NONLINEAR ASYMPTOTIC STABILITY OF INHOMOGENEOUS STEADY SOLUTIONS TO BOUNDARY PROBLEMS OF VLASOV-POISSON EQUATION

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Abstract. We consider an ensemble of mass collisionless particles, which interact mutually either by an attraction of Newton’s law of gravitation or by an electrostatic repulsion of Coulomb’s law, under a background downward gravity in a horizontally-periodic 3D half-space, whose inflow distribution at the boundary is prescribed. We investigate a nonlinear asymptotic stability of its generic steady states in the dynamical kinetic PDE theory of the Vlasov-Poisson equations. We construct Lipschitz continuous space-inhomogeneous steady states and establish exponentially fast asymptotic stability of these steady states with respect to small perturbation in a weighted Sobolev topology. In this proof, we crucially use the Lipschitz continuity in the velocity of the steady states. Moreover, we establish well-posedness and regularity estimates for both steady and dynamic problems.

1. Introduction

We consider the Vlasov-Poisson equations ([14, 33]) subjected to a vertical downward gravity of a fixed gravitational constant $g > 0$ in a 3D half space $(x_1, x_2, x_3) \in \Omega := T^2 \times (0, \infty)$ with a periodic cube $T^2 = \{(x_1, x_2) : -\frac{1}{2} \leq x_i < \frac{1}{2}, i = 1, 2\}$:

$$\partial_t F + v \cdot \nabla_x F - \nabla_x (\phi_F + gx_3) \cdot \nabla_v F = 0 \quad \text{in } \mathbb{R}_+ \times \Omega \times \mathbb{R}^3,$$

$$F(0, x, v) = F_0(x, v) \quad \text{in } \Omega \times \mathbb{R}^3.$$  \hfill (1.1)

Here, the potential of an intermolecular force solves Poisson equations as

$$\Delta_x \phi_F = \eta \int_{\mathbb{R}^3} F \, dv \quad \text{in } \Omega,$$  \hfill (1.3)

for either $\eta = +1$ (an attractive potential of the Newton’s law of gravitation) or $\eta = -1$ (a repulsive potential of the Coulomb’s law). In our paper, all results hold for $\eta = 1$ and $\eta = -1$. For the sake of simplicity, we have set physical constants such as masses and size of charges of identical particles to be 1. At the incoming boundary $\gamma_- := \{(x, v) : x_3 = 0 \text{ and } v_3 > 0\}$, the distribution $F$ satisfies an in-flow boundary condition: For a given function $G(x, v) \geq 0$ on $\gamma_-$,

$$F = G \quad \text{on } \gamma_- := \{(x, v) : x_3 = 0 \text{ and } v_3 > 0\};$$  \hfill (1.4)

while the potential satisfies the zero Dirichlet boundary condition at the boundary:

$$\phi_F|_{x_3=0} = 0.$$  \hfill (1.5)

About the application of this problem, we refer to [6, 5, 23, 24, 25, 26]. For example, see an application in the stellar atmosphere such as the solar wind theory of the Pannekoek-Rosseland condition in [6] and the references therein.

In the contents of nonlinear Vlasov systems, constructing steady states ([19, 32, 13, 15]) and studying their stability ([20, 12]) or instability ([21, 29]) have been important subjects. Several boundary problems have been studied in [12, 13, 15, 17, 18, 22, 33]. Among others, we discuss some literature concerning the asymptotic stability of the Vlasov-Poisson system in a confining setting. In [28], Landau looked into analytical solutions of the linearized Vlasov-Poisson system around the Maxwellian and observed that the self-consistent field $\nabla_x \phi_F$ is subject to temporal decay even in the
absence of collisions (cf. Boltzmann equation [7]). A rigorous justification of the Landau damping in a nonlinear dynamical sense has been a long-standing major open problem. In [22, 4], it was shown that there exist certain analytical perturbations for which the fields decay exponentially at the nonlinear level. Recently, Mouhot-Villani settled the nonlinear Landau damping affirmatively for general real-analytical perturbations of stable space-homogeneous equilibria with exponential decay in [31]. Bedrossian-Masmoudi-Mouhot establishes the theory in the Gevrey regular perturbations in [3]. We also refer to [16] for a very recent result in this direction.

In this celebrated justification of nonlinear Landau damping, the high regularity such as \textit{real-analyticity} or \textit{Gevrey regularity} of perturbation seems crucial as some counterexamples are constructed for Sobolev regular perturbations ([30, 2]). Moreover, the theories of [31, 3, 16] strictly apply to \textit{space-homogeneous} equilibria but not space-inhomogeneous states. However, in many physical cases, the boundary problems do not allow these two constraints in general. Any steady solution to (1.1)–(1.3), if exists, is space-inhomogeneous unless the boundary datum \(G\) in (1.4) is space-homogeneous. Moreover, derivatives of any solution \(F\) to (1.1) are singular in general ([18, 6]).

In this paper, we establish a different stabilizing effect of downward gravity and the boundary in the content of the nonlinear Vlasov-Poisson system. Namely, we construct \textit{space-inhomogeneous steady states} \(h(x,v)\), which are Lipschitz continuous, and establish exponentially fast asymptotic stability of these steady states with respect to \textit{small (in a weighted \(L^\infty\) topology) perturbation} \(f\):

\[
F(t,x,v) = h(x,v) + f(t,x,v).
\]

For the initial datum in (1.1), we set \(F_0(x,v) = h(x,v) + f_0(x,v)\).

1.1. Illustration of the Bootstrap argument. We shall illustrate how a strong gravity may stabilize the Vlasov system for a certain class of steady solutions. As far as the author knows, this stability mechanism is \textit{new in the nonlinear contents}. For simplicity, we pick a simplified toy PDE (the real PDE in (2.5)): for given \(E = E(t,x)\),

\[
\partial_t f + v \cdot \nabla_x f - g \partial_{v_3} f = E \cdot \nabla_x h \quad \text{in } \mathbb{R}^+ \times \Omega \times \mathbb{R}^3 \quad \text{in } \mathbb{R}^+ \times \Omega \times \mathbb{R}^3.
\]

We add \(E \cdot \nabla_v h\) to count the nonlinear contribution of the Vlasov-Poisson equation (cf. [24]). The natural boundary condition of \(f\) is the absorption boundary condition \(f = 0\) on \(\gamma_-\) as in (2.7).

The characteristics of (1.7) is explicitly given by \(\dot{X} = \nabla\) and \(\dot{V} = - \nabla g\); or \(\dot{X}_i(s; t, x, v) = x_i - (t - s) v_i - g \delta_{i3} \frac{(t-s)^2}{2}\) and \(\dot{V}_i(s; t, x, v) = v_i + g \delta_{i3} (t - s)\). The unique non-negative time lapse \(t_B(t, x, v) \geq 0\), satisfying \(X_3(t - t_B(t, x, v); t, x, v) = 0\), is given by \(t_B(t, x, v) = \frac{1}{g} (\sqrt{|v_3|^2 + 2gx_3} - v_3)\). Therefore, due to a crucial effect of gravity, we can control \(t_B\) by the total energy of the particle:

\[
t_B(t, x, v) \leq \frac{2}{g} \sqrt{|v|^2 + 2gx_3}.
\]

Now we can bound a local density \(\rho(t, x) = \int_{\mathbb{R}^3} f(t, x, v) dv\) (see (2.8)) by

\[
|\rho(t, x)| \leq \int_{\mathbb{R}^3} \int_{t - t_B(t, x, v)}^t |E(s, X(s; t, x, v))| |\nabla_v h(s, X(s; t, x, v), V(s; t, x, v))| ds dv + \cdots.
\]

A major challenge is to achieve an \(E \cdot \nabla_v h\)-control, which corresponds to the nonlinear contribution for the real problem (2.5). Let us impose a crucial condition, namely \(\nabla_v h(x, v)\) has some Gaussian upper bound with respect to the total energy: for a universal positive constant \(C_0\),

\[
|\nabla_v h(x, v)| \leq C e^{-\beta |v|^2 + 2gx_3}.
\]

From the fact that the energy is conserved along the characteristics, we can derive that \(|\rho(t, x)|\) is bounded above by

\[
e^{-\lambda t} \left( \int_{\mathbb{R}^3} t_B(t, x, v) e^{\lambda t_B(t, x, v)} e^{\beta |v|^2 + 2gx_3} dv \right)^{1/2} e^{\lambda s} |E(s, x, v)| \sup_{(s, x)} e^{\beta |v|^2 + 2gx_3} |\nabla_v h(x, v)|.
\]
Now using the crucial bound (1.7) of $t_B$ with respect to the total energy, we can control the term in the parenthesis above by $O\left(\frac{1}{g^2}\text{e}^{-\frac{\lambda t}{C}}\right)$. Therefore we might hope that

$$\sup_t e^{\lambda t} \|\varrho(t)\|_{L^\infty(\Omega)} \leq \frac{C}{g^2} e^{\frac{\lambda t}{C}} \|e^{\beta(t^2 + 2gx^3)} \nabla v h\|_{L^\infty(\Omega \times \mathbb{R}^3)} \sup_t e^{\lambda t} \|E(t)\|_{L^\infty(\Omega)} + \cdots.$$  

On the other hand, for the real nonlinear problem (2.5), the Poisson equation (2.9) of $E = \nabla_x \Psi$ might suggest a control of $E(t, x)$ pointwise (at least locally) by some weighted pointwise bound of $\varrho(t, x)$ mainly. Therefore, as far as we have chosen $g\beta^2$ large enough, depending on our possible control of $\|e^{\beta(t^2 + 2gx^3)} \nabla v h\|_{L^\infty(\Omega \times \mathbb{R}^3)}$ and $\lambda$, we find “a small factor” in the nonlinear contribution.

Applying this idea to a real nonlinear problem is challenging as the steady and dynamic characteristics are governed by the different self-contained fields. Moreover, as the total energy is not conserved along the dynamic characteristics, we cannot simply deduce a crucial Gaussian upper bound of the underbraced term in (1.8), even (1.9) is granted. Indeed, we need a fine control of the nonlinear characteristics for both steady and dynamic problems. To realize the idea in the real nonlinear problem, we ought to overcome two major difficulties: nonlinear regularity estimate of the nonlinear characteristics for both steady and dynamic problems. To realize the idea in the real nonlinear problem, we ought to overcome two major difficulties: nonlinear regularity estimate of the nonlinear characteristics for both steady and dynamic problems. To realize the idea in the real nonlinear problem, we ought to overcome two major difficulties: nonlinear regularity estimate of the nonlinear characteristics for both steady and dynamic problems. To realize the idea in the real nonlinear problem, we ought to overcome two major difficulties: nonlinear regularity estimate of the nonlinear characteristics for both steady and dynamic problems. To realize the idea in the real nonlinear problem, we ought to overcome two major difficulties: nonlinear regularity estimate of the nonlinear characteristics for both steady and dynamic problems.

**Notations:** Throughout this paper, we often use the following notations: $A > 0$ is a universal positive constant unless it is specified; $x \lesssim y$ means $x \leq Cy$.

2. **Main Results**

Consider the steady problem (for $h = h(x, v)$):

$$v \cdot \nabla_x h - \nabla_x (\Phi + gx_3) \cdot \nabla_v h = 0 \quad \text{in} \quad \Omega \times \mathbb{R}^3,$$

$$h = \Phi \quad \text{on} \quad \gamma_- := \{(x, v) : x_3 = 0 \text{ and } v_3 > 0\}.$$  

We define a steady local density (whenever $h(x, \cdot) \in L^1(\mathbb{R}^3)$)

$$\rho(x) = \int_{\mathbb{R}^3} h(x, v) dv.$$  

Then the potential of a steady distribution solves (let $\eta \Delta_{0}^{-1} \rho$ denote $\Phi$)

$$\Delta_x \Phi(x) = \eta \rho(x) \quad \text{in} \quad \Omega, \quad \text{and} \quad \Phi = 0 \quad \text{on} \quad \partial \Omega.$$  

Next we consider the dynamical problem (1.1)-(1.5) as a perturbation $f(t, x, v)$ in (1.6) around the steady solution $(h, \Phi)$ to (2.1)-(2.4):

$$\partial_t f + v \cdot \nabla_x f - \nabla_x (\Psi + \Phi + gx_3) \cdot \nabla_v f = \nabla_x \Psi \cdot \nabla_v h \quad \text{in} \quad \mathbb{R}_+ \times \Omega \times \mathbb{R}^3,$$

$$f(0, x, v) = f_0(x, v) \quad \text{in} \quad \Omega \times \mathbb{R}^3,$$

$$f = 0 \quad \text{on} \quad \gamma_-.$$  

We define a local density of the dynamical fluctuation

$$\varrho(t, x) = \int_{\mathbb{R}^3} f(t, x, v) dv.$$  

Then the electrostatic potential $\Psi = \eta \Delta_{0}^{-1} \varrho$ solves

$$\Delta_x \Psi(t, x) = \eta \varrho(t, x) \quad \text{in} \quad \mathbb{R}_+ \times \Omega, \quad \text{and} \quad \Psi(t, x) = 0 \quad \text{on} \quad \mathbb{R}_+ \times \partial \Omega.$$  

Often we let $\eta \Delta_{0}^{-1} \varrho$ denote $\Psi$. The evolution of $\varrho$ is determined by a continuity equation

$$\partial_t \varrho + \nabla_x \cdot b = 0 \quad \text{in} \quad \mathbb{R}_+ \times \Omega.$$  

(2.10)
2.1. Lagrangian approach. Consider the characteristics $Z(s; x, v) = (X(s; x, v), V(s; x, v))$ for the steady problem (2.1):

$$\frac{dX(s; x, v)}{ds} = V(s; x, v), \quad \frac{dV(s; x, v)}{ds} = -\nabla_x \Phi(X(s; x, v)) - g e_3,$$

(2.12)

where $\Phi \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ solves (2.4) and $e_3 = (0, 0, 1)^T$. The data at $s = 0$ is given by $Z(0; x, v) = (X(0; x, v), V(0; x, v)) = (x, v) = z$.

We also define the characteristics $Z(s; t, x, v) = (\mathcal{X}(s; t, x, v), \mathcal{V}(s; t, x, v))$ for the dynamical problem (2.5) solving

$$\frac{d\mathcal{X}(s; t, x, v)}{ds} = \mathcal{V}(s; t, x, v),$$

$$\frac{d\mathcal{V}(s; t, x, v)}{ds} = -\nabla_x \Psi(s, \mathcal{X}(s; t, x, v)) - \nabla_x \Phi(\mathcal{X}(s; t, x, v)) - g e_3,$$

(2.13)

and satisfying $Z(t; t, x, v) = (\mathcal{X}(t; t, x, v), \mathcal{V}(t; t, x, v)) = (x, v) = z$. Here, $\Psi(t, \cdot) \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ and $\Phi \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ solve (2.9) and (2.4), respectively. Note that the Picard theorem ensures that the unique solutions $Z$ and $\mathcal{Z}$ to ODEs (2.12) and (2.13) exist, respectively.

**Definition 2.1.** (1) Suppose $\nabla_x \Phi \in C^1(\Omega)$ for $p > 2$. Then $Z(s; x, v)$ is well-defined as long as $X(s; x, v) \in \Omega$. There exists a backward/forward exit time

$$t_b(x, v) := \sup \{s \in [0, \infty) : X_3(-\tau; x, v) > 0 \text{ for all } \tau \in (0, s) \} \geq 0,$$

$$t_f(x, v) := \sup \{s \in [0, \infty) : X_3(+\tau; x, v) > 0 \text{ for all } \tau \in (0, s) \} \geq 0.$$

(2.14)

In particular, $X_3(-t_b(x, v); x, v) = 0$. Moreover, $Z(s; x, v)$ is continuously extended in a closed interval of $s \in [-t_b(x, v), 0]$.

We also define backward exit position and velocity:

$$x_B(x, v) = X(-t_B(x, v); x, v) \in \partial \Omega, \quad v_B(x, v) = V(-t_B(x, v); x, v).$$

(2.15)

(2) Suppose $\nabla_x \Phi, \nabla_x \Psi(t, \cdot) \in C^1(\Omega)$ for $p > 1$. Then $Z(s; t, x, v)$ is well-defined as long as $\mathcal{X}(s; t, x, v) \in \Omega$. There exists a backward/forward exit time

$$t_B(t, x, v) := \sup \{s \in [0, \infty) : \mathcal{X}_3(-\tau; t, x, v) > 0 \text{ for all } \tau \in (0, s) \} \geq 0,$$

$$t_F(t, x, v) := \sup \{s \in [0, \infty) : \mathcal{X}_3(+\tau; t, x, v) > 0 \text{ for all } \tau \in (0, s) \} \geq 0,$$

(2.16)

such that $X_3(t - t_B(t, x, v); t, x, v) = 0$ and backward exit position and velocity are defined

$$x_B(t, x, v) = \mathcal{X}(t - t_B(t, x, v); t, x, v) \in \partial \Omega, \quad v_B(t, x, v) = \mathcal{V}(t - t_B(t, x, v); t, x, v).$$

(2.17)

Then $Z(s; t, x, v)$ is continuously extended in a closed interval of $s \in [t - t_B(t, x, v), t]$.

**Definition 2.2** (Mild solution). For given $C^2$ potentials and their characteristics, it is well-known that any weak solutions are the Lagrangian solution. For the steady problem (2.1)-(2.4) and dynamic problem (2.5)-(2.9), they take the form of

$$h(x, v) = 1_{t \leq t_b(x, v)} h(X(-t; x, v), V(-t; x, v)) + 1_{t > t_b(x, v)} G(x_B(x, v), v_B(x, v));$$

(2.18)

and

$$f(t, x, v) = 1_{t \leq t_b(t, x, v)} f(0, \mathcal{X}(0; t, x, v), V(0; t, x, v))$$

$$+ \int_{\max\{0, t - t_B(t, x, v)\}}^t \nabla_x \Psi(s, \mathcal{X}(s; t, x, v)) \cdot \nabla_v h(\mathcal{X}(s; t, x, v), \mathcal{V}(s; t, x, v)) ds.$$

(2.19)
As we have described across (1.9) in the introduction, Gaussian weight functions have a crucial role in our analysis.

**Definition 2.3.** For an arbitrary $\beta > 0$, we set weight functions for the steady problem (2.1) and for dynamic problem (1.1) and (2.5)

$$w(x, v) = w_\beta(x, v) = e^{\beta(|v|^2 + 2\Phi(x) + 2g x_3)},$$

$$w_0(t, x, v) = w_\beta(t, x, v) = e^{\beta(|v|^2 + 2\Phi(x) + 2\Psi(t, x) + 2g x_3)}.$$

**Remark 2.4.** Few basic properties: (i) the steady weight $w(x, v)$ in (2.20) is invariant along the steady characteristics (2.12):

$$w(X(s; x, v), V(s; x, v)) = w(x, v) \text{ for all } s \in [-t_B(x, v), 0].$$

(ii) The dynamic weight $w_0(t, x, v)$ is not invariant along the dynamic characteristics, as the dynamic total energy is not invariant:

$$\frac{d}{ds} ([V(s; t, x, v)]^2 + 2\Phi(X(s; t, x, v)) + 2\Psi(s, X(s; t, x, v)) + 2gX_3(s; t, x, v))$$

$$= 2\partial_t \Psi(s, X(s; t, x, v)).$$

(iii) If the Dirichlet boundary conditions (2.4) and (2.9) hold then we have that at the boundary

$$w_\beta(x, v) = w_\beta(t, x, v) \equiv e^{\beta|v|^2} \text{ at } x_3 = 0.$$

(iv) At the initial plan $t = 0$,

$$w_\beta(0, x, v) = e^{\beta(|v|^2 + 2\Phi(x) + 2\Psi(0, x) + 2g x_3)} = w_\beta(0, x, v).$$

Here, $\Delta^{-1} \int_{\mathbb{R}^3} f_0(x, v) dv = \int_{\Omega} G(x, y) \int_{\mathbb{R}^3} f_0(y, v) dv dy$ as in (3.2).

As we have described across (1.7) in the introduction, the next lemma is crucial in our analysis.

**Lemma 2.5.** (i) Recall the steady characteristics (2.12) and its self-consistent potential $\Phi \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ in (2.4). Suppose the condition (2.44) holds. Then the backward exit time (2.14) is bounded above as

$$t_B(x, v) \leq 2 \min \left\{ \sqrt{|v_3|^2 + gx_3} - v_3, \sqrt{|v_{b,3}(x, v)|^2 - gx_3 + v_{b,3}(x, v)} \right\}.$$  

(ii) Recall the dynamic characteristics (2.13) and its self-consistent potentials $\Phi \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ and $\Psi(t, \cdot) \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ in (2.4) and (2.9), respectively. Suppose the condition (2.35) holds. Then the backward/forward exit time (2.16) is bounded above as, for all $(t, x, v) \in [0, T] \times \Omega \times \mathbb{R}^3$,

$$t_B(t, x, v) \leq 2 \min \left\{ \sqrt{|v_3|^2 + gx_3 - v_3}, \sqrt{|v_{B,3}(t, x, v)|^2 - gx_3 + v_{B,3}(t, x, v)} \right\},$$

$$t_B(t, x, v) + t_F(t, x, v) \leq \frac{4}{g} \sqrt{|v_3|^2 + gx_3}.$$  

**Proof.** We only prove the dynamical part (2.27) as the steady part (2.26) can be proved similarly. From the bootstrap assumption (2.35), the vertical acceleration is bounded from above as

$$\frac{d}{ds} V_3(s; t, x, v) \leq -g/2.$$  

Note that

$$X_3(s; t, x, v) = x_3 + \int_s^t \left( v_3 + \int_\tau^t \frac{dV_3(\tau'; t, x, v)}{d\tau'} d\tau' \right) d\tau \leq x_3 - v_3(t - s) - \frac{g}{4} |t - s|^2.$$
The zeros of the above quadratic form are \( \{ -2v_3 \pm 2\sqrt{|v_3|^2 + gx_3} \}/g \). Then, from the definition of \( t_B \) at (2.16), we can prove that
\[
t_B(t, x, v) \leq 2\left( \sqrt{|v_3|^2 + gx_3} - v_3 \right)/g.
\] (2.30)

By expanding (2.29) at \( t - t_B(t, x, v) \) and using (2.28), we derive that
\[
X_3(t - t_B(t, x, v); t, x, v) = x_3 + \int_t^{t_B(t,x,v)} \left( v_{B,3}(t, x, v) + \int_t^{t_B(t,x,v)} \frac{dV_3(\tau'; t, x, v)}{d\tau'}d\tau' \right)d\tau
\]
\[
\leq x_3 - v_{B,3}(t, x, v)t_B + \frac{g}{4}|t_B|^2.
\] (2.31)

The zeros of the above quadratic form (of \( t_B \)) are \( \{ 2v_{B,3}(t, x, v) \pm 2\sqrt{|v_{B,3}(t, x, v)|^2 - gx_3} / g \} \). Hence we conclude
\[
t_B(t, x, v) \leq \frac{2}{g} \left( \sqrt{|v_{B,3}(t, x, v)|^2 - gx_3 + v_{B,3}(t, x, v)} \right).
\] (2.32)

Combining (2.30) and (2.32) together, we conclude that the first bound of (2.27). Following the same argument we can have the bound for \( t_F(t, x, v) \).

2.2. Asymptotic Stability Criterion. As the main purpose of this paper, we establish a bootstrap machinery of starting with linear decay due to gravity effect to prove nonlinear decay.

