THE GARDEN OF QUANTUM SPHERES

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Abstract

A list of known quantum spheres of dimension one, two and three is presented.

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0. Introduction.

Recently, examples of quantum spheres cropped in abundance in the literature. The goal of this note is to aid the book-keeping of these newly emerged species by systematically comparing their basic properties.

As is customary in noncommutative geometry, these quantum spaces are described and studied in terms of certain noncommutative algebras, generalizing the usual correspondence between spaces and function algebras. Here I am concerned mainly with ‘deformations’ of the ∗-algebra of polynomials on the sphere $S^n$ and their enveloping $C^*$-algebras. The ∗-algebras are usually given in terms of generators and relations. Some of these relations can be regarded as deformations of the commutation relations and some as deformations of the sphere relation $\sum_{j=1}^{n+1} x_j^2 = 1$.

The classical spheres are often particular members of the family, or ‘limit’ cases. Here, by the dimension of such a quantum sphere I understand just the number $n$.

In this note the examples in lowest dimensions (one, two and three) are listed (in alphabetic order), which appeared in the literature known to the author, without pretending to be complete or exhaustive. Most of them have a $C^*$-algebraic version and often the deformation forms a continuous field of $C^*$-algebras. The smooth structure is described only in a few cases. Trying to uniformize the notation somewhat I mention some of their properties such as classical subspaces, (C∗-algebraic) K-groups and other remarks.

It turns out that among basic building blocks of quantum spheres are noncommutative tori and discs. The $C^*$-algebra of the noncommutative torus $T_\theta$, $0 \leq \theta < 1$, is generated by two unitaries $U, V$ with the relation $UV = e^{2\pi i \theta} VU$ [49]. The $C^*$-algebra of a quantum disc $D_{\mu,q}$, $0 \leq \mu \leq 1, 0 < q \leq 1$, is generated by $z$ with the relation $qzz^* - z^*z = q - 1 + \mu (1 - zz^*) (1 - z^*z)$ [31]. For $0 \leq \mu < 1 - q$ and for $q = 1, 0 < \mu < 1$, they are known [31, 51] (see also [25] and references therein) to be all isomorphic to the Toeplitz algebra $\mathcal{T}$ and, in turn, also to the $C^*$-algebra generated by the raising (shift) operator $e_j \mapsto e_{j+1}$ on $l^2(\mathbb{N})$, [51].

In the sequel I denote the algebra unit as $I$ and the compact operators on a (infinite dimensional, separable) Hilbert space as $K$. 

1
1. Quantum circle.

1.1. Markov [34].

Generators: $A, B$. Relations:

$$A = A^*, \quad B = B^*, \quad A^2 + B^2 = I.$$  

Classical subset: $S^1$. K-groups: $K_0 = Z, K_1 = 0$ [40]. Remarks:

1. The universal (unital) $C^*$-algebra for these relations can be represented as the algebra of all continuous $2 \times 2$ matrix-valued functions on the rectangle $[0, \pi/2] \times [0, \pi]$ satisfying certain boundary conditions.

2. The isomorphic universal unital $C^*$-algebra of the quantum disc at $q = -1, \mu = 0$ (generated by $z$ with the relation $z^* + z^2 = 2I$) has been independently studied in [41].

3. Note that this example does not really fit to our list (of deformations). However at least it could be supplemented by a commutation relation, e.g. $AB = qBA$, with $q = \pm 1$. Then the case $q = 1$ corresponds to the classical $S^1$ while in the case $q = -1$ the $C^*$-algebra is isomorphic to the universal $C^*$-algebra of the free product of groups $Z_2 \ast Z_2$ with $K_0 = Z$, $K_1 = 0$ and the classical subspace consisting of four points.

2. Quantum 2-spheres.

2.1. Bratteli, Elliott, Evans, Kishimoto [3, 4, 5].

Generators: $A, B$. Parameter: $-1 < \lambda = \cos(2\pi\theta) < 1, \ (0 < \theta < 1/2)$. Relations:

$$A = A^*, \quad B = B^*,$$

$$BABA = (4\lambda^2 - 1) ABAB - 2\lambda A^2B^2 + 8\lambda (1 - \lambda^2) (A^2 + B^2 - I),$$

$$A^2B + BA^2 = 2\lambdaABA + 4(1 - \lambda^2)B,$$

$$AB^2 + B^2A = 2\lambdaBAB + 4(1 - \lambda^2)A.$$  

Classical subset: $0$. K-groups: $K_0 = Z^6, K_1 = 0$ [45, 32]. Remarks:

1. Has been introduced, via $A = U + U^{-1}, V + V^{-1}$, as a fixed point algebra of the ‘flip’ automorphism $\sigma : U \mapsto U^{-1}, V \mapsto V^{-1}$ of the noncommutative torus $T_\theta$.

