Jensen Inequality and the Second Law

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Abstract

Jensen’s Inequality (JIEQ) has proved to be a major tool to prove the consistency of various fluctuation theorems with the second law in microscopic thermodynamics. We show that the situation is far from clear and the reliance on the JIEQ may be quite misleading in general.

1. Introduction The Jensen inequality (JIEQ) has become popularized recently in modern nonequilibrium (NEQ) thermodynamics as a tool to imply that various (integral) fluctuation theorems (FT) of the form
\[ \langle e^{-\Delta \Phi} \rangle_{TE} = 1, \]  
(1)
such as the Jarzynski identity, Crooks theorem, Seifert’s entropy generation theorem, etc. are consistent with the second law. These FTs are determined by the trajectories \{\gamma_k\} and their probabilities \{p_{\gamma_k}\}; see various reviews. The collection \{\gamma_k\} forms the trajectory ensemble (TE) and defines the average \langle \bullet \rangle_{TE}. Jensen’s inequality, a purely mathematical result to obtain the inequality \langle \Delta \Phi \rangle_{TE} \geq 0 from Eq. (1), is extensively used to argue for the conformity of the FTs with the second law, a macroscopic result in physics. It thus follows that the only use of the JIEQ is to justify that the FTs can describe NEQ processes so as to make them quite suitable to gain insight into the second law. As the choice of \{p_{\gamma}\} is not unique, see below, the relationship of \langle \Delta \Phi \rangle_{TE} with its thermodynamic average, to be denoted simply by \langle \Delta \Phi \rangle, is not clear, since the second law must only refer to thermodynamic averages such as \langle \Delta S \rangle for the entropy change of an isolated system \Sigma for which the second law results in the inequality \langle \Delta S \rangle \geq 0. Another consequence of the second law is the dissipated work in an isothermal process
\[ \Delta R_{\text{diss}} \equiv \langle \Delta R \rangle - \Delta F = T_0 \Delta S \geq 0, \]  
(2)
where \langle \Delta R \rangle is the thermodynamic average work (see Eq. 7 for a proper definition) done on the system \Sigma during a process, \Delta F is the free energy change, and \Delta R_{\text{diss}} is the dissipated work [11]. As an example of a FT, Jarzynski [12] derived
\[ \langle e^{-\beta_0 \Delta R_{\text{diss}}} \rangle_0 \equiv \sum_k p_{k0} e^{-\beta_0 \Delta R_{\gamma_k}} = 1, \]  
(3)
where \Delta R_{\text{diss}} \equiv \Delta R_{\gamma_k} - \Delta F, \Delta R_{\gamma_k} denotes the work done on the system \Sigma along the trajectory \gamma_k, the suffix 0 refers to a special averaging with respect to the initial equilibrium (EQ) microstate probabilities \{p_{k0}\}, and k refers to the initial microstate of \gamma_k. The above identity is known as the Jarzynski identity (JE). We will refer to \langle \bullet \rangle_0 as the Jarzynski average in this work.

It is implicitly assumed in the current literature that \langle \Delta \Phi \rangle_{TE} \equiv \langle \Delta \Phi \rangle. What is the significance of \langle \Delta \Phi \rangle_{TE} \geq 0 if \langle \Delta \Phi \rangle does not refer to a quantity that must obey the second law (see the discussion later for such a quantity)? Indeed, we will establish here that using the JIEQ can be misleading in suggesting that the FT applies to NEQ processes or that \langle \Delta \Phi \rangle_{TE} satisfies the second law, while in fact they do not. To the best of our knowledge, this issue has not been discussed in the literature despite the wide use of the JIEQ. We establish that (i) the trajectory average \langle \bullet \rangle_{TE} may or may not be the same as the thermodynamic average \langle \bullet \rangle, and (ii) even when the two are the same, \langle \Delta \Phi \rangle may have nothing to do with the second law. As a consequence, the consequence of the JIEQ need not refer to the second law, thus casting doubts on its utility for FTs.