**Theorem 2.6** (Asymptotic Stability Criterion). Suppose \((h, \Phi)\) solves (2.1)-(2.4), and \((f, g, \Psi)\) solves (2.5)-(2.9) globally-in-time in the sense of Definition 2.2. Suppose \( \nabla_x \cdot b \in L^\infty_0(\mathbb{R}^+ \times \Omega) \).

Assume that the following three conditions hold, for \( g, \beta > 0 \)
\[
\|w_\beta \nabla_v h\|_{L^\infty(\Omega)} < \infty,
\] (2.33)
\[
\sup_{t \geq 0} \|e^{\frac{2}{g}(|v|^2 + gx_3)} f(t)\|_{L^\infty(\Omega \times \mathbb{R}^3)} \leq \frac{(\ln 2)^{\frac{3}{2}} g^{\frac{1}{2}} \beta^{\frac{3}{2}}}{64\pi (1 + \frac{1}{\beta g})},
\] (2.34)
\[
\|\nabla_x \Phi\|_{L^\infty(\Omega)} + \sup_{t \geq 0} \|\nabla_x \Psi(t)\|_{L^\infty(\Omega)} \leq \frac{g}{2}.
\] (2.35)

Then there exists a computable number \( \lambda_\infty = \lambda_\infty(g, \beta, \|w_\beta \nabla_v h\|_{L^\infty(\Omega)}) > 0 \) such that \((f(t), g(t))\) decays exponentially fast as \( t \to \infty \):
\[
\sup_{t \geq 0} e^{\lambda_\infty t} \|g(t)\|_{L^\infty(\Omega)} \lesssim \|w_\beta,0f_0\|_{L^\infty(\Omega \times \mathbb{R}^3)},
\] (2.36)
\[
\sup_{t \geq 0} e^{\lambda_\infty t} \|e^{\frac{2}{g}(|v|^2 + gx_3)} f(t)\|_{L^\infty(\Omega \times \mathbb{R}^3)} \lesssim \left( 1 + \|w_\beta \nabla_v h\|_{L^\infty(\Omega)} \right) \|w_\beta,0f_0\|_{L^\infty(\Omega \times \mathbb{R}^3)}.
\] (2.37)

**Remark 2.7.** An exponent, which we will derive in (4.10), depends on \( g \) and \( \beta \) as \( \lambda_\infty \sim g^2 \beta \) roughly. This is somewhat intuitive: larger \( \beta \) implies lesser particles of high momentum while a large gravity \( g \) would trap the particles rapidly.

2.3. Construction of a steady solution. To carry out the idea of stabilizing effect in Section 1.1, it is important to construct steady solutions that satisfy the same in-flow boundary condition (1.4) as the perturbation, so that the zero in-flow boundary condition is exactly satisfied. Although some previous constructions have been made in bounded domains ([32, 15]), there seems to be no result in the half-space, which is relevant to the solar wind model (e.g., corona-heating problem).

In general, the uniqueness theorem plays an important role in asymptotic stability. We prove the uniqueness of the solution to the nonlinear problem by establishing the regularity theorem. Moreover, in a proof of asymptotic stability, it is crucial to establish some Gaussian upper bound of the derivatives of the steady solutions (see an explanation across (1.9) in the introduction). Generally speaking, the regularity estimate is difficult, as the derivatives blow up at the grazing set \( \gamma_0 = \{ x_3 = 0 \text{ and } v_3 = 0 \} \).
Theorem 2.8 (Construction of Steady Solutions). Suppose the inflow boundary solutions satisfy
\[ \|e^{\beta |x|^2} G\|_{L^\infty(\gamma_-)} + \|e^{\tilde{\beta} |x|^2} \nabla_{x,v} G(x,v)\|_{L^\infty(\gamma_-)} < \infty, \]
for \(\beta, \tilde{\beta} > 0\). For \(g > 0\), assume that \(\beta > \tilde{\beta} > \max\{1, \frac{4}{g}\}\). We also assume that
\[ \frac{g^2}{\beta} \leq \frac{\mathcal{C}_1}{\beta^{3/2}} \left(1 + \frac{1}{g}\right) \|e^{\beta |x|^2} G\|_{L^\infty(\gamma_-)} \leq \frac{g}{2}, \]  \[ \left(1 + \frac{1}{g^2/\beta}\right) \left(1 + \frac{1}{g}\right) \|e^{\tilde{\beta} |x|^2} \nabla_{x,v} G\|_{L^\infty(\gamma_-)} \leq \frac{g^2}{16}. \] Here, \(\mathcal{C}, \mathcal{C}_1 > 0\) are the computable constants, which appeared in (5.48) and (5.49). For sufficiently small \(\varepsilon_1 > 0\), suppose the following bound also hold:
\[ \frac{\mathcal{C}}{g^{3/2}} \left(1 + \frac{1}{g\beta}\right) \left(1 + \frac{1}{g^{3/2}}\right) \|e^{\tilde{\beta} |x|^2} \nabla_{x,v} G\|_{L^\infty(\gamma_-)} \leq \varepsilon_1. \]

Then there exists a unique strong solution \((h, \rho, \Phi)\) to (2.1)-(2.4). Moreover, we have
\[ \|h\|_{L^\infty(\Omega \times R^3)} \leq \|G\|_{L^\infty(\gamma_-)}, \]  \[ \|e^{\beta |x|^2} \|_{L^\infty(\gamma_-)} \leq \|\rho\|_{L^\infty(\gamma_-)}, \]  \[ \|e^{\tilde{\beta} |x|^2} \rho\|_{L^\infty(\gamma)} \leq \frac{\beta^{3/2}}{\beta^{3/2}} \|e^{|x|^2} h\|_{L^\infty(\gamma_-)} \leq \frac{\beta^{3/2}}{\beta^{3/2}} \|e^{\beta |x|^2} G\|_{L^\infty(\gamma_-)}, \]  \[ \|\nabla_x \Phi\|_{L^\infty(\gamma)} \leq \frac{g}{2}. \]

Furthermore,
\[ e^{\tilde{\beta} |x^3|} \partial_x \rho(x) \leq \frac{\beta^{3/2}}{\beta^{3/2}} \left(1 + \frac{\delta_3}{\alpha(x,v)} \right) \|e^{\tilde{\beta} |x|^2} \nabla_{x,v} G\|_{L^\infty(\gamma_-)}, \]  \[ \|\nabla^2_x \Phi\|_{L^\infty(\gamma)} \leq \frac{\beta^{3/2}}{\beta^{3/2}} \left(1 + \frac{\delta_3}{\alpha(x,v)} \right) \|e^{\tilde{\beta} |x|^2} \nabla_{x,v} G\|_{L^\infty(\gamma_-)}, \]  \[ \|e^{\tilde{\beta} |x|^2} \nabla_{x,v} h\|_{L^\infty(\gamma \times R^3)} \leq \frac{\beta^{3/2}}{\beta^{3/2}} \left(1 + \frac{\delta_3}{\alpha(x,v)} \right) \|e^{\tilde{\beta} |x|^2} \nabla_{x,v} G\|_{L^\infty(\gamma_-)}, \]  \[ e^{\tilde{\beta} |x|^2} e^{\tilde{\beta} |x^3|} \partial_x h(x,v) \leq \frac{\beta^{3/2}}{\beta^{3/2}} \left(1 + \frac{\delta_3}{\alpha(x,v)} \right) \|e^{\tilde{\beta} |x|^2} \nabla_{x,v} G\|_{L^\infty(\gamma_-)}. \]

Here, a kinetic distance for a steady problem is defined as
\[ \alpha(x,v) = \sqrt{|v_3|^2 + |x_3|^2 + 2 \partial_x \Phi(x,v) + 2g} x_3. \] In particular, \(\alpha(x,v) = |v_3|\) when \(x \in \partial \Omega\) (i.e. \(x_3 = 0\)).

Remark 2.9. An exponential decay-in-\((x,v)\) result of (2.47) is crucially important in our later proof of an asymptotic stability of a dynamical perturbation.

2.4. Construction of a global-in-time dynamical solution and Asymptotic stability.

Theorem 2.10 (Construction of Dynamic Solutions). Assume a compatibility condition:
\[ F_0(x,v) = G(x,v) \text{ on } (x,v) \in \gamma_. \]  \[ \frac{1}{\beta^2} \left\{ \|w\|_{L^\infty(\Omega \times R^3)} + \|e^{\beta |x|^2} G\|_{L^\infty(\gamma_-)} \right\} \leq \varepsilon g, \]  \[ \frac{1}{\beta^2} \|w\|_{L^\infty(\Omega \times R^3)} + \frac{1}{\beta^2} \|e^{\tilde{\beta} |x|^2} \nabla_{x,v} G\|_{L^\infty(\gamma_-)} \leq \varepsilon g^{1/2}, \]
\[
\frac{1}{\beta^2} \left\{ \|w_{\beta,0}F_0\|_{L^\infty(\Omega \times \mathbb{R}^3)} + \|e^{\beta|v|^2}G\|_{L^\infty(\gamma^-)} \right\} \\
\times \log \left( e + \frac{1}{\beta^{3/2}}\|w_{\beta,0} \nabla_x vF_0\|_{L^\infty(\Omega \times \mathbb{R}^3)} + \frac{1}{\beta}\|e^{\beta|v|^2} \nabla_{x,\|v\|}G\|_{L^\infty(\gamma^-)} \right) \leq \varepsilon g.
\] \tag{2.53}

Then there exists a unique global-in-time strong solution \((f, g, \Psi)\) to (2.5)-(2.9). Moreover, for all \((t, x, v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^3\),

\[
e^{\frac{\beta}{2}|v|^2 + gx_3}|f(t, x, v)| + \frac{1}{\beta^{3/2}}e^{\frac{\beta}{2}gx_3}|\phi(t, x)| \lesssim \|w_{\beta,0}F_0\|_{L^\infty(\Omega \times \mathbb{R}^3)} + \|e^{\beta|v|^2}G\|_{L^\infty(\gamma^-)},
\] \tag{2.54}

\[
e^{\frac{\beta}{2}|v|^2 + gx_3} |\nabla_v F(t, x, v)| \lesssim \|w_{\beta,0} \nabla_x vF_0\|_{L^\infty(\Omega \times \mathbb{R}^3)} + \left(1 + \frac{1}{g\beta^{1/2}}\right)\|e^{\beta|v|^2} \nabla_{x,\|v\|}G\|_{L^\infty(\gamma^-)},
\] \tag{2.55}

\[
e^{\frac{\beta}{2}|v|^2 + gx_3} |\partial_{x,\|v\|} F(t, x, v)| \lesssim \|w_{\beta,0} \nabla_x vF_0\|_{L^\infty(\Omega \times \mathbb{R}^3)} + \left(1 + \frac{1}{g\beta^{1/2}}\right)\|e^{\beta|v|^2} \nabla_{x,\|v\|}G\|_{L^\infty(\gamma^-)},
\] \tag{2.56}

\[
|\nabla_x \phi_{F+1}(t, x)| \lesssim \frac{1}{\beta^{3/2}} \left(1 + \frac{1}{\beta g}\right) \left\{ \|w_{\beta,0}F_0\|_{L^\infty(\Omega \times \mathbb{R}^3)} + \|e^{\beta|v|^2}G\|_{L^\infty(\gamma^-)} \right\} \leq \frac{g}{2},
\] \tag{2.57}

and

\[
\sup_{t \geq 0} \|\nabla_x^2 \phi_{F}(t)\|_{L^\infty(\Omega)} \leq \frac{c_1}{\beta^{3/2}} \left\{ \|w_{\beta,0}F_0\|_{L^\infty(\Omega \times \mathbb{R}^3)} + \|e^{\beta|v|^2}G\|_{L^\infty(\gamma^-)} \right\} \\
\times \left\{ \frac{1}{g\beta} \log \left( e + \frac{1}{\beta^{3/2}}\|w_{\beta,0} \nabla_x vF_0\|_{L^\infty(\Omega \times \mathbb{R}^3)} + \frac{1}{\beta}\left(1 + \frac{1}{\beta^{3/2}} + \frac{1}{g\beta}\right)\|e^{\beta|v|^2} \nabla_{x,\|v\|}G\|_{L^\infty(\gamma^-)} \right) \right\}.
\] \tag{2.58}

Here, \(\alpha_F(t, x, v) = \left|\|v\|_3^2 + |x_3|^2 + 2\partial_{x,\|v\|} \phi_F(t, x, 0)x_3 + 2gx_3\right|^{1/2}\) is a kinetic distance for a dynamical problem.

As a direct consequence of Theorem 2.6, Theorem 2.8 and Theorem 2.10, we conclude the following asymptotic stability.

**Theorem 2.11 (Asymptotic Stability).** Suppose all conditions in Theorem 2.8 and Theorem 2.10 hold. Then the dynamical solution \((F(t), \phi_F(t)) \to (h, \Phi)\) converges exponentially fast to the steady solution of Theorem 2.10 as in (2.36)-(2.37).

**Structure of the paper:** In Section 3, we construct a Green function of the Poisson equation (2.4) in \(\Omega\); in Section 4, we prove the asymptotic stability criterion (Theorem 2.6) using the Lagrangian proof; in Section 5, we establish the steady theorem (Theorem 2.8); and in Section 6, we prove the dynamic theorem (Theorem 2.10).

**Notation:** We will use an abbreviation of \(\|g\|_{L^\infty_{t,x}} = \sup_{t \in [0, \tau]} \|g(t)\|_{L^\infty(\Omega)}\) for some \(t \geq 0\). \tag{2.59}

3. **Green function**

In this section, we construct and study the Green’s function \(G(x, y)\) of the following Poisson equation in the horizontally-periodic 3D half-space:

\[
\Delta \phi(x) = \rho(x) \quad \text{in} \ x \in \Omega := \mathbb{T}^2 \times [0, \infty), \\
\phi(x) = 0 \quad \text{on} \ x \in \partial \Omega := \mathbb{T}^2 \times \{0\},
\] \tag{3.1}

such that \(\phi\) solving (3.1) takes the form

\[
\phi(x) = \int_{\mathbb{T}^2 \times [0, \infty)} G(x, y)\rho(y)dy.
\] \tag{3.2}

We will construct \(G(x, y)\) and prove its properties in Proposition 3.1.
The 2D Green’s function in $T \times [0, \infty)$ has an explicit formula. It is so-called the Green’s function for the one-dimensional grating in $\mathbb{R}^2$ (see [1]). However, there seems no known explicit form in 3D. In this section, we utilize a classical argument of multiple Fourier series (e.g. Theorem 2.17 in [34]) to study the 3D problem.

**Theorem 3.1.** The Green’s function for (3.1) takes a form of

$$G(x, y) = \frac{|x_3 - y_3|}{2} - \frac{|x_3 + y_3|}{2} - G(x, y), \quad \text{for } x, y \in \Omega := T^2 \times [0, \infty).$$

(3.3)

When $|x_3 - y_3| \geq 1$ and $x, y \in \Omega$, $G(x, y)$ satisfies

$$|\nabla_{x, y}^k \partial_{x_3}^i \partial_{y_3}^j G(x, y)| \lesssim e^{-|x_3 - y_3|} \quad \text{for } 0 \leq k, i + j \leq 2 \text{ and } |x_3 - y_3| \geq 1. \quad (3.4)$$

When $|x_3 - y_3| \leq 1$ and $x, y \in \Omega$,

$$G(x, y) := \frac{c_2}{|x - y|} - \frac{c_2}{|\hat{x} - y|} + G_0(x, y) \quad \text{for } |x_3 - y_3| \leq 1, \quad (3.5)$$

$$G_0(\cdot, x_3, \cdot, y_3), \partial_{x_3} G_0(\cdot, x_3, \cdot, y_3), \partial_{y_3} G_0(\cdot, x_3, \cdot, y_3) \in C^\infty(T^2 \times T^2) \text{ for } |x_3 - y_3| \leq 1. \quad (3.6)$$

Here, $\hat{x} = (x_1, x_2, -x_3)$, and $c_2 := \frac{1}{2\pi^{3/2}} \Gamma(3/2)$ with the Gamma function $\Gamma$. Moreover,

$$\partial_{x_3}^2 G_0(x, y) = \delta_0(x_3 - y_3)G_1(x, y) + G_2(x, y) \quad \text{for } |x_3 - y_3| \leq 1,$n

$$G_1(x_3, \cdot, y_3, \cdot), G_2(x_3, \cdot, y_3, \cdot) \in C^\infty(T^2 \times T^2) \text{ for } |x_3 - y_3| \leq 1. \quad (3.7)$$

Once we have the following lemma, the proof of Theorem 3.1 is straightforward.

**Lemma 3.2.** We construct a function in $T^2 \times \mathbb{R}$, which solves the following equation

$$\Delta \tilde{G}(x) = \sum_{m \in \mathbb{Z}^2} \delta_0(x + (m, 0)) = \sum_{m_1, m_2 \in \mathbb{Z}} \delta_0(x_3 + m_2, x_3) \quad \text{in } T^2 \times \mathbb{R}. \quad (3.8)$$

This Green’s function takes the form of

$$\tilde{G}(x, x_3) = \frac{1}{2} |x_3| + c - \tilde{G}(x, x_3). \quad (3.9)$$

Here, $\tilde{G}$ is defined in (3.26) and $c$ is an arbitrary constant. When $|x_3| \geq 1$, $\tilde{G}$ satisfies that

$$|\nabla_{x_3}^n \partial_{x_3}^i \tilde{G}(x, x_3)| \lesssim e^{-|x_3|} \quad \text{for } |x_3| \geq 1, \text{ and } n \in \mathbb{N}, i = 0, 1, 2.$$

(3.10)

When $|x_3| \leq 1$, the function can be decomposed as

$$\tilde{G}(x) = \frac{c_2}{\|x\|^2 + |x_3|^2} + \tilde{d}(x) + \tilde{r}(x), \quad \text{for } |x_3| \leq 1. \quad (3.11)$$

Then $\tilde{d}$ and $\tilde{r}$ satisfy

$$\tilde{d}(\cdot, x_3), \partial_{x_3} \tilde{d}(\cdot, x_3) \in C^\infty(\mathbb{R}^2); \quad (3.12)$$

$$\partial_{x_3}^2 \tilde{d}(\cdot, x_3) = \delta_0(x_3)\tilde{d}_1(\cdot, x_3) + \tilde{d}_2(\cdot, x_3), \quad \tilde{d}_1(\cdot, x_3), \tilde{d}_2(\cdot, x_3) \in C^\infty(\mathbb{R}^2); \quad (3.13)$$

$$\sum_{j=0,1} |\nabla_{x_3}^n \partial_{x_3}^j \tilde{d}(x, x_3)| + \sum_{i=1,2} |\nabla_{x_3}^n \tilde{d}_i(x, x_3)| \lesssim_{n,N} (1 + |x_3|)^{-N} \quad \text{for } |x_3| \leq 1, \quad (3.14)$$

and

$$\tilde{d}(\cdot, x_3), \partial_{x_3} \tilde{d}(\cdot, x_3) \in C^\infty(\mathbb{R}^2); \quad (3.15)$$

$$\partial_{x_3}^2 \tilde{d}(\cdot, x_3) = \delta_0(x_3)\tilde{d}_1(\cdot, x_3) + \tilde{d}_2(\cdot, x_3), \quad \tilde{d}_1(\cdot, x_3), \tilde{d}_2(\cdot, x_3) \in C^\infty(\mathbb{R}^2); \quad (3.13)$$

$$\sum_{j=0,1} |\nabla_{x_3}^n \partial_{x_3}^j \tilde{d}(x, x_3)| + \sum_{i=1,2} |\nabla_{x_3}^n \tilde{d}_i(x, x_3)| \lesssim_{n} e^{-|x_3|}. \quad (3.14)$$
**Proof of Theorem 3.1.** With the Green’s function \( \tilde{G}(x) \) to (3.8) in our hand, it is straightforward to construct the Green’s function of (3.1) by setting

\[
G(x, y) = \tilde{G}(x - y) - \tilde{G}(\tilde{x} - y),
\]

where \( \tilde{x} = (x_1, x_2, -x_3) \). We can easily show (3.4)-(3.7) from Lemma 3.2. \( \square \)

We postpone the proof of Theorem 3.1 and first study some elliptic estimates.

**Lemma 3.3.** Suppose \( |\rho(x)| \leq Ae^{-Bx_3} \) for \( A > 0 \) and \( B > 1 \).

Then \( \phi(x) \) in (3.2) satisfies that, for some \( \mathcal{C} > 0 \),

\[
|\partial_x \phi(x)| \leq \mathcal{C} A \left\{ e^{-B\min\{0,x_3-1\}} + \frac{e^{-Bx_3}}{B} + \min \left\{ B^{-1}, \frac{1}{B-1}e^{-x_3} \right\} \right\} \quad \text{for } x \in T^2 \times [0, \infty). \tag{3.17}
\]

Moreover, for any \( \delta > 0 \), \( \phi(x) \) in (3.2) satisfies that

\[
\|\nabla_2^2 \phi\|_{L^\infty(\Omega)} \lesssim \delta \|\rho\|_{L^\infty(\Omega)} \log(e + |\rho|_{C^{0,\delta}(\Omega)}) + AB^{-1}. \tag{3.18}
\]

**Proof.** Proof of (3.17). From (3.3) and (3.2), we have

\[
|\partial_{x_j} \phi(x)| = |(\partial_{x_j} G \ast \rho)(x)| \leq I_1 + I_2 + I_3 + I_4
\]

\[
= \delta_3 \int_0^\infty \int_{T^2} \left| \frac{x_3 - y_3}{2} \right| - \partial_{x_3} \left( \frac{x_3 + y_3}{2} \right) |Ae^{-Bx_3}dy_3|
\]

\[
+ \int_0^\infty \int_{T^2} 1_{|x_3 - y_3| \leq 1} \left| \partial_{x_j} \left( \frac{c_2}{|x - y|} \right) - \partial_{x_j} \left( \frac{c_2}{|\tilde{x} - y|} \right) \right| Ae^{-Bx_3}dy_3 \tag{3.19}
\]

\[
+ \int_0^\infty \int_{T^2} 1_{|x_3 - y_3| \geq 1} |\partial_{x_j} b(x, y)| Ae^{-Bx_3}dy_3
\]

\[
+ \int_0^\infty \int_{T^2} 1_{|x_3 - y_3| \leq 1} |\partial_{x_j} b_0(x, y)| Ae^{-Bx_3}dy_3.
\]

The first term can be easily bounded as

\[
I_1 = \frac{\delta_3}{2} \int_{T^2} \int_0^\infty \left| 1_{x_3 > y_3} - 1_{x_3 < y_3} - 1_{x_3 < y_3} + 1_{x_3 < y_3} \right| Ae^{-Bx_3}dy_3
\]

\[
= \delta_3 \int_{T^2} \int_{x_3}^\infty Ae^{-Bx_3}dy_3 \leq \delta_3 AB^{-1}e^{-Bx_3}. \tag{3.20}
\]

For the second term, using \( |\tilde{x} - y| \geq |x - y| \) for \( x_3, y_3 \geq 0 \), we derive that

\[
I_2 \leq 2 \int_{x_3 - 1}^{x_3 + 1} \int_{T^2} \frac{1}{|x - y|^2} Ae^{-Bx_3}dy_3 \leq 2 \int_{x_3 - 1}^{x_3 + 1} Ae^{-Bx_3} \int_0^{\sqrt{y_3}} \frac{2\pi r dr}{r^2 + |x_3 - y_3|^2} \tag{3.21}
\]

\[
= 2\pi A \int_{x_3 - 1}^{x_3 + 1} e^{-Bx_3} \ln \left( 1 + \frac{1}{2|x_3 - y_3|^2} \right) dy_3 \leq Ae^{-B\min\{0,x_3-1\}}.
\]

For the third term, using (3.4), we derive that

\[
I_3 \lesssim \int_0^\infty \int_{T^2} 1_{|x_3 - y_3| \geq 1} e^{-|x_3 - y_3|} Ae^{-Bx_3}dy_3
\]

\[
\lesssim \int_0^{x_3 - 1} e^{-(x_3 - y_3)} Ae^{-Bx_3}dy_3 + \int_{x_3 + 1}^\infty e^{-(y_3 - x_3)} Ae^{-Bx_3}dy_3 \tag{3.22}
\]

\[
\lesssim \min \left\{ AB^{-1}, A \frac{e^{Bx_3}}{B-1} \right\} + \frac{A}{(B+1)e^{B+1}}e^{-Bx_3}.
\]
Lastly, using (3.6), we derive that
\[ I_4 \lesssim \int_{x_3-1}^{x_3+1} A e^{-B y_3} dy_3 \lesssim A e^{-B \min\{0,x_3-1\}}. \]  
(3.23)

Combining (3.20)-(3.23), we conclude (3.17).

