POSITIVE SOLUTIONS OF A NONLINEAR SCHRÖDINGER SYSTEM WITH NONCONSTANT POTENTIALS

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ABSTRACT. Existence of a solution of the nonlinear Schrödinger system
\[\begin{align*}
-\Delta u + V_1(x)u &= \mu_1(x)u^3 + \beta(x)uv^2 \quad \text{in } \mathbb{R}^N, \\
-\Delta v + V_2(x)v &= \beta(x)u^2v + \mu_2(x)v^3 \quad \text{in } \mathbb{R}^N, \\
u > 0, v > 0, \quad u, v \in H^1(\mathbb{R}^N),
\end{align*}\]
where N = 1, 2, 3, and V_j, \mu_j, \beta are continuous functions of x \in \mathbb{R}^N, is proved provided that either V_j, \mu_j, \beta are invariant under the action of a finite subgroup of O(N) or there is no such invariance assumption. In either case the result is obtained both for \beta small and for \beta large in terms of V_j and \mu_j.

1. Introduction and main results. In recent years, much attention has been paid to the following 2-coupled system of nonlinear Schrödinger equations
\[\begin{align*}
-\Delta u + \lambda_1 u &= \mu_1 u^3 + \beta u v^2 \quad \text{in } \mathbb{R}^N, \\
-\Delta v + \lambda_2 v &= \beta u^2 v + \mu_2 v^3 \quad \text{in } \mathbb{R}^N, \\
u, v \in H^1(\mathbb{R}^N),
\end{align*}\]
where N = 1, 2, 3, and \lambda_j, \mu_j, \beta are constants. This type of system appears when one focuses on standing wave solutions of the time-dependent nonlinear Schrödinger system
\[\begin{align*}
-i \frac{\partial }{\partial t} \Phi_1 &= \Delta \Phi_1 + \mu_1 |\Phi_1|^2 \Phi_1 + \beta |\Phi_2|^2 \Phi_1 \quad \text{for } x \in \mathbb{R}^N, \ t > 0, \\
-i \frac{\partial }{\partial t} \Phi_2 &= \Delta \Phi_2 + \mu_2 |\Phi_2|^2 \Phi_2 + \beta |\Phi_1|^2 \Phi_2 \quad \text{for } x \in \mathbb{R}^N, \ t > 0, \\
\Phi_j(x, t) &\to 0 \quad \text{as } |x| \to \infty, \ t > 0, \ j = 1, 2,
\end{align*}\]
which stems from many physical problems, especially in the Hartree-Fock theory for a double condensate and nonlinear optics. See [2, 18, 19, 20, 36] and the references therein for more information on physical background of this system.

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A solution of (1) is said to be nontrivial if both of its components are nonzero and is called semitrivial if one component is zero and the other is nonzero. A solution of (1) is said to be positive if both of its components are positive. Note that (1) has infinitely many semitrivial solutions \((w_{n1}, 0)\) and \((0, w_{n2})\), where \(w_{nj}\) are nonzero solutions of the equation

\[-\Delta u + \lambda_j u = \mu_j u^3 \quad \text{in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N).\]

Because of physical meaning, only nontrivial solutions of (1) are of interest. The main difficulty in studying nontrivial solutions of (1) lies in distinguishing them from semitrivial solutions. The first attempt is the work of Lin and Wei [24] for small interaction coefficient \(\beta\). Following the work [24], a large amount of papers have been devoted to the study of existence and multiplicity of nontrivial solutions of (1) in various different regimes of the nonlinear coupling coefficient \(\beta\). See, for instance, [4, 5, 9, 10, 24, 34] for existence of a ground state or a bound state solution, and [8, 14, 27, 28, 30, 33, 35, 37, 38] for multiplicity of nontrivial solutions.

In all the papers mentioned above the coefficients in (1) are constants. (1) with nonconstant coefficients has not been much studied, and we are only aware of a few papers in this direction ([29, 32, 41]). Recently, in the case where either the potentials \(V_j\), \(\mu_j\) and \(\beta\) are periodic or \(V_j\) are well-shaped and \(\mu_j\) and \(\beta\) are anti-well-shaped, existence of a positive ground state of the nonlinear Schrödinger system

\[
\begin{aligned}
-\Delta u + V_1(x)u &= \mu_1(x)u^3 + \beta(x)uv^2 \quad \text{in } \mathbb{R}^N, \\
-\Delta v + V_2(x)v &= \beta(x)u^2v + \mu_2(x)v^3 \quad \text{in } \mathbb{R}^N, \\
\end{aligned}
\]

(3)

where \(N = 1, 2, 3\), was proved in [26] by the authors both for \(\beta\) small and for \(\beta\) large in terms of \(V_j\) and \(\mu_j\). In this paper, we continue the study on existence of positive solutions of (3) both for \(\beta\) large and for \(\beta\) small in terms of \(V_j\) and \(\mu_j\). In contrast to [26], neither periodicity nor well-shaped structure is imposed on \(V_j, \mu_j, \beta\), except in Theorem 1.6 where \(V_j\) are well-shaped and \(\mu_j\) are anti-well-shaped.

Assume that \(V_j, \mu_j, \beta \in C(\mathbb{R}^N, \mathbb{R})\). Two cases will be considered in this paper: (i) \(V_j, \mu_j, \beta\) are invariant under the action of a finite subgroup \(G\) of \(O(N)\) and (ii) there is no such invariance assumption.

In the first case, a function \(Q\) is said to be \(G\)-invariant if \(Q(gx) = Q(x)\) for all \(x \in \mathbb{R}^N\) and \(g \in G\). Assume, for any \(x \in S^{N-1} = \{x \in \mathbb{R}^N \mid |x| = 1\}\), there exists \(g \in G\) such that \(gx \neq x\). Set

\[m = \min_{x \in S^{N-1}} \#\{gx \mid g \in G\} \geq 2,
\]

where \(\#\{gx \mid g \in G\}\) denotes the cardinal number of the set \(\{gx \mid g \in G\}\). Let \(x_0 \in S^{N-1}\) be such that \(m = \#\{gx_0 \mid g \in G\}\) and denote

\[\{e_1, e_2, \ldots, e_m\} = \{gx_0 \mid g \in G\}, \quad \sigma_0 = \min_{i \neq j} |e_i - e_j| \in (0, 2].
\]

The following assumptions will be made use of.

\((H_1)\) \(V_j(x), \mu_j(x), \beta(x)\) are positive and

\[
\lim_{|x| \to \infty} V_j(x) = 1, \quad \lim_{|x| \to \infty} \mu_j(x) = \mu_j \infty > 0, \quad j = 1, 2
\]

and

\[
\lim_{|x| \to \infty} \beta(x) = \beta \infty > 0.
\]
$$(H_2)$$ \(V_j(x), \mu_j(x), \beta(x)\) are \(G\)-invariant.

$$(H_3)$$ There exist \(C > 0\) and \(\sigma > \sigma_0\) such that, for all \(x \in \mathbb{R}^N\),

\[
V_j(x) - 1 \leq Ce^{-\sigma|x|}, \quad \mu_j \leq Ce^{-\sigma|x|}, \quad j = 1, 2
\]

and

\[
\beta - \beta(x) \leq Ce^{-\sigma|x|}.
\]

Taking \(\sigma\) in \((H_3)\) smaller if necessary, it can be assumed that \(\sigma \in (\sigma_0, 4)\). Without loss of generality, it can also be assumed that \(\sigma \neq 2\). Let \(\|\cdot\|_{\infty}\) be the usual norm in \(L^\infty(\mathbb{R}^N)\). The first main result in this paper is as follows and is for the case where \((3)\) is \(G\)-invariant and \(\beta\) is large.

**Theorem 1.1.** Assume \((H_1), (H_2), \text{ and } (H_3)\) hold. If

\[
\beta(x) \geq \max\{1, \|V_1^{-1}V_2\|_{\infty}\} \mu_1(x), \quad \beta(x) \geq \max\{1, \|V_1V_2^{-1}\|_{\infty}\} \mu_2(x)
\]

and \(\beta \geq \max\{\mu_1, \mu_2\}\), then \((3)\) has a \(G\)-invariant positive solution.

As a direct consequence of Theorem 1.1, the following corollary follows.

**Corollary 1.2.** Assume \((H_1), (H_2), \text{ and } (H_3)\) hold. If \(V_1 = V_2\),

\[
\beta(x) \geq \max\{\mu_1(x), \mu_2(x)\}
\]

and \(\beta \geq \max\{\mu_1, \mu_2\}\), then \((3)\) has a \(G\)-invariant positive solution.

Under the assumptions \((H_1), (H_2), \text{ and } (H_3)\), the single nonlinear Schrödinger equation

\[
-\Delta u + V_j(x)u = \mu_j(x)u^3 \quad \text{in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N)
\]

has a \(G\)-invariant positive solution (see Theorem 4.1 below), denoted by \(w_j\), for \(j = 1, 2\), such that

\[
\int_{\mathbb{R}^N} (|\nabla w_j|^2 + V_j w_j^2) = \int_{\mathbb{R}^N} \mu_j w_j^4 = S_j^2,
\]

where

\[
S_j = \inf_{u \in \mathcal{M}_j} \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 + V_j u^2)}{(\int_{\mathbb{R}^N} \mu_j u^4)^{1/2}}
\]

and

\[
\mathcal{M}_j = \left\{ u \in H^1(\mathbb{R}^N) \mid u \neq 0, u \text{ is } G\text{-invariant}, \int_{\mathbb{R}^N} (|\nabla u|^2 + V_j u^2) = \int_{\mathbb{R}^N} \mu_j u^4 \right\}.
\]

The following theorem is the second main result which is for the case where \((3)\) is \(G\)-invariant and \(\beta\) is small.

**Theorem 1.3.** Assume \((H_1), (H_2), \text{ and } (H_3)\) hold. If

\[
\xi := \|\mu_1 \mu_2\|^{1/2} < \min \left\{ \frac{S_{1G} S_{2G}^2 - \eta_G^4}{S_{1G} S_{2G} (S_{1G}^2 + S_{2G}^2 - 2 \eta_G^2)}, \frac{S_{1G}}{S_{2G}}, \frac{S_{2G}}{S_{1G}} \right\},
\]

where \(\eta_G = (\int_{\mathbb{R}^N} \beta u_{1j}^2 u_{2j}^2)^{1/2}\), and \(\beta \leq \min\{\mu_1, \mu_2\}\), then \((3)\) has a \(G\)-invariant positive solution.

It is easy to see that \((6)\) is satisfied if \(\|\beta\|_{\infty}\) is sufficiently small. Moreover, in the special case where \(V_1 = V_2\), the two numbers \(\frac{S_{2G}}{S_{1G}}\) and \(\frac{S_{1G}}{S_{2G}}\) satisfy

\[
\min_{x \in \mathbb{R}^N} \mu_1^{1/2}(x)/\mu_2^{1/2}(x) \leq \frac{S_{2G}}{S_{1G}} \leq \|\mu_1^{1/2}/\mu_2^{1/2}\|_{\infty}
\]
and
\[ \min_{x \in \mathbb{R}^N} \mu_2^{1/2}(x)/\mu_1^{1/2}(x) \leq \frac{S_{1G}}{S_{2G}} \leq \|\mu_2^{1/2}/\mu_1^{1/2}\|_\infty. \]

Then, since the Hölder inequality implies \( \eta_G^2 \leq \xi_{1G} S_{2G}, \)
\[ \frac{S_{1G}^2 S_{2G}^2 - \eta_G^4}{S_{1G} S_{2G}(S_{1G}^2 + S_{2G}^2 - 2\eta_G^2)} \geq \frac{1 - \xi^2}{S_{1G}/S_{2G} + S_{2G}/S_{1G}} \geq 1 - \xi^2 \]
Therefore, (6) holds if \( \tau = 4 \)
\[ \xi < \min \left\{ \frac{1 - \xi^2}{\max\{\|V_j\|_1^2, \|V_2\|_1^2\} + \min\{\|\mu_1\|_1^2(\xi), \min\{\|\mu_2\|_1^2(x) \right\} \right\}. \]

This means the following is a direct corollary of Theorem 1.3.

**Corollary 1.4.** Assume \((H_1), (H_2), \) and \((H_3)\) hold. If \( V_1 = V_2, \) (7) is satisfied, and \( \beta_\infty < \min\{\mu_{1\infty}, \mu_{2\infty}\}, \) then (3) has a \( G \)-invariant positive solution.

Now we consider the second case where the coefficients \( V_j, \mu_j, \beta \) may not have any symmetry. In this case, the following assumption for \( \beta \) large will be used.

\( (H_4) \) Either of the following is satisfied.
\begin{align*}
(H_{4.1}) & \quad \beta(x) \geq \frac{\tau_1}{2} \max\{\|V_1\|_2^2, \|V_2\|_2^2\} \quad \text{and} \quad \mu_j(x) \geq \frac{\mu_j^\infty}{2} \max\{\|V_1\|_\infty^2, \|V_2\|_\infty^2\}, \quad j = 1, 2, \quad \text{for all} \quad x \in \mathbb{R}^N. \\
(H_{4.2}) & \quad \text{There exists} \quad \kappa \in (\frac{1}{2}, 1) \quad \text{such that} \quad \beta(x) \geq \kappa \beta_\infty \max\{\|V_1\|_\infty^2, \|V_2\|_\infty^2\} \quad \text{for all} \quad x \in \mathbb{R}^N.
\end{align*}

The next theorem is the third main result which is for the case where (3) may not have any group invariance and \( \beta \) is large. A more general result Theorem 5.6 is stated at the end of Section 5.

**Theorem 1.5.** Let \((H_1) \) and \((H_4) \) be satisfied. Assume
\[ \beta_\infty \geq \tau \left( \|\mu_1\|_\infty \|V_1^{-1}\|_\infty^{-2} \|V_2^{-1}\|_\infty^{-2} \right), \]
where \( \tau = 4 \) if \((H_{4.1}) \) holds and \( \tau = \max\left\{ \frac{1}{\tau_1 - 1}, 4 \right\} \) if \((H_{4.2}) \) holds. Then (3) has a positive solution.

Denote \( u^- = \min\{u, 0\}. \) To state the last main result which is for the case where (3) may not have any group invariance and \( \beta \) is small, it will be assumed that \((H_5) \) there exists \( \nu > 0 \) such that, for all \( x \in \mathbb{R}^N, \)
\[ V_j(x) \leq 1, \quad (\mu_j(x) - \mu_j^\infty) + (1 + \nu)(\beta(x) - \beta_\infty) \geq 0, \quad j = 1, 2. \]

Note that \((H_5) \) implies \( \mu_j \geq \mu_j^\infty \) for \( j = 1, 2. \) Then, under the assumptions \((H_1) \) and \((H_5), \) (5) has a positive ground state solution ([16, 25, 40]), denoted by \( w_j, \) for \( j = 1, 2, \) such that
\[ \int_{\mathbb{R}^N} (|\nabla w_j|^2 + V_j w_j^2) = \int_{\mathbb{R}^N} \mu_j w_j^4 = S_j^2, \]
where
\[ S_j = \inf_{u \in \mathcal{M}_j} \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 + V_j u^2)}{\left( \int_{\mathbb{R}^N} \mu_j u^4 \right)^{1/2}}. \]
Corollary 1.7. Assume \( \text{ground state solution.} \)
\( \beta \)
and for \( \beta \) for all bounded and positive
Finally, special attention has to be paid to distinguishing nontrivial solutions from
negative terms in the numerators as well as in the denominators should be compared.
be more difficult than the case of a single equation. This is the case as showed
sequences at appropriate levels so that compactness holds, and this turns out to
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interested in solutions with both components being nonzero, new difficulties arise.
under the action of a finite group. Since we are concerned with system and are
used in \([1, 21]\) to study existence of positive solutions of a single equation invariant
of positive solutions of a single semilinear elliptic equation. These assumptions were
of system. If \( c \) is achieved then the solution in Theorem 1.5 is a positive ground
state solution.

Theorem 1.1 has the following direct consequence.

Corollary 1.7. Assume \((H_1)\) and \((H_5)\) hold. If \( V_1 = V_2, \) (7) is satisfied, and \( \beta_\infty < \min\{\mu_1, \mu_2\}, \) then (3) has a positive ground state solution.

The above theorems assert existence of positive solutions of (3) only for \( \beta \) large
and for \( \beta \) small. It can not be expected to have existence of positive solutions of (3)
for all bounded and positive \( \beta, \) as proved in [9] in the case of constant coefficients.

