Circular Polarization from Inhomogeneous Synchrotron Sources

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Abstract

Inhomogeneities can influence the polarization emerging from a synchrotron source. However, it is shown that the frequency distribution of circular polarization is only marginally affected, although its magnitude may change substantially. This is used to argue that the observed properties of compact radio sources imply a radiating plasma in which the characteristic waves are nearly circular. As a result, restrictions can be put on the low-energy part of the energy distribution of the relativistic electrons as well as the presence of electron–positron pairs. It is emphasized that this constrains theoretical modeling of the acceleration process for the relativistic electrons; for example, some of the currently popular scenarios seem to need modifications to become consistent with observations.

Unified Astronomy Thesaurus concepts: Non-thermal radiation sources (1119); Radio continuum emission (1340); Radio active galactic nuclei (2134); Extragalactic radio sources (508); Radiative transfer (1335); Radiative transfer equation (1336); Radiative processes (2055); Active galaxies (17); Radio galaxies (1343); Magnetic fields (994); Radio jets (1347); Relativistic jets (1390)

1. Introduction

It is commonly believed that the launching of jets in AGNs is driven by magnetic fields (Blandford & Znajek 1977; Blandford & Payne 1982). At some distance down-stream of the jet, some fraction of the Poynting flux needs to be converted into kinetic energy and relativistic particles in order to give rise to the observed radiation. The understanding of the processes by which this occurs is still rather limited.

Several mechanisms have been suggested for the acceleration of the particles; for example, diffusive shock acceleration, second-order Fermi acceleration in a turbulent medium (e.g., Zhdankin et al. 2018; Blandford et al. 2019), and magnetic reconnection (e.g., Romanova & Lovelace 1992; Giannios et al. 2009). Any of these mechanisms can give rise to a particle energy distribution, which, at the high end, is consistent with observations. On the other hand, their low-energy part is expected to differ, since it reflects the injection of particles into the acceleration process. Unfortunately, this part is normally hidden from view by optical depth effects.

Another related issue concerns the composition of the plasma, in particular, the presence of electron–positron pairs. As discussed by Sikora et al. (1997) and Sikora & Madejski (2000), a pure electron–positron plasma is likely to overproduce X-ray emission through the bulk comptonization of low-energy photons in the most luminous sources. At the same time, the kinetic energy of a jet dominated by electron–protons is often deduced to exceed that released through the accretion process (Ghisellini et al. 2014; Madejski et al. 2016). In order to be consistent with both of these constraints, it has been argued that the plasma needs to contain, roughly, 10 electron–positron pairs per proton (Ghisellini et al. 2010; Madejski et al. 2016). Furthermore, these two issues are connected, since the relative number of electron–positron pairs is expected to influence both particle heating and the efficiency of the acceleration process (Petropoulou et al. 2019).

Although neither the low-energy electrons nor the presence of electron–positron pairs can be observed directly, they can significantly affect the propagation of polarized light through a medium; this is particularly true for circular polarization. The potential importance of circular polarization in compact radio sources was realized early on (e.g., Pacholczyk 1973). However, the low observed value limited its role as a plasma diagnostic. To some extent, this has now changed with the advent of more sensitive observations (Macquart et al. 2000; Rayner et al. 2000), a larger frequency range (O’Sullivan et al. 2013; Agudo et al. 2018), and larger spatial resolution through very long baseline interferometry (Homan & Wardle 2004; Homan et al. 2009). This was used in Björnsson (2019) to argue that the observed properties of the circular polarization indicate that the characteristic waves are nearly circularly polarized rather than nearly linearly in the emitting plasma. This puts quite strong constraints on the combined properties of the electron energy distribution and the presence of electron–positron pairs.

The conclusion in Björnsson (2019) relied, mainly, on the frequency distribution of the circular polarization from a homogeneous source. The aim of the present paper is to determine to what extent inhomogeneities may alter the polarization properties emerging from a synchrotron source. The transport of polarized light is expressed as a coupling between the propagating characteristic waves ( Försterling 1942), which is in contrast to the standard way of using Stokes parameters. This allows for a more transparent discussion of the physical effects and, in particular, the solution of the transport equations can be expressed in terms of the polarization properties of the characteristic waves.

The paper is structured as follows: The coupling of characteristic waves is introduced in Section 2. It is shown that the ensuing transport equation can be obtained in a simpler and more concise way than is usually done. Its solution is discussed in Section 3. A constant coupling approximation is used to bring forth the general properties resulting from inhomogeneities. The main conclusion is that they can significantly affect the emerging circular polarization. However, this applies mainly to its magnitude, and the frequency dependence is only marginally affected. In Section 4, this approximation is contrasted with another one in which a given
change in plasma properties is modeled as occurring instantaneously. Although the resulting polarization can be quite different, again, the frequency distribution of the circular polarization should be a good discriminator between various plasma properties. The main points of the paper are summarized in Section 6.

2. A More Concise Derivation of the Transport Equation for Polarized Light in an Inhomogeneous Medium

Locally, the interaction between a propagating electromagnetic field and plasma can be described by \( J_i = \sigma_{l,m} E_m \). Here, \( J_i \) is the current, \( E_m \) is the electric field, and \( \sigma_{l,m} \) is the dielectric tensor. The indices \((l,m)\) run over the three spatial coordinates, i.e., \((l,m) = (x, y, z)\), and a repeated index indicates summation. The notation in this paper follows that in Björnssonn (2019; see also Jones & O’Dell 1977a). The amplitude of the electric field in the direction of its propagation (the \( z \)-direction, see Figure 1) is only a fraction \( |\sigma_{l,m}|/\nu| \sim \kappa \lambda \) of that in the perpendicular direction, where \( \nu \) and \( \lambda \) are the frequency and wavelength of the electromagnetic wave, respectively, while \( \kappa \) is the absorptivity of the medium. Since this is usually a very small number, the propagation of the electromagnetic field can be approximated by a second-order partial differential equation for a two-dimensional plane wave (i.e., \( l, m = x, y \)). Furthermore, as discussed in Björnsson (2019), without any additional approximations, this wave equation can be further reduced to a first-order ordinary differential equation

\[
\frac{d}{ds} E_i = -\frac{2\pi}{c} \sigma_{l,m} E_m,
\]

where \( s \) is the distance along a ray path.

The local properties of the plasma can be used to define two characteristic waves (1 and 2)

\[
J_{1,2}^l = \eta_{1,2} E_{1,2}^l,
\]

where \( \eta_{1,2} \) are the two eigenvalues obtained by diagonalizing \( \sigma_{l,m} \). Furthermore, \( J_i = J_i^1 + J_i^2 \) and \( E_i = E_i^1 + E_i^2 \). The plasma properties can be described by \( \xi_L = \xi_U + i\xi_V \), which accounts for the circular birefringence and absorption, and the corresponding linear quantity \( \xi_L = \xi_U + i\xi_L \) (see Björnsson 2019). The eigenvalues can then be expressed as

\[
\eta_{1,2} = \frac{c}{4\pi} (1 \pm i\sqrt{\xi_V^2 + \xi_L^2}).
\]

Furthermore,

\[
K_{1,2}^l = \frac{\pm \sqrt{1 - \rho^2 - \sin(2\varphi)}}{\rho + \cos(2\varphi)},
\]

where \( K_{1,2}^l \equiv E_{y,1,2}^l/E_{x,1,2}^l \) are the polarization of the two characteristic waves (Jones & O’Dell 1977b) and \( \rho \equiv i\xi_V/\xi_L \). Here, the azimuthal angle vary along the ray path as \( \phi = -\pi/4 + \varphi \), with \( \varphi = 0 \) for \( s = 0 \) (see Figure 1).

The standard transport equation expressed in terms of the Stokes parameters can be obtained directly from Equation (1). There are two aspects of this equation that should be noted; namely, (1) it is valid also for inhomogeneous media and (2) the notion of characteristic waves does not enter in its formulation. However, a not so attractive property is its low physical transparency. It was Försterling (1942) who first suggested the use of characteristic waves to elucidate the effects that inhomogeneities have on the polarization properties of a propagating electromagnetic wave. The reason for this approach is that for a homogeneous medium, the characteristic waves propagate independently, and the solution to Equation (1) can be written \( \hat{E} \equiv (E_x, E_y) = (E_x^1 + E_x^2, K^1 E_x^1 + K^2 E_x^2) \), where

\[
E_{1,2}^\ell = E_0^{1,2} \exp \left( -\frac{2\pi}{c} \eta_{1,2}^2 s \right).
\]

In this formulation, the effects of inhomogeneities manifest themselves as a coupling between the two characteristic waves. This idea was further developed by Cohen (1960). The WKB-approximation to the original wave equation was discussed by Ginzburg (1961). Physically, in this approximation, the polarization of the propagating characteristic waves adjusts to their local values; hence, no coupling between them occurs. However, the conditions for its applicability can easily be violated. Instead, it has been used as an “ansatz” to derive the coupling between the characteristic waves (see, e.g., Jones & O’Dell 1977b).

The derivation usually takes as its starting point the wave equation, i.e., a second-order differential equation. This leads to a long and rather tedious calculation, where, in the end, only terms of lowest order in the small quantity \( \kappa \lambda \) are retained. However, as discussed above, without loss of accuracy, one may instead start with Equation (1). Since this is a first-order
ordinary differential equation, a substantially shorter derivation should result. Its solution for a homogeneous medium (Equation (5)) contains two constant $E_{1,2}$. A suitable “ansatz” for the general solution is then

$$E_{1,2} = E_{1,2}^1 \exp\left(-\frac{2\pi i}{c} \int_0^\xi \eta_1^2 ds\right), \quad (6)$$

where the spatial variations of $\eta_{1,2}$ (and $K_{1,2}$) now imply that also $E_{1,2}$ vary with distance along a ray path.

