MONOIDAL CATEGORIFICATION OF CLUSTER ALGEBRAS II

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Abstract. We prove that the quantum unipotent coordinate algebra $A_q(n(w))$ associated with a symmetric Kac-Moody algebra and its Weyl group element $w$ has a monoidal categorification as a quantum cluster algebra. As an application of our earlier work, we achieve it by showing the existence of a quantum monoidal seed of $A_q(n(w))$ which admits the first-step mutations in all the directions. As a consequence, we solve the conjecture that any cluster monomial is a member of the upper global basis up to a power of $q^{1/2}$.

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INTRODUCTION

The quantum unipotent coordinate ring $A_q(n)$, which is isomorphic to the negative half $U_q^{-}(g)$ of a quantized enveloping algebra $U_q(g)$ associated with a symmetrizable Kac-Moody algebra $g$, has very interesting bases so called upper global basis and lower global basis ([11]). In particular, the upper global basis $B_{up}$ has been studied emphasizing on its multiplicative structure. For example, Berenstein and Zelevinsky ([1]) conjectured that, in the case $g$ is of type $A_n$, the product $b_1b_2$ of two elements $b_1$ and $b_2$ in $B_{up}$ is again an element of $B_{up}$ up to a multiple of a power of $q$ if and only if they are $q$-commuting; i.e., $b_1b_2 = q^mb_2b_1$ for some $m \in \mathbb{Z}$. This conjecture turned out to be not true in general, because Leclerc ([23]) found examples of an imaginary element $b \in B_{up}$ such that $b^2$ does not belong to $B_{up}$. Nevertheless, the idea of considering subsets of $B_{up}$ whose elements are $q$-commuting with each other and studying the relations between those subsets has survived and becomes one of the motivations of (quantum) cluster algebras ([4]).

A cluster algebra is a $\mathbb{Z}$-subalgebra of a rational function field given by a set of generators, called the cluster variables. They are grouped into overlapping subsets, called clusters and there is a procedure called mutation which produces new clusters successively from a given initial cluster. A quantum cluster algebra is a non-commutative $q$-deformation of a cluster algebra, and a quantum cluster is a family of mutually $q$-commuting elements of it ([2]). There are many examples of algebras which turned out to be (quantum) cluster algebras. In particular, Geiß, Leclerc and Schröer showed that the quantum unipotent coordinate algebra $A_q(n(w))$ has a quantum cluster algebra structure ([6]). Here $A_q(n(w))$ is a subalgebra of the $\mathbb{Q}(q)$-algebra $A_q(n) \simeq U_q^{-}(g)$, which is associated with a symmetric Kac-Moody algebra $g$ and its Weyl group element $w$.

It is shown by Kimura [18] that $B_{up} \cap A_q(n(w))$ is a basis of $A_q(n(w))$. Then, in terms of quantum cluster algebras, Berenstein-Zelevinsky’s ideas can be generalized and reformulated in the following form:
Conjecture 1 ([6, Conjecture 12.9], [18, Conjecture 1.1(2)]). When \( g \) is of symmetric type, every quantum cluster monomial in \( A_{q^{1/2}}(n(w)) := \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Q}(q)} A_q(n(w)) \) belongs to the upper global basis up to a power of \( q^{1/2} \).

There are some partial results of this conjecture. It is proved for \( g = A_2, A_3, A_4 \) and \( A_q(n(w)) = A_q(n) \) in [1] and [5, Section 12], for \( g = A_1^{(1)}, A_n \) and \( w \) is a square of a Coxeter element in [21] and [22], when \( g \) is symmetric and \( w \) is a square of a Coxeter element in [19].

In this paper, we prove the above conjecture completely by showing that there exists a monoidal categorification of \( A_{q^{1/2}}(n(w)) \) along the lines of our previous work [10]. Note that Nakajima proposed a geometric approach of this conjecture via quiver varieties ([27]).

Let us briefly recall the notion of monoidal categorifications of quantum cluster algebras. Let \( C \) be an abelian monoidal category equipped with an auto-equivalence \( q \) and a tensor product which is compatible with a decomposition \( C = \bigoplus_{\beta \in \Omega} C_\beta \). Fix a finite index set \( J = J_{\text{ex}} \sqcup J_{\text{fr}} \) with a decomposition into the exchangeable part and the frozen part. Let \( \mathcal{S} \) be a quadruple \( (\{ M_i \}_{i \in J}, L, \tilde{B}, D) \) of a family of simple objects \( \{ M_i \}_{i \in J} \) in \( C \), an integer-valued skew-symmetric \( J \times J \)-matrix \( L = (\lambda_{i,j}) \), an integer-valued \( J \times J_{\text{ex}} \)-matrix \( \tilde{B} = (b_{i,j}) \) with skew-symmetric principal part, and a family of elements \( D = \{ d_i \}_{i \in J} \) in \( \mathbb{Q} \). If those data satisfy the conditions in Definition 5.2 below, then we call it a quantum monoidal seed in \( C \). For each \( k \in J_{\text{ex}} \), we have mutations \( \mu_k(L), \mu_k(\tilde{B}) \) and \( \mu_k(D) \) of \( L, \tilde{B} \) and \( D \), respectively. We say that a quantum monoidal seed \( \mathcal{S} = (\{ M_i \}_{i \in J}, L, \tilde{B}, D) \) admits a mutation in direction \( k \in J_{\text{ex}} \), if there exists a simple object \( M'_k \in C_{\mu_k(D)k} \) which fits into two short exact sequences (0.1) below in \( C \) reflecting the mutation rule in quantum cluster algebras, and thus obtained quadruple \( \mu_k(\mathcal{S}) := (\{ M_i \}_{i \neq k} \cup \{ M'_k \}, \mu_k(L), \mu_k(\tilde{B}), \mu_k(D)) \) is again a quantum monoidal seed in \( C \). We call \( \mu_k(\mathcal{S}) \) the mutation of \( \mathcal{S} \) in direction \( k \in J_{\text{ex}} \).

Now the category \( C \) is called a monoidal categorification of a quantum cluster algebra \( A \) over \( \mathbb{Z}[q^{\pm 1/2}] \) if (i) the Grothendieck ring \( \mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} K(C) \) is isomorphic to \( A \), (ii) there exists a quantum monoidal seed \( \mathcal{S} = (\{ M_i \}_{i \in J}, L, \tilde{B}, D) \) in \( C \) such that \( [\mathcal{S}] := (\{ q^{m_i}[M_i] \}_{i \in J}, L, \tilde{B}) \) is a quantum seed of \( A \) for some \( m_i \in \frac{1}{2}\mathbb{Z} \), and (iii) \( \mathcal{S} \) admits successive mutations in all directions in \( J_{\text{ex}} \). Note that if \( C \) is a monoidal categorification of \( A \), all the quantum cluster monomials in \( A \) are the classes of simple objects in \( C \) up to a power of \( q^{1/2} \).

In the case of quantum unipotent coordinate ring \( A_q(n) \), there is a natural candidate for monoidal categorification, the category of finite-dimensional graded modules over a Khovanov-Lauda-Rouquier algebras ([16, 17], [28]). The Khovanov-Lauda-Rouquier algebras (abbreviated by KLR algebras) are a family of \( \mathbb{Z} \)-graded algebras \( \{ R(\beta) \}_{\beta \in \Omega^+} \) such that the Grothendieck ring of \( R\)-gmod := \( \bigoplus_{\beta \in \Omega^+} R(\beta) \)-gmod, the direct sum of the categories of finite-dimensional graded \( R(\beta) \)-modules, is isomorphic to the integral form \( A_q(n)_{\mathbb{Z}[q^{\pm 1}]} \) of \( A_q(n) \). Here \( \Omega^+ \) denotes the positive root lattice of the corresponding symmetrizable Kac-Moody algebra \( g \). The multiplication of \( K(R\text{-gmod}) \) is given by
the convolution product of modules, and the action of $q$ is given by the grading shift functor. In [30, 29], Varagnolo–Vasserot and Rouquier proved that the upper global basis $B^{up}$ of $A_q(n)$ corresponds to the set of the classes of all self-dual simple modules of $R$-gmod under the assumption that $R$ is a symmetric KLR algebra with the base field of characteristic 0. For each Weyl group element $w$, let $C_w$ be the full subcategory of $R$-gmod consisting of objects $M$ such that $[M]$ belongs to $A_q(n(w))$. Then, $C_w$ is an abelian monoidal category whose Grothendieck ring is isomorphic to $A_q(n(w))$ and its simple objects correspond to the basis $B^{up} \cap A_q(n(w))$ of $A_q(n(w))$. In particular, Conjecture 1 is equivalent to saying that when $g$ is of symmetric type, every quantum cluster monomial in $\mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1}]} K(C_w)$ belongs to the class of self-dual simple modules up to a power of $q^{1/2}$.

In [10], to simplify the conditions of quantum monoidal seeds and their mutations, we introduce the notion of admissible pairs in $C_w$. A pair $\{(M_i)_{i \in I}, \widetilde{B}\}$ is called admissible in $C_w$ if (i) $\{M_i\}_{i \in I}$ is a commuting family of self-dual real simple objects of $C_w$, (ii) $\widetilde{B}$ is an integer-valued $J \times J_{\text{ex}}$-matrix with skew-symmetric principal part, and (iii) for each $k \in J$, there exists a self-dual simple object $M'_k$ in $C_w$ such that $M'_k$ commutes with $M_i$ for all $i \in J \setminus \{k\}$ and there are exact sequences in $C_w$

$$0 \to q \bigcirc_{b_{i,k} > 0} M_i^{\otimes b_{i,k}} \to q^{\Lambda(M_k,M'_k)} M_k \otimes M'_k \to \bigcirc_{b_{i,k} < 0} M_i^{\otimes (-b_{i,k})} \to 0 \quad (0.1)$$

$$0 \to q \bigcirc_{b_{i,k} < 0} M_i^{\otimes (-b_{i,k})} \to q^{\Lambda(M'_k,M_k)} M'_k \otimes M_k \to \bigcirc_{b_{i,k} > 0} M_i^{\otimes b_{i,k}} \to 0$$

where $\Lambda(M_k,M'_k)$ and $\Lambda(M'_k,M_k)$ are prescribed integers and $\bigcirc$ is a tensor product up to a power of $q$.

For an admissible pair $\{(M_i)_{i \in I}, \widetilde{B}\}$, let $\Lambda = (\Lambda_{i,j})_{i,j \in I}$ be the skew-symmetric matrix where $\Lambda_{i,j}$ is the homogeneous degree of $r_{M_i,M_j}$, the $r$-matrix between $M_i$ and $M_j$, and let $D = \{d_i\}_{i \in J}$ be the family of elements in $\mathbb{Q}$ given by $M_i \in R(-d_i)$-gmod.

Then, together with the result of [6], the main theorem of [10] reads as follows:

**Theorem 2** (Theorem 6.3 and Corollary 6.4 in [10]). If there exists an admissible pair $(\{M_i\}_{i \in I}, \widetilde{B})$ in $C_w$ such that $\mathcal{S} := \{q^{-\langle\text{wt}(M_i),\text{wt}(M_i')\rangle/4}[M_i]_{i \in J}, -\Lambda, \widetilde{B}, D\}$ is an initial seed of $A_{q^{1/2}}(n(w))$, then $C_w$ is a monoidal categorification of $A_{q^{1/2}}(n(w))$.

This paper is mainly devoted to show that there exists an admissible pair in $C_w$ for every symmetric Kac-Moody algebra $g$ and its Weyl group element $w$. In [6], Geiß, Leclerc and Schröer provided an initial quantum seed in $A_q(n(w))$ whose quantum cluster variables are unipotent quantum minors. The unipotent quantum minors are elements in $A_q(n)$, which are a $q$-analogue of a generalization of the minors of upper triangular matrices. In particular, they are elements in $B^{up}$. We define the determinantal module $M(\mu, \zeta)$ to be the simple module in $R$-gmod corresponding to the unipotent quantum minor $D(\mu, \zeta)$ under the isomorphism $A_q(n(w))_{\mathbb{Z}[q^{\pm 1}]} \simeq K(R$-gmod). Here $(\mu, \zeta)$ is a pair of elements in the weight lattice of $g$ satisfying certain conditions.

Our main theorem is as follows.
Main Theorem 3. Let \( \{D(k,0)\}_{1 \leq k \leq r} \) be the initial quantum seed of \( A_q(n(w)) \) in [6] with respect to a reduced expression \( \tilde{w} = s_{i_1} \cdots s_{i_r} \) of \( w \). Let \( M(k,0) \) be the determinantal module corresponding to the unipotent quantum minor \( D(k,0) \). Then the pair
\[
(\{M(k,0)\}_{1 \leq k \leq r}, \widetilde{B})
\]
is admissible in \( C_w \).

The most essential condition for an admissible pair is that there exists the first mutation \( M(k,0)' \) in (0.1) for each \( k \in J_{ex} \). To obtain it, we investigate the properties of determinantal modules and those of their convolution products. Note that a unipotent quantum minor is the image of a global basis element of the quantum coordinate ring \( A_q(g) \) under the projection \( A_q(g) \to A_q(n) \). Since there exists a bicrystal embedding from the crystal basis \( B(A_q(g)) \) of \( A_q(g) \) to the crystal basis \( B(\tilde{U}_q(g)) \) of the modified quantum groups \( \tilde{U}_q(g) \), this investigation amounts to the study on the interplays among the crystal and global bases of \( A_q(g), \tilde{U}_q(g) \) and \( A_q(n) \).

This paper is organized as follows:

In Section 1, we review on quantum groups, their integrable modules, and crystal and global bases. In Section 2, we review the algebras \( A_q(g), \tilde{U}_q(g) \) and \( A_q(n) \), and study relations among them. In Section 3, we study the properties of quantum minors including T-systems and generalized T-systems. In Section 4, we review and study further about the modules over Khovanov-Lauda-Rouquier algebras along the lines of [9, 10]. In Section 5, we review the quantum cluster algebras and their monoidal categorifications by symmetric KLR algebras. In the last section, we establish our main theorem.

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1. Preliminaries

This is a continuation of [10], and we sometimes omit the definitions and notations employed there. We refer the reader to loc. cit. In this section, we briefly recall the crystal and global bases theory for \( U_q(g) \). We refer to [11, 12, 15] for materials in this section.

1.1. The quantum enveloping algebras and their integrable modules. Let \( g \) be a Kac-Moody algebra. We denote by \( I \) the index set which parametrizes the set of simple roots \( \Pi = \{\alpha_i \mid i \in I\} \) and the set of simple coroots \( \Pi' = \{h_i \mid i \in I\} \). We also denote by \( P \) the weight lattice, by \( P' := \text{Hom}_{\mathbb{Z}}(P, \mathbb{Z}) \) the co-weight lattice, by \( A = \langle \langle h_i, \alpha_j \rangle \rangle_{i,j \in I} \) the generalized Cartan matrix of \( g \), by \( W \) the Weyl group of \( g \), and by \( (\ , \ ) \) a \( W \)-invariant symmetric bilinear form on \( P \). The free abelian group \( Q := \bigoplus_{i \in I} \mathbb{Z} \alpha_i \).
is called the root lattice. Set $Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i \subset Q$ and $Q^- = \sum_{i \in I} \mathbb{Z}_{\leq 0} \alpha_i \subset Q$. For $\beta = \sum_{i \in I} m_i \alpha_i \in Q$, we set $|\beta| = \sum_{i \in I} |m_i|$.

Let $U_q(\mathfrak{g})$ be the quantum enveloping algebra of $\mathfrak{g}$, which is the $\mathbb{Q}(q)$-algebra generated by $e_i, f_i$ ($i \in I$) and $q^h$ ($h \in \mathbb{P}^+$) with certain defining relations (see e.g. [10, Definition 1.1]). We set $t_i = q^{(\frac{1}{2} \alpha_i \cdot \alpha_i)} h_i$. We denote by $U_q^-(\mathfrak{g})$ (resp. $U_q^+(\mathfrak{g})$) the subalgebra of $U_q(\mathfrak{g})$ generated by $f_i$ ($i \in I$) (resp. $e_i$ ($i \in I$)).

We define the divided powers by

$$
e^{(n)}_i = e_i^n / [n]_!, \quad f^{(n)}_i = f_i^n / [n]_! \quad (n \in \mathbb{Z}_{\geq 0}),$$

where $q_i = q^{(\alpha_i, \alpha_i)/2}$, $[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$ and $[n]_! = \prod_{k=1}^n [k]_i$ for $n \in \mathbb{Z}_{\geq 0}$, $i \in I$. Let us denote by $U_q(\mathfrak{g})_{Z[q^{\pm 1}]}$ the $\mathbb{Z}[q^{\pm 1}]$-subalgebra of $U_q(\mathfrak{g})$ generated by $e^{(n)}_i$, $f^{(n)}_i$ ($i \in I$, $n \in \mathbb{Z}_{\geq 0}$) and $q^h$ ($h \in \mathbb{P}^+$), by $U_q^-(\mathfrak{g})_{Z[q^{\pm 1}]}$ the $\mathbb{Z}[q^{\pm 1}]$-subalgebra of $U_q^-(\mathfrak{g})$ generated by $f^{(n)}_i$ ($i \in I$, $n \in \mathbb{Z}_{\geq 0}$), and by $U_q^+(\mathfrak{g})_{Z[q^{\pm 1}]}$ the $\mathbb{Z}[q^{\pm 1}]$-subalgebra of $U_q^+(\mathfrak{g})$ generated by $e^{(n)}_i$ ($i \in I$, $n \in \mathbb{Z}_{\geq 0}$).

We denote by $\mathcal{O}_{\text{int}}(\mathfrak{g})$ the category of integrable left $U_q(\mathfrak{g})$-modules $M$ satisfying the following conditions: (i) $M = \bigoplus_{\eta \in \mathbb{P}} M_{\eta}$ where $M_{\eta} := \{ m \in M \mid q^h m = q^{(h, \eta)} m \}$, (ii) dim $M_{\eta} < \infty$, and (iii) there exist finitely many weights $\lambda_1, \ldots, \lambda_m$ such that wt$(M) \subset \cup \lambda_j + \mathbb{Q}^-$. The category $\mathcal{O}_{\text{int}}(\mathfrak{g})$ is semisimple with its simple objects being isomorphic to the highest weight modules $V(\lambda)$ with highest weight vector $u_\lambda$ of highest weight $\lambda \in \mathbb{P}^+$, the set of dominant integral weight.

Let us recall the $\mathbb{Q}(q)$-anti-isomorphism $\varphi$ and $* \, \varphi$ of $U_q(\mathfrak{g})$ given as follows:

\begin{equation}
\varphi(e_i) = f_i, \quad \varphi(f_i) = e_i, \quad \varphi(q^h) = q^{-h},
\end{equation}

\begin{equation}
e^{\ast}_i = e_i, \quad f^{\ast}_i = f_i, \quad (q^{h})^{\ast} = q^{-h}.
\end{equation}

Let us recall also the $\mathbb{Q}$-antimorphism $\overline{\cdot}$ of $U_q(\mathfrak{g})$ given by

\begin{equation}
\overline{e}_i = e_i, \quad \overline{f}_i = f_i, \quad \overline{q}^h = q^{-h}, \quad \overline{q} = q^{-1}.
\end{equation}

For $M \in \mathcal{O}_{\text{int}}(\mathfrak{g})$, let us denote by $D_{\varphi} M$ the left $U_q(\mathfrak{g})$-module $\bigoplus_{\eta \in \mathbb{P}} \text{Hom}(M_{\eta}, \mathbb{Q}(q))$ with the action of $U_q(\mathfrak{g})$ given by:

$$(av)(m) = \psi(\varphi(a)m) \quad \text{for} \ \psi \in D_{\varphi} M, \ m \in M \ \text{and} \ a \in U_q(\mathfrak{g}).$$

Then $D_{\varphi} M$ belongs to $\mathcal{O}_{\text{int}}(\mathfrak{g})$.

For a left $U_q(\mathfrak{g})$-module $M$, we denote by $M^r$ the right $U_q(\mathfrak{g})$-module $\{ m^r \mid m \in M \}$ with the right action of $U_q(\mathfrak{g})$ given by

$$(m^r) x = (\varphi(x)m)^r \quad \text{for} \ m \in M \ \text{and} \ x \in U_q(\mathfrak{g}).$$

We denote by $\mathcal{O}^r_{\text{int}}(\mathfrak{g})$ the category of right integrable $U_q(\mathfrak{g})$-modules $M^r$ such that $M \in \mathcal{O}_{\text{int}}(\mathfrak{g})$.

There are two comultiplications $\Delta_+$ and $\Delta_-$ on $U_q(\mathfrak{g})$ defined as follows:

\begin{equation}
\Delta_+(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta_+(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \quad \Delta_+(q^h) = q^h \otimes q^h,
\end{equation}

\begin{equation}
\Delta_-(e_i) = e_i \otimes 1 + 1 \otimes e_i, \quad \Delta_-(f_i) = f_i \otimes t_i^{-1} + t_i \otimes f_i, \quad \Delta_-(q^h) = q^h \otimes q^h.
\end{equation}
(1.5) $\Delta_-(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i, \quad \Delta_-(f_i) = f_i \otimes 1 + t_i \otimes f_i, \quad \Delta_-(q^h) = q^h \otimes q^h.$

For two $U_q(\mathfrak{g})$-modules $M_1$ and $M_2$, let us denote by $M_1 \otimes_+ M_2$ and $M_1 \otimes_- M_2$ the vector space $M_1 \otimes_{\mathbb{Q}(q)} M_2$ endowed with $U_q(\mathfrak{g})$-module structure induced by the comultiplications $\Delta_+$ and $\Delta_-$, respectively. Then we have

$$\mathbf{D}_\varphi(M_1 \otimes_- M_2) \simeq (\mathbf{D}_\varphi M_1) \otimes_+ (\mathbf{D}_\varphi M_2).$$

The simple $U_q(\mathfrak{g})$-module $V(\lambda)$ and the $B_q(\mathfrak{g})$-module $U_q^-(\mathfrak{g})$ have a unique non-degenerate symmetric bilinear form $(\cdot, \cdot)$ such that

\begin{align*}
(1.6) \quad (u_\lambda, u_\lambda) &= 1 \quad \text{and} \quad (xu, v) = (u, \varphi(x)v) \quad \text{for} \ u, v \in V(\lambda) \quad \text{and} \quad x \in U_q(\mathfrak{g}), \\
(1.7) \quad (1, 1) &= 1 \quad \text{and} \quad (xu, v) = (u, \varphi(x)v) \quad \text{for} \ u, v \in U_q^-(\mathfrak{g}) \quad \text{and} \quad x \in B_q(\mathfrak{g}).
\end{align*}

Recall that $B_q(\mathfrak{g})$ is the quantum boson algebra generated by $e'_i$ and $f_i$, and $\varphi$ is the anti-automorphism of $B_q(\mathfrak{g})$ sending $e'_i$ to $f_i$ and $f_i$ to $e'_i$.

Note that $(\cdot, \cdot)$ induces the non-degenerated bilinear form

$$\langle \cdot, \cdot \rangle : V(\lambda)^r \otimes_{U_q(\mathfrak{g})} V(\lambda) \to \mathbb{Q}(q)$$

given by $\langle u^r, v \rangle = (u, v)$, by which $\mathbf{D}_\varphi V(\lambda)$ is canonically isomorphic to $V(\lambda)$.

1.2. **Crystal bases and global bases.** For a subring $A$ of $\mathbb{Q}(q)$, we say that $L$ is an $A$-lattice of a $\mathbb{Q}(q)$-vector space $V$ if $L$ is a free $A$-submodule of $V$ such that $V = \mathbb{Q}(q) \otimes_A L$.

Let us denote by $A_0$ (resp. $A_\infty$) the ring of rational functions in $\mathbb{Q}(q)$ which are regular at $q = 0$ (resp. $q = \infty$). We set $A := A_0 \cap A_\infty = \mathbb{Q}[q^{\pm 1}]$.

Let $M$ be a $U_q(\mathfrak{g})$-module in $\mathcal{O}_{\text{int}}(\mathfrak{g})$. Then any $u \in M$ can be uniquely written as

$$u = \sum_{n=0}^{\infty} f_i^{(n)} u_n \quad \text{with} \ e_i u_n = 0.$$ 

We define the **lower Kashiwara operators** by

$$e_i^\text{low}(u) = \sum_{n=1}^{\infty} f_i^{(n-1)} u_n \quad \text{and} \quad f_i^\text{low}(u) = \sum_{n=0}^{\infty} f_i^{(n+1)} u_n,$$

and the **upper Kashiwara operators** by

$$e_i^\text{up}(u) = e_i^\text{low} q_i^{-1} t_i u \quad \text{and} \quad f_i^\text{up}(u) = f_i^\text{low} q_i^{-1} t_i^{-1} u.$$ 

We say that an $A_0$-lattice of $M$ is a lower (resp. upper) crystal lattice of $M$ if $L$ is $\mathbb{P}$-graded and invariant by the lower (resp. upper) Kashiwara operators.

