Balanced Phase Field model for Active Surfaces

Jozsef Molnar¹ and Peter Horvath²

¹Synthetic and Systems Biology Unit, Biological Research Centre,
Hungarian Academy of Sciences, Szeged, Hungary, Email: jmolnar64@digikabel.hu
²Synthetic and Systems Biology Unit, Biological Research Centre,
Hungarian Academy of Sciences, Szeged, Hungary, Email: horvath.peter@brc.mta.hu

September 13, 2018

Abstract

In this paper we present a balanced phase field model for active surfaces. This work is devoted to the generalization of the Balanced Phase Field Model for Active Contours devised to eliminate the often undesirable curvature-dependent shrinking of the zero level set while maintaining the smooth interface necessary to calculate the fundamental geometric quantities of the represented contour. As its antecedent work, the proposed model extends the Ginzburg-Landau phase field energy with a higher order smoothness term. The relative weights are determined with the analysis of the level set motion in a curvilinear system adapted to the zero level set. The proposed model exhibits strong shape maintaining capability without significant interference with the active (e.g. a segmentation) model.

1 Introduction

Geometric active contours and surfaces [1-4] are widely used for image segmentation where the representation of contours/surfaces are mainly implicit: the zero level set of an appropriately constructed function discretized on a fixed grid (Eulerian description). The evolution of the level set function is governed by the Euler-Lagrange equation associated with the appropriately designed functional for the segmentation problem. Strict criteria are to be fulfilled by an adequate level set representation. The most important one is that it needs to be reasonably smooth across a certain neighborhood of the zero level set to provide the basis of the accurate calculation of fundamental geometric quantities of the contour/surface, the building blocks of the equation(s) associated with the segmentation problem. On the other hand, the segmentation equation deteriorates
the shape of the level set function - measurements must be taken to correct it periodically.

During the decades several methods were elaborated to cope with this problem. The two main approaches are a) reinitialization and b) extension of the PDE associated with the original problem with an extra term that penalizes the deviations from the smooth (usually distance) function. Reinitializing the level set function by calculating the distance to the contours/surfaces on the whole domain is slow and may cause instability at discontinuous locations of the distance function. The partial remedy for this problems is the narrow band technique [5] for the price of higher complexity. The extension of the original PDE with a distance regularizing term [3] may add instability too (see [9]) or increase complexity [7][8]. More importantly, these approaches may move the zero level set away from the expected stopping location, which is rarely acceptable. From theoretical perspective, any method dedicated to this shape maintaining should have the least possible interference with the segmentation PDE.

The Ginzburg-Landau phase field model was introduced in the image segmentation literature in [6][2]. It possesses interesting advantages over the earlier level set frameworks as greater topological freedom; the possibility of a ‘neutral’ initialization; and a purely energy-based formulation. It also automatically forms a narrow band around the zero level set with fast shape recovery owing to a double well potential term incorporated to its functional; but it still moves the level sets due to the energy proportional to the length of the contour (or the surface area of the zero level set surface). This problem was treated with high efficiency in [XXX] for active contours. This work is devoted to the generalisation of the proposed balanced phase field model for active surfaces.

The structure of the paper is the following. In section 2 we summarize the Ginzburg-Landau and the balanced phase field model. Then we examine the balanced phase field model for active surfaces in section 3. Section 4 concludes the paper by discussion.

2 Phase field models

In the level set framework, the contours (2D) and surfaces (3D) are represented by a constant (usually the zero) level set of a function of two \( \phi(x, y) \) and three variables \( \Phi(x, y, z) \) respectively. The quantities of the segmentation problem are extracted from these functions, such as the unit normal vector \( \mathbf{n} = \frac{\nabla \phi}{|\nabla \phi|} \) or the curvature \( \kappa = -\nabla \cdot \left( \frac{\nabla \phi}{|\nabla \phi|} \right) \) for contours and \( \mathbf{n} = \frac{\nabla \Phi}{|\nabla \Phi|} \) or the sum curvature \( K_S = -\nabla \cdot \left( \frac{\nabla \Phi}{|\nabla \Phi|} \right) \) for surfaces, where \( \nabla \) is the gradient operator of the appropriate dimensions, “.” stands for the scalar (dot) product, \( i.e. \nabla \cdot \mathbf{v} \) is the divergence of the vector field \( \mathbf{v} \). The level set function is usually maintained on a uniform grid and its derivatives are approximated by finite differences. This manner of calculation requires the level set function to be approximately linear locally, across a small neighborhood of the zero level set. Phase field is one of the possible realizations of the level set frameworks. Its energy functional is designed
to form regions with $\pm 1$ field values (with the help of a double well potential term) and a smooth transition between these regions adding smoothness term(s), naturally representing a narrow band around the zero level set.

