Effective field theories and inflationary magnetogenesis

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Abstract

The effective approach is applied to the analysis of inflationary magnetogenesis. Rather than assuming a particular underlying description, all the generally covariant terms potentially appearing with four space-time derivatives in the effective action have been included and weighted by inflaton-dependent couplings. The higher derivatives are suppressed by the negative powers of a typical mass scale whose specific values ultimately depend on the tensor to scalar ratio. During a quasi-de Sitter stage the corresponding corrections always lead to an asymmetry between the hypermagnetic and the hyperelectric susceptibilities. After presenting a general method for the estimate of the gauge power spectra, the obtained results are illustrated for generic models and also in the case of some non-generic scenarios where either the inflaton has some extra symmetry or the higher-order terms are potentially dominant.
The dynamical evolution of a large class of inflationary models is conventionally described in terms of a scalar-tensor theory of gravity whose effective Lagrangian density is characterized by a single inflaton field $\varphi$

$$\mathcal{L}_{\text{inf}} = \sqrt{-G} \left[ \frac{M_P^2}{2} R + \frac{1}{2} G^{\alpha\beta} \partial_{\alpha} \varphi \partial_{\beta} \varphi - V(\varphi) \right], \quad (1)$$

where $M_P = M_P/\sqrt{8\pi}$ is the reduced Planck mass while $V(\varphi)$ denotes the inflaton potential. Equation (1) can be regarded as the first term of a generic effective field theory where the higher derivatives are suppressed by the negative powers of a large mass $M$ associated with the fundamental theory that underlies the effective description. For practical reasons it will be useful to deal with an appropriate dimensionless scalar $\phi = \varphi/M$. Following the lucid discussion of Ref. [1] (see also [2]) the leading correction to Eq. (1) consists of all possible terms containing four derivatives and it can be parametrized in the following manner:

$$\Delta \mathcal{L}_{\text{inf}} = \sqrt{-G} \left[ c_1(\phi) (G^{\alpha\beta} \partial_{\alpha} \phi \partial_{\beta} \phi)^2 + c_2(\phi) G^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi \Box \phi + c_3(\phi) (\Box \phi)^2 \right.$$

$$+ c_4(\phi) R^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi + c_5(\phi) R G^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi + c_6(\phi) R \Box \phi + c_7(\phi) R^2 + c_8(\phi) R_{\mu\nu} R^{\mu\nu}$$

$$+ c_9(\phi) R_{\mu\alpha\nu\beta} R^{\mu\alpha\nu\beta} + c_{10}(\phi) C_{\mu\alpha\nu\beta} C^{\mu\alpha\nu\beta} + c_{11}(\phi) R_{\mu\alpha\nu\beta} \tilde{R}^{\mu\alpha\nu\beta} + c_{12}(\phi) C_{\mu\alpha\nu\beta} \tilde{C}^{\mu\alpha\nu\beta} \right], \quad (2)$$

where $R_{\mu\alpha\nu\beta}$ and $C_{\mu\alpha\nu\beta}$ denote the Riemann and Weyl tensors while $\tilde{R}^{\mu\alpha\nu\beta}$ and $\tilde{C}^{\mu\alpha\nu\beta}$ are the corresponding duals. From Eq. (2) various interesting conclusions can be drawn. For instance the leading correction to the two-point function of the scalar mode of the geometry comes from the terms containing four-derivatives of the inflaton field while in the case of the tensor modes the leading corrections stem from $C_{\mu\alpha\nu\beta} \tilde{C}^{\mu\alpha\nu\beta}$ and $R_{\mu\alpha\nu\beta} \tilde{R}^{\mu\alpha\nu\beta}$ which are typical of Weyl and Riemann gravity [3, 4]. Incidentally both terms break parity and are therefore capable of polarizing the stochastic backgrounds of the relic gravitons [5] by ultimately affecting the dispersion relations of the two circular polarizations.

The same logic shall now be extended to the description of magnetogenesis scenarios based on the evolution of the gauge coupling so that the Lagrangian density [1] will now be complemented by the contribution of the hypercharge fields:

$$\mathcal{L}_{\text{gauge}} = -\sqrt{-G} \left[ \frac{\lambda(\phi)}{16\pi} Y_{\alpha\beta} Y^{\alpha\beta} + \frac{\lambda(\phi)}{16\pi} Y_{\alpha\beta} \tilde{Y}^{\alpha\beta} \right], \quad \tilde{Y}^{\alpha\beta} = E^{\alpha\beta\mu\nu} Y_{\mu\nu}/2, \quad (3)$$

where $E^{\alpha\beta\mu\nu} = e^{\alpha\beta\mu\nu}/\sqrt{-G}$ and $e^{\alpha\beta\mu\nu}$ is the four-dimensional Levi-Civita symbol; within the notations of Eq. (3) the gauge coupling is $g = \sqrt{4\pi/\lambda}$. For the Lagrangian density Eq. (3) the analog of $\Delta \mathcal{L}_{\text{inf}}$ will have to include the collection of all possible terms containing four derivatives and combining the inflaton, the gauge fields and the metric tensor:

$$\Delta \mathcal{L}_{\text{gauge}} = \sqrt{-G} \left[ \frac{16\pi}{M^2} \lambda_1(\phi) R Y_{\alpha\beta} Y^{\alpha\beta} + \lambda_2(\phi) R_{\mu}^{\quad \nu} Y_{\mu\alpha} Y_{\nu\beta} + \lambda_3(\phi) R_{\mu\alpha\nu\beta} Y^{\mu\alpha} Y^{\nu\beta} \right.$$