**Lemma 1.1.** Let $L$ be a lower crystal lattice of $M \in \mathcal{O}_{\text{int}}(\mathfrak{g})$. Then we have

(i) $\bigoplus_{\lambda \in \mathbb{P}} q^{-\langle \lambda, \lambda \rangle/2} L_\lambda$ is an upper crystal lattice of $M$.

(ii) $L^\vee := \{ \psi \in \mathbf{D}_\varphi M \mid \langle \psi, L \rangle \in A_0 \}$ is an upper crystal lattice of $\mathbf{D}_\varphi M$. 

Proof. (i) Let $\phi_M$ be the endomorphism of $M$ given by $\phi_M|_{M_\lambda} = q^{-(\lambda,\lambda)/2}i|_{M_\lambda}$. Then we have $\tilde{e}_i^{up} = \phi_M \circ \epsilon_i^{low} \phi_M^{-1}$ and $\tilde{f}_i^{up} = \phi_M \circ \tilde{f}_i^{low} \phi_M^{-1}$.

(ii) follows from the fact that $\tilde{e}_i^{up}$ and $\tilde{f}_i^{up}$ are the adjoint operators of $\tilde{f}_i^{low}$ and $\tilde{e}_i^{low}$, respectively.

**Definition 1.2.** A lower (resp. upper) crystal basis of $M$ consists of a pair $(L, B)$ satisfying the following conditions:

(i) $L$ is a lower (resp. upper) crystal lattice of $M$,
(ii) $B = \sqcup \eta B_\eta$ is a basis of the $\mathbb{Q}$-vector space $L/qL$, where $B_\eta = B \cap (L_\eta/qL_\eta)$,
(iii) the induced maps $\tilde{e}_i$ and $\tilde{f}_i$ on $L/qL$ satisfy

$$\tilde{e}_i B, \tilde{f}_i B \subset B \sqcup \{0\},$$

and $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$ for $b, b' \in B$.

Here $\tilde{e}_i$ and $\tilde{f}_i$ denote the lower (resp. upper) Kashiwara operators.

It is shown in [11] that $V(\lambda)$ has the lower crystal basis $(L^{low}(\lambda), B^{low}(\lambda))$. Using the non-degenerate symmetric bilinear form $(\ , \ )$, $V(\lambda)$ has the upper crystal basis $(L^{up}(\lambda), B^{up}(\lambda))$ where

$$L^{up}(\lambda) := \{u \in V(\lambda) \mid (u, L^{low}(\lambda)) \subset A_0\},$$

and $B^{up}(\lambda) \subset L^{up}(\lambda)/qL^{up}(\lambda)$ is the dual basis of $B^{low}(\lambda)$ with respect to the induced non-degenerate pairing between $L^{up}(\lambda)/qL^{up}(\lambda)$ and $L^{low}(\lambda)/qL^{low}(\lambda)$. An (abstract) crystal is a set $B$ together with maps

$$\text{wt}: B \to \mathbb{P}, \quad \varepsilon_i, \varphi_i: B \to \mathbb{Z} \sqcup \{\infty\}$$

and $\tilde{e}_i, \tilde{f}_i: B \to B \sqcup \{0\}$ for $i \in I$, such that

(C1) $\varphi_i(b) = \varepsilon_i(b) + (h_i, \text{wt}(b))$ for any $i$,
(C2) if $b \in B$ satisfies $\tilde{e}_i(b) \neq 0$, then

$$\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1, \quad \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1, \quad \text{wt}(\tilde{e}_i b) = \text{wt}(b) + \alpha_i,$$

(C3) if $b \in B$ satisfies $\tilde{f}_i(b) \neq 0$, then

$$\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \quad \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1, \quad \text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i,$$

(C4) for $b, b' \in B$, $b' = \tilde{f}_i b$ if and only if $b = \tilde{e}_i b'$,
(C5) if $\varphi_i(b) = -\infty$, then $\tilde{e}_i b = \tilde{f}_i b = 0$.

Recall that, with the notions of morphisms and tensor product rule of crystals, the category of crystal becomes a monoidal category ([14]). If $(L, B)$ is a crystal of $M$, then $B$ is an abstract crystal. Since $B^{low}(\lambda) \simeq B^{up}(\lambda)$, we drop the superscripts for simplicity.

Let $V$ be a $\mathbb{Q}(q)$-vector space, and let $L_0$ be an $A_0$-lattice of $V$, $L_\infty$ an $A_\infty$-lattice of $V$ and $V_A$ an $A$-lattice of $V$. We say that the triple $(V_A, L_0, L_\infty)$ is balanced if the following canonical map is a $\mathbb{Q}$-linear isomorphism:

$$E := V_A \cap L_0 \cap L_\infty \to L_0/qL_0.$$
Hence, if $B$ forms an $\mathbb{G}$ by $G$ Then the sets $\mathbb{Q}$ and called the $Q$, such that $(\mathbb{Z})$ form $\mathbb{G}$ and the $\mathbb{Z}$.

Later on, it will be convenient to use Sweedler’s notation $\Delta^+$ multiplication $\Delta^+ \mathbb{V} \circ \mathbb{V}$ ($\mathbb{A}$) $\mathbb{Q}$ forms the dual $\mathbb{Z}$ of $B$. Similarly, there exists a crystal basis $(\mathbb{Q} \otimes U_q^{-}(\mathfrak{g})$, $L(\infty), \mathbb{L}(\infty))$ is balanced. Let us denote the globalizing map by $G^{\mathbb{G}}$. Then the set $B^{\mathbb{G}}(\lambda) := \{G^{\mathbb{G}}(b) \mid b \in B^{\mathbb{G}}(\lambda) \}$ form $\mathbb{Z}[q^{\pm1}]$-bases of $V^{\mathbb{G}}(\lambda)$ and $V^{\mathbb{G}}(\lambda)$, respectively. They are called the lower global basis and the upper global basis of $V(\lambda)$.

Similarly, there exists a crystal basis $(L(\infty), B(\infty))$ of the simple $B_q(\mathfrak{g})$-module $U_q^{-}(\mathfrak{g})$ such that $(\mathbb{Q} \otimes U_q^{-}(\mathfrak{g})$, $L(\infty), \mathbb{L}(\infty))$ is balanced. Let us denote the globalizing map by $G^{\mathbb{G}}$. Then the set $B^{\mathbb{G}}(U_q^{-}(\mathfrak{g})) := \{G^{\mathbb{G}}(b) \mid b \in B^{\mathbb{G}}(\infty) \}$ forms an $\mathbb{Z}[q^{\pm1}]$-basis of $U_q^{-}(\mathfrak{g})$ and is called the lower global basis of $U_q^{-}(\mathfrak{g})$.

Let us denote by $B^{\mathbb{G}}(U_q^{-}(\mathfrak{g})) := \{G^{\mathbb{G}}(b) \mid b \in B^{\mathbb{G}}(\infty) \}$ the dual basis of $B^{\mathbb{G}}(U_q^{-}(\mathfrak{g}))$ with respect to $\langle , \rangle$. Then it is a $\mathbb{Z}[q^{\pm1}]$-basis of $U_q^{-}(\mathfrak{g})$.

and called the upper global basis of $U_q^{-}(\mathfrak{g})$.

2. Quantum coordinate rings and modified quantized enveloping algebras

2.1. Quantum coordinate ring. Let $U_q(\mathfrak{g})^*$ be Hom$_{\mathbb{Q}(q)}(U_q(\mathfrak{g}), \mathbb{Q}(q))$. Then the comultiplication $\Delta_+$ induces the multiplication $\mu$ on $U_q(\mathfrak{g})^*$ as follows:

$$\mu: U_q(\mathfrak{g})^* \otimes U_q(\mathfrak{g})^* \to (U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}))^* \xrightarrow{(\Delta_+)^*} U_q(\mathfrak{g})^*.$$ 

Later on, it will be convenient to use Sweedler’s notation $\Delta_+(x) = x(1) \otimes x(2)$. With this notation,

$$(fg)(x) = f(x(1)) \ g(x(2)) \text{ for } f, g \in U_q(\mathfrak{g})^* \text{ and } x \in U_q(\mathfrak{g}).$$
The $U_q(\mathfrak{g})$-bimodule structure on $U_q(\mathfrak{g})$ induces a $U_q(\mathfrak{g})$-bimodule structure on $U_q(\mathfrak{g})^*$. Namely,
\[(x \cdot f)(v) = f(vx) \quad \text{and} \quad (f \cdot x)(v) = f(xv) \quad \text{for} \quad f \in U_q(\mathfrak{g})^* \quad \text{and} \quad x, v \in U_q(\mathfrak{g}).\]

Then the multiplication $\mu$ is a morphism of a $U_q(\mathfrak{g})$-bimodule, where $U_q(\mathfrak{g})^* \otimes U_q(\mathfrak{g})^*$ has the structure of a $U_q(\mathfrak{g})$-bimodule via $\Delta_+$: for $f, g \in U_q(\mathfrak{g})^*$ and $x, y \in U_q(\mathfrak{g})$,
\[x(fg)y = (x_1fy_1)(x_2gy_2),\]
where $\Delta_+(x) = x_1 \otimes x_2$ and $\Delta_+(y) = y_1 \otimes y_2$.

**Definition 2.1.** We define the quantum coordinate ring $A_q(\mathfrak{g})$ as follows:

\[A_q(\mathfrak{g}) = \{ u \in U_q(\mathfrak{g})^* \mid U_q(\mathfrak{g})u \text{ belongs to } \mathcal{O}_{\text{int}}(\mathfrak{g}) \text{ and } uU_q(\mathfrak{g}) \text{ belongs to } \mathcal{O}_{\text{int}}^r(\mathfrak{g}) \}.\]

Then, $A_q(\mathfrak{g})$ is a subring of $U_q(\mathfrak{g})^*$ because (i) $\mu$ is $U_q(\mathfrak{g})$-bilinear, (ii) $\mathcal{O}_{\text{int}}(\mathfrak{g})$ and $\mathcal{O}_{\text{int}}^r(\mathfrak{g})$ are closed under the tensor product.

We have the weight decomposition: $A_q(\mathfrak{g}) = \bigoplus_{\eta, \zeta \in \mathbb{P}} A_q(\mathfrak{g})_{\eta, \zeta}$ where
\[A_q(\mathfrak{g})_{\eta, \zeta} := \{ \psi \in A_q(\mathfrak{g}) \mid q^{h_i} \psi \cdot q^{h_r} = q^{(h_i, \eta) + (h_r, \zeta)} \psi \text{ for } h_i, h_r \in \mathbb{P}^V \},\]

For $\psi \in A_q(\mathfrak{g})_{\eta, \zeta}$, we write
\[\text{wt}_\lambda(\psi) = \eta \quad \text{and} \quad \text{wt}_r(\psi) = \zeta.\]

For any $V \in \mathcal{O}_{\text{int}}$, we have the $U_q(\mathfrak{g})$-bilinear homomorphism
\[\Phi_V : V \otimes (D_\varphi V)^r \to A_q(\mathfrak{g})\]
given by
\[\Phi_V(v \otimes \psi^r)(a) = \langle \psi^r, av \rangle = \langle \psi^r, a \rangle \quad \text{for} \quad v \in V, \psi \in D_\varphi V \text{ and } a \in U_q(\mathfrak{g}).\]

**Proposition 2.2** ([12, Proposition 7.2.2]). We have an isomorphism $\Phi$ of $U_q(\mathfrak{g})$-bimodules
\[(2.1) \quad \Phi : \bigoplus_{\lambda \in \mathbb{P}^+} V(\lambda) \otimes V(\lambda)^r \xrightarrow{\sim} A_q(\mathfrak{g})\]
given by $\Phi|_{V(\lambda) \otimes q(\mathfrak{g}) V(\lambda)^r} = \Phi_\lambda := \Phi_{V(\lambda)}$ : Namely,
\[\Phi(u \otimes v^r)(x) = \langle v^r, xu \rangle = \langle v^r, x \rangle \quad \text{for any} \quad v, u \in V(\lambda) \quad \text{and} \quad x \in U_q(\mathfrak{g}).\]

We introduce the crystal basis $(L_{up}(A_q(\mathfrak{g})), B(A_q(\mathfrak{g})))$ of $A_q(\mathfrak{g})$ as the images by $\Phi$ of
\[\bigoplus_{\lambda \in \mathbb{P}^+} L_{up}(\lambda) \otimes L_{up}(\lambda)^r \text{ and } \bigsqcup_{\lambda \in \mathbb{P}^+} B(\lambda) \otimes B(\lambda)^r.\]

Hence it is a crystal base with respect to the left action of $U_q(\mathfrak{g})$ and also the right action of $U_q(\mathfrak{g})$. We sometimes write by $e_i^*$ and $f_i^*$ the operators of $A_q(\mathfrak{g})$ obtained by the right actions of $e_i$ and $f_i$.

We define the $\mathbb{Z}[q^{\pm 1}]$-form of $A_q(\mathfrak{g})$ by
\[A_q(\mathfrak{g})_{\mathbb{Z}[q^{\pm 1}]} := \{ \psi \in A_q(\mathfrak{g}) \mid \langle \psi, U_q(\mathfrak{g})_{\mathbb{Z}[q^{\pm 1}]} \rangle \subset \mathbb{Z}[q^{\pm 1}] \}.\]
We define the bar-involution $\bar{}$ of $A_q(\mathfrak{g})$ by
\[
\bar{\psi}(x) = \overline{\psi(x)} \quad \text{for } \psi \in A_q(\mathfrak{g}), \ x \in U_q(\mathfrak{g}).
\]
Note that the bar-involution is not a ring homomorphism but it satisfies
\[
\bar{\psi \theta} = q^{\langle \text{wt}_t(\psi), \text{wt}_t(\theta) \rangle - \langle \text{wt}_t(\psi), \text{wt}_t(\theta) \rangle} \bar{\theta} \bar{\psi} \quad \text{for any } \psi, \theta \in A_q(\mathfrak{g}).
\]
Since we do not use this formula and it is proved similarly to Proposition 2.4 below, we omit its proof.

The triple $(\mathbb{Q} \otimes A_q(\mathfrak{g})_{\mathbb{Z}[q^\pm 1]}, L^{up}(A_q(\mathfrak{g})), L^{up}(\overline{A_q(\mathfrak{g})}))$ is balanced ([12, Theorem 1]), and hence there exists an upper global basis of $A_q(\mathfrak{g})$
\[
B^{up}(A_q(\mathfrak{g})) := \{C^{up}(b) \mid b \in B^{up}(A_q(\mathfrak{g}))\}.
\]

For $\lambda \in \mathbb{P}^+$ and $\mu \in W\lambda$, we denote by $u_{\mu}$ a unique member of the upper global basis of $V(\lambda)$ with weight $\mu$. It is also a member of the lower global basis.

**Proposition 2.3.** Let $\lambda \in \mathbb{P}^+$, $w \in W$ and $b \in B(\lambda)$. Then, $\Phi(G^{up}(b) \otimes u_{w\lambda}^r)$ is a member of the upper global basis of $A_q(\mathfrak{g})$.

**Proof.** The element $\bar{\psi} := \Phi(G^{up}(b) \otimes u_{w\lambda}^r)$ is bar-invariant, and a member of crystal basis modulo $L^{up}(A_q(\mathfrak{g}))$. For any $P \in U_q(\mathfrak{g})_{\mathbb{Z}[q^\pm 1]}$,
\[
\langle \psi, P \rangle = (u_{w\lambda}, PG^{up}(b))
\]
belonging to $\mathbb{Z}[q^\pm 1]$ because $PG^{up}(b) \in V^{up}(\lambda)_{\mathbb{Z}[q^\pm 1]}$ and $u_{w\lambda} \in V^{low}(\lambda)_{\mathbb{Z}[q^\pm 1]}$. Hence $\psi$ belongs to $A_q(\mathfrak{g})_{\mathbb{Z}[q^\pm 1]}$. \hfill $\square$

The $\mathbb{Q}(q)$-algebra anti-automorphism $\varphi$ of $U_q(\mathfrak{g})$ induces a $\mathbb{Q}(q)$-linear automorphism $\varphi^*$ of $A_q(\mathfrak{g})$ by
\[
(\varphi^*(\psi))(x) = \psi(\varphi(x)) \quad \text{for any } x \in U_q(\mathfrak{g}).
\]
We have
\[
\varphi^*(\Phi(u \otimes v^r)) = \Phi(v \otimes u^r),
\]
and
\[
\text{wt}_t(\varphi^* \psi) = \text{wt}_t(\psi) \quad \text{and} \quad \text{wt}_t(\varphi^* \psi) = \text{wt}_t(\psi).
\]
It is obvious that $\varphi^*$ preserves $A_q(\mathfrak{g})_{\mathbb{Z}[q^\pm 1]}$, $L^{up}(A_q(\mathfrak{g}))$ and $B^{up}(A_q(\mathfrak{g}))$.

**Proposition 2.4.**
\[
\varphi^*(\psi \theta) = q^{\langle \text{wt}_t(\psi), \text{wt}_t(\theta) \rangle - \langle \text{wt}_t(\psi), \text{wt}_t(\theta) \rangle} (\varphi^* \psi)(\varphi^* \theta).
\]

In order to prove this proposition, we prepare a sublemma.

Let $\xi$ be the algebra automorphism of $U_q(\mathfrak{g})$ given by
\[
\xi(e_i) = q_i^{-1} t_i e_i, \quad \xi(f_i) = q_i t_i^{-1} f_i, \quad \xi(q^h) = q^h.
\]
We can easily see
\[
(\xi \otimes \xi) \circ \Delta_+ = \Delta_- \circ \xi.
\]
Let \( \xi^* \) be the automorphism of \( A_q(\mathfrak{g}) \) given by
\[
(\xi^* \psi)(x) = \psi(\xi(x)) \quad \text{for } \psi \in A_q(\mathfrak{g}) \text{ and } x \in U_q(\mathfrak{g}).
\]

**Sublemma 2.5.** We have
\[
(2.3) \quad \xi^*(\psi) = q^{A(wt_{1}(\psi), wt_{1}(\psi))} \psi.
\]

Here \( A(\lambda, \mu) = \frac{1}{2}(\langle \mu, \mu \rangle - \langle \lambda, \lambda \rangle) \).

**Proof.** Let us show that, for each \( x \), the following equality
\[
(2.4) \quad \psi(\xi(x)) = q^{A(wt_{1}(\psi), wt_{1}(\psi))} \psi(x)
\]
holds for any \( \psi \).

The equality (2.4) is obviously true for \( x = q^h \). If (2.4) is true for \( x \), then
\[
\xi^*(\psi)(xe_{i}) = \psi(\xi(xe_{i})) = \psi(\xi(x)e_{i}t_{i})q_{i}
= \frac{q}{2}^{(\alpha_{i}, wt_{1}(\psi)) + (\alpha_{i}, \alpha_{i})/2} \psi(\xi(x)e_{i})
= q^{(\alpha_{i}, wt_{1}(\psi)) + (\alpha_{i}, \alpha_{i})/2} (\xi^*(e_{i} \psi))(x)
= q^{(\alpha_{i}, wt_{1}(\psi)) + (\alpha_{i}, \alpha_{i})/2 + A(wt_{1}(\psi) + \alpha_{i}, wt_{1}(\psi))} (e_{i} \psi)(x).
\]

Since \( \| \lambda + \alpha_{i} \|^2 = \| \lambda \|^2 + 2(\alpha_{i}, \lambda) + (\alpha_{i}, \alpha_{i}) \), (2.4) holds for \( x e_{i} \). Similarly if (2.4) holds for \( x \), then it holds for \( x f_{i} \).

**Proof of Proposition 2.4.** We have
\[
(2.5) \quad (\varphi \circ \varphi) \circ \Delta_{-} = \Delta_{+} \circ \varphi.
\]

Hence, we have
\[
\langle \varphi^*(\psi\theta), x \rangle = \langle \psi\theta, \varphi(x) \rangle
= \langle \psi \otimes \theta, \Delta_{+}(\varphi(x)) \rangle
= \langle \psi \otimes \theta, (\varphi \circ \varphi) \circ \Delta_{-}(x) \rangle
= \langle \varphi^*(\psi) \otimes \varphi^*(\theta), \Delta_{-}(x) \rangle.
\]

Hence we have
\[
\langle \xi^*(\varphi^*(\psi\theta)), x \rangle = \langle \varphi^*(\psi\theta), \xi(x) \rangle = \langle \varphi^*(\psi) \otimes \varphi^*(\theta), \Delta_{-}(\xi(x)) \rangle
= \langle \varphi^*(\psi) \otimes \varphi^*(\theta), (\xi \otimes \xi) \circ \Delta_{+}x \rangle
= \langle \xi^* \varphi^*(\psi) \otimes \xi^* \varphi^*(\theta), \Delta_{+}x \rangle
= \langle (\xi^* \varphi^*(\psi)) (\xi^* \varphi^*(\theta)), x \rangle
= q^{A(wt_{1}(\psi), wt_{1}(\psi)) + A(wt_{1}(\psi), wt_{1}(\psi))} ((\varphi^*(\psi)) (\varphi^*(\theta)), x).
\]

Hence we obtain
\[
\varphi^*(\psi\theta) = q^c (\varphi^*(\psi)) (\varphi^*(\theta))
\]
with
\[
c = A(wt_{1}(\psi), wt_{1}(\psi)) + A(wt_{1}(\psi), wt_{1}(\psi)) - A(wt_{1}(\psi) + wt_{1}(\psi), wt_{1}(\psi) + wt_{1}(\psi))
\]
Lemma 2.6. For \( \Delta_n \) be the algebra homomorphism \( U_q^+(g) \to U_q^+(g) \otimes U_q^+(g) \) given by
\[
\Delta_n(e_i) = e_i \otimes 1 + 1 \otimes e_i.
\]
Set
\[
A_q(n) = \bigoplus_{\beta \in Q^-} A_q(n)_\beta \quad \text{where} \quad A_q(n)_\beta := (U_q^+(g)_{-\beta})^*.
\]
Defining the bilinear form \( \langle \cdot, \cdot \rangle : (A_q(n) \otimes A_q(n)) \times (U_q^+(g) \otimes U_q^+(g)) \) by
\[
\langle \psi \otimes \theta, x \otimes y \rangle \equiv \theta(x) \psi(y),
\]
we define the algebra structure on \( A_q(n) \) by
\[
(\psi \cdot \theta)(x) = \langle \psi \otimes \theta, \Delta_n(x) \rangle = \theta(x(1)) \psi(x(2))
\]
where \( \Delta_n(x) = x(1) \otimes x(2) \).
Since \( U_q^+(g) \) has a \( U_q^+(g) \)-bimodule structure, so does \( A_q(n) \).
We define the \( \mathbb{Z}[q^{\pm 1}] \)-form of \( A_q(n) \) by
\[
A_q(n)_{\mathbb{Z}[q^{\pm 1}]} = \left\{ \psi \in A_q(n) \mid \psi \left( U_q^+(g)_{\mathbb{Z}[q^{\pm 1}]} \right) \subset \mathbb{Z}[q^{\pm 1}] \right\}.
\]
We define the bar-operator \( \overline{\cdot} \) on \( A_q(n) \) by
\[
\overline{\psi}(x) = \overline{\psi(x)} \quad \text{for} \quad \psi \in A_q(n) \quad \text{and} \quad x \in U_q^+(g).
\]
Note that the bar-operator is not a ring homomorphism but it satisfies
\[
\overline{\psi \theta} = q^{(\text{wt}(\psi), \text{wt}(\theta))} \overline{\theta} \overline{\psi} \quad \text{for any} \quad \psi, \theta \in A_q(n).
\]
For \( i \in I \), we denote by \( e_i^* \) the right action of \( e_i \) on \( A_q(n) \).