### 2.1 Summary of the balanced phase field model for active contours

The two dimensional balanced phase field model was introduced in [XXX] with the aim to eliminate the undesired shrinking effect of the Ginzburg-Landau phase field model. The proposed model extended the Ginzburg-Landau phase field energy

$$\int_{\Omega} \frac{D_o}{2} |\nabla \phi|^2 + \lambda \left( \frac{\phi^4}{4} - \frac{\phi^2}{2} \right) dA. \quad (1)$$

with a higher order smoothness term $\frac{D_o}{2} |\nabla \phi|^2$ (plus a constant term $\frac{\lambda}{4}$ such that the extended functional expresses the energy of the transitional regions). The relative weights were determined by the analysis of the extended energy and the constant level set motion in a curvilinear system adapted to the zero level set. Two conditions could be set: a) one for the width of the transition (hereinafter denoted by $W$) between the field values $\pm 1$ (where both the Ginzburg-Landau and the balanced functionals take their energy minima) and b) another for the elimination of the curvature dependent term of the motion equation (i.e. the Euler-Lagrange equation expressed in the adapted system) of the zero level set invoking an adequately chosen ansatz. The approach led to two equations for the weights in the extended functional as the functions of the width of the transition. Note that the (minimal value of the) width required is a priori known by the highest order of derivative occur in the segmentation model.

Here we assess the most important results. Since two constraints have to be satisfied, one of the weights can be arbitrarily set ($D_o$ is chosen to be $-1$). The balanced phase field functional and the Euler-Lagrange equation then become

$$\int_{\Omega} \frac{W^2}{16} |\nabla \phi|^2 - \frac{1}{2} |\nabla \phi|^2 + \frac{21}{W^2} \left( \frac{\phi^4}{4} - \frac{\phi^2}{2} \right) dA \quad (2)$$

and

$$\frac{W^2}{16} \Delta \Delta \phi + \Delta \phi + \frac{21}{W^2} (\phi^3 - \phi) = 0 \quad (3)$$

respectively. The gradient descent of the Euler-Lagrange equation is recommended to be used for reinitialization using fix iteration number $n \geq 10$; with this value the balanced phase field is stable, ensure smooth transition without significantly affecting the motion of the constant level sets.

The question arises naturally: how can these results be extended for the three dimensional case (for active surfaces).
2.2 The three dimensional Ginzburg-Landau functional

The energy of the simplest three dimensional Ginzburg-Landau phase field level set representation: $\Phi(x, y, z)$ is defined by the functional:

$$E(\Phi) = \iiint_{\Omega} \frac{1}{2} |\nabla \Phi|^2 + \lambda \left( \frac{\Phi^4}{4} - \frac{\Phi^2}{2} + \frac{1}{4} \right) \, dx dy dz,$$

where $\nabla \Phi$ is the gradient of the field $\Phi$, $\Omega$ represents the volume (the whole voxel image) of the integration. The origin of the energy scale can be chosen freely. Term $\frac{1}{4}$ is added such that at field values $\Phi = \pm 1$ (where the functional has its minima) $E = 0$. This constant term does not influence the Euler-Lagrange equation associated with the functional, which is:

$$- \Delta \Phi + \lambda (\Phi^3 - \Phi) = 0,$$

where $\Delta \Phi$ is the Laplacian of $\Phi$. As in the 2D case it is easy to prove that energy (4) is proportional to the surface area of the enclosed volume and as a consequence the gradient descent of (5) is driven by the sum curvature of the zero level set surface at every point.

3 The balanced phase field model for active surfaces

3.1 The balanced phase field functional

By analogy to the 2D version we propose the three dimensional balanced phase field $\Phi(x, y, z)$ for level set representation with the energy functional defined as:

$$E(\Phi) = \iiint_{\Omega} \frac{D}{2} |\Delta \Phi|^2 - \frac{1}{2} |\nabla \Phi|^2 + \lambda \left( \frac{\Phi^4}{4} - \frac{\Phi^2}{2} + \frac{1}{4} \right) \, dx dy dz.$$

Again, term $\frac{1}{4}$ is added such that at field values $\Phi = \pm 1$ (where this functional has its minima) the energy becomes zero and any deviation from the zero value is identified as the energy of the transitional stripes between field values $-1$ and $1$. The associated Euler-Lagrange equation is:

$$D \Delta \Delta \Phi + \Delta \Phi + \lambda (\Phi^3 - \Phi) = 0.$$

We wish to determine the weights $D$ and $\lambda$ such that the motion of the level sets governed by (7) are to be independent of the curvatures of the surfaces determined by the level sets.

