EQUATIONS AND SYZYGIES OF THE FIRST SECANT VARIETY TO A SMOOTH CURVE

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Abstract. We show that if $C$ is a linearly normal smooth curve embedded by a line bundle of degree at least $2g + 3 + p$, then the secant variety to the curve satisfies $N_{3,p}$.

1. Introduction

We work throughout over an algebraically closed field $k$ of characteristic zero. If $X \subset \mathbb{P}^n$ is a smooth variety, then we let $\Sigma_i(X)$ (or just $\Sigma_i$ if the context is clear) denote the (complete) variety of $(i+1)$-secant $i$-planes. Though secant varieties are a classical subject, the majority of the work done involves determining the dimensions of secant varieties to well-known varieties. Perhaps the two most well-known results in this direction are the solution by Alexander and Hirschowitz (completed in [1]) of the Waring problem for homogeneous polynomials and the classification of the Severi varieties by Zak [16].

More recently there has been great interest, e.g. related to algebraic statistics and algebraic complexity, in determining the equations defining secant varieties (e.g. [2], [4], [5], [11], [13], [19], [21]). In this work, we use the detailed geometric information concerning secant varieties developed by Bertram [3], Thaddeus [22], and the author [23] to study not just the equations defining secant varieties, but the syzygies among those equations as well.

It was conjectured in [8] and it was shown in [18] that if $C$ is a smooth curve embedded by a line bundle of degree at least $2g + 2k + 3$, then $\Sigma_k$ is set theoretically defined by the $(k+2) \times (k+2)$ minors of a matrix of linear forms. This was extended in [12], where the degree bound was improved to $4g + 2k + 2$ and it was shown that the secant varieties are scheme theoretically cut out by the minors. It was further shown in [24] that if $X \subset \mathbb{P}^n$ satisfies condition $N_2$, then $\Sigma_1(v_d(X))$ is set theoretically defined by cubics for $d \geq 2$.

In [25] it was shown that if $C$ is a smooth curve embedded by a line bundle of degree at least $2g + 3$, then $T_{\Sigma_1}$ is 5-regular, and under the same hypothesis it was shown in [20] that $\Sigma_1$ is arithmetically Cohen-Macaulay. Together with the analogous well-known facts for the curve $C$ itself ([10], [13], [17]), this led to the following conjecture, extending the one found in [24].
**Conjecture 1.1** \((\text{[20]})\). Suppose that \(C \subset \mathbb{P}^n\) is a smooth linearly normal curve of degree \(d \geq 2g + 2k + 1 + p\), where \(p, k \geq 0\). Then:

1. \(\Sigma_k\) is ACM and \(\mathcal{I}_{\Sigma_k}\) has regularity \(2k + 3\) unless \(g = 0\), in which case the regularity is \(k + 2\).
2. \(\beta_{n-2k-1,n+1}(\Sigma_k) = (\frac{g+k}{k+1})\).
3. \(\Sigma_k\) satisfies \(N_{k+2,p}\). \(\square\)

**Remark 1.2.** Recall \([7]\) that a variety \(Z \subset \mathbb{P}^n\) satisfies \(N_{r,p}\) if the ideal of \(Z\) is generated in degree \(r\) and the syzygies among the equations are linear for \(p - 1\) steps. Note that the better-known condition \(N_p\) \([13]\) implies \(N_{2,p}\).

By the work of Green and Lazarsfeld \([13, 15]\), the conjecture holds for \(k = 0\). Further, by \([9]\) and by \([27]\) it holds for \(g \leq 1\), and by \([20]\) parts (1) and (2) hold for \(k = 1\). In this work, we show that part (3) holds for \(k = 1\) (Theorem 3.5). Some analogous results for higher dimensional varieties can be found in \([26]\).

