THE \((p,q)\) PROPERTY IN FAMILIES OF \(d\)-INTERVALS AND \(d\)-TREES

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Abstract. Given integers \(p \geq q > 1\), a family of sets satisfies the \((p,q)\) property if among any \(p\) members of it some \(q\) intersect. We prove that for any fixed integer constants \(p \geq q > 1\), a family of \(d\)-intervals satisfying the \((p,q)\) property can be pierced by \(O(d^{q-1})\) points, with constants depending only on \(p\) and \(q\). This extends results of Tardos, Kaiser and Alon for the case \(q = 2\), and of Kaiser and Rabinovich for the case \(p = q = \lceil \log_2(d+2) \rceil\). We further show that similar bounds hold in families of subgraphs of a tree or a graph of bounded tree-width, each consisting of at most \(d\) connected components, extending results of Alon for the case \(q = 2\). Finally, we prove an upper bound of \(O(d^{1/p-1})\) on the fractional piercing number in families of \(d\)-intervals satisfying the \((p,p)\) property, and show that this bound is asymptotically sharp.

1. INTRODUCTION

A homogenous \(d\)-interval is a union of \(d\) disjoint closed intervals on the real line. A separated \(d\)-interval is a union of \(d\) non-empty closed intervals, one on each of \(d\) fixed pairwise disjoint segments on the real line. It is easy to see that a family of separated \(d\)-intervals is, in particular, a family of homogeneous \(d\)-intervals. Thus, when discussing families of homogeneous \(d\)-intervals, we will sometimes omit the word ‘homogeneous’.

Given a finite family of (homogeneous, separated) \(d\)-intervals, we are interested in properties of the hypergraph \(H\) whose vertex set is the real line, and whose edges correspond to the \(d\)-intervals in the family, viewed as sets of points. We call \(H\) a hypergraph of (separated, homogeneous) \(d\)-intervals.

A matching in a hypergraph \(H\) with vertex set \(V\) and edge set \(E\) is a subset of disjoint edges in \(E\). A cover is a subset of \(V\) intersecting all edges. The matching number \(\nu(H)\) is the maximal size of a matching in \(H\), and the covering number \(\tau(H)\) is the minimal size of a cover in \(H\). The fractional relaxations of \(\nu\) and \(\tau\) are denoted, as usual, by \(\nu^*\) and \(\tau^*\), respectively. By linear programming duality, \(\nu \leq \nu^* = \tau^* \leq \tau\).
An old result of Gallai is that if $H$ is a hypergraph of intervals, then $	au(H) = \nu(H)$. Tardos [12] and Kaiser [8] used topological methods to prove that a hypergraph $H$ of homogenous $d$-intervals has $\frac{\tau(H)}{\nu(H)} \leq d^2 - d + 1$, and this bound improves to $\frac{\tau(H)}{\nu(H)} \leq d^2 - d$ in the separated case.

Another approach was taken by Alon [2] who proved a slightly weaker upper bound in the homogenous case, namely $\tau(H) \leq 2d^2$, by breaking his bound into two results: $\tau^* \leq 2d\nu$, and $\tau \leq d\tau^*$. In [1] examples for the sharpness of these two fractional bounds as well as an alternative proof of the latter are given. Matoušek [11] showed that the quadratic bound in $d$ on the ratio $\tau/\nu$ in hypergraphs of $d$-intervals is not far from being optimal: there are examples of intersecting hypergraphs of $d$-intervals in which $\tau = \nu = \Omega\left(\frac{d^2}{\log d}\right)$.

A hypergraph $H$ is said to satisfy the $(p, q)$ property if among every $p$ edges in $H$ some $q$ have a non-empty intersection. In this terminology Gallai’s theorem asserts that if an interval hypergraph $H$ satisfies the $(p, 2)$ property for some fixed integer constant $p$, then $\tau(H) = p - 1$. This result generalizes to all pairs of integer constants $p \geq q > 1$: by a theorem of Hadwiger and Debrunner from 1957 [7], if an interval hypergraph $H$ satisfies the $(p, q)$ property, then $\tau(H) \leq p - q + 1$.

