THE SASSENFELD CRITERION REVISITED

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ABSTRACT. The starting point of this article is a decades-old yet little-noticed sufficient condition, presented by Sassenfeld in 1951, for the convergence of the classical Gauß-Seidel method. The purpose of the present paper is to shed new light on Sassenfeld’s criterion and to demonstrate that the original work can be perceived as a special case of a far more extensive concept in the context of preconditioners and iterative linear solvers. Our main result is a classification theorem for the set of all matrices which this general framework applies to.

1. INTRODUCTION

The Gauß-Seidel method is one of the most classical examples for the iterative solution of linear systems in many numerical analysis textbooks. Convergence is typically established for strictly diagonally dominant as well as for symmetric positive definite matrices. Only a few authors (see, e.g., [Ven17, Thm. 4.16]), however, point to a less standard convergence criterion for the Gauß-Seidel scheme introduced by Sassenfeld in his paper [Sas51]: Given a matrix \( A = [a_{ij}] \in \mathbb{R}^{m \times m} \) with non-vanishing diagonal entries, i.e. \( a_{ii} \neq 0 \) for each \( i = 1, \ldots, m \), define non-negative real numbers \( s_1, \ldots, s_m \) iteratively by

\[
s_i = \frac{1}{|a_{ii}|} \left| \sum_{j < i} |a_{ij}|s_j + \sum_{j > i} |a_{ij}| \right|, \quad i = 1, \ldots, m. \tag{1}
\]

Sassenfeld has proved that \( \max_{1 \leq i \leq m} s_i < 1 \) is a sufficient condition for the convergence of the Gauß-Seidel scheme. For matrices that satisfy this property, the notion of a Sassenfeld matrix was recently introduced in [BW17] as a generalization of (strict) diagonal dominance.

The purpose of the present paper is to show that there is a general principle behind Sassenfeld’s original work that applies far beyond the Gauß-Seidel method. To illustrate this observation, we note that (1) can be written in matrix form as

\[
(\|D\| - |L|)s = |U|e, \tag{2}
\]

where the matrix \( A = L + D + U \) is decomposed in the usual way into the (strict) lower and upper triangular parts \( L = \text{tril}(A) \) and \( U = \text{triu}(A) \), respectively, and the diagonal part \( D = \text{diag}(A) \); furthermore, \( || \cdot || \) signifies the modulus of a matrix \( \cdot \) taken entry-wise, \( s = (s_1, \ldots, s_m) \) contains the iteratively defined real numbers \( s_1, \ldots, s_m \) from (1), and

\[
e = (1, \ldots, 1)^T \in \mathbb{R}^m \tag{3}
\]

is the (column) vector containing only components 1. More generally, for an appropriate invertible matrix \( P \in \mathbb{R}^{m \times m} \), which will be called a Sassenfeld preconditioner, we consider the splitting

\[
A = \text{off}(P) + \text{diag}(P) + (A - P), \tag{4}
\]

\[1\]

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where $\text{diag}(\{\ast\})$ and $\text{off}(\{\ast\})$ denote the diagonal and off-diagonal parts of a matrix, respectively. Then, define the vector $s \in \mathbb{R}^m$ to be the solution (if it exists) of the system

$$
(\{\text{diag}(P)\} - \{\text{off}(P)\})s = |A - P|e.
$$

(5)

For instance, in the context of the Gauss-Seidel scheme, letting $P := \mathbf{L} + \mathbf{D}$, with $\mathbf{L}$ and $\mathbf{D}$ as above, we notice that (5) translates into (2). In this work, we will focus on matrices $A$ and $P$ for which the components of the solution vector $s$ of the linear system (5) satisfy

$$
0 \leq s_i < 1 \quad \forall \ i = 1, \ldots, m.
$$

(6)

We begin our work by introducing a class of preconditioners $P$, for which the system (5) is invertible; our approach is based on some classical results from the Perron-Frobenius theory of non-negative matrices. Subsequently, we will focus on all matrices for which the bounds (6) for the solution vector $s$ of (5) can be achieved: such matrices will be termed \textit{Sassenfeld matrices}. We will discuss a number of properties, and prove that Sassenfeld matrices give rise to convergent iterative splitting methods for linear systems (Prop. 4.3). In addition, a characterization theorem for Sassenfeld matrices (Thm. 4.3) will be provided.

\textbf{Outline.} We begin by introducing the set of Sassenfeld preconditioners \[\mathbb{S}\] in §2 and give a number of examples. We then continue to present the Sassenfeld index of a matrix (with respect to a Sassenfeld preconditioner) in §3 before turning to the definition and characterization of Sassenfeld matrices in §4. We also elaborate on the application to iterative linear solvers. Moreover, some spectral properties of Sassenfeld matrices are devised in §5. Finally, a few elementary considerations on the construction of preconditioners are provided in §6.

\textbf{Notation.} For any vectors or matrices $x, y \in \mathbb{R}^{m \times n}$, we use the notation $x \geq y$ (or $x > y$) to indicate that all entries of the difference $x - y \in \mathbb{R}^{m \times n}$ are non-negative (resp. positive). Furthermore, for a matrix $A = [a_{ij}] \in \mathbb{R}^{m \times n}$, we denote by

$$
\|A\|_{\infty} := \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}|
$$

the standard infinity matrix norm. Moreover, we signify by $\varrho(A)$ the spectral radius of a square-matrix $A \in \mathbb{R}^{m \times m}$.

