SOME COMPANIONS OF OSTROWSKI TYPE INEQUALITY
FOR FUNCTIONS WHOSE SECOND DERIVATIVES ARE
CONVEX AND CONCAVE WITH APPLICATIONS

M.EMIN ÖZDEMİR ♦ AND MERVE AVCI ARDIC ♦

ABSTRACT. In this paper, we obtain some companions of Ostrowski type in-
equality for absolutely continuous functions whose second derivatives absolute
value are convex and concave. Finally, we gave some applications for special
means.

1. INTRODUCTION

The following inequality is well known as Ostrowski’s inequality in the literature
[4]:

Theorem 1. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on $I^\circ$, the interior of
the interval $I$, such that $f^' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'(x)| \leq M$, then
the following inequality,

$$
|f(x) - \frac{1}{b - a} \int_a^b f(x)\, dx| \leq M \left(\frac{1}{4} + \frac{(x - \frac{a + b}{2})^2}{(b - a)^2}\right)
$$

holds for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it can
not be replaced by a smaller constant.

In [5], Set et al. proved some inequalities for $s$–concave and concave functions
via following Lemma:

Lemma 1. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function on $I^\circ$ with
$f'' \in L_1[a, b]$, then

$$
\frac{1}{b - a} \int_a^b f'(x)\, dx + \left(x - \frac{a + b}{2}\right) f'(x) = \frac{(x - a)^3}{2 (b - a)} \int_0^1 t^2 f''(tx + (1 - t) a)\, dt + \frac{(b - x)^3}{2 (b - a)} \int_0^1 t^2 f''(tx + (1 - t) b)\, dt.
$$

Theorem 2. Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a twice differentiable function on $I^\circ$ such
that $f'' \in L_1[a, b]$ where $a, b \in I$ with $a < b$. If $|f''|^q$ is $s$–concave in the second
sense on $[a, b]$ for some fixed $s \in (0, 1]$, $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following

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♦ Corresponding Author.
Inequality holds:

\[(1.1)\]
\[
\left| \frac{1}{b-a} \int_a^b f(u)du - f(x) + \left( x - \frac{a+b}{2} \right) f'(x) \right| 
\leq \frac{2^{\frac{p-1}{q}}}{(2p+1)^{1+\frac{s}{q}} (b-a)} \left( (x-a)^3 |f''(\frac{x+a}{2})| + (b-x)^3 |f''(\frac{a+b}{2})| \right)^{\frac{1}{2}}
\]

for each \( x \in [a,b] \).

**Corollary 1.** If in \( (1.1) \), we choose \( x = \frac{a+b}{2} \), then we have

\[
\left| \frac{1}{b-a} \int_a^b f(u)du - f \left( \frac{a+b}{2} \right) \right| \leq \frac{2^{\frac{p-1}{q}}}{16 (2p+1)^{1+\frac{s}{q}}} \left[ \left| f'' \left( \frac{3a+b}{4} \right) \right| + \left| f'' \left( \frac{a+3b}{4} \right) \right| \right].
\]

For instance, if \( s = 1 \), then we have

\[
\left| \frac{1}{b-a} \int_a^b f(u)du - f \left( \frac{a+b}{2} \right) \right| \leq \frac{(b-a)^2}{16 (2p+1)^{1+\frac{s}{q}}} \left[ \left| f'' \left( \frac{3a+b}{4} \right) \right| + \left| f'' \left( \frac{a+3b}{4} \right) \right| \right].
\]

In [1], Liu introduced some companions of an Ostrowski type inequality for functions whose first derivative are absolutely continuous. In [2], Barnett et al. established some companions for the Ostrowski inequality and the generalized trapezoid inequality. In [3], Alomari et al. introduced some companions of Ostrowski inequality for functions whose first derivatives absolute value are convex.

In this paper, we established some companions of Ostrowski type inequality for absolutely continuous functions whose second derivatives absolute value are convex and concave.

