STABILIZING THE MONODROMY OF AN OPEN BOOK DECOMPOSITION

VINCENT COLIN AND KO HONDA

Abstract. We prove that any mapping class on a compact oriented surface with nonempty boundary can be made pseudo-Anosov and right-veering after a sequence of positive stabilizations.

1. Introduction

Let \( S \) be a compact oriented surface with nonempty, possibly disconnected, boundary \( \partial S \), and let \( h : S \to S \) be a diffeomorphism for which \( h|_{\partial S} = \text{id} \). By a theorem of Giroux [Gi], there is a 1-1 correspondence between isomorphism classes of contact structures on closed 3-manifolds and equivalence classes of pairs \((S, h)\), up to positive stabilization and conjugation. The pair \((S, h)\) can be interpreted as an open book decomposition of some 3-manifold \( M \), as will be explained later on in the introduction.

A positive stabilization of a pair \((S, h)\) is defined as follows. Let \( S' \) be the oriented union of the surface \( S \) and a band \( B \) attached along the boundary of \( S \), i.e., \( S' \) is obtained from \( S \) by attaching a 1-handle along \( \partial S \). Let \( \gamma \) be a simple closed curve in \( S' \) which intersects the core of \( B \) at exactly one point and let \( \text{id}_B \cup h \) be the extension of \( h \) by the identity map to \( S' \). Then define \( h' = R_\gamma \circ (\text{id}_B \cup h) \), where \( R_\gamma \) is a positive Dehn twist about \( \gamma \). The pair \((S', h')\) is called an elementary positive stabilization of \((S, h)\). More generally, we say that \((S', h')\) is a positive stabilization of \((S, h)\) if it is obtained from \((S, h)\) by a sequence of elementary positive stabilizations. In this paper, a stabilization will always mean “positive stabilization”. By conjugation we mean replacing \((S, h)\) by \((S, ghg^{-1})\), where \( g \) is a diffeomorphism of \( S \) which is not necessarily the identity on the boundary.

Next we describe the notion of a right-veering diffeomorphism \( h : S \to S \), introduced in [HKM]. Let \( \alpha \) and \( \beta \) be properly embedded oriented arcs \([0, 1] \to S\) with a common initial point \( x \in \partial S \). Denote by \( \pi : \tilde{S} \to S \) the universal cover of \( S \). We pick lifts \( \tilde{\alpha} \) and \( \tilde{\beta} \) of \( \alpha \) and \( \beta \) in \( \tilde{S} \), starting at the same lift \( \tilde{x} \) of \( x \). The arc \( \tilde{\alpha} \) divides \( \tilde{S} \) into two regions, one to the left and the other to the right of \( \tilde{\alpha} \); the left-right convention is such that at \( \tilde{x} \) the (tangential) orientation of \( \partial \tilde{S} \) points towards the right-hand component of \( \tilde{S} \setminus \tilde{\alpha} \). We say that \( \beta \) is to the

\begin{itemize}
  \item Date: This version: June 5, 2007.
  \item 1991 Mathematics Subject Classification. Primary 57M50; Secondary 53C15.
  \item Key words and phrases. tight, contact structure, open book decomposition, fibered link, mapping class group, stabilization.
  \item KH supported by an Alfred P. Sloan Fellowship and an NSF CAREER Award (DMS-0237386).
\end{itemize}
right of \( \alpha \) if \( \tilde{\beta}(1) \) belongs to the region to the right of \( \tilde{\alpha} \) or equals \( \tilde{\alpha}(1) \). (Observe that this definition clearly does not depend on the choice of representatives in the isotopy classes of \( \alpha \) and \( \beta \), rel boundary.) Now, if \( h \) is a diffeomorphism of a compact surface \( S \) with \( \partial S \neq \emptyset \) and \( h|_{\partial S} = id \), we say that \( h \) is right-veering if for every properly embedded oriented arc \( \alpha \) in \( S \), the arc \( h(\alpha) \) is to the right of \( \alpha \). A typical example of a right-veering diffeomorphism is a positive (= right-handed) Dehn twist around a simple closed curve in \( S \).

The goal of this note is to prove the following:

**Theorem 1.1.** Let \( S \) be a compact oriented surface with nonempty boundary and \( h \) be a diffeomorphism of \( S \) which is the identity on \( \partial S \). Then there exists a stabilization \((S', h')\) of \((S, h)\), where \( \partial S' \) is connected and \( h' \) is right-veering and freely homotopic to a pseudo-Anosov homeomorphism.