Lemma 2.6. For \( u, v \in A_q(n) \), we have \( q \)-boson relations
\[
e_i(uv) = (e_i u)v + q^{(\alpha_i, \text{wt}(u))} u(e_i v) \quad \text{and} \quad e_i^*(uv) = u(e_i^* v) + q^{(\alpha_i, \text{wt}(v))}(e_i^* u)v.
\]

Proof. 
\[
\langle e_i(uv), x \rangle = \langle uv, xe_i \rangle = \langle u \otimes v, \Delta_n(xe_i) \rangle.
\]
If we set \( \Delta_n x = x(1) \otimes x(2) \), then we have
\[
\Delta_n(xe_i) = (x(1) \otimes x(2))(e_i \otimes 1 + 1 \otimes e_i) = q^{-(\alpha_i, \text{wt}(x(2)))}(x(1)e_i) \otimes x(2) + x(1) \otimes (x(2)e_i).
\]
Hence, we have
\[
\langle u \otimes v, \Delta_n(xe_i) \rangle = q^{-(\alpha_i, \text{wt}(x(2)))} u(x(2)) v(x(1)e_i) + u(x(2)e_i) v(x(1))
\]
\[
= q^{(\alpha_i, \text{wt}(u))} u(x(2)) \cdot (e_i v)(x(1)) + (e_i u)(x(2)) \cdot v(x(1))
\]
below, we have

\[ = (q^{(\alpha_i, \text{wt}(u))}u \otimes (e_i v) + (e_i u) \otimes v, \Delta_n x). \]

The second identity follows in a similar way. \(\square\)

We define the map \(\iota: U_q^-(\mathfrak{g}) \to A_q(\mathfrak{n})\) by

\[ \langle \iota(u), x \rangle = (u, \varphi(x)) \quad \text{for any } u \in U_q^-(\mathfrak{g}) \text{ and } x \in U_q^+(\mathfrak{g}). \]

Since \((\ , \ )\) is a non-degenerate bilinear form on \(U_q^-(\mathfrak{g})\), \(\iota\) is injective. The relation

\[ \langle \iota(e'_iu), x \rangle = (e'_iu, \varphi(x)) = (u, f_i \varphi(x)) = \langle e_iu, x \rangle, \]

implies that

\[ \iota(e'_iu) = e_i\iota(u). \]

**Lemma 2.7.** \(\iota\) is an algebra isomorphism.

**Proof.** The map \(\iota\) is an algebra homomorphism because \(e'_i\) and \(e_i\) both satisfy the same \(q\)-boson relation. \(\square\)

Hence, the algebra \(A_q(\mathfrak{n})\) has an upper crystal basis \((L^{\text{up}}(A_q(\mathfrak{n})), B(A_q(\mathfrak{n})))\) such that \(B(A_q(\mathfrak{n})) \simeq B(\infty)\). Furthermore, \(A_q(\mathfrak{n})\) has an upper global basis

\[ B^{\text{up}}(A_q(\mathfrak{n})) = \{ G^{\text{up}}(b) \}_{b \in B(A_q(\mathfrak{n}))} \]

induced by the balanced triple \((\mathbb{Q} \otimes A_q(\mathfrak{n})_{\mathbb{Z}[q^{\pm1}]}, L^{\text{up}}(A_q(\mathfrak{n})), \overline{L^{\text{up}}(A_q(\mathfrak{n}))})\).

**Remark 2.8.** Note that the multiplication on \(A_q(\mathfrak{n})\) given in [6] is different from ours. Indeed, by denoting the product of \(\psi\) and \(\phi\) in [6, §4.2] by \(\psi \cdot \phi\), we have, for \(x \in U_q^+(\mathfrak{g})\)

\[ (\psi \cdot \phi)(x) = \psi(x^{(1)})\phi(x^{(2)}), \]

where \(\Delta_n(x) = x^{(1)}q^{h^{(1)}} \otimes x^{(2)}q^{h^{(2)}}\) for \(x^{(1)}, x^{(2)} \in U_q^+(\mathfrak{g})\), \(h^{(1)}, h^{(2)} \in P^\vee\). By Lemma 2.17 below, we have

\[ (\psi \cdot \phi)(x) = q^{(\text{wt}(x^{(1)}), \text{wt}(x^{(2)}))}(x^{(1)})\phi(x^{(1)}) = q^{(\text{wt}(x^{(1)}), \text{wt}(x^{(2)}))}\psi(x^{(2)})\phi(x^{(1)}) = q^{(\text{wt}(\psi), \text{wt}(\phi))}(\psi \phi)(x), \]

for \(x \in U_q^+(\mathfrak{g})\), where \(\Delta_n(x) = x^{(1)} \otimes x^{(2)}\). In particular, we have a \(\mathbb{Q}(q)\)-algebra isomorphism from \((A_q(\mathfrak{n}), \cdot)\) to \(A_q(\mathfrak{n})\) given by

\[ (2.7) \quad x \mapsto q^{-\frac{1}{2}(\beta, \beta)}x \quad \text{for } x \in A_q(\mathfrak{n})_\beta. \]

Note also that the bar-operator \(-\) is a ring anti-isomorphism between \(A_q(\mathfrak{n})\) and \((A_q(\mathfrak{n}), \cdot)\).
2.3. **Modified quantum enveloping algebra.** For the materials in this subsection we refer the reader to [24], [14]. We denote by \( \text{Mod}(\mathfrak{g}, P) \) the category of left \( U_q(\mathfrak{g}) \)-modules with the weight space decomposition. Let (forget) be the functor from \( \text{Mod}(\mathfrak{g}, P) \) to the category of vector spaces over \( \mathbb{Q}(q) \), forgetting the \( U_q(\mathfrak{g}) \)-module structure.

Let us denote by \( R \) the endomorphism ring of (forget). Note that \( R \) contains \( U_q(\mathfrak{g}) \). For \( \eta \in P \), let \( a_\eta \in R \) denotes the projector \( M \to M_\eta \) to the weight space of weight \( \eta \). Then the defining relation of \( a_\eta \) (as a left \( U_q(\mathfrak{g}) \)-module) is

\[
q^h \cdot a_\eta = q^{\langle h, \eta \rangle} a_\eta.
\]

We have

\[
a_\eta a_\xi = \delta_{\eta, \xi} a_\eta, \quad a_\eta P = Pa_{\eta-\xi} \quad \text{for } \xi \in \mathbb{Q} \text{ and } P \in U_q(\mathfrak{g})_\xi.
\]

Then \( R \) is isomorphic to \( \prod_{\eta \in P} U_q(\mathfrak{g}) a_\eta \). We set

\[
\tilde{U}_q(\mathfrak{g}) := \bigoplus_{\eta \in P} U_q(\mathfrak{g}) a_\eta \subset R.
\]

Then \( \tilde{U}_q(\mathfrak{g}) \) is a subalgebra of \( R \). We call it the modified quantum enveloping algebra. Note that any \( U_q(\mathfrak{g}) \)-module in \( \text{Mod}(\mathfrak{g}, P) \) has a natural \( \tilde{U}_q(\mathfrak{g}) \)-module structure.

The (anti-)automorphisms \( * \), \( \varphi \) and \( \overline{\cdot} \) of \( U_q(\mathfrak{g}) \) extend to the ones of \( \tilde{U}_q(\mathfrak{g}) \) by

\[
a_\eta^* = a_{-\eta}, \quad \varphi(a_\eta) = a_\eta, \quad \overline{a}_\eta = a_\eta.
\]

For a dominant integral weight \( \lambda \in P^+ \), let us denote by \( V(\lambda) \) (resp. \( V(-\lambda) \)) the irreducible module with highest (resp. lowest) weight \( \lambda \) (resp. \( -\lambda \)). Let \( u_\lambda \) (resp. \( u_{-\lambda} \)) be the highest (resp. lowest) weight vector.

For \( \lambda \in P^+, \mu \in P^- := -P^+ \), we set

\[
V(\lambda, \mu) := V(\lambda) \otimes_- V(\mu).
\]

Then \( V(\lambda, \mu) \) is generated by \( u_\lambda \otimes u_\mu \) as a \( U_q(\mathfrak{g}) \)-module, and the defining relation of \( u_\lambda \otimes_- u_\mu \) is

\[
q^h (u_\lambda \otimes_- u_\mu) = q^{\langle h, \lambda+\mu \rangle} (u_\lambda \otimes_- u_\mu),
\]

\[
e_i^{1-\langle h_i, \mu \rangle} (u_\lambda \otimes_- u_\mu) = 0, \quad f_i^{1+\langle h_i, \lambda \rangle} (u_\lambda \otimes_- u_\mu) = 0.
\]

Let us define the automorphism \( \overline{\cdot} \) of \( V(\lambda, \mu) \) by

\[
\overline{P(u_\lambda \otimes_- u_\mu)} = P(u_\lambda \otimes_- u_\mu).
\]

We set

(i) \( L^\text{low}(\lambda, \mu) := L^\text{low}(\lambda) \otimes_{A_0} L^\text{low}(\mu) \),

(ii) \( V(\lambda, \mu)_{\mathbb{Z}[q^\pm 1]} := V(\lambda)_{\mathbb{Z}[q^\pm 1]} \otimes_{\mathbb{Z}[q^\pm 1]} V(\mu)_{\mathbb{Z}[q^\pm 1]} \),

(iii) \( B(\lambda, \mu) := B(\lambda) \otimes B(\mu) \).
Proposition 2.9 ([24]). \((L^\text{low}(\lambda, \mu), B(\lambda, \mu))\) is a lower crystal basis of \(V(\lambda, \mu)\). Furthermore, \((\mathbb{Q} \otimes V(\lambda, \mu))_{\mathbb{Z}[q^{\pm 1}]}\), \(L^\text{low}(\lambda, \mu), \overline{L^\text{low}(\lambda, \mu)}\) is balanced, and there exists a lower global basis \(B^\text{low}(V(\lambda, \mu))\) obtained from the lower crystal basis \((L^\text{low}(\lambda, \mu), B(\lambda, \mu))\).

Theorem 2.10 ([24]). The algebra \(\tilde{U}_q(\mathfrak{g})\) has a lower crystal basis \((L^\text{low}(\tilde{U}_q(\mathfrak{g})), B(\tilde{U}_q(\mathfrak{g})))\) satisfying the following properties:

(i) \(L^\text{low}(\tilde{U}_q(\mathfrak{g})) = \bigoplus_{\lambda \in \mathcal{P}} L^\text{low}(\tilde{U}_q(\mathfrak{g}))_{a_\lambda}\) and \(B(\tilde{U}_q(\mathfrak{g})) = \bigsqcup_{\lambda \in \mathcal{P}} B(\tilde{U}_q(\mathfrak{g}))_{a_\lambda}\) where

- \(L^\text{low}(\tilde{U}_q(\mathfrak{g}))_{a_\lambda} = L^\text{low}(\tilde{U}_q(\mathfrak{g})) \cap U_q(\mathfrak{g})_{a_\lambda}\) and
- \(B(\tilde{U}_q(\mathfrak{g}))_{a_\lambda} = B(\tilde{U}_q(\mathfrak{g})) \cap (L^\text{low}(\tilde{U}_q(\mathfrak{g}))_{a_\lambda}) / qL^\text{low}(\tilde{U}_q(\mathfrak{g}))_{a_\lambda}\).

(ii) Set \(\tilde{U}_q(\mathfrak{g})_{\mathbb{Z}[q^{\pm 1}]} := \bigoplus_{\eta \in \mathcal{P}} U_q(\mathfrak{g})_{\mathbb{Z}[q^{\pm 1}]} a_\eta\). Then \((\mathbb{Q} \otimes \tilde{U}_q(\mathfrak{g})_{\mathbb{Z}[q^{\pm 1}]}, L^\text{low}(\tilde{U}_q(\mathfrak{g})), \overline{L^\text{low}(\tilde{U}_q(\mathfrak{g}))})\) is balanced, and \(\tilde{U}_q(\mathfrak{g})\) has the lower global basis \(B^\text{low}(\tilde{U}_q(\mathfrak{g})) := \{G^\text{low}(b) \mid b \in B(\tilde{U}_q(\mathfrak{g}))\}\).

(iii) For any \(\lambda \in \mathcal{P}^+\) and \(\mu \in \mathcal{P}^-\), let \(\Psi_{\lambda, \mu}: U_q(\mathfrak{g})_{a_{\lambda+\mu}} \to V(\lambda, \mu)\) be the \(U_q(\mathfrak{g})\)-linear map \(a_{\lambda+\mu} \mapsto u_\lambda \otimes u_\mu\). Then we have \(\Psi_{\lambda, \mu}(L(\tilde{U}_q(\mathfrak{g})_{a_{\lambda+\mu}})) = L^\text{low}(\lambda, \mu)\).

(iv) Let \(\overline{\Psi}_{\lambda, \mu}\) be the induced homomorphism

\[
L^\text{low}(\tilde{U}_q(\mathfrak{g})_{a_{\lambda+\mu}})/qL^\text{low}(\tilde{U}_q(\mathfrak{g})_{a_{\lambda+\mu}}) \to L^\text{low}(\lambda, \mu)/qL^\text{low}(\lambda, \mu).
\]

Then we have

(a) \(\{b \in B(\tilde{U}_q(\mathfrak{g})_{a_{\lambda+\mu}}) \mid \overline{\Psi}_{\lambda, \mu}(b) \neq 0\} \to B(\lambda, \mu)\),

(b) \(\Psi_{\lambda, \mu}(G^\text{low}(b)) = G^\text{low}(\overline{\Psi}_{\lambda, \mu}(b))\) for any \(b \in B(\tilde{U}_q(\mathfrak{g})_{a_{\lambda+\mu}})\).

(v) \(B(\tilde{U}_q(\mathfrak{g}))\) has a structure of crystal such that the injective map induced by (iv) (a)

\(B(\lambda, \mu) \to B(\tilde{U}_q(\mathfrak{g})_{a_{\lambda+\mu}}) \subset B(\tilde{U}_q(\mathfrak{g}))\)

is a strict embedding of crystals for any \(\lambda \in \mathcal{P}^+\) and \(\mu \in \mathcal{P}^-\).

For \(\lambda \in \mathcal{P}\), take any \(\zeta \in \mathcal{P}^+\) and \(\eta \in \mathcal{P}^-\) such that \(\lambda = \zeta + \eta\). Then \(B(\zeta) \otimes B(\eta)\) is embedded into \(B(\tilde{U}_q(\mathfrak{g})_{a_{\eta}})\).

For \(\mu \in \mathcal{P}\), let \(T_\mu = \{t_\mu\}\) be the crystal with

\[\text{wt}(t_\mu) = \mu, \quad \varepsilon_i(t_\mu) = \varphi_i(t_\mu) = -\infty, \quad \tilde{e}_i(t_\mu) = \tilde{f}_i(t_\mu) = 0.\]

Since we have

\(B(\zeta) \to B(\infty) \otimes T_\zeta, \ B(\eta) \to T_\eta \otimes B(-\infty)\) and \(T_\zeta \otimes T_\eta \simeq T_{\zeta+\eta}\),

\(B(\zeta) \otimes B(\eta)\) is embedded into the crystal \(B(\infty) \otimes T_\lambda \otimes B(-\infty)\). Taking \(\zeta \to \infty\) and \(\eta \to -\infty\), we have

Lemma 2.11 ([14]). For any \(\lambda \in \mathcal{P}\), we have a canonical crystal isomorphism

\(B(\tilde{U}_q(\mathfrak{g})_{a_\lambda}) \simeq B(\infty) \otimes T_\lambda \otimes B(-\infty).\)
Hence we identify

\[ B(\widetilde{U}_q(\mathfrak{g})) = \bigsqcup_{\lambda \in \mathcal{P}} B(\infty) \otimes T_\lambda \otimes B(-\infty). \]

**Theorem 2.12 ([14]).**

(i) \( L^{\text{low}}(\widetilde{U}_q(\mathfrak{g})) \) is invariant under the anti-automorphisms \(*\) and \(\varphi\).

(ii) \( B(\widetilde{U}_q(\mathfrak{g}))^* = \varphi(B(\widetilde{U}_q(\mathfrak{g}))) = B(\widetilde{U}_q(\mathfrak{g})). \)

(iii) \( (G^{\text{low}}(b))^* = G^{\text{low}}(b^*) \) and \(\varphi(G^{\text{low}}(b)) = G^{\text{low}}(\varphi(b)) \) for \(b \in B(\widetilde{U}_q(\mathfrak{g})). \)

**Corollary 2.13 ([14]).** For \(b_1 \in B(\infty)\), \(b_2 \in B(-\infty)\), we have

1. \( (b_1 \otimes t_\mu \otimes b_2)^* = b_1^* \otimes t_{\mu - \text{wt}(b_1) - \text{wt}(b_2)} \otimes b_2^* \).
2. \( \varphi(b_1 \otimes t_\mu \otimes b_2) = \varphi(b_2) \otimes t_{\mu + \text{wt}(b_1) + \text{wt}(b_2)} \otimes \varphi(b_1) \).

We define, for \(b \in B \) with \( B = B(\widetilde{U}_q(\mathfrak{g})), B(\infty) \) or \( B(-\infty) \),

\[ \varepsilon_i^*(b) = \varepsilon_i(b^*), \quad \varphi_i^*(b) = \varphi_i(b^*), \quad \text{wt}^*(b) = \text{wt}(b^*), \quad \tilde{\varepsilon}_i^*(b) = \tilde{\varepsilon}_i(b^*) \quad \text{and} \quad \tilde{f}_i^*(b) = \tilde{f}_i(b^*). \]

This defines another crystal structure on \( \widetilde{U}_q(\mathfrak{g}) \): For \(b_1 \in B(\infty)\) and \(b_2 \in B(-\infty)\) and \(\eta \in \mathcal{P}\), we have

\[ \varepsilon_i^*(b_1 \otimes t_\eta \otimes b_2) = \max(\varepsilon_i^*(b_1), \varphi_i^*(b_2) + \langle h_i, \eta \rangle), \]
\[ \varphi_i^*(b_1 \otimes t_\eta \otimes b_2) = \max(\varepsilon_i^*(b_1) - \langle h_i, \eta \rangle, \varphi_i^*(b_2)), \]
\[ \text{wt}^*(b_1 \otimes t_\eta \otimes b_2) = -\eta, \]
\[ \tilde{\varepsilon}_i^*(b_1 \otimes t_\eta \otimes b_2) = \begin{cases} (\tilde{\varepsilon}_i^*(b_1) \otimes t_{\eta - h_i} \otimes b_2) & \text{if } \varepsilon_i^*(b_1) \geq \varphi_i^*(b_2) + \langle h_i, \eta \rangle, \\ b_1 \otimes t_{\eta - h_i} \otimes (\tilde{\varepsilon}_i^*(b_2)) & \text{if } \varepsilon_i^*(b_1) < \varphi_i^*(b_2) + \langle h_i, \eta \rangle, \end{cases} \]
\[ \tilde{f}_i^*(b_1 \otimes t_\eta \otimes b_2) = \begin{cases} (\tilde{f}_i^*(b_1) \otimes t_{\eta + h_i} \otimes b_2) & \text{if } \varepsilon_i^*(b_1) > \varphi_i^*(b_2) + \langle h_i, \eta \rangle, \\ b_1 \otimes t_{\eta + h_i} \otimes (\tilde{f}_i^*(b_2)) & \text{if } \varepsilon_i^*(b_1) \leq \varphi_i^*(b_2) + \langle h_i, \eta \rangle. \end{cases} \]

We have

\[ \tilde{\varepsilon}_i \circ \varphi = \varphi \circ \tilde{f}_i^* \quad \text{and} \quad \tilde{f}_i \circ \varphi = \varphi \circ \tilde{f}_i^* \quad \text{for every } i \in I. \]

For \(\xi \in \mathcal{Q}_-\) and \(\eta \in \mathcal{Q}_+\), we shall denote by

\[ U^+_q(\mathfrak{g})_{\eta} := \bigoplus_{\eta' \in \mathcal{Q}_+ \cap (\eta + \mathcal{Q}_-) \setminus \{\eta\}} U^+_q(\mathfrak{g})_{\eta'}. \]

Then for any \(\lambda \in \mathcal{P}, b_- \in B(\infty)_\xi\) and \(b_+ \in B(-\infty)_\eta\), we have

\[ G^{\text{low}}(b_- \otimes t_\lambda \otimes b_+) - G^{\text{low}}(b_-)G^{\text{low}}(b_+)a_\lambda \in U^-_q(\mathfrak{g})_{\xi}U^+_q(\mathfrak{g})_{\eta}a_\lambda. \]
2.4. **Relationship of** $A_q(\mathfrak{g})$ **and** $\tilde{U}_q(\mathfrak{g})$. There exists a canonical pairing $A_q(\mathfrak{g}) \times \tilde{U}_q(\mathfrak{g}) \to \mathbb{Q}(q)$ by

\begin{equation}
\langle \psi, \alpha \rangle = \delta_{\mu, \alpha} \psi(x) \quad \text{for any } x \in U_q^-(\mathfrak{g}) \text{ and } \mu \in \mathbf{P}.
\end{equation}

**Theorem 2.14** ([14]). There exists a bi-crystal embedding

\begin{equation}
\tau_q : B(A_q(\mathfrak{g})) \to B(\tilde{U}_q(\mathfrak{g}))
\end{equation}

which satisfies:

\begin{equation}
\langle G_{\text{up}}(b), \varphi(G_{\text{low}}(b')) \rangle = \delta_{\tau_q(b), b'}
\end{equation}

for any $b \in B(A_q(\mathfrak{g}))$ and $b' \in B(\tilde{U}_q(\mathfrak{g}))$.

2.5. **Relationship of** $A_q(\mathfrak{g})$ **and** $A_q(\mathfrak{n})$.

**Definition 2.15.** Let $p_n : A_q(\mathfrak{g}) \to A_q(\mathfrak{n})$ be the homomorphism induced by $U_q^+(\mathfrak{g}) \to U_q(\mathfrak{g})$:

\begin{equation}
\langle p_n(\psi), x \rangle = \psi(x) \quad \text{for any } x \in U_q^+(\mathfrak{g}).
\end{equation}

Then we have

\begin{equation}
\text{wt}(p_n(\psi)) = \text{wt}(\psi) - \text{wt}(\psi).
\end{equation}

Note that the map $p_n$ sends the upper global basis of $A_q(\mathfrak{g})$ to the upper global basis of $A_q(\mathfrak{n})$ or zero. Hence we have a map

\begin{equation}
\overline{\tau}_n : B(A_q(\mathfrak{g})) \to B(A_q(\mathfrak{n}))(\{0\}).
\end{equation}

It is obvious that $p_n$ sends all $\Phi(u_{\omega \lambda} \otimes u_{\nu}^*) (\lambda \in \mathbf{P}^+ \text{ and } \nu \in \mathbf{W})$ to 1. Note that $\tilde{\tau}_n(u_{\omega \lambda} \otimes u_{\nu}^*) = b_\infty \otimes t_{\nu} \otimes b_{-\infty} \in B(\tilde{U}_q(\mathfrak{g}))$.

**Proposition 2.16.** For $b \in B(A_q(\mathfrak{g}))$, set $\overline{\tau}_n(b) = b_1 \otimes t_{\zeta} \otimes b_2 \in B(\infty) \otimes T_{\zeta} \otimes B(-\infty) \subset B(\tilde{U}_q(\mathfrak{g})) (\zeta \in \mathbf{P})$. Then we have

\begin{equation}
p_n(G_{\text{up}}(b)) = \delta_{b_2, b_{-\infty}} G_{\text{up}}(b_1).
\end{equation}

**Proof.** Set $\eta := \text{wt}(b_1) + \zeta + \text{wt}(b_2) = \text{wt}(b)$. Then for any $\tilde{b} \in B(\infty)$, we have

\begin{align*}
\langle p_n(G_{\text{up}}(b)), \varphi(G_{\text{low}}(\tilde{b})) \rangle &= \langle G_{\text{up}}(b), G_{\text{low}}(\varphi(\tilde{b})) \rangle a_\eta \\
&= \langle G_{\text{up}}(b), G_{\text{low}}(b_\infty \otimes t_\eta \otimes \varphi(\tilde{b})) \rangle = \delta_{\overline{\tau}_n(b)} \tilde{b} \otimes t_{\eta} \otimes b_{-\infty} \\
&= \delta(b_2 = b_{-\infty}, b_1 = \tilde{b}).
\end{align*}

Although the map $p_n$ is not an algebra homomorphism, $p_n$ preserves the multiplications up to a power of $q$, as we will see below.

