On strictly 2-maximal subgroups of finite groups

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1 Introduction

All groups in this paper are finite. We write $H \leq G$ ($H < G$) if $H$ is a (proper) subgroup of a group $G$. A subgroup $M$ of a group $G$ is called a maximal subgroup if $M < G$ and $M \leq H \leq G$ implies that either $M = H$ or $H = G$. If $M$ is a maximal subgroup of a group $G$, then we write $M \lhd G$.

Let $G$ be a group and let $H$ be a subgroup of $G$. We use the following notation

$$\text{Max}(G, H) = \{ M \lhd G \mid H \leq M \}.$$ 

If $H = 1$ is the unit subgroup of $G$, then we write $\text{Max}(G)$ instead of $\text{Max}(G, 1)$. It is clear that $\text{Max}(G)$ is the set of all maximal subgroups of $G$. It should be noted that $\text{Max}(G) = \emptyset$ exactly when $G = 1$.

Definition 1. A subgroup $H$ of a group $G$ is called

- a 2-maximal subgroup of $G$ if there is $M \in \text{Max}(G, H)$ such that $H \lhd M$;
- an $n$-maximal subgroup of $G$ for $n \geq 3$ if there is $M \in \text{Max}(G, H)$ such that $H$ is an $(n - 1)$-maximal subgroup in $M$.

Example 1. In $L_2(8)$ $[1, \text{IdGroup}(504,156)]$, a subgroup $C_2$ is a 2-, 3- and 4-maximal subgroup:

- $C_2 \lhd D_{14} \lhd L_2(8)$,
- $C_2 \lhd S_3 \lhd D_{18} \lhd L_2(8)$,
- $C_2 \lhd C_2^2 \lhd C_2^3 \lhd C_2^3 : C_7 \lhd L_2(8)$.

In view of [2, example 3], for any $n > 2$ there is a group in which a 2-maximal subgroup is $n$-maximal.

Definition 2. A subgroup $H$ of a group $G$ is called a strictly 2-maximal subgroup of $G$ if $H \lhd M$ for all $M \in \text{Max}(G, H)$. Clearly, a strictly 2-maximal subgroup of a group $G$ is 2-maximal in $G$ and is not $n$-maximal in $G$ for any $n > 2$.

By $\text{Max}_2(G)$ we denote the set of all 2-maximal subgroups of a group $G$, $\text{Max}_2^\star(G)$ denotes the set of all strictly 2-maximal subgroups of $G$. It is clear that $\text{Max}_2(G) = \emptyset$ exactly when $G = 1$ or $|G|$ is a prime. From the indices lemma, it follows that a 2-maximal subgroup of least index is strictly 2-maximal, see Lemma $[1]$. Therefore $\text{Max}_2^\star(G) \neq \emptyset$ for any $G \neq 1$ of nonprime order.

The first author of this paper proper the following problem $[3, 19.54]$

What are the chief factors of a finite group in which every 2-maximal subgroup is not $n$-maximal for any $n \geq 3$?
This problem is researched in \[3\].

If every 2-maximal subgroup of a group $G$ is not $n$-maximal for all $n \geq 3$, then $\text{Max}_2(G) = \text{Max}_2^*(G)$, i.e. every 2-maximal subgroup of $G$ is strictly 2-maximal. Hence the noted problem could be formulated as follows.

What are the chief factors of a finite group in which $\text{Max}_2(G) = \text{Max}_2^*(G)$?

The examples of groups with $\text{Max}_2(G) = \text{Max}_2^*(G)$ are supersoluble groups, the nonsupersoluble group $C_3^2 : C_8$ [4, remark 4], the group $U_3(2)$, the simple groups $U_3(3)$ and $L_2(17)$, see examples [4, 5].

In this paper, we conclude from the results of Hanguang Meng and Xiuyun Guo [4] some corollaries about the existence of strictly 2-maximal subgroups in groups. We give examples of groups that illustrate properties of strictly 2-maximal subgroups.

2 On groups with $\text{Max}_2(G) = \text{Max}_2^*(G)$

Lemma 1. If $G \neq 1$ is a group of nonprime order, then $\text{Max}_2^*(G) \neq \emptyset$.

