Recursion for Masur-Veech Volumes of Moduli Spaces of Quadratic Differentials

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Abstract We derive a quadratic recursion relation for the linear Hodge integrals of the form \(\langle \tau_{m}^2 \lambda_k \rangle\). These numbers are used in a formula for Masur-Veech volumes of moduli spaces of quadratic differentials discovered by Chen, Möller and Sauvaget. Therefore, our recursion provides an efficient way of computing these volumes.

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1. The recursion

We use Witten’s notation for linear Hodge integrals over the moduli space \(\overline{M}_{g,n}\) of stable genus \(g\) curves with \(n\) marked points,

\[ \langle \tau_{i_1} \ldots \tau_{i_n} \lambda_k \rangle = \int_{\overline{M}_{g,n}} \lambda_k \psi_{i_1}^{n_1} \ldots \psi_{i_n}^{n_n}, \]

where \(\psi_i\) and \(\lambda_k\) are the corresponding standard characteristic classes and \(g\) is recovered from dimension count \(3g - 3 + n = k + \sum i_j\). In this note we derive a quadratic recurrent relation for the numbers of the form \(\langle \tau_{2}^m \lambda_k \rangle\). The numbers of this sort are involved in a formula for Masur-Veech volume of the principal stratum of moduli space of quadratic meromorphic differentials derived in [5]. Namely, with the above notation this formula reads

\[
V_{g,n} = \text{vol}(Q_{g,4g-4+2n}(1^{4g-4+n}, -1^n))
\]

\[= 2^{2g+1} \pi^{6g-6+2n} \frac{(4g-4+n)!}{(6g-7+2n)!} \sum_{k=0}^{g} \frac{(\tau_0^n \tau_2^{3g-3+n-k} \lambda_k)}{(3g-3+n-k)!} \]

\[= 2^{2g+1} \pi^{6g-6+2n} \frac{(4g-4+n)!}{(6g-7+2n)!} \sum_{k=0}^{g} \frac{(5g-7+2n-k)!! \tau_2^{3g-3-k} \lambda_k)}{(5g-7-k)!! (3g-3-k)!} \]
The second equality holds for \( g \geq 2 \) and follows from the first one by the string and dilaton relations allowing one to eliminate \( \tau_0 \) and \( \tau_1 \) from correlators

\[
\langle \tau_0^{m+1} \tau_2^{n+1} \lambda_k \rangle = (n+1) \langle \tau_1 \tau_0^m \tau_2^n \lambda_k \rangle = (n+1)(2g-2+m+n) \langle \tau_0^m \tau_2^n \lambda_k \rangle,
\]

where \( n-m = 3g-3-k \). There are some corrections in the string and dilaton relations in the cases \( g = 0 \) and \( g = 1 \) due to absence of the moduli spaces \( \overline{\mathcal{M}}_{0, \leq 2} \) and \( \overline{\mathcal{M}}_{1,0} \). Fortunately, the corresponding intersection numbers for \( g \leq 1 \) are easy to compute explicitly: By the string and dilaton they are reduced to just \( \langle \tau_0^3 \rangle = 1 \) and \( \langle \tau_1 \rangle = \langle \tau_0 \lambda_1 \rangle = \frac{1}{2\pi} \) and we recover the known values (cf. [3] for the case \( g = 0 \) and [5] for the case \( g = 1 \))

\[
V_{0,n} = \frac{\pi^{2n-6}}{2^{n-5}}, \quad V_{1,n} = \frac{\pi^{2n} n!}{3(2n-1)!}((2n-3)!! + (2n-2)!!).
\]

The recursion presented below for the numbers \( \langle \tau_2^m \lambda_k \rangle \) looks somewhat nicer with the following normalization. Denote

\[
c_{g,k} = \frac{\langle \tau_0^2 \tau_2^{3g-1-k} \lambda_k \rangle}{(3g-1-k)!} = (5g-3-k)(5g-5-k) \langle \lambda_k \tau_2^{3g-3-k} \rangle/(3g-3-k)!.\]

These numbers are nonzero only if \( g \geq 1 \) and \( 0 \leq k \leq g \).

