Effect of variable thermal conductivity and viscosity on Casson nanofluid flow with convective heating and velocity slip

J.A. Gbadeyan *, E.O. Titiloye, A.T. Adeosun

Department of Mathematics, University of Ilorin, Ilorin, Nigeria

A R T I C L E   I N F O

Keywords: Applied mathematics Computational mathematics Mechanical engineering Nanotechnology Geometry Slip Casson nanofluid Weighted residual method Gauss-Laguerre formulae Variable properties

A B S T R A C T

This work investigates the effects of combined variable viscosity and thermal conductivity, nonlinear radiation and non-Darcian porous medium on a boundary layer MHD Casson nanofluid flow over a vertical flat plate with convective heating and velocity slip boundary conditions. The governing transport nonlinear partial differential equations and the boundary conditions are non-dimensionalized. The resulting system of coupled partial differential equations is then reduced to a set of coupled nonlinear ordinary differential equations using similarity transformation. Galerkin weighted residual method (GWRM) is then employed to solve the resulting set of equations. Numerical results are obtained for dimensionless velocity, temperature and nanoparticle volume fraction (nanoparticle concentration). It is found that the velocity increases, while both temperature and nanoparticle volume fraction decrease with increased values of variable thermal conductivity and viscosity. Comparisons are carried out with published data in the literature thereby validating the numerical results. An excellent agreement is observed. Furthermore, this present study can find applications in the process involving nanofluid operations.

1. Introduction

In the past several decades, the study of non-Newtonian fluids model has gained a lot of attention due to its applications in industries and engineering such as petroleum industry, aerodynamic heating, paper production, coating and polymer processing, hot rolling and so forth. Some materials such as mud, blood, paint, polymers solutions exhibit non-Newtonian fluid characteristics. However, no single model in literature describes all the properties of non-Newtonian fluid due to the complexity of its physical nature. Casson fluid is one of the examples of non-Newtonian fluids. It is a type of non-Newtonian fluid that behaves like an elastic solid. Casson model constitutes a fluid model that exhibits shear thinning characteristics, yield stress and high shear viscosity [1].

Saidulu and Venkata [2] used numerical method (Keller box method) to study the effect of slip on MHD flow of a Casson fluid over an exponentially stretching sheet in the presence of thermal radiation, heat source/sink, and chemical reaction. It was found that the temperature and concentration profile increase when the Casson parameter increases, but the reverse was the case for the velocity profile. Unsteady MHD slip flow of a Casson fluid due to a stretching sheet with suction or blowing effect was analyzed by Mahdi [3]. It was observed that increasing the slip parameter increases the fluid flow, and the thermal boundary layer becomes thinner in case of suction or blowing. Nadeem et al. [4] used the Adomian decomposition method to obtain the solution for MHD boundary layer flow of a Casson fluid over an exponentially shrinking sheet. It is noticed that when the fluid parameter approaches infinity their problem reduced to the Newtonian case.

Nanoparticles are known to be made from various materials, such as metallic oxide (Al₂O₃, CuO), nitride ceramics (AlN, SiN), Carbide ceramics (SiC, TiC), metals (Cu, Ag, Au), semiconductors (TiO₂, SiC), etc. [5] and have significant effects when it is combined with base fluids for the enhancement of thermal conductivity in heat transfer. Nanofluids are very useful in some engineering (electronics cooling, vehicles cooling, and so on), biomedical applications (cancer therapy and saver surgery by cooling) and process industries (paper, chemical, textile and detergent manufacturing, as well as foods and drinks production) [5]. Ibukun et al. [6] used spectral relaxation method to study unsteady Casson nanofluid flow over a stretching sheet with thermal radiation subjected to convective and slip boundary condition. However, the radiation term used is a linearized Rosseland radiation term, and a porous medium was not considered as well. They found that an increase in fluid unsteadiness has a significant influence on fluid flow, temperature

* Corresponding author.
E-mail address: shiteq91@gmail.com (J.A. Gbadeyan).

https://doi.org/10.1016/j.heliyon.2019.e03076
Received 23 July 2019; Received in revised form 18 November 2019; Accepted 16 December 2019

2405-8440/© 2019 Published by Elsevier Ltd. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).
and concentration profile. Gireesha et al. [7] discussed MHD flow of nonlinear radiative heat transfer of a Casson nanofluid past a nonlinearly stretching sheet in the presence of chemical reaction. The effect of a porous medium is not put into consideration. It is discovered that the Casson parameter tends to control the fluid flow and the nonlinear thermal radiation enhances the thermal boundary layer thickness. Arshad et al. [8] used Laplace transform method to examine MHD flow of sodium Alginate-Based Casson nanofluid passing through a porous medium with Newtonian heating. Darcian porous medium, as well as Rosseland linear radiation terms, are considered, while viscous dissipation term is neglected. From their results, it was observed that both the velocity and temperature profiles of the fluid are decreasing with increasing nanoparticle volume fraction.

In the aforementioned studies, the fluid viscosity and thermal conductivity were assumed constant. However, the physical properties of fluids may change significantly when exposed to temperature. For the fluids, which are important in the theory of lubrication, the heat generated by the internal friction and the corresponding rise in temperature does affect the viscosity and thermal conductivity of the fluid [9]. According to Batchelor [10], Anyakoha [11] and Meyers et al. [12], the viscosity and thermal conductivity are most sensitive to temperature rises. The increase of temperature leads to a local increase in the transport phenomena by reducing the viscosity across the momentum boundary layer and also the heat transfer rate at the wall is also affected greatly [1]. Due to the above facts, the fluid viscosity and thermal conductivity can no longer be assumed to be constant. Afify and Bazid [13] adopted Dybbbs [14] and Chiam [15] models for temperature dependent viscosity and thermal conductivity respectively to study the effect of variable properties on the natural convective boundary layer flow of a nanofluid past a vertical plate. It was discovered that an increase in variable viscosity parameter leads to a decrease in Nusselt and Sherwood numbers whereas opposite results were obtained for the Skin-friction.

