Achieving Proportional Representation via Voting

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Abstract Proportional representation (PR) is often discussed in voting settings as a major desideratum. For the past century or so, it is common both in practice and in the academic literature to jump to STV (Single Transferable Vote) as the solution for achieving PR. Some of the most prominent electoral reform movements around the globe are pushing for the adoption of STV.

It has been termed a major open problem to design a voting rule that satisfies the same PR properties as STV and better monotonicity properties. We present a rule called EAR (Expanding Approvals Rule) that satisfies properties stronger than the central PR axiom satisfied by STV, can handle indifferences in a convenient and computationally efficient manner, and also satisfies better candidate monotonicity properties. In view of this, our proposed rule seems to be a compelling solution for achieving proportional representation in voting settings.

Keywords committee selection · multiwinner voting · proportional representation · single transferable vote.

JEL Classification: C70 · D61 · D71

1 Introduction

Of all modes in which a national representation can possibly be constituted, this one [STV] affords the best security for the intellectual qualifications desirable in the representatives—John Stuart Mill (Considerations on Representative Government, 1861).

A major unsolved problem is whether there exist rules that retain the important political features of STV and are also more monotonic—Woodall (1997).
We consider a well-studied voting setting in which $n$ voters express ordinal preferences over $m$ candidates and based on the preferences $k \leq m$ candidates are selected. The candidates may or may not be from particular parties but voters express preferences directly over individual candidates. This kind of voting problem is not only encountered in parliamentary elections but to form any kind of representative body. When making such a selection by a voting rule, a desirable requirement is that of proportional representation. Proportional representation stipulates that voters should get representation in a committee or parliament according to the strengths of their numbers. It is widely accepted that proportional representation is the fairest way to reflect the diversity of opinions among the voters.

For the last 120 years or so, the most widely used and accepted way to achieve it is via STV—single transferable vote (Black, 1958; Tideman and Richardson, 2000) and its several variants. In fact STV is used for elections in several countries including Australia, Ireland, India, and Pakistan. It is also used to select representative committees in hundreds of settings including professional organisations, scientific organizations, political parties, schools groups, and university student councils all over the globe.

The reason for the widespread adoption of STV is partly due to the fact that it has been promoted to satisfy proportional representation axioms. In particular, STV satisfies a key PR axiom called PSC which is the abbreviation for Proportionality for Solid Coalition (Tideman and Richardson, 2000; Woodall, 1994). Tideman (1995) argues that “It is the fact that STV satisfies PSC that justifies describing STV as a system of proportional representation.” Woodall (1997) also calls the property the “essential feature of STV, which makes it a system of proportional representation.” Dummett (1984) motivated PSC as a minority not requiring to coordinate its reports and that it should deserve some high preferred candidates to be selected as long as enough voters are ‘solidly committed to such candidates.’ PSC captures the idea that as long as voters have the same top candidates (possibly in different ordering), they do not need to coordinate their preferences to get a justified number of such candidates selected. Voters from the same party not having to coordinate their reports as to maximize the number of winners from their own party can be viewed as a weak form of group-strategyproofness. PSC can also be seen as a voter’s vote not being wasted due to lack of coordination with like-minded voters. PSC has been referred to as “a sine qua non for a fair election rule” by Woodall (1994).

Although STV is not necessarily the only rule satisfying PR properties, it is synonymous with proportional representation in academia and policy circles. The out-

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1 The setting is referred to as a preferential voting system.
2 Proportional representation may be the fairest way for representation but it also allows for extreme group to have some representation at least when the group is large enough. PR also need not be the most effective approach to a stable government. Black (1958) wrote that “It [PR] makes it difficult to form a cabinet which can command a parliamentary majority and so makes for weak government.”
3 Notable uses of STV include Oscar nominations, internal elections of the British Liberal Democrats, and selection of Oxford Union, Cambridge Union, and Harvard/Radcliffe Undergraduate Councils.
4 There are two PSC axioms that differ in only whether the Hare quota is used or whether the Droop quota is used. The one with respect to the Droop quota has also been referred to as DPC (Droop’s proportionality criterion) (Woodall, 1992). Woodall (1994) went so far as saying that “I assume that no member of the Electoral Reform Society will be satisfied with anything that does not satisfy DPC.”
come of STV can also be computed efficiently which makes it suitable for large scale elections. Another reason for the adoption of STV is historical. Key figures proposed ideas related to STV or pushed for the adoption of STV. The ideas behind STV can be attributed to several thinkers including C. Andrae, T. Hare, H. R. Droop, and T. W. Hill. For a detailed history of the development of STV family of rules, please see the article by Tideman (1995). In a booklet, Aiyar (1930) explains the rationale behind different components of the STV rule. STV was supported by influential intellectuals such as John Stuart Mill who placed STV “among the greatest improvements yet made in the theory and practice of government.” Bowler and Grofman (2000) note the British influence on the spread of STV among countries with historical association with Great Britain.

With historical, normative, and computational motivation behind it, STV has become the ‘go to’ rule for PR and has strong support. It is also vigorously promoted by prominent electoral reform movements across the globe including the Proportional Representation Society of Australia (http://www.prsa.org.au) and the Electoral Reform Society (https://www.electoral-reform.org.uk).

Despite the central position of STV, it is not without some flaws. It is well-understood that it violates basic monotonicity properties even when selecting a single candidate (see e.g., Zwicker, 2016). Increasing the ranking of the winning candidate may result in the candidate not getting selected. STV is also typically defined for strict preferences which limits its ability to tackle more general weak orders. There are several settings where voters may be indifferent between two candidates because the candidates have the same characteristics that the voter cares about. It could also be that the voter does not have the cognitive power or time to distinguish between two candidates and does not wish to break ties arbitrarily. It is not clearly resolved in the literature how STV can be extended to handle weak orders without compromising on its computational efficiency or some of the desirable axiomatic properties it satisfies. The backdrop of this paper is that improving upon STV in terms of both PR as well as monotonicity has been posed as a major challenge (Woodall, 1997).

Contributions. We propose a new voting rule called EAR (Expanding Approvals Rule) that has several advantages. (1) It satisfies an axiom called Generalised PSC that is stronger than PSC. (2) It satisfies some natural monotonicity criteria that are not satisfied by STV. (3) It is defined on general weak preferences rather than just for strict preferences and hence constitutes a flexible and general rule that finds a suitable outcome in polynomial time for both strict and dichotomous preferences. Efficient computation of a rule is an important concern when we deal with election of large committees.

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5 One notable exception was philosopher Michael Dummett who was a stringent critic of STV. He proposed a rival PR method called the Quota Borda System (QBS) and pushed its case (Dummett, 1984, 1997). However, even he agreed that in terms of achieving PR, “[STV] guarantees representation for minorities to the greatest degree to which any possible electoral system is capable of doing” (page 137, Dummett, 1997).

6 Hill (2001) and Meek (1992) propose one way to handle indifferences but which leads to an algorithm that may take time $O(m^2)$.)
Our work also helps understand the specifications under which different variants of STV satisfy different PR axioms. Apart from understanding how far STV and EAR satisfy PR axioms, one of the conceptual contributions of this paper is to define a taxonomy of PR axioms based on PSC and identify their relations with each other. In particular, we propose a new axiom for weak preferences called Generalised PSC that simultaneously generalises PSC (for strict preferences) and proportional justified representation (for dichotomous preferences).

2 Model and Axioms

In this section, we lay the groundwork of the paper by first defining the model and then formalizing the central axioms by which proportional representation rules are judged.

