Dynamics of Massive Scalar Fields in dS Space and the dS/CFT Correspondence

Zhe Chang\footnote{Email: changz@mail.ihep.ac.cn.} and Cheng-Bo Guan\footnote{Email: guancb@mail.ihep.ac.cn.}

Institute of High Energy Physics, Academia Sinica

P.O.Box 918(4), Beijing 100039, China

Global geometric properties of dS space are presented explicitly in various coordinates. A Robertson-Walker like metric is deduced, which is convenient to be used in study of dynamics in dS space. Singularities of wavefunctions of massive scalar fields at boundary are demonstrated. A bulk-boundary propagator is constructed by making use of the solutions of equations of motion. The dS/CFT correspondence and the Strominger’s mass bound is shown.
1 Introduction

Recent astronomical observations of supernovae and cosmic microwave background [1]-[10] indicate that the universe is accelerating and can be well approximated by a world with a positive cosmological constant. If the universe would accelerate indefinitely, the standard cosmology leads to an asymptotic de Sitter (dS) universe. De Sitter space expands so rapidly that inertial observers see an event horizon. Like the Bekenstein-Hawking entropy[11] of a black hole, the dS entropy [12]-[15] can be written as

\[ S = \frac{A}{4G} , \]

where \( G \) is Newton’s constant, and \( A \) is area of the event horizon. The holographic principle then implies that the Hilbert space of quantum gravity in dS is finite dimensional. As pointed out by Witten[16], if the Hilbert space of quantum gravity really has a finite dimension this gives a strong hint that the general relativity cannot be quantized and must be derived from a more fundamental theory. However, persistent efforts by many researchers have so far all failed to find any clear-cut way to get dS from superstring theory or M-theory. In fact, there is no positive conserved energy and there cannot be unbroken supersymmetry in dS space. Thus, a universe with an positive cosmological constant poses serious challenges for superstring theory and M-theory. However, string theory may not be the only route to understanding of dS space. In fact, the AdS/CFT correspondence was first discussed from a general analysis of the asymptotic symmetries of AdS[17]. One can suspect that the AdS/CFT correspondence is a manifestation of the more fundamental holographic principle. Along this direction, in the similar sense of the AdS/CFT duality, Strominger [18] has suggested dS/CFT correspondence to relates quantum gravity on dS space with boundary conformal field theory [19]-[21].

In this paper, we study dynamics for massive scalar fields in dS by solving exactly the equations of motion. Global geometric properties of dS is presented in various
coordinates. A Robertson-Walker like metric is deduced from the familiar $SO(1, n+1)$ invariant one in an $(n+2)$-dimensional embedding space. Singularities of wavefunctions at the boundary are demonstrated explicitly. A bulk-boundary propagator is constructed explicitly by making use of the solutions of equations of motion for massive scalar fields. The dS/CFT correspondence and the Strominger’s mass bound is shown.

The paper is organized as follows. In Section 2, global geometric properties of dS in various coordinates are presented. The Penrose diagram is drawn. Relations between the explicit $SO(1, n+1)$ invariant metric and the Hua’s metric\cite{22, 23} are shown. In Section 3, We solve exactly the equations of motion for massive scalar fields by making use of the variable-separating method\cite{24}. A bulk-boundary propagator is constructed in section 4. The dS/CFT correspondence and the Strominger’s mass bound is discussed.

## 2 Penrose diagram and global geometric properties

In an $(n + 2)$-dimensional embedding space, the $(n + 1)$-dimensional dS space can be written as

$$\xi^0\xi^0 - \sum_{i=1}^{n} \xi^i\xi^i - \xi^{n+1}\xi^{n+1} = -1 , \quad (1)$$

with induced metric

$$ds^2 = d\xi^0 d\xi^0 - \sum_{i=1}^{n} d\xi^i d\xi^i - d\xi^{n+1} d\xi^{n+1} . \quad (2)$$

It is easy to check that the isometric symmetry of dS$_{n+1}$ is $SO(1, n+1)$. It is not difficult to see that, at least, $(n + 1)$ charts of coordinates $U_\alpha$ ($\xi^\alpha \neq 0$, $\alpha = 1, 2, \cdots, n + 1$) should be needed to describe dS$_{n+1}$ globally. In the chart dS$_{n+1} \cap U_\alpha$, we introduce a coordinate
\[ x^\mu = \frac{\xi^\mu}{\xi^\alpha}, \quad (\mu \neq \alpha; \quad \xi^\alpha \neq 0) \]  

(3)

for both the parts \( \mathcal{U}_+^\alpha (\xi^\alpha > 0) \) and \( \mathcal{U}_-^\alpha (\xi^\alpha < 0) \).

