Controlled onset of low-velocity collisions in a vibro-impacting system with friction

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This paper investigates the onset of low-velocity, near-grazing collisions in an example vibro-impacting system with dry friction with particular emphasis on feedback control strategies that regulate the grazing-induced bifurcation behaviour. The example system is characterized by a twofold degeneracy of grazing contact along an extremal stick solution that is shown to result in a locally one-dimensional and piecewise-linear description of the near-grazing dynamics. Explicit control strategies are derived that ensure a persistent, low-impact-velocity, steady-state response across the critical parameter value corresponding to grazing contact even in instances where the dynamics in the absence of control exhibit a sudden transition to a high-impact-velocity response.

Keywords: hybrid dynamical systems; grazing bifurcations; stick-slip solutions; feedback

1. Introduction

It is well documented that the onset of low-velocity impacts in a vibro-impacting mechanical system introduces instabilities in the system dynamics that may cause discontinuous transitions between distinct steady-state responses (Shaw 1985a, b; Nordmark 1991, 1992, 1997, 2001; Chin et al. 1994; Foale & Bishop 1994; Fredriksson & Nordmark 1997; Fredriksson et al. 1999; de Weger et al. 2000; Dankowicz et al. 2001; Molenaar et al. 2001; Dankowicz & Zhao 2005; Thota & Dankowicz 2006; Zhao & Dankowicz 2006b). Specifically, this occurs when a parameter of the system is varied across some critical value at which parts of the system, undergoing periodic motion, come into zero-relative-velocity, grazing, contact. In contrast to smooth systems in the absence of impacts, the associated loss of structural stability is sudden and unanticipated.

In this work, the source and nature of the bifurcation behaviour past such a point of grazing contact is investigated in an example vibro-impacting system with dry friction, along with a proposed feedback formulation to regulate this behaviour (see Batako & Piirainen (2008), Pavlovskaja et al. (2001) and Virgin & Begley (1999) for other studies of systems with impacts and friction). The analysis is distinct from those referenced above in that the grazing trajectory here lies on the boundary of an embedded submanifold that is invariant and globally

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attracting in the absence of impacts. In particular, the analysis of the bifurcation behaviour in the absence of feedback extends that in Svahn & Dankowicz (2008) by providing a complete unfolding in an open region of parameter space.

Low-cost control strategies for preventing a loss of structural stability owing to grazing in vibro-impacting systems without dry friction have been successfully developed by the present authors in a series of publications (Dankowicz & Jerrelind 2005; Zhao & Dankowicz 2006a; Dankowicz & Svahn 2007; see also Misra & Dankowicz (submitted) for an application to compliant impacts). As shown here, the present system again permits the formulation of feedback control strategies that regulate the grazing-induced bifurcation scenarios. Suitably designed, the application of such control results in a persistent attractor locally to the critical parameter value, consistent with steady-state dynamics with sustained low-velocity impacts.

The paper is organized as follows. In §2, the example mechanical system is described in a hybrid system formulation along with a qualitative and numerical characterization of its dynamics. Under a set of open conditions on the system parameters, §3 derives a discrete map that captures the dynamics of the open-loop system, and proceeds to analyse the existence and stability of periodic points in terms of system parameters. Linear and nonlinear feedback control strategies are then presented in §4, which rely on the discrete description of the system dynamics for ensuring a persistent attractor across the condition of grazing contact. The paper concludes with a discussion of the results and an appendix that motivates the delimitations of the analysis.

2. A vibro-impacting system with friction

(a) Mechanical model

Of concern in this work is the one-degree-of-freedom, idealized vibro-impact mechanism in the presence of dry friction shown in figure 1. Here, a movable object of mass $m$ is connected by a linear spring of stiffness $m\omega_0^2$ to a rigid
frame and pushed against a rough surface. The lateral motion of the object is limited by a harmonically oscillating unilateral constraint, referred to below as the impactor.

Denote by \( q \) the signed displacement of the object relative to some reference position, such that the spring is unstretched when \( q = 0 \) and is in compression for \( q > 0 \). Let

\[
q_c(t) = -b + a \sin \omega t
\]

(2.1)

for \( a, b, \omega > 0 \) describe the position of the impactor, such that \( q - q_c \) equals the gap between the movable object and the impactor. Then, as long as \( q - q_c(t) > 0 \), the forward-in-time dynamics of the movable object are governed by the equation of motion

\[
\ddot{q} + \omega_0^2 q = \frac{F}{m},
\]

(2.2)

where

\[
F = \begin{cases} 
-F_f & \text{when } \dot{q} > 0 \text{ or when } \dot{q} = 0 \text{ and } q > \frac{F_f}{m \omega_0^2}, \\
F_f & \text{when } \dot{q} < 0 \text{ or when } \dot{q} = 0 \text{ and } q < -\frac{F_f}{m \omega_0^2}, \\
m \omega_0^2 q & \text{when } \dot{q} = 0 \text{ and } |q| \leq \frac{F_f}{m \omega_0^2}.
\end{cases}
\]

(2.3)

Now suppose that

\[
\lim_{t \to t_0^-} q(t) = q_c(t_0) \text{ and } \lim_{t \to t_0^-} \dot{q}(t) \leq \dot{q}_c(t_0)
\]

(2.4)

for some time \( t_0 \), corresponding to the onset of collisional contact between the impactor and the object. Then, assuming an instantaneous contact phase, conservation of linear momentum implies that

\[
\lim_{t \to t_0^+} q(t) = q_c(t_0) \text{ and } \lim_{t \to t_0^+} \dot{q}(t) = -e \lim_{t \to t_0^-} \dot{q}(t) + (1 + e) \dot{q}_c(t_0),
\]

(2.5)

where \( 0 < e \leq 1 \) is a kinematic coefficient of restitution. Unless otherwise stated, we omit from consideration trajectories with degenerate contact between the impactor and the object, for example sustained contact over a finite interval of time.

