Suppression of $1/f$ noise in quantum simulators of gauge theories

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In the current drive to quantum-simulate evermore complex gauge-theory phenomena, it is necessary to devise schemes allowing for the control and suppression of unavoidable gauge-breaking errors on different experimental platforms. Although there have been several successful approaches to tackle coherent errors, comparatively little has been done in the way of decoherence. By numerically solving the corresponding Bloch–Redfield equations, we show that the recently developed method of linear gauge protection suppresses the growth of gauge violations due to $1/f^\beta$ noise as $1/V^\beta$, where $V$ is the protection strength and $\beta > 0$, in Abelian lattice gauge theories, as we show through exemplary results for $U(1)$ quantum link models and $Z_2$ lattice gauge theories. We support our numerical findings with analytic derivations through time-dependent perturbation theory. Our findings are of immediate applicability in modern analog quantum simulators and digital NISQ devices.

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I. INTRODUCTION

Quantum simulators are quantum systems implementable in the laboratory onto which quantum many-body models of interest can be mapped and studied [1–4]. Due to its promise as a probe of phenomena relevant for high-energy and nuclear physics on easily accessible table-top quantum devices, and its potential to calculate time evolution from first principles, the quantum simulation of lattice gauge theories [5] has come at the forefront of research in several fields ranging from condensed matter to subatomic physics [6–13]. Thanks to the advent of high-control and precision synthetic quantum devices, recent years have seen various groundbreaking quantum-simulation experiments of gauge theories [14–27].

Of particular interest in this endeavor are gauge theories with both dynamical matter and gauge fields. The characteristic property of gauge theories is their gauge symmetry [28–30], which imposes local constraints that enforce specific configurations of matter and electric fields, such as Gauss’s law from quantum electrodynamics. A major issue in quantum simulations is stabilizing gauge symmetry against gauge-breaking terms that will unavoidably arise due to higher orders in the perturbative mapping or due to experimental imperfections [31]. These terms allow for processes driving the system dynamics out of the physical gauge sector of Gauss’s law, in which it should stay in an ideal scenario where such terms are not present. Even when perturbative in strength, gauge-breaking terms can be quite detrimental to gauge-theory quantum simulations, leading to gauge-noninvariant dynamics that cannot be directly related to the target model [32–34].

Various methods have been proposed to suppress coherent gauge-breaking errors [31, 35–54], but there has been little work done on suppressing incoherent errors due to decoherence, which can be quite adverse to the stability of gauge-theory implementations [34, 55]. Indeed, decoherence [56, 57] poses a major roadblock to achieving long evolution times in quantum simulations.
of quantum many-body models in general, whose key properties of quantum entanglement and superposition are particularly sensitive to interactions with the environment. Prominent examples of the detrimental effects of decoherence on quantum many-body systems include $1/f$ noise in superconducting quantum interference devices (SQUIDs) that undermines superconducting qubits [58–63]. Given that superconducting qubits, as well as other platforms, have been of great recent interest in the quantum simulation of gauge theories [26, 27], suppressing $1/f$ noise sources using efficient and experimentally feasible schemes becomes of central importance.

In this work, using exact diagonalization calculations and time-dependent perturbation theory, we demonstrate how the principle of linear gauge protection, initially devised to control coherent gauge-breaking errors [51], can be employed to suppress the growth of the gauge violations due to incoherent errors with spectral form $1/f^\beta$ ($\beta > 0$) as $1/V^\beta$, where $V$ is the protection strength. The rest of this paper is organized as follows: We briefly review the concept of linear gauge protection in Sec. II, and $1/f^\beta$ noise and the corresponding Bloch–Redfield formalism in Sec. III. We present our main numerical results in Sec. IV. We finally conclude and provide an outlook in Sec. V. We include Appendix A for a derivation of the Bloch–Redfield equation employed for our analysis, Appendix B for our derivations in time-dependent perturbation theory, in addition to Appendix C where we provide supplemental numerical results.

II. LINEAR GAUGE PROTECTION

Let us consider an Abelian gauge theory described by the Hamiltonian $\hat{H}_0$, and whose gauge symmetry is generated by the operator $\hat{G}_j$, where $j$ denotes a site on a lattice of length $L$. The gauge invariance of $\hat{H}_0$ is encoded in the commutation relations $[\hat{H}_0, \hat{G}_j] = 0, \forall j$. The set of gauge-invariant states $\{|\psi\rangle\}$ is defined as the simultaneous eigenstates of the generators: $\hat{G}_j |\psi\rangle = g_j |\psi\rangle, \forall j$. A set of these eigenvalues $\mathbf{g} = (g_1, g_2, \ldots, g_L)$ over the volume of the system defines a unique gauge superselection sector, the projector onto which is $\hat{P}_\mathbf{g}$. One can further define a target or physical gauge superselection sector $\mathbf{g}^\mathrm{tar} = (g_1^\mathrm{tar}, g_2^\mathrm{tar}, \ldots, g_L^\mathrm{tar})$ in which one wishes to restrict the dynamics in an experiment, for example.