$$+ \lambda_4(\phi) C_{\mu\alpha\nu\beta} Y^{\mu\alpha} Y^{\nu\beta} + \lambda_5(\phi) \Box Y_{\alpha\beta} Y^{\alpha\beta} + \lambda_6(\phi) \partial_{\mu} \phi \partial^{\mu} \phi Y^{\mu\alpha} Y_{\nu\alpha} + \lambda_7(\phi) \Box Y_{\mu\nu} \partial_{\nu} \phi Y_{\mu\alpha} Y^{\alpha\beta}$$

$$+ \lambda_8(\phi) R Y_{\alpha\beta} \tilde{Y}^{\alpha\beta} + \lambda_9(\phi) R_{\mu}^{\quad \nu} Y_{\mu\nu} \tilde{Y}^{\alpha\beta} + \lambda_{10}(\phi) R_{\mu\alpha\nu\beta} Y^{\mu\alpha} \tilde{Y}^{\nu\beta} + \lambda_{11}(\phi) C_{\mu\alpha\nu\beta} \tilde{Y}^{\mu\alpha} \tilde{Y}^{\nu\beta}$$

$$+ \lambda_{12}(\phi) \Box \phi Y_{\alpha\beta} \tilde{Y}^{\alpha\beta} + \lambda_{13}(\phi) \partial_{\mu} \phi \partial^{\mu} \phi Y^{\mu\alpha} Y_{\nu\alpha} + \lambda_{14}(\phi) \partial_{\mu} \phi \partial^{\mu} \phi Y_{\mu\alpha} Y_{\nu\alpha} \tilde{Y}^{\alpha\beta} \right]. \quad (4)$$

Equation (4) contains 14 distinct terms; 7 of them do not break parity and are weighted by the couplings $\lambda_i(\phi)$ (with $i = 1, \ldots, 7$). The remaining 7 contributions are weighted by the prefactors $\lambda_j(\phi)$ (with $j = 8, \ldots, 14$) and contain parity-breaking terms. The contributions containing the dual Riemann or Weyl tensors (e.g. $R_{\mu\alpha\nu\beta} Y^{\mu\alpha} Y^{\nu\beta}$ and $C_{\mu\alpha\nu\beta} Y^{\mu\alpha} Y^{\nu\beta}$) are fully equivalent to the ones already present in Eq. (4).
by recalling the explicit definition\(^2\) of \(\tilde{R}_{\mu\nu\alpha\beta}\) and \(\tilde{C}_{\mu\nu\alpha\beta}\). Various particular case implicitly contained in Eq. (4) have been separately discussed in specific physical contexts but they have never been concurrently studied together with the inflaton coupling. For instance when the \(\phi\)-dependent couplings disappear (i.e. \(\lambda_i(\phi) \to 1\)) the first three terms have been analyzed by Drummond and Hathrell \(^6\) in the curved version of quantum electrodynamics. Always in the absence of scalar couplings the considerations of Ref. \(^6\) have been applied to the analysis of large-scale magnetism long ago mostly with negative conclusions. More recently the same terms (in a similar approximation) have been considered in Ref. \(^7\) for the analysis of photon propagation in curved space-times. Even more recently the Riemann coupling associated with \(\tilde{\lambda}_{10}(\phi)\) has been proposed in Ref. \(^5\); this term may ultimately polarize the relic graviton background. The contributions containing the gradients of the inflaton (i.e. \(\lambda_5(\phi), \lambda_6(\phi), \lambda_7(\phi)\) and their corresponding duals) arise in the relativistic theory of Van der Waals (or Casimir-Polder) interactions in flat \(^9\) \(^10\) and curved \(^11\) backgrounds. It is finally appropriate to stress that we shall be interested in the situation where the gauge fields are amplified from their quantum fluctuations so that the gauge fields vanish on the background and Eq. (4) does not include terms like \((Y_{\mu\nu}Y^{\mu\nu})^2\) typically appearing in the Euler-Heisenberg Lagrangian. When a classical gauge field background is present these terms should be however included and may play a relevant role as argued\(^3\) in Ref. \(^12\).

The full Lagrangian density \(L_{gauge} + \Delta L_{gauge}\) encompassing Eqs. (3) and (4) does not necessarily imply that the electric and magnetic susceptibilities must coincide. Let us in fact consider, for the sake of concreteness, a conformally flat background geometry \(g_{\mu\nu} = a^2(\tau)\ \eta_{\mu\nu}\) where \(\eta_{\mu\nu}\) is the Minkowski metric and \(a(\tau)\) is the scale factor written in terms of the conformal time coordinate \(\tau\). In this case the full gauge action is:

\[
S_{gauge} = \int d^3x \int d\tau \left( L_{gauge} + \Delta L_{gauge} \right) = \frac{1}{2} \int d^3x \int d\tau \left( \chi_E^2 E^2 - \chi_B^2 B^2 + \chi^2 \tilde{E} \cdot \tilde{B} \right),
\]

where \(\tilde{E}\) and \(\tilde{B}\) denote the comoving fields that are related to their physical counterparts as \(\tilde{B} = a^2 \ B_{phys}\) and as \(\tilde{E} = a^2 \ E_{phys}\). The hyperelectric and the hypermagnetic susceptibilities \(\chi_E^2\), \(\chi_B^2\) and \(\chi^2\) are instead defined as:

\[
\chi_E^2 = \frac{\lambda}{4\pi} \left[ 1 + \frac{6(\mathcal{H}^2 + \mathcal{H}')}{M^2 a^2} \left( \frac{\lambda_1}{\lambda} \right) + \frac{(2 \mathcal{H} + 2 \mathcal{H}')}{{M^2 a^2}} \left( \frac{\lambda_2}{\lambda} \right) + \frac{2 \mathcal{H}'}{M^2 a^2} \left( \frac{\lambda_3}{\lambda} \right) \right] - \frac{\phi'' + 2 \mathcal{H} \phi'}{M^2 a^2} \left( \frac{\lambda_5}{\lambda} \right) + \frac{1}{2} \frac{\mathcal{H}}{M^2 a^2} \left( \frac{\lambda_6}{\lambda} \right) - \frac{\phi'}{2 M^2} \left( \frac{\lambda_7}{\lambda} \right),
\]

\[
\chi_B^2 = \frac{\lambda}{4\pi} \left[ 1 + \frac{6(\mathcal{H}^2 + \mathcal{H}')}{M^2 a^2} \left( \frac{\lambda_1}{\lambda} \right) + \frac{(2 \mathcal{H} + 2 \mathcal{H}')}{{M^2 a^2}} \left( \frac{\lambda_2}{\lambda} \right) + \frac{2 \mathcal{H}^2}{M^2 a^2} \left( \frac{\lambda_3}{\lambda} \right) \right] - \frac{(\phi'' + 2 \mathcal{H} \phi')}{2 M^2 a^2} \left( \frac{\lambda_5}{\lambda} \right) + \frac{2 \mathcal{H} \phi'}{M^2 a^2} \left( \frac{\lambda_6}{\lambda} \right),
\]

\[
\chi^2 = \frac{\lambda}{4\pi} \left[ 1 + \frac{6(\mathcal{H}^2 + \mathcal{H}')}{M^2 a^2} \left( \frac{\lambda_1}{\lambda} \right) + \frac{3}{2} \left( \frac{\lambda_9}{\lambda} \right) + \frac{\lambda_{10}}{\lambda} \right] - \frac{2(\phi'' + 2 \mathcal{H} \phi')}{M^2 a^2} \left( \frac{\lambda_{12}}{\lambda} \right) - \frac{(\phi'' + 4 \mathcal{H} \phi')}{4 M^2 a^2} \left( \frac{\lambda_{13}}{\lambda} \right) - \frac{\phi'}{4 M^2 a^2} \left( \frac{\lambda_{14}}{\lambda} \right),
\]

where the prime denotes a derivation with respect to the conformal time coordinate \(\tau\), while, as usual, \(\mathcal{H} = a H = a'/a\). From Eq. (5) the evolution equations for the hyperelectric and for the hypermagnetic

\(^2\)The same kind of comment holds for terms containing a pair of dual fields of different nature (e.g. \(\tilde{R}_{\mu\nu\alpha\beta} Y^{\mu\nu} Y^{\alpha\beta}\)); in these cases the resulting expression will ultimately contain two four-dimensional Levi-Civita symbols whose contraction leads to a string of contributions that are already included in Eq. (4).

\(^3\)While the inclusion of a gauge background is clearly contrary to the logic of magnetogenesis (where the gauge fields should be dynamically generated) it is interesting to remark the the effects on the effective gauge couplings are somehow similar to the ones produced by the inflaton background and by the geometry.

\(^4\)In terms of the physical fields we obviously have \(Y_\alpha = a^2 E_\alpha^{phys}\) and \(Y^{ij} = -\epsilon^{ijk} B_k^{phys}/a^2\).
couplings; the result of this twofold manipulation is the following:

\[ d^{(1)}_E = 12(\lambda_1/\lambda) + 3(\lambda_2/\lambda) + 2(\lambda_3/\lambda) \]
\[ d^{(3)}_E = (\lambda_7/\lambda)/8\pi \]
\[ d^{(1)}_B = 12(\lambda_1/\lambda) + 3(\lambda_2/\lambda) + 2(\lambda_3/\lambda) \]
\[ d^{(3)}_B = 3(\lambda_5/\lambda)/\sqrt{4\pi} \]
\[ d^{(1)}_d = 12(\vec{\lambda}_8/\vec{\lambda}) + 3(\vec{\lambda}_9/\vec{\lambda}) + 2(\vec{\lambda}_{10}/\vec{\lambda}) = 2d^{(2)}_d \]
\[ d^{(4)} = [6(\vec{\lambda}_{12}/\vec{\lambda}) + (5/4)(\vec{\lambda}_{13}/\vec{\lambda})]/\sqrt{4\pi} \]

It is relevant to mention that when \( \vec{\chi} \to 0 \) (i.e. in the absence of parity-breaking terms) Eqs. (9) and (10) are invariant for a generalised duality symmetry: when the susceptibilities are exchanged (i.e. \( \chi_E \to \chi_B \)) the underlying equations are invariant provided \( \vec{B} \to \vec{E} \) and \( \vec{E} \to -\vec{B} \). In the limit \( \chi_E \to \chi_B \) this is exactly the standard duality symmetry [13,14] here analyzed in a conformally flat background.

