Egocentric physics: Just about Mie

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Abstract – We show that the physics of anapole excitations can accurately be described in terms of a truncated resonant state expansion formulation of standard Mie theory as an alternative to the recently advocated “toroidal” current description. In the purely Mie framework, the anapole behavior arises as a result of a Fano-type interference effect between different resonant states with complex wave numbers, provided that non-singular phase factors associated with causality are properly taken into account.

Introduction. – Modern research in metamaterials and nanotechnology often seeks to exploit the wide variety in the behaviors of open nanoresonator systems. Despite the complexity of such systems, there is a growing appreciation that their behavior near a resonance can often be described in terms of a small number of parameters within frameworks satisfying general physical principles like energy conservation, reciprocity, and time reversal [1,2]. There is also an increasing interest in exploiting the resonances of high-index dielectrics for metamaterial and sensing applications in order to circumvent the problems caused by losses in metallic resonators at optical and near-optical frequencies.

An interesting topic arising in the applications and theory of high-index optical nanosystems is that of anapole excitations, in which the fields generated by polarization currents inside high-index dielectric spheres interfere in such a manner that their superposition radiates little energy outside the sphere [3–5]. Recently, doubt has been raised as to whether or not the conventional multipole basis of Mie scattering theory (truncated to a manageable number of terms) is well adapted to describe these anapole excitations [6], or whether it is better to describe fields in terms of a combination of the Mie basis and an alternative “toroidal” basis [1,3,5,7].

Here we will show that the slow convergence of the Mie basis reported by Powell [6] can be readily remedied by incorporating an extra, analytically determined causality-based phase factor, arising in the theory of product representations of functions of a complex variable [8,9]. With this extra factor included, we show that just two resonant states from Mie theory are sufficient to give an accurate Fano interference representation of the anapole state in a case from the literature [3].

It is worth noting that the method we describe fits into the framework of the Singularity Expansion Method (SEM) [10]. While this method has been widely applied by applied mathematicians in the context of time-dependent scattering theory, it appears to be less well known among physicists studying time-independent scattering problems. It works in terms of modes with complex wave numbers, yet avoids the problems of normalizability of such modes that have long been remarked upon [11].

That the anapole phenomenon has a Fano-type profile (cf. fig. 1) and can be explained in these terms has been remarked before [12,13]. The novel aspects of our work are the resonant state treatment within a purely Mie theory context including the causality-based phase factor, which, in addition to providing a compact description of the physical response, can be extended to achieve essentially arbitrary accuracy. The casual factor provides an essential element in the accurate prediction of the position of the anapole state, so that it is moot whether one should indeed use the description “Fano resonance” for it. Working purely within the Mie basis, we will show that a more complete description would be “Fano resonance with a causal contribution”.

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We remark that although we chosen the wording “resonant states” since it appears to be the one of the oldest terminologies, in more recent literature such descriptions have often gone by other names like “Quasi-Normal Modes” (QNMs) and “leaky modes”. We also remark that we restrict our attention here to dielectric structures, so avoiding the question as to the distinction between scattering suppression states and anapole states (see [1,14]).

Anapole “states” – Anapole states of a scatterer have generally been described in the context of high-index dielectrics. As a first example, we consider a spherical particle model with a permittivity index near that of silicon. As an incident wavelength of $\lambda = 550$ nm that has already been shown to possess an anapole behavior [3]. In this lossless dielectric example, the scattering and extinction cross-sections are identical, $\sigma_e = \sigma_s = \sigma$, and scale invariance renders it convenient to express formulas in terms of the particle size parameter, i.e. $z \equiv kR$ with $k \equiv \frac{2\pi}{\lambda} \sqrt{\varepsilon_s \mu_s^\prime} R \equiv \sqrt{\varepsilon_s} N_0 \frac{\pi}{\lambda}$, the wave number in the background media.

The total scattering cross-section for this particle, together with its multipole decomposition in terms of both electric or magnetic wave types is plotted in fig. 1. The “anapole” condition for the electric dipole mode occurs when its contribution to the scattering cross-section is zero, which as can be seen from the figure occurs at $D \approx 204$ nm ($z \simeq 1.165$).

Each multipole contribution to the scattering cross-section in fig. 1 is calculated from its corresponding $S_n$ function via the extinction cross-section formula,

$$
\sigma_n = \frac{2n + 1}{z^2} R \{1 - S_n\},
$$

with a Mie theory expression for $S_n$ given in eq. (A.6).

