Abstract

We present the results of an extensive study of the effect of a filter upon the performance of a resistive SQUID noise thermometer used to define an absolute temperature scale below 1 K. Agreement between the model for this effect and the experimental results indicates that the temperature scale defined by this noise thermometer is accurate to 0.1%.

Introduction

At the NIST we have been developing a temperature scale below 1 K. The absolute temperature is defined by a Josephson-junction noise thermometer (or, R-SQUID) which consists of a resistor R at temperature T connected in parallel with a Josephson junction. The bandwidth of the electronic system used to measure the noise is limited by a filter. The influence of this filter on the noise will be discussed as well as an assessment of the overall performance of the noise thermometer.

Theory

The purpose of the Josephson junction is to convert voltages V, generated across R, into frequencies, \( f = V / (\hbar/2e) = V/\phi_0 \), where \( \phi_0 \) is the flux quantum. There are two contributions to the total voltage drop across the resistor. The first is a dc voltage generated by a dc current \( b \) which is introduced into the circuit. This bias causes the Josephson junction to oscillate at a frequency \( f_0 = bR/\phi_0 \). The second consists of voltage fluctuations generated by the Johnson noise in the resistor. These are converted by the Josephson junction into frequency fluctuations which are superimposed on the carrier frequency \( f_0 \). This signal from the R-SQUID is processed by a circuit comprised of several amplifiers in series, a band pass filter tuned to \( f_0 \), and a frequency counter. Connected to the output of the counter is a computer programmed to calculate the variance \( \sigma_f^2 \) of the frequency, defined by the equation

\[
\sigma_f^2 = \frac{1}{N} \sum_{i=1}^{N} (f_i - \bar{f})^2
\]

where \( f_i \) is an individual frequency reading, and \( \bar{f} \) is the average frequency for N frequency counts. The measurement time for each count is \( t \). The statistical uncertainty of the variance is \( (2N)^{-1/2} \).

The theoretical expression relating the measured variance to the spectral power density, \( S(\omega) \), and the filter characteristic, \( A(\omega) \), is

\[
\sigma_f^2 = \frac{1}{t} \left[ 1 - \frac{1}{\tau} \right] \left[ \frac{1}{C} - \frac{1}{\phi_0} \right] \int_{-\infty}^{\infty} A(\omega) S(\omega) d\omega
\]

Several possible variances may be defined, depending on N. The Allen variance, \( \sigma_f^2(2,\tau) \), for N=2, is most amenable to on-line computer processing and is less sensitive to long-term drifts in \( f_0 \). The more familiar definition (limit as N approaches infinity) is easier to treat theoretically, but is more difficult to compute on line. We have shown elsewhere that the two may be considered equivalent for the circumstances prevailing in our experiments. As standard practice, we measured the Allen variance as well as an approximation for the variance in the large N limit. Unless specifically noted, the variance referred to for the remainder of this article is \( \sigma_f^2(2,\tau) \).

In order to integrate Eqn. 2, we need to specify \( S(\omega) \) and \( A(\omega) \). We assume that the spectral power density has two components: the first is \( S_T \) due to Johnson noise generated by R, and the second is \( S_A \) due to amplitude noise generated by the first stage of amplification. Their sum is given as

\[
S(\omega) = S_T + \omega^2 S_A = 4k_BRT + \omega^2 S_A
\]

We have previously studied the influence of this filter on the variance and compared the results with these equations, but since then we made several changes which allowed the much more stringent tests summarized here. The improvements are listed as follows. (1) \( C \) is now given in Eqn. 6b in terms of the measured signal-to-noise voltage ratio, \( V_s/V_N \) whereas previously it was a fitted parameter. (2) We varied the temperature by two orders of magnitude. (3) We reduced the statistical uncertainty of the variance from 1% to 0.1% (this requires that N=2x10^6) by reducing the measurement time. (4) We improved the computer program so that many more values of \( f_0 \) could be taken (previously we were restricted to decade values of \( T \)). All of these factors promote a much more rigorous test of the model, enhance our understanding of the behavior of the circuit, and lead to a reduced uncertainty ascribed to the temperatures derived from the measurements.

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Experiments

Since the variance depends critically on the character of the filter, we carefully measured its frequency response before proceeding to use it to measure the noise from the R-SQUID. This was done by driving the input of the filter with a frequency-synthesized, constant-amplitude oscillator and registering the output voltage with a digital voltmeter while the frequency was varied. $A(o)$ is defined as the square of the output voltage. A comparison of such data and the fit to Eqn. 4 is shown in Fig. 1. We see that the agreement is very good when the frequency lies within $\pm 40$ kHz of $f_o$, while it deteriorates progressively as the frequency is increased outside this range. Inspection of the Eqn. 2 leads us to conclude, however, that the observed deviation in $A(o)$ will only cause a deviation in the variance given by Eqn. 5 for gate times shorter than the inverse of the frequency limit where $A(o)$ deviates from Eqn. 4, i.e., for $\tau < 25\mu s$ (i.e., $\tau = (40kHz)^{-1}$).

