String-Like BTZ on Codimension-2 Braneworlds in the Thin Brane Limit

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We consider five-dimensional gravity with a Gauss-Bonnet term in the bulk and an induced gravity term on a 2-brane of codimension-2. We show that this system admits BTZ black holes on the 2-brane which are extended into the bulk with regular horizons.

A growing interest in codimension-2 braneworlds, i.e. a brane embedded in a bulk with two extra dimensions, has recently appeared. The most attractive feature of these models is that the vacuum energy (tension) of the brane instead of curving the brane world-volume, merely induces a deficit angle in the bulk around the brane [1]. This property was used to solve the cosmological constant problem [2]. However, soon it was realized [3] that one can only find nonsingular solutions if the brane stress tensor is proportional to its induced metric. To obtain the Einstein equation on the brane one has to introduce a cut-off (brane thickness)[4], loosing the predictability of the theory, or alternatively, one can modify the gravitational action by including a Gauss-Bonnet term [5] or a scalar curvature term (induced gravity) on the brane [6].

We still lack an understanding of time dependent cosmological solutions in codimension-2 braneworlds. In the thin brane limit, because the energy momentum tensors on the brane and in the bulk are related, we cannot get the standard cosmology on the brane [7, 8]. One then has to regularize the codimension-2 branes by introducing some thickness and then consider matter on them [9, 10]. A cosmological evolution on the regularized branes requires an expanding brane world-volume and in general also a time evolving bulk. An alternatively approach was followed in [11] by considering a codimension-1 brane moving in the regularized static background. The resulting cosmology, however, was unrealistic having a negative Newton’s constant (for a review see [12]).

Moreover, the issue of localization of a black hole on the brane and its extension to the bulk is not fully understood. In codimension-1 braneworlds, a first attempt was to consider the black string extension in the bulk of a Schwarzschild metric [13]. Unfortunately, this string is unstable to classical linear perturbations [14] (for a review see [15]). Further attempts deal with the Einstein equations projected on the brane, which include an unknown bulk dependent term, the Weyl tensor projection. Due to this reason the system is not closed, and some assumptions have to be made either in the form of the metric or in the Weyl term [16]. The stability and thermodynamics of these solutions were worked out in [17].

A lower dimensional version of a black hole living on a (2+1)-dimensional braneworld was considered in [18] by Emparan, Horowitz, and Myers. They based their analysis on the so-called C-metric [19] modified by a cosmological constant term. They found a BTZ black hole [20] on the brane which can be extended as a BTZ string in a four-dimensional AdS bulk. Their thermodynamical stability analysis showed that the black string remains a stable configuration when its transverse size is comparable to the four-dimensional AdS radius, being destabilized by the Gregory-Laflamme instability above that scale, breaking up to a BTZ black hole on a 2-brane.

In the codimension-2 scenario, a six-dimensional black hole on a 3-brane was proposed in [21] and extended in [22] to include also rotations. This is a generalization of the 4D Aryan, Ford, Vilenkin [23] black hole pierced by a conical string adjusted to the codimension-2 branes, with a conical structure in the bulk and deformations accommodating the deficit angle. However, it is not clear how to realize these solutions in the thin 3-brane limit.

In this work we study black holes on an infinitely thin conical 2-brane and their extension into the five-dimensional bulk. We consider the following gravitational action in five dimensions with a Gauss-Bonnet term [24] in the bulk and an induced three-dimensional curvature term on the brane

\[
S_{\text{grav}} = \frac{M_5^3}{2} \left\{ \int d^5x \sqrt{-g^{(5)}} \left[ R^{(5)} + \alpha (R^{(5)})^2 - 4 R_{MN}^{(5)} R^{(5)MN} + R_{MNKL}^{(5)} R^{(5)MNKL} \right] \right. \\
+ r_c^2 \int d^2x \sqrt{-g^{(2)}} R^{(3)} \frac{\delta (\rho)}{2\pi b} \right. \\
+ \int d^5x L_{\text{bulk}} + \int d^2x L_{\text{brane}} \frac{\delta (\rho)}{2\pi b},
\]