In order to prove our main results we need the following Lemma [1]:

**Lemma 2.** Let \( f : [a,b] \to \mathbb{R} \) be such that the derivative \( f' \) is absolutely continuous on \( [a,b] \). Then we have the equality

\[
\frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{2} [f(x) + f(a+b-x)] + \frac{1}{2} \left( x - \frac{3a+b}{4} \right) [f'(x) - f'(a+b-x)]
\]

\[= \frac{1}{2(b-a)} \left[ \int_a^x (t-a)^2 f''(t)dt + \int_x^{a+b-x} \left( t - \frac{a+b}{2} \right)^2 f''(t)dt \right.
\]

\[+ \left. \int_{a+b-x}^b (t-b)^2 f''(t)dt \right] \]

for all \( x \in [a, \frac{a+b}{2}] \).

2. **MAIN RESULTS**

We will start with the following theorem:

**Theorem 3.** Let \( f : [a,b] \to \mathbb{R} \) be a function such that \( f' \) is absolutely continuous on \( [a,b] \), \( f'' \in L_1[a,b] \). If \( |f''| \) is convex on \( [a,b] \), then we have the following...
inequality:

\[
\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{2} [f(x) + f(a + b - x)] + \frac{1}{2} \left( x - \frac{3a + b}{4} \right) [f'(x) - f'(a + b - x)] \right| \\
\leq \frac{(x-a)^3}{24(b-a)} \left[ |f''(a)| + |f''(b)| \right] + \frac{6(x-a)^3 + (a+b-2x)^3}{48(b-a)} \left[ |f''(x)| + |f''(a+b-x)| \right]
\]

for all \( x \in [a, \frac{a+b}{2}] \).

Proof. Using Lemma 2 and the property of the modulus we have

\[
\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{2} [f(x) + f(a + b - x)] + \frac{1}{2} \left( x - \frac{3a + b}{4} \right) [f'(x) - f'(a + b - x)] \right| \\
\leq \frac{1}{2(b-a)} \left[ \int_a^x (t-a)^2 |f''(t)| dt + \int_x^{a+b-x} \left( t - \frac{a+b}{2} \right)^2 |f''(t)| dt \\
+ \int_{a+b-x}^b (t-b)^2 |f''(t)| dt \right].
\]

Since \(|f''|\) is convex on \([a, b]\), we have

\[
|f''(t)| \leq \frac{t-a}{x-a} |f''(x)| + \frac{x-t}{x-a} |f''(a)|, \quad t \in [a, x];
\]

\[
|f''(t)| \leq \frac{t-x}{a+b-2x} |f''(a+b-x)| + \frac{a+b-x-t}{a+b-2x} |f''(x)|, \quad t \in (x, a+b-x]
\]

and

\[
|f''(t)| \leq \frac{t-a-b+x}{x-a} |f''(b)| + \frac{b-t}{x-a} |f''(a+b-x)|, \quad t \in (a+b-x, b],
\]
Therefore we can write
\[
\left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{2} \left[ f(x) + f(a + b - x) \right] \right| \\
+ \frac{1}{2} \left( x - 3a + b \right) \left| f'(x) - f'(a + b - x) \right| \\
\le \frac{1}{2(b-a)} \left\{ \int_a^x (t-a)^2 \left[ \frac{t-a}{x-a} |f''(x)| + \frac{x-t}{x-a} |f''(a)| \right] dt \\
+ \int_a^{a+b-x} (t-a+b)^2 \left[ \frac{t-x}{a+b-2x} |f''(a+b-x)| + \frac{a+b-x-t}{a+b-2x} |f''(x)| \right] \\
+ \int_{a+b-x}^b (t-b)^2 \left[ \frac{t-a-b+x}{x-a} |f''(b)| + \frac{b-t}{x-a} |f''(a+b-x)| \right] \right\} \\
= \frac{1}{2(b-a)} \left\{ \frac{1}{4} (x-a)^3 |f''(x)| + \frac{1}{12} (x-a)^3 |f''(a)| \\
+ \frac{1}{24} (a+b-2x)^3 |f''(a+b-x)| + \frac{1}{24} (a+b-2x)^3 |f''(x)| \\
+ \frac{1}{12} (x-a)^3 |f''(b)| + \frac{1}{4} (x-a)^3 |f''(a+b-x)| \right\} \\
= \frac{(x-a)^3}{24(b-a)} [ |f''(a)| + |f''(b)| ] \\
+ \frac{6 (x-a)^3 + (a+b-2x)^3}{48(b-a)} [ |f''(x)| + |f''(a+b-x)| ],
\]
which is the desired result. \(\square\)