**Theorem 1.** We have \( c_{1,0} = \frac{1}{12} \) and for all other pairs \((g,k)\) the following recursion relation holds:

\[
c_{g,k} = \frac{g+1-k}{5g-2-k} c_{g,k-1} + \frac{(5g-6-k)(5g-4-k)}{12} c_{g-1,k} + \frac{1}{2} \sum_{\substack{q_1+q_2=g \\ k_1+k_2=k}} c_{q_1,k_1} c_{q_2,k_2}.
\]

The computer allows one to find the constants \( c_{g,k} \) by this recursion and, thus, the volume

\[
V_{g,0} = \frac{2^{g+1} \pi^{6g-6}(4g-4)!}{(6g-7)!} \sum_{k=0}^{g} \frac{c_{g,k}}{(5g-3-k)(5g-5-k)}
\]

up to, say, \( g = 100 \) in just few seconds, which helps to study experimentally the large genus asymptotics of volumes. For example, one can observe numerically that the ratio between the exact value of \( V_{g,0} \) and its conjectural asymptotic value

\[
V_{g,0} \approx \frac{4}{\pi} \left( \frac{8}{3} \right)^{4g-4}
\]

for \( g = 100 \) is equal to 0.9993, supporting a conjecture formulated in [2] and [6] and proved in [1].

Our proof of the recursion is based on KP hierarchy for linear Hodge integrals [8]. Another integrable hierarchy of partial differential equations for Hodge integrals, namely, the intermediate long-wave hierarchy, is discovered in [4]. Based on the equations of that hierarchy, a number of other efficient recursions for Masur-Veech volumes are derived in [12]. The recursions of [12] have an advantage in that they are formulated in terms of the volumes themselves rather than single Hodge integrals.
2. Proof of Theorem 1

To simplify the arguments, consider first the case $k = 0$; that is, we first compute the numbers

$$c_g = c_{g,0} = \frac{\langle \tau_0^2 \tau_2^{3g-1} \rangle}{(3g-1)!} = (5g-3)(5g-5)\frac{\langle \tau_2^{3g-3} \rangle}{(3g-3)!}. $$

(The rightmost equality holds for $g > 1$ only, whereas for $g = 1$ we have $c_1 = c_{1,0} = \frac{1}{12}$. ) Consider the Kontsevich-Witten potential, which is the generating function for $\tau$-correlators

$$F(h; t_0, t_1, \ldots) = \sum_{g,n} \hbar^g \sum_{i_1 + \cdots + i_n = 3g-3+n} \langle \tau_{i_1} \cdots \tau_{i_n} \rangle \frac{t_{i_1} \cdots t_{i_n}}{n!}.$$ 

By the Kontsevich-Witten theorem, it satisfies the equations of Korteweg-de Vries (KdV) hierarchy, in particular, the KdV equation itself (see [11], [9], [7]),

$$\frac{\partial^2 F}{\partial t_0 \partial t_1} = \frac{1}{2} \left( \frac{\partial^2 F}{\partial t_0^2} \right)^2 + \hbar \frac{\partial^4 F}{\partial t_0^4}. $$

Let us restrict this equation to the point $(t_0, t_1, t_2, \ldots) = (0, 0, 1, 0, \ldots)$. Namely, set

$$f_{k_1 \ldots k_n} = \left. \frac{\partial^n F}{\partial t_{k_1} \cdots \partial t_{k_n}} \right|_{t_i = \delta_{i,2}}.$$ 

Then we have

$$f_{01} = \frac{1}{2} f_{00}^2 + \frac{\hbar}{12} f_{0000}. $$

(1)

