Non-exponential decoherence of radio-frequency resonance rotation of spin in storage rings

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Precision experiments, such as the search for electric dipole moments of charged particles using radiofrequency spin rotators in storage rings, demand for maintaining the exact spin resonance condition for several thousand seconds. Synchrotron oscillations in the stored beam modulate the spin tune of off-central particles, moving it off the perfect resonance condition set for central particles on the reference orbit. Here we report an analytic description of how synchrotron oscillations lead to non-exponential decoherence of the radiofrequency resonance driven up-down spin rotations. This non-exponential decoherence is shown to be accompanied by a nontrivial walk of the spin phase. We also comment on sensitivity of the decoherence rate to the harmonics of the radiofrequency spin rotator and a possibility to check predictions of decoherence-free magic energies.

1. INTRODUCTION

Radiofrequency (RF) spin rotator driven resonance up-down oscillations of polarization of charged particles in storage rings are of interest in a broad class of spin experiments inculding the search for electric dipole moments of charged particles [1, 2, 3]. Inherent to particles in storage rings is a modulation of the spin-tune by synchrotron oscillations (SO) which drives the spin precession and RF frequencies apart [4] and decoheres evolution of the polarization of particles in an ensemble. It becomes still more important at large spin rotation times imperative in searches for extremely small EDM’s [5, 6, 7]. In this paper we develop a fully analytic theory of SO driven spin decoherence and show that SOs entail a non-exponential attenuation of resonance driven up-down oscillations averaged over an ensemble. We also derive the associated nontrivial spin phase walk and the dependence of non-exponential decoherence on the harmonics of the RF spin rotator. These analytic results are new and do substantially extend the early considerations by Benati et al. [4].

SO driven decoherence depends on the spread of SO amplitudes $a_z$ in an ensemble. Hence first we relate a distribution of SO amplitudes, $F(a_z)$, to longitudinal density $N(z)$ in the bunch. A particle with SO amplitude $a_z$ oscillates around the center of bunch, $z = a_z \cos(2\pi \nu_z f_R t + \lambda)$, where $f_R$ is the ring frequency, $\nu_z$ is the synchrotron tune, $\lambda \in [0, 2\pi]$ is random phase and we only need to consider $a_z > 0$. Evidently, the one-particle density $N(z) \propto 1/\Delta\beta_z$, where $\Delta\beta_z c = 2\pi \nu_z f_R (a_z^2 - z^2)^{1/2}$ is a velocity of synchrotron oscillations, $c$ is velocity of light. For an ensemble

$$N(z) = \frac{1}{\pi} \int_0^\infty \frac{d a_z F(a_z)}{\sqrt{a_z^2 - z^2}}$$

(1)

and has a form of the Abel transform with solution [8]

$$F(a_z) = -2a_z \int_{a_z}^{\infty} \int \frac{d z N'(z)}{\sqrt{z^2 - a_z^2}}.$$  

(2)

With Gaussian $N(z) \propto \exp(-z^2/2B^2)$, suggested by the experimental studies [9] for the sinusoidal potential RF-cavity, a solution for $F(a_z)$ is readily obtained in the closed analytic form:

$$F(a_z) = \frac{a_z}{B^2} \exp \left( -\frac{a_z^2}{2B^2} \right),$$  

(3)

$$\approx \frac{1}{\sqrt{2\pi B}} \exp \left( -\frac{(a_z - B)^2}{2B^2} \right).$$  

(3)

We can relate $B$ to the momentum spread in the bunch

$$B = \frac{c}{f_R} \gamma \sqrt{\beta} \pi \frac{1}{\nu_z} \frac{\Delta p^2}{p^2}^{1/2},$$  

(4)
especially for deuterons with small magnetic anomaly

SOs modulate the revolution time RF spin rotator is described by the spin transfer matrix

\[
\xi k_{\text{stored particle per revolution}}
\]

and \( \xi = a_z/B \). Typically the spin phase slip parameter

\[
\psi_s = \frac{2\pi G\beta^3 f_R B}{c} \ll 1 ,
\]

especially for deuterons with small magnetic anomaly \( G = (g - 2)/2 \).

