Integrated Lax Formalism for PCM

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Abstract

By solving the first-order algebraic field equations which arise in the dual formulation of the $D = 2$ principal chiral model (PCM) we construct an integrated Lax formalism built explicitly on the dual fields of the model rather than the currents. The Lagrangian of the dual scalar field theory is also constructed. Furthermore we present the first-order PDE system for an exponential parametrization of the solutions and discuss the Frobenious integrability of this system.

1 Introduction

As an integrable system the $D = 2$ principal chiral model (PCM) \[1, 2\] has been extensively studied both in physics and mathematics literature. PCM in general is an example of a sigma model \[3\] with a lie group target space (G). From the integrability point of view it can be formulated as the compatibility or the integration conditions (flat curvature conditions for the relative current) of a pair of linear matrix equations and some additional constraint which fixes the gauge among the other integrable systems which
can also be derived within the same context [2]. For this reason it has gauge relations to the other integrable systems. Alternatively it can directly be derived from again the zero curvature integrability condition of a pair of linear equations known as the Lax pair via an introduction of a complex spectral parameter. From geometrical point of view it corresponds to the theory of (pseudo)-harmonic maps from (pseudo)-Riemannian manifolds into compact Lie groups [4]. Certain class of fermionic models can also be formulated as PCM [5, 6, 7]. The reductions derived from the PCM lead to the coset sigma models and in particular to the symmetric space sigma models [1, 2]. Therefore from this standpoint PCM is also essential to study a major branch of the sigma models. On the other hand by taking the target group manifold as infinite dimensional (large N limit) the four dimensional self-dual Einstein [8, 9, 10, 11, 12, 13, 14] and also the self-dual Yang-Mills [9, 15, 16, 17, 18] equations can be formulated as PCM. On the string theory side the string dynamics on product manifolds containing Lie group factors lead to PCM with a critical Wess-Zumino (WZ) term which is needed for conformal anomaly cancellation in quantizing the theory [7, 19]. For the type IIB Green-Schwarz superstring on $AdS(5) \times S^5$ the field space can be formulated as a coset and the dynamics can be described via a coset sigma model, a fermionic WZ term and the $\kappa$ symmetry [20, 21, 22]. Moreover the bosonic sectors of the type IIA or IIB superstrings on $AdS_3 \times S^3 \times T^4$ can be formulated as worldsheet-PCM [23, 24] when the background current is purely R-R otherwise again as a PCMWZ model. Here we should remark that the non-critical PCMWZ models can always be derived from the PCM via field transformations based on the extended solutions of the Lax formalism [25]. The symmetries of these superstrings as integrable systems have been studied extensively [26, 27]. Apart from these superstrings in the low energy limit also nearly all the scalar sectors of the supergravity theories can be formulated as symmetric space sigma models [25, 28, 29]. Thus as a result of its appearance in string theory and geometry studying PCM has become crucial in understanding and possibly refining the dynamical or the algebraic geometrical features and extending the structures of the relative models or the theories. We can refer the reader to a moderate selection from the rich literature studying different aspects of PCM in [1, 2, 6, 30, 31, 32, 33, 34].

In [35] by considering the components of the invariant right-Noether cur-
rent of the $D$-dimensional model which has a global left-right acting $G \times G$ symmetry as independent fields we have formulated a dual theory. This led us to the first-order algebraic equations of these current components in terms of the dual fields which appear in the dual construction as Lagrange multipliers. In this work we will solve these equations for the $2D$ case. We will show that the solutions to these first-order equations exactly determine the form of the right-current in terms of the dual fields which become scalars in the $2D$ case. The solution appears to be in the form of a Lax connection. For this reason we realize that this solution together with the dual field equation which becomes the zero curvature condition for this connection define an integrated$^2$ Lax formalism for the theory. As we obtain the current components explicitly in terms of the dual fields we will also present the dual lagrangian whose independent fields are solely the dual scalars. In Section five we will construct the first-order PDE system of the theory by equating the explicitly calculated Noether-currents both in terms of the original$^{36,37,38}$ and the dual scalar fields of the theory for the case of a lie algebra parametrization of the dynamic $G$-valued map which constitutes the original lagrangian. Finally in Section six we will discuss the Frobenious integrability of these first-order equations.

