Classification of reflection symmetry protected topological semimetals and nodal superconductors

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While the topological classification of insulators, semimetals, and superconductors in terms of nonspatial symmetries is well understood, less is known about topological states protected by crystalline symmetries, such as mirror reflections and rotations. In this work, we systematically classify topological semimetals and nodal superconductors that are protected, not only by nonspatial (i.e., global) symmetries, but also by a crystal reflection symmetry. We find that the classification crucially depends on (i) the codimension of the Fermi surface (nodal line or point) of the semimetal (superconductor), (ii) whether the mirror symmetry commutes or anticommutes with the nonspatial symmetries and (iii) how the Fermi surfaces (nodal lines or points) transform under the mirror reflection and nonspatial symmetries. The classification is derived by examining all possible symmetry-allowed mass terms that can be added to the Bloch or Bogoliubov-de Gennes Hamiltonian in a given symmetry class and by explicitly deriving topological invariants. We discuss several examples of reflection symmetry protected topological semimetals and nodal superconductors, including topological crystalline semimetals with mirror \(Z_2\) numbers and topological crystalline nodal superconductors with mirror winding numbers.

Introduction video: http://www.youtube.com/watch?v=9jTBJ7pYpw

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I. INTRODUCTION

Inspired by the recent experimental discovery of two- and three-dimensional topological insulators\(^{[1,2]}\) a multitude of novel topological states protected by different symmetries has been predicted over the last few years\(^{[3–7]}\). One of the main hallmarks of these topological materials is the appearance of protected zero-energy surface states, which arise as a consequence of the nontrivial topological characteristics of the bulk wave functions. For fully gapped topological phases protected by general nonspatial symmetries a complete classification, the tenfold way, has been obtained for arbitrary dimensions\(^{[8,9]}\). This scheme classifies fully gapped noninteracting systems in terms of nonspatial symmetries, i.e., symmetries that act locally in position space, namely time-reversal symmetry (TRS), particle-hole symmetry (PHS), and chiral or sublattice symmetry (SLS).

However, over the last few years it has become apparent that besides nonspatial symmetries, also crystalline symmetries, i.e., symmetries that act nonlocally in position space, can lead to nontrivial topological properties of bulk insulating states\(^{[10,27]}\). A prime example of a topological material protected by a crystalline symmetry is the topological crystalline insulator SnTe\(^{[28,29]}\). This band insulator exhibits Dirac-cone surface states that are protected by a mirror reflection symmetry of the crystal. Other than reflection symmetry, inversion\(^{[28,29]}\) and rotation\(^{[21,22]}\) can also give rise to topologically nontrivial quantum states of matter. In fact, it is expected that for any given discrete space group symmetry there is a distinct topological classification of band insulators and fully gapped superconductors, and that each of these space-group-symmetry protected topological states can be characterized in terms of an associated crystalline topological number.

Parallel to these developments, the concept of topological band theory has been extended to semimetals with Fermi points or Fermi lines, and nodal superconductors with point nodes or line nodes\(^{[30–44]}\). Although a global topological number cannot be defined for these gapless systems, it is nevertheless possible to determine their topological characteristics and the stability of their Fermi points or Fermi lines in terms of momentum-dependent topological numbers. Notable examples of gapless topological phases include Weyl semimetals\(^{[35–37]}\), Weyl superconductors\(^{[42,43]}\), and nodal noncentrosymmetric superconductors\(^{[44–46]}\). Similar to fully gapped topological materials, the topological characteristics of gapless topological states manifest themselves at the surface in the form of either linearly dispersing boundary modes (i.e., Dirac or Majorana states) or dispersionless states, forming two-dimensional surface flat-bands or one-dimensional surface arcs. While a complete topological classification of semimetals and nodal superconductors in terms of nonspatial symmetries has been established recently\(^{[26,33–35]}\), the characterization of gapless topological materials protected by crystalline symmetries has remained an open problem.

In this paper, we present a complete classification of topological semimetals and nodal superconductors protected by crystal reflection symmetries and possibly one or two nonspatial (i.e., global) symmetries. We find that the topological classification of these reflection symmetry protected gapless states sensitively depends on (i) the codimension of the Fermi surface, (ii) whether the reflection symmetry commutes or anticommutes with the nonspatial symmetries, and (iii) whether the Fermi points or Fermi lines are left invariant by the mirror symmetry or the nonspatial symmetries. The outcome of this classification scheme is summarized in Tables\(^{[11]}\) and\(^{[13]}\).
which constitute the main results of this paper. Similar to the ten-fold classification in terms of nonspatial symmetries\cite{10}; these tables exhibit two-fold and eight-fold Bott periodicities as a function of spatial dimension. Two complementary methods are used to derive these classification tables. The first approach is based on classifying all possible symmetry-allowed mass terms that can be added to the Bloch or Bogoliubov-de Gennes (BdG) Hamiltonian in a given symmetry class. The second method relies on the explicit derivation of different types of topological invariants that guarantee the stability of the Fermi surfaces (superconducting nodes). In order to illustrate the new topological phases predicted by these classification schemes, we discuss several specific examples of reflection symmetry protected topological semimetals and nodal superconductors, see Sec. V.

The remainder of this article is organized as follows. In Sec. II we briefly review the classification of gapless topological materials in terms of nonspatial symmetries. The classification of insulators and fully gapped superconductors in terms of mirror symmetries is surveyed in Sec. III. This is followed by the derivation of the topological classification of reflection symmetry protected semimetals and nodal superconductors in Sec. IV, which is the principal result of this paper. We present some explicit examples of topological semimetals and nodal superconductors protected by reflection symmetries in Sec. V and conclude with a brief summary in Sec. VI. Some technical details have been relegated to two appendices.

II. GAPLESS TOPOLOGICAL MATERIALS PROTECTED BY NONSPATIAL SYMMETRIES

Since the classification of reflection symmetry protected topological semimetals and nodal superconductors is closely related to the topological classification of gapless states protected by global symmetries, we first briefly review the ten-fold classification of gapless topological materials (cf. Appendix A)\cite{11,12}. This ten-fold scheme classifies gapless fermionic systems in terms of three fundamental global symmetries, i.e., antiunitary time-reversal and particle-hole symmetry, as well as chiral (i.e., sublattice) symmetry\cite{13,14}. In momentum space, TRS and PHS of the Bloch or BdG Hamiltonian $H(k)$ are implemented by antiunitary operators $T$ and $C$, which act on $H(k)$ as

$$T^{-1}H(-k)T = +H(k) \text{ and } C^{-1}H(-k)C = -H(k),$$

respectively. Both $T$ and $C$ can square either to $+1$ or $-1$, depending on the type of the symmetry (see last three columns of Table I). Chiral symmetry, on the other hand, is implemented by

$$S^{-1}H(k)S = -H(k),$$

where $S$ is a unitary operator.

![Diagram](https://via.placeholder.com/150)

**FIG. 1.** (Color online) The ten-fold classification of gapless topological materials depends on the location of the Fermi surfaces in the Brillouin zone, which in turn determines how the Fermi surfaces transform under global antiunitary symmetries, see Table I (a) Each Fermi surface (red point/line) is left invariant under global (i.e., non-spatial) symmetries. The contour, on which the topological invariant is defined, is indicated by blue circles/spheres. Here, $d$ denotes the spatial dimension and $p = d - d_{FS}$ is the codimension of the Fermi surface. (b) Different Fermi surfaces are pairwise related to each other by global symmetries ($k \leftrightarrow -k$).

### A. Ten-fold classification of gapless topological materials

As it turns out, the topological classification of gapless materials depends not only on the symmetry class of the Hamiltonian and the codimension $p$ of the Fermi surface

$$p = d - d_{FS},$$

where $d_{FS}$ is the codimension of the Fermi surface.
where $d$ and $d_{FS}$ denote the dimension of the Brillouin zone (BZ) and the Fermi surface, respectively, but also on how the Fermi surface transforms under the global symmetries. Regarding the symmetry properties of the Fermi surfaces, two different cases have to be distinguished: (i) each individual Fermi surface is left invariant under nonspatial symmetries, and (ii) different Fermi surfaces are pairwise related to each other by nonspatial symmetries, see Fig. 1.

1. Fermi surfaces at high-symmetry points

As shown in Refs. 26, 33, 35, Fermi surfaces located at high-symmetry points in the BZ, can be protected by either Z-type or $Z_2$-type invariants. The complete ten-fold classification of Fermi surfaces that are left invariant under global symmetries is shown in Table I, where the second row indicates the codimension $p$ of the Fermi surface at a high-symmetry point. This result has been obtained using a dimensional reduction procedure and an approach based on K-theory. In Appendix A, we present yet another derivation of this classification scheme by considering all possible symmetry-allowed mass terms that can be added to a representative Dirac-matrix Hamiltonian in a given symmetry class. It is important to note that for a given symmetry class and codimension $p$, a $Z$-type topological invariant guarantees the stability of the Fermi surface independent of the Fermi surface dimension $d_{FS}$. A $Z_2$-type topological number, on the other hand, only protects Fermi surfaces of dimension zero, i.e., Fermi points. We can see from Table I that the ten-fold classification of global-symmetry invariant Fermi points (i.e., $d_{FS} = 0$) is related to the original ten-fold classification of topological insulators and superconductors by a dimensional shift, i.e., $d \rightarrow d - 1$. Due to a bulk-boundary correspondence, gapless materials with nontrivial topology support protected surface states, which, depending on the case, are either Dirac or Majorana states or are dispersionless, forming flat bands or arc surface states.

Let us illustrate some of the gapless topological states listed in Table I by considering specific lattice models.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
Top. insul. & Top. SC & FS at high-sym. point & FS off high-sym. point & T & C & S \\
\hline
AI & 0 & 0 & 0 & Z & 0 & Z & 0 & Z & 0 & Z & 0 & 0 & 0 & 0 & 1 \\
BDI & 0 & 0 & 0 & 2Z & 0 & Z & 0 & Z & 0 & 2Z & 0 & + & 0 & 0 \\
D & 0 & 0 & 0 & 0 & Z & 0 & 2Z & 0 & Z & 0 & + & 0 & 0 \\
DIII & 0 & 0 & 0 & 0 & Z & 0 & 0 & 0 & 2Z & 0 & + & 0 & 0 \\
All & 0 & 0 & 0 & 0 & 0 & Z & 0 & 0 & 0 & 2Z & 0 & + & 0 & 0 \\
CII & 2Z & 0 & 0 & Z & 0 & Z & 0 & 0 & 0 & 0 & + & 0 & 0 \\
C & 2Z & 0 & 0 & Z & 0 & Z & 0 & 0 & 0 & 0 & + & 0 & 0 \\
CI & 0 & 0 & 0 & Z & 0 & Z & 0 & 0 & 0 & 0 & + & 0 & 0 \\
\hline
\end{tabular}
\caption{Ten-fold classification of topological insulators and fully gapped superconductors, as well as of Fermi surfaces and nodal point lines in semimetals and nodal superconductors, respectively. The first row indicates the spatial dimension $d$ of topological insulators and superconductors, whereas the second and third rows specify the codimension $p = d - d_{FS}$ of the Fermi surfaces (nodal lines) at high-symmetry points (Fig. 1(a)) and away from high-symmetry points of the Brillouin zone (Fig. 1(b), respectively. The first column gives the name of the symmetry classes. The labels $T$, $C$, and $S$ in the last three columns indicate the presence (“+”) or absence (“0”) of time-reversal, particle-hole and chiral symmetries, respectively, as well as the sign of the squared topological invariant.}
\end{table}

\[ \nu = \frac{1}{2\pi} \int_{C} q^{+} dq, \]

(5)

where $q = (\sin k_x - i \sin k_y)/\sqrt{\sin^2 k_x + \sin^2 k_y}$, is quantized to $\pm 1$ for closed contours $C$ encircling one of the four nodal points. Specifically, for an anticlockwise-oriented contour we obtain $\nu = +1$ for the nodes at $(0, 0)$ and $(\pi, \pi)$, whereas $\nu = -1$ for the nodes at $(0, \pi)$ and $(\pi, 0)$. The topological nature of these point nodes results in the appearance of protected flat-band edge states for all edge orientations, except the (10) and (01) faces. As demonstrated in Fig. 2(a), these flat-band states connect two projected nodal points with different topological charge (i.e., different winding number $\nu$) in the edge BZ. The BdG Hamiltonian (4) can be converted in a straightforward manner to a three-dimensional topological superconductor with protected line nodes ($d_{FS} = 1$) by including an extra momentum-space coordinate. Similar to the two-dimensional example, Eq. (4), the stability of these nodal lines is guaranteed by the quantized winding number $\nu$. 

\[ H^{\text{DIII}} = \sin k_x \sigma_x + \sin k_y \sigma_y, \]

(4)

which describes a nodal superconductor with point nodes ($d_{FS} = 0$) at the four time-reversal invariant momenta $(0, 0)$, $(0, \pi)$, $(\pi, 0)$, and $(\pi, \pi)$. Hamiltonian (4) preserves time reversal symmetry, with $T = \sigma_y K$, and particle-hole symmetry, with $C = \sigma_z K$. Here, $K$ denotes the complex conjugation operator. Since $T^2 = -1$ and $C^2 = +1$, the Hamiltonian belongs to symmetry class DIII, where 1 is the $2 \times 2$ identity matrix. According to Table I, superconducting nodes with codimension $p = 2$ in class DIII are protected by a $Z_2$-type topological invariant. Indeed, we find that the winding number

\[ \nu = \frac{1}{2\pi} \int_{C} q^{+} dq, \]
Eq. (5).

b. Semimetal with TRS (class AII) As stated above, \( \mathbb{Z}_2 \)-type invariants only protect Fermi surfaces of dimension zero (\( d_{PS}=0 \)) at high-symmetry points of the BZ and cannot give rise to topologically stable Fermi surfaces with \( d_{PS} > 0 \). To exemplify this, we consider the following two-dimensional Bloch Hamiltonian on the square lattice

\[
H_{s}^{\text{AII}} = \sin k_x \sigma_x + \sin k_y \sigma_y + \sin(k_x + k_y)\sigma_z \tag{6}
\]

describes a semimetal with Fermi points at the four time-reversal-invariant momenta of the two-dimensional BZ. Hamiltonian (6) preserves time-reversal symmetry, with \( T = \sigma_y \mathcal{K} \), but breaks particle-hole symmetry, thus belonging to symmetry class AII. The four Fermi points are protected by a binary \( \mathbb{Z}_2 \) invariant, which can be defined in terms of an extension of \( H_{s}^{\text{AII}} \) to three dimensions \( \tilde{H}_s \)

\[
\tilde{H}_s^{\text{AII}}(k, \theta) = \left[ \sin k_x \sigma_x + \sin k_y \sigma_y \right. \\
+ \sin(k_x + k_y)\sigma_z \left. \right] \sin \theta + \sigma_z \cos \theta,
\]

where \( \theta \in [0, \pi] \) is the parameter for the extension in the third direction. The extended Hamiltonian (7) is required to preserve TRS

\[
T^{-1} \tilde{H}_s^{\text{AII}}(-k, \pi - \theta) T = \tilde{H}_s^{\text{AII}}(k, \theta). \tag{8}
\]

Performing a small-momentum expansion around a given Fermi point, we find that the \( \mathbb{Z}_2 \) invariant is expressed as

\[
n_{\mathbb{Z}_2} = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi d\theta \tilde{g} \cdot (\partial_\phi \tilde{g} \times \partial_\theta \tilde{g}) \mod 2, \tag{9}
\]

with \( \tilde{g} = g/|g| \) and

\[
g = (\pm \hat{k}_x, \pm \hat{k}_y, \hat{k}_x + \hat{k}_y + \Delta), \quad \text{for } k = (0, 0), (\pi, \pi), \\
g = (\pm \hat{k}_x, \mp \hat{k}_y, -\hat{k}_x - \hat{k}_y + \Delta), \quad \text{for } k = (0, \pi), (\pi, 0),
\]

(10)

where \( \hat{k}_x = k \cos \phi \sin \theta, \hat{k}_y = k \sin \phi \sin \theta, \Delta = \Delta_0 \cos \theta \), and \((k, \Delta_0)\) are positive constants. The integral (9) is evaluated along the sphere that surrounds the Fermi point and is required to preserve TRS. We observe that the \( \mathbb{Z}_2 \) invariant (9) is nontrivial (i.e., \( n = 1 \)) for all four Fermi points, hence indicating the topological protection of these two-dimensional Dirac points. By the bulk-boundary correspondence, the topological characteristics of these Fermi points lead to linearly dispersing edge modes, which connect two projected Dirac points in the edge BZ (see Fig. 2(b)). Importantly, we find that Hamiltonian (6) cannot be converted to a three-dimensional semimetal with Fermi lines, since it is possible to gap out the Fermi lines located at \((0, 0, k_z), (0, \pi, k_z), (\pi, 0, k_z)\), and \((\pi, \pi, k_z)\) by the symmetry preserving term \( \sin k_z \sigma_z \). That is, in the presence of Fermi lines along the \( k_z \) direction, the topological invariant (9) is ill-defined for \( k_z \neq 0, \pi \), since it breaks TRS.

c. Unstable semimetal with TRS and PHS (class BDI) As an example of an unstable semimetal in two-dimensions we consider the square-lattice Hamiltonian

\[
H_{s}^{\text{BDI}} = \sin k_x \sigma_x \otimes \sigma_y + \sin k_y \sigma_y \otimes \mathbb{1}, \tag{11}
\]

which represents a four-band semimetal with Fermi points at the four time-reversal-invariant momenta. Hamiltonian (11) belongs to class BDI, since it is both time-reversal and particle-hole symmetric with \( T = \mathbb{1} \otimes \mathbb{1} \mathcal{K} \) and \( C = \sigma_z \otimes \mathbb{1} \mathcal{K} \), respectively. In agreement with the classification of Table 1, the four Fermi points of \( H_{s}^{\text{BDI}} \) are unstable, as they can be gapped out by the symmetry-preserving mass \( \sigma_x \otimes \sigma_z \). This is in accordance with the fact that the winding number

\[
\nu = \frac{i}{2\pi} \int_C \text{Tr} (q^1 dq), \tag{12}
\]

where

\[
q = -i \left[ \frac{\sin k_y}{\sqrt{\sin^2 k_x + \sin^2 k_y}} \begin{pmatrix} \sin k_y & \sin k_x \\ -\sin k_x & \sin k_y \end{pmatrix} \right], \tag{13}
\]

vanishes identically for any closed contour \( C \).

2. Fermi surfaces off high-symmetry points

Second, we discuss the topological classification of semimetals and nodal superconductors with Fermi surfaces (or superconducting nodes) that are located away from high-symmetry points of the BZ. In this case, global antunitary symmetries pairwise relate different Fermi surfaces with each other, see Fig. 1(b). Interestingly, only \( \mathbb{Z}_2 \)-type invariants can guarantee the stability of Fermi surfaces off high-symmetry points. \( \mathbb{Z}_2 \)-type numbers, on the other hand, cannot protect these Fermi surfaces, but may nevertheless lead to the appearance of zero-energy surface states at time-reversal invariant momenta of the surface BZ. The complete classification of Fermi surfaces that are pairwise related by global symmetries is shown in Table 1 where the third row indicates the codimension \( p \) of the Fermi surface located away from high-symmetry points (cf. Appendix A). We observe that the classification for the two complex symmetry classes A and AIII is identical to the one of Fermi surfaces that are left invariant by global symmetries, while the classification for the eight real symmetry classes is different. As before, we notice that this classification scheme is related to the original ten-fold classification of topological insulators and superconductors by a dimensional shift, i.e., in this case \( d \rightarrow d + 1 \).