**Proof of (3.18).** First we note that
\[ \int_{T^2 \times [0,\infty)} G(x,y)\rho(y)dy = \int_{-\infty}^{\infty} \int_{T^2} G(x,y)\tilde{\rho}(y)dydy_3 \text{ in } x \in T^2 \times [0, \infty), \]  
where \( \tilde{\rho} \) is defined in \( y \in T^2 \times \mathbb{R} \) as
\[ \tilde{\rho}(y||,y_3) := \frac{1}{2} \left\{ [\rho(y||,y_3) - \rho(y||,0)] - [\rho(y||,-y_3) - \rho(y||,0)] \right\}. \]  
(3.25)

Then we apply a standard result of the potential theory ([33] or Section 4.2.5 of [14]). The last term in (3.18) comes from \( \|\rho\|_{L^1(\Omega)} \lesssim AB^{-1}. \)

**Proof of Lemma 3.2.** Step 1. We claim that \( \hat{G} \) takes the following form: for some constant \( c \),
\[ \hat{G}(x||,x_3) = \frac{1}{2} |x_3| + c - \hat{G}(x||,x_3) \text{ with } \hat{G}(x||,x_3) := \sum_{m \in \mathbb{Z}^2 \setminus \{0\}} \frac{e^{-2\pi|m||x|}}{4\pi|m|} e^{i2\pi m \cdot x}. \]  
(3.26)

For any \( x = (x_1,x_2,x_3) \in T^2 \times \mathbb{R} \), we have
\[ \sum_{m_1,m_2 \in \mathbb{Z}} \delta_0(x + (m_1,m_2,0)) = \sum_{m_1 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \delta_0(x_1 + m_1)\delta_0(x_2 + m_2)\delta_0(x_3). \]

Recall the Poisson summation formula \( \sum_{n \in \mathbb{Z}} \delta_0(y + n) = \sum_{n \in \mathbb{Z}} e^{i2\pi ny} \) for \( y \in \mathbb{R} \). Thus, we have
\[ \sum_{m_1 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \delta_0(x_1 + m_1)\delta_0(x_2 + m_2)\delta_0(x_3) = \sum_{m_1,m_2 \in \mathbb{Z}} \delta_0(x_3)e^{i2\pi m_1 x_1}e^{i2\pi m_2 x_2}. \]  
(3.27)

Now we try the following Ansatz to solve (3.8): With unknown functions \( w_m = w_{m_1,m_2} : \mathbb{R} \mapsto \mathbb{R} \),
\[ \hat{G}(x) = \sum_{m_1,m_2 \in \mathbb{Z}} w_{m_1,m_2}(x_3)e^{i2\pi m_1 x_1}e^{i2\pi m_2 x_2}. \]  
(3.28)

By inserting (3.28) in (3.8), we compute that
\[ \Delta \hat{G}(x) = \sum_{m_1,m_2 \in \mathbb{Z}} \left( w''_{m_1,m_2}(x_3) + (i2\pi m_1)^2 + (i2\pi m_2)^2 \right) w_{m_1,m_2}(x_3) e^{i2\pi m_1 x_1}e^{i2\pi m_2 x_2} \]
\[ = \sum_{m_1,m_2 \in \mathbb{Z}} \left( w''_{m_1,m_2}(x_3) + (i2\pi)^2 (m_1^2 + m_2^2) w_{m_1,m_2}(x_3) \right) e^{i2\pi m_1 x_1}e^{i2\pi m_2 x_2}. \]

To solve (3.8), we ought to solve a second order linear ODE with the Dirac delta source term:
\[ w''_{m_1,m_2}(x_3) + (i2\pi)^2 (m_1^2 + m_2^2) w_{m_1,m_2}(x_3) = \delta_0(x_3). \]

Explicit solutions are given by
\[ w_m(x_3) = w_{m_1,m_2}(x_3) := \begin{cases} \frac{1}{2} |x_3| + c, & \text{if } m_1 = m_2 = 0, \\ \frac{e^{-2\pi \sqrt{m_1^2 + m_2^2} |x_3|}}{4\pi \sqrt{m_1^2 + m_2^2}}, & \text{otherwise}, \end{cases} \]  
(3.29)

where \( c \) is a constant. Finally, inserting (3.29) in (3.28), we complete the proof of (3.26).
Step 2. Define \( \iota = \iota(x_\parallel) = \iota(|x_\parallel|) \in C^\infty(\mathbb{R}^2) \) such that

\[
\iota(x_\parallel) = \begin{cases} 
0 & \text{for } |x_\parallel| \leq 1/2, \\
\text{an increasing function of } |x_\parallel| & \text{for } 1/2 \leq |x_\parallel| \leq 1, \\
1 & \text{for } |x_\parallel| \geq 1.
\end{cases}
\]  

(3.30)

Define \( Q \) and its inverse horizontal Fourier transform \( q \): for \((x_\parallel, x_3) \in \mathbb{R}^2 \times \mathbb{R}^\ast\),

\[
Q(x_\parallel, x_3) := \iota(x_\parallel) \frac{e^{-2\pi |x_\parallel| |x_3|}}{4\pi |x_\parallel|}, \quad q(x_\parallel, x_3) := \int_{\mathbb{R}^2} Q(\xi, x_3) e^{2\pi \xi \cdot x_\parallel} d\xi. 
\]  

(3.31)

From the Poisson summation formula and (3.26), (3.31), we obtain that

\[
\mathcal{G}(x_\parallel, x_3) = \sum_{m \in \mathbb{Z}^2} Q(m, x_3) e^{2\pi i x_\parallel \cdot m} = \sum_{m \in \mathbb{Z}^2} q(x_\parallel + m, x_3). 
\]  

(3.32)

Step 3. We claim that

\[
|\nabla_{x_\parallel} q(x_\parallel, x_3)| \lesssim_{n, N} |x_\parallel|^{-N} e^{-|x_3|} \quad \text{for all } n, N \in \mathbb{N}^2 \text{ such that } N \geq |n| + 2,
\]  

(3.33)

\[
|\nabla_{x_\parallel} \partial_{x_3} q(x_\parallel, x_3)| \lesssim_{n, N} |x_\parallel|^{-N} e^{-|x_3|} \quad \text{for all } n, N \in \mathbb{N}^2 \text{ such that } N \geq |n| + 3.
\]  

(3.34)

As \( Q \) vanishes for \( |\xi| \leq 1/2 \), we take derivatives to (3.31) and derive that, for \( n' = (n_1', n_2') \in \mathbb{N}^2 \),

\[
(-2\pi i x_\parallel)^{n'} \nabla_{x_\parallel} q(x_\parallel, x_3) = \int_{\mathbb{R}^2} \partial_{\xi_1}^{n_1'} \partial_{\xi_2}^{n_2'} \left( \iota(\xi) \frac{e^{-2\pi |\xi| |x_3|}}{4\pi |\xi|} \right) \nabla_{x_\parallel}^{n'} e^{2\pi i \xi \cdot x_\parallel} d\xi. 
\]  

(3.35)

Using the fact that \( \iota(\xi) = 0 \) if \( |\xi| \leq 1/2 \) and \( \nabla_{\xi} \iota(\xi) = 0 \) if \( |\xi| \geq 1 \) from (3.30), we bound the above underlined term in (3.35) by

\[
C(n')||\iota||_{C^{n'}(\mathbb{R}^2)} |\xi| \geq \frac{1}{2} \left\{ \frac{1}{|\xi||n'|} + \frac{(|\xi||x_3|)^{n'}}{|\xi||n'|} + 1_{|\xi| \leq 1} \left( \sum_{m=0}^{n'-1} \frac{(\xi||x_3||m)}{|\xi||n'|} \right) \right\} e^{-2\pi |\xi||x_3|} 
\]  

\[
\lesssim_{n', \xi} \left( \frac{1}{|\xi||n'|+1} + 1_{|\xi| \leq 1} \right) e^{-|x_3|}. 
\]

Suppose \( |n'| \geq |n| + 2 \), then

\[
|\nabla_{x_\parallel} q(x_\parallel, x_3)| \lesssim_{n, n', \xi} e^{-|x_3|} \left( \int_{|\xi| \geq \frac{1}{2}} \frac{d\xi}{|\xi||n'|+1} + \int_{\frac{1}{2} \leq |\xi| \leq 1} \frac{|\xi||n|}{|\xi|} d\xi \right) \lesssim_{n, n', \xi} e^{-|x_3|}. 
\]  

(3.36)

Summing (3.36) over all possible \( n' \in \mathbb{Z}^2 \) such that \( |n'| = \max\{N, |n| + 2\} \), we conclude (3.33).

From (3.35) we compute that

\[
(-2\pi i x_\parallel)^{n'} \nabla_{x_\parallel} \partial_{x_3} q(x_\parallel, x_3) = -\int_{\mathbb{R}^2} \partial_{\xi_1}^{n_1'} \partial_{\xi_2}^{n_2'} \left( \iota(\xi) \frac{x_3}{|x_3|} \frac{e^{-2\pi |\xi| |x_3|}}{2} \right) \nabla_{x_\parallel}^{n'} e^{2\pi i \xi \cdot x_\parallel} d\xi. 
\]  

(3.37)

We bound the underlined terms of (3.37) respectively by

\[
C_{n', n, \xi} |\xi| \geq \frac{1}{2} \left\{ \frac{(|\xi||x_3|)^{n'}}{|\xi||n|+1} + 1_{|\xi| \leq 1} \left( \sum_{m=0}^{n'-1} \frac{(|\xi||x_3||m)}{|\xi||n|} \right) \right\} e^{-2\pi |\xi||x_3|} \lesssim \left( \frac{1}{|\xi||n|+1} + 1_{|\xi| \leq 1} \right) e^{-|x_3|}. 
\]  

(3.38)

Choose \( |n'| = \max\{N, |n| + 3\} \). Then the above upper bounds are integrable-in-\( \xi \) in \( \mathbb{R}^2 \). This allows us to prove (3.47).
Step 4. We claim that

\[ \nabla^n x_q(x||x_3) = \frac{c_2}{2^{n/2}} \frac{e^{-2\pi|\xi||x_3|}}{4\pi|\xi|} \int e^{2\pi i x \xi} \, dx. \]

From (3.37) we compute that

\[ (-2\pi i x||x_3)^{n} \nabla^n x_q(x||x_3) = -\delta_0(x_3) \int_{\mathbb{R}^2} \nabla_x^2 \left( \tau(\xi) e^{-2\pi|\xi||x_3|} \right) \nabla_x^{n} e^{2\pi i \xi \cdot x} \, d\xi \]

Following the argument of the previous step, we bound first underlined term by (3.38); and bound the second underlined term of (3.40) by

\[ C_{n', m, 1} \left| \xi \right|^{n/2} \left( \frac{|\xi||x_3|}{|\xi|^n - |n - 1|} \right) e^{-2\pi|\xi||x_3|} \lesssim \left( \frac{1}{|\xi|^n - |n - 1|} + 1 \right) e^{-|x_3|}. \]

Choose \(|n'| = \max\{N, |n| + 4\}\). Then the above upper bounds are integrable-in-\(\xi\) in \(\mathbb{R}^2\). This allows us to prove (3.39).

Step 5. Define

\[ \tilde{r}(x||x_3) := \sum_{|m| > 0} q(x||m, x_3). \]

From (3.33), (3.47), and (3.39), we conclude that the series (3.41) is absolutely convergent and hence (3.15) holds.

Step 6. We claim that when \(|x_3| \leq 1\), we can decompose \(\tilde{b}\) as (3.11) where \(\tilde{r}\) satisfies (3.41)- (3.15), and (3.12)-(3.14) hold. Recall the following horizontal Fourier transform:

\[ e^{-2\pi|\xi||x_3|} = \frac{c_2}{2^{n/2}} \frac{e^{-2\pi i x \xi} \, dx}{\pi \left| \int e^{2\pi i x \xi} \, dx \right|^2}. \]

Here, \(c_2 = 1/2\pi^{n/2} \Gamma(n/2)\) where \(\Gamma\) is the Gamma function.

We decompose \(q\) and use the duality of Fourier transform with (3.42) to get that

\[ q(x||x_3) = \frac{c_2}{2^{n/2}} \frac{e^{-2\pi i x \xi} \, dx}{\pi \left| \int e^{2\pi i x \xi} \, dx \right|^2} + \tilde{d}(x||x_3), \quad \tilde{d}(x) := \int_{\mathbb{R}^2} \left( \tau(\xi) - 1 \right) e^{-2\pi|\xi||x_3|} \frac{e^{2\pi i \xi \cdot x}}{4\pi|\xi|} \, d\xi. \]

Now we only need to prove the properties of \(\tilde{d}\), which are (3.12)- (3.14). Note that \(\tilde{d}(\cdot, x_3)\) is the inverse horizontal Fourier transforms of an integrable function with bounded support in \(\mathbb{R}^2\) horizontally. Hence \(\tilde{d}(\cdot, x_3) \in C^\infty(\mathbb{R}^2)\). Next, we compute its derivatives of \(\tilde{d}\). For any \(n = (n_1, n_2) \in \mathbb{N}^2\) and \(n' = (n_1', n_2') \in \mathbb{N}^2\),

\[ \nabla^n x_q(x||x_3) = \int_{\mathbb{R}^2} \nabla^n x_q(x||x_3) \, dx + \tilde{d}(x||x_3), \quad \tilde{d}(x) := \int_{\mathbb{R}^2} \left( \tau(\xi) - 1 \right) e^{-2\pi|\xi||x_3|} \frac{e^{2\pi i \xi \cdot x}}{4\pi|\xi|} \, d\xi. \]
Here, we have used two functions defined as
\[
\tilde{d}_1(x_\parallel, x_3) := -\int_{\mathbb{R}^2} (\iota(\xi) - 1) e^{-2\pi|\xi||x_3|} e^{2\pi i \xi \cdot x_\parallel} d\xi,
\]
\[
\tilde{d}_2(x_\parallel, x_3) := \pi \int_{\mathbb{R}^2} (\iota(\xi) - 1) |\xi| e^{-2\pi|\xi||x_3|} e^{2\pi i \xi \cdot x_\parallel} d\xi.
\]
(3.45)
(3.46)

Note that the three underlined integrals in the right hand side of (3.44) are the inverse horizontal Fourier transform of integrable functions with bounded support in \(\mathbb{R}^2\) horizontally. By summing (3.44) over \(|n'| \leq N\) in \(n' \in \mathbb{Z}^2\), we conclude (3.12)-(3.14).

**Step 7.** We consider the case of \(|x_3| \geq 1\). We claim that (3.10) holds.

We first prove that
\[
|\nabla_{x_3} \partial_{x_3} q(x_\parallel, x_3)| \lesssim (1 + |x_\parallel|)^{-N} e^{-|x_3|} \quad \text{for } |x_3| \geq 1 \text{ and } n, N \in \mathbb{N}.
\]
(3.47)

We compute that, for \(n' = (n'_1, n'_2) \in \mathbb{N}^2\),
\[
(-2\pi i x)^{n'} \nabla_{x_3} \partial_{x_3} q(x) = \int_{\mathbb{R}^2} \nabla_{x_\parallel} \left((\iota(\xi) - 1) e^{-2\pi|\xi||x_3|} \right) \nabla_{x_\parallel} e^{2\pi i \xi \cdot x_\parallel} d\xi,
\]
\[
(-2\pi i x)^{n'} \nabla_{x_3} \partial_{x_3} q(x) = \int_{\mathbb{R}^2} \pi \nabla_{\xi} \left((\iota(\xi) - 1) e^{-2\pi|\xi||x_3|} \right) \nabla_{x_\parallel} e^{2\pi i \xi \cdot x_\parallel} d\xi.
\]
(3.48)

We bound each of underlined terms in (3.48) as follows: for \(|x_3| \geq 1\),
\[
I \lesssim_{n, t} 1_{|\xi| \geq 2} |\xi|^{n_1} (1 + |x_3| |n|) e^{-2\pi|\xi||x_3|} \lesssim_{n, t} e^{-|\xi||x_3|},
\]
\[
II \lesssim_{n, n', t} 1_{|\xi| \geq 2} |\xi|^{n_1} (1 + |\xi| |x_3| |n'|) e^{-2\pi|\xi||x_3|} \lesssim_{n, n', t} e^{-|\xi||x_3|}.
\]

Then we follow the argument to prove (3.36) and derive that
\[
|\nabla_{x_\parallel} \partial_{x_3} q(x)| + |\nabla_{x_3} \partial_{x_3} q(x)| \lesssim_{n, n', t} |x_\parallel|^{-|n'|} e^{-|x_3|} \quad \text{for } |x_3| \geq 1.
\]

Now by choosing \(|n'| = N\), we conclude (3.47).

Using (3.47) with \(N \geq 3\), we conclude that the summation (3.32) is absolutely convergent. Using this together with (3.26) and (3.47), we conclude (3.10).

### 4. Asymptotic Stability Criterion

The goal of current section is to give a proof of Theorem 2.6. In this section we always assume all conditions of Theorem 2.6 hold. For example, global-in-time self-consistent potentials \(\Phi(\cdot), \Psi(t, \cdot) \in C^1(\Omega) \cap C^2(\Omega)\), and \(f(t, x, v)\) is a global-in-time Lagrangian solution in the sense of Definition 2.2 and (2.19). We also assume that \(\nabla_v h \in L^\infty\). Recall the Lagrangian formulation of \(f\) solving (2.5)-(2.9):
\[
f(t, x, v) = I(t, x, v) + \mathcal{N}(t, x, v),
\]
(4.1)

where
\[
I(t, x, v) := 1_{t < t_\Omega(t, t, x, v)} f_0(\mathcal{Z}(0; t, x, v)),
\]
(4.2)
\[
\mathcal{N}(t, x, v) := \int_t^{t_{\max}} \nabla_x \Psi(s, \mathcal{X}(s; t, x, v)) \cdot \nabla_v h(\mathcal{Z}(s; t, x, v)) ds.
\]
(4.3)

Recall \(b(t, x)\) in (2.11) and the continuity equation (2.10). Assume that \(\nabla_x : b \in L^\infty_{loc}(\mathbb{R} \times \Omega)\). Then a weak solution \(\varrho\) of the continuity equation is absolutely continuous in time. Therefore we can take a time derivative to the Poisson equation (2.9). This leads to
\[
\partial_t \Psi(t, x) = \eta \Delta_0^{-1} \partial_t \varrho(t, x) = -\eta \Delta_0^{-1} (\nabla_x : b)(t, x).
\]
(4.4)
Recall the dynamic weight function \( w_\beta \) in (2.21). Using (2.23) and (4.4), we have that
\[
\frac{d}{ds} \left( |\mathcal{V}(s; t, x, v)|^2 + 2\Phi(\mathcal{X}(s; t, x, v)) + 2\Psi(s, \mathcal{X}(s; t, x, v)) + 2g\chi(s; t, x, v) \right) \\
= 2\partial_t \Psi(s, \mathcal{X}(s; t, x, v)) = 2\Delta_0^{-1} \partial_t b(s, \mathcal{X}(s; t, x, v)) = -2\Delta_0^{-1}(\nabla_x \cdot b)(s, \mathcal{X}(s; t, x, v)).
\]
(4.5)

The forcing term \(-2\Delta_0^{-1}(\nabla_x \cdot b)(s, \mathcal{X}(s; t, x, v))\) is bounded pointwisely if a distribution \( f \) decays fast with respect to \( v \) and \( x_3 \) in \( L^\infty \):

**Lemma 4.1.** Assume that \( f \) and \( b \) are related as in (2.11). Suppose \( b \in L^\infty_{loc}(\mathbb{R}_+; C^1(\bar{\Omega})) \). Then we have that
\[
\|\Delta_0^{-1}(\nabla_x \cdot b)(t, x)\|_{L^\infty(\Omega)} \lesssim 1 + \frac{1}{\beta^2} \|e^{\frac{\beta}{2}(|v|^2 + gx_3)} f(t)\|_{L^\infty(\Omega)}.
\]
(4.6)

**Proof.** Recall the Green function \( G(x, y) \) constructed in Lemma 3.1. By the integration by parts, we derive that
\[
\Delta_0^{-1}(\nabla \cdot b)(t, x) = \int_{\mathbb{T}^2 \times \mathbb{R}_+} G(x, y) \nabla_x b(y) dy \\
= -\int_{\mathbb{T}^2 \times \mathbb{R}_+} b(y) \cdot \nabla_y G(x, y) dy - \int_{\mathbb{T}^2} G(x, y|0)b_3(y|0)dy|.
\]
(4.7)

From (3.3), we have \( G(x, y|0) = -\mathcal{G}(x, y|0) \) at \( y_3 = 0 \). From (3.5), when \( |y_3| \leq 1 \) then \( \mathcal{G}(x, y|0) = \tilde{G}_0(x, y|0) \). We also note that
\[
|b(t, x)| \leq \int_{\mathbb{R}^3} |v||f(t, x, v)|dv \leq \left( e^{-\frac{1}{2}gx_3} \int_{\mathbb{R}^3} |v|e^{-\frac{1}{4}|v|^2} dv \right) \|e^{\frac{\beta}{2}(|v|^2 + gx_3)} f(t)\|_{L^\infty(\Omega)}
\]
\[
\leq \frac{8\pi e^{-\frac{1}{2}gx_3}}{\beta^2} \|e^{\frac{\beta}{2}(|v|^2 + gx_3)} f(t)\|_{L^\infty(\Omega)}.
\]

Now we utilize Lemma 3.1, and follow the proof of Lemma 3.3 to conclude the lemma. \( \square \)

In the stability analysis, it is important to compare weight functions along the characteristics.

**Lemma 4.2.** Suppose the assumption (2.35) holds. Recall \( w_\beta(x, v) \) in (2.20), \( w_\beta(t, x, v) \) in (2.21), and \( (\mathcal{X}, \mathcal{V}) \) solving (2.13). Then, for \( s, s' \in [t - t_B(t, x, v), t + t_F(t, x, v)] \),
\[
\frac{w_\beta(s', \mathcal{Z}(s'; t, x, v))}{w_\beta(s, \mathcal{Z}(s; t, x, v))} \leq e^{\frac{9}{2} \Delta_0^{-1}(\nabla_x \cdot b)\|L_{\infty}^\omega \sqrt{|v|^2 + gx_3}}
\]
\[
\frac{1}{w_\beta(s, \mathcal{Z}(s; t, x, v))} \leq e^{\frac{16\beta}{9} \Delta_0^{-1}(\nabla_x \cdot b)\|L_{\infty}^\omega e^{-\frac{\beta}{2}|v|^2} e^{-\frac{\beta}{4}gx_3}}.
\]
(4.8)

and
\[
\frac{w_\beta(\mathcal{Z}(s; t, x, v))}{1} \leq e^{\frac{16\beta}{25} \Delta_0^{-1}(\nabla_x \cdot b)\|L_{\infty}^\omega e^{-\frac{\beta}{4}|v|^2} e^{-\frac{\beta}{8}gx_3}}.
\]
(4.9)

Here, we have used the notation \( L_{\omega}^\infty \) defined in (2.59).

