Chiral orbital current and anomalous magnetic moment in gapped graphene

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We present a low-energy effective-mass theory to describe chiral orbital current and anomalous magnetic moment in graphene with band gap and related materials. We explicitly derive a quantum mechanical current distribution in general Bloch electron systems, which describes a chiral current circulation supporting the magnetic moment. We apply the formulation to gapped graphene monolayer, bilayer and ABC-stacked multilayers, to show that the chiral current is opposite between different valleys, and corresponding magnetic moment accounts for valley splitting of Landau levels. In gapped bilayer and ABC multilayer graphenes, in particular, the valley-dependent magnetic moment is responsible for huge paramagnetic susceptibility at low energy, which enables a full valley polarization up to relatively high electron density. The formulation also applies to the gapped surface states of three-dimensional topological insulator, where the anomalous current is related to the magneto-electric response in spatially-modulated potential.

I. INTRODUCTION

The magnetic moment in an electronic system consists of two distinct factors due to spin and orbital motion of electrons. In solids, the spin magnetic moment is enhanced by anomalous factor caused by the orbital effect, resulting in increase of g-factor.\cite{12} Graphene,\cite{13, 14} has an intriguing counterpart of spin, which is associated with valley pseudo-spins, i.e., degree of freedom corresponding to different points in the Brillouin zone called $K_+$ and $K_-$ valleys. Specifically, when the band gap is opened by an asymmetric potential breaking the sublattice symmetry, the graphene electrons have anomalous magnetic moment opposite in different valleys similarly to real spin.\cite{15} Generally the anomalous magnetic moment is closely related to the geometric nature of the Bloch band, and has been argued in relation to Berry phase.\cite{16, 17} Previously we calculated the orbital susceptibility in gapped monolayer and bilayer graphenes, and showed that the susceptibility near $K_\pm$ point, where the dispersion is quadratic, is contributed from the Pauli paramagnetism caused by the valley pseudo-spin.\cite{18}

In this paper, to understand the physical origin of pseudo-spin magnetic moment, and also to investigate the pseudo-spin magnetic moment in various electronic structures other than quadratic dispersion, we develop a general low-energy effective-mass theory to describe anomalous current density supporting the magnetic moment. We explicitly derive a quantum mechanical current distribution in general Bloch electron systems, which describes chiral current circulation for each eigenstate. Using the formula, we actually calculate the valley-dependent chiral current in gapped graphene monolayer, bilayer,\cite{19, 20} and ABC-stacked multilayers.\cite{21, 22} The valley-dependent magnetic moment exactly gives the valley splitting of Landau levels, generalizing our previous results limited to the quadratic dispersion.\cite{18} In gapped bilayer\cite{23} and ABC multilayers, in particular, the valley splitting and diverging density of states at the band bottom result in a huge paramagnetic susceptibility, enabling a full valley polarization up to relatively high electron density of the order of $10^{12}$ cm$^{-2}$ at a magnetic field of $\sim 1$ T.

The formulation also allows to include the external potential field within the low-energy approximation, and thus useful to investigate the chiral current in disordered systems and also finite systems bound by potential barrier. It also applies to the gapped surface states of the three-dimensional topological insulator, where the anomalous current describes the magneto-electric response in a spatially-modulated potential.\cite{23, 24}

Paper is organized as follows. In Sec. II we present the general effective mass description of the anomalous current density for Bloch electrons. We apply this to asymmetric monolayer, bilayer and ABC multilayer graphenes in Sec. III, Sec. IV and Sec. V, respectively, to describe the chiral current circulation, magnetic moment and valley splitting of Landau levels. In Sec. VI we calculate the magnetic susceptibility and argue the role of the anomalous magnetic moment. We describe in Sec. VII the current distribution in spatially modulated external potential, and formulate it in terms of a response function analogous to the Hall conductivity. The conclusion is given in Sec. VIII.

II. ANOMALOUS ORBITAL CURRENT

We consider a Bloch electron system described by an effective-mass Hamiltonian matrix $H_{mm'}(p)$, where $p$ is the crystal momentum, and $m$ and $m'$ are band indeces. We assume that the Hamiltonian is diagonalized at $p = 0$ as

$$H_{mm'}(0) = \varepsilon_m^0 \delta_{mm'}, \quad (1)$$

and, for simplicity, that there are no degeneracy at $p = 0$. In presence of the external potential $V(r)$, the effective-mass wavefunction $F(r)$ obeys the Schrödinger equation

$$\sum_{m'} H_{mm'}(p) F_{m'}(r) = (\varepsilon - V(r)) F_m(r), \quad (2)$$

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$$\sum_{m'} H_{mm'}(p) F_{m'}(r) = (\varepsilon - V(r)) F_m(r), \quad (2)$$
where \( p = -i\hbar \nabla \), and \( \varepsilon \) is the eigen energy. We assume \( |V| \ll |\varepsilon_m^{(0)} - \varepsilon_n^{(0)}| \), so that the states of different bands are not strongly mixed.

We focus on an eigenstate near \( \varepsilon = \varepsilon_n^{(0)} \) of the particular band \( n \). Then the wavefunction mainly has its amplitude on \( F_n \). By the first-order perturbation, the amplitude at \( F_{m \neq n} \) can be written in terms of \( F_n \) as

\[
F_m(\mathbf{r}) \approx \frac{\mathcal{H}_{mn}(\mathbf{p})}{\varepsilon_n^{(0)} - \varepsilon_m^{(0)}} F_n(\mathbf{r}).
\]

The Schrödinger equation, Eq. (2), then becomes

\[
[\mathcal{H}_n^{(\text{eff})}(\mathbf{p}) + V(\mathbf{r})] F_n(\mathbf{r}) = \varepsilon F_n(\mathbf{r}),
\]

with the effective Hamiltonian

\[
\mathcal{H}_n^{(\text{eff})}(\mathbf{p}) = \mathcal{H}_n(\mathbf{p}) + \sum_{m \neq n} \frac{\mathcal{H}_{nm}(\mathbf{p})\mathcal{H}_{mn}(\mathbf{p})}{\varepsilon_n^{(0)} - \varepsilon_m^{(0)}}.
\]