**Lemma 2.17.** For $x \in U_q^+(\mathfrak{g})$, $\Delta_n(x) = x_{(1)} \otimes x_{(2)}$, then

\begin{equation}
\Delta_+(x) = q^{\text{wt}(x_{(1)})} x_{(2)} \otimes x_{(1)}.
\end{equation}
Proposition 2.19. For \( x \in U_q^+(\mathfrak{g}) \). Note that

\[
\Delta_n(e_i x) = (e_i \otimes 1 + 1 \otimes e_i)(x(1) \otimes x(2)) = e_i x(1) \otimes x(2) + q^{-(\alpha_i, \omega t(x(1)))} x(1) \otimes (e_i x(2)).
\]

On the other hand, we have

\[
\Delta_+(e_i x) = (e_i \otimes 1 + q^{\alpha_i} \otimes e_i)(q^{\omega t(x(1))} x(2) \otimes x(1))
\]

\[
= (e_i q^{\omega t(x(1))} x(2) \otimes x(1)) + (q^{\alpha_i + \omega t(x(1))} x(2) \otimes (e_i x(1))
\]

\[
= q^{-(\alpha_i, \omega t(x(1)))}(q^{\omega t(x(1))} e_i x(2) \otimes x(1)) + (q^{\omega t(e_i x(1))} x(2) \otimes (e_i x(1)))
\]

Hence (2.11) holds for \( e_i x \).

Proof. Assume that (2.11) holds for \( x \in U_q^+(\mathfrak{g}) \). Note that

\[
\Delta_n(e_i x) = (e_i \otimes 1 + 1 \otimes e_i)(x(1) \otimes x(2)) = e_i x(1) \otimes x(2) + q^{-(\alpha_i, \omega t(x(1)))} x(1) \otimes (e_i x(2)).
\]

Proposition 2.18. For \( \psi, \theta \in A_q(\mathfrak{g}) \), we have

\[
p_n(\psi \theta) = q^{(\omega t(\psi), \omega t(\theta) - \omega t(\theta))} p_n(\psi)p_n(\theta).
\]

Proof. For \( x \in U_q^+(\mathfrak{g}) \), set \( \Delta_n(x) = x(1) \otimes x(2) \). Then, we have

\[
\langle p_n(\psi \theta), x \rangle = \langle \psi \theta, x \rangle = \langle \psi \otimes \theta, q^{\omega t(x(1))} x(2) \otimes x(1) \rangle = \langle \psi, q^{\omega t(x(1))} x(2) \rangle \langle \theta, x(1) \rangle
\]

\[
= q^{(\omega t(\psi), \omega t(x(1)))} \langle \psi, x(2) \rangle \langle \theta, x(1) \rangle = q^{(\omega t(\psi), \omega t(x(1)))} \langle p_n(\psi), x(2) \rangle \langle p_n(\theta), x(1) \rangle
\]

\[
= q^{(\omega t(\psi), \omega t(\theta) - \omega t(\theta))} \langle p_n(\psi) \otimes p_n(\theta), \Delta_n(x) \rangle
\]

\[
= q^{(\omega t(\psi), \omega t(\theta) - \omega t(\theta))} \langle p_n(\psi) \otimes p_n(\theta), x \rangle.
\]

Here, we used \( \omega t(x(1)) = -\omega t(p_n(\theta)) \) in (a).

There exists an injective map

\[
(2.12) \quad \tau_\lambda : B(\lambda) \rightarrow B(\infty)
\]

induced by the \( U_q^+(\mathfrak{g}) \)-linear homomorphism \( \iota_\lambda : V(\lambda) \rightarrow A_q(\mathfrak{n}) \) given by

\[
v \mapsto (U_q(\mathfrak{g})^+ \ni a \mapsto (av, u_\lambda)).
\]

It commutes with \( \tilde{e}_i \). We have

\[
G^{low}(b) = G^{low}(\tau_\lambda(b)) u_\lambda \quad \text{and} \quad \iota_\lambda G^{up}(b) = G^{up}(\tau_\lambda(b)) \quad \text{for any} \ b \in B(\lambda).
\]

Proposition 2.19 ([14]). Let \( \lambda, \mu \in P^+ \) and \( w \in W \). Then for any \( b \in B(\tilde{U}_q(\mathfrak{g}) a_{\lambda + w \mu}) \),

\[
G^{low}(b)(u_\lambda \otimes u_{w \mu}) \text{ vanishes or is a member of the lower global basis of } V(\lambda) \otimes V(\mu).
\]

Hence we have a crystal morphism

\[
(2.13) \quad \pi_{\lambda, w \mu} : B(\tilde{U}_q(\mathfrak{g}) a_{\lambda + w \mu}) \rightarrow B(\lambda) \otimes B(\mu)
\]

by \( G^{low}(b)(u_\lambda \otimes u_{w \mu}) = G^{low}(\pi_{\lambda, w \mu}(b)) \).

3. Quantum minors and T-systems

Hereafter, we assume that the generalized Cartan matrix \( A \) is symmetric, although many of the results hold also in the non-symmetric case. Hence we assume that \( A = ((\alpha_i, \alpha_j))_{i,j \in I} \).
3.1. Quantum minors. Using the isomorphism $\Phi$ in (2.1), for each $\lambda \in P^+$ and $\mu, \xi \in W\lambda$, we define the elements

$$\Delta(\mu, \xi) := \Phi(u_\mu \otimes u_\xi) \in A_q(g)$$

and

$$D(\mu, \xi) := p_n(\Delta(\mu, \xi)) \in A_q(n).$$

The element $\Delta(\mu, \xi)$ is called a (generalized) quantum minor and $D(\mu, \xi)$ is called a unipotent quantum minor.

**Lemma 3.1.** $\Delta(\mu, \xi)$ is a member of the upper global basis of $A_q(g)$. Moreover $D(\mu, \xi)$ is either a member of the upper global basis of $A_q(n)$ or zero.

**Proof.** It follows from Proposition 2.3 and Proposition 2.16. $\square$

**Lemma 3.2 ([2, (9.13)]).** For $u,v \in W$ and $\lambda,\mu \in P^+$, we have

$$\Delta(u\lambda, v\lambda)\Delta(u\mu, v\mu) = \Delta(u(\lambda + \mu), v(\lambda + \mu)).$$

By Proposition 2.18, we have the following corollary:

**Corollary 3.3.** For $u,v \in W$ and $\lambda,\mu \in P^+$, we have

$$D(u\lambda, v\lambda)D(u\mu, v\mu) = q^{-(v\lambda, v\mu-u\mu)}D(u(\lambda + \mu), v(\lambda + \mu)).$$

Note that

$$D(\mu, \mu) = 1 \quad \text{for } \mu \in W\lambda.$$

Then $D(\mu, \xi) \neq 0$ if and only if $\mu \preceq \xi$. Recall that for $\mu, \xi$ in the same $W$-orbit, we say that $\mu \preceq \xi$ if there exists a sequence $\{\beta_k\}_{1 \leq k \leq l}$ of positive real roots such that, defining $\lambda_0 = \xi$, $\lambda_k = s_{\beta_k}\lambda_{k-1}$ ($1 \leq k \leq l$), we have $(\beta_k, \lambda_{k-1}) \geq 0$ and $\lambda_l = \mu$.

More precisely, we have

**Lemma 3.4.** Let $\lambda \in P^+$ and $\mu, \xi \in W\lambda$. Then the following conditions are equivalent:

(a) $D(\mu, \xi)$ is an element of upper global basis of $A_q(n)$,

(b) $D(\mu, \xi) \neq 0$,

(c) $u_\mu \in U_q^-(g)u_\xi$,

(d) $u_\xi \in U_q^+(g)u_\mu$,

(e) $\mu \preceq \xi$,

(f) for any $w \in W$ such that $\mu = w\lambda$, there exists $u \leq w$ (in the Bruhat order) such that $\xi = u\lambda$,

(g) there exist $u,v \in W$ such that $\mu = w\lambda$, $\xi = u\lambda$ and $u \leq w$.

**Proof.** The equivalence of (b), (c) and (d) is obvious. The equivalence of (e), (f), (g) is well-known. The equivalence of (d) and (f) is proved in [13]. $\square$

For any $u \in A_q(n) \setminus \{0\}$ and $i \in I$, we set

$$\varepsilon_i(u) := \max\{n \in \mathbb{Z}_{\geq 0} \mid e_i^n u \neq 0\},$$

$$\varepsilon_i^*(u) := \max\{n \in \mathbb{Z}_{\geq 0} \mid e_i^*^n u \neq 0\}. $$
Then for any \( b \in B(A_q(n)) \), we have
\[
\varepsilon_i(G^\up(b)) = \varepsilon_i(b) \quad \text{and} \quad \varepsilon^*_i(G^\up(b)) = \varepsilon^*_i(b).
\]

**Lemma 3.5.** Let \( \lambda, \mu, \zeta \in W \lambda \) such that \( \mu \preceq \zeta \) and \( i \in I \).

(i) If \( n := \langle h_i, \mu \rangle \geq 0 \), then
\[
\varepsilon_i(D(\mu, \zeta)) = 0 \quad \text{and} \quad \varepsilon^*_i(D(s_i \mu, \zeta)) = D(\mu, \zeta).
\]

(ii) If \( \langle h_i, \mu \rangle \leq 0 \) and \( s_i \mu \preceq \zeta \), then \( \varepsilon_i(D(\mu, \zeta)) = -\langle h_i, \mu \rangle \).

(iii) If \( m := -\langle h_i, \zeta \rangle \leq 0 \), then
\[
\varepsilon^*_i(D(\mu, \zeta)) = 0 \quad \text{and} \quad \varepsilon^*_i(D(s_i \mu, \zeta)) = D(\mu, \zeta).
\]

(iv) If \( \langle h_i, \zeta \rangle \geq 0 \) and \( \mu \preceq s_i \zeta \), then \( \varepsilon^*_i(D(\mu, \zeta)) = \langle h_i, \zeta \rangle \).

**Proof.** We have \( \varepsilon_i(\Delta(\mu, \zeta)) = \max(\langle h_i, \mu \rangle, 0) \) and \( \varepsilon^*_i(\Delta(\mu, \zeta)) = \max(\langle h_i, \zeta \rangle, 0) \). Moreover, \( p_n \) commutes with \( \varepsilon^*_i(n) \) and \( \varepsilon^*_i(n) \).

Let us show (ii). Set \( \ell = -\langle h_i, \mu \rangle \). Then we have \( \varepsilon^*_{\ell+1}\Delta(\mu, \zeta) = 0 \), which implies \( \varepsilon^*_{\ell+1}D(\mu, \zeta) = 0 \). Hence \( \varepsilon_i(D(\mu, \zeta)) \leq \ell \). We have
\[
\varepsilon^*_{\ell}(\Delta(\mu, \zeta)) = \Delta(s_i \mu, \zeta).
\]

Hence we have \( \varepsilon^*_{\ell}(D(\mu, \zeta)) = D(s_i \mu, \zeta) \). By the assumption \( s_i \mu \preceq \zeta \), \( D(s_i \mu, \zeta) \) does not vanish. Hence we have \( \varepsilon_i(D(\mu, \zeta)) \geq \ell \).

The other statements can be proved similarly. \( \square \)

**Proposition 3.6** ([2, (10.2)]). Let \( \lambda, \mu, \sigma \in P^+ \) and \( s, t, s', t' \in W \) such that \( \ell(s's) = \varepsilon_i(D(s's, t't)) = \ell(s') + \ell(t') \). Then we have

(i) \( \Delta(s's, t't)
\Delta(s's') = q^{(s's', t't)}D(s's', t't)
\Delta(s's, t't).
\]

(ii) If we assume further that \( s's, t't \preceq s't' \), then we have
\[
D(s's, t't)D(s's', t't') = q^{(s's, t't)}D(s's, t't)D(s's', t't),
\]

or equivalently
\[
q^{(s's', t't')D(s's, t't)} = q^{(s's', t't')D(s's, t't)}.
\]

Note that (ii) follows from by Proposition 2.18 and (i). Note also that the both sides of (3.2) are bar-invariant, and hence they are members of the upper global basis as seen by [10, Corollary 3.5].

**Proposition 3.7.** For \( \lambda, \mu \in P^+ \) and \( s, t \in W \), set \( \bar{\omega}(u_{s\lambda} \otimes (u_{\lambda})^t) = b_{-} \otimes t_{\lambda} \otimes b_{-} \) and \( \bar{\omega}(u_{\mu} \otimes (u_{\mu})^t) = b_{s_{\lambda}} \otimes t_{\mu} \otimes b_{s_{\lambda}} \) with \( b_{\pm} \in B(\pm \infty) \). Then we have
\[
\Delta(s, \lambda)\Delta(\mu, t) = G^{\up}((\bar{\omega}_b^{-1}(\omega_{s_{\lambda}} \otimes t_{\lambda} \otimes b_{})\otimes b_{}))
\]

**Proof.** Let \( (\cdot, \cdot): (V(\lambda) \otimes V(\mu))^t \Rightarrow (V(\lambda) \otimes V(\mu)) \to Q(q) \) be the coupling defined
\[
(P(u \otimes v, u' \otimes v')) = (u \otimes v', \varphi(P)(u' \otimes v')) \quad \text{for any } P \in U_q(g).
\]
For $u, u' \in V(\lambda)$ and $v, v' \in V(\mu)$, we have
\[
\langle \Phi(u \otimes u'), \Phi(v \otimes v') \rangle = \langle u' \otimes_+ v', P(u \otimes_+ v) \rangle
= (\varphi(P)(u' \otimes_+ v'), u \otimes_+ v).
\]
Hence for $P \in U_q(\mathfrak{g})$, we have
\[
\langle \Delta(s\lambda, \lambda) \Delta(\mu, t\mu), P \rangle = \delta(\zeta = s\lambda + \mu)(\varphi(P)(u_\lambda \otimes_+ u_\mu), u_{s\lambda} \otimes_+ u_\mu).
\]
If $P \varphi = G_{\text{low}}(\varphi(b))$ for $b \in B(\tilde{U}_q(\mathfrak{g}))$, then we have
\[
\langle \Delta(s\lambda, \lambda) \Delta(\mu, t\mu), \varphi(G_{\text{low}}(b)) \rangle = \delta(\zeta = s\lambda + \mu)(G_{\text{low}}(b)(u_\lambda \otimes_+ u_\mu), u_{s\lambda} \otimes_+ u_\mu).
\]
The element $G_{\text{low}}(b)(u_\lambda \otimes_+ u_\mu)$ vanishes or is a global basis of $V(\lambda) \otimes_+ V(\mu)$ by Proposition 2.19. Since $u_{s\lambda} \otimes_+ u_\mu$ is a member of the upper global basis of $V(\lambda) \otimes_+ V(\mu)$, we have
\[
\langle \Delta(s\lambda, \lambda) \Delta(\mu, t\mu), \varphi(G_{\text{low}}(b)) \rangle = \delta(\zeta = s\lambda + \mu) \delta(\pi_{\lambda, \mu}(b) = u_{s\lambda} \otimes u_\mu).
\]
Here $\pi_{\lambda, \mu} : B(\tilde{U}_q(\mathfrak{g})) a_{\lambda+\mu} \to B(\lambda) \otimes B(\mu)$ is the crystal morphism given in (2.13).

Hence we obtain
\[
\Delta(s\lambda, \lambda) \Delta(\mu, t\mu) = G_{\text{up}}(\bar{t}_q^{-1})(b)
\]
where $b \in B(\tilde{U}_q(\mathfrak{g}))$ is a unique element such that $(G_{\text{low}}(b)(u_\lambda \otimes_+ u_\mu), u_{s\lambda} \otimes_+ u_\mu) = 1$.

On the other hand, we have $G_{\text{low}}(b_+) u_{t\mu} = u_\mu$ and $G_{\text{low}}(b_-) u_{s\lambda} = u_{s\lambda}$. The last equality implies $\varphi(G_{\text{low}}(b_-)) u_{s\lambda} = u_\lambda$ because $(\varphi(G_{\text{low}}(b_-)) u_{s\lambda}, u_\lambda) = (u_{s\lambda}, G_{\text{low}}(b_-) u_\lambda) = (u_{s\lambda}, u_{s\lambda}) = 1$. As seen in (2.8), we have
\[
G_{\text{low}}(b_-) G_{\text{low}}(b_+) a_{\lambda+\mu} - G_{\text{low}}(b_- t_{\lambda+\mu} b_+) \in U_q(\mathfrak{g})_{s\lambda-\lambda} U_q(\mathfrak{g})_{-t\mu} a_{\lambda+\mu}.
\]
Hence we obtain
\[
(G_{\text{low}}(b_- t_{\lambda+\mu} b_+) (u_\lambda \otimes_+ u_\mu), u_{s\lambda} \otimes_+ u_\mu) = (G_{\text{low}}(b_-) G_{\text{low}}(b_+) (u_\lambda \otimes_+ u_\mu), u_{s\lambda} \otimes_+ u_\mu) = (G_{\text{low}}(b_-) (u_\lambda \otimes_+ u_\mu), \varphi(G_{\text{low}}(b_-)) (u_{s\lambda} \otimes_+ u_\mu)) = 1.
\]
In the last equality, we used $G_{\text{low}}(b_+) (u_\lambda \otimes_+ u_\mu) = u_\lambda \otimes_+ (G_{\text{low}}(b_+) u_\mu) = u_\lambda \otimes_+ u_\mu$ and
\[
\varphi(G_{\text{low}}(b_-))(u_{s\lambda} \otimes_+ u_\mu) = (\varphi(G_{\text{low}}(b_-)) u_{s\lambda}) \otimes_+ u_\mu = u_{s\lambda} \otimes_+ u_\mu.
\]
Hence we conclude that $b = b_- t_{\lambda+\mu} b_+$.

Let
\[
i_{\lambda, \mu} : V(\lambda + \mu) \hookrightarrow V(\lambda) \otimes V(\mu)
\]
be the canonical embedding and
\[
i_{\lambda, \mu} : B(\lambda + \mu) \hookrightarrow B(\lambda) \otimes B(\mu)
\]
the induced crystal embedding.

**Lemma 3.8.** For $\lambda, \mu \in \mathcal{P}^+$ and $x, y \in W$ such that $x \geq y$, we have
\[
u_{x_\lambda \otimes y_\mu} \in i_{\lambda, \mu}(B(\lambda + \mu)) \subset B(\lambda) \otimes B(\mu).
Proof. Let us show by induction on $\ell(x)$, the length of $x$ in $W$. We may assume that $x \neq 1$. Then there exists $i \in I$ such that $s_i x < x$. If $s_i y < y$, then $s_i x \geq s_i y$ and $e_i^{\max}(u_{x,\lambda} \otimes u_{y,\mu}) = u_{s_i x,\lambda} \otimes u_{s_i y,\mu}$.

If $s_i y > y$, then $s_i x \geq y$ and $e_i^{\max}(u_{x,\lambda} \otimes u_{y,\mu}) = u_{s_i x,\lambda} \otimes u_{y,\mu}$. In both cases, $u_{x,\lambda} \otimes u_{y,\mu}$ is connected with an element of $T_{\lambda,\mu}(B(\lambda + \mu))$. \hfill $\square$

Lemma 3.9. For $\lambda, \mu \in \mathbb{P}^+$ and $w \in W$, we have

$$\Delta(w\lambda, \lambda)\Delta(\mu, \mu) = G^{\text{up}}(\tilde{\iota}_{\lambda,\mu}^{-1}(u_{w\lambda} \otimes u_{\mu}) \otimes u_{\lambda+\mu}^{-1}).$$

Proof. We have

$$\tilde{\iota}_g(u_{w\lambda} \otimes u_\lambda^r) = b_{w\lambda} \otimes t_{\lambda} \otimes b_{-\infty},$$

$$\tilde{\iota}_g(u_\mu \otimes u_\mu^r) = b_{-\infty} \otimes t_{\mu} \otimes b_{-\infty},$$

where $b_{w\lambda} := \tilde{\iota}_\lambda(u_{w\lambda})$. Hence Proposition 3.7 implies that

$$\Delta(w\lambda, \lambda)\Delta(\mu, \mu) = G^{\text{up}}(\tilde{\iota}_g^{-1}(b_{w\lambda} \otimes t_{\lambda+\mu} \otimes b_{-\infty})).$$

Then, $\tilde{\iota}_g(\tilde{\iota}_{\lambda,\mu}^{-1}(u_{w\lambda} \otimes u_{\mu}) \otimes u_{\lambda+\mu}^{-1}) = b_{w\lambda} \otimes t_{\lambda+\mu} \otimes b_{-\infty}$ implies the desired result. \hfill $\square$

3.2. $T$-systems. In this subsection, we record the $T$-system among the (unipotent) quantum minors for later use (see [20] for $T$-system).

Proposition 3.10 ([6, Proposition 3.2]). Assume that $u, v \in W$ and $i \in I$ satisfy $u < us_i$ and $v < vs_i$. Then

$$\Delta(us_i \varpi_i, vs_i \varpi_i)\Delta(u \varpi_i, v \varpi_i) = q^{-1} \Delta(us_i \varpi_i, v \varpi_i)\Delta(u \varpi_i, vs_i \varpi_i) + \prod_{j \neq i} \Delta(u \varpi_j, v \varpi_j)^{-a_{i,j}},$$

$$\Delta(u \varpi_i, v \varpi_i)\Delta(us_i \varpi_i, vs_i \varpi_i) = q \Delta(u \varpi_i, vs_i \varpi_i)\Delta(us_i \varpi_i, v \varpi_i) + \prod_{j \neq i} \Delta(u \varpi_j, v \varpi_j)^{-a_{i,j}},$$

and

$$q^{(us_i \varpi_i, v \varpi_i - u \varpi_i)}D(us_i \varpi_i, vs_i \varpi_i)D(u \varpi_i, v \varpi_i)$$

$$= q^{-1+(vs_i \varpi_i - u \varpi_i)}D(us_i \varpi_i, v \varpi_i)D(u \varpi_i, vs_i \varpi_i) + D(u \lambda, v \lambda),$$

$$q^{(v \varpi_i, vs_i \varpi_i - u \varpi_i)}D(u \varpi_i, v \varpi_i)D(us_i \varpi_i, vs_i \varpi_i)$$

$$= q^{1+(v \varpi_i, us_i \varpi_i - u \varpi_i)}D(u \varpi_i, vs_i \varpi_i)D(us_i \varpi_i, v \varpi_i) + D(u \lambda, v \lambda),$$

where $\lambda = s_i \varpi_i + \varpi_i = -\sum_{j \neq i} a_{j,i} \varpi_j$. 


3.3. Revisit of crystal bases and global bases. In order to prove Theorem 3.13 below, we first investigate the upper crystal lattice of $D_\varphi V$ induced by an upper crystal lattice of $V \in \mathcal{O}_{\text{int}}(g)$.

Let $V$ be a $U_q(g)$-module in $\mathcal{O}_{\text{int}}$. Let $L^\text{up}$ be an upper crystal lattice of $V$. Then we have (see Lemma 1.1)

$$\bigoplus_{\xi \in \mathcal{P}} q^{(\xi,\xi)/2} (L^\text{up})_\xi$$

is a lower crystal lattice of $V$.

Recall that, for $\lambda \in \mathcal{P}^+$, the upper crystal lattice $L^\text{up}(\lambda)$ and the lower crystal lattice $L^\text{low}(\lambda)$ of $V(\lambda)$ are related by

$$L^\text{up}(\lambda) = \bigoplus_{\xi \in \mathcal{P}} q^{((\lambda,\lambda)-2(\xi,\xi))/2} L^\text{low}(\lambda)_\xi \subset L^\text{low}(\lambda).$$

Write

$$V \simeq \bigoplus_{\lambda \in \mathcal{P}^+} E_\lambda \otimes V(\lambda)$$

with finite-dimensional $\mathbb{Q}(q)$-vector spaces $E_\lambda$. Accordingly, we have a canonical decomposition

$$L^\text{up} \simeq \bigoplus_{\lambda \in \mathcal{P}^+} C_\lambda \otimes A_0 L^\text{up}(\lambda),$$

where $C_\lambda \subset E_\lambda$ is an $A_0$-lattice of $E_\lambda$.

On the other hand, we have

$$D_\varphi V \simeq \bigoplus_{\lambda \in \mathcal{P}^+} E^*_\lambda \otimes V(\lambda).$$

Note that we have

$$\Phi_V ((a \otimes u) \otimes (b \otimes v)^r) = (a, b)\Phi_\lambda(u \otimes v^r)$$

for $u, v \in V(\lambda)$ and $a \in E_\lambda$, $b \in E^*_\lambda$.

