Energy Conditions in $f(R, T, R_{\mu\nu}T^{\mu\nu})$ Gravity

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Abstract

We discuss the validity of the energy conditions in a newly modified theory named as $f(R, T, R_{\mu\nu}T^{\mu\nu})$ gravity, where $R$ and $T$ represent the scalar curvature and trace of the energy-momentum tensor. The corresponding energy conditions are derived which appear to be more general and can reduce to the familiar forms of these conditions in general relativity, $f(R)$ and $f(R, T)$ theories. The general inequalities are presented in terms of recent values of Hubble, deceleration, jerk and snap parameters. In particular, we use two specific models recently developed in literature to study concrete application of these conditions as well as Dolgov-Kawasaki instability. Finally, we explore $f(R, T)$ gravity as a specific case to this modified theory for exponential and power law models.

Keywords: $f(R, T, R_{\mu\nu}T^{\mu\nu})$ gravity; Raychaudhuri equation; Energy conditions.

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1 Introduction

In current scenario, dark energy (DE) is referred as an active agent which tends to accelerate the expansion in cosmos. The expanding paradigm of
the universe has been affirmed from various observational measurements [1]. Modified theories have received much attention to count with the issue of cosmic acceleration. In these theories, modified gravity models have been formulated to recognize the origin of DE as modification to the Einstein-Hilbert action. One of the fascinating class of models is named as $f(R)$ gravity where the generic nonlinear function $f$ succeeds the Ricci scalar (for review see [2, 3]). In this theory, one can reproduce the cosmological constant scenario, i.e., the classic $\Lambda$ cold dark matter ($\Lambda$CDM) model by choosing the $f(R)$ Lagrangian as $f = R - 2\Lambda$. The $\Lambda$CDM expanding paradigm has been tested and confirmed from the recent results [4]. One interesting fact about this theory is that it develops equivalence with the Brans-Dicke (BD) [5] theory for a specific BD parameter. The BD theory involves nonminimal coupling between geometry and scalar field which has further been formulated in $f(R)$ theory [6, 7].

Bertolami et al. [7] put a new twist on $f(R)$ gravity by considering the Lagrangian as a function of scalar curvature explicitly coupled with matter Lagrangian density. Bertolami and Paramos [8] developed the correspondence between this modified theory and scalar-tensor theory in which nonminimal curvature matter coupling would yield two scalar fields. The interaction between the matter components and curvature terms results in non conservation of matter energy-momentum tensor [9] which may describe the cosmic acceleration [10]. Later, Harko [11] extended this theory by inserting a general function of matter Lagrangian. Nesseris [12] studied matter density perturbations to constrain this theory from growth factor as well as weak lensing observations. Wu [13] established the laws of thermodynamics in this modified theory and discussed some forms of curvature components. Harko and Lobo [14] suggested a more generalized form of $f(R)$ theory by taking Lagrangian as a generic function of $R$ and matter Lagrangian $L_m$.

In modified $f(R)$ theories of the type involving nonminimal coupling with matter Lagrangian suffer issue related to the choice of matter Lagrangian density. If $L_m = p$ is considered then extra force would be vanished out so that natural conservation of matter exist in such case [15]. One can still get the effective nonminimal coupling if $L_m = -\rho$ [16]. Another way of modifying the Einstein Lagrangian is to consider the function of trace of the energy-momentum tensor $T$ [17] such that $\Lambda$CDM model can be considered of the form $R + 2\Lambda(T)$. Harko et al. [18] implemented this idea to generate a new Lagrangian $f(R, T)$ where the matter geometry coupled system is introduced with arbitrary function of $R$ and $T$. The corresponding effective
matter geometry coupling favors the non-geodesic motion of test particles leading to extra force as suggested in other modified theories [7, 11, 15].

This theory has drawn significant attention and some cosmological features have been studied comprehensively. We have investigated the validity of first and second laws of thermodynamics in $f(R, T)$ gravity. It is shown that equilibrium picture of thermodynamics may not be achieved due to matter geometry interaction [19]. The reconstruction of $f(R, T)$ Lagrangian is executed under various considerations likewise, considering an auxiliary scalar field [20], family of holographic DE models in the background of FRW universe [21] and anisotropic solutions [22]. Alvarenga [23] discussed scalar matter perturbations for a particular model which assures the standard continuity equation and obtained matter density perturbed equations.

It is shown that for $f(R, T)$ Lagrangian unlikely consequences are obtained from quasistatic approximation as compared to those derived in agreement with the $ΛCDM$ model. Jamil et al. [24] explored the reconstruction of $f(R, T)$ theory corresponding to cosmological solutions like $ΛCDM$, phantom as well as non-phantom matter fluids and Einstein static universe. However, they used the standard continuity equation without any additional constraint on $f(R, T)$ gravity [23]. We have used a significant approach for cosmological reconstruction in terms of e-folding reproducing different cosmological eras and the stability of $f(R, T)$ models is also analyzed [25].

Recently, a more complicated modified theory is developed [26, 27] which involves the nonminimal coupling through contraction of the Ricci and energy-momentum tensors referred to $f(R, T, R_{\mu\nu}T^{\mu\nu})$ gravity whose action is the extended Lagrangian of $f(R, T)$ gravity. It would be interesting to explore different cosmic features in this theory. The classical energy conditions of general relativity (GR) are profound to the Hawking-Penrose singularity theorems and classical black hole laws of thermodynamics [28]. These conditions have been used to address several important issues in GR and cosmology [29].

Energy conditions have been investigated in modified theories with different considerations such as $f(R)$ gravity [30], $f(R)$ gravity with nonminimal coupling to matter [31], $f(R, L_m)$ gravity [32], $f(T)$ gravity [33], scalar-tensor theory [34], modified Gauss-Bonnet gravity [35] and $f(R, T)$ gravity [36]. We have also discussed the energy conditions in $f(R, T)$ gravity and investigated the stability of power law solutions for particular class of models [36].

In this work, we are interested to develop the energy conditions bounds in $f(R, T, R_{\mu\nu}T^{\mu\nu})$ gravity and analyze some specific models. The energy conditions permit to set bounds for attractiveness property of gravity as
well as energy density being positive. The paper has the following format. In the next section, the fundamental formulation of the field equations is presented. Section 3 comprises a brief review of energy conditions in GR and also the respective inequalities in this modified theory. In section 4, we consider some specific forms of \( f(R, T, R_{\mu\nu}T^{\mu\nu}) \) gravity and illustrate the energy conditions bounds as well as analyze the Dolgov-Kawasaki instability. Section 5 concludes our findings.

