A NONLINEAR THEORY OF TENSOR DISTRIBUTIONS

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Abstract. The coordinate invariant theory of generalised functions of Colombeau and Meril is reviewed and extended to enable the construction of multi-index generalised tensor functions whose transformation laws coincide with their counterparts in classical distribution theory.

1. Introduction

Colombeau’s theory of new generalised functions (Colombeau, 1984) has increasingly had an important role to play in General Relativity, enabling a distributional interpretation to be given to products of distributions which would otherwise be undefined in the framework of Classical distribution theory. Recent applications of Colombeau’s theory to Relativity have included the calculation of distributional curvatures which correspond to metrics of low differentiability, such as those which occur in space-times with thin cosmic strings (Clarke et al, 1996; Wilson 1997) and Kerr singularities (Balasin, 1997), and the electromagnetic field tensor of ultra-relativistic Reissner-Nordstrøm solution (Steinbauer, 1997).

The theory of General Relativity is built upon the Principle of General Covariance which means that the measurement of all physical quantities do not depend on what local coordinate system the observer is using; therefore they must transform locally as tensors under \( C^\infty \) diffeomorphisms. Unfortunately, Colombeau’s theory is not naturally suited for representing such a covariant physical theory because it is heavily built upon the linear structure. Colombeau’s algebra \( G(\Omega) \) is a quotient algebra on the algebra \( \mathcal{E}_M(\Omega) \) of smooth functionals \( \mathcal{A}_0 \times \Omega \to \mathbb{R} \), where one defines the kernel spaces \( \mathcal{A}_k \) as being the subspaces of functions \( \Phi \in \mathcal{D} \) such that

\[
\int \Phi(\xi) d\xi = 1 \\
\int \xi^\alpha \Phi(\xi) d\xi = 0 \quad \alpha \in \mathbb{N}^n, \quad 1 \leq |\alpha| \leq k,
\]

These spaces are not invariant under a smooth diffeomorphism \( x' = \mu(x) \), unless \( \mu \) is linear. Since these spaces are used in both the definitions of the spaces \( \mathcal{E}_M(\Omega) \) (moderate functions) and \( \mathcal{N}(\Omega) \) (null functions) it makes the definition of a mapping \( \tilde{\mu}^* : \mathcal{E}(\Omega') \to \mathcal{E}(\Omega) \) which preserves both \( \mathcal{E}_M(\Omega) \) and \( \mathcal{N}(\Omega) \) very problematic.

As a consequence of this, one must apply Colombeau’s theory with great caution in that coordinate transformations should be handled at the distributional level and Colombeau’s algebra is to be used as a tool for assigning a distributional interpretation to nonlinear operations involving distributions. It was shown by Vickers and Wilson (1998) that the calculation of Clarke et al. (1996), which shows that the distributional Ricci scalar density of a conical singularity with a deficit angle of \( 2\pi(1 - A) \) is \( 4\pi(1 - A)\delta^{(2)} \), is invariant under \( C^\infty \) transformations in the sense that the following diagram commutes

\[
g'_{ab} \xrightarrow{\iota'} \tilde{g}'_{ab} \xrightarrow{\tilde{R}'\sqrt{\tilde{g}'}} \tilde{R}'4\pi(1 - A)\delta^{(2)} \\
\downarrow \mu^* \quad \downarrow \mu^* \quad \\
g_{ab} \xrightarrow{\iota} \tilde{g}_{ab} \xrightarrow{\tilde{R}\sqrt{\tilde{g}}} \tilde{R}4\pi(1 - A)\delta^{(2)}
\]

where the maps \( \mu^* \) denote the relevant tensor distribution transformations. It should be noted that Colombeau’s theory is applied to both coordinate systems and that no attempt is made to compare...
generalised functions in one coordinate system with their counterparts until the distributional curvature has been recovered via weak equivalence.

The problem resulting from the inability of the spaces $A_k$ to transform invariantly has been overcome by the use of the ‘simplified’ algebra $G_s(\Omega)$ which is formed from using a base algebra $E_s(\Omega)$ consisting of smooth functionals on $(0,1] \times \Omega$ rather than on $A_k \times \Omega$ (Biagioni, 1990; Colombeau 1992), so one does not need to worry about how $A_k(\Omega)$ transforms under $\mu$, and hence is able to define a natural map $\tilde{\mu}^* : E_s(\Omega') \rightarrow E_s(\Omega)$ which preserves its moderate subalgebra $E_{M,s}(\Omega')$ and its null ideal $N_s(\Omega')$. However there is still a problem which undermines the suitability of using $G(\Omega)$ as a coordinate invariant representation of distribution theory, which itself is a coordinate invariant construction, in that the embedding $\iota : D'(\Omega) \rightarrow E_M(\Omega)$ provided by the smoothing convolution integral does not commute with the diffeomorphism $\mu$.

A solution to this problem was proposed by Colombeau and Meril (1994), in which they constructed a coordinate invariant $G(\Omega)$ together with a scalar transformation that commutes with the embedding. It essentially involves weakening the moment conditions used in defining the smoothing kernel space $A_k$ which is then defined in a coordinate invariant manner.

In this paper we shall begin by reviewing the algebra of Colombeau and Meril and extend the definitions so that multi-index tensor transformations which commute with the embedding may also be defined.

2. Scalars

Let $\Omega$ and $\Omega'$ be open subsets of $\mathbb{R}^n$ that are diffeomorphic to each other by $\mu : \Omega \rightarrow \Omega'$. In general the map $\mu$ will induce a map $\mu^* : C(\Omega') \rightarrow C(\Omega)$, between the spaces of continuous functions on $\Omega$ and $\omega_{\mu}$, given by $\mu^*(f') = f' \circ \mu$, enabling $f'$ to transform as a scalar between the two coordinate systems.

Functions from $C(\Omega)$ may be embedded into Colombeau’s generalised function algebra $G(\Omega)$ by the smoothing convolution

$$\iota : C(\Omega) \rightarrow E_M(\Omega)$$

$$\iota(f) = \tilde{f}$$

where

$$\tilde{f}(\Phi, x) = \int f(x + \xi)\Phi(\xi) d\xi \quad \Phi \in A_k$$

We would like to be able to construct a map $\tilde{\mu}^* : E_M(\Omega') \rightarrow E_M(\Omega)$, an analogue of $\mu^*$, such that

$$C(\Omega') \xrightarrow{\iota'} E_M(\Omega')$$

$$\downarrow \mu^* \downarrow \tilde{\mu}^*$$

$$C(\Omega) \xrightarrow{\iota} E_M(\Omega)$$

commutes, and furthermore we would like $\tilde{\mu}^*$ to map $N'(\Omega')$ to $N(\Omega)$ so that we can construct $G(\Omega')$ and $G(\Omega)$ consistently.

Given $\tilde{f}' \in E_M(\Omega')$, the obvious way to construct $\tilde{\mu}^* \tilde{f}'$ is by defining

$$\tilde{\mu}^* \tilde{f}'(\Phi, x) = \tilde{f}(\tilde{\mu}_* \Phi, \mu(x))$$

where $\tilde{\mu}_*$ is some mapping on the kernel space $A_k$. If we take $\tilde{\mu}_*(\Phi) = \Phi \circ \mu^{-1} \circ \mu$ then $\tilde{\mu}_*(\Phi)$ will no longer satisfy the normalisation condition

$$\int \tilde{\mu}_* \Phi(\lambda) d\lambda = 1$$

However if we take

$$\tilde{\mu}_*(\Phi) = \frac{1}{|J_\mu|} \Phi \circ \mu^{-1}$$

then the normalisation is preserved. With the choice above,

$$\tilde{\mu}^* \tilde{f}'(\Phi, x) = \int f'((\mu(x) + \lambda) \frac{1}{|J_\mu(\mu^{-1}(\lambda))|} \Phi(\mu^{-1}(\lambda)) d\lambda$$

$$= \int f'(\mu(x) + \mu(\xi))\Phi(\xi) d\xi$$
On the other hand
\[
\hat{f}' \circ \mu(\Phi, x) = \int f'(\mu(x + \xi))\Phi(\xi) d\xi
\]
so \(\hat{\mu} f'\) and \(\hat{f}' \circ \mu\) are different.

In order for \(\hat{\mu} f'\) and \(\hat{f}' \circ \mu\) to represent the same generalised function, their difference must be null; however, the problem is that there is no obvious relation between \(\hat{\mu}_*(A_k)\) and \(A_p\). This prevents us from having a coordinate invariant definition in the framework of Colombeau's original theory.

This problem was resolved by Colombeau and Meril (1994) in which they gave a new definition of the space of smoothing kernels, \(\hat{A}_k\) together with a map \(\hat{\mu}_*\) in such a way that \(\hat{\mu} f' = f' \circ \mu\) and the notion of a null function was preserved.

