Semicausal operations are semilocalizable

T. Eggeling*, D. Schlingemann† and R.F. Werner‡
Institut für Mathematische Physik, TU Braunschweig,
Mendelssohnstr.3, 38106 Braunschweig, Germany.

We prove a conjecture by DiVincenzo, which in the terminology of Preskill et al. [quant-ph/0102043] states that “semicausal operations are semilocalizable”. That is, we show that any operation on the combined system of Alice and Bob, which does not allow Bob to send messages to Alice, can be represented as an operation by Alice, transmitting a quantum particle to Bob, and a local operation by Bob. The proof is based on the uniqueness of the Stinespring representation for a completely positive map. We sketch some of the problems in transferring these concepts to the context of relativistic quantum field theory.

I. INTRODUCTION

In a recent paper [1] Preskill et al. focus on the constraints that quantum operations must fulfill in order to be compatible with relativistic quantum theory. They introduce the notions of causal, localizable, semicausal and semilocalizable quantum operations for bipartite systems. The prefix *semi* refers to “directed” properties, which are not invariant under exchanging Alice’s and Bob’s system.

An operation on a bipartite system is called *semilocalizable* for Alice, if it can be decomposed into two local operations with one way quantum communication from Alice to Bob as illustrated by Figure 1: First Alice performs a local operation $G$ on her system and sends quantum information (via C) to Bob. Then Bob performs a local operation $F$ on his system, depending on the information he got from Alice.

An obvious consequence of this setup is that Bob cannot send messages to Alice: the only possible carrier of information (quantum or classical) is the system C, which goes from Alice to Bob. Operations with this property will be called “semicausal”. In other words, if we consider only measurements on Alice’s output of a semicausal operation, expectations do not depend on Bob’s initial preparation. That is, as far as Alice is concerned, the device can be described by an operation on her system alone.

The main result of this paper is the proof of the converse of the above remark: if a device allows no signalling from Bob to Alice (semicausality) we can represent it explicitly as a device involving possibly a particle sent from Alice to Bob but none in the other direction.

Causality and localizability are defined as the analogous symmetric properties. From this it would seem that these are also equivalent (just use the proof twice, with an exchange of the roles of Alice and Bob). However, full localizability is defined to be stronger than the two semilocalizabilities: the latter would mean only that there are two representations, each involving only one-way communication, whereas localizability means the absence of all communication. Indeed, in [1] an example for a causal operation which is not localizable has been given.

Thus all proven implications between causal, semicausal, localizable and semilocalizable operations (including the results of this paper) can be visualized in the following diagram:

\[
\text{localizable} \implies \text{causal} \downarrow \downarrow
\]

\[
\text{semilocalizable} \iff \text{semicausal}
\]

(1)

As in [1], the concepts introduced so far only require the standard setup of quantum information theory, in which “localization” is phrased entirely in terms of the Hilbert space tensor product of Alice’s and Bob’s respective Hilbert space. This kind of localization is a priori unrelated to relativistic locality, and it turns out that for building a fully relativistic quantum theory it is too narrow. Detailed knowledge about the relativistic localization structures has been accumulated during the last three decades in a research programme known variously as “algebraic quantum field theory” or “local quantum physics” [2–4]. Quantum information theoretical aspects have also been studied within this framework (See e.g. [5–7] and references cited therein). Technically, the main change is related to the fact that one has to deal with infinitely many degrees of freedom, and occurs in a similar
way when discussing statistical mechanics in the thermodynamic limit: the observable algebra of a local subsystem can no longer be taken as the operators of the form $A \otimes 1$ with respect to some tensor product decomposition of the underlying Hilbert space. Instead, more general von Neumann algebras must appear as observable algebras. We will briefly comment on the changes this introduces in the concepts of localized operations in the last section.

