Abstract

An action principle technique is used to examine the infra-red problem in the effective action for gauge field theories. It is shown by means of a dynamical analogy that the renormalization group and the ansatz of non-local sources can be simultaneously understood through generalized variations of an action supplemented by sources in the manner of the Schwinger action principle.

It is shown that indiscriminate resummations of the effective potential can lead to erroneous conclusions about phase transitions in a gauge theory if they correspond to a partial resummation of matter self-energies at the expense of the gauge sector. The action principle method illuminates the reason for this and shows a way of proceeding, without having to go to non-zero momentum. Some examples are computed to lowest order to compare to results previously obtained by renormalization group analysis as well as to prepare for future work. The utility of the present formalism is a greater insight into symmetries of the effective action which can be exploited to regulate the theory perturbatively in the infrared. The dynamical structure suggests a connection with Chern-Simons theory in describing the infrared limit of gauge theories.

Finally the dynamical analogy leads to an obvious comparison with a class of phenomenological non-equilibrium lagrangians in which time-dependent couplings are used to model the influence of an external system. These models and some of their implications are discussed in terms of the action principle.

1 Introduction

Understanding of the massless limit of field theories is essential for the description of many physical phenomena. Progress has been made in recent years through a variety of techniques for resumming the perturbation series and regulating infrared divergences. Of the many techniques available for dealing with
non-perturbative behaviour in massless theories, two main schools of thought have emerged: that of ring or daisy summations and that of the renormalization group. The former represents a method for systematically including a class of contributions from all orders of conventional perturbation theory while the latter concerns the redefinition of variables in the theory so as to most effectively reorganize the conventional perturbation series.

Attention has focussed particularly on a class of theories which undergo second order (continuous) phase transitions. In such systems, information about phase structure and the order of transitions can be obtained directly by use of field theoretical renormalization group arguments. Their phases and phase transitions correspond to scale-invariant points in the parameter space. The success of this program is partly due to its preoccupation with the simplest scalar field theories; in particular, it seems that in systems exhibiting a fluctuation driven first order phase transitions there are no such perturbatively accessible fixed points on which to infer the phase structure. An important reason for this is the very fact that, while the continuous transition is essentially classical (weak coupling), the discontinuous (first order) transition is an essentially quantum (strong coupling) phenomenon and there is no particular reason why it should be amenable to normal perturbative methods.

A few authors have pursued a parallel line of development, generalizing the notion of the effective action to include certain non-local sources [17, 18]. These sources capture some of the non-linear, non-local structure of the effective action. Here the preoccupation has been with the bi-local corrections which sum classes of daisy or superdaisy diagrams in the perturbation expansion, thus the method falls into the first of the two classes discussed above. The two philosophies appear conceptually different but can be related if one understands the effect of a renormalization on the perturbation series.

In the present paper, it is shown that both approaches can be considered as arising from an application of the Schwinger action principle [1, 3] written in such a way that renormalization group transformations appear as canonical transformations. The result is a pseudo-dynamical problem which can be straightforwardly examined and compared to the above alternatives. The principal advantage of this viewpoint is that a generalized variation of the action leads to a combined canonical and renormalization group transformation, providing a renormalization group improved effective action more directly. Furthermore, the action principle is a unifying object which reveals the structure of the renormalization problem in a way which is easily adapted to problems in curved spacetime [4, 5], finite temperature and non-equilibrium systems [22]. It also makes a connection with Padé approximants which in certain regimes can lead to exact results [6, 7].
The renormalization group as an infrared regulator

The issue of ‘resummation’ and renormalization can be illustrated simply when vertex conditions are used to implement the renormalization procedure. The vertex method has two formally independent motivations. First of all, the phenomenological coupling constants (including the mass) are generally regarded as being given by the curvatures of the full effective potential, including all radiative corrections, rather than by the parameters which appear in the unquantized Lagrangian; secondly, it is advantageous in the context of perturbation theory to organize the expansion parameters in such a way that the convergence of the perturbation series (be it asymptotic or otherwise) be optimal. The latter remark can be understood quite easily from the following observation about the Taylor expansion. Consider a function \( f(S) \). Around a point \( s \) one may expand the function

\[
f(S) = f(s + \Delta s) = f(s) + \frac{df}{ds} \bigg|_s \Delta s + \frac{1}{2!} \frac{d^2 f}{ds^2} \bigg|_s (\Delta s)^2 + \ldots \tag{1}
\]

where \( \Delta s \) is small in some appropriate sense. The expansion is clearly not unique, since adding a small counterterm \( \delta s \)

\[
S = (s + \delta s) + (\Delta s - \delta s) \tag{2}
\]

one has

\[
f(S) = f(s + \delta s) + \frac{df}{ds} \bigg|_{s+\delta s} (\Delta s - \delta s) + \frac{1}{2!} \frac{d^2 f}{ds^2} \bigg|_{s+\delta s} (\Delta s - \delta s)^2 + \ldots \tag{3}
\]

This series is somewhat different. Its convergence properties might be better or worse than those of (1), depending on the magnitude of \( \Delta s - \delta s \). The relationship between the two series may be found by expanding the new coefficients and expansion parameters in powers of \( \delta s \); it is then seen that the two are identical provided one works to a consistent order in \( \delta s \) and \( \Delta s \). The function of \( \delta s \) is to reorganize the size of terms in the series. In field theory one uses this property to advantage by defining the renormalized coupling constants as the coefficients in the Taylor expansion of the quantum effective potential in powers of \( \overline{\phi} \).

\[
V_{\text{eff}}(\overline{\phi}) = \Lambda + \sigma \overline{\phi} + \frac{1}{2} m^2 \overline{\phi}^2 + \frac{1}{3} \eta \overline{\phi}^3 + \frac{1}{4!} \lambda \overline{\phi}^4 \ldots \tag{4}
\]

After a redefinition of the coefficients it is often said that one has ‘resummed’ the perturbation series since certain contributions have been formally absorbed from ‘unknown’ higher orders into ‘known’ parameters, by analogy with the above Taylor series. In practice only the leading contributions to higher orders are accessible in field theory. Moreover, the situation is complicated by the fact that the Taylor expansion of the effective potential may not exist around \( \overline{\phi} = 0 \).

\[1\] The quotes refer to the fact that this phrase is only intended heuristically, since no explicit sum is ever performed.
An example is $\lambda \phi^4$ theory in $3 + 1$-dimensions, where the coefficient is proportional to $\ln \bar{\phi}^2$ which diverges in the limit. In spite of this, one can use the same argument as before to regularize the Taylor expansion by rewriting it about a different point and working to a consistent order in some appropriate expansion parameter. The invariance of the final result under reparameterizations is expressed by a ‘renormalization group’ equation

$$\frac{d}{d(\delta s)} f(S) = 0,$$

or the appropriate generalization for $n$ counterterms. The freedom to define the expansion points of the Taylor coefficients leads to a method of application for the renormalization group. To use the property to maximal advantage one notices that every term in the Taylor series can formally be expanded around a different point, provided they all lie within a radius of the smallness parameter of each other.

$$f(S) = \sum_{n=0}^{m} f_n(\delta s_n) \Delta s_n(\delta s_n) + O(\delta s^m)$$

where $\delta s_n \sim \delta s (\forall n)$. In that case, the difference between the regularized expansion and the unregularized expansion is formally of higher order than the consistent order in $\delta s$ to that which one works. Although this viewpoint is not obviously conventional, it will become apparent in forthcoming sections that such a regularization scheme corresponds both to an application of the usual field theoretical renormalization group and to a variation the $n$-point functions of a theory independently of one another. In appendices A and B, it is shown explicitly how the method can be used to regulate infra-red divergences in $\lambda \phi^4$ theory and in scalar electrodynamics without the need to calculate beta functions.

One of the motivations for such a procedure is to optimize, in some sense, the asymptotic perturbation series sufficiently to determine the type of phase transition a given theory would undergo from a perturbative calculation. For the two examples, it is known that $\lambda \phi^4$ theory has a second order (continuous) transition, while scalar electrodynamics has a first order transition. However, the latter result does not follow simply from the type of renormalized calculation discussed above. Indeed it can readily be used to prove that the transition is of second order. This is disturbing, particularly since essentially the same type of resummation is regularly used in the literature in the guise of a diversity of schemes. The reason for the false conclusion is that the scalar mass renormalization leads to an effective regularization of only the scalar sector; the scalar sector dominates and the gauge sector is neglected. Regrettably there is no immediately analogous way of generalizing this method to define finite regulating counterterms for the gauge sector, owing both to the gauge invariant structure of the results and the general absence of a background photon field. One approach which has been used is to define the theory at non-zero photon momentum. One then finds a renormalization group flow towards an apparent fixed point at negative infinity $[13, 14]$.

In view of the preceding argument, it is natural to ask whether one might construct a dynamical principle for locating the optimal perturbation series.
Can the problem be solved by finding the minimum of some effective energy functional? The purpose of the following section is to show that such a principle can be obtained in a limited sense by analogy with the Schwinger action principle. The extent to which such an approach is useful is naturally limited by the extent to which it can be calculated in practice, though in the present paper the method also yields interesting formal comparisons. In the case of gauge theories it illustrates that the full BRST symmetry might be utilized to improve perturbative calculations.

In what follows it should be borne in mind that even an optimal improvement of the conventional perturbation series might not be enough to obtain the desired information. Some results may be intrinsically non-perturbative. Nevertheless, once an appropriate improvement procedure has been identified, approximations can be made without prejudicing a special sector of the field space and the advantage of obtaining a formal expression is that it should be possible to see when the approximation is valid. It proves convenient to proceed from the Schwinger action principle.

3 Sources, the action principle and renormalization group flow

Schwinger’s action principle is a differential statement of changes in the transition amplitudes of a theory under appropriate variations of the action. The principle states that, given a complete set of states characterized by $|q^i\rangle$, the variation of the amplitude or transformation function is given by

$$\delta \langle q^j(t')|q^i(t)\rangle = i\langle q^j(t')|\delta S|q^i(t)\rangle$$

(7)

where $S$ is the action of the system. If the variations are such that the end points of the variations are fixed, then one has that

$$\delta \langle q^j(t')|q^i(t)\rangle = 0$$

(8)

from which one infers both the operator equations of motion and the quantum equations of motion of the physical expectation values $\langle S, i \rangle = 0$. The comma denotes the functional derivative of the action with respect to the field $q^i$. If the endpoints are not rigid, then the wisdom of the action principle lies in the statement that the variation of any dynamical operator $F$ is given by

$$\delta F = -i\alpha[F,G]$$

(9)

where $G$ is the generator of the transformation. The form of $G$ follows from the total derivatives incurred on varying the symmetrized action operator. $\alpha$ is a constant which must be fixed by demanding that the Hamiltonian operator be the generator of time translations; it depends on the order of time derivatives in the theory and their symmetry with respect to the field variables. This statement leads directly to the fundamental commutation relations of the system. The reader is referred to refs. [1, 2, 3] for a careful exposition of the action principle.
The utility of the action principle becomes apparent when one introduces external sources which test the linear response of the theory to small disturbances \( J \). In the simplest instance one lets \( S_{\text{tot}} = S + Jq \). Repeated functional differentiation with respect to the source generates the \( n \)-point Green functions of the theory

\[
\frac{\delta^n \langle q_2 | q_1 \rangle}{\delta J_{i_1} \cdots \delta J_{i_n}} = i^n \langle q_2 | T(q^{i_1} \cdots q^{i_n}) | q_1 \rangle
\]

which implies that

\[
\langle q_2(t') | q_1(t) \rangle_J = \langle q_2 | T \exp(iJ^i q^i) | q_1 \rangle,
\]

thus the source serves as both a convenient generator of Green functions as well as a deformant of the transformation function. Variation of (7), including source term, with respect to \( q^i \) now leads to

\[
\Gamma, i = \frac{\langle q^i | S_{\alpha} | q^i \rangle}{\langle q^i | q^i \rangle} = -J_i
\]

where \( \Gamma \) is the quantum action functional or effective action\(^9\). A path integral form is readily obtained from (11) by writing

\[
\langle q^i(t') | q^i(t) \rangle = \exp iW[J] = \int d\mu[\Phi] \exp i\{S[\Phi] + J\Phi\}
\]

where \( d\mu[\Phi] \) is an appropriate measure. The effective action is then obtained by writing the field \( \Phi = \bar{\phi} + \varphi \) and integrating over \( \varphi \). The Legendre transform \( \Gamma(\bar{\phi}) = W[J] - J\bar{\phi} \) now yields the effective action satisfying (12).

If the variables in the theory act in the presence of an invariance group, then a change which corresponds to the action of the group leaves the transformation function unchanged. For instance, using a condensed notation in which the summation over indices includes an integration over a corresponding continuous spacetime label,

\[
\delta q^i = R^i_\alpha[q] \delta \xi^\alpha
\]

\[
\delta \langle 0 | 0 \rangle = \delta \langle 0 | \exp(iJ_i R^i_\alpha \delta \xi^\alpha) | 0 \rangle = 0.
\]

The parameters \( \xi^\alpha \) which generate infinitesimal transformations represent an arbitrary symmetry of the system and the combination \( R^i_\alpha[q] \delta \xi^\alpha \) is formally a ‘counterterm’ in the sense of an infinitesimal change in renormalization. Defining \( R^i_\alpha = \overline{R^i_\alpha} + R^i_{\alpha j} q^j \), one has infinitesimally\(^9\)

\[
\delta \langle 0 | 0 \rangle = iJ_i \delta \xi^\alpha \left[ \overline{R^i_\alpha} - iR^i_{\alpha j} \frac{\delta}{\delta J^j} \right] \langle 0 | T \exp(iJ_j q^j) | 0 \rangle = 0.
\]

The vanishing of the coefficient in this equation leads to an analogous ‘renormalization group’ equation for the system as we shall see below.