Jawali et al. [9] analyzed the combined effect of variable viscosity and thermal conductivity on free convection flow of a viscous fluid in a vertical channel. He used Attia’s [16] model for both temperature dependent viscosity and thermal conductivity. It was found that the fluid flow and heat transfer increase as the variable viscosity parameter increases, while an increase in variable thermal conductivity reduces both the heat transfer and the fluid flow. Bagai and Nishad [17] used numerical method (i.e. shooting method) to study the effect of temperature dependent viscosity on natural convective boundary layer flow over a horizontal plate embedded in a nanofluid saturated porous medium. The viscosity of the fluid is assumed to vary exponentially with temperature. The thermal conductivity was assumed constant and radiation term is neglected. It was observed that the heat and mass transfer rate increase as the viscosity parameter increases. Casson fluid flow with variable thermo-physical properties along exponentially stretching sheet with suction and exponentially decaying internal heat generation using the homotopy analysis method (HAM) was studied by Anisamsan et al. [18]. However, the radiation term used was linear and the effect of porosity was not put into consideration. Also, only Casson fluid, as opposed to Casson nanofluid, was studied in this work. They found that an increase in the variable plastic dynamic viscosity parameter of Casson fluid leads to an increase in velocity profile and a decrease in temperature profile throughout the boundary layer.

Heat carrier fluids (e.g. water, mineral oil, ethylene glycol, etc.) are very useful in industrial sectors such as power generation, chemical production, microelectronics, air conditions, etc. However, the performance of these fluids is limited, in themselves, due to their low thermal conductivities which obstruct their applications in heat exchangers. Thus there is a need to increase their thermal conductivity by, for example, introducing nanoparticles into them.

Motivated by the high demand, in these modern days, of heat carrier fluids with high thermal conductivity by the engineers and scientists for intensification and miniaturization, attention is focused, in this present study, on the influence of fluid properties on Casson nanofluid (as opposed to Casson fluid). Interestingly, it is noted from the above reviewed literature that the study under consideration on Casson nanofluid with variable properties, radiation, and a porous medium has not been carried out. Hence, the main goal of the present work is to address the lack of research in this direction. Therefore, the effects of a non-Darcian porous medium, nonlinear radiation as well as temperature dependent thermal conductivity and viscosity (which are, from a practical point of view, very important but ignored in the previous works) on the flow of Casson nanofluid are examined. The emerging governing equations of the problem are highly nonlinear partial differential equations which according to literature survey may be solved using finite element method, the Laplace transformation, perturbation technique, finite different approach, homotopy perturbation technique, Haar wavelets scheme method [19] and Runge-Kutta Fehlberg method.

However, in this study, the equations governing the flow are first simplified to a set of coupled systems of nonlinear ordinary differential equations. These are then solved using Galerkin weighted residual method (GWRM). This technique is a powerful method for the computation of solutions to nonlinear boundary value problems. It consists of the following three major steps: namely (i) The unknown dependent functions in the differential equations are firstly assumed as linear combinations of shape or trial functions with unknown coefficients (ii) These assumed solutions are then inserted into the governing equations yielding residuals or errors (iii) The errors are then forced to become as small as possible using some weight functions thereby obtaining the unknown coefficients. The main features which make this technique (GWRM) attractive are (a) its simplicity in handling boundary value problems involving semi-infinite domain, (b) its high accuracy, effectiveness and rapid convergence and (c) the fact that the associated domain within zero and infinity $(0, \infty)$ is minimized directly without division.

The rest of the article is organized as follows. The mathematical formulation of the problem is presented in the next section. Section 3 deals with the concept of Galerkin weighted residual method and its application in solving the present nonlinear boundary value problems. In section 4, the obtained numerical results are presented graphically and discussed. Concluding remarks are given in the last section.

2. Problem formulation

In Fig. 1 (i, stands for the momentum boundary layer, while ii, stands for both thermal and nanoparticle volume fraction boundary layers) below, the two-dimensional steady laminar natural convective flow of viscous electrically conducting incompressible Casson nanofluid over a vertical flat plate is considered. All the properties of the fluid are assumed to be constant except the density, viscosity and thermal conductivity of the fluid.

It is assumed that the surface of the plate is subjected to convective heating with temperature $T_f$. $\bar{z}$ is the distance along with the plate, while $\bar{y}$ is the distance perpendicular to the plate. A local magnetic field
The fluid temperature and nanoparticle volume fraction (concentration) are denoted by \( T \) and \( C \) respectively. It is also assumed that the induced magnetic field is neglected as a result of small magnetic Reynolds number. The nanoparticle volume fraction at the wall is taken as \( C_w \), while the temperature and nanoparticle volume fraction far from the wall are denoted by \( T_w \) and \( C_w \) respectively. The rheological equations of an isotropic and incompressible flow of a Casson fluid can be written as (Rao et al. [20], Anisamasa [1])

\[
\begin{align*}
\tau_{ij} &= 2(\mu_B + \frac{\rho_f}{\sqrt{2\pi}}) e_{ij} \quad \text{when } \eta > \eta_c, \\
\tau_{ij} &= 2(\mu_B + \frac{\rho_f}{\sqrt{2\pi}}) e_{ij} \quad \text{when } \eta < \eta_c, \\
P_f &= \frac{\mu_B \sqrt{2\pi}}{2\pi} \\
\end{align*}
\]

where \( \tau_{ij} \) is the component of stress tensor, \( P_f \) is the fluid yield stress, \( \mu_B \) is the plastic dynamic viscosity of the non-Newtonian fluid, \( \beta \) is Casson parameter, \( e_{ij} = \frac{1}{2}(\nabla u \cdot \nabla u) \) is the rate of strain tensor, \( \eta = \epsilon_j e_{ij} \) is the product of rate of strain tensor with itself, \( \eta_c \) is the critical value of the product of rate of strain tensor with itself. Some fluids require a gradually increasing shear stress to maintain a constant strain rate, such fluids are called Rheopetic. For Casson fluid flow \( \eta > \eta_c \). The dynamic viscosity is defined as

\[
\mu_f = \mu_B + \frac{P_f}{\sqrt{2\pi}}
\]

Substituting equation (2) into equation (3) we obtain the kinematic viscosity as

\[
\nu_f = \frac{\mu_B}{\beta}(1 + \frac{1}{\beta})
\]