2.1 Model

We consider the standard social choice setting with a set of voters \( N = \{1, \ldots, n\} \), a set of candidates \( C = \{c_1, \ldots, c_m\} \) and a preference profile \( \succsim = (\succsim_1, \ldots, \succsim_n) \) such that each \( \succsim_i \) is a complete and transitive relation over \( C \). Based on the preference profile, the goal is to select a committee \( W \subset C \) of size \( k \). Since our new rule is defined over weak orders rather than strict orders, we allow the voters to express weak orders. We write \( a \succsim_i b \) to denote that voter \( i \) values candidate \( a \) at least as much as candidate \( b \) and use \( >_i \) for the strict part of \( \succsim_i \), i.e., \( a >_i b \) if \( a \succsim_i b \) but not \( b \succsim_i a \). Finally, \( \sim_i \) denotes \( i \)'s indifference relation, i.e., \( a \sim_i b \) if and only if both \( a \succsim_i b \) and \( b \succsim_i a \). The relation \( \succsim_i \) results in equivalence classes \( E^1_i, E^2_i, \ldots, E^k_i \) for some \( k_i \) such that \( a >_i a' \) if and only if \( a \in E^l_i \) and \( a' \in E^{l'}_i \) for some \( l < l' \). Often, we will use these equivalence classes to represent the preference relation of a voter as a preference list \( i : E^1_i, E^2_i, \ldots, E^k_i \). If candidate \( c \) is in \( E^j_i \), we say it has rank \( j \) in voter \( i \)'s preference. For example, we will denote the preferences \( a \sim_i b >_i c \) by the list \( i : \{a, b\}, \{c\} \). If each equivalence is of size 1, the preferences will be called strict preferences or linear orders. Strict preferences will be represented by a comma separated list of candidates. If for each voter, the number of equivalence classes is at most two, the preferences are referred to as dichotomous preferences. When the preferences of the voters are dichotomous, the voters can be seen as approving a subset of voters. In this case for each voter \( i \in N \), the first equivalence class \( E^1_i \) is also referred to as the approval set \( A_i \). The vector \( A = (A_1, \ldots, A_n) \) is referred to as the ballot profile.

The model allows for voters to express preference lists that do not include some candidates. In that case, the candidates not included in the list will be assumed to form the last equivalence class.
2.2 PR under Strict Preferences

In order to understand the suitability of voting rules for proportional representation, we recap the central PR axiom from the literature. It was first mentioned and popularised by Dummett (1984). It is defined for strict preferences.

**Definition 1 (Solid coalition)** A set of voters \( N' \) is a solid coalition for a set of candidates \( C' \) if every voter in \( N' \) ranks (strictly prefers) every candidate in \( C' \) ahead of every candidate in \( C \setminus C' \). That is, for all \( i \in N' \) and for any \( c' \in C' \)

\[ \forall c \in C \setminus C' \quad c' >_{i} c. \]

**Definition 2 (q-PSC)** Let \( W \) be an election outcome, \( q \)-PSC is satisfied if for all solid coalitions \( N' \) of size \( |N'| \geq \ell q \) supporting candidate subset \( C' \) then

\[ |W \cap C'| \geq \min\{\ell, |C'|\}. \]

If \( q = n/k \), then we refer to the property as Hare-PSC. If \( q = n/(k + 1) + \epsilon \) for small \( \epsilon > 0 \), then we refer to the property as Droop-PSC.

There are some reasons to prefer the 'Droop' quota \( n/(k + 1) + \epsilon \) for small \( \epsilon > 0 \). Firstly, for \( k = 1 \) the use of the Droop quota leads to rules that return a candidate that is most preferred by more than half of the voters. Secondly, STV defined with respect to the Droop quota ensures slight majorities get slight majority representation. Hare-PSC was stated as an essential property that a rule designed for PR should satisfy (Dummett, 1984). Droop-PSC is referred to as the Droop Proportionality Criterion in the literature. When preferences are strict and \( k = 1 \), Woodall (1997) refers to the restriction of Droop-PSC under these conditions as the majority principle. The majority principle requires that if a majority of voters are solidly committed to a set of candidates, then one of the candidates must be selected.

**Remark 1** If \( N' \) is a solid coalition for \( C' \) with \( |N'| > n/2 \) then \( C' \) is a Condorcet committee according to Gehrlein (1985).

**Example 1** Consider the profile with 9 voters and where \( k = 3 \). Then the voters in set \( N' = \{1, 2, 3\} \) form a solid coalition with respect to Hare quota who solidly support candidates in \( \{c_1, c_2, c_3, c_4\} \). The voters in set \( N'' = \{4, 5, 6, 7, 8, 9\} \) form a solid coalition with respect to Hare quota who solidly support three candidate subsets.

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\[ \text{Droop PSC is also referred to as Droop's proportionality criterion (DPC). Technically speaking the Droop quota is } n/(k + 1) + 1. \text{ The exact value } n/(k + 1) \text{ is referred to as the Hagenbach-Bischoff quota.} \]
\{e_1\}, \{e_1, e_2\} and \{e_1, e_2, e_3\}.

\begin{align*}
1 : & \quad c_1, c_2, c_3, c_4, \ldots \\
2 : & \quad c_4, c_1, c_2, c_3, \ldots \\
3 : & \quad c_2, c_3, c_4, c_1, \ldots \\
4 : & \quad e_1, e_2, e_3, \ldots \\
5 : & \quad e_1, e_2, e_3, \ldots \\
6 : & \quad e_1, e_2, e_3, \ldots \\
7 : & \quad e_1, e_2, e_3, \ldots \\
8 : & \quad e_1, e_2, e_3, \ldots \\
9 : & \quad e_2, e_1, e_3, \ldots 
\end{align*}

One can also define a weak version of PSC. In some works (see e.g., Elkind et al., 2014, 2017; Faliszewski et al., 2017), the weaker version has been attributed to the original definition of PSC as defined by Dummett.

**Definition 3 (weak q-PSC)** Let \( W \) be an election outcome, weak q-PSC is satisfied if for all solid coalitions \( N' \) of size \(|N'| \geq \ell q\) supporting a candidate subset \( C' \) with \(|C'| \leq \ell\) then

\[ |W \cap C'| \geq \min\{\ell, |C'|\}. \]

Note that q-PSC implies weak q-PSC but the reverse need not hold. We also note that under strict preferences and \( k = 1, \) if a majority of the voters have the same most preferred candidate, then Weak Droop-PSC implies that the candidate is selected.

We will generalise the PSC property to weak preferences which has not been done in the literature.

### 2.3 Candidate Monotonicity Axioms

PR captures the requirement that cohesive groups of voters should get sufficient representation. Another desirable property is candidate monotonicity that requires that increased support for a candidate should result in increased chance of the candidate being selected. Candidate monotonicity involves the notion of a candidate being reinforced. We say a candidate is reinforced if its relative preference is improved while not changing the relative preferences of all other candidates. We are now in a position to define standard candidate monotonicity properties of voting rules (see e.g., Elkind et al., 2017; Sanver and Zwicker, 2013). The definitions apply not just to strict preferences but also to weak preferences.

**Definition 4 (Candidate Monotonicity)**

- **Candidate Monotonicity (CM)**: if a winning candidate is reinforced, it remains a winning candidate.
– Rank Respecting Candidate Monotonicity (RRCM): if a winning candidate $c$ is reinforced without changing the respective ranks of other winning candidates in each voter’s preferences, then $c$ remains a winning candidate.

– Non-Crossing Candidate Monotonicity (NCCM): if a winning candidate $c$ is reinforced without ever crossing over another winning candidate, then $c$ remains a winning candidate.

– Weak Candidate Monotonicity (WCM): if a winning candidate is reinforced, then some winning candidate still remains winning.

NCCM and WCM are extremely weak properties but STV violates them even for $k = 1$. We observe the following relations between the properties.

**Proposition 1** The following relations hold.

– $CM \implies RRCM \implies NCCM$
– $CM \implies WCM$
– Under $k = 1$, RRCM is equivalent to CM
– Under dichotomous preferences, RRCM is equivalent to CM.