The chart \( \text{dS}_{n+1} \cap \mathcal{U}_\alpha \), in the coordinate \((x^\mu)\), is described by

\[
\text{dS}_{n+1} \cap \mathcal{U}_\alpha : \quad \sigma(x^\mu, x^\nu) > 0 ,
\]

\[
\sigma(x^\mu, x^\nu) \equiv 1 + \sum_{\mu, \nu \neq \alpha} \eta_{\mu\nu} x^\mu x^\nu , \quad \eta = \text{diag}(-1, 1, 1, \cdots, 1) .
\]

(4)

At the overlap region \( \text{dS}_{n+1} \cap \mathcal{U}_\alpha \cap \mathcal{U}_\beta \) of the two charts \( \mathcal{U}_\alpha \) and \( \mathcal{U}_\beta \), we have relations between the coordinates \((x^\mu) \in \mathcal{U}_\alpha \) and \((y^\nu) \in \mathcal{U}_\beta \)

\[
y^\alpha = \frac{1}{x^\beta} , \quad y^\nu = \frac{x^\nu}{x^\beta} \quad (\nu \neq \alpha). \]

(5)

This shows clearly a differential structure of \( \text{dS} \) space.

The boundary \( \overline{\mathcal{M}}^\alpha \) of the slice \( \text{dS}_{n+1} \cap \mathcal{U}_\alpha^+ \) of \( \text{dS} \) is

\[
\overline{\mathcal{M}}^\alpha : \quad 1 + \eta_{\mu\nu} x^\mu x^\nu = 0 .
\]

(6)

In the coordinate \((x^\mu)\), the metric is reduced as

\[
ds^2 = -\frac{dx J(I - x' J)^{-1} dx'}{1 - x J x'},
\]

(7)

where \( J \equiv \text{diag}(-1, 1, 1, \cdots, 1) \), \( x \equiv (x^0, x^1, \cdots, x^{\alpha-1}, x^{\alpha+1}, \cdots, x^n) \), and \( x' \) the transport of the vector \( x \). This is obviously a generalization of the Hua’s metric for manifolds with Lorentz signature.

It should be noticed that the transformations \( \mathcal{D} \) among the coordinate variables \((x^\mu, \mu \neq 0)\) form a group \( \mathcal{D} \in SO(n) \) as subgroup of \( SO(1, n + 1) \). Thus, we can refer to these transformations as space-like ones. The \( x^0 \) element of \((x^\mu)\) is not invariant under the so-called space-like transformations. We can not got a time-like Killing vector using of the \( x^0 \). To draw explicitly time-like geodesics, one should investigate space-like
transformation invariance variables. In fact, we find that under the transformations 
\( D, \xi^0(= \sigma^{-\frac{1}{2}}(x^\mu, x^\nu)x^0) \) is invariant. Therefore, it is convenient to use the coordinate 
\((\xi^0, x^1, \cdots, x^n)\).

By making use of the relations between the coordinates \((x^\mu)\) and \((\xi^0, x^i)\)

\[
\sigma(x^\mu, x^\nu) = \frac{1 + xx'}{1 + \xi^0 \xi^0}, \quad x^0 x^0 = \frac{\xi^0 \xi^0}{1 + \xi^0 \xi^0} (1 + xx'),
\]

we get a deduced Robertson-Walker like metric in terms of the coordinate \((\xi^0, x^i)\)

\[
ds^2 = \frac{d\xi^0 d\xi^0}{1 + \xi^0 \xi^0} - (1 + \xi^0 \xi^0) \frac{d x(I + x' x)^{-1} d x'}{1 + xx'},
\]

where the vector \(x\) denotes \((x^1, x^2, \cdots, x^{\alpha-1}, x^{\alpha+1}, \cdots, x^{n+1})\) and \(x'\) the transport of the vector \(x\).

