Optimal Multithreaded Batch-Parallel 2-3 Trees

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Abstract
This paper presents a batch-parallel 2-3 tree $T$ in an asynchronous dynamic multithreading model that supports searches, insertions and deletions in sorted batches and has essentially optimal parallelism, even under the restrictive QRMW (queued read-modify-write) memory contention model where concurrent accesses to the same memory location are queued and serviced one by one.

Specifically, if $T$ has $n$ items, then performing an item-sorted batch (given as a leaf-based balanced binary tree) of $b$ operations on $T$ takes $O(b \cdot \log (\frac{n}{b} + 1) + b)$ work and $O(\log b + \log n)$ span (in the worst case as $b, n \to \infty$). This is information-theoretically work-optimal for $b \leq n$, and also span-optimal for pointer-based structures. Moreover, it is easy to support optimal intersection, union and difference of instances of $T$ with sizes $m \leq n$, namely within $O(m \cdot \log (\frac{n}{m} + 1))$ work and $O(\log m + \log n)$ span. Furthermore, $T$ supports other batch operations that make it a very useful building block for parallel data structures.

To the author’s knowledge, $T$ is the first parallel sorted-set data structure that can be used in an asynchronous multi-processor machine under a memory model with queued contention and yet have asymptotically optimal work and span. In fact, $T$ is designed to have bounded contention and satisfy the claimed work and span bounds regardless of the execution schedule.

Since all data structures and algorithms in this paper fit into the dynamic multithreading paradigm, all their performance bounds are directly composable with those of other data structures and algorithms in the same model. Finally, the pipelining techniques in this paper are also likely to be very useful in asynchronous parallelization of other recursive data structures.

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1 Introduction

The dynamic multithreading paradigm (see [7] chap. 27) is a common parallel programming model underlying many parallel languages and libraries such as Java [1], OpenMP [17], Cilk dialects [10, 16], Intel Thread Building Blocks [21] and the Microsoft Task Parallel Library [23]. In this paradigm, programs can use programming primitives such as threads, fork/join (also spawn/sync), parallel loops and synchronization primitives, but cannot stipulate how the subcomputations are scheduled for execution on the processors.

We consider a multithreaded procedure (which can be an algorithm or a data structure operation) to be correct if and only if it has the desired behaviour regardless of the execution schedule. Moreover, we wish to obtain good bounds on the work and span of the procedure, preferably independent of the execution schedule.

Unfortunately, many data structures and algorithms are designed in theoretical computation models with synchronous processors, such as the (synchronous) PRAM models, and so they can be difficult or impossible to implement in dynamic multithreading with the same asymptotic work/time bounds once we take memory contention into account, such as under the QRMW (queued read-modify-write) contention model described in Section 5, which captures both the asynchronicity and contention costs inherent in running multithreaded procedures on most real multi-processor machines. Thus it is desirable to have as many useful algorithms and data structures as possible designed in computation models compatible with dynamic multithreading.

One indispensable data structure is the map (or dictionary) data structure, which supports searches/updates, inserts and deletes (collectively referred to as accesses) of items from a linearly ordered set. Balanced binary trees such as the AVL tree or the red-black tree are commonly used to implement a sequential map, taking \( O(\log n) \) worst-case cost (in the comparison model) per access for a tree with \( n \) items. A related data structure is the sorted-set data structure, which supports intersection, union and difference of any two sets. Using maps based on balanced binary trees to implement sorted-sets yields \( O(\min(m,n) \cdot \log \max(m,n)) \) worst-case cost of each set operation where \( m, n \) are the sizes of the input sets.

The obvious question is whether we can have an efficient multithreaded parallel map, or an efficient multithreaded sorted-set, or both. In this paper we describe a pointer-based multithreaded batch-parallel 2-3 tree \( \mathbb{T} \) that is both information-theoretically work-optimal and span-optimal even under the QRMW contention model. Here the input batch is given as a leaf-based balanced binary tree. Specifically, performing a sorted batch of \( b \) accesses on an instance of \( \mathbb{T} \) with \( n \) items takes \( O( b \cdot \log(\frac{n}{b} + 1) + b ) \) work and \( O(\log b + \log n) \) span. This is superior to the work and span bounds of the PVW 2-3 tree [19] despite not having the luxury of lock-step synchronous processors.

Furthermore, since \( \mathbb{T} \) is a multithreaded data structure whose performance bounds are independent of the schedule, it is trivially composable, which means that we can use \( \mathbb{T} \) as a black-box data structure in any multithreaded algorithm and easily obtain composable performance bounds. Indeed, the parallel working-set map in [2] and the parallel finger structure in [13] both rely such a parallel 2-3 tree as a key building block.

2 Related Work

To illustrate the difficulty of converting data structures designed in the PRAM models to efficient multithreaded implementations, consider the PVW 2-3 tree [19] that supports performing an item-sorted batch of searches, insertions or deletions, which was designed in the EREW PRAM model.

Performing a sorted batch of searches in the PVW 2-3 tree involves splitting the batch into contiguous subbatches and pushing them down the tree in non-overlapping waves. This is easy with synchronous processors, but it is non-trivial to translate that pipelining technique to an asynchronous setting with queued memory contention, since a naive use of locking to prevent overlapping waves would cause the worst-case span to increase from \( O(\log b + \log n) \) to \( O(\log b \cdot \log n) \). Section 7.1 shows how this can be done. But performing a sorted batch of \( b \) insertions or deletions on the PVW 2-3 tree with \( n \) items involves spawning \( O(\log b) \) non-overlapping waves of structural changes from the bottom of the 2-3 tree upwards to the root, and for this there does not seem any way to eliminate the reliance on the processors’ lock-step synchronicity.

Other map data structures in the PRAM models include parallel B-trees by Higham et al. [15], parallel red-black trees by Park et al. [18] and parallel \((a,b)\)-trees by Akhremtsev et al. [3], all of which crucially rely on lock-step synchronous processors as well.
A different approach of pipelining using futures by Blelloch et al. [6] yields an implementation of insertion into a variant of PVW 2-3 trees that requires not only a CREW/EREW PRAM but also a unit-time plus-scan (all-prefix-sums) operation. The multithreaded parallel sorted-sets presented by Blelloch et al. in [4] take $O(b \cdot (\log \frac{n}{b} + 1))$ work but up to $\Theta(n \log b \cdot \log n)$ span per operation between two sets of sizes $n, b$ where $n \geq b$. The span was reduced to $O(\log b + \log n)$ by Blelloch et al. in [5], but that algorithm as written relies on $O(1)$-time concurrent reads and may take $\Omega(\sqrt{n})$ span in a queued memory contention model.

This paper shows that, using special pipelining schemes, it is actually possible to design a multithreaded sorted-set data structure variant of PVW 2-3 trees that requires not only a CREW/EREW PRAM but also a unit-time plus-scan (all-prefix-sums) operation. The multithreaded parallel sorted-sets presented by Blelloch et al. in [4] take $O(b \cdot (\log \frac{n}{b} + 1) + b)$ work and $O(\log b + \log n)$ span, even if only bounded memory contention is permitted. The techniques shown here can likely be adapted to pipeline top-down operations on many other tree-based data structures.

### 3 Main Results

This paper presents, to the author’s best knowledge, the first multithreaded sorted-set data structure that can be run on an asynchronous parallel pointer machine and achieves optimal work and span bounds even under queued memory contention.

Specifically, the underlying data structure $\mathbb{T}$ is a pointer-based batch-parallel 2-3 tree that works in the QRMW contention model (see Section 5) and supports performing an item-sorted batch of $b$ accesses within $O(b \cdot \log(\frac{n}{b} + 1) + b)$ work and $O(\log b + \log n)$ span (in the worst case as $b, n \to \infty$) where $n$ is the current size of $\mathbb{T}$ (Section 8.3). Here we of course assume that we are given an $O(1)$-step comparison function on pairs of items (i.e. the comparison model), but there is no loss of generality. This is information-theoretically work-optimal for $b \leq n$, and also span-optimal in the parallel pointer machine model. Furthermore, the input batch can be any balanced binary tree, including even another instance of $\mathbb{T}$, and hence $\mathbb{T}$ can be used to implement optimal persistent sorted sets supporting intersection, union and difference of sets with sizes $m \leq n$ in $O(m \cdot (\log \frac{n}{m} + 1))$ work and $O(\log m + \log n)$ span (Section 9).

$\mathbb{T}$ also supports performing an unsorted batch of $b$ searches within $O(b \cdot \log n)$ work and $O(\log b \cdot \log n)$ span (Section 8.2), or an unsorted batch of $b$ accesses within $O(b \cdot \log \max(n, n'))$ work and $O((\log b)^2 + \log n)$ span where $n'$ is the size of $\mathbb{T}$ after the batch operation (Section 8.5). These are useful when $b \gg n$. Additionally, $\mathbb{T}$ supports performing a reverse-indexing on an unsorted batch of $b$ direct pointers to distinct items in it, which yields a sorted batch of those items within $O(b \cdot \log \frac{n}{b} + b)$ work and $O(\log n)$ span (Section 8.4).

Actually, $\mathbb{T}$ is designed to have bounded contention, meaning that there is some constant $c$ such that every operation on $\mathbb{T}$ never makes more than $c$ concurrent accesses to the same memory location.

### 4 Key Ideas

$\mathbb{T}$ uses a pipelined splitting scheme to partition the 2-3 tree itself around the operations in the input batch, and then performs each operation on its associated part, and then uses a pipelined joining scheme to join the parts back up. Both pipelining schemes are top-down. The splitting scheme is similar to the search in the PVW 2-3 tree, except that we push the 2-3 tree down the input batch, rather than the input batch down the 2-3 tree. But the joining scheme is completely different from the bottom-up restructuring in the PVW 2-3 tree.

The main difficulty in both the splitting phase and joining phase is in finding a top-down procedure that can be decomposed into ‘independent’ local procedures each of which runs in $O(1)$ span, which can then be pipelined. This is not so hard for the splitting phase, but for the joining phase this seems to require using integers to maintain the structure of the spine nodes over a sequence of joins but without actually performing the joins (see the outline in Section 8).
5 Parallel Computation Model

In this section, we describe the programming model as well as the underlying memory model chosen in this paper.

5.1 Programming Model

We shall work within a high-level object-oriented dynamic multithreading model that supports procedures and threads with standard multithreading primitives (terminate, suspend, fork, join, resume). In this model, a thread $\tau$ can terminate itself, or fork a new thread (obtaining a reference to it), or join to another thread $v$ (i.e. wait until $v$ terminates). Or $\tau$ can suspend itself (i.e. temporarily stop running), and another thread (with a reference to $\tau$) can resume $\tau$ (i.e. make it continue running after the suspension). $\tau$ can also obtain a reference to itself. Each of these takes $O(1)$ steps.

This programming model also supports the standard RMW (read-modify-write) operations (including read, write, test-and-set, fetch-and-add, compare-and-swap), as in almost all modern architectures. To capture contention costs, we adopt the QRMW (queued read-modify-write) contention model, as described in [9], in which RMW operations on each memory location are FIFO-queued to be serviced, with only one RMW operation on that memory location serviced per time step. The thread making each memory request is blocked until the request has been serviced. Additionally, we require each memory location to be a named field of an object, denoted by “$x.d$” where $x$ is a reference to the object and “d” is the name of the field.

The actual complete execution of a multithreaded computation is captured by its execution DAG $E$ (which may be schedule-dependent), in which each node is a primitive instruction weighted by the time taken to execute it, and the directed edges represent the computation dependencies. Specifically, each thread $\tau$ executes a sequence of instructions, the first one depending on the fork instruction that forked $\tau$, and every subsequent instruction depending on the one just before it. The first instruction after a join instruction to join with thread $v$ depends also on the last instruction executed by $v$. And the first instruction after a suspend instruction executed by $\tau$ depends also on the resume instruction that resumed $\tau$. But there are no dependencies between concurrent accesses to the same memory location, even though they are linearized during the actual execution under the QRMW contention model.

We can view $E$ as being dynamically generated as the computation proceeds, where at any point during execution, a ready node in $E$ (i.e. node whose parents have been executed) corresponds to a running thread, and a scheduler is used to assign ready nodes to available processors (i.e. processors that are not executing any nodes) for execution. A greedy scheduler on each step assigns as many unassigned ready nodes as possible to available processors for execution.

We can now define work and span of a (terminating) multithreaded computation. This allows us to capture the intrinsic costs incurred by the computation itself, separate from the costs of any multithreaded program using it.

Definition 1 (Subcomputation Work/Span). Take any multithreaded computation on $p$ processors with execution DAG $E$, and any subcomputation $C$ (identified with a subset of the nodes in $E$). The work taken by $C$ is the total weight of $C$. The span taken by $C$ is the maximum possible total weight of the nodes in $C$ that lie on any (directed) path in $E$.

Note that work/span is subadditive across subcomputations, so performance bounds for algorithms and data structures in this model are composable. Moreover, all the algorithms and data structures in this paper achieve the stated work/span bounds independent of the scheduler.

In our analysis of a computation with execution DAG $E$, we will often reason about procedure run fragments, meaning a contiguous sequence of instructions executed during some procedure run. We will also reason about the relations between events that happened during the computation, where each such event $X$ is associated with some set $\Gamma_X$ of nodes in $E$ whose execution was required for $X$ to happen, where $\Gamma_X$ is closed under ancestors. Events are often described in terms of the start or end of certain fragments, such as the start or end of a procedure run. For this we define the following notion of span between events.

Definition 2 (Inter-Event Span). Take any events $X, Y$ in a multithreaded computation with execution DAG $E$. We say that $Y$ happens within $s$ span after $X$ iff $\Gamma_Y \setminus \Gamma_X$ has at most $s$ span in $E$.

For convenience, we shall also say that a fragment $C$ takes $s$ steps iff $C$ has no suspend node and takes $s$ work, in which case $C$ also ends within at most $s$ span after it starts.
5.2 Memory Model

The QRMW contention model was chosen because the synchronous PRAM model makes unrealistic assumptions including lock-step synchronicity of processors and lack of collision on concurrent memory accesses to the same locations [11, 12, 20]. For example, the load latency in the Cray XMT increases roughly linearly with number of concurrent accesses to the same address but stays roughly constant when the concurrent accesses are to random addresses [22].

In the QRMW PPM model, generalizing the PPM model in [14] to cater to the QRMW contention model, processors are asynchronous and all accesses to shared memory are done via pointers, which can be locally stored or tested for equality (but no pointer arithmetic). More precisely, each pointer (if not null) is to a memory node, which has a fixed number of memory cells. Each memory cell can hold a single field, which is either an integer or a pointer. Each processor also has a fixed number of local registers, each of which can hold a single field. At each step, each processor (that has finished its previous operation) can start any one of the following operations, which except for RMW operations finishes in one step:

1. Perform a basic arithmetic operation on integers in its registers, storing the result in another register.
2. Perform an equality-test between pointers in its registers, storing the result (0 or 1) in an integer register.
3. Perform an RMW operation on a memory cell via a pointer to the memory node that it belongs to.
4. Create a new memory node, storing a pointer to it in a register.

It is easy to see that the high-level programming model in Section 5.1 can be implemented in the QRMW PPM model if we have a greedy scheduler. It also turns out that a greedy scheduler can be approximated in the QRMW PPM model (with local RAM of size $p$) by a suitable work-stealing scheduler such as the one in [13], in the sense that any multithreaded computation that takes work and span on $p$ processors takes $O(\frac{w}{p} + s)$ expected time when run using that work-stealing scheduler. All these results hold in the QRMW PRAM model (i.e. asynchronous PRAM with the QRMW contention model) as well, though it is worth noting that the QRMW PPM model requires more sophisticated techniques because we cannot use pointer arithmetic.

6 Higher Synchronization Primitives

In this model we can implement the following synchronization primitives:

1. **Non-blocking locks**: A lock $L$ can be acquired and released over time, and is said to be held by a thread $r$ iff it has been acquired by $r$ but not yet released. A non-blocking lock $L$ is implemented as a boolean field. Operations on $L$ appear to be serialized (i.e. queued and performed one by one). TryLock($L$) attempts to acquire $L$ and succeeds if $L$ is not currently held but fails otherwise, and returns a boolean indicating whether it is successful. Unlock($L$) releases $L$. If at most $O(1)$ threads concurrently access $L$ (via any operation), then each access takes $O(1)$ steps (hence “non-blocking”).