In experimental implementations of gauge theories, $\hat{H}_0$ is mapped onto the microscopic degrees of freedom of a quantum simulator. In general, unavoidable gauge symmetry-breaking errors $\lambda \hat{H}_1$ at strength $\lambda$ will arise in this process either due to higher orders in the perturbation theory used to perform the mapping, or in experimental imperfections in equipment. Even when perturbative, these errors can generate gauge violations that grow as $\lambda^2 t^2$ over evolution time $t$, which in turn lead to a complete departure from faithful gauge-theory dynamics beyond timescales $t \propto 1/\lambda$ [31].

In order to suppress these errors in a controlled way, the concept of linear gauge protection was introduced in Ref. [51]. It entails adding the protection term

$$V \hat{H}_G = V \sum_j c_j \hat{G}_j,$$

where $V$ is the protection strength. The sequence $c_j$ can be chosen to be rational and satisfying the condition $\sum_j c_j (g_j - g_j^\mathrm{tar}) = 0 \iff g_j = g_j^\mathrm{tar}, \forall j$. In this case, the sequence is said to be compliant, and, for a volume-independent and sufficiently large $V$, the gauge violation is controlled up to times exponential in $V$ [51, 64]. Although $V$ is volume-independent, the sequence $c_j$ would have to grow (not faster than) exponentially with system size in order to satisfy the compliance condition. This renders the compliant sequence somewhat inconvenient for large-scale gauge-theory quantum simulators such as those realized in recent cold-atom setups [23, 24].

However, reality turns out to be more forgiving, and even simple noncompliant sequences such as $c_j = (-1)^j$ can give excellent protection in the target sector against gauge errors up to all accessible evolution times in both finite systems [51] and the thermodynamic limit [65]. This can be explained through the coherent quantum Zeno effect [66–69], which guarantees that upon adding the protection term (1) an effective Zeno Hamiltonian $\hat{H}_Z = \hat{H}_0 + \lambda \hat{P}_\mathbf{g}^\mathrm{tar} \hat{H}_1 \hat{P}_\mathbf{g}^\mathrm{tar}$ emerges that faithfully reproduces the dynamics of the faulty gauge theory $\hat{H}_0 + \lambda \hat{H}_1 + V \hat{H}_G$ up to timescales linear in $V$ in a worst-case scenario [51].

For certain gauge theories, the full local generator $\hat{G}_j$ may be too challenging to realize in an experiment [19], in which case the linear gauge protection as given in Eq. (1) becomes impractical. Nevertheless, a powerful workaround exists based on local pseudogenerators $\hat{W}_j$, which are identical to the full local generators $\hat{G}_j$ in the target sector, but not necessarily outside of it [52]. Formally, they satisfy the relation

$$\hat{W}_j |\phi\rangle = g_j^\mathrm{tar} |\phi\rangle \iff \hat{G}_j |\phi\rangle = g_j^\mathrm{tar} |\phi\rangle .$$

One can then extend the principle of linear gauge protection to one in terms of the local pseudogenerator, with protection term

$$V \hat{H}_W = V \sum_j c_j \hat{W}_j, $$

where the same rules apply for the sequence $c_j$ as in the case of Eq. (1). Note that even though $\hat{H}_0$ commutes with $\hat{G}_j$, it generally does not commute with $\hat{W}_j$, with the latter associated with a local symmetry richer than that generated by $\hat{G}_j$ [70]. The resulting Zeno Hamiltonian when protecting with Eq. (3) is $\hat{H}_Z = \hat{P}_\mathbf{g}^\mathrm{tar} (\hat{H}_0 + \lambda \hat{H}_1) \hat{P}_\mathbf{g}^\mathrm{tar}$, under which the dynamics of the faulty gauge theory $\hat{H}_0 + \lambda \hat{H}_1 + V \hat{H}_W$ can be faithfully reproduced up to times at least linear in $V$ [52].

In terms of purely unitary errors, extensive numerical simulations in exact diagonalization (ED) and inf-
nite matrix product states (iMPS) based on the time-dependent variational principle [71–73] have shown that for a compliant or properly chosen noncompliant sequence, linear gauge protection in the full local generator or the local pseudogenerator leads to stabilized gauge-theory dynamics up to all accessible evolution times with the gauge violation settling at a timescale $\propto 1/V$ into a plateau of value $\propto \lambda^2/V^2$ [51, 52, 65, 74]. Importantly, the linear gauge protection terms (1) and (3) are composed of single and two-body terms at most, and they are local, which renders them experimentally highly feasible.

It is a relevant open question whether linear gauge protection can be employed to protect against incoherent errors due to noise in an experiment. When left unchecked, these errors lead to gauge violations growing $\propto \gamma t$, where $\gamma$ is the strength of the incoherent errors. Even just slowing down the growth of gauge violations due to them would be greatly desirable in near-term quantum simulators.

