Impossible process of noninteracting identical particles

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In the framework of General probability theories (GPT), we illustrate a statistical process in case of two non-interacting identical particles in two modes that satisfies a well motivated notion of physicality conditions proposed by Karczewski et. al. (Phys. Rev. Lett. 120, 080401 (2018)), which can’t be realized through quantum mechanical process. This statistical process is ruled out by an additional requirement imposed on two particle evolution.

I. INTRODUCTION

Out of many features due to which quantum mechanics (QM) deviates from classical probability theory, indistinguishable nature of identical particles, and non-local nature [1] of quantum correlations remain as prominent features. The statistical nature of indistinguishable particles plays a central role in our every day understanding starting from the molecules, atoms, solids to many astronomical events. The non-local nature exhibited by quantum theory via violation of Bell-type inequality [2] has revolutionized the fundamental understanding of Nature, and also has contributed to development of quantum information theory and computation.

GPT framework begins with adopting the formalism of operational quantum mechanics which was advocated by quantum logic community [3–5] in pre-quantum information era. Later, due to the advancement in the quantum information theory, GPT was formulated using Euclidean real probabilistic vector spaces to describe the state space [6–8]. It is considered as one of the fundamental frameworks to study the quantum correlations in any probabilistic theory [6–14], and to find the operationally motivated information theoretic axioms for quantum theory [15–20].

The quest for the physical axioms to characterize QM began with an attempt by Popescu and Rohrlich in which they constructed a probabilistic model (later called PR-box) for maximally nonlocal correlations that violate CHSH inequality to its algebraic maximum. The impossibility of realizing PR-box correlations in QM was due to Tsirelson [21], even though the correlations are non-local as well as non-signalling. This prompted the search for different physical principles to reproduce quantum mechanical bound in case of CHSH inequality [22–26].

GPT framework, which can accommodate any probabilistic physical theory facilitated such constructions, thus provided a deeper understanding of the nature of QM, and in turn that of Nature. Motivated by these developments, GPT framework has been extended to accommodate general relativity [27] and indefinite causal structure [28]. Recently, Hardy has initiated program to extend GPT framework to field theories, and has provided a manifesto for constructing quantum gravity [29].

Recently, Karczewski et. al. [30], proposed a general probabilistic framework to deal with non-interacting identical particles. In this work, the authors formulated a well motivated physical principle called consistency condition similar to no-signaling condition that any theory with non-interacting identical particles should satisfy, and provided an example of a statistical process with three identical particles in three mode, that satisfies consistency condition but cannot be realized in QM.

Following the framework of Karczewski et. al. [30], in this work we show that their exists much simpler configuration, i.e, two non-interacting identical particles in two mode which satisfies consistency condition, yet fails to produce such process in QM. We also show that an extra physicality condition proposed by Karczewski et. al., recovers quantum mechanical statistics in case of three particles in three modes, which can also be used to rule out proposed impossible process in two identical particles in two mode case.

II. GENERALIZED PROBABILITY THEORY FRAMEWORK

The basic ingredients of GPT’s are states, transformations and measurements, and are formulated in the language independent of Hilbert space formalism so that it can accommodate any probabilistic physical theory. In case of distinguishable particles, the framework is very well developed, and has been used as a cornerstone for both foundational understanding as well as for applications. In case of foundations, the framework is used to provide the physicality conditions that characterizes QM [22–26], to understand the limits and advantages of general correlations [9, 10, 31–35], and to find an axiomatic formulation of QM [6, 18, 36–38]. On the application side, it has provided a methodology [39] of device independence certification of many quantum information theoretic and quantum computational tasks.

Recently Karczewski et al. [30] developed a GPT framework for noninteracting identical particles, which we briefly review here.
Consider $N$ particles in $M$ modes. The state of identical particles is described by particle occupation number in each mode, $\phi = \{n_1, n_2, \ldots, n_M\}$, where $n_1, n_2, \ldots, n_M$ are occupation number in mode $\{1, 2, \ldots, M\}$ with $\sum_{i=1}^{M} n_i = N$. The state probability vector $\Phi$ is a d-dimensional vector representing the probability distribution of particles over all modes. The GPT framework consists of an initial state $\Phi_i$, an evolution (or transformation) $T$ given by a stochastic matrix, and a final state $\Phi_f$.