It is convenient to start by considering generalised functions on \(\mathbb{R}^n\) together with diffeomorphisms \(\mu : \mathbb{R}^n \to \mathbb{R}^n\). We will later restrict the algebra to open sets \(\Omega\) and consider diffeomorphisms between open sets. We begin by considering the embedding of \(f \circ \mu\) using the \(\varepsilon\)-dependent kernel \(\Phi_\varepsilon(x) = \frac{1}{\varepsilon} \Phi\left(\frac{x}{\varepsilon}\right)\)

\[
\hat{f}' \circ \mu(\Phi_\varepsilon, x) = \frac{1}{\varepsilon^n} \int f'(\mu(x + \xi))\Phi(\xi/\varepsilon) d\xi
\]

\[
= \int f'(\mu(x + \varepsilon \eta))\Phi(\eta) d\eta
\]

\[
= \int \frac{f'(\mu(x) + \varepsilon \lambda)}{|J_\mu(\mu^{-1}(\mu(x) + \varepsilon \lambda))|} \Phi\left(\frac{\mu^{-1}(\mu(x) + \varepsilon \lambda) - x}{\varepsilon}\right) d\lambda
\]

where \(\lambda\) is defined by \(\mu(x + \varepsilon \eta) = \mu(x) + \varepsilon \lambda\).

If we now define
\[
\hat{\mu}_* \Phi(\lambda) = \frac{1}{|J_\mu(\mu^{-1}(\mu(x) + \varepsilon \lambda))|} \Phi\left(\frac{\mu^{-1}(\mu(x) + \varepsilon \lambda) - x}{\varepsilon}\right)
\]

then
\[
\hat{f}' \circ \mu(\Phi_\varepsilon, x) = \int f'(\mu(x) + \varepsilon \lambda))(\hat{\mu}_* \Phi(\lambda)) d\lambda
\]

\[
= \frac{1}{\varepsilon^n} \int f'(\mu(x) + \lambda))\left(\hat{\mu}_* \Phi\left(\frac{\lambda}{\varepsilon}\right)\right) d\lambda
\]

\[
= \hat{f}\left((\hat{\mu}_* \Phi)_\varepsilon, \mu(x)\right)
\]

However \(\hat{\mu}_* \Phi\) now depends upon \(\varepsilon\) (although unlike \(\Phi_\varepsilon\) it is bounded as \(\varepsilon \to 0\)) so we replace \(A_0\) by the space \(\hat{A}_0\) of bounded paths in \(D(\mathbb{R}^n)\).

**Definition 1.** The space \(\hat{A}_0\) is the set of all smooth maps

\[
(0, 1] \to D(\mathbb{R}^n)
\]

\[
\varepsilon \mapsto \Phi(\varepsilon)
\]

such that \(\{\Phi(\varepsilon) : \varepsilon \in (0, 1]\}\) is bounded in \(D(\mathbb{R}^n)\) and

\[
\int \Phi(\varepsilon)(\xi) d\xi = 1 \quad \forall \varepsilon \in (0, 1]
\]

However as well as an \(\varepsilon\)-dependence we see that \(\hat{\mu}_*\) also introduces an \(x\)-dependence. We therefore modify the definition of \(\hat{A}_0\) as follows:

**Definition 2.** The space \(\hat{A}_0(\mathbb{R}^n)\) is defined to be the set of all \(\Phi_x(\varepsilon) \in D(\mathbb{R}^n)\) where \(\Phi_x(\varepsilon) = \hat{\mu}_* \Phi(\varepsilon)\) for some \(\Phi(\varepsilon) \in \hat{A}_0(\mathbb{R}^n)\) and some \(\mu \in \text{Diff}(\mathbb{R}^n, \mathbb{R}^n)\).

Note that elements of \(\hat{A}_0(\mathbb{R}^n)\) satisfy the normalisation condition

\[
\int \Phi_x(\varepsilon)(\xi) d\xi = 1 \quad \forall (\varepsilon, x) \in (0, 1] \times \mathbb{R}^n
\]

and that this condition is preserved by \(\hat{\mu}_*\). Unfortunately the moment condition (2) is not preserved by \(\hat{\mu}_*\). We therefore weaken the moment condition as follows:
3. If \( \mu \Phi \) of moderate functions the sense of calculus on 'convenient vector spaces' (Kriegl and Michor, 1997). We next define a subalgebra (Colombeau, 1982). An alternative, and possibly simpler requirement, is to demand differentiability in smoothing kernels. Colombeau and Meril require Silva-differentiability of \( \tilde{f} \) with respect to the second argument of \( \tilde{f} \) which are \( C^\infty \) functions. The space \( \tilde{A}_0(\Omega) \) which have the property

\[
\int \xi^\alpha \Phi^{(\varepsilon)}(\xi)\,d\xi = O(\varepsilon^k) \quad \forall \alpha \in \mathbb{N}^n, \quad 1 \leq |\alpha| \leq k
\]  

(4)

The diffeomorphism \( \mu : \mathbb{R}^n \to \mathbb{R}^n \) induces a map

\[
\hat{\mu}_* : \tilde{A}_0(\mathbb{R}^n) \to \tilde{A}_0(\mathbb{R}^n)
\]

\[
\hat{\mu}_* \Phi^{(\varepsilon)}(\lambda) = \frac{1}{|J_\mu(\mu^{-1}(\mu(x) + \varepsilon \lambda) - x)|} \Phi^{(\varepsilon)}(\mu(x) + \varepsilon \lambda - x)
\]

(5)

**Theorem 1.** \( \hat{\mu}_* \) exhibits the following properties
1. \( \Phi \in \tilde{A}_0(\mathbb{R}^n) \implies \hat{\mu}_* \Phi \in \tilde{A}_0(\mathbb{R}^n) \)
2. \( \Phi \in \tilde{A}_k(\mathbb{R}^n) \implies \hat{\mu}_* \Phi \in \tilde{A}_k(\mathbb{R}^n) \)
3. If \( \mu : \mathbb{R}^n \to \mathbb{R}^n \) and \( \nu : \mathbb{R}^n \to \mathbb{R}^n \) are diffeomorphisms then \( (\nu \circ \mu)_* = \nu_* \circ \mu_* \).

**Proof.** Refer to Colombeau and Meril (1994).

**Definition 4.** The unbounded path \( (\Phi_{x,\varepsilon}) \) corresponding to \( \Phi^{(\varepsilon)} \) is defined by

\[
\Phi_{x,\varepsilon}(\xi) = \frac{1}{\varepsilon^n} \Phi^{(\varepsilon)}(\xi/\varepsilon)
\]

The unbounded path may be regarded as a regularisation for the delta distribution since \( \forall \Psi \in \mathcal{D}(\mathbb{R}^n) \);

\[
\lim_{\varepsilon \to 0} \int \Phi_{x,\varepsilon}(\xi)\Psi(\xi)\,d\xi = \Psi(0)
\]

hence its alternative name, the delta-net. Here the unbounded path \( (\Phi_{x,\varepsilon}) \) is a new \( \tilde{A}_0(\mathbb{R}^n) \), unlike its counterpart in Colombeau's original theory. A transformation law for these unbounded paths may be formed from (5): Writing \( \Phi^{(\varepsilon)}_{\mu(x)} = \hat{\mu}_* \Phi^{(\varepsilon)}_x \) we have that

\[
\Phi^{(\varepsilon)}_{\mu(x)}(\lambda) = \frac{1}{\varepsilon^n} \Phi^{(\varepsilon)}_{\mu(x)}(\lambda)
\]

\[
= \frac{1}{\varepsilon^n} \frac{1}{|J_\mu(\mu^{-1}(\mu(x) + \lambda) - x)|} \Phi^{(\varepsilon)}(\mu^{-1}(\mu(x) + \lambda) - x)
\]

which we shall also express as

\[
\hat{\mu}_* \Phi_{x,\varepsilon}(\lambda) = \frac{1}{|J_\mu(\mu^{-1}(\mu(x) + \lambda) - x)|} \Phi_{x,\varepsilon}(\mu^{-1}(\mu(x) + \lambda) - x)
\]

It should be noted that in this context \( \hat{\mu}_* \) is a map acting on the space of unbounded paths and not on elements of \( \tilde{A}_0 \).

