Polygamy of multi-party $q$-expected quantum entanglement

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We characterize the polygamy nature of quantum entanglement in multi-party systems in terms of $q$-expectation value for the full range of $q \geq 1$. By investigating some properties of generalized quantum correlations in terms of $q$-expectation value and Tsallis $q$-entropy, we establish a class of polygamy inequalities of multi-party quantum entanglement in arbitrary dimensions based on $q$-expected entanglement measure. As Tsallis $q$-entropy is reduced to von Neumann entropy, and $q$-expectation value becomes the ordinary expectation value when $q$ tends to 1, our results encapsulate previous results of polygamy inequalities based on von Neumann entropy as special cases.

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I. INTRODUCTION

Quantum entanglement is a quintessential phenomenon of quantum mechanics showing the non-local nature of quantum states in multi-party quantum systems. As a quantum correlation among distinct parties, entanglement plays a central role in quantum information and computation theory with many applications [1–3]. Thus it has been an important and even challenging task to have a proper way of quantifying entanglement to understand full characteristics of quantum entanglement in various quantum systems.

One property that makes quantum entanglement fundamentally different from other classical correlations is the restricted shareability and distribution of entanglement in multi-party quantum systems, namely, monogamy and polygamy relations of quantum entanglement [4, 5]. Mathematically, the monogamy of quantum entanglement has been characterized as monogamy inequalities using various entanglement measures [6–11]. These monogamy inequalities of entanglement show the mutually exclusive structures of entanglement shareability in multi-party quantum systems. The polygamy relation of quantum entanglement was also quantitatively characterized as polygamy inequalities in multi-party quantum systems [12–14].

By using the concept of $q$-expectation value for any nonnegative real parameter $q$, von Neumann entropy can be generalized into a one-parameter class of entropy functions, namely Tsallis $q$-entropy [15, 16]. Because $q$-expectation value is theoretically consistent with the minimum cross-entropy principle, Tsallis $q$-entropy is considered to be more relevant to nonextensive statistical mechanics [17, 18]. Tsallis $q$-entropy can also be used to characterize classical statistical correlations inherent in quantum states [19, 20].

In quantum entanglement theory, Tsallis $q$-entropy can be used to define a faithful entanglement measure because the property of entanglement monotone is guaranteed by the concavity of Tsallis $q$-entropy for $q > 0$ [21]. It is also known that some conditions on separability criteria of quantum states can be established based on Tsallis $q$-entropy [22, 24].

Here, we characterize the polygamy relation of quantum entanglement in multi-party systems in terms of $q$-expectation value for the full range of $q \geq 1$. We first recall the generalized definitions of various classical and quantum correlations in terms of Tsallis $q$-entropy and $q$-expectation value. By investigating some properties of generalized correlations in relation to classical-classical-quantum(ccq) states, we establish a class of polygamy inequalities of multi-party quantum entanglement in arbitrary dimensions in terms of the $q$-expected entanglement measure.

Due to the existence of equivalence among the monogamy and polygamy inequalities of $q$-expected quantum correlations [23], our results also guarantee several classes of monogamy and polygamy inequalities about $q$-expected entanglement and discord distributed in three-party quantum systems. As Tsallis $q$-entropy is reduced to von Neumann entropy, and $q$-expectation value becomes the ordinary expectation value when $q$ tends to 1, our results encapsulate previous results of polygamy inequalities based on von Neumann entropy as special cases.

This paper is organized as follows. In Sec. II, we recall the definitions and properties of $q$-expected classical and quantum correlations in terms of Tsallis $q$-entropy and $q$-expectation value. In Sec. IIIA we provide the definition of a ccq state in four-party quantum systems, and its properties related with the $q$-expected correlations. In Sec. IIIB we provide analytic upper and lower bounds of $q$-expected correlations in accordance with the ccq states for $q \geq 2$. In Sec. IV we establish a class of polygamy inequalities of multi-party quantum entanglement in arbitrary dimensions based on $q$-expected entanglement measure $q \geq 1$. Finally, we summarize our results in Sec. V.
II. q-EXPECTED QUANTUM CORRELATIONS

Based on the generalized logarithmic function

\[ \ln_q x = \frac{x^{1-q} - 1}{1 - q}, \]  

(1)

of the real parameter \( q \) with \( q \geq 0 \) and \( q \neq 1 \), the Tsallis \( q \)-entropy of a quantum state \( \rho \) is defined as \[15, 16\]

\[ S_q (\rho) = -\text{tr}\rho^q \ln \rho. \]  

(2)

Tsallis-\( q \)-entropy is concave for any nonnegative real parameter \( q \geq 0 \), and it converges to von Neumann entropy as \( q \) tends to 1,

\[ \lim_{q \to 1} S_q (\rho) = -\text{tr}\rho \ln \rho =: S (\rho). \]  

(3)

For a quantum state \( \rho \) with the spectrum \( \{ \lambda_i\} \) \[20\], its Tsallis \( q \)-entropy can be written as

\[ S_q (\rho) = -\sum_i \lambda_i^q \ln \lambda_i, \]  

(4)

that is, the \( q \)-expectation value of the generalized logarithms. Thus, Tsallis \( q \)-entropy is a one-parameter generalization of von Neumann entropy based on the concept of \( q \)-expectation value for nonnegative real parameter \( q \).