Tardos’ and Kaiser’s results are that in any hypergraph of $d$-intervals $H$ satisfying the $(p, 2)$ property for some fixed integer constant $p$, $\tau(H) \leq (p - 1)(d^2 - d + 1)$. Kaiser and Rabinovich [9] proved that if a hypergraph $H$ of separated $d$-intervals satisfies the $(p, p)$ property for $p = \lceil \log_2(d + 2) \rceil$, then $\tau(H) \leq d$. Their proof does not apply in the non-separated case. Björner, Matoušek and Ziegler [5] then raised the question whether the $\lceil \log_2(d + 2) \rceil$ factor can be replaced by some constant bound on $p$ independent of $d$. Matoušek’s example shows that such a constant must be greater than 2. For a survey on the $(p, q)$ property in families of convex sets and $d$-intervals see [6].

Viewed as a discrete object, a family of $d$-intervals is a collection of subgraphs of a path $G$, each consisting of at most $d$ connected components. Alon [3] extended this setting to collections of subgraphs of a tree $G$. Let $G$ be a connected graph and let $H = H(G)$ be a collection of subgraphs of $G$, each consisting of at most $d$ connected components. If $G$ is a tree, we call $H(G)$ a hypergraph of $d$-trees. As before, $H(G)$ satisfies the $(p, q)$ property if in every $p$ edges of it there exists some $q$ edges sharing a common vertex. Alon proved that if a hypergraph of $d$-trees $H$ satisfies the $(p, 2)$ property for some fixed integer $p > 1$, then $\tau(H) \leq 2(p - 1)d^2$. In the case $G$ is an arbitrary graph of bounded tree-width $k$ (see definition in Section 7) satisfying the $(p, 2)$ property, he showed $\tau(H(G)) \leq 2(k + 1)(p - 1)d^2$. 


In this paper we extend the above mentioned results, by proving upper bounds on the piercing numbers in families of \(d\)-intervals or \(d\)-trees satisfying the \((p, q)\) property, for any two fixed integer constants \(p \geq q > 1\). In all our results we are interested in the asymptotic order of the piercing numbers as \(d\) goes to infinity, and make no attempt to optimize the involved constants depending on fixed \(p, q\).

For families of separated \(d\)-intervals we prove the following.

**Theorem 1.1.** If a hypergraph \(H\) of separated \(d\)-intervals satisfies the \((p, p)\) property for some fixed integer constant \(p > 1\), then \(\tau(H) \leq d^{\frac{p}{p-1}}\).

The proof of Theorem 1.1 uses a topological theorem due to Komiya [10] together with a counting argument generalizing [9]. In the case \(p = \lceil \log_2(d + 2) \rceil\) it produces \(\tau(H) \leq d\), as is obtained in [9].

For families of homogenous \(d\)-intervals we have:

**Theorem 1.2.** Let \(H\) be a hypergraph of homogeneous \(d\)-intervals satisfying the \((p, p)\) property, for some fixed integer constant \(p > 1\). Then

\[
\tau^*(H) < p^{\frac{1}{p-1}}d^{\frac{1}{p-1}} + 1, \quad \tau(H) < p^{\frac{1}{p-1}}d^{\frac{1}{p-1}} + d, \quad \text{and the bound on } \tau^*(H) \text{ is asymptotically sharp: for every integer } p > 1, \text{ there exists a hypergraph of homogeneous } d\text{-intervals } H \text{ satisfying the } (p, p) \text{ property with } \tau^*(H) = \Omega(d^{\frac{1}{p-1}}).
\]

**Theorem 1.3.** If a hypergraph \(H\) of homogeneous \(d\)-intervals satisfies the \((p, q)\) property for some fixed integer constants \(p \geq q > 1\), then

\[
\tau(H) \leq \max\left\{\frac{2^{\frac{1}{p-1}}(ep)^{\frac{q}{p}}}{q}d^{\frac{q}{p}} + d, \ 2p^2d\right\}.
\]

Our next two theorems extend the above to families of \(d\)-trees. Although Theorems 1.2 and 1.3 are special cases, their proofs are simpler and have geometric nature, and thus we consider them separately.

For families of homogeneous \(d\)-intervals we have:

**Theorem 1.4.** Let \(G\) be a tree, and let \(H = H(G)\) be a hypergraph of \(d\)-trees satisfying the \((p, p)\) property for some fixed integer constant \(p > 1\). Then \(\tau(H) < p^{\frac{1}{p-1}}d^{\frac{1}{p-1}} + d\).