For example, consider a single boson in two modes with a symmetric beam splitter (BS) transformation. The set of occupation number states are $\{1,0\}$ and $\{0,1\}$. The state probability vector $\Phi = \{P(\{1,0\}), P(\{0,1\})\}$, where $P(\{1,0\})$ is the probability distribution of particles. The transformation matrix $T_{BS}^{(1)}$ for a BS for single particle in two modes is given by

$$T_{BS}^{(1)} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \quad (1)$$

Similarly, a GPT framework for two bosons in symmetric BS is given by representing the state $\Phi = \{P(\{2,0\}), P(\{1,1\}), P(\{0,2\})\}$ and the transformation $T_{BS}^{(2)}$ by

$$T_{BS}^{(2)} = \frac{1}{4} \begin{pmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 0 & 1/4 \\ 1/4 & 1/2 & 1/4 \end{pmatrix}. \quad (2)$$

### III. NO-INTERACTION CONDITION

In case of characterizing nonlocal correlations in Bell-type scenario, the natural physicality condition is to satisfy relativistic causality i.e., impossibility of instantaneous communication. The no-signaling condition states that the marginal probability distribution of one party should not be affected by the choice of observables by any another spatially separated party. Similarly, in case of contextuality [40] it is Gleason’s no-signaling (also called no-disturbance in literature) which acts as the physicality or consistency condition.

The consistency condition for non-interacting identical particles has to consider the non-interacting character. This requirement as formulated in Ref. ([30]), which we call here no-interaction criteria.

**No-interaction:** The transformation of a single particle distribution should not be affected by the presence of any other particle.

The precise mathematical characterization requires defining a transition matrix $R^{(N)}$ with elements $R_{ij}^{(N)}$ which specify the transition of $N$-particle state $\Phi_j$ to $N-1$ particle state $\Phi_i$, by randomly removing one particle from $N$-particle state:

$$\Phi_i^{(N-1)} = R^{(N)} \Phi_i^{(N)}. \quad (3)$$

Any $K$-particle state can be obtained from $N$-particle state by sequentially removing single particle. For example, a transition from 3 particle state to single particle state can be obtained as $\Phi^{(1)} = R^{(2)} R^{(3)} \Phi^{(3)}$. Thus a transition matrix $R^{(N-K)}$ for $N$-particle state to $K$-particle state is given by

$$R^{(N-K)} = R^{(K+1)} \cdots R^{(N-1)} R^{(N)}. \quad (4)$$

With these mathematical devices, the no-interaction condition constrain the allowed transformations $T$ as,

$$R^{(N-K)} T^{(N)} \Phi_i = T^{(K)} R^{(N-K)} \Phi_i, \forall \Phi. \quad (5)$$

This means that the state probability vector obtained by first reducing an $N$-particle state to $K$-particle state and then transferring a $K$-particle state must be same as first transferring an $N$-particle state and then reducing it to a $K$-particle state.

This will be evident by considering an elementary system with $N=2$ and $M=2$, for which $R^{(2)}$ is given by

$$R^{(2)} = \frac{1}{2} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}. \quad (6)$$

The no-interaction condition constraints any $T^{(2)}$ that satisfies,

$$R^{(2)} T^{(2)} \Phi_i^{(2)} = T^{(1)} R^{(2)} \Phi_i^{(2)}, \quad (7)$$

for all states $\Phi_i$. It is very clear from Eq. (1) and Eq. (6), that the symmetric BS transformation given in Eq. (2) satisfies no-interaction condition (7).

Along with no-interaction condition, the no increase of entropy after transformation demands that the transformation matrix $T$ has to be doubly stochastic.