Note that if a rule fails CM for $k = 1$, then it also fails RRCM, NCCM, and WCM.

### 3 PR under generalised preference relations

The notion of a solid coalition and PSC can be generalised to the case of weak preferences. In this section, we propose a new axiom called generalised PSC which not only generalises PSC (that is only defined for strict preferences) but also PJR (proportional justified representation) a PR axiom that is only defined for dichotomous preferences.

**Definition 5** ((Generalised) solid coalition) A set of voters $N'$ is a (generalised) solid coalition for a set of candidates $C'$ if every voter in $N'$ ranks (weakly prefers) every candidates in $C'$ at least as high as every candidate in $C\setminus C'$. That is, for all $i \in N'$ and for any $c' \in C'$

$$\forall c \in C \setminus C' \quad c' \succ_i c.$$  

**Definition 6** ((Generalised) q-PSC) Let $W$ be an election outcome, (generalised) q-PSC is satisfied if for all (generalised) solid coalitions $N'$ of size $|N'| \geq \ell q$ supporting candidate subset $C'$ then there exists a set $C'' \subseteq W$ with size at least $\min\{\ell, |C'|\}$ such that for all $c'' \in C''$

$$\exists i \in N' : c'' \succ_i c^{(i|C'|)} ,$$

where $c^{(i|C'|)}$ denotes voter $i$’s $|C'|$-th most preferred candidate.

When $q = n/k$ this defines the (generalised) Hare quota, when $q = n/(k + 1) + \varepsilon$ for small $\varepsilon > 0$ this defines the (generalised) Droop quota.

The following example shows that generalised q-PSC is a weak property when solid coalitions equal, or just barely exceed, the quota $q$. 


Example 2 Let \( N = \{1, 2, 3, 4\} \), \( C = \{a, b, \ldots, j\} \) and \( k = 2 \),
\[
\begin{align*}
\succeq_1 &: c \sim c' \quad \forall c, c' \in C \\
\succeq_2 &: a >_2 b >_2 \ldots >_2 j \\
\succeq_3 &: \text{define in anyway,} \\
\succeq_4 &: \text{define in anyway.}
\end{align*}
\]

We consider PSC with respect to the Hare quota; that is, \( q = n/k = 2 \). There is a (generalised) solid coalition \( N' = \{1, 2\} \) with \( |N'| \geq q \) supporting candidate subset \( C' = \{a\} \). The generalised PSC axiom requires the election of \( \ell = 1 \) candidates into \( W \) who are at least as preferred as either voter \( \{1, 2\} \)'s most preferred candidate. Since voter 1 is indifferent between all candidates, electing any candidate such as \( j \in C \), will satisfy the axiom - this is despite candidate \( j \) being voter 2’s strictly least preferred candidate.

Definition 7 ((Generalised) weak \( q \)-PSC) Let \( W \) be an election outcome, weak (generalised) \( q \)-PSC is satisfied if for all (generalised) solid coalitions \( N' \) of size \( |N'| \geq \ell q \) supporting a candidate subset \( C' : |C'| \leq \ell \) then there exists a set \( C'' \subseteq W \) with size at least \( \min(\ell, |C'|) \) such that for all \( c'' \in C'' \)
\[
\exists i \in N' : c'' \succ_i c^{(i,|C'|)},
\]
where \( c^{(i,|C'|)} \) denotes voter \( i \)'s \( |C'| \)-th most preferred candidate.

Generalising PSC to the case of weak preferences is important because it provides a useful link with PR properties defined on dichotomous preferences. PJR (Proportional Justified Representation) (Sánchez-Fernández et al., 2017b, Aziz and Huang, 2016) is a proportional representation property for dichotomous preferences (Aziz et al., 2017).

Recall the following definition of PJR:

Definition 8 (PJR) A committee \( W \) with \( |W| = k \) satisfies PJR for a ballot profile \( A = (A_1, \ldots, A_n) \) over a candidate set \( C \) if for every positive integer \( \ell \leq k \) there does not exist a set of voters \( N^* \subseteq N \) with \( |N^*| \geq \ell n/k \) and both
\[
\big| \bigcap_{i \in N^*} A_i \big| \geq \ell \quad \text{but} \quad \big| \bigcup_{i \in N^*} A_i \cap W \big| < \ell.
\]

Proposition 2 Under dichotomous preferences, generalised weak Hare-PSC implies PJR.

Proof. Under dichotomous preferences every voter, say \( i \), has precisely two equivalence classes which partition the candidate set \( C \). Thus, we can naturally represent every voters preference by an (possibly empty) approval ballot \( A_i \) representing the subset of strictly preferred candidates.

For the purpose of a contradiction, let \( W \) be a committee of size \( k \) and suppose that Hare-PSC holds but PJR does not. If PJR does not hold, then there must exist a set \( N^* \) of voters and a positive integer \( \ell \) such that \( |N^*| \geq \ell n/k \) and both
\[
\big| \bigcap_{i \in N^*} A_i \big| \geq \ell \quad \text{and} \quad \big| \bigcup_{i \in N^*} A_i \cap W \big| < \ell.
\]
But note that $N^*$ is a (generalised) solid coalition for each candidate subset $C' \subseteq \bigcap_{i \in N^*} A_i$, since every candidate in $C'$ is weakly preferred to every candidate in $C$ for $i \in N^*$. Since $|\bigcap_{i \in N^*} A_i| \geq \ell$, we can select a $C'$ with exactly $\ell$ candidates so that $|C'| = \ell$.

Thus, if generalised weak Hare-PSC holds then there exists a set $C'' \subseteq W$ with size $\geq \min\{\ell, |C'|\}$ such that for all $i \in N^*$, $c'' \in c(i,|C'|)$. But note that for any $i \in N^*$ we have $c(i,|C'|) \in A_i$ and hence $c'' \in A_i$. It follows that $c'' \in W \cap A_i$ and $C'' \subseteq (\bigcup_{i \in N^*} A_i) \cap W$ and so

$$|\bigcup_{i \in N^*} A_i \cap W| \geq |C'' \cap W| \geq \ell,$$

which contradicts (1).

\[ \square \]

**Proposition 3** Under dichotomous preferences, PJR implies generalised Hare-PSC.

**Proof.** For the purpose of a contradiction, suppose that $W$ is a committee of size $k$ which satisfies PJR but not generalised Hare-PSC. If generalised Hare-PSC does not hold, then for some positive integer $\ell$ there exists a solid coalition $N^*$ with $|N^*| \geq \ell n^k$ supporting a candidate subset $C'$ such that for every subset $C'' \subseteq W$ with size at least $\min\{\ell, |C'|\}$ we have for all $i \in N^*$ and for all $c'' \in C''$

$$c(i,|C'|) \succ c''. \tag{2}$$

Now since $N^*$ is a solid coalition for $C'$ of size at least

$$\ell n^k \geq \min\{\ell, |C'|\} n^k,$$

and so it must be that

$$C' \subseteq \bigcap_{i \in N^*} A_i \Rightarrow |\bigcap_{i \in N^*} A_i| \geq |C'|.$$

By the PJR condition we then have

$$|\bigcup_{i \in N^*} A_i \cap W| \geq \min\{\ell, |C'|\}.$$

But then defining

$$C'' = (\bigcup_{i \in N^*} A_i) \cap W,$$

provides a contradiction to (2), as required.

\[ \square \]

**Corollary 1** Under dichotomous preferences, PJR, weak generalised Hare-PSC, and generalised Hare-PSC are equivalent.

Since it is known that testing PJR is coNP-complete [Aziz and Huang, 2017], it follows that testing generalised PSC and generalised weak PSC is coNP-complete.