Using Tsallis \( q \)-entropy and the concept of \( q \)-expectation value, a class of bipartite entanglement measures has been introduced; for \( q \geq 0 \) and a bipartite pure state \( |\psi\rangle_{AB} \), its \( q \)-expected entanglement \( (q\text{-E}) \) is defined as \[23\]

\[ E_q (|\psi\rangle_{AB}) = S_q (\rho_A), \]  

(5)

where \( \rho_A = \text{tr}_B |\psi\rangle_{AB} \langle \psi| \) is the reduced density matrix of \( \rho_{AB} \) on subsystem \( A \). For a bipartite mixed state \( \rho_{AB} \), its \( q \)-E is defined as the minimum \( q \)-expectation value

\[ E_q (\rho_{AB}) = \min \sum_i p_i^q E_q (|\psi_i\rangle_{AB}), \]  

(6)

over all possible pure state decompositions of \( \rho_{AB} \),

\[ \rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i| . \]  

(7)

As a dual quantity to \( q \)-E, \( q \)-expected entanglement of assistance \( (q\text{-EOA}) \) was also defined as

\[ E_{q}^a (\rho_{AB}) = \max \sum_i p_i^q E_q (|\psi_i\rangle_{AB}), \]  

(8)

where the maximum is taken over all possible pure state decompositions of \( \rho_{AB} \).

Tsallis \( q \)-entropy converges to von Neumann entropy and the \( q \)-expectation value becomes ordinary expectation value when \( q \) tends to 1, therefore we have

\[ \lim_{q \to 1} E_q (\rho_{AB}) = E_1 (\rho_{AB}) \]  

(9)

and

\[ \lim_{q \to 1} E_{q}^a (\rho_{AB}) = E^a (\rho_{AB}), \]  

(10)

where \( E_1 (\rho_{AB}) \) is the entanglement of formation \( (\text{EOF}) \) \[27\], and \( E^a (\rho_{AB}) \) is the entanglement of assistance \( (\text{EOA}) \) of \( \rho_{AB} \) \[28\].

Let us consider more generalized quantum correlations based on \( q \)-expectation value and Tsallis \( q \)-entropy. For \( q \geq 0 \) and a probability ensemble \( \mathcal{E} = \{ p_i, \rho_i \} \) of a quantum state \( \rho \) (equivalently, a probability decomposition \( \rho = \sum_i p_i \rho_i \) denoted by \( \mathcal{E} \)), its Tsallis-\( q \) difference is defined as \[29\]

\[ \chi_q (\mathcal{E}) = S_q (\rho) - \sum_i p_i S_q (\rho_i). \]  

(11)

Tsallis-\( q \) difference is nonnegative for \( q \geq 1 \) due to the concavity of Tsallis \( q \)-entrop, and it converges to the Holevo quantity

\[ \lim_{q \to 1} \chi_q (\mathcal{E}) = S (\rho) - \sum_i p_i S (\rho_i) =: \chi (\mathcal{E}), \]  

(12)

as \( q \) tends to 1.

For a bipartite quantum state \( \rho_{AB} \), each measurement \( \{ M_{pB}^x \} \) applied on subsystem \( B \) induces a probability ensemble \( \mathcal{E} = \{ p_x, \rho_{A}^x \} \) of the reduced density matrix \( \rho_{A} = \text{tr}_B \rho_{AB} \) in the way that

\[ p_x = \text{tr}[ (I_A \otimes M_{pB}^x ) \rho_{AB} ] \]  

(13)

is the probability of the outcome \( x \) and

\[ \rho_{A}^x = \text{tr}_B [ (I_A \otimes M_{pB}^x ) \rho_{AB} ] / p_x \]  

(14)

is the state of system \( A \) when the outcome was \( x \). The one-way classical \( q \)-correlation \( (q\text{-CC}) \) \[22\] of a bipartite state \( \rho_{AB} \) is defined as the maximum Tsallis-\( q \) difference

\[ \mathcal{J}_q^\rightarrow (\rho_{AB}) = \max_\mathcal{E} \chi_q (\mathcal{E}) \]  

(15)

over all possible ensemble representations \( \mathcal{E} \) of \( \rho_{A} \) induced by measurements on subsystem \( B \).

As a dual quantity to \( q\text{-CC} \), the one-way unlocalizable \( q \)-entanglement \( (q\text{-UE}) \) \[24\] is defined by taking the minimum Tsallis-\( q \) difference

\[ \mathbf{u} E_{q}^\rightarrow (\rho_{AB}) = \min_\mathcal{E} \chi_q (\mathcal{E}), \]  

(16)

over all probability ensembles \( \mathcal{E} \) of \( \rho_{A} \) induced by rank-1 measurements on subsystem \( B \). Due to the continuity of Tsallis-\( q \) difference with respect to \( q \), we have

\[ \lim_{q \to 1} \mathcal{J}_q^\rightarrow (\rho_{AB}) = \mathcal{J}_1^\rightarrow (\rho_{AB}), \]  

(17)

and

\[ \lim_{q \to 1} \mathbf{u} E_{q}^\rightarrow (\rho_{AB}) = \mathbf{u} E_1^\rightarrow (\rho_{AB}), \]  

(18)
where $\mathcal{I}^+(\rho_{AB})$ is the one-way classical correlation (CC) [30] and $uE^+(\rho_{AB})$ is one-way unlocalizable entanglement (UE) [13] of the bipartite state $\rho_{AB}$.