A spin kick per revolution depends on the phase of the RF field at exactly when a particle passes the spin rotator. SOs modulate the spin tune

\[
\theta_k \text{ modulates the revolution time } \tau,
\]

where \( \eta \) is the phase slip factor. This \( \Delta \tau \) changes a time

\[
t(n) = \frac{n}{f_R} + \sum_{k=1}^{n} \Delta t_k .
\]

Consequently, a phase of the RF spin rotator, \( \theta_{rf}(n) \), acquires the slip term, \( \Delta \theta_{rf}(n) \),

\[
\theta_{rf}(n) = 2\pi f_{rf} t(n) = 2\pi (\nu_{rf} + K) n + \Delta \theta_{rf}(n)
\]

\[
= 2\pi (\nu_{rf} + K) n + \frac{\nu_{rf} + K}{\nu_s} \eta \frac{\beta^2}{2} \Delta \theta_s(n)
\]

\[
= 2\pi (\nu_{rf} + K) n + (C_{rf} + 1) \Delta \theta_s(n) ,
\]

where \( K = 0, \pm 1, \pm 2, \ldots \) is the harmonics number in the RF frequency \( f_{rf} = (\nu_{rf} + K) f_R \).

Take note of correlated spin and RF phase slips [10].

In further derivation of the impact of SO on the RF driven rotation of spin of stored particles, we follow closely the formalism of Appendix A in Ref. III.

RF spin rotator is described by the spin transfer matrix

\[
t_{rf}(k) = \cos \left( \frac{1}{2} \chi_{rf} - i(\vec{\sigma} \cdot \vec{\omega}) \sin \left( \frac{1}{2} \chi_{rf} \right) \right) .
\]

where \( \vec{\omega} \) is the spin rotation axis, \( \vec{\sigma} \) are the Pauli matrices and the spin kick for revolution \( k \) equals \( \chi_{rf} \cos (\theta_{rf}(k)) \).

Evolution of the spinor wave function \( \phi \) of the stored particle per revolution \( k \) is described by spin transfer matrices of the ring, \( T_k \), and the RF rotator,

\[
\phi(k + 1) = t_{rf}(k) T_k \phi(k) .
\]

We pass to the conventional interaction representation with \( T(n) = \prod_{k=1}^{n} T_k \),

\[
\phi(n) = T(n) \xi(n)
\]

\[
= \left \{ \cos \left( \frac{1}{2} \theta_s(n) - i(\vec{\sigma} \cdot \vec{\omega}) \sin \left( \frac{1}{2} \theta_s(n) \right) \right) \right \} \xi(n) .
\]

Here \( \vec{\omega} \) is the stable spin axis and the spinor \( \xi(n) \) describes the envelope over the rapid oscillations of the spin, \( \zeta(0) = \phi(0) \). The evolution equation for \( \zeta(n) \) becomes

\[
\zeta(n) = T^{-1}(n) t_{rf}(n) T(n) \xi(n - 1)
\]

\[
= \exp \left \{ -i \frac{1}{2} \vec{\omega} \cdot \vec{U} \right \} \zeta(n - 1) .
\]

Here

\[
\vec{U}(n) = 2 \sin \left( \frac{1}{2} \chi_{rf} \right) \times \left \{ \cos \theta_s(n) \left[ \vec{\omega} \times \vec{\sigma} \right] \times \vec{\omega} \right \} - \sin \theta_s(n) \left[ \vec{\sigma} \times \vec{\omega} \right] + (\vec{\sigma} \cdot \vec{\omega}) \vec{\omega}
\]

is the spin rotation axis which lies in the plane rotating with frequency locked to the spin precession frequency, for details see [11].

Equation Eq. [12] has a solution in terms of the n-ordered exponential

\[
\zeta(n) = T^n \exp \left \{ -i \sum_{k=1}^{n} \vec{\omega} \cdot \vec{U}(k) \right \} \xi(0) .
\]

