FINSLER SPACES OF CONSTANT FLAG CURVATURE AND THEIR PROJECTIVE GEOMETRY

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Abstract. In Theorem 3.1, a Finslerian extension of Schur’s Lemma that includes dimension 2 as well, we provide three necessary and sufficient conditions (CC-conditions) for a Finsler space to be of constant flag curvature. Depending on the dimension of the manifold, one of these three CC-conditions is automatically satisfied.

First and third CC-conditions are projectively invariant and they have Riemannian correspondents. Third CC-condition is given by a projectively invariant 2-form that plays the role of Cotton (Liouville) tensor.

The second CC-condition is purely Finslerian and it is projectively invariant if and only if the projective factor is a Hamel function. This condition restricts the validity of the Beltrami Theorem in the Finslerian context. Theorem 3.3 represents the Finslerian version of Beltrami’s Theorem that includes dimension 2 and extends a previous version valid in dimension greater than 2, [3, Theorem 5.2].

1. Introduction

Flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry. While in Riemannian geometry, metrics of constant curvature are well understood and classified, in Finsler geometry the problem is far from being solved, [9,14]. In Finsler geometry, there are many characterisations for metrics of constant flag curvature, [1,2,3]. In this paper, in Theorem 3.2, we provide a new characterisation for Finsler spaces of constant curvature in terms of three conditions, which we call CC-conditions. There are two motivations for this formulation of the three CC-conditions: (3.6), (3.7), (3.8). First, we can view Theorem 3.2 as a Finslerian extension of Schur’s Lemma that includes dimension 2 as well. Secondly, it is easy to check the projective invariance of the three CC-conditions, to determine under what additional requirements these conditions are invariant and to decide how to formulate a Finslerian version of Beltrami Theorem.

In Theorem 3.2, one of the CC-conditions is always satisfied, depending on the dimension of the manifold. In the 2-dimensional case, the first CC-condition (the isotropy condition) is automatically satisfied, while in dimension grater than 2, the third CC-condition is satisfied due to differential Bianchi identities. The first and third CC-conditions are invariant under projective deformations and they have corresponding quantities in the Riemannian context. The second CC-condition is not a projective invariant unless the projective factor is a Hamel function. Therefore, the second CC-condition gives, for a Finsler metric of constant curvature, the class of projectively related Finsler metrics that also have constant curvature.

Beltrami’s Theorem states that a Riemannian metric projectively equivalent to a Riemannian metric of constant curvature has constant curvature. A classic proof of Beltrami’s Theorem uses
the projective Weyl tensor if $n \geq 3$ and the Liouville (Cotton) tensor if $n = 2$, see [12] for a recent survey on the projective geometry of affine sprays.

There are two important aspects to be considered for appropriate generalisations of Beltrami’s Theorem from Riemann to Finsler spaces. On the one hand the generalisation should be a result about Finsler spaces of constant curvature, and this is the line we followed in this work and [3]. On the other hand the generalisation should be a result about projectively equivalent spray spaces, as it has been done by Crampin in [5, 6].

In this work we prove the following Finslerian version of Beltrami’s Theorem, for dimension $n \geq 2$, see Theorem 3.3.

*If two Finsler metrics over the same base manifold have projectively equivalent geodesic sprays and one is of constant flag curvature then the other is also of constant curvature if and only if the projective factor is a Hamel function.*

The proof we provide uses the projective transformations of the three CC-conditions proposed in Theorem 3.2, first two conditions for $n \geq 3$ and the last two conditions for $n = 2$. The projective invariant 2-form from the third CC-condition (3.8) can be viewed as the Cotton (Liouville) tensor of the spray. The Cotton tensor has been introduced very recently by Crampin in the geometry of a spray, [6], as an obstruction to the $R$-flatness of a 2-dimensional spray.

In dimension $n \geq 3$, the Finslerian version of Beltrami Theorem, [3, Theorem 5.2], has been proved using a Weyl-type curvature tensor that characterise Finsler metrics of constant curvature.