Consider the non-dimensionalized, three-dimensional state vector

\[
x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} m \omega_0^2 q \\ \frac{F_f}{m \omega_0} \dot{q} \\ \omega t \mod 2\pi \end{pmatrix} \in \mathbb{R}^2 \times S^1,
\]

(2.6)

such that, for \( x_1 + \tilde{b} - \tilde{a} \sin x_3 > 0 \), the forward-in-time dynamics are governed by

\[
\frac{dx}{d\tau} = \begin{cases} 
\mathbf{f}_+(x) & \overset{\text{def}}{=} (x_2 - 1 - x_1 \tilde{\omega})^T \text{ when } x_2 > 0 \text{ or } x_2 = 0 \text{ and } x_1 > 1, \\
\mathbf{f}_-(x) & \overset{\text{def}}{=} (x_2 1 - x_1 \tilde{\omega})^T \text{ when } x_2 < 0 \text{ or } x_2 = 0 \text{ and } x_1 < -1, \\
\mathbf{f}_0(x) & \overset{\text{def}}{=} (0 0 \tilde{\omega})^T \text{ when } x_2 = 0 \text{ and } |x_1| \leq 1,
\end{cases}
\]

(2.7)
where
\[\tilde{a} = \frac{a m \omega_0^2}{F_f}, \quad \tilde{b} = \frac{b m \omega_0^2}{F_f} \quad \text{and} \quad \tilde{\omega} = \frac{\omega}{\omega_0}, \quad (2.8)\]
and differentiation is performed with respect to the non-dimensionalized time variable \(\tau = \omega_0 t\). Then, contact between the impactor and the object,
\[
\lim_{\tau \to \tau_0^-} x_1(\tau) = -\tilde{b} + \tilde{a} \sin x_3(\tau_0) \quad \text{and} \quad \lim_{\tau \to \tau_0^-} x_2(\tau) \leq \tilde{a} \tilde{\omega} \cos x_3(\tau_0), \quad (2.9)
\]
for some time \(\tau_0\) implies that
\[
\lim_{\tau \to \tau_0^+} x_1(\tau) = -\tilde{b} + \tilde{a} \sin x_3(\tau_0) \quad (2.10)
\]
and
\[
\lim_{\tau \to \tau_0^+} x_2(\tau) = -e \lim_{\tau \to \tau_0^-} x_2(\tau) + (1 + e) \tilde{a} \tilde{\omega} \cos x_3(\tau_0). \quad (2.11)
\]

(b) Hybrid system formulation

The integral curves of the \(f_0\) vector field are described by the flow
\[
\phi_0(x, \tau) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 + \tilde{\omega} \tau \mod 2\pi \end{pmatrix} \quad (2.12)
\]
corresponding to straight lines parallel to the \(x_3\)-axis (figure 2). Similarly, the integral curves of the \(f_+\) and \(f_-\) vector fields are described by the flows
\[
\phi_+(x, \tau) = \begin{pmatrix} -1 + (x_1 + 1) \cos \tau + x_2 \sin \tau \\ x_2 \cos \tau - (x_1 + 1) \sin \tau \\ x_3 + \tilde{\omega} \tau \mod 2\pi \end{pmatrix} \quad (2.13)
\]
and
\[
\phi_-(x, \tau) = \begin{pmatrix} 1 + (x_1 - 1) \cos \tau + x_2 \sin \tau \\ x_2 \cos \tau - (x_1 - 1) \sin \tau \\ x_3 + \tilde{\omega} \tau \mod 2\pi \end{pmatrix}, \quad (2.14)
\]
corresponding to families of circular helices with identical pitch and forming concentric cylinders with symmetry axes given by \(x_2 = 0\) and \(x_1 = -1\) and \(+1\), respectively (figure 2). Forward-in-time trajectories of this hybrid dynamical system are then obtained by concatenating segments of the integral curves of \(f_0, f_+\) and \(f_-\) at points along the switching surface
\[
S = \left\{ h_S(x) \overset{\text{def}}{=} x_2 = 0 \right\}, \quad (2.15)
\]
accounting, when necessary, for discontinuous jumps between points on the impact surface
\[
I = \left\{ h_I(x) \overset{\text{def}}{=} x_1 + \tilde{b} - \tilde{a} \sin x_3 = 0 \right\} \quad (2.16)
\]
given by the impact map
\[
g_I(x) = \begin{pmatrix} x_1 \\ -\epsilon x_2 + (1 + \epsilon) \tilde{a} \tilde{\omega} \cos x_3 \end{pmatrix}. \quad (2.17)
\]
Figure 2. Example trajectories corresponding to the three flows $\phi_0$ (grey surface at $x_2 = 0$), $\phi_+$ (for $x_2 > 0$) and $\phi_-$ (for $x_2 < 0$), respectively.

It follows from the explicit expressions for the partial flows $\phi_+$, $\phi_-$ and $\phi_0$ that recurrent dynamics of the hybrid dynamical system take one of the two distinct forms. Specifically, stick trajectories coincide with integral curves of $f_0$ on the forward-in-time invariant set

$$E = S \cap \{\max(-1, \tilde{a} - \tilde{b}) \leq x_1 \leq 1\}. \quad (2.18)$$

These correspond to a stationary object in the absence of collisional contact with the impactor. An example stick trajectory is shown in figure 3a. In contrast, impacting trajectories include repeated intersections with $I$, i.e. repeated occurrences of collisional contact between the movable object and the impactor. A typical impacting trajectory is shown in figure 3b.