In order to exemplify some of the gapless topological states with Fermi surfaces away from high symmetry points we consider a few specific lattice modes.

a. Two-dimensional semimetal with SLS (class AIII) To demonstrate that \( \mathbb{Z}_2 \)-type invariants protect Fermi surfaces at non-high-symmetry points of the BZ, we study the following sublattice symmetric Hamiltonian on the square lattice

\[
H_{n}^{\text{AIII}} = X \sigma_x + Y \sigma_y, \tag{14}
\]

where \( X = 1 + \cos k_x + A \sin k_x + B \cos k_x \) and \( Y = \sin k_y \). Sublattice symmetry acts on \( H_{n}^{\text{AIII}} \) as \( SH_{n}^{\text{AIII}} + H_{n}^{\text{AIII}} S = 0 \), with the unitary matrix \( S = \sigma_y \). Hamiltonian (14) exhibits two Fermi points located at \((\delta, \pi)\) and \((\delta - \pi, \pi)\), where \( \delta = \arctan(-B/A) \) and we require that \( \sqrt{A^2 + B^2} < 2 \).
Note that, in agreement with the fermion-doubling theorem by Nielsen and Ninomiya,\(^\text{19}\) the number of Fermi points is even. Since there exists no symmetry-allowed mass term that can be added to Hamiltonian\(^\text{14}\), the two Fermi points are stable and, according to Table\(^\text{1}\), protected by the \(\mathbb{Z}\) topological number Eq.\(^\text{5}\), with \(q = (X - Y)/\sqrt{X^2 + Y^2}\) and \(\mathcal{C}\) a closed contour. Choosing \(\mathcal{C}\) to be parallel to the \(k_y\) axis, we find that \(\nu = +1\) for \(\delta - \pi < k_x < \delta\), and zero otherwise. Due to an index theorem\(^\text{23}\) a nonzero value of the winding number\(^\text{5}\) implies the existence of flat-band edge states at zero energy. At the \((01)\) edge, the zero-energy flat-band states appear within the interval \(k_x \in [\delta - \pi, \delta]\) of the edge BZ, see Fig.\(^\text{2}\)\(\text{(c)}\).

b. Three-dimensional semimetal with TRS and PHS (class BDI) \(\mathbb{Z}\)-type numbers can protect Fermi surfaces of arbitrary dimension \(d_{\text{BZ}}\). To demonstrate this for the case of Fermi surfaces located away from high-symmetry points, we consider the following three-dimensional tight-binding model on the cubic lattice

\[
H_{\text{BDI}}^n = (1 + \cos k_x + \cos k_y)\sigma_x + \sin k_y\sigma_y,
\]

which realizes a topological semimetal with two Fermi lines at \((\pm \pi/2, \pi, k_z)\). Hamiltonian\(^\text{15}\) belongs to symmetry class BDI, since it satisfies both TRS and PHS with \(T = \sigma_y \otimes \mathbb{1}_K\) and \(C = \sigma_x \otimes \mathbb{1}_K\), respectively. We observe that the two Fermi lines, which are located away from the time-reversal invariant momenta of the BZ, transform into each other under particle-hole and time-reversal symmetry [cf. Fig.\(^\text{1}\)\(\text{(b)}\)]. As indicated in Table\(^\text{1}\) the Fermi lines are protected by a \(\mathbb{Z}\)-type topological invariant, which for the tight-binding model\(^\text{15}\) takes the form of Eq.\(^\text{5}\), with \(q = (1 + \cos k_y + \cos k_z) - i \sin k_y\). The integration contour in Eq.\(^\text{5}\) can be chosen to be any circle enclosing the Fermi line. (The integration contour does not need to be time-reversal or particle-hole symmetric.) Similar to the class AIII model\(^\text{14}\), a nonzero value of this winding number leads to zero-energy flat-band surface states that connect the two projected Fermi lines in the surface BZ.

c. Unstable nodal superconductor with TRS (class DIII) As indicated in Table\(^\text{4}\) \(\mathbb{Z}_2\)-type topological numbers do not guarantee the topological stability of Fermi surfaces (superconducting nodes) at non-high-symmetry points of the BZ. Nevertheless, \(\mathbb{Z}_2\)-type invariants, which are defined on time-reversal symmetric contours, can give rise to protected gapless surface states. To demonstrate this, we consider an example of an unstable nodal superconductor given by the four-band BdG Hamiltonian

\[
H_{\text{DIII}}^n = (1 + \cos k_x + \cos k_y)\sigma_x \otimes \sigma_y + \sin k_x\sigma_y \otimes \mathbb{1}.
\]

This superconductor belongs to symmetry class DIII, as it preserves both time-reversal and particle-hole symmetry, with \(T = \sigma_y \otimes \mathbb{1}_K\) and \(C = \sigma_x \otimes \mathbb{1}_K\), respectively. Hamiltonian\(^\text{16}\) exhibits two point nodes at \((\pi, \pm \pi/2)\). These two point nodes, which are positioned away from the high-symmetry points of the BZ, are unstable, since the symmetry-preserving extra kinetic term \(\sin k_z\sigma_z \otimes \sigma_x\) opens up a gap in the entire bulk BZ (cf. Table\(^\text{1}\)). This is corroborated by the fact that the winding number\(^\text{16}\) for model Hamiltonian\(^\text{16}\) is identically zero for any closed contour \(\mathcal{C}\), which follows from a similar argument as the one given in the example of Eq.\(^\text{11}\). In contrast, the one-dimensional \(\mathbb{Z}_2\) number\(^\text{16}\)

\[
n_{\mathbb{Z}_2} = \prod_{K \in \mathcal{C}} \text{Pr}[\omega(K)]/\sqrt{\det[\omega(K)]}
\]

for Hamiltonian\(^\text{16}\) can take on nontrivial values, which however does not lead to a protection of the point nodes of the superconductor (cf. Table\(^\text{1}\) and Appendix\(^\text{A}\)). In Eq.\(^\text{17}\) the product is over the two time-reversal invariant momenta \(K\) (high-symmetry points) of the contour \(\mathcal{C}\) and \(\omega(K)\) denotes the \(2 \times 2\) sewing matrix

\[
\omega_{ab}(k) = \langle u_a^-(-k)|T u_b^-(k)\rangle,
\]

with \(|u_a^-(k)\rangle\) the negative-energy BdG wave functions of Hamiltonian\(^\text{16}\). Even though \(\mathbb{Z}_2\) number\(^\text{17}\) does not stabilize point nodes in the bulk, it nevertheless leads to protected...
zero-energy surface states at time-reversal invariant momenta of the surface BZ. To exemplify this, we consider two time-reversal invariant contours \( C \) oriented along the \( k_x \) axis with \( k_y \) held fixed at \( k_y = 0 \) or \( k_y = \pi \). With these contours, the \( \mathbb{Z}_2 \) number takes on the values \( n = +1 \) and \( n = -1 \) at \( k_y = 0 \) and \( k_y = \pi \), respectively, indicating the existence of a zero-energy edge state at \( k_y = \pi \) of the \( (10) \) edge BZ of the superconductor. We observe that the unstable nodal superconductor \([16]\) can be connected to a fully gapped topological superconductor without removing the zero-energy edge-states. That is, the edge-states of Hamiltonian \([16]\) are inherited from the fully gapped topological phase \([22]\).

III. FULLY GAPPED TOPOLOGICAL MATERIALS PROTECTED BY REFLECTION SYMMETRIES

To set the stage for deriving the classification of reflection symmetry protected topological semimetals and nodal superconductors, we briefly survey in this section the classification of fully gapped topological materials protected by crystal reflection symmetries. As we will see in Sec. IV, the classification of reflection symmetry protected topological semimetals and nodal superconductors can be related to the classification of reflection symmetry protected insulators/superconductors (fully gapped superconductors) by dimensional reduction. Both of these classification schemes crucially depend on whether the crystal reflection symmetry commutes or anticommutes with the global nonspatial symmetries.

A. Reflection symmetries

Crystal reflection is a spatial symmetry, which acts nonlocally in position space. For concreteness, let us consider a \( d \)-dimensional Bloch or BdG Hamiltonian \( H(\mathbf{k}) \) in momentum space which is invariant under reflection in the first direction. The invariance of \( H(\mathbf{k}) \) under this mirror symmetry implies

\[
R^{-1}H(-k_1, \mathbf{k})R = H(k_1, \mathbf{k}),
\]

\( R^{-1} \) being \( R \) the reflection operator. Here, \( R \) is a unitary matrix. Due to a phase ambiguity in the definition of the reflection operator \( R \), we can assume without loss of generality that \( R \) is Hermitian (at least for electronic insulators). With this assumption, the commutation or anticommutation relations between \( R \) and the global nonspatial symmetry operators \( T, C \), and \( S \),

\[
SRS^{-1} = \eta_S R, \quad TRT^{-1} = \eta_T R, \quad CRC^{-1} = \eta_C R,
\]

(20)
can be determined in an unambiguous way, which in turn simplifies the classification of reflection symmetry protected insulators and superconductors. The three indices \( \eta_S, \eta_T, \) and \( \eta_C \) in Eq. (20) take values \( +1 \) or \( -1 \) and specify whether \( R \) commutes \( (+1) \) or anticommutes \( (-1) \) with the corresponding global symmetry operator. These different possibilities are label by \( R_{\eta_T}, R_{\eta_S}, \) and \( R_{\eta_C} \) for the five symmetry classes AII, AIII, C, and D, respectively, which contain only one global symmetry operation. For the remaining four symmetry classes BDI, CI, CII, and DIII, which contain two nonspatial symmetries, the four different possible \((anti)\)commutation relations are denoted by \( R_{\eta_T\eta_C} \). Hence, there is a total of 27 different symmetry classes for reflection symmetry protected topological insulators and fully gapped superconductors, see Table I. We observe that since the reflection operator \( R \) is both Hermitian and unitary, \( R^2 = 1 \) and all eigenvalues of \( R \) are either \( +1 \) or \( -1 \). Here, \( \mathbb{I} \) denotes the identity matrix with unspecified matrix dimension.

B. Classification of reflection symmetry protected topological insulators and fully gapped superconductors

The classification of reflection symmetry protected topological insulators and fully gapped superconductors is summarized in Table I where the first row indicates the dimension \( d \) of the fully gapped system. In even (odd) spatial dimension \( d \), ten (seventeen) out of the 27 symmetry classes allow for the existence of nontrivial topological insulators/superconductors protected by reflection symmetry. The different topological sectors within a given class of reflection symmetry protected topological insulators/superconductors can be labeled by an integer \( \mathbb{Z} \) number, a binary \( \mathbb{Z}_2 \) quantity, a mirror Chern or winding number \( MZ \), a mirror binary \( \mathbb{Z}_2 \) quantity \( M\mathbb{Z}_2 \), or a binary \( \mathbb{Z}_2 \) quantity with translation symmetry \( T\mathbb{Z}_2 \). Interestingly, reflection symmetric topological states belonging to symmetry classes with chiral symmetry, can be protected in some cases by both an integer \( \mathbb{Z} \) number (binary \( \mathbb{Z}_2 \) quantity) and a mirror Chern or winding number \( MZ \) (mirror \( \mathbb{Z}_2 \) quantity \( M\mathbb{Z}_2 \)), as indicated by the label \( M\mathbb{Z} + \mathbb{Z} (M\mathbb{Z}_2 + \mathbb{Z}_2) \) in Table I. The nontrivial bulk topology characterized by these invariants manifests itself at the boundary in terms of protected Dirac or Majorana surface states, which, depending on the type of the invariant, appear either at any surface (for \( \mathbb{Z} \) and \( \mathbb{Z}_2 \)) or only at surfaces that are left invariant under the reflection symmetry (for \( M\mathbb{Z} \) and \( M\mathbb{Z}_2 \)). We will see in Sec. IV that by use of a dimensional reduction procedure these surface states of a \( d \)-dimensional fully gapped system can be interpreted as a reflection symmetry protected topological semimetal (or nodal superconductor) in \( d-1 \) dimensions.

Before discussing in detail the different invariants that characterize reflection symmetry protected topological materials, we remark that the recently discovered topological crystalline insulator SnTe is included in Table II. Specifically, SnTe belongs to symmetry class AII with \( T^2 = -1 \) in \( d = 3 \) dimensions and exhibits a reflection symmetry \( R \) that anticommutes with the time-reversal symmetry operator \( T \). As indicated by Table II, this crystalline topological insulator is described by a mirror Chern number \( M\mathbb{Z} \) and hence supports Dirac-cone states at reflection-symmetric surfaces. These Dirac surface states have recently been observed in angle-resolved photoemission experiments. \([22,23,24]\)
1. MZ and MZ\(_2\) invariants

The mirror Chern or winding numbers and mirror Z\(_2\) invariants, denoted by MZ and MZ\(_2\) in Table I, respectively, are defined on the hyperplanes in the BZ that are symmetric under reflection \(R\), i.e., the two hyperplanes \(k_1 = 0\) and \(k_1 = \pi\). Since \(R\) is Hermitian and anticommutes with the Hamiltonian \(H(k)\) restricted to the hyperplanes \(k_1 = 0\) and \(k_1 = \pi\), \(H(k)\)\(_{k_1=0,\pi}\) can be block diagonalized with respect to the two eigenspaces \(R = \pm 1\) of the reflection operator. We observe that each of the two blocks of \(H(k)\)\(_{k_1=0,\pi}\) is left invariant only under those global symmetries that commute with the reflection operator \(R\). Hence, depending on the non-spatial symmetries of the \(R = \pm 1\) blocks of \(H(k)\)\(_{k_1=0,\pi}\), it is possible to define a mirror Chern or winding invariant\(^\[22\]\)

\[
\nu_{MZ} = \text{sgn} \left[ \nu_{k_1=0} - \nu_{k_1=\pi} \right] \left( |\nu_{k_1=0}^d| - |\nu_{k_1=\pi}^d| \right),
\]

where \(\nu_{k_1=0(\pi)}\) denotes the mirror or winding number of the \(R = +1\) block of \(H(k)\)\(_{k_1=0(\pi)}\)\(^\[23\]\) Similarly, the mirror Z\(_2\) quantity MZ\(_2\) is defined by

\[
n_{MZ2} = 1 - \left| \nu_{k_1=0}^d - \nu_{k_1=\pi}^d \right|,
\]

with \(\nu_{k_1=0(\pi)}^d \in \{-1, +1\}\) the Z\(_2\) invariant of the \(R = +1\) block of \(H(k)\)\(_{k_1=0(\pi)}\). A nontrivial value of these mirror indices indicates the appearance of Dirac or Majorana states at reflection symmetric surfaces, i.e., at surfaces that are perpendicular to the reflection hyperplane \(x_1 = 0\). At surfaces that break reflection symmetry, however, the boundary modes are in general gapped. Some illustrative examples of topological crystalline insulators with mirror Chern or winding numbers are given in Ref.]\(^{12}\)

2. Z and Z\(_2\) invariants

For symmetry classes with at least one nonspatial symmetry that anticommutes with the reflection operator \(R\), it is possible in certain cases to define a global Z or Z\(_2\) number even in the presence of reflection. These Z and Z\(_2\) indices are identical to the ones of the original ten-fold classification in the absence of mirror symmetry (cf. Table I) and lead to the appearance of linearly dispersing Dirac or Majorana states at any surface, independent of the surface orientation.

3. MZ \(\oplus\) Z and MZ\(_2\) \(\oplus\) Z\(_2\) invariants

Topological properties of reflection symmetric insulators (superconductors) with chiral symmetry are described in some cases by both a global Z or Z\(_2\) invariant and a mirror index MZ or MZ\(_2\). The global invariant and the mirror invariant are independent of each other. At surfaces which are perpendicular to the mirror plane the number of protected gapless states is given by \(\max\left\{ |n_Z|, |n_{MZ}| \right\}\) where \(n_Z\) denotes the global Z invariant, whereas \(n_{MZ}\) is the mirror Z invariant.

In Sec.]\(^{VA1}\) we provide an examples of a gapless topological phases with nontrivial MZ and Z\(_2\) invariants. Examples of gapless topological phases with nontrivial MZ\(_2\) and Z\(_2\) invariants are given in Secs.]\(^{VA3}\) and \([^{VB4}\)]\.

4. TZ\(_2\) invariant

In symmetry classes where the reflection operator \(R\) anticommutes with the global antiunitary symmetries TRS and PHS \((R_+\) and \(R_-\) in Table I) the second descendant Z\(_2\) invariant\(^\[20\]\) are only well defined in the presence of translation symmetry. That is, the edge or surface states of these phases can be gapped out by density-wave type perturbations, which preserve reflection and global symmetries but break translation symmetry. Hence, these topological states are protected by a combination of reflection, translation, and global antiunitary symmetries. Therefore we denote their topological indices by “TZ\(_2\)” in Table II.

To exemplify the properties of reflection symmetric insulators (superconductors) with a TZ\(_2\) invariant we consider a two-dimensional superconductor with \(R_-\) reflection symmetry in class CII given by the \(8 \times 8\) BdG Hamiltonian

\[
H_{\text{bulk}}^{\text{CII}} = M\gamma_0 + \sin k_x\gamma_1 + \sin k_y\gamma_2,
\]

where \(M = 1 + \cos k_x + \cos k_y\), \(\gamma_0 = \sigma_z \otimes 1\otimes 1\), \(\gamma_1 = \sigma_x \otimes \sigma_y \otimes 1\), and \(\gamma_2 = \sigma_y \otimes \sigma_x \otimes 1\). Superconductor \(^{23}\) preserves TRS and PHS with \(T_{\text{bulk}} = 1 \otimes \gamma_y \otimes \mathbb{I}K\) and \(\mathcal{C}_{\text{bulk}} = \sigma_x \otimes 1 \otimes \sigma_y K\), respectively. Reflection symmetry is implemented as \(R_{\text{bulk}} = H_{\text{bulk}}^{\text{CII}}(k_x, k_y)\), with \(R_{\text{bulk}} = 1 \otimes \sigma_y \otimes 1\). This topological crystalline superconductor is characterized by a TZ\(_2\) invariant (cf. Table II), which indicates that the helical Majorana states at the \((01)\) edge are only stable in the presence of translation symmetry. We find that these Majorana-cone edge states appear at \(k_z = \pm \delta\) of the edge BZ and are described by the following edge Hamiltonian\(^\[23\]\)

\[
h_{\text{edge}}^{\text{CII}} = k_x\sigma_x + k_x + \frac{\delta}{2} \sigma_z \otimes \sigma_y.
\]

The edge Hamiltonian satisfies TRS, PHS, and reflection symmetry with \(T_{\text{edge}} = \sigma_y \otimes 1 K\), \(C_{\text{edge}} = \sigma_y \otimes \sigma_z K\), and \(R_{\text{edge}} = \sigma_z \otimes 1\), respectively. In the absence of reflection symmetry the gap opening mass term \(m\sigma_z \otimes \sigma_x\), which preserves both TRS and PHS, can be added to Eq. \([24]\). Therefore, Hamiltonian \([23]\) is topologically trivial according to the ten-fold classification of Table I. However, with reflection and translation symmetry \(h_{\text{edge}}^{\text{CII}}\) cannot be gapped since \(m\sigma_z \otimes \sigma_y\) breaks reflection symmetry \(R_{\text{edge}}\). Considering two copies of the edge Hamiltonian, i.e., \(H_{\text{edge}} = h_{\text{edge}}^{\text{CII}} \otimes 1\), we find that the symmetry preserving mass term \(m\sigma_z \otimes \sigma_x \otimes \sigma_y\) opens up a gap in the spectrum of the doubled Hamiltonian \(H_{\text{edge}}^{\text{CII}}\). Hence, BdG Hamiltonian \([23]\) exhibits a nontrivial \(Z_2\) type topological characteristic (cf. Appendix A\(_2\)). To demonstrate that the two Majorana edge modes, Eq. \([24]\), are unstable against translation symmetry breaking we consider the density wave type mass term

\[
\mathcal{M} = m \sum_{-\eta \leq k_x < \eta} \left( |c_{k_x + \eta + \delta}^{+}\mathcal{M} c_{k_x - \eta + \delta}|\right)
\]
TABLE II. Classification of reflection symmetry protected topological insulators and fully gapped superconductors as well as of Fermi surfaces and nodal points/lines in reflection symmetry protected semimetals and nodal superconductors, respectively. The first row specifies the spatial dimension $d$ of reflection symmetry protected topological insulators and fully gapped superconductors, while the second and third rows indicate the codimension $p = d - d_{FS}$ of the reflection symmetric Fermi surfaces (nodal lines) at high-symmetry points [Fig. 3(a)] and away from high-symmetry points of the Brillouin zone [Fig. 3(b)], respectively.

| Reflection | top. insul. and top. SC | $d$=1 | $d$=2 | $d$=3 | $d$=4 | $d$=5 | $d$=6 | $d$=7 | $d$=8 |
|------------|------------------------|------|------|------|------|------|------|------|------|
| $R$        | $M_z$                  | 0    | $M_z$| 0    | $M_z$| 0    | $M_z$| 0    | $M_z$|
| $R_+ $     | $M_z$                  | 0    | $M_z$| 0    | $M_z$| 0    | $M_z$| 0    | $M_z$|
| $R_- $     | $M_z$                  | 0    | $M_z$| 0    | $M_z$| 0    | $M_z$| 0    | $M_z$|
| $R_+ R_{++}$ | $M_z$                  | 0    | $M_z$| 0    | $M_z$| 0    | $M_z$| 0    | $M_z$|
| $R_- R_{--}$ | $M_z$                  | 0    | $M_z$| 0    | $M_z$| 0    | $M_z$| 0    | $M_z$|

$^{a}$ $Z_2$ and $MZ_2$ invariants only protect Fermi surfaces of dimension zero ($d_{FS} = 0$) at high-symmetry points of the Brillouin zone.