The following proposition shows the trade-off relations between $q$-CC and $q$-E as well as $q$-UE and $q$-EOA distributed in three-party quantum systems.

**Proposition 1.** [22] For $q \geq 1$ and a three-party pure state $|\psi\rangle_{ABC}$ with its reduced density matrices $\rho_{AB} = \text{tr}_C|\psi\rangle_{ABC}\langle\psi|$, $\rho_{AC} = \text{tr}_B|\psi\rangle_{ABC}\langle\psi|$ and $\rho_A = \text{tr}_{BC}|\psi\rangle_{ABC}\langle\psi|$, we have
\begin{align}
S_q(\rho_A) = J_q^+(\rho_{AB}) + E_q(\rho_{AC}) \tag{19}
\end{align}
and
\begin{align}
S_q(\rho_A) = uE_q^+(\rho_{AB}) + E^a_q(\rho_{AC}). \tag{20}
\end{align}

The concept of $q$-expectation and Tsallis-$q$ entropy are also used to generalize quantum discord [31], a different kind of quantum correlation. For $q \geq 0$ and a bipartite quantum state $\rho_{AB}$, its Tsallis-$q$ mutual entropy is defined as
\begin{align}
\mathcal{I}_q(\rho_{AB}) = S_q(\rho_A) + S_q(\rho_B) - S_q(\rho_{AB}), \tag{21}
\end{align}
which generalizes the quantum mutual information
\begin{align}
\mathcal{I}(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}) \tag{22}
\end{align}
in a way that
\begin{align}
\lim_{q \to 1} \mathcal{I}_q(\rho_{AB}) = \mathcal{I}(\rho_{AB}). \tag{23}
\end{align}

For a bipartite state $\rho_{AB}$, its quantum $q$-discord (q-D) [32] is defined by the difference between its Tsallis-$q$ mutual entropy and $q$-CC,
\begin{align}
\delta_q^-(\rho_{AB}) = \mathcal{I}_q(\rho_{AB}) - \mathcal{J}_q^-(\rho_{AB}). \tag{24}
\end{align}
We note that $q$-D is a generalization of the quantum discord $\delta^-(\rho_{AB})$ as
\begin{align}
\lim_{q \to 1} \delta_q^-(\rho_{AB}) = \mathcal{I}(\rho_{AB}) - \mathcal{J}^-(\rho_{AB}) =: \delta^-(\rho_{AB}). \tag{25}
\end{align}
Moreover, the duality between $q$-CC and $q$-UE provides us with a dual definition to q-D,
\begin{align}
u\delta_q^-(\rho_{AB}) = \mathcal{I}_q(\rho_{AB}) - uE_q^-(\rho_{AB}). \tag{26}
\end{align}
Eq. (26) is referred to as the one-way unlocalizable quantum $q$-discord (q-UD) of $\rho_{AB}$ [23], which is a generalization of one-way unlocalizable quantum discord (UD) [33],
\begin{align}
u\delta^-(\rho_{AB}) = \mathcal{I}(\rho_{AB}) - uE^-(\rho_{AB}). \tag{27}
\end{align}

The following proposition provides a trade-off relation between quantum entanglement (q-UE) and quantum discord (q-UD) distributed in three-party quantum systems.

**Proposition 2.** [23] For $q \geq 1$ and a three-party pure state $|\psi\rangle_{ABC}$ with its reduced density matrices $\rho_{AB} = \text{tr}_C|\psi\rangle_{ABC}\langle\psi|$, $\rho_{AC} = \text{tr}_B|\psi\rangle_{ABC}\langle\psi|$ and $\rho_A = \text{tr}_{BC}|\psi\rangle_{ABC}\langle\psi|$, we have
\begin{align}
S_q(\rho_A) = u\delta_q^-(\rho_{BA}) + uE_q^-(\rho_{CA}). \tag{28}
\end{align}

### III. SOME PROPERTIES OF $q$-EXPECTED QUANTUM CORRELATIONS

In this section, we first consider a class of four-party classical-classical-quantum (ccq) states and their $q$-expected correlations. By assuming the subadditivity of Tsallis-$q$ mutual entropy for this class of ccq states, we provide an analytic upper bound for $q$-UE as well as a lower bound for $q$-UD.