The $S_n$ functions can alternatively be expressed in terms of a Weierstrass-type product expansion in terms of the resonant state frequencies, $\omega_{n,\alpha}$, or equivalently resonant state size parameters, $z_n \equiv k_n R \equiv N_0 \omega_{n,R}/c$. For each multipolar order and wave type, the $z_n$ can be readily determined as zeros of the transcendental equations in eq. (A.8a) or eq. (A.8b), with the $S_n$ function taking the form

$$
S_n(z) = (-)^{n+1} e^{-2iz} \prod_{\alpha=-\infty}^{\infty} \frac{z - z_{n,\alpha}^*}{z - z_{n,\alpha}} = (-)^{n+1} e^{-2iz} \prod_{\alpha=-\infty}^{\infty} \frac{z^2 - |z_{n,\alpha}|^2 + 2iz \text{Im}(z_{n,\alpha})}{z^2 - |z_{n,\alpha}|^2 - 2iz \text{Im}(z_{n,\alpha})},
$$

where the $e^{-2iz}$ phase factor is associated with causality. (This phase factor can be expressed either in terms of a sum over the difference between the $z$ values of zeros and poles, summed over the entire set, or obtained in a more general way using causality — see King [9].) In the first line of eq. (2), time-reversal symmetry requires a symmetry between negative and positive resonant state eigenvalues,

$$
z_{n,-\alpha} = -z_{n,\alpha}^*,
$$

while in the last line of eq. (2) this symmetry is exploited to remove explicit reference to the negative energy resonant states. We remark that eq. (2) is only valid as it stands for lossless scatterers (the zeros of $S_n$ are not the complex conjugates of the resonant state values for temporally dispersive media), and that eq. (2) is the form applicable to electric (TM) multipoles (magnetic (TE) product expressions are in fact identical but of opposite sign, $(-)^n$ prefactor).

Note the following identity which connects the exponential prefactor in eq. (2) with the sum over all the poles:

$$
\frac{S_n'(0)}{S_n(0)} = -2i + 2 \sum_{\alpha} \left( \frac{1}{z_{n,\alpha}^*} - \frac{1}{z_{n,\alpha}} \right).
$$

Fig. 1: (Colour online) Cross-section vs. particle diameter in nanometers for a sphere of permittivity, $\varepsilon_s = 16$ with a background wavelength of $\lambda = 550$ nm. Total scattering cross-section is plotted as a red solid line. The electric dipole cross-section exhibiting anapole behavior is plotted in blue (long-dashed), with magnetic dipole contribution in green and magnetic quadrupole in cyan (both short-dashed).

Fig. 2: (Colour online) Size parameter eigenvalues of the electric dipole resonant states, $z_n^{(\varepsilon)}$, for a sphere of permittivity, $\varepsilon_s = 16$. Vertical dashed lines delimit the size parameter range evaluated in fig. 1.
Fig. 3: (Colour online) Electric dipole cross-section, (a)-(b), and average internal field enhancement, (c). Total electric dipole contributions are plotted as blue (solid) curves. Dashed red curves in (a) and (c) correspond to including only the resonant states. The green curves correspond to a superposition of the non-resonant contribution and the resonant state, while in (b) it corresponds to a superposition of the non-resonant contribution and the resonant state. The dashed purple lines in (a)-(c) correspond to an inclusion of only the \( z_{1,1}^{(e)} \) contributions. The green curves correspond to a coherent sum of \( z_{1,1}^{(e)} \) and \( z_{1,2}^{(e)} \) resonant states.

The convergence of the sum over poles is assured by the following asymptotic expression for the \( z_{1,m} \):

\[
z_{1,m} \approx -\frac{1}{n_s} \left( \frac{m\pi}{2} + i \frac{\log (n_s + 1)}{n_s - 1} \right) - \frac{i}{(m - 1/2)\pi} - i \frac{1}{(m - 1/2)^2 n_s},
\]

where \( n_s \) is the refractive index of the sphere. This estimate can be used to show the quadratic convergence of the sum in (4) and the completeness of the Mie basis of eigenfunctions.

