A DERIVED DECOMPOSITION FOR EQUIVARIANT D-MODULES

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Abstract. We show that the adjoint equivariant coherent derived category of $D$-modules on a reductive Lie algebra $\mathfrak{g}$ carries an orthogonal decomposition into blocks indexed by cuspidal data (in the sense of Lusztig). Each block admits a monadic description in terms of a certain differential graded algebra related to the homology of Steinberg varieties, which resembles a “triple affine” Hecke algebra. Our results generalize the work of Rider and Rider–Russell on constructible complexes on the nilpotent cone, and the earlier work of the author on the abelian category of equivariant $D$-modules on $\mathfrak{g}$. However, the algebra controlling the entire derived category of $D$-modules appears to be substantially more complicated than either of these special cases, as evidenced by the non-splitting of the Mackey filtration on the monad controlling each block.

Main Results. Let $G$ be a connected, complex reductive group with Lie algebra $\mathfrak{g}$, and let $D_{\text{coh}}^G(\mathfrak{g})$ denote the equivariant derived category of bounded complexes of $D$-modules with coherent cohomology on $\mathfrak{g}$. The goal of this paper is to understand the structure of this category.

Recall that a cuspidal datum $(L, \mathcal{E})$, consists of a Levi subgroup $L$ of $G$ together with a cuspidal local system $\mathcal{E}$ on a nilpotent orbit of $L$. As part of the Generalized Springer Correspondence, Lusztig showed that cuspidal data index blocks of the category of equivariant perverse sheaves on the nilpotent cone $N_G$. In this paper we extend these results to the category $D_{\text{coh}}^G(\mathfrak{g})$.

Theorem A. There is an orthogonal decomposition:

$$D_{\text{coh}}^G(\mathfrak{g}) \cong \bigoplus_{(L, \mathcal{E})} D_{\text{coh}}^G(L, \mathcal{E})$$

Concretely, this means that every object of $D_{\text{coh}}^G(\mathfrak{g})$ can be written as a finite direct sum where each summand belongs to one of the blocks, and there are no Ext’s in either direction between the objects in distinct blocks.

Lusztig also described the block of the category of perverse sheaves corresponding to $(L, \mathcal{E})$ in terms of representations of the relative Weyl group $W_{(G, L)} = N_G(L)/L$, which is known to be a Coxeter group acting by reflections on $\mathfrak{z} = \text{Lie}(Z(L))$. Our next result concerns the blocks of $D_{\text{coh}}^G(\mathfrak{g})$. For motivation, let us first consider the following subcategories:

- The abelian category of coherent equivariant $D$-modules $M_{\text{coh}}^G(\mathfrak{g})$.
- The subcategory of complexes with support on the nilpotent cone $D_{\text{coh}}(N_G)^G$. 

The intersection of these two subcategories is equivalent to the abelian category of equivariant perverse sheaves on $N_G$. In these two special cases, we have

- By earlier work of the author [Gün], the abelian category admits an orthogonal decomposition, and the blocks take the following form:
  \[ M(\mathfrak{g})|^{G}_{(L, E)} \simeq M(\mathfrak{z})|^{W(\mathfrak{g}, L)} \simeq \mathcal{D}_3 \times W(\mathfrak{g}, L) - \text{mod} \]
  where $\mathcal{D}_3$ denotes the ring of differential operators on $\mathfrak{z}$.

- By work of Achar [Ach], Rider [Rid] and Rider–Russell [RR], the category of complexes on the nilpotent cone admits an orthogonal decomposition, and the blocks take the following form:
  \[ \mathcal{D}_{\text{coh}}(N_{\mathfrak{g}})|^{G}_{(L, E)} \simeq S_{3*} \times W(\mathfrak{g}, L) - \text{Perf} \]
  where $S_{3*}$ denotes the formal differential graded (dg) algebra $\text{Sym}(\mathfrak{z}^*[-2])$.

We would like to combine these results to obtain a description of all of the abelian category of perfect dg-$\mathfrak{g}$-modules. Naively, one might hope that these blocks would be controlled by the dg-algebra $(\mathcal{D}_3 \otimes S_{3*}) \times W(\mathfrak{g}, L)$ which is some kind of “triple affine” Hecke algebra (see Section 1.6 for further comments on this idea). This naive guess is not quite right, as clarified by the following result.

**Theorem B.** For each cuspidal datum $(L, E)$, choose a parabolic subgroup $P$ containing $L$ as a Levi fact. There is a dg-ring $\mathbf{A} = \mathbf{A}_{(P, L, E)}$ such that:

1. There is an equivalence $\mathcal{D}_{\text{coh}}(\mathfrak{g})|^{G}_{(L, E)} \simeq \mathbf{A} - \text{Perf}$, where the right hand side denotes the category of perfect dg-$\mathbf{A}$-modules.
2. There is a homomorphism of dg-rings $\mathbf{A}$ as $\mathfrak{g}$-dg-bimodules, indexed by $W(\mathfrak{g}, L)$.
3. There is a filtration (the Mackey filtration) of $\mathbf{A}$ by $W(\mathfrak{g}, L)$.
4. The Mackey filtration does not split as long as $L \neq G$.
5. If $L = G$ (so $E$ is a cuspidal local system on $G$), then $\mathbf{A} \simeq \mathcal{D}_3(\mathfrak{g})$.

Categorically, the dg-algebra $\mathbf{A}$ represents a monad acting on the cuspidal block of $\mathcal{D}_{\text{coh}}(\mathfrak{g})|^{G}_{(L, E)}$ corresponding to $E$, which arises from the adjoint functors of parabolic induction and restriction. Geometrically, $\mathbf{A}$ can be thought of as a complex of $D$-modules on $\mathfrak{z} \times \mathfrak{z}$, and as such it can be identified with the relative equivariant Borel-Moore homology of a version of the Steinberg variety adapted to the data $(P, L, E)$. The Mackey filtration arises from a certain stratification of this variety. Algebraically, $\mathbf{A}$ is given by the derived endomorphism algebra of a parabolic induction of the $D$-module $\mathcal{D}_3 \boxtimes E$ on $\mathfrak{z} \times N_L \hookrightarrow \mathfrak{g}$.

**Organization of the paper.**

- In the introduction below, we discuss some of the background for these results (Section 1.1), explore some related ideas (Sections 1.6, 1.7, 1.8), present the example of $SL_2$ (Section 1.5), and give an outline of proof of the main results (Sections 1.2 and 1.3).

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1 There are many coherent equivariant $D$-modules on $\mathfrak{g}$ which are not holonomic. However, the condition of being supported on $N_G$ forces the $D$-modules to be regular holonomic and thus we can identify the category of such objects with the equivariant derived category of constructible complexes on $N_G$ by the Riemann-Hilbert correspondence.

2 It should be noted that these earlier results and the ones in this paper rely on the cleanness of cuspidal character sheaves, which was established by Lusztig [Lus].
In Section 2 we give an overview of some of the tools we will need from the theory of stable \( \infty \)-categories and \( D \)-modules.

In Section 3 we review some of the ideas from [Gun], and show that the category of \( D \)-modules decomposes into orthogonal blocks indexed by Levis.

In Section 4 we study the cuspidal blocks of the derived category, and complete the proof of Theorem A.

Finally, in Section 5, we study the blocks of the derived category via the Steinberg monad, and complete the proof of Theorem B.

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1. Introduction

1.1. Background and Motivation. Lusztig’s paper [Lusc] introduced the Generalized Springer Correspondence, which gave a block decomposition of the equivariant perverse sheaves on the nilpotent cone \( \mathcal{N}_G \), where the blocks are indexed by cuspidal data \( (L, \mathcal{E}) \) and each block is equivalent to the category of representations of the relative Weyl group \( W_{(G,L)} \). This work extends the Generalized Springer Correspondence in two directions:

- replacing perverse sheaves on the nilpotent cone with (not-necessarily holonomic) \( D \)-modules on the whole of \( g \);
- replacing the abelian category with the derived category.

The first direction was treated in [Gun], in which further motivation for studying the category of \( D \)-modules on \( g \) is given. For example, this category and its elliptic, quantum, and mirabolic variants are related to certain Cherednik and double affine Hecke algebras. One can think of the passage from orbital sheaves (or character sheaves) to all \( D \)-modules as allowing the infinitesimal character parameter to vary continuously.

The second direction (i.e. derived complexes on the nilpotent cone) has been studied by Achar [Ach], Rider [Rid], and Rider-Russell [RR]. The Ext algebras of certain induced objects of this category are related to graded affine Hecke algebras by the work of Lusztig [Lusa] (see Section 1.7). As explained by Ginzburg [Gin], closely related Ext algebras for character sheaves in the group setting appear in a version of Koszul-Langlands duality.

More generally, one can consider the study of equivariant \( D \)-modules on \( g \) as part of the Langlands program for a genus 1 curve with a single cusp. The good behavior of the functors of parabolic induction and restriction on the derived category as shown in [Gun] \((t\text{-exactness, preservation of})\)

\[ 3 \text{Character sheaves on the Lie algebra } g \text{ in the sense of Mirković [Mir] are equivalent to sheaves on the nilpotent cone via Fourier transform.} \]

\[ 4 \text{The stack } g/G \text{ can be identified with the degree 0 semistable locus of the moduli of } G \text{-bundles on such a curve. The category of } D \text{-modules on the corresponding moduli space for other genus 1 curves (e.g. a smooth elliptic curve) will enjoy similar properties to } D(g)^G. \]
coherence) are encouraging signs of extra structure, perhaps making the genus 1 case more accessible than the general case.

Another source of motivation for considering $D$-modules “beyond” character sheaves is the theory of categorical harmonic analysis. Just as one may want to express a general function as an integral of characters, one may also want to express a general complex of $D$-modules on $\mathfrak{g}$ (for example, the fiberwise cohomology of a family of spaces over $\mathfrak{g}$) as an integral of character sheaves.5 Particular examples of complexes of $D$-modules over $\mathfrak{g}$ (or better, $G$) arise in the study of the cohomology of character varieties of curves and related spaces, as studied by Hausel, Letellier, and Rodriguez-Villegas [HRV, HLRV]. Ongoing work of the author with David Ben-Zvi and David Nadler aims to replace the arithmetic harmonic analysis with categorical harmonic analysis, to obtain a deeper understanding of these cohomology groups (see [BZGN]). Essentially, one would like to treat the category $D(G)^G$ (the group version of the category considered in this paper) as a categorical analogue of the finite group theoretic Frobenius formula for the number of Galois covers of a curve. The present paper grew out of a desire to understand this category more concretely.

1.2. Outline of the proof of Theorem A. We have adjoint functors of parabolic induction and restriction on the unbounded derived category of all $D$-modules:$$\text{Ind}_{P,L}^G : D(l)^L \leftarrow \leftarrow \leftarrow \text{Res}_{P,L}^G : D(\mathfrak{g})^G.$$In [Gun] it was shown that these functors are $t$-exact and preserve coherent complexes. We defined a filtration (the Mackey filtration) on the composite functor $\text{Res}_{Q,M}^G \circ \text{Ind}_{P,L}^G$ for a pair of parabolics $Q, P$ with Levi factors $M, L$ respectively.