Our approach combines the geometric knowledge of secant varieties mentioned above with the well-known Koszul approach of Green and Lazarsfeld. To fix notation, if \(L\) is a vector bundle on a smooth curve \(C\), then we let \(\mathcal{E}_L = d_*(L \boxtimes \mathcal{O})\), where \(d: C \times C \to \mathcal{S}^2 C\) is the natural double cover, and if \(\mathcal{F}\) is a globally generated coherent sheaf on a variety \(X\), then we have the coherent sheaf \(\mathcal{M}_F\) defined via the exact sequence \(0 \to \mathcal{M}_F \to \Gamma(\mathcal{O}(1)) \otimes \mathcal{O}_\Sigma(b)\).

As we will be interested only in the first secant variety for the remainder of the paper, we write \(\Sigma\) for \(\Sigma_1\).

## 2. Preliminaries

Our starting point is the familiar:

**Proposition 2.1.** Let \(C \subset \mathbb{P}^n\) be a smooth curve embedded by a line bundle \(L\) of degree at least \(2g + 3\). Then \(\Sigma\) satisfies \(N_{3,p}\) if and only if \(H^1(\Sigma, \wedge^a M_L(b)) = 0\), \(2 \leq a \leq p + 1, \ b \geq 2\).

**Proof.** Because \(L\) also induces an embedding \(\Sigma \subset \mathbb{P}^n\), we abuse notation and denote the associated vector bundle on \(\Sigma\) by \(M_L\). Letting \(F = \bigoplus \Gamma(\mathcal{O}(n))\) and applying \([6, 5.8]\) to \(\mathcal{O}_\Sigma\) give the exact sequence:

\[
0 \to \text{Tor}_{a-1}(F, k)_{a+b} \to H^1(\Sigma, \wedge^a M_L(b)) \to H^1(\Sigma, \wedge^a \Gamma(\mathcal{O}(1)) \otimes \mathcal{O}_\Sigma(b))
\]

As \(\Sigma\) is ACM \([20]\), the term on the right vanishes. \(\square\)

**Notation and Terminology 2.2.** Under the hypothesis that \(\deg(L) \geq 2g + 3\), the reader should keep in mind the following morphisms \([23]\):

\[
\begin{array}{c}
\xymatrix{ & S^2 C 
\ar[dr]^d & \\
 Z = C \times C \ar[r]^i \ar[ur]^\varphi & \Sigma \\
 C \ar[rr]^\pi \ar[ru]_{\pi_1=\pi|z} & & \Sigma \ar[u]_{\pi_2}
}\end{array}
\]

where

- \(\pi\) is the blow up of \(\Sigma\) along \(C\).
- \(i\) is the inclusion of the exceptional divisor of the blow-up.
• $d$ is the double cover; $\pi_i$ are the projections.
• $\varphi$ is the morphism induced by the linear system $|2H - E|$ which gives $\tilde{\Sigma}$ the structure of a $\mathbb{P}^1$-bundle over $S^2C$; note in particular that $\tilde{\Sigma}$ is smooth.

We make frequent use of the rank 2 vector bundle $E_L = \varphi_* \mathcal{O}(H) = d_* (L \boxtimes \mathcal{O})$ and the fact [25, Proposition 9] that $R^i \pi_* \mathcal{O}_{\tilde{\Sigma}} = H^i(C, \mathcal{O}_C) \otimes \mathcal{O}_C$ for $i \geq 1$.

\textbf{Proposition 2.3.} If $C$ is a smooth curve embedded by a line bundle $L$ with $\text{deg}(L) \geq 2g + 3$, then $\Sigma$ satisfies $N_{3,p}$ if and only if

$$H^1(\tilde{\Sigma}, \pi_* \bigwedge^a M_L(b)) \to H^0(\Sigma, \bigwedge^a M_L(b) \otimes R^1 \pi_* \mathcal{O}_{\tilde{\Sigma}})$$

is injective for $2 \leq a \leq p + 1$, $b \geq 2$.

