On some sums involving the counting function of nonisomorphic Abelian groups*

Haihong Fan¹ and Wenguang Zhai

Department of Mathematics, China University of Mining and Technology, Beijing 100083, China
(e-mail: fanhaihong1@hotmail.com; zhaiwg@hotmail.com)

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Abstract. Let \( a(n) \) denote the number of nonisomorphic Abelian groups with \( n \) elements. In 1991, Ivić proved an asymptotic formula for the sum \( \sum_{n \leq x} a(n + a(n)) \). In this paper, we prove a sharper asymptotic formula for this sum. Also, we extend this process to the divisor function and construct a similar result.

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1 Introduction

Let \( a(n) \) denote the number of nonisomorphic Abelian groups of order \( n \). The Dirichlet series of \( a(n) \) is

\[
\sum_{n=1}^{\infty} a(n)n^{-s} = \zeta(s)\zeta(2s)\zeta(3s)\cdots \quad (\Re s > 1),
\]

where \( \zeta(s) \) is the Riemann zeta-function.

It is well known that the arithmetical function \( a(n) \) is multiplicative and satisfies the equality \( a(p^\alpha) = P(\alpha) \) for any prime \( p \) and integer \( \alpha \geq 1 \), where \( P(\alpha) \) is the number of partitions of \( \alpha \). Hence, for each prime number \( p \), we have \( a(p) = 1 \), \( a(p^2) = 2 \), \( a(p^3) = 3 \), \( a(p^4) = 5 \), \( a(p^5) = 7 \).

A vast literature exists on the asymptotic properties of \( a(n) \). See, for example, [10] and [14] for historical surveys. The classical problem is to study the summatory function

\[
A(x) := \sum_{n \leq x} a(n).
\]

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¹ Corresponding author.
In 1935, Erdős and Szekeres [7] proved that

\[ A(x) = A_1 x + O(x^{1/2}), \]

where \( A_1 = \prod_{v=2}^{\infty} \zeta(v) \). Schwarz [23] showed that

\[ A(x) = A_1 x + A_2 x^{1/2} + A_3 x^{1/3} + R(x) \]

with \( R(x) \ll x^{3/10-7/(30\cdot 23)} (\log x)^{21/23} \) and \( A_j = \prod_{v \neq j} \zeta(v/j) \) \((j = 1, 2, 3)\). Many authors investigate the upper bound of \( R(x) \). For later improvements, see [13, 21, 22]. The best result to date is

\[ R(x) \ll x^{1/4+\varepsilon} \]

for every \( \varepsilon > 0 \), proved by Robert and Sargos [20].

For an arithmetic function \( f : \mathbb{N} \to \mathbb{N} \) and integer \( r > 1 \), define

\[ f^{(r)}(n) = f(f(\ldots f(n)\ldots)) \]

as the \( r \)th iterate of \( f \). If \( r \geq 2 \) is fixed, then two among the most natural problems concerning \( f^{(r)}(n) \) are evaluation of sums of \( f^{(r)}(n) \) and the determination of the maximal order of \( f^{(r)}(n) \). In the case of \( f(n) = d(n) \), representing the Dirichlet divisor function, these problems were investigated by Erdős and Kátai [5, 6]. In [6], it was shown that

\[ \sum_{n \leq x} d^{(r)}(n) = (1 + o(1)) D_r x \log^r x \quad (D_r > 0, \ x \to \infty) \]

for \( r = 4 \), which was also proved earlier by Kátai for \( r = 2, 3 \). Additionally, there has been work on the analogue of this problem for \( a(n) \). Ivić [9] considered the second of \( a(n) \) and proved that

\[ \sum_{n \leq x} a(a(n)) = C x + O(x^{1/2} \log^4 x) \]

for a suitable \( C > 0 \).

In 1986, Spiro [25] studied a new iteration problem involving the divisor function and proved that

\[ \sum_{n \leq x} d(n) \frac{x}{(\log x)^7} \gg \frac{x}{(\log x)^7}. \quad (1.1) \]

In view of the work of Spiro, we can conjecture that for some \( D > 0 \),

\[ \sum_{n \leq x} d(n + d(n)) = D x \log x + O(x). \quad (1.2) \]

However, at present, it seems very difficult to prove (1.2). A result analogous to (1.2) is much less difficult if \( d(n) \) is replaced by \( a(n) \) or by a suitable prime-independent multiplicative function \( f(n) \) such that \( f(p) = 1 \). This is roughly because \( d(p) = 2 \) and \( a(p) = 1 \).
Inspired by (1.1), Ivić [11] pointed out an asymptotic formula for the sum
\[ Q(x) := \sum_{n \leq x} a(n + a(n)) \]
and derived that
\[ Q(x) = C_1 x + O(x^{11/12+\varepsilon}) \] (1.3)
for a positive constant \( C_1 \).

In this paper, we use a different approach to improve (1.3). We have the following theorem.

Theorem 1. For any \( \varepsilon > 0 \), we have the asymptotic formula
\[ Q(x) = C_1 x + O(x^{3/4+\varepsilon}) , \]
where the \( O \)-constant depends only on \( \varepsilon \).