The renormalized coupling constants of a quantum field theory are constant with respect to position and time in any fundamental theory, but they are not independent of the renormalization scale \( \mu \) at which they are defined. The
effect of a change of $\mu$ is to vary the values of the renormalized couplings and expectation values in such a way that the amplitude $\langle 0 + \infty | 0 - \infty \rangle$ is left unchanged. In the contemporary literature a change of $\mu$ is often referred to as a ‘flow’. Clearly this is not a physical flow in the sense of a time variation, but a flow with respect to another parameter which is non-physical. In any given equilibrium scenario the value of $\mu$ is fixed by convenience and any measurable is independent of $\mu$ when computed exactly. However the perturbation series depends upon the renormalized values of the couplings since the effect of a renormalization is to absorb the effect of certain composite field operators into new effective couplings, thereby grouping together like-terms up to a shift in the vacuum energy.

Based on this straightforward observation it is now asserted that the action principle can be employed to exploit the formal analogy between the appearance of expectation values of the quantum field $\Phi^i \rightarrow \varphi^i + \phi^i$ and the arbitrary redefinition of the couplings in the action due to renormalization e.g. $m^2 \rightarrow m_{R}^2 + \delta m^2 (\mu^2)$. Such a procedure exposes a canonical or symplectic structure in the symmetry.\footnote{After submitting this paper for publication a related preprint\cite{39} appeared demonstrating this connection nicely from a geometrical viewpoint. I was also informed of unpublished work by C. Stephens and D. O’Conner on which this reference was based\cite{40} and which preceeds this paper.}

To make the dynamical analogy most explicit one can make the unusual step of introducing a fictitious set of states, quite separate from the Fock space, whose only purpose is to rewrite the renormalization group symmetry in a new form. One then represents the coupling constants in the usual Lagrangian as fictitious operators (matrices) which act only on these states in such a way as to reproduce the renormalized values of the couplings as their eigenvalues. This is the essence of the dynamical analogy described in the next section.

The couplings can be thought of as varying under a renormalization group transformation in accord with a scale parameter $\mu$ which takes on the role of a ‘proper time’ variable in the manner of ref. \footnote{I am grateful to C. Stephens for his subsequent comments on this paper.}. This step is purely artificial and implies no reinterpretation of the theory – nor does the introduction of states imply anything other than an extra formal step. One should not, for example, impose any probabilistic interpretation on such states, but rather treat them as fictitious states whose role is rather trivial. It does however carry with it the implicit assumption that the variable $\mu$ generates a true symmetry of the system which need not be the case in an arbitrary renormalization scheme. In particular we require the symmetry to be a group since it will be necessary to assume that the combination of infinitesimal transformations leads to a finite transformation.

The resulting dynamical analogy, aside from being interesting from a pedagogical perspective, has a useful role to play in the investigation of phenomenological non-equilibrium Lagrangians in which the couplings are regarded as depending on the real time. We shall return to this issue in section 7.

\footnotetext[2]{After submitting this paper for publication a related preprint\cite{39} appeared demonstrating this connection nicely from a geometrical viewpoint. I was also informed of unpublished work by C. Stephens and D. O’Conner on which this reference was based\cite{40} and which preceeds this paper.}
4 The dynamical analogy

To facilitate a better understanding of the above remarks, consider the addition of a source $J$ to the action, in the form

$$S[\Phi] \rightarrow S[\Phi] + \int dV_x J\Phi$$  \hspace{1cm} (17)

The classical equations of motion are clearly given by

$$\frac{\delta S}{\delta \Phi} = -J$$  \hspace{1cm} (18)

If the potential described by $S$ has a global minimum corresponding to the stationary value of the field $\Phi$, then the role of the source term is to administer a generalized ‘force’ which shifts the position of the minimum. The potential must be at least of quadratic order if the field is to be non-degenerate, otherwise the source simply generates a symmetry transformation on the variables.

The source works by formally varying the quantity in the action which is conjugate to the field, i.e. the first derivative of the action with respect the the field variable. If the potential is not infinitely degenerate in $\Phi$, this will yield an effective change in the stationary value of $\Phi$ when the equations of motion are solved, which is equivalent to the claim that $J$ varies the stationary solutions of $\Phi$ up to a rescaling of the action. It is incidentally noteworthy that, in the context of perturbation theory, the introduction of a source may affect the order of perturbation theory to which one must work to achieve a certain accuracy.

In passing to the quantum theory the latter remark transfers to the expectation values of operator ‘observables’. It is now advantageous to make the parametric dependence of the couplings appear in a manner which is analogous to the dynamics of the system. Let the couplings be represented by operators acting on quasi-states, as discussed in the previous section.

$$m_R^2 \rightarrow m^2(\mu)|\mu\rangle_Q$$  \hspace{1cm} (19)

This is a purely formal substitution and implies no physical interpretation. Clearly the fictitious operators only have an appropriate meaning inside a scalar of the form $\langle \mu'|m^2|\mu\rangle_Q$. The field operator plays an important role in mediating between the physical quantum states of the field and the quasi-states. This is expressed in the present formalism by defining the field operator to operate only on the physical states while its expectation value acts only on the quasi-states. In this way the two sets decouple at the formal level.

Let the set of all couplings and non-zero expectation values be denoted by $\{q^i\}$. We shall see below that an infinitesimal change in the renormalization point for all $q^i(\mu)$ can be written in the form of a Hamiltonian flow

$$q^i(\mu) = q^i(0)e^{-i \int_0^\mu d\mu^i H_i}.$$  \hspace{1cm} (20)

We begin by introducing sources for each of the $q^i$. Define

$$\Delta S = \int_{\mu_1}^{\mu_2} d\mu \ J_i(\mu)q^i(\mu).$$  \hspace{1cm} (21)
then by analogy with the Schwinger action principle, we may consider a change in the quasi-amplitude as arising from the action of a generator $G$ on the end states in the following manner.

$$\delta\langle \mu' | \mu \rangle_Q = i\langle \mu' | G' - G | \mu \rangle_Q$$

$$= i\langle \mu' | \int_\mu^{\mu'} d\mu' G d\mu | \mu \rangle_Q$$  \hspace{1cm} (23)

(23) rests on the tacit assumption that a given $\mu$ generates a unique set of couplings given appropriate initial conditions – i.e. that the renormalization group flows do not cross. There is no general, rigorous justification for this assumption, except that this is consistent with the other essential assumption, namely that the symmetry forms a group. For an infinitesimal transformation of the form (20), we have

$$|q\rangle_Q \rightarrow (1 - iH_Q d\mu)|q\rangle_Q$$

which, on comparison with (22) implies that

$$H_Q \delta \mu = G = J \delta q.$$ \hspace{1cm} (25)

Thus infinitesimally we have

$$\frac{\delta}{\delta \mu} \langle \mu' | \mu \rangle_Q = i\langle \mu' | H_Q | \mu \rangle_Q$$

where

$$H_Q = \beta^i J_i$$ \hspace{1cm} (27)

where

$$\beta_i = \frac{dq_i(\mu)}{d\mu}$$

$$\gamma_i = \beta_i q^{-1}_i$$ \hspace{1cm} (28)

and $H_i = i\gamma_i$ in (20). The analogy with the proper time method in ref. [8] is apparent. The corresponding ‘proper time’ equations of motion are formally (24) and the quasi-amplitude satisfies the boundary condition that

$$\lim_{\mu \rightarrow 0} \langle q_i(\mu) | q_j(0) \rangle = \delta(q_i, q_j).$$ \hspace{1cm} (30)

We can now proceed to define the generalized effective action by supplementing the physical states in (7) by the quasi-states. Rather than introducing further new notation, we shall simply refer to these states by $|q^A\rangle$, where uppercase roman characters to represent the sum of all the indices $q^A = \{\Phi^a, q^i\}$, and it will be understood that this means the combination of the two sets. Since there is no overlap between the operators which act on the two types of state, there is no ambiguity in this procedure.

The generalized action operator including sources is now

$$S[\Phi] \rightarrow S[\Phi] + J_a \Phi^a + J_i q^i$$ \hspace{1cm} (31)
where the repeated indices are assumed to include an integration over the continuous label: $x$ for indices $a$ and $\mu$ for indices $i$. The expectation value of the field variable $\bar{\phi}$ has two sources as written, but these may be combined to form a single source.

The effective action is now given by

$$\Gamma = W[J] - \overline{q^i}J_i - \overline{\phi^a}J_a$$  \hspace{1cm} (32)

$$\overline{q^i} = \langle q^i \rangle = \frac{\delta W}{\delta J_i}$$  \hspace{1cm} (33)

$$\overline{\phi^a} = \langle \phi^a \rangle = \frac{\delta W}{\delta J_a}$$  \hspace{1cm} (34)

where the barred quantities represent renormalized values and the part analogous to the quantum field in the split $q \to \overline{q} + \delta q$ is to be eliminated. One therefore expects the present formalism to yield a set of relations expressing the counterterms in terms of only the barred variables. Since the additional sources only generate a symmetry transformation, this apparently modified effective action must correspond simply to a reparameterization of the usual renormalized effective action. Hence, in spite of appearances, $\Gamma$ is not a new object.

Varying (32) with respect to $q^i$ involves all the related generators $R^i_\alpha$. For example, for $\lambda \phi^4$ scalar field theory, writing explicitly, one has

$$\delta \Gamma \delta \phi = -J_\phi$$  \hspace{1cm} (38)

$$\delta \Gamma \delta m^2 = -J_m$$  \hspace{1cm} (39)

$$\delta \Gamma \delta \lambda = -J_\lambda$$  \hspace{1cm} (40)

leading to the simple equation set
From these variational equations it is now straightforward to show that the action principle is consistent with a ‘renormalization group’ equation for the system. In particular one sees that the effect of the sources is to induce a renormalization group transformation on the effective action. Consider the case in which the variables \( q^i \) are parametrically dependent on the variable \( \mu \). Then

\[
\frac{\delta}{\delta \mu} \langle q^A | q^B \rangle = i \frac{\delta}{\delta \mu} \langle q^A | S + J_\delta \Phi + J_m m^2 + J_\lambda \lambda | q^B \rangle
\]

where the sources are not objects of variation. Integrating the equation

\[
\Gamma_{,a} = \langle S, a \rangle
\]

and using the equations of motion for the mean quantities (38)-(40), one obtains

\[
\left\{ \frac{\partial}{\partial \mu} - \int dV \beta_\phi \frac{\partial}{\partial \phi} - \beta_m \frac{\partial}{\partial m^2} - \beta_\lambda \frac{\partial}{\partial \lambda} \right\} \Gamma = \frac{\partial \Lambda(m^2, \mu)}{\partial \mu} \tag{43}
\]

where \( \Lambda(m^2, \mu) \) is formally the constant of integration in (42) and \( \beta_q = \langle q | \frac{\partial}{\partial \mu} | q \rangle_Q \) etc. This is normally chosen so that \( \Gamma[\bar{\phi} = 0] = 0 \). In curved spacetime \( \Lambda \) acquires the additional significance of being the cosmological constant. Its presence has an important formal role, though in practice its only function in flat space is to shift the zero point energy.

The completeness of the operator picture can be seen by noting as in [39] that eqn (43) has the form of a Hamilton-Jacobi relation. In the following relations (45-48) the variable \( i \) is not summed over.

\[
\frac{\partial \Gamma}{\partial \mu} + H = 0 \tag{44}
\]

where

\[
H = -\beta^i \frac{\partial \Gamma}{\partial q^i} = \beta^i J_i = \sum_i H_i. \tag{45}
\]

Consistency with the Hamiltonian flow of the couplings is obtained on noting that

\[
H_i = i \beta^i q_i^{-1} = i \beta^i \frac{\partial}{\partial q^i} \tag{46}
\]

where \( q^{-1} \) is the left-inverse of \( q^i \). In particular this gives the quasi-canonical commutation relations at equal \( \mu \). From

\[
\frac{\partial}{\partial \mu} q^i = -i[H_i, q^i] = \beta^i \tag{47}
\]

and, on substituting for \( H_i \) one obtains

\[
[q_i^{-1}, q_i] = 1 \tag{48}
\]

or

\[
[p_i, q_i] = -i \tag{49}
\]
where $p_i = i \frac{\partial}{\partial q_i}$, so that the Hamiltonian projection $H_i = p_i \dot{q}^i$ from which is should be apparent that $H_i$ is the generator of a quasi-canonical transformation. Finally,

$$\langle p|q \rangle_Q = e^{i \frac{H_i}{\hbar}q}.$$  

(50)

An interpretation of this dynamical analogy may be seen in a simple exam-
ple. Consider the effective variation of the coupling constant due to a change
in the source. If the parameter space of the effective action has local maxima
and minima on varying the couplings then they may be located by considering

$$\frac{\delta \lambda(J_\lambda)}{\delta J_\lambda} = 0$$  

(51)

From (53) this may be written

$$\frac{\delta^2 W[J_i]}{\delta J_\lambda} = \langle \lambda(\mu)\lambda(\mu') \rangle - \langle \lambda(\mu) \rangle \langle \lambda(\mu') \rangle = 0$$  

(52)

which from the equation of motion

$$i \frac{d}{d\mu^2} \langle \lambda(\mu)|\lambda(\mu') \rangle = \langle \lambda(\mu)|H_\lambda|\lambda(\mu') \rangle$$  

(53)

implies that the beta-function $\beta_\lambda$ is vanishing. Thus the stationary points refer
to renormalization group fixed points. The two pictures are related through
a reparameterization, up to a shift in $\Lambda$. The dynamical appeareance of these
equations should not lead to confusion, $\mu$ is not a true dynamical parameter: it
simply characterizes a change in dependent variables; the symbolism is intended
to inspire obvious analogies, which we shall find especially significant when
turning to deal with with phenomenological non-equilibrium problems with
time-dependent effective couplings. No physical evolution is implied.

Although the dynamical analogy is only a formality, it provides us with one
useful trick. Normally one considers variations of the dynamical variables at a
fixed spacetime point. Additional variations of the coordinate system lead to
terms proportional to the energy momentum tensor and angular momentum.
Such variations can be generalized still further to include a class of variations
pertaining to renormalization group transformations. This equips us with the
possibility of obtaining a so-called renormalization group improved effective
action directly from the action principle. This remark will be amplified in the
next section.