\section{Currents in terms of Dual Fields}

The first-order field equations of a generic $D$-dimensional PCM was derived in$^{35}$ by means of introducing a set of Lagrange multipliers. They read

\[ (-1)^D T_{ml} \ast F^m = -dA_l - C_{ln}^k F^m \wedge A_k. \]  

(2.1)

Here $\{F^m\}$ are the components of the Noether right-current which is a flat connection of the half $G$-global symmetry of the model. $\{A_l\}$ are the $(D-2)$-forms which are the free Lagrange multipliers of the theory $D$ being the dimension of the base manifold which for the superstring case becomes the worldsheet. When $\{F^m\}$ are solved via (2.1) in terms of $\{A_l\}$ the permissable dual fields $\{A_l\}$ appearing in (2.1) are the ones which enable $\{F^m\}$ to satisfy the Bianchi identities

\[ dF^l = -\frac{1}{2} C^l_{mn} F^m \wedge F^n. \]  

(2.2)

$^2$In the sense that it is explicitly composed of scalar fields rather than currents.
In other words they must be solved from these equations. As a consequence of the construction of the dual theory in [35] (2.1) and (2.2) replace the original PCM field equations. All the Latin indices in the above formulation run from 1 to the dimension of the target group \( G \). \( C^k_{ln} \) are the structure constants of the corresponding Lie algebra. \( T_{ml} \) is the Cartan-Killing metric or more generally the trace convention of the representation chosen for the Lie algebra of \( G \). In this section we will systematically derive the solutions of \( F \) in terms of \( A \) when \( D = 2 \) with signature \( s = 1 \). In other words we will solve (2.1) for \( D = 2 \). When \( D = 2 \) the Lagrange multipliers \( A \) become scalar fields. First let us observe that in \( D = 2 \) on a one-form for which \( p = 1 \)

\[
\ast \ast \ast = (-1)^{(p(D-p)+s)} = (-1)^{(1(2-1)+1)} = 1. \tag{2.3}
\]

Before attacking on (2.1) let us first consider a similar non-matrix equation

\[
\ast F = -dA - F \wedge A, \tag{2.4}
\]

which does not contain group indices. In solving (2.4) as a first trial if one assumes

\[
F = - \ast dA, \tag{2.5}
\]

from (2.4) one obtains

\[
- \ast \ast dA = -dA \neq -dA + A \ast dA. \tag{2.6}
\]

Then a second trial

\[
F = - \ast dA + AdA, \tag{2.7}
\]

yields

\[
- \ast \ast dA + A \ast dA \neq -dA + A \ast dA - A^2 dA. \tag{2.8}
\]

Adding again the excessive term to the solution anzats

\[
F = - \ast dA + AdA - A^2 \ast dA, \tag{2.9}
\]

via (2.4) yields

\[
- \ast \ast dA + A \ast dA - \ast A^2 dA \neq -dA + A \ast dA - A^2 dA + A^3 \ast dA. \tag{2.10}
\]

One more trial

\[
F = - \ast dA + AdA - A^2 \ast dA + A^3 dA, \tag{2.11}
\]

