In this paper we study the regularity of stationary and minimizing harmonic maps $f : B_2(p) \subseteq M \to N$ between Riemannian manifolds. If $S^k(f) \equiv \{ x \in M : \text{no tangent map at } x \text{ is } k + 1\text{-symmetric} \}$ is $k^{th}$-stratum of the singular set of $f$, then it is well known that $\dim S^k \leq k$, however little else about the structure of $S^k(f)$ is understood in any generality. Our first result is for a general stationary harmonic map, where we prove that $S^k(f)$ is $k$-rectifiable. In fact, we prove for $k$-a.e. point $x \in S^k(f)$ that there exists a unique $k$-plane $V^k \subseteq T_xM$ such that every tangent map at $x$ is $k$-symmetric with respect to $V$.

In the case of minimizing harmonic maps we go further, and prove that the singular set $S(f)$, which is well known to satisfy $\dim S(f) \leq n - 3$, is in fact $n - 3$-rectifiable with uniformly finite $n - 3$-measure. An effective version of this allows us to prove that $|\nabla f|$ has estimates in $L^3_{\text{weak}}$, an estimate which is sharp as $|\nabla f|$ may not live in $L^3$. More generally, we show that the regularity scale $r_f$ also has $L^3_{\text{weak}}$ estimates.

The above results are in fact just applications of a new class of estimates we prove on the quantitative stratifications $S^k_\epsilon(f)$ and $S^k_{\epsilon_n}(f)$. Roughly, $S^k_\epsilon \subseteq M$ is the collection of points $x \in S^k$ for which no ball $B_r(x)$ is $\epsilon$-close to being $k + 1$-symmetric. We show that $S^k_\epsilon$ is $k$-rectifiable and satisfies the Minkowski estimate $\text{Vol}(B_r S^k_\epsilon) \leq Cr^k$.

The proofs require a new $L^2$-subspace approximation theorem for stationary harmonic maps, as well as new $W^{1,p}$-Reifenberg and rectifiable-Reifenberg type theorems. These results are generalizations of the classical Reifenberg, and give checkable criteria to determine when a set is $k$-rectifiable with uniform measure estimates. The new Reifenberg type theorems may be of some independent interest. The $L^2$-subspace approximation theorem we prove is then used to help break down the quantitative stratifications into pieces which satisfy these criteria.
1. INTRODUCTION

Let \( f : B_2(p) \subseteq M \to N \) be a stationary harmonic map between two Riemannian manifolds with \( N \) compact without boundary. Then one can use \( f \) to give a stratification of \( M \) by the singular sets

\[
S^0(f) \subseteq \cdots \subseteq S^k(f) \subseteq \cdots S(f) \subseteq M,
\]

defined by

\[
S^k(f) \equiv \{ x \in M : \text{no tangent map at } x \text{ is } k+1\text{-symmetric} \},
\]

see Definition 1.2 for a precise definition and more detailed discussion. A key result of [SU82] is that we have the Hausdorff dimension estimate

\[
\dim S^k \leq k.
\]
However, little else is known about the structure of the singular sets $S^k(f)$. In the stationary case, nothing is known in general. In the case of a minimizing harmonic map, and under the additional assumption that $N$ is analytic, it follows from the wonderful work of [Sim96] that the top stratum $S^{n-3}$ is rectifiable. In low dimensions and when the targets are spheres, these results have been further refined in [LW06] and [HL90].

The goal of this paper is to study additional properties of the stratification and the associated quantitative stratification for stationary and minimizing harmonic maps. That is, a first goal is to see for a stationary harmonic map with a general compact target space that $S^k(f)$ is $k$-rectifiable for all $k$. In fact, the proof allows us to see the following stronger result. For $k$-a.e. $x \in S^k$ there will exist a unique $k$-plane $V^k \subseteq T_xM$ such that every tangent map at $x$ will be $k$-symmetric with respect to $V$. Let us observe a subtlety to this statement. We are not claiming that the tangent maps at such $x$ are unique. We are simply claiming that for $k$-a.e. point there is a maximal dimensional $k$-plane of symmetry which exists, and that this plane of symmetry is unique independent of which tangent map is chosen. Theorem 1.5 gives the precise results for the classical stratification of a stationary harmonic map.

For minimizing harmonic maps we can improve on this result in two ways. First, we can show that the top stratum is not only $n-3$ rectifiable, but has an a priori bound on its $n-3$ measure. That is, if

$$\int_{B_2(p)} |\nabla f|^2 \leq \Lambda, \quad (1.4)$$

is the $L^2$ bound on the energy, then we have the $n-3$-Hausdorff measure estimate

$$\lambda^{n-3}(S^{n-3}(f) \cap B_1) \leq C. \quad (1.5)$$

Indeed, we show a much stronger Minkowski version of this estimate, see Theorem 1.6 for the complete statement. In fact, we can prove what turns out to be a much more effective analytic version of the above. Namely, in Theorem 1.8 we show that $|\nabla f|$, and in fact the regularity scale $r_f$, have a priori weak $L^3$ bounds. That is,

$$\text{Vol}(|\nabla f| > r^{-1}) \cap B_1 \leq \text{Vol}(B_r(|\nabla f| > r^{-1}) \cap B_1) \leq Cr^3. \quad (1.6)$$

Notice that this is indeed sharp, in that there are counterexamples showing that $|\nabla f| \notin L^3$, see Example 2.6.1. Let us also point out that this sharpens results from [CN13b], where it was proven for minimizing harmonic maps that $|\nabla f| \in L^p$ for all $p < 3$. We refer the reader to Section 1.2 for the precise and most general statements.

Now the techniques of this paper all center around estimating the quantitative stratification, not the standard stratification itself. In fact, it is for the quantitative stratification that the most important results of the paper hold, everything else can be seen to be corollaries of these statements.

The quantitative stratification was first introduced in [CN13a] with the goal of proving new estimates on noncollapsed manifolds with Ricci curvature bounded below and in particular Einstein manifolds, and then extended in [CN13b] to give effective and $L^p$ estimates on minimal and stationary harmonic maps and minimal submanifolds. It has since been used in [CHN13b], [CHN13a], [CVN15],[BL14] to prove
similar results in the areas of mean curvature flow, critical sets of elliptic equations, harmonic map flow, and biharmonic maps.

Before describing the results in this paper on the quantitative stratification, let us give more precise definitions of everything. To begin with, to describe the stratification and quantitative stratification we need to discuss the notion of symmetry associated to a solution. Specifically:

**Definition 1.1.** We define the following:

1. A mapping $f : \mathbb{R}^n \to N$ is called $k$-symmetric if $f(\lambda x) = f(x)$ \forall $\lambda > 0$ and if there exists a $k$-plane $V^k \subseteq \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$ and $y \in V^k$ we have that $f(x + y) = f(x)$.

2. Given a mapping $f : M \to N$ and $\epsilon > 0$, we say a ball $B_r(x) \subseteq M$ with $r \in \text{inj}(x)$ is $(k, \epsilon)$-symmetric if there exists a $k$-symmetric mapping $\tilde{f} : T_x M \to N$ such that $\int_{B_r(x)} |f - \tilde{f}|^2 dv_x < \epsilon$, where we have used the exponential map to identify $\tilde{f}$ as a function on $M$.

Thus, a function is $k$-symmetric if it only depends on $n - k$ variables and is radially invariant. A $k$-symmetric function can therefore be naturally identified with a function on a $n - k - 1$ sphere. A function is $(k, \epsilon)$-symmetric on a ball $B_r(x)$ if it is $L^2$ close to a $k$-symmetric function on this ball.

With the notion of symmetry in hand we can define precisely the quantitative stratification associated to a solution. The idea is to group points together based on the amount of symmetry that balls centered at those points contain. In fact, there are several variants which will play a role for us, all of which are important for the applications to the standard singular set. Let us introduce them and briefly discuss them:

**Definition 1.2.** For a stationary harmonic map $f : B_2(p) \to N$ we make the following definitions:

1. For $\epsilon, r > 0$ we define the $k^{th}$ $(\epsilon, r)$-stratification $S^k_{\epsilon, r}(f)$ by

   \[
   S^k_{\epsilon, r}(f) = \{ x \in B_1(p) : \text{for no } r \leq s < 1 \text{ is } B_s(x) \text{ a } (k + 1, \epsilon)-\text{symmetric ball} \}. \tag{1.7}
   \]

2. For $\epsilon > 0$ we define the $k^{th}$ $\epsilon$-stratification $S^k_{\epsilon}(f)$ by

   \[
   S^k_{\epsilon}(f) = \bigcap_{r>0} S^k_{\epsilon, r}(f) = \{ x \in B_1(p) : \text{for no } 0 < r < 1 \text{ is } B_r(x) \text{ a } (k + 1, \epsilon)-\text{symmetric ball} \}. \tag{1.8}
   \]

3. We define the $k^{th}$-stratification $S^k(f)$ by

   \[
   S^k(f) = \bigcup_{\epsilon>0} S^k_{\epsilon} = \{ x \in B_1(p) : \text{no tangent cone at } x \text{ is } k + 1-\text{symmetric} \}. \tag{1.9}
   \]

**Remark 1.1.** It is a small but important exercise to check that the standard stratification $S^k(f)$ as defined in (1.1) agrees with the set $\bigcup_{\epsilon>0} S^k_{\epsilon}$. We do this carefully in Section 6.3.

Let us discuss in words the meaning of the quantitative stratification, and how it relates to the standard stratification. As discussed at the beginning of the section, the stratification $S^k(f)$ of $M$ is built by separating points of $M$ based on the infinitesimal symmetries of $f$ at those points. The quantitative stratifications $S^k_{\epsilon}(f)$ and $S^k_{\epsilon, r}(f)$ are, on the other hand, instead built by separating points of $M$ based on how many symmetries exist on balls of definite size around the points. In practice, the quantitative stratification has two advantages to the standard stratification. First, for applications to minimizing harmonic maps the quantitative
stratification allows one to prove effective estimates. In particular, in [CN13b] the weaker $L^p$ estimates
\[
\int_{B_1(p)} |\nabla f|^{3-\delta}, \int_{B_1(p)} |\nabla^2 f|^{3-\delta} < C_{\delta} \forall \, \delta > 0,
\]
on solutions were obtained by exploiting this fact. The second advantage is that the estimates on the quantitative stratification are much stronger than those on the standard stratification. Namely, in [CN13b] the Hausdorff dimension estimate (1.3) on $S^k(f)$ was improved to the Minkowski content estimate
\[
\text{Vol}(B_{r/\delta} S^k_{x,\epsilon}) \leq C_{\delta}^n r^{-k-\delta} \forall \delta > 0.
\]

One of the key technical estimates of this paper is that in Theorem 1.3 we drop the $\delta$ from the above estimate and obtain an estimate of the form
\[
\text{Vol}(B_{r/\epsilon} S^k_{x,\epsilon}) \leq C r^{n-k}.
\]
From this we are able conclude in Theorem 1.4 an estimate on $S^k_{x,\epsilon}$ of the form
\[
\text{Vol}(B_{r/\epsilon} S^k_{x,\epsilon}) \leq C r^{n-k}.
\]

In particular, this estimate allows us to conclude that $S^k_{x,\epsilon}$ has uniformly finite $k$-dimensional measure. In fact, the techniques will prove much more for us. They will show us that $S^k_{x,\epsilon}$ is $k$-rectifiable, and that for $k$-a.e. point $x \in S^k_{x,\epsilon}$ there is a unique $k$-plane $V^k \subseteq T_x M$ such that every tangent map at $x$ is $k$-symmetric with respect to $V$. By observing that $S^k(f) = \bigcup S^k_{x,\epsilon}(f)$, this allows us to prove in Theorem 1.5 our main results on the classical stratification. This decomposition of $S^k$ into the pieces $S^k_{x,\epsilon}$ is crucial for the proof.

On the other hand, (1.12), combined with the $\epsilon$-regularity theorems of [SU82],[CN13b], allow us to conclude in the minimizing case both the weak $L^3$ estimate on $|\nabla f|$, and the $n - 3$-finiteness of the singular set of $f$. Thus we will see that Theorems 1.6 and 1.8 are fairly quick consequences of (1.13).

Thus we have seen that (1.12) and (1.13), and more generally Theorem 1.3 and Theorem 1.4, are the main challenges of the paper. We will give a more complete outline of the proof in Section 1.4, however let us mention for the moment that two of the new ingredients to the proof are a new $L^2$-subspace approximation theorem for stationary harmonic maps, proved in Section 4, and new $W^{1,p}$-Reifenberg and rectifiable-Reifenberg theorems, proved in Section 3. The $L^2$-approximation result roughly tells us that the $L^2$-distance of a measure from being contained in a $k$-dimensional subspace may be estimated by integrating the energy drop of a stationary harmonic map over the measure. To exploit the estimate we prove a new $W^{1,p}$-Reifenberg type theorem. The classical Reifenberg theorem states that if we have a set $S$ which is $L^\infty$-approximated by a subspace at every point and scale, then $S$ is bi-Hölder to a manifold, see theorem 2.18 for a precise statement. It is important for us to improve on this bi-Hölder estimate, at least enough that we are able to control the $k^{th}$-dimensional measure of the set and prove rectifiability. In particular, we want to improve the $C^\alpha$-maps to $W^{1,p}$-maps for $p > k$, and we will want to do it using a condition which is integral in nature. More precisely, we will only require a form of summable $L^2$-closeness of the subset $S$ to the approximating subspaces. We will see in Theorem 4.1 that by using the $L^2$-subspace approximation theorem that the conditions of this new rectifiable-Reifenberg are in fact controllable for the quantitative stratifications $S^k_{x,\epsilon}$, as least after we break it up into appropriate pieces.
1.1. Results for Stationary Harmonic Maps. We now turn our attention to giving precise statements of the main results of this paper. In this subsection we focus on those concerning the singular structure of stationary harmonic maps. That is, we will be considering stationary harmonic maps \( f : B_2(p) \subseteq M \rightarrow N \).

In order to write the constants involved in some explicit form, let us choose \( K_M, K_N, n > 0 \) to be the smallest number such that

\[
|\sec_{B_2(p)}| \leq K_M, \quad \text{inj}(B_2(p)) \geq K_M^{-1},
\]

\[
|\sec_{B_2(p)}| |\sec_{B_2(p)}| \leq K_N, \quad \text{inj}(N) \geq K_N^{-1}, \quad \text{diam}(N) \leq K_N,
\]

\[
\dim(M), \dim(N) \leq n.
\] (1.14)

Now let us begin by discussing our main theorem for the quantitative stratifications \( S_{\epsilon, r}(f) \):

**Theorem 1.3** \((\epsilon, r)\)-Stratification of Stationary Harmonic Maps. Let \( f : B_2(p) \subseteq M \rightarrow N \) be a stationary harmonic mapping satisfying (1.14) with \( \frac{1}{2} \int_{B_2(p)} |\nabla f|^2 \leq \Lambda \). Then for each \( \epsilon > 0 \) there exists \( C_\epsilon(n, K_M, K_N, \Lambda, \epsilon) \) such that

\[
\text{Vol}(B_r(S_{\epsilon, r}(f))) \leq C_\epsilon r^{n-k}.
\] (1.15)

When we study the stratum \( S_{\epsilon, r}(f) \), we can refine the above to prove structure theorems on the set itself. For the definition of \( k \)-rectifiability, we refer the reader to the standard reference [Fed69].

**Theorem 1.4** \((\epsilon)\)-Stratification of Stationary Harmonic Maps. Let \( f : B_2(p) \subseteq M \rightarrow N \) be a stationary harmonic mapping satisfying (1.14) with \( \frac{1}{2} \int_{B_2(p)} |\nabla f|^2 \leq \Lambda \). Then for each \( \epsilon > 0 \) there exists \( C_\epsilon(n, K_M, K_N, \Lambda, \epsilon) \) such that

\[
\text{Vol}(B_r(S_{\epsilon}(f))) \leq C_\epsilon r^{n-k}.
\] (1.16)

In particular, we have the \( k \)-dimensional Hausdorff measure estimate \( \lambda^k(S_{\epsilon}(f)) \leq C_\epsilon \). Further, \( S_{\epsilon}(f) \) is \( k \)-rectifiable, and for \( k \)-a.e. \( x \in S_{\epsilon}^k \) there exists a unique \( k \)-plane \( V^k \subseteq T_x M \) such that every tangent cone of \( x \) is \( k \)-symmetric with respect to \( V^k \).

**Remark 1.2.** In fact, the techniques will prove an estimate even stronger than the Minkowski estimate of (1.16). That is, one can prove a uniform \( k \)-dimensional packing content estimate. More precisely, let \( \{B_{\epsilon, i}(x_i)\} \) be any collection of disjoint balls such that \( x_i \in S_{\epsilon}^k \), then we have the content estimate \( \sum r_i^k \leq C_\epsilon \).

Finally, we end this subsection by stating our main results when it comes to the classical stratification \( S^k(f) \). The following may be proved from the previous Theorem in only a few lines given the formula \( S^k(f) = \bigcup S_{\epsilon}^k(f) \):

**Theorem 1.5** (Stratification of Stationary Harmonic Maps). Let \( f : B_2(p) \subseteq M \rightarrow N \) be a stationary harmonic mapping satisfying (1.14) with \( \frac{1}{2} \int_{B_2(p)} |\nabla f|^2 \leq \Lambda \). Then for each \( k \) we have that \( S^k(f) \) is \( k \)-rectifiable.
Further, for k-a.e. \( x \in S^k(f) \) there exists a unique k-plane \( V^k \subseteq T_xM \) such that every tangent cone of \( x \) is k-symmetric with respect to \( V^k \).

### 1.2. Results for Minimizing Harmonic Maps.

In this section we record our main results for minimizing harmonic maps. Most of the results of this section follow quickly by combining the quantitative stratification results of Section 1.1 with the \( \epsilon \)-regularity of [SU82, CN13b], see Section 2.4 for a review of these points.

Our first estimate is on the singular set \( \text{Sing}(f) \) of a minimizing harmonic map. Recall that \( \text{Sing}(f) \) is the set of points where \( f \) is not smooth. Our first estimate on the singular structure of a minimizing harmonic map is the following:

**Theorem 1.6 (Structure of Singular Set).** Let \( f : B_2(p) \subseteq M \to N \) be a minimizing harmonic mapping satisfying \((1.14)\) with \( \int_{B_2(p)} |\nabla f|^2 \leq \Lambda \). Then \( \text{Sing}(f) \) is \( n-3 \)-rectifiable and there exists \( C(n, K_M, K_N, \Lambda) \) such that

\[
\text{Vol}(B_r(\text{Sing}(f)) \cap B_1(p)) \leq Cr^3. \tag{1.17}
\]

In particular, \( \lambda^{n-3}(\text{Sing}(f)) \leq C \).

**Remark 1.3.** As in Theorem 1.4, the same techniques prove a packing content estimate on \( \text{Sing}(f) \). That is, if \( \{B_r(x_i)\} \) is any collection of disjoint balls such that \( x_i \in \text{Sing}(f) \cap B_1 \), then we have the content estimate

\[
\sum r_i^k \leq C.
\]

The above can be extended to effective Schauder estimates on \( f \). To state the results in full strength let us recall the notion of the regularity scale associated to a function. Namely:

**Definition 1.7.** Let \( f : B_2(p) \to N \) be a mapping between Riemannian manifolds. For \( x \in B_1(p) \) we define the regularity scale \( r_f(x) \) by

\[
r_f(x) \equiv \max \{0 \leq r \leq 1 : \sup_{B_r(x)} |\nabla f| \leq r^{-1}\}. \tag{1.18}
\]

By definition, \( r_f(x) \equiv 0 \) if \( f \) is not Lipschitz in a neighborhood of \( x \).

**Remark 1.4.** The regularity scale is scale invariant. That is, if \( r \equiv r_f(x) \) and we rescale \( B_r(x) \to B_1(x) \), then on the rescaled ball we will have that \( |\nabla f| \leq 1 \) on \( B_1(x) \).

**Remark 1.5.** We have the easy estimate \( |\nabla f|(x) \leq r_f(x)^{-1} \). However, a lower bound on \( r_f(x) \) is in principle much stronger than an upper bound on \( |\nabla f|(x) \).

**Remark 1.6.** Notice that the regularity scale is a Lipschitz function with \( |\nabla r_f(x)| \leq 1 \).

**Remark 1.7.** If \( f \) satisfies an elliptic equation, e.g. is weakly harmonic, then we have the estimate

\[
\sup_{B_{r_f/2}(x)} |\nabla f| \leq C_k r_f(x)^{-k}. \tag{1.19}
\]
In particular, control on \( r_f \) gives control on all higher order derivatives.

Now let us state our main estimates for minimizing harmonic maps:

**Theorem 1.8** (Estimates on Minimizing Harmonic Maps). Let \( f : B_2(p) \subseteq M \to N \) be a minimizing harmonic mapping satisfying (1.14) with \( \int_{B_2(p)} |\nabla f|^2 \leq \Lambda \). Then there exists \( C(n, K_M, K_N, \Lambda) \) such that

\[
\text{Vol}(\{x \in B_1(p) : |\nabla f| > r^{-1}\}) \leq \text{Vol}(\{x \in B_1(p) : r_f(x) < r\}) \leq Cr^3.
\]  

(1.20)

In particular, both \(|\nabla f|\) and \(r_f^{-1}\) have uniform bounds in \(L^3_{\text{weak}}(B_1(p))\), the space of weakly \(L^3\) functions. In fact, we also have the Hessian estimate

\[
\text{Vol}(\{x \in B_1(p) : |\nabla^2 f| > r^{-2}\}) \leq Cr^3,
\]

(1.21)

which in particular gives us uniform bounds for \(|\nabla^2 f|\) in \(L^{3/2}_{\text{weak}}(B_1(p))\), the space of weakly \(L^{3/2}\) functions.

**Remark 1.8.** The \(L^3_{\text{weak}}\) estimates are sharp in that there exist examples for which \(|\nabla f|\) does not live in \(L^3\), see Section 2.6.1.

1.3. **Results under Additional Hypothesis.** The previous two subsections have focused on results for completely general stationary or minimizing harmonic maps. In this subsection we would like to see how these results may be improved under further assumptions. Specifically, for stationary harmonic maps \( f : M \to N \) we would like to see that the regularity results may be improved to match those of minimizing harmonic maps if we assume there are no smooth harmonic maps from \(S^k\) into \(N\). The idea behind this follows that of [Lin99], [CHN13a]. Additionally, though the regularity results of Sections 1.1 and 1.2 are sharp in complete generality, they may be improved if we assume that \(N\) has no other stationary or minimizing harmonic maps from \(S^k\) into \(N\). Precisely, the main result of this subsection for stationary harmonic maps is the following:

**Theorem 1.9** (Improved Estimates for Stationary Maps). Let \( f : B_2(p) \subseteq M \to N \) be a stationary harmonic mapping satisfying (1.14) with \( \int_{B_2(p)} |\nabla f|^2 \leq \Lambda \). Assume further that for some \( k \geq 2 \) there exists no smooth nonconstant stationary harmonic maps \( S^\ell \to N \) for all \( \ell \leq k \). Then there exists \( C(n, K_M, N, \Lambda) \) such that

\[
\text{Vol}(\{x \in B_1(p) : |\nabla f| > r^{-1}\}) \leq \text{Vol}(\{x \in B_1(p) : r_f(x) < r\}) \leq Cr^{2+k}.
\]

(1.22)

In particular, both \(|\nabla f|\) and \(r_f^{-1}\) have uniform bounds in \(L^{2+k}_{\text{weak}}(B_1(p))\), the space of weakly \(L^{2+k}\) functions.

**Remark 1.9.** The proof of Theorem 1.9 follows verbatim the proof of Theorem 1.8, except one replaces the \(\epsilon\)-regularity of theorem 2.8 with the \(\epsilon\)-regularity of theorem 2.11.

In the context where \( f \) is minimizing we have a similar improvement, though in this case we only need to assume there exists no minimizing harmonic maps from \(S^k \to N\):
**Theorem 1.10** (Improved Estimates for Minimizing Maps). Let \( f : B_2(p) \subseteq M \to N \) be a minimizing harmonic mapping satisfying \((1.14)\) with \( \int_{B_2(p)} |\nabla f|^2 \leq \Lambda \). Assume that for some \( k \geq 2 \) that there exists no smooth nonconstant minimizing harmonic maps \( S^\ell \to N \) for all \( \ell \leq k \). Then there exists \( C(n, K_M, N, \Lambda) \) such that
\[
\text{Vol}(\{ x \in B_1(p) : |\nabla f| > r^{-1} \}) \leq \text{Vol}(\{ x \in B_1(p) : r_f(x) < r \}) \leq Cr^{2+k}.
\] (1.23)

In particular, both \( |\nabla f| \) and \( r_f^{-1} \) have uniform bounds in \( L^{2+k}_{\text{weak}}(B_1(p)) \), the space of weakly \( L^{2+k} \) functions.

**Remark 1.10.** The proof of Theorem 1.10 follows verbatim the proof of Theorem 1.8, except one replaces the \( \epsilon \)-regularity of theorem 2.8 with the \( \epsilon \)-regularity of theorem 2.10.

### 1.4. Outline of Proofs and Techniques.

In this subsection we give a brief outline of the proof of the main theorems. To describe the new ingredients involved it will be helpful to give a comparison to the proofs of previous results in the area, in particular the dimension estimate \((1.3)\) of [SU82] and the Minkowski and \( L^p \) estimates \((1.10)\) of [CN13b].

Indeed, the starting point for the study of singular sets for solutions of geometric equations typically looks the same, that is, one needs a monotone quantity. In the case of harmonic maps \( f : M \to N \) between Riemannian manifolds we consider the normalized Dirichlet energy
\[
\theta_r(x) \equiv r^{2-n} \int_{B_r(x)} |\nabla f|^2.
\] (1.24)

For simplicity sake let us take \( M \equiv \mathbb{R}^n \) in this discussion, which is really of no loss except for some small technical work (it is the nonlinearity of \( N \) that is the difficulty). Then \( \frac{\partial}{\partial r} \theta_r \geq 0 \), and \( \theta_r(x) \) is independent of \( r \) if and only if \( f \) is 0-symmetric, see Section 2.1 for more on this. Interestingly, this is the only information one requires to prove the dimension estimate \((1.3)\). Namely, since \( \theta_r(x) \) is monotone and bounded, it must converge as \( r \) tends to zero. In particular, if we consider the sequence of scales \( r_\alpha = 2^{-\alpha} \) then we have for each \( x \) that
\[
\lim_{\alpha \to \infty} \left| \theta_{r_\alpha}(x) - \theta_{r_{\alpha+1}}(x) \right| \to 0.
\] (1.25)

From this one can conclude that every tangent map of \( f \) is 0-symmetric. This fact combined with some very general dimension reduction arguments originating with Federer from geometric measure theory [Sim83], which do not depend on the harmonic behavior of \( f \) at all, yield the dimension estimate \((1.3)\) from [SU82].

The improvement in [CN13b] of the Hausdorff dimension estimate \((1.3)\) to the Minkowski content estimate \((1.11)\), and therefore the \( L^p \) estimate of \((1.10)\), requires exploiting more about the monotone quantity \( \theta_r(x) \) than that it limits as \( r \) tends to zero. Indeed, an effective version of \((1.25)\) says that for each \( \delta > 0 \) there exists \( N(\Lambda, \delta) > 0 \) such that
\[
\left| \theta_{r_\alpha}(x) - \theta_{r_{\alpha+1}}(x) \right| < \delta,
\] (1.26)
holds for all except for at most \( N \) scales \( \alpha \in \{\alpha_1, \ldots, \alpha_N\} \subseteq \mathbb{N} \). These \textit{bad} scales where \((1.26)\) fails may differ from point to point, but the number of such scales is uniformly bound. This allows one to conclude that for all but at most \( N \)-scales that \( B_{r_\alpha}(x) \) is \((0, \epsilon)\)-symmetric, see Section 2.2. To exploit this information a
new technique other than dimension reduction was required in [CN13b]. Indeed, in [CN13b] the quantitative 0-symmetry of (1.26) was instead combined with the notion of cone splitting and an energy decomposition in order to conclude the estimates (1.10), (1.11). Since we will use them in this paper, the quantitative 0-symmetry and cone splitting will be reviewed further in Section 2.2.

Now let us begin to discuss the results of this paper. The most challenging aspect of this paper is the proof of the estimates on the quantitative stratifications of Theorems 1.3 and 1.4, and so we will focus on these in our outline. Let us first observe that it might be advantageous to replace (1.26) with a version that forces an actual rate of convergence, see for instance [Sim96]. More generally, if one is in a context where an effective version of tangent cone uniqueness can be proved then this may be exploited. In fact, in the context of critical sets of elliptic equations one can follow exactly this approach, see the authors work [NV] where versions of Theorems 1.3 and 1.4 were first proved in this context. However, in the general context of this paper such an approach fails, as tangent cone uniqueness is not available, and potentially not correct.

Instead, we will first replace (1.26) with the following relatively simple observation. Namely, for each \( x \) there exists \( N(\Lambda, \delta) \) and a finite number of scales \( \{\alpha_1, \ldots, \alpha_N\} \subseteq \mathbb{N} \) such that

\[
\sum_{\alpha_j < \alpha < \alpha_{j+1}} \left| \phi_{\alpha}(x) - \phi_{\alpha+1}(x) \right| < \delta.
\]

That is, not only does the energy drop by less than \( \delta \) between these scales, but the sum of the all the energy drops is less than \( \delta \) between these scales.

Unfortunately, exploiting (1.27) turns out to be substantially harder to use than exploiting either (1.25) or even (1.26). In essence, this is because it is not a local assumption in terms of scale, and one needs estimates which can see many scales simultaneously, but which do not require any form of tangent cone uniqueness statements. Accomplishing this requires two new ingredients, a new rectifiable-Reifenberg type theorem, and a new \( L^2 \)-best subspace approximation theorem for stationary harmonic maps, which will allow us to apply the rectifiable-Reifenberg. Let us discuss these two ingredients separately.

We begin by discussing the new \( W^{1,p} \)-Reifenberg and rectifiable-Reifenberg theorems, which are introduced and proved in Section 3. Recall that the classical Reifenberg theorem, reviewed in Section 2.7, gives criteria under which a set becomes \( C^\alpha \)-Hölder equivalent to a ball \( B_1(0^k) \) in Euclidean space. In the context of this paper, it is important to improve on this result so that we have gradient and volume control of our set.

Let us remark that there have been many generalizations of the classical Reifenberg theorem in the literature, see for instance [Tor95] [DT12], however those results have hypotheses which are much too strong for the purposes of this paper. Instead, we will focus on improving the \( C^\alpha \)-equivalence to a \( W^{1,p} \)-equivalence. This is strictly stronger by Sobolev embedding, and if \( p > k \) then this results in volume estimates and a rectifiable structure for the set. More generally, we will require a version of the theorem which allows for more degenerate structural behavior, namely a rectifiable-Reifenberg theorem. In this case, the assumptions will conclude that a set \( S \) is rectifiable with volume estimates. Of course, what is key about this result is that the criteria will be checkable for our quantitative stratifications, thus let us discuss this criteria briefly. Roughly, if \( S \subseteq B_1(0^n) \) is a subset equipped with the \( k \)-dimensional Hausdorff measure \( \mathcal{H}^k \), then let us define...
the $k$-dimensional distortion of $S$ by
\[ D^k_S(y, s) = s^{-2} \inf_{L^k} \int_{S \cap B(x)} d^2(y, L^k) \, d\lambda^k(y), \tag{1.28} \]
where where the inf over all $k$-dimensional affine subspaces of $\mathbb{R}^n$. That is, $D^k$ measures how far $S$ is from being contained in a $k$-dimensional subspace. Our rectifiable-Reifenberg then requires this be small on $S$ in an integral sense, more precisely that
\[ r^{-k} \int_{S \cap B(x)} \sum_{r_a \leq r} D^k(y, r_a) \, d\lambda^k(x) < \delta^2. \tag{1.29} \]
For $\delta$ sufficiently small, the conclusions of the rectifiable-Reifenberg Theorem 3.3 are that the set $S$ is rectifiable with effective bounds on the $k$-dimensional measure. Let us remark that one cannot possibly conclude better than rectifiable under this assumption, see for instance the Examples of Section 2.6.