From here on we concentrate attention on the electric dipole response since it exhibits anapole behavior. The electric dipole resonant states with the six lowest real parts are plotted in fig. 2 where the \( z_{1,m}^{(e)} \) are obtained by solving for the zeros of the transcendental expression in eq. (A.8a). We immediately see from fig. 2 that there are two electric dipole resonant states in the diameter range explored in fig. 1 (region between the dashed vertical lines). Their values are \( z_{1,1}^{(e)} \approx 1.039 - i0.501 \), with a rather large imaginary part and \( z_{1,2}^{(e)} \approx 1.053 - i0.0723 \) close to the real z-axis.

The product expansion of the electric dipole cross-section, \( \sigma_{1}^{(e)} \), obtained from eqs. (1), (2) is graphically illustrated in fig. 3(a) (same size range as fig. 1). The dashed curves are obtained by only including the \( z_{1,1}^{(e)} \) (red) and \( z_{1,2}^{(e)} \) (purple) contributions to the \( S_{1}^{(e)} \) product expansion. The green solid curve is obtained by the superposition of \( z_{1,1}^{(e)} \) and \( z_{1,2}^{(e)} \) terms in eq. (2), and the addition of higher resonant states would produce a rather slow convergence to the exact blue solid curve.

We can furthermore compare the expression (2) restricted to just the two dipole terms of the previous paragraph,

\[
\sigma_{1}^{(e)} \propto 2 \Re \left[ 1 - \exp(-2iz) \right] \left( \frac{z - z_{1,1}^{(e)}}{z - z_{1,2}^{(e)}} \right), \tag{6}
\]

with the calculations of Miroshnichenko A.E. et al [3] incorporating both Cartesian electric and toroidal dipoles —see fig. 4. The simple expression (6) with the exponential factor present does very well in predicting the whole of the curve after the resonant maximum, through the anapole minimum, and right on to the large diameter end of the figure. Without the exponential factor, the two-term theory is totally inadequate. With the exponential factor, there is no need to incorporate toroidal terms in the Mie theory, in predicting the anapole minimum.

Although figs. 3(a) and 4 establish that both the \( z_{1,1}^{(e)} \) and \( z_{1,2}^{(e)} \) resonant states are required to reproduce the anapole behavior, product expansions do not readily lend themselves to the more familiar explanations of the Fano shape as an interference effect. This more traditional approach prefers different approximations. We shall return to this in a forthcoming article.
reasoning is obtainable by reformulating the scattering cross-section contributions as resonant state sums of the Mie “coefficients”, \( a_n \) and \( b_n \), (which are opposite in sign from the \( T_n \)-functions), given in eq. (A.6) of the appendix. The \( T_n \)-functions by definition relate the coefficients of the incident field to those of the scattered field as in eq. (A.2), and are consequently related to the \( S_n \) functions via \( S_n = 1 + 2T_n \), and contribute to the scattering cross-section via the relation
\[
\frac{\sigma_n}{\pi R^2} = \frac{2n+1}{2} - \frac{2}{\pi R^2} |T_n|^2 . \tag{7}
\]
The \( T_n \)-functions can be expressed as a the sum of a non-resonant term plus a sum over the resonant state responses,
\[
T_n(z) = \frac{(-1)^{n+1} e^{-2iz} - 1}{2} + \frac{(-1)^{n+1} e^{-2iz}}{2} \sum_{\alpha=-\infty}^{\infty} \frac{r_{n,\alpha}}{z - z_{n,\alpha}}
\]
where in the last line we used eq. (3) and the fact that time reversal symmetry also imposes a relation,
\[
r_{n,-\alpha} = -r_{n,\alpha} . \tag{9}
\]

We remark that unlike the product form of eq. (2), the sum formulation of eq. (8) requires a calculation of the resonant state contributions as resonant state sums of the \( S_n \)-functions, given in eq. (A.6) of the appendix.

The sum form of the resonant state expansion of the electric dipole cross-section using eqs. (7), (8) for \( T_1^{(c)} \) is graphically illustrated in fig. 3(b). The red curve is obtained by summing the first two resonant states as \( z_{1,1}^{(c)} = -0.474 - i1.740 \) and \( r_{1,1}^{(c)} = 0.0376 + i0.1996 \) so that even though the \( z_{1,1}^{(c)} \) mode is far from the real axis, it cannot be neglected on account of its large residue strength.