$^{b}$ Fermi surfaces located within the mirror plane but away from high symmetry points cannot be protected by a $Z_2$ or $MZ_2$ topological invariant. Nevertheless, the system can exhibit gapless surface states that are protected by a $Z_2$ or $MZ_2$ topological invariant.

$^{c}$ For gapless topological materials the presence of translation symmetry is always assumed. Hence, there is no distinction between $TZ_2$ and $Z_2$ for gapless topological materials.

\[ +ic_{-k_x+\eta-\delta} \mathbb{M} c_{-k_x-\eta+\delta} + \text{H.c.}, \]  

(25)

which is invariant under TRS, PHS, and reflection $\tilde{R} = \sum_k c_{k_x+\sigma} \otimes 1 c_k$. In Eq. (25), $\mathbb{M} = m \sigma_x \otimes \sigma_y$ and $\eta$ is a constant with $0 < \eta < \delta$. For $m > \eta$ the translation symmetry breaking mass term (25) fully gaps out all edge modes.

In closing, we remark that for the classification of gapless topological materials presented in Sec. IV the presence of translation symmetry is always assumed. In particular, density-wave type mass terms are disregarded, since these can gap out the bulk by coupling Fermi surfaces (nodal lines) located at different parts of the BZ. Thus, the distinction between $Z_2$ and $TZ_2$ invariants is irrelevant for the topological classification of reflection symmetric semimetals and nodal superconductors.

IV. CLASSIFICATION OF REFLECTION SYMMETRY PROTECTED GAPLESS TOPOLOGICAL MATERIALS

Having discussed the classification of fully gapped reflection symmetric topological materials, we are now ready to classify reflection symmetric topological semimetals and nodal superconductors. As for fully gapped systems, reflection symmetries lead to an enrichment of the ten-fold classification of topological semimetals (nodal superconductors) with new topological phases. The classification depends on the codimension $p = d - d_{FS}$ of the Fermi surface (nodal
line/point) and on whether the reflection operator commutes or anticommutes with the nonspatial symmetries. Moreover, we need to distinguish how the Fermi surface (nodal line/point) transforms under the mirror reflection and nonspatial symmetries. There are three different cases to be considered: (i) The Fermi surface is invariant under both reflection and global symmetries [Fig. 3(a) and Table III], (ii) Fermi surfaces are invariant under reflection, but transform pairwise into each other by the global antunitary symmetries [Fig. 3(b) and Table III], and (iii) different Fermi surfaces are pairwise related to each other by reflection and nonspatial symmetries [Fig. 3(c) and Table III].

Our derivation of these classification schemes, which are presented in Tables III and IV, relies primarily on the so-called minimal Dirac-matrix Hamiltonian method. This method is based on considering reflection symmetric Dirac-matrix Hamiltonians with the smallest possible matrix dimension for a given symmetry class of the ten-fold way. The topological properties of the Fermi surfaces (nodal lines) described by these Dirac-matrix Hamiltonians is then determined by the existence or non-existence of symmetry-preserving gap-opening terms (SPGTs), i.e., symmetry-allowed terms that fully gap out the bulk Fermi surfaces. The existence of such an SPGT indicates that the Fermi surface is topologically trivial and hence unstable. This is denoted by the label “0” in Tables III and IV. On the other hand, if no SPGT exists, then the Fermi surface is topologically stable and protected by a topological invariant (for more details see Appendix A and Ref. [12]).

The minimal Dirac-matrix Hamiltonian approach is complemented by a discussion of different types of topological invariants (i.e., , , , , and -type invariants) that guarantee the stability of these Fermi surfaces. For some concrete examples we derive explicit expressions for these topological numbers in Sec. V. The classification of reflection symmetric gapless materials in terms of topological invariants is consistent with the Dirac-matrix Hamiltonian method.

### A. Fermi surfaces at high-symmetry points within mirror planes

Fermi points that are invariant under both reflection and global symmetries [red points in Fig. 3(a)], can be protected by , , , , or -type topological numbers. The topological classification of these Fermi points ( ) in dimensions is related to the classification of reflection symmetric fully gapped systems in dimensions. To demonstrate this relation, let us consider a-dimensional Dirac Hamiltonian of a reflection symmetric insulator (or fully gapped superconductor) in a given symmetry class

\[ H^{\text{TI}}_{\text{Dirac}} = \sum_{i=1}^{d} k_i \gamma_i + m \gamma_0. \]  

(26)

Reflection symmetry is implemented by \( R^{-1} H^{\text{TI}}_{\text{Dirac}}(-k_i, k) R = H^{\text{TI}}_{\text{Dirac}}(k_i, k) \). Here and in the following, denote Dirac matrices which anticommute (commute) with the time-reversal operator \( T \) (particle-hole operator \( C \)) of the given symmetry class, whereas \( \gamma_i \) are Dirac matrices that commute (anticommute) with \( T \) (C), see Appendix A. By considering the reflection symmetric surface states of \( H^{\text{TI}}_{\text{Dirac}} \) we can derive from Eq. (26) a Dirac Hamiltonian describing a reflection symmetric Fermi point in the same symmetry class as Eq. (26) but in one dimension lower

\[ H^R = \sum_{i=1}^{d-1} k_i P \gamma_i P, \]  

(27)

with the projection operator \( P = \left( \mathbb{1} - i \gamma_0 \gamma_d \right) / 2 \). The topological property of \( H^{\text{TI}}_{\text{Dirac}} \) is signaled by the existence or nonexistence of an extra symmetry-allowed mass term \( \Gamma \), i.e., a symmetry preserving Dirac matrix that anticommutes with all Dirac matrices \( \gamma_i \) and \( \gamma_0 \) of Eq. (26).

Whenever such an extra mass term \( \Gamma \) exists, it is possible to construct an SPGT for \( H^{\text{TI}}_{\text{Dirac}} \), Eq. (27), by \( \Gamma P = \Gamma P \gamma_0 \) which is nonzero since \( \Gamma \) anticommutes with both \( \gamma_0 \) and \( \gamma_d \). Vice versa, one can show that whenever there exists an SPGT for \( H^{\text{TI}}_{\text{Dirac}} \), i.e., a symmetry-allowed Dirac matrix \( \gamma_i \) that anticommutes with \( H^{\text{TI}}_{\text{Dirac}} \), there is a corresponding extra symmetry-allowed mass term for \( H^{\text{TI}}_{\text{Dirac}} \). Hence, the classification of Fermi points (i.e., ) at high-symmetry positions within the mirror plane follows from the classification of reflection symmetric fully gapped systems by the dimensional shift \( d \rightarrow d - 1 \) (Table III). The dimensional shifting is in agreement with the classification of Fermi points located at symmetry points described by Eq. 9.5 of Ref. [26].

For Fermi surfaces with , on the other hand, the classification differs from the one of Fermi points ( ). That is, only -type invariants (i.e., , , , and -type topological numbers) can protect Fermi surfaces with . This is because for a gapless -dimensional system with, e.g., Fermi lines along the direction (described by Eq. (27)), we can add to the Hamiltonian the additional symmetry-preserving kinetic term \( k_4 \gamma_4 \), which gaps out the Fermi lines (except at high-symmetry points). For gapless systems with a -type invariant such an extra kinetic term always exists, whereas for Fermi surfaces with a -type topological number this extra kinetic term is absent (cf. Appendix A for more details and Sec. V A for some examples).

The classification of Fermi surfaces that are located within the mirror plane at high-symmetry positions is summarized in Table III where the second row indicates the codimension \( p \) of the Fermi surface. The prefix “M” in Table III indicates that the corresponding topological invariant is defined on a \( (p-2) \)-dimensional contour within the reflection plane [blue points/lines in Fig. 3(a)]. The topological invariants labeled by and , on the other hand, are defined on a \( (p-1) \)-dimensional contours that intersect with the mirror plane (same invariants as in the absence of reflection symmetry, cf. Table I).
B. Fermi surfaces within mirror planes but off high-symmetry points

Second, we classify Fermi surfaces that are located within the mirror plane but away from high-symmetry points [Fig. 3(b)]. These Fermi surfaces are invariant under reflection, but transform pairwise into each other by the nonspatial antiunitary symmetries. We discuss this classification by considering the following reflection symmetric Dirac-matrix Hamiltonian

\[ H_n^R = \sum_{i=1}^{p-1} \sin k_i \gamma_i + (1 - p + \sum_{i=1}^{p} \cos k_i) \gamma_0, \tag{28} \]

which describes a semimetal (nodal superconductor) with a \((d-p)\)-dimensional Fermi surface (superconducting node) located at

\[ k = (0, \ldots, 0, \pm \pi/2, k_{p+1}, \ldots, k_d). \tag{29} \]

Reflection symmetry acts on Hamiltonian (28) as \(R^{-1} H_n^R(-k_1, \ldots, k_d) R = H_n^R(k_1, \ldots, k_d)\). We observe that Fermi surface (29) lies within the mirror plane \(k_1 = 0\), but away from the high-symmetry points \((0, 0, 0, \ldots, 0)\), \((\pi, 0, 0, \ldots, 0)\), \((0, \pi, 0, \ldots, 0)\), etc. of the BZ. Comparing Eq. (28) to Eq. (26) we find that \( H_n^R \), with \(k_p \neq \pm \pi/2\) and \(k_{p-1}, \ldots, k_d\) held fixed, can be interpreted as a reflection symmetric insulator (fully gapped superconductor) in \(d = p - 1\) dimensions. Hence, the existence (or non-existence) of an extra symmetry-allowed mass term \(\Gamma\) for \(H_n^{\text{Tr}}\), Eq. (26), implies the existence (or non-existence) of a momentum-independent SPGT for \(H_n^R\), Eq. (28). However, Fermi surface (29) can also be gapped out by an additional symmetry-allowed kinetic term, i.e., by the momentum-dependent SPGT \(\sin k_p \gamma_p\). It turns out that for symmetry classes with a \(\mathbb{Z}_2\)- or \(M\mathbb{Z}_2\)-type invariant this extra kinetic term is always allowed by symmetry, whereas for classes with a \(\mathbb{Z}_r\) or \(M\mathbb{Z}_r\)-type number this term is symmetry forbidden (cf. Appendix A). With this, it follows that the classification of \(p\)-dimensional Fermi surfaces (superconducting nodes) within the reflection plane but off high-symmetry points is given by the classification of reflection symmetric topological insulators (fully gapped superconductors) in \(d = p - 1\) dimensions which are protected by a \(\mathbb{Z}_r\) or \(M\mathbb{Z}_r\)-type invariant (cf. Table III). We note that while \(\mathbb{Z}_2\)- or \(M\mathbb{Z}_2\)-type invariants cannot protect Fermi surfaces that are located within the mirror plane but away from high-symmetry points, they nevertheless might give rise to protected gapless surface states (see Sec. V B 4 for an example).

C. Fermi surfaces outside mirror planes

Finally, we discuss the classification of Fermi surfaces (superconducting nodes) that are located outside the mirror plane. These Fermi surfaces are pairwise related to each other by both reflection and nonspatial antiunitary symmetries, see Fig. 3(c). Reflection symmetry alone cannot protect Fermi surfaces that lie outside the reflection plane, since the reflection symmetry does not restrict the form of the mass term at the position of the Fermi surface. However, a combination of reflection and global antiunitary symmetries can give rise to topologically stable Fermi points (or point nodes in the superconducting gap) [26]. In order to study this possibility we introduce the combined symmetry operators

\[ \tilde{T} = RT \quad \text{and} \quad \tilde{C} = RC, \tag{30a} \]

which are antiunitary. These combined symmetry operators act on the \(d\)-dimensional Bloch or BdG Hamiltonian as fol-
lows

\[ \tilde{T}^{-1} H(k_1, -\tilde{k}) \tilde{T} = + H(k_1, \tilde{k}) \]  

(30b)

and

\[ \tilde{C}^{-1} H(k_1, -\tilde{k}) \tilde{C} = - H(k_1, \tilde{k}) \].  

(30c)

Hence, \( \tilde{T} \) (\( \tilde{C} \)) can be viewed as an effective time-reversal (particle-hole) symmetry acting within \((d-1)\)-dimensional planes that are perpendicular to the \( k_1 \) direction [blue lines/planes in Fig. 3(c)]. For each of these planes it is possible to define a topological number and study its evolution as a function of the parameter \( k_1 \). These \( k_1 \)-dependent topological numbers can only change across gap closing points. Hence, the stability of Fermi points or superconducting point nodes (i.e., gap closing points) can be discussed in terms of these topological invariants which are defined in the presence of the combined symmetry \( \tilde{T} \) and/or \( \tilde{C} \), Eq. (30). Moreover, at surfaces that are parallel to the \( k_1 \) direction, these \( k_1 \)-dependent topological numbers give rise to arc surface states that connect two projected Fermi points in the surface BZ.

In this section, we derive the classification of Fermi surfaces outside the mirror plane, by examining which types of topological invariants can be defined within the \((d-1)\)-dimensional planes perpendicular to the \( k_1 \) axis. For this, we have to distinguish between two different kinds of invariants: (i) mirror invariants that are defined within the mirror plane for a given eigenspace of the reflection operator \( R \) and (ii) invariants which are defined for any given plane perpendicular to the \( k_1 \) axis [green and blue lines/planes in Fig. 3(c), respectively]. Since these two kinds of invariants are constrained differently by symmetry, they can in principle give rise to different classifications. However, it turns out that the Fermi points are only protected by the “weaker” of these two invariants. That is, if one invariant is of \( \mathbb{Z} \)-type whereas the other one is of \( \mathbb{Z}_2 \)-type, then the Fermi points only exhibit a \( \mathbb{Z}_2 \)-type topological characteristic. This follows from the fact that the topological invariant cannot change as a function of \( k_1 \) as long as the bulk gap does not close. Hence, the invariant defined in the mirror plane must equal the invariant defined in a plane that is perpendicular to \( k_1 \) and infinitesimally close to the mirror plane. This condition can only be satisfied if the “stronger” of the two invariants reduces to the “weaker” one. In Appendix B we present a complementary derivation of the classification scheme of Table III using the Dirac-matrix Hamiltonian approach.

Let us now discuss in detail for which of the 27 symmetry classes listed in Tables II and III there exist topologically stable Fermi points (point nodes) protected by the combined symmetry \( \tilde{T} \) and/or \( \tilde{C} \).

1. \( R_+ \) and \( R_{++} \)

First, we study the situation where the reflection symmetry operator \( R \) commutes with all global antisymmetric symmetries, which is denoted by \( R_+ \) and \( R_{++} \) in Table III. Since \([R, T] = 0\) and \([R, C] = 0\), we have \( \tilde{T}^2 = T^2 \) and \( \tilde{C}^2 = C^2 \), from which it follows that the ten-fold symmetry class defined in terms of \( T \) and \( C \) is the same as the one defined in terms of the combined symmetries \( \tilde{T} \) and \( \tilde{C} \). Hence, the classification of \( R_+ \) \( (R_{++}) \) reflection symmetric systems with Fermi points outside the reflection plane is almost the same as the classification of Fermi points off high-symmetry momenta in the absence of reflection symmetry (compare Table II with Table III and see Appendix B). The only difference is that the \( \mathbb{Z}_2 \)-type invariants of Table III which are defined in terms of the combined symmetries (30), lead to stable Fermi points outside the reflection plane, whereas the \( \mathbb{Z}_2 \)-type invariants of Table II do not protect Fermi points that are located away from high symmetry momenta (cf. Sec. II.A.2). We observe that for systems with \( R_+ \) \( (R_{++}) \) reflection symmetry in Table III the mirror invariants which are defined in the mirror planes for a given eigenspace of \( R \) yield the same classification as the invariants which are defined in the planes perpendicular to \( k_1 \) with \( k_1 \neq 0, \pi \).

2. \( R_- \) and \( R_{--} \)

Second, we study the case where the reflection operator \( R \) anticommutes with the nonspatial symmetries \( T \) and \( C \), which is labeled by \( R_- \) and \( R_{--} \) in Table III. Here, we find that \( \tilde{T}^2 = -T^2 \) and \( \tilde{C}^2 = -C^2 \) which implies that the symmetry class defined in terms of \( T \) and \( C \) is shifted by four positions on the “Bott clock” \( R \) with respect to the symmetry class defined in terms of \( T \) and \( C \). Note that since the “Bott clock” has periodicity eight, the direction of the shift is irrelevant. Therefore, the types of invariants that can be defined in \((d-1)\)-dimensional planes with fixed \( k_1 \neq 0, \pi \) can be inferred from column \( p = d + 4 \) of the classification of Fermi surfaces that are away from high-symmetry points (Table I). This, however, is inconsistent with the invariants that can be defined within the mirror planes \( k_1 = 0, \pi \). That is, since \([H(k_1 = 0, \pi; k), R] = 0 \) and \([S = TC, R] = 0 \), it is possible to block-diagonalize \( H \) within the mirror plane with respect to \( R \), and for each block one can define a Chern number (class BDI, DIII, CII, and CI) or a winding number (class AI, D, AII, and C). For example, for three-dimensional systems, there are the following invariants that can be defined within the mirror planes (fixed \( k_1 = 0, \pi \)) or within planes with fixed \( k_1 \neq 0, \pi \)

| \( d = 3 \) | AI | BDI | D | DIII | AII | CII | CI |
|---|---|---|---|---|---|---|---|
| mirror plane | \( k_1 \neq 0, \pi \) - plane | \( \mathbb{Z} \) | 0 | \( \mathbb{Z} \) | 0 | \( \mathbb{Z} \) | 0 |
| \( Z_2 \) | 0 | \( \mathbb{Z} \) | 0 | \( \mathbb{Z} \) | 0 | \( \mathbb{Z} \) | \( \mathbb{Z}_2 \) |

As discussed above, the Fermi points are only protected by the “weaker” of these two invariants \( \mathbb{Z}_2 \). Extending these arguments to other dimensions yields the classification shown in Table III. The derivation of this result using the Dirac-matrix Hamiltonian approach is given in Appendix B. We observe that the classification for classes with \( \mathbb{Z}_2 \)-type invariants almost agrees with the classification of Fermi points located away from high-symmetry momenta in the absence of reflection symmetry (Table II). The only difference is that reflection symmetry requires that the \( \mathbb{Z} \) invariants are even (in-
TABLE III. Classification of Fermi points and superconducting point nodes of reflection symmetric semimetals and nodal superconductors, respectively, where the Fermi points (point nodes) are located outside the mirror plane [see Fig. 3(c)]. The first row indicates the spatial dimension \(d\) of the semimetal (nodal superconductor). The prefix “C” indicates that the corresponding topological invariant is defined in terms of the combined symmetries \(T\) and/or \(C\) [see Eq. (29)] on a \((d-1)\)-dimensional plane which is perpendicular to the \(k_1\) axis (blue line/plane in Fig. 3(c)). The \(Z_2\) and \(Z_2\)-type invariants, on the other hand, are identical to the ones of the original ten-fold classification in the absence of mirror symmetry (cf. Table I) and are defined on \((d-1)\)-dimensional hyperspheres surrounding the Fermi point.

| Reflection | FS off mirror plane and off high-sym. point | \(d=1\) | \(d=2\) | \(d=3\) | \(d=4\) | \(d=5\) | \(d=6\) | \(d=7\) | \(d=8\) |
|------------|---------------------------------------------|--------|--------|--------|--------|--------|--------|--------|--------|
| \(R\) \(R^+\) | \(A\) | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 | 0 0 0 0 0 0 0 0 |
| \(R_{+},R_{++}\) | \(AII\) | \(CZ_2\) \(Z\) 0 0 0 2\(Z\) 0 \(CZ_2\) \(Z\) | \(CZ_2\) \(Z\) 0 0 0 2\(Z\) 0 \(CZ_2\) \(Z\) | \(CZ_2\) \(Z\) 0 0 0 2\(Z\) 0 \(CZ_2\) \(Z\) | \(CZ_2\) \(Z\) 0 0 0 2\(Z\) 0 \(CZ_2\) \(Z\) | \(CZ_2\) \(Z\) 0 0 0 2\(Z\) 0 \(CZ_2\) \(Z\) | \(CZ_2\) \(Z\) 0 0 0 2\(Z\) 0 \(CZ_2\) \(Z\) | \(CZ_2\) \(Z\) 0 0 0 2\(Z\) 0 \(CZ_2\) \(Z\) | \(CZ_2\) \(Z\) 0 0 0 2\(Z\) 0 \(CZ_2\) \(Z\) |
| \(R_{-},R_{--}\) | \(AII\) | \(2\(Z\) 0 \(CZ_2\) \(Z\) 0 \(CZ_2\) \(Z\) 0 \(CZ_2\) \(Z\) | \(2\(Z\) 0 \(CZ_2\) \(Z\) 0 \(CZ_2\) \(Z\) 0 \(CZ_2\) \(Z\) | \(2\(Z\) 0 \(CZ_2\) \(Z\) 0 \(CZ_2\) \(Z\) 0 \(CZ_2\) \(Z\) | \(2\(Z\) 0 \(CZ_2\) \(Z\) 0 \(CZ_2\) \(Z\) 0 \(CZ_2\) \(Z\) | \(2\(Z\) 0 \(CZ_2\) \(Z\) 0 \(CZ_2\) \(Z\) 0 \(CZ_2\) \(Z\) | \(2\(Z\) 0 \(CZ_2\) \(Z\) 0 \(CZ_2\) \(Z\) 0 \(CZ_2\) \(Z\) | \(2\(Z\) 0 \(CZ_2\) \(Z\) 0 \(CZ_2\) \(Z\) 0 \(CZ_2\) \(Z\) | \(2\(Z\) 0 \(CZ_2\) \(Z\) 0 \(CZ_2\) \(Z\) 0 \(CZ_2\) \(Z\) |

By a similar logic as above, we find by use of Eq. (31) and Table I that, e.g., for three-dimensional systems, the following invariants can be defined within the mirror planes (fixed \(k_1 = 0, \pi\)) or within planes with fixed \(k_1 \neq 0, \pi\):

\[
\begin{array}{ccc|ccc}
 d \ = \ 3 & DIII & CI & BDII & CII \\
 \text{mirror plane} & \(Z_2\) & 0 & \(Z\) & \(Z\) \\
 \(k_1 \neq 0, \pi\) - plane & 0 & 0 & \(Z_2\) & 0
\end{array}
\]

As before we find that only the “weaker” of these two types of invariants leads to a protection of the Fermi point (cf. Appendix B3). Extending these arguments to other dimensions gives the classification of Table III.

