$ is a 2-skeletal expansion of $\Delta$. However, $F$ is in fact a closed surface, thus $\chi(\Delta) \leq 2$. Because $\Delta$ is a regular 3-valent graph without loop edge, $\nu_1(\Delta) = 3\nu_0(\Delta)/2$. Thus,

$$\nu_2(\Delta) - \nu_0(\Delta)/2 \leq 2.$$

Because any $k$ edges ($k = 0, 1, 2$, or 3) sharing a common vertex determine a unique $k$-nest, we know that every vertex is the face of exactly four 3-nests, and every 2-nest is the face of exactly two 3-nests. Therefore summing up the above inequality over all 3-nests, one obtains

$$2\nu_2 - 2\nu_0 \leq 2\nu_3.$$

Our assumption implies that the inequalities here must all be equalities. Consequently those $F^2(e_0, e_1, e_2)$ must all be spheres. Hence by Lemma 2.3, one can fill these spheres with some 3-cells and get a 3-skeletal expansion $(M, K)$. By Theorem 2.4, $M$ must be a closed 3-manifold. \hfill $\Box$

To understand intuitively why $M$ must be a closed 3-manifold (without invoking Theorem 2.4), we can consider the local picture near a vertex $v$ (see Fig. 3). Choose some local coordinates such that $v$ becomes $(0, 0, 0)$ and the adjacent edges become (locally) $e_0 \sim \{(t, 0, 0) \mid 0 < t < 1\}$, $e_1 \sim \{(t, 0, 0) \mid -1 < t < 0\}$, $e_2 \sim \{(0, t, 0) \mid 0 < t < 1\}$, $e_3 \sim \{(0, t, 0) \mid -1 < t < 0\}$ respectively. Suppose $\alpha(e_i) = x_i$. 
For each $i \neq j \in \{0, 1, 2, 3\}$, there is exactly one 2-cell $\sigma_{ij}$ adjacent to both $e_i$ and $e_j$, let us glue it to $e_i$ and $e_j$ (locally) as follows:

$\sigma_{01} \mapsto \{(t, 0, s) \mid -1 < t < 1, -1 < s < 0\}$
$\sigma_{02} \mapsto \{(t, s, 0) \mid 0 < t < 1, 0 < s < 1\}$
$\sigma_{03} \mapsto \{(t, s, 0) \mid 0 < t < 1, -1 < s < 0\}$
$\sigma_{12} \mapsto \{(t, s, 0) \mid -1 < t < 0, 0 < s < 1\}$
$\sigma_{13} \mapsto \{(t, s, 0) \mid -1 < t < 0, -1 < s < 0\}$
$\sigma_{23} \mapsto \{(0, t, s) \mid -1 < t < 1, 0 < s < 1\}$

For each $i \neq j \neq k \in \{0, 1, 2, 3\}$, there is exactly one 3-cell $\Delta_{ijk}$ adjacent to three edges $e_i$, $e_j$ and $e_k$, and we can glue it to $e_i$, $e_j$ and $e_k$ (locally) as follows:

$\Delta_{012} \mapsto \{(t, s, u) \mid -1 < t < 1, 0 < s < 1, -1 < u < 0\}$
$\Delta_{013} \mapsto \{(t, s, u) \mid -1 < t < 1, -1 < s < 0, -1 < u < 0\}$
$\Delta_{023} \mapsto \{(t, s, u) \mid 0 < t < 1, -1 < s < 1, 0 < u < 1\}$
$\Delta_{123} \mapsto \{(t, s, u) \mid -1 < t < 0, -1 < s < 1, 0 < u < 1\}$

The possibility of this arrangement implies that every vertex $v$ has a neighborhood in $M$ homeomorphic to $\mathbb{R}^3$. Thus the combinatorial nature of $(M, K)$ implies that $M$ is a closed 3-manifold.

![Figure 3. Local picture near a vertex of 3-skeletal expansion](image)

On the other hand, one can also use this canonical method to verify the $G$-colorability of a general 3-manifold, by finding cell decomposition that locally looks like the above canonical picture near each vertex.

**Theorem 4.2.** Every closed 3-manifold is $(\mathbb{Z}_2)^4$-colorable. Moreover, one can use only four 3-cells to do this.
Remark. Notice that the first half of the statement is also a corollary of Theorem 2.5 since every closed 3-manifold has a (in fact unique up to diffeomorphism) differential structure (see e.g. [M]), which implies that every closed 3-manifold is a closed combinatorial manifold. Notice also that for 3-manifolds the graph may not be planar anymore, and may contain double-edges.