**Theorem 1.5.** Let \(G\) be a tree, and let \(H = H(G)\) be a hypergraph of \(d\)-trees satisfying the \((p, q)\) property for some fixed integer constants \(p \geq q > 1\). Then

\[
\tau(H) \leq \max\left\{\frac{2^{\frac{1}{q-1}}(ep)^{\frac{q}{q-1}}}{q}d^{\frac{q}{q-1}} + d, \ 2p^2d\right\}.
\]
The proofs of Theorems 1.1, 1.2 and 1.3 are given in Sections 2, 3 and 4, respectively. In Section 5 we prove a lemma concerning collections of subtrees of a tree $G$, and the proofs of Theorems 1.4 and 1.5 are given in Section 6. In Section 7 we further extend our results to collections of subgraphs of an arbitrary graph $G$ of bounded tree-width.

**Theorem 1.6.** Let $G$ be a graph of tree-width at most $k$, and let $H$ be a collection of subgraphs of $G$, each consisting of at most $d$ connected components. If $H$ satisfies the $(p, q)$ property for some fixed integer constants $p \geq q > 1$, then

$$\tau(H) \leq (k + 1) \cdot \max \left\{ \frac{2 \exp (ep)^{q-1}}{q} d^{q-1} + d, \ 2p^2 d \right\}.$$ 

In our proofs we apply some of the methods used in [2, 3, 4, 1] together with additional counting arguments. In most cases we first prove that the fractional covering number $\tau^*(H)$ cannot exceed a certain bound, and then use Alon’s upper bound on the ratio $\tau(H)/\tau^*(H)$:

**Theorem 1.7** (Alon, [2, 3]). Let $G$ be a tree, and let $H = H(G)$ be a hypergraph of $d$-trees. Then $\tau(H) \leq d \tau^*(H)$. In particular, if $H$ is a hypergraph of $d$-intervals then $\tau(H) \leq d \tau^*(H)$.

2. **The $(p, p)$ property in families of separated $d$-intervals**

For a positive integer $d$, we denote by $[d]$ the set of integers $\{1, \ldots, d\}$. A $d$-uniform hypergraph $D = (V, E)$ is $d$-partite if its vertex set $V$ is a disjoint union of $d$ sets $V = V_1 \cup \cdots \cup V_d$, and every edge $e \in E$ has $|e \cap V_i| = 1$ for all $i \in [d]$. For every subset $S \subseteq V$, denote by $\chi_S \in \{0, 1\}^V$ the characteristic vector of $S$, namely, $\chi_S(v) = 1$ if $v \in S$, and 0 otherwise. A hypergraph $D = (V, E)$ is said to be balanced if there exist positive weights $\alpha_e, e \in E$, such that $\sum_{e \in E} \alpha_e \chi_e = \chi_V$.

For the proof of Theorem 1.1 we will need the following lemma, generalizing a lemma proved in [9].

**Lemma 2.1.** Let $D = (V, E)$ be a balanced $d$-partite hypergraph on vertex set $V = V_1 \cup \cdots \cup V_d$, where $|V_i| = t$ for all $i$. If every $\ell + 1$ edges of $H$ have at least $m$ vertices in common, then every $\ell$ edges of $H$ have at least $tm + 1$ vertices in common.

**Proof.** Assume to the contrary that there exists a set $A$ of $\ell$ edges in $E$ such that $|\bigcap A| \leq tm$. Write $C = \bigcap A$, and note that $|C \cap V_i| \leq 1$ for every $i \in [d]$. Let $I \subset [d]$ be the set of indices $i$ such that $|C \cap V_i| = 1$. Write $C = (\bigcup_{i \in I} V_i) \setminus C$. Since $|V_i| = t$ for all $i$, we have $|C| = (t - 1)|C|$. 

By the conditions of the lemma, for every edge \( e \notin A \), the set \( A \cup \{e\} \) have at least \( m \) vertices in common, and these vertices must be in \( C \). Since \( D \) is balanced this implies
\[
|C| \geq |C| \sum_{e \in A} \alpha_e + m \sum_{e \notin A} \alpha_e > m \sum_{e \notin A} \alpha_e.
\]
Therefore, from \( |C| \leq tm \) we get
\[
|\bar{C}| \leq (|C| - m) \sum_{e \notin A} \alpha_e \leq (t - 1)m \sum_{e \notin A} \alpha_e < (t - 1)|C|,
\]
a contradiction. \( \square \)

We will further need a topological theorem due to Komiya \([10]\). The application of Komiya’s theorem in the proof of Theorem \([11]\) will be similar to its application in the proof of Theorem 6.3 in \([1]\).

