SINUSOIDAL FREQUENCY ESTIMATION BY GRADIENT DESCENT

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ABSTRACT
Sinusoidal parameter estimation is a fundamental task in applications from spectral analysis to time-series forecasting. Estimating the sinusoidal frequency parameter by gradient descent is, however, often impossible as the error function is non-convex and densely populated with local minima. The growing family of differentiable signal processing methods has therefore been unable to tune the frequency of oscillatory components, preventing their use in a broad range of applications. This work presents a technique for joint sinusoidal frequency and amplitude estimation using the Wirtinger derivatives of a complex exponential surrogate and any first order gradient-based optimizer, enabling end-to-end training of neural network controllers for unconstrained sinusoidal models.

Index Terms— differentiable signal processing, machine learning, sinusoidal parameter estimation

1. INTRODUCTION
Estimating sinusoidal parameters from a signal is a crucial step in numerous signal processing algorithms, and a wealth of techniques have been proposed in both the single and multiple sinusoid formulations. Most seek the maximum likelihood (ML) estimate of sinusoidal model parameters in the presence of white Gaussian noise, the statistical properties of which are well established [1].

Estimators of sinusoidal frequency must circumvent the non-linearity of the model and non-convexity of the corresponding objective as a function of the frequency parameter. The most common approach is thus to apply a multi-stage algorithm in which an initial frequency estimate is obtained through search heuristics [2, 3, 4], spectral peak interpolation [5], discrete-time Fourier transform (DTFT) decorrelation [6], or other procedures [7, 8, 9], and then refined using an optimization method. Alternate approaches include iteratively updating a model-based relaxation of the problem [10, 8], linearizing the problem using delay operators [11, 12], or defining a surrogate model where an equivalence can be drawn between solutions [13].

Such methods achieve accurate estimates but are unsuitable for use in the context of end-to-end models fit by gradient descent, where integrating derivative-free operations or complex heuristics is challenging and often unstable. In particular, the recent proliferation of models applying differentiable digital signal processing (DDSP) [14] – a family of techniques which allow neural networks to directly control digital signal processors – highlights the need for a method for sinusoidal frequency estimation by gradient descent.

Applications of DDSP have included providing high level controls for harmonic-plus-noise synthesizers [14], controlling digital synthesis methods with neural networks [15, 16], modelling [17] and controlling [18] audio effects and direct filter design [19]. Yet, despite success at these complex tasks, DDSP-based models have so far been unable to predict sinusoidal frequency parameters. Aspects of the problem have been acknowledged in the literature. Turian & Henry [20] showed that frequency domain distances lack a stable and informative frequency gradient, whilst Engel et al. [21] used a parameter regression pretraining scheme to circumvent issues with local minima when optimizing sinusoidal frequencies. Caspe et al. [16] similarly note that gradient descent fails to tune the modulations frequencies of a differentiable FM synthesizer due to ripple in the error function.

In this work, we propose a simple surrogate to the sinusoidal oscillator with gradients that allow first-order gradient based optimization. With this approach, we take a first step towards end-to-end learning of neural network controllers for a broader family of differentiable audio synthesizers and signal processors.

2. SINUSOIDAL FREQUENCY ESTIMATION
We are concerned with modelling the class of discrete-time signals that can be expressed as:

\[ x_n = v_0 + \sum_{k \in K} a_k \cos(\omega_k n + \phi_k), \] (1)

where \( v_0 \sim \mathcal{N}(0, \sigma^2) \), and \( a_k, \omega_k, \phi_k \) are the amplitude, frequency, and phase parameters, respectively, of unordered sinusoidal components with index set \( K \subseteq \mathbb{N} \). Following the standard ML derivations, finding estimates \( \tilde{a}_k, \tilde{\omega}_k, \tilde{\phi}_k \) is equivalent to minimizing the mean squared error of the model. In many applications of machine learning to audio, we are concerned with other formulations of the error. These can be accounted for by expressing the likelihood in terms of other signal representations, such as the discrete Fourier transform (DFT).

It is well established that when \( \omega_k \) and \( \phi_k \) are known, this problem is linear in \( a_k \) [22] – a property which, for example, allows DDSP models to directly predict harmonic amplitudes [14]. When \( \phi \) is unknown, an optimal estimate can be found by evaluating the DFT at the known frequencies \( \omega_k \). In the case where fre-
Our proposed technique circumvents these issues by defining a surrogate for a differentiable sinusoidal model. The surrogate produces an exponentially decaying sinusoid as the real part of an exponentiated complex number:

\[ s_u(z_k) \triangleq \Re \left( z_k^N \right) = |z_k|^N \cos N \angle z_k \]  

where \( Z = \{ z_k \in \mathbb{C} \mid k \in K \} \) is a set of specific surrogate parameters with index set \( K \), and \( \angle z \) denotes the argument of \( z \). As the surrogate maps \( s_u : \mathbb{C} \rightarrow \mathbb{R} \), it does not have a complex derivative. However, its partial derivatives can be computed using Wirtinger’s calculus. A detailed explanation of these operators is beyond the scope of this paper, but we refer the interested reader to the work of Kreuz-Delgado [23] for an introduction.