Equation (1) can then be rewritten as

$$\frac{d}{ds}(E_1^1 + E_2^2) = -\frac{2\pi i}{c} (\eta_1^1 E_1^1 + \eta_2^2 E_2^2), \quad (7)$$

for the $x$-component, and similarly for the $y$-component

$$\frac{d}{ds}(K_1^1 E_1^1 + K_2^2 E_2^2) = -\frac{2\pi i}{c} (\eta_1^1 K_1^1 E_1^1 + \eta_2^2 K_2^2 E_2^2). \quad (8)$$

With the use of Equation (6), one then finds

$$\frac{d}{ds}(E_1^1) + \exp\left(-\int_0^\xi \Delta k ds\right) \frac{d}{ds}(E_2^2) = 0,$$

$$\frac{d}{ds}(K_1^1 E_1^1) + \exp\left(-\int_0^\xi \Delta k ds\right) \frac{d}{ds}(K_2^2 E_2^2) = 0, \quad (9)$$

where $\Delta k \equiv -(2\pi i/c)(\eta_1 - \eta_2)$ is the phase difference between the two characteristic waves.

Substituting the expression for $d/ds E_1$ from the first part into the second part of Equation (9) yields

$$\frac{d}{ds} E_1^2 + \frac{d}{ds} \frac{K_1^2}{K_1^2 - K_1^1} \frac{E_1^1}{E_1^1} = -\frac{d}{ds} \frac{K_1^1}{K_1^2 - K_1^1} \exp\left(\int_0^\xi \Delta k ds\right), \quad (10)$$

which can be rewritten as

$$\frac{d}{ds} \left[ E_1^2 \exp\left(\int_0^\xi \frac{dK_1^1}{ds} \frac{ds}{(K_1^2 - K_1^1)}\right) \right] = -\frac{d}{ds} \frac{K_1^1}{K_1^2 - K_1^1} \exp\left(\int_0^\xi \frac{dK_2^2}{ds} \frac{ds}{(K_2^2 - K_1^1)}\right) \times \frac{1}{(K_2^2 - K_1^1)} \exp\left(\int_0^\xi \Delta k ds\right). \quad (11)$$

The complementary equation is obtained by instead substituting $(d/ds) E_2$ from the first part into the second part of Equation (9). It is seen that the corresponding equation can be obtained directly from Equation (11) by interchanging 1 and 2 (i.e., $1 \leftrightarrow 2$) and letting $\Delta k \rightarrow -\Delta k$.

The coupling between the characteristic waves is normally expressed in terms of the amplitudes of the WKB-approximation, which are given by

$$E_{WKB}^{1,2} = E_{1,2}^1 \exp\left(\int_0^\xi \frac{dK_{1,2}^1}{ds} \frac{ds}{(K_{1,2}^1 - K_{1,2}^1)}\right). \quad (12)$$

The standard formulation of the propagation of electromagnetic waves in an inhomogeneous medium is then obtained from Equation (12) and its complement as

$$\frac{dE_{WKB}^1}{ds} = \psi_1^2 E_{WKB}^2 \exp\left(-\int_0^\xi \Delta k ds\right),$$

$$\frac{dE_{WKB}^2}{ds} = \psi_1^1 E_{WKB}^1 \exp\left(\int_0^\xi \Delta k ds\right). \quad (13)$$

where

$$\psi_1^i = \frac{d}{ds} \frac{K_1^i}{K_1^2 - K_1^1} \exp\left(\int_0^\xi \frac{dK_1^i}{ds} \frac{ds}{(K_1^2 - K_1^1)}\right) \frac{ds}{(K_1^2 - K_1^1)} \frac{1}{K_1^1 K_1^2},$$

$$\psi_2^i = \frac{d}{ds} \frac{K_2^i}{K_1^2 - K_1^1} \exp\left(\int_0^\xi \frac{dK_1^i}{ds} \frac{ds}{(K_1^2 - K_1^1)}\right) \frac{ds}{(K_1^2 - K_1^1)} \frac{1}{K_1^1 K_1^2}. \quad (14)$$

are the two coupling parameters.

The relative ease with which Equations (13) and (14) are derived does not only depend on the starting point. It may also be noticed that no higher-order terms in $k\lambda$ occur in the derivation, i.e., all higher-order terms disappear when starting from Equation (1) rather than the wave equation. Furthermore, the calculations are made less cumbersome by a functionally simpler “ansatz” (Equation (6) instead of the WKB-approximation).

Although Equation (13) explicitly shows how the inhomogeneities couple the characteristic waves, the underlying physics can be made more transparent by introducing

$$E_{1,2} = E_{1,2}^1 \exp\left(\pm \int_0^\xi \frac{\Delta k}{2} ds\right) \quad (15)$$

so that Equation (13) can be written

$$\frac{dE_1^1}{ds} - \frac{\Delta k}{2} E_1^1 = \psi_1^2 E_2^2,$$

$$\frac{dE_2^1}{ds} + \frac{\Delta k}{2} E_2^1 = \psi_1^1 E_1^1. \quad (16)$$

The solution to Equation (1) is then

$$E_1^1 = \tilde{E}_1^1 \exp\left(\int_0^\xi \frac{dK_1^1}{ds} \frac{ds}{(K_1^2 - K_1^1)} \exp\left(-\int_0^\xi \frac{\kappa}{2} ds\right)\right) \quad (17)$$

$$E_2^1 = \tilde{E}_2^1 \exp\left(\int_0^\xi \frac{dK_2^1}{ds} \frac{ds}{(K_1^2 - K_1^1)} \exp\left(-\int_0^\xi \frac{\kappa}{2} ds\right)\right).$$

It is directly seen from Equation (16) that the coupling between the characteristic waves is determined by the ratio $|\psi_{1,2}|/|\Delta k|$. The limit $|\psi_{1,2}| \gg |\Delta k|$ implies that the coupling is so strong that the plasma properties vary more rapidly along a ray path than does the relative phase between the characteristic waves. Effectively, then, the plasma is isotropic, and the polarization stays roughly constant. Apart from this limit, the emerging polarization is due to an interplay between the adiabatically changing properties of the characteristic waves (i.e., the WKB-approximation) and their coupling. In order to highlight the effects of the inhomogeneities, the result of this interplay will be presented as deviations from that of a homogeneous medium.

The interaction between these two independent effects may lead to the conclusion that the polarization is only rarely seriously affected by inhomogeneities. However, it is important to note that the relative change of a small quantity can be significant over a much wider range of conditions. For
3. Solution to the Transport Equation

Numerical solutions to Equation (13) were discussed in Björnsson (1990). Since the aim of the present paper is to bring forth the underlying physics governing the effects of inhomogeneities, a different approach is followed below. Two limiting situations will be considered. The first assumes $\phi = \text{constant}$. This makes it possible to choose $K^3 = -K^2$ (see Equation (4)), which implies $\Psi^2 = \Psi^4 = (1/2)(d \ln K^{1/2}/ds)$ (Equation (14)). In the general case, $\phi$ also varies. When the transport effects are dominated by variations in $\phi$, it is shown that they are well described by $\Psi^2 = -\Psi^2$ over a limited range in $\phi$.

3.1. Magnetic Field with a Constant Azimuthal Angle

This case was discussed in some detail in Björnsson (1990). The focus in this section is therefore limited to its generic properties and how inhomogeneities may modify the conclusions drawn from the homogeneous solution. In order to simplify the notation, $K = K^2$ and $\Psi = \Psi^2$ will be used. The solution to Equation (1) can then be written

$$ E_x = E_x^1 + E_x^2 = (E^1 + E^2) \sqrt{\frac{K^2}{K}} \exp \left( - \int_0^s \frac{K}{2} \frac{d\bar{s}}{d\bar{s}} \right) $$

$$ E_y = E_y^1 + E_y^2 = -(E^1 - E^2) \sqrt{\frac{K^2}{K}} \exp \left( - \int_0^s \frac{K}{2} \frac{d\bar{s}}{d\bar{s}} \right), $$

where $K_0$ is the value of $K$ at $s = 0$. Introducing, $X \equiv E^1 + E^2$ and $Y \equiv E^1 - E^2$ together with $\alpha = \Delta k/2 \Psi$ and $d\chi = \Psi \, ds$, Equation (16) can be written

$$ \frac{dX}{d\chi} = X + \alpha Y $$

$$ \frac{dY}{d\chi} = \alpha X - Y. $$

With $\alpha = \text{constant}$, Equation (19) is analogous to the equation for propagation in a homogeneous medium and can be solved in a similar way. This is done in Appendix A, where it is shown that

$$ E_x = \frac{I_0}{2(1 + q_o)} \left( 1 + q_o - \alpha \frac{\sinh (\beta \chi)}{\beta} \right) $$

$$ + (1 + q_o) \cosh (\beta \chi) \exp (-\chi - \tau / 2) $$

$$ E_y = -K_0 \frac{I_0}{2(1 + q_o)} \left( \alpha (1 + q_o) + \sigma_o \frac{\sinh (\beta \chi)}{\beta} \right) $$

$$ - \sigma_o \cosh (\beta \chi) \exp (\chi - \tau / 2). $$

The emitted emission at $s = 0$ is assumed to be 100% polarized and described by the Stokes parameters $I_o$, $Q_o$, $U_o$, and $V_o$. Here, $q_o = Q_o/K_0$, $u_o = U_o/I_o$, $v_o = V_o/I_o$, so that $q_o^2 + u_o^2 + v_o^2 = 1$. Furthermore, $\beta \equiv \sqrt{1 + \alpha^2}$, $\sigma_o \equiv (u_o - i v_o)/K_0$, $\chi = \int_0^s \Psi d\bar{s} = (\ln K/K_0)/2$ and $\tau = \int_0^s d\bar{s}$ is the optical depth along the ray path. With $\alpha$ assumed to be a constant, it can be expressed as $\alpha = \delta k\tau / 2\chi$, where $\delta k\tau = \int_0^s \Delta k d\bar{s}$. It is also shown in Appendix A that the approximation $\alpha = \text{constant}$ should be a good one as long as $(1 + \alpha^2)^{-1}(d \ln \alpha / d\chi) = 2(1 + \alpha^2)^{-1}(d\alpha / d\chi) < 1$.

With

$$ \Psi = \frac{1}{2} \frac{d \ln K}{ds} = -\frac{1}{2} \frac{1}{(1 - \rho^2)} \frac{d\rho}{ds}, $$

it is seen that a necessary condition for inhomogeneities to produce large values of $\chi$ (i.e., $|\chi| \approx 1$) is that $|\rho| \approx 1$. As discussed in Appendix C, this, in turn, implies large values for the circular polarization in a homogeneous source. However, the observed values in compact radio sources are usually quite small ($\lesssim 1\%$). It is argued in Section 5.2 that this low value is unlikely to be the result of large-scale cancellation due to an inhomogeneous source structure. If so, one may conclude that $|\chi|$ is substantially smaller than unity and, hence, that either $|K| \ll 1$ (i.e., $|\rho| < 1$, nearly linear characteristic waves) or $|K| \ll 1$ (i.e., $|\rho| \gg 1$, nearly circular characteristic waves).

