The Regularity of Edge Rings and Matching Numbers

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Abstract: Let \( K[G] \) denote the edge ring of a finite connected simple graph \( G \) on \([d]\) and \( \text{mat}(G) \) the matching number of \( G \). It is shown that \( \text{reg}(K[G]) \leq \text{mat}(G) \) if \( G \) is non-bipartite and \( K[G] \) is normal, and that \( \text{reg}(K[G]) \leq \text{mat}(G) - 1 \) if \( G \) is bipartite.

Keywords: edge ring; edge polytope; regularity; matching number

Let \( G \) be a finite connected simple graph on the vertex set \([d] = \{1, \ldots, d\} \) and let \( E(G) \) be its edge set. Let \( S = K[x_1, \ldots, x_d] \) denote the polynomial ring in \( d \) variables over a field \( K \). The edge ring of \( G \) is the toric ring \( K[G] \subset S \) which is generated by those monomials \( x_i x_j \) with \( \{i, j\} \in E(G) \). The systematic study of edge rings originated in [1]. It has been shown that \( K[G] \) is normal if and only if \( G \) satisfies the odd cycle condition ([2], p. 131). Thus, particularly if \( G \) is bipartite, \( K[G] \) is normal.

Let \( e_1, \ldots, e_d \) denote the canonical unit coordinate vectors of \( \mathbb{R}^d \). The edge polytope \( P_G \subset \mathbb{R}^d \) which is the convex hull of the finite set \( \{ e_i + e_j : \{i, j\} \in E(G) \} \). One has \( \dim P_G = d - 1 \) if \( G \) is non-bipartite and \( \dim P_G = d - 2 \) if \( G \) is bipartite. We refer the reader to ([2], Chapter 5) for the fundamental materials on edge rings and edge polytopes.

A matching of \( G \) is a subset \( M \subset E(G) \) for which \( e \cap e' = \emptyset \) for \( e \neq e' \) belonging to \( M \). The matching number is the maximal cardinality of matchings of \( G \). Let \( \text{mat}(G) \) denote the matching number of \( G \).

When \( K[G] \) is normal, the upper bound of regularity of \( K[G] \) can be explicitly described in terms of \( \text{mat}(G) \). Our main result in the present paper is as follows:

**Theorem 1.** Let \( G \) be a finite connected simple graph. Then

(a) If \( G \) is non-bipartite and \( K[G] \) is normal, then \( \text{reg}(K[G]) \leq \text{mat}(G) \);

(b) If \( G \) is bipartite, then \( \text{reg}(K[G]) \leq \text{mat}(G) - 1 \).

Lemma 1 stated below, which provides information on lattice points belonging to the interiors of dilations of edge polytopes, is indispensable for the proof of Theorem 1.

**Lemma 1.** Suppose that \( (a_1, \ldots, a_d) \in \mathbb{Z}^d \) belongs to the interior \( q(P_G \setminus \partial P_G) \) of the dilation \( qP_G = \{ qa : a \in P_G \} \), where \( q \geq 1 \), of \( P_G \). Then \( a_i \geq 1 \) for each \( 1 \leq i \leq d \).

**Proof.** The facets of \( P_G \) are described in ([1], Theorem 1.7). When \( W \subset [d] \), we write \( G_W \) for the induced subgraph of \( G \) on \( W \). Since \( K[G] \) is normal, it follows that \( P_G \) possesses the integer decomposition property ([2], p. 91). In other words, each \( a \in qP_G \cap \mathbb{Z}^d \) is of the form

\[
a = (e_{i_1} + e_j) + \cdots + (e_{i_q} + e_{j_q}),
\]

where \( q \geq 1 \), \( 1 \leq i_1 < \cdots < i_q \leq d \), and \( j \neq i_1, \ldots, i_q \).

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where \(\{i_1, j_1\}, \ldots, \{i_q, j_q\}\) are edges of \(G\).

**First Step** Let \(G\) be non-bipartite. Let \(i \in [d]\). Let \(H_1, \ldots, H_s\) and \(H'_1, \ldots, H'_{s'}\) denote the connected components of \(G_{[d] \setminus \{i\}}\), where each \(H_j\) is bipartite and where each \(H'_j\) is non-bipartite. If \(s = 0\), then \(i \in [d]\) is regular ([1], p. 414) and the hyperplane of \(\mathbb{R}^d\) defined by the equation \(x_i = 0\) is a facet of \(q\mathcal{P}_G\). Hence \(a_i > 0\).