Proof. Recall the idea of Heegaard splitting (see e.g. [H]): suppose $H^\pm$ are two genus-$g$ handle-bodies (here we require $g \geq 2$) surfaces, and $f : F^- \to F^+$ is a homeomorphism, then one can glue $H^-$ to $H^+$ along their boundaries using $f$ and get a closed 3-manifold $M = H^+ \cup_f H^-$. Every closed 3-manifold $M$ can be obtained in this way and the corresponding $(H^+, H^-, f)$ is called a Heegaard splitting for $M$.

![Heegaard Splitting for $S^3$](image)

Each genus-$g$ handle-body $H$ can be further decomposed into two balls $B_\pm (H)$ in a canonical way along $(g + 1)$ disjoint non-separating discs $D_0 (H), \ldots, D_g (H)$. Let us decompose $H^+$ and $H^-$ in this way. For notational convenience, let $D^\pm_i = D_i (H^\pm)$

$$B_0 = B_+(H^+), B_1 = B_-(H^+), B_2 = B_+(H^-), B_3 = B_-(H^-),$$

and let $a^\pm_i = \partial (D^\pm_i) \subseteq \partial H^\pm$ (see Fig. 4).

Notice that the topological type of $M = H^+ \cup_f H^-$ remains unchanged when changing $f$ isotopically, while these $f(a^-_j)$ will be moved around on $F^+$ when doing so. Clearly one can isotopically adjust $f$ a little bit, such that each $f(a^-_j)$ intersects these $a^+_i$ transversely in a finite number of points. Moreover, one can isotopically adjust $f$ a little bit further, adding some redundant turns to those $f(a^-_j)$, such that for any $i, j \in \{0, 1, \ldots, g\}$,

$$a^+_i \cap (f(a^+_0) \cup \cdots \cup f(a^+_g)) \neq \emptyset, \ (a^+_0 \cup \cdots \cup a^+_j) \cap f(a^-_j) \neq \emptyset,$$

and all these $a^+_i \cap f(a^-_j)$ together cut $F^+$ into small disc regions.

Now the points in $(a^+_0 \cup \cdots \cup a^+_g) \cap (f(a^-_0) \cup \cdots \cup f(a^-_g))$ cut these $a^+_i$ and $f(a^-_j)$ into edges, and these edges together form a finite connected regular 4-valent graph $\Gamma \subseteq F^+$. Suppose $\{t_0, t_1, t_2, t_3\}$ is a basis for $G$, and $\{x_0, x_1, x_2, x_3\}$ is the dual basis in $\text{Hom}(G, \mathbb{Z}_2)$, namely $x_i (t_j) = \delta_{ij}$. Define a characteristic function $\lambda : B_i \mapsto t_i$ and then define the edge coloring $\alpha : E_\Gamma \to \text{Hom}(G, \mathbb{Z}_2)$ as follows: $\alpha (e) (\lambda (B_i)) = 0$ if and only if $e \subseteq \partial B_i$. In fact, each $e \in E_\Gamma$ lies within the boundary of exactly three of $B_0, B_1, B_2, B_3$. If the
fourth ball is $B_i$, then $\alpha(e) = x_i$. It is easy to see that the set of colors of the four edges around each vertex of $\Gamma$ is exactly $\{x_0, x_1, x_2, x_3\}$, so $\alpha$ is a “pure” coloring.

Let us calculate the 3-skeletal expansion of $(\Gamma, \alpha)$. Since $\alpha$ is a “pure” coloring, the set of all 2-nests of $(\Gamma, \alpha)$ consists of exactly all the bi-colored circles of $\Gamma$. By the construction of $\Gamma$ as described above, it is easy to see that each bi-colored circle of $\Gamma$ is either the boundary of a disc region of

\[ F^+ \setminus (a_0^+ \cup \cdots \cup a_g^+ \cup f(a_0^-) \cup \cdots \cup f(a_g^-)) \]

or one of those $a_i^+$ (bounds $D_i^+$ in $H^+$) or $f(a_i^-)$ (bounds $D_i^-$ in $H^-$). Let those $D_i^\pm$ and disc regions be the 2-cells. Then one may construct a 2-skeletal expansion $(M_2, K_2)$ of $(\Gamma, \alpha)$, where $M_2$ is obtained by gluing each $D_i^-$ to $F^+ \cup D_0^+ \cup \cdots \cup D_g^+$ along the boundary, such that $a_i^-$ is identified with $f(a_i^-)$. Since an edge $e \subseteq \partial B_i$ if and only if $\alpha(e) \neq x_i$, all edges in each $\partial B_i$ form a connected regular 3-valent subgraph $\Delta_i \subseteq \Gamma$ with $\dim \Delta_i = 3$, and $\Delta_0, \Delta_1, \Delta_2, \Delta_3$ are all the 3-nests. Hence by adding $B_0, B_1, B_2, B_3$ as the 3-cells, one obtains a 3-skeletal expansion $(M_3, K_3)$. Clearly $M_3 = \bigcup_{\sigma \in K_3} \sigma$ is just the original closed 3-manifold $M$. □

As an example, Fig. 5 (a) shows the colored graph corresponding to the Heegaard splitting of $S^3$ in Fig. 4. For a more fancy picture of this graph see Fig. 5 (b).