For each fixed \( l \geq 2 \), let \( d_l(n) \) denote the number of ways \( n \) can be written as a product of \( l \) natural numbers. A classical problem in analytic number theory is the study of the counting function \( D_l(x) := \sum_{n \leq x} d_l(n) \). We have the asymptotic formula
\[ D_l(x) = xP_l(\log x) + O(x^{\alpha_l+\varepsilon}) , \]
where \( P_l(t) \) is a polynomial in \( t \) of degree \( l - 1 \), and \( 0 < \alpha_l < 1 \) is a real constant (see Ivić [10] for more detail). For example, Bourgain and Watt [1] proved that \( \alpha_2 \leq 517/1648 \), Kolesnik [12] proved that \( \alpha_3 \leq 43/96 \), and Ivić [10] proved that \( \alpha_4 \leq 1/2 \).

By the same approach we can prove the following theorem.

Theorem 2. Let \( l \geq 2 \) be an integer. Then we have
\[ Q_l(x) := \sum_{n \leq x} d_l(n + a(n)) = xQ_l(\log x) + O(x^{1-1/(2l)+\varepsilon}) , \]
where \( Q_l(t) \) is a polynomial in \( t \) of degree \( l - 1 \), and the \( O \)-constant depends only on \( \varepsilon \).

The structure of this paper is as follows. In Section 2, we quote some lemmas needed for our proof. In Section 3, we study a sum of \( a(n) \) in arithmetic progression. The proof of Theorem 1 is given in Section 4. In Section 5, we give a sketch of the proof of Theorem 2.

Notations. Throughout this paper, \( \mathbb{N} \) and \( \mathbb{C} \) denote the sets of positive integers and complex numbers, respectively. We always use \( q \) to denote square-free numbers and \( s \) to denote square-full numbers. Further, \( \varphi \) is Euler’s totient function, \( \mu \) is the Möbius function, \( \chi \) denotes a Dirichlet character modulo \( r \), and \( L(z, \chi) \) denotes the Dirichlet \( L \)-function corresponding to \( \chi \). For \( l \geq 2 \), \( d_l(n) \) denotes the number of ways \( n \) can be written as a product of \( l \) natural numbers, and \( d(n) = d_2(n) \). In this paper, \( \varepsilon \) always denotes a small enough positive constant.

2 Preliminary lemmas

To prove the theorem, we need the following lemmas.
Lemma 1. Suppose \( g(n) \in \mathbb{C} \ (n \geq 1) \) is such that the Dirichlet series \( G(s) := \sum_{n \geq 1} g(n)n^{-s} \) absolutely converges for \( \sigma > \sigma_a \) and
\[
\sum |g(n)| n^{-\sigma} \leq B(\sigma), \quad \sigma > \sigma_a.
\]
Suppose further that \( |g(n)| \leq H(u) (n \geq 1) \), where \( H(u) > 0 \) is a function on \([1, \infty)\) such that \( H(u) \asymp H(v) \) for \( u \asymp v \). Suppose \( b > \sigma_a \), \( T \geq 1 \), \( x \geq 1 \), and \( x \notin \mathbb{N} \). Then we have
\[
\sum_{n \leq x} g(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} G(s) x^s \frac{dx}{s} + O\left(\frac{x B(b)}{T}\right) + O\left(x H(2x) \min\left(1, \frac{\log x}{T}\right) + O\left(x H(N) \min\left(1, \frac{x}{T \|x\|}\right)\right)\right),
\]
where \( N \) is the integer nearest to \( x \), and \( \|x\| = |x - N| \).

Proof. This is the well-known Perron formula. See formula (A.10) in Appendix of Ivić [10]. \( \square \)

Lemma 2. Let \( \chi \) be a Dirichlet character modulo \( r \). Then we have
\[
L(\sigma + it, \chi) \ll \log(r(|t|+2)) \quad (\sigma \geq 1).
\]

Proof. See, for example, [17]. \( \square \)

Lemma 3. Let \( \chi \) be a Dirichlet character modulo \( r \). Then we have
\[
L(\sigma + it, \chi) \ll d(r)(r(|t|+2))^{(1-\sigma)/2} \log(r(|t|+2)) \quad \left(\frac{1}{2} \leq \sigma \leq 1\right).
\]

Proof. If \( \chi \) is a primitive Dirichlet character, then we have [17]
\[
L(\sigma + it, \chi) \ll (r(|t|+2))^{(1-\sigma)/2} \log(r(|t|+2)) \quad \left(\frac{1}{2} \leq \sigma \leq 1\right).
\]

If \( \chi \) is not a primitive Dirichlet character, denote by \( \chi^* \) the primitive character inducing \( \chi \), that is,
\[
\chi \ (\text{mod } q) \iff \chi^* (\text{mod } q^*).
\]

Then Lemma 3 follows from the well-known relation
\[
L(z, \chi) = L(z, \chi^*) \prod_{p \mid r} \left(1 - \frac{\chi^*(p)}{p^z}\right) \quad (\Re z > 1). \quad \square
\]