2. **Barriers**: A single thread can wait at a barrier $B$ until another thread has notified $B$ to allow any waiting thread to continue. Notification takes $O(1)$ steps, and waiting at $B$ takes $O(1)$ work and will return within $O(1)$ span after the wait has started and $B$ has been notified. There must be only one waiting thread and one notifying thread for each barrier.

3. **Reactivation calls**: A procedure $P$ with no input/output can be encapsulated by a reactivation wrapper, in which it can be run only via reactivations. If there are always at most $O(1)$ concurrent reactivations of $P$, then the following appear to hold (see Theorem 7 for more precise guarantees):
   (a) Each reactivation call $C$ takes $O(1)$ steps and occurs at a single point during $C$, called a reactivation point. All reactivation points are distinct. At each reactivation point, if $P$ is not currently running then it starts running (in another thread), otherwise it will run again after its current run finishes.
   (b) Each run of $P$ starts within $O(1)$ span after either the end of the previous run of $P$ or after the start of a reactivation call with reactivation point after all previous runs.
   (c) The total work done by the reactivation wrapper is $O(1)$ times the number of reactivations.

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1 In this paper we use only integer addition, subtraction, multiplication, modulo and equality.
All data structures and algorithms in this paper will be expressed in procedural form using only the standard multithreading primitives, these synchronization primitives and read/write (i.e. we will not directly use any other RMW operations).

We now explain how to implement the higher synchronization primitives with proofs of their correctness and cost bounds.

6.1 Non-Blocking Lock

The non-blocking lock is trivially implemented using test-and-set on a boolean field used to store the lock’s state.

**Definition 3 (Non-Blocking Lock Operations).**

- **TryLock (Bool Field x):**
  
  Return ¬TestAndSet(x).

- **Unlock (Bool Field x):**
  
  Set x := false.

If we wish, we can of course have a non-blocking lock object with a boolean field for its state, and implement the lock operations on it in the same manner.

6.2 Wait-Notify Barrier

Using threads and test-and-set, we can also implement a barrier, where a condition C is initially false, and one thread can wait for C to become true (getting blocked if it is not), and another thread can notify that C has become true. ²

**Definition 4 (Barrier).**

Private Bool C := false.
Private Bool pass := false.
Private Thread t := null.

Public Wait():
  Set t := CurrentThread.
  If ¬TestAndSet(pass), suspend (current thread).

Public Notify():
  Set C := true.
  If TestAndSet(pass), resume t.

Public Notified():
  Return C.

**Theorem 5 (Barrier Properties).** If there is only one call to Wait() and one call to Notify(), then the following hold:

1. Wait() will not return before Notify() is called.
2. Wait() takes $O(1)$ work and will return within $O(1)$ span after both Wait() and Notify() have been called.
3. Notify() takes $O(1)$ steps, after which Notified() will always return true.

**Proof.** It is easy to see that expression “TestAndSet(pass)” is evaluated exactly twice, once by Wait() and once by Notify(), the first time to false and the second time to true. If Wait() evaluates that expression to false, then it suspends the thread calling it after having stored (a pointer to) that thread in t, and so the Notify() would subsequently evaluate that expression to true and resume that thread, within $O(1)$ span after Wait() and Notify() have been called. If instead Wait() evaluates that expression to true, Notify() must have already evaluated that expression to false, and Wait() would return within $O(1)$ span.

² This is different from Java’s semantics, because in Java a notify() before a wait() would fail to trigger the waiting thread.
6.3 Reactivation Wrapper

Next is the reactivation wrapper for an inputless procedure \( P \), which guarantees roughly that the runs of \( P \) will never overlap if \( P \) is run only via reactivations (i.e. by calling Reactivate()), and that there is always a complete run of \( P \) as soon as possible after each reactivation of \( P \), and moreover each reactivation triggers at most one run of \( P \).

**Definition 6 (Reactivation Wrapper).**

Private Procedure \( P \). // \( P \) is the procedure to be guarded by the wrapper.

Private Int \( count := 0 \).

Public Reactivate():

If \( \text{FetchAndAdd}(count, 1) = 0 \):

- Fork:
  - Do:
    - Set \( count := 1 \).
    - Call \( P() \).
  - While \( \text{FetchAndAdd}(count, -1) > 1 \).

We now give the precise guarantees of the reactivation wrapper along with an illustrated example.

![Diagram of reactivation calls and runs of Procedure P](image)

*Each rectangle represents the time interval for a reactivation call \( C_i \) or a run \( R_i \) of \( P \).*

- "\( \downarrow \)" represents a reactivation point, and "\( \downarrow \)" represents an initiation point.
- \( R_1, R_2, R_3, R_4 \) are triggered by \( C_1, C_2, C_4, C_5 \) respectively.
- \( R_1 \) must start within \( O(1) \) span after the start of \( C_1 \).
- \( R_2, R_3 \) must each start within \( O(1) \) span after the end of the previous run.
- \( R_4 \) starts within \( O(1) \) span after the previous run or after the start of \( C_5 \).

**Figure 1:** Example subcomputation generated by the reactivation wrapper for \( P \)

**Theorem 7 (Reactivation Wrapper Properties).** Consider only the reactivation calls to \( P \) (i.e. calls to Reactivate()), and only the runs of \( P \) initiated by them (i.e. via calls of \( P() \) from Reactivate()). Then the following hold if \( P \) is reactivated by at most \( k \) threads concurrently (see Figure 1):

1. No two (such) runs of \( P \) overlap (in time).
2. Each reactivation call \( C \) takes \( O(k) \) span.
3. Each reactivation call \( C \) can be associated with some **reactivation point** during \( C \), and each run \( R \) of \( P \) can be associated with some **initiation point** before \( R \) but after any previous run of \( P \), such that the following hold:
   (a) For every reactivation call with reactivation point \( t \), there is some run of \( P \) with initiation point after \( t \).
   (b) For every run \( R \) of \( P \) with initiation point \( t \), there is some reactivation call \( C \) with reactivation point \( s \) before \( t \), such that \( s \) is the earliest reactivation point after all initiation points before \( t \), and we say that \( C \) is the reactivation call that **triggers** \( R \). Furthermore, \( R \) starts within \( O(k) \) span after **either** the end of the previous run of \( P \) or after the start of \( C \), and in the latter case \( s \) is after the end of any previous run of \( P \).
4. The reactivation wrapper does \( O(k) \) work per reactivation call.

**Proof.** At any time, we say that a thread is at a line of code iff the thread is currently executing that line (not yet finished) or will execute that line next. For the sake of this analysis, we consider the execution of a line involving a FetchAndAdd operation to be finished once the FetchAndAdd operation itself has modified \( count \) (i.e. after the FetchAndAdd operation has modified \( count \), the thread will immediately be at the appropriate next line). And we say that a thread is in the **critical section** iff it is at some line after “If FetchAndAdd(\( count, 1 \) = 0;)”.
We shall establish the invariant that \( \text{count} \geq 0 \) and \( \text{count} \neq 0 \) iff there is a thread in the critical section, and that there is at most one such thread. Note that any thread that executes the “Fork:” line exits the critical section immediately after that, at the same time as the forked thread enters the critical section at the “Do:” line. Note also that a thread that enters or exits the critical section does so immediately after executing a line involving “\( \text{FetchAndAdd(count, 1)} \)” or “\( \text{FetchAndAdd(count, -1)} \)”. Thus it suffices to show that the invariant is preserved whenever \( \text{count} \) is changed (i.e. a thread finishes executing some line involving \( \text{count} \)):

\[ \diamond \quad \text{Whenever \( \text{count} \) is changed from zero: By the invariant, there was no thread in the critical section before that change, thus it must be due to some thread \( \tau \) executing “\( \text{FetchAndAdd(count, 1)} \)” resulting in \( \text{count} \) being changed to 1, upon which \( \tau \) immediately enters the critical section, preserving the invariant.} \]

\[ \diamond \quad \text{Whenever \( \text{count} \) is changed from nonzero: By the invariant, before that change \( \text{count} > 0 \) and there was a unique thread \( \tau \) in the critical section.} \]

\[ \diamond \quad \text{If the change was due to \( \tau \) executing “\( \text{FetchAndAdd(count, -1)} \)”': After that change, either \( \text{count} = 0 \) and \( \tau \) exits the critical section, or \( \text{count} > 0 \) and \( \tau \) remains in the critical section, in either case preserving the invariant.} \]

\[ \diamond \quad \text{If the change was due to another thread executing “\( \text{FetchAndAdd(count, 1)} \)”': That thread does not enter the critical section, and after that change \( \text{count} > 0 \) still, and \( \tau \) is still in the critical section, preserving the invariant.} \]

\[ \diamond \quad \text{If the change was due to } \tau \text{ executing “\( \text{count} := 1 \)”}, then after the change we have that \( \text{count} = 1 \) and \( \tau \) is still in the critical section, preserving the invariant.} \]

We can now use the invariant to prove the desired properties. Note that there are at most \( k \) concurrent evaluations of “\( \text{FetchAndAdd(count, 1)} \)” and by the invariant at most one thread in the critical section modifying \( \text{count} \), and so every access to \( \text{count} \) takes \( O(k) \) work/span. Property 1 is satisfied, since the only calls to \( P \) are from the critical section. Property 2 is also satisfied, since it accesses \( \text{count} \) once and performs at most one fork.

We now turn to Property 3. Define the reactivation point for a reactivation call \( C \) to be when it finishes evaluating “\( \text{FetchAndAdd(count, 1)} \)”, and the initiation point for a run \( R \) of \( P \) due to executing “\( \text{Call P()}. \)” to be when that instruction is reached (i.e. at the end of the execution of the previous instruction “\( \text{count} := 1 \)”). Clearly the reactivation point for \( C \) is during \( C \), and the initiation point for \( R \) is before \( R \) but after any previous run of \( P \).

To verify Property 3a, consider any thread \( \tau \) that makes a reactivation call \( C \) and finishes evaluating “\( \text{FetchAndAdd(count, 1)} \)” at time \( t \). The result of that evaluation satisfies \( r \geq 0 \) by the invariant, so there are two cases:

\[ \diamond \quad \text{If } r = 0: \ \tau \text{ enters the critical section and reaches “Call P()” after time } t. \]

\[ \diamond \quad \text{If } r > 0: \ By the invariant, there is a unique thread } \sigma \text{ in the critical section just before time } t, \text{ and } \text{count} > 1 \text{ at time } t. \text{ If } \sigma \text{ was at a line before “Call P()” at time } t, \text{ then it reaches “Call P()” later. If not, then } \sigma \text{ was at a line after “count := 1” at time } t, \text{ so no thread can decrease count after time } t \text{ until } \sigma \text{ evaluates “\( \text{FetchAndAdd(count, -1)} \)”}, \text{ upon which it remains in the critical section and reaches “Call P()” again later.} \]

To verify Property 3b, note that the reactivation call \( C_0 \) with the earliest reactivation point evaluates “\( \text{FetchAndAdd(count, -1)} \)” to 0 and the thread that executes \( C_0 \) is the first to enter the critical section, so the first run \( R_0 \) of \( P \) starts within \( O(k) \) span after \( C_0 \) starts, and hence Property 3b holds for \( R_0 \). Thus it suffices to verify Property 3b for any run \( R \) that has a previous run \( R' \). Let \( t, t' \) be the initiation points for \( R, R' \) respectively, and let \( \tau \) be the thread that executed \( R' \). Since \( \text{count} = 1 \) at both time \( t' \) and time \( t \), and \( \tau \) decreases \( \text{count} \) via “\( \text{FetchAndAdd(count, -1)} \)” at some time \( u \) between \( t' \) and \( t \), and by the invariant \( \text{count} \) can only be increased by an evaluation of “\( \text{FetchAndAdd(count, 1)} \)” there must be some reactivation point between \( t' \) and \( t \). Let \( C \) be the reactivation call with the earliest reactivation point \( s \) between \( t' \) and \( t \). There are two cases:

\[ \diamond \quad \text{If } s \text{ is before } u: \text{ count} > 1 \text{ just before } u, \text{ so } \tau \text{ remains in the critical section at time } u, \text{ and hence } R \text{ starts within } O(k) \text{ span after } R'. \]

\[ \diamond \quad \text{If } s \text{ is after } u: \text{ s is after the end of } R', \text{ and } C \text{ evaluates “\( \text{FetchAndAdd(count, 1)} \)” to 0 at time } s, \text{ so } R \text{ starts within } O(k) \text{ span after the start of } C. \]

Property 4 follows from Property 3b, since each reactivation call (excluding the forked subcomputations) takes at most \( O(k) \) work, and the total work done by the resulting forked subcomputations excluding the work done by the calls to \( P \) is at most \( O(k) \) times the number of runs of \( P \).
7 Basic Parallel Batch Operations

In this section we shall show how to do some basic operations on batches in the QRMW PPM model:

- **Filter** a batch of \( n \) items based on an \( O(1) \)-time condition, within \( O(n) \) work and \( O(\log n) \) span.
- **Partition** a sorted batch of \( n \) items around a sorted batch of \( k \) pivots, within \( O(k \cdot \log(\frac{k}{n} + 1) + k) \) work and \( O(\log n + \log k) \) span.
- **Join** a batch of \( b \) batches of items, with \( n \) items in total, within \( O(b \cdot n) + O(\log b \cdot \log n) \) span.
- **Merge** two sorted batches of items, with \( n \) items in total, within \( O(n) \) work and \( O(\log n) \) span.
- **Sort** a batch of \( n \) items within \( O(n \cdot \log n) \) work and \( O((\log n)^2) \) span.

We will always store any (non-empty) **batch** of items in a **BBT**, namely a leaf-based height-balanced binary tree (i.e. with the items only at its leaves). Each binary tree \( T \) is identified with its root node \( T.\text{root} \), and each node \( v \) of \( T \) stores the following:

- \( v.\text{left} \) and \( v.\text{right} \) are its left and right child nodes respectively.
- \( v.\text{height} \) and \( v.\text{size} \) are the height and number of leaves respectively of the subtree at \( v \).
- \( v.\text{first} \) and \( v.\text{last} \) are the first item and last item respective in the subtree at \( v \).

Note that the item at each leaf \( v \) of \( T \) is stored in \( v.\text{first} = v.\text{last} \). For convenience, we shall also use \( V(T) \) to denote the nodes of \( T \), and \( L(T) \) to denote the leaves of \( T \), and \( H(T, h) \) to denote the nodes of \( T \) with subtree height \( h \) (i.e. \( v \in H(T, h) \) iff \( v.\text{height} = h \)). Also, for any leaves \( v, w \) of \( T \) we shall write \( v \preceq_T w \) to mean that \( v \) is before or equal to \( w \) in \( T \), and we shall drop the subscript if it is clear from the context.

Each node of the BBT for a batch has additional fields that store various values and pointers to objects needed for the parallelization technique, which will be specified in the description of each parallel batch operation.

### 7.1 Pipelined Splitting

The **pipelined splitting scheme** is the key technique employed here for these parallel batch operations. This scheme is used to solve the problem of efficiently distributing the leaves of a binary tree \( T \) to the leaves of another binary tree \( U \) in a weak-order-preserving manner, meaning that each leaf \( v \) of \( T \) is sent to some leaf \( f(v) \) of \( U \), and for every leaves \( v, w \) of \( T \) such that \( v \preceq_T w \) we have \( f(v) \preceq_U f(w) \). This scheme can be used as long as we can always determine within \( O(1) \) span whether \( f(v) \preceq_U f(w) \) given any leaf \( v \) of \( T \) and any leaf \( w \) of \( U \).

The basic idea is that we can **push** \( T \) **down** \( U \) in a **pipelined** fashion: We first push the entire tree \( T \) to the root of \( U \), and whenever a subtree \( B \) of \( T \) arrives at an internal node \( v \) of \( U \), we push the whole \( B \) down to \( v.\text{left} \) if \( f(B.\text{first}) \preceq v.\text{left.last} \), or to \( v.\text{right} \) if \( f(B.\text{first}) \nless v.\text{right.last} \), but if neither holds then we **split** \( B \), each time pushing the appropriate half (i.e. left or right subtree) down to a child of \( v \) and continuing to split the other half. Note that the subtrees that arrive at each node of \( U \) form a slice of \( T \) (Definition 8), and at most one of them will be split.

**Definition 8 (Binary Tree Slice).** A **slice** of a binary tree \( T \) is a sequence of disjoint non-sibling subtrees of \( T \) that contain a set of consecutive leaves of \( T \). An **ordered slice** of \( T \) is a slice of \( T \) that has the subtrees listed in rightward order in \( T \).