The assumptions \((H_1)\) and \((H_3)\) were first used in [6, 7] to investigate existence
of positive solutions of a single semilinear elliptic equation. These assumptions were
used in [1, 21] to study existence of positive solutions of a single equation invariant
under the action of a finite group. Since we are concerned with system and are
interested in solutions with both components being nonzero, new difficulties arise.
First of all, compactness of Palais-Smale sequences of the functional \( I \) associated
with (3) has to be taken into account. Second, one has to compare variant infimum
values of \( I \) with those of \( I_\infty \) associated with the limit system to find Palais-Smale
sequences at appropriate levels so that compactness holds, and this turns out to
be more difficult than the case of a single equation. This is the case as showed
in the proof of Lemma 4.6, where two fractions both having positive terms and
negative terms in the numerators as well as in the denominators should be compared.
Finally, special attention has to be paid to distinguishing nontrivial solutions from
semi-trivial solutions.

The solutions obtained in Theorems 1.1, 1.3 and 1.6 are minimizers of the infimum
values \( c \), \( c_G \) and \( \hat{c} \) defined in (17), (30) and (45) below, respectively. The assumptions (6) and (9) are from [26] where such an assumption was also used to
obtain a positive solution of (3) for \( \beta \) relatively small. Theorem 1.6 generalizes [26, Theorem 1.1] where \( \beta \) was assumed to satisfy \( \beta(x) \geq \beta_\infty \) for all \( x \in \mathbb{R}^N \). Once \( c \) defined in (39) is not achieved, the solution in Theorem 1.5 is a critical point
corresponding to the critical value \( c_1 \) defined in (42) and is obtained in a similar way as in [3, 11], but the argument will be more involved here due to the character
of system. If \( c \) is achieved then the solution in Theorem 1.5 is a positive ground
state solution.

It is in the same way as below to extend the above results to the more general
system
\[
\begin{align*}
- \Delta u + V_1(x)u &= \mu_1(x)|u|^{p-2}u + \beta(x)|u|^\frac{p-2}{2}u|v|^\frac{p}{2} \quad \text{in } \mathbb{R}^N, \\
- \Delta v + V_2(x)v &= \beta(x)|u|^\frac{p}{2}|v|^\frac{p-2}{2}v + \mu_2(x)|v|^{p-2}v \quad \text{in } \mathbb{R}^N, \\
u, \ v &\in H^1(\mathbb{R}^N),
\end{align*}
\]
where \( N \geq 1, \ 2 < p < 2^*, \ 2^* = 2N/(N-2) \) for \( N > 2 \) and \( 2^* = +\infty \) for \( N = 1, 2 \).
The linearly coupled Schrödinger system
\[
\begin{aligned}
- \Delta u + u &= \mu_1(x)|u|^{p-2}u + \lambda v & \text{in } \mathbb{R}^N, \\
- \Delta v + v &= \mu_2(x)|v|^{p-2}v + \lambda u & \text{in } \mathbb{R}^N, \\
&\text{in } \mathbb{R}^N, \\
u, v &\in H^1(\mathbb{R}^N),
\end{aligned}
\] (10)
where \( \lambda \in (0, 1), N \geq 2, 2 < p < 2^*, \mu_1, \mu_2 \in L^\infty(\mathbb{R}^N), \inf_{\mathbb{R}^N} \mu_1(x) > 0, \inf_{\mathbb{R}^N} \mu_2(x) > 0, \) and \( \lim_{|x| \to \infty} \mu_1(x) = \lim_{|x| \to \infty} \mu_2(x) = 1, \) was studied in [3].

Existence of a positive ground state of (10) was also proved in [3] both for \( \lambda \) close to 0 and for \( \lambda \) close to 1. The main difference between (3) and (10) is that (3) may have semitrivial solutions while (10) does not have any semitrivial solution, and this difference makes (3) harder to study.

The paper is organized as follows. In section 2, we give a global compactness result, which is applied in the proof of the main results. Sections 3, 4, 5 and 6 are devoted to the proof of Theorems 1.1, 1.3, 1.5 and 1.6 respectively.

2. A compactness result. Let the Sobolev space \( H^1(\mathbb{R}^N) \) be endowed with the two equivalent norms
\[
\|u\|_j = \left( \int_{\mathbb{R}^N} (|\nabla u|^2 + V_j u^2) \right)^{1/2}, \quad j = 1, 2.
\]
For \( (u, v) \in \mathcal{H} := H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N), \) denote
\[
\|(u, v)\|_* = \left( \|u\|_1^2 + \|v\|_2^2 \right)^{1/2}.
\]
If \( V_j \) are replaced with 1 then \( \| \cdot \|_j \) become the usual norm \( \| \cdot \| \) in \( H^1(\mathbb{R}^N) \) and \( \|(\cdot, \cdot)\|_* \) becomes the usual norm \( \|(\cdot, \cdot)\| \) in \( \mathcal{H} \). It is known that solutions of (3) correspond to critical points of the functional \( I : \mathcal{H} \to \mathbb{R} \) defined by
\[
I(u, v) = \frac{1}{2} \|(u, v)\|^2 - \frac{1}{4} \int_{\mathbb{R}^N} (\mu_1 u^4 + 2\beta uv^2 + \mu_2 v^4).
\]
Under \( (H_1), \) the limit system
\[
\begin{aligned}
- \Delta u + u &= \mu_1 u^3 + \beta_{\infty} uv^2 & \text{in } \mathbb{R}^N, \\
- \Delta v + v &= \beta_{\infty} u^2 v + \mu_{2\infty} v^3 & \text{in } \mathbb{R}^N, \\
u, v &\in H^1(\mathbb{R}^N)
\end{aligned}
\]
and its corresponding functional
\[
I_{\infty}(u, v) = \frac{1}{2} \|(u, v)\|^2 - \frac{1}{4} \int_{\mathbb{R}^N} (\mu_1 u^4 + 2\beta_{\infty} u^2 v^2 + \mu_{2\infty} v^4)
\]
play important roles in searching solutions of (3).

Let \( H^{-1}(\mathbb{R}^N) \) be the dual space of \( H^1(\mathbb{R}^N) \) and \( \mathcal{H}^* \) be the dual space of \( \mathcal{H} \). A solution of (3) or (11) is said to be nonzero if at least one of its components is nonzero. Thus nonzero solutions are either nontrivial solutions or semitrivial solutions. The main result in this section is the following theorem.
Theorem 2.1. Let \((H_1)\) hold and assume \(\{(u_n, v_n)\} \subset H\) to be a \((PS)_c\) sequence for \(I\), that is

\[
I(u_n, v_n) \to c, \quad I'(u_n, v_n) \to 0 \quad \text{in } H'.
\]

Then, replacing \(\{(u_n, v_n)\}\) by a subsequence if necessary, there exist a nonnegative integer \(k\), a solution \((u^0, v^0)\) of (3), nonzero solutions \((u^1, v^1), \ldots, (u^k, v^k)\) of the limit system (11) and \(k\) sequences \(\{y^j_n\} \subset \mathbb{R}^N\) such that, as \(n \to \infty,\)

\[
|y^j_n| \to \infty, \quad |y^j_n - y^{j'}_n| \to \infty, \quad j \neq j',
\]

\[
\left\|u_n - u^0 - \sum_{j=1}^k u^j(\cdot - y^j_n)\right\| \to 0, \quad \left\|v_n - v^0 - \sum_{j=1}^k v^j(\cdot - y^j_n)\right\| \to 0,
\]

\[
\|u_n\|^2 \to \sum_{j=0}^k \|u^j\|^2, \quad \|v_n\|^2 \to \sum_{j=0}^k \|v^j\|^2, \quad I(u^0, v^0) + \sum_{j=1}^k I_\infty(u^j, v^j) = c.
\]

Moreover, if \(u_n \geq 0, v_n \geq 0\) a.e. in \(\mathbb{R}^N\) for all \(n\) then \(u^j \geq 0, v^j \geq 0\) in \(\mathbb{R}^N\) for all \(j\).

In Theorem 2.1 the superscript \(j\) in \((u^j, v^j)\) means an index, not a power, while it stands for a power in the expression of \(I(u, v)\). This will not cause any confusion since the exact meaning of superscripts will be clear from the context.

Such a result for a single equation is due to Benci and Cerami [11]. The result stated here is in the form of [40, Theorem 8.4], and for completeness of the paper a proof will be given. For that purpose some lemmas are needed, and the first one is the following lemma which is an analog of the Brézis-Lieb lemma in [13].

Lemma 2.2. Let \(\Omega\) be an open subset of \(\mathbb{R}^N\) and \(r \geq 1, s \geq 1, p \geq 1, q \geq 1\) with \(\frac{1}{p} + \frac{1}{q} = 1\). Assume \(\{u_n\} \subset L^{r_p}(\Omega)\) and \(\{v_n\} \subset L^{s_q}(\Omega)\) are bounded, \(u_n \to u\) a.e. in \(\Omega\), and \(v_n \to v\) a.e. in \(\Omega\). Then

\[
\lim_{n \to \infty} \int_\Omega (|u_n|^r |v_n|^s - |u_n - u|^r |v_n - v|^s) = \int_\Omega |u|^r |v|^s.
\]

Proof. The proof is similar to the argument in [13]. Firstly, Fatou’s lemma yields that \(u \in L^{r_p}(\Omega)\) and \(v \in L^{s_q}(\Omega)\). Given any \(\varepsilon > 0\), there exists \(C_\varepsilon > 0\) such that

\[
\|1 + (1 + |y|^s)^{-1}\| \leq \varepsilon + C_\varepsilon (|x|^r + |y|^s + |x|^r |y|^s)
\]

for all \(x, y \in \mathbb{R}\). Taking \(x = \frac{b}{a}\) and \(y = \frac{d}{c}\) yields, for \(a, b, c, d \in \mathbb{R}\),

\[
|a + b|^r |c + d|^s - |a|^r |c|^s + C_\varepsilon (|b|^r |c|^s + |a|^r |d|^s + |b|^r |d|^s)
\]

\[
\leq \varepsilon (|a|^r |c|^s + |a|^r |d|^s + |b|^r |d|^s).
\]

Define \(u^+ = \max\{u, 0\}\). Then

\[
\left|\int (|a + b|^r |c + d|^s - |a|^r |c|^s + \varepsilon (|a|^r |c|^s + |a|^r |d|^s + |b|^r |d|^s)) \right|
\]

\[
\leq C_\varepsilon (|b|^r |c|^s + |a|^r |d|^s + |b|^r |d|^s).
\]

Setting \(a = u_n - u, \ b = u, \ c = v_n - v\) and \(d = v\) leads to

\[
\int (|u_n|^r |v_n|^s - |u_n - u|^r |v_n - v|^s - |u|^r |v|^s)
\]

\[
- \varepsilon (|u_n - u|^r |v_n - v|^s + |u_n - u|^r |v_n - v|^s) \right| \leq C_\varepsilon (|u|^r |v|^s + |u|^r |v|^s) \in L^1(\Omega).
\]
By the Lebesgue dominated convergence theorem, $\int_{\Omega} f_n^\varepsilon \to 0$ as $n \to \infty$. Since
\[ ||u_n||^r v_n^s - |u_n - u|^r v_n^s - |u|^r v^s|| 
\leq f_n^\varepsilon + \varepsilon (|u_n - u|^r v_n^s + |u_n - u|^r + |v_n - v|^s),
\]
it follows that
\[ \lim_{n \to \infty} \int_{\Omega} ||u_n||^r v_n^s - |u_n - u|^r v_n^s - |u|^r v^s|| \leq C\varepsilon. \]
Letting $\varepsilon \to 0$, we finish the proof. \( \Box \)

The next lemma is a special case of [40, Lemma 8.1].

**Lemma 2.3.** If $u_n \to u$ in $H^1(\mathbb{R}^N)$, then $v_n^3 - (u_n - u)^3 \to u^3$ in $H^{-1}(\mathbb{R}^N)$.

**Lemma 2.4.** Assume $u_n \to u$ in $H^1(\mathbb{R}^N)$ and $v_n \to v$ in $H^1(\mathbb{R}^N)$. Then
\[ u_n v_n^2 - (u_n - u)(v_n - v)^2 \to uv^2 \text{ in } H^{-1}(\mathbb{R}^N). \] (12)

**Proof.** Define $f(u, v) = uv^2$. Then
\[ |f(u_n, v_n) - f(u_n - u, v_n - v) - f(u, v)| \leq (v_n^2 + v^2)|u| + 2(|u_n| + |u|)(|v_n| + |v|)|v|. \] (13)
From (13), it is easy to see that (see the proof of [40, Lemma 8.1]), for any $\varepsilon > 0$, there exists $R > 0$ such that
\[ \int_{|x| > R} |f(u_n, v_n) - f(u_n - u, v_n - v) - f(u, v)| \varphi \leq \varepsilon \|\varphi\| \] (14)
for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Since $u_n \to u$ and $v_n \to v$ in $L^4_{\text{loc}}(\mathbb{R}^N)$, for $R$ given above and for $n$ large,
\[ \int_{|x| \leq R} |f(u_n, v_n) - f(u_n - u, v_n - v) - f(u, v)| \varphi \\
= \int_{|x| \leq R} [(u_n - u)(2v_n - v)v + u(v_n + v)(v_n - v)] \varphi \\
\leq \varepsilon \|\varphi\|. \] (15)
Combining (14) with (15) completes the proof. \( \Box \)

Note that Lemmas 2.2-2.4 also hold if some weight function $Q$ with $Q \in L^\infty(\mathbb{R}^N)$ is appropriately added. We are now ready to prove Theorem 2.1.

**Proof of Theorem 2.1.** Since $\{(u_n, v_n)\}$ is a $(PS)_c$ sequence for $I$, it is bounded in $\mathcal{H}$. Assume, passing to a subsequence if necessary,
\[ (u_n, v_n) \to (u^0, v^0) \text{ in } \mathcal{H}, \]
\[ (u_n, v_n) \to (u^0, v^0) \text{ a.e. in } \mathbb{R}^N. \]
Then $I'(u^0, v^0) = 0$ and $(u^0, v^0)$ is a solution of (3). Set $u_n^1 = u_n - u^0$ and $v_n^1 = v_n - v^0$. It is easy to see that
\[ \|u_n^1\|^2 = \|u_n\|^2 - \|u^0\|^2 + o(1), \quad \|v_n^1\|^2 = \|v_n\|^2 - \|v^0\|^2 + o(1). \]
Making use of assumption $(H_1)$, the Brézis-Lieb lemma and Lemma 2.2 gives
\[ I_\infty(u_n^1, v_n^1) = I(u_n^1, v_n^1) + o(1) = I(u_n, v_n) - I(u^0, v^0) + o(1) = c - I(u^0, v^0) + o(1). \]
Using $(H_1)$ again together with Lemmas 2.3 and 2.4 leads to
\[ I'_\infty(u_n^1, v_n^1) = I'(u_n, v_n) - I'(u^0, v^0) + o(1) = I(u_n, v_n) - I(u^0, v^0) + o(1) = o(1). \]
Define
\[
\delta_1 = \lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |u_n^1|^2, \quad \delta_2 = \lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |v_n^1|^2,
\]
where \(B_1(y)\) is the unit ball centered at \(y \in \mathbb{R}^N\). If \(\delta_1 = \delta_2 = 0\), then the P.L. Lions lemma (see [25] or [40, Lemma 1.21]) implies
\[
(u_n^1, v_n^1) \to (0, 0) \quad \text{in} \quad L^4(\mathbb{R}^N) \times L^4(\mathbb{R}^N).
\]
Combining this with the fact \(I_\infty(u_n^1, v_n^1) \to 0\) in \(H^*\) yields \((u_n^1, v_n^1) \to (0, 0)\) in \(H\), and then the proof terminates.