2. Classically (for $\theta = 0$) this is a ‘pillow’ (a smooth 2-sphere with four corners). After the deformation (for $0 < \theta < 1/4$) this geometry manifests in $K_0 = Z^6$ (with four generators besides $I$ and the Bott projector).

3. For $\lambda = 1$ ($\theta = 0$) this $C^*$-algebra indeed corresponds to the classical $S^2$ as it turns out that in this case either of the last two relations implies that $AB = BA$.

4. For irrational $\theta$ these $C^*$-algebras are simple, approximately finite-dimensional and with a unique trace state.

5. A closely related (strongly Morita equivalent for $\theta \neq 1/2$) is the crossproduct $C^*$-algebra $T_\theta \times_{\sigma} Z_2$, where the generator of $Z_2$ acts on the noncommutative torus $T_\theta$ by the ‘flip’ automorphism $\sigma : U \mapsto U^{-1}, V \mapsto V^{-1}$, cf. [3, 4, 5, 32, 54]. It is generated by three unitaries $U, V, W$ with the relations $VU = e^{2\pi i \theta} UV, WUW = U^*, WVW = V^*, W^2 = I$. For irrational $\theta$ the $C^*$-algebras are isomorphic if $\theta = \theta'$ or $\theta = 1 - \theta'$. For rational $\theta = p/q, \theta' = p'/q'$, with $p, p'$ and also $q, q'$ relatively prime, they are isomorphic if $q = q'$.
6. It admits other presentations [32] with three selfadjoint unitaries \(X, Y, Z\) and the relation 
\[XYZ = e^{2\pi i \theta} ZYX,\]
or with four selfadjoint unitaries \(X, Y, Z, T\) and the relation 
\[XY = e^{\pi i \theta} T Z.\]

2.2. Calow, Matthes [11].

Generators: \(A, B\). Parameters: \(0 < p, q < 1\). Relations:

\[
\begin{align*}
A &= A^* , \\
B^* B - qBB^* &= (p-q)A + I - p , \\
AB - pBA &= (1-p)B , \\
(I - A)(BB^* - A) &= 0.
\end{align*}
\]

Classical subset: \(S^1\). K-groups: \(K_0 = \mathbb{Z}^2\), \(K_1 = 0\) [11]. Remarks:

1. Obtained (at the \(\ast\)-algebra level) by glueing the quantum disc \(D_{0,p}\) with \(D_{0,q}\).
2. Classically (for \(p = 1 = q\)) this \(\ast\)-algebra describes a closed cone with one vortex and one circular edge.
3. It is non \(\ast\)-isomorphic [11] with any of the Podleś quantum spheres. However, its universal \(C^*\)-algebra is isomorphic to the \(C^*\)-algebra of the generic \((s > 0)\) Podleś quantum sphere, and as such can be realized as the Cuntz-Krieger algebra of a certain graph or as a quantum double suspension of two points [30].

2.3. Gurevich, Leclercq, Saponov [23].

Generators: \(A, B\). Parameters: \(h \geq 0, q > 0\). Relations:

\[
\begin{align*}
A &= A^* ,
qB^* B + q^{-1}BB^* + q^2(q + q^{-1})A^2 &= q + q^{-1} ,
q^2 AB - BA &= hB ,
BB^* - B^* B &= (1 - q^4)A^2 + h(1 + q^2)A.
\end{align*}
\]

Classical subset: \(S^2\) for \(h = 0, q = 1\); \(S^1\) for \(h = 0, q \neq 1\); a point for \(\pm(q^4 - 1)/q = h \neq 0\) and \(\emptyset\) otherwise. Remarks:

1. I have set the (radius) parameter \(-\alpha = 1\) in [23] by overall rescaling of the generators, which relate to those used in [23] as \(B = b\) and \(A = g/(1 + q^2)\), and the parameters as \(h = q^{-1}h (q \text{ is unchanged})\).
2. This example deforms the universal enveloping algebra of \(su(2)\) with a constrained value of the quadratic Casimir element.
3. The \(C^*\)-algebra completion is in general not possible but can be accomplished in particular cases.
4. The one-dimensional subfamily with \(h = 0\) coincides with the one parameter subfamily of equatorial quantum Podleś [48] spheres.
5. Another particular one-dimensional subfamily with \(q = 1\) coincides, using the variables 
\(Z = B/\sqrt{h^2 + 4}, H = A/\sqrt{h^2 + 4}\) and \(\mu = h/\sqrt{h^2 + 4}\) (when treated as a formal parameter) with the quantum 2-sphere of Omori, Maeda, Miyazaki, Yoshioka [46]. Any member of this subfamily forms a basis of a quantum principal \(U(1)\)-bundle with a total space being a formal deformation of \(S^1\) (with a noncentral formal parameter \(\mu\)), c.f. Section 4.
2.4. Natsume [42].