This filter was then used to measure the variance generated by the R-SQUID. This was accomplished by attaching the R-SQUID to a $^{3}$He-$^{4}$He dilution refrigerator which was enclosed in a carefully shielded environment. The system was cooled, the refrigerator was stabilized at a chosen temperature, and the computer program was activated to measure the variances: $\sigma^2(2,\tau)$ and $\sigma^2(N,\tau)$. Each temperature was maintained for a sufficiently long time to permit the measurement of the variance with an imprecision of 0.1% at each of several values of $\tau$ ranging from 12us to 10ms. This group of variances and measurement times taken at a single temperature was defined as a data set which was assigned a four-digit identification number. Several auxiliary measurements were made at each temperature as well. The average frequency and $I_o$ were measured in order to calculate $R$ from Eqn. 1. The signal-to-noise voltage ratio, $\xi = \langle V_n^2\rangle / \langle V_s^2\rangle$, was measured in order to calculate $C$ from Eqn. 6b.

The cryostat was cooled several times over the past several years, and the noise was measured in most of them, but the most extensive studies of $\sigma^2(2,\tau)$ and $\sigma^2(N,\tau)$ were taken during Runs 50 and 53. In Run 50 over 80 complete data sets were taken, at temperatures from 8mK to 520mK. Run 53 was similar: over 60 data sets were taken over a somewhat larger temperature range (6.3mK to 520 mK). In addition to providing data to test the model for the filter, data sets also were taken to study the reproducibility of the R-SQUID during the same run and between runs to better assess the reproducibility of the noise temperature scale.

Data Analysis and Interpretation

Some data sets from Run 53 are shown in Fig. 2 to give examples of the variance as a function of $\tau$ and $T$. To emphasize the fact that the variance approaches the limit $\sigma^2 = 0$ for long $\tau$, we plot $\sigma^2(2,\tau)^{1/2}$ rather than $\sigma^2(2,\tau)$ versus $\tau$. Eqn. 5 was fitted to the data shown in Fig. 2 to obtain values for the parameters $\omega_0$, $\omega_1$ and $C$. The absolute temperature was calculated from the first parameter using Eqn. 6a (the value of $R$ was calculated with an inaccuracy of 10ppm using the measured value for $I_o$ and $C$).

The quality of the fit for one of the data sets shown in Fig. 2 may be seen in the next figure where the percentage difference between the data and fit is displayed.
We can see from the data in Fig. 3 that 50% of the points lie within 1-$\sigma$ of the average, 92% lie within 2-$\sigma$, and all 12 points lie within 3-$\sigma$. We interpret this as indicating that the fit of Eqn. 5 to this particular data set was quite consistent with the 0.1% statistical measurement imprecision of the variance. Furthermore, the agreement between the fitted and calculated values for $\omega_1$ is within the estimated uncertainty. The agreement between the fitted and calculated values of $C$ is rather poor (37%), however, since it exceeds our estimate of the leading measurement uncertainty in $C$—the measurement error in the voltage signal-to-noise voltage ratio (5%).

Analysis identical to that described above was conducted for all data sets to obtain values for the parameters $\sigma^2_1$, $T$, $\omega_1$, and $C$. The fitted values for $\omega_1$ were compared to that obtained from the fit of $A(\omega)$ to Eqn. 4. Furthermore, the quantity $V_s^2/V_\lambda^2$ was measured independently in order to calculate a value for $C$ (Eqn. 6b) which was compared with the one obtained by fitting. Summarizing these comparisons, we found that the differences between fitted and calculated values for these three parameters were hierarchical: Discrepancies were the largest for $C$ (5-50%), were smaller for $\omega_1$ (1-5%), and were still less for $\sigma^2_1$ (0.05-0.2%). The overall fits of Eqn. 5 to the data were often as good as the statistical imprecision (viz. Fig. 3), but deviations in excess of 0.2% were occasionally found for variances at short $\tau$ (sometimes 1% for some of the 12 ps and 24 ps points). As was discussed above, deviations are to be expected for these gate times due to the deviation of $A(\omega)$ from ideal Butterworth behavior. A more accurate model expected to encompass even these data points is under development.

We comment on the relevance of these observations on use of the R-SQUID as a noise thermometer. The fitted values of $\sigma^2_2$ (and thus, $T$) are largely determined by the $\sigma^2(2,\tau)$ data at large $\tau$, whereas $\omega_1$ and $C$ are determined by the variation of $\sigma^2(2,\tau)$ for small $\tau$. Consequently, $T$ was found to be very insensitive (less than 0.1%) to fitting even though the other parameters varied by much larger amounts.

The computer program used to accumulate the Allen variance also included an algorithm to calculate an approximation to $\sigma^2(N,\tau)$ data at large $\tau$. Consider first the reproducibility of data taken with the R-SQUID. Consider first the reproducibility of data taken with the R-SQUID. We turn next to the issue of reproducibility of data taken with the R-SQUID. Consider first the reproducibility of data taken with the R-SQUID. The results are shown in Table II.