Correspondingly, we can define the effective velocity operator

\[
v_n^{(\text{eff})} = \frac{\partial \mathcal{H}_n^{(\text{eff})}(\mathbf{p})}{\partial \mathbf{p}},
\]

and the local current density operator

\[
j_n^{\mu(\text{eff})}(\mathbf{R}) = -\frac{e}{2} \left\{ v_n^{\mu(\text{eff})}, \delta(\mathbf{r} - \mathbf{R}) \right\}.
\]

\[
j_n^{\mu(\text{eff})} = -\frac{e}{2} \left[ \{ v_n^{\mu}, \delta(\mathbf{r} - \mathbf{R}) \} + \frac{1}{\varepsilon_n^{(0)} - \varepsilon_m^{(0)}} \times \right.
\]

\[
\left. \left\{ (v_{nm}^{\mu} \mathcal{H}_{mn} + \mathcal{H}_{nm} v_{mn}^{\mu}), \delta(\mathbf{r} - \mathbf{R}) \right\} \right].
\]

where \( \{a, b\} = ab + ba \) is the anti-commutator, and

\[
v_{\mu mn}^{\mu} = \frac{\partial \mathcal{H}_{mn}(\mathbf{p})}{\partial \mathbf{p}}.
\]

\( j_n^{\mu(\text{eff})} \) actually covers only a part of the total current density even in the low-energy limit. The original current density operator is given by

\[
j_\mu(\mathbf{R}) = -\frac{e}{2} \left\{ v_\mu, \delta(\mathbf{r} - \mathbf{R}) \right\}.
\]

where \( v_\mu \) is a matrix defined by Eq. (3). The expectation value of \( j_\mu \) for a given state \( \mathbf{F} \) near \( \varepsilon_0^{(0)} \) is written as

\[
\langle j_\mu(\mathbf{R}) \rangle = \sum_{mn} \int d\mathbf{r} \mathcal{F}_\mu^*(\mathbf{r}) [j_\mu(\mathbf{R})]_{mn} \mathcal{F}_m(\mathbf{r}) \approx \int d\mathbf{r} \mathcal{F}_\mu(\mathbf{r}) j_n^{\mu(\text{eff})}(\mathbf{R}) F_n(\mathbf{r}).
\]

III. MONOLAYER GRAPHENE

Graphene is composed of a honeycomb network of carbon atoms, where a unit cell contains a pair of sublattices, denoted by \( A \) and \( B \). Low-energy electronic states
are described by the effective Hamiltonian,
\[ \mathcal{H}(p) = \begin{pmatrix} \Delta & vp_z \\ vp_z & -\Delta \end{pmatrix}, \] (15)
where \( p_\pm = \xi p_x \pm ip_y, \xi = \pm \) is the valley index corresponding to \( K_\pm \) point in the Brillouin zone, and \( p \) is the momentum measured from the \( K_\xi \). The matrix works on two-component envelope wave function \( (F_A(r), F_B(r)) \) at the \( A \) and \( B \) sublattices, respectively. The diagonal terms \( \pm \Delta \) opening the energy gap at Dirac point, is given by the potential asymmetry between \( A \) and \( B \) sites, which can arise in a certain substrate material for instance.\[13,14]\] The band velocity is \( v \approx 1 \times 10^6 \text{ m/s} \).

The surface states of the three-dimensional topological insulator of Bi\(_2\)Se\(_3\) family is also described by a similar Hamiltonian to Eq. \((15)\), where \((p_x, p_y)\) is rotated to \((p_y, -p_x)\).\[23,24]\] The rotation of vector \( p \) is compensated by the spinor rotation and does not affect the following argument. There is only single valley index, and the diagonal term \( \Delta \) appears only when the time-reversal symmetry is broken, for instance, by attaching a ferromagnetic insulator of Bi\(_2\)Se\(_3\).\[18\]

We assume \( \Delta > 0 \) and consider a state near the electron band bottom \( \varepsilon = \Delta \). The wave amplitude is then mainly concentrated on the first component \( F \equiv F_A \). The reduced Hamiltonian for \( F \) becomes apart from the constant energy,
\[ \mathcal{H}^{(\text{eff})}(p) = \frac{p^2}{2m^*}, \] (16)
with the effective mass,
\[ m^* = \frac{\Delta}{v^2}. \] (17)
Applying Eq. \((11)\), the local current density is written as
\[ j(r) = -\frac{e\hbar}{m^*} \text{Im}(F^* \nabla F) - \frac{e\hbar}{2m^*} (\varepsilon z \times \nabla)|F|^2, \] (18)
where \( \nabla = (\partial/\partial x, \partial/\partial y, 0) \), and \( \varepsilon z = (0, 0, 1) \). The first term is the usual current density, corresponding to \( j^{(\text{eff})} \) of Eq. \((11)\). The second term is the anomalous component, and denoted as \( j_\varepsilon \), in the following. It flows perpendicularly to the gradient of the density \( |F|^2 \), and thus it circulates on a closed loop and does not contribute to the electron transport. The direction is opposite between \( \varepsilon = \pm \). It is written in terms of equivalent local magnetic moment \( \mu \) as
\[ j_\varepsilon(r) = e \nabla \times \mu(r), \]
\[ \mu(r) = -\frac{e\hbar}{2m^*c} |F|^2 \varepsilon z. \] (19)

For the valence band electron, a similar calculation shows that the first term of Eq. \((18)\) flips the sign while the second term remains unchanged.

The expression of the magnetic moment operator, Eq. \((14)\), becomes
\[ m = -\frac{e}{2m^*c} (xp_y - yp_x) - \frac{\xi e\hbar}{2m^*c}, \] (20)
where the first and second terms corresponds to those of Eq. \((18)\), respectively. The second term, now denoted as \( m_\varepsilon \), is the magnetic moment induced by the anomalous current and coincides with the integral of \( \mu(r) \) of Eq. \((14)\) over the space. It should be noted that \( m_\varepsilon \) is constant regardless of the detail of the wavefunction. This is analogous to spin magnetic moment of bare electron system with \( \xi \) being the spin index, while in graphene this is mimicked by the valley-dependent chiral orbital current. The expression agrees with an intrinsic magnetic moment in the semi-classical picture, that attributed to the self-rotation of the wave packet.\[22\]