The preoccupation with the renormalization parameter $\mu$ is no particular re-
striction, since the foregoing equations can easily be generalized to include a de-
pendence of the $q^i$ on a variety of parameters. Recent work on non-equilibrium
physics motivates the choice of time-dependent couplings and it has been sug-
gested that such a dependence corresponds to a real-time renormalization group
transformation[28]. However, for a system which is not at equilibrium, such
a dynamical change in couplings will not express a true invariance of the sys-
tem and thus more thought must be given to the meaning of such a procedure.
Another invariance parameter which is known to distribute non-perturbatively
is the gauge fixing multiplier. Here the action principle asserts that the expectation value of the gauge fixing constraint vanishes in an exact calculation. The Ward identity plays an analogous role to the renormalization group equation. For now, we note that the utility of these formal manipulations lies in ability to unravel symmetries of the effective action in a dynamical form. In particular, it gives rise to self-consistent relations between the Green functions and renormalized variables in the manner of the gap equation and Schwinger-Dyson equations. The addition source terms makes themselves felt through the Legendre transformation in (31) and it should be sufficient to consider these in order to determine the formal expressions for the counterterms.

An alternative possibility is to consider the present problem in the conjugate representation in which one attempts to extract the implicit operator valuedness from the couplings by introducing generators (sources) which directly resum particular classes of Feynman diagrams. This is the method of multi-local sources and is discussed further in section 6.

5 Applications and examples

A useful feature of the dynamical analogy is that it yields an indication of the gains and limitations of the renormalization group. It is particularly useful to know whether a renormalization of the couplings in the theory is enough to resum all of the required diagrams in perturbation theory for a given purpose. The purpose of this section is to use the recursive structure implied by the dynamical analogy in the preceding section to analyse the simplest gauge theory (scalar electrodynamics) in the massless limit. This model has been examined previously using more conventional methods[11, 13, 14]. The present formalism confirms previous results with an important corollary and specifically indicates how one might improve on the simplest calculation.

The central object of interest for many applications is the effective action. Using the functional evaluation scheme due to Jackiw[10], the effective action may be cast, for \(\phi^4\) theory, in the form

\[
\Gamma[\phi] = S[\phi] - i \int D\phi \exp \left\{ \frac{1}{2} \phi^a S_{ab} \phi^b + S_{int} - \phi^a \Gamma[\phi],_a - \delta m^2 \frac{\delta \Gamma}{\delta m^2} - \delta \lambda \frac{\delta \Gamma}{\delta \lambda} \right\}
\]

where it is assumed that a background scalar field is present. The term involving \(\Lambda\) can be safely ignored for present purposes since only variations of the action (potential differences) have a physical significance. From the effective equations of motion one has

\[
\frac{\delta \Gamma}{\delta m} = \int dV x \left\{ \frac{1}{2} \phi(x) \phi(x) + G(x, x) \right\} - \delta m^2 \frac{\delta^2 \Gamma}{\delta m^2} - \delta \lambda \frac{\delta^2 \Gamma}{\delta \lambda} = 0
\]

\[
\frac{\delta \Gamma}{\delta \lambda} = \int dV x \left\{ \frac{1}{4!} \phi(x) \phi(x) \phi(x) \phi(x) + \frac{1}{4} G(x, x) \phi(x) \phi(x) \right\} - \delta q \frac{\delta \Gamma}{\delta q} = 0
\]
where, for example,
\[
G(x, x') = \frac{\int D\phi \varphi(x)\varphi(x')e^{i\frac{1}{2}\varphi^a S_{ab}\varphi^b + S_{int} - q'\Gamma, i}}{\int D\phi e^{i\frac{1}{2}\varphi^a S_{ab}\varphi^b + S_{int} - q'\Gamma, i}}
\]  
(57)
and there are a number of relations of the form
\[
\frac{\delta^2 \Gamma}{(\delta m^2)^2} = \int dV_x \frac{1}{2} \frac{\delta G(x, x)}{\delta m^2} + \cdots
\]  
(58)
\[
\frac{\delta^2 \Gamma}{\delta m^2 \delta \lambda} = \int dV_x \left\{ \frac{1}{4} \varphi(x)\varphi(x) \frac{\delta G(x, x)}{\delta m^2} + \frac{1}{4} \delta G(x, x, x, x) \right\}
\]  
(59)
\[
\frac{\delta G(x, x)}{\delta m^2} = \frac{1}{2} \int dV'_{x'} \{ G(x, x', x') - G(x, x)G(x', x') \}
\]  
(60)
Using these expressions in the equations of motion for vanishing quantum source part, one obtains iterative differential equations which can be solved for and used to eliminate \(\delta q_i\); to leading order in the loop counting parameter (which is set to 1 in this paper) this can be separated
\[
\int \frac{\delta G}{G(x, x)} = \int \frac{\delta (\delta m^2)}{\delta m^2}
\]  
(61)
\[
\int \frac{\delta G}{G(x, x, x, x)} = \int \frac{\delta (\delta \lambda)}{\delta \lambda}
\]  
(62)
To zeroth order one verifies only the ‘classical equation of motion’ for the couplings. The remaining conditions relate therefore to the perturbative corrections to the standard effective action – in other words fluctuations in the true dynamical fields (not the quasi-operators). They therefore correspond, in a limited sense, to a minimization of perturbative corrections, or an optimal perturbation series. With the partial boundary condition that \(\delta m^2 = 0\), \(\delta \lambda = 0\) when \(\lambda = 0\), one may write on dimensional grounds
\[
\delta m^2 = \lambda G(x, x)\xi
\]  
(63)
\[
\delta \lambda = \lambda^2 \text{Tr} G(x, x, x, x)\xi'
\]  
(64)
where \(\xi, \xi'\) are dimensionless constants. These forms make the source terms in the action into an invariant form. These arbitrary constants reflect the residual non uniqueness of the coupling-counterterm split and must therefore still be chosen by some additional condition. This is to be expected since we have so far imposed no renormalization conditions on the parameters of the theory.

The coincidence limit of the Green functions \(G(x, x') \to G(x, x)\) is to be regarded in a formal sense, since in reality this may diverge\(^3\). These expressions can be compared with the usual vertex conditions used in the method of renormalization by ‘oversubtraction’. Differentiating (54) in the zero momentum

\(^3\)In principle one might absorb the ultraviolet divergences by redefinition of the source terms. Here I am ignoring the ultraviolet behaviour which may be dealt with by minimal subtraction, for instance, and treating only the infra-red behaviour in the present scheme.
limit leads to

\[
\frac{1}{\Omega} \frac{\partial^2 \Gamma}{\partial \phi^2} \bigg|_{\phi=0} = m^2 + \frac{\lambda}{2} G(x, x) \tag{65}
\]

\[
\frac{1}{\Omega} \frac{\partial^4 \Gamma}{\partial \phi^4} \bigg|_{\phi=0} = \lambda + \frac{3\lambda^2}{2} \mathrm{Tr} G(x, x, x, x) \tag{66}
\]

where only the leading order corrections are included. \(\Omega\) is a spacetime volume scale. It is seen that the introduction of the sources generates a term precisely of the form due to a more conventional renormalization method. Moreover, it is now possible to see why the approximate regularization prescription in appendix A has the desired effect. The arbitrary renormalization point \(\Phi_2\) plays the role of a crude approximation to \(G(x, x)\). In that particular calculation, no great accuracy was required, only the presence of a regulating contribution from \(G(x, x)\). In this example nothing is gained from the new approach, except perhaps the satisfaction of knowing that an old result can be shown in a new way. It is more interesting to consider a gauge theory.

The inclusion of gauge fields presents new subtleties for the renormalization approach, the first of which arises from the non-linear dependence of the action on the gauge coupling in a second order derivative theory – even at the classical level. Given the gauge-covariant derivative defined by \(D_\mu = \partial_\mu + ieA_\mu\), one considers an action of the form

\[
S = \int dV_x \left\{ (D^\mu \Phi)^\dagger (D_\mu \Phi) + V(\Phi) \right\}. \tag{67}
\]

In the quantized theory one is forced to break the invariance group so as to count only physical fields. Also, since the gauge parameter distributes non-perturbatively in the usual definition of the effective action, the specific choice of gauge will effectively lead to different perturbative expansions, some of which may be better than others. In a functional integral representation, this is implemented by a gauge fixing condition and possibly the introduction of ghost terms \(^4\) which must be taken into account by the renormalization. A particular example is adopted for the remainder of this section, namely scalar electrodynamics in 3 + 1-dimensions, as described in appendix B. The addition of a source term \(J e\) now varies the conjugate quantity

\[
\frac{\delta \Gamma}{\delta e} = -J e = J^\mu A_\mu + 2e e^2 \langle A^\mu A_\mu \rangle + 2e e^2 \langle \overline{\eta} \eta \rangle + 2e e^2 \langle A_\mu \varphi_a \varphi_a \rangle + e e^2 \langle A^\mu A_\mu \rangle + e \langle A^\mu A_\mu \varphi_a \varphi_a \rangle - e e^2 \frac{\delta^2 \Gamma}{\delta e^2} \tag{68}
\]

where

\[
J^\mu = i[\Phi(D^\mu \Phi)^\dagger - \Phi^\dagger (D^\mu \Phi)]. \tag{69}
\]

and the expectation symbols refer to the exact correlators, defined in terms of the full action. These contributions arise specifically from the vacuum polarization in the theory, as would be expected. The renormalization of the gauge

\(^4\)While the introduction of ghosts is not a necessity at one loop, practical calculations in most gauges will demand their introduction at higher loops. Moreover, the widespread belief that the Jacobian contribution is unity in scalar electrodynamics is mistaken in general [3].
theory is seen to be a complicated non-linear problem and it is apparent that the full BRST symmetry might be utilized to generate an effective resummation.

The determination of the counterterm proceeds as before. Further approximation is now required owing to the complexity of the gauge symmetry. Earlier work using the renormalization group calculated by traditional methods\cite{13, 14} shows that a charge renormalization is necessary in order to reveal the possibility of a first order phase transition. This is generated by renormalizing away from zero momentum in order to compute a beta-function for the electric charge. Here we shall attempt to investigate this possibility more directly using the action principle and the structure generated by the sources.

Information about the phase transitions of the theory in contained in the infrared limit of the effective action. The extraction of this information presupposes that the infrared limit is well behaved. This is not the case in the usual perturbation expansion since one has to be able to determine the form of the effective potential for the background $\phi$ field in the limit as one approaches the origin. The perturbative expression for the effective potential does not exist at the origin and thus it must be regulated by some appropriate resummation.

The regulated form of the effective potential around the origin determines the order of the transition in the following way. In a second order phase transition, the addition of a small mass to the $m^2 = 0$ potential makes the potential curve monotonically upward implying a single minimum at the origin. In a first order transition a minimum away from $\phi = 0$ at $m^2 = 0$ survives for some finite mass correction and the potential curves downwards at some small value of $\phi$ below the minimum. One is therefore interested in regulating the infrared behaviour of the effective potential, namely the small $\phi$ regime.

It ought to be remarked at this juncture that the non-convex form of the effective potential is a perturbative artefact\cite{12} which has led to much confusion in the past. The present arguments relate primarily to the approach to phase transitions through perturbation theory on which the majority of the literature is based. It is understood that one is working at the edge of perturbative credibility here. The distinction between first and second order is in whether the potential curves monotonically upward for an infinitesimal mass perturbation\cite{31}.

To obtain an infrared regularization one would clearly like a mass term in the various functional determinants to supplement $e^{2\phi^2}$ in the limit of vanishing $\phi$. To determine whether such a mass can be induced by a renormalization group transformation, one can follow the formalism of the sources and determine the appropriate behaviour from the definition of the perturbative propagator – which must be regarded as a small quantity if the diagrammatic expansion is to make sense. For the scalar field this was straightforwardly obtained by using the trick in appendix A for renormalizing the mass. However in the case of the gauge field no direct analogue of such a mass term exists in the classical Lagrangian and thus there is no source for such a mass. It is interesting however that, in the present case, a charge renormalization is partially successful and this explains why the renormalization group\cite{14} argument enables the infrared behaviour to be regulated in an asymptotic limit. The regulating mass arises
from vacuum polarization.

Since the gauge field mass is given by $e^2 \phi^2$, one is interested in the behaviour of the theory approaching the symmetric phase ($\phi \to 0$) where perturbation theory is weakest.

$$\frac{\delta^2 \Gamma}{\delta e^2} \sim \langle A^\mu A_\mu \varphi_a \varphi_a \rangle + e \frac{\delta \langle A^\mu A_\mu \varphi_a \varphi_a \rangle}{\delta (\delta e)}.$$  

(70)

In the neighbourhood of $\bar{\phi} = 0$ one has

$$\frac{\delta \Gamma}{\delta e} \bigg|_{\overline{T, A}=0} \sim (\overline{\tau} + \delta e) \langle A^\mu A_\mu \varphi_a \varphi_a \rangle - \delta e \left\{ \langle A^\mu A_\mu \varphi_a \varphi_a \rangle + e \frac{\delta \langle A^\mu A_\mu \varphi_a \varphi_a \rangle}{\delta (\delta e)} \right\} = 0$$  

(71)

for the first order corrections to $\Gamma$, and one is thus led to the following formal equation for $\delta e$

$$\int \frac{\pi \delta (\delta e)}{\delta e (\overline{\tau} + \delta e)} = \frac{\delta \langle A^\mu A_\mu \varphi_a \varphi_a \rangle}{\langle A^\mu A_\mu \varphi_a \varphi_a \rangle}.$$  

(72)

Expressing this in partial fractions and integrating leads to the approximate result

$$\delta e \sim e^3 \text{Tr} \langle A^\mu A_\mu \varphi_a \varphi_a \rangle \xi'$$  

(73)

for some $\xi'$. The action principle leads therefore to a leading order gauge mass of the form

$$e^2 \phi^2 \sim e^2 \phi^2 + 2e \delta e + \delta e^2 \phi^2.$$  

(74)

The regularization of the effective potential near $\overline{\phi} \to 0$ requires a solution of the form $\delta e \sim \overline{\phi}^{-1}$, securing finiteness in the limit. To investigate the possibility for this one may deduce the limiting behaviour of the perturbative Green functions from their definitions after a renormalization group transformation of the kind implied by the sources.