\[4\]
gives
\[-**dA + A*dA = -**A^2dA + A^3*dA \neq -dA + A*dA - A^2dA + A^3*dA - A^4dA,\]
\[(2.12)\]
which suggests the form of the series that solves (2.4). Therefore we will assume that
\[F = - * dA + A*dA - A^2*dA + A^3*dA - A^4*dA + \cdots\]
\[= -(1 + A^2 + A^6 + \cdots)*dA + (A + A^3 + A^5 + A^7 + \cdots)dA.\]
\[(2.13)\]
Since for \(-1 < x < 1\)
\[
\frac{1}{1+x} + \frac{1}{1-x} = 2(1 + x^2 + x^4 + \cdots),
\]
\[
\frac{1}{1-x} - \frac{1}{1+x} = 2(x + x^3 + x^5 + \cdots),
\]
\[(2.14)\]
we have
\[F = -\frac{1}{2}(\frac{1}{1+A} + \frac{1}{1-A})*dA + \frac{1}{2}(\frac{1}{1-A} - \frac{1}{1+A})dA,\]
\[(2.15)\]
which yields
\[F = -\frac{1}{1-A^2}*dA + \frac{A}{1-A^2}dA.\]
\[(2.16)\]
Although this solution is derived via a series expansion by inspecting term by term one can show that by simply substituting (2.16) in (2.4) its functional form satisfies (2.4) and thus it is a particular solution. However it needs to be checked whether (2.16) is the most general form of solution to the algebraic equation (2.4). Now being equipped with the form of solutions for the corresponding non-matrix equation we will focus on (2.1) which for \(D = 2\) can be written as
\[T * F = -dA - CF,\]
\[(2.17)\]
where we start using the matrix notation with definitions
\[(T^t)^m \_l = T_{ml}, \quad (C^t)^m \_n = C^{k}_{tn}A_k, \quad (F)^m = F^m, \quad (A)^k = A_k.\]
\[(2.18)\]
Here again \( \{ A_k \} \) are scalar fields. In the following instead of (2.17) we will prefer to work on
\[
* F = -T^{-1}dA - T^{-1}CF. \tag{2.19}
\]
Now like we have done above the trials of solution
\[
F = -T^{-1}dA, \\
F = -T^{-1}dA + T^{-1}CT^{-1}dA, \\
F = -T^{-1}dA + T^{-1}CT^{-1}dA - T^{-1}CT^{-1}CT^{-1}dA, \\
F = -T^{-1}dA + T^{-1}CT^{-1}dA - T^{-1}CT^{-1}CT^{-1}dA
+ T^{-1}CT^{-1}CT^{-1}CT^{-1}dA, \tag{2.20}
\]
in (2.19) suggests that the series expansion of the general solution has the form
\[
F = -T^{-1}(1 + (CT^{-1})^2 + (CT^{-1})^4 + \cdots) \ast dA
\]
\[
+ T^{-1}(CT^{-1} + (CT^{-1})^3 + (CT^{-1})^5 + \cdots) \ast dA. \tag{2.21}
\]
With the help of the Taylor expansion
\[
(1 + X)^{-1} = 1 - X + X^2 - X^3 + \cdots, \tag{2.22}
\]
for matrices we can express (2.21) as
\[
F = -T^{-1}(1 - (CT^{-1})^2)^{-1} \ast dA + T^{-1}CT^{-1}(1 - (CT^{-1})^2)^{-1} \ast dA. \tag{2.23}
\]
Again we have derived this solution as a series expansion however as before we can once more show that its functional form (2.23) solves (2.19) exactly \(^3\). We can prove this by directly substituting (2.23) in (2.19). If one does so the left hand side (lhs) becomes
\[
- T^{-1}(1 - (CT^{-1})^2)^{-1}dA + T^{-1}CT^{-1}(1 - (CT^{-1})^2)^{-1} \ast dA, \tag{2.24}
\]
and the right hand side (rhs) yields
\[
-(T^{-1} + T^{-1}CT^{-1}CT^{-1}(1 - (CT^{-1})^2)^{-1}dA + T^{-1}CT^{-1}(1 - (CT^{-1})^2)^{-1} \ast dA. \tag{2.25}
\]
Now by using the binomial inverse theorem
\[
(A + B)^{-1} = A^{-1} - A^{-1}B(B + BA^{-1}B)^{-1}BA^{-1}, \tag{2.26}
\]
\(^3\)One may again check whether (2.23) is the general solution of (2.19).
one can show that
\[
(1 - (CT^{-1})^2)^{-1} = -TC^{-1}TC^{-1} + ((CT^{-1})^2)^{-1}(1 - (CT^{-1})^2)^{-1}.
\] (2.27)
Thus when substituted in (2.25) this proves that the rhs namely (2.25) and the lhs (2.24) become equal.

3 The Integrated Lax Formulation

In the previous section we have exactly derived the solutions of (2.1) which express the components of the Noether current (which is conserved due to the half of the $G$-global symmetry of the PCM) in terms of the dual (Lagrange multiplier) fields in (2.23). Now referring to [35] we should face the fact that as a coarse of the first-order formulation, in addition to (2.23) these current components which were considered as independent fields in [35] must also satisfy the second set of field equations (2.2) of the dualised theory. These Bianchi identities originate from the zero curvature condition
\[
dG' + G' \wedge G' = 0,
\] (3.1)
of the right-current $G' = F^mT_m$. Here $\{T_m\}$ are the generators of the Lie algebra of the target group manifold $G$. By using (2.23) we can express $G'$ as
\[
G' = [T^{-1}CT^{-1}(1 - (CT^{-1})^2)^{-1}dA]^mT_m - [T^{-1}(1 - (CT^{-1})^2)^{-1} \ast dA]^mT_m.
\] (3.2)
This relation whose components are the general solutions of (2.1) together with (3.1) replace the original second-order field equations of the PCM. In this section we will show that (3.1) and (3.2) are indeed forming an integrated Lax formalism for the theory. First let us observe that (3.1) has the general solution
\[
G' = g^{-1}dg,
\] (3.3)
where $g$ is the Lie group valued map. Independently we also owe the pure gauge form of (3.3) to the construction of the dual theory in [35] where we start with directly the Noether current (3.3). At this stage we should remark that owing to the formulation of [35] (2.1) and (3.1) are independent of the form or the parametrization of $g$ thus the results there and here are valid for the entire solution space. Now let us call
\[
\lambda(A^n) = CT^{-1},
\] (3.4)
which is a matrix function of the dual scalar fields. Substituting (3.3) in (3.2), and putting it in a more explicit form, also giving it a new name we get

