Registration of hydrogen–like leptonic bound states \((e^-\mu^+)\) and \((e^+\mu^-)\) in reactions of high–energy scattering of polarized electrons and positrons by nuclei with \(Z \sim 100\) and analysis of CPT invariance

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Abstract

The cross sections for the reactions of muonium(anti–muonium) production in high energy electron(positron) scattering by nuclei \(e^-(e^+) + Z \rightarrow Z + M^0(M^0) + \mu^-(\mu^+)\) are calculated in dependence on an energy and polarization of an initial electron(positron) and a polarization of a final \(\mu^-(\mu^+)\)–meson. Due to coherent phenomenon the cross sections are proportional to \(Z^2\). For \(Z \sim 100\) due to the factor \(Z^2\) the cross sections are large enough to be measured at energies available for the HERA Collider at DESY. The results are discussed in connection with a test of CPT invariance.

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1 Introduction

The Standard Model represents the Lagrangian approach to the description of strong, electromagnetic and weak interaction of elementary particles, based on the assumptions of locality and Lorentz invariance. Due to the Lüders–Pauli theorem (or the CPT theorem) locality and Lorentz invariance of the Lagrangian of a quantum system lead to the invariance of a quantum system under CPT transformation which contains (i) a charge conjugation (C), a replacement of all particles by their anti–particles, (ii) a parity transformation (P), a reflection of spatial coordinates \((t, \vec{x}) \rightarrow (t, -\vec{x})\), and (iii) a time reversal (T), a reflection of time \((t, \vec{x}) \rightarrow (-t, \vec{x})\). A simplest consequence of the CPT theorem is the equality of masses and lifetimes of particles and their anti–particles. At present these are the most well verified experimentally requirements of the CPT theorem. Nevertheless, theoretical and experimental test for CPT invariance is still a well motivated problem of Elementary particle and Nuclear Physics. This is related to the development of modern quantum field theories of strings and superstrings, which are more fundamental than the Standard Model and include it in the low–energy limit. Since string theories deal with extended non–local objects, the Lüders–Pauli theorem is not valid for these theories. A direct consequence of this can be a violation of CPT invariance for high energy reactions of elementary particles and nuclei.

The problem of a test of CPT and Lorentz invariance has been recently discussed by Kostelecký with co–workers. They suggested to check CPT and Lorentz invariance analysing a microwave spectroscopy of muonium \(M^0\). Muonium is a leptonic hydrogenlike bound state of a positively charged muon \(\mu^+\) and electron \(e^-\). It was discovered in 1960 through the observation of its characteristic Larmor precession in a magnetic field. The mean lifetime of muonium \(\tau_{M^0}\) is approximately equal to the lifetime of a positively charged muon \(\tau_{\mu^+} \approx 2.197 \times 10^{-6} \text{s}\). Due to absence of strong interactions muonium is an ideal system for determining of the properties of muons, for testing of quantum electrodynamics, and for searching for effects of unknown interactions in the electron–muon bound state. Anti–muonium \(\bar{M}^0\) is a leptonic analog of anti–hydrogen. It is a bound state of a negatively charged muon \(\mu^-\) and positron \(e^+\).

A hydrogenlike structure of muonium allows to use atomic notations for the classification of its quantum states. For example, \(2S+1L_J\) corresponds to the quantum state of muonium (or anti–muonium) with a total angular momentum (or a total spin) \(J\), an angular momentum \(L\) and a spin \(S\).

The use of muoniums \(M^0\) and anti–muoniums \(\bar{M}^0\) as a laboratory for a test of CPT invariance has been recently suggested by Choban and Kazakov. In their approach muoniums and anti–muoniums are produced with a total angular momentum \(J = 0\) in the reactions \(e^- + Z \rightarrow Z + M^0 + \mu^-\) and \(e^+ + Z \rightarrow Z + \bar{M}^0 + \mu^+\) of high energy scattering of electrons and positrons by nuclei with a number of protons \(Z\). According to atomic classification muonium (or anti–muonium) with a total angular momentum \(J = 0\) can be in two bound states: (i) a ground 1s state \(^1S_0\) with \(L = S = 0\) and (ii) an excited 2p state \(^3P_0\) with \(L = S = 1\). Due to principle of superposition muonium and anti–muonium should be produced in the reactions \(e^- + Z \rightarrow Z + M^0 + \mu^-\) and \(e^+ + Z \rightarrow Z + \bar{M}^0 + \mu^+\) in both states \(^1S_0\).

\(^1\)Anti–muonium \(\bar{M}^0\) is a bound state of a negatively charged muon \(\mu^-\) and a positron \(e^+\). It is a leptonic analogy of the anti–hydrogen.

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1. Anti–muonium \(\bar{M}^0\) is a bound state of a negatively charged muon \(\mu^-\) and a positron \(e^+\). It is a leptonic analogy of the anti–hydrogen.
and $^{3}\text{P}_0$. The interference of these states should lead to time–oscillations of a probability of muonium (anti–muonium) detected at a moment $t$. A comparison of time–oscillations of probabilities of the detected muonium and anti–muonium should testify whether CPT invariance conserved or not. This is Choban–Kazakov’s idea of a test of CPT invariance in the high–energy reactions $e^- + Z \rightarrow Z + M^0 + \mu^-$ and $e^+ + Z \rightarrow Z + M^0 + \mu^+$. In terms of formulas it can be represented as follows.

Let the wave function of muonium produced in the reaction $e^- + Z \rightarrow Z + M^0 + \mu^-$ be defined by

$$\Psi_{M^0}(t, \vec{x}) = \sqrt{\frac{m_{M^0}}{|\vec{k}|}} e^{i \vec{k} \cdot \vec{x} - i E t} \psi_{M^0}(t), \quad (1.1)$$

where $E$ and $\vec{k}$ are an energy and 3–momentum of muonium, $m_{M^0}$ is a mass of muonium. Note, that the energy $E$ does not contain the contributions of the binding energies $E_{1s}$ and $E_{2p}$ of the bound $1s$ and $2p$ states. The wave function $\psi_{M^0}(t)$ can be described by

$$\psi_{M^0}(t) = C_{1s} \exp \left( -i \frac{m_{M^0}}{E} E_{1st} \right) + C_{2p} \exp \left( -i \frac{m_{M^0}}{E} E_{2pt} \right). \quad (1.2)$$

The coefficients $C_{1s}$ and $C_{2p}$ describe the contributions of the $1s$ and $2p$ states, respectively.

Introducing a parameter $\varepsilon = |C_{2p}|^2/|C_{1s} + C_{2p}|^2$, related to a fraction of the excited $2p$ state in the wave function of muonium $\psi_{M^0}(t, \vec{x})$ \cite{12}, the probability to find muonium at the moment $t$ can be given by

$$P_{M^0}(t) = P_{M^0}(0) \left[ 1 - 4 \sqrt{\varepsilon} (1 - \sqrt{\varepsilon}) \sin^2(\Omega t) \right], \quad (1.3)$$

where $\Omega = m_{\mu}(E_{2p} - E_{1s})/2E = 5.103 \times 10^{-6} (m_{\mu}/E)$ MeV \cite{7}.

It is seen that the probability $P_{M^0}(t)$ is an oscillating function. A period of oscillations $T_{M^0}$ is determined by

$$T_{M^0} = \frac{2\pi}{\Omega} = \frac{4\pi}{E_{2p} - E_{1s}} \left( \frac{E}{m_{\mu}} \right) = 1.232 \times 10^6 \left( \frac{E}{m_{\mu}} \right) \text{MeV}^{-1}. \quad (1.4)$$

In order to get $T_{M^0}$ in seconds we have to multiply the r.h.s. of (1.4) by $h = 6.582 \times 10^{-22}$ MeV s \cite{7}. This yields $T_{M^0} = 8.106 \times 10^{-16} (E/m_{\mu})$ s. The period of oscillations $T_{M^0}$ should be compared with the lifetime of muonium in the laboratory frame $t_{M^0}$ which is related to the mean lifetime $\tau_{M^0}$ by the relativistic relation

$$t_{M^0} = \left( \frac{E}{m_{\mu}} \right) \tau_{M^0}. \quad (1.5)$$

Taking into account that $\tau_{M^0} \approx 2.197 \times 10^{-6}$ s we are able to estimate the number of oscillations $\nu_{M^0}$:

$$\nu_{M^0} = \frac{t_{M^0}}{T_{M^0}} \approx 2.710 \times 10^9. \quad (1.6)$$

\footnote{The account for a constant relative phase $2\varphi$ of coefficients $C_{1s}$ and $C_{2p}$, changes the probability (1.3) as follows $P_{M^0}(t) = P_{M^0}(0) \left[ 1 - 4 \sqrt{\varepsilon} (\sqrt{1 - \varepsilon} \sin^2 \varphi - \sqrt{\varepsilon} \cos \varphi) \sin(\Omega t + \varphi) \sin(\Omega t) \right]$}
The analogous expression can be written down for the probability $P_{M^0}(t)$ to detect antimuonium at moment $t$ with parameters $\varepsilon$ and $\Omega$. The result reads

$$P_{M^0}(t) = P_{M^0}(0) \left[ 1 - 4\sqrt{\varepsilon} (1 - \sqrt{\varepsilon}) \sin^2(\Omega t) \right].$$  \hspace{1cm} (1.7)

A relation of the probabilities $P_{M^0}(t)$ and $P_{M^0}(t)$ to the experimental analysis of the violation of CPT invariance in the reactions $e^- + Z \rightarrow Z + M^0 + \mu^-$ and $e^+ + Z \rightarrow Z + M^0 + \mu^+$ is the following.

