Novel Theory for Topological Structure of Vortices in a Bose-Einstein condensate

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By making use of the $\phi$-mapping topological current theory, a novel expression of $\nabla \times \vec{V}$ in BEC is obtained, which reveals the inner topological structure of vortex lines characterized by Hopf indices and Brouwer degrees. This expression is just that formula Landau and Feynman expected to find out long time ago. In the case of superconductivity, the decomposition theory of $U(1)$ gauge potential in terms of the condensate wave function gives a rigorous proof of London assumption, and shows that each vortex line should carry a quantized flux. The $\phi$-mapping topological current theory of $\nabla \times \bar{V}$ also provides a reasonable way to study the bifurcation theory of vortex lines in BEC.

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It is well known that, as semi-phenomenological scenarios of low dimensional BEC continuum, the Gross-Pitaevskii (GP) equation and the Ginzburg-Landau (GL) equations are of great importance. For neutral superfluid, GP equation is given by

$$i\hbar \frac{\partial}{\partial t} \psi = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi + V(\bar{\vec{x}}) \psi + \frac{4\pi \hbar^2 a}{m} |\psi|^2 \psi, \tag{1}$$

and the velocity field coming from the current $J^i = \rho V^i$ ($\rho = |\psi|^2$) is defined as

$$V^i = \frac{\hbar}{2im} \frac{(\psi^* \partial_i \psi - \partial_i \psi^* \psi)}{\psi^* \psi}, \tag{2}$$

where $i = 1, 2, 3$ denotes the 3-dimensional space coordinates. For superconductor, the GL equations are known as

$$\frac{1}{2m} (-i\hbar \partial_t - \frac{e}{c} A_i)^2 \psi + a \psi + b |\psi|^2 \psi = 0, \tag{3}$$

$$(\nabla \times \vec{B})^i = \frac{4\pi}{c} J^i, \tag{4}$$

where the current $J^i$ is covariant under $U(1)$ gauge transformation

$$J^i = e \rho V^i - \frac{e^2}{mc} \rho A_i, \tag{5}$$

with $A_i$ denoting the external magnetic vector potential. In GL theory the velocity take the same form as Eq. (2); $m$ and $e$ should be regarded as the effective mass and the effective electric charge (especially for Cooper pair, $m$ and $e$ should be replaced by $2m$ and $2e$ respectively). In all these formulas $\psi$ denotes the order parameter, i.e., the condensate wave-function, which is a section of complex line bundle.

In theoretical and experimental studies, the curl of $\vec{V}$ is paid much attention to. For a long time, up to now, the wave-function is usually expressed in the form

$$\psi = |\psi| e^{i\Theta(\bar{\vec{x}})}; \tag{6}$$

then $\vec{V}$ becomes the gradient of a velocity potential $\Theta(\bar{\vec{x}})$ ($\vec{V} = \frac{\hbar}{m} \nabla \Theta$), which directly leads to a trivial curl-free result:

$$\nabla \times \vec{V} = 0. \tag{7}$$

But nearly half a century ago Onsager and Feynman found that this statement must be modified, and Landau predicted $\delta$-functions in it, namely, $\nabla \times \vec{V}$ can be non-zero at a singular line, the core of a quantized vortex line. Therefore, it is indispensable to study: what is the exact expression for $\nabla \times \vec{V}$ in topology theory?

In this paper, based on our $\phi$-mapping topological current theory a novel and precise expression for $\nabla \times \vec{V}$ is obtained, which is just the topological current with inside the $\delta$-function of the order parameter. Thus isolated vortices in BEC, i.e., the topological excitation, can be naturally created from the zero points of condensate wave-function, and be characterized by the quantum numbers: Hopf indices and Brouwer degrees of $\phi$-mapping. Using the $U(1)$ gauge potential decomposition theory, the composed intrinsic electromagnetic gauge potential in terms of the wave-function is studied; so a rigorous proof of London assumption ($\vec{V} = \frac{\hbar}{m} A_i$) is given, and the essence of this relation is revealed. A step further, the $\phi$-mapping topological current theory also provides a reasonable way to study the spatial bifurcation of the vortex lines, including intersection, splitting and merge. Being different from the others, the bifurcation theory of this paper does not need to deal with the concrete form of the wave function. At last, it should be pointed out that all the conclusions of this paper do not matter with the concrete form of the nonlinear terms in Eqs. (3) and (4); the nonlinearity may even be generalized to a form $f(|\psi|^2)$, and all the conclusions are the same.