Let us briefly recall the 3-dimensional context for Theorem 1.1, which is our primary motivation. An oriented, positive contact structure on a closed oriented 3-manifold \( M \) is a plane field \( \xi \) given as the kernel of a global 1-form \( \alpha \), whose exterior product with \( d\alpha \) is a volume form on \( M \). An open book decomposition of \( M \) is a pair \((K, \theta)\), where \( K \) is a link in \( M \) and \( \theta : M \setminus K \to S^1 \) is a fibration given by \((k, r, \phi) \mapsto \phi \) in a neighborhood \( N(K) \simeq \{(k, r, \phi) \in S^1 \times [0,1) \times S^1 \} \) \((r, \phi) \) are polar coordinates\) of \( K \simeq \{k \in S^1, r = 0\} \). The link \( K \) is called the binding and the fibers of \( \theta \) are called the pages of the open book. Notice that the orientations given on \( M \) and \( S^1 \) induce a co-orientation of the pages and thus an orientation of \( K \) as the boundary of the closure of one page. An open book decomposition of \( M \) is completely determined, up to a diffeomorphism of \( M \), by a compact retraction \( S \) of a page, and a monodromy map \( h : S \to S \) for \( \theta \) which is the identity on \( \partial S \). Conversely, any pair \((S, h)\) gives rise to an open book decomposition of some manifold, called the relative suspension of \((S, h)\). Moreover, the open book decompositions given by \((S, h)\) and a stabilization \((S', h')\) of \((S, h)\) are conjugated by a diffeomorphism which induces the natural inclusion \( i : S \to S' \) on one page.

The link between contact structures and open books can be expressed via the following definition, introduced by Giroux. A contact structure \( \xi \) is carried by an open book \((K, \theta)\) if there exists a 1-form \( \alpha \) with kernel \( \xi \), such that \( d\alpha \) gives an area form on the pages and \( \alpha \) a length form on \( K \). The main theorem of Giroux \([Gi]\) states that every contact structure \( \xi \) on a closed oriented 3-manifold \( M \) is carried by an open book, and that two open books of \( M \) carry isotopic contact structures if and only if they have isotopic stabilizations. We then have the following immediate translation of Theorem 1.1:

**Corollary 1.2.** On a closed oriented 3-manifold \( M \), every oriented, positive contact structure is carried by an open book whose binding is connected, and whose monodromy is right-veering and freely homotopic to a pseudo-Anosov homeomorphism.

Although the primary motivation of this paper is 3-dimensional, our perspective, as well as the proof of Theorem 1.1, will mostly be 2-dimensional.
2. The curve complex of $S$

The proof of Theorem 1.1 uses distances in the *curve complex* $\mathcal{C}(S)$, introduced by Harvey [Har], as well as its hyperbolicity properties, due to Masur and Minsky [MM], in an essential way. In this section we briefly review the necessary background on the 1-skeleton of the curve complex $\mathcal{C}(S)$. We assume the reader is familiar with the basics of the Nielsen-Thurston theory of surface homeomorphisms. (See, for example, [Bo, CB, FLP, Th].)

Suppose the genus $g(S) > 1$. Since we are free to stabilize in Theorem 1.1 this is not a serious restriction. (When $g(S) = 1$ and $\partial S$ is connected, the definition of $\mathcal{C}(S)$ is slightly different.) The vertices of $\mathcal{C}(S)$ are isotopy classes of non-peripheral (i.e., not isotopic to a component of $\partial S$) simple closed curves on $S$. There is an edge of length one connecting each pair $\{\alpha, \beta\}$ of vertices, if $\alpha$ and $\beta$ are distinct isotopy classes which can be realized disjointly, i.e., have geometric intersection $i(\alpha, \beta) = 0$. Denote the $k$-skeleton of the curve complex by $\mathcal{C}_k(S)$. The 1-skeleton $\mathcal{C}_1(S)$ is a geodesic metric space, and the distance between $\alpha, \beta \in \mathcal{C}_0(S)$ is denoted by $d_{\mathcal{C}(S)}(\alpha, \beta)$, or simply by $d(\alpha, \beta)$. (We have no need for the higher-dimensional simplices of $\mathcal{C}(S)$ in this paper.)

The following useful facts can be found in [MM]:

**Lemma 2.1.**

1. If $\alpha, \beta \in \mathcal{C}_0(S)$, then $d(\alpha, \beta) \leq 2i(\alpha, \beta) + 1$.
2. If $d(\alpha, \beta) \geq 3$, then $\alpha$ and $\beta$ fill $S$, i.e., any non-peripheral simple closed curve $\gamma$ must intersect $\alpha$ or $\beta$.

We learned the following from Yair Minsky, at least for the case when $h$ is pseudo-Anosov:

**Lemma 2.2.** For any $h \in Map(S, \partial S)$ which is not freely isotopic to a periodic diffeomorphism, there exists $\alpha \in \mathcal{C}_0(S)$ such that $d(\alpha, h(\alpha))$ is arbitrarily large.

**Proof.** Fix a reference hyperbolic metric on $S$ with geodesic boundary. A geodesic lamination $\mu$ on $S$ is *minimal* if it does not contain any proper sublamination. In particular, $\mu$ does not have any closed leaves, and components of $\partial S$ cannot be leaves of $\mu$. The lamination $\mu$ is *filling* if each component of $S - \mu$ (called a *complementary region*) is an ideal polygon or a “once-punctured ideal polygon”. (Strictly speaking, the latter is a half-open annulus, whose boundary component is a component of $\partial S$.) Let $\mathcal{E}(S)$ be the set of minimal filling laminations, viewed as a subset of $ML(S)$/measures, where $ML(S)$ is the space of measured geodesic laminations. (A *measured geodesic lamination* is defined to be a compact geodesic lamination $\lambda$, which is endowed with a transverse measure whose support is all of $\lambda$. In particular, $\lambda$ has no infinite isolated leaf. An example of such a leaf is a diagonal $l$ of an ideal $n$-gon complementary region of a lamination $\lambda$, with $n \geq 4$.)