**Proof.** The proof follows (4.5) and (4.6). For the detail, we refer to the proof of Lemma 6.3. \( \square \)

**Lemma 4.3.** Suppose (2.35) and (2.34) hold. Set
\[
\lambda_\infty = \frac{g^{2\beta}}{2\pi} \ln \left( 2 + \frac{g\beta^2}{\pi e^{\frac{\beta}{2}|v|^2} \|w_\beta \nabla v h\|_{L^\infty(\Omega)}} \right),
\]
(4.10)
Then
\[\mathcal{I}(t, x, v) \leq 2e^{-\frac{\beta}{2}|v|^2}\lambda t e^{-\frac{4\lambda^2}{g^2} \beta t} \|\omega_{\beta, 0} f_0\|_{L^\infty(\Omega \times \mathbb{R}^3)}, \quad (4.11)\]
\[\mathcal{N}(t, x, v) \leq e^{-\frac{\beta}{9}|v|^2} e^{-\frac{\alpha}{8} |x|^2} e^{-\frac{16\lambda^2}{g^2} t} \sup_{s \in [0, t]} \|e^{\lambda s} g(s)\|_{L^\infty(\Omega)} \|w_{\beta} \nabla v h\|_{L^\infty(\Omega \times \mathbb{R}^3)}. \quad (4.12)\]

Proof. Using Lemma 4.1, applying (4.7) to (2.34), we derive that
\[\frac{648}{g^2} \sup_{0 \leq t \leq T} \|\Delta_0^{-1}(\nabla x - b)\|_{L^\infty(\Omega \times \mathbb{R}^3)}^2 \leq e^{\frac{648}{g^2} (1 + \frac{1}{g}) \lambda^2} \sup_{0 \leq t \leq T} \|e^{\frac{\beta}{g^2}} (|v|^2 + 9x^2) f(t)\|_{L^\infty(\Omega \times \mathbb{R}^3)}^2 \leq 2. \quad (4.13)\]

First we prove (4.11). For that, we will apply Lemma 2.5, Lemma 4.2 (since (2.35) holds), and Lemma 4.1. Then we derive that
\[\mathcal{I}(t, x, v) / \|\omega_{\beta, 0} f_0\|_{L^\infty(\Omega \times \mathbb{R}^3)} \leq \frac{1}{gt/4} \sqrt{|v|^2 + 9x^2} e^{-\frac{\beta}{9} |v|^2} e^{-\frac{\alpha}{8} |x|^2} e^{-\frac{16\lambda^2}{g^2} t} \leq 2e^{-\frac{g^2}{4} t} e^{-\frac{\beta}{9} (|v|^2 + 9x^2)}. \quad (4.14)\]

We have used (2.27) and (4.8) from the first to second line; and from the second to third line we have used (4.13) and the fact that if \( t \leq \frac{4}{g} \sqrt{|v|^2 + 9x^2} \) then
\[1 \leq \frac{4}{g} \sqrt{|v|^2 + 9x^2} e^{-\frac{\beta}{9} (|v|^2 + 9x^2)} \leq \frac{4}{g} e^{-\frac{9}{4}|v|^2 - \frac{9}{4} |x|^2} \leq e^{-\frac{9}{4} |v|^2 - \frac{9}{4} |x|^2}. \]

Now we have, by completing the square,
\[e^{-\frac{9}{4} |v|^2} = e^{-\frac{9}{4} (|v|^2 + 9x^2) - \frac{9}{4} |x|^2} e^{-\frac{16\lambda^2}{g^2} t} \leq e^{-\frac{16\lambda^2}{g^2} t}. \]

Combining this with (4.14), we derive (4.11).

Next we prove (4.12). Using (4.9) and (1.3), we obtain
\[\nabla v h(\mathcal{Z}(s; t, x, v)) \leq \frac{1}{w_{\beta}(\mathcal{V}(s; t, x, v), \mathcal{V}(s; t, x, v))} \|w_{\beta} \nabla v h\|_{L^\infty(\Omega \times \mathbb{R}^3)} \leq e^{\frac{16\lambda^2}{g^2} \Delta_0^{-1}(\nabla x - b)\|_{L^\infty(\Omega \times \mathbb{R}^3)}^2} \leq e^{\frac{16\lambda^2}{g^2} \Delta_0^{-1}(\nabla x - b)\|_{L^\infty(\Omega \times \mathbb{R}^3)}^2} \leq e^{|v|^2 + 9x^2} \|w_{\beta} \nabla v h\|_{L^\infty(\Omega \times \mathbb{R}^3)}. \quad (4.15)\]

where we have used that \( e^{|v|^2 + 9x^2} \leq e^{\frac{1}{4} |v|^2} \leq e^{\frac{1}{4} |v|^2} \leq e^{|v|^2 + 9x^2} \) from (4.13).

Using (4.15), we now bound \( \mathcal{N} \):
\[|\mathcal{N}(t, x, v)| \leq 4^{1/g} \int_{t-t_{B}(t, x, v)}^{t} e^{-\lambda s} \sup_{s \in [0, t]} |e^{\lambda s} g(s)|_{L^\infty(\Omega \times \mathbb{R}^3)} e^{-\frac{\beta}{4} |v|^2} e^{-\frac{\alpha}{4} |x|^2} \|w_{\beta} \nabla v h\|_{L^\infty(\Omega \times \mathbb{R}^3)} \leq t_{B}(t, x, v) e^{\lambda s} \sum_{s \in [0, t]} e^{-\frac{\beta}{4} |v|^2} e^{-\frac{\alpha}{4} |x|^2} \|w_{\beta} \nabla v h\|_{L^\infty(\Omega \times \mathbb{R}^3)}. \]

Then using (2.27) for \( t_{B}(t, x, v) \), we bound the above underlined term as
\[t_{B}(t, x, v) e^{\lambda s} \sum_{s \in [0, t]} \leq 4 g \sqrt{|v|^2 + 9x^2} e^{4 \lambda |v| + \sqrt{g^2 x^2}} \leq 8 g^{-1/2} e^{-\frac{\beta}{8} |v|^2 + \frac{\alpha}{8} |x|^2} e^{-\frac{\beta}{8} |x|^2} \leq 8 g^{-1/2} e^{\frac{16\lambda^2}{g^2} \beta^2 |v|^2 + \frac{16\lambda^2}{g^2} \beta^2 x^2}. \]

Therefore we get (4.12).
**Proof of Theorem 2.6.** Taking $v$-integration to (4.1) and using (4.11)-(4.12), we derive that
\[
\begin{align*}
egthinspace
\left|e^{\lambda t}g(t,x)\right| & \leq e^{\lambda t}\left\{ \int_{\mathbb{R}^3} f(t,x,v)\,dv + \int_{\mathbb{R}^3} g(t,x,v)\,dv\right\} \\
egthinspace & \leq \frac{1}{\beta^{3/2}}\left\{ (4\pi)^{3/2}2e^{-\frac{\beta}{2}g}\frac{16\lambda^2}{g}\left\|w_{\beta,0}f_0\right\|_{L^\infty_x,v} \\
egthinspace & \quad + (8\pi)^{3/2}e^{-\frac{\beta}{2}g}\frac{8\cdot 4^{1/g}}{g^{3/2}}e^{\frac{16\lambda^2}{g^{3/2}}\nu^2}\sup_{s\in[0,t]}\left|e^{\lambda \infty}g(t^*\nu(s))\right|\right\} \\
egthinspace & \leq \frac{16\pi^{3/2}}{\beta^{3/2}}\frac{16\lambda^2}{g}\left\|w_{\beta,0}f_0\right\|_{L^\infty_x,v} + \frac{1}{2}\sup_{s\in[0,t]}\left|e^{\lambda \infty}g(t^*\nu(s))\right|\infty.
\end{align*}
\]

Here, at the last line, we have used (2.33). By absorbing the last term, we conclude (2.36). We can prove (2.37) using (2.36) and (4.11)-(4.12). □

5. **Steady solutions**

For the construction of a solution to the steady problem (2.1)-(2.4), we use a sequence of solutions. For an arbitrary number $\ell \in \mathbb{N}$, we suppose that
\[
\Phi^\ell \in C^2(\Omega) \cap C^1(\bar{\Omega}), \quad \Phi^\ell = 0 \quad \text{on } \partial\Omega, \quad |\nabla x \Phi^\ell| \leq g/4 \quad \text{in } \bar{\Omega} := \Omega \cup \partial\Omega.
\]

Then we can solve the characteristics $(X^{\ell+1}(t; x, v), V^{\ell+1}(t; x, v))$, as in (2.12), to
\[
\begin{align*}
egthinspace
\frac{dX^{\ell+1}}{dt} & = V^{\ell+1}, \quad \frac{dV^{\ell+1}}{dt} = -\nabla x \Phi^\ell(X^{\ell+1}) + ge_3, \\
V^{\ell+1}(0; x, v) & = v,
\end{align*}
\]
with $X^{\ell+1}(t; x, v)|_{t=0} = x$ and $V^{\ell+1}(t; x, v)|_{t=0} = v$ and $e_3 = (0, 0, 1)^T$. A continuous-in-$(t, x, v)$ solution exists uniquely due to the Picard theorem. As long as $(X^{\ell+1}, V^{\ell+1}) \in \Omega \times \mathbb{R}^3$ exists, then
\[
\begin{align*}
egthinspace
V^{\ell+1}(t; x, v) & = v + \int_0^t \left( -\nabla x \Phi^\ell(X^{\ell+1}(s; x, v)) + ge_3 \right)\,ds, \\
egthinspace
X^{\ell+1}(t; x, v) & = x + vt + \int_0^t \int_0^s \left( -\nabla x \Phi^\ell(X^{\ell+1}(\tau; x, v)) + ge_3 \right)\,d\tau\,ds.
\end{align*}
\]

With $(X^{\ell+1}, V^{\ell+1})$, we define the backward exit time, position, and velocity as in Definition 2.1:
\[
\begin{align*}
egthinspace
\tau^{\ell+1}_b(x, v) & = \sup\{s \in [0, \infty) : X^{\ell+1}_3(-\tau; x, v) \text{ for all } \tau \in (0, s) \} \geq 0, \\
x^{\ell+1}_b(x, v) & = X^{\ell+1}(-\tau^{\ell+1}_b(x, v); x, v), \quad v^{\ell+1}_b(x, v) = V^{\ell+1}(-\tau^{\ell+1}_b(x, v); x, v).
\end{align*}
\]

From (2.2), it is easy to check that $\tau^{\ell+1}_b(x, v) < \infty$ for each $(x, v) \in \bar{\Omega} \times \mathbb{R}^3$.

Now we define that, for a given in-flow boundary datum $G$ in (1.4),
\[
h^{\ell+1}(x, v) = G(x^{\ell+1}_b(x, v), v^{\ell+1}_b(x, v)).
\]

Note that this $h^{\ell+1}(x, v)$ is a unique solution for given $\Phi^\ell$ to
\[
\begin{align*}
egthinspace
v \cdot \nabla x h^{\ell+1} - \nabla x (\Phi^\ell + gx_3) \cdot \nabla x h^{\ell+1} & = 0 \quad \text{in } \Omega, \\
h^{\ell+1} & = G \quad \text{on } \gamma_-.
\end{align*}
\]

Then we define the density
\[
\rho^{\ell+1}(x) = \int_{\mathbb{R}^3} h^{\ell+1}(x, v)\,dv \quad \text{in } \Omega.
\]
Next, as (2.20), we define a weight function which is invariant along the characteristics
\[ w^\ell_{\beta}(x, v) = e^\beta(|v|^2 + 2\Phi^\ell(x) + 2gx_3), \]
(5.10)
Note that, as (2.24), at the boundary
\[ w^\ell_{\beta}(x, v) = e^{\beta|v|^2} \text{ on } x \in \partial \Omega. \]
(5.11)
Using (5.6), (5.8), and (5.11), as long as \( t^{\ell+1}_b(x, v) \leq \infty \), then we have
\[ w^\ell_{\beta}(x, v)h^\ell+1(x, v) = w^\ell_{\beta}(x^b_\ell(x, v), t^{\ell+1}_b(x, v))G(x^b_\ell(x, v), v^\ell+1_b(x, v)) \]
\[ = e^{\beta|t^b_\ell(x, v)|^2}G(x^b_\ell(x, v), v^\ell+1_b(x, v)). \]
(5.12)

**Lemma 5.1.** For an arbitrary \( g > 0 \) and a given \( \nabla_x \Phi^\ell \), we assume that (5.1) and (5.2) hold. Then \( h^{\ell+1} \) solving (5.7)-(5.8) and \( \rho^{\ell+1} \) defined (5.9) satisfy the following estimates:
\[ \|h^{\ell+1}\|_{L^\infty(\bar{\Omega} \times \mathbb{R}^3)} \leq \|G\|_{L^\infty(\gamma_-)}, \]
(5.13)
\[ \|w^{\ell+1}_{\beta}h^{\ell+1}\|_{L^\infty(\bar{\Omega} \times \mathbb{R}^3)} \leq \|w_{\beta}G\|_{L^\infty(\gamma_-)}, \]
(5.14)
\[ \|e^{\beta gx_3}\rho^{\ell+1}\|_{L^\infty(\Omega)} \leq \pi^{3/2}\beta^{-3/2}\|w_{\beta}G\|_{L^\infty(\gamma_-)}. \]
(5.15)

Furthermore, \( \Phi^\ell(x_3) \in L^\infty \) and
\[ w^{\ell+1}_{\beta}(x, v) \geq e^{\beta(|v|^2 + gx_3)}. \]
(5.16)

**Proof.** From (5.6) and (5.12), we prove that (5.13) and (5.14), respectively.
Using (5.9) and (5.14), we have
\[ |\rho^\ell(x)| = \left| \int_{\mathbb{R}^3} w^\ell_{\beta}h^\ell(x, v) \frac{1}{w^\ell_{\beta}(x, v)} dv \right| \leq \|w_{\beta}G\|_{L^\infty(\gamma_-)} \int_{\mathbb{R}^3} \frac{dv}{w^\ell_{\beta}(x, v)}. \]
Since (5.2) holds, using (5.1) we derive that
\[ |\Phi^\ell(x_\parallel, x_3)| = \left| \frac{\Phi^\ell(x_\parallel, 0)}{\partial_3 \Phi^\ell(x_\parallel, y_3)dy_3} \right| \leq \frac{g}{2}x_3. \]
Therefore we deduce that
\[ w^{\ell}_{\beta}(x, v) \geq e^{\beta(|v|^2 + 2\Phi^\ell(x) + 2gx_3)} \geq e^{\beta(|v|^2 + gx_3)}, \]
which implies
\[ \int_{\mathbb{R}^3} \frac{1}{w^\ell_{\beta}(x, v)} dv \leq \int_{\mathbb{R}^3} e^{-\beta(|v|^2 + gx_3)} dv = \frac{\pi^{3/2}}{\beta^{3/2}} e^{-\beta gx_3}. \]
Therefore we conclude (5.15). In addition, we have derived (5.16). \( \square \)

Next, we move to construct \( h^{\ell+2} \). The starting point is defining \( \Phi^{\ell+1}(x) := \eta \int_{\Omega} G(x, y)\rho^{\ell+1}(y)dy \) as (3.2). From (3.1), we deduce that \( \Phi^{\ell+1} \) is a weak solution to
\[ \Delta \Phi^{\ell+1} = \eta \rho^{\ell+1} \text{ in } \Omega, \Phi^{\ell+1} = 0 \text{ on } \partial \Omega. \]
(5.17)
To repeat the process to construct \( h^{\ell+2} \) as in (5.3), we verify that \( \Phi^{\ell+1} \in C^2(\Omega) \cap C^1(\bar{\Omega}) \). We achieve this by establishing the regularity estimate of \( h^{\ell+1} \) and \( \rho^{\ell+1} \) and then using an elliptic estimate to (5.17).
5.1. Regularity Estimate. In this section, we establish a regularity estimate of \((h^{\ell+1}, \rho^{\ell+1})\) which are given in (5.6) and (5.9). We utilize the kinetic distance function (2.49):

\[
\alpha^{\ell+1}(x, v) = \sqrt{|v_3|^2 + |x_3|^2 + 2\partial x_3 \Phi(x, 0)x_3 + 2gx_3}.
\]  

(5.18)

Lemma 5.2. Suppose a condition (5.2) holds. For all \((x, v) \in \Omega \times \mathbb{R}^3\) and \(t \in [-t_b^{\ell+1}(x, v), 0]\),

\[
\alpha^{\ell+1}(X^{\ell+1}(t; x, v), V^{\ell+1}(t; x, v)) \\
\leq \alpha^{\ell+1}(x, v)e^{\left((1+\|\partial x_3 \Phi\|_{L^\infty(\Omega)})|t|\right)\frac{1}{1+\|\partial x_3 \Phi\|_{L^\infty(\Omega)}}} \int_0^t |V^{\ell+1}(s; x, v)|ds,
\]

\[
\alpha^{\ell+1}(x, v)e^{-\left((1+\|\partial x_3 \Phi\|_{L^\infty(\Omega)})|t|\right)\frac{1}{1+\|\partial x_3 \Phi\|_{L^\infty(\Omega)}}} \int_0^t |V^{\ell+1}(s; x, v)|ds
\]

(5.19)

In particular, the last inequality implies that

\[
|v_{b,3}^{\ell+1}(x, v)| \geq \alpha^{\ell+1}(x, v)e^{-\left((1+\|\partial x_3 \Phi\|_{L^\infty(\Omega)})|t_{b}^{\ell+1}(x, v)|\right)\frac{1}{1+\|\partial x_3 \Phi\|_{L^\infty(\Omega)}}} \int_0^t |V^{\ell+1}(s; x, v)|ds.
\]

(5.20)

Proof. Note that

\[
[v \cdot \nabla x - \nabla_x (\Phi^\ell(x) + gx_3) \cdot \nabla_v] \sqrt{|v_3|^2 + |x_3|^2 + 2\partial x_3 \Phi(x, 0)x_3 + 2gx_3}
\]

\[
= -(\partial_x \Phi^\ell(x) + g)v_3 + v_3 x_3 + \partial_x \Phi^\ell(x, 0)v_3 + v_3x_3 + v_3 \cdot \nabla_x \partial_x \Phi^\ell(x, 0)x_3
\]

\[
\leq \left(1 + \|\partial_x \partial_x \Phi^\ell\|_{L^\infty(\Omega)}\right)|v_3||x_3| + |v_3|\|\nabla_x \partial_x \Phi^\ell\|_{L^\infty(\partial \Omega)}|x_3|
\]

\[
\sqrt{|v_3|^2 + |x_3|^2 + 2\partial x_3 \Phi(x, 0)x_3 + 2gx_3}.
\]

where we have used \(-\partial_x \Phi^\ell(x, x_3) + \partial_x \Phi^\ell(x, x_3) = \int_{x_3}^{x_3} \partial_x \partial_x \Phi^\ell(x, y_3)|y_3|d_3\). Using (2.44), we deduce that

\[
[|v \cdot \nabla x - \nabla_x (\Phi^\ell(x) + gx_3) \cdot \nabla_v] \alpha^{\ell+1}(x, v)
\]

\[
\leq \left(1 + \|\partial_x \partial_x \Phi^\ell\|_{L^\infty(\Omega)}\right) + \frac{1}{g}|v_3|\|\nabla_x \partial_x \Phi^\ell\|_{L^\infty(\partial \Omega)}\alpha^{\ell+1}(x, v).
\]

By the Gronwall’s inequality, we conclude both inequalities of (5.19). The inequality (5.20) is a direct consequence of (5.19).

\]

Proof. The proof of (5.21) follows one for (2.26). Now we prove (5.22). Using (5.20), (5.3), and (5.21), we have

\[
\int_{-t_b^{\ell+1}(x, v)}^0 |V^{\ell+1}(s; x, v)|ds \leq |v_{b,3}^{\ell+1}| \frac{|t_b^{\ell+1}|^2}{2} |\nabla x \Phi^\ell|_{L^\infty(\Omega)} \leq \frac{4}{g} \left(1 + \frac{2}{g} |\nabla x \Phi^\ell|_{L^\infty(\Omega)} \right) |t_b^{\ell+1}|^2,
\]

and therefore we prove (5.22).
Lemma 5.4. Suppose (5.4) holds for all \((t, x, v)\). Then
\[
\partial_x X_j^{t+1}(t; x, v) = \delta_{ij} + \int_0^t \partial_x V_j^{t+1}(s; x, v) ds
\]
\[
= \delta_{ij} - \int_0^t \int_0^s \partial_x X^{t+1}(\tau; x, v) \cdot \nabla_x \partial_x \Phi(X^{t+1}(\tau; x, v)) d\tau ds,
\]  
(5.23)
\[
\partial_v V_j^{t+1}(t; x, v) = - \int_0^t \partial_x X^{t+1}(s; x, v) \cdot \nabla_x \partial_x \Phi(X^{t+1}(s; x, v)) ds,
\]
and
\[
\partial_v X_j^{t+1}(t; x, v) = t \delta_{ij} - \int_0^t \int_0^s \partial_v X^{t+1}(\tau; x, v) \cdot \nabla_x \partial_x \Phi(X^{t+1}(\tau; x, v)) d\tau ds,
\]  
(5.24)
\[
\partial_v V_j^{t+1}(t; x, v) = \delta_{ij} - \int_0^t \partial_v X^{t+1}(s; x, v) \cdot \nabla_x \partial_x \Phi(X^{t+1}(s; x, v)) ds.
\]
Moreover,
\[
|\nabla_x X^{t+1}(t; x, v)| \leq \min \left\{ e^{\frac{t^2}{2} \|\nabla^2 \Phi\|_{L\infty(\Omega)}} e^{(1+\|\nabla^2 \Phi\|_{L\infty(\Omega)})t} \right\},
\]  
(5.25)
\[
|\nabla_x V^{t+1}(t; x, v)| \leq \min \left\{ t \|\nabla^2 \Phi\|_{L\infty(\Omega)} e^{\frac{t^2}{2} \|\nabla^2 \Phi\|_{L\infty(\Omega)}} e^{(1+\|\nabla^2 \Phi\|_{L\infty(\Omega)})t} \right\},
\]  
(5.26)
\[
|\nabla_v X^{t+1}(t; x, v)| \leq \min \left\{ t e^{\frac{t^2}{2} \|\nabla^2 \Phi\|_{L\infty(\Omega)}} e^{(1+\|\nabla^2 \Phi\|_{L\infty(\Omega)})t} \right\},
\]  
(5.27)
\[
|\nabla_v V^{t+1}(t; x, v)| \leq \min \left\{ e^{\frac{t^2}{2} \|\nabla^2 \Phi\|_{L\infty(\Omega)}} e^{(1+\|\nabla^2 \Phi\|_{L\infty(\Omega)})t} \right\}.
\]  
(5.28)
Proof. From (5.4), we directly compute (5.23) and (5.24).
For \(\partial_x X_j^{t+1}\) and \(\partial_v X_j^{t+1}\), we change the order of integrals in each last double integral to get
\[
\partial_x X_j^{t+1} = \delta_{ij} - \int_0^t (t - \tau) \partial_x X^{t+1}(\tau; x, v) \cdot \nabla_x \partial_x \Phi(X^{t+1}(\tau; x, v)) d\tau,
\]
\[
\partial_v X_j^{t+1} = t \delta_{ij} - \int_0^t (t - \tau) \partial_v X^{t+1}(\tau; x, v) \cdot \nabla_x \partial_x \Phi(X^{t+1}(\tau; x, v)) d\tau.
\]
Now applying the Gronwall’s inequality, we derive that
\[
|\partial_x X^{t+1}(t; x, v)| \leq e^{\int_0^t (t - \tau) \|\nabla_x^2 \Phi\|_{L\infty(\Omega)} d\tau} \leq e^{\frac{t^2}{2} \|\nabla_x^2 \Phi\|_{L\infty(\Omega)}},
\]  
(5.29)
\[
|\partial_v X^{t+1}(t; x, v)| \leq t e^{\int_0^t (t - \tau) \|\nabla_x^2 \Phi\|_{L\infty(\Omega)} d\tau} \leq t e^{\frac{t^2}{2} \|\nabla_x^2 \Phi\|_{L\infty(\Omega)}}.
\]
Next using (5.29) and the second identities of (5.23), we have
\[
|\partial_x V_j^{t+1}(t; x, v)| \leq t \|\nabla_x^2 \Phi\|_{L\infty(\Omega)} e^{\frac{t^2}{2} \|\nabla_x^2 \Phi\|_{L\infty(\Omega)}}.
\]  
(5.30)
From the second line of (5.24) and the \(v_1\)-derivative to the first line of (5.4), we obtain that
\[
|\partial_v V_j^{t+1}(t; x, v)| = \left| \delta_{ij} - \int_0^t \int_0^s \partial_v V_j^{t+1}(\tau; x, v) \cdot \nabla_x \partial_x \Phi(X^{t+1}(\tau; x, v)) d\tau ds \right|
\]
\[
= \left| \delta_{ij} - \int_0^t \partial_v V_j^{t+1}(\tau; x, v) \cdot \int_0^t \nabla_x \partial_x \Phi(X^{t+1}(\tau; x, v)) d\tau ds \right|
\]
\[
\leq \delta_{ij} + \int_0^t (t - \tau) \|\nabla_x^2 \Phi\|_{L\infty(\Omega)} |\partial_v V_j^{t+1}(\tau; x, v)| d\tau,
\]
and hence, by the Gronwall’s inequality,
\[
|\partial_v V^{\ell+1}(t; x, v)| \leq e^t \|\nabla^2 \Phi\|_{L^\infty(\Omega)}.
\] (5.31)