(c) Characteristic dynamics

As long as $\tilde{a} < \tilde{b} + 1$, the analysis in Svaan & Dankowicz (2008) shows that the set $E$ is locally attracting. Figures 4 and 5 show two examples of changes in the steady-state response, obtained using numerical forward simulation for fixed values of the frequency ratio $\tilde{\omega}$, the coefficient of restitution $e$ and $\tilde{b} = 3$, as the parameter $\tilde{a}$ is varied quasi-statically near $\tilde{b} + 1$. In each case, the starting point for the analysis is a stick trajectory with $x_1 = 0.9$ and $\tilde{a} = 3.8$. Each point in figure 4 corresponds to the minimum value of $x_1$ for each period of the impactor motion. Similarly, each point in figure 5 corresponds to the maximum value of $x_2$ for each period of the impactor motion. Independently of the value of $\tilde{\omega}$, the original stick trajectory is unaffected by the value of $\tilde{a}$ as long as $\tilde{a} \leq \tilde{b} + x_1 = 3.9$. Consistent with the results in Svaan & Dankowicz (2008), the steady-state response remains in $E$ under further increases in the value of $\tilde{a}$ with $x_1 \approx \tilde{a} - \tilde{b}$ until $\tilde{a} = \tilde{b} + 1 = 4$ (cf. the steady-state trajectory shown in figure 3a).
Figure 3. Example (a) stick and (b) impacting trajectories. Here, \( \mathcal{I} \) and \( \mathcal{S} \) denote the impact surface and the switching surface, respectively. Similarly, \( \mathcal{E} \) denotes the forward-in-time invariant set of stick trajectories in the case that \( \tilde{a} \leq \tilde{b} + 1 \). (b) The filled circles (labelled by \( x_i, x_b, x_a \) and \( x_f \) as per the notation used in subsequent sections and in the appendix) denote endpoints of distinct trajectory segments with \( x_2 > 0 \), \( x_2 < 0 \) and \( x_2 = 0 \), respectively.

In figures 4(a) and 5(a), the steady-state response subsequent to further increases in the value of \( \tilde{a} \) lies on a branch of periodic impacting trajectories that emanates from \( x_1 = 1 \) at \( \tilde{a} = \tilde{b} + 1 \) (cf. the steady-state trajectory at \( A \) in figures 4(a) and 5(a), for which the post-impact absolute velocity of the movable object is approx. 0.3). The steady-state response remains on this branch even as \( \tilde{a} \) is again decreased to \( \tilde{a} = \tilde{b} + 1 \). Subsequent decreases in the value of \( \tilde{a} \) result in a steady-state stick trajectory with \( x_1 = 1 \). In contrast, in figures 4(b) and 5(b), the steady-state response...
subsequent to further increases in the value of $\tilde{a}$ lies on a disjoint branch of periodic impacting trajectories that persists even as $\tilde{a}$ is reduced below $\tilde{b} + 1$ and that terminates at some $\tilde{a} < \tilde{b} + 1$ (cf. the steady-state trajectory at B in figures 4b and 5b, for which the post-impact absolute velocity of the movable object is approx. 4). Subsequent decreases in the value of $\tilde{a}$ again result in a steady-state stick trajectory for some $x_1 \leq 1$.

The critical condition $\tilde{a} = \tilde{b} + 1$ corresponds to a situation in which the movable object, in a stationary position of maximal spring compression, is grazed by the impactor. The bifurcation scenario in figures 4a and 5a exhibits a persistent attractor. Here, the steady-state response undergoes a continuous transition across the critical point $\tilde{a} = \tilde{b} + 1$ and the corresponding stick trajectory with $x_1 = 1$ for increasing as well as decreasing values of $\tilde{a}$. In contrast, the scenario in figures 4b and 5b is characterized by a loss of a local attractor as $\tilde{a}$ is increased past the critical point with a subsequent discontinuous transition to
a distinct steady-state response. Indeed, for \( \tilde{a} > \tilde{b} + 1 \), no steady-state attractor exists locally to \( x_1 = 1 \). Instead, after an initial transient, the system response settles on a periodic impacting trajectory with a relatively high post-impact velocity. As shown in figures 4b and 5b, the loss of a local attractor is associated with parameter hysteresis and the coexistence of stick and impacting steady-state trajectories on an interval in \( \tilde{a} \) containing the critical point. The objective of the next section is to derive inequality conditions on the parameter values that differentiate between the case of a persistent attractor and the case of a loss of a local attractor at the critical point.

### 3. The open-loop dynamics

(a) An equivalent map of the dynamics

Consider \( \tilde{b} = \tilde{a}^* - 1 \) for some \( \tilde{a}^* \). Then, as shown below and further elaborated upon in the appendix, provided that

\[
\gamma \overset{\text{def}}{=} \frac{\tilde{a}^*(1 + e)^2 \tilde{\omega}^2}{2} > 1,
\]

\( \tilde{\omega} < 2 \) and \( \tilde{a} \approx \tilde{a}^* \), the system response for admissible initial conditions with \( x_1 \approx 1 \) can be fully described by iterates of the piecewise-smooth, one-dimensional map

\[
P : x \mapsto \begin{cases} 
-1 + \sqrt{(1 + e)^2 \tilde{\omega}^2 (\tilde{a}^2 - (x + \tilde{b})^2) + (x + 1)^2} & \text{if } x \leq \min(1, \tilde{a} - \tilde{b}), \\
2 - x & \text{if } \tilde{a} - \tilde{b} < x \leq 1,
\end{cases}
\]

applied to the \( x_1 \) value at successive intersections of system trajectories with \( S \) or, in the case of stick trajectories, to the value of \( x_1 \) along the trajectory. In particular, that \( P \) restricted to the interval \([1 + \tilde{a} - \tilde{a}^*, 1]\) equals the identity is consistent with the condition that \( x_1 \) remains constant for all time along the corresponding stick trajectories.