\section{Sassenfeld Preconditioners}

We define the mapping

$$
\begin{align*}
\{\ast\}^\circ : \mathbb{R}^{m \times m}_{\geq 0} &\to \mathbb{R}^{m \times m}_{\geq 0} \\
P &\mapsto P^\circ := |\text{diag}(P)|^{-1} |\text{off}(P)| = |\mathbf{I} - \text{diag}(P)|^{-1} P,
\end{align*}
$$

(7)

where $\mathbf{I} = \text{diag}(1, \ldots, 1)$ is the identity matrix in $\mathbb{R}^{m \times m}$, and $\mathbb{R}^{m \times m}_{\geq 0}$ and $\mathbb{R}^{m \times m}_{\geq 0}$ are the sets of all real $m \times m$ matrices with non-vanishing diagonal entries and with non-negative entries, respectively.

\textbf{Definition 2.1} (Sassenfeld preconditioners). A matrix $P \in \mathbb{R}^{m \times m}_{\geq 0}$ is called a Sassenfeld preconditioner if the matrix $P^\circ \in \mathbb{R}^{m \times m}_{\geq 0}$ from (7) satisfies $\varrho(P^\circ) < 1$.

\textbf{Remark 2.2.} Observe that $P \in \mathbb{R}^{m \times m}_{\geq 0}$ is a Sassenfeld preconditioner if and only if the matrix $\mathbf{I} - P^\circ$ is an $M$-matrix. In particular, we notice that $|\mathbf{I} - P^\circ|^{-1} \succeq 0$; cf. \cite{Mey00}, Expl. 7.10.7).

The following result provides a useful tool to verify Def. 2.1 practically.
Lemma 2.3. A matrix \( P \in \mathbb{R}^{m \times m} \) is a Sassenfeld preconditioner if and only if
\[
|\text{off}(P)|z < |\text{diag}(P)|z,
\]
for some positive vector \( z > 0 \) in \( \mathbb{R}^m \).

Before proving the above lemma, we recall an instrumental fact from the Perron-Frobenius theory of non-negative matrices (see, e.g., [Mey 00, §8]), which will be repeatedly used in this paper.

Lemma 2.4. Given a non-negative matrix \( B \geq 0 \) in \( \mathbb{R}^{m \times m} \), with \( r := \rho(B) \). Then, for any \( \epsilon > 0 \), there exists a positive vector \( z > 0 \) in \( \mathbb{R}^m \) such that it holds \( Bz < (r + \epsilon)z \).

Proof. For given \( \epsilon > 0 \), choose \( \delta > 0 \) sufficiently small such that \( \rho(B + \delta E) < r + \epsilon \), where \( E = ee^\top \in \mathbb{R}^{m \times m} \), with \( e \in \mathbb{R}^m \) from (3), is the matrix with all entries 1. Due to Perron's theorem, there exists a (right Perron) vector \( z > 0 \) such that
\[
(B + \delta E)z = \rho(B + \delta E)z < (r + \epsilon)z,
\]
which shows the claim. \( \Box \)

Proof of Lemma 2.3 If \( P \) is a Sassenfeld preconditioner with \( r := \rho(P^\circ) < 1 \), then by Lem. 2.4, for \( \epsilon = (1-r)/2 \), there exists a vector \( z > 0 \) such that
\[
P^\circ z < \frac{1}{2} (r + 1)z < z,
\]
which is equivalent to (8). Conversely, suppose that there is \( z > 0 \) in \( \mathbb{R}^m \) such that (8) is satisfied. Then, we immediately see that \( P \in \mathbb{R}^{m \times m} \). Furthermore, owing to the Perron-Frobenius theory there exists a (right Perron) vector \( q \succeq 0 \) in \( \mathbb{R}^m \), \( q \neq 0 \), such that \( q^\top P^\circ = \rho(P^\circ)q^\top \). Hence, exploiting (9), we obtain
\[
\rho(P^\circ)q^\top z = q^\top P^\circ z < q^\top z,
\]
from which we infer that \( \rho(P^\circ) < 1 \). \( \Box \)

Proposition 2.5. Any Sassenfeld preconditioner is invertible.

Proof. Let \( P \in \mathbb{R}^{m \times m} \) be a Sassenfeld preconditioner. Suppose that there exists a vector \( x \in \mathbb{R}^m \) with \( \|x\|_\infty = 1 \) and \( Px = 0 \). Then it follows that
\[
x - (1 - \text{diag}(P)^{-1}P)x = 0.
\]
Taking moduli, we obtain \( |x| \leq P^\circ |x| \). Iteratively, for any \( n \in \mathbb{N} \), we infer that \( |x| \leq (P^\circ)^n |x| \). Exploiting that \( \rho(P^\circ) < 1 \) and letting \( n \to \infty \), we deduce that \( x = 0 \), which is a contradiction. \( \Box \)

We present a few examples.

Examples 2.6. Any matrix \( P = [p_{ij}] \in \mathbb{R}^{m \times m} \) of one of the following types is a Sassenfeld preconditioner:

(i) All invertible upper and lower triangular matrices;
(ii) Any strictly diagonally dominant matrix (by rows or by columns);
(iii) All \( M \)-matrices;
(iv) Any symmetric positive definite matrix \( P \) for which it holds that
\[
\rho(1 - \alpha B) \geq \beta \rho(|B|),
\]
for some \( \alpha > 0 \) and \( \beta \geq \max(1/2, \alpha) \), where we let
\[
B := \text{diag}(P)^{-1/2}P \text{diag}(P)^{-1/2}.
\]
(v) All symmetric positive definite matrices \( P \) with a symmetric sign pattern of the form
\[
\text{sign}(p_{ij}) = -\xi_i \xi_j \quad 1 \leq i < j \leq m,
\]
for a vector \( \xi \in [\pm 1]^m \).
**Proof.** We begin by noticing that all matrices $\mathbf{P} \in \mathbb{R}^{m \times m}$ in (i)–(v) belong to $\mathbb{R}_{u}^{m \times m}$, i.e. $\mathbf{P}^\circ$ is well-defined. For each of the examples, we need to prove that $\rho(\mathbf{P}^\circ) < 1$.

