Deformed Galilei symmetry

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Abstract

A particular deformation of central extended Galilei group is considered. It is shown that the deformation influences the rules of constructing the composed systems while one particle states remain basically unaffected. In particular the mass appeared to be non additive.

1 Introduction

Quantum groups have emerged in physics in connection with an attempt to understand the symmetries underlying the exact solvability of certain quantum–mechanical and statistical models [1].

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In this respect they appeared to be quite powerful. It is therefore natural to pose the question whether quantum groups could provide a suitable tool for describing another symmetries in physics, in particular the space–time ones. From the mathematical point of view one can define a variety of deformations of classical space–time symmetry groups. They possess some attractive features. For example the deformation of Poincare symmetry, constructed in [2] provides a ten–parameter symmetry “group” with dimensional parameter naturally built in; one can speculate that this parameter is a natural cut–off or reflects the change of properties of space–time at very small scale. However once we take seriously the very idea of quantum symmetries we are faced with many conceptual problems: what is the meaning of non–commuting group parameters, how to deal with non–cocommutative coproduct on the Lie algebra level, etc. There exists no general, commonly acceptable solution to these questions. An order to shed some light on them in the present paper we describe a particular example of rather mild deformation of Galilei group which admits a nice and simple physical interpretation (on the Lie algebra level this deformation was given in [3]). The quantum group we are considering is a deformation of the standard central extension of classical Galilei group constructed in such a way that only the relations concerning the additional group variable are modified. Our basic assumption is that the projective representations of (classical) Galilei group arise from vector representations of our deformed group. It appears then that the one–particle representations remain unmodified except that the physical mass cannot be directly identified with the eigenvalue of mass operator and, moreover is bounded from above. The quantum structure influences the way the many–particles states are constructed. The main difference is the way the total mass of composed system is defined; the mass is no longer additive. We address also to the problem of non–commutativity of coproduct for generators and show that both direct and transpose coproduct lied to physically equivalent theories. We hope that the results presented below show that, at least in cases it is possible to give a coherent physical interpretation to the
2 The deformed centrally extended Galilei group $G_k$

Our starting point is the $\kappa$–Poincare group defined in Ref.[2]. Its defining relations read

\begin{align*}
[\Lambda^\mu_\nu, \Lambda^\alpha_\beta] &= 0 \\
[\Lambda^\mu_\nu, a^\rho] &= -\frac{i}{\kappa} \left((\Lambda^\mu_0 - \delta^\mu_0)\Lambda^\rho_\nu + (\Lambda^0_\nu - \delta^0_\nu)g^{\mu\rho}\right) \\
[a^\mu, a^\nu] &= \frac{i}{\kappa} (\delta^\mu_0 a^\nu - \delta^\nu_0 a^\mu) \\
\Delta(\Lambda^\mu_\nu) &= \Lambda^\mu_\alpha \otimes \Lambda^\alpha_\nu \\
\Delta(a^\mu) &= \Lambda^\mu_\alpha \otimes a^\alpha + a^\mu \otimes I \\
(\Lambda^\mu_\nu)^* &= \Lambda^\mu_\nu, \quad (a^\mu)^* = a^\mu
\end{align*}

In order to obtain the deformed central extension of Galilei group, $G_k$, we follow the same procedure as in the classical case. First, we define a trivial extension of $\kappa$–Poincare by adding new unitary generator $\xi$,

\begin{align*}
\xi^* \xi = \xi \xi^* &= I \\
[\xi, \Lambda^\mu_\nu] = 0, \quad [\xi, a^\mu] = 0
\end{align*}
$$\Delta(\xi) = \xi \otimes \xi$$

Then we redefine

$$\zeta = \xi e^{-imc\alpha^0}$$

$\zeta$ is again unitary and obeys

$$a^0\zeta = \zeta a^0$$

$$a^k\zeta = e^{-\frac{mc}{c^2}}\zeta a^k$$

$$\Lambda^0\zeta = \frac{\zeta \frac{mc}{c^2} + \Lambda^0_0 \frac{mc}{c^2}}{\frac{mc}{c^2} + \Lambda^0_0 \frac{mc}{c^2}}$$

$$\Lambda^0_i\zeta = \frac{\zeta \frac{mc}{c^2} + \Lambda^0_0 \frac{mc}{c^2}}{\frac{mc}{c^2} + \Lambda^0_0 \frac{mc}{c^2}}$$

$$\Lambda^i_0\zeta = \frac{\zeta \frac{mc}{c^2} + \Lambda^i_0 \frac{mc}{c^2}}{\frac{mc}{c^2} + \Lambda^i_0 \frac{mc}{c^2}}$$

$$\Lambda^i_j\zeta = \zeta \left( \Lambda^i_j - \frac{\Lambda^i_0 \Lambda^0_j}{1 - (\Lambda^0_0)^2 (c^2 + \alpha) - \Lambda^0_0} \right), \ c^2 \alpha \equiv \Lambda^0_0$$