![Figure 5. Colored graph for $S^3$](image)

Another example is $S^2 \times S^1$. Fig. 6 shows a nice Heegaard splitting, such that every small region on $F^+$ is a disc (the picture for $H^-$ is omitted), and Fig. 7 is the corresponding colored graph.

Remark. Similarly to the case of simple convex polytopes, the colorings here are also induced by characteristic functions on the $n$-dimensional faces. However, the cell decompositions here are not dual to any triangulation, so this is different from the case of simple convex polytopes or the graphs constructed in the proof of Theorem 2.5. In fact a triangulation for a closed 3-manifold has at least two 3-simplices, so it has at least five vertices, which implies that its dual regular cell decomposition has at least five 3-cells.
5. HIGH DIMENSIONAL CASES

Proof of Theorem 2.4. Because of the combinatorial nature of \((M, K)\), for any cell \(\sigma \in K\) with dimension \(\geq 1\) and any two points \(u, v\) inside \(\sigma\) (away from \(\partial \sigma\)), \(u\) has a neighborhood homeomorphic to \(\mathbb{R}^n\) if and only if \(v\) does so. Therefore we only need to verify that every point in \(M\) sufficiently close to a vertex has a neighborhood homeomorphic to \(\mathbb{R}^n\). Now choose a vertex \(v\), we will show that \(v\) has a neighborhood homeomorphic to the space
\[
C_n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid t_0 \cdots t_n = 0, 0 \leq t_0 < 1, \ldots, 0 \leq t_n < 1\},
\]
which is homeomorphic to \(\mathbb{R}^n\).

The argument is quite similar to the case \(n = 3\) in section 3. Choose a local coordinate such that \(v\) becomes \((0, \ldots, 0)\) and the adjacent edges become (locally) \(e_0 \sim \cdots \sim \cdots \sim e_n\).
\{(t,0,\ldots,0) \mid 0 < t < 1\}, e_1 \rightsquigarrow \{(0,t,0,\ldots,0) \mid 0 < t < 1\}, \ldots, e_n \rightsquigarrow \{(0,\ldots,0,t) \mid 0 < t < 1\}\) respectively. Suppose \(\alpha(e_j) = x_i\).

For each \(i_1 < \cdots < i_k \in \{0,\ldots,n\}\), there is a unique \(k\)-nest \(\Delta_{i_1 \cdots i_k}\) containing \(e_{i_1}, \ldots, e_{i_k}\), and by Lemma 2.1 these \(\Delta_{i_1 \cdots i_k}\) enumerate all \(k\)-nests containing \(v\). Suppose \(\sigma_{i_1 \cdots i_k}\) is the \(k\)-cell corresponding to \(\Delta_{i_1 \cdots i_k}\). By an induction on \(k\) one can adjust the local coordinates such that each of these \(\sigma_{i_1 \cdots i_k}\) becomes (locally)

\[\{(t_0,\ldots,t_n) \mid 0 < t_{i_1} < 1, \ldots, 0 < t_{i_k} < 1, t_j = 0 \text{ for all other } j\}\]

The possibility of this arrangement implies that every vertex \(v\) has a neighborhood in \(M\) homeomorphic to the cone \(C_n\), which is homeomorphic to \(\mathbb{R}^n\). Thus the combinatorial nature of \((M,K)\) implies that every point in \(M\) has a neighborhood homeomorphic to \(\mathbb{R}^n\). \(M\) is connected since \(\Gamma\) is connected, and \(M\) is compact because it has only finitely many cells. Therefore \(M\) is a closed \(n\)-manifold. \(\square\)

Before proving Theorem 2.5, let us recall some notation. If a cell \(\sigma\) is a face of cell \(\sigma'\) and \(\sigma \neq \sigma'\), we write \(\sigma \prec \sigma'\). Now suppose that \(K\) is a finite regular cell decomposition of \(M\), then the barycentric subdivision \(S_d K\) is a simplicial complex, such that every \(k\)-simplex in \(S_d K\) corresponds to a sequence of cells \(\sigma_0 \prec \cdots \prec \sigma_k \in K\), we denote such a simplex by \([\sigma_0 \cdots \sigma_k]\), and call it a flag. Notice that such a simplicial complex always exists and is a triangulation of \(M\). When \(K\) itself is a triangulation of \(M\), \([\sigma_0 \cdots \sigma_k]\) is just the convex hull spanned by \(\tilde{\sigma}_0, \ldots, \tilde{\sigma}_k\), where \(\tilde{\sigma}\) denotes the barycenter of \(\sigma\).