It turns out that we can use queues \( v.\text{queue}[1] \) and \( v.\text{queue}[2] \) to store the unprocessed subtrees of \( T \) at each node \( v \) of \( U \), and maintain the **splitting invariant** that reverse(\( v.\text{queue}[1] \)) + \( v.\text{queue}[2] \) forms an ordered slice of \( T \). To do so, when we process a subtree of \( T \) from \( v.\text{queue}[i] \), if we push it down whole to a child \( w \) of \( v \) then we push it onto \( w.\text{queue}[i] \), otherwise if we are splitting it then we always push the split subtrees onto \( v.\text{left.}\text{queue}[2] \) or \( v.\text{right.}\text{queue}[1] \). **Figure 2** illustrates this.
This scheme can be carried out using a procedure \(v\text{-pushdown}[i]\) for each \(v\text{-queue}[i]\) that is run only via reactivation calls (see \textbf{Section 6}) and is reactivated whenever a subtree is pushed onto \(v\text{-queue}[i]\). We start by pushing \(T\) onto \(U\text{-root.queue}[1]\). Each run of \(v\text{-pushdown}[i]\) processes one subtree \(B\) from \(v\text{-queue}[i]\) (if any) and then reactivates itself if \(v\text{-queue}[i]\) was non-empty. Processing \(B\) means either pushing it down whole or forking a separate thread to split it (repeatedly pushing down the appropriate half and continuing to split the other half). Since at most one subtree that arrives at \(v\) is split, the splitting invariant guarantees that every subtree that arrives at \(v\) after that will only be pushed onto the outer queues \(v\text{-left.queue}[1]\) or \(v\text{-right.queue}[2]\). Therefore no concurrent pushes or concurrent pops are ever performed on any queue.

Thus each queue can be implemented using a \textbf{dedicated queue}, which is a wait-free single-producer single-consumer queue implemented by a linked list \(L\) in the following manner. \(L\) maintains pointers to both the first node \(L\text{-head}\) and the last node \(L\text{-tail}\), and every node \(v\) in \(L\) except \(L\text{-tail}\) stores an item \(v\text{-value}\) and a pointer to the next node \(v\text{-next}\), and \(L\text{-tail.next} = \text{null}\). Initially \(L\text{-head} = L\text{-tail}\). The dedicated queue operations are implemented as follows:

- **Push**( \(x\)): Create Node \(w\) with \(w\text{-next} = \text{null}\). Set \(L\text{-tail}.\text{value} := x\). Set \(L\text{-tail}.\text{next} := w\). Set \(L\text{-tail} = w\).
- **Pop**( ): Set \(h := L\text{-head}\). If \(h\text{-next} \neq \text{null}\), set \(L\text{-head} := h\text{-next}\). Return \(h\text{-value}\).

It can be shown that the total span is just \(O(T\text{-height} + U\text{-height})\) (see \textbf{Theorem 13}). But we must know what exactly the pipelined splitting scheme is used for to get a good bound on the total work.

### 7.2 Parallel Filtering

Parallel filtering an (unordered) batch \(T\) according to a condition \(C\), without changing the order in the batch, is done in 3 phases:

1. **Preprocessing phase**: Each item in \(T\) that satisfies \(C\) has a rank in the sublist of \(T\) that satisfies \(C\), which we shall call its filtered-rank. Recursively compute for each node \(v\) in \(T\) the number \(v\text{-count}\) of filtered items (i.e. items that satisfy \(C\)) in the subtree at \(v\), as well as the range \(v\text{-range}\) of filtered-ranks of the filtered items in the subtree at \(v\). Then construct a blank batch \(U\) of size \(T\text{-root.count}\) that is a complete BBT (i.e. every level is full except perhaps the last), and compute for each node \(w\) of \(U\) the number \(w\text{-count}\) of leaves in its subtree and the range \(w\text{-range}\) of their ranks in \(U\). And place a barrier \(w\text{-done}\) at each leaf \(w\) of \(U\).

2. **Push-down phase**: Use the pipelined splitting scheme (\textbf{Section 7.1}) to push \(U\) down \(T\), where a subtree \(B\) of \(U\) is pushed down whole to a node \(v\) of \(T\) iff \(B\text{-range} \subseteq v\text{-range}\). Then clearly each leaf of \(U\) will be pushed down to a unique leaf of \(T\) that has an item satisfying \(C\), and the order of those leaves in \(U\) is the same as the order of those items in \(T\). Thus when a leaf \(w\) of \(U\) reaches a leaf \(v\) of \(T\), we can simply copy the item from \(v\) to \(w\) and then notify \(w\text{-done}\).

3. **Collating phase**: After initiating the push-down phase, wait on \(w\text{-done}\) for each leaf \(w\) of \(U\), before returning \(U\). Then clearly \(U\) is only returned after the push-down phase has finished.

We shall now give the technical details, including the specific push-down phase obtained by applying the pipelined-splitting scheme here, and the splitting invariant involved.
Definition 9 (Parallel Filtering). Parallel filtering an (unsorted) batch (or more generally a leaf-based binary tree) $T$ according to a condition $C$, without changing the order of items in the batch, is done via the following procedure:

First preprocess the input batch and prepare the output batch $U$:

1. Recursively for each node $v$ of $T$, compute the number $v$.count of items in the subtree at $v$ that satisfy $C$. Then recursively compute $v$.rstart and $v$.rend defined by $T$.root.rstart = 0 and $v$.left.rstart = $v$.rstart and $v$.right.rstart = $v$.rstart + $v$.left.count and $v$.rend = $v$.rstart + $v$.count for each internal node $v$ of $T$.
2. If $T$.root.count = 0, return a blank output batch (skipping all the other phases).
3. Construct a blank complete BBT $U$ of size $T$.root.count, and compute for each node $v$ of $U$ the number $v$.count of leaves in its subtree, and $v$.rstart and $v$.rend defined in exactly the same way as for $T$.
4. In parallel place at each leaf $v$ of $U$ a Barrier $v$.done. // see Definition 4

Then push $U$ down $T$ to the appropriate leaf nodes via a pipelined splitting scheme:

1. In parallel place at each node $v$ of $T$ two newly created (empty) DedicatedQueues $v$.queue[1..2].
2. Define feeding $B$ to $v$.queue[i] to be pushing $B$ onto $v$.queue[i] and then reactivating $v$.pushdown[i].
3. Start by feeding $U$.root to $T$.root.queue[1].
4. Whenever $v$.pushdown[i] is reactivated for some node $v$ of $T$, it does the following:
   Pop subtree $B$ off $v$.queue[i]. If $B$ = null (i.e. $v$.queue[i] was empty), return.
   Reactivate $v$.pushdown[i].
   If $v$ is a leaf, copy the item from $v$ into $B$ and then call $B$.done.notify() and return.
   If $B$.rend $\leq$ $v$.left.rend, feed $B$ to $v$.left.queue[i] and return.
   If $B$.rstart $\geq$ $v$.right.rstart, feed $B$ to $v$.right.queue[i] and return.
Fork the following splitting process:

While $B$ is not a leaf:
   If $B$.left.rend $\leq$ $v$.left.rend:
      Feed $B$.left to $v$.left.queue[2] and set $B$ := $B$.right.
   Otherwise:
      Feed $B$.right to $v$.right.queue[1] and set $B$ := $B$.left.
   If $B$.rend $\leq$ $v$.left.rend, feed $B$ to $v$.left.queue[2], otherwise feed $B$ to $v$.right.queue[1].

And (after starting the push-down phase) collate the results by the following steps:
1. Call $v$.done.wait() for each leaf $v$ of $U$ in parallel (and wait for all to finish).
2. Recursively update $v$.first and $v$.last for each node $v$ of $U$.
3. Return $U$.

In our subsequent analysis we shall frequently use the fact that runs of $v$.pushdown[i] do not overlap (by Theorem 7 Property 1), and so we shall not mention it.

Lemma 10 (Parallel Filtering Invariants). Parallel Filtering satisfies the following for each node $v$ of $T$:

1. The subtrees fed to $v$ (i.e. to either $v$.queue[1] or $v$.queue[2]) form a slice of $U$.
2. The subtrees fed to $v$.queue[1] are (strictly) on the left in $U$ of those fed to $v$.queue[2].
3. The subtrees fed to $v$.queue[1] are in (strictly) leftward order and increasing depth in $U$.
4. The subtrees fed to $v$.queue[2] are in (strictly) rightward order and increasing depth in $U$.
5. At most one subtree fed to $v$ is split, after which only the splitting process $(*)$ will ever push onto $v$.left.queue[2] or $v$.right.queue[1].

Proof. We use structural induction on $T$. Invariants 1,2 follow from themselves for $v$.parent (i.e. the parent of $v$).
Invariants 3,4 follow from themselves and Invariants 2,5 for $v$.parent. To establish Invariant 5, observe that by Invariant 1 at most one subtree fed to $v$ is not pushed down whole, and if it was a subtree $B$ from $v$.queue[1] then the following hold at that point:

$\Diamond$ Every subtree from $v$.queue[2] is on the right of $B$, by Invariant 2, and so none of it gets fed to $v$.left.queue[2].
Hence from then on only the splitting process feeds to $v$.left.queue[2].

$\Diamond$ Every subsequent subtree from $v$.queue[1] is on the left of $B$, by Invariant 3, and so none of it gets fed to $v$.right.queue[1]. Also, every preceding subtree from $v$.queue[1] had already been fed to $v$.right.queue[1] in a preceding run of $v$.pushdown[1]. Hence from then on only the splitting process feeds to $v$.right.queue[1].

Likewise if it was a subtree $B$ from $v$.queue[2], by symmetry. Therefore Invariant 5 holds. $\Diamond$
Using these invariants we can prove both the correctness and costs of parallel filtering. In particular, by Invariant 5 there are no concurrent pushes performed on the dedicated queues, as required, and hence the pipelined splitting scheme runs correctly. And now we shall bound the parallel filtering costs.

**Lemma 11 (Tree Path Length Sum).** Given any tree $T$ with $e$ edges, for each node $v$ of $T$ let $m(v)$ be the length of the shortest path from $v$ to a leaf if $v$ has at least 2 children, but 0 otherwise. Then $\sum_{v \in V(T)} m(v) \leq e$.

**Proof.** Let $C$ be the nodes of $T$ with at least 2 children. For each node $v$ of $T$, let $P(v)$ be the downward path from $v$ that first takes the leftmost edge and then takes rightmost edges all the way to a leaf. Observe that $P(v)$ and $P(w)$ have disjoint edges for any distinct $v,w$ in $C$, since if they pass through a common node $x$ then it must be that one of them starts at $x$, so that path takes the leftmost edge, which is not the rightmost edge taken by the other path. Therefore $\sum_{v \in V(T)} m(v) \leq \sum_{v \in C} P(v).\text{length} \leq e$.

**Definition 12 (Log-Splitting Property).** We say that a binary tree $T$ is c-log-splitting if every slice of $T$ containing $k$ leaves of $T$ has at most $c \cdot \log_2 (k+1)$ subtrees of $T$. Note that every BBT is 4-log-splitting.

**Theorem 13 (Parallel Filtering Costs).** Parallel filtering a batch $T$ of size $n$ according to a condition $C$ takes $O(n \cdot w)$ work and $O(\log n + s)$ span if every evaluation of $C$ takes $O(w)$ work and $O(s)$ span.

**Proof.** The preprocessing phase clearly takes $O(n \cdot w)$ work and $O(\log n + s)$ span. And the collating phase clearly takes $O(n)$ work and ends within $O(\log n)$ span after it has started and the push-down phase has ended. So it only remains to show that the push-down phase takes $O(n)$ work and $O(\log n)$ span. Clearly initializing the nodes of $T$ takes $O(n)$ work and $O(T.\text{height}) \leq O(\log n)$ span.

Now observe that the work taken by the push-down phase is $O(1)$ times the number of feedings, since every self-reactivation of $v.\text{pushdown}[i]$ corresponds to a unique subtree that had been popped off $v.\text{queue}[1]$ or $v.\text{queue}[2]$, and every while-loop iteration in the splitting process at $v$ feeds a subtree to a child of $v$. Each node $v$ of $T$ has $O(\log (v.\text{count}+1)) \leq O(\log v.\text{size}+1)$ subtrees fed to it (by Lemma 10 Invariant 1 since $U$ is 4-log-splitting), and $\log v.\text{size}$ is $O(1)$ times the length of the shortest path from $v$ to a leaf (since $T$ is a BBT). Thus the number of feedings is $O(1)$ times the number of edges of $T$ (by Lemma 11), which is $O(n)$.

To bound the span taken by the push-down phase, we set forth some convenient definitions. Call a run of $v.\text{pushdown}[i]$ a $(v,i)$-run, and a reactivation of $v.\text{pushdown}[i]$ a $(v,i)$-reactivation. Call a $(v,i)$-run effective iff it process some subtree $X$ (i.e. it pops $X$ off $v.\text{queue}[i]$ and $X \neq \text{null}$). Call a $(v,i)$-run a $(v,i,X)$-run iff it processes the subtree $X$. Partition each run of the splitting process $(\ast)$ into fragments around the while-loop boundaries, and call each such fragment a $(v,X)$-splitter iff it splits the subtree $X$ (i.e. $B = X$ at the start of the loop). And for each subtree $X$ of $U$ let $d(X)$ be its depth in $U$, and observe that $d(X') < d(X)$ for any $(v,i,X')$-run that precedes a $(v,i,X)$-run (by Lemma 10 Invariants 3,4).

Note that each $(v,i)$-run performs a $(v,i)$-reactivation iff it is effective. Also, there cannot be three consecutive ineffective $(v,i)$-runs $R_1,R_2,R_3$ in that order, otherwise (by Theorem 7 Property 3b) there are distinct $(v,i)$-reactivations $C_1,C_2$ in that order such that $C_1$ triggers $R_2$ and $C_2$ triggers $R_3$, so $C_1$ ends after all $(v,i)$-runs preceding $R_1$ and $C_2$ starts before $R_1$, and hence $C_1$ and $C_2$ must be executed by feedings to $v$ and cannot overlap, which implies that some subtree $X$ is fed to $v.\text{queue}[i]$ between $C_1$ and $C_2$, but then $X$ must be processed by one of $R_1,R_2,R_3$, contradicting their ineffectiveness.

Now consider any $(v,i)$-run $R$. Let $X$ be the last subtree processed by the $(v,i)$-runs up to $R$ (which exists since the first $(v,i)$-run processes the first tree fed to $v.\text{queue}[i]$). There are two cases (by Theorem 7 Property 3b):

- $R$ starts within $O(1)$ span after the end of the previous $(v,i)$-run $R'$. Note that the last subtree $X'$ processed by the $(v,i)$-runs up to $R'$ satisfies $d(X') \leq d(X)$.
- $R$ starts within $O(1)$ span after the start of the $(v,i)$-reactivation $C$ that triggers $R$, and the reactivation point for $C$ is after the end of any previous $(v,i)$-run. In this case, the previous $(v,i)$-run $R'$ (if any) is ineffective, and hence $C$ must be executed by the feeding of some subtree $X'$ to $v.\text{queue}[i]$. Observe that any feeding of a subtree $X''$ to $v.\text{queue}[i]$ before $X'$ must push $X''$ onto $v.\text{queue}[i]$ before executing some $(v,i)$-reactivation $C'$ before $C$, and $C'$ must have reactivation point before the start of $R'$ (since $C$ triggers $R$), and hence $X''$ must have been processed by some $(v,i)$-run preceding $R$ since $R'$ is ineffective. Thus $R$ would process $X'$ if it had not already been processed by an earlier $(v,i)$-run, and hence $d(X') \leq d(X)$. 

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From this and the fact that there cannot be three consecutive ineffective \((v,i)\)-runs, we can deduce that for every \((v,i,X)\)-run \(R\), at least one of the following holds:

\[\diamond \quad R \text{ starts within } O(1) \text{ span after the end of some preceding } (v,i,X')\)-run where \(d(X') < d(X)\).
\[\diamond \quad R \text{ starts within } O(1) \text{ span after the start of some } (v,i)\)-reactivation executed by the feeding of some subtree \(X'\) to \(v.queue[i]\) where \(d(X') \leq d(X)\).