### IV. IMPOSSIBLE PROCESS

Karczewski et.al.[30] provided an example of a process involving $N=3$ in $M=3$ system with a transformation $T^{(3)}$ that exceeds the bunching probability bound given in QM, even though the transformation $T^{(3)}$ satisfies no-interaction condition (5), as well the matrix representing $T^{(3)}$ is doubly stochastic. The given transformation (a.k.a. process) is not realizable in QM, similar to maximally non-local PR-box which cannot be realized in QM. In this work, we present a more elementary example, involving $N=2$ in $M=2$, which satisfies no-interaction condition (7) and doubly stochastic but no quantum mechanical unitary transformation can realize such process.

The impossible transformation $T_{(imp)}^{(2)}$ involving $N=2$ in $M=2$ is given by

$$T_{(imp)}^{(2)} = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}. \quad (8)$$
It can be easily verified that $T_{(\text{imp})}^{(2)}$ is doubly stochastic, and satisfies no-interaction condition (7).

In QM, the transformation $T_{QM}^{(2)}$ for $N = 2$ in $M = 2$ bosonic systems is given by [41, 42]

$$T_{QM}^{(2)} = \begin{pmatrix} |t|^2 & 2|rt|^2 & |r|^2 \\ 2|rt|^2 & |t|^2 + 2|rt|^2 & 2|rt|^2 \\ |r|^2 & 2|rt|^2 & |t|^2 \end{pmatrix},$$  \hspace{1cm} (9)

where $r$ and $t$ are complex reflection and transmission coefficients, which satisfy $|r|^2 + |t|^2 = 1$ and $rt^* + tr^* = 0$.

The impossibility of realizing the transformation $T_{(\text{imp})}^{(2)}$ in QM can be easily seen by comparing it with $T_{QM}^{(2)}$. Comparing Eq. (8) and Eq. (9), we find that $|rt|^2 = 0, |r|^2 = \frac{1}{\sqrt{2}}$ and $|t|^2 = \frac{1}{\sqrt{2}}$, which is contradictory.

In order to rule out a transformation that cannot be realized in QM in case of $N = 3, M = 3$, Karczewski et.al.[30] provided an additional principle which states that, the evolution of the states in which all the particles are in same mode should be equal to the evolution generated by its single particle counterpart. By using this principle, they showed the impossibility of realization of the transformation exceeding QM bound on bosonic bunching.

In case of two particle this principle is written as:

$$T^{(2)} \Phi^{(2)} = T^{(1)} \Phi^{(1)} \times T^{(1)} \Phi^{(1)},$$  \hspace{1cm} (10)

We can use the same principle to rule out $T_{(\text{imp})}^{(2)}$, as the transformation $T_{(\text{imp})}^{(2)}$ violates it. This can be seen by noting that $P^{(20)} = P^{(10)} \times P^{(10)}$.

V. CONCLUSION

Non-local nature of quantum correlations is one of the salient features that dramatically deviates from our understanding of classical correlations which purely originate from human ignorance. Maximal nonlocality beyond QM exhibited by PR-box, has laid the foundation for exploring the possibility of finding physical principles behind many properties of QM, and provided a methodological application in terms of device independent characterization of information theoretic tasks.

Indistinguishable nature of quantum identical particles deviates our world view of understanding Nature from a classical description. Recent computational advantages in case of Boson sampling [43] have solely emerged from the indistinguishable nature of identical quantum particles. Formulating and studying identical particles in GPT, and identification of more general principles that single out quantum mechanical statistics is important.

In this work, we provided an elementary example of two noninteracting identical particles in two modes, which satisfies well defined notion of physicality conditions, yet cannot be realized in QM. A principle proposed by Karczewski et. al.[30], where they considered three particles in three modes, has been applied in this work for the case of a more elementary system of two identical particles in two mode to rule out such an impossible process. This provides a greater insight to identify physicality condition which will provide a necessary and sufficient condition for bounding the quantum mechanical probabilities in case of many identical particles in multi mode settings.

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