**Theorem 2.2** (Komiya, \([10]\)). Let \( P \) be a polytope, and for every face \( F \) of \( P \) choose a point \( q(F) \in F \) and an open set \( B_F \subseteq P \). If \( G \subseteq \bigcup_{F \subseteq G} B_F \) for every face \( G \) of \( P \), then there exists a collection \( Q \) of faces of \( P \) satisfying \( q(P) \in \text{conv}\{q(F) \mid F \in Q\} \) and \( \bigcap_{F \subseteq Q} B_F \neq \emptyset \).

We are now ready to prove Theorem \([11]\).

**Proof of Theorem \([11]\)** Since \( H \) is finite, we may assume that the vertex set of \( H \) is the union of \( d \) disjoint copies of open unit segment \((0, 1)\). Write \( k = \lfloor d^{\frac{1}{d-1}} \rfloor \). We apply Komiya’s theorem to the polytope \( P = \Delta_k \times \Delta_k \times \ldots \times \Delta_k \), the \( d \)-fold product of the \( k \)-dimensional simplex \( \Delta_k \) by itself. A point \( \bar{x} \in P \) has the form:
\[
\bar{x} = ((x_1^1, x_2^1, \ldots, x_k^1), (x_1^2, x_2^2, \ldots, x_k^2), \ldots, (x_1^d, x_2^d, \ldots, x_k^d)),
\]
where \( x_i^j \geq 0 \) and \( \sum_{i=1}^{k+1} x_i^j = 1 \) for every \( 1 \leq j \leq d \). For \( \bar{x} \in P \) and \( (i, j) \in [k + 1] \times [d] \) let \( p_\bar{x}(i, j) = \sum_{\ell \leq i} x_{\ell}^j \). A 0-dimensional face of \( P \) is a point \( \bar{y} \in P \) consisting of only 0 and 1 components.

For the application of Theorem \([2,2]\) define sets \( B_F \) and points \( q(F) \) for every face \( F \) of \( P \) as follows. For a 0-dimensional face \( \bar{y} \in P \) let \( B_{\bar{y}} \) be the set of all points \( \bar{x} \in P \) for which there exists a separated \( d \)-interval \( h \in H \) satisfying \( h^j \subseteq (p_\bar{x}(i^j(\bar{y}), j) - 1, p_\bar{x}(i^j(\bar{y}) + j)) \) for all \( j \), where \( h^j \) is the \( j \)-th interval component of \( h \), and \( i^j(\bar{y}) \in [k + 1] \) is the unique index \( i \) such that \( y_i^j = 1 \). For all other faces \( F \) of \( P \) let \( B_F = \emptyset \). Since every \( d \)-interval in \( H \) consists of closed interval components, the sets \( B_F \) are open. For every face \( F \) of \( P \) choose the point \( q(F) \) to be the barycenter of \( F \).

Assume to the contrary that \( \tau(H) > d^{\frac{1}{d-1}} \). Then for every \( \bar{x} \in P \), the set of points \( \{p_{\bar{x}}(i, j) \mid 1 \leq i \leq k, 1 \leq j \leq d\} \subseteq V(H) \) is not a cover.
for $H$. Hence $\bigcup B_{\vec{y}} = P$, where the union is over all 0-dimensional faces $\vec{y}$ of $P$. Moreover, for every face $F = \text{conv}(T)$, where $T$ is a set of 0-dimensional faces in $P$, we have $F \subseteq \bigcup\{B_{\vec{y}} \mid \vec{y} \in T\}$, because an empty segment cannot contain a non-empty interval component of a $d$-interval in $H$. Thus by Theorem 2.2, there exists a set $Q$ of 0-dimensional faces of $P$, such that $q(P) \in \text{conv}\{q(\vec{y}) \mid \vec{y} \in Q\}$ and $\bigcap_{\vec{y} \in Q} B_{\vec{y}} \neq \emptyset$.