$$L = g^{-1}dg = [T^{-1}\lambda \frac{1}{1 - \lambda^2}]^m dA^n T_m - [T^{-1}\frac{1}{1 - \lambda^2}]^m * dA^n T_m. \quad (3.5)$$

Here we prefer the notation

$$\frac{1}{1 - \lambda^2} \equiv (1 - \lambda^2)^{-1}. \quad (3.6)$$

(3.5) is certainly a Lax connection whose consistency or integration conditions namely (3.1) lead to the field equations of the theory in the dual language. The reader should realize that in (3.5) although $\lambda$ appears at the place of a spectral parameter which is a complex variable of the ordinary Lax connection

$$L = \frac{\lambda}{1 - \lambda^2} J - \frac{1}{1 - \lambda^2} * J, \quad (3.7)$$

where $J$ happens to be the Noether current of the PCM lagrangian. In (3.5) it is a matrix functional of the dual scalar fields. This is a bare consequence of the overall integration of the system of field equations of the theory by degree one. Therefore we may simply call (3.5) an integrated Lax formalism of the PCM. We can furthermore express (3.5) as

$$L = \left[ [T^{-1}\lambda \frac{1}{1 - \lambda^2}]^m \partial_\alpha A^n T_m - [T^{-1}\frac{1}{1 - \lambda^2}]^m \partial_\beta A^n \epsilon_\beta \epsilon_\alpha T_m \right] dx^\alpha, \quad (3.8)$$

where on the 2D base manifold $M$ we have introduced $\epsilon_{\alpha \beta} = \sqrt{-\det(h)} \epsilon_{\alpha \beta}$ with $\epsilon_{01} = 1$ and $h$ is the Minkowski signature metric on $M$. Thus the components of $L$ become

$$L_\alpha = [T^{-1}\lambda \frac{1}{1 - \lambda^2}]^m \partial_\alpha A^n T_m - [T^{-1}\frac{1}{1 - \lambda^2}]^m \partial_\beta A^n \epsilon_\beta \epsilon_\alpha T_m, \quad (3.9)$$

for which $\alpha = 1, 2$. Finally we can define the integrated Lax pair of the theory as

$$\partial_\alpha g = gL_\alpha, \quad (3.10)$$

whose compatibility conditions $\partial_\alpha \partial_\beta g = \partial_\beta \partial_\alpha g$ which are equivalent to (3.1) can be expressed in component form as

$$\partial_\alpha L_\beta - \partial_\beta L_\alpha + [L_\alpha, L_\beta] = 0. \quad (3.11)$$

As a result of the construction of the dual theory these equations are equivalent to the original field equations of the PCM.
4 The Dual Lagrangian

Now as we have explicitly solved (2.1) for $D = 2$ we can again explicitly construct the lagrangian of the dual theory in terms of the dual fields $\{ A_k \}$. The Bianchi-term containing lagrangian was introduced in [35] in terms of $\{ A_k \}$ and the flat connection components $\{ F^m \}$. Since we have also expressed $\{ F^m \}$ in terms of the dual scalar fields the pure form of the lagrangian of the dual scalar field theory can now be given as

$$L_{\text{Dual}} = -\frac{1}{2} tr(*G' \wedge G') + F^l A_l,$$  (4.1)

where $G'$ should be taken as (3.2) and the field strength in the Chern-Simon type term is

$$F = dG' + \frac{1}{2}[G', G'],$$  (4.2)

whose components can be given as

$$F^l = dF^l + \frac{1}{2} C^l_{mn} F^m \wedge F^n.$$  (4.3)