Generators: A, B. Parameter: \( t \in \mathbb{R} \). Relations:

\[
A = A^*, \quad B^* B + A^2 = I, \quad BB^* + (tBB^* + A)^2 = I, \quad BA - AB = tBB^* B.
\]

Classical subset: two points for \( t \neq 0 \). K-groups: \( K_0 = \mathbb{Z}^2, K_1 = 0 \) [44].

Remarks:
1. Motivated by Poisson geometry.
2. For \( t \in [0, 1/2[ \) the enveloping \( C^* \)-algebras are of type I and are isomorphic [44] to certain extension of \( C^2 \) by the crossproduct \( C^* \)-algebra \( C_0([1 - 1, 1]) \rtimes \mathbb{Z} \), or equivalently by \( K \otimes C(S^1) \) (the generator of \( \mathbb{Z} \) acts by automorphism \( \alpha_t \), given by a pull back of the homeomorphism \( x \mapsto tx^2 + x - t, \forall x \in ]-1, 1[ \), which is topologically conjugate to the translation by 1 on \( \mathbb{R} \)) [44].
3. They form a continuous field of \( C^* \)-algebras over \( ]0, 1/2[ \), which is trivial over \( ]0, 1/2[ \) (in particular they are all isomorphic for \( t \in ]0, 1/2[ \) ). They also constitute a strong deformation of \( C(S^2) \) [44].

2.5. Podleś [48].

Generators: A, B. Parameters: \( 0 \leq q < 1, \quad 0 \leq s \leq 1 \). Relations:

\[
A^* = A, \quad BA = q^2 AB,
\]

\[
BB^* = -q^4 A^2 + (1 - s^2)q^2 A + s^2 I, \quad B^* B = -A^2 + (1 - s^2)A + s^2 I.
\]

Classical subset: a point if \( c = 0; \quad S^1 \) if \( c \in ]0, \infty[ \). K-groups: \( K_0 = \mathbb{Z}^2, K_1 = 0 \) [36].

Remarks:
1. Discovered as homogeneous \( SU_q(2) \)-spaces.
2. In order to write the relations in a uniform way I parametrize the whole family as in [28] by the parameter \( 0 \leq s \leq 1 \), related to the parameter \( 0 \leq c \leq \infty \) in [48] by \( s = 2\sqrt{c}/(1 + \sqrt{1 + 4c}) \) (and \( c = (s^{-1} - 1)^2 \)). Also, I use the Podleś generators \( A, B \) rescaled, iff \( 0 \leq s < 1 \) (i.e., \( 0 \leq c < \infty \), by \( 1 - s^2 \).
3. Any members of this family describes a ‘round’ quantum sphere, in the sense that the Cartesian coordinates can be found i.e., three selfadjoint elements which generate the \( s \)-algebra, commute among themselves for \( q = 1, s = 0 \) and whose squares sum up to \( I \).
4. The case \( s = 0 \) (i.e., \( c = 0 \)) when the last two relations read \( BB^* = q^4 A - q^4 A^2 \) and \( B^* B = A - A^2 \), is known as the standard Podleś sphere. It can be viewed as a quotient sphere \( SU_q(2)/U(1) \) in the spirit of the Hopf fibration (c.f. Example 3.4). For all \( 0 \leq q < 1 \), the corresponding \( C^* \)-algebras are isomorphic to the minimal unitization of the compacts \( K \).
5. For \( 0 < s \leq 1 \) (i.e., \( 0 < c \leq \infty \)) the related \( C^* \)-algebras are all isomorphic [51] to certain extension of \( C(S^1) \) by \( K \oplus K \), or extension of \( T \) by \( K \). They can also be described, at the \( C^* \)-algebra level, as two quantum discs glued along their boundaries \( S^1 \) [51], see also [10, 11]; as the Cuntz-Krieger algebra of a certain graph or as a quantum double suspension of two points [30].
6. The case \( s = 1 \) (i.e. \( c = \infty \)) when the last two relations read \( BB^* = -q^4A^2 + I \) and \( B^*B = -A^2 + I \), is known as the \textit{equatorial Podleś sphere}. It is easily seen to be \( * \)-isomorphic to the two dimensional Euclidean sphere, introduced in [21]. As such, it admits a higher (even) dimensional generalization. Also, it contains the \( * \)-algebra of quantum disk, which can be geometrically interpreted as collapsing this quantum 2-sphere by the reflection with respect to the equatorial plane [25]. Moreover, it is isomorphic to the quotient of the underlying \( * \)-algebra of Example 3.4. (with parameter \( q^2 \)) by the relation \( b = b^* \). The geometric meaning of this is that the equatorial Podleś sphere embeds as an equator in \( S^3 \) thought of as a quantum 3-sphere [27]. Hence for fixed \( q \), the path \( 0 \leq s \leq 1 \) of Podleś spheres can be viewed as an interpolation between the quotient sphere \( SU_q(2)/U(1) \) and the embedded (equator) 2-sphere in \( SU_q(2) \).