Proof. Let $H$ be a 2-maximal subgroup in $G$ of least index. Suppose that $H$ is not a strictly 2-maximal subgroup. Then there is $M \in \text{Max}(G, H)$ such that $H$ is not a maximal subgroup in $M$. So, in $M$ there is a subgroup $K$ such that $H < K < M$. By the indices lemma,

$$|G : H| = |G : K||K : H|, \quad |K : H| \neq 1, \quad |G : K| < |G : H|.$$ 

Thus, $K$ is 2-maximal in $G$ and $|G : K| < |G : H|$, this contradicts the choice of $H$. Hence we conclude that $H$ is a strictly 2-maximal subgroup of $G$. \hfill \Box

Lemma 2. Let $H$ be a 2-maximal subgroup of a group $G$, $H \triangleleft M \triangleleft G$. If the indices $|G : M|$ and $|M : H|$ are primes, then $H$ is a strictly 2-maximal subgroup of $G$. In particular, if $G$ is a supersoluble group, then $\text{Max}_2(G) = \text{Max}_2^*(G)$.

Proof. Assume that $H$ is a 2-maximal subgroup of $G$, $H \triangleleft M \triangleleft G$, and the indices $|G : M|$ and $|M : H|$ are primes. Suppose that $H$ is not a strictly 2-maximal subgroup of $G$. Hence there is a subgroup $K$ of $G$ such that $H < K < G$ and $H$ is 2-maximal in $K$. Therefore there is a subgroup $L$ such that $H \triangleleft L \triangleleft K < G$. By the indices lemma,

$$|G : H| = |G : K||K : L||L : H|, \quad |G : K| \neq 1, \quad |K : L| \neq 1, \quad |L : H| \neq 1,$$

so $|G : H|$ is divided by three primes, a contradiction. Consequently, $H$ is a strictly 2-maximal subgroup of $G$.

Let $H \triangleleft M \triangleleft G$ and let $G$ be a supersoluble group. By the Huppert Theorem [5, VI.9.5], $|G : H|$ is divided by exactly two not necessarily different primes. If $H \triangleleft X \triangleleft G$, then $|X : H|$ is a prime and $H$ is a maximal subgroup in $X$. Since $X$ is an arbitrary maximal subgroup of $G$ containing $H$, we obtain that $H$ is a strictly 2-maximal subgroup of $G$. \hfill \Box

We give examples nonsupersoluble groups with $\text{Max}_2(G) = \text{Max}_2^*(G)$. In examples, we based on [1, 6, 8] and build a graph for each group, whose vertices are representatives of the classes of conjugate subgroups and two vertices $A$ and $B$ are joined by an edge whenever $B \triangleleft A$, at that $B$ is located below $A$. We follow the notation of [6]. Besides, $C_q$ denotes a cyclic group of order $q$, $G_q^n$ denotes a direct product of $n$ copies of $C_q$.
Example 2 (remak 4]). $C_2^3 : C_8$ (IdGroup(72,39)], [S]

\[\text{Max}(C_2^3 : C_8) = \{C_8, C_2^3 : C_4\}, \]
\[\text{Max}_2(C_2^3 : C_8) = \{C_4, C_3 : S_3\}, \]
\[\text{Max}^*_2(C_2^3 : C_8) = \text{Max}_2(C_2^3 : C_8). \]

Example 3. $U_3(2)$ (IdGroup(72,41)], [S]

\[\text{Max}(U_3(2)) = \{Q_8, C_3^2 : C_4\}, \]
\[\text{Max}_2(U_3(2)) = \{C_4, C_3 : S_3\}, \]
\[\text{Max}^*_2(U_3(2)) = \text{Max}_2(U_3(2)). \]

Example 4. $L_2(17)$ ([1], [6] p. 9], [7]}

\[\text{Max}(L_2(17)) = \{Q_8, C_3^2 : C_4\}, \]
\[\text{Max}_2(L_2(17)) = \{C_4, C_3 : S_3\}, \]
\[\text{Max}^*_2(L_2(17)) = \text{Max}_2(L_2(17)). \]
Example 5. For $U_3(3)$ [6, p. 9] it follows from [1, 7] that

$$\text{Max}(L_2(17)) = \{C_{17} : C_8, S_4, D_{18}, D_{16}\},$$

$$\text{Max}_2(L_2(17)) = \{C_{17} : C_4, C_8, A_4, D_8, S_3, C_9\} = \text{Max}_2^*(L_2(17)).$$