On the other hand, we could derive the following inequalities by adding the first equalities of (5.23) and (5.24):
\[
|\nabla_x X^{\ell+1}(t; x, v)| + |\nabla_x V^{\ell+1}(t; x, v)| \leq 1 + \int_0^t \left\{1 + |\nabla^2 \Phi(X(s; x, v))|\right\}\{|\nabla_x X^{\ell+1}| + |\nabla_x V^{\ell+1}|\}ds,
\]
\[
|\nabla_v X^{\ell+1}(t; x, v)| + |\nabla_v V^{\ell+1}(t; x, v)| \leq 1 + \int_0^t \left\{1 + |\nabla^2 \Phi(X^{\ell+1}(s; x, v))|\right\}\{|\nabla_v X^{\ell+1}| + |\nabla_v V^{\ell+1}|\}ds.
\]

Using the Gronwall’s inequality, we derive that
\[
|\nabla_x X^{\ell+1}(t; x, v)| + |\nabla_x V^{\ell+1}(t; x, v)| \leq e^{(1+\|\nabla^2 \Phi\|_{L^\infty(\Omega)}t),
\]
\[
|\nabla_v X^{\ell+1}(t; x, v)| + |\nabla_v V^{\ell+1}(t; x, v)| \leq e^{(1+\|\nabla^2 \Phi\|_{L^\infty(\Omega)}t)}.
\] (5.32)

Finally we finish the proof of (5.25)-(5.28) by combining (5.29), (5.30), (5.31), and (5.32).

**Lemma 5.5.** Recall \((f^{\ell+1}_b(x, v), x^{\ell+1}_b(x, v), v^{\ell+1}_b(x, v))\) in Definition 2.1. The following identities hold:

\[
\partial_{x_i}f^{\ell+1}_b(x, v) = \frac{\partial_{x_i}X_3^{\ell+1}(-f^{\ell+1}_b(x, v); x, v)}{v^{\ell+1}_b(x, v)}
\]
\[
= \frac{1}{v^{\ell+1}_b(x, v)}\left\{\delta_{3j} - \int_0^{t_b^{\ell+1}(x, v)} \int_0^s \partial_{x_i}X^{\ell+1}(\tau; x, v) \cdot \nabla_x \partial_{x_j}X^{\ell+1}(\tau; x, v)d\tau ds\right\},
\] (5.33)
\[
\partial_{x_i}x^{\ell+1}_b(x, v) = \frac{\partial_{x_i}X_3^{\ell+1}(-f^{\ell+1}_b(x, v); x, v)}{v^{\ell+1}_b(x, v)}
\]
\[
= \frac{1}{v^{\ell+1}_b(x, v)}\left(f^{\ell+1}_b(x, v)\delta_{3j} - \int_0^{t_b^{\ell+1}(x, v)} \int_0^s \partial_{x_i}X^{\ell+1}(\tau; x, v) \cdot \nabla_x \partial_{x_j}X^{\ell+1}(\tau; x, v)d\tau ds\right),
\] (5.34)

and
\[
\partial_{x_i}v^{\ell+1}_b(x, v) = \frac{\partial_{x_i}X_3^{\ell+1}(-f^{\ell+1}_b(x, v); x, v)}{v^{\ell+1}_b(x, v)}\nabla_x \Phi(X^{\ell+1}_b(x, v))
\]
\[
- \int_0^{t_b^{\ell+1}(x, v)} (\partial_{x_i}X^{\ell+1}(s; x, v) \cdot \nabla_x \Phi(X^{\ell+1}(s; x, v))ds,
\] (5.35)
\[
\partial_{x_i}v^{\ell+1}_b(x, v) = \frac{\partial_{x_i}X_3^{\ell+1}(-f^{\ell+1}_b(x, v); x, v)}{v^{\ell+1}_b(x, v)}\nabla_x \Phi(X^{\ell+1}_b(x, v))
\]
\[
+ e_1 - \int_0^{t_b^{\ell+1}(x, v)} (\partial_{x_i}X^{\ell+1}(s; x, v) \cdot \nabla_x \Phi(X^{\ell+1}(s; x, v))ds.
\] (5.36)
Proof. By taking derivatives to $X^\ell_3(-t^\ell_3(x,v); x,v) = 0$ (see (2.1)), we get
\[-\partial_x t^\ell_3(x,v) V^\ell_3(-t^\ell_3(x,v); x,v) + \partial_x X^\ell_3(-t^\ell_3(x,v); x,v) = 0,
\]
which imply the first identities of (5.33) and (5.34). Then using the first identity of (5.23), we derive (5.34). Similarly, using the first identity of (5.24), we have (5.34).

By taking derivatives to $x^\ell_b(x,v)$ in (2.17), we derive that
\[
\partial_x x^\ell_b(x,v) = -\partial_x t^\ell_b(x,v) + \partial_x X^\ell_b(-t^\ell_b(x,v); x,v),
\]
\[
\partial_v x^\ell_b(x,v) = -\partial_v t^\ell_b(x,v) + \partial_v X^\ell_b(-t^\ell_b(x,v); x,v),
\]
Then using the first identities of (5.23) and (5.24), we conclude (5.35) and (5.36).

Finally we take derivatives to $v^\ell_b(x,v)$ in (2.17)
\[
\partial_x v^\ell_b(x,v) = -\partial_x t^\ell_b(x,v) V^\ell_b(x,v) + \partial_x X^\ell_b(-t^\ell_b(x,v); x,v)
\]
\[
= \partial_x t^\ell_b(x,v) \nabla \Phi(x^\ell_b(x,v)) + \partial_v X^\ell_b(-t^\ell_b(x,v); x,v),
\]
\[
\partial_v v^\ell_b(x,v) = \partial_v t^\ell_b(x,v) \nabla \Phi(x^\ell_b(x,v)) + \partial_v X^\ell_b(-t^\ell_b(x,v); x,v).
\]
Finally we use the second identities of (5.23) and (5.24) and conclude (5.37) and (5.38). \hfill \Box

Lemma 5.6.
\[
|\partial_x x^\ell_b(x,v)| \leq \frac{|v^\ell_b(x,v)|}{|v^\ell_b(x,v)|} \|\nabla^2 \Phi^\ell_b\|_\infty |||t^\ell_b t^\ell_b||^2 \min \left\{ e^{\frac{|t^\ell_b t^\ell_b|}{2}} ||\nabla^2 \Phi^\ell_b\|_\infty, e^{(1+\|\nabla^2 \Phi^\ell_b\|_\infty) t_b} \right\},
\]
(5.39)
\[
|\partial_v x^\ell_b(x,v)| \leq \frac{|v^\ell_b(x,v)|}{|v^\ell_b(x,v)|} \|\nabla^2 \Phi^\ell_b\|_\infty |||t^\ell_b t^\ell_b||^2 \min \left\{ t_b e^{\frac{|t^\ell_b t^\ell_b|}{2}} ||\nabla^2 \Phi^\ell_b\|_\infty, e^{(1+\|\nabla^2 \Phi^\ell_b\|_\infty) t_b} \right\},
\]
(5.40)
\[
|\partial_x v^\ell_b(x,v)| \leq \frac{|\partial_x \Phi^\ell_b(x^\ell_b(x,v))|}{|v^\ell_b(x,v)|} \|\nabla^2 \Phi^\ell_b\|_\infty |||t^\ell_b t^\ell_b|| \min \left\{ e^{\frac{|t^\ell_b t^\ell_b|}{2}} ||\nabla^2 \Phi^\ell_b\|_\infty, e^{(1+\|\nabla^2 \Phi^\ell_b\|_\infty) t_b} \right\},
\]
(5.41)
\[
|\partial_v v^\ell_b(x,v)| \leq \frac{|\partial_x \Phi^\ell_b(x^\ell_b(x,v))|}{|v^\ell_b(x,v)|} \|\nabla^2 \Phi^\ell_b\|_\infty |||t^\ell_b t^\ell_b|| \min \left\{ t_b e^{\frac{|t^\ell_b t^\ell_b|}{2}} ||\nabla^2 \Phi^\ell_b\|_\infty, e^{(1+\|\nabla^2 \Phi^\ell_b\|_\infty) t_b} \right\},
\]
(5.42)

Proof. Lemma 5.4 and Lemma 5.5 yield the estimates. \hfill \Box

Lemma 5.7. Recall $(h^\ell_1, h^\ell_1)$, which are constructed in (5.6) and (5.9). Suppose the condition (5.2) holds. For arbitrary numbers $\beta > \tilde{\beta} > 0$, we assume that
\[
\|e^{\beta |v|} G(x,v)\|_{L^\infty(\Omega_\gamma)} + \|e^{\tilde{\beta} |v|} \nabla_x G(x,v)\|_{L^\infty(\Omega_\gamma)} < \infty
\]
and
\[
\frac{16}{g^2} \|\nabla^2 \Phi^\ell_b\|_{L^\infty(\Omega)} \leq \tilde{\beta},
\]
(5.43)
Then for \((x, v) \in \bar{\Omega} \times \mathbb{R}^3\)

\[
\begin{align*}
  w^{\ell+1}_{\beta/2}(x, v)|\nabla_x h^{\ell+1}(x, v)| &\lesssim \left(1 + \frac{1}{g^\beta/2}\right) \|e^{\tilde{\beta}|v|^2/2} \nabla_{\|, v} G\|_{L^\infty(\gamma_-)}, \\
  w^{\ell+1}_{\beta/2}(x, v)|\partial_x h^{\ell+1}(x, v)| &\lesssim \left[\left(1 + \frac{1}{g^\beta/2}\right) + \left(1 + \frac{1}{\beta^3/2}\right) \delta_\beta\right] \|e^{\tilde{\beta}|v|^2/2} \nabla_{\|, v} G\|_{L^\infty(\gamma_-)},
\end{align*}
\]  

(5.44)

(5.45)

where \(w^{\ell+1}_{\beta/2}(x, v) \geq e^{\tilde{\beta}|v|^2/2} x^2 x_3\).

Moreover, for all \(x \in \bar{\Omega}\),

\[
\begin{align*}
  e^{\tilde{\beta} x_3} |\partial_x h^{\ell+1}(x)| &\lesssim \frac{1}{\beta^3/2} (1 + \frac{1}{\beta^3/2}) \|e^{\tilde{\beta}|v|^2/2} \nabla_{\|, v} G\|_{L^\infty(\gamma_-)} \\
  &\quad + \frac{\delta_\beta^3}{\beta} \left(1 + \frac{1}{\beta^3/2}\right) (1 + 1_{|x_3| \leq 1} \ln(|x_3|^2 + g x_3)) + \frac{1}{\beta^3/2} \|e^{\tilde{\beta}|v|^2/2} \nabla_{\|, v} G\|_{L^\infty(\gamma_-)},
\end{align*}
\]  

(5.46)

For \(0 < \delta < 1\),

\[
[r^{\ell+1}]_{C^0, \delta} \lesssim \delta \frac{1}{\beta} \left(1 + \frac{1}{\beta^3/2} + \frac{1}{g^\beta}\right) \|e^{\tilde{\beta}|v|^2/2} \nabla_{\|, v} G\|_{L^\infty(\gamma_-)}.
\]  

(5.47)

Furthermore, \(\Phi^\ell \in C^1(\bar{\Omega}) \cap C^2(\Omega)\) and

\[
\begin{align*}
\|\nabla_x \Phi^{\ell+1}\|_{L^\infty(\Omega)} &\leq c_1 x^{3/2}_3 \frac{1}{\beta^3/2} \left(1 + \frac{2}{\beta^3/2}\right) \|e^{\tilde{\beta}|v|^2/2} G\|_{L^\infty(\gamma_-)} \\
&\leq c_1 \frac{\delta_\beta^3}{\beta} \|e^{\tilde{\beta}|v|^2/2} G\|_{L^\infty(\gamma_-)} \left\{\frac{1}{g^\beta} + \log \left(e + \frac{1}{\beta} \left(1 + \frac{1}{\beta^3/2} + \frac{1}{g^\beta}\right) \|e^{\tilde{\beta}|v|^2/2} \nabla_{\|, v} G\|_{L^\infty(\gamma_-)}\right)\right\}.
\end{align*}
\]  

(5.48)

(5.49)

**Remark 5.8.** To iterate this construction of sequence of solutions, we need to verify the iteration assumption (5.2) and (5.43).

**Proof.** Taking a derivative to (5.12), we derive that

\[
\nabla_{x, v} h^{\ell+1}(x, v) = \nabla_{x, v} x^{\ell+1} \cdot \nabla_x G(z^{\ell+1}_b) + \nabla_{x, v} v^{\ell+1} \cdot \nabla_v G(z^{\ell+1}_b),
\]  

(5.50)

where \(z^{\ell+1}_b = (x^{\ell+1}_b, v^{\ell+1}_b)\) and their derivatives in the right hand side were evaluated at \((x, v)\).

**Step 1. Proof of (5.44).** From (5.50),

\[
\|\nabla_{x, v} h^{\ell+1}(x, v)\| \leq \frac{\|\nabla_{x, v} x^{\ell+1}(x, v)\|}{e^{\beta|v^{\ell+1}(x, v)|/2}} \|e^{\tilde{\beta}|v|^2/2} \nabla_{\|, v} G\|_{L^\infty(\gamma_-)}.
\]  

(5.51)

Using (5.40), (5.21), and (5.43), we have

\[
\frac{\|\partial_{v_b} x^{\ell+1}_b(x, v)\|}{e^{\beta|v^{\ell+1}_b(x, v)|/2}} \leq \frac{4}{g} |v^{\ell+1}_b| e^{-\beta|v^{\ell+1}_b|^2} \delta_\beta + \left(1 + \frac{8}{g^2} \|\nabla^2 \Phi\|_{\infty} |v^{\ell+1}_b| |v^{\ell+1}_b, v^{\ell+1}_b|\right) e^{-\beta|v^{\ell+1}_b|^2/2} \\
\times \min \left\{\frac{4|x^{\ell+1}_b|}{g}, \frac{8}{g^2} \|\nabla^2 \Phi\|_{\infty} |v^{\ell+1}_b, v^{\ell+1}_b|, \frac{8}{g} \right\} \leq \frac{16}{g^2 \beta} \left(1 + \frac{8}{g^2 \beta} \|\nabla^2 \Phi\|_{\infty} \right) e^{-\beta|v^{\ell+1}_b|^2/2} \leq \frac{24}{g^2 \beta} w^{\ell+1}(x, v),
\]  

(5.52)

Here, we have used the following lower bound, from (5.2),

\[
\frac{|v^{\ell+1}_b(x, v)|^2}{2} = \frac{|v|^2}{2} + \Phi^\ell(x) + g x_3 \geq \frac{|v|^2}{2} + (g - \|\nabla \Phi\|_{\infty}) x_3 \geq \frac{|v|^2}{2} + \frac{g}{2} x_3.
\]  

(5.53)
Similarly, using (5.42), (5.21), and (5.53), we derive that
\[
\frac{|\partial_x v_b^{\ell+1}(x,v)|}{e^{\tilde{\beta}|v_b^{\ell+1}|^2}} \leq \left(1 + \frac{4}{g} \|
abla_x \Phi\|_\infty\right)e^{-\tilde{\beta}|v_b^{\ell+1}|^2} + \left(1 + \frac{4}{g} \|
abla_x \Phi\|_\infty\right)\frac{4}{g} \|
abla_x^2 \Phi\|_\infty |v_b^{\ell+1}(x,v)|e^{-\tilde{\beta}|v_b^{\ell+1}|^2}
\]
\[\times \min \left\{ \frac{4|v_b^{\ell+1}|}{g} e^{\frac{4}{g} \|
abla_x^2 \Phi\|_\infty |v_b^{\ell+1}|^2}, e^{\frac{4}{g} (1 + \|
abla_x^2 \Phi\|_\infty) |v_b^{\ell+1}|^2} \right\}
\]
\[\leq \left(1 + \frac{4}{g} \|
abla_x \Phi\|_\infty\right)\left(1 + \frac{32}{g^2 \beta} \|
abla_x^2 \Phi\|_\infty\right)e^{-\frac{\tilde{\beta}}{2}|v_b|^2} \leq 9w_b^{\ell+1}(x,v).
\]

Finally we complete the prove of (5.44) using (5.51), (5.52), and (5.54) altogether.

Step 2. From (5.50),
\[
|\partial_x h_b^{\ell+1}(x,v)| \leq \frac{|\partial_x x_b^{\ell+1}(x,v)|}{e^{\tilde{\beta}|v_b^{\ell+1}|^2}} \left\|e^{\tilde{\beta}|v|} \nabla_x G\right\|_{L^\infty(\gamma_\cdot)} + \frac{|\partial_x v_b^{\ell+1}(x,v)|}{e^{\tilde{\beta}|v_b^{\ell+1}|^2}} \left\|e^{\tilde{\beta}|v|} \nabla_v G\right\|_{L^\infty(\gamma_\cdot)}.
\]

Using (5.22), (5.39), and (5.21), we derive that
\[
\frac{|\partial_x x_b^{\ell+1}(x,v)|}{e^{\tilde{\beta}|v_b^{\ell+1}|^2}} \leq \frac{|v_b^{\ell+1}(x,v)|}{|v_b^{\ell+1}(x,v)|} e^{-\tilde{\beta}|v_b^{\ell+1}|^2} \delta_{13} + \left(1 + \frac{8}{g} \|
abla_x^2 \Phi\|_\infty |v_b^{\ell+1}| |v_b^{\ell+1}|\right)e^{-\tilde{\beta}|v_b^{\ell+1}|^2}
\]
\[\times \min \left\{ e^{\frac{8}{g} \|
abla_x^2 \Phi\|_\infty |v_b^{\ell+1}|^2}, e^{\frac{4}{g} (1 + \|
abla_x^2 \Phi\|_\infty) |v_b^{\ell+1}|^2} \right\}
\]
\[\leq \frac{|v_b^{\ell+1}(x,v)|}{\alpha^{\ell+1}(x,v)} e^{\frac{4}{g} (1 + \|
abla_x^2 \Phi\|_\infty) |v_b^{\ell+1}|^2} e^{-\frac{\tilde{\beta}}{2}|v_b|^2} \leq \frac{16}{g^{3/2}} \left(1 + \frac{8}{g^2 \beta} \|
abla_x^2 \Phi\|_\infty\right) e^{-\frac{\tilde{\beta}}{2}|v_b|^2}
\]
\[\leq \left(\frac{\delta_{13}}{\beta^{1/2} \alpha^{\ell+1}(x,v)} + \frac{16}{g^{3/2}} \left(1 + \frac{8}{g^2 \beta} \|
abla_x^2 \Phi\|_\infty\right) \right) e^{-\frac{\tilde{\beta}}{2}|v|^2} e^{-\frac{\tilde{\beta}}{2}|x|^2}
\]
where we have used \(\frac{4}{g} (1 + \|
abla_x^2 \Phi\|_\infty |v_b|) + (\tilde{\beta} - \tilde{\beta} + \frac{4}{g} \|
abla_x^2 \Phi\|_\infty (1 + \frac{2}{g} \|
abla_x \Phi\|_\infty)) |v_b|^2 \leq \frac{4}{g} |v_b|^2\).