2 \( f(R, T, R_{\mu\nu}T^{\mu\nu}) \) Gravity

The \( f(R, T, R_{\mu\nu}T^{\mu\nu}) \) gravity is an interesting candidate among the modified theories which are based on nonminimal coupling between matter and geometry. The action of this modified theory is of the form \[ A = \frac{1}{2\kappa^2} \int dx^4 \sqrt{-g} [f(R, T, R_{\mu\nu}T^{\mu\nu}) + \mathcal{L}_m], \] where \( \kappa^2 = 1 \), \( f(R, T, R_{\mu\nu}T^{\mu\nu}) \) is an arbitrary function in all of its contents, the Ricci scalar \( R \), trace of the energy-momentum tensor \( T = T^\mu_\mu \) and contraction of the Ricci tensor with \( T_{\mu\nu} \), \( \mathcal{L}_m \) denotes the Lagrangian density of matter part. The matter energy-momentum tensor is given by \[ T_{\mu\nu} = -2\sqrt{-g} \frac{\delta}{\delta{g_{\mu\nu}}} \mathcal{L}_m. \] If the matter action depends only on the metric tensor rather than on its derivatives then the energy-momentum tensor yields \[ T_{\mu\nu} = g_{\mu\nu} \mathcal{L}_m - 2\frac{\partial \mathcal{L}_m}{\partial g_{\mu\nu}}. \] The field equations in \( f(R, T, R_{\mu\nu}T^{\mu\nu}) \) gravity can be found by varying the action (1) with respect to \( g_{\mu\nu} \) as

\[
R_{\mu\nu}f_R - \left\{ \frac{1}{2} f - \mathcal{L}_m f_T - \frac{1}{2} \nabla_\alpha \nabla_\beta (f_Q T^{\alpha\beta}) \right\} g_{\mu\nu} + \left( g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) f_R \\
+ \frac{1}{2} \Box (f_Q T_{\mu\nu}) + 2f_Q R_{\alpha(\mu} T^{\alpha\beta)}_{\nu)} - \nabla_\alpha \nabla_\beta (g_{\mu\nu} T_{\alpha\beta}) - G_{\mu\nu} \mathcal{L}_m f_Q - 2f_T g^{\alpha\beta} \\
+ f_Q R^{\alpha\beta} \frac{\partial^2 \mathcal{L}_m}{\partial g_{\mu\nu} \partial g^{\alpha\beta}} = (1 + f_T + \frac{1}{2} R f_Q) T_{\mu\nu},
\]
where we set \( Q = R_{\mu\nu}T^{\mu\nu} \) to make the equations more convenient while the subscripts indicate the derivatives with respect to \( R, T \) and \( Q \).

One can obtain the field equations in \( f(R, T) \) and \( f(R) \) theories from the above expression by substituting some particular functions of Lagrangian. For vacuum, this leads to the field equations in \( f(R) \) gravity. The field equation (4) can be rearranged in the following form

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = T^{\text{eff}}_{\mu\nu}, \tag{5}
\]

which is analogous to the standard field equations in GR. Here \( T^{\text{eff}}_{\mu\nu} \), the effective energy-momentum tensor in \( f(R, T, Q) \) gravity is defined as

\[
T^{\text{eff}}_{\mu\nu} = \frac{1}{f_R - f_QL_m} \left[ (1 + f_T + \frac{1}{2}RFQ)T_{\mu\nu} + \left\{ \frac{1}{2}(f - RF_R) - L_mf_T \right\} T_{\mu\nu} - \frac{1}{2} (f_Q R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}) \right]. \tag{6}
\]

Applying the covariant divergence to the field equation (4), we obtain

\[
\nabla_{\mu} T_{\mu\nu} = \frac{2}{2(1 + f_T) + RF_Q} \left[ \nabla_{\mu} (f_Q R_{\alpha\mu} T_{\alpha\nu}) + \nabla_{\nu} (L_m f_T) - \frac{1}{2} (f_Q R_{\sigma\zeta}) \nabla_{\mu} (f_Q R_{\alpha\sigma} T_{\alpha\zeta} - G_{\mu\nu} \nabla^{\mu} (f_Q L_m) - \frac{1}{2} [\nabla^{\mu} (R f_Q) + 2 \nabla^{\mu} f_T] T_{\mu\nu} \right]. \tag{7}
\]

It is significant to see that ideal continuity equation does not agree in this modified theory which is also true in other modified theories involving non-minimal matter geometry coupling [7]-[16].

### 3 Energy Conditions

#### 3.1 Raychaudhuri Equation

To discuss the energy conditions in modified theories one needs to adopt the procedure originally developed in Einstein gravity. We first discuss these conditions in GR and search a way to express them in this modified theory. In fact, Raychaudhuri equation plays a key role to prove singularity theorems
and explain the congruence of timelike and null geodesics. Raychaudhuri’s equation for the congruence of timelike geodesics is defined as

$$\frac{d\theta}{d\tau} = -\frac{1}{3} \theta^2 - \sigma^{\mu\nu} \sigma_{\mu\nu} + \omega^{\mu\nu} \omega_{\mu\nu} - R_{\mu\nu} u^\mu u^\nu,$$

(8)

where $\theta$ denotes the expansion parameter (if $\theta > 0$ then congruence will be diverging and for $\theta < 0$, it will be converging), $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$ measure the distortion of volume and rotation of curves linked to the congruence set by the vector field $u^\mu$. In case of null geodesics characterized by the vector field $\kappa^\mu$, the temporal variation of expansion is given by

$$\frac{d\theta}{d\tau} = -\frac{1}{2} \theta^2 - \sigma^{\mu\nu} \sigma_{\mu\nu} + \omega^{\mu\nu} \omega_{\mu\nu} - R_{\mu\nu} \kappa^\mu \kappa^\nu.$$

(9)

It is significant to remark that Raychaudhuri equation is exclusively geometric and hence develops no deal to any theory of gravity under discussion. Actually, the energy-momentum tensor can have contribution from different sources and it is convenient to set some constraints to deal it on physical grounds. There are certain inequalities which may limit the arbitrariness in the energy-momentum tensor based on Raychaudhuri equation with attractiveness property of gravity. The association of Raychaudhuri equation can be set from the fact that the variation of expansion parameter is related to $T_{\mu\nu}$ if one finds the Ricci tensor from the field equations. Hence, one can develop the physical constraints on the energy-momentum tensor through the connection between Raychaudhuri equation and the field equations.

As $\sigma^{\mu\nu} \sigma_{\mu\nu} \geq 0$ (shear tensor is purely spatial), so for any hypersurface orthogonal congruence ($\omega_{\mu\nu} = 0$), the condition of attractive gravity takes the form

SEC: $R_{\mu\nu} u^\mu u^\nu \geq 0$, \quad NEC: $R_{\mu\nu} \kappa^\mu \kappa^\nu \geq 0$. \quad (10)

Using the field equations, one can relate $R_{\mu\nu}$ to $T_{\mu\nu}$ so that the above conditions become

$$R_{\mu\nu} u^\mu u^\nu = (T_{\mu\nu} - \frac{T}{2} g_{\mu\nu}) u^\mu u^\nu \geq 0, \quad R_{\mu\nu} \kappa^\mu \kappa^\nu = T_{\mu\nu} \kappa^\mu \kappa^\nu \geq 0.$$

(11)