Moreover, except for the first feeding at the start of the push-down phase, every feeding of a subtree \(X\) to \(v.queue[i]\) is executed by either some \((v,parent,i,X)\)-run or some \((v,parent,X)\)-splitter. And every \((v,X)\)-splitter starts within \(O(1)\) span after either the start of some \((v,i,X)\)-run or the end of some \((v,X')\)-splitter where \(d(X') < d(X)\).

Therefore by induction every \((v,i,X)\)-run \(R\) starts within \(O(k+m+1)\) span after the push-down phase starts, where \(k\) is the depth of \(v\) in \(T\) and \(m\) is the depth of \(X\) in \(U\). Thus the whole push-down phase finishes within \(O(T.height+U.height+1) \subseteq O(\log n)\) span.

\[\diamond\]

Sometimes, it is also useful to use parallel filtering on a leaf-based binary tree that may not be a BBT but is sufficiently balanced. One suitable notion is as follows.

**Definition 14** \((c\text{-balanced Binary Trees})\). We say that a binary tree \(T\) is **\(c\text{-balanced}\)** (where \(c \in \mathbb{R}^+\)) iff \(v.height \leq c \cdot \log_2 (v.size)\) for every node \(v\) of \(T\).

**Remark.** Every BBT is \(2\)-balanced. Every red-black tree is also \(2\)-balanced.

We now prove a combinatorial lemma for \(c\)-balanced binary trees, which we will then use to prove the cost bounds for performing parallel filtering on such trees.

**Lemma 15** \((Balanced Tree Bound)\). \(\sum_{v \in V(T)} \log_2 v.size \leq 4c \cdot n\) for every \(c\)-balanced binary tree \(T\) with \(n\) leaves.

**Proof.** For each leaf \(x\) of \(T\), the \(d\)-th ancestor \(v\) of \(x\) satisfies \(v.size \geq 2^{\text{v.height}/c} \geq 2^{d/c} > 3\) if \(d \geq 2c\), since \(T\) is \(c\)-balanced. Let \(A(v)\) be the ancestors of any node \(v\) of \(T\) (including \(v\)). Then \(\sum_{v \in V(T)} \log_2 v.size = \sum_{x \in L(T)} \sum_{v \in A(x)} \frac{\log_2 v.size}{v.size} < \sum_{x \in L(T)} \left( \sum_{d=0}^{2c-1} \frac{\log_2^3}{3} + \sum_{d=2c}^{\infty} \frac{d/c}{2^{d/c}} \right) \) because \(\frac{\log_2 k}{k} \leq \frac{\log_2^3}{3}\) for every \(k \in \mathbb{N}^+\) and \(\frac{\log_2 k}{k} \leq \frac{\log_2 m}{m}\) for every real \(k \geq m > 3\). Since we have \(\sum_{d=0}^{2c-1} \frac{\log_2^3}{3} \leq 2c\) and \(\sum_{d=2c}^{\infty} \frac{d/c}{2^{d/c}} = \sum_{k=2}^{\infty} \sum_{i=0}^{k-1} \frac{k(c+i)/c}{2^{k(c+i)/c}} < \sum_{k=2}^{\infty} (c \cdot \frac{k+1}{2^k}) = 2c\), the desired claim follows.

**Theorem 16** \((General Parallel Filtering Costs)\). Parallel filtering an \(O(1)\)-balanced leaf-based binary tree \(T\) of size \(n\) and height \(h\) according to a condition \(C\) takes \(O(n \cdot w)\) work and \(O(h+s)\) span if every evaluation of \(C\) takes \(O(w)\) work and \(O(s)\) span.

**Proof.** To show the work bound, we just need to bound the number of readings, which is \(O(\sum_{v \in V(T)} (\log v.size + 1)) \subseteq O(n)\) by Lemma 15. The proof of the span bound is the same as before.

Parallel filtering can also be used to efficiently change the shape of the underlying BBT of any batch, such as to a complete binary tree.

**Definition 17** \((Parallel Balancing)\). Parallel balancing a batch \(T\) of items to make the underlying BBT a complete binary tree, without changing the order in the batch, can be done by parallel filtering (Section 7.2) with no condition (i.e. the condition \(C\) always returns true).

**Remark.** In general, we can obtain any desired shape of the underlying BBT of \(T\), by simply constructing the output batch to have the desired shape.

**Theorem 18** \((Parallel Balancing Costs)\). Parallel balancing a batch of \(n\) items takes \(O(n)\) work and \(O(\log n)\) span.

**Proof.** The claim follows immediately from Parallel Filtering Costs (Theorem 13).

**Remark.** Since a complete binary tree can be easily transformed into a 2-3 tree represented by a red-black tree, parallel balancing can be used to transform any batch of \(n\) items into a 2-3 tree within \(O(n)\) work and \(O(\log n)\) span.
7.3 Parallel Partitioning

Exactly the same technique allows us to do parallel multi-way partitioning of a sorted batch $T$ of items around a sorted batch $P$ of pivot items, in 3 similar phases:

1. **Preprocessing phase:** Insert $\infty$ into $P$ (treating $\infty$ as more than every item).
2. **Push-down phase:** Use the pipelined splitting scheme (Section 7.1) to push $T$ down $P$, where a subtree $B$ of $T$ is pushed down whole from a node $v$ of $P$ to $v$.left iff $B$.last $\leq v$.left.last and to $v$.right iff $B$.first $> v$.left.last. Then clearly each item $x$ in $T$ will be pushed down (in some subtree of $T$) to the leftmost leaf $v$ of $P$ such that $x \leq v$.last. We also close $v$ when no more subtrees will be fed to $v$. Once $v$ is closed and both $v$.queue[1] and $v$.queue[2] are empty, we wait for the splitting process at $v$ to finish before closing both $v$.left and $v$.right.
3. **Collating phase:** After starting the push-down phase, for each leaf $v$ of $P$, wait for $v$ to be closed before joining the subtrees in each $v$.queue[i] and then tagging $v$ with the join of the results.

As before, we shall give the technical details of the whole parallel partitioning algorithm here.

**Definition 19 (Parallel Partitioning).** Parallel partitioning a sorted batch $T$ of items around a sorted batch $P$ of pivot items is done via the following procedure:

1. Insert $\infty$ into $P$ (as the rightmost leaf).
2. Then **push $T$ down $P** by essentially the same pipelined splitting scheme as in **Parallel Filtering** (Section 7.2):
   1. In parallel place at each node $v$ of $P$:
      - DedicatedQueues $v$.queue[1..2].
      - Bool $v$.qclear[i] := false for each $i \in [1..2]$.
      - Bool $v$.frozen := false.
      - Pointer $v$.split := null.
      - Barrier $v$.fed. // see Definition 4
   2. Define feeding $B$ to $v$.queue[i] to be pushing $B$ onto $v$.queue[i] and then reactivating $v$.pushdown[i].
   3. Define closing $v$ to be calling $v$.fed.notify() and then reactivating both $v$.pushdown[1] and $v$.pushdown[2].
   4. Start by feeding $T$.root to $P$.root.queue[1] and then closing $P$.root.
   5. Whenever $v$.pushdown[i] is reactivated for some node $v$ of $P$, it does the following:
      - If $v$ is a leaf, return.
      - Set done := $v$.fed.notified().
      - Pop subtree $B$ off $v$.queue[i].
      - If $B$ = null (i.e. $v$.queue[i] was empty):
         - If done: // if no more subtrees will be fed to $v$ before $v$.queue[i] is found empty //
            - Set $v$.qclear[i] := true.
            - If $v$.qclear[3-i] and TryLock($v$.frozen):
               - If $v$.split $\neq$ null, call $v$.split.wait().
               - Close $v$.left and close $v$.right.
      - Return.
   6. Reactivate $v$.pushdown[i].
   7. If $B$.last $\leq v$.left.last, feed $B$ to $v$.left.queue[i] and return.
   8. If $B$.first $> v$.left.last, feed $B$ to $v$.right.queue[i] and return.
   9. Set $v$.split := new Barrier. // see Definition 4
   10. Fork the following **splitting process**:
       - While $B$ is not a leaf:
          - If $B$.left.last $\leq v$.left.last:
              - Feed $B$.left to $v$.left.queue[2] and set $B := B$.right.
          - Otherwise:
              - Feed $B$.right to $v$.right.queue[1] and set $B := B$.left.
              - If $B$.last $\leq v$.left.last, feed $B$ to $v$.left.queue[2], otherwise feed $B$ to $v$.right.queue[1].
              - Call $v$.split.notify().
   11. And (after starting the push-down phase) **collate** the results by doing the following for each leaf $v$ of $P$ in parallel:
       1. Call $v$.fed.wait().
       2. Join the batches in each $v$.queue[i] (by repeatedly joining the last two until there is only one left).
       3. Tag $v$ with the join of the results.
The correctness of the push-down phase in this parallel partitioning algorithm follows in the same way as for parallel filtering. To check the correctness of the whole algorithm, we just need to observe the following invariants:

1. If at any time \( v.\text{fed.}\text{notified}() \) is true, then there will be no more feeding to any queue of \( v \).
2. If at any time \( v.\text{fed.}\text{notified}() \) is true and \( v \) is an internal node that has empty queues and no ongoing splitting process, then eventually both \( v.\text{left.}\text{fed.}\text{notified}() \) and \( v.\text{right.}\text{fed.}\text{notified}() \) will be true and remain true.

These invariants imply that eventually \( v.\text{fed.}\text{notified}() \) will become and remain true for every leaf of \( P \), after which the results will be collated.

We now prove two lemmas, which we shall not only use to prove the cost bounds for parallel partitioning but also employ later for the Batch-Parallel 2-3 Tree (Section 8).

**Lemma 20 (BBT Slice Joining).** Any ordered slice \( S \) of a BBT \( T \) containing \( k \) leaves can be joined into a single BBT in \( O(\log(k+1)) \) sequential time.

*Proof.* \( S \) is the concatenation of two ordered slices such that in each of them the subtrees have monotonic height with at most one pair of the same height. Thus we can join the subtrees in each ordered slice from shortest to tallest, taking \( O(1) \) time per join, and then join the two results in \( O(\log(k+1)) \) time.

**Lemma 21 (BBT Log Sum Bound).** Take any real \( n \), and any BBT \( T \) with \( k \) leaves and a non-negative real weight-function \( m \) on its nodes such that \( \sum_{v \in H(T,h)} m(v) \leq n \) for every \( h \in [0..T.\text{height}] \). Then
\[
\sum_{v \in V(T)} \log(m(v) + 1) \leq O(k \cdot \log(k^2 + 1) + k).
\]

*Proof.* Let \( c(h) \) be the size of \( H(T,h) \). Each node in \( H(T,h) \) has at least \((\frac{3}{2})^h\) leaves by induction, and hence \( c(h) \leq k \cdot r^h \) where \( r = \frac{2}{3} \). And \( \sum_{v \in H(T,h)} \log(m(v) + 1) \leq c(h) \cdot \log(\frac{n}{c(h)} + 1) \) by Jensen’s inequality. Since \( c \cdot \log(\frac{n}{c(h)} + 1) \) increases when \( c \) increases, \( \sum_{v \in V(T)} \log(m(v) + 1) \leq \sum_{h=0}^{\infty} (k \cdot r^h \cdot \log(\frac{n}{k \cdot r^h} + 1)) \leq \sum_{h=0}^{\infty} \log(\frac{n}{k \cdot r^h} + 1) \in O(k \cdot \log(k^2 + 1) + k). \)

**Theorem 22 (Parallel Partitioning Costs).** Parallel partitioning a sorted batch \( T \) of \( n \) items around a sorted batch \( P \) of \( k \) pivots takes \( O(k \cdot \log(k^2 + 1) + k) \) work and \( O(n \log k) \) span.

*Proof.* First we bound the work taken by the push-down phase. Preparing \( P \) (i.e. inserting \( \infty \) and initializing the nodes) takes \( O(n + k) \) work and \( O(k) \) span, and \( \log n < \log(n+1) + (k-1) \cdot \log 1 \leq k \cdot \log(k^2 + 1) \) by Jensen’s inequality. Observe that the remaining work is \( O(1) \) times the number of feedings plus \( O(1) \) times the number of times \( v.\text{fed.}\text{notified}() \) changes from false to true for some node \( v \). The latter is clearly \( O(k) \), so it suffices to bound the number of feedings.

The subtrees fed to each node \( v \) of \( P \) form a slice of \( T \) (Definition 8), so the number of feedings to \( v \) is at most \( 4 \cdot \log_2(m(v) + 1) \) where \( m(v) \) is the total number of items in that slice (since \( T \) is 4-log-splitting). And clearly \( \sum_{v \in H(T,h)} m(v) \leq n \) for every \( h \in [0..T.\text{height}] \). Thus by the BBT Log Sum Bound (Lemma 21) the total number of feedings is \( O(k \cdot \log(k^2 + 1) + k) \).

Next we bound the span taken by the push-down phase. By the same proof as for Parallel Filtering Costs (Theorem 13), every run of \( v.\text{pushdown}[i] \) that processes some subtree of \( T \) ends within \( O(T.\text{height} + P.\text{height} + 1) \leq O(n + k) \) span after the push-down phase starts, since every such run must come before any run of \( v.\text{pushdown}[i] \) that reaches \( \text{TryLock}(v.\text{fed.}\text{notified}()) \) (at which point no more subtree will be processed at \( v \)). And every splitting process ends within \( O(n) \) span after the run that forked it ends. And each node will be closed within \( O(1) \) span after all these runs and splitting processes have ended and its parent (if any) has been closed. Therefore the push-down phase takes \( O(n + k) \) span.

Finally, we bound the work and span taken by the collation phase. The waiting clearly takes \( O(k) \) work and \( O(n) \) span. Joining the subtrees at each leaf \( v \) of \( P \) takes \( O(m(v) + 1) \leq O(n) \) work/span (see Lemma 10 and Lemma 20). Thus the collation phase takes \( O(1) \) times the work taken by the push-down phase, and we are done.
7.4 Parallel Joining

Incidentally, parallel joining (concatenating) of a batch of batches can be done efficiently by using a simple parallel filtering.

**Definition 23 (Parallel Joining).** Parallel joining a batch $T$ of $b$ batches is done via the following 2-phase procedure:

1. In parallel replace each leaf $v$ of $T$ by the BBT for the batch at $v$ if that batch is non-empty (i.e. if the batch $B$ stored at $v$ is stored in a BBT with root $r$, then copy the fields of $r$ to $v$), otherwise put a dummy item (null) at $v$.
2. Parallel filter (Section 7.2) $T$ to remove dummy items and hence obtain the desired output batch $U$.

**Theorem 24 (Parallel Joining Costs).** Parallel joining a batch of $b$ batches with total size $n$ takes $O(b+n)$ work and $O(b\log b + log n)$ span.

**Proof.** Phase 1 clearly takes $O(b+n)$ work and $O(b\log b + log n)$ span, after which $T$ is 4-balanced since every BBT is 2-balanced. By the General Parallel Filtering Costs (Theorem 16), we are done.

**Remark.** Alternatively, in phase 2 we can push $T$ down $U$ instead, after which every non-dummy item of $T$ will have been pushed to a unique leaf $v$ of $U$ and will be at one end of the slice of $T$ that reaches $v$, and hence we can in $O(1)$ steps store that item at $v$ as desired. $T$ may not be a BBT after phase 1, but still has $O(b\log b + log n)$ height and is 8-log-splitting. Thus the same proof as for Parallel Filtering Costs (Theorem 13) works.

Note that if we have a batch of unsorted instances of $T$, rather than just a batch of plain batches, then there is a more efficient algorithm to join them (Section 8.6).

7.5 Parallel Merging

Another useful operation is parallel merging of two sorted batches.

**Definition 25 (Parallel Merging).** Parallel merging two sorted lists $A, B$ is done via the following 3-phase procedure:

1. Parallel partition (Section 7.3) $B$ around $A$, resulting in a part of $B$ at each leaf of $A$.
2. For each leaf $v$ of $A$ in parallel, insert the item of $A$ at $v$ into the part of $B$ at $v$, optionally combining duplicates of the same item. (This combining procedure can be any $O(1)$-time procedure.)
3. Parallel join (Section 7.4) the resulting batches at the leaves of $v$.

**Theorem 26 (Parallel Merging Costs).** Parallel merging sorted lists $A, B$ with total size $n$ takes $O(n)$ work and $O(\log n)$ span.