Lemma 4. Let \( T \geq 2 \). Then we have
\[
\sum_{\chi (\text{mod } r)} \int_1^T \left|L\left(\frac{1}{2} + it, \chi\right)\right|^2 dt \ll \varphi(r) T \log(rT).
\]

Proof. See, for example, [18]. \( \square \)

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Lemma 5. Let \( T \geq 3 \) be a large real number. Suppose \( z = \sigma + it, |t| \leq T, |\sigma - 1/2| \leq 1/\log T \). Then we have
\[
\sum_{\chi \neq \chi_0} |L(z, \chi)|^2 \ll \varphi(r)|z|\log^2(r(|z| + 2)).
\]

Proof. See, for example, [17]. \( \square \)

Lemma 6. We have
\[
\limsup_{n \to \infty} \frac{\log a(n)}{\log n} = \frac{\log 5}{4}.
\]

Proof. See Krätzel [14]. \( \square \)

3 A sum of \( a(n) \) in arithmetic progression

In this section, we study a sum of \( a(n) \) in arithmetic progression. Let \( x \geq 3 \) be a large parameter, and let \( r \) and \( k \) be natural numbers such that \( 2 \leq r \ll x^{1/2} \) and \( 1 \leq k \ll x^{1/2} \). Define
\[
T(x; k, r) := \sum_{m \leq x \atop m \equiv k \pmod{r}} a(m),
\]
which plays an important role in the proof of our theorem. This sum has been studied by several authors, for example, by Richert [19], Duttlinger [4], and Ivić [11]. Ivić stated that
\[
T(x; k, r) = B(r, k)x + O((rx)^{1/2+\varepsilon}), \quad B(r, k) = O\left(\frac{1}{r}\right).
\] (3.1)

Ivić gave a detailed proof of (3.1) for the case \( (r, k) = 1 \).

In this section, we give a sharper asymptotic formula than (3.1). We will prove the following proposition.

Proposition 1. Uniformly for \( r \ll x^{1/2} \), we have
\[
T(x; k, r) = \frac{c(r, k)}{r}x + O(d(r)x^{1/2\log 2.5}),
\]
where \( c(r, k) \) defined by (3.13) is such that \( c(r, k) \ll d(r) \), and the \( O \)-constant is absolute.

Proof. Let \( u = (r, k), r = ur_1, k = uk_1, (r_1, k_1) = 1 \). Then \( m \equiv k \pmod{r} \) implies that \( m = rn + k = u(r_1n + k_1) \). So we have
\[
T(x; k, r) = \sum_{u(nr_1+k_1) \leq x} a(u(nr_1 + k_1)) = \sum_{n \leq x/u \atop n \equiv k_1 \pmod{r_1}} a(un)
\]
\[
= \frac{1}{\varphi(r_1)} \sum_{\chi \pmod{r_1}} \overline{\chi}(k_1) \left( \sum_{n \leq x/u} \chi(n)a(un) \right),
\] (3.2)
where in the last step, we used the orthogonality property of Dirichlet characters. Note that (3.2) holds for \( u = 1 \) or \( r_1 = 1 \).
It suffices for us to evaluate the innermost sum \( X = x/u \)

\[
M(X; \chi, u) := \sum_{n \leq X} \chi(n)a(un)
\]

for any Dirichlet character \( \chi \) modulo \( r_1 \). Suppose that \( u = \prod_p p^l \). Since both \( a(n) \) and \( \chi(n) \) are multiplicative, we have by the Euler product that \( (\Re z > 1) \)

\[
D_u(z, \chi) = \sum_{n=1}^{\infty} \frac{a(un)\chi(n)}{n^z} = \prod_p \left( \sum_{\alpha=0}^{\infty} \frac{a(p^\alpha)\chi(p^\alpha)}{p^{\alpha z}} \right) = \prod_{p|u} \left( \sum_{\alpha=0}^{\infty} \frac{a(p^\alpha)\chi(p^\alpha)}{p^{\alpha z}} \right) \prod_{p \not| u} \left( \sum_{\alpha=0}^{\infty} \frac{a(p^\alpha)\chi(p^\alpha)}{p^{\alpha z}} \right) = \prod_{p} \left( \sum_{\alpha=0}^{\infty} \frac{a(p^\alpha)\chi(p^\alpha)}{p^{\alpha z}} \right) \prod_{p \not| u} \left( \sum_{\alpha=0}^{\infty} \frac{a(p^\alpha)\chi(p^\alpha)}{p^{\alpha z}} \right) \prod_{p \not| u} \left( \sum_{\alpha=0}^{\infty} \frac{a(p^\alpha)\chi(p^\alpha)}{p^{\alpha z}} \right)
\]

where

\[
H(z, \chi) := \prod_p \left( \sum_{\alpha=0}^{\infty} \frac{a(p^\alpha)\chi(p^\alpha)}{p^{\alpha z}} \right),
\]

\[
F_u(z, \chi) := \prod_{p|u} \left( \sum_{\alpha=0}^{\infty} \frac{a(p^\alpha)\chi(p^\alpha)}{p^{\alpha z}} \right) \prod_{p \not| u} \left( \sum_{\alpha=0}^{\infty} \frac{a(p^\alpha)\chi(p^\alpha)}{p^{\alpha z}} \right).
\]