III. 1/f NOISE AND THE BLOCH–REDFIELD MASTER EQUATION

We focus here on 1/f noise, a decohering process with a noise power spectrum

$$S(\omega) = \frac{\gamma}{|\omega|^{\beta}},$$

where $\gamma$ is the system–environment coupling strength, $\omega$ is the frequency, and $0 < \beta < 2$. This type of noise is ubiquitous in nature, especially in condensed matter systems in quasi-equilibrium (for $\beta \approx 1$) and electronic equipment, but this signal can also be found in biological systems, music, and even in economics [75, 76]. In particular, as mentioned above, it is present in SQUIDs, which can lead to adverse effects on quantum simulation platforms based on superconducting qubits [58–63].

Since we a priori know the noise power spectrum of the environment, we employ the Bloch–Redfield formalism [77, 78] to derive a master equation from a microscopic perspective. We consider a system $\hat{H}_S$ coupled to a bath (the environment) $\hat{H}_B$ with the interaction Hamiltonian $\hat{H}_{SB} = \sqrt{\gamma} \sum_\alpha \hat{A}_\alpha \otimes \hat{B}_\alpha$, where $\hat{A}_\alpha$ and $\hat{B}_\alpha$ are system and bath operators, respectively, with system-environment coupling strength $\gamma$. In general, the system operators $\hat{A}_\alpha$ do not preserve Gauss’s law. Under the assumption of weak system-environment coupling, we obtain a master equation in terms of system operators and correlation functions that characterize the statistical properties of the bath.

To obtain the master equation in terms of a noise power spectrum that can be numerically implemented, we write the bath correlation function $C_{\alpha\nu}(\tau) = \gamma \mathcal{Tr}_B [\hat{B}_\alpha(t)\hat{B}_\nu(t-\tau)\rho_B]$—here, we denote tilde on quantities written in the interaction picture—in terms of the spectral function $S_{\alpha\nu}(\omega)$, after neglecting a small energy shift arising due to the imaginary part in the Fourier transform of $C_{\alpha\nu}(\tau)$:

$$S_{\alpha\nu}(\omega) = 2 \int_0^\infty d\tau e^{i\omega\tau} C_{\alpha\nu}(\tau).$$

Hence, one can show that the final form of the Bloch–Redfield master equation, describing the evolution of the reduced density matrix for the system, after employing the Born, Markov, and the secular approximation as detailed in Appendix A can be written explicitly as,

$$d_t\rho_{ab}(t) = -i\omega_{ab}\rho_{ab}(t) + \sum_{c,d} R_{abcd}\rho_{cd}(t),$$

where $R_{abcd}$ is the Bloch–Redfield relaxation tensor, which can be written in matrix form with $\hat{A}_\alpha$ assumed to be Hermitian for ease of numerical implementation,

$$R_{abcd} = -\frac{1}{2} \sum_\alpha \left[ \delta_{bd} \sum_n A_{an}^\alpha A_{cn}^\alpha S_\alpha(\omega_{en}) - A_{ac}^\alpha A_{db}^\alpha S_\alpha(\omega_{ca}) + \delta_{ac} \sum_n A_{dn}^\alpha A_{nb}^\alpha S_\alpha(\omega_{dn}) - A_{ac}^\alpha A_{db}^\alpha S_\alpha(\omega_{db}) \right].$$

The Redfield tensor contains all the information about the dissipative processes that arise due to the coupling of the system with the bath degrees of freedom.

One requirement for the validity of the Bloch–Redfield approach is the smallness of the Bloch–Redfield decay rates that describe the effective incoherent coupling between two eigenlevels $i$ and $f$ against the relevant transition frequencies $\omega_{if}$ [79]. The Bloch–Redfield decay rates, also known as the golden rule rates, are defined as $\Gamma_{if} \propto \sum_\alpha \frac{\delta_{\alpha i} \delta_{\alpha f}}{\omega_{\alpha f}} S_\alpha(\omega_{\alpha f})$. We checked for the numerical models we describe throughout our paper that the condition $\Gamma_{if} \ll \omega_{if}$ was always satisfied. In particular, as the system operators $\hat{A}_\alpha$ violate Gauss’s law, the relevant incoherent transitions happen on large energy scales of order $V$, where the noise spectrum becomes weak, thus further solidifying our approach for employing this formalism.

As 1/f noise and other types of decoherence can drastically undermine performance in an experimental setup, it becomes important to find ways that may ameliorate its effect. Left unchecked, decoherence can lead to a fast buildup in the gauge violation, which renders the quantum simulation of true gauge-theory dynamics unfaithful [34, 55].