We claim that there is a minimal filling geodesic lamination $\mu \subset S$ so that $h(\mu) \neq \mu$. If $h$ is (freely homotopic to) a pseudo-Anosov homeomorphism, then we simply pick a minimal filling $\mu$ which is not the stable lamination or the unstable lamination of $h$. If $h$ is reducible, then, after taking a sufficiently large power $h^n$ of $h$, we may assume that there is a collection of disjoint homotopically nontrivial annuli $A_1, \ldots, A_m$ so that $h^n$ on each component $S_j$ of $S - \cup_i A_i$ is either the identity or pseudo-Anosov, and $h^n$ on $A_i$ is $R_{\gamma_i}^n$, $n_i \in \mathbb{Z}$, where $\gamma_i$ is...
the core curve of $A_i$. In this case, any minimal filling lamination $\mu$ with ideal 3-gon and once-punctured ideal monogon complementary regions would work, as we explain in the next two paragraphs.

Suppose there is a pseudo-Anosov piece $S_j$. The restriction of $\mu$ to $S_j$ consists of a finite number of disjoint non-parallel arc types which cut up $S_j$ into 3-gons and once-punctured monogons. By a slight abuse of notation, let $\mu|_{S_j}$ denote the union of arcs, one from each arc type. We then claim that the minimum geometric intersection $i(h(\mu)|_{S_j}, \mu|_{S_j}) \neq 0$, where the endpoints of the arcs are free to move around $\partial S_j$. Suppose $i(h(\mu)|_{S_j}, \mu|_{S_j}) = 0$. Since $S_j - \mu|_{S_j}$ consists of 3-gons and once-punctured monogons, it follows that $\mu|_{S_j} = h(\mu)|_{S_j}$, and $h$ must be periodic on $S_j$, a contradiction. Thus, $\mu$ and $h(\mu)$ will have nontrivial intersection, and are distinct.

If there are no pseudo-Anosov components, then $h^n$ consists of disjoint (high multiples of) Dehn twists. One can also verify that $\mu$ and $h(\mu)$ nontrivially intersect in this case. Indeed, let $\overline{S}_j$ be the union of $S_j$ and all its adjacent annuli $A_i$. For simplicity, assume that no $A_i$ bounds $S_j$ along both boundary components. Let $\delta$ be a properly embedded arc of $\overline{S}_j$ from $A_i$ to another $A_{i'}$, and let $\delta'$ be a parallel push-off of $\delta$. Then $h(\delta')$ and $\delta$ intersect efficiently if $n_i$ and $n_{i'}$ have the same sign; if they have opposite signs, $h(\delta')$ and $\delta$ intersect efficiently for a push-off to one side and have two extraneous intersections for a push-off to the other side. In either case, $\delta$ and $h(\delta')$ intersect nontrivially, provided $|n_i| + |n_{i'}| \gg 0$. The same argument holds for $\mu$.

Next, for any $\mu$ which is minimal and filling, we claim there exists a sequence of (connected) simple closed curves which converges to $\mu$. Pick a point $p \in \mu$ and an arbitrarily short transversal $\delta$ to $\mu$ so that $p \in \text{int}(\delta)$. Take a neighborhood $N(p)$ of $p$ of the form $[-1, 1] \times [-1, 1]$, where leaves of $\mu \cap N(p)$ are $[-1, 1] \times \{pt\}$, $\delta = \{0\} \times [-1, 1]$, and $p = (0, 0)$. Let $\beta$ be an arc which starts at $p$ and follows along $\mu$ in one direction, until the first return to $\delta$. If $\mu$ is orientable, then we can close up $\beta$ and “stretch it tight” to obtain a closed curve which is arbitrarily close to $\mu$. If $\mu$ is not orientable, then the arc $\beta$ could return to $\delta$ at $p_1$ from the “same side”, i.e., the same component of $N(p) - (\{0\} \times [-1, 1])$, from which it left $p$. Next consider the subarc $\delta_1 \subset \delta = \delta_0$ which contains $p$ and has $p_1$ as an endpoint. Continuing $\beta$ past $p_1$, let $p_2$ be the first return to $\delta_1$. If the approach is from the “same side” again, then we take a shorter $\delta_2 \subset \delta_1$, and continue. If further closest returns to $\delta$ are always from the “same side”, then we can naturally orient $\mu$ at the point $p$. If we can do this at every point on $\mu$, then $\mu$ would be orientable, a contradiction.