#### A. Classical-Classical-Quantum States

For a two-qudit state $\rho_{AB}$, let us consider a spectral decomposition of the reduced density matrix $\rho_B = \text{tr}_A\rho_{AB}$ such that
\begin{align}
\rho_B = \sum_{i=0}^{d-1} \lambda_i |e_i\rangle_B \langle e_i|. \tag{29}
\end{align}
Based on the eigenvectors $\{|e_j\rangle_B\}$ of $\rho_B$, generalized $d$-dimensional Pauli operators can be defined as
\begin{align}
Z = \sum_{j=0}^{d-1} \omega_d^j |e_j\rangle \langle e_j|, \quad X = \sum_{j=0}^{d-1} \omega_d^{-j} |\tilde{e}_j\rangle \langle \tilde{e}_j|, \tag{30}
\end{align}
where $\omega_d = e^{2\pi i / d}$ is the $d$th-root of unity, and $\{|\tilde{e}_j\rangle\}$ is the $d$-dimensional Fourier basis,
\begin{align}
|\tilde{e}_j\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \omega_d^{jk} |e_k\rangle, \quad j = 0, \ldots, d-1, \tag{31}
\end{align}
with respect to the eigenvectors $\{|e_j\rangle_B\}$ of $\rho_B$. By using the Pauli operators in Eq. (30), we define two quantum operations acting on any $d$-dimensional quantum state $\sigma$ as
\begin{align}
M_0(\sigma) = \frac{1}{d} \sum_{b=0}^{d-1} Z^b \sigma Z^{-b}, \quad M_1(\sigma) = \frac{1}{d} \sum_{a=0}^{d-1} X^a \sigma X^{-a}. \tag{32}
\end{align}
For the two-qudit state $\rho_{AB}$ whose reduced density matrix is $\rho_B$ in Eq. (29), the actions of the channels $M_0$ and $M_1$ applied on the subsystem $B$ are
\begin{align}
(I_A \otimes M_0)(\rho_{AB}) = \sum_{i=0}^{d-1} \sigma_A^i \otimes \lambda_i |e_i\rangle_B \langle e_i| \tag{33}
\end{align}
and

\[
(I_A \otimes M_1)(\rho_{AB}) = \sum_{j=0}^{d-1} \tau_A^j \otimes \frac{1}{d} |\tilde{e}_j\rangle_B \langle \tilde{e}_j|,
\]

where \(\lambda_i \sigma_A^i = \text{tr}_B[(I_A \otimes |e_i\rangle_B \langle e_i|)\rho_{AB}]\) and \(\tau_A^j/d = \text{tr}_B[(I_A \otimes |\tilde{e}_j\rangle_B \langle \tilde{e}_j|)\rho_{AB}]\) for \(i, j \in \{0, \cdots, d-1\}\). Thus the ensembles of subsystem \(A\) induced by the action of the channels \(M_0\) and \(M_1\) from Eqs. (33) and (34) are

\[
\mathcal{E}_0 = \{\lambda_i, \sigma_A^i\}_i, \quad \mathcal{E}_1 = \{\frac{1}{d}, \tau_A^j\}_j,
\]

respectively. Equivalently, we can say that each of the rank-1 measurements \(|e_i\rangle_B \langle e_i|\) and \(|\tilde{e}_j\rangle_B \langle \tilde{e}_j|\) on subsystem \(B\) of \(\rho_{AB}\) induces the ensembles \(\mathcal{E}_0\) and \(\mathcal{E}_1\) of subsystem \(A\), respectively.

Now, let us consider a four-qudit ccq-state \(\Omega_{XY:AB}\)

\[
\Omega_{XY:AB} = \frac{1}{d^2} \sum_{x,y=0}^{d-1} |x\rangle_X \langle x| \otimes |y\rangle_Y \langle y| \otimes (I_A \otimes X_B^x Z_B^y) \rho_{AB} (I_A \otimes Z_B^{-y} X_B^{-x}),
\]

with the reduced density matrices

\[
\Omega_{X:AB} = \frac{1}{d} \sum_{x=0}^{d-1} |x\rangle_X \langle x| \otimes X_B^x \left( \sum_{i=0}^{d-1} \sigma_A^i \otimes \lambda_i |e_i\rangle_B \langle e_i| \right) X_B^{-x}
\]

(37)

and

\[
\Omega_{Y:AB} = \frac{1}{d} \sum_{y=0}^{d-1} |y\rangle_Y \langle y| \otimes Z_B^y \left( \sum_{j=0}^{d-1} \tau_A^j \otimes \frac{1}{d} |\tilde{e}_j\rangle_B \langle \tilde{e}_j| \right) Z_B^{-y}.
\]

(38)

It is straightforward to verify that the Tsallis-q mutual entropies of \(\Omega_{XY:AB}\), \(\Omega_{X:AB}\) and \(\Omega_{Y:AB}\) in Eqs. (36), (37) and (38) are

\[
\mathcal{I}_q(\Omega_{XY:AB}) = \frac{d^{1-q} - 1}{1-q} + d^{1-q} S_q(\rho_A) - d^{2(1-q)} S_q(\rho_{AB}),
\]

(39)

\[
\mathcal{I}_q(\Omega_{X:AB}) = \frac{d^{1-q} - 1}{1-q} - d^{1-q} S_q(\rho_B) + d^{1-q} \chi_q(\mathcal{E}_0)
\]