We can now express (4.1) as

$$L_{\text{Dual}} = -\frac{1}{2} * F^m \wedge F^n T_{mn} + (dF^l + \frac{1}{2} C^l_{mn} F^m \wedge F^n) \wedge A_l,$$  (4.4)

where $F^l$ must be read from (2.23). We may further split the lagrangian (4.4) into two parts by eliminating the cross terms coming from the connection components in (2.23). First let us introduce the vector notation

$$G = T^{-1}(1 - (CT^{-1})^2)^{-1} dA \equiv MdA,$$  (4.5)

so that we can write (2.23) as

$$F = -*G + T^{-1} CG.$$  (4.6)

Now inserting (4.6) in the first term in (4.4) after some algebra we get

$$-\frac{1}{2} * F^m \wedge F^n T_{mn} = -\frac{1}{2} G^m \wedge *G^n T_{mn} + \frac{1}{2} C_{mn} G^m \wedge G^n + \frac{1}{2} C_{mn} * G^m \wedge *G^n$$

$$-\frac{1}{2} (T^{-1} C * G)^m \wedge (T^{-1} CG)^n T_{mn}.$$  (4.7)
Furthermore the second part of the second term becomes

\[
\frac{1}{2} C_{mn} F^m \wedge F^n \wedge A_t = \frac{1}{2} C_{mn} G^m \wedge *G^n - \frac{1}{2} C_{mn} G^m \wedge (T^{-1}CG)^n \\
- \frac{1}{2} C_{mn} G^m \wedge (T^{-1}CG)^n \\
+ \frac{1}{2} C_{mn} (T^{-1}CG)^m \wedge (T^{-1}CG)^n.
\]  

(4.8)

Now since

\[- \frac{1}{2} (T^{-1}C * G)^m \wedge (T^{-1}CG)^n T_{mn} = \frac{1}{2} C_{mn} G^m \wedge (T^{-1}CG)^n, \]  

(4.9)

Combining (4.7) with (4.8) in (4.4) and simplifying more we can express the dual lagrangian as

\[
\mathcal{L}_{\text{Dual}} = -\frac{1}{2} tr(\tilde{G} \wedge *\tilde{G}) - \frac{1}{2} tr(\tilde{G}' \wedge *\tilde{G}') + \frac{1}{2} C_{mn} \tilde{G}^m \wedge \tilde{G}^n \\
+ \frac{1}{2} C_{mn} \tilde{G}'^m \wedge \tilde{G}'^n - d(*\tilde{G}^l)A_l + d(\tilde{G}'^l)A_l.  
\]  

(4.10)

Here we have introduced the dual field strengths

\[
\tilde{G} = G^m T_m, \quad \tilde{G}' = (T^{-1}CG)^m T_m,
\]  

(4.11)

with \(\tilde{G}^m = G^m\) and \(\tilde{G}'^m = (T^{-1}CG)^m\).

5 Field Equations as a First-order PDE System

In this section we will comment on some methods of solving the PCM equations from different points of views. Firstly to solve the dual theory one may simply derive the Euler-Lagrange equations of the dual lagrangian which we have derived in the previous section. However this would not bring more ease as like the original PCM equations one would again obtain second-order field equations. Another route can be to consider the equivalent system (3.1) and (3.2) or their combined version (3.5). In this respect a direct method of

\footnote{We should remark that we freely raise and lower Latin indices when necessary for a compact notation.}
obtaining general solutions of the dual theory is to substitute (3.2) in (3.1) and seek solutions for \{A_n\}. By doing so one gets the equation

\[ d(-\ast \tilde{G} + \tilde{G}') + (\ast \tilde{G} + \tilde{G}') \wedge (\ast \tilde{G} + \tilde{G}') = 0. \quad (5.1) \]

In fact the system of equations achieved from (5.1) are more simplified than the field equations which can be obtained from the dual lagrangian (4.10). Alternatively one may consider the first-order Lax pair (3.10) where as we have mentioned before the role of the spectral parameter is replaced by a matrix functional of dual fields. At this point we may discuss that the inverse scattering method of [2, 39] which is based on the Riemann-Hilbert problem via the spectral parameter can be adopted by complexifying the dual fields and considering a spectral functional of the fields of the theory. One may even generate generalized forms of soliton solutions in this manner. Moreover other methods can also be derived to solve (3.10) as unlike the original Lax pair associated with (3.7) whose building block \( J \) is yet to be determined the integrated Lax pair of (3.10) is exactly resolved in terms of the dual scalar fields. In one sense these fields can be considered as dynamically free since they are only subject to the algebraic integrability conditions of (3.10).