Hereafter we work to the lowest order in a small parameter \( \chi_{rf} \ll 1 \). For central particles the resonance condition is \( \nu_{rf} = \nu_s \) and \( f_{rf} = (\nu_s + K) f_R \), where the integer \( K = 0, \pm 1, \pm 2, \ldots \) is the harmonics number. The large-n behavior of \( \zeta(n) \) is evaluated using the Bogolyubov-Krylov-Mitropolsky averaging method [12]. It amounts to keeping in the sum \( \sum_{k=1}^{n} \vec{\omega} \cdot \vec{U}(k) \) only the linearly rising terms

\[
\sum_{k=1}^{n} 2 \chi_{rf} \cos \theta_s(k) = \chi_{rf} \sum_{k=1}^{n} \cos [\theta_s(k) - \theta_s(k)]
\]

\[
= \chi_{rf} \sum_{k=1}^{n} \cos \left \{ C_{rf} \psi_s \xi [\cos (2\pi \nu_s n + \lambda) - \cos \lambda] \right \}
\]

\[
= n \chi_{rf} \cos \left \{ C_{rf} \psi_s \xi \cos \lambda \right \} J_0 \left (C_{rf} \psi_s \xi \right) ,
\]

\[
\sum_{k=1}^{n} 2 \chi_{rf} \sin \theta_s(k) =
\]

\[
= -n \chi_{rf} \sin \left \{ C_{rf} \psi_s \xi \cos \lambda \right \} J_0 \left (C_{rf} \psi_s \xi \right) ,
\]

where \( J_0(x) \) is the Bessel function and, as derived in [10],

\[
C_{rf} = \eta \beta^2 \left (1 + \frac{K}{G\gamma} \right ) - 1 .
\]
The SO corrected large-n spin evolution law becomes
\[
\zeta(n) = \exp \left\{ -\frac{i}{2} n \epsilon(\xi) \vec{u} \cdot \vec{A} \right\} \phi(0),
\]
where \( \epsilon(\xi) = \epsilon_0 J_0 \left( C_{rf} \psi_s \xi \right) \) and
\[
\epsilon_0 = \frac{1}{2} \lambda_{rf} \sqrt{1 - (\vec{e} \cdot \vec{w})^2},
\]
is a spin resonance strength at vanishing SO’s, when polarization along the stable spin axis \( \vec{c} \) oscillates \( \propto \cos(\epsilon_0 n) \). The spin rotation axis \( \vec{u} \) in Eq. (18) equals
\[
\vec{u} = \frac{\cos \Delta_{rf} [\vec{c} \times \vec{w}] \times \vec{c} + \sin \Delta_{rf} [\vec{c} \times \vec{w}]}{\sqrt{1 - (\vec{e} \cdot \vec{w})^2}}
\]
and \( \Delta_{rf} = C_{rf} \psi_s \xi \cos \lambda \). Hereafter the up-down oscillations are defined with respect to the closed spin orbit vector \( \vec{e} \). The above derived analytic factor \( J_0 \left( C_{rf} \psi_s \xi \right) \) describes a SO driven spread of the spin resonance strength and spin phase of an individual particle over an ensemble of particles in a bunch.

A final step is averaging of the up-down oscillations of one-particle polarizations over an ensemble. We define the ensemble averaged polarization \( A(n) \) as \( \langle \vec{S}(n) \cdot \vec{e} \rangle = \langle \vec{S}(0) \cdot \vec{e} \rangle A(n) \). It does not depend on the SO phase \( \lambda \). Upon averaging polarizations of individual particles over the SO amplitudes with the distribution of Eq. (3), we obtain our main result:
\[
A(n) = \Re \left\{ \exp \left\{ -i n \epsilon(\xi) \right\} \right\} \xi \\
\approx \Re \left\{ \exp \left\{ -i n \epsilon_0 \left[ 1 - \frac{1}{4} C_{rf}^2 \psi_s^2 \xi^2 \right] \right\} \right\} \xi \\
= (1 - i \eta n)^{-1/2} \exp \left\{ -i \epsilon_0 n + \frac{i \eta n}{1 - i \eta n} \right\} \\
= \exp \left\{ -\sin^2 \varphi(n) \right\} \frac{\cos \left\{ \epsilon_0 n - \kappa(n) \right\}}{(1 + \rho^2 n^2)^{1/4}} \cos \{ \epsilon_0 n - \kappa(n) \},
\]
where
\[
\rho = \frac{1}{4} \epsilon_0 C_{rf}^2 \psi_s^2,
\]
\[
\varphi(n) = \arctan(\rho n),
\]
\[
\kappa(n) = \frac{1}{2} \left[ \varphi(n) + \sin 2\varphi(n) \right].
\]
In the above averaging over the SO amplitudes over the distribution given by Eq. (3), we resorted to a steepest descent approximation.