For simplicity, we consider a ”work-process” \mathcal{P}_0 on a system \Sigma as proposed by Jarzynski [12]. It is an arbitrary process over \{(0, \tau_{eq})\} between two EQ macrostates A and B at the same inverse temperature \beta_0 = 1/T_0; here, \tau_{eq} is the time needed to reach the EQ macrostate B. The system is driven (the driving stage \mathcal{P}) over \{(0, \tau)\}, \tau \leq \tau_{eq}, by \Sigma_{eq} and is then allowed to equilibrate (the reequilibration stage \mathcal{P}) due to interaction with \Sigma_{eq} only over \{(\tau, \tau_{eq})\}. For simplicity, we assume that during \mathcal{P}, \Sigma is not in thermal contact with \Sigma_{eq}. We denote by \Sigma the combination \Sigma_{eq} \cup \Sigma_{eq} and the combination \Sigma \cup \Sigma by \Sigma_0, which is an isolated system. All quantities pertaining to \Sigma have no suffix, and those pertaining to \Sigma (\Sigma_0) with a tilde (suffix 0). For concreteness, we assume the work process to change the volume \mathcal{V}(\tau) of the system by applying an external pressure \mathcal{P}_0, but the arguments are valid for any external ”work” process. The system-intrinsic (SI) pressure for the kth microstate \mathcal{m}_k will be denoted by \mathcal{P}_k \equiv -\partial \mathcal{E}_k/\partial \mathcal{V}, where \mathcal{E}_k is the microstate energy, an SI-quantity. The difference \Delta \mathcal{P}_k \equiv \mathcal{P}_k - \mathcal{P}_0 denotes the ubiquitous force imbalance (FI) between the external and induced internal forces that is normally nonzero even in equilibrium [13][15]. Therefore, to discard FI in an irreversible process is counter-productive. We find it convenient to use Prigogine’s modern notation, which is highly suitable in NEQ thermodynamics [16][20].

2. Jensen’s Inequality Consider a convex function \Phi(X) of a random variable X, and let \mathcal{E} be an expectation operator such as \langle \bullet \rangle_0, \langle \bullet \rangle_{TE}, etc. Then, the inequality
\[ \mathcal{E}(\Phi(X)) \geq \Phi(\mathcal{E}(X)) \]  
(4)
is known as Jensen’s inequality (JIEQ) for \( \Phi(X) \). For the JE, \( \mathcal{E}(t) \) represents \( \mathbb{E} \) so the JIEQ results in \( \langle R \rangle_0 - \Delta F \geq 0 \). By exploiting an ad-hoc assumption \( \langle R \rangle_0 = \langle R \rangle \) without offering any justification, Jarzynski \[12\] has argued that the JE results in \( \langle R \rangle \geq \Delta F \) in accordance with the second law; see Eq. (2). The use of the JIEQ has become widespread to establish consistency with the second law by exploiting a similar ad-hoc assumption \( \langle R \rangle_{\text{TE}} = \langle R \rangle \) such as by Crooks \[21\], Seifert \[14, 15, 20\] and many others. The argument is crucial since it indirectly "justifies" the results to be nonequilibrium results. The assumption \( \langle R \rangle_{\text{TE}} = \langle R \rangle \) is never ever explicitly mentioned but seems to have been accepted by all workers without ever been justified.

### 3. Thermodynamic Ensemble Averages

In general, an EQ or NEQ ensemble average (EA) is defined instantaneously, and requires identifying (a) the elements (microstates \( \{m_k\} \)) of the ensemble and (b) their instantaneous probabilities \( \{p_k\} \). The average is uniquely defined over \( \{m_k\} \) using \( \{p_k\} \) at each instant, which we identify as the instantaneous ensemble average (IEA). Let \( O_k \) be some extensive quantity pertaining to \( m_k \). The instantaneous thermodynamic average \( \langle O \rangle \) is defined \[8, 17\] as

\[
\langle O(t) \rangle = \sum_k O_k(t)p_k(t).
\]

We will usually not show the time \( t \) unless clarity is needed. In thermodynamics, it is common to simply use \( O \) for the average but it may cause confusion in some cases. We will use macroquantity for the average \( O \) and microquantity for \( O_k \). The average energy \( E \equiv \langle E \rangle \) is such an average system-intrinsic (SI) macroquantity. The infinitesimal thermodynamic work \( dR \equiv \langle dR \rangle \) done on the system and the work done by the system \( dW \equiv \langle dW \rangle \) represent such an average instantaneous macroquantities; the former is medium-intrinsic (MI) quantity and the latter a SI quantity. The first law of thermodynamics as a sum of two system-intrinsic (SI) contributions

\[
d\langle E \rangle = \sum_k E_k dp_k + \sum_k p_k dE_k.
\]