If the matter part is considered as perfect fluid

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu - pg_{\mu\nu},$$

(12)

where energy density and pressure are denoted by $\rho$ and $p$, then these conditions reduce to the most familiar form of strong and null energy conditions in GR,

$$\rho + 3p \geq 0, \quad \rho + p \geq 0.$$

(13)
3.2 Energy Conditions in $f(R, T, Q)$ Gravity

Here, we adopt the procedure developed in [31, 32] for $f(R)$, $f(R, \mathcal{L}_m)$ and $f(R)$ gravity with arbitrary and nonminimal matter geometry coupling to extend it to a more general $f(R, T, Q)$ gravity. The Ricci tensor in Eq. (5) can be represented in terms of $T^\text{eff}_{\mu\nu}$ and its trace $T^\text{eff}$ as

$$R_{\mu\nu} = T^\text{eff}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\text{eff},$$

where the contraction of Eq.(6) yields the trace of the energy-momentum tensor

$$T^\text{eff} = \frac{1}{f_R - f_Q\mathcal{L}_m} \left[(1 + f_T + \frac{1}{2}Rf_Q)T + 2(f - Rf_R) - 4\mathcal{L}_mf_Tight. \\
- \nabla_\alpha \nabla_\beta (f_Q T^{\alpha\beta}) - 3f_T - \frac{1}{2}\Box (f_Q T) - 2f_Q R^{\alpha\beta} T + 2g^{\mu\nu}(f_T g^{\alpha\beta} \\
+ f_Q R^{\alpha\beta}) \frac{\partial^2 \mathcal{L}_m}{\partial g^{\mu\nu} \partial g^{\alpha\beta}} \right].$$

(15)

The attractive nature of gravity needs to satisfy the following additional constraint

$$\frac{1 + f_T + \frac{1}{2}Rf_Q}{f_R - f_Q\mathcal{L}_m} > 0$$

(16)

which does not depend on the conditions (10) derived from the Raychaudhuri equation. In fact this condition corresponds to the effective gravitational coupling in $f(R, T, Q)$ gravity.

We take the homogeneous and isotropic flat FRW metric defined as

$$ds^2 = dt^2 - a^2(t)dx^2,$$

where $a(t)$ represents the scale factor and $dx^2$ is the spatial part of the metric. The corresponding effective energy density and pressure can be taken such that $T^\text{eff}_{\mu\nu}$ assumes the form of perfect fluid. In FRW background, $\rho_{\text{eff}}$ and $p_{\text{eff}}$ can be obtained in this modified theory as

$$\rho_{\text{eff}} = \frac{1}{f_R - f_Q\mathcal{L}_m} \left[\rho + (\rho - \mathcal{L}_m)f_T + \frac{1}{2}(f - Rf_R) - 3H\partial_t f_R - \frac{3}{2}(3H^2 \\
- \dot{H})\rho f_Q - \frac{3}{2}(3H^2 + \dot{H})p f_Q + \frac{3}{2}H \partial_t [(p - \rho)f_Q]\right],$$

(17)
\[ p_{\text{eff}} = \frac{1}{f_R - f_Q L_m} \left[ p + (p + L_m) f_T + \frac{1}{2}(R f_R - f) + \frac{1}{2}(\dot{H} + 3H^2)\rho f_Q \right] \\
+ \frac{1}{2}(3H^2 - \dot{H})p f_Q + \partial_{tt} f_R + 2H\partial_t f_R + \frac{1}{2} \partial_{tt}[(\rho - p)f_Q] + 2H\partial_t[(\rho + p)f_Q] \],
\]

(18)

where \( R = -6(\dot{H} + 2H^2) \), \( H = \frac{\dot{a}}{a} \) being Hubble parameter and over dot refers to time derivative. Here, we neglect the terms involving second derivative of matter Lagrangian with respect to the metric tensor. As we are dealing with perfect fluid, so matter Lagrangian can either be \( L_m = p \) or \( L_m = -\rho \) which makes it obvious to ignore such term.

In this modified theory, we can employ an approach analogous to that in GR and combine Eqs. (11) and (14) so that SEC is of the form

\[ T_{\mu\nu}^{\text{eff}} u^\mu u^\nu - \frac{1}{2} T^{\text{eff}} \geq 0, \]

(19)

where \( g_{\mu\nu} u^\mu u^\nu = 1 \). Using Eqs. (6) and (15), it follows that

\[ \rho_{\text{eff}} + 3p_{\text{eff}} = \frac{1}{f_R - f_Q L_m} \left[ (\rho + 3p) + (\rho + 3p + 2L_m) f_T + R f_R - f \right] \\
+ 3[\dot{H}(\rho - p)]f_Q + 3H\partial_t[f_R + \frac{1}{2}(3\rho + 5p)f_Q] \\
+ 3\partial_{tt}[f_R + \frac{1}{2}(\rho - p)f_Q] \geq 0, \]

(20)

which is the SEC in \( f(R, T, Q) \) gravity. One can represent the NEC in \( f(R, T, Q) \) gravity in the form

\[ T_{\mu\nu}^{\text{eff}} k^\mu k^\nu \geq 0. \]

Inserting Eq. (6) in the above relation results the following inequality

\[ \rho_{\text{eff}} + p_{\text{eff}} = \frac{1}{f_R - f_Q L_m} \left[ (1 + f_T)(\rho + p) - 3H^2(\rho + p)f_Q + 2\dot{H}(\rho - p)f_Q \right] - H\partial_t[f_R - \frac{1}{2}(\rho + 7p)f_Q] \]

\[ \partial_{tt}[f_R + \frac{1}{2}(\rho - p)f_Q] \geq 0. \]

(21)

It is remarked that one can obtain the NEC and SEC in \( f(R) \) and \( f(R, T) \) modified theories by taking \( f(R, T, Q) = f(R) \) and \( f(R, T, Q) = f(R, T) \),
respectively. Moreover, the traditional structures for the NEC \((\rho + p \geq 0)\) and SEC \((\rho + 3p \geq 0)\) can be found in the framework of GR as a specific case with \(f(R, T, Q) = R\). In determining the WEC and DEC, we consider the modified form of energy conditions in GR which are obtained under the transformations \(\rho \rightarrow \rho_{\text{eff}}\) and \(p \rightarrow p_{\text{eff}}\). We would like to mention here that the null and strong energy conditions given by Eqs. (20) and (21) are derived from the Raychaudhuri equation. One can obtain equivalent results following the same procedure as that in GR with conditions \(\rho_{\text{eff}} + p_{\text{eff}} \geq 0\) and \(\rho_{\text{eff}} + 3p_{\text{eff}} \geq 0\).

We extend this approach to develop the constraints for WEC and DEC so that these conditions for \(f(R, T, Q)\) gravity are given by \(\rho_{\text{eff}} \geq 0\) and \(\rho_{\text{eff}} - p_{\text{eff}} \geq 0\). Using Eqs. (17) and (18), we can obtain the constraints on WEC and DEC. The WEC requires the condition (21) and the following inequality

\[
\rho_{\text{eff}} = \frac{1}{f_R - f_Q L_m}\left[\rho + (\rho - L_m) f_T + \frac{1}{2} (f - R f_R) - 3 H \partial_t f_R - \frac{3}{2} (3 H^2 \\
- \dot{H}) \rho f_Q - \frac{3}{2} (3 H^2 + \dot{H}) p f_Q + \frac{3}{2} H \partial_t [(p - \rho) f_Q]\right] \geq 0, \tag{22}
\]

whereas the DEC is satisfied by meeting the inequalities (21), (22) and the condition

\[
\rho_{\text{eff}} - p_{\text{eff}} = \frac{1}{f_R - f_Q L_m}\left[(\rho - p) + (\rho - p) - 2 L_m f_T + f - R f_R \right.
+ \left\{\dot{H}(\rho - p) - 6 H^2 (\rho + p)\right\} f_Q - H \partial_t \left[\frac{1}{2}(7\rho + p) f_Q + 5 f_R\right] \right.
- \partial_t [f_R + \frac{1}{2} (\rho - p) f_Q] \right] \geq 0. \tag{23}
\]

When we take \(f(R, T, Q) = f(R, T)\), the above expressions reduce to the WEC and DEC in \(f(R, T)\) gravity which are similar to that in [36]. Also, by neglecting the dependence on the trace of energy-momentum tensor, we can have the energy conditions in \(f(R)\) gravity which are consistent with the results in [30]. If the variation of Lagrangian with respect to \(T\) and \(Q\) is null then such conditions constitute \(\rho \geq 0\) and \(\rho + p \geq 0\), i.e., the WEC and DEC in GR.