IV. RESULTS AND DISCUSSION

We now present our numerical results on the quench dynamics of gauge theories subjected to 1/f noise, which we have computed using the exact diagonalization toolkit.
where on site $j$ the matter field is represented by the Pauli operator $\sigma_j^\mu$, with $\mu$ denoting the fermionic mass, the gauge (electric) field on the link between sites $j$ and $j+1$ is denoted by the spin-1/2 operator $\sigma_j^z$, $L$ is the total number of sites with periodic boundary conditions enforced, and the overall energy scale is set by the coupling strength $J = 1$. The generator of the U(1) gauge symmetry of Hamiltonian (9) is given by

$$\hat{G}_j = (-1)^{j} \left( \frac{\sigma_j^z s_{j-1,j} + s_{j,j+1}^z + \sigma_j^z + 1}{2} \right). \quad (10)$$

The model (9) is a quantum link formulation \cite{82} of lattice quantum electrodynamics in 1 + 1D, and is experimentally very relevant as it has been the subject of recent large-scale cold-atom quantum simulations \cite{23, 24}.

We now prepare the system in a vacuum state, which is one of two doubly degenerate eigenstates of Hamiltonian (9) at $\mu/J \to \infty$. This initial state is in the target sector $g_{\text{tar}} = 0$, $\forall j$, i.e., $\text{Tr}\{\hat{\rho}_0 \hat{G}_j\} = 0$, $\forall j$, where its sites host no matter and the local electric fields are in a staggered formation. We then quench this vacuum state with $H_0 + V H G$ at $\mu/J = 0.5$ in the presence of $1/f$ noise with power spectrum (4) and jump operators $\hat{A}_j^m = \sigma_j^m$ and $\hat{A}_j^{s} = \sigma_j^z$, which couple the matter and gauge fields to the environment, respectively. Let us first consider the case without protection, i.e., $V = 0$, shown in Fig. 1 setting $\beta = 1$. The dynamics of the gauge violation (8) is shown for various values of the system-environment coupling strength $\gamma$ in Fig. 1(a). At early times, the violation grows $\propto \gamma t$, as can be shown in time-dependent perturbation theory, until it begins to settle into a maximal violation plateau at a timescale $\propto 1/\gamma$. We observe similar behavior in the chiral condensate, a measure of how strongly the dynamics spontaneously breaks the chiral symmetry associated with fermions in the theory,

$$C(t) = \frac{1}{2} + \frac{1}{2L} \sum_{j=1}^{L} \{ \hat{\rho}(t) \sigma_j^2 \}, \quad (11)$$

shown in Fig. 1(b). The error with respect to the ideal case, shown in the inset, grows $\propto \gamma t$ before settling into a maximal value at late times for sufficiently large $\gamma$. These results demonstrate the pernicious effect of $1/f$ noise on quantum simulations of gauge theories when left unprotected.

We now repeat the same quench protocol as in Fig. 1, but with fixed $\gamma = 0.1 J$ and the addition of the gauge protection (1) at strength $V$, with $c_j = \{-115, 116, -118, 122\}/122$ chosen to be a compliant sequence. The corresponding dynamics of the gauge violation is shown in Fig. 2(a), where we see a robust suppression in the growth of the gauge violation such that $\varepsilon(t) \propto \gamma t/V$ at short times, in agreement with time-dependent perturbation theory; see Appendix B. This suppression is also seen in the dynamics of the chiral condensate, shown in Fig. 2(b). Indeed, whereas the unprotected case (red curve) quickly and significantly diverges from the ideal case (green curve), at sufficiently

\[ \varepsilon(t) = \frac{1}{L} \sum_{j=1}^{L} \text{Tr} \left\{ \hat{\rho}(t) \left( \hat{G}_j - g_{\text{tar}} \right)^2 \right\}, \quad (8) \]

where $\hat{\rho}(t)$ is the time-evolved density operator of the system at time $t$, in addition to calculating the dynamics of relevant local observables. Due to the large evolution times we investigate, we restrict our system size to $L = 4$ sites due to computational overhead, and we employ periodic boundary conditions.

\section{U(1) quantum link model}

We first consider the U(1) quantum link model \cite{40, 82-84}

$$\hat{H}_0 = \sum_{j=1}^{L} \left[ J \left( \sigma_j^z \sigma_{j+1}^{z} + \text{H.c.} \right) + \mu \sigma_j^x \right], \quad (9)$$

QuTiP \cite{80, 81}. In all cases, we prepare our system in an initial state $\hat{\rho}_0$ in the target gauge sector $g_{\text{tar}}$, and monitor its quench dynamics in the presence of $1/f$ noise with and without linear gauge protection. In particular, we will focus on the dynamics of the gauge violation,

$$\varepsilon(t) = \frac{1}{L} \sum_{j=1}^{L} \text{Tr} \left\{ \hat{\rho}(t) \left( \hat{G}_j - g_{\text{tar}} \right)^2 \right\}, \quad (8)$$

\[ \hat{\rho}(t) \approx 1 + \frac{1}{2L} \sum_{j=1}^{L} \{ \hat{\rho}(t) \sigma_j^2 \}, \quad (11) \]
large $V$ the agreement with the ideal case is excellent. The inset shows the deviation from the ideal case for the various considered values of $V$, where we find that the error grows roughly $\propto \gamma t/V$. These results show, therefore, that linear gauge protection extends the timescale of the dynamics during which one can perturbatively connect to a gauge theory from $\propto 1/\gamma$ to $\propto V/(J\gamma)$. Even though linear gauge protection does not suppress the gauge violation into a long-lived plateau of constant value as it does in the case of purely coherent errors [51], this is nevertheless a positive result that can allow one to significantly enhance the achievable coherent evolution times, and which can thus be of significant benefit to current and near-term gauge-theory quantum simulators.