The valley pseudo-spin magnetic moment \( m_\varepsilon \) produces the pseudo-spin Zeeman energy in presence of a magnetic field, and this accounts for the valley splitting of Landau levels in graphene.\[25\] This can be checked by considering the Hamiltonian in a uniform external field \( \mathbf{B} \), or \( \mathcal{H}(\pi) \) in Eq. \((15)\), where \( \pi = \mathbf{p} + e\mathbf{A}/c \) with the vector potential \( \mathbf{A} \) giving \( \mathbf{B} = \nabla \times \mathbf{A} \). Noting the relation \( [\pi_x, \pi_y] = -i\hbar eB/c \), the reduced Hamiltonian for the \( A \) site near \( \varepsilon = \Delta \) is written as
\[ \mathcal{H}^{(\text{eff})}(\pi) = \frac{\hbar^2}{2\Delta} \pi_{-\pi_+} = \hbar \omega_c \left( \hat{n} + \frac{1}{2} + \xi \pi \right), \] (21)
where \( \omega_c = eB/(mc) \), \( \pi_{\pm} = \xi \pi_x \pm i\pi_y \), \( \hat{n} = a^\dagger a \), \( a = (2\hbar eB/c)^{-1/2}(\pi_x - i\pi_y) \) is the annihilation operator of Landau level, and we used the relation \( \pi^2 = (2\hbar eB/c)(\hat{n} + 1/2) \). The term depending on \( \xi \) is the pseudo-spin Zeeman energy, and actually coincides with \(-m_\varepsilon \cdot \mathbf{B} \). In graphene, the pseudo-spin Zeeman splitting is equal with the Landau level spacing, so that the \( n \)-th Landau level at the valley \( K_+ \) has the same energy \((n+1)\)-th level at \( K_- \).

The two terms in the current distribution of Eq. \((18)\) can be distinguished by change in the two-dimensional mirror reflection,
\[ F(r) \rightarrow F'(r) = \begin{pmatrix} F(r) \\ \pi \end{pmatrix}, \] (22)
where \( r = (x, y) \) and \( r' = (-x, y) \). Let \( j(r) \) and \( j'(r) \) be the expectation values of the current density for the wavefunctions \( F \) and \( F' \), respectively. Each current component changes with either of \( s = \pm \) in
\[ \begin{pmatrix} j_s(r) \\ j_s'(r) \end{pmatrix} = s \begin{pmatrix} j_{-s}(r) \\ j_{-s}(r) \end{pmatrix}, \] (23)
or equivalently,
\[ r' \times j'(r) = -s r \times j(r). \] (24)
In Eq. \((18)\), the first term \( j^{(\text{eff})} \) yields to \( s = + \), i.e., the current map is just mirror-reflected in the same way as
This is a natural consequence, since $\mathcal{H}^{(\text{eff})}$ is invariant in the mirror reflection.

The second term $j'$ has an opposite sign $s = -$, or $j'_c$ goes against the mirror reflection of $j_c$, and can be called chiral in this sense. In gapped graphene, having this term may look counter-intuitive since the system is originally mirror symmetric with respect to a line containing an $AB$ bond. But this “real” reflection exchanges valleys $\xi = \pm$ at the same time in addition to Eq. (22), so that $j_c$ is then simply mirror-reflected as it should. Therefore the chiral term is necessarily accompanied by the factor $\xi$.

Two current components behave also differently in the effective time reversal operation $F \rightarrow F^\ast$ within single valley. The first term obviously reverses in this operation, as a consequence of the effective time-reversal symmetry for $\mathcal{H}^{(\text{eff})}$. The second term depends only on the absolute value of the wave amplitude and thus remains unchanged in the same operation. But it reverses in the real time-reversal operation which switches $\xi = \pm$. We will see that the same argument applies to bilayer graphene as well.

IV. BILAYER GRAPHENE

Bilayer graphene\textsuperscript{14–17} is a pair of graphene layers arranged in AB (Bernal) stacking and includes $A_1$ and $B_1$ atoms on layer 1 and $A_2$ and $B_2$ on layer 2.\textsuperscript{35–42} The states at $B_1$ and $A_2$ are coupled by $\gamma_1 \approx 0.39$ eV.\textsuperscript{43} The low-energy states are described by the Hamiltonian matrix for the basis $\{|A_1\rangle, |B_1\rangle, |A_2\rangle, |B_2\rangle\}$\textsuperscript{35,36}

$$\mathcal{H}(p) = \begin{pmatrix}
\Delta & vp_- & 0 & 0 \\
vp_+ & -\Delta & \gamma_1 & 0 \\
0 & \gamma_1 & -\Delta & vp_- \\
0 & 0 & vp_+ & -\Delta
\end{pmatrix}, \quad (25)$$

where $\Delta$ describes potential asymmetry between layer 1 and 2 (not $A$ and $B$ sites), which gives rise to an energy gap.\textsuperscript{35–39,41,44,45} Experimentally the potential asymmetry can be induced by applying an electric field perpendicular to the layer and the asymmetry as large as $\Delta \sim 0.1$ eV was actually observed in spectroscopic measurements.\textsuperscript{15,46,47} For simplicity, we neglected the trigonal warping effect due to the extra band parameter.\textsuperscript{35,40}

Let us assume $\Delta > 0$ in the following. At $p = 0$, the Hamiltonian gives four eigen energies

$$\varepsilon^0_1 = -\sqrt{\gamma_1^2 + \Delta^2}, \quad \varepsilon^0_2 = -\Delta, \quad \varepsilon^0_3 = \Delta, \quad \varepsilon^0_4 = \sqrt{\gamma_1^2 + \Delta^2}. \quad (26)$$

We consider a state near the conduction band bottom $\varepsilon = \varepsilon^0_3$, of which wave amplitude is mostly concentrated on the first component $F_{A1} \equiv F$. The effective Hamiltonian
The second components of momentum and energy are
\[ g = \frac{p^2}{4m_0\gamma_1^2} - \frac{p^2}{2m_0}, \] (27)
where the energy is measured from \( \varepsilon = \Delta \) and
\[ m_0 = \frac{\gamma_1^2}{4e^2\Delta}, \quad p_0 = \hbar k_0 = \frac{\sqrt{2\Delta}}{\nu}. \] (28)
The term with \( p^2 \) comes from the off-diagonal elements of the Hamiltonian matrix for \( p = 0 \). To have \( p^4 \) term, we need in Eq. (11) the higher order term for the off-diagonal matrix element between \( j = 2 \) and \( 3 \); i.e., use instead of \( H_{32} \)
\[ \tilde{H}_{32} = H_{32} + \sum_{m=1,4} H_{3m} \frac{1}{\varepsilon_0 - \varepsilon_n} H_{n2}. \] (29)
The dispersion is plotted in Fig. 1(a). It is non-monotonic function of \( p \), and the band minimum appears at off-center momentum \( p = p_0 \) and energy \( \varepsilon = -\varepsilon_0 \), where
\[ \varepsilon_0 = \frac{2\Delta^3}{\gamma_1^2}. \] (30)
For instance, the asymmetric energy of \( \Delta = 0.1 \) eV gives \( \varepsilon_0 = 13 \) meV. The density of states is given by
\[ D(\varepsilon) = g_s g_v \frac{m_0}{2\pi\hbar^2} \frac{1}{\sqrt{1 + \varepsilon/\varepsilon_0}} \times \begin{cases} 0 & (\varepsilon < -\varepsilon_0) \\ 2 & (-\varepsilon_0 < \varepsilon < 0) \\ 1 & (\varepsilon > 0), \end{cases} \] (31)
where \( g_s = g_v = 2 \) is spin and valley degeneracies.