**Corollary 2** Testing generalised PSC and generalised weak PSC is coNP-complete even under dichotomous preferences.

On the other hand, PSC and weak PSC can be tested efficiently (please see the appendix).

Figure depicts the relations between the different PR axioms.
4 The Case of STV

In this section, we define the family of STV rules for instances where voters submit strict preferences. The family is formalised as Algorithm 1. STV is a multi-round rule in which in each round either a candidate is selected as a winner or one candidate is eliminated from the set of potential winners. Depending on the quota $q$ and the reweighting rule applied, one can obtain particular STV rules (see e.g., Aleskerov and Karpov 2013). One of the most common rules is attained when the quota is set to the Hare quota and the discrete reweighting is used, this implies that a subset of voters of size $n/k$ is removed from the profile once their most preferred candidate in the current profile has been selected. STV modifies the preference profile $\succ$ by deleting candidates. We will denote by $C(\succ)$ the current set of candidates in the profile $\succ$. When $k = 1$, STV is referred to as Instant-Runoff voting (IRV) or as the Alternative Vote (AV).

Example 3 (Illustration of STV) Consider the following profile with 9 voters, $k = 3$ and suppose we use Hare-STV with fractional reweighting.

1 : $c_1, c_2, c_3, e_1, e_2, e_3, e_4, d_1$
2 : $c_2, c_3, c_1, e_1, e_2, e_3, e_4, d_1$
3 : $c_3, c_1, d_1, c_2, e_1, e_2, e_3, e_4$
4 : $e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1$
5 : $e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1$
6 : $e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1$
7 : $e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1$
8 : $e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1$
9 : $e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1$

In the first round $e_1$ is selected and the total weight of the voters in set $\{4, 5, 6, 7, 8, 9\}$ goes down by slightly above 3. Candidate $e_1$ is then removed from
Algorithm 1 STV family of Rules

Input: \((N, C, \succ, k)\) parametrised by quota \(q \in \left(\frac{1}{k+1}, \frac{1}{k}\right]\). \(\succ\) is profile of strict preferences

Output: \(W \subseteq C\) such that \(|W| = k\)

1: \(W \leftarrow \emptyset\);
2: \(w_i \leftarrow 1\) for each \(i \in N\)
3: \(j \leftarrow 1\)
4: while \(|W| < k\) do
5: if \(|W| + |C(\succ)| = k\) then
6: return \(W \cup C(\succ)\)
else
7: if there is a candidate \(c\) with plurality support (weight of voters who have \(c\) as the first ranked candidate) at least \(q\) then
8: Letting the set of voters supporting \(c\) be denoted by \(N'\). Modify the weights of voters in \(N'\) so the total weight of voters in \(N'\) decreases by at least \(q\).
9: Remove \(c\) from the profile \(\succ\).
10: \(W \leftarrow W \cup \{c\}\)
11: else
12: Remove a candidate with the lowest plurality support from the current preference profile \(\succ\)
13: end if
14: end if
15: end while
16: return \(W\)

the preference profile. In the second round, \(e_2\) is selected and the total weight of the voters in set \(\{4, 5, 6, 7, 8, 9\}\) is now zero. Candidate \(e_2\) is then removed from the preference profile. After that since no candidate has support, with respect to current weights, of at least \(q = 3\) one candidate is deleted. Candidates \(e_3\) and \(e_4\) are removed in succession as they have plurality score zero with respect to the current voting weights. Then candidates \(c_1, c_2, c_3\) all have equal and lowest plurality scores of one, due to lexicographic tie-breaking candidates \(c_3\) and \(c_2\) are removed in succession. At that point, \(c_1\) has plurality score \(q = 3\) so it is the last candidate selected.

STV has been claimed to satisfy Proportionality for Solid Coalitions/Droop Proportionality Criterion [Dummett 1984; Woodall 1994]. On the other hand, STV violates just about every natural monotonicity axiom that has been proposed in the literature.

STV can use fractional reweighting or discrete reweighting. We will show that fractional reweighting is crucial for some semblance of PR. Incidentally, fractional reweighting is not necessarily introduced to achieve better PR but primarily to minimize the “stochastic aspect” of tie-breaking in STV (pp 32, Tideman 1995). The following result shows that if STV resorts to discrete reweighting then it does not even satisfy weak PSC. Discrete reweighting refers to the modification of voter weights in Line 9 of Algorithm I such that the total weight of voters in \(N'\) decreases by some integer greater or equal to \(q\).
Proposition 4  Under strict preferences, STV with discrete reweighting does not satisfy Weak $q$-PSC irrespective to whether we use the Hare quota or Droop quota, for any $q \in \left(\frac{1}{8}, \frac{5}{12}\right]$.

Proof. Let $N = \{1, 2, \ldots, 10\}$, $C = \{c_1, \ldots, c_8\}$, $k = 7$ and consider the following profile:

1 : $c_1, c_5, c_6, c_7, c_8, c_2, c_3, c_4$
2 : $c_2, c_5, c_6, c_7, c_8, c_1, c_3, c_4$
3 : $c_3, c_5, c_6, c_7, c_8, c_1, c_2, c_4$
4 : $c_4, c_5, c_6, c_7, c_3, c_1, c_2, c_3$
5 : $c_5, c_6, c_7, c_8, c_1, c_2, c_3, c_4$
6 : $c_5, c_6, c_7, c_8, c_1, c_2, c_3, c_4$
7 : $c_5, c_6, c_7, c_8, c_1, c_2, c_3, c_4$
8 : $c_5, c_6, c_7, c_8, c_1, c_2, c_3, c_4$
9 : $c_5, c_6, c_7, c_8, c_1, c_2, c_3, c_4$
10 : $c_5, c_6, c_7, c_8, c_1, c_2, c_3, c_4$

We use STV with discrete reweighting to select the candidates. Under discrete reweighting, the total weights of voters are modified by some integer $p \geq 2$ (refer to Line 9 of Algorithm 1). In this proof we focus on the case where $p = 2$, a similar argument can be applied to prove the proposition for larger integer values.

Applying the STV process, first $c_5, c_6$ and $c_7$ are selected. Each time we select these candidates, the voting weight of voters in the set $\{5, 6, 7, 8, 9, 10\}$ goes down by 2. Thus the remaining four candidates are to be selected from $c_1, c_2, c_3, c_4$ and $c_8$. At this stage candidate $c_8$ has the lowest support of zero and is removed from all preference profiles and the list of potentially elected candidates, and hence $c_8 \not\in W$.

Weak $q$-PSC requires that at least four candidates from $\{c_5, c_6, c_7, c_8\}$ be selected, since $6 \geq 4 \times q$ for all $q \in \{10/8, 10/7\}$, but using discrete reweighting only three candidates are selected by STV.

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The proof above has a similar argument as Example 1 in (Sánchez-Fernández et al., 2017) that concerns an approval voting setting.

Next we show that STV satisfies weak PSC with fractional reweighting. We provide a formal proof of this statement, under the assumption that the reweighting in Line 9 of Algorithm 1 decreases the total voter weight by precisely $q$.

Proposition 5  For any $q \in \left(\frac{1}{8}, \frac{5}{12}\right]$, under strict preferences STV satisfies weak $q$-PSC.

Proof. Let $N' \subseteq N$ be a solid coalition supporting a set of candidates $|C'| \leq \ell$ such that $|N'| \geq \ell q$.

Now consider the first approval election. Let $C^{(1)}$ be the set of candidates involved in this election (i.e. the union of all voters’ first preference). The total weight of all voters in $N'$ is $T' \geq \ell q$ and the size of $C' \cap C^{(1)}$ is $\leq \ell$, hence there must exists at
least one candidate in $C' \cap C^{(1)}$ with support at least $q (\ast)$. Thus, the deletion aspect of STV (Line 13, Algorithm 1) is not applied.