3. \(DIII \& CI\) with \(R_{+}\) and \(BDII \& CII\) with \(R_{-}\)

Third, we discuss the case where the reflection operator \(R\) commutes with one of the global antiunitary symmetries but anticommutes with the other one, i.e., class \(DIII \& CI\) with \(R_{+}\)-type reflection symmetry and class \(BDII \& CII\) with \(R_{-}\)-type reflection symmetry. From the (anti-)commutation relations of \(R\) with the nonspatial symmetries we find that the symmetry class defined in terms of \(T\) and \(C\) (symmetry class for plane with fixed \(k_1 \neq 0, \pi\)) is shifted with respect to the symmetry class defined in terms of \(T\) and \(C\) (symmetry class of entire system) as follows:

\[
\begin{align*}
DIII & \rightarrow CII, \ CII & \rightarrow CI, \ CI & \rightarrow BDII, \ BDII & \rightarrow DIII. \quad (31a)
\end{align*}
\]

On the other hand, since only one global symmetry commutes with the reflection operator \(R\), the symmetry class within the mirror plane is reduced in the following way:

\[
\begin{align*}
DIII & \rightarrow AII, \ CI & \rightarrow AI, \ BDII & \rightarrow D, \ CII & \rightarrow C. \quad (31b)
\end{align*}
\]
fixed $k_1 \neq 0, \pi$

\[
\begin{array}{c|cccc}
\text{d = 3} & \text{AIII} & \text{DIII} & \text{CI} & \text{BDI} & \text{CII} \\
\hline
\text{mirror plane} & Z & Z & 0 & 2Z & Z_2 \\
(\text{k_1 \neq 0, \pi}) \cdot \text{plane} & 0 & 0 & 0 & 0 & Z_2
\end{array}
\]

which suggests that Fermi points in three-dimensional systems with class CII symmetries are protected by a $Z_2$-type invariant. However, this is in contradiction with the result obtained from the Dirac-matrix Hamiltonian approach, which shows that all Fermi points have trivial topology (Appendix B 3). It turns out that even though some nontrivial $Z_2$-type invariants can in principle be defined, these invariants do not protect Fermi points outside the mirror plane. We conclude that Fermi points outside the mirror plane in class AIII with $R_-$-type reflection symmetry, class DIII & CI with $R_+$-type reflection symmetry, and class BDI & CII with $R_{-+}$-type reflection symmetry have trivial topology in all spatial dimensions (Table III).

V. EXAMPLES OF REFLECTION SYMMETRY PROTECTED TOPOLOGICAL SEMIMETALS AND NODAL SUPERCONDUCTORS

In this section we present several examples of gapless topological phases protected by reflection symmetry. As in Sec. IV we consider three different types of Fermi surface positions, which are defined by how the Fermi surface transforms under the mirror reflection and nonspatial symmetries (see Fig. 3).

A. Fermi surfaces at high-symmetry points within mirror planes

We start by discussing four examples of reflection symmetry protected Fermi surfaces (superconducting nodes) that are left invariant under both reflection and global symmetries. These Fermi surfaces are located at high symmetry points within the reflection plane, see Fig. 3(a).

1. Reflection symmetric nodal spin-triplet superconductor with TRS (class DIII with $R_{-+}$ and $p = 2$)

As indicated in Table II point nodes ($d_{FS} = 0$) in two-dimensional spin-triplet superconductors with TRS and $R_{-+}$-type reflection symmetry (class DIII with $R_{-+}$) are protected by an $MZ \oplus Z$ invariant. That is, the number of protected point nodes at high symmetry points within the mirror plane is given by $\max \{|n_Z|, |n_{MZ}|\}$, where $n_Z$ denotes the one-dimensional winding number, whereas $n_{MZ}$ is the mirror invariant. Let us illustrate this type of reflection symmetric nodal superconductor by considering the following continuum model

\[ H^{DIII}_{s,1} = k_x \sigma_x \otimes \sigma_z + k_y \sigma_y \otimes \sigma_z \]  

(32a)

and

\[ H^{DIII}_{s,2} = k_x \sigma_x \otimes \sigma_z + k_y \sigma_y \otimes \mathbb{1}, \]  

(32b)

which have the same symmetry properties as Eq. (32) with $T = \sigma_y \otimes \mathbb{1}$, $C = \sigma_x \otimes \mathbb{1}$, and $R = \sigma_y \otimes \mathbb{1}$. Eqs. (32a) and (32b) have different topological characteristics: While the topology of $H^{DIII}_{s,1}$ is given by $n_Z = 2$ and $n_{MZ} = 0$, for $H^{DIII}_{s,2}$ we find that $n_Z = 0$ and $n_{MZ} = \mp 2$. Hence, both Hamiltonians in Eq. (32) exhibit two stable gapless modes at $k = 0$. We now form a direct product between $H^{DIII}_{s,1}$ and $H^{DIII}_{s,2}$, which yields an $8 \times 8$ Hamiltonian, $H^{DIII}_{s,3} = \text{diag}(H^{DIII}_{s,1}, H^{DIII}_{s,2})$, with four gapless modes. However, only two of these four modes are topologically stable, since it is possible to gap out two states by the symmetry preserving mass term

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & \text{im} \sigma_y & 0 \\
0 & 0 & 0 & 0 \\
0 & -\text{im} \sigma_y & 0 & 0
\end{pmatrix}.
\]  

(35)

Thus, in accordance with the formula $\max \{|n_Z|, |n_{MZ}|\} = 2$, $H^{DIII}_{s,3}$ exhibits only two stable gapless modes at $k = 0$.

In closing, we observe that by including an extra momentum-space coordinate we can convert Hamiltonian (32) to a three-dimensional reflection symmetric superconductor with a protected line node ($d_{FS} = 1$) located at $k = (0, 0, k_z)$. The stability of this nodal line is guaranteed by the quantized winding number $n_Z$, Eq. (5), and the mirror invariant $n_{MZ}$, Eq. (35).
2. Reflection symmetric Dirac semimetal with TRS (class AII with \( R_+ \) and \( p = 3 \))

Next, we study a reflection symmetric three-dimensional Dirac semimetal with TRS, which is described by

\[
H^{\text{AII}}_s = k_z \sigma_z \otimes \sigma_z + k_y \sigma_y \otimes 1 + k_z \sigma_z \otimes 1. \tag{36}
\]

Time-reversal and reflection symmetry operators are given by \( T = \sigma_y \otimes 1K \) and \( R = 1 \otimes \sigma_z \), respectively. Because \( T^2 = 1 \) and \( [T, R] = 0 \), Hamiltonian (36) belongs to symmetry class AII with \( M \).

Indeed, according to Table II, this Fermi point is protected by an extension of Hamiltonian (36) with an \( M \)-type topological invariant, which is defined on the mirror line \( k_y = 0 \) for each eigenspace of the reflection operator \( R \). Focusing on the eigenspace \( R = +1 \), we find that

\[
h^{\text{AII}}_{R=+1} = k_y \sigma_y + k_z \sigma_z. \tag{37}
\]

The \( \mathbb{Z}_2 \) invariant is defined in terms of an extension of Eq. (37) to three dimensions [cf. Eq. (7)]

\[
\tilde{h}^{\text{AII}}_{R=+1} = (k_y \sigma_y + k_z \sigma_z) \cos \theta + \Delta \sigma_x \sin \theta, \tag{38}
\]

where \( \Delta \) is a positive constant and \( \theta \in [0, \pi] \) is the parameter for the extension in the third dimension. With this, we find that the stability of the single Dirac point at \( \mathbf{k} = (0, 0, 0) \) is guaranteed by the invariant (38) with \( g = (\Delta \sin \theta, k \cos \phi \cos \theta, k \sin \phi \cos \theta) \), which evaluates to \( n_{\mathbb{Z}_2} = 1 \). However, as indicated by the \( \mathbb{Z}_2 \)-type invariant, a doubled version of this Dirac point is unstable. This can be seen by considering two copies of Hamiltonian (36), i.e.,

\[
H^{\text{AII}}_s \otimes 1. \tag{39}
\]

The doubled Dirac point of this \( 8 \times 8 \) Hamiltonian can be gapped out by the momentum-independent SPGT \( \sigma_z \otimes \sigma_z \otimes \sigma_y \), which is in agreement with the value of the topological number \( n_{\mathbb{Z}_2} = 0 \) for \( H^{\text{AII}}_s \).

\( \mathbb{Z}_2 \)-type invariants only protect Fermi surfaces of dimension zero (\( d_{\mathbb{BZ}} = 0 \)) at high-symmetry points of the BZ. To illustrate this, we consider an extension of Hamiltonian (36) to four spatial dimensions with a Fermi line along the fourth momentum direction \( k_w \). This Fermi line, which is located at \( (0, 0, 0, k_w) \), can be gapped out by the symmetry-preserving kinetic term \( k_w \sigma_x \otimes \sigma_x \). Only the Fermi point at \( (0, 0, 0, 0) \) remains gapless; it is protected by the non-zero \( \mathbb{Z}_2 \) invariant which is well-defined only for \( k_w = 0 \).

3. Nodal spin-singlet superconductor with TRS and \( R_+ \)-type reflection symmetry (class CII with \( R_+ \) and \( p = 2 \))

Let us now discuss an example of a nodal superconductor with an \( \mathbb{Z}_2 \otimes \mathbb{Z}_2 \)-type index. According to Table II point nodes of time-reversal invariant spin-singlet superconductors with an \( R_+ \)-type reflection symmetry are protected by an \( \mathbb{Z}_2 \otimes \mathbb{Z}_2 \) topological invariant. A simple example of such a reflection symmetric topological superconductor is provided by the \( 4 \times 4 \) Hamiltonian

\[
H^{\text{CII}}_s = k_z \sigma_y \otimes 1 + k_y \sigma_y \otimes 1, \tag{39}
\]

which preserves time-reversal and particle-hole symmetry with \( T = \sigma_y \otimes 1K \) and \( C = \sigma_x \otimes \sigma_y K \), respectively. \( H^{\text{CII}}_s \) is invariant under reflection \( k_z \rightarrow -k_z \) with \( R = \sigma_x \otimes \sigma_y \). Since \( T^2 = 1 \), \( C^2 = -1 \), \( [T, R] = 0 \), and \( \{C, R\} = 0 \), Hamiltonian (39) belongs to symmetry class CII with \( R_+ \).

The two-dimensional superconductor (39) exhibits a point node at \( \mathbf{k} = (0, 0) \) whose stability is guaranteed by a \( \mathbb{Z}_2 \otimes \mathbb{Z}_2 \) topological index. To demonstrate this, we compute both the global invariant \( n_{\mathbb{Z}_2} \) and the mirror invariant \( n_{\mathbb{Z}_2} \). From Table II we find that the global invariant \( n_{\mathbb{Z}_2} \) in column \( p = 2 \) is a second descendant of a \( \mathbb{Z}_2 \)-type invariant in column \( p = 4 \). Hence, the topological number \( n_{\mathbb{Z}_2} \) can be defined in terms of an extension of \( H^{\text{CII}}_s \) to four dimensions

\[
\tilde{H}^{\text{CII}}_s = [k_x \sigma_y \otimes 1 + k_y \sigma_y \otimes 1] \sin \theta \sin \phi + \sigma_x \otimes \sigma_y \sin \theta \cos \phi + \sigma_x \otimes \sigma_x \cos \theta, \tag{40}
\]

where \( \psi, \theta \in [0, \pi] \) are the parameters for the extension to four dimensions. Just as Eq. (39), Hamiltonian (40) satisfies both time-reversal and particle-hole symmetry with

\[
T^{-1} \tilde{H}(-\mathbf{k}, \mathbf{p}, -\psi, -\theta) = -\tilde{H}(\mathbf{k}, \mathbf{p}, \psi, \theta), \tag{41a}
\]

and

\[
C^{-1} \tilde{H}(-\mathbf{k}, \mathbf{p}, -\psi, -\theta) = -C = -\tilde{H}(\mathbf{k}, \mathbf{p}, \psi, \theta), \tag{41b}
\]

respectively. We note that for the definition of the global invariant \( n_{\mathbb{Z}_2} \) we do not need to consider the restrictions imposed by reflection symmetry. Using the extension (40), the \( n_{\mathbb{Z}_2} \) invariant is expressed as

\[
n_{\mathbb{Z}_2} = \frac{1}{48\pi^2} \oint_C \text{Tr} \left[ S \left( \tilde{H}^{\text{CII}}_s \partial_\phi [\tilde{H}^{\text{CII}}_s]^{-1} \right) \right] \mod 2, \tag{42}
\]

with the chiral symmetry operator \( S = \sigma_z \otimes \sigma_y \) and \( C \) a three-dimensional contour which encloses the point node and which is mapped onto itself by both TRS and PHS [see Fig. 1a]. Choosing \( C \) to be the unit three-sphere \( S^3 \), we parametrize the momenta as \( k_x = \cos \phi \) and \( k_y = \sin \phi \), which yields

\[
n_{\mathbb{Z}_2} = \frac{1}{8\pi^2} \int_0^{2\pi} d\phi \int_0^\pi d\psi \int_0^\pi d\theta \text{Tr} \left[ S \left( \tilde{H}^{\text{CII}}_s \partial_\phi [\tilde{H}^{\text{CII}}_s]^{-1} \right) \right] \times \left( \tilde{H}^{\text{CII}}_s \partial_\phi [\tilde{H}^{\text{CII}}_s]^{-1} \right) \text{mod 2} = 1, \tag{43}
\]

indicating that the point node at \( \mathbf{k} = (0, 0) \) is protected by the nontrivial value of \( n_{\mathbb{Z}_2} \).

As opposed to the global invariant \( n_{\mathbb{Z}_2} \), the mirror invariant \( n_{\mathbb{Z}_2} \) is defined in the reflection plane \( k_z = 0 \) for a given eigenspace of the reflection operator \( R \). Focusing on the eigenspace \( R = +1 \), we find that the extended Hamiltonian (40) in this eigenspace within the mirror plane is given by

\[
\tilde{h}^{\text{CII}}_{R=+1} = k_y \sigma_y \sin \psi \sin \theta - \sigma_z \cos \psi \sin \theta + \sigma_x \cos \theta, \tag{44}
\]
where $\psi \in [0, \pi]$ and $\theta \in [0, \pi]$. Hamiltonian (44) is invariant under TRS
\[ T_R^{-1} h_{n=1}^{C(1)} (-k, \pi - \psi, \pi - \theta) T_R = h_{n=1}^{C(1)} (k, \psi, \theta), \tag{45} \]
with $T_R = i \sigma_y K$. The mirror invariant $n_{MZ_2}$ is of the same form as Eq. (9) with an integration contour that preserves TRS, that lies within the mirror plane, and that surrounds the nodal point [see Fig. 3(a)]. As the integration contour we choose a two-sphere $S^2$ which intersects the $(k_x, k_y)$-plane at $k = (0, \pm a)$, such that the Fermi point at $k = (0, 0)$ on the mirror line is enclosed by $k_y = \pm a$, see Fig. 3(a). That is, to perform the contour integration $k_y = 0$ in $h_{n=1}^{C(1)}$ is replaced by $a$ and $\psi$ is integrated over the interval $[0, 2\pi]$, whereas $\theta$ is integrated over $[0, \pi]$. With this integration contour we find that $n_{MZ_2}$ is given by Eq. (9) with $g = (\cos \theta, a \sin \psi \sin \theta, -\cos \psi \sin \theta)$, which evaluates to $n_{MZ_2} = 1$. Hence, the point node at $k = (0, 0)$ is protected also by the mirror invariant $n_{MZ_2}$.

As indicated in Table 1, $MZ_2 \oplus Z_{2}$-type indices only protect Fermi surfaces (superconducting nodes) of dimension zero, i.e., $d_{FS} = 0$. To exemplify this, we consider a trivial extension of Hamiltonian (39) to three spatial dimensions by including the extra momentum component $k_z$, which yields a three-dimensional superconductor with a line node at $(0, 0, k_z)$. However, this line node is unstable, since it can be gapped out by the symmetry-preserving kinetic term $k_z \sigma_z \otimes \sigma_z$. Only the point node at $k = (0, 0, 0)$ is topologically stable. Moreover, we find that the global invariant $n_{Z_2}$, Eq. (42), as well as the mirror invariant $n_{MZ_2}$ cannot be defined for the three-dimensional superconductor with a line node along the $k_z$ direction, since it is impossible to choose a time-reversal-invariant integration contour that surrounds this nodal line (except for $k_z = 0$ and $k_z = \pi$).