In order to apply contraction we redefine [4]

$$\Lambda^0_0 = \gamma \equiv (1 - v^2/c^2)^{-\frac{1}{2}}$$

$$\Lambda^0_i = \frac{\gamma}{c} v^k R^k_i$$

$$\Lambda^i_0 = \frac{\gamma}{c} v^i$$

(4)
\[ \Lambda^i_j = \left( \delta^i_k + (\gamma - 1) \frac{v^i v^k}{v^2} \right) R^k_j \]

\[ a^0 = c\tau, \quad a^i \equiv a^i \]

where \( R^i_j \) represent rotations, \( R R^T = R^T R = I \). Then we put \( \kappa \to \infty, \ c \to \infty, \ \frac{2}{\kappa} \equiv k \)-fixed. The resulting structure reads

\[
[v^i, v^j] = [v^i, R^k_j] = [v^i, a^j] = [v^i, \tau] = 0 \\
[a^i, R^k_j] = [a^i, a^j] = [a^i, \tau] = 0 \\
[R^i_j, R^k_l] = [v^i_j, \tau] = 0 \tag{5} \\
\Delta(v^i) = R^i_j \otimes v^j + v^i \otimes I \\
\Delta(a^i) = R^i_j \otimes a^j + v^i \otimes \tau + a^j \otimes I \\
\Delta(R^i_j) = R^l_k \otimes R^k_j
\]

as well as

\[
\tau \zeta = \zeta \tau \\
a^k \zeta = e^{-\frac{m}{\kappa}} \zeta a^k \\
v^k \zeta = e^{-\frac{m}{\kappa}} \zeta v^k \tag{6} \\
R^i_j \zeta = \zeta R^i_j \\
\Delta(\zeta) = (\zeta \otimes \zeta) e^{-im\left(\frac{2}{\kappa} \otimes \tau + v^k R^k_i \otimes a^i\right)}
\]
It follows from eq. (5) that, as long as the additional generator $\zeta$ is neglected, we are dealing here with the classical Galilei group. Therefore we expected that the one–particle states are described by a standard theory. However, the description of many–particle states will be certainly affected.

In order to find, via duality, the corresponding Lie algebra we define $\Phi(\zeta, \tau, \vec{a}, \vec{v}, \vec{\Theta})$, where $\Theta^i$ parametrise the rotation matrices $R$, to be a function on group manifold under the proviso that $\zeta$ stands leftmost. The Lie algebra generators are then defined by the following duality rules:

\[
< H, \Phi(\zeta, \tau, \vec{a}, \vec{v}, \vec{\Theta}) > = i \frac{\partial \Phi}{\partial \tau} |_e
\]

\[
< P_k, \Phi(\zeta, \tau, \vec{a}, \vec{v}, \vec{\Theta}) > = -i \frac{\partial \Phi}{\partial a^k} |_e
\]

\[
< J_k, \Phi(\zeta, \tau, \vec{a}, \vec{v}, \vec{\Theta}) > = -i \frac{\partial \Phi}{\partial \Theta^k} |_e
\]

\[
< K_k, \Phi(\zeta, \tau, \vec{a}, \vec{v}, \vec{\Theta}) > = -i \frac{\partial \Phi}{\partial v^k} |_e
\]

\[
< M, \Phi(\zeta, \tau, \vec{a}, \vec{v}, \vec{\Theta}) > = \mu \zeta \frac{\partial \Phi}{\partial \zeta} |_e
\]

where $\mu$ is an arbitrary mass unit (see below).

The standard Hopf algebra duality rules imply then

\[
[J_i, P_k] = i \epsilon_{ikl} P_l, \quad [J_i, K_k] = i \epsilon_{ikl} K_l
\]

\[
[K_i, P_k] = i \delta_{ik} \frac{\mu}{1 - e^{-2\mu/k}} (1 - e^{-\frac{2M}{k}})
\]

\[
[K_i, H] = i P_i,
\]
the remaining commutators being vanishing and

\[ \Delta P_i = P_i \otimes a^{-M/K} + I \otimes P_i \]  
\[ \Delta K_i = K_i \otimes e^{-M/k} + I \otimes K_i \]  

(9)

the remaining coproducts being primitive.