Now consider an admissible initial point \( x_i \) on \( S \) with \( x_1 \approx \min(1, 1 + \tilde{a} - \tilde{a}^*) \). The subsequent trajectory follows an integral curve of \( f_0 \) until an intersection with \( S \cap I \) at a point \( x_b \), where

\[
x_b = g(x_b) \quad \text{where } \quad x_b = \begin{cases} 
1 - \tilde{a}^* + \tilde{a} \sin x_b, & \text{if } x_b \leq \min(1, \tilde{a} - \tilde{b}), \\
1 + (1 + e) \tilde{a} \tilde{\omega} \cos x_b, & \text{if } \tilde{a} - \tilde{b} < x_b \leq 1,
\end{cases}
\]

and \( x_b \leq \pi/2 \). The subsequent trajectory follows the integral curve of \( f_+ \) based at the point \( x_a = g(x_b) \), where \( x_{a1} = x_{b1} \),

\[
x_{a2} = (1 + e) \tilde{a} \tilde{\omega} \cos x_b \geq 0
\]

and \( x_{a3} = x_{b3} \). It follows from the analysis in the appendix that the trajectory terminates on \( S \) at a point \( x_f \), where

\[
x_f = -1 + \sqrt{x_{a2}^2 + (x_{a1} + 1)^2} \geq 1 + \tilde{a} - \tilde{a}^* \quad \text{and } x_3 \approx \pi/2 \text{. It follows that } x_1 = P(x_{b1}).
\]

1 Here, and in the following, \( \overset{\text{\( > \)}}{\sim}, \overset{\text{\( \geq \)}}{\sim}, \overset{\text{\( < \)}}{\sim} \) and \( \overset{\text{\( \leq \)}}{\sim} \) are equivalent to \( \approx \) together with \( >, \geq, < \text{ and } \leq \), respectively.
Finally, consider an admissible initial point \( x_i \) on \( S \) where \( x_{i1} \gtrsim 1 \) and, without loss of generality, \( x_{i3} \approx \pi/2 \). The subsequent trajectory follows an integral curve of \( f_- \) and terminates on \( S \) at a point \( x_f \), where \( x_{f1} = 2 - x_{i1} \) and \( x_{f3} \approx \pi/2 + \bar{\omega}\pi \), before reaching \( I \) (see appendix). Again, \( x_{f1} = \bar{P}(x_{i1}) \) in agreement with the claim regarding the map \( P \). Iterates of \( P \) thus capture the dynamics of all admissible initial points on \( S \) sufficiently close to \( x_1 = 1 \). Trajectories outside of this neighbourhood, after evolution along integral curves of \( f_- \), typically reach \( I \) before \( S \) or reach \( S \) with \( x_1 < -1 \), in which case \( P \) does not capture the dynamics.

For \( \tilde{a} \approx \bar{a}^* \) and \( x \approx 1 \), it follows that

\[
P: x \mapsto \begin{cases} 
1 + \gamma(\tilde{a} - \bar{a}^*) + (1 - \gamma)(x - 1) & \text{if } x \lesssim 1 + \tilde{a} - \bar{a}^*, \\
2 - x & \text{if } 1 + \tilde{a} - \bar{a}^* < x \leq 1, \\
\end{cases} \tag{3.6}
\]

In the case that \( \tilde{a} \lesssim \bar{a}^* \), the map \( P \) is thus continuous across \( x = 1 + \tilde{a} - \bar{a}^* \) and \( x = 1 \) and its graph takes the form shown in figure 6a. Similarly, in the case that \( \tilde{a} \gtrsim \bar{a}^* \), the map \( P \) is discontinuous across \( x = 1 \) with

\[
\lim_{x \to 1^-} P(x) - \lim_{x \to 1^+} P(x) = \gamma(\tilde{a} - \bar{a}^*), \tag{3.7}
\]

and its graph takes the form shown in figure 6b.

(b) Fixed points and stability

For \( \tilde{a} \lesssim \bar{a}^* \), all points with \( 1 + \tilde{a} - \bar{a}^* \leq x \leq 1 \) (and only these points) are fixed points of \( P \) and, consequently, of all its iterates. These trivial fixed points correspond to stick trajectories in \( \mathcal{E} \). There are no trivial fixed points of \( P \) in the case when \( \tilde{a} \gtrsim \bar{a}^* \). Indeed, as \( P(\{x < \min(1, 1 + \tilde{a} - \bar{a}^*)\}) \cap \{x < \min(1, 1 + \tilde{a} - \bar{a}^*)\} = P(\{x > 1\}) \cap \{x > 1\} = \emptyset \), it follows that there are no non-trivial fixed points of odd iterates of \( P \).

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Denote by
\[ \hat{x} \approx 1 + \frac{\gamma}{\gamma - 1} (\tilde{a} - \tilde{a}^*) \lesssim 1 + \tilde{a} - \tilde{a}^* \] (3.8)

(figure 6a) the unique local solution to the equation \( P(x) = 1 \) in the case that \( \tilde{a} \lesssim \tilde{a}^* \) and let \( \hat{x} = \infty \) for \( \tilde{a} \gtrsim \tilde{a}^* \). It follows from equation (3.2) that
\[
P^2 : x \mapsto \begin{cases} 
3 - \sqrt{(1 + e)^2\omega^2(\tilde{a}^2 - (x + \tilde{a}^* - 1)^2)} + (x + 1)^2 & \text{if } x \leq \min(1, \hat{x}), \\
-1 + \sqrt{(1 + e)^2\omega^2(\tilde{a}^2 - (x + \tilde{a}^* - 1)^2)} + (x + 1)^2 & \text{if } \hat{x} < x \leq 1 + \tilde{a} - \tilde{a}^*, \\
x & \text{if } 1 + \tilde{a} - \tilde{a}^* < x \leq 1, \\
2 - x & \text{if } 1 < x \leq 1 + \tilde{a}^* - \tilde{a}, \\
-1 + \sqrt{(1 + e)^2\omega^2(\tilde{a}^2 - (1 - x + \tilde{a}^*)^2)} + (3 - x)^2 & \text{if } \max(1, 1 + \tilde{a}^* - \tilde{a}) < x.
\end{cases}
\]