The local current density of Eq. (11) is written in the same level of approximation as
\[ \langle j(r) \rangle = \text{Im} \mathbf{u} + \xi(-\mathbf{e}_z \times \text{Re} \mathbf{u}) \] (32)
where the vector \( \mathbf{u} \) is defined by
\[ u_{\mu} = -\frac{e\hbar}{2m_0 k_0^2} \sum_{\nu=x,y} \left[ 2(\partial_{\nu} F^*) \partial_{\mu} (\partial_{\nu} F) - \partial_{\mu} (F^* \partial_{\nu}^2 F) \right] + \frac{e\hbar F^* \partial_{\mu} F}{m_0}. \] (33)
The second components of \( \langle j(r) \rangle \) is the chiral current and expressed as
\[ \langle j_z(r) \rangle = e\nabla \times \mathbf{\mu}(r), \]
\[ \mathbf{\mu}(r) = \xi e^2 \frac{\hbar}{2m_0 c} \left\{ -\frac{1}{k_0^2} \left[ (\nabla F)^2 - \text{Re}(F^* \nabla^2 F) \right] + |F|^2 \right\}. \] (34)
The equivalent magnetic moment \( \mathbf{\mu}(r) \) now depends on \( F \) and its derivative. The magnetization of Eq. (14) becomes
\[ m = -\frac{e}{2c} (xy^y^{(eff)} - yv^e^{(eff)}) - \xi \frac{e\hbar}{m_0 c} \left( \frac{p^2}{p_0^2} - \frac{1}{2} \right). \] (35)
The second term, \( m_{c} \), is the valley magnetic moment induced by the chiral current. The valley splitting energy at the band bottom can be estimated by inserting \( p = p_0 \),
\[ 2|m_{c}(p_0)| B = \frac{\hbar e B}{m_0 c} \approx \hbar \omega_0. \] (36)
The effective \( g \)-factor for this pseudo-spin splitting is given by \( g^* = 2m/m_0 \) where \( m \) is the bare electron mass. \( g^* \) is proportional to \( \Delta \) and it approximately 30 at \( \Delta = 0.1 \) eV.

When the valley splitting exceeds \( \varepsilon_F \), the system is fully valley-polarized with single kind of chiral particles. Using the density of states of Eq. (31), the condition for full valley polarization is estimated in low \( B \)-field limit,
\[ n < n_{\text{crit}} = \frac{1}{\pi \hbar v} \sqrt{2eB/(\gamma eB)}, \] (37)
where \( n \) is the electron density. We have \( n_{\text{crit}} \approx 5 \times 10^{11} \text{ cm}^{-2} \) at \( \Delta = 0.1 \) eV and \( B = 1 \) T. For the gapped monolayer graphene, the condition is
\[ n < n_{\text{crit}} = g_s eB/\hbar, \] (38)
which is approximately \( 5 \times 10^{10} \text{ cm}^{-2} \) at \( B = 1 \) T. In bilayer, the critical density is proportional to \( \sqrt{B} \) rather than \( B \), and thus the valley polarization is achieved in much lower magnetic fields than in monolayer, in a small electron density. This property is owing to the divergence of the density of states at the band bottom.

Similarly to monolayer, the valley splitting of Landau levels in asymmetric bilayer graphene\(^{16,35,48}\) is correctly given by the pseudo-spin Zeeman energy due to the magnetic moment \( m_{c} \). The original Hamiltonian in a magnetic field is given by Eq. (25) with \( p \) replaced by \( \pi \). Near \( \varepsilon = \Delta \), it is reduced to
\[ \mathcal{H}^{(eff)}(\pi) \approx \frac{1}{2\Delta} \left( \frac{(v\pi_+)^2}{\gamma_1^2} - 2\Delta \frac{(v\pi_-)(v\pi_+)}{\gamma_1^2} \right) \]
\[ = \frac{(\hbar \omega_0)^2}{4\varepsilon_0} \left[ \left( n + \frac{1}{2} + \xi \right)^2 - \frac{1}{4} \right] - \hbar \omega_0 \left( n + \frac{1}{2} + \frac{\xi}{2} \right), \] (39)
where \( \omega_0 = eB/(m_0 c) \). The pseudo-spin Zeeman energy, i.e., half of the energy difference between \( \xi = \pm \), is transformed to
\[ E_{\text{Zeeman}} = \xi \frac{e\hbar}{m_0 c} \left( \frac{n^2}{p_0^2} - \frac{1}{2} \right) B, \] (40)
which coincides with \(-m_{c} \cdot B \) in the limit of \( B = 0 \).

The first and second terms in Eq. (39) correspond to \( p^4 \) and \( p^2 \) terms in the zero-field Hamiltonian, respectively, and become dominant when \( \hbar \omega_0 (n + 1/2) \gg \varepsilon_0 \) and \( \ll \varepsilon_0 \), respectively. In the lower Landau levels where the second term dominates, the \( n \)-th level at the valley \( K_+ \) and \((n+1)\)-th level at \( K_- \) approximately degenerate.
In higher levels where the first term becomes dominant, the \( n \)-th Landau level at the valley \( K_\pm \) and \((n+2)\)-th level of \( K_- \) degenerate. Fig. 1(b) plots the Landau level energy of Eq. (39) as a function of magnetic field, where dashed and solid lines represent the valley \( \xi = + \) and \(- \), respectively. At \( \Delta = 0.1 \) eV, for instance, the characteristic magnetic field corresponding to \( h\omega_0/\varepsilon_0 = 1 \) is 7.6T. A pair of dotted lines represent the energy of the band bottom shifted by pseudo-spin Zeeman energy, i.e., \( -\varepsilon_0 + \xi h\omega_0/2 \). In small \( B \)-field, they actually serve as the envelope curves for Landau levels of \( \xi = \pm \). Full valley polarization occurs below the upper slope.

