Phase diagrams of the three-flavor NJL model with color superconductivity and pseudoscalar condensation

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Abstract
We present results of the calculation of phase diagrams of the three-flavor NJL model as a function of temperature and different quark chemical potentials. These phase diagrams are an extension of earlier calculations in the literature in which either only the possibility of color superconductivity or the possibility of pseudoscalar condensation was investigated. This study takes into account both options and hence allows one to investigate their competition. It turns out that the color superconducting phase and the phase in which the pseudoscalar mesons condense are separated by a first order transition.

1 Introduction
The NJL model is a low-energy effective theory for QCD and as such widely used for investigating the phase structure of QCD as a function of temperature and chemical potentials. Many studies have been carried out for equal quark chemical potentials, that is \( \mu_u = \mu_d = \mu_s \). This constraint does not necessarily hold in realistic situations. For example, in a compact star in equilibrium under the weak interactions \( \mu_d = \mu_s \), and \( \mu_u \) is tuned in such a way that the star becomes electrically neutral \([1, 2, 3]\). Therefore, it is useful and interesting to investigate the phase diagram of the NJL model as a function of different chemical potentials. It turns out that the NJL model has a rich phase structure with phases in which chiral symmetry is broken, different color superconducting phases and phases in which the pseudoscalar mesons condense. In the last mentioned phase a field with the same quantum numbers as the pion or the kaon obtains a vacuum expectation value, that is the condensates \( \langle \bar{u} i \gamma_5 d \rangle \), \( \langle \bar{u} i \gamma_5 s \rangle \) or \( \langle \bar{d} i \gamma_5 s \rangle \) are nonzero. The interesting feature of these phases is that parity is broken there. The existence of a pion condensate \([4]\) is also confirmed on the lattice at finite isospin chemical potential with zero baryon chemical potential \([5]\). The analysis presented in this article (and in more detail in Ref. \([6]\)) is an extension of previous studies in which either only pseudoscalar condensation \([7, 8, 9]\) or only color superconductivity \([10, 11, 12]\) was taken into account. Here both possibilities will be investigated simultaneously. The more complete phase diagrams which we have calculated are not merely a superposition of the phase diagrams obtained earlier, since it turns out that there is competition between the color superconducting phases and the phases in which the pseudoscalar mesons condense. If the chemical potentials of the quarks are large and approximately equal, for example \( \mu_u \approx \mu_d \), it is possible that an up and a down quark form a diquark condensate giving rise to a two-flavor color superconducting (2SC) phase. If the chemical potentials of the quarks are large and approximately opposite to each other, for example \( \mu_u \approx -\mu_s \), it is possible that an up and a strange quark form a charged kaon condensate \( \langle \bar{u} i \gamma_5 s \rangle \). So one might wonder what would happen if \( \mu_u \approx \mu_d \approx -\mu_s \), could a 2SC phase appear together with a phase in which the charged kaon meson condenses? We find that the color superconducting phases and the phase in which the pseudoscalar mesons condense are always separated by a first order transition for any temperature for a set of realistic parameters chosen.

The three-flavor NJL model we use is derived from a 4-point color current-current interaction and
given by the following Lagrangian density (see for example Ref. [13])

\[ \mathcal{L} = \bar{\psi} (i\gamma^\mu \partial_\mu - M_0 + \mu \gamma_0) \psi + G (\bar{\psi} \lambda_a \psi)^2 + G (\bar{\psi} \lambda_a i\gamma_5 \psi)^2 + \frac{3}{4} G (\bar{\psi} t_A \lambda_B C i\gamma_5 \bar{\psi}^T) (\psi^T t_A \lambda_B C i\gamma_5 \psi), \]