By definition, cuspidal objects of $D(\mathfrak{g})^G$ are those for which parabolic restriction to any proper Levi subgroup is zero. The blocks $D(\mathfrak{g})^{Q,M}_{L,E}$ of Theorem A correspond to those objects which are generated by parabolic induction from cuspidal objects in $D(l)^L$ of the form $\mathfrak{N} \boxtimes \mathcal{E}$, where $\mathfrak{N} \in D(\mathfrak{g})$.

The proof of Theorem A proceeds as follows:

1. First we show that the functors of parabolic induction and restriction define a recollement situation$^6$ for $D(\mathfrak{g})^G$. In other words, $D(\mathfrak{g})^G$ can be glued together from the subcategories $D(\mathfrak{g})^{Q,M}_{L,E}$. We must show that the gluing is trivial, i.e. the recollement is split.

2. One shows that parabolic induction from non-conjugate cuspidal data gives orthogonal objects of $D_{\text{coh}}(\mathfrak{g})^G$. This essentially follows from the Mackey theorem, together with the cleanness of cuspidal local systems. It follows that we have orthogonality for objects which are direct summands of parabolic inductions from non-conjugate cuspidal data.

3. In [Gun] it was shown that the cohomology objects of any complex in $D(\mathfrak{g})^{Q,M}_{L,E}$ are direct summands of parabolic induction from a cuspidal object of $D(l)^L$. This fact essentially boils down to the statement that in the abelian category, parabolic induction and restriction are controlled by the relative Weyl group, and the invariants for a finite group action on an object in a $\mathbb{C}$-linear category is a direct summand.

4. The required orthogonality then follows from the Ext spectral sequence.

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5The preference for $D$-modules over constructible sheaves (which only correspond to a much smaller subcategory via the Riemann-Hilbert correspondence) is justified by the nice form of the Fourier transform for $D$-modules—the first example of categorical Harmonic analysis.

6There is also a geometric interpretation of this recollement situation; see Section 1.8.
1.3. Monadic description of the blocks. The proof of Theorem B uses the Barr-Beck-Lurie Theorem [Lur] in a fundamental way. It is for this reason that we work in the setting of stable ∞-categories rather than the more tradition setting of triangulated categories.

Let us outline how the Barr-Beck-Lurie Theorem is used in Theorem B. The functor of parabolic restriction maps $D(p)_{GL,E}^G$ conservatively to the cuspidal block $D(l)_{GL,E}^L$. This monad is called the Steinberg monad as it is given by a pull-push formula with respect to a correspondence given by a (relative, parabolic) version of the Steinberg variety. The theory of integral transforms for $D$-modules relates the Steinberg monad to the of relative Borel-Moore homology of this variety.

Note that a monad acting on the category of modules for a (dg)-algebra $B$ is the same thing as another dg-algebra $A$ together with a homomorphism $B \to A$; such an object will be called a dg $B$-ring. We will give an identification:

$$D_{coh}(l)_{GL,E} \cong D_{coh}(j)^Z \cong \mathbb{D}_p \otimes \text{Perf}$$

Thus the Steinberg monad is expressed as a dg $\mathbb{D}_p$-ring $A_{(P,L,E)}$.

1.4. Non-splitting of the Mackey filtration. One of the key results in the abelian category case is that the Mackey filtration is canonically split, expressing the functor as a direct sum of restriction and induction functors in the usual pattern of the Mackey formula [Gun]. In this paper we will see that the corresponding result fails in the derived setting (see Section 1.5).

Essentially, the result boils down to the topological fact that the exact triangle on the Borel-Moore homology of $SL_2$ induced by the Bruhat decomposition is not split.

The non-splitting of the derived Mackey filtration has a number of interesting consequences, which demonstrate the differences between the abelian and the derived setting. For example:

- The abelian category functors $\text{ind}_L^G$, $\text{res}_L^G$ are independent of the choice of parabolic subgroup containing $L$ as a Levi. However, the derived category functors do not enjoy this independence.
- In the abelian category setting, the functors of induction and restriction are bi-adjoint; in the derived setting, there is a cycle of adjoints of length 4:

  $$\cdots \to \text{Ind}_{P,L}^G \to \text{Res}_{P,L}^G \to \text{Ind}_{P,L}^G \to \text{Res}_{P,L}^G \to \text{Ind}_{P,L}^G \to \cdots$$

- Every object of the abelian category block $M(g)_{GL,E}^G$ is a direct summand of some parabolic induction from $l$; by contrast, an object of the derived block can only be obtained from such summands by a sequence of iterated cones.

Remark 1.1. The statement about independence of the choice of parabolic has meaning even in the case when all parabolics containing a given Levi are conjugate. For example, in the case $G = SL_2$ (see Section 1.5), fixing the maximal torus $H$ (say as diagonal matrices), there are two Borel subgroups containing $H$, $B^+$ and $B^-$. There is no equivalence of functors between $\text{Res}_{B^+,H}^G$ and $\text{Res}_{B^-,H}^G$. One can see this by precomposing with $\text{Ind}_{B^+,H}$ and comparing the Mackey filtrations (they turn out to be Verdier dual exact triangles). One must be careful in formulating this

If the reader is happy to take the Barr-Beck-Lurie Theorem as a black box, then they will lose nothing by thinking in terms of triangulated categories.
statement: conjugating by any element \( s \in N_G(H) - H \) defines a canonical equivalence between \( B^+/[B^+, B^+] \) and \( B^-/[B^-, B^-] \) (each of which is naturally identified with \( H \)), and the functors \( \text{Res}^G_{B^+ \cdot H} \) and \( \text{Res}^G_{B^- \cdot H} \) become equivalent if we use this identification (which is nothing more than the action of \( s : H \to H \)).

1.5. Example: \( SL_2 \). Most of the main features of our results can be seen in the case \( G = SL_2 \). There are two conjugacy classes of cuspidal data: the Springer datum \( (T, C_q, T) \), where \( T \) is a maximal torus of \( G \), and the cuspidal datum \( (G, E_q, C) \), where \( E \) is the unique non-trivial local system on the regular nilpotent orbit of \( G \). In this case Theorem A gives a decomposition of the form:

\[
\text{D}_{\text{coh}}(\mathfrak{sl}_2)^{SL_2} \simeq \text{D}_{\text{coh}}(\mathfrak{sl}_2)^{SL_2}_{\text{Spr}} \oplus \text{D}_{\text{coh}}(\mathfrak{sl}_2)^{SL_2}_{\text{cusp}}
\]

The cuspidal block is given by

\[
\text{D}_{\text{coh}}(\mathfrak{sl}_2)^{SL_2}_{\text{cusp}} \simeq \mathbb{C} - \text{Perf}
\]

and the Springer block is given by

\[
\text{D}_{\text{coh}}(\mathfrak{sl}_2)^{SL_2}_{\text{Spr}} \simeq A - \text{Perf}
\]

where \( A \) is a certain dg-algebra, which we construct below.

Consider the following relative version of the Steinberg variety:

\[
\mathcal{S}t = \{(x, g_1 B, g_2 B) \in G \times G/B \times G/B \mid x \in g_1 b \cap g_2 b\}
\]

where \( B \simeq \mathbb{P}^1 \) denotes the variety of Borel subalgebras. This variety has a partition \( \mathcal{S}t = \mathcal{S}t_c \sqcup \mathcal{S}t_s \) indexed by the Weyl group \( W = \{e, s\} \) (\( \mathcal{S}t_c \) is the subvariety where \( g_1 B = g_2 B \)). There is a \( G \)-invariant map

\[
f : \mathcal{S}t \longrightarrow t \times t
\]

which takes \( (x, b_1, b_2) \) to the image of \( g_1^{-1} x \) and \( g_2^{-1} x \) in the quotient \( b/[b, b] \simeq t \). The fibers of \( f \) are empty unless the pair \( (t_1, t_2) \in t \times t \) is in the diagonal or antidiagonal. The fiber over \((0,0)\) consists of the subvariety of \( \mathcal{S}t \) for which \( x \) is nilpotent (this is what is usually called the Steinberg variety, e.g. in [CG]).

The dg-vector space \( A \) is given by global sections of the complex of (regular holonomic) \( D \)-modules \( f_* (\omega_{\mathcal{U}/G}) \) on \( t \times t \), which measures the equivariant Borel-Moore homology of the fibers of \( f \) (shifted in to appropriate degree). The dg-vector space \( A \) carries an algebra structure coming from convolution relative to \( t \times t \). It also carries a commuting action of \( S_t \otimes S_t \); thus we can consider \( A \) as a bimodule for \( \mathcal{D}_t \). The partition of \( \mathcal{S}t \) defines an exact triangle of \( \mathcal{D}_t \)-bimodules:

\[
(1) \quad A_c \longrightarrow A \longrightarrow A_s \longrightarrow A_c^+ \quad (\text{shifted by } 1)
\]

where \( A_c \) (respectively \( A_s \)) is the diagonal bimodule \( \mathcal{D}_t \) (respectively the bimodule \( \mathcal{D}_t \) where the left action is as usual and the right action is twisted by \( s \in W \)).

Part I of Theorem B means that the exact triangle \( 1 \) is not split as \( \mathcal{D}_t \)-bimodules.
1.6. **A triple affine Hecke algebra?** Theorem B gives a sense in which entire block $D^\text{coh}\left(\mathfrak{g}^G\right)_{(L,E)}$ has the flavor of a “triple affine Hecke algebra”: two of the affine directions are in degree zero as represented by the copy of $D^2 = \text{Sym}^3(\mathfrak{g}) \rtimes \text{Sym}(\mathfrak{g}^*)$ sitting in $H^0(A_{(P,L,E)})$, and the third is in even cohomological degrees, represented by copy of $S^4$.

The non-splitting of the Mackey filtration means that this algebra is not just a semidirect product of $W_{(L,E)}$ with $D^2$. This raises the following question, which the author hopes to return to in future work.

**Question 1.2.** Is there a combinatorial description of the dg-algebra $A_{(P,L,E)}$ in terms of the Coxeter system $(W_{(L,E)},\mathfrak{g})$?

**Remark 1.3.** It is not clear to the author if the dg-algebra is independent of the choice of parabolic $P$, or if it is formal (both of which seem to be necessary prerequisites to having any kind of reasonable combinatorial description).

It is natural to look for deformations of the category $D^\text{coh}(\mathfrak{g})^G$ which realize Hecke-type deformations of the algebras controlling the blocks. There are two flavors of such deformations.

The first corresponds to deforming $H^0(A_{(P,L,E)}) = D^2 \rtimes W_{(L,E)}$ to a rational Cherednik algebra (“turning on the $c$ parameter”); the corresponding deformation of the abelian category $M^\text{coh}(\mathfrak{g})^G$ has only been understood geometrically in the case $G = GL_n$, in which case one studies the category of mirabolic $D$-modules $M(\mathfrak{gl}_n \times C^n)^{GL_n \times C}$ (moreover, only a generic block of that category, seen by Hamiltonian reduction, has been related to a Cherednik algebra). Even the abelian category story (as opposed to the derived categories studied in this paper) is a rich and active topic of research in type $A$ (see e.g. [BG]) and there appears to be no analogue of the mirabolic deformation outside of type $A$.