The velocity of the fluid in \( \overrightarrow{x} \) and \( \overrightarrow{y} \) directions is denoted by \( \overrightarrow{u} \) and \( \overrightarrow{v} \) respectively. The buoyancy effect sets in as a result of temperature and concentration gradient between the plate surface and the fluid. Following the above assumptions, Boussinesq and boundary layer approximation, the equations governing the Casson nanofluid flow are written as ([6], [21] and [22])

\[
\begin{align*}
\frac{\partial \overrightarrow{u}}{\partial t} + \nabla \cdot \overrightarrow{u} &= 0, \\
\frac{\partial \overrightarrow{u}}{\partial t} + \nabla \cdot \overrightarrow{u} &= \frac{1}{\rho_f} \left[ \frac{\partial P_f}{\partial \overrightarrow{x}} - \nabla \cdot (\mu_f \nabla \overrightarrow{u}) + \frac{\sigma_0 B^2(T)\overrightarrow{u}}{\rho_f} \right. \\
&\left. + \frac{\rho_f}{\rho_f}(1 - C_w) \left( T - T_w \right) \frac{\partial C_w}{\partial \overrightarrow{x}} \right] - \frac{\mu_f(T)}{\beta \rho_f} \left[ 1 + \frac{1}{\beta} \frac{\partial \overrightarrow{u}}{\partial \overrightarrow{x}} \right] - \frac{1}{\beta} \frac{\partial \overrightarrow{u}}{\partial \overrightarrow{y}} \\
\frac{\partial \overrightarrow{v}}{\partial t} + \nabla \cdot \overrightarrow{v} &= D_p \frac{\partial^2 \overrightarrow{v}}{\partial \overrightarrow{y}^2} + \frac{D_T}{T_w} \left( \frac{\partial T}{\partial \overrightarrow{y}} \right)^2 \\
\end{align*}
\]

subjected to the following appropriate boundary conditions

\[
\begin{align*}
\overrightarrow{n} &\cdot \overrightarrow{u} = 0, \quad \frac{\partial \overrightarrow{u}}{\partial \overrightarrow{n}} = h_f(T(T - T_w)), \quad C = C_w, \quad \text{at } \overrightarrow{n} = 0, \\
\overrightarrow{n} &\cdot \overrightarrow{v} = 0 \quad T = T_w, \quad \overrightarrow{C} = \overrightarrow{C}_w, \quad \text{as } \overrightarrow{n} \rightarrow \infty.
\end{align*}
\]

The term representing joule heating which is supposed to be in the energy equation is assumed to be neglected since it is very small in slow motion natural convective flow considered in this problem. \( \beta_b \) is the volumetric thermal expansion coefficient, \( \tau = \frac{\beta_b}{(\gamma T)} \) is the ratio of nanoparticle heat capacity to the base fluid heat capacity, \( \rho_f \) is the density of the base fluid, \( \mu_f(T) \) is the temperature dependent dynamic viscosity of Casson fluid, \( k(T) \) is the temperature dependent thermal conductivity, \( \sigma = \sigma_0 \frac{\mu_B}{\sqrt{2\pi}} \) is the variable electric conductivity, \( \sigma_0 \) is the constant electric conductivity, \( \mathcal{R}_T \frac{\partial T}{\partial \overrightarrow{y}} \) is the local magnetic field, \( B_0 \) is the constant magnetic field, \( \eta_c \) is the density of nanoparticle, \( g \) is the acceleration due to gravity, \( k_p \) is the permeability of the porous medium, \( \pi_{\mu_B} = \frac{N_1 \mu_B}{T_w} \) is the slip velocity, \( h_f(T) \) is the heat transfer coefficient, \( b^* \) is the Forchheimer’s inertia coefficient, \( D_p \) is the Brownian diffusion coefficient, \( D_T \) is the thermophoresis diffusion coefficient. Radiative heat flux is considered by assuming that the fluid is gray, radiatively absorbing and emitting but non-scattering [21]. The radiative heat flux follows Rosseland approximation and is given as

\[
q_r = \frac{4a_1}{3k_1} \frac{\partial T^4}{\partial \overrightarrow{y}}
\]

where \( a_1 \) is the Stefan-Boltzmann constant, and \( k_1 \) is the Rosseland mean absorption coefficient.

The assumed variable plastic dynamic viscosity and thermal conductivity used for the non-Newtonian fluid are [9]

\[
\mu_f(T) = \mu_B e^{-a(T - T_w)} \quad \text{and} \quad k(T) = k^* e^{-b(T - T_w)}
\]

Thermal conductivity varies linearly with temperature in the range of 0 to 400 F [23]. Therefore, variable thermal conductivity is approximated as

\[
k(T) = k^* e^{-b(T - T_w)} \approx k^* (1 - b(T - T_w))
\]

where \( \mu_f \) is the constant value of the coefficient of viscosity far from the plate, \( k^* \) is the constant value of the coefficient of thermal conductivity far from the plate, \( a^* \) and \( b \) are the empirical constants. The constants \( a^* \) and \( b \) (0 < \( a^* \), \( b \) << 1) may be positive values for fluids and negative values for gases [24].

Using the following boundary layer variables

\[
\frac{x}{\overrightarrow{L}} = \frac{\overrightarrow{y}Ra_t}{L}, \quad u = \frac{u}{a^* Ra_t}, \quad v = \frac{v}{a^* Ra_t}, \quad \theta = \frac{T - T_w}{T_f - T_w}, \quad = \frac{\phi - \phi_w}{\phi_u - \phi_w}
\]

and the stream function \( \psi \) defined as

\[
\frac{\partial \psi}{\partial \overrightarrow{y}} = u \text{ and } v = -\frac{\partial \psi}{\partial \overrightarrow{x}}
\]

in equations (5)-(9), we obtained the following dimensionless equations:

\[
\begin{align*}
\frac{\partial \psi}{\partial \overrightarrow{x}} - \frac{\partial^2 \psi}{\partial \overrightarrow{y}^2} &= Pr_\omega \left[ 1 + \frac{1}{\beta} \frac{\partial \psi}{\partial \overrightarrow{x}} \right] \frac{\partial^2 \psi}{\partial \overrightarrow{y}^2} + \frac{\partial^2 \psi}{\partial \overrightarrow{y}^2} - \frac{\sigma_0 B^2(T)}{\rho_f} \frac{\partial \psi}{\partial \overrightarrow{x}} \\
&+ Pr_\omega (\theta - N \overrightarrow{r}) - \frac{Pr_p L^2}{k_p V \overrightarrow{Ra}} \left[ 1 + \frac{1}{k_p L} \frac{\partial \psi}{\partial \overrightarrow{x}} \right] \frac{\partial \psi}{\partial \overrightarrow{x}} \\
&+ 4 \frac{Pr_p L}{k_p V T} (1 - T) \frac{\partial \psi}{\partial \overrightarrow{x}} + \frac{4}{3} \frac{Pr_p L}{k_p V T} (1 - T)(1 - \epsilon_b) \frac{\partial \psi}{\partial \overrightarrow{x}} + \frac{4}{3} \frac{Pr_p L}{k_p V T} \left[ \frac{\nabla a^*}{\nabla} \frac{\partial^2 \psi}{\partial \overrightarrow{y}^2} \right] \\
&+ \frac{Le \overrightarrow{r}}{k_p V T} \frac{\partial \psi}{\partial \overrightarrow{x}} - \overrightarrow{Le} \frac{\partial \psi}{\partial \overrightarrow{x}} + \frac{\partial \psi}{\partial \overrightarrow{y}} + \frac{N_1 \overrightarrow{r}}{k_p V T} \frac{\partial^2 \psi}{\partial \overrightarrow{y}^2}
\end{align*}
\]