For each cell \(\sigma \in K\), the link complex is

\[\text{Lk } \sigma = \{[\sigma_0 \cdots \sigma_k] \in S_d K \mid \sigma \prec \sigma_0 \prec \cdots \prec \sigma_k\}\].

If \(M\) is connected and has a finite regular cell decomposition \(K\), and

\[\forall \sigma \in K, \ |\text{Lk } \sigma| \cong S^{n-\dim \sigma-1}\]

(|\text{Lk } \sigma| is the union of all cells in \text{Lk } \sigma), then \(M\) is called an \(n\)-dimensional closed combinatorial manifold. Notice that if \(K\) satisfies this condition, then \(S_d K\) also satisfies this condition.

Finally for each cell \(\sigma \in K\), the dual block is

\[\mathcal{D} \sigma = \{[\sigma_0 \cdots \sigma_k] \in S_d K \mid \sigma = \sigma_0 \prec \cdots \prec \sigma_k\}\].

If \(K\) satisfies the previous condition, then these dual blocks are all cells (in fact \(\overline{\mathcal{D} \sigma}\) is the topological cone on \text{Lk } \sigma), and they also form a finite regular cell decomposition of \(M\), we denote it by \(\overline{\mathcal{D} K}\). In particular, the 1-skeleton of this dual cell decomposition is a regular graph. It is clear that \(\mathcal{D} : K \to \overline{\mathcal{D} K}\) is a one-to-one correspondence reversing face relations, and \(\dim \mathcal{D} \sigma = n - \dim \sigma\).

Proof of Theorem 2.5. Suppose that \(K'\) is a finite regular cell decomposition of \(M\) satisfying the condition

\[\forall \sigma \in K', \ |\text{Lk } \sigma| \cong S^{n-\dim \sigma-1}\].

Let \(K'' = S_d K'\) be its barycentric subdivision. Then \(K''\) also satisfies the above condition, thus one can define a dual regular cell decomposition \(K = \mathcal{D} K''\). Now let us define a coloring \(\alpha\) for the 1-skeleton of \(K\) (denoted by \(\Gamma\)) as follows.
Let $x_0, \ldots, x_n$ be a linear basis for $\text{Hom}(G, \mathbb{Z}_2)$. For each 1-cell $\sigma = \mathcal{D}_K[\sigma_0 \cdots \sigma_{n-1}] \in \Gamma$ (here $\sigma_0 \prec \cdots \prec \sigma_{n-1} \in K'$), suppose that

$$\{0, \ldots, n\} \setminus \{\dim \sigma_0, \ldots, \dim \sigma_{n-1}\} = \{k\},$$

then let $\alpha(\sigma) = x_k$. Because $\mathcal{D}_K$ reverses face relations, for each vertex $v \in K$, say $v = \mathcal{D}_K[\sigma_0 \cdots \sigma_n]$ (here $\sigma_0 \prec \cdots \prec \sigma_n \in K'$), there are exactly $(n + 1)$ edges in $K$ adjacent to $v$, namely

$$e_k = \mathcal{D}_K[\sigma_0 \cdots \sigma_{k-1}\sigma_k+1 \cdots \sigma_n], \ k = 0, \ldots, n.$$ 

Since $\alpha(e_k) = x_k$, this implies that $(\Gamma, \alpha)$ is a finite regular $(n + 1)$-valent graph with “pure” $G$-coloring (see Fig 8).

![Diagram](image_url)