**Corollary 2.** Let \( f \) as in Theorem 3. Additionally, if \( f'(x) = f'(a + b - x) \), we have
\[
\left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{2} \left[ f(x) + f(a + b - x) \right] \right| \\
\le \frac{(x-a)^3}{24(b-a)} [ |f''(a)| + |f''(b)| ] \\
+ \frac{6 (x-a)^3 + (a+b-2x)^3}{48(b-a)} [ |f''(x)| + |f''(a+b-x)| ],
\]
\(\text{Corollary 3.} \) In Corollary 2, if \( f \) is symmetric function, \( f(a + b - x) = f(x) \), for all \( x \in [a, \frac{a+b}{2}] \) we have
\[
\left| \frac{1}{b-a} \int_a^b f(t) dt - f(x) \right| \\
\le \frac{(x-a)^3}{24(b-a)} [ |f''(a)| + |f''(b)| ] \\
+ \frac{6 (x-a)^3 + (a+b-2x)^3}{48(b-a)} [ |f''(x)| + |f''(a+b-x)| ],
\]
which is an Ostrowski type inequality.
Corollary 4. In Corollary 3, if we choose \( x = \frac{a + b}{2} \), we have

\[
\left| \frac{1}{b - a} \int_a^b f(t)dt - f \left( \frac{a + b}{2} \right) \right| 
\leq \frac{(b - a)^2}{192} \left[ |f''(a)| + 6 \left| f'' \left( \frac{a + b}{2} \right) \right| + |f''(b)| \right].
\]

Corollary 5. In Theorem 3, if we choose \( x = \frac{3a + b}{4} \), we have

\[
\left| \frac{1}{b - a} \int_a^b f(t)dt - \frac{1}{2} \left[ f \left( \frac{3a + b}{4} \right) + f \left( \frac{a + 3b}{4} \right) \right] \right| 
\leq \frac{(b - a)^2}{1536} \left[ |f''(a)| + 7 \left| f'' \left( \frac{3a + b}{4} \right) \right| + 7 \left| f'' \left( \frac{a + 3b}{4} \right) \right| + |f''(b)| \right].
\]

Theorem 4. Let \( f : [a, b] \to \mathbb{R} \) be a function such that \( f' \) is absolutely continuous on \([a, b]\), \( f'' \in L_1[a, b] \). If \( |f''|^q \) is convex on \([a, b]\), for all \( x \in \left[ a, \frac{a + b}{2} \right] \) and \( q > 1 \), then we have the following inequality:

\[
\left| \frac{1}{b - a} \int_a^b f(t)dt - \frac{1}{2} [f(x) + f(a + b - x)] 
+ \frac{1}{2} \left( x - \frac{3a + b}{4} \right) \left[ f'(x) - f'(a + b - x) \right] \right| 
\leq \frac{1}{2^{1 + \frac{1}{q}} (b - a) (2p + 1)^{\frac{1}{p}}} \left[ (x - a)^3 (|f''(a)|^q + |f''(x)|^q)^{\frac{1}{q}} 
+ \frac{(a + b - 2x)^3}{4} \left( |f''(x)|^q + |f''(a + b - x)|^q \right)^{\frac{1}{q}} 
+ (x - a)^3 \left( |f''(a + b - x)|^q + |f''(b)|^q \right)^{\frac{1}{q}} \right],
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).
Proof. Using Lemma\(\text{2}\) Hölder inequality and convexity of \(|f''|^q\), we have