The first sum represents the generalized heat \( dQ = TdS \) while the second sum represents \(-dW\), the generalized work \[14, 15, 20\] in terms of the SI microwork \( dW_k = -dE_k \) done by \( m_k \). These generalized macroquantities should not be confused with the exchanged macroheat and macrowork \( dQ \) and \( dW \), respectively. Their differences are \( dQ \) and \( dW \), respectively, with an important identity of their magnitudes \( dQ = dW \geq 0 \). We also observe that during generalized work, \( \{E_k\} \) change but not \( \{p_k\} \); during generalized heat, \( \{p_k\} \) change but not \( \{E_k\} \). This allows us to treat work and heat separately.

### 4. Trajectory Ensemble Averages

The uniqueness inherent in Eq. \[4\] may not hold for the TEA \( \mathcal{E}(t) \), which we now discuss. Let \( \gamma_k \) denote the trajectory followed by \( m_k \) during its evolution along \( \mathcal{P}_0 \). The average cumulative change \( \langle \Delta O_{\gamma_k} \rangle \) along a process \( \mathcal{P}_0 \), we suppress the suffix TE for simplicity, is obtained by integrating \( dO(t) \) over the process between \( t = 0 \) and \( t' = \tau_{eq} \):

\[
\langle \Delta O_{\gamma_k}(t) \rangle = \int_{\mathcal{P}_0} (dO) = \sum_k \int_{\gamma_k} p_k(t') dO_k(t');
\]

we will use \( \Delta O_{\gamma_k}(\tau_{eq}) \) or simply \( \Delta O_{\gamma_k} \) or \( \Delta O \) for \( \langle \Delta O_{\gamma_k} \rangle \) unless clarity is needed. We note that \( m_k \) retains its identity during its evolution along \( \mathcal{P}_0 \) as indicated by the sum; no transition between different microstates is allowed. We can also introduce the cumulative change \( \Delta O_{\gamma_k}(t) = \int_{\gamma_k} dO_k \) along \( \gamma_k \) over the interval \( (0, t) \), and rewrite the above equation as

\[
\Delta O = \sum_{\gamma_k} p_{\gamma_k}^{(O)}(\tau_{eq}) \Delta O_{\gamma_k}(\tau_{eq}),
\]

where we have introduced the trajectory probability \( p_{\gamma_k} = p_{\gamma_k}^{(O)}(\tau_{eq}) \) in terms of \( \Delta O_{\gamma_k}(\tau_{eq}) \):

\[
p_{\gamma_k}^{(O)}(\tau_{eq}) = \int_{\gamma_k} p_k(t) dO_k(t) / \Delta O_{\gamma_k}(\tau_{eq}) = \int_{\gamma_k} p_k(t) d\gamma_k^{(O)}(t);
\]

and \( \Delta W(k) \), respectively, in Eq. \[4\]; here, we have used the fact that \( dR_k(t) \) is nonzero only during the driving stage \( (0, \tau \leq \tau_{eq}) \). One can similarly define a trajectory probability \( p_{\gamma_k}^{(W)}(\tau_{eq}) \) by using \( \gamma_k^{(t)}(t) \) instead of \( \gamma_k \):

\[
p_{\gamma_k}^{(W)}(\tau_{eq}) = \int_{\gamma_k} p_k(t) d\gamma_k^{(t)}(t).
\]

This average (over time) probability is determined by \( \gamma_k \) alone and can be identified as the intrinsic trajectory probability. We observe that \( \Delta R(\tau) = \Delta R(\tau_{eq}) \). It should be evident that the three probabilities are not the same. In other words, there is no unique trajectory probability \( p_{\gamma_k} \) as said earlier.

