The random transverse field Ising model in $d = 2$: analysis via boundary strong disorder renormalization

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Abstract. To avoid the complicated topology of surviving clusters induced by standard strong disorder RG in dimension $d > 1$, we introduce a modified procedure called ‘boundary strong disorder RG’ where the order of decimations is chosen a priori. We apply this modified procedure numerically to the random transverse field Ising model in dimension $d = 2$. We find that the location of the critical point, the activated exponent $\psi \simeq 0.5$ of the infinite-disorder scaling, and the finite-size correlation exponent $\nu_{FS} \simeq 1.3$ are compatible with the values obtained previously using standard strong disorder RG. Our conclusion is thus that strong disorder RG is very robust with respect to changes in the order of decimations. In addition, we analyze the RG flows within the two phases in more detail, to show explicitly the presence of various correlation length exponents: we measure the typical correlation exponent $\nu_{typ} \simeq 0.64$ for the disordered phase (this value is very close to the correlation exponent $\nu_0^{\text{pure}}(d = 2) \simeq 0.63$ of the pure two-dimensional quantum Ising model), and the typical exponent $\nu_h \simeq 1$ for the ordered phase. These values satisfy the relations between critical exponents imposed by the expected finite-size scaling properties at infinite-disorder critical points. We also measure, within the disordered phase, the fluctuation exponent $\omega \simeq 0.35$ which is compatible with the directed polymer exponent $\omega_{\text{DP}}(1+1) = \frac{1}{3}$ in $(1+1)$ dimensions.

Keywords: finite-size scaling, quantum phase transitions (theory), renormalization group, disordered systems (theory)

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1. Introduction

Strong disorder renormalization (see [1] for a review) was first introduced for one-dimensional quantum spin chains [2]–[4], where exact solutions can be obtained because the renormalized lattice of surviving degrees of freedom remains one-dimensional. In dimension $d > 1$, the strong disorder RG procedure cannot be carried out analytically, because the topology of the lattice changes upon renormalization, but it has been implemented numerically, in particular for the quantum Ising model [5]–[15]. Nevertheless, the complicated topology that emerges between renormalized degrees of freedom in dimension $d > 1$ tends to obscure the physics and slow down the numerics, because a large number of very weak bonds are generated during the RG, that will eventually not be important for the subsequent RG steps. Various kinds of simplifications have hence been proposed, like the ‘maximum rule’ [5]–[10] possibly supplemented by some very efficient algorithm [12]–[15], the introduction of a cutoff within the full sum rules [16] and the planar approximation [17]. Recently we have proposed another strategy: the idea is to allow some changes in the order of decimations with respect to the full procedure in order to maintain a simple spatial renormalized structure. We have already applied

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this idea in two ways: (i) in [18], we have proposed including strong disorder RG ideas within the more traditional fixed-length-scale real space RG framework that preserves the topology upon renormalization, with numerical results for various kinds of fractal lattices; (ii) in [19], for the Cayley tree geometry, we have proposed a ‘boundary strong disorder RG procedure’ that preserves the tree structure, so that one can write simple recursions with respect to the number of generations. In both cases, we have checked that in dimension $d = 1$, these modified procedures correctly capture all critical exponents except for the magnetic exponent $\beta$ which is related to persistence properties of the full RG flow. In the present paper, we adapt this idea of a ‘boundary strong disorder RG procedure’ to the two-dimensional case and present the corresponding numerical results, which we compare with the results from standard strong disorder RG [5]–[15] and with quantum Monte Carlo results [20, 21].

The paper is organized as follows. In section 2, we define the boundary strong disorder RG procedure for the two-dimensional square lattice. In the following sections, we discuss the numerical results obtained by this procedure. For the disordered phase (section 3), we measure the typical correlation exponent $\nu_{\text{typ}}$, the fluctuation exponent $\omega$ and the essential singularity exponent $\kappa$. For the ordered phase (section 4), we measure the typical correlation exponent $\nu_h$. We find that the location of the critical point in the critical region (section 5), the activated exponent $\psi$ and the finite-size correlation exponent $\nu_{\text{FS}}$ are compatible with the results obtained previously by using standard strong disorder RG [5]–[15]. Our conclusions are summarized in section 6.

2. The boundary strong disorder RG procedure in $d = 2$

As recalled in appendix A, the strong disorder renormalization for the quantum Ising model is an energy-based RG, where the strongest ferromagnetic bond or the strongest transverse field is iteratively eliminated. In this section, we introduce a modified procedure, called boundary strong disorder RG, that preserves a simple spatial structure.

