Geodesic distance: A descriptor of geometry and correlator of pre-geometric density of spacetime events

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Abstract
Classical geometry can be described either in terms of a metric tensor $g_{ab}(x)$ or in terms of the geodesic distance $\sigma^2(x, x')$. Recent work, however, has shown that the geodesic distance is better suited to describe the quantum structure of spacetime. This is because one can incorporate some of the key quantum effects by replacing $\sigma^2$ by another function $S[\sigma^2]$ such that $S[0] = L_0^2$ is non-zero. This allows one to introduce a zero-point-length in the spacetime. I show that the geodesic distance can be an emergent construct, arising in the form of a correlator $S[\sigma^2(x, y)] = \langle J(x) J(y) \rangle$, of a pregeometric variable $J(x)$, which, in turn, can be interpreted as the quantum density of spacetime events. This approach also shows why null surfaces play a special role in the interface of quantum theory and gravity. I describe several technical and conceptual aspects of this construction and discuss some of its implications.

1 Geodesic distance: A replacement for the metric

The geometry of spacetime (or space, since I will work with D-dimensional Euclidean manifold in this section) is conventionally described in terms of a metric $g_{ab}(x)$ which is a local, second rank symmetric tensor. The distance between two infinitesimally separated events (or points) is then given by $ds^2 = g_{ab} dx^a dx^b$. All other geometrical features of the space(time) are then related to the metric.

There is, however, another way of describing the geometry which is conceptually far superior. This is in terms of the geodesic distance $\sigma^2(x, x')$ [also called Synge’s world function [1]; but I will use the terminology geodesic distance] which is related to the metric tensor $g_{ab}$ by two equations. The first one is:

$$
\sigma = \int_{x'}^x \sqrt{g_{ab} dx^a dx^b} = \int_{\lambda_0}^{\lambda} \sqrt{g_{ab} n^a n^b} d\lambda \quad (1)
$$

where $n^a = dx^a/d\lambda$ is the tangent vector to the geodesic. This equation tells you how the metric tensor $g_{ab}$ determines the geodesic distance $\sigma^2(x, x')$. The second equation is the differential version of the same, given by:

$$
\frac{1}{2} \left[ \nabla_a \nabla_b \sigma^2(x, x') \right] = g_{ab}(x) - \frac{1}{3} R_{abcd} n^c n^d \sigma^2 + \ldots = g_{ab}(x) + O(R \sigma^2) \quad (2)
$$

This one tells you how $\sigma^2(x, x')$ determines the metric tensor $g_{ab}$ when you take the limit of $x \to x'$ on both sides of this equation. (In a way $\sigma^2(x, x')$ actually encodes more information than the metric; the
Taylor series expansion of the function $\sigma^2(x, x')$ gives the components of curvature tensor etc.) These two equations together imply that the metric $g_{ab}(x)$ and the geodesic distance $\sigma^2(x, x')$ contain the same amount of information about the geometry. Classical gravity can, therefore, be described entirely in terms of the single \textit{biscalar} function $\sigma^2(x, x')$ instead of the ten components of the local metric tensor $g_{ab}(x)$.

While both $g_{ab}(x)$ and $\sigma^2(x, x')$ contain the same amount of information classically, the metric is a lot easier to work with in technical computations and hence is usually considered as \textit{the} descriptor of spacetime geometry. (In fact, conventional textbooks in general relativity hardly mention the geodesic distance as a descriptor of geometry!) It turns out, however, that $\sigma^2(x, x')$ is conceptually far better suited to describe the quantum microstructure of spacetime. I will show that there is a natural way of expressing the geodesic distance $\sigma^2(x, y)$ as a correlator of a pregeometric \textit{density} of spacetime events, $J(x)$, in the form

$$S[\sigma^2(x, y)] = \langle J(x)J(y) \rangle$$

(4)