While Eqs. (6), (7) and (8) only assume a conformally flat background geometry, in view of the inflationary applications it is desirable to rephrase Eqs. (6), (7) and (8) by introducing the slow-roll parameters \( \epsilon = -\dot{H}/H^2 \) and \( \eta = \ddot{\phi}/(H \dot{\phi}) \) (see for instance [15,16]) and by also rescaling the gauge couplings; the result of this twofold manipulation is the following:

\[ \chi_E^2 = \frac{\lambda}{4\pi} \left( 1 + \frac{H^2}{M^2} d^{(1)}_E - \frac{\epsilon}{M^2} d^{(2)}_E - \frac{H^2}{M^2} d^{(3)}_E + \sqrt{\epsilon} \eta \frac{H^2 M_P}{M^3} d^{(4)}_E + \frac{\sqrt{\epsilon} \eta}{M} \frac{H^2 M_P}{M^3} d^{(5)}_E \right), \]
\[ \chi_B^2 = \frac{\lambda}{4\pi} \left( 1 + \frac{H^2}{M^2} d^{(1)}_B - \frac{\epsilon}{M^2} d^{(2)}_B - \frac{H^2 M_P}{M^3} d^{(3)}_B + \sqrt{\epsilon} \eta \frac{H^2 M_P}{M^3} d^{(4)}_B \right), \]
\[ \vec{\chi}^2 = \frac{\vec{\chi}}{4\pi} \left( 1 + \frac{H^2}{M^2} d^{(1)}_d - \frac{\epsilon}{M^2} d^{(2)}_d - \frac{H^2 M_P^2}{M^4} d^{(3)}_d + \sqrt{\epsilon} \eta \frac{H^2 M_P^2}{M^4} d^{(4)}_d + \sqrt{\epsilon} \eta \frac{M_P H^2}{M^3} d^{(5)}_d \right). \]

Equations (2) and (4) have in fact the same content since they represent the lowest terms of an expansion in inverse powers of \( M \). In Tab. 1 the explicit expressions of the \( \phi \)-dependent couplings appearing in Eqs. (11), (12) and (13) have been collected by directly employing the Planck mass \( M_P \) and not its reduced counterpart. Depending on the value of the slow-roll parameters the explicit evaluation of the various corrections follows in two complementary limits. The first limit is the one where \( \epsilon \) is smaller than 1 but not too small. Since the change of \( \phi \) during a Hubble time \( H^{-1} \) follows from the background evolution in this limit we can safely estimate that \( M \simeq \sqrt{2\epsilon} M_P \). For generic theories of inflation (i.e. when \( \varphi \) is not constrained by symmetry principles) \( M \) cannot be much smaller than \( \sqrt{2\epsilon} M_P \), otherwise \( \hat{\phi}/H \) would diverge. If \( M = \sqrt{2\epsilon} M_P \) then \( H/M \) will be slightly larger than \( H/M_P \). In the case of conventional inflationary scenarios the physical wavenumber \( k/a \) and the Hubble rate coincide at horizon exit and, more precisely, we will have that \( M_P H^2/M_k^4 = \epsilon A_R/8 \) where \( A_R = 2.41 \times 10^{-9} \) is the amplitude of the curvature inhomogeneities assigned at the pivot scale \( k_p = 0.002 \text{Mpc}^{-1} \) (see, for instance, [16,17]). If we keep track of the various factors the standard result \( H/M_P = \sqrt{\pi \epsilon A_R} \) is readily obtained. Introducing

\[ \text{[In particular it follows from } 2(M_P^2/M^2)H = -\ddot{\phi}^2; \text{this condition can obviously be rephrased as } \ddot{\phi}/H = \sqrt{2\epsilon} M_P/M. \]
then $A_0 = 8\pi^3A_R \simeq 6 \times 10^{-7}$ the leading contributions to $\chi^2_E$, $\chi^2_B$ and $\chi^2$ are all $O(A_0/\epsilon)$ while the subleading terms are $O(\epsilon A_0/\epsilon)$ and $O(A_0)$.

The current observational determinations of the tensor to scalar ratio $r_T$ range between $r_T < 0.07$ \cite{17} and $r_T < 0.01$ \cite{18,19}. Since the consistency relations stipulate that $\epsilon \simeq r_T/16$, we have to acknowledge that $\epsilon < 10^{-3}$ so that we are not in the situation discussed in the previous paragraph. For consistency we should then require, in the present context, that $M \gg \sqrt{2}M_P$ which implies that $M \simeq M_P$ and $\epsilon \ll 1$. Consequently, the leading contributions appearing in Eqs. (11), (12) and (13) will be associated with $d_E^{(1)}$, $d_B^{(1)}$ and $d_0^{(1)}$. The latter terms are all $O(8\pi A_0\epsilon)$ while the former contain further powers of the slow-roll parameters. The suppressions coming from the inflationary evolution have to be combined with possible hierarchies of the different $d_E^{(i)}(\phi)$, $d_B^{(i)}(\phi)$ and $d_0^{(i)}(\phi)$. All in all we two complementary situations emerge.

In the first case the naturalness of the couplings would imply that all the $\lambda_i(\phi)$ and similarly for the $\bar{\lambda}_i(\phi)$ which should all be $O(\bar{\lambda})$. In this situation the leading contribution to the gauge power spectra will be arguably given by the leading-order action. The same conclusion follows if $\lambda_i(\phi) \ll \lambda(\phi)$ and $\bar{\lambda}_i(\phi) \ll \bar{\lambda}(\phi)$. In the opposite situation $\lambda_i(\phi) \gg \lambda(\phi)$ and $\bar{\lambda}_i(\phi) \gg \bar{\lambda}(\phi)$ the hyperelectric and the hypermagnetic susceptibilities may evolve at different rates.