(3.9)

For \( \tilde{a} \approx \tilde{a}^* \) and \( x \approx 1 \), it follows that
\[
P^2 : x \mapsto \begin{cases} 
1 - \gamma(\tilde{a} - \tilde{a}^*) + (\gamma - 1)(x - 1) & \text{if } x \lesssim \min(1, \hat{x}), \\
1 + \gamma(\tilde{a} - \tilde{a}^*) + (1 - \gamma)(x - 1) & \text{if } \hat{x} < x \leq 1 + \tilde{a} - \tilde{a}^*, \\
x & \text{if } 1 + \tilde{a} - \tilde{a}^* < x \leq 1, \\
2 - x & \text{if } 1 < x \leq 1 + \tilde{a}^* - \tilde{a}, \\
1 + \gamma(\tilde{a} - \tilde{a}^*) + (\gamma - 1)(x - 1) & \text{if } \max(1, 1 + \tilde{a}^* - \tilde{a}) \lesssim x.
\end{cases}
\]

(3.10)

In the case that \( \tilde{a} \lesssim \tilde{a}^* \), the map \( P^2 \) is thus everywhere continuous and its graph takes the form shown in figure 7a,c. Similarly, in the case that \( \tilde{a} \gtrsim \tilde{a}^* \), the map \( P^2 \) is discontinuous across \( x = 1 \) with
\[
\lim_{x \to 1^-} P^2(x) - \lim_{x \to 1^+} P^2(x) = -2\gamma(\tilde{a} - \tilde{a}^*),
\]

(3.11)

and takes the form shown in figure 7b,d.

From the piecewise-linear approximation of \( P^2 \) for \( \tilde{a} \approx \tilde{a}^* \) and \( x \approx 1 \), it follows that a pair of non-trivial fixed points of \( P^2 \) exist when \( \tilde{a} \lesssim \tilde{a}^* \) provided that \( \gamma > 2 \) and when \( \tilde{a} \gtrsim \tilde{a}^* \) provided that \( \gamma < 2 \), corresponding to intersections with \( S \) of the same trajectory for decreasing and increasing velocity, respectively. In each case, the fixed points converge to the single trivial fixed point at \( x = 1 \) as \( \tilde{a} \) limits on \( \tilde{a}^* \). Specifically, from the explicit expression for \( P^2 \), it follows that fixed points for \( x \lesssim \min(1, \hat{x}) \) are among the solutions to the equation
\[
\tilde{a} = \frac{\sqrt{8(1 - x) + (1 + e)^2\omega^2(x + \tilde{a}^* - 1)^2}}{(1 + e)\omega}.
\]

(3.12)

Indeed, for \( \gamma > 2 \), there exist two solutions to this equation provided that
\[
\tilde{a} - \tilde{a}^* > \frac{4\sqrt{\gamma - 1} - 2\gamma}{(1 + e)^2\omega^2},
\]

(3.13)
Figure 7. Representative graphs of $P^2$ with $\bar{a} = 4$, $e = 0.9$ and $(a,b) \, \bar{a} = 0.6$ and consequently $\gamma = 2.5992$, $(c,d) \, \bar{a} = 0.4$ and consequently $\gamma = 1.1552$, $(a,c) \, \tilde{\omega} = 3.99$, and $(b,d) \, \tilde{\omega} = 4.01$. $P^2$ describes the dynamics for a full period of the excitation. Fixed points thus correspond to periodic impacting trajectories. (a) Two unstable fixed points are located at $x \approx 1 \pm 0.044$. The two fixed points represent the intersection of the same trajectory with $S$ for decreasing and increasing velocity, respectively. (b) There is no local attractor. (c) $E$ is an attracting set for the entire neighbourhood shown. (d) Two asymptotically stable fixed points at $x \approx 1 \pm 0.014$, again corresponding to intersections with $S$ of the same trajectory for decreasing and increasing velocity, respectively.

It follows directly from the piecewise-linear approximation of $P^2$ for $\bar{a} \approx \tilde{a}^*$ and $x \approx 1$ and the schematic cobweb diagram in figure 7 that the non-trivial fixed points that converge on $x = 1$ as $\bar{a} \to \tilde{a}^*$ are asymptotically stable in the case that $\gamma < 2$ and unstable in the case that $\gamma > 2$. Indeed, for $\gamma \geq 2$, it further follows that the second pair of non-trivial fixed points are asymptotically stable, such that the coincidence of the non-trivial fixed points when equality holds in equation (3.13) corresponds to a saddle–node bifurcation point. Finally, it is clear that there cannot exist any additional non-trivial fixed points of $P^2$ or its iterates for $\bar{a} \approx \tilde{a}^*$ and $x \approx 1$.

The non-trivial fixed points of $P^2$ found above correspond to periodic impacting trajectories of the hybrid dynamical system with a single impact per period of the impactor motion. In the limit as $\tilde{a} \to \tilde{a}^*$, these periodic trajectories converge
to the grazing trajectory $x^*(\tau) = (1 \ 0 \ \tilde{\omega} \tau)^T$, which corresponds to a grazing, zero-relative-velocity collision with the impactor at the point $x^* = (1 \ 0 \ \pi/2)^T$. It follows from the above analysis that the grazing trajectory is asymptotically stable provided that $\gamma \leq 2$ and unstable in the case that $\gamma > 2$.