Similarly, using (5.22), (5.41), and (2.26), we derive that
\[
\frac{|\partial_v v_b^{\ell+1}(x,v)|}{e^{\tilde{\beta}|v_b^{\ell+1}|^2}} \leq \frac{\|
abla_x \Phi\|_\infty}{\|
abla_x \Phi\|_\infty} e^{-\tilde{\beta}|v_b^{\ell+1}|^2} \delta_{13}
\]
\[\left(1 + \frac{4}{g} \|
abla_x \Phi\|_\infty\right)\frac{4}{g} \|
abla_x^2 \Phi\|_\infty |v_b, b_3(x,v)|e^{-\tilde{\beta}|v_b^{\ell+1}|^2} \min \left\{ e^{\frac{4}{g} \|
abla_x^2 \Phi\|_\infty |v_b^{\ell+1}|^2}, e^{\frac{4}{g} (1 + \|
abla_x^2 \Phi\|_\infty) |v_b^{\ell+1}|^2} \right\}
\]
\[\leq \left(\frac{\delta_{13}}{\alpha^{\ell+1}(x,v)} + \left(1 + \frac{4}{g} \|
abla_x \Phi\|_\infty\right) \right) e^{-\frac{\tilde{\beta}}{2}|v|^2} e^{-\frac{\tilde{\beta}}{2}|x|^2} \leq \left(\frac{1/2}{\alpha^{\ell+1}(x,v)} \delta_{13} + 3\right) e^{-\frac{\tilde{\beta}}{2}|v|^2} e^{-\frac{\tilde{\beta}}{2}|x|^2}.
\]

Step 3. Proof of (5.46). From (5.2), we obtain a lower bound of \(\alpha^{\ell+1}(x,v) \geq \sqrt{|v_3|^2 + |x|^2} + g x_3\). We derive that
\[
\int_{\mathbb{R}^3} \frac{1}{\alpha^{\ell+1}(x,v)} e^{-\tilde{\beta}\left(\frac{|v_3|^2 + g x_3}{\alpha^{\ell+1}(x,v)}\right)} dv \leq \int_{\mathbb{R}^3} \frac{1}{\sqrt{|v_3|^2 + |x|^2 + g x_3}} e^{-\tilde{\beta}\left(\frac{|v_3|^2 + g x_3}{\alpha^{\ell+1}(x,v)}\right)} dv
\]
\[\leq \frac{2\pi}{\beta} e^{-\tilde{\beta}|x|^2} \int_{\mathbb{R}} \frac{e^{-\frac{4}{\beta} |v_3|^2}}{\sqrt{|v_3|^2 + |x|^2 + g x_3}} dv \leq \frac{8\pi}{\beta} e^{-\tilde{\beta}|x|^2} \left(1 + \mathbf{1}_{|x_3| \leq 1} \ln(|x|^2 + g x_3) + \tilde{\beta}^{-1/2}\right),
\]

(5.58)
where we have used the following computation of \( A = 1 \):

\[
\int_{\mathbb{R}} \frac{e^{-\frac{\hat{\delta}}{2} |v_3|^2}}{\sqrt{|v_3|^2 + |x_3|^2 + gx_3}} dv_3 = \int_{|v_3| \leq A} + \int_{|v_3| \geq A}
\]

\[
\leq 4 \int_0^A \frac{dr}{\sqrt{r^2 + |x_3|^2 + gx_3}} + \frac{4e^{-\frac{\hat{\delta} r^2}{4}}}{\sqrt{A^2 + |x_3|^2 + gx_3}} \int_A^\infty e^{-\frac{\hat{\delta} r^2}{4}} dr
\]

\[
\leq 4 \left( \ln |A + \sqrt{A^2 + |x_3|^2 + gx_3}| - \ln \sqrt{|x_3|^2 + gx_3} \right) + \frac{\sqrt{\frac{\hat{\delta}}{\beta}}}{\sqrt{A^2 + |x_3|^2 + g|x_3|}}
\]

\[
\leq 4 \{ 1 + 1|_{x_3} \leq 1 \} \ln((|x_3|^2 + gx_3)) \} + 4\tilde{\beta}^{-1/2}.
\]

Then it is straightforward to derive (5.46) using (5.45) and (5.58).

**Step 4. Proof of (5.49).** We will use (3.18). For \( 0 < |h| < 1 \), using (5.46) we derive that

\[
\frac{|\rho^{\ell+1}(x + he_i) - \rho^{\ell+1}(x)|}{|h|^\delta} \leq \frac{1}{|h|^\delta} \int_0^{|h|} |\nabla_x \rho^{\ell+1}(x + \tau e_i)| d\tau
\]

\[
\leq \| e^{\tilde{\beta} |v|^2} \nabla_{x |v| G} \|_{L^\infty(\gamma_-)} \left\{ |h|^{1-\delta} \frac{1}{\beta^{3/2}} \left( 1 + \frac{1}{g^{3/2}} \right) + \frac{\delta_3}{\beta} \left( 1 + \frac{1}{\beta^{1/2}} \right) \right\} \int_0^{|h|} \left( 1 + 1|_{x_3 + h|_{x_3} \leq 1} \right) \ln(|x_3 + \tau|^2 + g(x_3 + \tau)) + |\tilde{\beta}^{-1/2}| d\tau,
\]

(5.59)

as long as \( x_3 + he_3 \geq 0 \).

Note that, for \( 0 < |h| < 1 \),

\[
|h|^{-\delta} \int_0^{|h|} \ln(x_3 + \tau) d\tau \leq |h|^{-\delta} \left| |h| \ln(x_3 + |h|) + x_3 \ln(x_3 + |h|) - \ln x_3 \right| - |h| \leq |h|^{1-\delta} |\ln(x_3 + |h|)| + 2|h|^{1-\delta} \lesssim 1,
\]

and

\[
|h|^{-\delta} \int_{-\min\{h, x_3\}}^0 \ln(x_3 + \tau) d\tau
\]

\[
\leq |h|^{-\delta} \left\{ x_3 \ln x_3 - \ln(x_3 - \min\{h, x_3\}) \right\} + |h|^{-\delta} \ln\min\{h, x_3\} \ln(x_3 - \min\{h, x_3\}) + \ln x_3 + |h|^{-\delta} \lesssim 1.
\]

Using these bounds, we bound (5.59), for all \( i = 1, 2, 3 \), above by

\[
\sup_{0 < |h| < 1} \frac{|\rho^{\ell+1}(x + he_i) - \rho^{\ell+1}(x)|}{|h|^\delta} \lesssim \delta \left\{ \frac{1}{\beta^{3/2}} \left( 1 + \frac{1}{g^{3/2}} \right) + \frac{\delta_3}{\beta} \left( 1 + \frac{1}{\beta^{1/2}} \right) \right\} \| e^{\tilde{\beta} |v|^2} \nabla_{x |v| G} \|_{L^\infty(\gamma_-)}.
\]

(5.60)

Using (5.60), (5.15), and (3.18), we conclude (5.49). \( \square \)

5.2. **Construction of Sequences and their Stability.** Let us go back to the discussion right after (5.17). Using \( \Phi^{\ell+1} \in C^2(\Omega) \cap C^1(\overline{\Omega}) \) in (5.49), now we can repeat the process to construct \( h^{\ell+2} \) as in (5.3). In order to achieve the uniform-in-\( \ell \) estimates, we make sure the bound (5.49) guarantees (5.43).
Theorem 5.9. Suppose
\[ \mathcal{C} \pi^{3/2} (1 + 1/\beta) \left\{ \frac{1}{g^3} \log \left( 1 + \frac{1}{\beta} \right) \| e^{\beta |v|^2} \nabla G \|_{L^\infty(\gamma_-)} \right\} \leq \frac{g}{2}, \] (5.61)
\[ \mathcal{C}_1 \pi^{3/2} \| e^{\beta |v|^2} G \|_{L^\infty(\gamma_-)} \left\{ \frac{1}{g^3} + \log \left( e + \frac{1}{\beta} \right) \left( 1 + \frac{1}{\beta} \right) \| e^{\beta |v|^2} \nabla_x v G \|_{L^\infty(\gamma_-)} \right\} \leq \frac{g^2}{16}. \] (5.62)
where \( \mathcal{C}, \mathcal{C}_1 > 0 \) are the computable constants, which appeared in (5.48) and (5.49). Then we can construct \( \Phi^\ell, h^\ell+1, \rho^\ell+1, X^\ell+1, V^\ell+1 \) solve (5.7), (5.8), (5.9), (5.17), (5.3). Moreover they satisfy (5.2) and (5.43)-(5.49).

Proof. We set \( \Phi^0 = 0 \) and \( h^0 \equiv 0 \). Then we solve the characteristics \((X^1, V^1)\) to (5.3) and initial condition with \( \ell = 0 \). Clearly \((X^1, V^1) \in C^1\). Then now we define \( h^1, t^1_b, x^1_b, v^1_b \) as in (5.6) and (5.5) with \( \ell = 0 \). Using Lemma 5.1 and (5.15), we derive that \( \| e^{\beta |x|^3} \|_{L^\infty(\Omega)} \leq \frac{\pi^{3/2}}{g^3} \| \nabla G \|_{L^\infty(\gamma_-)} \). Then using (5.48) and (5.49), we verify the iteration assumptions (5.2) and (5.43) for \( \ell = 1 \). Therefore using Lemma 5.7, we can iterate this process to construct \( \Phi^\ell \), then \((X^\ell+1, V^\ell+1)\) and \( h^\ell+1 \) for \( \ell = 1, 2, \ldots \). \( \square \)

To pass a limit of the sequences we prove a stability lemma, which is very helpful to prove both the stability a la Cauchy and uniqueness of a limiting solution.

Lemma 5.10. For given \( \bar{h}_i(x, v) \) such that \( \bar{p}_i := \int \bar{h}_i dv \in C^{0, \delta}(\Omega) \) for some \( \delta > 0 \), suppose \( \Phi_i \in C^1(\Omega) \times C^2(\Omega) \) solves
\[ \Delta \Phi_i = \eta \bar{p}_i \text{ in } \Omega \times \mathbb{R}^3, \quad \Phi_i = 0 \text{ on } \partial \Omega. \]
Now we consider \( h_i(x, v) \) solving, in the sense of (2.18),
\[ v \cdot \nabla_x h_i - \nabla_x (\Phi_i + gx_3) \cdot \nabla_v h_i = 0 \text{ in } \Omega \times \mathbb{R}^3, \quad h_i = G \text{ on } \gamma_. \] (5.63)
Suppose the following two condition hold for \( g, \beta, \varepsilon_0 > 0 \)
\[ \frac{2\bar{\beta}}{\pi^{3/2}} \frac{|\Phi_1(x)|}{g^3/2} \left\{ 1 + \frac{4}{\beta} \right\} \| \nabla_v h_2 \|_{L^\infty(\Omega \times \mathbb{R}^3)} \leq \frac{\varepsilon_0}{2}, \] (5.64)
\[ \frac{2\bar{\beta}}{\pi^{3/2}} \frac{|\Phi_1(x)|}{g^3/2} \left\{ 1 + \frac{4}{\beta} \right\} \| \nabla_v h_2 \|_{L^\infty(\Omega \times \mathbb{R}^3)} \leq \frac{\varepsilon_0}{2}, \] (5.65)
where \( \mathcal{C} \) had appeared in (5.48).

Then for a small number \( \varepsilon_0 > 1 \), the following stability holds
\[ \| e^{\beta/2 (|v|^2 + gx_3)} (h_1(x, v) - h_2(x, v)) \|_{L^\infty(\Omega \times \mathbb{R}^3)} \leq \frac{1}{2} \| e^{\beta/2 (|v|^2 + gx_3)} (\bar{h}_1(x, v) - \bar{h}_2(x, v)) \|_{L^\infty(\Omega \times \mathbb{R}^3)}. \] (5.66)

Proof. Clearly the difference of two solutions solves
\[ v \cdot \nabla_x (h_1 - h_2) - \nabla_x (\Phi_1 + gx_3) \cdot \nabla_v (h_1 - h_2) = \nabla_x (\Phi_1 - \Phi_2) \cdot \nabla_v h_2 \text{ in } \Omega \times \mathbb{R}^3, \]
\[ h_1 - h_2 = 0 \text{ on } \partial \Omega \times \{ v_3 > 0 \}. \] (5.67)
Let \((X_1, V_1)\) be the characteristics solving (2.12) with \( \nabla_x \Phi = \nabla_x \Phi_1 \) and \( t_{b,1}(x, v) \) (as (2.1)) is the backward exit time of this characteristics \((X_1, V_1)\). Then, as \((h_1 - h_2)(X_1(-t_{b,1}(x,v);x,v)) \equiv 0, \)
\[ (h_1 - h_2)(x,v) = \int_{-t_{b,1}(x,v)}^0 (\nabla_x \Phi_1(X_1(s;x,v)) - \nabla_x \Phi_2(X_1(s;x,v))) \cdot \nabla_v h_2(X_1(s;x,v), V_1(s;x,v)) ds. \] (5.68)
Now we bound above the right hand side of (5.68) by
\[ t_{b,1}(x,v) \sup_{s \in [-t_{b,1}(x,v),0]} \left( \frac{1}{w_{\bar{\beta},1}(X_1(s;x,v), V_1(s;x,v))} \right) \| w_{\bar{\beta},1} \nabla_v h_2 \|_{L^\infty(\Omega)} \| \nabla_x \Phi_1 - \nabla_x \Phi_2 \|_{L^\infty(\Omega)}. \] (5.69)
Note that (5.64) implies
\[ w_{\hat{\beta},1}(x, v) = e^{\beta(|v|^2+2\Phi_1(x)+2gx_3)} \geq e^{\beta(|v|^2+gx_3)}. \quad (5.70) \]

Using Lemma 2.5 (and (2.26)), (2.22) and (5.70), we bound that
\begin{align*}
(5.69)_* & \leq \frac{2g^{-1}(\sqrt{|v_3|^2 + gx_3} - v_3)}{w_{\hat{\beta},1}(x, v)} \leq \frac{4}{g} \sqrt{|v_3|^2 + gx_3} e^{-\beta(|v|^2+gx_3)} \\
& \leq \frac{1}{g^{\beta/2}} \sqrt{\beta(|v|^2 + gx_3)} e^{-\beta(|v|^2+gx_3)} \leq \frac{1}{g^{\beta/2}} e^{-\frac{\beta}{2}(|v|^2+gx_3)}. \quad (5.71)
\end{align*}

On the other hand, using Lemma 3.3 ((3.17) with \( A = \|e^{\beta'gx_3}(\hat{\rho}_1 - \hat{\rho}_2)\|_{L^\infty} \) and \( B = \beta'g \) for \( \beta' < \beta \)), we derive that
\[ \|\nabla_x \Phi_1 - \nabla_x \Phi_2\|_{L^\infty(\Omega)} \leq C \left\{ 1 + \frac{2}{\beta'g} \right\} \left\| e^{\beta'gx_3}(\hat{\rho}_1 - \hat{\rho}_2) \right\|_{L^\infty(\Omega)}. \quad (5.72) \]

Using (2.43), we bound the underbraced term above by
\[ \left\| e^{\beta'gx_3}(\hat{\rho}_1 - \hat{\rho}_2) \right\|_{L^\infty(\Omega)} \leq \frac{\pi^{3/2}}{(\beta')^{3/2}} \left\| e^{\beta'(|v|^2+gx_3)}(\hat{h}_1(x,v) - \hat{h}_2(x,v)) \right\|_{L^\infty(\Omega)}. \quad (5.73) \]

Now combining above bounds together with (5.69) and (5.71), we conclude that
\[ |h_1(x,v) - h_2(x,v)| \leq \frac{C \pi^{3/2}}{g^{\beta/2} (\beta')^{3/2}} \left\{ 1 + \frac{2}{\beta'g} \right\} \left\| w_{\beta} \nabla_v h_2 \|_{L^\infty(\Omega)} e^{-\frac{\beta}{2}(|v|^2+gx_3)} \left\| e^{\beta'(|v|^2+gx_3)}(\hat{h}_1(x,v) - \hat{h}_2(x,v)) \right\|_{L^\infty(\Omega \times \mathbb{R}^3)}. \]

With a choice of \( \beta' = \tilde{\beta}/2 \) and (5.65), we bound the underbraced term for a sufficiently small \( \varepsilon_0 > 0 \) to get (5.66).

Finally, we prove the existence of a unique solution by passing a limit of the sequences in Theorem 5.9 and using the stability in Lemma 5.10.

**Proof of Theorem 2.8.** Let us first check that, if (5.44) and (2.40) hold then for \( \varepsilon_1 \ll \varepsilon_0 \) we have that
\[ \frac{2^{\beta/2} \pi^{3/2}}{g^{\beta/2}} \left\{ 1 + \frac{4}{\beta'g} \right\} w_{\beta}(x,v)|\nabla_v h_{\ell+1}(x,v)| \leq \frac{\varepsilon_0}{2} \]

Note that this bound guarantees (5.65) in Lemma 5.10. Therefore now we can apply Lemma 5.10 to the sequences of Theorem 5.9 with \( \tilde{\beta} = \frac{\beta}{2} \); \( \left\| e^{\frac{\beta}{2}(|v|^2+gx_3)}|h_{\ell+1}(x,v) - h_{\ell}(x,v)| \right\|_{L^\infty} \leq \frac{1}{2} \left\| e^{\frac{\beta}{2}(|v|^2+gx_3)}h_{\ell}(x,v) \right\|_{L^\infty} \). Then \( h_{\ell} \) is Cauchy: for all \( \ell, m \in \mathbb{N} \),
\[ \left\| e^{\frac{\beta}{2}(|v|^2+gx_3)}|h_{\ell}(x,v) - h_{m}(x,v)| \right\|_{L^\infty(\Omega \times \mathbb{R}^3)} \leq \frac{2}{2^{\min\{\ell,m\}}} \left\| e^{\frac{\beta}{2}(|v|^2+gx_3)}h_{\ell}(x,v) \right\|_{L^\infty(\Omega \times \mathbb{R}^3)}. \quad (5.74) \]

With this strong convergence together with uniform-upper-bounds of Theorem 5.9, it is standard to prove the convergence of the sequences and prove that their limiting function \((h, \rho, \Phi)\) is a strong solution to (2.1)-(2.4). Moreover, every upper bound of Theorem 5.9 is valid for the limiting function. Finally Lemma 5.10 implies the uniqueness of solution.

\[ \Box \]

6. Dynamic Solutions

In this section, we construct a global-in-time strong solution to the dynamic problem (2.5)-(2.9), and study their properties such as regularity and uniqueness.
6.1. **Construction of Sequences.** We construct a solution to the dynamic problem (2.5)-(2.9) via the following sequences: starting with \( f^0 \equiv 0 \), we set \( (\varphi^0, \Psi^0) = (0, 0) \); and then \( f^1 \) solves
\[
\partial_t f^1 + v \cdot \nabla_x f^1 - \nabla_x (\Phi + gx_3) \cdot \nabla_v f^1 = 0, \quad f^1|_{\gamma_-} = 0, \quad f^1|_{t=0} = f_0.
\]
(6.1)
Since \( \Phi + gx_3 \in C^1(\bar{\Omega}) \cap C^2(\Omega) \), the characteristics to (2.13) equals the steady characteristics \((X, V)\) of (2.12) and hence \( f^1 \) is defined as in (2.19) along the characteristics.

Suppose that \( \Psi^\ell \in C^1(\bar{\Omega}) \cap C^2(\Omega) \) satisfies
\[
\Delta \Psi^\ell = g^\ell := \eta \int_{\mathbb{R}^3} f^\ell dv, \quad \Psi^\ell|_{\partial \Omega} = 0.
\]
(6.2)
Note that \( \phi^\ell = \Psi^\ell + \Phi \).

The corresponding characteristics is
\[
Z^{\ell+1}(s; t, x, v) = (X^{\ell+1}(s; t, x, v), Y^{\ell+1}(s; t, x, v)),
\]
(6.3)
solving
\[
\frac{dX^{\ell+1}}{ds} = \gamma^{\ell+1}, \quad \frac{dY^{\ell+1}}{ds} = -\nabla_x \Psi^\ell - \nabla_x \Phi - g e_3,
\]
(6.4)
We define \( t_B^{\ell+1}(t, x, v), t_F^{\ell+1}(t, x, v), t_B^{\ell+1}(t, x, v), \) and \( t_B^{\ell+1}(t, x, v) \) as in Definition 2.1 but for the characteristics \( Z^{\ell+1} = (X^{\ell+1}, Y^{\ell+1}) \) in (6.3).

Then we successively construct solutions in the sense of Definition 2.2 along the characteristics as in (2.19) to the problem
\[
\partial_t f^{\ell+1} + v \cdot \nabla_x f^{\ell+1} - \nabla_x (\Psi^\ell + \Phi + gx_3) \cdot \nabla_v f^{\ell+1} = \nabla_x \Psi^\ell \cdot \nabla_v h,
\]
(6.5)
\[
f^{\ell+1}|_{\gamma_-} = 0,
\]
(6.6)
\[
f^{\ell+1}|_{t=0} = f_0 := F_0 - h.
\]
(6.7)
From (2.11), (2.10), and (4.4), we have
\[
b^\ell(t, x) := \int_{\mathbb{R}^3} v f^\ell(t, x, v)dv \quad \text{in} \quad \mathbb{R}_+ \times \Omega,
\]
(6.8)
\[
\partial_t g^\ell + \nabla_x \cdot b^\ell = 0 \quad \text{in} \quad \mathbb{R}_+ \times \Omega.
\]
(6.9)

**Remark 6.1.** The continuity equation (6.9) should hold in a weak sense against smooth test function with compact support. As what we have done for the steady solution construction, we will prove that the sequence \((f^{\ell+1}, \varphi^\ell, \Psi^\ell)\) belongs to some regularity space. Then, in Lemma 6.7 and Remark 6.8, we will derive that the continuity equation (6.9) holds in a strong sense so that the following identity is valid:
\[
\partial_t \Psi^\ell(t, x) = \eta \Delta_0^{-1} \partial_t \varphi^\ell(t, x) = -\eta \Delta_0^{-1} (\nabla_x \cdot b^\ell)(t, x) \quad \text{in} \quad \mathbb{R}_+ \times \Omega.
\]
(6.10)

Applying Lemma 2.5, we have the following result:

**Lemma 6.2.** Assume a bootstrap assumptions \( \Psi^\ell(t, \cdot) \in C^1(\bar{\Omega}) \cap C^2(\Omega) \) and
\[
\sup_{0 \leq \tau \leq t} \| \nabla_x \phi^\ell(\tau) \|_{L^\infty(\Omega)} = \sup_{0 \leq \tau \leq t} \| \nabla_x \Psi^\ell(\tau) + \nabla_x \Phi \|_{L^\infty(\Omega)} \leq \frac{g}{2}.
\]
(6.11)
Then we have that for all \( 0 \leq s \leq t \)
\[
t_B^{\ell+1}(s, x, v) \leq \frac{2}{g} \min \left\{ \sqrt{|v_3|^2 + gx_3} - v_3, \sqrt{|v_B^{\ell+1}(s, x, v)|^2 - gx_3} + t_B^{\ell+1}(s, x, v) \right\},
\]
(6.12)
\[
t_B^{\ell+1}(s, x, v) + t_F^{\ell+1}(s, x, v) \leq \frac{4}{g} \sqrt{|v_3|^2 + gx_3}.
\]
Define a dynamic weight for the sequence (cf. \( w_\beta \) in (2.21))

\[
\text{w}_\beta^{\ell+1}(s, x, v) = w_\beta(|v|^2 + 2\Phi(x) + 2\Psi^\ell(s, x) + 2g x_3) = e^{\beta(|v|^2 + 2\Phi(x) + 2\Psi^\ell(s, x) + 2g x_3)}. \tag{6.13}
\]

As (2.23), we have

\[
d\frac{d}{ds} \left( |\chi^{\ell+1}(s; t, z)|^2 + 2 \Phi_{tv}(X^{\ell+1}(s; t, z)) + 2g \chi_3^{\ell+1}(s; t, z) \right) = 2\partial_t \Psi^\ell(s, X^{\ell+1}(s; t, z)). \tag{6.14}
\]

**Lemma 6.3.** Suppose the assumption (6.11) holds. Then, for \( s, s' \in \{\max\{0, t - t_B^\ell(t, x, v)\}, t\} \) and \( \beta > 0 \),

\[
\frac{w^{\ell+1}_\beta(s', Z^{\ell+1}(s'; t, x, v))}{w^{\ell+1}_\beta(s, Z^{\ell+1}(s; t, x, v))} \leq e^{\frac{6\beta^3}{7} \|\partial_t \Psi^\ell\|_{L^\infty_{t,x}} \sqrt{|v|^2 + gx_3}}, \tag{6.15}
\]

and

\[
\frac{1}{w^{\ell+1}_\beta(s, Z^{\ell+1}(s; t, x, v))} \leq e^{\frac{64\beta^3}{7} \|\partial_t \Psi^\ell\|_{L^\infty_{t,x}} - \frac{\alpha}{\sqrt{|v|^2 + gx_3}}}. \tag{6.16}
\]

Here, we have used the notation \( L^\infty_{t,x} \) defined in (2.59).