(1)

where \( \alpha (\geq 0) \) is the GB coupling constant, \( r_c = M_{(3)}/M_{(5)}^3 \) is the induced gravity “cross-over” scale, \( M_{(3)} \) is the five-dimensional Planck mass, and \( M_{(3)} \) is the three-dimensional one. The above induced term has been written in the particular coordinate system in which the metric is

\[
ds_5^2 = g_{\mu\nu}(x, \rho)dx^\mu dx^\nu + a^2(x, \rho)dr^2 + b^2(x, \rho)d\theta^2,
\]

(2)

where \( g_{\mu\nu}(x, 0) \) is the braneworld metric, and \( x^\mu \) denotes three dimensions, \( \mu = 0, 1, 2 \), whereas \( \rho, \theta \) denote the radial and angular coordinates of the two extra dimensions, and we have assumed an azimuthal symmetry in the system. Capital \( M, N \) indices take values in the five-dimensional space.

The Einstein equations resulting from the variation of the action (1) are...
brane. The brane metric is given by

\[ G^{(5)N}_M + r_c^2 G^{(3)\nu} \partial_M g^N \delta(\rho) - \alpha H^N_M \]

where \( H^N_M \) is the Gauss-Bonnet contribution to the bulk equations [5]. To obtain the braneworld equations we expand the metric around the brane as \( b(x, \rho) = \beta(x) \rho + O(\rho^2) \).

At the boundary of the internal two-dimensional space where the 2-brane is situated, the function \( b \) behaves as \( b^\dagger(x,0) = \beta(x) \), where a prime denotes derivative with respect to \( \rho \). We also demand that the space in the vicinity of the conical singularity is regular, which imposes the supplementary conditions \( \partial_\rho \beta = 0 \) and \( \partial_\rho g_{\mu\nu}(x,0) = 0 \) [5].

The extrinsic curvature in the particular gauge \( g_{\mu\nu} = 1 \) we consider is given by \( K_{\mu\nu} = g_{\rho\nu} \). We will now use the fact that the second derivatives of the metric functions contain \( \delta \)-function singularities at the position of the brane. The nature of the singularity then gives the following relations [5]

\[ \frac{b''}{b} = -(1-b') \frac{\delta(\rho)}{b} + \text{non-singular terms} \]  

\[ \frac{K'_{\mu\nu}}{b} = K_{\mu\nu} \frac{\delta(\rho)}{b} + \text{non-singular terms} \]  

From the above singularity expressions and using the Gauss-Codacci equations, we can match the singular parts of the Einstein equations (3) and get the following “boundary” Einstein equations

\[ G^{(3)}_{\mu\nu} = \frac{1}{M^3_5} \left[ T^{(br)}_{\mu\nu} + \frac{2\pi(1-\beta)}{r_c^2 + 8\pi(1-\beta)\alpha} g_{\mu\nu} \right]. \]  

We assume that there is a (2+1) black hole on the brane. The brane metric is

\[ ds_b^2 = \left( -n(r)^2 dt^2 + n(r)^{-2} dr^2 + r^2 d\phi^2 \right), \]

where \( 0 \leq r < \infty \) is the radial coordinate, \( \phi \) has the usual periodicity \((0,2\pi)\) and \( l \) is the length scale of the \( AdS_3 \) space. We will look for string-like solutions of the Einstein equations (3) using the five-dimensional metric (2) in the form

\[ ds_b^2 = f^2(\rho) \left( -n(r)^2 dt^2 + n(r)^{-2} dr^2 + r^2 d\phi^2 \right) \]

\[ + a^2(r,\rho) d\rho^2 + b^2(r,\rho) d\theta^2. \]  

Since the space outside the conical singularity is regular, the warp function \( f(\rho) \) must also be regular everywhere. We assume that there is only a cosmological constant \( \Lambda_5 \) in the bulk, and we take \( a(r,\rho) = 1 \). Then combining the \((rr, \phi\phi)\) Einstein equations we get

\[ \left( \dot{n}^2 + n\ddot{n} - \frac{n\ddot{n}}{r} \right) \left( 1 - 4\alpha \frac{b''}{b} \right) = 0, \]  

while a combination of the \((\rho\rho, \theta\theta)\) equations gives

\[ \left( f'' - \frac{f' b'}{b} \right) \left[ 3 - 4\alpha \frac{f''}{f} \left( \frac{n^2}{r} + n\ddot{n} + 2\frac{n\ddot{n}}{r} + 3f'^2 \right) \right] = 0, \]  

where a dot denotes derivatives with respect to \( r \). We will consider first \( \dot{n}^2 + n\ddot{n} - \frac{n\ddot{n}}{r} = 0 \), which has as a solution the simplest BTZ black hole without charge or angular momentum [20]

\[ n^2(r) = -M + \frac{r^2}{l^2}. \]  