One comment on the above derivation is in order. In generic case, the synchrotron tune \( \nu_z \) depends on the synchrotron amplitude. We observe that \( \nu_z \) only enters the principal spin phase slip through the parameter \( \psi_s \) of Eq. (3) in the combination \( \xi / \nu_z \). Consequently, weak dependence of \( \nu_z \) on \( \xi \) only would mimic a slight modification of the synchrotron amplitude distribution and would affect none of our principal conclusions.

Now we turn to discussion of our principal findings. Synchrotron oscillations do clearly decohere the RF driven resonance up-down oscillations of polarization of stored particles \([3]\). The last line of Eq. (21) is our final analytic result for the SO caused decoherence of RF driven up-down spin oscillations in storage rings. Typical decoherence pattern is shown by solid line in Fig. 1. Here we evaluated \( A(n) \) for deuterons of momentum \( p = 970 \) MeV/c, \( \eta = -0.61 \) \([4]\), the momentum spread \( (\Delta p^2 / p^2)^{1/2} = 3 \cdot 10^{-4} \) and the RF driven up-down spin oscillations with the period \( \tau_s = 2.4 \) s. This corresponds to RF solenoid with \( \int Bdl = 0.0264 \) T mm. Then in this example we have \( \rho = 1.37 \cdot 10^{-7} \).

The overall attenuation of the envelope of the up-down spin oscillations comes from two non-exponential factors. The first one, \( \exp \left\{ -\sin^2 \varphi(n) \right\} \) tends to a constant \( 1/e \) as soon as \( \varphi(n) > 1 \). A significance of this attenuation factor can be judged from a comparison of the solid line with the dot-dashed (blue) line in Fig. 1. In the latter case we took out the factor \( \exp \left\{ -\sin^2 \varphi(n) \right\} \). As such, the dot-dashed (blue) line in Fig. 1 shows the effect of the second attenuation factor, \( (1 + \rho^2 n^2)^{-1/4} \), which decreases continuously \( \propto 1 / \sqrt{n} \). Attempts to describe this decoherence by conventional exponential attenuation of the oscillation envelope will be entirely misleading.

Besides the non-exponential damping of spin oscillations, we predict a nontrivial walk of the spin phase
κ(n). This phase walk, shown in Fig. 2 is clearly seen from a comparison of the solid and dashed (red) curves in Fig. 1 where the dashed curve (red) shows plane \( \cos(\epsilon_0 n) \) oscillation law omitting the phase walk \( \kappa(n) \). We predict a large-n limiting phase \( \kappa(n) \rightarrow \pi/4 \).

Our predictions can be tested experimentally. Our point is that the spin phase motion and attenuation of the up-down spin oscillations to our form of \( A(n) \), which only depends on two free parameters: \( \epsilon_0 \) and \( \rho \).

We emphasize that \( \rho \) must follow a generic pattern of dependence on the harmonics \( K \) as it was predicted in Ref. [10], including the decoherence-free magic energies of dependence on the harmonics \( K \).

These extensions and detailed numerical simulations of the technique exposed in Appendix A of Ref. [11], one can readily extend the above considerations to the off-resonance case. Of particular interest will be side band resonances at \( \nu_{rf} = \nu_v \pm \nu_z \). These extensions and detailed numerical simulations of dependences on the synchrotron tune and the beam momentum spread, based on the orbit and spin trackers, will be reported elsewhere.

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Fig. 2. Phase walk \( \kappa(n) \) of vertical polarization oscillations.

The factor \( C_{rf} \) is readily calculable in terms of the known slip-factor \( \eta \), and one can readily convert the fitted \( \rho \) into the r.m.s. bunch length parameter \( B \). On the other hand, one can determine \( B \) experimentally from the time distribution of interactions in the internal target [9], what would offer important crosscheck of our analytic results. All these tests, including the dependence on the synchrotron tune and on the momentum spread in the beam, both only entering via the bunch length, see Eq. (4), can be performed in future experiments at COSY.

In this short communication we restricted ourselves to analytic results only for up-down oscillations of polarization for the resonance case, \( \nu_{rf} = \nu_v \). Following the technique exposed in Appendix A of Ref. [11], one can readily extend the above considerations to the off-resonance case. Of particular interest will be a decoherence of polarization in the ring plane. Also of practical interest will be side band resonances at \( \nu_{rf} = \nu_v \pm \nu_z \).