As a matter of fact, by means of the $\phi$-mapping theory great progress has been made in studying the topological

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invariants and the topological structures in many mathematical and physical topics besides here [4–13].

I. \(\phi\)-MAPPING TOPOLOGICAL CURRENT THEORY FOR \(\nabla \times \vec{V}\)

The basic field of condensate wave-function \(\psi(\vec{x})\) is a section of complex line bundle, i.e., a section of 2-dimensional real vector bundle on \(\mathbb{R}^3\):

\[
\psi(\vec{x}) = \phi^1(\vec{x}) + i \phi^2(\vec{x}).
\]

Following \(\phi\)-mapping theory a 2-dimensional unit vector is defined as \(n_a = \frac{\partial}{\partial x_a}\), where \(\phi = |\phi|^2 = \phi^a \phi^a = \psi^* \psi\). Substituting these formulas into Eq. (2), it is easy to find out that \(V^i = \frac{1}{m} \epsilon_{abc} n^a \partial_i n^b\), and the curl of \(V\) can be expressed in terms of \(n^a\):

\[
(\nabla \times \vec{V})^i = \frac{\hbar}{m} \epsilon^{ijk} \epsilon_{abc} \partial_i n^a \partial_j n^b.
\]

Using \(\partial_i n^a = \frac{\partial}{\partial x^a} + \phi^a \phi^b \partial_i \phi^b\) and the Green function relation in \(\phi\)-space, \(\frac{\partial}{\partial \phi} \frac{\partial}{\partial \phi} \ln ||\phi|| = 2 \pi \delta^2(\phi)\), one can directly prove a novel expression for \(\nabla \times \vec{V}\):

\[
(\nabla \times \vec{V})^i = \frac{\hbar}{m} \delta^2(\phi) D^i(\phi) \frac{\partial}{\partial x^i} = \frac{\hbar}{m} j^i,
\]

where \(D^i(\phi) = \frac{1}{2\pi} \epsilon^{ijk} \epsilon_{abc} \partial_j n^a \partial_k n^b = \delta^2(\phi) D^i(\phi)\), and

\[
j^i = \frac{1}{2\pi} \epsilon^{ijk} \epsilon_{abc} \partial_j n^a \partial_k n^b = \delta^2(\phi) D^i(\phi)\frac{\partial}{\partial x^i},
\]

is just a simple 2-dimensional case of the \(N\)-dimensional \(\phi\)-mapping topological current \([13]\). This formula including \(\delta^2(\phi)\) to describe the singularities of \(\phi\) is just the precise topological expression for \(\nabla \times \vec{V}\) that Landau and Feynman expected to find out long time ago. Therefore an important conclusion is reached: \(\nabla \times \vec{V} = 0\), \(\text{iff } \phi \neq 0\); \(\nabla \times \vec{V} \neq 0\), \(\text{iff } \phi = 0\).

The implicit function theory shows that \([4]\), under the regular condition \(\vec{D}(\frac{\phi}{x}) \neq 0\), the general solutions of

\[
\phi^1(x, y, z) = 0, \quad \phi^2(x, y, z) = 0
\]

can be expressed as

\[
x = x_j(s), \quad y = y_j(s), \quad z = z_j(s), \quad (j = 1, 2, \cdots, N)
\]

which represent \(N\) isolated singular strings \(L_j\) with parameter \(s\). These strings are just known as the vortex lines.