with the corresponding dimensionless boundary conditions

\[
\begin{align*}
\frac{\partial \psi}{\partial \overrightarrow{y}} &= N_1 \frac{\partial \psi}{\partial \overrightarrow{y}} \left[ 1 + \frac{1}{k_p} \frac{\partial \psi}{\partial \overrightarrow{x}} \right], \quad \psi = 0, \quad \text{at } \overrightarrow{y} = 0, \\
\overrightarrow{\psi} &= 0, \quad T = T_w, \quad \overrightarrow{C} = \overrightarrow{C}_w, \quad \text{as } \overrightarrow{y} \rightarrow \infty.
\end{align*}
\]
In equation (13)-(17), we have \( Ra = \frac{(1-C_w)\alpha_k(T_f - T_w) L^3}{\nu_f \gamma} \) being the Rayleigh number based on the characteristic length \( L \), \( \alpha^* = \frac{k^*}{\beta^*} \) is the thermal diffusivity number from the characteristic plate, \( Pr_w = \frac{\nu_f}{\alpha^*} \) is the ambient Prandtl number, \( N_t = \frac{D_T(T_f - T_w)}{T_{ref} \beta^*} \) is the thermophoresis parameter, \( N_b = \frac{(\rho_D c^*_P - C_w)}{T_{ref}} \) is the Brownian motion parameter, \( L_e = \frac{a^*}{\beta^*} \) is the Lewis number, \( N_r = \frac{(\alpha_x \rho_p c^*_P - C_w)}{T_{ref} (1-C_w)} \) is the buoyancy ratio parameter and \( T_r = \frac{T_f}{T_{ref}} \) is the excess wall temperature parameter, \( R = \frac{a^* k^*}{\beta^*} \) is the conduction radiation parameter, \( \gamma = a^*(T_f - T_{ref}) \) is the temperature dependent viscosity parameter and \( \epsilon = b(T_f - T_{ref}) \) is the temperature dependent thermal conductivity parameter.

The following similarity transformations [22]:

\[
\eta = \frac{y}{x^2}, \quad \psi = \frac{3}{2} f(\eta), \quad \phi = \phi(\eta), \quad D_1 = x \frac{1}{v} (D_1)_w,
\]

\[
N_l = \frac{1}{x^2} (N_l)_w, \quad k_p = \frac{1}{x} (k_p)_w, \quad h_j(x) = \frac{1}{x} (h_j)_w, \quad b^* = x^{-1} (b^*)_w
\]

were also introduced in equations (14)-(17) to obtain

\[
(1 + \frac{1}{\beta^*}) e^{-\epsilon \phi} f'''' - 4 e^{-\epsilon \phi} f''' \theta' + \frac{1}{4 Pr_w} (4 f'''' - 2 f'' - 4 M f'' - 4 F_s f''')
\]

\[
+ \theta - N_r \phi = \frac{1}{D_0} (1 + \frac{1}{4}) e^{-\epsilon \phi} f' = 0
\]

\[
(1 - \epsilon \phi) f'' - \epsilon \phi (\theta')' + \frac{4}{3 R} ((1 + (T_f - 1) \phi)' \theta')' + \frac{3}{4} f \theta' + N \phi \theta' + N \phi \theta' + E_c Pr_w (1 + \frac{1}{4}) e^{-\epsilon \phi} f'''' = 0
\]

\[
\phi'' + \frac{3}{2} L e f \phi' + \frac{N_l}{N_b} \theta'' = 0
\]

with boundary conditions

\[
f'(0) = \delta e^{-\epsilon \phi}(1 + \frac{1}{4}) f''(0), \quad f(0) = 0, \quad \theta'(0) = \frac{-B_l}{(1-e\theta(0))}(1 - \theta(0)), \quad \phi(0) = 1 \quad f''(\infty) \rightarrow 0, \quad \theta'(\infty) \rightarrow 0, \quad \phi(\infty) \rightarrow 0
\]

We denoted similarity independent variable by \( \eta \), dimensionless velocity, temperature and nanoparticle volume fraction functions by \( f(\eta) \), \( \theta(\eta) \) and \( \phi(\eta) \) respectively, the heat transfer coefficient constant factor by \( (R_f)_w \), the velocity slip constant factor by \( (N_l)_w \), the constant permeability of the porous media by \( (k_p)_w \) and the constant inertial Forchheimer coefficient by \( (b^*)_w \). Further, more, in equation (19)-(22), the prime symbol stands for ordinary differentiation with respect to \( \eta \),

\[
Ec = \frac{a^* g a^*_w}{L^2 C_w (1 - C_w)} \quad \text{is the local Eckert number,} \quad F_s = \frac{(\beta^*)_w L}{(b^*)_w} \quad \text{is the Forchheimer parameter,} \quad D_0 = \frac{k_p \alpha_k \rho_D c^*_P}{L^2 (1-C_w)} \quad \text{is Darcy number,} \quad \delta = \frac{(N_l)_w \nu_f}{\alpha^*_w L}
\]

\[\begin{align*}
\beta_l &= \frac{(N_l)_w L}{k^*_w \alpha^*_w} \quad \text{is Biot number,} \quad M = \frac{m R_i^2}{\gamma^*} \quad \text{is the Magnetic Field parameter,} \quad \gamma = a^*(T_f - T_{ref}) \quad \text{is the variable viscosity parameter and} \quad \epsilon = b(T_f - T_{ref}) \quad \text{is the variable thermal conductivity parameter.}
\end{align*}\]

2.1. Variable Prandtl number

It is remarked that Prandtl number is a function of specific heat, viscosity, and thermal conductivity. Since the thermal conductivity and viscosity vary across the boundary layer, the Prandtl number varies as well. According to Pantokratoras [25], Rahman et al. [26] and Mohammad et al. [27], the assumption of constant Prandtl number inside the boundary layer when the viscosity and thermal conductivity are temperature dependent leads to an unrealistic result. Prandtl number is therefore represented in this present work as

\[
Pr = \frac{\nu_f}{\alpha^*} \epsilon^{-\nu_f} (1 - \epsilon \theta(\eta)) - Pr_w \epsilon^{-\nu_f} (1 - \epsilon \theta(\eta))
\]