The second flavor of Hecke deformation corresponds to deforming $A_{(P,L,E)} \rtimes W_{(L,E)}$ to a graded Hecke algebra. We discuss this in Section 1.7 below.

1.7. **Constructible complexes on the nilpotent cone, and graded affine Hecke algebras.** In the work of Rider [Rid], formality of Springer block of the constructible derived category $D^\text{con}(N_G)^G$ had to be established first, before one could give a description of the category in terms of dg-modules. This was achieved by defining a mixed version of the category, which involves some intricate and technical constructions in the theory of triangulated categories.

The Barr-Beck-Lurie theorem allows us to construct equivalences as in Theorem B without having any a priori formality results.

For example, it follows from the techniques in this paper that the Springer block of $D^\text{con}(N_G)^G$ is given by the dg-algebra of equivariant Borel-Moore chains on the Steinberg variety. Once this fact is established, it is not hard to prove that this dg-algebra is formal (e.g. by using Hodge theory), recovering Rider’s identification of the Springer block. Note that these techniques do not address the construction of mixed enhancements of the category, which is of significant independent interest.

More generally, let us consider a block of the category $D^\text{con}(N_G)^G$ corresponding to a cuspidal datum $(L,E)$. It follows from the Barr-Beck-Lurie Theorem that this block is given by dg-modules for the dg-algebra $R\text{End}(\text{Ind}^G_{L,E})$ Once again this algebra is formal, and thus is given by the corresponding Ext algebra. As computed in [RR], this algebra is given by the semidirect product $S^4 \rtimes W_{(L,E)}$. 
It is interesting to note that these semidirect product algebras which control the blocks of $D_{\text{coh}}(N_G)^{G \times \mathbb{G}_m}$ deform to graded affine Hecke algebras once one considers $\mathbb{G}_m$-equivariant objects for the scaling $\mathbb{G}_m$-action on $\mathfrak{g}$. This is implicit in the work of Lusztig [Lusa, Lusb], where such algebras were first defined and identified with the Ext algebras of the parabolic induction of cuspidal local systems on Levi subgroups, or equivalently the twisted equivariant homology of certain Steinberg varieties. Putting Lusztig’s results in the language of this paper, we obtain:

**Theorem 1.4.** There is an orthogonal decomposition

$$D_{\text{coh}}(N_G)^{G \times \mathbb{G}_m} = \bigoplus_{(L, \mathcal{E})} D_{\text{coh}}(N_G)^{G \times \mathbb{G}_m}_{(L, \mathcal{E})}$$

where each block is given by the category of dg-modules for the graded Hecke algebra $H_{(L, \mathcal{E})}$ with parameters as specified in [Lusa].

**Remark 1.5.** The motivation for considering these graded Hecke algebras comes from a relation with the representation theory of $p$-adic groups. The intersection cohomology complexes of local systems on nilpotent orbits for $G$ give examples of modules for the graded Hecke algebra.

1.8. **Split recollement arising from the partition of the commuting variety.** As explained in [Gun], the decomposition of Theorem A (or rather the coarser decomposition indexed by the conjugacy class of the Levi $L$) has a geometric interpretation as follows. The variety $\text{comm}(\mathfrak{g})$ of commuting elements of $\mathfrak{g}$ has a locally closed partition indexed by conjugacy classes of Levi subgroups. The singular support of an object in $D_{\mathfrak{g}}^G$ is a closed subvariety of $\text{comm}(\mathfrak{g})$, and thus the category $D_{\mathfrak{g}}^G$ carries a filtration indexed by conjugacy classes of Levis, according to the singular support. The somewhat surprising conclusion is that this filtration is, in fact, an orthogonal decomposition.

**Remark 1.6.** Work of McGerty and Nevins identifies certain stratifications of the cotangent bundle of a stack which give rise to a recollement; although the results of this paper do not immediately apply in their setting, we expect that they are closely related (note that an orthogonal decomposition is a very special case of recollement).

2. **Preliminaries**

In this section we set up the category theoretic framework, and describe the category of Ind-coherent $D$-modules on a stack.

2.1. **Stable $\infty$-categories.** In this paper we will make use of the theory of $\mathbb{C}$-linear, stable, presentable, $\infty$-categories, as developed by Lurie [Lurc, Lurb], or alternatively, pretriangulated differential graded categories (see [Coh] for the relationship between the two theories). We refer the reader to [BZFN] or [Gaib] for an overview of the main results and techniques, and directions towards further references. Below we outline some key properties.

To each $\mathbb{C}$-linear, stable, presentable $\infty$-category $\mathcal{C}$, the homotopy category $h\mathcal{C}$ is a triangulated category; one thinks of $\mathcal{C}$ as an enhancement of the triangulated category $h\mathcal{C}$. For much of this paper, the reader may replace stable $\infty$-categories with their homotopy categories without any loss of understanding. However, the extra structure of these enhancements allow for much cleaner and more natural proofs of many of the results in this paper.
Definition 2.1. Let $C, D$ be stable, presentable, $\mathbb{C}$-linear $\infty$-categories. Given objects $c, d \in C$, we have a complex $R\text{Hom}(c, d)$ of morphisms from $c$ to $d$.

1. A functor $F : C \to D$ is called \textit{continuous} if it preserves all small colimits.
2. An object $c \in C$ is called \textit{compact} if the functor $R\text{Hom}(c, -)$ is continuous (it is equivalent to check that the functor preserves direct sums).
3. We say $F$ is \textit{quasi-proper} if it sends compact objects to compact objects.

The collection of stable presentable $\infty$-categories forms an $(\infty, 1)$-category $\text{Cat}_C$, in which the morphisms are continuous functors. We also have $\infty$-categories $\text{Fun}(C, D)$ and $\text{Fun}^L(C, D)$, of functors and continuous functors respectively; both of these categories are themselves stable, presentable, and $\mathbb{C}$-linear.

A category $C$ is called compactly generated if there is a subset $S$ of compact objects in $C$ such that the right orthogonal to $S$ vanishes. Given $C \in \text{Cat}_C$, we write $C_c$ for the subcategory of compact objects. We can recover $C$ from $C_c$ as the Ind-category: $C \Rightarrow \text{Ind}(C_c)$. All the categories arising in this paper will be compactly generated.

Example 2.2. Given a differential graded (dg) algebra $A$, we have the stable $\infty$-category of perfect complexes $\text{Perf}$, and $A \Rightarrow \text{dgMod} = \text{Ind}(A \Rightarrow \text{Perf})$ is the category of unbounded complexes of $A$-modules. In the special case $A = \mathbb{C}$, we write $\text{Vect} := \mathbb{C} \Rightarrow \text{dgMod}$.

The category $\text{Cat}_C$ carries a monoidal product $\otimes$, which is characterized by the property that continuous functors from $C \otimes D$ to $\text{Vect}$ are the same thing as functors from $C \times D$ to $\text{Vect}$ which are continuous in each argument separately. Given dg-algebras $A$ and $B$, we have

$$A \Rightarrow \text{dgMod} \otimes B \Rightarrow \text{dgMod} = A \otimes B \Rightarrow \text{dgMod}.$$

Proposition 2.3. Suppose $C \in \text{Cat}_C$ is compactly generated category. Then $C$ is dualizable, with dual $C^\perp := \text{Ind}(C_c^\text{op})$.

Note that if $C$ is compactly generated then we have an equivalence

$$\text{Fun}^L(C, D) = C^\perp \otimes D.$$

Suppose $C, D \in \text{Cat}_C$ are compactly generated, and

$$L : C \rightleftarrows D : R$$

are an adjoint pair of functors (i.e. $L$ is left adjoint to $R$). Then $R$ is continuous (i.e. $R$ preserves small colimits) if and only if $L$ is proper (i.e. $L$ sends compact objects to compact objects). In that case, there is an adjunction

$$L_c : C_c \rightleftarrows D_c : R_c.$$

Conversely, if

$$L_c : C_c \rightleftarrows D_c : R_c$$

is an adjoint pair of functors between the subcategories of compact objects, then we have an adjunction:

$$\text{Ind}(L_c) : C \rightleftarrows D : \text{Ind}(R_c).$$

Definition 2.4. We say that a diagram

$$L : C \rightleftarrows D : R$$
in $\mathbf{Cat}_C$ is a continuous adjunction if $L$ is left adjoint to $R$, and $R$ is continuous (equivalently, $L$ is proper).

The following concept will be useful for us later.

**Definition 2.5.** A filtration of an object $a$ in a stable category $C$, indexed by a poset $(I, \leq)$, is a functor

$$\left( I, \leq \right) \to C/a$$

$$i \mapsto a_{\leq i}.$$  

In the cases of interest to us, $I$ will be a finite poset with a maximal element $i_{max}$, and we demand in addition that $a_{\leq i_{max}} \to a$ is an isomorphism. Given a closed subset $Z$ of $I$ (i.e. if $j \in Z$ and $i \leq j$, then $i \in Z$), we define $a_Z$ to be $\text{colim}_{j \in Z} a_{\leq j}$. For any subset $J$ of $I$, we define $a_J$ to be the cone of $a_{\leq J} \to a_{\leq J}$. In particular, for any $i \in I$, we set $a_i$ to be the cone of $a_{\leq i} \to a_{\leq i}$. Thus, we think of the object $a$ as being built from $a_i$ by a sequence of cones. The associated graded object is defined to be $\bigoplus_{i \in I} a_i$.

Note that there is no requirement for the maps to be injective (indeed, it is not clear what this would mean in this setting).

2.2. **The Barr-Beck-Lurie Theorem.** Recall that a monad acting on a category $C \in \mathbf{Cat}_C$ is an algebra object $T$ in the monoidal category $\text{Fun}(C, C)$. The category of $T$-modules is denoted $C_T$.

**Example 2.6.** Suppose $B$ is a dg-algebra, $C = B - \text{dgMod}$, and $T$ is a continuous monad acting on $C$. The monad $T$ can be thought of as an algebra object $A$ in the category of $B$-bimodules, or equivalently, a dg $B$-ring (i.e. a dg-algebra $A$ together with a morphism $B \to A$). The category of modules $C_T$ is equivalent $A - \text{dgMod}$ (i.e. $A$-modules in $B - \text{dgMod}$ are the same thing as $A$-modules in $\text{Vect}$).

Let $F : D \to C$ be a functor with a left adjoint $F^L$ such that:

1. $C$ and $D$ are Grothendieck abelian categories, and the functors $F$ and $F^L$ are exact and preserve direct sums; or,
2. $C$ and $D$ are compactly generated, stable, presentable $\infty$-categories, and the functors $F$ and $F^L$ are continuous.

**Remark 2.7.** If we are in context (2), and in addition, the categories $C$ and $D$ carry a $t$-structure which the functors $F$ and $F^L$ preserve, then taking the heart of the $t$-structure gives an example of context (1).

Let $T = FF^L$ denote the corresponding monad. We denote by $C_T$ the category of $T$-modules (also known as $T$-algebras) in $C$. Note that for any object $d \in D$, $F(d)$ is a module for $T$. Thus we have the following diagram:

- Sometimes $T$-modules in $C$ are referred to as $T$-algebras.
Definition 2.8. A functor \( F : \mathcal{D} \to \mathcal{C} \) is called **conservative** if whenever \( F(x) \simeq 0 \) then \( x \simeq 0 \).