We define the induced upper crystal lattice $D_\varphi L^\text{up}$ of $D_\varphi V$ by

$$D_\varphi L^\text{up} := \bigoplus_{\lambda \in \mathcal{P}^+} C^\vee_\lambda \otimes A_0 L^\text{up}(\lambda) \subset D_\varphi V,$$

where $C^\vee_\lambda := \{u \in E^*_\lambda \mid \langle u, C_\lambda \rangle \subset A_0\}$. Then we have

$$\Phi_V (L^\text{up} \otimes (D_\varphi L^\text{up})^r) \subset L^\text{up}(A_q(g)).$$

Indeed, we have

$$D_\varphi L^\text{up} = \{u \in D_\varphi V \mid \Phi_\lambda(L^\text{up} \otimes u^r) \subset L^\text{up}(A_q(g))\}.$$

Since $(L^\text{up}(\lambda))^\vee = L^\text{low}(\lambda)$, we have

$$(L^\text{up})^\vee = \bigoplus_{\lambda \in \mathcal{P}^+} C^\vee_\lambda \otimes A_0 L^\text{low}(\lambda).$$

The properties $L^\text{up}(\lambda) \subset L^\text{low}(\lambda)$ and $L^\text{up}(\lambda)_\lambda = L^\text{low}(\lambda)_\lambda$ imply the following lemma.

**Lemma 3.11.** $D_\varphi L^\text{up}$ is the largest upper crystal lattice of $D_\varphi V$ contained in the lower crystal lattice $(L^\text{up})^\vee$. 

Let $\lambda, \mu \in P^+$. Then $(L_{\text{up}}(\lambda) \otimes L_{\text{up}}(\mu))^\vee = L_{\text{low}}(\lambda) \otimes L_{\text{low}}(\mu)$ is a lower crystal lattice of $D_\varphi(V(\lambda) \otimes V(\mu)) \simeq V(\lambda) \otimes V(\mu)$. Let $\Xi_{\lambda,\mu} : V(\lambda) \otimes V(\mu) \xrightarrow{\sim} V(\lambda) \otimes V(\mu) \simeq D_\varphi(V(\lambda) \otimes V(\mu))$ be the $U_q(g)$-module isomorphism defined by

$$\Xi_{\lambda,\mu}(u \otimes v) = q^{(\lambda,\mu) - (\xi,\eta)}(u \otimes v) \text{ for } u \in V(\lambda)_{\xi} \text{ and } v \in V(\mu)_{\eta}.$$  

Then

$$\tilde{L} := \bigoplus_{\xi,\eta \in P} q^{(\lambda,\mu) - (\xi,\eta)} L_{\text{up}}(\lambda) \otimes L_{\text{up}}(\mu)_{\eta}.$$  

is an upper crystal lattice of $V(\lambda) \otimes V(\mu)$. Since we have $(\lambda, \mu) - (\xi, \eta) \geq 0$ for any $\xi \in \text{wt}(V(\lambda))$ and $\eta \in \text{wt}(V(\mu))$, Lemma 3.11 implies that $\tilde{L} \subset D_\varphi(L_{\text{up}}(\lambda) \otimes L_{\text{up}}(\mu)).$

**Lemma 3.12.** Let $\lambda, \mu \in P^+$ and $x_1, x_2, y_1, y_2 \in W$ such that $x_k \geq y_k \ (k = 1, 2)$. Then we have

$$q^{(\lambda,\mu) - (x_2\lambda, y_2\mu)} \Delta(x_1\lambda, x_2\lambda) \Delta(y_1\mu, y_2\mu) \equiv G_{\text{up}}(\tilde{\tau}_{\lambda,\mu}^{-1}(u_{x_1\lambda} \otimes u_{y_1\mu}) \otimes \tilde{\tau}_{\lambda,\mu}^{-1}(u_{x_2\lambda} \otimes u_{y_2\mu})^\tau) \mod qL_{\text{up}}(A_q(g)).$$

**Proof.** By the definition, we have

$$\Delta(x_1\lambda, x_2\lambda) \Delta(y_1\mu, y_2\mu) = \Phi_{V(\lambda) \otimes V(\mu)}((u_{x_1\lambda} \otimes u_{y_1\mu}) \otimes (u_{x_2\lambda} \otimes u_{y_2\mu})^\tau).$$

Hence we have

$$q^{(\lambda,\mu) - (x_2\lambda, y_2\mu)} \Delta(x_1\lambda, x_2\lambda) \Delta(y_1\mu, y_2\mu) = \Phi_{V(\lambda) \otimes V(\mu)}((u_{x_1\lambda} \otimes u_{y_1\mu}) \otimes q^{(\lambda, \mu) - (x_2\lambda, y_2\mu)}(u_{x_2\lambda} \otimes u_{y_2\mu})^\tau) = \Phi_{V(\lambda) \otimes V(\mu)}((u_{x_1\lambda} \otimes u_{y_1\mu}) \otimes (\Xi_{\lambda,\mu}(u_{x_2\lambda} \otimes u_{y_2\mu})^\tau)).$$

The right-hand side of (3.6) can be calculated as follows. Let us take $v_k \in L_{\text{up}}(\lambda + \mu)$ such that $\iota_{\lambda,\mu}(v_k) - u_{x_k\lambda} \otimes u_{y_k\mu} \in qL_{\text{up}}(\lambda) \otimes L_{\text{up}}(\mu)$ for $k = 1, 2$. Here $\iota_{\lambda,\mu} : V(\lambda + \mu) \to V(\lambda) \otimes V(\mu)$ denotes the canonical $U_q(g)$-module homomorphism. Then we have

$$G_{\text{up}}(\tilde{\iota}_{\lambda,\mu}^{-1}(u_{x_1\lambda} \otimes u_{y_1\mu}) \otimes (\tilde{\iota}_{\lambda,\mu}^{-1}(u_{x_2\lambda} \otimes u_{y_2\mu})^\tau)) = \Phi_{V(\lambda) \otimes V(\mu)}((v_1 \otimes \iota_{\lambda,\mu}(v_2)) \mod qL_{\text{up}}(A_q(g)))$$

$$= \Phi_{V(\lambda) \otimes V(\mu)}(\iota_{\lambda,\mu}(v_1) \otimes (\Xi_{\lambda,\mu}(v_2))^\tau).$$

On the other hand, we have

$$\iota_{\lambda,\mu}(v_1) \equiv u_{x_1\lambda} \otimes u_{y_1\mu} \mod qL_{\text{up}}(\lambda) \otimes L_{\text{up}}(\mu)$$

and

$$\Xi_{\lambda,\mu}(\iota_{\lambda,\mu}(v_2)) \equiv \Xi_{\lambda,\mu}(u_{x_2\lambda} \otimes u_{y_2\mu}) \mod q\tilde{L}.$$
Hence
\[
\Phi_{V(\lambda) \otimes V(\mu)} \left( \left( u_{x_{1}\lambda} \otimes u_{y_{1}\mu} \right) \otimes \Xi_{\lambda,\mu} \left( u_{x_{2}\lambda} \otimes u_{y_{2}\mu} \right)^{t} \right) \\
\equiv \Phi_{V(\lambda) \otimes V(\mu)} \left( \left( t_{\lambda,\mu} (v_{1}) \otimes (\Xi_{\lambda,\mu} t_{\lambda,\mu} (v_{2}))^{t} \right) \right) \mod qL^{up}(A_{q}(g)).
\]

\[\Box\]

**Theorem 3.13.** Let \( \lambda \in P^{+} \) and \( x, y \in W \) such that \( x \geq y \). Then we have
\[
D(x, \lambda)D(y, \lambda) \equiv D(x, \lambda) \mod qL^{up}(A_{q}(n)).
\]

**Proof.** Applying \( p_{n} \) to (3.6), we have
\[
D(x, y)D(y, \lambda) \equiv p_{n} \left( G^{up}(\tau_{x,\lambda}^{-1}(u_{x} \otimes u_{y}) \otimes \tau_{y,\lambda}^{-1}(u_{y} \otimes u_{\lambda})) \right) \mod qL^{up}(A_{q}(n)).
\]
Hence the desired result follows from Proposition 2.16 and Lemma 3.14 below. \[\Box\]

**Lemma 3.14.** Let \( \lambda \in P^{+} \) and \( x, y \in W \) such that \( x \geq y \). Then we have
\[
\bar{t}_{\lambda} \left( \bar{t}_{\lambda,\lambda}^{-1}(u_{x} \otimes u_{y}) \otimes (\bar{t}_{\lambda,\lambda}^{-1}(u_{y} \otimes u_{\lambda}))^{t} \right) = b_{x_{\lambda}} \otimes t_{y_{\lambda}+\lambda} \otimes b_{-\infty}.
\]

**Proof.** We shall argue by induction on \( \ell(x) \). We set \( b_{x_{\lambda}} = \bar{t}_{\lambda}(u_{x_{\lambda}}) \). Since the case \( x = 1 \) is obvious, assume that \( x \neq 1 \). Take \( i \in I \) such that \( x' := s_{i}x < x \)

(a) First assume that \( s_{i}y > y \). Then we have \( y \leq x' \). Hence by the induction hypothesis,
\[
\bar{t}_{\lambda} \left( \bar{t}_{\lambda,\lambda}^{-1}(u_{x'} \otimes u_{y}) \otimes (\bar{t}_{\lambda,\lambda}^{-1}(u_{y} \otimes u_{\lambda}))^{t} \right) = b_{x'} \otimes t_{y_{\lambda}+\lambda} \otimes b_{-\infty}.
\]
We have \( \varphi_{i}(u_{x_{\lambda}}) = \langle h_{i}, x' \lambda \rangle \) and \( \varphi_{i}(b_{x_{\lambda}} \otimes t_{y_{\lambda}+\lambda}) = \langle h_{i}, x' \lambda \rangle + \langle h_{i}, y \lambda \rangle \geq \langle h_{i}, x' \lambda \rangle \).
Hence, applying \( f_{i}^{(h_{i}, x') \lambda} \) to (3.8), we obtain
\[
\bar{t}_{\lambda} \left( \bar{t}_{\lambda,\lambda}^{-1}(u_{x} \otimes u_{y}) \otimes (\bar{t}_{\lambda,\lambda}^{-1}(u_{y} \otimes u_{\lambda}))^{t} \right) = b_{x_{\lambda}} \otimes t_{y_{\lambda}+\lambda} \otimes b_{-\infty}.
\]

(b) Assume that \( y' := s_{i}y < y \). Then we have \( y' \leq x' \), and the induction hypothesis implies that
\[
\bar{t}_{\lambda} \left( \bar{t}_{\lambda,\lambda}^{-1}(u_{x'_{\lambda}} \otimes u_{y}) \otimes (\bar{t}_{\lambda,\lambda}^{-1}(u_{y} \otimes u_{\lambda}))^{t} \right) = b_{x'} \otimes t_{y'_{\lambda}+\lambda} \otimes b_{-\infty}.
\]
We shall apply \( e_{i}^{*} \langle h_{i}, y' \lambda \rangle \bar{f}_{i}^{(h_{i}, x'_{\lambda} + y' \lambda)} \) to the both sides. Then the left-hand side yields
\[
\bar{t}_{\lambda} \left( \bar{t}_{\lambda,\lambda}^{-1}(u_{x} \otimes u_{y}) \otimes (\bar{t}_{\lambda,\lambda}^{-1}(u_{y} \otimes u_{\lambda}))^{t} \right).
\]
Since \( \varphi_{i}(b_{x_{\lambda}} \otimes t_{y_{\lambda}+\lambda}) = \langle h_{i}, x' \lambda \rangle + \langle h_{i}, y' \lambda + \lambda \rangle \geq \langle h_{i}, x' \lambda + y' \lambda \rangle \), the right-hand side yields
\[
\bar{e}_{i}^{*} \langle h_{i}, y' \lambda \rangle \bar{f}_{i}^{(h_{i}, x' \lambda + y' \lambda)} (b_{x_{\lambda}} \otimes t_{y'_{\lambda}+\lambda} \otimes b_{-\infty}) = \bar{e}_{i}^{*} \langle h_{i}, y' \lambda \rangle (\langle \bar{f}_{i}^{(h_{i}, x' \lambda + y' \lambda)} b_{x_{\lambda}} \otimes t_{y'_{\lambda}+\lambda} \otimes b_{-\infty})
\]
\[
= \bar{e}_{i}^{*} \langle h_{i}, y' \lambda \rangle (\langle \bar{f}_{i}^{(h_{i}, y' \lambda)} b_{x_{\lambda}} \otimes t_{y'_{\lambda}+\lambda} \otimes b_{-\infty}).
\]

Since $\varepsilon_i^* (b_{x, \lambda}) = - \varphi_i (b_{x, \lambda}) = \langle h_i, \lambda \rangle$ and $f_i^{(h, y/\lambda)} (b_{x, \lambda}) = \tilde{f}_i^{* (h, y/\lambda)} (b_{x, \lambda})$, we have

$$
\varepsilon_i^{(h, y/\lambda)} (f_i^{(h, y/\lambda)} (b_{x, \lambda}) \otimes t_{y, \lambda+\lambda} \otimes b_{-\infty}) = b_{x, \lambda} \otimes t_{y, \lambda+\lambda} \otimes b_{-\infty}.
$$

\[\square\]

3.4. **Generalized T-systems.** The $T$-system in \(3.2\) can be interpreted as an equation among the three products of elements in $B^{up}(A_q(\mathfrak{g}))$ or $B^{up}(A_q(\mathfrak{n}))$. In this subsection, we introduce another equation among the three products of elements in $B^{up}(A_q(\mathfrak{g}))$, called a **generalized T-system**.

**Proposition 3.15.** Let $\mu \in W \varpi_i$ and set $b = \tau_{\varpi_i} (u_\mu) \in B(\infty)$. Then we have

$$
\Delta(\mu, s_i \varpi_i) \Delta(\varpi_i, \varpi_i) = q^{-1} G^{up} \left( \tau_{\varpi_i, i}^{-1} (u_\mu \otimes u_{\varpi_i}) \otimes (\tau_{\varpi_i, i}^{-1} (u_{s_i \varpi_i} \otimes u_{\varpi_i}))^r \right) + G^{up} \left( \tau_{\varpi_i, i}^{-1} (\tilde{e}_i^* b) \otimes u_{\varpi_i, i}^r \right).
$$

(3.9)

Note that if $\mu = \varpi_i$, then $b = 1$ and the last term in (3.9) vanishes. If $\mu \neq \varpi_i$, then $\varepsilon_i^* (b) = 1$ and $\tau_{\varpi_i, i}^{-1} (\tilde{e}_i^* b) \in B(\varpi_i + s_i \varpi_i)$, and $u_\mu \otimes u_{\varpi_i} \in \tau_{\varpi_i, i} B(2 \varpi_i)$.

**Proof.** In the sequel, we omit $\tau_{\varpi_i, i}$ for the sake of simplicity. Set

$$
u = \Delta(\mu, s_i \varpi_i) \Delta(\varpi_i, \varpi_i) - q^{-1} G^{up} \left( (u_\mu \otimes u_{\varpi_i}) \otimes (u_{s_i \varpi_i} \otimes u_{\varpi_i})^r \right).
$$

Then $\nu \otimes u^i$ for $j \neq i$. Since $\tilde{e}_i (u_{s_i \varpi_i} \otimes u_{\varpi_i}) = u_{\varpi_i} \otimes u_{s_i \varpi_i}$, we have

$$
G^{up} \left( (u_\mu \otimes u_{\varpi_i}) \otimes (u_{s_i \varpi_i} \otimes u_{\varpi_i})^r \right) f_i = G^{up} \left( (u_\mu \otimes u_{\varpi_i}) \otimes (u_{\varpi_i} \otimes u_{s_i \varpi_i})^r \right) f_i = G^{up} (u_\mu \otimes u_{\varpi_i}^r) G^{up} \left( u_{s_i \varpi_i} \otimes u_{\varpi_i}^r \right)
$$

$$
= \Delta(\mu, \varpi_i) \Delta(\varpi_i, \varpi_i).
$$

Here the second equality follows from Corollary 3.9. On the other hand, we have

$$
(\Delta(\mu, s_i \varpi_i) \Delta(\varpi_i, \varpi_i)) f_i = (\Delta(\mu, s_i \varpi_i) f_i) \left( \Delta(\varpi_i, \varpi_i) t_i^{-1} \right)
$$

$$
= q_i^{-1} \Delta(\mu, \varpi_i) \Delta(\varpi_i, \varpi_i).
$$

Hence we have $u f_i = 0$. Thus, $u$ is a lowest weight vector of weight $\lambda$ with respect to the right action of $U_q(\mathfrak{g})$. Therefore there exists some $v \in V(\lambda)$ such that

$$
u = \Phi (v \otimes u^i).
$$

Hence we have $p_n (u) = \iota_\lambda (v) \in A_q(\mathfrak{g})$. On the other hand, we have

$$
p_n (\Delta(\mu, s_i \varpi_i) \Delta(\varpi_i, \varpi_i)) = p_n (\Delta(\mu, s_i \varpi_i)) p_n (\Delta(\varpi_i, \varpi_i)) = D(\mu, s_i \varpi_i) = G^{up} (\tilde{e}_i^* b)
$$

$$
= \iota_\lambda (G^{up} (\tilde{e}_i^* b)).
$$

Note that since $\varepsilon_i^* (\tilde{e}_i^* b) = 0$ and $\varepsilon_j^* (\tilde{e}_i^* b) \leq - \langle h_j, \alpha_i \rangle$ for $j \neq i$, we have $\tilde{e}_i^* b \in \tau_\lambda (B(\lambda))$.

Hence in order to prove our assertion, it is enough to show that

$$
p_n \left( G^{up} \left( (u_\mu \otimes u_{\varpi_i}) \otimes (u_{s_i \varpi_i} \otimes u_{\varpi_i})^r \right) \right) = 0.
$$
This follows from Proposition 2.16 and
\[
3.10 \quad \tau_g((u_\mu \otimes u_{\alpha i}) \otimes (u_{s_{\alpha i}} \otimes u_{\alpha i})^t) = b \otimes t_\lambda \otimes \tilde{e}_i b_{-\infty}.
\]

Let us prove (3.10). Since \[
(u_\mu \otimes u_{\alpha i}) \otimes (u_{s_{\alpha i}} \otimes u_{\alpha i})^t = \tilde{e}_i^* ((u_\mu \otimes u_{\alpha i}) \otimes (u_{\alpha i} \otimes u_{\alpha i})^t),
\]
the left-hand side of (3.10) is equal to
\[
\tilde{e}_i^* (\tau_g((u_\mu \otimes u_{\alpha i}) \otimes (u_{\alpha i} \otimes u_{\alpha i})^t)) = \tilde{e}_i^* (b \otimes t_{\alpha i} \otimes b_{-\infty}).
\]

Since \( \varepsilon_i^*(b) = 1 < \langle h_i, 2\alpha_i \rangle = 2 \), we obtain
\[
\tilde{e}_i^* (b \otimes t_{\alpha i} \otimes b_{-\infty}) = b \otimes t_{\alpha i} \otimes \tilde{e}_i^* b_{-\infty} = b \otimes t_\lambda \otimes \tilde{e}_i b_{-\infty}.
\]
\[\square\]

4. KLR algebras and their modules

4.1. KLR algebras and \( R \)-matrices. In this subsection, we briefly recall the basic materials of symmetric KLR algebras and \( R \)-matrices following [10]. For precise definitions in this subsection, we refer [8, 9, 10]. In this paper, for the sake of simplicity, KLR algebras are always assumed to be symmetric although some of results in this paper hold also in the non-symmetric case. We consider only graded modules over KLR algebras. Hence we write “a module” instead of “a graded module”. We sometimes omit the grading shift if there is no afraid of confusion.

Let the quintuple \((A, P, \Pi, P^\vee, \Pi^\vee)\) be a symmetric Cartan datum and let \( k \) be a base field. Hence we assume that the generalized Cartan matrix \( A = (a_{i,j})_{i,j \in I} \) is equal to \((\langle a_i, \alpha_j \rangle)_{i,j \in I} \). For \( i, j \in I \) such that \( i \neq j \), let us take a family of polynomials \((Q_{i,j})_{i,j \in I} \) in \( k[u, v] \) such that \( Q_{i,i} = 0 \) and
\[
Q_{i,j}(u, v) = c_{i,j}(u - v)^{-a_{i,j}} \quad \text{for} \ i \neq j
\]
with \( c_{i,j} \in k^\times \). We assume
\[
Q_{i,j}(u, v) = Q_{j,i}(v, u).
\]

For \( n \in \mathbb{Z}_{\geq 0} \) and \( \beta \in \mathbb{Q}^+ \) such that \( |\beta| = n \), we set
\[
I^\beta = \{ \nu = (\nu_1, \ldots, \nu_n) \in I^n \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \beta \}.
\]

For \( \beta \in \mathbb{Q}^+ \), we denote by \( R(\beta) \) the KLR algebra at \( \beta \) associated with \((A, P, \Pi, P^\vee, \Pi^\vee)\) and \((Q_{i,j})_{i,j \in I} \). It is a \( \mathbb{Z} \)-graded \( k \)-algebra generated by the generators \( \{e(\nu)\}_{\nu \in I^\beta}, \{x_k\}_{1 \leq k \leq n}, \{\tau_m\}_{1 \leq m \leq n-1} \) with certain defining relations (e.g., see [10, Definition 1.2]). Note that the isomorphism class of \( R(\beta) \) does not depend on the choice of \( c_{i,j} \)’s.

We denote by \( R(\beta)\text{-Mod} \) the category of \( R(\beta) \)-modules and by \( R(\beta)\text{-mod} \) the category of \( R(\beta) \)-modules \( M \) which are finite-dimensional over \( k \) and the action of \( x_k \) on \( M \) is nilpotent for any \( k \).

Similarly, we also denote by \( R(\beta)\text{-gMod} \) and by \( R(\beta)\text{-gmod} \) the category of graded \( R(\beta) \)-modules and the category of graded \( R(\beta) \)-modules finite-dimensional over \( k \), respectively.
For $\beta \in \mathbb{Q}^+$ and $M \in R(\beta)$-Mod, we set
\[
\text{wt}(M) = -\beta.
\]
We set
\[
R\text{-gmod} = \bigoplus_{\beta \in \mathbb{Q}^+} R(\beta)\text{-gmod}.
\]
Then $R\text{-gmod}$ is a $k$-linear abelian monoidal category, whose tensor product is given by the convolution product $\circ$.

Let us denote by $K(R\text{-gmod})$ the Grothendieck group of $R\text{-gmod}$. Then $K(R\text{-gmod})$ becomes a $\mathbb{Z}[q^{\pm 1}]$-algebra whose multiplication is induced by the convolution product and the $\mathbb{Z}[q^{\pm 1}]$-action is induced by the grading shift functor $q$.

For $M \in R(\beta)$-mod, the dual space $M^* := \text{Hom}_k(M, k)$ admits an $R(\beta)$-module structure via
\[
(r \cdot f)(u) := f(\psi(r)u) \quad (r \in R(\beta), \ u \in M),
\]
where $\psi$ denotes the $k$-algebra anti-involution on $R(\beta)$ which fixes the generators $e(\nu)$, $x_m$ and $\tau_k$ for $\nu \in I^\beta$, $1 \leq m \leq |\beta|$ and $1 \leq k \leq |\beta| - 1$.

It is known that (see [26, Theorem 2.2 (2)])
\[
(M_1 \circ M_2)^* \simeq q^{\text{wt}(M_1),\text{wt}(M_2)}(M_2^* \circ M_1^*) \quad \text{for } M_1, M_2 \in R\text{-gmod}.
\]

A simple module $M$ is called self-dual if $M^* \simeq M$. Every simple module is isomorphic to a grading shift of a self-dual simple module ([16, §3.2]).

**Theorem 4.1** ([16, 29]). There exists a $\mathbb{Z}[q^{\pm 1}]$-algebra isomorphism
\[
\text{ch} : \bigoplus_{\beta \in \mathbb{Q}^+} K(R(\beta)\text{-gmod}) \xrightarrow{\sim} A_q(\mathbb{n})_{\mathbb{Z}[q^{\pm 1}]}.
\]

**Theorem 4.2** ([29, 30]). We assume further that $k$ has characteristic 0. Then under the isomorphism $\text{ch}$ in (4.3), the upper global basis $\mathbf{B}^\text{up}(A_q(\mathbb{n}))$ corresponds to the set of the isomorphism classes of self-dual simple $R$-modules.