Now we observe that the coefficients of the series involved in this equation contain only intersection numbers of the form $\langle \tau_0^i \tau_j^2 \tau_2^p \rangle$ with small $i$ and $j$ and, hence, they can be expressed in terms of $c_g$. Explicitly, applying repeatedly the string and dilaton relations, we get

$$f_{00} = \sum_{g \geq 1} \frac{\langle \tau_0^2 \tau_2^{3g-1} \rangle}{(3g-1)!} \hbar^g = \sum_{g \geq 1} c_g \hbar^g, $$

$$f_{01} = \sum_{g \geq 1} \frac{\langle \tau_0 \tau_1 \tau_2^{3g-2} \rangle}{(3g-2)!} \hbar^g = \sum_{g \geq 1} \frac{\langle \tau_0^2 \tau_2^{3g-1} \rangle}{(3g-1)!} \hbar^g = f_{0,0}. $$

$$\hbar f_{0000} = \sum_{g \geq 1} \frac{\langle \tau_0^4 \tau_2^{3g-2} \rangle}{(3g-2)!} \hbar^g = \sum_{g \geq 2} (5g-4)(5g-6)\frac{\langle \tau_0^2 \tau_2^{3g-4} \rangle}{(3g-4)!} \hbar^g + \hbar$$

$$= \sum_{g \geq 2} g(5g-4)(5g-6)c_{g-1} \hbar^g + \hbar = (5G-4)(5G-6)(\hbar f_{0,0}) + \hbar, $$

where $G = \frac{\hbar^2}{36}$. Substituting in the KdV equation (1), we get an equation on the generating series $f_{00}$ for the numbers $c_g$,

$$f_{00} = \frac{(5G-4)(5G-6)}{12} (\hbar f_{00}) + \frac{1}{2} f_{00}^2 + \frac{\hbar}{12}. $$
which is equivalent to the recursion
\[ c_g = \frac{(5g-4)(5g-6)}{12} c_{g-1} + \frac{1}{2} \sum_{g_1+g_2=g} c_{g_1} c_{g_2}. \]
(See [7], [10], [13] where this recursion is derived by essentially the same arguments.)

In the general case, we consider similarly the generating function for linear Hodge integrals,
\[ H(h, s; t_0, t_1, \ldots) = \sum_{g,n} \sum_{i_1+\ldots+i_n+k=3g-3+n} h^g \langle \tau_{i_1} \ldots \tau_{i_n} \lambda_k \rangle s^k \frac{t_1 \ldots t_n}{n!}. \]
Set \( s = -\epsilon^2 \) and consider a sequence of linear functions \( T_k(p_1, p_2, \ldots) \) defined recursively by
\[ T_0 = p_1, \quad T_{k+1} = \sum_m m \left( p_{m+2} + \epsilon p_{m+1} \right) \frac{\partial T_k}{\partial p_m}. \]
We have
\[ T_1 = \epsilon p_2 + p_3, \quad T_2 = 2\epsilon^2 p_3 + 5\epsilon p_4 + 3 p_5, \quad T_3 = 6\epsilon^3 p_4 + 26\epsilon^2 p_5 + 35\epsilon p_6 + 15 p_7, \quad T_4 = 24\epsilon^4 p_5 + 154\epsilon^3 p_6 + 340\epsilon^2 p_7 + 315\epsilon p_8 + 105 p_9, \]
and so on. The substitution \( t_k = T_k(p) \) is a linear triangular invertible (for \( \epsilon \neq 0 \)) change of variables.

**Theorem 2** ([8]). The function \( H \) written in \( p \)-coordinates satisfies equations of KP hierarchy (for any value of \( \epsilon \)).

**Remark 3.** We used this fact in [8] to provide one of the shortest proofs of the Kontsevich-Witten theorem. Indeed, setting \( \epsilon = 0 \) we get \( T_k|_{\epsilon=0} = (2k-1)!!p_{2k+1} \). Therefore, \( F = H \big|_{\epsilon=0} \) is a solution of the KP hierarchy. Moreover, this solution is independent of even times \( p_{2k} \); that is, it is a solution of the KdV hierarchy.