Thus, in order to prove the quantitative stratification estimates of Theorems 1.3 and 1.4, we will need to verify that the integral conditions (1.29) hold for the quantitative stratifications $S^k_\lambda(f)$, $S^k_{\lambda,2}(f)$ on all balls $B_r(x)$. In actuality the proof is more complicated. We will need to apply a discrete version of the rectifiable-Reifenberg, which will allow us to build an iterative covering of the quantitative stratifications, and each of these will satisfy (1.29). This will allow us to keep effective track of all the estimates involved. However, let us for the moment just focus on the main estimates which allows us to turn (1.29) into information about our harmonic maps, without worrying about such details.

Namely, in Section 4 we prove a new and very general approximation theorem for stationary harmonic maps. As always in this outline, let us assume $M \equiv \mathbb{R}^n$, the general case is no harder and we simply work on an injectivity radius ball. Thus we consider a stationary harmonic map $f : B_{16}(0^n) \to N$, as well as an arbitrary measure $\mu$ which is supported on $B_1(0^n)$. We would like to study how closely the support of $\mu$ can be approximated by a $k$-dimensional affine subspace $L^k \subseteq \mathbb{R}^n$, in the appropriate sense, and we would like to estimate this distance by using properties of $f$. Indeed, if we assume that $B_8(0^n)$ is not $(k + 1, \epsilon)$-symmetric with respect to $f$, then for an arbitrary $\mu$ we will prove in Theorem 4.1 that
\[ \inf_{L^k \subseteq \mathbb{R}^n} \int_{L^k} d^2(x, L^k) \, d\mu \leq C \int |\theta_0(x) - \theta_1(x)| \, d\mu, \tag{1.30} \]
where $C$ will depend on $\epsilon$, energy bound of $f$, and the geometry of $M$ and $N$. That is, if $f$ does not have $k + 1$ degrees of symmetry, then how closely the support of an arbitrary measure $\mu$ can be approximated by a $k$-dimensional subspace can be estimated by looking at the energy drop of $f$ along $\mu$. In applications, $\mu$ will be the restriction to $B_1$ of some discrete approximation of the $k$-dimensional Hausdorff measure on $S^k_\epsilon$, and thus the symmetry assumption on $f$ will hold for all balls centered on the support of $\mu$.

In practice, applying (1.30) to (1.29) is subtle and must be done inductively on scale. Additionally, in order to prove the effective Hausdorff estimates $\lambda^k(S^k_\lambda \cap B_r) \leq Cr^k$ we will need to use the Covering Lemma 5.1 to break up the quantitative stratification into appropriate pieces, and we will apply the estimates to these. This decomposition is based on a covering scheme first introduced by the authors in [NV]. Thus for the purposes of our outline, let us assume we have already proved the Hausdorff estimate $\lambda^k(S^k_\lambda \cap B_r) \leq Cr^k$, for
and use this to be able to apply the rectifiable-Reifenberg in order to conclude the rectifiability of the singular set. This may feel like a large assumption, however it turns out the proof of the Hausdorff estimate will itself be proved by a very similar tactic, though will require an inductive argument on scale, and will use the discrete rectifiable-Reifenberg of Theorem 3.4 in replace of the rectifiable-Reifenberg of Theorem 3.3.

Thus let us choose a ball \( B_r \) and let \( E \equiv \text{sup}_{B_r} \theta_\epsilon(y) \). Let us consider the subset \( \tilde{S}_\epsilon^k \subseteq S_\epsilon^k \cap B_r \) defined by

\[
\tilde{S}_\epsilon^k \equiv \{ y \in S_\epsilon^k \cap B_r : \theta_\epsilon(y) > E - \eta \},
\]

where \( \eta = \eta(n, K_M, K_N, \Lambda, \epsilon) \) will be chosen appropriately later. We will show now that \( \tilde{S}_\epsilon^k \) is rectifiable. Since \( \eta \) is fixed and the ball \( B_r \) is arbitrary, the rectifiability of all of \( S_\epsilon^k \) follows quickly from a covering argument. Thus, let us estimate (1.29) by plugging in (1.30) and the Hausdorff estimate to conclude:

\[
\begin{align*}
& r^{-k} \int_{\tilde{S}_\epsilon^k} \sum_{\alpha \leq r} D^k(x, r_\alpha) \, d\lambda ^k \\
& = r^{-k} \int_{\tilde{S}_\epsilon^k} \sum_{\alpha \leq r} \left( \inf_{r_\alpha} r_{\alpha}^{-2-k} \int_{\tilde{S}_\epsilon^k \cap B_{r_\alpha}(x)} d^k(y, L^k) d\lambda ^k(y) \right) d\lambda ^k(x) \\
& \leq C r^{-k} \int_{\tilde{S}_\epsilon^k} \sum_{\alpha \leq r} \left( r_{\alpha}^{-k} \int_{\tilde{S}_\epsilon^k \cap B_{r_\alpha}(x)} |\theta_{B_{r_\alpha}}(y) - \theta_{r_\alpha}(y)| d\lambda ^k(y) \right) d\lambda ^k(x) \\
& = C r^{-k} \sum_{\alpha \leq r} r_{\alpha}^{-k} \int_{\tilde{S}_\epsilon^k} \lambda ^k(\tilde{S}_\epsilon^k \cap B_{r_\alpha}(y)) |\theta_{B_{r_\alpha}}(y) - \theta_{r_\alpha}(y)| d\lambda ^k(y), \\
& \leq C r^{-k} \int_{\tilde{S}_\epsilon^k \cap B_{r}(x)} \sum_{\alpha \leq r} |\theta_{B_{r_\alpha}}(y) - \theta_{r_\alpha}(y)| d\lambda ^k(y), \\
& \leq C r^{-k} \int_{\tilde{S}_\epsilon^k} |\theta_{B_{r}}(y) - \theta_{r}(y)| d\lambda ^k(y) \\
& \leq C r^{-k} \lambda ^k(\tilde{S}_\epsilon^k) \cdot \eta \\
& < \delta^2.
\end{align*}
\]

where in the last line we have chosen \( \eta = \eta(n, K_M, K_N, \Lambda, \epsilon) \) so that the estimate is less than the required \( \delta \) from the rectifiable-Reifenberg. Thus we can apply the rectifiable-Reifenberg of Theorem 3.3 in order to conclude the rectifiability of the set \( \tilde{S}_\epsilon^k \), which in particular proves that \( S_\epsilon^k \) is itself rectifiable, as claimed.

2. Preliminaries

2.1. Stationary Harmonic Maps and Monotonicity. The key technical tool available to a stationary harmonic map \( f : B_2(p) \subseteq M \to N \), which is not available to a weakly harmonic map, is that of a monotone quantity. Namely, given \( x \in B_1(p) \) and \( r \leq 1 \) we can consider the normalized Dirichlet energy defined by

\[
\theta_r(x) \equiv r^{-2-n} \int_{B_r(x)} |\nabla f|^2.
\]
If $M \equiv \mathbb{R}^n$, then for each $x$ fixed we have that $\theta_r(x)$ is exactly a monotone increasing function of $r$. More precisely, we have that
\[
\frac{d}{dr} \theta_r(x) = 2r^{2-n} \int_{\partial B_r(x)} |\frac{\partial f}{\partial r}|^2 = 2r^{2-n} \int_{\partial B_r(x)} (\nabla f, n_x)^2.
\] (2.2)
Hence, we see that if $\theta_r(x)$ is independent of $r$ then we have that $f$ is 0-symmetric with respect to $x$. More generally, if $\theta_r(x) = \theta_s(x)$ then $f$ is radially symmetric on the annulus $A_{s,r}(x) = B_s(x) \setminus B_r(x)$. Motivated by this, we see that what we are really interested in is the amount the Dirichlet energy drops from one scale to the next, and thus we define
\[
W_{s,r}(x) \equiv \theta_s(x) - \theta_r(x) \geq 0.
\] (2.3)
Often times we will want to enumerate our choice of scale, so let us define the scales $r_\alpha \equiv 2^{-\alpha}$ for $\alpha \geq 0$, and the corresponding Dirichlet energy drop:
\[
W_{\alpha}(x) \equiv W_{r_{\alpha-3}, r_{\alpha}}(x) \equiv \theta_{r_{\alpha-3}}(x) - \theta_{r_{\alpha}}(x) \geq 0.
\] (2.4)
In the general case, i.e., when $M \neq \mathbb{R}^n$, essentially the same statements may be made, however $\theta_r(x)$ is now only almost monotone, meaning that $e^{Cr_r} \theta_r(x)$ is monotone for some constant which depends only on the geometry of $M$. From this one can prove that at every point, every tangent map is 0-symmetric, which is the starting point for the dimension estimate (1.3) of [SU82]. In Section 2.2 we will discuss quantitative versions of this point, first introduced in [CN13b] and used in this paper as well, and also generalizations which involve higher degrees of symmetry. These points were first used in [CN13b] to prove Minkowski estimates on the quantitative stratification of a stationary harmonic map. They will also play a role in our arguments, though in a different manner.

2.2. Quantitative 0-Symmetry and Cone Splitting. In this subsection we review some of the quantitative symmetry and splitting results of [CN13b], in particular those which will play a role in this paper.

The first result we will discuss acts as an effective formulation of the fact that every tangent map is 0-symmetric. Namely, the quantitative 0-symmetry of [CN13b] says that for each $\epsilon > 0$ and point $x \in B_1$, that away from a finite number of scales every ball $B_{r_\alpha}(x)$ is $(0, \epsilon)$-symmetric, where $r_\alpha \equiv 2^{-\alpha}$. Precisely:

**Theorem 2.1** (Quantitative 0-Symmetry [CN13b]). Let $f : B_2(p) \subseteq M \rightarrow N$ be a stationary harmonic map satisfying (1.14) with $\int_{B_2(p)} |\nabla f|^2 \leq \Lambda$ and let $\epsilon > 0$ be fixed. Then the following hold:

1. There exists $\delta(n, K_N, \Lambda, \epsilon) > 0$ such that for each $x \in B_1(p)$ if $|\theta_r(x) - \theta_{r\delta}(x)| < \delta$ and $K_M < \delta$, then $B_r(x)$ is $(0, \epsilon)$-symmetric.
2. For each $x \in B_1(p)$ there exists a finite number of scales $\{\alpha_1, \ldots, \alpha_K\} \subseteq \mathbb{N}$ with $K \leq K(n, K_M, K_N, \Lambda, \epsilon)$ such that for $r \notin \{r_{\alpha_j}/2, 2r_{\alpha_j}\}$ we have that $B_r(x)$ is $(0, \epsilon)$-symmetric.

**Remark 2.1.** In [CN13b] it is stated that the constant depends on the manifold $N$, not just $K_N$, however it is easy to check that only $K_N$ is important in the proof. See the proof of Lemma 4.2 for a relevant argument.

**Remark 2.2.** The assumption $K_M < \delta$ is of little consequence, since this just means focusing the estimates on balls of sufficiently small radius after rescaling.
Another technical tool that played an important role in [CN13b] was that of cone splitting. This will be used in this paper when proving the existence of unique tangent planes of symmetry for the singular set, so we will discuss it here. In short, cone splitting is the idea that multiple 0-symmetries add to give rise to a \( k \)-symmetry. To state it precisely let us give a careful definition of the notion of independence of a collection of points:

**Definition 2.2.** We say a collection of points \( \{x_1, \ldots, x_\ell\} \in \mathbb{R}^n \) is independent if they are linearly independent. We say the collection is \( \tau \)-independent if \( d(x_{k+1}, \text{span}\{x_1, \ldots, x_k\}) > \tau \) for each \( k \). If \( \{x_1, \ldots, x_\ell\} \in M \) then we say the collection is \( \tau \)-independent with respect to \( x \) if \( d(x, x_j) < \text{inj}(x) \) and the collection is \( \tau \)-independent when written in exponential coordinates at \( x \).

Now we are in a position to state the effective cone splitting of [CN13b]:

**Theorem 2.3** (Cone Splitting [CN13b]). Let \( f : B_2(p) \subseteq M \rightarrow N \) be a stationary harmonic map satisfying (1.14) with \( \int_{B_2(p)} |\nabla f|^2 \leq \Lambda \) and let \( \epsilon, \tau > 0 \) be fixed. Then there exists \( \delta(n, K_N, \Lambda, \epsilon, \tau) > 0 \) such that if \( K_M < \delta \) and \( x_0, \ldots, x_k \in B_1(p) \) are such that

1. \( B_2(x_j) \) are \((0, \delta)\)-symmetric,
2. \( \{x_0, \ldots, x_k\} \) are \( \tau \)-independent at \( p \),

then \( B_1(p) \) is \((k, \epsilon)\)-symmetric.

We end with the following, which one can view as a quantitative form of dimension reduction. The proof is standard and can be accomplished by a contradiction argument:

**Theorem 2.4** (Quantitative Dimension Reduction). Let \( f : B_2(p) \subseteq M \rightarrow N \) be a stationary harmonic map satisfying (1.14) with \( \int_{B_2(p)} |\nabla f|^2 \leq \Lambda \). Then for each \( \epsilon > 0 \) there exists \( \delta(n, K_M, K_N, \Lambda, \epsilon), \tau(n, K_M, K_N, \Lambda, \epsilon) > 0 \) such that if \( B_2(p) \) is \((k, \delta)\)-symmetric with respect to some \( k \)-plane \( V^k \), then for each \( x \in B_1(p) \setminus B_\epsilon(V^k) \) we have that \( B_\epsilon(x) \) is \((k+1, \epsilon)\)-symmetric.

2.3. **Defect Measure.** Since it will play a role for us in one technical aspect of this paper, let us recall in this section the defect measure of a sequence of stationary harmonic maps. We begin with a definition:

**Definition 2.5.** Let \( f_i : B_2(p_i) \rightarrow N \) be a sequence of stationary harmonic maps satisfying (1.14) with \( \int_{B_2(p_i)} |\nabla f_i|^2 \leq \Lambda \). Then, after possibly passing to a subsequence, \( f_i \) has a weak \( W^{1,2} \) limit \( f \), and we can consider the measure

\[
|\nabla f|_p^2 \, dv_x 
\rightarrow |\nabla f|^2 \, dv_x + \nu ,
\]

where this last convergence is intended in the weak sense of Radon measures. The measure \( \nu \) is nonnegative by Fatou’s lemma, and it is called the defect measure associated to \( f_i \rightarrow f \).

**Remark 2.3.** We can also allow \( f_i : B_2(p_i) \rightarrow N \) to be defined on different manifolds. In practice, this will occur under the assumption \( K_M \rightarrow 0 \), so that \( \nu \) becomes a measure on \( B_2(0^\circ) \).
The following theorem is one of the main accomplishments of [Lin99]:

**Theorem 2.6** (Rectifiability of Defect Measure). [Lin99, lemma 1.7] If $\nu$ is a defect measure as in definition 2.5, then it is $n-2$ rectifiable.

A key tool in the proof of the above, which will be useful in this paper as well, is the following 0-symmetry result:

**Theorem 2.7.** [Lin99, lemma 1.7 (ii)] Let $f_i : B_2(p_i) \subseteq M_i \rightarrow N$ be a sequence of stationary harmonic maps satisfying (1.14) with $\int_{B_2(p_i)} |\nabla f_i|^2 \leq \Lambda$. Assume $K_{M_i} < \delta_i \rightarrow 0$ and that $|\theta_{i,2}(p_i) - \theta_{i,0}| < \delta_i \rightarrow 0$. Then the defect measure $|\nabla f_i|^2 dv_g \rightarrow |\nabla f|^2 dv_g + \nu$ is 0-symmetric, that is, both the function $f$ and $\nu$ are invariant under dilation around the origin.

2.4. $\epsilon$-regularity for Minimizers. In this subsection we quickly review the $\epsilon$-regularity theorems of [CN13b], which themselves build on the difficult work of [SU82]. This will be our primary technical tool in upgrading the structural results on stationary harmonic maps to the regularity results for minimizing harmonic maps.

Recall the definition of the regularity scale given in Definition 1.7, then the main theorem of this subsection is the following:

**Theorem 2.8** (Minimizing $\epsilon$-Regularity [CN13b]). Let $f : B_2(p) \subseteq M \rightarrow N$ be a minimizing harmonic map satisfying (1.14) with $\int_{B_2(p)} |\nabla f|^2 \leq \Lambda$. Then there exists $\epsilon(n, K_N, \Lambda) > 0$ such that if $K_M < \epsilon$ and $B_2(p)$ is $(n-2, \epsilon)$-symmetric, then

$$r_f(p) \geq 1.$$  

**Remark 2.4.** As previously remarked, the assumption $K_M < \epsilon$ is of little consequence, since this just means focusing the estimates on balls of sufficiently small radius after rescaling.

The following stationary version was proved in [CHN13a], and is essentially just a combination of the defect measure ideas of [Lin99], the $\epsilon$-regularity of [Bet93], and a contradiction argument:

**Theorem 2.9** (Stationary $\epsilon$-Regularity [CHN13a]). Let $f : B_2(p) \subseteq M \rightarrow N$ be a minimizing harmonic map satisfying (1.14) with $\int_{B_2(p)} |\nabla f|^2 \leq \Lambda$. Assume that there exists no nonconstant stationary harmonic map $S^2 \rightarrow N$, then there exists $\epsilon(n, N, \Lambda) > 0$ such that if $K_M < \epsilon$ and $B_1(p)$ is $(n-2, \epsilon)$-symmetric, then

$$r_f(p) \geq 1.$$  

Let us also discuss some improvements of the above $\epsilon$-regularity theorems in the case where there are no nonconstant harmonic maps from $S^k$ into $N$. These are the key $\epsilon$-regularity results needed for Theorems 1.9 and 1.10. We begin with the minimizing case:
Theorem 2.10 (Improved Minimizing $\epsilon$-Regularity [CHN13a]). Let $f : B_2(p) \subseteq M \to N$ be a minimizing harmonic mapping satisfying (1.14) with $\int_{B_2(p)} |\nabla f|^2 \leq \Lambda$. Assume further that for some $k \geq 2$ there exists no nonconstant minimizing harmonic maps $S^\ell \to N$ for all $\ell \leq k$. Then there exists $\epsilon(n,N,\Lambda)$ such that if $K_M < \epsilon$ and $B_1(p)$ is $(n-k-1, \epsilon)$-symmetric, then

$$r_f(p) \geq 1.$$ 

Finally, we end with the stationary version of the above theorem:

Theorem 2.11 (Improved Stationary $\epsilon$-Regularity [CN13b]). Let $f : B_2(p) \subseteq M \to N$ be a stationary harmonic mapping satisfying (1.14) with $\int_{B_2(p)} |\nabla f|^2 \leq \Lambda$. Assume further that for some $k \geq 2$ there exists no stationary harmonic maps $S^\ell \to N$ for all $\ell \leq k$. Then there exists $\epsilon(n,N,\Lambda)$ such that if $K_M < \epsilon$ and $B_1(p)$ is $(n-k-1, \epsilon)$-symmetric, then

$$r_f(p) \geq 1.$$ 

Remark 2.5. Note that in both theorems above, nothing would change if we only required that $N$ admits no continuous (or even smooth) nonconstant stationary (or minimizing) harmonic map $S^\ell \to N$ for all $k \leq \ell$. Indeed, the tangent maps at singular points of non-smooth stationary (or minimizing) harmonic maps give rise to smooth stationary (or minimizing) harmonic maps from lower dimensional spheres.

2.5. Hausdorff, Minkowski, and packing Content. In this subsection we give a brief review of the notions of Hausdorff, Minkowski, and packing content. We will also use this to recall the Hausdorff measure. The results of this subsection are completely standard, but this gives us an opportunity to introduce some notation for the paper. Let us begin with the notions of content:

Definition 2.12. Given a set $S \subseteq \mathbb{R}^n$ and $r > 0$ we define the following:

1. The $k$-dimensional Hausdorff $r$-content of $S$ is given by

$$\lambda^k_r(S) \equiv \inf \left\{ \sum \omega_k r_i^k : S \subseteq \bigcup B_r(x_i) \text{ and } r_i \leq r \right\}.$$  \hfill (2.6)

2. The $k$-dimensional Minkowski $r$-content of $S$ is given by

$$m^k_r(S) \equiv \inf \left\{ \sum \omega_k r_i^k : S \subseteq \bigcup B_r(x_i) \right\}.$$ \hfill (2.7)

3. The $k$-dimensional packing $r$-content of $S$ is given by

$$p^k_r(S) \equiv \sup \left\{ \sum \omega_k r_i^k : x_i \in S \text{ and } \{B_r(x_i)\} \text{ are disjoint} \right\}.$$ \hfill (2.8)

These definitions make sense for any $k \in [0, \infty)$, though in this paper we will be particularly interested in integer valued $k$. Notice that if $S$ is a compact set then $\lambda^k_r(S), m^k_r(S) < \infty$ for any $r > 0$, and that we always have the relations

$$\lambda^k_r(S) \leq m^k_r(S) \leq p^k_r(S).$$ \hfill (2.9)

In particular, bounding the Hausdorff content is less powerful than bounding the Minkowski content, which is itself less powerful than bounding the packing content. Notice that $\lambda^k_r, m^k_r, p^k_r$ are honest contents in
that they are finitely additive measures. In particular, integration with respect to bounded functions is well defined. Let us also note that $m^k_r(S)$ is uniformly equivalent to $r^{k-n}\text{Vol}(B_r(S))$ in the sense that there exists a dimensional constant $C(n) > 0$ such that
\begin{equation}
C^{-1}r^{k-n}\text{Vol}(B_r(S)) \leq m^k_r(S) \leq Cr^{k-n}\text{Vol}(B_r(S)).
\end{equation}

Primarily in this paper we will be mostly interested in content estimates, because these are the most effective estimates. However, since it is classical, let us go ahead and use the Hausdorff content to define a measure. To accomplish this, let us more generally observe that if $r \leq r'$ then we have the monotone relations:
\begin{align}
\lambda^k_r(S) &\geq \lambda^k_{r'}(S), \\
m^k_r(S) &\geq m^k_{r'}(S), \\
p^k_r(S) &\leq p^k_{r'}(S).
\end{align}

In particular, in each case there is a well defined limit
\begin{align}
\lambda^k_0(S) &\equiv \lim_{r \to 0} \lambda^k_r(S), \\
m^k_0(S) &\equiv \lim_{r \to 0} m^k_r(S), \\
p^k_0(S) &\equiv \lim_{r \to 0} p^k_r(S).
\end{align}

The case of the 0-Hausdorff content is particularly special, as it defines a genuine measure. In the case of the Minkowski and packing contents, these only define pre-measures, though we can use the standard procedure to define a measure out of them. Since we will have no need for it we will only consider the Hausdorff measure:

**Definition 2.13.** Given a set $S \subseteq \mathbb{R}^n$ we define its $k$-dimensional Hausdorff measure by $\lambda^k(S) \equiv \lambda^k_0(S).

Finally, let us use the contents to define the notion of dimension. To accomplish this notice that if $k \leq k'$ then we have the relations
\begin{align}
\lambda^k_r(S) &\geq \lambda^{k'}_{r'}(S), \\
m^k_r(S) &\geq m^{k'}_{r'}(S),
\end{align}
for any $r \geq 0$. In particular, as a function of $k$ we have that the contents of a set $S$ are monotone decreasing functions. Also observe that if $S \subseteq \mathbb{R}^n$ and if $k > n$, then
\begin{equation}
\lambda^k_0(S) = m^k_0(S) = 0.
\end{equation}

Now we define dimension:

**Definition 2.14.** Given a set $S \subseteq \mathbb{R}^n$ we define its Hausdorff and Minkowski dimension (or box-dimension) by
\begin{align}
\dim_H S &\equiv \inf \left\{ k \geq 0 : \lambda^k_0(S) = 0 \right\}, \\
\dim_M S &\equiv \inf \left\{ k \geq 0 : m^k_0(S) = 0 \right\}.
\end{align}
As an easy example consider the rationals \( \mathbb{Q}^n \subseteq \mathbb{R}^n \). Then it is a worthwhile exercise to check that \( \dim_H \mathbb{Q}^n = 0 \), while \( \dim_M \mathbb{Q}^n = n \).

A very important notion related to measures is the \textit{density} at a point. Although this is standard, for completeness we briefly recall the definition of Hausdorff density, and refer the reader to [Mat95, chapter 6] for more on this subject.

\textbf{Definition 2.15.} Given a set \( S \subset \mathbb{R}^n \) which is \( \lambda^k \)-measurable, and \( x \in \mathbb{R}^n \), we define the \( k \)-dimensional upper and lower density of \( S \) at \( x \) by

\[
\theta^k(S, x) = \limsup_{r \to 0} \frac{\lambda^k(S \cap B_r(x))}{\omega_k r^k}, \quad \theta^k_*(S, x) = \liminf_{r \to 0} \frac{\lambda^k(S \cap B_r(x))}{\omega_k r^k}.
\]

(2.16)

In the following, we will use the fact that for almost any point in a set with finite \( \lambda^k \)-measure, the density is bounded from above and below.

\textbf{Proposition 2.16} ([Mat95]). Let \( S \subset \mathbb{R}^n \) be a set with \( \lambda^k(S) < \infty \). Then for \( k \)-a.e. \( x \in S \):

\[
2^{-k} \leq \theta^k(S, x) \leq 1,
\]

(2.17)

while for \( k \)-a.e. \( x \in \mathbb{R}^n \setminus S \)

\[
\theta^k_*(S, x) = 0.
\]

(2.18)

\subsection*{2.6. Examples.}

In this subsection we present a few examples which motivate the sharpness of our results.

\textbf{2.6.1. Sharpness of \( L^p \) Estimates for Minimizers.} This example is completely standard, we wish to simply point out some of its properties. Namely, consider the mapping \( f : B_2(0^3) \to S^2 \) given by the projection

\[
f(x) = \frac{x}{|x|}.
\]

(2.19)

This is a minimizing harmonic map (see [Lin87, CG89]). It is easy to compute that

\[
|\nabla f|(x) \approx r_f^{-1}(x) \approx \frac{1}{|x|}.
\]

(2.20)

We therefore get that \( |\nabla f|, r_f^{-1}(x) \) have uniform estimates in \( L^3_{\text{weak}} \), however neither belong to \( L^3 \). In particular, this shows that the estimates of Theorem 1.8 are sharp.

\textbf{2.6.2. Rectifiable-Reifenberg Example I.} Let us begin with an easy example, which shows that the rectifiable conclusions of Theorem 3.3 is sharp. That is, one cannot hope for better structural results under the hypothesis. Indeed, consider any \( k \)-dimensional subspace \( V^k \subseteq \mathbb{R}^n \), and let \( S \subseteq V^k \cap B_2(0^n) \) be an arbitrary measurable subset. Then clearly \( D(x, r) \equiv 0 \) for each \( x \) and \( r > 0 \), and thus the hypotheses of Theorem 3.3 are satisfied, however \( S \) clearly need not be better than rectifiable. In the next example we shall see that \( S \) need not even come from a single rectifiable chart, as it does in this example.
2.6.3. Rectifiable-Reifenberg Example II. With respect to the conclusions of Theorem 3.3 there are two natural questions regarding how sharp they are. First, is it possible to obtain more structure from the set $S$ than rectifiable? In particular, in Theorem 3.2 there are topological conclusions about the set, is it possible to make such conclusions in the context of Theorem 3.3? In the last example we saw this is not the case. Then a second question is to ask whether we can at least find a single rectifiable chart which covers the whole set $S$. This example taken from [DT12, counterexample 12.4] shows that the answer to this question is negative as well.

To build our examples let us first consider a unit circle $S^1 \subseteq \mathbb{R}^3$. Let $M^2 \supset S^1$ be a smooth Möbius strip around this circle, and let $S_\varepsilon \subseteq M^2 \cap B_\varepsilon (S^1) \equiv M^2_\varepsilon$ be an arbitrary $\lambda^2$-measurable subset of the Möbius strip, contained in a small neighborhood of the $S^1$. In particular, $\text{Area}(S_\varepsilon) \leq C\varepsilon \to 0$ as $\varepsilon \to 0$. It is not hard, though potentially a little tedious, to check that assumptions of Theorem 3.3 hold for $\delta \to 0$ as $\varepsilon \to 0$.

However, we have learned two points from these example. First, since $S_\varepsilon$ was an arbitrary measurable subset of a two dimensional manifold, we have that it is 2-rectifiable, however that is the most which may be said of $S_\varepsilon$. That is, structurally speaking we cannot hope to say better than 2-rectifiable about the set $S_\varepsilon$. More than that, since $S_\varepsilon$ is a subset of the Möbius strip, we see that even though $S_\varepsilon$ is rectifiable, we cannot even cover $S_\varepsilon$ by a single chart from $B_1(0^2)$, as there are topological obstructions, see [DT12] for more on this.

2.7. The Classical Reifenberg Theorem. In this Section we recall the classical Reifenberg Theorem, as well as some more recent generalizations. The Reifenberg theorem gives criteria on a closed subset $S \subseteq B_2 \subseteq \mathbb{R}^n$ which determine when $S \cap B_1$ is bi-Hölder to a ball $B_1(0^k)$ in a smaller dimensional Euclidean space. The criteria itself is based on the existence of good best approximating subspaces at each scale. We start by recalling the Hausdorff distance.

**Definition 2.17.** Given two sets $A, B \subseteq \mathbb{R}^n$, we define the Hausdorff distance between these two by
\[
d_H(A, B) = \inf \{ r \geq 0 \text{ s.t. } A \subseteq B_r(A) \text{ and } B \subseteq B_r(B) \}.
\] (2.21)

Recall that $d_H$ is a distance on closed sets, meaning that $d_H(A, B) = 0$ implies $\overline{A} = \overline{B}$.

The classical Reifenberg theorem says the following:

**Theorem 2.18** (Reifenberg Theorem [Rei60, Sim]). For each $0 < \alpha < 1$ and $\varepsilon > 0$ there exists $\delta(n, \alpha, \varepsilon) > 0$ such that the following holds. Assume $0^\alpha \subseteq S \subseteq B_2 \subseteq \mathbb{R}^n$ is a closed subset, and that for each $B_\delta(x) \subseteq B_1$ with $x \in S$ we have
\[
\inf_{L^k} d_H(S \cap B_\delta(x), L^k \cap B_\delta(x)) < \delta r,
\] (2.22)
where the $\inf$ is taken over all $k$-dimensional affine subspaces $L^k \subseteq \mathbb{R}^n$. Then there exists $\phi : B_1(0^k) \to S$ which is a $C^\alpha$ bi-Hölder homeomorphism onto its image with $[\phi]_{C^\alpha}, [\phi^{-1}]_{C^\alpha} < \varepsilon$ and $S \cap B_1 \subseteq \phi(B_1(0^k))$. 

Remark 2.6. In fact, one can prove a little more. In particular, under the hypothesis of the previous theorem, there exists a closed subset $S' \subset \mathbb{R}^n$ such that $S' \cap B_1(0) = S \cap B_1(0)$ and which is homeomorphic to a $k$-dimensional subspace $0^k \in T_0 \subseteq \mathbb{R}^n$ via the $C^\alpha$ bi-Hölder homeomorphism $\phi : T_0 \to S'$. Moreover, $|\phi(x) - x| \leq C(n)\delta$ for all $x \in T_0$ and $\phi(x) = x$ for all $x \in T_0 \setminus B_2(0)$.

One can paraphrase the above to say that if $S$ can be well approximated on every ball by a subspace in the $L^\infty$-sense, then $S$ must be bi-Hölder to a ball in Euclidean space.

Let us also mention that there are several more recent generalizations of the classic Reifenberg theorem. In [Tor95], the author proves a strengthened version of $(2.22)$ that allows one to improve bi-Hölder to bi-Lipschitz. Unfortunately, for the applications of this paper the hypotheses of [Tor95] are much too restrictive. We will require a weaker condition than in [Tor95], which is more integral in nature, see Theorem 3.2. In exchange, we will improve the bi-Hölder of the classical Reifenberg to $W^{1,p}$.

We will also need a version of the classical Reifenberg which only assumes that the subset $S$ is contained near a subspace, not conversely that the subspace is also contained near $S$. In exchange, we will only conclude the set is rectifiable. A result in this direction was first proved in [DT12], but again the hypothesis are too restrictive for the applications of this paper, and additionally there is a topological assumption necessary for the results of [DT12], which is not reasonable in the context in this paper. We will see how to appropriately drop this assumption in Theorem 3.3.