In order to estimate the importance of inhomogeneities, it is instructive to expand Equation (20) to lowest order in $|\beta \chi|$, and this yields

$$ E_x = \frac{I_0}{2(1 + q_o)} \left( 1 + q_o - \frac{\delta k\tau}{2}(1 - \chi) \exp (-\tau / 2) \right) $$

$$ \times \left( 1 + q_o - \sigma_o \frac{\delta k\tau}{2}(1 - \chi) \exp (-\tau / 2) \right), $$

with $|\chi| \ll 1$, the limit $|\beta \chi| \approx 1$ corresponds to $|\alpha| \approx |\chi|^{-1} \approx 1$ and $|\delta k\tau| \approx 1$. It should be noted that the first-order terms in $\chi$ (i.e., $u_o \chi$) have canceled. Equation (22) is a valid approximation for all values of $|\alpha|$, since $|\alpha| < 1$ implies $\beta \approx 1$ and $|\alpha| > 1$ leads to $\chi \delta k\tau / 2 = \alpha^2$ so that $|\chi|^2$-terms can be neglected in this limit.

Since $2E_x E_y^* \equiv U + iV$, where $^*$ denotes complex conjugate, it is shown in Appendix A that Equation (22) leads to

$$ V = I_0 \nu_0 - \xi_\nu \tau - q_o \xi_\nu \tau - \xi_\nu \chi \tau + \xi_\nu \chi \tau $$

$$ - q_o (\xi_\nu \tau + \xi_\nu \chi \tau), $$

which gives the circular polarization in the limit $|\delta k\tau| < 1$. The subscripts “r” and “i” are used to denote the real and imaginary parts, respectively, of a quantity. Furthermore, the various $\xi$-parameters in Equation (23) are now quantities integrated over the ray path, so that $\xi \tau$ stands for $\int_0^s \xi d\bar{s}$. With this redefinition of the $\xi$-parameters, the polarization can be considered to
consist of one “homogeneous” part and one due to the inhomogeneities (i.e., $\chi$). For synchrotron radiation, $|\xi_\nu| \ll 1$ and $q_\nu = 0$. Although the homogeneous terms contributing to the circular polarization are $\sim |\delta k|$, it is seen from Equation (23) that cancellation occurs, and their net result is only $|\xi_\nu|$ (see Björnsson 2019, for a more detailed discussion).

A similar cancellation also takes place for the second-order inhomogeneous terms. For $|\rho| \ll 1$, the inhomogeneous terms are $\sim |\chi|$. Writing $K = -1 + \Delta K$, with $|\Delta K| \ll 1$, one finds $\chi = (\Delta K_0 - \Delta K)/2$. It was shown in Björnsson (2019) that locally $-\xi_\nu \Delta K_0 + \xi_\nu \Delta K = \xi_\nu$. Hence, the inhomogeneous terms in Equation (23) correspond, roughly, to the variations of $\xi_\nu$ along the ray path (i.e., a very small number). Furthermore, in the limit $|\rho| \gg 1$, $|\xi_\nu| \gg 1$ and dominates all of the other $\xi$-parameters. Since Equation (23) is valid for $|\tau| \lesssim |\xi_\nu|^{-1}$, one finds, in this case, that the contribution from the inhomogeneities are $\sim |\chi|/|\xi_\nu|$. This shows that for frequencies such that $|\tau| \lesssim |\xi_\nu|^{-1}$, transport effects are likely to only marginally affect the emerging circular polarization (i.e., its value is given by $\sim v_\nu$). This is true whether or not the plasma is inhomogeneous.

As shown below, these cancellations do not occur for $|\delta k|\tau \gtrsim 1$; hence, one expects a rapid increase in the circular polarization for frequencies corresponding to the range where $\tau \sim |\delta k|^{-1}$.

Another useful expression for the circular polarization can be obtained from Equation (20) for large values of $|\alpha|$. Expansion to first order in $|\alpha|^{-1}$ yields (see Appendix A)

$$
U + iV = I_\nu \exp(-\tau - 2i\chi_i) \times \left[ u_{\text{hom}} + iv_{\text{hom}} + \frac{K^*_i}{|\alpha|^2} \right. \\
\times \left\{ \frac{\alpha(1 - q_\nu)}{|\alpha|^2} - c^i(1 + q_\nu) \right\} \sinh(\delta k\tau/2)^2 \\
+ \left. \frac{2i(u_{\text{hom}} + iv_{\text{hom}})}{|\alpha|^2} \left( c^i(1 + q_\nu) \sinh(\delta k\tau/2)cosh^i(\delta k\tau/2) \right) \right].
$$  

Here, $u_{\text{hom}} + iv_{\text{hom}}$ is the part corresponding to the homogeneous case with an optical depth $\tau$. The different effects of $\chi$ (the WKB-approximation) and $\alpha$ (the coupling between the characteristic waves) are clearly seen. While $\chi$ gives rise to circular polarization through a process similar to conversion of linear polarization in a homogeneous plasma, $\alpha$ accounts for the circular polarization induced by the interaction with the local medium. One may notice two things from Equation (24):

1. It describes the emerging polarization of a light ray with optical depth $\tau$ and initial values $I_\nu$, $Q_\nu$, $U_\nu$, $V_\nu$, and $K_\nu$. The polarization of the total radiation is obtained by integrating along the line of sight, i.e., over $\tau$ and taking the variations of the initial values into account.
2. It is a linear function of the initial Stokes parameters. Since Stokes parameters are additive, Equation (24) is valid also for partially polarized light rays. This is generally true (Björnsson 1988).

The range of validity of Equation (24) overlaps that of Equation (22) ($1 < |\alpha| < |\chi|^{-1}$); hence, the latter can be obtained by expanding the former for $|\delta k|\tau < 1$. In the opposite limit (i.e., $|\delta k|\tau > 1$), the terms $\alpha \cos(\delta k\tau)$ and $\alpha \sin(\delta k\tau)$ are unlikely to be important, since integration along the line of sight tends to cancel out their contributions. With this simplification, Equation (24) can be written to first order in $|\chi|$:

$$
U + iV = I_\nu \exp(-\tau) \times \left[ u_{\text{hom}} + iv_{\text{hom}} - 2i\chi_i u_{\text{hom}} \right. \\
\times \left\{ \frac{\alpha(1 - q_\nu)}{|\alpha|^2} \right\} \sinh(\delta k\tau/2)^2 \\
+ \left. \frac{2i(u_{\text{hom}} + iv_{\text{hom}})}{|\alpha|^2} \left( c^i(1 + q_\nu) \sinh(\delta k\tau/2)cosh^i(\delta k\tau/2) \right) \right].
$$  

where, also, $|\alpha| \ll |\alpha|$ together with $|K_\nu| = 1$ has been used. It may be noted that the latter approximation cannot be used in the limit $|\delta k|\tau < 1$, since all of the first-order terms cancel, and the expression for $|K_\nu|^2 - 1$ is important in order to get the correct second-order terms. With

$$
\frac{\alpha}{|\alpha|^2} = \frac{2\delta k\chi^*}{\tau|\delta k|^2},
$$  

the various contributions to the circular polarization can be directly estimated from Equation (25).

Consider first the case $|\rho| \ll 1$ (nearly linear characteristic waves), for which $u_{\text{hom}}(\tau) = u_\nu \cos(\delta k\tau) + q_\nu \sin(\delta k\tau)$, $K_\nu = -1$ and $\delta k = -\xi_\nu + i\xi_\nu$ (Björnsson 2019). Since $u_\nu \approx 1$, it is seen from Equation (26) that the WKB-term $\chi_i u_{\text{hom}}$ in Equation (25) is a factor $\sim |\delta k|\tau (\sim 1)$ larger than the coupling terms; hence,

$$
V = I_\nu \exp(-\tau)(v_{\text{hom}} - 2\chi_i[u_\nu \cos(\delta k\tau) + \sin(\delta k\tau)]).
$$  

Furthermore, $v_{\text{hom}} = K_{\nu,i}[u_\nu \cos(\delta k\tau) + \sin(\delta k\tau)]$ and $2\chi_i = K_{\nu,i} - K_\nu$, which give the simple expression

$$
V = I_\nu \exp(-\tau)K_\nu[u_\nu \cos(\delta k\tau) + \sin(\delta k\tau)],
$$  

or $V = (K_{\nu,i}V_{\text{hom}})$ and $K_\nu$ is the polarization of the characteristic waves appropriate for the surface and, hence, is independent of $\tau$ for a given line of sight. One may note that in this limit, none of the terms $\propto |\alpha|^{-1}$ in Equation (25) contributes to the circular polarization. Hence, this result corresponds to the WKB-approximation discussed in Section 2. Since $K_i \approx \hat{\xi}_V/\hat{\xi}_U \propto \nu/B$, where $B$ is the strength of the magnetic field, the inhomogeneities change the value of the circular polarization by a factor $B_\nu/B$. The main conclusions for nearly linear characteristic waves are then: (1) The inhomogeneities can substantially affect the circular polarization at frequencies for which $|\delta k|\tau \gtrsim 1$. (2) The amplitude of the change is independent of frequency, and hence, the frequency distribution of the circular polarization remains the same as for a homogeneous source. Both of these features can be seen explicitly in the numerical solutions to the transport equation shown in Björnsson (1990).