Masur and Minsky [MM] have shown that the curve complex $\mathcal{C}(S)$ is hyperbolic in the sense of Gromov; hence it makes sense to talk about its boundary at infinity $\partial_{\infty} \mathcal{C}(S)$. Now, according to a theorem of Klarreich [KL], there is a homeomorphism between $\mathcal{E}\mathcal{L}(S)$ and $\partial_{\infty} \mathcal{C}(S)$. Moreover, a sequence $\beta_i \in \mathcal{C}_0(S)$ converges to $\beta \in \partial_{\infty} \mathcal{C}(S)$ if and only if it converges to $\beta \in \mathcal{E}\mathcal{L}(S)$ in the topology of $\mathcal{M}\mathcal{L}(S)/\text{measures}$. (Klarreich’s theorem, in turn, relies on the results of Masur-Minsky [MM].) Let $\mu$ be a minimal filling geodesic lamination so that $h(\mu) \neq \mu$, and let $\alpha_n$, $n = 1, 2, \ldots$, be a sequence of simple closed curves which converges to
μ in the topology of \( \mathcal{ML}(S) \)/measures. Clearly \( h(\alpha_n) \to h(\mu) \). Since \( \mu \) and \( h(\mu) \) are distinct points on \( \partial_\infty \mathcal{C}(S) \), it immediately follows that \( d(\alpha_n, h(\alpha_n)) \to \infty \).

Here is a more direct proof, which the authors learned from Yair Minsky. The technique apparently can be traced back to Kobayashi [Ko], and has been used by Hempel [He] and Abrams-Schleimer [AS] to study distances of Heegaard splittings. Suppose \( d(\alpha_i, h(\alpha_i)) \) does not approach \( \infty \). Then, after passing to subsequences, we may assume that \( d(\alpha_i, h(\alpha_i)) = N \) for a constant \( N \). Consider a geodesic in the curve complex which connects \( \alpha_i \) to \( h(\alpha_i) \) — write it as \( \alpha_i, A_{i,1}, A_{i,2}, \ldots, A_{i,N} = h(\alpha_i) \). Then \( A_{i,1} \) is disjoint from \( \alpha_i \), and hence, after taking a subsequence, \( A_{i,1} \) converges to a lamination \( \mu' \) which has zero geometric intersection with \( \mu \). Similarly, \( A_{i,N-1} \) converges to a lamination \( \nu \) which has zero geometric intersection with \( h(\mu) \). Since \( \mu \) is minimal, we may conclude that \( \mu' = \mu \) and \( \nu = h(\mu) \). We now have a new pair of sequences converging to \( \mu \) and \( h(\mu) \), but at a shorter distance. By repeating the procedure, we eventually conclude that \( \mu = h(\mu) \), a contradiction. \( \square \)

3. Proof of Theorem 1.1

Using the technique in ([HKM], Proposition 6.1), any monodromy map \((S, h)\) may be stabilized so that it becomes right-veering. In the proof, a four-times punctured sphere is attached onto a boundary component of \( \partial S \). This construction in [HKM] makes the monodromy map reducible, so we may additionally assume that \( h \) is not periodic. Observe that, once \((S, h)\) is right-veering, stabilizations of it will remain right-veering.

Throughout the proof, \( d \) will always mean \( d_\mathcal{C}(S) \). We will write \( i_\Sigma(\cdot, \cdot) \) to indicate the geometric intersection number on the surface \( \Sigma \).

Case 1: \( \partial S \) is connected. Suppose that \( \partial S \) is connected. By Lemma 2.2, there exists a non-peripheral \( \gamma_0 \in \mathcal{C}_0(S) \) satisfying \( d(h(\gamma_0), \gamma_0) = N \gg 0 \). Let \( \gamma \) be a properly embedded arc in \( S \) which becomes \( \gamma_0 \) after concatenating with an arc of \( \partial S \) which connects the endpoints of \( \gamma \). Stabilize along the arc \( \gamma \) to obtain \((S', h' = R_{\gamma'} \circ h)\). Here \( \gamma' \) is the extension of \( \gamma \) to \( S' \) so that \( \gamma' \) intersects the co-core \( a \) of the 1-handle at exactly one point. We will show that \( h' \) is pseudo-Anosov. Although \( \partial S' \) will have two boundary components, this will be remedied in Case 2.

We first prove that \( h'(\delta) \neq \delta \) for any multicurve (i.e., closed embedded 1-manifold) \( \delta \) of \( S' \) without peripheral components. This would imply that \( h' \) is not reducible. The basic idea of the proof is to distinguish \( h'(\delta) \) and \( \delta \) by intersecting with the co-core \( a \) and the closed curve \( \gamma' \).