### 2. Isotropic sprays

We start this section with a brief introduction to the geometric setting associated to a spray (nonlinear connection and curvature tensors). Then, we focus on isotropic sprays and show, in Lemma 2.3, that such sprays can be characterised in terms of a semi-basic 1-form that we call the curvature 1-form. In Lemma 2.4 we reformulate the differential Bianchi identities in terms of the curvature 1-form.

We consider $M$ a smooth, $n$-dimensional, connected manifold, with $n \geq 2$. $TM$ is the tangent bundle and $T_0M = TM - \{0\}$ the tangent bundle with the zero section removed. Local coordinates on $M$ are denoted by $(x^i)$, while induced local coordinates on $TM$ are denoted by $(x^i, y^i)$ for $i \in \{1, 2, \cdots, n\}$. On $TM$ there are two canonical structures that we will use further, the Liouville vector field and the tangent endomorphism, whose expressions in local coordinates are given by

$$C = y^i \frac{\partial}{\partial y^i}, \ J = dx^i \otimes \frac{\partial}{\partial y^i}$$

A system of second order ordinary differential equations on $M$, \begin{equation}
\frac{d^2 x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0
\end{equation}
can be identified with a special vector field on $TM$

\begin{equation}
S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i},
\end{equation}

that is called a semispray. If additionally, $S \in \mathfrak{X}(T_0M)$ and satisfies the homogeneity condition $[C, S] = S$ we say that $S$ is a spray. In this work we deal with sprays only.

A curve $c : I \subset \mathbb{R} \to M$, solution of the system (2.1), is called a geodesic of the corresponding spray (2.2).

An orientation preserving reparametrisation $t \to \tilde{t}(t)$ of the system (2.1) leads to a new spray $\tilde{S} = S - 2PC$, where $P \in C^\infty(T_0M)$ is a positively 1-homogeneous scalar function.
Definition 2.1. Two sprays $S$ and $\tilde{S}$ are projectively related if their geodesics coincide up to an orientation preserving reparametrisation.

We will refer to the map $S \rightarrow \tilde{S} = S - 2PC$, for $P \in C^\infty(T_0M)$ a positively 1-homogeneous function, as to a projective deformation of the spray $S$.

In this work we will use the Frölicher-Nijenhuis formalism to associate a geometric framework to any given spray, [7, 8]. First geometric structure associated to a spray is the canonical nonlinear connection, that determines a horizontal and a vertical projector:

$$h = \frac{1}{2}(\text{Id} - [S, J]), \quad v = \frac{1}{2}(\text{Id} + [S, J]).$$

Locally, the two projectors $h$ and $v$ can be expressed as follows

$$h = \frac{\delta}{\delta x^i} \otimes dx^i, \quad v = \frac{\partial}{\partial y^i} \otimes \delta y^i,$$

$$\delta = \frac{\partial}{\partial x^i} - N^j_i(x, y) \frac{\partial}{\partial y^j}, \quad \delta y^i = dy^i + N^j_i(x, y)dx^j, \quad N^j_i(x, y) = \frac{\partial G^i_j}{\partial y^j}(x, y).$$

For a spray $S$ consider the vector valued semi-basic 1-form

$$(2.3) \quad \Phi = v \circ [S, h] = R^i_j(x, y)\frac{\partial}{\partial y^j} \otimes dx^i, \quad R^i_j = 2\frac{\delta G^i_j}{\delta x^j} - S \left( \frac{\partial G^i_j}{\partial y^j} \right) + \frac{\partial G^i_j}{\partial y^i} \frac{\partial G^k_j}{\partial y^k},$$

which will be called the Jacobi endomorphism. Another important geometric structure induced by a spray $S$ is the curvature tensor:

$$(2.4) \quad R = \frac{1}{2}[h, h].$$

The curvature tensors $\Phi$ and $R$ are related by:

$$(2.5) \quad 3R = [J, \Phi], \quad \Phi = i_S R.$$ 

The Ricci scalar $\rho \in C^\infty(T_0M)$ given by:

$$(2.6) \quad \rho = \frac{1}{n-1} R^i_i = \frac{1}{n-1} \text{Tr}(\Phi),$$

is another important quantity that we will use in this work.