4. Reflection symmetric nodal spin-singlet superconductor (class C with $R_-$ and $p = 2$)

As a fourth example we consider a two-dimensional nodal spin-singlet superconductor with reflection symmetry, which is described by the $4 \times 4$ Hamiltonian
\[ H_s^C = k_x \sigma_x \otimes \sigma_y + k_y \sigma_y \otimes \sigma_y. \tag{46} \]
Eq. (46) satisfies PHS with $C = \sigma_y \otimes 1 K$ and is invariant under reflection $k_z \rightarrow -k_z$ with $R = \sigma_y \otimes 1$. Because $C^2 = -1$ and $\{C, R\} = 0$, Hamiltonian (46) belongs to symmetry class C with an $R_-$-type reflection symmetry. This superconductor has a point node at $k = (0, 0)$, which, according to Table 1, is protected by a $T_{Z_2}$ invariant. Indeed, there exists no SPGT that can gap out this point node. To demonstrate the $Z_{2}$-type property of Eq. (46), we consider different doubled versions of the Hamiltonian. Using $H_s^C$, there are four possibilities to construct an $8 \times 8$ Hamiltonian in the symmetry class C with $R_-$
\[ H_{s, \pm}^C = H_s^C \otimes 1, \quad H_{s, \pm}^C = H_s^C \otimes \sigma_z, \tag{47a} \]
\[ H_{s, \pm}^C = k_x \sigma_x \otimes \sigma_y \otimes 1 + k_y \sigma_y \otimes \sigma_y \otimes \sigma_y, \tag{47b} \]
and
\[ H_{s, \pm}^C = k_x \sigma_x \otimes \sigma_y \otimes \sigma_z + k_y \sigma_y \otimes \sigma_y \otimes 1. \tag{47c} \]

We find that the first three Hamiltonians can be fully gapped out by the momentum-independent SPGTs $1 \otimes \sigma_z \otimes \sigma_y$, $1 \otimes 1 \otimes \sigma_y$, and $\sigma_y \otimes \sigma_y \otimes \sigma_y$, respectively. Interestingly, the fourth Hamiltonian $H_{s, \pm}^C$ has a stable point node at $k = 0$, i.e., there exists no SPGT for $H_{s, \pm}^C$. However, if we consider quadrupled versions of $H_{s, \pm}^C$, Eq. (46), we find that for each quadrupled Hamiltonian there exists at least one SPGT which gaps out all the point nodes. (In a sense, the Hamiltonian has a $Z_4$-property rather than a $Z_2$-property.)

B. Fermi surfaces within mirror planes but off high-symmetry points

Second, we present some examples of Fermi surfaces (superconducting nodes) that are left invariant by the mirror symmetry but transform pairwise into each other under the global symmetries. These Fermi surfaces are located within the mirror plane but away from the time-reversal invariant momenta, see Fig. 3(b).

1. Reflection symmetric Dirac semimetal with TRS (class AII with $R_+$ and $p = 2$)

We begin by considering the following two-orbital tight-binding Hamiltonian $H_{n=1}^{AII} = \sum_k \Psi_k^d h_n^{AII} (\Psi_k^d)$ with the spinor $\Psi_k = [\psi_{11}(k), \psi_{12}(k), \psi_{21}(k), \psi_{22}(k)]^T$ and
\[ h_n^{AII} (k) = t_x \sin k_z \sigma_z \otimes \tau_x + [1 - t_y \cos k_y] \sigma_y \otimes \tau_z, \tag{48} \]
where $\sigma_i$ operates in spin grading and $\tau_i$ in orbital grading. This Hamiltonian satisfies TRS, with $T = \sigma_y \otimes \tau_0 K_0$ and reflection symmetry $k_z \rightarrow -k_z$, with $R = \sigma_0 \otimes \tau_0$. Because $T^2 = -1$ and $[R, T] = 0$, semimetal (48) belongs to symmetry class AII with $R_+$. The spectrum of the Hamiltonian is given by
\[ E = \pm \sqrt{t_x^2 \sin^2 k_x + (1 - t_y \cos k_y)^2}. \tag{49} \]
For $t_y > 1$ Hamiltonian (48) has four Dirac points at $(k_x, k_y) = (0, \pm \arccos(1/t_y))$ and $(\pi, \pm \arccos(1/t_y))$, for $t_y = 1$ there are two Dirac points at $(k_x, k_y) = (0, 0)$ and $(\pi, 0)$, and for $t_y < 1$ there is a full gap in the BZ. The reflection symmetry $R$ maps each Dirac point onto itself, i.e., $R$.

This shows that the Fermi points are located within the mirror lines $k_x = 0$ and $k_x = \pi$, see Fig. 3(b). Since there does not exist any SPGT that can be added to Eq. (48), the four Dirac points of Hamiltonian (48) with $t_y > 1$ are topologically stable and protected against gap opening by TRS and reflection symmetry. This is in agreement with the classification of Table 1 (column $p = 2$), which shows that the Fermi points are protected by a mirror invariant of type $2MZ_2$, where the prefix “$2$” indicates that the mirror invariant only takes on even values. To exemplify this for semimetal (48), we evaluate the mirror
number \( n_{2MZ} \) for the reflection line \( k_x = 0 \). We find that \( h_n^{AI} \) in the eigenspace \( R = \pm 1 \) for \( k_x = 0 \) reads
\[
h_{R=\pm 1}^{AI} = \pm (1 - t_x \cos k_y) \mathbb{1}.
\]
(50)

The mirror index \( n_{2MZ}^{\pm} \) for the eigenspace \( R = \pm 1 \) is given by the difference of occupied states (i.e., states with \( E < 0 \)) of Hamiltonian \( h_{R=\pm 1} \) on either side of the Dirac point, i.e.,
\[
n_{2MZ}^{\pm} = n_{\text{occ}}^{\pm}(|k_y| < k_0) - n_{\text{occ}}^{\pm}(|k_y| > k_0) = \pm 2,
\]
(51)
where \( k_0 = \arccos(1/t_y) \) and
\[
n_{\text{occ}}^{+}(k_y) = \left\{ \begin{array}{ll}
2, & |k_y| < k_0 \\
0, & |k_y| > k_0
\end{array} \right.
\]
and
\[
n_{\text{occ}}^{-}(k_y) = \left\{ \begin{array}{ll}
0, & |k_y| < k_0 \\
2, & |k_y| > k_0
\end{array} \right.
\]
(52)
denotes the number of occupied states at \( \mathbf{k} = (0, k_y) \) in the eigenspace or \( R \) with eigenvalue \( +1 \) and \( -1 \), respectively. Hence, the two Dirac points at \( (0, \pm k_0) \) are protected by the invariant \( \langle 51 \rangle \). The index \( n_{2MZ}^{\pm} \) for the \( k_x = \pi \) line, which guarantees the stability of the Fermi points at \( (\pi, \pm k_0) \), can be computed in a similar fashion.

2. Reflection symmetric tight-binding model on the honeycomb lattice (class \( AI \) with \( R_+ \) and \( p = 2 \))

As a second example we discuss a tight-binding model of spinless fermions on the honeycomb lattice, which describes the electronic properties of graphene\( ^{30} \) (ignoring any spin-dependent terms). Considering both first- and second-neighbor hopping the tight-binding Hamiltonian can be written as
\[
H_n^{AI} = \sum_{\mathbf{k}} \Psi_\mathbf{k}^\dagger h_n^{AI}(\mathbf{k}) \Psi_\mathbf{k}
\]
with the spinor \( \Psi_\mathbf{k} = (a_\mathbf{k}, b_\mathbf{k})^T \) and
\[
h_n^{AI}(\mathbf{k}) = \left( \begin{array}{cc}
\Theta_\mathbf{k} & \Phi_\mathbf{k} \\
\Phi_\mathbf{k}^* & \Theta_\mathbf{k}
\end{array} \right),
\]
(53)
where \( a_\mathbf{k} \) and \( b_\mathbf{k} \) denote the fermion annihilation operators with momentum \( \mathbf{k} \) on sublattice \( A \) and \( B \), respectively. The hopping terms are given by \( t_1 \sum_{i=1}^3 e^{i \mathbf{d}_i \cdot \mathbf{k}} \) and \( \Theta_\mathbf{k} = t_2 \sum_{i=1}^3 e^{i \mathbf{d}_i \cdot \mathbf{k}} \), where \( \mathbf{s}_i \) and \( \mathbf{d}_i \) denote the nearest- and second-neighbor bond vectors, respectively [Fig. 4(a)]. The hopping integrals \( t_1 \) and \( t_2 \) are assumed to be positive.

Hamiltonian \( 53 \) satisfies TRS with \( T = \sigma_y K \) and is invariant under the mirror symmetry \( k_x \rightarrow -k_x \) with \( R = \sigma_x \). (Incidentally, Eq. \( 53 \) is also symmetric under \( k_y \rightarrow -k_y \). However, we shall ignore this symmetry, since it does not play any role for the protection of the Dirac points.) Because \( T^2 = +1 \) and \( [R, T] = 0 \) we find that Hamiltonian \( 53 \) belongs to symmetry class \( AI \) with \( R_+ \). The energy spectrum
\[
E_n^{\pm} = +2 t_2 \left[ 2 \cos \left( \frac{3k_x}{2} \right) \cos \left( \sqrt{3}k_y \right) \right] + \cos(\sqrt{3}k_y)
\]
\[
\pm t_1 \left[ 3 + 4 \cos \left( \frac{3k_x}{2} \right) \cos \left( \sqrt{3}k_y \right) + 2 \cos(\sqrt{3}k_y) \right]^{\frac{3}{2}}
\]
(54)
equations two Dirac points, which are located on the mirror line \( k_x = 0 \), i.e., at \( (k_x, k_y) = (0, \pm k_0) \) in the BZ, with
\[
k_0 = 4\pi/(3\sqrt{3}).
\]
These two Dirac points transform pairwise into each other under TRS. Because there does not exist any SPGT that can be added to Eq. \( 53 \), we find that the Dirac points are topologically stable and protected against gap opening by TRS, reflection symmetry, and \( SU(2) \) spin-rotation symmetry. In particular, we note that the TRS preserving mass term \( \sigma_3 \) is forbidden by reflection symmetry \( R \). This finding is confirmed by the classification of Table III which indicates that the stability of the Dirac points is guaranteed by an \( MZ \)-type invariant.

To compute this mirror invariant \( n_{MZ} \) we determine the eigenstates \( \psi_k^{\pm} \) of \( h_n^{AI}(\mathbf{k}) \) with energy \( E_k^{\pm} \)
\[
\psi_k^- = \frac{1}{\sqrt{2}} \begin{pmatrix} -e^{i \varphi_k} \\ 1 \end{pmatrix}, \quad \psi_k^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i \varphi_k} \\ 1 \end{pmatrix},
\]
(55)
where \( \varphi_k = \arg[\Phi_k] \). On the mirror line \( k_x = 0 \) we have
\[
e^{i \varphi(0,k_y)} = \left\{ \begin{array}{ll}
+1, & |k_y| < k_0 \\
-1, & |k_y| > k_0
\end{array} \right.
\]
(56)
Hence, \( \psi_{(0,k_y)}^{\pm} \) are simultaneous eigenstates of the reflection operator \( R = \sigma_x \) with opposite eigenvalue \((+1 \text{ or } -1)\), which prohibits the hybridization between them. The mirror invariant \( n_{MZ}^{\pm} \) is given by the difference of the number of states with energy \( E_k^{\pm} \) and reflection eigenvalue \( R = \pm 1 \) on either side of the Dirac point, i.e.,
\[
n_{MZ}^{\pm} = n_{\text{neg}}^{\pm}(|k_y| > k_0) - n_{\text{neg}}^{\pm}(|k_y| < k_0),
\]
(57)
where \( n_{\text{neg}}^{\pm}(k_y) \) denotes the number of states with energy \( E_k^{\pm} \) and reflection eigenvalue \( R = \pm 1 \). Using Eq. \( 56 \) we find

\[\text{FIG. 4.} \quad (a) \text{ The honeycomb lattice of graphene is a bipartite lattice composed of two interpenetrating triangular sublattices. The two sublattices are marked ‘‘A’’ (black dots) and ‘‘B’’ (blue dots). The nearest-neighbor bond vectors (green arrows) are given by } s_1 = (-1, 0), s_2 = (1, \sqrt{3}), \text{ and } s_3 = \frac{2}{3}(1, -\sqrt{3}). \text{ The second-nearest-neighbor bond vectors (red arrows) are } d_1 = -d_4 = \frac{1}{3}(3, \sqrt{3}), d_2 = -d_5 = \frac{1}{3}(3, -\sqrt{3}), \text{ and } d_3 = -d_6 = (0, -\sqrt{3}). \text{ The mirror line } x \rightarrow -x \text{ is indicated by the green line.} (b) \text{ Energy spectrum of a graphene ribbon with } (10) \text{ edges (i.e., zigzag edges) and } (t_1, t_2) = (1.0, 0.1). \text{ A linearly dispersing edge state (red trace) connects the Dirac points, which are located at } k_{||} = 2\pi/3 \text{ and } k_{||} = 4\pi/3 \text{ in the edge BZ and are projected from the bulk Dirac points at } (0, \pm k_0).
\]
that $n_{MZ}^\pm = \pm 1$, and hence the Dirac points are protected by the mirror invariant \[(57)\]. By the bulk-boundary correspondence, the nontrivial topology of the Dirac points leads to a linearly dispersing edge mode, which connects the projected Dirac points in the (10) edge BZ, see Fig.\[3\](b).

3. Reflection symmetric semimetal (class $A$ with $R$ and $p = 3$)

To exemplify that $MZ$-type invariants can give rise to topologically stable Fermi surfaces with $d_R > 0$, we consider the following three-dimensional semimetal on the square lattice $H_n^A = \sum k \Psi^d_h(k)\Psi_k$, with the spinor $\Psi_k = [c_1(k), c_2(k), c_3(k), c_4(k)]^T$ and
\[ h_n^A(k) = M(k)\tau_0 \otimes \sigma_z + m_2 \tau_z \otimes \sigma_z + \sin k_x \tau_0 \otimes \sigma_x. \] \[(58)\]

Here, $M(k) = m_1 - \cos k_x - \cos k_y - \cos k_z$ is a momentum dependent mass term, and $m_1$ and $m_2$ are positive constants. Eq. \[(58)\] breaks both TRS and PHS, but is symmetric under $k_x \rightarrow -k_x$ with $\tau_0 \otimes \sigma_z$. Incidentally, Eq. \[(58)\] also exhibits a chiral symmetry with $S = \mathbb{I} \otimes \sigma_y$ and $\{R, S\} = 0$, which corresponds to class $AIII$ with $\mathbb{R}$ in Table \[II\]. However, chiral symmetry can be broken by including a staggered chemical potential
\[ V_s = \mu_s \sum_{i=1}^{N} (-1)^i \Psi^d(x_i) \mathbb{I} \otimes \sigma_y \Psi_s(x_i), \] \[(59)\]

with $N$ the number of lattice sites in the $z$ direction. For simplicity we assume that $N$ is an even number. The Hamiltonian with the staggered chemical potential, i.e., $H_n^A + V_s$, is still reflection symmetric about the mirror plane $x = (x_1 + x_N)/2$, and hence belongs to class $A$ with $R$ in Table \[II\].

The energy spectrum of $H_n^A$ in the absence of $V_s$ is given by
\[ E_{\pm,\mu} = \pm \sqrt{|M + (-1)^\mu m_2|^2 + \sin^2 k_x}, \] \[(60)\]

with $\mu \in \{1, 2\}$. Assuming that $m_2 > 0$ and $m_1 - m_2 > 1$, we find that Hamiltonian \[(58)\] exhibits two Fermi rings (i.e., two Fermi surfaces with $d_R = 1$) located within the mirror plane $k_x = 0$, which are described by
\[ \cos k_y + \cos k_z = m_1 - 1 \pm m_2. \] \[(61)\]

These Fermi rings are topologically stable, since there does not exist any reflection symmetric mass term nor any reflection symmetric kinetic term that can be added to Eq. \[(58)\] (cf. Appendix A). This finding is in agreement with Table \[III\], which shows that the Fermi rings \[(61)\] are protected by an $MZ$-type invariant (in the presence of $V_s$) or an $MZ$ $\otimes Z$-type invariant (in the absence of $V_s$). To demonstrate this, let us compute the corresponding mirror and winding numbers.

The mirror number $n_{MZ}$ is defined within the mirror plane $k_x = 0$ for a given eigenspace of the reflection operator $R$. Focusing on the eigenspace $R = +1$, we find that $h_n^A(0, k_y, k_z)$ in this subspace reads
\[ h_{R=+1} = (m - 1 - \cos k_y - \cos k_z) \mathbb{I} - m_2 \sigma_z. \] \[(62)\]

The mirror topological number $n_{MZ}$ is given by the difference of occupied states (i.e., states with negative energy) on either side of the Fermi ring
\[ n_{MZ}^+ = n_{occ}^+(k_y^\prime, k_z^\prime) - n_{occ}^+(k_y^\prime, k_z^\prime), \] \[(63)\]

where $(k_y^\prime, k_z^\prime)$ and $(k_y, k_z)$ are two momenta on either side of the Fermi ring and
\[ n_{occ}^+(k_y, k_z) = \begin{cases} 2, & \tilde{m}(k_y, k_z) < -m_2 \\ 1, & -m_2 < \tilde{m}(k_y, k_z) < +m_2 \\ 0, & \tilde{m}(k_y, k_z) > +m_2 \end{cases}, \] \[(64)\]

with $\tilde{m}(k_y, k_z) = m_1 - 1 - \cos k_y - \cos k_z$, represents the number of occupied states in the eigenspace with $R = +1$.

In the absence of the staggered chemical potential $V_s$, Hamiltonian \[(58)\] satisfies chiral symmetry and the Fermi rings are also protected by a winding number $n_z$, which takes the form of Eq. \[(12)\] with
\[ q = \begin{pmatrix} \sin k_x - i(M(k)+m_2) & r_+ \\ r_- & \sin k_x - i(M(k)-m_2) \end{pmatrix}, \] \[(65)\]

where $r_\pm = \sqrt{(M(k) \pm m_2)^2 + \sin^2 k_x}$, and an integration contour $C$ that encircles the Fermi ring. Choosing the contour along the $k_x$ direction we find
\[ n_z(k_y, k_z) = \begin{cases} 2, & \tilde{m}(k_y, k_z) < -m_2 \\ 1, & -m_2 < \tilde{m}(k_y, k_z) < +m_2 \\ 0, & \tilde{m}(k_y, k_z) > +m_2 \end{cases}. \] \[(66)\]

By the bulk-boundary correspondence, a nontrivial value of $n_z$, Eq. \[(66)\], leads to zero-energy flat bands at the surface of the semimetal. These zero-energy states appear within regions of the surface BZ that are bounded by the projection of the bulk Fermi rings, see Figs.\[5\](a) and \[5\](b). When chiral symmetry is broken, for example by a finite staggered chemical potential $V_s$, the surface flat bands acquire a finite dispersion, see Fig.\[5\](c).

4. Unstable reflection symmetric nodal superconductors (class $DIII$ with $R_++$ and $p = 2$, class $D$ with $R_+$ and $p = 2$)

As shown in Table \[III\], $Z_2$-type topological invariants (i.e., $Z_2, MZ_2$, and $MZ_2 \oplus Z_2$) do not protect Fermi surfaces (superconducting nodes) that are located within the mirror planes but away from high-symmetry points (cf. Sec. \[II\]A.2c). However, these $Z_2$-type invariants can lead to protected gapless surface states. To exemplify this behavior we study in this subsection two-dimensional unstable nodal superconductors belonging to class $DIII$ with $R_+$-type reflection and class $D$ with $R_+$-type reflection, which are classified as $MZ_2 \oplus Z_2$ and $MZ_2$, respectively, in Table \[III\]. For this purpose, we borrow an example from Sec. \[II\]A.2c i.e., $H_{DIII}^A = \sum_k \Psi_k^d h_n^A \Psi_k$ with the Nambu spinor $\Psi_k = (a_k^\dagger, b_k^\dagger, a_k^\dagger, b_k^\dagger)^T$ and
\[ h_n^D = (1 + \cos k_x + \cos k_y)\sigma_x \otimes \sigma_y + \sin k_x \sigma_y \otimes \mathbb{I}, \] \[(67)\]
which describes a time-reversal symmetric superconductor with point nodes located at \((\pi, \pm \pi/2)\). Here, \(a_k^\dagger\) and \(b_k^\dagger\) represent fermionic creation operators with momentum \(k\). Hamiltonian \((67)\) preserves TRS and PHS with \(T = \sigma_y \otimes 1 K\) and \(C = \sigma_z \otimes 1 K\), respectively, and is invariant under \(k_x \to -k_x\) with \(R = \sigma_x \otimes 1\). Because \(T^2 = -1\), \(C^2 = +1\), \(\{R, T\} = 0\), and \([R, C] = 0\), \(H\) belongs to class DIII with \(R_\pi\). According to Table II the point nodes of Hamiltonian \((67)\), which transform pairwise into each other by TRS and PHS, are topologically unstable, even though the topological numbers \(n_{Z2}\) [cf. Eq. \((17)\)] and \(n_{MZ2}\) for Hamiltonian \((67)\) take on nontrivial values. Indeed, we find that the symmetry-preserving extra kinetic term \(\delta t \sin k_y \sigma_x \otimes \sigma_x\) gaps out the Fermi points at \((\pi, \pm \pi/2)\) and turns Eq. \((67)\) into a fully gapped reflection symmetric topological superconductor

\[
H_{ig}^{DIII} = H_{ig}^{DIII} + \delta t \sum_k \Psi_k^\dagger \sum_i k_i \sigma_x \otimes \sigma_x \Psi_k. \tag{68}
\]

That is, the unstable nodal superconductor \((67)\) is connected to the fully gapped reflection symmetric topological superconductor \((68)\) and inherits topological edge states from the fully gapped phase \((22)\).