In \(\delta\)-function theory \([15]\), one can prove

\[
\delta^2(\phi) = \sum_{j=1}^{N} \beta_j \int_{L_j} \frac{\delta^3(\vec{x} - \vec{x}_j(s))}{|D(\frac{\phi}{x})| \Sigma_j} ds,
\]

where \(D(\frac{\phi}{x}) \Sigma_j = (\frac{1}{\rho} \epsilon_{mn} \frac{\partial \phi_m}{\partial x_n} + \frac{1}{\rho} \epsilon_{mn} \frac{\partial \phi_n}{\partial x_m})\), and \(\Sigma_j\) is the \(j\)th planer element transversal to \(L_j\) with local coordinates \((u^1, u^2)\). The positive integer \(\beta_j\) is the Hopf index of \(\phi\)-mapping. Meanwhile it can be proved that the direction vector of \(L_j\) is

\[
\left(\frac{d\vec{x}}{ds}\right)_{x_j} = [\vec{D}(\frac{\phi}{x})/D(\frac{\phi}{x}) \Sigma_j]_{x_j}.
\]

Then from Eqs. (14) and (13) we find the important inner topological structure of \(\nabla \times \vec{V}\):

\[
\nabla \times \vec{V} = \frac{\hbar}{m} \sum_{j=1}^{N} \beta_j \eta_j \int_{L_j} \frac{ds}{ds} \delta^3(\vec{x} - \vec{x}_j(s)) ds,
\]

where the positive integer \(\beta_j\) is the Hopf index of \(\phi\)-mapping, and \(\eta_j\) is the Brouwer degree, \(\eta_j = \pm 1\). And the winding number of \(\phi\) around \(L_j\) is \(W_j = \beta_j \eta_j\). Therefore the vorticity of vortex line \(L_j\) is \(\Gamma_j = \oint_{\Sigma_j} \nabla \times \vec{V} \cdot ds = \frac{\hbar}{m} W_j\), where \(\Sigma_j\) is the \(j\)th planar element transversal to \(L_j\); and the total vorticity on a surface \(\Sigma\) should be

\[
\Gamma = \oint_{\Sigma} \nabla \times \vec{V} \cdot ds = \frac{\hbar}{m} \sum_{j=1}^{N} W_j.
\]

We stress that there are no hypothesis in the deduction above. Eqs. (14) and (13) are called the differential forms of the quantization condition, which cannot be derived from the single-valued principle of wave function, and are more essential than the integral form (Eq. (17)).

For the GL theory, Eqs. (14) and (13) lead to

\[
\vec{A} + \lambda^2 \nabla \times \vec{B} = \frac{mc}{e} \vec{V},
\]

where \(\lambda\) is the penetration depth, \(\lambda^2 = \frac{1}{\rho} \frac{mc^2}{4\pi e^2}\). In London approximation, \(\rho\) and therefore \(\lambda\) are treated as constants; hence when noticing \(\vec{B} = \nabla \times \vec{A}, \nabla \cdot \vec{B} = 0\) and Eq. (10), we find a topological equation for \(\vec{B}\):

\[
\vec{B} - \lambda^2 \nabla^2 \vec{B} = \Phi_0 \sum_{j=1}^{N} W_j \int_{L_j} \frac{d\vec{x}}{ds} \delta^3(\vec{x} - \vec{x}_j(s)) ds,
\]

This formula directly leads to

\[
\vec{B} - \lambda^2 \nabla^2 \vec{B} = \Phi_0 \sum_{j=1}^{N} W_j \int_{L_j} \frac{d\vec{x}}{ds} \delta^3(\vec{x} - \vec{x}_j(s)) ds,
\]

where \(\Phi_0 = \frac{hc}{2}\pi\) is the unit flux quantum. We see that in simple case \(W_j = 1\), the above equation is just the so-called modified London equation \([14,15]\). This expression says that, when the condensate wave function \(\psi\) has no zero values, \(\phi \neq 0\), i.e., \(\delta^2(\phi) = 0\), and \(\vec{B} - \lambda^2 \nabla^2 \vec{B} = 0\), which just corresponds to the Meissner state; while in the case of mixed state, \(\phi\) possesses \(N\) isolated zeros, \(\delta^2(\phi) \neq 0\), thus a type-II superconductor is penetrated by an array of \(N\) vortices, with each one carrying a quantum flux proportional to the winding number \(W_j\).
II. DECOMPOSITION OF U(1) GAUGE
POTENTIAL AND THE VORTEX WITH
QUANTIZED FLUX