For $|\rho| \gg 1$ (nearly circular characteristic waves), $u_{\text{hom}}(\tau) = u_\nu \cos(\delta k\tau) + q_\nu \sin(\delta k\tau)$, $K_\nu = 1$ and $\delta k = -(\xi_\nu + \xi_\nu \hat{\xi}_U/\hat{\xi}_V) + i\xi_\nu$. As already discussed, integration
along the line of sight is expected to give $u_{\text{hom}} \approx 0$ so that

$$V = I_0 \exp(-\tau) \times \left[ v_{\text{hom}} + \frac{2u_0(\delta k_t \chi_t - \delta k_1 \chi_1) \sinh(\delta k_t \tau)}{|\delta k|} + \frac{2q_0(\delta k_1 \chi_1 + \delta k_1 \chi_1) \cosh(\delta k_1 \tau)}{|\delta k\tau|} \right].$$  \hfill (29)

Since $\hat{\xi}_1 \gg 1$ and $|\delta k_t| \ll 1$, the magnitude of the two inhomogeneous terms are roughly $|u_0| \delta k_t / |\delta k_1|$ and $|q_0| \chi_1 / (|\delta k_1| \tau)$. As compared to $|v_{\text{hom}}| \approx u_0 \hat{\xi}_1 / \hat{\xi}$, the first term is smaller by a factor $|\xi_1 / \chi_t| \gg 1$, while the second one could be of the same order of magnitude. However, for synchrotron radiation, $q_0 = 0$. Hence, for nearly circular characteristic waves, the emerging polarization is given to a good approximation by the homogeneous value, and the inhomogeneities will affect the result only marginally.

### 3.2. Magnetic Field with a Varying Azimuthal Angle

When the azimuthal angle varies along the ray path, the expressions for $K^1$ and $K^2$ lead to a more complex relation between $\Psi^1$ and $\Psi^2$. This is caused by the term $\sin(2\varphi)$ in Equation (4). In general, under such conditions, no simple solutions can be found for the transport equation. However, as discussed in Section 3.1, the low value of the circular polarization observed in compact radio sources suggests that the relative variations of $K^1$ and $K^2$ along the ray path is rather small. Defining

$$K^{1,2} = K_0 (\mp 1 + 2\chi^{1,2})$$  \hfill (30)

and assuming $|\chi^{1,2}| \ll 1$, one finds to lowest order that

$$\chi^{1,2} = \chi_c \pm \chi_i = \frac{\sin(2\varphi)}{2\sqrt{1 - \rho^2}} \pm \left\{ \frac{\Delta \rho}{2(1 - \rho^2)} - \frac{1 - \cos(2\varphi)}{2(1 + \rho^2)} \right\}.$$  \hfill (31)

Here, $\rho = \rho_0 + \Delta \rho$ and, again, $K_0 \equiv K_0^2 = -\sqrt{(1 - \rho^2)/(1 + \rho^2)}$. When, $|\chi_c| \ll |\chi_i|$, $\Psi^1 = \Psi^2$, while $|\chi_c| \gg |\chi_i|$ leads to $\Psi^1 = -\Psi^2$. Since the former situation is treated in Section 3.1, the focus in this section is on the latter case.

From Equation (30), one deduces

$$\exp \int_0^\tau \frac{dK^{1,2}}{ds} (K^{2,1} - K^{1,2}) = 1 \pm \chi_c.$$  \hfill (32)

Equation (17) then leads to

$$E_x = E_0^x + E_0^2 = (\hat{E}_1 + \hat{E}_2 + \chi_c (\hat{E}_1 + \hat{E}_2)) \exp(-\tau/2)$$

$$E_y = K_0^1 E_0^1 + K_0^2 E_0^2 = -K_0 (\hat{E}_1 - \hat{E}_2 - \chi_c (\hat{E}_1 + \hat{E}_2)) \exp(-\tau/2).$$  \hfill (33)

Furthermore, the coupling constants (see Equation (14)) are given by $\Psi^{1,2} = \mp d\chi_c / ds$. In order to emphasize the similarities to Section 3.1, the same notation will be used; hence, $\chi \equiv \chi_c$ and $\alpha = \Delta \chi / (2d\chi / ds)$. Furthermore, with $\alpha = \text{constant}$, it can be expressed as $\alpha = \delta k \tau / 2\chi$. The equations for $X$ and $Y$ are then

$$\frac{dX}{d\chi} = (\alpha - 1)Y$$

$$\frac{dY}{d\chi} = (\alpha + 1)X.$$

(34)

These equations resemble those in Section 3.1 and can be solved in a similar manner. This is done in Appendix B. The solution is in this case

$$E_x = \sqrt{\frac{I_0}{2(1 + q_o)}} \times \left\{ (1 + q_o) \sinh(\beta \chi) + (1 - \alpha\sigma_o \sinh(\beta \chi)) \right\} \exp(-\tau/2)$$

$$E_y = \sqrt{\frac{I_0}{2(1 + q_o)}} \times \left\{ \sigma_o \sinh(\beta \chi) - (1 + \alpha)(1 + q_o) \right\} \exp(-\tau/2),$$  \hfill (35)

where, now, $\beta = \sqrt{\alpha^2 - 1}$.

Expansion of Equation (35) to lowest order in $|\beta \chi|$ gives

$$E_x = \sqrt{\frac{I_0}{2(1 + q_o)}} \times \left\{ 1 + q_o + \frac{\delta k \tau}{2} \sinh(\beta \chi) \right\} \exp(-\tau/2)$$

$$E_y = \sqrt{\frac{I_0}{2(1 + q_o)}} \times \left\{ \sigma_o - \frac{\delta k \tau}{2} \sinh(\beta \chi) \right\} \exp(-\tau/2),$$  \hfill (36)

which leads to the simple expression

$$U + iV \equiv 2E_x E_y^* = I_0 \exp(-\tau) \times \left[ u_{\text{hom}} + iu_{\text{hom}} + (u_o + iv_o)ir(\delta k \chi) \right].$$  \hfill (37)

Furthermore, since $\delta k \chi = \delta k \sin(2\varphi)/(2\sqrt{1 - \rho^2})$ and $\varphi$ is real, one finds from Equation (37)

$$V = I_0 \exp(-\tau) |u_{\text{hom}} + u_o \hat{\xi}_1 \tau \sin(2\varphi)/2|,$$  \hfill (38)

which is valid for both $|\rho| \ll 1$ and $|\rho| \gg 1$. Similarly to the situation in Section 3.1, the first-order terms vanish, and the impact of the inhomogeneities is given by the second-order term $\tau \delta k \chi$. However, in contrast to the $\Psi^1 = \Psi^2$ case, the inhomogeneous term here cannot be neglected.

Equation (38) is valid for $|\beta \chi| \approx |\delta k \tau| < 1$. For $|\rho| \ll 1$, this implies that in the transition region (i.e., $\hat{\xi}_1 \tau \sim 1$), the inhomogeneous term approaches the value $u_o \varphi$. This should be compared to the corresponding value of the homogeneous term,
which is \( \sim u_0 \frac{\hat{\xi}_V}{\xi_U} \). Since \( |\hat{\xi}_V/\xi_U| \ll 1 \) in this limit, the inhomogeneities could dominate the circular polarization. Likewise, for \(|\rho| \gg 1\), the value in the transition region (i.e., \( |\hat{\xi}| \tau \sim 1 \)) is \( \sim u_0 (\xi_U/\xi_U)(\sin(2\varphi)/2) \), which should be compared to the corresponding expression for the homogeneous part, \( \sim u_0 \xi_U \). Since \(|\varphi| \sim 1\) is allowed in this limit, inhomogeneities may significantly affect the circular polarization in the transition region also in this case, although the change in \( \varphi \) must be much larger than for \(|\rho| \ll 1\).

In analogy with Section 3.1, it is useful to expand Equation (35) to first order in \(|\alpha|^{-1}\), which yields (see Appendix B)

\[
U + iV = I_0 \exp(-\tau) \left[ u_{\text{hom}} + i\nu_{\text{hom}} + 2i(u_0 + i\nu_0) \times \{ \chi \cosh(\hat{\xi}k_T/2)\sinh(\delta k_T/2) \}ight.
\]

\[
+ 2i\sigma_0 K^*_0 \left\{ \frac{\alpha}{|\alpha|^2} \sinh(\delta k_T/2)^2 \right\}
\]

\[
+ \chi \cosh(\hat{\xi}k_T/2) \sinh(\delta k_T/2) \left\{ -2K^*_0 \{ \chi \cosh(\delta k_T) \}ight.
\]

\[
+ \left\{ \frac{\alpha \cosh(\delta k_T/2) \sinh(\delta k_T/2)}{|\alpha|^2} \right\}
\]

\[
\left. + 2q_0 K^*_0 \{ \chi \cosh(\delta k_T) \} \right\} \right]
\]

\[
\frac{- \{ \alpha \cosh(\delta k_T/2) \sinh(\delta k_T/2) \} \left\{ \frac{\alpha}{|\alpha|^2} \sinh(\delta k_T) \right\} \right] \right].
\]

(39)

Equation (39) is written so as to highlight the cancellations that occur for \(|\delta k| \tau < 1\); namely, the last three inhomogeneous terms vanish to second order (i.e., \(|\delta k\tau|\alpha|\)), so that the result in Equation (38) corresponds to the first term only. On the other hand, all of the terms contribute in the limit \(|\delta k| \tau > 1\),

\[
U + iV = I_0 \exp(-\tau) \left[ u_{\text{hom}} + i\nu_{\text{hom}} + 2i(u_0 + i\nu_0) \right.
\]

\[
\times \{ \chi \cosh(\hat{\xi}k_T/2)\sinh(\delta k_T/2) \}
\]

\[
+ 2i\sigma_0 K^*_0 \left\{ \frac{\alpha}{|\alpha|^2} \sinh(\delta k_T/2)^2 \right\}
\]

\[
+ \chi \cosh(\hat{\xi}k_T/2) \sinh(\delta k_T/2) \left\{ -2K^*_0 \{ \chi \cosh(\delta k_T) \}ight.
\]

\[
+ \left\{ \frac{\alpha \cosh(\delta k_T/2) \sinh(\delta k_T/2)}{|\alpha|^2} \right\}
\]

\[
\left. + 2q_0 K^*_0 \{ \chi \cosh(\delta k_T) \} \right\} \right]
\]

\[
\frac{- \{ \alpha \cosh(\delta k_T/2) \sinh(\delta k_T/2) \} \left\{ \frac{\alpha}{|\alpha|^2} \sinh(\delta k_T) \right\} \right] \right].
\]

(40)

With the use of Equation (26), this can be written

\[
V = I_0 \exp(-\tau) \left[ V_{\text{hom}} + \frac{2\varphi \hat{\xi}_U}{\tau(\xi^2 + \hat{\xi}^2)} \times \{ u_0 \cosh(\xi_T \tau) - \sinh(\xi_T \tau) \} \right].
\]

(42)

where \( \delta k = -\xi_T + i\hat{\xi}_U \) has been used.