3. Quantum 3-spheres.

3.1. Calow, Matthes [12].

**Generators:** \( a, b \). **Parameters:** \( 0 < p, q < 1 \). **Relations:**

\[
a^*a - qaa^* = 1 - q, \quad b^*b - pbb^* = 1 - p, \quad ab = ba, \quad a^*b = ba^*, \quad (I - aa^*)(I - bb^*) = 0.
\]

**Classical subset:** \( S^1 \times S^1 \). **K-groups:** \( K_0 = \mathbb{Z}, \quad K_1 = \mathbb{Z} \) [26]. **Remarks:**

1. As a \( * \)-algebra obtained by glueing the quantum solid torus \( D_{0,p} \times S^1 \) with \( D_{0,q} \times S^1 \).
2. As a \( C^* \)-algebra isomorphic to \( (T \otimes T)/(K \otimes K) \), or also to the Cuntz-Krieger algebra of a certain graph [26].
3. Forms a locally trivial, globally nontrivial [12], in fact nonclosed [26], quantum principal \( U(1) \)-bundle (Hopf-Galois extension) over the quantum \( S^3 \) of Example 2.2.

3.2. Connes, Dubois-Violette [16].

**Generators:** \( x^0, x^1, x^2, x^3 \). **Parameters:** \( \pi > \varphi_1 \geq \varphi_2 \geq \varphi_3 \geq 0 \). **Relations:**

\[
\begin{align*}
\cos(\varphi_1)(x^0x^1 - x^1x^0) &= i\sin(\varphi_2 - \varphi_3)(x^2x^3 + x^3x^2), \\
\cos(\varphi_2)(x^0x^2 - x^2x^0) &= i\sin(\varphi_3 - \varphi_1)(x^0x^3 + x^1x^2), \\
\cos(\varphi_3)(x^0x^3 - x^3x^0) &= i\sin(\varphi_1 - \varphi_2)(x^0x^2 + x^2x^1), \\
\cos(\varphi_2 - \varphi_3)(x^2x^3 - x^3x^2) &= -i\sin(\varphi_1)(x^0x^1 + x^1x^0), \\
\cos(\varphi_3 - \varphi_1)(x^0x^3 - x^3x^0) &= -i\sin(\varphi_2)(x^0x^2 + x^2x^1), \\
\cos(\varphi_1 - \varphi_2)(x^0x^2 - x^2x^0) &= -i\sin(\varphi_3)(x^0x^3 + x^3x^0), \\
\cos(\varphi_2)(x^1x^2 - x^2x^1) &= 1.
\end{align*}
\]

**Classical subset:** generically discrete. **K-groups:** will be studied in part II of [16]. **Remarks:**

1. Obtained as a unital \( * \)-algebra generated by four generators \( u_{jk}, \ j, k \in \{1, 2\} \) such that \( u \) as a \( 2 \times 2 \) matrix is unitary and \( ch_{3/2}(u) := \sum_{j,k} (u_{jk}u_{kj}^* - u_{jk}^*u_{kj}) = 0 \).
2. The particular one parameter subfamily \( \varphi_1 = \varphi_2 = -\frac{i}{2}\theta \) and \( \varphi_3 = 0 \) coincides, using the variables \( Z = x^0 + ix^3, W = x^1 + ix^2 \), with the particular one parameter subfamily of Matsumoto quantum spheres 3.3, when \( \Theta = \theta \) is a constant function, and thus also with the Natsume, Olsen [43] family. It fulfills all the properties of a noncommutative manifold in the sense of [14] and has a higher (odd) dimensional generalization.
3.3. Matsumoto [37, 38].