The left hand side gives the quantum corrected geodesic distance at mesoscopic scales; the probability distribution defining the correlator in the right hand side will be provided as we go along. From Eq. (3) we can also express the quantum corrected qmetric also as a correlator:

$$q_{ab}(x, x') = \frac{1}{2} \langle J(x')\nabla_a \nabla_b J(x) \rangle$$

(5)

which is valid to the lowest order in $RS$. While there is no simple way to introduce the metric tensor as an emergent construct, we can directly obtain the quantum corrected qmetric at mesoscopic scales, as an emergent variable, if we adopt the geodesic distance as the fundamental descriptor of geometry and use Eq. (4) and Eq. (5). Clearly, $\sigma^2(x, y)$ is better suited for an emergent description of geometry compared to the metric tensor. This is the most important reason for using $\sigma^2(x, x')$ rather than $g_{ab}(x)$ as a descriptor

\footnote{Me and my collaborators have emphasized the superiority of $\sigma^2$ over $g_{ab}$ in modeling quantum microstructure in several of our publications (see e.g., [2, 4]; for different but related point-of-view, see [5] and references therein.). This point of view is slowly gaining some acceptance in the later works by others and I hope this trend continues!}

\footnote{While most of the ideas described here will will work for arbitrary $S(\sigma^2)$, I will illustrate the results for the simple choice $S(\sigma^2) = \sigma^2(x, y) + L_0^2$ when appropriate; on analytic continuation to a spacetime with a mostly positive signature, the zero-point-length adds to the spatial distance.}
of geometry. In other words, the classical geodesic distance is the descriptor of spacetime geometry while the quantum corrected geodesic distance $S(\sigma^2)$ can be related to the description of pregeometric variables through Eq. (4). Let me summarize this point-of-view:

- It is better to describe classical geometry in terms of $\sigma^2(x, y)$ rather than the metric $g_{ab}(x)$.
- Spacetime geometry is an emergent phenomenon and QG effects change $\sigma^2(x, y)$ to $S[\sigma^2(x, y)]$.
- One can describe the geodesic distance $\sigma^2(x, y)$ and $S[\sigma^2]$ as correlators of a pregeometric variable $J(x)$ which can be thought of as the density of spacetime events.

I will now describe how such a picture emerges.

## 2 Geodesic distance emerges as a correlator of a pregeometric variable

Let me first outline the algebraic aspects and then take up the physical interpretation. Consider a stochastic variable $J(x)$, described by the probability function $P[J(x)]$, with the leading order behaviour given by:

$$P[J(x)] = \exp \left[ -\frac{L_0^2}{2} \int \frac{J(x)J(y)}{S[\sigma^2(x, y)]} dV_x dV_y + \cdots \right]$$

(6)

It is then obvious that the correlator $\langle J(x)J(y) \rangle$ is indeed given by $S[\sigma^2(x, y)]/L_0^2$. In such a description, $(x, y)$ etc. are thought of as variables in an abstract space and could even be discrete. In the semi-classical limit, these are treated as coordinates of a differential manifold with the geodesic distance $\sigma^2(x, y)$ in the limit of $L_0 \to 0$. The relation in Eq. (4) will hold even in this classical limit for events $(x, y)$ for which $\sigma^2(x, y) \gg L_0^2$. In Eq. (6), $J$ is dimensionless and $S$ has dimensions of square of length; $dV_x$ etc. are dimensionless and could correspond to e.g., $d^4x/L_P^4$ etc. in the appropriate limit. (Alternatively, one can think of $J$ having the dimensions of inverse volume, so that $JdV$ is dimensionless; these two descriptions are related by a simple rescaling.)

While the above ideas work for an arbitrary $S[\sigma^2(x, y)]$, let me illustrate them in the special case in which $S[\sigma^2(x, y)] = \sigma^2(x, y) + L_0^2$ where $L_0$ is the zero point length in spacetime (which, as I said before, is taken to be of the order of the Planck length $L_P$). In this specific case Eq. (6) becomes

$$P[J(x)] = \exp \left[ -\frac{L_0^2}{2} \int \frac{J(x)J(y)}{\sigma^2(x, y) + L_0^2} dV_x dV_y + \cdots \right]$$

(7)

In both Eq. (6) and Eq. (7) $J(x)$ can be thought of as the pregeometric density of spacetime events. Let me now introduce this concept.

