Non-negatively Curved 6-Manifolds with Almost Maximal Symmetry Rank

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Abstract We classify closed, simply connected, non-negatively curved 6-manifolds of almost maximal symmetry rank up to equivariant diffeomorphism.

Keywords Almost maximal symmetry rank · Equivariant diffeomorphism · 6-Manifolds · Non-negative curvature

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1 Introduction

For the class of closed, simply connected Riemannian manifolds, there are no known obstructions that allow us to distinguish between positive and non-negative sectional curvature, in spite of the fact that the number of known examples of manifolds of non-negative sectional curvature is vastly larger than those known to admit a metric of positive sectional curvature.

The introduction of symmetries, however, allows us to distinguish between such classes. An important first case to understand is that of maximal symmetry rank, where the symmetry rank is defined to be the rank of the isometry group: \( \text{symrk}(M^n) = \text{rk}(\text{Isom}(M^n)) \). For manifolds of strictly positive sectional curvature, a classification up to equivariant diffeomorphism was obtained by Grove and Searle [10]. They showed...
that for such manifolds, the maximal symmetry rank is equal to \( \lfloor (n + 1)/2 \rfloor \). For closed, simply connected manifolds of non-negative sectional curvature, the maximal symmetry rank is conjectured to be \( \lfloor 2n/3 \rfloor \) (see Galaz-García and Searle [7] and Escher and Searle [3]). A classification for the latter has been obtained, but only in dimensions less than or equal to nine (see [7] and Galaz-García and Kerin [6], for dimensions less than or equal to 6 and [3] for dimensions 7 through 9) and the upper bound for the symmetry rank has been verified for dimensions less than or equal to 12 (see [3, 7]).

A natural next step is the case of almost maximal symmetry rank. In positive curvature, a homeomorphism classification was obtained by Rong [24] in dimension 5, and Fang and Rong [4] for dimensions greater than or equal to 8, using work of Wilking [29]. In non-negative curvature, a homeomorphism classification was obtained independently by Kleiner [14] and Searle and Yang [25], in dimension 4. This classification was later improved to equivariant diffeomorphism by [10], Galaz-García [5], [6] and Grove and Wilking [11]. A diffeomorphism classification in dimension 5 was obtained by Galaz-García and Searle [8].

In this article, we consider closed, simply connected Riemannian 6-manifolds admitting a metric of non-negative sectional curvature and an effective, isometric torus action of almost maximal symmetry rank and prove the following classification theorem.

**Main Theorem** Let \( T^3 \) act isometrically and effectively on \( M^6 \), a closed, simply connected, non-negatively curved Riemannian manifold. Then \( M^6 \) is diffeomorphic to \( S^3 \times S^3 \) or equivariantly diffeomorphic to a torus manifold with a linear torus action.

Closed, orientable manifolds of dimension \( 2n \) admitting a smooth \( T^n \)-action with non-empty fixed point set are called torus manifolds. Non-negatively curved torus manifolds were classified up to equivariant diffeomorphism by Wiemeler [28] (see Theorem 2.13). In dimension 6, they are equivariantly diffeomorphic to \( S^6 \cong \mathbb{C}P^3 = S^7/T^1 \), or the quotient by (1), a free linear circle action on \( S^3 \times S^4 \), (2), a free linear \( T^2 \)-action on \( S^3 \times S^5 \), or (3), a free linear \( T^3 \)-action on \( S^3 \times S^3 \times S^3 \). In the process of classifying bi-quotients of dimension 6, De Vito [2] has given a classification of these manifolds up to diffeomorphism. For case (1), one obtains \( S^2 \times S^4 \) or the non-trivial \( S^4 \)-bundle over \( S^2 \), whereas cases (2) and (3) consist of infinite families. It is worth noting that Kuroki [15], using torus graphs, has obtained an Orlik–Raymond type classification of 6-dimensional torus manifolds with vanishing odd degree cohomology without curvature restrictions.

**Remark** In the Main Theorem, in all cases except one, \( M^6 \) is actually equivariantly diffeomorphic to \( S^3 \times S^3 \) or a torus manifold with a linear torus action. In the case where \( M^6 \) admits only rank one isotropy, then we can only say that \( M^6 \) is diffeomorphic to \( S^3 \times S^3 \).

The paper is organized as follows. Section 2 covers preliminary material required for the proof of the Main Theorem. Section 3 contains general topological results about manifolds that decompose as disk bundles without curvature restrictions. Section 4 contains the proof of the Main Theorem.
2 Preliminaries

In this section, we will gather basic results and facts about transformation groups, the topological classification of six-dimensional manifolds, and $G$-invariant manifolds of non-negative curvature.

2.1 Transformation Groups

Let $G$ be a compact Lie group acting on a smooth manifold $M$. We denote by $G_x = \{ g \in G : gx = x \}$ the isotropy group at $x \in M$ and by $G(x) = \{ gx : g \in G \} \cong G/G_x$ the orbit of $x$. Orbits are called principal, exceptional, or singular, depending on the relative size of their isotropy subgroups; that is, principal orbits correspond to those orbits with the smallest possible isotropy subgroup, an orbit is called exceptional when its isotropy subgroup is a finite extension of the principal isotropy subgroup, and singular when its isotropy subgroup is of strictly larger dimension than that of the principal isotropy subgroup.