Parameters: real valued continuous functions \( \Theta \) on the closed interval \([0, 1]\).

Relations:
\[
ZW = e^{2\pi i \Theta(Z^* Z)} W Z, \quad Z^* Z + W^* W = I,
\]
where \( \Theta(Z^* Z) \) stands for the self-adjoint operator obtained by the functional calculus of the operator \( Z^* Z \) from the continuous function \( \Theta \).

Classical subset: \((\Theta^{-1}(Z) \cap [0, 1]) \times T^2) \cup ((\Theta^{-1}(Z) \cap \{0, 1\}) \times S^1\).

K-groups: \( K_0 = \mathbb{Z}, \quad K_1 = \mathbb{Z} \).

Remarks:
1. Obtained by glueing two quantum solid tori described by crossproduct \( C^* \)-algebra \( \mathcal{C}(D) \rtimes_\theta \mathbb{Z} \),
   where the generator of \( Z \) acts on the 2-disk \( D \) as a rotation by angle \( \Theta(r) \), and \( r \in [0, 1] \) is the radial coordinate on \( D \).
2. Forms a quantum principal \( U(1) \)-Hopf fibration over the usual 2-sphere.
3. When the function \( \Theta \) is a constant number \( \theta \), the \( C^* \)-algebra generated by the relations above has been studied by Natsume and Olsen [43] and shown to be isomorphic to the universal \( C^* \)-algebra generated by two normal operators \( T, S \) satisfying
   \[
   (I - T^* T)(I - S^* S) = 0 \quad \text{and} \quad \|T\| = 1 = \|S\|,
   \]
   introduced in [37]. Its classical subset is \( S^1 \sqcup S^3 \) and it has odd-dimensional generalization [43]. It coincides with the particular one parameter subfamily \( \varphi_1 = \varphi_2 = -\frac{1}{2}\theta \) and \( \varphi_3 = 0 \) of Connes, Dubois-Violette [16].

3.4. Woronowicz [55].

Parameters: \( q \in \mathbb{C} \).

Relations:
\[
ba = qab, \quad a^* b = qba^*, \quad a a^* + b b^* = I, \quad a^* a + |q|^2 b b^* = I, \quad b b^* = b^* b.
\]

Classical subset: \( S^1 \) for \( |q| \neq 1 \), \( S^1 \sqcup S^3 \) for \( q = 1 \) and \( q \neq 1 \), \( S^3 \) for \( q = 1 \).

K-groups: \( K_0 = \mathbb{Z}, \quad K_1 = \mathbb{Z} \) for \( q > 0 \) [35].

Remarks:
1. Has been discovered for \(-1 \leq q \leq 1, q \neq 0 \), as a family of quantum groups \( SU_q(2) \).
2. Here I generalize the range of the parameter to \( q \in \mathbb{C} \), this ‘interpolates’ between the original Woronowicz family \( q \in \mathbb{R} \), and the one-parameter subfamily |\( q | = 1 \) which coincides (if \( q = e^{i\theta} \)) with Natsume and Olsen [43] family and also with the particular one parameter subfamily \( \varphi_1 = \varphi_2 = -\frac{1}{2}\theta \) and \( \varphi_3 = 0 \) of the example 3.2 of Connes, Dubois-Violette [16].
   The transformation \( a \mapsto a^* \), \( b \mapsto -qb \), and \( q \mapsto 1/q \) defines a \( * \)-isomorphism for \( q \neq 0 \), hence it suffices to restrict to the range \( |q| \leq 1 \).
3. For \( 0 \leq q < 1 \) the members of this family are easily seen to be \( * \)-isomorphic to the three dimensional Euclidean spheres introduced in [21] and also to the three dimensional unitary spheres introduced as quantum homogeneous spaces of \( SU_q(n) \) in [53]. It turns out that also their higher dimensional generalizations, the quantum Euclidean spheres and the quantum unitary spheres, are \( * \)-isomorphic at any given odd dimension.\(^1\)
4. For all \( q \in \mathbb{C} \) these \( * \)-algebras have a \( C^* \)-algebraic version. For \( q = 1 \) this is just \( C(S^3) \). The case \( q = -1 \) has been studied in [56]. For \( 0 \leq q < 1 \) these \( C^* \)-algebras are all isomorphic to certain extension of \( C(S^3) \) by \( C(S^1) \otimes K \), which can also be described as the Cuntz-Krieger algebra of certain graph [30] or as a quantum double suspension of the circle [30].