Let us now make few concrete examples by first assuming the case of generic inflationary models and by positing, for the sake of simplicity, that all the $\lambda_i$ and $\bar{\lambda}_i$ are proportional to $\lambda$ and $\bar{\lambda}$ through some numerical constants of order 1. In this case the coefficients of Tab. 1 become, in practice, $\phi$-independent and the leading-order expressions of the susceptibilities, as established above, is obtained by setting $M \sim M_P$ and $\epsilon \ll 1$:

$$\chi_X = \sqrt{\frac{\lambda}{4\pi}} \sqrt{1 + \alpha_X (\frac{H}{M_P})^2}, \quad \bar{\chi} = \sqrt{\frac{\bar{\lambda}}{4\pi}} \sqrt{1 + \bar{\alpha} (\frac{H}{M_P})^2},$$

where $X = E$, $B$ so that $\alpha_X$ and $\bar{\alpha}$ do not depend on $\phi$. As a second illustrative example we shall consider the more general version of the vertex considered in Ref. [5] where a term $f(\phi)R_{\mu\alpha\nu\beta}Y^{\mu\alpha} Y^{\nu\beta}$ has been considered in the context of polarized backgrounds of relic gravitons. This term corresponds to $\bar{\lambda}_{10}$ in Eq. (4) however, as we saw above, it does not make much sense to consider only $\bar{\lambda}_{10}$ for magnetogenesis considerations since $\bar{\lambda}_8$ a and $\bar{\lambda}_9$ give exactly the same kind of contribution. For this reason as a further less generic model we could consider the case

$$\chi_E = \chi_B = \sqrt{\frac{\lambda}{4\pi}}, \quad \bar{\chi} = \sqrt{\frac{\bar{\lambda}}{4\pi}} \sqrt{1 + q (2 + \epsilon) (\frac{H}{M_P})^2},$$

where $q = O(1)$ is just a numerical constant since we have assumed, for the sake of simplicity, that all the $\lambda_i$ vanish while $\bar{\lambda}_8 = \bar{\lambda}_9 = \bar{\lambda}_{10} = \bar{\lambda} = \bar{\lambda}$. The various $\lambda_i$ and $\bar{\lambda}_i$ might also be much larger than $\lambda$ and $\bar{\lambda}$ and perhaps dominate the expressions of the susceptibilities. From the viewpoint of the underlying inflationary model it could also happen that the inflaton has some particular symmetry (like a shift symmetry $\varphi \rightarrow \varphi + e$) or that the rate of inflaton roll defined by $\eta$ remains constant (and possibly larger than 1), as it happens in certain fast-roll scenarios \cite{20} (see also, for instance, \cite{21,22}). In all these cases $\chi_E$ and $\chi_B$ may have rather different evolution and can be generically parametrized, in conformal time, as

$$\chi_E = \left(-\frac{\tau}{\tau_1}\right)^{\gamma_E}, \quad \chi_B = \left(-\frac{\tau}{\tau_1}\right)^{\gamma_B}.$$  

The non-generic classes of scenarios suggested by the present considerations can be multiplied and so far they not have been specifically analyzed.

While examples of Eqs. (14), (15) and (16) are only illustrative what matters, for the present ends, is that the general problem can be treated by adopting a new time parametrization and a consequent\

\footnote{We are here considering that since $\eta = \epsilon - \tau$ (with $\tau = M_P^2 (V_{\varphi\varphi}/V)/2$, and since the scalar spectral index is $n_s = 1 - 6\epsilon + 2\tau$, $\epsilon$ and $\eta$ can be approximately of the same order of magnitude.}
redefinition of the susceptibilities, namely:
\[ \tau \to s = s(\tau), \quad d\tau = n(s) \, ds, \quad n^2 = \chi_E^2/\chi_B^2, \quad \chi = \sqrt{\chi_E \chi_B}. \]  
(17)

It is relatively straightforward to rearrange Eqs. [9]–[10] in the \( s \)-parametrization of Eq. [17], but probably the simplest way to discuss the problem without unnecessary details is to appreciate that in terms of \( \chi \) and \( n \) the comoving fields are given by \( \vec{B} = \nabla \times \vec{A}/\sqrt{n} \) and by \( \vec{E} = - (\chi/\sqrt{n}) \partial_s (\vec{A}/\chi) \) where \( \vec{A} \) is the comoving vector potential defined in the Coulomb gauge [23] which is invariant under conformal rescaling. If the latter expressions are inserted into Eq. (5) the full action takes the following simple form:
\[ S_{\text{gauge}} = \frac{1}{2} \int d^3x \int ds \left[ \dot{A}_a^2 + \left( \frac{\dot{\chi}}{\chi} \right)^2 A_a^2 - 2 \left( \frac{\dot{\chi}}{\chi} \right) \dot{A}_a \dot{A}_a - C(s) A_a \partial_s A_m \partial^a b^m \right], \]
(18)

where the overdots now denote a derivation\(^7\) with respect to the new time coordinate \( s \). The canonical Hamiltonian associated with Eq. (18) easily follows. The classical fields and the conjugate momenta can then be promoted to the status of quantum operators so that the mode expansion of the hyperelectric and hypermagnetic fields in the circular basis turns out to be:
\[ \hat{E}_i(\vec{x},s) = -i \int \frac{d^3k}{(2\pi)^{3/2} \sqrt{n(s)}} \sum_{a=\pm,} \left[ g_{k,\alpha}(s) \hat{\alpha}_{k,\alpha} \hat{\epsilon}_i^*(\vec{k}) e^{-i\vec{k} \cdot \vec{x}} + g_{k,\alpha}^*(s) \hat{\alpha}_{k,\alpha}^\dagger \hat{\epsilon}_i(\vec{k}) e^{i\vec{k} \cdot \vec{x}} \right], \]
(19)
\[ \hat{B}_k(\vec{x},s) = -i \int \frac{\epsilon_{ij}k_i d^3k}{(2\pi)^{3/2} \sqrt{n(s)}} \sum_{a=\pm,} \left[ f_{k,\alpha}(s) \hat{\alpha}_{k,\alpha} \hat{e}^*_j(\vec{k}) e^{-i\vec{k} \cdot \vec{x}} + f_{k,\alpha}^*(s) \hat{\alpha}_{k,\alpha}^\dagger \hat{e}_j(\vec{k}) e^{i\vec{k} \cdot \vec{x}} \right], \]
(20)