To demonstrate this, let us compute the global \(n_{Z2}\), invariant and the mirror invariant \(n_{MZ2}\) for Hamiltonian \((67)\) and \((68)\). The computation of the global invariant \(n_{Z2}\), which is given by Eq. \((17)\), follows along similar lines as in the example of Sec. II 2 c (Note that for the definition of a \(Z_2\)-type invariant the reflection symmetry does not play any role; the \(Z_2\) number \(n_{Z2}\) is defined solely in terms of the global symmetries.) We find that for a contour \(C\) oriented along the \(k_x\) axis with \(k_y\) held fixed at \(k_y = 0\) (or \(k_y = \pi\)), the topological index evaluates to \(n_{Z2} = +1\) (or \(n_{Z2} = -1\)) both for the nodal superconductor \(H_{ig}^{DIII}\) and the fully gapped superconductor \(H_{ig}^{DIII}\). This indicates that there appear zero-energy edge states at \(k_y = \pi\) of the \((10)\) edge BZ of both the fully gapped and the nodal system.

To calculate the mirror number \(n_{MZ2}\) we focus on the eigenspace of the reflection operator with eigenvalue \(R = +1\) and transform Hamiltonian \((68)\) to a Majorana basis \((33)\). On the mirror lines \(k_x = 0\) and \(k_x = \pi\), \(H_{ig}^{DIII}\) in the eigenspace \(R = +1\) can be expressed as

\[
H_{R=+1}^{DIII} = \sum_{k_y} M_{\nu}(k_y) \begin{pmatrix} d_{\nu,k_y}^\dagger & d_{\nu,-k_y} \end{pmatrix} \begin{pmatrix} 1 & -i \delta T \nu \nu, k \nu, k \nu, k \nu, k \nu, k \nu, k \nu, k \nu, k \nu, k \nu, k \nu, k \nu, k \nu, k \nu, k \nu, k
\]

with \(\nu \in \{0, \pi\}\) and where \(M_{\nu}(k_y) = 1 + (-1)^{\nu \nu} \cos k_y\) and \(\delta T(k_y) = \delta t \sin k_y\). In Eq. \((69)\) the transformed fermion operators \(d_{\nu,k_y}\) are given by

\[
d_{\nu,k_y} = \frac{1}{2} \left[ a_{\nu,k_y}^\dagger + a_{\nu,-k_y} + i (b_{\nu,k_y}^\dagger + b_{\nu,-k_y}) \right]. \tag{70}
\]

Using Eq. \((70)\) we can construct real Majorana operators \(\lambda_{\nu,k_y} = (\lambda_{\nu,k_y}, \lambda_{\nu,k_y}^*)^T\), with

\[
\lambda_{\nu,k_y} := a_{\nu,k_y}^\dagger + d_{\nu,k_y}, \quad \lambda_{\nu,k_y}^* := i (d_{\nu,k_y} - a_{\nu,k_y}), \tag{71}
\]

and rewrite the Hamiltonian in the \(R = +1\) eigenspace as

\[
H_{R=+1}^{DIII,\nu} = \frac{i}{2} \sum_{k_y} \Lambda_{\nu,-k_y} B_{\nu}(k_y) \Lambda_{\nu,k_y}, \tag{72a}
\]

with

\[
B_{\nu}(k_y) = \begin{pmatrix} \delta T(k_y) & M_{\nu}(k_y) \\ -M_{\nu}(k_y) & \delta T(k_y) \end{pmatrix}. \tag{72b}
\]

It follows that the mirror invariant \(n_{MZ2}\) on the two mirror lines \(k_y = 0\) and \(k_y = \pi\) is given by

\[
n_{MZ2}^{\nu} = \text{sgn}[\text{Pf} B_{\nu}(0)] \text{sgn}[\text{Pf} B_{\nu}(\pi)] = \begin{cases} +1, & \nu = 0 \\ -1, & \nu = \pi \end{cases}, \tag{73}
\]
Interestingly, the value of \( n_{MZ} \) does not depend on the extra kinetic term \( \delta t \sin k_y \sigma_y \otimes \sigma_x \). Hence, we conclude that the unstable nodal superconductor \( H^{\text{DIII}}_u \) can be connected to the fully gapped topological superconductor \( H^{\text{DIII}}_g \) (whose bulk topology is described by \( n^0_{MZ}, n^\pi_{MZ} \)) without changing the values of the invariants \( n_{2z} \) and \( n^\pi_{2z} \). Both \( n_{2z} \) and \( n^\pi_{2z} \) lead to protected zero-energy states at the edge of the nodal (or fully gapped) superconductor.

We observe that in systems that are classified as \( M\mathbb{Z}_2 \oplus \mathbb{Z}_2 \) in Table I the two invariants \( n_{2z} \) and \( n^\pi_{2z} \) always take on the same values. This is in contrast to topological materials with an \( M\mathbb{Z} \oplus \mathbb{Z} \) classification, where the two invariants \( n_{MZ} \) and \( n_Z \) can be distinct, see example in Sec. [V A 1]. That is, the presence of reflection symmetry in \( M\mathbb{Z}_2 \oplus \mathbb{Z}_2 \)-type systems does not lead to any new topological characteristics, but it simplifies the calculation of the topological index. I.e., the topological characteristics can be inferred from the wavefunctions at reflection planes alone. (This situation is in a sense similar to the \( \mathbb{Z}_2 \) time reversal symmetric topological insulator with inversion symmetry of Ref. [82], where the inversion symmetry does not lead to new topological features, but simplifies the formula for the topological index.)

A similar analysis as above can be preformed for a two-dimensional unstable nodal superconductor in class \( D \) with \( R_+ \)-type reflection symmetry. In the absence of TRS the global \( \mathbb{Z}_2 \) number \( n_{2z} \) is ill defined, however the mirror invariant \( n_{MZ} \) is still well defined and takes on nontrivial values (cf. Table I). This mirror index leads to stable zero-energy modes at edges that are invariant under reflection. As before, we find that a reflection symmetric nodal superconductor in class \( D \) with \( R_+ \) can be connected to a fully gapped topological superconductor without removing the zero-energy edge-states.

5. Reflection symmetric nodal spin-triplet superconductor with TRS (class DIII with \( R_{--} \) and \( p = 3 \))

As the last example of this subsection, we study a three-dimensional reflection symmetric superconductor in class DIII

\[
H^{\text{DIII}}_{3D} = M(k) \sigma_z \otimes 1 \sin k_x \sigma_y \otimes \sigma_x + \sin k_z \sigma_x \otimes \sigma_z,
\]

(74)

which exhibits point nodes at \( k = (0, \pm \pi/3, 0) \). The k-dependent mass \( M(k) \) is given by \( M(k) = -2.5 + \cos k_x + \cos k_y + \cos k_z \). Hamiltonian (74) satisfies TRS and PHS with \( T = 1 \otimes \sigma_y K \) and \( C = \sigma_x \otimes 1 K \), respectively, and is reflection symmetric under \( k_x \rightarrow -k_x \) with \( R = \sigma_x \otimes \sigma_x \). Because \( T^2 = -1 \), \( C^2 = +1 \), \( \{T, R\} = 0 \), and \( \{C, R\} = 0 \), Eq. (74) is classified as DIII with \( R_{--} \). The two point nodes, which are located within the mirror plane at \( k = (0, \pm \pi/3, 0) \) are protected by TRS, PHS, and reflection symmetry, since there does not exist any SPGT that can be added to Eq. (74). We note that the gap opening term sin \( k_x \sigma_y \otimes \sigma_x \) is symmetric under TRS and PHS but breaks mirror symmetry, which shows that the reflection symmetry \( R \) is crucial for the protection of the point nodes. Indeed, as indicated by Table I the point nodes are unstable in the absence of reflection symmetry.

Let us now compute the mirror invariant \( n_{MZ} \) which, as listed in Table I, protects the point nodes. Since the chiral symmetry operator \( S = T C = \sigma_y \otimes \sigma_y \) commutes with \( R \), the mirror number \( n_{MZ} \) can be expressed as a one-dimensional winding number, i.e., for the eigenspace \( R = +1 \) it takes the form of Eq. (12) with

\[
q = \frac{M(k) - \sin k_z i}{\sqrt{M(k)^2 + \sin^2 k_z}},
\]

(75)

and a contour \( C \) that lies within the mirror plane and encloses one of the point nodes [see Fig. 3(b)]. Choosing the contour along the \( k_z \) axis with \( k_x = 0 \) and \( k_y \) a fixed parameter, we find that the mirror number evaluates to

\[
n_{MZ}^+(k_y) = \begin{cases} 1, & 0 \leq |k_y| < \frac{\pi}{3} \\ 0, & \frac{\pi}{3} < |k_y| \leq \pi \end{cases}.
\]

(76)

By the bulk boundary correspondence, the nontrivial value of Eq. (76) leads to zero-energy arc states on surfaces that are perpendicular to the mirror plane. As shown in Fig. 6 these zero-energy arc states connect two projected point nodes in the surface BZ.

C. Fermi surfaces outside mirror planes

Third, we discuss three examples of Fermi surfaces (superconducting nodes) that lie outside the mirror plane. These Fermi surfaces are pairwise related to each other by both reflection and nonspatial symmetries, see Fig. 3(c). Their topological properties are classified by Table I.

1. Reflection symmetric Dirac semimetal with TRS (class AII with \( R_+ \) and \( p = 3 \))

We start by studying an example of a three-dimensional Dirac semimetal with an \( R_+ \)-type reflection symmetry, which is described by[29,33,38]

\[
H^{\text{AII}}_{\text{off}} = \sin k_y \tau_z \otimes \sigma_z + \sin k_z \tau_y \otimes 1 + M(k) \tau_z \otimes \sigma_y
\]

(77)

Here, \( M(k) = M - \cos k_x - \cos k_y - \cos k_z \) and \( M \) is a positive constant, which we set to \( M = 2.0 \). The Pauli matrices \( \sigma_i \) and \( \tau_i \) operate in spin and orbital grading, respectively. Hamiltonian (77) preserves TRS \( T = 1 \otimes \sigma_y K \) and is symmetric under \( k_x \rightarrow -k_x \) with \( R = \sigma_x \otimes \sigma_x \). Since \( T^2 = -1 \) and \( |T, R| = 0 \), the Hamiltonian belongs to class AII with \( R_+ \).

By computing the energy spectrum we find that the semimetal exhibits two doubly degenerate Dirac points that are located outside the reflection plane \( k_x = 0 \), i.e., at \( k = (\pm \pi/2, 0, 0) \). These Fermi points are protected by a combination of time-reversal and reflection symmetry, because there does not exist any SPGT that can be added to Eq. (77). We note, however, that in the absence of reflection symmetry, the Dirac points can be gapped out by the time-reversal invariant term \( \sin k_x \tau_x \otimes \sigma_x \), which turns Hamiltonian (77) into a class AII...
topological insulator. This finding is in agreement with the ten-fold classification of gapless topological materials shown in Table I. To determine whether the Dirac points have a Z2 or Z2-type character, we consider a doubled version of \( H_{\text{off}}^{\text{III}} \), i.e., \( H_{\text{off}}^{\text{III}} \otimes \mathbb{1} \). For the doubled Hamiltonian there exist a momentum-independent SPGT (i.e., \( \tau_x \otimes \sigma_x \otimes \sigma_y \)), demonstrating that the Dirac points are protected by a Z2-type invariant, which is denoted as “CZ2” in Table III.

The CZ2 invariant \( n_{\text{CZ2}} \) is defined in terms of the combined symmetry (30b), i.e., \( T^{-1}H_{\text{off}}^{\text{III}}(k_x, -\mathbf{k}) T = H_{\text{off}}^{\text{III}}(k_x, \mathbf{k}) \). Since each plane perpendicular to the \( k_x \) axis is left invariant by the combined symmetry (30b), we can define the topological number \( n_{\text{CZ2}} \) for any given plane \( E_{k_x} \) with fixed \( k_x \) [see Fig. 3(c)]. We find that

\[
n_{\text{CZ2}}(k_x) = \begin{cases} 
+1, & \frac{\pi}{2} < |k_x| \leq \pi \\
-1, & 0 \leq |k_x| < \frac{\pi}{2} 
\end{cases}
\]

(78)

Due to the bulk-boundary correspondence, the nontrivial value of \( n_{\text{CZ2}}(k_x) \) in the interval \([-\pi/2, +\pi/2]\) gives rise to helical Fermi arcs on surfaces that are perpendicular to the reflection plane. These helical arc states connect the project bulk Dirac points in the surface BZ.

2. Reflection symmetric nodal spin-triplet superconductor (class D with \( R_- \) and \( d = 3 \))

Next, we consider a reflection symmetric nodal spin-triplet superconductor, which is described by the BdG Hamiltonian

\[
H_{\text{off}}^{\text{D}} = \sin k_y \tau_y \otimes \sigma_y + \sin k_z \tau_z \otimes \sigma_z + \mathcal{M}(k) \tau_z \otimes \mathbb{1}, \quad (79)
\]

where \( \mathcal{M}(k) = 2 - \cos k_x - \cos k_y - \cos k_z \). Here, the Pauli matrices \( \sigma_x, \sigma_y, \sigma_z \), \( \tau_x, \tau_y, \tau_z \), respectively, \( H_{\text{off}}^{\text{D}} \) satisfies PHS with \( C = \tau_y \otimes \mathbb{1} \mathcal{K} \) and is invariant under \( k_x \rightarrow -k_x \) with \( R_- = \tau_x \otimes \sigma_y \). Because \( C^2 = +\mathbb{1} \) and \( \{ R, C \} = 0 \), the BdG Hamiltonian belongs to class D with \( R_- \). As an aside, we note that reflection symmetry \( k_x \rightarrow -k_x \) for spin-\( \frac{1}{2} \) systems is usually implemented by the operator \( R'_p = i\sigma_y (R'_p = -i\sigma_y) \) for particle-like (hole-like) degrees of freedom, i.e., by the operator \( R' = i\tau_x \otimes \sigma_x \) in particle-hole space. However, in order to correctly categorize the Hamiltonian with respect to the 27 symmetry classes of Table III, we need to ensure that the reflection operator \( R \) is Hermitian (cf. Sec. III A). Therefore we have dropped the factor \( i \) in the above definition of \( R \).

The spectrum of Hamiltonian (79) exhibits two doubly degenerate point nodes, which are located outside the mirror plane at \( k = (\pm\pi/2, 0, 0) \). These point nodes are topologically stable, since there does not exist any SPGT that can be added to Eq. (79). According to Table III the point nodes of \( H_{\text{off}}^{\text{D}} \) are protected by an invariant of type “2Z2” (i.e., a Chern number), where the prefix “2” indicates that the topological number only takes on even values. Choosing the two-dimensional integration contour to be a plane perpendicular to the \( k_x \) axis, we find that the Chern number for Hamiltonian (79) is given by

\[
n_{\text{C}}(k_x) = \frac{i}{2\pi} \int \sum_{i=1}^{2} d \langle u^-_i | u^-_i \rangle 
\]

\[
= \int \frac{1}{2\pi R^3} (ZdX \wedge dY + XdY \wedge dZ + YdZ \wedge dX),
\]

(80)

where \( X = \sin k_x, \quad Y = \sin k_y, \quad Z = \mathcal{M}(k), \quad \) and \( R = \sqrt{X^2 + Y^2 + Z^2} \). Evaluating the integral, we obtain

\[
n_{\text{C}}(k_x) = \begin{cases} 
0, & 0 < |k_x| \leq \pi, \\
-2, & 0 \leq |k_x| < \frac{\pi}{2} 
\end{cases}
\]

(81)

Note that for the definition of the Chern number (81), the combined symmetry \( C = RC \), Eq. (30c), does not play any role, except to ensure that there are an even number of point nodes on either side of the reflection planes. By the bulk-boundary correspondence, the nontrivial value of \( n_{\text{C}} \), Eq. (81), gives rise to arc surface states, which connect the projected point nodes in the surface BZ.  

![FIG. 6. (Color online) Surface band structure of the reflection symmetric nodal superconductor \( (74) \) (class DIII with \( R_- \)) for the (001) face as a function of (a) surface momentum \( k_y \) with \( k_x = 0 \) and (b) surface momentum \( k_x \) with \( k_y = 0 \). A zero-energy arc surface state (red trace) connects the projected point nodes in the surface BZ. (c) Surface spectrum on the (001) face as a function of both \( k_x \) and \( k_y \). The surface states and bulk states are indicated in green and grey, respectively.](image)
3. Unstable reflection symmetric nodal superconductor with TRS (class DIII with \( R_{-+} \) and \( d = 2 \))

As stated in Section IV C 3, superconducting nodes outside the mirror plane in systems of class DIII with \( R_{-+} \)-type reflection symmetry are unstable, even though a nontrivial \( \mathbb{M} Z_2 \)-type invariant can be defined for these systems. To illustrate this, we consider the following BdG Hamiltonian

\[
H_{\text{off}}^{\text{DIII}} = \sin k_y \sigma_x \otimes 1 + (1 + \cos k_x + \cos k_y) \sigma_z \otimes \sigma_y,
\]

which describes a superconductor with unstable point nodes. Eq. (82) preserves TRS and PHS with \( T = \sigma_y \otimes 1 \) and \( C = \sigma_x \otimes \sigma_z \), respectively, and is symmetric under \( k_x \to -k_x \) with \( R = \sigma_x \otimes \sigma_z \). Because \( T^2 = -1 \), \( C^2 = +1 \), \( \{T, R\} = 0 \), and \( [C, R] = 0 \), Hamiltonian (82) is classified as DIII with \( R_{-+} \). We find that the spectrum of Eq. (82) exhibits point nodes located away from the mirror lines \( k_x = 0 \) and \( k_x = \pi \), i.e., at \( k = (\pm \pi/2, 0) \). These point nodes are topologically unstable, since there exists a momentum-dependent SPGT (i.e., \( \sin k_x \sigma_y \otimes 1 \)), which opens up a full gap.

Let us now examine topological invariants for Hamiltonian (82). First, we consider a winding number \( \nu_{\sigma} \), which is defined by chiral symmetry with \( S = TC = -\sigma_x \otimes \sigma_z \) on a line perpendicular to the \( k_x \) direction. Since chiral symmetry is momentum independent, combining reflection and chiral symmetries is not required to define the winding number \( \nu_{\sigma} \).

We find that this one-dimensional winding number is given by Eq. (5) with

\[
q = \frac{1}{\sqrt{\sin^2 k_y + M^2}} \left( \frac{\sin k_y}{-iM} \right) - \frac{iM}{-iM \sin k_y} \right),
\]
where \( M = 1 + \cos k_x + \cos k_y \). Evaluating the integral, one obtains that \( \nu_{\sigma} \) is trivial for any fixed \( k_x \) (i.e., \( \nu_{\sigma} = 0 \)), in agreement with the fact that the point nodes are unstable. Second, we consider the mirror invariant, which is defined within the mirror lines \( k_x = 0 \) and \( k_x = \pi \) for a given eigenspace of \( R \). Since \( H_{\text{off}}^{\text{DIII}} \) restricted to the mirror lines satisfies PHS, a mirror invariant of type \( \mathbb{M} Z_2 \) can be defined. By a similar calculation as in example V B 2, we find that the mirror invariant \( n_{\mathbb{M} Z_2} \) is given by \( n_{\mathbb{M} Z_2} = 1 \) for \( k_x = 0 \) and \( n_{\mathbb{M} Z_2} = -1 \) for \( k_x = \pi \). However, even though \( n_{\mathbb{M} Z_2} \) takes on a nontrivial value, this \( \mathbb{M} Z_2 \)-type invariant does not protect the point nodes that are located at \( k = (\pm \pi/2, 0) \) (see Appendix B 3).