2.1. The initial model

In this paper, we consider the quantum Ising model defined in terms of Pauli matrices

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij} \sigma_i^z \sigma_j^z - \sum_i h_i \sigma_i^x \tag{1}$$

on the square lattice in dimension $d = 2$ where the initial nearest-neighbor couplings $J_{ij}^{\text{ini}}$ are independent random variables drawn with the box distribution on the unit interval $[0, 1]:$

$$\pi_J(J_{ij}^{\text{ini}}) = \theta(0 \leq J_{ij}^{\text{ini}} \leq 1) \tag{2}$$

and where the initial transverse fields $h_i^{\text{ini}} > 0$ are independent random variables drawn with the box distribution on the interval $[0, h]:$

$$\pi_h(h_i^{\text{ini}}) = \frac{1}{h} \theta(0 \leq h_i^{\text{ini}} \leq h), \tag{3}$$

so the parameter $h$ is the control parameter of the quantum phase transition as in [7]–[15], [20, 21].

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For the numerical results, we consider more precisely a square lattice containing \((2L - 1)^2\) spins of coordinates \((x = 1, 2, \ldots, 2L - 1; y = 1, 2, \ldots, 2L - 1)\). Each spin has its random initial transverse field drawn with the distribution of equation (3) and is connected to four neighbors via random ferromagnetic couplings drawn with the distribution of equation (2). All exterior sites situated along the boundaries at \(x = 0, x = 2L\) or \(y = 0, y = 2L\) are identified with a single formal ‘external spin’ to keep track of the coupling to the boundary of the finite sample.

2.2. Boundary strong disorder RG spatial structure

We wish to eliminate sites in a simple deterministic order, starting from the boundary: we will first eliminate sites that are at distance 1 from the boundary having coordinates \((x = 1)\) or \((x = 2L - 1)\) or \((y = 1)\) or \((y = 2L - 1)\); then sites that are at distance 2 from the boundary having coordinates \((x = 2)\) or \((x = 2L - 2)\) or \((y = 2)\) or \((y = 2L - 2)\); and so on, up to sites that are at distance \((L - 1)\) from the boundary having \((x = L - 1)\) or \((x = L + 1)\) or \((y = L - 1)\) or \((y = L + 1)\). At a given stage of the RG, we have a renormalized spatial structure containing a ‘corona of renormalized boundary sites’ (the number of sites in the corona scales as the surface \((L - l)^{d-1}\) when the corona is at distance \(l\) from the boundary) and the ‘interior of the corona’ (the number of sites in the corona scales as the volume \((L - l)^d\) when the corona is at distance \(l\) from the boundary). We have the following properties (see figure 1).

- The ‘interior of the corona’ contains sites that have not yet been modified with respect to the initial model, i.e. the sites are characterized by their initial random fields \(h_{\text{ini}}(i)\), and are connected to their initial neighbors by their initial ferromagnetic coupling \(J_{\text{ini}}^{ij}\).
- The ‘corona’ contains renormalized boundary sites \((i)\) that have renormalized transverse fields \(h(i)\) and that are connected to the formal external spin via some renormalized coupling \(J_{\text{ext}}^{ij}(i)\). These corona sites are connected to ‘interior spins’ via their initial ferromagnetic coupling \(J_{\text{ini}}^{ij}\). Finally, there may exist renormalized ferromagnetic couplings \(J_{ij}\) between any two pair \((i, j)\) of sites belonging the corona.
- The sites outside the ‘corona’ have been already eliminated, and these eliminations are responsible for the renormalized variables characterizing the corona sites.

2.3. Boundary strong disorder RG rules for a corona site

To eliminate a given site \(i\) of the corona, we determine the maximum between its renormalized transverse field \(h(i)\) and its renormalized ferromagnetic couplings \(J_{ij}\) with the other sites of the corona or interior sites:

\[
\Omega_i = \max \left[ h_i, J_{ij} \right].
\]

(Note that the renormalized external coupling \(J_{\text{ext}}^{ij}(i)\) is excluded, since it is just a ‘passive’ variable used to measure the effective coupling to the initial boundary.)

