THE LOCAL GROSS-PRASAD CONJECTURE OVER ARCHIMEDEAN LOCAL FIELDS

CHENG CHEN

Abstract. Following the approach of C. Mœglin and J.L.-Waldspurger, this article proves the generic cases of the local Gross-Prasad conjecture over \( \mathbb{R} \) and \( \mathbb{C} \).

Contents

1. Introduction 1
2. Local Gross-Prasad Conjecture 5
  2.1. Gross-Prasad triples 5
  2.2. Vogan \( L \)-packet 6
  2.3. The conjecture 8
3. Proof for the complex case 9
4. Representations in generic packets 11
5. Proof for the real case 15
  5.1. Main results 15
  5.2. Schwartz homology 17
  5.3. Reduction to codimension-one case 19
  5.4. Multiplicity formula: first inequality 25
  5.5. Multiplicity formula: second inequality 30
  5.6. An expedient: a trick using Schwartz homology 32

References 34

1. Introduction

This paper completes our project for the proof of the local Gross-Prasad conjecture over \( \mathbb{R} \) and \( \mathbb{C} \), which was formulated by B. Gross and D. Prasad in \([GP92]\) \([GP94]\) for all local fields of characteristic 0. The Gross-Prasad conjecture over a local field is one of the most important problems in the interaction between Representations of Local Groups and Number Theory via the local Langlands conjecture. Let us

2020 Mathematics Subject Classification. Primary 22E50 22E45; Secondary 20G20.
Key words and phrases. Gross-Prasad conjecture, Schwartz homology.
mention two recent important applications of the local Gross-Prasad conjecture. The paper [JZ20] takes it as an input local information in order to prove one direction of the global Gross-Prasad conjecture, and the paper [JLZ22] uses the local Gross-Prasad conjecture to develop the theory of Arithmetic Wavefront Sets for irreducible admissible representations of classical groups. In general, we refer to the ICM report of R. Beuzart-Plessis [BP22] for a general discussion of the arithmetic significance of the local Gross-Prasad conjecture.

Let $\mathbb{F}$ be a local field of characteristic 0, and $(W, V)$ be a pair of non-degenerate quadratic spaces over $\mathbb{F}$ such that the orthogonal complement $W^{\perp}$ of $W$ in $V$ is odd-dimensional and split. Let $G = \text{SO}(W) \times \text{SO}(V)$ and its subgroup $H = \Delta \text{SO}(W) \rtimes N$, where $\Delta \text{SO}(W)$ is the image of the diagonal embedding and $N$ is the unipotent group defined by a totally isotropic flag on $W^{\perp}$. Fix a generic character $\xi_N$ of $N(\mathbb{F})$ that uniquely extended to a character $\xi$ of $H(\mathbb{F})$. For every irreducible Casselman-Wallach representation $\pi$, we set

$$m(\pi) = \dim \text{Hom}_H(\pi, \xi).$$

It was proved in [AGRS10] [GGP12] [Wal12d] over non-archimedean fields and in [SZ12] [JSZ10] over archimedean fields that

$$m(\pi) \leq 1.$$ 

The local Gross-Prasad conjecture is a refinement of the behavior of $m(\pi)$ that takes all representations in a relevant Vogan $L$-packet into consideration.

The pure inner forms of $\text{SO}(W)$ are parameterized by $H^1(\mathbb{F}, \text{SO}(W)) = H^1(\mathbb{F}, H)$, which classifies quadratic spaces with the same dimension and discriminant of $W$. For every $\alpha \in H^1(\mathbb{F}, H)$, there is a pair of quadratic spaces $(W, V) = (W, W^{\perp})$ has parallel properties as $(W, V)$. We define $(G_{\alpha}, H_{\alpha}, \xi_{\alpha})$ accordingly. Moreover, the Langlands dual group $L G_{\alpha}$ of $G_{\alpha}$ is isomorphic to that of $G$. For every local $L$-parameter $\varphi : W_F \to L G$, we denote by $\Pi_{\varphi}(G)$ the corresponding $L$-packet of $G_{\alpha}$.