3. THE $W^{1,p}$-REIFENBERG AND RECTIFIABLE-REIFENBERG THEOREMS

In this Section we prove very general Reifenberg type theorems for subsets of Euclidean spaces. Although this is a key ingredient of the proof of our main theorems, the results of this Section are of an independent nature, and may be of some separate interest. Therefore, we will attempt to write them so that this Section may be read independently of the rest of the paper, with the exception of the appropriate Preliminary Sections.

As described in Section 2.7, the classical Reifenberg theorem tells us that if a closed subset $S \subseteq B_2 \subseteq \mathbb{R}^n$ can be well approximated on every ball by some $k$-dimensional affine space in the $L^\infty$ sense, then $S \cap B_1$ must itself be homeomorphic, in fact bi-Hölder, to a ball $B_1(0^k)$ in a smaller dimensional Euclidean space.

For the applications of this paper, we will need to improve on the classical Reifenberg theorem in several ways. At the most basic level, $C^\alpha$-equivalence is not strong enough. We will require control on the gradient of the mappings, since this is the essence of rectifiability and volume control. For this purpose we will obtain $W^{1,p}$-control on our mappings for $k < p < \infty$. We will deal with this first in Theorem 3.2, where we will also replace the $L^\infty$ closeness of the Reifenberg theorem with an $L^2$ closeness condition, which turns out to be more natural in the applications.

A second manner in which we will need improvement is that we will need to allow for the existence of holes. That is, in applications our set $S$ may be rectifiable, but it may not be homeomorphic to a ball, this
is much too strong. To deal with this we can weaken the hypothesis of the Reifenberg theorem and only assume that on each ball $B_r(x)$ there is a $k$-dimensional subspace $V$ such that each point of our set $S$ is close to $V$, but not conversely that each point of the subspace $V$ is close to $S$. We study this case in Theorem 3.3, where we will show under natural conditions that such a set $S$ is rectifiable with volume bounds, which is easily seen to be a sharp result under the assumptions of Theorem 3.3.

The last version of the Reifenberg theorem that we prove has a more discrete nature, and it is the one we will use to obtain the volume bounds in the rest of the paper. In this case, our set $S = \{x_i\}$ will be a discrete set of points paired with radii $r_i > 0$ such that $\{B_r(x_i)\}$ are disjoint balls. We will associate to this collection the measure $\mu = \sum \omega_i r_i^k \delta_{x_i}$, and under the appropriate assumptions show in Theorem 3.4 that the volume of $\mu$ continues to enjoy the packing upper bound $\mu(B_r(x)) < Cr^k$ for all radii $r > r_i$. In the applications, we will prove the main theorems of this paper by building a series of covers of the quantitative stratifications $S^k$, and we will apply the discrete $W^{1,\mu}-$Reifenberg to each of this to obtain volume bounds. Only at the last stage will we apply Theorem 3.3 in order to obtain the rectifiability of the sets as well.

3.1. Statement of Main $W^{1,\mu}$-Reifenberg and rectifiable-Reifenberg Results. There are three main results we wish to discuss in this subsection, each more general than the last. In order to keep the statements as clean and intuitive as possible we will introduce the following definitions, and discuss them briefly.

Definition 3.1. Let $\mu$ be a measure on $B_2$ with $r > 0$ and $k \in \mathbb{N}$. Then we define the $k$-dimensional displacement by

$$D^k_\mu(x, r) \equiv \inf_{l^k} \int_{B_r(x)} d^2(y, l^k) \, d\mu(y),$$

if $\mu(B_r(x)) \geq \varepsilon_k r^k = \omega_k 20^{-k} r^k$, and $D^k_\mu(x, r) \equiv 0$ otherwise, where the inf’s are taken over all $k$-dimensional affine subspaces $L^k \subseteq \mathbb{R}^n$. If $S \subseteq B_2$, then we can define its $k$-displacement $D^k_S(x, r)$ by associating to $S$ the $k$-dimensional Hausdorff measure $\lambda^k_S$ restricted to $S$.

Remark 3.1. One can replace $\varepsilon_k$ by any smaller lower bound, and the proofs and statements will all continue to hold.

Remark 3.2. Notice that the definitions are scale invariant. In particular, if we rescale $B_r \to B_1$ and let $\tilde{S}$ be the induced set, then $D^k_{\tilde{S}}(x, r) \to D^k_S(x, 1)$.

Remark 3.3. Notice the monotonicity given by the following: If $S' \subseteq S$, then $D^k_{S'}(x, r) \leq D^k_S(x, r)$.

Remark 3.4. It is easily seen from the definition that, up to dimensional constants, $D^k_S(x, r)$ is controlled above and below by $D^k_S(x, r/2)$ and $D^k_S(x, 2r)$. In particular, if $\mu(B_r(x)) \geq 2^k \varepsilon_k r^k$, then for all $y \in S \cap B_r(x)$, $D^k_S(x, r) \leq 2^{k+2} D^k_S(y, 2r)$. As a corollary we have the estimate

$$D^k_S(x, r) \leq 2^{k+2} \int_{S \cap B_r(x)} D^k_S(y, 2r) \, d\lambda^k(y).$$

The first of the main results of this part of the paper is not strictly used elsewhere in the paper, but it is a natural generalization of the Reifenberg and gives intuition and motivation for the rest of the statements,
which are essentially more complicated versions of it. Thus, we have decided it is worth discussing independently. The first theorem of this section is the following:

**Theorem 3.2** ($W^{1,p}$-Reifenberg). For each $\epsilon > 0$ and $p \in [1, \infty)$ there exists $\delta(n, \epsilon, p) > 0$ such that the following holds. Let $S \subseteq B_2 \subseteq \mathbb{R}^n$ be a closed subset with $0^p \in S$, and assume for each $x \in S \cap B_1$ and $B_r(x) \subseteq B_2$ that

$$\inf_{L^k} d_H(S \cap B_r(x), L^k \cap B_r(x)) < \delta r, \quad (3.3)$$

and

$$\int_{S \cap B_r(x)} \left( \int_0^r D^k_S(x, s) \frac{ds}{s} \right) d\lambda^k(x) < \delta^2 r^k. \quad (3.4)$$

Then the following hold:

1. there exists a mapping $\phi : \mathbb{R}^k \to \mathbb{R}^n$ which is a $1 + \epsilon$ bi-$W^{1,p}$ map onto its image and such that $S \cap B_1(0^p) = \phi(B_1(0^p)) \cap B_1(0^p)$.
2. $S \cap B_1(0^p)$ is rectifiable.
3. For each ball $B_r(x) \subseteq B_1$ with $x \in S$ we have

$$\int_{S \cap B_r(x)} \left( \int_0^r D^k_S(x, s) \frac{ds}{s} \right) d\lambda^k(x) < \delta^2 r^k. \quad (3.6)$$

**Remark 3.5.** The result (2) both follows from (1). We get (3) by applying the result of (1) to all smaller balls $B_r(x) \subseteq B_1$, since the assumptions of the theorem hold on these balls as well.

**Remark 3.6.** Note that a bi-$W^{1,p}$ map is a bi-$C^\alpha$ map, in particular we see that $\phi(B_1(0^p))$ is homeomorphic to the ball $B_1(0^p)$.

**Remark 3.7.** As it is easily seen, the requirement that $S$ is closed is essential for this theorem, and in particular for the lower bound on the Hausdorff measure. As an example, consider any set $S \subseteq \mathbb{R}^k$ which is dense but has zero Hausdorff measure. In the following theorems, we will not be concerned with lower bounds on the measure, and we will be able to drop the closed assumption.

Let us now consider the case when we drop the assumption (3.3) from the result. The key distinction now is that $S$ only needs to be locally near a piece of a $k$-dimensional subspace, but not conversely. Thus, we cannot hope to obtain topological information about the set $S$, see Section 2.6.3. The precise statement is the following:

**Theorem 3.3** (Rectifiable-Reifenberg). For every $\epsilon > 0$, there exists $\delta(n, \epsilon) > 0$ such that the following holds. Let $S \subseteq B_2 \subseteq \mathbb{R}^n$ be a $\lambda^k$-measurable subset, and assume for each $B_r(x) \subseteq B_2$ with $\lambda^k(S \cap B_r(x)) \geq \epsilon r^k$ that

$$\int_{S \cap B_r(x)} \left( \int_0^r D^k_S(x, s) \frac{ds}{s} \right) d\lambda^k(x) < \delta^2 r^k. \quad (3.6)$$

Then the following hold:
(1) For each ball $B_r(x) \subseteq B_1$ with $x \in S$ we have
\[
\lambda^k(S \cap B_r(x)) \leq (1 + \epsilon)\omega_k r^k. \tag{3.7}
\]

(2) $S \cap B_1(0)$ is $k$-rectifiable.

Remark 3.8. Notice that for the statement of the theorem we do not need control over balls which already have small measure. This will be quite convenient for the applications.

Remark 3.9. Instead of (3.6) we may assume the essentially equivalent estimate
\[
\sum_{r_\alpha \leq 2r} \int_{S \cap B_r(x)} D_\delta^k(x, r_\alpha) d\lambda^k(x) < \delta^2 r^k. \tag{3.8}
\]
In the applications, this will be the more convenient phrasing.

Finally, we end by stating a version of the above theorem which is more discrete in nature. This result will be particularly important in the proof of the main theorems of this paper:

**Theorem 3.4** (Discrete Rectifiable-Reifenberg). There exists $\delta(n) > 0$ such that the following holds. Let $\{B_r(x_j)\} \subseteq B_2$ be a collection of disjoint balls, and let $\mu \equiv \sum_j \omega_k r^k \delta_{S_j}$ be the associated measure. Assume that for each $B_r(x) \subseteq B_2$ with $\mu(B_r(x)) \geq \epsilon_k r^k = \omega_k(r/20)^k$ we have
\[
\int_{B_r(x)} \left( \int_0^r D_\mu^k(x, s) \frac{ds}{s} \right) d\mu(x) < \delta^2 r^k. \tag{3.9}
\]
Then we have the estimate
\[
\sum_{x_j \in B_1} r_j^k < D(n). \tag{3.10}
\]

Remark 3.10. As in Theorem 3.3, instead of (3.6) we can may assume the estimate
\[
\sum_{r_\alpha \leq 2r} \int_{B_r(x)} D_\mu^k(x, r_\alpha) d\mu(x) < \delta^2 r^k. \tag{3.11}
\]
In the applications, this will be the more convenient phrasing.

For many of the applications of this paper, it is this version of the Reifenberg which will be most important. The reasoning is that to keep uniform control on all estimates our inductive covering will need to cover at all scales. It is only at the last scale that we begin to cover the singular sets $S^k$ on sets of positive measure.
3.2. **Technical Constructions.** In this section, we prove some technical lemmas needed for dealing with the relation between best $L^2$ subspaces. We start by recalling some standard facts about affine subspaces in $\mathbb{R}^n$ and Hausdorff distance.

**Definition 3.5.** Given two linear subspaces $L, V \subseteq \mathbb{R}^n$, we define the Grassmannian distance between these two as

$$d_G(L, V) = d_H(L \cap B_1(0), V \cap B_1(0)),$$

(3.12)

Note that if $\dim(L) \neq \dim(V)$, then $d_G(L, V) = 1$.

**Remark 3.11.** In practice we will want to consider the Grassmannian distance of affine subspaces, not just linear. In this case, we are by definition considering the Grassmannian distance of the induced linear subspaces.

For general subsets in $\mathbb{R}^n$, it is evident that $A \subseteq B_{\delta}(B)$ does not imply $B \subseteq B_{c\delta}(A)$. However, if $A$ and $B$ are affine spaces, then it is not difficult to see that this property holds. More precisely:

**Lemma 3.6.** Let $V, W$ be two $k$-dimensional affine subspaces in $\mathbb{R}^n$, and suppose that $V \cap B_{1/2}(0) \neq \emptyset$. There exists a constant $c(k, n)$ such that if $V \cap B_1(0) \subseteq B_{\delta}(W \cap B_1(0))$, then $W \cap B_1(0) \subseteq B_{c\delta}(V \cap B_1(0))$. Thus in particular $d_H(V \cap B_1(0), W \cap B_1(0)) \leq c\delta$.

**Proof.** The proof relies on the fact that $V$ and $W$ have the same dimension. Let $x_0 \in V$ be the point of minimal distance from the origin. By assumption, we have that $\|x_0\| \leq 1/2$. Let $x_1, \ldots, x_k \in V$ be a sequence of points such that

$$\|x_i - x_0\| \geq 1/2 \quad \text{and for } i \neq j, \quad \langle x_i - x_0, x_j - x_0 \rangle = 0.$$

(3.13)

In other words, $\{x_i - x_0\}_{i=1}^k$ is an affine base for $V$. Let $\{y_i\}_{i=0}^k \subseteq W \cap B_1(0)$ be such that $d(x_i, y_i) \leq \delta$. Then

$$\|y_i - y_0\| \geq 1/2 - 2\delta \quad \text{and for } i \neq j, \quad \|y_i - y_0, y_j - y_0\| \leq 2\delta + 4\delta^2.$$

(3.14)

This implies that for $\delta \leq \delta_0(n)$, $\{y_i - y_0\}_{i=1}^k$ is an affine base for $W$ and for all $y \in W$

$$y = y_0 + \sum_{i=1}^k \alpha_i (y_i - y_0), \quad |\alpha_i| \leq 10\|y - y_0\|.$$

(3.15)

Now let $y \in W \cap B_1(0)$ be the point of maximum distance from $V$, and let $\pi$ be the projection onto $V$ and $\pi^\perp$ the projection onto $V^\perp$. Then

$$d(y, V) = d(y, \pi(y)) = \|\pi^\perp(y)\| \leq \sum_{i=1}^k |\alpha_i| \|\pi^\perp(y_i - y_0)\| \leq c'(n, k)\delta.$$

(3.16)

Since $y \in B_1(0)$, then $\pi(y) \in V \cap B_{1+c\delta}(0)$, and thus $d(y, V \cap B_1(0)) \leq 2c'\delta \equiv c\delta$, which proves the claim. \qed

Next we will see that the Grassmannian distance between two subspaces is enough to control the projections with respect to these planes.
Lemma 3.7. There exists $c(n) > 0$ such that if $V, W$ are $k$-dimensional linear subspaces with $d_G(V, W) \leq \epsilon$, then for every $x \in \mathbb{R}^n$,

$$\| \pi_V(x) - \pi_W(x) \| \leq c(n) \epsilon \| x \|. \quad (3.17)$$

Proof. By choosing $c(n) > \epsilon_0^{-1}$, we can assume without loss that $\epsilon \leq \epsilon_0(n)$. Let $\{e_i\}_{i=1}^k$ be an orthonormal basis for $V$. Then evidently

$$\pi_V(x) = \sum_{i=1}^k \langle x, e_i \rangle e_i. \quad (3.18)$$

Let also $w_1, \cdots, w_k$ be points in $W$ with $d(e_i, w_i) \leq \epsilon$. It is easy to see that $|1 - \|w_i\| | \leq \epsilon$. Now let $f_i$ be the result of Gram-Schmidt orthonormalization process on $w_i$. In particular

$$f_1 = \frac{w_1}{\| w_1 \|}, \quad f_2' = w_2 - \langle f_1, w_2 \rangle f_1, \quad f_2 = \frac{f_2'}{\| f_2' \|},$$

$$\cdots$$

$$f_k' = w_k - \sum_{i=1}^{k-1} \langle f_i, w_k \rangle f_i, \quad f_k = \frac{f_k'}{\| f_k' \|}. \quad (3.19)$$

It is clear that if we choose $\epsilon_0$ small enough, then this process is well-defined and there exists a constant $c$ such that for all $i$:

$$\| e_i - f_i \| \leq c \epsilon. \quad (3.20)$$

As an immediate corollary we obtain that for all $i$

$$\| \langle x, e_i \rangle e_i - \langle x, f_i \rangle f_i \| \leq 2 \| x \| \| e_i - f_i \| \leq 2c \| x \| \epsilon. \quad (3.21)$$

The result then follows. \qed

Distance between $L^2$ best planes.

Let us begin by fixing our notation for this subsection. Throughout, $\rho = \rho(n) > 0$ will be fixed according to Lemma 3.8. We will consider a positive Radon measure $\mu$ supported on $S \subseteq B_1(0)$, and use $D(x, r) \equiv D^k_\mu(x, r)$ to bound the distances between best $L^2$ planes at different points and scales. By definition let us denote by $V(x, r)$ a best $k$-dimensional plane on $B_r(x)$, i.e., a $k$-dimensional affine subspaces minimizing $\int_{B_r(x)} d(x, V)^2 d\mu$. We want to prove that, under reasonable hypothesis, the distance between $V(x, r)$ and $V(y, r')$ is small if $d(x, y) \sim r$ and $r' \sim r$.

In order to achieve this, we will need to understand some minimal properties of $\mu$. First, we need to understand how concentrated $\mu$ is on any given ball. For this reason, for some $\rho > 0$ and all $x \in B_1(0)$ we will want to consider the upper mass bound
Lemma 3.8. There exists a \( \rho \) independently of \( x \) such that

\[
\mu(B_\rho(x)) \leq M\rho^k \quad \forall x \in B_1(0) .
\]  

However, an upper bound on the measure is not enough to guarantee best \( L^2 \)-planes are close, as the following example shows.

Example 3.1. Let \( V, V' \) be \( k \)-dimensional subspaces, \( 0 \in V \cap V' \), and set \( S = V \cap B_1(0) \setminus B_{1/10}(0) \cup S' \), where \( S' \subseteq V' \cap B_{1/10}(0) \) with \( \lambda^k(S') < \epsilon_k 10^{-k} \), and \( \mu = \lambda^k \). Then evidently \( D(0, 1) \leq \lambda^k(S') \) and \( D(0, 1/10) = 0 \), independently of \( V \) and \( V' \). However, \( V(0, 1) \) will be close to \( V' \), while \( V(0, 1/10) = V' \). Thus, in general, we cannot expect \( V(0, 1) \) and \( V(0, 1/10) \) to be close if \( \mu(B_{1/10}(0)) \) is too small.

Thus, in order to prove that the best planes are close, we need to have some definite amount of measure on the set, in such a way that \( S \) “effectively spans” a \( k \)-dimensional subspace. Precisely, we have the following:

**Lemma 3.9.** There exists a \( \rho_0(n, M) \) such that if (3.22) holds for some \( \rho \leq \rho_0 \) and if \( \mu(B_1(0)) \geq \epsilon_k \), then for every affine subspace \( V \leq \mathbb{R}^n \) of dimension \( \leq k-1 \), there exists an \( x \in S \) such that \( B_\rho(x) \subseteq B_1(0) \), \( B_{10\rho}(x) \cap V = \emptyset \), and \( \mu(B_\rho(x)) \geq c(n, \rho) = c(n)\rho^n > 0 \).

**Proof.** Let \( V \) be any \( k-1 \)-dimensional subspace, and consider the set \( B_{11\rho}(V) \). Let \( B_i = B_{\rho}(x_i) \) be a sequence of balls that cover the set \( B_{11\rho}(V) \cap B_1(0) \) and such that \( B_i/2 \) are disjoint. If \( N \) is the number of these balls, then a standard covering argument gives

\[
N \omega_n \rho^n / 2^n \leq \omega_{k-1}(1 + \rho)^{k-1} \omega_{n-k+1}(12\rho)^{n-k+1} \leq 24^n \omega_{k-1} \omega_{n-k+1} \rho^{n-k+1}
\]

\[
\implies N \leq 48^n \frac{\omega_{k-1} n^{-1} \omega_{n-k+1}}{\omega_n} (\omega_{n-k+1} \rho^{n-k+1}).
\]  

By (3.22), the measure of the set \( B_{11\rho}(V) \) is bounded by

\[
\mu(B_{11\rho}(V)) \leq \sum_i \mu(B_i) \leq MN\omega_k \rho^k \leq c(n)M\rho.
\]  

Thus if \( \rho < c(n)\epsilon_k/(4M) \), then \( \mu(B_{11\rho}(V)) \leq \epsilon_k/4 \). In particular, we get that there must be some point of \( S \) not in \( B_{11\rho}(V) \). More effectively, let us consider the set \( B_1(0) \setminus B_{11\rho}(V) \). This set can be covered by at most \( c(n, \rho) = 2^n \rho^{-n} \) balls of radius \( \rho \), and we also see that

\[
\mu(B_1(0) \setminus B_{11\rho}(V)) \geq \frac{3\epsilon_k}{4}.
\]  

Thus, there must exist at least one ball of radius \( \rho \) which is disjoint from \( B_{10\rho}(V) \) and such that

\[
\mu(B_\rho(x)) \geq \frac{3\epsilon_k}{4} 2^{-n} \rho^n \geq c(n)\rho^n.
\]  

\( \square \)

Now if at two consecutive scales there are some balls on which the measure \( \mu \) effectively spans \( k \)-dimensional subspaces, we show that these subspaces have to be close together.

**Lemma 3.9.** Let \( \mu \) be a positive Radon measure and assume \( \mu(B_1(0)) \geq \epsilon_k \) and that for each \( B_r(y) \subseteq B_1(0) \) with \( r \in (\rho, \rho^2) \) we have \( \mu(B_r(y)) \leq Mr^k \), where \( \rho \leq \rho_0 \). Additionally, let \( B_\rho(x) \) be a ball such that
We prove this statement by induction on $V$. Then if $A = V(0, 1) \cap B_\rho(x)$ and $B = V(x, \rho) \cap B_\rho(x)$ are $L^2$-best subspace approximations of $\mu$ with $d(x, B) < \rho/2$, then

$$d_H(A, B)^2 \leq c(n, \rho, M) \left(D^k_\mu(x, \rho) + D^k_\mu(0, 1)\right).$$

(3.27)

Proof. Let us begin by observing that if $c(n, \rho, M) > \delta^{-1}(n, \rho, M)$, which will be chosen later, then we may assume without loss that

$$D^k_\mu(x, \rho) + D^k_\mu(0, 1) \leq \delta = \delta(n, \rho, M),$$

(3.28)

since otherwise (3.27) is trivially satisfied.

We will estimate the distance $d_H(A, B)$ by finding $k + 1$ balls $B_{\rho^2}(y_i)$ which have enough mass and effectively span in the appropriate sense $V(x, \rho)$. Given the upper bounds on $D^k_\mu$, we will then be in a position to prove our estimate.

Consider any $B_{\rho^2}(y) \subseteq B_1(0)$ and let $p(y) \in B_{\rho^2}(y)$ be the center of mass of $\mu$ restricted to this ball. Let also $\pi(p)$ be the orthogonal projection of $p$ onto $V(x, \rho)$. By Jensen’s inequality:

$$d(p(y), V(x, \rho))^2 = d(p, \pi(p))^2 = d\left(\int_{B_{\rho^2}(y)} z d\mu(z), V(x, \rho)\right)^2 \leq \frac{1}{\mu(B_{\rho^2}(y))} \int_{B_{\rho^2}(y)} d(z, V(x, \rho))^2 d\mu(z).$$

(3.29)

Using this estimate and Lemma 3.8, we want to prove that there exists a sequence of $k + 1$ balls $B_{\rho^2}(y_i) \subseteq B_\rho(x)$ such that

(i) $\mu\left(B_{\rho^2}(y_i)\right) \geq c(n, \rho, M) > 0$

(ii) $\{\pi(p(y_i))\}_{i=0}^k \equiv \{\pi_i\}_{i=0}^k$ effectively spans $V(x, \rho)$. In other words for all $i = 1, \cdots, k$, $\pi_i \in V(x, \rho)$ and

$$\pi_i \notin B_{\rho^2}(\text{span}(\pi_0, \cdots, \pi_{i-1})).$$

(3.30)

We prove this statement by induction on $i = 0, \cdots, k$. For $i = 0$, the statement is trivially true since $\mu(B_\rho(x)) > \epsilon \rho^2$. In order to find $y_{i+1}$, consider the $i$-dimensional affine subspace $V^{(i)} = \text{span}(\pi_0, \cdots, \pi_{i-1}) \subseteq V(x, \rho)$. By Lemma 3.8 applied to the ball $B_\rho(x)$, there exists some $B_{\rho^2}(y_{i+1})$ such that $\mu\left(B_{\rho^2}(y_{i+1})\right) \geq c(n, \rho, M) > 0$ and

$$y_{i+1} \notin B_{10\rho^2}(\text{span}(\pi_0, \cdots, \pi_{i-1})).$$

(3.31)

By definition of center of mass it is clear that $d(y_i, p(y_i)) \leq \rho^2$. Moreover, by item (i) and equation (3.29), we get

$$d(p(y_{i+1}), V(x, \rho))^2 \leq c \int_{B_{\rho^2}(y)} d(z, V(x, \rho))^2 d\mu(z) \leq cD^k_\mu(x, \rho) \leq c\delta.$$

(3.32)

Thus by the triangle inequality we have $d(y_i, \pi_i) \leq 2\rho^2$ if $\delta \leq \delta_0(n, \rho, M)$ is small enough. This implies (3.30). Using similar estimates, we also prove $d(p(y_i), V(0, 1))^2 \leq c'D^k_\mu(0, 1)$ for all $i = 0, \cdots, k$. Thus by the triangle inequality

$$d(\pi_i, V(0, 1)) \leq d(\pi_i, p(y_i)) + d(p(y_i), V(0, 1)) \leq c(n, \rho, M) \left(D^k_\mu(x, \rho) + D^k_\mu(0, 1)\right)^{1/2}.$$

(3.33)
Now consider any \( y \in V(x, \rho) \). By item (ii), there exists a unique set \( \{ \beta_i \}_{i=1}^k \) such that
\[
y = \pi_0 + \sum_i \beta_i (\pi_i - \pi_0), \quad |\beta_i| \leq c(n, \rho) \|y - \pi_0\|.
\] (3.34)

Hence for all \( y \in V(x, \rho) \cap B_\rho(x_i) \), we have
\[
d(y, V(0, 1)) \leq d(\pi_0, V(0, 1)) + \sum_i |\beta_i| d(\pi_i - \pi_0, V(0, 1)) \leq c(n, \rho, M) \left(D_\rho^k(x, \rho) + D_\rho^k(0, 1)\right)^{1/2}.
\] (3.35)

By Lemma 3.6, this completes the proof of (3.27).

**Comparison between \( L^2 \) and \( L^\infty \) planes.** Given \( B_r(x) \), we denote as before \( V(x, r) \) to be the \( k \)-dimensional subspace minimizing \( \int_{\mathbb{R}^n} d(y, V)^2 d\mu \). Suppose that the support of \( \mu \) satisfies a uniform one-sided Reifenberg condition, i.e. suppose that there exists a \( k \)-dimensional plane \( L(x, r) \) such that \( x \in L(x, r) \) and
\[
supp(\mu) \cap B_r(x) \subseteq B_{\delta r}(L(x, r)).
\] (3.36)

Then, by the same technique used in Lemma 3.9, we can prove that

**Lemma 3.10.** Let \( \mu \) be a positive Radon measure with \( \mu(B_1(0)) \geq \varepsilon_k \) and such that for all \( B_\rho(y) \subseteq B_1(0) \) we have \( \mu(B_\rho(y)) \leq M \rho^k \) and (3.36). Then
\[
d_H(L(0, 1) \cap B_1(0), V(0, 1) \cap B_1(0))^2 \leq c(n, \rho, M) \left(\delta^2 + D_\rho^k(0, 1)\right).
\] (3.37)

**bi-Lipschitz equivalences.** In this subsection, we study a particular class of maps with nice local properties. These maps are slightly modified version of the maps which are usually exploited to prove Reifenberg’s theorem, see for example [Reif60, Tor95, DT12], [Mor66, section 10.5] or [Sim].

We start by defining the functions \( \sigma \). For some \( r > 0 \), let \( \{x_i\} \) be an \( r \)-separated subset of \( \mathbb{R}^n \), i.e., let \( d(x_i, x_j) \geq r \). Let \( p_i \in B_{10r}(x_i) \) and \( V_i \) be a sequence of \( k \)-dimensional linear subspaces.

By standard theory, it is easy to find a locally finite smooth partition of unity \( \lambda_i : \mathbb{R}^n \rightarrow [0, 1] \) such that

(i) \( \text{supp}(\lambda_i) \subseteq B_{3r}(x_i) \) for all \( i \),
(ii) for all \( x \in \bigcup_i B_{2r}(x_i) \), \( \sum_i \lambda_i(x) = 1 \),
(iii) \( \sup_i \|\nabla \lambda_i\|_\infty \leq c(n)/r \),
(iv) if we set \( 1 - \psi(x) = \sum_i \lambda_i(x) \), then \( \psi \) is a smooth function with \( \|\nabla \psi\|_\infty \leq c(n)/r \).

Note that by (i), and since \( x_i \) is \( r \)-separated, there exists a constant \( c(n) \) such that for all \( x \), \( \lambda_i(x) > 0 \) for at most \( c(n) \) different indexes.

**Definition 3.11.** Given \( x_i, \lambda_i, p_i, V_i \) as above, define a smooth function \( \sigma : \mathbb{R}^n \rightarrow \mathbb{R}^n \) by
\[
\sigma(x) = x + \sum_i \lambda_i(x) \pi_{V_i} (p_i - x) = \psi(x) x + \sum_i \lambda_i(x) (p_i + \pi_{V_i} (x - p_i)),
\] (3.38)

where \( \pi_{V_i}(x) \) is the orthogonal projection of the vector \( x \) onto \( V_i \).
By local finiteness, it is evident that \( \sigma \) is smooth. Moreover, if \( \psi(x) = 1 \), then \( \sigma(x) = x \). It is clear that philosophically \( \sigma \) is a form of “smooth interpolation” between the identity and the projections onto the subspaces \( V_i \). It stands to reason that if \( V_i \) are all close together, then this map \( \sigma \) is close to being an orthogonal projection in the region \( \bigcup_i B_{2r}(x_i) \).

**Lemma 3.12.** Suppose that there exists a \( k \)-dimensional affine subspace \( V \leq \mathbb{R}^n \) such that for all \( i \)

\[
d_G(V_i, V) \leq \delta, \quad d(p_i, V) \leq \delta, \quad (3.39)
\]

Then the map \( \sigma \) restricted to the set \( U = \psi^{-1}(0) = (\sum_i \lambda_i)^{-1} (1) \) can be written as

\[
\sigma(x) = q + \pi_V(x - q) + e(x),
\]

where \( q \) is any point in \( V \), and \( e(x) \) is a smooth function with \( \|e\|_\infty + \|\nabla e\|_\infty \leq c(n)\delta / r = c(n)\delta \).

**Remark 3.12.** Thus, on \( U \) we have that \( \sigma \) is the affine projection onto \( V \) plus an error which is small in \( C^1 \).

**Proof.** On the set \( U \), we can define

\[
e(x) = \sigma(x) - q - \pi_V(x - q) = -q - \pi_V(x - q) + \sum_i \lambda_i (x) \cdot (p_i + \pi_V(x - p_i)) = \\
= \sum_i \lambda_i (x) \cdot (p_i - q - \pi_V(p_i - q) + \pi_V(x) - \pi_V(x)). \quad (3.41)
\]

By (3.39) and Lemma 3.7, we have the estimates

\[
\|p_i - q - \pi_V(p_i - q)\| < \delta, \quad \|\pi_V(x - p_i) - \pi_V(x - p_i)\| \leq c(n)\delta \|x - p_i\| \leq 13c(n)r\delta. \quad (3.42)
\]

This implies

\[
\|e\|_{L^\infty(U)} \leq c(n)(1 + 13r)\delta \leq c(n)\delta. \quad (3.43)
\]

As for \( \nabla e \), we have

\[
\nabla e = \sum_i \nabla \lambda_i (x) \cdot (p_i - q - \pi_V(p_i - q) + \pi_V(x) - \pi_V(x)) = \sum_i \lambda_i (x) \nabla (\pi_V(x) - \pi_V(x)). \quad (3.44)
\]

The first sum is easily estimated, and since \( \langle \nabla (\pi_W)|_x, w \rangle = \pi_W(w) \), we can still apply Lemma 3.7 and conclude:

\[
\|\nabla e\|_{L^\infty(U)} \leq \frac{c(n)}{r} \delta. \quad (3.45)
\]

□

As we have seen, \( \sigma \) is in some sense close to the affine projection to \( V \). In the next lemma, which is similar in spirit to the [Sim, squash lemma], we prove that the image through \( \sigma \) of a graph over \( V \) is again a graph over \( V \) with nice bounds.
Lemma 3.13 (squash lemma). Fix some $B_{r/p}(y) \subseteq \mathbb{R}^n$, and let $I = \{x_i\} \cap B_{r/p}(y)$. Suppose that there exists a $k$-dimensional affine subspace $V$ such that $d(y, V) \leq \delta r$ and for all $x_i \in I$:

$$d(p_i, V) \leq \delta r \quad \text{and} \quad d_G(V_i, V) \leq \delta.$$  

(3.46)

Suppose also that there exists a $C^1$ function $g : V \to V^\perp$ such that $G \subseteq \mathbb{R}^n$ is the graph $G = \{x + g(x) \mid x \in V\} \cap B_{r/p}(y)$, and $r^{-1}\|g\|_\infty + \|\nabla g\|_\infty \leq \delta'$. Then

(i) $\forall z \in G$, $r^{-1}|\sigma(z) - z| \leq c(n)\delta r^{-1}$, and $\sigma$ is a $C^1$ diffeomorphism from $G$ to its image,

(ii) the set $\sigma(G)$ is contained in a $C^1$ graph $\{x + \tilde{g}(x) \mid x \in V\}$ with

$$r^{-1}\|\tilde{g}\|_\infty + \|\nabla \tilde{g}\|_\infty \leq c(n)(\delta + \delta')\rho^{-1}.$$  

(3.47)

(iii) moreover, for $x \in U' = \bigcup_i B_r(x_i)$, the previous bound is independent of $\delta'$, in the sense that $r^{-1}\|\tilde{g}\|_{L^\infty(U' \cap V)} + \|\nabla \tilde{g}\|_{L^\infty(U' \cap V)} \leq c(n)\delta \rho^{-1}$.  