To lowest order in perturbation theory one may write

$$\langle A^\mu A_\mu \varphi_a \varphi_a \rangle \sim \langle A^\mu A_\mu \rangle \langle \varphi_a \varphi_a \rangle$$

thus, to a first approximation, we must obtain self-consistent forms for the scalar propagator $\Delta$ and the gauge propagator $G$ in the coincidence limit, which incorporate the appropriate counterterms generated by the sources. The scalar propagator may be written

$$\Delta = \int \frac{d^n k}{(2\pi)^n} (k^2 + \frac{\lambda}{2} \phi^2 + \lambda \xi \Delta)^{-1}$$  

(75)

from which one obtains, using dimensional regularization

$$\Delta = \lambda \frac{1}{2} \phi^2 + \xi \Delta \ln \left( \frac{\lambda \frac{1}{2} \phi^2 + \xi \Delta}{4\pi \mu^2} \right) - \frac{\lambda \frac{1}{2} \phi^2 + \xi \Delta}{4\pi}$$  

(76)

To regulate the divergences as $\overline{\phi}^2 \to 0$ it is tempting to let $\mu^2 \sim \frac{1}{2} \lambda \overline{\phi^2}$ reproducing the procedure in appendix A. One then finds that $\Delta \sim \lambda \overline{\phi^2}$ as the background field vanishes, in accordance with the appendix. However in the gauge theory the same procedure does not regulate all the divergences and one
is forced to adopt a different tactic. If one holds \( \mu^2 \) fixed and considers the \( \phi \to 0 \) limit of the propagator, it is found that

\[
\Delta \to \frac{4\pi \mu^2}{\lambda \xi}.
\]

(77)

This result will be used shortly. The gauge propagator in the Landau-deWitt gauge may be written

\[
G = G^\mu_\mu(x,x) = \int \frac{d^n k}{(2\pi)^n} \frac{(n-1)}{k^2 + e^{2\phi}}.
\]

(78)

Using the form in equation (74) and employing dimensional regularization one obtains the following implicit equation for \( G \)

\[
4\pi G = (e^{\phi^2} + e^{4\phi^2} G \Delta \Omega \xi) \left( 1 + 3 \ln \left( \frac{e^{\phi^2} + e^{4\phi^2} G \Delta \Omega \xi}{4\pi \mu^2} \right) \right)
\]

(79)

where \( \Omega \) is a spacetime volume scale. If one uses the form for \( \Delta \) from appendix A, i.e. \( \Delta \sim \lambda \phi^2 \), and the implied behaviour of \( \mu^2 \), then it appears that one can regulate the infrared divergences in precisely the procedure analogous to appendix B. The argument in appendix B yields a second order phase transition, thus the present procedure also indicates a second order transition, since the curvature of the effective potential appears to curve upward even for \( m^2 = 0 \).

However in the steps above one is forced to make the implicit assumption that \( \lambda \sim \xi^2 \) in order to consistently avoid large logarithmic corrections in \( G \) and it is known, from Coleman and Weinberg’s original argument[11] that the one-loop calculation in appendix B is not consistent if this is the case. Thus what the action principle reveals here if this inconsistency in making a thoughtless resummation of the scalar sector.

One may now proceed according to the second possibility (77) above. Considering the \( \phi \to 0 \) limit of (79), one obtains the approximate form

\[
\exp \left[ \frac{4\pi G}{e^{\phi^2} G \Delta \xi' \Omega} \right] = \frac{e^{\phi^2} G \Delta \xi' \Omega}{4\pi \mu^2}
\]

(80)

and on substituting the asymptotic form of the scalar propagator one obtains

\[
\exp \left[ \frac{\lambda \xi}{e^{\phi^2} \mu^2 \xi' \Omega} \right] = e^{\phi^2} G \xi' \Omega \lambda \xi.
\]

(81)

One may now proceed in one of two ways. The first method parallels the conventional renormalization group approach in refs [13, 14]. Here we note that, as \( \phi \to 0 \), \( \delta e \) must become large in order to prevent the vanishing of the gauge propagator mass. However, since the effective action is invariant under the placement of such a counterterm

\[
\delta \Gamma = -J_m \delta m^2 - J_e \delta e - J_\lambda \delta \lambda \sim 0
\]

(82)
where $\sim$ indicates that the result is true up to a shift in the zero point energy. Since both $\delta m^2$ and $\delta e$ must be positive to avoid sicknesses in the theory, one has that $\delta \lambda < 0$, since $\Gamma$ varies approximately symmetrically with respect to these quantities. If the initial value of $\lambda$ is sufficiently small, it is thus possible that $\lambda$ itself can become negative. This is indeed what happens by conventional methods and one sees from (81) that this is in fact necessary if this equation is to be satisfied as $\bar{\phi}$ vanishes. However, it is noteworthy that the equation is only satisfied asymptotically as $\bar{\phi}$ reaches zero owing to the exponential behaviour. This parallels the asymptotic fixed point at $\lambda = -\infty$ in refs [13, 14]. The fact that $\lambda$ turns negative shows that the effective action has negative curvature at $m^2 = 0$, $\bar{\phi} = 0$ and this is strongly suggestive of a first order phase transition. It is noteworthy, however, that this conclusion is reached by desperate means – as an asymptotic regularization of infrared divergences, thus the conclusion rests of the very edge of what one might hope to extract from perturbation theory.

The second method of proceeding from (81) is to attempt to cure infrared divergences by finding a solution of the form

$$\frac{\Omega \bar{\Phi}^2 G \xi'}{\xi} = c$$

(83)

for some constant $c$ which is assumed to be of order unity. Then one obtains the limiting form

$$\exp \left[ \frac{\lambda G}{\bar{\tau}^4 c \mu^2} \right] = \frac{\bar{\tau}^4}{\lambda c}.$$  

(84)

This may be solved by taking $\lambda \sim \bar{\tau}^4$, which provides a motivation for this choice in Coleman and Weinberg’s argument. It is known that this choice leads to a reliable perturbation theory which indicates a first order transition. One must again be cautious however. In writing (83) one is stretching the privilege to redefine the free parameters $\xi$ to limit of credibility and thus this result is no more trustworthy than that obtained from the first approach.

The procedure adopted here has been to make the minimally acceptable approximation. It is natural to wonder whether a more accurate representation of $\delta e$ would lead to a better result. Would it, for example, lead to the appearance of a fixed point associate with a first order phase transition? Examining the form of (68) it would appear that any improvement would be difficult to obtain unless $A_\mu \neq 0$. One way to understand this could be that the phase transition is on the very edge between first and second order for $A_\mu = 0$, but that it is first order for $A_\mu \neq 0$, i.e. with an electromagnetic background field – since the present conclusion of fluctuation induced symmetry breaking can be potentially more reliably inferred in this case. One might attempt to consider the present theory as the $A_\mu \rightarrow 0$ limit of the same theory in an external field. This is essentially the procedure which must be adopted in [13, 14] since there is no charge renormalization according to usual regularization schemes unless $A_\mu \neq 0$. Either way, it appears that the phase transition is weakly first order. No more reliable conclusion has thus far been obtained by other means to the present author’s knowledge.
The conclusion of the above pragmatical approach via the action principle is not only that familiar renormalization group results can be reproduced by considering the self-consistent properties of the propagators, but that the reason for these results can be seen clearly in the way in which the different Green functions contribute to the calculation in the infrared limit.

As a further illustration, the present method can be used to calculate the magnetic vacuum polarization energy in a non-Abelian gauge theory. This calculation is known to be perturbatively unreliable and leads to the so-called Nielsen-Olesen ‘instability’[13], which consists of an imaginary contribution to the energy and hence vacuum decay in an external Abelianized magnetic field. In contrast to the imposition of a constant electric field, one would not expect magnetically induced vacuum decay since the magnetic field does no work on particles. The only apparent alternative is that the imaginary part is an infrared artifact. The previous computations and conventions of reference[16] can be used to obtain an improved one-loop result which includes an estimate of leading order non-perturbative corrections for an arbitrary semi-simple gauge group.

Consider the action

\[ S_{YM}[A] = \frac{1}{4I_2(G_{adj})} \int dv_x \text{Tr} \left( F^{\mu\nu} F_{\mu\nu} \right), \]  

for Dynkin index \( I_2 \) in the Hermitian adjoint representation.

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] \]  

The gauge-fixing term is

\[ S_{GF} = \frac{1}{2\alpha I_2(G_{adj})} \int dv_x \text{Tr} \left( D_\mu A^\mu \right)^2, \]  

where

\[ D_\mu A^\mu = \partial_\mu A^\mu + ig[A_\mu, A^\mu] \]  

and the ghost action is

\[ S_{GH} = \int dv_x \bar{\eta} \left[ -D^2 - gA^\mu D_\mu - g(D_\mu A^\mu) \right] \eta. \]

A \( d + 2 \) dimensional Euclidean metric is used. The one-loop operator in the Feynman gauge is given by

\[ \Gamma^{(1)} = -\frac{1}{2} \text{Tr} \ln(-D^2 \delta^{\mu\nu} + 2igF^{\mu\nu}) \]

and the magnetic background field \( A_\mu \) is a linear combination of the generators of the Cartan subalgebra \( \mathbf{B} \cdot \mathbf{H} \). The source of the imaginary part can be seen by replacing the quadratic derivative operator by its eigenvalues

\[ -D^2 \to p^2 + 2g|\mathbf{B} \cdot \mathbf{H}|(n+1/2), \quad \text{degeneracy per unit volume} \quad \frac{g|\mathbf{B} \cdot \mathbf{H}|}{2\pi} \]

Note that \( D^2 \) is a diagonal matrix in the Lie algebra and the eigenvalues of \( F^{\mu\nu} \) are straightforwardly

\[ 2iF^{\mu\nu} \to (+2g|\mathbf{B} \cdot \mathbf{H}|, -2g|\mathbf{B} \cdot \mathbf{H}|, 0 \ldots 0) \]
where 0 occurs with degeneracy \( d \), the one loop contribution is

\[
\Gamma_{YM}^{(1)} = -\frac{1}{4\pi} \sum_{n=0}^{\infty} \int \frac{d^d p}{(2\pi)^d} \left[ d \text{Tr} g|B \cdot H| \ln(p^2 + 2g|B \cdot H|(n + 1/2)) + \text{Tr} g|B \cdot H| \ln(p^2 + 2g|B \cdot H|(n + 3/2)) + \text{Tr} g|B \cdot H| \ln(p^2 + 2g|B \cdot H|(n - 1/2)) \right].
\]

When \( p = 0, n = 0 \) the last logarithm suffers a negative argument. A quantum mass counterterm of the order \( g|B \cdot H| \) would eliminate this problem. Since the non-Abelian theory in non-linear in \( A_\mu \), it is possible to determine that such a mass term exists from the self-interactions and does not contradict the structure of the theory. Whilst the classical gauge mass necessarily vanishes for gauge invariance, a finite mass counterterm is allowed provided it only resums existing parts of the theory. Note that a charge \( g \) renormalization can never change the minus sign in (92) into a plus sign, so a straightforward renormalization group solution for this does not exist. The variation of the effective action with respect to the postulated gauge mass leads to

\[
\delta m_{YM}^2 = \frac{1}{2} \text{Tr}(A^{\mu a} A_\mu^a) \xi
\]

analogously to (63). Factors of the Dynkin invariants cancel owing to the normalization in (87). The calculation of the value of this counterterm is extremely difficult. An estimate only can be obtained from the results in ref.[16]. Considering only orders of magnitude, one has after some calculation

\[
\Gamma_{YM}^{(1)} = (4\pi)^2 \text{Tr}(2g|B \cdot H|)^2 \left\{ -\ln(2g|B \cdot H|) \zeta_H(-1, \rho) + \zeta_H(-1, \rho) + \zeta_H(-2, \rho) \right\}
\]

where \( \zeta_H(a, b) \) is the Hurwitz zeta function and \( \rho = \text{Tr} \delta m^2 / (2g|B \cdot H|) \). Assuming \( \rho \leq 1 \) and using the formula

\[
\zeta_H(-s, \nu) = \frac{2\Gamma(s+1)}{(2\pi)^{s+1}} \sum_{n=1}^{\infty} \frac{\sin(2\pi n \nu - s\pi/2)}{n^{s+1}}, \quad \text{Re} \, (s) > 0, \quad 0 < \nu \leq 1
\]

one may write \( \delta m_{YM}^2 \sim \frac{\partial \Gamma}{\partial (\delta m_{YM}^2)} \), whose leading order approximation leads to

\[
\delta m_{YM}^2 \sim \frac{-4g|B \cdot H|(4\pi)^2 + \sqrt{8(4\pi)^4(2g|B \cdot H|)^2 + 8(2g|B \cdot H|)^2(\ln(\frac{2g|B \cdot H|}{\mu^2}) - \frac{1}{2})^2}}{2(\ln(\frac{2g|B \cdot H|}{\mu^2}) - \frac{1}{2})}.
\]

In the region of validity of this crude approximation one has

\[
\delta m^2 = g|B \cdot H| \xi.
\]

Since this estimate arises from a perturbative calculation there is naturally a dependence of the arbitrary scale \( \mu^2 \), which is transferred above to the parameter \( \xi \). There is freedom within the bounds of the approximations to choose the value of this parameter. In the present instance a value of \( \xi = 1 \) is desirable.
This corresponds roughly to logarithms \( \ln\left(\frac{2g|\mathbf{B}\cdot\mathbf{H}}{\mu^2}\right) \) of the order \((4\pi)^2\) which lies very far outside of the range of one-loop perturbation theory. The best estimate in a good perturbative regime is \( \xi \sim 0.1 \) which is hardly adequate. However, a larger value does not necessarily cause a problem for \((\mathbf{98})\) which is an order of magnitude estimate, obtained by self-consistently including a subset of higher order contributions. One can however only conclude in this instance that it is plausible that the ‘unstable mode’ is an infrared artifact. Its eventual cancellation in mean-field perturbation theory requires presently impossible feats of calculation. The inclusion of correlations in the self-interacting gauge field is nonetheless essential, as the action principle indicates.