Let $z$ be an indeterminate which is homogeneous of degree 2, and let $\psi_z$ be the algebra homomorphism
\[
\psi_z : R(\beta) \to k[z] \otimes R(\beta)
\]
given by
\[
\psi_z(x_k) = x_k + z, \quad \psi_z(\tau_k) = \tau_k, \quad \psi_z(e(\nu)) = e(\nu).
\]
For an $R(\beta)$-module $M$, we denote by $M_z$ the $(k[z] \otimes R(\beta))$-module $k[z] \otimes M$ with the action of $R(\beta)$ twisted by $\psi_z$.

For a non-zero $R(\beta)$-module $M$ and a non-zero $R(\gamma)$-module $N$,
\[
\text{let } s \text{ be the order of zero of } R_{M_z,N} : M_z \circ N \to N \circ M_z; \text{ i.e., the largest non-negative integer such that the image of } R_{M_z,N} \text{ is contained in } z^s N \circ M_z,
\]
where $R_{M,N}$ is the R-matrix from $M \circ N$ to $N \circ M$ constructed in [8].
Definition 4.3. For a non-zero $R(\beta)$-module $M$ and a non-zero $R(\gamma)$-module $N$,

$$R_{M,N}^{\text{ren}}: M \circ N \to N \circ M \quad \text{and} \quad r_{M,N}: M \circ N \to N \circ M$$

by

$$R_{M,N}^{\text{ren}} = z^{-s}R_{Mz,N} \quad \text{and} \quad r_{M,N} = (z^{-s}R_{Mz,N})|_{z=0},$$

where $s$ is the integer in (4.4).

For $M, N \in R$-gmod, $\Lambda(M, N)$ denotes the homogeneous degree of the morphisms $R_{Mz,N}^{\text{ren}}$ and $r_{M,N}$. Hence we have a morphism in $R$-gmod:

$$q^{\Lambda(M, N)} M \circ N \longrightarrow N \circ M.$$ 

We say that a simple $R$-module $M$ is real if $M \circ M$ is simple again.

The following theorem is the main result of [9]

Theorem 4.4 ([9, Theorem 3.2]). Let $M$ and $N$ be simple modules. We assume that one of them is real. Then we have:

(i) $M \circ N$ has a simple head and a simple socle,
(ii) $\text{Im}(r_{M \circ N})$ coincides with the head of $M \circ N$ and the socle of $N \circ M$ (up to grading shifts).

We also have the following

Proposition 4.5. Let $M$ and $N$ be simple modules. We assume that one of them is real. Then we have

$$\text{Hom}_{R\text{-mod}}(M \circ N, N \circ M) = k r_{M,N}.$$ 

Proof. Since the other case can be proved similarly, we assume that $M$ is real. Let $f: M \circ N \to N \circ M$ be a morphism. Then we have a commutative diagram (up to a constant multiple)

$$
\begin{array}{ccc}
M \circ M \circ N & \xrightarrow{\text{Mor}_{M,N}} & M \circ N \circ M \\
M \circ f & \downarrow & \circ f M \\
M \circ N \circ M & \xrightarrow{r_{M,N} \circ M} & N \circ M \circ M
\end{array}
$$

Note that $r_{M, M \circ N} = M \circ r_{M,N}$ and $r_{M, N \circ M} = r_{M,N} \circ M$. Hence we have

$$M \circ \text{Im}(r_{M,N}) \subseteq f^{-1}(\text{Im}(r_{M,N})) \circ M.$$ 

Hence there exists a submodule $K$ of $N$ such that $\text{Im}(r_{M,N}) \subseteq K \circ M$ and $M \circ K \subseteq f^{-1}(\text{Im}(r_{M,N}))$. Since $K \neq 0$, we have $K = N$. Hence $f(M \circ N) \subseteq \text{Im}(r_{M,N})$, which means that $f$ factors as $M \circ N \to \text{soc}(N \circ M) \to N \circ M$. It remains to remark that $\text{Hom}_{R\text{-mod}}(M \circ N, \text{soc}(N \circ M)) = k r_{M,N}$. $\square$

For simple modules $M, N$, let us denote by $M \diamond N$ the head of $M \circ N$. 

4.2. Properties of $\tilde{\Lambda}(M, N)$ and $\vartheta(M, N)$.

**Lemma 4.6 ([10], Lemma 2.9).** Let $M$ and $N$ be non-zero modules. Then we have

1. $\Lambda(M, N) + \Lambda(N, M) \in 2\mathbb{Z}_{\geq 0}$.
2. Setting $\Lambda(M, N) + \Lambda(N, M) = 2m$ with $m \in \mathbb{Z}_{\geq 0}$, we have
   
   $R_{M,N}^{\text{ren}} \circ R_{N,M}^{\text{ren}} = z^m \text{id}_N \circ M$ and $R_{N,M}^{\text{ren}} \circ R_{M,N}^{\text{ren}} = z^m \text{id}_M \circ N$

   up to constant multiples.

By [10, Lemma 2.5] and the above lemma, we can define the following integers:

**Definition 4.7 ([10]).** Let $M$ and $N$ be non-zero modules.

\[
\tilde{\Lambda}(M, N) := \left( \langle \text{wt}(M), \text{wt}(N) \rangle + \Lambda(M, N) \right)/2 \in \mathbb{Z},
\]

\[
\vartheta(M, N) := \left( \Lambda(M, N) + \Lambda(N, M) \right)/2 \in \mathbb{Z}_{\geq 0}.
\]

**Lemma 4.8 ([10]).** Let $M$ and $N$ be simple modules. Assume that one of them is real. Then the following conditions are equivalent:

(i) $\vartheta(M, N) = 0$,

(ii) $r_{M,N}$ and $r_{N,M}$ are inverse to each other up to a constant multiple,

(iii) $M \circ N \simeq N \circ M$ up to a grading shift,

(iv) $M \circ N \simeq N \circ M$ up to a grading shift,

(v) $M \circ N$ is simple.

**Definition 4.9.** Let $M$ and $N$ be simple modules.

(i) We say that $M$ and $N$ commute if $\vartheta(M, N) = 0$.

(ii) We say that $M$ and $N$ are simply linked if $\vartheta(M, N) = 1$.

**Definition 4.10.** Let $M_1, \ldots, M_m$ be simple modules. Assume that they commute with each other. We set

\[
M_1 \bigodot M_2 := q^{\tilde{\Lambda}(M_1, M_2)} M_1 \circ M_2,
\]

\[
\bigodot_{1 \leq k \leq m} M_k := (\cdots (M_1 \bigodot M_2) \cdots) \bigodot M_{m-1} \bigodot M_m
\]

\[
\simeq q^{\sum_{1 \leq i < j \leq m} \tilde{\Lambda}(M_i, M_j)} M_1 \circ \cdots \circ M_m.
\]

It is invariant by the permutations of $M_1, \ldots, M_m$.

**Lemma 4.11 ([10], Lemma 2.15).** Let $M_1, \ldots, M_m$ be real simple modules commuting with each other. Then for any $\sigma \in S_m$, we have

\[
\bigodot_{1 \leq k \leq m} M_k \simeq \bigodot_{1 \leq k \leq m} M_{\sigma(k)} \text{ in } R\text{-gmod}.
\]

Moreover, if $M_k$’s are self-dual, so is

\[
\bigodot_{1 \leq k \leq m} M_k.
\]

**Lemma 4.12.** Let $\{M_i\}_{1 \leq i \leq n}$ and $\{N_i\}_{1 \leq i \leq n}$ be a pair of commuting families of real simple modules. We assume that
(a) \( \{M_i \diamond N_i\}_{1 \leq i \leq n} \) is a commuting family of real simple modules,
(b) \( M_i \diamond N_i \) commutes with \( N_j \) for any \( 1 \leq i, j \leq n \).

Then we have

\[
\bigotimes_{1 \leq i \leq n} M_i \diamond \bigotimes_{1 \leq j \leq n} N_j \simeq \bigotimes_{1 \leq i \leq n} (M_i \diamond N_i) \text{ up to a grading shift}
\]

Since the proof is elementary and similar to the proof of [10, Lemma 2.23], we omit it.

**Proposition 4.13.** Let \( L, M \) and \( N \) be simple modules. We assume that \( L \) is real and one of \( M \) and \( N \) is real.

(i) If \( \Lambda(L, M \diamond N) = \Lambda(L, M) + \Lambda(L, N) \), then \( L \circ M \circ N \) has a simple head and \( N \circ M \circ L \) has a simple socle.

(ii) If \( \Lambda(M \diamond N, L) = \Lambda(M, L) + \Lambda(N, L) \), then \( M \circ N \circ L \) has a simple head and \( L \circ N \circ M \) has a simple socle.

(iii) If \( \delta(L, M \diamond N) = \delta(L, M) + \delta(L, N) \), then \( L \circ M \circ N \) and \( M \circ N \circ L \) have a simple head, and \( N \circ M \circ L \) and \( L \circ N \circ M \) have a simple socle.

**Proof.** (i) Denote \( k = \Lambda(L, M \diamond N) = \Lambda(L, M) + \Lambda(M, N) \) and \( m = \Lambda(M, N) \). Then

\[
\begin{array}{ccc}
L \circ M \circ N & \xrightarrow{r_{L,M \circ N}} & q^{-k} M \circ N \circ L \\
L \circ (M \diamond N) & \xrightarrow{r_{L,M \diamond N}} & q^{-k} (M \diamond N) \circ L \\
q^{-m} L \circ N \circ M & \xrightarrow{r_{L,N \circ M}} & q^{-k-m} N \circ M \circ L
\end{array}
\]

commutes. Hence [10, Proposition 2.1, Proposition 2.2] implies that \( L \circ M \circ N \) has a simple head and \( N \circ M \circ L \) has a simple socle. (ii) are proved similarly.

(iii) If \( \delta(L, M \diamond N) = \delta(L, M) + \delta(L, N) \), the we have \( \Lambda(L, M \diamond N) = \Lambda(L, M) + \Lambda(L, N) \) and \( \Lambda(M \diamond N, L) = \Lambda(M, L) + \Lambda(N, L) \). Thus the third assertion follows from the first and second assertion. \( \square \)

**Proposition 4.14.** Let \( M \) and \( N \) be simple modules. Assume that one of them is real and \( \delta(M, N) = 1 \). Then we have an exact sequence

\[
0 \rightarrow q^{\Lambda(N,M)} N \diamond M \rightarrow M \circ N \rightarrow M \diamond N \rightarrow 0.
\]

In particular, \( M \circ N \) has length 2.

**Proof.** In the course of the proof, we ignore the grading.

Set \( X = M \diamond N \) and \( Y = N \circ M \). By \( R_{N,M_2}^{\text{ren}} : Y \hookrightarrow X \) let us regard \( Y \) as a submodule of \( X \). By the condition, we have \( R_{N,M_2}^{\text{ren}} \circ R_{M_2,N}^{\text{ren}} = \text{id}_X \) up to a constant multiple, and hence we have

\[
zX \subset Y \subset X.
\]
We have an exact sequence
\[ 0 \to \frac{Y}{zX} \to \frac{X}{zX} \to \frac{X}{Y} \to 0. \]
Since
\[ M \circ N \simeq \frac{X}{zX} \to \frac{X}{Y} \to \frac{z^{-1}Y}{Y} \simeq N \circ M, \]
we have \( \frac{X}{Y} \simeq M \circ N \). Similarly
\[ N \circ M \simeq \frac{Y}{zY} \to \frac{Y}{zX} \to \frac{X}{zX} \simeq M \circ N \]
implies that \( \frac{Y}{zX} \simeq N \circ M \). □

Lemma 4.15. Let \( M \) and \( N \) be simple modules. Assume that one of them is real. If the equation
\[ [M][N] = q^m[X] + q^n[Y] \]
holds in \( K(R\text{-gmod}) \) for self-dual simple modules \( X, Y \) and integers \( m, n \) such that \( m \geq n \), then we have
(i) \( \mathfrak{b}(M, N) = m - n > 0 \),
(ii) there exists an exact sequence
\[ 0 \to q^mX \to M \circ N \to q^nY \to 0, \]
(iii) \( q^mX \) is a socle of \( M \circ N \) and \( q^nY \) is a head of \( M \circ N \).

Proof. First note that \( \mathfrak{b}(M, N) > 0 \) since \( M \circ N \) is not simple. By the assumption there exists either an exact sequence
\[ 0 \to q^mX \to M \circ N \to q^nY \to 0, \]
or
\[ 0 \to q^nY \to M \circ N \to q^mX \to 0. \]
The last sequence cannot exist by [10, Corollary 2.24] because \( \mathfrak{b}(M, N) = n - m \leq 0 \). Hence the first sequence exists, and the assertions follow from loc.cit. □

4.3. Chevalley and Kashiwara operators. Let us recall the definition of several functors used to categorify \( U_q^{-}(g) \) by using KLR algebras.

Definition 4.16. Let \( \beta \in \mathbb{Q}^+ \).
(i) For \( i \in I \) and \( 1 \leq a \leq |\beta| \), set
\[ e_a(i) = \sum_{\nu \in P^\beta, \nu_a = i} e(\nu) \in R(\beta). \]
(ii) We take conventions:

\[ E_i^* M = e_i(i)M, \]
\[ E_i^* M = e_{|\beta|}(i)M, \]

which are functors from \( R(\beta)\)-gmod to \( R(\beta - \alpha_i)\)-gmod.

(iii) For a simple module \( M \), we set

\[ \varepsilon_i(M) = \max \{ n \in \mathbb{Z}_{\geq 0} \mid E_i^n M \neq 0 \}, \]
\[ \varepsilon_i^*(M) = \max \{ n \in \mathbb{Z}_{\geq 0} \mid E_i^* n M \neq 0 \}, \]
\[ F_i M = q^{\varepsilon_i(M)}L(i) \otimes M, \]
\[ F_i^* M = q^{\varepsilon_i^*(M)}M \otimes L(i), \]
\[ \tilde{E}_i M = q^{1-\varepsilon_i(M)}soc(E_i M) \cong q^{\varepsilon_i(M)-1}hd(E_i M), \]
\[ \tilde{E}_i^* M = q^{1-\varepsilon_i^*(M)}soc(E_i^* M) \cong q^{\varepsilon_i^*(M)-1}hd(E_i^* M), \]
\[ \tilde{E}_i^{\max} M = \tilde{E}_i^{\varepsilon_i(M)}M \quad \text{and} \quad \tilde{E}_i^{\max} = \tilde{E}_i^{\varepsilon_i^*(M)}M. \]

(iv) For \( i \in I \) and \( n \in \mathbb{Z}_{\geq 0} \), we set

\[ L(i^n) = q^{n(n-1)/2}L(i) \circ \cdots \circ L(i). \]

Here \( L(i) \) denotes the \( R(\alpha_i)\)-module \( R(\alpha_i)/R(\alpha_i)x_1 \). Then \( L(i^n) \) is a self-dual real simple \( R(n\alpha_i)\)-module.

In the course of the following propositions, we use the following notations.

\[ \overline{Q}_{i,j}(x_a, x_{a+1}, x_{a+2}) := \frac{Q_{i,j}(x_a, x_{a+1}) - Q_{i,j}(x_{a+2}, x_{a+1})}{x_a - x_{a+2}}. \]

Then we have

\[ \tau_a \tau_a \tau_{a+1} - \tau_a \tau_{a+1} \tau_a = \sum_{i,j \in I} \overline{Q}_{i,j}(x_a, x_{a+1}, x_{a+2})e_a(i)e_{a+1}(j)e_{a+2}(i). \]

**Proposition 4.17.** Let \( \beta \in \mathbb{Q}^+ \) with \( n = |\beta| \). Assume that an \( R(\beta)\)-module \( M \) satisfies \( E_i M = 0 \). Then the morphism \( R(\alpha_i) \otimes M \to q^{(\alpha_i, \beta)}M \circ R(\alpha_i) \) given by

\[ e(i) \otimes u \mapsto \tau_1 \cdots \tau_n (u \otimes e(i)) \]

extends uniquely to an \( (R(\alpha_i + \beta), R(\alpha_i))\)-bilinear homomorphism

\[ R(\alpha_i) \circ M \to q^{(\alpha_i, \beta)}M \circ R(\alpha_i) \]

**Proof.** (i) First note that, for \( 1 \leq a \leq n \),

\[ \tau_1 \cdots \tau_{a-1} e_a(i) \tau_{a+1} \cdots \tau_n (u \otimes e(i)) = \tau_{a+1} \cdots \tau_n (e_1(i) \tau_1 \cdots \tau_{a-1} (u \otimes e(i))) = 0 \]
since $E_i M = 0$.
(ii) In order to see that (4.7) is a well-defined $R(\alpha + \beta)$-linear homomorphism, it is enough to show that (4.6) is $R(\beta)$-linear.
(a) Commutation with $x_a \in R(\beta)$ ($1 \leq a \leq n$): We have
\[ x_{a+1} \tau_1 \cdots \tau_n (u \otimes e(i)) = \tau_1 \cdots \tau_{a-1} x_{a+1} \tau_a \cdots \tau_n (u \otimes e(i)) \]
\[ = \tau_1 \cdots \tau_{a-1} (\tau_a x_a + e_a(i)) \tau_{a+1} \cdots \tau_n (u \otimes e(i)) \]
\[ = \tau_1 \cdots \tau_n x_a (u \otimes e(i)) \]
by (4.8).
(b) Commutation with $\tau_a \in R(\beta)$ ($1 \leq a < n$): Then we have
\[ \tau_{a+1} \tau_1 \cdots \tau_n (u \otimes e(i)) \]
\[ = \tau_1 \cdots \tau_{a-1} (\tau_a \tau_{a+1}) \tau_{a+2} \cdots \tau_n (u \otimes e(i)) \]
\[ = \tau_1 \cdots \tau_{a-1} (\tau_a \tau_{a+1} + \sum_j Q_{i,j}(x_a, x_{a+1}, x_{a+2}) e_a(i) e_{a+1}(j) \tau_{a+2} \cdots \tau_n (u \otimes e(i)) \]
\[ = \tau_1 \cdots \tau_n \tau_a (u \otimes e(i)) \]
\[ + \sum_j \tau_1 \cdots \tau_{a-1} Q_{i,j}(x_a, x_{a+1}, x_{a+2}) e_a(i) e_{a+1}(j) \tau_{a+2} \cdots \tau_n (u \otimes e(i)). \]
The last term vanishes because $E_i M = 0$ implies that
\[ \tau_1 \cdots \tau_{a-1} f(x_a, x_{a+1}) g(x_{a+2}) e_a(i) \tau_{a+2} \cdots \tau_n (u \otimes e(i)) \]
\[ = g(x_{a+2}) \tau_{a+2} \cdots \tau_n e_1(i) \tau_1 \cdots \tau_{a-1} f(x_a, x_{a+1}) (u \otimes e(i)) = 0 \]
for any polynomial $f(x_a, x_{a+1})$ and $g(x_{a+2})$.
(iii) Now let us show that (4.7) is right $R(\alpha)$-linear. By (4.8), we have
\[ \tau_1 \cdots \tau_{a-1} x_a \tau_a \cdots \tau_n (u \otimes e(i)) = \tau_1 \cdots \tau_{a-1} (\tau_a x_a + e_a(i)) \tau_{a+1} \cdots \tau_n (u \otimes e(i)) \]
\[ = \tau_1 \cdots \tau_a x_a \tau_{a+1} \cdots \tau_n (u \otimes e(i)) \]
for $1 \leq a \leq n$. Therefore we have
\[ x_1 \tau_1 \cdots \tau_n (u \otimes e(i)) = \tau_1 \cdots \tau_n x_{n+1} (u \otimes e(i)) = \tau_1 \cdots \tau_n (u \otimes e(i) x_1). \]
\[ \square \]

For $m, n \in \mathbb{Z}_{\geq 0}$, let us denote by $w[m, n]$ the element of $\mathcal{S}_{m+n}$ defined by
\begin{equation}
(4.9) \quad w[m, n](k) = \begin{cases} 
  k + n & \text{if } 1 \leq k \leq m, \\
  k - m & \text{if } m < k \leq m + n.
\end{cases}
\end{equation}
Set $\tau_{w[m, n]} := \tau_{i_1} \cdots \tau_{i_v}$, where $s_{i_1} \cdots s_{i_v}$ is a reduced expression of $w[m, n]$. Note that $\tau_{w[m, n]}$ does not depend on the choice of reduced expression ([8, Corollary 1.4.3]).
Proposition 4.18. Let $M \in R(\beta)$-gmod and $N \in R(\gamma)$-gmod, and set $m = |\beta|$ and $n = |\gamma|$. If $E_iM = 0$ for any $i \in \text{supp}(\gamma)$, then

$$v \otimes u \mapsto \tau_{w,m,n}(u \otimes v)$$

(4.10)

gives a well-defined $R(\beta + \gamma)$-linear homomorphism $N \circ M \rightarrow q^{(\beta,\gamma)}M \circ N$.

Proof. The proceeding proposition implies that

$$v \otimes u \mapsto \tau_{w,m,n}(u \otimes v)$$

for $u \in M, v \in R(\gamma)$

gives a well-defined $R(\beta + \gamma)$-linear homomorphism $R(\gamma) \circ M \rightarrow M \circ R(\gamma)$. Hence it is enough to show that it is right $R(\gamma)$-linear. Since we know that it commutes with the right multiplication of $x_k$, it is enough to show that it commutes with the right multiplication of $\tau_k$. For this, we may assume that $n = 2$ and $k = 1$. Set $\gamma = \alpha_i + \alpha_j$.

Thus we have reduced the problem to the equality

$$(\tau_1 \tau_{2\tau_1}) \cdots (\tau_{m+1}\tau_m)(u \otimes e(i) \otimes e(j)) = (\tau_2 \tau_1) \cdots (\tau_{m+1}\tau_m)(u \otimes e(i) \otimes e(j))$$

(4.11)

for $u \in M$. It is a consequence of

$$(\tau_2 \tau_1) \cdots (\tau_{a+1}\tau_a)(\tau_{a+2}\tau_{a+1}) \cdots (\tau_{m+1}\tau_m)(u \otimes e(i) \otimes e(j))$$

(4.12)

for $1 \leq a \leq m$. Note that

$$\tau_a(\tau_{a+1}\tau_a) \cdots (\tau_{m+1}\tau_m)(u \otimes e(i) \otimes e(j)) = \tau_a(\tau_{a+1}\tau_a)e_{a+1}(i)e_{a+2}(j)(\tau_{a+2}\tau_{a+1}) \cdots (\tau_{m+1}\tau_m)(u \otimes e(i) \otimes e(j))$$

and

$$\tau_a(\tau_{a+1}\tau_a)e_{a+1}(i)e_{a+2}(j) = (\tau_{a+1}\tau_a)\tau_{a+1}e_{a+1}(i)e_{a+2}(j) - \overline{Q}_{j,i}(x_a, x_{a+1}, x_{a+2})e_a(j)e_{a+1}(i)e_{a+2}(j)$$

Hence it is enough to show

$$(\tau_2 \tau_1) \cdots (\tau_{a+1}\tau_a)\overline{Q}_{j,i}(x_a, x_{a+1}, x_{a+2})e_a(j)(\tau_{a+2}\tau_{a+1}) \cdots (\tau_{m+1}\tau_m)(u \otimes e(i) \otimes e(j)) = 0.$$ 

This follows from

$$(\tau_2 \tau_1) \cdots (\tau_{a+1}\tau_a)f(x_a)g(x_{a+1}, x_{a+2})e_a(j)(\tau_{a+2}\tau_{a+1}) \cdots (\tau_{m+1}\tau_m)(u \otimes e(i) \otimes e(j))$$

$$= (\tau_2 \cdots \tau_a)(\tau_1 \cdots \tau_{a-1})f(x_a)g(x_{a+1}, x_{a+2})e_a(j)(\tau_{a+2}\tau_{a+1}) \cdots (\tau_{m+1}\tau_m)(u \otimes e(i) \otimes e(j))$$

$$= (\tau_2 \cdots \tau_a)g(x_{a+1}, x_{a+2})(\tau_{a+2}\tau_{a+1}) \cdots (\tau_{m+1}\tau_m)e_1(j)(\tau_1 \cdots \tau_{a-1})f(x_a)(u \otimes e(i) \otimes e(j))$$

$$= 0$$

for $1 \leq a \leq m$ and $f(x_a) \in k[x_a]$, $g(x_{a+1}, x_{a+2}) \in k[x_{a+1}, x_{a+2}]$. \qed
Corollary 4.19. Let $i \in I$ and $M$ a simple module. Then we have
\[
\tilde{\Lambda}(L(i), M) = \varepsilon_i(M),
\]
\[
\Lambda(L(i), M) = 2\varepsilon_i(M) + \langle h_i, \text{wt}(M) \rangle = \varepsilon_i(M) + \varphi_i(M).
\]

Proof. Set $n = \varepsilon_i(M)$ and $M_0 = E_i^{(n)}(M)$. Then the preceding proposition implies
\[
\Lambda(L(i), M_0) = (\alpha_i, \text{wt}(M_0)).
\]
Hence we have
\[
\tilde{\Lambda}(L(i), M_0) = 0,
\]
which implies
\[
\tilde{\Lambda}(L(i), M_0) = 0.
\]
□

Let $P(i^n)$ be a projective cover of $L(i^n)$. The functor
\[
E_i^{(n)} : R(\beta)\text{-Mod} \to R(\beta - n\alpha_i)\text{-Mod}
\]
is defined by
\[
E_i^{(n)}(M) := P(i^n) \psi \otimes_{R(n\alpha_i)} E_i^n M,
\]
where $P(i^n) \psi$ denotes the right $R(n\alpha_i)$-module obtained from the left $R(\beta)$-module $P(i^n)$ via the anti-automorphism $\psi$. Similarly we define $E_i^{* (n)}$. We have
\[
E_i^n \simeq [n]!E_i^{(n)}.
\]

Proposition 4.20. Let $M$ and $N$ be modules, and $m, n \in \mathbb{Z}_{\geq 0}$.

