SEMIDEFINITE PROGRAMMING RELAXATIONS FOR LINEAR
SEMI-INFINITE POLYNOMIAL PROGRAMMING

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Abstract. This paper studies a class of so-called linear semi-infinite polynomial programming (LSIPP) problems. It is a subclass of linear semi-infinite programming problems whose constraint functions are polynomials in parameters and index sets are basic semialgebraic sets. When the index set of an LSIPP problem is compact, a convergent hierarchy of semidefinite programming (SDP) relaxations is constructed under the assumption that the Slater condition and the Archimedean property hold. Compared with existing methods, in dealing with LSIPP problems, the SDP relaxation method is more computationally efficient. When the index set is noncompact, we use the technique of homogenization to equivalently convert the LSIPP problem into compact case under some generic assumption. As a byproduct, a new SDP relaxation method is derived for solving the class of polynomial optimization problems whose objective polynomials are stably bounded from below on noncompact feasible sets.

Key words  linear semi-infinite programming, semidefinite programming relaxations, sum of squares, polynomial optimization

1. INTRODUCTION

We consider the following linear semi-infinite polynomial programming (LSIPP) problem

\[
\begin{aligned}
  p^* \coloneqq \inf_{x \in \mathbb{R}^m} & \ c^T x \\
  \text{s.t.} & \ a(y)^T x + b(y) \geq 0, \ \forall y \in S \subseteq \mathbb{R}^n,
\end{aligned}
\]

where \( c \in \mathbb{R}^m, b(Y) \in \mathbb{R}[Y] := \mathbb{R}[Y_1, \ldots, Y_n] \) the polynomial ring in \( Y \) over the real field, \( a(Y) = (a_1(Y), \ldots, a_m(Y))^T \in \mathbb{R}[Y]^m \), and the index set \( S \) is a basic semialgebraic set defined by

\[
S \coloneqq \{ y \in \mathbb{R}^n \mid g_1(y) \geq 0, \ldots, g_s(y) \geq 0 \},
\]

where \( g_j(Y) \in \mathbb{R}[Y], j = 1, \ldots, s \). Lowercase letters (e.g. \( x, y \)) are hereinafter used for denoting points in a space while uppercase letters (e.g. \( X, Y \)) for variables. In this paper, we assume that \( 1.1 \) is feasible and bounded from below, i.e., \(-\infty < p^* < \infty\). Note that the problem \( 1.1 \) is NP-hard. Indeed, it is obvious that the problem of minimizing a polynomial \( f(Y) \in \mathbb{R}[Y] \) over \( S \) can be regarded as a special LSIPP problem (see Section 3.2.2). As is well known, the polynomial optimization problem is NP-hard even when \( n > 1, f(Y) \) is a nonconvex quadratic polynomial and \( g_j(Y) \)'s are linear \[15\]. Hence, a general LSIPP problem cannot be expected to be solved in polynomial time unless \( \text{P}=\text{NP} \).

LSIPP can be seen as a special branch of linear semi-infinite programming (LSIP), or more general, of semi-infinite programming (SIP), in which the involved
functions are not necessarily polynomials. Numerically, SIP problems can be solved by different approaches including, for instance, discretization methods, local reduction methods, exchange methods, simplex-like methods and so on. See the surveys [13, 14, 15, 22, 57, 53, 61] and the references therein for details. One of main difficulties in numerical treatment of general SIP problems is that the feasibility test of \( \bar{u} \in \mathbb{R}^m \) is equivalent to globally solve the problem of minimizing the constraint function with fixed \( \bar{u} \) over the index set, which is called the lower level problem. Some algorithms even require the computation of all global minima of the lower level problems which is only possible under strong assumptions on the constraint functions and index sets, e.g., that the index sets are box-shaped. Generally, the stopping rules of the above four kinds of methods provide infeasible points and hence lower bounds of SIP problems. By replacing the semi-infinite constraint by some overestimation, or successively increasing restrictions imposed on the right-hand side of discretized constraints, some approaches which generate feasible points and upper bounds are recently proposed in [9, 11, 33, 62, 71]. Typically, in applying existing methods in the literature on an SIP problem, the difficulty is mainly determined by the difficulty of NLP subproblems which need to be solved in each iteration.

While LSIPP, as a special case of SIP, has many applications like minmax problems, functional approximation problems (see examples in the Appendix), to the best of our knowledge, few of the numerical methods mentioned above are specially designed by exploiting features of polynomial optimization problems. Parpas and Rustem [49] proposed a discretization-like method to solve minimax polynomial optimization problems, which can be reformulated as semi-infinite polynomial programming (SIPP) problems. Using polynomial approximation and an appropriate hierarchy of semidefinite programming (SDP) relaxations, Lasserre presented an algorithm to solve the generalized SIPP problems in [29]. Based on an exchange scheme, an SDP relaxation method for solving SIPP problems was proposed in [70]. By using representations of nonnegative polynomials in the univariate case, an SDP method was given in [75] for LSIPP problems [1, 1] with \( S \) being closed intervals.

For SIPP or LSIPP problems, the feasibility test of a given point is equivalent to a global polynomial optimization problem. Due to representations of nonnegative polynomials as sums of squares and the dual theory of moments, a hierarchy of SDP relaxations can be constructed to approximate polynomial optimization problems, see [28, 62, 41, 50, 54] and the references therein. More precisely, we associate the set \( S \) with a so-called quadratic module which is a set of polynomials generated by \( g_j(Y) \)'s. If the Archimedean property (Definition 2.3) holds, then Putinar’s Positivstellensatz [52] states that any polynomial positive over \( S \) belongs to the quadratic module. According to Schmüdgen’s Positivstellensatz [60], if \( S \) is compact but the Archimedean property fails, we can replace in the above statement the quadratic module by the preordering generated by \( g_j(Y) \)'s. These representations of positive polynomials can be reduced to SDP feasibility problems. An SDP problem can be solved by interior-point method to a given accuracy in polynomial time [68, 74]. For this reason and in the light of the wide applications of LSIPP, it is worthy extending the powerful SDP relaxation approach to LSIPP problems.

**Contribution.** The main contribution of this paper is the following.

(i) When \( S \) is compact, under the assumptions of the Archimedean property and the Slater condition, we obtain a convergent hierarchy of SDP relaxations of LSIPP
by replacing the constraint that $a(y)^T x + b(y) \geq 0$ on $S$ by that the polynomial $a(Y)^T u + b(Y)$ belongs to the quadratic module associated with $S$. Compared with existing methods designed for general SIP problems, in dealing with LSIPP problems, the main features of the SDP relaxation method are the following: (a) It is a feasible point method and a decreasing sequence of upper bounds convergent to $p^*$ can be computed for arbitrary index sets (not necessarily box-shaped) without any initial bounds on the $x$-variables of (1.1). The practical importance of these upper bounds is that by combining the lower bounds of $p^*$ gained by, for instance, discretization methods, a desired $\varepsilon$-optimal solution of (1.1) can be obtained; (b) We say that the finite convergence of the proposed SDP relaxations occurs if the optimal value of the SDP relaxation of some finite order equals $p^*$. We point out that a rank condition on the dual moment matrices of the SDP relaxations can be used for certifying the finite convergence. Numerical experiments show that the finite convergence usually happens and can be certified in relaxations of the first few orders; (c) Without solving lower level problems (usually nonlinear) or subdivision of the index set as in each iteration of some existing feasible methods, the SDP relaxation method is more computationally efficient by solving a sequence of SDP problems as shown in the numerical experiments.

(ii) As a byproduct in coping with the case when $S$ is noncompact by the homogenization technique, a new SDP relaxation approach is obtained for solving the class of polynomial optimization problems whose objective polynomials are stably bounded from below on noncompact feasible sets, which was first proposed and investigated by Marshall in [40]. Note that the classic Lasserre’s SDP relaxation method might fail for this class of problems, see Example 3.24.

This paper is organized as follows. We introduce some notation and preliminaries in Section 2. Depending on whether $S$ is compact or not, two SDP relaxation methods for (1.1) and a stopping criterion are proposed in Section 3. Some numerical experiments are given in Section 3 to show the efficiency of the SDP relaxations. A conclusion is made in Section 5.

2. Notation and Preliminaries

Here is some notation used in this paper. The symbol $\mathbb{N}$ (resp., $\mathbb{R}$) denotes the set of nonnegative integers (resp., real numbers). For any $t \in \mathbb{R}$, $[t]$ denotes the smallest integer that is not smaller than $t$. For $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $\|y\|_2$ denotes the standard Euclidean norm of $y$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $\|\alpha\|_1 = \alpha_1 + \cdots + \alpha_n$. For $k \in \mathbb{N}$, denote $\mathbb{N}_k^n = \{\alpha \in \mathbb{N}^n \mid \|\alpha\|_1 \leq k\}$. For $y \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$, $y^\alpha$ denotes $y_1^{\alpha_1} \cdots y_n^{\alpha_n}$. $\mathbb{R}[Y] = \mathbb{R}[Y_1, \ldots, Y_n]$ denotes the ring of polynomials in $(Y_1, \ldots, Y_n)$ with real coefficients. For $k \in \mathbb{N}$, denote by $\mathbb{R}[Y]_k$ the set of polynomials in $\mathbb{R}[Y]$ of total degree up to $k$. For a symmetric matrix $W$, $W \succeq 0 (> 0)$ means that $W$ is positive semidefinite (definite). For two symmetric matrices $A, B$ of the same size, $\langle A, B \rangle$ denotes the inner product of $A$ and $B$.

For any feasible point $x \in \mathbb{R}^m$ of (1.1), the active index set of $x$ is

$$\{y \in S \mid a(y)^T x + b(y) = 0\}.$$ 

Let $\mathcal{Y}$ be the space of all functions $\gamma : S \to \mathbb{R}$ equipped with natural algebraic operations of addition and multiplication by a scalar. Associate $\mathcal{Y}$ with the linear space $\mathcal{Y}^* = \{\lambda = (\lambda_y)_{y \in S} \mid \lambda_y \in \mathbb{R}, \lambda_y \neq 0 \text{ for finitely many } y \in S\}$. For $\gamma \in \mathcal{Y}$ and $\lambda \in \mathcal{Y}^*$, define the scalar product $\langle \gamma, \lambda \rangle = \sum_{y \in S} \lambda_y \gamma(y)$. For any $x \in \mathbb{R}^m$, 

the semi-infinite constraint in (1.1) is equivalent to \( a(Y)^T x + b(Y) \in K \) where \( K = \{ \gamma \in \mathcal{Y} \mid \langle \gamma, y \rangle \geq 0, \forall y \in S \} \). The dual cone of \( K \) is
\[
K^* = \{ \lambda \in \mathcal{Y}^* \mid \langle \lambda, \gamma \rangle \geq 0, \forall \gamma \in K \} = \{ \lambda \in \mathcal{Y}^* \mid \lambda_y \geq 0, \forall y \in S \}.
\]
Then, the Lagrangian dual of (1.1) is (cf. [61])
\[
\begin{cases}
d^* := \sup_{y \in S} - \sum_{y \in S} \lambda_y b(y) \\
\text{s.t.} \quad \sum_{y \in S} \lambda_y a(y) = c, \\
\lambda_y \geq 0, \forall y \in S,
\end{cases}
\]
where only finitely many dual variables \( \lambda_y, y \in S \), take positive values. The problem (D) is known as the Haar dual problem [4] of (1.1).

**Definition 2.1.** We say that the Slater condition holds for the problem (1.1) if there exists \( \bar{x} \in \mathbb{R}^n \) such that \( a(y)^T \bar{x} + b(y) > 0 \) for all \( y \in S \).

**Proposition 2.2.** [5] If \( S \) is compact and the Slater condition holds for (1.1), then \( p^* = d^* \) and \( d^* \) is attainable.

Next we recall some background about sums of squares (s.o.s) of polynomials and the dual theory of moment matrices. For any \( f(Y) \in \mathbb{R}[Y]_k \), let \( f \) denote its column vector of coefficients in the canonical monomial basis of \( \mathbb{R}[Y]_k \). A polynomial \( f(Y) \in \mathbb{R}[Y] \) is said to be a sum of squares of polynomials if it can be written as \( f(Y) = \sum_{i=1}^t f_i(Y)^2 \) for some \( f_1(Y), \ldots, f_t(Y) \in \mathbb{R}[Y] \). The symbol \( \Sigma^2 \) denotes the set of polynomials that are s.o.s.