By using the classical solution methods one may expect to generate richer classes of solutions since the degrees of freedom for manipulating the solutions increase from a single parameter to a functional due to the integration of the system by degree one. Another route to seek solutions is to construct the first-order system of partial differential equations of the theory for a particular parametrization of the Lie group valued map \( g \). If we consider the parametrization

\[ g = e^{\varphi^m(x)T_m}, \quad (5.2) \]

of \( g \) with scalars \( \varphi^i \) then we have [36, 37, 38]

\[ F^m = W^m_n(\varphi^i)d\varphi^n, \quad (5.3) \]

where the \( \dim G \times \dim G \) matrix \( W \) is

\[ W = (I - e^{-M})M^{-1}. \quad (5.4) \]

Here we define the matrix \( M \) as

\[ M^m_n = C^m_{im}\varphi^l. \quad (5.5) \]
By definition the rhs of (5.3) automatically satisfies the zero curvature conditions (2.2). Substituting (5.3) in (4.6) one obtains

\[ W^m_n(\phi^l)d\phi^n = -*G^m + (T^{-1}CG)^m. \] (5.6)

Now one is entirely free to start with any choice of \( \{A_n\} \), the unique constraint on them is the existence of a transformation from the fields \( \{\phi^m\} \) to the dual ones \( \{A_n\} \) via (5.6). Thus if one can find explicit or implicit relations between \( \{\phi^m\} \) and \( \{A_n\} \) which will satisfy (5.6) then one automatically by-passes (2.2) since the lhs of (5.6) trivially satisfies (2.2) so does the rhs due to the constructed transformation. Such a field transformation method which is seeking algebraic relations between matrix exponential and matrix polynomial forms may partially replace the differential equation solving methodology of the original \( D = 2 \) PCM. Furthermore starting from (5.6) we can also construct the first-order partial differential equation system of the theory. First let us assume a local coordinate bases \( \{dx^1, dx^2\} \) on the two-dimensional base manifold \( M \). We have already introduced the Minkowski signatured pseudo-Riemannian metric \( h \) on \( M \) it can be expressed as

\[ h = h_{\alpha\beta}dx^\alpha \otimes dx^\beta, \] (5.7)

where \( \alpha, \beta = 0, 1 \). We have \( [40] \)

\[ *dx^0 = h'(h^{00}dx^1 + h^{01}dx^0), \quad *dx^1 = h'(h^{10}dx^1 + h^{11}dx^0), \] (5.8)

where \( h' = \sqrt{|\text{Det}(h)|} \). Now (5.6) becomes

\[
\left[ W^m_n \partial_0 \phi^n + h'M^m_n \partial_0 A^n h^{01} + h'M^m_n \partial_1 A^n h^{11} - (T^{-1}C)^m_l M^n_l \partial_0 A^l \right] dx^0 + \\
\left[ W^m_n \partial_1 \phi^n + h'M^m_n \partial_0 A^n h^{00} + h'M^m_n \partial_1 A^n h^{10} - (T^{-1}C)^m_l M^n_l \partial_1 A^l \right] dx^1 = 0.
\] (5.9)

Here we have used the \( M \)-matrix notation introduced in (5.5). (5.9) is a first-order partial differential equation system for the fields \( \{\phi^m(x^\alpha), A^n(x^\alpha)\} \).

We should state that the dual fields enter into this system as free fields without any additional constraints the integrability conditions of (5.9) will bring algebraic constraints on them which will also shape the solution space of the dual theory.

### 6 Frobenius Integrability

In this section we will inspect the Frobenius integrability conditions of (5.6) and we will present the special set of solutions which follow these conditions.
Now starting from (3.6) let us construct

\[ W^m_n(\varphi) d\varphi^n T_m = (-* G^m + (T^{-1} CG)^m) T_m = \mathcal{G}'(A_n). \] (6.1)

If now we assume that \( \mathcal{G}'(A_n) \) is closed

\[ d\mathcal{G}' = 0, \] (6.2)
then locally it is an exact form namely there exists a Lie algebra-valued function \( \omega(A_n) \) such that

\[ \mathcal{G}'(A_n) = d\omega(A_n). \] (6.3)

