THE GAMMA-CONSTRUCTION AND PERMANENCE PROPERTIES OF THE (RELATIVE) $F$-RATIONAL SIGNATURE

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Abstract. We study some permanence properties of the relative $F$-rational signature defined and studied by Smirnov–Tucker. We show that this invariant is compatible with the gamma-construct ion, and then derive other main results from the $F$-finite case established by Smirnov–Tucker. We also obtain limited results about the $F$-rational signature defined and studied by Hochster–Yao. We explore some features of the gamma-construction along the way, which may be of independent interest.

Conventions. Throughout this paper, $p$ is a fixed prime number. All rings contain $\mathbb{F}_p$. For an ideal $\mathfrak{a}$ of a ring $R$ and a positive integer $e$, $\mathfrak{a}^{[p^e]}$ denotes the ideal generated by $\{a^{p^e} \mid a \in \mathfrak{a}\}$, or equivalently, the ideal generated by $\{a^{p^e} \mid a \in \Sigma\}$ where $\Sigma$ is a set of generators of $\mathfrak{a}$. The length of a finite length module $M$ over a ring $R$ is denoted $l_R(M)$, and usually just $l(M)$ if $R$ is clear from the context. An étale-local map of local rings is a local map of local rings that is a localization of an étale ring map. The completion of a local ring $A$ is denoted $A^\wedge$.

1. Introduction

For a Noetherian local ring $(R, \mathfrak{m})$ and an $\mathfrak{m}$-primary ideal $I$, the Hilbert–Kunz function of $I$ is the function $e \mapsto l(R/I^{[p^e]})$. It is studied by Kunz [Kun76] (at least in the case $I = \mathfrak{m}$), and the Hilbert–Kunz multiplicity $e_{HK}(R, I) = \lim_{e \to \infty} \frac{l(R/I^{[p^e]})}{p^e \text{dim } R}$ is shown to exist by Monsky [Mon83].

The Hilbert–Kunz multiplicity has a direct connection to tight closure theory. In a Noetherian reduced equidimensional complete local ring $(R, \mathfrak{m})$, given two $\mathfrak{m}$-primary ideals $I \subseteq J$, $J$ is contained in the tight closure of $I$ if and only if $e_{HK}(I) = e_{HK}(J)$ [HH90, Theorem 8.17]. Here we omitted the ring $R$ from the notations. Hochster–Yao [HY] and Smirnov–Tucker [ST19] defined the $F$-rational signature $s_{\text{rat}}(R)$ and the relative $F$-rational signature $s_{\text{rel}}(R)$ of a Noetherian local ring $(R, \mathfrak{m})$,

$s_{\text{rat}}(R) = \inf \{e_{HK}(I_0) - e_{HK}(I_0 + uR) \mid I_0 \text{ is a parameter ideal of } R, u \not\in I_0\}$

$s_{\text{rel}}(R) = \inf \left\{ \frac{e_{HK}(I_0) - e_{HK}(I)}{l(R/I_0) - l(R/I)} \mid I_0 \text{ is a parameter ideal of } R, I \supseteq I_0 \right\}$.
These invariants are known to characterize the $F$-rationality of $R$ under mild conditions [HY, Theorem 4.1; ST19, Proposition 2.1]. Various permanence properties for $s_{\text{rat}}$ are studied in [HY]; in [ST19], via the alternative characterization of $s_{\text{rel}}$ as the dual $F$-signature [ST19, Corollary 5.9] in the $F$-finite case, more permanence properties of $s_{\text{rel}}$ are obtained.

In this paper, we continue the study in [ST19] on properties of the relative $F$-rational signature, especially for non-$F$-finite rings. The classical way of moving from the non-$F$-finite setting to the $F$-finite setting is via the $\Gamma$-construction of Hochster–Huneke, which we review in §§2.1. Our main result is the following; for explicit statements, see Theorems 7.1, 7.2, and 7.3.

**Theorem 1.1.** The relative $F$-rational signature is compatible with the $\Gamma$-construction, ascends along a flat extension of local rings with geometrically regular closed fiber, and defines a lower semi-continuous function on the spectrum of a finite type algebra over a local $G$-ring.\(^1\)

Weaker results and difficulties in the case of the $F$-rational signature are discussed in §8.

The following result is used for the proof of Theorem 7.2 and may be of independent interest. For an explicit statement see Corollary 6.3.

**Theorem 1.2.** A regular local map between Noetherian complete local rings can be approximated by regular local maps between $F$-finite Noetherian local rings.

The strategy is as follows. Modulo Corollary 6.3, which is proved in §6, we can apply the $F$-finite case in [ST19] to prove our main theorem, as soon as we prove compatibility with the $\Gamma$-construction. To do so, we use a uniform boundedness result of Polstra [Pol18] to replace the Hilbert-Kunz multiplicity $e_{HK}(R, I)$ with an individual value of the Hilbert-Kunz function $l(R/ [p^e])$ normalized by a factor of $p^{-e \dim R}$. See §3, where a slightly stronger version of Polstra’s theorem is recorded.

For a fixed $e$, we prove compatibility of the “truncated” invariant in §5. The method is more or less model-theoretic; ideals of our concern are parameterized by matrices, and various lengths are virtually constructible functions of the entries of the matrices. (We do not require any model-theoretic knowledge, and the argument is carried out using the language of abstract and linear algebra.) We are then facing the following question: if a system of polynomial equations defined over a field $k$ has a solution over $k^G$ for all small $G$, does it necessarily have a solution over $k$? The answer is yes, at least up to a finite separable extension of $k$, see Lemma 2.3. This extra complication is settled in §4, owing to a trick that only works for the relative $F$-rational signature and not the $F$-rational signature.

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\(^1\) Readers unfamiliar of $G$-rings can refer to [Stacks, Tag 0766] and [Mat80, (34.A)].
2. Preparation

2.1. $\Gamma$-construction. We recall the classical $\Gamma$-construction of Hochster–Huneke [HH94].

**Construction 2.1** ([HH94, (6.11)]). Let $A$ be a Noetherian complete local ring, $k$ a coefficient field of $A$. Let $\Lambda$ be a $p$-basis of $k$ and $\Gamma \subseteq \Lambda$ a cofinite subset.

For a positive integer $e$, let $k^\Gamma,e$ be the field obtained from adjoining the $p^e$th root of all elements in $\Gamma$, and let $k^\Gamma = \bigcup_e k^\Gamma,e$ and $A^\Gamma = \bigcup_e k^\Gamma,e \otimes_k A$.

Finally, for a finite type $A$-algebra $R$ and $Q \in \text{Spec}(R)$, let $R^\Gamma = A^\Gamma \otimes_A R$ and $Q^\Gamma = \sqrt{QR^\Gamma}$.

We shall use the following basic facts about the $\Gamma$-construction.

**Lemma 2.2** (Hochster–Huneke). Let $A$ be a Noetherian complete local ring, $k$ a coefficient field of $A$. Let $\Lambda$ be a $p$-basis of $k$. Let $R$ be a finite type $A$-algebra. Then the followings hold.

1. For all cofinite subset $\Gamma$ of $\Lambda$, $R^\Gamma$ is Noetherian and $F$-finite, hence excellent; the map $R \to R^\Gamma$ is flat with Gorenstein fibers.
2. For all cofinite subsets $\Gamma$ of $\Lambda$, $\text{Spec}(R^\Gamma) \to \text{Spec}(R)$ is a universal homeomorphism, thus for all $Q \in \text{Spec}(R)$, $Q^\Gamma$ is a prime ideal.
3. For every $Q \in \text{Spec}(R)$, there exists a cofinite subset $\Gamma_0$ of $\Lambda$, such that for all cofinite subsets $\Gamma$ of $\Gamma_0$, $Q^\Gamma = QR^\Gamma$.
4. For every $Q \in \text{Spec}(R)$ and every finite extension $F/\kappa(Q)$ of fields, there exists a cofinite subset $\Gamma_0$ of $\Lambda$, such that for all cofinite subsets $\Gamma$ of $\Gamma_0$, $\kappa(Q^\Gamma)$ is linearly disjoint with $F$ over $\kappa(Q)$.