II. DEFINITION OF LOCALIZATION PROPERTIES

As is well known, physical operations can be described either in the Schrödinger picture, by a map acting on states or density matrices or, equivalently, in the Heisenberg picture, by an operator acting on observables. In spite of this equivalence, however, some things and especially localization properties and the Stinespring dilation are stated more easily in the Heisenberg picture. Therefore we will work in the Heisenberg picture.

We recall some basic notions and notations. In the Heisenberg picture an operation $\mathcal{E}$ taking systems with Hilbert space $\mathcal{H}_\text{in}$ to systems with Hilbert space $\mathcal{H}_\text{out}$ is a completely positive operator $E : \mathcal{B}(\mathcal{H}_\text{out}) \to \mathcal{B}(\mathcal{H}_\text{in})$ satisfying $E(1) \leq 1$. The corresponding map $E_\star$ in the Schrödinger picture acts on a density matrix $\rho$ on $\mathcal{H}_\text{in}$ such that

$$\text{tr}(E_\star(\rho) A) = \text{tr}(\rho E(A)) \quad \forall A \in \mathcal{B}(\mathcal{H}_\text{out}). \quad (2)$$

The conditions on $E$ are equivalent to $E_\star$ being likewise completely positive, and satisfying the normalization condition $\text{tr}(E_\star(\rho)) \leq 1$ for all density operators $\rho$. If $E_\star$ is even trace preserving (equivalently: $E(1) = 1$), we call the operation non-selective, or a channel [11]. As the name suggests, selective operations typically occur, when part of the operation is a selection of a sub-ensemble according to measuring results obtained by the device.

The four properties in diagram [10] all refer to an operation $E$ with $\mathcal{H}_\text{in} = \mathcal{H}_\text{out} = \mathcal{H}_{AB} \equiv \mathcal{H}_A \otimes \mathcal{H}_B$.

Definition: A completely positive map (not necessarily a channel) $E : \mathcal{B}(\mathcal{H}_{AB}) \to \mathcal{B}(\mathcal{H}_{AB})$ is called

1. semicausal if it can be written as

$$E(a \otimes 1_B) = T(a) \otimes 1_B \quad (3)$$

for all $a \in \mathcal{B}(\mathcal{H}_A)$ and for some completely positive map $T : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_A)$.

2. causal if it semicausal in both directions, i.e. if

$$E(a \otimes 1_B) = T(a) \otimes 1_B \quad \text{and} \quad (4)$$

$$E(1_A \otimes b) = 1_A \otimes T'(b) \quad (5)$$

for all $a \in \mathcal{B}(\mathcal{H}_A)$, $b \in \mathcal{B}(\mathcal{H}_B)$ and for some completely positive maps $T : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_A)$ and $T' : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_B)$.

3. localizable if it can be decomposed into

$$E = G \otimes F \quad (6)$$

where $F : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_B)$ is a channel and $G : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_A)$ a completely positive map.

4. semilocalizable if it can be decomposed into

$$E = (G \otimes \text{id}_B) \circ (\text{id}_A \otimes F) \quad (7)$$

where $F : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_B)$ is a channel, $G : \mathcal{B}(\mathcal{H}_{AC}) \to \mathcal{B}(\mathcal{H}_A)$ a completely positive map and $\mathcal{H}_C$ a finite dimensional Hilbert space.

III. SEMICAUSAL OPERATIONS ARE SEMILOCAL

We are now prepared to give the proof of the conjecture by DiVincenzo, of which the special case of complete measurements was treated in [11].

Theorem: A completely positive map (not necessarily a channel) is semilocal if and only if it is semicausal.