First suppose that \( \delta \subset S \), i.e., \( \delta \) does not intersect the co-core \( a \). If \( i_{S'}(h(\delta), \gamma') = m \), then we claim that \( i_{S'}(R_{\gamma'} \circ h(\delta), a) = m \). (Note that \( i_{S'}(\alpha, \gamma') = i_{S}(\alpha, \gamma) \) if \( \alpha \) is a closed curve on \( S \).) Clearly, there is a representative \( g \) of \( R_{\gamma'} \circ h(\delta) \) which intersects \( a \) at \( m \) points. If \( i_{S'}(R_{\gamma'} \circ h(\delta), a) < m \), then there must exist a bigon consisting of a subarc of \( a \) and an arc of \( g \). It follows (without too much difficulty) that there is a bigon consisting of an arc of \( \gamma \) and an arc of \( h(\delta) \), a contradiction. Since \( i_{S'}(\delta, a) = 0 \) and \( i_{S'}(h'(\delta), a) = m \), we have \( h'(\delta) \neq \delta \) if \( m > 0 \). Observe that this case covers the possibility that a component of \( \delta \) is \( \partial S \). On the other hand, if \( m = 0 \), then \( i_{S}(h(\alpha), \gamma_0) = 0 \) and \( d(h(\alpha), \gamma_0) = 1 \) for every component \( \alpha \) of
δ. Since $d(h(\gamma_0), \gamma_0) = N$, it follows that $d(h(\alpha), h(\gamma_0)) \approx N$ for every $\alpha$. (Here $\approx$ means “approximately equal to”.) Composing with $h^{-1}$, we obtain $d(\alpha, \gamma_0) \approx N$. With the help of Lemma 2.1, we find that $i_S(\alpha, \gamma_0) \geq \frac{N-1}{2}$. Hence $i_S(\alpha, \gamma) = i_S(\alpha, \gamma') \geq \frac{N-1}{2}$ for each component $\alpha$ of $\delta$. Since $m = 0$, $R_{\gamma'} \circ h(\delta) = h(\delta)$ and $i_S(h'(\delta), \gamma') = 0$. By comparing $i_S(\cdot, \gamma')$, we obtain $h'(\delta) \neq \delta$ for $m = 0$.

Next suppose that $\delta \not\subset S$. Let $k = i_S(\delta, a)$. Write $B = S' - \text{int}(S) = [-1, 1] \times [-1, 1]$, so that $\{\pm 1\} \times [-1, 1] \subset \partial S'$, $a = [-1, 1] \times \{0\}$, and $\gamma' \cap B = \{0\} \times [-1, 1]$. We now explain how to normalize $\delta$. Isotop $\delta$ so that it intersects $\gamma'$ and $a$ transversely and efficiently, for example by realizing $\gamma'$, $a$, and $\delta$ as geodesics. Then subdivide $\delta$ into arcs $\delta_1, \ldots, \delta_k, \delta'_1, \ldots, \delta'_k$, where $\delta_i \subset S$ and $\delta'_i \subset B$. (The $\delta_i$ and $\delta'_i$ are not ordered in any particular way.) The $\delta'_i$ are linear arcs in $B$, each with an endpoint on $[-1, 1] \times \{1\}$ and on $[-1, 1] \times \{-1\}$. If $\delta'_i$ does not intersect $\gamma' \cap B$, then we assume that $\delta'_i$ is vertical, i.e., $\{pt\} \times [-1, 1]$. Moreover, we may normalize $\delta$ so that there is no triangle in $S$ whose boundary consists of (i) a subarc of $\delta_i$, (ii) a subarc of $\gamma$, and (iii) a subarc of $[-1, 1] \times \{\pm 1\}$. If there is such a triangle, then we can isotop $\delta$ and push the triangle into $B$. Note that if $\delta_i$ is parallel to $\gamma$, then the isotopy may not be unique. We similarly normalize $h(\delta)$ and subdivide $h(\delta)$ into arcs $(h(\delta))_1, \ldots, (h(\delta))_k, (h(\delta))'_1, \ldots, (h(\delta))'_k$.

We claim that if any $\delta_i$ is a boundary-parallel arc in $S$, then $\delta$ must have a component which is parallel to $\partial S'$. Indeed, $\delta_i$ either has endpoints on both components of $[-1, 1] \times \{1\}$ or begins on a single component (but does not form a bigon together with some arc of $[-1, 1] \times \{\pm 1\}$). In the former case, if the endpoints of $\delta_i$ are not connected by a single $\delta'_j$, then $\delta$ will be spiraling towards one component of $\partial S'$, a contradiction. In the latter case, one of the two $\delta'_j$, $\delta'_k$, which begin at the endpoints of $\delta_i$ continues on to spiral around one component of $\partial S''$, also a contradiction. Since we are assuming that $\delta$ has no peripheral components, it follows that no $\delta_i$ is parallel to $\partial S$.

Consider the case where some $(h(\delta))'_i$ has negative slope $\neq -\infty$. Let $m > 0$ be the number of arcs with negative slope. The rest of the arcs $(h(\delta))'_i$ will have slope $\infty$. Also let $n = \sum_{j=1}^{k} i_S((h(\delta))_j, \gamma)$, i.e., the number of intersections between $h(\delta)$ and $\gamma$, away from $B$. Figure 1 depicts this situation. We then claim that $i_S(h'(\delta), a) = i_S(R_{\gamma'} \circ h(\delta), a) = k + m + n$. Since $i_S(\delta, a) = k$, this would show that $\delta \neq h'(\delta)$. The representative of $h'(\delta)$ shown in Figure 1 intersects $a$ at $k + m + n$ points, and we need to show that the intersection is efficient. In other words, no subarc of $(h'(\delta))_i$ bounds a bigon together with a subarc of $a$.