Let \(s \geq 1\) and \(s' \geq 0\). For each \(1 \leq j \leq s\), we write \(W_j \cup U_j\) for the vertex set of the bipartite graph \(H_j\) for which there is \(a \in W_j\) with \(\{a, i\} \in E(G)\), where \(U_j = \emptyset\) if \(H_j\) is a graph consisting of a single vertex. Then \(T = W_1 \cup \cdots \cup W_s\) is independent ([1], p. 414). In other words, no edge \(e \in E(G)\) satisfies \(e \in T\). Let \(G'\) denote the bipartite graph induced by \(T\). Thus the edges of \(G'\) are \(\{b, c\} \in E(G)\) with \(b \in T\) and \(c \in T' = U_1 \cup \cdots \cup U_s \cup \{i\}\). Since each induced subgraph \(G_{[d] \setminus (W_j \cup U_j \cup \{i\})}\) is connected, it follows that \(G'\) is connected with \(V(G') = T \cup T'\) as its vertex set. Since the connected components of \(G_{[d] \setminus V(G')}\) are \(H'_i, \ldots, H'_{s'}\), it follows that \(T\) is fundamental ([1], p. 415) and the hyperplane of \(\mathbb{R}^d\) defined by \(\sum_{\xi \in T} a_\xi = \sum_{\xi' \in T'} a_{\xi'}\) is a facet of \(q\mathcal{P}_G\). Now, suppose that \(a_i = 0\). Since \(\mathcal{P}_G\) possesses the integer decomposition property, one has \(\sum_{\xi \in T} a_\xi = \sum_{\xi' \in T'} a_{\xi'}\). Hence \((a_1, \ldots, a_d) \in Z^d\) cannot belong to \(q(\mathcal{P}_G \setminus \partial \mathcal{P}_G)\). Thus \(a_i > 0\), as desired.

**Second Step** Let \(G\) be bipartite. If \(G\) is a star graph with, say, \(E(G) = \{\{1, 2\}, \{1, 3\}, \ldots, \{1, d\}\}\), then \(\mathcal{P}_G\) can be regarded to be the \((d - 2)\) simplex of \(\mathbb{R}^{d-1}\) with the vertices \((1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\). Thus, since each \((a_1, \ldots, a_d) \in q\mathcal{P}_G \cap Z^d\) satisfies \(a_i = q\), the assertion follows immediately. In the argument below, one will assume that \(G\) is not a star graph.

Let \(i \in [d]\) and \(H_1, \ldots, H_s\) be the connected components of \(G_{[d] \setminus \{i\}}\). If \(s = 1\), then \(i \in [d]\) is ordinary ([1], p. 414) and the hyperplane of \(\mathbb{R}^d\) defined by the equation \(x_i = 0\) is a facet of \(q\mathcal{P}_G\). Hence \(a_i > 0\).

Let \(s \geq 2\). Let \(W_j \cup U_j\) denote the vertex set of \(H_j\) for which there is \(a \in W_j\) with \(\{a, i\} \in E(G)\). Since \(G\) is not a star graph, one can assume that \(U_j \neq \emptyset\). Then \(T = W_2 \cup \cdots \cup W_s\) is independent and the bipartite graph induced by \(T\) is \(G_{[d] \setminus (W_1 \cup U_1)}\). Hence \(T\) is acceptable ([1], p. 415) and the hyperplane of \(\mathbb{R}^d\) defined by \(\sum_{\xi \in W_1} a_\xi = \sum_{\xi' \in U_1} a_{\xi'}\) is a facet of \(q\mathcal{P}_G\). Now, suppose that \(a_i = 0\). Since \(\mathcal{P}_G\) possesses the integer decomposition property, one has \(\sum_{\xi \in W_1} a_\xi = \sum_{\xi' \in U_1} a_{\xi'}\). Hence \((a_1, \ldots, a_d) \in Z^d\) cannot belong to \(q(\mathcal{P}_G \setminus \partial \mathcal{P}_G)\). Thus \(a_i > 0\), as required.

We say that a finite subset \(L \subset E(G)\) is an **edge cover** of \(G\) if \(\cup_{e \in L} e = [d]\). Let \(\mu(G)\) denote the minimal cardinality of edge covers of \(G\).

**Corollary 1.** When \(K[G]\) is normal, one has \(q \geq \mu(G)\) if \(q(\mathcal{P}_G \setminus \partial \mathcal{P}_G) \cap Z^d \neq \emptyset\).

**Proof.** Since \(\mathcal{P}_G\) possesses the integer decomposition property, Lemma 1 guarantees that, if \(a \in q(\mathcal{P}_G \setminus \partial \mathcal{P}_G) \cap Z^d\), one has \(q \geq \mu(G)\).