In the spherical coordinate \((x^1, x^2, \cdots, x^{\alpha-1}, x^{\alpha+1}, \cdots, x^{n+1}) \rightarrow (\rho, \theta_1, \cdots, \theta_{n-1})\), the Robertson-Walker like metric is of the form

\[
ds^2 = \frac{d\xi^0 d\xi^0}{1 + \xi^0 \xi^0} - (1 + \xi^0 \xi^0) \left[ (1 + \rho^2)^{-2} d\rho^2 + (1 + \rho^2)^{-1} \rho^2 d\Omega^2_{(n-1)} \right],
\]

where \(d\Omega^2_{(n-1)}\) is the metric on \((n - 1)\)-sphere.

The boundary \(\sigma(x^\mu, x^\nu) = 0\) of dS space is represented by \(\xi^0 = \pm \infty\) in the coordinate \((\xi^0, x)\), and so called future and past boundary, respectively.

To further investigate global geometric properties of dS space, we would like to introduce other sets of coordinates.

A global coordinate can be introduced by relations

\[
\xi^0 = \sinh t, \quad \xi^i = \cosh t \cdot u^i, \quad (i = 1, 2, \cdots, n + 1),
\]

where \(u^i\) are coordinates of \(n\)-sphere. It is easy to see that dS space is topologically equivalent to \(\mathbb{R} \times S^n\). With this global coordinate, the metric deduced from the
$SO(1, n+1)$ invariant one is of the form

$$
 ds^2 = d\xi^0 d\xi^0 - \sum_{i=1}^{n} d\xi^i d\xi^i - d\xi^{n+1} d\xi^{n+1} 
 = dt^2 - \cosh^2 t d\Omega^2_{(n)} .
$$

(11)

Generally, the metric on $n$-sphere can be written as

$$
 d\Omega^2_{(n)} = dr^2 + \sin^2 r d\Omega^2_{(n-1)} ,
$$

with $u^{n+1} = \cos r$ and $0 \leq r \leq \pi$ , thus, we have

$$
 ds^2 = dt^2 - \cosh^2 t dr^2 - \cosh^2 t \sin^2 r d\Omega^2_{(n-1)} .
$$

Suppressing the dimension of $S^{n-1}$ and using the diffeomorphism transformation

$$
 \left\{ \begin{array}{l}
 \eta = 2 \tan^{-1} (e^t) , \\
 \theta = r
\end{array} \right. 
(0 < \eta < \pi), \\
(0 \leq \theta \leq \pi),
$$

we obtain

$$
 ds^2 = f(\eta)(d\eta^2 - d\theta^2) ,
$$

$$
 f(\eta) = \frac{1}{\sin^2 \eta} = \cosh^2 t .
$$

(12)

The Penrose diagram of dS space is shown in figure 1.

Let

$$
 \left\{ \begin{array}{l}
 \xi^0 + \xi^{n+1} = e^t , \\
 \xi^i = e^t \cdot x^i , \\
 \xi^0 - \xi^{n+1} = r^2 e^t - e^{-t} ,
\end{array} \right. 
(i = 1, 2, \cdots, n), \\
(r^2 = \sum x^i x^i)
$$

the metric can be written as

$$
 ds^2 = dt^2 - e^{2t} \sum_{i=1}^{n} dx^i dx^i ,
$$

(13)

where the subspace of $(x^i)$ or $(t = \text{constant})$ is Euclidean.

With relation

$$
 e^t = \xi^0 + \xi^{n+1} = \frac{\cos \theta - \cos \eta}{\sin \eta} > 0 ,
$$
Fig. 1 Penrose diagram for de-Sitter space

Fig. 2 Planar coordinate for de-Sitter space
we can see clearly that this coordinate describes upper-left half of dS space as shown in figure 2.

