Phase Transitions in Parallel Replication Process

P. N. Timonin

Physics Research Institute at Southern Federal University, 344090 Rostov-on-Don, Russia

G. Y. Chitov

Department of Physics, Laurentian University, Sudbury, Ontario P3E 2C6, Canada

(Dated: April 20, 2008)

The one-dimensional kinetic contact process with parallel update is introduced and studied by Monte Carlo simulations. This process is proposed to describe the plant population replication and epidemic disease spreading among them. The phase diagram of the model features the line of the second order transitions between absorbing and active phases. The numerical results for the critical index $\beta$ demonstrate its continuous variation along the transition line accompanied by the variations of the structural characteristics of limiting steady states. We conjecture the non-universality of the critical behavior of the model.

PACS numbers: 05.20.Dd, 64.60.De, 64.60.F-

There has been a steadily growing interest during the last two decades or so in the kinetics and phase transitions in non-equilibrium systems. The applications of those systems range from physics, like, e.g., statistical physics, critical phenomena, condensed matter to less conventional fields, like biology, ecology or quantitative finance. [1, 2, 3] The essentially non-equilibrium feature of these systems is due to their possibility to irreversibly enter an absorbing (dead, empty) state. The mere existence of such a state violates the detailed balance. The central problem in studies of non-equilibrium systems is the transitions they undergo between various active phases and the inactive (absorbing) state.

The kinetic contact processes which model, e.g., the epidemic disease spreading or population replication, are the simplest kinetic models exhibiting such non-equilibrium phase transitions under variation of their parameters. [1, 2, 3] These variations consist in the change of limiting probability distribution for the site occupation numbers in replication process, or the infected site numbers for epidemic models. Mostly these models possess two phases: the absorbing state with the population or viruses extinct and the active phase where some sites are populated or infected.

Usually the sequential update formalism is used to study these models. It is based on the differential kinetic equation for the probability distribution function. In this framework the generic second order transition into absorbing state of the directed percolation (DP) universality class is established for the majority of such contact processes. [1, 2, 3] Another approach is the parallel update scheme when the model is represented as a discrete-time probabilistic cellular automaton (PCA). [4, 5, 6] It is known that these two approaches can give rather different results [1] and one may reasonably suppose that in such case parallel update gives more adequate description of real kinetics just because in Nature there is no queue for the fulfillment of the state transformation processes. So the studies of PCA implementation of replication (epidemic) processes may give a more realistic picture of them and reveal some new types of critical behavior.

In this paper we consider a one-dimensional PCA representing the contact process of population replication or disease spreading. Imagine a line of plants which maintain population by spreading the seeds to the nearby sites so the new plants can grow on them if these sites are empty. Let $q$ be the probability of such event for empty site having only one neighboring plant. For the case of two plants around the empty one we can choose this probability $r > q$ to be $r = 1 - (1 - q)^2 = q(2 - q)$ assuming the independence of the two plants’ seeds “not spreading effects”. Some other choices are also possible but here we consider only the case $r = q(2 - q)$. Let also $p$ be the probability for plant to survive in the one time step. Thus we have the PCA on the line of two-state sites (empty and full $\rightarrow S_i = 0, 1$ resp., where $i$ is the intrachain site index) with the evolution step probabilities defined for the local three-site configurations in Table I.

| $S_{i-1}, S_i, S_{i+1}$ | 111, 110, 011, 010 | 100, 001 | 101, 000 |
|-------------------------|--------------------|---------|---------|
| Probability ($S_i = 1$) | $p$                | $q$     | $q(2 - q)$ |

Obviously, this model describes also the epidemic process among the plants, since instead of the plant seeds we can consider the virus spreading. Note also that this process differs essentially from the one of the cell replication considered in Ref. [3].

We have performed Monte Carlo simulations of this PCA starting from several random initial states for the
chains of \( N = 10000 \) sites for up to 5000 time steps. The periodic boundary conditions were implemented. The resulting \( p-q \) diagram (Fig. 1) consists of a single transition line between absorbing (all occupation numbers \( S_i = 0 \) in the infinite-time limit) and active (some \( S_i \neq 0 \)) phases. The curve has the following approximate analytical representation \( q = 0.66 - 0.913(p - 0.15)^2 \), determined from a direct fit.