As a final corollary to the dynamical analogy, it is interesting to speculate on the role of the Chern-Simons action in the infrared limit of field theory. Recent work shows that a Chern-Simons term involving a fictitious gauge field can be used to effect so-called hard thermal loop resummations in QCD\(^{38}\). The Chern-Simons term in this reference has essentially the form

\[
L_{CS} \sim \frac{1}{2} \mu (a_+ \partial a_- - a_- \partial a_+). \tag{99}
\]

This is of the general canonical form \(p\dot{q}\). In reference \(^{38}\) the fictitious gauge field is composed of essentially the pair \(A_\mu\) and \(\frac{\delta}{\delta A_\mu}\) which are the natural conjugate variables in the dynamical analogy of the renormalization group in section 3 if the derivative (dot) is the renormalization scale rather than the true time. At equilibrium the true time plays no role as is evident in the imaginary time formalism. One wonders from the structure in section 3 whether the Chern-Simons term can simply be regarded as the generator of a renormalization group transformation which induces a gauge invariant mass for the gauge field. The linear form of the Lagrangian is strongly suggestive of such an interpretation. This will be pursued elsewhere.

### 6 Non-local sources

Referring to the equations of motion \((\mathbf{38}-\mathbf{40})\), it is apparent that one might extract the field dependence implicit in the sources by rewriting their expectation values as field operators multiplying new sources. This would lead to the treatment of the conjugate problem in place of the finite renormalization problem\(^{13}\). The contact with renormalization theory becomes more remote in such a scheme, but the separation of the problem in terms of field operators is logical in the sense that the need for extra formal apparatus disappears.

From \((\mathbf{38}-\mathbf{40})\), the sources can be formally expanded in a Taylor series in powers of the field, leading to a set of relations of the form

\[
\frac{\delta^\alpha J_a}{\delta \phi_{b_1} \cdots \delta \phi_{b_n}} \bigg|_{\phi=0} = \alpha_1 \delta(x_{b_1}, x_{b_2}, \ldots, x_{b_n}) + \cdots + g_n(x_{b_1}, x_{b_2}, \ldots, x_{b_n}) \tag{100}
\]

where the dots indicate all possible mixtures of delta functions with non-diagonal ‘matrices’ \(g_n(x, x', x'' \ldots)\). The existence of non-zero \(g'\)s follows from the non-locality of the functional integral. This arises both from its description
in terms of multiple integrals over spacetime quantities and also through its
dependence on the Feynman boundary conditions for propagators which depend
on the field at widely separated points. For example
\[
\frac{\delta J_m}{\delta \phi(x) \delta \phi(x')} \bigg|_{x=0} = \alpha \delta(x, x') + g(x, x').
\] (101)

One observes from the action principle that the sources can be generated di-
rectly from an operator valued kernel \( K \) on replacing \( S_{\text{tot}} = S + K \), such that
in the condensed notation,
\[
S_{\text{tot}}[\Phi] = S[\Phi] + \Lambda + J^a(x) \Phi_a(x) + \frac{1}{2!} \Phi^a(x) J_{ab}(x, x') \Phi^b(x') + \frac{1}{3!} J^{abc}(x, x', x'') \Phi_a(x) \Phi_b(x') \Phi_c(x'') + \frac{1}{4!} J^{abcd}(x, x', x'', x''') \Phi_a(x) \Phi_b(x') \Phi_c(x'') \Phi_d(x''') + \ldots
\] (102)
and the example considered is of a scalar field theory. Note that the opera-
tor valuedness is only of the quantum type and has no direct connection with
renormalization or quasi-states. This effect of this rewriting is to generalize
the notion of sources from linear to non-linear response in the manner sug-
gested by Dahmen and Jona-Lasinio[17], and computed by Cornwall, Jackiw
and Tomboulis[18]. Given a particle interpretation, the sources are \( n \)-particle
sources; they bear no one to one relation to the old ones.

The effective action for the theory in this form is straightforwardly gener-
ated. Defining
\[
\frac{\delta W[J]}{\delta J^a} = \overline{\Phi}_a
\]
\[
\frac{\delta W[J]}{\delta J^{ab}} = \frac{1}{2} [\overline{\Phi}_a \overline{\Phi}_b + G_{ab}]
\]
\[
\frac{\delta W[J]}{\delta J^{abc}} = \frac{1}{3!} [\overline{\Phi}_a \overline{\Phi}_b \overline{\Phi}_c + 3 \overline{\Phi}_a (G_{bc}) + G_{abc}]
\]
\[
\frac{\delta W[J]}{\delta J^{abcd}} = \frac{1}{4!} [\overline{\Phi}_a \overline{\Phi}_b \overline{\Phi}_c \overline{\Phi}_d + 6 \overline{\Phi}_a (G_{bc}) \Phi_d + 4 \overline{\Phi}_a (G_{bcd}) + G_{abcd}]
\] (103)
and performing the Legendre transformation as before,
\[
\Gamma[\overline{\Phi}_a, G_{ab}, G_{abc}, G_{abcd}] = W[J] - J^a \overline{\Phi}_a - \frac{1}{2} J^{ab} [\overline{\Phi}_a \overline{\Phi}_b + G_{ab}]
\]
- \( \frac{1}{3!} J^{abc} [\overline{\Phi}_a \overline{\Phi}_b \overline{\Phi}_c + 3 \overline{\Phi}_a (G_{bc}) + G_{abc}] \)
- \( \frac{1}{4!} J^{abcd} [\overline{\Phi}_a \overline{\Phi}_b \overline{\Phi}_c \overline{\Phi}_d + 6 \overline{\Phi}_a (G_{bc}) \Phi_d + 4 \overline{\Phi}_a (G_{bcd}) + G_{abcd}] \)

yields the equations of motion
\[
\frac{\delta \Gamma}{\delta \overline{\Phi}} = -J_a - J_{ab} \overline{\Phi}_b - \frac{1}{2} J_{abc} (G_{bc} + \overline{\Phi} \overline{\Phi} \overline{\Phi}) - \frac{1}{6} J_{abcd} (G_{abcd} + 3 G_{(bc)} \overline{\Phi} + \overline{\Phi} \overline{\Phi} \overline{\Phi} \overline{\Phi})
\]
\[
\frac{\delta \Gamma}{\delta G^{ab}} = -\frac{1}{2} J_{ab} - \frac{1}{2} \frac{\phi}{\phi} J_{abc} - \frac{1}{4} \frac{\phi}{\phi} J_{abcd}
\]
\[
\frac{\delta \Gamma}{\delta G^{abc}} = -\frac{1}{6} J_{abc} - \frac{1}{6} \frac{\phi}{\phi} J_{abcd}
\]
\[
\frac{\delta \Gamma}{\delta G^{abcd}} = -\frac{1}{24} J_{abcd}
\]

(105)

and the respective Taylor coefficients

\[
\frac{\delta^2 \Gamma}{\delta \phi^2} \bigg|_{\phi=0} = m^2 \delta(x_a, x_b) + \frac{\lambda}{2} G_{ab} - \frac{1}{2} J_{abc} G_{cd}
\]
\[
\frac{\delta^3 \Gamma}{\delta \phi^3} \bigg|_{\phi=0} = -J_{abc}
\]
\[
\frac{\delta^4 \Gamma}{\delta \phi^4} \bigg|_{\phi=0} = \lambda \delta(x_a, x_b, x_c, x_d) + \frac{3 \lambda^2}{2} G_{abcd} - J_{abcd}.
\]

(106)

The latter relations imply that

\[
\frac{\delta^2 \Gamma}{\delta \phi^2} \bigg|_{\phi=0} = m^2 \delta(x_a, x_b) + \frac{\lambda}{2} G_{ab} + 2 \frac{\delta \Gamma}{\delta G_{ab}} - \frac{1}{2} G_{cd} \frac{\delta^4 \Gamma}{\delta \phi^4} \bigg|_{\phi=0}
\]
\[
\frac{\delta^3 \Gamma}{\delta \phi^3} \bigg|_{\phi=0} = 3 \frac{\delta \Gamma}{\delta G_{abc}}
\]
\[
\frac{\delta^4 \Gamma}{\delta \phi^4} \bigg|_{\phi=0} = \lambda \delta(x_a, x_b, x_c, x_d) + \frac{3 \lambda^2}{2} G_{abcd} + 4 \frac{\delta \Gamma}{\delta G_{abcd}}.
\]

(107)

which illustrates nicely the triality between \( G^{(n)} \), \( \frac{\partial \Gamma}{\partial \phi} \) and \( \frac{\delta \Gamma}{\delta G^{(n)}} \). The new source terms \( J^{(n)} \equiv J^{a_1...a_n} \) generate non-local counterterms \( R_{ij} \xi^\alpha \) and the preceding methodology is apparent in a modified form. See (63) and (138). In particular, if one considers the action principle for variations with respect to some common generator \( \xi^\alpha = \mu^2 \), one has

\[
\delta \langle 0 \ldots \infty | 0 \ldots - \infty \rangle = \langle 0 \ldots \infty | \delta \left( S + \Lambda + J^a \Phi_a + \frac{1}{2} \Phi^a J_{ab} \Phi_b + \ldots \right) | 0 \ldots - \infty \rangle
\]

(108)

which, using the equations of motion (105), should lead to the corresponding renormalization group equation in this new scheme

\[
\left[ \frac{\partial}{\partial \mu} - \beta_i^{(n)} \frac{\delta}{\delta G^{(n)}} \right] \Gamma = 0.
\]

(109)

Where \( \beta_i^{(n)} = \frac{\partial G^{(n)}}{\partial \mu} \), \( G^{(1)} = \bar{\phi} \). This result is what one would naively expect on differentiating a function of \( N \) \( n \)-point functions. On performing the variation however, one obtains

\[
\left[ \frac{\partial}{\partial \mu} - \beta^a_{ab} \frac{\delta}{\delta \phi^a} - \beta_{G,ab} \frac{\delta}{\delta G_{ab}} - \beta_{G,abc} \frac{\delta}{\delta G_{abc}} - \beta_{G,abcd} \frac{\delta}{\delta G_{abcd}} + A(\bar{\phi}^a, G^{ab} \ldots) \right] \Gamma = 0
\]

(110)
where,

\[
A = 4\beta^a \phi G^{abcd} \frac{\delta \Gamma}{\delta G^{abcd}} + \beta^b \phi (\frac{\delta \Gamma}{\delta G^{ab}} + 6 \phi \frac{\delta \Gamma}{\delta G^{abc}} - 13 \phi \frac{\delta \Gamma}{\delta G^{abcd}}) \tag{111}
\]

The anomalous term \( A \) arises due to the truncation of the series of sources at some level. The interpretation of the result is as follows. When \( G^{(n)} \equiv G^{(i_1 \ldots i_n)} \) is the exact solution to the equation \( \frac{\delta \Gamma}{\delta G^{(n)}} = 0 \), the invariance of the transformation function follows automatically from the vanishing of the sources in \( \Gamma \). Suppose now that \( G^{(n)} \) is not an exact solution; this means that the source is non-zero, implying in turn that an effective non-zero counterterm is in the theory in order to satisfy \( \Gamma \) exactly. When the result is exact there is clearly no computational advantage in adding finite counterterms. The normal situation is that all of the \( G^{(n)} \) are approximations, in which case one obtains a renormalization group type equation relating equivalent theories to vertex rescalings. It is worth noting that the truncation of this series at quadratic order corresponds to a self-consistently defined mass counterterm, but no quartic coupling counterterms. While this is clearly the most important contribution for inhibiting infra-red divergences, it does not take into account the fact that the other couplings in the theory are also effectively altered by higher order corrections. This deficiency will be most apparent in models where the ratio of two couplings determines some crucial physical property; the phase transition in a gauge theory is an example of this.

One notes briefly that to extend the discussion to encompass Abelian gauge theories, it is necessary to consider mixed sources in an arbitrary gauge. The discussion in section 5 indicates that the sources must take the general form

\[
S_{\text{source}} = \int dV_x dV_{x'} \left\{ \frac{1}{2} \Phi^\dagger(x) \Delta(x, x') \Phi(x') + \Phi(x) C^{\mu}(x, x') A_{\mu} + \Phi^\dagger(x) C^{\mu}(x, x') A_{\mu}(x') + \frac{1}{2} A^{\mu}(x) J_{\mu \nu}(x, x') A^{\nu}(x') \right\} \tag{112}
\]

Under a gauge transformation \( \Phi \rightarrow e^{i\theta(x)} \Phi(x), \ A_{\mu} \rightarrow A_{\mu}(x) + \partial_{\mu} \theta(x) \), one finds that

\[
\frac{\delta S}{\delta \theta(x)} = \int dV_x \left\{ i C^{\mu}(x, x') (A_{\mu}(x') + \partial_{\mu} \theta(x')) - i C^{\mu}(x, x') (A_{\mu}(x) + \partial_{\mu} \theta(x')) \right. \\
- i \theta(x') \bar{x} \partial_{\mu} C^{\mu}(x, x') + i \theta(x) \bar{x} \partial_{\mu} C^{\mu}(x, x') \\
- 2 \bar{x} \partial_{\mu} J^{\mu \nu}(x, x') A_{\nu}(x') + 2 \partial_{\mu} \bar{x} J^{\mu \nu}(x, x') \theta(x') \right\} \tag{113}
\]

and thus gauge invariance requires

\[
C^{\mu}(x, x') = C^{\mu}(x, x') \]
\[
\bar{x} \partial_{\mu} J^{\mu \nu}(x, x') = 0 \tag{114}
\]

where all the objects are symmetrical in \( x, x' \). While \( C^{\mu}(x, x') = 0 \) is a solution to the gauge invariance condition, it is clear from the preceding section that the \( C^{\mu} \)'s play an important role in maintaining the structure of the gauge coupling.
A note is in order concerning the relationship between the renormalization group and the present method. Since the renormalization group can only reorganize information is already in the loop expansion, it must be completely equivalent to a resummation in the final analysis. Differences are nonetheless apparent. The renormalization group focusses on the couplings of a theory in a democratic way, while most resummation schemes favour effective masses for practical reasons. On the other hand, the beta-functions used in the RG analysis are normally derived from low order results and thus the validity of conclusions arrived at in the RG is limited by the extent to which the perturbation series can be trusted. Once does not escape the bounds of self-consistency. Both methods resum only particular terms which fit their formal structure – there is no guarantee that the terms excluded from such privileged resummations are not important, so this must be argued independently. The method described in section 4 has the distinct advantage of demonstrating the formal relationship between the two methods, but as long as non-perturbative calculation demands a certain cunning, it will be important to cross check the different approaches.