(i) If $E_i^{m+1}M = 0$ and $E_i^{n+1}N = 0$, then we have
\[
E_i^{(m+n)}(M \circ N) \simeq q^{mn+\langle h_i, \text{wt}(M) \rangle} E_i^{(m)} M \circ E_i^{(n)} N.
\]

(ii) If $E_i^{* m+1}M = 0$ and $E_i^{* n+1}N = 0$, then we have
\[
E_i^{* (m+n)}(M \circ N) \simeq q^{mn+\langle h_i, \text{wt}(N) \rangle} E_i^{* (m)} M \circ E_i^{* (n)} N.
\]

Proof. It follows from the shuffle lemma ([16, Lemma 2.20]). □

The following corollary is an immediate consequence of Proposition 4.20.

Corollary 4.21. Let $i \in I$ and let $M$ be a real simple module. Then $\tilde{E}_i^{\text{max}} M$ is also real simple.

Proposition 4.22. Let $M$ and $N$ be simple modules. We assume that one of them is real. Assume further that $\varepsilon_i(M \circ N) = \varepsilon_i(M)$. Then we have an isomorphism in $R$-gmod.
\[
\tilde{E}_i^{\text{max}} (M \circ N) \simeq (\tilde{E}_i^{\text{max}} M) \circ N.
\]
Proof. Set \( n = \varepsilon_i(M \otimes N) = \varepsilon_i(M) \) and \( M_0 = \widetilde{E}_i^{\text{max}}M \). Then \( M_0 \) or \( N \) is real. Now we have
\[
L(i^n) \otimes M_0 \otimes N \hookrightarrow F_i^n(M \otimes N) \cong L(i^n) \otimes \widetilde{E}_i^{\text{max}}(M \otimes N),
\]
which induces a non-zero map \( M_0 \otimes N \rightarrow \widetilde{E}_i^{\text{max}}(M \otimes N) \). Hence we have a surjective map
\[
M_0 \otimes N \rightarrow \widetilde{E}_i^{\text{max}}(M \otimes N).
\]
Since \( M_0 \) or \( N \) is real by Corollary 4.21, \( M_0 \otimes N \) has a simple head and we obtain the desired result. \( \Box \)

4.4. Determinantal modules and T-systems. In the rest of this paper, we assume further that the base field \( k \) is of characteristic 0. Under this condition, the family of self-dual simple \( R \)-modules corresponds to the upper global basis of \( A_q(n) \) by Theorem 4.2.

Definition 4.23. For \( \lambda \in \mathbb{P}^+ \) and \( \mu, \zeta \in W\lambda \) such that \( \mu \preceq \zeta \), let \( M(\mu, \zeta) \) be a simple \( R(\zeta - \mu) \)-module such that \( \text{ch}(M(\mu, \zeta)) = D(\mu, \zeta) \).

Since \( D(\mu, \zeta) \) is a member of the upper global basis, such a module exists uniquely due to Theorem 4.2. The module \( M(\mu, \zeta) \) is self-dual and we call it the determinantal module.

Lemma 4.24. \( M(\mu, \zeta) \) is a real simple module.

Proof. It follows from \( \text{ch}(M(\mu, \zeta) \circ M(\mu, \zeta)) = \text{ch}(M(\mu, \zeta))^2 = q^{-\langle \zeta, \zeta - \mu \rangle}D(2\mu, 2\zeta) \) which is a member of the upper global basis up to a power of \( q \). Here the last equality follows from Corollary 3.3. \( \Box \)

Proposition 4.25. Let \( \lambda, \mu \in \mathbb{P}^+ \), and \( s, s', t, t' \in W \) such that \( \ell(s's) = \ell(s') + \ell(s) \), \( \ell(t't) = \ell(t') + \ell(t) \), \( s's\lambda \preceq t\lambda \) and \( s'\mu \preceq t't\mu \). Then
(i) \( M(s's\lambda, t\lambda) \) and \( M(s'\mu, t't\mu) \) commute,
(ii) \( \Lambda(M(s's\lambda, t\lambda), M(s'\mu, t't\mu)) = (s's\lambda + t\lambda, t't\mu - s'\mu) \).
(iii) \( \widetilde{\Lambda}(M(s's\lambda, t\lambda), M(s'\mu, t't\mu)) = (t\lambda, t't\mu - s'\mu) \) and
\( \widetilde{\Lambda}(M(s'\mu, t't\mu), M(s's\lambda, t\lambda)) = (s'\mu - t't\mu, s's\lambda) \).

Proof. It is a consequence of Proposition 3.6 (ii) and [10, Corollary 3.4]. \( \Box \)

Proposition 4.26. Let \( \lambda \in \mathbb{P}^+ \), \( \mu, \zeta \in W\lambda \) such that \( \mu \preceq \zeta \) and \( i \in I \).
(i) If \( n := \langle h_i, \mu \rangle \geq 0 \), then
\[ \varepsilon_i(M(\mu, \zeta)) = 0 \quad \text{and} \quad M(s_i\mu, \zeta) \simeq \widetilde{F}_i^mM(\mu, \zeta) \simeq L(i^n) \circ M(\mu, \zeta) \text{ in } R\text{-gmod}. \]
(ii) If \( \langle h_i, \mu \rangle \leq 0 \) and \( s_i\mu \preceq \zeta \), then \( \varepsilon_i(M(\mu, \zeta)) = -\langle h_i, \mu \rangle \).
(iii) If \( m := -\langle h_i, \zeta \rangle \leq 0 \), then
\[ \varepsilon_i^*(M(\mu, \zeta)) = 0 \quad \text{and} \quad M(\mu, s_i\zeta) \simeq \widetilde{F}_i^{-m}M(\mu, \zeta) \simeq M(\mu, \zeta) \circ L(i^m) \text{ in } R\text{-gmod}. \]
(iv) If \( \langle h_i, \zeta \rangle \geq 0 \) and \( \mu \preceq s_i\zeta \), then \( \varepsilon_i^*(M(\mu, \zeta)) = \langle h_i, \zeta \rangle \).
Proof. It is a consequence of Lemma 3.5. □

Proposition 4.27. Assume that $u, v \in W$ and $i \in I$ satisfy $u < us_i$ and $v < vs_i \leq u$.

(i) We have exact sequences

\[ 0 \rightarrow M(u, v) \rightarrow q^{(us_i v, vs_i v)} M(us_i, vs_i) \circ M(u, v) \rightarrow 0, \]

and

\[ 0 \rightarrow q^{1+(us_i v, vs_i v)} M(us_i, vs_i) \circ M(u, v) \rightarrow 0, \]

where $\lambda = s_i v + v = \sum_{j \neq i} -a_{j, i} w_j$.

(ii) $b(M(u, v), M(us_i, vs_i)) = 1$.

Proof. Since the proof of (4.15) is similar, let us only prove (4.16). (Indeed, they are dual to each other.)

Set

\[ X = q^{(us_i v, vs_i v)} M(us_i, vs_i) \circ M(u, v), \]

\[ Y = q^{1+(us_i v, vs_i v)} M(us_i, vs_i) \circ M(u, v), \]

\[ Z = M(u, v). \]

Then Proposition 3.10 implies that

\[ \text{ch}(Y) = \text{ch}(qX) + \text{ch}(Z). \]

Since $X$ and $Z$ are simple and self-dual, the assertion follows from Lemma 4.15. □

4.5. Generalized T-system on determinantal modules.

Theorem 4.28. Let $\lambda \in \mathcal{P}^+$ and $\mu_1, \mu_2, \mu_3 \in W \lambda$ such that $\mu_1 \leq \mu_2 \leq \mu_3$. Then there exists a canonical epimorphism

\[ M(\mu_1, \mu_2) \circ M(\mu_2, \mu_3) \rightarrow M(\mu_1, \mu_3), \]

which is equivalent to saying that $M(\mu_1, \mu_2) \circ M(\mu_2, \mu_3) \simeq M(\mu_1, \mu_3)$.

In particular, we have

\[ \Lambda(M(\mu_1, \mu_2), M(\mu_2, \mu_3)) = 0 \quad \text{and} \quad \Lambda(M(\mu_1, \mu_2), M(\mu_2, \mu_3)) = -(\mu_1 - \mu_2, \mu_2 - \mu_3). \]

Proof. (a) Our assertion follows from Theorem 3.13 and [10, Theorem 3.6] when $\mu_3 = \lambda$.

(b) We shall prove the general case by induction on $|\lambda - \mu_3|$. By (a), we may assume that $\mu_3 \neq \lambda$. Then there exists $i$ such that $\langle h_i, \mu_3 \rangle < 0$. The induction hypothesis implies that

\[ M(\mu_1, \mu_2) \circ M(\mu_2, s_i \mu_3) \simeq M(\mu_1, s_i \mu_3). \]

Since $\mu_1 \leq \mu_2 \leq \mu_3 \leq s_i \mu_3$, Proposition 4.26 (iv) implies that

\[ \varepsilon_i^*(M(\mu_2, s_i \mu_3)) = \varepsilon_i^*(M(\mu_1, s_i \mu_3)) = -\langle h_i, \mu_3 \rangle. \]
Then Proposition 4.22 implies that
\[ \tilde{E}^{\gamma_{\text{max}}}(M(\mu_1, \mu_2) \triangledown M(\mu_2, s_i \mu_3)) \simeq M(\mu_1, \mu_2) \triangledown (\tilde{E}^{\gamma_{\text{max}}}M(\mu_2, s_i \mu_3)), \]
from which we obtain
\[ M(\mu_1, \mu_3) \simeq M(\mu_1, \mu_2) \triangledown M(\mu_2, \mu_3). \]

Lemma 4.11 implies that \( \widetilde{\Lambda}(M(\mu_1, \mu_2), M(\mu_2, \mu_3)) = 0 \). Hence we have
\[ \Lambda(M(\mu_1, \mu_2), M(\mu_2, \mu_3)) = -(\text{wt}(M(\mu_1, \mu_2), \text{wt}(M(\mu_2, \mu_3))). \]

\[ \square \]

**Proposition 4.29.** Let \( i \in I \) and \( x, y, z \in W \).

(i) If \( \ell(xy) = \ell(x) + \ell(y) \), \( z s_i > z, y > y s_i \), and \( x \geq z \), then we have
\[ \mathfrak{b}(M(xy z s_i, z s_i z) , M(x z s_i, z s_i z)) \leq 1. \]

(ii) If \( \ell(zy) = \ell(z) + \ell(y) \), \( x s_i > z, x s_i \geq z y \), and \( x \geq z \), then we have
\[ \mathfrak{b}(M(x s_i z s_i, z y s_i) , M(x s_i z s_i) \leq 1. \]

**Proof.** In the course of proof, we omit \( \tau_i^{-1} \) for the sake of simplicity. Set \( y' = y s_i < y \).

Let us show (i). By Proposition 3.15, we have
\[ \Delta(y z s_i, z s_i z) \Delta(w z s_i, s_i z) = q^{-1}G_{\text{up}} \left( (u y z s_i \otimes u z s_i) \otimes (u s_i z s_i \otimes u z s_i) \right) \]
\[ + G_{\text{up}}(\tau_i^{-1}(s_i^* b) \otimes u_i^*), \]
where \( \lambda = z s_i + s_i z \) and \( b = \tau_i^{-1}(u y z s_i) \in B(\infty) \). Let \( S_i^* \) be the operator on \( A_q(\mathfrak{g}) \) given by the multiplication of \( e_{(a_1)}^{(r)} \cdots e_{(a_r)}^{(r)} \) from the right, where \( z = s_j \cdots s_{j_i} \) is a reduced expression of \( z \) and \( a_k = \langle h_{j_k} , s_{k-1} \cdots s_{j_i} \rangle \). Then applying \( S_i^* \) to (4.17), we obtain
\[ \Delta(y z s_i, z s_i z) \Delta(w z s_i, z s_i z) = q^{-1}G_{\text{up}} \left( (u y z s_i \otimes u z s_i) \otimes (u s_i z s_i \otimes u z s_i) \right) \]
\[ + G_{\text{up}}(\tau_i^{-1}(s_i^* b) \otimes u_i^*). \]

Recall that \( \mu \in \mathcal{P} \) is called \( x \)-dominant if \( c_k \geq 0 \). Here \( x = s_i \cdots s_{i_k} \) is a reduced expression of \( x \) and \( c_k = \langle h_{i_k}, s_{i_k-1} \cdots s_{i_1} \rangle \) \((1 \leq k \leq r)\). Recall that an element \( v \in A_q(\mathfrak{g}) \) with \( w_{t_i}(v) = \mu \) is called \( x \)-highest if \( x \)-dominant and
\[ f_{ik}^{c_k} f_{ik-1}^{c_{k-1}} \cdots f_{i_1}^{c_1} v = 0 \]
for any \( k \) \((1 \leq k \leq r)\).

If \( v \) is \( x \)-highest, then \( v \) is a linear combination of \( x \)-highest \( G_{\text{up}}(b) \)'s. Moreover, \( S_{x,\mu}G_{\text{up}}(b) := f_{ik}^{c_k} \cdots f_{i_1}^{c_1}G_{\text{up}}(b) \) is either a member of the upper global basis or zero. Since \( \Delta(y z s_i, z s_i z) \Delta(w z s_i, z s_i z) \) is \( x \)-highest of weight \( \mu := z s_i + w z s_i \), we obtain
\[ \Delta(x y z s_i, z s_i z) \Delta(x z s_i, z s_i z) = q^{-1}G_{\text{up}} \left( (u x y z s_i \otimes u x z s_i) \otimes (u s_i z s_i \otimes u z s_i) \right) \]
\[ + S_{x,\mu}G_{\text{up}}(\tau_i^{-1}(s_i^* b) \otimes u_i^*). \]

Applying \( p_n \), we obtain
\[ q^n D(x y z s_i, z s_i z) D(x z s_i, z s_i z) = q^{-1}p_n G_{\text{up}} \left( (u x y z s_i \otimes u x z s_i) \otimes (u s_i z s_i \otimes u z s_i) \right) \]
for some integer $c$. Hence we obtain (i) by Lemma 4.15.

(ii) is proved similarly. By applying $\varphi^*$ to (4.17), we obtain

$$q^{(s_1w_{1i}, w_{1i})} - (y_{1i}, w_{1i}) \Delta(s_1w_{1i}, y_{1i}) \Delta(w_{1i}, w_i) = q^{-1} G^{ap} \left( (u_{s_1w_{1i}} \otimes u_{w_{1i}}) \otimes (u_{y_{1i}} \otimes u_{w_{1i}})^r \right) \ + \ G^{ap} \left( u_{\lambda} \otimes (\tau^{-1}_\lambda \epsilon^*_i b)^r \right).$$

Here we used Proposition 2.4. Then the similar arguments as above show (ii). \hfill \Box

**Proposition 4.30.** Let $x \in W$ such that $xs_i > x$ and $xw_i \neq w_i$. Then we have

$$\mathfrak{b}(M(xs_iw_{1i}, xw_i), M(xw_i, w_i)) = 1.$$

**Proof.** By Proposition 4.29 (ii), we have $\mathfrak{b}(M(xs_iw_{1i}, xw_i), M(xw_i, w_i)) \leq 1$. Assuming $\mathfrak{b}(M(xs_iw_{1i}, xw_i), M(xw_i, w_i)) = 0$, let us derive a contradiction.

By Theorem 4.28 and the assumption, we have

$$M(xs_iw_{1i}, xw_i) \circ M(xw_i, w_i) \simeq M(xs_iw_{1i}, w_i).$$

Hence we have

$$\varepsilon^*_j(M(xs_iw_{1i}, w_i)) = \varepsilon^*_j(M(xs_iw_{1i}, xw_i)) + \varepsilon^*_j(M(xw_i, w_i))$$

for any $j \in I$. Since $xs_iw_{1i} \leq xw_i \leq s_1w_i$, Proposition 4.26 implies that

$$\varepsilon^*_j(M(xs_iw_{1i}, w_i)) = \varepsilon^*_j(M(xw_i, w_i)) = (h_j, w_i).$$

It implies that

$$\varepsilon^*_j(M(xs_iw_{1i}, xw_i)) = 0 \quad \text{for any } j \in I.$$

It is a contradiction since $\text{wt}(M(xs_iw_{1i}, xw_i)) = xs_iw_{1i} - xw_i$ does not vanish. \hfill \Box

5. **Quantum cluster algebras and monoidal categorifications**

5.1. **Quantum cluster algebras.** In this subsection, we briefly recall the definition of a quantum cluster algebra following [2], [6, §8] and [10, §4.1].

Let us fix a finite index set $J$ with a decomposition $J = J_{\text{ex}} \sqcup J_{\text{fr}}$ into the set $J_{\text{ex}}$ of exchangeable indices and the set $J_{\text{fr}}$ of frozen indices. Let $L = (\lambda_{i,j})_{i,j \in J}$ be a skew-symmetric integer-valued $J \times J$-matrix. We denote by $\mathcal{P}(L)$ the $\mathbb{Z}[q^{\pm1/2}]$-algebra generated by $\{X_i\}_{i \in J}$ subject to the following defining relations:

$$X_iX_j = q^{\lambda_{i,j}}X_jX_i \quad (i, j \in J).$$

For $a := (a_i)_{i \in J} \in \mathbb{Z}_{\geq 0}^J$, we define the element $X^a$ of $\mathcal{P}(L)$ as

$$X^a := q^{1/2 \sum_{i>j} a_ia_j \lambda_{i,j}} \prod_{i \in J} X_i^{a_i}.$$
where $<$ is a total order on $J$ and $\prod_{i \in J} X^a_i = X^{a_1}_{i_1} \cdots X^{a_r}_{i_r}$ with $J = \{i_1, \ldots, i_r\}$ such that $i_1 < i_2 < \cdots < i_r$. Note that $X^a$ does not depend on the choice of a total order on $J$. We have

$$X^a X^b = q^{1/2 \sum_{i,j \in J} a_i b_j} X^{a+b} \quad \text{for all } a, b \in \mathbb{Z}_{\geq 0}^J.$$  

It is well-known that $\{X^a \mid a \in \mathbb{Z}_{\geq 0}^J\}$ forms a $\mathbb{Z}[q^{\pm 1/2}]$-basis of $\mathcal{P}(L)$.

For a $\mathbb{Z}[q^{\pm 1/2}]$-algebra $A$, we say that a family of elements $\{x_i\}_{i \in J}$ of $A$ is $L$-commuting if it satisfies the relation (5.1), i.e., $x_i x_j = q^{\lambda_{i,j}} x_j x_i$.

Let $\widetilde{B} = (b_{i,j})$ be an integer-valued $J \times J_{\text{ex}}$ matrix whose principal part $B := (b_{i,j})_{i,j \in J_{\text{ex}}}$ is skew-symmetric. To the matrix $\widetilde{B}$ we can associate the quiver $Q_{\widetilde{B}}$ without loops, 2-cycles and arrows between frozen vertices such that its vertices are labeled by $J$ and the arrows are given by

$$b_{i,j} = (\text{the number of arrow from } i \text{ to } j) - (\text{the number of arrow from } j \text{ to } i).$$

Here we extend the $J \times J_{\text{ex}}$-matrix $\widetilde{B}$ to the skew-symmetric $J \times J$-matrix $\widetilde{B}' = (b_{i,j})_{i,j \in J}$ by setting $b_{i,j} = 0$ for $i, j \in J_{\text{fr}}$.

Conversely, whenever we have a quiver with vertices labeled by $J$ and without loops, 2-cycles and arrows between frozen vertices, we can associate a $J \times J_{\text{ex}}$-matrix $\widetilde{B}$ by (5.3).

We say that the pair of matrices $(L, \widetilde{B})$ is compatible if there exists a positive integer $d$ such that

$$\sum_{k \in J} \lambda_{i,k} b_{k,j} = \delta_{i,j} d, \quad (i \in J, \ j \in J_{\text{ex}}).$$

For a $\mathbb{Z}[q^{\pm 1/2}]$-algebra $A$ and a compatible pair $(L, \widetilde{B})$, we say that the datum $\mathcal{S} = (\{x_i\}_{i \in J}, L, \widetilde{B})$ is a quantum seed if $\{x_i\}_{i \in J}$ is an algebraically independent $L$-commuting family of elements in $A$.

The set $\{x_i\}_{i \in J}$ is called a cluster of $\mathcal{S}$ and its elements are called cluster variables.

In particular, the elements in $\{x_i\}_{i \in J_{\text{fr}}}$ are called frozen variables. The elements in $\{x^a \mid a \in \mathbb{Z}_{\geq 0}^J\}$ are called cluster monomials.

See [10] for the definition of

$$\mu_k(\mathcal{S}) := (\{\mu_k(x)^i\}_{i \in J}, \mu_k(L), \mu_k(\widetilde{B})), $$

the mutation of $\mathcal{S}$ in direction $k \in J_{\text{ex}}$.

**Definition 5.1.** Let $\mathcal{S} = (\{x_i\}_{i \in J}, L, \widetilde{B})$ be a quantum seed in a $\mathbb{Z}[q^{\pm 1/2}]$-algebra $A$ which is contained a skew field $K$. The quantum cluster algebra $\mathcal{A}_{q^{1/2}}(\mathcal{S})$ associated with the quantum seed $\mathcal{S}$ is the $\mathbb{Z}[q^{\pm 1/2}]$-subalgebra of the skew field $K$ generated by all the quantum cluster variables in the quantum seeds obtained from $\mathcal{S}$ by any sequence of mutations.

We call $\mathcal{S}$ the initial quantum seed of the quantum cluster algebra $\mathcal{A}_{q^{1/2}}(\mathcal{S})$. 
5.2. Monoidal categorifications of quantum cluster algebras. In this subsection, we shall review monoidal categorifications of cluster algebras. In this paper, we shall restrict ourselves to the case when monoidal categories are full subcategories of $R$-gmod. Here $R$-gmod is the category of finite-dimensional graded modules over a symmetric KLR algebra $R$ with the base field $k$ of characteristic 0. We refer to [9] in a more general setting.

Let $C$ be a full subcategory of $R$-gmod stable by taking convolution products, subquotients, extensions and grading shifts. Hence $C$ is a $k$-linear abelian monoidal category with the decomposition

$$C = \bigoplus_{\beta \in \mathbb{Q}^*} C_\beta \text{ with } C_\beta := C \cap R(-\beta)\text{-gmod}.$$  

**Definition 5.2.** We call a quadruple $\mathcal{S} = (\{M_i\}_{i \in J}, L, \tilde{B}, D)$ a quantum monoidal seed in $C$ if it satisfies the following conditions:

(i) $\tilde{B} = (b_{i,j})$ is an integer-valued $J \times J_{\text{ex}}$-matrix whose principal part is skew-symmetric.

(ii) $L = (\lambda_{i,j})$ is an integer-valued skew-symmetric $J \times J$-matrix.

(iii) $D = \{d_i\}_{i \in J}$ is a family of elements in $\mathbb{Q}$.

(iv) $\{M_i\}_{i \in J}$ is a family of simple objects such that $M_i \in C_{d_i}$ for $i \in J$.

(v) $M_i \circ M_j \simeq q^{\lambda_{i,j}} M_j \circ M_i$ for all $i, j \in J$.

(vi) $M_{i_1} \circ \cdots \circ M_{i_t}$ is simple for every sequence $(i_1, \ldots, i_t)$ in $J$.