For the calculation of the amplitude of muonium and anti–muonium production in the reactions $e^- + Z \rightarrow Z + M^0 + \mu^-$ and $e^+ + Z \rightarrow Z + M^0 + \mu^+$ we use the effective Lagrangian of the $M^0\mu^+e^-$ interaction which can be defined as

$$\mathcal{L}_{M^0\mu^+e^-} (x) = g_{1s} \bar{\psi}_{\mu^-} (x) \gamma^5 \psi_{e^-} (x) \Phi_{1s}^\dagger (x) + g_{2p} \bar{\psi}_{\mu^-} (x) \psi_{e^-} (x) \Phi_{2p}^\dagger (x),$$  \hspace{1cm} (1.8)

where $\bar{\psi}_{\mu^-} (x)$ and $\psi_{e^-} (x)$ are local interpolating fields of the $\mu^+$–meson and electron $e^-$, $\Phi_{1s} (x)$ and $\Phi_{2p} (x)$ are local operators of interpolating fields of muonium in the states $1s$ and $2p$, respectively. They are expanded into plane waves and operators of creation and annihilation.

The wave functions of the relative motion of the muon $\mu^+$ and the electron $e^-$ contribute to the coupling constants $g_{1s}$ and $g_{2p}$, which define the interaction of muonium in the $1s$ and $2p$ states with $\mu^+e^-$ pair, respectively. For the calculation of the effective coupling constant we would use the wave functions of muonium in the states $^1S_0$ and $^3P_0$ with a total momentum $\vec{P}$ defined by [13] [14]

$$|M^0 (\vec{P}) ; ^1S_0 \rangle = \frac{1}{(2\pi)^3} \int \frac{d^3k}{\sqrt{2E_{e^-}(\vec{k})}} \frac{d^3q}{\sqrt{2E_{\mu^+}(\vec{q})}} \delta^{(3)} (\vec{P} - \vec{k} - \vec{q}) \varphi_{1s}(\vec{k})$$
\[ \times \frac{1}{\sqrt{2}} \left[ b_{e^-}^\dagger (\vec{k}, +1/2) d_{\mu^+}^\dagger (\vec{q}, -1/2) - b_{e^-}^\dagger (\vec{k}, -1/2) d_{\mu^+}^\dagger (\vec{q}, +1/2) \right] |0\rangle, \]

$$|M^0 (\vec{P}) ; ^3P_0 \rangle = \frac{1}{(2\pi)^3} \int \frac{d^3k}{\sqrt{2E_{e^-}(\vec{k})}} \frac{d^3q}{\sqrt{2E_{\mu^+}(\vec{q})}} \delta^{(3)} (\vec{P} - \vec{k} - \vec{q}) \varphi_{2p}(\vec{k})$$
\[ \times \frac{1}{\sqrt{2}} \left[ b_{e^-}^\dagger (\vec{k}, +1/2) d_{\mu^+}^\dagger (\vec{q}, -1/2) + b_{e^-}^\dagger (\vec{k}, -1/2) d_{\mu^+}^\dagger (\vec{q}, +1/2) \right] |0\rangle, \hspace{1cm} (1.9)\]

where $|0\rangle$ is the vacuum wave function; $b_{e^-}^\dagger (\vec{k}, \sigma)$ ($b_{e^-} (\vec{k}, \sigma)$) and $d_{\mu^+}^\dagger (\vec{k}, \sigma)$ ($d_{\mu^+} (\vec{k}, \sigma)$) are operators of creation (annihilation) of electron and muon $\mu^+$ with a momentum $\vec{k}$ and polarization $\sigma = \pm 1/2$. These operators obey the covariant canonical anti–commutation relations

$$\{ b_{e^-} (\vec{k}, \sigma), b_{e^-}^\dagger (\vec{k}', \sigma') \} = (2\pi)^3 2E_{e^-} (\vec{k}) \delta^{(3)} (\vec{k} - \vec{k}') \delta_{\sigma\sigma'},$$
$$\{ d_{\mu^+} (\vec{k}, \sigma), d_{\mu^+}^\dagger (\vec{k}', \sigma') \} = (2\pi)^3 2E_{\mu^+} (\vec{k}) \delta^{(3)} (\vec{k} - \vec{k}') \delta_{\sigma\sigma'}. \hspace{1cm} (1.10)$$

Then, $\varphi_{1s}(\vec{k})$ and $\varphi_{2p}(\vec{k})$ are the wave functions of the $1s$ and $2p$ states in the momentum representation. They are normalized to unity

$$\int \frac{d^3k}{(2\pi)^3} |\varphi_{1s}(\vec{k})|^2 = \int \frac{d^3k}{(2\pi)^3} |\varphi_{2p}(\vec{k})|^2 = 1. \hspace{1cm} (1.11)$$
The wave functions (1.9) are normalized by
\[\langle 1S_0; M^0(\vec{P}) | M^0(\vec{P}'); 1S_0 \rangle \] = \( (2\pi)^32E^{(1s)}_{M^0}(\vec{P}) \delta^{(3)}(\vec{P} - \vec{P}'), \]
\[\langle 1P_0; M^0(\vec{P}) | M^0(\vec{P}'); 3P_0 \rangle \] = \( (2\pi)^32E^{(2p)}_{M^0}(\vec{P}) \delta^{(3)}(\vec{P} - \vec{P}'), \) \hspace{1cm} (1.12)
where \( E^{(n)}_{M^0}(\vec{P}) = \sqrt{(m_{\mu^+} + m_{e^-} + E_n)^2 + \vec{P}^2} \) is the total energy of muonium with \( E_n = E_{1s} \) and \( E_n = E_{2p} \) for the 1s and 2p state, respectively.

In the limit \( m_e \rightarrow 0 \) due to invariance of the interpolating electron field \( \psi_{e^-}(x) \) under \( \gamma^5 \)-transformation, \( \psi_{e^-}(x) \rightarrow \gamma^5\psi_{e^-}(x) \), the effective Lagrangian (1.8) can be transcribed into the form
\[ \mathcal{L}_{M^0}\mu^+e^-(x) = \bar{\psi}_{\mu^-}(x)\gamma^5\psi_{e^-}(x) \left( g_{1s} \Phi_{1s}^+(x) + g_{2p} \Phi_{2p}^+(x) \right). \] \hspace{1cm} (1.13)

Through the loop diagrams in Fig.1 the coupling constants \( g_{1s} \) and \( g_{2p} \) are related to the constants \( C_{1s} \) and \( C_{2p} \) (1.2).

Since one cannot distinguish experimentally the 1s and 2p states of muonium and of anti–muonium, the number of favourable events \( N_{M^0}(T) \) and \( N_{\bar{M}^0}(T) \), detected during an interim \( T \), should be proportional to \( \sigma^{(e^-Z)}_{M^0}(E_1)P_{M^0}(T) \) and \( \sigma^{(e^+Z)}_{M^0}(E_1)P_{\bar{M}^0}(T) \)
\[ N_{M^0}(T) = \sigma^{(e^-Z)}_{M^0}(E_1)P_{M^0}(T)LT, \]
\[ N_{\bar{M}^0}(T) = \sigma^{(e^+Z)}_{\bar{M}^0}(E_1)P_{\bar{M}^0}(T)LT, \] \hspace{1cm} (1.14)
where \( \sigma^{(e^-Z)}_{M^0}(E_1) \) and \( \sigma^{(e^+Z)}_{M^0}(E_1) \) are the cross sections for the reactions \( e^- + Z \rightarrow Z + M^0 + \mu^- \) and \( e^+ + Z \rightarrow Z + \bar{M}^0 + \mu^+ \), respectively, \( E_1 \) is the energy of the initial electron and positron in the laboratory frame, and \( L \) is a luminosity of the Collider.