(3.48)

The same bound holds for all $U' \subset U = \psi^{-1}(0) \subseteq \mathbb{R}^n$ such that $B_{c\delta \rho^{-1}}(U') \subseteq U$.

(iv) the map $\sigma$ is a bi-Lipschitz equivalence between $G$ and $\sigma(G)$ with bi-Lipschitz constant $\leq \sqrt{1 + c(n)(\delta + \delta')^2\rho^{-2}}$.

Proof. Since we are only interested in the local behaviour of $\sigma$ inside $B_{r/p}(y)$, we can assume wlog that $\sigma(x + g(x)) = \psi(z)(x + g(x)) + \sum_{i \in I} \lambda_i(z)(p_i + \pi_V(x + g(x) - p_i))$, $1 - \psi(x) = \sum_{i \in I} \lambda_i(x)$, (3.49)

where we have set for convenience $z = z(x) = x + g(x)$. Set by definition

$$(1 - \psi(z))[x + h(x)] \equiv \sum_i \lambda_i(z)(p_i + \pi_V(x + g(x) - p_i)).$$  

(3.50)

By projecting the function $\sigma(x + g(x))$ onto $V$ and its orthogonal complement we obtain

$$\sigma(x + g(x)) = \sigma^T(x) + \sigma^\perp(x), \quad \sigma^T(x) = x + (1 - \psi(z))h^T(x), \quad \sigma^\perp(x) = \psi(z)g(x) + (1 - \psi(z))h^\perp(x).$$  

(3.51)

By adapting the proof of Lemma 3.12, we obtain

$$r^{-1}\left|(1 - \psi(z))h^\perp(x)\right| + \left|\nabla (1 - \psi(z))h^T(x)\right| \leq \frac{c\delta}{\rho}.$$  

(3.52)

Note that this bound is independent of $\delta'$, as long as $\delta' \leq 1$.

Thus we can apply the inverse function theorem on the function $\sigma^T(x) : V \to V$ and obtain a $C^1$ inverse $Q$ such that for all $x \in V$, $r^{-1}|Q(x) - x| + |\nabla Q - id| \leq c(n)\delta r^{-1}$, and if $\psi(x + g(x)) = 1$, then $Q(x) = x$. So we can write that for all $x \in V$

$$\sigma(x + g(x)) = \sigma^T(x) + \tilde{g}(\sigma^T(x)) \quad \text{where} \quad \tilde{g}(\cdot) = \sigma^\perp(Q(\cdot)).$$  

(3.53)

Arguing as before, we see that $(1 - \psi(z))h^\perp(x)$ is a $C^1$ function with

$$r^{-1}\left|(1 - \psi(z))h^\perp(x)\right| + \left|\nabla (1 - \psi(z))h^\perp(x)\right| \leq \frac{c\delta}{\rho}.$$  

(3.54)

and this bound is independent of $\delta'$ (as long as $\delta' \leq 1$).
Thus the function $\tilde{g} : V \to V^\perp$ satisfies for all $x$ in its domain
\begin{equation}
    r^{-1} |\tilde{g}(x)| + |\nabla \tilde{g}(x)| \leq c(n)(\delta + \delta')r^{-1},
\end{equation}
Moreover, for those $x$ such that $\psi(\sigma^T(x) + g(\sigma^T(x))) = 0$, the estimates on $\tilde{g}$ are independent of $\delta'$, in the sense that $r^{-1} |\tilde{g}(x)| + |\nabla \tilde{g}(x)| \leq c(n)\rho r^{-1}$.

This proves items (ii), (iii). As for item (i), it is an easy consequence of the estimates in (3.52), (3.54).

Now since both $G$ and $\sigma(G)$ are Lipschitz graphs over $V$, we can estimate the bi-Lipschitz constant of $\sigma$ by decomposing it as
\begin{equation}
    \sigma : x + g(x) \to x \to \sigma^T(x) \to \sigma^T(x) + \tilde{g}(\sigma^T(x)).
\end{equation}
\[\square\]

**Pointwise Estimates on $D$.** We wish to see in this subsection how (3.4) implies pointwise estimates on $D$, which will be convenient in the proof of the generalized Reifenberg results. Indeed, the following is an almost immediate consequence of Remark 3.4:

**Lemma 3.14.** Assume $B_r(x) \subseteq B_2(0)$ is such that $\mu(B_r(x)) \geq 2^k \rho^k r^k$ and $\int_{B_r(x)} D^k_{\mu}(y, 2r) \, d\mu(y) < \delta^2(2r)^{\rho}$. Then there exists $c(n)$ such that $D(x, r) < c\delta^2$. In particular, if (3.4) holds then for every $B_r(x) \subseteq B_2(0)$ such that $\mu(B_r(x)) \geq 2^k \rho^k r^k$ we have that $D(x, r) < c\delta^2$.

3.3. **Proof of Theorem 3.3: The Rectifiable-Reifenberg.** In the proof, we will fix the constant $M = C_1(n) \leq 20^n$ and therefore the a positive scale $\rho(n, C_1(n)) = \rho(n) < 1$ according to Lemma 3.8. For convenience, we will assume that $\rho = 2^q$, $q \in \mathbb{N}$, so that we will be able to use the sum bounds (3.8) more easily.

The constant $C_1(n)$ will be defined by the end of the proof, however it is enough to know that it is itself bounded by $20^n$. Let us also define the scales $r_j = \rho^j$.

3.3.1. **Weak Upper bound** $\mathcal{L}^k(S \cap B_r(x)) \leq C(n)r^k$. We start by proving a uniform upper bound weaker than (3.7), in particular we want to show that for all $x \in \mathbb{R}^n$ and $r > 0$:
\begin{equation}
    \mathcal{L}^k(S \cap B_r(x)) \leq C(n)r^k.
\end{equation}

Given the scale invariance of the quantities involved, we do not lose generality if we prove (3.57) only for $x = 0$ and $r = 1$. The strategy for the proof is the following: first we prove that $S$ has $\sigma$-finite $k$-dimensional Hausdorff measure, and use this information to build a suitable covering by balls with controlled $\mathcal{L}^k$. Then we fix any $A$ and $r_A$ and show by induction on $j = A, \cdots, 0$ that the measure of quasi-balls $B_{r_j}(y)$, defined in (3.66), is bounded above as in (3.57). In particular, we will see for $r = 1$ that the quasi-ball $B_1(x)$ agrees with the set $S \cap B_1(x)$, up to a set of measure zero, which will prove (3.57). In order to prove the measure statement on the quasi-balls, we will use a second downward induction on $i = j, \cdots, A$, which is the heart of the construction.
\textbf{\(\sigma\)-finiteness of the measure} As a first step towards the proof, we remark that \(S\) must have \(\sigma\)-finite \(k\)-dimensional Hausdorff measure. We will use this to reduce the proof to the case when \(\lambda^k(S) < \infty\). Indeed, in order for (3.6) to be true, we need in particular
\[
\int_{S \cap B(x)} D_3^k(x, 1) \, d\lambda^k(x) \, ds < c\delta^2.
\] (3.58)

Define for \(a > 0\) the sets \(S_a = \{x \in S \text{ s.t. } D_3^k(x, 1) \geq a\}\) and \(S_0 = \{x \in S \text{ s.t. } D_3^k(x, 1) = 0\}\). Then evidently for all \(a > 0\), \(\lambda^k(S_a) < c\delta^2/a < \infty\). Moreover, if \(x \in S_0\), then either \(\lambda^k(S \cap B_1(x)) < \epsilon_k \leq 1\), or up to sets of \(k\)-measure zero, \(S \cap B_1(x)\) is contained in a \(k\)-dimensional plane, and therefore \(\lambda^k(S_0) < \infty\). Since \(S = S_0 \cup_{i=1}^\infty S_{r_i}\), \(S\) has \(\sigma\)-finite \(k\)-dimensional measure.

Now in order to prove the uniform bounds on \(\lambda^k(S)\), we can therefore assume without loss that \(\lambda^k(S) < \infty\). Indeed, by the monotonicity of Remark 3.3, all subsets of \(S\) satisfy (3.6). Therefore if we show that (3.57) holds for all subsets of finite \(k\)-dimensional measure, then the estimate will hold also on \(S\). Thus, we will assume throughout the remainder of the proof that \(\lambda^k(S) < \infty\).

\textbf{Covering of the set} \(S^*\) Let \(S^* \subseteq S \cap B_1(0)\) be the set of points with controlled upper density, i.e.
\[
S^* = \left\{ x \in S \cap B_1(0) \text{ s.t. } 2^{-k} \leq \theta^k(S, x) \leq 1 \right\}.
\] (3.59)

By proposition 2.16, \(S \setminus S^*\) has zero \(k\)-dimensional measure, thus it is sufficient to give bounds on \(S^*\).

We want to cover the set \(S^*\) by balls which have uniform upper and lower bounds on their Hausdorff measure and such that their best \(L^2\) plane is not too far away from their center. In order to achieve this, for all \(x \in S^*\) let \(r_x \in (0, \rho^2]\) be such that
\[
\lambda^k(B_{r_x/4}(x) \cap S) \geq \frac{1}{2} \omega_k \left(\frac{r_x}{8}\right)^k,
\]
\[
\lambda^k(B_r(x) \cap S) \leq 2 \omega_k r^k, \quad \forall r \leq r_x \quad (3.60)
\]

Let \(p_x\) be the center of mass of \(B_{r_x/4}(x) \cap S\) with respect to \(\lambda^k|_{B_{r_x/4}(x) \cap S}\). In particular, we have that \(p_x \in B_{r_x/8}(x)\). By (3.6) applied to \(r_x\), and by Jensen’s inequality, we have
\[
\lambda^k(B_{r_x/8}(x) \cap S) \, d(p_x, V(x, r_x))^2 \leq \int_{B_{r_x/10}(x) \cap S} d(y, V(x, r_x))^2 \leq r_x^{k+2} D(x, r_x). \quad \ (3.61)
\]

Using Lemma 3.14 we therefore get that
\[
\lambda^k(B_{r_x/8}(x) \cap S) \, d(p_x, V(x, r_x))^2 \leq cr_x^{k+2} \delta^2. \quad \ (3.62)
\]

In particular, using our lower bound on \(\lambda^k(B_{r_x/8}(x) \cap S)\) we obtain that \(d(p_x, V(x, r_x)) \leq c\delta r_x\), which implies for \(\delta\) small enough that
\[
d(x, V(x, r_x)) \leq r_x/10. \quad \ (3.63)
\]
Without loss of generality, we can assume for simplicity that \( r_x = \rho^{n_x} \) with \( n_x \in \mathbb{N} \). Consider the open covering of \( S^* \) given by \( \cup_{x \in S^*} B_{r_x}(x) \), and extract a countable Vitali subcovering. Thus
\[
S^* \subseteq \bigcup_{x \in S^*} B_{r_x}(x),
\]
(3.64)
where if \( x, y \in S^* \), \( x \neq y \), then \( B_{r_x/5}(x) \cap B_{r_y/5}(y) = \emptyset \).

Fix any \( A \in \mathbb{N} \) and \( \tilde{r} = \rho^A \), and define the sets \( S^*_\tilde{r} = \{ x \in S \; s.t. \; \tilde{r} \leq r_x \leq \tilde{r} \} \), \( \tilde{S}_\tilde{r} = \tilde{S}^*_{\tilde{r}} \) and
\[
S_\tilde{r} = \bigcup_{x \in S^*_\tilde{r}} B_{r_x}(x) \cap S^*.
\]
(3.65)
It is clear that \( S_\tilde{r} \not\supseteq S^* \) as \( \tilde{r} \to 0 \), so if we have bounds on \( \lambda^k(S_\tilde{r}) \) which are independent of \( \tilde{r} \), we are done. Thus from here on we will consider \( \tilde{r} = \rho^A \) to be positive and fixed.

**First induction: upwards** We are going to prove inductively on \( j = A, \ldots, 0 \) that for all \( x \in \mathbb{R}^n \) and \( r_j = \rho^j \leq 1 \) the measure of
\[
\tilde{B}_{r_j}(x) = \bigcup_{y \in \tilde{S}^*_{r_j+1} \cap B_{r_j}(x)} B_{r_j}(y) \cap S^*,
\]
(3.66)
is bounded by
\[
\lambda^k(\tilde{B}_{r_j}(x)) \leq C_1(n)r_j^k \leq 20^k r_j^k.
\]
(3.67)
Note that by construction \( \tilde{B}_r(x) \subseteq B_{r+\rho}\rho^A) \) (x) and \( S^* \cap B_1(0) \subseteq \tilde{B}_1(0) \). Moreover, for \( j = A \), this bound follows from the definition of \( r_A \) and (3.60).

The reason why we have to introduce and estimate \( \tilde{B}_{r_j}(x) \) instead of \( B_{r_j}(x) \) is that we have no a priori control of what happens inside any of the balls \( B_{r_j}(y) \). The bounds (3.60) are valid only on each ball as a whole. However, since our primary goal is to estimate \( S^* \cap B_1(0) \), and we have the inclusion \( S^* \cap B_1(0) \subseteq \tilde{B}_1(0) \), there is no loss in this restriction.

**Rough estimate** Fix some \( j \), and suppose that (3.67) holds for all \( \tilde{r} \leq r \leq r_{j+1} \).

Let us first observe that we can easily obtain a bad upper bound on \( \lambda^k(\tilde{B}_{r_j}(x)) \) for some fixed \( \chi > 1 \). Consider the points \( y \in \tilde{S}_\tilde{r} \cap B_{r_j}(x) \), and divide them into two groups: the ones with \( r_y \leq r_{j+1} \) and the ones with \( r_y \geq r_j \).

For the first group, cover them by balls \( B_{r_{j+1}}(z_i) \) such that \( B_{r_{j+1}/2}(z_i) \) are disjoint. Since there can be at most \( c(n, \chi) \) balls of this form, and for all of these balls the upper bound (3.67) holds, we have an induced upper bound on the measure of this set.

As for the points with \( r_y > r_{j+1} \), by construction there can be only \( c(n, \chi) \) many of them, and we also have the bound \( r_y \leq \chi r_{j+1} \), which by (3.60) implies \( \lambda^k(\tilde{B}_{r_y}(y)) \leq c(n, \chi)r_{j+1}^k \). Summing up the two contributions, we get the very rough estimate
\[
\lambda^k(\tilde{B}_{r_{j+1}}(z_i)) \leq C_2(n, \chi)r_{j+1}^k,
\]
(3.68)
where \( C_2 >> C_1 \). Note that, as long as the inductive hypothesis holds, \( C_1 \) is independent of \( j \). However, it is clear that successive repetitions of the above estimate will not lead to (3.67).
Second induction: downwards.

Suppose that (3.67) is true for all \( x \in B_1(0) \) and \( i = j + 1, \ldots, A \). Fix \( x \in \mathbb{R}^n \), and consider the set \( \tilde{B} = \tilde{B}_{r_j}(x) \subseteq B_{r_j(1+\rho)}(x) \). We are going to build by induction on \( i \geq j \) a sequence of smooth maps \( \sigma_i : \mathbb{R}^n \to \mathbb{R}^n \) and smooth \( k \)-dimensional manifolds \( T_{ji} = T_i \) which satisfy nine properties, which we will use to eventually conclude the proof of (3.67) for \( \tilde{B} \). Let us outline the inductive procedure now, and introduce all the relevant terminology. Everything described in the remainder of this subsection will be discussed more precisely over the coming pages. To begin with, we will have at the first step that

\[
\sigma_j = \text{id}, \quad T_{jj} = \tilde{V}(x, r_j) \subseteq \mathbb{R}^n ,
\]

where \( \tilde{V}(x, r_j) \) is one of the \( k \)-dimensional affine subspaces which minimizes \( \int_{\tilde{B}_{r_j}(x)} d^2(y, \tilde{V}) \, d\lambda^k \). Thus, the first manifold \( T_j \) is a \( k \)-dimensional affine subspace which best approximates \( \tilde{B}_{r_j}(x) \). At future steps we can recover \( T_i \) from \( T_{i-1} \) and \( \sigma_i \) from the simple relation

\[
T_i = \sigma_i(T_{i-1}) .
\]  

We will see that \( \sigma_i \) is a diffeomorphism when restricted to \( T_{i-1} \), and thus each additional submanifold \( T_j \) is also diffeomorphic to \( \mathbb{R}^k \). As part of our inductive construction we will build at each stage a Vitali covering of \( T_i \) given by

\[
B_{r_j}(T_i) \cap \tilde{B}_{r_j}(x) \sim \bigcup_{s=j}^{i} \left( \bigcup_{y \in I_g} \tilde{B}_{r_s}(y) \bigcup_{y \in I_b} \tilde{B}_{r_s}(y) \bigcup_{y \in I_f} \tilde{B}_{r_s}(y) \right) \tilde{B}_{r_j}(y) ,
\]

where \( I_g, I_b, \) and \( I_f \) represent the good, bad, and final balls in the covering. A final ball \( \tilde{B}_{r_i}(y) \) with \( y \in I_f^i \) is a ball such that \( y \in \tilde{S} \) and \( r_j = r_i \), and the other balls in the covering are characterized as good or bad according to how much measure they carry. Good balls are those with large measure, bad balls the ones with small measure. More precisely, we have

\[
\lambda^k(\tilde{B}_{r_i}(y)) \geq 2^k \epsilon_k r_i^k , \quad \text{if} \quad y \in I_g^i ,
\]

\[
\lambda^k(\tilde{B}_{r_i}(y)) < 2^k \epsilon_k r_i^k , \quad \text{if} \quad y \in I_b^i .
\]

We will see that, over each ball \( \tilde{B}_{r_i}(y) \) in this covering, \( T_i \) can be written as a graph over the best approximating subspace \( \tilde{V}(y, r_i) \) with good estimates.

Our goal in these constructions is the proof of (3.67) for the ball \( \tilde{B} = \tilde{B}_{r_j}(x) \), and thus we will need to relate the submanifolds \( T_i \), and more importantly the covering (3.71), to the set \( \tilde{B} \). Indeed, this covering of \( T_i \) almost covers the set \( \tilde{B} \), at least up to an excess set \( \tilde{E}_{i-1} \). That is,

\[
\tilde{B} \subseteq \tilde{E}_{i-1} \cup \bigcup_{s=j}^{i} \left( \bigcup_{y \in I_g} \tilde{B}_{r_s}(y) \bigcup_{y \in I_b} \tilde{B}_{r_s}(y) \bigcup_{y \in I_f} \tilde{B}_{r_s}(y) \right) \tilde{B}_{r_j}(y) .
\]
We will see that the set \( \tilde{E}_{i-1} \) consists of those points of \( \tilde{B} \) which do not satisfy a uniform Reifenberg condition. Thus in order to prove (3.67) we will need to estimate the covering (3.71), as well as the excess set \( \tilde{E}_{i-1} \).

Let us now outline the main properties used in the inductive construction of the mapping \( \sigma_{j+1} : \mathbb{R}^n \to \mathbb{R}^n \), and hence \( T_{j+1} = \sigma_{j+1}(T_j) \). As is suggested in (3.71), it is the good balls and not the bad and final balls which are subdivided at further steps of the induction procedure. In order to better understand this construction let us begin by analyzing the good balls \( \tilde{B}_{r_j}(y) \) more carefully. On each such ball we may consider the best approximating \( k \)-dimensional subspace \( \tilde{V}(y, r_j) \). Since \( \tilde{B}_{r_j}(y) \) is a good ball, one can check that most of \( \tilde{B} \cap \tilde{B}_{r_j}(y) \) must satisfy a uniform Reifenberg and reside in a small neighborhood of \( \tilde{V}(y, r_j) \). We denote those points which do not lie in \( \tilde{E}(y, r_j) \), see (3.85) for the precise definition. Then we can define the next step of the excess set by

\[
\tilde{E}_i = \tilde{E}_{i-1} \bigcup_{y \in \tilde{I}^i} \tilde{E}(y, r_j).
\]

Thus our excess set represents all those points which do not lie in an appropriately small neighborhood of the submanifolds \( T_i \). With this in hand we can then find a submanifold \( T'_i \subseteq T_i \), which is roughly defined by

\[
T'_i \approx T_i \cap \bigcup_{y \in \tilde{I}^i} \tilde{B}_{r_j}(y),
\]

see (3.95) for the precise inductive definition, such that

\[
\tilde{B} \subseteq \tilde{E}_i \bigcup_{s=j}^{i} \left( \bigcup_{y \in \tilde{I}^s} \tilde{B}_{r_j}(y) \bigcup \tilde{B}_{r_j}(y) \bigcup B_{\tilde{r}_{s+1}/4}(T'_i) \equiv \tilde{R}_i \bigcup B_{\tilde{r}_{s+1}/4}(T'_i),
\]

where \( \tilde{R}_i \) represents our remainder term, and consists of those balls and sets which will not be further subdivided at the next stage of the induction. Now in order to finish the inductive step of the construction, we can cover \( B_{\tilde{r}_{s+1}/4}(T'_i) \) by some Vitali set

\[
B_{\tilde{r}_{s+1}/4}(T'_i) \subseteq \bigcup_{y \in I} \tilde{B}_{r_j}(y),
\]

where \( y \in I \subseteq T'_i \). We may then decompose the ball centers

\[
l^{i+1} = l^{i+1}_g \cup l^{i+1}_b \cup l^{i+1}_f,
\]

based on (3.72). Now we will use Definition 3.11 and the best approximating subspaces \( V(y, r_{i+1}) \) to build \( \sigma_{i+1} : \mathbb{R}^n \to \mathbb{R}^n \) such that

\[
supp(\sigma_{i+1} - Id) \subseteq \bigcup_{y \in \tilde{I}^{i+1}} B_{3r_{i+1}}(y).
\]

This completes the outline of the inductive construction. Let us now record the precise properties of all of these sets which will be proved during the inductive process, and will eventually lead to the proof of (3.67):

(i) \( \sigma_{j} = id, T_{j,j} = T_j = \tilde{V}(x, r_j) \leq \mathbb{R}^n \)

(ii) \( T_i = \sigma_i(T_{i-1}) \)
The points in $\tilde{E}$ are in some sense what prevents the set $S$ from satisfying a uniform Reifenberg condition at this scale. By construction, all points in $\tilde{E}$ have a uniform lower bound on the distance from $\tilde{V}$, so that if we assume $\lambda^k(\tilde{B}_{\rho_i}(x)) \geq 2^k \epsilon_k r_i^k$, i.e. $B_{\rho_i}(x)$ is a good ball, then we can estimate

$$
\int_{B_{\rho_i}(x) \cap \tilde{E}(x, r_i)} d(y, \tilde{V}(x, r_i))^2 d\lambda^k(y) + \lambda^k(\tilde{E}(x, r_i))(r_{i+1}/4)^2 \leq \int_{B_{\rho_i}(x)} d(y, \tilde{V}(x, r_i))^2 d\lambda^k(y) \\
\leq \int_{B_{\rho_i}(x)} d(y, V(x, r_i))^2 d\lambda^k(y) \leq \int_{B_{\rho_i}(1+y\rho_i)(x)} d(y, V(x, r_i(1 + \rho)))^2 \\
= ((1 + \rho)r_i)^{k+2} D^k_S(x, (1 + \rho)r_i) \leq C(n)D^k_S(x, 2r_i). 
$$

**Good, bad and final balls** At our base step let us set $I^j_g = \{ x \}$, $I^j_b = I^j_f = \emptyset$. Inductively, let us define the

(iii) for every $i \geq j + 1$ and $y \in I^j_{g', i}$, $d(y, \tilde{V}(x, r_i)) \leq c\delta r_i$, the set $T_i \cap B_{2r_i}(y)$ is the graph over $\tilde{V}(y, r_i)$ of a smooth function $f$ with

$$
\frac{\|f\|_{\infty}}{r_i} + \|\nabla f\|_{\infty} \leq c\delta, \quad (3.80)
$$

(iv) for $y \in T_i$, $d(\sigma_{i+1}(y), y) \leq c\delta r_{i+1}$, and $\sigma_{i+1}|T_i$ is a diffeomorphism

(v) for every $y \in T_i$, $T_i \cap B_{2r_i}(y)$ is the graph over some $k$-dimensional affine subspace of a smooth function $f$ satisfying (3.80).

(vi) for all $i$, we have the inclusion

$$
\tilde{B} \subseteq B_{r_{i+1}/2} \left( T_i \right) \bigcup \tilde{R}^i 
$$

(vii) for every $i \geq j + 1$ and all $y \in I^j_{b', i}$, the set $T'_i \cap B_{2r_i}(y)$ is a Lipschitz graph over the plane $\tilde{V}(y, r_i)$ with $d(y, \tilde{V}(y, r_i)) \leq c\delta r_i$

(viii) we can estimate

$$
\lambda^k(\sigma_{i}^{-1}(T'_i)) + \#(I^j_g) + \#(I^j_f) \mid \omega_k(r_i/10)^k \leq \lambda^k(T'_{i-1}). \quad (3.82)
$$

(ix) we can estimate the excess set by

$$
\lambda^k(\tilde{E}(y, r_i)) r^2_{i+1} \leq C(n)D^k_S(y, 2r_i)) \quad (3.83)
$$

**Excess set.** Let us begin by describing the construction of the excess set. We will only be interested here in a ball $\tilde{B}_{\rho_i}(x)$ which is a good ball, in the sense that $\lambda^k(\tilde{B}_{\rho_i}(x)) \geq 2^k \epsilon_k r_i^k$.

Thus define $\tilde{V}(x, r)$ to be (one of) the $k$-dimensional plane minimizing $\int_{B_{\rho_i}(x)} d(y, \tilde{V})^2 d\lambda^k$, and define also the excess set to be the set of points which are some definite amount away from the best plane $\tilde{V}$. Precisely,

$$
E(x, r_i) = \left( B_{\rho_i}(x) \setminus B_{r_{i+1}/4}(\tilde{V}) \right) \cap S, \quad \tilde{E}(x, r_i) = \bigcup_{y \in S_{\rho_i}^{i+1} \cap E(x, r_i)} B_{\rho_i}(y) \cap S^* \quad (3.84)
$$

The points in $\tilde{E}$ are in some sense what prevents the set $S$ from satisfying a uniform Reifenberg condition at this scale. By construction, all points in $\tilde{E}$ have a uniform lower bound on the distance from $\tilde{V}$, so that if we assume $\lambda^k(\tilde{B}_{\rho_i}(x)) \geq 2^k \epsilon_k r_i^k$, i.e. $B_{\rho_i}(x)$ is a good ball, then we can estimate

$$
\int_{B_{\rho_i}(x) \cap \tilde{E}(x, r_i)} d(y, \tilde{V}(x, r_i))^2 d\lambda^k(y) + \lambda^k(\tilde{E}(x, r_i))(r_{i+1}/4)^2 \leq \int_{B_{\rho_i}(x)} d(y, \tilde{V}(x, r_i))^2 d\lambda^k(y) \\
\leq \int_{B_{\rho_i}(x)} d(y, V(x, r_i))^2 d\lambda^k(y) \leq \int_{B_{\rho_i}(1+y\rho_i)(x)} d(y, V(x, r_i(1 + \rho)))^2 \\
= ((1 + \rho)r_i)^{k+2} D^k_S(x, (1 + \rho)r_i) \leq C(n)D^k_S(x, 2r_i). 
$$

Good, bad and final balls At our base step let us set $I^j_g = \{ x \}$, $I^j_b = I^j_f = \emptyset$. Inductively, let us define the
remainder set to be the union of all the previous bad balls, final balls, and the excess sets:

\[
R^i = \bigcup_{s=j}^i \left( \bigcup_{y \in I_j^p} B_{r_i}(y) \cup B_{r_i}(y) \bigcup_{I_j} E(y, r_s) \right), \quad \tilde{R}^i = \bigcup_{s=j}^i \left( \bigcup_{y \in I_j^p} \tilde{B}_{r_i}(y) \cup \tilde{B}_{r_i}(y) \bigcup_{I_j} \tilde{E}(y, r_s) \right). \tag{3.86}
\]

The set \(\tilde{R}^i\) represents everything we want to throw out at the inductive stage of the proof. We will see later in the proof how to estimate this remainder set itself.

Now consider the balls in the covering outside the remainder set, and separate the balls with radius \(r_{i+1}\) from the others by defining for \(y \in I_g^1\) the sets

\[
I_f^{i+1}(y) = \left\{ z \in (\tilde{S}_y \setminus R^i) \cap B_{r_i}(y) \right\}, \quad \text{s.t.} \quad r_z = r_{i+1}, \tag{3.87}
\]

and

\[
J_{i+1}(y) = \left\{ z \in (\tilde{S}_y \setminus R^i) \cap B_{r_i}(y) \cap B_{r_{i+1/5}}(\tilde{V}(y, r_i)) \right\}. \tag{3.88}
\]

From this we can construct the sets

\[
I_f^{i+1} = \bigcup_{y \in I_f} I_f^{i+1}(y) \quad \text{and} \quad J_{i+1} = \bigcup_{y \in I_g} J_{i+1}(y), \tag{3.89}
\]

Note that by construction we have

\[
S^* \setminus \tilde{R}^i \subseteq \bigcup_{y \in I_f} \left( \tilde{B}_{r_i}(y) \setminus \tilde{E}(y, r_i) \right) \subseteq \left( \bigcup_{z \in I_f} B_{r_z}(z) \bigcup_{z \in J_{i+1}} B_{r_z}(z) \right) \bigcap S^*. \tag{3.90}
\]

Let us now consider a covering of (3.90) given by

\[
S^* \setminus \tilde{R}^i \subseteq \bigcup_{z \in I_f} B_{r_z}(z) \bigcup_{z \in I} B_{r_{i+1}}(z), \tag{3.91}
\]

where \(I \subseteq T'_i\), and for any \(p \neq q \in I_f \cup I, B_{r_{i+1/5}}(x) \cap B_{r_{i+1/5}}(q) = \emptyset\). Note that this second property is true by definition for \(p, q \in I_f\), we only need to complete this partial Vitali covering with other balls of the same size. Since the set we need to cover is contained in \(\bigcup_{y \in I_r} B_{r_{i+1/5}}(\tilde{V}(y, r_i))\) and since \(T'_i\) is locally a Lipschitz graph over \(\tilde{V}\) with (3.80), then we can also choose \(I \subseteq T'_i\).

