Non-standard loop quantum cosmology

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We present results concerning the nature of the cosmological big bounce (BB) transition within the loop
group geometry underlying loop quantum cosmology (LQC). Our canonical quantization method is an alternative
to the standard LQC. An evolution parameter we use has clear interpretation both at classical and quantum
levels. The physical volume operator has discrete spectrum which is bounded from below. The minimum
gap in the spectrum defines a quantum of the volume. The spectra of operators are parametrized by a free
parameter to be determined.

1 Introduction

There are two alternative methods of quantization of a Hamiltonian system with constraints: (i) Dirac’s
method - ‘first quantize, then impose constraints’, and (ii) non-Dirac’s method - ‘first solve constraints,
then quantize’. We have two corresponding methods in quantization of cosmological models of general
relativity (GR) which make use of the so-called loop geometry: (i) standard LQC - Dirac’s method [1, 2],
and (ii) non-standard LQC - method proposed recently [3, 4]. The latter method corresponds to the reduced
phase space quantization of loop quantum gravity [5].

In what follows we present quantization of flat FRW model with massless scalar field by making use
of the non-standard LQC. Available results are the following: (1) within standard LQC - classical Big-
Bang is replaced by quantum Big-Bounce due to strong quantum effects at the Planck scale [6, 7, 8]; (2)
within non-standard LQC - modification of GR by loop geometry is responsible for the resolution of the
singularity, quantization may lead to discrete spectra of physical observables [9, 10].

2 Classical Level

2.1 Modified Hamiltonian

The gravitational part of the Hamiltonian of the flat FRW universe with massless scalar field (in special
gauges) is found to be [3]

\[ H_g = -\gamma^{-2} \int_V d^3x \,Ne^{-1} \varepsilon_{ijk} E^{a_j} E^{b_k} F^{i}_{ab}, \]

where \(\gamma\), Barbero-Immirzi parameter; \(V \subset \Sigma\), elementary cell; \(N\), lapse function; \(\varepsilon_{ijk}\), alternating tensor;
\(E_i^a\), density weighted triad; \(F^{i}_{ab} = \partial_a A^k_b - \partial_b A^k_a + \epsilon_{ij} A^i_a A^j_b\), curvature of \(SU(2)\) connection \(A^i_a\); \(e := \sqrt{|\det E|}\);

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Modification by loop geometry means approximation of $F_{ab}^k$ as follows

$$F_{ab}^k(\lambda) \approx -2 \text{Tr} \left( \frac{h^{(\lambda)}_{B_{ij}} - 1}{\lambda^2 V_0^{1/3}} \right) \tau^k \omega^i_\alpha \omega^j_\beta, \quad F_{ab}^k = \lim_{\lambda \to 0} F_{ab}^k(\lambda),$$

where the holonomy of the connection around the square loop $B_{ij}$, with sides length $\mu V_0^{1/3}$, reads

$$h_{B_{ij}}^{(\mu)} = h_i^{(\mu)} h_j^{(\mu)} (h_i^{(\mu)})^{-1} (h_j^{(\mu)})^{-1}, \quad h_k^{(\mu)}(c) = \cos(\mu c/2) + 2 \sin(\mu c/2) \tau_k,$$

where $\tau_k = -i \sigma_k/2$ ($\sigma_k$ are the Pauli spin matrices).

Making use of Thie mann’s identity leads finally to

$$H_g = \lim_{\lambda \to 0} H_g^{(\lambda)} \quad \text{(4)}$$

where

$$H_g^{(\lambda)} = -\frac{\text{sgn}(p)}{2 \pi G \gamma^3 \lambda^3} \sum_{ijk} N \epsilon^{ijk} \text{Tr} \left( h_i^{(\lambda)} h_j^{(\lambda)} (h_i^{(\lambda)})^{-1} (h_j^{(\lambda)})^{-1} h_k^{(\lambda)} \{ (h_k^{(\lambda)})^{-1}, V \} \right).$$