Maximality of subgroups and normal subgroups of $G$ is a maximal subgroup of $N$. Hence $N$ is a maximal subgroup of $G$. Maximal subgroup $H$ is the normal subgroup of $G/N$, $K$ is a minimal subgroup of $G$. Suppose that $K < H$ is a maximal subgroup of $G$ such that $K ≤ M ≤ H$, $K$ is maximal in $M$ and $K$ is not maximal in $H$. The following statements hold:

1. $K = M \cap H$;
2. $K_G = M_G < M$;
3. either $K_G = H_G$ or $KH_G = H$;
4. if $G$ is soluble, $K ≤ X < G$ and $K$ is not maximal in $X$, then $X = H$, $K_G = M_G < H_G$ and $KH_G = H$.

Proof. (1) Since $K ≤ (H \cap M) < M$ and $K$ is maximal in $M$, then $K = H \cap M$.

(2) This statement is Lemma 1 [4]. We can assume that $K_G = 1$ and $M_G \neq 1$. Choose a minimal normal subgroup $N$ of $G$ such that $N ≤ M_G$. Since $K_G = 1$, we get $N \nleq K$ and $KN = M$. In view of (1), $K = (H \cap M)$, therefore $H \cap N ≤ K$ and $H \cap N = K \cap N$. From $K < H \neq M$ and $KN = M$, we conclude that $N \nleq H$ and $G = HN$. The maximal normal subgroup $KN/N = M/N$ is maximal in $G/N = HN/N$, consequently,

$$K/(H \cap N) = K/(K \cap N) \cong KN/N < G/N = HN/N \cong H/(H \cap N).$$

Hence $K$ is maximal in $H$, a contradiction, and $M_G = 1 = K_G$. Since $K < M$, it follows from $M_G = K_G$ that $M_G \neq M$.

(3) Since $K < H$, we get $K_G ≤ H_G$. Suppose that $K_G < H_G$ and $N/K_G$ is a minimal normal subgroup of $G/K_G$, $N/K_G ≤ H_G/K_G$. In view of (2), $M_G = K_G$, therefore $N \nleq M$ and $G = NM$. Since $NK \nleq H$ and $K$ is maximal in $M$, we conclude that $NK$ is a maximal subgroup of $G$ and $H = NK = H_GK$.

(4) This statement is Theorem B [4]. We can assume that $K_G = 1$. By the hypothesis $K < M < G$. Suppose that there are two subgroups $H \in \text{Max}(G, K)$ and $X \in \text{Max}(G, K)$ such that $K \leq H \cap X$, $H \neq X$, $K$ is not maximal in $H$ and $K$ is not maximal in $X$. In view of (2), $M_G = 1$, $K = M \cap H = M \cap X$. Since $G$ is a soluble primitive group, we obtain

$$G = N : M, \ N = F(G), \ \Phi(G) = 1,$$

$N$ is the unique minimal normal subgroup of $G$. If $N \leq H \cap X$, then

$$H = H \cap (NM) = N(H \cap M) = N(X \cap M) = X \cap (NM) = X,$$
So $G = N : H$. From $K < M < G$, it follows that $NK < G$. By the hypothesis, $K$ is not maximal in $H$, therefore there is a subgroup $T$ such that $K < T < H$. Now $NK < NT < G$, a contradiction. Hence we conclude that $H = X$.

If $K_G = H_G$, then $K$ and $H$ are conjugate. Since $K_G = M_G$ and $K/M_G < M/M_G$, we get $K/H_G < H/H_G$, and $K$ is maximal in $H$, a contradiction. Therefore $K_G \neq H_G$, and it follows from (3) that $K_H = G$.

**Corollary 1.1.** Let $G$ be a group, $K < M < G$. If $K_G \neq M_G$, then $K$ is a strictly $2$-maximal subgroup of $G$. In particular, if a maximal subgroup $M$ of $G$ is normal in $G$, then every maximal subgroup of $M$ is a strictly $2$-maximal subgroup of $G$.

**Note 1.** Let $G$ be a soluble group and let $M$ be a maximal subgroup in $G$ of least index. According to [9, Lemma 1], $M$ is normal in $G$, and in view of Corollary 1.1 all maximal subgroups of $M$ are strictly $2$-maximal subgroups of $G$. In insoluble groups, it is not true.