Before going into the proof, we point out some salient facts about the Stinespring representation of a completely positive map [11].

a. The Stinespring representation: The Stinespring representation theorem, as adapted to maps between finite dimensional quantum systems, states that any completely positive map $E : \mathcal{B}(\mathcal{H}_\text{out}) \to \mathcal{B}(\mathcal{H}_\text{in})$ can be written as

$$E(a) = V^* (a \otimes 1_K) V \quad (8)$$

with a linear operator $V : \mathcal{H}_\text{in} \to \mathcal{H}_\text{out} \otimes \mathcal{K}$, where $\mathcal{K}$ is a finite dimensional Hilbert space, called the dilation space. The representation (8) is called minimal if (and only if) the set of vectors

$$(a \otimes 1_K) V \varphi \quad (9)$$

with $a \in \mathcal{B}(\mathcal{H}_\text{out})$ and $\varphi \in H_\text{in}$ spans $H_\text{out} \otimes \mathcal{K}$.

b. Uniqueness: The main step of the proof will be to get a factorization of the given operation into two operations with suitable localization properties. It turns out that such a factorization is provided precisely by the uniqueness statement for the Stinespring dilation. We therefore explain this uniqueness in more detail.

Suppose that $E$ has a minimal Stinespring representation (8) as well as a further (not necessarily minimal) one
E(a) = V_1^*(a \otimes 1_{K_1})V_1 \quad (10)

with another linear map V_1: \mathcal{H} \to \mathcal{H} \otimes \mathcal{K}_1. Since representation (\ref{eq:V1}) is taken to be minimal, we conclude \dim(K) \leq \dim(K_1) and the prescription

\hat{U}(a \otimes 1_{K_1})V\psi := (a \otimes 1_{K_1})V_1\psi \quad (11)

eyields a well defined isometry \hat{U}: \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}_1. This can be easily verified by observing that all scalar products between vectors such as the ones on the right hand side of (11) are fixed by the relation (\ref{eq:V1}). From this definition of \hat{U} we find that the intertwining relation

\hat{U}(a \otimes 1_{K_1}) = (a \otimes 1_{K_1})\hat{U} \quad (12)

holds for each a ∈ \mathcal{B}(\mathcal{H}). Hence \hat{U} must be decomposable into

\hat{U} = 1_{\mathcal{H}} \otimes U \quad (13)

with an isometry U: \mathcal{K} \to \mathcal{K}_1. If both representations (\ref{eq:V1}) and (13) are minimal the dimensions of the dilation spaces coincide and U must be a unitary operator. The minimal Stinespring representation is thus unique up to a unitary transformation.

e. Kraus operators: In most of the current literature the Stinespring dilation is used only in the form of a corollary, called the Kraus representation of a completely positive map. If we introduce an orthonormal basis (ε_1, ..., ε_k) of the dilation space \mathcal{K} and define the “Kraus operators” \(K_\alpha: \mathcal{H}_\text{in} \to \mathcal{H}_\text{out}\) by

\[ V\psi = \sum_\alpha (K_\alpha \psi) \otimes \varepsilon_\alpha \quad \forall \psi \in \mathcal{H}_\text{in}, \quad (14) \]

we can write the dilation formula (\ref{eq:V1}) as \(E(a) = \sum_{\alpha=1}^k K_\alpha^*aK_\alpha\) for all \(a \in \mathcal{B}(\mathcal{H}_\text{out})\). Of course, everything we do in this paper could be formulated in terms of Kraus operators. We found this less practical, however, because the above uniqueness statement becomes more involved: The choice of the basis ε_\alpha always introduces some arbitrariness, so even in the minimal case the collection of Kraus operators is only unique up to a unitary transformation acting on the index \alpha.

Proof of Theorem: As noted in the introduction the implication “semilocalizable⇒semilocal” is trivial. In the notation of Section (\ref{sec:proof}) it becomes

\[ E(a \otimes 1_B) = G(a \otimes 1_C) \otimes 1_B \]

so (\ref{eq:proof}) follows with \(T(a) := G(a \otimes 1_C)\).