(40)

and

\[
\mathcal{I}_q(\Omega_{Y:AB}) = (1 - d^{1-q}) \frac{d^{1-q} - 1}{1-q} + d^{1-q} \chi_q(\mathcal{E}_1).
\]

(41)

B. Upper and Lower Bounds

\textbf{Theorem 1.} For \(q \geq \log_d \left(\frac{1+\sqrt{5}}{2}\right) + 1\) and any two-qudit state \(\rho_{AB}\), we have

\[
\mathbf{u} E_q^+ (\rho_{AB}) \leq \frac{\mathcal{I}_q(\rho_{AB})}{2}
\]

(42)

and

\[
\mathbf{u} E_q^- (\rho_{AB}) \geq \frac{\mathcal{I}_q(\rho_{AB})}{2}
\]

(43)

conditioned on the subadditivity of Tsallis-q mutual entropy for the ccq states in Eq. (37), that is,

\[
\mathcal{I}_q(\Omega_{XY:AB}) \geq \mathcal{I}_q(\Omega_{X:AB}) + \mathcal{I}_q(\Omega_{Y:AB}).
\]

(44)

\textbf{Proof.} For a two-qudit state \(\rho_{AB}\) and its four-party ccq state defined in Eq. (39), Eqs. (39), (40) and (41) enable us to rewrite Inequality (44) as

\[
\chi_q(\mathcal{E}_0) + \chi_q(\mathcal{E}_1) \leq S_q(\rho_A) + S_q(\rho_B)
\]

\[
- d^{1-q} S_q(\rho_{AB}) + \frac{(d^{1-q} - 1)^2}{d^{1-q}(1-q)}.
\]

Now we note that the Tsallis-q differences \(\chi_q(\mathcal{E}_0)\) and \(\chi_q(\mathcal{E}_1)\) for the ensembles \(\mathcal{E}_0\) and \(\mathcal{E}_1\) in Eq. (35) can be obtained from \(\rho_{AB}\) by measuring its subsystem \(B\) with respect to the rank-1 measurements \(|e_i\rangle_B \langle e_i|\) and \(|\tilde{e}_j\rangle_B \langle \tilde{e}_j|\), respectively. From the definition of \(q\)-UE in Eq. (49), we also note that each of rank-1 measurement \(|e_i\rangle_B \langle e_i|\) and \(|\tilde{e}_j\rangle_B \langle \tilde{e}_j|\) provides an upperbound of \(q\)-UE as

\[
\mathbf{u} E_q^+ (\rho_{AB}) \leq \chi_q(\mathcal{E}_0)
\]

(46)

and

\[
\mathbf{u} E_q^- (\rho_{AB}) \leq \chi_q(\mathcal{E}_1).
\]

(47)

Because Inequalities (46) and (47) imply

\[
\mathbf{u} E_q^- (\rho_{AB}) \leq \frac{\chi_q(\mathcal{E}_0) + \chi_q(\mathcal{E}_1)}{2},
\]

(48)

Inequalities (46) and (47) enable us to have

\[
\mathbf{u} E_q^+ (\rho_{AB}) \leq \frac{\mathcal{I}_q(\rho_{AB})}{2}
\]

(49)

\[
+ \frac{1}{2} \left( 1 - d^{1-q} S_q(\rho_{AB}) + \frac{(d^{1-q} - 1)^2}{d^{1-q}(1-q)} \right).
\]

To prove Inequality (48), it is now sufficient to show that

\[
(1 - d^{1-q}) S_q(\rho_{AB}) + \frac{(d^{1-q} - 1)^2}{d^{1-q}(1-q)} \leq 0.
\]

(50)
For $q \geq 1$, we have $1 - d^{1-q} \geq 0$. Moreover, the Tsallis $q$-entropy attains its maximal value for the maximally mixed state,

$$S_q(\rho_{AB}) \leq S_q\left(\frac{I_{AB}}{d^2}\right) = \frac{1 - d^{2(1-q)}}{q-1},$$

therefore we have

$$(1 - d^{1-q})S_q(\rho_{AB}) + \frac{(d^{1-q} - 1)^2}{d^{1-q}(1-q)} \leq \left(1 - d^{1-q}\right)^2 \left[1 + d^{1-q} - d^{q-1}\right].$$

The non-positivity of the right-hand side of Inequality (52) is equivalent to

$$1 + d^{1-q} - d^{q-1} \leq 0,$$

which can be rewritten as

$$q \geq \log_d\left(\frac{1 + \sqrt{5}}{2}\right) + 1$$

for nonnegative $q$.

Inequality (53) is then a one step consequence of Inequality (52) together with the definition of $q$-UD in Eq. (26).