Note that if \( u = 1 \), then \( F_u(z, \chi) \equiv 1 \). If \( u > 1 \), then it is easy to see that the Dirichlet series for \( F_u(z, \chi) \) converges absolutely for \( \Re z > 0 \), and we have the estimate

\[
F_u(z, \chi) \ll \prod_{p|u} (1 + a(p^l)p^{-1/2}) \ll d(u) \quad (\Re z \geq \frac{1}{2}, u = \prod_p p^l).
\]

By the formula \( 1 + t + 2t^2 = (1-t)^{-1}(1-t^2)^{-1}(1+O(t^3)) \) we have

\[
H(z, \chi) = \prod_p \left( 1 + \frac{\chi(p)}{p^z} + \frac{2\chi(p)^2}{p^{2z}} + \cdots \right) = \prod_p \left( 1 - \frac{\chi(p)}{p^z} \right)^{-1} \left( 1 - \frac{\chi(p)^2}{p^{2z}} \right)^{-1} \times \prod_p \left( 1 - \frac{\chi(p)}{p^z} \right) \left( 1 - \frac{\chi(p)^2}{p^{2z}} \right) \left( 1 + \frac{\chi(p)}{p^z} + \frac{2\chi(p)^2}{p^{2z}} + \cdots \right) = L(z, \chi)L(2z, \chi^2)G(z, \chi) \quad (\Re z > 1), \quad (3.6)
\]
where

\[ G(z, \chi) := \prod_p \left( 1 - \frac{\chi(p)}{p^2} \right) \left( 1 - \frac{\chi(p)^2}{p^{2z}} \right) \left( \sum_{\alpha=0}^{\infty} a(p^\alpha) \frac{\chi(p^\alpha)}{p^{\alpha z}} \right) \]

\[ = \prod_p \left( 1 + \frac{\chi(p)^3}{p^{3z}} + \cdots \right), \]

which absolutely converges for \( \Re z > 1/3 \). So from (3.3), (3.4), and (3.6) we have

\[ D_u(z, \chi) = L(z, \chi) L(2z, \chi^2) G(z, \chi) F_u(z, \chi). \]

Let \( 1 \leq T \leq X^{1000} \) be a parameter to be determined. By Lemma 1 we have

\[ M(X; \chi, u) = \frac{1}{2\pi i} \int_{1+\varepsilon+iT}^{1+\varepsilon+iT} D_u(z, \chi) \frac{X^z}{z} dz + O \left( \frac{X^{1+\varepsilon \log X}}{T} \right) \]

\[ = \frac{1}{2\pi i} \int_{1+\varepsilon-iT}^{1+\varepsilon-iT} L(z, \chi) L(2z, \chi^2) G(z, \chi) F_u(z, \chi) \frac{X^z}{z} dz \]

\[ + O \left( \frac{X^{1+\varepsilon \log X}}{T} \right). \]

Let

\[ C_1 = \left\{ z = \sigma + iT: \frac{1}{2} \leq \sigma \leq 1 + \varepsilon \right\}, \]

\[ C_2 = \left\{ z = \frac{1}{2} + it: \frac{1}{\log X} \leq t \leq T \right\}, \]

\[ C_3 = \left\{ z = \frac{e^{i\theta}}{\log X}: -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \right\}, \]

\[ C_4 = \left\{ z = \frac{1}{2} + it: -T \leq t \leq -\frac{1}{\log X} \right\}, \]

\[ C_5 = \left\{ z = \sigma - iT: \frac{1}{2} \leq \sigma \leq 1 + \varepsilon \right\}. \]

By the residue theorem we have

\[ M(X; \chi, u) = \text{Res}_{z=1} L(z, \chi) L(2z, \chi^2) G(z, \chi) F_u(z, \chi) \frac{X^z}{z} \]

\[ + \sum_{j=1}^{4} \int_{C_j} + O \left( \frac{X^{1+\varepsilon \log X}}{T} \right), \]

where

\[ \int_{C_j} := \frac{1}{2\pi i} \int_{C_j} L(z, \chi) L(2z, \chi^2) G(z, \chi) F_u(z, \chi) \frac{X^z}{z} dz. \]
If $\chi = \chi_0$ is the principal character, then we have
\[
\text{Res}_{z=1} L(z, \chi) L(2z, \chi^2) G(z, \chi) F_u(z, \chi) \frac{X^z}{z} = \frac{\varphi(r_1)}{r_1} L(2, \chi_0^2) G(1, \chi_0) F_u(1, \chi_0) X.
\]

If $\chi$ is not a principal character, then
\[
\text{Res}_{z=1} L(z, \chi) L(2z, \chi^2) G(z, \chi) F_u(z, \chi) \frac{X^z}{z} = 0.
\]