**Definition 5.** The space \( \mathcal{E}(\mathbb{R}^n) \) is the set of all maps

\[
\tilde{f} : \tilde{A}_0 \times \mathbb{R}^n \to \mathbb{R}
\]

\[
(\Phi^{(\varepsilon)}_x, x) \mapsto \tilde{f}(\Phi^{(\varepsilon)}_x, x)
\]

which are \( C^\infty \) as functions of \( x \)

Since the smoothing kernels now depend upon \( x \), it is no longer sufficient to require smoothness with respect to the second argument of \( \tilde{f} \). Instead we also require some kind of smooth dependence upon the smoothing kernels. Colombeau and Meril require Silva-differentiability of \( \tilde{f} \) as a function of the mollifier (Colombeau, 1982). An alternative, and possibly simpler requirement, is to demand differentiability in the sense of calculus on 'convenient vector spaces' (Kriegl and Michor, 1997). We next define a subalgebra of moderate functions.
Definition 6. $\mathcal{E}_M(\mathbb{R}^n)$ is the space of $\hat{f} \in \mathcal{E}(\mathbb{R}^n)$ such that $\forall K \subset \subset \mathbb{R}^n, \forall \alpha \in \mathbb{N}^n, \exists N \in \mathbb{N}$ such that $\forall \Phi_{x}^{(\varepsilon)} \in \hat{A}_n(\mathbb{R}^n)$,

$$
\sup_{x \in K} \left| D^\alpha \hat{f}(\Phi_{x, \varepsilon}, x) \right| = O(\varepsilon^{-N}) \quad \text{(as } \varepsilon \to 0) \n$$

Note that this differs from the definition in Colombeau and Meril because our $\hat{A}_0$ have an $x$-dependence.

We now define an ideal of negligible (null) functions

Definition 7. $\mathcal{N}(\mathbb{R}^n)$ is the space of functions $\tilde{f} \in \mathcal{E}_M(\mathbb{R}^n)$ such that $\forall K \subset \subset \mathbb{R}^n, \forall \alpha \in \mathbb{N}^n, \exists N \in \mathbb{N}$, $\exists \sigma \in \mathcal{S}$ such that $\forall \Phi_{x}^{(\varepsilon)} \in \hat{A}_k(\mathbb{R}^n)$ ($k \geq N$),

$$
\sup_{x \in K} \left| D^\alpha \tilde{f}(\Phi_{x, \varepsilon}, x) \right| = O(\varepsilon^{\sigma(k)}) \quad \text{(as } \varepsilon \to 0) \n$$

where

$$
\mathcal{S} = \{ \sigma : \mathbb{N} \to \mathbb{R}^+ \mid \sigma(k+1) > \sigma(k), \sigma(k) \to \infty \} \n$$

The space of generalised functions is then defined as the quotient

$$
\mathcal{G}(\mathbb{R}^n) = \frac{\mathcal{E}_M(\mathbb{R}^n)}{\mathcal{N}(\mathbb{R}^n)}.
$$

We may then define a map

$$
\hat{\mu}^* : \mathcal{E}(\mathbb{R}^n) \to \mathcal{E}(\mathbb{R}^n)
\hat{\mu}^* f = \hat{f} = f(\mu(x))
$$

The fact that $\hat{\mu}_* \hat{f}$ maps the kernel spaces $\hat{A}_k(\mathbb{R}^n)$ into $\hat{A}_{k/2}(\mathbb{R}^n)$ will imply that

$$
\hat{\mu}^* \hat{f} \in \mathcal{E}_M(\mathbb{R}^n) \iff \hat{f} \in \mathcal{E}_M(\mathbb{R}^n)
\hat{\mu}^* \hat{f} \in \mathcal{N}(\mathbb{R}^n) \iff \hat{f} \in \mathcal{N}(\mathbb{R}^n)
$$

which will induce a well defined map $[\hat{\mu}^*] : \mathcal{G}(\mathbb{R}^n) \to \mathcal{G}(\mathbb{R}^n)$. Furthermore if we define an embedding by

$$
\iota : \mathcal{C}(\mathbb{R}^n) \to \mathcal{E}_M(\mathbb{R}^n)
\iota(f) = \tilde{f}
\tilde{f}(\Phi_{x}^{(\varepsilon)}, x) = \int f(x + \varepsilon \xi) \Phi_{x}^{(\varepsilon)}(\xi) d\xi
$$

then if $f \in \mathcal{C}(\mathbb{R}^n)$ we have

$$
\hat{\mu}^* \tilde{f}(\Phi_{x}^{(\varepsilon)}, x) = \int f(\mu(x) + \varepsilon \xi) \Phi_{x}^{(\varepsilon)} \left( \frac{\mu^{-1}(\mu(x) + \varepsilon \lambda) - x}{\varepsilon} \right) \frac{d\lambda}{|J_\mu(\mu^{-1}(\mu(x) + \varepsilon \lambda))|}
$$

$$
= \int f(\mu(x) + \varepsilon \xi) \Phi_{x}^{(\varepsilon)}(\xi) d\xi
$$

giving us that

$$
\hat{\mu}^* \circ \iota(f) = \iota \circ \mu^*(f)
$$

as we required.

We now define the algebra $\mathcal{G}(\Omega)$ on open sets $\Omega \subset \subset \mathbb{R}^n$. $\mathcal{E}_M(\Omega)$ and $\mathcal{N}(\Omega)$ are defined in the obvious way by restricting $x$ to be in $\Omega$. Then

$$
\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)
$$

If we now only have a diffeomorphism between open sets $\mu : \Omega \to \Omega'$ rather than the whole of $\mathbb{R}^n$ we see that $\hat{\mu}_* : \hat{A}_0(\mathbb{R}^n) \to \hat{A}_0(\mathbb{R}^n)$ is not always well defined by (5). However it is always well defined for $x \in \Omega$. 

\text{...}
and sufficiently small \( \varepsilon \). Since this is all that is needed for the theory we may define \( \tilde{\mu}_x \Phi_x^{(\varepsilon)} \) for other values in some arbitrary manner without effecting the definition of \( \tilde{\mu}^* f(\Phi_x^{(\varepsilon)}, x) \). Thus a diffeomorphism \( \mu : \Omega \to \Omega' \) induces a well defined map \( \tilde{\mu}^* : \mathcal{G}(\Omega') \to \mathcal{G}(\Omega) \).

We now consider the embedding. Given \( f \in \mathcal{D}(\Omega) \) we extend \( f \) to some function \( g \in \mathcal{D}(\mathbb{R}^n) \) such that \( g|\Omega = f \). We then embed \( g \) into \( \mathcal{E}_M(\mathbb{R}^n) \) using (6). Again one can show that for all \( x \in \Omega \) and sufficiently small \( \varepsilon \) the answer does not depend upon the extension. We therefore have a well defined embedding \( \iota : \mathcal{D}(\Omega) \to \mathcal{G}(\Omega) \). Furthermore we still have the result that

\[
\tilde{\mu}^* \circ \iota(f) = \iota \circ \mu^*(f)
\]  

(7)
as the previous result is valid for sufficiently small \( \varepsilon \).

We are finally in a position to consider generalised functions on manifolds. Let \( M \) be a manifold with atlas \( \mathcal{A} = \{(\psi_\alpha, V_\alpha) : \alpha \in A\} \). Given a function \( f \in \mathcal{D}(M) \) we define \( f_\alpha : \psi_\alpha(V_\alpha) \to \mathbb{R} \) by \( f_\alpha = f \circ \psi_\alpha^{-1} \). The set \( \{f_\alpha\}_{\alpha \in A} \) then satisfies

\[
f_\beta|\psi_\beta(V_\alpha \cap V_\beta) = f_\alpha|\psi_\beta(V_\alpha \cap V_\beta) \circ \psi_\alpha \circ \psi_\beta^{-1} = (\psi_\beta^{-1})^* \psi_\alpha f_\alpha|\psi_\beta(V_\alpha \cap V_\beta) \quad \forall \alpha, \beta \in A \text{ with } V_\alpha \cap V_\beta \neq \emptyset
\]

(8)

Conversely any set of functions \( \{f_\alpha \in \mathcal{D}(\psi_\alpha(V_\alpha))\}_{\alpha \in A} \) which satisfies (8) defines an element of \( \mathcal{D}(M) \). In exactly the same way we make the following definition

**Definition 8.** A generalised function \( \tilde{f} \in \mathcal{G}(M) \) is a set of generalised functions \( \{\tilde{f}_\alpha \in \mathcal{G}(\psi_\alpha(V_\alpha))\}_{\alpha \in A} \) which satisfies

\[
\tilde{f}_\beta|\psi_\beta(V_\alpha \cap V_\beta) = (\psi_\beta^{-1})^* \tilde{f}_\alpha|\psi_\beta(V_\alpha \cap V_\beta) \quad \forall \alpha, \beta \in A \text{ with } V_\alpha \cap V_\beta \neq \emptyset
\]

(9)

If we define \( \tilde{f}_\alpha = \iota_\alpha \circ f_\alpha \) where \( \iota_\alpha \) is the embedding \( \iota_\alpha : \mathcal{D}(\psi_\alpha(V_\alpha)) \to \mathcal{G}(\psi_\alpha(V_\alpha)) \), then one has an embedding

\[
\iota : \mathcal{D}(M) \to \mathcal{G}(M)
\]

**3. Smoothing of vector and covector fields**

We now have a coordinate invariant theory of generalised functions at the level of scalars. We would like to extend this theory to enable vectors, covectors and, ultimately, multi-index tensors to be defined as generalised functions whose transformation laws will coincide with those of distributions.