For present purposes, the conjugate Wirtinger derivative of the surrogate is:

\[ \frac{\partial}{\partial z} s_u (z) = \frac{1}{2} \left( \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} \right) s_u (z) = \frac{n}{2} z^{n-1} \]  

for \( z = x + jy \). Where \( L \) is the loss between a signal produced by \( s \) and a target, \( -\frac{\partial L}{\partial \theta} \) is then the direction of steepest descent [23].

Unlike the frequency parameter of a real sinusoid, the surrogate parameter allows both the frequency and amplitude decay of the signal to be varied. As illustrated in Fig. 2, an optimizer can thus move the parameter inside the unit circle, creating an exponential amplitude decay, before moving it back out at the correct angle from the real line.

The frequency estimate represented by the minimum is given by the complex argument of the surrogate parameter \( \alpha_k \), where \( Z^* = \{ z_k \in \mathbb{C} \mid k \in K \} \) minimizes the loss:

\[ Z^* = \arg \min_{z \in Z} \sum_{n=1}^{N} \left( x_n - \sum_{k \in K} s_u (z_k) \right)^2 \]  

It follows that when target amplitudes \( \alpha_k = 1 \), this gives \( |z_k^*| = 1 \) and \( \angle z_k^* = \omega_k \), which is equivalent to the ML estimate for frequency. In other cases, a linear amplitude parameter \( \alpha_k \) can be introduced to the surrogate, i.e. \( \alpha_k s_u (z_k) \). In this way, \( \omega_k \) can give the ML estimate when \( \alpha_k \neq 1 \) and \( \alpha_k \) is known, by simply setting \( \alpha_k = \alpha_k \). In the case where target amplitudes are unknown, \( \alpha_k \) can be learned jointly with surrogate parameters.

2.1.1. Amplitude estimation

Even when an amplitude coefficient is jointly learned, the complex exponential surrogate and the standard sinusoid may differ in the evolution of their amplitudes across time. However, assuming \( z_k^* \) minimizes the least squares objective, we can recover an amplitude estimate by solving the opposite least squares problem. Specifically, to recover amplitudes from a surrogate model consisting of \( |K| \) components \( z_k \), we define \( U \in \mathbb{R}^{N \times K} \) where \( u_{nk} = \cos \angle z_k n \) and \( v \in \mathbb{R}^N \) where \( v = \sum_{k=1}^{K} s_u (z_k) \). For some linear signal representation \( h : \mathbb{R}^N \rightarrow \mathbb{R}^M \), such as the

\[ \sum_{n=1}^{N} (x_n - \hat{x}_n)^2 \text{ (MSE)} \]

\[ \sum_{n=1}^{N} |x_n - \hat{x}_n| \text{ (MAE)} \]

\[ \sum_{n=1}^{N} \left( |X_k| - |\hat{X}_k| \right)^2 \text{ (DFT-MSE)} \]
A single starting parameter estimate was uniformly sampled from synthesized using the real valued sinusoidal model in Eqn 1. A dB, for a total of 2000 targets. The corresponding signals were the signal-to-noise ratio (SNR) at 20 steps in the interval amplitude

In the single sinusoid case, we generated target signals with fixed estimator was evaluated by fitting the model to sinusoidal signals by gradient descent. All signals were of length

The performance of the surrogate model as a sinusoidal parameter appears to yield acceptable estimates.

3. EVALUATION

The performance of the surrogate model as a sinusoidal parameter estimator was evaluated by fitting the model to sinusoidal signals by gradient descent. All signals were of length $N = 4096$.

3.1. Single sinusoid frequency estimation

In the single sinusoid case, we generated target signals with fixed amplitude $\alpha = 1$ and initial phase $\phi = 0$. The frequency parameter was sampled at 100 equal steps in the interval $[0, 1, 0.9\pi]$, and the signal-to-noise ratio (SNR) at 20 steps in the interval $[0, 40]$ dB, for a total of 2000 targets. The corresponding signals were synthesized using the real valued sinusoidal model in Eqn 1. A single starting parameter estimate was uniformly sampled from within the unit circle used for all 2000 targets, and the procedure was repeated with 10 different pseudo-random number generator seeds. Optimization proceeded for 50k steps using the Adam optimizer with a learning rate of 0.0001 and the mean squared error loss on either the time-domain signal or DFT magnitude spectrum.

Fig. 3 displays the results of this experiment. The mean and median squared error between the predicted and ground truth frequency parameters are plotted on a decibel scale (i.e. $10 \log_{10}(\text{MSE})$). The dotted black line plots the Cramér-Rao lower bound (CRLB) on variance for an unbiased estimator of sinusoidal frequency in Gaussian white noise, as given by Kay [1]. Whilst enquiry into the convergence properties of our method – and therefore the underlying bias of the estimator – is beyond the scope of this paper, this bound is representative of the performance of other sinusoidal parameter estimation algorithms. We thus plot it here to facilitate comparison and to illustrate that the surrogate model with time domain MSE loss is capable of achieving an error comparable with non-gradient based estimators.