Calculating the cross sections in the CPT invariant approximation, \( \sigma^{(e^+Z)}_{M^0}(E_1) = \sigma^{(e^-Z)}_{M^0}(E_1) \), the ratio of the numbers of favourable events \( R(T) = N_{M^0}(T)/N_{\bar{M}^0}(T) \) should be defined only by the ratio \( P_{M^0}(T)/P_{\bar{M}^0}(T) \). It reads
\[ R(T) = \frac{N_{M^0}(T)}{N_{\bar{M}^0}(T)} = \frac{P_{M^0}(T)}{P_{\bar{M}^0}(T)}. \] \hspace{1cm} (1.15)
Thus, measuring the ratio \( R(T) \) of favourable events one can conclude that (i) CPT invariance is broken if \( R(t) \) depends on time of the observation and oscillates in time, and (ii) CPT invariance is unbroken if \( R(T) \) does not depend on the time of observation. Of course, this is only a qualitative test.

A practical realization of an experimental test of CPT invariance in high–energy reactions \( e^- + Z \rightarrow Z + M^0 + \mu^- \) and \( e^+ + Z \rightarrow Z + \bar{M}^0 + \mu^+ \) depends on the statistics of favourable events \( N = \sigma LT \) which can be detected during a certain interim of observation \( T \). Nowadays the HERA Collider at DESY operates 27.5 GeV electron and positron beams with luminosities \( L_{e^-} = (15 - 17) \times 10^{30} \text{cm}^{-2}\text{s}^{-1} = (15 - 17) \text{pb}^{-1} \) (H1 – ZEUS) and \( L_{e^+} = (65 - 68) \times 10^{30} \text{cm}^{-2}\text{s}^{-1} = (65 - 68) \text{pb}^{-1} \) (H1 – ZEUS), respectively [15]. For these luminosities the number of events detected during one year for the production of muonium and anti–muonium are equal to \( N_{M^0} = 500 \sigma_{M^0} \) and \( N_{\bar{M}^0} = 2100 \sigma_{\bar{M}^0} \), where cross sections \( \sigma_{M^0} \) and \( \sigma_{\bar{M}^0} \) are measured in \( 1 \text{pb} = 10^{-36} \text{cm}^2 \).

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Thus, the problem of an experimental realization of a test of CPT invariance suggested by Choban and Kazakov [12] is related to (i) the values of the cross sections for the reactions $e^- + Z \to Z + X^0 + \mu^-$ and $e^+ + Z \to Z + \bar{X}^0 + \mu^+$, defining total number of favourable events and (ii) a distinct signal that in the reactions $e^- + Z \to Z + X^0 + \mu^-$ and $e^+ + Z \to Z + \bar{X}^0 + \mu^+$ the states $X^0$ and $\bar{X}^0$ should be identified with muonium $M^0$ and $\bar{M}^0$ anti-muonium, i.e. $X^0 = M^0$ and $\bar{X}^0 = \bar{M}^0$, respectively.

It is well-known that a more detailed information about nuclear reactions can be obtained investigating polarizations of coupled particles. Therefore, in this paper we focus on the calculation of the cross sections for the high-energy reactions $e^- + Z \to Z + M^0 + \mu^-$ and $e^+ + Z \to Z + \bar{M}^0 + \mu^+$ in dependence on polarizations of initial electron and positron and final muons $\mu^-$ and $\mu^+$. Following [10] we denote these reactions as $\bar{e}^- + Z \to Z + M^0 + \bar{\mu}^-$ and $\bar{e}^+ + Z \to Z + \bar{M}^0 + \bar{\mu}^+$. We suppose that a dependence on polarizations of final muons relative to polarizations of initial electrons and positrons should provide a necessary distinct signal confirming the production of muonium and anti-muonium with a total spin $J = 0$ in the reactions $\bar{e}^- + Z \to Z + X^0 + \bar{\mu}^-$ and $\bar{e}^+ + Z \to Z + \bar{X}^0 + \bar{\mu}^+$. Indeed, the processes competing with $\bar{e}^- + Z \to Z + X^0 + \bar{\mu}^-$ and $\bar{e}^+ + Z \to Z + \bar{X}^0 + \bar{\mu}^+$ are the reactions $\bar{e}^\mp + Z \to Z + e^\mp + \mu^+ + \mu^-$ of the production of the $\mu^+\mu^-$ pairs. In these reactions the momenta and polarizations of $\mu^+$ and $\mu^-$ mesons are strongly correlated each other and decorrelated with the polarization of the initial electron (or positron). Therefore, the detection of longitudinally polarized muons in the final state of the scattering of longitudinally polarized electrons (or positrons) by a nucleus $Z$ should be a distinct signal for the production of muonium (or anti-muonium) with a total spin $J = 0$.

The paper is organized as follows. In section 2 we calculate the energy spectrum of the final muon and the cross section for the reaction $\bar{e}^- + Z \to Z + M^0 + \bar{\mu}^-$. Since it is obvious that the CPT violation for the cross sections is negligible small effect which can be hardly measured, the cross section is calculated assuming CPT invariance. This implies that the cross section for the reaction $\bar{e}^- + Z \to Z + M^0 + \bar{\mu}^-$ amounts to the cross section for the reaction $\bar{e}^+ + Z \to Z + \bar{M}^0 + \bar{\mu}^+$, i.e. $\sigma_{\bar{e}^-Z}(E_1) = \sigma_{\bar{e}^+Z}(E_1)$. In Section 3 we estimate the contributions of the finite nuclear radius and the distortion of the wave functions of incoming and outcoming leptons caused by the strong Coulomb field induced by the electric charge Ze with $Z \sim 100$. We estimate that the contribution of the finite radius of the nucleus is of order of a few percents. We show that the strong Coulomb field can hardly destroy the production of bound states of $\mu^+e^-$ and $\mu^-e^+$ pairs, i.e. muoniums and anti-muoniums, in the reactions under consideration. This is by virtue of the time of the decays $M^0 \to \mu^+e^-$ and $M^0 \to \mu^-e^+$ induced by the strong Coulomb field is much greater than the time of the production of muonium and anti-muonium. In the Conclusion we discuss the obtained results and a practical realization of experiments on the test of CPT invariance for the HERA Collider at DESY.

## 2 Cross sections for reactions $\bar{e}^- + Z \to Z + M^0 + \bar{\mu}^-$ and $\bar{e}^+ + Z \to Z + \bar{M}^0 + \bar{\mu}^+$

Feynman diagrams describing the amplitude of the reaction $\bar{e}^- + Z \to Z + M^0 + \bar{\mu}^-$ are depicted in Fig.1. The amplitude of the reaction $\bar{e}^- + Z \to Z + M^0 + \bar{\mu}^-$ has been
calculated in Ref. [12] and reads

\[
M(\bar{e}^- (p_1) Z(p_2) \rightarrow Z(p'_2) M^0(k) \bar{\mu}^-(p'_1)) = \frac{\alpha^2}{q^2} \frac{16\pi^2}{m_e} \frac{\Psi_{1s}(0)}{m_{\mu}^{3/2}} \frac{\ell^\mu L_\mu}{(q^2 - 2q \cdot k)},
\]

(2.1)

where \(\ell^\mu\) is the electromagnetic current of a nucleus and \(L_\mu\) denotes the leptonic current

\[
L_\mu = \bar{u}(p'_1, \sigma'_1) \gamma_5 (\gamma_\mu \bar{q} p'_{1\mu} - q \cdot p'_1 \gamma_\mu) u(p_1, \sigma_1),
\]

(2.2)

where \(u(p_1, \sigma_1)\) and \(\bar{u}(p'_1, \sigma'_1)\) are bispinorial wave functions of an initial electron and final muon \(\mu^\pm\), \(\Psi_{1s}(0) = 1/\sqrt{\pi a_B^3}\) is the wave function of the muonium in the ground state, \(a_B = 1/m_e \alpha = 268.173\text{ MeV}^{-1}\) is the Bohr radius of muonium, \(\alpha = 1/137.036\) is the fine structure constant.

