Specializing Cubulated Relatively Hyperbolic Groups

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Abstract. In [1], Agol proved the Virtual Haken and Virtual Fibering Conjectures by confirming a conjecture of Wise: Every cubulated hyperbolic group is virtually special. We extend this result to cocompactly cubulated relatively hyperbolic groups with minimal assumptions on the parabolic subgroups. Our proof proceeds by first recubulating to obtain an improper action with controlled stabilizers (a weakly relatively geometric action), and then Dehn filling to obtain many cubulated hyperbolic quotients. We apply our results to prove the Relative Cannon Conjecture for certain cubulated or partially cubulated relatively hyperbolic groups.

One of our main results (Theorem A) recovers via different methods a theorem of Oregón-Reyes [29].

1. Introduction

If \( G \) is a group with a proper, cellular cocompact action on a CAT(0) cube complex \( X \), we say \( G \) is cubulated by \( X \). (Some authors drop the cocompactness assumption.) Actions on cube complexes typically arise via a construction of Sageev [33] from collections of codimension-one subgroups.

A cube complex is special if it admits a locally isometric immersion to the Salvetti complex of some right-angled Artin group (see [19], where this notion was introduced, and called A-special). The group \( G \) is specially cubulated by \( X \) if the action can be chosen to be free, with quotient a special cube complex. Such a group \( G \) embeds into a finitely generated right-angled Artin group, so in particular it is linear over \( \mathbb{Z} \). It also inherits many useful separability properties from the right-angled Artin group.

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The group $G$ is \textit{virtually specially cubulated by $X$} if there is a finite index subgroup $G_0$ of $G$ which acts freely on $X$ with quotient a special, compact cube complex. The $\mathbb{Z}$–linearity and nice separability properties of $G_0$ are passed on to $G$, so this is nearly as useful a property as being specially cubulated. Indeed, the Virtual Haken and Virtual Fibering Theorems of Agol \cite{1} are proved by showing that any hyperbolic group which is cubulated is virtually specially cubulated.

We note that not all cubulated groups are virtually specially cubulated. Indeed Wise showed there are infinite cubulated groups with no finite index subgroups \cite{38}; Burger and Mozes showed there are even infinite simple cubulated groups \cite{7, 8}. By taking free products of such groups, one can produce cubulated \textit{relatively} hyperbolic groups which are not virtually specially cubulated. This suggests that to extend Agol’s result about cubulated hyperbolic groups to relatively hyperbolic groups it is necessary to make some assumptions about the parabolic subgroups.

1.1. \textbf{Main results.} The following appear to us to be the minimal possible assumptions on a relatively hyperbolic pair $(G, P)$ acting geometrically on a CAT(0) cube complex $X$, which might allow us to conclude that $G$ acts virtually co-specially on $X$.

\begin{assumption}
For each hyperplane stabilizer $S$ and each $P \in \mathcal{P}$ the intersection $P \cap S$ is separable in $P$.
\end{assumption}

\begin{assumption}
For each pair of hyperplane stabilizers $S_1, S_2$ and each $P \in \mathcal{P}$, the double coset $(S_1 \cap P) (S_2 \cap P)$ is separable in $P$.
\end{assumption}

The following is our main result, which recovers via different methods a theorem of Oregón-Reyes \cite{29}.

\begin{theorem}
\cite[Theorem 1.2]{29}
Suppose that $(G, \mathcal{P})$ is relatively hyperbolic and that $G$ acts properly and cocompactly on a CAT(0) cube complex $X$ so Assumptions 1.1 and 1.2 are satisfied. Then $G$ acts virtually co-specially on $X$.
\end{theorem}

\begin{remark}
Suppose that $(G, \mathcal{P})$ is relatively hyperbolic and that $G$ acts properly and cocompactly on a CAT(0) cube complex $X$. It follows from Sageev–Wise \cite[Theorem 1.1]{35} that for each $P \in \mathcal{P}$ there exists a convex $P$–cocompact sub-complex $Y_P$ in $X$. It follows from Haglund–Wise \cite[Corollary 4.3]{20} that each $P \setminus Y_P$ is virtually special if and only if $(G, \mathcal{P})$ satisfies Assumptions 1.1 and 1.2. If $G \setminus X$ is virtually special then $P \setminus Y_P$ certainly is. This justifies our belief that Assumptions 1.1 and 1.2 are minimal.
\end{remark}
Remark 1.4. In case $G$ is cubulated and the elements of $\mathcal{P}$ are virtually abelian, Assumptions 1.1 and 1.2 always hold. This applies in particular to cubulated fundamental groups of finite volume hyperbolic manifolds and orbifolds.

We use the special case where elements of $\mathcal{P}$ are abelian to prove the Relative Cannon Conjecture for groups which are cubulated (Corollary 6.2) and for those which admit a weakly relatively geometric action on a CAT(0) cube complex (Theorem 6.1). This second result strengthens one of Einstein–Groves [13]. See Definition 1.5 for the definition of a weakly relatively geometric action.

While both assumptions hold whenever $G$ acts virtually co-specially, it is unclear whether Assumption 1.2 is really necessary (see [39, Problem 13.38]). In case the parabolics are hyperbolic, Assumption 1.2 follows from Assumption 1.1 by work of Minasyan [27].

Much of our analysis does not depend on Assumption 1.2, and in particular under only Assumption 1.1 we are able to prove that various relatively quasi-convex subgroups are separable. In particular Theorem 4.7 and Theorem C have the following consequence, which is not obviously accessible via Oregón-Reyes’ methods.

**Theorem B.** Suppose $(G, \mathcal{P})$ is relatively hyperbolic and that $G$ acts properly cocompactly on a CAT(0) cube complex so that Assumption 1.1 is satisfied. Then every hyperplane stabilizer is separable in $G$.

More generally Theorem 4.7 states that relatively quasi-convex subgroups are separable whenever their intersections with parabolics are separable in those parabolics.

1.2. Our strategy. Our proof of Theorem A has two steps. The first, which takes up the bulk of this paper, is to replace the given cubulation of $G$ by a weakly relatively geometric cubulation. In the following definition, recall that a full subgroup of a relatively hyperbolic group is one whose intersection with any parabolic subgroup is either finite or finite index in the parabolic.

**Definition 1.5.** If $(G, \mathcal{P})$ is relatively hyperbolic, and $X$ is a CAT(0) cube complex, an action $G \acts X$ is weakly relatively geometric if the following hold:

1. each $P \in \mathcal{P}$ fixes some point of $X$;
2. $G$ acts cocompactly on $X$; and
3. if $\sigma$ is a cell of $X$ with infinite stabilizer, then
   a. $\text{Stab}(\sigma)$ is full relatively quasi-convex, and
(b) Stab(σ) is the fundamental group of a graph of groups, where each edge group is finite and each vertex group is either finite or full parabolic.

The notion of a relatively geometric action was defined in [13]; it can be obtained from the above by strengthening (3) to the requirement that every cell stabilizer is either finite or full parabolic.

Under Assumption 1.1 we put a new wallspace structure on the vertices of X. We first find appropriate sub-complexes \( \{ X_P \mid P \in \mathcal{P} \} \) stabilized by the peripheral subgroups and having certain “superconvexity” properties. Translating these around we get a \( G \)-equivariant family of parabolic sub-complexes with uniformly bounded pairwise intersection. For each \( G \)-orbit of hyperplane H of X, we amalgamate the hyperplane stabilizer with finite index subgroups of the stabilizers of all the parabolic sub-complexes it meets, obtaining a relatively quasi-convex subgroup B. Now we associate a wall \( W_H \) to H, declaring two vertices \( v, w \) to be on the same “side” of this wall if there is a path joining them which crosses \( B \cdot H \) an even number of times. These finitely many walls are completed to a \( G \)-equivariant wallspace structure \( W \) on \( X^{(0)} \). This is described in more detail in Section 2. In Section 3 we prove that the action on the cube complex dual to this wallspace structure is weakly relatively geometric, establishing:

**Theorem C.** Suppose \( (G, \mathcal{P}) \) is relatively hyperbolic, and that \( G \) acts properly cocompactly on a CAT(0) cube complex \( X \), and that this action satisfies Assumption 1.1.

There exists a weakly relatively geometric action of \( G \) on a CAT(0) cube complex \( \hat{X} \).

For the second step in the proof of Theorem A, we use tools developed in [16, 15] to show separability of the hyperplane subgroups and double cosets in the original cubulation. By a criterion of Haglund–Wise [20, Corollary 4.3], this implies virtual specialness.

1.3. **Outline and conventions.** In Section 2 we recall and setup the basic properties of the geometry of a CAT(0) space \( X \) upon which a relatively hyperbolic group acts properly and cocompactly. Under Assumption 1.1 we define augmented walls from the hyperplanes of \( X \) which will be used to define a new wallspace structure on \( X \). In Section 3 the cube complex associated to this wallspace structure is analyzed and Theorem C is proved. In Section 4 Dehn fillings are used to deduce Theorem A as well as Theorem B. In Section 5 we provide an example which is virtually special because of Theorem A, but not obviously because of existing results. In Section 6 we provide
the promised applications towards the Relative Cannon Conjecture. Finally, in Appendix A we prove a technical generalization of a result from [15] which is required for the proof of Theorem A.

Convention 1.6. In this paper, whenever we speak of a relatively hyperbolic group pair \((G, \mathcal{P})\) we always assume that each element of \(\mathcal{P}\) is infinite. We may always ensure this by removing the finite elements from \(\mathcal{P}\), which does not affect relative hyperbolicity. The removal of these finite parabolics also does not affect whether a given action is weakly relatively geometric in the sense of Definition 1.5.

Note also that we use the notation \(H^g = gHg^{-1}\).

2. A NEW WALLSPACE STRUCTURE

Throughout this section and the next we fix a relatively hyperbolic group pair \((G, \mathcal{P})\) and a proper cocompact \(G\)–action on a CAT(0) cube complex \(X\) satisfying Assumption 1.1.

The goal of this section and the next is to prove Theorem C. In the current section we describe a wallspace structure which gives rise to a new CAT(0) cube complex \(\hat{X}\); in the next we prove that the \(G\)–action on \(\hat{X}\) is weakly relatively geometric. Our wallspace has underlying set \(X^{(0)}\); the new walls come from amalgamations of the original hyperplanes with carefully chosen finite index subgroups of parabolic groups. They are similar in spirit to the augmented hyperplanes used in [3], but we do not work in an augmented cube complex as in that paper.

Definition 2.1. A wall in a set \(S\) is a partition of \(S\) into two nonempty subsets usually written \(W = \{W^+, W^-\}\). The two elements of the partition are called the halfspaces from the wall. A wall separates \(a\) from \(b\) in \(S\) if \(a\) and \(b\) lie in different halfspaces from the wall. A space with walls is a pair \((S, \mathcal{W})\) where \(\mathcal{W}\) is a collection of nonempty subsets of \(S\) which is closed under complementation and which satisfies the finiteness condition:

\[
\forall x, y \in S, \#\{W \subseteq \mathcal{W} | W \text{ separates } x \text{ from } y\} < \infty \quad (*)
\]

Two walls \(W = \{W^+, W^-\}\) and \(V = \{V^+, V^-\}\) are said to cross if for every choice of \(\epsilon_V, \epsilon_W \in \{+, -\}\), the intersection \(W^{\epsilon_W} \cap V^{\epsilon_V}\) is non-empty.

This is a special case of a wallspace as studied in [25] (see also the references therein, including [33, 28, 9, 18]). In the current paper we use the term “wallspace” in this paper interchangeably with “space with walls”. A wallspace gives rise to a CAT(0) cube complex in a way
which we revisit in the next section. Conversely, the hyperplanes of the CAT(0) cube complex $X$ give a wallspace structure on $X^{(0)}$ in an obvious way. In the current section we put a less obvious wallspace structure on $X^{(0)}$ which includes information about the peripheral structure. To begin, we need to see that peripheral structure in the cube complex $X$.

2.1. Peripheral complexes. In this subsection we find some sub-complexes of $X$ associated to the peripheral subgroups $\mathcal{P}$ and record some of their properties. In contrast to the peripheral sub-complexes used in [3], our peripheral complexes are not necessarily disjoint, though they have bounded overlap with each other.

The following definition is a slight variant of [12, Definition 2.5] (the difference being that we do not consider actually thin triangles to be relatively thin).

**Definition 2.2.** Suppose that $M$ is a geodesic metric space, let $a, b, c \in M$ and let $\Delta(a, b, c)$ be a geodesic triangle. Let $\pi: \Delta(a, b, c) \to Y_{abc}$ be the map to the comparison tripod. For $\nu \geq 0$, we say that $\Delta(a, b, c)$ is $\nu$–thin if for all $p \in Y_{abc}$ we have $\text{diam} (\pi^{-1}(p)) \leq \nu$.

If $U \subseteq M$ is a subset then $\Delta(a, b, c)$ is $\nu$–thin relative to $U$ if it is not $\nu$–thin and for every $p \in Y_{abc}$ either

1. $\text{diam} (\pi^{-1}(p)) \leq \nu$; or
2. $\pi^{-1}(p) \subseteq N_\nu(U)$.

**Definition 2.3** (Thin and fat parts of triangles). Let $\Delta(a, b, c)$ be a geodesic triangle in a metric space $M$, and let $\nu > 0$. Let $\pi: \Delta(a, b, c) \to Y_{abc}$ be the map to the comparison tripod. Let $o$ be the central point of the tripod. The *internal points* of $\Delta(a, b, c)$ are the points in $\pi^{-1}(o)$. The $\nu$–fat part of $\Delta(a, b, c)$ is the union of those fibers of $\pi$ with diameter $\geq \nu$. Let $x \in \{a, b, c\}$, and let $l_x$ be the leg of the tripod joining $\pi(x)$ to $o$. Let $t_x \subseteq l_x$ be a maximal connected subset containing $\pi(x)$ and so that $\text{diam} \pi^{-1}(p) < \nu$ for all $p \in t_x$. The set $\pi^{-1}(t_x)$ is a union of two segments starting at $x$, called the $\nu$–corner segments at $x$.

We remark that the $\nu$–fat part of a triangle is always closed. If the metric on $M$ is convex (for example CAT(0)) then any triangle is the union of its $\nu$–fat part and $\nu$–corner segments.

**Lemma 2.4.** If $\Delta$ is a triangle in a convex metric space which is $\nu$–thin relative to a set $Z$, and the fat part of some side has length at most $L$, then $\Delta$ is $2(L + \nu)$–thin.

**Proof.** In the terminology of Definition 2.3 $\text{insize}(\Delta) = \text{diam}(\pi^{-1}(o))$. In a convex metric space, $\text{insize}(\Delta) \leq K$ implies that $\Delta$ is $K$–thin, for any $K \geq 0$. It therefore suffices to bound $\text{insize}(\Delta)$. 
If the $\nu$–fat part of $\Delta$ is nonempty, then, using convexity of the metric, $\pi^{-1}(o)$ is in the fat part. Let $s$ be the side of $\Delta$ whose fat part has length at most $L$, and let $x$ be the point of $\pi^{-1}(o)$ contained in $s$. Let $y_1$ and $y_2$ be the other two points. We have $d(x, y_1) \leq 2L_1 + \nu$, and $d(x, y_2) \leq 2L_2 + \nu$, where $L_1 + L_2 = L$. Thus $d(y_1, y_2) \leq 2L + 2\nu$, and $\text{insize}(\Delta) \leq 2L + 2\nu$. □

**Definition 2.5.** Let $M$ be a geodesic metric space and $U$ a collection of subspaces. We say that the pair $(M, U)$ has $\nu$–relatively thin triangles if for every geodesic triangle $\Delta(a, b, c)$, either $\Delta(a, b, c)$ is $\nu$–thin or else there exists $U \in U$ so that $\Delta(a, b, c)$ is $\nu$–thin relative to $U$.

The following is an immediate consequence of [12, Proposition 2.7] (see also the proofs of [35, Theorem 4.1, Proposition 4.2] and [11, §8]).

**Proposition 2.6.** Fix $x \in X$ and let $U = \{gP \cdot x \mid P \in \mathcal{P}, gP \in G/P\}$. There exists $\nu$ so that $(X, U)$ has $\nu$–relatively thin triangles.

The following lemma is obvious.

**Lemma 2.7.** Suppose $(M, U)$ has $\nu$–relatively thin triangles, and that $V$ is a collection of subspaces so that every $U \in U$ is contained in some $V \in V$. Then $(M, V)$ has $\nu$–relatively thin triangles.

The next theorem is an immediate consequence of Sageev–Wise [35, Theorem 1.1].

**Theorem 2.8.** Let $x \in X$ be as in Proposition 2.6. For each $P \in \mathcal{P}$ there exists a convex $P$–invariant sub-complex $Z_P \subseteq X$ with $x \in Z_P$ so that $P \setminus Z_P$ is compact.

For each $P \in \mathcal{P}$ fix some $Y_P$ as in Theorem 2.8 and let $Z = \{g \cdot Z_P \mid P \in \mathcal{P}, g\}$, where $g$ ranges over a set of coset representatives of $G/P$. Combining Proposition 2.6 with Lemma 2.7 and Theorem 2.8 yields the following.

**Corollary 2.9.** There exists $\delta \geq 0$ so that $(X, Z)$ has $\delta$–relatively thin triangles.

The constant $\delta$ from Corollary 2.9 will remain fixed for the remainder of this section and the next.

The following definition is a slight variant of [12, Definition 5.1].

**Definition 2.10.** Let $\nu \geq 0$ and let $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a function. Suppose that $M$ is a geodesic metric space, and that $U$ is a collection of subspaces of $M$. The pair $(M, U)$ is a $(\nu, \phi)$–relatively hyperbolic pair if
(1) \((M, \mathcal{U})\) has \(\nu\)-relatively thin triangles; and
(2) For all \(r \geq 0\) and all \(F_1, F_2 \in \mathcal{U}\) with \(F_1 \neq F_2\), we have
\[
\text{diam} (N_r(F_1) \cap N_r(F_2)) \leq \phi(r).
\]
The subspaces \(\mathcal{U}\) are called peripheral subspaces.

The following is an immediate consequence of [11, Theorems 4.1, 5.1 and A.1] and Proposition 2.6.

Lemma 2.11. Let \(x \in X\) and \(\mathcal{U}\) be as in Proposition 2.6 and let \(\delta\) be as in Corollary 2.9. There is a function \(f_0: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\) so that \((X, \mathcal{U})\) is a \((\delta, f_0)\)-relatively hyperbolic pair.

The following easy fact is left to the reader.

Lemma 2.12. Let \(M\) be a geodesic metric space and suppose that \(\mathcal{U}\) is a set of subspaces so that \((M, \mathcal{U})\) is a \((\nu, \phi)\)-relatively hyperbolic pair (for some \(\nu \geq 0\) and function \(\phi\)). Suppose further that \(\mathcal{V}\) is a collection of subspaces of \(M\) and that there is \(r \geq 0\) and a bijection \(\rho: \mathcal{U} \to \mathcal{V}\) so that for all \(U \in \mathcal{U}\) we have \(U \subseteq \rho(U) \subseteq N_r(U)\). Then \((M, \mathcal{V})\) is \((\nu, \phi')\)-relatively hyperbolic pair where \(\phi'(x) = \phi(x + r)\).

The following is an immediate consequence of Lemmas 2.11 and 2.12.

Corollary 2.13. Let \(Z\) be as in Theorem 2.8, let \(r_0\) be the maximum diameter of \(P \setminus Z_P\) for \(P \in \mathcal{P}\), let \(f_0\) be the function from Lemma 2.11, let \(f_1\) be the function defined by \(f_1(x) = f_0(x + r_0)\), and let \(\delta\) be the constant from Corollary 2.9. Then \((X, Z)\) is a \((\delta, f_1)\)-relatively hyperbolic pair.