The ineffective kernel of the action is the subgroup $K = \cap_{x \in M} G_x$. We say that $G$ acts effectively on $M$ if $K$ is trivial. The action is called almost effective if $K$ is finite.

We will sometimes denote the fixed point set $M^G = \{ x \in M : gx = x, g \in G \}$ of the $G$-action by $\text{Fix}(M; G)$. One measurement for the size of a transformation group $G \times M \rightarrow M$ is the dimension of its orbit space $M/G$, also called the cohomogeneity of the action. This dimension is clearly constrained by the dimension of the fixed point set $M^G$ of $G$ in $M$. In fact, $\dim(M/G) \geq \dim(M^G) + 1$ for any non-trivial action with fixed points. In light of this, the fixed point cohomogeneity of an action, denoted by $\text{cohomfix}(M; G)$, is defined by

$$\text{cohomfix}(M; G) = \dim(M/G) - \dim(M^G) - 1 \geq 0.$$

A manifold with fixed point cohomogeneity 0 is also called a $G$-fixed point homogeneous manifold.

2.2 Topological Classification of 6-Manifolds

Note that throughout the paper, we will use the convention that all homology groups have integer coefficients, unless otherwise specified.

The topological classification of simply connected, closed, oriented 6-manifolds has been completed in a sequence of articles by Wall [27], Jupp [13], and Žubr [30–32]. We will narrow our focus to the classification of closed, simply connected, oriented 6-manifolds with torsion-free homology. The classification theorem below is due to Wall in the case of smooth spin manifolds, [27], and in its final form due to Jupp [13]. We first describe the basic invariants used to classify 6-dimensional, closed, simply connected, oriented, smooth manifolds, $M$, with torsion-free homology [13].
Theorem 2.1 ([13]) Let $M$ be a 6-dimensional, closed, simply connected, oriented, smooth manifold with torsion-free homology. The basic invariants used to classify $M$ are as enumerated below.

1. $H := H^2(M)$, a finitely generated free abelian group;
2. $b := b_3(M) = \text{rk}_\mathbb{Z}(H^3(M)) \in 2\mathbb{Z}$ since $H^3(M)$ admits a non-degenerate symplectic form;
3. $F := F_M : H^2(M) \otimes H^2(M) \otimes H^2(M) \to \mathbb{Z}$ a symmetric trilinear form given by the cup product evaluated on the orientation class;
4. $p := p_1(M) \in H^4(M)$, the first Pontrjagin class;
5. $w := w_2(M) \in H^2(M; \mathbb{Z}_2)$, the second Stiefel–Whitney class.

We now use Poincaré duality to identify $H^4(M)$ with $\text{Hom}_\mathbb{Z}(H_2(M); \mathbb{Z})$ so that $p_1(M)$ can be interpreted as a linear form on $H^2(M)$ and we let $x \cdot y \cdot z$ denote $F_M(x \otimes y \otimes z)$ for $x, y, z \in H^2(M)$.

Definition 2.2 (Admissibility) The system of invariants $(H, b, w, F, p)$ is called admissible if and only if for every $\omega \in H$ and $T \in H^* := \text{Hom}_\mathbb{Z}(H; \mathbb{Z})$ with $\rho_2(\omega) = w$ and $\rho_2(T) = 0$, where $\rho_2 : \mathbb{Z} \to \mathbb{Z}_2$ is reduction modulo 2, the following congruence holds:

$$\omega^3 \equiv (p + 24T) \omega \mod 48.$$

Definition 2.3 (Equivalence) Two systems $(H, b, w, F, p)$ and $(H', b', w', F', p')$ are called equivalent if and only if $b = b'$ and there exists an isomorphism $\alpha : H \to H'$ such that $\alpha(w) = w'$, $\alpha^*(F') = F$, $\alpha^*(p') = p$.

We are now ready to state the classification result:

Theorem 2.4 ([13]) The assignment

$$M \mapsto \left( \frac{b}{2}, H^2(M), w_2(M), F_M, p_1(M) \right)$$

induces a 1–1 correspondence between oriented diffeomorphism classes of simply connected, closed, oriented, 6-dimensional, smooth manifolds with torsion-free homology, and equivalence classes of admissible systems of invariants.

Note that Žubr generalized Wall’s theorem in a different direction: he proved a classification theorem for simply connected, smooth spin manifolds with not necessarily torsion-free homology [30], and then in [31, 32] also obtained Jupp’s theorem and proved that algebraic isomorphisms of systems of invariants can always be realized by orientation preserving diffeomorphisms.

Observe that the first invariant, $b/2$, of Theorem 2.4 is completely independent of the other invariants which implies that the following splitting theorem holds.