This Hopf algebra, modulo simple redefinitions, was obtained firstly in Ref. [3].

3 One–particle dynamics

Due to the fact that the classical Galilei group is a Hopf subalgebra one can define in a standard way its projective representations [5]

\[ \Phi_\sigma(\vec{p}) \rightarrow e^{i(-\frac{\vec{p}^2}{2m_f} + \vec{p} \cdot \vec{a})} \sum_{\sigma'} D_{\sigma\sigma'}^s(R) \Phi_{\sigma'}(R^{-1} \vec{p} - m_f \vec{v}); \]  

(10)

here \( m_f \) denotes the physical mass of a particle. In the classical case the projective representations can be converted to the vector representation of a central extension of Galilei group [5]. Usually, the mass parameter enters the definition of this extension so that it depends on the projective representation we have selected. However, it is not difficult to rephrase the whole construction in such a way that there is a universal central extension which produces all projective representations. In fact, we choose an arbitrary reference mass \( \mu \) to define the central extension and consider the following representation

\[ \Phi_\sigma(\vec{p}) \rightarrow \zeta \frac{m_f}{\mu} e^{i(-\frac{\vec{p}^2}{2m_f} + \vec{p} \cdot \vec{a})} \sum_{\sigma'} D_{\sigma\sigma'}^s(R) \Phi_{\sigma'}(R^{-1} \vec{p} - m_f \vec{v}); \]  

(11)
It is obvious that one obtains the projective representation (10).

We follow the same strategy in the deformed case. Namely, we demand that

\[ \rho \Phi_s(\vec{p}) = (I \otimes \zeta^n) e^{i(-\frac{\vec{p}}{2m_f} \otimes \tau + \vec{p} \otimes \bar{a})} \]

\[ \sum_{\sigma'} (I \otimes D^{s}_{\sigma',}(R)) \Phi_{\sigma'}(\vec{p} \otimes R^{-1} - m_f I \otimes \bar{v}) \]  

be the representation of \( G_k \). It follows from the duality rules that \( m \) is an eigenvalue of \( M \). However, contrary to the classical case, it differs in general from \( m_f \). In fact, it is easy to check that \( \rho \), as given by eq.(12), provides a representation of \( G_k \),

\[ (\rho \otimes I) \circ \rho = (I \otimes \Delta) \circ \rho \]  

only if the following relation holds

\[ m_f = \frac{\mu}{1 - e^{-2\mu/k}}(1 - e^{-2m/k}) \]  

In the limit \( k \to \infty \) we get \( m_f = m \), as expected.

At first it seems that the condition that we are dealing with the representations of \( G_k \) does not give anything new: eq.(14) is merely a parametrization of the physical mass \( m_f \) in terms of some parameter \( m \). However, it is \( m \) which is an eigenvalue of primitive generator \( M \) so it is \( m \) and not \( m_f \) which is additive.

It is easy to see that by a dimensionless (and \( m_f \)-independent!) redefinition
of the $K_i$ and $P_i$ generators eq.(14) can be put in form

$$m_f = \frac{k}{2}(1 - e^{-2m/k}) \quad (15)$$

It follows now that the addition formula for physical masses reads

$$M_f = m_f + m'_f - \frac{2m_fm'_f}{k} \quad (16)$$

Eqs.(15) and (16) have interesting properties. Eq.(13) implies $m_f < \frac{k}{2}$ so that $\frac{k}{2}$ is a maximal possible mass value. This is the only modification for one–particle theory. It is then easy to check that this result agrees with the addition formula: $m_f < \frac{k}{2}$, $m'_f < \frac{k}{2}$ imply $M_f < \frac{k}{2}$; moreover, if $m_f = \frac{k}{2}$ then $M_f = \frac{k}{2}$; the value $\frac{k}{2}$ plays the role of infinite mass in the theory (see also below).

### 4 Two–particle dynamics

In order to describe two–particle states we demand that they should also transform according to the representations of $G_k$. Let us first remind briefly the relevant construction in the undeformed case. The Galilean generators are then simply the sums of generators corresponding to both particles, $P_i = P_{1i} + P_{2i}$, etc. Applying this assumption to the energy operators one obtain $H = H_1 + H_2$ which implies that the particles do not interact. However, due to the specific form of Galilei Lie algebra this situation can be cured in a simple way (contrary to the relativistic case). The energy operator does not appear on the right–hand side of commutation rules. Therefore, one can redefine $H = H_1 + H_2 \rightarrow H_1 + H_2 + V$, where $V$ commutes with all
generators. Due to the fact that two–particle representation is reducible, \( V \) can be nontrivial. In fact, it can be any function of “relative” dynamical variables \((\vec{p}_1 - \vec{p}_2)^2, (\frac{\vec{K}_1}{m_{f1}} - \frac{\vec{K}_2}{m_{f2}})^2\), etc.; these relative variables form, together with Galilean generators, the complete set of dynamical variables.