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and \( E \) is a field amplitude parameter [15]. The resonant state expansion of \( E_{n,m}^{(\text{int})} \) is

\[
E_{n,m}^{(\text{int},e)} = \frac{(-1)^n e^{-2iz}E_{n,m}}{2} \sum_{\alpha=-\infty}^{\infty} \frac{f_{n,\alpha}^{(1)}}{z - z_{n,\alpha}} E_{n,m,\alpha}^{(\text{int},e)},
\]

where \( E_{n,m,\alpha}^{(\text{int},e)} \) are the resonant state wave functions of the electric \( n, m \) multipole mode, for which the analytical expressions in the case of a homogeneous sphere are

\[
E_{n,m,\alpha}^{(\text{int},e)}(r) = \left[ \sqrt{n(n+1)}j_n(\rho_s z_{n,\alpha} r) Y_{n,m}(\theta, \phi) + \psi'_{\alpha}(\rho_s z_{n,\alpha} r) Z_{n,m}(\theta, \phi) \right] e^{-\frac{1}{2} \frac{k_s^2}{n_s} z_{n,\alpha} r},
\]

where \( Y_{n,m} \) and \( Z_{n,m} \) are normalized vector spherical harmonics [16]. In eq. (14), \( \rho_s \) is the refractive index contrast of the sphere (cf. eq. (A.10)), and \( r \equiv r/R \) is a dimensionless radial coordinate normalized with respect to the sphere’s radius \( (r \leq 1) \).

The real part of the electric field lines from Mie theory are plotted in the bottom of fig. 5 on a background of an anapole field lines. One sees that there is indeed a “toroidal” aspect to the induced polarization currents, but these are completely described within the context of Mie theory without the need for any additional basis functions. The field lines of the low and high-\( Q \) resonant states, \( z_1^{(e)} \) and \( z_2^{(e)} \), which are determined by using the resonant state wave functions described in eq. (13), are plotted in the top line of this figure. Although their respective field profiles are quite distinct, their superposition does indeed largely reproduce the exact Mie anapole calculations on the bottom line of the figure.

\[ \text{Conclusions.} \] – The approach described here for understanding the scattering properties of high-index dielectric spheres, and anapole excitations in particular, has a number of advantages. Firstly, it relies only on the knowledge of the positions of the relevant complex resonant frequencies calculated from Mie theory. Secondly, it incorporates an analytically determined phase term, without which a very large number of terms in the product over resonances would be required to achieve accuracy. Thirdly, it does not require the evaluation of inner products or normalisation factors for different resonant modes, which have proved problematic in other approaches. We have shown that for a given sphere diameter range, a small number of relevant terms yields information about the scattering behaviour to a good accuracy, with the simplicity of the formulae being beneficial to a physical understanding of the phenomena under investigation. This approach is quite general, can readily be extended to both dispersive media and non-spherical geometries, [17] and will no doubt find applications in a wide range of situations in addition to those studied here.

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\[ \text{Appendix: Mie theory.} \] – The excitation, \( E_e \), and scattered, \( E_s \), fields can respectively be developed as a multipolar decomposition,

\[
E_e(kr) = E \sum_{n,m} \left[ e_{nm}^{(e)} N_{nm}^{(1)}(kr) + e_{nm}^{(h)} M_{nm}^{(1)}(kr) \right],
\]

\[
E_s(kr) = E \sum_{n,m} \left[ f_{nm}^{(e)} N_{nm}^{(+)}(kr) + f_{nm}^{(h)} M_{nm}^{(+)}(kr) \right],
\]

where \( n \) is the electromagnetic angular quantum number, and \( m \) the projection quantum number. The real parameter, \( E \), determines the strength of the incident field, while the multipole wave functions, \( N_{n,m}(kr) \) and \( M_{n,m}(kr) \), are respectively of the “electric” and “magnetic” types [17], and their coefficients are respectively distinguished with \( (e) \) and \( (h) \) superscripts. The superscript, \( ^{(1)} \) on the multipolar fields in eq. (A.1) denotes regular type fields described by spherical Bessel functions, whereas \( ^{(+)}, ^{(-)} \) refer to fields with outgoing (incoming) boundary conditions (Hankel-type fields).
The $T_n$-functions of Mie theory are meromorphic functions of frequency and constitutive parameters that relate the scattered field coefficients, $f_{nm}$, to excitation field coefficients, $e_{nm}$,

$$f_{nm} = T_n e_{nm}.$$  \hspace{1cm} (A.2)