Proof. For (1)/(2), see [HH94, (6.11)]. (3) is part of [HH94, Lemma 6.13(b)]. (4) follows from the proof of [HH94, Lemma 6.13(b)], but let us formally deduce it as follows. Let $S$ be a finite $(R/Q)$-algebra that is an integral domain with $F = \text{Frac}(S)$. From (2)/(3) we see that there exists a $\Gamma_0$ such that for all cofinite subsets $\Gamma$ of $\Gamma_0$, both $QR^\Gamma$ and $0 \subseteq S^\Gamma$ are prime. Then we see from (2) that

$$F \otimes_{\kappa(Q)} \kappa(Q^\Gamma) = F \otimes_R R^\Gamma = F \otimes_S S^\Gamma = \text{Frac}(S^\Gamma)$$

is a field, as desired. \qed

2.2. Solutions to polynomial equations. We wish to show that if a system of polynomial equations has a solution over all sufficiently small $k^\Gamma$, then it has a solution over $k$. We can only prove this after taking a finite separable extension of $k$, see below. The result is not stated with the $\Gamma$-construction, but in a more general form, in view of Lemma 2.2(4).

**Lemma 2.3.** Let $K/k$ be a field extension, $X$ a $k$-scheme of finite type, and let $\{K_\alpha\}_\alpha$ be a cofiltered family of subextensions of $K/k$, such that for every finite purely inseparable extension $F/k$, there exists an $\alpha$ such that $F$ and $K_\alpha$ are linearly disjoint over $k$. 


Assume that $X(K_{\alpha}) \neq \emptyset$ for all sufficiently small $K_{\alpha}$. Then $X(E) \neq \emptyset$ for some finite separable extension $E/k$.

**Proof.** By Noetherian induction we may assume that $X$ is integral and that for all nonempty opens $U \subseteq X$, $U(K_{\alpha}) \neq \emptyset$ for all sufficiently small $K_{\alpha}$.

Let $F/k$ be finite purely inseparable such that $(X_F)_{\text{red}}$ is geometrically reduced over $F$. Take $\alpha$ such that $X(K_{\alpha}) \neq \emptyset$ and that $K_{\alpha}$ is linearly disjoint with $F$ over $k$. Then $K_{\alpha} \otimes_k F$ is a field and we have a commutative diagram of schemes

$$
\begin{array}{ccc}
\Spec(K_{\alpha} \otimes_k F) & \longrightarrow & X_F \\
\downarrow & & \downarrow \\
\Spec(K_{\alpha}) & \longrightarrow & X
\end{array}
$$

Let $\{x\}$ be the image of the bottom horizontal arrow. Then the diagram shows that $X_F \times_X \Spec(\kappa(x)) \times_{\Spec(\kappa(x))} \Spec(K_{\alpha}) = \Spec(K_{\alpha} \otimes_k F)$ is reduced, thus $X_F \times_X \Spec(\kappa(x))$ is reduced. Since $X$ is reduced, $X_F$ is reduced at preimages of $x$ (cf. [Stacks, Tag 0C21]), and shrinking $X$ we may assume $X_F$ reduced. Then $X_F$ is geometrically reduced over $F$, so $X$ is geometrically reduced over $k$. But every geometrically reduced $k$-scheme of finite type contains a nonempty smooth open subscheme, thus an $E$-point for some finite separable $E/k$. \hfill \Box

3. Uniform control

In this section we define and begin our study on some invariants “truncated” from the definition of relative $F$-rational signature.

**Definition 3.1.** Let $(R, \mathfrak{m})$ be a Noetherian local ring, $I_0$ an $\mathfrak{m}$-primary ideal, and $e$ a positive integer. For an ideal $I$ of $R$ properly containing $I_0$ we set

$$s_{I_0}^e(R, I) := \frac{l(I^{[p^e]}/I_0^{[p^e]})}{p^e \dim R/I_0} = \frac{l(R/I_0^{[p^e]}) - l(R/I^{[p^e]})}{p^e \dim R(l(R/I_0) - l(R/I))};$$

and we set

$$S_{I_0}^e(R) := \{s_{I_0}^e(R, I) \mid \mathfrak{m}I \subseteq I_0 \subset I\},$$

$$s_{I_0}^e(R) := \min S_{I_0}^e(R).$$

Note that $S_{I_0}^e(R)$ is contained in the finite set $[0, l(R/I_0^{[p^e]})] \cap \frac{1}{p^e \dim R/I_0} \mathbb{Z}$, so the minimum is attained and is in the same set.

**Lemma 3.2.** Let $(R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ be a flat map of Noetherian local rings such that $\mathfrak{m}S$ is $\mathfrak{n}$-primary. Let $I_0$ be an $\mathfrak{m}$-primary ideal.

Then for an ideal $I$ of $R$ properly containing $I_0$ and any positive integer $e$, $IS$ is an ideal of $S$ properly containing $I_0S$, and $s_{I_0}^e(R, I) = s_{I_0S}^e(S, IS)$. Consequently, if $\mathfrak{m}S = \mathfrak{n}$, then $S_{I_0}^e(R) \subseteq S_{I_0S}^e(S)$ and $s_{I_0}^e(R) \geq s_{I_0S}^e(S)$; and if $R/I_0 \cong S/I_0S$, then $S_{I_0}^e(R) = S_{I_0S}^e(S)$ and $s_{I_0}^e(R) = s_{I_0S}^e(S)$.
Proof. One observes that for all ideals \(a\) of \(R\) and all positive integers \(e\), \(a^{[pe]}S = (aS)^{[pe]}\) by definition. Moreover, by flatness, \(l_S(M \otimes_R S) = l_R(M)l_S(S/mS)\) for all \(R\)-modules \(M\) of finite length. Thus the lemma is clear. \(\square\)

**Definition 3.3.** Let \(R\) be a Noetherian ring, \(P \in \text{Spec}(R)\). Let \(C\) be a positive real number.

We say \(C\) controls \(F\)-colength differences for \(P \in \text{Spec}(R)\) if for all \(PR_P\)-primary ideals \(I_0 \subsetneq I\), and all positive integers \(e < e'\),

\[
|s_{I_0}^e(R_P, I) - s_{I_0}^{e'}(R_P, I)| \leq Cp^{-e},
\]

and consequently

\[
\left| s_{I_0}^e(R_P, I) - \frac{e_{HK}(I_0) - e_{HK}(I)}{l(R_P/I_0) - l(R_P/I)} \right| \leq Cp^{-e}.
\]

We say \(C\) controls \(F\)-colength differences for \(R\) if \(C\) controls \(F\)-colength differences for all \(P \in \text{Spec}(R)\).

The relation between our truncated invariants and the (relative) \(F\)-rational signature is the following.

**Lemma 3.4.** Let \((R, m)\) be a Noetherian local ring whose completion is equidimensional.\(^2\)

Let \(I_0\) be a parameter ideal of \(R\), and let \(C\) be a real number that controls \(F\)-colength differences for \(m \in \text{Spec}(R)\).