7 Open dynamical systems and non-equilibrium

In a many particle theory one is naturally led to consider the effect of disturbances which drive the field into a state of disequilibrium\(^5\). Such a disturbance may lead to one of two formally distinct situations: (i) homogeneous, time-dependent energy distributions, (ii) inhomogenous, time-dependent mass-energy distributions. In a closed system one may only have the latter type, since energy cannot escape, it can only be redistributed about the system. In the absence of external forces or sources a system returns, by means of collision and random motion to a homogeneous, time-independent state within some characteristic relaxation time \(\tau\). If the initial state is inhomogenous, this decay process involves transport.

In recent work (see for example [25, 31, 28]) time dependent coupling constants have been used to model non-equilibrium systems. It is interesting in the light of the dynamical analogy presented in this paper to ask whether such a time variation of couplings might simply be regarded as a renormalization group flow enacted in real time. This has been tacitly assumed by some authors. To answer this question it is important to understand that a system with time-dependent couplings represents at best a phenomenological description of a physical system in the same way that renormalized coupling constants are phenomenological in the sense that they simply absorb the effect of virtual interactions into ‘effective’ parameters. Thus one is interested in knowing: to what extent is a theory with time dependent coupling constants simply a time dependent rescaling of a certain renormalizable theory?

The answer to this question depends on the extent to which the time variation of the coupling constants can be thought of as describing a symmetry of the system. The answer is evidently that, if the values of the couplings are governed by a Hamiltonian, then the answer to the above question must be yes.

\(^5\)For recent work with reference to various approaches, see ref \([26, 27, 28]\).
However, this is dependent on the way in which the notion of time-dependent couplings is used. One could imagine simply specifying that all the couplings should increase linearly with time – but this would not be a local physical system in the normal sense. In systems possessing a conserved current one cannot \textit{a priori} expect to simply substitute time dependent couplings into the action with impunity. General invariance under symmetry transformations will dictate the class of variations of the action for which the field equations and evolution generators preserve those symmetries and can therefore be considered physical\[24\].

To summarize: in a closed system one is guided by principles of energy conservation, gauge invariance and other conservation laws, but in an open system any one of these bastions of principle can be locally violated. Indeed it is often stated that Lagrangian or Hamiltonian descriptions of open systems do not exist, though this is an exaggeration since certain open systems can in fact be regarded as constrained systems\[24\] if there is sufficient symmetry on which to intuit legal behaviour. The present discussion will be restricted to those limited cases in which a Lagrangian methodology is appropriate.

Consider the closed total system,

\[
S_{\text{tot}} = S_1(x_1) + S_{12}(x_1, x_2) + S_2(x_2)
\]

(115)

composed of two subsystems which interact through the contact term \(S_{12}\). \(S_{12}\) serves as a source both supplying perturbations from \(S_2\) to \(S_1\) and the backreaction of \(S_1\) on \(S_2\). In thermal field theory, \(S_2\) is often identified with a heat bath which is sufficiently large that this backreaction problem can be neglected in a first approximation. Since \(S_{\text{tot}}\) is closed one may write

\[
\frac{d\langle H_{\text{tot}} \rangle}{dt} = 0, \quad \frac{\delta S_{\text{tot}}}{\delta \theta_i(x)} = 0 \quad \ldots
\]

(116)

for symmetry generators \(\theta_i(x)\) and \(x\) covers the whole system. However, for the subsystems in general\[1\]

\[
\frac{d\langle H_a \rangle}{dt} \neq 0, \quad \frac{\delta S_a}{\delta \theta_i(x)} \neq 0.
\]

(117)

In the above notation this involves a restriction to \(S_1\). The effect of \(S_2\) on the system is then modelled by the time variation of the coupling constants in \(S_1\). This bears of course the additional assumption that such a division leads to a phenomenological description in terms of these couplings. This point conceals an important and fundamental ambiguity which will be discussed below.

---

\[6\] This lesson applies also to an isolated unitary field theory. If one truncates part of the system by working to some order in perturbation theory, the perturbative result will behave like a partial system and will exhibit apparent dissipative characteristics. The remainder of the uncalculated terms behave as the external part. The issue of gauge fixing dependence is also known to be related in some cases to the truncation of the generating functional at some perturbative order. The question of sources and conservation of degrees of freedom is also connected with ghosts and unitarity\[2\].
The physical conservation laws of the reduced system are not those of the total system, but they must lead internally to a well defined set of observables in $S_1$. It is noteworthy that, if one assumes the existence of time dependent couplings without explicitly introducing a physical source which generates them, then they may also have to be spatially dependent in order not to violate conservation laws. The analogy with polarons with interaction-induced position dependent masses is evident; in this case the interaction with the local system acts as a source. If one insists on purely time dependent couplings then conservation laws can only be upheld if they represent true fields in an extended Hamiltonian, or if one explicitly introduces external sources which generate the change.

An elementary example in which a difficulty arises is in a leaky gauge theory such as might be obtained by connecting a closed system to a battery (an external system) using leads (a contact term). Consider the minimal coupling of the gauge field to a current $J_\mu$,

$$S = \int dV_x J^\mu A_\mu. \quad (118)$$

Let the integral over $x$ contain some boundary $\Sigma$ which is permeable to $J^\mu$. The system is open, but the observable coordinates $x_1$ are restricted to the interior of the boundary. Under a gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu \theta$,

$$\delta S_1 = \int dV_{x_1} \{-(\partial^\mu J_\mu)\theta + \partial^\mu (J_\mu \theta)\}. \quad (119)$$

Since the current is not conserved within the domain of $x_1$, this quantity is non-vanishing in the vicinity of the boundary $\Sigma$. It should be clear that the source of the deficiency is in the construction of the theory and not in its calculation. The only solution is to postulate the remainder of the system (in this case the battery) on the basis of conservation arguments or to include a dissipative ‘integrating factor’ in the Lagrangian, which fixes the formal consistency at the expense of the introduction of new variables. If one insists on describing a partial system with time dependent couplings, difficulties in the construction of non-equilibrium systems cannot be entirely eschewed: they involve a fundamental ambiguity in the manner in which the partitioning of the subsystems is achieved.

The action principle for the direct computation of statistical expectation values at equilibrium or otherwise, has been given by Schwinger and leads to quantities of the form

$$\langle t_2 | \mathcal{O}(t_1) | t_2 \rangle$$

where $| t_2 \rangle$ describes some macrostate of the system. Since the time-dependence of the operator is now an issue, it matters that the bra and ket states be described by the same basis. Then gives the expectation value of the operator $\mathcal{O}$ at time $t_1$ given that the system was in the prescribed state $| t_2 \rangle$ at $t_2$. Now

$$\langle t_2 | \mathcal{O}(t_1) | t_2 \rangle = \sum_{i,i'} \langle t_2 | i \rangle \langle i | \mathcal{O}(t_1) | i' \rangle \langle i' | t_2 \rangle \quad (121)$$
which involves the mutually conjugate forms of the action principle
\begin{align*}
\delta \langle t_2| i \rangle &= i \langle t_2| \delta S|i \rangle \\
\delta \langle i| t_2 \rangle &= -i \langle i| \delta S|t_2 \rangle.
\end{align*}
(122)

Since the time-dependence of an operator advances by the unitary rule \( U O U^\dagger \), for which one may solve
\begin{align*}
U(t_2, t_1) = T \exp(-i \int_{\Sigma_2}^{\Sigma_1} J' \Phi),
\end{align*}
for spacelike hypersurfaces \( \Sigma_a \), the expectation value with respect to equal in-out states involves both time-ordered and anti-time-ordered transformation amplitudes. Both forms are available through (122) and the complete expression can be written down without reference to the individual amplitudes or their conjugates:
\begin{align*}
\delta \langle t_2| t_2 \rangle = i \langle \int_{\Sigma_2}^{\Sigma_1} dt \left( L^+ \phi \mathfrak{R} + L^- \phi \mathfrak{L} \right) \rangle.
\end{align*}
(123)

The result generalizes for other expectation values.

Whilst there is no ambiguity in the usage of ref. [22], issues of definition inevitably haunt the fringes of the calculational procedure for open systems. The situation of interest here is that in which the phenomenological couplings are time- or spacetime-dependent. Appealing to the discussion in section 4, one understands that the couplings are themselves then fields. Indeed, this is no longer a device but a phenomenological ‘reality’. In particular one must understand how to define \( \langle t_2| t_1 \rangle \) when, in passing from \( t_1 \) to \( t_2 \) or vice-versa, the values of the couplings have changed. While in certain adiabatic circumstances it might be possible to neglect this change, in general the elevation of this remark to a precise statement is the source of the fundamental ambiguity in the construction of a phenomenological model of an open system. In the partitioning of a system into subsystem \( S_1 \) and ‘reservoir’ \( S_2 \) one is led to consider the effect of \( S_1 \) on \( S_2 \), but often neglecting the effect that such change has on the \( S_1 \) – the so-called back-reaction problem. Often one is interested in the case where \( S_1 \) is a large reservoir so that the back-reaction is negligible, nevertheless it is this neglect of the back-reaction which gives the illusion of dissipation. To take account of this backreaction in a phenomenological system, it is necessary to complete the logical system in some way. Since in general one does not possess any detailed knowledge about such a reservoir, it is necessary to do this in a fairly arbitrary manner. Consider the action
\begin{align*}
S_s = \int dV_x \left\{ \frac{1}{2} (\partial^\mu \phi)(\partial_\mu \phi) + \frac{1}{2} m^2 \phi^2 + \xi R(x) \phi^2 + \frac{\lambda}{4!} \phi^4 \right\}
\end{align*}
(126)
where \( R(x) \) might be regarded as the extrinsic curvature of some lower dimensional subsystem embedded in a higher-dimensional space. (This could pertain to excitations trapped in a boundary layer, for instance\[29\].) Let \( R(x) \) satisfy the phenomenological equation

\[
(-\Box + \omega_0^2) R(x) = J_{R\phi}.
\] (127)

If the source \( J_{R\phi} \) is vanishing in (127), an asymmetry is introduced into the formulation. From (126) it is evident that \( R(x) \) is a source for \( \phi^2 \), however the converse is only true if \( J_{R\phi} \neq 0 \) in (127). The latter corresponds to neglecting the back-reaction of the system defining \( R(x) \). It is clearly incorrect to neglect this change in a physical system: even though it may be small in real terms, it has an important formal function. Internal consistency requires that \( R(x) \) satisfy an equation of motion which is compatible with the remainder of the system. It must be “on-shell”. A more realistic approach is to rewrite (126) and (127) to give the local form

\[
S \rightarrow S + \int dV_x \left\{ \alpha R(-\Box + \omega_0^2)R \right\}
\] (128)

where \( \alpha/\xi \) measures the relative strength of the contact. The implication is that a time dependent effective mass is not a complete description in itself, but a better simulation of a physical system is obtainable on including the kinematical development through a source coupling as in a local theory. Clearly this theory is different from the one in which one ignores changes dependent changes in \( R(x) \), but it shares more of the characteristics of a fundamental system than the alternative. As emphasized by Schwinger, sources play an important conceptual role. A source may either be fictitious, as a generator of some transformation, or physical in the manner of an external generalized force. In an open system, this distinction is blurred.

Given a phenomenological open system, one must proceed to calculate it. Infrared behaviour must be regularized in an analogous way to that in section 4. Renormalized couplings must now be defined at a given energy and at a given time (this distinction is no longer clear). The previous methods in section 4 can be imported. Sources for the couplings have an exactly equivalent role and the action principle takes on the form

\[
\delta\langle t_2|t_2\rangle = i\langle t_2|\delta\tilde{S}|t_2\rangle
\] (129)

where

\[
\tilde{S} = \int_{\Sigma_1}^{\Sigma_2} \left\{ \frac{1}{2} q^i D_{ij} q^j + \ldots \right\}
\] (130)

and \( q^i = \{ \Phi_+, \Phi_-, m_+^2, m_-^2, \ldots \} \). \( D_{ij} \) is a generalized derivative operator, which encodes the appropriate signs in (124). For the couplings, this must be determined from their equations of motion. If not all the \( q^i \) are true degrees of freedom in the reduced system, there might be subsidiary conditions to satisfy; in the above covariant form the problem is then reminiscent of a gauge theory. The counterterms in \([63]\) etc. are now matrix valued and may contain terms...
relating to dissipation if one uses a model in which back-reaction is neglected. These two will contribute to screening infra-red divergences. The method of Lawrie[31] appears to coincide with that described here but neglects the back-reaction problem discussed above; it provides an example of the computation of a mass counterterm when correlations between the time dependent masses are not coupled to the subsystem minimally, that is, when the subsequent development of the system has no effect on the effective mass itself. A further pleasing demonstration can be found in ref [30]. The calculations of this paper will not be repeated here, but it is useful to discuss the result from the present perspective. The authors consider particle production (vacuum dissipation) due to an isolated single-particle source. Starting with the action (in the original conventions)

$$S = \int dV \left\{ \frac{1}{2} (\partial^\mu \phi) (\partial_\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{1}{6} g \phi^3 - J \phi \right\},$$

(131)

the closed time path effective action can be written

$$\Gamma[\phi_+, \phi_-] = -i \ln \int D\varphi_+ D\varphi_- \exp(iS[\phi_+ + \varphi_+] - iS[\phi_- + \varphi_-] - \varphi_+ \Gamma[\phi_+],_+- \varphi_- \Gamma[\phi_-],_-).$$

(132)

Whereas the usual effective action can often be interpreted literally as an effective action, the present object has no immediate interpretation. Only its variation with respect to one of the arguments can be interpreted after setting the + fields and sources equal to the − ones. From (12) one has effective field equations for the physical expectation value

$$\Gamma[\phi_+, \phi_-],_{i+} \bigg|_{+-} = \langle t_2 | S_{i}(t_1) | t_2 \rangle,$$

(133)

if the functional differentiation is with respect to $\phi_+(t_1)$. The linearized field equations to order $g^2$ were deduced to be of the form

$$(\square + m^2 + g^2 Z) \phi(x) = J(x)$$

(134)

for some complex operator $Z$. The source term in this expression plays an analogous role to the time-varying mass in the previous example. It acts as a source for the expectation value of the field. Since no source-source correlations are included, there is no backreaction on the source and an imaginary part is developed due to the non-conservation of the incomplete problem. This is, by the discussion in previous sections, equivalent to the claim by the authors that higher non-local sources will rectify the source of the dissipation. Some examples will be presented in a separate publication.