We split the balls with centers in \(I\) into two subsets, according to how much measure they carry. In particular, let

\[
I_g^{i+1} = \left\{ y \in I \right\}, \quad \text{s.t.} \quad \lambda^k \left( \tilde{B}_{r_{i+1}}(y) \right) \geq 2^k e_k r^k, \quad I_h^{i+1} = \left\{ y \in I \right\}, \quad \text{s.t.} \quad \lambda^k \left( \tilde{B}_{r_{i+1}}(y) \right) < 2^k e_k r^k. \tag{3.92}
\]

**Map and manifold structure.** Let \(\{\lambda^{i+1}_s\} = \{\lambda_s\}\) be a partition of unity such that for each \(y_s \in I_g^{i+1}\)

- \(\text{supp}(\lambda_s) \subseteq B_{3r_{i+1}}(y_s)\)
- for all \(z \in \bigcup_{y_s \in I_g^{i+1}} B_{2r_{i+1}}(y_s)\), \(\sum_s \lambda_s(z) = 1\)
- \(\max_s \|\nabla \lambda_s\|_{L^\infty} \leq C(n)/r_{i+1}\).
For every \( y_s \in I^{i+1}_g \), let \( \tilde{V}(y_s, r_{i+1}) \) to be (one of) the \( k \)-dimensional subspace that minimizes \( \int_{B_{r_{i+1}}(y_s)} d(z, V)^2 d\lambda^k \). By Remark 3.3 and by (3.66), we can estimate
\[
 r^{-k-2}_{i+1} \int_{B_{r_{i+1}}(y_s)} d(z, \tilde{V}(y_s, r_{i+1}))^2 d\lambda^k(z) \leq (1 + \rho)^{k+2} D(y_s, (1 + \rho) r_{i+1}) .
\]

Let \( p_s \in B_{(1+\rho) r_{i+1}}(y_s) \) be the center of mass of \( \lambda^k |_{B_{r_{i+1}}(y_s)} \). It is worth observing that \( p_s \in \tilde{V}(y_s, r_{i+1}) \).

Define the smooth function \( \sigma_{i+1} : \mathbb{R}^n \to \mathbb{R}^n \) as in Definition 3.11, i.e.,
\[
\sigma_{i+1}(x) = x + \sum y_i \lambda_{i+1}^k (y_i) \pi_{\tilde{V}(y_s, r_{i+1})} (p_s - x) .
\]

With this function, we can define the sets
\[
T_{i+1} = \sigma_{i+1}(T_i), \quad T'_{i+1} = \sigma_{i+1} \left( T_i \setminus \bigcup_{y \in T_i} B_{\rho_{i+1}}(y) \right).
\]

Fix any \( y \in I^{i+1}_g \), and let \( z \in I^i_g \) be such that \( y \in B_{r_i}(z) \). By induction, \( T_i \cap B_{10r_{i+1}}(y) \subseteq T_i \cap B_{2r_i} (z) \) is the graph of a \( C^1 \) function over \( \tilde{V}(z, r_i) \). Consider the points \( \{ y_s \} = I^{i+1}_g \cap B_{5r_{i+1}} (y) \). By construction and using an estimate similar to (3.63), it is easy to see that \( d(y_s, \tilde{V}(z, r_i)) \leq c \delta r_{i+1} \), and so we can apply the estimates in Lemma 3.9 with \( M = C_1 \) by the first induction. Using condition (3.6), we obtain that for all \( y_s \):
\[
r^{-1}_{i+1} d_H \left( \tilde{V}(z, r_i) \cap B_{r_{i+1}}(y_s), \tilde{V}(z, r_{i+1}) \cap B_{r_{i+1}}(y_s) \right) \leq c \left( D_5^k(y_s, r_{i+1}) + D_5^k(z, r_i) \right) \leq c(n, \rho, C_1) \delta .
\]

This implies that, if \( \delta(n, \rho, C_1) \) is small enough, \( T_i \cap B_{10r_{i+1}}(y) \) is a graph also over \( \tilde{V}(y, r_{i+1}) \) satisfying the same estimates as in (3.80), up to a worse constant \( c \). That is, if \( \delta \) is sufficiently small, we can apply Lemma 3.13 and prove induction point (iii).

Points (iv) and (v) Points (iv) and (v) are proved with similar methods. We briefly sketch the proofs of these two points.

Let \( y \in T_{i+1} \), and recall the function \( \psi_{i+1} \equiv 1 - \sum \lambda_s \). If \( \psi_{i+1}|_{B_{2r_{i+1}}(y)} \) is identically 1, then \( \sigma_{i+1}|_{B_{2r_{i+1}}(y)} = id \), and there is nothing to prove.

Otherwise, there must exist some \( z' \in I^{i+1}_g \cap B_{5r_{i+1}}(y) \), and thus there exists a \( z \in I^i_g \) such that \( B_{3r_{i+1}}(y) \subseteq B_{r_i} (z) \). By our induction hypothesis, \( T_i \cap B_{r_i} (z) \) is a Lipschitz graph over \( \tilde{V}(z, r_i) \). Proceeding as before, by the estimates in Lemma 3.9 and Lemma 3.13, we obtain that \( T_{i+1} \cap B_{2r_{i+1}}(y) \) is also a Lipschitz graph over \( \tilde{V}(z, r_i) \) with small Lipschitz constant, and that \( |\sigma_{i+1}(p) - p| \leq c \delta r_{i+1} \) for all \( p \in T_i \). Moreover, \( \sigma_{i+1}|_{T_i} \) is locally a diffeomorphism at scale \( r_{i+1} \). From this we see that \( \sigma_{i+1} \) is a diffeomorphism on the whole \( T_i \). Now we turn our attention to the items (vi), (vii), (viii).

Properties of the manifolds \( T'_{i} \) Here we want to prove the measure estimate in (3.82). The basic idea is that bad and final balls correspond to holes in the manifold \( T_i \), and each of these holes carries a \( k \)-dimensional measure which is proportionate to the measure inside the balls. In particular, let \( B_{r_{i+1}}(y) \) be a bad or a final
ball. In either case, we will see that \( y \) must be \( \sim r_{i+1} \)-close to \( T_i \), which is a Lipschitz graph at scale \( r_i \). This implies that \( \lambda^k(B_{r_i}(y) \cap T_i) \sim r_i^k \), and thus we can bound the measure of a bad or final ball with the measure of the hole we have created on \( T_i \).

In detail, point (vi) is an immediate consequence of the definition of \( \tilde{R}_i \). As for point (vii), if \( y \in I_{g_i}^i \), then by definition, there exists \( z \in I_{g_i}^{i-1} \) such that \( y \in B_{5r_i+6}(z) \), and \( y \notin \tilde{R}_i^{i-1} \). This implies that \( y \) is far away from the balls we discard while building \( T_i' \), in particular

\[
B_{3r_i}(y) \cup \bigcup_{y \in I_{g_i}^{i-1} \cup I_{g_i}^{i+1}} B_{r_i+6/7}(y) = \emptyset
\]  

(3.97)

This proves that \( T_i \cap B_{2r_i}(y) = T_i' \cap B_{2r_i}(y) \), and in turn point (vii).

In order to prove the volume measure estimate, consider that

\[
T_i' \setminus \sigma_i^{-1}(T_i') \subseteq \bigcup_{y \in I_{g_i}^{i-1} \cup I_{g_i}^{i+1}} B_{r_i+6/7}(y)
\]  

(3.98)

Note that the balls in the collection \( \{B_{r_i+5/6}(y)\}_{y \in I_{g_i}^{i-1} \cup I_{g_i}^{i+1}} \) are pairwise disjoint. Pick any \( y \in I_{g_i}^{i+1} \), and let \( z \in I_{g_i}^i \) be such that \( y \in B_{r_i}(z) \). By definition, \( y \in T_i' \) and \( \lambda^k(B_{r_i+1}(y)) < 2^k e_k r_i^k < 10^{-k} \omega_k r_i^k \). Since \( T_i' \cap B_{2r_i}(z) \) is a graph over \( \tilde{V}(z, r_i) \) with \( y \in T_i' \), then

\[
\lambda^k(T_i' \cap B_{r_i+6/7}(y)) \geq \omega_k 7^{-k} r_i^k.
\]  

(3.99)

A similar estimate holds for the final balls. The only difference is that if \( y \in I_{g_i}^{i+1} \), then it is not true in general that \( y \in T_i \). However, by construction of the balls \( B_{r_i}(y) \), using an argument similar to the one in the proof of (3.62), we obtain that \( d(y, \tilde{V}(z, r_i)) \leq c\delta r_i + r_i+1/10 \). Given (3.80), we can conclude

\[
\lambda^k(T_i' \cap B_{r_i+7/7}(y)) \geq \omega_k 10^{-k} r_i^k
\]  

(3.100)

Now it is evident from the definition of \( T_i' \) that

\[
\lambda^k(\sigma_i^{-1}(T_i')) + \#(I_{g_i}^i) + \#(I_{g_i}^i) \omega_k(r_i/10)^k \leq \lambda^k(T_i').
\]  

(3.101)

**Volume estimates on the manifold part** Here we want to prove that for every measurable \( \Omega \subseteq T_i \)

\[
\lambda^k(\sigma_{i+1}(\Omega)) \leq \lambda^k(\Omega) + c(n, \rho, C_1) \int_{S \cap B_{r_i+1}(x)} D(p, 2r_i+1) d\lambda^k(p).
\]  

(3.102)

The main applications will be with \( \Omega = T_i \) and \( \Omega = T_i' \). In order to do that, we need to analyze in a quantitative way the bi-Lipschitz correspondence between \( T_i \) and \( T_{i+1} \) given by \( \sigma_{i+1} \).

As we already know, \( \sigma_{i+1} = id \) on the complement of the set \( G = \cup_{y \in I_{g_i}^{i+1}} B_{r_i+1}(y) \), so we can concentrate only on this set.
Using the same techniques as before, and in particular by Lemmas 3.9 and 3.13, we can prove that for each \( y \in I^i_g \), the set \( T_i \cap B_{5r_{i+1}}(y) \) is a Lipschitz graph over \( \bar{V}(y, r_{i+1}) \) with Lipschitz constant bounded by

\[
|c(n, \rho, C_1)| \left( D(y, r_{i+1}) + \sum_{z \in \mathcal{I} \cap B_{5r_{i+1}}(y)} D(z, r_i) \right)^{1/2}.
\]

In a similar manner, we also have that \( T_{i+1} \cap B_{y}(5r_{i+1}) \) is a Lipschitz graph over \( \bar{V}(y, r_{i+1}) \) with Lipschitz constant bounded by

\[
|c(n, \rho, C_1)| \left( \sum_{z \in \mathcal{I} \cap B_{5r_{i+1}}(y)} D(z, r_{i+1}) + \sum_{z \in \mathcal{I} \cap B_{5r_{i+1}}(y)} D(z, r_i) \right)^{1/2}.
\]

Summing up, we obtain that \( \sigma_{i+1} \) restricted to \( T_i \cap B_{5r_i}(y) \) is a bi-Lipschitz equivalence with bi-Lipschitz constant bounded by

\[
L(y, 5r_{i+1}) \leq 1 + |c(n, \rho, C_1)| \left( \sum_{z \in \mathcal{I} \cap B_{5r_{i+1}}(y)} D(z, r_{i+1}) + \sum_{z \in \mathcal{I} \cap B_{5r_{i+1}}(y)} D(z, r_i) \right)
\]

In order to estimate this upper bound, we use (3.2) and the definition of good balls to write

\[
D(z, r) \leq c \int_{\bar{B}_{\rho, j}(c)} D(p, 2r) d\lambda_k(p) \leq c(n, \rho, C_1) r^{-k} \int_{\bar{B}_{\rho, j}(c)} D(p, 2r) d\lambda_k(p).
\]

Since by construction any point \( x \in \mathbb{R}^n \) can be covered by at most \( c(n) \) different good balls at different scales, we can bound

\[
L(y, 5r_{i+1}) \leq 1 + \frac{(c(n, \rho, C_1))}{\# \mathcal{I}^i_{i+1}} \int_{B_{5r_i}(y)} [D(p, 2r_{i+1}) + D(p, 2r_i)] d\lambda_k(p).
\]

We can also badly estimate

\[
D(p, 2r_{i+1}) + D(p, 2r_i) \leq c(n, \rho) D(p, 2r_i).
\]

Now let \( P_s \) be a measurable partition of \( \Omega \cap G \) such that for each \( s, P_s \subseteq B_{5r_{i+1}}(y_s) \). By summing up the volume contributions of \( P_s \), and since evidently \( \lambda^k(P_s) \leq (1 + c\delta) r^k_{i+1} \), we get

\[
\lambda^k(\sigma_{i+1}(\Omega)) = \sum_s \lambda^k(\sigma_{i+1}(\Omega \cap P_s)) \leq \sum_s \lambda^k(P_s) \left( 1 + \frac{c}{r^k_{i+1}} \int_{B_{5r_i}(y_s)} D(p, 2r_i) d\lambda_k(p) \right) \leq \lambda^k(\Omega) + \frac{c}{\# \mathcal{I}^i_{i+1}} \int_{B_{5r_i}(y_s)} D(p, 2r_i) d\lambda_k(p) \leq \lambda^k(\Omega) + c(n, \rho, C_1) \int_{B_{r_i}(\Omega \cap S)} D(p, 2r_i) d\lambda_k(p).
\]

**Estimates on the excess set** In this paragraph, we estimate the total measure of the excess set, which is
defined by

\[ \tilde{E}_T = \bigcup_{i=j}^A \bigcup_{y \in I_i^j} \tilde{E}(y, r_i). \]  \hfill (3.110)

At each \( y \) and at each scale, we have by (3.85) and (3.2)

\[ \lambda^k(E(y, r_i)) \leq c(n, \rho) r_i^k D_S^k(y, 2r_i) \leq c(n, \rho)r_i^k \int_{B_{r_i}(x) \cap S^*} D_S^k(p, 4r_i) dA^k(p). \]  \hfill (3.111)

Since by definition of excess set \( \tilde{B}_{r_i}(y) \) must be a good ball, then

\[ \lambda^k(E(y, r_i)) \leq c(n, \rho) \int_{B_{2r_i}(y)} D_S^k(p, 4r_i) dA^k(p). \]  \hfill (3.112)

Now by construction of the good balls, there exists a constant \( c(n) \) such that at each step \( i \), each \( x \in \mathbb{R}^n \) belongs to at most \( c(n) \) many balls. Thus for each \( i \geq j \), we have

\[ \sum_{y \in I_i^j} \lambda^k(E(y, r_i)) \leq c(n, \rho) \int_{\bigcup_{y \in I_i^j} B_{2r_i}(y)} D_S^k(p, 4r_i) dA^k(p) \leq c(n, \rho) \int_{B_{r_i}(x) \cap S^*} D_S^k(p, 4r_i) dA^k(p). \]  \hfill (3.113)

If we sum over all scales, we get

\[ \lambda^k(\tilde{E}_T) \leq c(n, \rho) \sum_{i=j}^A \int_{B_{r_i}(x) \cap S^*} D_S^k(p, 4r_i) dA^k(p). \]  \hfill (3.114)

Since \( \rho = 2^q \), it is clear that

\[ \lambda^k(\tilde{E}_T) \leq c(n, \rho) \sum_{i=j}^A \int_{B_{r_i}(x) \cap S^*} D_S^k(p, 2^{qj-2}) dA^k(p) \leq c(n, \rho) \delta r_j^k, \]  \hfill (3.115)

since the sum in the middle is clearly bounded by (3.8).

This estimate is exactly what we want from the excess set.

**Volume estimates.**

By adding (3.102), with \( \Omega = \sigma_{i+1}^{-1}(T_{i+1})' \), and (3.82), we prove that for all \( i = j, \cdots, A + 1, \ldots \)

\[ \lambda^k(T_{i+1}') + |\#(I_i^j) + |\#(I_j^j)| \omega_k(r_i/10)^k \leq \lambda^k(T_i') + c(n, \rho, C_1) \int_{B_{2r_i}(y) \cap S^*} D(p, 2r_i) dA^k(p). \]  \hfill (3.116)
Adding the contributions from all scales, by (3.6) we get
\[
\lambda^k(T_{i+1}) + \sum_{s=j}^i \left[ \#(I_s^j) + \#(I_j^i) \right] \omega_k(r_s/10)^k \leq \\
\leq \lambda^k(T_j \cap B_{2r_j}(x)) + c(n, \rho, C_1) \sum_{s=j}^i \int_{B_{2r_j}(y) \cap S} D(p, 2r_s) d\lambda^k(p) \leq \\
\leq \lambda^k(T_j \cap B_{2r_j}(x))[1 + c(n, \rho, C_1)\delta] ,
\]
where in the last line we estimated \( \lambda^k(T_j \cap B_{2r_j}(x)) \sim r_j^k \), since \( T_j \) is a \( k \)-dimensional subspace, and we bounded the sum using (3.8).

In the same way, we can also bound the measure of \( T_k \) by
\[
\lambda^k(T_{i+1}) \leq \lambda^k(T_j \cap B_{2r_j}(x))[1 + c(n, \rho, C_1)\delta] \leq c(n)r_j^k .
\] (3.118)

**Upper estimates for \( \lambda^k \).** Given the definition of \( \tilde{S}_p \), all the balls \( B_{r_y}(y) \) inside this set have \( r_y \geq r_A \). Thus by (3.81)
\[
\tilde{B} = \tilde{B}_{r_j}(x) \subseteq \tilde{R}^k .
\] (3.119)

In particular, this and the estimates in (3.115) and (3.117) imply
\[
\lambda^k(\tilde{B}) \leq \sum_{s=j}^{A} \left[ 2^k e_k \#(I_s^j) + 2\#(I_j^i) \right] \omega_k(r_s)^k + \lambda^k(\tilde{E}_T) \leq \\
\leq C_3(n) \sum_{j}^i \left[ \#(I_s^j) + \#(I_j^i) \right] \omega_k(r_s/10)^k + \lambda^k(\tilde{E}_T) \leq C_3(n)(1 + c(n, \rho, C_1)\delta)r_j^k.
\] (3.120)

In this last estimate, we can fix \( C_1(n) = 2C_3(n) \leq 20^6 \), and \( \rho(n, C_1) \) according to Lemma 3.8. Now, it is easy to see that if \( \delta(n, \rho, C_1) \) is sufficiently small, then
\[
\lambda^k(\tilde{B}) \leq C_1(n)r_j^k ,
\] (3.121)
which finishes the proof of the downward induction, and hence the actual ball estimate (3.57).

3.3.2. *Improved estimates.* We have now proved a mass bound for the set \( S \). In this subsection, we wish to improve this mass bound, as well as prove the rectifiability conditions on \( S \). At several stages we will have to repeat the arguments of the upward and downward inductions, which we will only sketch since the arguments will be almost verbatim, though in some cases technically much easier.

**Rectifiability and \( W^{1,p} \) estimates** Now that we have proved the bound (3.57), we can use a construction similar to the one just described to sharpen the upper estimate and prove rectifiability of \( \lambda^k|_S \). The main difference with the previous case is that we do not need to be concerned any more with any fixed covering \( B_{r_y}(y) \) of our set, and so at every step we can estimate directly the measure \( \lambda^k|_S \) of a whole Euclidean ball...
\( B_r(z) \) without the need to limit our estimate to \( \tilde{B}_r(z) \). For the same reason, we do not need to introduce the subspaces \( \tilde{V}(z,r) \) and the sets \( \tilde{E}, \tilde{R} \), we will only deal with \( V(z,r), E, R \). Moreover, since we don’t have to stop our construction at any positive scale \( r_j \), we do not need to introduce and study the set of final balls \( I_j' \).

As a consequence, the construction we need here is technically less involved than the one used before.

However, as opposed to the previous construction, we are concerned about what happens at an infinitesimal scale, and in particular we want to have uniform estimates also for the limit \( \tilde{r} \to 0 \) of various quantities.

Fix any \( x \in \mathbb{R}^n \) and \( r > 0 \) such that \( B_r(x) \subseteq B_1(0) \). We are going to show that

\[
\lambda^k (S \cap B_r(x)) \leq (1 + \varepsilon)\omega_k r^k. \tag{3.122}
\]

For convenience and wlog, we assume \( x = 0 \) and \( r = 1 \). Using the same technique as before, we build a sequence of smooth maps \( \{\sigma_i\}_{i=0}^{\infty} \) on \( \mathbb{R}^n \) and manifolds \( T_i \) such that

(i) \( \sigma_0 = id, T_0 = V(x, r) = V(0,1) \subseteq \mathbb{R}^n \)

(ii) \( \phi_i = \sigma_i \circ \sigma_{i-1} \circ \cdots \circ \sigma_0 \)

(iii) \( T_i = \sigma_i(T_{i-1}) = \phi_i(T_0) \) is a smooth \( k \)-dimensional submanifold of \( \mathbb{R}^n \)

(iv) for every \( y \in T_i, T_i \cap B_{2r_i}(y) \) is the graph over some \( k \)-dimensional affine subspace of a smooth function \( f \) satisfying

\[
\frac{\|f\|_\infty}{r_i} + \|\nabla f\|_\infty \leq c\delta, \tag{3.123}
\]

(v) for \( y \in T_i, d(\sigma_{i+1}(y), y) \leq c\delta r_{i+1}, \) and \( \sigma_{i+1}|_{T_i} \) is a diffeomorphism. In a similar way for all \( y \in T_0, d(\phi_i(y), y) \leq c\delta \) and \( \phi_i|_{T_0} \) is a diffeomorphism.

In order to do so, we define inductively on \( i = 0, \cdots, \infty \) a sequence of sets \( E(y, r_i), I_g^i, I_b^i, R^i \) and manifolds \( T'_i \subseteq T_i \) such that

(vi) for all \( i \), we have the inclusion

\[
S \cap B_1(0) \subseteq B_{r_{i+1}/2}(T'_i) \cup R^i \tag{3.124}
\]

(vii) for every \( i \geq 1 \) and all \( y \in I_g^i \), the set \( T'_i \cap B_{2r_i}(y) \) is a Lipschitz graph over the plane \( V(y, r_{i+1}) \) with \( d(y, V(y, r_{i+1})) \leq c\delta r_i \)

(viii) we can estimate

\[
\lambda^k(\sigma_i^{-1}(T'_i)) + \#(I_b^i) \omega_k(r_i/10)^k \leq \lambda^k(T'_{i-1}). \tag{3.125}
\]

Up to minor modifications (actually simplifications), all these properties are proved in the same way as in the downward induction from the previous subsection.

The key extra-property we need is some form of control over the \( W^{1,p} \) norm of \( \phi_i \). In particular we will prove inductively that for all \( j \in \mathbb{N} \)

\[
\left\| \nabla \phi \right\|_{L^p}^p = \int_{T_0 \cap B_1(0)} \left\| \nabla \phi \right\|_p^p d\lambda^k \leq 1 + c(n)2^p\delta, \tag{3.126}
\]

where here \( \lambda^k \) is also Lebesgue measure on \( \mathbb{R}^k \).

As we will see in (3.137), in order for this estimate to work, we will need to choose \( \delta(n, p, C_1, p) \) small enough, and in particular \( \delta \to 0 \) as \( p \to \infty \).
**W** \textsuperscript{1-} \textit{p} \ estimates \quad As \ part \ of \ the \ downward \ induction \ in \ the \ proof \ of \ (i)-(viii), \ let \ us \ define \ the \ maps \ \( \tau_{i,j} = \sigma_j \circ \sigma_{j-1} \circ \cdots \circ \sigma_i \), \ so \ that \ \( \tau_{i,j}(T_i) = T_{j+1} \) \ and \ \( \tau_{0,j} = \phi_j \). \ We \ will \ prove \ inductively \ for \ \( i = j, \ldots , 0 \) \ that \ for \ all \ \( x \in T_i \),

\[
 r_j^{-k} \int_{B_j(x) \cap T_i} |\nabla \tau_{i,j}|^p \, d\lambda^k \leq 2C_1(n). \tag{3.127}
\]

Here \ the integration is simply the integration on a smooth \( k \)-dimensional subset \textit{wrt} the \( k \)-dimensional Lebesgue measure \( \lambda^k \), and the gradient of the functions \( \sigma_i \) \ and \( \tau_{i,j} \) is the restriction of the gradient in \( \mathbb{R}^n \) to the corresponding manifold \( T_i \).

Suppose that the statement is true for \( i + 1 \). \ Consider a covering of this ball by \( \{B_{r_{i+1}}(y)\} \), then by the chain rule

\[
 \int_{B_i(x) \cap T_i} |\nabla \tau_{i,j}|^p \, d\lambda^k(y) \leq \sum_s \int_{B_{i+1}(y) \cap T_i} |\nabla \tau_{i+1,j} \circ \sigma_i(y)|^p |\nabla \sigma_i(y)|^p \, d\lambda^k(y) \leq C(n, p, \rho). \tag{3.128}
\]

This gives us a first rough estimate.

In order to obtain a better estimate, we will prove by induction on \( s = i, \ldots , j \) that

\[
 \int_{B_i(x) \cap T_i} |\nabla \tau_{i,j}|^p \, d\lambda^k(y) \leq \int_{B_i} |\nabla \tau_{s,x}|^p \, d\lambda^k(y) + C(n, \rho)2^p \sum_{s = i}^j \int_{B_{10r_i}(x) \cap S} D(p, 10r_i) \, d\lambda^k(p), \tag{3.129}
\]

where \( B_s = \tau_{i+1}^{-1}(T_i \cap B_{r_i}(x)) \subseteq T_s \cap B_{10r_i}(x) \). \ Indeed, suppose that the statement is true for \( s \), then we have

\[
 \int_{B_i} |\nabla \tau_{s,x}|^p \, d\lambda^k(y) = \int_{B_i} |\nabla \tau_{s+1,j} \circ \sigma_i(y)|^p |\nabla \sigma_i(y)|^p \, d\lambda^k(y). \tag{3.130}
\]

Since \( \sigma_i \) is a diffeomorphism, we can change the variables and write

\[
 \int_{B_i} |\nabla \tau_{s,x}|^p \, d\lambda^k(y) = \int_{B_{s+1}} |\nabla \tau_{s+1,j} |^p |\nabla \sigma_i \circ \tau_{i,s}(y)|^p \, d\lambda^k(y). \tag{3.131}
\]

Now consider the partial covering of this set given by \( \{B_{s_{r_i}}(y)\}_{y \in I_s^i} \). \ As we have seen before, outside of this set, \( \sigma_i \) is the identity, so we don’t need to make any estimates on it.

Arguing in a manner verbatim to the proof of (3.102), we can prove that \( \sigma_s \) restricted to \( T_s \cap B_{s_{r_{s+1}}}(y) \) is a bi-Lipschitz map with

\[
 \left\| \nabla \sigma_s |_{T_s} - id |_{T_s} \right\|_{L^\infty(T_s \cap B_{s_{r_{s+1}}}(y))} \leq c(n, \rho) \frac{1}{r_s^k} \int_{B_{s_{r_s}}(y) \cap S} D(p, 10r_s) \, d\lambda^k(p). \tag{3.132}
\]

Let \( \{P_t, Q\} \) be a measurable partition of \( B_s \) \ with \( P_t \subseteq B_{s_{r_{s+1}}}(y) \) \ and \( Q \subseteq \cap B_{s_{r_{s+1}}}(y)^c \), then \( \{\sigma_s(P_t), \sigma_s(Q)\} \) is a measurable partition of \( B_{s+1} \) \ with \( \sigma_s(Q) = Q \). \ By arguing in a similar way to (3.109), we obtain

\[
 \int_{B_i} |\nabla \tau_{s,x}|^p \, d\lambda^k(y) = \int_Q |\nabla \tau_{s,x}|^p \, d\lambda^k(y) + \sum_t \int_{P_t} |\nabla \tau_{s+1,j}|^p |\nabla \sigma_i \circ \tau_{i,s}(y)|^p \, d\lambda^k(y) \leq \int_Q |\nabla \tau_{s+1,j}|^p \, d\lambda^k(y) + \sum_t \int_{P_t} |\nabla \tau_{s+1,j}|^p \, d\lambda^k(y) \left( 1 + \frac{c}{r_{s+1}^k} \int_{B_{s_{r_s}}(y) \cap S} D(p, 10r_s) \, d\lambda^k(p) \right) \tag{3.133}
\]
By the first induction, we have the upper bound $\int_{B_i} |\nabla \tau_{s+1,j}|^p d\lambda^k(y) \leq c(n)_{s+1}$, and thus
\begin{equation}
\int_{B_i} |\nabla \tau_{s,j}|^p d\lambda^k(y) \leq \int_{B_{s+1}} |\nabla \tau_{s+1,j}|^p d\lambda^k(y) + c(n,\rho)2^p \sum_i \int_{B_{s+1}(y) \cap S} D(p, 10r_j) d\lambda^k(p) .
\end{equation}

Since all points in $\mathbb{R}^n$ are covered at most $c(n,\rho)$ times by $\{B_{s+1}(y)\}$, we can simply estimate
\begin{equation}
\int_{B_i} |\nabla \tau_{s,j}|^p d\lambda^k(y) \leq \int_{B_{s+1}} |\nabla \tau_{s+1,j}|^p d\lambda^k(y) + c(n,\rho)2^p \int_{B_{s+1}(y) \cap S} D(p, 10r_j) d\lambda^k(p) .
\end{equation}

This proves (3.129). In order to complete the first induction, observe that
\begin{equation}
\int_{B_{s+1}(y) \cap T_j} |\nabla \tau_{s,j}|^p d\lambda^k(y) \leq \int_{B_{s+1}} |\nabla \tau_{s+1,j}|^p d\lambda^k(y) + c(n,\rho)2^p \sum_j \int_{B_{s+1}(y) \cap S} D(p, 10r_j) d\lambda^k(p) ,
\end{equation}

Since $\tau_{s,j}$ is the identity on $T_{j}$, and given the measure bounds (3.118), we get the result if we choose $\delta(n,\rho,p)$ such that the rhs is small enough, and in particular
\begin{equation}
c(n,\rho)2^p \sum_j \int_{S \cap B_{s+1}(y) \cap T_j} D(p, 10r_j) d\lambda^k(p) \leq c(n,\rho)2^p \delta \leq C_1(n) .
\end{equation}

Note that, once the improved estimate (3.7) is proven, which will be done in the next subsection, it will likewise be possible to improve this $W^{1,p}$ estimate to
\begin{equation}
r_j^{-k} \int_{B_{s+1}(y) \cap T_j} |\nabla \tau_{s,j}|^p d\lambda^k \leq 1 + \epsilon .
\end{equation}

**Improved measure estimates** Consider the maps $\phi_j : T_0 \to T_{j+1}$. By construction, these maps converge uniformly on $\mathbb{R}^n$ to a map $\phi_{\infty}$. Moreover, since $\phi_j$ has uniform $W^{1,p}$ estimates, then $\phi_j \to \phi_{\infty}$ uniformly, with
\begin{equation}
\int_{T_0 \cap B_2(0)} |\nabla \phi_{\infty}|^p \leq 2C_1(n) .
\end{equation}

By (3.124), we have the inclusion for all $i$:
\begin{equation}
S \subseteq E_{i} \bigcup_{j=0}^{i} B_{r_{i+1}} \bigcup_{j=0}^{i} B_{r_{j}}(y) , \quad \Rightarrow \quad S \subseteq E_{\infty} \bigcup_{j=0}^{\infty} B_{r_{j}}(y) .
\end{equation}

By arguing as in the proofs of (3.115), (3.117) and (3.82), we obtain $\lambda^k(E_{\infty}) \leq c\delta$, and since $\epsilon_k < \omega_k^{-1}$
\begin{equation}
\lambda^k(T_{j}^i) + \sum_{j=0}^{i} \sum_{y \in I_{j}^i} \lambda^k(B_{r_{j}}(y)) \leq \lambda^k(T_{j}) .
\end{equation}

In order to give better estimates on $T_{\infty}$, define the function $f : T_0 \to \mathbb{R}$ by
\begin{equation}
f(x) = \sum_{i=0}^{\infty} \sup_{B_{r_{i}}(y) \cap T_{j}} |\nabla \sigma_{i} - Id| (\phi_{j}(x)) .
\end{equation}
By similar arguments to the ones used to prove (3.121) and (3.127), we can estimate
\[
\int_{T_0} f(x) d\lambda^k \leq c\delta. \tag{3.143}
\]

Define also the sets \( U_a \subseteq T_0 \) to be
\[
U_a = \{ x \in T_0 \text{ s.t. } f(x) > a \}. \tag{3.144}
\]

By simple \( L^1 \) estimates, we know that \( \lambda^k(U_a) \leq c\delta \) and
\[
\lambda^k(\phi_\infty(U_a)) \leq C(n, p) \| \phi \|_{W^{1,p}}^k \lambda^k(U_a)^{\frac{n-k}{p}} \leq C \left( \frac{c\delta}{a} \right)^{1-k/p}. \tag{3.145}
\]

On the complement of \( U_a \), \( \phi_\infty \) is a bi-Lipschitz function with \( \| \nabla \phi_\infty \|_\infty \leq \sqrt{1+a^2} \), thus
\[
\lambda^k(T_\infty \cap B_{1+c\delta}(0)) = \lambda^k(T_\infty \cap B_{1+c\delta}(0) \cap U_a) + \lambda^k(T_\infty \cap B_{1+c\delta}(0) \cap U_a^C) \leq C \left( \frac{c\delta}{a} \right)^{1-k/p} + (1+a^2)\omega_k r_k. \tag{3.146}
\]

Again, by choosing \( a(\epsilon) \) sufficiently small, and \( \delta(n, \rho, p, \epsilon) \) sufficiently small as well, we get
\[
\lambda^k(T_\infty \cap B_{1+c\delta}(0)) \leq \left( 1 + \frac{1}{2} \epsilon \right) \omega_k. \tag{3.147}
\]

Summing up,
\[
\lambda^k(S) \leq \lambda^k(E_\infty) + \lambda^k(\phi_\infty(T_0 \cap B_{1+c\delta}(0))) \leq (1 + \epsilon)\omega_k. \tag{3.148}
\]

**Rectifiability** As for the rectifiability, we can restrict ourselves to \( S^* \) and consider a covering of \( B_{r_x}(x) \) such that \( \lambda^k(B_{r_x}(x) \cap S) \geq 2^{-k-1} \omega_k r_x^k \).