This is best done by noting that such a description (and terminology) is completely analogous to what you do in the description of a fluid made of discrete molecules. In such a case, one often talks about

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3 May be one could add a third reason: It is very unlikely that physicists would have been tempted to quantize a biscalar $\sigma^2(x, x')$! On the other hand, a second rank symmetric tensor $g_{ab}(x)$ lured them to try their luck with creating a quantum theory for the metric tensor (with repeated failures). There is sufficient evidence that metric tensor should not be thought of as a garden-variety field (e.g. Yang-Mills field) and subjected to some kind of quantization. If general relativity was taught in terms of $\sigma^2$ may be one would have realized this earlier.

4 I will ignore normalisation factors and will not display them.
a function \( n(x) = n(t, x) \) which is supposed to give the number density of molecules at an event \( x \). Obviously an event \( x \) of geometrically zero size cannot host a single molecule of finite size, let alone several of them. But we never worry about this aspect when we do fluid mechanics. This mathematical abstraction is based on the idea \( dN = n(x)d^3x \) can describe the number of molecules in a ‘small’ volume \( d^3x \). This volume, however, should be large enough to contain sufficient number of molecules (i.e., it is significantly larger than \( \lambda^3 \) where \( \lambda \) is the mean free path) and hence cannot be strictly infinitesimal. At the same time, it is taken to be small enough to be treated as infinitesimal for mathematical purpose. This is precisely what I do while introducing the density of spacetime events \( J(x) \); the idea is that

\[
dN_e = J(x)dV_x
\]

gives the number of pregeometric events around \( x \), just as \( dN = n(x)d^3x \) gives the number of molecules around \( x \). The argument \( x \) of the function \( J(x) \) refers to a coordinate label in a coarse grained, mesoscopic description of the spacetime, treated like a fluid, with \( L_0 \) being analogous to the mean-free-path. I stress that there are no conceptual ambiguities in using such a description while claiming that \( S[\sigma^2(x, y)] \) is an emergent variable.

The relation in Eq. \( 4 \) suggests that \( G(x, y) \equiv (1/S[\sigma^2(x, y)]) \) can be interpreted as a propagator for a theory with \( J(x) \) acting as the sources. The coincidence limit of this propagator \( G(x, x) = (1/L_0^2) \) remains finite due to the existence of the zero-point-length. Such a source \( J(x) \) could also be thought of as generating a field \( \phi(x) \) with \( \mathcal{P}[J(x)] \) becoming the partition function for the theory. That is, we can also write:

\[
\mathcal{P}[J(x)] \propto \int \mathcal{D}\phi \exp \left[ -\frac{1}{2} \int dV_x dV_y \phi(x) M(x, y) \phi(x) + \int dV_x J(x) \phi(x) \right]
\]

where \( G(x, y) \) is the inverse of the operator \( M(x, y) \). When \( \sigma^2 \) is only a function of \( (x - y) \), as in the flat spacetime, the Lagrangian for \( \phi(x) \) will have the structure \( L \propto \phi(x) G^{-1}(k = i\nabla) \phi(x) \) where \( G(k) \) is the Fourier transform of \( G(x) \). In general, the field theory for \( \phi(x) \) will be non-local.