(a) the initial $K'$

(b) dual of $\text{Sd} K'$

**Figure 8.** The dual graph of barycentric subdivision

Now for any $m$-cell $\Delta = \mathcal{D}_K[\sigma_{j_0} \cdots \sigma_{j_{m-1}}]$ (here $\sigma_{j_0} \prec \cdots \prec \sigma_{j_{m-1}} \in K'$ and $\dim \sigma_{j_k} = j_k$), let $\kappa(\Delta)$ be the 1-skeleton of $\partial \Delta$. Then a vertex $v' = \mathcal{D}_K[\sigma'_0 \cdots \sigma'_n]$ (here $\sigma'_0 \prec \cdots \prec \sigma'_n \in K'$) is a vertex of $\kappa(\Delta)$ if and only if

$$\sigma'_j = \sigma_{j_0}, \ldots, \sigma'_{j_{m-1}} = \sigma_{j_{m-1}},$$

and an edge $e' = \mathcal{D}_K[\sigma'_0 \cdots \sigma'_{k-1}\sigma'_{k+1} \cdots \sigma'_n]$ adjacent to $v'$ is an edge of $\kappa(\Delta)$ if and only if $k \not\in \{j_0, \ldots, j_{m-1}\}$, namely $\alpha(e') \in \{x_{i_1}, \ldots, x_{i_m}\}$

$$(i_1 < \cdots < i_m \in \{0, \ldots, n\} \setminus \{j_0, \ldots, j_{m-1}\}).$$

This implies that $\kappa(\Delta)$ is a connected regular $m$-valent subgraph of $\Gamma$ with edge-colors in $\{x_{i_1}, \ldots, x_{i_m}\}$, hence $\kappa(\Delta)$ is an $m$-nest in $K(\Gamma, \alpha)$.

Therefore the correspondence $\kappa : K \to K(\Gamma, \alpha)$, sending each cell in $K$ to the 1-skeleton of its boundary complex, is a well defined map which preserves dimensions and face relations. $\kappa$ is also a one-to-one correspondence. In fact, for any $m$-nest $\Delta$, suppose $v = \mathcal{D}_K[\sigma_0 \cdots \sigma_n]$ is a vertex in $\Delta$ and $e_{i_1}, \ldots, e_{i_m}$ are the $m$ edges in $\Delta$ adjacent to $v$, with each $e_{i_k} = \mathcal{D}_K[\sigma_0 \cdots \sigma_{i_k-1}\sigma_{i_k+1} \cdots \sigma_n]$. Then any inverse image of $\Delta$ under $\kappa$ must contain $v$ and these edges, and there does exists a unique $m$-cell containing $e_{i_1}, \ldots, e_{i_m}$, namely $\mathcal{D}_K[\sigma_{j_0} \cdots \sigma_{j_{m-1}}]$ (here $j_0 < \cdots < j_{m-1} \in \{0, \ldots, n\} \setminus \{i_1, \ldots, i_m\}$).

Clearly
\( \mathcal{D}_K[\sigma_{j_0} \cdots \sigma_{j_{n-m}}] \) is the one and only cell that is sent to \( \Delta \) by \( \kappa \). Thus \( \kappa \) must be a one-to-one correspondence. \( \square \)

As an example of calculation, let us reproduce the “cubic” graph for \( S^n \) as a \( \mathcal{D}(S^d K') \).

Take a regular cell decomposition \( K' \) for \( S^n \) with two \( i \)-cells at each dimension \( i \in \{0, 1, \ldots, n\} \):
\[
\sigma_i^{(1)} = \{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} | t_0^2 + \cdots + t_i^2 = 1, t_i > 0, t_{i+1} = \cdots = t_n = 0 \};
\sigma_i^{(2)} = \{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} | t_0^2 + \cdots + t_i^2 = 1, t_i < 0, t_{i+1} = \cdots = t_n = 0 \}.
\]

Notice that \( \sigma_i^{(e_i)} \prec \sigma_j^{(e_j)} \) if and only if \( i < j \). Therefore \( K = \mathcal{D}(S^d K') \) contains \( 2^{n+1} \) vertices, each of which has the form
\[
\mathcal{D}[\sigma_0^{(e_0)} \cdots \sigma_n^{(e_n)}], \ e_0, \ldots, e_n \in \{1, 2\}.
\]

To give the graph a more fancy appearance, identify each such vertex with
\[
v(\epsilon) = ((-1)^{e_0}, \ldots, (-1)^{e_n}) \in \mathbb{R}^{n+1}.
\]

Two vertices \( v(\epsilon) \) and \( v(\tau) \) are connected by an edge in \( K \) of the form
\[
\mathcal{D}[\sigma_0^{(e_0)} \cdots \sigma_i^{(e_i-1)} \sigma_{i+1}^{(e_{i+1})} \cdots \sigma_n^{(e_n)}]
\]
(colored by \( x_i \)) if and only if \( \tau_i \neq e_i \) while for all other \( j, \tau_j = e_j \). This implies that we can also identify each such edge with the line segment
\[
\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} | -1 \leq t_i \leq 1, \forall j \neq i, t_j = (-1)^{e_j} \}.
\]

Hence one obtains the 1-skeleton of the \( (n+1) \)-dimensional cube
\[
C = [-1, 1]^{n+1} \subset \mathbb{R}^{n+1},
\]
and an edge of this graph is colored by \( x_i \) if and only if it is parallel to the \( t_i \)-axis. When \( n = 2 \), this equals to the graph for \( S^2 \) in Fig. 2.