If we prove that for all such balls there exists a subset \( G \subseteq S \cap B_{r_x}(x) \) which is rectifiable and for which \( \lambda^k(G) \geq \lambda^k(B_{r_x}(x) \cap S)/2 \), then an easy covering argument gives us the rectifiability of all of \( S \).

By scale invariance, we assume for simplicity \( x = 0 \) and \( r_x = 1 \). As we have seen before,
\[
S \subseteq R^\infty \cup T_\infty. \tag{3.149}
\]

Since \( T_\infty \) is the image of a \( W^{1,p} \) map with \( p > k \), this set is rectifiable. Moreover, by (3.141), we know that
\[
\lambda^k(S \cap T_\infty) \geq \lambda^k(S) - \lambda^k(R^\infty) \geq 2^{-k-1} \omega_k - \epsilon_k 10^k - c\delta \geq 2^{-k-2} \omega_k, \tag{3.150}
\]
which therefore completes the proof.
3.4. **Proof of Theorem 3.2: The \(W^{1,p}\)-Reifenberg.** The proof of this theorem is almost a corollary of Theorem 3.3 and the classic Reifenberg theorem 2.18. In addition to the upper volume bound proved in Theorem 3.3, we also need to prove a lower volume bound on each ball.

In fact, one could just trace through the argument of Theorem 3.3, and see inductively that there exists no bad balls to conclude this. Instead, we will use a standard argument (see for example [DT12, lemma 13.2]) to prove the lower bound on \(\lambda^k(S \cap B_r(x))\) given in (3.5) directly by using only the uniform Reifenberg condition.

**Lemma 3.15.** Under the assumptions of Theorem 2.18, for all \(x \in S\) such that \(B_r(x) \subseteq B_1(0)\),

\[
\lambda^k(S \cap B_r(x)) \geq (1 - c\delta)\omega_k r^k.
\] (3.151)

**Proof.** By scale invariance, we assume \(x = 0\) and \(r = 1\). The classic Reifenberg theorem proves that there exists a bi-Hölder continuous map \(\phi : L \to \mathbb{R}^n\) where \(L\) is a \(k\)-dimensional plane and

1. \(|\phi(x) - x| \leq c\delta\) for all \(x \in L\)
2. \(\phi(x) = x\) for \(|x| \geq 1 + c\delta\)
3. \(S \cap B_1(0) = \phi(L')\), where \(B_{1-c\delta}(0) \cap L \subseteq L' \subseteq B_{1+c\delta}(0) \cap L\).

Now let \(f = \pi_L \circ \phi : L \to L\). This map is continuous and it is the identity outside \(B_{1+c\delta}(0)\), and thus by topological reasons (degree theory) it is also surjective from \(L\) to itself.

In particular, the set \(A = B_{1-3c\delta}(0) \cap L\) is contained in the image of \(f\). By the uniform Reifenberg condition, \(\pi^{-1}(A) \cap S \subseteq B_{1-2c\delta}(0)\), and by the properties of \(\phi\), \(f^{-1}(A) = \phi^{-1}(\pi^{-1}(A) \cap S) \subseteq B_{1-c\delta}(0)\). Thus \(\phi(f^{-1}(A)) \subseteq S\).

Now, since \(\pi\) has Lipschitz constant 1, by a standard result (see [Fed69, 2.10.11])

\[
\lambda^k(S) \geq \lambda^k(\phi(f^{-1}(A))) \geq \lambda^k(\pi \circ \phi(f^{-1}(A))) = \lambda^k(A) \geq (1 - 3c\delta^2)\omega_k r^k.
\] (3.152)

For \(\delta\) small enough, we have the thesis.

\(\square\)

The upper bound on \(\lambda^k(S)\) and the rectifiability are direct consequences of Theorem 3.3. The only thing left to prove is the statement about \(W^{1,p}\) bounds on the map. This can be obtained by simple modifications (actually simplifications) in the proof of Theorem 3.3.

As before, we denote by \(V(x, r)\) (one of) the \(k\)-dimensional subspace that minimizes \(\int_{B_r(x)} d(y, V)^2\), and we set \(L(x, r)\) to be one of the \(k\)-dimensional subspaces satisfying the Reifenberg condition, i.e., a \(k\)-dimensional subspace such that \(d_H(S \cap B_r(x), L \cap B_r(x)) < \delta r\).

First of all, note that for any \(x \in S\), \(r > 0\), \(\lambda^k(B_r(x)) > \omega_k r^k/2\), and so there are no bad balls in our covering. Thus, by Lemma 3.10, we can conclude that

\[
d_H(V(x, r) \cap B_r(x), L(x, r) \cap B_r(x)) \leq c\delta.
\] (3.153)

As a consequence, for \(\delta(n, p)\) small enough, all the excess sets at all scales are empty. Indeed, by (3.84),

\[
E(x, r) = B_r(x) \setminus B_{pr/3} \cap (V(x, r)) \cap S \subseteq B_r(x) \setminus B_{(pr/3 - c\delta)r} \cap (L(x, r)) \cap S = 0.
\] (3.154)

Proceeding with the same construction as in Section 3.3.2, we obtain a sequence of maps \(\phi_j : T_0 \to \mathbb{R}^n\), \(\phi_j(T_0) \equiv T_{j+1}\) converging in \(W^{1,p}\) to some \(\phi_\infty\) such that for all \(i\), \(S \subseteq B_{r_i}(T_i) \subseteq B_{r_{i-1}/2}(T_{i-1})\), thus proving that \(S \subseteq T_\infty \equiv \phi_\infty(T_0)\).
\section*{3.5. Proof of Theorem 3.4: The Discrete Rectifiable-Reifenberg.} Up to minor differences, the proof of this theorem is essentially the same as the proof of (3.121), actually simpler from the technical point of view. Indeed, we do not need to define the sets $\tilde{E}, \tilde{B}, \tilde{V}$, but we can just deal with the sets $E, V, B$.

\section*{3.6. Statement of secondary results.} As is clear from the proofs, the results in this section can be adapted to situations slightly more general than the ones just described. For the sake of clarity, and since they are not used elsewhere in this paper, we keep these results separated from the others.

For example, in all the theorems one can relax the assumptions by replacing the integral condition
\begin{equation}
\int_0^r s^{-1} \int_{S \cap B_r(x)} D_k^k(x, s) d\lambda^k(x) ds < \delta^2 r^k
\end{equation}
with the $L^\infty + L^2$ condition
\begin{equation}
\forall x, r, \quad D_k^k(x, r) \leq \delta^2 \quad \text{and} \quad \int_0^r s^{-1} \int_{S \cap B_r(x)} D_k^k(x, s) d\lambda^k(x) ds \leq I < \infty .
\end{equation}
Clearly in this case one can prove only upper bounds depending on $I$. To be more precise, here we state an alternative version of 3.3.

\begin{theorem}[Rectifiable-Reifenberg] For every $\epsilon > 0$, there exists $\delta(n) > 0$ such that the following holds. Let $S \subseteq B_2 \subseteq \mathbb{R}^n$ be a $\lambda^k$-measurable subset, and assume for each $B_r(x) \subseteq B_2$ with $\lambda^k(S \cap B_r(x)) \geq \epsilon_k r^k$ that
\begin{equation}
\sup_{x,r} D_k^k(x, r) \leq \delta \quad \text{and} \quad \int_0^r s^{-1} \int_{S \cap B_r(x)} D_k^k(x, s) d\lambda^k(x) ds < 4r^k .
\end{equation}
Then the following hold:
\begin{enumerate}
\item For each ball $B_r(x) \subseteq B_1$ with $x \in S$ we have
\begin{equation}
\lambda^k(S \cap B_r(x)) \leq \omega_k C(n, I) r^k ,
\end{equation}
where $C(n, I) \to 1$ as $I \to 0$;
\item $S \cap B_1(0^n)$ is $k$-rectifiable.
\end{enumerate}
\end{theorem}

\begin{proof}
The proof of Theorem 3.3 carries over to this case immediately. \hfill \Box
\end{proof}

In the discrete case, we can also replace $r^k \int_{B_r(x)} d\mu$ with the slightly more powerful estimate on $\int_{B_r(x)} d\mu$. 

**W^{1,p} estimates for the inverse** Given (3.138), we know that the set $S$ is $k$-rectifiable, and thus we can write
\begin{equation}
\int_S |\nabla \phi_{\infty}^{1/p} d\lambda^k(y) = \int_{\phi^{-1}(S)} |\nabla \phi_{\infty}^{1/p} | J(\phi) |_{\phi^{-1}(S)} | d\lambda^k(x) .
\end{equation}
By an induction similar to the one used to prove (3.127), one can prove also the bound
\begin{equation}
\int_S |\nabla \phi_{\infty}^{1/p} d\lambda^k(y) \leq (1 + \epsilon) \omega_k .
\end{equation}
**Theorem 3.17** (Discrete Rectifiable-Reifenberg with average). For any finite positive Radon measure $\mu$, define

$$F^k_\mu(x, r) \equiv \inf_{L^k} \int_{B_\rho(x)} d^2(y, L^k) \, d\mu(y)$$

(3.161)

if $\mu(B_r(x)) \geq \epsilon kr^k$, and $F^k_\mu(x, r) = 0$ otherwise.

There exists $\delta(n) > 0$ such that the following holds. Let $\{B_{r_i}(x_i)\}_{x_i \in S} \subseteq B_2$ be a collection of disjoint balls with $r_i \geq \tilde{r} > 0$, and let $\mu \equiv \sum_{x_i \in S} \omega_{r_i}^k \delta_{x_i}$ be the associated measure. Assume for each $B_i(x) \subseteq B_2$ with $\mu(B_r(x)) \geq \epsilon kr^k = \omega_k (r/20)^k$ that

$$F^k_\mu(x, r) \leq \delta \quad \text{and} \quad \int_0^\rho s^{-1} \int_{B_\rho(x)} F^k_\mu(x, s) \, d\mu(x) \, ds < I.$$  

(3.162)

Then we have the estimate

$$\sum_{x_i \in B_1} r_i^k < C(n, I).$$  

(3.163)

**Proof.** The proof of this theorem is very similar to the previous ones. The only points which require a different argument are the estimates involving $r^{-k} \int_{B_i(y)} F(p, r) \, d\mu(p)$ and $\int_{B_i(y)} F(p, r) \, d\mu(p)$. Even though a priori we do not know whether these two integrals are comparable, at each induction step we have the rough bound (3.68), moreover it will only be interesting in estimating these quantities for “good balls”, i.e., for balls which satisfy $\mu(B_r(p)) \geq 2^k \epsilon kr$. This allows us to estimate inductively on $r_i = \rho^j$ for $i = A, \cdots, 1$, where $\tilde{r} = \rho^1$ that

$$c(n, \rho) F(p, r) \leq D(p, r) \leq c(n, \rho) F(p, r),$$  

(3.164)

$$c(n, \rho) \int_{B_i(y)} F(p, r) \, d\mu(p) \leq r^{-k} \int_{B_i(y)} F(x, r) \, d\mu(p) \leq c(n, \rho) \int_{B_i(y)} F(p, r) \, d\mu(p),$$

and this is all we need to carry over the proof. \hfill \Box

### 4. $L^2$-Best Approximation Theorems

In this Section we prove the main estimate necessary for us to be able to apply the rectifiable-Reifenberg of Theorems 3.3 and 3.4 to the singular sets $S^k_\epsilon(f)$ of the stratification induced for stationary harmonic maps.

Namely, we need to understand how to estimate on a ball $B_r(x)$ the $L^2$-distance of $S^k_\epsilon$ from the best approximating $k$-dimensional subspace. When $r < \text{inj}(M)$ this means we would like to consider subspaces $L^k \subseteq T_x M$ and estimate $d^2(x, L^k)$ for $x \in S^k \cap B_r(x)$, where the distance is taken in the normal coordinate charts. After rescaling this is equivalent to looking at a ball of definite size, but assuming $K_M << 1$. The main Theorem of this Section is stated in some generality as we will need apply it with some care when proving Theorems 1.3 and 1.4. We recall that by definition

$$W_\alpha(x) \equiv W_{r_\alpha, r_{\alpha-3}}(x) \equiv \theta_{r_{\alpha-3}}(x) - \theta_{r_\alpha}(x) \geq 0,$$

(4.1)

where $r_\alpha = 2^{-\alpha}$. 


Theorem 4.1 (L^2-Best Approximation Theorem). Let \( f : B_8(p) \to N \) be a stationary harmonic map satisfying (1.14) with \( \frac{1}{f_{B_8(p)}}|\nabla f|^2 \leq \Lambda \), and let \( \epsilon > 0 \). Then there exists \( \delta(n, K_N, \Lambda, \epsilon) > 0 \) such that if \( K_M < \delta \), and \( B_8(p) \) is (0, \( \delta \))-symmetric but not \((k + 1, \epsilon)\)-symmetric, then for any finite measure \( \mu \) on \( B_1(p) \) we have that

\[ D_\mu(p, 1) = \inf_{L^1} \int_{B_1(p)} d^2(x, L^k) \, d\mu(x) \leq C \int_{B_1(p)} W_0(x) \, d\mu(x), \tag{4.2} \]

where the inf is taken over all \( k \)-dimensional affine subspaces \( L^k \subseteq T_p M \).

Remark 4.1. The assumption \( K_M < \delta \) is of little consequence, since given any \( K_M \) this just means focusing the estimates on balls of sufficiently small radius after rescaling.

4.1. Symmetry and Gradient Bounds. In this subsection we study stationary harmonic maps which are not \((k + 1, \epsilon)\)-symmetric on some ball. In particular, we show that this forces the gradient to be of some definite size when restricted to any \( k + 1 \)-dimensional subspace. More precisely:

Lemma 4.2. Let \( f : B_4(p) \subseteq M \to N \) be a stationary harmonic map satisfying (1.14) with \( \frac{1}{f_{B_4(p)}}|\nabla f|^2 \leq \Lambda \). Then for each \( \epsilon > 0 \) there exists \( \delta(n, K_N, \Lambda, \epsilon) > 0 \) such that if \( K_M < \delta \), and \( B_4(p) \) is (0, \( \delta \))-symmetric but is not \((k + 1, \epsilon)\)-symmetric, then for every \( k + 1 \)-subspace \( V^{k+1} \subseteq T_p M \) we have

\[ \int_{A_{3,4}(p)} |\langle \nabla f, V \rangle|^2 \, d\lambda \geq \delta, \tag{4.3} \]

where \( |\langle \nabla u, V \rangle|^2 = \sum_{i=1}^{k+1} |\langle \nabla u, v_i \rangle|^2 \) for any orthonormal basis \( \{v_i\} \) of \( V \), and \( \lambda \) is the volume measure on \( M \) (which is equivalent to \( \lambda^p \)).

Proof. The proof is by contradiction. So with \( n, K_N, \Lambda, \epsilon > 0 \) fixed let us assume the result fails. Then there exists a sequence \( f_i : B_4(p_i) \to N_i \) for which \( K_{M_i} < \delta_i \), with \( B_4(p_i) \) being \((0, \delta_i)\)-symmetric but not \((k + 1, \epsilon)\)-symmetric, and such that for some subspaces \( V_i^{k+1} \) we have that

\[ \int_{A_{3,4}(p_i)} |\langle \nabla f_i, V_i^{k+1} \rangle|^2 \, d\lambda_i \leq \delta_i \to 0. \tag{4.4} \]

Then after passing to subsequences we have that \( V_i \to V^k \subseteq \mathbb{R}^n \) with

\[ N_i \overset{C^{1,\alpha}}{\to} N, \tag{4.5} \]

and

\[ f_i \to f : B_4(0^p) \to N, \tag{4.6} \]

where the convergence is in \( L^2 \cap H^1_{\text{weak}} \). Now (4.4) and the \( H^1_{\text{weak}} \) convergence guarantees that

\[ \int_{A_{3,4}(0^p)} |\langle \nabla f, V^{k+1} \rangle|^2 = 0. \tag{4.7} \]
On the other hand, the \((0, \delta_i)\)-symmetry of the \(f_i\), combined with the \(L^2\) convergence, tells us that \(f\) is 0-symmetric. Combining these tells us that

\[
\int_{B_i(0^p)} |\langle \nabla f, V^{k+1} \rangle|^2 = 0,
\]

and hence we have that \(f\) is \(k + 1\)-symmetric. Because the convergence \(f_i \to f\) is in \(L^2\), this contradicts that the \(f_i\) are not \((k + 1, \epsilon)\)-symmetric for \(i\) sufficiently large, which proves the Lemma.

\[\blacksquare\]

4.2. Best \(L^2\)-Subspace Equations. In order to prove Theorem 4.1 we need to identify which subspace minimizes the \(L^2\)-energy, and the properties about this subspace that allow us to estimate the distance. We begin in Section 4.2.1 by studying some very general properties of the second directional moments of a general probability measure \(\mu \subseteq B_1(p)\). We will then study in Section 4.2.2 the Dirichlet energy of a stationary harmonic map when restricted to these directions.

4.2.1. Second Directional Moments of a Measure. Let us consider a probability measure \(\mu \subseteq B_1(0^n)\), and let

\[
x_i^{cm} = x_i^{cm}(\mu) \equiv \int x^i \, d\mu(x),
\]

be the center of mass. Let us inductively consider the maximum of the second directional moments of \(\mu\). More precisely:

**Definition 4.3.** Let \(\lambda_1 = \lambda_1(\mu) \equiv \max_{|v|=1} \int |(x - x^{cm}, v)|^2 \, d\mu(x)\) and let \(v_1 = v_1(\mu)\) with \(|v_1| = 1\) (any of) the vector obtaining this maximum. Now let us define inductively the pair \((\lambda_{k+1}, v_{k+1})\) from \(v_1, \ldots, v_k\) by

\[
\lambda_{k+1} = \lambda_{k+1}(\mu) \equiv \max_{v_{k+1}} \int |(x - x^{cm}, v)|^2 \, d\mu(x),
\]

where \(v_{k+1}\) is (any of) the vector obtaining this maximum.

Thus \(v_1, \ldots, v_n\) defines an orthonormal basis of \(\mathbb{R}^n\), ordered so that they maximize the second directional moments of \(\mu\). Let us define the subspaces

\[
V^k = V^k(\mu) \equiv x^{cm} + \text{span}\{v_1, \ldots, v_k\}.
\]

The following is a simple but important exercise:

**Lemma 4.4.** If \(\mu\) is a probability measure in \(B_1(0^p)\), then for each \(k\) the functional

\[
\min_{L^k \subseteq \mathbb{R}^n} \int d^2(x, L^k) \, d\mu(x),
\]

where the min is taken over all \(k\)-dimensional affine subspaces, attains its minimum at \(V^k\). Further, we have that

\[
\min_{L^k \subseteq \mathbb{R}^n} \int d^2(x, L^k) \, d\mu(x) = \int d^2(x, V^k) \, d\mu(x) = \lambda_{k+1}(\mu) + \cdots + \lambda_n(\mu).
\]
Note that the best affine subspace $V_k$ will necessarily pass through the center of mass $x_{cm}$ by Steiner’s formula, or equivalently by Jensen’s inequality and the definition of $x_{cm}$.

Now let us record the following Euler-Lagrange formula, which is also an easy computation:

**Lemma 4.5.** If $\mu$ is a probability measure in $B_1(0^n)$, then we have that $v_1(\mu), \ldots, v_n(\mu)$ satisfy the Euler-Lagrange equations:

\[
\int \langle x - x_{cm}, v_k \rangle (x - x_{cm})^j \, d\mu(x) = \lambda_k v_k^j + \sum_{\ell < k} \lambda_{k,\ell} v_{\ell}^j ,
\]

where

\[
\lambda_k = \int |\langle x - x_{cm}, v_k \rangle|^2 \, d\mu(x),
\]

\[
\lambda_{k,\ell} = \int \langle x - x_{cm}, v_\ell \rangle \langle x - x_{cm}, v_k \rangle \, d\mu(x).
\]

**Proof.** The proof is a simple application of the Lagrange multipliers method. Inductively on $k$, consider the function $f(v_k, \lambda_k, \lambda_{k,1}, \ldots, \lambda_{k,k-1}) : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ given by

\[
f(v_k, \lambda_k, \lambda_{k,\ell}) = \int \langle x - x_{cm}, v_k \rangle^2 \, d\mu(x) - \lambda_k \langle |v_k|^2 - 1 \rangle - 2 \sum_{\ell=0}^{k-1} \lambda_{k,\ell} \langle v_k, v_\ell \rangle.
\]

By the multipliers method, we obtain that $v_k$ and $\lambda_k$ satisfy the equations

\[
\frac{1}{2} \nabla^{(v_k)} f(v_k, \lambda_k, \lambda_{k,\ell}) = \int \langle x - x_{cm}, v_k \rangle (x - x_{cm}) d\mu(x) - \lambda_k v_k - \sum_{\ell=0}^{k-1} \lambda_{k,\ell} v_\ell = 0,
\]

which is exactly (4.14). Moreover, $\langle v_k, v_\ell \rangle = \delta_{k,\ell}$. The other equations are obtained by taking scalar products of (4.14) with $v_\ell$ for $\ell = 1, \ldots, k$.

\(\square\)

4.2.2. **Restricted Dirichlet Energies.** Our goal is to now study the Dirichlet energy of a stationary harmonic map when restricted to the directions $v_1(\mu), \ldots, v_n(\mu)$ associated to a probability measure. The main result of this subsection is the following, which holds for a general stationary harmonic map:

**Proposition 4.6.** Let $f : B_8(p) \subseteq M \to N$ be a stationary harmonic map satisfying (1.14) with $K_M < 2^{-4}$ and such that $\int_{B_8(p)} |\nabla f|^2 \leq \Lambda$. Let $\mu$ be a probability measure on $B_1(p)$ with $\lambda_k(\mu), v_k(\mu)$ defined as in Definition 4.3. Then there exists $C(n, K_N) > 0$ such that

\[
\lambda_k \int_{A_3(z(p))} |\nabla f(z, v_k)|^2 \, dv_g(z) \leq C \int W_0(x) \, d\mu(x).
\]

**Proof.** Note first that there is no harm in assuming that $x_{cm} \equiv 0$. If not we can easily translate to make this so, in which case we still have that supp$(\mu) \subseteq B_2$. Additionally, we will simplify the technical aspect of the proof by assuming that $M \equiv \mathbb{R}^n$. By working in normal coordinates the proof of the general case is no
different except up to some mild technical work.

Now let us fix any $z \in A_{3,4}$. By Lemma 4.5, we can inner product both sides of (4.14) by $\nabla f(z)$ to obtain for each $k$ and $z \in A_{3,4}$:

$$
\lambda_k \langle \nabla f(z), v_k \rangle = \int \langle x, v_k \rangle \langle \nabla f(z), x \rangle \; d\mu(x) - \sum_{\ell \neq k} \lambda_{k,\ell} \langle \nabla f(z), v_\ell \rangle = \int \langle x, v_k \rangle \left[ \langle \nabla f(z), x \rangle - \sum_{\ell \neq k} \langle \nabla f(z), v_\ell \rangle \langle v_\ell, x \rangle \right] \; d\mu(x) = \int \langle x, v_k \rangle \langle \nabla^{(k)} f(z), x \rangle \; d\mu(x), \tag{4.19}
$$

where by definition $\nabla^{(k)} f(z) = \nabla f(z) - \sum_{\ell \neq k} \langle \nabla f(z), v_\ell \rangle v_\ell$ is the component of $\nabla f(z)$ orthogonal to $V^{k-1}$. Observe that, by definition of center of mass,

$$
\int \langle x, z \rangle \; d\mu(x) = \langle x_{cm}, z \rangle = 0. \tag{4.20}
$$

Thus we can write

$$
\lambda_k \langle \nabla f(z), v_k \rangle = \int \langle x, v_k \rangle \langle \nabla^{(k)} f(z), x - z \rangle \; d\mu(x). \tag{4.21}
$$

By Hölder inequality, we can then estimate

$$
\lambda_k^2 |\langle \nabla f(z), v_k \rangle|^2 \leq \lambda_k \int \left| \langle \nabla^{(k)} f(z), x - z \rangle \right|^2 \; d\mu(x). \tag{4.22}
$$

If $\lambda_k \neq 0$, integrating with respect to $z$ on both sides we get the estimate

$$
\lambda_k \int_{A_{3,4}} |\langle \nabla f(z), v_k \rangle|^2 \; dv_g(z) \leq \lambda_k \int_{A_{3,4}} \left| \langle \nabla^{(k)} f(z), x - z \rangle \right|^2 \; dv_g(z) \; d\mu(x). \tag{4.23}
$$

Set for convenience $n_z(z) = (z - x)/|z - x|$, i.e., $n_z(z)$ is the radial vector from $x$ to $z$. Now for $x \in \text{supp}(\mu)$ we can estimate

$$
\int_{A_{3,4}} \left| \langle \nabla^{(k)} f(z), x - z \rangle \right|^2 \; dv_g(z) = \int_{A_{3,4}} \left| \langle \nabla^{(k)} f(z), n_z(z) \rangle \right|^2 |x - z|^{2-n} \; dv_g(z)
\leq 6^n \int_{A_{3,4}} |\langle \nabla f(z), n_z(z) \rangle|^2 |x - z|^{2-n} \; dv_g(z)
\leq C(n) \int_{A_{3,4}} |\langle \nabla f(z), n_z(z) \rangle|^2 |x - z|^{2-n} \; dv_g(z)
= C(n) W_0(x). \tag{4.24}
$$

Applying this to (4.23) we get as desired

$$
\lambda_k \int_{A_{3,4}} |\langle \nabla f(z), v_k \rangle|^2 \; dv_g(z) \leq C(n) \int W_0(x) \; d\mu(x). \tag{4.25}
$$

□
4.3. Proof of Theorem 4.1. Let us now combine the results of this Section in order to prove Theorem 4.1. Indeed, let \( \mu \) be a measure in \( B_1(p) \subseteq T_p M \). We can assume that \( \mu \) is a probability measure without any loss of generality, since both sides of our estimate scale. Let \( (\lambda_1(\mu), v_1(\mu)), \ldots, (\lambda_n(\mu), v_n(\mu)) \) be the directional second moment as defined in Definition 4.3, with \( V^k \) the induced subspaces defined as in (4.11). Using Lemma 4.4 we have that

\[
\min_{L^k \subseteq \mathbb{R}^k} \int d^2(x, L^k) \, d\mu(x) = \int d^2(x, V^k) \, d\mu(x) = \lambda_{k+1}(\mu) + \cdots + \lambda_n(\mu) \leq (n-k)\lambda_{k+1}(\mu),
\]

where we have used that \( \lambda_j \leq \lambda_i \) for \( j \geq i \). Therefore our goal is to estimate \( \lambda_{k+1} \). To begin with, Proposition 4.6 tells us that for each \( j \)

\[
\lambda_j \int_{A_3,\lambda(p)} |\nabla f(z, v_j)|^2 \, dv_g(z) \leq C \int W_0(x) \, d\mu(x).
\]

Let us sum the above for all \( j \leq k + 1 \) in order to obtain

\[
\sum_{j=1}^{k+1} \lambda_j \int_{A_3,\lambda(p)} |\nabla f(z, v_j)|^2 \, dv_g(z) \leq (k+1)C \int W_0(x) \, d\mu(x),
\]

or by using that \( \lambda_{k+1} \leq \lambda_j \) for \( k + 1 \geq j \) we get

\[
\lambda_{k+1} \int_{A_3,\lambda(p)} |\nabla f(z, V^{k+1})|^2 \, dv_g(z) = \lambda_{k+1} \sum_{j=1}^{k+1} \int_{A_3,\lambda(p)} |\nabla f(z, v_j)|^2 \, dv_g(z) \leq C \int W_0(x) \, d\mu(x).
\]

Now we use that \( B_8(p) \) is \((0, \delta)\)-symmetric, but not \((k + 1, \epsilon)\)-symmetric in order to apply Lemma 4.2 and conclude that

\[
\int_{A_3,\lambda(p)} |\nabla f(z, V^{k+1})|^2 \, dv_g(z) \geq \delta.
\]

Combining this with (4.29) we obtain

\[
\delta \lambda_{k+1} \leq \lambda_{k+1} \int_{A_3,\lambda(p)} |\nabla f(z, V^{k+1})|^2 \, dv_g(z) \leq C \int W_0(x) \, d\mu(x),
\]

or that

\[
\lambda_{k+1} \leq C(n, K_N, \Lambda, \epsilon) \int W_0(x) \, d\mu(x).
\]

Combining this with (4.26) we have therefore proved the Theorem. \( \square \)

5. The Inductive Covering Lemma

This Section is dedicated to the basic covering lemma needed for the proof of the main theorems of the paper. The covering scheme is similar in nature to the one introduced by the authors in [NV] in order to prove structural theorems on critical sets. Specifically, let us consider a stationary harmonic map \( f \) between Riemannian manifolds. We wish to build a covering of the quantitative stratification

\[
S_{\epsilon,\ell}^k(f) \cap B_1(p) \subseteq U_r \cup U_+ = U_r \cup \bigcup B_{r}(x_i),
\]

(5.1)
which satisfies several basic properties. To begin with, the set \( U_+ \) is a union of balls satisfying \( r_i > r \geq 0 \), and should satisfy the packing estimate \( \omega_k \sum r_i^k \leq C \). Each ball \( B_r(x_i) \) should have the additional property that there is a definite energy drop of \( f \) when compared to \( B_1(p) \). To describe the set \( U_r \) we should distinguish between the case \( r > 0 \) and \( r \equiv 0 \). In the case \( r > 0 \) we will have that \( U_r = \bigcup B_r(x_i) \) is a union of \( r \)-balls and satisfies the Minkowski estimate \( \text{Vol}(U_r) \leq C r^{n-k} \). In the case when \( r \equiv 0 \) we will have that \( U_0 \) is \( k \)-rectifiable with the Hausdorff estimate \( \mathcal{H}^k(U_0) \leq C \). Let us be more precise:

**Lemma 5.1** (Covering Lemma). Let \( f : B_{16}(p) \subseteq M \to N \) be a stationary harmonic map satisfying (1.14) with \( \int_{B_{16}(p)} |\nabla f|^2 \leq \Lambda \) and \( K_M < \delta(n, K_N, \Lambda, \epsilon) \). Let \( E = \sup_{x \in B_1(p)} \theta_1(x) \) with \( \epsilon > 0 \), \( r \geq 0 \), and \( k \in \mathbb{N} \). Then for all \( \eta \leq \eta(n, K_N, \Lambda, \epsilon) \), there exists a covering \( S^{k}(f) \cap B_1(p) \subseteq U = U_r \cup U_+ \) such that

1. \( U_+ = \bigcup B_r(x_i) \) with \( r_i > r \) and \( \sum r_i^k \leq C(n, K_N, \Lambda, \epsilon) \)
2. \( \sup_{y \in B_r(x_i)} \theta_1(y) \leq E - \eta \).
3. If \( r > 0 \) then \( U_r = \bigcup B_r(x_i^r) \) with \( N \leq C(n) r^{-k} \).
4. If \( r = 0 \) then \( U_0 \) is \( k \)-rectifiable and satisfies \( \text{Vol}(B_s(U_0)) \leq C(n) s^{n-k} \) for each \( s > 0 \).