Then equation (10) becomes

\[ \left( f'' - \frac{f' b'}{b} \right) \left[ 1 - 4\alpha \frac{f''}{f} \left( \frac{1}{l^2} + f'^2 \right) \right] = 0. \]

From the above equation we have two cases. The first case is \( f'^2 = \frac{4\alpha}{l^2} + \frac{1}{l^2} = 0 \), which has the following solution

\[ f_1(\rho) = C_1 e^{\frac{\rho}{2\alpha}} + C_2 e^{-\frac{\rho}{2\alpha}}, \]

where \( C_1 \) and \( C_2 \) are integration constants and satisfy the relation \( C_1 C_2 = \alpha/l^2 \). The function \( f(\rho) \) is regular if and only if we require that on the position of the brane the boundary condition is \( f'(\rho = 0) = 1 \), then the integration constants can be expressed in terms of \( \alpha \) and \( l \)

\[ C_1 = \pm \frac{1 + \epsilon \sqrt{1 - 4\alpha^2}}{2}, \]

\[ C_2 = \pm \frac{1 - \epsilon \sqrt{1 - 4\alpha^2}}{2}, \]

where \( \epsilon = \pm 1 \) independently of the \( \pm \) sign in \( C_1 \) and \( C_2 \). If we impose also the condition \( \partial_\rho g_{\mu\nu}(x,0) = 0 \) we obtain \( C_1 = C_2 = 1/2 \) and the solution (13) simplifies to \( f_1(\rho) = \cosh(\rho/2\sqrt{\alpha}) \). Substituting the above solutions back to the five-dimensional equations we get a fine-tuned relation between \( \alpha \) and \( \Lambda_5 \)

\[ \Lambda_5 = -\frac{3}{4\alpha}. \]  

Because of the positivity of \( \alpha \) the five-dimensional bulk space is Anti-de-Sitter. The choices we made in solving (9) and (10) determined only the functions \( n(r) \) and \( f(\rho) \) and although they solve equations (3) they leave \( b(\rho) \) undetermined making the bulk metric arbitrary [24].

The second case is to consider

\[ f'' - \frac{f' b'}{b} = 0 \Rightarrow b(\rho) = b_0 f'(\rho). \]

Then, back into the bulk equation we notice that the \((\rho\rho)\) equation can only be solved when \( \Lambda \) takes the same value as in the first case given in (15), thus we have

\[ \left( 1 - 4\alpha \frac{f''}{f} \right) \left[ 1 - 4\alpha \frac{f''}{f} \left( \frac{1}{l^2} + f'^2 \right) \right] = 0, \]

from which we have two subcases. The first subcase is \( 1 - 4\alpha \frac{f''}{f} = 0 \), and with (15) it gives the same solution for \( f(\rho) \) as in equation (13) where \( C_1 \) and \( C_2 \) are
The size of the horizon is defined by the scale and the fact that at the position of the brane \( b(\rho = 0) = 0 \) and \( b'(\rho = 0) = \beta \) we get \( C_1 = C_2 = \pm \frac{1}{2} \) and in (16) \( b_0 = 4\alpha \beta \). Therefore, for this case \( f(\rho) \) and \( b(\rho) \) can be written as

\[
f_2(\rho) = \pm \cosh \left( \frac{\rho}{2 \sqrt{\alpha}} \right), \quad b_2(\rho) = \pm 2 \beta \sqrt{\alpha} \sinh \left( \frac{\rho}{2 \sqrt{\alpha}} \right).
\] (18)

One can check that the above solution is consistent with all bulk equations.

For the second subcase we get as in the first case the same solution for \( f(\rho) \) (Eq.(13)) with \( C_1 C_2 = \alpha/l^2 \) but the function \( b(\rho) \) is given by \( b(\rho) = b_0 f'(\rho) \). Imposing again the boundary conditions for \( f_2(\rho) \) and \( b(\rho) \) we get \( b_0 = 2\alpha \beta \) and \( C_1 = C_2 = 1/2 \). Then \( f(\rho) \) and \( b(\rho) \) are given by (18), relation (15) still holds, and we get an extra constraint \( l^2 = 4\alpha \).