The subspaces \(Z\) from Theorem 2.8 are not the ones that we want. Instead, we want them to satisfy the following condition, which is a slightly stronger condition than that of ‘attractive’ in [12, Definition 5.2] in that we insist that most of the geodesic \([a, b]\) is contained in \(Z\), not just some of it.

Definition 2.14. Let \(M\) be a complete CAT(0) space and \(\eta \geq 0\). A convex subspace \(Z \subseteq M\) is \(\eta\)-super-attractive if for any \(K \geq 0\) and any pair of points \(a, b \in N_K(Z)\), all but the initial and terminal segments of \([a, b]\) of length \((\eta + K)\) are contained in \(Z\).

Remark 2.15. Let \(K: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\) be a function. Einstein [12, Definition 5.2] defines a subspace \(Z\) of a geodesic space to be \(K\)-attractive if, for every \(m \geq 0\), every geodesic with endpoints in the \(m\)-neighborhood of \(Z\) either meets \(Z\) or has length at most \(K(m)\). If \(Z\) is \(\eta\)-super-attractive, it is not hard to see that \(Z\) is \(K\)-attractive, for \(K(m) = 2(\eta + m)\).
We now fix notation for closest point projections.

**Notation 2.16.** Suppose that $M$ is complete and CAT(0), and that $A$ is a closed convex subset of $M$. We write $\pi_A : M \to A$ for the closest point projection map.

**Lemma 2.17.** Suppose that $M$ is complete CAT(0), $C$ is a collection of closed convex subspaces, and $(M, C)$ is $(\nu, \phi)$–relatively hyperbolic. Let $Z \in C$, let $\pi_Z : M \to Z$ be closest point projection, and let $a, b \in X$. Let $Q$ be the geodesic quadrilateral with vertices $\pi_Z(a), a, b, \pi_Z(b)$. Suppose

$$\nu^\prime = 4\nu + 2\phi(\nu), \quad \nu^{\prime\prime} = 4\nu + 2\phi(\nu^\prime), \quad \Delta(\nu, \phi) = \nu^\prime + \nu^{\prime\prime}. \quad (1)$$

Then either $d(\pi_Z(a), \pi_Z(b)) \leq \phi(\nu^\prime) + 2\nu + 3\nu^\prime$ or the quadrilateral $Q$ is $\Delta(\nu, \phi)$–slim.

**Proof.** We assume that $d(\pi_Z(a), \pi_Z(b)) > \phi(\nu^\prime) + \nu + 3\nu^\prime$. (2)

We divide $Q$ by a diagonal into an “upper” triangle with vertices $a, b, \pi_Z(b)$ and a “lower” triangle with vertices $a, \pi_Z(a), \pi_Z(b)$. See Figure 1.

![Figure 1](fig:slimquad)

We show that $Q$ is $\Delta(\nu, \phi)$–slim by showing that the lower triangle is $\nu^\prime$–thin and the upper triangle is $\nu^{\prime\prime}$–thin.

We examine the lower triangle first. We may suppose it is not $\nu$–thin. There is therefore some $W \in C$ so that it is $\nu$–thin relative to $W$.

If $W = Z$, we argue as follows. Let $a_1$ be the point of the fat part of $[a, \pi_Z(a)]$ closest to $a$. Since $d(a_1, Z) \leq \nu$, the segment $[a_1, \pi_Z(a)]$ can have length at most $\nu$. The segment $[a_1, \pi_Z(a)]$ contains the $\nu$–fat part of $[a, \pi_Z(a)]$, and so the lower triangle is $4\nu$–thin by Lemma 2.4.

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1Recall a polygon is $\mu$–slim if each side is contained in the $\mu$–neighborhood of the union of the other sides.
In case $W \neq Z$, we argue that the $\nu$-fat part of $[\pi_Z(a), \pi_Z(b)]$ is bounded. Indeed, the entire segment is contained in $Z$, so the part which is also in a $\nu$-neighborhood of $W$ has diameter bounded by $\phi(\nu)$. Since the $\nu$-fat part of one side of the lower triangle has length at most $\phi(\nu)$, Lemma 2.4 shows that the lower triangle is $2(\phi(\nu) + \nu)$-thin. In either case, we conclude that the lower triangle is $4\nu + 2\phi(\nu)$-thin.

We now show the upper triangle is $\nu''$-thin. Again, we may assume that the upper triangle is not $\nu$-thin, so it is $\nu$-thin relative to some $V \in C$.

If $V = Z$, then we argue as for the lower triangle, concluding that the $\nu$-fat part of the side $[b, \pi_Z(b)]$ must have length at most $\nu$, so the upper triangle is $4\nu$-thin by Lemma 2.4.

If $V \neq Z$, we argue as follows. We first note something about the lower triangle: Since $[\pi_Z(a), \pi_Z(b)]$ lies entirely in $Z$, and $\pi_Z(a)$ is the closest point of $Z$ to $a$, the Gromov product $(a \mid \pi_Z(b))_{\pi_Z(a)}$ is at most $\nu'$. Let $x$ be the point on the diagonal $[a, \pi_Z(b)]$ which is in the preimage of the central point of the comparison tripod for the lower triangle. The segment $[x, \pi_Z(b)]$ lies in $N_{\nu'}(Z)$ and has length in the interval $[d(\pi_Z(a), \pi_Z(b)) - \nu', d(\pi_Z(a), \pi_Z(b))]$.

Now consider the location of $x$ in the upper triangle. There are three cases, depending on whether $x$ is in the $\nu$-corner segment at $a$, the $\nu$-fat part, or the $\nu$-corner segment at $\pi_Z(b)$.

Suppose $x$ lies in the $\nu$-corner segment at $a$. Then the $\nu$-fat part of the upper triangle meets the diagonal in a subsegment of $[x, \pi_Z(b)] \subseteq N_{\nu'}(Z)$. Since $V \neq Z$, this subsegment has length at most $\phi(\nu')$. By Lemma 2.4, the upper triangle is $\nu'' = (2\nu + 2\phi(\nu'))$-thin.

Suppose next that $x$ lies in the $\nu$-fat part of the upper triangle. This $\nu$-fat part can only extend $\phi(\nu')$ past $x$. Consider the point $w$ on the $\nu$-fat part of $[b, \pi_Z(b)]$ closest to $\pi_Z(b)$. The corresponding point on $[a, \pi_Z(b)]$ is within $\nu'$ of $[\pi_Z(a), \pi_Z(b)] \subseteq Z$ (and exactly $\nu$ away from $w$), so $d(w, \pi_Z(b)) = d(w, Z) \leq \nu + \nu'$. This is also the distance from the $\nu$-fat part of $[a, \pi_Z(b)]$ to $\pi_Z(b)$, so we have $d(x, \pi_Z(b)) \leq \phi(\nu') + \nu + \nu'$. Since $d(x, \pi_Z(a)) \leq 2\nu'$, we have $d(\pi_Z(a), \pi_Z(b)) \leq \phi(\nu') + \nu + 3\nu'$, a contradiction to [2].

Finally suppose $x$ lies in the corner segment of the upper triangle adjacent to $\pi_Z(b)$. Let $x'$ be the point on $[\pi_Z(a), \pi_Z(b)]$ which is in the preimage of the central point of the comparison tripod for the lower triangle. The lower triangle is $\nu'$-thin, so $d(x, x') \leq \nu'$. Thus the point $x'$ is at most $\nu + \nu'$ from the corner segment of $[b, \pi_Z(b)]$ adjacent to $\pi_Z(b)$. Since $d(x', b) \geq d(\pi_Z(b), b)$, this corner segment must have length at most $\nu + \nu'$. Since $d(x', a) \geq d(\pi_Z(a), a)$ and the lower triangle is $\nu'$-thin, the distance from $\pi_Z(a)$ to $x'$ is at most $\nu'$. We
conclude that $d(\pi_Z(a), \pi_Z(b)) \leq d(\pi_Z(a), x') + d(x', \pi_Z(b)) \leq \nu + 2\nu'$, again contradicting \[2\]. □

**Corollary 2.18.** Suppose that $M$ is complete and CAT(0), $C$ is a collection of convex subspaces, and $(M, C)$ is $(\nu, \phi)$–relatively hyperbolic. There exists a constant $C(\nu, \phi)$, depending only on $\nu$ and $\phi$ so that for any distinct $Z, Z' \in C$ we have

$$\text{diam}(\pi_Z(Z')) \leq C(\nu, \phi).$$

**Proof.** Let $\nu', \nu''$ and $\Delta(\nu, \phi)$ be as in Lemma 2.17. We set $C(\nu, \phi) = \max \{\phi(\nu') + 2\nu + 3\nu', 2\Delta(\nu, \phi) + \phi(\Delta(\nu, \phi))\}$.

Fix $a, b \in Z'$. We have to prove that $d(\pi_Z(a), \pi_Z(b)) \leq C(\nu, \phi)$.

Let $Q$ be the geodesic quadrilateral with vertices $\pi_Z(a), a, b, \pi_Z(b)$. By Lemma 2.17 either $d(\pi_Z(a), \pi_Z(b)) \leq \phi(\nu') + 2\nu + 3\nu'$ or $Q$ is $\Delta(\nu, \phi)$–slim.

In the first case, we are done. Thus, suppose that $Q$ is $\Delta(\nu, \phi)$–slim. Therefore, the geodesic $\pi_Z(a), \pi_Z(b)$ is contained in the $\Delta(\nu, \phi)$–neighborhood of the other three sides. Using CAT(0) geometry, we see that the only points on $[\pi_Z(a), \pi_Z(b)]$ which lie within $\Delta(\nu, \phi)$ of $[a, \pi_Z(a)]$ lie within $\Delta(\nu, \phi)$ of $\pi_Z(a)$. Similarly, the only points on $[\pi_Z(a), \pi_Z(b)]$ lying within $\Delta(\nu, \phi)$ of $[b, \pi_Z(b)]$ lie within $\Delta(\nu, \phi)$ of $\pi_Z(b)$. Since $[a, b] \subseteq Z'$, the diameter of the set of points on $[\pi_Z(a), \pi_Z(b)]$ lying within $\Delta(\nu, \phi)$ of $[a, b]$ is at most $\phi(\Delta(\nu, \phi))$. Thus, in case $Q$ is $\Delta(\nu, \phi)$–slim, we have

$$d(\pi_Z(a), \pi_Z(b)) \leq 2\Delta(\nu, \phi) + \phi(\Delta(\nu, \phi)),$$

as required. □

The following is a strengthened version of [12, Proposition 3.3].

**Proposition 2.19.** There exists $\eta$ so that for each $P \in \mathcal{P}$ there exists a convex $\eta$–super-attractive $P$–invariant sub-complex $Y_P \subseteq X$ so that $Z_P \subseteq Y_P$ and $P \setminus Y_P$ is compact.

**Proof.** Let $Z$, $\delta$, and $f_1$ be as in the conclusions to Theorem 2.8 and Corollary 2.13. That is, for each $P \in \mathcal{P}$, there is a convex $P$–invariant $P$–cocompact sub-complex $Z_P$ of $X$, and the family $Z$ is the family of distinct translates of these $Z_P$ under the action of $G$. Corollary 2.13 says the pair $(X, Z)$ is a $(\delta, f_1)$–relatively hyperbolic pair. Let $\Delta_0 = \Delta(\delta, f_1)$ be the constant from Lemma 2.17.

If $E$ is a subset of a CAT(0) cube complex the *combinatorial hull* $\text{Hull}(E)$ is defined to be the intersection of the convex subcomplexes containing $E$. 
The argument at the beginning of Section 5 of [35] can be adapted to show that there is an $S$ (depending only on $\dim(X)$, $\delta$, $f_1$ and $\Delta_0 = \Delta(\delta, f_1)$) so that

$$\text{Hull}(N_{\Delta_0}(Z)) \subseteq N_S(Z) \quad \text{(3)}$$

for every $Z \in \mathcal{Z}$. For $P \in \mathcal{P}$, we take $Y_P = \text{Hull}(N_{\Delta_0}(Z_P))$. We set

$$\eta = \frac{1}{2} (2S + f_1(\delta') + 2\delta + 3\delta').$$

From now on we fix $P$, and set $Z = Z_P$, $Y = Y_P$.

Let $K \geq 0$, and let $a, b \in N_K(Y)$. By (3), $a, b \in N_{K+S}(Z)$. Let $\pi_Z$ be the closest point projection to $Z$, and consider the geodesic quadrilateral $Q$ with vertices $\pi_Z(a), a, b, \pi_Z(b)$.

Given Lemma 2.17, we argue as follows. In case

$$d(\pi_Z(a), \pi_Z(b)) \leq f_1(\delta') + 2\delta + 3\delta',$$

we have

$$d(a, b) \leq d(a, \pi_Z(a)) + d(\pi_Z(a), \pi_Z(b)) + d(\pi_Z(b), b) \leq 2(K + S) + f_1(\delta') + 2\delta + 3\delta' \leq 2(K + \eta),$$

and there is nothing to show.

Otherwise, every point on $[a, b]$ is within $\Delta_0$ of some point on $[a, \pi_Z(a)] \cup [\pi_Z(a), \pi_Z(b)] \cup [b, \pi_Z(b)]$.

Let $[a', b'] \subseteq [a, b]$ be the subsegment beginning and ending on $N_{\Delta_0}([a, \pi_Z(a)] \cup [b, \pi_Z(b)])$.

The lengths of the remaining subsegments $[a, a']$ and $[b, b']$ are at most $K + S + \Delta_0$. Since $[\pi_Z(a), \pi_Z(b)] \subseteq Z$, the subsegment $[a', b']$ lies in $N_{\Delta_0}(Z) \subseteq Y$, as desired. \[\square\]

**Definition 2.20.** For each $P \in \mathcal{P}$, fix a $P$–invariant sub-complex $Y_P$ of $X$ satisfying the conclusions of Proposition 2.19. Any sub-complex of $X$ of the form $gY_P$ for $g \in G, P \in \mathcal{P}$ is called a peripheral complex. We let $\mathcal{B}$ be the collection of all peripheral complexes in $X$.

The following is an immediate consequence of Corollary 2.13 and Lemma 2.12.

**Lemma 2.21.** Let $\delta$ be as in Corollary 2.9 and define the function $f: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by $f(x) = f_1(x + r_1)$, where $f_1$ is the function from Corollary 2.13 and $r_1$ is the maximum diameter of the spaces $p\backslash Y_P$ from Proposition 2.19. Then the pair $(X, \mathcal{B})$ is $(\delta, f)$–relatively hyperbolic.
For the remainder of this section and the next we fix the collection $\mathcal{B}$ of convex $\eta$–super-attractive peripheral complexes from Definition 2.20 so that $(X, \mathcal{B})$ is a $(\delta, f)$–relatively hyperbolic pair, for $\eta$ as in Proposition 2.19 and $\delta, f$ as in Lemma 2.21.

2.2. Augmented walls. Up to the $G$–action, there are finitely many hyperplanes in $X$. In this subsection we fix some such hyperplane $H$, and describe a partition $\mathcal{W}_H = \{W^+_H, W^-_H\}$ of $X^{(0)}$ associated to that hyperplane together with some extra data we now specify.

**Definition 2.22.** Suppose that $H$ is a hyperplane in $X$. Let $\text{Stab}(H)$ be the stabilizer of $H$, and let $Y_1, \ldots, Y_k$ be representatives of the $\text{Stab}(H)$–orbits of peripheral complexes that intersect $H$.

A **broadcaster** for $H$ is a subgroup of $G$ of the form $B = \langle \text{Stab}(H), B_1, \ldots, B_k \rangle$, where for each $i$ the subgroup $B_i$ is a finite-index subgroup of $\text{Stab}(Y_i)$.

Given a broadcaster $B$, the associated collection $B \cdot H$ of hyperplanes of $X$ is called the **scattering of $H$ by $B$**. Given $R > 0$, we say that $B \cdot H$ is an $R$–separated scattering if any two carriers of distinct hyperplanes in $B \cdot H$ are distance at least $R$ from each other.

We make use of the following result of Martínez-Pedroza, which we have slightly rephrased.

**Theorem 2.23.** [26, Theorem 1.1] Let $A < G$ be relatively quasi-convex, and let $P \in \mathcal{P}$. There is a finite set $F \subseteq P \setminus A$ so that if $\hat{P} < P$ satisfies $A \cap P \subseteq \hat{P} \subseteq P \setminus F$, then:

1. $\hat{A} := \langle A, \hat{P} \rangle$ is relatively quasi-convex;
2. $\hat{A} = A \ast_{A \cap P} \hat{P}$; and
3. Every parabolic subgroup of $\hat{A}$ is $\hat{A}$–conjugate either to a subgroup of $\hat{P}$ or a subgroup of $A$.

We will apply Theorem 2.23 to hyperplane stabilizers. This requires the (well-known) fact that they are relatively quasi-convex.

**Lemma 2.24.** Let $H$ be a hyperplane of $X$. Then $\text{Stab}(H)$ is relatively quasi-convex in $(G, \mathcal{P})$.

**Proof.** The quasi-isometry of pairs $(X, H) \to (G, \text{Stab}(H))$ implies that $\text{Stab}(H)$ is undistorted in $G$. Therefore by [24, Theorem 1.5] $\text{Stab}(H)$ is relatively quasi-convex in $(G, \mathcal{P})$. \qed
Proposition 2.25. Let $H$ be a hyperplane in $X$, let $\{Y_1, \ldots, Y_k\}$ be as above, and let $R > 0$.

There exists a broadcaster $B$ for $H$ so that $B$ is full relatively quasi-convex in $G$ and the associated scattering $B \cdot H$ is $R$–separated.

Proof. We use Theorem 2.23 to construct a sequence of relatively quasi-convex subgroups $\text{Stab}(H) = A_0 < A_1 < \cdots < A_k = B$ so that $A_i$ is an amalgam of $A_{i-1}$ with a finite index subgroup of $\text{Stab}(Y_i)$.

By Lemma 2.24, $A_0$ is relatively quasi-convex in $(G, P)$. We inductively assume that $A_{i-1}$ is relatively quasi-convex and that

$$A_{i-1} \cap \text{Stab}(Y_j) = A_0 \cap \text{Stab}(Y_j), \; \forall j \geq i.$$  \hspace{1cm} (†_i–1)

Applying Theorem 2.23 with $A = A_{i-1}$ and $P = \text{Stab}(Y_i)$, we obtain $F_i = F \subseteq \text{Stab}(Y_i) \setminus A_{i-1}$ as in the theorem. Assumption 1 guarantees the existence of a finite index $B_i < \text{Stab}(Y_i)$ which contains $A_0 \cap \text{Stab}(Y_i)$ but misses $F_i$. We can then apply Theorem 2.23 with $\check{P} = B_i$ to obtain a relatively quasi-convex $A_i = A_{i-1} \ast_{A_0 \cap \text{Stab}(Y_i)} B_i$.

In order to continue the induction we must establish Condition (†_i); this follows easily from [26, Lemma 5.4]. It also follows from that result that the final amalgam $B = A_k$ is full.