Theorem 2.9. \((\text{Barr-Beck } [\text{BW}], \text{Lurie Theorem 6.2.0.6})\). The functor \( \tilde{F} : \mathcal{D} \to \mathcal{C}^T \) has a fully faithfiful left adjoint, \( J \). If \( F \) is conservative, then \( \tilde{F} \) and \( J \) are inverse equivalences.

Remark 2.10. (1) The essential image of \( J \) is given by the subcategory of \( \mathcal{D} \) generated under colimits by the essential image of \( F \).

(2) Given a \( T \)-module \( c \) in \( \mathcal{C} \), we have a simplicial diagram in \( \mathcal{D} \):

\[
\begin{array}{c}
F^L c \rightharpoonup F^L F^L c \rightharpoonup F^L F^L F^L c \rightharpoonup \ldots
\end{array}
\]

The object \( J(y) \) is given by the colimit of Diagram 2. Indeed, applying the functor \( F \) to diagram 2 is the canonical simplicial resolution of \( y \in \mathcal{C} \) (the bar construction).

Example 2.11 (Koszul duality). Let \( V \) be a vector space, and consider the graded algebras \( \Lambda = \text{Sym}(V[1]) \) and \( S = \text{Sym}(V^*[\leq 2]) \) (note that \( \Lambda \) is an exterior algebra when considered as an ungraded algebra). Each of these algebras is equipped with an augmentation module which we denote simply by \( C \). This defines a functor

\[
R \text{Hom}_\Lambda(C, -) : \Lambda - \text{dgMod} \to \text{Vect}
\]

One can check that this functor is conservative (\( \mathcal{C} \) is a generator) but is not continuous (\( \mathcal{C} \) is not compact). One can fix this defect by considering the category of \( \text{Ind-Coherent} \ \Lambda \text{-modules} \): by definition, this is given by \( \text{Ind}(\Lambda - \text{Coh}) \), where \( \Lambda - \text{Coh} \) is the subcategory of \( \Lambda - \text{dgMod} \) consisting of complexes with bounded cohomology, where each cohomology group is finite dimensional. By construction, \( \mathcal{C} \) is now a compact object of \( \text{Ind}(\Lambda - \text{Coh}) \), and we obtain a functor

\[
R = R \text{Hom}_{\text{Ind}(\Lambda - \text{Coh})}(\mathcal{C}, -) : \text{Ind}(\Lambda - \text{Coh}) \to \text{Vect}
\]

which now satisfies the conditions of the Barr-Beck-Lurie theorem. The corresponding monad on \( \text{Vect} \) is precisely the dg-algebra \( S \). Thus we obtain an equivalence

\[
\text{Ind}(\Lambda - \text{Coh}) \simeq S - \text{dgMod}
\]

Restricting to the subcategory of compact objects gives the more familiar presentation

\[
\Lambda - \text{Coh} \simeq S - \text{Perf}.
\]

2.3. From adjunctions to recollement situations. Let us consider the case when the functor \( F \) is not necessarily conservative (keeping the notation from the previous subsection). Let \( K \) denote the kernel of \( F \), i.e. the full subcategory of \( \mathcal{D} \) consisting of objects \( d \) such that \( F(d) \simeq 0 \). Let \( Q \) denote the quotient category \( \mathcal{D}/K \), which is the localization of \( \mathcal{D} \) with respect to the multiplicative system of morphisms that are taken to isomorphisms under the functor \( F \).

Let us denote the quotient morphism \( \mathcal{D} \to Q \) by \( j^* \). By the assumption that \( F \) is continuous, \( j^* \) has a fully faithful right adjoint, denoted \( j_* \). On the other hand, \( F \) descends to a conservative functor on the quotient \( Q \), and thus by the Barr-Beck-Lurie Theorem, we can identify \( Q \) with the category of \( T \)-modules, \( \mathcal{C}^T \). Using this identification, the bar construction defines a fully faithful left adjoint, which we denote \( j_! \). This gives rise to a **recollement situation**:

---

9 The usual definition of a conservative functor is a functor \( F \) such that if \( F(\phi) \) is an isomorphism, then \( \phi \) is an isomorphism. This definition is equivalent to the one above, in our context(s), by considering the cone (or the kernel and cokernel) of \( \phi \).
Theorem 2.12. We have a diagram:

\[ \begin{align*}
K & \xrightarrow{i^*} D \xrightarrow{j^*} Q, \\
K & \xrightarrow{i_*} D \xrightarrow{j_*} Q,
\end{align*} \]

where:

1. The functor \( j^* \) is left adjoint to \( j_* \), and right adjoint to \( i_* \).
2. The functors \( j_* \), \( j^* \), and \( i_* \) are fully faithful.
3. There are distinguished triangles (respectively, exact sequences) of functors
   \[ i_* i^! \to i_D \to j_* j^* \to j_! j^! \]
   \[ j^! i_* \to 1_D \to j_* i^* \to j_! i_* \]
4. The essential image \( K \) is the kernel of \( j^* \). The essential image of \( j_* \) (respectively, \( j^* \)) is the right (respectively, left) orthogonal to \( K \).

Remark 2.13. If the functor \( F \), in addition, takes compact objects in \( D \) to compact objects in \( C \), then the right adjoint \( FR \) preserves colimits. In that case, the recollement of Theorem 2.12 restricts to one on the level of small categories of compact objects.

2.4. \( D \)-modules on stacks. See [GR14, BZN, GR17] and the references therein for an outline of the theory of \( D \)-modules on (pre)stacks using the theory of stable \( \infty \)-categories (earlier formulations of the equivariant derived category can be found in Bernstein–Lunts [BL] and Beilinson–Drinfeld [BD]). In this section we just give an outline of the ideas and establish notation.

Let \( X \) be a stack of the form \( Y/K \), where \( Y \) is a smooth algebraic variety, and \( K \) an affine algebraic group (for the remainder of this section we will refer to such a stack as a quotient stack). We denote by \( \mathbf{D}(X) \) the (unbounded) derived category of \( D \)-modules on \( X \), or equivalently, the \( K \)-equivariant derived category of \( D \)-modules on \( Y \). If \( X \) is a scheme, then this is the derived category of sheaves of \( \mathcal{D}_X \)-modules, where \( \mathcal{D}_X \) is the sheaf of differential operators on \( X \). In general, \( \mathbf{D}(X) \) can be defined as the limit (in \( \text{Cat}_C \)) of the cosimplicial diagram

\[ \mathbf{D}(Y) \to \mathbf{D}(Y \times K) \ldots \]

where the structure maps are induced by taking the upper \(!\) functor.

Example 2.14. Given an affine algebraic group \( K \), we have

\[ \mathbf{D}(pt/K) \cong C^* (K) - \text{dgComod} \cong C_*(K) - \text{dgMod} \]

The category \( \mathbf{D}(X) \) is stable, presentable, and compactly generated, and the category of compact objects is denoted \( \mathbf{D}_{\text{com}}(X) \). Thus we have \( \mathbf{D}(X) = \text{Ind} \mathbf{D}_{\text{com}}(X) \). Denote by \( \mathbf{D}_{\text{coh}}(X) \) the category of bounded complexes each of whose cohomology objects is a coherent \( D \)-module. It is known that \( \mathbf{D}_{\text{com}}(X) \subset \mathbf{D}_{\text{coh}}(X) \), with equality when \( X \) is a variety, but for a non-safe stack the inclusion is strict (see Example 2.21 below).

Let us recall the functoriality properties of \( D \)-modules.
Proposition 2.15 ([GR17] [HTT]). Given a safe\footnote{The notion of a safe morphism of stacks was introduced in [DG]. Safety guarantees that the D-module pushforward is continuous. The safe morphisms appearing in this paper will always be composites of a representable morphism and a unipotent gerbe (such morphisms are called very safe [Gun]).} morphism of quotient stacks $f : X \to Y$, we have functors:

$$f_* : D(X) \to D(Y),$$

and

$$f^! : D(Y) \to D(X).$$

1. If $f$ is proper, then $f_* \simeq D_Y f_* D_X$ preserves coherence and is right adjoint to $f^!$. We sometimes write $f_!$ instead of $f_*$. In that case.

2. If $f$ is smooth of relative dimension $d$, then $f^!$ preserves coherence and $f^* := f^![-2d]$ is left adjoint to $f_*$. The functor $f^\circ = f^![-d]$ is $t$-exact, and $f^\circ \simeq D_X f^\circ D_Y$.

3. If

$$X \times_W V \xrightarrow{\tilde{f}} V \xrightarrow{g} X \xrightarrow{f} W$$

is a cartesian diagram of stacks, then the base change morphism is an isomorphism: $g^! f_* \simeq \tilde{g}_* \tilde{f}^!$.

4. We have the projection formula:

$$f_* (f^! M \otimes N) \simeq M \otimes f_* (N).$$

5. The category $D(X)$ carries a symmetric monoidal tensor product

$$M \otimes N := \Delta ! (M \boxtimes N) \simeq M \boxtimes \mathcal{O}_X N[-\dim(X)].$$

6. We have an internal Hom:

$$\text{Hom}(M, N) := D(M) \otimes N.$$

Remark 2.16. For a non-safe morphism of stacks $f$ (e.g. the projection $pt/G \to pt$ for a reductive group $G$), to obtain a continuous functor one should take the continuous extension of the restriction of $f_*$ to compact objects. This is called the renormalized pushforward, and is denoted $f^\wedge$. Alternatively, one can replace $D$-modules with Ind-coherent $D$-modules as defined in Section 2.5.

The continuous extension of the Verdier duality functor defines a self-duality of the category $D(X)$. The functor $f_*^\wedge$ is dual to $f^!$ with respect to this self-duality. This gives rise to a good theory of integral transforms for $D$-modules.

Proposition 2.17 ([Gai], [BZN]). Given quotient stacks $X$ and $Y$, we have an equivalence:

$$D(X \times Y) \leftarrow \leftarrow \sim \text{Fun}_L(D(X), D(Y)) \sim \leftarrow \leftarrow D(X) \otimes D(Y).$$

$$\mathcal{R} \leftarrow \leftarrow \Phi_{\mathcal{R}} := q_{Y \wedge} (\mathcal{R} \otimes q_X^! (-))$$
where

\[ X \xleftarrow{g_X} X \times Y \xrightarrow{g_Y} Y, \]

are the projections. We refer to \( \mathcal{R} \) as the kernel corresponding to the functor \( \Phi_{\mathcal{R}} \).

If

\[ X \leftarrow Z \twoheadrightarrow Z, \]

is a diagram of smooth quotient stacks, then the functor \( f_* g_{!} \) is represented by the integral kernel \( \mathcal{R}_Z = (f \times g)_* \omega_Z \) (this follows from the projection formula).

**Remark 2.18.** If \( Z \) has a partition \( Z = \bigsqcup_{i \in I} Z_i \) in to locally closed substacks, then any object of \( D(Z) \) on \( Z \) is filtered by \( I \). In that case the functor \( f_* g_{!} \) (or equivalently, the object \( \mathcal{R}_Z \in D(X \times Y) \)) has a filtration indexed by \( I \): the functor \( (f_* g_{!})_i \) is given by \( f_J g_j \), where \( f_J \) (respectively \( g_j \)) is the restriction of \( f \) (respectively \( g \)) to \( Y_j = \bigsqcup_{i \in I} Y_j \).