These solutions extent the BTZ black hole on the brane into the bulk. Calculating the curvature invariants we find no \( r = 0 \) curvature singularity for the BTZ string-like solution as expected. The warp function \( f_2(\rho) \) gives the shape of the horizon of the BTZ string-like solution. The size of the horizon is defined by the scale \( \sqrt{\alpha} \). Using the fine-tuned relation (15) and the relation \( \Lambda_5 = -6/L^2 \), this scale can be fine-tuned to the length scale \( L \) of the five-dimensional AdS space. Then, the warp function \( f(\rho) \) is finite at the boundary of the AdS space, and depending on the integration constants of the various cases, it gives the shape of a ‘throat’ to the horizon.

There is also a constant solution for \( f(\rho) \) which we show in Table I (with \( \gamma = \sqrt{\frac{l^2 - \rho_0^2}{2}} \) and \( \Lambda = -3/l^2 \)).

We have also studied more general solutions without the restriction that \( n \) is chosen as BTZ black hole, which means that in equation (9) we chose \( 1-4\alpha \frac{\rho_0^2}{l^2} = 0 \) , from which \( b(\rho) \) is obtained as

\[
b_5(\rho) = b_0 \left( C_1 e^{\rho/2\sqrt{\alpha}} + C_2 e^{-\rho/2\sqrt{\alpha}} \right).
\] (19)

The first case is to consider from (10) the relation (16). Then we get

\[
f_5(\rho) = f_0 \left( C_1 e^{\frac{\rho}{2\sqrt{\alpha}}} + C_2 e^{-\frac{\rho}{2\sqrt{\alpha}}} \right) + C_3,
\] (20)

where \( C_1, C_2, \) and \( C_3 \) are integration constants and the bulk equations gives \( C_3 = 0 \) and the relation (15). Imposing again the boundary conditions for \( f_2(\rho) \) and \( b(\rho) \) we get \( C_1 = C_2 = \pm \frac{1}{2} \), \( f_0 = 1 \) and \( b_0 = 2\beta \sqrt{\alpha} \) we get \( f(\rho) \) and \( b(\rho) \) as in (18). The function \( n(\rho) \) remains undetermined connected with the brane matter \( T_0^1 = T_0^2 = mn' \rho - \Lambda_5, \ T_3^3 = n'^2 + n'' + \Lambda_3 \) from (6). For the second case we analyse from (10) the term

\[
3 \left( f^2 - 4\alpha f'^2 \right) = \kappa , \ 4\alpha \left( \frac{n^2}{r} + \dot{n} + \ddot{n} \right) = \kappa.
\] (21)

which give

\[
f_9(\rho) = C_3 e^{\frac{\rho}{2\sqrt{\alpha}}} + \frac{\kappa}{12C_3} e^{-\frac{\rho}{2\sqrt{\alpha}}} , \quad (22)
\]

\[
n(\rho) = \sqrt{C_5 + \frac{\kappa}{12\alpha} r^2 + \frac{\kappa}{r}} . \quad (23)
\]

Imposing that \( f^2(\rho = 0) = 1 \) we get for the function \( f(\rho) \) as in the first case of the BTZ solution

\[
f_9(\rho) = C_3 e^{\frac{\rho}{2\sqrt{\alpha}}} + C_4 e^{-\frac{\rho}{2\sqrt{\alpha}}} , \quad (24)
\]

where

\[
C_3 = \pm 1 + \frac{\varepsilon}{2} \sqrt{\frac{l^2 - \rho_0^2}{2}} , \quad C_4 = \pm 1 - \varepsilon \sqrt{1 - \frac{\varepsilon^2}{2}} \ . \quad (25)
\]

Moreover, imposing the boundary conditions for \( b(\rho) \) its solution is given by equation (18). If we impose also the condition \( \partial_\rho g_{\mu\nu}(x, 0) = 0 \) then \( \kappa = 3 \) and the solution (24) simplifies to \( f_9(\rho) = \cosh(\rho/2\sqrt{\alpha}) \). These solutions represent BTZ-corrected black string with the usual \( r = 0 \) curvature singularity. If we redefine \( C_5 = -M, \ C_6 = -\zeta \) we use \( l^2 = 4\alpha \), we get the BTZ black hole solution with a short distance correction term which corresponds to the BTZ conformally coupled to a scalar field [25]

\[
n(\rho) = \sqrt{\frac{\rho}{l^2} - \frac{\zeta}{2}}. \quad (26)
\]