For \( \mathcal{G}'(A_n) \) to be exact we must formally have

\[ \partial_{\mu} \left( [T^{-1} \lambda \frac{1}{1 - \lambda^2}]_m A^n - [T^{-1} \frac{1}{1 - \lambda^2}]_m \partial^\beta A^n \varepsilon_{\beta\alpha} \right) \]

\[ = \partial_\alpha \left( [T^{-1} \lambda \frac{1}{1 - \lambda^2}]_m \partial_\nu A^n - [T^{-1} \frac{1}{1 - \lambda^2}]_m \partial^\beta A^n \varepsilon_{\beta\mu} \right). \] (6.4)

However we should also state that instead of solving these equations by inspection one may seek special forms of closed 1-forms in (6.1) by assuming special combinations of \( \{A_n\} \). Apart from (6.4) if furthermore \( \mathcal{G}' \) satisfies

\[ [\omega, \mathcal{G}'] = 0, \] (6.5)
then by applying an exterior derivative on (6.5) one can show that

\[ \mathcal{G}' \wedge \mathcal{G}' = 0. \] (6.6)

In this case via (6.2) and (6.6) we have \( d\mathcal{G}' + \mathcal{G}' \wedge \mathcal{G}' = 0 \) which being the zero curvature condition (3.1) is the second set of equations of the dual form of the PCM. Thus the set of fields \( \{A_n\} \) which would satisfy (6.6) are the solutions of the dual theory. Furthermore since \(35\)

\[ W^m_n d\varphi^n T_m = e^{-\varphi^m T_m} d e^{\varphi^m T_m} \]

\[ = d(\varphi^m T_m) - \frac{1}{2}[\varphi^m T_m, d(\varphi^n T_n)] + \frac{1}{6}[\varphi^m T_m, [\varphi^n T_n, d(\varphi^l T_l)]] - \cdots, \] (6.7)
if one finds fields \( \{ A_n \} \) which satisfy equations (6.4) and (6.5) then choosing

\[ \varphi^m = \omega^m, \]  

(6.8)

with \( \omega = \omega^m T_m \) gives

\[ W^m_n d\varphi^n = d\omega^m = G^m, \]  

(6.9)

via (6.7) where due to the choice of (6.8) (6.5) can be used to eliminate all the terms but the first. Under these conditions as (6.9) is the same equation with (6.1) the first set of equations of the theory are also satisfied. Thus (6.8) become the solutions of the original \( D = 2 \) PCM. In this respect equations (6.4) and (6.5) give the dual solutions whereas by calculating \( G' \) and then \( \omega^m \) via (6.3) in terms of these dual solutions (6.8) provide the original solutions of the Lie algebra parametrization of the map \( g \).

7 Conclusion

In this work we have solved the algebraic first-order equations of the dual formulation of the PCM which were derived in [35]. Therefore by expressing the right-Noether current in terms of the dual fields we have obtained an integrated Lax formulation of the theory. We call it integrated since the formulation presented in [35] and the solutions derived here enable us to reduce the degree of the system of equations by one. This reflects itself in the Lax connection and the associated Lax pair. The ordinary Lax pair contains a complex spectral parameter however although the usual form of the Lax connection again appears here this time the constant parameter is substituted with a matrix functional of the fields of the dual theory. Furthermore we have also taken the advantage of reaching the above mentioned solutions in constructing the dual lagrangian which is the most general dual scalar field theory equivalent of the PCM. Following a concise discussion on the system of equations of the dual theory, by the means of the matrix exponential-polynomial transformation of the current components to the scalar field strengths for the exponential parametrization of the group map we have mentioned various solution techniques, and also obtained the first-order PDE system of the \( D = 2 \) PCM. In connection with this we have discussed the Frobenious integrability of this system.