Using equation (23) in equations (19) and (20), non-dimensionless velocity and temperature equations with variable Prandtl number (Pr) becomes, respectively

\[
(1 + \frac{1}{\beta^*}) e^{-\epsilon \phi} f'''' - 4 e^{-\epsilon \phi} f''' \theta' + \frac{1}{4 Pr(1 - e\theta)} (3 f'''' - 2 f'' - 4 F_s f''')
\]

\[
+ \theta - N_r \phi - \frac{1}{D_0} (1 + \frac{1}{4}) e^{-\epsilon \phi} f' = 0
\]

\[
\text{and}
\]

\[
(1 - \epsilon \phi) f'' - \epsilon \phi (\theta')' + \frac{4}{3 R} ((1 + (T_f - 1) \phi)' \theta')' + \frac{3}{4} f \theta' + N \phi \theta' + N \phi \theta' + E_c Pr(1 + \frac{1}{4}) e^{-\epsilon \phi} f'''' = 0
\]

It is noted in equation (23) that when \( \gamma \) and \( \epsilon \) are set to zero, the ambient Prandtl number \( (Pr_w) \) equals to variable Prandtl number \( (Pr) \), therefore equations (24) and (25) reduce to equations (19) and (20). Also, when \( \eta \rightarrow \infty, Pr_w = Pr \) irrespective of the values of \( \gamma \) and \( \epsilon \). 2.2. Nusselt and Sherwood numbers

The quantities which are of the engineering interest in this study are Nusselt number \( (Nu) \) and Sherwood number \( (Sh) \) and these are defined as [21]

\[
Nu_w = \frac{-\tau}{T_f - T_w} \left[ 1 + \frac{16 Bo}{3 k_i (T_f - T_w)} \left( \frac{\partial \theta}{\partial \tau} \right) \right]_{\tau = 0} \quad \text{and}
\]

\[
Sh_w = \frac{-\tau}{C_w - C_w} \frac{\partial \theta}{\partial \tau} \left| \right. _{\tau = 0}
\]

Using equations (12), (13) and (18) in equation (26), we get

\[
Nu_w = Nu_{\eta r} Ra_w^{-\frac{1}{2}} = -[1 + \frac{4}{3 R(1 + \epsilon \theta)} ((1 + (T_r - 1) \theta(0))^{\frac{1}{2}} - 0)
\]

\[
\text{and}
\]

\[
Sh_w = Sh_{\eta r} Ra_w^{-\frac{1}{2}} = -\psi'(0)
\]

where \( Ra_w = \frac{(1-C_w)\alpha_k (T_f - T_w) L^3}{\nu_f \gamma} \) is the local Rayleigh number.

3. Galerkin weighted residual method

Galerkin weighted residual method (GWRM) is employed to solve the above governing differential equations (21), (24) and (25) with the associated boundary conditions (22).

The techniques of GWRM is to seek an approximate solution to a differential equation of the form

\[
L(\phi(x)) + f(x) = 0 \quad \text{in} \ D,
\]

where \( \phi(x) \) is the unknown dependent variable, \( f(x) \) is the independent function in domain \( D \) and \( L \) is the differential operator. An approximate solution

\[
\phi(x) = \phi_0 + \sum_{k=1}^{n} \phi_k(x)
\]

is assumed in such a way that it satisfies the given boundary conditions. Substituting eqn. (30) into eqn. (29) resulted in a residual function \( R(x) \), \( R(x) \) is minimized as small as possible in the domain \( D \) by setting the integral of the product of the weight functions \( \phi_k(x) \) and residue \( R(x) \) over the entire domain \( D \) to zero for \( k \geq 0 \). That is

\[
\frac{1}{D} \int R(x) \phi_k(x) dx = 0, \quad k = 0, 1, \ldots, n.
\]
Since the boundary condition ranges from zero to infinity, the Gauss-Laguerre formula (see section 3.1) is used to integrate each of the equations in (31) thereby obtaining a system of algebraic equations. The values of \( a_k \) are obtained by solving the resulting algebraic equations.

### 3.1. The Gauss-Laguerre formula

Gauss-Laguerre formula is applied as follows [28]:

\[
\int_0^\infty e^{-x} f(x) \, dx = \sum_{k=1}^{\infty} A_k f(x_k)
\]

(32)

where the coefficients \( A_k \) are defined as [29]

\[
A_k = \frac{1}{L_n'(x_k)} \int_0^\infty L_n(x) e^{-x} \, dx = \frac{(n!)^2}{x_k (L_n'(x_k))^2},
\]

(33)

and \( x_k \) are the zeroes of the \( n \)th Laguerre polynomial

\[
L_n = e^{x} \frac{d^n}{dx^n}(e^{-x}x^n).
\]

(34)

For \( n = 10 \), for example, Table 1 shows the values of \( x_k \) and corresponding values of \( A_k \).

### 3.2. Application of the GWRM to the present problem

By using Galerkin weighted residual method, the following trial solutions are assumed for \( f(\eta) \), \( \theta(\eta) \) and \( \phi(\eta) \) as follows [28]

\[
f(\eta) = \sum_{j=0}^{N} a_j e^{-\eta^2/2}, \quad \theta(\eta) = \sum_{j=0}^{N} b_j e^{-\eta^2/2} \quad \text{and} \quad \phi(\eta) = \sum_{j=0}^{N} c_j e^{-\eta^2/2}
\]

(35)

Choosing \( N = 15 \), equation (35) is substituted into the boundary conditions in equation (22) to obtain

\[
\exp \left( -\gamma (b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 + b_8 + b_9 + b_{10} + b_{11} + b_{12} + b_{13} + b_{14} + b_{15}) \right) = 0
\]

(36)

\[
a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} + a_{15} = 0
\]

(37)

\[
\left( - b_1 - b_2 - b_3 - b_4 - b_5 - b_6 - b_7 - b_8 - b_9 - b_{10} - b_{11} - b_{12} - b_{13} - b_{14} - b_{15} + 1 \right) \left( 1 - (b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 + b_8 + b_9 + b_{10} + b_{11} + b_{12} + b_{13} + b_{14} + b_{15}) \right) e^{-1 - b_3 - b_5} = 0
\]

(38)