\[
\frac{1}{b-a} \left| \int_a^b f(t) dt - \frac{1}{2} [f(x) + f(a + b - x)] \right| \\
+ \frac{1}{2} \left( x - \frac{3a + b}{4} \right) \left| f'(x) - f'(a + b - x) \right| \\
\leq \frac{1}{2} \left( \int_a^b (t - a)^{2p} dt \right)^{\frac{1}{2}} \left( \int_a^b |f''(t)|^q dt \right)^{\frac{1}{2}} \\
+ \left( \int_a^b (t - a)^{2p} dt \right)^{\frac{1}{2}} \left( \int_a^b \left| f''(t) \right|^q dt \right)^{\frac{1}{2}} \\
+ \left( \int_a^b (t - b)^{2p} dt \right)^{\frac{1}{2}} \left( \int_a^b \left| f''(t) \right|^q dt \right)^{\frac{1}{2}} \\
= \frac{1}{2} \left( \frac{(x - a)^{2p+1}}{(2p+1)} \right)^{\frac{1}{2}} \left( \frac{x - a}{2} \right)^{\frac{1}{2}} \left( |f''(a)|^q + |f''(x)|^q \right)^{\frac{1}{2}} \\
+ \left( \frac{2}{2p+1} \left( \frac{a + b - x}{2} \right)^{2p+1} \right)^{\frac{1}{2}} \left( \frac{a + b}{2} - x \right)^{\frac{1}{2}} \left( |f''(a)|^q + |f''(x)|^q \right)^{\frac{1}{2}} \\
+ \left( \frac{(x - a)^{2p+1}}{(2p+1)} \right)^{\frac{1}{2}} \left( \frac{x - a}{2} \right)^{\frac{1}{2}} \left( |f''(a + b - x)|^q + |f''(b)|^q \right)^{\frac{1}{2}}.
\]

When we arrange the statements above, we obtain the desired result. \(\square\)
Corollary 6. Let \( f \) as in Theorem 4. Additionally, if \( f'(x) = f'(a + b - x) \), we have

\[
\left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{2} [f(x) + f(a + b - x)] \right| \\
\leq \frac{1}{2^{1+\frac{3}{q}} (b-a) (2p+1)^\frac{1}{q}} \left[ (x-a)^3 (|f''(a)|^q + |f''(x)|^q)^{\frac{1}{q}} \\
+ \frac{(a+b-2x)^3}{4} (|f''(x)|^q + |f''(a+b-x)|^q)^{\frac{1}{q}} \\
+ (x-a)^3 (|f''(a+b-x)|^q + |f''(b)|^q)^{\frac{1}{q}} \right],
\]

for all \( x \in [a, \frac{a+b}{2}] \).

Corollary 7. In Corollary 6 if \( f \) is symmetric function, \( f(a + b - x) = f(x) \), we have

\[
\left| \frac{1}{b-a} \int_a^b f(t) dt - f(x) \right| \\
\leq \frac{1}{2^{1+\frac{3}{q}} (b-a) (2p+1)^\frac{1}{q}} \left[ (x-a)^3 (|f''(a)|^q + |f''(x)|^q)^{\frac{1}{q}} \\
+ \frac{(a+b-2x)^3}{4} (|f''(x)|^q + |f''(a+b-x)|^q)^{\frac{1}{q}} \\
+ (x-a)^3 (|f''(a+b-x)|^q + |f''(b)|^q)^{\frac{1}{q}} \right],
\]

for all \( x \in [a, \frac{a+b}{2}] \).

Corollary 8. In Corollary 6 if we choose \( x = a \) we have

\[
\left| \frac{1}{b-a} \int_a^b f(t) dt - f(a) + f(a+b) \right| \leq \frac{(b-a)^2}{2^{1+\frac{3}{q}} (2p+1)^\frac{1}{q}} \left[ |f''(a)|^q + |f''(b)|^q \right]^{\frac{1}{q}}.
\]