Then we apply the strong disorder RG rules as follows (see appendix A):

(i) If \(\Omega_i = h_i\), then the site \(i\) is decimated, and all couples \((j, k)\) of neighbors of \(i\) are now linked via the renormalized ferromagnetic coupling

\[
J_{\text{new}}^{jk} = J_{jk} + \frac{J_{ji}J_{ik}}{h_i}.
\]

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Figure 1. Illustration of the spatial renormalized structure during the ‘boundary strong disorder RG’ in $d = 2$: any site $k$ belonging to the interior of the corona (denoted here by a circle) still has its initial transverse field $h_{\text{ini}}(k)$ and its initial coupling $J_{\text{ini}} = J$ to its neighbors; any site $i$ of the ‘corona’ (denoted here by a square) has a renormalized transverse field $h_i$, a renormalized coupling $J_{i\text{ext}}$ to the formal external spin, and possibly a renormalized coupling $J_{ij}$ to any other spin $j$ of the corona; finally the sites outside the corona have already been eliminated, and these eliminations are responsible for the renormalized variables characterizing the corona sites.

Accordingly, the external couplings of all neighbors $j$ of $i$ are renormalized according to

$$J_{j,\text{new}} = J_j + J_{j,i} \frac{h_i}{h_j}. \quad (6)$$

(ii) If $\Omega_i = J_{ij}$, then the site $i$ is merged with the site $j$. The new renormalized site $j$ has a reduced renormalized transverse field

$$h_{j,\text{new}} = \frac{h_i h_j}{J_{ij}}. \quad (7)$$

This renormalized site $j$ is connected to other sites $k$ via the renormalized couplings

$$J_{j,k,\text{new}} = J_{jk} + J_{jk}. \quad (8)$$

In particular, the external coupling of the renormalized site $j$ becomes

$$J_{j,\text{ext,new}} = J_{j,\text{ext}} + J_{j,\text{ext}}. \quad (9)$$

Note that when the site $i$ is eliminated, all interior sites that were connected to the site $i$ become sites of the new corona.

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In the final state of the RG procedure, only the center site of coordinates \((x_c = L, y_c = L)\) remains. The observables of interest are its final renormalized transverse field \(h_L\) and its ferromagnetic coupling \(J_{L}^{\text{ext}}\) to the formal ‘external spin’, i.e. to the initial boundary of the square sample. To simply the notation, \(J_{L}^{\text{ext}}\) will be denoted simply by \(J_L\) from now on. In the following, we will concentrate on the typical values \(h_L^{\text{typ}}\) and \(J_L^{\text{typ}}\) defined by
\[
\ln h_L^{\text{typ}} \equiv \overline{\ln h_L}
\]
and on the widths of the distributions of \(\ln h_L\) and \(\ln J_L\) defined by
\[
\Delta_{\ln h_L} \equiv \left( \left( \ln h_L \right)^2 - \left( \overline{\ln h_L} \right)^2 \right)^{1/2}
\]
\[
\Delta_{\ln J_L} \equiv \left( \left( \ln J_L \right)^2 - \left( \overline{\ln J_L} \right)^2 \right)^{1/2}
\]
where the overbar denotes an average over the disordered samples.

2.4. Numerical details

We have followed numerically the boundary RG rules for square samples containing \(N_L = (2L - 1)^2\) spins, for various sizes \(L \leq 100\) corresponding to \(N_L \leq 39601\). For a given size \(L\), the number \(n_s(L)\) of independent disordered samples that we have been able to study depends on the value \(h\) of the initial disorder distribution of equation (3). So let us give some typical values that we have used for the critical region (we were able to study more samples in the ordered phase \(h < h_c\) and fewer samples in the disordered phase):
\[
L = 6, 10, 20, 40, 60, 80, 100
\]
\[
n_s(L) = 10^8, 2 \times 10^7, 3 \times 10^6, 2 \times 10^5, 2 \times 10^4, 5 \times 10^2, 2 \times 10^2.
\]
(12)
Our various data shown below are compatible with a critical point located around the value (see the definition of the control parameter \(h\) in the initial disorder distribution of equation (3)):
\[
h_c \simeq 5.15,
\]
(13)
which is slightly lower than but close to the values found previously using the standard strong disorder RG rules with the maximum rule, namely \(h_c \simeq 5.3\) [7] and \(h_c \simeq 5.35\) [10]–[12], [14, 15]. Let us first discuss the properties of the two phases, before we turn to the critical region.