Let

\[
\begin{align*}
\xi_0 + \xi_{n+1} &= (1 - r^2)^{1/2} e^t, \quad (>0), \\
\xi_0 - \xi_{n+1} &= -(1 - r^2)^{1/2} e^{-t}, \quad (<0), \\
\sum \xi_i \xi_i &= r^2, \quad (<1), \quad (i = 1, 2, \cdots, n),
\end{align*}
\]

then, we get

\[
ds^2 = (1 - r^2)dt^2 - \frac{dr^2}{1 - r^2} - r^2 d\Omega_{(n-1)}^2.
\]

(14)

With relations

\[
\begin{align*}
\xi_0 + \xi_{n+1} &= \frac{\cos \theta - \cos \eta}{\sin \eta} > 0 , \\
\xi_0 - \xi_{n+1} &= -\frac{\cos \theta + \cos \eta}{\sin \eta} < 0 ,
\end{align*}
\]

we can see that this coordinate describes a quarter of dS space as shown in figure 3.
3 Dynamics of massive scalar fields

The equation of motion for massive scalar fields is of the form

\[(\Box + m_0^2)\phi(x) = 0,\]  

where \(\Box\) denotes the Laplace operator in dS space

\[
\Box = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j) = (1 + \xi^0 \xi^0) \partial^2_{\xi^0} + (n + 1) \xi^0 \partial_{\xi^0} \\
- (1 + \xi^0 \xi^0)^{-1} \left[ (1 + \rho^2)^2 \partial^2_{\rho} + \rho^{-1}(1 + \rho^2)(n - 1 + 2\rho^2) \partial_\rho \right] \\
- (1 + \xi^0 \xi^0)^{-1}(1 + \rho^2)\rho^{-2} \Delta_{(n-1)},
\]

here \(\Delta_{(n-1)}\) is the Laplace operator on \(S^{n-1}\).

Rewriting the scalar field \(\phi\) into variable-separating form

\[
\phi(\xi^0, \rho, \Theta) = T(\xi^0) R(\rho) Y_{lm}(\Theta),
\]

we can transform the equation (15) as

\[
(1 + \xi^0 \xi^0)^2 T''(\xi^0) + (n + 1) \xi^0(1 + \xi^0 \xi^0) T'(\xi^0) + [m_0^2(1 + \xi^0 \xi^0) + (\epsilon - m_0^2)] T(\xi^0) = 0, \\
\rho^2(1 + \rho^2) R''(\rho) + (n - 1 + 2\rho^2) \rho R'(\rho) + [\rho^2(1 + \rho^2)^{-1} - (\epsilon - m_0^2) - l(l + n - 2)] R(\rho) = 0, \\
[\Delta_{(n-1)} + l(l + n - 2)] Y_{lm}(\Theta) = 0,
\]

where \(Y_{lm}(\Theta)\) is the spherical harmonic function on \(S^{n-1}\) and \(\epsilon\) an arbitrary constant.

Solutions for the time and radial parts are, respectively,

\[
T(\xi^0) = (1 + \xi^0 \xi^0)^{(1-n)/4} \left\{ \begin{array}{c} P^\mu_\nu(i\xi^0) \\ Q^\mu_\nu(i\xi^0) \end{array} \right\},
\]

\[
R(\rho) = \rho^l(1 + \rho^2)^{k/2} F\left(\frac{1}{2}(l + k + 1), \frac{1}{2}(l + k), l + \frac{n}{2}; -\rho^2\right),
\]

where we have used the notations
\begin{equation}
\begin{aligned}
\mu^2 &= \frac{1}{4}(n-1)^2 + (\epsilon - m_0^2), \\
\nu(\nu + 1) &= \frac{1}{4}(n^2 - 1) - m_0^2, \\
k^2 - (n-1)k - (\epsilon - m_0^2) &= 0.
\end{aligned}
\tag{20}
\end{equation}

Now we consider properties of \( T(\xi^0) \) near the boundary \( \xi^0 \xi^0 = \infty \).