Remark. There is in fact a more general way of doing skeletal expansions, which we call the “generalized skeletal expansion”. For example, take the colored graph in Fig. 1 (a), one can also glue Möbius bands instead of discs to each of the bi-colored circles, and then the resulting manifold will be a closed non-orientable surface with genus 4.

To describe this expansion more accurately, we will use an induction on the dimension. The generalized 1-skeletal expansion of a regular graph \( \Gamma \) with “good” coloring \( \alpha \) is just \( \Gamma \) itself. For \( k > 1 \), a generalized \( k \)-skeletal expansion of \( (\Gamma, \alpha) \) is a space \( M^k \) together with a defining sequence \( \mathcal{E}^k \) of gluing operations such that \( \Gamma \) can be extended to \( M^k \) by \( \mathcal{E}^k \) and each \( \ell \)-nest with \( \ell \leq k+1 \) can be extended to a closed \( (\ell - 1) \)-dimensional manifold. If \( (\Gamma, \alpha) \) admits a generalized \( k \)-skeletal expansion \( (M^k, \mathcal{E}^k) \), then \( \mathcal{E}^k \) also induces a generalized \( k \)-skeletal expansion for each \( (k+1) \)-nest of \( (\Gamma, \alpha) \). By \( (F^k, \mathcal{E}^k) \) we denote the disjoint union of the generalized \( k \)-skeletal expansions of all \( (k+1) \)-nests in \( \Gamma \). Suppose that \( (\Gamma, \alpha) \) admits a generalized \( k \)-skeletal expansion \( (M^k, \mathcal{E}^k) \) such that \( F^k \) is the boundary of a \( (k+1) \)-dimensional compact manifold \( M^{k+1} \) (may not be connected).
Then one can glue $\tilde{M}^{k+1}$ onto $M^k$ to form a new space $M^{k+1}$. Add this operation to $E^k$ to form a new sequence of operations $E^{k+1}$, and we call $(M^{k+1}, E^{k+1})$ a generalized $(k + 1)$-skeletal expansion of $(\Gamma, \alpha)$. Clearly, in each dimension, ordinary skeletal expansions (if they exist) are uniquely determined by the colored graphs up to cell isomorphisms, while the generalized skeletal expansions are not so.

As an example, let us study the colored regular 4-valent graph $\Gamma$ shown in Fig. 9. It admits a 2-skeletal expansion but admits no 3-skeletal expansion, since by Theorem 4.1
\[
\#\{3\text{-nests}\} - \#\{2\text{-nests}\} + \#\{\text{vertices}\} = 5 - 12 + 8 \neq 0.
\]
In fact $\Gamma$ has five 3-nests. When doing (ordinary) 2-skeletal expansion on their disjoint union, two of them (both colored by $x_1x_2x_3$) yield $P^2$, while the other three (colored by $x_0x_1x_2$, $x_0x_1x_3$ and $x_0x_2x_3$ respectively) yield $S^2$. However, one can still glue three 3-cells to those $S^2$, glue a $P^2 \times I$ to the two $P^2$, and finally glue the boundary surfaces of all these blocks together according to the chosen (ordinary) 2-skeletal expansion. The resulting space of this generalized 3-skeletal expansion is homeomorphic to $P^2 \times S^1$. This graph $\Gamma$ even has a generalized 4-skeletal expansion, since $P^2 \times S^1$ is the boundary of the compact 4-manifold $P^2 \times B^2$.

![Figure 9. Graph admitting generalized 3-skeletal expansion](image)

For the generalized skeletal expansion, we would like to propose the following conjecture:

**Conjecture.** Suppose that $\Gamma$ is a regular $(n + 1)$-valent graph with a “good” $(\mathbb{Z}_2)^{n+1}$-coloring $\alpha$. Then $(\Gamma, \alpha)$ always admits a generalized $(n + 1)$-skeletal expansion $(M, E)$ such that $M$ is a compact $(n + 1)$-manifold with boundary.