and where $V = |p| 3/2 = a^3 V_0$ is the volume of the elementary cell $V$. Variables $c$ and $p$ determine connections $A^k_i$ and triads $E^k_{ij}$: $A^k_i = \omega^k_\alpha c V_0^{-1/3}$ and $E^k_i = \omega^k_\alpha \sqrt{\text{Tr} q_0} p V_0^{-2/3}$, where $c = \gamma a V_0^{1/3}$ and $|p| = a^2 V_0^{2/3}$, $\{c, p\} = 8 \pi G \gamma^3 / 3$.

The total Hamiltonian for FRW universe with a massless scalar field $\phi$ reads

$$H = H_g + H_\phi,$$

where $H_g$ is defined by (4) and $H_\phi = p_\phi \cos(\lambda \beta)/2$, and where $\phi$ and $p_\phi$ are elementary variables satisfying $\{\phi, p_\phi\} = 1$. The relation $H \approx 0$ defines the physical phase space.

Making use of (3) we calculate (5) and get the modified total Hamiltonian corresponding to (6)

$$H^{(\lambda)}/N = -\frac{3}{8 \pi G \gamma^2} \frac{\sin^2(\lambda \beta)}{\lambda^2} v + \frac{p_\phi^2}{2 v}, \quad \beta := \frac{c}{|p|^{1/2}}, \quad v := |p|^{3/2}.$$

Equation (7) presents a modified classical Hamiltonian.

### 2.2 Observables

A function, $O : F_{kin}^{(\lambda)} \to R$, is a Dirac observable if

$$\{O, H^{(\lambda)}\} = 0, \quad \{\cdot, \cdot\} := 4 \pi G \gamma \left[ \frac{\partial}{\partial \beta} \frac{\partial}{\partial v} - \frac{\partial}{\partial v} \frac{\partial}{\partial \beta} \right] + \frac{\partial}{\partial \phi} \frac{\partial}{\partial p_\phi} - \frac{\partial}{\partial p_\phi} \frac{\partial}{\partial \phi}. \quad \text{(8)}$$

Thus, $O$ is solution to the equation

$$\frac{\sin(\lambda \beta)}{\lambda} \frac{\partial O}{\partial \beta} - v \cos(\lambda \beta) \frac{\partial O}{\partial v} - \frac{\kappa \text{sgn}(p_\phi)}{4 \pi G} \frac{\partial O}{\partial \phi} = 0.$$ \quad \text{(9)}

Solutions to (9) are found to be

$$O_1 := p_\phi, \quad O_2 := \phi - \frac{\text{sgn}(p_\phi)}{3 \kappa} \arctan(\cos(\lambda \beta)), \quad O_3 := \text{sgn}(p_\phi) v \frac{\sin(\lambda \beta)}{\lambda}. \quad \text{(10)}$$

Observables satisfy the Lie algebra

$$\{O_2, O_1\} = 1, \quad \{O_1, O_3\} = 0, \quad \{O_2, O_3\} = \gamma \kappa. \quad \text{(11)}$$
Due to the constraint $H^{(λ)} = 0$, we have $O_3 = γκ O_1$. Thus, in the physical phase space, $F^{(λ)}_{phys}$, we have only two observables which satisfy the algebra

$$\{O_2, O_1\} = 1.$$  \hspace{1cm} (12)

In what follows we consider functions which can be expressed in terms of observables and an evolution parameter $φ$ so they are not observables. They do become observables for each fixed value of $φ$, since in such case they are only functions of observables:

The energy density of matter field is found to be $ρ(λ, φ) = \frac{1}{2} (κγλ)^2 \cosh^2 3κ(φ - O_2)$, \hspace{1cm} (13)

The volume operator may be expressed as $v(φ, λ) = κγλ |O_1| \cosh 3κ(φ - O_2)$. \hspace{1cm} (14)

## 3 Quantum Level

The energy density operator has been considered recently in [9]. The spectrum of the quantum operator corresponding to $ρ$ turns out to coincide with (13). The energy density operator is bounded and has continuous spectrum.