A final comment is in order concerning the infrared problem in non-equilibrium systems. In the present formalism, the cure proceeds by direct analogy with the previous discussion, prior to taking the limit $+ \rightarrow -$ in the auxiliary fields. The essential difference is the matrix valued nature of the propagators. All of the previous limitations and cautions apply. More explicit calculations for open systems will be deferred until a later publication.
8 Concluding remarks

The action principle, as given by Schwinger, is an elegant way of obtaining amplitudes and expectation values in field theory. Emphasis is placed on the role of unitary or canonical transformations. By introducing sources for the couplings, renormalization group transformations can be written in a form which is obtainable from the action principle. The structure one generates in this way enables familiar results from renormalization group analysis to be obtained through self-consistent equations for the propagators. This is strongly reminiscent of the method of non-local sources and can be related to it by a reparameterization.

The present formalism shows that phenomenological models with time-dependent couplings can be thought of as realtime enactments of renormalization group transformations only if the time variation is in accordance with a generalized Hamiltonian for both the fields and couplings. In other instances such models must be regarded as open systems and one cannot expect them to respect conservation laws or their associated symmetries.

As remarked in section 4 and demonstrated in reference [38], the Chern-Simons Lagrangian has a symplectic form which leads to the thermal mass resummation in the the high temperature infrared regime of QCD. It is interesting to speculate as to whether a more general connection with infrared behaviour can be established for gauge theories. The present dynamical analogy suggests that the Chern-Simons action is simply the generator of a renormalization group transformation, but a more careful investigation is needed to formalize a connection. This and other issues will be considered in subsequent work.

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This paper is dedicated warmly to the memory of Julian Schwinger.

A Regularized $\lambda \phi^4$ interaction

Consider the Euclideanized action

$$S = \int dV_x \left\{ \frac{1}{2} (\partial^\mu \varphi)(\partial_\mu \varphi) + \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4!} \varphi^4 \right\}$$

(135)
To one loop order, the effective potential is given by

\[ V_{\text{eff}} = \frac{1}{2} m^2 \Phi^2 + \frac{\lambda}{4!} \Phi^4 + \frac{(m^2 + \frac{\lambda}{2} \Phi^2)^2}{64\pi^2} \left[ \ln \left( \frac{m^2 + \frac{\lambda}{2} \Phi^2}{\mu^2} \right) - \frac{\alpha}{2} \right] \]  

(136)

where \( \alpha \) is a divergent constant, the details of which vary for different regularization schemes. The following (re)normalization conditions are imposed:

\[ V(0) = 0 \]  

(137)

\[ \frac{\partial^2 V_{\text{eff}}}{\partial \phi^2} \bigg|_{\phi = \Phi_m} = m^2 + \frac{\lambda}{2} \Phi_m^2 \equiv M^2 \]  

(138)

\[ \frac{\partial^4 V_{\text{eff}}}{\partial \phi^4} \bigg|_{\phi = \Phi_L} = \lambda \]  

(139)

Let \( m^2 \to m^2 + \delta m^2 \) and \( \lambda \to \lambda + \delta \lambda \). The above conditions now fix the counterterms. Specifically, to leading order, one finds

\[ \delta \lambda = \frac{3\lambda^2}{32\pi^2} \left[ \ln \left( \frac{m^2 + \frac{\lambda}{2} \Phi^2}{\mu^2} \right) - \frac{\alpha}{2} \right] + \frac{9\lambda^2}{64\pi^2} - \frac{\lambda^4 \Phi^4}{32\pi^2 (m^2 + \frac{\lambda}{2} \Phi^2)^2} + \frac{3\lambda^3 \Phi^2}{16\pi^2 (m^2 + \frac{\lambda}{2} \Phi^2)^2} \]  

(140)

\[ \delta m^2 = \left[ -\frac{\lambda \Phi^2}{32\pi^2} - \frac{\lambda M^2}{32\pi^2} + \frac{3 \lambda^2 \Phi^2}{64\pi^2} \right] \left[ \ln \left( \frac{m^2 + \frac{\lambda}{2} \Phi^2}{\mu^2} \right) - \frac{\alpha}{2} \right] - \frac{3 \lambda^2 \Phi^2}{64\pi^2} \]  

(141)

For the investigation of critical phenomena one is interested in the massless limit \( m^2 = 0 \), whereupon the effective potential becomes

\[ V_R(\Phi) = \frac{\lambda}{4!} \Phi^4 + \frac{9 \lambda \Phi^2}{128\pi^2} \Phi^2 + \frac{3 \lambda^2 \Phi^2}{128\pi^2} \ln \left( \frac{\Phi^2}{\Phi_m^2} \right) + \frac{\lambda^2 \Phi^2}{256\pi^2} \left[ \ln \left( \frac{\Phi^2}{\Phi_L^2} \right) - \frac{25}{6} \right] \]  

(142)

There is no particular loss from taking \( \Phi_m = \Phi_L \equiv \Phi \) and one sees that, in the limit that \( \Phi \to 0 \) one must regularize the logarithms by taking \( \Phi \sim \Phi \). This has the effect of resumming the logs to leading order and results in a potential which goes purely like \( \Phi^4 \), a result which may be shown from the renormalization group. Clearly the larger \( \Phi^2 \) becomes, the less reliable the original Taylor series expansion becomes and thus one cannot draw conclusions about the large \( \Phi \) region with any certainty. The present result only shows that there is no discontinuous change in the potential (the potential rises only positively) as the minimum moves away from the origin, implying a continuous, second order phase transition.

In \( 2 + 1 \) dimensions the unrenormalized potential is given by

\[ V_{\text{eff}}(\Phi) = \frac{1}{2} m^2 \Phi^2 + \frac{\lambda}{4!} \Phi^4 + \frac{(m^2 + \frac{\lambda}{2} \Phi^2)^2}{12\pi^2} \]  

(143)
to one loop order. Logarithmic corrections to this order are conspicuous by their absence; an arbitrary scale now lies hidden in the dimensionful couplings. This follows in turn from the absence of ultraviolet divergences although at two loops there is a divergent contribution. To show this, we note that

$$\Gamma^{(2)} = \langle S^{(4)} \rangle - \frac{1}{2} \langle (S^{(3)})^2 \rangle$$

(144)

where \( S^{(n)} \) is the part of the classical action of \( n \)-th order in the quantum fields after making the background field split \( \varphi \to \varphi + \bar{\phi} \). The operator product average may be defined by

$$\langle F[\varphi] \rangle = \frac{\int d\mu[\varphi]F[\varphi]e^{-\frac{1}{\hbar}S^{(2)}}}{\int d\mu[\varphi]e^{-\frac{1}{\hbar}S^{(2)}}},$$

(145)

and only the one-particle irreducible terms survive. Using Wick’s theorem it is straightforward to show that

$$\langle S^{(4)} \rangle = \int dV_{x} \frac{\lambda}{8} \Delta(x, x) \Delta(x, x)$$

(146)

$$\langle (S^{(3)})^2 \rangle = \int dV_{x} dV_{x'} \frac{\lambda \phi^2}{6} \Delta(x, x') \Delta(x, x') \Delta(x, x')$$

(147)

The first ‘graph’ is finite after regularization

$$\Delta(x, x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 + m^2 + \frac{\lambda}{2} \phi^2} = -\frac{|m^2 + \frac{\lambda}{2} \phi^2|}{4\pi},$$

(148)

whereas the latter diverges:

$$I = \int dV_{x} dV_{x'} \Delta(x, x') \Delta(x, x') \Delta(x, x')$$

$$= \int dV_{x} \int d^n k_1 d^n k_2 \frac{\mu^{-2\epsilon}}{(2\pi)^n (k_1^2 + M^2)(k_2^2 + M^2)((k_1 + k_2)^2 + M^2)}$$

(149)

Following Collins, the denominators are combined using

$$\frac{1}{a_1 a_2 a_3} = 2 \int_0^1 d^3 z \delta(1 - z_1 - z_2 - z_3)[a_1 z_1 + a_2 z_2 + a_3 z_3]^{-3}$$

(150)

whereupon

$$I = (4\pi)^{-n} \Gamma(3 - n) \int dV_{x} (M^2)^{n-3} G(n) \mu^{-2\epsilon}$$

(151)

$$G(n) = \int_0^1 d^3 z \delta(1 - z_1 - z_2 - z_3)(z_1 z_2 + z_2 z_3 + z_3 z_1)^{-\frac{n}{2}}.$$
\( G(3) \) is evaluated on making the substitution \( z_1 = \rho x, z_2 = \rho (1 - x), z_3 = 1 - \rho \), which implements the delta-function constraint explicitly, giving

\[
G(3) = \int_0^1 \frac{\rho d\rho dx}{(\rho - \rho(1 - x + x^2))^2}
\]

(153)

Note that although the integrand is singular, the divergence is illusory and the integral itself is finite. Evaluating this integral leads to

\[
G(3) = 2 \int_0^1 \frac{dx}{\sqrt{(x - x^2)}} = 2\pi
\]

(154)

Thus, one obtains

\[
I = -\frac{1}{32\pi^2} \left[ \frac{1}{\epsilon} + \ln \left( \frac{M^2}{4\pi\mu^2} \right) + \gamma \right]
\]

(155)

where \( n = 3 + \epsilon, M^2 = m^2 + \frac{\lambda^2}{2}\phi^2 \) and \( \gamma \) is Euler’s constant. This shows that there is a single logarithm of the effective mass at two loop order.

Considering only the one-loop potential (nonetheless mindful of the two-loop logarithm), one proceeds to apply the renormalization conditions (137-139) which leads, in the massless limit, to

\[
\delta \lambda = 0 \quad \text{(156)}
\]

\[
\delta m^2 = \frac{\lambda^{3/2} \Phi_m}{4\pi\sqrt{2}} \quad \text{(157)}
\]

\[
V_R(\phi) = \frac{\lambda^{3/2} \Phi_m^2}{4!} + \frac{\lambda^{3/2} \Phi_m^2}{8\pi\sqrt{2}} \phi^2 - \frac{\lambda^2 \Phi^3}{24\sqrt{2}\pi} \phi^3.
\]

(158)

There is a subtlety here, namely that \( \lambda \) is not a dimensionless quantity, and so it is not possible to discuss its smallness directly. To this end one may introduce \( \lambda = \Phi^2_s \lambda \) where \( \Phi_s \) has dimensions of the field. The effective expansion parameter

\[
\frac{\lambda}{M} \sim \frac{\lambda^* \Phi^2_s}{\sqrt{\lambda^* \Phi^2_s \Phi^2_m}}
\]

(159)

There is clearly no particular advantage to not choosing \( \Phi_s = \Phi_m \) and one therefore has

\[
V_R(\phi) = \frac{\lambda^* \Phi^2_m \phi^4}{4!} + \frac{\lambda^* \Phi^4_m}{8\pi\sqrt{2}} \phi^2 - \frac{\lambda^* \Phi^3_m}{24\sqrt{2}\pi} \phi^3
\]

(160)

The presence of the two loop logarithm indicates that, in order to regulate infrared divergences, it will be necessary to scale \( \Phi_m \sim \phi \), giving the leading behaviour

\[
V_R \propto \left( \frac{\lambda^*}{4!} + k \lambda^* \frac{3}{2} \right) \phi^6
\]

(161)

which may be compared with the renormalization group improved potential given by Lawrie(34).
The above technique, quite useful in the scalar theory, is easily extended to encompass finite temperature states in equilibrium. Then from the imaginary time formalism one may write [35] in the massless case

\[
V = \frac{\lambda + \delta \lambda}{4!} \varphi^4 + \frac{1}{2}(m^2 + \delta m^2)\varphi^2 + \frac{1}{4} \left( \frac{\varphi^2}{4\pi} + \delta m_T^2 \right)^2 \left[ \ln \left( \frac{\varphi^2}{\mu^2} \right) - \frac{\alpha}{2} \right] + 2\beta^{-4}\pi^\frac{3}{2} \Gamma \left( \frac{3}{2} \right) \sin \left( \frac{3\pi}{2} \right) \int_\nu^\infty \frac{(u^2 - \nu^2)^\frac{3}{2} du}{\exp(2\pi u) - 1}
\]

(162)

where \(\nu^2 = \frac{\beta^2}{4\pi^2} \left( \frac{\lambda}{2} \varphi^2 + \delta m_T^2 \right)\) and \(\delta m_T^2\) is the temperature dependent part of the mass counterterm which is not of order \(\hbar\) in the sense of the loop counting parameter and therefore cannot be neglected. Dependence on \(\delta \lambda_T\) cancels in this scheme. Here it is assumed that the renormalization is performed at temperature \(T = \beta^{-1}\) and not at zero temperature. Since the renormalization is a resummation of self-energies, it is natural to implement this at the temperature of interest. Let

\[
I(\nu) = \int_\nu^\infty \frac{(u^2 - \nu^2)^\frac{3}{2} du}{\exp(2\pi u) - 1}.
\]

(163)

Appropriate renormalization conditions are now

\[
V(0) = 0
\]

\[
\frac{\partial^2 V}{\partial \varphi^2} \bigg|_{\Phi,\beta} = \frac{\lambda}{2} \Phi_R^2
\]

\[
\frac{\partial^4 V}{\partial \varphi^4} \bigg|_{\Phi,\beta} = \lambda
\]

(164)

giving a set of non-linear equations for the counterterms, whose temperature dependent part must be solved self-consistently. Although there is no rigid formal distinction between the temperature dependent and temperature independent counterterms, it is advantageous to keep these parts separate, since the \(\beta\)-independent terms contain all the divergences. One is thus led to define

\[
\delta \lambda_T = \frac{8}{3} \pi^2 \beta^{-4} \frac{\partial^4 I(\nu)}{\partial \varphi^4} \bigg|_{\Phi,\beta}
\]

\[
\delta m_T^2 = \frac{8}{3} \pi^2 \beta^{-4} \frac{\partial^2 I(\nu)}{\partial \varphi^2} \bigg|_{\Phi,\beta} - \frac{\delta \lambda_T}{2} \Phi_R^2.
\]

(165)

In a regime where \(\varphi \sim \Phi_R\), \(\delta \lambda_T\) has no non-trivial effect and only the mass counterterm is important. In a high temperature regime, it is straightforward to check that these equations solve to give the leading order behaviour \(\delta m_T^2 \sim \lambda T^2\). This may be pursued in greater detail as required [36].