In the following, we consider the case where \(\delta_1 > 0\) or \(\delta_2 > 0\). Assume \(\delta_1 > 0\) without loss of generality. Then, passing to a subsequence if necessary, there exists \(\{y_n\} \subset \mathbb{R}^N\) such that, for all \(n,\)
\[
\int_{B_1(y_n)} |u_n^1|^2 > \frac{\delta_1}{2}.
\]
Define \(\hat{u}_n^1 = u_n^1(\cdot + y_n^1)\) and \(\hat{v}_n^1 = v_n^1(\cdot + y_n^1)\) and assume that
\[
(\hat{u}_n^1, \hat{v}_n^1) \rightharpoonup (u^1, v^1) \quad \text{in} \quad H,
\]
\[
(\hat{u}_n^1, \hat{v}_n^1) \to (u^1, v^1) \quad \text{a.e. in} \quad \mathbb{R}^N.
\]
Then (16) and the weak convergence \(\hat{u}_n^1 \rightharpoonup u^1\) in \(H^1(\mathbb{R}^N)\) imply \(u^1 \neq 0\). In addition, (16) and the weak convergence \(u_n^1 \rightharpoonup 0\) in \(H^1(\mathbb{R}^N)\) imply \(|y_n^1| \to \infty\). It follows from the weak convergence \((\hat{u}_n^1, \hat{v}_n^1) \rightharpoonup (u^1, v^1)\) in \(H\) and \(I'_\infty(u_n^1, v_n^1) \to 0\) in \(H^*\) that \(I'_\infty(u^1, v^1) = 0\), and thus \((u^1, v^1)\) is nonzero solution of (11). Set \(u_n^2 = u_n^1 - u^1(\cdot - y_n^1)\) and \(v_n^2 = v_n^1 - v^1(\cdot - y_n^1)\). Then
\[
\|u_n^2\|^2 = \|u_n\|^2 - \|u^0\|^2 - \|u^1\|^2 + o(1), \quad \|v_n^2\|^2 = \|v_n\|^2 - \|v^0\|^2 - \|v^1\|^2 + o(1).
\]
The Brézis-Lieb lemma and Lemma 2.2 imply
\[
I_\infty(u_n^2, v_n^2) = I_\infty(u_n^1, v_n^1) - I_\infty(u^1, v^1) + o(1) = c - I(u^0, v^0) - I_\infty(u^1, v^1) + o(1).
\]
Moreover, it follows from Lemmas 2.3 and 2.4 that
\[
\|I'_\infty(u_n^2, v_n^2)\| = \|I'_\infty(u_n^1(\cdot + y_n^1) - u^1(\cdot + y_n^1) - v^1(\cdot - y_n^1))\|
\]
\[
= \|I'_\infty(u_n^1(\cdot + y_n^1), v_n^1(\cdot + y_n^1)) - I'_\infty(u^1, v^1) + o(1)\|
\]
\[
= \|I'_\infty(u_n^1(\cdot + y_n^1), v_n^1(\cdot + y_n^1))\| + o(1)
\]
\[
= \|I'_\infty(u_n^1, v_n^1)\| + o(1)
\]
\[
= o(1),
\]
that is, \(I'_\infty(u_n^2, v_n^2) = o(1)\).

Iterating the above procedure, suppose \(k\) nonzero solutions \((u^j, v^j)\) of (11) and \(k\) sequences \(\{y_n^j\} \subset \mathbb{R}^N\) with \(|y_n^j| \to \infty, j = 1, 2, \ldots, k\), have been constructed such that, for \((u_n^{k+1}, v_n^{k+1})\) defined by \(u_n^{k+1} = u_n - u^0 - \sum_{j=1}^k u^j(\cdot - y_n^j)\) and \(v_n^{k+1} = v_n - v^0 - \sum_{j=1}^k v^j(\cdot - y_n^j),\)
\[
\|u_n^{k+1}\|^2 = \|u_n\|^2 - \sum_{j=0}^k \|u^j\|^2 + o(1), \quad \|v_n^{k+1}\|^2 = \|v_n\|^2 - \sum_{j=0}^k \|v^j\|^2 + o(1),
\]
\[
I_\infty(u_n^{k+1}, v_n^{k+1}) = c - I(u^0, v^0) - \sum_{j=1}^k I_\infty(u^j, v^j) + o(1),
\]
and

\[ I'_\infty(u^{k+1}_n, v^{k+1}_n) = o(1). \]

For any \( j = 1, 2, \ldots, k \), since \( (u^j, v^j) \) is a nonzero critical point of \( I_\infty \), the Sobolev embedding \( H^1(\mathbb{R}^N) \to L^4(\mathbb{R}^N) \) implies

\[ \|u^j, v^j\| \geq S^2 \min \{ \mu^{-1}_{1,\infty}, \mu^{-1}_{2,\infty}, \beta^{-1}_\infty \} > 0, \]

where \( S \) is the optimal constant defined by

\[ S = \inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 + u^2 \mid u \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} u^4 = 1 \right\}. \]

Therefore, the iterating process must terminate in some finite, say \( k \), steps. That is, the two numbers

\[ \delta'_1 = \lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |u^{k+1}_n|^2, \quad \delta'_2 = \lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |v^{k+1}_n|^2 \]

are both zero. Then the P. L. Lions lemma implies \( (u^{k+1}_n, v^{k+1}_n) \to (0, 0) \) in \( L^4(\mathbb{R}^N) \times L^4(\mathbb{R}^N) \). This fact together with \( I'_\infty(u^{k+1}_n, v^{k+1}_n) = o(1) \) yields \( (u^{k+1}_n, v^{k+1}_n) \to (0, 0) \) in \( \mathcal{H} \).

To see that \( |y^j_n - y^j'_n| \to \infty \) for \( j \neq j' \), assume for induction that \( |y^j_n - y^j'_n| \to \infty \) holds for all \( j < j' \leq l \) and for some \( l \leq k - 1 \). Now, let \( j < j' = l + 1 \) and assume, without loss of generality, \( \int_{B_1(y^{l+1}_n)} |u^{l+1}_n|^2 \geq \delta \) for some \( \delta > 0 \). Since

\[ u^{l+1}_n = u^l_n - u^l(y^l_n) = \cdots = u^l_n - \sum_{i=j}^l u^l(y^l_n - y^l_i), \]

it follows that

\[ \int_{B_1(y^{l+1}_n - y^l_i)} |u^l_n(x + y^l_n) - u^l - \sum_{i=j+1}^l u^l(x + y^l_n - y^l_i)|^2 \geq \delta. \]

Since \( u^l_n(x + y^l_i) \to u^l \) in \( H^1(\mathbb{R}^N) \) and \( |y^l_n - y^l'_n| \to \infty \) for \( i = j + 1, \ldots, l \), it follows that \( |y^{l+1}_n - y^{l+1}_n| \to \infty \). Therefore, \( |y^j_n - y^j'_n| \to \infty \) for all \( j \neq j' \).

It remains to prove the last sentence of the conclusions of the theorem. In fact, if \( u_n \to 0, \ v_n \to 0 \) a.e. in \( \mathbb{R}^N \) for all \( n \), then the functionals

\[ I^+(u, v) = \frac{1}{2} \| (u, v) \|^2 - \frac{1}{4} \int_{\mathbb{R}^N} (\mu_1 u^4 + 2\beta(u^+)^2(v^+)^2 + \mu_2 v^4), \]

\[ I^+_{\infty}(u, v) = \frac{1}{2} \| (u, v) \|^2 - \frac{1}{4} \int_{\mathbb{R}^N} (\mu_1 u^4 + 2\beta_\infty(u^+)^2(v^+)^2 + \mu_2 v^4) \]

can be used to obtain the conclusion. \( \square \)

3. Proof of Theorem 1.1. Set

\[ \mathcal{H}_G = \{(u, v) \in \mathcal{H} \mid u \text{ and } v \text{ are } G\text{-invariant}\}. \]

By the principle of symmetric criticality due to Palais [31], any critical point of the restriction functional \( I|_{\mathcal{H}_G} \) is a critical point of \( I \). Therefore, finding \( G\)-invariant solutions of (3) is reduced to seeking critical points of \( I|_{\mathcal{H}_G} \).

Define the Nehari manifolds

\[ \mathcal{N}_G = \{(u, v) \in \mathcal{H}_G \mid (u, v) \neq (0, 0), \ J(u, v) = 0 \}, \]

where \( J(u, v) = (I(u, v), (u, v) = \| (u, v) \|^2 - \int_{\mathbb{R}^N} (\mu_1 u^4 + 2\beta u^2 v^2 + \mu_2 v^4), \) and

\[ \mathcal{N}_\infty = \{(u, v) \in \mathcal{H} \mid (u, v) \neq (0, 0), \ J'_\infty(u, v), (u, v) = 0 \}. \]
Consider the two infimums
\[ c_G = \inf_{(u,v) \in N_G} I(u,v) \] (17)
and
\[ c_\infty = \inf_{(u,v) \in N_\infty} I(\infty, u,v). \]
Clearly, \( c_G > 0 \) and \( c_\infty > 0 \). It will be showed that \( c_G < mc_\infty \), and for this purpose some lemmas are needed.

**Lemma 3.1.** Let \( m_0 \in \mathbb{N} \) and \( \xi_j \geq 0, j = 1, 2, \cdots, m_0 \). Then
(i) \[
\sum_{j=1}^{m_0} \xi_j^4 - \left( \sum_{j=1}^{m_0} \xi_j \right)^4 \leq -3 \sum_{i \neq j} \xi_i \xi_j;
\]
(ii) for any compact set \( K \subset (0, \infty) \), there exists \( C_K > 0 \) such that
\[
\sum_{j=1}^{m_0} \xi_j^4 - \left( \sum_{j=1}^{m_0} \xi_j \right)^4 \leq -2 \sum_{i \neq j} \xi_i \xi_j - C_K \sum_{j \neq n} \xi_j,
\]
provided some \( \xi_n \in K \).

**Proof.** The first inequality is from [23, Lemma 3.3] and the second one follows directly from the first one. \( \square \)

The next lemma is a well known result and its proof can be found, say, in [17, 22].

**Lemma 3.2.** The equation
\[ -\Delta u + u = u^3 \text{ in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N) \] (18)
has a unique positive radial solution \( w \in C^\infty(\mathbb{R}^N) \) which satisfies, for some \( A > 0 \),
\[
w(r) r^{N-1} e^r \to A, \quad w'(r) r^{N-1} e^r \to -A, \quad \text{as } r = |x| \to \infty.
\]
Moreover, every positive solution of (18) has the form \( w(\cdot - y) \) for some \( y \in \mathbb{R}^N \).

Let \( e_i, \sigma_0, \sigma \) be as in Section 1. Recall that the number \( \sigma \) in (H3) satisfies \( \sigma \in (\sigma_0, 4) \) and \( \sigma \neq 2 \). The next lemma was proved in [1] via Lemma 3.2 and standard estimates.

**Lemma 3.3.** (i) There exists \( C > 0 \) such that
\[
\int_{\mathbb{R}^N} e^{-\sigma|x|} w^2(x - se_i) \leq C \max \left\{ e^{-\sigma s}, e^{-2s} s^{-(N-1)} \right\},
\]
\[
\int_{\mathbb{R}^N} e^{-\sigma|x|} w^4(x - se_i) \leq Ce^{-\sigma s},
\]
for all \( i = 1, 2, \cdots, m \) and \( s \geq 1 \).
(ii) For any \( \zeta \in (\frac{1}{2}, 1) \), there exists \( C_\zeta > 0 \) such that
\[
\int_{\mathbb{R}^N} w(x - se_i) w(x - se_j) \leq C_\zeta e^{-\zeta |e_i - e_j| s}
\]
for all \( i, j = 1, 2, \cdots, m \) with \( i \neq j \) and \( s \geq 1 \).
(iii) For any \( R > 0 \), there exists \( C_R > 0 \) and \( s_R > 0 \) such that
\[
\int_{B_R(s)} w(x - se_j) \geq C_R e^{-|x_j - e_j|s} s^{-\frac{2-\beta}{4}}
\]
for all \( i, j = 1, 2, \ldots, m \) with \( i \neq j \) and \( s \geq s_R \).

**Lemma 3.4.** Assume \((H_1), (H_2), \) and \((H_3)\) hold. If \( \beta_\infty > \max\{\mu_1, \mu_2\} \), then \( c_G < mc_\infty \).

**Proof.** Recall that \( w \) is the unique positive radial solution of \((18)\). Set
\[
k_1 = \frac{\mu_2 - \beta_\infty}{\mu_1 \mu_2 - \beta_\infty^2}, \quad k_2 = \frac{\mu_1 - \beta_\infty}{\mu_1 \mu_2 - \beta_\infty^2}.
\]
(19)

Since \( \beta_\infty > \max\{\mu_1, \mu_2\} \), there must be \( c_\infty = I_\infty(\sqrt{k_1} w, \sqrt{k_2} w) \) (see [9]). Thus
\[
c_\infty = \frac{1}{4} (k_1 + k_2) \|w\|^2.
\]
(20)

For simplicity of notations, for \( s > 0 \), denote \( W = W(s) = \sum_{i=1}^m w(\cdot - se_i) \) and define two functions
\[
F_1(s) = \sum_{j=1}^2 k_j \int_{\mathbb{R}^N} (|\nabla W|^2 + V_j W^2)
\]
and
\[
F_2(s) = \int_{\mathbb{R}^N} (k_1^2 \mu_1 + 2k_1k_2 \beta + k_2^2 \mu_2) W^4.
\]

Then, it is easy to see that
\[
\sqrt{\frac{F_1(s)}{F_2(s)}} (\sqrt{k_1} W, \sqrt{k_2} W) \in N_G,
\]
and thus \( c_G \) can be estimated as
\[
c_G \leq I \left( \frac{k_1 F_1(s)}{F_2(s)} W, \sqrt{\frac{k_2 F_1(s)}{F_2(s)}} W \right) = \frac{F_1^2(s)}{4 F_2(s)}.
\]
(21)

According to (20) and (21), to conclude it suffices to prove that
\[
\frac{F_1^2(s)}{F_2(s)} < m(k_1 + k_2) \|w\|^2
\]
(22)
for \( s \) sufficiently large.

Then \( F_1(s) \) and \( F_2(s) \) are to be estimated. Setting
\[
E = E(s) = \sum_{i \neq j} \int_{\mathbb{R}^N} w^3(x - se_i) w(x - se_j)
\]
and using \((H_3)\), \( F_1(s) \) is estimated as
\[
F_1(s) \leq (k_1 + k_2) \|W\|^2 + \sum_{j=1}^2 k_j \int_{\mathbb{R}^N} (V_j - 1) W^2
\]
\[
\leq (k_1 + k_2) (m \|w\|^2 + E) + C \sum_{i=1}^m \int_{\mathbb{R}^N} e^{-\sigma |x|} w^2(x - se_i),
\]
where the fact that \( w \) is a solution of \((18)\) has been used. By Lemma 3.3 (i),
\[
F_1(s) \leq (k_1 + k_2) (m \|w\|^2 + E) + C e^{-\sigma s} + C e^{-2s \mu_{\infty}^{-1}},
\]
(18)
which together with Lemma 3.3 (ii) implies
\[ F_2^2(s) \leq (k_1 + k_2)^2(m^2 \|w\|^4 + 2m\|w\|^2 E) + C e^{-2s} e^\sigma + C e^{-\sigma s} + C e^{-2s} s^{-(N-1)}, \]  \hspace{1cm} (23)
where the number \( \zeta \in (\frac{1}{2}, 1) \) is fixed from now on. Using \( (H_3) \) again \( F_2(s) \) can be estimated as
\[ F_2(s) = \int_{\mathbb{R}^N} (k_1^2 \mu_1 \infty + 2k_1 k_2 \beta \infty + k_2^2 \mu_2 \infty) W^4 \]
\[ - \int_{\mathbb{R}^N} [k_1^2 (\mu_1 \infty - \mu_1) + 2k_1 k_2 (\beta \infty - \beta) + k_2^2 (\mu_2 \infty - \mu_2)] W^4 \]
\[ \geq (k_1 + k_2) \int_{\mathbb{R}^N} W^4 - C \sum_{i=1}^m \int_{\mathbb{R}^N} e^{-\sigma |x|} w^4(x - se_i), \]
where the fact that \( k_1^2 \mu_1 \infty + 2k_1 k_2 \beta \infty + k_2^2 \mu_2 \infty = k_1 + k_2 \) has been used. Using Lemma 3.3 (i) yields
\[ F_2(s) \geq (k_1 + k_2) \int_{\mathbb{R}^N} W^4 - C e^{-\sigma s}. \]

Denote \( U = U(s) = W^4(s) - \sum_{i=1}^m w^4(\cdot - se_i). \) Then, since \( \int_{\mathbb{R}^N} w^4 = \|w\|^2, \)
\[ F_2(s) \geq (k_1 + k_2) \left( m \|w\|^2 + \int_{\mathbb{R}^N} U \right) - C e^{-\sigma s}. \]  \hspace{1cm} (24)

To estimate \( \int_{\mathbb{R}^N} U, \) fix an \( R > 0, \) take \( s_R \) as in Lemma 3.3 (iii) and enlarge it so that
\[ B_R(se_i) \cap B_R(se_j) = \emptyset \]
for \( i \neq j \) and \( s \geq s_R. \) Decompose \( \mathbb{R}^N \) into \( \bigcup_{n=1}^m B_R(se_n) \) and \( \mathbb{R}^N \setminus \bigcup_{n=1}^m B_R(se_n). \)