(1)

where \( G \) is a coupling constant, \( M_0 \) is a diagonal matrix containing the up, down and strange quark masses and \( \mu \) a diagonal matrix containing the up, down and strange quark chemical potentials. The constants \( A, B \in \{2, 5, 7\} \), since the one gluon exchange interaction is only attractive in the antisymmetric color and flavor triplet channel. The matrices \( \lambda_a \) are the 9 generators of U(3) and act in flavor space. They are normalized as \( \text{Tr} \lambda_a \lambda_b = 2\delta_{ab} \). The matrices \( t_a \) are the generators of U(3) and act in color space. Their normalization is \( \text{Tr} t_a t_b = 2\delta_{ab} \). To remind the reader, the antisymmetric flavor matrices \( \lambda_2, \lambda_3 \) and \( \lambda_5 \) couple up to down, up to strange and down to strange quarks, respectively. The charge conjugate of a field \( \psi \) is denoted by \( \psi_c = C \psi^T \) where \( C = i\gamma_0 \gamma_2 \). The coupling strength \( 3G/4 \) of the diquark interaction is fixed by the Fierz transformation (see for example Ref. [13]). However, other choices for this coupling strength are also made in the literature.

The NJL model has a symmetry structure similar to QCD, except for the local color symmetry. The NJL model is only invariant under global SU(3) color transformations since it does not contain gauge fields. In absence of quark masses and chemical potentials, the Lagrangian density of the NJL model has a global SU(3)\(_c\) \( \times \) U(3)\(_L\) \( \times \) U(3)\(_R\) symmetry. Due to the non-vanishing quark masses, the symmetry is like in QCD broken down to SU(3)\(_c\) \( \times \) U(3)\(_V\). Since the masses of the quark and the chemical potentials are chosen to be different, the symmetry of the Lagrangian density is further reduced to SU(3)\(_c\) \( \times \) U(1)\(_B\) \( \times \) U(1)\(_I\) \( \times \) U(1)\(_Y\), where \( B, I, Y \) stand respectively for baryon, isospin (the \( z\)-component) and hypercharge number. These remaining symmetries can be used to rotate away several possible condensates which simplifies the analysis in the end.

The results that will be presented here are obtained with the following choice of parameters which were also used in Ref. [13]

\[ m_{u_d} = m_{u_d} = 5.5 \text{ MeV}, \quad m_{s} = 112 \text{ MeV}, \quad G = 2.319/\Lambda^2, \quad \Lambda = 602.3 \text{ MeV}. \]

These are the precise values used in the calculations, but clearly not all digits are significant implying that a small change of parameters will not have a big influence on the results. This choice of parameters gives rise to constituent quark masses \( M_u = M_d = 368 \text{ MeV} \) and \( M_s = 550 \text{ MeV} \).

2 Effective potential

To obtain a phase diagram one has to investigate the behavior of order parameters like the chiral condensates \( \alpha_a = -2G \langle \bar{\psi} \lambda_a \psi \rangle \), the pseudoscalar condensates \( \beta_a = -2G \langle \bar{\psi} i\gamma_5 \lambda_a \psi \rangle \) and the diquark condensates \( \Delta_{AB} = \frac{3}{2} G \langle \bar{\psi} t_A \lambda_B C i\gamma_5 \psi \rangle \). The values of these order parameters can be found by minimizing the effective potential. The effective potential can be obtained by the introduction of auxiliary fields which are then shifted in such a way that the action of the NJL becomes quadratic in the fermion fields. After integration over the fermion fields and replacing the auxiliary fields by their vacuum expectation values, it turns out [6] that the effective potential in the mean field approximation is given by

\[ \mathcal{V} = \frac{\alpha_a^2 + \beta_a^2}{4G} + \frac{|\Delta_{AB}|^2}{3G} \sum_{p_0=(2n+1)\pi T} \log \det \left( \begin{array}{cc} \mathbb{I}_c \otimes \mathcal{D}_1 & \Delta_{AB} t_A \otimes \lambda_B \otimes \gamma_5 \\ \mathbb{I}_c \otimes \mathcal{D}_2 & \mathcal{D}_1 \end{array} \right), \]

(3)

where

\[ \mathcal{D}_1 = \mathbb{I}_f \otimes (i\gamma_0 p_0 + \gamma_i p_i) - \mu \otimes \gamma_0 - (M_0 + \alpha_a \lambda_a) \otimes \mathbb{I}_d - \beta_a \lambda_a \otimes i\gamma_5, \]

(4)

\[ \mathcal{D}_2 = \mathbb{I}_f \otimes (i\gamma_0 p_0 + \gamma_i p_i) + \mu \otimes \gamma_0 - (M_0 + \alpha_a \lambda_a^T) \otimes \mathbb{I}_d - \beta_a \lambda_a^T \otimes i\gamma_5. \]

(5)
The matrix $\mathbb{1}$ is the identity matrix in color ($c$), flavor ($f$), or Dirac ($d$) space.