**B Scalar Electrodynamics**

Consider now the use of vertex conditions for the renormalization of the gauge theory. The one loop effective potential may be computed in an invariant
gauge \textsuperscript{32} provided one uses a Cartesian parameterization of the scalar field. The results will only be given for 3 + 1-dimensions; results in 2 + 1-dimensions have previously been given in \textsuperscript{32}. The action is given by

\[ S = \int dV_x \left\{ (D^\mu \Phi)^\dagger (D_\mu \Phi) + m^2 \Phi^\dagger \Phi + \frac{\lambda}{6} (\Phi^\dagger \Phi)^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right\} \]  

(166)

which is to be parametrized in terms of \( \Phi = \frac{1}{\sqrt{2}} (\varphi_1 + i \varphi_2) \), or \( \varphi_a \) where \( a = 1, 2 \). Consider the gauge transformation \( \delta \varphi^i = R^i_a [\varphi] \delta \xi^a \) in condensed notation, where \( \varphi^i = \{ \varphi^a(x), A^\mu(x) \} \). The components of the Killing vector \( R_i[\varphi](x, x') \) may be found from

\[
\begin{align*}
\delta \varphi_\mu &= \delta A_\mu = -\partial_\mu \delta \theta(x) \\
\delta \varphi_a &= -e_{ab} \varphi_b(x) \delta \theta(x)
\end{align*}
\]

(167)

where \( \theta \) is a gauge transformation. Then

\[
\begin{align*}
R_\mu[\varphi](x, x') &= -\partial_\mu \delta(x, x') \\
R_a[\varphi](x, x') &= -e_{ab} \varphi_b(x') \delta(x, x')
\end{align*}
\]

(168)

(169)

On expanding \( \varphi_a = \bar{\varphi}_a + \varphi_a \), the appropriate gauge fixing condition becomes \( \chi_a = R_{i\alpha} \varphi^i = 0 \), or

\[
\chi(x) = \partial^\mu A_\mu - e_{ab} \varphi_b(x) (\bar{\varphi}_a(x) + \varphi_a(x)) = 0
\]

(170)

and the one-loop effective action is given by

\[
\Gamma^{(1)}[\bar{\varphi}] = -i \ln \int d\mu[\varphi_a, A_\mu] \left| \det Q \right| \delta[\partial^\mu A_\mu - e_{ab} \varphi_\mu \varphi_b] e^{\frac{1}{2} S^{(2)}}
\]

(171)

where \( Q = \Box + e^2 \bar{\varphi}^2 \) and \( S^{(2)} \) is the leading part of the action, quadratic in the quantum fields. Evaluating the functional integral, one obtains

\[
\begin{align*}
\Gamma^{(1)} &= \frac{1}{2} \text{Tr} \ln(\Box + m^2 + \frac{\lambda}{2}) + \frac{1}{2} \text{Tr} \ln(-\Box + e^2 \bar{\varphi}^2) + \frac{1}{2} \text{Tr} \ln A + \frac{1}{2} \text{Tr} \ln B \\
\{ A, B \} &= -\Box + \frac{(m^2 + \frac{\lambda}{6} \bar{\varphi}^2 + 2e^2 \bar{\varphi}^2) \pm \sqrt{(m^2 + \frac{\lambda}{6} \bar{\varphi}^2 + 2e^2 \bar{\varphi}^2) - 4e^4 \bar{\varphi}^2}}{2}
\end{align*}
\]

(172)

(173)

In spontaneous phase transitions one is specifically interested in the massless limit, whereupon the one-loop contribution to the effective potential becomes

\[
\begin{align*}
64\pi^2 \bar{\varphi}^4 V^{(1)} &= e^4 \left[ \ln \left( \frac{e^2 \bar{\varphi}^2}{\mu^2} \right) - \frac{\alpha}{2} \right] + \frac{\lambda^2}{4} \left[ \ln \left( \frac{\frac{2}{3} \bar{\varphi}^2}{\mu^2} \right) - \frac{\alpha}{2} \right] \\
&+ \left( \frac{\lambda^2}{72} + \frac{\lambda e^2}{3} + e^4 \right) \left[ \ln \left( \frac{\bar{\varphi}^2}{\mu^2} \right) + e^2 + \frac{1}{2} \sqrt{\frac{\lambda}{6} \left( \frac{\lambda}{6} + 4e^2 \right)} \right] \\
&+ \ln \left( \frac{\bar{\varphi}^2}{\mu^2} \right) + e^2 - \frac{1}{2} \sqrt{\frac{\lambda}{6} \left( \frac{\lambda}{6} + 4e^2 \right)} \right] \\
&+ \frac{1}{4} \left( \frac{\lambda}{6} + 2e^2 \right) \sqrt{\frac{\lambda}{6} \left( \frac{\lambda}{6} + 4e^2 \right)} \ln \left[ \frac{\lambda}{6} + 2e^2 + \sqrt{\frac{\lambda}{6} \left( \frac{\lambda}{6} + 4e^2 \right)} \right] \\
&+ \frac{1}{4} \left( \frac{\lambda}{6} + 2e^2 \right) \left[ \frac{\lambda}{6} + 2e^2 - \sqrt{\frac{\lambda}{6} \left( \frac{\lambda}{6} + 4e^2 \right)} \right]
\end{align*}
\]

(74)
Renormalizing, using the vertex conditions in (137-139) one obtains for the effective potential

\[ V_R = \frac{\lambda\phi^4}{4!} + \frac{3e^2 + \frac{5}{12}\lambda^2 + \frac{5}{3}\lambda^2\phi^2}{64\pi^2} \ln \left( \frac{\phi^2}{\Phi^2} \right) - \frac{25}{6} + \frac{6\Phi^2_m\phi^2}{64\pi^2} (3e^2 + \frac{5}{12}\lambda^2 + \frac{5}{3}\lambda^2\phi^2) \left[ \ln \left( \frac{\phi^2}{\Phi^2_m} \right) + 3 \right] \]

This is the result obtained by Coleman and Weinberg, up to gauge dependent terms. It is noted that, in spite of the gauge sector, the form of the effective potential is the same as that in the scalar theory, so that the leading behaviour for small \( \phi \) is apparently identical to the scalar case. The argument for the order of the phase transition also follows through identically and one concludes, erroneously, that the phase transition is of second order (continuous). The source of the error lies in the tacit assumption that the dimensionless quantities \( \lambda \sim e^2 \). This arises in neglecting the logarithm of the ratio of the couplings when going from (174) to (175) above. The important point is that the relevant mass scales are \( e^2\phi \) and \( \frac{\lambda}{2}\phi^2 \) and not merely \( \phi^2 \). The vertex condition (137), insensitively applied, gives preference to the scalar mass at the expense of the effective working gauge mass. The same problem occurs in the large-\( N \) expansion in which the gauge contribution is suppressed by \( 1/N \).

It should be remarked that the correct result can be obtained from the renormalization group provided one includes counterterms for the electric charge and for a background gauge field. Although these are formally zero in the zero momentum limit, the additional parametric dependence on the gauge sector prevents the coarse mass renormalization from washing out the effect of the gauge fields. One wonders, on the other hand, why it is necessary to work at non-zero momentum to obtain a zero momentum result. The answer is simply that one needs to renormalize the gauge sector on an equal basis with the scalar sector, but that there is no natural way to do this at zero momentum using vertex conditions as a renormalization scheme, since the beta function for the electric charge vanishes in this limit. It is sufficient to compute the effective action to quadratic order in the background fields \( \overline{\phi}, \overline{A}_\mu \). Defining for the purposes of minimal subtraction \( \overline{\sigma}_B = Z_{\overline{\phi}}^\frac{1}{2} \overline{\phi} \) and \( \overline{A}_\mu^i = Z_{\overline{A}_\mu}^\frac{1}{2} \overline{A}_\mu \), and additive counterterms for the remainder one can after lengthy calculation show that

\[ \Gamma^{(1)}_{quad} = \int dV_x \left\{ e^2 \overline{A}^\mu \overline{A}_\mu \Delta(x, x) + \frac{1}{2} e^2 \overline{\phi}^2 G_{\mu}^\mu (x, x) + \frac{\lambda}{3} \overline{\phi}^2 \Delta(x, x) + \frac{1}{\alpha} \right\} 
\]

\[ - \frac{e^2}{2} \int dV_x dV_{x'} \left\{ 2 \overline{A}^\mu \overline{A}_\mu \Delta(x, x') \overline{\partial}_\mu \overline{\partial}_\nu \Delta(x, x') - 2 \overline{A}^\mu \overline{A}_\nu \overline{\partial}_\mu \Delta(x, x') \overline{\partial}_\nu \Delta(x, x') \right\} 
\]

\[ + \frac{\overline{\phi}^2}{\alpha} G_{\mu}^\mu (x, x') \overline{\partial}_\mu \Delta(x, x') + \overline{\partial}_\mu \overline{\partial}_\nu G_{\mu}^\mu (x, x') \Delta(x, x') \]

\[ + \frac{1}{\alpha} \overline{\phi}^2 \Delta(x, x') \overline{\partial}_\mu \overline{\partial}_\nu G_{\mu}^\mu (x, x') - 2 \overline{\phi} \overline{\partial}_\nu \Delta(x, x') \Delta(x, x') \]

\[ + \frac{2}{\alpha} \overline{\phi}^2 \overline{\partial}_\nu G_{\mu}^\mu (x, x') \Delta(x, x') + \frac{2}{\alpha} \overline{\partial}_\mu \overline{\phi} \overline{\partial}_\nu \overline{\phi} G_{\mu}^\mu (x, x') \Delta(x, x') \right\} \]

(176)
\begin{align*}
\left(-\Box + m^2\right)\Delta(x, x') &= \delta(x, x') \\
\left(-\Box \delta_{\mu\nu} + (1 - \frac{1}{\alpha})\partial^\mu \partial_\nu\right)G_{\lambda}^{\nu} &= \delta^\mu_\lambda \delta(x, x') 
\end{align*}

and the pole-parts of the above products of Green functions can be extracted using dimensional regularization. See for instance the method used in \cite{37}. It is noted briefly that terms varying like negative powers of \(\alpha\) cancel as they must for gauge fixing independence in the Landau-DeWitt gauge \((\alpha \to 0)\) and terms quadratic in the background photon field cancel, confirming renormalizability. The computations are rather lengthy and will not be given here. On comparison to the quadratic counterterm action,

\begin{align*}
S^{(1)}_{CT} &= \int dV \left\{ \frac{1}{2}\delta Z_{\phi} \overline{\phi}(-\Box \overline{\phi} + \frac{1}{2}(\delta m^2 + m^2 \delta Z_{\phi})\overline{\phi}^2 + \frac{1}{2}\delta Z_{A} \overline{A}^\mu(-\Box \delta_{\mu\nu} - \partial_\mu \partial_\nu \overline{A}^\nu) \right\} 
\end{align*}

and using the relation \((e + \delta e)\delta Z_A^2 = e\), one obtains the one-loop counterterms for the gauge theory.

\begin{align*}
\delta Z_{\phi} &= -6e^2\epsilon^{-1}(4\pi)^{-2} \\
\delta Z_{A} &= \frac{2}{3}e^2\epsilon^{-1}(4\pi)^{-2} \\
\delta e &= -\frac{1}{3}e^3\epsilon^{-1}(4\pi)^{-2} \\
\delta m^2 &= = (2m^2e^2 - \frac{4}{3}\lambda m^2)\epsilon^{-1}(4\pi)^{-2} \\
\delta \lambda &= -12\left(\frac{5}{18}\lambda^2 + 3e^4 - \frac{4}{3}\lambda e^2\right)\epsilon^{-1}(4\pi)^{-2} 
\end{align*}

where \(\epsilon = n - 4\). Thus there are non-zero counterterms for the gauge sector which will lead to a renormalization group flow. The move to non-zero momentum can be construed as artificial; a better approach would be desirable. One can speculate as to whether a different scheme for finding beta functions at zero momentum would result in the appearance of fixed points associated with a first order transition.

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