V. ABC MULTILAYER GRAPHENES

For the structure of bulk graphite, there are two known forms called ABA (AB, hexagonal, or Bernal) and ABC (rhombohedral) with different stacking manners.\(^{18-20}\) The ABA phase is thermodynamically stable and common, while it is known that some portion of natural graphite takes the ABC form.\(^{20-53}\) The low-energy band structure of a finite ABC graphene multilayer is given by a pair for the surface bands localized at outer-most layers.\(^{20,49,50,52,53}\) and the interlayer potential asymmetry opens an energy gap between those bands.\(^{20-53}\)

Now we attempt to argue the chiral magnetic moment of gapped low-energy bands of ABC \( N \)-layered graphene, in a parallel way to the bilayer graphene. If the basis is taken as \( |A_1\rangle, |B_2\rangle, |A_2\rangle, |B_2\rangle, \ldots, |A_N\rangle, |B_N\rangle \), the low-energy effective Hamiltonian can be written as:\(^{26,49,50,52,53}\)

\[
\mathcal{H}_{\text{ABC}} = \begin{pmatrix}
H_1 & V & V \dagger & V \\
V & H_2 & V & V \\
V \dagger & V & H_3 & V \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{pmatrix},
\]

and

\[
H_j = \begin{pmatrix}
U_j & v_p \gamma_1 & v_p \gamma_1 & v_p \gamma_1 \\
v_p \gamma_1 & U_j & v_p \gamma_1 & v_p \gamma_1 \\
v_p \gamma_1 & v_p \gamma_1 & U_j & v_p \gamma_1 \\
v_p \gamma_1 & v_p \gamma_1 & v_p \gamma_1 & U_j \\
\end{pmatrix},
\]

where \( U_j \) is the electrostatic potential at \( j \)-th layer. For simplicity, we neglected the trigonal warping effect due to the extra band parameter.\(^{23}\)

The potential asymmetry \( U_j \) can be induced by applying an electric field \( E \) perpendicular to the layer. When \( E \) is uniform, the potential energy with respect to the middle of the stack is written as

\[
U_j = \left( \frac{N + 1}{2} - j \right) e\xi d,
\]

where \( d \approx 0.334 \) nm is the interlayer spacing. The bilayer graphene of Eq. (25) is a special case of Eq. (41) with \( N = 2 \) and \( e\xi d = 2\Delta \). The actual field \( E \) can be smaller than externally applied electric field due to the screening by the electrons in the graphene.\(^{28}\) We assume \( E > 0 \) and \( |U_j| \ll \gamma_1 \) in the following.

At \( p = 0 \), there are two low-energy eigenenergies at \( \varepsilon = U_1 \) and \( U_N \) originating from \( |A_1\rangle \) and \( |B_N\rangle \), while all other states appear near \( \varepsilon = \pm \gamma_1 \) through the dimerization between \( |B_j\rangle \) and \( |A_{j+1}\rangle \) for each of \( j = 1, \ldots, N-1 \). The effective Hamiltonian for the states near \( \varepsilon = U_1 \), is derived as

\[
\mathcal{H}^{(\text{eff})}(p) \approx \frac{\gamma_1^2}{(N-1)e\xi d} \left( \frac{v_p}{\gamma_1} \right)^{2N} - e\xi d \left( \frac{v_p}{\gamma_1} \right)^2 \approx \frac{1}{N} \frac{p_0^2}{2m_0} - \frac{p^2}{2m_0},
\]

where the energy is measured from \( \varepsilon = U_1 \)

\[
m_0 = \frac{\gamma_1}{2 \sqrt{e\xi d}}, \quad p_0 = \frac{\gamma_1}{v} \left( \sqrt{\frac{N - 1}{N}} \frac{e\xi d}{\gamma_1} \right)^{1/2}.
\]

The density of states diverges at \( \varepsilon = -\varepsilon_0 \) as,

\[
D(\varepsilon) \approx g_s g_v \frac{m_0}{\pi \hbar^2} \sqrt{2N} \frac{1}{N-1} \sqrt{\frac{1}{\varepsilon/\varepsilon_0} - 1}.
\]

For example, we show the energy dispersion of \( N = 3 \) and 4 in Fig. 2(a). Note that the unit \( p_0 \) and \( \varepsilon_0 \) depend on \( N \). At \( e\xi d = 0.2 \) eV, for instance, the characteristic energy scale is \( \varepsilon_0 = 54 \) meV and 86 meV for \( N = 3 \) and 4, respectively.

The magnetization of Eq. (14) becomes

\[
m = -\frac{e}{2c} (x_p y^{(\text{eff})} - y_p x^{(\text{eff})}) + \frac{e\hbar}{m_0 c} \left( \frac{N}{2} \frac{p}{p_0} \right)^{2(N-1)} - \frac{1}{2},
\]

where the second term, \( m_v \), is the valley magnetic moment. The valley splitting energy at the band bottom can be estimated by inserting \( p = p_0 \),

\[
2|m_v(p_0)| B = (N-1)\hbar\omega_0,
\]

where \( \omega_0 = eB/(m_0 c) \). The splitting is greater for larger \( N \) under the same electric field \( E \). The condition for full valley polarization in low \( B \)-field limit is

\[
n < n_{\text{crit}} = g_s \frac{1}{\pi} \sqrt{\frac{eB}{\gamma_1}} \left( \frac{N - 1}{N} \right)^{1/2}. \]

In the small field region \( e\xi d \ll \gamma_1 \), which is currently assumed, \( n_{\text{crit}} \) increases for larger \( N \), i.e., the valley polarization is achieved up to higher electron density in
larger stack. In the large $N$ limit, $n_{\text{crit}}$ approaches a value independent of $E$,

$$n_{\text{crit}} = g_s \frac{1}{\pi} \sqrt{\frac{eB}{\hbar \omega}},$$  \hspace{1cm} (51)$$

which approximates $1.5 \times 10^{12}$ cm$^{-2}$ at $B = 1$T.