One can utilize the energy conditions constraints (20)-(23) to restrict some specific models in \(f(R, T, Q)\) gravity in the framework of FRW metric.
To be more definite about these energy constraints, we define deceleration, jerk and snap parameters as \[39\]

\[
q = -\frac{1}{H^2} \frac{\ddot{a}}{a}, \quad j = \frac{1}{H^3} \frac{\dddot{a}}{a}, \quad \text{and} \quad s = \frac{1}{H^4} \frac{\ddddot{a}}{a}.
\]

and express the Hubble parameter and its time derivatives in terms of these parameters

\[
\dot{H} = -H^2(1 + q), \quad \ddot{H} = H^3(j + 3q + 2),
\]

\[
\dddot{H} = -H^4(5q + 2j - s + 3).
\]

Since \(R, \dot{R}\) and \(\ddot{R}\) are represented in terms of the above relations, so using these parameters the energy conditions (20)-(23) can be constituted as

\[
\begin{align*}
(\rho + p)(1 + f_T) &+ \frac{1}{2} \left\{ \dot{\rho} - \dot{\rho} + H(\dot{\rho} + 7\dot{\rho}) - 4(1 + q)H^2(\rho - p) - 6H^2 \\
\times (\rho + p) \right\} f_Q - 6H^2(s - j + (q + 1)(q + 8))f_{RR} + (\dot{\dot{T}} - HT)f_{RT} + \{\dot{\dot{Q}} \\
- H\dot{Q} - 3H^3(j - q - 2)(2\dot{\rho} - \dot{\rho}) + H(\rho + 7\rho) - 3H^4(\rho - p)(s + q^2 + 8q \\
+ 6)\} f_{RQ} + \frac{1}{2} \{2\dot{T} + (2\dot{\rho} - \dot{\rho}) + H(\rho + 7p))\dot{T}f_{TQ} + \frac{1}{2} \{2\dot{Q} + (2\dot{\rho} - \dot{\rho}) \\
+ H(\rho + 7p))\dot{Q}\} f_{QQ} + [6H^3(j - q - 2)]^2 f_{RRR} - 12H^3(j - q - 2)H f_{RRT} \\
+ (18(\rho - p)[H^3(j - q - 2)]^2 - 12H^3(j - q - 2)\dot{Q}f_{RQQ} + 2\dot{T}[\dot{Q} - 3H^3(\rho \\
- p)(j - q - 2)]f_{RTQ} + \dot{T}^2 f_{RRT} + \dot{Q}[\dot{Q} - 6H^3(\rho - p)(j - q - 2)]f_{RQQ} + \frac{1}{2} \\
\times (\rho - p)\dot{T}[\dot{Q}f_{TTQ} + \dot{T}f_{TQQ}] + \frac{1}{2} (\rho - p)\dot{Q}[\dot{T}f_{TQQ} + \dot{Q}f_{QQQ}] \geq 0, \quad (\text{NEC})
\end{align*}
\]

\[
\rho(1 + f_T) - L_m f_T + \frac{1}{2} f + 3H^2(1 - q)f_R + \frac{3}{2}H \{\dot{\rho} - \dot{\rho} - 2H(2\rho + p) \\
- H(\rho - p)q\} f_Q - 3H \{\dot{Q} + 3H^3(\rho - p)(j - q - 2)\} f_{RQ} + \frac{3}{2}H(\rho - p)(\dot{Q} \\
\times f_{QQ} + \dot{T}f_{TQ}) + 18H^4(j - q - 2)f_{RRR} - 3H\dot{T}f_{RT}) \geq 0, \quad (\text{WEC})
\]
To illustrate how these energy conditions put limits on $f$ consider some specific functional forms for the Lagrangian \((1)\) namely \((27)\),

\[
\begin{align*}
(\rho + 3p)(1 + f_T) + 2\mathcal{L}_m f_T - f - 6H^2(1 - q)f_R + \frac{3}{2}\left(\dot{\rho} - \ddot{p} + H(3\dot{\rho} + 5\dot{p})\right) \\
+ 6H^2[(p - \rho)(1 + q)]f_Q - 18H^4(s + j + q^2 + 7q + 4)f_{RR} + 3(\ddot{T} - H\dot{T})f_{RT} + 3\{Q + H\dot{Q} - 3H^3(j - q - 2)[2(\rho - p) + H(3\rho + 5p)] - 3H^4 \\
\times (\rho - p)(s + q^2 + 8q + 6)\}f_{RQ} + \frac{3}{2}\{(\rho - p)\dddot{T} + [2(\dot{\rho} - \dot{p}) + H(3\rho + 5p)\dot{T}]\} \\
\times f_{TQ} + \frac{3}{2}\{(\rho - p)\dddot{Q} + [2(\dot{\rho} - \dot{p}) - H(3\rho + 5p)\dot{Q}]\}f_{QQ} + 3[6H^3(j - q - 2)]^2 \\
\times f_{RRR} - 36H^3(j - q - 2)\dddot{T}f_{RRT} + 3\{-12H^3(j - q - 2)\dot{Q} + 18(\rho - p)\}H^3(j - q - 2)\}f_{RTQ} + 3\dot{T}^2 \\
\times f_{RTT} + 3\dot{Q}\{\dot{Q} - 6H^3(\rho - p)(j - q - 2)\}f_{RQQ} + \frac{3}{2}(\rho - p)\dot{T}\{2\dot{Q}f_{TQQ} \\
+ \dot{T}f_{TTQ}\} + \frac{3}{2}(\rho - p)\dot{Q}^2f_{QQQ} \geq 0. \quad \text{(SEC)}, \quad (26)
\end{align*}
\]