3.2 The metric of the adapted system

To get quantitative insight, we examine the system energy and the motion of the zero level set in the curvilinear system adapted to the zero level set.
\( S(u, v) \) be the zero level set surface, using Gaussian description. The space in the vicinity of \( S \) can be parameterized as \( R(u, v, w) = S(u, v) + wn(u, v) \), where \( n = \frac{S_u \times S_v}{|S_u \times S_v|} \) is the unit normal vector of the surface at point identified with general coordinates \( u, v \); lower indices stand for the partial derivatives, i.e. \( S_u, S_v \) are the local (covariant) basis vectors. The length of the zero level set surface normal vector \(|S_u \times S_v|\) is equivalent to the square root of the determinant of the metric tensor \( G_{ik} = [S_i \cdot S_k] \) which is denoted by \( \sqrt{G} \) (i.e. \( G = \det (G_{ik}) \)). It is used to define the parameterization independent infinitesimal surface element \( dS = \sqrt{G} du dv \). The square root of the determinant of the metric tensor \( g_{ik} = [R_i \cdot R_k] \), \( i, k \in \{u, v, w\} \) is denoted by \( \sqrt{g} \) and used to define the parameterization independent infinitesimal volume element \( dV = \sqrt{g} du dv dw \). It can be expressed as the determinant of the matrix constructed from the covariant basis vectors

\[
R_u = S_u + wn_u
\]

\[
R_v = S_v + wn_v
\]

\[
R_w = n
\]

such that \( \sqrt{g} = n \cdot (R_u \times R_v) \). Expanding this expression we have:

\[
\sqrt{g} = \sqrt{G} + w [n_u \cdot (S_v \times n) + n_v \cdot (n \times S_u)] + w^2 |n_u \times n_v|.
\]

In the second line \( S_v \times n = \sqrt{G} S^u \), \( n \times S_u = \sqrt{G} S^v \), where \( S^u, S^v \) are the contravariant basis vectors of the surface \( S \) with definition \( S^i \cdot S_k = \delta^i_k \), \( i, k \in \{u, v, w\} \) (\( \delta^i_k \) is the Kronecker delta). The second line of (8) is therefore the \( w \sqrt{G} \) times the divergence of the unit normal vector which is in turn the negative of the sum curvature \( -K_S \). \(|n_u \times n_v|\) in the third line is the integrand of the total curvature expression equivalent with \( \sqrt{G} K_G \) where \( K_G \) is the Gaussian curvature (see also appendix A). The square root of the metric therefore can be expressed by a quadratic function of \( w \) with coefficients being the sum and Gaussian curvatures of the zero level set:

\[
\sqrt{g} = \sqrt{G} \left( 1 - wK_S + w^2 K_G \right).
\]

### 3.3 Energy terms in the adapted system

First we examine the constituents of energy (6) in the curvilinear system adapted to the level sets surfaces. In this case \( \Phi(u, v, w) \) takes constant values regardless the parameter values \( u, v \), hence its partial derivatives wrt these parameters are all zero, that is \( \frac{\partial \Phi}{\partial u} = 0, \frac{\partial \Phi}{\partial v} = 0, m, n \) are arbitrary. The gradient

\[
\frac{\partial \Phi}{\partial u} n_u + \frac{\partial \Phi}{\partial v} n_v + \frac{\partial \Phi}{\partial w} n_w \bigg|_{\Phi=\text{const}} = \frac{\partial \Phi}{\partial w} n_w,
\]

\(^1\) Also known as first fundamental form.
and the Laplacian

\[ \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^i} \partial (\sqrt{g} g^{ik} \frac{\partial \Phi}{\partial u^k}) \bigg|_{\Phi=\text{const}} = \frac{\partial^2 \Phi}{\partial w^2} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial w} \frac{\partial}{\partial w} \frac{\partial \Phi}{\partial w}, \]

where

\[ \frac{1}{\sqrt{g}} \frac{\partial}{\partial w} = \frac{-K_S + wK_G}{1 - wK_S + w^2K_G}. \]

Note that in the general expression (left of (12)) the Einstein summation convention is used. These expressions are dependent only on the geometric quantities of the zero level set \(K_S, K_G\) and the derivatives of the level set function in normal direction. To simplify the notation, from now on we use primes to denote the derivatives in the normal direction: \(\Phi' \equiv \frac{\partial \Phi}{\partial w}\), \(\Phi'' \equiv \frac{\partial^2 \Phi}{\partial w^2}\); notice that both the gradient and the Laplacian expressions contain derivatives only in normal direction explicitly. Implicitly the derivatives wrt \(u\) and \(v\) occur in the geometric quantities of the zero level set surface only \((K_S, K_G)\).

At this point it is tempting to assume the following

1. \(\Phi (u, v, w) \equiv \Phi (w)\), that is the constant level sets are equidistant to each other (hence \(\frac{\partial^l+m+n \Phi}{\partial u^l \partial v^m \partial w^n} = 0\));

2. They are arranged symmetrically around the zero level set \(\Phi (0)\) i.e. \(\Phi (-w) = -\Phi (w)\);

3. The transition between \(\Phi = -1, \Phi = 1\) (representing the minimal energy states) is confined to a stripe with constant width \(W\).