\textbf{Proof.} This follows immediately from the 5-term sequence associated to the Leray-Serre spectral sequence,

$$0 \to H^1(\Sigma, \bigwedge^a M_L(b)) \to H^1(\tilde{\Sigma}, \pi_* \bigwedge^a M_L(b)) \to H^0(\Sigma, \bigwedge^a M_L(b) \otimes R^1 \pi_* \mathcal{O}_{\tilde{\Sigma}}),$$

and Proposition 2.1. \hfill \square

We will need a cohomological result:

\textbf{Lemma 2.4.} Let $C \subset \mathbb{P}^n$ be a smooth curve embedded by a line bundle $L$ with $\text{deg}(L) \geq 2g + 3$. Then $H^i(\tilde{\Sigma}, \mathcal{O}_{\tilde{\Sigma}}(bH - E)) = 0$ for $i, b \geq 1$.

\textbf{Proof.} Because $C$ is projectively normal we have $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(bH - E)) = 0$ for $i, b \geq 1$. Thus $H^i(\tilde{\Sigma}, \mathcal{O}_{\tilde{\Sigma}}(bH - E)) = H^{i+1}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(bH - E) \otimes \mathcal{I}_{\tilde{\Sigma}})$, but by [20, 2.4(6)], we know that $H^{i+1}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(bH - E) \otimes \mathcal{I}_{\tilde{\Sigma}}) = H^{i+1}(\mathbb{P}^n, \mathcal{I}_{\Sigma}(b)) = 0$ for $i \geq 0$, $b \in \mathbb{Z}$. \hfill \square

3. Main result

We first reinterpret the injection in Proposition 2.3 as a cohomological vanishing statement on $\tilde{\Sigma}$ (Proposition 3.1), then on $S^2C$ (Corollary 3.3), and finally on $C \times C$ (Theorem 3.5).

\textbf{Proposition 3.1.} Let $C \subset \mathbb{P}^n$ be a smooth curve satisfying $N_p$ embedded by a line bundle $L$ with $\text{deg}(L) \geq 2g + 3$. Then $\Sigma$ satisfies $N_{3,p}$ if

$$H^i(\tilde{\Sigma}, \pi^* \bigwedge^{a-1+i} M_L \otimes \mathcal{O}(2H - E)) = 0$$

for $2 \leq a \leq p + 1$, $i \geq 1$.

\textbf{Proof.} We use Proposition 2.3. Consider the sequence on $\tilde{\Sigma},$

$$0 \to \pi^* \bigwedge^a M_L(bH - E) \to \pi^* \bigwedge^a M_L(bH) \to \pi^* \bigwedge^a M_L(bH) \otimes \mathcal{O}_Z \to 0.$$

We know that

$$H^1(Z, \pi^* \bigwedge^a M_L(bH) \otimes \mathcal{O}_Z) = H^1 \left( Z, \left( \bigwedge^a M_L \otimes L^b \right) \boxtimes \mathcal{O}_C \right)$$

$$= H^1(C, \mathcal{O}_C) \otimes H^0(C, \bigwedge^a M_L \otimes L^b)$$

$$= H^0(\Sigma, \bigwedge^a M_L(b) \otimes R^1 \pi_* \mathcal{O}_{\tilde{\Sigma}}).$$
The first equality follows as the restriction of $\pi^* \bigwedge^a M_L(bH)$ to $Z$ is $\bigwedge^a M_L(bH) \otimes \mathcal{O}_C$. For the second we use the Künneth formula together with the fact that $h^1(C, \bigwedge^a M_L \otimes L^b) = 0$ as $C$ satisfies $N_p \geq 13$. The third is the last part of subsection 2.2.

Thus

$$h^1(\Sigma, \bigwedge^a M_L(b)) = \text{Rank} \left( H^1(\tilde{\Sigma}, \pi^* \bigwedge^a M_L(bH - E)) \rightarrow H^1(\tilde{\Sigma}, \pi^* \bigwedge^a M_L(bH)) \right),$$

and so by Proposition 2.3 it is enough to show that $H^1(\tilde{\Sigma}, \pi^* \bigwedge^a M_L \otimes \mathcal{O}(bH - E)) = 0$ for $2 \leq a \leq p + 1, b \geq 2$.