We note that the mean squared error of the frequency domain loss (DFT-MSE) does not fall below roughly $-83$ dB, but the median squared error continues to fall, implying an increasingly skewed error distribution as the SNR rises. We speculate that this occurs due to the loss of phase information in taking the modulus of the spectrum, and will investigate this hypothesis in future work.

3.2. Multi-sinusoid frequency and amplitude estimation

In the multi-sinusoid case, targets were generated with phase $\phi_k = 0$, and frequency and amplitude sampled from uniform dis-
clearly achieves superior performance, outperforming the base-
sampled sinusoidal parameters. Here, as expected, the surrogate
For comparison, we also plot the errors achieved with randomly
both our surrogate model and the real sinusoidal model baseline.
tween target and predicted magnitude spectra using a dB scale for
ation 3.2. We plot the distributions of mean squared errors be-
estimates, to a differentiable real valued sinusoidal model.
was repeated for
of starting surrogate parameter estimates was uniformly sampled
synthesized using the real valued sinusoidal model. A random set
of target parameters were sampled, and the corresponding signals
tributions, \( \omega_k \sim \mathcal{U}(0.1\pi, 0.9\pi) \) and \( \alpha_k \sim \mathcal{U}(0.1, 1.0) \). 2000 sets of
target parameters were sampled, and the corresponding signals
thesized using the real valued sinusoidal model. A random set
of starting surrogate parameter estimates was uniformly sampled
from the unit circle for each target, and linear amplitude was
initialized for each component at \( \alpha_k = \frac{1}{|k|} \). The experiment
was repeated for \( |K| \in \{2, 8, 32\} \). Optimization ran for 300k
steps using the same optimizer and losses. As a baseline,
the same procedure was applied, using the same targets and starting
estimates, to a differentiable real valued sinusoidal model.
Fig. 4 displays the results of the experiment described in sec-
tion 3.2. We plot the distributions of mean squared errors be-
tween target and predicted magnitude spectra using a dB scale for
both our surrogate model and the real sinusoidal model baseline.
For comparison, we also plot the errors achieved with randomly
sampled sinusoidal parameters. Here, as expected, the surrogate
clearly achieves superior performance, outperforming the base-
line in all configurations.
We note that the performances of both the baseline and ran-
domly sampled parameters improve as the number of components
increases. We speculate that this effect is due to the proportion-
ally smaller expected distance between each model component
and any target component for higher values of \( |K| \). Indeed, the
decrease observed in the metric for both the baseline and random
models is almost exactly proportional to the increase in the num-
er of components – that is, on the decibel scale we observe a
change of \( 10\log_{10} \frac{1}{|K|} \approx -6.02 \) for a \( 4 \times \) increase in components.
Conversely, the surrogate model’s performance slightly de-
grades as \( |K| \) increases. Through informal observation of con-
verged models, we hypothesize that this occurs due to a greater
number of a specific class of local minimum, wherein multiple
model components combine to match a single component in the
target signal. This phenomenon also appears to be responsible for
the wide distribution of surrogate performances. This is perhaps
best illustrated by the bimodal distribution observed in the two
sinusoid case, where inspection of model fits suggests that the two
modes correspond to surrogates matching either one or both tar-
get frequencies. We leave formal study of this behaviour to future
work, but note that it seems to occur primarily at the expense of
quieter signal components.
To illustrate the optimization dynamics of the surrogate, Fig.
5 plots the evolution of the metric throughout optimization for
both the surrogate and baseline model for a randomly selected
set of target parameters. Here we see that the baseline metric
either does not fall, or falls imperceptibly, as should be expected
given the properties described in Section 2. The surrogate metric,
however, does clearly fall before converging on a final value. It
appears to solve the multi-sinusoid problem sequentially – that
is, it seems to resolve each component one-by-one, causing the
metric to fall to a series of plateaus. This observation may have
implications for training strategies in DDSP deep learning tasks,
where a plateau in a metric is typically taken as a signifier that a
model has converged.
4. CONCLUSION
This work presented a technique for matching the frequency and
amplitude parameters of a single- and multi-component sinusoidal
model to a target signal by gradient descent. We evaluated
the performance of our method on single and multiple sinusoid
signals and demonstrated that it clearly outperforms a standard
sinusoidal model in the multi-sinusoid case, whilst approaching
the performance of other, non-gradient based estimators in the
single sinusoid case.
This problem was previously intractable using differentiable
signal processing techniques, preventing a variety of applications
of this family of methods, including the modelling of inharmonic
audio signals, unsupervised fundamental frequency detection,
and more. Our approach now paves the way for these applica-
tions to be explored. In particular, we believe our surrogate model
is suitable for use as a drop-in replacement for a differentiable
sinusoidal model, and in future work will explore its capabili-
ties in end-to-end learning with differentiable signal processing.
We will also conduct further study into the surrogate model’s
optimization characteristics.
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