6. **Reconstruction of manifolds with $(\mathbb{Z}_2)^{n+1}$-actions**

Finally, let us consider the realization problem of reconstructing $(n + 1)$-dimensional $G$-manifolds from $(n + 1)$-valent $G$-colored graphs, where $G = (\mathbb{Z}_2)^{n+1}$. 
In the previous section, we showed that under certain conditions, a $G$-colored graph $(\Gamma, \alpha)$ admits a skeletal expansion into a closed manifold $M$. If $M$ can be used as the boundary of a simple convex polytope $P$, then $\alpha$ determines a characteristic function $\lambda$ on $P$. Furthermore, by the reconstruction technique of small covers in [D.J], one can use $\lambda$ and the product bundle $P \times G$ to construct a small cover $X$ such that the moment graph of $X$ is exactly the given $G$-colored graph $(\Gamma, \alpha)$. This observation provided much insight into the study of the realization problem.

Now let us prove Theorem 2.6.

Proof of Theorem 2.6. Let $\Gamma$ be a finite connected regular $(n + 1)$-valent graph with a “good” $G$-coloring $\alpha$. Suppose that $(\Gamma, \alpha)$ admits an $n$-skeletal expansion $(M, K)$. Then by Theorem 2.4, $M$ is an $n$-dimensional closed manifold.

First, let us prove the special case in which $M$ is the boundary of a compact $(n + 1)$-dimensional manifold $X_0$. Suppose that $\Delta \subset \Gamma$ is a $k$-nest with $k \leq n$. We recall that the color of an edge $e$ is a nontrivial element $\alpha(e)$ in $\text{Hom}(G, \mathbb{Z}_2)$. Let $E_\Delta$ denote the set of all edges in $\Delta$. Then $\mathfrak{g}_\Delta = \cap_{e \in E_\Delta} \ker \alpha(e)$ is a co-rank $k$ subgroup of $G$. Notice that $\mathfrak{g}_\Delta = G$ if $\Delta$ is a 0-nest (i.e., a single vertex). On the other hand, the $(k - 1)$-skeletal expansion of $\Delta$ bounds a ball, and it corresponds to a $k$-dimensional open cell $C_{\Delta}$ in $M$. Moreover, for each point $x \in M$, there is a unique nest $\Delta$ such that $x \in C_{\Delta}$.

Now suppose that $X_0$ admits a trivial action of $G$. Consider the trivial principal $G$-bundle $\xi = X_0 \times G$ over $X_0$, where the bundle projection is denoted by $\pi : \xi \rightarrow X_0$, $\pi(x, h) = x$, and the action of $G$ on $\xi$ is defined by $\phi : G \rightarrow \text{End}(\xi)$ as follows: $\phi(g)(x, h) = (x, gh)$. Then we define an equivalence relation $\sim$ on $\xi$ as follows: $p \sim p'$ if and only if

1. $\pi(p) = \pi(p') \in X_0 \setminus M$; or
2. there exists a nest $\Delta$ and some $g \in \mathfrak{g}_\Delta$ such that $\pi(p) = \pi(p') \in C_{\Delta}$ and $p = \phi(g)p'$.

By $[p]$ we denote the equivalence class of $p \in \xi$. Set $X = \xi/\sim$.

Clearly $X$ is obtained by gluing $2^{n+1}$ copies of $X_0$ together along their boundaries. To prove that $M$ is indeed an $(n + 1)$-dimensional manifold, it suffices to show that each point $[p]$ with $\pi(p) \in M$ has a neighborhood homeomorphic to $\mathbb{R}^{n+1}$. Let $[p] \in X$ with $\pi(p) \in M$. Then there is a $k$-nest $\Delta$ such that $\pi(p) \in C_{\Delta}$. Since all points inside a component of $\pi^{-1}(C_{\Delta})$ can have homeomorphic neighborhoods, we need only to prove that $[p]$ has a neighborhood homeomorphic to $\mathbb{R}^{n+1}$ when $\pi(p)$ is very close to a vertex.

Without loss of generality, one assumes that $v = \pi(p)$ is a vertex of $\Gamma$. Then there are $n + 1$ $n$-nests containing $v$, denoted by $\Delta_0, \ldots, \Delta_n$ respectively. Choose a local coordinate chart $(U, \varphi)$ near $v$ in $X_0$ such that $v$ is mapped into the origin of $\mathbb{R}^{n+1}$, each $C_{\Delta_j} \cap U$ is mapped into an $n$-dimensional open cone

$$\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | x_j = 0, x_i > 0 \text{ when } i \neq j\},$$

and $U$ is mapped into an $(n + 1)$-dimensional cone

$$\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | x_i \geq 0, i = 0, \ldots, n\}.$$
Each $g_{\Delta_j}$ is a subgroup of rank 1, so we can write $g_{\Delta_j} = \{\text{id}, g_j\}$. Obviously, $\{g_0, \ldots, g_n\}$ forms a basis of $G$. Then one can define a linear action $\rho : G \rightarrow \text{End}(\mathbb{R}^{n+1})$ by