**Proof.** Let $k, m$ be the sizes of $A, B$ respectively. Then phase 1 takes $O(k \cdot \log(\frac{n}{k} + 1) + k) \subseteq O(k + m)$ work and $O(k \log k + \log m)$ span (Theorem 22). Let $c_i$ be the size of the part of $B$ that was at the $i$-th leaf of $A$ after phase 1. Then phase 2 takes $O(\sum_{i=1}^k \log(c_i + 1) + k) \subseteq O(k \cdot \log(\frac{n}{k} + 1) + k)$ work (by Jensen’s inequality) and $O(\log k + \log m)$ span. And phase 3 takes $O(n)$ work and $O(\log n)$ span (Theorem 24).

**Remark.** Note that it is actually possible to merge two sorted lists of lengths even more efficiently in some cases if they are stored in instances of a specialized sorted-set data type, such as if they are stored in parallel sorted-sets based on the batch-parallel 2-3 tree (Section 9) and their sizes $k, m$ satisfy $k \ll m$.

7.6 Parallel Sorting

Now we come to the problem of sorting a batch of items from an arbitrary set $S$ with a given (strict) linear ordering $<$ on $S$. As is standard, let $S^n$ be the set of all length-$n$ sequences from $S$. In our setting, a sorting algorithm is a procedure that given any input batch of items from $S$ will output a batch containing exactly the same items sorted in (non-strict) $<$-increasing order. Each item in the input batch may have an arbitrary tag, and all copies of an item are considered equal under the ordering regardless of their tags, but these tags must be preserved in the output batch.

We shall first give the obvious parallel merge-sort PMSort that we obtain from parallel merging, reproduced from [13]. Incidentally, PMSort outperforms the multithreaded sorting algorithm in [8] (which takes $O(n \cdot \log n)$ work and $O((\log n)^3)$ span).
**Definition 27 (Parallel Merge-Sort).** Let PMSort be the procedure that does the following on an input batch $I$ of items:

If $I$.size $\leq 1$, return $I$. Otherwise, compute in parallel $A = \text{PMSort}(I, \text{left})$ and $B = \text{PMSort}(I, \text{right})$, and then parallel merge (Section 7.5) $A$ and $B$ into an item-sorted batch $C$, and then return $C$.

**Theorem 28 (PMSort Costs).** PMSort($I$) takes $O(n \cdot \log n)$ work and $O((\log n)^2)$ span for every sequence $I$ in $S^n$.

**Proof.** The claim follows directly from the work/span bounds for parallel merging (Theorem 26) and $I.\text{height} \in O(\log n)$.

PESort, also first given in [13], is obtained from PMSort by simply tweaking it to combine multiple copies of the same item into bundles, by combining bundles in each merging step. PESort that achieves the entropy bound for work but yet takes only $O((\log n)^2)$ span on an input list of $n$ items.

**Definition 29 (Parallel Entropy-Sort).** Define a bundle of an item $x$ to be a BT (binary tree) in which every leaf has a tagged copy of $x$. Let PESort be the parallel merge-sort variant that does the following on an input batch $I$ of items:

If $I$.size $\leq 1$, return $I$. Otherwise, compute in parallel $A = \text{PESort}(I, \text{left})$ and $B = \text{PESort}(I, \text{right})$, and then parallel merge (Section 7.5) $A$ and $B$ into an item-sorted batch $C$ of bundles, combining bundles of the same item into one by simply making them the child subtrees of a new bundle, and then return $C$.

Then PESort($I$) returns an item-sorted batch of bundles, with one bundle (of all the tagged copies) for each distinct item in $I$, and clearly each bundle has height at most $I$.height.

**Theorem 30 (PESort Costs).** PESort($I$) takes $O(H+n) \subseteq O(n \cdot \log n + n)$ work and $O((\log n)^2)$ span for every sequence $I$ in $S^n$ with item frequencies $q_1, \ldots, q_n$, where $H = \sum_{i=1}^{n}(q_i \cdot \ln \frac{n}{q_i})$.

**Proof.** The full proof can be found in [13], but note that the key lemma is simply that for each item $x$ with frequency $k$ in $I$ there are at most $O(k \cdot \log \frac{n}{k} + k)$ merging steps whose result contains $x$, and that key lemma can be generalized to Lemma 52.

PESort is not a full sorting algorithm, because PESort($I$) where $I$ is in $S^n$ returns a batch of bundles rather than a batch of items. Nevertheless, this suffices for many applications because each bundle in the output batch has height at most $H$.height $\in O(\log n)$. But we can in fact obtain a full parallel entropy-sorting algorithm. Specifically, we can convert each bundle in PESort($I$) to a batch (Definition 31), and then parallel join (Section 7.4) all those batches to obtain the desired output.

**Definition 31 (Bundle Balancing).** A bundle $B$ of size $b$ and height $h$ is balanced as follows:

Recursively construct a linked list through all the leaves of $B$, and mark the leaves of $B$ with (1-based) rank of the form $(i \cdot h + 1)$, and then extract those marked leaves as a batch $P$ by parallel filtering (Section 7.2). Then at each leaf $v$ in $P$, construct and store at $v$ a batch of the items in $B$ with ranks $i \cdot h + 1$ to $(i+1) \cdot h$, obtained by traversing the linked list forward.

Now $P$ is essentially a batch of size-$h$ batches (except perhaps the last smaller batch), which we then recursively join to obtain the batch of all items in $G$.

**Theorem 32 (Bundle Balancing Costs).** Balancing a bundle $B$ of size $b$ and height $h$ takes $O(b)$ work and $O(h)$ span.

**Proof.** Note that $B$ has less internal nodes than leaves, and so constructing the linked list takes $O(b)$ work and $O(h)$ span. Extracting the batch $P$ of items of $B$ with ranks at intervals of $h$ takes $O(b + P.\text{size}.h) = O(b)$ work and $O(h)$ span, by essentially the same proof as Theorem 13 since the number of feedings is at most $P.\text{size}.h$. Constructing the batches of items in-between those in $P$ takes $O(b)$ work and $O(P.\text{height}+h) \subseteq O(h)$ span, and recursively joining them takes $O(1)$ work and span per node of $P$ (except $O(h)$ span for the first joining involving the last batch).
8 Batch-Parallel 2-3 Tree

We now present the batch-parallel 2-3 tree $T$ and explain how to support the following operations on $T$:

- **Unsorted batch search/update:** Perform an unsorted batch of $b$ searches/updates within $O(b \cdot \log n)$ work and $O(\log b \cdot \log n)$ span, tagging each search with the result and a direct pointer to the item in $T$ (if it exists).

- **Sorted batch access:** Perform an item-sorted batch of $b$ accesses (i.e. searches/updates, inserts and deletes) to distinct items within $O(b \cdot \log (2^v + 1) + b) + O(\log b + \log n)$ span, tagging each access with the result and a direct pointer to the item in $T$ (if it exists).

- **Batch reverse-indexing:** Given an unsorted batch of $b$ direct pointers to distinct items in $T$, return a sorted batch of those items within $O(b \cdot \log \frac{n}{b} + b)$ work and $O(\log n)$ span.

Here $n$ is the current number of items in $T$, and a **direct pointer** is an object that allows retrieving the item and its attached value in $T$ at the time of the search in $O(1)$ steps. It must also be used in the batch reverse-indexing operation. All these operations must be performed sequentially (i.e. one batch operation at a time).

Note that the sorted batch access requires that the accesses are to distinct items, but there is no actual disadvantage to that constraint. Suppose we are given an item-sorted input batch $I$ of $b$ accesses that may have multiple accesses to be the same item. We can perform an easy parallel recursion on $I$ to compute which accesses to an item $x$ are the leftmost access to $x$ in $I$. Then we can recursively join all the accesses to $x$ into a single batch $B_x$ (see Lemma 20), store it at the leftmost access to $x$, and compute the effective result of $B_x$ (if they are performed in order), within $O(1)$ work per access and $O(\log b)$ span. After that, we can parallel filter (Section 7.2) out those leftmost accesses from $I$ to obtain an item-sorted batch $I'$ of the effective accesses, which are to distinct items, within $O(b)$ work and $O(\log b)$ span (Theorem 13). We can now perform the usual sorted batch access on $I'$, and perform one more parallel recursion to tag the original accesses in $I$ with the appropriate results.

The sorted batch access also requires the input batch to be item-sorted. But we can clearly also support unsorted batch access within $O(b \cdot \log \max(b, n))$ work and $O((\log b)^2 + \log n)$ span, by using parallel sorting (Definition 27) to sort the input batch. This is more efficient than unsorted batch search/update when $1 \ll b \ll n$. However, unsorted batch search/update is useful when $b \gg n$, since sorting the input batch would take more work. In fact, we shall see later that if we judiciously combine both unsorted batch search/update and sorted batch access with parallel entropy-sorting (Definition 29), we can support **unsorted batch access** within $O(b \cdot \log \max(n, n'))$ work and $O((\log b)^2 + \log n)$ span where $n'$ is the number of items in $T$ after that batch operation (Section 8.5).

The batch reverse-indexing operation is useful in maintaining synced instances of $T$ with the same items but sorted differently; we can tag each item with direct pointers into other instances of $T$, and after performing an $<=$-sorted batch access on an $<=$-sorted batch $S_1$, we can use the obtained batch of direct pointers into an $<=$-sorted instance $S_2$ to obtain the corresponding $<=$-sorted batch of items, which we can then use to perform the same accesses on $S_2$.

8.1 Preliminaries & Notation

$T$ stores the items in a leaf-based 2-3 tree encoded as a leaf-based red-black tree $T$ (i.e. every red node has two black child nodes, and every black node is either a leaf or has two child nodes at most one of which is red, and the black nodes correspond to the nodes of the 2-3 tree). From now on we shall drop the adjective “leaf-based” since we only use leaf-based trees.

For any 2-3 tree $X$ we shall also denote the children of a node $v$ of $X$ by $v$.left and $v$.right and $v$.mid (if it exists), and denote the height of $v$ in $X$ by $v$.height. If $X$ is encoded as a red-black tree $X'$ and $v$ corresponds to the node $v'$ in $X'$, then $v$.left would correspond to the first black descendant of $v'$ (and not necessarily $v'$.left), and likewise for $v$.right, and $v$.height would be the number of black nodes excluding $v'$ in any path from $v'$ to a leaf in $X'$. These apparent ambiguities will always be resolved by the context, which will always specify whether we treat a node as in a 2-3 tree or in a red-black tree.

For convenience, let $X + Y$ denote the standard join of 2-3 trees $X, Y$ in that order, and identify a 2-3 tree with its root node. Also we shall write “$X \sim Y$” and “$X \gg Y$” as short for “$X$.height = Y$.height” and “$X$.height > Y$.height” respectively.
8.2 Unsorted Batch Search

Performing an unsorted search/update on an input batch $I$ of $b$ searches/updates is done by calling USearch($T$.root,$I$) (Definition 33). Note that we cannot simply spawn a thread for each search that traverses $T$ from root to leaf, as it would incur $\Omega(b)$ span at the root of $T$ in the queued contention model (Section 5.1).

Definition 33 (Unsorted Search/Update).

Private USearch( Node $v$ of BBT $T$, Access Batch $B$ ):

// treat $T$ as a BBT

   If $B$ is empty, return.
   If $v$ is a leaf:
      In parallel tag each access in $B$ to an item $x$ such that $x = v$.last with a direct pointer to $v$.
      Check whether $B$ has an update, and if so execute the leftmost update in $B$ on $v$.
   Return.

   Parallel partition $B$ around pivot $v$.last into a lower part $L$ and an upper part $R$ (see Section 7.2).
   In parallel call USearch($v$.left,$L$) and USearch($v$.right,$R$) (and wait for both to finish).

Theorem 34 (Unsorted Search Costs). USearch($T$.root,$I$) takes $O(b \cdot \log n)$ work and $O(\log b \cdot \log n)$ span.

Proof. Clearly we can ignore any call USearch($v$, $B$) where $B$ is empty. Each call USearch($v$, $B$) with non-empty $B$ at an internal node $v$ of $T$ takes $O(B$.size$)$ work and $O(\log b)$ span to partition $B$ into $L$ and $R$ (Theorem 13). Hence the entire unsorted batch search takes $O(b \cdot \log n)$ work and $O(\log b \cdot \log n)$ span.

8.3 Sorted Batch Access

Performing a sorted access on an item-sorted input batch $I$ of accesses to distinct items is done in 3 phases:

1. Splitting phase: Split the items in $T$ (treated as a BBT) around the items in $I \setminus I$.last, using Parallel Partitioning (Section 7.3) but without inserting $\infty$ and without collating. Then every item $x$ in $T$ will end up in some subtree of $T$ at a leaf $v$ of $I$ such that if $x$ exists in $I$ then $x$ is at $v$.

2. Execution phase: At each leaf $v$ of $I$, join the subtrees of $T$ at $v$ into a single 2-3 tree (see Theorem 41), and execute the access at $v$ on that 2-3 tree.

3. Joining phase: Recursively join the 2-3 trees at the leaves of $I$ via a pipelined joining scheme that pushes those 2-3 trees down each other. Here is a high-level overview and explanation of the algorithm:

   (a) Define the spine structure (Definition 35) of a non-root spine node $v$ (i.e. along the leftmost/rightmost path) of a 2-3 tree $X$ as the $v$.height-bit binary number where the $k$-th (most significant) bit is 1 if the $k$-th spine node from $v$ downwards (along the spine) has 3 children, and is 0 otherwise. Then given any 2-3 trees and their left/right children’s spine structures, we can within $O(1)$ steps determine whether the join overflows (i.e. is taller than the original trees) and compute the left/right children’s spine structures for the join (Theorem 36, and see Figure 3 below).

   (b) Augmenting every 2-3 tree with spine structure (i.e. the spine structure of every non-root spine node $v$ is stored in $v$.spine) allows us to join any 2-3 tree $Y$ into $X$ top-down, if $X \backslash Y$ or $X \gg Y$, where we view the joining as pushing $Y$ down the spine of $X$, and at each node $v$ we perform a local adjustment that has the desired effect, based on $Y$ and $v$.left, $v$.mid, $v$.right and $v$.left.spine, $v$.right.spine alone. Specifically, we immediately update $v$.left, $v$.mid, $v$.right and $v$.left.spine, $v$.right.spine to their final values after the join. This includes when $c + Y$ overflows where $c$ is the next node along the way, in which case we also create a blank child $w$ of $v$ and tag $Y$ with $w$, so that at $c$ we can move the overflowed subtrees to $w$ without having to access $v$ again. (See Table 1 below for all needed local adjustments.)

   (c) Observe that each local adjustment in the above top-down joining procedure is independent from other local adjustments made at any other node in the 2-3 tree (Lemma 42), and hence multiple join operations can be pipelined (Lemma 43), as long as the local adjustments at each node remain in the same order.

   (d) This order constraint is easily achieved by using a dedicated queue $v$.queue at each spine node $v$ of $X$ to maintain the 2-3 trees currently at $v$, and using a procedure $v$.joinin that is run only via reactivation calls (see Section 5.1) to process each 2-3 tree $X$ in $v$.queue one by one and perform the appropriate local adjustment at $v$ before pushing $X$ down to a child of $v$ if appropriate. To push a 2-3 tree down to a node $w$, we push it onto (the back of) $w$.queue and then reactivate $w$.joinin. $v$.joinin also reactivates itself after it has processed each 2-3 tree from $v$.queue.
(e) Putting everything above together: We just need to prepare each 2-3 tree \( X \) at a leaf of \( I \) by augmenting it with spine structure, and then at each internal node of \( I \) recursively compute the join of the 2-3 trees computed by its children, pipelined in the above manner. Then the root of \( I \) would effectively compute the join of all the 2-3 trees at the leaves of \( I \), in the sense that its final state after all queued trees have been processed is the desired join (Theorem 45).

(f) So at the end we just have to wait for all queued trees to be processed, which can be done by waiting on a barrier \( \text{v}.\text{done} \) at every internal node of \( I \) (see Section 5.1), where \( \text{v}.\text{done} \) is notified when the corresponding joining has finished. If that joining was of \( Y \) into \( X \), it finishes after the local adjustment that makes \( Y \) a subtree of the resulting 2-3 tree, so we just have to tag \( Y \) with \( \text{v}.\text{done} \) so that the local adjustment that finishes that joining can notify \( \text{v}.\text{done} \).