VI. SUMMARY AND CONCLUSIONS

In this paper we have performed an exhaustive classification of reflection symmetry protected topological semimetals and nodal superconductors. We have shown that the classification depends on (i) the codimension \( p = d - d_{\text{BS}} \) of the Fermi surface (nodal line) of the semimetal (nodal superconductor), (ii) how the Fermi surface (nodal line) transforms under the crystal reflection and the global symmetries, and (iii) whether the reflection symmetry operator \( R \) commutes or anticommutes with the global (i.e., nonspatial) symmetries. The result of this classification scheme is summarized in Tables II and III which show that the presence of reflection symmetries leads to an enrichment of the ten-fold classification of gapless topological materials (cf. Table I) with additional topological states. The reflection symmetry \( R \) together with the three nonspatial symmetries, time-reversal, particle-hole, and chiral symmetry, define a total of 27 different symmetry classes. For Fermi surfaces with even (odd) codimension \( p \) located within the mirror plane, 17 (10) out of these 27 classes allow for nontrivial topological characteristics of the Fermi surface (Table I). For Fermi surfaces located outside the mirror plane, on the other hand, there are 9 symmetry classes which permit the existence of nontrivial topological properties (Table III).

To illustrate the general principles of the classification schemes, we have discussed in Sec. V concrete examples of reflection symmetry protected topological semimetals and nodal superconductors. The topological properties of these gapless materials manifest themselves at the surface in the form of linearly dispersing Dirac or Majorana modes, or dispersionless states, which form two-dimensional flat-bands or one-dimensional arcs (see Figs. I 5 and 6). These different types of surface states are protected by different types of topological invariants. For the examples of Sec V we have derived explicit expressions for these topological numbers.

Probably, the most prominent example of a reflection symmetric topological semimetal is graphene\(^{80}\) whose Dirac points are protected against gap opening by time-reversal symmetry together with reflection and \( SU(2) \) spin-rotation symmetry. In the classification scheme of Table II, graphene belongs to class AI with \( R_{-+} \)-type reflection symmetry. Hence, the Dirac points of graphene, which are located within the reflection line but away from time-reversal invariant points, are protected by a mirror invariant (\( \mathbb{M} Z_2 \)), see Sec. V B 3. The classifications of Tables II and III predict several new reflection symmetric topological semimetals and nodal superconductors, for which realistic physical systems have yet to be found. For example, a reflection symmetric topological nodal superconductor with spin-triplet pairing is predicted to exist in three spatial dimensions (class DIII with \( R_{-+} \)), see Sec. V B 5. This nodal superconductor, which exhibits two point nodes within the reflection plane (but away from the time-reversal invariant momenta) is a three-dimensional superconducting analogue of graphene.

Recently, several examples of space group symmetry protected topological semimetals have been theoretically proposed\(^{81,82}\). The surface states of \( \text{Na}_3\text{Bi} \)\(^{85,86}\) and \( \text{Cd}_3\text{As}_2 \)\(^{83,84}\) which are two topological Dirac materials protected by rotation symmetry, have been experimentally observed using angle-resolve photoemission and scanning tunneling measurements. We hope that these recent discoveries will spur the experimental search for other types of topological phases. The results of this paper will be useful for the search and design of new gapless topological materials that are protected by reflection symmetry.
The classification of topological insulators and superconductors expressed by the homotopy group is given by
\[ K^C(s, d) = \pi_0(C_{s-d}), \quad (A3) \]
\[ K^R(s, d) = \pi_0(R_{s-d}). \quad (A4) \]

The classification of gapless modes at high-symmetry points for the global symmetries corresponding to \( d + 1 \) dimensional of the original ten-fold classification reads
\[ G_{\pi}^C(s, d) = \pi_0(C_{s-d-1}), \quad (A5) \]
\[ G_{\pi}^R(s, d) = \pi_0(R_{s-d-1}). \quad (A6) \]

Table IV shows the presence or absence of symmetry-allowed kinetic and mass terms without enlarging the minimal Hamiltonian, which preserves all system symmetries and is written in minimal matrix dimension. For the minimal Hamiltonian described by \( \mathbb{Z} \) symmetry-allowed kinetic (\( \gamma_i \)) and mass (\( \tilde{\gamma}_j \)) terms are absent. For the trivial symmetry class, which is labeled by "0", the presence of \( m\gamma_i \) as SPEMT keeps the system in the trivial phase. For the \( \mathbb{Z}_2 \), an SPEMT (\( m\gamma_i \)) is absent but an extra kinetic term (\( k_i\gamma_{d+1} \)) is present in the minimal Dirac Hamiltonian. Although the presence of the extra kinetic term does not guarantee trivial topology, when the merge of two non-trivial systems, this kinetic term plays an important role to construct an SPEMT and trivialize the merging system.

For the topological gapless phases of the AZ symmetry classes are described by three types of topological invariants: 0, \( \mathbb{Z}_2 \), \( \mathbb{Z} \). We review the topological behaviors of these invariants for the gapless Dirac Hamiltonian at the high-symmetry point (\( k = 0 \)), which is written as
\[ H_{s}^{\text{Dirac}} = \sum_{i} d \gamma_i. \quad (A7) \]

That is, the gapless Hamiltonian is identical to the insulator Dirac Hamiltonian in Eq. (26) without the mass term (\( m\gamma_i \)). The Hamiltonian preserves symmetries corresponding to the symmetry class so the kinetic gamma matrix obeys Eq. (A1). Furthermore, \( H_{s}^{\text{Dirac}} \) in \( d \) dimensions can be treated as the boundary states of \( H_{\text{Dirac}}^{\text{TI}} \) in \( d + 1 \) dimensions. According to Clifford algebra, the behaviors of gamma matrices are identical to the same system, however, with one extra mass and kinetic terms. The topology of \( d \) dimensional gapless systems, which is determined by SPEMTs (\( m\gamma_i \)), is the same with \( (d + 1) \)-dimensional TI and SC in the same symmetry class. For gapless systems, \( m\gamma_i \) is realized as a symmetry-preserving gap-opening term (SPGT), which gaps out the gapless modes. Thus, the classification of gapless modes at high-symmetry points in \( d \) dimensions corresponds to the ten-fold classification of TIs and SCs in \( d + 1 \) dimensions as shown in Table IV. Bulk gapless modes are classified by three types of topological invariants (0, \( \mathbb{Z}_2 \), \( \mathbb{Z} \)). Their characteristics are discussed in the following.

Similarly, for Fermi surfaces outside high-symmetry points the Hamiltonian is given by Eq. (28). The Hamiltonian can be

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**Appendix A: Review of ten-fold classification scheme of gapless topological materials**

Topological properties of gapless materials can be classified by two different methods\(^{22,23,24}\): (i) the minimal Dirac-matrix Hamiltonian method and (ii) the derivation of topological invariants. For the former, the topological property is determined by the presence or absence of a symmetry preserving mass term (SPEMT), which preserves symmetries and prevents insulators and superconductors from passing through quantum phase transition by keeping the spectrum gapped. The presence of this term implies trivial topology of such systems. However, in the absence of this term non-trivial topology emerges. Topological invariants, on the other hand, classify topology of quantum systems. If non-zero topological invariant rises, this system is classified as non-trivial topology. In Appendix B, we mainly use the minimal Dirac-matrix Hamiltonian method to derive the classification of reflection symmetric gapless modes off reflection planes and then the classification can be confirmed by topological invariant method in Sec. IV C.

The Dirac Hamiltonian (\( H_{\text{Dirac}}^{\text{TI}} \)) that classifies topological insulators and superconductors is given by Eq. (26), where \( \gamma_i \) is a kinetic term and \( \tilde{\gamma}_j \) is a mass term. For real symmetry classes, they obey
\[
\begin{align*}
\{T, \gamma_i\} &= 0, & [C, \gamma_i] &= 0, \quad (A1) \\
\{T, \tilde{\gamma}_j\} &= 0, & [C, \tilde{\gamma}_j] &= 0, \quad (A2)
\end{align*}
\]
to preserve TRS and PHS. The two types of the gamma matrices anticommute with chiral symmetry operator \( S = CT \).

---

**TABLE IV.** The presence of the gamma matrices without enlarging the minimal Dirac Hamiltonians. Due to the periodicity of two and eight for complex and real symmetry classes, respectively, \( l = 0, 1 \) mod 2 for \( C_l \) and \( l = 0, 1, \ldots, 7 \) mod 8 for \( R_l \).

| \( s \) | AZ class \((d = 0)\) | Topological invariant | gamma matrix |
|---|---|---|---|
| 0 | A | \( \pi_0(C_0) = \mathbb{Z} \) | \( \gamma_{d+1} \) or \( \tilde{\gamma}_1 \) |
| 1 | AIII | \( \pi_0(C_1) = 0 \) | |
| 0 | AI | \( \pi_0(R_0) = \mathbb{Z} \) | |
| 1 | BDI | \( \pi_0(R_1) = \mathbb{Z}_2 \) | \( \gamma_{d+1} \) |
| 2 | D | \( \pi_0(R_2) = \mathbb{Z}_2 \) | \( \gamma_{d+1, \gamma_{d+2}} \) |
| 3 | DIII | \( \pi_0(R_3) = 0 \) | \( \gamma_{d+1, \gamma_{d+2, \gamma_{d+3}} \gamma_{d+4}} \) |
| 4 | AII | \( \pi_0(R_4) = 2\mathbb{Z} \) | |
| 5 | CII | \( \pi_0(R_5) = 0 \) | \( \tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3 \) |
| 6 | C | \( \pi_0(R_6) = 0 \) | \( \tilde{\gamma}_1, \tilde{\gamma}_2 \) |
| 7 | CI | \( \pi_0(R_7) = 0 \) | \( \tilde{\gamma}_1 \) |
interpreted as \((p - 1)\)-dimensional \(H^{\text{TI}}_{\text{Dirac}}\), while
\[
(1 - p + \sum_{i=1}^{p} \cos k_{ii}) \gamma_{0} \quad (A8)
\]
is treated as the mass term \(m\gamma_{0}\). Hence, the topology of Fermi surface systems with \(p\) dimension corresponds to \((p - 1)\)-dimensional TI and SC in the same symmetry class. However, the extra mass term \(\gamma_{1}\) is not the only SPGT. An extra kinetic term \(\sin k_{iy} \gamma_{y}\) is able to gap the Fermi surfaces at the locations shown in Eq. (A7). For non-trivial topology, according to Table [IV] this kinetic term is allowed to be present by symmetries in \(Z_{2}\) systems, whereas this term is symmetry forbidden in \(Z\) systems. In the presence of \(\sin k_{iy} \gamma_{y}\), the Fermi surfaces are unstable; nevertheless, a \(Z_{2}\) invariant is well-defined in \((p - 1)\)-dimensional Brillouin zone manifold that is invariant under TRS or PHS. This \(Z_{2}\) invariant still leads to protected gapless surface states.

1. **Topological invariant “0”**

For a given set of symmetries and spatial dimension, we write down a Dirac Hamiltonian of the minimal matrix dimension, which is in the form of Eq. (A7). If an SPGT \(\mathcal{M}\), which gaps out the gapless modes, is present in the Hamiltonian, the system is always in the trivial phase; we can classify this phase as topological invariant “0”.

For example, consider a one-dimensional Dirac Hamiltonian in class D. A Dirac Hamiltonian in the form of the minimal matrix dimension reads
\[
H_{a}^{D} = k_{x} \sigma_{x} + k_{y} \sigma_{y}. \quad (A9)
\]
PHS is preserved with PHS operator \(C = \sigma_{z} K\). An extra mass term \(m\sigma_{z}\), which preserves TRS, plays a SPGT role \(\mathcal{M}\) so the nodal point at \(k = 0\) is gapped.

For the other two cases (\(Z_{2}\) and \(Z\)) because the gapless modes are protected, any extra symmetry-allowed mass term does not exist in the minimal model. To distinguish \(Z_{2}\) and \(Z\), we need to enlarge the Hamiltonian and then check the presence of a SPGT.

2. **Topological invariant “\(Z_{2}\)”**

While enlarging the Dirac Hamiltonian, we consider in the new system the merge of two minimal Dirac Hamiltonians, which may have the same or opposite orientations. That is, one is given by Eq. (A7) and the other is in the form of Eq. (A7) with some \(\gamma_{i} \rightarrow -\gamma_{i}\). Moreover, each new merging gamma matrix in the enlarged Hamiltonian must anticommute with each other and keep the original symmetries. The expression of the enlarged Hamiltonian of the two minimal Dirac Hamiltonians can be written as
\[
\mathcal{H}_{2} = \sum_{i} k_{n_{i}} \gamma_{n_{i}} \otimes \sigma_{z} + \sum_{\text{remain}} k_{n_{j}} \gamma_{n_{j}} \otimes \mathbb{1}. \quad (A10)
\]
The orientation of the second minimal Dirac Hamiltonian is determined by \(\sigma_{z}\). The first summation is over arbitrary set of \(\gamma_{n_{i}}\) \(\left(n_{i} = 1, 2, ..., d - 1, \text{ or } d\right)\) and the second summation is over \(\gamma_{n_{j}}\)’s that are not picked up by the first summation. For the system with a \(Z_{2}\) topological invariant, an SPGT can always be added to the enlarged Hamiltonian in Eq. (A10) so the system is in the trivial phase. The SPGT can be constructed by considering even and odd numbers of terms in the first summation separately. For the odd number, the SPGT is given by \(\mathcal{M} = m(i) \prod_{d} \gamma_{n_{i}} \otimes \sigma_{z}\) and for the even number, the SPGT is given by \(\mathcal{M} = m(i) \prod_{d} \gamma_{n_{i}} \otimes \sigma_{u}\). The presence or absence of \(i\) keeps \(\mathcal{M}\) being Hermitian and choosing \(u = x, y\) lets \(\mathcal{M}\) preserve TRS and PHS. The reason is that according to Table [IV] an extra kinetic term \(\gamma_{d+1}\) for \(Z_{2}\) system exists in the minimal Dirac Hamiltonians. Therefore, the SPGT is always present in any type of the first summation in \(\mathcal{H}_{2}\).

We provide an example to explain \(Z_{2}\) characteristics for Dirac Hamiltonians. Consider the low-energy Hamiltonian of a 2D semimetal
\[
\begin{align*}
\mathcal{H}_{a}^{\text{AII}} &= k_{x} \sigma_{x} + k_{y} \sigma_{y}, \quad (A11) \\
\text{which is identical to the surface Hamiltonian in a 3D strong topological insulator. The Hamiltonian } \\
\mathcal{H}_{a}^{\text{AII}} &\text{ in class AII preserves TRS } T = i\sigma_{y} K. \text{ In this case, only possible extra mass term, which anticommutes with } \sigma_{x} \text{ and } \sigma_{y}, \text{ is } \sigma_{z}. \text{ However, this term, which breaks TRS, is not allowed to be added to the Hamiltonian so SPEMTs are absent. Therefore, the gapless mode is protected.}
\end{align*}
\]
Since this system is classified as \(Z_{2}\), a new system that is constructed by the two gapless Hamiltonians can be gapped. The Hamiltonian for the merging system is in form of
\[
\begin{align*}
\mathcal{H}_{a}^{\text{AII}} &= \begin{pmatrix}
\mathcal{H}_{a}^{\text{AII}} & 0 \\
0 & \mathcal{H}_{a}^{\text{AII}}
\end{pmatrix}, \quad (A12)
\end{align*}
\]
where \(\mathcal{H}_{a}^{\text{AII}}' = \pm k_{x} \sigma_{x} \pm k_{y} \sigma_{y}\) or \(\mathcal{H}_{a}^{\text{AII}}' = \pm k_{x} \sigma_{x} \mp k_{y} \sigma_{y}\). The new Hamiltonian might be in four possible forms, which are determined by the signs. It is not difficult to show for each form at least one SPEMT can be present in the new Hamiltonian. For example, for \(\mathcal{H}_{a}^{\text{AII}}'\) the SPGTs can be \(\sigma_{z} \otimes \sigma_{y}\). Thus, the gapless mode is unstable.

3. **Topological invariant “\(Z(2Z)\)”**

For the system with a \(Z\) (or \(2Z\)) topological invariant, the gapless mode of the Hamiltonian \(H_{s}^{D}\) in the minimal matrix dimension is protected. The Hamiltonian has to be enlarged to Eq. (A10) to study topology. When the first summation in Eq. (A10) includes odd number of \(\gamma_{n_{i}}\)’s, the presence of a SPGT \(\mathcal{M} = m(i) \prod_{d} \gamma_{n_{i}} \otimes \sigma_{u}\) opens gaps. However, when there are even number of \(\gamma_{n_{i}}\)’s in the first summation, an SPGT does not exist due to the absence of an extra kinematic term. On the other hand, the orientations of the minimal Hamiltonian are distinguished by the number of the gamma matrices that have a minus sign in front. The two gapless
modes only with the same orientations are protected. Similarly, when the system is extended to \( n \) gapless modes with the same orientations, in the absence of an SPGT the gapless modes are protected. This behavior reveals the signature of the \( Z \) invariant.

To explain \( Z \) invariant, we consider the Hamiltonian of Weyl semimetals [29] as an example. This two-dimensional system, which does not preserve any symmetry, belongs to class A. One of the simplest Hamiltonians, which is also a system, which does not preserve any symmetry, belongs to Weyl semimetal [54]. The same orientations, in the absence of an SPGT the gapless system with \( Z \) is given by

\[
h_A^s = k_x \sigma_x + k_y \sigma_y + k_z \sigma_z. \tag{A13}\]

It is impossible to find an extra gap term because only three gamma matrices can be present in the \( 2 \times 2 \) matrix dimension. Therefore, the gapless mode is stable. This is similar with a system with \( Z \) invariant. To distinguish \( Z \) and \( Z_2 \), introducing two copies of \( h_A^s \) is necessary.

First, the twice-as-big Hamiltonian with two identical \( h_A^s \)'s is given by

\[
H_A^s = k_x \sigma_x \otimes 1 + k_y \sigma_y \otimes 1 + k_z \sigma_z \otimes 1. \tag{A14}\]

An extra mass term is still absent so the two identical gapless modes are stable. Secondly, we change the sign of one of the Pauli matrices in the second \( h_A \). The Hamiltonian is written as

\[
H_A^{s'} = k_x \sigma_x \otimes \sigma_z + k_y \sigma_y \otimes 1 + k_z \sigma_z \otimes 1. \tag{A15}\]

Two extra mass terms (\( \sigma_x \otimes \sigma_z \) and \( \sigma_z \otimes \sigma_z \)), which are also SPGTs, can be found. Hence, the gapless modes are gapped.

### Appendix B: Derivation of classification scheme of Fermi points off reflection planes

Consider reflection symmetry at the \( k \) direction. The Dirac Hamiltonian that describes the gapless modes off the reflection planes is similar with the Hamiltonian in Eq. \([28]\) that describes Fermi surfaces off high-symmetry but in reflection planes. We simply let \( d_{G\Sigma} = 0 \) and replace \( \sin k_1 \gamma_1 \) by \( \sin k_d \gamma_d \) in Eq. \([28]\). This Dirac Hamiltonian reads

\[
H_{\text{off}} = \sum_{i=2}^{d} \sin k_i \gamma_i + (d - 1 + \sum_{i=1}^{d} \cos k_i) \gamma_0, \tag{B1}\]

so the Fermi points are located at \( k = (\pm \pi / 2, 0, \ldots) \). In the absence of \( \sin k_1 \gamma_1 \) we require \([R, H] = 0\) to preserve reflection symmetry. This Hamiltonian can be realized as \((d - 1)\)-dimensional insulator system while the last term is treated as a mass term. Such systems are different from the original ten-fold classification in which only \( m \gamma_j \) is considered as an SPGT that brings trivial topology. In the classification of these gapless modes, two types of SPGTs, which can gap out the gapless modes and implies trivial topology, are given by

\[
m \gamma_1, \quad \sin k_1 \gamma_1. \tag{B2}\]

The latter as a kinetic term changes the original pattern of the ten-fold classification.

| \( s - d \) | \( 0 \) | \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) | \( 5 \) | \( 6 \) | \( 7 \) |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| \( G^c_{\text{off}}(R^+, s - d) \) | \( CZ_2 \) | \( CZ_2 \) | \( 0 \) | \( 2 \) | \( 0 \) | \( 0 \) | \( Z \) |
| \( G^c_{\text{off}}(R^-, s - d) \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 2 \) | \( 0 \) | \( 2 \) |
| \( G^c_{\text{off}}(R^{\pm \mp}, s - d) \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) |

TABLE V. The classification of Fermi points outside reflection planes. The prefix “\( C \)” indicates the \( Z_2 \) invariant only can be defined by combined symmetries from topological invariant method in Sec. IV C. Nevertheless, Dirac Hamiltonian only can show \( Z_2 \) properties. \( R^{\pm \mp} \) represents \( R_+ \) in class BD1 and CI for and \( R_{++} \) in class CI and DIII. Similarly, \( R^{\pm} \) represents \( R_{++} \) in class BD1 and CI and \( R_{++} \) in class CI and DIII.