Then for all positive integers \(e\), we have

\[
|s_{I_0}^e(R) - s_{\text{rel}}(R)| \leq Cp^{-e}.
\]

Proof. This follows from [ST19, Corollary 3.7].\(^3\) \(\square\)

**Theorem 3.5** (Polstra). The followings hold.

1. If \(R\) is essentially of finite type over a Noetherian local ring, then there exists a \(C\) that controls \(F\)-colength differences for \(R\).
2. If \(R \to R'\) is a flat map of Noetherian rings, \(C\) controls \(F\)-colength differences for some \(P' \in \text{Spec}(R')\), and \(\text{ht} P' = \text{ht}(P' \cap R)\), then \(C\) controls \(F\)-colength differences for \(P' \cap R \in \text{Spec}(R)\).
3. If \(R \to R'\) is faithfully flat map of Noetherian rings, \(C\) controls \(F\)-colength differences for \(R'\), then \(C\) controls \(F\)-colength differences for \(R\).

\(^2\)By a result of Ratliff, this is equivalent to that \(R\) is equidimensional and universally catenary, see [Stacks, Tags 0AW4 and 0AW6]. In particular, if \(R\) is equidimensional and a homomorphic image of a Cohen-Macaulay ring, then \(R^\wedge\) is equidimensional, see [Stacks, Tag 00NM].

\(^3\)The author believes that the mild assumption on \(R\) that is necessary for [HH94, Theorem 7.15] to apply was overseen in the proof of [ST19, Corollary 3.7]. One can also apply [HH94, Theorem 4.2(c)(d)] to get \(R\) Cohen-Macaulay when the right hand side in [ST19, Corollary 3.7] is not 0; this also requires formal equidimensionality.
Proof. (2) follows from Lemma 3.2: \( s_{I_0}^e(R_P, I) = s_{I_0}^e(R_{P'}, I R_{P'}) \) for all \( I_0, I, e \), thus if \( C \) controls \( F \)-colength differences for \( P' \in \text{Spec}(R') \), then it does for \( P \in \text{Spec}(R) \).

Now (3) follows from (2), since for all \( P \in \text{Spec}(R) \), a minimal prime \( P' \) of \( PR' \) is such that \( P' \cap R = P \) and that \( \text{ht} P' = \text{ht} P \). Finally, (1) follows from (3) and [Poll18, Theorem 3.6] by taking a faithfully flat extension of the local ring that is Noetherian and \( F \)-finite. \qed

4. Étale-local extensions

We use the following trick to show that étale-local extensions preserve our invariant \( s_{I_0}^e \); Corollary 4.2.

Lemma 4.1. Given \( R, I_0, e \) as in Definition 3.1, let \( I_1, I_2 \) be two ideals that satisfy \( mI_1 \subseteq I_0 \not\subset I_1 \) and \( mI_2 \subseteq I_0 \not\subset I_2 \). If \( s_{I_0}^e(R, I_1) = s_{I_0}^e(R, I_2) = s_{I_0}^e(R) \), then \( s_{I_0}^e(R, I_1 + I_2) = s_{I_0}^e(R) \).

Proof. Let \( a = p^e \dim_{R} s_{I_0}^e(R) \). Then for all ideals \( I \) that contains \( I_0 \) and satisfies \( mI \subseteq I_0 \), we have \( a((I/I_0) \leq l(I_{[p^e]}/I_0[pr^e]), \) with equality if and only if \( I_0 = I \) or \( s_{I_0}^e(R, I) = s_{I_0}^e(R) \). Thus

\[
\begin{align*}
   a(l(I_1/I_0) + l(I_2/I_0)) = l(I_1[pr^e]/I_0[pr^e]) + l(I_2[pr^e]/I_0[pr^e])
   &= l\left(\frac{I_1[pr^e] + I_2[pr^e]}{I_0[pr^e]}\right) + l\left(\frac{I_1[pr^e] \cap I_2[pr^e]}{I_0[pr^e]}\right)
   \geq l\left(\frac{(I_1 + I_2)[pr^e]}{I_0[pr^e]}\right) + l\left(\frac{(I_1 \cap I_2)[pr^e]}{I_0[pr^e]}\right)
   \geq a\left(l\left(\frac{I_1 + I_2}{I_0}\right) + l\left(\frac{I_1 \cap I_2}{I_0}\right)\right)
\end{align*}
\]

Now equality holds everywhere. Since \( I_1 + I_2 \neq I_0 \) we see \( s_{I_0}^e(R, I_1 + I_2) \) is well-defined and equal to \( s_{I_0}^e(R) \). \qed

Corollary 4.2. Let \((R, m) \to (S, n)\) be an étale-local map of Noetherian local rings and let \( I_0 \) be \( m \)-primary. Then for all \( e \), \( s_{I_0}^e(R) = s_{I_0S}^e(S) \).

Proof. We may replace \( R \) and \( S \) by \( R/I_0[pr^e] \) and \( S/(I_0S)[pr^e] \) to assume \( R \) and \( S \) Artinian, so \( R \to S \) is finite étale. There is a compatible choice of coefficient fields of \( R \) and \( S \) (see for example [Mat80, (28.3) Theorem 60]), and thus \( S \cong R \otimes_k l \) where \( l/k \) is finite separable and \( k \) is a coefficient field of \( R \). By Lemma 3.2 we may enlarge \( l \) and assume \( l/k \) finite Galois with group \( G \), so \( G \) acts on \( S = R \otimes_k l \) on the second factor.

Let \( J \) be an ideal of \( S \) such that \( mJ \subseteq I_0S \subseteq J \) and that \( s_{I_0S}^e(S, J) = s_{I_0S}^e(S) \). Then \( \sum_{g \in G} g(J) \) satisfies the same by Lemma 4.1 and is of the form \( IS \) where \( I \) is an ideal of \( R \) (cf. [Stacks, Tag \texttt{CDR}]). By flatness, \( mI \subseteq I_0 \not\subset I \). Our equality now follows from Lemma 3.2. \qed
5. Γ-Construction and the Truncated Invariants

The goal of this section is to prove the following. Consequences will be discussed in Section 7.

**Theorem 5.1.** Let \( A \) be a Noetherian complete local ring, \( k \) a coefficient field of \( A \), and \( \Lambda \) a \( p \)-basis of \( k \). Let \( R \) be a finite type \( A \)-algebra. Let \( Q \in \text{Spec}(R) \) and let \( I_0 \) be a \( QR \)-primary ideal of \( R \). Let \( e \) be a positive integer.

Then there exists a cofinite subset \( \Gamma_0 = \Gamma_0(Q,I_0,e) \) of \( \Lambda \) such that for all cofinite subsets \( \Gamma \) of \( \Gamma_0 \), \( s^e_{I_0}(R_Q) = s^e_{I_0 R_{Q}^{\Gamma}}(R_{Q}^{\Gamma}) \).

Before we go into the argument, let us discuss the idea of the proof. To give an ideal \( I \) with \( mI \subseteq I_0 \subseteq I \) is the same as to give (an equivalence class of) a non-zero \( n \times n \) matrix if we fix a basis of the \( \kappa \)-dimensional \( \kappa(Q) \)-vector space \( (I_0 : R_Q Q) / I_0 \). The corresponding invariant is then “algebraically computed” by the entries of the matrix. The set where the invariant takes a certain value should then be constructible, and if a constructible subset of the variety of matrices has points over small \( \kappa(Q) \)-field, then “almost” has a \( \kappa(Q) \)-point, see Lemma 2.3. However, we run into the problem that \( R_Q / I_0[p^e] \) and \( R_{Q}^{\Gamma} / (I_0 R_{Q}^{\Gamma})[p^e] \) may not have compatible coefficient fields. We remedy this by choosing and fixing a coefficient field in a ring \( B \) that dominates all those we shall consider, and showing a weaker constructibility result (Lemma 5.2).

Now we begin to prove the theorem. We fix \( A, k, \Lambda, R, Q, I_0, e \) as in the theorem. Let \( d = \text{ht} Q \).