(i) For any invertible upper or lower triangular matrix $\mathbf{P}$ it is straightforward to see that $\rho(\mathbf{P}^\circ) = 0$.

(ii) If $\mathbf{P}$ is strictly diagonally dominant by rows, then we have

$$\rho(\mathbf{P}^\circ) \leq \|\mathbf{P}^\circ\|_{\infty} < 1.$$ 

Moreover, if $\mathbf{P}$ is strictly diagonally dominant by columns, then we observe that the spectral radii of the two matrices $\mathbf{P}^\circ$ and $(\mathbf{P}^\circ)^T$ are equal, and we can repeat the same argument.

(iii) If $\mathbf{P}$ is an $M$-matrix then it can be expressed in the form $\mathbf{P} = r\mathbf{1} - \mathbf{B}$, where $\mathbf{B} \succeq \mathbf{0}$ and $r > \rho(\mathbf{B})$; cf. [Mey00, Expl. 7.10.7]. In light of Lem. [2.4] for $c = \frac{1}{2}(r - \rho(\mathbf{B}))$, we can find $\mathbf{z} > \mathbf{0}$ such that

$$\mathbf{B}\mathbf{z} < \frac{r + \rho(\mathbf{B})}{2}\mathbf{z} < r\mathbf{z}.$$ 

Furthermore, we have

$$|\text{off}(\mathbf{P})| = \text{off}(\mathbf{B}) = \mathbf{B} + \text{diag}(\mathbf{P}) - r\mathbf{1}.$$ 

Hence, it follows that

$$|\text{off}(\mathbf{P})|\mathbf{z} = \mathbf{B}\mathbf{z} - rz + \text{diag}(\mathbf{P})\mathbf{z} < |\text{diag}(\mathbf{P})|\mathbf{z}.$$ 

Then, applying Lem. [2.3] yields the claim.

(iv) Let $r := \rho((1 - \alpha\mathbf{B}) \triangleright \beta \rho(|\mathbf{B}|))$. Due to the symmetry of $\mathbf{P}$, the spectrum of the matrix $1 - \alpha\mathbf{B}$ is real, and contains either $+r$ or $-r$, or both. Suppose that there exists $\mathbf{\xi} \in \mathbb{R}^m$, $\mathbf{\xi} \neq \mathbf{0}$, with $(1 - \alpha\mathbf{B})\mathbf{\xi} = -r\mathbf{\xi}$. Then, we obtain $\mathbf{B}\mathbf{\xi} = \alpha^{-1}(r + 1)\mathbf{\xi}$, which leads to

$$\rho(|\mathbf{B}|) \triangleright \rho(|\mathbf{B}|) \geq \frac{r + 1}{\alpha} > \frac{r}{\alpha}.$$ 

Noticing that $\alpha \leq \beta$ yields $\rho(|\mathbf{B}|) \triangleright \beta / \rho$, which constitutes a contradiction. Hence, there is $\mathbf{\xi} \in \mathbb{R}^m$, with $\mathbf{\xi}^\top\mathbf{\xi} = 1$, such that

$$(1 - \alpha\mathbf{B})\mathbf{\xi} = r\mathbf{\xi}.$$ 

Defining the vector $\mathbf{\eta} := \text{diag}(\mathbf{P})^{-1/2}\mathbf{\xi}$, it follows that

$$\text{diag}(\mathbf{P})^{1/2}(1 - \alpha\mathbf{B})\text{diag}(\mathbf{P})^{1/2}\mathbf{\eta} = r\text{diag}(\mathbf{P})\mathbf{\eta},$$ 

and therefore,

$$\text{diag}(\mathbf{P})\mathbf{\eta} - \alpha\mathbf{P}\mathbf{\eta} = r\text{diag}(\mathbf{P})\mathbf{\eta}.$$ 

Using that $\mathbf{P}$ is positive definitive, and noticing that $\mathbf{\eta}^\top\text{diag}(\mathbf{P})\mathbf{\eta} = \mathbf{\xi}^\top\mathbf{\xi} = 1$, we deduce that

$$r = 1 - \alpha\mathbf{\eta}^\top\mathbf{P}\mathbf{\eta} < 1.$$ 

Note that the diagonal entries of $\mathbf{P}$ are all positive. Hence, exploiting that $\rho(|\mathbf{B}|) \triangleright \beta / \rho$, and applying Lem. [2.4] for $c = (1 - r)/\rho > 0$, there exists $\mathbf{z} \in \mathbb{R}^m$, $\mathbf{z} > \mathbf{0}$, scaled such that $\mathbf{z}^\top\text{diag}(\mathbf{P})\mathbf{z} = 1$, with $|\mathbf{B}|\mathbf{z} < \beta^{-1}\mathbf{z}$. Then, upon defining the (positive) vector $\mathbf{y} = \text{diag}(\mathbf{P})^{-1/2}\mathbf{z}$, and applying $\beta \geq 1/2$, it holds that

$$|\text{off}(\mathbf{P})|\mathbf{y} < (\beta^{-1} - 1)|\text{diag}(\mathbf{P})|\mathbf{y} \leq |\text{diag}(\mathbf{P})|\mathbf{y}.$$ 

Applying Lem. [2.3] completes the argument.