In what follows we present quantization of the volume observable [10]. The classical volume operator, $v$, reads

$$v = |w|, \ w := κγλ O_1 \cosh 3κ(φ - O_2).$$ \hspace{1cm} (15)

Thus, quantization of $v$ reduces to the quantization problem of $w$:

$$\hat{w} f(x) := κγλ \frac{1}{2} (\hat{O}_1 \cosh 3κ(φ - \hat{O}_2) + \cosh 3κ(φ - \hat{O}_2) \hat{O}_1) f(x),$$ \hspace{1cm} (16)

where $f \in L^2(\mathbb{R})$. For $O_1$ and $O_2$ we use the Schrödinger representation

$$O_1 \rightarrow \hat{O}_1 f(x) := -i \hbar \frac{d}{dx} f(x), \quad O_2 \rightarrow \hat{O}_2 f(x) := \hat{x} f(x) := xf(x).$$ \hspace{1cm} (17)

Thus, an explicit form of $\hat{w}$ is

$$\hat{w} = i \frac{κγλ\hbar}{2} (2 \cosh 3κ(φ - x) \frac{d}{dx} - 3κ \sinh 3κ(φ - x)).$$ \hspace{1cm} (18)

An explicit form of $\hat{w}$ is

$$\hat{w} = i \frac{κγλ\hbar}{2} (2 \cosh 3κ(φ - x) \frac{d}{dx} - 3κ \sinh 3κ(φ - x)).$$ \hspace{1cm} (19)

Solution to the eigenvalue problem

$$\hat{w} f_a(x) = a f_a(x), \ a \in \mathbb{R},$$ \hspace{1cm} (20)

is found to be $f_a(x) := \sqrt{\frac{3κ}{π}} \exp \left( i \frac{2a}{3κγλ\hbar} \arctan \frac{\cosh 3κ(φ - x)}{\cosh^2 3κ(φ - x)} \right)$, $a = b + 8πGγλ\hbar m$, \hspace{1cm} (21)
where $b \in \mathbb{R}$ and $m \in \mathbb{Z}$. Completion of the span of

$$
\mathcal{F}_b := \{ f_a \mid a = b + 8\pi G\gamma \bar{\hbar} m \} \subset L^2(\mathbb{R}),
$$

(22)
in the norm of $L^2(\mathbb{R})$ leads to $\infty$-dim separable Hilbert space $\mathcal{H}_b$. One may show [4] that the operator $\hat{\omega}$ is essentially self-adjoint on each orthonormal space $\mathcal{F}_b$.

Due to the relation (14) and the spectral theorem on self-adjoint operators we get the solution of the eigenvalue of the volume operator

$$
v = |w| \rightarrow \hat{v} f_a := |a| f_a.
$$

(23)
The spectrum is bounded from below and discrete. There exists the minimum gap $\Delta := 8\pi G\gamma \bar{\hbar} \lambda$ in the spectrum, which defines a quantum of the volume. In the limit $\lambda \rightarrow 0$, corresponding to the classical FRW model, there is no quantum of the volume.

The discreteness may translate at the semi-classical level into a foamy structure of spacetime. Our results suggest that the foamy structure of space is a real property of the Universe so its identification via astro-cosmo observations has sound motivation and is important for the fundamental physics.

## 4 Conclusions

Modification of gravitational part of classical Hamiltonian, realized by making use of the loop geometry (parameterized by $\lambda$), turns big bang into big bounce (BB). Since there is no specific $0 < \lambda \in \mathbb{R}$, there is no specific energy density at BB, and no specific quantum of the volume. The spectra of operators are parametrized by a free parameter $\lambda$ that has not been determined theoretically, but is expected to be fixed by the data of observational cosmology.

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