It remains to show that by expanding the finite sets $F_i$ if necessary we can ensure that the scattering $B \cdot H$ is $R$–separated. Since $B \cdot H$ consists of the $B$–orbit of a single hyperplane, it suffices to ensure that the distance from $H$ to $B \cdot H \setminus H$ is at least $R + 1$. For fixed $R$, there exists a finite set $\{g_1, \ldots, g_k\} \subseteq G \setminus \text{Stab}(H)$ so that any hyperplane within $R + 1$ of $H$ is of the form $a \cdot g_i \cdot H$ for some $a \in \text{Stab}(H)$. Since $\text{Stab}(H) \leq B$, if $a \cdot g_i \cdot H \in B \cdot H$ then $g_i \cdot H \in B \cdot H$ also. Therefore, we need to exclude the finite set $\{g_1 \cdot H, \ldots, g_k \cdot H\}$ from $B \cdot H$. It is easy to see (again using $\text{Stab}(H) \leq B$) that this is the same as excluding the finite set $\{g_1, \ldots, g_k\}$ from $B$. That this can be done (by choosing $F_i$ appropriately, depending on $R$) is an easy consequence of [26, Lemma 3.6 and Lemma 5.2] – we find a short geodesic between 1 and $g_i$ which has no large $\check{P}$–components, but on the other hand if $g_i \in B$ then there is some path in $B$ which is not contained in $\text{Stab}(H)$ and necessarily contains a large $\check{P}$–component. This contradicts [26, Lemma 3.6]. This argument completes the proof of the proposition. \hfill \Box

We are now ready to describe the augmented walls $W_H$. There is some freedom in this description; further constraints are imposed in Assumption 2.40 and Definition 2.41 where we pick the particular augmented walls for our wallspace.
Definition 2.26. Given a hyperplane $H$, and $R > 0$, fix a broadcaster $B$ whose associated scattering $B \cdot H$ is $R$–separated as in Proposition 2.25. Define an equivalence relation on $X^{(0)}$ by declaring $v \sim w$ if there is a path in the 1–skeleton of $X$ which crosses $B \cdot H$ an even number of times. Let $W_{H,B}$ be the pair of equivalence classes.

We observe:

Lemma 2.27. Let $B$ and $W_{H,B} = \{W_{H,B}^+, W_{H,B}^-\}$ be as above. Then both $W_{H,B}^+$ and $W_{H,B}^-$ are nonempty, $\text{Stab}(W_{H,B}) = B$, and $\text{Stab}(W_{H,B}^+) = \text{Stab}(W_{H,B}^-)$ is a subgroup of $B$ of index at most two.

In particular, $\text{Stab}(W_{H,B})$ is full relatively quasi-convex in $(G, \mathcal{P})$.

Remark 2.28. The hyperplane $H$ descends to an embedded hyperplane of $B \setminus X$. The index of $\text{Stab}(W_{H,B}^+)$ in $B$ is one if this hyperplane is separating; otherwise the index is two.

2.3. Connected quasi-convex sub-complexes associated to walls.

Later we fix an $R$–separated scattering $B \cdot H$ of $H$ with $R$ very large. Therefore, though $B \cdot H$ is quasi-convex, the quasi-convexity constant is very large. In order to associate a connected uniformly quasi-convex subspace of $X$ to the wall $W_{H,B}$, we adjoin peripheral complexes to the hyperplane carriers.

There are three results in this subsection used extensively in the sequel. In Proposition 2.37 we establish the uniform quasi-convexity of the thick carriers defined below in Definition 2.29. In Lemma 2.36 we quantify the statement that peripheral complexes which come near to a thick carrier for a long time must be contained in the thick carrier. In Lemma 2.32 we quantify the fact that two peripheral complexes in a thick carrier can only come near to each other if they are both near to a hyperplane of the scattering. It is crucial for our applications that all these results are uniform for $R$–scatterings, so long as $R$ is sufficiently large.

Definition 2.29 (Thick carrier). Let $H$ be a hyperplane in $X$, $R > 0$ and $B \cdot H$ an $R$–separated scattering of $H$.

Let $\mathcal{A}$ be the collection of carriers of translates $bH$ where $b \in B$. As before $\mathcal{B}$ is the set of peripheral complexes. Form an abstract cube complex

$$Y_{H,B} = \bigcup \mathcal{A} \bigsqcup \mathcal{B} / \sim.$$  

The equivalence relation $\sim$ is generated by the following identifications: For each $bH$ with $b \in B$ and carrier $A \in \mathcal{A}$, and each $Z \in \mathcal{B}$ meeting $bH$, glue $A$ to $Z$ along their intersection in $X$. Now let $S_{H,B}$ be the
component of \( Y_{H,B} \) containing \( H \). (The other components are just elements of \( B \).) There is a canonical map \( \phi_{H,B} : S_{H,B} \to X \) which restricts to the inclusion on any hyperplane carrier or peripheral complex. We define the thick carrier of \( W_{H,B} \) to be \( T(W_{H,B}) = \phi_{H,B}(S_{H,B}) \). The thick carrier \( T(W_{H,B}) \) is a union of convex sub-complexes.

We observe:

**Lemma 2.30.** \( T(W_{H,B}) \) is a connected sub-complex of \( X \) on which \( B \) acts properly cocompactly.

The next result uses work of Einstein [12] to show that \( T(W_{H,B}) \) is quasi-isometrically embedded and quasi-convex.

**Proposition 2.31.** There exist constants \( R_0 \geq 0 \), \( \lambda \geq 1 \), and \( \epsilon \geq 0 \) depending only on \((X,B)\) so that the following holds. Let \( R > R_0 \), let \( H \) be a hyperplane, and let \( B \cdot H \) be an \( R \)-separated scattering of \( H \). Let \( S = S_{H,B} \) and \( \phi_{H,B} \) be as in Definition 2.29. Then for any \( p,q \in S \) we have

\[
 d_X(\phi_{H,B}(p), \phi_{H,B}(q)) \leq d_S(p,q) \leq \lambda d_X(\phi_{H,B}(p), \phi_{H,B}(q)) + \epsilon. \quad (*)
\]

**Proof.** We would like to apply Einstein’s [12, Proposition 5.16], with \( A \) equal to the carriers of hyperplanes in the \( R \)-scattering \( B \cdot H \), and \( B_0 \) equal to the collection of peripheral complexes meeting some \( A \in A \). We have fixed a \((\delta, f)\) so that our pair \((X, B)\) is \((\delta, f)\)-relatively hyperbolic, but we have to temporarily modify \( f \) in order to apply Einstein’s results. This is allowable because \((X, B)\) is also \((\delta, f')\)-relatively hyperbolic for any function \( f' \) which dominates \( f \).

Hypothesis 5.5 of [12] requires that the pair \((X, B)\) is \((\delta, f')\)-relatively hyperbolic, and every \( B \in B \) is \( K \)-attractive, where \( K(m) = 3f'(5\delta) + 21\delta + 6m \). Each \( B \in B \) is \( \eta \)-super-attractive. Thus by Remark 2.15, each \( B \in B \) is \( K_0 \)-attractive, where \( K_0(m) = 2\eta + 2m \). As long as \( K \) dominates \( K_0 \), each \( B \) will also be \( K \)-attractive. In order to ensure \( K \) dominates \( K_0 \), we set

\[
 f'(r) = \max\{\eta, f(r)\}.
\]

Since \( f' \) dominates \( f \), the pair \((X, B)\) is still \((\delta, f')\)-relatively hyperbolic.

Now [12, Hypotheses 5.15] amounts to the assertion that \( A \) is an \( R \)-scattering for \( R \geq 500f'(5\delta) + 10000\delta \). We therefore take

\[
 R_0 = 500f'(5\delta) + 10000\delta.
\]

Then [12, Proposition 5.16] implies that \( \phi_{H,B} \) is a \((\lambda, \epsilon)\)-quasi-isometry, for \((\lambda, \epsilon) = (2, 110f'(5\delta) + 1592\delta) \). These constants only depend on
\((X, \mathcal{B})\). Since \(\phi_{H, \mathcal{B}}\) is obviously distance non-increasing, we have the inequalities \([\star]\).

We fix the constants \(R_0, \lambda, \epsilon\) from Proposition 2.31 for the remainder of this section and the next.

The next lemma says that peripheral complexes in a thick carrier can only be close to each other if they are close to a hyperplane associated to the carrier.

**Lemma 2.32.** There exists a function \(F_{2.32}: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\), depending only on \((X, \mathcal{B})\), so that the following holds whenever \(R > R_0\).

Let \(H\) be a hyperplane, let \(B\) be a broadcaster for \(H\) so that the associated scattering \(B \cdot H\) is \(R\)-separated, and let \(T(W_{H, \mathcal{B}})\) be the corresponding thick carrier. For any \(m \geq 0\) and any distinct peripheral complexes \(Z_1, Z_2\) in \(T(W_{H, \mathcal{B}})\), there is some \(g \in B\) so that \(N_m(Z_1) \cap N_m(Z_2) \subseteq N_{F_{2.32}(m)}(gH)\).

**Proof.** Fix \(p \in N_m(Z_1) \cap N_m(Z_2)\). For \(i = 1, 2\), let \(p_i\) be a point of \(Z_i\) with \(d_X(p_i, p) \leq m\), and let \(\hat{p}_i\) be the corresponding point of the complex \(S_{H, \mathcal{B}}\). Using the second inequality of \([\star]\) in Proposition 2.31, we have \(d_S(\hat{p}_1, \hat{p}_2) \leq \lambda d_X(p_1, p_2) + \epsilon \leq 2\lambda m + \epsilon\). Choose \(gH\) so that the hyperplane carrier for \(gH\) separates \(Z_1\) from \(Z_2\) in the complex \(S_{H, \mathcal{B}}\). Any path from \(\hat{p}_1\) to \(\hat{p}_2\) in \(S_{H, \mathcal{B}}\) must pass through the carrier of \(gH\), so in particular \(d_S(\hat{p}_1, gH) \leq 2\lambda m + \epsilon + \frac{1}{2}\). Thus (using the first inequality of \([\star]\)) the distance between \(p\) and \(gH\) is at most \(2\lambda m + \epsilon + m + \frac{1}{2}\). Since the diameter of \(N_m(Z_1) \cap N_m(Z_2)\) is at most \(f(m)\), we can set \(F_{2.32}(m) = 2\lambda m + \epsilon + m + \frac{1}{2} + f(m)\). \(\square\)

For the next two results we prove, we need another result from Ein-\hspace{1pt}stein \[12\]. Note that the result there uses peripheral cosets, whereas we will use the peripheral complexes \(Z\). Since their Hausdorff distance is finite, Proposition 2.34 follows from \[12\] Theorem 4.2.

**Definition 2.33** (Relative fellow traveling). Let \(l \geq 0\). Let \(\sigma_1\) and \(\sigma_2\) be two paths in \(X\) with the same endpoints, and suppose that these paths can be subdivided into an equal number of sub-paths, occurring in the same order. Suppose further that if \(J_{\sigma_1} \subseteq \sigma_1\) and \(J_{\sigma_2} \subseteq \sigma_2\) are corresponding subpaths (called paired subpaths), then they begin and end within \(l\) of one another and that either

1. the Hausdorff distance between \(J_{\sigma_1}\) and \(J_{\sigma_2}\) is at most \(l\); or
2. there exists some \(Z \in \mathcal{B}\) so \(N_l(Z)\) contains both \(J_{\sigma_1}\) and \(J_{\sigma_2}\).

Then \(\sigma_1\) and \(\sigma_2\) are said to \(l\)-relatively fellow travel, relative to \(\mathcal{B}\).
Proposition 2.34. (cf. [12, Theorem 4.2], [22, Proposition 4.6]) Let \( \mu \geq 1 \) and \( \kappa \geq 0 \). There exists \( l \), depending only on \( \mu , \kappa \), and the pair \( (X, \mathcal{B}) \), so that \((\mu , \kappa )\)-quasi-geodesics with the same endpoints in \( X \) must \( l \)-relatively fellow travel, relative to \( \mathcal{B} \).

The following lemma says that if a geodesic spends any time in a peripheral complex, then a quasi-geodesic with the same endpoints must spend a similar amount of time close to that peripheral complex. (Recall that \( \lambda , \epsilon \) are fixed constants coming from Proposition 2.31. If they were allowed to vary the lemma would still be true, but \( l' \) would have to be allowed to depend on them.)

Lemma 2.35. There is an \( l' \) depending only on \( (X, \mathcal{B}) \) so that the following holds. Let \( \gamma \) be a geodesic in \( X \), and let \( \sigma \) be a \((\lambda , \epsilon )\)-quasi-geodesic with the same endpoints as \( \gamma \). Let \( Z \) be some peripheral complex so that \( \gamma' = \gamma \cap Z \) is nonempty. Then there is a subpath \( \sigma' \) of \( \sigma \) whose endpoints are within \( l' \) of the endpoints of \( \gamma' \), and which is contained in an \( l' \)-neighborhood of \( Z \).

Proof. Let \( l \) be the constant of relative fellow travelling from Proposition 2.34 applied to \((\lambda , \epsilon )\)-quasi-geodesics.

We claim first that there are points on \( \sigma \) within

\[
l_1 = \max\{2l + \eta, f(l) + l\}
\]

of the endpoints of \( \gamma' \). Indeed, let \( a \) be an endpoint of \( \gamma' \), and let \( J_\gamma, J_\sigma \) be paired subpaths of \( \gamma \) and \( \sigma \) so that \( J_\gamma \) contains \( a \). There are three possibilities:

1. \( J_\gamma \) and \( J_\sigma \) are Hausdorff distance at most \( l \) from each other;
2. \( J_\gamma \) and \( J_\sigma \) are both contained in the \( l \)-neighborhood of \( Z \), or
3. \( J_\gamma \) and \( J_\sigma \) are both contained in the \( l \)-neighborhood of some peripheral complex \( W \neq Z \).

In case (1) we find a point of \( \sigma \) at most \( l \) away from \( a \). In case (2), we note that any component of \( J_\gamma \setminus Z \) has diameter at most \( l + \eta \), by \( \eta \)-super-attractiveness. Thus an endpoint of \( J_\gamma \) is at most \( l + \eta \) from \( a \). The corresponding endpoint of \( J_\sigma \) is at most \( 2l + \eta \) from \( a \). In case (3), we note that \( J_\gamma \cap Z \) has diameter at most \( f(l) \), so there is an endpoint of \( J_\gamma \) at most \( f(l) \) from \( a \). The corresponding endpoint of \( J_\sigma \) is at most \( f(l) + l \) from \( a \). In any case we have found a point of \( \sigma \) at most \( l_1 \) away from \( a \).

We let \( \sigma' \) be a subsegment of \( \sigma \) whose endpoints are at most \( l_1 \) from the endpoints of \( \gamma' \). Let \( \beta \) be the path obtained from \( \sigma' \) by adjoining
geodesics between the endpoints of \( \gamma' \) and those of \( \sigma' \). Then \( \beta \) is a \((\lambda, \epsilon + 4l_1)\)-quasi-geodesic with the same endpoints as \( \gamma \). Let \( l_2 \) be the constant of relative fellow travelling from Proposition 2.34 applied to \((\lambda, \epsilon + 4l_1)\)-quasi-geodesics. We claim that \( \beta \) (and hence \( \sigma' \)) lies in an \( l_3 \)-neighborhood of \( Z \), where

\[
l_3 = \frac{\lambda^2}{2} (f(l_2) + 2l_2 + \epsilon + 4l_1) + \epsilon + 4l_1 + l_2.
\]

Indeed, let \( b \) be a point of \( \beta \), and let \( J_{\gamma'} \) and \( J_{\beta} \) be paired subpaths of \( \gamma' \) and \( \beta \) so that \( b \in J_{\beta} \). As before there are three possibilities:

1. \( J_{\gamma'} \) and \( J_{\beta} \) are Hausdorff distance at most \( l_2 \) from each other;
2. \( J_{\gamma'} \) and \( J_{\beta} \) are both contained in the \( l_2 \)-neighborhood of \( Z \), or
3. \( J_{\gamma'} \) and \( J_{\beta} \) are both contained in the \( l_2 \)-neighborhood of some peripheral complex \( W \neq Z \).

In cases (1) and (2) there is clearly a point of \( Z \) within \( l_2 \) of \( b \). In case (3), the diameter of \( N_{l_2}(W) \cap N_{l_2}(Z) \) is at most \( f(l_2) \). Thus the segment \( J_{\gamma'} \) has length at most \( f(l_2) \). The endpoints of \( J_{\beta} \) are thus at most \( f(l_2) + 2l_2 \) apart from one another. A computation shows that \( b \) is at most

\[
\frac{\lambda^2}{2} (f(l_2) + 2l_2 + \epsilon + 4l_1) + \epsilon + 4l_1
\]

from one of these endpoints, and thus at most \( l_3 \) from some point on \( \gamma' \subseteq Z \).

Setting \( l' = l_3 \), we have established the lemma.

The next lemma shows that a thick carrier which has large coarse intersection with a peripheral complex must contain that peripheral complex.

**Lemma 2.36.** There exists a function \( F_{2.36} \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) and a constant \( R_1 \geq R_0 \) depending only on \((X, \mathcal{B})\) so that the following holds. Let \( B \cdot H \) be an \( R \)-separated scattering of some hyperplane \( H \), where \( R > R_1 \). Let \( T = T(W_{H, B}) \) be the associated thick carrier, and let \( Z \) be a peripheral complex.

For any \( d \geq 0 \), if

\[
\text{diam}(Z \cap N_d(T)) \geq F_{2.36}(d),
\]

then \( Z \subseteq T \).

**Proof.** We take \( R_1 = \max\{R_0, f(l')\} \), where \( l' \) is the constant from Lemma 2.35 and suppose \( R \geq R_1 \). Our proof is based on the following claim.
Claim. There is an $L_0$ depending only on $(X, B)$ so that if $a, b \in T$ and $[a, b]$ has a subsegment of length $> L_0$ in some peripheral complex $Z$, then $Z \subseteq T$.

Given the claim, we argue as follows. Fix $d > 0$. We define $F_{2350}(d) = L_0 + 4d + 2\eta$. Suppose that $x$ and $y$ satisfy the following three properties:

1. $x, y \in Z$;
2. $d(x, y) > F_{2350}(d)$; and
3. $d(x, T)$ and $d(y, T)$ are at most $d$.

Let $x', y'$ be closest points on $T$ to $x, y$ respectively. We have $d(x', y') > L_0 + 2d + 2\eta$. By $\eta$–super-attractiveness, a subsegment of $[x', y']$ of length greater than $L_0$ lies in $Z$. Applying the Claim with $a = x', b = y'$, we conclude $Z \subseteq T$.

It remains to prove the claim.

Proof of Claim. We prove the claim with $L_0 = 2f(l') + 4l' + 2\eta$. Let $T$ be the thick carrier as in the statement of the lemma and $S = S(W_{H, B})$ the associated abstract complex. Let $\sigma$ be the $(\lambda, \epsilon)$–quasi-geodesic joining $a$ to $b$ which is the image of a geodesic in $S$. Suppose that $\gamma = [a, b]$ is the $X$–geodesic with the same endpoints and that $\gamma$ intersects the peripheral complex $Z$ in a segment $\gamma'$ of length greater than $L_0$. Lemma 2.35 gives a subpath $\sigma'$ of $\sigma$ starting and ending within $l'$ of the endpoints of $L_0$, so that $\sigma'$ is contained in the $l'$–neighborhood of $Z$.

This subpath $\sigma'$ must have length bigger than $L_0 - 2l' = 2f(l') + 2l' + 2\eta$; it is a broken geodesic made alternately of segments inside peripheral complexes and carriers of hyperplanes.

If there are four or more such subsegments, one is a peripheral segment between two different hyperplane carriers in the $R$–scattering; in particular it has length at least $R > f(l')$. But since the endpoints of this peripheral segment are within $l'$ of $Z$, the peripheral segment must actually be a $Z$–segment, and so $Z$ was in $T$ to begin with. A similar argument shows $Z \subseteq T$ if $\sigma'$ begins or ends with a peripheral segment of length bigger than $f(l')$.

If there is a hyperplane segment of length greater than $2l' + 2\eta$, then the $\eta$–super-attractiveness of $Z$ implies that this hyperplane segment meets $Z$. In particular a hyperplane of the scattering meets $Z$, and again we conclude that $Z \subseteq T$.