Note that for all $q \in \left( \frac{\ell}{\ell q}, \frac{\ell q}{2} \right]$, if there exist a candidate with support at least $q$, at some stage, then it must be that strictly less than $k$ candidates have been elected into $W$ at this stage. To see this, simply observe that every candidate elected reduces the total weight of voters $n$ by precisely $q$. Thus if at least $k$ candidates were elected and there exists a candidate with support at least $q$ this would imply a total initial weight of all voters at least $kq + q = (k + 1)q$ which is strictly greater than $n$. This is a contradiction since the total initial weight of all voters is $n$.

Returning to $(\ast)$ it must be that some $c \in C' \cap C^{(1)}$ is elected into $W$ at the first stage and the total weight of voters in $N'$ is reduced to $T' - q \geq (\ell - 1)q$. Consider the new set of plurality votes of voters in $N'$, say $C'' \subset C' \cap C^{(1)}$. The size of this set $C''$ is bounded above by $(\ell - 1)$ and the total weight of voters in $N'$ is at least $(\ell - 1)q$, hence once again, there must exist at least one candidate in the election with support at least $q$. Noting the previous argument, it must be that $|W| < k$ at this stage and so another candidate from $C' - \{c\}$ is elected.

The above process can be iteratively applied $\ell$ times - since the first $\ell$ preferences of every voter in $N'$ is contained in $C'$. This ensures that $\ell$ candidates from $C'$ are elected into $W$. Thus, weak $q$-PSC is satisfied. $\square$

**Remark 2** Within the proof above, it can be observed that the deletion aspect of STV (Line 13, Algorithm 1) is never applied in the first $\ell$ rounds.

The argument for weak $q$-PSC is extended to that for $q$-PSC as follows. Again we assume that the reweighting in Line 9 of Algorithm 1 decreases the total voter weights by precisely $q$.

**Proposition 6** For any $q \in \left( \frac{\ell}{\ell q}, \frac{\ell q}{2} \right]$, under strict preferences STV satisfies $q$-PSC.

**Sketch.** Let $N' \subseteq N$ be a solid coalition supporting a set of candidates $|C'|$ such that $|N'| \geq \ell q$. Note that at most $k - \ell$ candidates can be selected that have no plurality contribution by voters in $N'$. The remaining $\ell$ candidates must be selected with some contribution by voters in $N'$. Also note that voters in $N'$ must contribute their weight to candidates in $C'$ before they contribute to candidates that come later. They can only contribute to candidates after $C'$ once all the candidates in $C'$ have been removed. However we argue that at most $|C'| - \ell$ candidates can be deleted from $C'$. At any point when the candidates in $N'$ have voting weight at least $\ell'q$ and the voters in $N'$ form a solid coalition for $\ell'$ candidates, then by a similar argument as in proof of the previous proposition, all the $\ell'$ candidates have to be selected. This implies that at most $|C'| - \ell$ candidates can be deleted from $C'$ and that $\ell$ candidates must be selected from $C'$. $\square$

### 5 EAR (Expanding Approvals Rule)

We now present EAR (Expanding Approvals Rule). The rule utilises the idea of $j$-approval voting whereby every voter is asked to approve their $j$ most preferred candidates, for some positive integer $j$. At a high level, EAR works as follows.
An index $j$ is initialised to 1. The voting weight of each voter is initially 1. We use a quota $q$ that is between $n/(k+1)$ and $n/k$. While $k$ candidates have not been selected, we do the following. We perform $j$-approval voting with respect to the voters’ current voting weights. If there exists a candidate $c$ with approval support at least a quota $q$, we select such a candidate with the highest support and reduce the weight of all voters who approved of $c$ by a total of $q$. If there are multiple such candidates, we use rank-maximality (defined below) to break ties. If there exists no such candidate, we increment $j$ by one and repeat until $k$ candidates have been selected.

The rule is formally specified as Algorithm 2. It is well-defined for weak preferences. EAR is a novel rule that has not been formalised or analyzed before. However it is based on a combination of several natural ideas that have been used in the design of voting rules.

(i) Candidates are selected in a sequential manner.
(ii) A candidate needs to have at least $n/(k+1)$ (roughly Droop quota) ‘support’ to be selected.
(iii) The voting weight of a voter is reduced if some of her voting weight has already been used to select some candidate. The way voting weight is reduced is fractional.
(iv) We use $j$-approval voting for varying $j$. When considering weak orders, we adapt $j$-approval voting so that in $j$-approval voting, a voter not only approves her $j$-most preferred candidates but also any candidate that is at least as preferred as the $j$-th most preferred candidate.
(v) Among candidates with support at least $q$, rank maximality is used for the order of selection (defined below).

For any candidate $a$, its corresponding rank vector is $r(a) = (r_1(a), \ldots, r_m(a))$ where $r_j(a)$ is the number of voters who have $a$ in his $j$-th most preferred equivalence class. We compare rank vectors lexicographically. One rank vector $r = (r_1, \ldots, r_m)$ is better than $r' = (r'_1, \ldots, r'_m)$ if for the smallest $i$ such that $r_i \neq r'_i$, it must hold that $r_i > r'_i$.

In EAR, we have specified how to perform the priority tie-breaking (Step 1) and how exactly to do the fractional reweighting (Step 11). However these specifications are not critical for satisfying the PR properties of EAR. On the other hand, the default of quota of $q = \frac{1}{k+1} + \frac{1}{n(k+1)} \left( \left\lfloor \frac{n}{k+1} \right\rfloor + 1 - \frac{n}{k} \right)$ is useful to satisfy (Generalised) Droop-PSC. The reason for choosing this quota is that $q$ can be viewed as $q = \frac{1}{k+1} + \epsilon$ where $\epsilon$ is small enough so that for any $\ell \leq k$, $\ell \cdot q < \ell \cdot \frac{1}{k+1} + 1$. The specific way we perform the priority tie-breaking (Step 11) is useful for ensuring that EAR satisfies RRCM.

In the following example, we demonstrate how EAR works.

---

8 Fractional reweighting in STV has been referred to as Gregory or ‘senatorial’ (see e.g., Janson, 2016; Tideman, 1995).
Algorithm 2 EAR

**Input:** \( (N, C, \succ, k) \) parametrised by quota \( q \in \left( \frac{n}{k+1}, \frac{2}{k+1} \right] \). [\( \succ \) can contain weak orders; if a voter \( i \) expresses her preferences over a subset \( C' \subset C \), then \( C \setminus C' \) is considered the last equivalence class of the voter.]

**Output:** \( W \subseteq C \) such that \( |W| = k \)

1. Construct a priority ordering \( L \) over \( C \) that is with respect to rank maximality under \( \succ \). In case of ties, use lexicographic tie-breaking.
2. Default quota \( q \) is set as follows.
   \[
   q = \frac{n}{k+1} + \frac{1}{m+1} \left( \left\lfloor \frac{n}{k+1} \right\rfloor + 1 - \frac{n}{k+1} \right) - \frac{n}{k+1}.
   \]
3. \( C' \leftarrow C; W \leftarrow \emptyset \)
4. \( w_i \leftarrow 1 \) for each \( i \in N \)
5. \( j \leftarrow 1 \)
6. while \( |W| < k \) do
7.   while there does not exist a candidate in \( C \setminus W \) with support at least \( q \) in a \( j \)-approval vote do
8.     \( j \leftarrow j + 1 \)
9.   end while
10. Among the candidates with weight support at least \( q \) in a \( j \)-approval election, select the candidate \( c \) from \( C' \) that has highest ranking wrt \( L \). [Voters are asked to approve their \( j \) most preferred candidates and any candidates that are at least as preferred as the \( j \)-th most preferred candidate.]
11. Modify the weights of voters who supported \( c \) in the \( j \)-approval election. If the total support for \( c \) is \( T \), then for each \( i \in N \) who supported \( c \):
   \[
   w_i \leftarrow w_i \times \frac{T - q}{T}.
   \]
12. \( W \leftarrow W \cup \{c\} \)
13. \( C' \leftarrow C \setminus \{c\} \)
14. end while
15. return \( W \)

Example 4 (Illustration of EAR) Consider the profile with 9 voters and where \( k = 3 \). Note that the Droop quota is \( 9/4 = 2.25 \).