\[
\begin{align*}
(\rho - p)(1 + f_T) - 2\mathcal{L}_m f_T + f + 6H^2(1 - q)f_R + \frac{1}{2}\left(\dddot{\rho} - \ddot{\rho} - H(7\dot{\rho} + \dot{p})\right) \\
- 12H^2(\rho + p) - 2H^2(\rho - p)(1 + q)\}f_Q - \{Q + 5HQ - 6H^3(j - q - 2) \\
\times (\rho - p) - 3H^4(\rho - p)\}H^4(s + q^2 + 8q + 6)(\rho - p)\} \\
- \frac{1}{2}\{(\rho - p)\dddot{T} + [2(\dot{\rho} - \dot{p}) + H(7\rho + p)\dot{T}]\}f_{TQ} - \frac{1}{2}\{(\rho - p)\dddot{Q} + [2(\dot{\rho} - \dot{p})\} \\
+ H(7\rho + p)\}f_{QQ} + 6H^4[s + 5j + (q - 1)(q + 4)]f_{RR} - (\dddot{T} + 5\dot{T}H)f_{RT} \\
- [6H^3(j - q - 2)]^2f_{RRR} + 12H^3(j - q - 2)\dddot{T}f_{RRT} + \{12H^3(j - q - 2)\}Q \\
- 18[H^3(j - q - 2)]^2(\rho - p)\}f_{RRQ} - 2\dddot{T}\{\dot{Q} - 6H^3(j - q - 2)(\rho - p)\}f_{RTQ} \\
- \dot{T}^2f_{RTT} - \dot{Q}\{\dot{Q} - 6H^3(j - q - 2)(\rho - p)\}f_{RQQ} - \frac{1}{2}(\rho - p)\dddot{T}\{\dot{Q}f_{TQQ} \\
+ \dot{T}f_{TTQ}\} + \frac{1}{2}(\rho - p)\dot{Q}\{\dot{T}f_{TQQ} + \dot{Q}f_{QQQ}\} \geq 0. \quad \text{(DEC)} \quad (27)
\end{align*}
\]

The results of energy conditions in terms of cosmographic parameters for $f(R)$ and $f(R, T)$ theories can be achieved from the constraints \((23)-(27)\).

### 4 Constraints on Class of $f(R, T, Q)$ Models

To illustrate how these energy conditions put limits on $f(R, T, Q)$ gravity, we consider some specific functional forms for the Lagrangian \((1)\) namely \((27)\),
1. $f(R, T, Q) = R + \alpha Q$,
2. $f(R, T, Q) = R(1 + \alpha Q)$,

where $\alpha$ is a coupling parameter. Recently, these models have been studied in [27] which suggest that exponential and de Sitter type solutions exist for these forms of $f(R, T, Q)$ gravity. Thus one can deduce that coupling between matter and geometry may cause the current cosmic acceleration.

4.1 $f(R, T, Q) = R + \alpha Q$

In the first place, we consider the Lagrangian given by $R + \alpha Q$. In FRW background, the energy conditions for such model can be represented as

$$\alpha A_1 + H\partial_t A_2 \geq A_3,$$

where $A_i$'s purely depend on the energy conditions under discussion. For NEC, one can have

$$A_1^{NEC} = (2\dot{H} - 3H^2)\rho - (2\dot{H} + 3H^2)p + \partial_{tt}[\alpha^{-1} + \frac{1}{2}(\rho - p)],$$
$$A_2^{NEC} = -(1 - \alpha^2)(\rho + 7p), \quad A_3^{NEC} = -(\rho + p).$$

For WEC, this yields

$$A_1^{WEC} = -3H^2\rho, \quad A_2^{WEC} = \frac{3\alpha}{2}(p - \rho) - 3, \quad A_3^{WEC} = -\rho.$$  \hspace{1cm} (30)

For SEC, one can find

$$A_1^{SEC} = 3\rho(2\dot{H} + H^2) - 3p(2\dot{H} + 3H^2) + 3\partial_{tt}[\alpha^{-1} + \frac{1}{2}(\rho - p)],$$
$$A_2^{SEC} = 3(1 + \frac{\alpha}{2}(3\rho + 5p)), \quad A_3^{SEC} = -(\rho + 3p).$$  \hspace{1cm} (31)

For DEC, it follows that

$$A_1^{DEC} = 2\dot{H}(p - \rho) + 6H^2(p - \rho) - \partial_{tt}[\alpha^{-1} + \frac{1}{2}(\rho - p)],$$
$$A_2^{DEC} = -\frac{\alpha}{2}(7\rho + p) - 5, \quad A_3^{DEC} = -(\rho - p).$$  \hspace{1cm} (32)
We can also find the condition of attractive gravity for this model from inequality (28) so that

\[ A_1^{AG} = (1 - \alpha L_m) \left( \frac{1}{\alpha} + \frac{R}{2} \right)^{-1}, \quad A_2^{AG} = \text{constant}, \quad A_3^{AG} = 0. \]

The energy conditions (28)-(32) can be expressed in terms of deceleration parameter (see appendix A.1). It can be seen that these conditions depend only on the parameters \( H, q \) and \( \alpha \). In our discussion, we set the present day values of cosmographic parameters as \( q_0 = -0.81^{+0.14}_{-0.11}, \ j_0 = 2.16^{+0.81}_{-0.75} \) \[40\] and \( H_0 = 73.8 \) \[41\], while matter is assumed to be pressureless. To exemplify how these conditions can constrain the above model, we consider the WEC given by the relation

\[ \rho - 6H^2 - 3\alpha H \dot{\rho} \geq 0. \] (33)

For the given \( H \) and \( q \), one can see that the above inequality relies on the measures of parameter \( \alpha \) and time derivative of energy density. Here, \( \dot{\rho} \) can be evaluated using Eq.(7) which takes the form

\[ \dot{\rho} = -6H\rho\{1 + \alpha H^2(2 + 5q - 3H(1 + q))\}/\{2 - 3\alpha(2 - q)H^2\}. \]

It can be seen that \( \dot{\rho} \) is always negative. Using this value of \( \dot{\rho} \) in Eq.(33), we find that WEC for the model \( f(R, T, Q) = R + \alpha R \) is satisfied if \( \alpha > 0 \) for present day values of \( q \) and \( H \).

4.2 \( f(R, T, Q) = R(1 + \alpha Q) \)

In second example, we consider the function \( f \) given by \( R(1+\alpha Q) \) and energy conditions for such model can be written as

\[ \hat{\alpha}B_1 + H\partial_t B_2 \geq B_3, \] (34)

where \( \hat{\alpha} = (-1)\alpha \) and \( B_i \)'s purely depend on the energy conditions under discussion. For NEC, one can have

\[ B_1^{NEC} = [-3H^2(\rho + p) + 2\dot{H}(\rho - p)]R + \partial_t[\hat{\alpha}^{-1} + Q + \frac{1}{2}(\rho - p)R], \]

\[ B_2^{NEC} = -(1 + \hat{\alpha}Q - \frac{\hat{\alpha}}{2}(\rho + 7p)R), \quad B_3^{NEC} = -(\rho + p). \] (35)
For WEC, one can have
\[
\begin{align*}
B_{1}^{WEC} & = -\frac{3}{2}(3H^{2} - \dot{H})\rho + (3H^{2} + \dot{H})p R, \\
B_{2}^{WEC} & = 3[\frac{\dot{\alpha}}{2}(p - \rho)R - (1 + \dot{\alpha}Q)], \quad B_{3}^{WEC} = -\rho. \quad (36)
\end{align*}
\]

For SEC, it follows that
\[
\begin{align*}
B_{1}^{SEC} & = 3[\dot{\alpha}(p - \rho)]R + \partial_{\tau}[\dot{\alpha}^{-1} + Q + \frac{1}{2}(\rho - p)R], \\
B_{2}^{SEC} & = 3[1 + \dot{\alpha}Q + \frac{\dot{\alpha}}{2}(3\rho + 5p)R], \quad B_{3}^{SEC} = -(\rho + 3p). \quad (37)
\end{align*}
\]