Let \( G := \{ g_1, \ldots, g_s \} \) be the set of polynomials that defines the semialgebraic set \( S \) [12]. We denote by
\[
Q(G) := \left\{ \sum_{j=0}^s \sigma_j g_j \mid g_0 = 1, \sigma_j \in \Sigma^2, j = 0, 1, \ldots, s \right\}
\]
the quadratic module generated by \( G \) and denote by
\[
Q_k(G) := \left\{ \sum_{j=0}^s \sigma_j g_j \mid g_0 = 1, \sigma_j \in \Sigma^2, \deg(\sigma_j g_j) \leq 2k, j = 0, 1, \ldots, s \right\}
\]
its \( k \)-th quadratic module. It is clear that if \( f \in Q(G) \), then \( f(y) \geq 0 \) for any \( y \in S \). However, the converse is not necessarily true, see Example 2.5. Note that checking \( f \in Q_k(G) \) for a fixed \( k \in \mathbb{N} \) is an SDP feasibility problem [25, 50].

For \( k \in \mathbb{N} \), denote \( s(k) := \binom{n+k}{n} \). A finite sequence of real numbers \( z := (z_\alpha)_{\alpha \in \mathbb{N}^n_{2k}} \in \mathbb{R}^{s(2k)} \) whose elements are indexed by \( n \)-tuples \( \alpha \in \mathbb{N}^n_{2k} \) is called a truncated moment sequence up to order \( 2k \). For \( z \in \mathbb{R}^{s(2k)} \), if there exists a Borel measure \( \mu \) on \( \mathbb{R}^n \) such that
\[
z_\alpha = \int Y^\alpha d\mu(y), \forall \alpha \in \mathbb{N}^n_{2k},
\]
then we say that \( z \) has a representing measure \( \mu \). The associated \( k \)-th moment matrix is the matrix \( M_k(z) \) indexed by \( \mathbb{N}^n_{2k} \), with \((\alpha, \beta)\)-th entry \( z_{\alpha+\beta} \) for \( \alpha, \beta \in \mathbb{N}^n_{2k} \). Given a polynomial \( f(Y) = \sum_{\alpha} f_{\alpha} Y^\alpha \), for \( k \geq d_f := \lceil \deg(f)/2 \rceil \), the \( (k-d_f) \)-th localizing moment matrix \( M_{k-d_f}(f(z)) \) is defined as the moment matrix of the shifted
vector \((f z)_{\alpha}\) for \(\alpha \in \mathbb{N}_2^{n(k-d_f)}\), with \(f z)_{\alpha} = \sum_{\beta} f_\beta z_{\alpha + \beta}\). \(\mathcal{M}_{2k}\) denotes the space of all truncated moment sequences with order at most \(2k\). For any \(z \in \mathcal{M}_{2k}\), the Riesz functional \(L_z\) on \(\mathbb{R}[Y]_{2k}\) is defined by
\[
L_z \left( \sum_{\alpha} q_\alpha Y_1^{\alpha_1} \cdots Y_n^{\alpha_n} \right) := \sum_{\alpha} q_\alpha z_\alpha, \quad \forall q(Y) \in \mathbb{R}[Y]_{2k}.
\]
From the definition of the localizing moment matrix \(M_{k-d_f}(f z)\), it is easy to check that
\[
q^T M_{k-d_f}(f z) q = L_z (f(Y) q(Y)^2), \quad \forall q(Y) \in \mathbb{R}[Y]_{k-d_f}.
\]
Let \(d_j := [\text{deg}(g_j)/2]\) for each \(j = 1, \ldots, s\). For any \(v \in S\), let \(\zeta_{2k,v} := [v^\alpha]_{\alpha \in \mathbb{N}_2^k}\) be the Zeta vector of \(v\) up to degree \(2k\), i.e.,
\[
\zeta_{2k,v} = [1, v_1, \ldots, v_n, v_1^2, v_1 v_2, \ldots, v_n^{2k}].
\]
Then, \(M_k(\zeta_{2k,v}) \geq 0\) and \(M_{k-d_j}(g_j \zeta_{2k,v}) \geq 0\) for \(j = 1, \ldots, s\). In fact, let \(g_0 = 1\), then for each \(j = 0, 1, \ldots, s\),
\[
q^T M_{k-d_j}(g_j \zeta_{2k,v}) q = L_{\zeta_{2k,v}} (g_j(Y) q(Y)^2) = g_j(v)q(v)^2 \geq 0, \quad \forall q(Y) \in \mathbb{R}[Y]_{k-d_j}.
\]
**Definition 2.3.** We say that \(Q(G)\) is Archimedean if there exists \(\psi \in Q(G)\) such that the inequality \(\psi(y) \geq 0\) defines a compact set in \(\mathbb{R}^n\).

Note that the Archimedean property implies that \(S\) is compact but the converse is not necessarily true. However, for any compact set \(S\) we can always force the associated quadratic module to be Archimedean by adding a redundant constraint \(M - \|y\|^2_2 \geq 0\) in the description of \(S\) for sufficiently large \(M\).

**Theorem 2.4.** [52] Putinar’s Positivstellensatz: Suppose that \(Q(G)\) is Archimedean.

(i) If a polynomial \(p \in \mathbb{R}[Y]\) is positive on \(S\), then \(p \in Q_k(G)\) for some \(k \in \mathbb{N}\);

(ii) If \(M_k(z) \geq 0\) and \(M_k(g_j z) \geq 0\) for all \(j = 1, \ldots, s\), and all \(k = 0, 1, \ldots\), then \(z\) has a representing measure \(\mu\) with support contained in \(S\).

The following is an illustrative example for the concepts and notation introduced in this section.

**Example 2.5.** Consider the LSIPP problem
\[
(2.3) \quad p^* := \inf_{x \in \mathbb{R}} - \frac{x}{2} \quad \text{s.t.} \quad (1 - 3y_2)x + 3y_1 \geq 0, \quad \forall y \in S,
\]
where
\[
S := \{ y \in \mathbb{R}^2 \mid y_1 \geq 0, y_1^2 - y_2^3 \geq 0 \}
\]
is the gray shadow below the right half of the cusp as shown in Figure 1. Since \((0, 0) \in S\), a feasible \(x\) must be nonnegative. Clearly, \(x = 0\) is a feasible point. \(x > 0\) is feasible if and only if
\[
0 \geq \max_{x \in S} \left\{ \frac{y_2}{3} - \frac{y_1}{x} \right\} = \max_{x \in S} \left\{ \frac{4}{9} - \frac{y_1}{x} \right\}.
\]
The latter maximum is attained at \(\frac{8x^3}{27}\) with optimal value \(\frac{4x^2}{27} - \frac{1}{3}\). Thus, the feasible set of (2.3) is \([0, 3/2]\) and the minimizer is \(x^* = \frac{3}{2}\). The dual problem of (2.3)
is
\[
\begin{align*}
\{ d^* := & \sup_{y \in S} - \sum_{y \in S} 3\lambda_y y_1 \\
\text{s.t. } & \sum_{y \in S} \lambda_y (1 - 3y_2) = -\frac{1}{2}, \\
& \lambda_y \geq 0, \forall y \in S,
\end{align*}
\]
where only finitely many \( \lambda_y \) take positive values. Because the Slater condition holds for any point \( x \in (0, \frac{3}{2}) \), we have \( p^* = d^* \).

Let \( G := \{ g_1, g_2 \} \) with \( g_1 = Y_1 \) and \( g_2 = Y_1^2 - Y_2^3 \). By definition,
\[
Q(G) := \left\{ \sum_{j=0}^{2} \sigma_j g_j \mid \sigma_j \in \Sigma^2, j = 0, 1, 2 \right\}.
\]
Clearly, \( Q(G) \) is not Archimedean since \( S \) is noncompact. For a finite sequence of real numbers \( z := (z_{i,j})_{(i,j) \in \mathbb{N}_2^2} \),
\[
M_1(z) = \begin{bmatrix}
  z_{(0,0)} & z_{(1,0)} & z_{(0,1)} \\
  z_{(1,1)} & z_{(2,0)} & z_{(1,1)} \\
  z_{(0,1)} & z_{(1,1)} & z_{(0,2)}
\end{bmatrix}
\]
is the associated 1-th moment matrix and
\[
M_1(g_2z) = \begin{bmatrix}
  z_{(2,0)} - z_{(3,0)} & z_{(3,0)} - z_{(1,3)} & z_{(2,1)} - z_{(0,4)} \\
  z_{(3,0)} - z_{(3,0)} & z_{(4,0)} - z_{(2,3)} & z_{(3,1)} - z_{(1,4)} \\
  z_{(2,1)} - z_{(4,0)} & z_{(3,1)} - z_{(1,4)} & z_{(2,2)} - z_{(0,5)}
\end{bmatrix}
\]
is the associated 1-th localizing moment matrix at \( g_2 \). \qed

3. SDP RELAXATIONS OF LSIPP

In this section, depending on whether the index set \( S \) is compact or not, we shall construct two hierarchies of SDP relaxations and provide a sufficient stopping criterion when the finite convergence occurs for solving the LSIPP problem \( (1.1) \).

3.1. **Compact case**. We assume that \( S \) in \( (1.1) \) is compact.
3.1.1. SDP relaxations of compact LSIPP problems. For a given feasible point \( x \in \mathbb{R}^m \) of the LSIPP problem (1.1), the constraint requires that the polynomial \( a(Y)^T x + b(Y) \in \mathbb{R}[Y] \) is nonnegative on \( S \). Since every polynomial in the quadratic module \( Q(G) \) of \( S \) generated by \( G \) is nonnegative on \( S \), we can relax the problem (1.1) as follows

\[
(3.1) \quad p^{\text{sos}} := \inf_{x \in \mathbb{R}^m} c^T x \quad \text{s.t.} \quad a(Y)^T x + b(Y) \in Q(G).
\]

Clearly, any feasible point of (3.1) is also feasible for (1.1). Hence, we have \( p^{\text{sos}} \geq p^* \).

**Theorem 3.1.** If \( Q(G) \) is Archimedean and the Slater condition holds for the LSIPP problem (1.1), then \( p^{\text{sos}} = p^* \).

**Proof.** See Appendix A \( \square \)

Note that we do not require that \( p^* \) is attainable in the proof. Define

\[
(3.2) \quad d_j := \lfloor \deg(g_j)/2 \rfloor, \quad d_S := \max\{1, d_1, \ldots, d_s\}, \quad d_P := \max\{d_S, \lfloor \deg(a_1)/2 \rfloor, \ldots, \lfloor \deg(a_m)/2 \rfloor, \lfloor \deg(b)/2 \rfloor \}.
\]

For \( k \geq d_P \), replacing \( Q(G) \) in (3.1) by its \( k \)-th truncation \( Q_k(G) \), we obtain

\[
(3.3) \quad \begin{align*}
p^{\text{sos}}_k := \inf_{x \in \mathbb{R}^m} & c^T x \\
\text{s.t.} & \quad a(Y)^T x + b(Y) = \sum_{j=0}^s \sigma_j(Y) g_j(Y), \\
& \quad g_0 = 1, \sigma_j \in \Sigma^2, \deg(\sigma_j g_j) \leq 2k, \quad j = 0, \ldots, s.
\end{align*}
\]

Now we reformulate (3.3) as an SDP problem. For any \( t \in \mathbb{N} \), let \( m_t(Y) \) be the column vector consisting of all the monomials in \( Y \) of degree up to \( t \). Recall that \( s(t) = \binom{n+t}{n} \) which is the dimension of \( m_t(Y) \). For each \( j = 0, 1, \ldots, s \), there exists a positive semidefinite matrix \( Z_j \in \mathbb{R}^{s(k-d_j) \times s(k-d_j)} \) such that

\[
\sigma_j(Y) = m_{k-d_j}(Y)^T \cdot Z_j \cdot m_{k-d_j}(Y).
\]

For each \( \alpha \in \mathbb{N}^n_{2k} \), we can find a symmetric matrix \( C_{j,\alpha} \in \mathbb{R}^{s(k-d_j) \times s(k-d_j)} \) such that the coefficient of \( \sigma_j g_j \) equals \( \langle Z_j, C_{j,\alpha} \rangle \) for each \( j = 0, 1, \ldots, s \). Let

\[
b(Y) = \sum_{\alpha \in \mathbb{N}^n_{2k}} b_{\alpha} Y^\alpha \quad \text{and} \quad a_i(Y) = \sum_{\alpha \in \mathbb{N}^n_{2k}} a_{i,\alpha} Y^\alpha, \quad i = 1, \ldots, m.
\]

Then (3.3) can be written as the SDP problem

\[
\begin{align*}
p^{\text{sos}}_k = \inf_{Z_j \succeq 0, x \in \mathbb{R}^m} & c^T x \\
\text{s.t.} & \quad \sum_{i=1}^m x_i a_{i,\alpha} + b_{\alpha} = \sum_{j=0}^s \langle Z_j, C_{j,\alpha} \rangle, \quad \forall \alpha \in \mathbb{N}^n_{2k}.
\end{align*}
\]

It follows that

**Theorem 3.2.** If \( Q(G) \) is Archimedean and the Slater condition holds for the LSIPP problem (1.1), then \( p^{\text{sos}}_k \) decreasingly converges to \( p^* \) as \( k \to \infty \).