For the reverse implication, let \(E: \mathcal{B}(\mathcal{H}_{AB}) \to \mathcal{B}(\mathcal{H}_{AB})\) be a semicausal operation, i.e. an operation fulfilling (\ref{eq:proof}) for some completely positive map \(T: \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_A)\). The Stinespring representation of \(E\) gives us a dilation space \(\mathcal{H}_D\) and a linear operator \(V: \mathcal{H}_{AB} \to \mathcal{H}_{ABD}, \mathcal{H}_{ABD} \equiv \mathcal{H}_{AB} \otimes \mathcal{H}_D\), such that

\[ E(a \otimes b) = V^*(a \otimes b \otimes 1_D)V \quad (15) \]

for each \(a \in \mathcal{B}(\mathcal{H}_A)\) and \(b \in \mathcal{B}(\mathcal{H}_B)\). Analogously we get a dilation space \(\mathcal{H}_C\) and a linear operator \(W: \mathcal{H}_A \to \mathcal{H}_{AC}\) from the minimal Stinespring representation of \(T:\)

\[ T(a) = W^*(a \otimes 1_C)W \quad (16) \]

for all \(a \in \mathcal{B}(\mathcal{H}_A).\) According to the relation (\ref{eq:proof}) we obtain

\[ V^*(a \otimes 1_{BD})V = (W^* \otimes 1_B)(a \otimes 1_C)(W \otimes 1_B) \quad (17) \]

for all \(a \in \mathcal{B}(\mathcal{H}_A)\) [2]. The uniqueness of the minimal Stinespring representation now implies the existence of an isometry \(U: \mathcal{H}_{CB} \to \mathcal{H}_{BD}\) (see Eq. (\ref{eq:proof})) such that

\[ (a \otimes 1_{BD})V = (1_A \otimes U)(a \otimes 1_C)(W \otimes 1_B) \quad (18) \]

and therefore

\[ V = (1_A \otimes U)(W \otimes 1_B). \quad (19) \]

From this we obtain a completely positive unital map \(F: \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_{CB})\) by taking

\[ F(b) := U^*(b \otimes 1_D)U \quad (20) \]

for every \(b \in \mathcal{B}(\mathcal{H}_B)\) and a completely positive (not necessarily unital) map \(G: \mathcal{B}(\mathcal{H}_C) \to \mathcal{B}(\mathcal{H}_A)\):

\[ G(a \otimes c) := W^*(a \otimes c)W \quad (21) \]

for every \(a \in \mathcal{B}(\mathcal{H}_A)\) and \(c \in \mathcal{B}(\mathcal{H}_C).\) Thus we conclude from (\ref{eq:proof}) and (\ref{eq:proof}) that the identity (\ref{eq:proof}) holds. In other words, \(E\) is semilocalizable.

IV. OUTLOOK

As already indicated in the introduction, quantum field theory requires a more general setup than the one used in this note and in (\ref{eq:proof}). Relativistic localization is then expressed by assigning to every spacetime region the algebra of observables, which can be measured in that region, typically a von Neumann algebra. Signal causality then means that whenever two regions are spacelike separated (no causal signals can be exchanged) the corresponding algebras commute elementwise. One might try replacing this by an assignment of “local Hilbert spaces” to spacetime regions, such that the union of spacelike separated regions corresponds to the Hilbert space tensor product. This is not possible, however, because spacetime regions can be split into finer and finer pieces, and this would create difficulties for the tensor products, especially when the overall Hilbert space is required to be

\[ \cdots \]
separable (to have a countable orthonormal basis), and relativistic invariance is imposed.

Surprisingly the von Neumann algebras of local regions are all isomorphic under mild axiomatic assumptions. More specifically they are all isomorphic to the unique hyperfinite type \( II_1 \)-factor (see [4] for an explanation of these terms, and the proof). Nevertheless the localization structure is far from trivial, and resides in the way these algebras are nested into each other. Already for the inclusion of two such algebras an amazing variety is possible.

As might be expected from the heuristic argument at the beginning of this section, the small distance localization structure in quantum field theory deviates dramatically from what would be expected from Hilbert space tensor products. Suppose Alice and Bob operate in spacelike separated regions, and \( \mathcal{A} \) and \( \mathcal{B} \) are the local von Neumann algebras assigned to these regions. Then when the regions are very close together, no physical product states exist, hence there are no separable states at all. In fact, all physical states violate the CHSH-Bell inequality maximally \([1]\).