Let $D$ be the $d$-partite hypergraph on vertex set $V = \bigcup_{j=1}^{d} V^j$, where each $V^j$ is a distinct copy of $[k+1]$, and with edge set $E = \{e_{\vec{y}} \mid \vec{y} \in Q\}$, where $e_{\vec{y}}$ is the subset of $V$ satisfying $\chi_{e_{\vec{y}}} = \vec{y}$. Then the fact that $q(F) \in \text{conv}\{q(\vec{y}) \mid \vec{y} \in Q\}$ means that $D$ is balanced. Moreover, since $H$ satisfies the $(p, p)$ property, $D$ must satisfy it too, that is, every $p$ edges of $D$ intersect at (at least) one vertex. Applying Lemma 2.1 $p - 1$ times, we obtain that every edge of $D$ intersects itself at more than $(k + 1)^{p-1} > d^{p-1} = d$ vertices. But this contradicts the fact that every edge of $D$ contains exactly $d$ vertices. □

3. The $(p, p)$ Property in Families of Homogeneous $d$-Intervals

Let $H$ be a hypergraph of homogeneous $d$-intervals. By performing small perturbations on the $d$-intervals of $H$, we may assume that no two endpoints of intervals in edges of $H$ coincide. We begin this section by proving a simple lemma which will play a crucial role in the sequel.

Lemma 3.1. If a finite family $\mathcal{I}$ of $k$ closed intervals on a line intersects, then there exist at least two pairs $(x, \mathcal{I} \setminus \{I\})$, where $x$ is an endpoint of the interval $I \in \mathcal{I}$ and $x$ lies in every interval $J \in \mathcal{I}$.

Proof. Let $K = \bigcap \mathcal{I}$. Then $K$ is a non-empty closed interval $[x_1, x_2]$, where $x_1, x_2$ are endpoints of (not necessarily distinct) intervals $I_1, I_2 \in \mathcal{I}$, respectively, and in particular, $x_1 \neq x_2$. Thus $(x_1, \mathcal{I} \setminus \{I_1\})$ and $(x_2, \mathcal{I} \setminus \{I_2\})$ are the two pairs in the lemma. □

Proof of Theorem 1.2 Let $f : H \to \mathbb{Q}^+$ be a maximal fractional matching. Let $r$ be a common denominator for the values of $f$. By duplicating edges and removing edges $h$ with $f(h) = 0$, we obtain a hypergraph $H'$ with the same relevant parameters as $H$, and with maximal degree $r$. Thus without loss of generality we may assume that $f(h) = \frac{1}{r}$ for all $h \in H$, where $r$ is the maximal size of a subset of edges in $H$ intersecting in a single point. Note that the removal and duplication of edges does not invalidate the $(p, p)$ property of $H$. In particular, we have $r \geq p$. Letting $n$ be the number of edges in $H$ we
have by LP duality that $\tau^*(H) = \nu^*(H) = \frac{n}{r}$, and thus we need to show that $\frac{n}{r} < p^{\frac{r-1}{r}} + 1$.

Since $H$ satisfies the $(p, p)$ property, it contains $\binom{n}{p}$ subsets of $p$ edges, each of which intersects at a vertex of $H$. By Lemma 3.1 each such subset contributes at least two different pairs $(x, \{h_1, \ldots, h_{p-1}\})$, where $x$ is an endpoint of an edge $h$ of $H$, the edges $h_1, \ldots, h_{p-1}$ of $H$ are distinct and all different from $h$, and $x$ lies in every $h_i$, $1 \leq i \leq p - 1$.

Since there are altogether at most $2dn$ possible choices for $x$, there is such a point $x$ that belongs to at least

$$X := \frac{2}{2dn} \left( \frac{n}{p} \right) = \frac{\prod_{i=1}^{p-1} (n-i)}{d \cdot p!} \geq \frac{(n-p+1)^{p-1}}{d \cdot p!}$$

different pairs $(x, \{h_1, \ldots, h_{p-1}\})$. If the number of edges $h \in H$ containing $x$ is $Z$, then, by the above, $\left( \frac{Z}{p-1} \right) \geq X$, from which it follows that

$$Z \geq (p-1)! \cdot X \lceil \frac{1}{p} \rceil \geq \frac{n-p+1}{(pd)^{\frac{1}{p-1}}}.$$

Thus, in particular,

$$r \geq \frac{n-p+1}{(pd)^{\frac{1}{p-1}}},$$

implying

$$\frac{n-p+1}{r(pd)^{\frac{1}{p-1}}} \leq 1.$$

Since $r \geq p$ this entails

$$\tau^*(H) = \frac{n}{r} \leq (pd)^{\frac{1}{p-1}} + \frac{p-1}{r} < (pd)^{\frac{1}{p-1}} + 1.$$

Now, by Theorem 1.7 we have $\tau(H) \leq d\tau^*(H)$, and therefore,

$$\tau(H) < (pd)^{\frac{1}{p-1}} d^{\frac{1}{p-1}} + d.$$

To see that the bound on $\tau^*(H)$ is asymptotically sharp, let $q$ be a prime power and let $P = \mathbb{P}^k(\mathbb{F}_q)$ denote the $k$-dimensional projective space over the field $\mathbb{F}_q$ of $q$ elements. Let $H = (V,E)$ be the hypergraph with vertex set $V = P$ whose edge set $E$ is the family of $(k-1)$-dimensional projective subspaces of $P$. Clearly, $H$ satisfies the $(k,k)$ property.