3. Analysis of the disordered phase

3.1. RG flow of the renormalized external coupling \(J_L\) in the disordered phase

In the disordered phase, the renormalized external coupling \(J_L\) is expected to present the following scaling:
\[
\ln J_L = -\frac{L}{\xi_{\text{typ}}} + L^\omega A(h)u.
\]
(14)
The first non-random term describing the exponential decay with the size $L$ (see figure 2(a)) defines the typical correlation length $\xi_{\text{typ}}$:

$$\ln J_L^{\text{typ}} \equiv \ln J_L \overset{L \to \infty}{\longrightarrow} -\frac{L}{\xi_{\text{typ}}}$$

(15)

In figure 2(b), we show how $1/\xi_{\text{typ}}$ varies as a function of the control parameter $h$ of the transition: our data are compatible with the power-law divergence (see the log–log plot in the inset of figure 2(b))

$$\xi_{\text{typ}} \propto (h - h_c)^{-\nu_{\text{typ}}}$$

(16)

with a typical correlation exponent of order

$$\nu_{\text{typ}} \simeq 0.64.$$  

(17)

To the best of our knowledge, this is the first numerical measure of this typical exponent $\nu_{\text{typ}}$ within the disordered phase, since previous studies have concentrated on the critical region where finite-size effects are governed by another correlation length exponent $\nu_{\text{FS}}$ (see section 5). We note that the value found here for the typical exponent of equation (17) turns out to be very close to the correlation exponent $\nu_{\text{pure}}^Q(d = 2) \simeq 0.63$ of the pure two-dimensional quantum Ising model. (The latter is known to coincide with the correlation exponent $\nu_{\text{pure}}^{\text{class}}(d + 1 = 3) \simeq 0.63$ of the pure three-dimensional classical Ising model as a consequence of the quantum–classical correspondence [23].)

Since in dimension $d = 1$, the typical exponent $\nu_{\text{typ}}(d = 1) = 1$ also coincides with the correlation exponent $\nu_{\text{pure}}^Q(d = 1) = 1$ of the pure one-dimensional quantum Ising model (and equivalently with the exponent $\nu_{\text{pure}}^{\text{class}}(d + 1 = 2) = 1$ of the pure two-dimensional

Figure 2. (a) Statistics of the logarithm of the external renormalized coupling ($\ln J_L$) in the disordered phase (here $h = 8$): the RG flows of the typical value $\ln J_L^{\text{typ}}$ and of the width $\Delta \ln J_L$ in a log–log plot display slopes 1 and $\omega \simeq 0.35$ respectively (equation (19)). (b) Inverse of the typical correlation length $\xi_{\text{typ}}$ of equation (15) as a function of the control parameter $h > h_c$: the correlation length exponent $\nu_{\text{typ}}$, of equation (16) is of order $\nu_{\text{typ}} \simeq 0.64$ (see the log–log plot in the inset).
classical Ising model), it would be interesting to determine whether these coincidences continue in higher dimensions \(d \geq 3\), i.e. whether the typical exponent takes the simple value \(\nu_{\text{typ}}(d \geq 3) = \nu_{\text{pure}}^\text{class}(d \geq 3) = \nu_{\text{pure}}^{d+1}(d \geq 4) = \frac{1}{2}\).

The second term in equation (14) contains an \(O(1)\) random variable \(u\), which is expected to be subleading with respect to the first term, i.e. the width \(\Delta_{\ln J_L}\) of the distribution of \(\ln J_L\) is of order \(L^\omega\) with some fluctuation exponent \(\omega < 1\):

\[
\Delta_{\ln J_L} \equiv \left( \left( \ln J_L \right)^2 - \left( \ln J_L \right)^2 \right)^{1/2} \xrightarrow{L \to +\infty} L^\omega.
\]

We have argued in [22] that this exponent \(\omega\) should coincide with the droplet exponent \(\omega_{\text{DP}}(D = d - 1)\) of the directed polymer with \(D = (d - 1)\) transverse directions. For our present case with \(d = 2\), the droplet exponent of the directed polymer is exactly known to be \(\omega_{\text{DP}}(D = 1) = \frac{1}{3}\):

\[
\omega = \omega_{\text{DP}}(D = 1) = \frac{1}{3}
\]

in agreement with our numerical results shown in figure 2(a). Again, to the best of our knowledge, this fluctuation exponent \(\omega\) within the disordered phase had not been measured previously, since previous studies have concentrated on the critical region.