\(^1\)To our knowledge this simple fact, which was observed during a conversation with G. Landi, E. Hawkins and F. Bonechi, has not been presented before.
5. For \(0 < q \leq 1\) forms a quantum principal \(U(1)\)-fibre bundle over the Podleš quantum 2-sphere of Example 2.5 in the sense of Hopf-Galois extensions (if \(s = 0\)) or coalgebra-Galois extensions (if \(s \in [0,1]\)) [9, 24, 6].

6. For some further noncommutative-geometric aspects see [15] and references therein.

4. Final comments.

Some finite dimensional algebras (in a sense corresponding to zero dimensional quantum spaces, which nevertheless possess certain properties of 2-spheres) have also been studied. For instance the classification [48] of \(SU_q(2)\)-homogeneous spaces, besides the family 2.5 of Podleš quantum spheres, also includes a discrete series of full matrix algebras Mat\(_N\). It has been observed in [22] that this family of ‘quantum spheres’ can be equipped with an additional structure, notably a sequence of injections \(\text{Mat}_N \rightarrow \text{Mat}_{N+1}\), which are morphisms in the category of \(U_q(\mathfrak{su}(2))\)-modules. For \(q = 1\), this agrees with the fuzzy-sphere philosophy of [33]. Therein, the \(N \times N\) matrix algebras are considered as \(U(\mathfrak{su}(2))\)-modules together with \(U(\mathfrak{su}(2))\)-module injections that form a direct system whose limit is the algebra of polynomials on \(S^2\) [29]. One can also show that the matrix algebras converge to the sphere for quantum Gromov–Hausdorff distance [50]. Furthermore, these matrix algebras can be viewed as representations of the universal enveloping algebra of \(\mathfrak{su}(2)\) with the value \(\frac{1}{2}\) of the quadratic Casimir element. This is why the discrete family of Podleš spheres can be thought of as the family of q-fuzzy spheres [22] (the aforementioned injections are \(U_q(\mathfrak{su}(2))\)-linear).

There are other examples of quantum spheres which do not fit exactly to our lists as they are not deformations in our sense (families of \(*\)-algebras). In [46, 47] a formal deformation of \(S^3\) (as a contact manifold) is provided with a invertible non-central deformation ‘parameter’ \(\mu\), generators \(a, b\) and relations

\[
\begin{align*}
\mu &= \mu^*, \\
\mu^{-1}a - a\mu^{-1} &= -a, \\
\mu^{-1}b - b\mu^{-1} &= -b, \\
ba &= ab, \\
ab^* &= (1 - \mu)b^*a, \\
(a^*a - (1 - \mu)a^*a &= \mu, \\
bb &= (1 - \mu)b^*b = \mu.
\end{align*}
\]

This deformation yields a certain ‘smooth’ algebra admitting a \(U(1)\)-action that is principal in the sense of Hopf-Galois theory [7] (the base space of the principal fibre bundle is given by the quantum 2-sphere of of Example 2.3.5.) Another example of a noncommutative Hopf fibration given by the principal \(U(1)\)-action on a super 3-sphere was studied in [18]. There are also examples of quantum complex spheres related to the Jordanian quantum group \(\text{SL}_h(2)\) [13, 57].

As far as four-dimensional quantum spheres are concerned, recently several examples together with instanton bundles over them, have been constructed. They indicate a wealth even greater than that of the known one, two and three dimensional examples. It should be mentioned that in general various principles were employed to proliferate the examples of quantum spheres, such as Poisson, contact or homogeneous structure. We have encountered also several types of glueing and quotienting. One of the tools is the usual suspension operation which can be used to find links between different examples and to produce new samples of one dimension greater. Note for instance that the suspension of Example 3.2.2 is just the Connes, Landi noncommutative 4-sphere [17], while the suspension of Example 3.4 occurs in [19], c.f. also [20]. A \(C^*\)-algebraic noncommutative double suspension has been also mentioned in examples 2.2, 2.5 and 3.4. A kind of quantum double suspension at the \(*\)-algebra level, which raises the dimension by two and yields different families of quantum 4-spheres, has been employed in [52] and [8]. The example of a quantum 4-sphere presented in [1], with its classical subspace being just a point, is yet another type of double suspension [2] of the standard Podleš quantum sphere, motivated by Poisson structure. Another method to obtain more examples employs the twisting (see e.g., [17, 16]).
However, it seems that it is quite premature yet to attempt any classification of quantum 4-spheres and that is certainly beyond the scope of this note.

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References

[1] F. Bonechi, N. Ciccoli, M. Tarlini, Noncommutative instantons and the 4-sphere from quantum groups, Commun. Math. Phys. 226 (2002) 419–432.