**Proposition 3.2.** $\tau^*(H) = q + \frac{1}{\sum_{i=0}^{k-1} q^i}$.

**Proof.** Since $H$ is a $d$-uniform $d$-regular hypergraph with $d = \frac{q^{k-1}-1}{q-1}$, the constant functions $f(e) = \frac{1}{d}$ for every $e \in E$ and $g(v) = \frac{1}{d}$ for
every $v \in V$ form a fractional matching and a fractional cover for $H$, respectively, of size $q + \frac{1}{\sum_{i=0}^{k-1} q^i}$ each. By linear programming duality, the proposition follows.

Since $H$ is a $d$-uniform hypergraph for $d = \frac{2^{k-1}}{q-1} = \sum_{i=0}^{k-1} q^i$, it can be realized as a family $F$ of $d$-intervals (where each interval is just a point in $\mathbb{R}$) with $\tau^*(F) = q + \frac{1}{\sum_{i=0}^{k-1} q^i} = \Omega(d^{\frac{1}{k-1}})$.

4. THE $(p, q)$-PROPERTY IN FAMILIES OF HOMOGENEOUS $d$-INTERVALS

**Theorem 4.1.** If a family $H$ of homogeneous $d$-intervals satisfies the $(p, q)$-property for some fixed integer constants $p \geq q > 1$, then

$$\tau^*(H) \leq \max \left\{ \frac{2^{1/p} \left( e \cdot p \right)^{q-1}}{q}, \frac{d^{1/p}}{q} + 1, \ 2p^2 \right\}.$$  

**Proof.** By removing and duplicating edges in $H$ we may assume that $H$ is a multiset $\{h_1, \ldots, h_n\}$ in which each $h_i$ has multiplicity $a_i$, with $m = \sum_{i=1}^{n} a_i$, and that $\tau^*(H) = \nu^*(H) = m/r$, where $r$ is the maximal number of edges in $H$ with a non-empty intersection. Then clearly $r \geq a_i$ for every $1 \leq i \leq n$. Moreover, since a subset of $p$ elements in $H$ in which no two elements are copies of the same edge must contain some $q$ elements that intersect, we have also $r \geq q$.

For $1 \leq i \leq n$ and $1 \leq j \leq a_i$ Denote by $h_{i,j}$ the $j$-th copy of $h_i$ in $H$. Let $T$ be the family of all subsets of cardinality $p$ of $H$ of the form $t = \{h_{i_{1,j_1}}, \ldots, h_{i_{p,j_p}}\}$, with $i_u \neq i_v$ for all $1 \leq u < v \leq p$. Since $r \geq a_i$ we have

$$|T| \geq \frac{1}{p!} m (m - r) (m - 2r) \cdots (m - (p-1)r) \geq \frac{1}{p!} m^p (1 - \frac{pr}{m})^p.$$  

The $(p, q)$-property implies that for every $t \in T$ there exists an intersecting subset $s \subset t$ of cardinality $q$. Moreover, $s$ is contained in at most $\binom{m-q}{p-q}$ members of $T$. It follows that the number of intersecting subsets of size $q$ in $H$ is at least

$$\frac{|T|}{\binom{m-q}{p-q}} \geq \binom{m}{q} \left( \frac{p}{q} \right) (1 - \frac{pr}{m})^p.$$  

By Lemma 3.1 each such intersecting subset admits at least two pairs $(x, \{f_1, \ldots, f_{q-1}\})$, where $x$ is an endpoint of an edge $f$ of $H$, $f_1, \ldots, f_{q-1}$ are $q - 1$ distinct edges of $H$, all different from $f$, and $x$
lies in every $f_i$, $1 \leq i \leq q - 1$. Since there are altogether $2dm$ endpoints of members in $H$, there must exist an endpoint $x$ that pierces at least

$$X := \frac{\binom{m}{q}}{dm \binom{p}{q}} \left(1 - \frac{pr}{m}\right)^p$$

subsets of cardinality $q - 1$ of $H$. If $\frac{m}{r} \geq 2p^2$, this implies that $x$ lies in at least