where \( \hat{\epsilon}^{(\pm)}(\vec{k}) \) denote the two complex polarization obeying \( \hat{\vec{k}} \times \hat{\epsilon}^{(\pm)} = \mp i \hat{\epsilon}^{(\pm)} \); the creation and annihilation operators are directly defined in the circular basis and they obey the standard commutation relation \( [\hat{\alpha}_{k,\alpha}, \hat{\alpha}_{p,\beta}] = \delta^{(3)}(\vec{k} - \vec{p}) \delta_{\alpha\beta} \). The mode functions appearing in Eqs. (19) and (20) obey:
\[ \hat{f}_{k,\pm} = g_{k,\pm} + \left( \frac{\dot{\chi}}{\chi} \right) f_{k,\pm}, \quad \hat{g}_{k,\pm} = -k^2 f_{k,\pm} - \left( \frac{\dot{\chi}}{\chi} \right) g_{k,\pm} \pm C(s) k f_{k,\pm}. \]
(21)

From Eqs. (19) and (20) the two-point functions in Fourier space become:
\[ \langle \hat{E}_i(\vec{k},s) \hat{E}_j(\vec{p},s) \rangle = \frac{2\pi^2}{k^3} \left[ P_E(k,s) p_{ij}(\vec{k}) + P_E^{(G)}(k,s) i \epsilon_{ij\ell} \vec{k}^\ell \right] \delta^{(3)}(\vec{p} + \vec{k}), \]
(22)
\[ \langle \hat{B}_i(\vec{k},s) \hat{B}_j(\vec{p},s) \rangle = \frac{2\pi^2}{k^3} \left[ P_B(k,s) p_{ij}(\vec{k}) + P_B^{(G)}(k,s) i \epsilon_{ij\ell} \vec{k}^\ell \right] \delta^{(3)}(\vec{p} + \vec{k}), \]
(23)

where \( p_{ij} = \delta_{ij} - \vec{k}_i \vec{k}_j \) is the usual divergenceless projector. In Eqs. (22)–(23) \( P_E(k,s) \) and \( P_B(k,s) \) denote the hyperelectric and the hypermagnetic power spectra while \( P_E^{(G)}(k,s) \) and \( P_B^{(G)}(k,s) \) are the corresponding gyrorotropic contributions:
\[ P_E(k,s) = \frac{k^3}{4\pi^2 n(s)} \left[ |g_{k,-}|^2 + |g_{k,+}|^2 \right], \quad P_B(k,s) = \frac{k^5}{4\pi^2 n} \left[ |f_{k,-}|^2 + |f_{k,+}|^2 \right], \]
(24)
\[ P_E^{(G)}(k,s) = \frac{k^3}{4\pi^2 n(s)} \left[ |g_{k,-}|^2 - |g_{k,+}|^2 \right], \quad P_B^{(G)}(k,s) = \frac{k^5}{4\pi^2 n(s)} \left[ |f_{k,-}|^2 - |f_{k,+}|^2 \right]. \]
(25)

\(^7\) The derivatives with respect to \( s \) and the derivations with respect to the cosmic time coordinate \( t \) never appear in the same context and, for this reason, we kept the overdot in the definitions of the slow-roll parameters (e.g. \( \epsilon = -\dot{H}/H^2 \)).

\(^8\) It is easy to show that Eqs. (21) also imply that \( f_{k,\pm} + |k^2 - \dot{\chi}/\chi| f_{k,\pm} \mp C(s) k f_{k,\pm} = 0 \) while the first of the two equations becomes a definition of \( g_{k,\pm} \), i.e. \( g_{k,\pm} = f_{k,\pm} - (\dot{\chi}/\chi) f_{k,\pm} \).
The results of Eqs. (17)–(18) lead directly to Eqs. (24)–(25) and are very convenient for estimating the magnitude of the corrections induced on the power spectra. To further illustrate the considerations developed so far we shall therefore analyze more specifically the cases of Eqs. (14), (15) and (16).