The first equation of the KP hierarchy is the KP equation itself,
\[ \frac{\partial^2 H}{\partial p_1 \partial p_3} = \frac{\partial^2 H}{\partial p_2^2} + \frac{h}{12} \frac{\partial^4 H}{\partial p_1^4} + \frac{1}{2} \left( \frac{\partial^2 H}{\partial p_1^2} \right)^2. \]
By the change of coordinates described in Theorem 2 we have
\[ \frac{\partial H}{\partial p_1} = \frac{\partial H}{\partial t_0}, \quad \frac{\partial H}{\partial p_2} = \frac{1}{2} \frac{\partial^2 H}{\partial p_0^2} - \epsilon \frac{\partial H}{\partial t_1}, \quad \frac{\partial H}{\partial p_3} = \frac{\partial H}{\partial t_1} - 2\epsilon^2 \frac{\partial H}{\partial t_2}. \]
Therefore, the KP equation takes in \( t \)-coordinates the following form (recall that \( s = -\epsilon^2 \)):
\[ \frac{\partial^2 H}{\partial t_0 \partial t_1} = 2s \frac{\partial^2 H}{\partial t_0 \partial t_2} - s \frac{\partial^2 H}{\partial t_1^2} + \frac{h}{12} \frac{\partial^4 H}{\partial t_0^4} + \frac{1}{2} \left( \frac{\partial^2 H}{\partial t_0^2} \right). \]
Let us restrict the KP equation to \( t = (0, 0, 1, 0, \ldots) \). Setting

\[
h_{k_1 \ldots k_n} = \left. \frac{\partial^n H}{\partial t_{k_1} \cdots \partial t_{k_n}} \right|_{t_i = \delta_{i,2}}
\]

we get

\[
h_{01} = 2s h_{02} - s h_{11} + \frac{h}{12} h_{0000} + \frac{1}{2} h_{00}^2.
\] (2)

The coefficients of all involved series include only intersection numbers of the form \( \langle \tau_0^i \tau_1^j \tau_2^k \lambda_0 \rangle \), with small \( i \) and \( j \) and, hence, they can be expressed in terms of \( c_{g,k} \). Applying repeatedly the string and dilaton relations, we get

\[
h_{00} = \sum_{g,k} \frac{(\tau_0^2 \tau_2^3 \lambda_0^{1-k})}{(3g-1-k)!} h^g s^k = \sum_{g,k} c_{g,k} h^g s^k,
\]

\[
h_{01} = \sum_{g,k} \frac{(\tau_0^2 \tau_2^3 \lambda_0^{1-k})}{(3g-1-k)!} h^g s^k = \sum_{g,k} \frac{(\tau_0^2 \tau_2^3 \lambda_0^{1-k})}{(3g-1-k)!} h^g s^k = h_{00,0}.
\]

\[
h_{0,2} = \sum_{g,k} \frac{(\tau_0 \tau_1^3 \lambda_0^{1-k})}{(3g-2-k)!} h^g s^k = \sum_{g,k} \frac{(3g-1-k)}{5g-2-k} c_{g,k-1} h^g s^k.
\]

\[
h_{1,1} = \sum_{g,k} \frac{(\tau_0 \tau_1^3 \lambda_0^{1-k})}{(3g-2-k)!} h^g s^k = \sum_{g,k} \frac{(5g-3-k)}{5g-2-k} h^g s^k.
\]

\[
h_{0,0,0,0} = \sum_{g,k} \frac{(\tau_0^4 \tau_2^3 \lambda_0^{1-k})}{(3g-2-k)!} h^g s^k = \sum_{g,k} (5g-4-k)(5g-6-k) c_{g-1,k} h^g s^k + h.
\]

Introduce commuting operators \( G = \frac{\partial}{\partial h}, \ K = \frac{\partial}{\partial s} \). Any operator of the form \( P(G, K) \), where \( P \) is a polynomial, acts on a series in \( h \) and \( s \) multiplying any monomial of the form \( h^g s^k \) by \( P(g, k) \). With these notations, the KP equation (2) takes the form of differential equation on the generating function \( h_{0,0} \) for the coefficients \( c_{g,k} \):

\[
h_{0,0} = (G - K + 1)(s h_0) + \frac{(5G - K - 4)(5G - K - 6)}{12} (h h_{0,0}) + \frac{1}{2} h_{0,0}^2 + \frac{h}{12},
\]

\[
s h_0 = (5G - K - 2)^{-1}(s h_{0,0}).
\]

Taking the coefficient of \( h^g s^k \) we get exactly the relation of Theorem 1.

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