It turns out that the same techniques used in the proof of the Parallel Partitioning Costs (Theorem 22) can be used to prove the desired work and span bounds for the sorted batch access (Theorem 39, Theorem 48, Theorem 49).

| Operation | Case | Local Adjustment |
|-----------|------|------------------|
| Join 2-3 trees \( L \) and \( R \) (where \( L \sim R \) or \( L \gg R \)) | \( L \sim R \) | ![Diagram 1] |
| \( L.\text{right} \sim R \) | ![Diagram 2] |
| \( L.\text{right} \gg R \) and \( L.\text{right}+R \) overflows | ![Diagram 3] |
| \( L.\text{right} \gg R \) and \( L.\text{right}+R \) does not overflow | ![Diagram 4] |
| Join 2-3 tree \( R \) to the right of 2-3 subtree \( L \) (where \( L \gg R \)) | \( L.\text{right} \sim R \) | ![Diagram 5] |
| \( L.\text{right} \gg R \) and \( L.\text{right}+R \) overflows | ![Diagram 6] |
| \( L.\text{right} \gg R \) and \( L.\text{right}+R \) does not overflow | ![Diagram 7] |

“\( L \leftarrow R_1 R_2 \ldots \)” denotes that \( R_1, R_2, \ldots \) are to be joined to the right of \( L \) in that order.

“?” denotes that we do not care whether the (middle) subtree exists (and we leave it as it is).

“\( X \)” denotes a newly created blank node, and “\( R[X] \)” denotes that \( R \) is tagged with \( X \).

**Table 1:** Local adjustments for 2-3 tree joining on the right (it is symmetrical on the left)
We shall now fill in the technical details. First is the precise definition of spine structure and the proof that it can be easily computed for the result of any join without actually performing the join.

**Definition 35 (2-3 Tree Spine Structure).** Take any 2-3 tree \( X \). The right spine structure of a node \( v \) of \( X \) is denoted by \( \text{rspine}(v) \) and defined as \( \sum_{k=1}^{\text{height}(v)} (c_k - 2) \cdot 2^{k-1} \) where \( c_k \) is the number of children of the node on the right spine of the subtree at \( v \) with distance \( k \) from the leaf. Symmetrically for the left spine structure of \( v \) denoted by \( \text{lspine}(v) \). The spine structure of a non-root node \( v \) of \( X \) is denoted by \( \text{spine}(v) \) and defined as \( \text{rspine}(v) \) if \( v \) is a right child and \( \text{lspine}(v) \) if \( v \) is a left child. Note that \( \text{spine}(v) = 0 \) if \( v \) is a leaf and is \( \text{spine}(v) = \text{spine}(v \text{.right}) + (c - 2) \cdot v \text{.right.weight} \) otherwise, where \( c \) is the number of children of \( v \) and \( v \text{.weight} = 2^v \text{.height} \), and symmetrically for \( \text{lspine}(v) \). Moreover, if \( v \) is a right child, then \( \text{spine}(v) = \text{spine}(v \text{.parent}) \% v \text{.weight} \), and symmetrically for a left child.

**Theorem 36 (2-3 Tree Join Spine Structure).** Take any 2-3 trees \( X, Y \). Given \( X \text{.weight}, \text{lspine}(X), \text{rspine}(X), Y \text{.weight}, \text{lspine}(Y), \text{rspine}(Y) \), within \( O(1) \) steps we can determine whether the join \( J := X + Y \) overflows (i.e. \( J \gg X, Y \)) and compute \( \text{lspine}(J), \text{rspine}(J) \).

**Proof.** If \( X \sim Y \), then \( X + Y \) overflows, and \( \text{lspine}(J) = \text{lspine}(X) \) and \( \text{rspine}(J) = \text{rspine}(Y) \), so we are done. So by symmetry we can assume \( X \gg Y \). Clearly \( X + Y \) overflows iff \( \text{rspine}(X) + Y \text{.weight} \geq X \text{.weight} \). During the joining of \( Y \) to \( X \), let \( J' \) be the tree just after adding \( Y \) to \( X \) as a right sibling of the right spine node \( v \) of \( X \) such that \( v \sim Y \), and let \( r \) be the root. Then \( \text{rspine}(J') = \text{rspine}(X) - \text{rspine}(X) \% Y \text{.weight} + Y \text{.weight} + \text{rspine}(Y) \). After that, whenever a non-root 4-child node \( w \) is split into two 2-child siblings, \( \text{rspine}(r) \) is unchanged, since \( w \text{.parent} \) gains 1 child while its rightmost child loses 2 children, and \( 2^2 = 10_2 \). If \( r \) is split, \( \text{lspine}(r) \) decreases by \( Y \text{.weight} \). Thus \( \text{rspine}(J) = \text{rspine}(J') \% X \text{.weight} \). Finally, we can check that \( \text{lspine}(J) = \text{spine}(X \text{.left}) + (\text{rspine}(J) - \text{rspine}(J) \% J \text{.right.weight}) \). (See Figure 3 for an illustrated example.)

![Figure 3: Example computation of r spine(X+Y) for X >> Y within O(1) steps given X.weight, r spine(X), Y.weight and r spine(Y). Here X.weight = 1000000_2, Y.weight = 100_2, r spine(X) = 101101_2 and r spine(Y) = 10_2, so X+Y does not overflow since 101101_2 + 100_2 < 10000000_2, and r spine(X+Y) = 101100_2 + 100_2 + 10_2 = 110010_2.](image-url)
Next is the algorithm for executing an input batch $I$ of $b$ accesses, which is done by calling $\text{Execute}(I)$ (Definition 37).

**Definition 37 (Sorted Access).**

Define **feeding** $B$ to $v.\text{queue}[i]$ to be pushing $B$ onto $v.\text{queue}[i]$ and then reactivating $v.\text{pushdown}[i]$.

Define **closing** $v$ to be calling $v.\text{fed.notify}()$ and then reactivating both $v.\text{pushdown}[1]$ and $v.\text{pushdown}[2]$.

**Public Execute (Item-Sorted Access Batch $I$):**

- If $I$ is empty, return.
  
  // Partition $T$ around $I \setminus I.\text{last}$ by pushing $T$ down $I$ //
  
  For each node $v$ of $I$ in parallel:
  
  1. Create DedicatedQueues $v.\text{queue}[1..2]$.
  2. Create Procedure $v.\text{pushdown}[i]$, which runs by calling $\text{PushDown}[i](v)$, for each $i \in [1..2]$.
  3. Initialize Bool $v.\text{qclear}[i] := \text{false}$ for each $i \in [1..2]$.
  4. Initialize Bool $v.\text{frozen} := \text{false}$.
  5. Initialize Pointer $v.\text{split} := \text{null}$.
  6. Create Barrier $v.\text{fed}$. // see Definition 4
  7. Create Barrier $v.\text{done}$.

  Feed $T.\text{root}$ to $I.\text{root}.\text{queue}[1]$ and then close $I.\text{root}$.

  // Rejoin the parts of $T$ after executing each access in $I$ on the correct part //

- Set $T := \text{Collate}(I.\text{root})$ and call $\text{Finalize}(I.\text{root})$.

**Private PushDown**[i](Node $v$ of BBT $I$):

- If $v$ is a leaf, return.
  
  Set $\text{done} := v.\text{fed}.\text{notified}()$.

  Pop subtree $B$ off $v.\text{queue}[i]$. // treat $B$ as a BBT

  If $B = \text{null}$ (i.e. $v.\text{queue}[i]$ was empty):
    
    - If $\text{done}$: // if no more subtrees will be fed to $v$ before $v.\text{queue}[i]$ is found empty //
      
      Set $v.\text{qclear}[i] := \text{true}$.

      If $v.\text{qclear}[3-i]$ and $\text{TryLock}(v.\text{frozen})$:
        
        If $v.\text{split} \neq \text{null}$, call $v.\text{split}.\text{wait}()$.

        Close $v.\text{left}$ and close $v.\text{right}$.

    Return.

  Reactivate $v.\text{pushdown}[i]$.

  If $B.\text{last} \leq v.\text{left}.\text{last}$, feed $B$ to $v.\text{left}.\text{queue}[i]$ and return.

  If $B.\text{first} > v.\text{left}.\text{last}$, feed $B$ to $v.\text{right}.\text{queue}[i]$ and return.

  Set $v.\text{split} := \text{new Barrier}$. // see Definition 4

  Fork the following splitting process:

  - While $B$ is not a leaf:
    
    - If $B.\text{left}.\text{last} \leq v.\text{left}.\text{last}$:
      
      Feed $B.\text{left}$ to $v.\text{left}.\text{queue}[2]$ and set $B := B.\text{right}$.

    Otherwise:
      
      Feed $B.\text{right}$ to $v.\text{right}.\text{queue}[1]$ and set $B := B.\text{left}$.

  If $B.\text{last} \leq v.\text{left}.\text{last}$, feed $B$ to $v.\text{left}.\text{queue}[2]$, otherwise feed $B$ to $v.\text{right}.\text{queue}[1]$.

  Call $v.\text{split}.\text{notify}()$. 

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The subsequent procedures involve 23Trees, where each 23Tree \( v \) is a 2-3 tree node in a weak sense; \( v \) is either a leaf node or an internal node with 2 or 3 child nodes that are 23Trees, and \( v\.height = w\.height + 1 \) for each child \( w \) of \( v \).

**Definition 38 (23Tree Joining).**

- **Private Collate( Node \( v \) of BBT \( L \) )**: If \( v \) is a leaf:
  - Call \( v\.fed\.wait() \).
  - // Join all 2-3 trees at \( v \) and execute the access at \( v \) //
  - Convert each subtree in each queue \( v\.queue[i] \) to a 2-3 tree (i.e. make its root node black).
  - Create empty 2-3 trees \( L,R \). // An empty 2-3 tree is stored as null.
  - For each 2-3 tree \( Q \) in \( v\.queue[1] \) in reverse order, sequentially join \( Q \) onto the right of \( L \).
  - For each 2-3 tree \( Q \) in \( v\.queue[2] \) in reverse order, sequentially join \( Q \) onto the left of \( R \).
  - Sequentially join \( L,R \) to obtain 2-3 tree \( X \).
  - Execute the access at \( v \) on \( X \).
  - // Prepare the resulting 2-3 tree for pipelined joining // see Definition 38
  - Call \( \text{InitLeft}(w) \) for each non-root left spine node \( w \) of \( X \) in order from leaf to root.
  - Call \( \text{InitRight}(w) \) for each non-root right spine node \( w \) of \( X \) in order from leaf to root.
  - Return \( X \).

- // Return the join of the results from the child nodes of \( v \) //
  - In parallel set \( L := \text{Collate}(v\.left) \) and \( R := \text{Collate}(v\.right) \) (and wait for both to finish).
  - Return Join(\( L,R,v\.done \)). // see Definition 38

- **Private Finalize( Node \( v \) of BBT \( L \) )**: If \( v \) is a leaf, return.
  - In parallel call \( \text{Finalize}(v\.left) \) and \( \text{Finalize}(v\.right) \) (and wait for both to finish).
  - Call \( v\.done\.wait() \).

The subsequent procedures involve 23Trees, where each 23Tree \( v \) is a 2-3 tree node in a weak sense; \( v \) is either a leaf node or an internal node with 2 or 3 child nodes that are 23Trees, and \( v\.height = w\.height + 1 \) for each child \( w \) of \( v \).
Private Join(23Tree L, 23Tree R, Barrier done):  // returns join of root 2-3 trees L,R; see Table 1
If L = null, call done.notify() and return R.
If R = null, call done.notify() and return L.
If L ~ R:
    Call done.notify() and return RJoin(L, R).
If L > R:
    Call InitRight(R).
    If L.right ~ R:
        Call done.notify().
        If L.mid = null:
            Set L.mid := L.right and L.right := R.
            Return L.
        Otherwise:
            Return RJoin(SJoin(L.left, L.mid), SJoin(L.right, R)).
    Otherwise:
        Set R.joined := done.
        Create Pointer R.overflow := null.
        If L.right.spine + R.weight ≥ L.right.weight:  // L.right + R overflows; see Theorem 36
            Create blank 23Tree (node) X with same height as L.right.
            Set R.overflow := X.
            FeedRight(R, L.right).
            If L.mid = null:
                Set L.mid := X.
                Return L.
            Otherwise:
                Return RJoin(SJoin(L.left, L.mid), SJoin(X, L.right)).
        Otherwise:
            FeedRight(R, L.right).
            Return L.
Symmetrically if L < R.

Private JoinRight(23Tree Node L) returns the Procedure that runs as follows:
Pop (the first) 23Tree R off L.queue.
If R = null, return.
If L.right ~ R:
    If L.mid = null:
        Set L.mid := L.right and L.right := R.
    Otherwise:
        Copy (root of) SJoin(L.left, L.mid) to R.overflow.
        Set L.left := L.right and L.mid := null and L.right := R.
        Call R.joined.notify().
    Otherwise:
        If L.right.spine + R.weight ≥ L.right.weight:  // L.right + R overflows; see Theorem 36
            Create blank 23Tree (node) X with same height as L.right.
            If L.mid = null:
                Set L.mid := X.
            Otherwise:
                Copy (root of) SJoin(L.left, L.mid) to R.overflow.
                Set L.left := X and L.mid := null.
                Set R.overflow := X.
            FeedRight(R, L.right).
        Reactivate L.joinin.

Private JoinLeft(23Tree Node R) is symmetrically defined.
First we deal with the correctness and performance bounds for the splitting phase and execution phase (i.e., the procedures Execute, PushDown, Collate in Definition 37, excluding the preparation for pipelined joining in Collate).

**Theorem 39 (Splitting+Execution Guarantees).** The splitting phase and execution phase take $O(b \cdot \log(\frac{2}{b} + 1) + b)$ work and $O(\log b + \log n)$ span, and their result is that the join of the 2-3 trees at the leaves of $I$ is the desired final 2-3 tree.

*Proof.* The claims follow from Theorem 41 below and the same reasoning as for Parallel Partitioning (Section 7.3), since they are essentially identical. In particular, the execution phase takes $O(\log (c + 1))$ work/span at each leaf $v$ of $I$ to collate the 2-3 trees there with total size $c$, and then takes another $O(\log (c + 1))$ work/span to perform the access at $v$ on the result.

**Definition 40 (2-3 Tree Slice).** A slice of a 2-3 tree $T$ is a minimal-length sequence of disjoint 2-3 subtrees of $T$ that contain a set of consecutive leaves of $T$. An ordered slice of $T$ is a slice of $T$ that has the subtrees listed in rightward order in $T$.

**Theorem 41 (2-3 Tree Slice Joining).** Any ordered slice $S$ of a 2-3 tree $T$ containing $k$ leaves can be joined into a single 2-3 tree in $O(\log k)$ sequential time.

*Proof.* $S$ is the concatenation of two ordered slices such that in each of them the subtrees have monotonic height and there are at most two subtrees of each height. Note that if 2-3 trees $X, Y, Z$ have height in the range $[h-1, h]$, then $(X + Y) + Z$ is a 2-3 tree of height in the range $[h, h+1]$ and takes $O(1)$ time. Thus we can join the subtrees in each ordered slice from shortest to tallest, taking $O(1)$ time per join, and then join the two results in $O(\log k)$ time. 

So we will devote the rest of this section to analyzing the joining phase (i.e., the pipelined joining preparation in Collate, and the procedures $SJoin$, $RJoin$, $FeedLeft$, $FeedRight$, $Join$, $JoinLeft$, $JoinRight$ in Definition 38, and Finalize in Definition 37). First we introduce some basic observations and terminology.

Observe that the 23Tree nodes involved always form a DAG (according to the left/mid/right child pointers) at any point in time (since children are always shorter). Based on this, we say that the subtree at a 23Tree node $v$ is the set of nodes reachable by following child pointers. We also say that a 23Tree $X$ is queued at $v$ iff $X$ is in $v.queue$, in which case we call $X$ a queued tree at $v$. Additionally, we can impose a partial ordering on the queued trees called the pipeline order, where a queued tree $X$ at $v$ is before a queued tree $Y$ at $w$ iff $v$ is a strict descendant of $w$ or both $v = w$ and $X$ is before $Y$ in $v.queue$.

Next observe that if 23Trees $L, R$ satisfy the property that $u.joinin$ is defined for both its left and right child $u$, then $Join(L, R, \_)$ also does. And $v.joinin$ is only defined via $InitLeft(v)$ or $InitRight(v)$, so we can check that it feeds to a node $w$ (i.e., calls $FeedLeft(\_, w)$ or $FeedRight(\_, w)$) only if $w.joinin$ is already defined. Hence the feedings done by $v.joinin$ are well-defined.