The low-energy Landau level spectrum near $\varepsilon = U_1$ is

$$\mathcal{H}^{(\text{eff})} \approx \frac{\gamma_1^2}{(N-1)\varepsilon \hbar d} \frac{\varepsilon}{\gamma_1} - e\hbar d \frac{\varepsilon}{\gamma_1} \left[ \hat{n} + j - \frac{1}{2} \right]$$

$$= \frac{(N-1)^{N-1}}{N^N} \frac{(\hbar \omega_0)^N}{\varepsilon_0^{N-1}} \prod_{j=1}^{\mathcal{N}} \left[ \hat{n} + j - \frac{1}{2} \right]$$

$$- \hbar \omega_0 \left( \hat{n} + 1 + \frac{\xi}{2} \right).$$  \hspace{1cm} (52)$$

The valley splitting in the limit of $B = 0$ is again shown to be equivalent with $-m_e \cdot B$ of Eq. (18). In higher Landau levels where the first term becomes dominant, the $n$-th Landau level at the valley $K_+$ and $(n+N)$-th level of $K_-$ degenerate. Fig. 2 (b) and (c) plot the Landau level spectra of Eq. (52) for the cases of $N = 3$ and 4, respectively. The Landau levels in small magnetic fields are well bound by dotted lines, or the energies of $-\varepsilon_0 + \xi \hbar \omega_0/2$. At $e\hbar d = 0.2$ eV, for instance, the magnetic field corresponding to $\hbar \omega_0/\varepsilon_0 = 1$ is 33T and 52T for $N = 3$ and 4, respectively. As argued above, we can see that, for greater $N$, the full valley polarization is possible up to larger electron density (i.e., more Landau levels) at the same magnetic field.

VI. PSEUDO-SPIN PARAMAGNETISM

The pseudo-spin Zeeman splitting causes the Pauli paramagnetism in analogous way to real spin. The magnetic susceptibility was previously calculated for gapped monolayer and bilayer graphenes, and it was shown that the susceptibility in the quadratic dispersion near $K_\pm$ point, is expressed as sum of valley pseudo-spin-paramagnetism and Landau diamagnetism similarly to a bare electron. In monolayer graphene, the pseudo-spin-paramagnetism diverges in the zero gap limit, leading to a singular orbital susceptibility where the strong diamagnetism suddenly disappears off the Dirac point $\pm 12$T.

Here we extend the argument to general electronic structures other than quadratic, and show that the pseudo-spin splitting always accompanies paramagnetic contribution in any part of the dispersion. Let us consider a system in a magnetic field $B$ with the Landau level sequence,

$$\varepsilon_n = \varepsilon(x_n, \delta) \quad (n = 0, 1, 2, \cdots),$$

$$x_n = \left( n + \frac{1}{2} \right) \delta, \quad \delta = \hbar eB \frac{m^*}{e^2}.$$  \hspace{1cm} (53)$$

where $n$ is the Landau level index, $m^*$ is the effective mass characterizing the system. The second argument $\delta$ in $\varepsilon(x_n, \delta)$ represents the dependence on $B$, which are not included in $x_n$. For example the low-energy Landau level of gapped monolayer graphene, Eq. (21), is given by

$$\varepsilon(x_n, \delta) = x_n + \frac{\xi}{2} \delta.$$  \hspace{1cm} (54)$$
and that of bilayer graphene, Eq. (59), by
\[
\varepsilon(x, \delta) = \frac{1}{4\varepsilon_0} \left[ (x_n + \delta)^2 - \frac{1}{4} \delta^2 \right] + \left( x_n + \frac{\delta}{2} \right),
\]
with \( m^* \) replaced by \( m_0 \).

By treating \( x(x_n) \) and \( \delta \) as independent variables, we can expand \( \varepsilon(x, \delta) \) as
\[
\varepsilon(x, \delta) = \varepsilon^{(0)}(x) + \varepsilon^{(1)}(x) \delta + \frac{1}{2} \varepsilon^{(2)}(x) \delta^2 + \cdots.
\]
(56)
The zero-th order term \( \varepsilon^{(0)} \) is related to the energy spectrum at \( B = 0 \). When the system is isotropic, in particular, the dispersion is given by \( \varepsilon^{(0)}(x) \) with \( x = p^2 / 2m^* \).

The first order shift \( \varepsilon^{(1)} \) can be regarded as pseudo-spin Zeeman term associated with magnetic moment \( -(eh/cm^*) \varepsilon^{(1)} \), which corresponds to \( m_c \) in previous arguments.

The thermodynamic potential becomes
\[
\Omega = -\frac{1}{\beta} \frac{1}{2N_{\beta}} \sum_{n=0}^{\infty} \varphi[\varepsilon(x_n, \delta)]
\]
\[
= -\frac{1}{\beta} \frac{m^*}{2\hbar^2} \left[ \int_{0}^{\infty} \varphi[\varepsilon(x, \delta)] dx + \frac{\delta^2}{24} \frac{\partial^2 \varphi[\varepsilon(x, 0)]}{\partial x} \bigg|_{x=0} \right] + O(\delta^3),
\]
(57)
where \( \varphi(\varepsilon) = \ln[1 + e^{-\beta(\varepsilon - \mu)}] \), \( \beta = 1/(k_B T) \), \( \mu \) is the chemical potential, and we used the Euler-Maclaurin formula in the second equation. Using Eq. (56), we can further expand \( \Omega \) in terms of \( \delta \propto B \). The magnetization is given by
\[
M = -\left( \frac{\partial \Omega}{\partial B} \right)_\mu, \tag{58}
\]
and the magnetic susceptibility by
\[
\chi = -\left( \frac{\partial^2 \Omega}{\partial B^2} \right)_\mu \bigg|_{B=0}. \tag{59}
\]
We end up with
\[
\chi(\mu, T) = \int_{-\infty}^{\infty} d\varepsilon \left( -\frac{\partial f}{\partial \varepsilon} \right) \chi(\varepsilon), \tag{60}
\]
with
\[
\chi(\varepsilon) = \left( \frac{e\hbar}{cm^*} \right)^2 \left[ D(\varepsilon)(\varepsilon^{(1)} - 2 - \int_{-\varepsilon}^{0} d\varepsilon D(\varepsilon)^2) \right. \]
\[
- \left. \frac{1}{12} \frac{m^*}{2\hbar^2} \theta(\varepsilon - \varepsilon^{(0)}(0)) \frac{\partial \varepsilon^{(0)}(x)}{\partial x} \bigg|_{x=0} \right]. \tag{61}
\]
where \( f(\varepsilon) = [1 + e^{\beta(\varepsilon - \mu)}]^{-1} \) is the Fermi distribution function, and \( \varepsilon^{(1)} \) and \( \varepsilon^{(2)} \) are regarded as functions of energy \( \varepsilon \) through \( \varepsilon = \varepsilon^{(0)}(x) \). \( D(\varepsilon) \) is the density of states given by
\[
D(\varepsilon) = \frac{m^*}{2\pi\hbar^2} \int_{0}^{\infty} \delta(\varepsilon - \varepsilon^{(0)}(x)) dx. \tag{62}
\]
The susceptibility at \( T = 0 \) is given by \( \chi(\mu) \). The first term in Eq. (61) is regarded as the Pauli paramagnetism induced by the pseudo-spin magnetic moment. It is always positive, and purely determined by the density of states and the magnetic moment at Fermi energy. The second term is the summation of the second order energy shift \( \varepsilon^{(2)} \) over all the states below Fermi level, and the third term gives a discrete jump at the energy corresponding to \( p = 0 \).