### 5.1. Parametrization of socle ideals.

Choose and fix \( \epsilon_1, \ldots, \epsilon_n \in (I_0 : R_Q Q) \) that map to a basis of \( (I_0 : R_Q Q) / I_0 \).

Let \( R_Q \rightarrow D \) be a flat map of local rings such that \( QD \) the maximal ideal of \( D \). Then \( \epsilon_1, \ldots, \epsilon_n \) are in \( (I_0 D : D Q) \) and map to a basis of \( (I_0 D : D Q) / I_0 D \). Let \( a_{ij} \) \((1 \leq i, j \leq n)\) be elements of the residue field \( D/QD \), and let \( \tilde{a}_{ij} \in D \) be arbitrary lifts. Then the ideal

\[
J = J(D; a_{ij} \ (1 \leq i, j \leq n)) := I_0 D + \left( \sum_j \tilde{a}_{ij} \epsilon_j \mid 1 \leq i \leq n \right)
\]

is independent of the choice of \( \tilde{a}_{ij} \) and \( QJ \subseteq I_0 \subseteq J \). We have \( I_0 \neq J \) if and only if not all \( a_{ij} \) are zero. Therefore we get a map

\[
J(D, -) : (\mathbb{A}^n_{\kappa(Q)} \setminus \{0\})(D/QD) \to \{ J \subseteq D \mid QJ \subseteq I_0 D \subseteq J \}.
\]

This map is surjective since \( \epsilon_1, \ldots, \epsilon_n \) map to a basis of \( (I_0 D : D Q) / I_0 D \).
If $D \to D'$ is another flat map of local rings with $QD'$ the maximal ideal of $D'$, then we have a commutative diagram

$$
\begin{array}{ccc}
(A_{\kappa(Q)}^n \setminus \{0\})(D/QD) & \xrightarrow{J(D,-)} & \{J \subseteq D \mid QJ \subseteq I_0D \subseteq J\} \\
\downarrow & & \downarrow J_{\to JD'} \\
(A_{\kappa(Q)}^n \setminus \{0\})(D'/QD') & \xrightarrow{J(D',-)} & \{J \subseteq D' \mid QJ \subseteq I_0D' \subseteq J\}
\end{array}
$$

5.2. Expressing the length using the entries of the matrix. If we choose a coefficient field of $R_Q/I_0[\overline{p}]$ and then a basis of this ring as a vector space over the chosen coefficient field, then we can write out elements explicitly and calculate the length $l(J[\overline{p}]/I_0[\overline{p}])$ as a constructible function of $a_{ij}$, where $J = J(R_Q; a_{ij})$, as explained below. The same can be done for each $R^\Gamma_Q$. However, if the choice of coefficient fields is not compatible, then the constructible functions are also not compatible. We get around this issue as follows.

Choose and fix a cofinite subset $\Gamma_1$ such that for all $\Gamma \subseteq \Gamma_1$, $QR^\Gamma$ is prime (Lemma 2.2(3)). Let $B$ be the strict Henselization of $R^\Gamma_{Q,\Gamma_1}$. Then $(B, QB)$ is a Noetherian Henselian local ring with $k_B := B/QB$ separably closed. Note that $k_B$ is a separable closure of $\kappa(Q_{\Gamma_1})$, thus algebraic over $\kappa(Q)$ by Lemma 2.2(2).

Choose and fix a coefficient field $\sigma : k_B \to B/(I_0B)[\overline{p}]$. We remind the reader again that $k_B$ does not necessarily contain a coefficient field of $R/I_0[\overline{p}]$. Choose and fix a basis $\epsilon'_1, \ldots, \epsilon'_m$ of $B/(I_0B)[\overline{p}]$ over this coefficient field. We may therefore write

$$
\epsilon'^p_k = \sum_i \sigma(c_{ik})\epsilon'_k
$$

where $c_{ik} \in k_B$. If we change the choice of $\sigma$ and $\epsilon'$, then $c_{ik}$ may change. Next, write

$$
\epsilon'_k \epsilon'_l = \sum_h \sigma(d_{khl})\epsilon'_h
$$

Again, the choice of $\sigma$ and $\epsilon'$ may affect $d_{khl}$.

For $a_1, \ldots, a_n \in k_B$, and any lifts $\tilde{a}_i \in B$, one has

$$
(\sum \tilde{a}_i \epsilon_i)^{\overline{p}} = \sum_k \sigma \left(\sum_i a_i^{\sigma} c_{ik}\right) \epsilon'_k \in B/(I_0B)[\overline{p}]
$$

This is because $\tilde{a}_i - \sigma(a_i) \in QB$, so $(\tilde{a}_i - \sigma(a_i))\epsilon_i \in I_0B$ and $\tilde{a}_i^{\overline{p}}\epsilon'^{\overline{p}}_i - \sigma(a_i)^{\overline{p}}\epsilon'^{\overline{p}}_i \in (I_0B)[\overline{p}]$.

Now let $M = (a_{ij})$ be a nonzero $n$-by-$n$ $k_B$-matrix, and consider the ideal $J = J(B; a_{ij})$ as in (5.1). By definition, the ideal $J[\overline{p}]/(I_0B)[\overline{p}]$ of
By previous calculations,
\[ \sum a_{ij}^{(p^e)} e_i^{(p^e)} (1 \leq i \leq n, 1 \leq l \leq m). \]

By previous calculations,
\[ \left( \sum_{j=1}^{n} a_{ij} \right)^{p^e} \epsilon_i^l = \sum_{k=1}^{m} \sigma \left( \sum_{j=1}^{n} a_{ij}^{p^e} c_{jk} \right) \epsilon_k^l \]
\[ = \sum_{h=1}^{m} \sigma \left( \sum_{k=1}^{m} \sum_{j=1}^{n} a_{ij}^{p^e} c_{jk} d_{kth} \right) \epsilon_h^l. \]

Let \( M' \) be the \( m \)-by-\( mn \) matrix whose \((n(l-1)+i)\)th column is the vector
\[ \left( \sum_{k=1}^{m} \sum_{j=1}^{n} a_{ij}^{p^e} c_{jk} d_{kth} \right)_{1 \leq h \leq m}. \]
Consider the function \( f : M \mapsto \frac{\text{rank} M'}{p^e \text{rank} M}. \)
We have \( f(M) = s_{I_0B}(B, J). \)

5.3. **Constructibility.** We now know that for \( M \in (\mathbb{A}^{n^2} \setminus \{0\})(k_B), \)
\[ s_{I_0B}(B, J(B; M)) = \frac{\text{rank} M'}{p^e \text{rank} M}. \]

Note that the rank of a matrix is detected by the (non-)vanishing of minors, and minors are polynomials of the entries of the matrix. The problem now is that the entries of \( M' \), being polynomials of the entries of \( M \) with \( k_B \)-coefficients, are not necessarily polynomials with \( \kappa(Q) \)-coefficients. We have the following remedy.

**Lemma 5.2.** Let \( L = \kappa(Q)[c_{jk}, d_{kth}] \), a finite extension of \( \kappa(Q) \).

For each \( s \in \mathbb{Q} \), there is a locally closed subset \( \mathcal{C}_s \subseteq \mathbb{A}^{n^2}_{\kappa(Q)} \setminus \{0\} \) with the following properties.

1. If \( M = (a_{ij}) \in \mathcal{C}_s(k_B) \), then \( f(M) \leq s \); and
2. If \( M = (a_{ij}) \in (\mathbb{A}^{n^2}_{\kappa(Q)} \setminus \{0\})(k') \), where \( k' \) is a subextension of \( k_B/\kappa(Q) \) linearly disjoint with \( L \) over \( \kappa(Q) \), then \( f(M) = s \) if and only if \( M \in \mathcal{C}_s(k') \).