For DEC, this yields
\[
\begin{align*}
B_{1}^{DEC} & = -6H^{2}(\rho + p)R - (\rho + p)\dot{H}R - \partial_{\tau}[\dot{\alpha}^{-1} + Q + \frac{1}{2}(\rho - p)R], \\
B_{2}^{DEC} & = -(5(1 + \alpha Q) + \frac{\alpha}{2}(7\rho + p)R), \quad B_{3}^{DEC} = -(\rho - p). \quad (38)
\end{align*}
\]

The condition of attractive gravity can be obtained from the inequality (34) and relevant components are
\[
B_{1}^{AG} = (1 + \dot{\alpha}Q - RL_{m}) \left( \frac{1}{\dot{\alpha}} + \frac{R^{2}}{2} \right)^{-1}, \quad B_{2}^{AG} = \text{constant}, \quad B_{3}^{AG} = 0.
\]

The viability of modified theories is under debate to develop the criteria for different modifications to the Einstein-Hilbert action. In this perspective, one of the important criterion is Dolgov-Kawasaki instability which has been developed to constrain the \( f(R) \) gravity and \( f(R) \) gravity with curvature matter coupling [42]. Recently, the authors [26, 27] have executed this instability analysis for \( f(R, T, Q) \) gravity which yields the condition of Dolgov-Kawasaki instability as
\[
3f_{RR} + \left( \frac{1}{2}T - T_{00}^{00} \right)f_{QR} \geq 0. \quad (39)
\]

For the model \( f = R(1 + \alpha Q) \), the inequality (39) takes the form
\[
\alpha(\rho - 3p) + 6\dot{\alpha}H(\rho + p)\partial_{\tau}\left( \frac{H}{R} \right) \geq 0,
\]
14
where

\[ \dot{\alpha} = \begin{cases} (-1)\alpha & \text{if } R, Q < 0 \\ \alpha & \text{if } R, Q > 0 \end{cases} \]

One can derive the above inequality using the relation (34) so that \( B_i \)'s are given by

\[ B_{1AG} = \frac{\rho - 3p}{\rho + p}, \quad B_{2AG} = \frac{6\dot{\alpha}H}{R}, \quad B_{3AG} = 0. \]

We check the validity of constraints (34)-(38) for this model. The constraint to ensure WEC is given by

\[ \rho [1 + 9\alpha H^4(2j - q^2 - 3q + 2)] + 9\alpha H^3(1 - 2q) \dot{\rho} \geq 0. \]

As in the previous case, we evaluate \( \dot{\rho} \)

\[ \dot{\rho} = -3H\rho \{1 + 6\alpha H^4(j - 4q + 2q^2 - 1)\}/\{1 + 9\alpha H^4(1 - q)(2 - q)\}. \]

Here, \( \dot{\rho} < 0 \) for any value of \( \alpha \) and hence the WEC is satisfied only if parameter \( \alpha \) is positive.

### 4.3 \( f(R, T) \) Models

Here we present \( f(R, T, Q) \) gravity models which involve null variation with respect to \( Q \) and corresponds to \( f(R, T) \) gravity. Although the energy conditions are examined in \( f(R, T) \) gravity but the constraints are developed for very particular cases. Alvarenga et al. [43] studied the energy conditions for some models of the type \( f(R, T) = R + 2f(T) \) and analyse their stability under matter perturbations. In our previous work [36], we established the energy condition constraints for those \( f(R, T) \) models which confirm the existence of power law solutions in this modified theory. Here we are interested in more general functional forms of \( f(R, T) \) models involving an exponential function and also the coupling between \( R \) and \( T \). We present the energy condition constraints for the following models

1. \( f(R, T) = \alpha \exp \left( \frac{R}{\alpha} + \lambda T \right) \)
2. \( f(R, T) = R + \eta R^m T^n \)

where \( \alpha, \lambda, \eta, m \) and \( n \) are arbitrary constants.
\[ f(R, T) = \alpha \exp \left( \frac{R}{\alpha} + \lambda T \right) \]

If \( \frac{R}{\alpha} + \lambda T \ll 1 \) then \( f(R, T) \approx \alpha + R + \frac{\lambda}{\alpha} T + \ldots \) representing the ΛCDM model. The energy constraints in \( f(R, T) \) gravity can be achieved by placing null variation of \( f \) with respect to \( Q \) in the results \([20]-[23]\) which are similar to that in \([36]\).

For the exponential model, these conditions take the form
\[ \exp \left( \frac{R}{\alpha} + \lambda T \right) \frac{1}{1 + \alpha \lambda \exp \left( \frac{R}{\alpha} + \lambda T \right)} (C_1 + C_2) \geq C_3, \] (40)

where \( C_i \)'s depend on the energy conditions given in Appendix A.2. The condition of attractive gravity in \( f(R, T) \) gravity is \( (1 + f_T)/f_R > 0 \) which becomes \( (1 + \alpha \lambda \exp \left( \frac{R}{\alpha} + \lambda T \right)) / \exp \left( \frac{R}{\alpha} + \lambda T \right) > 0 \) for the exponential model. We can obtain this inequality from Eq. (40) for \( C_1 = 1, C_2 = C_3 = 0 \). It is suggested \([27]\) that Dolgov-Kawasaki instability in \( f(R, T) \) gravity would be identical to that in \( f(R) \) gravity so that one can check the viability of \( f(R, T) \) models on similar steps as in \( f(R) \) theory. Thus for \( f(R, T) \) theory, we have
\[ f_R(R, T) > 0, \quad f_{RR}(R, T) > 0, \quad R \geq R_0. \]

For this model, the instability conditions are formulated as \( \exp \left( \frac{R}{\alpha} + \lambda T \right) > 0 \) and \( \frac{1}{\alpha} \exp \left( \frac{R}{\alpha} + \lambda T \right) > 0 \) which can be derived from relation \( [40] \) by taking \( C_1 = 1, C_2 = \alpha \lambda \exp \left( \frac{R}{\alpha} + \lambda T \right), C_3 = 0 \) and \( C_1 + C_2 = \frac{1}{\alpha} (1 + \alpha \lambda \exp \left( \frac{R}{\alpha} + \lambda T \right)) \), \( C_3 = 0 \), respectively. One can represent energy conditions \( [A.2] \) in the form of cosmographic parameters. It is mentioned here that the measurement of present day value of snap parameter ‘s’ has not been reported from reliable sources so far.