To prove the claim, consider the subsurface $S'' \subset S'$ which is the union of $B$ and a small neighborhood of $\gamma'$ containing the support of the Dehn twist $R_{\gamma'}$. The intersection of our representative of $h'(\delta)$ with $S''$ consists of non-boundary-parallel arcs with the exception of vertical arcs in $B$. Observe that the restriction of $h'(\delta)$ to $S' - S''$ is the same as the restriction of $h(\delta)$ to $S' - S''$. Hence, it suffices to prove that there is no boundary-parallel component of $(S' - S'') \cap h(\delta)$ which cobounds a bigon together with a subarc of $\partial S'' - \partial S'$. Such a boundary-parallel arc contradicts our normalization.

Next consider the case where no $(h(\delta))'_i$ has negative slope. Let $m$ be the number of arcs $(h(\delta))'_i$ with positive slope and let $n = \sum_{j} i_S((h(\delta))_j, \gamma)$, as before. We have $i_S(h'(\delta), a) = \sum_{i=1}^{k} i_S((h(\delta))_i, \gamma)$.
STABILIZING THE MONODROMY OF AN OPEN BOOK DECOMPOSITION

Figure 1. The case where some \((h(\delta))'_i\) have negative slope. The shaded region is the 1-handle \(B\). In the picture, \(k = 6\), \(m = 1\), and \(n = 3\) for \(h(\delta)\). The right-hand diagram depicts the effect of \(R_{\gamma'}\) on \(h(\delta)\).

\(k - m + n\), which is precisely the number of intersection points between our particular representative of \(h'(\delta)\) and \(a\). (The proof is the same as in the previous case.) If \(m \neq n\), then \(i_S(\delta, a) \neq i_S(h'(\delta), a)\). It remains to consider the case \(m = n\). In this case, \(i_S(h'(\delta), \gamma') = m + n = 2m \leq 2k\). We argue by contradiction that \(i_S(\delta, \gamma') \neq i_S(h'(\delta), \gamma')\). Suppose \(i_S(\delta, \gamma') = i_S(h'(\delta), \gamma') \leq 2k\). Since \(1 \leq i \leq k\), this means that there is some \(i\) for which \(i_S(\delta_i, \gamma_i) \leq \frac{2k}{2} = k\). If we close up \(\delta_i\) by concatenating with an arc of \(\partial S\) to obtain the simple closed curve \(\tilde{\delta}_i \subset S\), then \(i_S(\tilde{\delta}_i, \gamma_0) \leq 3\). (Recall that \(\delta_i\) is not boundary-parallel on \(S\), so \(\tilde{\delta}_i\) is neither homotopically trivial nor parallel to \(\partial S\).) By Lemma 2.1, \(d(\tilde{\delta}_i, \gamma_0) \leq 7\). For simplicity, we write \(d(\tilde{\delta}_i, \gamma_0) \approx 0\). Since \(\delta_i\) and \(\delta_j\) are disjoint, it is not hard to see that \(d(\tilde{\delta}_i, \tilde{\delta}_j) \approx 0\) for all \(i, j\). Hence \(d(\tilde{\delta}_j, \gamma_0) \approx 0\) for any \(j\). Acting by \(h\), we have \(d(h(\tilde{\delta}_j), h(\gamma_0)) \approx 0\). Since \(d(h(\tilde{\gamma}_0), \gamma_0) = N \gg 0\), we have \(d(h(\tilde{\delta}_j), \gamma_0) \approx N\) for all \(j\). This, in turn, implies that \(i_S(h(\tilde{\delta}_j), \gamma) \gtrapprox \frac{N}{2}\) and \(i_S(h(\delta), \gamma') \gtrapprox \frac{N}{2}k \gg 2k\). Since \(i_S(h'(\delta), \gamma') = i_S(h(\delta), \gamma')\), we have a contradiction and \(\delta \neq h'(\delta)\).

Now that we know \(h'\) is not reducible, it remains to show that \(h'\) is not periodic, i.e., there is some \(\delta\) such that \((h')^j(\delta) \neq \delta\) for any \(j \in \mathbb{N}\). For simplicity, take \(\delta = \partial S\). Normalizing with respect to \(B\) and \(\gamma\) as before, \(h'(\delta) = R_{\gamma'}(\delta)\) has \(k_1 = 2\) intersections with \(a\) and \(n_1 = 0\) intersections with \(\gamma'\) away from \(B\). (Clearly, \(h'(\delta) \neq \delta\).) Suppose inductively that \((h')^i(\delta)\) has \(k_i\) intersections with \(a\) and \(n_i\) intersections with \(\gamma'\) away from \(B\), and that \(\frac{n_i}{k_i} \ll 1\) and \(k_i \to \infty\). Then, for each component \(((h')^i(\delta))_j\) of \((h')^i(\delta)\cap S\), we have \(d(((h')^i(\delta))_j, \gamma_0) \approx 0\) and \(d(h(((h')^i(\delta))_j), \gamma_0) \approx N\) for all \(j\). Hence \((h \circ (h')^i)(\delta)\) has \(k_i\) intersections with \(a\).
Figure 2. The case where some \((h(\delta))'_i\) have positive slope. The left-hand diagram depicts \(h(\delta)\) and the right-hand diagram depicts \(R_{\gamma'} \circ h(\delta)\). The shaded region is the 1-handle \(B\). In the picture, \(k = 6, m = 2,\) and \(n = 3\) for \(h(\delta)\).

and \(\gtrsim \frac{N}{2} k_i\) intersections with \(\gamma'\) away from \(B\). Using the same method as in the previous paragraphs, \((h')^{i+1}(\delta)\) has \(\gtrsim \frac{N}{2} k_i\) intersections with \(a\) and at most \(k_i\) intersections with \(\gamma'\) away from \(B\). Hence \(\frac{n_{i+1}}{k_{i+1}} \ll 1, k_{i+1} > k_i,\) and \((h')^i(\delta) \neq \delta\) for all \(i \in \mathbb{N}\).