Definition 2.2. A spray $S$ is said to be isotropic if there exists a semi-basic 1-form $\alpha \in \Lambda^1(T_0M)$ such that the Jacobi endomorphism can be written as follows:

$$(2.7) \quad \Phi = \rho J - \alpha \otimes C.$$

We mention that, for an isotropic spray $S$, due to the second formula (2.5), we have that $\Phi(S) = i^S_S R = 0$ and hence $\rho = i_S \alpha$.

The two conditions (2.5) allow to reformulate the isotropy condition (2.7) in terms of the curvature tensor $R$.

Lemma 2.3. A spray $S$ is isotropic if and only if there exists a semi-basic 1-form $\xi \in \Lambda^1(T_0M)$ such that its curvature tensor $R$ is given by:

$$(2.8) \quad R = \xi \wedge J - d(j \xi) \otimes C.$$
Proof. Assume that the spray $S$ is isotropic and hence its Jacobi endomorphism $\Phi$ is given by formula (2.7). Using first formula (2.5), the curvature tensor $R$ will have the form:

$$3R = [J, \Phi] = [J, \rho J - \alpha \otimes C] = [J, \rho J] - [J, \alpha \otimes C]$$

(2.9)

$$= \rho [J, J] + d_J \rho \wedge J - (d\rho) J \wedge J - d_J \alpha \otimes C + d\alpha \otimes JC + \alpha \wedge [J, C]$$

$$= d_J \rho \wedge J - d_J \alpha \otimes C + \alpha \wedge J = (d_J \rho + \alpha) \wedge J - d_J \alpha \otimes C,$$

which gives formula (2.8), for the semi-basic 1-form

$$\xi := \frac{1}{3} (\alpha + d_J \rho).$$

(2.10)

Conversely, we will assume that the curvature tensor $R$ is given by formula (2.8). Using the second formula (2.5) we can recover the Jacobi endomorphism as follows:

$$\Phi = i_S R = i_S (\xi \wedge J - d_J \xi \otimes C) = i_S \xi J - (i_S d_J \xi + \xi) \otimes C.$$

By taking the trace of this endomorphism we obtain $\text{Tr}(\Phi) = (n - 1) i_S \xi$ and hence the Ricci scalar is given by $\rho = i_S \xi$. With this, the Jacobi endomorphism satisfies (2.7) and the spray $S$ is isotropic. $\square$

The characterisation (2.8) for isotropic spray has been used by Crampin in [5, 6]. We will call the semi-basic 1-form $\xi$ in formula (2.8) the curvature 1-form of the isotropic spray.

Lemma 2.4 (Differential Bianchi identities). In dimension $n \geq 3$, the curvature 1-form of an isotropic spray satisfies $d h \xi = 0$.

Proof. Consider $S$ a spray with the curvature tensor given by (2.4). Using the Jacobi identity for the vector-valued 1-form $h$ we obtain the differential Bianchi identity

$$[h, R] = \frac{1}{2} [h, [h, h]] = 0.$$

We assume now that $S$ is an isotropic spray, with curvature tensor given by formula (2.8). We will use that the tension of the spray vanishes, $[h, J] = 0$, and the fact that the connection is 1-homogeneous, $[h, C] = 0$, see [7], to write the Bianchi identity as follows:

$$0 = [h, R] = [h, \xi \wedge J - d_J \xi \otimes C] = dh \xi \wedge J - dh d_J \xi \otimes C.$$

In the above formula we take the trace of the vector valued 3-forms and obtain

(2.11)

$$(n - 2) dh \xi - i_S dh d_J \xi = 0.$$

In formula (2.11) we apply the inner product $i_S$ and obtain $(n - 2)i_S dh \xi = 0$. Since $n \geq 3$ it follows that $i_S dh \xi = 0$. In formula (2.11) we evaluate $i_S dh d_J \xi$. We have

$$-i_S dh d_J \xi = i_S d_J dh \xi = -d_J i_S dh \xi + L_J i_S dh \xi + i_{[J, S]} dh \xi = L_C dh \xi + d_J - 2v dh \xi = dh \xi + 2 dh \xi = 3 dh \xi.$$

Last calculations, together with formula (2.11) imply that $dh \xi = 0$. $\square$

The semi-basic 2-form $dh \xi$ appears in [3] as an obstruction for an isotropic spray to be $R$-flat. In his paper [5], Crampin shows, using local coordinates, that in dimension $n \geq 3$, this obstruction always vanishes, while in [6] he shows that a 2-dimensional spray is projectively $R$-flat if and only if $dh \xi = 0$. 