### 3.2. RG flow of the renormalized transverse field \(h_L\) in the disordered phase

In the disordered phase, the renormalized transverse field \(h_L\) is expected to remain a finite random variable as \(L \to +\infty\). In particular, the typical value remains finite:

\[
\ln h_L^{\text{typ}} \equiv \ln h_L \xrightarrow{L \to +\infty} \ln h_\infty.
\]

To analyze more clearly the statistics of the finite renormalization with respect to the initial random fields \(h_\infty\) drawn with the box distribution for equation (3) corresponding to

\[
\bar{\ln h_\infty} = \int_0^h \frac{dh_\infty}{h} \ln h_\infty = \ln h - 1
\]

we show in figure 3 the difference \((\ln h_\infty - \ln h_\infty)\) as a function of \(h > h_c\). Our data are compatible with the expected essential singularity

\[
\ln \frac{h_\infty}{h_\infty} \propto -(h - h_c)^{-\kappa}
\]

with an exponent of order

\[
\kappa \simeq 0.65.
\]

Since the dynamical exponent \(z\) is expected to have the same singularity as the averaged value of equation (22), we may compare our estimate of \(\kappa\) with the measure given after equation (31) of [12] based on the standard RSRG procedure: \(z \propto (h - h_c)^{-0.60(6)}\).
4. Analysis of the ordered phase

4.1. RG flow of the renormalized transverse field $h_L$ in the ordered phase

For the ordered phase, the logarithm of the renormalized transverse field is expected to behave extensively in the volume $L^d$:

$$\ln h_L^{-\text{typ}} \equiv \ln h_{L \rightarrow \infty} \propto \frac{L}{\xi_h}$$

with $d = 2$, in agreement with our data shown in figure 4(a). The length scale $\xi_h$ represents the characteristic size of finite disordered clusters within this ordered phase. It is expected to diverge as a power law near the transition:

$$\xi_h \propto (h_c - h)^{-\nu_h}.$$  \hfill (25)

The corresponding correlation exponent $\nu_h$ plays in the ordered phase a role similar to that of $\nu_{\text{typ}}$ in the disordered phase (equation (16)). Our data are compatible with a value of order (see figure 4(b))

$$\nu_h \simeq 1.$$  \hfill (26)

Again, to the best of our knowledge, this is the first numerical measure of the typical exponent $\nu_h$ for within the ordered phase, since previous studies have concentrated on the critical region where finite-size effects are governed by another correlation length exponent $\nu_{\text{FS}}$ (see section 5). We note that this exponent also takes the same value $\nu_h(d = 1) = 1$ in $d = 1$, but we are not aware of any argument in favor of this simple constant value as $d$ varies.

Figure 4. Statistics of the asymptotic finite renormalized transverse field $h_{L \rightarrow \infty}$ in the disordered phase as a function of the control parameter $h > h_c$: the critical exponent $\kappa$ of equation (22) is of order $\kappa \simeq 0.65$ (see the log–log plot in the inset).

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Figure 4. (a) Statistics of the logarithm of the renormalized transverse field \((\ln h_L)\) in the ordered phase (here \(h = 1\)): the RG flows of the typical value \((\ln h_L)_{\text{typ}}\) and of the width \(\Delta \ln h_L\) in a log–log plot display slope \(d = 2\) and slope \(d/2 = 1\) respectively. (b) Inverse of the correlation length \(\xi_h\) of equation (24) as a function of the control parameter \(h < h_c\): the correlation length exponent \(\nu_h\) of equation (16) is of order \(\nu_h \simeq 1\).

As shown in figure 4(a), the width of the distribution of the logarithm of the renormalized transverse field grows linearly in \(L\):

\[
\Delta \ln h_L \equiv \left( (\ln h_L)^2 - (\ln h_L)_{\text{typ}}^2 \right)^{1/2} \propto L \rightarrow +\infty L.
\]  

4.2. RG flow of the renormalized external coupling \(J_L\) in the ordered phase

In dimension \(d = 1\) where there is no underlying classical ferromagnetic transition, the typical renormalized coupling remains finite in the ordered phase, and presents the same essential singularity as in equation (22). However in dimension \(d > 1\) where there exists an underlying classical ferromagnetic transition, the renormalized coupling \(J_L\) is expected to grow at large \(L\) with the scaling of the classical random ferromagnetic model (see figure 5):

\[
\ln J_L^{\text{typ}} \equiv \ln J_L \propto \ln \left( \sigma(h) L^{d_s} \right)
\]  

where \(d_s = d - 1 = 1\) represents the interface dimension, and where \(\sigma\) represents the surface tension. We are able to measure the asymptotic behavior of equation (28) only sufficiently far \((h \leq 4.25)\) from the critical point \(h_c \simeq 5.15\), so we cannot measure the critical behavior of the surface tension \(\sigma(h)\).