**Proof.** See Appendix A \( \square \
Remark 3.4. Proof. See Appendix A.

Theorem 3.3. If $Q(G)$ is Archimedean and the Slater condition holds for the LSIPP problem (1.1), then $p_k^{\text{mom}}$ decreases converges to $p^*$ as $k \to \infty$.

Proof. See Appendix A.

Condition 3.5. An optimizer $z^*$ of the $k$-th SDP relaxation (3.4) satisfies the Rank Condition when

$$\exists t \in \mathbb{N} \quad \text{s.t.} \quad d_p \leq t \leq k \quad \text{and} \quad \text{rank} M_{t-d_p}(z^*) = \text{rank} M_t(z^*).$$
This condition can be used for certifying the finite convergence of Lasserre’s SDP relaxations [25] of polynomial optimization problems [28, 32, 44]. Under this condition, by [7, Theorem 1.1], \( z^* \) has a unique \( r \)-atomic measure supported on \( S \), i.e., there exist \( r \) positive real numbers \( \lambda_1, \ldots, \lambda_r \) and \( r \) distinct points \( v_1, \ldots, v_r \in S \) such that

\[
z^* = \lambda_1 \zeta_{2k, v_1} + \cdots + \lambda_r \zeta_{2k, v_r},
\]

where \( \zeta_{2k,v} \) is the Zeta vector of \( v \) up to degree \( 2k \).

**Theorem 3.6.** Suppose that \( Q(G) \) is Archimedean and the Slater condition holds for the LSIPP problem (1.1). If the Rank Condition holds for an optimizer \( z^* \) of the \( k \)-th SDP relaxation (3.4), then \( p^p = p^* \) and \( v_1, \ldots, v_r \) belong to the active index set of each minimizer \( x^* \) of (1.1).

**Proof.** See Remark 2 and Theorem 2 in [27]. \( \square \)

**Remark 3.7.** (i) The extraction procedure of the points \( v_i \)'s can be found in [20] and has been implemented in GloptiPoly.

(ii) Note that the Rank Condition is only a sufficient condition which means that it might not hold when the finite convergence happens. Indeed, we will see in Section 3.2.2 that when applying (3.3) and (3.4) to the LSIPP problems reformulated from polynomial optimization problems, we can get their classic Lasserre’s SDP relaxations whose finite convergence can be certified by the Rank Condition. In fact, [32, Example 6.24] shows that the Rank Condition is only sufficient not necessary. \( \square \)

3.1.3. Some extensions. Inspired by Lasserre and Putinar’s work [30], we would like to point out that the SDP relaxation method proposed here is applicable to a more general subclass of LSIP problems. Denote by \( X \subseteq \mathbb{R}^m \) a convex polyhedron defined by finitely many linear inequalities in the variables \( X \). Denote by \( \mathcal{A} \) the algebra consisting of functions generated by finitely many of the dyadic operations \{+, -, /, \vee, \wedge\} and monadic operations \{|\cdot|, (\cdot)^{1/p}, p \in \mathbb{N}\} on polynomials in \( \mathbb{R}[Y] \), where \( f \vee g := \max\{f, g\} \) and \( f \wedge g := \min\{f, g\} \) for \( f, g \in \mathbb{R}[Y] \). For example,

\[
\sqrt{|f(Y)| + g(Y)^2} \wedge \left( \frac{1}{g(Y)} \vee f(Y) \right) \in \mathcal{A}.
\]

Note that every function in \( \mathcal{A} \) has a lifted basic semi-algebraic representation [30, Definition 1]. Then, the SDP relaxations (3.3) and (3.4) can be extended for more general LSIP problems of the form (e.g., Problem B.6, B.8, B.11, and B.13–B.17)

\[
p^* := \inf_{x \in X} c^T x
\]

s.t. \( a^l(Y)^T x + b_l(Y) \geq 0, \forall y \in S \) and \( l = 1, \ldots, t \),

(3.6)

where \( c \in \mathbb{R}^m, a^l(Y) \in \mathcal{A}^m, b^l(Y) \in \mathcal{A}, l = 1, \ldots, t \) and

\[
S := \{ y \in \mathbb{R}^m \mid g_1(y) \geq 0, \ldots, g_s(y) \geq 0 \},
\]

(3.7)

where \( g_j(Y) \in \mathcal{A}, j = 1, \ldots, s \). In fact, as shown in [30], the nonnegativity of semi-algebraic functions in \( \mathcal{A} \) on the set (3.7) can be reduced to an equivalent polynomial problem case in a lifted space by adding some new variables. For instance, with \( f, h, g_1, g_2 \in \mathbb{R}[Y_1] \),

\[
\sqrt{f(y_1)} - 1/h(y_1) \geq 0 \quad \text{on} \quad \{ y_1 \in \mathbb{R} \mid |g_1(y_1)|g_2(y_1) \geq 1 \}. 
\]
can be written as \( y_2 - y_3 \geq 0 \) on
\[
\{ y \in \mathbb{R}^4 \mid f(y_1) = y_2^2, \ y_2 \geq 0, \ h(y_1)y_3 = 1, \ y_4g_2(y_1) \geq 1, \ g_1(y_1)^2 = y_4^2, \ y_4 \geq 0 \}.
\]
Consequently, like in the polynomial problem case, the extension [30] Theorem 2] to \( A \) of Putinar’s Positivstellensatz provides us representations of each nonnegativity constraint in (3.7) via s.o.s and the dual theory of moments. Notice that the constraint \( x \in A \) is linear in \( X \). Hence, SDP relaxations as (3.3) and its dual (3.4) can be similarly derived for (3.6) by lifting the parameter space. Moreover, the convergence results and stopping criterion, as Theorem 3.2, 3.3 and 3.6, can also be analogously established. As might be expected, additional parameters in the lifted space can cause more computational burden in resulting SDP problems. However, as pointed out in [30], the running intersection property holds true for these lifted parameters. Hence, like for polynomial optimization problems [17, 26, 54, 69], some sparse SDP relaxations for (3.6) can be explored to reduce the computational cost.

Notice that one may also replace Putinar’s Positivstellensatz on which (3.3) and (3.4) are based by other recently proposed representations of nonnegative polynomials, like in [8, 23-31, 45-69], which can yield more computationally efficient SDP relaxations of (1.1). We omit the details of these extensions for the sake of brevity.

3.1.4. Comparison with some other LSIP methods. When \( n = 1 \) and \( S \) is a closed interval in (1.1), an SDP relaxation method is proposed in [75] based on representations of nonnegative polynomials in the univariate case. Now we compare the SDP relaxations (3.3) and (3.4) restricted to the case \( n = 1 \) with the method given in [75]. Without loss of generality, we may assume that \( S = [-1, 1] \). The following representation result can be found in [32, 51].

**Theorem 3.8** (Fekete, Markov-Lukácz). Let \( f \in \mathbb{R}[Y] \) and \( f \geq 0 \) on \([-1, 1] \), then
(i) \( f = \sigma_0 + \sigma_1(1 - Y^2) \), where \( \sigma_0, \sigma_1 \in \Sigma^2 \) and \( \deg(\sigma_0), \deg(\sigma_1(1 - Y^2)) \leq \deg(f) \) (resp. \( \deg(f) + 1 \) when \( \deg(f) \) is even (resp. odd).
(ii) If \( \deg(f) \) is odd, \( f = \sigma_1(1 - Y) + \sigma_2(1 + Y) \), where \( \sigma_1, \sigma_2 \in \Sigma^2 \) and \( \deg(\sigma_1(1 - Y), \deg(\sigma_2(1 + Y)) \leq \deg(f) \).

For a fixed \( x \in \mathbb{R}^m \), one can replace the constraint \( a(y_1)^T x + b(y_1) \geq 0 \) on \( S \) in (1.1) by the s.o.s representations of \( a(Y_1)^T x + b(Y_1) \) as in Theorem 3.8. Since the degrees of s.o.s polynomials in the representations are bounded, a single SDP problem equivalent to (1.1) can be derived [75]. As the number of s.o.s polynomials and their degrees in the representations of \( a(Y_1)^T x + b(Y_1) \) determine the size of the resulting SDP problem, when \( d := \max\{\deg(a_1), \ldots, \deg(a_m), \deg(b)\} \) is even (resp. odd), the representation in (i) (resp. (ii)) is used in [75]. If we describe \( S \) as \( \{ y_1 \in \mathbb{R} \mid 1 - y_1^2 \geq 0 \} \), then by the definition of quadratic modules, we in fact use the same representation of \( a(Y_1)^T x + b(Y_1) \) in (3.3) as in Theorem 3.8 (i). Moreover, the relaxation (3.3) of the first order is equivalent to (1.1) by Theorem 3.8. Consequently, when \( d \) is even, the two SDP methods in this paper and in [75] are just the same. Now assume that \( d \) is odd. Resulting from the representation in Theorem 3.8 (ii), the associated SDP problem in [75] contains two positive semidefinite matrices of the same size \( (d - 1) \times (d - 1) \) (corresponding to the number of monomials in \( Y_1 \) of degree up to \( d - 1 \)). Using the relaxation (3.3) in this case, there are two positive semidefinite matrices of the size \( (d + 1)^2 \times (d + 1)^2 \) and \( (d + 1) \times (d + 1) \), respectively. Hence, when \( d \) is odd, the SDP data in (3.3) is only slightly bigger than in the SDP problem from [75]. As the algorithms
for SDP problems output the optimal value up to an additive error $\varepsilon$ in time that is polynomial in the program description size and $\log(1/\varepsilon)$ [24], we can expect that the CPU consuming time of these two methods does not differ too much in practice when $S$ is a closed interval. More importantly, the method of SDP relaxations (3.3) and (3.4) can deal with LSIPP problems with $S$ in higher dimensional spaces.

The cutting-plane based method is another popular and effective method for solving LSIP problems. For example, an accelerated central cutting plane (ACCP) algorithm, which tries to add an deeper objective cut at each iteration, is given in [3] and an interior point constraint generation algorithm, which is inspired by the logarithmic barrier decomposition method [38], is proposed in [46]. At each iteration, these methods require to solve a so-called lower level problem of globally minimizing $a(Y)^T\bar{x} + b(Y)$ over $S$ with $\bar{x}$ being the optimal solution of the current iteration to check the feasibility of $\bar{x}$ or find the violated constraints. Although the discretization of the LSIP at each iteration is an LP problem, the lower level problem could be quite nonlinear. Thus, it is only possible to solve the subproblem and find the minimizers under some strong assumptions on $a_i(Y)$’s, $b(Y)$ and $S$. The main advantage of the SDP methods in this paper and [13] is that the lower level problems are avoided. It is reported in [70] that the SDP method therein is more efficient than the ACCP method. Of course, the SDP relaxation method in this paper is more suitable for LSIP problems of small or medium size since the size of the resulting SDP problems grows exponentially as the order $k$ increases.

3.2. Noncompact case. In this subsection, we consider the LSIPP problem (1.1) with noncompact index set $S$. Since the Archimedean property is violated in this case, the optima of the SDP relaxations (3.3) and (3.4) might not converge to $P^\ast$.