From the sequence

$$0 \rightarrow \pi^* \bigwedge^{a+1} M_L \otimes \mathcal{O}(bH - E) \rightarrow \bigwedge^a \Gamma \otimes \mathcal{O}(bH - E)$$

and the fact (Lemma 2.4) that $H^i(\tilde{\Sigma}, \mathcal{O}(bH - E)) = 0$, we see that

$$H^1(\tilde{\Sigma}, \pi^* \bigwedge^a M_L \otimes \mathcal{O}(2H - E)) = H^{b-2}(\tilde{\Sigma}, \pi^* \bigwedge^{a+b-2} M_L \otimes \mathcal{O}(2H - E))$$

for $b \geq 2$. □

**Lemma 3.2.** Let $C \subset \mathbb{P}^n$ be a smooth curve embedded by a line bundle $L$ with $\text{deg}(L) \geq 2g + 3$ and consider the morphism $\varphi : \tilde{\Sigma} \to S^2C \subset \mathbb{P}^s$ induced by the linear system $|2H - E|$. Then $\varphi_* \bigwedge^a M_L = \bigwedge^a M_{\mathcal{E}_L}$, and hence

$$H^i(\tilde{\Sigma}, \varphi_* \bigwedge^a M_L \otimes \mathcal{O}(2H - E)) = H^i(S^2C, \bigwedge^a M_{\mathcal{E}_L} \otimes \mathcal{O}_{S^2C}(1)).$$

**Proof.** Consider the diagram on $\tilde{\Sigma}$:

\[
\begin{array}{ccccccc}
0 & \\
\downarrow & \\
0 & \varphi^* M_{\mathcal{E}_L} & \rightarrow & \Gamma(S^2C, \mathcal{E}_L) \otimes \mathcal{O}_{\tilde{\Sigma}} & \rightarrow & \varphi^* \mathcal{E}_L & \rightarrow 0 \\
\downarrow & \\
0 & \rightarrow & \pi^* M_L & \rightarrow & \Gamma(C, \mathcal{L}) \otimes \mathcal{O}_{\tilde{\Sigma}} & \rightarrow & \pi^* L & \rightarrow 0 \\
\downarrow & \\
K & \\
\downarrow & \\
0 &
\end{array}
\]
The vertical map in the middle is surjective, as we have $\Gamma(S^2 C, \mathcal{E}_L) = \Gamma(\overline{\Sigma}, \mathcal{O}(H)) = \Gamma(C \times C, L \boxtimes \mathcal{O}) = \Gamma(C, L)$. Therefore, surjectivity of the lower right horizontal map and commutativity of the diagram show that the right-hand vertical map is surjective.

Note that $R^i \varphi_* \varphi^* \mathcal{E}_L = \mathcal{E}_L \otimes R^i \varphi_* \mathcal{O}_{\overline{\Sigma}}$ by the projection formula and that the higher direct image sheaves $R^i \varphi_* \mathcal{O}_{\overline{\Sigma}}$ vanish as $\overline{\Sigma}$ is a $\mathbb{P}^1$-bundle over $S^2 C$. For the higher direct images, we have $R^i \varphi_* \pi^* L = 0$ as the restriction of $L$ to a fiber of $\varphi$ is $\mathcal{O}(1)$, and hence the cohomology along the fibers vanishes. From the rightmost column, we see that $R^i \varphi_* K = 0$. From the leftmost column, we have the sequence

$$0 \rightarrow \varphi^* \bigwedge^a M_{\mathcal{E}_L} \rightarrow \pi^* \bigwedge^a M_L \rightarrow \varphi^* \bigwedge^{a-1} M_{\mathcal{E}_L} \otimes K \rightarrow 0,$$

but as $R^i \varphi_* \left( K \otimes \varphi^* \bigwedge^{a-1} M_{\mathcal{E}_L} \right) = R^i \varphi_* K \otimes \bigwedge^{a-1} M_{\mathcal{E}_L} = 0$, we have $\varphi_* \bigwedge^a M_L = \bigwedge^a M_{\mathcal{E}_L}$. \[\square\]