Although incident and scattered fields are very useful from a response function standpoint, it is also to work with the total fields that are actually present in the particle plus field system. The partial-wave representations of the total field inside the particle, $E_{\text{int}}$, and the total external, field, $E_t$, are, respectively,

$$E_{\text{int}} = E \sum_{n,m} \left[ s_{nm}^{(e)} N_{nm}^{(1)}(kr) + s_{nm}^{(h)} M_{nm}^{(1)}(kr) \right],$$  \hspace{1cm} (A.3a)

$$E_t = E \sum_{n,m} \left[ a_{+nm}^{(e)} N_{nm}^{(+)}(kr) + a_{+nm}^{(h)} M_{nm}^{(+)}(kr) 
+ a_{-nm}^{(e)} N_{nm}^{(-)}(kr) + a_{-nm}^{(h)} M_{nm}^{(-)}(kr) \right].$$  \hspace{1cm} (A.3b)

The linear relationship between the internal field coefficients and the incident field coefficients is denoted by the matrix, $\Omega$, which is diagonal for spherically symmetric problems, For spheres, the internal field response functions, $\Omega_n$, relate the total fields, $s_{nm}$, to the excitation field coefficients, $e_{nm}$, as

$$s_{nm} = \Omega_n e_{nm},$$  \hspace{1cm} (A.4)

while the $S$-matrix relates incoming to outgoing components of the total external field,

$$a_{+nm} = S_n a_{-nm},$$  \hspace{1cm} (A.5)

The $T$, $S$, and $\Omega$ functions have exact expressions in Mie theory,

$$\Omega_n = \frac{1}{D_n}, \quad S_n = -\frac{N_n S_n}{D_n}, \quad T_n = -\frac{N_n T_n}{D_n},$$  \hspace{1cm} (A.6)

where the numerator functions can be expressed as

$$N^{(e)}_n = \frac{\varepsilon_s j_n (\rho_s z) \xi_{-n} (z) - \psi_n' (\rho_s z) h_{-n} (z)}{i \mu_s},$$  \hspace{1cm} (A.7a)

$$N^{(h)}_n = \frac{\mu_s j_n (\rho_s z) \xi_{-n} (z) - \psi_n' (\rho_s z) h_{-n} (z)}{i \mu_s},$$  \hspace{1cm} (A.7b)

$$N^{(e)}_{T_n} = \frac{\varepsilon_s j_n (\rho_s z) \psi_n' (z) - \psi_n' (\rho_s z) j_n (z)}{i \mu_s},$$  \hspace{1cm} (A.7c)

$$N^{(h)}_{T_n} = \frac{\mu_s j_n (\rho_s z) \psi_n' (z) - \psi_n' (\rho_s z) j_n (z)}{i \mu_s},$$  \hspace{1cm} (A.7d)

and the “denominator” functions as

$$D^{(e)}_n = \frac{\varepsilon_s j_n (\rho_s z) \xi_{-n} (z) - \psi_n' (\rho_s z) h_{-n} (z)}{i \mu_s},$$  \hspace{1cm} (A.8a)

$$D^{(h)}_n = \frac{\mu_s j_n (\rho_s z) \xi_{-n} (z) - \psi_n' (\rho_s z) h_{-n} (z)}{i \mu_s}.$$  \hspace{1cm} (A.8b)

The $\psi_n$ and $\xi_{n,\pm}$ are Riccati spherical Bessel functions, $\psi_n(z) \equiv z j_n(z)$, and $\xi_{n,\pm}(z) \equiv z h_{n}^{(\pm)}(z)$. Here, the $\varepsilon_s$ and $\mu_s$ are the relative constitutive parameters with respect to the background medium,

$$\mu_s \equiv \frac{\mu_r s}{\mu_r b}, \quad \varepsilon_s \equiv \frac{\varepsilon_r s}{\varepsilon_r b},$$  \hspace{1cm} (A.9)

and $\rho_s$ is the refractive contrast with respect to the background medium,

$$\rho_s \equiv \sqrt{\varepsilon_s \mu_s} = \sqrt{\frac{\varepsilon_r s \mu_r s}{\varepsilon_r b \mu_r b}}.$$  \hspace{1cm} (A.10)