We shall call a run of $v.joinin$ effective iff it pops off a (non-null) tree $X$ from $v.queue$, in which case we say that it processes $X$. Note that each queued tree $X$ will be processed by $v.joinin$ for each node $v$ that it is fed to. Clearly ineffective runs have no effect, and we can from now on view all the runs of $Join, JoinLeft, JoinRight$ as atomic, because runs of $Join$ clearly do not interfere with each other, and because $v.joinin$ is guarded by the reactivation wrapper (Definition 6) and each effective run of $v.joinin$ is independent of other processes (Lemma 42).

**Lemma 42 (Joinin Run Independence).** For each 23Tree node $v$ involved in the joining phase, what the sequence of all effective runs of $v.joinin$ do is independent of any other runs of $Join$ or $JoinLeft$ or $JoinRight$. (In other words, the effect of those runs only depends on the initial subtree at $v$ and the sequence of queued trees processed by $v.joinin$.)

*Proof.* Note that every 23Tree node $w$ has constant $w.height$ (and hence $w.weight$) that was fixed at its creation. Thus each run of $v.joinin$ that processes a queued tree $X$ depends only on $X$ and the fields $v.left, v.mid, v.right, v.left.spine, v.right.spine$, and so it suffices to show that these fields are modified only by $v.joinin$. Other runs of $JoinLeft$ or $JoinRight$ besides those of $v.joinin$ will not modify these fields, since they can only do so if $v = X.overflow$ for some queued tree $X$, but that is impossible because $X.overflow$ is always a blank 23Tree node at the point when it is set, and neither $InitLeft$ nor $InitRight$ is ever called on it after that, so $X.overflow.joinin$ is never defined. And a run of $Join$ can only modify these fields if it returns $v$, but clearly all runs of $v.joinin$ can only start after that run of $Join$ has returned.

This independence lemma also implies that given any 23Tree $X$, the result of processing all queued subtrees in $X$ (without ever feeding $X$) is uniquely determined by the current state of $X$, and we can define that result to be the joined state of $X$, as made precise in the next lemma.
Lemma 43 (23Tree Joined State). Define a **joining sequence** for a 23Tree \( X \) to be a sequence of effective joinin runs on the subtree at \( X \) (i.e. each is an effective run of \( v.\text{joinin} \) for some node \( v \) in the subtree at \( X \)) that processes queued trees in (the subtree at) \( X \) until there are none left. Then every joining sequence for \( X \) terminates and yields the same resulting subtree at \( X \), which we call the **joined state** of \( X \).

**Proof.** Observe that each queued tree \( Y \) in \( X \) that is processed by \( v.\text{joinin} \) is either fed to a child of \( v \) if \( v \gg Y \) or stops being a queued tree if \( v \sim Y \). Thus \( Y \) eventually stops being a queued tree, and so every joining sequence eventually terminates. Now observe that each effective run (viewed atomically) does not change the pipeline order on the (remaining) queued trees, and hence all effective runs of \( v.\text{joinin} \) at any particular node \( v \) of \( X \) process exactly the same queued trees in exactly the same (pipeline) order regardless of which joining sequence for \( X \) is executed, hence yielding the same result because of their independence (Lemma 42).

Note that during the joining phase, the joined state of a node \( v \) involved may change (if \( v \) is fed).

We can now state and prove the correctness of Join (Theorem 45), and then the correctness of sorted accesses (Theorem 47).

**Definition 44 (2-3 Tree With Spine Structure).** We say that a 2-3 tree \( X \) is **with spine structure** iff every non-root spine node \( v \) of \( X \) has spine structure (Definition 35) \( v.\text{spine} \).

**Theorem 45 (23Tree Joining Correctness).** Take any run of Join on 23Trees \( L, R \) that returns the 23Tree \( J \) during the joining phase. Let \( L', R' \) be the joined states of \( L, R \) respectively just before that run, and let \( J' \) be the joined state of \( J \) just after that run. If \( L', R' \) are 2-3 trees with spine structure, then \( J' \) is also a 2-3 tree with spine structure, and furthermore \( J' = L' + R' \) (i.e. \( J' \) is the standard join of \( L' \) and \( R' \)).

**Proof.** Assume as given that \( L', R' \) are 2-3 trees with spine structure. By symmetry we can also assume that \( L \sim R \) or \( L \gg R \). Let \( S \) be the global state just before the Join(\( L, R, done \)) run. We shall now consider two possible sequences of runs of Join, JoinLeft, JoinRight (treating them as atomic) that can be executed starting from the same state \( S \).

By definition, \( J' \) is the resulting state of \( J \) upon performing the following in order:

1. Execute \( J := \text{Join}(L,R,done) \) (which may make \( R \) a queued tree in \( J \)).
2. Process all queued trees in \( J \).

By Joinin Run Independence (Lemma 42), \( J' \) is also the resulting state of \( J \) upon performing the following 3 stages in order:

1. Process all queued trees in \( L \).
2. Execute \( J := \text{Join}(L,R,done) \), and then process \( R \) until it is not a queued tree in \( J \).
3. Process all (remaining) queued trees in \( J \).

Henceforth we shall assume this second sequence of runs. Clearly, stage 1 makes \( L \) become \( L' \), and stage 3 makes \( R \) (which is a subtree on the right spine of \( J \) after stage 2) become \( R' \). \( L' \) is a 2-3 tree with spine structure, so stage 2 is effectively performing the standard 2-3 tree joining algorithm to join \( R \) on the right of \( L' \) (based on Theorem 36), treating \( R \) as a 2-3 tree with height \( R.\text{height} \). Since we always have \( R.\text{height} \approx R'.\text{height} \), the result \( J' \) of these 3 stages is indeed \( L' + R' \).

It remains to verify that \( J' \) is with spine structure as well. The non-root spine nodes of \( J' \) comprise:

- The non-root left spine nodes of \( L' \).
- The non-root right spine nodes of \( R' \).
- The root of \( R' \).
- The non-root right spine nodes of \( L' \) that \( R \) was fed to (via FeedRight).
- The nodes \( X,Y \) if the Join(\( L, R, done \)) run calls RJoin(\( X,Y \)).

First observe that Join is run only on 23Trees that have no queued tree at the root. Thus before the Join(\( L, R, done \)) run, \( L \) has the same children as \( L' \), each child \( v \) of \( L \) having the same \( v.\text{spine} \) as in \( L' \), and likewise for \( R \). Moreover, stages 2,3 do not feed any child of \( R \), so RSpine(\( R' \)) = rSpine(\( R' \)) throughout all stages. Thus if \( L \gg R \), then the Join(\( L, R, done \)) run sets \( R.\text{spine} \) to RSpine(\( R' \)), and it is never changed throughout stages 2,3.
With these observations, we can then check that if the \( v \) is a 2-3 tree with spine structure \( v.\text{spine} \) just before the call, then just after the call \( v.\text{spine} \) is the spine structure \( s \) of \( v \) in \( J' \). To see why, let \( R_0 \) be the state of \( R \) during \( C \), and let \( v_0 \) and \( v_1 \) be the state of \( v \) just before and just after \( C \) respectively, and consider \( v_0 \) to also denote the subtree at \( v_0 \) just before \( C \). Then \( v_1.\text{spine} = (v_0.\text{spine} - v_0.\text{spine}\%R'.\text{weight} + R'.\text{weight})\%v_0.\text{weight} + \text{rspine}(R') \), because \( R_0.\text{weight} = R'.\text{weight} \) and \( \text{RSpine}(R_0) = \text{rspine}(R') \) since \( R_0.\text{right}\text{.spine} = R'.\text{right}\text{.spine} = \text{spine}(R'.\text{right}) \). Also \( \text{rspine}(v_0 + R') = v_0.\text{spine} - v_0.\text{spine}\%R.\text{weight} + R.\text{weight} + \text{rspine}(R') \) by Theorem 36. If \( v_0 + R' \) does not overflow, then \( v_0.\text{spine} + R'.\text{weight} < v_0.\text{weight} \), and hence \( v_1.\text{spine} = \text{rspine}(v_0 + R') = s \). But if \( v_0 + R' \) does overflow, then \( v_1.\text{spine} = \text{rspine}(v_0 + R') \%v_0.\text{weight} = \text{spine}((v_0 + R').\text{right}) \).

With these observations, we can then check that if the \( \text{Join}(L,R,done) \) run calls \( \text{RJoin}(X,Y) \) on some nodes \( X,Y \), then at that point \( X.\text{left}\text{.spine} \) and \( Y.\text{right}\text{.spine} \) have been set to \( \text{spine}(X.\text{left}) \) and \( \text{spine}(Y.\text{right}) \) respectively, and hence \( X.\text{spine} \) and \( Y.\text{spine} \) will also be set to their spine structure in \( J' \).

Therefore we can now easily verify that every non-root spine node \( v \) in \( J' \) has spine structure \( v.\text{spine} \).

Definition 46 (Finished 23 Tree). We say that a 23Tree \( X \) is a finished iff it is a 2-3 tree with spine structure (Definition 44) and with no queued trees (which implies that \( X \) is its own joined state).

Theorem 47 (Sorted Access Correctness). Each call to \( \text{Execute}(I) \) (i.e. the sorted batch access on input batch \( I \)) eventually returns, at which point \( T \) is a finished 23Tree that matches the result of performing \( I \) on the original \( T \) just before the call.

Proof. The call to \( \text{Execute}(I) \) eventually returns, because each \( \text{Join}(L,R,done) \) run either calls \( done.\text{notify}() \) or makes some 23Tree \( X \) a queued tree after setting \( X.\text{joined} := done \), and whenever \( X \) is processed either it remains a queued tree or \( X.\text{joined.} \text{notify}() \) is called, so \( \text{Finalize}(I.\text{root}) \) eventually returns. And the call to \( \text{Execute}(I) \) does not return until the joining phase is done, because every queued tree \( X \) is fed to a node by a unique \( \text{Join}(L,R,done) \) run, before \( \text{Finalize}() \) calls \( done.\text{wait}() \), and \( X.\text{joined.} \text{notify}() \) is called only when \( X \) is no longer a queued tree. The rest of the claim follows from Theorem 45.

Now we establish the work and span bounds for the joining phase.

Theorem 48 (Joining Phase Work). The joining phase takes \( O(b \cdot \log(\frac{2}{b}+1)+b) \) work.

Proof. Observe that the work taken by the joining phase is \( O(1) \) times the total number of runs of \( \text{Join} \) or \( \text{JoinLeft} \) or \( \text{JoinRight} \). The number of runs of \( \text{Join} \) is clearly at most \( b-1 \). The number of runs of \( \text{JoinLeft} \) or \( \text{JoinRight} \) is at most twice the number of feedings (via \( \text{FeedLeft} \) or \( \text{FeedRight} \)), because each reactivation of \( \text{JoinLeft}(v) \) or \( \text{JoinRight}(v) \) is done either by \( \text{FeedLeft} \) or \( \text{FeedRight} \) or by itself, and the number of self-reactivations is at most the number of queued trees fed to \( v \). So we shall bound the total number of feedings.

Each node \( v \) of \( I \) corresponds to a call \( \text{Join}(L,R,done) \) for some 2-3 trees \( L,R \). If \( L \approx R \) then \( \text{Join}(L,R,done) \) does not feed any queue. By symmetry it will suffice to analyze the case that \( L \gg R \). If \( L.\text{right} \approx R \) or \( L+R \) overflows, then \( \text{Join}(L,R,done) \) also does not feed any queue. In the remaining cases \( R \) is pushed onto the queue at \( L.\text{right} \). Observe that \( R \) is pushed during \( L.\text{height} \) times before it is no longer a queued tree. Thus the number of feedings of \( R \) to a node is at most \( L.\text{height} \leq \log((L+R).\text{size}+1) = \log(m(v)+1) \) where \( m(v) \) is the total number of items in all the 2-3 trees that were prepared by \( \text{Collate}(w) \) for some leaf \( w \) of the subtree at \( v \). And clearly \( \sum_{v \in H(l,h)} m(v) \leq n \) for every \( h \in [0..L.\text{height}] \). Thus by the BBT Log Sum Bound (Lemma 21) the total number of feedings is \( O(b \cdot \log(\frac{2}{b}+1)+b) \), and we are done.

Theorem 49 (Joining Phase Span). The joining phase takes \( O(\log b + \log n) \) span.

Proof. It is clear that \( \text{Finalize}() \) takes \( O(\log b + \log n) \) span after all queued trees have been joined, because it only waits at \( v.\text{done} \) at every node \( v \) of the final tree \( T \), whose height is at most \( O(\log n + L.\text{height}) \in O(\log b + \log n) \). Hence we just have to show that all the \( v.\text{join} \) runs in \( O(\log b + \log n) \) span after \( \text{Collate}(I.\text{root}) \) finishes.

We shall use the same technique as in the proof of Parallel Filtering Costs (Theorem 13). Call a run of \( v.\text{join} \) a \( v.-run \), and a reactivation of \( v.\text{join} \) a \( v.-\text{reactivation} \). Call a \( v.-run \) a \( (v,X)-run \) iff it processes the 23Tree \( X \) (i.e. it pops \( X \) off \( v.\text{queue} \)). As before, each \( v.-run \) performs a \( v.-\text{reactivation} \) iff it is effective, and there cannot be three consecutive ineffective \( v.-runs \).

Also, each queued 23Tree \( X \) during the joining phase is first pushed onto a queue by some run of \( \text{Join} \) that corresponds to some internal node \( u(X) \) of \( I \). Let \( d(X) \) be the depth of \( u(X) \) in \( I \), and observe that \( d(X') < d(X) \) for any \( (v,X') \)-run that precedes a \( (v,X)-run \).
Now consider any 𝑣-run 𝑅. Let 𝑋 be the last 23Tree processed by the 𝑣-runs up to 𝑅 (which exists since the first 𝑣-run processes the first tree fed to 𝑣.queue). There are two cases (by Theorem 7 Property 3b):

- 𝑅 starts within 𝑂(1) span after the end of the previous 𝑣-run 𝑅’. Note that the last 23Tree 𝑋’ processed by the 𝑣-runs up to 𝑅’ satisfies 𝑑(𝑋’) ≤ 𝑑(𝑋).
- 𝑅 starts within 𝑂(1) span after the start of the 𝑣-reactivation 𝐶 that triggers 𝑅, and the reactivation point for 𝐶 is after the end of any previous 𝑣-run. In this case, the previous 𝑣-run 𝑅’ (if any) is ineffective, and hence 𝐶 must be executed by the feeding of some 23Tree 𝑋’ to 𝑣.queue. Observe that any feeding of a 23Tree 𝑋’’ to 𝑣.queue before 𝑋’ must push 𝑋’’ onto 𝑣.queue before executing some 𝑣-reactivation 𝐶’ before 𝐶, and 𝐶’ must have reactivation point before the start of 𝑅’ (since 𝐶 triggers 𝑅), and hence 𝑋’’ must have been processed by some 𝑣-run preceding 𝑅 since 𝑅’ is ineffective. Thus 𝑅 would process 𝑋’ if it had not already been processed by an earlier 𝑣-run, and hence 𝑑(𝑋’) ≤ 𝑑(𝑋).

From this and the fact that there cannot be three consecutive ineffective 𝑣-runs, we can deduce that for every (𝑣, 𝑋)-run 𝑅, at least one of the following holds:

- 𝑅 starts within 𝑂(1) span after the end of some preceding (𝑣, 𝑋’)-run where 𝑑(𝑋’) < 𝑑(𝑋).
- 𝑅 starts within 𝑂(1) span after the start of some 𝑣-reactivation executed by the feeding of some 23Tree 𝑋’ to 𝑣.queue where 𝑑(𝑋’) ≤ 𝑑(𝑋).

Moreover, every feeding of a 23Tree 𝑋 to 𝑣.queue is executed by either some (𝑣.parent, 𝑋)-run or some run of Join. Therefore by induction every (𝑣, 𝑋)-run 𝑅 starts within 𝑂(𝑘+𝑚+1) span after Collate(𝑖.root) finishes, where 𝑘 is the depth of 𝑣 in 𝑇 and 𝑚 = 𝑑(𝑋). Thus the whole joining phase finishes within 𝑂(𝑇.height+𝑖.height+1) ≤ 𝑂(𝑙𝑜𝑔 𝑏+𝑙𝑜𝑔 𝑛) span after Collate(𝑖.root) finishes.