The \( D = 2 \) PCM has a deeper and a more extensive symmetry scheme than the global \( G \times G \) left-right symmetry. Firstly as an integrable system it
has infinitely many local left-right conserved charges \[41, 42, 43, 44, 45, 46, 47, 48, 49, 50\] which form \(W\)-algebras \[48, 49\]. Moreover apart from these usual symmetries of the integrable systems there are non-local conserved charges which extend the \(G \times G\) global symmetry algebras to the infinite dimensional Kac-Moody algebras which lead to the Yangians at the quantum level as well as the Virasoro algebras which on the total are called the hidden symmetries. The Lax formalism is at the heart of studying these symmetries. Starting from the Lax pair which serves as the generating functional for the monodromy matrices one studies the Poisson brackets of these matrices to obtain the Yang-Baxter type infinite algebras for the ultra-local cases and the more general ones for the non-ultra-local cases which are both related to the Kac-Moody algebras \[61, 60, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63\]. In either case in constructing the infinite dimensional symmetry algebras the explicit form of the Lax pair and the inverse scattering method of \[2, 39\] play the central role. In the ultra-local models, field independent \(r\)-matrices appear in the commutation relations of the spatial parts of the Lax pair whereas in the more generalized non-ultra-local models the field dependent \(s\)-matrices accompany the \(r\)-matrices. In addition the monodromy matrices are also defined as the ordered exponentials of the integrals of these Lax pair components satisfying similar Poisson brackets. Moreover an infinite number of conserved charges can be derived from the spectral parameter mode expansions of the monodromy matrices with the regularization provided by the infinite volume limit of the inverse scattering method. Equivalently in another direction the same Kac-Moody symmetries can be realized as infinitesimal transformation algebras (loop algebras) acting on the solution space of the PCM \[64, 65, 66, 67, 68, 69, 70, 71, 72, 73\]. In either of these approaches the hierarchy of continuity equations of the infinitely many global symmetries or the infinitesimal transformations are direct consequences of the Lax equation. On the other hand the energy-momentum tensor as a result of the backstage canonical structure also gives rise to Virasoro symmetries \[74, 75, 76, 77, 78\] for the PCM. In recent years in connection with the research programme on the integrability of the related string models the hidden symmetries of the PCM have been revisited \[25, 27, 33, 34, 48, 49, 50, 58, 79, 80, 81\]. The supersymmetric extensions of the PCMWZ are also studied \[48, 50, 82, 83, 84, 85, 86, 87\]. We may

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\(^5\) The reader may find a complete historical survey of the hidden symmetries of the PCM in \[33\].

\(^6\) Which for example appear in the strings on group manifolds with world-sheet super-
easily state that for all these above mentioned hidden symmetries the underlying framework is a natural consequence of the two distinct Hamiltonian formulations and the associated canonical structures provided by the Lax pair. Moreover the algebra structures as well as the explicit representations of the building blocks of these symmetries depend on the explicit form of the Lax connection components.

Therefore we may conclude that the Lax formalism is at the center of exploring the hidden symmetries of the PCM. On the other hand in this work we have derived a Lax connection and a Lax pair for a general base manifold and its Minkowski signature pseudo-Riemannian structure barely in terms of the fields of the dual theory by performing an overall integration of the field equations. Although the form of this integrated Lax formulation resembles the ordinary one (on which the entire above-mentioned analysis is based on) in our formulation the role of the spectral parameter is replaced by a matrix functional of the dual fields. Therefore following a complexification the degrees of freedom of the spectral analysis can be made a continuum. By means of these spectral matrix functionals the dynamics of the theory may directly enter into the canonical structure whose implications can be read from the field-dependent $s$-matrices of the more generalized non-ultra-local cases. At this point we should state that an integrated Lax pair may lead to a more refined and generalized hidden symmetry formulations as it is one degree of derivation close to the general solutions of the theory. Besides the explicit form of the connection components will enable one to compute explicitly the elements and the representations of the infinite-dimensional symmetry algebras. In this respect one may hope to find extensions of the already existing structures in the literature. Further mode expansions in the fields of the spectral matrix functional will certainly lead to richer structures. Apart from this new form of integrated Lax connection our results also contribute simplifications to solution generation. The approach of Section five reduces the problem of finding general solutions of the theory into finding field transformations between the original and the dual fields. This requires to find implicit or explicit forms of transformations relating the matrix exponential and polynomial dressings of the differentials on either side. Apart from being a tool to search for general solutions the first-order form of the field equations may also provide a framework for the study of the transformations of the solutions such as the Backlund ones and their possible generalizations.
One may also hope to build new solution generating techniques other than the inverse scattering method built on the new Lax formalism we have derived as we have discovered a new notion for the spectral degrees of freedom. Before ending we should remark the resemblance of the hierarchy of mode expansions [25] and the sequence of the solution ansatz namely the Taylor expansion terms of the coefficients of our integrated Lax connection. We also believe that the integrated dual Lax formalism constructed here should be covering the special cases of the T-dual string Lax formulations studied in [89, 90, 91]. One may also use the formalism introduced here to extend and generalize the results of [79] which studies the solutions of the $U(N)$ PCM. Finally further research may be performed to derive similar derivations of Section two and three for the supersymmetric extensions of PCM as well as the PCMWZ models.

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