We shall first consider the smoothing of covector field \( \omega \), which may be represented in both coordinate systems \( x \in \Omega \) and \( x' \in \Omega' \) as

\[
\omega = \omega_a(x) dx^a = \omega'_a(x') dx'^a,
\]

with the components \( \omega_a \) and \( \omega'_a \) related by \( \omega_a = \mu^a \omega'_a \) where

\[
\mu^* : \mathcal{C}^0(\Omega') \to \mathcal{C}^0(\Omega),
\mu^* \omega'_a(x) = \mu^b_s(x) \omega'_a(\mu(x)).
\]

We could use a componentwise smoothing,

\[
\iota : \mathcal{C}^0(\Omega) \to \mathcal{E}^0(\Omega)
\]

\[
\iota(\omega) = \tilde{\omega}
\]

\[
\tilde{\omega}_a(\Phi, x) = \int \omega_a(x + \varepsilon \xi) \Phi_x^{(\varepsilon)}(\xi) \, d\xi
\]

(9)

and relate the smoothings by \( \tilde{\mu}^* : \mathcal{E}^0(\Omega') \to \mathcal{E}^0(\Omega) \)

\[
\tilde{\mu}^* \tilde{\omega}'_a(\Phi, x) = \mu^b_s(x) \tilde{\omega}'_a(\mu_x \Phi, x)
\]
in which case we would have that

\[ \hat{\omega}_a(\Phi, x) = \int \mu_a^b(x + \varepsilon \xi) \omega_b^c(\mu(x + \varepsilon \xi)) \Phi_x^{(\varepsilon)}(\xi) \, d\xi \]

\[ = \int \mu_a^b(\mu^{-1}(\mu(x) + \varepsilon \lambda)) \omega_b^c(\mu(x) + \varepsilon \lambda) \mu_* \Phi_x^{(\varepsilon)}(\lambda) \, d\lambda \]

and

\[ \hat{\mu}^* \hat{\omega}_a(\Phi, x) = \int \mu_a^b(x) \omega_b^c(\mu(x) + \varepsilon \lambda) \mu_* \Phi_x^{(\varepsilon)}(\lambda) \, d\lambda \]

If it is the case that \( \omega'_a \) is \( C^\infty \), we can expand \( \mu_a^b \) and \( \omega_a^b \) in powers of \( \varepsilon \lambda \) which will reveal that

\[ \tilde{\omega}_a(\Phi, x) - \tilde{\mu}^* \tilde{\omega}_a(\Phi, x) = O(\varepsilon^k) \quad \text{for} \quad \tilde{\mu}_a \in \mathcal{N}(\Omega') \]

so \( \tilde{\omega}_a(\Phi, x) \) and \( \tilde{\mu}^* \tilde{\omega}_a \) may be regarded as representing the same element of \( \mathcal{G}(\Omega) \). On the other hand if \( \omega'_a \) admitted only a finite level of differentiability, then \( \tilde{\omega}_a \) and \( \tilde{\mu}^* \tilde{\omega}_a \) would not be equivalent as elements of \( \mathcal{G}(\Omega) \) although they may well be equivalent at the level of association, given that the level of differentiability was high enough.

A possible remedy for this problem is to introduce smoothing kernels with indices, thus we replace (9) by

\[ \tilde{\omega}^a(x) = \int \omega_b(x + \varepsilon \xi) \Phi_x^{(\varepsilon)}(\xi) \, d\xi \]

where

\[ \int \Phi_x^{(\varepsilon)}(\xi) \, d\xi = \delta_a^b + O(\varepsilon^k) \]

\[ \int \xi^\alpha \Phi_x^{(\varepsilon)}(\xi) \, d\xi = O(\varepsilon^k) \quad \alpha \in \mathbb{N}^n, \quad 1 \leq |\alpha| \leq k \]

and define

\[ \hat{\mu}_a \Phi_x^{(\varepsilon)}(\lambda) = \frac{\eta_a^b(x)}{\mu_n(\mu^{-1}(\mu(x) + \varepsilon \lambda))} \Phi_x^{(\varepsilon)} \int \frac{d}{\varepsilon} \left( \frac{\mu^{-1}(\mu(x) + \varepsilon \lambda) - x}{\varepsilon} \right) \]

where

\[ \eta_a^b(x) = \mu^{-1}a_b(\mu(x)) \]

**Proposition 2.** The moment conditions (10) are preserved under \( \hat{\mu}_a \) in that

\[ \int \hat{\mu}_a \Phi_x^{(\varepsilon)}(\lambda) \, d\lambda = \delta_a^b + O(\varepsilon^k) \]

\[ \int \lambda^\alpha \hat{\mu}_a \Phi_x^{(\varepsilon)}(\lambda) \, d\lambda = O(\varepsilon^{[k/2]}) \quad \alpha \in \mathbb{N}^n, \quad 1 \leq |\alpha| \leq [k/2] \]

**Proof.** The first condition holds since

\[ \int \Phi_x^{(\varepsilon)}(\xi) \, d\lambda = \int \eta_a^b(x) \mu_a^b(x + \varepsilon \xi) \Phi_x^{(\varepsilon)}(\xi) \, d\xi \]

\[ = \eta_a^b(x) \mu_a^b(x) \int \Phi_x^{(\varepsilon)}(\xi) \, d\xi + \sum_{|\alpha| = 1}^k \frac{D^\alpha \eta_a^b(x)}{|\alpha|!} \int \xi^\alpha \Phi_x^{(\varepsilon)}(\xi) \, d\xi \]

\[ + \sum_{|\alpha| = k+1}^k D^\alpha \eta_a^b(x + \varepsilon \theta \xi) \mu_a^b(x) \frac{\varepsilon^{|\alpha|}}{|\alpha|!} \int \xi^\alpha \Phi_x^{(\varepsilon)}(\xi) \, d\xi \]

\[ = \delta_a^b + O(\varepsilon^{k+1}) \]

A similar calculation (See Colombeau and Meril, 1994) will show that the second condition is preserved with \( k \) replaced by \( [k/2] \).
Proposition 3. \( \tilde{\mu}^* \tilde{\omega}'_a = \tilde{\omega}_a \)

Proof.

\[
\tilde{\omega}_a(\Phi, x) = \int \mu^b_a(x + \varepsilon \xi) \omega^c_b(\mu(x + \varepsilon \xi)) \Phi^{(c)}_x b^{(a)}(\xi) \, d\xi \\
= \int \mu^b_a(\mu^{-1}(\mu(x) + \varepsilon \lambda)) \omega^c_b(\mu(x) + \varepsilon \lambda) \Phi^{(c)}_x b^{(a)} \left( \frac{\mu^{-1}(\mu(x) + \varepsilon \lambda) - x}{\varepsilon} \right) \, d\lambda \\
= \int \mu^b_a(x) \omega^c_b(\mu(x) + \varepsilon \lambda) \tilde{\mu}_* \Phi^{(c)}_x b^{(a)}(\varepsilon, \mu(x), \lambda) \, d\lambda \\
= \tilde{\mu}^* \tilde{\omega}'_a(\Phi, x)
\]

The need for smoothing kernels with indices is clear, because we are integrating the coefficients of the covector field at the point \( x + \varepsilon \xi \) to give a covector at the point \( x \). The kernel \( \Phi^{(c)}_x b^{(a)} \) therefore has the effect of transporting a covector at the point \( x + \varepsilon \xi \) to the point \( x \). Rather than considering general kernels with indices \( \Phi^{(c)}_x b^{(a)} \) we shall consider those which have the form

\[
\Phi^{(c)}_x b^{(a)}(\xi) = \Gamma^{(c)}_a b^{(a)}(x, x + \varepsilon \xi) \Phi^{(c)}_x b^{(a)}(\xi)
\]

where

\[
\Gamma : \Omega \times \Omega \to \text{T}^*(\Omega) \times \text{T}(\Omega)
\]

is a smooth map with \( \Gamma^{(c)}_a b^{(a)}(x, y) = \delta^b_a \) and \( \Phi^{(c)}_x b^{(a)} \in A_0(\Omega) \). The respective transformation laws (5) and (11) for \( \Phi^{(c)}_x b^{(a)} \) and \( \Phi^{(c)}_a b^{(a)} \) will imply that \( \Gamma^{(c)}_a b^{(a)}(x, y) \) has to transform as

\[
\tilde{\mu}_* \Gamma^{(c)}_a b^{(a)}(\mu(x), \mu(y)) = \eta^c_b(x) \mu^d_a(y) \Gamma^{(c)}_d b^{(a)}(x, y)
\]

so \( \Gamma^{(c)}_a b^{(a)} \) may be regarded as components of a covector (with index \( a \)) located at \( x \) and a vector (with index \( b \)) located at \( y \).