The trajectory \( \gamma_k \) is determined by a single microstate \( m_k \) and proves useful in the thermodynamic macroworks \( \Delta W \) or \( \Delta R \). We can also consider a mixed trajectory (MT) \( \gamma_{k-m} \) as a sequence \( \{m_j\}_{j=0,1,2,\ldots,n} \) of microstates starting at \( m_0 = k \) at \( t_0 = 0 \) and terminating at \( m_n = m \) at time \( t = t_n = \tau_{eq} \); the microstate \( m_i, i < n \), appears at time \( t = t_i \), while \( m_i, i < n \), appears at time \( t = t_i \). Consider the time interval \( \delta_i = (t_i, t_i+1) \), which we divide into an earlier interval \( \delta_i' = (t_i, t_i+1) \) and a later interval \( \delta_i'' = (t_i', t_i+1) \).
During $\delta_t'$, $m_t$ does not change as microwork $\delta R_{m_t}$ is performed by $\tilde{\Sigma}_w$; no microheat is transferred. During $\delta_t''$, no microwork $\delta R_{m_t}$ is performed by $\tilde{\Sigma}_w$ but microheat is transferred, which changes $m_t$ to $m_{t+1}$. For $m = k, \forall l$, $\gamma_{k\rightarrow l}(\tau)$ reduces to $\gamma_k$. The probability $p_{\gamma_k \rightarrow m}$ is given by

$$p_{\gamma_k \rightarrow m} = p_{k0} T(\{m_l\}_{l=1,2,\ldots,n} | k)$$

in terms of the multistate conditional probability $T(\{m_l\}_{l=1,2,\ldots,n} | k)$ and the initial probability $p_{k0}$. The corresponding $(\bullet)_{TE}$ with respect to $p_{\gamma_k \rightarrow m}$ is obtained by replacing $\gamma_k$ and $p_k$ by $\tau_{\gamma_k \rightarrow m}$ and $p_{\tau_{\gamma_k \rightarrow m}}$, respectively, in Eq. (7) and summing over all $\tau_{\gamma_k \rightarrow m}$. It is very common to assume that the sequence $\{m_l\}_{l=0,1,\ldots,n}$ forms a (memoryless) Markov (M) chain so that $T(\{m_l\}_{l=1,2,\ldots,n} | k)$ can be expressed as a product of two-state transition probabilities $T_{ij}$ to determine the Markov approximate $p_{\tau_{\gamma_k \rightarrow m}}^{(M)}$. Using $p_{\tau_{\gamma_k \rightarrow m}}^{(M)}$, the Markov average external work over $\tau \in (0, \tau_{eq})$ is

$$\Delta R^{(M)}(\tau_{eq}) = \sum_{\tau_{eq}} \sum_{m_k} \sum_{m_{k-1}} \cdots \sum_{m_1} \delta R_{m_k}(\delta_{ij}) \cdot \frac{p_{\tau_{\gamma_k \rightarrow m}}^{(M)}}{p_{\tau_{\gamma_k \rightarrow m}}},$$

(11)

where $\Delta R_{\tau_{\gamma_k \rightarrow m}}(\tau_{eq}) = \sum_{\tau_{eq}} \sum_{m_k} \cdots \sum_{m_1} \delta R_{m_k}(\delta_{ij})$ and $\delta R_{m_k}(\delta_{ij})$ is the external work done on $m_k$. Thus, in the Markov chain approximation, $\Delta R^{(M)}(\tau_{eq})$ gives a discrete approximation of the macrowork $\Delta R(\tau_{eq}) \equiv \langle \Delta R_{\gamma} \rangle$ in Eq. (10a) for which we require $\delta_{ij}$ to be extremely short. Otherwise, $\Delta R^{(M)}(\tau_{eq})$ and $\Delta R(\tau_{eq})$ are very different as we have stated earlier. For a non-Markovian process, $T(\{m_l\}_{l=1,2,\ldots,n} | k)$ cannot be expressed as a product of two-state transition probabilities and we must resort to the generalization noted above of Eq. (11).

**The Jarzynski Equality** As our first example of a TEA $(\bullet)_{TE}$ different from $(\bullet)$ in a FT, we consider the one proposed by Jarzynski [12] noted above. Jarzynski uses the external microwork $\Delta R_k$ done on $m_k$ during $\tau$ to prove the JE in Eq. (3). Here, $\Delta R_k = F_k(\beta_k) - F_k(\beta_0)$, and $\Delta R_k \neq 0$ only during the driving stage $\tau$; $\Delta R_k = 0$ over $\tilde{\tau}$. If the system at $t = \tau$ is out of equilibrium, we denote it by $b$. The interaction with $\tilde{\Sigma}_b$ during $\tilde{\tau}$ is to ensure that $b$ turns into $B$.