In particular, \( \mathcal{H}^k(U_0) \leq C(n) \).

**Remark 5.1.** The assumption \( K_M < \delta \) is of little consequence, since given any \( K_M \) this just means focusing the estimates on balls of sufficiently small radius after rescaling.

To prove the result let us begin by outlining the construction of the covering, we will then spend the rest of this section proving the constructed cover has all the desired properties.

Thus let us consider some \( \eta > 0 \) fixed, and then define the energy scale for \( x \in B_1(p) \) by

\[
s_x = s_x^{E, \eta} \equiv \inf \{ r \leq s \leq 1 : \sup_{B_r(x)} \theta_1(y) \geq E - \eta \}.
\]

(5.2)

Note that the energy scale implicitly depends on many constants. If \( r = 0 \) let us define the set \( U_0 \) by

\[
U_0 = \{ x \in S^k_{E, \eta}(f) \cap B_1(p) : s_x = 0 \},
\]

(5.3)

while if \( r > 0 \) let us define

\[
U_r = \bigcup B_r(x_i^r),
\]

(5.4)

where

\[
\{ x_i^r \} \subseteq \{ x \in S^k_{E, \eta}(f) \cap B_1(p) : s_x = r \},
\]

(5.5)

is any minimal \( r/2 \)-dense set. In particular, note that the collection of balls \( \{ B_{r/2}(x_i^r) \} \) are disjoint.

In order to define the covering \( U_+ = \{ B_1(x_i) \} \) let us first consider the covering

\[
\{ x \in S^k_{E, \eta}(f) \cap B_1(p) : s_x > r \} \cup \bigcup_{s_x > r} B_{s_x/5}(x),
\]

(5.6)
and choose from it a Vitali subcovering and set
\[ U_+ \equiv \bigcup_{i \in I} B_{r_i}(x_i), \tag{5.7} \]
where \( r_i \equiv s_{x_i} \). In particular, we have that the collection of balls \( \{B_{r_i/15}(x_i)\} \) are all disjoint. It is clear from the construction that we have built a covering
\[ S_{\varepsilon,r}^k(f) \cap B_1(p) \subseteq U_r \cup U_+, \tag{5.8} \]
we now need to study the properties of \( U_r \) and \( U_+ \).

The outline of this Section is as follows. In Sections 5.1 and 5.2 we will deal with estimating the set \( U_r \). In fact, from a technical standpoint \( U_r \) is much easier than \( U_+ \) to deal with, and for the rectifiability and local \( A^k \)-finiteness of \( S_{\varepsilon,r}^k \) this is the set which is most important. Indeed, for \( U_r \) we will be able to almost directly use the rectifiable-Reifenberg of Theorems 3.3 and 3.4, at least when combined with an additional induction argument. However, for the global Minkowski estimates and \( k \)-dimensional Hausdorff measure bounds on \( S_{\varepsilon,r}^k \) it is crucial to deal with the set \( U_+ \) as well. To do this we first prove a variety of technical lemmas in Section 5.3, these will allow us to exchange \( U_+ \) for a more manageable collection of balls without losing much content. Then in Section 5.4 we will be able argue as in the \( U_r \) case by applying the rectifiable Reifenberg of Theorem 3.4, however we will apply it not to \( U_+ \) but to a more carefully chosen covering obtained by exploiting the results of Section 5.3.

5.1. Estimating \( U_r \) in Lemma 5.1 for \( r > 0 \). Let us begin by altering the collection \( U_r = \bigcup B_r(x'_j) \) slightly. Indeed, by definition of \( U_r \), we have that for each ball \( B_r(x'_j) \) there exists a point \( x_i \in B_r(x'_j) \) such that \( \theta_r(x'_j) \geq E - \eta \). Then we have the covering
\[ U_r \subseteq \bigcup_{1}^{N} B_{2r}(x'_j), \tag{5.9} \]
and let \( U'_r \equiv \bigcup_{1}^{N'} B_{2r}(x'_j) \) be a maximal disjoint subset of these balls. Let us first observe that by a standard covering argument we have \( N \leq C(n)N' \), and thus to estimate \( U_r \) it is enough to estimate \( N' \). Indeed, by the maximality of \( U'_r \) we have that for every ball center \( x'_j \in U_r \) there exists a ball center \( x'_j \in U'_r \) such that \( x'_j \in B_{4r}(x'_j) \). Since the balls \( \{B_{r/8}(x'_j)\} \) are disjoint, we have that
\[ N \cdot \omega_n \left(\frac{r}{8}\right)^n \leq \text{Vol}(U_r) \leq \sum_{1}^{N'} \text{Vol}(B_{4r}(x'_j)) \leq N' \cdot \omega_n (4r)^n, \tag{5.10} \]
which in particular gives us the desired estimate \( N \leq 32^n N' \).

Now let us proceed to estimate \( N' \). Thus consider the measure associated to \( U'_r \) given by
\[ \mu = \sum \omega_n r^d \delta_{x'_j}. \tag{5.11} \]
Let us consider the sequence of radii \( r_\alpha = 2^{-\alpha} \). We now wish to prove that for each \( x \in B_1 \) and all \( r \leq r_\alpha \leq 1 \) that

\[
\mu(B_{r_\alpha}(x)) \leq D(n)r_\alpha^k ,
\]

where \( D(n) \) is from Theorem 3.4. Let us first observe that once (5.12) has been proved then we have the desired estimate on \( N' \). Indeed, by applying the estimate to \( x = 0 \) and \( r_0 = 1 \) we have that

\[
N' \omega_n r^k = \sum_{i=1}^{N'} \omega_n r^k = \mu(B_1) \leq D(n) ,
\]

which gives the claimed estimate \( N' \leq C(n)r^{-k} \).

Thus let us now concentrate on proving (5.12). We will prove it by induction on \( \alpha \). Let \( \alpha_0 \) be such that \( r \leq r_{\alpha_0} < 2r \). Let us begin by observing that the result clearly holds for \( r_{\alpha_0} \). Therefore let us now assume we have proved (5.12) for some \( r_{\alpha+1} \), and then proceed to prove it for \( r_\alpha \).

Let us first observe a rough estimate in this direction through a covering argument. Namely, let us consider the radii \( r_{\alpha+1} \leq s \leq 4r_\alpha \). We can cover \( B_s(x) \) by a collection of at most \( C(n) \) balls \( \{B_{r_{\alpha+1}}(y_i)\} \), and thus by the inductive assumption we have for all \( x \in B_1 \) and \( s \leq 4r_\alpha \) that

\[
\mu(B_s(x)) \leq \sum \mu(B_{r_{\alpha+1}}(y_i)) \leq D(n)C(n)r_{\alpha+1}^k \leq D'(n)s^k ,
\]

where of course \( D'(n) >> D(n) \).

Now let us consider a ball \( B_s(y) \subseteq B_1 \) with \( s \leq 2r_\alpha \). If \( \mu(B_s(y)) \leq \epsilon_k s^k \) then \( D_\mu(y, s) \equiv 0 \) by definition, while if \( s \leq r \) then we also have that \( D_\mu(y, s) \equiv 0 \), since the support of \( \mu \) in \( B_s(x) \) contains at most one point and thus is precisely contained in a \( k \)-dimensional subspace. Thus let us consider the case when \( s > r \) and \( \mu(B_s(y)) > \epsilon_k s^k \). In this case notice by theorem 2.1 that there is certainly at least one point \( z \in \text{supp}(\mu) \cap B_s(x) \) which is \( (0, \delta) \)-symmetric, where \( \delta = \delta(\eta n, K_N, \Lambda) \rightarrow 0 \) as \( \eta \rightarrow 0 \). Thus for \( \eta \leq \eta(n, K_N, \Lambda) \) we can apply Theorem 4.1 to see that

\[
D_\mu(y, s) \leq C(n, K_M, K_N, \Lambda, \epsilon)s^{-k} \int_{B_s(y)} W_s(z) d\mu(z) .
\]

By applying this to all \( r \leq t \leq s \) we have

\[
s^{-k} \int_{B_s(x)} D_\mu(y, t) d\mu(y) \leq C s^{-k} \int_{B_s(x)} t^{-k} \int_{B_t(y)} W_t(z) d\mu(z) d\mu(y)
\]

\[
= C s^{-k} t^{-k} \int_{B_s(x)} \mu(B_t(y)) W_t(y) d\mu(y)
\]

\[
\leq C s^{-k} \int_{B_s(x)} W_t(y) d\mu(y) ,
\]

(5.16)
where we have used our rough estimate (5.14) in the last line. Let us now consider the case when \( t = r_\beta \leq s \leq 2r_\alpha \). Then we can sum to obtain:
\[
\sum_{r_\beta \leq s} s^{-k} \int_{B_\delta(x)} D_\mu(y, r_\beta) \, d\mu(y) \leq C \sum_{r_\beta \leq r_\delta \leq s} s^{-k} \int_{B_\delta(x)} W_{r_\delta}(y) \, d\mu(y) \\
= Cs^{-k} \int_{B_\delta(x)} \sum_{r_\beta \leq r_\delta \leq s} W_{r_\delta}(y) \, d\mu(y) \\
\leq Cs^{-k} \int_{B_\delta(x)} \| \theta_s(y) - \theta_r(y) \| \, d\mu(y) \leq C(n, K_N, \Lambda, \epsilon)\eta, \tag{5.17}
\]
where in the last line we have used both the rough estimate (5.14) on \( \mu(B_s(x)) \) and that in the support of \( \mu \) we have that \( \| \theta_s(y) - \theta_r(y) \| \leq \eta \) by the construction of \( U_r \). Now let us choose \( \eta \leq \eta(n, K_N, \Lambda, \epsilon) \) such that we have
\[
\sum_{r_\beta \leq s} s^{-k} \int_{B_\delta(x)} D_\mu(y, r_\beta) \, d\mu(y) \leq \delta^2, \tag{5.18}
\]
where \( \delta \) is taken from Theorem 3.4. Since the estimate (5.18) holds for all \( B_s \subseteq B_{2r_\alpha}(x) \), we can therefore apply Theorem 3.4 to conclude the estimate
\[
\mu(B_{r_\alpha}(x)) \leq D(n) r_\alpha^k. \tag{5.19}
\]
This finishes the proof of (5.12), and hence the proof of estimate of \( U_r \) for \( r > 0 \) in Lemma 5.1. \( \Box \)

5.2. Estimating \( U_0 \) in Lemma 5.1. Let us begin by proving the Minkowski estimates on \( U_0 \). Indeed, observe that for any \( r > 0 \) that \( U_0 \subseteq U_r \), and thus we have the estimate
\[
\text{Vol}(B_r(U_0)) \leq \text{Vol}(B_r(U_r)) \leq c_\omega n^{n-1} \cdot N \leq C(n, K_N, \Lambda, \epsilon) r^{n-k}, \tag{5.20}
\]
which proves the Minkowski claim. In particular, we have as a consequence the \( k \)-dimensional Hausdorff measure estimate
\[
\mathcal{H}^k(U_0) \leq C(n, K_N, \Lambda, \epsilon). \tag{5.21}
\]
In fact, let us conclude a slightly stronger estimate, since it will be a convenient technical tool in the remainder of the proof. If \( B_s(x) \) is any ball with \( x \in B_1 \) and \( s < \frac{1}{2} \), then by applying the same proof to the rescaled ball \( B_s(x) \rightarrow B_1(0) \), we can obtain the Hausdorff measure estimate
\[
\mathcal{H}^k(U_0 \cap B_s(x)) \leq C s^k. \tag{5.22}
\]
To finish the construction we need to see that \( U_0 \) is rectifiable. We will in fact apply Theorem 3.3 in order to conclude this. To begin with, let \( \mu \equiv \mathcal{H}^k|_{U_0} \) be the \( k \)-dimensional Hausdorff measure, restricted to \( U_0 \). Let \( B_s(y) \) be a ball with \( y \in B_1 \) and \( s < \frac{1}{2} \), now we will now argue in a manner similar to Section 5.1. Thus, if \( \mu(B_s(y)) \leq \epsilon s^k \) then \( D_{U_0}(y, s) \equiv 0 \), and otherwise we then have by Theorem 4.1 that
\[
D_\mu(y, s) \leq C(n, K_N, \Lambda, \epsilon) s^{-k} \int_{B_s(y)} W_s(z) \, d\mu(z). \tag{5.23}
\]
By applying this to all \( t \leq s \) we have

\[
S^{-k} \int_{B_t(x)} D_\mu(y, t) \, d\mu(y) \leq C S^{-k} \int_{B_t(x)} \tau^{-k} \int_{B_t(y)} W_t(z) \, d\mu(z) \, d\mu(y)
= CS^{-k} \tau^{-k} \int_{B_t(x)} \mu(B_t(y)) W_t(y) \, d\mu(y)
\leq CS^{-k} \int_{B_t(x)} W_t(y) \, d\mu(y),
\]

where we have used our estimate (5.22) in the last line. Let us now consider the case when \( t = r_\beta = 2^{-\beta} \leq s \). Then we can sum to obtain:

\[
\sum_{r_{\beta} \leq s} S^{-k} \int_{B_t(x)} D_\mu(y, r_{\beta}) \, d\mu(y) \leq C \sum_{r_{\beta} \leq s} S^{-k} \int_{B_t(x)} W_{r_{\beta}}(y) \, d\mu(y)
= CS^{-k} \int_{B_t(x)} \sum_{r_{\beta} \leq s} W_{r_{\beta}}(y) \, d\mu(y)
\leq CS^{-k} \int_{B_t(x)} |\theta_s(y) - \theta_0(y)| \, d\mu(y) \leq C(n, K_N, \Lambda, \epsilon) \eta, \tag{5.25}
\]

where we have used two points in the last line. First, we have used our estimate \( \mu(B_{s}(x)) \leq C s^{k} \). Second, we have used that by the definition of \( U_{0} \), for each point in the support of \( \mu \) we have that \( |\theta_s(y) - \theta_0(y)| \leq \eta \).

Now let us choose \( \eta \leq \eta(n, K_N, \Lambda, \epsilon) \) such that we have

\[
\sum_{r_{\beta} \leq s} S^{-k} \int_{B_t(x)} D_\mu(y, r_{\beta}) \, d\mu(y) \leq \delta^2, \tag{5.26}
\]

where \( \delta \) is chosen from Theorem 3.3. Thus, by applying Theorem 3.3 we see that \( U_{0} \) is rectifiable, which finishes the proof of Lemma 5.1 in the context of \( U_{0} \). \( \square \)

5.3. Technical Constructions for Estimating \( U_{+} \). To estimate the set \( U_{+} \) we want to follow the same morals as the estimates for \( U_{r} \) and \( U_{0} \), unfortunately the estimates on the set \( U_{+} \) turn out to be rather delicate. The reasoning is that we cannot apply the rectifiable Reifenberg of Theorem 3.4 directly to this set. Instead, we will need to replace \( U_{+} \) with a different covering at each stage, which will be more adaptable to Theorem 3.4. Thus this subsection is dedicated to proving a handful of technical results which are important in the construction of this new covering.

Throughout this subsection we are always working under the assumptions of Lemma 5.1. Let us begin with the following point, which is essentially a consequence of the continuity of the energy:

**Lemma 5.2.** For each \( \eta' > 0 \) there exists \( R(n, K_N, \Lambda, \eta') > 1 \) such that with \( \eta \leq \eta(n, K_N, \Lambda, \eta') \) we have for each \( z \in B_{r_\eta}(x_i) \) the estimate

\[
\theta_{Rr_\eta}(z) > E - \eta'. \tag{5.27}
\]
Proof. The proof relies on a straightforward energy comparison. Namely, let \( x, y \in B_{1/2} \) with \( s < 1 \) and let us denote \( d \equiv d(x, y) \). Then we have the estimate
\[
\theta_i(y) = \left( \int_{B_i(y)} |\nabla f|^2 \right)^{2-n} \leq s^{2-n} \int_{B_{s+d}(x)} |\nabla f|^2 = \left( \frac{s}{s + d} \right)^{2-n} \theta_{s+d}(x) .
\] (5.28)
To apply this in our context, let us note for each \( i \) in our covering, that by our construction of \( U_+ \) there must exist \( y_i \in B_r(x_i) \) such that \( \theta_{(R-1)r_i}(y_i) \geq \theta_i(y_i) = E - \eta \). Let us now apply (5.28) to obtain
\[
\theta_{Rr_i}(z) \geq \left( \frac{Rr_i}{(R-2)r_i} \right)^{2-n} \theta_{(R-2)r_i}(y_i) \geq \left( \frac{R}{R-2} \right)^{2-n} (E - \eta) .
\] (5.29)
If \( R = R(n, K_N, \Lambda, \eta') > 0 \) and \( \eta \leq \eta(n, K_N, \Lambda, \eta') \), then we obtain from this the claimed estimate. \( \square \)

In words, the above Lemma is telling us that even though we have no reasonable control over the size of \( \theta_i(x_i) \), after we go up a controlled number of scales we can again assume that the energy density is again close to \( E \).

In the last Lemma the proof was based on continuity estimates on the energy \( \theta \). In the next Lemma we wish to show an improved version of this continuity under appearance of symmetry. Precisely:

**Lemma 5.3** (Improved Continuity of \( \theta \)). Let \( f : B_4(p) \to N \) be a stationary harmonic map satisfying (1.14) with \( \int_{B_4(p)} |\nabla f|^2 \leq \Lambda \). Then for each \( 0 < \tau, \eta < 1 \) there exists \( \delta(n, K_N, \Lambda, \eta, \tau) > 0 \) such that if

1. we have \( K_M < \delta \),
2. there exists \( x_0, \ldots, x_k \in B_1(p) \) which are \( \tau \)-independent with \( |\theta_{0r_i}(x_j) - \theta_3(x_j)| < \delta \),

then if \( V^k \) is the \( k \)-dimensional subspace spanned by \( x_0, \ldots, x_k \), then for all \( x, y \in B_1(p) \cap B_\delta(V^k) \) and \( 10^{-4} \leq s \leq 1 \) we have that \( |\theta_s(x) - \theta_s(y)| < \eta \).

**Proof.** The proof is by contradiction. Thus, imagine no such \( \delta \) exists. Then there exists a sequence of stationary harmonic maps \( f_i : B_4(p_i) \to N_i \) satisfying \( \int_{B_4(p_i)} |\nabla f|^2 \leq \Lambda \) such that

1. \( K_M < \delta_i \to 0 \)
2. there exists \( x_{i,0}, \ldots, x_{i,k} \in B_1(p) \) which are \( \tau \)-independent with \( |\theta_{0r_i}(x_{i,j}) - \theta_3(x_{i,j})| < \delta_i \to 0 \),

however we have that there exists \( x_i, y_i \in B_1(p) \cap B_\delta(V^k) \) and \( 10^{-4} \leq s_i \leq 1 \) such that \( |\theta_{s_i}(x_i) - \theta_{s_i}(y_i)| \geq \eta \).

Let us now pass to a subsequence to obtain the defect measure
\[
|\nabla f_i|^2 dv_g \to |\nabla f|^2 dv_g + \nu ,
\]
\[
x_{i, \beta} \to x_\beta \in B_1(0^p) ,
\]
\[
x_i \to s , \quad y_i \to y \in V = \text{span}\{x_0, \ldots, x_k\} ,
\]
\[
s_i \to s ,
\] (5.30)
where \( \nu \) is a measure on \( B_4(0^p) \), and
\[
 s^{2-n} \left| \int_{B_{s}(x)} (|\nabla f|^2 dv_g + dv) - \int_{B_{s}(y)} (|\nabla f|^2 dv_g + dv) \right| \geq \eta .
\] (5.31)
However, we have by theorem 2.7 that $f$ and $v$ are 0-symmetric on $B_2$ with respect to each of the points $x_0, \ldots, x_k$. In particular, by the standard cone splitting arguments we have that $v$ is $k$-symmetric with respect to the $k$-plane $V$ on $B_1$. In particular, $v$ is invariant under translation by elements of $V$. However, this is a contradiction to (5.31), and therefore we have proved the result.

\[ \square \]

Now the first goal is to partition $U_+$ into a finite collection, each of which will have a few more manageable properties than $U_+$ itself. More precisely:

**Lemma 5.4.** For each $R > 1$ there exists $N(n, R) > 1$ such that we can break up $U_+$ as a union

\[ U_+ = \bigcup_{a=1}^{N} U^a_+ = \bigcup_{a=1}^{N} \bigcup_{i \in I^a} B_{r_i}(x_i), \quad (5.32) \]

such that each $U^a_+$ has the following property: if $i \in I^a$, then for any other $j \in I^a$ we have that if $x_j \in B_{Rr_i}(x_i)$, then $r_j < R^{-2}r_i$.

**Proof.** Let us recall that the balls in the collection $\{B_{r_i/S}(x_i)\}$ are pairwise disjoint. In particular, given $R > 1$ if we fix a ball $B_{r_i}(x_i)$ then by the usual covering arguments there can be at most $N(n, R)$ ball centers $\{x_j\}_1^N \subset U_+ \cap B_{Rr_i}(x_i)$ with the property that $r_j \geq R^{-6}r_i$. Indeed, if $\{x_j\}_1^N$ is such a collection of balls then we get

\[ \omega_n(6R^3r_i)^n = \text{Vol}(B_{6R^3r_i}(x_i)) \geq \sum_{I^a} \text{Vol}(B_{Rr_i/S}(x_i)) \geq N\omega_n(R^{-6}r_i/S)^n, \quad (5.33) \]

which by rearranging gives the estimate $N \leq N(n, R)$ as claimed.

Now we wish to build our decomposition $U_+ = \bigcup_{a=1}^{N} U^a_+$, where $N$ is from the first paragraph. We shall do this inductively, with the property that at each step of the inductive construction we will have that the sets $U^a_+$ will satisfy the desired property. In particular, for every $a$ and $i \in I^a$, if $j \in I^a$ is such that $x_j \in B_{Rr_i}(x_i)$, then $r_j < R^{-2}r_i$.

Begin by letting each $I^a$ be empty. We are going to sort the points $\{x_i\}_{i \in J}$ into the sets $I^a$ one at a time. At each step let $i \in I \setminus \bigcup_{a=1}^{N} I^a$ be an index such that $r_i = \max r_j$, where the max is taken over all indexes in $I \setminus \bigcup_{a=1}^{N} I^a$, i.e., over all indexes which haven’t been sorted out yet. Now let us consider the collection of ball centers $\{y_j\}_{j \in J}$ such that $J \subset I$, $x_i \in B_{Rr_i}(y_j)$ and $r_i \leq r_j \leq R^2r_i$. Note that, by construction, either $y_j$ has already been sorted out in some $I^a$, or $r_j = r_i$. Now evidently $y_j \in B_{Rr_i}(x_i)$ for all $j \in J$ and so by the first paragraph the cardinality of $J$ is at most $N(n, R)$. In particular, there must be some $U^a_+$ such that $U^a_+ \cap \{y_j\} = 0$. Let us assign $x_i \in U^a_+$ to this piece of the decomposition. Clearly the decomposition $\bigcup U^a_+$ still satisfies the inductive hypothesis after the addition of this point, and so this finishes the inductive step of construction. Since at each stage we have chosen $x_i$ to have the maximum radius, this process will continue indefinitely to give the desired decomposition of $U_+$.

\[ \square \]
Now with a decomposition fixed, let us consider for each $1 \leq a \leq N$ the measures
\[ \mu^a = \sum_{x_i \in U^a_i} \omega_k r_i^k \delta_{x_i}. \] (5.34)

The following is a crucial point in our construction. It tells us that each ball $B_{10r_i}(x_i)$ either has small $\mu^a$-volume, or the point $x_i$ must have large energy at scale $r_i$. Precisely:

**Lemma 5.5.** Let $\eta', D > 0$ be fixed. There exists $R = R(n, K_N, \Lambda, \eta', D, \varepsilon) > 0$ such that if we consider the decomposition (5.32), and if

1. $K_M < \delta(n, K_N, \Lambda, \eta', \varepsilon)$ and $\eta \leq \eta(n, K_N, \Lambda, \eta', \varepsilon)$, $r_i < 10^{-4}$,
2. we have $\mu^a(B_{10r_i}(x_i)) \geq 2\omega_k r_i^k$,
3. for all ball centers $y_j \in A_{r_i/5, 10r_i}(x_i) \cap U^a_i$ and for $s = 10^{-2n}D^{-1}r_i$, we have that $\mu^a(B_s(y_j)) \leq Ds^k$,

then we have that $\theta(x_i, r_i/10) \geq E - \eta'$.

**Proof.** Let us begin by choosing $\eta'' < \eta' < \eta(n, K_N, \Lambda, \varepsilon)$, which will be fixed later in the proof, and let us also define $\tau \equiv 10^{-2n}D^{-1}$. Let $\delta(\eta(n, K_N, \Lambda, \eta', \varepsilon))$ be from Lemma 5.3 so that the conclusions hold with $10^{-1}\eta'$, and let $\delta'(n, K_N, \Lambda, \eta'', \varepsilon)$ be chosen so that conclusions of Lemma 5.2 are satisfied with $\eta''$ if $\eta \leq \eta(n, K_N, \Lambda, \eta'', \varepsilon)$. Now throughout we will assume $\eta'' \leq \delta$ and $K_M < \min(\delta, \delta')$. We will also choose $R = R(n, K_N, \Lambda, \eta'', D, \varepsilon) > \max(\tau^{-1}, \delta^{-1}, \delta'^{-1})$ so that Lemma 5.2 is satisfied with $\eta''$.

Since it will be useful later, let us first observe that $r_i \geq 2r$. Indeed, if not then for each ball center $y_j \in A_{r_i/5, 10r_i}(x_i) \cap U^a_i$ we would have $r \leq r_j < R^2r_i < r$. This tells us that there can be no ball centers in $A_{r_i/5, 10r_i}(x_i) \cap U^a_i$. However, by our volume assumption we have that
\[ \mu^a(A_{r_i/5, 10r_i}(x_i)) = \mu^a(B_{10r_i}(x_i)) - \mu^a(B_{r_i/5}(x_i)) \geq 2\omega_k r_i^k - \omega_k r_i^k \geq \omega_k r_i^k, \] (5.35)
which contradicts this. Therefore we must have that $r_i \geq 2r$.

Now our first real claim is that under the assumptions of the Lemma there exists ball centers $y_0, \ldots, y_k \in U^a_i \cap A_{r_i/5, 10r_i}(x_i)$ which are $\tau r_i$-independent in the sense of Definition 2.2. Indeed, assume this is not the case, then we can find a $k - 1$-plane $V^{k-1}$ such that
\[ \{y_i\}_{i \in E} \cap A_{r_i/5, 10r_i}(x_i) \subseteq B_{\tau r_i}(V^{k-1}). \] (5.36)
In particular, by covering $B_{\tau r_i}(V) \cap B_{10r_i}(x_i)$ by $C(n)\tau^{-1-k} \leq 10^n\tau^{-1-k}$ balls of radius $\tau r_i$, and using our assumption that $\mu^a(B_{\tau r_i}(y)) \leq \mu^a(B_{\tau r_i}(V) \cap B_{10r_i}(x_i)) \leq 10^n\tau r_i^k \leq \omega_n r_i^k$. (5.37)

On the other hand, our volume assumption guarantees that
\[ \mu^a(A_{r_i/5, 10r_i}(x_i)) = \mu^a(B_{10r_i}(x_i)) - \mu^a(B_{r_i/5}(x_i)) \geq 2\omega_k r_i^k - \omega_k r_i^k \geq \omega_k r_i^k, \] (5.38)
which leads to a contradiction. Therefore there must exist $k + 1$ ball centers $y_0, \ldots, y_k \in A_{r_i/5, 10r_i}(x_i)$ which are $\tau r_i$-independent points, as claimed.

Let us now remark on the main consequences of the existence of these $k + 1$ points. Note first that for each $y_j$ we have that $\theta_{10r_i}(y_j) > E - \eta''$, since by the construction of $U^a_i$ we have that $r_j \leq R^{-2}r_i$, and
Let us observe that once we have proved (5.1) we can apply Lemma 5.2, and whose energies are \( \eta'' \)-pinched. To exploit this, let us first apply Lemma 5.3 in order to conclude that for each \( x \in B_{\delta}(V) \) we have

\[
\theta_{n/10}(x) \geq \theta_{n/10}(y_j) - |\theta_{n/10}(x) - \theta_{n/10}(y_j)| \geq E - \eta'' - 10^{-1}\eta' > E - \eta'.
\] (5.39)

In particular, if we assume that \( x_i \) is such that \( \theta_{n/10}(x_i) < E - \eta' \), then we must have that \( r_i^{-1}d(x_j, V) \geq \delta = \delta(n, K_N, \Lambda, \eta') \).

Therefore, let us now assume \( \theta_{n/10}(x_i) < E - \eta' \), and thus to prove the Lemma we wish to find a contradiction. To accomplish this notice that we have our \( k + 1 \) points \( y_0, \ldots, y_k \in B_{10r_i} \) which are \( \tau r_i \)-independent and for which \( |\theta_{20r_i}(y_j) - \theta_{R^{-1}r_i}(y_j)| < \eta'' \). Therefore, by applying the cone splitting of Theorem 2.3 we have for each \( \epsilon' > 0 \) that if \( \eta'' \leq \eta''(n, K_N, \Lambda, \epsilon') \) then \( B_{10r_i}(x_i) \) is \( (k, \epsilon') \)-symmetric with respect to the \( k \)-plane \( V^k \). However, since \( d(x_i, V) > \delta r_i \), we have by Theorem 2.4 that if \( \epsilon' \leq \epsilon(n, K_N, \Lambda, \epsilon) \) then there exists some \( \tau' = \tau'(n, K_N, \Lambda, \epsilon) \) such that \( B_{\tau r_i}(x_i) \) is \( (k + 1, \epsilon) \)-symmetric. However, we can assume after a further increase that \( R = R(n, K_N, \Lambda, D, \epsilon) > 4r^{-1} \), and thus we have that \( \tau' r_i > 4R^{-1}r_i > 4r \). This contradicts that \( x_i \in S_{\epsilon, r} \), and thus we have contradicted that \( \theta_{n/10}(x_i) < E - \eta' \), which proves the Lemma.

### 5.4. Estimating \( U_+ \) in Lemma 5.1

Now we proceed to finish the proof of Lemma 5.1 by estimating the set \( U_+ \). First let us pick \( D' = D'(n) \equiv 2^{4n} D(n) \), where \( D(n) \) is from Theorem 3.4. Then for some \( \eta' \) fixed we can choose \( R \) as in Lemma 5.5. It is then enough to estimate each of the sets \( U^d_+ \), as there are at most \( N = N(n, K_N, \Lambda, \epsilon, \eta, \eta') \) pieces to the decomposition. Thus we will fix a set \( U^d_+ \) and focus on estimating the content of this set. Let us begin by observing that if \( r > 0 \) then we have the lower bound \( r_i \geq r \). Otherwise, let us fix any \( r > 0 \) and restrict ourselves to the collection of balls in \( U^d_+ \) with \( r_i \geq r \). The estimates we will prove in the end will be independent of \( r \), and thus by letting \( r \to 0 \) we will obtain estimates on all of \( U^d_+ \).