[2] F. Bonechi, N. Ciccoli, M. Tarlini, Quantum even spheres Σ_q^{2n} from Poisson double suspension, preprint math.QA/0211462.

[3] O. Bratteli, G. A. Elliott, D. E. Evans, A. Kishimoto, Noncommutative spheres. I, Internat. J. Math. 2 (1991) 139–166.

[4] O. Bratteli, G. A. Elliott, D. E. Evans, A. Kishimoto, Noncommutative spheres. II. Rational rotations, J. Operator Theory. 27 (1992) 53–85.

[5] O. Bratteli, A. Kishimoto, Noncommutative spheres. III. Irrational rotations, Commun. Math. Phys. 147 (1992) 605–624.

[6] T. Brzeziński, Quantum homogeneous spaces as quantum quotient spaces, J. Math. Phys. 37 (1996) 2388–2399.

[7] T. Brzeziński, L. Dąbrowski, B. Zieliński, Hopf fibration and monopole connection over the contact quantum spheres, in preparation.

[8] T. Brzeziński, C. Gonera, Non-commutative 4-spheres based on all Podleś 2-spheres and beyond, Lett. Math. Phys. 54 (2000/2001), 315–321.

[9] T. Brzeziński, S. Majid, Quantum group gauge theory on quantum spaces, Commun. Math. Phys. 157 (1993) 591–638. Erratum 167 (1995) 235.

[10] R. J. Budzyński, W. Kondracki, Quantum principal fiber bundles: Topological aspects, Rep. Math. Phys. 37 (1996) 365–385, preprint 517 PAN Warsaw 1993, hep-th/9401019.

[11] D. Calow, R. Matthes, Covering and gluing of algebras and differential algebras, J. Geom. Phys. 32 (2000) 364–396.

[12] D. Calow, R. Matthes, Connections on locally trivial quantum principal fibre bundles, J. Geom. Phys. 41 (2002) 114–165.

[13] R. Chakrabarti, J. Segar, Jordanian quantum spheres, Modern Phys. Lett. A 16 (2001) 1731–1740.

[14] A. Connes, Gravity coupled with matter and the foundation of noncommutative geometry, Commun. Math. Phys. 182 (1996) 155–176.

[15] A. Connes, Cyclic Cohomology, Quantum group Symmetries and the Local Index Formula for SUq(2), preprint math.QA/0209142.

[16] A. Connes, M. Dubois-Violette, Noncommutative finite-dimensional manifolds. I. Spherical manifolds and related examples, Commun. Math. Phys. 230 (2002) 539–579.

[17] A. Connes, G. Landi, Noncommutative manifolds, the instanton algebra and isospectral deformations, Commun. Math. Phys. 221 (2001) 141–159.