The use of the JIEQ with $(\bullet)_{TE}$ for $E$ in Eq. (3) immediately results in $\langle \Delta R_{\gamma} \rangle_0 \geq \Delta F$. Jarzynski assumes that $\langle \Delta R_{\gamma} \rangle_0 = \langle \Delta R_{\gamma} \rangle$ and argues that the JE represents a NEQ result so that $\langle \Delta R_{\gamma} \rangle_{eq} = \Delta F$ for a reversible process and $\langle \Delta R_{\gamma} \rangle_0 > \Delta F$ for an irreversible process.

We now consider a reversible process between $A$ and $B$, for which the thermodynamic macrowork $\Delta R \equiv \langle \Delta R_{\gamma} \rangle$ is the reversible macrowork $\Delta R_{rev} = \Delta F$, and demonstrate by a simple example that $\langle \Delta R_{\gamma} \rangle$ is not the same as $\langle \Delta R_{\gamma} \rangle_0$.

$$\langle \Delta R_{\gamma} \rangle_0 = \sum_k p_{k0} \Delta R_{\gamma_k}(\tau_{eq})$$

(12)

the Jarzynski average, except when $p_{k0} \equiv p^{(R)}_{\gamma_k}(\tau), \forall \tau$. For the calculation, we consider an ideal gas in a 1-dimensional box of length $L$, which expands quasistatically from $L_A$ to $L_B$; we let $x = L_A/L_B$ between $A$ and $B$. As there are no interparticle interactions, we can treat each particle by itself. The microstates in the exclusive approach are those of a particle in the box with energies determined by an integer $k : E_k = \alpha(k/L)^2, \alpha = \pi^2 kT/2m$. Let $\beta_0$ denote the inverse temperature of the heat bath. The gas remains in equilibrium at all times and $R_{\text{diss}} = 0$. The partition function at any $x$ is given by

$$Z(\beta_0, L) = \sum_n \exp(-\beta_0 n/L)^2 \approx \sqrt{L^2\pi/4\alpha\beta_0}$$

for any $L \in [L_A, L_B]$; in the last equation, we have made the standard integral approximation for the sum. We then have

$$\beta_0 F = -(1/2) \ln(L^2\pi/4\alpha\beta_0), E = 1/2\beta_0.$$  

(13)

We can now compute the two work averages with $\Delta R_k = E_k(L_B) - E_k(L_A)$. For the Jarzynski average, we have

$$\langle \Delta R_{\gamma} \rangle_0 = \sum_k p_{k0} [E_k(L_B) - E_k(L_A)] = (x^2 - 1)/2\beta_0,$$

(13)

where we have used $E_k(L_B) - E_k(L_A) = (x^2 - 1)E_k(L_A)$. For the thermodynamic average, we use $dE_k = -2E_k dL/L$ in Eq. (7) to obtain

$$\Delta R \equiv \langle \Delta R_{\gamma} \rangle = 1/\beta_0 \ln x = \Delta F.$$

(14)

It should be clear that it is the thermodynamic average work $\langle \Delta R_{\gamma} \rangle$ that satisfies the condition of EQ and not $\langle \Delta R_{\gamma} \rangle_0$, which is evidently different from $\langle \Delta R_{\gamma} \rangle_0$. This, thus, contradicts the conventional assumption $\langle \Delta R_{\gamma} \rangle_0 = \langle \Delta R_{\gamma} \rangle_0$. We evaluate the difference $\langle \Delta R_{\gamma} \rangle_0 - \Delta F$. Introducing $y = 1 - x^2 \geq 0$ for expansion, we have

$$\langle \Delta R_{\gamma} \rangle_0 - \Delta F = [\ln(1 - y) - y]/2\beta_0 > 0.$$  

(14)

The Jensen inequality is satisfied as expected, but the above nonnegative difference $\langle \Delta R_{\gamma} \rangle_0 - \Delta F$ makes no statement about any dissipation in the system, which is most certainly absent. Thus, the JIEQ makes no statement about the second law and casts doubts on the usefulness of the indiscriminate application of the JIEQ in FTs.

We now consider two more FTs, where the JIEQ has been used to justify consistency with the second law.