The decomposition theory of gauge potential in
SO(N) and SU(N) gauge theories is now playing a more
and more important role in theoretical studies, because
it virtually inputs topological information and other important
information to the gauge potential. In the theory of
superconductivity, \( \psi \) is a condensate wave function
describing the charged continuum, so the covariant derivative
in U(1) gauge theory is introduced to describe the interaction
between \( \psi \) and the electromagnetic field:

\[
D_i \psi = \partial_i \psi - i e \frac{\hbar}{c} A_i \psi, \quad (i = 1, 2, 3)
\]

where \( A_i \) is the magnetic gauge potential vector. The complex conjugate of \( D_i \psi \) is \( D^*_i \psi^* = \partial_i \psi^* + i e \frac{\hbar}{c} A_i \psi^* \). And the magnetic field tensor is given by

\[
f_{ij} = \partial_i A_j - \partial_j A_i.
\]

Multiplying \( D_i \psi \) with \( \psi^* \) and \( D^*_i \psi^* \) with \( \psi \) respectively, we can deduce the decomposition formula for U(1) gauge potential:

\[
A_i(\psi) = \frac{\hbar}{2ie} \psi^* \left( \partial_i \psi - \partial_i \psi^* \right) - (\psi^* D_i \psi - D^*_i \psi^* \psi).
\]

The above expression \( A_i = A_i(\psi) \) means that the magnetic gauge potential possesses an inner structure in terms of charged condensate wave function \( \psi \) and \( \psi^* \). The inner structure of \( A_i(\psi) \) with Eq. (24) gives a theory that in superconductivity how the stationary motion of condensate wave function creates an intrinsic magnetic field. This is the important physical meaning of decomposition of U(1) gauge potential in quantum mechanics.

Furthermore it has been proved that the covariant derivative part \( \left[ -\psi^* D_i \psi - D^*_i \psi^* \psi \right] \) corresponds to the gradient of a phase factor: \( (\partial_i \lambda) \). Thus this covariant derivative part contributes nothing to the field tensor \( f_{ij} \), so it can be ignored, and

\[
A_i(\psi) = \frac{\hbar}{2ie} \psi^* \left( \partial_i \psi - \partial_i \psi^* \right).
\]

It should be emphasized that the above U(1) gauge potential decomposition theory together with \( \phi \)-mapping theory has been successfully used to study many other topological problems in physics.

In \( \phi \)-mapping theory \( A_i(\psi) \) can be rewritten in terms of \( n^a \) as \( A_i(\psi) = \frac{\hbar}{2ie} \epsilon_{abc} n^a \partial_j n^b \), and \( f_{ij} \) becomes \( f_{ij} = 2 \frac{\hbar}{e} \epsilon_{abc} \partial_j n^a \partial_i n^b \). Therefore the intrinsic magnetic field vector from \( A_i(\psi) \) is expressed as

\[
B_i(\psi) = \frac{1}{2} \epsilon_{ijk} f_{jk} = \Phi_0 \frac{1}{2\pi} \epsilon_{ijk} \epsilon_{abc} \partial_j n^a \partial_k n^b.
\]

Using Eq. (14) we have \( B_i(\psi) = \Phi_0 \delta^2(\phi) D^i(\frac{\psi}{x}) \), which gives the topological structure of intrinsic magnetic field \( B_i(\psi) \). \( B_i(\psi) \) does not matter with the external magnetic field. As before, the zero points of \( \phi(x) \), i.e., the singular vortex lines in superconductivity contribute to intrinsic magnetic field as

\[
B_i(\psi) = \Phi_0 \sum_{j=1}^{N} W_j \int_{L_j} \frac{dx^i}{ds} \delta^3(\vec{x} - \vec{x}_j(s)) ds.
\]

This leads to an important phenomenon that, the magnetic flux coming from the stationary motion of \( \psi \) itself is quantized

\[
\Phi = \int_{L_j} \vec{B}(\psi) \cdot ds = \Phi_0 \sum_{j=1}^{N} W_j,
\]

and each singular vortex line \( L_j \) carries a magnetic flux \( \Phi_j = W_j \Phi_0 \).

The above decomposition theory of U(1) gauge potential naturally arises at the conclusion that, in superconductivity continuum, the \( N \) isolated singular vortices are just \( N \) isolated topological elementary excitations carrying with magnetic fluxoid, while their quantum numbers are characterized by topological numbers \( W_j = \beta_j \eta_j \). We see that the \( \phi \)-mapping topological theory in this paper is independent of concrete physical models, that gives a profound understanding to the nature of the creation of the vortex lines and the flux quantization in BEC.