(v) We apply (iv). To this end, by the Perron-Frobenius theorem, we note that there is $\mathbf{z} \neq \mathbf{0}$, $\mathbf{z} \neq \mathbf{0}$, such that $|\mathbf{B}|\mathbf{z} = rz$, with $r = \rho(|\mathbf{B}|)$. Equivalently, since the diagonal entries of $\mathbf{P}$ are all positive, we have $|\mathbf{P}|\mathbf{y} = r\text{diag}(\mathbf{P})\mathbf{y}$, with $\mathbf{y} = \text{diag}(\mathbf{P})^{-1/2}\mathbf{z}$. Hence, for $1 \leq i \leq m$, using (11), it holds

$$rp_{ii}y_i = \sum_{j=1}^{m} |p_{ij}|y_j = p_{ii}y_i + \sum_{j \neq i} \text{sign}(p_{ij})p_{ij}y_j = p_{ii}y_i - \sum_{j=1}^{m} \xi_j p_{ij}y_j.$$
Remark 2.7. It is easy to see that symmetric positive matrices fail to be Sassenfeld preconditioners in general. An example is given by the symmetric positive definite matrix \( \mathbf{A} \) for which it holds \( \mathbf{A} \preceq \mathbf{I} \), see Prop. 3.3 below. This is crucial, for instance, in the convergence analysis of iterative linear solvers, where \( \mathbf{A} \) takes the role of a preconditioner; see Prop. 4.3 later on.

We note that the vector \( \mathbf{s} \) from (12), contains only non-negative components.

**Definition 3.1** (Sassenfeld index). The Sassenfeld index of a matrix \( \mathbf{A} \in \mathbb{R}^{m \times m} \) with respect to a Sassenfeld preconditioner \( \mathbf{P} \) is defined by 
\[
\mu(\mathbf{A}, \mathbf{P}) := \|\mathbf{s}(\mathbf{A}, \mathbf{P})\|_{\infty},
\]
with \( \mathbf{e} \in \mathbb{R}^{m} \) from (8), and a Sassenfeld preconditioner \( \mathbf{P} \) from (12).

The essence of the Sassenfeld index defined above is that it allows to control the norm \( \|\mathbf{1} - \mathbf{P}^{-1}\mathbf{A}\|_{\infty} \); see Prop. 2.2 below. This is crucial, for instance, in the convergence analysis of iterative linear solvers, where \( \mathbf{P} \) takes the role of a preconditioner; see Prop. 4.3 later on.

We note that the vector \( \mathbf{s}(\mathbf{A}, \mathbf{P}) \) from (12) can be computed approximately by iteration. Indeed, if \( \mathbf{P} \) is a Sassenfeld preconditioner, then the iterative scheme given by
\[
\mathbf{s}^{k+1} = \mathbf{P}^{s}\mathbf{s}^{k} + |\text{diag}(\mathbf{P})|^{-1}\mathbf{A} - \mathbf{P}|\mathbf{e}, \quad k \geq 0,
\]
converges to the vector \( \mathbf{s}(\mathbf{A}, \mathbf{P}) \) from (12) for any initial vector \( \mathbf{s}^{0} \in \mathbb{R}^{m} \). The following result provides a practical upper bound for the Sassenfeld index.

**Proposition 3.2** (Iterative estimation). Consider a matrix \( \mathbf{A} \in \mathbb{R}^{m \times m} \), and a Sassenfeld preconditioner \( \mathbf{P} \in \mathbb{R}^{m \times m} \). Then, there exists a vector \( \mathbf{s}^{0} \in \mathbb{R}^{m} \) such that
\[
|\text{diag}(\mathbf{P})|^{-1}|\mathbf{A} - \mathbf{P}|\mathbf{e} \leq (\mathbf{1} - \mathbf{P}^{s})\mathbf{s}^{0}.
\]
Furthermore, if the iteration (13) is initiated by \( \mathbf{s}^{0} \) (for \( k = 0 \)), then it holds the bound 
\[
\mu(\mathbf{A}, \mathbf{P}) \leq \|\mathbf{s}^{k}\|_{\infty} \quad \text{for all} \quad k \geq 0.
\]

**Proof.** We proceed in two steps.
1. We first establish the existence of $s^0$. To this end, choose $c > 0$ sufficiently small such that, for the vector $z_c := ce + (1 - P^\tau)^{-1}e$, it holds

$$(1 - P^\tau)z_c = c(I - P^\tau)e + e \geq \frac{1}{2}e.$$ 

Furthermore, let $\alpha > 0$ be large enough so that

$$|\text{diag}(P)|^{-1}|A - P|e \leq \alpha e.$$ 

Then, defining $s^0 := 2az_c$, we obtain the estimate

$$(1 - P^\tau)s^0 \geq \alpha e \geq |\text{diag}(P)|^{-1}|A - P|e,$$ 

which is (14).

2. Next, from (13) with $k = 0$, we have

$$s^1 - s^0 = -(1 - P^\tau)s^0 + |\text{diag}(P)|^{-1}|A - P|e \leq 0.$$ 

Hence, by induction, since $P^\tau \geq 0$, from (13) we note that

$$s^{k+1} - s^k = P^\tau(s^k - s^{k-1}) \geq 0 \quad \forall \ k \geq 1.$$ 

Using that $\rho(P^\tau) < 1$, we infer that (13) converges to $s(A, P)$ from (12). Moreover, from (12) and (13) we deduce the identity

$$(1 - P^\tau)s(A, P) = |\text{diag}(P)|^{-1}|A - P|e$$

$$(15) = s^{k+1} - P^\tau s^k
= (1 - P^\tau)s^{k+1} + P^\tau(s^{k+1} - s^k),$$

for all $k \geq 0$. Exploiting that $(1 - P^\tau)^{-1}P^\tau \geq 0$, this implies that

$$s(A, P) = s^{k+1} + (1 - P^\tau)^{-1}P^\tau(s^{k+1} - s^k) \leq s^{k+1}.$$ 

Since $s(A, P)$ and $s^{k+1}$ are both non-negative, the asserted bound follows.