It is important to note that I have defined \( S[\sigma^2(x, y)] \) as a correlator of the source \( J(x) \) rather than as a correlator of the field \( \phi(x) \). The correlator of a field, in the limit of \( L_P \to 0 \), will usually behave as an inverse function of \( \sigma^2(x, y) \); for e.g., a massless free field in \( D = 4 \) will have a behaviour \( 1/\sigma^2 \). But in the limit of \( L_P \to 0 \) we want the correlator in Eq. \( 4 \) to behave as \( \sigma^2(x, y) \). This is achieved by defining the geodesic distance in terms of the correlator of the source rather than as correlator of the field. That is, the geodesic distance emerges from a pregeometric source \( J(x) \) rather than from a pregeometric field \( \phi(x) \). It is also clear from Eq. \( 4 \) that the correlator \( \langle J(x) J(y) \rangle \) increases without bound (rather than decrease) with the separation \((x - y)^2 \); obviously we are dealing with a somewhat counter-intuitive, but well-defined, stochastic variable.

Since the field theory of \( \phi(x) \) is difficult to handle, we need another route to reach the propagator \( G(x, y) \). Fortunately, the relevant formalism for propagators with finite coincidence limit has already been developed \( 8 \) in terms of path integrals (without introducing any field). In this approach, it is relatively straightforward to obtain the propagator \( G(x, y) \equiv (1/S[\sigma^2(x, y)]) \) directly through the introduction of zero-point length in spacetime. To do this, consider a spacetime with a classical metric \( g_{ab} \) and a corresponding Laplacian \( \Box_g \). The heat kernel for this spacetime is defined in the standard manner as

\footnote{The switching between \( J(x) \) and \( \phi(x) \) can also be thought of as functional Fourier transforms, if we replace \( J(x) \phi(x) \) by \( iJ(x)\phi(x) \) in Eq. \( 4 \). In this sense, the pair \( J(x) \) and \( \phi(x) \) are Fourier conjugate variables. But note that it is the correlator of \( J(x) \) that gives the geodesic distance directly.}
\[ K_{\text{std}}(x, y; s) \equiv \langle x | e^{-sL_0^2} | y \rangle. \]

The propagator, incorporating the zero-point length, can now be defined as
\[ G(x, y) = \int_0^\infty ds \ e^{-(L_0^2/4s)} K_{\text{std}}(x, y; s) \]  
(10)

This construction has been extensively discussed in the literature [7] and allows the modification of \( \sigma^2(x, y) \) to \( \sigma^2(x, y) + L_0^2 \) at the lowest order. The probability distribution in Eq. (11), governing the density of spacetime events, can now be expressed in the form
\[ \mathcal{P}[J(x)] = \exp \left[ -\frac{L_0^2}{2} \int_0^\infty ds e^{-(L_0^2/4s)} \int dV_x \ dV_y \ J(x) J(y) K_{\text{std}}(x, y; s) + \cdots \right] \]  
(11)

In classical geometry the propagation amplitude from \( x \) to \( y \) is governed by \( K_{\text{std}}(x, y; s) \). In the above expression we are using a heat kernel with two modifications: (1) The events \( x \) and \( y \) are weighted by the density of spacetime events \( J(x) \) and \( J(y) \). (2) The propagation at scales less than \( L_0^2 \) is suppressed by the factor \( e^{-L_0^2/4s} \).

It is possible to arrive at the modified propagator from a different route as well [8]. This is done by postulating that the action for a relativistic particle, propagating in a spacetime, should remain invariant under the transformation \( \sigma(x, y) \rightarrow L_0^2/\sigma(x, y) \). This can be achieved by defining the (Euclidean) propagator for a particle of mass \( m \), propagating in a given spacetime by the sum over paths:
\[ G(x, y) = \sum_\sigma \exp \left[ -m \left( \sigma + \frac{L_0^2}{\sigma} \right) \right] \]  
(12)