Likewise, for \(|\rho| \gg 1\) \((K_\alpha = i, |\hat{\xi}| \tau \gg 1)\) and \(|\xi_U| \ll 1\),

\[
V = I_0 \exp(-\tau) \left[ V_{\text{hom}} - u_0 \frac{\alpha_1}{|\alpha|^2} \cosh(\delta k_T) \right].
\]

(43)

Again, Equation (26) can be used to find

\[
V = I_0 \exp(-\tau) \left[ V_{\text{hom}} - u_0 \frac{\sin(2\varphi) \hat{\xi}_U}{\tau\hat{\xi}_V^2} \cosh(\xi_T \hat{\xi}_U \tau / \hat{\xi}_V) \right].
\]

(44)

where \( \delta k = -\xi_T \hat{\xi}_U/\xi_U + i\hat{\xi}_V \) and \( \delta k/\delta k \approx |\hat{\xi}_V|^{-2} \ll 1 \) have been used.

It is seen that in the transition region (i.e., \(|\delta k| \tau \sim 1\)), the inhomogeneities induce a circular polarization \( \sim u_0 |\alpha| \) (Equation 37). As discussed above, this may correspond to a significant fraction of the homogeneous value. When this is the case, the transition between the regimes \(|\delta k| \tau < 1\) and \(|\delta k| \tau > 1\) is smoother than the more abrupt one expected for \( \Psi = \Psi^2 \) (see discussion in Section 3.1). Even so, since the frequency dependence of \( \hat{\xi}_U \) is rather weak, the increase in circular polarization toward lower frequencies is still rather steep, \( V \propto \tau \propto \nu^{-3} \) (see Equation 38). It should also be noted that the frequency range over which the circular polarization is enhanced by inhomogeneities is rather narrow, since it declines as \( V \propto \tau^{-3} \) for \(|\delta k| \tau > 1\). In addition, for \(|\rho| \ll 1\), this frequency range is further narrowed down by the factor that the circular polarization changes sign not too far from the transition region (see Equation 42), while for \(|\rho| \gg 1\), it is somewhat broadened by the frequency dependence of \( \hat{\xi}_V \) \((\tau\hat{\xi}_V^2 \propto \nu^{-1} \); see Equation 44). The detailed spectral properties of the circular polarization can be seen in Hodge (1982), who solved the transport equation numerically for \( \rho = 0 \).

4. Comparison with a Piecewise Constant Approximation

The validity of the approximation used in Section 3 constrains the allowed variation of \( \alpha \) (see Appendix A). One may note that this \( \alpha = \text{constant} \) description has much wider applicability than the WKB-approximation discussed in Section 2, since the latter corresponds to \( \alpha \to \infty \) (i.e., no coupling between the characteristic waves). The smoothly varying inhomogeneities, assumed in the \( \alpha = \text{constant} \) description, can be contrasted by the piecewise constant approximation in which inhomogeneities are modeled as instantaneous changes of the plasma properties interspersed by homogeneous regions (e.g., Ruszkowski & Begelman 2002; MacDonald & Marscher 2018). The instantaneous changes do not affect the polarization, since they correspond to \( \alpha \to 0 \). These two approximations are each others opposites, i.e.,
mutually exclusive. It is, therefore, useful to compare their results for a given situation. This will give an estimate of the sensitivity of the emerging polarization to the approximation used to calculate it.

Consider a region of length \( s \), where the azimuthal angle of the magnetic field changes by \( \varphi \). Assume further that \( d\phi/ds \) is constant, which implies \( \varphi = s\alpha/\kappa \). Let \( s \) be small enough so that \( \Delta k \approx 1 \) and \( s_2 \approx 1 \). Divide the region in two, with \( s = s_1 + s_2 \) and \( \varphi = \varphi_1 + \varphi_2 \). For a light ray emitted at \( s = 0 \), one finds from Equation (38) that at the end of region 1,

\[
v_1^\alpha = v_{\text{hom}} + u_0\hat{\kappa}\kappa\frac{d\phi}{ds}s_1^2,
\]

where, for convenience, \( v_1^\alpha \equiv V/(I_0 \exp(-\tau)) \) has been introduced. For synchrotron radiation, \( q_0 \) can be chosen at the beginning of region 1. However, at the beginning of region 2, the rotation of the magnetic field implies \( q_2 = -2\varphi_1\mu_0 \), while, to first order in \( \varphi_2 \), \( u_0 \) remains the same. With the use of Equation (38) a second time, the circular polarization at the end of region 2 can be written

\[
v_2^\alpha = v_1^\alpha + u_0\hat{\kappa}\kappa\frac{d\phi}{ds}s_2^2 - q_0\hat{\kappa}\kappa s_2
\]

\[
= v_{\text{hom}} + u_0\hat{\kappa}\kappa\frac{d\phi}{ds}(s_1 + s_2)^2,
\]

where the term \( \propto q_0 \) comes from the homogeneous solution (see Equation 23). Hence, the expression for \( v_{\text{hom}} \) in Equation (46) is that for a homogeneous source with \( q_0 = 0 \), just as in Equation (45). It is seen that the circular polarization is additive, as it should be, since a given region can be divided up in any number of subregions without affecting the resulting value of \( V \).

If the same region is approximated by a piecewise constant medium, there will be two abrupt changes in the azimuthal angle of the magnetic field, one at the beginning of region 1 and a second one at the beginning of region 2, given by \( \varphi_1 \) and \( \varphi_2 \), respectively. The equations corresponding to Equations (45) and (46) are

\[
v_1^{pw} = v_{\text{hom}} + 2\varphi_1u_0\hat{\kappa}\kappa s_1,
\]

and

\[
v_2^{pw} = v_1^{pw} + 2(\varphi_1 + \varphi_2)u_0\hat{\kappa}\kappa s_2
\]

\[
= v_{\text{hom}} + 2u_0\hat{\kappa}\kappa\frac{d\phi}{ds}(s_1 + s_2)^2 - s_1s_2.
\]

It is seen that the transport-induced circular polarization calculated with the use of the two different approximations differs by a factor

\[
\frac{v_2^{pw} - v_{\text{hom}}}{v_2^{pw} - v_{\text{hom}}} = 2\left(1 - \frac{s_1s_2}{(s_1 + s_2)^2}\right)
\]

Hence, depending on how the division of the region is done, the piecewise constant approximation can give circular polarization up to a factor of two larger than the \( \alpha \)-approximation. Likewise, in situations where the main changes of the magnetic field are so abrupt that the \( \alpha \)-approximation is not applicable, its use would give an artificially low circular polarization.

Another aspect of the two approximations is how they account for variations in the sign of \( d\phi/ds \) along a ray path. In order to illustrate this, let the sign of \( d\phi/ds \) change between region 1 and 2 (i.e., \( s_2 \approx -s_2 \) in the equations above). This lowers the value of \( v_2^\circ \). The reason is that the value of \( \varphi \) in the \( \alpha \)-approximation is the integrated change of the azimuthal angle of the magnetic field along a ray path; e.g., \( s_2 = s_1 \) gives \( v_2^\circ - v_{\text{hom}} = 0 \). Hence, the detailed properties of the medium along a ray path can vary substantially without affecting the emerging polarization. This is not so for the piecewise constant approximation; for example, with \( s_2 = s_1 \), one finds from Equation (48)

\[
v_2^{pw} = v_{\text{hom}} + 2u_0\hat{\kappa}\kappa\frac{d\phi}{ds}s_1^2.
\]

This shows explicitly that in this case, the circular polarization is also sensitive to the detailed properties of the medium.

The polarization of the emerging radiation is obtained by adding up all of the light rays along the line of sight. Hence, the circular polarization results from a combination of an integration over the initial conditions of the light rays and their propagation through the medium. Consider, for example, a turbulent medium in which the sign of \( d\phi/ds \) changes repeatedly along a sight line. Although the circular polarization for a given light ray is likely to be quite different depending on whether the \( \alpha \)-approximation or the piecewise constant approximation is used, both approximations are \( \propto d\phi/ds \) so the statistical properties of the medium will affect them in a similar manner. The relative importance of the two effects depends on the detailed properties of the medium. However, one may expect the larger fluctuations between different light rays in the piecewise constant approximation to give rise to a higher circular polarization than that resulting from the \( \alpha \)-approximation.

5. Discussion

Before addressing the observed polarization of compact radio sources, it is useful to discuss a few general properties of the transport equation for polarized light. Normally, this equation is expressed using the Stokes parameters. Alternatively, it can be written in terms of the electric field. In this latter formulation, the Stokes parameters are then calculated from the solution to the transport equation.

As argued in Björnsson (2019), the interaction between the electromagnetic wave and the plasma is more transparently described in terms of the electric field directly rather than via the Stokes parameters. This is particularly true when the concept of characteristic waves is introduced. In a homogeneous source, these waves propagate independently and allow both a straightforward solution to the transport equation as well as a simple formulation of the full result in terms of the polarization properties of the characteristic waves.

In the inhomogeneous case, there exists a WKB-approximation, which, physically, corresponds to negligible coupling between the characteristic waves. Instead, the polarization of the propagating characteristic waves change in tune with the local properties of the plasma along a given ray path. However, this solution has limited applicability. The general description of the effects of inhomogeneities is instead formulated in terms of the coupling between the characteristic waves.

The standard derivation of the equations accounting for this coupling is rather tedious. Furthermore, it is normally written in
a form that is not so physically transparent. A shorter and more straightforward derivation is presented in Section 2. In addition, the equations can be written in a way so as to highlight the main physical effects. Most importantly, as shown in Section 3, these equations have a constant coupling solution (α = constant) with an applicability much wider than the WKB-approximation; for example, the latter is recovered in the limit of no coupling (i.e., α → ∞). When this α-approximation is valid, the calculation of the emerging polarization is much simplified; instead of solving coupled differential equations, one needs only to integrate over the conditions along a given line of sight.

This integration consists of two parts. The constant transport coefficients in the homogeneous case are substituted by their average values appropriate for a given light path. After this, the total polarization is obtained by integrating over the varying initial conditions along the line of sight. Furthermore, the solution for a given light ray can be represented as the sum of two terms. The first corresponds to the solution for the homogeneous case but with the constant phase difference between the characteristic waves substituted by their average value, while the second one accounts for the varying polarization properties of the characteristic waves. Moreover, it is shown that under a rather wide range of circumstances, the first, “homogeneous” term dominates the resulting polarization.