Let us follow the same procedure in the deformed case. Using the coproduct formulae (8) we define the generators of two–particle representation of \( G_k \):

\[
\vec{P} = e^{-\frac{\omega'}{k}} \vec{p} + \vec{p}'
\]
\[
\vec{K} = e^{-\frac{\omega'}{k}} \vec{k} + \vec{k}'
\]

the remaining generators being simply the sums of their one–particle counterparts. Let us put for a moment aside the problem concerning noncommutativity of the coproduct (the construction of total dynamical variables seems to depend on numbering of particles). In order to construct the two–particle dynamics we supply the generators of \( G_k \) by additional ones (relative variables) such that (i) together with the generators of \( G_k \) they form a complete set of operators, (ii) they have definite properties under rotations and (iii) the standard Heisenberg commutation rules among coordinate and momenta hold.

It is easy to check that these conditions are met by the following choice

\[
\vec{p} = e^{-m'/k} \vec{p} + \vec{p}' \quad \text{(total momentum)}
\]
\[
\vec{R} = \frac{1}{M_f} (e^{-m'/k} \vec{K} + \vec{K}') \quad \text{ (“center–of–mass” coordinate)}
\]
\[
\vec{\Pi} = \frac{1}{M_f} (m'_f \vec{p} - m_f e^{-m'/k} \vec{p}') \quad \text{(relative momentum)}
\]
\[ \vec{\rho} = \frac{K}{m_f} - e^{-m'/k} \frac{K'}{m'_f} \]  
(relative coordinate)

Let us note the following identity

\[ H = \frac{\vec{p}^2}{2m_f} + \frac{\vec{p}'^2}{2m'_f} = \frac{\vec{P}^2}{2M_f} + \frac{\vec{\Pi}^2}{2v_f} \]

where

\[ v_f = \frac{m_f m'_f}{M_f} \]  
(19)

is the deformed reduced mass. Let us note that \( v_f = m_f \) if \( m'_f = \frac{k}{2} \) which supports the point of view that \( \frac{k}{2} \) plays a role of infinite mass. Now, following the standard procedure we can introduce the interaction by adding to the total kinetic energy on arbitrary function \( V(|\vec{\rho}|, |\vec{\Pi}|, \vec{\rho} \cdot \vec{\Pi}) \). If, as in the undeformed case we restrict ourselves to the coordinate–dependent potential functions we arrive at the following form of total hamiltonian

\[ H = \frac{\vec{P}^2}{2M_f} + \left( \frac{\vec{\Pi}^2}{2v_f} + V(|\vec{\rho}|) \right) \]  
(20)

As an illustration consider a simple toy model of the k–deformed hydrogen atom. Its hamiltonian reads:

\[ H = \frac{\vec{p}^2}{2m_f} + \frac{\vec{p}'^2}{2m'_f} - \frac{e^2}{\sqrt{\vec{r}^2 - \sqrt{1 - \frac{2m'_f}{k} \vec{r}^2}}} \]  
(21)
where \( m_f \) and \( m'_f \) are the masses of electron and proton, respectively (their role can be exchanged – see below). It follows from eq. (20) that in the relative coordinates \( H \) takes a standard form

\[
H = \frac{\vec{p}^2}{2M_f} + \frac{\vec{p}^2}{2v_f} - \frac{e^2}{|\vec{\rho}|}
\]

so the energy spectrum reads

\[
E_n = -\frac{v_f e^4}{2\hbar^2 n^2} = -\frac{ve^4}{2\hbar^2 n^2} \left( 1 + \frac{2v}{k} + \ldots \right)
\]

where \( v \) is standard reduced mass \( \left( = \frac{m_f m'_f}{m_f + m'_f} \right) \). The question can be posed whether this correction is in principle observable. This depends on whether \( m_f \) and \( m'_f \) are observable masses of electron resp. proton. They should be measured by making them interacting with infinitely heavy source of forces which, in our theory means that its mass equals \( \frac{k}{2} \). But in this case the reduced mass is just the mass of the particle under consideration. Therefore, the masses \( m_f \) and \( m'_f \) are, in principle, measurable and so is the correction to the Bohr formula.