We are now going to discuss some explicit solutions in those examples that we regard as more generic from the viewpoint of the theory. Inserting Eq. (14) into Eq. (17) we have that the effect of the generic corrections on $n$ and $\chi$ is $\mathcal{O}(H^2/M_P^2)$:

$$n = 1 + \frac{\alpha_E - \alpha_B}{2} \left( \frac{H}{M_P} \right)^2, \quad \chi = \sqrt{\frac{\lambda}{4\pi}} \left[ 1 + \frac{\alpha_E + \alpha_B}{2} \left( \frac{H}{M_P} \right)^2 \right].$$

(26)

We remind that we are here considering the situation where $\epsilon \ll 1$; this means that the corrections in Eq. (26) will be typically smaller than $\mathcal{O}(10^{-10})$. To deduce the correction on the power spectrum it is sufficient to solve Eq. (21) directly in the $s$-parametrization; setting for simplicity $C(s) = 0$ in the action Eq. (18) the WKB solution of Eq. (21) is given by:

$$f_k(s) = \frac{\chi(s)}{\chi_{ex}} \left\{ f_k(s_{ex}) + \left[ f_k(s_{ex}) - f_{ex}f_k(s_{ex}) \right] \int_{s_{ex}}^{s} \frac{\lambda_{ex}^2}{\xi^2(s_1)} ds_1 \right\},$$

$$g_k(s) = \frac{\chi_{ex}}{\chi(s)} \left\{ g_k(\tau_{ex}) + \left[ g_k(\tau_{ex}) + f_{ex}g_k(s_{ex}) \right] \int_{s_{ex}}^{s} \frac{\lambda^2(s_1)}{\xi^2} ds_1 \right\},$$

(27)

where $f_{ex} = (\dot{\chi}/\chi)_{ex}$. Since the leading terms in Eq. (27) are given by $(\chi(s)/\chi_{ex}) f_k(s_{ex})$ and by $(\chi_{ex}/\chi(s)) g_k(s_{ex})$, for typical wavelengths larger than the Hubble radius during inflation Eq. (26) implies that $P_B(k, s) = P_B(k, \tau_1)[1 + \mathcal{O}(H^2/M_P^2)]$ where $P_B(k, \tau_1)$ is the spectrum obtained when $\alpha_B = \alpha_E = 0$. A conclusion similar to the one of Eqs. (26)–(27) follows after inserting Eq. (15) into Eq. (21):

$$\ddot{f}_{k, \pm} + \left\{ k^2 - \frac{\sqrt{\lambda}}{\sqrt{s}} \mp k \left[ \frac{\dot{\lambda}}{\lambda} - 2q\epsilon \left( \frac{H}{M_P} \right)^2 \mathcal{H} \right] \right\} f_{k, \pm} = 0.$$  

(28)

This equation is nothing but the standard equation for the Whittaker’s functions [21]. Indeed by rescaling the coordinates as $z = 2 \pm i k$, Eq. (28) becomes:

$$\frac{d^2 f_{k, \pm}}{dz^2} + \left\{ \frac{1}{4} - \frac{\mu^2 - 1/4}{z^2} + \zeta \left[ 1 + 2q\epsilon^2 A_R/2\mu + 1 \right] \right\} f_{k, \pm} = 0,$$

(29)

where $\zeta = i(\mu + 1/2)$. In Eq. (29) we assumed a power-law dependence for $\sqrt{s}$ but the relevant aspect concerns the comparison of the two terms in the squared bracket. Since $\epsilon \ll 1$ (and typically $\mathcal{O}(10^{-3})$) the second term is completely negligible with respect to $1$: a simple estimate implies that $\epsilon^2 A_R < 10^{-15}$. Note, in this respect, that the effect goes as $\epsilon^2$ since the first $\epsilon$ comes from the derivative of $H^2$ while the second one follows from the numerical value of $H$ in terms of $A_R$.

Let us finally come to Eq. (16) which is interesting since it can be realized in the context of some non-generic models of inflation and anyway when the $\lambda_i(\phi) \gg \mathcal{O}(\lambda)$. As before, inserting Eq. (16) into Eq. (17) we obtain, after simple algebra,

$$\left( -\frac{\tau}{\tau_1} \right)^{\gamma_B - \gamma_E + 1} = \left( -\frac{s}{s_1} \right), \quad n(s) = \left( -\frac{s}{s_1} \right)^{\gamma_B - \frac{\gamma_E - \gamma_B}{(\gamma_B - \gamma_E + 1)}}, \quad \chi(s) = \chi_1 \left( -\frac{s}{s_1} \right)^{\frac{\gamma_E - \gamma_B}{(\gamma_B - \gamma_E + 1)}},$$

(30)

where $s_1 = \tau_1/(1 - \gamma_E + \gamma_B)$. In what follows we shall assume $\gamma_B > 0$ and $\gamma_E > 0$. The solution of the evolution for the mode functions during the inflationary stage follows from Eq. (21) and it can be directly obtained in the $s$-parametrization:

$$f_k(s) = \frac{N_A}{\sqrt{2k}} \sqrt{-k} s H_p^{(1)}(-k s), \quad g_k(s) = -N_A \sqrt{\frac{k}{2}} \sqrt{-k} s H_p^{(1)}(-k s)$$

(31)

Note, as a side remark, that in this particular example $s$ and $\tau$ coincide exactly.
where $|N_\mu| = |N_\nu| = \sqrt{\pi/2}$ while $\mu = |(2\gamma_E - 1)/[2(\gamma_B - \gamma_E + 1)]|$ and $\nu = (2\gamma_B + 1)/[2(\gamma_B - \gamma_E + 1)]$ (note the absolute value in the expression of $\mu$). From Eqs. (22) and (31) the inflationary power spectra easily follow and they are

$$P_B(k, s, \tau) = \frac{a^4H^4}{n(s)}\left(\frac{\tau}{s}\right)^4 D(\mu)(-k s)^{5-2\mu}, \quad P_E(k, s, \tau) = \frac{a^4H^4}{n(s)}\left(\frac{\tau}{s}\right)^4 D(\nu)(-k s)^{5-2\nu},$$

where, for a generic argument $z$, $D(z) = 2^{2z-3}\Gamma^2(z)/\pi^3$. Note that the two power spectra can be usefully viewed, for practical purposes, as functions of $\tau$ and $s$; depending on the specific necessity Eq. (30) will be used to eliminate one of the two time variables. Various considerations restrain the variability of $\gamma_E$ and $\gamma_B$; for instance to have $\tau^4/[n(s) s^4] < 1$ throughout the whole inflationary stage we must require $\gamma_E < \gamma_B$; to have $\chi$ increasing during inflation we must demand $\gamma_E < \gamma_B + 1$; finally to avoid that the electric and magnetic fields will be overcritical during inflation we must have $\gamma_E < (3\gamma_B/4)/5$. Taking into account