**Case 2:** \(\partial S\) has two components. Suppose that \(\partial S\) has two components.

Let \(\gamma\) be an arc which connects the two components of \(\partial S\). We assign to \(\gamma\) a closed curve \(\alpha_{\gamma}\) as follows: Take a pair-of-pants neighborhood \(N \subset S\) of \(\gamma \cup \partial S\). Then let \(\alpha = \alpha_{\gamma}\) be the component of \(\partial N\) which is not a subset of \(\partial S\). On the other hand, given a closed curve \(\alpha\) which separates off a pair-of-pants \(N,\) we can recover \(\gamma,\) provided we allow the endpoints of \(\gamma\) to freely move on \(\partial S\).

We explain how to pick a suitable arc of stabilization \(\gamma\) which connects the two components of \(\partial S\). Let \(\gamma_0\) be any arc which connects the two components of \(\partial S,\) and \(\alpha_0 = \alpha_{\gamma_0}\). Let \(\mu\) be a stable lamination of a pseudo-Anosov \(g,\) where \(h\) is not freely homotopic to \(g.\) Then the sequence of iterates \(g^i(\alpha_0)\) converges to \(\mu\) as \(i \to \infty.\) Hence, by Lemma \ref{22}, there exists \(\alpha = g^n(\alpha_0), n \gg 0,\) so that \(d(\alpha, h(\alpha)) = N \gg 0.\) Now let \(\gamma\) be the arc for which \(\alpha_{\gamma} = \alpha.\) Then \(i_S(\gamma, h(\gamma))\) and \(i_S(\alpha, h(\alpha))\) are roughly proportional, with a proportionality factor of \(4.\) Here we are freely homotoping \(\gamma\) and \(h(\gamma)\) along \(\partial S\) so their minimal intersection number is realized.

Once we have picked \(\gamma\) so that \(\alpha_{\gamma}\) satisfies \(d(\alpha_{\gamma}, h(\alpha_{\gamma})) = N,\) the rest of the proof is almost identical to the case where \(\partial S\) has one boundary component. The only difference is the definition of \(\delta_i,\) when \(\delta_i\) has endpoints on the distinct components of \(\partial S.\) In this case we set \(\delta_i = \alpha_{\delta_i},\) as defined above.
This completes the proof of Theorem 1.1.

Using the same techniques we can prove the following:

**Corollary 3.1.** If $\partial S$ is connected, $h$ is pseudo-Anosov, and the fractional Dehn twist coefficient $c > 2$, then any elementary stabilization along a non-boundary-parallel arc $\gamma \subset S$ is pseudo-Anosov.

The fractional Dehn twist coefficient $c$ of a pseudo-Anosov $h$ is defined as follows: Let $H : S \times [0, 1] \to S$ be the free homotopy from $h(x) = H(x, 0)$ to its pseudo-Anosov representative $\psi(x) = H(x, 1)$. Define $\beta : \partial S \times [0, 1] \to \partial S \times [0, 1]$ by sending $(x, t) \mapsto (H(x, t), t)$, i.e., $\beta$ is the trace of the isotopy $H$ along $\partial S$. If we choose an oriented identification $\partial S \simeq \mathbb{R}/\mathbb{Z}$, then we can lift $\beta$ to $\tilde{\beta} : \mathbb{R} \times [0, 1] \to \mathbb{R} \times [0, 1]$ and set $f(x) = \tilde{\beta}(x, 1) - \tilde{\beta}(x, 0) + x$. The fractional Dehn twist coefficient $c \in \mathbb{Q}$ is the rotation number of $f$, i.e.,

$$c = \lim_{n \to \infty} \frac{f^n(x) - x}{n},$$

for any $x$.

The inequality $c > 2$ is certainly not optimal, but we will not pursue the optimal constant here.

**Proof.** We use the same notation as in the proof of Theorem 1.1 and highlight only the differences. If $\delta \subset S$, then $h(\delta) \neq \delta$ since $h$ is pseudo-Anosov. If $h(\delta)$ does not intersect $\gamma'$, then $h'(\delta) = h(\delta) \neq \delta$. Otherwise, $i_{\partial S}(h'(\delta), a) > 0$ whereas $i_{\partial S}(\delta, a) = 0$.