The only remaining possibility is that $\sigma'$ consists of three or fewer segments, at most one of which is a (short) hyperplane segment. The total length of $\sigma'$ is thus at most $2f(l') + 2l' + 2\eta$, contrary to our choice of $L_0$. This completes the proof of the claim. \qed
Once the claim is proved, the lemma is proved. \hfill \square

**Proposition 2.37.** There is a $Q$ depending only on $(X, \mathcal{B})$ so that the following holds. Let $R > R_1$, where $R_1$ is the constant from Lemma 2.36. Let $\mathcal{B} \cdot H$ be an $R$–separated scattering of some hyperplane $H$ of $X$. The thick carrier $T(W_{H,B})$ is $Q$–quasi-convex in $X$.

**Proof.** We prove the Proposition with $Q = 2l + \eta + \frac{1}{2} F_{2,36}(2l + \eta)$, where $l$ is the constant of relative fellow-traveling of $(\lambda, \epsilon)$–quasi-geodesics in $(X, \mathcal{B})$, $\eta$ is the super-attractiveness constant of elements of $\mathcal{B}$, and $F_{2,36}$ is the function from Lemma 2.36.

Let $T = T(W_{H,B})$, and let $S = S(W_{H,B})$. Recall that $T$ is the image of $S$ under a $(\lambda, \epsilon)$–quasi-isometric embedding (Proposition 2.31). Let $\gamma$ be a geodesic with endpoints in $T$, and let $\sigma$ be the image of an $S$–geodesic with the same endpoints. Since $S$ is $(\lambda, \epsilon)$–quasi-isometrically embedded, $\sigma$ is a $(\lambda, \epsilon)$–quasi-geodesic in $X$. In particular it $l$–relatively fellow travels $\gamma$ relative to $B$.

Let $J_\gamma$ and $J_\sigma$ be paired subpaths of $\gamma$ and $\sigma$ respectively. In case they are Hausdorff distance $l$ from each other, we clearly have $J_\gamma \subseteq N_Q(T)$. Otherwise, they are both in the $l$–neighborhood of some peripheral complex $Z$. If the length of $J_\gamma$ is at most $2(l + \eta) + F_{2,36}(2l + \eta)$, it is also contained in the $Q$–neighborhood of $T$, since its endpoints are distance at most $l$ from the endpoints of $J_\sigma \subseteq T$. Otherwise, the $\eta$–super-attractiveness of $Z$ implies that there is a subsegment of $J_\gamma$ of length at least $F_{2,36}(2l + \eta)$, whose endpoints are distance at most $l + \eta$ from the endpoints of $J_\gamma$, hence at most $2l + \eta$ from the endpoints of $J_\sigma$. Lemma 2.36 then shows that $Z \subseteq T$, so $J_\gamma \subseteq N_Q(T)$ in this case as well. \hfill \square

### 2.4. The wallspace.

We now construct the wallspace which we use to build the cubulation for Theorem C. In order to do this, we first fix some constants, and our assumptions on the constant of separatedness of the scatterings we use.

**Notation 2.38.** Let $\eta$ be as in Proposition 2.19 and suppose that $\mathcal{B}$ is a collection of convex $\eta$–super-attractive complexes as in Definition 2.20. Let $\delta$ and $f$ be so that $(X, \mathcal{B})$ is $(\delta, f)$–relatively hyperbolic, as in the conclusion of Lemma 2.21. Let $n = \dim(X)$.

We now recall and fix some more constants, and note that like $\eta, \delta, f$ and $n$, all of them depend only on the pair $(X, \mathcal{B})$.

The constant $R_1$ previously appeared in Lemma 2.36 and Proposition 2.37. We will assume that all scatterings are $R$–scatterings for some $R > R_1$. 

---

**Not: understandable**
(1) $C = C(\delta, f)$ is the bound on the diameter of projections of peripheral complexes to one another (see Corollary 2.18).

(2) $Q$ is the constant of quasi-convexity of thick carriers of scatterings (see Proposition 2.37);

(3) $\theta_n = \arcsin\left(\frac{1}{\sqrt{n}}\right)$ (a geodesic in $X$ which is not too short must meet some hyperplane at this angle or greater, see Lemma 3.19 below).

**Notation 2.39.** The following constants also depend only on the pair $(X, \mathcal{B})$. They may seem somewhat mysterious but we need to choose them carefully now in order to fix $R_\star$ below.

(1) (first used in Lemma 3.21):

$$\kappa = F_{2.36}(Q + 1) + 2Q + C + \frac{\sqrt{n}}{2} + 2$$

(2) (first used in Lemma 3.17)

$$\omega = 2F_{2.36}(0) + \left(8 + \frac{2}{\sin\left(\frac{\theta_n}{2}\right)}\right)Q + 20\eta + 8f(\delta) + 16\delta + 2\sqrt{n};$$

(3) (first used in Lemma 3.17)

$$\tau = F_{2.36}(Q) + \left(3 + \frac{1}{\sin\left(\frac{\theta_n}{2}\right)}\right)Q + 4\eta + f(\delta) + 2\delta + \frac{\sqrt{n}}{2};$$

(4) (first used in Definition 3.22)

$$D = \max\{2\sqrt{n}\kappa + \sqrt{n}, F_{2.36}(0) + 6\eta + 2f(\delta) + 6\delta, 2C + 1\} + 1$$

**Assumption 2.40.** Fix once and for all some number $R_\star$ so that $R_\star$ is strictly bigger than all of the following:

(1) (Required so Propositions 2.31 and 2.37 may be applied)

$$R_1$$

(2) (required in Proposition 3.19)

$$F_{2.36}(0) + 4(\eta + \delta) + 2f(\delta) + \tau$$

(3) (required in Proposition 3.23)

$$8(Q + F_{2.36}(Q) + 2\eta + \delta + \frac{1}{2}f(\delta))$$

(4) (required in Proposition 3.23)

$$\frac{8}{3}(D + 2\eta + 2\delta + f(\delta))$$
The constant $R_*$ is the same for the remainder of this section and the next; we only consider $R_*$-scatterings from now on.

**Definition 2.41** (The wallspace $(X^{(0)}, W)$). Choose representatives $H_i$ of the finitely many $G$-orbits of hyperplanes of $X$, and for each $i$ choose a broadcaster $B_i$ so that $B_i \cdot H_i$ is an $R_*$-separated scattering. Such a broadcaster exists by Proposition 2.25. Let $W_i = W_{H_i, B_i}$ be the wall from Definition 2.26 and let $\phi_i = \phi_{H_i, B_i}$ be the map defined in Definition 2.29. It follows that $T(W_i)$, $\phi_i$ satisfy the conclusion of Propositions 2.31 and 2.37. Let $$W = \{ gA \mid g \in G, A \in W_i \text{ for some } i \}.$$ Since $W$ is defined $G$-equivariantly, there is an action of $G$ on the wallspace $(X^{(0)}, W)$.

The following is immediate from the fact that walls are unions of hyperplanes in $X$ and from the equivalence relation defined in Definition 2.26.

**Lemma 2.42.** Let $v, w$ be adjacent vertices of $X$. Then there is exactly one wall of $W$ separating $v$ from $w$.

**Definition 2.43.** Lemma 2.42 says in particular that each hyperplane of $X$ is associated to a single wall $W$ of $W$. Such a hyperplane is called a $W$-hyperplane. Let $W = gW_{H_i, B_i}$ for some $g \in G$ and one of the hyperplanes $H_i$ discussed in Definition 2.41. We write $T(W)$ for the complex $gT(W_{H_i, B_i})$ and note that it is $Q$-quasi-convex and a $(\lambda, \epsilon)$-quasi-isometric image of $S_{H_i}$ (see Proposition 2.31). We refer to $T(W)$ as the thick carrier for $W$. If $Z$ is one of the peripheral complexes used to build $S_{H_i}$, then $gZ$ is called a $W$-peripheral complex.

**Lemma 2.44.** Let $Z$ be a peripheral complex and let $W$ be a wall. Then $Z$ is a $W$-peripheral complex if and only if $W$ separates two vertices of $Z$.

**Lemma 2.45.** The collection $W$ defined in Definition 2.41 makes $X^{(0)}$ into a space with walls.
Proof. We must show the finiteness condition \((\star)\) from Definition 2.1. But Lemma 2.42 implies that the number of walls in \(W\) separating two vertices is at most their distance in \(X^{(1)}\).

\[\square\]

3. Ultrafilters and Stabilizers for the action

The goal of this section is to prove Theorem C. To that end, we fix a wallspace structure \((X^{(0)}, W)\) as described in Definition 2.41. By Sageev’s construction this gives rise to a new CAT(0) cube complex \(\hat{X} = C(W)\) with a \(G\)-action. Many properties of this cube complex (for instance dimension and the precise vertex stabilizers) depend strongly on the choices made in the last section. However, as long our cube complex is built from \(R^\star\)-scatterings where \(R^\star\) satisfies Assumption 2.40, this cube complex will satisfy the conclusions of Theorem C.

We briefly recall the Sageev construction (see [33, 34, 25] for more details). Given a wallspace \((S, V)\), an \textit{ultrafilter} on \((S, V)\) is a subset \(U\) of \(V\) satisfying the following two properties:

1. (Consistency) If \(A \subseteq B\) are two halfspaces and \(A \in U\), then \(B \in U\); and
2. (Completeness) If \(A\) is a halfspace, then either \(A\) or \(A^c\) is in \(U\).

A \textit{DCC ultrafilter} is one which contains no infinite descending chain \(A_1 \supseteq A_2 \supseteq \cdots\).

\textbf{Convention 3.1.} All ultrafilters considered below are assumed to be DCC.

An ultrafilter is \textit{principal} if, for some \(x \in S\), it is of the form:

\[\Omega[x] := \{A \in V \mid x \in A\}.\]

\textbf{Notation 3.2.} Completeness means we can also think of an ultrafilter \(\Omega\) as a function from the set of walls to the set of halfspaces. Abusing notation slightly, if \(W = \{A, A^c\}\) is a wall and \(\Omega\) is an ultrafilter so \(A \in \Omega\), we write \(\Omega(W) = A\).

We build a cube complex \(C'(\mathcal{V})\) as follows. The 0–cells are the DCC ultrafilters on \(\mathcal{V}\). Two ultrafilters are connected by an edge if they differ on a single wall. (In other words \(\Omega_1 = (\Omega_2 \setminus \{A\}) \cup \{A^c\}\).) Higher dimensional cubes are added wherever their 1–skeleta appear. The principal ultrafilters all lie in the same component of \(C'(\mathcal{V})\), and it is this component which we name \(C(\mathcal{V})\). We say \(C(\mathcal{V})\) is the \textit{cube complex obtained by applying the Sageev construction} to \(\mathcal{V}\).
The following result is essentially due to Sageev, though he did not phrase things in terms of wallspaces. See [25, Section 3], and the references therein. See also [34, 32] for a slightly more general discussion of the cube complex dual to a “pocset” (poset with complementation).

**Theorem 3.3.** \[33\] For any wallspace \((S, \mathcal{V})\), the cube complex \(C(\mathcal{V})\) is CAT(0).

### 3.1. Parabolic sub-complexes of \(\hat{X}\) are finite.

In [25, Section 3.4], Hruska and Wise introduce the notion of a hemiwallspace, a subcollection of the halfspaces of a wallspace containing at least one from every wall. When \((S, \mathcal{V})\) is a wallspace and \(T \subseteq S\), one obtains the hemiwallspace induced by \(T\) (see [25, Example 3.20]):

\[
\mathcal{V}_T = \{ U \in \mathcal{V} \mid U \cap T \neq \emptyset \}.
\]

This hemiwallspace also gives rise to a cube complex \(C(\mathcal{V}_T)\) defined to be the cube complex obtained by applying the Sageev construction to the collection of halfspaces in \(\mathcal{V}_T\) whose complements are also in \(\mathcal{V}_T\). We have the following result of Hruska–Wise.

**Lemma 3.4.** [25, Lemmas 3.23, 3.24] The cube complex \(C(\mathcal{V}_T)\) embeds naturally into \(C(\mathcal{V})\) as a convex sub-complex.

We apply this result with \(S = X(0), \mathcal{V} = \mathcal{W}, T = Y(0)\) for \(Y\) equal to some peripheral complex. In this case we have:

**Proposition 3.5.** \(C(\mathcal{W}_{Y(0)})\) is a finite cube complex.

**Proof.** We show that there are only finitely many walls \(W \subseteq \mathcal{W} \) so that \(W \subseteq \mathcal{W}_{Y(0)}\). The Sageev construction applied to a wallspace with finitely many walls yields a finite cube complex, so this establishes the proposition.

Suppose that \(W = \{W^+, W^-\} \subseteq \mathcal{W}_{Y(0)}\). This means both \(W^+ \cap Y(0)\) and \(W^- \cap Y(0)\) are nonempty. We first show that \(W\) is determined by the partition

\[
\{W^+ \cap Y(0), W^- \cap Y(0)\}.
\]

Indeed, since \(Y\) is connected, there is a pair of adjacent vertices \(v, w \in Y(0)\) lying in different halfspaces. But Lemma 2.42 shows that \(W\) is the only wall in \(\mathcal{W}\) separating \(v\) from \(w\).

We next claim that there is a fixed finite index subgroup of \(\text{Stab}(Y)\) which preserves the partition \(\{W^+ \cap Y(0), W^- \cap Y(0)\}\) for every \(W \subseteq \mathcal{W}_{Y(0)}\). Each such partition is preserved by some finite index subgroup of \(\text{Stab}(Y)\) conjugate in \(G\) to one of the subgroups of the \(\text{Stab}(Y_j)\) in the definition of the broadcasters for the \(H_i\). There are only finitely many such subgroups of \(\text{Stab}(Y)\), so some finite index subgroup \(N_Y\) of...
Stab(Y) is contained in all of them. In particular, W descends to a partition of \( N_Y \setminus Y^{(0)} \), and is determined by that partition.

Since there are only finitely many partitions of a finite set, the set of walls in \( \mathcal{W}_{Y^{(0)}} \) is finite.

Work of Hruska–Wise [25] gives the following.

**Corollary 3.6.** The action of \( G \) on \( C(W) \) is cocompact. In particular \( C(W) \) is finite dimensional. Moreover each \( P \in \mathcal{P} \) fixes a point of \( C(W) \).

**Proof.** By Lemma 2.27 the wall stabilizers for \( G \triangleright W \) are full relatively quasi-convex.

Theorem 7.12 of [25] (with \( \heartsuit = \frac{1}{4} \)) shows that for \( (G, \mathcal{P}) \) relatively hyperbolic acting on a wallspace with relatively quasi-convex wall stabilizers, the \( G \)-action on the associated cube complex is “relatively cocompact.” Since the sub-complexes associated to the peripheral subgroups are compact (cocompact under the action of the peripheral subgroups is enough), the \( G \)-action is cocompact.

Let \( P \in \mathcal{P} \). Then \( P \) preserves a peripheral complex \( Y \subseteq X \), so it preserves the convex sub-complex \( C(\mathcal{W}_{Y^{(0)}}) \subseteq \hat{X} \). As this is a complete bounded diameter \( \text{CAT}(0) \) space (Proposition 3.5), there is a fixed point for the action of \( P \) (see [6, II.2.8]).

3.2. **Associating a characteristic subspace of \( X \) to an ultrafilter.** In general not every ultrafilter \( U \) on \( W \) is principal. In other words the intersection of all the halfspaces in \( U \) may be empty. We show in this subsection that by appropriately augmenting and thickening the halfspaces of \( U \) we do obtain a nonempty intersection (see Proposition 3.12).

**Definition 3.7 (Halfspace carriers).** In Definition 2.43 we defined thick carriers for walls. If \( W = \{A, A^c\} \), then define the *halfspace carrier* \( \overline{A} \) to be the full sub-complex of \( X \) on \( A \cup (T(W) \cap X^{(0)}) \).

The full sub-complex on a halfspace \( A \) is not connected, but the halfspace carrier is connected, as the following lemma shows. We will use the observations in this lemma repeatedly in the rest of the section.

**Lemma 3.8.** Let \( A \in \mathcal{W} \) be a halfspace. Then \( \overline{A} \) is a connected subcomplex of \( X \). Moreover, \( \overline{A} \cap \overline{A^c} = T(W) \) and \( \overline{A} \cup \overline{A^c} = X \).

**Proof.** Every component of the full subcomplex on \( A \) meets \( T(W) \), which is connected. Thus \( \overline{A} \) is connected.
The intersection of full subcomplexes is a full subcomplex, so \(A \cap A^c\) is the full subcomplex on the zero-skeleton of \(T(W)\). This is \(T(W)\).

Any cube of \(X\) with all its vertices in \(A\) is in the full subcomplex on \(A\), so it is contained in \(\overline{A}\). Similarly any cube of \(X\) with all its vertices in \(A^c\) is contained in \(\overline{A^c}\). A cube with vertices in both \(A\) and \(A^c\) must be in the carrier of some hyperplane of the scattering associated to \(W\), hence in \(T(W)\). So \(\overline{A} \cup \overline{A^c} = X\).

Next we show that halfspace carriers have the same “attractiveness” property as we showed for thick carriers in Lemma 2.36.

**Lemma 3.9.** Let \(m > 0\) and let \(F_{m-\Theta}(m)\) be as in Lemma 2.36. Let \(\overline{A}\) be a halfspace carrier and let \(Z\) be a peripheral complex. If 
\[
\text{diam}(Z \cap N_m(\overline{A})) \geq F_{m-\Theta}(m),
\]
then \(Z \subseteq \overline{A}\).

*Proof.* Let \(W = \{A, A^c\}\) be the associated wall and let \(T = T(W)\). If \(Z\) contains vertices of both \(A\) and \(A^c\), it must meet some \(W\)–hyperplane, so \(Z \subseteq T \subseteq \overline{A}\). If \(Z\) contains only vertices of \(A\) then \(Z \subseteq \overline{A}\).

We are left with the case that every vertex of \(Z\) is in \(A^c\). Let \(x, y \in Z \cap N_m(\overline{A})\). Since \(x, y\) are not in \(A\), a shortest path to \(\overline{A}\) must pass through \(T\), so they are in \(N_m(T)\). Using the case of wall carriers, if 
\[
d(x, y) \geq F_{m-\Theta}(m),
\]
then \(Z \subseteq T \subseteq \overline{A}\), and we are done.

**Lemma 3.10.** Suppose \(W, V\) are walls of \(W\) so that \(V^+ \subseteq W^+\). Then 
\(T(V) \subseteq \overline{W^+}\).

*Proof.* It is enough to show that if \(v \in T(V)^{(0)}\), then \(v \in \overline{W^+}\). If \(v\) is in the carrier of a \(V\)–hyperplane, it is distance \(\leq 1\) from a vertex in \(V^+\). Since the \(W\)–hyperplanes and the \(V\)–hyperplanes are disjoint, this means \(v\) must actually be in \(W^+\).

Suppose then that \(v\) is contained in a \(V\)–peripheral complex \(Z\), and that \(v \in W^-\). Since \(Z\) is a \(V\)–peripheral complex, by Lemma 2.44 it must contain vertices in both \(V^+\) and \(V^-\). In particular it contains some \(w \in V^+ \subseteq W^+\). Since \(Z\) contains vertices of both \(W^+\) and \(W^-\), it must also be a \(W\)–peripheral complex, and so \(Z \subseteq \overline{W^+}\).

We next prove a lemma about coarse intersections of thick carriers of crossing walls.

**Lemma 3.11.** For each positive integer \(m\) there exists a \(\Theta_m \geq 0\) so that for any collection of pairwise crossing walls \(\{W_0, \ldots, W_m\}\) of \(W\),
\[
\bigcap_{i=0}^{m} N_{\Theta_m}(T(W_i)) \neq \emptyset.
\]
**Proposition 3.12.** Let $\mathcal{U}$ be a DCC ultrafilter. There is some $K \geq 0$ so that $\bigcap_{A \in \mathcal{U}} N_K(A) \neq \emptyset$. 