Note that from condition 2, \(\Phi (0) = 0\). These assumptions are certainly true for the plane (for symmetry reason) and violated only wherever curvatures are present; for this reason the low curvature condition

\[ 1 - wK_S + w^2K_G \approx 1 \]

needs to be assumed.

### 3.4 Ansatz for the level set function

The simplest possible ansatz satisfying the assumption taken in 3.3 is the cubic function \(\Phi = aw^3 + bw\) with boundary conditions:

\[ \Phi \left( \frac{W}{2} \right) = 1 \]
\[ \Phi \left( -\frac{W}{2} \right) = -1 \]
\[ \Phi' \left( \pm \frac{W}{2} \right) = 0. \]

6
With these, the function and its derivatives involved in the system energy are:
\[
\Phi (w) = -\frac{4}{W^3} w^3 + \frac{3}{W} w
\]
\[
\Phi' (w) = -\frac{12}{W^3} w^2 + \frac{3}{W}
\]
\[
\Phi'' (w) = -\frac{24}{W^3} w
\]  
(16)

### 3.5 Energy expression in the adapted system

With the one-dimensional ansatz, the energy (6) becomes:
\[
\mathcal{E} = \int \int \left[ \frac{D}{2} \left( \Phi'' + \frac{1}{\sqrt{g}} \partial \sqrt{g} \Phi' \right)^2 - \frac{1}{2} \left( \Phi' \right)^2 + \lambda \left( \frac{\Phi^4}{4} - \frac{\Phi^2}{2} + \frac{1}{4} \right) \right] \sqrt{g} dw du dv .
\]  
(17)

Substituting the ansatz (16) into energy (17) and using the low-curvature approximation (14), the energy, as the function of the width of the transition, is
\[
\tilde{E} (W) = \left( D \frac{24}{W^3} - \frac{12}{5W} + \lambda \frac{W}{10} \right) A
\]  
(18)

(see appendices B, C and D).

### 3.6 Optimal width

Handling the energy expression (18) as extreme value problem\(^2\) one can get an equation for the width of the transition as the function of two parameters - the weights of the constituents of (7):
\[
\frac{d\tilde{E}}{dW} = \left( \frac{72D}{W^4} + \frac{12}{5W^2} + \frac{\lambda}{10} \right) A \equiv 0 .
\]  
(19)

Rearranging wherever surface area is not zero \((A > 0)\) we obtain to the first equation we need:
\[
\lambda W^4 + 24W^2 - 720D = 0 \rightarrow W^2 = \frac{1}{\lambda} \left( -12 \pm \sqrt{144 + 720D} \right) .
\]  
(20)

### 3.7 Euler-Lagrange equation in the adapted system

Under the three conditions stated in point 3.3 the (approximate) Euler-Lagrange equation (7) in the adapted system becomes a fourth order ordinary differential

---

\(^2\)This is rational, because the zero level set surface practically static (does not move) during the time necessary to form the shape of transition.
equation:

\[
D \left\{ \frac{\partial^4 \Phi}{\partial w^4} + \frac{2}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial w} \frac{\partial^3 \Phi}{\partial w^3} + \left[ \left( \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial w} \right)^2 + 2 \frac{\partial}{\partial w} \left( \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial w} \right) \right] \frac{\partial^2 \Phi}{\partial w^2} \right. \\
+ \left. \left[ \frac{\Delta_T}{1} \frac{\partial \sqrt{g}}{\partial w} + \left( \frac{\partial^2}{\partial w^2} + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial w} \frac{\partial^{\prime}}{\partial w} \right) \left( \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial w} \right) \frac{\partial \Phi}{\partial w} \right] \right. \\
+ \left. \frac{\partial^2 \Phi}{\partial w^2} + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial w} \frac{\partial \Phi}{\partial w} + \lambda \left( \Phi^3 - \Phi \right) \right) = 0 \quad (21)
\]

(see appendix E).