Let us now investigate the case of a fractional coefficient $\beta$ in the spectrum $S(\omega) = \gamma/|\omega|^\beta$. For this purpose, we repeat the above quench protocols for $\beta = 1.7$. The protection-free case is shown in Fig. 3. The result is qualitatively similar to that of $\beta = 1$ in Fig. 1. Indeed, the gauge violation grows $\propto \gamma t$ until a timescale $\propto V/(J\gamma)$, where it begins to settle into a maximal-violation plateau, as can be seen for large enough values of $\gamma$; see Fig. 3(a). This type of behavior is replicated in the chiral condensate, as depicted in Fig. 3(b), where the deviation from the ideal case grows $\propto \gamma t$ at short times before beginning to plateau at $t \propto 1/\gamma$. We can thus conclude that the effect of $\beta$ is merely quantitative in the case of no protection.

Upon employing gauge protection, the qualitative picture changes significantly. The gauge violation grows $\propto \gamma t/V^{1.7}$, as shown in Fig. 4(a) at fixed $\gamma = 0.1J$. In other words, the suppression in the growth of the gauge violation directly depends on $\beta$, with greater suppression at larger $\beta$. This also happens in the case of the chiral condensate, shown in Fig. 4(b). We find that even though the unprotected case vastly deviates from the ideal one ($\gamma = V = 0$), upon adding linear gauge protection, the chiral condensate faithfully reproduces the ideal case up to all accessible evolution times at sufficiently large $V$, with the deviation from the ideal case $\propto \gamma t/V^{1.7}$ (see inset).

This behavior can be explained in the following way. The spectral function of the considered decoherence process is $S(\omega) = \gamma/|\omega|^\beta$, where the relevant frequencies $\omega$ governing the system dynamics are those that create transitions between the target gauge sector and the other gauge sectors. Upon switching on the linear gauge protection, the undesired sectors are energetically separated from the target gauge sector proportionally to $V$. Hence, the relevant transition frequencies are on the order $\omega \sim V$. The strength of the spectral function thus scales as $S(\omega) \sim \gamma/V^\beta$ and becomes weaker as $V$ increases.

It is worth noting that we have also checked that our conclusions hold for different jump operators, quench parameters (different values of $\mu/J$), and initial states, as
well as for noncompliant sequences. See Appendix C for supplemental numerical results.

B. $Z_2$ lattice gauge theory

To check the generality of the above findings, we now turn our attention to a different model, namely a $Z_2$ lattice gauge theory that has been of recent theoretical [85–90] and experimental relevance [19, 20]. Its Hamiltonian reads

$$\hat{H}_0 = J \sum_{j=1}^{L} (\hat{a}_{j}^{\dagger} \hat{a}_{j+1} + H.c.) - \hbar \sum_{j=1}^{L} \hat{\pi}^{x}_{j,j+1},$$

(12)

where the bosonic ladder operators $\hat{a}_j, \hat{a}_j^{\dagger}$ on site $j$ represent the annihilation and creation of matter, respectively. The electric (gauge) field on the link between sites $j$ and $j + 1$ is represented by the Pauli operator $\hat{\pi}^{x}_{j,j+1}$ ($\hat{\pi}^{x}_{j,j+1}$), where the electric field strength is given by $\hbar$. The overall energy scale is set by $J = 1$. The generator of the $Z_2$ gauge symmetry of Hamiltonian (12) is given by

$$\hat{G}_j = (-1)^{\hat{a}^{\dagger}_j \hat{a}_j} \hat{\pi}^{x}_{j-1,j} \hat{\pi}^{x}_{j,j+1},$$

(13)

and its eigenvalues are $\pm 1$, where, due to the $Z_2$ gauge symmetry, $\hat{G}_j^2 = \hat{1}_j$. Unlike the generator (10) of the U(1) quantum link model (9), which is composed of one-body terms, the generator (13) of the $Z_2$ lattice gauge theory (12) is a three-body term that mixes matter and gauge degrees of freedom. This renders it significantly impractical in experimental implementations. As described in Sec. II, one can then utilize the concept of the local pseudogenerator [52], where in this case it takes the form

$$\hat{W}_j = \hat{\pi}^{x}_{j-1,j} \hat{\pi}^{x}_{j,j+1} + 2\hbar g_{\text{tar}}^{a} \hat{a}^{\dagger}_j \hat{a}_j.$$  

(14)

Note that even though $[\hat{H}_0, \hat{G}_j] = 0$, $\forall j$, on account of the $Z_2$ gauge symmetry of Hamiltonian (12), $[\hat{H}_0, \hat{W}_j] \neq 0$. However, when working in the target sector $g_{\text{tar}}$, then $\hat{W}_j$ and $\hat{G}_j$ are indistinguishable. It is interesting to note that the local symmetry associated with $\hat{W}_j$ contains the $Z_2$ gauge symmetry generated by $\hat{G}_j$. In fact, one can prove for a given Hamiltonian $\hat{H}$ that $[\hat{H}, \hat{W}_j] = 0 \Rightarrow [\hat{H}', \hat{G}_j] = 0$.