**Proof.** The proof is by induction, so we start by establishing the base case $m = 1$. We prove the contrapositive with $\Theta_1 = 0$. Suppose that there are walls $W_0, W_1$, so that $T(W_0) \cap T(W_1) = \emptyset$. Since $X = W_{1}^{\pm} \cup W_{1}^{-}$, we have $T(W_0) \subseteq W_{1}^{\pm} \cup W_{1}^{-}$. On the other hand $T(W_0)$ does not meet $T(W_1) = W_{1}^{\pm} \cap W_{1}^{-}$. It is therefore contained in a connected component of $X \setminus T(W_1)$. Each such connected component is contained either in the full subcomplex on $W_1^{\pm}$ or the full subcomplex on $W_1^{-}$. In particular we have $T(W_0)^{(0)} \subseteq W_1^{\pm}$ or $T(W_0)^{(0)} \subseteq W_1^{-}$.

Arguing similarly for $T(W_1)$, there are $\epsilon_0, \epsilon_1 \in \{\pm\}$ so that

$$T(W_0)^{(0)} \subseteq W_1^{\epsilon_1}, \ T(W_1)^{(0)} \subseteq W_0^{\epsilon_0},$$

but this implies $W_0^{-\epsilon_1} \cap W_0^{-\epsilon_0} = \emptyset$. In other words $W_0$ and $W_1$ do not cross.

For the inductive step suppose that $W_0, \ldots, W_m$ pairwise cross, and let $T_i = T(W_i)$ for each $i$. The base case implies that $T_0 \cap T_m \neq \emptyset$, whereas the inductive hypothesis implies that

$$I_1 := N_{\Theta_{m-1}}(T_0) \cap \cdots \cap N_{\Theta_{m-1}}(T_{m-1}) \text{ and } I_2 := N_{\Theta_{m-1}}(T_1) \cap \cdots \cap N_{\Theta_{m-1}}(T_m)$$

are nonempty. Take $a \in I_1$, $b \in I_2$, and $c \in T_0 \cap T_m$, and consider the geodesic triangle $\Delta$ with corners $a, b, c$. Let $L_m = F_{2L_m - 3}(Q + \Theta_{m-1})$. We may assume without loss of generality that $L_m \geq \delta$, where $\delta$ is the relatively thin triangles constant for $(X, \mathcal{B})$. We argue that we can take $\Theta_m = Q + \Theta_{m-1} + 2L_m + 2\delta$.

Suppose first that the $\delta$–fat part of $\Delta$ contains a segment of length at least $L_m$ in each side. Then in particular there is some peripheral complex $Z$ so that $\Delta$ is $\delta$–thin relative to $Z$. Lemma 2.34 applied to the side $[a, c]$ implies that $Z$ is contained in $T_0$; the same lemma applied to $[b, c]$ implies that $Z$ is contained in $T_m$; and the same lemma applied to $[a, b]$ implies that $Z$ is contained in each of $T_1, \ldots, T_{m-1}$. In particular $Z \subseteq \bigcap_{i=0}^{m} T_i$.

If on the other hand some $\delta$–fat part of a side of $\Delta$ has length bounded above by $L_m$, then Lemma 2.4 shows that $\Delta$ is $(2L_m + 2\delta)$–thin. In particular if $x$ is the point on $[a, b]$ which is in the preimage of the central point of the comparison tripod for $\Delta$, then $x$ is contained in the intersection of the $\Theta_m$–neighborhoods of the thick carriers $T_0, \ldots, T_m$. 

The main result of this subsection is the following.

**Proposition 3.12.** Let $\mathcal{U}$ be a DCC ultrafilter. There is some $K \geq 0$ so that $\bigcap_{A \in \mathcal{U}} N_K(A) \neq \emptyset$. 

Proof. We inductively define a certain collection of pairwise crossing walls $W_0, \ldots, W_k$, and show that the coarse intersection of their carriers is coarsely contained in every halfspace of $\mathcal{U}$.

To begin, choose $A_0$ minimal in $\mathcal{U}$ (with respect to inclusion) and let $W_0 = \{A_0, cA_0\}$. Note that such an $A_0$ exists because $\mathcal{U}$ is DCC. Suppose now we have chosen $W_0, \ldots, W_{s-1}$, and let $I_s$ be the set of walls which cross $W_i$ for all $i < s$. If $I_s$ is nonempty, let $A_s$ be a minimal element of $\{\mathcal{U}(W) \mid W \in I_s\}$, and let $W_s = \{A_s, cA_s\}$. By Corollary 3.6, the dimension of $C(W)$ is finite. For $s$ bigger than this dimension, the set $I_s$ is empty, so the process of choosing the $W_i$ eventually terminates.

Let $W_0, \ldots, W_k$ be the resulting collection, and let $T_i = T(W_i)$. Now we decompose $\mathcal{U}$ as $\mathcal{U}_0 \sqcup \cdots \sqcup \mathcal{U}_k$, where

$$\mathcal{U}_0 = \{A \in \mathcal{U} \mid \{A, A^c\} \text{ does not cross } W_0\},$$

and for $i > 1$,

$$\mathcal{U}_i = \{A \in \mathcal{U} \mid \{A, A^c\} \text{ crosses } W_0, \ldots, W_{i-1} \text{ but does not cross } W_i\}.$$ The minimality assumptions on the $W_i$ imply that if $A \in \mathcal{U}_i$, then $\mathcal{U}(W_i) \subseteq A$. By Lemma 3.10, this implies that $T_i \subseteq A$. In particular, we have

$$\bigcap_{i=1}^k N_{\Theta_k}(T_i) \subseteq \bigcap_{A \in \mathcal{U}} N_{\Theta_k}(A).$$

Lemma 3.11 shows that the left hand side is nonempty, and the Proposition follows, with $K = \Theta_k$. $\square$

3.3. Classification of stabilizers. Given a DCC ultrafilter $\mathcal{U}$, fix $K$ so that

$$N_K(\mathcal{U}) := \bigcap_{A \in \mathcal{U}} N_K(A)$$

is nonempty. The existence of such a $K$ is guaranteed by Proposition 3.12. We observe:

Lemma 3.13. The stabilizer of $N_K(\mathcal{U})$ contains the stabilizer of $\mathcal{U}$.

Since the action of $G$ on $X$ is proper, we immediately obtain:

Corollary 3.14. If $N_K(\mathcal{U})$ is bounded, then $\text{Stab}(\mathcal{U})$ is finite.
Assumption 2.40, this constant will disappear from the discussion once we have found this peripheral complex in $N_0(U)$.

We need a lemma about the geometry of CAT(0) cube complexes, which is very similar to Remark 3.2 of Sageev–Wise [35].

**Lemma 3.15.** Let $\Xi$ be a $k$–dimensional CAT(0) cube complex, and let $\theta_k = \arcsin(\frac{1}{\sqrt{k}})$. Suppose that $d_\Xi(x, y) > 2\sqrt{k}$, and let $m$ be the midpoint of $[x, y]$. Then there is a hyperplane $H$ meeting $[x, y]$ within $\frac{\sqrt{k}}{2}$ of $m$, and making an angle of at least $\theta_k$ with $[x, y]$. This hyperplane satisfies $d_\Xi(H, \{x, y\}) > d_\Xi(x, y) - \frac{\sqrt{k}}{2\sqrt{k}}$.

**Proof.** The idea here is from Section 3 of [35]. In [35, Lemma 3.1] it is shown that if $v$ is a nonzero vector in $\mathbb{R}^k$, and $E$ is the standard basis of $\mathbb{R}^k$, then there is a subset $E' \subseteq E$ of cardinality $k - 1$ so that every vector in the hyperplane spanned by $E'$ makes an angle of at least $\theta_k$ with $v$. It is observed in [35, Remark 3.2] that if $\gamma$ is a ray starting at a corner of a cube $\sigma$ and with a nondegenerate initial segment in $\sigma$, then $\gamma$ meets some midcube of $\sigma$ at an angle of at least $\theta_k$.

The difference in our situation is that $m$ need not be a vertex. There is however some nondegenerate segment $\sigma = [m, m']$ contained in a cube $\sigma$ of $\Xi$, which we suppose is of minimal dimension to contain such a segment. Let $\sigma'$ be a maximal length parallel segment beginning at a vertex of $\sigma$. By [35, Remark 3.2], there is a midcube of $\sigma$ meeting $\sigma'$ at an angle of at least $\theta_k$. Let $H$ be the hyperplane determined by this midcube. We show that $[x, y]$ meets $H$ as in the conclusion of the lemma.

The carrier $C$ of $H$ can be identified with $H \times [0, 1]$, where $H$ is identified with $H \times \{\frac{1}{2}\}$. Let $\lambda: [0, T] \to H \times [0, 1]$ be a unit speed geodesic with image the intersection of $[x, y]$ with $C$. The geodesic $\lambda$ projects to constant speed geodesics $\lambda_H: [0, T] \to H$ and $\lambda_I: [0, T] \to [0, 1]$. Any geodesic in $C$ makes a well-defined constant angle $\arctan(\frac{\lambda_I}{\lambda_H})$ with $H$, and this angle is invariant under translation inside a cube. Since the angle is at least $\theta_k = \arcsin(\frac{1}{\sqrt{k}})$, the length $T$ of $\lambda$ is at most $\sqrt{k}$. In particular neither $x$ nor $y$ lies in $C$, and $\lambda_I$ is surjective. Let $t \in [0, T]$ satisfy $\lambda_I(t) = \frac{1}{2}$, so $\lambda(t)$ lies on the hyperplane $H$. The distance from $\lambda(t)$ to $m$ is at most $\frac{T}{2} \leq \frac{\sqrt{k}}{2}$.

The last assertion is an easy calculation.

In the next lemma we return to the setting of our $(\delta, f)$–relatively hyperbolic pair $(X, B)$.

**Lemma 3.16.** (Pushing across triangles) Suppose that $a, b, c \in X$, $Z \in B$, and that there is a subsegment $[x, y] \subseteq [a, b] \cap Z$ satisfying:
(1) The length of \([x, y]\) is at least \(2\delta + 2\eta + f(\delta)\); and
(2) \([x, y]\) does not cross the internal point of \([a, b]\), in other words
\(d(a, x) < d(a, y) \leq (b, c)_a\).

Then there is a subsegment \([x', y']\) \([a, c] \cap Z\) with \(|d(a, x') - d(a, x)| \leq \delta + \eta\) and \(|d(a, y') - d(a, y)| \leq \delta + \eta + f(\delta)\).

**Figure 2.** Lemma 3.16. The small points are the points in the preimage of the central point of the comparison tripod.

**Proof.** Let \(\Delta\) be the triangle with corners \(a, b, c\). Let \(\pi: \Delta \rightarrow T_\Delta\) be the canonical map to a comparison tripod \(T_\Delta\). Let \(I = [x, y]\). The assumptions on \(d(a, x)\) and \(d(a, y)\) imply that \(\pi(I)\) lies entirely in one leg of \(T_\Delta\). Let \(I'\) be the subsegment of \([a, c]\) with the same image in \(T_\Delta\). There are three cases.

In case \(\Delta\) is \(\delta\)-thin, the segment \(I'\) has endpoints \(\delta\)-close to those of \(I\). In particular it begins and ends within \(\delta\) of \(Z\), so it intersects \(Z\) in a subsegment whose endpoints are at most \(\eta + \delta\) from those of \(I'\), and we are finished.

In case \(\Delta\) is \(\delta\)-thin relative to \(Z\), then every point of \(I'\) is within \(\delta\) of some point of \(I\) or some point of \(Z\). Since \(I \subseteq Z\), the segment \(I'\) begins and ends within \(\delta\) of \(Z\) as before.

In case \(\Delta\) is \(\delta\)-thin relative to some \(W \neq Z\), we note that the \(\delta\)-fat part of \([a, b]\) can intersect \(I\) in a segment of length at most \(f(\delta)\). Removing the \(\delta\)-fat part thus leaves a subsegment \(I_0\). The corresponding subsegment \(I_0' \subseteq I'\) begins and ends within \(\delta\) of \(Z\), and so we can find a subsegment of \(I_0'\) with the desired properties.

The main use of the next two lemmas is in the proof of Proposition 3.19 which says that whenever \(N_K(\mathcal{U})\) is unbounded \(N_0(\mathcal{U})\) contains a peripheral complex. Both will be used again later in slightly simpler circumstances.

In the following statement \(\omega\) and \(\tau\) are the constants which were fixed in Notation 2.39 their definitions are recalled in the proof.
Lemma 3.17. For any $J \geq 0$, if $x, y \in N_J(\mathcal{U})$ satisfy $d(x, y) > \left(\frac{2\pi}{\theta_n} + 2\right)J + \omega$, then there is a peripheral complex $Z$ which intersects $[x, y]$ in a subsegment $I$ so that

1. the length of $I$ is at least $\min\{R_* - \tau, \frac{1}{2}d(x, y) - \tau - J\}$, and
2. the distance from $I$ to the midpoint of $[x, y]$ is at most $\tau$.

Proof. We first remark that it is easy to check that $\left(\frac{2\pi}{\theta_n} + 2\right) \geq 2\sqrt{n} + 2$.

Recall from Notation 2.39 the definition of the following two constants:

$$\omega = 2F_{2.36}(0) + \left(8 + \frac{2}{\sin\left(\frac{\theta_n}{2}\right)}\right)Q + 20\eta + 8f(\delta) + 16\delta + 2\sqrt{n},$$

and

$$\tau = F_{2.32}(Q) + \left(3 + \frac{1}{\sin\left(\frac{\theta_n}{2}\right)}\right)Q + 4\eta + f(\delta) + 2\delta + \sqrt{n}.$$ 

Since $d(x, y) \geq \omega \geq 2\sqrt{n}$, we can apply Lemma 3.15 to find a hyperplane $H$ meeting $[x, y]$ at an angle of at least $\theta_n$, at a point $z \in [x, y]$ satisfying

$$d(z, \{x, y\}) > \frac{d(x, y) - \sqrt{n}}{2}. \quad (4)$$

Since $\left(\frac{2\pi}{\theta_n} + 2\right) > 2\sqrt{n}$ and $\omega > \sqrt{n}$, the last assertion of Lemma 3.15 implies that $H$ misses the $J$-balls around $x$ and $y$:

$$d(H, \{x, y\}) \geq \frac{\left(\frac{2\pi}{\theta_n} + 2\right)J + \omega - \sqrt{n}}{2\sqrt{n}} = \frac{\left(\frac{2\pi}{\theta_n} + 2\right)}{2\sqrt{n}}J + \frac{\omega}{2\sqrt{n}} - \frac{1}{2} > J.$$

Let $W$ be the wall corresponding to $H$, and let $A = \mathcal{U}(W)$. There are points $x_A, y_A$ in $A$ within $J$ of $x, y$ respectively.

![Figure 3](fig:NJU1)

Figure 3. The points $x_A$ and $y_A$ are within $J$ of $x, y$ respectively, so they are separated by the hyperplane $H$. The unlabeled point is the midpoint of $[x, y]$, which is distance at most $\sqrt{n}/2$ from $z$. 

[lem:NJU]

Since $H$ misses the $J$–balls around $x$ and $y$, it must separate $x_A$ from $y_A$. Observe that $z$ lies in the thick wall carrier $T = T(W)$.

**Claim 1.** Either one of $x_A, y_A$ lies in $T$, or there is a point $z' \in T \cap ([x_A, z] \cup [z, y_A])$ lying in a $W$–hyperplane other than $H$.

**Proof.** If neither $x_A$ nor $y_A$ is in $T$, they both lie in the full sub-complex on $A$. It follows that $W$ does not separate $x_A$ from $y_A$, and hence there is a second $W$–hyperplane $H'$ separating $x_A$ from $y_A$. The hyperplane $H'$ meets either $[x, x_A]$ or $[y, y_A]$ at some point $z'$.

In case one of $x_A, y_A$ lies in $T$, we relabel so that $y_A$ lies in $T$. If not, we similarly relabel so that the point $z'$ found in Claim 1 lies in the segment $[z, y_A]$.

**Claim 2.** There is a $W$–peripheral complex $Z$ intersecting $[z, y_A]$ in a segment $I_0$ of length at least

$$\min \left\{ R_\ast - (R_{n, 2\pi} Q + 4Q + 2\eta), \frac{d(x, y) - \sqrt{n}}{2} - \left( \frac{Q}{\sin(\frac{\theta_n}{2})} + 2Q + 2\eta \right) - J \right\},$$

and so that $d(I_0, z) \leq \frac{Q}{\sin(\frac{\theta_n}{2})} + Q + \eta$.

**Proof.** Note that the angle $\psi$ between $[z, y]$ and $[z, y_A]$ is less than $\frac{\theta_n}{2}$. Indeed, the hypothesis on $d(x, y)$, and the inequalities (4), $\omega > \sqrt{n}$ and $d(y, y_A) \leq J$ together imply (since at any rate $\psi < \frac{\pi}{2}$):

$$\psi \leq \frac{\pi}{2} \sin(\psi) \leq \frac{\pi}{2} \left( \frac{d(y, y_A)}{\min\{d(z, y), d(z, y_A)\}} \right) < \frac{\pi}{2} \left( \frac{J}{\pi J/\theta_n} \right) = \frac{\theta_n}{2}.$$

(Here we are using the fact that angles in a CAT(0) triangle are dominated by the angles in the comparison triangle.) Since the angle $[z, y]$ makes with $H$ is at least $\theta_n$, the angle $[z, y_A]$ makes with $H$ is at least $\theta_n - \psi > \frac{\theta_n}{2}$. It follows that the subsegment $[z, p]$ of $[z, y_A]$ lying in $N_Q(H)$ has length at most $\frac{Q}{\sin(\frac{\theta_n}{2})}$.

The point $p$ just defined must lie in the $Q$–neighborhood of some $W$–peripheral complex $Z$. This is the peripheral complex in which we will find the segment $I_0$.

In case all of $[p, y_A]$ lies in $N_Q(Z)$, we use $\eta$–super-attractiveness to obtain a segment $I_0 \subseteq [p, y_A] \cap Z$ so that

$$|I_0| \geq d(p, y_A) - (2Q + 2\eta) \geq \frac{d(x, y) - \sqrt{n}}{2} - \left( \frac{Q}{\sin(\frac{\theta_n}{2})} + 2Q + 2\eta \right) - J.$$
as desired.