Corollary 2.5 ([27]) Every simply connected, closed, oriented, 6-dimensional, smooth manifold $M$ admits a splitting $M = M_0 \# \frac{b}{2}(S^3 \times S^3)$ as a connected sum of a core $M_0$ with $b = b_3(M_0) = 0$ and $\frac{b}{2}$ copies of $S^3 \times S^3$. 
The following corollary is an immediate consequence of Corollary 2.5.

**Corollary 2.6** Let $M$ be a simply connected, closed, oriented, 6-dimensional, smooth manifold with

$$H_i(M^6) \cong H_i(S^3 \times S^3) \text{ for all } i.$$ 

Then $M^6$ is diffeomorphic to $S^3 \times S^3$.

### 2.3 $G$-manifolds with Non-negative Curvature

We now recall some general results about $G$-manifolds with non-negative curvature which we will use throughout. Recall that fixed point homogeneous manifolds of positive curvature were classified in [10]. More recently, Spindeler [26] proved the following theorem which characterizes non-negatively curved $G$-fixed point homogeneous manifolds.

**Theorem 2.7** ([26]) Assume that $G$ acts fixed point homogeneously on a complete non-negatively curved Riemannian manifold $M$. Let $F$ be a fixed point component of maximal dimension. Then there exists a smooth submanifold $N$ of $M$, without boundary, such that $M$ is diffeomorphic to the normal disk bundles $D(F)$ and $D(N)$ of $F$ and $N$ glued together along their common boundaries;

$$M = D(F) \cup \partial D(N).$$

Further, $N$ is $G$-invariant and contains all singularities of $M$ up to $F$.

Let $\text{Isom}_F(M)$ be the subgroup of the isometry group of $M$ that leaves $F$ invariant. The following lemmas from [26] will also be important.

**Lemma 2.8** ([26]) Let $M$ be a non-negatively curved fixed point homogeneous $G$-manifold, with $M$, $F$, and $N$ as in Theorem 2.7 and $K = \text{Isom}_F(M)$. Then there exists a $K$-equivariant diffeomorphism $b : \partial D(N) \to \partial D(F)$ and $M$ is $K$-equivariantly diffeomorphic to $D(F) \cup_\partial D(N)$.

**Lemma 2.9** ([26]) Let $M$ and $N$ be as in Theorem 2.7 and assume that $\pi_1(M) = 0$ and $G$ is connected. Then $N$ has codimension greater than or equal to 2 in $M$.

The next theorem from Galaz-García and Spindeler [9] covers the special case when both $F$ and $N$ are fixed point sets of the $G$-action and generalizes the Double Soul Theorem for $S^1$-fixed point homogeneous actions of [25].

**Double Soul Theorem 2.10** ([9, 25]) Let $M$ be a non-negatively curved $G$-fixed point homogeneous Riemannian manifold, where the principal isotropy group of the $G$-action is $H$. If $\text{Fix}(M, G)$ contains at least two connected components $F$ and $N$ with maximal dimension, one of which is compact, then $F$ and $N$ are isometric and $M$ is diffeomorphic to an $S^{k+1}$-bundle over $F$, where $S^k = G/H$. 

\[ \text{Springer} \]
Since fixed point homogeneous manifolds with either positive or non-negative lower curvature bounds decompose as unions of disk bundles, the following purely topological lemma from [3] will be useful.

**Lemma 2.11** ([3]) Let $M$ be a manifold with $\text{rk}(H_1(M)) = k$, $k \in \mathbb{Z}^+$. If $M$ admits a disk bundle decomposition

$$M = D(N_1) \cup_E D(N_2),$$

where $N_1, N_2$ are smooth submanifolds of $M$ and $N_1$ is orientable and of codimension two, then both $\text{rk}(H_1(N_1))$ and $\text{rk}(H_1(N_2))$ are less than or equal to $k + 1$.

An important subclass of manifolds admitting an effective torus action is the subclass of torus manifolds.

**Definition 2.12** (Torus Manifold) A torus manifold $M$ is a $2n$-dimensional closed, connected, orientable, smooth manifold with an effective smooth action of an $n$-dimensional torus $T$ such that $MT \neq \emptyset$.

A related concept is that of an isotropy-maximal $T^k$-action on $M^n$, when there exists a point in $M$ whose isotropy group is maximal, namely of dimension $n - k$ (see [3], cf. Ishida [12]). Note that a torus manifold, $M^{2n}$, is an example of a manifold admitting an isotropy-maximal $T^n$-action. In fact, we may characterize torus manifolds as $2n$-dimensional closed, connected, orientable, smooth manifolds with an effective and isotropy-maximal smooth $T^n$-action.

The following important theorem from [28] gives a classification up to equivariant diffeomorphism of non-negatively curved torus manifolds.

**Theorem 2.13** ([28]) Let $M^{2n}$ be a simply connected, non-negatively curved Riemannian manifold admitting an isometric, effective, and isotropy-maximal $T^n$-action. Then $M$ is equivariantly diffeomorphic to a quotient of a free linear torus action of

$$\mathcal{Z} = \prod_{i < r} S^{2n_i} \times \prod_{i \geq r} S^{2n_i - 1}, \quad n_i \geq 2.$$

Finally, we recall Theorem A from [3], which generalizes Theorem 2.13.