If $z = \sigma + Ti (1/2 \leq \sigma \leq 1)$, then from Lemmas 2 and 3 and from (3.5) we have
\[
L(z, \chi) L(2z, \chi^2) G(z, \chi) F_u(z, \chi) \frac{X^z}{z} \ll d(r_1) d(u) T^{-1} X^{\sigma} (r_1 T)^{(1-\sigma)/2} \log^2(r_1 T). \tag{3.8}
\]

If $z = \sigma + Ti (1 \leq \sigma \leq 1 + \varepsilon)$, then
\[
L(z, \chi) L(2z, \chi^2) G(z, \chi) F_u(z, \chi) \frac{X^z}{z} \ll d(u) T^{-1} X^{\sigma} \log^2(r_1 T). \tag{3.9}
\]

So from (3.8) and (3.9) we have
\[
\int_1 \ll \frac{X^{1+\varepsilon}}{T} + \frac{r_1^{1/4} X^{1/2+\varepsilon}}{T^{3/4}}. \tag{3.10}
\]

Similarly,
\[
\int_5 \ll \frac{X^{1+\varepsilon}}{T} + \frac{r_1^{1/4} X^{1/2+\varepsilon}}{T^{3/4}}. \tag{3.11}
\]

From (3.2), (3.7), (3.10), and (3.11) we obtain
\[
T(x, k, r) = \frac{c(r, k)}{r_1} X + \frac{1}{\varphi(r_1)} \sum_{\chi \equiv 1 \pmod{r_1}} \overline{\chi}(k_1) \left( \int_2 + \int_3 + \int_4 \right) + O \left( \frac{X^{1+\varepsilon}}{T} + \frac{r_1^{1/4} X^{1/2+\varepsilon}}{T^{3/4}} \right), \tag{3.12}
\]

where
\[
c(r, k) := L(2, \chi_0^2) G(1, \chi_0) F_u(1, \chi_0). \tag{3.13}
\]

It is easy to see that
\[
c(r, k) \ll |F_u(1, \chi_0)| \ll d(u) \ll d(r).
\]

We have
\[
\frac{1}{\varphi(r_1)} \sum_{\chi \equiv 1 \pmod{r_1}} \overline{\chi}(k_1) \left( \int_2 + \int_3 + \int_4 \right) \ll W_1 + W_2 + W_3, \tag{3.14}
\]

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where
\[
W_1 := X^{1/2}d(u) \log (r_1 T) \cdot \frac{1}{\varphi(r_1)} \sum_{\chi \pmod{r_1}} \int_{1/2}^{T} L \left( \frac{1}{2} + it, \chi \right) \frac{dt}{t},
\]
\[
W_2 := X^{1/2}d(u) \log (r_1 T) \cdot \frac{1}{\varphi(r_1)} \sum_{\chi \pmod{r_1} \frac{1}{\log X}} \int_{1/2}^{1} L \left( \frac{1}{2} + it, \chi \right) dt,
\]
\[
W_3 := X^{1/2}d(u) \log (r_1 T) \cdot \frac{1}{\varphi(r_1)} \sum_{\chi \pmod{r_1} \frac{1}{2}} \int_{0}^{\pi/2} L \left( \frac{1}{2} + e^{i \theta}, \chi \right) d\theta.
\]

From Lemma 4 by Cauchy’s inequality we have
\[
\frac{1}{\varphi(r_1)} \sum_{\chi \pmod{r_1}} \int_{1/2}^{T} L \left( \frac{1}{2} + it, \chi \right) \frac{dt}{t} \ll \log T,
\]
which implies that
\[
W_1 \ll X^{1/2}d(u) \log^{5/2} X. \tag{3.15}
\]

Similarly, from Lemma 5 we can get
\[
\frac{1}{\varphi(r_1)} \sum_{\chi \pmod{r_1} \frac{1}{\log X}} \int_{1/2}^{1} L \left( \frac{1}{2} + it, \chi \right) dt \ll \log T,
\]
which implies that
\[
W_2 \ll X^{1/2}d(u) \log^{2} X. \tag{3.16}
\]

Similarly, we have
\[
W_3 \ll X^{1/2}d(u) \log^{2} X. \tag{3.17}
\]

Inserting (3.14)–(3.17) into (3.12), we get
\[
T(x,k) = \frac{c(r,k)}{r_1} X + O \left( X^{1/2}d(u) \log^{2.5} X \right) = \frac{c(r,k)}{r} x + O \left( x^{1/2}d(r) \log^{2.5} x \right)
\]
by choosing \( T = X^{3.2} \). This completes the proof of the proposition. \( \Box \)
4 Proof of Theorem 1