Next we suppose that not all of \([p, y_A]\) lies in \(N_Q(Z)\). It follows that \(y_A\) does not lie in \(Z\), since the \(Q\)-neighborhood of \(Z\) is convex. Claim 1 implies that either \(y_A\) or some point \(z' \in [p, y_A]\) lies on a \(W\)-hyperplane or \(W\)-peripheral complex not equal to \(Z\), so either \([z, y_A]\) or \([z, z']\) is contained in \(N_Q(T)\) but not entirely contained in \(N_Q(Z)\). Let \(q\) be the first point on \([p, y_A]\) within \(Q\) of some \(W\)-peripheral complex \(Z' \neq Z\). Note the point \(q\) must also lie in \(N_Q(Z)\). In the case \(q\) is within \(Q\) of a \(W\)-peripheral complex \(Z' \neq Z\), there exists a hyperplane \(H'' \neq H\) so that \(d(q, H'') \leq \Gamma_{\text{max}}(Q)\). In either case there is a hyperplane \(H'' \neq H\) so that \(d(q, H'') \leq \max\{Q, \Gamma_{\text{max}}(Q)\}\), and so we have

\[
d(p, q) \geq R_* - (Q + \max\{Q, \Gamma_{\text{max}}(Q)\}) \geq R_* - (\Gamma_{\text{max}}(Q) + 2Q).
\]

The \(\eta\)-super-attractiveness gives us a subsegment \(I_0\) of \([p, q] \cap Z\) of length at least

\[
d(p, q) - (2Q + 2\eta) \geq R_* - (\Gamma_{\text{max}}(Q) + 4Q + 2\eta).
\]

In either case \(d(I_0, z) \leq d(p, z) + Q + \eta \leq \frac{Q}{\sin(\frac{\theta}{2})} + Q + \eta\), as desired.

\[
\Box
\]

We now wish to apply Lemma 3.16 (Pushing across triangles) to the segment \(I_0\) from the last claim, but it may extend past the internal point of the side \([z, y_A]\) of the triangle with corners \([z, y, y_A]\). We note however that \(d(y_A, z) \geq \frac{d(x, y) - \sqrt{n}}{2} - J\) and \(d(z, I_0) \leq \frac{Q}{\sin(\frac{\theta}{2})} + Q + \eta\) by Claim 2, so \(I_0\) contains a subsegment \(I_1\) as in the hypothesis of that lemma with

\[
|I_1| \geq \min\left\{|I_0|, \frac{d(x, y) - \sqrt{n}}{2} - \left(\frac{Q}{\sin(\frac{\theta}{2})} + Q + \eta\right) - J\right\}.
\]

This is a better lower bound than that already given for \(I_0\) in the statement of Claim 2, so we may assume that \(I_0\) already satisfies the hypotheses of Lemma 3.16. Pushing across the triangle we obtain a subsegment \(I\) in \([z, y] \subseteq [x, y]\) which is contained in \(Z\) and has length at least \(|I_0| - (2\delta + 2\eta + f(\delta))\). Moreover the distance from \(I\) to \(z\) is no more than \(d(I_0, z) + \eta + \delta\).

Adding up the constants we obtain the inequalities asserted in the lemma.

\[
\Box
\]

**Lemma 3.18.** Let \(J \geq 0\), and \(x, y \in N_J(\mathcal{U})\). Suppose that \(Z\) is a peripheral complex so that \(Z \cap [x, y]\) contains a segment \(\sigma\) so that
length(σ) > \( F_{2.36}(0) + 4(\eta + \delta) + 2f(\delta) \), and \( d(\sigma, \{x, y\}) > J + \delta + \eta \). Then \( Z \subseteq N_0(\mathfrak{U}) \).

**Proof.** Let \( B \in \mathfrak{U} \). There are points \( x_B, y_B \) of \( B \) within \( J \) of \( x, y \) respectively. We claim that there is a subsegment \( \sigma' \) of \([x_B, y_B] \cap Z \) of length at least \( F_{2.36}(0) \). Lemma 3.9 then implies that \( Z \subseteq B \). Since \( B \in \mathfrak{U} \) was arbitrary, we obtain \( Z \subseteq \bigcap_{A \in \mathfrak{U}} A = N_0(\mathfrak{U}) \).

Finding the subsegment \( \sigma' \) is a matter of applying Lemma 3.16 (Pushing across triangles) twice; first the Lemma is applied to a triangle whose corners are \( x, y, y_B \), and then to a triangle whose corners are \( x, x_B, y_B \). The lower bound on \( d(\sigma, \{x, y\}) \) implies the hypothesis about Gromov products in Lemma 3.16. Each time we push across a triangle, we can lose only up to \( 2(\eta + \delta) + f(\delta) \) in length, so we are left with at least \( F_{2.36}(0) \) in the end. \( \square \)

**Proposition 3.19.** If \( N_K(\mathfrak{U}) \) is unbounded then there is a peripheral complex contained in \( N_0(\mathfrak{U}) := \bigcap_{A \in \mathfrak{U}} A \).

**Proof.** Assuming \( N_K(\mathfrak{U}) \) is unbounded, we may choose two vertices \( x, y \in N_K(\mathfrak{U}) \) so \( d(x, y) \) is as large as we like. In particular we make the following assumption:

\[
d(x, y) > \max \left\{ \left( \frac{2\pi}{\theta_n} + 2 \right) K + \omega, 2R_* + 2K + 2\delta + 2\eta \right\}. \tag{6} \]

In particular, we have \( \frac{1}{2}d(x, y) > R_* + K \), so Lemma 3.17 with \( J = K \) gives us a peripheral complex \( Z \) and a segment \( I \subseteq [x, y] \) so that

1. the length of \( I \) is at least \( R_* - \tau \), and
2. the distance from \( I \) to the midpoint of \([x, y]\) is at most \( \tau \).

Note also that we have the following inequalities. (The first line follows from Assumption 2.40[2]; the second from (6).)

\[
|I| \geq R_* - \tau > F_{2.36}(0) + 4(\eta + \delta) + 2f(\delta)
\]

\[
\frac{1}{2}d(x, y) > R_* + (K + \delta + \eta).
\]

Since \( d(x, y) > 2(R_* + K + \delta + \eta) \) we can choose such an \( I \) to satisfy \( d(I, \{x, y\}) > K + \delta + \eta \). Applying Lemma 3.18 we see \( Z \subseteq N_0(\mathfrak{U}) \). \( \square \)

**Proposition 3.20.** Let \( Z \) be a peripheral complex. The following are equivalent:

1. \( Z \subseteq N_0(\mathfrak{U}) \).
2. \( \text{Stab}(\mathfrak{U}) \) contains a finite index subgroup of \( \text{Stab}(Z) \).
3. \( \text{Stab}(\mathfrak{U}) \) contains an infinite subgroup of \( \text{Stab}(Z) \).
Lemma 3.21. \textbf{lem:weaklyprincipal} since \( Z \subseteq N_0(\mathcal{U}) \) implies that for every wall \( W \), either \( Z^{(0)} \subseteq \mathcal{U}(W) \) or \( Z \) is a \( W \)-peripheral complex. Let \( \mathcal{U}' \subseteq \mathcal{U} \) consist of those halfspaces \( A \in \mathcal{U} \) so that \( Z^{(0)} \subseteq A \). Note that every element of \( \text{Stab}(Z) \) preserves \( \mathcal{U}' \). If \( \mathcal{U}(W) \notin \mathcal{U}' \), then \( Z \) is a \( W \)-peripheral complex. There are only finitely many such \( W \), so \( \mathcal{U} \setminus \mathcal{U}' \) is finite. Let \( S = (\mathcal{U} \setminus \mathcal{U}') \cup \{ A' \mid A \in \mathcal{U} \setminus \mathcal{U}' \} \). The finite set \( S \) is preserved by \( \text{Stab}(Z) \) and the kernel of \( \text{Stab}(Z) \rightarrow \text{Sym}(S) \) lies in \( \text{Stab}(\mathcal{U}) \).

The implication \( \text{[2]} \implies \text{[3]} \) is obvious, since the peripheral subgroups of \((G, \mathcal{P})\) are assumed to be infinite (recall Convention 1.6).

Finally we show \( \text{[3]} \implies \text{[1]} \). Let \( Z \) be a peripheral complex so that \( \text{Stab}(Z) \cap \text{Stab}(\mathcal{U}) \) is infinite. Let \( x \) be any point of \( N_0(\mathcal{U}) \). Choose \( g \in \text{Stab}(Z) \cap \text{Stab}(\mathcal{U}) \) so that \( d(x, gx) = E \geq F_{\mathbb{Z},2\mathbb{Z}}(Q) + 2(d(x, Z) + \eta) \).

Let \( A = \mathcal{U}(W) \) where \( W \) is a wall. Both \( x \) and \( gx \) lie in \( \overline{A} \), which is \( Q \)-quasi-convex, so \([x, gx]\) lies in \( N_Q(\overline{A}) \). The super-attractiveness of \( Z \) ensures that a subsegment of \([x, gx]\) of length at least \( E - 2(d(x, Z) + \eta) \) lies in \( Z \). The endpoints are distance at most \( Q \) from \( \overline{A} \), so Lemma 3.9 implies that \( Z \subseteq \overline{A} \). Since \( A \in \mathcal{U} \) was arbitrary, \( Z \subseteq N_0(\mathcal{U}) \).

3.4. A \( \text{Stab}(\mathcal{U}) \)-tree. For the rest of this section we focus on the case that \( \text{Stab}(\mathcal{U}) \) is infinite and not parabolic. By Corollary 3.14 and Proposition 3.19, there must be at least one peripheral complex contained in \( N_0(\mathcal{U}) \). If there is only one, then its stabilizer obviously contains \( \text{Stab}(\mathcal{U}) \), so we can assume there is more than one peripheral complex in \( N_0(\mathcal{U}) \). We consider projections of these peripheral complexes to each other. These have diameter bounded by \( C \) (see Notation 2.38 and Corollary 2.18). For any pair \( Z, Z' \) of peripheral complexes in \( N_0(\mathcal{U}) \), the next lemma shows that the ultrafilter \( \mathcal{U} \) is “weakly principal” for any point in \( \pi_Z(Z') \).

Lemma 3.21. For \( \kappa \) as in Notation 2.39, the following holds: If \( Z, Z' \) are peripheral complexes contained in \( N_0(\mathcal{U}) \), then

\[
\pi_Z(Z') \subseteq \bigcap_{A \in \mathcal{U}} N_\kappa(A).
\]

Proof. Recall \( \kappa = F_{\mathbb{Z},2\mathbb{Z}}(Q + 1) + 2Q + C + \frac{\sqrt{n}}{2} + 2 \), where \( F_{\mathbb{Z},2\mathbb{Z}} \) is the function from Lemma 2.32 and \( C \) is the constant from Corollary 2.18. Fix \( A \in \mathcal{U} \) and suppose that \( A = \mathcal{U}(W) \). Our goal is to show that \( \pi_{Z}(Z') \subseteq N_\kappa(A) \).

First suppose that \( Z \) is not a \( W \)-peripheral complex. In this case, since \( Z \subseteq N_0(\mathcal{U}) \subseteq N_0(\overline{A}) \) we see that \( Z^{(0)} \subseteq A \), so \( Z \subseteq N_\kappa(A) \). Since \( \pi_{Z}(Z') \subseteq Z \) we are done in this case.
Therefore, we suppose henceforth that $Z$ is a $W$–peripheral complex, so in particular $Z \subseteq T(W)$. Let $b \in Z'$ be arbitrary, and consider the geodesic $[b, \pi_Z(b)]$.

We remark that peripheral complexes intersect, so it is possible that $b = \pi_Z(b)$.

**Case 1.** $b = \pi_Z(b)$.

If $Z'$ is also a $W$–peripheral complex, Lemma 2.32 implies that $\pi_Z(b)$ is within $F_{\le W}(0) \le F_{\le W}(Q + 1) < \kappa$ of some $T(W)$–hyperplane. It follows that in this case $\pi_Z(Z') \subseteq N_\kappa(A)$, as required.

If $Z'$ is not a $W$–peripheral complex, then we have $(Z')^{(0)} \subseteq A$, and hence $b \in N_1(A)$.

We suppose for the remainder of the proof that $b \neq \pi_Z(b)$, and so $[b, \pi_Z(b)]$ is a non-degenerate geodesic which only intersects $Z$ at $\pi_Z(b)$.

**Case 2.** $b \neq \pi_Z(b)$ and $[b, \pi_Z(b)] \cap T(W) \neq \{\pi_Z(b)\}$.

In this case, let $x \in [b, \pi_Z(b)] \cap (T(W) \setminus Z)$. Since $Z \subseteq T(W)$ and $T(W)$ is $Q$–quasi-convex (Notation 2.38), the geodesic $[x, \pi_Z(b)]$ lies in the $Q$–neighborhood of $T(W)$. There are two subcases:

**Subcase 2.1.** $d(x, \pi_Z(b)) \ge Q + 1$.

In this subcase let $y$ be the point on $[x, \pi_Z(b)]$ at distance $Q + 1$ from $\pi_Z(b)$. Since $\pi_Z(b)$ is the closest point on $Z$ to $b$, the point $y$ is not within $Q$ of $Z$, so it is within $Q$ of some $W$–hyperplane or some other $W$–peripheral complex $Z_1$. If it is within $Q$ of a $W$–hyperplane, we are done, since $Q < \kappa$. Otherwise it is within distance $Q + 1$ of both $Z$ and $Z_1$, and so it is distance at most $F_{\le W}(Q + 1)$ from some $W$–hyperplane (Lemma 2.32). Since $F_{\le W}(Q + 1) < \kappa$ we are finished in this case as well.

**Subcase 2.2.** $d(x, \pi_Z(b)) < Q + 1$.

In this case the point $x$ itself is either on a hyperplane or on a $W$–peripheral complex $Z_1 \neq Z$, and can argue as in the previous subcase, substituting $x$ for $y$.

**Case 3.** $b \neq \pi_Z(b)$ and $[b, \pi_Z(b)] \cap T(W) = \pi_Z(b)$.

In this case, since $(Z')^{(0)} \subseteq A$, it must be that the geodesic $[b, \pi_Z(b)]$ lies entirely in the full sub-complex on $A$, meaning that $\pi_Z(Z')$ lies in $N_\kappa(A)$, as required. \[\square\]

We want to build a bipartite $\text{Stab}(\varnothing)$–tree in order to analyze the structure of $\text{Stab}(\varnothing)$ and prove Property (3b) of Definition 1.5. The
equivalence classes under the relation defined in Definition 3.22 will form one color of vertex of this tree. The other color will be the set of peripheral complexes in $N_0(\mathcal{U})$.

Recall from Notation 2.39 that

\[ D = \max \left\{ 2\sqrt{n}k + \sqrt{n} \cdot R(\delta(0) + 6\eta + 2f(\delta) + 6\delta, 2C + 1) \right\} + 1. \]

**Definition 3.22.** Define a relation on pairs $(Z, Z')$ where $Z \neq Z'$ are contained in $N_0(\mathcal{U})$ by saying $(Z_1, Z'_1) \sim (Z_2, Z'_2)$ if for every geodesic $\gamma$ joining $\pi_{Z_1}(Z'_1)$ to $\pi_{Z_2}(Z'_2)$, and every peripheral complex $Z$, we have $\text{diam}(\gamma \cap Z) < D$.

**Proposition 3.23.** The relation defined in Definition 3.22 is an equivalence relation.

**Proof.** Symmetry is obvious, and reflexivity follows from Corollary 2.18 because $D > C$ (the upper bound on the diameter of $\pi_Z(Z')$).

To show transitivity, we argue by contradiction. Suppose $(Z_1, Z'_1) \sim (Z_2, Z'_2) \sim (Z_3, Z'_3)$, but $(Z_1, Z'_1) \not\sim (Z_3, Z'_3)$. For each $i$, let $S_i = \pi_{Z_i}(Z'_i)$. Since $(Z_1, Z'_1) \not\sim (Z_3, Z'_3)$, there is a geodesic $\gamma$ joining $S_1$ to $S_3$ and a peripheral complex $Z_0$ so that $\text{diam}(\gamma \cap Z_0) \geq D$. Denote the endpoints of $\gamma$ by $x_1, x_3$, so $x_i \in S_i$. We first show that for either this or some other peripheral complex, the diameter must be much larger.

**Claim.** There exists a peripheral complex $Z$ so that $\text{diam}(\gamma \cap Z) > \frac{3}{4} R_*$. 

**Proof of Claim.** Let $m$ be the midpoint of $\gamma \cap Z_0$, and let $x'_1, x'_3$ be the endpoints of $\gamma \cap Z_0$. By Lemma 3.15 there exists a hyperplane $H$ cutting $\gamma$ at a point $m_0$ within $\sqrt{n}$ of $m$ so that

\[ d(H, \{x_1, x_3\}) > d(H, \{x'_1, x'_3\}) > \frac{d(x'_1, x'_3) - \sqrt{n}}{2\sqrt{n}} > \kappa. \]

(The last inequality holds because $d(x'_1, x'_3) \geq D > 2\sqrt{n}k + \sqrt{n}$.) Let $W$ be the wall determined by $H$. Since $m_0 \in \gamma \cap Z_0$ it is clear that $Z_0$ is a $W$–peripheral complex.

We claim some $W$–hyperplane $H' \neq H$ comes within $\kappa$ of $\gamma$. Indeed, if no such hyperplane crosses $\gamma$, then one endpoint of $\gamma$ is not in the full subcomplex on $\mathcal{U}(W)$. From Lemma 3.21 we see that the endpoint must nonetheless lie in $N_\kappa(\mathcal{U}(W))$, so it must lie within $\kappa$ of some $W$–hyperplane $H'$.

Let $y \in H'$ and $y_0 \in \gamma$ satisfy $d(y, y_0) \leq \kappa$. Consider the geodesic $\gamma' = [y, m_0]$. Since $y$ and $m_0$ lie in distinct $W$–hyperplanes and $T(W)$ is $Q$–quasi-convex, $[y, m_0] \subseteq N_0(T(W))$. Moreover, distinct hyperplanes are distance at least $R_*$ apart, so there must be a segment of
Lemma 3.24. If \( Z, Z' \subseteq N_0(\Omega) \), then
\[
\text{diam} \left( \bigcup \{ \pi_{Z_1}(Z'_1) \mid (Z_1, Z'_1) \sim (Z, Z') \} \right) < \max \{ \omega, 2(D + \tau) \}.
\]

Proof. Recall \( \omega \) and \( \tau \) are the constants defined in Notation 2.32 and used in Lemma 3.17.

Suppose that \( Z, Z', Z_1 \) and \( Z'_1 \) are peripheral complexes in \( N_0(\Omega) \), and that \( x \in \pi_Z(Z') \) and \( y \in \pi_{Z_1}(Z'_1) \) satisfy \( d(x, y) \geq \max \{ \omega, 2(D + \tau) \} \).

Our goal is to show \( (Z, Z') \not\sim (Z_1, Z'_1) \).

By Lemma 3.17 (with \( J = 0 \)), since \( d(x, y) \geq \omega \) and \( x, y \in N_0(\Omega) \), there is a peripheral complex \( Z_0 \) which intersects \([x, y]\) in a segment \( I \) of length at least \( \min \{ R_* - \tau, \frac{1}{2} d(x, y) - \tau \} \). Since \( R_* > D + \tau \) because of Assumption 2.40(5), we see that \( |I| \geq D \). It follows from the definition of the relation that \( (Z, Z') \not\sim (Z_1, Z'_1) \), as required.

Definition 3.25 (The graph \( \Lambda \)). Let \( V_0 \) be the set of peripheral complexes contained in \( N_0(\Omega) \), and let \( V_1 \) be the set of equivalence classes \([Z, Z']\) where both \( Z, Z' \) are contained in \( N_0(\Omega) \). To form the graph \( \Lambda \), connect a vertex \( Z \) of \( V_0 \) to every vertex of the form \([Z, Z']\) by an edge.

Lemma 3.26. The graph \( \Lambda \) is connected.
Lemma 3.27. The graph \( \Lambda \) is a bipartite tree.

Proof. That \( \Lambda \) is bipartite follows immediately from the definition. We now prove that \( \Lambda \) is a tree.

In order to obtain a contradiction, suppose that \( \sigma \) is an oriented cycle of length \( 2n \) in \( \Lambda \). The vertices of \( \sigma \) are labeled (with subscripts to be understood modulo \( n \)):

\[
Z_0, [Z_0, Z'_0] = [Z_1, Z'_1], Z_1, [Z_1, Z''_1] = [Z_2, Z'_2], \ldots, [Z_n, Z''_n] = [Z_0, Z'_0].
\]

For each \( i \) let \( b_i \) be a geodesic segment in \( Z_i \) from \( p_i \in \pi_{Z_i}(Z'_i) \) to \( q_i \in \pi_{Z_i}(Z''_i) \), and let \( a_i \) be the geodesic from \( q_i \) to \( p_{i+1} \). Let \( \gamma \) be the broken geodesic loop \( b_0a_0 \cdots a_{n-1} \). By Lemma 3.24 we have \( |a_i| < \max \{ \omega, 2(D + \tau) \} \).
We claim that the path $\gamma$ satisfies the hypothesis of [12, Theorem 5.6], and is therefore a good quality quasi-geodesic (and hence not a loop). In order to make our situation line up with the constants in [12], we apply a similar trick as in the proof of Proposition 2.31. Namely we temporarily modify the function $f$. To that end, define the function $f'(m) = \max\{f(m), \eta, \frac{1}{3}\max\{\omega, 2(D + \tau)\}\}$, and note that $(X, \mathcal{B})$ is $(\delta, f')$–relatively hyperbolic. Let $M = f'(5\delta)$ as in [12]. Also note that since the peripheral complexes are $\eta$–super-attractive, they are $2(\eta + m)$–attractive in the sense of Einstein (see Remark 2.15), and thus satisfy the attractiveness hypothesis of [12, Hypotheses 5.5]. We now check that the six conditions from [12, Theorem 5.6] hold.