Example 2.5 revisited. Recall that $Q(G)$ is not Archimedean. For any $k \in \mathbb{N}$, we know from [18] Example 2.10] that $(1 - 3Y_2)x + 3Y_1 \in Q\{G\}$ if and only if $x = 0$, i.e., $p_k^{\text{mom}} = 0$ for each $k \geq d_p$. Now we show that $p_k^{\text{mom}} = p_k^{\text{con}}$ for each $k \geq d_p$.

In fact, for the SDP relaxation (3.4) of the problem (2.3), let $\mu$ be a probability measure with uniform distribution in the following subset of $S$:

$$S_1 := \{(y_1, y_2) \in \mathbb{R}^2 \mid 1 \leq y_1 \leq 2, \ 0 \leq y_2 \leq 1\}$$

and $z(\mu)$ be the truncated moment sequence with representing measure $\mu$ up to order $2k$. It can be verified that $z(\mu)$ is a feasible point of (3.4) and its corresponding truncated moment matrix and localizing moment matrices are positive definite since $S_1$ has nonempty interior. Then $p_k^{\text{mom}} = p_k^{\text{con}}$ follows by the conic duality theorem. Hence, both SDP relaxations (3.3) and (3.4) do not converge to the optimum. □

3.2.1. Homogenization. In [70], we used the technique of homogenization to convert a general SIPP problem with noncompact index set into compact case. In the following, we apply this technique to (1.1).

For a polynomial $f(Y) \in \mathbb{R}[Y]$, denote its homogenization by $f^h(\tilde{Y}) \in \mathbb{R}[\tilde{Y}]$, where $\tilde{Y} = (Y_0, Y_1, \ldots, Y_n)$, i.e., $f^h(\tilde{Y}) = Y_0^{D_f} f(Y/Y_0)$, $D_f = \text{deg}(f)$. For the basic semialgebraic set $S$ in (1.1), define

$$\tilde{S}_\ast := \{\tilde{y} \in \mathbb{R}^{n+1} \mid g^h(\tilde{y}) \geq 0, \ y_0 > 0, \ ||\tilde{y}||_2^2 = 1\},$$

$$\tilde{S} := \{\tilde{y} \in \mathbb{R}^{n+1} \mid g^h(\tilde{y}) \geq 0, \ y_0 \geq 0, \ ||\tilde{y}||_2^2 = 1\}.$$

Proposition 3.10. [70] Proposition 4.2] For any $f(Y) \in \mathbb{R}[Y]$, $f(y) \geq 0$ on $S$ if and only if $f^h(\tilde{y}) \geq 0$ on closure($\tilde{S}_\ast$).
Define
\[ \omega := \max\{\deg(a_1), \ldots, \deg(a_m), \deg(b)\}. \]
We homogenize the polynomials \( a_i(Y), \) \( i = 1, \ldots, m, \) and \( b(Y) \) to the same degree \( \omega \) and still denote the resulting polynomials as \( a_i^h(\tilde{Y}) \) and \( b^h(\tilde{Y}) \) for simplicity. Denote
\[ a^h(\tilde{Y}) = (a_1^h(\tilde{Y}), \ldots, a_m^h(\tilde{Y})). \]
It follows that the problem \( (1.1) \) is equivalent to
\[
\begin{align*}
\inf_{x \in \mathbb{R}^m} & \ c^T x \\
\text{s.t.} & \ a^h(\tilde{y})^T x + b^h(\tilde{y}) \geq 0, \ \forall \tilde{y} \in \text{closure}(\tilde{S}_>) .
\end{align*}
\]
Replacing \( \text{closure}(\tilde{S}_>) \) by the basic semialgebraic set \( \tilde{S} \), we get the following problem
\[
(3.9) \quad \tilde{p}^* := \inf_{x \in \mathbb{R}^m} \ c^T x
\[
\text{s.t.} \ a^h(\tilde{y})^T x + b^h(\tilde{y}) \geq 0, \ \forall \tilde{y} \in \tilde{S} .
\]
It is obvious that \( \tilde{p}^* \geq p^* \) since \( \text{closure}(\tilde{S}_>) \subseteq \tilde{S} \).

For any polynomial \( f(Y) \in \mathbb{R}[Y] \), denote \( f(Y) \) as its homogeneous part of the highest degree. Define
\[
(3.10) \quad \tilde{S} := \{ y \in \mathbb{R}^n \mid \hat{g}_1(y) \geq 0, \ldots, \hat{g}_s(y) \geq 0, \|y\|_2^2 = 1 \} .
\]
Specially, denote \( \hat{a}_i(Y), i = 1, \ldots, m, \) and \( \hat{b}(Y) \) as the homogeneous parts of \( a_i(Y), i = 1, \ldots, m, \) and \( b(Y) \) of the same degree \( \omega \). Let \( \hat{a}(Y) := (\hat{a}_1(Y), \ldots, \hat{a}_m(Y)) \).

**Condition 3.11.** For any \( \varepsilon > 0 \), there exists a feasible point \( x^{(\varepsilon)} \) of \( (1.1) \) such that
\[ c^T x^{(\varepsilon)} - p^* \leq \varepsilon \quad \text{and} \quad \hat{a}(y)^T x^{(\varepsilon)} + \hat{b}(y) \geq 0, \ \forall y \in \tilde{S} .\]

**Theorem 3.12.** \( \tilde{p}^* = p^* \) if and only if Condition \( 3.11 \) holds for \( (1.1) \).

**Proof.** By Proposition \( 3.10 \) and the fact that \( \tilde{S} \setminus \text{closure}(\tilde{S}_>) \subseteq \{0\} \times \tilde{S} \), it is straightforward to verify the conclusion. \( \square \)

**Definition 3.13.** \( \tilde{S} \) is said to be closed at \( \infty \) if \( \text{closure}(\tilde{S}_>) = \tilde{S} \).

**Remark 3.14.** Clearly, \( \tilde{p}^* = p^* \) when \( S \) is closed at \( \infty \). Note that not every set \( S \) of form \( (1.2) \) is closed at \( \infty \) even when it is compact \( \in [33] \) Example 5.2. However, it is shown in \( [70] \) Theorem 4.10 that the closedness at \( \infty \) is a generic property. Namely, if we consider the space of all coefficients of generators \( g_j \)'s of all possible sets \( S \) of form \( (1.2) \) in the canonical monomial basis of \( \mathbb{R}[Y]_d \), coefficients of \( g_j \)'s of those sets \( S \) which are not closed at \( \infty \) are in a Zariski closed set of the space. It follows that the problems \( (1.1) \) and \( (3.9) \) are equivalent in general. Note that \( \tilde{S}_> \) depends only on \( S \), while \( S \) depends only on \( S \) but also on the choice of the inequalities \( g_1(y) \geq 0, \ldots, g_s(y) \geq 0 \). In some cases, we can add some redundant inequalities in the description of \( S \) to force it to be closed at \( \infty \) \( [18] \).

Next we construct the non-compact counterparts of the SDP relaxations \( (3.3) \) and \( (3.4) \) for the problem \( (3.9) \). Let
\[
(3.11) \quad G^h := \{ g_1^h, \ldots, g_s^h, Y_0, \|\tilde{Y}\|_2^2 - 1, 1 - \|\tilde{Y}\|_2^2 \}
\]
and denote by $Q(G^h)$ the quadratic module of $\tilde{S}$ generated by $G^h$. Then (3.3) becomes

$$\begin{aligned}
\tilde{p}_k^{\text{as}} := \inf_{x \in \mathbb{R}^m} c^T x \\
\text{s.t. } a^h(\tilde{Y})^T x + b^h(\tilde{Y}) \in Q_k(G^h).
\end{aligned}
$$

(3.12)

For $k \in \mathbb{N}$, denote $\tilde{s}(k) := \left(\begin{array}{c} (n+k+1) \\ n \end{array}\right)$. Let $z := (z_{\tilde{a}})_{\tilde{a} \in \mathbb{N}^{n+1}_{2k}} \in \mathbb{R}^n(2k)$ be a truncated moment sequence up to order $2k$ whose elements are indexed by $(n+1)$-tuples $\alpha := (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathbb{N}^{n+1}_{2k}$. Let

$$b^h(\tilde{Y}) = \sum_{\tilde{a} \in \mathbb{N}^{n+1}_{2k}} b_{\tilde{a}}^h \tilde{Y}^\tilde{a} \quad \text{and} \quad a^h_{i,\tilde{a}}(\tilde{Y}) = \sum_{\tilde{a} \in \mathbb{N}^{n+1}_{2k}} a_{i,\tilde{a}}^h \tilde{Y}^\tilde{a}, \ i = 1, \ldots, m.$$

According to (3.4), the dual of (3.12) is

$$\begin{aligned}
\tilde{p}_k^{\text{mom}} := \sup_{z \in \mathbb{N}^{n+1}_{2k}} - \sum_{\tilde{a} \in \mathbb{N}^{n+1}_{2k}} b_{\tilde{a}}^h z_{\tilde{a}} \\
\text{s.t. } \sum_{\tilde{a} \in \mathbb{N}^{n+1}_{2k}} a_{i,\tilde{a}}^h z_{\tilde{a}} = c_i, \ i = 1, \ldots, m, \\
M_k(z) \geq 0, \ M_{k-1}(g_j^h z) \geq 0, \ j = 1, \ldots, s, \\
M_{k-1}(Y_0 z) \geq 0, \ M_{k-1}(\|\tilde{Y}\|_2^2 - 1)z = 0.
\end{aligned}
$$

(3.13)

**Definition 3.15.** We say that the extended Slater condition holds for (1.1) if there exists a point $\tilde{x} \in \mathbb{R}^m$ of (1.1) such that $a(y)^T \tilde{x} + b(y) > 0$ for all $y \in S$ and $\hat{a}(y)^T \tilde{x} + \hat{b}(y) > 0$ for all $y \in \tilde{S}$. We call $\tilde{x}$ the extended Slater point of (1.1).

**Proposition 3.16.** The Slater condition holds for (3.9) if and only if the extended Slater condition holds for (1.1).

**Proof.** Suppose that $\tilde{x}$ is an extended Slater point of (1.1). For any $\tilde{v} = (v_0, v) \in \tilde{S}$, we have $v \in \tilde{S}$ if $v_0 = 0$ and $v/v_0 \in S$ otherwise. It is straightforward to verify that the Slater condition also holds for (3.9) at $\tilde{x}$.

Suppose that the Slater condition holds for (3.9) at $\tilde{x} \in \mathbb{R}^m$. For any point $v \in \mathbb{R}^n$, we have $(0, v) \in \tilde{S}$ if $v \in \tilde{S}$ and $\left(\frac{1}{\sqrt{1 + \|v\|_2^2}}, \frac{v}{\sqrt{1 + \|v\|_2^2}}\right) \in \tilde{S}$ if $v \in S$. Then similarly, it implies that the extended Slater condition holds for (1.1) at $\tilde{x}$. \(\square\)

**Theorem 3.17.** If the extended Slater condition holds for (1.1), then both $\tilde{p}_k^{\text{as}}$ and $\tilde{p}_k^{\text{mom}}$ decreasingly converge to $\tilde{p}^*$ as $k \to \infty$. Moreover, they both converge to $p^*$ if $S$ is closed at $\infty$ or Condition 3.11 holds for (1.1).