The boundary conditions at infinity in equation (22) are satisfied automatically. Substituting equation (35) into equations (21), (24) and (25) resulted in a residual functions \( R_f(a_i, b_i, c_i, \eta) \), \( R_\theta(a_i, b_i, c_i, \eta) \) and \( R_\phi(a_i, b_i, c_i, \eta) \) for \( i = 0, 1, \ldots, 15 \), \( k = 1, 2, \ldots, 15 \). We first minimize the residual errors by setting the integral of the product of residual func-
Table 2 Comparison of Nusselt number values of the present work with those of Uddin et al. [22] for \( M = F s = c = \gamma = E c = 0, \) \( L e = B i = 10, \) \( D a = R = \beta \rightarrow \infty, \) \( N t = 0.1. \)

| \( N b \) | \( N r \) | \( Pr = 1 \) | Present work | \( Pr = 5 \) | Present work | \( Pr = 10 \) | Present work |
|--------|--------|------------|--------------|------------|--------------|------------|-------------|
| 0.1    | 0      | 0.34257    | 0.34257      | 0.3895     | 0.38959      | 0.3953     | 0.395348    |
| 0.1    | 0.2    | 0.33659    | 0.336593     | 0.37734    | 0.377351     | 0.38856    | 0.388615    |
| 0.1    | 0.4    | 0.33012    | 0.330127     | 0.37024    | 0.370246     | 0.38133    | 0.381387    |
| 0.3    | 0      | 0.2960     | 0.295999     | 0.33288    | 0.332884     | 0.34301    | 0.343050    |
| 0.3    | 0.2    | 0.29178    | 0.291778     | 0.32821    | 0.328211     | 0.33826    | 0.338297    |
| 0.3    | 0.4    | 0.28724    | 0.287244     | 0.32322    | 0.323225     | 0.33319    | 0.333234    |

![Graphs](https://via.placeholder.com/150)

Fig. 6. Effects of \( \epsilon, \gamma \) and \( \delta \) on (a) dimensionless velocity (b) dimensionless temperature (c) dimensionless nanoparticle concentration.

4. Results and discussion

This section deals with the analysis of the effects of various parameters on the fluid flow. The values ([1], [21] and [30]) of the parameters \( (L e = 1, N r = 0.1, N b = 0.1, N T = 0.1, P r = 6.8, \) \( \delta = 0.1, \) \( D a = 10, \) \( F s = 0.1, R = 10, \) \( Tr = 2, \) \( \beta = 0.5, M = 0.5, E c = 0.01, \) \( e = 0.3, \) \( \gamma = 1 \) and \( B i = 0.5 \) ) are kept unchanged throughout the study unless otherwise stated. The graph of the residual functions \( R_{\eta}(\eta), R_{\phi}(\phi) \) and \( R_{\psi}(\psi) \) are shown in Fig. 2. It is seen that the residuals are minimized in the domain \( (0 \) to \( \infty) \). To ascertain the accuracy of our method, the present results of rate of heat transfer \( (-\theta'(0)) \) are compared with those of Uddin et al. [22] in Table 2. An excellent agreement is observed. Also, a simulation is provided to see the efficiency of the used method. This is carried out by comparing the graphical results of dimensionless velocity, temperature and nanoparticle volume fraction (see Figs. 3, 4 and 5) obtained using Galerkin weighted residual method (GWRM) and spectral collocation method (SCM). An excellent agreement is observed in each of the cases.

Figs. 6a-6c show the effects of velocity slip, variable thermal conductivity and viscosity parameters on the dimensionless velocity, temperature and nanoparticle volume fraction (nanoparticle concentration). It is found that the fluid velocity is an increasing function of velocity slip parameter \( (\delta) \) (see Fig. 6a). This is perhaps because when velocity slip occurs, the fluid particles move away from the plate and this leads to the reduction of shear force which in turn accelerates the nanofluid velocity. On the other hand, the corresponding temperature and nanoparticle volume fraction are reducing functions of velocity slip parameter \( (\delta) \) (see Figs. 6b and 6c). It is also observed that an increase in fluid variable physical properties \( (\epsilon \) and \( \gamma) \) leads to an increase in the dimensionless fluid flow (see Fig. 6a), while reduces both dimensionless temperature and nanoparticle volume fraction (see Figs. 6b and 6c). This may happen as a result of an increase in \( \epsilon \) and \( \gamma \) which tends to an increase in temperature difference \( (T_{w} - T_{\infty}) \) thereby weakening Casson fluid bond and reducing the strength of Casson plastic dynamic viscosity.

Figs. 7a-7c depict the effects of Casson and buoyancy ratio parameters on dimensionless velocity, temperature, and nanoparticle volume fraction. It is observed that an increase in Casson parameter \( (\beta) \) accelerates the fluid flow (see Fig. 7a) and reduces both fluid temperature and nanoparticle volume fraction (see Figs. 7b and 7c). In the true sense, an increase in Casson parameter \( (\beta) \) resists the fluid flow because it decreases the yield stress of Casson fluid and increases plastic dynamic viscosity. However, this decrease in velocity is overcome by the high amount of temperature being injected into the fluid due to the presence of temperature dependent viscosity and thermal conductivity.
Fig. 7. Effects of $\beta$ and $N_r$ on (a) dimensionless velocity (b) dimensionless temperature (c) dimensionless nanoparticle concentration.

which in turn accelerate the fluid flow. It is also discovered in Fig. 7a that an increase in buoyancy ratio parameter ($N_r$) leads to a decrease in the velocity of the nanofluid. The buoyancy ratio parameter ($N_r$) represents the relationship between species buoyancy force and thermal buoyancy force. When $N_r = 1$, the buoyancy forces contribute equally. When $N_r > 1$, species buoyancy dominates and vice versa for $N_r < 1$. Concentration buoyancy disappears when $N_r = 0$ and this leads to an increase in Casson nanofluid velocity. This shows that as the buoyancy ratio parameter becomes larger, the dimensionless velocity reduces. Furthermore, increasing buoyancy ratio parameter ($N_r$) enhances both dimensionless temperature and nanoparticle volume fraction in Figs. 7b and 7c respectively.