We would like to emphasize that the leptonic current \(L_\mu\) is calculated in the ultra-relativistic limit, when masses of leptons are set zero. According to [12] this corresponds to the kinematical region, where the squared invariant mass of the pair \(M^0\mu^\pm\), \(\omega^2 = (p'_1 + k)^2\), is much greater than the squared mass of the \(\mu^-\)-meson \(m_\mu^2\), i.e. \(\omega^2 \gg m_\mu^2\). In this kinematical region muonium with a total spin \(J = 0\) behaves like a massless neutral scalar point-like particle.

The cross section for the reaction \(\bar{e}^- + Z \rightarrow Z + M^0 + \bar{\mu}^-\) is defined by

\[
\sigma_{M^0}^{(\bar{e}^- Z)}(E_1) = \frac{\alpha^7}{4\pi^2} \frac{m_e}{m_\mu^3 m_Z E_1} \int \frac{T_{\mu\nu} R_{\mu\nu}}{q^2 (p_1 \cdot p_1')^2} \delta^{(4)}(p_2 + p_1' + k - p_2 - p_1) \frac{d^3k d^3p_1' d^3p_2'}{E E_1 E_2'},
\]

(2.3)

where \(E_1\) is the energy of the initial electron in the laboratory frame coinciding with the rest frame of a target nucleus \(p_{2\mu} = (m_Z, 0)\), then \(E, E_1'\) and \(E_2'\) are the energies of muonium, a \(\mu^-\)-meson and a final nucleus, respectively. The tensors \(R_{\mu\nu}\) and \(T_{\mu\nu}\) are determined by

\[
R_{\mu\nu} = \frac{1}{4} \text{Sp} \{ (\hat{p}_2 + m_Z) \ell^\mu_\mu (\hat{p}_2' + m_Z) \ell^\nu_\nu \} = F_{1Z}^2(q^2) \left[ 2 p_{2\mu} p_{2\nu} - (p_{2\mu} q_\nu + p_{2\nu} q_\mu) + \frac{1}{2} q^2 g_{\mu\nu} \right] + F_{2Z}^2(q^2) \left[ 2 q^2 m_Z^2 g_{\mu\nu} + q^2 (p_{2\mu} q_\nu + p_{2\nu} q_\mu) - q_\mu q_\nu \left( \frac{1}{2} q^2 + 2 m_Z^2 \right) - 2 q^2 p_{2\mu} p_{2\nu} \right].
\]

(2.4)
and
\[
T_{\mu\nu} = \frac{1}{4} \text{Sp}\{ (\hat{p}_1 - \gamma_5 \hat{w}_1) L^\dagger_\mu (\hat{p}_1' - \gamma_5 \hat{w}_1') L_\nu \} = \\
= \frac{1}{4} \text{Sp}\{ (\hat{p}_1 - \gamma_5 \hat{w}_1) \gamma_5 (q \cdot p_1' \gamma_\mu - \hat{q} p_{1,\mu}' ) (\hat{p}_1' - \gamma_5 \hat{w}_1') \gamma_5 (q \cdot p_1' \gamma_\nu - \hat{q} p_{1,\nu}' ) \},
\]
where \( F_{1Z}(q^2) \) and \( F_{2Z}(q^2) \) are form factors of a nucleus with a number of protons \( Z \).

The polarization matrices \((\hat{p}_1 - \gamma_5 \hat{w}_1)\) and \((\hat{p}_1' - \gamma_5 \hat{w}_1')\) are obtained in the zero-mass limit from the standard polarization matrices \((\hat{p}_1 + m_e)(1 - \gamma_5 \hat{a})\) and \((\hat{p}_1' + m_\mu)(1 - \gamma_5 \hat{b})\) [10], where \( a_\mu \) and \( b_\mu \), 4-vectors of polarization of the initial electron and the final muon, are defined by
\[
a_\mu = \left( \frac{\vec{p}_1 \cdot \vec{\xi}_1}{m_e}, \vec{\xi}_1 + \frac{\vec{p}_1 (\vec{p}_1 \cdot \vec{\xi}_1)}{m_e (E_1 + m_e)} \right),
\]
\[
b_\mu = \left( \frac{\vec{p}_1' \cdot \vec{\xi}_1'}{m_\mu}, \vec{\xi}_1' + \frac{\vec{p}_1' (\vec{p}_1' \cdot \vec{\xi}_1')}{m_\mu (E_1' + m_\mu)} \right). \tag{2.6}
\]

The 4-vectors of polarization \( a_\mu \) and \( b_\mu \) are normalized by \( a_\mu a^\mu = a_0^2 - \vec{a}^2 = -1 \) and \( b_\mu b^\mu = b_0^2 - \vec{b}^2 = -1 \). In turn, the 3-vectors of polarization \( \vec{\xi}_1 \) and \( \vec{\xi}_1' \) are normalized by \( \vec{\xi}_1^2 = \vec{\xi}_1'^2 = 1 \). Recall that \( p_1 \cdot a = p_1' \cdot b = 0 \).

According to definitions (2.6) the 4-vectors \( w_{1\mu} \) and \( w'_{1\mu} \) are equal to
\[
w_{1\mu} = (\vec{\eta}_1 \cdot \vec{\xi}_1, \vec{\eta}_1 (\vec{p}_1 \cdot \vec{\xi}_1)) = (\vec{\eta}_1 \cdot \vec{\xi}_1) p_{1\mu},
\]
\[
w'_{1\mu} = (\vec{\eta}'_{1} \cdot \vec{\xi}'_{1}, \vec{\eta}'_{1} (\vec{p}'_{1} \cdot \vec{\xi}'_{1})) = (\vec{\eta}'_{1} \cdot \vec{\xi}'_{1}) p'_{1\mu}. \tag{2.7}
\]

where \( \vec{\eta}_1 = \vec{\eta}_1 / E_1 \) and \( \vec{\eta}'_{1} = \vec{\eta}'_{1} / E_1' \) and \( p_1 \cdot w_1 = p_1' \cdot w_1' = 0 \) due to \( p_1^2 = p_1'^2 = 0 \). The analytical expression of \( T_{\mu\nu} \) is given by
\[
T_{\mu\nu} = -2 \left[ 1 + (\vec{\eta}_1 \cdot \vec{\xi}_1)(\vec{\eta}'_{1} \cdot \vec{\xi}'_{1}) \right] \\
\times (p_1 \cdot p_1') \left[ (q \cdot p_1') q_{\mu\nu} - (q \cdot p_1')(q_{\mu\nu}' + q_{\mu\nu} q_{\mu\nu}') + q^2 p_{1,\mu}' p_{1,\nu}' \right]. \tag{2.8}
\]

Due to conservation of electric charge the tensors \( T_{\mu\nu} \) and \( R_{\mu\nu} \) are gauge invariant
\[
q^\mu T_{\mu\nu} = T_{\mu\nu} q^\nu = 0, \\
q^\mu R_{\mu\nu} = R_{\mu\nu} q^\nu = 0. \tag{2.9}
\]

The cross section for the reaction under consideration is then defined by
\[
s_{M_0}^{(e^-Z)}(E_1) = \\
= \frac{\alpha^2}{\pi^2} \frac{m_e m_Z}{m_\mu^3 E_1} \int \frac{(-1)}{q^4 (p_1 \cdot p_1')} \left[ 1 + (\vec{\eta}_1 \cdot \vec{\xi}_1)(\vec{\eta}'_{1} \cdot \vec{\xi}'_{1}) \right] \left\{ (F_1^{2Z}(q^2) - q^2 F_2^{2Z}(q^2))(q \cdot p_1')^2 \\
+ \frac{q^2}{m_Z^2} \left[ (F_1^{2Z}(q^2) - q^2 F_2^{2Z}(q^2))(p_2 \cdot p_1')^2 - (q \cdot p_1')(p_2 \cdot p_1') + \frac{1}{2} (F_1^{2Z}(q^2) + 4 m_Z^2 F_2^{2Z}(q^2)) \right] \right\} \\
\times (q \cdot p_1')^2 \delta^{(4)}(p_2' + p_1' + k - p_2 - p_1) \frac{d^3 k}{E} \frac{d^3 p_1'}{E_1} \frac{d^3 p_2'}{E_2'}, \tag{2.10}
\]
8
The integration over the phase volume of the final state $ZM^0\mu^-$ we suggest to carry out in the non-relativistic limit of motion of a final nucleus $^{17}$. In this approximation the 4-momentum of the final nucleus is equal to $p_{2\mu} = (m_Z + q^2/2m_Z, -\vec{q}) = (m_Z + T_2, -\vec{q})$, then the transferred 4-momentum $q_\mu = (-T_2, \vec{q})$ and $q^2 = -\vec{q}^2$.