**Proof.** We can stratify \( \mathbb{A}^{n^2}_{\kappa(Q)} \setminus \{0\} \) according to rank, so it suffices to prove the statement for the function \( g(M) = \text{rank} M' \). We may assume \( s \in \mathbb{Z} \), \( 0 \leq s \leq m \), otherwise we can take \( \mathcal{C}_s = \emptyset \).

Fix \( \mathbb{Z} \)-polynomials \( G_1, \ldots, G_u, G_{u+1}, \ldots, G_{u+v} \) of \( n^2 + mn + m^3 \) variables such that \( G_1(a_{ij}, c_{jk}, d_{kth}) \), \( \ldots \), \( G_u(a_{ij}, c_{jk}, d_{kth}) \) are the \( s \) by \( s \) minors of \( M' \) and that \( G_{u+1}(a_{ij}, c_{jk}, d_{kth}) \), \( \ldots \), \( G_{u+v}(a_{ij}, c_{jk}, d_{kth}) \) are the \( s+1 \) by \( s+1 \) minors of \( M' \). Here \( u = \binom{m}{s} \) and \( v = \binom{m}{s+1}. \)

Let \( \lambda_1, \ldots, \lambda_w \) be a basis of \( L \) over \( \kappa(Q) \). Then for each \( r \leq u + v \) we can find \( k \)-polynomials \( G_{r_1}, \ldots, G_{rw} \) of \( n^2 \) variables such that
\[ G_r(X_{ij}, c_{jl}, d_{kth}) = \lambda_1 G_{r_1}(X_{ij}) + \ldots + \lambda_w G_{rw}(X_{ij}), \]
by writing every monomial of $c_{jk}, d_{klh}$ as a $k$-linear combination of $\lambda_1, \ldots, \lambda_w$.

Finally, set

$$C_s = \left( \bigcup_{1 \leq r \leq u, 1 \leq p \leq w} D(G_{rp}) \right) \cap \bigcup_{u+1 \leq r \leq u+v, 1 \leq p \leq w} Z(G_{rp}).$$

For $M = (a_{ij}) \in C_s(k_B)$, it is clear that all $(s + 1)$-by-$(s + 1)$ minors of $M'$ are zero, so $\text{rank } M' \leq s$. If $k'$ is linearly disjoint over $L$ over $\kappa(Q)$, then $\lambda_1, \ldots, \lambda_w$ are linearly independent over $k'$, so $M \in C_s(k')$ if and only if all $(s + 1)$-by-$(s + 1)$ minors of $M'$ are zero and some $s$-by-$s$ minor of $M'$ is nonzero, that is, $\text{rank } M' = s$.

5.4. Conclusion of the proof. Choose and fix a cofinite subset $\Gamma_2$ of $\Gamma_1$ such that $\kappa(Q^F)$ is linearly disjoint with the field $L$ in Lemma 5.2 over $\kappa(Q)$ for all $\Gamma \subseteq \Gamma_2$ (Lemma 2.2(4)). We now take $\Gamma_0$ to be any cofinite subset of $\Gamma_2$ such that for all $\Gamma \subseteq \Gamma_0$, the finite sets $S^{e_{l_0R'Q_f}}(R^F_Q) \subseteq [0,1](L_0[0][0]) \cap \frac{1}{p \in \pi(L_0[0])}(Z)$ are all the same, which we denote by $S$. This is possible because for $\Gamma \subseteq \Gamma' \subseteq \Gamma_2$, $S^{e_{l_0R'Q_f}}(R^F_Q) \subseteq S^{e_{l_0R'Q_f}}(R^F_Q)$ by Lemma 3.2. Let $s = \min S$, so $s = s^{e_{l_0R'Q_f}}(R^F_Q)$ for all $\Gamma \subseteq \Gamma_0$. We will show $s^{e_{l_0}}(R_Q) = s$, finishing the proof. Note that $s^{e_{l_0}}(R_Q) \geq s$ by Lemma 3.2, so it suffices to show $s^{e_{l_0}}(R_Q) \leq s$.

Let $C_s$ be as in Lemma 5.2. For each $\Gamma \subseteq \Gamma_0$, since $s \in S$, there exists an ideal $J$ of $R^F_Q$ such that $QJ \subseteq l_0R^F_Q \subseteq J$ and that $s^{e_{l_0R'Q_f}}(R^F_Q, J) = s$. $J$ is of the form $J(R^F_{Q_f}, M)$ for some $M \in (A^{n^2} \setminus \{0\})(\kappa(Q^F))$, see (5.1). $M$ is then in $C_s(\kappa(Q^F))$ by Lemma 3.2 and Lemma 5.2(2). Thus we have $C_s(\kappa(Q^F)) \neq \emptyset$ for all $\Gamma \subseteq \Gamma_0$. Lemma 2.3 applies by Lemma 2.2(4), so we can find a finite separable extension $E/\kappa(Q)$ such that $C_s(E) \neq \emptyset$.

Fix an embedding $E \rightarrow k_B$, possible as $k_B$ separably closed. Let $R_Q \rightarrow S$ be an arbitrary étale-local map whose closed fiber is Spec$(E)$. Then since $B$ is Henselian, $S$ maps uniquely into $B$ compatible with the chosen $E \rightarrow k_B$. This makes $B$ a local $S$-algebra, and $S \rightarrow B$ is flat by for example [Stacks, Tag 00MK]. A point $M \in C_s(E)$ gives rise of an ideal $J = J(S; M)$ with $QJ \subseteq l_0S \subseteq J$, see (5.1). By Lemma 3.2 $f(M) = s_{l_0S}(S, J)$. By Lemma 5.2(1) we have $s_{l_0S}(S, J) \leq s$, so $s^{e_{l_0}}(S) \leq s$. By Corollary 4.2, $s^{e_{l_0}}(R_Q) \leq s$, as desired.

6. Approximation of regular maps

In this section we use differential module calculations and some diagram chasing to prove a result about “approximating” a regular map, Theorem 6.2. Its consequence, Corollary 6.3, is a useful dévissage tool for our Theorem 7.2, and may be such in other situations. The proofs of Theorems 7.1 and 7.3 do not directly depend on the materials in this section.
Lemma 6.1. Let \((A, m)\) be a Noetherian complete local ring, \(k\) a coefficient field of \(A\). Let \(\Lambda\) be a \(p\)-basis of \(k\) and \(\Gamma \subseteq \Lambda\) be a cofinite subset.

Then the kernel of the canonical map \(\Omega_{A/\mathbb{F}_p} \otimes_A k^\Gamma \to \Omega_{A^\Gamma/\mathbb{F}_p} \otimes_{A^\Gamma} k^\Gamma\) is spanned by \(d\lambda\) (\(\lambda \in \Gamma\)).

Proof. First put \(A^\Gamma = A \otimes_k k^\Gamma\), a possibly non-Noetherian ring. By definition, there is a canonical map \(A^\Gamma \to A^\Gamma\) that is an isomorphism modulo \(m^2\).