The energy conditions corresponding to the above model depend on ‘s’ except the WEC, so we explore the validity of WEC. The inequality to fulfill the WEC is given by
\[ \rho \left( 1 + \alpha \lambda \exp \left( \frac{R}{\alpha} + \lambda T \right) \right) / \exp \left( \frac{R}{\alpha} + \lambda T \right) + \alpha(0.5 - \lambda \mathcal{L}_m) + 3H^2 \{ (1 - q) + 6 \alpha^{-1} H^2 \} \times (j - q - 2) \} - 3 \lambda H \dot{T} \geq 0. \]

Using the WEC results in GR, i.e., \( \rho > 0 \) and also the condition of attractive gravity \( (1 + \alpha \lambda \exp \left( \frac{R}{\alpha} + \lambda T \right)) / \exp \left( \frac{R}{\alpha} + \lambda T \right) > 0 \), the above inequality is reduced to
\[ \alpha(0.5 - \lambda \mathcal{L}_m) + 3H^2 \{ (1 - q) + 6 \alpha^{-1} H^2 (j - q - 2) \} - 3 \lambda H \dot{T} \geq 0. \]
We set the matter Lagrangian density as $\mathcal{L}_m = p$ and assume the pressureless matter so that
\begin{equation}
0.5\alpha + 3H^2\{(1 - q) + 6H^2(j - q - 2)\alpha^{-1}\} - 3\lambda H \dot{\rho} \geq 0. \tag{41}
\end{equation}
If we consider the present day values of the parameters like Hubble, deceleration and jerk then the above inequality depends on $\dot{\rho}$ and values of constants $(\alpha, \lambda)$. We find $\dot{\rho}$ from the energy conservation equation in $f(R, T)$ gravity
\begin{equation}
\dot{\rho} + 3H(\rho + p) = \frac{-1}{1 + f'_{2T}} \left[ (\rho - \mathcal{L}_m)\dot{f}_T - \mathcal{L}_m \dot{f}_T + \frac{1}{2} \dot{T} f_T \right]. \tag{42}
\end{equation}
For exponential model, this takes the form
\begin{equation}
\dot{\rho} = -\frac{3H\rho\{1 + \lambda(\alpha - 2H^2(j - q - 2))\exp\left(\frac{R}{\alpha} + \lambda T\right)\}}{1 + (1.5 + \lambda\rho)\alpha\lambda\exp\left(\frac{R}{\alpha} + \lambda T\right)},
\end{equation}
If $\alpha > 0$ then the first two terms in inequality (41) are positive whereas for the last term we need to have $-\dot{\rho} > 0$. From the above expression, we see that $-\dot{\rho} > 0$ if $\lambda > 0$ and $\alpha > 2H_0^2(j_0 - q_0 - 2)$. Thus, the WEC for exponential $f(R, T)$ model is satisfied if $\lambda > 0$ and $\alpha > 2H_0^2(j_0 - q_0 - 2)$. We consider another form of matter Lagrangian $\mathcal{L}_m = -\rho$ for which the continuity equation and constraint to fulfill the WEC are given by
\begin{equation}
\dot{\rho} = -\frac{3H\rho\{1 + \lambda(\alpha - 4H^2(j - q - 2))\exp\left(\frac{R}{\alpha} + \lambda \rho\right)\}}{1 + (1.5 + 2\lambda\rho)\alpha\lambda\exp\left(\frac{R}{\alpha} + \lambda \rho\right)},
\end{equation}
\begin{equation}
\alpha(0.5 + \lambda\rho) + 3H^2\{(1 - q) + 6\alpha^{-1}H^2(j - q - 2)\} - 3\lambda H \dot{\rho} \geq 0.
\end{equation}
As in the previous case, we find a constraint for which $\dot{\rho} < 0$ which is only possible if $\alpha > 4H^2(j - q - 2)$. It is to be noted that we set the present values of $H$ and other parameters so that the WEC is satisfied if $\lambda > 0$ and $\alpha > 4H^2(j - q - 2)$.

- $f(R, T) = R + \eta R^m T^n$

Here, we consider the power law type $f(R, T)$ model which involves coupling between $R$ and $T$. Such functional form of $f(R, T)$ matches to the form of Lagrangian $f(R, T) = f_1(R) + f_2(R)f_3(T)$ with $f_1(R) = R$, $f_2(R) = R^m$ and $f_3(T) = T^n$ which involves the explicit nonminimal gravitational matter geometry coupling. In a recent work [25], we have reconstructed such type...
of $f(R,T)$ models corresponding to power law solutions. The attractiveness of gravity implies that \( \{1 + \dot{\eta}mR^mT^n\} / \{1 + \dot{\eta}mR^{m-1}T^n\} > 0 \). The energy condition constraints for this model can be represented as

\[
\frac{\dot{\eta}|R|^m|T|^n}{1 - \dot{\eta}|R|^m|T|^n-1}[D_1 + \mathcal{L}_mT^{-1}D_2] \geq D_3,
\]

(43)

where $D_i$'s can have particular relations depending on the energy conditions which are shown in appendix A.3.

We are concerned to study the WEC inequality for this model and develop the constraints as in exponential model. The condition to meet the WEC is given by

\[
\rho + \eta R^m\rho^n \left[ n(1 - \mathcal{L}_m\rho^{-1}) + \frac{1}{2}(1 - m) - 3m(m - 1)H\dot{R}R^{-2} \right.
\]

\[
- 3mnH\dot{\rho}R^{-1}\rho^{-1} \right] \geq 0.
\]

Initially, we consider $\mathcal{L}_m = p$ for which the above inequality can be represented in the form of deceleration and jerk parameters as

\[
2\rho + 2\eta[6H^2(1 - q)]^m\rho^n \left[ 2n + 1 - m + m(m - 1)\frac{j - q - 2}{(1 - q)^2} \right.
\]

\[
+ \frac{mn\dot{\rho}\rho^{-1}}{2(1 - q)H} \right] \geq 0.
\]

(44)

The coupling parameter $\eta$ is assumed to be positive so that the above constraint is satisfied if one can meet the condition in square bracket. For this purpose, $\dot{\rho}$ can be obtained using Eq.(42) in the form

\[
\dot{\rho} = -\frac{3H\rho\{1 + n\eta[6H^2(1 - q)]^m\rho^{-1}\left(1 + \frac{m(j - q - 2)}{3(1 - q)}\right)\}}{1 + n(n + 0.5)\eta[6H^2(1 - q)]^m\rho^{-1}}.
\]

Substituting $\dot{\rho}$ in Eq.(44), it is found that WEC is satisfied if both the constants $m$ and $n$ are positive. One can also examine the WEC constraint for $\mathcal{L}_m = -\rho$ for which WEC can be met if $m, n > 0$ with coupling parameter being positive.

5 Conclusions

The late time accelerated cosmic expansion is a major issue in cosmology. Modified gravity has appeared as a handy candidate to address such issues.
which predict the destiny of the universe. Recently, a more generalized modified theory is established on the basis of curvature matter coupling named as $f(R, T, R_{\mu\nu}T^{\mu\nu})$ gravity [26, 27]. This theory is an extensive form of $f(R, T)$ gravity where Lagrangian not only depends on $R$ and $T$ but also involves the contribution from contraction of the Ricci and the energy-momentum tensors. In this modified theory, extra force is always present due to the coupling between matter and geometry leading to motion of test particles as nongeodesic. In previous studies of nonminimal coupling [15, 16], it has been encountered that extra force disappears if one chooses $\mathcal{L}_m = -p$ whereas it does not vanish even for this $\mathcal{L}_m$ in $f(R, T, R_{\mu\nu}T^{\mu\nu})$ theory. The expression of extra force in $f(R, T, R_{\mu\nu}T^{\mu\nu})$ gravity [27] explicitly depends on $R_{\mu\nu}$ as compared to $f(R, T)$ gravity. Thus the deviation from geodesic motion can be more significant in this case.