In the non-relativistic limit of motion of a final nucleus the cross section $^{2.10}$ reduces to the form

$$\sigma_{M^0}^{(\vec{e}^-Z)}(E_1) = Z^2 \frac{\alpha^2}{\pi^2} \frac{m_e}{m_\mu^2} E_1 \int \frac{1}{E_1 E_1' - \vec{p}_1 \cdot \vec{p}_1'} \left[ 1 + \left( \vec{n}_1 \cdot \vec{\xi}_1 \right) \left( \frac{\vec{p}_1' \cdot \vec{\xi}_1'}{E_1'} \right) \right]$$

$$\times \left( E_1'^2 - \frac{2(\vec{q} \cdot \vec{p}_1')^2}{q^2 E_1'^2} \right) \delta(E_1 - E_1' - |\vec{p}_1' - \vec{p}_1 - \vec{q}|) d^3p_1' d^3q d^3\vec{q},$$

where we have taken into account that $F_{1Z}(0) = Z$ $^{17}$ and that the main contribution comes from transferred momenta $\vec{q}$ comeasurable with zero. The former corresponds to the Weizsäcker–Williams approximation $^{18}$ $^{23}$.

For simplification of the calculation of the phase volume we neglect the contribution of a kinetic energy of a final nucleus, which is small compared with typical transferred energies of coupled leptons. Integrating over $\vec{k}$, a 3-momentum of muonium, we get

$$\sigma_{M^0}^{(\vec{e}^-Z)}(E_1) = Z^2 \frac{\alpha^2}{\pi^2} \frac{m_e}{m_\mu^2} E_1 \int \frac{E_1'}{E_1 E_1' - \vec{p}_1 \cdot \vec{p}_1'} \left[ 1 + \left( \vec{n}_1 \cdot \vec{\xi}_1 \right) \left( \frac{\vec{p}_1' \cdot \vec{\xi}_1'}{E_1'} \right) \right]$$

$$\times \left( 1 - \frac{(\vec{q} \cdot \vec{p}_1')^2}{q^2 E_1'^2} \right) \delta(E_1 - E_1' - |\vec{p}_1' - \vec{p}_1 - \vec{q}|) \frac{d^3p_1'}{E_1'} \frac{d^3q}{q^2},$$

where we have denoted

$$I(\vec{p}_1, \vec{p}_1') = \int \left( 1 - \frac{(\vec{q} \cdot \vec{p}_1')^2}{q^2 E_1'^2} \right) \delta(E_1 - E_1' - |\vec{p}_1' - \vec{p}_1 - \vec{q}|) \frac{1}{|\vec{p}_1' - \vec{p}_1 - \vec{q}|} \frac{d^3q}{q^2}.$$  \hspace{1cm} (2.13)

The integration over $\vec{q}$ we carry out assuming that $|\vec{p}_1' - \vec{p}_1| \gg |\vec{q}|$ that is valid for the Weizsäcker–Williams approximation. Using a vector $\vec{z} = \vec{q}/|\vec{p}_1' - \vec{p}_1|$ we obtain

$$I(\vec{p}_1, \vec{p}_1') = \int \left( 1 - \frac{(\vec{z} \cdot \vec{p}_1')^2}{\vec{z}^2 E_1'^2} \right) \delta(E_1 - E_1' - |\vec{p}_1' - \vec{p}_1| + (\vec{p}_1' - \vec{p}_1) \cdot \vec{z}) d^3\vec{z}. \hspace{1cm} (2.14)$$

Now it is convenient to introduce new variables $x = E_1'/E_1$, $\vec{n}_1' = \vec{p}_1'/E_1'$ and $\vec{n}_1 = \vec{p}_1/E_1$. In these variables the function $I(\vec{p}_1, \vec{p}_1')$ reads

$$I(\vec{p}_1, \vec{p}_1') = \frac{1}{E_1} \int \left( 1 - \frac{(\vec{z} \cdot \vec{n}_1')^2}{\vec{z}^2} \right) \delta(1 - x - |x\vec{n}_1' - \vec{n}_1| + (x\vec{n}_1' - \vec{n}_1) \cdot \vec{z}) d^3\vec{z}. \hspace{1cm} (2.15)$$

The next step in the integration over $\vec{z}$ is to rewrite the integral in the following form

$$I(\vec{p}_1, \vec{p}_1') =$$

$$= \frac{1}{\pi E_1} \text{Re} \int_0^\infty d\lambda e^{i\lambda(1 - x - |x\vec{n}_1' - \vec{n}_1|)} \int \left( 1 + \frac{1}{\lambda^2 \vec{z}^2} \frac{\partial^2}{\partial x^2} \right) e^{i\lambda(x\vec{n}_1' - \vec{n}_1) \cdot \vec{z}} \frac{d^3\vec{z}}{\vec{z}^2}. \hspace{1cm} (2.16)$$
Since the integrals over \( \tilde{z} \) are equal to
\[
\int e^{i\lambda(x\tilde{n}'_1 - \tilde{n}_1)} \cdot \frac{\tilde{z} \, d^3\tilde{z}}{\tilde{z}^2} = \frac{4\pi}{\lambda|x\tilde{n}'_1 - \tilde{n}_1|} \int_0^\infty \frac{\sin z}{z} \, dz = \frac{4\pi}{\lambda|x\tilde{n}'_1 - \tilde{n}_1|} \lim_{\alpha \to 1} \int_0^\infty \frac{\sin z}{z^\alpha} \, dz = \frac{4\pi}{\lambda|x\tilde{n}'_1 - \tilde{n}_1|} \frac{\Gamma(1 - \alpha)}{\Gamma(2 - \alpha)} \frac{\sin(\pi/2(1 - \alpha))}{1 - \alpha} = \frac{2\pi^2}{\lambda|x\tilde{n}'_1 - \tilde{n}_1|}
\]
and
\[
\int e^{i\lambda(x\tilde{n}'_1 - \tilde{n}_1)} \cdot \frac{\tilde{z} \, d^3\tilde{z}}{\tilde{z}^4} = 4\pi|\lambda x\tilde{n}'_1 - \tilde{n}_1| \int_0^\infty \frac{\sin z}{z^3} \, dz = 4\pi\lambda|x\tilde{n}'_1 - \tilde{n}_1| \lim_{\alpha \to 3} \int_0^\infty \frac{\sin z}{z^\alpha} \, dz = 4\pi\lambda|x\tilde{n}'_1 - \tilde{n}_1| \lim_{\alpha \to 3} \frac{\Gamma(1 - \alpha)}{(1 - \alpha)(2 - \alpha)(3 - \alpha)} = 4\pi\lambda|x\tilde{n}'_1 - \tilde{n}_1| \lim_{\alpha \to 3} \frac{\sin(\pi/2(1 - \alpha))}{(1 - \alpha)(2 - \alpha)(3 - \alpha)} = -\pi^2\lambda|x\tilde{n}'_1 - \tilde{n}_1|,
\]
the function \( I(\tilde{p}_1, \tilde{p}'_1) \) is defined by the integral over \( \lambda \)
\[
I(\tilde{p}_1, \tilde{p}'_1) = \frac{\pi}{E_1} \left( \frac{1}{|x\tilde{n}'_1 - \tilde{n}_1|} + \frac{(x - \tilde{n}'_1 \cdot \tilde{n}_1)^2}{|x\tilde{n}'_1 - \tilde{n}_1|^3} \right) \int_0^\infty \frac{d\lambda}{\lambda} \cos(\lambda(1 - x - |x\tilde{n}'_1 - \tilde{n}_1|)).
\]
The integral over \( \lambda \) is divergent. However, it can be regularized by following the theory of generalized functions [21]. The result reads
\[
I(\tilde{p}_1, \tilde{p}'_1) = \frac{\pi}{E_1} \left( \frac{1}{|x\tilde{n}'_1 - \tilde{n}_1|} + \frac{(x - \tilde{n}'_1 \cdot \tilde{n}_1)^2}{|x\tilde{n}'_1 - \tilde{n}_1|^3} \right) \ln \left( \frac{1}{|x\tilde{n}'_1 - \tilde{n}_1|} - (1 - x) \right).
\]
Substituting (2.20) in (2.12) and proceeding to variables \( x \) and \( \tilde{n}'_1 \) we define the energy spectrum of a final \( \mu^- \)-meson
\[
\frac{1}{x^2} \frac{d\sigma(\mu^- \rightarrow Z)(E_1)}{dx} = Z^2 \frac{\alpha^7}{\pi} \frac{m_e}{m^3_{\mu}} \int \frac{1}{1 - \tilde{n}_1 \cdot \tilde{n}'_1} \left( \frac{1}{|x\tilde{n}'_1 - \tilde{n}_1|} + \frac{(x - \tilde{n}'_1 \cdot \tilde{n}_1)^2}{|x\tilde{n}'_1 - \tilde{n}_1|^3} \right) \times \ln \left( \frac{1}{|x\tilde{n}'_1 - \tilde{n}_1|} - (1 - x) \right) d\Omega_{\tilde{n}'_1}.
\]
For the subsequent integration over a unit vector \( \vec{n}_1' \) we introduce angular variables as follows:

\[
\begin{align*}
\vec{n}_1 \cdot \vec{n}_1' &= \cos \vartheta_1', \\
\vec{n}_1' \cdot \vec{\xi}_1' &= \cos \vartheta_1' \cos \Theta_1' + \sin \vartheta_1' \sin \Theta_1' \cos (\varphi_1' - \Phi_1'), \\
d\Omega_{\vec{n}_1'} &= \sin \vartheta_1' d\vartheta_1' d\varphi_1',
\end{align*}
\]