![FIG. 1: (Color online) Phase diagram.](image)

We assume the second order transition between the phases, since the concentration of active sites defined as

\[
n_1 \equiv \frac{1}{N} \sum_{i=1}^{N} S_i
\]

appears to be continuous. The examples of such behavior are presented in Fig. 2 where the (continuous) \( q \)-dependence of \( n_1 \) is shown for \( p = 0.2, 0.5, 0.8 \). We found that the points obtained in simulations follow closely the power law \( n_1 \propto (q - q_c)^\beta \) with the indices \( \beta \) given in Table II.

| \( p \) | 0.2 | 0.5 | 0.8 |
|-------|-----|-----|-----|
| \( q_c \) | 0.645 | 0.535 | 0.275 |
| \( \beta \) | 0.332 | 0.286 | 0.254 |

![FIG. 2: (Color online) \( q \)-dependence of \( n_1 \) for \( p = 0.2 \) (×), 0.5 (○), 0.8 (□). Solid lines show power law dependencies with the indices given in Table II.](image)

One can notice the unusual variation the index \( \beta \) with \( q \) instead of its more conventional universality. The index varies closely to the directed percolation value \( \beta_{DP} = 0.276 \). A naive explanation of these deviations could have been as an artefact of low data precision, while the PCA studied belongs actually to the DP class. Indeed, there are small variations of \( n_1 \) for different trials (and initial states) as well as its fluctuations in the nominally steady state at large times. Not too close to the transition point (\( |q - q_c| > 0.05 \)) these variations of \( n_1 \) are of the order of several percents and could not possibly account for the differences in \( \beta \) of the order of ten percents.

Therefore we are lead to the assumption that this replication process does not belong to the DP universality class, and it is characterized by the variable critical indices. To date the signs of the non-universal behavior are found only in the 1d pair contact process with diffusion [2] but the type of its critical anomalies is still debated; see Ref. [8] and references therein. The non-universality of some equilibrium spin models (i.e., the coupling-dependent critical exponents) is well known. For example, it is established for the classical XY model [3], the eight-vertex model solved by Baxter [10], or for several two-dimensional Ising models with competing nearest-neighbor (nn) and next-nearest-neighbor (nnn) interactions, see Refs. [11, 12, 13, 14] and more references therein. More examples and discussions on the equivalence between the two-dimensional classical spin models (Ising, Potts, Ashkin-Teller, XY) and \((1+1)\) quantum models (quantum spin chains, Luttinger, Gaussian), their critical properties and (non)universality can be found in Refs. [15, 16].

From the renormalization group (RG) point of view the non-universality of the PCA implies that the RG flow for the equivalent equilibrium spin model would have a manifold of fixed points instead of a unique fixed point. A paradigmatic example of such behavior is the Kosterlitz-Thouless picture of RG flow for the XY model, which appears also in many \((1+1)\) quantum models. According to Kadanoff and Wegner, non-universality can be traced back to the presence of an RG marginal operator.

To corroborate our conjecture of non-universality we point out that the structure of the active state in this model undergoes considerable qualitative changes under
p and q variations at constant \( n_1 \). The space-time patterns of the process which has reached a steady state with \( n_1 \approx 0.5 \) are presented in Fig. 3 for different \( p \) and \( q \). The evolution of 100 sites chosen out of 10000 is shown there during 100 time steps in the steady state.

The pattern difference, quite distinct visually, can be assessed numerically. It appears that along with \( n_1 \), the concentration of clusters of adjacent active and inactive sites in the steady states

\[
n_c \equiv \frac{1}{N} \sum_{i=1}^{N} \delta(S_i + S_{i+1}, 1)
\]

stays almost constant with negligible fluctuations, see Fig. 4. In the above equation and throughout, we use \( \delta(m, n) \) as a notation for the conventional Kronecker delta.