**Crooks’ Approach** Crooks [21] assumes the evolution along $\tau_{\gamma_k \rightarrow m}$ as a Markov process satisfying the principle of detailed balance and divides $(0, \tau_{eq})$ into $n$ intervals $\delta_l = \delta_{l-1} \cup \delta'_l, l = 0, 1, 2, \ldots, n-1$ as described above. Microwork is performed during $\delta_l'$ and microheat is exchanged during $\delta_l''$. We will not follow Crooks’ derivation of the JE, which we have carried out elsewhere [22], but follow the consequences of the detailed balance here. The transition probability matrix $T^{(l)}$ in $\delta_l$ takes a very simple form under detailed balance, which we denote by
\( \mathbf{T}^{(l)} \). From the Fundamental Limit Theorem or Doebelin’s theorem about Markov chains \([20\text{]}\), we know that such a transition matrix is uniquely determined with its matrix elements corresponding to \( i \to j \) given by

\[
\mathbf{T}^{(l)}_{i,j} = p_{\text{eq}}(V_{i+1}), \forall l < n,
\]

where \( p_{\text{eq}}(V_{i+1}) \) is the EQ probability at fixed \( V_{i+1} \) of the \( j \)th microstate at time \( t_{i+1} \) and ensures that the end-microstate \( m_{i+1} \) at the end of \( \delta_t \) belongs to an EQ macrostate. Thus, at the end of \( \delta_0 \), the EQ-microstate \( m_0 \) turns into an EQ-microstate \( m_1 \) but the microwork done on \( m_0 \) is precisely \( \delta R_{m_0}^{(\delta_0)} \). By induction, we have a sequence of EQ-microstates \( m_1, m_2, \ldots, m_n \) and microworks \( \delta R_{m_1}^{(\delta_1)}, \delta R_{m_2}^{(\delta_2)}, \ldots, \delta R_{m_{n-1}}^{(\delta_{n-1})} \) and the total microwork is given by \( \Delta R_{\tau_{k-\rightarrow n}}^{(\text{eq})} \) used in Eq. (11). Using the above transition matrix \( \mathbf{T}^{(l)} \), we can easily evaluate \( \tau_{k-\rightarrow m} \):

\[
p_{\tau_{k-\rightarrow m}} = p_{\text{m}_0,\text{eq}} \prod_{l=1}^{n} p_{\text{m}_l,\text{eq}}
\]

so that

\[
\langle e^{-\beta_0 \Delta R_l} \rangle^{(M)}_{\tau} = \prod_{l<n} \left( \sum_{m_0} p_{m_0,\text{eq}} e^{-\beta_0 \delta R_{m_0}^{(\delta_l)}} \right).
\]

We see that the Crooks process during \( \delta_0 \) is no different from the Jarzynski process and the quantity within the parentheses denotes a Jarzynski averaging of the exponential microwork distribution over the probabilities of the initial EQ-microstates \( \{ m_l \} \) in the interval \( \delta_l, l = 0, 1, 2, \ldots, n - 1 \). In other words, the Crooks process is a sequence of \( n \) non-overlapping mini-Jarzynski processes \( \{ \delta P_{0,l} \} \), each over \( \delta_l, l = 0, 1, 2, \ldots, n - 1 \). For each mini-Jarzynski process, we have

\[
\langle e^{-\beta_0 \delta R_l} \rangle^{(M)}_{\tau} = e^{-\beta_0 \delta F_l},
\]

where the suffix \( l \) denotes averaging over the initial microstate probabilities \( p_{m_0,\text{eq}} \) and \( \delta F_l \) is the change over \( \delta_l \) between EQ-macrostates. It is obvious now that \( \langle \delta R_l \rangle^{(M)}_{\tau} \) is not the same as the thermodynamic average \( \langle \delta R \rangle \), just as it was for the Jarzynski process, unless \( \delta_l \) is extremely small. This means that the application of the JIEQ on \( \langle e^{-\beta_0 \delta R_l} \rangle \) does not give an inequality involving thermodynamic averages so no connection with the second law is possible.