As shown in figure 5, the width \(\Delta \ln J_L\) of the distribution of the logarithm of the external renormalized coupling decays as

\[
\Delta \ln J_L \propto L \rightarrow +\infty L^{-0.5}.
\]
5. Analysis of the critical region

5.1. RG flow of the renormalized transverse field at criticality

At the infinite-disorder critical point, the renormalized transverse field $h_L$ is expected to display an activated scaling in $L$ with some exponent $\psi$:

$$\ln h_L \propto -L^\psi v_c$$  \hspace{1cm} (30)

where $v_c$ is an $O(1)$ random variable. We show in figure 6(a) our data concerning the RG flows of the typical value ($\ln h_L^{\text{typ}}$) and the width $\Delta^{\ln \ln h_L}$ of the distribution of the logarithm of the renormalized transverse field at criticality, $h = h_c \simeq 5.15$. Our data are consistent with the same scaling for both, with an exponent in the region

$$0.4 \leq \psi \leq 0.5$$  \hspace{1cm} (31)

in agreement with previous estimates based on standard strong disorder RG [5]–[15] or on quantum Monte Carlo methods [20, 21]. The numerical estimate of equation (31) from the scaling at criticality is not precise as a consequence of the uncertainty of the exact location of the critical point (equation (13)), and of the curvature of the data in log–log plots. It is thus interesting to discuss the finite-size scaling in the critical region to relate $\psi$ to other critical exponents of the ordered and disordered phases measured in previous sections.

In the critical region around this infinite-disorder fixed point, one expects the following finite-size scaling form for the typical values [6]:

$$\ln h_L^{\text{typ}} \equiv \ln h_L = -L^\psi F_h \left(L^{1/\nu_S} |h - h_c| \right)$$  \hspace{1cm} (32)
where $\nu_{FS}$ is the correlation length exponent that governs all finite-size effects in the critical region. (The exponent $\nu_{FS}$ is expected [3, 6] to correspond to the exponent $\nu_{av}$ of the averaged two-point correlation function $\nu_{FS} = \nu_{av}$.)

The compatibility of the finite-size scaling form of equation (32) with the essential singularity of equation (22) as regards the disordered phase yields the relation

$$\kappa = \psi \nu_{FS}$$

whereas the compatibility with the behavior of equation (24) as regards the ordered phase yields the relation

$$\nu_h = \left(1 - \frac{\psi}{d}\right) \nu_{FS}.$$  

Eliminating $\nu_{FS}$, we may thus obtain a numerical estimate of $\psi$ from our previous measures of $\kappa \simeq 0.65$ (equation (23)) and $\nu_h \simeq 1$ (equation (26)):

$$\psi = \frac{\kappa}{\nu_h + \kappa/d} \simeq 0.49$$

and the corresponding finite-size exponent then reads

$$\nu_{FS} = \nu_h + \frac{\kappa}{d} \simeq 1.32.$$  

These two values are close to the values $\psi \simeq 0.48$ and $\nu \simeq 1.25$ obtained from standard strong disorder RG [11, 12, 14, 15]. To test the validity of the finite-size scaling form of equation (32), we show in figure 6(b) the satisfactory data collapse obtained with $\psi \simeq 0.5$ and $\nu_{FS} \simeq 1.3$. 

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5.2. RG flow of the renormalized external coupling $J_L$ at criticality

We show in figure 7(a) our data concerning the RG flows for the typical value $\ln J_L^{\text{typ}}$ and for the width $\Delta_{\ln J_L}$ of the distribution of the logarithm of the renormalized external coupling at criticality, $h = h_c \simeq 5.15$. Our data are again consistent with the same scaling for both, with an exponent $\psi$ again in the interval of equation (31).