### Table II. Reproducibility of R-SQUID noise thermometer at the $T_c$ of AuIn$_2$

| Data Set | $T$ (mK) | $\delta T$ (mK) |
|----------|----------|-----------------|
| 3406A    | 204.386  | 0.204           |
| 3406C    | 204.836  | 0.222           |
| 3406D    | 204.954  | 0.219           |
| 3417     | 204.490  | 0.133           |
| 3419A    | 204.265  | 0.210           |
| 3419B    | 204.804  | 0.143           |
| 3421A    | 204.455  | 0.167           |
| 3421B    | 204.435  | 0.167           |
| 3423A    | 204.522  | 0.167           |
| 3424A    | 204.763  | 0.167           |
| 3424B    | 204.663  | 0.200           |
| 3426     | 204.678  | 0.176           |
| 3428A    | 204.439  | 0.167           |
| 3428B    | 204.463  | 0.167           |
| 3434A    | 204.826  | 0.199           |
| 3434B    | 204.463  | 0.125           |
| 3438     | 204.558  | 0.130           |
| 3452A    | 204.296  | 0.538           |
| 3452B    | 204.814  | 0.175           |
| 3452C    | 204.882  | 0.170           |
| 3452G    | 204.481  | 0.143           |
| 3452H    | 204.256  | 0.197           |
| 3452I    | 204.601  | 0.143           |
| 3452G    | 204.127  | 0.285           |
| 3453H    | 204.593  | 0.144           |

|= | | |

**MEAN** | 204.560 | 0.179 |

**std error** | 0.044 | 0.000 |

An example of our data is shown below in Fig. 4. We see that the variance does converge to within 0.1% to a limiting value which is independent of N. This is precisely what is expected for a white noise spectrum generated solely by a resistor. Clearly, if another, non-white spectrum were present, the divergence of $\sigma^2(N,\tau)$ at large value of $N$ would be very apparent. Using the equations given in Table I, we can use these observations to calculate limits on the maximum amplitude of the spectral power density of extraneous noise present.
Each data set shown in Table I was taken over a 12 hour period, and each set is separated by a 12-hour period. The temperature $T$ was calculated from Eqn. 6a using the measured values of $\sigma^2$ and $R$, while the statistical standard deviation of the temperature expected for $N=2\times10^6$ is 8T. The mean temperature for the 25 measurements is 204.56 mK, and the mean standard deviation is 0.179 mK. The latter is in good agreement with individual values of 8T. The standard error of the ensemble of 25 measurements is found to be 44mK, which is in excellent agreement with the calculated value $(0.179/25)^{1/2}=36\mu$K). We take this as good indication that the noise observed by the system is controlled by a white noise source which is invariant over at least a 25 day period. We also interpret the data as supporting the assertion that the statistical imprecision may be reduced to even less than 0.1%.

We next comment on the reproducibility of the temperatures measured by the R-SQUID noise thermometer between experiments. We have two sets of data to explore this. The first is a comparison of the noise thermometer with the temperatures defined by the $T_c$ of a superconductive fixed point device

We have two sets of data to explore this. The first is a comparison of the noise thermometer with the temperatures defined by the $T_c$ of a superconductive fixed point device. Data is available which covers a seven-year period, and it tests the combined reproducibilities of the R-SQUID and fixed point device. These data for the comparison of the R-SQUID and $T_c$ device are not shown here, but they do indicate that the reproducibility is 0.3% at 15 mK, and 0.1% at 204 mK. The second is a comparison defined by a He melting curve thermometer and the R-SQUID. This only covers a two-year period, but the reproducibility of the R-SQUID from 8 mK to 520 mK for these two experiments separated by one year is generally better than 0.1%. These results are shown below in Table III.

### Table III. Reproducibility of an R-SQUID Noise Thermometer versus a He Melting Curve Thermometer

| $T_N$ (mK) | $T_N$ (mK) | $\Delta T$ (%) |
|-----------|-----------|----------------|
| Run 50    | Run 53    |                |
| 8.149     | 8.157     | -0.098         |
| 9.908     | 9.915     | -0.071         |
| 13.062    | 13.069    | -0.054         |
| 15.498    | 15.502    | -0.026         |
| 22.769    | 22.771    | -0.009         |
| 43.585    | 43.530    | +0.126         |
| 71.262    | 71.183    | +0.111         |
| 99.568    | 99.507    | +0.061         |
| 160.35    | 160.30    | +0.031         |
| 204.65    | 204.60    | +0.024         |
| 520.55    | 519.97    | +0.112         |

**Conclusion**

We have measured the effect of a bandpass filter on the noise of an R-SQUID. Comparison of the data with a model for the effect indicates good agreement to within the measurement imprecision, although the fitted parameters show some discrepancies. Fortunately the observed discrepancies influence the temperatures derived from the data by less than 0.1%. Additional studies indicate that the observed noise spectrum is white and uncontaminated to the level of 0.1% by extraneous noise. Finally, the reproducibility of repeated measurements of the noise within an experimental run is at least 0.1%, while the reproducibility between runs as judged by superconducting fixed points and a He melting curve thermometer are no worse than 0.3%. We conclude that a R-SQUID noise thermometer is capable of producing a temperature scale from 6.3 mK to 520 mK which is accurate to 0.1%.

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