Certainly the first is satisfied, and the second is satisfied because $\sigma$ is embedded. The third is satisfied because $(X, \mathcal{B})$ is $(\delta, f')$–relatively hyperbolic and $M = f'(5\delta)$. The fourth is satisfied because by the proof of Proposition 3.23 between inequivalent projections there is a peripheral path of length at least $\frac{3}{4}R_\star$, and Assumption 2.40(6) forces $\frac{4}{5}R_\star \geq 26M + 250\delta$. Since $|a_i| < \max\{\omega, 2(D + \tau)\}$, and by the choice of $f'$, we have $|a_i| \leq 3M + 63\delta$, which trivially imply that the fifth and sixth conditions from [12, Theorem 5.6] hold.

It follows from the conclusion of [12, Theorem 5.6] that a geodesic joining the endpoints of $\gamma$ must be long, and in particular $\gamma$ is not a loop, contrary to hypothesis. It follows that $\Lambda$ is a tree.

\section{Infinite non-parabolic cell stabilizers}

The goal of this subsection is the following:

**Theorem 3.28.** Suppose that $\text{Stab}(\mathcal{U})$ is infinite and non-parabolic. Then $\text{Stab}(\mathcal{U})$ is a finite graph of groups where the vertex groups are either full parabolic or finite, and the edge groups are finite.

For the proofs of Theorems 3.28 and C we need some facts about various kinds of quasi-convexity in relatively hyperbolic groups.

**Definition 3.29.** [36] Let $G$ be finitely generated, and let $\Gamma$ be a Cayley graph for $G$. The subgroup $G_0$ is strongly quasi-convex if for every $\lambda \geq 1$, $\epsilon \geq 0$ there is an $r$ so that any $(\lambda, \epsilon)$–quasi-geodesic in $\Gamma$ with endpoints in $G_0$ lies in an $r$–neighborhood of $G_0$.

Notice that strongly quasi-convex subgroups are always finitely generated and undistorted.

We deduce the next lemma from theorems of Tran and Hruska.

**Lemma 3.30.** Let $(G, \mathcal{P})$ be relatively hyperbolic, and let $G_0 < G$ be full. The following are equivalent:

1. $G_0$ is finitely generated and undistorted in $G$. 

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\( G_0 \) is relatively quasi-convex in \((G, \mathcal{P})\).

\( G_0 \) is strongly quasi-convex in \(G\).

**Proof.** Theorem 1.5 of \[24\] gives (1) \(\Rightarrow\) (2) (even without the assumption that \(G_0\) is full). Theorem 1.4 of \[24\] gives (2) \(\Rightarrow\) (1) when \(G_0\) is full.

Theorem 1.9 of \[36\] gives (1) \(\Rightarrow\) (3) when \(G_0\) is full, since finite index and finite subgroups are strongly quasi-convex. We have already noted that strongly quasi-convex subgroups are finitely generated and undistorted, so (3) \(\Rightarrow\) (1).

\(\square\)

**Corollary 3.31.** The subgroup \( \text{Stab}(\mathfrak{U}) \) is full relatively quasi-convex in \((G, \mathcal{P})\). In particular it is finitely generated.

**Proof.** The hyperplane stabilizers for \( G \rtimes \hat{X} \) are full relatively quasi-convex (Lemma 2.27) so by Lemma 3.30 they are strongly quasi-convex. By \[16, \text{Theorem 3.26}\] the cube stabilizers of \( G \rtimes \hat{X} \) are also strongly quasi-convex. In particular \( \text{Stab}(\mathfrak{U}) \) is strongly quasi-convex in \(G\). It is full by Proposition 3.20. Now apply Lemma 3.30 to deduce that \( \text{Stab}(\mathfrak{U}) \) is relatively quasi-convex.

\(\square\)

**Proof of Theorem 3.28.** The subgroup \( \text{Stab}(\mathfrak{U}) \) acts on the tree \( \Lambda \) described in the last subsection. A vertex corresponding to a peripheral complex in \( N_0(\mathfrak{U}) \) has full parabolic stabilizer by Proposition 3.20. The stabilizer of a vertex corresponding to an equivalence class \([([Z, Z'])]\) must also stabilize the bounded set from Lemma 3.24, so it must be finite. Since every edge is connected to some vertex of this type, edge stabilizers are also finite.

Since \( \text{Stab}(\mathfrak{U}) \) is finitely generated there exists a \( \text{Stab}(\mathfrak{U}) \)-invariant sub-tree \( \Lambda_0 \) of \( \Lambda \) so that \( \text{Stab}(\mathfrak{U}) \backslash \Lambda_0 \) is finite.

\(\square\)

3.6. **Proof of Theorem C.** See Definition 1.5 for the definition of a weakly relatively geometric action.

**Theorem C.** Suppose \((G, \mathcal{P})\) is relatively hyperbolic, and that \(G\) acts properly cocompactly on a CAT(0) cube complex \(X\), and that this action satisfies Assumption 1.1.

There exists a weakly relatively geometric action of \(G\) on a CAT(0) cube complex \(\hat{X}\).

**Proof.** We take the cube complex \(\hat{X}\) to be the one dual to the wallspace fixed in Definition 2.41.

The cocompactness of \( G \rtimes \hat{X} \) and ellipticity of peripheral subgroups is Corollary 3.6.
It suffices to prove the statement about cube stabilizers for vertex stabilizers, since each cube stabilizer is an intersection of vertex stabilizers, and the description passes to intersections.

So, let $\mathcal{U}$ be an ultrafilter so that $\text{Stab}(\mathcal{U})$ is infinite. Corollary 3.31 implies that $\text{Stab}(\mathcal{U})$ is full relatively quasi-convex.

Theorem 3.28 implies that $\text{Stab}(\mathcal{U})$ is a graph of finite and full parabolic subgroups as in the conclusion.

4. Proof of Theorem A

In this section we prove Theorem A. The strategy is to use the recubulation given by Theorem C and relatively hyperbolic Dehn filling results. See [30, 14, 2, 1, 15, 16] for more information on relatively hyperbolic Dehn filling. We first recall the definition.

Definition 4.1 (Dehn filling). Suppose that $(G, P)$ is a group pair, and that $\mathcal{N} = \{N_P \leq P \mid P \in P\}$ is a collection of normal subgroups of the elements of $P$. The Dehn filling induced by $\mathcal{N}$ is

$$G(\mathcal{N}) := G/\langle\langle \bigcup N \rangle \rangle_G.$$ 

Definition 4.2 (Wide subgroups and fillings). If $P$ is a group, $B < P$, and $F \subseteq P$ a finite set, then $N \leq P$ is $(B, F)$–wide if, for all $b \in B$ and $f \in F \setminus B$, the product $bf$ does not lie in $N$.

Let $H$ be a relatively quasi-convex subgroup of $(G, P)$, so there is a collection of infinite maximal parabolic subgroups $\mathcal{D}$ of $H$ so that $(H, \mathcal{D})$ is relatively hyperbolic. For each $D \in \mathcal{D}$ there is some $c_D \in G$ and some $P_D \in P$ so that $D^{-1} c_D D \leq P_D$.

Let $F \subseteq G \setminus \{1\}$ be finite. Let $G(\mathcal{N})$ be a Dehn filling, with $K = \langle\langle \bigcup \mathcal{N} \rangle \rangle_G$. We say the filling $G(\mathcal{N})$ is $(H, F)$–wide if for every $D \in \mathcal{D}$, the intersection $K \cap P_D$ is $(D^{-1} c_D, F \cap P_D)$–wide.

Definition 4.3 (Sufficiently long and wide Dehn fillings). We say that a statement $\mathcal{S}$ holds for all sufficiently long Dehn fillings if there is a finite set $F \subseteq G \setminus \{1\}$ so that $\mathcal{S}$ holds for all $G(\mathcal{N})$ so that $\bigcup \mathcal{N}$ contains no element of $F$.

Let $H < G$ be a quasi-convex subgroup. The statement $\mathcal{S}$ holds for all sufficiently $H$–wide Dehn fillings if there is a finite set $F \subseteq G \setminus \{1\}$ so that $\mathcal{S}$ holds for all $G(\mathcal{N})$ which are $(H, F)$–wide.

For the Dehn filling results, we make the following assumption (weakly relatively geometric was defined in Definition 1.5).

Assumption 4.4. The pair $(G, P)$ is relatively hyperbolic, and admits a weakly relatively geometric action on a CAT(0) cube complex $\hat{X}$. Further, each element of $P$ is residually finite.
A key reason for our interest in weakly relatively geometric actions is the following result.

**Theorem 4.5.** Under Assumption 4.4, there exist finite-index subgroups \( \{ L_P \trianglelefteq P \mid P \in \mathcal{P} \} \) so that the following holds:

1. \( N_P \trianglelefteq L_P \), and
2. \( P/N_P \) is hyperbolic and virtually special.

Then the Dehn filling \( G(\mathcal{N}) \) is hyperbolic and virtually special.

**Proof.** Since the \( G \)-action on \( \hat{X} \) is cocompact, there are cells \( \sigma_1, \ldots, \sigma_k \) in \( \hat{X} \) forming a collection of representatives of the \( G \)-orbits of cells. For \( 1 \leq i \leq k \) let \( Q_i \) be the finite-index subgroup of \( \text{Stab}(\sigma_i) \) consisting of elements which fix \( \sigma_i \) pointwise, and let \( Q = \{ Q_1, \ldots, Q_k \} \). Note that each infinite \( Q_i \) is a full relatively quasi-convex subgroup which admits a graph of groups decomposition with finite edge groups and finite or full parabolic vertex groups. By combining [30, Theorem 1.1], [2, Propositions 4.3, 4.4] and [16, Corollary 6.5], and the assumption that elements of \( \mathcal{P} \) are residually finite, we see that there are finite-index subgroups \( \{ L_P \trianglelefteq P \mid P \in \mathcal{P} \} \) so that for any \( N_P \leq L_P \) so \( P/N_P \) is hyperbolic and virtually special in \( P \), the Dehn filling

\[
\overline{G} = G/K := G(\{ N_P \mid P \in \mathcal{P} \})
\]

satisfies

1. \( \overline{G} \) is hyperbolic;
2. The image of each \( Q_i \) is quasi-convex in \( \overline{G} \), and splits as a graph of virtually special hyperbolic groups with finite edge groups (and in particular the image of each \( Q_i \) is hyperbolic and virtually special); and
3. The space \( \overline{X} := K \setminus \hat{X} \) is a CAT(0) cube complex.

Let \( \overline{G} \) be any such Dehn filling. Then \( \overline{G} \) acts cocompactly on the CAT(0) cube complex \( \overline{X} \). By [11] \( \overline{G} \) is hyperbolic. By [2], the cell stabilizers for the \( \overline{G} \)-action on \( \overline{X} \) are quasi-convex and virtually special. It now follows from [16, Theorem D] that \( \overline{G} \) is virtually special, as required. \( \square \)

The following is an immediate consequence of results of Haglund–Wise [19, Corollary 7.4] and Minasyan [27, Theorem 1.1].

**Theorem 4.6.** Suppose that \( \Gamma \) is a hyperbolic virtually special group. Then every quasi-convex subgroup of \( \Gamma \) is separable, and every finite product of quasi-convex subgroups of \( \Gamma \) is separable.
Given Theorem C, the next two results are the technical core of the proof of Theorem A.

**Theorem 4.7.** Under Assumption 4.4, suppose further that $S$ is a relatively quasi-convex subgroup of $(G, \mathcal{P})$ so that for all $P \in \mathcal{P}$ and $g \in G$ the subgroup $P \cap S^g$ is separable in $P$.

Then $S$ is separable in $G$.

**Proof.** Let $g_0 \in G \setminus S$, and let $\{L_P \mid P \in \mathcal{P}\}$ be as in Theorem 4.5. By [13, Proposition 4.5, 4.7], for sufficiently long and $S$–wide fillings of $(G, \mathcal{P})$, the image of $S$ is quasi-convex and does not contain $g_0$. However, by [13, Lemma 5.2], and the assumption on separability of the subgroups $S^g \cap P$ in $P$, there are such long and $S$–wide fillings.

Taking these fillings to also have filling kernels contained in the $L_P$ as above, and we obtain a virtually special hyperbolic quotient $\pi : G \to \overline{G}$ so that the image $\overline{S}$ of $S$ is quasi-convex and $\pi(g_0) \notin \overline{S}$. By applying Theorem 4.6, there is a finite quotient $\lambda : \overline{G} \to \overline{Q}$ so that $\lambda(\pi(g_0)) \notin \lambda(\pi(S))$. Since $g_0 \in G \setminus S$ was arbitrary, this shows that $S$ is separable in $G$, as required. \hfill $\square$

The following result relies on Proposition A.4 which is proved in Appendix A, using techniques from [13].

**Theorem 4.8.** Under Assumption 4.4, suppose further that $S_1, S_2$ are relatively quasi-convex subgroups of $(G, \mathcal{P})$ so that for all $P \in \mathcal{P}$ and $g_1, g_2 \in G$ the double coset $(P \cap S_1^{g_1})(P \cap S_2^{g_2})$ is separable in $P$.

Then the product $S_1S_2$ is separable in $G$.

**Proof.** Let $g \in G \setminus S_1S_2$. Applying Proposition A.4 and proceeding similarly to the proof of Theorem 4.7, there exists a Dehn filling $\pi : G \to \overline{G}$ so that

1. $\overline{G}$ is a virtually special hyperbolic group;
2. $\pi(S_1)$ and $\pi(S_2)$ are quasi-convex in $\overline{G}$; and
3. $\pi(g) \notin \pi(S_1)\pi(S_2)$.

Since double cosets of quasi-convex subgroups are separable in virtually special hyperbolic groups, as noted in Theorem 4.6, there is a further finite quotient of $\overline{G}$ which can be used to separate $g$ from $S_1S_2$ in $G$. \hfill $\square$

Finally, we restate and prove Theorem A.

**Theorem A.** [29, Theorem 1.2] Suppose that $(G, \mathcal{P})$ is relatively hyperbolic and that $G$ acts properly and cocompactly on a CAT(0) cube complex $X$ so Assumptions 1.1 and 1.2 are satisfied. Then $G$ acts virtually co-specialy on $X$. 

**Proof of Theorem A.** That each element of \( \mathcal{P} \) is residually finite is part of the hypothesis of Theorem A, and the rest of Assumption 4.4 follows from Theorem C.

Let \( S_1, S_2 \) be hyperplane stabilizers for the \( G \)-action on \( X \). By Lemma 2.24 these are relatively quasi-convex. The separability and double coset separability assumptions in Theorems 4.7 and 4.8 are part of the hypotheses of Theorem A. Therefore, it follows from Theorem 4.7 that \( S_1 \) is separable in \( G \), and from Theorem 4.8 that \( S_1S_2 \) is separable in \( G \). It now follows from [20, Corollary 4.3] that the \( G \)-action on \( X \) is virtually special, as required.

\[ \Box \]

**5. An example**

In this section we give an example of a group to which Theorem A applies, and for which it is at least not obvious how to apply previous results to deduce virtual specialness. Note that this example is merely indicative of many examples that could be built using this technique, or others. That the example is virtually special follows from the following application of our main result. (By a *standard cubical torus* we mean a unit cube of some dimension with opposite faces identified via translation.)

**Proposition 5.1.** Let \( Y \) be a 1-vertex non-positively curved cell complex built from a finite wedge of standard cubical tori by attaching finitely many regular right-angled hyperbolic polygons rescaled to have side-length 1. Then \( Y \) is homeomorphic to a cube complex \( C \), and this cube complex is virtually special.

**Proof.** We first note that the complex \( Y \) is negatively curved away from the tori, any two of which meet only in a point. It follows that the universal cover of \( Y \) is \( \text{CAT}(0) \) with isolated flats, and that \( \pi_1Y \) is hyperbolic relative to the free abelian subgroups represented by the cubical tori (see [23]).

Now we describe a cube complex homeomorphic to \( Y \), obtained by subdividing the cells of \( Y \). The complex \( Y \) has a single vertex (the wedge point), so all corners of all the polygons are glued at the wedge point. The only based loops in a cubical torus of length 1 are the edges of the standard cellulation of that torus. To obtain \( C \), we subdivide each \( n \)-torus into \( 2^n \) cubes of dimension \( n \), and each \( p \)-sided polygon into \( p \) squares. The link at the wedge point is unchanged, but now there are additional vertices. The link of one of the new vertices in an \( n \)-torus is a standard \( (n-1) \)-sphere, together possibly with some arcs of length \( \pi \) joining its north and south poles. (This is if the new vertex lies in the middle of an edge of the standard cellulation which
is traversed by some of the polygons.) The new vertex at the center of a \( p \)-gon has link which is a circle of length \( \frac{p\pi}{2} \). Since \( p \geq 5 \), the link condition is satisfied here as well.

The universal cover of \( C \) is therefore a \( \text{CAT}(0) \) cube complex on which \( \pi_1 Y \) acts geometrically. Since the parabolic subgroups are abelian, the Assumptions 1.1 and 1.2 both hold, and we may apply our main theorem to conclude that \( G \) acts virtually co-specially.

\[ \square \]

**Example 5.2.** Let \( G \) be given by the following presentation:

\[
\left\langle a_1, \ldots, a_6, b_1, \ldots, b_6 \mid [a_1, b_1], \ldots, [a_6, b_6], a_1a_2a_3a_4a_5a_6, b_1b_2b_3b_4b_5b_6, a_1a_2a_3a_4a_5b_1b_2b_3b_4b_5b_6, a_3a_2a_5b_1b_3b_4b_5b_6, a_4b_2a_3a_1a_2a_4b_1a_3a_2a_5b_1b_4b_5b_6, a_3a_2a_5b_1b_3b_4b_5b_6 \right\rangle.
\]

A presentation complex for \( G \) can be built from the wedge of six two-dimensional tori by attaching two right-angled hexagons and two right-angled pentagons. One can check (by hand or with a computer) that the link of the vertex is a graph of girth 4. This link is highly non-planar, suggesting that the boundary at infinity is most likely non-planar (but see \[10, 5\] for some cautionary tales). If the boundary at infinity is non-planar, this gives a proof that \( G \) is not virtually a 3–manifold group, and therefore not covered by previous theorems about 3–manifolds \[31, 37, 1\].

**6. Application to the Relative Cannon Conjecture**

To any relatively hyperbolic pair \((G, \mathcal{P})\) is associated its *Bowditch boundary* \( \partial(G, \mathcal{P}) \) \[4\]. The Relative Cannon Conjecture asserts that if \( \mathcal{P} \) is a non-empty collection of free abelian groups and \( \partial(G, \mathcal{P}) \) is homeomorphic to a 2–sphere, then \( G \) is Kleinian. The usual Cannon Conjecture makes the same assertion when \( \mathcal{P} \) is empty.

The following result is a generalization of \[13\], Theorem 1.1. Given the work we have already done, the proof is very similar.