**Remark 2.19.** The category \( D_{hol}(X) \) of holonomic complexes is the subcategory of \( D_{coh}(X) \) consisting of complexes whose cohomology objects are holonomic. Although coherent complexes are not always preserved by the \( D \)-module functors, the holonomic subcategory is preserved (for safe morphisms). By Verdier duality it follows that we have the full six functors, including adjoint pairs \( (f^*, f_*) \) and \( (f_!, f^!) \) for any safe morphism \( f \).

### 2.5. Ind-Coherent \( D \)-modules

We define the category of Ind-coherent \( D \)-modules on \( X \) by

\[ \hat{D}(X) = \text{Ind} D_{coh}(X) \]

By construction, the compact objects of \( \hat{D}(X) \) are exactly the coherent complexes.\footnote{This variant of the category of \( D \)-modules was considered by Arinkin and Gaitsgory in the context of geometric Satake [AG15], where it is referred to as the renormalized category. The construction is closely related to the theory of Ind-coherent sheaves as developed by Gaitsgory and Rozenblyum [Gra11, Gra17]. Note that the relationship between Ind-coherent sheaves and Ind-Coherent \( D \)-modules is not so clear: the usual category \( D(X) \) is already given by \( \text{IndCoh}(X_{dR}) \) (which is equivalent to \( QC(X_{dR}) \) as the de Rham stack of anything is trivially smooth). In some sense the difference between Ind-coherent and usual \( D \)-modules is measuring the singularities in \( T^*(Y/K) \) generated by the non-flatness of the moment map.} Both \( D(X) \) and \( \hat{D}(X) \) carry a \( t \)-structure whose heart is the (same) abelian category \( \mathcal{M}(X) \) of \( D \)-modules on \( X \). There are adjoint functors \( \hat{D}(X) \subset \hat{D}(X) \) which exhibit \( D(X) \) as a co-localization of \( \hat{D}(X) \), which restrict to an equivalence on the positive part of the \( t \)-structure.

**Remark 2.20.** If \( X \) is a variety, or more generally if \( X \) is safe in the sense of [DG], then \( \hat{D}(X) = D(X) \).

**Example 2.21.** Let \( X = pt/T \), where \( T \) is an algebraic torus, and consider \( \Lambda = H^*(T) = \text{Sym}(t \in T[1]) \), and \( S = H^*(pt/T) = \text{Sym}(t \in [-2]) \). One can see by descent that \( D(X) \) is equivalent to \( \Lambda - \text{dgMod} \), and thus \( D_{\text{coh}}(X) = \Lambda - \text{Perf} \). On the other hand, \( D_{\text{coh}}(X) = \Lambda - \text{Coh} \) consists of bounded dg \( \Lambda \)-modules whose cohomology objects are finite dimensional. As explained in Example 2.11 Koszul duality defines equivalence \( \Lambda - \text{Coh} \simeq S - \text{Perf} \), and thus \( \hat{D}(X) = S - \text{dgMod} \).

**Remark 2.22.** When \( X \) is a finite orbit stack (for example, a quotient stack \( Y/K \) where \( K \) acts on \( Y \) with finitely many orbits), every coherent complex on \( X \) is regular holonomic. Thus, via the Riemann-Hilbert correspondence, \( \hat{D}(X) \) can be identified with the Ind category of \( K \)-equivariant constructible complexes on \( Y \) in the sense of Bernstein–Lunts [BL].
Given a proper (representable) morphism of quotient stacks \( f : X \to Y \), the functor \( f_* \) preserves coherence and thus defines a continuous, quasi-proper functor \( \mathbf{D}(X) \to \mathbf{D}(Y) \) which (by abuse of notation) we still write as \( f_* \). By the remarks in Section 2.1, this functor has continuous right adjoint which we denote \( f^! \). Similarly, if \( f \) is smooth of relative dimension \( d \), then \( f^! \) preserves coherent complexes. We consider the induced functor \( f_* = f^![\dim X] : \mathbf{D}(X) \to \mathbf{D}(Y) \) and write \( f_* \) for the continuous right adjoint of \( f^! \).

This allows us to define a pair of functors

\[
\begin{align*}
\mathbf{f} : \mathbf{D}(X) & \xrightarrow{i} \mathbf{D}(Y) : f^! \\
\mathbf{f}^* : \mathbf{D}(Y) & \xleftarrow{\mathbf{f}^!} \mathbf{D}(X)
\end{align*}
\]

for every smooth or proper morphism of quotient stacks. One can use this to construct such functors for any representable morphism of quotient stacks by factoring such a morphism as a smooth morphism composed with a proper morphism.

The utility of Ind-Coherent \( D \)-modules stems from the fact that the following observation: given a quotient stack \( X \) with projection morphism \( p : X \to \text{pt} \), the functor \( p^! : \text{Vect} \to \mathbf{D}(X) \) does not preserve compact objects, but it does preserve coherent objects. Thus it defines a quasi-proper functor on Ind-coherent \( D \)-modules; we write \( f^* = f^![\dim X] \), and define \( f_* \) to be the (continuous) right adjoint of \( f^* \). Thus a smooth quotient stack behaves just as a smooth algebraic variety from the perspective of Ind-coherent \( D \)-modules.

**Remark 2.23.** The Verdier duality functor defines an equivalence \( D_{\text{coh}}(X) \to D_{\text{coh}}(X)^{\text{op}} \) and thus defines a self duality of \( D(X) \). This gives rise to a theory of integral transforms for Ind-coherent \( D \)-modules as in Proposition 2.17, except one does not have to consider the renormalized pushforward, as the non-renormalized version is already continuous on Ind-coherent \( D \)-modules.

### 3. Mackey Theory and Decomposition by Levis

Most of the material in this section is a recollection from [Gun]. Using these results we will show that the recollement situation for the category \( D(X) \) induced by induction and restriction functors gives an orthogonal direct sum.

#### 3.1. Parabolic induction and restriction.

Recall that we have functors:

\[
\begin{align*}
\text{Ind}^G_{P,L} = r_* s^! : D^G(I) & \xleftarrow{\text{Res}^G_{P,L}} D^G(g) : s_* r^! = \text{Res}^G_{P,L}
\end{align*}
\]

given by the diagram

\[
\begin{align*}
\mathfrak{g} & \xrightarrow{r} \mathfrak{p} \xrightarrow{s} I.
\end{align*}
\]

In [Gun] Proposition 3.14, Corollary 3.18, we showed that the functors \( \text{Ind}^G_{P,L} \) and \( \text{Res}^G_{P,L} \) are \( t \)-exact, and preserve coherent \( D \)-modules in the heart of the \( t \)-structure (this latter property is clear for induction as it is the composite of a smooth pullback and proper pushforward, but not obvious for restriction). It follows that parabolic induction and restriction define an adjoint pair of functors on Ind-coherent \( D \)-modules (for which we use the same notation):

\[
\begin{align*}
\text{Ind}^G_{P,L} : D^G(I) & \xleftarrow{\text{Res}^G_{P,L}} D^G(g) : \text{Res}^G_{P,L}
\end{align*}
\]
3.2. Steinberg Stacks and Functors. The fiber product $q \times_p \mathcal{P}$ will be denoted by $\mathcal{Q}st_P$ and referred to as the Steinberg stack. It is equipped with projections

$$
\mathcal{M} \xrightarrow{\alpha} \mathcal{Q}st_P \xrightarrow{\beta} \mathcal{P}
$$

The Steinberg stack is stratified by the (finitely many) orbits of $Q \times P$ on $G$ and all of the strata have dimension zero (in the stacky sense). For each orbit $w$ in $Q\backslash G/P$, we denote by $\mathcal{Q}st^w_P$ the corresponding strata in $\mathcal{Q}st_P$. Given any lift $\tilde{w} \in G$ of $w$, we have an equivalence of stacks

$$
\mathcal{Q}st^w_P \simeq (q \cap w P)/(Q \cap w P).
$$

We define the functor

$$
\text{St} = \text{Ind}^G_{M,Q} \text{St}_{P,L} := \text{Res}^G_{Q,M} \text{Ind}^G_{P,L} : \mathcal{D}(\mathcal{M}) \to \mathcal{D}(\mathcal{M}),
$$

(we will often drop the subscripts and superscripts when the context is clear). By base change, we have $\text{St} \simeq \alpha \ast \beta^!$, where:

$$
\begin{align*}
\alpha : & \quad \mathcal{Q}st_P \\
\beta : & \quad \mathcal{P}
\end{align*}
$$

The functor $\text{St}$ has a filtration (in the sense of Definition 2.5) indexed by the poset $Q\backslash G/P$ (we will refer to this filtration as the Mackey filtration). The following result identifies the components of the associated graded functor (see [Gun] for the notation).

**Proposition 3.1** (Mackey Filtration, [Gun], Proposition 1.6). For each lift $\tilde{w} \in G$ of $w \in Q\backslash G/P$, there is an equivalence

$$
\text{St}^w \simeq \text{Ind}^M_{M \cap \tilde{w} P,M \cap \tilde{w} P} \text{Res}^L_{Q \cap \tilde{w} P,Q \cap \tilde{w} P} \tilde{w}^* : \mathcal{D}(\mathcal{M}) \to \mathcal{D}(\mathcal{M}).
$$

3.3. The Recollement by Levis. In [Gun], we saw how the functors of parabolic induction and restriction give rise to a recollement situation of the abelian category of $D$-modules. Here we describe an analogous situation, but in the setting of stable/triangulated categories. The notion of recollement situations in this setting is discussed in Section 2.3.

Given a Levi subgroup $L$ of $G$, there are various closed subsets $\text{comm}(\mathfrak{g})_J \subseteq \text{comm}(\mathfrak{g})$ corresponding to closed subsets $J$ of the poset $\text{Levi}^G_P$ of Levis up to conjugacy. Thus we can define subcategories $\mathbf{D}(\mathfrak{g})_J$ consisting of objects with singular support in $\text{comm}(\mathfrak{g})_J$. By Theorem C in [Gun], we can identify these subcategories in terms of parabolic restriction. For example:

- $\mathbf{D}(\mathfrak{g})_{\mathfrak{l}(L)}$ consists of objects killed by parabolic restriction to $L$.
- $\mathbf{D}(\mathfrak{g})_{\mathfrak{c}(L)}$ consists of objects for which parabolic restriction to $L$ is cuspidal (note: the zero object is always cuspidal).
- $\mathbf{D}(\mathfrak{g})_{\mathfrak{g}(L)}$ consists of objects killed by parabolic restriction to $M$ whenever $(M) \nmid (L)$.
- $\mathbf{D}(\mathfrak{g})_{\mathfrak{g}(L)}$ consists of objects killed by parabolic restriction to $M$ whenever $(M) \nmid (L)$.
Remark 3.2. Even though the functor of parabolic restriction to $L$ depends on a choice of parabolic subgroup $P$ which contains $L$ as a Levi factor, the subcategories are independent of the choice.