Following the work of D. Vogan ([Vog93]), we can define the Vogan $L$-packet associate to $\varphi$ as

$$\Pi_{\varphi}^{\text{Vogan}} = \bigsqcup_{\alpha \in H^1(\mathbb{F}, G)} \Pi_{\varphi}(G_{\alpha}).$$

It was conjecture by Vogan and known over archimedean local fields ([Vog93, Theorem 6.3]), that, fixing a Whittaker datum of $(G_{\alpha})_{\alpha \in H^1(\mathbb{F}, G)}$, there is a bijection

$$\pi \in \Pi_{\varphi}^{\text{Vogan}} \longleftrightarrow \chi_{\pi} \in \widehat{S}_{\varphi}$$

Here $\widehat{S}_{\varphi}$ is the set of character of component group $S_{\varphi}$ of $S_{\varphi}$, where $S_{\varphi}$ is the centralizer of the image $\text{Im}(\varphi)$ in the dual group $\widehat{G}$. Gross and Prasad suggested that
one may consider the relevant Vogan packet

\[ \Pi^\text{Vogan}_{\varphi,\text{rel}} = \bigsqcup_{\alpha \in H^1(F,G)} \Pi_{\varphi}(G_\alpha) \]

**Conjecture 1** ([GP92] [GP94]). The following two statements hold.

1. **(Multiplicity-one)** For every generic parameter \( \varphi \), we have

\[ \sum_{\pi \in \Pi^\text{Vogan}_{\varphi,\text{rel}}} \quad \] m(\( \pi \)) = 1

2. **(Epsilon-dichotomy)** Fix the Whittaker datum of \( \{G_\alpha\}_{\alpha \in H^1(F,G)} \) as [GP94, (6.3)]. The unique representation \( \pi \in \Pi^\text{Vogan}_{\varphi,\text{rel}} \) such that \( m(\pi) = 1 \) can be characterized as

\[ \chi_{\pi} = \chi_{\varphi} \]

where \( \chi_{\varphi} \) is defined in (2.3.2).

When \( F \) is non-archimedean, Conjecture 1 for tempered parameters was proved by Waldspurger ([Wal10] [Wal12a] [Wal12b] [Wal12c] [Wal12d]). Mœglin and Waldspurger completed the proof of Conjecture 1 for generic parameters based on the results in the tempered cases.

When \( F = \mathbb{R}, \mathbb{Z} \). Luo proved Conjecture 1(1) in [Luo21] following the work of R. Beuzart-Plessis in [BP19]. The author and Luo proved Conjecture 1(2) in [CL22] by a simplification of Waldspurger’s approach.

The main result of the paper is the following.

**Theorem 1.0.1.** When \( F = \mathbb{R} \) or \( \mathbb{C} \), Conjecture 1 holds for generic parameters.

The proof over \( \mathbb{C} \) is done by construction based on results in [GSS19].

The proof over \( \mathbb{R} \) follows from the strategy in [MW12], using a structure theorem (Proposition 4.0.5) for representations in generic packets and a multiplicity formula (Theorem 5.1.1), we can reduce all situations into the tempered cases.

We prove the structure theorem in Section 4. To prove the multiplicity formula, we apply parallel arguments as in [Xue20b] in Section 5.3 to reduce the multiplicity to the codimension-one cases. In the codimension-one case, we prove one inequality of the multiplicity formula using orbit analysis (Section 5.4). The proof for the other inequality is expected to be complete using harmonic analysis. The codimension-one case of the proof is written in Section 5.5. Other cases require more subtle harmonic analysis, which will be treated in future work. For completeness of the proof, in Section 5.6, we use a trick in [Xue20b] to give a complete proof for the second inequality.

One can find the non-archimedean counterpart for these steps (except Section 5.6) in [MW12, Section 2], [GGP12, Theorem 15.1], [MW12, Section 1.4-1.6] and
[MW12, Section 1.7-1.8]. Some modifications are necessary to fit in the setting of the archimedean cases.

It is worth mentioning that there is a parallel conjecture for unitary groups formulated by W. Gan, Gross, and Prasad. Over non-archimedean local fields, the conjecture for tempered parameters was treated by Beuzart-Plessis in [BP14] [BP16]; based on the tempered cases, Gan and A. Ichino proved the conjecture for generic parameters in [GI16]. Over archimedean local fields, Beuzart-Plessis proved the multiplicity-one part of the conjecture in [BP19] for tempered parameters using local trace formula and endoscopy. Xue complete the proof for tempered cases in [Xue20a] using theta correspondence and proved the generic cases in [Xue20b].

The unitary cases and special orthogonal cases are called the Bessel cases of the Gan-Gross-Prasad conjecture. There are also Fourier-Jacobi cases of the conjecture, namely, skew-unitary cases and symplectic-metaplectic cases.

Over non-archimedean local fields, Gan and Ichino proved in [GI16] the conjecture for skew-hermitian cases. In [Ato18], H. Atobe proved the conjecture for symplectic-metaplectic cases.