1 : c₁, c₂, c₃, e₁, e₂, e₃, e₄, d₁
2 : c₂, c₃, c₁, e₁, e₂, e₃, e₄, d₁
3 : c₃, c₁, d₁, c₂, e₁, e₂, e₃, e₄
4 : e₁, e₂, e₃, e₄, c₁, c₂, c₃, d₁
5 : e₁, e₂, e₃, e₄, c₁, c₂, c₃, d₁
6 : e₁, e₂, e₃, e₄, c₁, c₂, c₃, d₁
7 : e₁, e₂, e₃, e₄, c₁, c₂, c₃, d₁
8 : e₁, e₂, e₃, e₄, c₁, c₂, c₃, d₁
9 : e₁, e₂, e₃, e₄, c₁, c₂, c₃, d₁
In the first round $e_1$ is selected and the total weight of the voters in set $\{4, 5, 6, 7, 8, 9\}$ goes down by slightly above $9/4 = 2.25$.

In the second round, since no other candidate has sufficient weight when we run 1-approval, we consider 2-approval voting. Under 2-approval, candidate $e_2$ receives more than 2.25 support. When $e_2$ is selected, the total weight of the voters in set $\{4, 5, 6, 7, 8, 9\}$ goes down by again slightly above $9/4 = 2.25$. At this point the total weight of voters in set $\{4, 5, 6, 7, 8, 9\}$ is at most 1 and no unselected candidate has approval support more than 2.25, under 2-approval. So EAR considers 3-approval whereby, candidates $c_1$ and $c_3$ both get support 3. Hence EAR selects $c_1$, due to the priority ordering $L$, and the winning committee is $\{e_1, e_2, c_1\}$.

We point out the rule’s outcome can be computed efficiently.

**Proposition 7** *EAR runs in polynomial time $O(n + m)$*.

**Proof.** In each round, the smallest $j$ is found for which there are some candidates in $C \setminus W$ that have an approval score of at least $q$. The candidate $c$ with the highest score is identified. All voters who approved of $c$ have their weight modified accordingly which takes linear time. Hence the whole algorithm takes polynomial time. \(\Box\)

A possible criticism of EAR is that the choice of quota as well as reweighting makes it complicated enough to not be usable by hand or to be easily understood by the general public. However we have shown that without resorting to fractional reweighting, STV already fails weak PSC. Secondly, even simpler versions of STV elections are best used by the help of computers.

Since EAR is designed for proportional representation which is only meaningful for large enough $k$, EAR may not be the ideal rule for $k = 1$. Having said that, we mention the following connection with a single-winner rule from the literature.

**Remark 3** For $k = 1$ and under linear orders with every candidate in the list, EAR is equivalent to the fall-back voting rule of Brams and Sanver (2009).

Under dichotomous preferences and using Hare quota, EAR bears similarity to Phragmén’s first method (also called Enström’s method) described by Janson (2016) (page 59). However the latter method when extended to strict preferences does not satisfy Hare-PSC. Although EAR has connections with previous rules, extending them to the case of multiple-winners and to handle dichotomous, strict and weak preferences simultaneously and satisfy desirable PR properties requires careful thought.

We observe some simple properties of the rule. It is anonymous (the names of the voters do not matter). It is also neutral as long as lexicographic tie-breaking is not required to be used. Under linear orders and when using EAR with the default quota, if more than half the voters most prefer a candidate, then that candidate is selected. We defer the proof the next section because the statement is implied by a stronger statement. We note that using the default quota of EAR is crucial for the majority property.
6 Proportional Representation and Candidate Monotonicity under EAR

6.1 Proportional Representation under EAR

We argue that EAR satisfies the central PR axioms. In order to build intuition, we first give an argument that EAR satisfies Droop-PSC under linear orders. We will then extend the argument for weak orders.

**Proposition 8** Under linear orders, EAR satisfies Droop-PSC.

**Proof.** For the purpose of a contradiction suppose that Droop-PSC is not satisfied for some EAR outcome \( W \). That is, there exists a solid coalition \( N' \) such that \(|N'| > \frac{ln}{k+1}\) for some candidate subset \( C' \) and \(|W \cap C'| < \min\{\ell, |C'|\}\). Let \( \ell \) be the smallest positive integer such that the previous statement holds and let \( C' \) be a minimally sized candidate subset which \( N' \) supports as a solid coalition.\(^*\)

First we claim that the \(|C'|\)-approval election must be reached. Suppose not, then it must be that \(|W| = k\) at some earlier \( j\)-approval election where \( j < |C'|\). But if \(|W| = k\) this implies that after reweighting

\[
\sum_{i \in N'} w_i = n - kq < \frac{n}{k+1} \quad \text{since } q > \frac{n}{k+1}.
\]

and since \( C' \) is minimal (recall \(^*\)) it follows that

\[
\sum_{i \in N'} w_i > \ell \frac{n}{k+1} - (\ell - 1) \frac{n}{k+1} = \frac{n}{k+1}.
\]

This contradicts (3).

Now, at the \(|C'|\)-approval election every voter in \( N' \) only supports candidates in \( C' \). Under the original weightings \( \sum_{i \in N'} w_i > \frac{ln}{k+1} \), since the EAR elects candidate with support at least \( q \) and scales down weightings after each elected candidate it is clear that at least \( \min\{\ell, |C'|\}\)-candidates from \( C' \) must be elected into \( W \), providing the contradiction required.\( \square \)

**Corollary 3** EAR satisfies Hare-PSC.

**Corollary 4** Under linear orders, EAR with the default quota satisfies the majority principle.

**Proof.** Under linear orders, Droop-PSC implies the majority principle.\( \square \)

**Proposition 9** Under weak orders, EAR satisfies generalised Droop-PSC.

**Proof.** Let \( W \) be an outcome of the EAR and suppose for the purpose of a contradiction that generalised Droop-PSC is not satisfied. That is, there exists a positive integer \( \ell \) and a solid coalition \( N' \) such that \(|N'| > \frac{ln}{k+1}\) supporting a candidate subset \( C' \) and for every set \( C'' \subseteq W \) with \(|C''| \geq \min\{\ell, |C'|\}\) there exists \( c'' \in C'' \) such that

\[
\forall i \in N' \quad c_{|C''|} >_i c''.
\]

(5)
Let \( \ell \) be the smallest positive integer such that the above statement holds and let \( C' \) be a minimally sized candidate subset which \( N' \) supports \((*)\).

Let \( j' \) be the smallest approval election which include \( C' \) in the approvals of all voters in \( N' \). We claim that the \( j' \)-approval election must be reached. Suppose not, then it must be that \( |W| = k \) at some earlier \( j \)-approval election where \( j < j' \). But if \( |W| = k \) this implies that after reweighting

\[
\sum_{i \in N} w_i = n - kq < \frac{n}{k + 1}
\]

since \( q > \frac{n}{k + 1} \). (6)

and since \( C' \) is minimal (recall \((*)\)) it follows that

\[
\sum_{i \in N'} w_i > \frac{\ell}{k + 1} - (\ell - 1) \frac{n}{k + 1} = \frac{n}{k + 1}.
\]

(7)

This contradicts (6).

Now, at the \( j' \)-approval election every voter in \( N' \) supports candidates in \( C' \) and possibly some other candidates whom are at least as preferred as their \( |C'| \)-th most preferred candidate. Let \( C'' \) be the super-set of \( C' \) which is equal to the set of candidates supported by voters in \( N' \). Note that \( C'' \) need not equal \( C' \).