On the other hand, if the regions are a finite distance apart, the so-called \textit{split property} holds, which is equivalent to the existence of many separable states, or to the possibility for Bob to prepare any state of his subsystem locally without disturbing Alice’s \([1]\). Algebraically this means that the von Neumann algebra generated by \( \mathcal{A} \) and \( \mathcal{B} \) is isomorphic to the von Neumann algebra tensor product \( \mathcal{A} \otimes \mathcal{B} \) \([5]\).

The notion of \textit{semicausality} is easy to express in this context: that \( E \) maps operators of the form \( a \otimes 1 \) to operators of the same form just means that Alice’s algebra is invariant in the sense that \( a \in \mathcal{A} \Rightarrow E(a) \in \mathcal{A} \), or briefly \( E(\mathcal{A}) \subset \mathcal{A} \).

It is not so clear what should be understood by \textit{semilocality}. The reason is that one has to decide what degree of independence should be postulated for the intermediate system \( C \), sent from Alice to Bob in Figure \([4]\). If we require the kind of independence valid for widely separated regions (split property), as suggested by the image of a system being “sent”, the Theorem is probably false. Although the Stinespring decomposition for semicausal maps is well understood, it is not clear what factorizations would be meaningful interpretations of Figure \([4]\).

\section*{ACKNOWLEDGEMENTS}

Funding by the European Union project EQUIP (contract IST-1999-11053) and financial support from the DFG (Bonn) are gratefully acknowledged.

[1] D. Beckman, D. Gottesman, M. A. Nielsen and John Preskill, quant-ph/0102043 (2001).
[2] H. Araki, Mathematical theory of quantum fields, Oxford University Press, (1999)
[3] H.-J. Borchers, Translation group and particle representations in quantum field theory, Lecture Notes in Physics (Springer Verlag, Berlin), 1996.
[4] R. Haag, Local quantum physics, Springer-Verlag, Berlin 1992. (A second edition was released in 1996.)
[5] R.F. Werner, Local preparability of states and the split property in quantum field theory, Lett. Math. Phys. 13 (1987) 325-329.
[6] S.J. Summers, R.F. Werner, Maximal violation of Bell’s inequalities for algebras of observables in tangent spacetime regions, Ann. Inst. H. Poincaré, A 49 (1988) 215-243.
[7] R. Clifton, H. Halvorson, Entanglement and open systems in algebraic quantum field theory, quant-ph/0001107.
[8] O. Bratteli and D.W. Robinson, Operator algebras and quantum statistical mechanics, two volumes, (Springer Verlag, New York) 1979 and 1981.
[9] E. B. Davies, Quantum Theory of Open Systems, Academic Press 1976.
[10] In \([6]\) the term \textit{superoperator} is used for (the dual map of) a \textit{channel}.
[11] W. F. Stinespring, Proc. Amer. Math. Soc. 6, 211-216 (1955).
[12] Note that if \( W \) is a minimal dilation of \( T \) then so is \( W \otimes 1_B \) for \( E \).
[13] Here “physical state” technically means a “normal” state, i.e., a state which can be represented by a density operator on the Hilbert space of the global system, as opposed to “singular” states. For the purposes of this discussion it is sufficient to require “local normality”, i.e., that the restriction to bounded spacetime regions can be represented in this way. Roughly speaking this requires that no infinite amount of energy needs to be invested in a finite region to prepare the state. This property allows also temperature states of finite density, which are globally different (singular) from the vacuum, but not locally.
[14] H. Baumgärtel and M. Wollenberg, Causal nets of operator algebras, Akademie Verlag Berlin, 1992.
[15] S. Sakai, \( C^* \)-algebras and \( W^* \)-algebras, Classics in Mathematics, Springer-Verlag Berlin Heidelberg New York, 1971.