Let us now consider the problem of apparent asymmetry in construction of two-particle states (the problem of coproduct in the context of particle interaction was also considered in [6]). It seems that our theory depends on the choice of order in which we add particles to the system; this is obviously related to the noncocommutativity of the coproduct. If it were true to the whole theory would make no sense. However, it is not difficult to show that the order in which we add particles is immaterial – both description are related by an unitary transformation. In fact, using the transposed coproduct we arrive at the following set of basic dynamical variables replacing those
given by eqs. (18)

\[ \tilde{\vec{p}} = \vec{p} + e^{-m/k} \vec{p}' \]
\[ \tilde{\vec{R}} = \frac{1}{M_f} (\vec{K} + e^{-m/k} \vec{K}') \]
\[ \tilde{\vec{\Pi}} = \frac{1}{M_f} (m'_f \vec{p} e^{-m/k} - m_f \vec{p}') \]
\[ \tilde{\vec{\rho}} = \frac{\vec{K} e^{-m/k}}{m_f} - \frac{\vec{K}'}{m'_f} \]

First two equations represent simply the transposed coproduct. The relative variables are chosen such that they obey Heisenberg commutation rules. The ambiguity in sign of \( \tilde{\vec{\Pi}} \) and \( \tilde{\vec{\rho}} \) is resolved by demanding that our transformation does not exchange particles (i.e. reduces to an identity for \( k \to \infty \)). It is easy to check that both sets of variables, eqs.(18) and (24) are related by the following unitary transformation

\[ U = e^{i \sqrt{m_f m'_f} \arctan \left( \frac{\sqrt{m_f m'_f} (1 - \sqrt{1 - \frac{m_f}{k}} \sqrt{1 - \frac{m'_f}{k}})}{m_f \sqrt{1 - \frac{2m_f}{k} + m'_f} \sqrt{1 - \frac{2m'_f}{k}}} \right) (\vec{R} \otimes \vec{p} - \vec{p} \otimes \vec{K})} \]

We conclude that whatever order of composing the system we choose, the result is, up to an unitary transformation, the same. Let us now consider a particular case of identical particles. Define the exchange operator \( S \) as

\[ S \Phi_{\sigma \sigma'}(\vec{p}, \vec{p}') = \Phi_{\sigma' \sigma}(\vec{p}', \vec{p}) \]
It easy to see that

\[(US)^2 = I\]  \hspace{1cm} (27)

Therefore, one can define bosonic vs. fermionic states as obeying

\[(US\Phi)_{\sigma\sigma'}(\vec{p}, \vec{p}') = \pm \Phi_{\sigma\sigma'}(\vec{p}, \vec{p}')\]  \hspace{1cm} (28)

In the limit \(k \to \infty\), \(U \to I\) and our definition coincides with the standard one. For any \(k\) it reverses the sign of relative coordinates. Consequently, it imposes the same restriction on admissible states and observables as classical Fermi–Bose symmetry.

5 Conclusions

We have presented simple nonrelativistic quantum mechanical model based on deformed centrally extended Galilei group. The deformation considered leaves the proper Galilei group unaffected. The main assumption was that the physical states form the vector representations of this central extension. The one–particle states do not differ from their undeformed counterparts and span the representation space of projective representation of standard Galilei group, the only difference being the relation between the physical mass and the eigenvalue of additional generator (mass operator). In particular, this relation implies an upper bound for particle mass, \(m_f < \frac{k}{2}\). The main difference concerns the way the many–particle states are constructed. The addition formula for masses is modified in the way it contains the deformation parameter \(k\). It obeys a consistency condition that adding two masses below
the $\frac{k}{2}$–bound one gets the total mass obeying the same inequality. Also, the definition of reduced mass is modified. However, we have shown that once the two–particle interaction is expressed in terms of relative coordinates (it should be expressible if the theory is Galilei–invariant) the theory takes standard form except the above — mentioned difference in the definition of reduced mass. Corrections to the reduced mass are, in principle, observable because the masses of individual particles can be determined by making them interacting with “infinitely heavy” ($m_f = \frac{k}{2}$) source.

The main difficulty inherent in the formalism is that the total as well as relative variables depend on the order in which particles are added to the system. However, the theories obtained by selecting different orders are unitary equivalent which makes the whole scheme consistent. Suitable changes should be made in the case of identical particles. The modified conditions defining bosons and fermions are given in eqs.(28). It appears that, once the wave function is expressed in terms of total and relative variables, they provide the same selection rules as in the standard case.

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