### 3.8 Motion of the level sets

According to the equidistance condition - assumed to be persistent during the evolution governed by the Euler-Lagrange equation \(7\) (or approximate equation \(21\)), it is sufficient to examine the motion of any constant level set. The simplest case is the zero level set; wrt this set the antisymmetry condition \(\Phi (w) - \Phi (-w) = 0\) (assumed in point \(3.3\)) and consequently \(\frac{\partial^4 \Phi}{\partial w^4} = 0\), \(k = 1, 2\ldots\) are satisfied. Moreover, from \(15\):

\[
\left. \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial w} \right|_{w=0} = -K_S \\
\left. \left( \frac{\partial^2}{\partial w^2} + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial w} \frac{\partial^{\prime}}{\partial w} \right) \left( \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial w} \right) \right|_{w=0} = -K_S \left( K_S^2 - 4K_G \right) \quad (22) \\
\left. \Delta_T \left( \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial w} \right) \right|_{w=0} = -\Delta_T K_S
\]

where \(\Delta_T\) denotes the tangential components of the Laplace operator. Substituting \(22\) to \(21\), the adapted Euler-Lagrange equation for the zero level set is reduced to:

\[
-2DK_S \frac{\partial^3 \Phi}{\partial w^3} - D \left[ \Delta_T K_S + K_S \left( K_S^2 - 4K_G \right) \right] \frac{\partial \Phi}{\partial w} - K_S \frac{\partial \Phi}{\partial w} = 0. \quad (23)
\]

The sum curvature dependency is therefore can be eliminated by the condition (involving the 1st and the 3rd term in \(23\)):

\[
-2 \frac{\partial^3 \Phi}{\partial w^3} - \frac{\partial \Phi}{\partial w} \bigg|_{w=0} \doteq 0, \quad (24)
\]

or using ansatz \(16\):

\[
2D \frac{24}{W^3} - \frac{3}{W} \doteq 0. \quad (25)
\]
3.9 Energy expression and Euler-Lagrange equation for curvature-independent motion

Equations (20) and (25) determine parameters $\lambda$ and $D$ in energy (6) with the curvature-driven shrinking effect removed from the gradient descent of its associated Euler-Lagrange equation (7) as the function of the width of transition $W$. The solution is:

$$D(W) = \frac{W^2}{16},$$

$$\lambda(W) = \frac{21}{W^2}.$$  \hspace{1cm} (26)

Note that the gradient descent of the balanced zero level set equation - the 2nd term in (23) - still describe dynamic surface, but with a motion of a very modest pace. In fact the remaining term $\triangle_T K_S + K_S (K_S^2 - 4K_G)$ is very close to the solution $\triangle_T K_S + \frac{1}{2} K_S (K_S^2 - 4K_G)$ of the functional derivative associated with the Euler’s elastica of surfaces $\frac{1}{2} \int K_S dA$.

With the determined weights, we have the energy (6)

$$E(\Phi) = \iiint_\Omega \frac{W^2}{32} |\triangle \Phi|^2 - \frac{1}{2} |\nabla \Phi|^2 + \frac{21}{W^2} \left( \frac{\Phi^4}{4} - \frac{\Phi^2}{2} + \frac{1}{4} \right) dx dy dz \hspace{1cm} (27)$$

and the Euler-Lagrange equation associated with it

$$\frac{W^2}{16} \triangle \triangle \Phi + \triangle \Phi + \frac{21}{W^2} (\Phi^3 - \Phi) = 0.$$  \hspace{1cm} (28)

4 Discussion

In this paper we generalized the 2D balanced field model to active surfaces. It is shown by the examination of the equations of motions - the Euler-Lagrange equations associated with the Ginzburg-Landau and the balanced phase field models in the adapted curvilinear systems - that (as usual) the sum curvature for active surfaces has the same role as the curvature for active contours and can be eliminated using the same constraints. This curvature/sum curvature correspondence holds for the constraints that can be imposed to determine the optimal widths of the transitions.

We concluded that the 3D equations expressed in Cartesian coordinates have exactly same form as their 2D counterparts. As in 2D, the gradient descent of the proposed model exhibits very fast shape recovery without moving the zero level set significantly. In fact the motion of level sets is similar to the motion associated with the Euler’s elastica. This remaining term contains a nonlinear expression of the sum and Gaussian curvatures (expressible with a cubic polynomial of the principal curvatures) and under the low curvature assumption its interference with the segmentation model is negligible, the property that makes this level set formulation suitable for accurate segmentation. As in 2D, this balancing could be used for any model that includes Laplacian smoothness term in their gradient descent equation like the reaction-diffusion model.
Appendices

Appendix A: The Gaussian term of the metric

The invariant surface element is defined with the infinitesimal area of parallelogram spanned by the covariant basis vectors $S_u, S_v$ as $dS = |S_u \times S_v| \, du \, dv$, where the factor $|S_u \times S_v|$ is:

$$|S_u \times S_v| = \sqrt{(S_u \times S_v) \cdot (S_u \times S_v)}$$
$$= \sqrt{S_u \cdot [S_v \times (S_u \times S_v)]}$$
$$= \sqrt{S_u \cdot [(S_v \cdot S_v) S_u - (S_u \cdot S_v) S_v]}$$
$$= \sqrt{S_u \cdot S_u} \cdot (S_v \cdot S_v) - (S_u \cdot S_v)^2 = \sqrt{G} \quad (29)$$

(For the derivation, the triple scalar product $a \cdot (b \times c) = b \cdot (c \times a)$ and the triple cross product $a \times (b \times c) = (a \cdot c) b - (a \cdot b) c$ equivalences are used.)