For the low-energy spectrum of the gapped monolayer graphene, Eq. (54), we obtain
\[
\chi = \chi_P + \chi_L
\]
\[
\chi_P = D\mu_B^2, \quad \chi_L = -\frac{1}{3} D\mu_B^2, \tag{63}
\]
where \( \chi_P \) and \( \chi_L \) come from the first and the third terms in Eq. (61), respectively, and the second term is zero. Here \( D = g_0 g_v m/(2\pi\hbar^2) \theta(\varepsilon) \) is the density of states, \( \mu_B^2 = e\hbar/(2m^*\varepsilon) \) is the effective Bohr magneton. Obviously \( \chi_P \) and \( \chi_L \) correspond to conventional Pauli paramagnetism and Landau diamagnetism, respectively. The susceptibility calculated above is the contribution from the conduction band, while the valence band gives the exactly opposite jump at the valence band top. The total susceptibility is diamagnetic at \( \chi = -\chi_P - \chi_L \) in the gap region, and disappears in conduction and valence bands.

For gapped bilayer graphene, Eq. (65), we get
\[
\chi(\varepsilon) = \frac{g_0 g_v e^2}{2\pi m_0 c^2} \times \begin{cases} 0 & (\varepsilon < -\varepsilon_0), \\ \frac{2}{\sqrt{1 + \varepsilon/\varepsilon_0}} & (-\varepsilon_0 < \varepsilon < 0), \\ \frac{1}{4\sqrt{1 + \varepsilon/\varepsilon_0}} + \frac{1}{6} & (\varepsilon > 0). \end{cases} \tag{64}
\]
The susceptibility diverges at the band bottom, \( \varepsilon = -\varepsilon_0 \). The physical meaning of the divergence is obvious, since the Pauli paramagnetism, i.e., the first term of Eq. (61), is proportional to the density of states, which diverges at the band bottom. The susceptibility of Eq. (64) is plotted in Fig. 3 together with and the density of states, Eq. (51).

The argument can be extended to ABC \( N \)-layer graphene in a straightforward fashion. Using Eqs. (47) and (48), the pseudo-spin paramagnetic susceptibility above and near the band bottom \( \varepsilon = -\varepsilon_0 \) is written as
\[
\chi(\varepsilon) \approx D(\varepsilon)m_c^2 = \frac{g_0 g_v e^2}{2\pi m_0 c^2} \frac{N(N - 1)}{2\sqrt{2N}} \frac{1}{\sqrt{1 + \varepsilon/\varepsilon_0}}. \tag{65}
\]
where \( m_0 \) and \( \varepsilon_0 \) are defined in Eqs. (45) and (46), respectively. The paramagnetic divergence is stronger for greater \( N \).
the Thomas-Fermi approximation gives

\[ M = \int_{\text{occupied}} m_c(p) d^2 p, \]

and \( m_c(p) \) being the anomalous magnetic moment at the momentum \( p \). Then Eq. (68) becomes

\[ J(r) = e \sigma \frac{\partial M_F}{\partial \varepsilon} \left[ e_z \times E(r) \right], \tag{69} \]

where \( E(r) = -\nabla V(r) / (-e) \) is the electric field, leading to a response function

\[ \sigma_{xy} = -e \sigma \frac{\partial M_F}{\partial \varepsilon}. \tag{70} \]

By applying Eq. (70) to the conduction band electrons of gapped monolayer graphene, where \( m_c(p) \) is given by the second term of Eq. (20), we have

\[ \sigma_{xy} \approx \frac{\xi_e^2}{2 \hbar}. \tag{71} \]

For gapped \( N \)-layer ABC graphenes including bilayer, of which \( m_c(p) \) is given by the second term of Eq. (18), the expression approximates in high energies \( \varepsilon \gg \varepsilon_0 \),

\[ \sigma_{xy} \approx \frac{N \xi_e^2}{2 \hbar}. \tag{72} \]

When the system is confined to a finite space, the above-mentioned Hall current gives a chiral edge current at the boundary. When the confining potential is slowly varying in space, the current circulation is

\[ I = -\frac{1}{e} \int_{\varepsilon_F}^{\varepsilon_F} \sigma_{xy}(\varepsilon) d\varepsilon = c M_F, \tag{73} \]

as a natural consequence. This is equally true in a sharp potential as well, where the current is distributed in a range of the Fermi wave length from the boundary. Fig. 4 illustrates the single-valley current distribution given by the conduction band electrons of gapped monolayer and bilayer graphenes. The detail of the derivation is presented in Appendix.
From its definition, the Hall conductivity argued here, Eq. (70), is the long wavelength limit of the static Hall conductivity, namely, \( \lim_{q \to 0} \lim_{\omega \to 0} \sigma_{xy}(q, \omega) \). For the original Hamiltonian of monolayer graphene, Eq. (15), this is evaluated as\(^{61}\)

\[
\lim_{q \to 0} \lim_{\omega \to 0} \sigma_{xy}(q, \omega) = -\frac{e^2}{2h} \theta(\Delta - |\varepsilon_F|), \tag{74}
\]

where \( \theta(x) = 1 \) \((x > 0)\), \( 0 \) \((x < 0)\) is the step function, and \( \Delta > 0 \) is assumed here. The low-energy result, Eq. (71), describes the contribution from the conduction band electrons, and indeed coincides with the discontinuous jump at \( \varepsilon = \Delta \) in Eq. (74). The valence band gives an exactly opposite jump at \( \varepsilon = -\Delta \), so that we have the half-integer Hall conductivity inside the gap, and zero in the conduction and valance bands.