Now let us make the precise statement we will prove in this subsection. Namely, consider any of ball centers \( \{x_i\}_{i \in I} \) and any radius \( 2^{-3} r_i \leq r_0 \leq 2 \), where \( r_0 = 2^{-\alpha} \). Then we will show that

\[
\mu^d(B_{r}(x_i)) \leq 2^{4n} D(n)r^k_0.
\] (5.40)

Let us observe that once we have proved (5.40) then we have finished the proof of the Covering Lemma, as we will then have the estimate

\[
\sum r^k_i = \mu^d(B_2(x_i)) \leq C(n).
\] (5.41)

We prove (5.40) inductively on \( \alpha \). To begin notice that for each \( x_i \) if \( \beta \) is the largest integer such that \( 2^{-3} r_i \leq r_{\beta} \), then the statement clearly holds, by the definition of the measure \( \mu^d \). In fact, we can go further than this. For each \( x_i \), let \( r'_i \in [r_i/5, 10r_i] \) be the largest radius such that for all \( r_i \leq s \leq r'_i \) we have

\[
\mu^d(B_s(x_i)) \leq 2 \omega_k s^k.
\] (5.42)
In particular, we certainly have the much weaker estimate \( \mu^\alpha(B_s(x_i)) \leq 2^{4n} D(n)s^k \), and hence (5.40) is also satisfied for all \( 2^{-3} r_i \leq r_\beta \leq r_i' \). Notice that we then also have the estimate
\[
\omega_k r_i^k \leq \mu^\alpha(B_{r_i/8}(x_i)) \leq \mu^\alpha(B_{r_i^*}(x_i)) \leq 2\omega_k(r_i^*)^k.
\] (5.43)

Now let us focus on proving the inductive step of (5.40). Namely, assume \( \alpha \) is such that for all \( x_i \) with \( 2^{-3} r_i \leq r_{\alpha + 1} < 2 \) we have that (5.40) holds. Then we want to prove that the same estimate holds for \( r_\alpha \). Let us begin by seeing that a weak version of (5.40) holds. Namely, for any index \( i \in I^d \) and any radius \( r_\alpha \leq s \leq 8r_\alpha \), by covering \( B_s(x_i) \) by at most \( 2^8 \) balls \( \{ B_{r_{\alpha+1}}(y_j) \} \) of radius \( r_{\alpha+1} \) we have the weak estimate
\[
\mu^\alpha(B_s(x_i)) \leq \sum \mu^\alpha(B_{r_{\alpha+1}}(y_j)) \leq D'(n)s^k,
\] (5.44)
where of course \( D'(n) >> 2^{4n} D(n) \).

To improve on this, let us fix an \( i \in I^d \) and the relative ball center \( x_i \in U^*_+ \) with \( 2^{-3} r_i \leq r_\alpha \). Now let \( \{ x_j \}_{j \in J} = \{ x_i \}_{i \in I} \cap B_{r_\alpha}(x_i) \) be the collection of ball centers in \( B_{r_\alpha}(x_i) \). Notice first that if \( r_j' > 2r_\alpha \) for any of the ball centers \( \{ x_j \} \), then we can estimate
\[
\mu^\alpha(B_{r_\alpha}(x_i)) \leq \mu^\alpha(B_{2r_\alpha}(x_i)) \leq 2\omega_k(2r_\alpha)^k \leq 2^{4n} D(n)r_\alpha^k,
\] (5.45)
so that we may fairly assume \( r_j' \leq 2r_\alpha \) for every \( j \in J \). Now for each ball \( B_{r_j'}(x_j) \) let us define a new ball \( B_{r_j}(x_j) \) which is roughly equivalent, but will have some additional useful properties needed to apply the discrete Reifenberg. Namely, for a given ball \( B_{r_j'}(x_j) \), let us consider the two options \( r_j' < 10r_j \) or \( r_j' = 10r_j \). If \( r_j' < 10r_j \), then we let \( y_j \equiv x_j \) with \( r_j \equiv r_j \). In this case we must have that \( \mu^\alpha(B_{10r_j}(x_j)) > 2\omega_k(r_j)^k \), and thus we can apply Lemma 5.5 in order to conclude that \( \theta_{r_j/10}(y_j) \geq E - \eta' \). In the case when \( r_j' = 10r_j \) is maximal, let \( y_j \in B_{r_j}(x_j) \) be a point such that \( \theta_{r_j}(y_j) = E - \eta \), such a point exists by the definition of \( r_j \), and let \( r_j \equiv 9r_j \). In either case we then have the estimates
\[
\theta_{r_j/8}(y_j) \geq E - \eta',
\]
\[
\omega_k 8^{-k} r_j^k \leq \mu(B_{r_j/8}(y_j)) \leq \mu^\alpha(B_{r_j}(y_j)) \leq 4\omega_k r_j^k,
\]
\[
B_{r_j}(x_j) \subseteq B_{r_j}(y_j).
\] (5.46)

Now let us consider the covering \( U^*_+ \cap B_{r_\alpha}(x_i) \subseteq \bigcup B_{r_j}(y_j) \), and choose from it a Vitali subcovering
\[
U^*_+ \cap B_{r_\alpha}(x_i) \subseteq \bigcup B_{r_j}(y_j),
\] (5.47)
such that \( \{ B_{r_j/8}(y_j) \} \) are disjoint, where we are now being lose on notation and referring to \( \{ y_j \}_{j \in J} \) as the ball centers from this subcovering. Let us now consider the measure
\[
\mu' \equiv \sum \omega_k \left( \frac{r_j}{8} \right)^k \delta_{y_j}.
\] (5.48)
That is, we have associated to the disjoint collection \( \{ B_{r_j/8}(y_j) \} \) the natural measure. Our goal is to prove that
\[
\mu'(B_{r_\alpha}(x_i)) \leq D(n)r_\alpha^k.
\] (5.49)
Let us observe that if we prove (5.49) then we are done. Indeed, using (5.46) we can estimate

\[ \mu^a(B_{r_o}(x_i)) \leq \sum \mu^a(B_{r_j}(y_j)) \leq 4 \omega_k \sum r_j^k = 4 \cdot 8^k \mu'(B_{r_o}(x_i)) \leq 2^{4n} D(n) r_o^k, \]  

which would finish the proof of (5.40) and therefore the Lemma.

Thus let us concentrate on proving (5.49). We will want to apply the discrete Reifenberg in this case to the measure \( \mu' \). Let us begin by proving a weak version of (5.49). Namely, for any ball center \( y_j \) from our subcovering and radius \( r_j < s \leq 4r_o \) let us consider the set \( \{ z_k \} = \{ y_j \}_{j \in J} \cap B_s(y_j) \) of ball centers inside \( B_s(y_j) \). Since the balls \( \{ B_{\bar{r}_j/4}(z_k) \} \) are disjoint we have that \( \bar{r}_k \leq 8s \). Using this, (5.43), and (5.46) we can estimate

\[ \mu'(B_s(y_j)) = \sum_{z_k \in B_s(y_j)} \omega_k 8^{-k} \leq C(n) \sum_{z_k \in B_s(y_j)} \mu^a(B_{\bar{r}_j/8}(z_k)) \leq C(n) \mu^a(B_{2s}(y_j)) \leq C(n)s^k, \]  

where of course \( C(n) \gg 2^{4n} D(n) \).

Now let us finish the proof of (5.49). Thus let us pick a ball center \( y_j \in B_{r_o}(x_i) \) and a radius \( s < 4r_o \). If \( \mu'(B_s(y)) \leq \epsilon_k s^k \) then \( D_{\mu'}(y, s) \equiv 0 \) by definition, and if \( s \leq \bar{r}_j/4 \) then \( D_{\mu'}(y, s) \equiv 0 \), since the support of \( \mu' \) in \( B_{\bar{r}_j/4}(y) \) contains at most one point and thus is precisely contained in a \( k \)-dimensional subspace. In the case when \( s > \bar{r}_j/4 \) and \( \mu(B_s(y)) > \epsilon_k s^k \) we then have by Theorem 4.1 that

\[ D_{\mu'}(y, s) \leq C(n, K_N, \Lambda, r, \epsilon) s^{-k} \int_{B_s(y)} W_s(z) d\mu'(z). \]  

By applying this to all \( r \leq t \leq s \) we can estimate

\[ s^{-k} \int_{B_s(x)} D_{\mu'}(y, t) d\mu'(y) \leq C s^{-k} \int_{B_s(x)} r^{-k} \int_{B_s(y)} W_t(z) d\mu'(z) d\mu'(y) = C s^{-k} t^{-k} \int_{B_s(x)} \mu'(B_s(y)) W_t(y) d\mu'(y) \leq C s^{-k} \int_{B_s(x)} W_s(y) d\mu'(y), \]  

where we have used our estimate on \( \mu'(B_s(y)) \) from (5.51) in the last line. Let us now consider the case when \( t = r_{\beta} \leq s \leq 2r_o \). Then we can sum to obtain:

\[ \sum_{r_{\beta} \leq s} s^{-k} \int_{B_s(x)} D_{\mu'}(y, r_{\beta}) d\mu'(y) \leq C \sum_{r_{\beta} \leq r \leq s} s^{-k} \int_{B_s(x)} W_{r_{\beta}}(y) d\mu'(y) = C s^{-k} \int_{B_s(x)} \sum_{r_{\beta} \leq r \leq s} W_{r_{\beta}}(y) d\mu'(y) \leq C s^{-k} \int_{B_s(x)} \left| \theta_{r_{\beta}}(y) - \theta_{r_{\beta}}(y) \right| d\mu'(y) \leq C(n, K_N, \Lambda, \epsilon) \eta', \]  

(5.54)
where we are using (5.46) in the last line in order to see that $|\theta_{s}(y) - \theta_{i}(y)| \leq \eta'$. Now let us choose $\eta' \leq \eta'(n, K_N, \Lambda, \varepsilon)$ such that

$$\sum_{r_i \leq s} s^{-k} \int_{B_i(y)} D\psi'(|y, r_i)| \mathcal{d}\mu(y) \leq \delta^2,$$  \hspace{1cm} (5.55)

where $\delta$ is chosen from the discrete rectifiable-Reifenberg of Theorem 3.4. Since the estimate (5.55) holds for all $B_s \subseteq B_{2n}(x)$, we can therefore apply Theorem 3.4 to conclude the estimate

$$\mu(B_{r_n}(x_i)) \leq D(n)\rho^k.$$  \hspace{1cm} (5.56)

This finishes the proof of (5.46), and hence the proof of Lemma 5.1. \hfill $\square$

6. PROOF OF MAIN THEOREMS FOR STATIONARY HARMONIC MAPS

In this section we prove the main theorems of the paper concerning stationary harmonic maps. With the tools of Sections 3, 4, and 5 developed, we will at this stage mainly be applying the covering of Lemma 5.1 iteratively to arrive at the estimates. When this is done carefully, we can combine the covering lemma with the cone splitting in order to check that for $k$-a.e. $x \in S^k$ there exists a unique $k$-dimensional subspace $V^k \subseteq T_x M$ such that every tangent map of $f$ at $x$ is $k$-symmetric with respect to $V$.

For the proofs of the Theorems’ of this section let us make the following remark. For any $\delta > 0$ we can cover $B_1(p)$ by a collection of balls

$$B_1(p) \subseteq \bigcup_{i=1}^{N} B_{K_M\delta}(p_i),$$  \hspace{1cm} (6.1)

where $N \leq C(n, K_M)\delta^{-n}$. Thus if $\delta = \delta(n, K_M, K_N, \Lambda, \varepsilon)$ and we can analyze each such ball, then this gives us estimates on all of $B_1(p)$. In particular, by rescaling $B_{K_M^{-1}n}(p_i) \to B_1(p_i)$, we see that we can assume in our analysis that $K_M < \delta$ without any loss of generality. We shall do this throughout this section.

6.1. Proof of Theorem 1.3. Let $f : B_2(p) \subseteq M \to N$ be a stationary harmonic map satisfying (1.14) with $\int_{B_2(p)} |\nabla f|^2 \leq \Lambda$. With $\varepsilon, r > 0$ fixed, let us choose $\eta(n, K_N, \Lambda, \varepsilon) > 0$ and $\delta(n, K_N, \Lambda, \varepsilon) > 0$ as in Lemma 5.1. By the remarks around (6.1) we see that we can assume that $K_M < \delta$, which we will do for the remainder of the proof.

Now let us begin by first considering an arbitrary ball $B_s(x)$ with $x \in B_1(p)$ and $r < s \leq 1$, potentially quite small. We will use Lemma 5.1 in order to build a special covering of $B_s(x)$. Let us define

$$E_{s,s} \equiv \sup_{y \in B_s(x)} |\theta_i(y)|,$$  \hspace{1cm} (6.2)

and thus if we apply Lemma 5.1 with $\eta(n, K_N, \Lambda, \varepsilon)$ fixed to $B_s(x)$, then we can build a covering

$$S^k_{x_i} \cap B_s(x) \subseteq U_r \cup U_+ \equiv \bigcup B_r(x_i) \cup \bigcup B_r(x_i),$$  \hspace{1cm} (6.3)

with $r_i > r$. Let us recall that this covering satisfies the following:
\[ r^{k-n} \text{Vol}(B_r(U_r)) + \omega_k \sum r_i^k \leq C(n) \lambda^k. \]

(b) \[ \sup_{y \in B_{r_i}(x_i)} \theta_{i,j}(y) \leq E_{x,s} - \eta. \]

Now that we have built our required covering on an arbitrary ball \( B_r(x) \), let us use this iteratively to build our final covering of \( S_{k,0}(f) \). First, let us apply it to \( B_1(p) \) in order to construct a covering

\[
S_{k,0}(f) \subseteq U_1 \cup U_{1^+} = \bigcup B_r(x_i^1) \bigcup B_{r_1}(x_i^1),
\]

such that

\[
r^{k-n} \text{Vol}(B_r(U_{1^+})) + \omega_k \sum (r_i^1)^k \leq C(n, K_N, \Lambda, \epsilon),
\]

and with

\[
\sup_{y \in B_{r_1}(x_i^1)} \theta_{1^+}(y) \leq \Lambda - \eta.
\]

Now let us tackle the following claim, which is our main iterative step in the proof:

**Claim:** For each \( \ell \) there exists a constant \( C_\ell(\ell, n, K_M, K_N, \Lambda, \epsilon) \) and a covering

\[
S_{k,\ell}(f) \subseteq U_\ell \cup U_{\ell^+} = \bigcup B_r(x_i^\ell) \bigcup B_{r_\ell}(x_i^\ell),
\]

with \( r_\ell^i > r \), such that the following two properties hold:

\[
r^{k-n} \text{Vol}(B_r(U_{\ell^+})) + \omega_k \sum (r_i^\ell)^k \leq C_\ell(\ell, n, K_N, \Lambda, \epsilon),
\]

\[
\sup_{y \in B_{r_\ell}(x_i^\ell)} \theta_{i,\ell}(y) \leq \Lambda - \ell \cdot \eta.
\]

To prove the claim let us first observe that we have shown this holds for \( \ell = 1 \). Thus let us assume we have proved the claim for some \( \ell \), and determine from this how to build the covering for \( \ell + 1 \) with some constant \( C_{\ell+1}(\ell + 1, n, K_M, K_N, \Lambda, \epsilon) \), which we will estimate explicitly.

Thus with our covering determined at stage \( \ell \), let us apply the covering of (6.3) to each ball \( B_{r_\ell^i}(x_i^\ell) \) in order to obtain a covering

\[
S_{k,\ell} \cap B_{r_\ell^i}(x_i^\ell) \subseteq U_{i,r} \cup U_{i,+} = \bigcup B_r(x_{i,j}) \bigcup B_{r_{i,j}}(x_{i,j}),
\]

such that

\[
r^{k-n} \text{Vol}(B_r(U_{i,\ell})) + \omega_k \sum (r_{i,j})^k \leq C(n, K_N, \Lambda, \epsilon)(r_i^\ell)^k,
\]

\[
\sup_{y \in B_{r_{i,j}}(x_{i,j})} \theta_{i,\ell}(y) \leq \Lambda - (\ell + 1) \eta.
\]
Let us consider the sets
\[ U_{\ell + 1}^r \equiv U_\ell^r \bigcup_i U_{i,r} , \]
\[ U_+^{\ell + 1} \equiv \bigcup_{i,j} B_{r_{i,j}}(x_{i,j}) . \] (6.11)

Notice that the second property of (6.8) holds for \( \ell + 1 \) by the construction, hence we are left analyzing the volume estimate of the first property. Indeed, for this we combine our inductive hypothesis (6.8) for \( U_\ell \) and (6.10) in order to estimate
\[ r^{k-n} \text{Vol} \left( B_r \left( U_{\ell + 1}^r \right) \right) + \omega_k \sum_{i,j} (r_{i,j})^k \leq r^{k-n} \text{Vol} \left( B_r \right) \left( U_\ell^r \right) \] \[ + \sum_i \left( r^{k-n} \text{Vol} \left( B_r \left( U_{i,r} \right) \right) + \omega_k \sum_j (r_{i,j})^k \right) \]
\[ \leq C_\ell + C \sum_i (r_i^k) \]
\[ \leq C(n, K_M, K_N, \Lambda, \epsilon) \cdot C_\ell(\ell, n, K_M, K_N, \Lambda, \epsilon) \]
\[ \equiv C_{\ell + 1} . \] (6.12)

Hence, we have proved that if the claim holds for some \( \ell \) then the claim holds for \( \ell + 1 \). Since we have already shown the claim holds for \( \ell = 1 \), we have therefore proved the claim for all \( \ell \).

Now we can finish the proof. Indeed, let us take \( \ell = \lceil \eta^{-1} \Lambda \rceil = \ell(\eta, \Lambda) \). Then if we apply the Claim to such an \( \ell \), we must have by the second property of (6.8) that
\[ U_+^{\ell} \equiv \emptyset , \] (6.13)
and therefore we have a covering
\[ S_{\epsilon,r}^k \subseteq U_\ell^r = \bigcup_i B_r(x_i) . \] (6.14)

But in this case we have by (6.8) that
\[ \text{Vol} \left( B_r \left( S_{\epsilon,r}^k(f) \right) \right) \leq \text{Vol} \left( B_r \left( U_\ell^r \right) \right) \leq C(n, K_N, \Lambda, \epsilon)r^{n-k} , \] (6.15)
which proves the Theorem. \( \square \)

### 6.2. Proof of Theorem 1.4
There are several pieces to Theorem 1.4. To begin with, the volume estimate follows easily now that Theorem 1.3 has been proved. That is, for each \( r > 0 \) we have that
\[ S_{\epsilon}^k(f) \subseteq S_{\epsilon,r}^k(f) , \] (6.16)
and therefore we have the volume estimate
\[ \text{Vol} \left( B_r \left( S_{\epsilon}^k(f) \right) \right) \leq \text{Vol} \left( B_r \left( S_{\epsilon,r}^k(f) \right) \right) \leq C(n, K_M, K_N, \Lambda, \epsilon)r^{n-k} . \] (6.17)
In particular, this implies the much weaker Hausdorff measure estimate
\[ \lambda^k \left( S_{\epsilon}^k(f) \right) \leq C(n, K_M, K_N, \Lambda, \epsilon) , \] (6.18)
which proves the first part of the Theorem.

Let us now focus on the rectifiability of $S^k_{\epsilon}$. We consider the following claim, which is the $r = 0$ version of the main Claim of Theorem 1.3. We will be applying Lemma 5.1, which requires $K_M < \delta$. As in the proof of Theorem 1.3 we can just assume this without any loss, as we can cover $B_1(p)$ by a controlled number of balls of radius $K^2_M/\delta$, so that after rescaling we can analyze each of these balls with the desired curvature assumption. Thus let us consider the following:

**Claim:** If $K_M < \delta$, then for each $\ell$ there exists a covering $S^k_{\epsilon}(f) \subseteq U^\ell_0 \cup U^\ell_+ = U^\ell_0 \cup B^\ell_i(x^\ell_i)$ such that

1. $\lambda^k(U^\ell_0) + \omega_k \sum \langle r^k_i \rangle^k \leq C(\ell, n, K_N, \Lambda, \epsilon)$.
2. $U^\ell_0$ is $k$-rectifiable.
3. $\sup_{y \in B^\ell_i(x^\ell_i)} \theta^\ell_i(y) \leq \Lambda - \ell \cdot \eta$

The proof of the Claim follows essentially the same steps as those for the main Claim of Theorem 1.3. For base step $\ell = 0$, we consider the decomposition $S^k_{\epsilon} \subseteq U^0_0 \cup U^0_+$ where $U^0_0 = \emptyset$ and $U^0_+ = B_1(p)$.

Now let us assume we have proved the claim for some $\ell$, then we wish to prove the claim for $\ell + 1$. Thus, let us consider the set $U^\ell_+$ from the previous covering step given by

$$U^\ell_+ = \bigcup B^\ell_i(x^\ell_i).$$

Now let us apply Lemma 5.1 to each of the balls $B^\ell_i(x^\ell_i)$ in order to write

$$S^k_{\epsilon} \cap B^\ell_i(x^\ell_i) \subseteq U^0_{i,0} \cup U^0_{i,+} = U^0_{i,0} \cup \bigcup_j B^\ell_{i,j}(x_{i,j}),$$

with the following properties:

1. $\lambda^k(U^0_{i,0}) + \omega_k \sum \langle r^k_{i,j} \rangle^k \leq C(n, K_N, \Lambda, \epsilon, p) \langle r^k_i \rangle^k$,
2. $\sup_{y \in B^\ell_{i,j}(x_{i,j})} \theta^\ell_{i,j}(y) \leq \Lambda - (\ell + 1) \eta$,
3. $U^0_{i,0}$ is $k$-rectifiable.

Now let us define the sets

$$U^\ell_{i,0} = \bigcup U^0_{i,0} \cup U^\ell_0,$$

$$U^\ell_{i,+} = \bigcup B^\ell_{i,j}(x_{i,j}).$$

Conditions (2) and (3) from the Claim are clearly satisfied. We need only check condition (1). Using (a) and the inductive hypothesis we can estimate that
\[ \lambda^k(U_0^{\ell+1}) + \omega_k \sum_{i,j} (r_{i,j})^k \leq \lambda^k(U_0^\ell) + \sum_{i} \left( \lambda^k(U_{i,0}) + \omega_k \sum_j (r_{i,j})^k \right), \]
\[ \leq C_{\ell} + C(n, K_N, \Lambda, \epsilon) \sum_j (r_{i,j})^k \]
\[ \leq C(n, K_N, \Lambda, \epsilon) \cdot C_{\ell} \]
\[ \equiv C_{\ell+1}. \]  
(6.22)

Thus, we have proved the inductive part of the claim, and thus the claim itself.

Let us now finish the proof that \( S_k^x(f) \) is rectifiable. So let us take \( \ell = [\eta^{-1} \Lambda] = \ell(\eta, \Lambda) \). Then if we apply the above Claim to \( \ell \), then by the third property of the Claim we must have that
\[ U_+^\ell \equiv \emptyset, \]
and therefore we have the covering
\[ S_k^x \subseteq U_0^\ell, \]
where \( U_0^\ell \) is \( k \)-rectifiable with the volume estimate \( \lambda^k(U_0^\ell) \leq C \), which proves that \( S_k^x \) is itself rectifiable.

Finally, we prove that for \( k \) a.e. \( x \in S_k^x \) there exists a \( k \)-dimensional subspace \( V_x \subseteq T_x M \) such that every tangent map at \( x \) is \( k \)-symmetric with respect to \( V_x \). To see this we proceed as follows. For each \( \eta > 0 \) let us consider the finite decomposition
\[ S_k^x = \bigcup_{\alpha=0}^{\lfloor \eta^{-1} \Lambda \rfloor} W_{\epsilon, \eta}^{k, \alpha}, \]
where by definition we have
\[ W_{\epsilon, \eta}^{k, \alpha} \equiv \{ x \in S_k^x : \theta_0(x) \in [\alpha \eta, (\alpha + 1)\eta) \}. \]  
(6.26)

Note then that each \( W_{\epsilon, \eta}^{k, \alpha} \) is \( k \)-rectifiable, and thus there exists a full measure subset \( \tilde{W}_{\epsilon, \eta}^{k, \alpha} \subseteq W_{\epsilon, \eta}^{k, \alpha} \) such that for each \( x \in \tilde{W}_{\epsilon, \eta}^{k, \alpha} \) the tangent cone of \( W_{\epsilon, \eta}^{k, \alpha} \) exists and is a subspace \( V_x \subseteq T_x M \).

Now let us consider such an \( x \in \tilde{W}_{\epsilon, \eta}^{k, \alpha} \), and let \( V_x^k \) be the tangent cone of \( W_{\epsilon, \eta}^{k, \alpha} \) at \( x \). For all \( r \ll 1 \) sufficiently small we of course have \( |\theta_r(x) - \theta_0(x)| < \eta \). Thus, by the monotonicity and continuity of \( \theta \) we have for all \( r \ll 1 \) sufficiently small and all \( y \in W_{\epsilon, \eta}^{k, \alpha} \cap B_r(x) \) that \( |\theta_r(y) - \theta_0(y)| < 2\eta \). In particular, by Theorem 2.1 we have for each \( y \in W_{\epsilon, \eta}^{k, \alpha} \cap B_r(x) \) that \( B_r(y) \) is \((0, \delta_\eta)\)-symmetric, with \( \delta_\eta \to 0 \) as \( \eta \to 0 \). Now let us recall the cone splitting of Theorem 2.3. Since the tangent cone at \( x \) is \( V_x^k \), for all \( r \) sufficiently small we can find \( k + 1 \) points \( x_0, \ldots, x_k \in B_r(x) \cap W_{\epsilon, \eta}^{k, \alpha} \) which are \( 10^{-r} \)-independent, see Definition 2.2, and for which \( B_{2r}(x_j) \) are \((0, \delta_\eta)\)-symmetric. Thus, by the cone splitting of Theorem 2.3 we have that \( B_r(x) \) is \((k, \delta_\eta)\)-symmetric with respect to \( V_x^k \) for all \( r \) sufficiently small, where \( \delta_\eta \to 0 \) as \( \eta \to 0 \). In particular, every tangent map at \( x \) is \((k, \delta_\eta)\)-symmetric with respect to \( V_x \), where \( \delta_\eta \to 0 \) as \( \eta \to 0 \).
Now let us consider the sets
\[ \tilde{W}_{k,\eta}^\alpha = \bigcup_{\alpha} \tilde{W}_{k,\eta}^\alpha. \] (6.27)

So \( \tilde{W}_{k,\eta}^\alpha \subseteq S^k_{\epsilon,\eta} \) is a subset of full \( k \)-dimensional measure, and for every point \( x \in \tilde{W}_{k,\eta}^\alpha \) we have seen that every tangent map of is \( (k,\delta,\eta) \)-symmetric with respect to some \( V_x \subseteq T_x M \), where \( \delta,\eta \to 0 \) as \( \eta \to 0 \). Finally let us define the set
\[ \tilde{S}^k_{\epsilon} = \bigcap_j \tilde{W}_{k,\epsilon,\eta}^{j-1}. \] (6.28)

This is a countable intersection of full measure sets, and thus \( \tilde{S}^k_{\epsilon} \subseteq S^k_{\epsilon} \) is a full measure subset. Further, we have for each \( x \in \tilde{S}^k_{\epsilon} \) that every tangent map must be \( (k,\delta,\eta) \)-symmetric with respect to some \( V_x \subseteq T_x M \), for all \( \delta > 0 \).

In particular, every tangent map at \( x \) must be \( (k,0) = k \)-symmetric with respect to some \( V_x \). This finishes the proof of the Theorem. \( \Box \)

6.3. **Proof of Theorem 1.5.** Let us begin by observing the equality
\[ S^k(f) = \bigcup_{\epsilon > 0} S^k_{\epsilon}(f) = \bigcup_{\beta \in \mathbb{N}} S^k_{2^{-\beta}}(f). \] (6.29)

Indeed, if \( x \in S^k_{\epsilon}(f) \), then no tangent map at \( x \) can be \( (k + 1,\epsilon/2) \)-symmetric, and in particular \( k + 1 \)-symmetric, and thus \( x \in S^k(f) \). This shows that \( S^k_{\epsilon}(f) \subseteq S^k(f) \). On the other hand, if \( x \in S^k(f) \) then we claim there is some \( \epsilon > 0 \) for which \( x \in S^k_{\epsilon}(f) \). Indeed, if this is not the case, then there exists \( \epsilon_i \to 0 \) and \( r_i > 0 \) such that \( B_{r_i}(x) \) is \( (k + 1,\epsilon_i) \)-symmetric. If \( r_i \to 0 \) then we can pass to a subsequence to find a tangent map which is \( k + 1 \)-symmetric, which is a contradiction. On the other hand, if \( r_i > r > 0 \) then we see that \( B_r(x) \) is itself \( k + 1 \)-symmetric, and in particular every tangent map at \( x \) is \( k + 1 \)-symmetric. In either case we obtain a contradiction, and thus \( x \in S^k_{\epsilon}(f) \) for some \( \epsilon > 0 \). Therefore we have proved (6.29).

As a consequence, \( S^k(f) \) is a countable union of \( k \)-rectifiable sets, and therefore is itself \( k \)-rectifiable. On the other hand, Theorem 1.4 tells us that for each \( \beta \in \mathbb{N} \) there exists a set \( \tilde{S}^k_{2^{-\beta}}(f) \subseteq S^k_{2^{-\beta}}(f) \) of full measure such that
\[ \tilde{S}^k_{2^{-\beta}} \subseteq \{ x : \exists V^k \subseteq T_x M \text{ s.t. every tangent map at } x \text{ is } k \text{-symmetric wrt } V \}. \] (6.30)

Hence, let us define
\[ \tilde{S}^k(f) = \bigcup S^k_{2^{-\beta}}(f). \] (6.31)

Then we still have that \( \tilde{S}^k(f) \) has \( k \)-full measure in \( S^k(f) \), and if \( x \in \tilde{S}^k(f) \) then for some \( \beta \) we have that \( x \in \tilde{S}^k_{2^{-\beta}} \), which proves that there exists a subspace \( V \subseteq T_x M \) such that every tangent map at \( x \) is \( k \)-symmetric with respect to \( V \). We have finished the proof of the theorem. \( \Box \)
7. Proof of Main Theorems for Minimizing Harmonic Maps

In this section we prove the main theorems of the paper concerning minimizing harmonic maps. That is, we finish the proofs of Theorem 1.6 and Theorem 1.8. In fact, the proofs of these two results are almost identical, though the first relies on Theorem 1.4 and the later on Theorem 1.3. However, for completeness sake we will include the details of both.

7.1. Proof of Theorem 1.6. We wish to understand better the size of the singular set Sing(f) of a minimizing harmonic map. Let us recall that the $\epsilon$-regularity of theorem 2.8 tells us that if $f$ is minimizing, then there exists $\epsilon(n, K_N, \Lambda) > 0$ with the property that if $x \in B_1(p)$ and $0 < r < r(n, K_M, K_N, \Lambda)$ is such that $B_{2r}(x)$ is $(n-2, \epsilon)$-symmetric, then $r_f(x) \geq r$. In particular, $x$ is a smooth point, and we have for $\epsilon(n, K_M, K_N, \Lambda) > 0$ that

$$\text{Sing}(f) \cap B_1(p) \subseteq S^{n-3}_\epsilon(f).$$  \hspace{1cm} (7.1)

Thus by Theorem 1.4 there exists $C(n, K_M, K_N, \Lambda) > 0$ such that for each $0 < r < 1$ we have

$$\text{Vol}(B_r(\text{Sing}(f)) \cap B_1(p)) \leq \text{Vol}(B_r(S^{n-3}_\epsilon(f))) \leq Cr^3.$$  \hspace{1cm} (7.2)

This of course immediately implies, though of course is much stronger than, the Hausdorff measure estimate

$$\lambda^{n-3}(\text{Sing}(f) \cap B_1(p)) \leq C,$$  \hspace{1cm} (7.3)

which finishes the proof of the Theorem. \hfill $\square$

7.2. Proof of Theorem 1.8. We begin again by considering the $\epsilon$-regularity of theorem 2.8. This tells us that if $f$ is minimizing, then there exists $\epsilon(n, K_N, \Lambda) > 0$ with the property that if $x \in B_1(p)$ and $0 < r < r(n, K_M, K_N, \Lambda)$ is such that $B_{2r}(x)$ is $(n-2, \epsilon)$-symmetric, then $r_f(x) \geq r$. In particular, we have for such $\epsilon, r$ that

$$\{x \in B_1(p): r_f(x) < r\} \subseteq S^{n-3}_{\epsilon, r}(f).$$  \hspace{1cm} (7.4)

Thus by Theorem 1.4 there exists $C(n, K_M, K_N, \Lambda) > 0$ such that for each $0 < r < 1$ we have

$$\text{Vol}(B_r\{x \in B_1(p): r_f(x) < r\}) \leq \text{Vol}(B_r(S^{n-3}_{\epsilon, r}(f))) \leq Cr^3,$$  \hspace{1cm} (7.5)

which proves the second estimate of (1.20). To prove the first we observe that $|\nabla f|(x) \leq r_f(x)^{-1}$, and to prove (1.21) we use the remark after Definition 1.7 to conclude that $|\nabla^2 f|(x) \leq C(n, K_M, K_N)r_f(x)^{-2}$. This concludes the proof of the Theorem. \hfill $\square$

8. Future Directions

The underlying ideas of this paper are quite general, and are applicable in many areas. Papers in progress include versions of these estimates for minimal surfaces [NV15], mean curvature flow, $p$-harmonic maps, harmonic map flow, and lower and bounded Ricci curvature.
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