**Theorem 2.14** ([3]) Let $M^n$, a closed, simply connected, non-negatively curved Riemannian manifold admitting an isometric, effective, and isotropy-maximal $T^k$-action, where $k \geq \lfloor (n + 1) / 2 \rfloor$. Then $M$ is equivariantly diffeomorphic to a quotient of a free linear torus action of

$$\mathcal{Z}^m = \prod_{i < r} S^{2n_i} \times \prod_{i \geq r} S^{2n_i - 1}, \quad n_i \geq 2, \quad n \leq m \leq 3n - 3k,$$

with a free linear $T^k$ action.
3 Disk Bundle Decompositions

In this section, we present general topological results about manifolds which decompose as unions of disk bundles. Note that these results are curvature independent.

The following theorem gives us information about the fundamental group of the base of each disk bundle and allows us to identify the fundamental group of $E$ in the disk bundle decomposition.

**Theorem 3.1** Let $M^n$ be a simply connected manifold that decomposes as the union of two disk bundles as follows:

$$M^n = D^{k_1}(N_1) \cup_E D^{k_2}(N_2).$$

If $k_1 = k_2 = 2$, then $\pi_1(N_1)$ and $\pi_1(N_2)$ are cyclic groups.

Moreover,

1. If $k_i = 2$, $\pi_2(N_i) = 0$, for $i = 1, 2$ and $\pi_1(N_i)$ is infinite for some $i \in \{1, 2\}$, then $\pi_1(E) \cong \mathbb{Z}^2$.
2. Let $i \in \{1, 2\}$. If $k_i \geq 3$, then $\pi_1(E) \cong \pi_1(N_i)$ and $\pi_1(N_{i+1}) = 0$. Here the indices are taken mod 2.

**Proof** Assume that $k_i = 2$, $i = 1, 2$. Then $E$ is a circle bundle over $N_i$, and we obtain the following exact sequences from the long exact sequence in homotopy:

$$\cdots \to \pi_1(S^1_j) \xrightarrow{i_j^*} \pi_1(E) \xrightarrow{f_j^*} \pi_1(N_j) \to 0, \text{ for } j \in \{1, 2\}.$$

Now let $U_1 = i_1^*(\pi_1(S^1_1))$, and $U_2 = i_2^*(\pi_1(S^1_2))$. By exactness, $\pi_1(N_i) \cong \pi_1(E)/U_i, i = 1, 2$. So we get the following commutative diagram:

$$\begin{array}{ccc}
\pi_1(E) & \xrightarrow{f_2^*} & \pi_1(N_2) \\
\downarrow{f_1^*} & & \downarrow{\phi_2} \\
\pi_1(N_1) & \xrightarrow{\phi_1} & \pi_1(E)/U_1U_2
\end{array}$$

The maps, $\phi_i$, are given by

$$\phi_i : \pi_1(N_i) \cong \pi_1(E)/U_i \to \pi_1(E)/U_1U_2, i = 1, 2.$$
Now by Seifert–van Kampen (universal property), there exists a morphism \( h : \pi_1(M) \longrightarrow \pi_1(E)/U_1U_2 \) making the following diagram commute:

\[
\begin{array}{ccc}
\pi_1(E) & \xrightarrow{f_2} & \pi_1(N_2) \\
\downarrow{f_1} & & \downarrow{\phi_2} \\
\pi_1(N_1) & \xrightarrow{h} & \pi_1(M) \\
& \downarrow{\phi_1} & \\
& \pi_1(E)/U_1U_2 & \\
\end{array}
\]

Since all the maps in (*) are surjective, \( h \) must be surjective. But since \( \pi_1(M) = 0 \), this implies that \( \pi_1(E) \cong U_1U_2 \). Further, \( \pi_1(N_1) \cong \pi_1(E)/U_2 \cong U_2/(U_1 \cap U_2) \) and both \( U_1 \) and \( U_2 \) are cyclic groups. Hence, \( \pi_1(N_1) \) is cyclic. The same holds true for \( \pi_1(N_2) \).

**Case 1** In the case where \( \pi_2(N_i) = 0, i = 1, 2 \), and \( \pi_1(N_1) \) is not a finite cyclic group, we argue as follows. Note that since \( \pi_2(N_i) = 0, i = 1, 2 \), this implies that \( U_i \cong \mathbb{Z}, i = 1, 2 \). Note that both \( U_1 \) and \( U_2 \) are normal in \( \pi_1(E) \). If, in addition, \( U_1 \cap U_2 = \{1\} \), then \( \pi_1(E) \cong U_1 \times U_2 \cong \mathbb{Z}^2 \) and the theorem follows. If \( U_1 \cap U_2 \neq \{1\} \), then \( \pi_1(N_1) \cong U_1U_2/U_1 \cong U_2/U_1 \cap U_2 \). But \( U_1 \cap U_2 \) is a normal subgroup of \( U_2 \cong \mathbb{Z} \), hence \( U_1 \cap U_2 \cong n \mathbb{Z} \) for some \( n \in \mathbb{Z} \). It follows that \( \pi_1(N_1) \cong U_2/U_1 \cap U_2 \cong \mathbb{Z}/n\mathbb{Z} \) which is a contradiction to the hypothesis that \( \pi_1(N_1) \) is not finite cyclic. Hence \( U_1 \cap U_2 = \{1\} \) and \( \pi_1(E) \cong U_1 \times U_2 \cong \mathbb{Z}^2 \).