$$\left( (q - 1)! \cdot X \right)^{\frac{1}{q - 1}} \geq \left( \frac{(q - 1)! \binom{m}{q}}{dm \binom{p}{q}} \left(1 - \frac{pr}{m}\right)^p \right)^{\frac{1}{q - 1}}$$

$$\geq \frac{m - q + 1}{(dq \binom{p}{q})^{\frac{1}{q - 1}}} \left(1 - \frac{p^2r}{m}\right)^{\frac{1}{q - 1}}$$

$$\geq \frac{m - q + 1}{(2dq \binom{p}{q})^{\frac{1}{q - 1}}} \geq \frac{q(m - q + 1)}{(2e^q p^q d)^{\frac{1}{q - 1}}}$$

different edges in $H$. It follows that

$$\frac{q(m - q + 1)}{2 \frac{1}{q - 1} (ep)^{\frac{1}{q - 1}} d^{\frac{1}{q - 1}}} \leq r,$$

which together with $r \geq q$ entails

$$\tau^*(H) = \frac{m}{r} < \frac{2 \frac{1}{q - 1} (ep)^{\frac{1}{q - 1}}}{q} \cdot d^{\frac{1}{q - 1}} + 1,$$

as we wanted to show. \hfill \square

The proof of Theorem 1.3 now follows by combining the bound in Theorem 4.1 with the bound $\tau(H) \leq d\tau^*(H)$ given in Theorem 1.7.

5. A lemma on collections of subtrees

**Lemma 5.1.** Let $H$ be a finite family of $n$ (not necessarily distinct) subtrees of a tree $G$ and suppose that $\binom{H}{p}$ contains at least $k$ intersecting subsets. Then there exists a vertex in $G$ that lies in at least

$$\left( (p - 1)! \frac{k}{n} \right)^{\frac{1}{p - 1}} + 1$$

members of $H$.

**Proof.** Let $H'$ be a sub-family of $H$ with the property that every element in $H'$ belongs to at least $k/n$ different intersecting subsets of size $p$ in $H'$. Such a sub-family exists since we can greedily remove from $H$ elements belonging to less than $k/n$ different intersecting subsets, and since in each step less than $k/n$ intersecting subsets of $p$ elements are removed, this process cannot terminate with an empty set.
Choose an arbitrary vertex \( u \in V(G) \) and consider \( G \) as a tree rooted in \( u \). Let \( \text{dist}(u, v) \) denote the length of the path from \( u \) to a vertex \( v \in V(G) \). Then every tree \( h \in H' \) is rooted at a vertex \( v_h \in V(h) \) with the property that \( \text{dist}(u, v_h) = \min\{\text{dist}(u, x) \mid x \in V(h)\} \). Let \( h \in H' \) be such that \( \text{dist}(u, v_h) = \max\{\text{dist}(u, v_f) \mid f \in H'\} \). Then if \( h \) belongs to an intersecting subset \( T \) of size \( p \) of \( H' \), then every one of the \( p - 1 \) members of \( T \) that are different from \( h \) must contain \( v_h \). Since \( h \) belongs to at least \( k/n \) different such subsets \( T \), we conclude that \( v_h \) belongs to at least \( k/n \) different subsets of \( p - 1 \) elements in \( H' \), all of which do not contain \( h \). It follows that \( v_h \) belongs to at least \( \left( (p - 1)! \frac{k}{n} \right)^{\frac{1}{p-1}} \) members of \( H \) all different from \( h \), which proves the lemma. \( \square \)

6. The \((p, p)\) and \((p, q)\) Properties in Families of \(d\)-Trees

As before, we first prove bounds on the fractional piercing numbers.

**Theorem 6.1.** Let \( G \) be a tree, and let \( H = H(G) \) be a hypergraph of \(d\)-trees satisfying the \((p, p)\) property for some fixed integer constant \( p > 1 \), then
\[
\tau^*(H) < (pd)^{\frac{1}{p-1}} + 1.
\]

**Proof.** Let \( f : H \to \mathbb{Q}^+ \) be a maximal fractional matching. As in the proof of Theorem 1.2 by removing and duplicating edges if necessary, one may assume without loss of generality that \( f(h) = \frac{1}{r} \) for all \( h \in H \), where \( r \) is the maximal size of an intersecting subset of \( H \). In particular, \( r \geq p \). Letting \( n = |H| \) we have by LP duality \( \tau^*(H) = \nu^*(H) = \frac{n}{r} \), and thus our aim is to show that \( \frac{n}{r} < (pd)^{\frac{1}{p-1}} + 1 \).