Lagrangian of $f(R, T, R_{\mu\nu}T^{\mu\nu})$ gravity is more comprehensive implying that different functional forms of $f$ can be suggested. The versatility in Lagrangian raises the question how to constrain such theory on physical grounds. In [26, 27], the authors have studied the validity of this theory and developed the notable Dolgov-Kawasaki instability (the condition of stability against local perturbations). In this paper, we have developed some constraints on general as well as specific forms of $f(R, T, R_{\mu\nu}T^{\mu\nu})$ gravity by examining the respective energy conditions. The NEC and SEC are derived using the Raychaudhuri equation along with the condition that gravity is attractive. These conditions are more general as compared to those derived in $f(R)$ and $f(R, T)$ theories. Moreover, these inequalities are equivalent to the results found from conditions $\rho + 3p \geq 0$ and $\rho + p \geq 0$ under the transformations $\rho \rightarrow p_{\text{eff}}$ and $p \rightarrow p_{\text{eff}}$, respectively. One can employ the similar procedure to derive the WEC and DEC by translating their counterpart in GR for effective energy-momentum tensor. The conditions of positive effective gravitational coupling and attractive nature of gravity are also obtained in this theory.

To illustrate how these conditions can constrain the $f(R, T, R_{\mu\nu}T^{\mu\nu})$ gravity, we have taken two functional forms of $f$ namely, $f = \bar{R} + \alpha Q$ and $f = R(1 + \alpha Q)$. It is shown that WEC for these models depends on the coupling parameter $\alpha$ and is satisfied only if $\alpha$ is positive. We have also set the Dolgov-Kawasaki criterion in this discussion. In section 4.3, the $f(R, T)$ gravity is addressed as a specific case to this modified theory. We have taken two interesting choices for the Lagrangian, one involving an exponential function and other having explicit coupling between $R$ and $T$. The validity of
WEC for both choices of matter Lagrangian $\mathcal{L}_m = p$ and $\mathcal{L}_m = -\rho$ have been explored. The WEC for $f(R, T) = \alpha \exp \left( \frac{R}{\alpha} + \lambda T \right)$ is met in both cases if coupling parameter $\lambda > 0$ and $\alpha > 2H_0^2(j_0 - q_0 - 2)(or \alpha > 4H_0^2(j_0 - q_0 - 2))$, respectively. For the model $f(R, T) = R + \eta R^m T^n$, the WEC is satisfied if both the constants $m$ and $n$ are positive.

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Appendix A

\begin{align*}
\text{NEC} & : \quad \rho + p + \frac{\alpha}{2} \{ \dot{\rho} - \ddot{p} + H(\dot{\rho} + 7\dot{p}) - H^2[\rho(5 + q) + p(1 - 2q)] \} \geq 0, \\
\text{WEC} & : \quad \rho + \frac{3\alpha}{2} \{ H(\dot{\rho} - \dot{p}) - 2H^2 \} \geq 0, \\
\text{SEC} & : \quad \rho + 3p + \frac{3\alpha}{2} \{ \dot{\rho} - \ddot{p} + H(3\dot{\rho} + 5\dot{p}) - 2H^2(1 + 2q)\rho - 2(1 - 2q) \\
& \qquad \times H^2p \} \geq 0, \\
\text{DEC} & : \quad \rho - p + \frac{\alpha}{2} \{ \ddot{\rho} - \ddot{p} - H(7\dot{\rho} + \dot{p}) + 4H^2\rho(q - 2)\rho + 4H^2(2 - q) \\
& \qquad \times p \} \geq 0. \quad (A.1)
\end{align*}

\begin{align*}
\text{NEC} & : \quad C_{\text{NEC}}^1 = \frac{1}{\alpha^2} \left( \dddot{\rho} - \alpha(\dddot{R} - H\dddot{R}) + 2\alpha\lambda\dddot{\lambda}\dddot{T} \right) + \lambda\{ \dddot{T} - H\dddot{T} + \lambda\dddot{\lambda}^2 \}, \\
& \quad C_{\text{NEC}}^2 = 0, \quad C_{\text{NEC}}^3 = -(\rho + p), \\
\text{WEC} & : \quad C_{\text{WEC}}^1 = 3(\dot{H} + 2H^2) - \frac{3}{\alpha} H\dddot{R} - 3\lambda H\dddot{T}, \quad C_{\text{WEC}}^2 = \alpha \left( \frac{1}{2} - \lambda\mathcal{L}_m \right), \\
& \quad C_{\text{WEC}}^3 = -\rho, \\
\text{SEC} & : \quad C_{\text{SEC}}^1 = R + \frac{3}{\alpha^2} \left( \dddot{R} + \alpha(\dddot{R} + H\dddot{R}) + 2\alpha\lambda\dddot{\lambda}\dddot{T} \right) + 3\lambda\{ \dddot{T} + \dddot{T}(H \\
& \qquad + \lambda\dddot{T}) \}, \quad C_{\text{SEC}}^2 = \alpha(2\lambda\mathcal{L}_m - 1), \quad C_{\text{SEC}}^3 = -(\rho + 3p), \\
\text{DEC} & : \quad C_{\text{DEC}}^1 = -R - \frac{1}{\alpha^2} \left( \dddot{R} + \alpha(\dddot{R} + 5H\dddot{R}) + 2\alpha\lambda\dddot{\lambda}\dddot{T} \right) - \lambda\{ \dddot{T} + \dddot{T}(5H \\
& \qquad + \lambda\dddot{T}) \}, \quad C_{\text{DEC}}^2 = \alpha(1 - 2\lambda\mathcal{L}_m), \quad C_{\text{DEC}}^3 = -(\rho - p). \quad (A.2)
\end{align*}
\[ D_{\text{NEC}}^1 = m(m - 1)R^{-2}\{\ddot{R} - H\dot{R} + (m - 2)\dot{R}^2 R^{-1} + 2n\dot{R}\dot{T}T^{-1}\} + nR^{-1}T^{-1}\{\ddot{T} - H\dot{T} + (n - 1)\dot{T}^2 T^{-1}\}, \quad D_{\text{NEC}}^2 = 0, \quad D_{\text{NEC}}^3 = -(\rho + p), \]

\[ D_{\text{WEC}}^1 = (1 - m)\{0.5 + 3mH\dot{R}R^{-2}\} - 3mnH\dot{T}R^{-1}T^{-1}, \quad D_{\text{WEC}}^2 = -n, \quad D_{\text{WEC}}^3 = -\rho \]

\[ D_{\text{SEC}}^1 = (m - 1)\{1 + 3mR^{-2}\{\ddot{R} + H\dot{R} + (m - 2)\dot{R}^2 R^{-1} + 2n\dot{R}\dot{T}\times T^{-1}\}\} + 3mnR^{-1}T^{-1}\{\ddot{T} + H\dot{T} + (n - 1)\dot{T}^2 T^{-1}\}, \quad D_{\text{SEC}}^2 = 2n, \quad D_{\text{SEC}}^3 = -3(\rho + 3p), \]

\[ D_{\text{DEC}}^1 = (1 - m)\{1 + mR^{-2}\{\ddot{R} + 5H\dot{R} + (m - 2)\dot{R}^2 R^{-1} + 2n\dot{R}\dot{T}\times T^{-1}\}\} - mnR^{-1}T^{-1}\{\ddot{T} + 5H\dot{T} + (n - 1)T^{-1}\}, \quad D_{\text{DEC}}^2 = -2n, \quad D_{\text{DEC}}^3 = -(\rho - p). \]  

(A.3)

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