(vii) The pair $(L, \tilde{B})$ is compatible in the sense of (5.4) with $d = 2$.

(viii) $\lambda_{i,j} - (d_i, d_j) \in 2\mathbb{Z}$ for all $i, j \in J$.

(ix) $\sum_{i \in J} b_{i,k}d_i = 0$ for all $k \in J_{\text{ex}}$.

By (vi), every $M_i$ is real simple. The integer $\lambda_{i,j}$ in (ii) is given by $-\Lambda(M_i, M_j)$. Note that (viii) is redundant since it follows from $\Lambda(M_i, M_j) \in \mathbb{Z}$. For a $J \times J_{\text{ex}}$-matrix $\tilde{B}$ with skew-symmetric principal part and $D = \{d_i\}_{i \in J}$, we define the mutation $\mu_k(D) \in \mathbb{Q}^{J}$ of $D$ in direction $k$ with respect to $\tilde{B}$ by

$$\mu_k(D)_i = d_i \ (i \neq k), \quad \mu_k(D)_k = -d_k + \sum_{b_{i,k} > 0} b_{i,k}d_i.$$  

Note that

- $\mu_k(\mu_k(D)) = D$ (for $k \in J_{\text{ex}}$),
- $(\mu_k(L), \mu_k(\tilde{B}), \mu_k(D))$ satisfies the conditions (viii) and (ix) for any $k \in J_{\text{ex}}$.

**Definition 5.3** ([10, Definition 5.6]). For $k \in J_{\text{ex}}$, we say that a quantum monoidal seed $\mathcal{S} = (\{M_i\}_{i \in J}, L, \tilde{B}, D)$ admits a mutation in direction $k$ if there exists a simple object $M'_k \in C_{\mu_k(D)_k}$ such that

(i) there exist exact sequences in $C$

$$0 \to q \bigoplus_{b_{i,k} > 0} M_{i}^{\odot b_{i,k}} \to q^{m_k} M_k \circ M'_k \to q^{m_{-k}} M_{i}^{\odot(-b_{i,k})} \to 0,$$  

(5.6)
Definition 5.5. We assume the following:

(i) the quadruple \( \mu_k(\mathcal{S}) := (\{M_i\}_{i \neq k} \cup \{M'_k\}, \mu_k(L), \mu_k(\tilde{B}), \mu_k(D)) \) is a quantum monoidal seed in \( \mathcal{C} \).

(ii) the quadruple \( \mu_k(\mathcal{S}) := (\{M_i\}_{i \neq k} \cup \{M'_k\}, \mu_k(L), \mu_k(\tilde{B}), \mu_k(D)) \) is a quantum monoidal seed in \( \mathcal{C} \).

We call \( \mu_k(\mathcal{S}) \) the mutation of \( \mathcal{S} \) in direction \( k \).

Definition 5.4 ([10, Definition 5.8]). The category \( \mathcal{C} \) is called a monoidal categorification of a quantum cluster algebra \( A \) over \( \mathbb{Z}[q^{1/2}] \) if

(i) the Grothendieck ring \( \mathbb{Z}[q^{1/2}] \otimes \mathbb{Z}[q^{\pm 1}] K(\mathcal{C}) \) is isomorphic to \( A \),

(ii) there exists a quantum monoidal seed \( \mathcal{S} = (\{M_i\}_{i \in J}, L, \tilde{B}, D) \) in \( \mathcal{C} \) such that

\[
[\mathcal{S}] := (q^{-\{(d_i,d_i)\}}[M_i])_{i \in J}, L, \tilde{B} \]

is the image of a quantum seed of \( A \) by the isomorphism in (i).

(iii) \( \mathcal{S} \) admits successive mutations in all directions,

Note that if \( \mathcal{C} \) is a monoidal categorification of \( A \), then any quantum seed in \( A \) obtained by mutations from the initial quantum seed is of the form \( (q^{-\{(d_i,d_i)\}}[M_i])_{i \in J}, L, \tilde{B}) \) for some monoidal seed \( (\{M_i\}_{i \in J}, L, \tilde{B}, D) \). Thus any quantum cluster monomial in \( A \) coincides with \( q^{-(\text{wt}(S),\text{wt}(S)))} \) for some real simple module \( S \) in \( \mathcal{C} \).

5.3. Main result of [10]. In this subsection, we briefly recall the main result of [10].

Definition 5.5 ([10, Definition 6.1]). A pair \( \{\{M_i\}_{i \in J}, \tilde{B}\} \) is called admissible if

\( \lambda \) is an integer-valued \( J \times J \)-matrix with skew-symmetric principal part, and

(3) for each \( k \in J \), there exists a self-dual simple object \( M_i^k \) in \( \mathcal{C} \) such that there is an exact sequence in \( \mathcal{C} \)

\[
0 \to q \bigoplus_{i\neq k} M_i^k \to q^{-1} M_k \otimes M'_k \to q^{-1} M'_k \to 0,
\]

and \( M'_k \) commutes with \( M_i \) for all \( i \neq k \in J \).

For an admissible pair \( \{\{M_i\}_{i \in J}, \tilde{B}\} \), let \( \Lambda = (\Lambda_{i,j})_{i,j \in J} \) be the skew-symmetric matrix given by \( \Lambda_{i,j} = \Lambda(M_i, M_j) \). and let \( D = \{d_i\}_{i \in J} \) be the family of elements of \( \mathbb{Q}^{-} \) given by \( d_i = \text{wt}(M_i) \).

Theorem 5.6 ([10, Theorem 6.3, Corollary 6.4]). Let \( \mathcal{S} = (\{M_i\}_{i \in J}, -\Lambda, \tilde{B}, D) \) be a quantum monoidal seed. We assume the following:

The \( \mathbb{Q}(q^{1/2}) \)-algebra \( \mathbb{Q}(q^{1/2}) \otimes \mathbb{Z}[q^{\pm 1}] K(\mathcal{C}) \) is isomorphic to \( \mathbb{Q}(q^{1/2}) \otimes \mathbb{Z}[q^{\pm 1}] [\mathcal{S}]^{1/2} \).
If the pair \((\{M_i\}_{i \in I}, \widetilde{B})\) is admissible, then \(\mathcal{F}\) admits successive mutations in all directions and the category \(\mathcal{C}\) is a monoidal categorification of \(A_{q/2}(\mathcal{F})\).

6. Monoidal categorification of \(A_q(n(w))\)

6.1. Quantum cluster algebra structure on \(A_q(n(w))\). In this subsection, we shall consider the Kac-Moody algebra \(\mathfrak{g}\) associated with a symmetric Cartan matrix \(A = (a_{i,j})_{i,j \in I}\). We shall recall briefly the definition of the subalgebra \(A_q(n(w))\) of \(A_q(\mathfrak{g})\) and its quantum cluster algebra structure by using the results of [6] and [18]. Remark that we bring the results in [6] through the isomorphism in (2.7).

For a given \(w \in W\), fix a reduced expression \(\tilde{w} = s_{i_1}\ldots s_{i_r}\). For \(s \in \{1,\ldots,r\}\) and \(j \in I\), we set

\[
s_+ := \min(\{k \mid s < k \leq r, \ i_k = i_s\} \cup \{r + 1\}) \quad (6.1)
\]

\[
s_- := \max(\{k \mid 1 \leq s < k, \ i_k = i_s\} \cup \{0\})
\]

\[
s^-(j) := \max(\{k \mid 1 \leq k < s, \ i_k = j\} \cup \{0\}).
\]

For \(1 \leq k \leq r\), set

\[
\lambda_k := u_k \overline{w}_{i_k} \text{ where } u_k := s_{i_1}\ldots s_{i_k}.
\]

Note that \(\lambda_{k_-} = u_{k-1} \overline{w}_{i_k}\). For \(0 \leq t \leq s \leq r\), we set

\[
D(s, t) = \begin{cases} 
D(\lambda_s, \lambda_t) & \text{if } 0 < t, \\
D(\lambda_s, \overline{w}_{i_s}) & \text{if } 0 = t < s \leq r, \\
1 & \text{if } t = s = 0.
\end{cases}
\]

The \(\mathbb{Q}(q)\)-subalgebra of \(A_q(n)\) generated by \(D(i, i_-)\) \((1 \leq i \leq r)\) is independent of the choice of a reduced expression of \(w\). We denote it by \(A_q(n(w))\). Then every \(D(s, t)\) \((0 \leq t \leq s \leq r)\) is contained in \(A_q(n(w))\) ([6, Corollary 12.4]). The set \(B^{op}(A_q(n(w))) := B^{op}(A_q(\mathfrak{g})) \cap A_q(n(w))\) is a basis of \(A_q(n(w))\) as a \(\mathbb{Q}(q)\)-vector space ([18, Theorem 4.2.5]). We call it the upper global basis of \(A_q(n(w))\). We denote by \(A_q(n)_{\mathbb{Z}[q^{\pm 1}]}\) the \(\mathbb{Z}[q^{\pm 1}]\)-module generated by \(B^{op}(A_q(n(w)))\). Then it is a \(\mathbb{Z}[q^{\pm 1}]\)-subalgebra of \(A_q(n(w))\)

\([18, \S 4.7.2]\). We set \(A_{q^{1/2}}(n(w)) := \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Q}(q)} A_q(n(w))\).

We set \(J = \{1,\ldots,r\}, \ J_{fr} := \{k \in J \mid k_+ = r + 1\}\) and \(J_{fr} := J \setminus J_{fr}\).

**Definition 6.1.** We define the quiver \(Q\) with the set of vertices \(Q_0\) and the set of arrows \(Q_1\) as follows:

\[
(Q_0) \quad Q_0 = J = \{1,\ldots,r\},
\]

\[
(Q_1) \text{ There are two types of arrows:}
\]

- **ordinary arrows** : \(s \xrightarrow{[a_{i_s,i_t}]_{\mathbb{Z}_{\geq 0}}} t \) if \(1 \leq s < t < s_+ < t_+ \leq r + 1\),

- **horizontal arrows** : \(s \rightarrow s_- \) if \(1 \leq s_- < s \leq r\).

Let \(\tilde{B} = (b_{i,j})\) be the integer-valued \(J \times J_{ex}\)-matrix associated to the quiver \(Q\) by (5.3).
Lemma 6.2. Assume that $0 \leq d \leq b \leq a \leq c \leq r$ and

- $i_b = i_a$ if $b \neq 0$,
- $i_d = i_c$ if $d \neq 0$.

Then $D(a, b)$ and $D(c, d)$ $q$-commute; that is, there exists $\lambda \in \mathbb{Z}$ such that

$$D(a, b)D(c, d) = q^\lambda D(c, d)D(a, b).$$

Proof. We may assume $a > 0$. Let $u_k$ be as in (6.2). Take $s' = u_a$, $s = u_a^{-1}u_c$, $t' = u_d$ and $t = u_d^{-1}u_b$. Then we have

$$D(s'\varpi_{i_a}, t'\varpi_{i_a}) = D(a, b) \quad \text{and} \quad D(s's\varpi_{i_c}, t'\varpi_{i_c}) = D(c, d).$$

From Proposition 3.6, our assertion follows.

Hence we have an integer-valued skew-symmetric matrix $L = (\lambda_{i,j})_{1 \leq i, j \leq r}$ such that

$$D(i, 0)D(j, 0) = q^{-\lambda_{i,j}}D(j, 0)D(i, 0).$$

Proposition 6.3 ([6, Proposition 10.1]). The pair $(L, \widetilde{B})$ is compatible with $d = 2$ in (5.4).

Theorem 6.4 ([6, Theorem 12.3]). Let $\mathcal{A}_q^{1/2}([\mathcal{F}])$ be the quantum cluster algebra associated to the initial quantum seed $[\mathcal{F}] := \{q^{-(d_s, d_s)/4}D(s, 0)\}_{1 \leq s \leq r}, L, \widetilde{B}$. Then we have an isomorphism

$$\mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} \mathcal{A}_q^{1/2}([\mathcal{F}]) \simeq A_q^{1/2}(\mathfrak{n}(w)),$$

where $d_s := \lambda_s - \varpi_{i_s} = \mathrm{wt}(D(s, 0))$ and $A_q^{1/2}(\mathfrak{n}(w)) := \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Q}(q)} A_q(\mathfrak{n}(w))$.

6.2. Admissible seeds in the monoidal category $\mathcal{C}_w$. For $0 \leq t \leq s \leq r$, we set $M(s, t) = M(\lambda_s, \lambda_t)$. It is a real simple module with $\mathrm{ch}(M(s, t)) = D(s, t)$.

Definition 6.5. For $w \in W$, let $\mathcal{C}_w$ be the smallest monoidal abelian full subcategory of $R$-gmod satisfying the following properties:

(a) It is stable under the subquotients, extensions and grading shifts.

(b) It contains $M(s, s_-)$ for all $1 \leq s \leq \ell(w)$.

Then by [6], $M \in R$-gmod belongs to $\mathcal{C}_w$ if and only if $\mathrm{ch}(M)$ belongs to $A_q(\mathfrak{n}(w))$.

Hence we have an $\mathbb{Z}[q^{\pm 1}]$-algebra isomorphism

$$K(\mathcal{C}_w) \simeq A_q(\mathfrak{n}(w))_{\mathbb{Z}[q^{\pm 1}]}.$$

We set

$$\Lambda := (\Lambda(M(i, 0), M(j, 0)))_{1 \leq i, j \leq r} \quad \text{and} \quad D = (d_i)_{1 \leq i \leq r} := (\mathrm{wt}(M(i, 0)))_{1 \leq i \leq r}.$$

Then, by Proposition 4.25, $\mathcal{F} := \{M(k, 0)\}_{1 \leq k \leq r}, -\Lambda, \widetilde{B}, D)$ is a quantum monoidal seed in $\mathcal{C}_w$. We are now ready to state the main theorem in this subsection:

Theorem 6.6. The pair $(\{M(k, 0)\}_{1 \leq k \leq r}, \widetilde{B})$ is admissible.

As we already explained, this theorem implies Theorem 2 in Introduction.
**Theorem 6.7.** The category $\mathcal{C}_w$ is a monoidal categorification of the quantum cluster algebra $A_{q^{1/2}}(n(w))$.

In the course of the proof of Theorem 6.6, we omit grading shifts if there is no afraid of confusion.

We shall start the proof of Theorem 6.6 by proving that, for each $s \in J_{ex}$, there exists a simple module $X$ such that

1. There exists a surjective homomorphism (up to a grading shift)
   $$X \circ M(s, 0) \twoheadrightarrow M(t, 0)^{ob_{t,s}};$$

2. There exists a surjective homomorphism (up to a grading shift)
   $$M(s, 0) \circ X \twoheadrightarrow M(t, 0)^{ob_{t,s}};$$

3. $\forall (X, M(s, 0)) = 1$.

We set

$$x := i_s \in I,$$

$$I_s := \{i_k \mid s < k < s_+\} \subset I \setminus \{x\},$$

$$A := \bigcirc_{t < s < t_+ < s_+} M(t, 0)^{ob_{t,s}} = \bigcirc_{y \in I_s} M(s^-(y), 0)^{ob_{a, y}}.$$

Then $A$ is a real simple module.

Now we claim that the following simple module $X$ satisfies the conditions in (6.4):

$$X := M(s_+, s) \diamond A.$$

Let us show (6.4) (a). The incoming arrows to $s$ are

- $t \xrightarrow{[a_x, i_t]} s$ for $1 \leq t < s < t_+ < s_+$,
- $s_+ \rightarrow s$.

Hence we have

$$\bigcirc_{t; b_{t,s} > 0} M(t, 0)^{ob_{t,s}} \simeq A \circ M(s_+, 0).$$

Then the morphism in (a) is obtained as the composition:

$$X \circ M(s, 0) \twoheadrightarrow A \circ M(s_+, s) \circ M(s, 0) \twoheadrightarrow A \circ M(s_+, 0).$$

Here the second epimorphism is given in Theorem 4.28, and [9, Corollary 3.11] asserts that the composition (6.5) is non-zero and hence an epimorphism.

Let us show (6.4) (b). The outgoing arrows from $s$ are

- $s \xrightarrow{[a_x, i_t]} t$ for $s < t < s_+ < t_+ \leq r + 1$.
- $s \rightarrow s_-$ if $s_- > 0$. 


Hence we have
\[(6.6) \quad \bigcirc_{t; b_{t,s} < 0} M(t, 0)^{\circ \cdot b_{t,s}} \simeq M(s_-, 0) \bigcirc \left( \bigcirc_{y \in I_s} M((s_+)^-(y), 0)^{\circ \cdot a_{x,y}} \right).\]

**Lemma 6.8.** There exists an epimorphism (up to a grading)
\[(6.7) \quad \Omega : M(s, 0) \circ M(s_+, s) \circ A \rightarrow \bigcirc_{t; b_{t,s} < 0} M(t, 0)^{\circ \cdot b_{t,s}}.\]

**Proof.** By the dual of Theorem 4.28 and the $T$-system (4.16) with $i = i_s$, $u = u_{s-1}$ and $v = u_{s-1}$, we have morphisms
\[(6.8) \quad M(s, 0) \rightarrow M(s_-, 0) \circ M(s, s_+),\]
\[(6.9) \quad M(s, s_+) \circ M(s_+, s) \rightarrow \bigcirc_{y \in I_s} M((s_+)^-(y), s^-(y))^{\circ \cdot a_{x,y}} \]
\[\simeq \bigcirc_{y \in I_s} M((s_+)^-(y), s^-(y))^{\circ \cdot a_{x,y}}.\]

Here the last isomorphism follows from the fact that $(s_+)^-(y) = s^-(y)$ for any $y \notin \{x\} \cup I_s = \{i_k | s \leq k < s_+\}$.

Thus we have a sequence of morphisms
\[M(s, 0) \circ M(s_+, s) \circ A \xrightarrow{\varphi_1} M(s_-, 0) \circ M(s, s_-) \circ M(s_+, s) \circ A \]
\[\xrightarrow{\varphi_2} M(s_-, 0) \circ \left( \bigcirc_{y \in I_s} M((s_+)^-(y), s^-(y))^{\circ \cdot a_{x,y}} \right) \circ A.\]

By [9, Corollary 3.11], the composition $\varphi := \varphi_2 \circ \varphi_1$ is non-zero.

Since $A = \bigcirc_{y \in I_s} M(s^-(y), 0)^{\circ \cdot a_{x,y}}$, Theorem 4.28 gives the morphisms
\[M(s, 0) \circ M(s_+, s) \circ A \xrightarrow{\varphi} M(s_-, 0) \circ \left( \bigcirc_{y \in I_s} M((s_+)^-(y), s^-(y))^{\circ \cdot a_{x,y}} \right) \circ A \]
\[\xrightarrow{\phi} M(s_-, 0) \circ \left( \bigcirc_{y \in I_s} M((s_+)^-(y), 0)^{\circ \cdot a_{x,y}} \right) \simeq \bigcirc_{t; b_{t,s} < 0} M(t, 0)^{\circ \cdot b_{t,s}}.\]

Here we have used Lemma 4.12 to obtain the morphism $\phi$. Note that the module $\bigcirc_{y \in I_s} M((s_+)^-(y), s^-(y))^{\circ \cdot a_{x,y}}$ is simple. By applying [9, Corollary 3.11] once again, $\phi \circ \varphi$ is non-zero, and hence it is an epimorphism. \qed

**Lemma 6.9.** We have $\vartheta(X, M(s, 0)) = 1$.

**Proof.** Since $A$ and $M(s, 0)$ commute and $\vartheta(M(s_+, s), M(s, 0)) = 1$ by Proposition 4.30, we have
\[\vartheta(X, M(s, 0)) \leq \vartheta(M(s_+, s), M(s, 0)) + \vartheta(A, M(s, 0)) \leq 1,\]
by [10, Corollary 2.17] and Lemma 4.8. If $X$ and $M(s, 0)$ commute, then (6.4) (a) would imply that $\text{ch} \left( \bigcirc_{t; b_{t,s} < 0} M(t, 0)^{\circ \cdot b_{t,s}} \right)$ belongs to $K(R\text{-gmod}) \text{ch}(M(s, 0))$. It contradicts the result in [7] that all the $\text{ch}(M(k, 0))$’s are prime at $q = 1$. \qed
Proposition 6.10. The map \( \Phi \) factors through \( M(s, 0) \circ X \); that is,

\[
\begin{array}{cccccc}
M(s, 0) \circ M(s_+, s) \circ A & \xrightarrow{\tau} & \Omega & \xrightarrow{t; b_{t,s} < 0} & M(t, 0)^{\circ - b_{t,s}}.
\end{array}
\]

Here \( \tau \) is the canonical surjection.

Proof. We have \( 1 = \delta \left( M(s, 0), M(s_+, s) \circ A \right) \) by Lemma 6.9, and

\[
\delta \left( M(s, 0), M(s_+, s) \right) + \delta \left( M(s, 0), A \right) = 1
\]

by Proposition 4.30 with \( x = u_{s_+, -1}, i = i_s \). Hence \( M(s, 0) \circ M(s_+, s) \circ A \) has a simple head by Proposition 4.13 (iii). \( \Box \)

End of the proof of Theorem 6.6. By the arguments above, we have proved the existence of \( X \) which satisfies (6.4). By Proposition 4.14 and (6.4) (c), \( M(s, 0) \circ X \) has composition length 2. Moreover, it has a simple socle and simple head. On the other hand, taking the dual of (6.4) (a), we obtain a monomorphism

\[
\bigcup_{t; b_{t,s} > 0} M(t, 0)^{\circ b_{t,s}} \hookrightarrow M(s, 0) \circ X
\]

in \( R \)-mod. Together with (6.4) (b), there exists a short exact sequence in \( R \)-gmod:

\[
0 \to q^{c} \bigcup_{t; b_{t,s} > 0} M(t, 0)^{\circ b_{t,s}} \to q^{\Lambda(M(s, 0), X)} M(s, 0) \circ X \to \bigcup_{t; b_{t,s} < 0} M(t, 0)^{\circ (-b_{t,s})} \to 0,
\]

for some \( c \in \mathbb{Z} \). By [10, Corollary 2.24], \( c \) must be equal to 1.

It remains to prove that \( X \) commutes with \( M(k, 0) \) \((k \neq s)\). For any \( k \in J \), we have

\[
\Lambda(M(k, 0), X) = \Lambda(M(k, 0), M(s, 0) \circ X) - \Lambda(M(k, 0), M(s, 0))
\]

\[
= \sum_{t; b_{t,s} < 0} \Lambda(M(k, 0), M(t, 0))(-b_{t,s}) - \Lambda(M(k, 0), M(s, 0))
\]

and

\[
\Lambda(X, M(k, 0)) = \Lambda(X \circ M(s, 0), M(k, 0)) - \Lambda(M(s, 0), M(k, 0))
\]

\[
= \sum_{t; b_{t,s} > 0} \Lambda(M(t, 0), M(k, 0))b_{t,s} - \Lambda(M(s, 0), M(k, 0)).
\]

Hence we have

\[
2 \delta(M(k, 0), X) = -2 \delta(M(k, 0), M(s, 0)) - \sum_{t; b_{t,s} < 0} \Lambda(M(k, 0), M(t, 0))b_{t,s}
\]

\[
- \sum_{t; b_{t,s} > 0} \Lambda(M(k, 0), M(t, 0))b_{t,s}
\]
\[
- \sum_{1 \leq t \leq r} \Lambda(M(k,0), M(t,0))b_{t,s} \\
= 2\delta_{k,s}, 
\]

We conclude that \(X\) commutes with \(M(k,0)\) if \(k \neq s\). Thus we complete the proof of Theorem 6.6.

As a corollary we obtain the following answer to the conjecture on the cluster monomials.

**Theorem 6.11.** Conjecture 1 in Introduction is true, i.e., every cluster variable in \(A_q(n(w))\) is a member of the upper global basis.

Theorem 6.6 also implies [6, Conjecture 12.7] in the refined form as follows:

**Corollary 6.12.** \(Z[q^{\pm 1/2}] \otimes_{Z[q^{\pm 1}]} A_q(n(w))\) has a quantum cluster algebra structure associated with the initial quantum seed \([\mathcal{B}] = (\{q^{-(d_i, d_i)/4}D(0, i, 0)\}_{1 \leq i \leq r}, L, \tilde{B})\); i.e.,

\[
Z[q^{\pm 1/2}] \otimes_{Z[q^{\pm 1}]} A_q(n(w)) \cong \mathcal{A}_{q^{1/2}}([\mathcal{B}]).
\]

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