The partial derivatives of the unit normal vector $n_u$ and $n_v$ are the elements of the tangent space hence can be decomposed such that

$$n_i = \left( n_i \cdot S_u \right) S_u + \left( n_i \cdot S_v \right) S_v, \quad i \in \{u, v\}$$

where $S_u, S_v$ are the local (covariant) basis, $S^u, S^v$ are the contravariant basis vectors with the property $S^i \cdot S^k = \delta^i_k$ ($\delta^i_k$ is the Kronecker delta). It can be seen by simple substitution that

$$S^u = \frac{1}{|S_u \times S_v|} S_u \times n$$
$$S^v = \frac{1}{|S_u \times S_v|} n \times S_u. \quad (30)$$

We also need

$$n \cdot S_k = 0 \quad \rightarrow \quad n_i \cdot S_k = -n \cdot S_{ik}$$
$$i, k \in \{u, v\} \quad (31)$$

Next we calculate $n_u \times n_v$ as

$$n_u \times n_v = [(n_u \cdot S^u) S_u + (n_v \cdot S^v) S_v] \times [(n_v \cdot S^u) S_u + (n_u \cdot S^v) S_v]$$
$$= (S^u \times S^v) \left( (n_u \cdot S_u) (n_v \cdot S_v) - (n_u \cdot S_v)^2 \right)$$
$$= (S^u \times S^v) \left( n \cdot S_{uu} (n \cdot S'_{vv}) - (n \cdot S_{uv})^2 \right). \quad (32)$$

In the third line $\texttt{33}$ is used. Substitution of $\texttt{33}$ leads to

$$n_u \times n_v = (S_v \times n) \left( n \times S_u \right) \frac{(n \cdot S_{uu})(n \cdot S'_{vv}) - (n \cdot S_{uv})^2}{|S_u \times S_v|^2}$$
$$= n |S_u \cdot (S_v \times n)| \frac{\det [\Pi]}{\det [G_{ik}]}$$
$$= |S_u \times S_v| K_G n. \quad (33)$$
where the Gaussian curvature is given as the ratio of the determinants of the second and first fundamental forms. The last line of equation (33) is a vector with length
\[ |\mathbf{n}_u \times \mathbf{n}_v| = |\mathbf{S}_u \times \mathbf{S}_v| K_G = \sqrt{G} K_G. \] (34)

Appendix B: Approximation of the gradient integral term

The second term of (17) is
\[ \frac{-1}{2} (\Phi')^2 (1 - wK_S + w^2 K_G) \, dwdA, \] (35)
where \( dA = \sqrt{G} \, dudv \). Substituting the corresponding ansatz \( (\Phi')^2 = \frac{12}{W^2} u^4 - \frac{7}{W} u^2 + \frac{9}{W^2} \) and performing the integration between the boundaries \(-\frac{W}{2}, \frac{W}{2}\), the result is:
\[ -\frac{12}{5W} A - \frac{W^3}{24} \frac{1}{R_1 R_2} dA. \] (36)

where \( A = \iint dA \) is the surface area of the zero level set (assumed to be closed, hence the notation \( \iint \)). The second term is:
\[ \frac{W^3}{24} \frac{1}{R_1 R_2} dA = \frac{W}{24} \iint \frac{W}{R_1 R_2} dA \ll \frac{W}{24} A, \] (37)
where \( R_1, R_2 \) are the rays of the osculating circles in the principal directions. If the ratio of \( \frac{W}{24} \) and the first \( \frac{12}{W^2} \) terms (\( \approx 0.132W \)) in (36) is not extremely big (a case for moderate curvature values) then the second term can be omitted from (36).

Appendix C: Approximation of the Laplacian integral term

Here we calculate the first term of (17) using the cubic ansatz (16), the metric (10) and the invariant surface element expression \( dA = \sqrt{G} \, dudv \).

\[ \iint \left( \Phi'' + \frac{-K_S + 2wK_G}{1 - wK_S + w^2 K_G} \Phi' \right)^2 (1 - wK_S + w^2 K_G) \, dwdA = \iint \left( 1 - wK_S + w^2 K_G \right) (\Phi'')^2 \]
\[ +2 \left( -K_S + 2wK_G \right) \Phi'' \Phi' \]
\[ + \left( -K_S + 2wK_G \right)^2 \Phi' \right)^2 \, dwdA \] (38)