Substituting the above solutions into the \((\rho \rho)\) bulk equation we get (15). Using the relation \( \Lambda_5 = -6/L^2 \) we get the fine-tuned relation \( L^2 = 2f^2 \). There is also a constant solution for \( f(\rho) \) with \( \Lambda_5 = -\frac{1}{4\alpha} \). We summarize our results in the following table.

| \( n(r) \) | \( f(\rho) \) | \( b(\rho) \) |
|-------|-------|-------|
| BTZ   | \cosh \left( \frac{\rho}{2\sqrt{\alpha}} \right) | \forall b(\rho) |
| \mbox{corrected} | \cosh \left( \frac{\rho}{2\sqrt{\alpha}} \right) | 2\beta \sqrt{\alpha} \sinh \left( \frac{\rho}{2\sqrt{\alpha}} \right) |
| BTZ   | \cosh \left( \frac{\rho}{2\sqrt{\alpha}} \right) | 2\beta \sqrt{\alpha} \sinh \left( \frac{\rho}{2\sqrt{\alpha}} \right) |

**TABLE I: Results**

To complete our solution with the introduction of the brane we must solve the corresponding junction conditions given by the Einstein equations on the brane (6) using the induced metric on the brane given by (7). For the case when \( n(r) \) corresponds to the BTZ black hole (11), and the brane cosmological constant is given by \( \Lambda_5 = -1/l^2 \), we found that the energy-momentum tensor is null. Therefore, the BTZ black hole is localized on the brane in vacuum.

When \( \Lambda(\rho) \) is of the form given in (26), we found the following traceless energy-momentum tensor

\[
T_\alpha^\beta = \text{diag} \left( \frac{\zeta}{2r^3}, \frac{\zeta}{2r^3}, -\frac{\zeta}{r^3} \right) , \quad (27)
\]
which is conserved on the brane, $\nabla_\alpha T^\alpha_\beta = 0$ [26]. The conformally coupled scalar field to the BTZ black hole does not introduce an independent conserved charge, it only modifies the energy-momentum tensor of the three-dimensional Einstein equations. If we consider the energy-momentum tensor in [25] necessary to sustain such solution, and we take the limit $r/l << 1$, we get the unexpected result that it reduces to (27) which is necessary to localize this black hole on the conical 2-brane. A way to understand this result is that because in this limit $r$ is very small, the black hole will be localized around the conical singularity and therefore, any matter will take a distributional form around this singularity. Note also, that this solution is a result of the presence of the Gauss-Bonnet term in the bulk. If we switch off the Gauss-Bonnet coupling, then from relations (9) and (10) it can be seen that only the BTZ black hole is a solution.

In conclusion we discussed black holes on an infinitely thin 2-brane of codimension-2 and their extension into a five-dimensional AdS bulk. To have a three-dimensional gravity on the brane we introduced a five-dimensional Gauss-Bonnet term in the bulk and an induced gravity term on the 2-brane. We showed that this system admits $(2+1)$ BTZ black holes solutions and their short distance extension on the 2-brane, while in the bulk these solutions describe BTZ-like strings. Consistency of the five-dimensional bulk equations requires a fine-tuned relation between the Gauss-Bonnet coupling constant and the length of the five-dimensional AdS space. The use of this fine-tuning gives to the non-singular horizon the shape of a throat up to the boundary of the AdS space.

We did not allow more severe singularities than conical. This assumption has fixed the deficit angle to a constant value. It is interesting to investigate how our solutions are modified in the presence of a variable deficit angle. Also, we did not discuss the thermodynamics and the stability issue of our solutions. We expect, however, similar behaviour of our solutions as the one found in four dimensions [18]. These issues are under study, and they will be reported elsewhere.

Of course, the important issue is if this analysis can be applied to a conical 3-brane. In our case the conical-like metric of the BTZ black hole helped us to localize it on the brane and further extent it into the bulk. A possible clue could be the BTZ short distance corrected solution (26). From a four-dimensional point of view it is a topological black hole in AdS space. It would be interesting to investigate the possibility of localization and further extension in the bulk of black holes with symmetries other than spherical.

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