We can now employ the concept of linear protection in terms of the local pseudogenerator according to Eq. (3) in order to protect against $1/f$ noise in the $Z_2$ lattice gauge theory. We prepare our system in a charge-density wave state in terms of the matter fields, with the electric fields aligned such that the system resides in the target state $g_{\text{tar}}^{a} = +1, \forall j$. We quench this state with Hamiltonian (12) at $h = 0.54 J$ in the presence of $1/f$ noise with the spectral function (4) and jump operators $A^m_j = \hat{a}_j + \hat{a}^{\dagger}_j$ and $A^g_{j,j+1} = \hat{\pi}^{x}_{j,j+1}$, coupling the matter and gauge fields, respectively, to the environment at a fixed value of $\gamma = 0.1 J$ and for several values of the protection strength $V$. The corresponding dynamics of the gauge violation is shown in Fig. 5(a,b) for $\beta = 1$ and 1.7, respectively. The qualitative picture is identical to that

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**FIG. 4.** (Color online). Same as Fig. 2, but where $\beta = 1.7$ in the noise spectral function $S(\omega) = \gamma |\omega|^{\beta}$. A qualitative difference arises whereby the gauge violation and the deviation of the chiral condensate from the ideal case both grow $\propto \gamma t/V^{1.7}$ instead of $\propto \gamma t/V$, showing that linear gauge protection suppresses errors more for a larger value of $\beta$.

**FIG. 5.** (Color online). Same as Figs. 2(a) and 4(a), but for the $Z_2$ lattice gauge theory (12) and with the linear gauge protection term (3) in the local pseudo generator, Eq. (14). The results are shown for the noncompliant sequence $\{(\pm 6)^3 + 5\}/11$ with quantum jump operators operators $A^m_j = \hat{a}_j + \hat{a}^{\dagger}_j$ and $A^g_{j,j+1} = \hat{\pi}^{x}_{j,j+1}$. The qualitative conclusions are identical to the corresponding cases of the U(1) quantum link model where errors evolve $\propto \gamma t/V^{\beta}$, showcasing the generality of our findings.
of the U(1) quantum link model, where we find that at sufficiently large $V$ the gauge violation evolves $\propto \gamma t/V^3$ at short to intermediate times, before eventually plateauing at a maximal value that is delayed from a timescale $\propto 1/\gamma$ in the unprotected case to a timescale $\propto V^3/\gamma$ under linear gauge protection.

We have also checked that these findings are valid for different initial states, quench parameters, and properly chosen sequences $c_j$. As such, our conclusions are not specific to a given model, and we expect our findings to be general and applicable to any Abelian gauge theory.

V. CONCLUSION AND OUTLOOK

We have demonstrated numerically how linear gauge protection schemes based on the local full generator or on the local pseudogenerator can suppress the growth of gauge violations due to $1/f$ noise with power spectrum $S(\omega) = \gamma/|\omega|^3$ as $\varepsilon(t) \propto \gamma t/V^3$ in gauge-theory quantum simulations, where $V$ is the protection strength. This extends coherent lifetimes by $V^3$ in experiments where $1/f$ noise is the dominant source of decoherence. As examples, we have used two paradigmatic Abelian systems: the U(1) quantum link model and the $Z_2$ lattice gauge theory. We have shown numerically, and argued analytically through time-dependent perturbation theory, that whereas without protection the gauge violation and errors in local observables evolve $\propto \gamma t$ in the presence of $1/f$ noise, under linear gauge protection this dynamics changes to $\propto \gamma t/V^3$.

Linear gauge protection may also help in suppressing $1/f$ noise sources in recent cold-atom experiments, where long coherent evolution times have been demonstrated [24]. This is due to the fact that the perturbative mapping of the U(1) quantum link model onto the Bose–Hubbard quantum simulator of Refs. [23, 24] gives rise to a leading order term that can be rearranged into a term equivalent to Eq. (1), with a site-dependent sequence $c_j$ [91].

Our findings offer the promising prospect of engineering experimentally feasible gauge protection terms that can suppress the growth of gauge violations due to $1/f$-like noise sources, and we expect our conclusions to hold in higher spatial dimensions, as well as for other generic Abelian gauge theories. An interesting avenue lies in studying how gauge violations can be further suppressed by making the time-independent sequence $c_j$ time-dependent. Indeed, for the limit of uncorrelated white noise sources it has been shown that leakage out of the target subspace can be delayed [41].