For the integral on \( \bigcup_{n=1}^m B_R(se_n), \) take \( K = \{w(x) \mid \|x\| \leq R\} \) and use Lemma 3.1 (ii) to see that
\[ \int_{\bigcup_{n=1}^m B_R(se_n)} U \geq 2 \sum_{i \neq j} \int_{\bigcup_{n=1}^m B_R(se_n)} w^3(x - se_i) w(x - se_j) \]
\[ + C_R \sum_{i \neq j} \int_{B_R(\{se_i\})} w(x - se_j). \]

Using Lemma 3.1 (i) for the integral on \( \mathbb{R}^N \setminus \bigcup_{n=1}^m B_R(se_n) \) yields
\[ \int_{\mathbb{R}^N \setminus \bigcup_{n=1}^m B_R(se_n)} U \geq 2 \sum_{i \neq j} \int_{\mathbb{R}^N \setminus \bigcup_{n=1}^m B_R(se_n)} w^3(x - se_i) w(x - se_j). \]

Putting the last two inequalities into (24), it then follows that
\[ F_2(s) \geq (k_1 + k_2)(m \|w\|^2 + 2E) + C \sum_{i \neq j} \int_{B_R(\{se_i\})} w(x - se_j) - C e^{-\sigma s}, \]
which combined with Lemma 3.3 (iii) leads to
\[ F_2(s) \geq (k_1 + k_2)(m \|w\|^2 + 2E) + C e^{-\sigma s} s^{-\frac{N-1}{2}} - C e^{-\sigma s}. \]

Note that it is in the proof of the last inequality where the assumption \( m \geq 2 \) is used. Then, for \( s \) large,
\[ F_2^{-1}(s) \leq (k_1 + k_2)^{-1}(m \|w\|^2)^{-1} \left( 1 - 2(m \|w\|^2)^{-1} E \right) \]
\[ - C e^{-\sigma s} s^{-\frac{N-1}{2}} + C e^{-\sigma s} + O(E^2). \]
$E^2$ can be estimated by using Lemma 3.3 (ii) and thus

$$F_2^{-1}(s) \leq (k_1 + k_2)^{-1}(m\|w\|^2)^{-1}\left(1 - 2(m\|w\|^2)^{-1}E\right)$$

$$- Ce^{-\sigma_0 s}s^{-\frac{N}{2} - 1} + Ce^{-\sigma s} + Ce^{-2\zeta s}.$$ (25)

Combining (23) and (25), we arrive at

$$F_1^2(s) \leq (k_1 + k_2)m\|w\|^2\left(1 - Ce^{-\sigma_0 s}s^{-\frac{N}{2} - 1} + Ce^{-\sigma s} + Ce^{-2\zeta s} + Ce^{-2\zeta s} - (N - 1)\right)$$

for $s$ sufficiently large. This last inequality implies (22) since $\sigma_0 \in (0, 2]$, $\sigma > \sigma_0$ and $\zeta \in (1, 2)$.

**Lemma 3.5.** If $(u, v) \in \mathcal{N}_G$ and $I(u, v) = c_G$, then $(u, v)$ is a critical point of $I|_{\mathcal{H}_G}$.

**Proof.** It is similar to the proof of [40, Theorem 4.3].

**Proof of Theorem 1.1.** By the Ekeland variational principle, there exist a sequence $\{\lambda_n\}$ of real numbers and $\{(u_n, v_n)\} \subset \mathcal{N}_G$ such that

$$I(u_n, v_n) \to c_G, \quad \nabla I|_{\mathcal{H}_G}(u_n, v_n) - \lambda_n \nabla J|_{\mathcal{H}_G}(u_n, v_n) \to 0 \quad \text{in } \mathcal{H}_G.$$ 

Since $I(u_n, v_n) = \frac{1}{2}\|\nabla(u_n, v_n)\|^2 \to c_G$,

$$o(1) = (\nabla I(u_n, v_n) - \lambda_n \nabla J(u_n, v_n), (u_n, v_n)) = 2\lambda_n\|\nabla(u_n, v_n)\|^2 = 2\lambda_n(4c_G + o(1)).$$

Then $\lambda_n = o(1)$ and $\nabla I|_{\mathcal{H}_G}(u_n, v_n) \to 0$ in $\mathcal{H}_G$. Since $I$ is $G$-invariant and $(u_n, v_n) \in \mathcal{H}_G$, $\nabla I|_{\mathcal{H}_G}(u_n, v_n) = \nabla I(u_n, v_n)$ (see the proof of [40, Theorem 1.28]). Therefore,

$$\nabla I(u_n, v_n) \to 0 \quad \text{in } \mathcal{H}.$$ 

Let $(u^0, v^0)$, $(u^j, v^j)$ and $\{y^j_n\}$, $j = 1, \cdots, k$, be as in Theorem 2.1. To show that $\{(u_n, v_n)\}$ converges, it is necessary to prove $k = 0$. Assume by contradiction that

$$k \geq 1. \quad \text{Then } k \geq m \quad \text{indeed, } \text{assume, passing to a subsequence if needed, } y^j_n / ||y^j_n|| \to z^j \in S^{N-1}. \quad \text{For any } l, \# \{g z^j \mid g \in G\} \geq m, \quad \text{there exists } g_1, g_2, \cdots, g_m \in G \quad \text{such that for any } R > 0 \text{ if } n \text{ is large enough then, for } i \neq j, \quad B(g y^j_n, R) \cap B(g y^i_n, R) = \emptyset.$$ 

Using Theorem 2.1 and the notations in its proof gives

$$\sum_{j=1}^{k} \|u^j\|^2 = \lim_{n \to \infty} \|u^j_n\|^2$$

$$\geq \lim_{n \to \infty} \sum_{i=1}^{m} \int_{B(y^j_n, R)} \left(\|
abla u^j_n\|^2 + |u^j_n|^2\right)$$

$$= m \lim_{n \to \infty} \int_{B(0, R)} \left(\|
abla u^j_n\|^2 + |u^j_n|^2\right)$$

$$= m \int_{B(0, R)} \left(\|
abla u^j\|^2 + |u^j|^2\right).$$

Letting $R \to \infty$, yields

$$\sum_{j=1}^{k} \|u^j\|^2 \geq m\|u^j\|^2, \quad l = 1, 2, \cdots, k.$$ (26)
Similarly,
\[ \sum_{j=1}^{k} \|v^j\|^2 \geq m\|v^l\|^2, \quad l = 1, 2, \ldots, k. \] (27)
This implies \( k \geq m \). Then
\[ c_G = I(u^0, v^0) + \sum_{j=1}^{k} I_N(u^j, v^j) \geq mc_N, \]
which contradicts the conclusion of Lemma 3.4. Thus \( k = 0 \) and \( (u_n, v_n) \to (u^0, v^0) \) in \( H \). Then \( c_G \) is achieved by \( (u^0, v^0) \in N_G \). Observing
\[ I(|u|, |v|) = I(u, v), \quad \langle I'(|u|, |v|), (|u|, |v|) \rangle = \langle I'(u, v), (u, v) \rangle, \]
we may assume \( u^0 \geq 0 \) and \( v^0 \geq 0 \). Lemma 3.5 and the principle of symmetric criticality imply that \( (u^0, v^0) \) is a nonnegative nonzero solution of (3).

It suffices to prove that \( u^0 \neq 0 \) and \( v^0 \neq 0 \). Assume, by contradiction, that \( v^0 = 0 \). Then the maximum principle implies \( u^0(x) > 0 \) for all \( x \in \mathbb{R}^N \). Since
\[ \beta(x) \geq \max\{1, \|V_1^{-1}V_2\|_\infty\} \mu_1(x), \]
there exists \( \rho > 0 \) sufficiently small such that
\[ 2 \int_{\mathbb{R}^N} (\max\{1, \|V_1^{-1}V_2\|_\infty\} \mu_1 - \beta) (u^0)^4 < \rho^2 \int_{\mathbb{R}^N} (\mu_2 - \max\{1, \|V_1^{-1}V_2\|_\infty\}^2 \mu_1) (u^0)^4, \]
which implies
\[ (1 + \rho^2 \max\{1, \|V_1^{-1}V_2\|_\infty\})^2 \int_{\mathbb{R}^N} (|\nabla u^0|^2 + V_1(u^0)^2) < \int_{\mathbb{R}^N} (\mu_1 + 2\beta \rho^2 + \mu_2 \rho^4) (u^0)^4. \] (28)
For \( t > 0 \) defined by
\[ t^2 = \frac{\|\langle u^0, \rho u^0 \rangle \|^2}{\int_{\mathbb{R}^N} (\mu_1 + 2\beta \rho^2 + \mu_2 \rho^4) (u^0)^4}, \]
\( (tu^0, t\rho u^0) \in N_G \). Then it follows from (28) that
\[ I(u^0, 0) = c_G \leq I(tu^0, t\rho u^0) \]
\[ \leq \frac{1}{4} \int_{\mathbb{R}^N} (\mu_1 + 2\beta \rho^2 + \mu_2 \rho^4) (u^0)^4 \]
\[ \leq \frac{1}{4} \frac{(1 + \rho^2 \max\{1, \|V_1^{-1}V_2\|_\infty\})^2 \int_{\mathbb{R}^N} (|\nabla u^0|^2 + V_1(u^0)^2))^2}{\int_{\mathbb{R}^N} (\mu_1 + 2\beta \rho^2 + \mu_2 \rho^4) (u^0)^4} \]
\[ < \frac{1}{4} \int_{\mathbb{R}^N} \mu_1(u^0)^4 = I(u^0, 0), \]
which is a contradiction. Therefore \( v^0 \neq 0 \). Similarly, \( u^0 \neq 0 \). The maximum principle implies \( (u^0, v^0) \) is a \( G \)-invariant positive solution of (3). \( \blacksquare \)
4. **Proof of Theorem 1.3.** The proof of Theorem 1.3, which is more delicate than the proof of Theorem 1.1, will be given in this section. First, existence of $w_G$ needed in the statement of Theorem 1.3 is proved in the following Theorem 4.1, which is closely related to the results in [1, 21]. The proof of Theorem 4.1 will be only sketched since it is similar to the proofs in [1, 21]. Let $G$ and $\sigma_0$ be as in Section 1.

**Theorem 4.1.** Assume that $V$ and $\mu$ are $G$-invariant positive functions,
\[
\lim_{|x| \to \infty} V(x) = V_{\infty} > 0, \quad \lim_{|x| \to \infty} \mu(x) = \mu_{\infty} > 0,
\]
and there exist $C > 0$ and $\sigma > \sigma_0$ such that, for all $x \in \mathbb{R}^N$,
\[
V(x) - V_{\infty} \leq Ce^{-\sigma|x|}, \quad \mu_{\infty} - \mu(x) \leq Ce^{-\sigma|x|}.
\]
Then the equation
\[
-\Delta u + V(x)u = \mu(x)u^3 \quad \text{in} \quad \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N)
\]
has a $G$-invariant positive solution, denoted by $w_G$, such that $w_G$ is a minimizer of the minimization problem
\[
\gamma_G = \inf_{u \in M_G} \phi(u),
\]
where
\[
\phi(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + Vu^2) - \frac{1}{4} \int_{\mathbb{R}^N} \mu u^4
\]
and
\[
M_G = \left\{ u \in H^1(\mathbb{R}^N) \mid u \neq 0, \ u \text{ is } G\text{-invariant, } \int_{\mathbb{R}^N} (|\nabla u|^2 + Vu^2) = \int_{\mathbb{R}^N} \mu u^4 \right\}.
\]

Note that $\gamma_G = \frac{1}{4} S_G^2$, where
\[
S_G = \inf_{u \in M_G} \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 + Vu^2)}{\left( \int_{\mathbb{R}^N} \mu u^4 \right)^{\frac{1}{2}}}
\]
Obviously,
\[
\gamma_G = \frac{1}{4} S_G^2 = \frac{1}{4} \int_{\mathbb{R}^N} (|\nabla w_G|^2 + Vw_G^2) = \frac{1}{4} \int_{\mathbb{R}^N} \mu w_G^4.
\]
Without loss of generality, assume $V_{\infty} = 1$. In the proof of Theorem 4.1, the following infimum will be used:
\[
\gamma_{\infty} = \inf_{u \in M_{\infty}} \phi_{\infty}(u),
\]
where
\[
\phi_{\infty}(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) - \frac{1}{4} \int_{\mathbb{R}^N} \mu_{\infty} u^4
\]
and
\[
M_{\infty} = \left\{ u \in H^1(\mathbb{R}^N) \mid u \neq 0, \ \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) = \int_{\mathbb{R}^N} \mu_{\infty} u^4 \right\}.
\]
If $V$, $\mu$ and $\mu_{\infty}$ are replaced with $V_j$, $\mu_j$ and $\mu_{j\infty}$, the corresponding $\gamma_G$ and $\gamma_{\infty}$ will be denoted by $\gamma_{jG}$ and $\gamma_{j\infty}$, respectively. The proof of Theorem 4.1 relies on the following lemma.

**Lemma 4.2.** Under the assumptions of Theorem 4.1, $\gamma_G < m \gamma_{\infty}$. 

Proof. A modification of the proof of Lemma 3.4 works. In the present case,
\[ \gamma_\infty = \frac{1}{4\mu_\infty} \|w\|^2. \]
Recall that \( W = W(s) = \sum_{i=1}^{m} w(-se_i) \) for \( s > 0 \). If the two functions \( F_1 \) and \( F_2 \) in the proof of Lemma 3.4 are redefined as
\[ F_1(s) = \int_{\mathbb{R}^N} (|\nabla W|^2 + VW^2) \]
and
\[ F_2(s) = \int_{\mathbb{R}^N} \mu W^4, \]
respectively, then
\[ \sqrt{\frac{F_1(s)}{F_2(s)}} W \in M_G, \]
and it suffices to prove that
\[ \frac{F_2(s)}{F_1(s)} < \frac{m}{\mu_\infty} \|w\|^2 \]
for \( s \) sufficiently large. This last inequality can be proved in the same way as in the proof of Lemma 3.4. \( \square \)

Proof of Theorem 4.1. Let \( \{u_n\} \subset M_G \) be a minimizing sequence for \( \gamma_G \). Replacing \( u_n \) with \( |u_n| \), we may assume \( u_n \geq 0 \). Using the Ekeland variational principle, it can be assumed that \( \phi'(u_n) \to 0 \) in \( H^{-1}(\mathbb{R}^N) \). By [40, Theorem 8.4], replacing \( \{u_n\} \) by a subsequence if necessary, there exist a nonnegative integer \( k \), a nonnegative solution \( u_0 \) of (29), nonzero solutions \( u_1, \ldots, u_k \) of the limit equation
\[ -\Delta u + u = \mu_\infty u^3 \text{ in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N) \]
and \( k \) sequences \( \{y^j_n\} \subset \mathbb{R}^N \) with
\[ |y^j_n| \to \infty, \quad |y^j_n - y^{j'}_n| \to \infty, \quad j \neq j', \quad n \to \infty, \]
such that
\[ u_n = u^0 + \sum_{j=1}^{k} u^j(-y^j_n) + o(1), \quad \phi(u^0) + \sum_{j=1}^{k} \phi(u^j) = \gamma_G. \]
Since \( u_n \) are \( G \)-invariant, if \( k \neq 0 \) then \( k \geq m \) which implies \( \gamma_G \geq m\gamma_\infty \), a contradiction to Lemma 4.2. Therefore, \( \|u_n - u^0\| \to 0 \), \( \phi(u^0) = \gamma_G \), and \( u^0 \) is a \( G \)-invariant positive solution of (29). \( \square \)

Denote \( J_1(u,v) = \langle I'(u,v),(u,0) \rangle \) and \( J_2(u,v) = \langle I'(u,v),(0,v) \rangle \). In order to prove Theorem 1.3, define
\[ \bar{c}_G = \inf_{(u,v) \in \bar{N}_G} I(u,v), \quad (30) \]
where \( \bar{N}_G \) is the generalized Nehari manifold
\[ \bar{N}_G = \{(u,v) \in H_G \mid u \neq 0, \quad \phi(u^0) = \gamma_G, \quad u \in H^1(\mathbb{R}^N)\} \]
with two constraints. To prove Theorem 1.3, it is necessary to compare \( \bar{c}_G \) with \( m \) multiple of the constant
\[ \bar{c}_\infty = \inf_{(u,v) \in \bar{N}_\infty} I_\infty(u,v), \]
where
\[ \tilde{N}_\infty = \{(u, v) \in \mathcal{H} \mid u \neq 0, \ v \neq 0, \ J_{1\infty}(u, v) = 0, \ J_{2\infty}(u, v) = 0\} , \]
and \( J_{1\infty}(u, v) = \langle I_{1\infty}(u, v), (u, 0) \rangle \) and \( J_{2\infty}(u, v) = \langle I_{2\infty}(u, v), (0, v) \rangle \). The following two lemmas are quoted from [26] and their proofs will be sketched just for convenience.