To estimate a more precise value, it is again interesting to consider the finite-size scaling properties in the critical region [6]:

$$\ln J_L^{\text{typ}} \equiv \ln J_L = -L^\psi F_J(L^{1/\nu_{FS}}|h - h_c|).$$

The compatibility of the finite-size scaling form with the behavior of equation (15) as regards the disordered phase implies the following relation between exponents:

$$\nu_{\text{typ}} = (1 - \psi)\nu_{FS}. \quad (38)$$

The comparison with the relation of equation (33) yields, with our previous measures of $\nu_{\text{typ}} \simeq 0.64$ (equation (17)) and of $\kappa \simeq 0.65$ (equation (23)),

$$\psi = \frac{1}{1 + \nu_{\text{typ}}/\kappa} \simeq 0.5 \quad (39)$$

in agreement with equation (35). The corresponding value

$$\nu_{FS} = \nu_{\text{typ}} + \kappa \simeq 1.29 \quad (40)$$

is close to the value of equation (36) and to the value $\nu \simeq 1.25$ obtained from standard strong disorder RG [11, 12, 14, 15]. To test the validity of the finite-size scaling form of equation (37), we show in figure 7(b) the satisfactory data collapse obtained with $\psi \simeq 0.5$ and $\nu_{FS} \simeq 1.3$. 

Figure 7. (a) Statistics of the renormalized external coupling $J_L$ at criticality, $h_c \simeq 5.15$: $(-\ln J_L^{\text{typ}})$ and $\Delta_{\ln J_L}$ as functions of $L$ (inset: a log–log plot of the same data). (b) Test of the finite-size scaling form of equation (37): the data collapse obtained with $\psi \simeq 0.5$ and $\nu_{FS} \simeq 1.3$ is satisfactory.
As a final remark, we have thus found that the activated exponent $\psi \simeq 0.5$ at criticality is greater than the fluctuation exponent $\omega = 1/3$ of the disordered phase (equation (19)), as already found for fractal lattices with $d > 1$ [18], whereas in dimension $d = 1$, the two exponents coincide: $\psi(d = 1) = 1/2 = \omega(d = 1)$. This means that the amplitude $A(h)$ of equation (14) should diverge as $A(h) \propto (h - h_c)^{-(\psi - \omega)\nu_{FS}}$, but we are not able to measure this singularity with our data.

6. Conclusion

To avoid the complicated topology of surviving clusters induced by standard strong disorder RG in dimension $d > 1$, we have introduced a modified procedure called ‘boundary strong disorder RG’ for the random transverse field Ising model in $d = 2$. The hope is that, as for the one-dimensional case discussed in [19], this simpler ‘boundary strong disorder RG’ that changes the order of decimations could be able to reproduce correctly all critical exponents except the magnetic exponent $\beta$ which is related to persistence properties of the RG flow. Note that within the standard RSRG procedure, the effects of ‘bad decimations’ in one dimension have been analyzed in detail in appendix E of [3], with the conclusion that ‘we recover exactly at a later stage from the errors made earlier’. This robustness of strong disorder RG rules against mistakes in the order of decimations has also been found in higher dimensional systems in another context (see section 3.6 of [24]). Here we thus hope that this phenomenon still occurs when one imposes even more bad decimations. However, since no exact result exists for RSRG in $d = 2$, we cannot really prove that this is indeed the case, but we have presented detailed numerical results obtained from ‘boundary strong disorder RG’ and compared them with previous results obtained via standard RSRG.

We have found that the location of the critical point, the activated exponent $\psi \simeq 0.5$ of the infinite-disorder scaling, and the finite-size correlation exponent $\nu_{FS} \simeq 1.3$ are compatible with the results obtained previously from standard strong disorder RG [5]–[15]. We thus believe that our modified simplified procedure captures correctly the critical properties. In addition, we have analyzed in detail the RG flows within the two phases. We have measured within the disordered phase: the typical correlation exponent $\nu_{typ} \simeq 0.64$, which is very close to the correlation exponent $\nu_{pure}^Q(d = 2) \simeq 0.63$ of the pure two-dimensional quantum Ising model; the fluctuation exponent $\omega \simeq 0.35$ which is compatible with the directed polymer exponent $\omega_{DP}(1 + 1) = \frac{1}{4}$ in $(1 + 1)$ dimensions, in agreement with the arguments of [22]; and the essential singularity exponent $\kappa \simeq 0.65$. We have measured within the ordered phase: the typical exponent $\nu_{h} \simeq 1$, which is close to the value $\nu_{h}(d = 1) = 1$ in $d = 1$.