Another approximation sometimes used is based on the assumption of a piecewise constant medium, in which a given variation of the plasma properties is modeled as an instantaneous change followed by a homogeneous region (Ruszkowski & Begelman 2002; MacDonald & Marscher 2018). This is in contrast to the α-approximation, which relies on smooth variations. The validity of these two approximations do not overlap, and hence, they apply to very different situations. As shown in Section 4, a given change of plasma properties can result in a variation of the value of the circular polarization differing by up to a factor two, depending on which of the approximations is used. Furthermore, while the α-solution is expressed in terms of integrated properties along the ray path only, the result from the piecewise constant approximation is more sensitive to the local properties of the plasma.

5.1. Polarization Properties of Compact Radio Sources

The low value of the circular polarization observed in compact radio sources makes it likely that the characteristic waves are either nearly linearly polarized or nearly circularly polarized. The main aim of the present paper is to use the observed polarization to distinguish between the two. In Björnsson (2019), the polarization properties of a homogeneous synchrotron source were used to argue that the properties of compact radio sources are such that the characteristic waves are nearly circularly polarized. In addition, qualitative arguments were given as to why this may also apply to inhomogeneous sources. Here, a quantitative estimate is done of the effects that inhomogeneities may have on the polarization emerging from a synchrotron source.

In Björnsson (2019), it was shown that the two types of characteristic waves result in very different frequency dependences of the circular polarization. This difference is due to the relative values of the circular and linear birefringence (ξc and ξl, respectively) for the two types. For nearly circular characteristic waves (|ξc/ξl| ≫ 1 and |ξc/ξl| ≪ 1), most of the circular polarization is emitted over a rather wide range of optically thin frequencies (ξc < ξl ≪ 1). In contrast, for nearly linear characteristic waves (|ξc/ξl| ≫ 1 and |ξc/ξl| ≪ 1), the circular polarization is emitted over a rather narrow range of mainly optically thick frequencies (ξc > ξl).

There are two instances that give rise to nearly linear characteristic waves. First, when the lower cutoff in the energy distribution of the relativistic electrons corresponds to synchrotron frequencies close to self-absorption and, second, the presence of a substantial amount of electron–positron pairs. Likewise, nearly circular characteristic waves imply a rather small value for the low-energy cutoff in the energy distribution. For example, consider the spectral region around the synchrotron self-absorption peak, where most of the circularly polarized flux is emitted. Here, ξc ≈ ξl ∼ 10^3(γ^3_\text{min}(1 + 2\hat{\alpha}))^{-1/3}, where γ^3_\text{min} is the lower cutoff in the energy distribution of the relativistic electrons, and \hat{\alpha} is the number of pairs per proton (Appendix C in Björnsson 2019).

It is shown in Section 3 that inhomogeneities affect the emerging circular polarization substantially more for nearly linear as compared to nearly circular characteristic waves. However, the most important point of this paper is that the main influence of the inhomogeneities is restricted to the amplitude of the circular polarization; its frequency dependence is only marginally affected. Hence, the conclusion in Björnsson (2019) that the observed properties of the polarization in compact radio sources are most directly understood as the result of nearly circular characteristic waves, also remains valid in the presence of inhomogeneities. The observations then imply ξc ≈ ξl ∼ 10^2 or γ^3_\text{min}(1 + 2\hat{\alpha}) ∼ 10^2. The degeneracy between γ^3_\text{min} and \hat{\alpha} may be broken by observations at frequencies for which τ ∝ |ξc|^−1. As argued in Section 3, in this frequency range, the observed circular polarization is likely dominated by the emission process itself, which is independent of γ^3_\text{min} but inversely proportional to \hat{\alpha}.

In the POLAMI survey (Thum et al. 2018), a sample of compact radio sources were observed multiple times at 1.3 mm and 3 mm. Except for some periods of increasing flux, the spectral index indicated that the emission was optically thin. Furthermore, the degrees of circular polarization at the two wavelengths were rather similar. This is consistent with nearly circular characteristic waves but hard to reconcile with nearly linear characteristic waves, since the latter are expected to show a steeply rising degree of circular polarization toward longer, optically thin wavelengths. Although it is shown in Section 3.2 that inhomogeneities may smooth this very steep rise for a homogeneous source, it is still hard to make the expected rise (V ∝ ν^−3) compatible with observations.

The degree of circular polarization at lower frequencies, where the flat spectra indicate optically thick emission, is smaller than that observed in the POLAMI survey. Both types of characteristic waves show a sign change of the circular polarization at optically thick frequencies. As discussed in Björnsson (2019), the relative magnitude of this contribution is substantially larger for nearly circular as compared to nearly linear characteristic waves. Since the flat spectra are likely due to an inhomogeneous jet (Blandford & Königl 1979), the emission at a given frequency is that obtained by integrating over a range of optical depths. Hence, the degree of circular polarization can be lowered by contributions from optically thick regions with different signs of the circularly polarized
flux. For nearly linear characteristic waves, this effect is quite small, while for nearly circular characteristic waves, it can be substantial. The observed clear decrease of circular polarization when going from optically thin to thick frequencies favors the presence of nearly circular characteristic waves. This will be discussed in more detail in a forthcoming paper.

When transport effects are important, the choice between nearly linear and nearly circular characteristic waves relies not only on the observed properties of the circular polarization but also on those of the flux and linear polarization. This is so because the polarization of the emerging radiation is determined by the low-energy electrons, which may be different from those giving rise to the bulk of the flux. Since the value of $\xi_{\perp}$ is independent of $\hat{n}$ and varies slowly with $\gamma_{\min}$, it is the value of $\xi_{\parallel}$ that distinguishes between the two types of characteristic waves. In principal then, determination of the amount of Faraday rotation alone would settle the issue. However, for an inhomogeneous source, this is not straightforward.

As discussed in Björnsson (2019), the observed properties of linear polarization in flat spectrum radio sources can be understood as the result of large Faraday depths. The longer timescale of variability for the linear polarization, as compared to the circular polarization, would be due to an emission site further out in the optically thin part of the jet ($\tau \lesssim |\xi_{\parallel}|^{-1}$). At the same time, this would lower the degree of linear polarization. Quantitatively, both of these effects are consistent with observations for $|\xi_{\parallel}| \sim 10^2$.

Large Faraday depths imply different frequency distributions for the circular and linear polarization. Since the circularly polarized flux comes mainly from the region close to the spectral peak, while the linearly polarized flux comes from the optically thin part of the spectrum where $\tau \lesssim |\xi_{\parallel}|^{-1}$, a broad minimum of the linear polarization is expected in the frequency range where the circular polarization peaks. In flat spectrum radio sources, the spectral peak usually occurs at $\sim 100$ GHz. At such large frequencies, the spectral resolution is not yet sufficient to establish the presence of such an anticorrelation between linear and circular polarization. The situation is different for Gigahertz-Peaked-Spectrum sources where the spectrum peaks at $\sim$ few GHz. For such sources, high-quality, multifrequency observations are now possible. A good example is PKS B2126-158 (O’Sullivan et al. 2013), which shows a clear anticorrelation between circular and linear polarization as expected for nearly circular characteristic waves.

5.2. Implications for the Acceleration Process

Numerical calculations based on first principles are now possible for the acceleration of particles. Although limited in scope, they are likely to give realistic insights to the injection of particles and their low-energy distribution. Hence, the observed properties of the circular polarization should provide useful constraints for the results from such PIC-simulations. In order to illustrate this, consider the distribution of electron energies calculated in Petropoulou et al. (2019) from magnetic reconnection. In general, a rising thermal tail is followed by a decreasing, roughly power-law distribution at higher energies, where the peak normally falls in a region around a Lorentz factor $\gamma_p \sim 10$. Jones & Hardee (1979) have shown that the transport coefficients for a relativistic Maxwellian are dominated by the particles around the peak energy, i.e., that the low-energy tail contributes negligibly (see also Björnsson 1990). Hence, the transport coefficients obtained from the energy distributions calculated in Petropoulou et al. (2019) should be approximately those of a power-law distribution with a low-energy cutoff at $\gamma_p$.

It has been argued that observations constrain the number of electron–positron pairs per proton to be around 10 (Ghisellini et al. 2010; Madejski et al. 2016). Together with $\gamma_p \sim 10$, this implies $|\xi_{\parallel}| \sim |\xi_{\perp}| \sim 1$ (see Appendix C in Björnsson 2019). In a homogeneous source, this leads to very high values for the circular polarization (several tens of percent, see Appendix C), which are at least an order of magnitude larger than observed ones.

An inhomogeneous source structure will affect the degree of circular polarization in two different, although related, ways: (1) the initial conditions for the light rays can vary along the line of sight; at the same time, (2) these variations will also influence the effective phase difference between the characteristic waves for a given light ray. However, it is important to note that these effects are not independent but are both induced by the varying plasma properties (see Appendix C). As an example, consider a situation where the component of the magnetic field changes sign repeatedly along the line of sight (see Figure 1). This causes both the linear conversion term $(x\hat{K}\hat{K})$ and the phase difference between the characteristic waves $(x\xi_{\parallel})$ to change sign. Although these sign changes will lower the degree of circular polarization of the emerging radiation, the effective value of $|\xi_{\parallel}|$ is lowered as well.

Hence, invoking varying initial conditions to lower the observed degree of circular polarization in the magnetic reconnection scenario discussed in Petropoulou et al. (2019) is likely to lead to an effective value of $|\xi_{\parallel}|$ substantially below unity. This, however, would be at odds with the conclusion reached above from the observed properties of the polarization, which is most readily understood as being due to a plasma with $|\xi_{\parallel}| \sim 10^2$. Moreover, there is another, independent argument against attributing the low observed value of the circular polarization to large-scale cancellation. The circular polarization varies more rapidly and with larger relative amplitude than either the flux or linear polarization. However, it only rarely changes sign in an individual source (Weiler & de Pater 1983; Komesaroff et al. 1984). To make these observations consistent with the needed large cancellation may require some fine tuning of the source properties.