3. Finsler spaces of constant flag curvature and their projective deformation

In this section we provide three necessary and sufficient conditions (CC-conditions: (3.6), (3.7), (3.8)) for a Finsler metric to have constant curvature, see Theorem 3.2. By studying the projective invariance of these CC-conditions we provide a Finslerian version of Beltrami Theorem for dimension $n \geq 2$ (Theorem 3.3).

**Definition 3.1.** By a Finsler metric we mean a continuous function $F : TM \to \mathbb{R}$ satisfying the following conditions

1. $F$ is smooth and strictly positive on $T_0M$;
2. $F$ is positively homogeneous of order 1, which means that $F(x, \lambda y) = \lambda F(x, y)$, for all $\lambda > 0$ and $(x, y) \in TM$;
3. The metric tensor with components
   $g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F}{\partial y^i \partial y^j}$
   has rank $n$ on $TM$.

Due to the regularity condition (3), the equation

$$i_Sdd_JF^2 = -dF^2$$

uniquely determines a vector field $S$ on $T_0M$, which is a spray and it is called the geodesic spray of the Finsler metric. The geometric setting for a Finsler metric is induced by its geodesic spray.

A Finsler metric has scalar flag curvature if there exists a 0-homogeneous function $\kappa \in C^\infty(T_0M)$, such that the Jacobi endomorphism can be expressed as follows

$$\Phi = \kappa(F^2 J - Fd_JF \otimes C).$$

When the scalar flag curvature $\kappa$ is constant, we say that the Finsler metric has constant curvature. For a Finsler metric of scalar flag curvature, its geodesic spray is isotropic, the Ricci scalar $\rho$ and the semi-basic 1-form $\alpha$ being given by:

$$\rho = \kappa F^2, \quad \alpha = \kappa F d_JF.$$

In view of Lemma 2.3, a Finsler metric has scalar flag curvature if and only if its curvature tensor takes the form

$$R = \frac{1}{3F}d_J(\kappa F^3) \wedge J - d_J \left( \frac{1}{3F} d_J(\kappa F^3) \right) \otimes C.$$

Using the formulae (2.10) and (3.2), the curvature 1-form $\xi$ is given by

$$\xi = \frac{1}{3F} d_J(\kappa F^3).$$

If the Finsler metric $F$ has constant curvature then its curvature tensor is given by

$$R = \kappa F d_JF \wedge J.$$

**Theorem 3.2** (Finslerian version of Schur’s Lemma for $n \geq 2$). Consider $S$ the geodesic spray of a Finsler metric $F$. Then $F$ has constant curvature if and only if:

1. $S$ is isotropic (this condition is always true for $n=2$);
2. the curvature 1-form satisfies:
   1. $d_J\xi = 0$;
   2. $d_B\xi = 0$ (this condition is always true for $n \geq 3$).
Proof. We assume that $F$ is a Finsler metric of constant curvature, which means that its curvature tensor is given by formula (3.3), which reduces to formula (3.5), when $\kappa$ is a constant. Therefore, its geodesic spray is isotropic and the curvature 1-form is given by

$$\xi = \kappa F dJ F.$$ (3.9)

It follows that $d_J \xi = 0$ and since $d_h F = 0$, the last CC-condition $d_h \xi = 0$ is also satisfied.

For the converse implication, we assume that $S$ is the geodesic spray of a Finsler metric that satisfies the three CC-conditions. First CC-condition tells us that the Finsler metric has isotropic geodesic sprays and, according to [13, Lemma 8.2.2], it has scalar flag curvature $\kappa$. It follows that the curvature 1-form is given by formula (3.4), which can be further written as

$$\xi = F^2 \frac{\kappa}{3} dJ F + \kappa F dJ F.$$ (3.10)

In order to show that $\kappa$ is constant we will prove that $d_J \kappa = 0$ and $d_h \kappa = 0$.