Comparing Eq. (24) with Eq. (2) it directly follows a simple relation between \( V_i \) and \( A_i \):

\[
V_i = \frac{e}{mc} A_i,
\]

which is just the London’s assumption. We stress that, the essence and the significance of this relation are not truly realized until now the inner structure of gauge potential is revealed and therefore the stationary motion of condensate wave function is naturally related to the intrinsic magnetic field.

III. SPATIAL BIFURCATION OF VORTEX LINES

The Solution \( \beta_j \) of Eq. (12) is based on the condition \( \vec{B}(\phi/x) \neq 0 \). When it fails, i.e.,

\[
\vec{B} \left( \frac{\phi}{x} \right) = 0
\]

(29)

at some points (marked as \( \vec{r}_j \)) along \( L_j \), the functional relationship between coordinate \( x \) and \( z \), or \( y \) and \( z \) is not unique in the neighborhood of \( \vec{r}_j \), because the direction of the zero line expressed by

\[
\frac{dx}{dz} = D^1 \left( \frac{\phi}{x} \right) / D^3 \left( \frac{\phi}{x} \right) \bigg|_{\vec{r}_j}, \quad \frac{dy}{dz} = D^2 \left( \frac{\phi}{x} \right) / D^3 \left( \frac{\phi}{x} \right) \bigg|_{\vec{r}_j}
\]

(30)
is indefinite at $\vec{r}_j^*$. Hence this very point $\vec{r}_j^*$ is called a bifurcation point of the two-component vector in 3-dimensional space.

According to the $\phi$-mapping theory, the Taylor expansion of the solution of Eq. (12) in the neighborhood of $\vec{r}_j^*$ can be generally expressed as (30): $A(x-x_j^*)^2 + B(x-x_j^*)(z-z_j^*) + C(z-z_j^*)^2 + \cdots = 0$, where $A$, $B$ and $C$ are constants. This leads to

$$A\left(\frac{dx}{dz}\right)^2 + 2B \frac{dx}{dz} + C = 0 \text{ or } C\left(\frac{dz}{dx}\right)^2 + 2B \frac{dz}{dx} + A = 0.$$  \hspace{1cm} (31)

The solutions of Eq. (31) give different branches of the zero lines, i.e., the vortex lines at bifurcation points. In following four main cases in the branch process are simply discussed (the detailed deduction and figures may be found in Ref. [10,5]):

Case 1 ($A \neq 0$): For $\Delta = 4(B^2 - AC) > 0$, from Eq. (31) we get two different spatial directions at the bifurcation point

$$\frac{dx}{dz} \Big|_{1,2} = -\frac{B \pm \sqrt{B^2 - AC}}{A}.$$  \hspace{1cm} (32)

This is the intersection of two vortex lines of different directions.

Case 2 ($A \neq 0$): For $\Delta = 4(B^2 - AC) = 0$, we get only one direction at the point

$$\frac{dx}{dz} \Big|_{1,2} = -\frac{B}{A}.$$  \hspace{1cm} (33)

which includes three sub-cases: (a) Two vortex lines tangentially intersect; (b) Two vortex lines merge into one line; (c) One vortex line splits into two lines.

Case 3 ($A = 0$, $C \neq 0$): For $\Delta = 4(B^2 - AC) > 0$, from Eq. (31) we have

$$\frac{dz}{dx} \Big|_{1,2} = -\frac{B \pm \sqrt{B^2 - AC}}{C} = 0, -\frac{2B}{C}.$$  \hspace{1cm} (34)

There are two sub-cases: (a) Three vortex lines merge into one line; (b) One vortex line splits into three lines.

Case 4 ($A = C = 0$): Eq. (31) gives respectively

$$\frac{dx}{dz} = 0, \frac{dz}{dx} = 0.$$  \hspace{1cm} (35)

This case shows that two curves normally intersect at the bifurcation point, which is similar to case 3.

It should be noted that, noticing the continuity of topological current $j$ from Eq. (11) ($\partial_j j_i^* = 0$), at the bifurcation point the sum of the topological charge of final vortex line(s) is required to be equal to that of the initial line(s) for a fixed index $j$: $\sum_j j_i \beta_i j_j = \sum_i j_i \eta_i j_j$.

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