This parameter, as well as \( n_1 \), is almost the same in the different trials. The values of \( n_c \) for three steady states in Fig. 3 are presented in Table III. The concentration of active adjacent pairs (11) in steady configurations \( n_{11} \) is also shown there with

\[
n_{11} \equiv \frac{1}{N} \sum_{i=1}^{N} \delta(S_i + S_{i+1}, 2)
\]

Note that an exact relation

\[
n_{11} = n_1 - n_c/2
\]
TABLE III: Parameters of three steady state configurations shown in Fig. 3.

| (p, q)  | (0.2, 0.89) | (0.5, 0.66) | (0.8, 0.32) |
|---------|-------------|-------------|-------------|
| n_1     | 0.499       | 0.506       | 0.501       |
| n_c     | 0.640       | 0.468       | 0.354       |
| n_{11}  | 0.179       | 0.272       | 0.324       |
| n_{111,1}| 0.016       | 0.072       | 0.179       |

holds. We have also calculated the average values of the supposed marginal operator

$$n_{111,1} = \frac{1}{N} \sum_{i=1}^{N} S_{i,t}S_{i+1,t}S_{i-1,t}S_{i,t+1}$$

This four-spin operator is present in the Hamiltonian of the corresponding equilibrium spin model and it can be responsible for the non-universality in (1+1) dimensions. The average n_{111,1} is also nearly constant in steady states (see Fig. 4), trial independent and varies with p and q, as shown in Table III. Its value defines the structural characteristics of emergent active state and, probably, their indices. Our data also show that n_{11} (and n_c) is proportional to n_1 near the transition points.

Summary & Discussion: The structural characteristics of steady states with nearly the same n_1 are considerably different, which is not only seen quite clearly from Fig. 5 but it manifests quantitatively via variations of numbers of clusters per site n_c [2] and the four-spin operator n_{111,1} [5]. Along with the proportionality of n_{11} (and n_c) to n_1 this suggests a multi-component order parameter. The latter violates one of the requirements of the Janssen-Grassberger hypothesis for a model to belong to the DP universality class [8]. We conjecture the non-universality in this model, most likely due to the marginal four-spin term of the Hamiltonian [cf. Eq. (5)], which manifests itself in variations of the critical index β.

However to confirm that this PCA does exhibit the non-universal critical behavior, the numerical values of other critical indices are needed. Although it is a rather difficult task for the present model to get them due to strong fluctuations which other quantities of interest such as correlation lengths exhibit near the transition point. Strong fluctuations also hinder determination of the index α = 2 - v⊥ - ν∥ [1] from the energy-like behavior of n_{111,1} ∼ (q - q_c)^{1-α}.

From the analytical side, the non-universality can be probed by analysis of the marginal perturbation around an integrable fixed point. For instance, the non-universality of the eight-vertex model or the Ising with competing nn and nnn couplings, can be demonstrated by the RG analysis of the flow generated by the nn or nnn interactions which couple two independent (integrable) Ising models. In the present case the RG analysis of non-universality, i.e., of the marginal perturbations, is more difficult problem, even since the (1+1) DP fixed point presumably controlling the universality, is not integrable. Note that the ε-expansions around the DP upper critical dimension d_c = 4 are not reliable for our case of d = 1. Clearly, an RG study of the field theory corresponding to our model is warranted. It can shed more light on the the critical properties of the model. In particular, the field theory formulation of the present PCA model could help to answer a natural question of how this PCA is distinct from others known to belong to the DP universality class. We plan to address all these issues in the forthcoming paper.

We acknowledge financial support from the Natural Science and Engineering Research Council of Canada (NSERC) and the Laurentian University Research Fund (LURF). We thank V. Oudovenko for useful discussions and his help with the numerical calculations on the earlier version of this project. P.N.T. thanks Laurentian University, where the initial stage of the work was done, for hospitality.

* Electronic address: timonin@aaanet.ru
† Electronic address: gchitov@laurentian.ca
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