In the second term, \( 2\Phi'' \Phi' = \left( (\Phi')^2 \right)' \). Applying integration by parts to this integrand, on of the term \( \left[ (-K_S + 2wK_G) \Phi' \right]^{\frac{W}{2}} \frac{W}{2} = 0 \) (according to the third
property (15) of the chosen ansatz). What remains is:

\[ \int \left( 1 - wK_S + w^2K_G \right) (\Phi'')^2 \]

\[ + \left[ \frac{(-K_S + 2wK_G)^2}{1 - wK_S + w^2K_G} - 2K_G \right] (\Phi')^2 \, dwdA = \]

\[ \int \left( 1 - wK_S + w^2K_G \right) (\Phi'')^2 \]

\[ + \frac{K_S^2 + 2K_G \left( w^2K_G - wK_S - 1 \right)}{1 - wK_S + w^2K_G} (\Phi')^2 \, dwdA. \]

Now we use approximation (14) and arrive to:

\[ \int (\Phi'')^2 + (K_S^2 - 2K_G) (\Phi')^2 \, dwdA = \]

\[ \int (\Phi'')^2 + (K_1^2 + K_2^2) (\Phi')^2 \, dwdA, \] (40)

where \( K_1 \) and \( K_2 \) are the principal curvatures. With the ansatz values (16) the first term of the energy (12) integrated between the boundary values \( -\frac{W}{2}, \frac{W}{2} \) becomes

\[ D \left( \frac{24}{W^3} A + \frac{12}{5W} \right) \left( \frac{1}{R_1} + \frac{1}{R_2} \right) dA. \] (41)

The second term is:

\[ \frac{12}{5W} \int \frac{1}{R_1} + \frac{1}{R_2} dA = \frac{12}{5W^3} \int \left( \frac{W}{R_1} \right)^2 + \left( \frac{W}{R_2} \right)^2 \, dA \ll \frac{12}{5W^3} A. \] (42)

Observing that the ratio of \( \frac{12A}{5W^2} \) and the first \( \frac{24A}{W^3} \) terms (= 0.5) is not an extremely big number, one can conclude that the second term can be omitted from (41).

**Appendix D: Approximation of the double potential well term**

The

\[ \int \lambda \left( \frac{\Phi^4}{4} - \frac{\Phi^2}{2} + \frac{1}{4} \right) \sqrt{g} \, dwdudv = \]

\[ \int \lambda \left( \frac{\Phi^4}{4} - \frac{\Phi^2}{2} + \frac{1}{4} \right) (1 - wK_S + w^2K_G) \, dwdA \] (43)

can be considered as the active term of the balanced phase field model. Substituting the ansatz (16) and carrying out the integration between the boundary values \( -\frac{W}{2}, \frac{W}{2} \) one can have:

\[ \lambda \left( \frac{W}{10} A + \frac{W^3}{12} \int \frac{1}{R_1} \frac{1}{R_2} dA \right) = \lambda W \left( \frac{1}{10} A + \frac{1}{12} \right) \left( \frac{W}{R_1} \frac{W}{R_2} dA \right). \] (44)
Here the approximation \( \int \frac{4}{\Phi^2} - \frac{\Phi'^2}{2} + \frac{1}{4}dw \approx 0.1W \) is used. With similar reasoning to the previous cases it is obvious that the second term can be ignored similarly. Note that the terms neglected in comparison with the dominant terms in the (gradient, Laplacian and active part) one by one. If there is magnitudes of differences between the relative weights, then this approximations can be invalid. The final result however justified this approach a posteriori for a wide range of realistic width selection.

Appendix E: Approximation of Laplacian and the Laplacian of Laplacian

It is expedient to decompose the Lagrangian operator in the direction tangential and perpendicular to the level sets

\[
\Delta = \frac{1}{\sqrt{g}} \partial \left( \sqrt{g} g^{ik} \frac{\partial}{\partial u^k} \right) \approx \Delta_T + \left( \frac{\partial^2}{\partial w^2} + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial w} \frac{\partial}{\partial w} \right),
\]

(45)

\( i, k \in \{ u, v, w \} \), the metric \( \sqrt{g} \) is given by (10) and

\[
\Delta_T = g^{uu} \frac{\partial^2}{\partial u^2} + 2g^{uv} \frac{\partial^2}{\partial u \partial v} + g^{vv} \frac{\partial^2}{\partial v^2}
+ \frac{1}{\sqrt{g}} \left( \frac{\partial g^{uu}}{\partial u} + \frac{\partial g^{uv}}{\partial v} \right) \frac{\partial}{\partial u}
+ \frac{1}{\sqrt{g}} \left( \frac{\partial g^{uv}}{\partial u} + \frac{\partial g^{vv}}{\partial v} \right) \frac{\partial}{\partial v}.
\]

(46)

It immediately follows that on level sets \( \Phi = \text{const} \), the derivatives \( \frac{\partial^{m+n} \Phi}{\partial u^m \partial v^n} \) are automatically zero, the tangential component \( \Delta_T \) has no effect, i.e.