**Example 6.** In $A_6$ ([1] IdGroup(360,118)], [8]), a maximal subgroup $A_5$ has the least index and $S_3$ is a maximal subgroup of $A_5$. Since

$$S_3 < C_3 : S_3 < C_3^2 : C_4 < G,$$

$S_3$ is a $3$-maximal subgroup of $A_6$. Hence $S_3$ is not a strictly $2$-maximal subgroup of $A_6$.

**Corollary 1.2.** Let $G$ be a soluble group and let $K$ be a $2$-maximal subgroup of $G$. If $K$ is not a strictly $2$-maximal subgroup of $G$, then there is a unique maximal subgroup $V$ of $G$ such that $K < V$ and $K$ is not maximal in $V$.

**Example 7.** In $L_2(3^3)$ ([6] p. 18], [7]), there is a maximal subgroup $M \cong D_{26}$. A subgroup $K$ of order 2 from $M$ is a $2$-maximal subgroup of $L_2(3^3)$. In $L_2(3^3)$, there are maximal subgroups $H \cong D_{28}$ and $U \cong A_4$. Since Sylow $2$-subgroups of $L_2(3^3)$ are of order 4 and conjugate, we can assume that $K \leq H \cap U$. As $K$ is $2$-maximal in $H$ and in $U$, we have $K$ is a $3$-maximal subgroup of $L_2(3^3)$. Consequently, the condition of group solubility could not be removed in Corollary 1.2.

**Lemma 3.** Let $H$ be a subgroup of a $p$-soluble group $G$ and $|G : H| = p$. Then $G/H_G$ is supersoluble.

**Proof.** We can assume $H_G = 1$. Since $O_p'(G) \leq H_G = 1$, we obtain that $O_p(G) \neq 1$ and $C_G(O_p(G)) \leq O_p(G)$. Since $H \cap O_p(G) \leq H_G = 1$, we have $G = O_p(G) : H$, $|O_p(G)| = p$ and $H$ is isomorphic to a subgroup of a cyclic group of order $p - 1$. Therefore $G$ is supersoluble.

**Corollary 1.3.** Let $G$ be a $p$-soluble group, $M < G$. If $|G : M| = p$, then every maximal subgroup in $M$ is a strictly $2$-maximal subgroup of $G$. In particular, in a soluble group, all maximal subgroups of a subgroup of prime index are strictly $2$-maximal subgroups of a group.

**Proof.** Assume that the assertion is false. Then there are a maximal subgroup $K$ in $M$ and a maximal subgroup $H$ in $G$ such that $K < H$ and $K$ is not maximal in $H$. According to Theorem 1, $M$ is not normal in $G$ and $K_G = M_G \leq H_G$. The quotient group $G/K_G$ is supersoluble by Lemma 3, therefore $K$ is a strictly $2$-maximal subgroup of $G$, a contradiction.
Note 2. We do not know whether the requirement of group \( p \)-solubility could be removed in Corollary 1.3.

Corollary 1.4. Let \( G \) be a group and \( K \leq G \). Suppose that there are two maximal subgroups \( M \) and \( H \) in \( G \) such that \( K \) is maximal in \( M \) and \( K \) is not maximal in \( H \). If \( K \) is subnormal in \( G \), then \( K = M_G \) and \( G/K = H/K : M/K \) is a nonprimary nonsupersoluble group in which all proper subgroups are primary.

Proof. According to Theorem 1, \( M \) is not normal in \( G \) and \( K_G = M_G \leq H_G \). By the hypothesis, \( K \) is subnormal in \( G \), therefore \( K \) is normal in \( G \) and \( K = M_G \). Since \( K \) is maximal in \( M \), we get \( |M/K| = p \) for a prime \( p \). As \( N_{G/K}(M/K) = M/K \) we deduce \( G/K \) is soluble \[5\], \[IV.7.4\], \( H/K \) is normal in \( G/K \) \[5\], \[II.3.2\] and \( H/K \) is a minimal normal subgroup. Hence \( H/K \) is a \( q \)-group for a prime \( q \neq p \). Since \( K \) is not maximal in \( H \), we have \( |H/K| > q \), and \( G/K = H/K : M/K \) is a nonprimary nonsupersoluble group. It is clear that all proper subgroups in \( G/K \) are primary.

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