Seifert’s Approach: Here, we will continue to use a discrete formulation for simplicity. According to Seifert \([4\text{]}\), \( S_k = -\ln p_k \) denotes the microscopic entropy and its thermodynamic average \( \langle S \rangle = \sum_k p_k S_k \) gives the (average) entropy, commonly written as \( S \). Seifert defines the average change \( dS^{(S)} \) (S indicating Seifert) in terms of \( dS_k \),

\[
dS^{(S)} \equiv \langle dS \rangle
\]

and conjectures that \( dS^{(S)} \) is nothing but \( d\langle S \rangle \). One can also determine \( \Delta S_k \) as the integral of \( dS_k \) along the trajectory \( \tau_{k-m} \) and introduce \( \Delta S^{(S)} = \langle dS \rangle_{\tau} \). Similarly, we also have \( dS_0^{(S)} = \langle dS_0 \rangle \) and \( \Delta S_0^{(S)} = \langle dS_0 \rangle_{\tau_0} \) for the isolated system \( \Sigma_0 \) and where \( \tau_0 \) denotes its set of trajectories. Seifert then derives the following equality \( \langle e^{-\Delta S_0^{(S)}} \rangle_{\tau_0} = 1 \). The use of the JIEQ then results in \( \Delta S_0^{(S)} = \langle dS_0 \rangle_{\tau_0} \geq 0 \), which is interpreted using the above conjecture that \( \Delta S_0^{(S)} \) denotes \( \langle S \rangle_0 = \langle S \rangle_{\tau_0} \). Using this interpretation, the inequality is considered a statement of the second law by taking \( \Delta S_0^{(S)} \) to mean \( \Delta S = \beta_0 \Delta R_{\text{diss}} \), see Eq. (2). With this, Seifert provides another proof of the JE in terms of the mixed trajectory average

\[
\langle e^{-\beta_0 \Delta R_{\text{diss}}} \rangle_{\tau_0} \geq \langle e^{-\beta_0 \Delta R_{\text{diss}}} \rangle_{\tau_0} = 1.
\]

Since

\[
dS_0^{(S)} \equiv \langle dS_0 \rangle = -\langle dp_0/p_0 \rangle = -\sum_k dp_{0k} \equiv 0,
\]

which is simply a statement of the conservation of probability, we conclude that \( dS_0^{(S)} = 0 \). To determine \( \Delta S_0^{(S)} = \langle dS_0 \rangle_{\tau_0} \), we follow Eqs. (7)-(9). Since \( \langle dS_0 \rangle_{\tau_0} \) is integral of \( dS_0^{(S)} \), it is clear that \( \langle dS_0 \rangle_{\tau_0} = 0 \). Thus, the above JIEQ conclusion \( \Delta S_0^{(S)} = \langle dS_0 \rangle_{\tau_0} \geq 0 \) does not prove that it encodes the second law. The second law requires considering the differentials \( dS, d\tilde{S} \) and \( dS_0 \). Recalling that \( dS = d\langle S \rangle = \sum_k p_k dS_k + \sum_k S_k dp_k \), compare with Eq. (9), and \( dS_0 = -dp_k/p_k \), we have

\[
dS = \langle dS \rangle - \langle S dS \rangle = -\langle S dS \rangle.
\]

Thus, \( dS^{(S)} \) is not the entropy differential \( dS \). Unfortunately, this point has been overlooked.

Conclusions: In summary, we have shown that the application of the Jensen inequality does not at all make any statement about the second law. It should be pointed out that while there is a consequence of the second law for \( \Delta R_{\text{diss}} \), there is no second law statement about \( dS^{(S)} = 0 \). Thus, while in the former case, the use of the JIEQ may provide a statement of the second law, its applications to \( \langle e^{-\Delta S_0^{(S)}} \rangle_{\tau_0} \) has no relationship to the second law. The conclusion is that care must be exercised to draw any conclusion about the second law by applying the JIEQ in general, a point that does not seem to have been appreciated.
\[ \Delta E = \Delta_{\text{ext}} + \Delta_{\text{int}} + \Delta_{\text{int}} \]

\[ \Delta R_{\text{diss}} = \Delta E - \Delta_{\text{ext}} \frac{d\Delta S}{dT} = \Delta_{\text{gen}} \Delta_S \]

\[ \langle R \rangle = -\Delta_{\text{int}} W \]

By definition, \( \langle \Delta R \rangle \) and \( \langle R \rangle \) differ from \( \Delta W \) due to the presence of irreversibility \( \Delta W \).