**Proof.** Since $Q(G^h)$ is Archimedean, the conclusion follows by combining Theorems 3.2, 3.3, 3.12 and Proposition 3.16. \(\square\)

**Example 2.5 revisited.** By definition, we have

$$\begin{aligned}
\tilde{S}_x &= \{(y_0, y_1, y_2) \in \mathbb{R}^3 \mid y_1 \geq 0, \ y_0 y_1 - y_2^2 \geq 0, \ y_0 > 0, \ \|\tilde{y}\|_2^2 = 1\}, \\
\tilde{S} &= \{(y_0, y_1, y_2) \in \mathbb{R}^3 \mid y_1 \geq 0, \ y_0 y_1 - y_2^3 \geq 0, \ y_0 \geq 0, \ \|\tilde{y}\|_2^2 = 1\}, \\
\tilde{S} &= \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq 0, \ y_2 \leq 0, \ y_1^2 + y_2^2 = 1\}.
\end{aligned}$$
After homogenization, the problem (2.3) is reformulated as

\[
\inf_{x \in \mathbb{R}} - \frac{x}{2} \quad \text{s.t.} \quad (y_0 - 3y_2)x + 3y_1 \geq 0, \quad \forall \tilde{y} = (y_0, y_1, y_2) \in \tilde{S}.
\]

Note that \( S \) is closed at \( \infty \). In fact, for every \((0, v_1, v_2) \in \tilde{S} \setminus \tilde{S}_\infty \), let

\[
v^{(c)} := \left( \varepsilon, v_1, \frac{3}{\varepsilon} v_1^2 + v_2^3 \right).
\]

Then \( \{v^{(c)}/\|v^{(c)}\|_2 \}_{\varepsilon > 0} \subseteq \tilde{S}_\infty \) and \( \lim_{\varepsilon \to 0} v^{(c)}/\|v^{(c)}\|_2 = (0, v_1, v_2) \). Hence, we have \( \tilde{S}_\infty \setminus \tilde{S}_\infty \subseteq \text{closure}(\tilde{S}_\infty) \) and so \( S \) is closed at \( \infty \). It is easy to check that the extended Slater condition holds for (3.14) if we let \( \tilde{x} = 1 \). Hence, the assumptions in Theorem 3.17 are satisfied. With GloptiPoly, we get the following numerical results: \( \tilde{p}_2^{\text{nom}} = -1.2124 \times 10^{-8} \) and \( \tilde{p}_3^{\text{nom}} = -0.7500 \). The Rank Condition is satisfied for \( k = 3 \) and we obtain the certified optimum \(-0.7500\). As noted in Remark 3.7, the extracted numerical active index set of the minimizer \( x^* = 3/2 \) is \( (0.5773, 0.5774, 0.5774) \) which corresponds to \((1, 1) \in S \). \( \square \)

3.2.2. A byproduct in polynomial optimization. Consider the general polynomial optimization problem

\[
\begin{align*}
\{ & f^* := \inf_{y \in \mathbb{R}^m} f(y) \\
& \text{s.t.} \quad g_1(y) \geq 0, \ldots, g_s(y) \geq 0. \}
\end{align*}
\]

Recall that the feasible set of (3.15) is denoted by \( S \). We assume that \(-\infty < f^* < \infty \). The problem (3.15) can be reformulated as an LSIPP problem

\[
\begin{align*}
- f^* = \inf_{x \in \mathbb{R}} -x \quad \text{s.t.} \quad f(y) - x \geq 0, \quad \forall y \in S.
\end{align*}
\]

As we will see, by applying the SDP relaxation approach to the special LSIPP problem (3.16), we can obtain: (i) the classic Lasserre’s SDP relaxation method [25] of (3.15) when \( S \) is compact, which can be expected from the way of reformulation (3.16) and relaxation (3.1); (ii) a new hierarchy of SDP relaxations of (3.15) when \( S \) is noncompact and \( f \) is stably bounded from below on \( S \), which is a class of polynomial optimization problems studied in [30]. Note that the classic Lasserre’s SDP relaxation method might fail for this kind of problems, see Example 3.24.

We first assume that \( S \) is compact. In the special LSIPP (3.16), we have

\[
m = 1, \quad a(Y) = -1, \quad b(Y) = f(Y) \quad \text{and} \quad c = -1.
\]

According to (3.3) and (3.4), by exchanging of ‘inf’ and ‘sup’, we obtain SDP relaxations of (3.15):

\[
\begin{align*}
\{ & f_k^{\text{opt}} := \sup_{x \in \mathbb{R}} x \\
& \text{s.t.} \quad f(Y) - x = \sum_{j=0}^s \sigma_j(Y)g_j(Y), \\
& g_0 = 1, \sigma_j \in \Sigma^k, \deg(\sigma_j g_j) \leq 2k, \quad i = 0, \ldots, s.
\end{align*}
\]
and

$$f^\text{mom}_k := \inf_{z \in \mathbb{R}^{n+1}} \sum_{\alpha \in \mathbb{N}^{n+1}_k} f_\alpha z_\alpha \quad \text{s.t. } z_0 = 1,$$

where $z_0$ denotes the element of $z$ indexed by the $n$-tuple $(0, \ldots, 0)$. They are just the classic Lasserre’s SDP relaxations of polynomial optimization problems [25]. Clearly, the Slater condition holds for (3.16) if and only if $f$ is bounded from below on $S$. Hence, by Theorem 3.2 and Theorem 3.3 when $Q(G)$ is Archimedean, both $f^\text{pos}$ and $f^\text{mom}_k$ converge to $\hat{f}$ as $k \to \infty$, which has already been proved in [25 Theorem 4.2]. Note that by Remark 3.7 the points in the active index set of the minimizer of (3.16) extracted when the Rank Condition is satisfied are just the global minimizers of (3.15).

Now we consider the polynomial optimization problem (3.15) with noncompact feasible set $S$. After homogenization, the problem (3.16) becomes

$$(3.19) \begin{cases} \hat{f}^* := \sup_{x \in \mathbb{R}} x \\ \text{s.t. } f^h(\bar{y}) - xy_0^D \geq 0, \forall \bar{y} \in \bar{S}, \end{cases}$$

where $D_f = \deg(f)$. According to (3.3) and (3.4), we obtain a hierarchy of SDP relaxations of (3.19):

$$(3.20) \begin{cases} \hat{f}^\text{pos}_k := \sup_{x \in \mathbb{R}} x \\ \text{s.t. } f^h(\bar{Y}) - xy_0^D \in Q_k(G^h), \end{cases}$$

where $G^h$ is defined in (3.11). For $k \in \mathbb{N}$, denote $\hat{s}(k) := (n+1)\binom{n+1}{n+1}$. Let $z := (z_\alpha)_{\alpha \in \mathbb{N}^{n+1}_k} \in \mathbb{R}^{n+1}$ be a truncated moment sequence of degree $2k$. Denote by $z_{D_f,0}$ the element of $z$ indexed by the $(n+1)$-tuple $(D_f, 0, \ldots, 0)$. The dual problem of (3.20) is

$$(3.21) \begin{cases} \hat{f}^\text{mom}_k := \inf_{z \in \mathbb{R}^{n+1}} \sum_{\alpha \in \mathbb{N}^{n+1}_k} f_\alpha z_\alpha \\ \text{s.t. } z_{D_f,0} = 1, M_k(z) \geq 0, M_{k-1}(Y_0 z) \geq 0, \\
M_{k-1}(g_j^h z) \geq 0, j = 1, \ldots, s, \\
M_k(\|\bar{Y}\|_2^2 - 1) z = 0. \end{cases}$$

Definition 3.19. [10] A polynomial $f$ is said to be stably bounded from below on $S$ if $f$ remains bounded from below on $S$ for all sufficiently small perturbations of the coefficients of $f, g_1, \ldots, g_s$.

Recall the notation $\hat{f}(Y)$ and $\hat{S}$ defined in (3.10).

Proposition 3.20. The following conditions are equivalent:

(i) $f$ is stably bounded from below on $S$;
(ii) $\hat{f}$ is strictly positive on $\hat{S}$;
(iii) the Slater condition holds for (3.19).
are obvious problems with this if polynomial optimization problems on compact semialgebraic sets and hence there
As pointed in [40, Notes 5.2], the lower bound
Figure 2. The global minimizers are
We consider the case when
Consider the following polynomial optimization problem
Example 3.24. Consider the following polynomial optimization problem
\[
\begin{align*}
\inf_{y \in \mathbb{R}^2} & \quad y_1^2 + y_2^2 \\
\text{s.t.} & \quad y_2^2 - 1 \geq 0, \\
& \quad y_1^2 - My_1y_2 - 1 \geq 0, \\
& \quad y_2^2 + My_1y_2 - 1 \geq 0,
\end{align*}
\]
where \(M\) is a positive constant. It was shown in [8][39][45] that the global minimizers and global minimum are
\[
\left( \frac{\pm M + \sqrt{M^2 + 4}}{2}, \pm 1 \right) \quad \text{and} \quad \frac{2 + M(M + \sqrt{M^2 + 4})}{2}.
\]
We consider the case when \(M = 1\). The feasible set \(S\) is depicted in gray in Figure 2. The global minimizers are \(\left( \frac{\pm 1 + \sqrt{5}}{2}, \pm 1 \right) \approx (\pm 1.618, \pm 1)\) and its global minimum is \(2 + \frac{(1 + \sqrt{5})}{2} \approx 3.618\). Because \(S\) is noncompact, by the argument in [8], the classic Lasserre’s SDP relaxations (3.17) of (3.22) can only provide lower
bounds $f_k^{sos} = 2$ no matter how large the order $k$ is. Since $S$ has nonempty interior and the relaxation (3.17) of (3.22) is feasible, by [25, Theorem 4.2], $f_k^{mom}$ equals $f_k^{sos}$ for each $k$ and therefore cannot converge to the optimum as $k \to \infty$, either.

Obviously, the condition (ii) in Proposition 3.20 holds. We compute the relaxations (3.21) with GloptiPoly. For $k = 3$, the Rank Condition is satisfied and we get the numerically certified optimum $f_3^{mom} = 3.6180$. The extracted active index set is $\{(0.4653, \pm 0.7529, \pm 0.4653)\}$ which, by Remark 3.22 provides the global minimizers $(\pm 1.6181, \pm 1)$. □

4. NUMERICAL EXPERIMENTS

In this section, we present some numerical experiments to show the efficiency of the SDP relaxation method (3.3) and (3.4) in solving LSIPP problems. We use GloptiPoly in MATLAB R2016a to manipulate the moment relaxations (3.4) and call the SDP solver SeDuMi [65] in GloptiPoly to solve the resulting SDP problems. The desired accuracy in SeDuMi is set to $10^{-8}$. For checking the Rank Condition and extracting the $r$ points in Theorem 3.6, the singular value decomposition is used in GloptiPoly with accuracy set to $10^{-3}$. For each problem, the consumed computer time is calculated as the total time of all moment relaxations (3.4) from the least order $k = d_P$ to the largest order we have tried. In order to extract minimizers $x^*$ of (1.1) if necessary, we use the software YALMIP to solve the relaxation problems (3.3).

To the best of our knowledge, few of the existing methods for SIP are specially designed for the LSIPP (1.1) where the index set $S$ is an arbitrary basic semi-algebraic set. Notice that the SDP relaxations (3.3) and (3.4) of each order provide decreasing upper bounds of $p^*$ and feasible points of (1.1). Hence, we show the efficiency of the SDP relaxation method in solving LSIPP problems by comparing its performance with some recently proposed methods designed for general SIP which can produce feasible points and upper bounds of the original problems. In the following, we consider the hybrid discretization algorithm (HDA for short) based on [9, 42, 67] and the adaptive convexification algorithm (ACA for short) based on [11, 62]. By employing the algorithm in [42] and an oracle problem adapted from [67], the HDA given in [9] can generate cheap lower bounds and tight upper bounds of SIP problems without a dense population of the discretization. The feasible point method ACA can solve SIP problems with arbitrary, not necessarily box-shaped, index sets by adaptively constructing convex relaxations of the lower level problems.
and subdivisions of index sets. The ACA was proposed in [11] for SIP problem with one-dimensional index sets and generalized in [62] to multidimensional index sets. Note that both of the HDA and the ACA require initial bounds of the \( x \)-variables of SIP problems.

All numerical experiments in this section were carried out on a PC with two 64-bit Intel Core i5-5200U 2.20 GHz CPUs and 8G RAM running Windows 7.

Example 4.1. In this example, there are 17 testing problems of the extended form (3.6) which are listed in Appendix B, including some important applications of LSIP such as minimax problems and functional approximation problems. Some of them are relatively small, but they are well-established and commonly tested in the literature: most of them are collected in the SIAMPL database and others can be found in the books [16, 24].