Examine the following commutative diagram of \(k^\Gamma\)-vector spaces with exact rows (cf. [Stacks, Tags 00S2 and 07BP])

\[
\begin{array}{cccccc}
H_1(L_{k^\Gamma/\mathbb{F}_p}) & \longrightarrow & mA^\Gamma/m^2A^\Gamma & \longrightarrow & \Omega_{A^\Gamma/\mathbb{F}_p}/m & \longrightarrow & \Omega_{k^\Gamma/\mathbb{F}_p} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_1(L_{k^\Gamma/\mathbb{F}_p}) & \longrightarrow & mA^\Gamma/m^2A^\Gamma & \longrightarrow & \Omega_{A^\Gamma/\mathbb{F}_p}/m & \longrightarrow & \Omega_{k^\Gamma/\mathbb{F}_p} \longrightarrow 0
\end{array}
\]

we see \(\Omega_{A^\Gamma/\mathbb{F}_p}/m = \Omega_{A^\Gamma/\mathbb{F}_p}/m\). Thus our desired kernel is the kernel of \(\Omega_{A^\Gamma/\mathbb{F}_p} \otimes_A k^\Gamma \to \Omega_{A^\Gamma/\mathbb{F}_p}/m\). Next, consider the exact sequence [Stacks, Tag 00S2]

\[
\begin{array}{cccccc}
H_1(L_{A^\Gamma/A}) & \longrightarrow & \Omega_{A^\Gamma/\mathbb{F}_p} \otimes_A k^\Gamma & \longrightarrow & \Omega_{A^\Gamma/\mathbb{F}_p} & \longrightarrow & \Omega_{A^\Gamma/A} \longrightarrow 0
\end{array}
\]

We know that \(A^\Gamma = A \otimes_k k^\Gamma = \bigcup_\lambda A[X_{\lambda,e}(\lambda \in \Gamma)]/(X_{\lambda,e}^e - \lambda)\), so \(\Omega_{A^\Gamma/A} = 0\). Thus the kernel of \(\Omega_{A^\Gamma/\mathbb{F}_p} \otimes_A k^\Gamma \to \Omega_{A^\Gamma/\mathbb{F}_p}/m\) is the image of the image of \(\delta\). Using the presentation once again we see that this vector space is spanned by \(d\lambda\) (\(\lambda \in \Gamma\)).

\[\square\]

Theorem 6.2. Let \(\varphi : (A, m, k) \to (B, n, l)\) be a flat local map of Noetherian complete local rings such that \(\overline{B} := B/mB\) is geometrically regular over \(k\). Fix a coefficient field \(k \subseteq A\) and a \(p\)-basis \(\Lambda\) of \(k\). Then the followings hold.

1. There exists a cofinite subset \(\Gamma_0 \subseteq \Lambda\) such that the image of \(\Gamma_0\) in \(l\) is \(p\)-independent.
2. Let \(\Gamma_0\) be as in (1). Then there exists a coefficient field \(l' \subseteq B\) that contains the set \(\varphi(\Gamma_0) \subseteq B\).
3. Let \(\Gamma_0, l'\) be as in (2), and extend \(\varphi(\Gamma_0)\) to a \(p\)-basis \(\Lambda'\) of \(l'\). Then for each cofinite subset \(\Gamma\) of \(\Gamma_0\) and each cofinite subset \(\Gamma'\) of \(\Lambda'\) with \(\varphi(\Gamma) \subseteq \Gamma'\), there exists a canonical \(A\)-algebra map \(A^\Gamma \to B^{\Gamma'}\) that is flat and local.
4. Let \(\Gamma_0, l', \Lambda'\) be as in (3). Then for each cofinite subset \(\Gamma\) of \(\Gamma_0\) there exists a cofinite subset \(\Gamma'_0\) of \(\Lambda'\) containing \(\varphi(\Gamma)\), such that for any cofinite subset \(\Gamma'\) of \(\Gamma'_0\) containing \(\varphi(\Gamma)\), the canonical map \(A^\Gamma \to B^{\Gamma'}\) in (3) is regular.

Proof. We shall use, without further mentioning, that \(p\)-independence is the same as differential independence, and a \(p\)-basis is the same as a differential basis; see [Stacks, Tag 07P2].

By [Stacks, Tag 07E5], the canonical map \(\Omega_{k/\mathbb{F}_p} \otimes_k l \to \Omega_{\overline{\mathbb{F}_p}/\mathbb{F}_p} \otimes_{\overline{\mathbb{F}_p}} l\) is injective. Let \(V\) denote its image. By the exact sequence [Stacks, Tag 00RU]
we see that $W := \ker(\Omega_{\overline{B}/F_p} \otimes_{\overline{B}} l \to \Omega_{l/F_p})$ is a finite dimensional $l$-vector space. Thus there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $V \cap W$ is contained in the linear span of the images of $d\lambda (\lambda \in \Lambda_0)$ in $V$. Thus we see that for $\Gamma_0 = \Lambda \setminus \Lambda_0$ the images of $d\lambda (\lambda \in \Gamma_0)$ in $\Omega_{l/F_p}$ are $l$-linearly independent, showing (1).

Let $\Gamma_0$ be as in (1). Note that $\Omega_{F_p(\Gamma_0)/F_p}$ has a basis $d\lambda (\lambda \in \Gamma_0)$ and they are mapped to $l$-linearly independent elements of $\Omega_{l/F_p}$. By [Stacks, Tag \texttt{07EL}], we see that $F_p(\Gamma_0) \to l$ is formally smooth, thus by definition [Stacks, Tag \texttt{00TI}] and the completeness of $B$ we can lift the identity map $l = B/n$ to an $F_p(\Gamma_0)$-algebra map $l \to B$, showing (2).

Now let $\Gamma_0, l'$ be as in (2) and take $\Gamma \subseteq \Gamma_0$. For any cofinite subset $\Gamma'$ of $\Lambda'$ containing $\varphi(\Gamma)$ and any positive integer $e$ we have a canonical map $k^{\Gamma,e} \to l'^{\Gamma,e}$ of $k$-algebras, see Construction \texttt{2.1}. This induces a canonical map $A^{\Gamma} \to B'^{\Gamma'}$ of $A$-algebras. This map is clearly local, and it is flat since both $A^{\Gamma}$ and $B'^{\Gamma'}$ are flat over $A$ (Lemma \texttt{2.2(1)}) and $A^{\Gamma}/mA^{\Gamma} = k^{\Gamma}$ is a field, see [Stacks, Tag \texttt{00MK}].

Finally, let $\Gamma_0, l', \Lambda'$ be as in (3) and fix a cofinite subset $\Gamma$ of $\Gamma_0$. Then $A^{\Gamma}$ is excellent (Lemma \texttt{2.2(1)}). Thus for any $\Gamma'$ containing $\varphi(\Gamma)$, the flat map $A^{\Gamma} \to B'^{\Gamma'}$ is regular if $k^{\Gamma} \to \overline{B'^{\Gamma'}}$ is formally smooth [And74]. Note that $\overline{B'^{\Gamma'}}$ is a regular local ring, since $B$ is regular, the map $\overline{B} \to \overline{B'^{\Gamma'}}$ is flat, and $\overline{B'^{\Gamma}}/n\overline{B'^{\Gamma}} = l'^{\Gamma}$ is a field. By [Stacks, Tag \texttt{07E5}], we must show that the elements $d\lambda (\lambda \in \Lambda \setminus \Gamma)$ are linearly independent in $\Omega_{\overline{B'^{\Gamma}}/F_p}/n$ for all small $\Gamma'$ containing $\varphi(\Gamma)$.