A similar procedure could be applied to smooth a vector

\[
X = X^a(x) \frac{\partial}{\partial x^a} = X'^b(x) \frac{\partial}{\partial x'^b}
\]

with the components \( X^a \) and \( X'^a \) related by \( X^a = \mu^a X'^a \) where

\[
\mu^a : C^0_b(\Omega') \to C^0_b(\Omega), \\
\mu^a X'^a(x) = \eta^c_b(x) X'^b(\mu(x)).
\]

In this case we use a smoothing of the form

\[
\tilde{X}^a(x) = \int X^b(x + \varepsilon \xi) \Phi^{(c)}_x b^{(a)}(\xi) \, d\xi
\]

By allowing \( \Phi^{(a)}_b \) to transform as

\[
\tilde{\mu}_* \Phi^{(c)}_x b^{(a)}(\lambda) = \frac{\mu^d_a(x) \eta^d_b(\mu^{-1}(\mu(x) + \varepsilon \lambda))}{|\mu(\mu^{-1}(\mu(x) + \varepsilon \lambda))|} \Phi^{(c)}_x b^{(a)} \left( \frac{\mu^{-1}(\mu(x) + \varepsilon \lambda) - x}{\varepsilon} \right)
\]

we are able to define a map \( \tilde{\mu}^* \) by

\[
\tilde{\mu}^* \tilde{X}'^a(\Phi, x) = \eta^c_b(x) \tilde{\omega}^{'b}(\tilde{\mu}_* \Phi, \mu(x))
\]

which will satisfy

\[
\tilde{\mu}^* \tilde{X}'^a = \tilde{X}^a
\]
A NONLINEAR THEORY OF TENSOR DISTRIBUTIONS

This time \( \Phi^{a \rightarrow b} \) transports a vector at \( x + \varepsilon \xi \) to a vector at \( x \), we may therefore consider kernels \( \Phi^{a \rightarrow b} \) which admit the form

\[
\Phi^{(\varepsilon) a \rightarrow b}(\xi) = \Gamma^{a b}(x, x + \varepsilon \xi) \Phi^{(\varepsilon)}(\xi)
\]

where \( \Phi^{(\varepsilon)} \in \hat{A}_0(\mathbb{R}^n) \) and the map

\[
\Gamma : \Omega \times \Omega \rightarrow T(\Omega) \times T^*(\Omega)
\]

\[
(x, y) \mapsto \Gamma^{a b}(x, y)
\]

is smooth with \( \Gamma^{a b}(x, x) = \delta^a_b \) and transforms as

\[
\hat{\mu}_a \Gamma^{a b}(\mu(x), \mu(y)) = \mu_a^c(x) \eta^d_b(y) \Gamma^{c d}(x, y)
\]

We shall formalise our transport operators \( \Gamma^{a b} \) and \( \Gamma_a^b \) by defining the following space of transport operators;

**Definition 9.** The space \( T(\Omega) \) is defined to be the set of maps

\[
\Gamma : \Omega \times \Omega \rightarrow T(\Omega) \times T^*(\Omega)
\]

\[
(x, y) \mapsto \Gamma^{a b}(x, y)
\]

which are smooth and are such that \( \Gamma^{a b}(x, x) = \delta^a_b \).

In this way we may simultaneously write down smoothings of vectors and covectors as functions

\[
\tilde{A}_0(\mathbb{R}) \times T(\Omega) \times \Omega \rightarrow \mathbb{R}
\]

\[
\tilde{X}^a(\Phi, \Gamma, x) = \int X^b(x + \varepsilon \xi) \Gamma^{a b}(x, x + \varepsilon \xi) \Phi(\xi) \, d\xi
\]

\[
\tilde{\omega}_a(\Phi, \Gamma, x) = \int \omega_b(x + \varepsilon \xi) \Gamma^{b a}(x, x + \varepsilon \xi) \Phi(\xi) \, d\xi
\]

where

\[
\Gamma^{a c} \Gamma^{c b} = \delta^a_b
\]

An important consequence of this arrangement is that \( \Gamma \) will preserve contractions of covectors with vectors; suppose that

\[
X^a(x) = \Gamma^{a b}(x, y) X^b(y)
\]

\[
\omega_a(x) = \Gamma_a^b(x, y) \omega_b(y)
\]

then

\[
\omega_a(x) X^a(x) = \Gamma_a^b(x, y) \Gamma^{a c}(x, y) \omega^c(y) X^c(y)
\]

\[
= \omega^c(y) X^c(y)
\]

4. **Remarks on the transport operator**

We now show that the derivative of the transport operator defines a connection and that conversely a connection defines a transport operator (in a normal neighbourhood). We first consider rules for differentiating \( \Gamma \). We may write

\[
\frac{\partial \Gamma^{a b}}{\partial x^c}(x, y) = \gamma^{d c}_a(x) \Gamma^{d b}(x, y)
\]

using the fact that

\[
\Gamma^{a b}(x, y) \Gamma^{b c}(y, x) = \delta^c_a
\]

\[
\Gamma^{a c}(x, y) \Gamma^{c b}(x, y) = \delta^a_b
\]

we also have

\[
\frac{\partial \Gamma^{a b}}{\partial y^c}(x, y) = -\gamma^{b c}_d(y) \Gamma^{a d}(x, y)
\]

\[
\frac{\partial \Gamma^{a b}}{\partial x^c}(x, y) = -\gamma^{a c}_d(x) \Gamma^{d b}(x, y)
\]

\[
\frac{\partial \Gamma^{a b}}{\partial y^c}(x, y) = \gamma^{d b}_c(y) \Gamma^{d a}(x, y)
\]
Proposition 4. $\gamma_{bc}^a$ transforms as a connection

Proof. The connector $\Gamma^a_{\ bc}$ transforms as

$$\Gamma'^{a}_{\ bc}(x', y') = \frac{\partial x^c}{\partial x'^c} \frac{\partial y^b}{\partial y'^d} \Gamma^d_{\ c}(x, y)$$

so

$$\frac{\partial \Gamma'^{a}_{\ bc}}{\partial x'^c} = \frac{\partial^2 x^c}{\partial x'^a \partial x'^c} \frac{\partial y^b}{\partial y'^d} \Gamma^d_{\ c} + \frac{\partial x^c}{\partial x'^a} \frac{\partial x'^f}{\partial x'^c} \frac{\partial y^b}{\partial y'^d} \frac{\partial \Gamma'^{d}_{\ f}}{\partial x'^f}$$

this implies that

$$\gamma'^{ed}_{\ ca} \Gamma'^{b}_{\ d} = \frac{\partial^2 x^c}{\partial x'^a \partial x'^c} \frac{\partial y^b}{\partial y'^d} \Gamma^d_{\ c} + \frac{\partial x^c}{\partial x'^a} \frac{\partial x'^f}{\partial x'^c} \frac{\partial y^b}{\partial y'^d} \gamma'^{e}_{\ f} \Gamma^d_{\ g}$$

and hence

$$\gamma'^{ed}_{\ ca} \frac{\partial x^g}{\partial x'^c} \frac{\partial y^b}{\partial y'^d} \Gamma^d_{\ g} = \frac{\partial^2 x^g}{\partial x'^a \partial x'^c} \frac{\partial y^b}{\partial y'^d} \Gamma^d_{\ g} + \frac{\partial x^c}{\partial x'^a} \frac{\partial x'^f}{\partial x'^c} \frac{\partial y^b}{\partial y'^d} \gamma'^{e}_{\ f} \Gamma^d_{\ g}$$

on multiplying this expression throughout by

$$\frac{\partial y^p}{\partial y'^b} \frac{\partial x'^r}{\partial x'^a} \Gamma^q_{\ p}$$

one obtains the standard connection transformation law

$$\gamma'^{ia}_{\ bc} = \frac{\partial x'^a}{\partial x'^c} \frac{\partial x'^f}{\partial x'^c} \gamma'^{d}_{\ f} + \frac{\partial x'^a}{\partial x'^d} \frac{\partial^2 x^d}{\partial x'^b \partial x'^c} \gamma'^{e}_{\ f}$$

If $\gamma$ is a connection then one can always choose a neighbourhood $U_x$ of any point $x \in \Omega$ such that the normal coordinates at $x$ are well defined. Moreover one can choose $U_x$ to be simply convex; that is that $U_x$ is also a normal neighbourhood for any point $y \in U_x$. In a simply convex neighbourhood one can define $\Gamma_{\ bc}^a(x, y)$ by parallel propagation with respect to $\gamma$ along the unique geodesic connecting $x$ and $y$, so in geodesic coordinates $\Gamma_{\ bc}^a(x, y) = \delta_{\ bc}^a$.