1. \( R_+ \) and \( R_{++} \)

We simply pick up \( R^+ = 1 \) presenting \( R_+ \) and \( R_{++} \), which commute with the Hamiltonian and all of the global symmetry operators. We note that even if the global symmetries allow the kinetic mass term \( \sin k_1 \gamma_1 \), the reflection symmetry forbids this term due to \( k_1 \rightarrow -k_1 \). Therefore, we discuss the presence or absence of the regular mass term \( \gamma_1 \) for topology of \( R^+ \) reflection systems. Thus, the classification of \( d \)-dimensional gapless modes is identical to \((d - 1)\)-dimensional original ten-fold TI and SC classification \([G^c_{\text{off}}(R^+, s, d) = \pi_0(C_{s-d+1})]\).

2. \( R_- \)

For the anticommutation case, three possible ways to construct the anticommuting reflection symmetry operator (\( R^- \)) are given by \( R^- = i \gamma_{d+1} \gamma_{d+2}, \quad R^- = i \gamma_1 \gamma_2 \), or \( R^- = 1 \otimes \sigma_y \). The homotopy group \( \pi_0(R_c) \), where \( l = s - d + 1 \) mod 8, describes the topology of the gapless Dirac Hamiltonian Eq. \([B1]\). Each symmetry class, which exhibits distinct topology, has to be discussed separately. Without enlarging the gapless Hamiltonian, \( R_{--} \) can be properly defined in four symmetry classes based on Table IV

\[
l = 2, 3, \quad R^- = i \gamma_{d+1} \gamma_{d+2}, \tag{B3}\]

\[
l = 5, 6, \quad R^- = i \gamma_1 \gamma_2. \tag{B4}\]

As \( l = 2, 3 \), \( \sin k_1 \gamma_{d+1} \), which preserves the symmetries and opens gaps, implies trivial topology. Similarly, as \( l = 5 \), the presence of the symmetry-allowed gapped term \( \sin k_1 \gamma_2 \) also implies trivial topology. As \( l = 6 \), the gapless modes are stable in the absence of SPGTs. To distinguish \( Z \) and \( Z_2 \) topology, consider the Hamiltonian with two identical gapless modes

\[
H_{\text{off}} = H_{\text{off}} \otimes 1. \tag{B5}\]

The gapless modes are unprotected in the presence of a SPGT \( \sin k_1 \gamma_1 \otimes \sigma_y \). Therefore, as \( l = 6 \), symmetry class is classified as \( Z_2 \). As \( l = 1, 7 \), a single extra kinetic or mass term cannot construct the proper reflection symmetry operator (\( R^- \)). The Hamiltonian has to be double sized in the form
of Eq. (B5) so that \( R^- = 1 \otimes \sigma_y \). For \( l = 1, 7 \), SPGTs can be found as \( m\gamma_1 \otimes I \) and \( m\gamma_{d+1} \otimes \sigma_y \) respectively. Similarly, as \( l = 0 \), the twice-as-big Hamiltonian \( H''_{\text{off}} \) allows the construction of \( R^- = 1 \otimes \sigma_y \). Gapped terms are forbidden by the symmetries so the size of the Hamiltonian \( H''_{\text{off}} \) is doubled again

\[
H''_{\text{off}} = H_{\text{off}} \otimes 1 \otimes 1. \tag{B6}
\]

SPGTs are still absent. Therefore, the system of \( l = 0 \) inherits \( \mathbb{Z} \) topology and is classified as \( 2\mathbb{Z} \) due to the doubled size of the minimal Hamiltonian \( (H_{\text{off}}) \). As \( l = 4 \), the system, which corresponds to \( 2\mathbb{Z} \), can be effectively treated as two identical copies of the \( \mathbb{Z} \) system in the spatial dimensions

\[
H''_{\text{off}} = H_{\text{off}} \otimes 1. \tag{B7}
\]

The relations of the global symmetry operators between \( \mathbb{Z} \) and \( 2\mathbb{Z} \) are given by \( T_{2Z} = T_Z \otimes \sigma_y \) and \( C_{2Z} = C_Z \otimes \sigma_y \). Therefore, we can simply define \( R^- = 1 \otimes \sigma_y \), which anticommutes with \( T_{2Z} \) and \( C_{2Z} \). Following the similar discussion of \( l = 0 \), we find the system of \( l = 4 \) inherits \( \mathbb{Z} \) topology.

3. AIII with \( R^- \), DIII & CI with \( R_{+-} \), and BDI & CI with \( R_{+-} \)

Considering symmetry classes possess chiral symmetry with the symmetry operator \( S \), we construct reflection symmetry operator \( R^{\mp} = i\gamma_{d+1}S \) by introducing another kinetic term \( \gamma_{d+1} \). The commutation and anticommutation relations between \( R^{\mp} \) and the global symmetry classes are different in the different symmetry classes, which are given by

\[
R_- \text{ for class AIII,} \tag{B8}
\]

\[
R_{+-} \text{ for class BDI and CI,} \tag{B9}
\]

The commutation and anticommutation relations can be verified as follows. Let us go back to the expressions of TRS and PHS operators

\[
T = U_T K, \quad C = U_C K, \tag{B10}
\]

where \( U_T \) and \( U_C \) are complex matrices. To simplify our problem, we assume that \( U_T \) and \( U_C \) are Hermitian and unitary. To define chiral operator \( S \), which is Hermitian, we let \( S = TC \) if \( \{U_C, U_T\} = 0 \) or \( S = iTC \) if \( \{U_C, U_T\} = 0 \). Therefore, \( R^{\mp} = i\gamma_{d+1}S \) is Hermitian. To determine the commutation and anticommutation relations of \( R^{\mp} \) with \( T \) and \( C \), we have to check the relations of \( S \) with \( T \) and \( C \). In the both cases \( \{U_C, U_T\} = 0 \) and \( \{U_C, U_T\} = 0 \), we have the same relations

\[
TST^{-1} = \pm S, \quad CSC^{-1} = \pm S, \tag{B11}
\]

where we pick up the plus sign in front of \( S \) when \( T^2 = \pm 1 \) and \( C^2 = \pm 1 \), whereas we pick up the minus sign when \( T^2 = \pm 1 \) and \( C^2 = \mp 1 \). The reason is that

\[
T^2 = U_T U_T^* = \pm 1, \quad C^2 = U_C U_C^* = \pm 1. \tag{B12}
\]

By using Hermitian and unitary properties of \( U_T \) and \( U_C \),

\[
U_T = \pm U_T^*, \quad U_C = \pm U_C^*. \tag{B13}
\]

By using Eq. (A1), which describes the relations between the kinetic term \( \gamma_{d+1} \) and the global symmetry operator, we obtain the commutation and anticommutation relations of \( R^{\pm} = i\gamma_{d+1}S \) exactly shown in Eqs. (B8) and (B9):

\[
\{T, R^{\pm}\} = 0 \quad \text{and} \quad \{C, R^{\pm}\} = 0 \text{ when } T^2 = C^2 = \pm 1 \quad \text{and} \quad \{T, R^{\mp}\} = 0 \quad \text{and} \quad \{C, R^{\mp}\} = 0 \text{ when } T^2 = C^2 = \mp 1.
\]

The kinetic gapped term \( \sin k_1 \gamma_{d+1} \) is allowed by not only the global symmetries but also reflection symmetry. Hence, such reflection systems are always classified as trivial topology. We label these classification types of the symmetry classes, which obey the anticommutation and commutation relations of reflection symmetry operator, as \( G_0^R(R^{\mp}, s - d) = 0 \).

4. DIII & CI with \( R_{+-} \) and BDI & CI with \( R_{+-} \)

Similarly, reflection symmetry operator can be constructed in the form of \( R^{\mp} = i\gamma_{d+1}S \) to satisfy \( \{T, R^{\mp}\} = 0 \) and \( \{C, R^{\mp}\} = 0 \) when \( T^2 = C^2 = \pm 1 \) and \( \{T, R^{\pm}\} = 0 \) and \( \{C, R^{\pm}\} = 0 \) when \( T^2 = C^2 = \mp 1 \). Topology of \( R^{\mp} \) for different symmetry classes \((s)\) and dimensions \((d)\) has to be discussed by considering different \( R_{\alpha} \), where \( l = s - d + 1 \).

As \( l = 5, 6 \), without enlarging the minimal Hamiltonian, according to Table [IV] at least two mass terms \( \gamma_1, \gamma_2 \) preserve system global symmetries. The reflection symmetry operator can be defined by \( R^{\mp} = i\gamma_1 S \). The other mass term \( \gamma_2 \), which preserves all of the system symmetries, gaps the Fermi points. Hence, the topology is classified as “0”.

As \( l = 4 \), three kinetic terms \( \gamma_{d+1}, \gamma_{d+2}, \gamma_{d+3} \) satisfying Eq. (A1) are present in the minimal Hamiltonian. These three kinetic terms form a mass term \( \gamma_{d+1}\gamma_{d+2}\gamma_{d+3} \), which preserves global symmetries. The reflection symmetry is given by \( R^{\mp} = i\gamma_{d+1}\gamma_{d+2}\gamma_{d+3} S \). The kinetic term \( \sin k_1 \gamma_{d+1} \), which also preserves reflection symmetry, is allowed to be added in Hamiltonian (B1) as a SPGT. It is the trivial phase.

As \( l = 2, 3 \), for the minimal Hamiltonian in the absence of the mass term \( \gamma_1 \) the reflection symmetry operator \( R^{\mp} \) cannot be constructed. To seek \( R^{\mp} \), the Hamiltonian has to be enlarged to the two identical copies of \( H_{\text{off}} \). Because a kinetic term \( \gamma_{d+1} \) is present in the minimal Hamiltonian, a mass term in the twice-as-big Hamiltonian \( (H_{\text{off}} \otimes I) \) can be defined as \( \gamma_1 = \gamma_{d+1} \otimes \sigma_y \). Therefore, the reflection symmetry operator is given by \( R^{\mp} = i\gamma_{d+1}\gamma_{d+2} \otimes \sigma_y \). It is easy to find a mass term \( \gamma_{d+1} \otimes \sigma_x \) that gaps the Fermi points and preserves all of the symmetries. It is classified as “0”.

As \( l = 7, 8 \), only one mass term \( \gamma_1 \) is allowed by the global symmetries so it is possible to construct the reflection symmetry operator \( R^{\mp} = i\gamma_1 S \). The reflection symmetry forbids \( \gamma_1 \), which is the only term that gaps the Fermi points. Although the Fermi points are stable in the minimal Hamiltonian, to distinguish \( \mathbb{Z}_2 \) and \( \mathbb{Z} \) we have to merge the two minimal Hamiltonians. For \( H_{\text{off}} \otimes I \), an extended mass term \( \sin k_1 \gamma_1 \otimes \sigma_y \) preserves global symmetries and reflection
symmetry with \( R^{±} = i\gamma_1 S \otimes 1 \). It exhibits \( \mathbb{Z}_2 \) characteristics.

As \( l = 0 \), to construct \( R^{±} \) in absence of mass and kinetic terms the minimal Hamiltonian in Eq. (B1) has to be enlarged to

\[
H'_{\text{off}} = \sum_{i=2}^{d} \sin k_i \gamma_i \otimes 1 + (d - 1 + \sum_{i=1}^{d} \cos k_i) \tilde{\gamma}_0 \otimes \sigma_z. \tag{B14}
\]

Although the minimal Hamiltonian can be enlarged in various ways, the enlarged Hamiltonians that have well-defined \( R^{±} \) are equivalent under unitary transformation. Only one mass (\( \gamma_0 \otimes \sigma_x \)) and one kinetic (\( \gamma_0 \otimes \sigma_y \)) terms that preserve global symmetries can be found. The former leads to the reflection symmetry operator \( R^{±} = i\gamma_0 S \otimes \sigma_x \). Both of the terms (\( m\gamma_0 \otimes \sigma_x \) and \( k_1 \gamma_0 \otimes \sigma_y \)), which are able to gap the Fermi points, break the reflection symmetry. The Fermi points are protected by symmetries. To distinguish \( \mathbb{Z}_2 \) and \( \mathbb{Z} \), the Hamiltonian has to be doubled \( H'_{\text{off}} \otimes 1 \). Without changing any symmetry operator, \( m\gamma_0 \otimes \sigma_y \otimes \sigma_y \), which destabilizes the Fermi points, is allowed by all of the symmetries. Thus, it is a \( \mathbb{Z}_2 \) system.

The minimal Hamiltonian of \( l = 4 \) can be written by the minimal Hamiltonian of \( l = 0 \) as

\[
H^{l=4}_{\text{off}} = H^{l=0}_{\text{off}} \otimes 1. \tag{B15}
\]

The global symmetry operators are given by

\[
T^{l=4} = T^{l=0} \otimes \sigma_y, \quad C^{l=4} = C^{l=0} \otimes \sigma_y. \tag{B16}
\]

As \( l = 4 \), mass and kinetic terms are absent, which is similar to \( l = 0 \). The enlarged Hamiltonian is given by \( H^{l=4}_{\text{off}} = H^{l=0}_{\text{off}} \otimes 1 \) from Eq. (B15). The reflection symmetry operator can be built as \( R^{±} = i\gamma_0 S \otimes \sigma_x \otimes 1_{2 \times 2} \), where \( \gamma_0 \) and \( S \) are the mass term and the chiral symmetry operator of \( H^{l=0}_{\text{off}} \), respectively. A mass term \( m\gamma_0 \otimes \sigma_y \otimes \sigma_y \), which preserve all symmetries, gaps the Fermi points. It is classified as “0”.

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1. M. König, H. Buhmann, L. W. Molenkamp, T. Hughes, C.-X. Liu, X.-L. Qi, and S.-C. Zhang, J. Phys. Soc. Japan 77, 031007 (2008).
2. M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010).
3. M. Z. Hasan and J. E. Moore, Annu. Rev. Condens. Matter Phys. 2, 55 (2011).
4. X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. 83, 1057 (2011).
5. B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, Science 314, 1757 (2006).
6. A. Kitaev, AIP Conf. Proc. 1134 (2009).
7. A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, AIP Conf. Proc. 1134, 22 (2009).
8. S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. W. Ludwig, New J. Phys. 12, 065010 (2010).
9. A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Phys. Rev. B 78, 195125 (2008).
10. J. C. Y. Teo, L. Fu, and C. L. Kane, Phys. Rev. B 78, 045426 (2008).
11. L. Fu, Phys. Rev. Lett. 106, 106802 (2011).
12. C.-K. Chiu, H. Yao, and S. Ryu, Phys. Rev. B 88, 075142 (2013).
13. T. Morimoto and A. Furusaki, Phys. Rev. B 88, 125129 (2013).
14. R.-J. Slager, A. Mesaros, V. Juricic, and J. Zaanen, Nat. Phys. 9, 98 (2013).
15. Y. Ueno, A. Yamakage, Y. Tanaka, and M. Sato, Phys. Rev. Lett. 111, 087002 (2013).
16. F. Zhang, C.-L. Kane, and E. J. Mele, Phys. Rev. Lett. 111, 056403 (2013).
17. W. A. Benalcazar, J. C. Y. Teo, and T. L. Hughes, Phys. Rev. B 89, 224503 (2014).
18. C. Fang, M. J. Gilbert, and B. A. Bernevig, Phys. Rev. B 86, 115112 (2012).
19. C. Fang, M. J. Gilbert, and B. A. Bernevig, Phys. Rev. B 87, 035119 (2013).
20. P. Jadaun, D. Xiao, Q. Niu, and S. K. Banerjee, Phys. Rev. B 88, 085110 (2013).
21. J. C. Y. Teo and T. L. Hughes, Phys. Rev. Lett. 111, 047006 (2013).
22. A. M. Turner, Y. Zhang, R. S. K. Mong, and A. Vishwanath, Phys. Rev. B 85, 165120 (2012).
23. A. M. Turner, Y. Zhang, and A. Vishwanath, Phys. Rev. B 82, 241102 (2010).
24. T. L. Hughes, E. Prodan, and B. A. Bernevig, Phys. Rev. B 83, 245132 (2011).
25. Y.-M. Lu and D.-H. Lee, arXiv:1403.5558 (2014).
26. K. Shiozaki and M. Sato, arXiv:1403.3331 (2014).
27. M. Koshino, T. Morimoto, and M. Sato, arXiv:1406.3094 (2014).
28. Y. Tanaka, Z. Ren, T. Sato, K. Nakayama, S. Souma, T. Takahashi, K. Segawa, and Y. Ando, Nat Phys 8, 800 (2012).
29. T. H. Hsieh, H. Lin, J. Liu, W. Duan, A. Bansil, and L. Fu, Nat. Commun. 3, 982 (2012).
30. S.-Y. Xu, C. Liu, N. Alidoust, M. Neupane, D. Qian, I. Belopolski, J. D. Denlinger, Y. J. Wang, H. Lin, L. A. Wray, et al., Nat. Commun. 3, 1192 (2012).
31. P. Dziawa et al., Nat. Mater. 11, 1023 (2012).
32. S. Ryu and Y. Hatsugai, Phys. Rev. Lett. 89, 077002 (2002).
33. S. Matsura, P.-Y. Chang, A. P. Schnyder, and S. Ryu, New J. Phys. 15, 065001 (2013).
34. Y. X. Zhao and Z. D. Wang, Phys. Rev. Lett. 110, 240404 (2013).
35. Y. X. Zhao and Z. D. Wang, Phys. Rev. B 89, 075114 (2014).
36. P. Holzmann, Phys. Rev. Lett. 95, 016405 (2005).
37. B. Béni, Phys. Rev. B 81, 134515 (2010).
38. G. E. Volovik, *Universe in a helium droplet* (Oxford University Press, 2003), ISBN 0521670535.
39. G. E. Volovik, *Topology of quantum vacuum*, vol. 870 of *Lecture Notes in Physics* (Springer Berlin, 2013).
40. J. L. Mañes, Phys. Rev. B 85, 155118 (2012).
41. A. Lau and C. Timm, Phys. Rev. B 88, 165402 (2013).
42. Y. Ran, F. Wang, H. Zhai, A. Vishwanath, and D.-H. Lee, Phys. Rev. B 79, 014505 (2009).
43. J.-M. Hou, Phys. Rev. Lett. 111, 130403 (2013).
44. S. Kobayashi, K. Shiozaki, Y. Tanaka, and M. Sato, Phys. Rev. B 90, 024516 (2014).
45. A. A. Burkov, M. D. Hook, and L. Balents, Phys. Rev. B 84, 235126 (2011).
46. A. A. Burkov and L. Balents, Phys. Rev. Lett. 107, 127205 (2011).
Note that the Chern or winding number of the Hamiltonian $H(k)|_{k_i=0,\pi}$ (i.e., both blocks together) is vanishing.

Note that the edge Hamiltonian $h_{\text{edge}}$, and the symmetry operators $T_{\text{edge}}, C_{\text{edge}},$ and $R_{\text{edge}}$ are only defined up to a similarity transformation, since one is free to choose a basis among the degenerate states of the Hamiltonian. The topological properties, however, are independent of this basis choice.

The distinction between $Z$ and $2Z$ classification follows from the Dirac-matrix Hamiltonian approach.

Note that this model is similar to the BHZ model describing the quantum spin Hall effect in HgTe quantum wells.

Due to this condition the “stronger” of the two invariants reduces to the “weaker” one.

The edge Hamiltonian $H_{\text{edge}}$ only depends on the mirror plane, and the symmetry operators $T_{\text{edge}}, C_{\text{edge}},$ and $R_{\text{edge}}$ are only defined up to a similarity transformation, since one is free to choose a basis among the degenerate states of the Hamiltonian. The topological properties, however, are independent of this basis choice.

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