**Case 2** Assume now that \( k_i \geq 3 \) for some \( i \in \{1, 2\} \). Then \( E \) is an \( S^{k_i-1} \) bundle over \( N_i \) and hence by the long exact sequence in homotopy \( \pi_1(E) \cong \pi_1(N_i) \). Note that since the codimension of \( N_i \) is greater than or equal to 3, it follows by transversality that \( \pi_1(M \setminus N_i) \cong \pi_1(M) \). Since \( \pi_1(N_{i+1}) \cong \pi_1(M \setminus N_i) \), the result follows. \( \Box \)

**Remark 3.2** We note that in the case where \( k_1 = k_2 = 2 \) in Theorem 3.1, we can actually say more about the fundamental groups of the \( N_i, i = 1, 2 \), in general, and in one special case about the fundamental group of \( E \). Namely, using the notation of the proof of Theorem 3.1, the following are true:

1. If \( U_1 \cap U_2 = \{1\} \), then \( \pi_1(N_i) \cong U_i, i = 1, 2 \) and \( \pi_1(E) \cong U_1 \times U_2 \), a product of cyclic groups.
2. If \( U_1 \cap U_2 \neq \{1\} \), \( U_1 \) and \( U_2 \) must be either both infinite or both finite cyclic.
   
   Hence this case splits into two further subcases:
   
   (a) \( U_1 \cong U_2 \cong \mathbb{Z} \) and so \( U_1 \cap U_2 \cong n\mathbb{Z} \) for some \( n \in \mathbb{Z}^+ \) and \( \pi_1(N_i) \cong \mathbb{Z}_n, i = 1, 2; \)
   
   (b) \( U_1 \cong \mathbb{Z}_n, U_2 \cong \mathbb{Z}_m \) for some \( n, m \in \mathbb{Z}^+ \), then \( U_1 \cap U_2 \cong \mathbb{Z}_d \) where \( (n, m) = d \), and \( \pi_1(N_1) \cong \mathbb{Z}_n^d, \pi_1(N_2) \cong \mathbb{Z}_m^d \). In the special case where \( d = 1, \pi_1(E) \cong \mathbb{Z}_n \times \mathbb{Z}_m. \)
4 Proof of Main Theorem

In this section, we present the proof of the Main Theorem. We first recall the following lemma from [8].

**Lemma 4.1** ([8]) Let $T^n$ act on $M^{n+3}$, a closed, simply connected smooth manifold. Then $T^n$ cannot act freely or almost freely; that is, some circle subgroup has non-trivial fixed point set.

Note that by Lemma 4.1, a $T^3$-action on a closed, simply connected $M^6$ must have circle isotropy. Therefore, we may break the proof of the Main Theorem into three cases, depending on the rank of the largest isotropy subgroup, which will be either 1, 2, or 3. Theorem 2.13 gives us the desired classification for those manifolds with $T^3$ isotropy. Thus, we have proven Part (1) of the following theorem.

**Theorem 4.2** Let $M^6$ be a closed, simply connected, non-negatively curved Riemannian 6-manifold admitting an isometric, effective $T^3$-action. Then the action has singular isotropy of rank 1, 2, or 3 and the following hold.

1. If the rank of the largest singular isotropy subgroup is equal to 3, then $M^6$ is equivariantly diffeomorphic to a torus manifold with a linear $T^3$-action, that is, to one of $S^6$, $CP^3$, $(S^3 \times S^4)/T^1$, $(S^3 \times S^5)/T^2$, or $(S^3 \times S^3 \times S^3)/T^3$.
2. If the rank of the largest singular isotropy subgroup is less than or equal to 2, then $M^6$ is diffeomorphic to $S^3 \times S^3$.

It remains to prove Part (2) of Theorem 4.2. We break the proof into two cases: Case (i), where the action is $T^1$-fixed point homogeneous for some $T^1 \subset T^3$ and Case (ii), where no circle subgroup acts fixed point homogeneously.

4.1 Proof of Case (i) of Part (2) of Theorem 4.2

We have two further subcases to consider: Case i(a), where the action admits only $T^1$ isotropy and Case i(b), where the actions admits $T^2$ isotropy.

We first consider Case i(a), where some circle acts fixed point homogeneously and the induced $T^2$-action on the codimension two fixed point set is either free or almost free. We will prove the following theorem.