The second CC-condition (3.7) implies

$$d_J \kappa \wedge d_J F = 0.$$ (3.11)

If we apply $i_S$ to both sides of this formula we get

$$S(\kappa) d_J F - F d_J S(\kappa) = 0.$$ (3.12)

Since $d_J \kappa = 0$ it follows that $d_h \kappa = 0$ and using the commutation rule for the derivations $L_S$ and $d_J$ we have

$$d_J S(\kappa) = L_S d_J \kappa + d_{[J,S]} \kappa = d_h \kappa.$$ (3.13)

In conclusion, we can rewrite (3.12) in the following way

$$S(\kappa) d_J F - F d_J S(\kappa) = 0 \iff S(\kappa) \frac{d_J F}{F^2} \cdot \frac{1}{F} d_J S(\kappa) = 0 \iff d_J \left( \frac{S(\kappa)}{F} \right) = 0.$$ (3.14)

Last formula implies $S(\kappa) = 0$ and, going back to formula (3.12), we obtain $d_h \kappa = 0$, which assures that $\kappa$ is constant and therefore the Finsler metric has constant curvature.

According to [13, Lemma 8.1.10], in the 2-dimensional case, sprays are always isotropic, and therefore the first CC-condition is automatically satisfied.

According to Lemma 2.4, in dimension $n \geq 3$, the curvature 1-form of an isotropic spray automatically satisfies the third CC-condition. □

The first two CC-conditions, (3.6) and (3.7), provide and equivalent characterisation for Finsler metrics of isotropic curvature (scalar curvature does not depend on the fiber coordinates) that were studied in [10]. These conditions were used to define a Weyl-type curvature tensor in [3, (4.1)] that characterises Finsler metrics of constant curvature in dimension $n \geq 3$.

When $n \geq 3$, there are various proofs of Theorem 3.2 in Finsler geometry, [11, Proposition 26.1], [15, Theorem 9.4.11].
Next, we study the projective invariance of the three CC-conditions: (3.10), (3.11) and (3.12). While the isotropy condition is known to be invariant, we prove that the third CC-condition is also invariant. We will show that the second CC-condition is invariant only for those projective deformations $P$ satisfying the Hamel equation
\begin{equation}
\tilde{h}_b d_J P = 0.
\end{equation}
A positive, 1-homogeneous function $P$ on $T_0 M$, which satisfies the equation (3.13) is called a Hamel function.

**Theorem 3.3** (Finslerian version of Beltrami’s Theorem for $n \geq 2$). If two Finsler metrics over the same base manifold have projectively equivalent geodesic sprays and one is of constant flag curvature then the other is also of constant curvature if and only if the projective factor is a Hamel function.

**Proof.** Consider $S$ and $\tilde{S}$ the geodesic sprays of the two projectively related Finsler metrics, which means that $\tilde{S} = S - 2PC$, for $P \in C^\infty(T_0 M)$, positively 1-homogeneous.