Note that usual Hall conductivity relevant in the transport is given by a different limit, \( \lim_{\omega \to 0} \lim_{q \to 0} \sigma_{xy}(q, \omega) \). This is calculated for gapped monolayer graphene as\(^{62}\)

\[
\lim_{\omega \to 0} \lim_{q \to 0} \sigma_{xy}(q, \omega) = \begin{cases} 
-\frac{e^2}{2h} \frac{\Delta}{|\varepsilon_F|} & (|\varepsilon_F| > \Delta), \\
-\frac{e^2}{2h} & (|\varepsilon_F| < \Delta),
\end{cases} \tag{75}
\]

which differs from Eq. (74) except for the value inside the gap. The Berry curvature is directly related to this transport Hall conductivity.\(^{13,63}\)

From the relationship between the local current and local magnetic moment, Eq. (66), the spatial-dependent static Hall conductivity \( \sigma_{xy}(q) \) can be formulated as a magnetization-density correlation function, i.e.,

\[
M(q) = \frac{1}{e} \sigma_{xy}(q) V(q). \tag{76}
\]

In the low-energy region of gapped monolayer graphene, it becomes a density-density correlation function, because the pseudospin magnetization Eq. (20) is constant for each eigenstate regardless of the detail of the wavefunction. This suggests that \( \sigma_{xy}(q) \) is insensitive to the disorder localization effect since the magnetic moment of each eigenstate remains even when the wavefunction is localized. This is in contrast to the transport Hall conductivity, where the localized eigenstates have zero contribution.

Lastly, we show that Hall conductivity Eq. (70) is directly related to index difference \( \Delta n \) between degenerated Landau levels of two valleys, which are argued in the previous sections. This is defined by the ratio of pseudo-spin Zeeman splitting to Landau level spacing, or

\[
\Delta n = \frac{2m_e B}{\hbar \omega_c}, \quad \hbar \omega_c = \frac{\hbar c}{e} \frac{2\pi}{2} \left( \frac{\partial S(\varepsilon_F)}{\partial \varepsilon_F} \right)^{-1} \tag{77}
\]

and \( S(\varepsilon_F) = \pi p_F^2 \) is the area of the momentum space at the Fermi energy \( \varepsilon_F \). Using Eqs. (65) and (70), we obtain,

\[
\Delta n = \frac{2\hbar c}{e} \frac{\partial M_F}{\partial \varepsilon_F} = -\frac{2\hbar}{e^2} \sigma_{xy}. \tag{78}
\]

Indeed, we have \( \Delta n = 1 \) for gapped monolayer graphene, and \( \Delta n \approx N \) for gapped \( N \)-layer ABC graphene (including bilayer graphene) in high energies.

\section{Conclusion}

We presented systematic analyses of anomalous chiral current and magnetic moment in gapped graphenes and related materials. Starting from the low-energy effective-mass theory, we formulate a description of local current distribution supporting anomalous magnetic moment in general Bloch systems. In gapped monolayer, bilayer and ABC multilayer graphenes, we showed that the chiral current circulation accounts for the valley-dependent magnetic moment and valley-splitting of Landau levels. The bilayer and ABC multilayer graphenes exhibit a large paramagnetism at the band bottom, and full valley polarization is possible in relatively high electron density.

There have been suggested various mechanisms for valley polarization or valley filtering which might be used to control electronic devices.\(^{6,7,64,67}\) The possibility of full valley polarization in graphene bilayer and ABC multilayers invokes a simple mechanism for valley-dependent transport. For example, if we could locally apply opposite magnetic fields to the left and right sides of a gapped bilayer or ABC-multilayer strip, and achieve different valley polarizations in two regions, then the transport between two regions would be killed, as long as the valley flipping is prohibited in the intermediate region, i.e., the impurity potential and the spacial magnetic field change are smooth compared to the atomic scale. On the contrary, electrons can travel almost freely when the same magnetic field is applied to two regions.

While we focus on the family of ABC-stacked multilayer graphenes in the present studies, the anomalous magnetic moment arises in ABA-stacked multilayer graphenes as well when the inversion symmetry is broken.\(^{28}\) In ABA multilayers with an odd number of layers, the lattice structure originally lacks in the inversion symmetry so that the valley splitting intrinsically exists even in absence of the external field.\(^{68}\) The present analysis applies to every subband comprising the total band structure, each of which is akin to gapped monolayer or bilayer graphenes.\(^{69,69}\)

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Appendix A: Chiral edge current

Here we calculate the edge current distribution of gapped monolayer and bilayer graphenes bound by a sharp confining potential. Let us consider a low-energy Hamiltonian gapped monolayer graphene, Eq. (10), bound by a potential barrier, $$V(x) = \begin{cases} \infty & (x < 0) \\ 0 & (x > 0) \end{cases}. \quad (A1)$$

The eigenstates are given by

$$F(r) \propto e^{ik_y y} \sin k_x x. \quad (A2)$$

The current density of Eq. (18) integrated over the occupied states is written in terms of the Bessel function as,

$$J_y(r) = \frac{e^\varepsilon_F}{\hbar} J_2(2k_F x). \quad (A3)$$

It oscillates and decays in the length scale of $2\pi / k_F$ as shown in Fig. 4. The total edge current is

$$I \equiv \int_0^\infty dx \, J_y(r) = \frac{e^\varepsilon_F}{2\hbar}, \quad (A4)$$

which coincides with $cM_F$.

The similar argument is available in bilayer graphene. For simplicity, we consider high energies $\varepsilon \gg \varepsilon_0$ and neglect $p^2$ term in Eq. (21). The Schrödinger equation becomes the fourth-order differential equation due to the $p^4$ term, and the boundary condition becomes $F(0) = F'(0) = 0$. The eigenstate then becomes

$$F(r) \propto e^{ik_y y} [\cos k_x x + \sin k_x x - e^{-k_x x}]. \quad (A5)$$

The total current density, Eq. (25), integrated over the occupied states is numerically calculated and plotted in Fig. 5. The length scale is again characterized by is Fermi wave length, but it decays more rapidly than in monolayer. The total edge current is shown to be

$$I = \int_0^\infty dx \, J_y(r) = \frac{e^\varepsilon_F}{\hbar}, \quad (A6)$$

which is twice as large as monolayer’s.

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