**Lemma 4.3.** Under the assumptions of Theorem 1.3, there exists \((u, v) \in \tilde{N}_G\) such that
\[ I(u, v) = \frac{S_{1G}^2S_{2G}^2(S_{1G}^2 + S_{2G}^2 - 2\eta_G^2)}{4(S_{1G}^2S_{2G}^2 - \eta_G^4)} . \]

**Proof.** Let \( w_{1G} \) be the solution of (5) as in Section 1. Then \((u, v)\) can be chosen as \((\sqrt{s_0w_{1G}}, \sqrt{t_0w_{2G}})\) for suitable \( s_0 > 0 \) and \( t_0 > 0 \).

**Lemma 4.4.** Under the assumptions of Theorem 1.3, for any \((u, v) \in \tilde{N}_G\) with
\[ I(u, v) \leq \frac{S_{1G}^2S_{2G}^2(S_{1G}^2 + S_{2G}^2 - 2\eta_G^2)}{4(S_{1G}^2S_{2G}^2 - \eta_G^4)} , \]
there hold
\[ \int_{\mathbb{R}^N} \mu_1u^4 \geq \alpha_{1G} := \left( \frac{S_{1G}S_{2G}(S_{1G}^2S_{2G}^2 - \eta_G^4) - \xi S_{1G}^2S_{2G}^2(S_{1G}^2 + S_{2G}^2 - 2\eta_G^2)}{(S_{1G}^2S_{2G}^2 - \eta_G^4)(S_{1G}^2 + S_{2G}^2 - \xi S_{1G}^2)} \right)^2 > 0 , \]
and
\[ \int_{\mathbb{R}^N} \mu_2v^4 \geq \alpha_{2G} := \left( \frac{S_{1G}S_{2G}(S_{1G}^2S_{2G}^2 - \eta_G^4) - \xi S_{1G}^2S_{2G}^2(S_{1G}^2 + S_{2G}^2 - 2\eta_G^2)}{(S_{1G}^2S_{2G}^2 - \eta_G^4)(S_{1G}^2 + S_{2G}^2 - \xi S_{1G}^2)} \right)^2 > 0 , \]
and
\[ \int_{\mathbb{R}^N} \mu_1u^4 \int_{\mathbb{R}^N} \mu_2v^4 - \left( \int_{\mathbb{R}^N} \beta u^2v^2 \right)^2 \geq (1 - \xi^2) \left[ S_{1G}S_{2G}(S_{1G}^2S_{2G}^2 - \eta_G^4) - \xi S_{1G}^2S_{2G}^2(S_{1G}^2 + S_{2G}^2 - 2\eta_G^2) \right]^4\left( S_{1G}^2S_{2G}^2 - \eta_G^4 \right)^4(\xi S_{1G}^2)^2(S_{1G}^2 + S_{2G}^2 - \xi S_{1G}^2)^2 \].

**Proof.** For any \((u, v) \in \tilde{N}_G\) with \( I(u, v) \leq \frac{S_{1G}^2S_{2G}^2(S_{1G}^2 + S_{2G}^2 - 2\eta_G^2)}{4(S_{1G}^2S_{2G}^2 - \eta_G^4)} \), the definition of \( S_{1G} \) in Section 1 and the Hölder inequality imply
\[ \left( \int_{\mathbb{R}^N} \mu_1u^4 \right)^\frac{1}{2} + \xi \left( \int_{\mathbb{R}^N} \mu_2v^4 \right)^\frac{1}{2} \geq S_{1G} , \]
\[ \xi \left( \int_{\mathbb{R}^N} \mu_1u^4 \right)^\frac{1}{2} + \left( \int_{\mathbb{R}^N} \mu_2v^4 \right)^\frac{1}{2} \geq S_{2G} , \]
and
\[ \frac{S_{1G}^2S_{2G}^2(S_{1G}^2 + S_{2G}^2 - 2\eta_G^2)}{(S_{1G}^2S_{2G}^2 - \eta_G^4)} \geq S_{1G} \left( \int_{\mathbb{R}^N} \mu_1u^4 \right)^\frac{1}{2} + S_{2G} \left( \int_{\mathbb{R}^N} \mu_2v^4 \right)^\frac{1}{2} . \]
From these three inequalities it is easy to prove the result.

To prove Theorem 1.3, \( \tilde{c}_G \) should be compared with \( \gamma_1 + \gamma_2 \) and \( m\tilde{c}_\infty \). This will be done in the following two lemmas.

**Lemma 4.5.** Under the assumptions of Theorem 1.3,
\[ \tilde{c}_G < \gamma_1 + \gamma_2 < \min\{\gamma_1 + m\gamma_\infty, \gamma_2 + m\gamma_\infty\} < m(\gamma_1 + \gamma_\infty) . \] (31)
Proof. Clearly, the second and the third inequalities are consequences of Lemma 4.2. Using (6), the fact that \( \int_{\mathbb{R}^N} \mu_j w_j^4 = S_{jG}^2 \) and the Hölder inequality gives

\[
\eta_G^2 = \int_{\mathbb{R}^N} \beta w_{1G}^2 w_{2G}^2 \leq \xi S_{1G} S_{2G} < \min\{S_{1G}^2, S_{2G}^2\},
\]

which implies

\[
\frac{S_{1G}^2 S_{2G}^2 (S_{1G}^2 + S_{2G}^2 - 2 \eta_G^2)}{4(S_{1G}^2 S_{2G}^2 - \eta_G^2)} < \frac{1}{4} (S_{1G}^2 + S_{2G}^2).
\]

Thus the first inequality in (31) is a consequence of Lemma 4.3, since \( \gamma_{jG} = \frac{1}{4} S_{jG}^2 \).

\[\square\]

Lemma 4.6. Under the assumptions of Theorem 1.3,

\[
\tilde{c}_G < m \tilde{c}_\infty.
\]

Proof. Let \( W = W(s) = \sum_{i=1}^{m} w(.-s e_i) \) be as in the proof of Lemma 3.4. Consider the linear system for \( l_1 \) and \( l_2 \)

\[
\begin{cases}
    l_1 \int_{\mathbb{R}^N} \mu_1 W^4 + l_2 \int_{\mathbb{R}^N} \beta W^4 = \|W\|_1^2 \\
    l_1 \int_{\mathbb{R}^N} \beta W^4 + l_2 \int_{\mathbb{R}^N} \mu_2 W^4 = \|W\|_2^2.
\end{cases}
\]

Since \((H_1)\) implies \( \int_{\mathbb{R}^N} \mu_1 W^4 = m \mu_j \|w\|^2 + o(1) \), \( \int_{\mathbb{R}^N} \beta W^4 = m \beta \|w\|^2 + o(1) \), and \( \|W\|_j^2 = m \|w\|^2 + o(1) \) as \( s \to \infty \) and since \( \beta \in \min\{\mu_1, \mu_2\} \), the linear system has a unique solution \( (l_1, l_2) \) with each component \( l_j = l_j(s) > 0 \) for \( s \) sufficiently large, which is given by

\[
\begin{align*}
    l_1 &= \frac{\|W\|_1^2 \int_{\mathbb{R}^N} \mu_1 W^4 - \|W\|_2^2 \int_{\mathbb{R}^N} \beta W^4}{\int_{\mathbb{R}^N} \mu_1 W^4 \int_{\mathbb{R}^N} \mu_2 W^4 - (\int_{\mathbb{R}^N} \beta W^4)^2}, \\
    l_2 &= \frac{\|W\|_2^2 \int_{\mathbb{R}^N} \mu_1 W^4 - \|W\|_1^2 \int_{\mathbb{R}^N} \beta W^4}{\int_{\mathbb{R}^N} \mu_1 W^4 \int_{\mathbb{R}^N} \mu_2 W^4 - (\int_{\mathbb{R}^N} \beta W^4)^2}.
\end{align*}
\]

Then \( (\sqrt{l_1} W, \sqrt{l_2} W) \in \tilde{N}_G \) and \( \tilde{c}_G \) can be estimated as

\[
\tilde{c}_G \leq \frac{1}{4} (l_1 \|W\|_1^2 + l_2 \|W\|_2^2)
= \frac{\|W\|_1^4 \int_{\mathbb{R}^N} \mu_1 W^4 + \|W\|_2^4 \int_{\mathbb{R}^N} \mu_1 W^4 - 2 \|W\|_1^2 \|W\|_2^2 \int_{\mathbb{R}^N} \beta W^4}{4 \int_{\mathbb{R}^N} \mu_1 W^4 \int_{\mathbb{R}^N} \mu_2 W^4 - (\int_{\mathbb{R}^N} \beta W^4)^2}.
\]

Let \( k_1, k_2 \) be as in the proof of Lemma 3.4. Since \( \beta \in \min\{\mu_1, \mu_2\} \), \( \tilde{c}_\infty \) has the expression (9)

\[
\tilde{c}_\infty = \frac{1}{4} (k_1 + k_2) \|w\|^2 = \frac{(\mu_1 + \mu_2 - 2 \beta \infty) \|w\|^2}{4(\mu_1 \mu_2 - \beta \infty^2)}.
\]

Therefore to prove (32), it suffices to prove

\[
(\mu_1 \mu_2 - \beta \infty^2) \left( \|W\|_1^4 \int_{\mathbb{R}^N} \mu_1 W^4 + \|W\|_2^4 \int_{\mathbb{R}^N} \mu_1 W^4 - 2 \|W\|_1^2 \|W\|_2^2 \int_{\mathbb{R}^N} \beta W^4 \right)
- (\mu_1 + \mu_2 - 2 \beta \infty) m \|w\|^2 \left( \int_{\mathbb{R}^N} \mu_1 W^4 \int_{\mathbb{R}^N} \mu_2 W^4 - (\int_{\mathbb{R}^N} \beta W^4)^2 \right) < 0,
\]

for \( s \) sufficiently large.
Recall that
\[ E = E(s) = \sum_{i \neq j} \int_{\mathbb{R}^N} w^3(x - se_i)w(x - se_j) \]
and
\[ U = U(s) = W^4(s) - \sum_{i=1}^m w^4(\cdot - se_i). \]
Then the Hölder inequality implies that \( \int_{\mathbb{R}^N} U = O(E) \). To prove (33), each quantity containing \( W \) should be estimated. Rewrite \( \|W\|^4 \int_{\mathbb{R}^N} \mu_2 W^4 \) as
\[
\|W\|^4 \int_{\mathbb{R}^N} \mu_2 W^4 = \left( m\|w\|^2 + E + \int_{\mathbb{R}^N} (V_1 - 1)W^2 \right)^2 \\
\times \left( \mu_2m\|w\|^2 + \mu_2 \int_{\mathbb{R}^N} U + \int_{\mathbb{R}^N} (\mu_2 - \mu_2\infty)W^4 \right) .
\]
Using (H1) and the fact that \( \int_{\mathbb{R}^N} U = O(E) \) leads to
\[
\|W\|^4 \int_{\mathbb{R}^N} \mu_2 W^4 = \mu_2m^3\|w\|^6 + 2\mu_2m^2\|w\|^4E \\
+ \mu_2\infty m^2\|w\|^4 \int_{\mathbb{R}^N} U + m^2\|w\|^4 \int_{\mathbb{R}^N} (\mu_2 - \mu_2\infty)W^4 \\
+ 2\mu_2\infty m^2\|w\|^4 \int_{\mathbb{R}^N} (V_1 - 1)W^2 + O(E^2) \\
+ o(1) \int_{\mathbb{R}^N} (\mu_2 - \mu_2\infty)W^4 + o(1) \int_{\mathbb{R}^N} (V_1 - 1)W^2 .
\]
Here, \( o(1) \) means a quantity which tends to 0 as \( s \to \infty \). In the same way,
\[
\|W\|^2 \int_{\mathbb{R}^N} \mu_1 W^4 = \mu_1m^3\|w\|^6 + 2\mu_1m^2\|w\|^4E \\
+ \mu_1\infty m^2\|w\|^4 \int_{\mathbb{R}^N} U + m^2\|w\|^4 \int_{\mathbb{R}^N} (\mu_1 - \mu_1\infty)W^4 \\
+ 2\mu_1\infty m^2\|w\|^4 \int_{\mathbb{R}^N} (V_2 - 1)W^2 + O(E^2) \\
+ o(1) \int_{\mathbb{R}^N} (\mu_1 - \mu_1\infty)W^4 + o(1) \int_{\mathbb{R}^N} (V_2 - 1)W^2 .
\]
To estimate the term \( \|W\|^2 \|W\|^2 \int_{\mathbb{R}^N} \beta W^4 \), rewrite it as
\[
\|W\|^2 \|W\|^2 \int_{\mathbb{R}^N} \beta W^4 \\
= \left( m\|w\|^2 + E + \int_{\mathbb{R}^N} (V_1 - 1)W^2 \right) \left( m\|w\|^2 + E + \int_{\mathbb{R}^N} (V_2 - 1)W^2 \right) \\
\times \left( \beta \infty m\|w\|^2 + \beta \infty \int_{\mathbb{R}^N} U + \int_{\mathbb{R}^N} (\beta - \beta \infty)W^4 \right) .
\]
Then, using \((H_1)\) and the fact that \(\int_{\mathbb{R}^N} U = O(E)\) again,
\[
\|W\|_2^2 = \beta_{\infty} m^3 \|w\|^6 + 2\beta_{\infty} m^2 \|w\|^4 E \\
+ \beta_{\infty} m^2 \|w\|^4 \int_{\mathbb{R}^N} U + m^2 \|w\|^4 \int_{\mathbb{R}^N} (\beta - \beta_{\infty})W^4 \\
+ \beta_{\infty} m^2 \|w\|^4 \sum_{j=1}^2 \int_{\mathbb{R}^N} (V_j - 1)W^2 + O(E^2) \\
+ o(1) \int_{\mathbb{R}^N} (\beta - \beta_{\infty})W^4 + o(1) \sum_{j=1}^2 \int_{\mathbb{R}^N} (V_j - 1)W^2.
\]

Write \(\int_{\mathbb{R}^N} \mu_1 W^4 \int_{\mathbb{R}^N} \mu_2 W^4 - \left( \int_{\mathbb{R}^N} \beta W^4 \right)^2\) in the form
\[
\int_{\mathbb{R}^N} \mu_1 W^4 \int_{\mathbb{R}^N} \mu_2 W^4 - \left( \int_{\mathbb{R}^N} \beta W^4 \right)^2 \\
= \left( \mu_{1\infty} m \|w\|^2 + \mu_{2\infty} \int_{\mathbb{R}^N} U + \int_{\mathbb{R}^N} (\mu_1 - \mu_{1\infty})W^4 \right) \\
\times \left( \mu_{2\infty} m \|w\|^2 + \mu_{2\infty} \int_{\mathbb{R}^N} U + \int_{\mathbb{R}^N} (\mu_2 - \mu_{2\infty})W^4 \right) \\
- \left( \beta_{\infty} m \|w\|^2 + \beta_{\infty} \int_{\mathbb{R}^N} U + \int_{\mathbb{R}^N} (\beta - \beta_{\infty})W^4 \right)^2.
\]

Using \((H_1)\) once more yields
\[
\int_{\mathbb{R}^N} \mu_1 W^4 \int_{\mathbb{R}^N} \mu_2 W^4 - \left( \int_{\mathbb{R}^N} \beta W^4 \right)^2 \\
= (\mu_{1\infty} \mu_{2\infty} - \beta_{\infty}^2) m^2 \|w\|^4 + 2(\mu_{1\infty} \mu_{2\infty} - \beta_{\infty}^2) m \|w\|^2 \int_{\mathbb{R}^N} U \\
+ \mu_{1\infty} m \|w\|^2 \int_{\mathbb{R}^N} (\mu_2 - \mu_{2\infty})W^4 + \mu_{2\infty} m \|w\|^2 \int_{\mathbb{R}^N} (\mu_1 - \mu_{1\infty})W^4 \\
- 2\beta_{\infty} m \|w\|^2 \int_{\mathbb{R}^N} (\beta - \beta_{\infty})W^4 + O(E^2) \\
+ o(1) \int_{\mathbb{R}^N} (\beta - \beta_{\infty})W^4 + o(1) \sum_{j=1}^2 \int_{\mathbb{R}^N} (\mu_j - \mu_{j\infty})W^2.
\]