\[
\Delta \Phi = \frac{\partial^2 \Phi}{\partial w^2} + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial w} \frac{\partial \Phi}{\partial w}.
\]

(47)

Applying the Laplace operator twice leads to the decomposition

\[
\Delta \Delta \Phi = \Delta_T \Delta_T \Phi + \Delta_T \left( \frac{\partial^2 \Phi}{\partial w^2} + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial w} \frac{\partial \Phi}{\partial w} \right)
+ \left( \frac{\partial^2}{\partial w^2} + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial w} \frac{\partial}{\partial w} \right) \Delta_T \Phi
+ \left( \frac{\partial^2}{\partial w^2} + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial w} \frac{\partial}{\partial w} \right) \left( \frac{\partial^2 \Phi}{\partial w^2} + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial w} \frac{\partial \Phi}{\partial w} \right).
\]

(48)

The first term is automatically zero. If the level sets are equidistant to each-other
(i.e. $\frac{\partial m+n+r}{\partial u m n r} = 0$) then the non-zero terms remained are

$$\Delta \Delta \Phi = \frac{\partial \Phi}{\partial w} \Delta T \left( \frac{1}{\sqrt{g}} \frac{\partial}{\partial w} \right)$$

$$+ \left( \frac{\partial^2}{\partial w^2} + \frac{1}{\sqrt{g}} \frac{\partial^3}{\partial w^3} \right) \left( \frac{\partial^2 \Phi}{\partial w^2} + \frac{1}{\sqrt{g}} \frac{\partial \Phi}{\partial w} \right), \quad (49)$$

or rearranging the equation by the orders of the derivatives of the phase field function

$$\Delta \Delta \Phi = \frac{\partial^4 \Phi}{\partial w^4} + \frac{2}{\sqrt{g}} \frac{\partial^3}{\partial w^3} \left( \frac{1}{\sqrt{g}} \frac{\partial}{\partial w} \right)^2 + 2 \frac{\partial}{\partial w} \left( \frac{1}{\sqrt{g}} \frac{\partial \Phi}{\partial w} \right) \frac{\partial^2 \Phi}{\partial w^2}$$

$$+ \left[ \Delta T \left( \frac{1}{\sqrt{g}} \frac{\partial}{\partial w} \right)^2 + \left( \frac{\partial^2}{\partial w^2} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial w} \frac{\partial}{\partial w} \right) \frac{1}{\sqrt{g}} \frac{\partial \Phi}{\partial w} \right] \frac{\partial \Phi}{\partial w}, \quad (50)$$

where the coefficient of the first derivative $\frac{\partial \Phi}{\partial w}$ is the complete (spatial) Laplacian of $\frac{1}{\sqrt{g}} \frac{\partial \Phi}{\partial w}$.

References

[1] Vicent Caselles, Francine Catté, Toméu Coll, and Françoise Dibos. A geometric model for active contours in image processing. Numerische matematik, 66(1):1–31, 1993.

[2] Péter Horváth and Ian H Jermyn. A ‘gas of circles’ phase field model and its application to tree crown extraction. In Signal Processing Conference, 2007 15th European, pages 277–281. IEEE, 2007.

[3] Chunming Li, Chenyang Xu, Changfeng Gui, and Martin D Fox. Distance regularized level set evolution and its application to image segmentation. IEEE transactions on image processing, 19(12):3243–3254, 2010.

[4] Ravi Malladi, James A Sethian, and Baba C Vemuri. Shape modeling with front propagation: A level set approach. IEEE transactions on pattern analysis and machine intelligence, 17(2):158–175, 1995.

[5] Danping Peng, Barry Merriman, Stanley Osher, Hongkai Zhao, and Myungjoo Kang. A pde-based fast local level set method. Journal of computational physics, 155(2):410–438, 1999.

[6] Marie Rochery, Ian Jermyn, and Josiane Zerubia. Phase field models and higher-order active contours. In Computer Vision, 2005. ICCV 2005. Tenth IEEE International Conference on, volume 2, pages 970–976. IEEE, 2005.

[7] Xuchu Wang, Jinxiao Shan, Yanmin Niu, Liwen Tan, and Shao-Xiang Zhang. Enhanced distance regularization for re-initialization free level set evolution with application to image segmentation. Neurocomputing, 141:223–235, 2014.
[8] Yongfei Wu and Chuanjiang He. Indirectly regularized variational level set model for image segmentation. *Neurocomputing*, 171:194–208, 2016.

[9] Kaihua Zhang, Lei Zhang, Huihui Song, and David Zhang. Reinitialization-free level set evolution via reaction diffusion. *IEEE Transactions on Image Processing*, 22(1):258–271, 2013.