Extreme Local Extrema of Two-Dimensional Discrete Gaussian Free Field

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Abstract: We consider the discrete Gaussian Free Field in a square box in $\mathbb{Z}^2$ of side length $N$ with zero boundary conditions and study the joint law of its properly-centered extreme values ($h$) and their scaled spatial positions ($x$) in the limit as $N \to \infty$. Restricting attention to extreme local maxima, i.e., the extreme points that are maximal in an $r_N$-neighborhood thereof, we prove that the associated process tends, whenever $r_N \to \infty$ and $r_N/N \to 0$, to a Poisson point process with intensity measure $Z(dx) e^{-\alpha h} dh$, where $\alpha := 2/\sqrt{g}$ with $g := 2/\pi$ and where $Z(dx)$ is a random Borel measure on $[0, 1]^2$.

In particular, this yields an integral representation of the law of the absolute maximum, similar to that found in the context of Branching Brownian Motion. We give evidence that the random measure $Z$ is a version of the derivative martingale associated with the continuum Gaussian Free Field.

1. Introduction

1.1. Main results. Consider a box $V_N := (0, N)^2 \cap \mathbb{Z}^2$ in the square lattice and let $G_N(x, y)$ denote the Green function of the simple symmetric random walk started from $x$ and killed upon exiting $V_N$. The two-dimensional Discrete Gaussian Free Field (DGFF) in $V_N$ is a collection of Gaussian random variables $\{h_x : x \in V_N\}$ with mean zero and covariance $\text{Cov}(h_x, h_y) := G_N(x, y)$. Another way to define the DGFF is by prescribing its full distribution; this is achieved by normalizing the measure

$$\exp\left\{-\frac{1}{8} \sum_{\langle x, y \rangle} (h_x - h_y)^2\right\} \prod_{x \in V_N} dh_x \prod_{x \in \partial V_N} \delta_0(dh_x).$$

(1.1)

Here the sum goes over unordered nearest-neighbor pairs with at least one vertex in $V_N$ and the product of Dirac deltas imposes a Dirichlet boundary condition on the outer
boundary $\partial V_N$ of $V_N$. By (1.1) the DGFF has the Gibbs–Markov property: Conditional on $\{h_z: z \neq x\}$, the field $h_x$ reduced by the average of $h_z$ over the nearest neighbors $z$ of $x$ has the law of a standard normal.

The aim of this paper is to study the statistics of extreme values of the DGFF in the limit $N \to \infty$. We will focus attention on large local maxima, i.e., those extreme points whose value dominates the configuration in an $r$-neighborhood thereof. Thus, for $r \geq 1$, let $A_r(x) := \{z \in Z^2: |z - x| \leq r\}$ and define a measure on $[0, 1]^2 \times \mathbb{R}$ by

$$\eta_{N,r}(A \times B) := \sum_{x \in V_N} 1_{\{x/N \in A\}} 1_{\{h_x - m_N \in B\}} 1_{\{h_x = \max_{z \in A_r(x)} h_z\}},$$

for Borel sets $A \subset [0, 1]^2$ and $B \subset \mathbb{R}$ and a suitable centering sequence $m_N$. A sample of $\eta_{N,r}$ is a Radon measure supported on a collection of points of the form $(x, h)$, where $x$ is the scaled location and $h$ is the reduced height of a large “peak” in the underlying field configuration.

To study distributional limits, we endow the space of point measures on $[0, 1]^2 \times \mathbb{R}$ with the topology of vague convergence. For the centering sequence $m_N$ we will take

$$m_N := 2\sqrt{g} \log N - \frac{3}{4} \sqrt{g} \log \log N,$$

where $g := 2/\pi$ links $m_N$ to the asymptotic growth of the Green function which for $x$ deep inside $V_N$ scales as $G_N(x, x) = g \log N + O(1)$. Anticipating Poisson limit laws, let us write $\text{PPP}(\lambda)$ for the Poisson point process on a Polish space $\Omega$ with sigma-finite intensity measure $\lambda$. We will use this notation even when $\lambda$ is itself random (i.e., when $\text{PPP}(\lambda)$ is a Cox process); the law of the points is then averaged over the law of $\lambda$. Our principal result is then:

**Theorem 1.1.** There is a random Borel measure $Z(dx)$ on $[0, 1]^2$ satisfying $Z([0, 1]^2) < \infty$ a.s. and $Z(A) > 0$ a.s. for any open set $A \subset [0, 1]^2$ such that for any $r_N$ with $r_N \to \infty$ and $r_N/N \to 0$,

$$\eta_{N,r_N} \xrightarrow{\text{law}}_{N \to \infty} \text{PPP}(Z(dx) \otimes e^{-\alpha h} dh),$$

where $\alpha := 2/\sqrt{g}$—which in present normalization reads $\alpha = \sqrt{2\pi}$.

As an immediate consequence, we get information about the joint law of the (a.s. unique) position and height of the absolute maximum:

**Corollary 1.2.** Let $\nu_N$ denote the law of $(N^{-1}\argmax h, \max_{x \in V_N} h_x - m_N)$ on $[0, 1]^2 \times \mathbb{R}$. For the random measure $Z(dx)$ from Theorem 1.1, define

$$\hat{Z}(A) := \frac{Z(A)}{Z([0, 1]^2)}.$$  

(1.5)

Then $\nu_N \xrightarrow{\text{law}} \nu$, where $\nu$ is for any Borel $A \subset [0, 1]^2$ given by

$$\nu(A \times (-\infty, t]) := E(\hat{Z}(A)e^{-\alpha^{-1}Ze^{-at}}), \quad t \in \mathbb{R},$$

(1.6)

and $Z := Z([0, 1]^2)$. 