If $\delta \not\subset S$, then we normalize $\delta$ with respect to $B$ as in the proof of Theorem 1.1 and consider the arcs $\delta_i$ of $S$; the $\delta_i$ are non-boundary-parallel. Let $\pi : \tilde{S} \to S$ be the universal covering map. Fix a connected component of $\pi^{-1}(\partial S)$, which we call $L$. Let $\tilde{\gamma}_j$, $j \in \mathbb{Z}$, be the preimages of $\gamma$ which have an endpoint on $L$. If $p$ is an endpoint of $\delta_i$, denote by $n(\delta_i, p)$ the geometric intersection number, relative to the endpoints, of $\cup_j \tilde{\gamma}_j$ and a lift of $\delta_i$ which has an endpoint on $\pi^{-1}(p) \cap L$.

Next, given the pair $(\delta_i, p)$ and a component of $\partial S \cap \partial B$ which contains $p$, we define what it means for $(\delta_i, p)$ to be to the left or to the right of $\gamma$. Using the boundary orientation for $\partial S$, if (the relevant component of) $\partial S \cap \partial B$ intersects $p$ before it intersects $\gamma$, then $(\delta_i, p)$ is to the left of $\gamma$; otherwise, $(\delta_i, p)$ is to the right of $\gamma$. (Remember that the $\delta_i$ are normalized with respect to $\gamma'$.)

If $(\delta_i, p)$ is to the right of $\gamma$, then we claim that $n((h(\delta))_i, q) > 4$, where $q$ is an endpoint of $(h(\delta))_i$. To see this, let $g = (R_{\partial S})^2$. The arc $g(\delta_i)$ satisfies $n(g(\delta_i), p) = 4$. Since $c > 2$, $h \circ g^{-1}$ is pseudo-Anosov with fractional Dehn twist coefficient $c - 2 > 0$, and hence is right-veering. Therefore $h(\delta_i)$ is to the right of $g(\delta_i)$, and $n(h(\delta_i), p) \geq n(g(\delta_i), p) = 4$. When $(\delta_i, p)$ is to the right of $\gamma$, we may take $(h(\delta))_i = h(\delta_i)$ and $p = q$. This proves $n((h(\delta))_i, q) \geq 4$. On the other hand, if $(\delta_i, p)$ is to the left of $\gamma$, then a similar calculation yields

$$(3.0.1) \quad n((h(\delta))_i, q) + n(\delta_i, p) > 3.$$
In either case, Equation 3.0.1 is satisfied. This means that, if $k$ is the number of components of $S \cap \delta$ and $m, m'$ are the number of intersections of $\delta, h(\delta)$ with $\gamma$ outside of $B$, then

$$m + m' \geq \sum (n(h(\delta)), q) + n(\delta, p) > 6k.$$  

(Remember that each $\delta_i$ has two endpoints.) In order for $i_s(\delta, a) = i_s(h'(\delta), a)$, we require $m' \leq k$. This implies that $m > 5k$, which contradicts $i_s(\delta, \gamma') = i_s(h'(\delta), \gamma')$.

We have shown that $h'$ is not reducible. To show that $h'$ is not periodic, consider $\delta = \partial S$. One easily verifies that the number of intersections with $a$ increases with each iterate of $h'$.

\textbf{Acknowledgements.} We thank Yair Minsky for helpful e-mail correspondence.

\textbf{References}

[AS] A. Abrams and S. Schleimer, \textit{Distances of Heegaard splittings}, Geom. Topol. \textbf{9} (2005), 95–110.

[Bo] F. Bonahon, \textit{Closed Curves on Surfaces}, monograph in progress.

[CB] A. Casson and S. Bleiler, \textit{Automorphisms of Surfaces after Nielsen and Thurston}, London Mathematical Society Student Texts \textbf{9}, Cambridge University Press, Cambridge, 1988.

[FLP] A. Fathi, F. Laudenbach and V. Poenaru, \textit{Travaux de Thurston sur les surfaces}, Astérisque \textbf{66-67}, Société Mathématique de France (1991/1971).

[Gi] E. Giroux, \textit{Géométrie de contact: de la dimension trois vers les dimensions supérieures}, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 405–414, Higher Ed. Press, Beijing, 2002.

[Har] W. Harvey, \textit{Boundary structure of the modular group}, Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), pp. 245–251, Ann. of Math. Stud., 97, Princeton Univ. Press, Princeton, N.J., 1981.

[He] J. Hempel, \textit{3-manifolds as viewed from the curve complex}, Topology \textbf{40} (2001), 631–657.

[HKM] K. Honda, W. Kazez and G. Matić, \textit{Right-veering diffeomorphisms of compact surfaces with boundary}, Invent. Math., to appear.

[Kl] E. Klarreich, \textit{The boundary at infinity of the curve complex and the relative Teichmüller space}, preprint 1999. \url{http://www.nasw.org/users/klarreich/publications.htm}.

[Ko] T. Kobayashi, \textit{Heights of simple loops and pseudo-Anosov homeomorphisms}, Braids (Santa Cruz, CA, 1986), 327–338, Contemp. Math., 78, Amer. Math. Soc., Providence, RI, 1988.

[MM] H. Masur and Y. Minsky, \textit{Geometry of the complex of curves I: Hyperbolicity}, Invent. Math. \textbf{138} (1999), 103–149.

[Th] W. Thurston, \textit{On the geometry and dynamics of diffeomorphisms of surfaces}, Bull. Amer. Math. Soc. (N.S.) \textbf{19} (1988), 417–431.