Each number \( n \) can be uniquely written as \( n = q s \) so that \( q \) is square-free, \( s \) is square-full, and \( (q, s) = 1 \). It is well known that \( a(\ell) \equiv 1 \) for any square-free \( \ell \). Thus we have

\[
Q(x) = \sum_{n \leq x} a(n + a(n)) = \sum_{q \leq x} a(q s + a(s))
\]

\[
= \sum_{k \leq A(x)} \sum_{a(s) = k} \sum_{q \leq x/s} a(q s + k)
\]

\[
= \sum_{k \leq A(x)} \sum_{a(s) = k} \sum_{d^2 s \leq x} \mu(d) a\left( d^2 ns + k \right)
\]

\[
= \sum_{k \leq A(x)} \sum_{a(s) = k} \sum_{d^2 s \leq x} \mu(d) \sum_{n \leq x/d^2 s} \sum_{(n,s) = 1} a\left( d^2 ns + k \right)
\]

where \( A(x) := \max_{n \leq x} a(n) \), and in the fourth equality, we used the familiar relation \( \mu^2(n) = \sum_{d \mid n} \mu(d) \).

Suppose \( x^\varepsilon \ll y \ll x^{1/2} \) is a parameter to be determined later. We write

\[
Q(x) = Q_1(x, y) + Q_2(x, y),
\]

where

\[
Q_1(x, y) := \sum_{k \leq A(x)} \sum_{s \leq x} \sum_{a(s) = k} \sum_{d^2 s \leq y} \mu(d) \sum_{n \leq x/d^2 s} \sum_{(n,s) = 1} a\left( d^2 ns + k \right)
\]

\[
Q_2(x, y) := \sum_{k \leq A(x)} \sum_{s \leq x} \sum_{a(s) = k} \sum_{y < d^2 s \leq x} \mu(d) \sum_{n \leq x/d^2 s} \sum_{(n,s) = 1} a\left( d^2 ns + k \right).
\]

If \( d^2 s \leq y \), then it follows that \( s \leq y \) and \( k \leq A(y) \). So \( Q_1(x, y) \) can be rewritten as

\[
Q_1(x, y) = \sum_{k \leq A(y)} \sum_{s \leq y} \sum_{a(s) = k} \sum_{d^2 s \leq y} \mu(d) \sum_{n \leq x/d^2 s} \sum_{(n,s) = 1} a\left( d^2 ns + k \right).
\]

By the well-known bound \( a(n) \ll n^\varepsilon \) we have (note that \( d^2 s \) is square-full)

\[
Q_2(x, y) \ll \sum_{k \leq A(x)} \sum_{s \leq x} \sum_{a(s) = k} \left\lfloor \mu(d) \right\rfloor x^{1+\varepsilon} d^2 s \ll x^{1+\varepsilon} \sum_{y < d^2 s \leq x} \left\lfloor \mu(d) \right\rfloor d^2 s
\]

\[
\ll x^{1+\varepsilon} \sum_{y < s \leq x} \frac{1}{s} \ll \frac{x^{1+\varepsilon}}{y^{1/2}}
\]

by using partial summation with the help of the familiar bound

\[
\sum_{s \leq u} 1 \ll u^{1/2}.
\]
Now we evaluate the sum \( Q_1(x, y) \). We first consider the innermost sum in \( Q_1(x, y) \). By the elementary formula \( \sum_{d|n} \mu(d) = [1/n] \) we have

\[
\sum_{\substack{n \leq x/d^2s \quad (n, s) = 1}} a(d^2ns + k) = \sum_{\delta|s} \mu(\delta) \sum_{\delta n_1 \leq x/d^2s} a(d^2\delta n_1 s + k). \tag{4.6}
\]

From (4.3) and (4.6) we have

\[
Q_1(x, y) = \sum_{k \leq A(y)} \sum_{s \leq y} \sum_{d^2s \leq y} \mu(d) \sum_{\delta|s} \mu(\delta) \sum_{\substack{n \leq x/d^2s \quad n \equiv k(d^2s)}} a(d^2\delta ns + k) = \sum_{k \leq A(y)} \sum_{s \leq y} \sum_{d^2s \leq y} \mu(d) \sum_{\delta|s} \mu(\delta) \sum_{\substack{n \leq x+k \quad n \equiv k(d^2s)}} a(n). \tag{4.7}
\]

By Proposition 1 we have

\[
\sum_{\substack{n \leq x+k \quad n \equiv k(d^2s)}} a(n) = \frac{c(d^2\delta s, k)}{d^2\delta s} (x + k) + O(x^{1/2} \log^4 x) = \frac{c(d^2\delta s, k)}{d^2\delta s} x + O(x^{1/2} \log^4 x). \tag{4.8}
\]

Inserting (4.8) into (4.7), we get

\[
Q_1(x, y) = xJ_1(y) + O(x^{1/2} \log^4 x \times J_2(y)), \tag{4.9}
\]

where

\[
J_1(y) := \sum_{k \leq A(y)} \sum_{s \leq y} \sum_{d^2s \leq y} \mu(d) \sum_{\delta|s} \mu(\delta) \frac{c(d^2\delta s, k)}{d^2\delta s},
\]

\[
J_2(y) := \sum_{k \leq A(y)} \sum_{s \leq y} \sum_{d^2s \leq y} |\mu(d)| \sum_{\delta|s} |\mu(\delta)|.
\]