Recall that $\Omega_{k/F_p} \otimes_k l \to \Omega_{\overline{B}/F_p} \otimes_{\overline{B}} l$ is injective. Therefore the images of the elements $d\lambda (\lambda \in \Lambda \setminus \Gamma)$ are linearly independent in $\Omega_{\overline{B}/F_p} \otimes_{\overline{B}} l'^{\Gamma'}$, and by base change linearly independent in $\Omega_{\overline{B'^{\Gamma'}}/F_p}$. Now, the ring map $\overline{B} \to \overline{B'^{\Gamma'}}$ induces a commutative diagram

$$
\begin{array}{ccc}
\Omega_{\overline{B}/F_p} \otimes_{\overline{B}} l'^{\Gamma'} & \longrightarrow & \Omega_{l/F_p} \otimes_{l} l'^{\Gamma'} \\
\downarrow \psi & & \downarrow \\
\Omega_{\overline{B'}}/F_p \otimes_{F_p} n & \longrightarrow & \Omega_{l'/F_p}
\end{array}
$$

and we know that $\ker(\psi)$ is spanned by $d\lambda' (\lambda' \in \Gamma')$ by Lemma \texttt{6.1}. Note that $\varphi(\Gamma)$ is part of the $p$-basis $\Lambda'$, and every element of $\Gamma$ becomes a $p$-power in $\overline{B'^{\Gamma'}}$. Thus if we factor out the images of all $d\lambda (\lambda \in \Gamma)$ in the diagram,
we get a commutative diagram

\[
\frac{\Omega_{\mathbb{P}_F/\mathbb{F}_p} \otimes \Omega_{\mathbb{P}^r}}{\operatorname{span}\{d\lambda|\lambda \in \Gamma\}} \xrightarrow{\psi} \bigoplus_{\lambda' \in \Lambda' \setminus \varphi(\Gamma)} I_{\Gamma'} d\lambda' \\
\downarrow \theta \\
\Omega_{B^r}/\mathbb{F}_p/\mathbb{n} \xrightarrow{\phi} \bigoplus_{\lambda' \in \Lambda' \setminus \gamma} I_{\Gamma'} d\lambda'
\]

where \(\ker(\psi)\) is spanned by \(d\lambda' (\lambda' \in \Gamma' \setminus \varphi(\Gamma))\), thus maps isomorphically onto \(\ker(\theta)\). Since the elements \(d\lambda (\lambda \in \Lambda \setminus \Gamma)\) are linearly independent in \(\frac{\Omega_{\mathbb{P}_F/\mathbb{F}_p} \otimes \Omega_{\mathbb{P}^r}}{\operatorname{span}\{d\lambda|\lambda \in \Gamma\}}\), for them to be linearly independent in \(\Omega_{B^r}/\mathbb{F}_p/\mathbb{n}\) it suffices that the space their images span in \(\bigoplus_{\lambda' \in \Lambda' \setminus \varphi(\Gamma)} I_{\Gamma'} d\lambda'\) has zero intersection with \(\ker(\theta)\) (diagram chase). Now the choice is clear: take a finite subset \(\Lambda_0'\) of \(\Lambda' \setminus \varphi(\Gamma)\) such that the images of \(d\lambda (\lambda \in \Lambda \setminus \Gamma)\) in \(\Omega_{\mathbb{P}_F/\mathbb{F}_p}/\mathbb{n}\{d\varphi(\lambda) \mid \lambda \in \Gamma\} = \bigoplus_{\lambda' \in \Lambda' \setminus \varphi(\Gamma)} I_{\Gamma'} d\lambda'\) are in \(\bigoplus_{\lambda' \in \Lambda_0'} I_{\Gamma'} d\lambda'\), and take \(\Gamma_0' = \Lambda \setminus \Lambda_0'\).

**Corollary 6.3.** Let \(\varphi : (A, m) \rightarrow (B, n)\) be a flat local map of Noetherian complete local rings such that \(\overline{B} := B/mB\) is geometrically regular over \(k\).

Fix a coefficient field \(k \subseteq A\) and a \(p\)-basis \(\Lambda\) of \(k\). Then there exists a coefficient field \(l\) of \(B\), not necessarily containing \(k\), and a \(p\)-basis \(\Lambda'\) of \(l\) that satisfy the following:

For all cofinite subsets \(\Gamma_1\) of \(\Lambda\) and \(\Gamma_1'\) of \(\Lambda'\), there exist cofinite subsets \(\Gamma\) of \(\Gamma_1\) and \(\Gamma'\) of \(\Gamma_1'\) and a local map \(A^{\Gamma} \rightarrow B^{\Gamma'}\) of \(A\)-algebras that is regular.

**Proof.** Let \(\Gamma_0\) a cofinite subset of \(\Lambda\), \(l = l'\) a coefficient field of \(B\), and \(\Lambda'\) a \(p\)-basis of \(l\) be as in Theorem 6.2(3); we shall show that this choice of \(l\) and \(\Lambda'\) works.

Let \(\Gamma\) be a cofinite subset of \(\Gamma_0 \cap \Gamma_1\) such that \(\varphi(\Gamma) \subseteq \Gamma_1'\), and let \(\Gamma_0'\) be as in Theorem 6.2(4) for \(\Gamma\). Then we have \(\varphi(\Gamma) \subseteq \Gamma_0' \cap \Gamma_1'\). Thus by Theorem 6.2(4) we see \(\Gamma\) and \(\Gamma' := \Gamma_0' \cap \Gamma_1'\) works. \(\square\)

7. Consequences

We are now able to answer a few questions in [ST19, §7]. These results recover known results about \(F\)-rationality [Vél95, Theorems 2.2, 3.1, and 3.5] via [ST19, Proposition 2.1], and can be viewed as quantitative versions of such.

**Theorem 7.1** ([ST19, Question 7.4]). Let \(A\) be a Noetherian complete local ring, \(k\) a coefficient field of \(A\). Let \(\Lambda\) be a \(p\)-basis of \(k\). Let \(R\) be a finite type \(A\)-algebra and let \(Q \in \operatorname{Spec}(R)\).

Then \(s_{\text{rel}}(R_Q) = \sup_{\Gamma} s_{\text{rel}}(R_{Q^\Gamma})\), where \(\Gamma\) ranges over all cofinite subsets of \(\Lambda\).

**Proof.** We know \(s_{\text{rel}}(R_Q) \geq \sup_{\Gamma} s_{\text{rel}}(R_{Q^\Gamma})\) by [ST19, Corollary 3.12], and it suffices to show the reversed inequality. We may assume \(s_{\text{rel}}(R_Q) > 0\), so \(R_Q\) is \(F\)-rational [ST19, Proposition 2.1], thus \(R_Q\) is Cohen-Macaulay
By Lemma 2.2(1), all $R^\Gamma_{Q'}$ are Cohen-Macaulay, hence Lemma 3.4 applies.

Let $I_0$ be a parameter ideal of $R_Q$, so $I_0 R^\Gamma_{Q'}$ is a parameter ideal of $R^\Gamma_{Q'}$ for all $\Gamma$. By Theorem 3.5(1) (or [Pol18, Theorem 3.6]) we can take a constant $C$ that controls $F$-colength differences for $R^\Lambda$ (Definition 3.3). By Theorem 3.5(3) we see that $C$ controls $F$-colength differences for $R$ and all $R^\Gamma$ where $\Gamma$ is a cofinite subset of $\Lambda$. Thus for all positive integers $e$, $|s^e_{I_0}(R_Q) - s_{rel}(R_Q)| \leq C p^{-e}$, and for all $\Gamma$, $|s^e_{I_0 R^\Gamma_{Q'}}(R^\Gamma_{Q'}) - s_{rel}(R^\Gamma_{Q'})| \leq C p^{-e}$. From Theorem 5.1 we see $\sup_\Gamma s_{rel}(R^\Gamma_{Q'}) \geq s_{rel}(R_Q) - 2C p^{-e}$ for all $e$, as desired.

**Theorem 7.2** ([ST19, Question 7.3]). Let $R \to S$ be a flat local map of Noetherian local rings with geometrically regular closed fiber. Then $s_{rel}(R) = s_{rel}(S)$.