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Appendix A: Further details on the derivation of the Bloch–Redfield master equation

In this Appendix, we supply additional details for deriving Eq. (6). Going into the interaction picture with respect to $\hat{H}_S + \hat{H}_B$ via the operators $\hat{U}_S = e^{-it\hat{H}_S}$ and $\hat{U}_B = e^{-it\hat{H}_B}$, we start by writing the von-Neumann equation

$$d_t \hat{\rho}_{SB}(t) = -i \left[ \hat{H}_{SB}(t), \hat{\rho}_{SB}(t) \right].$$

(A1)

After substituting the integrated solution into the equation of motion for the combined system, we can obtain the evolution of the reduced density matrix of the system in the interaction picture as

$$d_t \hat{\rho}(t) = - \text{Tr}_B \left\{ \left[ \hat{H}(t), \int_0^t ds \left[ \hat{H}(s), \hat{\rho}_{SB}(s) \right] \right] \right\},$$

(A2)

where $\hat{H}(t) = \sqrt{\gamma} \sum_{n} \hat{A}_n(t) \otimes \hat{B}_n(t)$. After the change of variables $\tau = t - s$, we get

$$d_t \hat{\rho}(t) = - \text{Tr}_B \left\{ \left[ \hat{H}(t), \int_0^t ds \left[ \hat{H}(t-\tau), \hat{\rho}_{SB}(t-\tau) \right] \right] \right\}.$$  

(A3)

We further proceed to use a Born approximation where we assume the state of the composite system is always uncorrelated and hence can be factorized as $\hat{\rho}_{SB} = \hat{\rho}(t) \otimes \hat{\rho}_B$, also assuming the bath is much larger than the system in question. Further, we introduce the Markov approximation, where we assume that the bath has a very short correlation time $\tau_B$, i.e., that the correlation function $C_{\alpha\nu}(\tau) = \gamma \text{Tr}_B \left[ \hat{B}_\alpha(t) \hat{B}_\nu(t-\tau) \hat{\rho}_B \right] = \gamma \left\langle \hat{B}_\alpha(\tau) \hat{B}_\nu(0) \right\rangle$ decays rapidly with some characteristic timescale $|C_{\alpha\nu}(\tau)| \sim e^{-\tau/\tau_B}$. In the limit of $\tau_B \to 0$ and replacing $\hat{\rho}(t-\tau)$ with $\hat{\rho}(t)$, which is possible due to the fact that correlation function is negligible for $\tau \gg \tau_B$, and under the assumption that $t \gg \tau_B$, one obtains a memory-less evolution of the density matrix. It then becomes also a good approximation to extend the integration to infinity as the integrand vanishes sufficiently fast.
for $\tau \gg \tau_B$, making it a fully Markovian equation. These approximations ensure the trace-preserving nature of the density matrix throughout the time evolution. However, the master equation that is obtained is still often times known to give rise to evolution which is not completely positive. Therefore, a secular approximation which is also known as the rotating wave approximation is then used to make the evolution of the resulting dynamical map completely positive (CPTP) [92–94]. Writing Eq. (A3) in terms of system operators and bath correlation functions, one obtains after evaluating the partial trace

$$d_t \hat{\rho}(t) = -\sum_{\alpha \nu} \int_0^\infty dt \left\{ C_{\alpha \nu}(\tau) \left[ \hat{A}_\alpha(t) \hat{A}_\nu(t - \tau) \hat{\rho}(t) - \hat{A}_\nu(t - \tau) \hat{A}_\alpha(t - \tau) \hat{\rho}(t) - \hat{A}_\alpha(t) \hat{A}_\nu(t - \tau) \hat{\rho}(t) \hat{A}_\nu(t - \tau) \right] \right\},$$

(A4)

Going into the frequency domain and expanding in the eigenbasis of the system Hamiltonian $\hat{H}_S$, we can write the operators acting on the system as

$$\hat{A}_\alpha(t) = \sum_{m,n} e^{-i(\epsilon_m - \epsilon_n)t} |\epsilon_n \rangle \langle \epsilon_m| A_{mn}(\omega) e^{-i\omega_{mn} t},$$

(A5)

where we have defined the transition frequencies $\omega_{mn} = \epsilon_m - \epsilon_n$. In the Schrödinger picture, we obtain the master equation in matrix form after substituting Eq. (A5) into Eq. (A4) as

$$d_t \rho_{ab}(t) = -i \omega_{ab} \rho_{ab}(t) - \sum_{\alpha \nu} \sum_{c,d} \int_0^\infty d\tau \left\{ C_{\alpha \nu}(\tau) \left[ \delta_{bd} \times \sum_n A_{\alpha n}^a A_{\nu n}^b e^{i\omega_{na} \tau} - A_{\alpha n}^a A_{\nu n}^b e^{i\omega_{ba} \tau} \right] - C_{\alpha \nu}(\tau) \sum_n A_{\alpha n}^a A_{\nu n}^b e^{i\omega_{na} \tau} - \rho_{cd}(t) \right\}.$$

(A6)

Further substituting the expression for spectral function of Eq. (5) in the above equation under the assumptions of vanishing cross correlations between different environment operators acting at different particle sites, i.e., $C_{\alpha \alpha}(\tau) = C_{\alpha \nu}(\tau) = \delta_{\alpha \nu} C_{\nu}(\tau)$ we obtain Eq. (6) of the main text.