Corollary 9. In Theorem 4, if we choose

(1) \( x = \frac{a+b}{2} \), we have

\[
\left| \frac{1}{b-a} \int_a^b f(t) dt - f \left( \frac{a+b}{2} \right) \right| \\
\leq \frac{(b-a)^2}{2^{1+\frac{3}{q}} (2p+1)^\frac{1}{q}} \left[ \left( f''(a) \right)^q + \left( f'' \left( \frac{a+b}{2} \right) \right)^q \right]^{\frac{1}{q}} \left( \left( f'' \left( \frac{a+b}{2} \right) \right)^q + |f''(b)|^q \right)^{\frac{1}{q}}.
\]
(2) \( x = \frac{3a+b}{4} \), we have
\[
\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{2} \left[ f\left( \frac{3a+b}{4} \right) + f\left( \frac{a+3b}{4} \right) \right] \right|
\leq \frac{(b-a)^2}{2^{p/q} (2p+1)^\frac{1}{q}} \left\{ \left( \left| f''(a) \right|^q + \left| f''\left( \frac{3a+b}{4} \right) \right|^q \right)^{\frac{1}{q}} + 2 \left( \left| f''\left( \frac{3a+b}{4} \right) \right|^q + \left| f''\left( \frac{a+3b}{4} \right) \right|^q \right)^{\frac{1}{q}} + \left( \left| f''\left( \frac{a+3b}{4} \right) \right|^q + \left| f''(b) \right|^q \right)^{\frac{1}{q}} \right\}.
\]

Remark 1. Using the well-known power-mean integral inequality one may get inequalities for functions whose second derivatives absolute value are convex. The details are omitted.

We obtain the following result for concave functions.

Theorem 5. Let \( f : [a, b] \to \mathbb{R} \) be a function such that \( f' \) is absolutely continuous on \([a, b], f'' \in L_1[a, b] \). If \( f'' \) is concave on \([a, b], \) for all \( x \in [a, \frac{a+b}{2}] \) and \( q > 1 \), then we have the following inequality:
\[
\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{2} \left[ f(x) + f(a + b - x) \right] \right|
\leq \frac{1}{(b-a) (2p+1)^\frac{1}{q}} \left\{ (x-a)^3 \left| f''\left( \frac{x+a}{2} \right) \right| + \frac{(a+b-2x)^3}{4} \left| f''\left( \frac{a+b}{2} \right) \right| + (x-a)^3 \left| f''\left( \frac{a+2b-x}{2} \right) \right| \right\}
\]
where \( \frac{1}{p} + \frac{1}{q} = 1 \).

Proof. From Lemma \[\text{2}\] and using Hölder inequality, we have
\[
\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{2} \left[ f(x) + f(a + b - x) \right] \right|
\leq \frac{1}{2(b-a)} \left\{ \left( \int_a^x (t-a)^{2p} dt \right)^{\frac{1}{p}} \left( \int_a^x \left| f''(t) \right|^q dt \right)^{\frac{1}{q}}
\right. \\
+ \left( \int_{a+b-x}^b \left( t - \frac{a+b}{2} \right)^{2p} dt \right)^{\frac{1}{p}} \left( \int_{a+b-x}^b \left| f''(t) \right|^q dt \right)^{\frac{1}{q}}
\right. \\
+ \left( \int_{a+b-x}^a \left( t-a \right)^{2p} dt \right)^{\frac{1}{p}} \left( \int_{a+b-x}^a \left| f''(t) \right|^q dt \right)^{\frac{1}{q}} \right\}.
\]
Let us write,
\[ \int_a^x |f''(t)|^q \, dt = (x - a) \int_0^1 |f''(\lambda x + (1 - \lambda) a)|^q \, d\lambda, \]
\[ \int_{x}^{a+b-x} |f''(t)|^q \, dt = (a + b - 2x) \int_0^1 |f''(\lambda(a + b - x) + (1 - \lambda) x)|^q \, d\lambda \]
and
\[ \int_{a+b-x}^b |f''(t)|^q \, dt = (x - a) \int_0^1 |f''(\lambda b + (1 - \lambda)(a + b - x))|^q \, d\lambda. \]

Since \(|f''|^q\) is concave on \([a, b]\), we use the Jensen integral inequality to obtain
\[ (x - a) \int_0^1 |f''(\lambda x + (1 - \lambda) a)|^q \, d\lambda \]
\[ = (x - a) \int_0^1 \lambda^0 |f''(\lambda x + (1 - \lambda) a)|^q \, d\lambda \]
\[ \leq (x - a) \left( \int_0^1 \lambda^0 d\lambda \right)^\frac{1}{q} \left( \int_0^1 \left| f'' \left( \frac{1}{\int_0^1 \lambda^0 d\lambda} \int_0^1 (\lambda x + (1 - \lambda) a) \, d\lambda \right) \right|^q \, d\lambda \right)^\frac{1}{q} \]
\[ = (x - a) \left| f'' \left( \frac{x + a}{2} \right) \right|^q \]

and analogously
\[ (a + b - 2x) \int_0^1 |f''(\lambda(a + b - x) + (1 - \lambda) x)|^q \, d\lambda \leq (a + b - 2x) \left| f'' \left( \frac{a + b}{2} \right) \right|^q, \]
\[ (x - a) \int_0^1 |f''(\lambda b + (1 - \lambda)(a + b - x))|^q \, d\lambda \leq (x - a) \left| f'' \left( \frac{a + 2b - x}{2} \right) \right|^q. \]