It can be shown that [8] the resulting propagator incorporates the zero-point length exactly as given by Eq. (10). For a massive particle, the modification in Eq. (12) has a very simple physical interpretation. Recall that the action for a relativistic particle of mass \( m \) is \( A = -m\sigma = -\sigma/\lambda_c \) where \( \sigma \) is the length of the path and \( \lambda_c = \hbar/mc \) is the Compton wavelength of the particle. When we introduce gravity, it makes no sense to sum over paths with length \( \sigma \) smaller than the Schwarzchild radius \( R_g = Gm/c^2 \) of the particle. This suggests suppressing the contribution from paths with \( \sigma \lesssim R_g \) in some suitable manner. Assuming that this suppression preserves a duality symmetry under \( \sigma \rightarrow 1/\sigma \), one arrives at the (unique) modification of the action to the form \( A_g = -\sigma(1/\lambda_c) - (R_g/\sigma) \) which can be written in the form \( A_g = -(1/\lambda_c)[\sigma + (L_0^2/\sigma)] \) where \( L_0 \) is of the order of Planck length. To obtain the results in Eq. (11) etc., we can take the limit \( m \rightarrow 0 \) at the end of the computation.

So the procedure for obtaining \( \sigma^2 \) as an emergent variable from pregeometric density of spacetime events can be summarized as follows:

- You start with a classical spacetime and metric \( g_{ab} \) valid at macroscopic scales. Compute the classical heat kernel for this metric, \( K_{\text{std}}(x, y; s) \equiv \langle x | e^{-sL_0^2} | y \rangle \).
- Postulate that the probability \( \mathcal{P}[J(x)] \) for the density of spacetime events \( J(x) \) is given by Eq. (11).
- Then the quantum corrected geodesic distance \( S[\sigma^2(x, y)] \) is given by the correlator \( \langle J(x) J(y) \rangle \). This provides an emergent description for the geodesic distance from which one can obtain an (emergent) QG corrected metric, \( q_{ab} \). The properties of such a metric has been explored in detail in Ref. [4].
3 Euclidean spacetime as a set of null surfaces

I will now take a closer look at the conceptual picture which emerges from Eq. (8), using the analogy of fluid mechanics. Let us ask what is the operational procedure for determining the number density of molecules $n(x)$ in fluid around an event $x_i$. One simple method will be to take a small, spherical, volume $V(\epsilon)$ of radius $\epsilon$ centered at $x$ (on a $t = \text{constant}$ hypersurface) and count the number $N(\epsilon)$ of molecules in it. We can then define the number density $n(x)$ as the limit of $N(\epsilon)/V(\epsilon)$ as $\epsilon \to 0$. One can introduce a similar procedure to define $J(x)$, in the Euclidean space, using a geodesic sphere of radius $\epsilon$. Interestingly enough, something very curious happens when we analytically continue to the Lorentzian spacetime. It turns out that, shifting the attention from the metric to geodesic distance, provides fresh insights into the nature of null surfaces in spacetime. Given the fact that null surfaces seem to play a vital role in the emergent gravity paradigm [6], let me highlight this alternative point of view.

Consider a $D$-dimensional, flat, Euclidean space described in Cartesian coordinates $x_E^i = (t_E, x_E^i, x_{\perp})$ where $x_{\perp}$ denotes $(D - 2)$ transverse coordinates. I will concentrate on the $t - x$ plane to illustrate the ideas which can be easily generalized to the $D$-dimensional space. Let us begin by asking how we can assign to a point $P$ in this plane the coordinates, say, $(T_E, X_E)$. The simple procedure is just to say that the coordinates of $P$ are specified by the equations:

$$t_E = T_E; \quad x_E = X_E$$

(13)

It is, however, possible to specify the coordinates of a point in a different — but completely equivalent — manner. One can say that the coordinates of $P$ are given by the solution to the equation:

$$\sigma^2_E(x, X) \equiv (x_E - X_E)^2 + (t_E - T_E)^2 = 0$$

(14)

That is, we set the geodesic distance between the two points to be equal to zero. In the Euclidean spacetime the procedures in Eq. (13) and Eq. (14) lead to identical results, because the Euclidean geodesic interval $\sigma^2_E(x, X)$ vanishes only if $x = X$.