(2.22)

where \( \Theta_1' \) and \( \Phi_1' \) are polar and azimuthal angles of the polarization vector \( \vec{\xi}_1' \) relative to the momentum \( \vec{p}_1 \). In (2.22) we have taken into account that \(|\xi_1'| = 1\). Integrating over \( \varphi_1' \) we get

\[
\frac{1}{x^2} \frac{d\sigma_{M^0}(\vec{e}^-Z)(E_1)}{dx} = 2Z^2 \alpha \frac{m_e}{m_\mu^3} \int_{0}^{1} \frac{1 + (\vec{n}_1 \cdot \vec{\xi}_1) \cos \vartheta_1' \cos \Theta_1'}{1 - \cos \vartheta_1'} \left( \frac{1}{\sqrt{1 - 2x \cos \vartheta_1' + x^2}} \right) \\
+ \frac{(x - \cos \vartheta_1')^2}{(1 - 2x \cos \vartheta_1' + x^2)^{3/2}} \ln\left( \frac{1}{\sqrt{1 - 2x \cos \vartheta_1' + x^2 - (1 - x)}} \right) \sin \vartheta_1' d\vartheta_1' 
\]

(2.23)

Now it is convenient to introduce a new variable \( t = \sqrt{1 - 2x \cos \vartheta_1' + x^2} \), which varies in the limits \( 1 - x \leq t \leq 1 + x \). In terms of \( t \) the energy spectrum (2.23) reads

\[
\frac{1}{x} \frac{d\sigma_{M^0}(\vec{e}^-Z)(E_1)}{dx} = 2Z^2 \alpha \frac{m_e}{m_\mu^3} \int_{1-x}^{1} \frac{2x + (1 + x^2 - t^2)(\vec{n}_1 \cdot \vec{\xi}_1) \cos \Theta_1'}{t^2 - (1 - x)^2} \\
\times \left( 1 + \frac{(1 - x^2 - t^2)^2}{4x^2t^2} \right) \ln\left( \frac{1}{t - (1 - x)} \right) dt.
\]

(2.24)

It is seen that the integral over \( t \) is concentrated in the vicinity of the lower limit. The singularity of the integrand in the vicinity of the lower limit can be easily regularized by making a change of the lower limit \( 1 - x \to 1 - x + \Lambda^2/E_1' \), where \( \Lambda \) is a cut-off restricting energies of a final \( \mu^- \)-meson from below. According to the kinematical region \( \omega^2 \gg m_\mu^2 \) [12] the cut-off \( \Lambda \) can be chosen of order of \( \Lambda \approx 1 \text{ GeV} \). Such a dependence on the cut-off \( \Lambda \) can be justified as follows: \( E_1' = |\vec{p}_1'| = \sqrt{(\vec{p}_1')^2 + \Lambda^2 - \Lambda^2} = \sqrt{(\vec{p}_1')^2 + \Lambda^2 - \Lambda^2/\sqrt{(\vec{p}_1')^2 + \Lambda^2}} \to E_1' - \Lambda^2/E_1' \approx E_1' - \Lambda^2/E_1 \).

Keeping only the dominant contributions to the integral over \( t \) we get

\[
\frac{d\sigma_{M^0}(\vec{e}^-Z)(E_1)}{dx} = 8Z^2 \alpha \frac{m_e}{m_\mu^3} \ln^2 \left( \frac{E_1}{\Lambda} \right) \frac{x^2}{1 - x} [1 + (\vec{n}_1 \cdot \vec{\xi}_1) \cos \Theta_1']
\]

(2.25)

Introducing the angle \( \Theta_1 \), defined by \( \vec{n}_1 \cdot \vec{\xi}_1 = \cos \Theta_1 \), where we have taken into account that \(|\xi_1'| = 1\), we obtain the energy spectrum of \( \mu^- \)-mesons for the reaction \( \vec{e}^- + Z \to Z + M^0 + \vec{\mu}^- \) in dependence on the polarizations of the initial electron and the final \( \mu^- \)-muon described by the angles \( \Theta_1 \) and \( \Theta_1' \)

\[
\frac{d\sigma_{M^0}(\vec{e}^-Z)(E_1)}{dx} = 8Z^2 \alpha \frac{m_e}{m_\mu^3} \ln^2 \left( \frac{E_1}{\Lambda} \right) \frac{x^2}{1 - x} (1 + \cos \Theta_1 \cos \Theta_1').
\]

(2.26)
Integrating over $x$ we arrive at the total cross section for the reaction $\bar{e}^- + Z \rightarrow Z + M^0 + \bar{\mu}^-$

$$\sigma_{M^0}(\bar{e}^-Z)(E_1) = 16Z^2\alpha^7\frac{m_e}{m_\mu} \ell n^3 \left( \frac{E_1}{\Lambda} \right) (1 + \cos \Theta_1 \cos \Theta'_1). \quad (2.27)$$

Assuming that electrons are longitudinally polarized electrons, $\cos \Theta_1 = 1$, one can see that for the fixed electron energy the cross section acquires the maximal value only for longitudinally polarized muons $\cos \Theta_1' = 1$. This agrees with the production of muonium with a total spin $J = 0$. Thus, we argue that the appearance of longitudinally polarized muons in the final state of the reaction $\bar{e}^- + Z \rightarrow Z + X + \bar{\mu}^-$ should testify the production of muonium $X \equiv M^0$.

For the numerical estimate of the cross sections at the energies available for the HERA Collider at DESY [15], i.e. $E_1 = 27.5$ GeV, we suggest to use Radon, $^{222}_{86}$Rn, as a target nucleus, since Radon has a spin 1/2. The cross sections for longitudinally polarized electrons and positrons scattering by $^{222}_{86}$Rn and longitudinally polarized muons are equal to

$$\sigma_{M^0}(\bar{e}^- \text{Rn})(E_1 = 27.5 \text{ GeV}) = \sigma_{M^0}(\bar{e}^+ \text{Rn})(E_1 = 27.5 \text{ GeV}) = 1.6 \text{ pb}. \quad (2.28)$$

In our calculation the cross section for the reaction $e^- + Z \rightarrow Z + M^0 + \mu^-$ has turned out to be dependent on a cut–off $\Lambda \simeq 1$ GeV. In this connection we would like to remind that the problem of the appearance of a cut–off in cross sections for some reactions calculated within the Weizsäcker–Williams approximation has been pointed out by Bertulani and Baur [20].

Now let us discuss the energy dependence of the cross section [20,22]. It is well–known that for the $e^+e^-$ pair production in heavy–ion collisions [20–22] and $p\bar{p}$ collisions [23] the cross section for a capture of a final electron in an atomic $K$–shell orbit is proportional to $\ell n(\gamma_{\text{coll}})$, where $\gamma_{\text{coll}}$ is a Lorentz factor of colliding particles in the center of mass frame. This factor is related to the corresponding Lorentz factor $\gamma_p$ of the projectile (for a fixed target machine) by $\gamma_p = 2\gamma_{\text{coll}}^2 - 1$ [20,22], where $\gamma_p \sim E_1$.

In turn, the cross section for the production of a point–like neutral scalar particle in high–energy heavy–ion collisions in the Weizsäcker–Williams approximation is proportional to $\ell n^3(\gamma_{\text{coll}})$ [20,22].