Inserting \((34)-(37)\) into \((33)\) and rearranging the terms, the left hand side of \((33)\) is written in the form
\[
\text{LHS} = (\mu_{1\infty} \mu_{2\infty} - \beta_{\infty}^2)(\mu_{1\infty} + \mu_{2\infty} - 2\beta_{\infty}) m^2 \|w\|^4 \left( 2E - \int_{\mathbb{R}^N} U \right) \\
+ [2(\mu_{1\infty} \mu_{2\infty} - \beta_{\infty}^2)(\mu_{2\infty} - \beta_{\infty}) m^2 \|w\|^4 + o(1)] \int_{\mathbb{R}^N} (V_1 - 1)W^2 \\
+ [2(\mu_{1\infty} \mu_{2\infty} - \beta_{\infty}^2)(\mu_{1\infty} - \beta_{\infty}) m^2 \|w\|^4 + o(1)] \int_{\mathbb{R}^N} (V_2 - 1)W^2
\]
By the Ekeland variational principle, there exists a sequence 

\[ u \] 

that 

\[ \int y \] 

According to the proof of Lemma 3.4, for a fixed 

\[ R > s \] 

integrals on the right hand side are all positive if \( s \) is sufficiently large. Using (H3) yields 

\[
\text{LHS} \leq (\mu_1 u_2 - \beta_2^2)(\mu_1 u_2 + \mu_2 u_2 - 2\beta_2^2)w^2 \left(2E - \int_{\mathbb{R}^N} U\right) + C \int_{\mathbb{R}^N} e^{-\sigma|z|}(W^2 + W^4) + O(E^2).
\]

According to the proof of Lemma 3.4, for a fixed \( R > 0 \) there exists \( C_R > 0 \) such that 

\[ \int_{\mathbb{R}^N} U \geq 2E + C_R \sum_{i \neq j} \int_{B_R(s e_j)} w(x - s e_j). \]

Then, combining the last two inequalities, 

\[
\text{LHS} \leq -C \sum_{i \neq j} \int_{B_R(s e_i)} w(x - s e_j) + C \int_{\mathbb{R}^N} e^{-\sigma|z|}(W^2 + W^4) + O(E^2).
\]

Invoking Lemma 3.3, we arrive at 

\[
\text{LHS} \leq -CE^{-\sigma s} s^{-\frac{N-1}{2}} + CE^{-\sigma s} + CE^{-2s}s^{-1} + CE^{-2\zeta s s} < 0,
\]

for \( s \) sufficiently large. \( \square \)

We are now ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** By the Ekeland variational principle, there exists a sequence \( \{(u_n, v_n)\} \subset N_G \) such that 

\[ I(u_n, v_n) \to \tilde{c} \quad \text{and} \quad \nabla I|_{N_G}(u_n, v_n) \to 0 \quad \text{as} \quad n \to \infty. \]

We may assume that \( u_n \geq 0 \) and \( v_n \geq 0 \). Then \( \{(u_n, v_n)\} \) is bounded in \( H \) and there exist \( \{s_n\}, \{t_n\} \subset \mathbb{R} \) such that 

\[ \nabla I(u_n, v_n) - s_n \nabla J_1(u_n, v_n) - t_n \nabla J_2(u_n, v_n) \to 0 \quad \text{as} \quad n \to \infty. \quad (38) \]

Testing the left side of (38) with \( (u_n, 0) \) and \( (0, v_n) \) yields 

\[
\begin{align*}
\left\{ s_n \int_{\mathbb{R}^N} \mu_1 u_n^2 + t_n \int_{\mathbb{R}^N} \beta u_n^2 v_n^2 \to 0 \quad \text{as} \quad n \to \infty, \\
\left. s_n \int_{\mathbb{R}^N} \beta_2^2 v_n^2 + t_n \int_{\mathbb{R}^N} \mu_2 u_n^4 \to 0 \quad \text{as} \quad n \to \infty. \right.
\end{align*}
\]

By Lemma 4.3, we may assume that, for all \( n \), 

\[ I(u_n, v_n) \leq \frac{S_1^2 S_2^2 (S_1^2 + S_2^2 - 2\eta_G^2)}{4(S_1^2 S_2^2 - \eta_G^2)}. \]
According to Lemma 4.4,
\[
\int_{\mathbb{R}^N} \mu_1 u_n^4 \int_{\mathbb{R}^N} \mu_2 v_n^4 - \left( \int_{\mathbb{R}^N} \beta u_n^2 v_n^2 \right)^2 \\
\geq (1 - \xi^2) \left[ S_{1G} S_{2G} (S_{1G}^2 - \eta_{1G}^2) - \xi S_{1G}^2 S_{2G}^2 (S_{1G}^2 + S_{2G}^2) - 2 \eta_{1G}^2 \right] > 0.
\]
Therefore, \( s_n \to 0 \) and \( t_n \to 0 \) as \( n \to \infty \). From (38) it can be deduced that \( \nabla I(u_n, v_n) \to 0 \) as \( n \to \infty \). Then, by Theorem 2.1, replacing \( \{(u_n, v_n)\} \) by a subsequence if necessary, there exist a nonnegative integer \( k \), a solution \((u^0, v^0)\) with \( u^0 \geq 0 \) and \( v^0 \geq 0 \) of (3), nonzero solutions \((u^1, v^1), \ldots, (u^k, v^k)\) of the limit system (11) and \( k \) sequences \( \{y^j_n\} \subset \mathbb{R}^N \) such that, as \( n \to \infty \),

\[
|y^j_n| \to \infty, \quad |y^j_n - y^{j'}_n| \to \infty, \quad j \neq j',
\]

\[
\left\| u_n - u^0 - \sum_{j=1}^k u^j (\cdot - y^j_n) \right\| \to 0, \quad \left\| v_n - v^0 - \sum_{j=1}^k v^j (\cdot - y^j_n) \right\| \to 0,
\]

and

\[
I(u^0, v^0) + \sum_{j=1}^k I(u^j, v^j) = \tilde{c}_G.
\]

If \( k = 0 \) then \( \|u_n - u^0\| \to 0 \) and \( \|v_n - v^0\| \to 0 \), and as a consequence of Lemma 4.4,

\[
\int_{\mathbb{R}^N} \mu_1 (u^0)^4 \geq \alpha_{1G} > 0, \quad \int_{\mathbb{R}^N} \mu_2 (v^0)^4 \geq \alpha_{2G} > 0.
\]

Then the maximum principle implies that \((u^0, v^0)\) is a \( G \)-invariant positive solution of (3) and \( I(u^0, v^0) = \tilde{c}_G \). Assume by contradiction that \( k \geq 1 \). Then there are four cases.

**Case 1.** \( u^j = 0 \) for all \( j = 1, 2, \cdots, k \) and \( v^j \neq 0 \) for some \( l \geq 1 \). In this case \( \|u_n - u^0\| \to 0 \) and \( u^0 \neq 0 \). If \( v^0 \neq 0 \) then

\[
\tilde{c}_G = I(u^0, v^0) + \frac{1}{4} \sum_{j=1}^k \|v^j\|^2 > \tilde{c}_G,
\]

which is a contradiction. If \( v^0 = 0 \) then using (27) yields

\[
\tilde{c}_G = I(u^0, 0) + \frac{1}{4} \sum_{j=1}^k \|v^j\|^2 \geq \gamma_{1G} + \frac{1}{4} m \|v^l\|^2 \geq \gamma_{1G} + m \gamma_{2G},
\]

which contradicts the result of Lemma 4.5.

**Case 2.** \( v^j = 0 \) for all \( j = 1, 2, \cdots, k \) and \( u^l \neq 0 \) for some \( l \geq 1 \). The argument is the same as in Case 1 and a contradiction is encountered.

**Case 3.** \( u^{l_1} \neq 0 \) and \( v^{l_2} \neq 0 \) for some \( l_1 \geq 1 \) and \( l_2 \geq 1 \), but there is no \( l \geq 1 \) such that \( u^l \neq 0 \) and \( v^l \neq 0 \). Then

\[
\tilde{c}_G \geq \frac{1}{4} \sum_{j=1}^k (\|w^j\|^2 + \|v^j\|^2) \geq \frac{1}{4} m (\|w^{l_1}\|^2 + \|v^{l_2}\|^2) \geq m (\gamma_{1G} + \gamma_{2G}),
\]
which is a contradiction to the result of Lemma 4.5.

**Case 4.** \( u^l \neq 0 \) and \( v^l \neq 0 \) for some \( l \geq 1 \). Then using (26) and (27) gives

\[
\tilde{c}_G \geq \frac{1}{4} \sum_{j=1}^{k} (\|u^j\|^2 + \|v^j\|^2) \geq \frac{1}{4} m (\|u^l\|^2 + \|v^l\|^2) \geq m \tilde{c}_\infty, 
\]

which is a contradiction to the result of Lemma 4.6.

\[\square\]

5. **Proof of Theorem 1.5.** This section is devoted to the proof of Theorem 1.5. Consider

\[c = \inf_{(u,v) \in \mathcal{N}_+} I(u,v),\] (39)

where

\[\mathcal{N}_+ = \{(u,v) \in \mathcal{H} \mid (u,v) \neq (0,0), \ u \geq 0, \ v \geq 0, \ J(u,v) = 0\}, \]

which is a closed subset of the Nehari manifold

\[\mathcal{N} = \{(u,v) \in \mathcal{H} \mid (u,v) \neq (0,0), \ J(u,v) = 0\}. \]

Note that \( \mathcal{N}_+ \) does not have interior in \( \mathcal{N} \).

**Lemma 5.1.** If \((H_1)\) is satisfied, then \( 0 < c \leq c_\infty \).

**Proof.** It is clear that \( c > 0 \). To prove \( c \leq c_\infty \), choose \( \{y_n\} \subset \mathbb{R}^N \) such that \( |y_n| \to \infty \). For any \((u,v) \in \mathcal{N}_\infty\), define \( u_n = |u(-y_n)| \) and \( v_n = |v(-y_n)| \). Then for \( t_n > 0 \) defined by

\[t_n^2 = \int_{\mathbb{R}^N} (\mu_1 u_n^4 + 2\beta u_n^2 v_n^2 + \mu_2 v_n^4)\]

\((t_n u_n, t_n v_n) \in \mathcal{N}_+\). Since \( |y_n| \to \infty \), using \((H_1)\) we see that

\[c \leq I(t_n u_n, t_n v_n) = \frac{1}{4} t_n^2 \|(u_n, v_n)\|^4 \]

\[= \frac{1}{4} \int_{\mathbb{R}^N} (\mu_1 u_n^4 + 2\beta u_n^2 v_n^2 + \mu_2 v_n^4) \]

\[= \frac{1}{4} \int_{\mathbb{R}^N} (\mu_1 u_n^4 + 2\beta u_n^2 v_n^2 + \mu_2 v_n^4) + o(1) \]

\[\to \frac{1}{4} \|(u,v)\|^4 = I_\infty(u,v),\]

as \( n \to \infty \). Thus \( c \leq c_\infty \). \[\square\]

**Remark 5.2.** According to Lemma 5.1, if \((H_1)\) holds then there are three possibilities:

(i) \( c < c_\infty \);

(ii) \( c = c_\infty \) and \( c \) is achieved;

(iii) \( c = c_\infty \) and \( c \) is not achieved.

It should be noted that each possibility may happen. To see this, first consider the case where, for any \( x \in \mathbb{R}^N \),

\[V_j(x) \leq 1 \ (j = 1, 2), \quad \mu_j(x) \geq \mu_j(\infty) \ (j = 1, 2), \quad \beta(x) \geq \beta(\infty), \]

\( \beta(\infty) > \max\{\mu_1(\infty), \mu_2(\infty)\} \), and at least one of the five functions \( V_j, \mu_j, \beta \) is not a constant. Then \( c_\infty \) is achieved at some \( (u,v) \in \mathcal{N}_\infty \) and it may be assumed that
$u(x) > 0$ and $v(x) > 0$ for all $x \in \mathbb{R}^N$ since $\beta_\infty > \max\{\mu_1, \mu_2\}$ (see [9]). Let $t > 0$ be such that $(tu, tv) \in \mathcal{N}_+$. Then

$$t^2 = \frac{\|(u, v)\|^2}{\int_{\mathbb{R}^N} (\mu_1 u^4 + 2\beta u^2 v^2 + \mu_2 v^4)} \leq \frac{\|(u, v)\|^2}{\int_{\mathbb{R}^N} (\mu_1 u^4 + 2\beta u^2 v^2 + \mu_2 v^4)} = 1,$$

and therefore (i) occurs as showed by

$$c \leq I(tu, tv) = \frac{1}{4} t^2 \|(u, v)\|^2 < \frac{1}{4} \|(u, v)\|^2 = I_\infty(u, v) = c_\infty.$$

Of course (ii) happens if the five functions $V_j, \mu_j, \beta_j$ are all constants and $\beta_\infty > \max\{\mu_1, \mu_2\}$. At last, consider the case in which, for any $x \in \mathbb{R}^N$,

$$V_j(x) \geq 1 \quad (j = 1, 2), \quad \mu_j(x) \leq \mu_j(x) \leq \beta_j \leq \beta_\infty, \quad \beta_\infty > \max\{\mu_1, \mu_2\},$$

and at least one of the five functions $V_j, \mu_j, \beta_j$ is not a constant. In this case, for any $(u, v) \in \mathcal{N}_+$, let $t_1 > 0$ be such that $(t_1 u, t_1 v) \in \mathcal{N}_\infty$. Then

$$t_1^2 = \frac{\|(u, v)\|^2}{\int_{\mathbb{R}^N} (\mu_1 u^4 + 2\beta u^2 v^2 + \mu_2 v^4)} \leq \frac{\|(u, v)\|^2}{\int_{\mathbb{R}^N} (\mu_1 u^4 + 2\beta u^2 v^2 + \mu_2 v^4)} = 1$$

and

$$c_\infty \leq I_\infty(t_1 u, t_1 v) = \frac{1}{4} t_1^2 \|(u, v)\|^2 \leq \frac{1}{4} \|(u, v)\|^2 = I(u, v),$$

which implies $c_\infty \leq c$. Combining this with Lemma 5.1 yields $c = c_\infty$. Now suppose, by contradiction, that $c$ is attained at $(u, v) \in \mathcal{N}_+$. Then $c_\infty$ is achieved at $(t_1 u, t_1 v)$. Since $\beta_\infty > \max\{\mu_1, \mu_2\}$, $u(x) > 0$ and $v(x) > 0$ for all $x$. Then $t_1 < 1$ and $c_\infty = I_\infty(t_1 u, t_1 v) < I(u, v) = c$. This yields a contradiction with Lemma 5.1.

Define a barycenter type continuous map $\theta : H^1(\mathbb{R}^N) \to \mathbb{R}^N$ by

$$\theta(u) = \int_{\mathbb{R}^N} u^2 \chi(|x|) x,$$

where $\chi(t) = 1$ if $0 \leq t \leq 1$ and $\chi(t) = t^{-1}$ if $t \geq 1$ (as in [11]). Consider

$$\hat{c} = \inf_{(u, v) \in \mathcal{N}} I(u, v)$$

where $\mathcal{N} = \{(u, v) \in \mathcal{N}_+ | \theta(u) = 0\}$. Then $c \leq \hat{c}$. Note that if (8) is satisfied then $\beta_\infty > \max\{\mu_1, \mu_2\}$.

**Lemma 5.3.** Assume $(H_1)$ and $\beta_\infty > \max\{\mu_1, \mu_2\}$. If $c = c_\infty$ and $c$ is not achieved, then $c < \hat{c}$.

**Proof.** Suppose, by contradiction, that $c = \hat{c}$. Then there exists $\{(u_n, v_n)\} \subset \mathcal{N}$ such that $I(u_n, v_n) \to c$. Applying the Ekeland variational principle to $I|_{\mathcal{N}_+}$, there exists $(u_n^*, v_n^*) \in \mathcal{N}_+$ such that $(I|_{\mathcal{N}_+})'(u_n^*, v_n^*) \to 0$ and $\|u_n^* - u_n\| + \|v_n^* - v_n\| \to 0$. Then it is easy to see that $I'(u_n, v_n) \to 0$. By Theorem 2.1 and the fact that $c = c_\infty$, we have

$$c = I(u^0, v^0) + I_\infty(u^1, v^1)$$

and

$$\|u_n - u^0 - u^1(- y_n^0)\| \to 0, \quad \|v_n - v^0 - v^1(- y_n^0)\| \to 0,$$

where $|y_n^0| \to \infty$, $(u^0, v^0)$ is a solution of (3) and $(u^1, v^1)$ is a solution of (11).