[18] L. Dąbrowski, H. Grosse, P. M. Hajac, Strong Connections and Chern-Connes Pairing in the Hopf-Galois Theory, Commun. Math. Phys. 220 (2001) 301–331.
[19] L. Dąbrowski, G. Landi, Instanton algebras and quantum 4-spheres, Diff. Geom. Appl. 16 (2002) 277–284.
[20] L. Dąbrowski, G. Landi, T. Masuda, Instantons on the Quantum 4-Spheres $S^4_q$, Commun. Math. Phys. 221 (2001) 161–168.
[21] L. D. Faddeev, N. Y. Reshetikhin, L. A. Takhtajan, Quantization Of Lie Groups And Lie Algebras, Alg. Anal. 1 (1990) 178. [Leningrad Math. J. 1 (1990) 193]
[22] H. Grosse, J. Madore, H. Steinacker, Field Theory on the q-deformed Fuzzy Sphere I, J. Geom. Phys. 38 (2001) 308–342.
[23] D. Gurevich, R. Leclercq, P. Saponov, Equivariant noncommutative index on braided sphere, preprint math.QA/0207268.
[24] P. M. Hajac, S. Majid, Projective module description of the q-monopole, Commun. Math. Phys. 206 (1999) 246–464.
[25] P. M. Hajac, R. Matthes, W. Szymański, Quantum real projective space, disc and sphere, Algebr. Represent. Theory, in press; preprint math.QA/0009185.
[26] P. M. Hajac, R. Matthes, W. Szymański, Locally Trivial Quantum Hopf Fibration, preprint math.QA/0112317.
[27] P. M. Hajac, R. Matthes, W. Szymański, Graph $C^*$-algebras and $\mathbb{Z}_q$-quotients of quantum spheres, preprint math.QA/0209268.
[28] P. M. Hajac, R. Matthes, W. Szymański, Fredholm index and noncommutative Hopf fibrations, in preparation.
[29] E. Hawkins, Quantization of Equivariant Vector Bundles, Commun. Math. Phys. 202 (1999) 517–546.
[30] J. H. Hong, W. Szymański, Quantum Spheres and Projective Spaces as Graph algebras, Commun. Math. Phys. 232 (2002) 157–188.
[31] S. Klimek, A. Leśniowski, A two-parameter quantum deformation of the unit disc, J. Funct. Anal. 115 (1993) 1–23.
[32] A. Kumjian, On the K-theory of the symmetrized non-commutative torus, C. R. Math. Rep. Acad. Sci. Canada 12 (1990) 87–89.
[33] J. Madore, The Fuzzy Sphere, Class. Quant. Grav. 9 (1992) 69.
[34] I. L. Markov, $C^*$-algebra generated by a noncommutative circle, in: Appl. Meth. Funct. Anal. in Math. Phys., Berezhanskii ed. p. 70–78, Kiev 1991 (in Russian).
[35] T. Masuda, Y. Nakagami, J. Watanabe, Noncommutative differential geometry on the quantum SU(2), I: An algebraic viewpoint, K-Theory 4 (1990) 157–180.
[36] T. Masuda, Y. Nakagami, J. Watanabe, Noncommutative differential geometry on the quantum two sphere of Podleś. I: An algebraic viewpoint, K-Theory 5 (1991) 151–175.
[37] K. Matsumoto, Non-commutative three dimensional spheres, Japan J. Math. 17 (1991) 333–356.
[38] K. Matsumoto, Non-commutative three dimensional spheres II—non-commutative Hopf fibering, Yokohama Math. J. 38 (1991) 103–111.
[39] K. Matsumoto, J. Tomiyama, Non-commutative lens spaces, J. Math. Soc. Japan 44 (1992) 13–41.
[40] G. Nagy, On the K-theory of the non-commutative circle, J. Oper. Theory 31 (1994) 303–309.
G. Nagy, A. Nica, On the “quantum disc” and a “non-commutative circle”, In: Algebraic Methods in Operator Theory, R. E. Curto, P. E. T. Jorgensen (Eds.), Birkhäuser, Basel, (1994) 276–290.

T. Natsume, On a continuous deformation of the 2-sphere, Talk at the MSRI, Apr 25 2001.

T. Natsume, C. L. Olsen, Toeplitz operators on noncommutative spheres and an index theorem, Indiana Univ. Math. J. 46 (1997) 1055–1112.

T. Natsume, C. L. Olsen, A new family of noncommutative 2-spheres, preprint (2002).

R. Nest, Cyclic cohomology of a noncommutative sphere, Kopenhagen Univ. preprint (1989).

H. Omori, Y. Maeda, N. Miyazaki, A. Yoshioka, Noncommutative 3-sphere: A model of noncommutative contact algebras, J. Math. Soc. Japan 50 (1998) 915–943.

H. Omori, N. Miyazaki, A. Yoshioka, Y. Maeda, Noncommutative 3-sphere as an example of noncommutative contact algebras, Banach Center Publications 40 (1997) 329–334.

P. Podleś, Quantum Spheres, Lett. Math. Phys. 14 (1987) 521–531.

M. A. Rieffel, C∗-algebras associated to irrational rotations, Pacific J. Math. 93 (1981) 415–429.

M. A. Rieffel, Matrix algebras converge to the sphere for quantum Gromov–Hausdorff distance, preprint math.OA/0108005.

A. J-L. Sheu, Quantization of the Poisson SU(2) and its Poisson homogeneous space – the 2-sphere, Commun. Math. Phys. 135 (1991) 217–232.

A. Sitarz, More noncommutative 4-spheres, Lett. Math. Phys. 55 (2001) 127–131.

L. L. Vaksman, Y. S. Soibelman, Algebra of functions on quantum SU(n+1) group and odd dimensional quantum spheres, Alg. Anal. 2 (1990) 101.

S. G. Walters, Projective modules over the Non-commutative Sphere, J. Lond. Math. Soc. 51 (1995) 589–602.

S. L. Woronowicz, Twisted SU(2) group. An example of a noncommutative differential calculus, Publ. RIMS, Kyoto University, 23 (1987) 117–181.

S. Zakrzewski, Matrix pseudogroups associated with anti-commutative plane, Lett. Math. Phys. 21 (1991) 309–321.

B. Zieliński, An idempotent for a Jordanian quantum sphere, this volume.