**Proof.** Using (6.14), we derive that

\[
d\frac{d}{ds} w^{\ell+1}_\beta(s, Z^{\ell+1}(s; t, z)) = 2\beta \partial_t \Psi^\ell(s, Z^{\ell+1}(s; t, z)) w^{\ell+1}_\beta(s, Z^{\ell+1}(s; t, z)). \tag{6.17}
\]

Hence, if \( \max\{0, t - t_B^\ell(t, x, v)\} \leq s, s' \leq t \) then

\[
w^{\ell+1}_\beta(s, Z^{\ell+1}(s; t, z)) = w^{\ell+1}_\beta(s', Z^{\ell+1}(s'; t, z)) e^{2\beta \int_s^{s'} \partial_t \Psi^\ell(r, X^{\ell+1}(r, t, z)) dr}. \tag{6.18}
\]

Now we estimate the underlined term in the exponent of (6.18): Using (2.10), (2.27), we bound it above by

\[
|s - s'| \|\partial_t \Psi^\ell\|_{L^\infty([s', s] \times \Omega)} \leq |t_B(t, x, v) + t_F(t, x, v)| \|\partial_t \Psi^\ell\|_{L^\infty_{t,x}} \leq \|\partial_t \Psi^\ell\|_{L^\infty_{t,x}} \frac{4}{g} \sqrt{|v|^2 + gx_3}. \tag{6.19}
\]

Finally, we conclude (6.15) by evaluating (6.18) at \( s' = t \) and using (6.19):

\[
w^{\ell+1}_\beta(s, X^{\ell+1}(s; t, x, v)) \geq w^{\ell+1}_\beta(t, t, v)) e^{-\frac{\beta |v|^2 - 8\beta^3 \|\partial_t \Psi^\ell\|_{L^\infty_{t,x}} \sqrt{|v|^2 + gx_3}}{7}} \geq e^{\frac{64\beta^3}{7} \|\partial_t \Psi^\ell\|_{L^\infty_{t,x}} - \frac{\alpha}{\sqrt{|v|^2 + gx_3}}}. \tag{6.20}
\]

Now we prove (6.16). Using (2.13) and (6.14), we compute that

\[
\frac{d}{ds} \Psi^\ell(x, X^{\ell+1}(s; t, x, v)) = \partial_t \Psi^\ell(s, X^{\ell+1}(s; t, x, v)) + V^{\ell+1}(s; t, x, v) \cdot \nabla_x \Psi^\ell(s, X^{\ell+1}(s; t, x, v)) \tag{6.20'}
\]

where we have also used

\[
\frac{d}{ds} \Psi^\ell(s, X^{\ell+1}(s; t, x, v)) = \partial_t \Psi^\ell(s, X^{\ell+1}(s; t, x, v)) + V^{\ell+1}(s; t, x, v) \cdot \nabla_x \Psi^\ell(s, X^{\ell+1}(s; t, x, v)). \tag{6.21}
\]

This implies

\[
w_\beta(Z^{\ell+1}(s; t, x, v)) = w_\beta(Z^{\ell+1}(s'; t, x, v)) e^{2\beta \int_0^t \partial_t \Psi^\ell(r, X^{\ell+1}(r, t, x, v)) dr - 2\beta \int_0^t V^{\ell+1}(r, t, x, v) dr}. \tag{6.22}
\]

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Using the Dirichlet boundary condition (2.9), we estimate the underlined term in the exponent:

\[
\left| \Psi^\ell(s, \mathcal{X}^{\ell+1}(s; t, x, v)) - \Psi^\ell(s', \mathcal{X}^{\ell+1}(s'; t, x, v)) \right| \leq 2\|\partial_x \Psi^\ell\|_{L^\infty} \max_{s \in [t, t + 1]} \mathcal{X}^{\ell+1}_s(s; t, x, v)
\]
\[
\leq 2\|\partial_x \Psi^\ell\|_{L^\infty_{t,x}} \left( x_3 + |v_3| \frac{4}{g} \right) \leq \frac{10}{g} \|\partial_x \Psi^\ell\|_{L^\infty_{t,x}} (|v_3|^2 + g x_3).
\]

Therefore, we conclude (6.16) from (6.22), (6.23), and (6.19):

\[
w_3(\mathcal{Z}^{\ell+1}(s; t, x, v)) \\
\geq e^{\frac{\beta}{2} |v|^2} e^{\frac{\beta}{2} g x_3} e^{-\frac{\beta}{g} s} e^{-\frac{\beta}{2} (|v|^2 + g x_3)} e^\frac{\beta}{2} (|v|^2 + g x_3) e^\frac{\beta}{2} (|v|^2 + g x_3)
\]
\[
\geq e^{\frac{\beta}{2} |v|^2} e^{\frac{\beta}{2} g x_3} e^{-\frac{\beta}{g} s} e^{-\frac{\beta}{2} (|v|^2 + g x_3)} e^\frac{\beta}{2} (|v|^2 + g x_3) e^\frac{\beta}{2} (|v|^2 + g x_3).
\]

**Lemma 6.4.** For an arbitrary \( \ell \in \mathbb{N} \), we suppose the bootstrap assumption (6.11) holds for \( \Psi^\ell \in C^1(\Omega) \cap C^2(\Omega) \). Then \( \mathcal{F}^{\ell+1}(s, x, v) \) solving (6.5)-(6.7) and (6.2) in the sense of Definition 2.2 satisfies that, for all \( s, x, v \in [0, t] \times \Omega \times \mathbb{R}^3 \),

\[
e^{\frac{\beta}{2} |v|^2} e^{\frac{\beta}{2} g x_3} |\mathcal{F}^{\ell+1}(s, x, v)| + \beta^{-3/2} \sqrt{\beta} \frac{e^{\frac{\beta}{2} g x_3}}{e^{\frac{\beta}{2} g x_3}} |\mathcal{F}^{\ell+1}(s, x, v)|
\]
\[
\leq \frac{e^{\frac{\beta}{2} |v|^2} e^{\frac{\beta}{2} g x_3} |\mathcal{F}^{\ell+1}(s, x, v)| + \beta^{-3/2} \sqrt{\beta} \frac{e^{\frac{\beta}{2} g x_3}}{e^{\frac{\beta}{2} g x_3}} |\mathcal{F}^{\ell+1}(s, x, v)|}{\|w_3 h\|_{\infty}},
\]

where we have used (6.11). Since we already have bounded \( \|w_3 h\|_{\infty} \) in (2.42), it suffices to estimate \( |\mathcal{F}^{\ell+1}(t, x, v)| \).

Along the characteristics (6.4), for \( s \in [\max \{0, t - t_B^{\ell+1}(s, t, x, v)\}, t - t^{\ell+1}_B(t, x, v)] \),

\[
\frac{d}{ds} \mathcal{F}^{\ell+1}(s, \mathcal{Z}^{\ell+1}(s; t, x, v)) = 0.
\]

Using the boundary condition (1.4) and the initial datum, we derive that

\[
\mathcal{F}^{\ell+1}(t, x, v) = \mathcal{F}^{\ell+1}_0(t, x, v) G(\mathcal{Z}^{\ell+1}(t, x, v); t, x, v).
\]

Using Lemma 6.3 and (6.15), we derive that \( |\mathcal{F}^{\ell+1}(t, x, v)| \leq I_1 + I_2 \), where

\[
I_1 = \frac{\|w_3 \gamma \mathcal{F}^{\ell+1}_0\|_{L^\infty_{x,v}}}{\mathcal{Z}^{\ell+1}_0(0, t, x, v)} \leq e^{\frac{\beta}{2} |v|^2} e^{-\frac{\beta}{2} g x_3} \|w_3 \mathcal{F}^{\ell+1}_0\|_{L^\infty_{x,v}};
\]

\[
I_2 = \frac{\|w_3 \gamma \mathcal{F}^{\ell+1}_0\|_{L^\infty_{x,v}}}{\mathcal{Z}^{\ell+1}_0(t - t^{\ell+1}_B, t, x, v)} \leq e^{\frac{\beta}{2} |v|^2} e^{-\frac{\beta}{2} g x_3} \|w_3 \gamma \mathcal{F}^{\ell+1}_0\|_{L^\infty_{x,v}}.
\]

Here we have used the following facts from (2.25) and (2.24):

\[
\mathcal{Z}^{\ell+1}_0(0, x, v) = \mathcal{Z}^{\ell+1}_0(t, x, v) \quad \text{in } (x, v) \in \Omega \times \mathbb{R}^3,
\]

\[
\mathcal{Z}^{\ell+1}_0(t, x, v) = \mathcal{Z}^{\ell+1}_0(t, x, v) \quad \text{on } (x, v) \in \partial \Omega \times \mathbb{R}^3.
\]

Using this bound of (6.27), together with (6.27) and (6.28), we derive a bound of \( \mathcal{F}^{\ell+1}(t, x, v) \).

Then combining with (6.25), we can conclude this lemma.
6.2. Regularity Estimate. In this section we study the higher regularity of $F^{\ell+1}(t, x, v) = h(x, v) + f^{\ell+1}(t, x, v)$ that we have constructed in the previous step. Note that

$$F^{\ell+1}(t, x, v) = 1_{t \leq t_B(t, x, v)} F_0(\mathcal{X}^{\ell+1}(0; t, x, v), \mathcal{V}^{\ell+1}(0; t, x, v)) + 1_{t > t_B^{\ell+1}(t, x, v)} G(x^{\ell+1}_B(t, x, v), v^{\ell+1}_B(t, x, v)).$$

(6.31)

Assume a compatibility condition (2.50). Due to this compatibility condition (2.50), weak derivatives of $F^{\ell+1}(t, x, v)$ in (6.31) are

$$\partial_x F^{\ell+1}(t, x, v) = 1_{t \leq t_B(t, x, v)} \{ \partial_x \mathcal{X}^{\ell+1}(0) \cdot \nabla_x F_0(\mathcal{Z}^{\ell+1}(0)) + \partial_x \mathcal{V}^{\ell+1}(0) \cdot \nabla_v F_0(\mathcal{Z}^{\ell+1}(0)) \} + 1_{t > t_B^{\ell+1}(t, x, v)} \{ \partial_x x^{\ell+1}_B \cdot \nabla_x G + \partial_x v^{\ell+1}_B \cdot \nabla_v G \},$$

(6.32)

$$\partial_v F^{\ell+1}(t, x, v) = 1_{t \leq t_B(t, x, v)} \{ \partial_v \mathcal{X}^{\ell+1}(0) \cdot \nabla_x F_0(\mathcal{Z}^{\ell+1}(0)) + \partial_v \mathcal{V}^{\ell+1}(0) \cdot \nabla_v F_0(\mathcal{Z}^{\ell+1}(0)) \} + 1_{t > t_B^{\ell+1}(t, x, v)} \{ \partial_v x^{\ell+1}_B \cdot \nabla_x G + \partial_v v^{\ell+1}_B \cdot \nabla_v G \},$$

(6.33)

where $\mathcal{Z}^{\ell+1}(0) = \mathcal{Z}^{\ell+1}(0; t, x, v)$ and $(\mathcal{X}^{\ell+1}(0), \mathcal{V}^{\ell+1}(0)) = (\mathcal{X}^{\ell+1}(0; t, x, v), \mathcal{V}^{\ell+1}(0; t, x, v))$; and every $G$ is evaluated at $(t - t_B^{\ell+1}(t, x, v), x^{\ell+1}_B(t, x, v), v^{\ell+1}_B(t, x, v))$.

Following the same proof of Lemma 5.4, we can derive the following estimate ([7, 10]):

$$|\nabla_x \mathcal{X}^{\ell+1}(s; t, x, v)| \leq \min \{ e^{-\frac{|t-s|^2}{2}} \| \nabla^2 \phi_F \| \infty, e^{(1+\| \nabla^2 \phi_F \| \infty)} |t-s| \},$$

(6.34)

$$|\nabla_v \mathcal{V}^{\ell+1}(s; t, x, v)| \leq \min \{ |t-s| \| \nabla^2 \phi_F \| \infty, e^{(1+\| \nabla^2 \phi_F \| \infty)} |t-s| \},$$

(6.35)

$$|\nabla_v \mathcal{V}^{\ell+1}(s; t, x, v)| \leq \min \{ |t-s| \| \nabla^2 \phi_F \| \infty, e^{(1+\| \nabla^2 \phi_F \| \infty)} |t-s| \},$$

(6.36)

Here, we have used the notation $L^\infty_{t,x}$ defined in (2.59).

We also follow the same proof of Lemma 5.6 to get ([7, 10])

$$|\partial_x x^{\ell+1}_B(t, x, v)| \leq \frac{|x^{\ell+1}_B(t, x, v)|}{v^{\ell+1}_B(t, x, v)} \frac{|\nabla_x \phi_F| \infty}{|v^{\ell+1}_B(t, x, v)|} \min \{ e^{-\frac{|t-s|^2}{2}} \| \nabla^2 \phi_F \| \infty, e^{(1+\| \nabla^2 \phi_F \| \infty)} |t_B| \},$$

(6.38)

$$|\partial_v x^{\ell+1}_B(t, x, v)| \leq \frac{|x^{\ell+1}_B(t, x, v)|}{v^{\ell+1}_B(t, x, v)} \frac{|\nabla_x \phi_F| \infty}{|v^{\ell+1}_B(t, x, v)|} \min \{ e^{-\frac{|t-s|^2}{2}} \| \nabla^2 \phi_F \| \infty, e^{(1+\| \nabla^2 \phi_F \| \infty)} |t_B| \},$$

(6.39)

$$|\partial_x v^{\ell+1}_B(t, x, v)| \leq \frac{|v^{\ell+1}_B(t, x, v)|}{v^{\ell+1}_B(t, x, v)} \frac{|\nabla_x \phi_F| \infty}{|v^{\ell+1}_B(t, x, v)|} \min \{ e^{-\frac{|t-s|^2}{2}} \| \nabla^2 \phi_F \| \infty, e^{(1+\| \nabla^2 \phi_F \| \infty)} |t_B| \},$$

(6.40)

$$|\partial_v v^{\ell+1}_B(t, x, v)| \leq \frac{|v^{\ell+1}_B(t, x, v)|}{v^{\ell+1}_B(t, x, v)} \frac{|\nabla_x \phi_F| \infty}{|v^{\ell+1}_B(t, x, v)|} \min \{ e^{-\frac{|t-s|^2}{2}} \| \nabla^2 \phi_F \| \infty, e^{(1+\| \nabla^2 \phi_F \| \infty)} |t_B| \},$$

(6.41)
where we have abbreviated \( \ell_{B,3}^\ell = \ell_{B}^\ell(t, x, v) \), \( \| \nabla_x^2 \phi_{F^\ell} \|_\infty = \sup_{\tau \in [\ell, t]} \| \nabla_x^2 \phi_{F^\ell}(\tau) \|_{L^\infty(\Omega \times \mathbb{R}^3)} \) and \( \| \nabla_x \phi_{F^\ell} \|_\infty = \sup_{\tau \in [\ell, t]} \| \nabla_x \phi_{F^\ell}(\tau) \|_{L^\infty(\Omega \times \mathbb{R}^3)} \).

Again we utilize a kinetic distance for the dynamic problem (1.1). We define a dynamic kinetic distance ([17, 6])

\[
\alpha_{F^\ell}^{\ell+1}(t, x, v) = \sqrt{|v_3|^2 + |x_3|^2 + 2\partial_{x_3} \phi_{F^\ell}(t, x_3, 0)x_3 + 2gx_3},
\]

(6.42)

In particular, \( \alpha_{F^\ell}^{\ell+1}(t, x, v) = |v_3| \) when \( x \in \partial \Omega \) (i.e. \( x_3 = 0 \)) due (1.5).

**Lemma 6.5.** Assume (6.11) holds for \( \Psi^\ell \in C^1(\Omega) \cap C^2(\Omega) \). Suppose the Dirichlet boundary condition (1.5) holds for \( \phi_{F^\ell} \). Recall the characteristics \( \mathcal{Z}^{\ell+1}(s; t, x, v) = (\mathcal{X}^{\ell+1}(s; t, x, v), \mathcal{Y}^{\ell+1}(s; t, x, v)) \) solving (2.13). For all \( (x, v) \in \Omega \times \mathbb{R}^3 \) and \( s \in [t - \ell_{B,3}^\ell(t, x, v), t] \),

\[
\begin{align*}
\alpha_{F^\ell}^{\ell+1}(s, \mathcal{X}^{\ell+1}(s; t, x, v), \mathcal{Y}^{\ell+1}(s; t, x, v)) \\
\leq \alpha_{F^\ell}^{\ell+1}(t, x, v)e^{\sup_{r \in [s, t]} 1 + \| \nabla_x \phi_{F^\ell}(r) \|_{L^\infty(\Omega)} + \frac{1}{g} \| \partial_x \phi_{F^\ell}(r) \|_{L^\infty(\Omega)}} \| \nabla_x \phi_{F^\ell}(r) \|_{L^\infty(\Omega)}} (t - s) + \frac{1}{g} \sup_{r \in [s, t]} \| \nabla_x \phi_{F^\ell}(r) \|_{L^\infty(\Omega)} \int_s^t \| \nabla_x \phi_{F^\ell}(s') \|_{L^\infty(\Omega)} ds' \] \\
\alpha_{F^\ell}^{\ell+1}(s, \mathcal{X}^{\ell+1}(s; t, x, v), \mathcal{Y}^{\ell+1}(s; t, x, v)) \\
\geq \alpha_{F^\ell}^{\ell+1}(t, x, v)e^{-\sup_{r \in [s, t]} 1 + \| \nabla_x \phi_{F^\ell}(r) \|_{L^\infty(\Omega)} + \frac{1}{g} \| \partial_x \phi_{F^\ell}(r) \|_{L^\infty(\Omega)}} \| \nabla_x \phi_{F^\ell}(r) \|_{L^\infty(\Omega)}} (t - s) + \frac{1}{g} \sup_{r \in [s, t]} \| \nabla_x \phi_{F^\ell}(r) \|_{L^\infty(\Omega)} \int_s^t \| \nabla_x \phi_{F^\ell}(s') \|_{L^\infty(\Omega)} ds'.
\end{align*}
\]

(6.43)

In particular, the last inequality implies that

\[
\begin{align*}
|\ell_{B,3}^{\ell+1}(t, x, v)| &\geq \alpha_{F^\ell}^{\ell+1}(t, x, v)e^{-\sup_{r \in [t - \ell_{B,3}^\ell, t]} 1 + \| \nabla_x \phi_{F^\ell}(r) \|_{L^\infty(\Omega)} + \frac{1}{g} \| \partial_x \phi_{F^\ell}(r) \|_{L^\infty(\Omega)}} (t - \ell_{B,3}^\ell) \\
&\times \frac{1}{g} \sup_{r \in [t - \ell_{B,3}^\ell, t]} \| \nabla_x \phi_{F^\ell}(r) \|_{L^\infty(\Omega)} \int_{t - \ell_{B,3}^\ell}^t \| \nabla_x \phi_{F^\ell}(s') \|_{L^\infty(\Omega)} ds' \] \\
&\geq \alpha_{F^\ell}^{\ell+1}(t, x, v)e^{-\sup_{r \in [t - \ell_{B,3}^\ell, t]} 1 + \| \nabla_x \phi_{F^\ell}(r) \|_{L^\infty(\Omega)} + \frac{1}{g} \| \partial_x \phi_{F^\ell}(r) \|_{L^\infty(\Omega)}} (t - \ell_{B,3}^\ell) \\
&\times \frac{1}{g} \sup_{r \in [t - \ell_{B,3}^\ell, t]} \| \nabla_x \phi_{F^\ell}(r) \|_{L^\infty(\Omega)} \int_{t - \ell_{B,3}^\ell}^t \| \nabla_x \phi_{F^\ell}(s') \|_{L^\infty(\Omega)} ds'.
\end{align*}
\]

(6.44)

(6.45)

**Proof.** Note that

\[
\begin{align*}
|\partial_t + v \cdot \nabla_x - \nabla_x (\phi_{F^\ell}(t, x) + gx_3) \cdot \nabla_v| &\leq \sqrt{|v_3|^2 + |x_3|^2 + 2\partial_{x_3} \phi_{F^\ell}(t, x_3, 0)x_3 + 2gx_3} \\
&= \partial_t \partial_{x_3} \phi_{F^\ell}(t, x_3, 0)x_3 - (\partial_{x_3} \phi_{F^\ell}(t, x_3, 0) + \partial_t \phi_{F^\ell}(t, x_3, 0))v_3 + v_3x_3 + v_3 \cdot \nabla_x \partial_{x_3} \phi_{F^\ell}(t, x_3, 0)x_3 \\
&\leq \alpha_{F^\ell}^{\ell+1}(t, x, v)
\end{align*}
\]

(6.46)

where we use \(-\partial_{x_3} \phi_{F^\ell}(t, x_3, 0) \neq 0\) and \( \partial_t \phi_{F^\ell}(t, x_3, 0) \neq 0 \).