Once Corollary 1 is established, to complete the proof of Theorem 1 is a routine job on computing the regularity of normal toric rings.

**Proof of Theorem 1.** In each of the cases (a) and (b), since the edge ring \(K[G]\) is normal, it follows that the Hilbert function of \(K[G]\) coincides the Ehrhart function ([2], p. 100) of the edge polytope \(\mathcal{P}_G\), which says that the Hilbert series of \(K[G]\) is of the form

\[
(1 + h_1 \lambda + \cdots + h_s \lambda^s) / (1 - \lambda)^{\dim \mathcal{P}_G + 1}
\]

with each \(h_i \in Z\) and \(h_s \neq 0\). One has

\[
s = (\dim \mathcal{P}_G + 1) - \min\{ q \geq 1 : q(\mathcal{P}_G \setminus \partial \mathcal{P}_G) \cap Z^d \neq \emptyset \}.\]
Now, Corollary 1 guarantees that
\[
s \leq (\dim \mathcal{P}_G + 1) - \mu(G).
\]
Finally, since \(\mu(G) = d - \text{mat}(G)\) ([3], Lemma 2.1), one has
\[
\text{reg } K[G] = s \leq \dim \mathcal{P}_G - (d - 1) + \text{mat}(G),
\]
as required.

Rafael H. Villarreal informed us that part (b) of Theorem 1 can also be deduced from ([4], Theorem 14.4.19).

When \(K[G]\) is non-normal, the behavior of regularity is curious.

**Proposition 1.** For given integers \(0 \leq r \leq m\), there exists a finite connected simple graph \(G\) such that \(\text{reg } K[G] = r\), and
\[
\text{mat}(G) = \begin{cases} m, & \text{if } G \text{ is non-bipartite}, \\ m + 1, & \text{if } G \text{ is bipartite.} \end{cases}
\]

**Proof.** In the non-bipartite case, let \(H\) be the complete graph with \(2r\) vertices. Its matching number is \(r\). We know from ([5], Corollary 2.12) that \(\text{reg } K[H] = r\). At one vertex of \(H\) we attach a path graph of length \(2(m - r)\) and call this new graph \(G\). Then \(\text{mat}(G) = m\) and \(\text{reg } K[G] = \text{reg } K[H] = r\), as \(K[G]\) is just a polynomial extension of \(K[H]\).

In the bipartite case, let \(H\) be the bipartite graph of type \((r + 1, r + 1)\). The matching number is \(r + 1\). Indeed, \(K[H]\) may be viewed as a Hibi ring whose regularity is well-known, see for example ([6], Theorem 1.1). At one vertex of \(H\) we attach a path graph of length \(2(m - r)\) and call this new graph \(G\). Then \(\text{mat}(G) = m + 1\) and \(\text{reg } K[G] = \text{reg } K[H] = r\), for the same reason as before.

These bounds for the regularity of \(K[G]\) are generally only valid if \(K[G]\) is normal. Consider, for example, the graph \(G\) which consists of two disjoint triangles combined as a path of length \(\ell\). Then the defining ideal of \(K[G]\) is generated by a binomial of degree \(\ell + 3\), and hence \(\text{reg } K[G] = \ell + 2\), while the matching number of \(G\) is \(2 + \lceil \ell / 2 \rceil\).

**Question 1.** Let \(m\) be a positive integer, and consider the set \(S_m\) of finite connected simple graphs with matching number \(m\).

- Is there a bound for \(\text{reg } K[G]\) with \(G \in S_m\)?
- If such a bound exists, is it a linear function of \(m\)?

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**References**
1. Ohsugi, H.; Hibi, T. Normal polytopes arising from finite graphs. *J. Algebra* 1998, 207, 409–426.
2. Herzog, J.; Hibi, T.; Ohsugi, H. *Binomial Ideals*; GTM 279; Springer: New York, NY, USA, 2018.
3. Herzog, J.; Hibi, T. Matching numbers and the regularity of the Rees algebra of an edge ideal. *arXiv* 2019, arXiv:1905.02141.
4. Villarreal, R.H. *Monomial Algebras*, 2nd ed.; Monographs and Research Notes in Mathematics, Taylor & Francis Group: Abingdon, UK, 2015.

5. Bruns, W.; Vasoncelos, W.V.; Villarreal, R.H. Degree bounds in monomial subrings. *Ill. J. Math.* 1997, 41, 341–353.

6. Ene, V.; Herzog, J.; Madani, S.S. A note on the regularity of Hibi rings. *Manuscripta Math.* 2015, 148, 501–506.

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