We first compute moment relaxations (3.4) for each problem and summarize the results in Table 1, where \( k_P \) denotes the largest order of (3.4) we have computed and the column certified indicates that whether the value \( p_{k_P}^{\text{mom}} \) is certified (Y) by the Rank Condition or not (N). Compared with results obtained in relevant papers, all of the 17 problems are successfully solved and for most of them the optimality is certified by the Rank Condition. We then compare our computational results with the HDA method implemented in the general algebraic modeling system (GAMS) 24.7.4 with the GAMS-F preprocessor. The nonlinear programming subproblems in HDA are solved with BARON 16.8.24. For all problems except B.8, we use the default optimality tolerances (10\(^{-3}\) for the solutions of SIP problems and 10\(^{-4}\) for finite subproblems in each iteration) set in the code of HDA. Problem B.8 was introduced in [19] to consider well-posedness in SIP. For this problem, small perturbations of the input data strongly effect the optimal value \( p^* \), which is poorly determined if \( |\varepsilon| \) is relatively small compared with the computational errors. Hence, for Problem B.8 we set all tolerances in HDA to 10\(^{-6}\) and test it for \( \varepsilon = 10^{-3}, 10^{-4}, 10^{-5} \). We list the best lower bounds (LBD) and upper bounds (UBD) obtained by HDA in the consumed time in Table 1. Problem B.7 was excluded due to the involvement of trigonometric functions which cannot be handled by BARON. As we can see, the moment relaxation method can always provide optimal values in less CPU time.

B.12–B.17 are problems of approximating functions \( b(Y) \) from the spans of \( a_i(Y) \)'s in some sense. Hence, it is more useful to give the minimizers \( x^* \) which are the corresponding optimal coefficients for the basis functions \( a_i(Y) \)'s in the approximations. These coefficients are obtained by the software YALMIP and listed in Table 2, where for Problem B.16 and B.17 the coefficients are listed in the order \( (x_{0,0}^*, x_{1,0}^*, x_{0,1}^*, x_{1,1}^*, x_{2,1}^*, x_{1,2}^*, x_{2,2}^*, x_{0,2}^*) \). For the univariate approximation problems B.12–B.15 over the interval [0, 1], we show the accuracy of the computed optimal approximations (denoted by \( f \)) of \( b(Y) \) in Figure 3.

Example 4.2. Notice that the LSIPP problems in the literature collected in Example 4.1 are relatively small. In order to show the efficiency of the SDP relaxation (3.4), we now construct a class of LSIPP problems of which we can control the size.

\[ \text{http://plato.la.asu.edu/ftp/sipampl.pdf} \]
\[ \text{available in the supplementary material at http://link.springer.com/article/10.1007\%2Fs10898-016-0476-7} \]
| Problem | $d_P$ | $k_P$ | $p^\text{mom}_{kp}$ | time | certified | LBD | UBD | time |
|---------|-------|-------|-----------------|------|-----------|-----|-----|------|
| B.1     | 1     | 1     | 0.6666666666   | 0.11s| Y         | 0.6664433| 0.6670252| 1.20s|
| B.2     | 2     | 3     | 1.00000000     | 0.20s| N         | 0.9995649| 1.0002923| 0.92s|
| B.3     | 1     | 1     | 0.32380150     | 0.12s| Y         | 0.3237544| 0.3243875| 1.07s|
| B.4     | 3     | 3     | -1.99999999    | 0.13s| Y         | -2.0004746| -1.9993772| 1.00s|
| B.5     | 2     | 2     | -11.99999999   | 0.14s| N         | -11.99277 | -12.00647 | 10.49s|
| B.6     | 1     | 1     | -99.66067255   | 0.13s| Y         | -99.707   | -99.626   | 0.81s|
| B.7     | 1     | 1     | -1.00000000    | 0.10s| Y         | —         | —         | —     |
| B.9     | 1     | 1     | 0.23606797     | 0.09s| Y         | 0.2354377 | 0.2362799 | 0.98s|
| B.10    | 2     | 2     | -0.33333333    | 0.15s| Y         | -0.3333980| -0.3328738| 0.95s|
| B.11    | $n=2$ | 2     | 0.04841442     | 0.18s| N         | 0.0484137 | 0.0490523 | 3.09s|
|         | $n=3$ | 2     | 0.00564715     | 0.49s| Y         | 0.0052813 | 0.0058925 | 2.43s|
| B.12    | 4     | 4     | -1.78689975    | 0.15s| Y         | -1.7877323| -1.7860611| 4.43s|
| B.13    | 4     | 4     | 0.69314814     | 0.23s| Y         | 0.6925085 | 0.6932714 | 4.89s|
| B.14    | 5     | 5     | -0.78539809    | 0.25s| Y         | -0.7860315| -0.7852433| 4.50s|
| B.15    | 2     | 3     | 0.00052199     | 0.75s| N         | 0.0001269 | 0.0007917 | 5.41s|
| B.16    | 2     | 3     | 0.05835920     | 1.01s| Y         | 0.0583573 | 0.0592570 | 15.42s|
| B.17    | 2     | 3     | 0.01140056     | 1.03s| N         | 0.0112178 | 0.0119224 | 13.23s|

| Problem | $p^*$ | $d_P$ | $k_P$ | $p^\text{mom}_{kp}$ | time | certified | LBD | UBD | time |
|---------|-------|-------|-------|-----------------|------|-----------|-----|-----|------|
| B.8     | -0.50025012 | 1     | 2     | -0.50025012    | 0.48s| Y         | -0.50025488| -0.50024857| 8.04s|
|         | -0.50002500 | 1     | 2     | -0.50002500    | 0.49s| Y         | -0.50002965| -0.50001996| 10.19s|
|         | -0.50000250 | 1     | 2     | -0.50000250    | 0.50s| Y         | -0.50000268| -0.49999719| 9.74s|

**Table 1.** Computational results of testing problems in Appendix B using moment relaxations (3.4) and HDA.
Table 2. Optimal coefficients for basis functions in some approximation problems in Appendix B.

| Problem | Optimal Coefficients                                      |
|---------|-----------------------------------------------------------|
| B.12    | $(-1.0000, 0.0448, -1.6585, 3.6660, -10.7319, 12.7851, -8.1017)$ |
| B.13    | $(0.5000, 0.2501, 0.1227, 0.0787, -0.0258, 0.1226, -0.0967, 0.0484)$ |
| B.14    | $(-1.0000, -0.0000, 1.0016, -0.0202, -0.8566, -0.6123, 2.6222, -2.6059, 1.1881, -0.2168)$ |
| B.15    | $(0.9995, 0.0311, -1.2981, 0.9885, -0.2205)$              |
| B.16    | $(0.2341, -0.0468, -0.1507, 0.1203, -0.0706, -0.1292, 0.0837, 0.0136, 0.0927)$ |
| B.17    | $(2.0043, 0.2494, 0.5125, -0.0747, 0.0194, 0.0350, -0.0133, -0.0178, -0.0708)$ |

Figure 3. Pictures for the univariate approximation problems B.12–B.15 over $[0, 1]$, where $f$ denotes the computed approximations.
Let $S = [-2, 2]^n$. Randomly pick $m$ distinct points $v^{(1)}, \ldots, v^{(m)}$ from $S$ whose coordinates are integers drawn from the discrete uniform distribution on $[-2, 2]$. Let $a_1(Y), \ldots, a_m(Y)$ be the Lagrange interpolation polynomials at these $m$ points, i.e., for each $v^{(i)}$ and $j \neq i$, choose the first index $k_{i,j} \in \{1, \ldots, n\}$ for which $v^{(i)}_{k_{i,j}} \neq v^{(j)}_{k_{i,j}}$ and define
\[
a_i(Y) = \prod_{j \neq i} \frac{Y_{k_{i,j}} - v^{(j)}_{k_{i,j}}}{v^{(i)}_{k_{i,j}} - v^{(j)}_{k_{i,j}}}, \quad i = 1, \ldots, m.
\]
Then, we have $a_i(v^{(i)}) = 1$ and $a_i(v^{(j)}) = 0$ for each $j \neq i$. Recall that for any $t \in \mathbb{N}$, $m_t(Y)$ denotes the column vector consisting of all the monomials in $Y$ of degree up to $t$ and $s(t)$ denotes its dimension. Let $N$ be an $s(t) \times s(t)$ real matrix and define $b(Y) = (N \cdot m_t(Y))^T (N \cdot m_t(Y)) + 1$. For each $i = 1, \ldots, m$, choose $c_i$ to be a positive real number. For the class of random LSIPP problems constructed above, we can see that the Slater condition holds if we let $\tilde{x} = 0$. Moreover, for each $i = 1, \ldots, m$, we have $x_i \geq -b(v_i)$ for every feasible point $x \in \mathbb{R}^m$. Therefore, the optimum $p^* > -\infty$.

For some tuples $(m, n, t)$, letting $N$ be real matrices containing elements randomly drawn from $\{-1, 0, 1\}$ and $c$ be vectors of all ones, we generate Problem C.1 [C.6] listed in the Appendix. The $m$ distinct points $v^{(i)}$’s are represented in the $m \times n$ matrices $V$ whose rows are the coordinates of the points. We apply the moment relaxation method (3.4) and HDA method (with the default tolerances) on these problems and report the results in Table 3. As mentioned in [B], the majority of the relatively large CPU time of HDA method on these problems is due to its relatively expensive NLP subproblems. For all these problems, as we can see, the moment relaxation method can give the certified optimal values in much less CPU time.

To show the performance of the moment relaxation method (3.4) on LSIPP problems of large size, we next apply it to several groups of random problems generated as above where $N$ are $s(t) \times s(t)$ matrices containing random elements drawn from the standard uniform distribution on the open interval $(0, 1)$ and each $c_i$ is chosen from the standard uniform distribution on the open interval $(0, 1)$. The results are listed in Table 4. For each group, the SDP relaxations (3.4) are computed from the order $k = d_P$ to $k = 8$. The inst. column denotes the number of randomly generated instances and the certi. column denotes the number of instances where certified finite convergence occurs, i.e., the Rank Condition holds at some order. Among all instances of each group, the min (max) order column shows the minimal (maximal) order of relaxations when the Rank Condition is satisfied and the min (max) time column shows the minimal (max) consumed computer time.

**Example 4.3.** Since the SDP relaxations (3.3) and (3.4) are applicable for LSIPP problems with index sets being arbitrary basic semialgebraic sets, not necessarily box-shaped, we consider in this example the problem
\[
\inf_{x \in \mathbb{R}^2} x_2 \quad \text{s.t. } x_1 y_1 + x_2 - y_2 \geq 0, \forall y \in S,
\]
where
\[
S := \{ y \in \mathbb{R}^2 \mid (y_1 + 5y_2)y_1^2 - (y_1^2 + y_2^2)^2 \geq 0 \}.
\]
Table 3. Computational results of testing problems in Appendix C using moment relaxations (3.4) and HDA.
which is the gray region in Figure 4. Clearly, it is equivalent to the bilevel problem

$$\min_{x_1 \in \mathbb{R}} \max_{y \in S} y_2 - x_1 y_1.$$ 

By replacing the lower level maximality condition by the KKT condition, it is easy to check that the minimizer $x^* = \left( \frac{1}{5}, \frac{125}{104} \right)$ with the active index set containing

$$\begin{align*}
\left( \frac{625}{2704} + \frac{1875}{2704} \sqrt{3}, \frac{3375}{2704} + \frac{375}{2704} \sqrt{3} \right) \approx (1.4322, 1.4884), \\
\left( \frac{625}{2704} - \frac{1875}{2704} \sqrt{3}, \frac{3375}{2704} - \frac{375}{2704} \sqrt{3} \right) \approx (-0.9699, 1.0079).
\end{align*}$$

Thus, the optimum is $\frac{125}{104} \approx 1.209$. Note that the current code of HDA method can only deal with SIP problems with box-shaped index sets. We compare the numerical results of the moment relaxation with ACA method implemented in SIPSOLVER\(^3\), which can provide feasible iterates and upper bounds of the optimal value. The Matlab toolbox Intlab 9.1 \(^{57}\) is used in SIPSOLVER to compute the convexification parameters. As shown in Table 5, we can obtain the certified optimal value by the moment relaxation of order 3 in much less CPU time. By YALMIP, we can extract the active index set $\{(1.4321, 1.4883), (-0.9699, 1.0079)\}$.