**Proof.** Note that if $k$ is a field and $A$ is a Noetherian local geometrically regular $k$-algebra, then so is $A^c$, since $A^c \otimes_k k'$ is the completion of $A \otimes_k k'$ for all finite purely inseparable extensions $k'$ of $k$. We may therefore assume $R, S$ complete since completion does not change the relative $F$-rational signature (cf. [HY, Observation 2.2(1)]). Fix a parameter ideal $I_0$ of $R$ and a parameter ideal $J_0$ of $S$. We do not require $I_0 R \subseteq J_0$. Since $0 \leq s_{rel}(S) \leq s_{rel}(R)$, we may assume $s_{rel}(R) > 0$, so $R$ is $F$-rational [ST19, Proposition 2.1], thus $R$ is Cohen-Macaulay [HH94, Theorem 4.2(c)]. Then $S$ is Cohen-Macaulay [Stacks, Tag 045J].

Let $k$ be a coefficient field of $R$ and $\Lambda$ a $p$-basis of $k$. Let $l$ be a coefficient field of $S$ and $\Lambda'$ a $p$-basis of $l$ as in Corollary 6.3. By Theorem 3.5(1) (or [Pol18, Theorem 3.6]) we can find a constant $C$ that controls $F$-colength differences for $S^{\Lambda'}$ (Definition 3.3).

Fix a positive integer $e$. By Theorem 5.1, there exists a cofinite subset $\Gamma_1$ of $\Lambda$ such that such that for all cofinite subsets $\Gamma$ of $\Gamma_1$, $s^e_{I_0}(R) = s^e_{I_0 R^\Gamma}(R^\Gamma)$; and a cofinite subset $\Gamma'_1$ of $\Lambda'$, such that for all cofinite subsets $\Gamma'$ of $\Gamma'_1$, $s^e_{J_0}(S) = s^e_{J_0 S'^{\Lambda'}}(S^{\Lambda'})$. Corollary 6.3 ensures that we can find $\Gamma \subseteq \Gamma_1$ and $\Gamma' \subseteq \Gamma'_1$ that admit an $R$-algebra map $R' := R^\Gamma \to S'^{\Lambda'} =: S'$ that is local and regular. Since $R$ and $S$ are Cohen-Macaulay, so are $R'$ and $S'$, see Lemma 2.2(1). Thus by [ST19, Corollaries 5.9 and 5.14], $s_{rel}(R') = s_{rel}(S')$.

By Theorem 3.5(3) we see that $C$ controls $F$-colength differences for $R, R', S,$ and $S'$. In particular, we have $|s^e_{\alpha}(D) - s_{rel}(D)| \leq C p^{-e}$ where $D$ is $R, R', S,$ or $S'$ and $\alpha$ is any parameter ideal of $D$ (Lemma 3.4). By our choices we see that $s^e_{I_0}(R) = s^e_{I_0 R'}(R')$ and $s^e_{J_0}(S) = s^e_{J_0 S'}(S')$. Now, $s_{rel}(R') = s_{rel}(S')$ implies $|s_{rel}(R) - s_{rel}(S)| \leq 4C p^{-e}$. Since $e$ was arbitrary, we see $s_{rel}(R) = s_{rel}(S)$, as desired.

A local ring $A$ is a $G$-ring if $A$ is Noetherian and the fibers of $A \to A^\wedge$ are geometrically regular (cf. [Stacks, Tag 07PT]).
Theorem 7.3 ([ST19, Question 7.1]). Let $R$ be a finite type algebra over a local $G$-ring. Then the function $p \mapsto s_{\text{rel}}(R_p)$ is lower semi-continuous on $\text{Spec}(R)$.

Proof. By Theorem 7.2 (and [Stacks, Tag 02JY]) it suffices to prove the result for a finite type algebra over a Noetherian complete local ring. Since the supremum of a family of lower semi-continuous functions is lower semi-continuous, by the $\Gamma$-construction and Theorem 7.1 it suffices to prove the result for an $F$-finite Noetherian ring $R$. Note that $R$ is excellent by Kunz’s theorem [Mat80, (42.B) Theorem 108]. In particular, the Cohen-Macaulay locus of $R$ is open [EGA IV$_2$, Proposition 6.11.8].

Note $s_{\text{rel}}(D) > 0$ for a Noetherian local ring $D$ implies $D^{\wedge}$ $F$-rational [ST19, Proposition 2.1], hence normal and Cohen-Macaulay [HH94, Theorem 4.2(b)(c)]. We see that if $s_{\text{rel}}(R_p) > 0$ for some $p \in \text{Spec}(R)$ then there exists an $f \in R \setminus p$ such that $R_f$ is a Cohen-Macaulay domain. Then on $\text{Spec}(R_f)$ the relative $F$-rational signature coincides with the dual $F$-signature, and is lower semi-continuous, see [ST19, Corollary 5.9 and Theorem 5.10]. Since $s_{\text{rel}}$ is always non-negative, this shows the lower semi-continuity of our function. □

8. The case of the $F$-rational signature

Our method can be used to study the $F$-rational signature defined by Hochster-Yao [HY]. To be precise, in the notations of Definition 3.1, let

$$s_{I_0}^{e,1}(R) = \min \{ s_{I_0}^e(R, I_0 + uR) \mid u \in R, (I_0 : u) = \mathfrak{m} \}.$$ 

Then if $R^{\wedge}$ is equidimensional, $I_0$ is a parameter ideal, and $C$ controls colength differences for $\mathfrak{m} \in \text{Spec}(R)$, we have

$$\left| s_{I_0}^{e,1}(R) - s_{\text{rat}}(R) \right| \leq Cp^{-e}$$

by [HY, Theorem 2.5]. Here, $s_{\text{rat}}(R)$ is the $F$-rational signature of $R$, denoted by $r_R(R)$ in [HY]. One can then apply the methods in §5 to study the truncated invariant $s_{I_0}^{e,1}(R)$: instead of parametrizing the ideal $J$ with a nonzero $n$-by-$n$ matrix, we parametrize the element $u$ by a nonzero 1-by-$n$ vector.

The problem here is that we do not know if étale-local extensions preserve this truncated invariant (or their limit $s_{\text{rat}}(R)$). So, we cannot get the full statement of Theorem 5.1 (or Theorem 7.1) in this case. However, the following holds.

Proposition 8.1. Let $A$ be a Noetherian complete local ring, $k$ a coefficient field of $A$, and $\Lambda$ a $p$-basis of $k$. Let $R$ be a finite type $A$-algebra. Let $Q \in \text{Spec}(R)$ and let $I_0$ be a $QRQ$-primary ideal of $R_Q$. Let $e$ be a positive integer.
Assume that $\kappa(Q)$ is separably closed. Then there exists a cofinite subset $\Gamma_0 = \Gamma_0(Q, I_0, e)$ of $\Lambda$ such that for every cofinite subset $\Gamma$ of $\Gamma_0$, $s_{I_0}^{e,1}(R_Q) = s_{I_0}^{e,1}(R_{Q_{\Gamma}})$.

Since $s_{\text{rat}}(R) > 0$ implies $R$ Cohen-Macaulay (in fact it implies $F$-rationality of $R^\Lambda$, [HY, Theorem 4.1]), we get from the proof of Theorem 7.1 the following.

**Corollary 8.2.** Let $A$ be a Noetherian complete local ring, $k$ a coefficient field of $A$, and $\Lambda$ a $p$-basis of $k$. Let $R$ be a finite type $A$-algebra and let $Q \in \text{Spec}(R)$.

Assume that $\kappa(Q)$ is separably closed. Then $s_{\text{rat}}(R_Q) = \sup_{\Gamma} s_{\text{rat}}(R_{Q_{\Gamma}})$, where $\Gamma$ ranges over all cofinite subsets of $\Lambda$.

However, even if one can remove the separable closedness condition, one does not necessarily get versions of Theorems 7.2 and 7.3 since the $F$-finite case is not known.

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4 Since we are in characteristic $p$, this is equivalent to say that $k$ is separably closed and that $Q$ is a maximal ideal of $R$ lying above the maximal ideal of $A$. 
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