Combining Proposition 3.1 with Lemma 3.2 yields:

**Corollary 3.3.** Let $C \subset \mathbb{P}^n$ be a smooth curve satisfying $N_p$ embedded by a line bundle $L$ with $\deg(L) \geq 2g + 3$. Then $\Sigma$ satisfies $N_{3, p}$ if

$$H^i(S^2 C, \bigwedge^a M_{\mathcal{E}_L} \otimes \mathcal{O}(1)) = 0$$

for $2 \leq a \leq p + 1$, $i \geq 1$. \[\square\]

We need a technical lemma, completely analogous to [15, 1.4.1].

**Lemma 3.4.** Let $X \subset \mathbb{P}^n$ be a smooth curve embedded by a non-special line bundle $L$ satisfying $N_{2, 2}$, let $x_1, \cdots, x_{n-2}$ be a general collection of distinct points, and let $D = x_1 + \cdots + x_{n-2}$. Then there is an exact sequence of vector bundles on $X \times X$,

$$0 \rightarrow L^{-1}(D) \boxtimes L^{-1}(D)(\Delta) \rightarrow d^* M_{\mathcal{E}_L} \rightarrow \bigoplus_i (\mathcal{O}(-x_i) \boxtimes \mathcal{O}(-x_i)) \rightarrow 0.$$