**Appendix B: Perturbation theory**

We can explain the initial growth of gauge violation under $1/f$ noise in our numerical results by perturbatively expanding the Bloch–Redfield master equation. It can be shown that Eq. (A3) can be written in the familiar Lindblad form [95], after employing the secular approximation and transforming back to the Schrödinger picture, as

$$d_t \hat{\rho} = -i \left[ \hat{H}_0 + V \hat{H}_G, \hat{\rho} \right] + \sum_{\omega} \sum_j S_j(\omega) \times \left[ \hat{A}_j(\omega) \hat{\rho} \hat{A}_j^\dagger(\omega) - \frac{1}{2} \{ \hat{A}_j(\omega) \hat{A}_j^\dagger(\omega), \hat{\rho} \} \right].$$

(B1)

One can write the above in the concise form

$$d_t \hat{\rho} = (S + D) \hat{\rho},$$

(B2)

where

$$S[\hat{\rho}] = -i \left[ \hat{H}_0 + V \hat{H}_G, \hat{\rho} \right],$$

(B3a)

$$D[\hat{\rho}] = \sum_{\omega} \sum_j S_j(\omega) \left[ \hat{A}_j(\omega) \hat{\rho} \hat{A}_j^\dagger(\omega) - \frac{1}{2} \{ \hat{A}_j(\omega) \hat{A}_j^\dagger(\omega), \hat{\rho} \} \right].$$

(B3b)

By Taylor expanding the solution to Eq. (B2), we can find the leading order incoherent term to explain the growth of the gauge violation in the regimes $V = 0$ and $V \gg J$. Choosing a target sector $g_{\text{tar}}^{\text{act}}$, the gauge violation is $\varepsilon(t) = \text{Tr} \{ \hat{G} \hat{\rho}(t) \}$ where we have introduced the abbreviation $\hat{G} = \sum_j (\hat{G}_j - g_{\text{tar}}^{\text{act}})^2 / L$. The contribution of the first-order term in the absence of gauge protection...
condensate, the suppression of errors also evolves $\propto \gamma t/V^3$. As seen in the quench dynamics of the (a) gauge violation and (b) the chiral condensate, the suppression of errors also evolves $\propto \gamma t/V^3$.

Once the gauge protection is switched on, in the limit $V \gg J$ the dominating coherent term is $\hat{H}_G = \sum_m V c_{m+1}^* c_m + \text{h.c.}$, where $c_m = \mathbf{g}^\dagger S_m$, where $\mathbf{g}$ is a gauge sector.

The relevant transition frequencies thus scale as $\omega_{mn} \sim V$. Taking this into account, neglecting corrections proportional to the energy scales of $\hat{H}_0$, and using the definition of the spectral function in Eq. (B4), we obtain $\epsilon(t) \sim \gamma t/V^3$, hence explaining the corresponding scaling in the results of the main text up to first order. Similar results can also apply to other contexts. E.g., in applications of error correction in adiabatic quantum computing, increasing the energy gap to the excited states can suppress the transition rate out of the code space if the noise power spectrum is decreasing with frequency [96].

Appendix C: Supplemental numerical results

The linear gauge protection scheme does not depend on the initial state, and will work effectively so long as the initial state is in the correct gauge sector(s) to be protected. We demonstrate this by repeating the results of Fig. 4 but for a charge-proliferated state, which has every site occupied with matter, and all its local electric fields pointing down. The corresponding dynamics of the gauge violation and chiral condensate are shown in Fig. 6(a,b), respectively, and the qualitative behavior is identical to that of the vacuum initial state in Fig. 4, with an error $\propto \gamma t/V^3$ in both cases.

Due to numerical overhead, we are limited in our ED calculations to small system sizes. However, in modern cold-atom quantum simulators, much larger sizes can be attained [23, 24]. This makes it difficult to construct a compliant sequence for such state-of-the-art quantum simulators, as the coefficients of the latter grow roughly exponentially with system size. However, we can use a simpler noncompliant sequence such as $c_j = (-1)^j$. We repeat the results of Fig. 4 using such a sequence, where the corresponding dynamics is shown in Fig. 7. We see that both the gauge violation and the chiral condensate show qualitatively identical behavior to the case of the compliant sequence of Fig. 7, with an error $\propto \gamma t/V^3$ in both cases.

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