In summary, the case when $\gamma < 2$ corresponds to that of a persistent attractor, because, as $\tilde{a}$ is increased past $\tilde{a}^*$, a branch of asymptotically stable periodic trajectories emanates from the grazing trajectory. In contrast, the case when $\gamma > 2$ corresponds to the loss of a local attractor because, as $\tilde{a}$ is increased past $\tilde{a}^*$, iterates of $P$ leave any small neighbourhood of the grazing trajectory, resulting in a discontinuous transition to a distinct, non-local steady-state response. These results are consistent with the numerical observations in figures 4 and 5.

4. The closed-loop plant

The objective of this section is to investigate the formulation of feedback control strategies that make discrete changes to the value of $\tilde{b}$ to ensure a persistent attractor across the point $(x, \tilde{b}) = (1, \tilde{a}^* - 1)$ as $\tilde{a}$ is increased past $\tilde{a}^*$ as long as $\tilde{\omega} < 2$ and $\gamma > 1$. Specifically, consider the formulation of a smooth closed-loop control strategy $\tilde{b}(x, \tilde{a})$, for which the system response for admissible initial conditions with $x \approx 1$ and $\tilde{a} \approx \tilde{a}^*$ can be fully described by iterates of the piecewise-smooth, one-dimensional map (3.2), where $\tilde{b} = \tilde{b}(x, \tilde{a})$. By the analysis of the previous section, this corresponds to the imposition of a small adjustment to the value of $\tilde{b}$ once every period of the impactor motion, at some moment in time when the movable object is away from the impactor, and based on the value of $x$ when the object is in stick.

We restrict attention to control strategies, for which the simultaneous pair of inequalities

$$\tilde{a} - \tilde{b}(x, \tilde{a}) \leq x \leq 1$$

have

(i) an interval of solutions with the upper bound at $x = 1$ for $\tilde{a} \lesssim \tilde{a}^*$,
(ii) a single solution at $x = 1$ for $\tilde{a} = \tilde{a}^*$, and
(iii) no solutions for $\tilde{a} \gtrsim \tilde{a}^*$.

In particular, these conditions are trivially satisfied provided that

$$\tilde{b}(1, \tilde{a}^*) = \tilde{a}^* - 1,$$  

and $\tilde{b}$ satisfies the conditions

$$\frac{\partial \tilde{b}}{\partial x}(1, \tilde{a}^*) > -1 \quad \text{and} \quad \frac{\partial \tilde{b}}{\partial \tilde{a}}(1, \tilde{a}^*) < 1.$$  

In this case, $P$ is a piecewise-smooth, continuous map with a single fixed point at $x = 1$ when $\tilde{a} = \tilde{a}^*$. Moreover, while the pre-grazing steady-state response for $\tilde{a} \lesssim \tilde{a}^*$ consists of stick trajectories, the post-grazing steady-state response for $\tilde{a} \gtrsim \tilde{a}^*$ must necessarily consist of impacting trajectories.
(a) Linear analysis

Without loss of generality, let

\[
\tilde{b}(x, \tilde{a}) = \tilde{a}^* - 1 + (1 - c_x)(\tilde{a} - \tilde{a}^*) + \left(1 - \frac{2c_x}{\gamma}\right)(1 - x)
\]

(4.4)

represent the linearization of some nonlinear update function \(\tilde{b}\) for \(x \approx 1\) and \(\tilde{a} \approx \tilde{a}^*\) with control parameters \(c_x, c_x > 0\), such that the absence of feedback on \(x\) in the linear limit corresponds to \(c_x = \gamma/2\).

It follows that

\[
P : x \mapsto \begin{cases} 
1 + \gamma c_a(\tilde{a} - \tilde{a}^*) + (1 - 2c_x)(x - 1) & \text{if } x \lesssim 1, \\
2 - x & \text{if } 1 < x,
\end{cases}
\]

(4.5)

is a linear approximation of \(P\) for \(x \approx 1\) and \(\tilde{a} \gtrsim \tilde{a}^*\). Denote by

\[
\hat{x} \approx 1 + \frac{c_a\gamma}{2c_x - 1}(\tilde{a} - \tilde{a}^*) \lesssim 1
\]

(4.6)

the unique local solution to the equation \(P(x) = 1\) in the case that \(\tilde{a} \gtrsim \tilde{a}^*\) and \(c_x < 1/2\) and let \(\hat{x} = -\infty\) when \(c_x \geq 1/2\). Then,

\[
P^2 : x \mapsto \begin{cases} 
1 + 2\gamma c_a(1 - c_x)(\tilde{a} - \tilde{a}^*) + (1 - 2c_x)^2(x - 1) & \text{if } x \lesssim \hat{x}, \\
1 - \gamma c_a(\tilde{a} - \tilde{a}^*) - (1 - 2c_x)(x - 1) & \text{if } \hat{x} < x \lesssim 1, \\
1 + \gamma c_a(\tilde{a} - \tilde{a}^*) - (1 - 2c_x)(x - 1) & \text{if } 1 \lesssim x,
\end{cases}
\]

(4.7)

is a linear approximation of \(P^2\) for \(x \approx 1\) and \(\tilde{a} \gtrsim \tilde{a}^*\). \(P\) and \(P^2\) for the cases \(c_x < 1/2\) and \(c_x > 1/2\) are illustrated in figure 8.