The following estimate, which provides a connection between the $\infty$-norm and the Sassenfeld index, is a crucial observation for some of our subsequent results.

**Proposition 3.3.** Let $A \in \mathbb{R}^{m \times m}$ be an invertible matrix, and $P \in \mathbb{R}^{m \times m}$ a Sassenfeld preconditioner. Then, it holds that

$$\|1 - P^{-1}A\|_\infty \leq \mu(A, P).$$

**Proof.** Consider an arbitrary vector $y \in \mathbb{R}^m$ with $\|y\|_\infty = 1$. Defining $R := P - A$, we let

$$x = P^{-1}Ry = P^{-1}(P - A)y = (1 - P^{-1}A)y.$$ 

Then, we have $\text{diag}(P)x + \text{off}(P)x = R y$. Taking moduli results in

$$|\text{diag}(P)| |1 - P^\tau| |x| = |\text{diag}(P) - \text{off}(P)| |x| \leq |R| |y| \leq |R| |e|.$$ 

Recalling Rem. 2.2 and (12), we deduce that

$$|x| \leq (1 - P^\tau)^{-1}|\text{diag}(P)|^{-1}|R|e = s(A, P).$$ 

Therefore, using (17), we infer that

$$\|1 - P^{-1}A\|_\infty = \|x\|_\infty \leq \|s(A, P)\|_\infty,$$ 

which yields (16).

**Corollary 3.4 (Invertibility).** Given a matrix $A$, and a Sassenfeld preconditioner $P$. Then, the matrix $A + \tau P$ is non-singular whenever $|\tau + 1| > \mu(A, P)$. 

\[\square\]
Proof. We apply a contradiction argument. To this end, suppose that there exists \( v \in \mathbb{R}^m, \|v\|_\infty = 1 \), such that \( A_rv = 0 \). Then, it holds that \( (r + 1)v = (P - A)v \), and thus \( (r + 1)v = P^{-1}(P - A)v \). Taking norms, and using (10), yields

\[
|r + 1| = \|1 - P^{-1}A\|v\|_\infty = \|1 - P^{-1}A\|_\infty \leq \mu(A, P),
\]

which causes a contradiction to the range of \( r \).

\[\square\]

4. Sassenfeld matrices

We are now ready to introduce the notion of Sassenfeld matrices. Our definition, see Def. 4.1 below, is motivated by the work [BW17], where the special case of all matrices \( A \in \mathbb{R}^{m \times m} \) with \( \mu(A, P) < 1 \), with \( P = \text{tril}(A) + \text{diag}(A) \) being the Gauß-Seidel preconditioner, has been discussed. In this specific situation, the system (12) takes the (lower-triangular) form

\[
|\text{diag}(A)|s = |\text{tril}(A)|s + |\text{triu}(A)|e,
\]

which is a simple forward solve for \( s \). Convergence of the Gauß-Seidel method is guaranteed if \( \|s\|_\infty < 1 \); this is the key observation in Sassenfeld’s original work [Sas51].

More generally, for Sassenfeld preconditioners in the current paper, we propose the following definition.

**Definition 4.1 (Sassenfeld matrices).** A matrix \( A \in \mathbb{R}^{m \times m} \) is called a Sassenfeld matrix if there exists a Sassenfeld preconditioner \( P \in \mathbb{R}^{m \times m} \) such that \( \mu(A, P) < 1 \).

From Cor. 3.3 for \( r = 0 \), we immediately deduce the following result.

**Proposition 4.2.** Every Sassenfeld matrix is invertible.

In the context of linear solvers, the following generalization of Sassenfeld’s result [Sas51] on the Gauß-Seidel scheme is an immediate consequence of Prop. 3.3.

**Proposition 4.3 (Iterative solvers).** For a Sassenfeld matrix \( A \in \mathbb{R}^{m \times m} \), and any given vector \( b \in \mathbb{R}^m \), consider the linear system

\[
Ax = b.
\]

Then, for a Sassenfeld preconditioner \( P \in \mathbb{R}^{m \times m} \) with \( \mu(A, P) < 1 \), and an arbitrary starting vector \( x_0 \in \mathbb{R}^m \), the iterative scheme

\[
P_{n+1} = (P - A)x_n + b, \quad n \geq 0,
\]

converges to the unique solution of (18). Furthermore, it holds the a priori bound

\[
\|x - x_n\|_\infty \leq \mu(A, P)^n \|x - x_0\|_\infty,
\]

for any \( n \geq 0 \).

The following proposition provides a condition number estimate for the preconditioned matrix \( P^{-1}A \) in terms of the Sassenfeld index.

**Proposition 4.4 (Condition number bound).** Suppose that \( A \in \mathbb{R}^{m \times m} \) is a Sassenfeld matrix, and \( P \in \mathbb{R}^{m \times m} \) a Sassenfeld preconditioner with \( \mu(A, P) < 1 \). Then, for the condition number with respect to the norm \( \|\cdot\|_\infty \) the bound

\[
\kappa_\infty(P^{-1}A) := \|P^{-1}A\|_\infty \|P^{-1}A\|_\infty \leq \frac{1 + \mu(A, P)}{1 - \mu(A, P)}
\]

holds true.