We let $D(\mathfrak{g})_L$ denote the quotient $D(\mathfrak{g})_{\succ(L)}/D(\mathfrak{g})_{\preceq(L)}$. According to Theorem 2.12 we have:

**Proposition 3.3.** There is a recollement situation:

![Diagram]

In what follows, we will identify $D(\mathfrak{g})_{L}$ with the fully faithful image of $i_{(L)}$, which is the left orthogonal to $D(\mathfrak{g})_{\succ(L)}$ in $D(\mathfrak{g})_{\preceq(L)}$, or equivalently, the cocompletion of the essential image of $\text{Ind}_{L,C}^{G}$ (for any choice of $P$).

Explicitly, Proposition 3.3 means that we can express any object $\mathcal{M} \in D(\mathfrak{g})$ as an iterated extension of objects taken from $D(\mathfrak{g})_{L}$ as $L$ varies over Levi subgroups of $G$. This can be seen by the following algorithm:

1. First choose a Levi subgroup $L$ which is minimal such that $\text{Res}_{L}^{G}(\mathcal{M}) \neq 0$ (thus $\mathcal{M} \in D(\mathfrak{g})_{L}$).
2. We have a distinguished triangle:
   
   $i_{(L)}^*(\mathcal{M}) \to \mathcal{M} \to i_{(L)}^! (\mathcal{M}) \xrightarrow{+1}$,
3. Replace $\mathcal{M}$ by $i_{(L)}^* (\mathcal{M})$ and repeat steps (1) and (2) (the algorithm halts after finitely many steps when $L = G$).

**Remark 3.4.** The exactness of parabolic induction and restriction imply that the recollement above is compatible with the corresponding one for the abelian category. In particular, we have that $D(\mathfrak{g})_{L}$ consists precisely of those complexes in $D(\mathfrak{g})$ each of whose cohomology objects are in $M(\mathfrak{g})_{L}$.

3.4. **Orthogonal Decomposition by Levi.** We would now like to show that the blocks $D(\mathfrak{g})_{L}$ are orthogonal, so all the triangles appearing in the recollement of Proposition 3.3 are split.

**Proposition 3.5 (Gin) Proposition 4.5.** Suppose $L$ and $M$ are non-conjugate Levi subgroups. Given $\mathcal{M} \in D(\mathfrak{g})_{L}$, and $\mathcal{N} \in D(\mathfrak{g})_{M}$ we have that $\text{Ind}_{L}^{G}(\mathcal{M})$ and $\text{Ind}_{M}^{G}(\mathcal{N})$ are orthogonal, i.e.

$R \text{Hom} (\text{Ind}_{L}^{G}(\mathcal{M}), \text{Ind}_{M}^{G}(\mathcal{N})) = R \text{Hom} (\text{Ind}_{M}^{G}(\mathcal{N}), \text{Ind}_{L}^{G}(\mathcal{M})) = 0$.

**Remark 3.6.** It follows immediately that (in the notation of Proposition 3.5) direct summands of $\text{Ind}_{L}^{G}(\mathcal{M})$ and of $\text{Ind}_{M}^{G}(\mathcal{N})$ will be orthogonal.

**Proposition 3.7 (Gin) Proposition 4.20.** Any object of the abelian category $M(\mathfrak{g})_{L}$ is a direct summand of an object of the form $\text{Ind}_{L}^{G}(\mathcal{M})$ for some $\mathcal{M} \in D(\mathfrak{g})_{L}$.

Now we are ready to prove:
Proposition 3.8. There is an orthogonal decomposition:
\[ D(\mathfrak{g}) = \bigoplus_{(L)} D(\mathfrak{g})(L) \]
as \((L)\) ranges over conjugacy classes of Levi subgroups of \(G\).

Proof. Given the recollement situation of Proposition 3.3, it remains to show that the subcategories \(D(\mathfrak{g})(L)\) are orthogonal for non-conjugate \(L\). In fact, it suffices to show that the corresponding subcategories of compact objects are orthogonal.

Suppose we have complexes \(M \overset{P}{\to} D(\mathfrak{g})(L)\) and \(N \overset{P}{\to} D(\mathfrak{g})(M)\), where \(L\) is not conjugate to \(M\).

We may assume that \(M\) and \(N\) are coherent complexes. We want to show that \(R\text{Hom}_M(N, P) \cong 0\).

Note that each cohomology object \(H^i_P M\) is orthogonal to \(H^j_P N\), by Proposition 3.7 and Remark 3.6. Thus the required orthogonality follows from the Ext spectral sequence. \(\square\)

4. Generalized Springer Decomposition

The goal of this section is to prove Theorem A.

4.1. Abelian category decomposition. Recall that a cuspidal datum for \(G\) is a pair \((L, E)\), where \(L\) is a Levi subgroup of \(G\) and \(E\) is a simple cuspidal local system on a nilpotent orbit of \(L\) (or equivalently, a simple cuspidal object of \(M(\mathfrak{g})L\)). Each cuspidal datum determines a subcategory \(M(\mathfrak{g})(L, E)\) of \(M(\mathfrak{g})L\) consisting of objects supported on \(L \times L\) of the form \(\mathfrak{m} \otimes E\), where \(\mathfrak{m} \in \mathfrak{M}(\mathfrak{z}(L))\).

Let \(M(\mathfrak{g})(L, E)\) denote the subcategory of \(M(\mathfrak{g})L\) consisting of direct summands of objects of the form \(\text{ind}_{\mathfrak{g}}^G(\mathfrak{m})\), where \(\mathfrak{m} \in M(\mathfrak{z}(L))\).

Theorem 4.1 (Gun Theorem A). There is an orthogonal decomposition:
\[ M(\mathfrak{g})G \cong \bigoplus_{(L, E)} M(\mathfrak{g})(L, E), \]
where
\[ M(\mathfrak{g})(L, E) \cong M(\mathfrak{z}(L))^{W(\mathfrak{g}, L)}. \]

4.2. The derived category of cuspidal objects. Let \(x \in \mathcal{N}_G\) be a nilpotent element, and \(\mathcal{O}\) the corresponding nilpotent orbit. As usual, we write \(\mathcal{O}/G\) for \(\mathcal{O}/G\), which is equivalent to the classifying stack \(pt/Z_G(x)\). We write \(Z^0\) for the neutral component of the center of \(G\). Let \(A = A_G(x) = \pi_0(Z_G(x)) = Z_G(x)/Z_G(x)^0\) be the equivariant fundamental group of \(\mathcal{O}\). Representations of \(A\) (or modules for \(\mathbb{C}[A]\)) are the same thing as \(G\)-equivariant local systems on \(\mathcal{O}\). More generally, we have the following description of derived local systems: \(D(G) \cong C_*(Z_G(x)) - \text{dgMod}\) (see Example 2.14).

Recall that if the orbit \(\mathcal{O}\) supports a cuspidal local system \(E\), then \(\mathcal{O}\) must be distinguished, i.e. \(Z^0\) is a maximal connected reductive subgroup of \(Z_G(x)\). The following lemma describes the derived local systems on a distinguished orbit. We denote by \(\Lambda_3\) the free graded-commutative algebra generated by \(\bigwedge[1]\) (considered as a dg-algebra with zero differential). There is an equivalence of dg-algebras
\[ C_*(Z^0) \cong \Lambda_3. \]
Lemma 4.2. Suppose $x$ is distinguished. Then there is an equivalence of dg-algebras

$$C_\bullet(Z_G(x)) \cong \Lambda_\chi \otimes \mathbb{C}[A]$$

Proof. Recall that by Jacobson-Morozov theory, there is an $\mathfrak{sl}_2$-triple, $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$ such that $\phi(e) = x$ (where $e, f, h$ are the standard basis for $\mathfrak{sl}_2$). Moreover, we have extensions

$$\xymatrix{ Z_G(x) & Z_G(x) \ar[r] & A \ar[l] \cr Z_G(\phi) & Z_G(\phi) \ar[r] \ar[u] & A \ar[l] \ar[u] }$$

where $Z_G(\phi)$ is the maximal reductive subgroup of $Z_G(x)$. In particular, the vertical maps are all homotopy equivalences, and thus $C_\bullet(Z_G(x)) \cong C_\bullet(Z_G(\phi))$. As $x$ is distinguished, $Z_G(\phi) = Z^\circ(G)$, so the lower line is in fact a central extension of the finite group $A_G(x)$ by the torus $Z^\circ(G)$. As such, it is determined by a cocycle

$$\mu : A \times A \to Z^\circ$$

Explicitly, $Z_G(\phi)$ is isomorphic to an algebraic group, whose underlying variety is $Z^\circ \times A$, but with group structure twisted by the cocycle:

$$(x_1, a_1) \cdot_{\mu} (x_2, a_2) = (x_1 x_2 \mu(a_1, a_2), a_1 a_2).$$

By the Künneth theorem, $C_\bullet(Z_G(\phi))$ is equivalent as a chain complex to $\Lambda_\chi \otimes \mathbb{C}[A]$. The convolution operation is determined by the induced map

$$\mu_* : \mathbb{C}[A] \otimes \mathbb{C}[A] \to \Lambda_\chi$$

which is necessarily trivial, as $Z^\circ$ is connected. This gives the required equivalence. \qed

Thus we have:

$$\mathbf{D}(\mathcal{O}) \cong \text{Rep}(A) \otimes \Lambda_\chi \text{-dgMod}$$

It follows that the category of $D$-modules on $\mathcal{O}$ decomposes over the the set $\hat{A}$ of irreducible representations of $A$.

Lemma 4.3. There is an orthogonal decomposition:

$$\mathbf{D}(\mathcal{O}) \cong \bigoplus_{i \in \hat{A}} \Lambda_\chi - \text{dgMod}$$

By Example 2.21 we have the corresponding result for Ind-coherent $D$-modules:

$$\mathbf{D}(\mathcal{O}) \cong \bigoplus_{i \in \hat{A}} S_{j^*}[2] - \text{dgMod}$$

where $S_{j^*} = \text{Sym}(j^*[-2])$. 


4.3. **Cleaness of cuspidal local systems.** Given an equivariant local system \( \mathcal{E} \) on \( \mathcal{O} \), we say \( \mathcal{E} \) has a clean extension to \( G \) (or simply, is clean) if the canonical maps \( j!(\mathcal{E}) \to j_*(\mathcal{E}) \to j!(\mathcal{E}) \) are equivalences, where \( j : \mathcal{O} \to \mathfrak{g} \) is the (locally closed) inclusion.

Lusztig has shown that all cuspidal local systems are clean. We were unable to give an independent proof of this fact using the results we have proved so far. The orthogonal decomposition by Levis, does at least give us this result:

**Proposition 4.4.** The following are equivalent:

1. All simple cuspidal local systems are clean.
2. Any two non-isomorphic simple cuspidal objects of \( \mathbf{M}(\mathcal{N}_G) \) are orthogonal in \( \mathbf{D}(\mathfrak{g}) \).