It is clear that \(P\) possesses no fixed points in the case that \(\tilde{a} \gtrsim \tilde{a}^*\). In contrast, from the piecewise linearizations of \(P^2\) for \(\tilde{a} \gtrsim \tilde{a}^*\) and \(x \approx 1\), it follows that there exists a pair of non-trivial asymptotically stable fixed points of \(P^2\) when \(\tilde{a} \gtrsim \tilde{a}^*\) provided that \(c_x < 1\). From the full expression for \(P^2\), it follows that the fixed points for \(x \lesssim 1\) are among the solutions to the equation

\[
3 - \sqrt{(1 + e)^2\tilde{a}^2 - (x + \tilde{b}(x, \tilde{a}))^2 + (x + 1)^2} = x,
\]

(4.8)

which for \(\tilde{a} \gtrsim \tilde{a}^*\) and \(x \lesssim 1\) yields

\[
x = 1 + \frac{\gamma c_a}{2(c_x - 1)}(\tilde{a} - \tilde{a}^*) + \mathcal{O}((\tilde{a} - \tilde{a}^*)^2) \geq \hat{x}
\]

(4.9)

in the case that \(c_x < 1\). This fixed point and the corresponding point for \(x > 1\) converge to the single non-trivial fixed point at \(x = 1\) as \(\tilde{a}\) limits on \(\tilde{a}^*\) and are the only non-trivial fixed points of \(P^2\) or its iterates for \(\tilde{a} \gtrsim \tilde{a}^*\), \(x \approx 1\) and \(0 < c_x < 1\). Finally, it follows from equation (4.9) that the initial slope with respect to changes in \(\tilde{a}\) of the corresponding branch of fixed points approaches \(-\infty\) as \(c_x \to 1\) and 0 as \(c_a \to 0\).

It follows from this analysis that the case when \(c_x = \gamma/2\) corresponds to a branch of asymptotically stable periodic impacting trajectories provided that \(\gamma < 2\). This is consistent with the results of the previous section because in this case a persistent attractor would result only if such a bifurcation scenario occurred already in the absence of control.

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A numerical illustration of the effect of the control strategy is given in Figure 9. The two panels show $P^2$ with $c_\tilde{a} = 1$, $\tilde{a} = 4.01$, $\tilde{a}^* = 4$, $\tilde{\omega} = 0.6$ and $e = 0.9$. (a) $c_x = \gamma/2$, corresponding to absence of control, and (b) $c_x = 1/2$.

Figure 8. Representative graphs of $P$ (a,c) and $P^2$ (b,d) for $c_\tilde{a} = 1$, $\tilde{a} = 4.01$, $\tilde{a}^* = 4$, $e = 0.9$ and $\tilde{\omega} = 0.6$. In (a) and (b) $c_x = 1/4$ and in (c) and (d) $c_x = 3/4$.

Figure 9. Representative graphs of $P^2$ with $c_\tilde{a} = 1$, $\tilde{a} = 4.01$, $\tilde{a}^* = 4$, $\tilde{\omega} = 0.6$ and $e = 0.9$. (a) $c_x = \gamma/2$, corresponding to absence of control, and (b) $c_x = 1/2$.
Figure 10. Steady-state responses of (a) the minimum value of $x_1$ (black line) and (b) the maximum velocity of the movable object, as a function of $\tilde{a}$, with $c = 0.8$, $\tilde{c}_a = 0$, $\tilde{a}^* = 4$, $\omega = 0.6$ and $e = 0.9$. (a) The grey line represents the linear prediction of the branch, equation (4.9). Arrows denote the direction of quasi-static changes in $\tilde{a}$.

Figure 10 shows the changes in steady-state response under variations in $\tilde{a}$ across $\tilde{a}^*$ in the presence of control with $c_{\tilde{a}} = 1$ and $c_x = 0.8$ for a choice of $\tilde{\omega}$, $e$ and $\tilde{a}^*$ such that $\gamma > 2$. Again, the bifurcation scenario includes a persistent attractor. This result can be compared with the jump in the solution for the bifurcation scenario of the original system in figures 4b and 5b.

In summary, given $c_{\tilde{a}} > 0$, a control parameter value $0 < c_x < 1$ guarantees a persistent attractor locally to $x = 1$ when $\tilde{a}$ is increased past $\tilde{a}^*$, independently of the value of $\gamma$.

(b) Nonlinear control strategies

Let $f$ be a smooth function such that $f(\tilde{a}^*) = 1$. The nonlinear control strategy

\[
\tilde{b}(x, \tilde{a}) = -x + \sqrt{\frac{4(1 - c_x x)(c_x x - 2 - x) + \tilde{a}^2(1 + e)^2\tilde{\omega}^2}{+4(1 - c_x)f(\tilde{a})(3 + (1 - 2c_x)x - (1 - c_x)f(\tilde{a}))}} \quad (4.10)
\]

corresponds to equation (4.4) with

\[
c_{\tilde{a}} = -\frac{2(1 - c_x)}{\gamma}f'(\tilde{a}^*). \quad (4.11)
\]

The conditions on the presence and absence of steady-state stick trajectories for $\tilde{a} \lesssim \tilde{a}^*$ and $\tilde{a} \gtrsim \tilde{a}^*$, respectively, are then trivially satisfied for $0 < c_x < 1$ and $f'(\tilde{a}^*) < 0$. Substitution of this control strategy into the full expression for $P^2$ for $\hat{x} < x \leq 1$ yields

\[
P^2(x) - f(\tilde{a}) = (2c_x - 1)(x - f(\tilde{a})), \quad (4.12)
\]

from which it follows that $x = f(\tilde{a})$ is a unique fixed point for $x \lesssim 1$ as long as $f(\tilde{a}) \leq 1$. Moreover, convergence to $x = f(\tilde{a})$ occurs at a rate equal to $|2c_x - 1|$.
In particular, for $c_x = 1/2$, convergence is instantaneous as, in this case,

$$P^2 : x \rightarrow \begin{cases} f(\tilde{a}) & \text{if } x \leq 1, \\ 2 - f(\tilde{a}) & \text{if } 1 < x, \end{cases}$$

for all $x$ in the region of validity of $P$ and $\tilde{a} \approx \tilde{a}^*$. The fixed point at $x = f(\tilde{a})$, and the corresponding periodic impacting trajectory, is thus a superstable attractor for the closed-loop plant in that any initial deviations disappear within one period of the impactor motion. Figure 11 illustrates $P$ and $P^2$ with the superstabilizing control strategy and $f(\tilde{a}) = 1 + \tilde{a}^* - \tilde{a}$.