If $(u^0, v^1) = (0, 0)$, then $(u_n, v_n) \to (u^0, v^0)$ in $\mathcal{H}$. Thus $(u^0, v^0) \in \mathcal{N}_+$ and $c = I(u^0, v^0)$, which contradicts the assumption that $c$ is not achieved. So $(u^0, v^0) \neq (0, 0)$. 


(0, 0) and \((u^0, v^0) = (0, 0)\). Since \(u_n \geq 0\) and \(v_n \geq 0\), we have \(u^1 \geq 0\) and \(v^1 \geq 0\). Since \(\beta_\infty > \max\{\mu_{1\infty}, \mu_{2\infty}\}\) and \(\mathcal{I}_\infty(u^1, v^1) = \epsilon_\infty\), \(u_1 \neq 0\) and \(v_1 \neq 0\) (see [9]), and the maximum principle implies that \(u^1(x) > 0\) and \(v^1(x) > 0\) for all \(x \in \mathbb{R}^N\). Since (11) has a unique positive solution up to translation (see [39]), \((u^1, v^1) = (\sqrt{k_1}w(-z), \sqrt{k_2}w(-z))\) for some \(z \in \mathbb{R}^N\), where \(k_1\) and \(k_2\) are given in (19). Therefore,

\[
\left\|u_n - \sqrt{k_1}w(-z - y_n^1)\right\| \to 0, \quad \left\|v_n - \sqrt{k_2}w(-z - y_n^1)\right\| \to 0.
\]

Similar to [11], it can be proved that \(\theta(w(-z - y_n^1))\) is bounded away from the origin of \(\mathbb{R}^N\) for \(n\) large. Therefore \(\theta(u_n) \neq 0\) for \(n\) large. This is also a contradiction.

Defining \(h_1 : \mathbb{R}^N \to H^1(\mathbb{R}^N)\) by

\[
h_1(y)(x) = \frac{\|\sqrt{k_1}w(x - y), \sqrt{k_2}w(x - y)\|_*\sqrt{k_1}w(x - y)}{\left(\int_{\mathbb{R}^N} (\mu_1(x)k_1^2 + 2\beta(x)k_1k_2 + \mu_2(x)k_2^2)w^4(x - y)\right)^{\frac{1}{2}}},
\]

and \(h_2 : \mathbb{R}^N \to H^1(\mathbb{R}^N)\) by

\[
h_2(y)(x) = \frac{\|\sqrt{k_1}w(x - y), \sqrt{k_2}w(x - y)\|_*\sqrt{k_2}w(x - y)}{\left(\int_{\mathbb{R}^N} (\mu_1(x)k_2^2 + 2\beta(x)k_1k_2 + \mu_2(x)k_2^2)w^4(x - y)\right)^{\frac{1}{2}}},
\]

and \(h : \mathbb{R}^N \to \mathcal{N}_+\) by \(h(y) = (h_1(y), h_2(y))\), where \(k_1\) and \(k_2\) are given in (19). Then it is easy to verify

\[
I(h(y)) = c_\infty + o(1) \quad \text{as } |y| \to \infty.
\]

From this and Lemma 5.3, the following lemma can be proved as in [11].

**Lemma 5.4.** Assume \((H_1)\) and \(\beta_\infty > \max\{\mu_{1\infty}, \mu_{2\infty}\}\). If \(c = c_\infty\) and \(c\) is not achieved, then there exists \(R > 0\) such that

\[
\mathcal{I}(h(y)) \in \left(c, \frac{c + \hat{c}}{2}\right), \quad \text{if } |y| \geq R
\]

and

\[
\theta(h_1(y), y) > 0, \quad \text{if } |y| = R.
\]

Fix \(R > 0\) such that Lemma 5.4 holds and define

\[
c_1 = \inf_{f \in \mathcal{F}} \max_{y \in B_R(0)} \mathcal{I}(f(y)), \quad \text{(42)}
\]

where

\[
\mathcal{F} = \{f \in C(\overline{B_R(0)}, \mathcal{N}_+) \mid f|_{\partial B_R(0)} = h|_{\partial B_R(0)}\}.
\]

The reason why \(\mathcal{N}_+\) is used in the definition of \(\mathcal{F}\) instead of \(\mathcal{N}\) is that a \((PS)_{c_1}\) sequence \(\{(u_n, v_n)\}\) with \(u_n \geq 0\) and \(v_n \geq 0\) has to exist if a positive solution of (3) at the \(c_1\) level is to be found.

**Lemma 5.5.** Let \((H_1)\) be satisfied and assume either \((H_{4.1})\) holds or \((H_{4.2})\) with

\[
\beta_\infty \geq \max \left\{ \frac{1}{2\kappa - 1}, 2 \right\} (\mu_{1\infty} + \mu_{2\infty})
\]

holds. If \(c = c_\infty\) and \(c\) is not achieved, then \(c_3 \in (c_\infty, 2c_\infty)\) and \(\max_{y \in \partial B_R(0)} \mathcal{I}(h(y)) < c_1\).
Proof. By (41) and degree theory, \( f(B_R(0)) \cap \mathbb{N} \neq \emptyset \) for any \( f \in \mathcal{F} \). As a consequence, \( c_1 \geq \hat{c} \). Then Lemma 5.3 implies \( c_1 \geq \hat{c} > c = c_\infty \). In the case where \((H_{4.1})\) holds,

\[
\max_{y \in B_R(0)} I(h(y)) = \frac{1}{4} \max_{y \in B_R(0)} \| h(y) \|_2^2 = \frac{1}{4} \max_{y \in B_R(0)} \int_{\mathbb{R}^N} \left( (\sqrt{k_1} w(x-y), \sqrt{k_2} w(x-y)) \right) w^4(x-y) \leq \frac{1}{2} \int_{\mathbb{R}^N} (\mu_1 \kappa k_1 + 2 \beta(x) k_1 k_2 + \mu_2(x) k_2^2) w^4(x-y) = 2 c_\infty,
\]

which implies \( c_1 < 2c_\infty \). In the case in which \((H_{4.2})\) and \((43)\) hold,

\[
4 \kappa \beta_\infty (\mu_1 - \beta_\infty)(\mu_2 - \beta_\infty) - (\mu_1 + \mu_2 - 2 \beta_\infty)(\mu_1 \mu_2 - \beta_\infty^2) = (4 \kappa - 2) \beta_\infty (\beta_\infty^2 - (\mu_1 + \mu_2 - 2 \beta_\infty)(\mu_1 \mu_2 - \beta_\infty^2)) - (\mu_1 + \mu_2 - 2 \beta_\infty)(\mu_1 \mu_2 - \beta_\infty^2)
\]

\[
\geq 2(\mu_1 + \mu_2 - 2 \beta_\infty)(\beta_\infty^2 - (\mu_1 + \mu_2 - 2 \beta_\infty)(\mu_1 \mu_2 - \beta_\infty^2)) - (\mu_1 + \mu_2 - 2 \beta_\infty)(\mu_1 \mu_2 - \beta_\infty^2)
\]

\[
= (\mu_1 + \mu_2 - 2 \beta_\infty^2) + 2(\mu_1 + \mu_2 - 2 \beta_\infty^2) \beta_\infty^2 + \mu_1 \mu_2 \beta_\infty + c_\infty
\]

which yields \( k_1 + k_2 < 4 \kappa \beta_\infty k_1 k_2 \), and then

\[
\max_{y \in B_R(0)} I(h(y)) = \frac{1}{4} \max_{y \in B_R(0)} \int_{\mathbb{R}^N} \left( (\sqrt{k_1} w(x-y), \sqrt{k_2} w(x-y)) \right) w^4(x-y) \leq \frac{1}{8 \kappa \beta_\infty k_1 k_2} \int_{\mathbb{R}^N} w^4 \leq \frac{1}{2(\mu_1 + \mu_2 - 2 \beta_\infty^2)} \beta_\infty^2 + \mu_1 \mu_2 \beta_\infty + c_\infty
\]

which also implies \( c_1 < 2c_\infty \). Finally, it follows from Lemma 5.3 and \((40)\) that

\[
\max_{y \in \partial B_R(0)} I(h(y)) < \frac{c + \hat{c}}{2} < \hat{c} \leq c_1.
\]

The proof is complete. \( \square \)

Note that if \((8)\) with \( \tau = \max \left\{ \frac{1}{4(\kappa - T)}, 1 \right\} \) is satisfied then \((43)\) is satisfied.

**Proof of Theorem 1.5.** For \( j = 1, 2 \), define

\[
\gamma_j = \inf_{u \in \mathcal{M}_j} \phi_j(u),
\]

where

\[
\phi_j(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V_j u^2) - \frac{1}{4} \int_{\mathbb{R}^N} \mu_j u^4
\]

and

\[
\mathcal{M}_j = \left\{ u \in H^1(\mathbb{R}^N) \mid u \neq 0, \int_{\mathbb{R}^N} (|\nabla u|^2 + V_j u^2) = \int_{\mathbb{R}^N} \mu_j u^4 \right\}.
\]
Denote $a_j = \min_{x \in \mathbb{R}N} V_j(x)$ and $b_j = \|\mu_j\|_{\infty}$, $j = 1, 2$. Then

$$
\gamma_j = \frac{1}{4} \inf_{u \in H^1(\mathbb{R}N) \setminus \{0\}} \left( \frac{\left( \int_{\mathbb{R}N} |\nabla u|^2 + V_j u^2 \right)^2}{\int_{\mathbb{R}N} \mu_j u^4} \right).
$$

$$
\geq \frac{1}{4} \inf_{u \in H^1(\mathbb{R}N) \setminus \{0\}} \left( \frac{\left( \int_{\mathbb{R}N} |\nabla u|^2 + a_j u^2 \right)^2}{\int_{\mathbb{R}N} b_j u^4} \right).
$$

$$
= \frac{a_j^2 - \frac{\gamma_j}{4b_j}}{\|w\|^2}.
$$

Since (8) implies $\beta_\infty \geq 4b_j/a_j^{2-\frac{\gamma_j}{2}}$ and since $b_j/a_j^{2-\frac{\gamma_j}{2}} \geq \mu_j\infty$,

$$
\beta_j^2 - \frac{4b_j}{a_j^{2-\frac{\gamma_j}{2}}} \beta_j + \frac{2b_j}{a_j^{2-\frac{\gamma_j}{2}}} \left( \mu_1\infty + \mu_2\infty \right) - \mu_1\infty \mu_2\infty > 0
$$

and then

$$
2c_\infty = \frac{1}{2} (k_1 + k_2) \|w\|^2 = \frac{2\beta_\infty - \mu_1\infty - \mu_2\infty}{2(\beta_\infty^2 - \mu_1\infty \mu_2\infty)} \|w\|^2 < \frac{a_j^{2-\frac{\gamma_j}{2}}}{4b_j} \|w\|^2 \leq \gamma_j.
$$

Therefore, once a nonzero solution of (3) with energy less than $2c_\infty$ is found, then it is a nontrivial solution of (3).

Recall that there are three cases stated in Remark 5.2. In Case (i), any minimizing sequence for $c$ is relatively compact according to Theorem 2.1 and thus it is standard to come to the conclusion. In Case (ii), the result is obvious. It remains to consider Case (iii). Since the functions in $\mathcal{F}$ take images in $\mathcal{N}_+$, in order to obtain a $(PS)_{c_1}$ sequence $\{(u_n, v_n)\} \subset \mathcal{N}_+$, $\mathcal{N}_+$ should be proved an invariant set for the descending flow $\varphi_t$ defined by

$$
d\frac{dt}{d} \varphi_t(u, v) = -\nabla I|_{\mathcal{N}}(\varphi_t(u, v)) \quad \varphi_0(u, v) = (u, v).
$$

For $(u, v) \in \mathcal{N}$, since $I(u, v) = \frac{1}{2}\|(u, v)\|_2^2$, the gradient $\nabla I|_{\mathcal{N}}(u, v)$ can be expressed as

$$
\nabla I|_{\mathcal{N}}(u, v) = \frac{1}{2} (u, v) - \frac{\langle \nabla J(u, v), (u, v) \rangle}{2 \|(\nabla J(u, v))\|_2^2} \nabla J(u, v)
$$

$$
= \frac{1}{2} (u, v) + \frac{\|(u, v)\|_2^2}{\|(\nabla J(u, v))\|_2^2} \nabla J(u, v)
$$

$$
= \pi_1(u, v)(u, v) - \pi_2(u, v)((-\Delta + V_1)^{-1}(\mu_1 u^3 + \beta u v^2), (-\Delta + V_2)^{-1}(\mu_2 v^3 + \beta u^2 v)),
$$

where $\pi_1, \pi_2 : \mathcal{N} \to \mathbb{R}$ defined by

$$
\pi_1(u, v) = \frac{\|(\nabla J(u, v))\|_2^2 + 4 \|(u, v)\|_2^2}{2 \|(\nabla J(u, v))\|_2^2}, \quad \pi_2(u, v) = \frac{4 \|(u, v)\|_2^2}{\|(\nabla J(u, v))\|_2^2}
$$

are positive $C^1$ functions. Since, by the maximum principle,

$$
(-\Delta + V_1)^{-1}(\mu_1 u^3 + \beta u v^2) \geq 0, \quad (-\Delta + V_2)^{-1}(\mu_2 v^3 + \beta u^2 v)) \geq 0
$$

for $u \geq 0$ and $v \geq 0$, it can be seen that $\mathcal{N}_+$ is an invariant set for the flow $\varphi_t$ according to [12] (see also [15]). With $\mathcal{N}_+$ being invariant for the flow $\varphi_t$, from
the definition of $c_1$ in (42) and the fact that $\max_{y \in \partial B_R(0)} I(h(y)) < c_1$, a standard argument leads to existence of $\{(u_n, v_n)\} \subset \mathcal{N}_+$ such that

$$I(u_n, v_n) \to c_1, \quad \nabla I|_{\mathcal{N}}(u_n, v_n) \to 0 \quad \text{in } \mathcal{H}.$$ 

Then $\nabla I(u_n, v_n) \to 0$ in $\mathcal{H}$. Let $(u^0, v^0)$, $(u^j, \nu^j)$ and $\{y_n\}$, $j = 1, \cdots, k$, be as in Theorem 2.1. We claim $k = 0$. Otherwise, since $c_1 < 2c_\infty$, $k = 1$ and there holds

$$c_1 = I(u^0, v^0) + I_\infty(u^1, v^1).$$

If $(u^0, v^0) \neq (0, 0)$, then $c_1 \geq c + c_\infty = 2c_\infty$ which is a contradiction. If $(u^0, v^0) = (0, 0)$, then, since $u_n \geq 0$ and $v_n \geq 0$, $u^1 \geq 0$ and $v^1 \geq 0$. According to [39], $(u^1, v^1)$ must be, up to translation, one of the three solutions $(\sqrt{k_1} w, \sqrt{k_2} w)$, $(\frac{1}{\sqrt{k_1}} w, 0)$, and $(0, \frac{1}{\sqrt{k_2}} w)$. Therefore, either $c_1 = 2c_\infty$, or

$$c_1 = \frac{1}{4\mu_1} \|w\|^2 > \frac{2\beta_\infty - \mu_1 - \mu_2}{2(\beta_\infty^2 - \mu_1 \mu_2)} \|w\|^2 = 2c_\infty,$$

or

$$c_1 = \frac{1}{4\mu_2} \|w\|^2 > \frac{2\beta_\infty - \mu_1 - \mu_2}{2(\beta_\infty^2 - \mu_1 \mu_2)} \|w\|^2 = 2c_\infty,$$

and in any case there is a contradiction. Thus $k = 0$ and $\|(u_n, v_n) - (u^0, v^0)\| \to 0$.

Now, $I(u^0, v^0) = c_1$, $I'(u^0, v^0) = 0$, and $(u^0, v^0)$ is a nonnegative nonzero solution of (3). Since $I(u^0, v^0) = c_1 < 2c_\infty < \gamma_j$, there must be that $u^0 \neq 0$ and $v^0 \neq 0$. Moreover, the maximum principle implies $(u^0, v^0)$ is a positive solution of (3).  

The discussion above in this section has in fact proved the following theorem which is a generalization of Theorem 1.5.