For $q = 1$, Inequalities (52) and (53) are reduced to

$$uE_q^{-}(\rho_{AB}) \leq \frac{I(\rho_{AB})}{2}, \quad \nuq^{-}\sigma(\rho_{AB}) \geq \frac{I(\rho_{AB})}{2},$$

respectively, whereas the condition in (52) is reduced to the subadditivity of quantum mutual information

$$I(\Omega_{XY:AB}) \geq I(\Omega_{X:AB}) + I(\Omega_{Y:AB}).$$

In fact, Inequality (56) was shown to be true for any ccq state in general [29]. Moreover, Inequalities (55) are also shown to be true for any quantum state $\rho_{AB}$ [12, 32]. Thus Theorem 1 is true for $q = 1$ without any condition.

The lower bound (the right-hand side) of Inequality (57) tends to 1 as $d$ is getting large. Thus Theorem 1 is true for the most range of $q \geq 1$ if $d$ is large enough, that is, large dimensional quantum systems. Although the proof method that we used here is not sufficient to guarantee the validity of Theorem 1 for $1 < q < \log_d\left(\frac{1 + \sqrt{5}}{2}\right)$, we conjecture that Theorem 1 is true for any $q$ larger than or equal to 1.

We also note that any bipartite quantum state can be considered as a two-qudit state where $d$ is the dimension of larger dimensional subsystem. Moreover, we also have $\log_d\left(\frac{1 + \sqrt{5}}{2}\right) \leq 1$ for any $d \geq 2$, Thus we have the following corollary.

**Corollary 1.** For $q \geq 2$ and any bipartite quantum state $\sigma_{AB}$, we have

$$uE_q^{-}(\sigma_{AB}) \leq \frac{I_q(\sigma_{AB})}{2}$$

and

$$\nuq^{-}\sigma(\sigma_{AB}) \geq \frac{I_q(\sigma_{AB})}{2},$$

conditioned on the subadditivity of Tsallis-$q$ mutual entropy for the ccq state in terms of $\sigma_{AB}$.

**IV. POLYGAMY OF $q$-EXPECTED ENTANGLEMENT IN MULTI-PARTY QUANTUM SYSTEMS**

In this section, we provide the polygamy inequalities of $q$-expected quantum entanglement distributed in multi-party quantum systems for $q \geq 1$ conditioned on the subadditivity of Tsallis-$q$ mutual entropy. The following theorem shows the polygamy inequality of $q$-EOA in three-party quantum systems.

**Theorem 2.** For $q \geq 1$, and any three-party pure state $|\psi\rangle_{ABC}$ with its two-party reduced density matrices $\text{tr}_C|\psi\rangle_{ABC}\langle\psi|_{ABC} = \rho_{AB}$ and $\text{tr}_B|\psi\rangle_{ABC}\langle\psi|_{ABC} = \rho_{AC}$, we have

$$E_q\left(|\psi\rangle_{A(BC)}\right) \leq E_q^a(\rho_{AB}) + E_q^a(\rho_{AC}),$$

conditioned on the subadditivity of Tsallis-$q$ mutual entropy for the ccq states in Eq. (56), where $E_q\left(|\psi\rangle_{A(BC)}\right)$ is the $q$-EOA of the pure state $|\psi\rangle_{ABC}$ with respect to the bipartition between $A$ and $BC$.

**Proof.** For a three-party quantum state $|\psi\rangle_{ABC}$, the universality of Eq. (20) of Proposition 1 leads us to

$$S_q(\rho_A) - uE_q^{-}(\rho_{AC}) = E_q^a(\rho_{AB})$$

and

$$S_q(\rho_A) - \nuq^{-}\sigma(\rho_{AB}) = E_q^a(\rho_{AC}),$$

therefore we have

$$2S_q(\rho_A) - (uE_q^{-}(\rho_{AB}) + uE_q^{-}(\rho_{AC})) = E_q^a(\rho_{AB}) + E_q^a(\rho_{AC}).$$

We also note that $|\psi\rangle_{ABC}$ can be assumed to be a three-qudit state, otherwise, we can always consider an imbedded image of $|\psi\rangle_{ABC}$ into a higher dimensional quantum system having the same dimensions of subsystems.

As we have already seen in the proof of Theorem 1 the subadditivity condition for ccq states leads us to Inequality (55), and this enables us to have an upper bound of
Because $E_{\psi} (\rho_{AB})$ as in Inequality (10), which can be rewritten as
\[
\mathbf{u} E_q^- (\rho_{AB}) = \frac{1}{2} |S_q (\rho_A) + S_q (\rho_B) - d^{1-q} S_q (\rho_B) + \left( \frac{d^{1-q} - 1}{d^{1-q} - 1} \right)|.
\]
For the two-qudit reduced density matrix $\rho_{AC}$, we also analogously have
\[
\mathbf{u} E_q^- (\rho_{AC}) = \frac{1}{2} |S_q (\rho_A) + S_q (\rho_C) - d^{1-q} S_q (\rho_C) + \left( \frac{d^{1-q} - 1}{d^{1-q} - 1} \right)|.
\]
From Inequality (62) together with inequalities (63) and (64), we have
\[
S_q (\rho_A) + \frac{1}{2} (\mathcal{E}_B + \mathcal{E}_C) \leq E_q^a (\rho_{AB}) + E_q^a (\rho_{AC})
\]
where
\[
\Xi_B = \frac{d^{q-1}}{d^{q-1}} \left[ \frac{d^{q-1} - 1}{q - 1} - S_q (\rho_B) \right]
\]
and
\[
\Xi_C = \frac{d^{q-1}}{d^{q-1}} \left[ \frac{d^{q-1} - 1}{q - 1} - S_q (\rho_C) \right].
\]
For $q \geq 1$, we have $\frac{d^{q-1}}{d^{q-1}} \geq 0$. Moreover, the Tsallis $q$-entropy attains its maximal value for the maximally mixed states,
\[
S_q (\rho_B) \leq S_q \left( \frac{I_q}{d} \right) = \frac{1 - d^{1-q}}{q - 1} \leq \frac{d^{q-1} - 1}{q - 1},
\]
and this implies the nonnegativity of $\Xi_B$. The nonnegativity of $\Xi_C$ can be analogously obtained, therefore
\[
\Xi_B \geq 0, \quad \Xi_C \geq 0.
\]
Because $E_q \left( |\psi\rangle_{A(BC)} \right) = S_q (\rho_A)$, Inequality (65) together with the inequalities in (69) implies Inequality (59), which completes the proof. □