**Proof.** Choose a general point $x_1 \in X$ and consider the following diagram on $X \times X$:

```
0 \rightarrow d^* M_{\mathcal{E}_L(-x_1)} \rightarrow M_{L(-x_1)} \boxtimes \mathcal{O} \rightarrow (\mathcal{O} \boxtimes L(-x_1))(-\Delta) \rightarrow 0
```

```
0 \rightarrow d^* M_{\mathcal{E}_L} \rightarrow M_L \boxtimes \mathcal{O} \rightarrow \mathcal{O} \boxtimes L(-\Delta) \rightarrow 0
```

```
0 \rightarrow \mathcal{O}(-x_1) \boxtimes \mathcal{O}(-x_1) \rightarrow \mathcal{O}(-x_1) \boxtimes \mathcal{O} \rightarrow \mathcal{O} \boxtimes (L \otimes \mathcal{O}_{x_1})(-\Delta) \rightarrow 0
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0 \rightarrow 0 \rightarrow 0 \rightarrow 0
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0 \rightarrow 0 \rightarrow 0 \rightarrow 0
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0 \rightarrow 0 \rightarrow 0 \rightarrow 0
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0 \rightarrow 0 \rightarrow 0 \rightarrow 0
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0 \rightarrow 0 \rightarrow 0 \rightarrow 0
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0 \rightarrow 0 \rightarrow 0 \rightarrow 0
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0 \rightarrow 0 \rightarrow 0 \rightarrow 0
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\[\square\]
where the center column comes from \cite[1.4.1]{15}. Following just as in that proof, we obtain

\[ 0 \to d^* \mathcal{E}_{P(-D)} \to d^* M \to \bigoplus_i (\mathcal{O}(-x_i) \boxtimes \mathcal{O}(-x_i)) \to 0 \]

from the left column. Note however that \( d^* M \mathcal{E}_{L(-D)} \) is a line bundle, and one checks that \( d^* M \mathcal{E}_{L(-D)} = \bigwedge^2 \mathcal{E}_{L(-D)} = L^{-1}(D) \boxtimes L^{-1}(D)(\Delta) \).

\[ \square \]

**Theorem 3.5.** Let \( C \subset \mathbb{P}^n \) be a smooth curve embedded by a line bundle \( L \) with \( \deg(L) \geq 2g + p + 3, p \geq 0 \). Then \( \Sigma \) satisfies \( N_{3,p} \).

**Proof.** First note \cite[13]{13} that such a curve satisfies \( N_{p+2} \). We verify the condition in Corollary 3.3. Pulling the sequence on \( S^2C \),

\[ 0 \to M \mathcal{E}_{L} \to \Gamma(S^2C, E_{L}) \to E_{L} \to 0, \]

back to \( Z = C \times C \) yields the diagram

As in \cite{20} we have \( d_* \mathcal{O}_Z = \mathcal{O}_{S^2C} \oplus M \) where \( d^* M = \mathcal{O}(-\Delta) \), \( d_* K = \mathcal{E}_{L} \otimes M \), and \( K = d^* \bigwedge^2 \mathcal{E}_{L} \otimes (L^* \boxtimes \mathcal{O}) = \mathcal{O} \boxtimes L(-\Delta) \). From the left vertical sequence we have

\[ 0 \to \bigwedge^a d^* M \mathcal{E}_{L} \to \bigwedge^a M \mathcal{L} \boxtimes \mathcal{O} \to \bigwedge^a d^* M \mathcal{E}_{L} \otimes K \to 0, \]

and pushing down to \( S^2C \) yields

\[ 0 \to \bigwedge^a M \mathcal{E}_{L} \oplus \left( \bigwedge^a M \mathcal{E}_{L} \otimes M \right) \to d_* \bigwedge^a M \mathcal{L} \boxtimes \mathcal{O} \]

\[ \to \bigwedge^a M \mathcal{E}_{L} \otimes \mathcal{E}_{L} \otimes M \to 0. \]

Twisting this sequence by \( \mathcal{O}_{S^2C}(1) \otimes M^* \) gives

\[ 0 \to \bigwedge^a M \mathcal{E}_{L}(1) \otimes M^* \oplus \bigwedge^a M \mathcal{E}_{L}(1) \mathcal{O}_{S^2C}(1) \otimes M^* \otimes d_* \bigwedge^a M \mathcal{L} \boxtimes \mathcal{O} \]

\[ \to \bigwedge^a M \mathcal{E}_{L}(1) \otimes \mathcal{E}_{L} \to 0. \]
Since \( d^* \mathcal{O}_{\text{Hilb}^2 X}(1) \otimes M^* = L \boxtimes L \otimes \mathcal{O}(-\Delta) \), it suffices to show that
\[
H^i(Z, \bigwedge^{a-1+i} d^* M_{E_L} \otimes L \boxtimes L \otimes \mathcal{O}(-\Delta)) = 0
\]
for \( 2 \leq a \leq p + 1, \ i = 1, 2 \).

Now, by Lemma 3.4 we have exact sequences
\[
0 \to \bigoplus_{i} (\mathcal{O}(-x_i) \boxtimes \mathcal{O}(-x_i)) \to 0,
\]
where \( Q = \bigoplus_{i} (\mathcal{O}(-x_i) \boxtimes \mathcal{O}(-x_i)) \).

On the right, we have a direct sum of vector bundles of the form \( F \boxtimes F(-\Delta) \), where \( F \) is a line bundle of degree \( \text{deg}(L) - r \). Thus \( H^1 \) and \( H^2 \) of the right side will vanish when \( \text{deg}(L) - r \geq 2g + 1 \).

On the left, we have a direct sum of vector bundles of the form \( F \boxtimes F \), where \( F \) is a line bundle of degree \( n - 2 - (r - 1) = \text{deg}(L) - g - r - 1 \). Because \( x_1, \ldots, x_{n-2} \) are general, \( H^1 \) and \( H^2 \) of the left side will vanish when \( \text{deg}(L) - g - r - 1 \geq g \).

Combining these, we see that
\[
H^i(Z, \bigwedge^{a-1+i} d^* M_{E_L} \otimes L \boxtimes L \otimes \mathcal{O}(-\Delta)) = 0
\]
for \( 2 \leq a \leq p + 1, \ i = 1, 2 \), as long as \( \text{deg}(L) \geq 2g + p + 3 \).

\[\square\]

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