**Proof.** Let \( B := P^{-1}A \). From Prop. 4.3 we deduce the bound

\[
\|B\|_\infty \leq 1 + \|B\|_\infty \leq 1 + \mu(A, P).
\]
Proof. Given a Sassenfeld matrix $A \in \mathbb{R}^{m \times m}$, any Sassenfeld matrix belongs to the characterization of Sassenfeld matrices. Before doing so, we notice the following fact.

This concludes the proof. □

Moreover, applying a Neumann series, see, e.g., [Mey00, §3.8], we deduce the estimate

$$
\|B^{-1}\|_{\infty} = \|(1 - (1 - B))^{-1}\|_{\infty} \leq \frac{1}{1 - \|1 - B\|_{\infty}} \leq \frac{1}{1 - \mu(A, P)}
$$

This provides a characterization of Sassenfeld matrices. Before doing so, we notice the following fact.

**Lemma 4.5.** Any Sassenfeld matrix belongs to $\mathbb{R}^{m \times m}_{s}.

\begin{proof}
Given a Sassenfeld matrix $A \in \mathbb{R}^{m \times m}$, and a Sassenfeld preconditioner $P \in \mathbb{R}^{m \times m}$ with $\mu(A, P) < 1$. Then, from (15) we have

$$
\left(|\text{diag}(P)| - |\text{off}(P)|\right)s(A, P) = |A - P|e,
$$

with $0 \leq s(A, P) = (s_1, \ldots, s_m) < e$. In components, this system reads

$$
|p_{ii}|s_i - \sum_{j \neq i} |p_{ij}|s_j = \sum_{j=1}^{m} |a_{ij} - p_{ij}|, \quad i = 1, \ldots, m.
$$

Letting

$$
e_i := \sum_{j=1}^{m} (1 - s_j)|a_{ij} - p_{ij}| \geq 0, \quad i = 1, \ldots, m,
$$

and rearranging terms, we observe the identity

$$
e_i + \sum_{j \neq i} (|p_{ij}| + |a_{ij} - p_{ij}|) s_j = (|p_{ii}| - |a_{ii} - p_{ii}|) s_i,
$$

for each $i = 1, \ldots, m$. Applying the triangle inequality on either side, it follows that

$$
e_i + \sum_{j \neq i} |a_{ij}|s_j \leq |a_{ii}|s_i, \quad i = 1, \ldots, m.
$$

Fix $i \in \{1, \ldots, m\}$. If $e_i > 0$ then it follows directly from (21) that $a_{ii} \neq 0$. Otherwise, if $e_i = 0$ then, from (20) and the fact that $s_j < 1$ for each $j = 1, \ldots, m$, we infer that $p_{ij} = a_{ij}$ for all $j = 1, \ldots, m$; in particular, for $j = i$, this shows that $a_{ii} = p_{ii} \neq 0$, which completes the proof. □

**Theorem 4.6.** (Characterization of Sassenfeld matrices.) A matrix $A \in \mathbb{R}^{m \times m}$ is a Sassenfeld matrix if and only if it is a Sassenfeld preconditioner.

\begin{proof}
If $A \in \mathbb{R}^{m \times m}$ is a Sassenfeld preconditioner then $\mu(A, A) = 0$, i.e., $A \in \mathbb{R}^{m \times m}_{s}$ is a Sassenfeld matrix. Conversely, suppose that $A \in \mathbb{R}^{m \times m}$ is a Sassenfeld matrix. Then, due to Lemma 4.5, we know that $A \in \mathbb{R}^{m \times m}_{s}$. Hence, it remains to prove that $\mathcal{g}(A^*) < 1$. To this end, select a Sassenfeld preconditioner $P \in \mathbb{R}^{m \times m}$ with $\mu(A, P) < 1$. Then, recalling (21), there are non-negative real numbers $0 \leq s_i < 1$, $i = 1, \ldots, m$, such that

$$
\sum_{j \neq i} |a_{ij}|s_j \leq |a_{ii}|(s_i - \delta_i), \quad i = 1, \ldots, m,
$$

with

$$
\delta_i := \frac{1}{|a_{ii}|} \sum_{j=1}^{m} (1 - s_j)|a_{ij} - p_{ij}| \geq 0, \quad i = 1, \ldots, m;
$$

cf., (20). Furthermore, since $\mathcal{g}(P^*) < 1$, by Lem., there is $z > 0$, scaled by $e^Tz = 1$, such that $P^*z < z$, cf. (9). Equivalently,

$$
\frac{1}{|p_{ii}|} \sum_{j \neq i} |p_{ij}|z_j < z_i,
$$

for each $i = 1, \ldots, m$. Introduce a positive vector $\theta = (\theta_1, \ldots, \theta_m) \in \mathbb{R}^m$ by

$$
\theta_i := a_{ii} + z_i > 0, \quad i = 1, \ldots, m,
$$

there are non-negative real numbers $0 \leq s_i < 1$, $i = 1, \ldots, m$, such that

$$
\sum_{j \neq i} |a_{ij}|s_j \leq |a_{ii}|(s_i - \delta_i), \quad i = 1, \ldots, m,
$$

with

$$
\delta_i := \frac{1}{|a_{ii}|} \sum_{j=1}^{m} (1 - s_j)|a_{ij} - p_{ij}| \geq 0, \quad i = 1, \ldots, m;
$$

cf. (20). Furthermore, since $\mathcal{g}(P^*) < 1$, by Lem., there is $z > 0$, scaled by $e^Tz = 1$, such that $P^*z < z$, cf. (9). Equivalently,
where $\alpha > 0$ will be specified later. Moreover, define the matrix $B = A^+ = \{b_{ij}\} \in \mathbb{R}^{m \times m}$ by

$$b_{ij} := \begin{cases} 0 & \text{for } i = j, \\ \frac{|a_{ij}|}{|a_{ii}|} |a_{jj}| & \text{for } i \neq j, \end{cases}$$

and

$$d_i := \sum_{j \neq i} \left( b_{ij} - \frac{|p_{ij}|}{|p_{ii}|} \right) z_j, \quad 1 \leq i \leq m.$$