When $q$ tends to 1, $q$-EOA is reduced to EOA as in Eq. (10), whereas the subadditivity of quantum mutual information in Inequality (56) was shown to be true for any ccq state in general [29]. Thus Theorem 2 encapsulates the results of general polygamy inequality of three-party entanglement in terms of EOA [13].

For any three-party quantum state $|\psi\rangle_{ABC}$, it was recently shown that the polygamy inequality of $q$-EOA in (59) is a necessary and sufficient condition for the monogamy inequality of $q$-UE as well as the polygamy inequality of $q$-UD for $q \geq 1$ [22]. Thus we have the following corollary.

**Corollary 2.** For $q \geq 1$, and any three-party pure state $|\psi\rangle_{ABC}$, we have
\[
\mathbf{u} E_q^- (|\psi\rangle_{A(BC)}) \geq \mathbf{u} E_q^- (\rho_{AB}) + \mathbf{u} E_q^- (\rho_{AC}),
\]
and
\[
\mathbf{u} \delta_q^- \left( |\psi\rangle_{A(BC)} \right) \leq \mathbf{u} \delta_q^- (\rho_{AB}) + \mathbf{u} \delta_q^- (\rho_{AC}),
\]
conditioned on the subadditivity of Tsallis-$q$ mutual entropy for the ccq state in Eq. (70).

Now, we generalize Theorem 2 for an arbitrary multi-party quantum system.

**Theorem 3.** For $q \geq 1$, and any multi-party quantum state $\rho_{A_1A_2\cdots A_n}$ with two-party reduced density matrices $\rho_{A_i}$, for $i = 2, \cdots, n$, we have
\[
E_q^a (\rho_{A_1A_2\cdots A_n}) \leq \sum_{i=2}^n E_q^a (\rho_{A_i}),
\]
conditioned on the subadditivity of Tsallis-$q$ mutual entropy for the ccq states in Eq. (72).

**Proof.** We first prove the theorem for any three-party mixed state $\rho_{ABC}$, then the validity of the theorem for an arbitrary $n$-party quantum state $\rho_{A_1A_2\cdots A_n}$ follows inductively.

For a three-party mixed state $\rho_{ABC}$, let us consider their optimal decompositions of $\rho_{ABC}$ for $q$-EOA with respect to the bipartition between $A$ and $BC$, that is,
\[
\rho_{ABC} = \sum_i p_i |\psi_i\rangle_{ABC} \langle \psi_i|,
\]
with
\[
E_q^a (\rho_{A(BC)}) = \sum_i p_i E_q |\psi_i\rangle_{A(BC)} \langle \psi_i|.
\]
From Theorem 2, each $|\psi_i\rangle_{ABC}$ in Eq. (74) satisfies
\[
E_q \left( |\psi_i\rangle_{A(BC)} \right) \leq E_q^a (\rho_{AB}^i) + E_q^a (\rho_{AC}^i)
\]
with $\rho_{AB}^i = \text{tr}_C |\psi_i\rangle_{ABC} \langle \psi_i|$ and $\rho_{AC}^i = \text{tr}_B |\psi_i\rangle_{ABC} \langle \psi_i|$. For each $i$ and the two-qudit reduced density matrices $\rho_{AB}^i$ and $\rho_{AC}^i$, let us consider their optimal decompositions for $q$-EOA, that is,
\[
\rho_{AB}^i = \sum_j r_{ij}^l |\phi_j^l\rangle_{AB} \langle \phi_j^l|, \quad \rho_{AC}^i = \sum_s r_{is}^s |\mu_s^l\rangle_{AC} \langle \mu_s^l|,
\]
such that
\[
E_q \left( \rho_{AB}^i \right) = \sum_j r_{ij}^l E_q \left( |\phi_j^l\rangle_{AB} \langle \phi_j^l| \right)
\]
and
\[
E_q \left( \rho_{AC}^i \right) = \sum_s r_{is}^s E_q \left( |\mu_s^l\rangle_{AC} \langle \mu_s^l| \right).
\]
\[
E_q^n \left( \rho_{A(BC)} \right) = \sum_i p_i^q E_q \left( \langle \psi_i \rangle_{A(BC)} \right) \\
\leq \sum_i p_i^q \left( E_q^n \left( \rho_{AB} \right) + E_q^n \left( \rho_{AC} \right) \right)
\]
\[
= \sum_i p_i^q \left( \sum_j r_{ij}^q E_q \left( |\phi_j^i \rangle_{AB} \right) + \sum_l s_{il}^q E_q \left( |\mu_l^i \rangle_{AB} \right) \right)
\]
\[
= \sum_{i,j} (p_i r_{ij})^q E_q \left( |\phi_j^i \rangle_{AB} \right) + \sum_{i,l} (p_i s_{il})^q E_q \left( |\mu_l^i \rangle_{AB} \right)
\]
\[
\leq E_q^n \left( \rho_{AB} \right) + E_q^n \left( \rho_{AC} \right),
\]
(78)