Over archimedean local fields, Xue proved the skew-unitary case in [Xue]. Z. Li and S. Wang are working on Xue’s approach for the symplectic-metaplectic cases. The Fourier-Jacobi cases of the conjecture are also treated in my ongoing working joint with R. Chen and J. Zou. We use a different approach based on a non-tempered reduction to the stable-range situations.

We also expect to prove a multiplicity formula (Conjecture 3) in all these situations, which also works for reducible representations obtained from parabolic induction. In particular, the unitary Bessel cases over archimedean local fields can be completed merely use the approach in this article. This conjecture is applied to the study of local descents in the ongoing project of D. Jiang, D. Liu, L. Zhang, and myself.

**Organization.** In Section 2, we fix conventions and notions and recall the statement of the local Gross-Prasad conjecture.

In Section 3, we work over the complex field \( \mathbb{C} \). We follow the observation in [GP92, §11] and prove the conjecture by constructing an explicit functional of the representation \( \pi_V \boxtimes \pi_W \) using the results in [GSS19].

In Section 4-5, we work over \( \mathbb{R} \). In Section 4, we use a sufficient condition for irreducibility to give a structure theorem of representations in generic packets. In Section 5, we follow the approach in [MW12] to prove the conjecture by reducing to the tempered cases. In Section 5.3, we modified Xue’s approach in [Xue20b] to fit the setting for special orthogonal groups and prove the reduction to the codimension-one cases. In the codimension-one case, we prove one inequality using representation theory and orbit analysis in Section 5.4. We prove the other inequality of the multiplicity formula using an approach in harmonic analysis in Section 5.5.
Acknowledgement. I would like to express my sincere thanks to my advisor Prof. Dihua Jiang for encouraging me to study this subject and giving me generous advice while I was working on this topic and writing this article. The work of this paper is supported in part by the Research Assistantship from the NSF grant DMS-1901802 and DMS-2200890. I thank Zhilin Luo for his patience during the collaboration for the proof of tempered cases. I thank Fangyang Tian for helpful discussions on a few technical issues related to Casselman-Wallach representations. I thank Chen Wan for important comments that leads my re-organization of Section 3. I thank Rui Chen and Jialiang Zou for their collaboration on the Fourier-Jacobi case which helps me work out Theorem 5.4.2 and understand the conjecture better. I am grateful to anonymous referees for giving me helpful suggestions to revise this paper.

2. Local Gross-Prasad Conjecture

In this section, we review the local Gross-Prasad conjecture over archimedean local fields following [GP92] and [GP94].

2.1. Gross-Prasad triples. In this section, we define Gross-Prasad triples over archimedean local fields following the definition of Gross-Prasad triples over \( \mathbb{R} \) in [Luo21, §6].

Let \( F = \mathbb{R} \) or \( \mathbb{C} \) and \((W, V)\) be a pair of non-degenerate quadratic spaces over \( F \). The pair \((W, V)\) is called admissible if and only if there exists an anisotropic line \( D \) and a split non-degenerate quadratic space \( Z \) over \( F \) such that

\[
V = W \oplus D \oplus Z
\]

In particular, for \( r = \frac{\dim V - \dim W - 1}{2} \), there exists a basis \( \{z_i\}_{i=\pm 1}^{\pm r} \) of \( Z \) such that

\[
q(z_i, z_j) = \delta_{i, -j}, \quad \forall i, j \in \{\pm 1, \ldots, \pm r\}
\]

where \( q \) is the quadratic form on \( V \).

Let \( P_V \) be the parabolic subgroup of \( \text{SO}(V) \) with \( P_V = M_V \cdot N \) stabilizing the following totally isotropic flag of \( V \)

\[
\langle z_r \rangle \subset \langle z_r, z_{r-1} \rangle \subset \cdots \subset \langle z_r, \ldots, z_1 \rangle
\]

Set \( G = \text{SO}(W) \times \text{SO}(V) \) and we identify \( N \) as a subgroup of \( G \) via \( \text{SO}(V) \hookrightarrow 1 \times \text{SO}(V) \) and \( \Delta \text{SO}(W) \) as the image of the diagonal embedding \( \text{SO}(W) \hookrightarrow G \). Then \( \Delta \text{SO}(W) \) acts on \( N \) by adjoint action of \( \text{SO}(W) \) and we set

\[
H = \Delta \text{SO}(W) \ltimes N.
\]

Define a morphism \( \lambda : N \to \mathbb{G}_a \) via

\[
\lambda(n) = \sum_{i=0}^{r-1} q(z_{-i-1}, nz_i), \quad n \in N.
\]
Then \( \lambda \) is \( \Delta \text{SO}(W) \)-conjugation invariant and hence \( \lambda \) admits a unique extension to \( H \) trivial on \( \Delta \text{SO}(W) \), which is still denoted by \( \lambda \). Let \( \lambda_F : H(F) \to F \) be the induced morphism on \( F \)-rational points. We define an unitary character of \( H(F) \) by

\[
\xi(h) = \lambda_F(h), \quad h \in H(F),
\]

where \( \psi \) is a fixed additive (unitary) character \( \psi \) of \( F \).