**Theorem 4.3** Let $T^3$ act isometrically and effectively on $M^6$, a closed, simply connected, non-negatively curved Riemannian manifold. Suppose that the action is $S^1$-fixed point homogeneous and that the largest isotropy subgroup of the $T^3$-action is of rank one. Then $M$ is diffeomorphic to $S^3 \times S^3$.

The strategy for the proof of Theorem 4.3 will be to show that $M^6$ decomposes as a union of two disk bundles, each a 2-disk bundle over a 4-manifold. One can then show that $M^6$ has the homology groups of $S^3 \times S^3$ and by Corollary 2.6, we then obtain a diffeomorphism classification.

We begin by establishing some notation. Let $F$ be the fixed point set component of the circle action of maximal dimension on $M^6$ and let $N$ be as in Theorem 2.7 such that $M^6$ is given as
where $E$ is the common boundary of the two disk bundles. Observe that $F$ is a closed, orientable, non-negatively curved 4-dimensional submanifold of $M^6$, admitting an isometric $T^2$-action. Among other things, we will show in Proposition 4.5 that under these hypotheses, $N$ is also 4-dimensional.

**Remark 4.4** For the remainder of this subsection, we will always assume that there is a $T^3$ isometric and effective action on $M^6$, a closed, simply connected, non-negatively curved Riemannian manifold, such that the action is $S^1$-fixed point homogeneous. As such, we will omit the statement of these hypotheses in what follows.

The following proposition shows that the topology of both $F$ and $N$ is strongly restricted when $M^6$ is $S^1$-fixed point homogeneous.

**Proposition 4.5** Let $M'$ denote either $F$ or $N$ and assume that the largest isotropy subgroup of the $T^3$-action is of rank one. Then the following are true:

1. $\pi_1(M') \cong \mathbb{Z};$
2. $\chi(M') = 0;$
3. $M'$ is orientable; and
4. $\dim(M') = 4.$

**Proof** We will first prove the proposition holds for $M' = F$. If we assume that $\chi(F) \neq 0$, then the induced $T^2$-action on $F$ would have non-empty fixed point set and thus there is a point in $M^6$ fixed by $T^3$, contrary to our hypothesis that the isotropy subgroups have rank at most 1. Thus, $\chi(F) = 0$.

It follows from Lemma 2.11 that $\operatorname{rk}(H_1(F)) \leq 1$. Suppose then that $\operatorname{rk}(H_1(F)) = 0$, to obtain a contradiction. $F$ is orientable, since it is a fixed point set of a circle action and $M^6$ is orientable. Therefore $\chi(F)$ is strictly positive, a contradiction. Thus $\operatorname{rk}(H_1(F)) = 1$.

By Lemma 2.9, $\dim(N) \leq 4$. Moreover, by Theorem 3.1, if $\dim(N) < 4$, then $\pi_1(F)$ is trivial. Hence $\dim(N) = 4$. By Theorem 3.1, it follows that both $F$ and $N$ have cyclic fundamental group. Hence $\pi_1(F) \cong \mathbb{Z}$ and we have proven the proposition for $F$.

We now proceed to prove the remainder of the results for $N$. Note that by Lemma 2.8, $N$ is $T^3$-invariant. We have two cases to consider: Case (1), $N$ is fixed by some circle subgroup of $T^3$ and Case (2), $N$ is not fixed by any circle subgroup of $T^3$. To prove Case (1), we note that since $M$ is simply connected, the Double Soul Theorem 2.10 implies that $N$ cannot be fixed by the same circle that fixes $F$, so we may apply Theorem 3.1 to show that the remainder of the results hold for $N$.

It remains to consider Case (2), that is where $N$ is not fixed by any circle subgroup. Then, since it is invariant under the $T^3$-action, it follows that it is a cohomogeneity one submanifold and hence diffeomorphic to $S^1 \times M^3$ (see Pak [19] and Parker [20]). Recall by Lemma 2.11 that $\operatorname{rk}(H_1(N)) \leq 1$, so by the Künneth formula, it follows that $H_1(M^3)$ is finite. Hence $M^3$ is one of $S^3$ or $L_{p,q}$ (see Mostert [17] and Neumann [18]). In particular, $N$ is an orientable submanifold with $\chi(N) = 0$ and we may apply Theorem 3.1 once again to show that $\pi_1(N) \cong \mathbb{Z}.$
We can now prove the following proposition, which tells us that $M^6$ has the same homology groups as $S^3 \times S^3$.