Let us now analytically continue from the Euclidean space to Lorentzian spacetime by the usual procedure of setting $t_E = it$ etc. The Eq. (13) will continue to work and one can specify the Lorentzian coordinates of the event $P$ by the relations $x = X$ and $t = T$. But the procedure in Eq. (14) now fails! The analytically continued version of Eq. (14)

$$\sigma^2(x, X) \equiv (x - X)^2 - (t - T)^2 = 0$$

(15)

gives the null surfaces originating at the event $(T, X)$. This is the direct consequence of the fact that the vanishing of the geodesic distance $\sigma^2(x, X) = 0$ in a Lorentzian spacetime specifies events connected by a null ray rather than a unique event.

Let us take this idea further. Let us assume that the pregeometric description and consequent QG effects are to be described primarily in the Euclidean space with an analytic continuation leading to the standard Lorentzian spacetime. In that case, an infinitely localized point in the Euclidean space will not have any operational significance. To tackle this issue, we can describe a point in the Euclidean space by the following procedure. Consider the equi-geodesic surfaces defined by the equation $\sigma^2_E(x, X) = \epsilon^2$. In the context of the 2-dimensional section of Euclidean flat space this equation describes a circle of radius $\epsilon$ centered on the point $(T, X)$. If we now take smaller and smaller equigeodesic surfaces by decreasing the value of $\epsilon$ the circles will approach the events $(T, X)$. So an event in Euclidean space can equivalently be thought of as the limit of the equigeodesic surface $\sigma^2_E(x, X) = \epsilon^2$ when the geodesic distance tends to zero. This is precisely the construction we should adopt to define $J(x)$.
Let us now repeat the same exercise after the analytic continuation. In the Lorentzian spacetime, the corresponding equation $\sigma^2(x, X) = \epsilon^2$ will represent a pair of hyperbola in the right and left wedges demarcated by the null surfaces $(x - X) = \pm(t - T)$. The limit $\epsilon \to 0$ will now give you the null surfaces $(x - X) = \pm(t - T)$ rather than a unique event. So if we choose to define events in Euclidean space by taking the limit $\epsilon \to 0$ of the equation $\sigma_E^2(x, X) = \epsilon^2$, then analytic continuation will associate a pair of null surfaces with each point in the Euclidean space.

This result, while algebraically elementary, has deep conceptual significance. As I described earlier, geodesic distance provides a natural link between pregeometry and geometry. It then makes sense to define the coordinates of an event in terms of the procedure in Eq. (14) in Euclidean space. On analytic continuation, this procedure associates a pair of null surfaces with pairs of points in the Euclidean space. Because the construction relies only on the $\epsilon \to 0$ limit, the same ideas carry over to a curved Euclidean space and curved spacetime. In the curved Euclidean space, one can always choose a locally flat coordinate system and perform the analytic continuation within that region leading to exactly the same results.

This approach links naturally with the idea of density of spacetime events $J(x)$. One can now consider an infinitesimal region around an event $X'$ defined through the relation $\sigma_E^2(x, X) \leq \epsilon^2$; this would represent a geodesic ball of radius $\epsilon$ in the Euclidean space and one can think of number of spacetime events inside such infinitesimal balls as encoded in the pregeometric variable $J(x)$. Analytic continuation will now lead to density of spacetime events in the infinitesimal neighbourhood of a pair of null surfaces in the spacetime. I would conjecture that the importance of null surfaces in the study of horizon thermodynamics and emergent gravity paradigm arises because infinitesimally localized points satisfying $\sigma^2(x, X) = 0$ in the Euclidean space are mapped to events along the null rays, satisfying $\sigma^2(x, X) = 0$ in the spacetime.

4 Summary and Highlights

Let me conclude by listing some of the key points described above.

- The geodesic distance is a far better descriptor of classical geometry and quantum pregeometry. Let us abandon the description in terms of the metric and concentrate on the geodesic interval, in the study of quantum spacetime!