For the first kind of associative Legendre function \( P_\nu^\mu(i\xi^0) \), by making use of the formula

\begin{equation}
P_\nu^\mu(z) = \frac{2^\nu \Gamma(\nu + \frac{1}{2})z^{\nu+\mu}(z^2 - 1)^{-\mu/2}}{\Gamma(\frac{1}{2})\Gamma(1 + \nu - \mu)} \times F\left(\frac{1 - \nu - \mu}{2}, -\frac{\nu + \mu}{2} + \nu, z^{-2}\right) + \frac{2^{-\nu-1}\Gamma(-\nu - \frac{1}{2})z^{-\nu+\mu-1}(z^2 - 1)^{-\mu/2}}{\Gamma(\frac{1}{2})\Gamma(-\nu - \mu)} \times F\left(\frac{2 + \nu - \mu}{2}, -\frac{1 + \nu - \mu}{2} + \nu, z^{-2}\right),
\end{equation}

we know that \( P_\nu^\mu(i\xi^0) \) is divergent near boundary as

\begin{equation}
P_\nu^\mu(i\xi^0) \sim (\xi^0)^\nu + (\xi^0)^{-\nu-1}. \tag{22}
\end{equation}

For the second kind of associative Legendre function \( Q_\nu^\mu(i\xi^0) \), by making use of the formula

\begin{equation}
Q_\nu^\mu(z) = \frac{e^{\mu \pi i}}{2^{\nu+1}} \frac{\Gamma(\nu + \mu + 1)\Gamma(\frac{1}{2})}{\Gamma(\nu + \frac{3}{2})} (z^2 - 1)^{\nu/2} z^{-\nu - \mu - 1} \times F\left(\frac{\nu + \mu + 1}{2}, \frac{\nu + \mu + 2}{2}, \nu + \frac{3}{2}, z^{-2}\right),
\end{equation}

we know that \( Q_\nu^\mu(i\xi^0) \) is divergent near boundary as

\begin{equation}
Q_\nu^\mu(i\xi^0) \sim (\xi^0)^{-\nu-1}. \tag{24}
\end{equation}
4 The dS/CFT correspondence

One of the keystones of the dS/CFT duality is the bulk-boundary propagator. To discuss the dS/CFT correspondence explicitly, we have to construct a bulk-boundary propagator for dS space. Here, by making use of the exact solutions of the equations of motion for massive scalar fields obtained in the last section, we write down a bulk-boundary propagator,

\[ G^\pm_{B\theta}(\xi^0, \rho, u; \varrho, v) \]

\[ = \int d\epsilon \sum_l \sum_m Y_{lm}(u - v)(1 + \xi^0 \xi^0)^{(1-n)/4}(\rho - \varrho)^l(1 + (\rho - \varrho)^2)^{k/2} \]

\[ \times P^\mu_\nu(i\xi^0) \cdot F\left(\frac{1}{2}(l + k + 1), \frac{1}{2}(l + k), l + \frac{n}{2}; -(\rho - \varrho)^2\right), \]

where \( \mu, \nu \) and \( k \) take the same values as in the equation (20). It is not difficult to see that this bulk-boundary propagator satisfies the equations of motion for a scalar field with mass \( m_0 \),

\[ \left( \Box + m_0^2 \right) G^\pm_{B\theta}(\xi^0, \rho, u; \varrho, v) = 0. \] (26)

The bulk field can be determined from a field living on the boundary by making use of the bulk-boundary propagator \( G^\pm_{B\theta}(\xi^0, \rho, u; \varrho, v) \),

\[ \Phi^\pm(\xi^0, \rho, u) = \frac{1}{\omega_n} \int dv d\varrho G^\pm_{B\theta}(\xi^0, \rho, u; \varrho, v)\phi(\varrho, v), \] (27)

where \( \omega_n \) denotes the normalization constant. This shows that \( G^\pm_{B\theta}(\xi^0, \rho, u; \varrho, v) \) is really a bulk-boundary propagator in dS space for massive scalar fields. The bulk field \( \Phi(\xi^0, \rho, u) \) got from this way does not approach to \( \phi(\varrho, v) \) when limited on the boundary, but goes divergent at boundary with dimension

\[ d = \frac{n}{2} \left( 1 + \sqrt{1 - \frac{4m_0^2}{n^2}} \right). \] (28)
This is in agreement with the mass bound given by Strominger.

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