6. Conclusions

The interaction between a propagating electromagnetic wave and inhomogeneous plasma can be formulated in, at least, two different ways. Normally, it is described in terms of the Stokes parameters, but an equivalent formulation can be made using the electromagnetic field itself. In the latter formulation, the concept of characteristic waves is central. The main results of the present paper are:

1. A shorter and more direct derivation of the equations describing the coupling of the characteristic waves is presented.
2. With constant coupling, these equations have a solution valid under a wide range of circumstances. This makes possible a much more simplified treatment of the effects...
of inhomogeneities on the emerging polarization. In addition, the use of the polarization properties of the characteristic waves allows a transparent formulation of the solution.

(3) The effects of inhomogeneities can be substantial for nearly linear characteristic waves but rather minor for nearly circular characteristic waves. Compared to the circular polarization from a homogeneous source, this affects mainly its magnitude and only marginally its frequency dependence. Hence, inhomogeneities have little effect on the frequency dependence of the circular polarization.

(4) The frequency dependence of the circular polarization differs significantly for plasma properties corresponding to nearly circular and nearly linear characteristic waves. It is argued that the observed polarization properties of compact radio sources fit nicely with nearly circular but are hard to reconcile with nearly linear characteristic waves. This, in turn, constrains the modeling of the acceleration process as well as the presence of electron–positron pairs; for example, some of the currently preferred parameter values do not easily match with observations.

**Appendix A**

**Propagation of a Polarized Light Ray in an Inhomogeneous Medium with a Constant Azimuthal Angle φ**

The appropriate matrix in Equation (19) can be diagonalized to give the eigenvalues \( \beta_\pm = \pm \sqrt{1 + \alpha^2} \). The two characteristic waves can then be written \( X_\pm = X_{o,\pm} \exp(\beta_\pm \chi) \) and \( Y_\pm = c_\pm X_\pm \), where \( c_\pm = \alpha/(1 + \beta_\pm) \). With \( X = X_+ + X_- \) and \( Y = Y_+ + Y_- \) together with the initial values (i.e., \( \chi = 0 \)),

\[
X_o = \left( \frac{\Lambda}{2(1 + q_o)} \right) (1 + q_o) \quad \text{and} \quad Y_o = -\left( \frac{\Lambda}{2(1 + q_o)} \right) \alpha \quad \text{(see Björnsson 2019)},
\]

one finds

\[
X_{o,\pm} = \sqrt{\frac{\Lambda}{2(1 + q_o)}} \frac{1 + q_o - c_\pm \alpha}{1 + c_\pm^2}, \tag{A1}
\]

where \( c_\pm c_\mp = -1 \) has been used. This leads to

\[
X = \sqrt{\frac{\Lambda}{2(1 + q_o)}} \frac{1}{2\beta} \times \left[ (1 + q_o) - \alpha \sigma_0 \exp(\beta \chi) \right] + \left[ (1 + q_o) + \alpha \sigma_0 \exp(-\beta \chi) \right]
\]

\[
Y = \sqrt{\frac{\Lambda}{2(1 + q_o)}} \frac{1}{2\beta} \times \left[ \alpha (1 + q_o) + (1 - \beta) \alpha \sigma_0 \exp(\beta \chi) \right] - \left[ \alpha (1 + q_o) + (1 + \beta) \alpha \sigma_0 \exp(-\beta \chi) \right], \tag{A2}
\]

where \( \beta \equiv \sqrt{1 + \alpha^2} \) has been introduced, which makes it possible to write \( 1 + c_\pm^2 = 2\beta/(\beta \pm 1) \) and \( c_\pm/(1 + c_\pm^2) = \pm \alpha/2\beta \). A more convenient form of Equation (A2) is given by

\[
X = \sqrt{\frac{\Lambda}{2(1 + q_o)}} \frac{1}{2\beta} \times \left[ (1 + q_o) - \alpha \sigma_0 \frac{\sinh(\beta \chi)}{\beta} \right] + (1 + q_o) \frac{\cosh(\beta \chi)}{\beta}, \tag{A3}
\]

\[
Y = \sqrt{\frac{\Lambda}{2(1 + q_o)}} \frac{1}{2\beta} \times \left[ \alpha (1 + q_o) + \sigma_0 \frac{\sinh(\beta \chi)}{\beta} - \sigma_0 \frac{\cosh(\beta \chi)}{\beta} \right].
\]

which then leads to Equation (20).

**A.1. Limiting Solution for \( |\beta \chi| < 1 \)**

The relevant Stokes parameters can be obtained from Equation (22).

\[
U + iV = 2E_\chi E_\chi^* = \frac{\Lambda}{2(1 + q_o)} \Lambda^* \times \left[ (1 + q_o) + \sigma_0 \frac{\sinh(\beta \chi)}{\beta} - \sigma_0 \frac{\cosh(\beta \chi)}{\beta} \right]. \tag{A4}
\]

The various terms in Equation (A4) are most conveniently evaluated using the relations \( \Lambda \delta k = -(\gamma_Y + i\gamma_L) \) and \( \delta k/\Lambda = (\gamma_Y - i\gamma_L) \), which can be obtained from Equations (3) and (4). The result is

\[
U + iV = \left( \frac{\Lambda}{2(1 + q_o)} \right) \left( \gamma_Y + i\gamma_L \right) + \frac{\sigma_0}{2} \left( \gamma_Y - i\gamma_L \right) + \left( \gamma_Y + i\gamma_L \right)^*.
\]

From the definitions of \( \gamma_Y \) and \( \gamma_L \), one finds

\[
U + iV = \left( \frac{\Lambda}{2(1 + q_o)} \right) \left( \gamma_Y + i\gamma_L \right) + \frac{\sigma_0}{2} \left( \gamma_Y - i\gamma_L \right) + \left( \gamma_Y + i\gamma_L \right)^*.
\]

which shows that the circular polarization is given by

\[
V = \left( \frac{\Lambda}{2(1 + q_o)} \right) \left( \gamma_Y + i\gamma_L \right) - \frac{\sigma_0}{2} \left( \gamma_Y - i\gamma_L \right) + \left( \gamma_Y + i\gamma_L \right)^*.
\]

**A.2. Limiting Solution for \( |\alpha| \gg 1 \)**

For \( |\alpha| \gg 1 \), one may expand the relevant expressions to first order in \( \alpha^{-1} \). Since \( \beta = \alpha \) in this limit so that \( \beta \chi = \delta k/2, \)
Equation (A3) yields
\[
2XK_a Y^* = (2XK_a Y^*)_{\text{hom}} + \frac{I_o K_a^*}{(1 + q_o)\alpha^2} \times \left[ \{\alpha^*(1 + q_o)\sinh(\delta k\tau/2)(1 + q_o)\sinh^* - \sigma_b\sinh(\delta k\tau/2) \} + \{\alpha^*(1 + q_o)\sinh(\delta k\tau/2)(1 + q_o)\sinh^*(\delta k\tau/2) - \sigma_b\sinh(\delta k\tau/2) \} \right].
\] (A8)

Here, \(2XK_a E_{\gamma}^*_{\text{hom}}\) is the part corresponding to a homogeneous source. With \(U + iV = -2XK_a Y^*\exp(-\tau - 2i\chi)\) (see Equation (18)),
\[
U + iV = I_o\exp(-\tau - 2i\chi)[u_{\text{hom}} + iv_{\text{hom}} + \frac{K_a^*\sinh(\delta k\tau/2)^2}{|\alpha|^2}\left\{\alpha(1 - q_o) - \alpha^*(1 + q_o)\right\} + \frac{2i(u_o + iv_o)}{|\alpha|^2}\{\alpha^*(\delta k\tau/2)\sinh^*(\delta k\tau/2)\}],
\] (A9)

where \(|\alpha|^2 = (1 - q_o^2)/|K_a|^2\) has been used.

### A.3. The Range of Validity for the \(\alpha = \text{Constant Solution}\)

The two eigenfunctions corresponding to Equation (19) are \(X + c_{\pm}Y\). The errors implied by assuming \(\alpha = \text{constant}\) can be estimated by letting \(c_{\pm}\) vary with \(\chi\). One can then write
\[
\frac{d(X + c_{\pm}Y)}{d\chi} = \beta(X + c_{\pm}Y)
\]
\[
\frac{d(X + c_{\pm}Y)}{d\chi} = -\beta(X + c_{\pm}Y).
\] (A10)

With the use of the expressions for \(c_{\pm}\), it is found that
\[
\frac{dX}{d\chi} + \frac{Y}{c_{\pm} - c_{\pm}}\left(\frac{d\ln c_{\pm}}{d\chi} - \frac{d\ln c_{\pm}}{d\chi}\right) = X + \alpha Y
\]
\[
\frac{dY}{d\chi} + \frac{Y}{c_{\pm} - c_{\pm}}\left(\frac{d\ln c_{\pm}}{d\chi} - \frac{d\ln c_{\pm}}{d\chi}\right) = X - Y.
\] (A11)

This shows explicitly how the variations of \(c_{\pm}\) affect the propagation of a light ray. Since \(c_{\pm}\) are functions of \(\alpha\) only, Equation (A11) can be rewritten as
\[
\frac{dX}{d\chi} = X + \alpha\left(1 - \frac{1}{1 + \alpha^2} \frac{d\ln \alpha}{d\chi}\right)Y
\]
\[
\frac{dY}{d\chi} = \alpha X - \left(1 - \frac{1}{1 + \alpha^2} \frac{d\ln \alpha}{d\chi}\right)Y.
\] (A12)

Comparison to Equation (19) makes it clear that so long as \(|(1 + \alpha^2)^{-1}(d\ln \alpha/d\chi)| \ll 1\), the approximation \(\alpha = \text{constant}\) is expected to be a good one.