### 8.4 Batch Reverse-Indexing

With the tools we have now, reverse-indexing is not too hard. A direct pointer 𝑋 to an item in 𝑇 stores a private pointer 𝑋.node to the leaf in 𝑇 that contains that item. We augment each node 𝑣 of 𝑇 with a pointer storing its parent node (null if it is the root), and update it accordingly during any of the other batch operations on 𝑇. We also augment 𝑣 with a boolean flag 𝑣.marked initialized to false. Reverse-indexing on an unsorted input batch 𝑃 of 𝑏 direct pointers to distinct items is done in 2 phases (treating 𝑇 as a BBT throughout):

1. **Tracing phase:** Recursively for each direct pointer 𝑋 in 𝑃, traverse the path from 𝑋.node to the root, where at each node 𝑣 the traversal is continued iff TryLock(𝑣.marked). The spawning takes 𝑂(𝑏) work and 𝑂(𝑙𝑜𝑔 𝑏) ≤ 𝑂(𝑙𝑜𝑔 𝑛) span since 𝑃 is a BBT and 𝑏 ≤ 𝑛. Note that at most one traversal will continue past each node and hence at most two traversals access each flag. After all the traversals are done, every node 𝑣 along the path from each desired leaf to the root of 𝑇 is marked (i.e. 𝑣.marked = true).

2. **Retrieving phase:** Recursively traverse 𝑇 top-down only through marked nodes, to find all the desired leaves and join them into a 2-3 tree 𝑈 := Retrieve(𝑇.root) via the same pipelined joining scheme as in the Sorted Batch Access (Section 8.3 joining phase). To wait for the joining to be done, we wait on a barrier 𝑣.done at every marked node 𝑣 of 𝑇 with 2 marked children, where 𝑣.done is notified after the corresponding joining has finished. After all the joining is done, recursively for each marked node 𝑣, call Unlock(𝑣.marked), and call 𝑣.done.wait() if 𝑣 has 2 marked children. After that, 𝑈 contains the desired items in sorted order, so return 𝑈 converted to a batch.

The same technique used to prove the Joining Phase Work (Theorem 48) and Joining Phase Span (Theorem 49) shows that each joining of 2-3 trees during the retrieving phase, and the subsequent waiting for the joining, takes 𝑂(𝑙𝑜𝑔 𝑛) work and 𝑂(𝑙𝑜𝑔 𝑛) span. Thus the total work taken is 𝑂(𝑏) plus 𝑂(1) per marked node plus 𝑂(𝑙𝑜𝑔 𝑛) per joining, amounting to 𝑂(𝑏 · 𝑙𝑜𝑔 𝑛). But a more careful analysis will show that the joinings also take only 𝑂(1) work per marked node, and that there are only 𝑂(𝑏 · 𝑙𝑜𝑔 2 3 + 𝑏) marked nodes, so in fact the total work is merely 𝑂(𝑏 · 𝑙𝑜𝑔 2 3 + 𝑏).

The technical details are as follows. Reverse-indexing on an unsorted input batch 𝑃 of direct pointers to distinct items is done by returning ReverseIndex(𝑃).
Definition 50 (Reverse-Indexing).

**Public ReverseIndex** (Batch \( P \) of DirectPointers into \( T \)):

- If \( P \) is empty, return new empty Batch of items.
- Call \( \text{Trace}(X.\text{node}) \) for each DirectPointer \( X \) in \( P \) in parallel (and wait for all to finish).
- Create 23Tree \( U := \text{Retrieve}(T.\text{root}) \).
- Call \( \text{Finalize}(U.\text{root}) \).
- Return \( U \) converted to a Batch.  // easy since a 2-3 tree is also a BBT

Private Trace (Node \( v \) of BBT \( T \)):

- If \( \text{TryLock}(v.\text{marked}) \) and \( v.\text{parent} \neq \text{null} \), call \( \text{Trace}(v.\text{parent}) \).

Private Retrieve (Node \( v \) of BBT \( T \)):

- If \( v \) is a leaf, return new 23Tree containing only the item at \( v \).
- If \( \neg v.\text{right}.\text{marked} \), return \( \text{Retrieve}(v.\text{left}) \).
- If \( \neg v.\text{left}.\text{marked} \), return \( \text{Retrieve}(v.\text{right}) \).
- Create Barrier \( v.\text{done} \).  // see Definition 4
- In parallel set \( L := \text{Retrieve}(v.\text{left}) \) and \( R := \text{Retrieve}(v.\text{right}) \) (and wait for both to finish).
- Return \( \text{Join}(L, R, v.\text{done}) \).

Private RFinalize (Node \( v \) of BBT \( T \)):

- Unlock(\( v.\text{marked} \)).
- If \( v \) is a leaf, return.
- If \( v.\text{right}.\text{marked} \), call RFinalize(\( v.\text{left} \)) and return.
- If \( v.\text{left}.\text{marked} \), call RFinalize(\( v.\text{right} \)) and return.
- In parallel call RFinalize(\( v.\text{left} \)) and RFinalize(\( v.\text{right} \)) (and wait for both to finish).
- Call \( v.\text{done.wait}() \).

First we prove a simple lemma (Lemma 51) that will be needed to prove the correctness and desired cost bounds of reverse-indexing, which is divided into the **tracing phase** (i.e. until just before the call to \( \text{Retrieve}(T.\text{root}) \)) and the **retrieving phase** (i.e. starting from the call to \( \text{Retrieve}(T.\text{root}) \)).

**Lemma 51 (Tracing Properties).** During the tracing phase, if there is a call to \( \text{Trace}(v) \), then the following properties hold:

1. Exactly one call to \( \text{Trace}(v) \) evaluates \( \text{TryLock}(v.\text{marked}) \) to true.
2. There is exactly one call from \( \text{Trace}(v) \) to \( \text{Trace}(v.\text{parent}) \) if \( v \) is not the root.

**Proof.** Property 1 is obvious from the definition of TryLock and the fact that (during the tracing phase) no \( v.\text{marked} \) is ever set to false. Property 2 is an immediate consequence of Property 1.

From Property 1 of the foregoing lemma it is clear that, after the tracing phase, every node \( v \) of \( T \) is marked (i.e. \( v.\text{marked} = \text{true} \)) iff \( v \) is along some path from a leaf \( X.\text{node} \) for some DirectPointer \( X \) in \( P \). Thus the correctness of the retrieving phase follows from the correctness of Join (Theorem 45) in the same manner as the correctness of sorted batch access (Theorem 47). So we are left with proving the desired cost bounds.

Next we bound the marked nodes using a combinatorial lemma, which in fact applies not just to BBTs as needed here but also to any \( O(1) \)-log-splitting full binary tree (see Definition 12).

**Lemma 52 (Subtree Size Bound).** Take any \( c \)-log-splitting full binary tree \( T \), with \( n \) leaves of which \( k \) are marked where \( k > 0 \), and with each internal node marked iff it is on a path from the root to a marked leaf. Then \( T \) has at most \( (c \cdot (k + 1) \cdot \log_2 \frac{n}{k} + 2k) \) marked nodes.

**Proof.** Order the nodes of \( T \) according to its in-order traversal. Let \( v_i \) be the \( i \)-th marked leaf, and let \( d_i \) be the number of unmarked leaves (strictly) between \( v_i \) and \( v_{i+1} \). Also let \( d_0 \) be the number of unmarked leaves before \( v_1 \) and \( d_k \) be the number of unmarked leaves after \( v_k \). Then there are at most \( (c \cdot \log_2 (d_i + 1) + 1) \) marked nodes between \( v_i \) and \( v_{i+1} \), since they are exactly the least common ancestor of \( v_i, v_{i+1} \) plus the parent of each subtree in the slice of \( T \) that contains the leaves between \( v_i \) and \( v_{i+1} \), and that slice has \( c \cdot \log_2 (d_i + 1) \) subtrees (since \( T \) is \( c \)-log-splitting). Similarly, there are at most \( c \cdot \log_2 (d_0 + 1) \) marked internal nodes before \( v_1 \) (since they are exactly the parents of subtrees in the slice that contains the leaves before \( v_1 \)), and at most \( c \cdot \log_2 (d_k + 1) \) marked internal nodes after \( v_k \). By Jensen’s inequality we have \( \sum_{i=0}^{k} \log_2 (d_i + 1) \leq (k+1) \cdot \log_2 \frac{n}{k+1} + (k-1) \). Thus the total number of marked internal nodes is at most \( c \cdot \sum_{i=0}^{k} \log_2 (d_i + 1) + (k-1) \leq c \cdot (k + 1) \cdot \log_2 \frac{n}{k+1} + (k-1) < c \cdot (k + 1) \cdot \log_2 \frac{n}{k} + k \).  \( \diamond \)
Theorem 53 (Reverse-Indexing Costs). Reverse-indexing on an input batch of \( b \) direct pointers to distinct items takes \( O(b \cdot \log \frac{n}{b} + b) \) work and \( O(\log n) \) span.

Proof. Clearly spawning the calls to Trace\( (X, \text{node}) \) for each DirectPointer \( X \) in \( P \) takes \( O(b) \) work and \( O(\log b) \subseteq O(\log n) \) span. And by Lemma 51 Property 2, there are at most 2 calls to Trace\( (v) \) for each node \( v \) in \( T \), so TryLock\( (v, \text{marked}) \) takes \( O(1) \) time, and hence the calls to Trace take \( O(m) \) total work where \( m \) is the number of marked nodes after the tracing phase.

After that, the call to Retrieve\( (T, \text{root}) \) takes \( O(m) \) work plus the work taken by the calls to Join, and takes \( O(\log n) \) span. Note that Retrieve\( (v) \) calls Join only if \( v \) has two marked children, and always returns a 23Tree with height at most \( v \cdot \text{height} \) by 23Tree Joining Correctness (Theorem 45) and induction. Thus the pipelined processing triggered by each call from Retrieve\( (v) \) to Join\( (L, R, v, \text{done}) \) (i.e. pushing \( L \) down \( R \) or vice versa) takes \( O(\max(L, \text{height}, R, \text{height}) + 1) \leq O(\text{height}) \) work by the same reasoning as in the proof of Joining Phase Work (Theorem 48), and hence the calls to Join take \( O(\sum_{v \in C} \text{height}) \) work in total, which is \( O(m) \) work by the Tree Path Length Sum (Lemma 11) since \( v \cdot \text{height} \) is at most twice the length of the shortest path from \( v \) to a leaf. Finally, RFinalize\( (T, \text{root}) \) takes \( O(m) \) work, and takes \( O(\log n) \) span by the same reasoning as in the proof of the Joining Phase Span (Theorem 49).

Therefore the total work taken is \( O(m) \), and \( m \in O(b \cdot \log \frac{n}{b} + b) \) by the Subtree Size Bound (Lemma 52), so we are done.

\( \diamond \)

8.5 Unsorted Batch Access

We now explain how to implement unsorted access on an input batch \( B \) of \( b \) accesses. How we perform \( B \) depends on the current size \( n \) of \( T \):

\( \diamond \) If \( b \leq n \):
- 1. Parallel sort (Definition 27) \( B \), taking \( O(b \cdot \log b) \) work and \( O((\log b)^2) \) span.
- 2. Use Sorted Batch Access (Section 8.3) to perform the now item-sorted \( B \) on \( T \), taking \( O(b \cdot \log n) \) work and \( O(\log b + \log n) \) span.

\( \diamond \) If \( b > n \):
- 1. Use Unsorted Batch Search (Section 8.2) to perform all search/updates in \( B \) on \( T \), including insertions on existing items (which are treated as updates), and to perform all deletions on non-existent items in \( T \), taking \( O(b \cdot \log n) \) work and \( O(\log b \cdot \log n) \) span.
- 2. Parallel filter (Section 7.2) out all the operations completed in the previous step from \( B \), taking \( O(b) \) work and \( O(\log b) \) span.
- 3. Parallel entropy-sort (Definition 29) the leftover batch \( B \), combining operations on the same item, taking \( O(b \cdot \log m + b) \) work and \( O((\log b)^2) \) span where \( m \) is the final size of \( B \). Let \( c \) be the final number of insertions in \( B \), and \( d \) be the final number of deletions in \( B \). Then the new size \( n' \) of \( T \) is at least \( c \), and \( d \leq n \) since there can only be \( n \) successful deletions. Thus \( m = c + d \leq n + n' \), and so \( O(b \cdot \log m) \subseteq O(b \cdot \log (n + n')) \subseteq O(b \cdot \log \max(n, n')) \).
- 4. Use Sorted Batch Access (Section 8.3) to perform the now item-sorted \( B \) on \( T \), taking \( O(m \cdot \log n) \) work and \( O(\log m + \log n) \) span.

In both cases, performing the batch \( B \) takes \( O(b \cdot \log \max(n, n')) \) work and \( O((\log b)^2 + \log n) \) span, where \( n' \) is the size of \( T \) after this batch operation.

8.6 Batch Joining

We have finished describing how to implement batch operations on sorted instances of \( T \), namely those whose items are in sorted order (when listed according to the order of the leaves from left to right). But we can also consider unsorted instances of \( T \), namely those whose items are not required to be in sorted order. It is easy to see that the joining phase of the Sorted Batch Access (Section 8.3) does not depend on the item ordering at all, and so we can perform a batch joining of any batch \( B \) of \( b \) unsorted instances of \( T \) with total size \( n \) by the same pipelined joining scheme (Section 8.3 joining phase). This takes \( O(b \cdot \log(\frac{n}{b}) + b) \) work and \( O(\log b + \log n) \) span, by the same proof as for Theorem 48 and Theorem 49.
9 Optimal Parallel Sorted-Set

The red-black tree encoding the 2-3 tree in $\mathbb{T}$ is a BBT, so it can be used as an input batch on another instance of $\mathbb{T}$. In particular, given (sorted) instances $X, Y$ of $\mathbb{T}$ with $m, n$ items respectively such that $m \geq n$, to compute $X \cap Y$, $X \cup Y$ or $X \setminus Y$ we can treat $Y$ as an input batch (of searches, insertions or deletions respectively) for $X$, taking $O(n \cdot \log(\frac{m}{n} + 1))$ work and $O(\log m + \log n)$ span. We can also compute $Y \setminus X = Y \setminus (X \cap Y)$ within the same bounds, since $i \cdot \log(\frac{X}{n} + 1) < n + i$ where $i = \#(X \cap Y) \leq n$. The work bound is information-theoretically optimal (in the comparison model), and the span bound is optimal in the PPM model.

As written, the sorted batch insertions and deletions (Section 8.3) are destructive, and so $\mathbb{T}$ is not persistent. However, if we want to use $\mathbb{T}$ only as a sorted-set data structure supporting intersections, unions and difference, then we do not need the parent pointers used in reverse-indexing (Section 8.4), and hence it is not hard to make $\mathbb{T}$ persistent. To do so, we modify Collate (Definition 37) to perform the sequential joins and execute the access non-destructively when computing the resulting 2-3 tree $X$ at each leaf of the input batch $I$, and then replace $X$ by a copy with just the left and right spine deep-copied. This deep-copying is necessary because Join (Definition 38) may modify the spine nodes of $X$.

10 Conclusions

This paper presents a batch-parallel 2-3 tree for the QRMW PPM model that is essentially optimal and can be trivially used to implement an optimal sorted-set data structure. It can be seen that clever pipelining can be used to achieve optimal work and span bounds that do not rely on any ‘tricks’ such as $O(1)$-time prefix-sums over all $p$ processors, or $O(1)$ concurrent memory accesses over all $p$ processors, or even pointer arithmetic. It raises the interesting question of just how much can be done in the QRMW PPM model, which can be argued to capture the intrinsic abstract costs of the problem.

Also, it is intriguing that the parallel sorted-set data structure described in [5], which is information-theoretically optimal under a different computation model, namely the binary-forking model with test-and-set, seems to crucially rely on concurrent reads taking $O(1)$ time even if the contention is arbitrarily high, and so cannot be translated to the QRMW model. However, that data structure does not use any RMW operations besides test-and-set, whereas the parallel sorted-set data structure in this paper (Section 9) implicitly uses fetch-and-increment and fetch-and-decrement (in the reactivation wrapper). Is such a trade-off necessary, or can we have optimal parallel sorted-sets in the QRMW model that only use read, write and test-and-set for memory accesses?

Lastly, this batch-parallel 2-3 tree supports sorted batch access much more efficiently than unsorted batch access, and there seems to be an intrinsic disparity. But is it possible to implement unsorted batch access with better performance bounds?
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