**Proof.** Suppose that all cuspidal local systems are clean, and let \( \mathcal{E}_1, \mathcal{E}_2 \) be nonisomorphic simple cuspidal local systems supported on nilpotent orbits \( \mathcal{O}_1, \mathcal{O}_2 \) respectively. We will denote by \( \mathcal{E}_1, \mathcal{E}_2 \) the clean extension to (simple) objects of \( \mathbf{M}(\mathfrak{g}) \). We have

\[
R\text{Hom}(\mathcal{E}_1, \mathcal{E}_2) \simeq R\text{Hom}(j_!\mathcal{E}_1, j_2*\mathcal{E}_2) \simeq R\text{Hom}(j_!\mathcal{E}_1, j_2*\mathcal{E}_2).
\]

If \( \mathcal{O}_1 \neq \mathcal{O}_2 \), then \( j_2^*j_!\mathcal{E}_1 \simeq 0 \); if \( \mathcal{O}_1 = \mathcal{O}_2 \), then \( j_2^*j_!\mathcal{E}_1 \simeq \mathcal{E}_1 \), and the right hand side again vanishes by Lemma 4.3.

Now suppose that any two simple cuspidal objects of \( \mathbf{M}(\mathcal{N}_G) \) are orthogonal in the derived category. Let \( \mathcal{E} \) be a simple cuspidal local system on a nilpotent orbit \( j : \mathcal{O} \to \mathcal{N}_G \). If \( \mathcal{E} \) were not clean, then \( i^*j_*\mathcal{E} \neq 0 \) for some other nilpotent orbit \( i : \mathcal{O}' \to \mathcal{N}_G \) in the closure of \( \mathcal{O} \). Thus we have that

\[
R\text{Hom}(j_*\mathcal{E}_1, i_*\mathcal{F}) \neq 0
\]

where \( \mathcal{F} = i^*j_*\mathcal{E} \in \mathbf{D}(\mathcal{N}_G) \). Decomposing \( \mathcal{F} = \bigoplus \mathcal{F}_i \) according to Proposition 3.8, we see that there must be a summand of \( \mathcal{F} \) which is cuspidal. It follows that there must be a simple cuspidal local system on \( \mathcal{O}' \) which is not orthogonal to \( \mathcal{E} \). \( \square \)

4.4. **Cuspidal Blocks.** From this point on we will freely use the fact that cuspidal local systems are clean.

Suppose \( \mathcal{E} \) is a simple cuspidal local system on \( \mathcal{O} \), corresponding to a certain irreducible representation of \( A \). Let \( \mathbf{D}(\mathcal{O})(\mathcal{E}) \simeq \Lambda_{\mathfrak{g}'(\mathcal{E})}^* \mathfrak{g}' \mathbf{Mod} \) denote the block of \( \mathbf{D}(\mathcal{O}) \) corresponding to \( \mathcal{E} \). Let \( \mathbf{D}(\mathfrak{g})(\mathfrak{g}'(\mathcal{E})) \) denote the corresponding subcategory of \( \mathbf{D}(\mathfrak{g}) \). Note that \( \mathfrak{g}' = \mathfrak{g}' / \mathfrak{g} \), where \( \mathfrak{g}' = \mathfrak{g} \times \mathfrak{g}' / \mathfrak{g} \). Consider the map:

\[
k : \mathfrak{g} \to \mathfrak{g}'.
\]

**Lemma 4.5.** The functor \( k_* : \mathbf{D}(\mathfrak{g})(\mathfrak{g}'(\mathcal{E})) \to \mathbf{D}(\mathfrak{g})(\mathfrak{g}) \) is fully faithful.

**Proof.** This follows immediately from the fact that objects of \( \mathbf{D}(\mathfrak{g})(\mathfrak{g}'(\mathcal{E})) \) have clean extensions to \( \mathfrak{g} \). \( \square \)

We denote the essential image of \( k_* : \mathbf{D}(\mathfrak{g})(\mathfrak{g}'(\mathcal{E})) \to \mathbf{D}(\mathfrak{g})(\mathfrak{g}) \) by \( \mathbf{D}(\mathfrak{g})(\mathfrak{g})(\mathcal{E}) \). In other words, these objects are of the form \( \mathcal{E} \boxtimes \mathfrak{g}' \), where \( \mathfrak{g}' \in \mathbf{D}(\mathfrak{g})(\mathfrak{g}) \) (and we identify \( \mathcal{E} \) with its clean extension to \( \mathcal{N}_G \)).

We have the following description of the category of cuspidal objects in \( \mathbf{D}(\mathfrak{g}) \):

\[12\] The orthogonality of non-isomorphic cuspidal local systems (and thus cleaness) follows from the fact (due to Lusztig) that such local systems admit distinct central characters. However, the proof of this fact uses some case-by-case analysis. It would be nice to have a more conceptual explanation for this phenomenon.
Proposition 4.6. We have an orthogonal decomposition

\[ D(\mathfrak{g})_{\text{cusp}} \simeq \bigoplus_{(\mathcal{E})} D(\mathfrak{g})(\mathcal{E}) \]

where each block \( D(\mathfrak{g})(\mathcal{E}) \) is equivalent to

\[ D(\mathfrak{g}) \simeq \mathcal{D}_1 \otimes \Lambda_1 - \text{dgMod}. \]

Proof. By Lemma 4.5, the category \( D(\mathfrak{g})_{\text{cusp}} \) decomposes in to orthogonal blocks indexed by distinguished orbits, where each block is equivalent to the subcategory generated by cuspidal local systems in \( D(\mathcal{O}) \). The result then follows from Lemma 4.3. □

Remark 4.7. The subcategory \( D(\mathfrak{g})(\mathcal{E}) \) consists precisely of those objects all of whose cohomology objects are contained in \( M(\mathfrak{g})(\mathcal{E}) \).

4.5. Proof of Theorem A. Let \( D(\mathfrak{g})(L, \mathcal{E}) \) denote the subcategory of \( D(\mathfrak{g})(L) \) consisting of complexes \( \mathfrak{M} \) each of whose cohomology objects are in \( M(\mathfrak{g})(L, \mathcal{E}) \).

Lemma 4.8. There is an orthogonal decomposition

\[ D(\mathfrak{g})(L) = \bigoplus D(\mathfrak{g})(L, \mathcal{E}) \]

where the sum ranges over isomorphism classes of simple cuspidal local systems on \( \mathcal{N}_L \).

Proof. Suppose \( \mathcal{E}, \mathcal{F} \) are two such cuspidal local systems on (nilpotent orbits for) \( L \), and let \( \mathfrak{M} \in D(\mathfrak{g})(L, \mathcal{E}) \), and \( \mathfrak{N} \in D(\mathfrak{g})(L, \mathcal{F}) \). As in Proposition 3.8, it suffices to take \( \mathfrak{N} \) and \( \mathfrak{M} \) to be bounded coherent complexes. Then by the Ext spectral sequence, it suffices to show that the cohomology objects are orthogonal in the derived category. Thus we may assume that \( \mathfrak{M} \) and \( \mathfrak{N} \) are coherent objects in the heart of the \( t \)-structure. In particular, they are given by direct summands of parabolic induction. Thus we reduce to showing that parabolic inductions from \( M(\mathfrak{g})(\mathcal{E}) \) and \( M(\mathfrak{g})(\mathcal{F}) \) are orthogonal (in the derived category).

Suppose \( \mathfrak{K} \in M(\mathfrak{g})(\mathcal{E}) \) and \( \mathfrak{L} \in M(\mathfrak{g})(\mathcal{F}) \). Then, by [\( \bigwedge \)]

\[ R \text{Hom}(\text{ind}^G_L(\mathfrak{K}), \text{ind}^G_L(\mathfrak{L})) \simeq R \text{Hom}(\text{res}_L^G(\mathfrak{N}), \text{ind}^G_L(\mathfrak{L})) \simeq R \text{Hom}(\mathfrak{N}, \bigoplus w^*(\mathfrak{L})). \]

where the sum is taken over elements of the relative Weyl group \( w \in W_{(G,L)} \). By Lusztig’s generalized Springer correspondence, the relative Weyl group fixes each simple cuspidal local system [Lusci]. In particular \( w^*(\mathfrak{L}) \in M(\mathfrak{g})(\mathcal{F}) \) for each \( w \in W_{(G,L)} \). Thus the result follows from the orthogonal decomposition of \( D(\mathfrak{g})_{\text{cusp}} \), Proposition 4.6. □

5. The Steinberg Monad

In this section we study the blocks \( D(\mathfrak{g})(L, \mathcal{E}) \) via parabolic restriction, and the corresponding Mackey filtration on the monad.
5.1. **Monadic description of the blocks.** Fix cuspidal data \((L, \mathcal{E})\), and let \(Z^\circ = Z^\circ(L)\), \(\mathfrak{z} = \mathfrak{z}(L)\), and \(\mathfrak{z}/Z^\circ \simeq \mathfrak{z} / pt / Z^\circ\). The functors of parabolic induction and restriction restrict to form a monadic adjunction:

\[
\text{Ind}_{P^L}^G : \mathcal{D}(\mathfrak{g})(L, \mathcal{E}) \xleftarrow{\sim} \mathcal{D}(\mathfrak{g})(L, \mathcal{E}) : \text{Res}_{P^L}^G
\]

By the Barr-Beck-Lurie Theorem [2.9] there is an equivalence

\[
\mathcal{D}(\mathfrak{g})(L, \mathcal{E}) \xrightarrow{\sim} \mathcal{D}(\mathfrak{g})(L, \mathcal{E})^{\text{St}_{P^L, \mathcal{E}}}
\]

According to Proposition [4.6], \(\mathcal{D}(\mathfrak{g})(L, \mathcal{E})\) is equivalent to \(\mathcal{D}(\mathfrak{g})\). Let us denote by \(\text{St} = \text{St}_E\) the corresponding monad acting on \(\mathcal{D}(\mathfrak{g})\) under this equivalence. The goal of this section is to understand this monad more concretely.

5.2. **The Steinberg functor as an integral transform.** For simplicity, let us first consider the case of the Springer block, i.e. where the cuspidal datum is given by \((T, \mathbb{C})\) for a maximal torus \(T\) of \(G\). Fix a Borel subgroup \(B\), and recall the Steinberg stack \(\text{st}_B\) from Subsection 3.2. There are maps

\[
\mathfrak{t} \xleftarrow{\alpha} \mathfrak{g} \xrightarrow{\beta} \mathfrak{t}
\]

Let \(f = \alpha \times \beta : \mathfrak{g} \to \mathfrak{t} \times \mathfrak{t}\).

Recall the notion of integral transforms for \(D\)-modules explained in Subsection 2.4. The following lemma is immediate from Proposition 2.17.

**Lemma 5.1.** The Steinberg monad

\[
\text{St} : \mathcal{D}(\mathfrak{g}) \to \mathcal{D}(\mathfrak{g})
\]

is represented by the integral kernel \(f_*(\omega_{\mathfrak{g}}) \in \mathcal{D}(\mathfrak{t} \times \mathfrak{t})\).

**Remark 5.2.** The monad structure on \(\text{St}\) translates in to an algebra structure on \(f_*(\omega_{\mathfrak{g}})\) with respect to the convolution monoidal product in \(\mathcal{D}(\mathfrak{t} \times \mathfrak{t})\). This corresponds to the structure of fiberwise convolution for the Steinberg stack \(\mathfrak{g}_B\). Over the fiber \((0, 0) \in \mathfrak{t} \times \mathfrak{t}\), this restricts to the usual convolution on the equivariant homology of the Steinberg variety as considered in [CG].