**Theorem 5.6.** Let $(H_1)$ be satisfied. Assume

$$(\beta_\infty - \mu_2)\mu_1(x) + 2(\beta_\infty - \mu_2)(\beta_\infty - \mu_1) \beta(x) + (\beta_\infty - \mu_1)\mu_2(x)$$

$\geq \frac{1}{2}(\beta_\infty^2 - \mu_1 \mu_2)(2\beta_\infty - \mu_1 - \mu_2) \max\{\|V_1\|_{\infty}^2, \|V_2\|_{\infty}^2\} = (44)$

for all $x \in \mathbb{R}^N$ and

$$\beta_\infty \geq 4 \left(\|\mu_1\|_{\infty}\|V_1^{-1}\|_{2^*-2}^{-\frac{2}{2^*}} + \|\mu_2\|_{\infty}\|V_2^{-1}\|_{2^*-2}^{-\frac{2}{2^*}}\right).$$

Then (3) has a positive solution.

Under $(H_1)$, (44) clearly implies $\max\{\|V_1\|_{\infty}^2, \|V_2\|_{\infty}^2\} \leq 2$. Moreover, if $(H_1)$ is satisfied and if $\max\{\|V_1\|_{\infty}^2, \|V_2\|_{\infty}^2\} < 2$ then (44) is automatically satisfied outside of a large ball in $\mathbb{R}^N$. Therefore, in the case where $\max\{\|V_1\|_{\infty}^2, \|V_2\|_{\infty}^2\} < 2$, (44) is only an assumption for $x$ in a large ball in $\mathbb{R}^N$.

6. **Proof of Theorem 1.6.** Consider

$$\tilde{c} = \inf_{(u,v) \in \mathcal{N}} I(u, v),$$

where $\mathcal{N}$ is the generalized Nehari manifold

$$\mathcal{N} = \{(u, v) \in \mathcal{H} \mid u \neq 0, v \neq 0, J_1(u, v) = 0, J_2(u, v) = 0\}$$

with two constraints, $J_1(u, v) = \langle I'(u, v), (u, 0) \rangle$ and $J_2(u, v) = \langle I'(u, v), (0, v) \rangle$. The following two lemmas from [26] will be used in the proof of Theorem 1.6.
Lemma 6.1. Under the assumptions of Theorem 1.6, there exists \((u, v) \in \mathcal{N}\) such that
\[
I(u, v) = \frac{S_1^2 S_2^2 (S_1^2 + S_2^2 - 2\eta^2)}{4(S_1^2 S_2^2 - \eta^4)}.
\]

Lemma 6.2. Under the assumptions of Theorem 1.6, for any \((u, v) \in \mathcal{N}\) with
\[
I(u, v) \leq \frac{S_1^2 S_2^2 (S_1^2 + S_2^2 - 2\eta^2)}{4(S_1^2 S_2^2 - \eta^4)},
\]
we have
\[
\int_{\mathbb{R}^N} \mu_1 u^4 \geq \alpha_1 := \left( \frac{S_1 S_2 (S_1^2 S_2^2 - \eta^4) - \xi S_1^2 S_2^2 (S_1^2 + S_2^2 - 2\eta^2)}{(S_1^2 S_2^2 - \eta^4)(S_2 - \xi S_1)} \right)^2 > 0,
\]
\[
\int_{\mathbb{R}^N} \mu_2 v^4 \geq \alpha_2 := \left( \frac{S_1 S_2 (S_1^2 S_2^2 - \eta^4) - \xi S_1^2 S_2^2 (S_1^2 + S_2^2 - 2\eta^2)}{(S_1^2 S_2^2 - \eta^4)(S_1 - \xi S_2)} \right)^2 > 0,
\]
and
\[
\int_{\mathbb{R}^N} \mu_1 u^4 \int_{\mathbb{R}^N} \mu_2 v^4 - \left( \int_{\mathbb{R}^N} \beta u^2 v^2 \right)^2 \geq (1 - \xi^2) \frac{[S_1 S_2 (S_1^2 S_2^2 - \eta^4) - \xi S_1^2 S_2^2 (S_1^2 + S_2^2 - 2\eta^2)]^4}{(S_1^2 S_2^2 - \eta^4)^4(S_1 - \xi S_2)^2(S_2 - \xi S_1)^2}.
\]

Recall the definition of \(\gamma_j\) in the proof of Theorem 1.5 and the definition of \(\gamma_{j\infty}\) before the statement of Lemma 4.2.

Lemma 6.3. Under the assumptions of Theorem 1.6,
\[
\tilde{c} < \gamma_1 + \gamma_2 \leq \min\{\gamma_1 + \gamma_{2\infty}, \gamma_{1\infty} + \gamma_2\} \leq \gamma_{1\infty} + \gamma_{2\infty}.
\]

Proof. The first inequality is proved in the same way as in the proof of Lemma 4.5 and the other two inequalities follow from the fact that \(\gamma_j \leq \gamma_{j\infty}\) for \(j = 1, 2\).

Lemma 6.4. Under the assumptions of Theorem 1.6, if at least one of the four functions \(V_j, \mu_j\) is not a constant, then \(\tilde{c} < \tilde{c}_{\infty}\).

Proof. The proof is similar to an argument in [26]. Since \(\beta_{\infty} < \min\{\mu_{1\infty}, \mu_{2\infty}\}\), \(\tilde{c}_{\infty} = I_{\infty}(v_1, v_2)\) where \(v_j = \sqrt{f_j} w\) for \(j = 1, 2\). Denote, for \(y \in \mathbb{R}^N\) and \(j = 1, 2\), \(v_{jy} = v_j (-y)\). Then, for \(|y|\) sufficiently large, the two numbers \(s_y\) and \(t_y\) defined by
\[
s_y = \frac{\|v_{1y}\|^2 \int_{\mathbb{R}^N} \mu_2 v_{2y}^2 - \|v_{2y}\|^2 \int_{\mathbb{R}^N} \mu_1 v_{1y}^2}{\int_{\mathbb{R}^N} \mu_1 v_{1y}^4 \int_{\mathbb{R}^N} \mu_2 v_{2y}^4 - \left( \int_{\mathbb{R}^N} \beta v_{1y}^2 v_{2y}^2 \right)^2},
\]
\[
t_y = \frac{\|v_{2y}\|^2 \int_{\mathbb{R}^N} \mu_1 v_{1y}^4 - \|v_{1y}\|^2 \int_{\mathbb{R}^N} \mu_2 v_{2y}^4}{\int_{\mathbb{R}^N} \mu_1 v_{1y}^4 \int_{\mathbb{R}^N} \mu_2 v_{2y}^4 - \left( \int_{\mathbb{R}^N} \beta v_{1y}^2 v_{2y}^2 \right)^2}
\]
are positive and \((\sqrt{s_y} v_{1y}, \sqrt{t_y} v_{2y}) \in \mathcal{N}\). Denote
\[
a_1 = \int_{\mathbb{R}^N} \mu_{1\infty} v_{1y}^4, \quad a_2 = \int_{\mathbb{R}^N} \beta_{\infty} v_{1y}^2 v_{2y}^2, \quad a_3 = \int_{\mathbb{R}^N} \mu_{2\infty} v_{2y}^4,
\]
\[
a_4 = \|v_{1y}\|^2, \quad a_5 = \|v_{2y}\|^2,
\]
\[
\chi_1(y) = \int_{\mathbb{R}^N} \mu_1 v_{1y}^4, \quad \chi_2(y) = \int_{\mathbb{R}^N} \beta v_{1y}^2 v_{2y}^2, \quad \chi_3(y) = \int_{\mathbb{R}^N} \mu_2 v_{2y}^4,
\]
\[
\chi_4(y) = \|v_{1y}\|^2, \quad \chi_5(y) = \|v_{2y}\|^2.
\]
and set
\[ \psi_1(y) = \chi_1(y) - a_1, \quad \psi_2(y) = \chi_2(y) - a_2, \quad \psi_3(y) = \chi_3(y) - a_3, \]
\[ \psi_4(y) = \chi_4(y) - a_4, \quad \psi_5(y) = \chi_5(y) - a_5. \]
Then it follows from (H1) and (H5) that
\[ \psi_1(y) \geq 0, \quad \psi_2(y) \geq 0, \quad \psi_3(y) \leq 0, \quad \psi_5(y) \leq 0 \] (46)
and, as \(|y| \to \infty, \)
\[ \psi_1(y) \to 0, \quad \psi_2(y) \to 0, \quad \psi_3(y) \to 0, \quad \psi_4(y) \to 0, \quad \psi_5(y) \to 0. \] (47)
The equation \( \tilde{c}_\infty = I_\infty(v_1, v_2) \) leads to the expression
\[ \tilde{c}_\infty = \frac{1}{4}(a_1 + 2a_2 + a_3) \]
and the relation \((\sqrt{s_0}v_{1y}, \sqrt{T_0}v_{2y}) \in \tilde{N}\) gives the estimate
\[ \tilde{c} \leq I(\sqrt{s_0}v_{1y}, \sqrt{T_0}v_{2y}) = \frac{\chi_1(1)\chi_3^2(y) + \chi_3(y)\chi_4^2(y) - 2\chi_2(y)\chi_4(y)\chi_5(y)}{4(\chi_1(y)\chi_3(y) - \chi_2^2(y))}, \]
which together imply
\[ \tilde{c}_\infty - \tilde{c} \geq \frac{A}{4(\chi_1(1)\chi_3(y) - \chi_2^2(y))}, \]
where \( A \) is a function of \( y \) and has the expression
\[
A = (a_1 + 2a_2 + a_3)((a_1 + \psi_1(y))(a_3 + \psi_3(y)) - (a_2 + \psi_2(y))^2)
\[ + 2(a_2 + \psi_2(y))(a_4 + \psi_4(y))(a_5 + \psi_5(y)) - (a_1 + \psi_1(y))(a_5 + \psi_5(y))^2
\[ - (a_3 + \psi_3(y))(a_4 + \psi_4(y))^2. \]
Thus to conclude it suffices to prove \( A > 0 \) for \(|y|\) large. Expanding \( A \) and using the fact that \( a_4 = a_1 + a_2, a_5 = a_2 + a_3 \) yield
\[
A = (a_1a_3 - a_2^2)(\psi_1(y) + 2\psi_2(y) + \psi_3(y) - 2\psi_4(y) - 2\psi_5(y))
\[ - (a_1 + 2a_2 + a_3)\psi_2^2(y) - a_1\psi_5^2(y) - a_3\psi_4^2(y) + (a_1 + 2a_2 + a_3)\psi_1(y)\psi_3(y)
\[ + 2a_2\psi_4(y)\psi_5(y) + 2a_4\psi_2(y)\psi_5(y) - 2a_4\psi_3(y)\psi_4(y) - 2a_5\psi_1(y)\psi_5(y)
\[ + 2a_5\psi_2(y)\psi_4(y) - \psi_1(y)\psi_5^2(y) - \psi_3(y)\psi_4^2(y) + 2\psi_2(y)\psi_4(y)\psi_5(y). \]
Let \( \nu \) be as in (H5) and choose a small \( \delta > 0 \) such that \((1 - \delta)(1 + \nu) > 1. \) Since \( \psi_2(y) \to 0 \) as \(|y| \to \infty, \) for \(|y|\) large
\[
\left(2 - \frac{a_1 + 2a_2 + a_3}{a_1a_3 - a_2^2}\psi_2(y)\right)\psi_2(y)
\[ \geq \left(2 - \frac{a_1 + 2a_2 + a_3}{a_1a_3 - a_2^2}\psi_2(y)\right) \int_{\mathbb{R}^N} (\beta - \beta_\infty)^{-v_{1y}^2v_{2y}^2}
\[ \geq 2(1 - \delta)(1 + \nu) \int_{\mathbb{R}^N} (\beta - \beta_\infty)^{-v_{1y}^2v_{2y}^2}
\[ \geq (1 - \delta)(1 + \nu) \int_{\mathbb{R}^N} (\beta - \beta_\infty)^{-(v_{1y}^4 + v_{2y}^4)}
\[ \geq -(1 - \delta)(\psi_1(y) + \psi_3(y)). \]
Combining this with (46) and (47) yields
\[
A \geq (a_1 a_3 - a_2^2) \times \left( \psi_1(y) + \left( 2 - \frac{a_1 + 2 a_2 + a_3}{a_1 a_3 - a_2^2} \psi_2(y) \right) \psi_2(y) + \psi_3(y) - \psi_4(y) - \psi_5(y) \right)
\geq (a_1 a_3 - a_2^2) (\delta \psi_1(y) + \delta \psi_3(y) - \psi_4(y) - \psi_5(y)) > 0
\]
for \(|y|\) large, which implies \(\hat{c} < \hat{c}_\infty\).

**Proof of Theorem 1.6.** If the four functions \(V_j, \mu_j\) are all constants, then \((H_5)\) implies that \(\beta(x) \geq \beta_\infty\) for all \(x \in \mathbb{R}^N\). Thus (3) has a positive ground state according to [26, Theorem 1.1]. In what follows we always assume that at least one of the four functions \(V_j, \mu_j\) is not a constant. The argument in the proof of Theorem 1.3 can be used to find a sequence \(\{(u_n, v_n)\} \subset \tilde{N}\) with \(u_n \geq 0, v_n \geq 0\) such that
\[
I(u_n, v_n) \to \hat{c}, \quad I'(u_n, v_n) \to 0 \quad \text{in} \quad \mathcal{H}^*.
\]
Then, by Theorem 2.1, replacing \(\{(u_n, v_n)\}\) by a subsequence if necessary, there exist a nonnegative integer \(k\), a solution \((u^0, v^0)\) with \(u^0 \geq 0\) and \(v^0 \geq 0\) of (3), nonzero solutions \((u^1, v^1), \ldots, (u^k, v^k)\) of the limit system (11) and \(k\) sequences \(\{y_n^j\} \subset \mathbb{R}^N\) such that, as \(n \to \infty\),
\[
|y_n^j| \to \infty, \quad |y_n^j - y_n^{j'}| \to \infty, \quad j \neq j',
\]
\[
\left\| u_n - u^0 - \sum_{j=1}^k u^j (\cdot - y_n^j) \right\| \to 0, \quad \left\| v_n - v^0 - \sum_{j=1}^k v^j (\cdot - y_n^j) \right\| \to 0
\]
and
\[
I(u^0, v^0) + \sum_{j=1}^k I_\infty (u^j, v^j) = \hat{c}.
\]
If \(k = 0\) then \(\|u_n - u^0\| \to 0\) and \(\|v_n - v^0\| \to 0\), and as a consequence of Lemma 6.2,
\[
\int_{\mathbb{R}^N} \mu_1 (u^0)^4 \geq \alpha_1 > 0, \quad \int_{\mathbb{R}^N} \mu_2 (v^0)^4 \geq \alpha_2 > 0.
\]
Then the maximum principle implies that \((u^0, v^0)\) is a positive solution of (3) and \(I(u^0, v^0) = \hat{c}\). Assume by contradiction that \(k \geq 1\). Then there are four cases.

**Case 1.** \(u^j = 0\) for all \(j = 1, 2, \ldots, k\) and \(u^l \neq 0\) for some \(l \geq 1\). In this case \(\|u_n - u^0\| \to 0\) and \(u^0 \neq 0\). If \(v^0 \neq 0\) then
\[
\hat{c} \geq I(u^0, v^0) + I_\infty (0, v^l) > \hat{c},
\]
which is impossible. If \(v^0 = 0\) then
\[
\hat{c} \geq I(u^0, 0) + I_\infty (0, v^l) \geq \gamma_1 + \gamma_2 \infty,
\]
which contradicts the result of Lemma 6.3.

**Case 2.** \(v^j = 0\) for all \(j = 1, 2, \ldots, k\) and \(u^l \neq 0\) for some \(l \geq 1\). The argument is the same as in Case 1.

**Case 3.** \(u^l \neq 0\) and \(v^{l'} \neq 0\) for some \(l_1 \geq 1\) and \(l_2 \geq 1\), but there is no \(l \geq 1\) such that \(u^l \neq 0\) and \(v^l \neq 0\). Then
\[
\hat{c} \geq I_\infty (u^l, 0) + I_\infty (0, v^{l'}) \geq \gamma_1 \infty + \gamma_2 \infty,
\]
which is a contradiction to the result of Lemma 6.3.

**Case 4.** \( u^l \neq 0 \) and \( v^l \neq 0 \) for some \( l \geq 1 \). Then we have
\[
\tilde{c} \geq I_\infty(u^l, v^l) \geq \tilde{c}_\infty,
\]
which is a contradiction to the result of Lemma 6.4.

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