The simple values found here for $\nu_{typ}$ and $\nu_{h}$ in $d = 2$, together with the exact solution in $d = 1$, raise the question of whether, in higher dimension $d \geq 3$, the typical exponent $\nu_{typ}$ still coincides with the correlation exponent of the pure quantum transition $\nu_{typ}(d \geq 3) = \nu_{pure}^Q(d \geq 3) = 1/2$, and whether the typical exponent $\nu_{h}$ still has the simple value $\nu_{h}(d \geq 3) \simeq 1$.

More generally, we hope that the idea of changing the order of decimations with respect to standard strong disorder RG in order to simplify the renormalized spatial structure will be useful for other kinds of models controlled by infinite-disorder scaling.
Appendix. A reminder of strong disorder RG rules on arbitrary lattices

For the random transverse field Ising model of equation (3), we recall that the standard strong disorder renormalization is formulated on arbitrary lattices as follows [5, 6]:

(0) Find the maximal value among all the transverse fields $h_i$ and all the ferromagnetic couplings $J_{jk}$:

$$\Omega = \max[h_i, J_{jk}].$$

(A.1)

(i) If $\Omega = h_i$, then the site $i$ is decimated and disappears, while all couples $(j, k)$ of neighbors of $i$ are now linked via the renormalized ferromagnetic coupling

$$J_{jk}^{\text{new}} = J_{jk} + \frac{J_{ji} J_{ik}}{h_i}.$$  

(A.2)

(ii) If $\Omega = J_{ij}$, then the site $j$ is merged with the site $i$. The new renormalized site $i$ has a reduced renormalized transverse field

$$h_i^{\text{new}} = \frac{h_i h_j}{J_{ij}}.$$  

(A.3)

and is connected to other sites via the renormalized couplings

$$J_{ik}^{\text{new}} = J_{ik} + J_{jk}.$$  

(A.4)

(iii) Return to (0).

These standard strong disorder RG rules should be compared with the modified procedure called ‘boundary strong disorder RG’ introduced in section 2.

References

[1] Igloi F and Monthus C, 2005 Phys. Rep. 412 277
[2] Ma S-K, Dasgupta C and Hu C-K, 1979 Phys. Rev. Lett. 43 1434
Dasgupta C and Ma S-K, 1980 Phys. Rev. B 22 1305
[3] Fisher D S, 1992 Phys. Rev. Lett. 69 534
Fisher D S, 1995 Phys. Rev. B 51 6411
Fisher D S, 1994 Phys. Rev. B 50 3799
Fisher D S, 1999 Physica A 263 222
[6] Motrunich O, Mau S-C, Huse D A and Fisher D S, 2000 Phys. Rev. B 61 1160
[7] Lin Y-C, Kawashima N, Igloi F and Rieger H, 2000 Prog. Theor. Phys. 138 479
[8] Karevski D, Lin Y C, Rieger H, Kawashima N and Igloi F, 2001 Eur. Phys. J. B 20 267
[9] Lin Y-C, Igloi F and Rieger H, 2007 Phys. Rev. Lett. 99 147202
[10] Yu R, Saleur H and Haas S, 2008 Phys. Rev. B 77 140402
[11] Kovacs I A and Igloi F, 2009 Phys. Rev. B 80 214416
Kovacs I A and Igloi F, 2010 Phys. Rev. B 82 054437
[13] Kovacs I A and Igloi F, 2011 Phys. Rev. B 83 174207
Kovacs I A and Igloi F, 2012 Europhys. Lett. 97 67009
[15] Kovacs I A and Igloi F, 2011 J. Phys.: Condens. Matter 23 404204
[16] Iyer D, Pekker D and Refael G, 2012 Phys. Rev. B 85 094202
[17] Lauman C R, Huse D A, Ludwig A W W, Refael G, Trebst S and Troyer M, 2012 Phys. Rev. B 85 224201
[18] Monthus C and Garel T, 2012 J. Stat. Mech. P09002
Monthus C and Garel T, 2012 arxiv:1205.4512
[20] Pich C, Young A P, Rieger H and Kawashima N, 1998 Phys. Rev. Lett. 81 5916
[21] Rieger H and Kawashima N, 1999 Eur. Phys. J. B 9 233
[22] Monthus C and Garel T, 2012 J. Phys. A: Math. Theor. 45 095002
[23] Sachdev S, 1999 Quantum Phase Transitions (Cambridge: Cambridge University Press)
[24] Monthus C and Garel T, 2008 J. Phys. A: Math. Theor. 41 375005

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