Recall that the monad \(\text{St}\) preserves the category of coherent objects of \(\mathcal{D}(\mathfrak{g}) \simeq \mathcal{D}_t \otimes \Lambda_t - \text{dgMod}\), which by Koszul duality can be identified with \(\mathcal{D}_t - \text{Perf}\), where \(\mathcal{D}_t = \mathcal{D}_t \otimes S^*_t\). Thus \(\text{St}\) is represented by an algebra object \(A\) in the monoidal category of \(\mathcal{D}_t\)-bimodules. By construction, we have:

**Proposition 5.3.** There is an equivalence:

\[
\mathcal{D}_{\text{coh}}(\mathfrak{g})(T, \mathbb{C}) \simeq A - \text{Perf}
\]

**Remark 5.4.** Equivalently, using the notion of Ind-coherent \(D\)-modules from Section 2.5 we have:

\[
\mathcal{D}(\mathfrak{g})(T, \mathbb{C}) \simeq A - \text{dgMod}
\]
5.3. **Non-splitness of the Mackey filtration.** In this subsection we will show that the Mackey filtration is non-split in general. We restrict attention to the rank 1 case, noting that an identical proof will show non-splitness whenever the corresponding relative Weyl group is non-trivial (by considering a single simple reflection).

For the remainder of this section we fix \( G = SL_2, B \) the standard Borel subgroup of upper triangular matrices and \( T \simeq \mathbb{C}^x \) the maximal torus of diagonal matrices. We identify \( T \) as the unipotent radical of \( B \). We have an adjunction

\[
\text{Res} : \text{Res}_{P,L}^G : D(g)^G \xrightarrow{\sim} D^T(t) : \text{Ind}_{P,L}^G =: \text{Ind},
\]

giving rise to the monad

\[
\text{St} = \text{Res} \circ \text{Ind} : D(\mathfrak{t}) \to D(\mathfrak{t}).
\]

Recall that \( \mathfrak{t} \) is the quotient stack \( t/T \); here \( T \simeq \mathbb{G}_m \) is acting trivially on \( t \). The Weyl group is generated by a single reflection \( s \) acting on \( t \) by \( t \mapsto -t \) and on \( T \) by \( a \mapsto a^{-1} \). The Mackey filtration is given by a single distinguished triangle in \( \text{Fun}^L(D(\mathfrak{t}), D(\mathfrak{t})) \):

\[
\text{St}_e \xrightarrow{\delta} \text{St} \xrightarrow{\delta} \text{St}_s \xrightarrow{\delta},
\]

Note that \( \text{St}_e \) is equivalent to the identity functor on \( D(\mathfrak{t}) \) and \( \text{St}_s \) is equivalent to \( s_\ast \). The following proposition means that the weak form of the Mackey theorem given by Proposition 3.1 cannot be improved to express \( \text{St} \) as a direct sum of Weyl group translations in general.

**Proposition 5.5.** The connection morphism \( \delta \) in the distinguished triangle \( 4 \) is non-zero.

In order to prove Proposition 5.5 recall that the category \( \text{Fun}^L(D(\mathfrak{t}), D(\mathfrak{t})) \) is equivalent to \( D(\mathfrak{t} \times \mathfrak{t}) \) (see Proposition 2.17), and thus we may replace the functors \( \text{St}, \text{St}_e, \text{St}_s \) by their integral kernels to obtain a distinguished triangle in \( D(\mathfrak{t} \times \mathfrak{t}) \):

\[
\text{St}_e \xrightarrow{\delta} \text{St} \xrightarrow{\delta} \text{St}_s \xrightarrow{\delta},
\]

By Lemma 5.1, \( \text{St} = f_\ast (\omega_{\mathfrak{t}B}) \). The stratification of the Steinberg stack \( \mathfrak{st} = \mathfrak{st}^e \cup \mathfrak{st}^s \) gives rise to the distinguished triangle \( 5 \). To make the computation more explicit, consider the following variant of the Steinberg variety:

\[
\mathfrak{st} = \{(x, g) \in \mathfrak{g} \times G \mid x \in \mathfrak{b}, g x \in \mathfrak{b}\}
\]

The Steinberg stack \( \mathfrak{st} = B \mathfrak{st}_B \) is the stack quotient \( \mathfrak{st}/(B \times B) \). The projection maps are given by:

\[
t \xleftarrow{a} \mathfrak{st} \xrightarrow{b} \mathfrak{t}
\]

\[
x + u \xleftarrow{a} (x, g) \xrightarrow{b} g x + u.
\]

The variety \( \mathfrak{st} \) is the union of a closed stratum \( \mathfrak{st}^e \), given by the locus where \( g \in B \), and an open stratum \( \mathfrak{st}^s \) where \( g \notin B \). We write \( f = (a \times b) : \mathfrak{st} \to \mathfrak{t} \times \mathfrak{t} \) and \( f_{e}, f_{s} \) for the restrictions to \( \mathfrak{st}^e, \mathfrak{st}^s \) respectively. We have a distinguished triangle

\[
\mathfrak{st}_e := f_{e \ast} \omega_{\mathfrak{st}^e} \to \mathfrak{st} := f_{\ast} \omega_{\mathfrak{st}} \to \mathfrak{st}_s := f_{s \ast} \omega_{\mathfrak{st}^s} \xrightarrow{\delta}.
\]
(Taking the fiber over a point \( (t_1, t_2) \in \mathfrak{t} \times \mathfrak{t} \), the distinguished triangle \( \triangleright \) computes the long exact sequence in Borel-Moore homology of the fiber \( \text{st}(t_1, t_2) \) associated to the partition \( \text{st}(t_1, t_2) = \text{st}^e_{(t_1, t_2)} \cup \text{st}^s_{(t_1, t_2)} \).

Up to a cohomological shift \( \triangleright \) the distinguished triangle \( \triangleright \) is obtained from \( \triangleright \) by forgetting the equivariant structure. Thus we are reduced to proving that the connecting morphism \( \delta \) in the distinguished triangle \( \triangleright \) is non-zero.

Let \( \Delta : t \mapsto t \times t \) denote the diagonal, and \( \nabla : t \mapsto t \times t \) the antidiagonal (i.e. the graph of \( s : t \mapsto t \)). Note that \( \mathfrak{f}_e \) is supported \( \Delta \) whereas \( \mathfrak{f}_s \) is supported in \( \nabla \), and that the images of \( \Delta \) and \( \nabla \) intersect at \((0,0)\).

Let \( p : t \mapsto pt \) and consider the distinguished triangle of complexes of vector spaces:

\[
p_*\nabla^!\mathfrak{f}_e \to p_*\nabla^!\mathfrak{f} \to p_*\nabla^!\mathfrak{f}_s \to
\]

This triangle gives rise to the long exact sequence in Borel-Moore homology associated to \( X \) with its partition into a closed subset \( X_e = \nabla^{-1}(\text{st}_e) \) and open complement \( X_s = \nabla^{-1}(\text{st}_s) \).

Explicitly we have

\[
X = \{(x, g) \in \mathfrak{g} \times G \mid x \in \mathfrak{b}, g^x \in \mathfrak{b}, x + u = -^g x + u\}.
\]

There are canonical maps \( h : X \to G \) and \( \tau : X \to \mathfrak{t} \). Note that \( X_e = B \times \mathfrak{u} \) and \( X_s \) is a (trivializable) line bundle over \( G - B \) whose fiber over \( g \) is \( b \cap ^g \mathfrak{b} \). In fact, we have:

**Lemma 5.6.** The variety \( X \) is isomorphic to \( G \times \mathbb{A}^1 \).

**Proof.** It is straightforward to check that the morphism

\[
\mathbb{A}^1 \times G \to \mathfrak{g} \times G
\]

\[
(t, \begin{bmatrix} a & b \\ c & d \end{bmatrix}) \to \begin{bmatrix} ct & 2dt \\ 0 & -ct \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

defines an isomorphism of \( G \times \mathbb{A}^1 \) onto \( X \). \( \square \)

To finish the proof of Proposition 5.5 we are reduced to showing that the connecting morphism in the long exact sequence

\[
H_*^{BM}(X_e) \to H_*^{BM}(X) \to H_*^{BM}(X_s) \to
\]

is non-zero. Using the above calculations we see that the topology of these spaces is as follows: \( X \) is homeomorphic to \( S^3 \times \mathbb{R}^5 \), \( X_e \) is homeomorphic to \( S^1 \times \mathbb{R}^5 \), and \( X_s \) is homeomorphic to \( S^1 \times \mathbb{R}^7 \).

---

\(^{13}\)Strictly speaking, the variety \( \text{st} \) is not obtained from the stack \( \mathfrak{a} \) by base change from \( t \times t \) to \( t \times t \), but it differs from the base change only by unipotent gerbes whose only effect on the category of \( D \)-modules is a cohomological shift.
Thus the long exact sequence in Borel-Moore homology takes the following form:

\[ 0 \rightarrow C[8] \rightarrow C[8] \]

\[ 0 \leftarrow 0 \rightarrow C[7] \]

\[ C[6] \leftarrow 0 \rightarrow 0 \]

\[ C[5] \rightarrow C[5] \rightarrow 0, \]

from which we deduce that the connecting morphism must be non-zero as required.

5.4. General Cuspidal Datum. In this subsection we indicate how to construct the dg algebra \( A_{(P,L,E)} \) in the case of a general cuspidal datum and spell out the proof of Theorem B.

Let \( st = \rho_{L,j} \) denote the Steinberg stack, as defined in Subsection 3.2. Recall from [Gun] that \( st \overset{\circ}{\rightarrow} \) denotes the substack \( j(l) \times N_L \). Consider the base change

\[ st \downarrow \downarrow st \overset{\circ}{\rightarrow} \leftarrow \leftarrow f \]

\[ \downarrow \downarrow \downarrow \downarrow \downarrow \]

\[ \downarrow \downarrow \downarrow \downarrow \downarrow \]

\[ l \times l \overset{\circ}{\rightarrow} l \overset{\circ}{\rightarrow} \]

Let \( \tilde{E} \in \tilde{D}(j \times N_L \times j \times N_L) \) denote the object \( \mathcal{E}^{\vee} \otimes \omega_j \otimes \mathcal{E} \otimes \omega_j \). Unwinding the definitions, we find:

**Lemma 5.7.** The integral kernel representing \( St \) is given by \( f_*(\tilde{E}) \in \tilde{D}(j \times j) \).

Note that

\[ \tilde{D}(j) \cong \mathcal{O}_j - \text{dgMod} \]

where \( \mathcal{O}_j = D_j \otimes S_{j*} \). Thus, we may represent the monad \( St \) as a dg-algebra \( A = A_{(P,L,E)} \), equipped with a morphism of dg-algebras \( D_{j(l)} \otimes S_{j*} \rightarrow A \).

Theorem B now follows by translating what we know about the Steinberg monad into statements about the dg-ring \( A \). In particular:

- Part 1 is immediate from the Barr-Beck-Lurie Theorem 2.9.
- Part 2 is what it means for \( A \) to represent a monad acting on \( \mathcal{O}_j - \text{dgMod} \) (see Example 2.6).
- Part 3 is the Mackey filtration of Proposition 3.1.
- Part 4 is the result of Subsection 5.3.
- Part 5 is given by Proposition 4.6.
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