Given the Cauchy residue theorem, a comparison of eqs. (10) and (A.6), leads to the following residues of the internal field coefficients, $r^{(\Omega,e,h)}_{n,\alpha}$:

$$r^{(\Omega,e,h)}_{n,\alpha} = \frac{2 \rho_s e^{2i\pi n \alpha}}{D_n (z)} \Big|_{z=z_{n,\alpha}},$$  \hspace{1cm} (A.11)

which following analytic manipulations results in

$$\frac{2i \rho_s e^{2i\pi n \alpha}}{r^{(\Omega,e,h)}_{n,\alpha}} = z_n^2 h_{-n+1} (z_{n,\alpha}) j_n (\rho_s z_{n,\alpha}) \varepsilon_s (\mu_s - 1) + (\varepsilon_s - 1) \left[ \xi_{+n} (z_{n,\alpha}) \psi_n' (\rho_s z_{n,\alpha}) + n (n+1) h_{n+1} (z_{n,\alpha}) j_n (\rho_s z_{n,\alpha}) \right],$$  \hspace{1cm} (A.12)

$$\frac{2i \rho_s e^{2i\pi n \alpha}}{r^{(\Omega,h)}_{n,\alpha}} = z_n^2 h_{n+1} (z_{n,\alpha}) j_n (\rho_s z_{n,\alpha}) \mu_s (\varepsilon_s - 1) + (\mu_s - 1) \left[ \xi_{+n} (z_{n,\alpha}) \psi_n' (\rho_s z_{n,\alpha}) + h_{n+1} (z_{n,\alpha}) j_n (\rho_s z_{n,\alpha}) n (n+1) \right],$$  \hspace{1cm} (A.13)

for electric and magnetic multipole resonant states, respectively. Analogous comparisons give for the residues, $r_{n,\alpha}$, of the $T$-matrix in eq. (8),

$$r_{n,\alpha} = -\frac{2 i \rho_s e^{2i\pi n \alpha} \varepsilon_s h_{n+1} (z_{n,\alpha})}{D_n (z)} \Big|_{z=z_{n,\alpha}} \equiv 2 i e^{2i\pi n \alpha} N^{(e)}_{n,\alpha},$$  \hspace{1cm} (A.14)

where $N^{(e)}_{n,\alpha}$ correspond to the resonant state “normalization” factors and for the electric and magnetic modes are respectively given by

$$\left[ N^{(e)}_{n,\alpha} \right]^2 = \xi_{+n} (z_{n,\alpha}) (\mu_s - 1) + \left\{ \xi_{+n} (z_{n,\alpha}) (\varepsilon_s - 1) + n (n+1) h_{n+1}^2 (z_{n,\alpha}) \right\},$$  \hspace{1cm} (A.15)

$$\left[ N^{(h)}_{n,\alpha} \right]^2 = \xi_{+n} (z_{n,\alpha}) (\varepsilon_s - 1),$$  \hspace{1cm} (A.15)

which given the differences in notation and conventions simplify to the normalization expressions derived in ref. [18] when restricted to the special case of null permeability contrast, i.e., $\mu_s = 1$. 

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It is important to keep in mind that resonant states have important and somewhat unfamiliar properties when compared with the much more common Hermitian spectral analysis of closed conserving systems. Notably, residue/normalisation factors like $N_{n,\alpha}$, are complex numbers so that $N_{n,\alpha}^2 \neq |N_{n,\alpha}|^2$.

The expressions $I_n$ and $J_n$ for determining the average internal field in eq. (3) are found by analytically evaluating the following integrals for the internal fields:

\[
\frac{2\rho^2}{z} I_n \equiv \frac{2\rho_0^2}{z} \int_0^z \left\{ j_n^2 (\rho_0 \eta) n (n + 1) + \psi_n' (\rho_0 \eta) \right\} d\eta \\
= \psi_n' (\rho_0 z) [\psi_n' (\rho_0 z) + j_n (\rho_0 z)] \\
+ (\rho_0^2 z^2 - n (n + 1)) j_n^2 (\rho_0 z) \\
\frac{2\rho^2}{z} J_n \equiv \frac{2\rho_0^2}{z} \int_0^z j_n^2 (\rho_0 \eta) \eta^2 d\eta \\
= \psi_n' (\rho_0 z) [\psi_n' (\rho_0 z) - j_n (\rho_0 z)] \\
+ x (\rho_0^2 z^2 - n (n + 1)) j_n^2 (\rho_0 z) .
\]

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