Combining all above inequalities, we obtain
\[ \left| \frac{1}{b - a} \int_a^b f(t) \, dt - \frac{1}{2} [f(x) + f(a + b - x)] \right| \]
\[ + \frac{1}{2} \left( x - \frac{3a + b}{4} \right) \left| f'(x) - f'(a + b - x) \right| \]
\[ \leq \frac{1}{2(b - a)} \left\{ \left( \frac{(x - a)^{2p+1}}{(2p + 1)} \right)^\frac{1}{p} (x - a)^\frac{1}{q} \left| f'' \left( \frac{x + a}{2} \right) \right| \right\} \]
\[ + \left( \frac{2}{2p + 1} \left( \frac{a + b}{2} - x \right)^{2p+1} \right)^\frac{1}{p} (a + b - 2x)^\frac{1}{q} \left| f'' \left( \frac{a + b}{2} \right) \right| \]
\[ + \left( \frac{(x - a)^{2p+1}}{(2p + 1)} \right)^\frac{1}{p} (x - a)^\frac{1}{q} \left| f'' \left( \frac{a + 2b - x}{2} \right) \right| \}
\[ \leq \frac{1}{2(b - a)(2p + 1)^\frac{1}{p}} \left[ (x - a)^3 \left| f'' \left( \frac{x + a}{2} \right) \right| \right.
\[ \left. + \frac{(a + b - 2x)^3}{4} \left| f'' \left( \frac{a + b}{2} \right) \right| + (x - a)^3 \left| f'' \left( \frac{a + 2b - x}{2} \right) \right| \right] \]
for all \( x \in [a, \frac{a+b}{2}] \) and \( \frac{1}{p} + \frac{1}{q} = 1. \)

**Corollary 10.** Let \( f \) as in Theorem 5. Additionally, if \( f'(x) = f'(a + b - x), \) we have

\[
\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{2} [f(x) + f(a+b-x)] \right| \\
\leq \frac{1}{2(b-a)} \left[ (x-a)^3 \left( |f'' \left( \frac{x+a}{2} \right)| + |f'' \left( \frac{a+2b-x}{2} \right)| \right) + \frac{(a+b-2x)^3}{4} \right],
\]

for all \( x \in [a, \frac{a+b}{2}] \).

**Corollary 11.** In Corollary 9, if \( f \) is symmetric function, \( f(a + b - x) = f(x) \), we have

\[
\left| \frac{1}{b-a} \int_a^b f(t)dt - f(x) \right| \\
\leq \frac{1}{2(b-a)} \left[ (x-a)^3 \left( |f'' \left( \frac{x+a}{2} \right)| + |f'' \left( \frac{a+2b-x}{2} \right)| \right) + \frac{(a+b-2x)^3}{4} \right],
\]

for all \( x \in [a, \frac{a+b}{2}] \).

**Corollary 12.** In Theorem 5, if we choose \( x = \frac{3a+b}{4} \), we have

\[
\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{2} \left[ f \left( \frac{3a+b}{4} \right) + f \left( \frac{a+3b}{4} \right) \right] \right| \\
\leq \frac{(b-a)^2}{128(2p+1)^2} \left[ 7a+b \right],
\]

for all \( x \in [a, \frac{a+b}{2}] \).