**Theorem 6.1.** Suppose that \((G, \mathcal{P})\) is relatively hyperbolic and \( \mathcal{P} \) is a non-empty collection of free abelian groups. Suppose further that \( \partial(G, \mathcal{P}) \cong \mathbb{S}^2 \), and that \((G, \mathcal{P})\) acts weakly relatively geometrically on a \( \text{CAT}(0) \) cube complex. Then \( G \) is Kleinian.

**Proof.** Note that each element \( P \in \mathcal{P} \) fixes a point \( \xi_P \) in \( \partial(G, \mathcal{P}) \cong \mathbb{S}^2 \) and acts properly cocompactly on the complement \[4\]. Since this complement is homeomorphic to \( \mathbb{R}^2 \), each \( P \in \mathcal{P} \) must have rank 2.

Since \((G, \mathcal{P})\) acts weakly relatively geometrically on a \( \text{CAT}(0) \) cube complex, we may apply Theorem \[4,5\]. Let \( \{L_P \mid P \in \mathcal{P}\} \) be the subgroups from the conclusion of that theorem. For any family of
infinite cyclic subgroups \(N = \{N_P \mid P \in \mathcal{P}\}\) with each \(N_P \leq L_P\) the quotient group \(G(N)\) is a hyperbolic virtually special group (each \(P/N_P\) is virtually cyclic, and hence hyperbolic and virtually special). By \([21, \text{Theorem 1.10}]\) any such \(G(N)\) is virtually a finite co-volume Kleinian group. Mostow rigidity then implies that there is a short exact sequence:

\[
1 \to F \to G(N) \to K \to 1
\]

where \(F\) is finite and \(K\) is Kleinian. Since the elements of \(\mathcal{P}\) are free abelian, the parent group \(G\) has no nontrivial finite normal subgroup. By \([14, \text{Theorem 7.2}]\), \(G(N)\) also has no finite normal subgroup (at least for sufficiently long fillings of this form) and hence is in fact Kleinian.

We now take a sequence of longer and longer fillings of this form, obtaining a collection \(\{G \to G_i\}\) of quotients so that each \(G_i\) is Kleinian. As in the proof of \([17, \text{Corollary 1.4}]\) this induces a sequence of representations \(\rho_i: G \to \text{Isom}(\mathbb{H}^3)\), and precisely as in \([17]\) this sequence converges to a discrete faithful representation of \(G\) into \(\text{Isom}(\mathbb{H}^3)\), so \(G\) is Kleinian, as required.

The following is an immediate consequence of Theorems \([C]\) and \([6.1]\).

**Corollary 6.2.** Suppose that \((G, \mathcal{P})\) is relatively hyperbolic, that elements of \(\mathcal{P}\) are free abelian, that \(\partial(G, \mathcal{P}) \cong S^2\), and that \(G\) acts properly and cocompactly on a CAT(0) cube complex. Then \(G\) is Kleinian.

**Appendix A. About double coset separability**

In this appendix we generalize \([15, \text{Proposition 6.2}]\), a technical result which helps to prove double coset separability. Most of the proof is the same as in \([15]\), and we largely keep the same notation from there.

**Definition A.1.** (cf. \([15, \text{Definition 3.1}]\)) Let \(P\) be a group, \(B_1, B_2 \subseteq P\) are subgroups, and \(S\) a finite subset. A normal subgroup \(N \leq P\) is \((B_1, B_2, S)\)-wide if whenever there are \(b_1 \in B_1, b_2 \in B_2\) and \(s \in S\) so that \(b_1 s b_2 \in N\) we have \(s \in B_1 B_2\).

**Remark A.2.** This is a generalization of wideness as in Definition \([\ast]\). Namely, if \(N\) is \((B, B, S)\)-wide, then it is \((B, S)\)-wide.

**Lemma A.3.** (cf. \([15, \text{Lemma 5.1}]\)) Suppose that \(P\) is a group and that \(B_1, B_2\) are subgroups so that \(B_1 B_2\) is a separable subset of \(P\). For any finite set \(S\) there exists a finite-index normal subgroup \(K_S \leq P\) so that for any \(N \leq P\) with \(N \leq K_S\), the subgroup \(N\) is \((B_1, B_2, S)\)-wide in \(P\).
Proof. For each \( s \in S \setminus B_1 B_2 \), choose some \( P_s \leq P \) finite-index so that \( B_1 B_2 \subseteq P_s \) and \( s \not\in P_s \). Let \( P_S = \bigcap \{ P_s \mid s \in S \setminus B_1 B_2 \} \) and let \( K_S \) be the normal core of \( P_S \). The subgroups \( K_S \leq P_S \) are both finite-index in \( P \). The double coset \( B_1 B_2 \) is contained in \( P_S \).

Choose \( N \leq P \) finite-index and contained in \( K_S \). It follows that \( N \) is contained in \( P_S \), which is all we will use to show \( N \) is \((B_1, B_2, S)\)-wide. Let \( b_1 \in B_1, b_2 \in B_2 \) and \( s \in S \) and suppose \( b_1 s b_2 \in N \). If \( s \in B_1 B_2 \) then there is nothing to prove so suppose that \( s \not\in B_1 B_2 \). Then \( s \not\in P_S \). However, \( b_1 s b_2 \in N \subseteq P_S \), and also \( b_1^{-1} \in B_1 \subseteq B_1 B_2 \subseteq P_S \) and similarly \( b_2^{-1} \in P_S \). Since \( P_S \) is a subgroup, we have \( s = b_1^{-1}(b_1 s b_2) b_2^{-1} \in P_S \), which contradicts \( s \not\in P_S \). \( \square \)

The following theorem is a generalization of [15, Proposition 6.2] and is the same as [29, Theorem 3.21]. Our proof is similar to the one there.

Proposition A.4. (cf. [15, Proposition 6.2]) Suppose \((G, \mathcal{P})\) is relatively hyperbolic. Let \( \mathcal{H} \) be a finite collection of relatively quasi-convex subgroups of \((G, \mathcal{P})\), so that for any \( H_1, H_2 \in \mathcal{H} \), any \( g_1, g_2 \in G \), and any \( P \in \mathcal{P} \) the double coset \((H_1^{g_1} \cap P) (H_2^{g_2} \cap P)\) is separable in \( P \).

Let \( F \subseteq G \) be a finite subset. There exist finite-index subgroups \( \{ K_P \leq P \mid P \in \mathcal{P} \} \) so that if \( N = \{ N_P \leq P \mid P \in \mathcal{P} \} \) is a collection with \( N_P \leq K_P \) for each \( P \), and \( K = \langle \bigcup N \rangle_G \), then for all \( f \in F \) and \( \Psi, \Theta \in \mathcal{H} \) for which \( 1 \not\in \Psi \Theta f \) we have \( K \cap \Psi \Theta f = \emptyset \).

Proof. We will follow the proof from [15, Proposition 6.2], keeping most of the notation and indicating only the changes which must be made.

We have the following setup from [15]. It suffices to deal with a single pair \((\Psi, \Theta)\) from \( \mathcal{H} \), for we can then take intersections of the \( K_P \) over each of the finitely many pairs to get the required result. We therefore fix such a pair.

The cusped space for \((G, \mathcal{P})\) is denoted \( X \), and is \( \delta \)-hyperbolic. Let \( \lambda \) be a quasi-convexity constant for each \( H \in \mathcal{H} \), let \( M = \max \{ d_X(1, f) \mid f \in F \} \) and let \( \alpha = 10\delta + 2M + 4\lambda \). Associated to \( \Psi \) is a finite collection \( \mathcal{D} \) of \( \Psi \)-conjugacy classes of maximal uniquely parabolic subgroups of \( \Psi \), and for each \( D \in \mathcal{D} \) there are \( P_D \in \mathcal{P} \) and \( c_D \in G \) so \( D \leq P_D^{c_D} \).

Similarly for \( \Theta \) we have a finite collection \( \mathcal{E} \) and for \( E \in \mathcal{E} \) there are \( P_E \in \mathcal{P} \) and \( d_E \in G \) so \( E \leq P_E^{d_E} \). For \( P \in \mathcal{P} \) choose the set \( S_P \subseteq P \) exactly as in [15]. Namely, we define

\[
S_P \supseteq \{ p \in P \mid d_X(1, p) \leq 44 \delta + 8M + 16\lambda + 2L_1 + 3 \}
\]

\(^2\)In particular (taking \( H_1 = H_2 \) and \( g_1 = g_2 \)) the subgroups \( H_1^{g_1} \cap P \) are separable in \( P \).
to be a finite set so that any filling which is \((H, S_P)\)–wide for every \(H \in \mathcal{H}\) will satisfy the conclusion of \([15]\) Lemma 4.2 with \(L_1, L_2\) as specified in the proof of \([15]\) Proposition 6.2.

We make a change from \([15]\) to the requirements on the filling kernels, as we now explain. In \([15]\) the additional requirements were indexed over certain pairs \((B_1, B_2)\) of parabolic subgroups of \(\Psi\) and \(\Theta\). We instead index our requirements over the finite set of triples \((c_D, d_E, p)\), where \(u_E = P_D\) and \(p \in P_D\) satisfies \(d_X(1, p) \leq 2\alpha + 4\delta\). For such a triple \(q = (c_D, d_E, p)\) choose a finite index normal subgroup \(K_q \leq P_D\) so that any \(N \leq P_D\) with \(N \leq K_q\) is \(\left( D^{c_D}, \left( E^{d_E} \right)^p, S_P \right)\)–wide. This is possible by Lemma \([A, 3]\). For each \(P \in \mathcal{P}\) for which \(P = P_D\) in one of these triples, let \(K_P\) be the intersection of all such \(K_q\).

Choose \(N_P \leq K_P\) and consider the filling

\[G \to G \left( \{ N_P \mid P \in \mathcal{P} \} \right) = G/K.\]

We remark that this filling must be \((H, S_P)\)–wide for every \(H \in \mathcal{H}\), by Remark \([A, 2]\).

There is no issue with the setup for proving \(K \cap \Psi \Theta f = \emptyset\). Moreover, Case 1 just uses \((\Psi, S_P)\)–width, and works verbatim in this setting.

For Case 2, the beginning of the proof works as written. The first issue with the proof from \([15]\) in this setting comes with the calculation about \(w^{-1}k_k\), which in \([15]\) uses the assumption that parabolics are abelian.

We pick up the argument at the reference to Figure 2 in that proof, and adjust it as follows (keeping the notation from \([15]\)):

Note that \(k \in K \cap \text{Stab}(A)\), where \(A\) is the horoball based on \(scP\) and \(w \in scP\), so \(w^{-1}k_w \in N_P \leq P\). Also

\[
w^{-1}kw = (w^{-1}u)(u^{-1}kv)(v^{-1}z)(z^{-1}w) = (w^{-1}u) \left[ (u^{-1}kv)(z^{-1}w) \right] \left( (z^{-1}w)^{-1}(v^{-1}z)(z^{-1}w) \right).
\]

Let \(B_1 = D^{c^{-1}}, B_2 = (E^{d^{-1}})(z^{-1}w)^{-1}, \) and \(s = (u^{-1}kv)(z^{-1}w)\). Note also that \(z^{-1}w \in P\), so \(B_1, B_2 \leq P\). Moreover, \(d_X(1, (z^{-1}w)^{-1}) \leq 2\alpha + 4\delta\) (see \([15]\) for details of this claim). Thus, the triple \((c, d, (z^{-1}w)^{-1})\) is one of the finitely many triples described above, so \(N_P\) is \((B_1, B_2, S_P)\)–wide.

As in \([15]\), the \(X\)–length of \((u^{-1}kv)\) is at most \(2\alpha + 2L_1 + 3\) and the \(X\)–length of \((z^{-1}w)\) is at most \(2\alpha + 4\delta\), so \([u^{-1}kv](z^{-1}w) \in S_P\). Let \(b_1 = w^{-1}u \in B_1, \) and \(b_2 = (z^{-1}w)^{-1}(v^{-1}z)(z^{-1}w)\). The above expression shows \(b_1[(u^{-1}kv)(z^{-1}w)]b_2 \in N_P\). Because \(N_P\) is \((B_1, B_2, S_P)\)–wide, we know that \([u^{-1}kv](z^{-1}w) \in B_1B_2\), so \(w^{-1}kw \in B_1B_2\). Choose
$\beta_1 \in B_1$ and $\beta_2 \in B_2$ so $w^{-1}kw = \beta_1 \beta_2$. From the definition of $B_2$ there is $e \in E$ so that $\beta_2 = w^{-1}zd^{-1}edz^{-1}w$.

Recall the following equations from [15]:

$$w = \psi_wc, \quad z = \psi_\theta d, \quad v = \psi_\theta d,$$

where $\psi_w \in \Psi, \theta_z, \theta_v \in \Theta$.

Now, similar to [15], but with a slightly different calculation:

$$k\psi\theta = w(w^{-1}kw)(w^{-1}z)(v^{-1}\psi)$$

$$= w (\beta_1 w^{-1}zd^{-1}edz^{-1}w) (w^{-1}z)(v^{-1}\psi)$$

$$= w\beta_1 (w^{-1}z)d^{-1}ed(z^{-1}v)(v^{-1}\psi)$$

$$= \psi_w(c\beta_1 c^{-1})c(w^{-1}z)(v^{-1}d^{-1}ed)(z^{-1}v)(v^{-1}\psi)$$

$$= [\psi_w(c\beta_1 c^{-1})\psi_w^{-1}] \cdot [\theta_z e \theta_z^{-1}]$$

giving an expression for $k\psi\theta$ as an element of $\Psi\Theta$. Thus, $k \cdot g = k\psi\theta f \in \Psi\Theta f$, the same contradiction as in [15]. $\square$

References

[VH][1] I. Agol. The Virtual Haken Conjecture. Doc. Math., 18:1045–1087, 2013. With an appendix by Agol, Daniel Groves, and Jason Manning.

[agm][2] I. Agol, D. Groves, and J. F. Manning. Residual finiteness, QCERF and fillings of hyperbolic groups. Geom. Topol., 13(2):1043–1073, 2009.

[AGM_msqt][3] I. Agol, D. Groves, and J. F. Manning. An alternate proof of Wise’s malnormal special quotient theorem. Forum Math. Pi, 4:e1, 54, 2016.

[bowditch:relhyp][4] B. H. Bowditch. Relatively hyperbolic groups. Internat. J. Algebra Comput., 22(3):1250016, 66, 2012.

[BMM][5] N. Brady, J. McCammond, and J. Meier. Local-to-asymptotic topology for cocompact CAT(0) complexes. Topology Appl., 131(2):177–188, 2003.

[bridhaef:book][6] M. R. Bridson and A. Haefliger. Metric Spaces of Non–Positive Curvature, volume 319 of Grundlehren der mathematischen Wissenschaften. Springer–Verlag, Berlin, 1999.

[BurgerMozes97][7] M. Burger and S. Mozes. Finitely presented simple groups and products of trees. C. R. Acad. Sci. Paris Sér. I Math., 324(7):747–752, 1997.

[BurgerMozes00][8] M. Burger and S. Mozes. Lattices in product of trees. Inst. Hautes Études Sci. Publ. Math., 92:151–194 (2001), 2000.

[ChatterjiNiblo][9] I. Chatterji and G.Niblo. From wall spaces to CAT(0) cube complexes. Internat. J. Algebra Comput., 15(5-6):875–885, 2005.

[DHW][10] P. Dani, M. Haulmark, and G. Walsh. Right-angled Coxeter groups with non-planar boundary. Preprint, arXiv:1902.01029, 2019.

[drutusapir][11] C. Drutu and M. Sapir. Tree-graded spaces and asymptotic cones of groups. Preprint, available at front.math.ucdavis.edu/math.GR/0405030, 2004.

[Einstein-Hierarchies][12] E. Einstein. Hierarchies for relatively hyperbolic virtually special groups. Preprint, arXiv:1903.12284, 2019.
Einstein and D. Groves. Relative cubulations and groups with a 2-sphere boundary. Compos. Math., 156:862–867, 2020.

D. Groves and J. F. Manning. Dehn filling in relatively hyperbolic groups. Israel Journal of Mathematics, 168:317–429, 2008.

D. Groves and J. F. Manning. Quasiconvexity and Dehn filling. Preprint, arXiv:1708.07968, 2017. To appear in Amer. J. Math.

D. Groves and J. F. Manning. Hyperbolic groups acting improperly. Preprint, arXiv:1808.02325, 2018.

D. Groves, J. F. Manning, and A. Sisto. Boundaries of Dehn fillings. Geom. Topol., 23:2929–3002, 2019.

F. Haglund and F. Paulin. Simplicité de groupes d’automorphismes d’espaces à courbure négative. In The Epstein birthday schrift, volume 1 of Geom. Topol. Monogr., pages 181–248. Geom. Topol. Publ., Coventry, 1998.

F. Haglund and D. T. Wise. Special cube complexes. Geom. Funct. Anal., 17(5):1551–1620, 2008.

F. Haglund and D. T. Wise. Coxeter groups are virtually special. Adv. Math., 224(5):1890–1903, 2010.

P. Haïssinsky. Hyperbolic groups with planar boundaries. Invent. Math., 201(1):239–307, 2015.

G. Hruska, B. Kleiner, and (with an appendix written jointly with M. Hindawi). Hadamard spaces with isolated flats. Geometry and Topology, 9:1501–1538, 2005.

G. C. Hruska. Geometric invariants of spaces with isolated flats. Topology, 44(2):441–458, 2005.

G. C. Hruska. Relative hyperbolicity and relative quasiconvexity for countable groups. Algebr. Geom. Topol., 10(3):1807–1856, 2010.

G. C. Hruska and D. T. Wise. Finiteness properties of cubulated groups. Compos. Math., 150(3):453–506, 2014.

E. Martínez-Pedroza. Combination of quasiconvex subgroups of relatively hyperbolic groups. Groups Geom. Dyn., 3(2):317–342, 2009.

A. Minasyan. Separable subsets of GFERF negatively curved groups. J. Algebra, 304(2):1090–1100, 2006.

B. Nica. Cubulating spaces with walls. Algebr. Geom. Topol., 4:297–309 (electronic), 2004.

E. Oregón-Reyes. On cubulated relatively hyperbolic groups. Preprint, arXiv:2003.12702, 2020.

D. V. Osin. Peripheral fillings of relatively hyperbolic groups. Invent. Math., 167(2):295–326, 2007.

P. Przytycki and D. T. Wise. Mixed 3-manifolds are virtually special. J. Amer. Math. Soc., 31(2):319–347, 2018.

M. Roller. Poc sets, median algebras and group actions. An extended study of Dunwoody’s construction and Sageev’s theorem. Habilitation, Universität Regensburg, 1998. [arXiv:1607.07747]

M. Sageev. Ends of group pairs and non-positively curved cube complexes. Proc. London Math. Soc. (3), 71(3):585–617, 1995.

M. Sageev. CAT(0) cube complexes and groups. In Geometric group theory, volume 21 of IAS/Park City Math. Ser., pages 7–54. Amer. Math. Soc., Providence, RI, 2014.
SageevWise15 [35] M. Sageev and D. T. Wise. Cores for quasiconvex actions. *Proc. Amer. Math. Soc.*, 143(7):2731–2741, 2015.

Tran [36] H. C. Tran. On strongly quasiconvex subgroups. *Geom. Topol.*, 23(3):1173–1235, 2019.

Wise [37] D. T. Wise. The structure of groups with a quasiconvex hierarchy. Unpublished manuscript, version dated October 29, 2012.

WiseCSC [38] D. T. Wise. Complete square complexes. *Comment. Math. Helv.*, 82(4):683–724, 2007.

Wise:ICM [39] D. T. Wise. The cubical route to understanding groups. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II*, pages 1075–1099. Kyung Moon Sa, Seoul, 2014.

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