**Proposition 4.6** Suppose that the largest isotropy subgroup of the $T^3$-action is of rank one. Then the homology groups of $M^6$ are isomorphic to those of $S^3 \times S^3$, that is,

$$H_i(M^6) \cong H_i(S^3 \times S^3) \text{ for all } i.$$

**Proof** Consider the Mayer–Vietoris sequence of the disk bundle decomposition for $M^6$. Using Poincaré Duality and the Universal Coefficient Theorem, one immediately concludes that $H_2(M^6) \cong H_4(M^6) = 0$ and that $H_3(M^6)$ has no torsion. So the only unknown homology group is $H_3(M^6)$. Using the Gysin sequence, we see that $\text{rk}(H_3(E)) \leq 1$. Further, using the Universal Coefficient Theorem, it follows that $H_3(E)$ has no torsion, thus $H_3(E)$ is either trivial or $\mathbb{Z}$, and $H_3(E) \cong \text{Hom}(H_2(E); \mathbb{Z})$. We then have the following exact sequence from the Mayer–Vietoris sequence:

$$0 \rightarrow H_3(E) \rightarrow \mathbb{Z}^2 \rightarrow H_3(M^6) \rightarrow H_2(E) \rightarrow 0.$$

Now, considering the two possibilities for $H_3(E)$, we find that in both cases $H_3(M^6) = \mathbb{Z}^2$. □

Combining the result of Proposition 4.6 with the fact that $\omega_2 = 0$, it follows by Corollary 2.6 that $M^6$ is diffeomorphic to $S^3 \times S^3$.

We now proceed to prove Case $i(b)$ of Part (2) of Theorem 4.2. We will prove the following theorem.

**Theorem 4.7** Let $T^3$ act on $M^6$, a 6-dimensional, closed, simply connected, non-negatively curved Riemannian manifold. Suppose that the action admits $T^2$ isotropy, but is not isotropy-maximal. Then $M^6$ is equivariantly diffeomorphic to $S^3 \times S^3$ with a linear $T^3$-action.

For Case $i(b)$, we note first that since there is $T^2$ isotropy, the smallest possible orbit is $T^1$ and we have the following nesting

$$T^1 \subset F^2 \subset F^4 \subset M^6,$$

where $F^2$ is fixed by $T^2$ and $F^4$ is fixed by a circle subgroup of $T^2$. In particular, since $F^4$ admits an induced $T^2$-action itself, but has no fixed points of this action, it is clear that $\chi(F^4) = 0$. Hence $\text{rk}(H_1(F^4)) = 1$ by Lemma 2.11, and applying Theorem 3.1, we obtain the following result.

**Lemma 4.8** Let $F^4$ be a 4-dimensional component of the fixed point set of some circle subgroup of the $T^3$-action on $M^6$. Suppose that $F^4$ admits a $T^2$-action with only circle isotropy. Then $\pi_1(F^4) \cong \mathbb{Z}$.

The proof of Proposition 4.5 is easily adapted to show that the submanifold $N$ of the disk bundle decomposition of $M^6$ in Display 4.1 must be 4-dimensional and $\pi_1(N) \cong \mathbb{Z}$. 
In order to complete the proof of Theorem 4.7, it suffices to show that \( M^6 \) is equivariantly diffeomorphic to \( S^3 \times S^3 \) with a linear \( T^3 \)-action. However, this follows from Lemma 2.1 and Theorem 2.3 of McGavran and Oh [16].

### 4.2 Proof of Case (ii) of Part 2 of Theorem 4.2

We now consider the case where there is only isolated circle isotropy, that is where the rank of the isotropy subgroups is at most one and the action is not \( S^1 \)-fixed point homogeneous. The goal of this subsection is to prove the following theorem.

**Theorem 4.9** Let \( T^3 \) act on \( M^6 \), a 6-dimensional, closed, simply connected, non-negatively curved Riemannian manifold. Suppose that the largest isotropy subgroup of the \( T^3 \)-action is of rank one. Then \( M^6 \) is diffeomorphic to \( S^3 \times S^3 \).

The argument is a straightforward generalization of the finite isotropy case for isometric \( T^2 \)-actions on closed, simply connected, non-negatively curved 5-manifolds with only isolated circle orbits that appears in [8]. We include it here for the sake of completeness.

First recall from Corollary 4.7 of Chapter IV of Bredon [1], that the quotient space, \( M^* \), of a cohomogeneity three \( G \)-action on a compact, simply connected manifold with connected orbits is a simply connected 3-manifold with or without boundary. Note that when there is only isolated circle isotropy for a cohomogeneity three torus action, the quotient space will not have boundary and thus, by the resolution of the Poincaré conjecture (see Perelman [21–23]), we have that \( M^* = S^3 \).

We first recall the following result from [8] (see Proposition 4.5 and Corollary 4.6), which gives us a lower bound for the number of isolated singular orbits of the action.

**Proposition 4.10** ([8]) Let \( T^n \) act on \( M^{n+3} \), a simply connected, smooth manifold. Suppose that \( M^* \) is homeomorphic to \( S^3 \) and that the rank of the largest isotropy subgroup is equal to one. Then there are at least \( n + 1 \) isolated singular orbits \( T^{n-1} \).

The non-negative curvature hypothesis gives us an upper bound on the number of isolated \( T^2 \) orbits. The following lemma from [11] is crucial:

**Lemma 4.11** ([11]) A three-dimensional non-negatively curved Alexandrov space \( X^3 \) has at most four points for which the space of directions is not larger than \( S^2(1/2) \).

Proposition 4.8 in [8] shows that if there is finite isotropy, it must be \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) or \( \mathbb{Z}_k \) and that in the latter case, those exceptional orbits are not isolated. Proposition 4.10 and Lemma 4.11 then imply that there are exactly 4 isolated \( T^2 \) orbits. This fact combined with the proof of Proposition 5.8 in [8] then tells us that \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) isotropy cannot occur.