The effects of conduction-radiation parameter ($R$) and Eckert number ($E_c$) on dimensionless velocity and temperature are shown in Figs. 8a and 8b. An increase in the value of ($R$) is found to increase the velocity profile (see Fig. 8a). Also, in Fig. 8a, it was observed that the Casson nanofluid velocity increases with an increase in Eckert number ($E_c$). The dimensionless temperature decreases with an increase in radiation parameter ($R$) (see Fig. 8b). It is noted in eqn. (20) that the parameter ($R$), denoting the contribution of thermal conduction heat
transfer tothermal radiation heat transfer \( (R = \frac{k^2 T^7}{4\epsilon_0 T^4}) \), is in the numerator. Therefore, an increase in the value of \((R)\) contributes to a decrease in a radiative mode of heat transfer which in turn reducing the fluid temperature near the wall. It is also discovered that the dimensionless temperature increases with an increase in Eckert number \((E_c)\) (see Fig. 8b). The reason for this is due to the work done by the fluid molecules in converting kinetic energy to heat energy.

Fig. 9 shows the effects of Darcy number \((D_o)\) and Forchheimer parameter \((F_s)\) on dimensionless velocity. It is observed that the fluid velocity increases with an increase in Darcy number \((D_o)\). The Darcian drag force in eqn. \((19)\) is inversely proportional to Darcy number \((D_o)\). This implies that the Darcian drag force reduces with an increase in Darcy number. Hence, this enhances fluid flow permeability and leads to an increase in Casson nanofluid velocity. It is also noticed in Fig. 9 that the dimensionless velocity decelerates with an increase in Forchheimer parameter \((F_s)\). This is due to the presence of an initial effect that is dragging the fluid backward. The effect of magnetic parameter \((M)\) on dimensionless velocity is shown in Fig. 10. An increase in the value of magnetic parameter \((M)\) leads to a reduction of dimensionless velocity. The reason for this is the presence of Lorentz force produced by a magnetic field placed in a transverse direction to the electrically conducting Casson nanofluid.

Figs. 11a and 11b depict the effects of Brownian motion and thermophoretic parameters on dimensionless temperature and nanoparticle volume fraction. It is found in Fig. 11a that the dimensionless temperature increases with an increase in Brownian motion parameter \((N_b)\), while the opposite trend is observed for the case of dimensionless nanoparticle volume fraction. This is because thermal energy is generated as a result of an increase in nanoparticle fluid collision which later boosts the Casson nanofluid temperature. Also, the nanoparticles tend to move away from the plate surface and lead to a decrease in nanoparticles volume fraction (see Fig. 11b). An increase in thermophoretic parameter \((N_t)\) increases the thermophoretic force (the force that the lower temperature nanoparticles exert on higher temperature nanoparticles to move it away from the surface of the sheet) which moves nanoparticles from the region of higher temperature to the region of lower temperature and this results to an increase in nanofluid temperature (see Fig. 11a) and corresponding nanoparticle volume fraction (see Fig. 11b).

The effect of convective heat transfer parameter \((B_i)\) on dimensionless velocity and temperature are shown in Figs. 12a and 12b respectively. It is found in Fig. 12a that an increase in convective heat transfer parameter \((B_i)\) increases dimensionless velocity as it is seen in the work of Uddin et al. [22]. This is because convective heat transfer parameter \((B_i = \frac{h/p C_p T}{\frac{7}{4} p V^2})\) in eqn. \((22)\) is directly proportional to the heat transfer coefficient \((h)\) and inversely proportional to the fourth root of the Rayleigh number \((R_o)\). An Increase in convective heat transfer parameter \((B_i)\) results to thermal convection enhancement at the wall and reduces buoyancy forces which in turn accelerates the fluid flow (see Fig. 12a) and the corresponding temperature (see Fig. 12b).

The effects of temperature ratio, Brownian motion, convective heat transfer, thermophoresis, and Casson parameters on the Nusselt number \((N_u)\) are shown in Figs. 13 and 14. It is noted that an increase in convective heat transfer and Casson parameters lead to an increase in Nusselt number, while an increase in Brownian motion, thermophoresis, buoyancy ratio, and temperature ratio parameters reduce it (see Figs. 13 and 14). Also, Sherwood number \((S_h)\) increases with an increase in Lewis number and convective heat transfer, Brownian motion,
fraction profiles for constant or variable fluid physical properties (thermal conductivity ($\kappa$) and viscosity ($\mu$)). It is also found that the velocity profile is increased, while temperature and nanoparticle profiles are decreased as the fluid properties increase.

- As the Casson parameter ($\beta$) increases, both the temperature and nanoparticle volume fraction are reduced, while the velocity profile is increased in the presence or absence of buoyancy ratio. Buoyancy ratio parameter ($N_r$) is found to enhance temperature and nanoparticle volume fraction profiles, while it impedes velocity profile.
- With increasing Eckert number ($Ec$), both velocity and temperature profiles are increased for a fixed value of conduction-radiation parameter ($R$). The velocity profile is found to increase with an increase in conduction-radiation parameter ($R$), while the temperature profile is decreased as increasing conduction-radiation parameter ($R$).
- An increase in Darcy number ($Da$) increases the velocity profile for a fixed value of the Forchheimer parameter ($Fs$). The velocity profile is found to be a decreasing function of the Forchheimer parameter ($Fs$).
- The velocity profile is reduced as the magnetic parameter ($M$) increases. Increasing the convective heating parameter ($Bi$) leads to an increase in both the velocity and temperature profiles.
- An increase in the thermophoresis parameter ($Nt$) enhances both temperature and nanoparticle volume fraction profiles for a fixed value of the Brownian motion parameter ($N_b$). An increase in the Brownian motion parameter increases the temperature profile and decreases nanoparticle volume fraction.
- Nusselt number profile is increased with increasing convective heating parameter ($Bi$) and Casson parameter ($\beta$), while it is reduced with increasing the Brownian motion parameter ($N_b$), thermophoresis parameter ($N_t$), buoyancy ratio parameter ($N_r$), and temperature ratio parameter ($Tr$).
- Sherwood number profile is found to be an increasing function of Lewis number ($Le$), convective heating parameter ($Bi$), Brownian motion parameter ($N_b$), and Casson parameter ($\beta$), while it is a decreasing function of thermophoresis parameter ($N_t$) and buoyancy ratio parameter ($N_r$).

Declarations

Author contribution statement

J.A. Gbadeyan: Conceived and designed the experiments; Analyzed and interpreted the data; Wrote the paper.
E.O. Titiloye: Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data.
A.T. Adeosun: Performed the experiments; Analyzed and interpreted the data; Wrote the paper.

Funding statement

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Competing interest statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

References

[1] I.L. Animasaun, Effect of thermophoresis, variable viscosity and thermal conductivity on free convective heat and mass transfer of non-Darcian MHD dissipative Casson fluid flow with suction and mb order chemical reaction, J. Niger. Math. Soc. 34 (2015) 11–31.
[2] N. Saidulu, A.I. Ventakata, Slip effect on MHD flow of Casson fluid over an exponentially stretching sheet in the presence of thermal radiation, heat source/sink and chemical reaction, Eur. J. Adv. Eng. Tech. 3 (2016) 47-55.