□

**Example 4.4.** In this example, we discuss the efficiency of the SDP relaxation method proposed for LSIPP problems in this section. Recalling (3.3) and (3.4), the SDP relaxations of order $k$ involve $O(n^{2k})$ variables and linear matrix inequalities

\(^{3}\)available at [http://kop.ior.kit.edu/english/downloads.php](http://kop.ior.kit.edu/english/downloads.php)
Table 5. Comparison of computational results for the LSIPP problem in Example 4.3 of moment relaxation and ACA method.

| $p^*$ | Moment relaxation | ACA |  |
|-------|-------------------|-----|---|
|       | $p_2^{\text{mom}}$ | $p_3^{\text{mom}}$ | time | certified | UBD | time |
| 1.2019 | 1.2982 | 1.2019 | 0.38s | Y | 1.3012 | 646s |

Figure 4. The semialgebraic set $S$ (gray) in Example 4.3 and the line $x_1^*y_1 + x_2^* - y_2 = 0$ (red).

of size $O(n^k)$. Hence, the size of resulting SDP problems grows very fast as $n$ and $k \geq d_P$ increase. Since SDP problems become computationally intractable when the size of the involved matrices goes beyond the order of a few hundreds, the SDP relaxation method is more suitable for LSIPP problems with small or medium $n$ and $d_P$. To illustrate this, we consider the following problem which is formulated from a portfolio problem [2, 63]

$$\max_{x \in \mathcal{X}, x_0 \in \mathbb{R}} x_0 \quad \text{s.t.} \quad y^T x - x_0 \geq 0, \quad \forall y \in S,$$

where

$$\mathcal{X} = \left\{ x \in \mathbb{R}^N \mid \sum_{i=1}^N x_i = 1, x \geq 0 \right\} \quad \text{and} \quad S = \left\{ y \in \mathbb{R}^N \mid \sum_{i=1}^N \frac{(y_i - \bar{y}_i)^2}{\sigma_i^2} \leq \theta^2 \right\}.$$

With particular choices $\theta = 1.5$,

$$\bar{y}_i = 1.15 + i \cdot \frac{0.05}{N} \quad \text{and} \quad \sigma_i = \frac{0.05}{N} \sqrt{2N(N+1)i}, \quad i = 1, \ldots, N,$$

the optimal solution was shown [2, 63] to be $x_i^* = 1/N$, $i = 1, \ldots, N$, with optimal value $x_0^* = 1.15$ independent of $N$. Since it can be checked that

$$\sum_{i=1}^N x_i^* Y_i - 1.15 = 10 \cdot \sum_{i=1}^N \frac{(Y_i - 1.15)^2}{i} + \frac{N + 1}{180N} \cdot \left( \theta^2 - \sum_{i=1}^N \frac{(Y_i - \bar{y}_i)^2}{\sigma_i^2} \right),$$

we have $p_{k^{\text{mom}}} = p_{k^{\text{mom}}}^{\text{mom}} = 1.15$ for all $k \geq 1$. Note that $N$ might be a large number in practice. We test the performance of the SDP relaxations (3.4) of order $k = 1$ for
10 different numbers $N = 10 \cdot i$, $i = 1, \ldots, 10$. The corresponding CPU time is the following

0.30s, 0.64s, 1.75s, 4.47s, 14.87s, 39.89s, 92.61s, 185.97s, 418.85s, 741.21s.

Compared with the computational results reported in [63], we use less time when $N \leq 60$ and more when $N$ is close to 100. Notice that the cost of computing (3.4) grows even faster as the order $k$ increases. For instance, our computer ran out of memory for $N = 30$ and $k = 2$. □

Remark 4.5. As a summary, in dealing with LSIPP problems, compared with the existing methods for general SIP in the literature, the SDP relaxation approach of (3.3) and (3.4) has the following properties: (a) Without any initial bounds of $x$-variables, sequences of feasible points of (1.1) and decreasing upper bounds of $p^*$ can be computed with $S$ being an arbitrary basic semi-algebraic set; (b) A checkable sufficient condition is available for certifying the finite convergence of the SDP relaxations when it occurs (Theorem 3.6); (c) Instead of solving lower level problems (usually nonlinear) and subdivision of the index set in each iteration, the SDP relaxation method is more computationally efficient by solving a sequence of SDP problems, especially for LSIPP problems of small or medium size (w.r.t. $n$ and $d_P$).

5. Conclusion

In this paper, we study a subclass of semi-infinite programming problems whose constraint functions are polynomials in parameters and index sets are basic semi-algebraic sets (LSIPP problems). When the index set of an LSIPP problem is compact, a convergent hierarchy of SDP relaxations is constructed based on Putinar’s Positivstellensatz. We extend this approach to the case when the index set is noncompact by the technique of homogenization. Applying our method to the LSIPP reformulation of a polynomial optimization problem, we obtain a new hierarchy of SDP relaxations for solving the class of polynomial optimization problems whose objective polynomials are stably bounded from below on noncompact feasible sets.

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Proof of Theorem 3.1. Fix an $\varepsilon > 0$ and a feasible $\bar{x} \in \mathbb{R}^m$ of (1.1) such that $a(y)^T \bar{x} + b(y) > 0$ for all $y \in S$. We next show that $p^{\text{cos}} - p^* < \varepsilon$. By Putinar’s Positivstellensatz, $\bar{x}$ is a feasible point of (1.1) and thus we can assume that $c \neq 0$ without loss of generality. If $c^T \bar{x} - p^* < \varepsilon$, then $p^{\text{cos}} - p^* \leq c^T \bar{x} - p^* < \varepsilon$ and we are done. Hence, we assume that $c^T \bar{x} - p^* \geq \varepsilon$ in the following. Then we can fix another feasible point $x' \in \mathbb{R}^m$ of (1.1) such that $c^T \bar{x} > c^T x'$ and $c^T x' - p^* < \varepsilon/2$. 

Appendix A.
Let 
\[ \delta := \frac{\varepsilon}{2c^T(x - x')} > 0 \quad \text{and} \quad \hat{x} := (1 - \delta)x' + \delta x. \]

Then we have \( 0 < \delta < 1 \) and hence 
\[ a(y)^T \hat{x} + b(y) = (1 - \delta)[a(y)^T x' + b(y)] + \delta[a(y)^T \hat{x} + b(y)] > 0, \quad \forall y \in S. \]

Since \( Q(G) \) is Archimedean, \( a(Y)^T \hat{x} + b(Y) \in Q(G) \) by Putinar’s Positivstellensatz. 
That is, \( \hat{x} \) is feasible for both (1.1) and (3.1). We have 
\[ p^{\text{soc}} - p^* \leq c^T \hat{x} - p^* \]
\[ = (1 - \delta)c^T x' + \delta c^T \hat{x} - p^* \]
\[ = (c^T x' - p^*) + \delta c^T (\hat{x} - x') \]
\[ < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \]
which means that \( p^{\text{soc}} \leq p^* \) since \( \varepsilon > 0 \) is arbitrary. As \( p^{\text{soc}} \geq p^* \), we can conclude that \( p^{\text{soc}} = p^* \). \( \square \)

Proof of Theorem 3.3. For any \( \varepsilon > 0 \), as shown in the proof of Theorem 3.1 there exists a feasible point \( \hat{x} \) of (1.1) such that \( a(Y)^T \hat{x} + b(Y) \in Q(G) \) and \( c^T \hat{x} - p^* < \varepsilon \). For some \( k \in \mathbb{N} \), we have \( a(Y)^T \hat{x} + b(Y) \in Q_k(G) \) and then \( p_k^{\text{soc}} - p^* \leq c^T \hat{x} - p^* < \varepsilon \). Since \( \varepsilon \) is arbitrary, \( p_k^{\text{soc}} \) decreasingly converges to \( p^* \) as \( k \to \infty \). \( \square \)

Proof of Theorem 3.3. By the ‘weak duality’ and Theorem 3.2, it suffices to prove that \( p_k^{\text{mom}} \geq p^* \) for each \( k \geq d_P \). Consider the Haar dual problem (2.1) of (1.1). Since \( S \) is compact and the Slater condition holds for (1.1), by Proposition 2.2 \( p^* = d^* \) and \( d^* \) is attainable. Denote by \( (\lambda^*_y)_{y \in S} \) an optimizer of \( d^* \) and \( S^* \) as the finite subset of \( S \) such that \( \lambda^*_y > 0 \) for every \( y \in S^* \). Let \( \bar{z} = \sum_{y \in S^*} \lambda^*_y z_{2k,y} \) where \( z_{2k,y} \) is the Zeta vector of \( y \) up to degree \( 2k \). Clearly, \( \bar{z} \) is feasible for (3.4) and then \( p_k^{\text{mom}} \geq -\sum_{\alpha} b_\alpha \bar{z}_\alpha = d^* = p^* \). Hence, we obtain that \( p_k^{\text{mom}} \downarrow p^* \) as \( k \to \infty \). \( \square \)

Appendix B. Testing problems from the literature

Problem B.1. \[ [65] \]
\[ \min_{x \in \mathbb{R}^2} 2x_1 + x_2 \]
\[ \text{s.t. } yx_1 + (1 - y)x_2 + y^2 - y \geq 0, \quad \forall y \in [0, 1]. \]

This problem is very sensitive with respect to the boundary of its level sets and has a unique optimal solution \( x^* = (1/9, 4/9) \).

Problem B.2. \[ [64] \]
\[ \min_{x \in \mathbb{R}^2} -x_1 + x_2 \]
\[ \text{s.t. } (y^2 - 1)x_1 + y^2 x_2 - y^4 \geq 0, \quad \forall y \in [-1, 1]. \]

It was shown that \( (0, 1) \) is the strict minimizer.

Problem B.3. \[ [35] \]
\[ \min_{x \in \mathbb{R}^2} \frac{1}{2} x_1 + x_2 \]
\[ \text{s.t. } (y + 1)^2 x_1 + (y - 2)^2 x_2 - 1 \geq 0, \quad \forall y \in [0, 1], \]
\[ x_1 \geq 0, \quad x_2 \geq 0. \]
Problem B.4. \(33 \ [73]\)
\[
\begin{align*}
\min_{x \in \mathbb{R}^3} & \quad x_1 - 2x_2 - x_3 \\
\text{s.t.} & \quad -x_1 - yx_2 - y^2x_3 + y^5 \geq 0, \quad \forall y \in [1, 2], \\
& \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0.
\end{align*}
\]
The optimal value is \(-2\).

Problem B.5. \(6\)
\[
\begin{align*}
\min_{x \in \mathbb{R}^6} & \quad -4x_1 - \frac{2}{3}(x_4 + x_6) \\
\text{s.t.} & \quad - (x_1 + y_1x_2 + y_2x_3 + y_1^2x_4 + y_1y_2x_5 + y_2^2x_6) + (y_1^2 - y_2^2)^2 + 3 \geq 0, \\
& \quad \forall y \in [-1, 1]^2.
\end{align*}
\]
The optimal value is \(-12\).

Problem B.6. \(35\)
\[
\begin{align*}
\max_{x \in \mathbb{R}^2} & \quad 3x_1 + \frac{5}{2}x_2 \\
\text{s.t.} & \quad \frac{1}{2}(y^2 - 7)x_1 + (2y^2 - 6) - 30y + 180 \geq 0, \\
& \quad (y^2 - 10)x_1 + \frac{1}{2}(y^2 - 15) - 20y + 320 \geq 0, \quad \forall y \in [0, 1], \\
& \quad x_1 \geq 0, \quad x_2 \geq 0.
\end{align*}
\]

Problem B.7. \(66\)
\[
\begin{align*}
\min_{x \in \mathbb{R}^2} & \quad x_2 \\
\text{s.t.} & \quad - \cos(t)x_1 - \sin(t)x_2 + 1 \geq 0, \quad \forall t \in [0, 2\pi].
\end{align*}
\]
Replacing \(\cos(t)\) and \(\sin(t)\) by \(y_1\) and \(y_2\) respectively, we can reformulate this problem as \([1.1]\) with \(S = \{y \in \mathbb{R}^2 \mid y_1^2 + y_2^2 = 1\}\). The optimal value is \(-1\).

Problem B.8. \(19\)
\[
\begin{align*}
\min_{x \in \mathbb{R}^2} & \quad x_1 + (1 + \varepsilon)x_2 \\
\text{s.t.} & \quad x_1 + yx_2 + \frac{1}{1 + y} \geq 0, \quad \forall y \in [0, 1].
\end{align*}
\]
The optimal value is \(-1/(2 + \varepsilon)\) for \(\varepsilon \leq 0\) and the problem is unbounded for \(\varepsilon > 0\).