We may summarize our results as follows.

**Proposition 4.12** Let \( T^3 \) act isometrically and effectively on \( M^6 \), a 6-dimensional, closed, simply connected Riemannian manifold as in Theorem 4.9. Suppose that \( M^6/T^3 = M^* = S^3 \). Then there are exactly 4 isolated \( T^2 \) orbits and if there is finite isotropy, then it must be cyclic and the corresponding exceptional orbits are not isolated.
We consider first the case where there is no finite isotropy. We have the following result.

**Proposition 4.13** Let $T^3$ act isometrically and effectively on $M^6$, a 6-dimensional, closed, simply connected Riemannian manifold. Suppose that $M^6/T^3 = M^* = S^3$. If there is no finite isotropy, then $M^6$ is diffeomorphic to $S^3 \times S^3$.

**Proof** The proof of Proposition 4.10 in [8] shows that $\pi_2(M^6) = 0$. By the Hurewicz isomorphism, it follows that $M^6$ has homology only in dimension 3 and by the Universal Coefficient Theorem there is no torsion. Since the fixed point set of the $T^3$-action is empty by hypothesis, it follows that $\chi(M^6) = 0$. This tells us that $b_3(M^6) = 2$ and thus $M^6$ has the homology groups of $S^3 \times S^3$, so by Corollary 2.6, we see that $M^6$ is diffeomorphic to $S^3 \times S^3$. $\square$

We now consider the case where the $T^3$-action on $M^6$ has non-trivial finite isotropy. There are just five admissible graphs corresponding to this case (see Fig. 1).

In the special case where the singular set in the orbit space contains a cycle, we have the following result which follows directly from work of [11] and its generalization in [8].

**Theorem 4.14** Let $M^6$ be a closed, simply connected, non-negatively curved 6-manifold with an isometric $T^3$-action and orbit space $M^* = S^3$. If the singular set in the orbit space $M^*$ contains a cycle $K^1$, then the following hold:

1. The cycle $K^1$ is the only cycle in the singular set in $M^*$.
2. $K^1$ comprises all of the singular set, i.e., $M^* \setminus K^1$ is smooth.
3. The cycle $K^1$ is unknotted in $M^*$.

We will now show in all cases where we have a cycle that we may decompose the manifold as a union of disk bundles, where at least one of the disk bundles is over one arc of the cycle.

![Possible weighted graphs when there is finite cyclic isotropy](image)
Fig. 2 How to complete a weighted graph with edges corresponding to principal orbits to obtain a cycle: the solid edge corresponds to orbits with finite cyclic isotropy, while the dotted edges correspond to principal orbits.

**Proposition 4.15** Let $T^3$ act on $M^6$ isometrically and effectively and suppose that $M^* = S^3$ and there is finite isotropy. Suppose further that the singular set in $S^3$ corresponds to graph $E$ in Fig. 1. Then we may decompose $M^6$ as a union of disk bundles over two disjoint 4-dimensional submanifolds, fixed by finite isotropy groups whose orders are relatively prime.

**Proof** We note first that it follows from Theorem 2.1 in Chapter VII of [1] that there can only be one codimension 2 submanifold of a given finite isotropy, and that in this setting all finite isotropy groups must be of relatively prime order. The proof of Proposition 4.15 now follows exactly as in [8] (see the proofs of Propositions 6.7 and 6.9).

In order to complete the proof of Case (ii) of Part 2 of Theorem 4.2, we must deal with the remaining graphs. For graphs (A) through (D), we may complete the weighted graph by joining disjoint isolated circle orbits or arcs via edges corresponding to shortest geodesics consisting of regular points in the orbit space. In this way we obtain a graph that is an unknotted circle (see Fig. 2) and now for all the possible graphs we may decompose $M^6$ as the union of two disk bundles over the 4-dimensional manifolds that correspond to opposite arcs of the circle. These 4-dimensional manifolds are invariant under the $T^3$-action and via the classification of torus actions of cohomogeneity one (see [19,20]), it follows that they are $T^1 \times M^3$, where $M^3$ is an orientable, cohomogeneity one manifold equal to one of $S^3$, $L_{p,q}$, $S^2 \times S^1$ by [17,18]. By Lemma 2.11, it follows that the 4-dimensional manifold may be one of $S^1 \times S^3$ or $S^1 \times L_{p,q}$.

As in Case (i) of Part 2 of Theorem 4.2, analyzing the Mayer–Vietoris sequence of the decomposition it is immediate that the 4-dimensional manifolds corresponding to opposite arcs for all the graphs must be $S^1 \times S^3$ and $M^6$ has the homology groups of $S^3 \times S^3$. Applying Corollary 2.6, it follows that $M^6$ is diffeomorphic to $S^3 \times S^3$.

This finishes the proof of Case (ii) of Part 2 of Theorem 4.2 and with it the proof of the Main Theorem.

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