In general if we allow any connection $\gamma$, the sets $\Omega$ in the atlas will not be normal neighbourhoods. However since $\Phi_{\ x}^{(\cdot)}$ has compact support, the integration will be confined to a normal neighbourhood for sufficiently small $\varepsilon$. Since we are only interested in results for sufficiently small $\varepsilon$ we could also work with $\gamma$ in place of $\Gamma$. However once we consider tensor fields on manifolds, rather than open sets $\Omega$, it is much less restrictive to work with a connection $\gamma$ rather than a global connector $\Gamma$. Even when working with subsets $\Omega$ of $\mathbb{R}^n$ it is preferable to work with connections on $\mathbb{R}^n$ rather than connectors on $\Omega$, since the set of algebras then has the structure of a (fine) sheaf. If one does this, then one defines the embedding by (14), where $\Gamma$ is defined by parallel transport along geodesics in a normal neighbourhood and arbitrarily elsewhere. Since one is in a normal neighbourhood for sufficiently small epsilon, the embedding into the algebra will be well defined.

5. General tensors

Having defined procedures for smoothing covectors and vectors in such a way that $\tilde{\mu}^* \circ \xi' = \xi \circ \mu^*$, it is now clear how to go about extending this smoothing operation to more general tensors; Suppose $T$ is a tensor field of type $(p, q)$ whose components in the two coordinate systems $x \in \Omega$ and $x' \in \Omega'$ are related by $T'^{a\cdots b}_{\ cd\cdots d} = \mu^* T^{a\cdots b}_{\ cd\cdots d}$ where

$$\mu^* T^{a\cdots b}_{\ cd\cdots d}(x) = \eta^a_{\ a}(x) \cdots \eta^b_{\ b}(x) \mu^q(x) \cdots \mu^p_{\ p}(x) T^{q\cdots f}_{\ g\cdots h}(\mu(x))$$

then we may define a smooth tensor field by

$$\tilde{T}^{a\cdots b}_{\ cd\cdots d}(\Phi, x) = \int T^{e\cdots f}_{\ g\cdots h}(x + \varepsilon \xi) \Phi^{(\cdot)}_{\ x}^{a\cdots b}_{\ c\cdots d\cdots g\cdots h}(\xi) \ d\xi$$

$$\Phi^{(\cdot)}_{\ x}^{a\cdots b}_{\ c\cdots d\cdots g\cdots h}(\xi) = \Gamma^{a}_{\ c}(x + \varepsilon \xi) \cdots \Gamma^{f}_{\ b}(x + \varepsilon \xi) \Gamma^{g}_{\ c}(x + \varepsilon \xi) \cdots \Gamma^{h}_{\ d}(x + \varepsilon \xi) \Phi^{(\cdot)}_{\ x}(\xi)$$

(16)
and a map $\tilde{\mu}^\ast$ by

$$\tilde{\mu}^\ast \tilde{T}^{a...b}_{c...d}(\Phi, \Gamma, x) = \eta^c_\epsilon(x) \ldots \eta^b_\epsilon(x) \mu^a_\epsilon(x) \ldots \mu^b_\epsilon(x) \tilde{T}^{e...f}_{g...h}(\hat{\mu}_\epsilon \Phi, \hat{\mu}_\epsilon \Gamma, \mu(x))$$

(17)

which will satisfy

$$\tilde{T}^{a...b}_{c...d} = \tilde{\mu}^\ast \tilde{T}^{a...b}_{c...d}$$

(18)

We are now in a position to formally define our generalised tensor fields on $\Omega$. The kernel space $\tilde{A}_0(\mathbb{R}^n)$ is defined as in Definition 2 and $K(\mathbb{R}^n)$ is the space of smooth connections on $\mathbb{R}^n$. We first define the base algebra $\mathcal{E}_q^p(\Omega)$

**Definition 10.** Let $\mathcal{E}_q^p(\Omega)$ be the set of all maps

$$\tilde{T} : \tilde{A}_0 \times K \times \Omega \to \mathbb{R}^{p+q}$$

$$(\Phi, \gamma, x) \mapsto \tilde{T}^{a...b}_{c...d}(\Phi, \gamma, x)$$

such that $x \mapsto \tilde{T}^{a...b}_{c...d}(\Phi, \gamma, x)$ is $C^\infty$.

As in the case of scalar fields we require some smooth dependence upon the smoothing kernel and connection to ensure smooth dependence in $x$. As usual we restrict to those of moderate growth

**Definition 11.** $\mathcal{E}_q^p(\Omega)$ is the space of tensors $\tilde{T} \in \mathcal{E}_q^p(\Omega)$ such that $\forall K \subset \subset \Omega$, $\forall \alpha \in \mathbb{N}^n$, $\exists N \in \mathbb{N}$ such that $\forall \Phi^{(\epsilon)} \in \tilde{A}_K(\mathbb{R}^n)$, $\forall \gamma \in K$,

$$\sup_{x \in K} \| D^\alpha \tilde{T}(\Phi_{x,\epsilon}, \gamma, x) \| = O(\epsilon^{-N}) \quad (\text{as } \epsilon \to 0)$$

We define an ideal of negligible (null) functions

**Definition 12.** $\mathcal{N}_q^p(\Omega)$ is the space of tensors $\tilde{T} \in \mathcal{E}_q^p(\Omega)$ such that $\forall K \subset \subset \Omega$, $\forall \alpha \in \mathbb{N}^n$, $\exists N \in \mathbb{N}$, $\exists \sigma \in S$ such that $\forall \Phi^{(\epsilon)} \in \tilde{A}_K(\mathbb{R}^n)$ ($k \geq N$), $\forall \gamma \in K$,

$$\sup_{x \in K} \| D^\alpha \tilde{T}(\Phi_{x,\epsilon}, \gamma, x) \| = O(\epsilon^\sigma(k)) \quad (\text{as } \epsilon \to 0)$$

where

$$S = \{ \sigma : \mathbb{N} \to \mathbb{R}^+ | \sigma(k+1) > \sigma(k), \ \sigma(k) \to \infty \}$$

We then define our generalised function space as a quotient

**Definition 13.**

$$\mathcal{G}_q^p(\Omega) = \frac{\mathcal{E}_q^p(\Omega)}{\mathcal{N}_q^p(\Omega)}$$

One may now define generalised tensor fields on manifolds as a collection of generalised tensor fields on $\psi_\alpha(V_\alpha)$ which transform in the appropriate way under $$(\psi^{-1}_\beta)^\ast (\psi_\alpha)^\ast$$ using (17). Because of (18) we also have a well defined embedding of $\mathcal{D}_q^p(M)$ into $\mathcal{G}_q^p(M)$.

Throughout the paper we have been using distributions that may be defined as locally integrable functions. It is possible to extend our results to cover the smoothing of more general tensors. We first recall how distribution theory is formulated in a coordinate invariant manner; suppose $\mu : \Omega \to \Omega'$ is a $C^\infty$ diffeomorphism, then we may define $\mathcal{D}_q^p(\Omega)$ to be the space of $C^\infty$ multi-index functions with compact support which transform as type $(q, p)$ densities with weight +1 under

$$\hat{\mu}_\epsilon \Psi^{a...d}_{c...e}(\mu(x)) = \frac{1}{|J_{\mu}(x)|} \eta^c_\epsilon(x) \ldots \eta^d_\epsilon(x) \mu^a_\epsilon(x) \mu^b_\epsilon(x) \Psi^{a...d}_{c...e}(x)$$
We simply define type \((p, q)\) tensor distributions as elements of the dual space \(\mathcal{D}'_{\mu}(\Omega)\) which then admit a transformation law \(\mu^* : \mathcal{D}'_{\mu}(\Omega') \rightarrow \mathcal{D}'_{\mu}(\Omega)\) of the form

\[
\langle \mu^* T', \Psi \rangle = \langle T, \mu \Psi \rangle
\]

This immediately becomes evident in case of \(T\) being a locally integrable tensor field, for the integral

\[
\langle T, \Psi \rangle = \int T^{a \cdots b}_{c \cdots d}(x) \Psi^{c \cdots d}_{a \cdots b}(x) \, dx
\]

is coordinate invariant.