By minimizing the effective potential it was found numerically that of the chiral condensates only $\alpha_0, \alpha_3$ and $\alpha_8$ can be nonzero. So the chiral condensates $\langle \bar{u}u \rangle, \langle \bar{d}d \rangle$ and $\langle \bar{s}s \rangle$ appear, while for example a $\langle \bar{u}d \rangle$ condensate is impossible. The chiral condensates can be found in the phase diagram at low chemical potentials and temperatures.

Furthermore, it turns out that the pseudoscalar condensates $\beta_0, \beta_3$ and $\beta_8$ are always nonzero. However, the other pseudoscalar condensates can appear. Using the $U(1)$ flavor symmetries one can reduce the possible set of pseudoscalar condensates to $\beta_2$, $\beta_5$ and $\beta_7$, which are respectively a pion ($\pi^\pm$), a charged kaon ($K^\pm$), and a neutral kaon $K^0/\bar{K}^0$ condensate. All these condensates break parity. Pion condensation is possible if $|\mu_u - \mu_d| > m_\pi$ [4] where $m_\pi$ is the mass of the pion, while kaon condensation is possible if $|\mu_{u,d} - \mu_s| > m_K$ [14] where $m_K$ is the kaon mass.

The different possible color-superconducting phases are named as follows (see for example [2])

$$
\begin{align*}
\Delta_{22} &\neq 0, \quad \Delta_{55} \neq 0, \quad \Delta_{77} \neq 0 \quad \text{CFL} , \\
\Delta_{77} & = 0, \quad \Delta_{22} \neq 0, \quad \Delta_{55} \neq 0 \quad \text{uSC} , \\
\Delta_{55} & = 0, \quad \Delta_{22} \neq 0, \quad \Delta_{77} \neq 0 \quad \text{dSC} , \\
\Delta_{22} & = 0, \quad \Delta_{55} \neq 0, \quad \Delta_{77} \neq 0 \quad \text{sSC} .
\end{align*}
$$

The abbreviation CFL stands for color-flavor locked phase. If there is exact $SU(3)_V$ flavor symmetry, the three diquark condensates in this phase have equal size and the vacuum is invariant under a combined rotation in color and flavor space [15]. In the uSC, dSC or sSC phase the up, down or strange quark always takes part in the diquark condensate, respectively. In the 2SC phase an up and a down quark form a diquark condensate, in the 2SCus and the 2SCds phase this condensate is formed by an up and and strange quark and a down and a strange quark, respectively. The color superconducting phases of the NJL model are not color neutral. In order to achieve color neutrality, one should introduce color chemical potentials [16] [17]. However, we leave this for further work.

To calculate the effective potential one needs to evaluate a determinant of a $72 \times 72$ matrix. Only in special cases such as when all masses and chemical potentials are equal and in absence of pseudoscalar condensation one can perform the sum over Matsubara frequencies analytically and hence simplify the effective potential somewhat further. But in the more general cases which will be discussed in this article this either is not possible or gives rise to very complicated equations. We therefore calculated and minimized the effective potential numerically.

### 3 Phase diagrams

Some of the phase diagrams we obtained in Ref. [6] are displayed in Fig. 1. Our results agree qualitatively with the phase diagrams with only pseudoscalar condensation presented in Refs. [8] [9] and the phase diagram with only superconductivity presented in Ref. [10].

One can clearly see that the phase diagrams evaluated at $T = 0$ are symmetric under reflection in the origin. This is because the free energy is invariant under the transformation $(\mu_u, \mu_d, \mu_s) \rightarrow (-\mu_u, -\mu_d, -\mu_s)$ from the symmetry between particles and anti-particles.