5. Discussion

This paper has provided a complete unfolding of the grazing-induced bifurcation behaviour in the example vibro-impacting system with dry friction for an open set of parameter values. In addition, a family of control strategies has been formulated that enable the regulation of the near-grazing dynamics, ensuring a persistent low-impact-velocity steady-state response across the critical parameter value corresponding to grazing contact.

For a similar vibro-impacting system without friction, the recurrent near-grazing dynamics are dominated by a two-dimensional, piecewise-defined, local map with an unbounded derivative in the limit of zero impact velocity. In contrast, the analysis in this paper shows that, in the presence of dry friction, the recurrent near-grazing dynamics are dominated by a one-dimensional, piecewise-linear, local map. Consequently, although the control strategies proposed here regulate the bifurcation behaviour using the same means of actuation as in previous studies of impacting systems, their successful derivation is here reduced to the analysis of a simple one-dimensional map. In fact, as shown in §4b, here the control strategy may be explicitly chosen so as to render a prescribed dependence of the near-grazing steady-state response on system parameters.
For an experimental verification of the proposed control strategies, it is desirable to conduct a robustness analysis to determine the sensitivity of the predicted response to uncertainties in the control law or the model parameters. This is a topic for a future publication.

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Appendix A

The objective of this section is to show the implications of the assumptions \( \tilde{\omega} < 2 \) and \( \gamma > 1 \). To this end, consider an admissible initial condition \( x_i \) such that \( x_i \approx 1 + \tilde{a} - \tilde{a}^* \) for \( \tilde{a} \approx \tilde{a}^* \). It follows directly from the statement regarding the integral curves of the vector fields \( f_+ \), \( f_- \) and \( f_0 \) that the subsequent trajectory will reach \( E \) in finite time at an admissible point near

\[
x^* = \begin{pmatrix} 1 + \tilde{a} - \tilde{a}^* & 0 & \frac{\pi}{2} \end{pmatrix}^T.
\]

Here, the condition on \( \tilde{\omega} \) guarantees that trajectory segments based on initial conditions \( x_i \) on \( S \) sufficiently close to \( x^* \) and with \( x_i > 1 + \tilde{a} - \tilde{a}^* \) terminate on \( E \) prior to the next intersection with \( I \).

Without loss of generality, denote by \( x_i \) an admissible initial condition on \( E \) near \( x^* \) and suppose that \( \tilde{a} = \tilde{a}^* \). The subsequent trajectory follows an integral curve of \( f_0 \) until an intersection with \( S \cap I \) at a point \( x_b \), where

\[
x_{a1} = x_{b1} = 1 - \tilde{a}^* + \tilde{a}^* \sin x_{b3}
\]

and \( x_{b3} \approx \pi/2 \). The subsequent trajectory follows the integral curve of \( f_+ \) based at the point \( x_a = g_I(x_b) \), where \( x_{a1} = x_{b1} \),

\[
x_{a2} = (1 + e) \tilde{a}^* \tilde{\omega} \cos x_{b3} \approx 0
\]

and \( x_{a3} = x_{b3} \). The condition on \( \gamma \) now guarantees that this trajectory terminates on \( S \) at a point \( x_f \), where

\[
x_{f1} = -1 + \sqrt{x_{a2}^2 + (x_{a1} + 1)^2} > 1.
\]

Indeed, substitution and expansion in the deviation of \( x_{b3} \) from \( \pi/2 \) yield

\[
x_{f1} = 1 + \frac{\tilde{a}^*}{2} (\gamma - 1) \left( x_{b3} - \frac{\pi}{2} \right)^2 + \mathcal{O}\left( \left( x_{b3} - \frac{\pi}{2} \right)^3 \right).
\]

We now claim that this trajectory does not intersect \( I \) prior to reaching \( x_f \). It is clear that trajectories based at initial conditions

\[
\begin{pmatrix} 1 & x_{a2} & \frac{\pi}{2} \end{pmatrix}^T
\]

will reach \( S \) prior to intersecting \( I \). By continuity, this conclusion follows for initial conditions of the form

\[
\begin{pmatrix} 1 - \tilde{a}^* + \tilde{a}^* \sin x_3 & x_{a2} & x_3 \end{pmatrix}^T
\]
for $x_3 \leq \pi/2$ and bounded below by a critical phase $\hat{x}_3$ for which the corresponding trajectory terminates on $S$ at a point with $x_1 = 1$. Direct computation yields

$$
\sin \hat{x}_3 = 1 - \frac{2 - \sqrt{4 - x_a^2}}{a^*} \approx 1 - \frac{x_a^2}{4a^*}.
$$

(A 8)

In contrast,

$$
\sin x_3 = \sqrt{1 - \frac{x_a^2}{(1 + e)^2\tilde{a}^2\omega^2}} \approx 1 - \frac{x_a^2}{2(1 + e)^2\tilde{a}^2\omega^2}.
$$

(A 9)

It follows that $x_3 > \hat{x}_3$ provided that $\gamma > 1$ and the claim follows.

The statements in the main text for the case when $\tilde{a} \approx \tilde{a}^*$ now follow by continuity.

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