Then

\[
|\partial_t + v \cdot \nabla_x - \nabla_x (\phi_{F^\ell} + gx_3) \cdot \nabla_v| \alpha_{F^\ell}^{\ell+1}(t, x, v)
\]

\[
\leq \left( 1 + \| \partial_{x_3} \phi_{F^\ell}(t) \|_{L^\infty(\Omega)} + \frac{1}{g} \| \partial_t \partial_{x_3} \phi_{F^\ell}(t) \|_{L^\infty(\Omega)} + \frac{1}{g} \| \nabla_x \partial_{x_3} \phi_{F^\ell}(t) \|_{L^\infty(\Omega)} \right) \alpha_{F^\ell}^{\ell+1}(t, x, v).
\]

By the Gronwall’s inequality, we conclude both inequalities of (6.43). Then (6.44) is a direct result of the second estimate in (6.43). For (6.45), we use (2.27).
Lemma 6.6. Assume that \((6.11)\) holds for \(\Psi^\ell \in C^1(\bar{\Omega}) \cap C^2(\Omega)\), and the compatibility condition \((2.50)\) holds. Let \((h, \Phi)\) and \((f^{\ell+1}, \Psi^\ell)\) be the solutions constructed in Theorem 2.8 and \((6.2)-(6.7)\) respectively. Recall that \(\Phi^\ell = \Phi + \Psi^\ell\). Suppose that
\[
\frac{\beta^{1/2}}{g^{1/2}} \left\| \partial_t \Phi^\ell \right\| _{L^\infty([0,t] \times \Omega)} + \frac{1}{g^{1/2}} \left\| \partial_t \partial_{x,v} \Phi^\ell \right\| _{L^\infty([0,t] \times \partial \Omega)} + \frac{1}{g^{1/2}} \left\| \nabla^2 \Phi^\ell \right\| _{L^\infty([0,t] \times \Omega)} \lesssim 1. \tag{6.46}
\]

Then, for \(F^{\ell+1} = h + f^{\ell+1}\) and \(\Phi^\ell = \Phi + \Psi^\ell\), we have that, for all \(s \in [0,t]\),
\[
e^{\frac{\delta}{g}(|v|^2 + gx_3)} |\nabla_x F^{\ell+1}(s, x, v)| \lesssim \|w_{\beta,0} \nabla_{x,v} F_0\|_{L^\infty(\Omega \times \mathbb{R}^3)} + \left(1 + \frac{1}{g^{1/2}}\right) \|e^{\beta|v|^2} \nabla_{x,v} G\|_{L^\infty(\gamma_-)}, \tag{6.47}
\]

\[
e^{\frac{\delta}{g}(|v|^2 + gx_3)} |\partial_{x,v} F^{\ell+1}(s, x, v)| \lesssim \left\| w_{\beta,0} \nabla_{x,v} F_0 \right\| _{L^\infty(\Omega \times \mathbb{R}^3)} + \left[1 + \frac{1}{g^{1/2}}\right] \left\| \frac{\delta_3}{\alpha^{\ell+1}_F(s, x, v)} \right\| \|e^{\beta|v|^2} \nabla_{x,v} G\|_{L^\infty(\gamma_-)}. \tag{6.48}
\]

Moreover, for all \((s, x) \in [0,t] \times \Omega\),
\[
e^{\frac{\delta}{g}|x_3|} |\partial_{x,v} F^{\ell+1}(s, x)| \lesssim \left[1 + \frac{1}{g^{1/2}}\right] \left\| w_{\beta,0} \nabla_{x,v} F_0 \right\| _{L^\infty(\Omega \times \mathbb{R}^3)} + \left(1 + \frac{1}{g^{1/2}}\right) \|e^{\beta|v|^2} \nabla_{x,v} G\|_{L^\infty(\gamma_-)} + \frac{\delta_3}{\beta} \left(1 + \frac{1}{g^{1/2}}\right) \left(1 + \frac{1}{g^{1/2}}\right) \left(1 + \frac{1}{g^{1/2}}\right) \frac{1}{|x_3|} \lesssim 1 \left\| |x_3|^2 + gx_3\right\| \|e^{\beta|v|^2} \nabla_{x,v} G\|_{L^\infty(\gamma_-)}. \tag{6.49}
\]

For \(0 < \delta < 1\) and for all \(s \in [0,t]\)
\[
\left| \Phi^{\ell+1}(s) \right| _{L^\infty(\Omega)} \lesssim \left[1 + \frac{1}{g^{1/2}}\right] \left\| w_{\beta,0} \nabla_{x,v} F_0 \right\| _{L^\infty(\Omega \times \mathbb{R}^3)} + \frac{\delta}{\beta} \left(1 + \frac{1}{g^{1/2}}\right) \|e^{\beta|v|^2} \nabla_{x,v} G\|_{L^\infty(\gamma_-)}. \tag{6.50}
\]

Furthermore, for all \(s \in [0,t]\), we have that \(\Phi^{\ell+1}(s) \in C^1(\bar{\Omega}) \cap C^2(\Omega)\) and
\[
\|\nabla_x \Phi^{\ell+1}(s)\|_{L^\infty(\Omega)} \leq \frac{1}{\beta^{1/2}} \left\{ \left| w_{\beta,0} F_0 \right| _{L^\infty(\Omega \times \mathbb{R}^3)} + \|e^{\beta|v|^2} G\|_{L^\infty(\gamma_-)} \right\}, \tag{6.51}
\]
\[
\|\nabla^2 \Phi^{\ell+1}(s)\|_{L^\infty(\Omega)} \leq \left\{ \left| w_{\beta,0} F_0 \right| _{L^\infty(\Omega \times \mathbb{R}^3)} + \|e^{\beta|v|^2} G\|_{L^\infty(\gamma_-)} \right\} \times \left\{ \frac{1}{\beta^{1/2}} + \frac{\log (e + \frac{1}{\beta^{1/2}} \|w_{\beta,0} \nabla_{x,v} F_0\|_{L^\infty(\Omega \times \mathbb{R}^3)})}{\beta^{1/2}} + \frac{\delta_3}{\beta} \left(1 + \frac{1}{g^{1/2}}\right) \frac{1}{|x_3|} \lesssim 1 \left\| |x_3|^2 + gx_3\right\| \|e^{\beta|v|^2} \nabla_{x,v} G\|_{L^\infty(\gamma_-)}. \tag{6.52}
\]

Proof. We bound \((6.32)\) and \((6.33)\) following the argument of the proof of Lemma 5.7. Recall \(w_{\beta,0}^{\ell+1}\) defined in \((6.33)\). From \((6.33)\),
\[

\|\nabla_v F^{\ell+1}(t, z)\| \leq 1_{t \leq t_B^{\ell+1}(t, z)} \|\nabla_v \Phi^{\ell+1}(0; t, z)\| \|w_{\beta,0}^{\ell+1} \nabla_{x,v} F_0\|_{L^\infty(\Omega \times \mathbb{R}^3)} + \frac{1}{\beta} \left(1 + \frac{1}{g^{1/2}}\right) \|e^{\beta|v|^2} \nabla_{x,v} G\|_{L^\infty(\gamma_-)}. \tag{6.53}
\]

Using \((6.36)-(6.37)\) and \((6.15)\), we bound
\[

\leq e^{\delta \frac{\beta}{g} |x_3|^2} \left\| \frac{\beta}{g} \right\| \left(1 + \left(\left\| v \right\|_{L^\infty(\Omega \times \mathbb{R}^3)}\right)^2 \right) \|w_{\beta,0} \nabla_{x,v} F_0\|_{L^\infty(\Omega \times \mathbb{R}^3)} \tag{6.55}
\]

where we use the abbreviation \(L^\infty_{t,x}\) of \((2.59)\). In \((6.55)\), we have used \((6.12)\) at the second line: \(e^{(1 + \|v\|^2_{L^\infty_{t,x}}) t_{\beta,0}(t,x,v)} \leq e^{\frac{\delta}{g} (1 + \left\| v \right\|_{L^\infty_{t,x}})} |t_3| + \sqrt{g |x_3|} \); and used the completing-square trick to get the last line of \((6.55)\).
Now we consider (6.54). We follow the same argument in (5.52) and (5.54): Using (6.39), (6.41), and (6.12), we derive that
\[
\frac{1}{\epsilon^{\beta}|v|^{1+1}} \leq \frac{16}{g^2\beta} \left(1 + \frac{8}{g^2\beta} \left\|\nabla_x^2 \phi_{F^\ell} \right\|_{L_{\infty}^\beta} \right) e^{-\frac{\beta}{2} |v|^2} e^{-\frac{\beta}{2} |x|},
\]
(6.56)
\[
\frac{1}{\epsilon^{\beta}|v|^{1+1}} \leq \left(1 + \frac{4}{g} \left\|\nabla_x \phi_{F^\ell} \right\|_{L_{\infty}^\beta} \right) \left(1 + \frac{32}{g^2\beta} \left\|\nabla_x^2 \phi_{F^\ell} \right\|_{L_{\infty}^\beta} \right) e^{-\frac{\beta}{2} |v|^2} e^{-\frac{\beta}{2} |x|}.
\]
(6.57)
Finally applying (6.56), (6.57), (6.55) to (6.53)-(6.54) we conclude (6.47) under the condition of (6.46).

Next, to get (6.48), we bound (6.32). We can bound the first line of (6.32), following the argument of (6.55) and using (6.34)-(6.35):
\[
1 \leq \epsilon^{\beta}|v|^{1+1} \left|\nabla_x^2 \phi_{F^\ell} \left(0; t, x, v \right) \right| + \left|\nabla_x \phi_{F^\ell} \left(0; t, x, v \right) \right| \left|\nabla_{x,v} F_0 \right|_{L_{\infty}^\beta} e^{-\frac{\beta}{2} |v|^2} e^{-\frac{\beta}{2} |x|},
\]
(6.58)
\[
\leq e^{\frac{64}{g}} \left|\nabla_x \phi_{F^\ell} \left(0; t, x, v \right) \right| \left|\nabla_{x,v} F_0 \right|_{L_{\infty}^\beta} e^{-\frac{\beta}{2} |v|^2} e^{-\frac{\beta}{2} |x|}.
\]
Now we consider the second line of (6.32). We follow the same argument of (5.56) and (5.57). Using (6.46), (6.45), and the completing-square trick, we have that
\[
\frac{1}{\epsilon^{\beta}|v|^{1+1}} \leq \left(1 + \frac{8}{g^2\beta} \left\|\nabla_x^2 \phi_{F^\ell} \right\|_{L_{\infty}^\beta} \right) e^{-\frac{\beta}{2} |v|^2} e^{-\frac{\beta}{2} |x|} \left|\nabla_x \phi_{F^\ell} \left(0; t, x, v \right) \right| \left|\nabla_{x,v} F_0 \right|_{L_{\infty}^\beta} e^{-\frac{\beta}{2} |v|^2} e^{-\frac{\beta}{2} |x|},
\]
(6.59)
\[
\leq \left(1 + \frac{8}{g^2\beta} \left\|\nabla_x^2 \phi_{F^\ell} \right\|_{L_{\infty}^\beta} \right) e^{-\frac{\beta}{2} |v|^2} e^{-\frac{\beta}{2} |x|} \left|\nabla_x \phi_{F^\ell} \left(0; t, x, v \right) \right| \left|\nabla_{x,v} F_0 \right|_{L_{\infty}^\beta} e^{-\frac{\beta}{2} |v|^2} e^{-\frac{\beta}{2} |x|},
\]
where we have abbreviated $v_{B}^{\ell+1} = v_{B}^{\ell+1}(t, x, v)$ and used (2.59).

Similarly, we derive that
\[
\frac{1}{\epsilon^{\beta}|v|^{1+1}} \leq \left(1 + \frac{8}{g^2\beta} \left\|\nabla_x^2 \phi_{F^\ell} \right\|_{L_{\infty}^\beta} \right) e^{-\frac{\beta}{2} |v|^2} e^{-\frac{\beta}{2} |x|} \left|\nabla_x \phi_{F^\ell} \left(0; t, x, v \right) \right| \left|\nabla_{x,v} F_0 \right|_{L_{\infty}^\beta} e^{-\frac{\beta}{2} |v|^2} e^{-\frac{\beta}{2} |x|},
\]
(6.60)
\[
\leq \left(1 + \frac{8}{g^2\beta} \left\|\nabla_x^2 \phi_{F^\ell} \right\|_{L_{\infty}^\beta} \right) e^{-\frac{\beta}{2} |v|^2} e^{-\frac{\beta}{2} |x|} \left|\nabla_x \phi_{F^\ell} \left(0; t, x, v \right) \right| \left|\nabla_{x,v} F_0 \right|_{L_{\infty}^\beta} e^{-\frac{\beta}{2} |v|^2} e^{-\frac{\beta}{2} |x|},
\]
Finally we conclude (6.48) using (6.58), (6.59), (6.60), under the condition of (6.46).

We prove (6.49)-(6.52) following the proof of (5.46)-(5.49).

**Lemma 6.7.** Suppose $g^{\frac{3}{2}} \geq 1$. Suppose (2.48) and (6.48) hold. Then (6.10) holds and $\nabla_x \cdot b^{\ell+1}(t, \cdot) \in L^\infty(\Omega)$ and $\partial_{x_3} \partial_t \phi_{F^{\ell+1}}(t, \cdot) \in C^{\delta}(\Omega)$ for some $\delta > 0$.

Moreover, for all $(s, x) \in [0, t] \times \Omega,$
\[
e^{\frac{\beta}{2} |x|^2} \left|\nabla_x \cdot b^{\ell+1}(s, x) \right| + \beta^{3/2} e^{\beta |x|^2} \left|\partial_{x_3} \partial_t \phi_{F^{\ell+1}}(s, x) \right| + \beta^{3/2} \left|\partial_t \phi_{F^{\ell+1}}(s, x) \right|
\leq \left(1 + \frac{1}{\beta^{3/2}} + \frac{1}{g^{\beta}} \right) e^{\frac{\beta}{2} |v|^2} \left|\nabla_{x,v} F_0 \right|_{L_{\infty}(\Omega \times x_3)} e^{\frac{\beta}{2} |v|^2} e^{\frac{\beta}{2} |x|} \left|\nabla_{x,v} F_0 \right|_{L_{\infty}(\Omega \times x_3)} G_{L^\infty(\gamma)}.
\]
(6.61)

**Remark 6.8.** Since $\nabla_x \cdot b^{\ell+1}$ is bounded, a weak solution $\phi^{\ell+1}$ to (6.9) should satisfy $\phi^{\ell+1}(t, x) = \eta \int_{0}^{T} f_{0}(x, v) dv - \int_{0}^{T} \nabla_x \cdot b^{\ell+1}(s, x) ds$. Therefore $\phi^{\ell+1}(\cdot, x)$ is absolutely continuous in time and hence (6.10) holds almost everywhere.
Proof of Lemma 6.7. Note that
\[ |\nabla_x \cdot b^{\ell+1}(t, x)| \leq \int_{\mathbb{R}^3} |v \cdot \nabla_x F^\ell(t, x, v)| dv + \int_{\mathbb{R}^3} |v \cdot \nabla_x h(x, v)| dv. \] (6.62)

Using (2.48) and (5.58), we bound
\[ \int_{\mathbb{R}^3} |v \cdot \nabla_x h(x, v)| dv \leq \|e^{\frac{v}{\alpha}} \|_{L^\infty(\gamma_-)} \int_{\mathbb{R}^3} e^{-\frac{\beta}{4}|v|^2} e^{-\frac{\beta}{2} x_3} \left( |v| + \frac{|v_3|}{\alpha(x, v)} \right) dv \]
\[ \lesssim \|e^{\frac{v}{\alpha}} \|_{L^\infty(\gamma_-)} e^{-\frac{\beta}{4} x_3} \int_{\mathbb{R}} (|v| + 1)e^{-\frac{\beta}{4}|v|^2} dv_3 \]
\[ \lesssim \|e^{\frac{v}{\alpha}} \|_{L^\infty(\gamma_-)} \left( 1 + \frac{1}{\beta^1/2} \right) e^{-\frac{\beta}{4} x_3}, \] (6.63)
where we have used that $|v_3| \leq \alpha(x, v)$.

Similarly, using (5.58) and (6.48), we bound
\[ \int_{\mathbb{R}^3} |v \cdot \nabla_x F^\ell(t, x, v)| dv \]
\[ \lesssim \left( \frac{1}{\beta^2} \right) \left( 1 + \frac{1}{\beta^1/2} + \frac{1}{\beta g} \right) \|e^{\frac{v}{\alpha}} \|_{L^\infty(\gamma_-)} \left( 1 + \frac{1}{\beta^1/2} \right) e^{-\frac{\beta}{4} x_3}. \] (6.64)

In summary, we can conclude a $\nabla_x \cdot b^{\ell+1}$-bound of (6.61) from (6.62)-(6.64).

From (6.10) and Lemma 3.1, we have
\[ \partial_x^3 \partial_t \phi_{F^\ell} = \partial_x^3 \partial_t \Psi^\ell(t, x) = \eta \int_{\Omega} \nabla \cdot b^{\ell+1}(y) \partial_x^3 G(x, y) dy. \] (6.65)

Now applying Lemma 3.3 and using (3.17) and the $\nabla_x \cdot b^{\ell+1}$-bound of (6.61), we prove the bound of $|\partial_x^3 \partial_t \phi_{F^\ell}|$ in (6.61). Next, using the Dirichlet boundary condition $\partial \phi_{F^\ell} |_{\partial \Omega} = 0$ and the bound of $|\partial_x^3 \partial_t \phi_{F^\ell}|$ in (6.61), we conclude the bound of $|\partial_t \phi_{F^\ell}|$. \[ \square \]

Theorem 6.9. Assume $\beta > \tilde{\beta} > \max\{1, \frac{3}{2}\}$. Suppose $\varepsilon > 0$ is sufficiently small such that (2.51), (2.52), and (2.53) hold. Then we can construct $\Psi^\ell, f^\ell, g^\ell, \psi^\ell, \phi^\ell$ solve (6.2)-(6.7) for all $\ell = 0, 1, 2, \cdots$. Moreover, they satisfy (6.11) and (6.46). Therefore all the results in Lemma 6.4, Lemma 6.6, and Lemma 6.7 hold.

Proof. The proof is a consequence of Lemma 6.4, Lemma 6.6, and Lemma 6.7. We ought to check the conditions (6.11) and (6.46) to iterate our construction of sequences in (6.1)-(6.7). If (2.51) holds then using (6.51) we can verify the condition (6.11). Moreover if (2.52) holds then using (6.52) and (6.61) we can verify the condition (6.46). \[ \square \]

6.3. Stability of the Sequence. The following lemma is useful to prove that i) the sequence in Theorem 6.9 is Cauchy; and ii) the solution (as a limit of the sequence) is unique.

Lemma 6.10. Suppose $\bar{F}_i(t, x, v)$ is defined in $\mathbb{R}_+ \times \bar{\Omega} \times \mathbb{R}^3$ and satisfies $\int_{\mathbb{R}^3} \bar{F}_i(t, x, v) dv \in C^{0,\delta}(\Omega)$ for some $\delta > 0$ and any $t \in \mathbb{R}_+$. Suppose $\phi_{\bar{F}_i}(t, \cdot) \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ for any $t \in \mathbb{R}_+$ and solves
\[ \Delta \phi_{\bar{F}_i} = \eta \int_{\mathbb{R}^3} \bar{F}_i dv, \quad \phi_{\bar{F}_i}|_{\partial \Omega} = 0. \]

Now we consider $F_i(t, x, v)$ solving that, in the sense of Definition 2.2,
\[ \partial_t F_i + v \cdot \nabla_x F_i - \nabla_x (\phi_{\bar{F}_i} + g x_3) \cdot \nabla_v F_i = 0, \quad F_i|_{t=0} = F_0, \quad F_i|_{\gamma_-} = G. \] (6.66)
Suppose the following condition hold for $g, \bar{\beta} > 0$

$$|\phi_{F_1}(t, x)| \leq \frac{g}{2} x_3, \quad (6.67)$$

$$\left\|e^{\frac{\bar{\beta}}{8}(\|v\|^2 + gx_3)} \nabla_v F_2\right\|_{L^\infty} < \infty.$$  \( (6.68) \)

Then there exists $C = \left\|e^{\frac{\bar{\beta}}{8}(\|v\|^2 + gx_3)} \nabla_v F_2\right\|_{L^\infty}(\Omega \times \mathbb{R}^3) \leq C \int_0^t \left\|e^{\frac{\bar{\beta}}{8}(\|v\|^2 + gx_3)}(\bar{F}_1(s) - \bar{F}_2(s))\right\|_{L^\infty(\Omega \times \mathbb{R}^3)} ds.$$

(6.69)

Proof. Set $\bar{\beta} = \frac{\bar{\beta}}{8}$ and $\mathbf{w}_{\bar{\beta}, 1}(t, x, v) = e^{\frac{\bar{\beta}}{8}(\|v\|^2 + 2\phi_{F_1}(t, x) + 2gx_3)}$. Note that the difference of solutions solves

$$[\partial_t + v \cdot \nabla_x - \nabla_x(\phi_{F_1} + gx_3) : \nabla_v]\left(\mathbf{w}_{\bar{\beta}, 1}(F_1 - F_2)\right) = 2\bar{\beta}\partial_v \phi_{F_1} \mathbf{w}_{\bar{\beta}, 1}(F_1 - F_2) + \nabla_x(\phi_{F_1} - \phi_{F_2}) : \mathbf{w}_{\bar{\beta}, 1} \nabla_v F_2,$$  \( (6.70) \)

$$\mathbf{w}_{\bar{\beta}, 1}(F_1 - F_2)|_{t=0} = 0, \quad \mathbf{w}_{\bar{\beta}, 1}(F_1 - F_2)|_{t=0} = 0.$$

From (6.67), we have that $\mathbf{w}_{\bar{\beta}, 1}(s, y, u) \geq e^{\frac{\bar{\beta}}{8}(\|v\|^2 + gy_3)}$ and $e^{\frac{\bar{\beta}}{8}(\|v\|^2 + gy_3)} \geq \mathbf{w}_{\bar{\beta}, 1}(s, y, u)$, and therefore

$$\mathbf{w}_{\bar{\beta}, 1}|\nabla_v F_2|(s, y, u) \leq \mathbf{w}_{\bar{\beta}, 1} \left|\mathbf{w}_{\bar{\beta}, 1}\left(|\nabla_v F_2|(s, y, u)\right)\right|.$$  \( (6.71) \)

Along the characteristics $Z_1 = (X_1, V_1)$ associated with a field $-\nabla_x(\phi_{F_1} + gx_3)$, we have a form

$$\mathbf{w}_{\bar{\beta}, 1}(F_1 - F_2)(t, z) = \int_{\max(0, t - t_{B_1}(t, z))}^t \mathbf{w}_{\bar{\beta}/8, 1}(s, Z_1(s, t, z)) \nabla_x(\phi_{F_1} - \phi_{F_2})(s, Z_1(s, t, z)) \cdot [e^{\frac{\bar{\beta}}{8}(V_1(s, t, z) + gx_3(s, t, z))} \nabla_v F_2(s, Z_1(s, t, z))] ds.$$  \( (6.71) \)

Here, from (6.68) we know that the second line of (6.71) is bounded.

We bound the first line of integrand in (6.71) term by term. Using Lemma 2.5 and (2.27), we derive that, for $s \in [t - t_{B_1}(t, z), t]$

$$\int_s^t 2\bar{\beta}\partial_v \phi_{F_1}(\tau, X_1(\tau, t, x, v), V_1(\tau, t, x, v)) d\tau \leq C' \frac{\bar{\beta}^{1/2}}{g^{1/2}} \sqrt{|v_3|^2 + gx_3} \frac{\bar{\beta}^{1/2}}{g^{1/2}} \left\|\partial_v \phi_{F_1}\right\|_{L^\infty(\Omega \times \mathbb{R}^3)}.$$  \( (6.72) \)

Then following a proof of Lemma 5.10 ((5.72) and (5.73), in particular), we derive that

$$\left\|\nabla_x(\phi_{F_1} - \phi_{F_2})(s)\right\|_{L^\infty(\Omega)} \leq \left\{1 + \frac{2}{\beta g}\right\} \frac{C e^{3/2}}{(\beta g)^{1/2}} \left|\nabla_x(\phi_{F_1} - \phi_{F_2})(s)\right|_{L^\infty(\Omega \times \mathbb{R}^3)}.$$  \( (6.73) \)

Finally, using (4.8) with $\bar{\beta}/8$ instead of $\beta$, we derive that

$$\frac{1}{\mathbf{w}_{\bar{\beta}, 1}(s, Z(s, t, x, v))} \leq e^{-\frac{\bar{\beta}}{32}\left|\nabla_x(\phi_{F_1} - \phi_{F_2})(s, t, x, v)\right|^2} e^{-\frac{\bar{\beta}}{32}|v|^2} e^{-\frac{\bar{\beta}}{32}g x_3}.$$  \( (6.74) \)
Then applying (6.73)-(6.74) to (6.71) we conclude (6.69) as
\[
\|e^{\beta(|v|^2+gx_3)}(F_1 - F_2)(t)\|_{L^\infty(\Omega \times \mathbb{R}^3)} \leq \int_0^t \|e^{\beta(|v|^2+gx_3)}(F_1 - F_2)(s)\|_{L^\infty(\Omega \times \mathbb{R}^3)} ds
\]
and Lemma 6.9 is valid for the limiting function. A proof of uniqueness is straightforward.

**Proof of Theorem 2.10.** Using Theorem 6.9 and Lemma 6.10, it is standard to deduce that, for \( \ell \geq m \),
\[
\|e^{\bar{\beta}(|v|^2+gx_3)}(F_\ell(t) - F_m(t))\|_{L^\infty(\Omega \times \mathbb{R}^3)} \leq \frac{(Ct)^m}{m!}e^{\frac{\bar{\beta}}{\beta}} \|\nu_{\beta,0}F_0\|_{L^\infty(\Omega \times \mathbb{R}^3)} + \|e^{\beta|v|^2}G\|_{L^\infty(\gamma^-)}
\]
where we have used (6.24) at the last step above. With this strong convergence together with uniform-upper-bounds of Theorem 6.9, it is standard to prove the convergence of the sequences and prove that their limiting function \((F, \phi_F)\) is a strong solution to (1.1)-(1.5). Moreover, every upper bound of Theorem 6.9 is valid for the limiting function. A proof of uniqueness is straightforward from Lemma 6.10. We omit the proof.

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