Figure 1: The common logarithm of the power spectrum is illustrated in the plane $(\gamma_E, \gamma_B)$ for $r_T = 0.01$ and for two different scales i.e. $k = 1$ Mpc$^{-1}$ (plot at the left) and $k = 0.01$ Mpc$^{-1}$ (plot at the right). All the relevant constraints the physical magnetic fields after inflation can be explicitly evaluated when the relevant scales reenter the Hubble radius, i.e. for $r_k \simeq 1/k$ where $k$ is of the order of the Mpc$^{-1}$ which is the typical scale for magnetogenesis considerations:

$$P_B^{(phys)}(k, \tau_k) \simeq H_k^4\left(\frac{a_1}{a_k}\right)^4 \mathcal{M}(\gamma_E, \gamma_B) \left(\frac{k}{a_1 H_1}\right)^{\alpha(\gamma_E, \gamma_B)}, \quad \alpha(\gamma_E, \gamma_B) = (4-5\gamma_E+3\gamma_B)/(1-\gamma_E+\gamma_B)$$

where $\mathcal{M}(\gamma_E, \gamma_B)$ is a numerical factor that varies between 0.2 and $5 \times 10^{-3}$ when $0 < \gamma_B < 4$ and $\gamma_E < (3\gamma_B + 4)/5$. In Fig. 1 the upper right corner is in fact excluded by the critical density constraint; in the remaining parts of the plots we illustrated the common logarithm of $\sqrt{P_B^{(phys)}(k, \tau_k)}$ expressed in nG. To achieve a successful magnetogenesis the least demanding requirement (i.e. $\sqrt{P_B^{(phys)}} \geq 10^{-16}$ nG) follows by assuming that, after compressional amplification, every rotation of the galaxy increases the initial magnetic field of one $e$-fold. According to some this requirement is not completely reasonable since it takes more than one $e$-fold to increase the value of the magnetic field by one order of magnitude and this is the rationale for the most demanding condition i.e. $\sqrt{P_B^{(phys)}} \geq 10^{-11}$ nG. In Fig. 2 the shaded areas denote the region where the spectral energy density is subcritical both during and after inflation while the magnetogenesis and the Cosmic Microwave Background constraints are all satisfied. In a conservative
perspective we required that the physical power spectrum after equality (but before decoupling) is smaller than $10^{-2}$ nG for typical wavenumbers comparable with the pivot scale $k_p = 0.002 \text{ Mpc}^{-1}$ at which the scalar and tensor power spectra are customarily assigned (see e.g. \cite{25}). In the two plots of Fig. 2, always in a conservative perspective, we required $\sqrt{P_B^{\text{phys}}} \geq 10^{-11}$ nG. It is finally interesting to remark that the allowed region of Fig. 2 naturally selects models that satisfy the approximate condition $\gamma_E < \gamma_B$ implying that $n > 1$.

Let us conclude with some comments and caveats on the overall logic of the present analysis. Since a generic spectator field $\psi$ may replace the inflaton and lead to plausible magnetogenesis scenarios (see \cite{25} and references therein), the considerations developed here also apply when the various $\lambda_i$ and $\tilde{\lambda}_i$ are $\psi$-dependent quantities. The contributions with four derivatives remain essentially the same but must be considered in conjunction with the supplementary restrictions associated with the different physical nature of the spectator fields. Furthermore if the couplings depend simultaneously on the inflaton $\phi$ and on $\psi$ further terms (containing the gradients of $\psi$) will have to be included in the effective Lagrangian.

The results of these additions will not crucially modify the general structure of the gauge action that will be always diagonalized in the $s$-time parametrization. There will certainly be some physical differences since the spectator fields induce entropic fluctuations that may also affect the consistency relations as well as other aspects of the CMB initial conditions \cite{25}; both themes are beyond the present discussion but have been analyzed in the past with fairly general conclusions (see e.g. \cite{26}). We also remark that for $\epsilon \ll 1$ and $M \simeq M_P$ the effect of the parity violating contributions coming from the corrections is always subleading and this result generalises the earlier discussions of Ref. \cite{8} where the coupling to the inflaton and the parity-violating terms of the type \cite{5} have been neglected altogether. If the higher-order corrections are non-generic either because of some symmetry of the inflaton or because of specific dynamical assumptions (like in the case of fast-roll models \cite{20, 21, 22}), it is plausible that the hyperelectric and the hypermagnetic susceptibilities will evolve at a different rate. All in all the systematic approach discussed here can be productively applied to slightly different situations while the overall logic and the general results will remain unchanged.

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Figure 2: The allowed region of the parameter space is illustrated with a shaded area for fixed $k$ and in the $(\gamma_E, \gamma_B)$ plane. The shaded regions correspond to the are where all the phenomenological requirements are satisfied.

\[\log \sqrt{P_B^{\text{phys}}} \text{ nG} \{n_r = 0.01, k = 1 \text{ Mpc}^{-1}\}\]
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