Minimax Problems
\[(B.1)\]
\[
\min_{x \in X} \max_{y \in S} \phi(x, y),
\]
which is equivalent to the LSIP problem
\[
\min_{x \in X} x_0 \\
\text{s.t.} \quad x_0 \geq \phi(x, y), \quad \forall y \in S.
\]

Problem B.9. \(47\) Problem \([1.1]\) with \(\phi(x, y) = (2y - 1)x + y(1 - y)(1 - x), \ X = \mathbb{R}\) and \(S = [0, 1]\). It was shown in \(47\) that the solution is given by \(x_0^* = \sqrt{5} - 2, x^* = 1 - 2\sqrt{5}/5\).
Problem B.10. [1] Problem (B.1) with \(x = (x_1, x_2), y = (y_1, y_2), \phi(x, y) = -y_1x_1 - y_2x_2 - \frac{1}{2}((y_1 - 1)^2 + y_2)(y_1 - y_2 + 2), X = \mathbb{R}^2\) and \(S = [0, 2]^2\). The minimizer was proved to be \(x^*_0 = -1/3, x^*_1 = 1/2, x^*_2 = 1/6\).

Problem B.11. [24] \[
\min_{x_0 \in \mathbb{R}, x \in \mathbb{R}^n} x_0 \quad \text{s.t.} \quad \sum_{i=1}^n a_i(y)x_i - y \geq 0,
\]
where \(a_i(y) = 2i(2i - 1)y^{i-1} + (1 + y)(1 - y^i), i = 1, \ldots, n\). This problem is derived in [24] to find an approximation solution of a linear boundary value problem of monotonic type \(L[v](t) = -v''(t) + (1 + t^2)v(t) = t^2, t \in [-1, 1]\) with boundary conditions \(v(-1) = v(1) = 0\).

One-sided \(L_1\) Approximation Problems

\[\min_{x \in \mathbb{R}^n} \sum_{i=1}^n x_i \quad \text{s.t.} \quad \sum_{i=1}^n \frac{y^{-1}x_i}{i} \geq 0, \quad \forall y \in [0, 1].\] (B.2)

Problem B.12. [10] Problem (B.2) with \(b(y) = -\sum_{i=0}^4 y^{2i}\) and \(n = 7\).

Problem B.13. [12] Problem (B.2) with \(b(y) = \frac{1}{2-y}\) and \(n = 8\).

Problem B.14. [12] Problem (B.2) with \(b(y) = -\frac{1}{1+y}\) and \(n = 10\).

Uniform Functional Approximation Problems

Problem B.15. [33, 34] \[
\min_{x_0 \in \mathbb{R}, x \in \mathbb{R}^n} x_0 \quad \text{s.t.} \quad \sum_{i=1}^n \frac{y^{-1}x_i}{i} \leq \frac{1}{1+y^2}, \quad \forall y \in [0, 1].
\]
To compare with the computational results given in [33], we set \(n = 5\).

\[\min_{x_0, x_1, x_2 \in \mathbb{R}} x_0 \quad \text{s.t.} \quad \sum_{i_1=0}^{t} \sum_{i_2=0}^{t} x_{i_1, i_2} y_{1}^{i_1} y_{2}^{i_2} - b(y) \leq x_0, \quad \forall y \in [-1, 1].\] (B.3)

Problem B.16. [72] Problem (B.3) with \(t = 2\) and \(b(y) = \frac{1}{y_1 + 2y_2 + 4}\).

Problem B.17. [72] Problem (B.3) with \(t = 2\) and \(b(y) = \sqrt{y_1 + 2y_2 + 4}\).
APPENDIX C. TESTING PROBLEMS

Problem C.1. \((m, n, t) = (4, 2, 3)\), \(m_t(Y) = (1, Y_2, Y_1^{2}, Y_1Y_2, Y_1^2Y_2, Y_1^{3}Y_2, Y_1^{4}Y_2, Y_1^{5}Y_2)\).

\[
V = \begin{pmatrix}
2 & 2 \\
1 & -1 \\
1 & 2 \\
-1 & 2 \\
\end{pmatrix},
N = \begin{pmatrix}
0 & -1 & -1 & 0 & 1 & 0 & -1 & -1 & 0 & 1 \\
-1 & -1 & 1 & 1 & 1 & 1 & 0 & -1 & 0 \\
0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 1 & 0 \\
1 & -1 & -1 & 1 & 1 & 0 & -1 & 1 & 0 & -1 \\
1 & -1 & 1 & -1 & -1 & 0 & 1 & -1 & 0 & -1 \\
-1 & 0 & 1 & 1 & 1 & -1 & 1 & 1 & 0 & 1 \\
1 & 1 & -1 & 1 & 1 & 0 & -1 & -1 & 1 & 0 \\
-1 & -1 & 1 & 0 & 0 & 1 & 1 & 1 & -1 & 0 \\
1 & 1 & 1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 \\
1 & -1 & -1 & 1 & 0 & -1 & -1 & 1 & 1 & 1 \\
\end{pmatrix}
\]

Problem C.2. \((m, n, t) = (3, 3, 2)\), \(m_t(Y) = (1, Y_3, Y_2, Y_1^{2}, Y_2Y_3, Y_1Y_3, Y_1^2Y_2, Y_1Y_2, Y_1^2)\).

\[
V = \begin{pmatrix}
1 & 2 & 0 \\
1 & 0 & 1 \\
2 & 1 & 0 \\
\end{pmatrix},
N = \begin{pmatrix}
1 & 0 & -1 & -1 & -1 & 0 & 0 & 1 & -1 \\
-1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 \\
-1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 \\
-1 & 1 & -1 & -1 & -1 & 0 & -1 & 0 & -1 \\
-1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & -1 & 0 & -1 & 1 & 1 & -1 & -1 \\
-1 & 1 & 0 & 0 & 1 & 1 & -1 & -1 & -1 & 0 \\
-1 & 0 & -1 & 0 & -1 & -1 & 1 & 1 & -1 & -1 \\
-1 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & -1 \\
0 & 0 & 1 & -1 & 0 & -1 & 1 & 1 & 0 & -1 \\
\end{pmatrix}
\]

Problem C.3. \((m, n, t) = (4, 5, 1)\), \(m_t(Y) = (1, Y_5, Y_4, Y_3, Y_2, Y_1)\).

\[
V = \begin{pmatrix}
-2 & -1 & -1 & 2 & -2 \\
2 & 1 & -1 & 1 & 0 \\
1 & 0 & -2 & 2 & -1 \\
0 & 1 & 0 & 2 & 2 \\
\end{pmatrix},
N = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & -1 & 1 & 1 & 0 & -1 & 1 \\
1 & 0 & -1 & 1 & 1 & -1 & 1 & -1 & 0 & 1 \\
-1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
-1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Problem C.4. \((m, n, t) = (4, 3, 2)\), \(m_t(Y) = (1, Y_3, Y_2, Y_1^{2}, Y_2Y_3, Y_1Y_3, Y_1^2Y_2, Y_1Y_2, Y_1^2)\).

\[
V = \begin{pmatrix}
1 & -1 & 1 \\
2 & -1 & 2 \\
-1 & -1 & 0 \\
1 & -2 & -1 \\
\end{pmatrix},
N = \begin{pmatrix}
1 & 1 & -1 & -1 & 0 & 0 & 1 & -1 & -1 & -1 \\
-1 & 1 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \\
1 & -1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\
-1 & 0 & 1 & 1 & -1 & -1 & -1 & 0 & -1 & -1 \\
0 & 0 & -1 & 0 & -1 & 0 & 1 & -1 & -1 & -1 \\
1 & -1 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & -1 & 1 & 1 & -1 & 0 & 0 & 1 \\
\end{pmatrix}
\]
Problem C.5. \((m, n, t) = (5, 5, 1)\), \(m_t(Y) = (1, Y_5, Y_3, Y_2, Y_1)\).

\[
V = \begin{pmatrix}
1 & 1 & -1 & 2 & 1 \\
0 & 2 & -2 & 2 & 2 \\
2 & -2 & 0 & 0 & -2 \\
1 & -1 & 0 & -2 & -2 \\
0 & 1 & 1 & -1 & 1
\end{pmatrix},
\quad
N = \begin{pmatrix}
-1 & 0 & -1 & 1 & -1 & 1 \\
-1 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & -1 & 0 & 1 \\
1 & 0 & 1 & -1 & 0 & 0 \\
-1 & -1 & -1 & 0 & 0 & -1 \\
-1 & 1 & 0 & -1 & 0 & -1
\end{pmatrix}.
\]

Problem C.6. \((m, n, t) = (2, 4, 2)\),

\(m_t(Y) = (1, Y_4, Y_3, Y_2, Y_1, Y_4^2, Y_3 Y_4, Y_2 Y_4, Y_1 Y_4, Y_1^2, Y_2 Y_3, Y_1 Y_3, Y_2^2, Y_1 Y_2, Y_1^2)\).

\[
V = \begin{pmatrix}
0 & 1 & 2 & 2 \\
-1 & 2 & 1 & 0
\end{pmatrix},
\quad
N = \begin{pmatrix}
1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \\
-1 & -1 & 1 & 1 & -1 & 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\
-1 & -1 & 1 & -1 & 1 & 0 & 1 & -1 & -1 & 0 & 1 & 1 & 0 \\
0 & -1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\
-1 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 1 & 1 & 0 & -1 \\
0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & -1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & -1 & -1 & 1 & -1 & 1 & 0 & 0 & -1 & 0 \\
0 & -1 & 1 & -1 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 1 & 1 & -1 \\
1 & 0 & 0 & -1 & 0 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 1 & 0 & -1 \\
-1 & -1 & 0 & 1 & -1 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & -1 & 0 \\
1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & -1
\end{pmatrix}.
\]

Problem C.7. \((m, n, t) = (3, 4, 2)\),

\(m_t(Y) = (1, Y_4, Y_3, Y_2, Y_1, Y_4^2, Y_3 Y_4, Y_2 Y_4, Y_1 Y_4, Y_1^2, Y_2 Y_3, Y_1 Y_3, Y_2^2, Y_1 Y_2, Y_1^2)\).

\[
V = \begin{pmatrix}
-2 & 2 & 0 & 1 \\
1 & 0 & 1 & -2 \\
-2 & -1 & 1 & -2
\end{pmatrix},
\quad
N = \begin{pmatrix}
0 & 1 & -1 & 1 & 1 & -1 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & -1 & 1 \\
-1 & 0 & -1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 0 \\
-1 & 0 & 0 & 1 & 1 & 0 & -1 & 1 & 1 & 0 & 1 & 1 & -1 & 1 \\
-1 & 0 & -1 & 1 & -1 & 1 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & -1 & 0 & 1 & 1 & -1 & -1 & 1 & 1 & 0 & -1 & 0 & -1 \\
0 & 0 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & 1 & 1 & -1 & -1 & 1 \\
-1 & 1 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 1 \\
-1 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 1 & -1 \\
0 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 1 & 0 & 1 & 1 \\
-1 & 1 & 0 & -1 & -1 & 1 & 0 & -1 & -1 & 1 & 0 & 1 & 1 & -1 \\
-1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & -1 \\
-1 & 0 & -1 & -1 & 1 & -1 & 0 & -1 & 0 & 1 & 1 & 1 & -1 & -1
\end{pmatrix}.
\]
Problem C.8. \((m, n, t) = (5, 3, 2), m_t(Y) = (1, Y_3, Y_2, Y_1, Y_3^2, Y_2Y_3, Y_1Y_3, Y_2^2, Y_1Y_2, Y_1^2)\),

\[
V = \begin{pmatrix}
0 & 1 & 2 \\
1 & 0 & -2 \\
-1 & -2 & -1 \\
0 & 0 & 0 \\
-1 & -2 & 0
\end{pmatrix}
\]

\[
N = \begin{pmatrix}
0 & 1 & -1 & 1 & -1 & 0 & 1 & -1 & 1 & 0 \\
0 & 0 & 1 & -1 & 0 & -1 & -1 & -1 & 1 & 1 \\
-1 & 0 & 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\
-1 & 1 & -1 & -1 & -1 & 1 & 1 & 0 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & -1 & 0 & -1 & -1 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
-1 & 1 & -1 & -1 & -1 & 0 & 0 & 1 & -1 & -1 \\
0 & -1 & 1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\
-1 & -1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
-1 & 0 & 1 & 1 & -1 & -1 & 1 & -1 & 0 & -1
\end{pmatrix}
\]