We now define

\[
t : \mathcal{D}'(\Omega) \rightarrow \mathcal{E}_M(\Omega)
\]

\[
t(T) = \hat{T}
\]

where

\[
\hat{T}^{a \cdots b}_{c \cdots d}(\Phi, \gamma, x) = \langle T, \Xi \Xi_{c \cdots d} \tau_x \Phi_{x, \varepsilon} \rangle
\]

\[
(\Xi_{c \cdots d} \tau_x \Phi_{x, \varepsilon})^{(a \cdots b)} f(y) = \Gamma_c^a(x, y) \ldots \Gamma_d^b(x, y) \gamma \ldots \Gamma_s^k(x, y)
\]

\[
\tau_x(y) = x - y
\]

It should be noted that the test function \(\Xi_{c \cdots d} \tau_x \Phi_{x, \varepsilon}\) behaves as a type \((p, q)\) tensor with respect to \(x\) and that the translated object \(\tau_x \Phi_{x, \varepsilon}\) will transform as

\[
\tau_{\mu(x)} \Phi_{x, \varepsilon}(\lambda) = \frac{1}{\lvert J_\mu(\mu^{-1} (\lambda)) \rvert} \tau_x \Phi_{x, \varepsilon}(\mu^{-1}(\lambda))
\]

**Proposition 5.** If \(T \in \mathcal{D}'_{\mu}(\Omega)\) then \(\hat{\mu}^* \hat{T}' = \hat{\mu}^* \hat{T}'\).

**Proof.**

\[
\hat{\mu}^* \hat{T}' \Xi_{c \cdots d} \Phi_{x, \varepsilon} = \hat{T}' \Xi_{c \cdots d} \Phi_{x, \varepsilon}(\mu(x)) \]

\[
= \langle T', \hat{\mu} \Phi_{x, \varepsilon} \rangle \Xi_{c \cdots d} \tau_x \Phi_{x, \varepsilon}
\]

\[
= \eta^a_c(x) \ldots \eta^b_f(x) \mu^\gamma_c(x) \ldots \mu^\gamma_s(x) \langle \mu^* T', \Xi \Xi_{g \cdots h} \tau_x \Phi_{x, \varepsilon} \rangle
\]

\[
= \eta^a_c(x) \ldots \eta^b_f(x) \mu^\gamma_c(x) \ldots \mu^\gamma_s(x) \mu^* \hat{T}' \Xi_{g \cdots h} \Phi_{x, \varepsilon}(\Phi, \gamma, x)
\]

This result enables us to define an embedding of \(\mathcal{D}'_{\mu}(M)\) into \(\mathcal{G}'_{\mu}(M)\).

Since elements of the generalised function space \(\mathcal{G}'_{\mu}(\Omega)\) are represented by smooth functions, we may define tensor operations on these generalised functions in the usual way. In particular we are able to define the following:

1. **Tensor products;** Suppose \([\hat{S}'] \in \mathcal{G}'_{\mu}(\Omega)\) and \([\hat{T}'] \in \mathcal{G}'_{\mu}(\Omega')\) are type \((p, q)\) and type \((r, s)\) tensors respectively then we may define \([\hat{S}' \otimes \hat{T}'] \in \mathcal{G}'_{\mu+r+s}(\Omega')\) by

\[
\langle \hat{S}' \otimes \hat{T}', \Phi_{x, \varepsilon} \rangle = \hat{S}' \Xi_{c \cdots d} \Phi_{x, \varepsilon}(\Phi, \gamma', x') \]

the resulting object \([\hat{S}' \otimes \hat{T}']\) will transform as a type \((p + r, q + s)\) tensor in that

\[
\hat{\mu}^* (\hat{S}' \otimes \hat{T}') = \hat{\mu}^* \hat{S}' \otimes \hat{\mu}^* \hat{T}'
\]

2. **Contractions;** Suppose \(\hat{T}' \in \mathcal{G}'_{\mu+r+1}(\Omega')\) is a type \((p, q)\) tensor then we may define \(\hat{S}' \in \mathcal{G}'_{\mu}(\Omega)\) by

\[
\hat{S}'^{a \cdots b}_{c \cdots d}(\Phi, \gamma', x') = \hat{T}'^{a \cdots b}_{c \cdots d}(\Phi', \gamma', x')
\]

\([\hat{S}']\) will transform as a type \((p, q)\) tensor because

\[
\hat{\mu}^* \hat{S}'^{a \cdots b}_{c \cdots d} = \hat{\mu}^* \hat{T}'^{a \cdots b}_{c \cdots d}
\]
3. Differentiation; Suppose that $[\hat{T}] \in G_q^0(\Omega')$ then we may define $[\partial^T] \in G_{q+1}^0(\Omega')$ by

$$\partial^T \tilde{T}^{a...b}_{c...de}(\Phi', \gamma', x') = \frac{\partial \tilde{T}^{a...b}_{c...de}(\Phi', \gamma', x')}{\partial x^e}$$

In general $[\partial^T]$ will not transform as a tensor for we do not necessarily have

$$\tilde{\mu}^*([\partial^T]) = \partial([\hat{T}])$$

However, if we are able to define a connection $[\tilde{\Gamma}^a_{bc}] \in G_q^1(\Omega')$ that transforms as

$$\tilde{\mu}^*\tilde{\Gamma}^a_{bc}(\Phi, \gamma, x) = \eta^a_d(x)\mu^b_e(x)\tilde{\mu}^f_d(\tilde{\mu}_*\Phi, \mu_*\gamma, \mu(x)) + \eta^a_d(x)\mu^b_e(x)$$

then we are able to define a covariant derivative $[\nabla^T]$ in $G_{q+1}^0(\Omega')$ by

$$\nabla^T \tilde{T}^{a...b}_{c...de} = \frac{\partial}{\partial x^e} \tilde{T}^{a...b}_{c...de} + \tilde{\Gamma}^a_{ef} \tilde{T}^{f...b}_{c...de} + \cdots + \tilde{\Gamma}^{ef}_{cd} \tilde{T}^{a...b}_{f...-d} - \tilde{\Gamma}^{ef}_{cd} \tilde{T}^{a...b}_{-f...d}$$

which will satisfy

$$\tilde{\mu}^*([\nabla^T]) = \nabla([\hat{T}])$$

4. Lie derivatives. For $[\tilde{T}] \in G_q^0(\Omega')$ and $[\hat{T}] \in G_{q}^0(\Omega)$, the Lie derivative $[\mathcal{L}_\hat{X}, \hat{T}] \in G_{q}^0(\Omega)$ may be defined as

$$\mathcal{L}_\hat{X} \hat{T}^{a...b}_{c...de} = \tilde{X}^a \hat{T}^{a...b}_{c...de} - \tilde{X}^a \hat{T}^{a...b}_{c...de} - \cdots - \tilde{X}^{ef}\hat{T}^{a...b}_{c...de} + \tilde{X}^{ef}\tilde{T}^{a...b}_{c...de}$$

which will satisfy

$$\tilde{\mu}^*([\mathcal{L}_\hat{X}^T]) = \mathcal{L}_{\tilde{\mu}^X}([\hat{T}])$$

The linear tensor operations such as addition, symmetrisation and antisymmetrisation of indices also extend to our generalised function valued tensors in a natural way.

6. DIFFERENTIATION

A $C^\infty$ tensor field $T$ may be embedded in to $G_q^0(\Omega)$ in two ways;

1. By smoothing: $T \mapsto \hat{T}$

2. By defining an element of $\mathcal{E}_q^0(\Omega)$ which is independent of $\Phi$ and $\gamma$; $\hat{S}(\Phi, \gamma, x) = T(x)$

By defining $\mathcal{G}_q^0(\Omega)$ as a quotient group we are able to guarantee that $\hat{S}$ and $\hat{T}$ represent the same generalised function. Here we shall verify that this is also the case for derivatives in that if $T \in C_q^{p,\infty}(\Omega)$ then it is the case that $\hat{T}^{a...b}_{e...c,d}$ and $T^{a...b}_{e...c,d}$ represent the same generalised function.

We begin with scalars; suppose $f \in C^\infty(\Omega)$ then we have,

$$\tilde{f}_a(x) = \int f_a(x + \varepsilon \xi) \Phi_x^{(\varepsilon)}(\xi) d\xi + \int f(x + \varepsilon \xi) \Phi_x^{(\varepsilon)}(a, \xi) d\xi$$

$$\tilde{f}_{\alpha}(x) = \int f_a(x + \varepsilon \xi) \Phi_x^{(\varepsilon)}(\xi) d\xi$$

where $\Phi_x^{(\varepsilon)}$ is the partial derivative of the operator $\Phi_x^{(\varepsilon)}$ with respect to $x^a$.

We now expand the difference $\tilde{f}_a - \tilde{f}_{\alpha}$ as a Taylor series (for some $\theta \in [0,1]$);

$$\tilde{f}_a - \tilde{f}_{\alpha} = \sum_{|\alpha| = 0}^{k} \varepsilon^{|\alpha|} |\alpha|! D^\alpha f(x) \int \xi^a \Phi_x^{(\varepsilon)}(\alpha) d\xi$$

$$+ \sum_{|\alpha| = k+1}^{\infty} \frac{\varepsilon^{k+1}}{(k+1)!} D^\alpha f(x + \varepsilon \theta \xi) \int \xi^a \Phi_x^{(\varepsilon)}(\alpha) d\xi$$
Proposition 7. Let 

\[ \gamma \]

By Taylor expanding and using the moment conditions (19) it may be shown that for \( \Phi \) Without loss of generality we shall consider the differentiation of vectors (the differentiation of covectors and \( \gamma \) at the level of association). This highlights an important difference from Colombeau’s original theory in which differentiation of distributions manifestly commutes with smoothing. This is a price we have to pay for introducing smoothing kernels that are dependent on \( x \).

The next step is to extend this to multi-index tensors; an added complication being that the transport operator \( \Gamma \) will depend on \( x \) so differentiating it will introduce extra terms.