In general, horizontal and vertical lines in the phase diagrams arise if the pairing of one type of quark is not changed after a transition. In this case, the location of the phase boundary is determined by the properties of other quarks. Therefore, changing the chemical potential of the unchanged quark species cannot have a big influence on the location of the phase boundary. This results in the horizontal and vertical lines. For $T = 0$, one always finds these lines near the values of the constituent quark masses, i.e. $\mu_u \approx M_u, \mu_d \approx M_d$ and $\mu_s \approx M_s$ (see for example Ref. [13]). The diagonal lines arise because at $T = 0$ pion condensation can occur if $|\mu_u - \mu_d| > m_\pi = 138$ MeV [4] and kaon condensation can occur if $|\mu_s - \mu_{u,d}| > m_K = 450$ MeV [14] (the chosen parameter set gives rise to a somewhat low kaon mass, but this is not relevant for the qualitative features of the phase diagram).
Fig. 1: Phase diagrams of the NJL model for $\mu_s = 0$ and $T = 0$ (upper left), $\mu_u = \mu_d + \epsilon$ and $T = 0$ (upper right), $\mu_d = \mu_s$ and $T = 0$ (lower left), and $\mu_d = \mu_s = 550$ MeV (lower right). First and second-order transitions are indicated by solid and dotted lines, respectively, while cross-overs are denoted by dashed-dotted lines. The letters denote the different phases, where a: $\bar{u}u + \bar{d}d + \bar{s}s$, b: $\bar{u}u + \bar{d}d$, c: $\bar{u}u + \bar{s}s$, d: $\bar{d}d + \bar{s}s$, e: $\bar{u}u$, g: $\bar{s}s$, h: $\pi^+ / \pi^-$, i: $\pi^+ / \pi^- + \bar{s}s$, j: $K^+ / K^-$, k: $K^+ / K^- + \bar{d}d$, l: $K^0 / \bar{K}^0$, m: $K^0 / \bar{K}^0 + \bar{u}u$, n: 2SC, p: 2SCd, q: 2SC + $\bar{s}s$, r: 2SCus + $\bar{d}d$, s: 2SCds + $\bar{u}u$, t: CFL and w: sSC.

The diagrams show that if the chemical potentials are different from each other, the transition to the color-superconducting phases (n/p/q/l/s) remains first order at $T = 0$ as was concluded in Ref. [18]. The lower right figure shows that this conclusion does no longer hold at finite temperature.

Moreover, one can see from the upper left diagram that if $\mu_u \neq \mu_d$ it is possible to go through two first-order transitions before entering the 2SC phase (q) (similar to the situation discussed in Ref. [7] without color superconductivity). We observe that to have such a scenario at zero temperature, a minimum difference between $\mu_u$ and $\mu_d$ is required. In the present case this is about 35 MeV. Pion condensation (i) and the 2SC phase (q) are in this diagram separated by two phase transitions in contrast to the estimated ($\mu_B, \mu_I$) phase diagram of Ref. [19]. The two lower phase diagrams in Fig. 1 (these phase diagrams were studied in Ref. [6] with $\mu_u \leftrightarrow \mu_d$) are of relevance for studying situations which are in weak equilibrium ($\mu_d = \mu_s$), but not necessarily electrically neutral. In that case there are situations in which a charged kaon condensate (j) is competing against a color superconducting phase (s). From the diagrams it can be seen that the color superconducting phases are separated from the phase in which the charged kaon condenses for any temperature and all values of the chemical potentials. The same conclusion holds for a pion condensate [6].
4 Summary and Conclusions

We presented phase diagrams of the three-flavor NJL model as a function of flavor chemical potentials and temperature. We found that the phases with pseudoscalar condensation are separated from the color superconducting phases by a first order transition for any temperature. Although many regions in the phase diagrams were or will never be realized in nature, these diagrams may be relevant for comparison with possible future lattice data. This work could be extended with the inclusion of the electric and color neutrality constraints, the discussion of gapless phases, and the investigation of the effect of ’t Hooft’s instanton-induced interaction on the phases with pseudoscalar condensation.

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