Then, for $1 \leq i \leq m$, we have

$$\sum_{j=1}^{m} b_{ij} \Theta_j = \sum_{j \neq i} b_{ij} \Theta_j = \frac{\alpha}{|a_{ii}|} \sum_{j \neq i} |a_{ij}| s_j + \frac{1}{|p_{ii}|} \sum_{j \neq i} |p_{ij}| z_j + d_i.$$

Employing (22) and (24), we derive the estimate

$$\sum_{j=1}^{m} b_{ij} \Theta_j < \alpha (s_i - \delta_i) + z_i + d_i.$$

Thus, we obtain

$$\sum_{j=1}^{m} b_{ij} \Theta_j < \alpha i - \alpha \delta_i + d_i, \quad (25)$$

for each $i = 1, \ldots, m$. Now let

$$\alpha \geq \max_{i \in J} \frac{|d_i|}{\delta_i} > 0,$$

where $J$ signifies the set of all indices $1 \leq i \leq m$ for which $\delta_i > 0$ in (23); we let $\alpha = 0$ if $J = \emptyset$. We distinguish two separate cases for each $1 \leq i \leq m$:

(i) If $\delta_i = 0$ then exploiting that $0 \leq s_j < 1$ for each $j = 1, \ldots, m$, we notice from (23) that $a_{ij} = p_{ij}$ for all $j = 1, \ldots, m$. Hence, we find $d_i = 0$.

(ii) Otherwise, if $\delta_i > 0$ then recalling $\alpha$ from (26), we infer that

$$-\alpha \delta_i + d_i = \delta_i \left( -\alpha + \frac{|d_i|}{\delta_i} \right) \leq \delta_i \left( -\alpha + \frac{|d_i|}{\delta_i} \right) \leq 0.$$

In summary, from (25), we obtain that

$$\sum_{j=1}^{m} b_{ij} \Theta_j < \Theta_i \quad \forall i = 1, \ldots, m.$$

Therefore, we have shown that there exists a positive vector $\Theta > 0$ such that $A^+ \Theta = B \Theta < \Theta$, which, by Lem. 2.3 implies that $A^+$ is a Sassenfeld preconditioner.

5. Spectral properties

As far as the eigenvalues of a Sassenfeld matrix are concerned, we establish a result that is related to the Gershgorin circle theorem (see, e.g., [Mey00, p. 498]). To this end, for a center point $a \in \mathbb{R}$, $a \neq 0$, we define the open ball

$$\mathcal{B}(a) := \{z \in \mathbb{C} : |z - a| < |a|\}$$

in the complex plane $\mathbb{C}$.

**Theorem 5.1** (Spectrum of Sassenfeld matrices). *Let $A \in \mathbb{R}^{m \times m}$ be a Sassenfeld matrix, and denote by $\sigma(A) \subset \mathbb{C}$ its spectrum. Then, for any Sassenfeld preconditioner $P \in \mathbb{R}^{m \times m}$ with $\mu(A, P) < 1$, it holds the inclusion

$$\sigma(A) \subset \bigcup_{i=1}^{m} \mathcal{B}(p_{ii}).$$*
Remark 5.3. From the above corollary, we can deduce a few interesting properties about $P$.

Proof. Suppose that $A \in \mathbb{R}^{m \times m}$ is a Sassenfeld matrix. Let $\lambda \in \sigma(A)$ be an eigenvalue, and $v = (v_1, \ldots, v_m) \in \mathbb{C}^m$ an associated eigenvector with $\|v\|_{\infty} = 1$. Recalling (3), we can write

$$\text{off}(P)v + (A - P)v = (\lambda 1 - \text{diag}(P))v.$$ 

Inverting by diag$(P)$, and taking moduli, we obtain

$$|\lambda \text{diag}(P)^{-1} - 1| |v| \leq |\text{diag}(P)|^{-1} |A - P| e + P^s |v|.$$ 

Moreover, recalling (15), we infer that

$$\|s(A, P) - |v| \|. 

Using that $(1 - P^s)^{-1} \geq 0$, cf. Rem. 2.2, it follows that

$$(1 - P^s)^{-1} [|\lambda \text{diag}(P)^{-1} - 1| |v| \leq s(A, P) - |v|.$$ 

Since $\|s(A, P)\|_{\infty} = \mu(A, P) < 1$ and $\|v\|_{\infty} = 1$, there is an index $i \in \{1, \ldots, m\}$ such that $s_i(A, P) - |v_i| < 0$. Therefore, the (diagonal) matrix $|\lambda \text{diag}(P)^{-1} - 1|$ has at least one negative diagonal entry, i.e. there exists $j \in \{1, \ldots, m\}$ with $|\lambda a_{ij} - 1| < 1$. This concludes the proof. 

□

Using Thm. 4.6 we may choose $P = A$ in Thm. 5.1 in order to draw the following conclusion.