**Remark 2.** In Theorem 5, if we choose \( x = \frac{a+b}{2} \), we have the second inequality in Corollary 9.

### 3. Applications for Special Means

We consider the means for nonnegative real numbers \( \alpha < \beta \) as follows:

1. The arithmetic mean:
   \[
   A(\alpha, \beta) = \frac{\alpha + \beta}{2}, \quad \alpha, \beta \in \mathbb{R}^+.
   \]

2. The logarithmic mean:
   \[
   L(\alpha, \beta) = \frac{\beta - \alpha}{\ln \beta - \ln \alpha}, \quad \alpha, \beta \in \mathbb{R}^+, \quad \alpha \neq \beta.
   \]

3. The generalized logarithmic mean:
   \[
   L_n(\alpha, \beta) = \left( \frac{\beta^{n+1} - \alpha^{n+1}}{(\beta - \alpha)(n+1)} \right)^{\frac{1}{n}}, \quad \alpha, \beta \in \mathbb{R}^+, \quad \alpha \neq \beta, \quad n \in \mathbb{Z} \setminus \{-1, 0\}.
   \]
(4) The identric mean:

$$I(\alpha, \beta) = \begin{cases} \frac{1}{e} \left( \frac{\alpha}{\beta} \right)^{1/\alpha} & \text{if } \alpha \neq \beta, \\ \alpha, & \alpha = \beta \end{cases} \alpha, \beta \in \mathbb{R}^+.$$ 

Now, using the results of Section 2, we give some applications to special means of real numbers.

**Proposition 1.** Let $a, b \in \mathbb{R}^+$, $a < b$. Then we have

$$\left| L^{-1}(a, b) - A \left( \frac{4}{3a + b}, \frac{4}{a + 3b} \right) \right| \leq \frac{(b - a)^2}{768} \left[ \frac{a^3 + b^3}{a^3b^3} + 448 \left( \frac{1}{(3a + b)^3} + \frac{1}{(a + 3b)^3} \right) \right].$$

**Proof.** The assertion follows from Corollary 5 applied to the convex mapping $f : [a, b] \to \mathbb{R}$, $f(x) = \frac{1}{x}$. □

**Proposition 2.** Let $a, b \in \mathbb{R}^+$, $a < b$ and $n \in \mathbb{Z}$, $|n| \geq 2$. Then we have

$$|I_n(a, b) - A^n(a, b)| \leq \frac{n(n - 1)(b - a)^2}{192} \left[ a^{n-2} + 6A^{n-2}(a, b) + b^{n-2} \right].$$

**Proof.** The assertion follows from Corollary 4 applied to the convex mapping $f : [a, b] \to \mathbb{R}$, $f(x) = x^n$. □

**Proposition 3.** Let $a, b \in \mathbb{R}^+$, $a < b$. Then we have

$$|A(\ln a, \ln b) - \ln I| \leq \frac{(b - a)^2}{2^{3+\frac{q}{2}}(2p + 1)\pi} \left[ \frac{1}{a^{2q}} + \frac{1}{b^{2q}} \right].$$

**Proof.** The assertion follows from Corollary 8 applied to the convex mapping $f : [a, b] \to [0, \infty)$, $f(x) = -\ln x$. □

**Proposition 4.** Let $a, b \in \mathbb{R}^+$, $a < b$. Then for all $q > 1$, we have

$$\left| L^{-1}(a, b) - A \left( \frac{4}{3a + b}, \frac{4}{a + 3b} \right) \right| \leq \frac{(b - a)^2}{2^{7+\frac{q}{2}}(2p + 1)\pi} \left\{ \left( \frac{2}{a^{3q}} + \frac{128}{(3a + b)^{3q}} \right)^{\frac{1}{q}} + 2 \left( \frac{128}{(3a + b)^{3q}} + \frac{128}{(a + 3b)^{3q}} \right)^{\frac{1}{q}} + \left( \frac{128}{(a + 3b)^{3q}} + \frac{2}{b^{3q}} \right)^{\frac{1}{q}} \right\}.$$ 

**Proof.** The assertion follows from second inequality in Corollary 9 applied to the convex mapping $f : [a, b] \to \mathbb{R}$, $f(x) = \frac{1}{x}$. □

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★Atatürk University, K.K. Education Faculty, Department of Mathematics, Erzurum 25240, Turkey
E-mail address: emos@atauni.edu.tr

★Adiyaman University, Faculty of Science and Arts, Department of Mathematics, Adiyaman 02040, Turkey
E-mail address: mavci@posta.adiyaman.edu.tr