where the first inequality is from Inequality (75), and the second inequality is due to

\[
\rho_{AB} = \sum_i p_i \rho_{AB}^i = \sum_{i,j} p_i r_{ij} |\phi_j^i \rangle_{AB} \langle \phi_j^i |,
\]
\[
\rho_{AC} = \sum_i p_i \rho_{AC}^i = \sum_{i,l} p_i s_{il} |\mu_l^i \rangle_{AB} \langle \mu_l^i |.
\]
(79)

and the definition of \( q \)-EOA in Eq. (8). Thus Inequality (72) is true for three-party mixed states.

For general multi-party quantum system, we use the mathematical induction on the number of parties \( n \); let us assume the polygamy inequality (72) is true for any \( (n - 1) \)-party quantum state, and consider an \( n \)-party quantum state \( \rho_{A_1 A_2 \cdots A_n} \) for \( n \geq 4 \). By considering \( \rho_{A_1 A_2 \cdots A_n} \) as a three-party state with respect to the partition \( A_1, A_2 \) and \( A_3 \cdots A_n \), Inequality (78) leads us to

\[
E_q^n \left( \rho_{A_1(A_2 \cdots A_n)} \right) \leq E_q^n \left( \rho_{A_1 A_2} \right) + E_q^n \left( \rho_{A_1(A_3 \cdots A_n)} \right)
\]
(80)

where \( \rho_{A_1 A_2 \cdots A_n} = \text{tr}_{A_3} \rho_{A_1 A_2 \cdots A_n} \).

Because \( \rho_{A_1 A_3 \cdots A_n} \) in Inequality (80) is a \( (n - 1) \)-party quantum state, the induction hypothesis assures that

\[
E_q^n \left( \rho_{A_1(A_3 \cdots A_n)} \right) \leq E_q^n \left( \rho_{A_1 A_3} \right) + \cdots + E_q^n \left( \rho_{A_1 A_n} \right).
\]
(81)

Thus Inequalities (80) and (81) lead us to the polygamy inequality of multi-party entanglement in terms of \( q \)-EOA in (72).

As \( q \)-EOA is reduced to EOA when \( q \) tends to 1, and due to the the subadditivity of quantum mutual information for CCQ states, Theorem 2 encapsulates the results of general polygamy inequality of multi-party entanglement in terms of EOA [14].

\section{Conclusion}

We have characterized the polygamy property of multi-party quantum entanglement in terms of \( q \)-expectation value for the full range of \( q \geq 1 \). By using the generalized definitions of various classical and quantum correlations in terms of Tsallis \( q \)-entropy and \( q \)-expectation value, we have first provided some properties of \( q \)-expected correlations in relation to classical-classical-quantum(\( cq \)) states. Based on these properties, we have established a class of polygamy inequalities of multi-party quantum entanglement in arbitrary dimensions in terms of \( q \)-EOA.

Due to the equivalence between the monogamy of \( q \)-UE and polygamy of \( q \)-EOA and \( q \)-UD, our results also guarantee the monogamy inequality of \( q \)-UE as well as the polygamy inequality of \( q \)-UD distributed in three-party quantum systems. We also note that, from the continuity of \( q \)-expectation value as well as Tsallis \( q \)-entropy, our results encapsulate the previous results of monogamy and polygamy inequalities based on von Neumann entropy as special cases.

Studying multi-party quantum correlations, especially in higher dimensions more than qubits, is important and necessary for various reasons. In many quantum information processing tasks such as quantum communication and quantum cryptography, higher-dimensional quantum systems are sometimes considered to be more useful because they can provide higher coding density and thus stronger security compared with qubit systems. We also note that the monogamy and polygamy properties of multi-party entanglement play a central role in quantum cryptography because they can bound the possible amount of correlation between the authenticated users and the eavesdropper; the fundamental concept of the security proof. Thus our results about monogamy and polygamy inequalities of \( q \)-expected quantum correlations in arbitrary high-dimensional quantum systems can provide good methods and rich references for the foundation of many secure quantum information processing tasks.

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