Corollary 5.2. For any Sassenfeld matrix $A \in \mathbb{R}^{m \times m}$ we have

$$\sigma(A) \subseteq \bigcup_{j=1}^{m} \mathcal{B}(a_{jj}).$$

Remark 5.3. From the above corollary, we can deduce a few interesting properties about Sassenfeld matrices.

(i) If the diagonal entries of a Sassenfeld matrix are all positive or negative, then its eigenvalues belong to the corresponding (open) half plane of $\mathbb{C}$.

(ii) In particular, from (i), we infer that every symmetric Sassenfeld matrix with positive diagonal entries is symmetric positive definite.

(iii) If $A = [a_{ij}] \in \mathbb{R}^{m \times m}$ is a Sassenfeld matrix then the spectral radius of $A$ satisfies the bound $\rho(A) < 2 \max_{1 \leq i \leq m} |a_{ii}|$.

6. APPLICATIONS

Our main Thm. 4.6 allows for a straightforward construction of preconditioners of a Sassenfeld matrix. Indeed, consider a Sassenfeld matrix $A \in \mathbb{R}^{m \times m}$, and define the set

$$\mathcal{P}(A) := \{ P \in \mathbb{R}^{m \times m} : \text{diag}(P) = \text{diag}(A) \text{ and } P \sqsubseteq A \},$$

where we write $P \sqsubseteq A$ to mean that any non-diagonal entry of $P$ is either zero or equals the corresponding entry of $A$. For any $P \in \mathcal{P}(A)$ observe that $P^s \preceq A^s$, and consequently $\rho(P^s) \leq \rho(A^s) < 1$, cf. Thm. 4.6. In particular, any $P \in \mathcal{P}(A)$ is a Sassenfeld preconditioner.

Proposition 6.1. Given a Sassenfeld matrix $A \in \mathbb{R}^{m \times m}$. Furthermore, let $P \in \mathcal{P}(A)$ and $0 \leq \delta < 1$ such that

$$(A^s - P^s)e \leq \delta (1 - P^s)e. \quad (27)$$

Then, it holds that $\mu(A, P) \leq \delta$. Note that (27) can be fulfilled trivially upon selecting $P = A$ and $\delta = 0$.

Proof. Since $\text{diag}(P) = \text{diag}(A)$, we notice that

$$|\text{diag}(A)|^{-1} |A - P| = |\text{diag}(A)|^{-1} A - \text{diag}(P)^{-1} P = A^s - P^s.$$ 

Thus, the vector $s(A, P) = (s_1, \ldots, s_m) \succeq 0$ from (12) satisfies

$$s(A, P) = (1 - P^s)^{-1} (A^s - P^s)e = -(1 - P^s)^{-1} (1 - A^s)e + e.$$

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In view of (27), we have the bound
\[(1 - A')e \succeq (1 - \delta)(1 - P')e,\]
which, upon recalling that \((1 - P')^{-1} \succeq 0\), cf. Rem. 2.2, implies that
\[(1 - P')^{-1}(1 - A')e \succeq (1 - \delta)e.\]
Therefore, we infer that \(s(A, P) \leq \delta e\), which shows that \(\mu(A, P) = \|s(A, P)\|_\infty \leq \delta\). This completes the proof.

For the diagonal preconditioner \(P = \text{diag}(A)\), the bound (27) simply expresses that \(A\) is strictly diagonally dominant (by rows). Thereby, in combination with Prop. 4.3, the above Prop. 6.1 recovers the well-known fact that the classical Jacobi iteration method is convergent for this type of matrices. We derive a slight generalization of this result for matrices \(A = [a_{ij}] \in \mathbb{R}^{m \times m}\), which are not necessarily strictly diagonally dominant. To this end, we define
\[\gamma_i := \sum_{j \neq i} |a_{ij}|, \quad i = 1, \ldots, m,\]
and suppose that the rows of \(A\) satisfy the bounds
\[\gamma_1 < |a_{11}|, \quad \text{and, for } i = 2, \ldots, m,\]
\[\gamma_i < |a_{ii}| \quad \text{if } a_{i,i-1} = 0,\]
\[\gamma_i < |a_{ii}| \quad \text{if } a_{i,i-1} \neq 0.\]
We claim that the lower bidiagonal preconditioner given by
\[P = \begin{pmatrix}
  a_{11} & a_{21} & \cdots & 0 \\
  a_{21} & a_{22} & \cdots & \vdots \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & \cdots & 0 & a_{mm}
\end{pmatrix}\]
renders the iteration (19) convergent. Indeed, we will prove the estimate \(\mu(A, P) < 1\); cf. Prop. 4.3. Due to the sparsity pattern of \(P\), we note that the system (12) takes the simple form
\[s_1 = \frac{\gamma_1}{|a_{11}|}, \quad \text{and} \quad s_i = \frac{1}{|a_{ii}|} \left(\gamma_i - (1 - s_{i-1})|a_{i,i-1}|\right) \quad \text{for } 2 \leq i \leq m.\]
From (28a), it follows that \(0 \leq s_1 < 1\). Furthermore, for an index \(i \in \{2, \ldots, m\}\), by induction, suppose that \(0 \leq s_j < 1\) for all \(j = 1, \ldots, i - 1\). If \(a_{i,i-1} = 0\) then, from (28b), we infer that
\[s_i \leq \gamma_i |a_{ii}| < 1.\]
Otherwise, if \(a_{i,i-1} \neq 0\) then, using (28c), we deduce that
\[s_i < \gamma_i |a_{ii}| \leq 1.\]
In conclusion, this shows that \(\mu(A, P) = \max_{1 \leq i \leq m} s_i < 1.\)

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