Obviously, we have \( \sum_{\delta|s} |\mu(\delta)| \leq d(s) \ll s^\varepsilon \). So by (4.5) we have that

\[
J_2(y) \ll \sum_{\substack{d^2s \leq y \quad (d, s) = 1}} |\mu(d)| \sum_{\delta|s} |\mu(\delta)| \ll y^\varepsilon \sum_{s \leq y} 1 \ll y^{1/2 + \varepsilon}. \tag{4.10}
\]

Now we consider \( J_1(y) \). We can write

\[
J_1(y) = \sum_{k \leq A(y)} \sum_{s \leq y} \frac{1}{s} \sum_{d \leq \sqrt{y/s}} \frac{\mu(d)}{d^2} \sum_{\delta|s} \frac{\mu(\delta)c(d^2\delta s, k)}{\delta} = J_{11} + O(J_{12}), \tag{4.11}
\]
where

\[ J_{11} := \sum_{k \leq A(y)} \sum_{s \leq y \atop a(s) = k} \frac{1}{s} \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} \sum_{\delta | s} \frac{\mu(\delta) c(d^2 \delta s, k)}{\delta}, \]

\[ J_{12} := \sum_{k \leq A(y)} \sum_{s \leq y \atop a(s) = k} \frac{s^\varepsilon}{s} \sum_{d > \sqrt{y/s}} \frac{d^\varepsilon}{d^2}, \]

where we used the bound \( c(r, k) \ll d(r) \ll r^\varepsilon \).

By (4.5) and partial summation we easily see that

\[ J_{12} \ll \sum_{s \leq y} s^\varepsilon \frac{1}{\sqrt{s \sqrt{y}}} \ll y^{-1/2+\varepsilon}. \]  

(4.12)

Let \( y_0 \) denote the smallest natural number not exceeding \( y \) such that \( A(y) = a(y_0) \). Then we can write

\[ J_{11} = C_1 + O(E_y), \]

(4.13)

where

\[ C_1 = \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \frac{1}{s} \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} \sum_{\delta | s} \frac{\mu(\delta) c(d^2 \delta s, k)}{\delta}, \]  

(4.14)

\[ E_y := \sum_{k > A(y)} \sum_{s > y_0 \atop a(s) = k} \frac{1}{s} \sum_{d=1}^{\infty} \frac{|\mu(d)|}{d^2} \sum_{\delta | s} \frac{|\mu(\delta)| c(d^2 \delta s, k)}{\delta}. \]  

(4.15)

From Lemma 6 we get that for any small positive constant \( \varepsilon > 0 \),

\[ (1 - 0.1 \varepsilon) \frac{\log 5}{4} \cdot \frac{\log y}{\log \log y} < \log A(y) < (1 + 0.1 \varepsilon) \frac{\log 5}{4} \cdot \frac{\log y}{\log \log y} \]

for \( y_0 > y^{1-\varepsilon} \). If \( y_0 \leq y^{1-\varepsilon} \), then

\[ \log A(y) < (1 + 0.1 \varepsilon) \frac{\log 5}{4} \frac{\log y^{1-\varepsilon}}{\log \log y} = (1 - 0.9 \varepsilon - 0.1 \varepsilon^2) \frac{\log 5}{4} \frac{\log y}{\log \log y + \log(1 - \varepsilon)}. \]

The above two formulas imply that

\[ (1 - 0.9 \varepsilon - 0.1 \varepsilon^2) \frac{\log 5}{4} \frac{\log y}{\log \log y + \log(1 - \varepsilon)} > (1 - 0.1 \varepsilon) \frac{\log 5}{4} \frac{\log y}{\log \log y}. \]

This is a contradiction if \( \varepsilon > 0 \) is small enough. So we have \( y_0 > y^{1-\varepsilon} \). This fact by (4.5) implies that

\[ E_y \ll \sum_{k > A(y)} \sum_{s > y_0 \atop a(s) = k} s^{\varepsilon} \sum_{d=1}^{\infty} \frac{d^\varepsilon}{d^2} \ll \sum_{s > y^{1-\varepsilon}} s^{\varepsilon} \ll y^{-1/2+\varepsilon}. \]

(4.16)
From (4.9)–(4.16) we get

\[ Q_1(x, y) = C_1 x + O\left( x^{1+\varepsilon} y^{-1/2} + x^{1/2+\varepsilon} y^{1/2} \right). \]  

(4.17)

Now Theorem 1 follows from (4.1), (4.4), and (4.17) by choosing \( y = x^{1/2} \).