[3] A. Mahdy, Unsteady slip flow of a non-Newtonian Casson fluid due to stretching with suction or blowing effect, J. Appl. Fluid Mech. 9 (2016) 785–793.

[4] S. Nadeem, U.I. Rizwan, S.A. Noreen, Z.H. Khan, MHD three dimensional Casson fluid flow past a porous linearly stretching sheet, Alex. Eng. J. 52 (2013) 577–582.

[5] B.K. Mahatha, R. Nandkeslyar, G. Nagaju, M. Das, MHD stagnation point flow of nanofluid with velocity slip, non-linear radiation and Newtonian heating, in: International Conference on Computational Heat and Mass Transfer, vol. 127, 2015, pp. 1010–1017.

[6] S.O. Ibufun, M. Sahiyasachi, S. Precious, Unsteady Casson nanofluid flow over a stretching sheet with thermal radiation, convective and slip boundary conditions, Alex. Eng. J. 55 (2016) 1025–1035.

[7] B.J. Gireesha, M.R. Krishnamurthy, B.C. Prasannakumara, S.R. Rama, MHD flow of nonlinear radiative heat transfer of a Casson nanofluid past a nonlinearly stretching sheet in the presence of chemical reaction, Nanosci. Nanotechnol. Int. J. 9 (2018) 207–219.

[8] K. Arshad, K. Dolat, K. Iyas, A. Farhad, K. Faizan, I. Muhammad, MHD flow of sodium alginate-based Casson type nanofluid passing through a porous medium with Newtonian heating, J. Sci. Rep. 8 (2018).

[9] C.U. Jiwali, A.J. Chamkha, Combined effect of variable viscosity and thermal conductivity on free convection flow of a viscous fluid in a vertical channel, Int. J. Numer. Methods Heat Fluid Flow 26 (2015) 18–39.

[10] G.K. Batchelor, An Introduction to Fluid Mechanics, Cambridge University Press, London, 1987.

[11] M.W. Anyaka, New School Physics, 3rd edition, Africana First Publisher PLC, 2010, pp. 36–51.

[12] T.G. Meyers, J.P.F. Champin, M.S. Tshela, The flow of variable viscosity fluid between parallel plates with shear heating, Appl. Math. Model. 30 (2006) 799–815.

[13] A.A. Afify, M.A.A. Bazid, Effect of variable fluid properties on the natural convection boundary layer flow of nanofluid past a vertical plate, J. Comput. Theor. Nanosci. 11 (2014) 210–218.

[14] A. Dybb, J.X. Ling, Force convection over a flat plate submerged in a porous medium: variable viscosity case, in: SME, Paper 87-WA-HT-23, ASME Winter Annual Meeting, Boston, Massachusetts, 1987.

[15] T.C. Chiam, Heat transfer in a fluid with variable thermal conductivity of a linearly stretching sheet, Acta Mech. 129 (1998) 63–72.

[16] H.A. Attia, Unsteady hydromagnetic channel flow of dusty fluid with temperature dependent viscosity and thermal conductivity, J. Heat Mass Transf. 42 (2006) 779–787.

[17] S. Bagai, C. Nishad, Effect of temperature dependent viscosity on natural convective boundary layer flow over a horizontal plate embedded in a nanofluid saturated porous medium, in: 5th International Conference on Porous Media and Their Applications in Science, Engineering and Industry, 2014, http://dc.engageintl.org/porousmediaV18.

[18] I.L. Animasaun, E.A. Adebile, A.I. Fagbade, Casson fluid flow with variable thermophysical property along exponentially stretching sheet with suction and exponentially decaying internal heat generation using the homotopy analysis method, J. Niger. Math. Soc. 35 (2016) 1–17.

[19] M.R. Ali, D. Baleanu, Haar wavelets scheme for solving the unsteady gas flow in four-dimensional, Therm. Sci. 13 (2019) 292–301.

[20] A.S. Rao, V.R. Prasad, N.B. Reddy, O.A. Beg, Heat transfer in Casson rheological fluid from semi-infinite vertical plate with partial slip, in: Heat Trans Asian Res., Wiley Periodicals, Inc., 2013, pp. 211–215.

[21] M.J. Uddin, O.A. Beg, A.I. Ismail, Radiative convective nanofluid flow past a stretching/shrinking sheet with slip effect, J. Thermophy. Heat Transf. (2015).

[22] M.J. Uddin, W.A. Khan, A.I. Ismail, MHD free convective boundary layer flow of a nanofluid past a flat vertical plate with Newtonian heating boundary condition, PLoS ONE 7 (2012) e49499.

[23] T.A. Savvas, N.C. Marktos, C.D. Papapnyrides, On the flow of non-Newtonian polymer solution, Appl. Math. Model. 18 (1994) 14–22.

[24] S. Jiangli, S.O. Adesanya, H.A. Ogunseye, R.S. Lebelo, Couple stress fluid flow with variable properties: a second law analysis, Math. Methods Appl. Sci. 42 (2019) 85–98.

[25] A. Pantokratoras, Further results on the variable viscosity on flow and heat transfer to a continuous moving flat plate, Int. J. Eng. Sci. 42 (2004) 1891–1896.

[26] M.M. Rahman, M.A. Rahman, M.A. Sama, M.S. Alam, Heat transfer in micropolar fluid along a non-linear stretching sheet with temperature dependent viscosity and variable surface temperature, Int. J. Thermophys. 30 (2009) 1649–1670.

[27] M.R. Mohammad, A. Aziz, A.A. Mohamed, Heat transfer in micropolar fluid along an inclined permeable plate with variable fluid properties, Int. J. Therm. Sci. 49 (2010) 993–1002.

[28] A.O. Razaz, Y.K.S. Aregbesola, Weighted residual method in a semi infinite domain using un-partitioned method, Int. J. Appl. Math. 25 (2012) 25–31.

[29] Francis Scheid, Numerical Analysis, Schaum’s Outline Series, McGraw-Hill Book Company, New York, 1964, pp. 135–139.

[30] P. Duali, C. Sewelli, MHD Non-Darcy mixed convection stagnation-point flow of a micropolar fluid towards a stretching sheet with radiation, J. Chem. Eng. Commun. 119 (2012) 1169–1193.