Under the original weightings \( \sum_{i \in N'} w_i > \frac{\ell n}{k + 1} \), since the EAR elects candidates with support at least \( q \) and scales down weightings after each elected candidate it is clear that at least \( \min \{\ell, |C''|\} \) candidates from \( C'' \) must be elected into \( W \). But for every candidate \( c'' \in C'' \) at least one voter weakly prefers \( c'' \) to \( \epsilon_{j'}(c') \) (by construction of \( C'' \)) which contradicts (5).

\[ \square \]

\textbf{Remark 4} Note that EAR satisfying PSC or generalised PSC does not depend on what priority tie-breaking is used (Step 1) or how the fractional reweighting is applied (Step 11).

\textbf{Corollary 5} Under weak orders, EAR satisfies generalised Hare-PSC.

We get the following corollary of the above corollary.

\textbf{Corollary 6} EAR satisfies proportional justified representation.

\textit{Proof.} We had observed that under dichotomous preferences, generalised Hare-PSC implies PJR. \[ \square \]

Since EAR satisfies generalised PSC, it implies that there exists a polynomial-time algorithm to compute a committee satisfying generalised PSC. Interestingly, we already observed that checking whether a given committee satisfies generalised PSC is coNP-complete.
6.2 Candidate Monotonicity under EAR

We show that EAR satisfies rank respecting candidate monotonicity (RRCM). In what follows we shall refer to the profile of all voter preferences (weak or strict) as simply the profile - often denoted by \( P \).

**Proposition 10** EAR satisfies rank respecting candidate monotonicity (RRCM).

**Proof.** Consider a profile \( P \) with election outcome \( W \) and let \( c_i \in W \). Now consider another modified profile \( P' \) in which \( c_i \)'s rank is improved while not harming the rank of other winning candidates, relative to \( P \), and denote the election outcome under \( P' \) by \( W' \). Let the order of candidates selected under \( P \) be \( c_1, \ldots, c_i, \ldots, c_{|W|} \). In the modified profile \( P' \), let us trace the order of candidates selected. For the first candidate \( c_1 \), either it is selected first for exactly the same reason as it is selected first under \( P \) or, alternatively, now \( c_i \) is selected. If \( c_i \) is selected, our claim has been proved. Otherwise, the same argument is used for candidates after \( c_1 \) until \( c_i \) is selected. \( \Box \)

This leads immediately to the following corollaries of the above proposition.

**Corollary 7** EAR satisfies non-crossing candidate monotonicity (NCCM).

**Corollary 8** For \( k = 1 \), EAR satisfies candidate monotonicity (CM).

**Corollary 9** For dichotomous preferences, EAR satisfies candidate monotonicity.

**Proof.** Consider dichotomous profile \( P \) and another profile dichotomous \( P' \) in which winning candidate \( c_i \)'s rank is improved while not harming the rank of other winning candidates. This implies that the rank of \( c_i \) is improved while not affecting the ranks of any other alternatives including the winning candidates. Since EAR satisfies rank respecting candidate monotonicity (RRCM), it follows that for dichotomous preferences, EAR satisfies candidate monotonicity. \( \Box \)

On the other hand, EAR does not satisfy CM or WCM for \( k > 1 \).

**Proposition 11** EAR does not satisfy WCM.

**Proof.** Let \( N = \{1, 2, 3, 4\} \), \( C = \{a, b, \ldots, f\} \) and let strict preferences be given by the following preference profile \( P \):

1 : \( a, c, f, d, ... \)
2 : \( d, b, f, c, ... \)
3 : \( a, d, c, f, ... \)
4 : \( c, e, f, d, ... \)

With this preference profile EAR will output the winning committee \( W = \{a, f\} \). Noting that \( q \in (\frac{1}{2}, 2] \), first \( a \) is selected into \( W \) and each weight of voters in \( [1, 2] \) is reduced to \( \frac{2 - q}{2} \in [0, \frac{1}{2}] \). Moving to the 2-approval election no candidate receives support of at least \( q \), finally in the 3-approval election the only candidate with support at least \( q \) is candidate \( f \) and hence \( f \) is also selected into \( W \).
Now consider a reinforcement of $f$ by voter 1 (shift $f$ from third to first place), this is described by the following preference profile $P'$:

1 : $f, a, c, d, ...$
2 : $d, b, f, c, ...$
3 : $a, d, c, f, ...$
4 : $c, e, f, d, ...$

With these preferences the winning committee is $W' = \{d, c\}$. In the 2-approval election both candidates $a$ and $d$ attain support of at least $q$, however due to the priority ordering $L$ (rank maximality) candidate $d$ is selected into $W$ and each voter in $\{2, 3\}$ has their weight reduced to $\frac{2(q - 2)}{2} \in [0, \frac{1}{3})$. In the 3-approval election both candidates $c$ and $f$ attain support of at least $2 \geq q$. Furthermore, $c$ and $f$ are equally ranked with respect to the priority ordering $L$ - applying lexicographic tie-breaking leads to $c$ being elected into $W'$.

\[ \square \]

Remark 5 The above proposition is for any $q \in (\frac{n}{k+1}, \frac{n}{k}]$, which includes the default quota of EAR.

7 Other Rules

In the literature, several rules have been defined for PR purposes. We explain how EAR is better in its role at achieving a strong degree of PR or has other relative merits.

7.1 QBS (Quota Borda System)

Dummett (1984) proposed a counterpart to STV called QPS (Quota Preference Score) or a more specific version QBS (Quota Borda System). The rule works for complete linear orders and is designed to obtain a committee that satisfies Droop-PSC. It does so by examining the prefixes (of increasing sizes) of the preference lists of voters and checking whether there exists a corresponding solid coalition for a set of voters. If there is such a solid set of voters, then the appropriate number of candidates with the highest Borda count are selected.

Although Dummett did not show that the rule satisfies some axiom which STV does not, he argued that QPS satisfies the Droop proportionality criterion and is somewhat less “chaotic” than STV. Schulze (2002) argues that QBS is chaotic as well and his Example 3 implicitly shows that QBS in fact violates WCM. Tideman also feels that QBS is overly designed to satisfy Droop-PSC but is not robust enough to go beyond this criterion especially if voters in a solid coalition perturb their preferences.

EAR has some important advantages over QBS: (1) it can easily handle indifferences whereas QBS is not well-defined for indifferences. In particular, in order for

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9 The description of the rule is somewhat informal and long in the original books of Dummett which may have lead to The Telegraph terming the rule as a “a highly complex arrangement” (Telegraph, 2013).

10 Geller (2002) wrongly claims that QBS satisfies CM.
QBS to be suitably generalised for indifferences and to still satisfy generalised PSC, it may become an exponential-time rule\footnote{QBS checks for PSC requirements and adds suitable number of candidates to represent the corresponding solid coalition of voters. In order to work for generalised PSC, QBS will have to identify whether solid coalitions of voters and meet their requirement which means that it will need to solve the problem of testing generalised-PSC which is coNP-complete.}.  

(2) EAR can easily handle voters expressing partial lists by implicitly having a last indifference class whereas QBS cannot achieve this, (3) EAR rule satisfies an established PR property called PJR for the case for dichotomous preferences. As said earlier, QBS is not well-defined for indifference and even for dichotomous preferences, (4) EAR addresses a criticism of Tideman (2006): “Suppose there are voters who would be members of a solid coalition except that they included an “extraneous” candidate, which is quickly eliminated among their top choices. These voters’ nearly solid support for the coalition counts for nothing which seems to me inappropriate.” We demonstrate the last flaw of QBS pointed out by Tideman in the explicit example below. EAR does not have this flaw.