## 5 Sketch of the proof of Theorem 2

In this section, we give a simple proof of Theorem 2. Since \( d_l(n) \) \( (l \geq 2) \) is a multiplicative function, \( Q_l(x) \) has a similar structure to \( Q(x) \). We still follow the idea of the proof of Theorem 1. As stated at the beginning of Section 3, the divisor problem in arithmetic progression is still an essential part of the proof. Define

\[ D_l(x; k, r) := \sum_{n \leq x \atop n \equiv k \mod r} d_l(n). \]

The divisor problem in the arithmetic progression is devoted to establishing an exact asymptotic formula for \( D_l(x; k, r) \). A large number of results have been obtained in many pieces of the literature; see, for example, [8, 15, 16, 24]. As is well known, we have the following form:

\[ D_l(x; k, r) = M_l(x; k, r) + \Delta_l(x; k, r). \]

The main term is expressed by Chace [3] as

\[ M_l(x; k, r) = x \sum_{n=0}^{l-1} c_{n+1}(k, r) L_n(x), \]

where

\[ L_n(x) = \sum_{j=0}^{n} (-1)^{n-j} \frac{\log^j x}{j!} \text{ and } c_n(k, r) \ll r^{-1}. \]

Additionally, the elementary bound

\[ \Delta_l(x; k, r) \ll x^{1-1/l} + r^{l-1} \]

is given in [2]. For \( r \leq x^{1/l} \), a combination of (5.1)–(5.3) implies that

\[ D_l(x; k, r) = x \sum_{n=0}^{l-1} c_{n+1}(k, r) L_n(x) + O\left( x^{1-1/l} \right). \]

(5.4)

Now we prove Theorem 2. By applying the same argument as in Section 4, we get

\[ Q_l(x) = \sum_{k \leq A(x)} \sum_{s \leq x \atop a(s) = k} \sum_{d^2 s \leq x \atop (d,s) = 1} \mu(d) \sum_{n \leq x/d^2 s \atop (n,s) = 1} d_l(d^2 ns + k). \]

Suppose \( x^\varepsilon \ll y \ll x^{1/l} \) is a parameter to be determined later. We write

\[ Q_l(x) = Q_l^{(1)}(x, y) + Q_l^{(2)}(x, y), \]

(5.5)
where

\[ Q_1^{(1)}(x, y) := \sum_{k \leq A(x)} \sum_{s \leq y} \sum_{d^2 s \leq y \atop a(s) = k} \mu(d) \sum_{n \leq x/d^2 s \atop (n, s) = 1} d_l(d^2 ns + k), \]

\[ Q_l^{(2)}(x, y) := \sum_{k \leq A(x)} \sum_{s \leq y} \sum_{y < d^2 s \leq x \atop a(s) = k} \mu(d) \sum_{n \leq x/d^2 s \atop (n, s) = 1} d_l(d^2 ns + k). \]

In view of the well-known bound \( d_l(n) \ll n^{\varepsilon} \), similarly to (4.4), we have

\[ Q_l^{(2)}(x, y) \ll \frac{x^{1+\varepsilon}}{y^{1/2}}. \tag{5.6} \]

So we only need to estimate \( Q_1^{(1)}(x, y) \). If \( d^2 s \leq y \), then it follows that \( s \leq y \) and \( k \leq A(y) \). So \( Q_1^{(1)}(x, y) \) can be rewritten as

\[ Q_1^{(1)}(x, y) = \sum_{k \leq A(y)} \sum_{s \leq y \atop a(s) = k} \sum_{d^2 s \leq y} \mu(d) \sum_{n \leq x/d^2 s \atop (n, s) = 1} d_l(d^2 ns + k). \]

Just similarly to (4.6) and (4.7), we obtain

\[ Q_1^{(1)}(x, y) = \sum_{k \leq A(y)} \sum_{s \leq y \atop a(s) = k} \sum_{d^2 s \leq y} \mu(d) \sum_{\delta \mid s} \mu(\delta) \sum_{n \leq x+k \atop n \equiv k(d^2 \delta s)} d_l(n). \tag{5.7} \]

Thus from (5.4) and (5.7) we have

\[
Q_1^{(1)}(x, y) = x \sum_{k \leq A(y)} \sum_{s \leq y \atop a(s) = k} \sum_{d^2 s \leq y} \mu(d) \sum_{\delta \mid s} \mu(\delta) \sum_{n=0}^{l-1} c_{n+1}(k, d^2 \delta s) L_n(x)
\]

\[ + O \left(x^{1-1/l} \sum_{k \leq A(y)} \sum_{s \leq y \atop a(s) = k} \sum_{d^2 s \leq y} \mu(d) \sum_{\delta \mid s} \mu(\delta) \right) \]

\[ = xQ_l(\log x) + O(x^{1+\varepsilon} y^{1/2}) + O(x^{1-1/l+\varepsilon} y^{1/2}) \]

\[ = xQ_l(\log x) + O(x^{1-1/(2l)+\varepsilon}) \tag{5.8} \]

for \( y = x^{1/l} \) with

\[ Q_l(\log x) = \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \sum_{d=1}^{\infty} \mu(d) \sum_{\delta \mid s} \mu(\delta) \sum_{n=0}^{l-1} c_{n+1}(k, d^2 \delta s) L_n(x). \]

Combing (5.5), (5.6), and (5.8) yields

\[ Q_l(x) = xQ_l(\log x) + O(x^{1-1/(2l)+\varepsilon}). \]

This completes the proof of Theorem 2.
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