- The geodesic interval $\sigma^2(x, y)$ is an emergent construct and can be thought of as a correlator $\langle J(x)J(y) \rangle$ of a pregeometric variable $J(x)$. This variable, which can be interpreted as the density of spacetime events, is completely analogous to density of molecules $n(x)$ of a fluid in standard fluid mechanics.

- The correlator $\langle J(x)J(y) \rangle$ is computed using a probability functional given by Eq. (11). This provides a systematic procedure for constructing the quantum corrected metric from the classical metric.

- Events in Euclidean space can be defined by taking the zero radius limit ($\epsilon \to 0$) of a geodesic ball $\sigma_E^2(x, X) = \epsilon^2$. The analytic continuation of this exercise allows us to associate a pair of null surfaces with every point in Euclidean plane.

- It is possible to introduce a pregeometric field $\phi(x)$ sourced by the density of spacetime events $J(x)$ by the procedure I have outlined. For reasons I have stressed, the geodesic interval should be thought of as the correlator of the source $J(x)$ and not of the field $\phi(x)$. 

One can investigate the pregeometric structure as well as specific examples (cosmological spacetimes, black hole spacetimes etc.) working with either $J(x)$ or $\phi(x)$. Standard Lorentz invariant, local, unitary QFT will not allow a propagator with finite coincidence limit. So the QFT of $\phi(x)$ will be non-local and rather unusual.

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References

[1] J.L. Synge, Relativity: the general theory (North-Holland, Amsterdam, 1960).

[2] Kothawala D and Padmanabhan T (2014) Phys. Rev. D 90 124060 [arXiv:1405.4967];
Kothawala D (2013) Phys. Rev. D 88 104029
Padmanabhan, T (2015) Entropy, 17, 7420 [arXiv:1508.06286]
D. Jaffino Stargen, D. Kothawala, (2015), Phys.Rev., D92 024046 [arXiv:1503.03793]

[3] B. S. DeWitt, (1964) Phys. Rev. Lett. 13, 114;
T.Padmanabhan, (1985), Gen. Rel. Grav., 17, 215;
T.Padmanabhan, (1985), Ann. Phys., 165, 38;
T.Padmanabhan, (1987), Class. Quan. Grav., 4, L107;
For a review, see L. Garay, Int. J. Mod. Phys. A 10, 145 (1995); S. Hossenfelder, Living Rev. Relativity 16, (2013), 2 [arXiv:1203.6191]

[4] Padmanabhan T, Chakraborty S and Kothawala D (2016) Gen. Rel. Grav., 48 55 [arXiv:1507.05669];
Kothawala D and Padmanabhan T (2015) Phys. Lett. B 748, 67;
Sumanta Chakraborty, D. Kothawala, Alessandro Pesci, (2019) Phys. Lett. B 797, 134877 [arXiv:1904.09053];
Alessandro Pesci, Class. Quantum Grav., 36 (2019) 075009 arXiv:1812.01275;
Alessandro Pesci, Looking at spacetime atoms from within the Lorentz sector, arXiv:1803.05726

[5] Alvarez, E. et al. Phys.Rev., D45 (1992) 2033

[6] T. Padmanabhan, Gen.Rel.Grav, 46, 1673 (2014) [arXiv:1312.3253];
T. Padmanabhan, Gravity and Quantum Theory: Domains of Conflict and Contact, (in press) [arXiv:1909.02015]

[7] K.Srinivasan, L.Sriramkumar and T. Padmanabhan, (1998) Phys. Rev. D 58, 044009 [gr-qc-9710104];
S. Shankaranarayanan and T. Padmanabhan, (2001) Int. Jour. Mod. Phys , 10 , 351 [gr-qc-0003058];
Dawood Kothawala, L. Sriramkumar, S. Shankaranarayanan, T. Padmanabhan, (2009), Phys.Rev., D 79, 104020 [arXiv:0904.3217]
[8] T. Padmanabhan, *Phys. Rev. Letts*, **78**, 1854 (1997) [hep-th-9608182];
T. Padmanabhan, *Phys. Rev.*, **D 57**, 6206 (1998)