**Example 5** Consider the profile with 9 voters and where \( k = 3 \).

1 : \( c_1, c_2, c_3, e_1, e_2, e_3, e_4, d_1 \)
2 : \( c_2, c_3, c_1, e_1, e_2, e_3, e_4, d_1 \)
3 : \( c_3, c_1, d_1, c_2, e_1, e_2, e_3, e_4 \)
4 : \( e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1 \)
5 : \( e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1 \)
6 : \( e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1 \)
7 : \( e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1 \)
8 : \( e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1 \)
9 : \( e_1, e_2, e_3, e_4, c_1, c_2, c_3, d_1 \)

In the example, \( \{e_1, e_2, e_3\} \) is the outcome of QBS. Although PSC is not violated for voters in \( \{1, 2, 3\} \) but the outcome appears to be unfair to them because they almost have a solid coalition. Since they form one-third of the electorate they may feel that they deserve that at least one candidate such as \( c_1, c_2 \) or \( c_3 \) should be selected.

### 7.2 Chamberlain-Courant and Monroe

There are other rules that have been proposed within the class of “fully proportional representation” rules such as Monroe (Monroe, 1995) and Chamberlain-Courant (CC) (Chamberlin and Courant, 1983). Recently, variants of the rules called Greedy Monroe, and Greedy CC (Elkind et al., 2014, 2017) have been discussed. However none of the rules satisfy even weak PSC (Elkind et al., 2014, 2017). Even though Monroe and CC have been extensively studied with respect to PR, they do not seem to satisfy any compelling PR axiom. A reason for this is that voters are assumed to not care about how many of their highly preferred candidates are in the committee...
as long the most preferred is present. Monroe and CC are also NP-hard to compute (Procaccia et al., 2008).

7.3 Phragmen’s Methods

A compelling rule is Phragmen’s Ordered Method (also called SeqPhragmen) that can even be generalised for weak orders. Under strict preferences, it satisfies weak Droop-PSC (Theorem 16.1 (ii), Janson, 2016). On the other hand, even under strict preferences, it does not satisfy Droop-PSC (page 51, Janson, 2016). If we are willing to forego PSC, then SeqPhragmen seems to be an exceptionally useful rule for strict preferences because unlike STV it satisfies both candidate monotonicity and committee monotonicity (Janson, 2016). Committee monotonicity requires that for any outcome $W$ of size $k$, there is a possible outcome $W'$ of size $k + 1$ such that $W' \supset W$.

7.4 Phragmén’s First Method

Phragmén’s first method was first considered by Phragmén but not published or pursued by him (Janson, 2016). In the method, voters approve of most preferred candidate that has not yet been selected. The candidate with highest weight of approval is selected. The total weight of the voters whose approved candidate was selected is reduced by the Hare quota if the total weight is more than the Hare quota. Otherwise all such voters’ weights are set to zero. The method does not satisfy Hare-PSC even under strict preferences. See the example below.

Example 6  Consider the profile with 9 voters and where $k = 3$.

| 1  | $c_1, c_2, c_3$ |
| 2  | $c_2, c_3, c_1$ |
| 3  | $c_3, c_1, c_3$ |
| 4  | $e_1, e_1, c_2, e_3$ |
| 5  | $e_2, e_1, c_2, c_3$ |
| 6  | $e_3, c_1, c_2, c_3$ |
| 7  | $e_4, c_1, c_2, c_3$ |
| 8  | $e_5, c_1, c_2, c_3$ |
| 9  | $e_6, c_1, c_2, c_3$ |

In the example, $\{e_1, e_2, e_3\}$ is a possible outcome of rule. When $e_1$ is selected, voter 4’s weight goes to zero. Then when $e_2$ is selected, voter 5’s weight goes to zero. Finally $e_3$ is selected. Hare-PSC requires that $c_1$ or $c_2$, or $c_3$ is selected.
7.5 Thiele’s Methods

Thiele’s methods are based on identifying candidates that are most preferred by the largest weight of voters. For any voter who has had $j$ candidates selected has current weight $1/(j+1)$. Janson (2016) presented Examples 13.14 and 13.15 that can be used to show that Thiele’s methods do not satisfy weak Droop-PSC even under strict or under dichotomous preferences.

7.6 CPO-STV rules

A class of STV related rules is CPO-STV that was proposed by Tideman (2006). The rules try to achieve a PR-type objective while ensuring that for $k = 1$, a Condorcet winner is returned if there exists a Condorcet winner. One particular rule within this class is Schulz-STV (Schulze, 2011). All of these rules are only defined for linear orders and hence do not satisfy generalised PSC. Furthermore, they all require enumeration of all possible committees and then finding pairwise comparisons between them. Hence they are exponential-time rules and impractical for large elections. Tideman (1995) writes that CPO-STV is “computationally tedious, and for an election with several winners and many candidates it may not be feasible.” Tideman (2006) also considered whether CPO-STV rules satisfy Droop-PSC but was unable to prove that they satisfy PSC (page 282). In any case, having an exponential-time rule satisfying PSC may not be compelling because there exists a trivial exponential-time algorithm that satisfies PSC: enumerate committees, check whether they satisfy PSC, and then return one of them.

8 Conclusions

|                      | STV       | QBS       | EAR       |
|----------------------|-----------|-----------|-----------|
| Generalised D-PSC / H-PSC | no        | no        | yes       |
| Generalised weak D-PSC / H-PSC | no        | no        | yes       |
| PJR                  | no        | no        | yes       |
| D-PSC / H-PSC        | yes       | yes       | yes       |
| Weak D-PSC / H-PSC   | yes       | yes       | yes       |
| CM                   | no        | no        | no        |
| CM for dichotomous preferences | no        | no        | yes       |
| CM for $k = 1$       | no        | yes       | yes       |
| RRCM                 | no        | no        | yes       |
| NCCM                 | no        | yes       | yes       |
| polynomial-time      | yes       | yes       | yes       |

Table 1 Properties satisfied by STV, QBS, and EAR. QBS and STV are the only rules from the literature that satisfy PSC and are polynomial-time. STV does not satisfy weak PSC if discrete reweighting is used.
In this paper, we undertook a formal study of proportional representation under weak preferences. The generalised PSC axiom we proposed generalises several well-studied PR axioms in the literature. We then devised a rule that satisfies the axiom. Since it has relative merits over STV and Dummet’s QBS (two known rules that satisfy PSC), it appears to be a compelling solution for achieving PR via voting. At the very least, it appears to be another useful option in the toolbox of representative voting rules and deserves further consideration and study. The relative merits of STV, QBS, and EAR are summarised in Table 1. Our work also sheds light on the complexity of computing committees that satisfy PR axioms as well the the complexity of testing whether a given a committee satisfies a given PR axiom. We found that whereas a polynomial-time algorithm such as EAR finds a committee that satisfies generalised PSC, testing whether a given committee satisfies properties such as generalised PSC or generalised weak PSC is computationally hard. These findings are summarised in Table 2.

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### A Complexity of Testing PSC

**Proposition 12** Under linear orders, it can be tested in polynomial time whether a committee satisfies PSC.

*Proof.* For each $i$ from 1 to $m$ one can look at prefixes of preference lists of sizes $i$. For these prefixes, we can see whether there exists a corresponding solid coalition. For such solid coalitions we can check whether the appropriate number of candidates are selected or not. \(\square\)

The same idea can be used for weak PSC.

**Proposition 13** Under linear orders, it can be tested in polynomial time whether a committee satisfies weak PSC.

**Proposition 14** Under dichotomous preference, the problem of testing generalised PSC is coNP-complete.

*Proof.* Under dichotomous preferences, generalised PSC is equivalent to PJR. Since [Aziz and Huang (2017)] showed that testing PJR is coNP-complete, it follows that that testing Generalised PSC is coNP-complete as well. \(\square\)