The isotropy condition is invariant under projective deformations and therefore both $S$ and $\tilde{S}$ are isotropic. According to [4, Proposition 4.4], the corresponding curvature 1-forms $\xi$ and $\tilde{\xi}$ are related by
\begin{equation}
\tilde{\xi} = \xi + Pd_J P - d_h P.
\end{equation}
If we apply $d_J$ to both sides of formula (3.14), we obtain:
\begin{equation}
d_J \tilde{\xi} = d_J \xi - d_J d_h P.
\end{equation}
In order to study the projective invariance of the third CC-condition, we use that the horizontal projectors are related by $\tilde{h} = h - PJ - d_J P \otimes C$, see [4, (4.8)], and hence
\begin{equation}
\begin{split}
d_k \tilde{\xi} &= d_k (\xi + Pd_J P - d_h P) = d_k \xi + d_k Pd_J P - d_k d_h P \\
&= d_h - PJ - d_J P \otimes C \xi + d_h - PJ - d_J P \otimes C Pd_J P - d_h - PJ - d_J P \otimes C d_h P \\
&\quad = d_h \xi - Pd_J \xi - d_J P \wedge \xi + d_h P \wedge d_J P + Pd_h d_J P - d_h P + Pd_J d_h P + d_J P \wedge d_h P \\
&= d_h \xi - Pd_J \xi + \xi \wedge d_J P - d_h P + Pd_J d_h P + Pd_h d_J P \\
&= d_h \xi - Pd_J \xi + \xi \wedge d_J P - d_h P.
\end{split}
\end{equation}
Now, using the form (2.8) of the curvature tensor $R$ we have
\begin{equation}
d_R P = d_\xi \wedge J - d_J \xi \otimes C P = \xi \wedge d_J P - P d_J \xi,
\end{equation}
which together with (3.16) allows us to conclude that
\begin{equation}
d_k \tilde{\xi} = d_k \xi.
\end{equation}
According to Theorem 3.2, the Finsler metric $F$ has constant curvature if and only if the three CC-conditions are satisfied: $S$ is isotropic, $d_J \xi = 0$ and $d_h \xi = 0$. Similarly, the projectively related Finsler metric $\tilde{F}$ has constant curvature if and only if it satisfies the corresponding three CC-conditions. In view of formulae (3.15) and (3.17) this is true if and only if $d_h d_J P = 0$, which means that $P$ is a Hamel function.

Formula (3.17) shows that the curvature 2-form $d_h \xi$ is a projective invariant of isotropic sprays. According to Lemma 2.4 this curvature 2-form vanishes in dimension $n \geq 3$ and therefore it is a useful projective invariant for 2-dimensional sprays, which are always isotropic. This corresponds to the Liouville (Cotton) projective invariant for affine sprays in 2-dimensional Riemannian geometry,
In Finsler geometry, Crampin [6] defines a projective Cotton tensor for arbitrary sprays, which reduces to the curvature 2-form $dh\xi$ for isotropic sprays.

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References

[1] Akbar-Zadeh, H.: Initiation to global Finslerian geometry, Elsevier, North-Holland Mathematical Library, 2006.
[2] Berwald, L.: Über die n-dimensionalen Geometrien konstanter Krümmung, in denen die Geraden die Kürzesten sind, Math. Z. 30 (1929), 449–469.
[3] Bucataru I., Creţu G.: A characterisation for Finsler metrics of constant curvature and a Finslerian version of Beltrami theorem, to appear in Journal of Geometric Analysis, DOI: 10.1007/s12220-019-00158-7, arXiv:1808.05001v2.
[4] Bucataru I., Muzsnay Z.: Projective and Finsler metrizability: parameterization-rigidity of geodesics, Int. J. Math., 23 (6) (2012), 1250099.
[5] Crampin M.: Isotropic and R-flat sprays, Houston Journal of Mathematics, 33 (2) (2007), 451–459.
[6] Crampin, M.: The Cotton tensor in the projective geometry of sprays, Publ. Math. Debrecen, 94 (1–2) (2019), 14.
[7] Grifone, J.: Structure presque-tangente et connexions I, Ann. Inst. Fourier, 22 (1972), 287–334.
[8] Grifone J., Muzsnay Z.: Variational Principles For Second-Order Differential Equations, World Scientific, 2000.
[9] Li, B.: On the classification of projectively flat Finsler metrics with constant flag curvature, Advances in Mathematics, 257 (2014), 266–288.
[10] Li, B., Shen, Z.: Sprays of isotropic curvature, Int. J. Math., 29 (1) (2018), 1850003.
[11] Matsumoto, M.: Foundations of Finsler geometry and special Finsler spaces, Kaiseisha Press, 1986.
[12] Matveev V.: Projectively Invariant Objects and the Index of the Group of Affine Transformations in the Group of Projective Transformations, Bull. Iran. Math. Soc. 44 (2018), 341-375.
[13] Shen Z.: Differential geometry of spray and Finsler spaces, Springer, 2001
[14] Shen Z.: Projectively flat Finsler metrics of constant flag curvature, Trans. of the American Mathematical Society, 355 (2002), 1713–1728.
[15] Szilasi, J., Lovas, R., Kertész, D.: Connections, sprays and Finsler structures, World Scientific, 2014.

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