$$\rho(g_j)(x_0, \ldots, x_j, \ldots, x_n) = (x_0, \ldots, -x_j, \ldots, x_n).$$

Using this action, one can extend $\varphi$ into a continuous surjection $\widetilde{\varphi} : \pi^{-1}(U) \rightarrow \mathbb{R}^{n+1}$ defined by

$$\widetilde{\varphi}(x, g) = \rho(g)(\varphi(x)).$$

It is not difficult to see that $\widetilde{\varphi}(q) = \widetilde{\varphi}(q')$ if and only if $q \sim q'$. Therefore, $\widetilde{\varphi}$ induces a homeomorphism from a neighborhood of $[p]$ to $\mathbb{R}^{n+1}$.

The action $\phi$ of $G$ on $\xi$ also induces a natural action $\widetilde{\phi}$ on $X$. If $\Delta$ is a $k$-nest, let $\Phi_{\Delta} = \pi^{-1}(C_{\Delta})/\sim$, then $G$ acts on $\Phi_{\Delta}$ with kernel $\ker \widetilde{\phi} = g_{\Delta}$. In fact, $\Phi_{\Delta}$ contains $2^k$ copies of $C_{\Delta}$ and the action of $G$ on $\Phi_{\Delta}$ permutes these copies. Also, the fixed point set of $g_{\Delta}$ acting on $X$ contains $\cup_{\Delta \subseteq \Delta'} \Phi_{\Delta'}$ as a component. In particular, for an edge $e$ with two ends $p, q$, the fixed point set of $g_e = \ker \alpha(e)$ acting on $X$ contains a circle which is the union of $\Phi_e$ (two arcs) and $\Phi_p, \Phi_q$ (two fixed points). This also implies that the moment graph of the $G$-manifold $X$ is just $(\Gamma, \alpha)$.

Now let us consider the general case in which $M$ is not the boundary of a compact $(n+1)$-dimensional manifold. Choose an $n$-nest $\Delta$, by the construction of $M$, then $\Delta$ corresponds to an open $n$-cell $C_{\Delta}$ of $(M, K)$ and the $n$-skeletal expansion of $\Delta$ is exactly the closure $\overline{C}_{\Delta}$ of $C_{\Delta}$, which is an $n$-disc. Now let us remove a very small open $n$-ball $B$ in the interior of $\overline{C}_{\Delta}$. Next, taking another copy $M'$ of $M$ and forgetting those combinatorial structures $(\Gamma, \alpha)$ and $K$ on it, one randomly removes a small open $n$-ball $B'$ in $M'$. Then one glues $M\backslash B$ and $M'\backslash B'$ together along their boundaries. This is actually a connected sum between two copies of $M$, so the resulting manifold $N$ becomes the boundary of an $(n+1)$-dimensional compact manifold $X_0$. In this operation, we see that the $n$-skeletal expansion $\overline{C}_{\Delta}$ of $\Delta$ is exactly replaced by the connected sum between $\overline{C}_{\Delta}$ and $M'$. Thus, $N$ becomes actually a generalized $n$-skeletal expansion of $(\Gamma, \alpha)$ such that $\Delta$ corresponds to the interior of $M'\backslash B'$ rather than $C_{\Delta}$. It is not difficult to see that the above argument still works for this $X_0$. This completes the proof.

Remark. It should be pointed out that if we start from any principal $G$-bundle over $X_0$ (not just the trivial one), then this argument still works, i.e., we can always obtain a $G$-manifold (c.f. [J], [D], [D1], and [LM]).

In the case where $n = 2$, Theorem 2.6 gives a complete answer to question (Q4). This is because $(\Gamma, \alpha)$ in this case always admits a 2-skeletal expansion, and the resulting surface bounds a 3-manifold if and only if its Euler characteristic is even.

Corollary 6.1. For every finite connected regular 3-valent graph $\Gamma$ with a “good” $G$-coloring $\alpha$ where $G = (\mathbb{Z}_2)^3$, there is a 3-dimensional $G$-manifold $X$ such that $(\Gamma, \alpha)$ corresponds to its moment graph.

The case where $n = 3$ is also very interesting because it is well-known that every closed 3-manifold is the boundary of a 4-manifold.
Corollary 6.2. For every finite connected regular 4-valent graph $\Gamma$ with a “good” $G$-coloring $\alpha$ where $G = (\mathbb{Z}_2)^4$, if $\#\{3\text{-nests}\} = \#\{2\text{-nests}\} - \#\{\text{vertices}\}$, then there is a 4-dimensional $G$-manifold $X$ such that $(\Gamma, \alpha)$ corresponds to its moment graph.

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