By the \((p, p)\) property, \( H \) contains \( \binom{n}{p} \) subsets of \( p \) elements, each of which intersect at a vertex. Since each subgraph in \( H \) contains at most \( d \) connected components, the multiset \( H' \) of all subtrees that appear as a connected component in an element of \( H \) is of size at most \( nd \), and it contains at least \( \binom{n}{p} \) intersecting subsets of \( p \) elements each.

Applying Lemma 5.1 to \( H' \), we conclude that there exists a vertex in \( G \) that lies in at least
\[
\left( (p - 1)! \frac{\binom{n}{p}}{nd} \right)^{\frac{1}{p-1}} + 1 > \frac{n - p + 1}{(dp)^{\frac{1}{p-1}}}
\]
members of \( H \), implying
\[
\frac{n - p + 1}{(dp)^{\frac{1}{p-1}}} \leq r.
\]
Now, as in the proof of Theorem 1.2, it follows that
\[ \frac{n}{r} < (dp)^{\frac{1}{p-1}} + 1. \]

□

Combining Theorems 6.1 and 1.7, the proof of Theorem 1.4 follows. Similarly, the proof of Theorem 1.5 follows by combining Theorem 1.7 with the following:

**Theorem 6.2.** Let \( G \) be a tree, and let \( H = H(G) \) be a hypergraph of \( d \)-trees satisfying the \((p,q)\) property for some fixed integer constants \( p \geq q > 1 \), then
\[
\tau^*(H) \leq \max \left\{ \frac{2\tau^*(ep)^{\frac{q}{q-1}}}{q} \cdot d^{\frac{1}{p-1}} + 1, \ 2p^2 \right\}.
\]

The proof of Theorem 6.2 is very similar to that of Theorem 4.1 (with Lemma 5.1 playing the role of Lemma 3.1 there), and therefore is omitted. We conclude this section by remarking that the constants in Theorems 1.3 and 1.5 can be improved by taking more careful estimations, but we avoid from doing so for the sake of simplicity.

7. Graphs of bounded tree-width

A tree decomposition of a graph \( G = (V,E) \) is a tree \( T \) with vertices \( X_1, \ldots, X_n \), where each \( X_i \) is a subset of \( V \), satisfying the following properties:

(i) \( \bigcup_{i=1}^{n} X_i = V \),
(ii) If \( v \in X_i \cap X_j \), then \( v \in X_k \) for all vertices \( X_k \) in the path connecting \( X_i \) and \( X_j \) in \( T \).
(iii) For every edge \( uv \in E \), there exists \( 1 \leq i \leq n \) so that \( \{u,v\} \subseteq X_i \).

The width of a tree decomposition is \( \max_i \{|X_i|\} - 1 \). The tree-width of a graph \( G \) is the minimum width among all possible tree decompositions of \( G \). Families of graphs with bounded tree-width include the cactus graphs, pseudoforests, series-parallel graphs, outerplanar graphs, Halin graphs and more.

**Proof of Theorem 7.6.** Let \( T \) be a tree decomposition of \( G \) with vertex set \( V(T) = \{X_1, \ldots, X_n\} \), where \( X_i \subset V \) and \( |X_i| \leq k + 1 \) for all \( 1 \leq i \leq n \). For each subgraph \( h \in H \) let \( h' \) be the subgraph of \( T \) induced on all vertices \( X_i \) for which \( X_i \) contains a vertex of \( h \). Let \( H' = \{h' \mid h \in H\} \). Then \( H' \) satisfies the \((p,q)\) property and each
member of $H'$ has at most $d$ connected components. By Theorem 1.5, there exists a set $C \subset V(T)$ of cardinality at most

$$\max \left\{ \frac{2^{\frac{1}{q-1}} (ep)^{\frac{q}{q-1}}}{q} \cdot d^{\frac{q}{q-1}} + d, \ 2p^2d \right\}$$

that forms a cover of $H'$. Then the set $\bigcup C$ is a cover of $H$, and its cardinality is at most

$$(k + 1) \cdot \max \left\{ \frac{2^{\frac{1}{q-1}} (ep)^{\frac{q}{q-1}}}{q} \cdot d^{\frac{q}{q-1}} + d, \ 2p^2d \right\}.$$ 

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