For very high energies, when masses of coupled leptons can be neglected, muonium with a total spin $J = 0$ can be treated as a point–like massless scalar neutral particle. Such a property of muonium is caused by an addition pole–singularity appearing at $(q - k)^2 = q^2 - 2k \cdot q = 0$ for $k^2 = m_\mu^2 = 0$ (see Eq.(2.1)). This makes the part of the diagram in Fig.1, responsible for creation of muonium, equivalent to an amplitude of a process $\gamma^* + \gamma^* \rightarrow M^0$, where $\gamma^*$’s are virtual photons. That is why the obtained cross section for the reaction $e^- + Z \rightarrow Z + M^0 + \mu^-$ has turned out to be proportional to $\ell n^3(\gamma_{\text{coll}})$.
3 Influence of a finite radius of a nucleus and a distortion of wave functions of coupled leptons

In this Section we estimate the influence of a finite radius of the nucleus $Z$. According to [17] the form factor of the nucleus with a mass number $A$ can be defined by the expansion

$$
\frac{1}{Z} F_{1Z}(q^2) = 1 - \frac{1}{6} r_A^2 \tilde{q}^2 + O(\tilde{q}^4)
$$

(3.1)

where we identify $r_A$ with a radius of a nucleus with a mass number $A$ given by [17]

$$
r_A = 1.2 A^{1/3} \text{ fm} = 6.1 A^{1/3} \text{ GeV}^{-1}.
$$

(3.2)

Due to the finite value of the nuclear radius the function $I(\vec{p}_1, \vec{p}_1')$ changes as follows

$$
\delta I(\vec{p}_1, \vec{p}_1') = -\frac{\pi}{E_1} \frac{1}{3} r_A^2 E_1^2 \left( -\frac{1}{|x\vec{n}_1 - \vec{n}_1'|} + 3 \frac{(x - \vec{n}_1' \cdot \vec{n}_1')^2}{|x\vec{n}_1' - \vec{n}_1'|^2} \right) (1 - x - |x\vec{n}_1' - \vec{n}_1|)^2 
\times \ln \left( \frac{1}{|x\vec{n}_1' - \vec{n}_1| - (1 - x)} \right).
$$

(3.3)

In the region of the integration over $t$, dominant for the leading term of the expansion of the form factor $F_{1Z}(\tilde{q}^2)$ into the powers of $\tilde{q}^2$, the contribution of the finite radius of the nucleus can be summarized as

$$
\sigma_{\mu^0}^{(e^- Z)}(E_1) = \frac{16 Z^2 \alpha^7}{(1 + \frac{1}{6} r_A^2 A^4 / E_1^2)} \frac{m_e}{m_\mu} \ell n^3 \left( \frac{E_1}{\Lambda} \right) (1 + \cos \Theta_1 \cos \Theta_1').
$$

(3.4)

For the electron (positron) scattering by $^{222}_{86}$Rn with the laboratory energy $E_1 = 27.5 \text{ GeV}$ the correction to the cross section, caused by the finite value of the nucleus radius [17], can be made of order of 4% varying the parameter $\Lambda$ from $\Lambda \simeq 1 \text{ GeV}$ to $\Lambda \simeq 0.8 \text{ GeV}$ in comparison with the value of the cross section [22,23] calculated for $E_1 = 27.5 \text{ GeV}$, $\Lambda \simeq 1 \text{ GeV}$ and $r_A = 0$. Hence, in the Weizsäcker–Williams approximation [18–23] without loss of generality we can treat a nucleus $Z$ in the reactions $e^- + Z \rightarrow Z + M^0 + \mu^-$ and $e^+ + Z \rightarrow Z + \bar{M}^0 + \bar{\mu}^+$ as a point–like particle with the electric charge $Ze$.

In the strong Coulomb field caused by a point–like charge $Ze$ for $Z \sim 100$ the wave functions of the initial electron (positron) and the final muon should be distorted. According to [17] at very high energies and in the eiconal approximation these wave functions can be written in the following form

$$
\Psi_{\mu^-}(\vec{r}_1; \vec{p}_1, \sigma_1)_{\text{in}} = u(\vec{p}_1, \sigma_1) \exp \left\{ +i \vec{p}_1 \cdot \vec{r}_1 + i \frac{E_1}{|\vec{p}_1|} \int_0^\infty \frac{Ze^2 ds}{\sqrt{\vec{p}_1^2 + (z - s)^2}} \right\},
$$

$$
\Psi_{\mu^+}(\vec{r}_1'; \vec{p}_1', \sigma'_1)_{\text{out}} = u(\vec{p}_1', \sigma'_1) \exp \left\{ +i \vec{p}_1' \cdot \vec{r}_1' - i \frac{E_1'}{|\vec{p}_1'|} \int_0^\infty \frac{Ze^2 ds}{\sqrt{\vec{p}_1'^2 + (z' + s)^2}} \right\},
$$

(3.5)

where $\vec{p}_1$ and $\vec{p}_1'$ are components of radius–vectors $\vec{r}_1$ and $\vec{r}_1'$ perpendicular to the momentum $\vec{p}_1$ and $\vec{p}_1'$, respectively.
In the limit \( m_e = m_\mu = 0 \) the wave functions change themselves as

\[
\Psi_{e^-}(\vec{r}; \vec{p}_1, \sigma_1)_{\text{in}} = u(\vec{p}_1, \sigma_1) \exp \left\{ + i \vec{p}_1 \cdot \vec{r} + i \int_0^\infty \frac{Z e^2 \, ds}{\sqrt{\rho^2 + (z - s)^2}} \right\},
\]

\[
\Psi_{\mu^-}(\vec{r}; \vec{p}_1^\prime, \sigma'_1)_{\text{out}} = u(\vec{p}_1^\prime, \sigma'_1) \exp \left\{ + i \vec{p}_1^\prime \cdot \vec{r} - i \int_0^\infty \frac{Z e^2 \, ds}{\sqrt{\rho^2 + (z + s)^2}} \right\},
\]

where we have taken into account the fact that at high energies effectively \( \tau \) production of the final muon occurs at the same spatial point \( \vec{r}_1 = \vec{r}'_1 = \vec{r} \), where the initial electron has been absorbed. The amplitude of the reaction \( e^- + Z \rightarrow Z + M^0 + \bar{\mu}^- \) is proportional to the product

\[
\Psi_{\mu^-}^\dagger(\vec{r}; \vec{p}_1^\prime, \sigma'_1)_{\text{in}} \Psi_{e^-}(\vec{r}; \vec{p}_1, \sigma_1)_{\text{out}} \sim \exp \left\{ i \int_0^\infty \frac{Z e^2 \, ds}{\sqrt{\rho^2 + (z + s)^2}} + i \int_0^\infty \frac{Z e^2 \, ds}{\sqrt{\rho^2 + (z - s)^2}} \right\} = \exp \left\{ i \int_{-\infty}^\infty \frac{Z e^2 \, ds}{\sqrt{\rho^2 + s^2}} \right\} = e^{-i Z e^2 \ln[C \rho^2]}, \tag{3.7}
\]

where \( C \) is an undefined constant related to the large-distance regularization of the integrals in (3.7). The spinorial factor has been taken already into account for the calculation of the cross section \( \sigma(e^-Z) \) or \( \sigma(e^-Z)(E_1)^{\text{max}} \sim n^3(E_1/\Lambda) \). This yields

\[
\Psi_{\mu^-}^\dagger(\vec{r}; \vec{p}_1^\prime, \sigma'_1) \Psi_{e^-}(\vec{r}; \vec{p}_1, \sigma_1) \sim e^{-i Z e^2 \ln[C' \ln^3(E_1/\Lambda)]}, \tag{3.8}
\]

As the cross section for the reaction is proportional to \( |\Psi_{\mu^-}^\dagger(\vec{r}; \vec{p}_1^\prime, \sigma'_1) \Psi_{e^-}(\vec{r}; \vec{p}_1, \sigma_1)|^2 \), the distortion of the wave functions of the initial and final leptons caused by the strong Coulomb field does not change crucially the cross section for the reaction \( e^- + Z \rightarrow Z + M^0 + \bar{\mu}^- \) calculated for the wave functions of the coupled leptons in the form of plane waves.