We shall use the notation \( T_{|e} \) to denote covariant differentiation with respect to the connection \( \gamma_{bc} \). Without loss of generality we shall consider the differentiation of vectors (the differentiation of covectors and more general tensors is achieved by adding the relevant terms), so for vectors we have

\[ X^a_{|b} = X^a_{,b} + \gamma^a_{bc} X^c \]

Lemma 6.

\[ [\tilde{X}^a_{|b}(\Phi, \gamma, x)] = \left[ \int X^c_{|b}(x + \varepsilon \xi) \Gamma^a_{c}(x, x + \varepsilon \xi) \Phi^{(e)}(\xi) \, d\xi \right] \]

Proof.

\[ \tilde{X}^a_{,b}(\Phi, \gamma, x) = \int X^c_{,b}(x + \varepsilon \xi) \Gamma^a_{c}(x, x + \varepsilon \xi) \Phi^{(e)}(\xi) \, d\xi \]

\[ + \int X^c(x + \varepsilon \xi) \left(-\gamma^d_{bd}(x) \Gamma^a_{d}(x, x + \varepsilon \xi) \right) \Phi^{(e)}(\xi) \, d\xi \]

\[ \quad + \int X^c(x + \varepsilon \xi) \Gamma^a_{c}(x, x + \varepsilon \xi) \Phi^{(e)}(\xi) \, d\xi \]

\[ = \int X^c_{|b}(x + \varepsilon \xi) \Gamma^a_{c}(x, x + \varepsilon \xi) \Phi^{(e)}(\xi) \, d\xi \]

\[ - \gamma^a_{bd}(x) \int X^c(x + \varepsilon \xi) \Gamma^d_{c}(x, x + \varepsilon \xi) \Phi^{(e)}(\xi) \, d\xi \]

\[ + \int X^c(x + \varepsilon \xi) \Gamma^a_{c}(x, x + \varepsilon \xi) \Phi^{(e)}(\xi) \, d\xi \]

By Taylor expanding and using the moment conditions (19) it may be shown that for \( \Phi \in \tilde{A}_k(\Omega) \),

\[ \int X^c(x + \varepsilon \xi) \Gamma^a_{c}(x, x + \varepsilon \xi) \Phi^{(e)}(\xi) \, d\xi = O(\varepsilon^k) \]

therefore

\[ [\tilde{X}^a_{|b}(\Phi, \gamma, x)] = \left[ \int X^c_{|b}(x + \varepsilon \xi) \Gamma^a_{c}(x, x + \varepsilon \xi) \Phi^{(e)}(\xi) \, d\xi \right] \]

Proposition 7. Let \( X^a \) and \( Y^a \) be smooth vector fields then covariant differentiation of \( X^a \) with respect to \( \gamma^a_{bc} \) in the \( Y^a \) direction commutes with the embedding in the sense that;

\[ \tilde{Y}^b \tilde{X}^a_{|b} = \tilde{X}^a_{|b} \tilde{Y}^b. \]
Proof. This follows from the above lemma together with the fact that we may identify \( Y^a \) and \( \tilde{Y}^a \) for a smooth vector field.

Suppose \( \Gamma^a_{bc} \) is another connection, then covariant differentiation with respect to it may be expressed as

\[
X^a_{,b} = X^a_{|b} + \Gamma^a_{bc} X^c
\]

this may also be expressed as

\[
X^a_{,b} = X^a_{|b} + \hat{\Gamma}^a_{bc} X^c
\]

where \( \hat{\Gamma}^a_{bc} = \Gamma^a_{bc} - \gamma^a_{bc} \). The symbol \( \hat{\Gamma}^a_{bc} \) will transform as a type \((1,2)\) tensor because it is the difference of two connections.

If \( X^a, Y^a \) and \( \Gamma^a_{bc} \) are all smooth then \( Y^b X^a_{|b} \) and \( \hat{\Gamma}^a_{bc} X^c Y^b \) commute with the embedding. This implies that the covariant derivative \( Y^b X^a_{|b} \) also commutes with the embedding.

We may define a torsion free connection \( \hat{\gamma}^a_{bc} \) by taking the symmetric part of the background connection

\[
X^a_{,b} = X^a_{|b} - \gamma^a_{[bc]} X^c
\]

\[
= X^a_{,b} + \hat{\gamma}^a_{bc} X^c
\]

For smooth tensor fields covariant differentiation with respect to the torsion-free connection \( \hat{\gamma}^a_{bc} \) will commute with the embedding. Rather than using the partial derivative to construct invariant objects, we shall use the covariant derivative of the background torsion free connection:

\[
\hat{\gamma}^a_{bc} = \frac{1}{2} g^{ad} (g_{bd,c} + g_{cd,b} - g_{cd,b})
\]

As a second example we consider a smooth metric \( g_{ab} \) and the Levi-Civita connection

\[
\Gamma^a_{bc} = \frac{1}{2} g^{ad} (g_{bd,c} + g_{cd,b} - g_{cd,b})
\]

Using

\[
g_{abc} = g_{ab,c} - \hat{\gamma}^d_{ac} g_{bd} - \hat{\gamma}^d_{bc} g_{ad}
\]

implies that

\[
\Gamma^a_{bc} = \frac{1}{2} g^{ad} (g_{bd,c} + g_{cd,b} - g_{cd,b})
\]

therefore we may write

\[
X^a_{,b} = X^a_{|b} + \hat{\Gamma}^a_{bc} X^c
\]

where

\[
\hat{\Gamma}^a_{bc} = \frac{1}{2} g^{ad} (g_{bd,c} + g_{cd,b} - g_{cd,b})
\]

Thus for a smooth metric the covariant derivative with respect to the Levi-Civita connection commutes with the embedding.

7. Association

The relation of association or weak equivalence for generalised functions is much the same as the normal definition;

**Definition 14.** We say that \( \tilde{T} \) is associated to zero (written as \( \tilde{T} \approx 0 \)) if \( \forall \Psi \in \tilde{D}_p^q(\Omega), \exists k > 0 \) such that

\[
\lim_{\varepsilon \to 0} \int \tilde{T}^{a}{}_{b,...,d}(\Phi_{x,\varepsilon},\gamma,x) \Psi^{c,...,d}(x) \, dx = 0 \quad \forall \Psi^{(x)} \in \tilde{A}_0(\mathbb{R}^n), \gamma \in \mathcal{K}.
\]

Two generalised functions \( \tilde{S} \) and \( \tilde{T} \) are associated to each other if \( \tilde{S} - \tilde{T} \approx 0 \)
Definition 15. We say that $\tilde{T} \in G_q^p(\Omega)$ is associated to the distribution $S \in D'^q_p(\Omega)$ (written as $\tilde{T} \approx S$) if $\forall \Psi \in \tilde{D}(\Omega)$, $\exists k > 0$ such that
$$\lim_{\varepsilon \to 0} \int \tilde{T}_{a...b}(\Phi_\varepsilon; \gamma, x) \Psi_{a...b}(x) \, dx = \langle S, \Psi \rangle \quad \forall \Phi_\varepsilon(x) \in \tilde{A}_0(\mathbb{R}), \gamma \in K$$

We finally verify that this assignment of a distributional interpretation to a generalised function also commutes with $\mu$.

Proposition 8. $[\tilde{T}] \approx S'$ implies $[\mu^* \tilde{T}'] \approx \mu^* S'$

Proof. Letting $\Psi \in D'^q_p(\Omega)$,
$$\lim_{\varepsilon \to 0} \mu_* \tilde{T}'_{e...f} (\Phi_\varepsilon; \gamma, x) \Psi_{e...f} \, dx$$
$$= \lim_{\varepsilon \to 0} \int \mu_*^a (x) \ldots \mu_*^b (x) \eta^a (x) \ldots \eta^b (x) \tilde{T}'_{e...f} (\mu_* \Phi_\varepsilon, \mu_* \gamma, \mu(x)) \Psi_{a...b} \, dx$$
$$= \lim_{\varepsilon \to 0} \int \tilde{T}'_{g...h} (\mu_* \Phi_\varepsilon, \mu_* \gamma, x') \mu_* \Psi_{a...b} \, dx'$$
$$= \langle S', \mu_* \Psi \rangle$$
$$= \langle \mu^* S', \Psi \rangle$$

8. Conclusion

We now have a covariant theory of generalised functions in which we are able to define tensors whose transformation laws coincide with those of distributions, both in the senses of convolution embedding and association.

This will mean that we may carry out calculations in General Relativity that involve the evaluation of products of distributions, such as the evaluation of the distributional curvature at the vertex of a cone with a deficit angle $2\pi(1 - A)$ by Clarke et al. (1996) in a coordinate independent way. One would need to first choose a convenient coordinate system, then embed the metric into the space $G^0_2(\mathbb{R}^2)$, which would transform as a type $(0,2)$ tensor using the definitions in this paper, calculate the distributional curvature $[\tilde{R} \sqrt{g}]$ as a generalised function, which will transform as a scalar density of weight $+1$ and show that it is associated to the distribution $4\pi(1 - A)\delta^{(2)}$ obtained in that paper.

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