### Appendix B

#### Propagation of a Polarized Light Ray in a Medium with a Varying Azimuthal Angle \(\phi\)

The transport equation in this case can be solved following the same procedure as in Appendix A. Diagonalizing the appropriate matrix for Equation (34) gives eigenvalues \(\beta_{\pm} = \pm \sqrt{\alpha^2 - 1}\). Likewise, the corresponding relation between \(X_{\pm}\) and \(Y_{\pm}\) is given by \(c_{\pm} = \beta_{\pm}/(\alpha - 1)\). Since the initial values are the same, this leads to
\[
X_{\alpha,\pm} = \frac{-I_o}{2(1 + q_o)} \times \frac{1}{2} \left(1 + q_o - \frac{\sigma_b}{c_{\pm}}\right).
\] (B1)

With \(\beta \equiv \sqrt{\alpha^2 - 1}\), the solution to Equation (34) can be written
\[
X = \frac{-I_o}{2(1 + q_o)} \times \left\{1 + q_o - \frac{(\alpha - 1)\sigma_b}{\beta}\exp(\beta\chi)\right\}
\]
\[
Y = \frac{-I_o}{2(1 + q_o)} \times \left\{1 + q_o - \frac{(\alpha - 1)\sigma_b}{\beta}\exp(\beta\chi) - \left(1 + q_o - \frac{\alpha + 1}{\beta} + \sigma_b\exp(\beta\chi)\right)\right\}.
\] (B2)

This can be rewritten as
\[
X = \frac{-I_o}{2(1 + q_o)} \times \left[1 + q_o - \frac{(\alpha - 1)\sigma_b}{\beta}\sinh(\beta\chi)\right]
\]
\[
Y = \frac{-I_o}{2(1 + q_o)} \times \left[\sigma_b\cosh(\beta\chi) + (1 + \alpha)(1 + q_o)\sinh(\beta\chi)\right].
\] (B3)

The corresponding electric field (i.e., Equation (35)) is then obtained by inserting these expressions into Equation (33).

#### B.1. Limiting Solution for \(|\alpha| \gg 1\)

For \(|\alpha| \gg 1\), the electric field in Equation (35) may be expanded to first order in \(\alpha^{-1}\). Similar to the \(\varphi = \text{const}\) case, in this limit, \(\beta = \alpha\) and \(\beta\chi = \delta k\tau/2\). Keeping first-order terms in \(\alpha^{-1}\) and \(\chi\), this yields
\[
U + iV = 2E_x E_{\gamma}^*\exp(-\tau)K_a^* \times [u_{\text{hom}} + iv_{\text{hom}}
\]
\[
+ \chi(-\sigma_b\sinh(\delta k\tau/2) + (1 + q_o)\sinh(\delta k\tau/2)\left\{\sigma_b\sinh(\delta k\tau/2) - (1 + q_o)\sinh(\delta k\tau/2)\right\}
\]
\[
- \frac{\alpha}{|\alpha|^2}(1 + q_o)\sinh^*(\delta k\tau/2) + \chi^*(1 + q_o)\cosh^*(\delta k\tau/2)(1 + q_o)\sinh^*(\delta k\tau/2)\right\}.
\] (B4)
where, again, the subscript “hom” refers to the corresponding homogeneous term. This can be rewritten as

\[
U + iV = I_0 \exp(-\tau) K_0^* \times [u_{\text{hom}} + iv_{\text{hom}} + \alpha - \alpha^*] \sigma_d \sinh(\delta k \tau / 2)^2
- \frac{\alpha}{|\alpha|^2} (1 + q_o) \cosh(\delta k \tau / 2) \sinh(\delta k \tau / 2)
+ \alpha^* (1 - q_o) \frac{|\alpha|^2}{|K_o|^2} \cos(\delta k \tau / 2) \sinh(\delta k \tau / 2)
- \chi \left\{ \frac{(1 - q_o)}{|K_o|^2} \cosh(\delta k \tau / 2)^2 \right\}
+ (1 + q_o) \sinh(\delta k \tau / 2)^2}
+ \chi^* \left\{ (1 + q_o) \cosh(\delta k \tau / 2)^2 \right\}
+ \chi \left\{ (1 - q_o) \cosh(\delta k \tau / 2)^2 \right\}
+ \chi^* \left\{ \sigma_o \cosh(\delta k \tau / 2) \sinh(\delta k \tau / 2) \right\}
+ \sigma_o \cosh(\delta k \tau / 2) \sinh(\delta k \tau / 2) \right\}. \tag{B5}
\]

In order to emphasize the role played by the initial conditions, the various terms in Equation (B5) can be rearranged as follows:

\[
U + iV = I_0 \exp(-\tau) [u_{\text{hom}} + iv_{\text{hom}}
+ \frac{2i}{\alpha} \sigma_o K_0^* \left\{ \frac{\alpha}{|\alpha|^2} \sinh(\delta k \tau / 2)^2 \right\}
+ \chi (\cos(\delta k \tau / 2) \sinh(\delta k \tau / 2))
- 2i K_0^* \left\{ \chi \cos(\delta k \tau) \right\}
+ \left\{ \alpha \cos(\delta k \tau / 2) \sinh(\delta k \tau / 2) \right\}
+ 2q_o K_0^* \left\{ \chi \cos(\delta k \tau) \right\}
- \left\{ \alpha \cos(\delta k \tau / 2) \sinh(\delta k \tau / 2) \right\}
+ K_o^* (1 - |K_o|^2) \left\{ (1 - q_o) \right\}
\times \left\{ \frac{\alpha^*}{|\alpha|^2} \cosh(\delta k \tau / 2) \sinh(\delta k \tau / 2) \right\}
- \chi (\cosh(\delta k \tau / 2)^2 + \chi^* \sinh(\delta k \tau / 2)^2) \right\}. \tag{B6}
\]

It is seen that in the limit |δkτ| ≪ 1, the last four inhomogeneous terms in Equation (B6) vanish to order |δkτχ|. Hence, only the first inhomogeneous term contributes to the polarization in this limit (see Equation (37)). Furthermore, for nearly linear or nearly circular characteristical waves, |1 - |K_o|^2| ≪ |K_o|^2 so that the last inhomogeneous term can be neglected also for |δkτ / 2| ≥ 1; i.e., this term will never contribute significantly to the polarization.

### Appendix C

#### Relating the Plasma Properties to the Polarization of the Characteristic Waves

As mentioned in the main text, the homogeneous solution is often also quite useful for inhomogeneous sources. Focusing on transport-induced effects (i.e., setting ν = 0) and a synchrotron plasma (ν = 0), the circular polarization for a given light ray can be written (Björnsson 2019)

\[
V = I_0 \exp(-\kappa s) \left[ -u_o K_i K_r^* \frac{\cosh(\delta k \tau) - \cos(\delta k \tau)}{|K|^2} \right]
+ \frac{K_i}{2} \left\{ \frac{|K|^2 + 1}{|K|^2} \right\} \sinh(\delta k \tau)
+ \frac{K_r}{2} \left\{ \frac{|K|^2 - 1}{|K|^2} \right\} \sin(\delta k \tau), \tag{C1}
\]

where, for an inhomogeneous source, δkτ = \int_0^s δkds (see Section 3.1), and K is the initial value (i.e., at s = 0) of the polarization for the characteristic waves.

With

\[
K^2 = \frac{1 - \rho_1}{1 + \rho_1}
= \frac{1 - |\rho|^2 - 2i\rho_1}{1 + |\rho|^2 + 2\rho_1}, \tag{C2}
\]

and

\[
K^2 = K^2 - K_i^2 + 2iK_i K_r, \text{ one identifies} \]

\[
K_i^2 - K_i^2 = \frac{1 - |\rho|^2}{1 + |\rho|^2 + 2\rho_1}, \tag{C3}
\]

so that

\[
K_i^2 - K_r^2
= \frac{1 - |\rho|^2}{\sqrt{1 + |\rho|^2}^2 - 4\rho_1^2}, \tag{C4}
\]

The first term in Equation (C1) accounts for the conversion of linear to circular polarization; its magnitude is determined by ρ_1 (Equation (C5)). The last term in Equation (C1) is zero for |K| = 1, which requires ρ_1 = 0 (Equation (C4)). This corresponds to orthogonal characteristic waves. Furthermore, the maximum value of |K_i K_r|/|K|^2 occurs for |K_i|/|K| = 1/\sqrt{2}, which leads to |K_i K_r|/|K|^2 = \sqrt{2}/3. This is close to where |K_i| = |K_r|, i.e., |\rho| = 1. Hence, the conversion of linear to circular polarization attains a maximum in the region where |\rho| ≈ 1 and can reach several tens of percent.

For a synchrotron plasma, ξ_L ≈ 1, while |ξ_v| ≪ 1. In order to be consistent with ν = 0, the circular absorptivity should be set to zero, i.e., |ξ_v| = 0. The plasma properties are then
described by

$$\rho = \frac{\hat{\xi}_V (\hat{\xi}_U + i \hat{\xi}^r_{SU})}{\xi^2_U + \xi^2_U}. \quad (C6)$$

This leads to

$$|\rho|^2 = \frac{\hat{\xi}_V^2}{\xi^2_U + \xi^2_U}, \quad \frac{\rho}{|\rho|^2} = \frac{\xi_U}{\xi_V} \quad \text{and} \quad \frac{\rho^*_1}{|\rho|^2} = \frac{\hat{\xi}_U}{\xi_V}. \quad (C7)$$

Furthermore, it is convenient to also express the phase difference between the characteristic waves in terms of $\rho$,

$$\Delta k = i \xi_V \sqrt{1 + \frac{(\rho^*_1 - \rho^2_1 + 2i \rho_1 \rho)}{|\rho|^4}}. \quad (C8)$$

It is seen from Equation (C7) that $\rho_r$ is a measure of the linear absorption. An important point to note from Equation (C8) is that the $\rho_r$-dependence of $\delta k_r$ implies $\delta k_r = 0$ when $\rho_r = 0$. Hence, neglect of absorption or, equivalently, assuming orthogonal characteristic waves causes the last two terms in Equation (C1) to become zero.

The contributions to the circular polarization in a synchrotron plasma from the various terms in Equation (C1) for $|\rho| \ll 1$ and $|\rho| \gg 1$ have been discussed in Björnsson (2019). When $|\rho| \sim 1$, Equation (C7) implies $|\hat{\xi}_V| \sim |\hat{\xi}_U| \gtrsim \xi_U$. Since $\rho^*_1 / \rho_1 = \xi_U / \hat{\xi}_U$, the above discussion shows that for $\xi_U / |\hat{\xi}_U| \ll 1$, the main contribution comes from linear conversion. Only when $\xi_U / |\hat{\xi}_U| \sim 1$ do all three terms contribute substantially. Actually, this latter case may be the relevant one for compact radio sources, since $|\xi_U| \sim 1$ is expected for a rather large range of plasma properties. This is due to the fact that, in contrast to $\xi_U$, the value of $\xi_V$ is rather insensitive to variations in the synchrotron plasma. Moreover, this suggests that the polarization of the characteristic waves is determined mainly by the value of $\xi_V$. 

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