For the estimate of the influence of the strong Coulomb field on the state of muonium we suggest to calculate the time of the decay \( M^0 \rightarrow \mu^+ + e^- \) induced by the external Coulomb field. The amplitude of the decay \( M^0 \rightarrow \mu^+ + e^- \) we define as

\[
\mathcal{M}(M^0 \rightarrow \mu^+ + e^-) = \int \frac{1}{\sqrt{V}} e^{-i \vec{p} \cdot \vec{r}} U(\vec{r}) \frac{1}{\sqrt{\pi a_B^3}} e^{-r/a_B} d^3r, \tag{3.9}
\]

where \( \vec{p} \) is a relative momentum of the \( \mu^+e^- \) pair, \( a_B = 268.173 \text{MeV}^{-1} \) is the Bohr radius of muonium, and \( V \) is a normalization volume. Then, \( U(\vec{r}) \) is the potential energy of
the dipole moment $\vec{d} = e \vec{r}$ of the $\mu^+e^-$ pair coupled to the strong Coulomb field of the nucleus $Z$

$$U(\vec{r}) = -\vec{d} \cdot \vec{E}(\vec{r}) = \frac{Ze^2}{r}.$$  \quad (3.10)

Integrating over $\vec{r}$ we get

$$\mathcal{M}(M^0 \rightarrow \mu^+e^-) = \frac{4\pi Z e^2}{\sqrt{\pi} a^3} \frac{a_B^2}{1 + a_B^2 p^2}. \quad (3.11)$$

The time of the decay $M^0 \rightarrow \mu^+e^-$ is equal to

$$\tau^{-1}(M^0 \rightarrow \mu^+e^-) = \frac{32Z^2\alpha^2}{a_B^2 E_{M^0}^2} = \frac{32Z^2\alpha^5}{E_{M^0}^2} \left( \frac{m_e m_\mu}{m_e + m_\mu} \right)^3, \quad (3.12)$$

where $E_{M^0}$ is the energy of the muonium in the rest frame of the nucleus $Z$. Since $E_{M^0} \gg 5 \text{ GeV}$, for $^{222}_{86}\text{Rn}$ we estimate $\tau(M^0 \rightarrow \mu^+e^-) \gg 2.6 \times 10^{-8} \text{ s}$. The time of the interaction of the electron scattering by Radon, during which muonium can be produced, is of order $\tau \sim 10^{-8} \text{ s}$. This means that the strong Coulomb field does not affect crucially the production of muonium or anti–muonium in the reactions $\vec{e}^- + Z \rightarrow Z + M^0 + \vec{\mu}^-$ and $\vec{e}^+ + Z \rightarrow Z + \bar{M}^0 + \vec{\mu}^+$. Of course, a more detailed analysis of the Coulomb distortion of the wave functions of leptons in the reactions $\vec{e}^- + Z \rightarrow Z + M^0 + \vec{\mu}^-$ and $\vec{e}^+ + Z \rightarrow Z + \bar{M}^0 + \vec{\mu}^+$ and the influence of this distortion on the production of muonium $M^0$ and anti–muonium $\bar{M}^0$ is required. We are planning to analyse this problem in our forthcoming investigations.

### 4 Conclusion

We have calculated the cross sections for the reactions $\vec{e}^- + Z \rightarrow Z + M^0 + \vec{\mu}^-$ and $\vec{e}^+ + Z \rightarrow Z + \bar{M}^0 + \vec{\mu}^+$ of the production of muonium $M^0$ and anti–muonium $\bar{M}^0$ with polarized $\mu^-$ and $\mu^+$ mesons by polarized electrons and positrons coupled at high energies to the nucleus $Z$.

The cross sections are calculated in dependence on (i) an energy $E_1$ of initial electron and positron in the laboratory frame, coinciding with the rest frame of a target nucleus $Z$, and (ii) polarizations of initial electron and positron and final muons in the kinematical region $\omega^2 = (p_1' + k)^2 \gg m_\mu^2$ making the massless limit of coupled leptons reasonable.

For the numerical estimate of the cross sections at the energies available for the HERA Collider at DESY [13], i.e $E_1 = 27.5 \text{ GeV}$, we suggest to use Radon, $^{222}_{86}\text{Rn}$, as a target nucleus, since Radon has a spin $1/2$. The theoretical values of the cross sections for longitudinally polarized electrons and positrons scattering by $^{222}_{86}\text{Rn}$ are equal to $\sigma_{M^0}(E_1 = 27.5 \text{ GeV}) = \sigma_{\bar{M}^0}(E_1 = 27.5 \text{ GeV}) = 1.6 \text{ pb}$. For these cross sections we predict the following numbers of favourable events: $N_{M^0} = 808$ and $N_{\bar{M}^0} = 3360$. Hence, the increase of luminosities of electron and positron beams should make the experiment for a test of CPT invariance, suggested by Choban and Kazakov in Ref.[12], feasible at DESY.

We have estimated the influence of the finite nuclear radius and the Coulomb distortion of the wave functions of the leptons. According to our estimate in the kinematical region
\[ \omega^2 = (p_1' + k)^2 \gg m^2_\mu \] the Weizsäcker–Williams approach, treating a nucleus as a point-like particle and neglecting the Coulomb distortion of the wave functions of incoming and outcoming leptons, is a rather well-defined approximation. The contribution of the finite nuclear radius can be kept at the level of a few percents. The distortion of the wave functions of the initial and final leptons caused by the strong Coulomb field does not change the cross sections for the reactions under consideration. During the time of the production of muonium or anti–muonium the strong Coulomb field induced by the charge of the nucleus Ze does not destroy bound states of \( \mu^+e^- \) or \( \mu^-e^+ \) pairs. Hence, the strong Coulomb field can hardly screen the phenomenon of the violation of CPT invariance in the reactions \( \vec{e}^- + Z \rightarrow Z + M^0 + \vec{\mu}^- \) and \( \vec{e}^+ + Z \rightarrow Z + \bar{M}^0 + \vec{\mu}^+ \).

We have shown that the test of CPT invariance in the reactions \( \vec{e}^- + Z \rightarrow Z + M^0 + \vec{\mu}^- \) and \( \vec{e}^+ + Z \rightarrow Z + \bar{M}^0 + \vec{\mu}^+ \) reduces to the experimental analysis of the ratio \( R(T) = \frac{N_{M^0}(T)}{N_{\bar{M}^0}(T)} \) of the numbers of favourable events detected during an interim \( T \). If \( R(T) \) is a constant in time – CPT invariance is conserved, and if \( R(T) \) is an oscillating function in time one can conclude that CPT invariance is violated.

We would like to accentuate that this is a qualitative analysis of CPT invariance. In the case of the ratio \( R(T) \) oscillating in time we can infer neither a strength nor a nature of a violation of CPT invariance.

We argue that the appearance of longitudinally polarized muons in the final states of the reactions \( \vec{e}^- + Z \rightarrow Z + X + \vec{\mu}^- \) and \( \vec{e}^+ + Z \rightarrow Z + X + \vec{\mu}^+ \) with longitudinally polarized electrons and positrons is a distinct signal for the production of muonium \( M^0 \) and anti–muonium \( \bar{M}^0 \) with a total spin \( J = 0 \). This should testify that \( X \equiv M^0 \) and \( \bar{X} \equiv \bar{M}^0 \) with a total spin \( J = 0 \).

Indeed, the creation of the \( \mu^+\mu^- \) pairs in the reactions \( \vec{e}^- + Z \rightarrow Z + \vec{e}^+ + \mu^+ + \mu^- \) seems to be the main process competing with the production of muonium and anti–muonium in the reactions \( \vec{e}^- + Z \rightarrow Z + M^0 + \vec{\mu}^- \) and \( \vec{e}^+ + Z \rightarrow Z + \bar{M}^0 + \vec{\mu}^+ \). The main distinction of the production of the \( \mu^+\mu^- \) pairs from the production of muonium and anti–muonium is a strong correlation between the momenta and polarizations of \( \mu^+ \) and \( \mu^- \) and a decorrelation of them with the initial electron or positron. In turn, a strong correlation between the polarizations of the final muons and the initial electron and positron is a feature of the production of muonium and anti–muonium with a total spin \( J = 0 \) in the reactions \( \vec{e}^- + Z \rightarrow Z + M^0 + \vec{\mu}^- \) and \( \vec{e}^+ + Z \rightarrow Z + \bar{M}^0 + \vec{\mu}^+ \). Hence, at first glimpse for the experimental realization of the test of CPT invariance in the reactions \( \vec{e}^- + Z \rightarrow Z + M^0 + \vec{\mu}^- \) and \( \vec{e}^+ + Z \rightarrow Z + \bar{M}^0 + \vec{\mu}^+ \) with longitudinally polarized electrons and positrons it suffices to count the number of longitudinally polarized \( \mu^- \) and \( \mu^+ \) mesons during an interim \( T \). Plotting the ratio of these numbers, which should coincide with \( R(T) \), one should obtain an experimental information about CPT invariance.
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