**Definition 2.1.1.** The triple \((G, H, \xi)\) is called the **Gross-Prasad triple** associated to the admissible \((W, V)\).

### 2.2. Vogan \( L \)-packet.

In this subsection, we are going to recall the notion of Vogan \( L \)-packets for special orthogonal groups over \( \mathbb{R} \) and \( \mathbb{C} \) following \([GP92]\) and \([Vog93]\).

Let \( L_F \) be the Weil group of the archimedean local field \( F \). By definition,

\[
L_C = \mathbb{C}^\times, \quad L_R = \mathbb{C}^\times \cup \mathbb{C}^\times \cdot j
\]

with \( j^2 = -1 \) and \( j \cdot z \cdot j = \bar{z} \) for \( z \in \mathbb{C}^\times \) (see \([Kna94]\)). In general, for any reductive algebraic group over \( \mathbb{R} \) or \( \mathbb{C} \), it was established by Langlands in \([Lan73]\) that there exists a bijective correspondence between local \( L \)-parameters for \( G \) and local \( L \)-packets \( \Pi^G(\varphi) \) consisting of a finite set of irreducible Casselman-Wallach (\([Cas89]\) \([Wal94]\)) representations of \( G \). A local \( L \)-parameter for \( G \), by definition, is a \( \hat{G} \)-conjugacy class of **admissible** homomorphisms

\[
\varphi : L_R \to \hat{L}G
\]

such that the elements in image \( \text{Im}(\varphi) \subset \hat{L}G \) are semi-simple. Here \( \hat{G} \), resp. \( \hat{L}G \) is the dual, resp. Langlands dual group of \( G \) (\([Lan70]\)). In particular, \( \varphi \) is called **tempered** if \( \text{Im}(\varphi) \) is bounded.

The pure inner forms of \( G \) share the same dual group, resp. Langlands dual group as \( G \). In particular, for any local \( L \)-parameter \( \varphi \) of \( G \), it can also be viewed as an \( L \)-parameter for any pure inner form of \( G \). Following \([Vog93]\), instead of working with a single \( L \)-packet \( \Pi^G(\varphi) \), one should work with the attached Vogan packet \( L \)-packet \( \Pi^{\text{Vogan}}(\varphi) \), which is the disjoint union of the \( L \)-packets for all pure inner forms of \( G \),

\[
\bigcup_{G'} \Pi^{G'}(\varphi)
\]

Here \( G' \) runs over the isomorphism classes of pure inner forms of \( G \).

A **Whittaker datum** \( \varpi \) for \( G \) is a triple \((G', B', \psi')\) where \( G' \) is a quasi-split pure inner form of \( G \), \( B' \) is a Borel subgroup of \( G' \), and \( \psi' \) is a generic character of the unipotent radical of \( B' \). An \( L \)-parameter \( \varphi \) of \( G \) is called **\( \varpi \)-generic** if there is a \( \varpi \)-generic representation in \( \Pi^{G'}(\varphi) \).

Now let us return to the special orthogonal groups situation.

Fix a non-degenerate quadratic space \((V, q)\) over \( F \). When \( F = \mathbb{C} \), there no non-trivial pure inner form of \( \text{SO}(V) \), so \( \Pi^{\text{Vogan}}(\varphi_V) = \Pi^{\text{SO}(V)}(\varphi_V) \) for every \( L \)-parameter.
$\varphi_V$ of $\text{SO}(V)$. When $F = \mathbb{R}$, the isomorphism classes of pure inner forms of $\text{SO}(V)$ are classified by the set $H^1(\mathbb{R}, \text{SO}(V))$, which in particular classifies quadratic spaces over $\mathbb{R}$ of the same dimension and discriminant as $V$ ([GP94, §8]). It is known that the quadratic spaces $V$ over $\mathbb{R}$ are classified by their signature $(p, q)$ with $p, q \in \mathbb{Z}_{\geq 0}$, where $p = PI(V)$ is the positive index and $q = NI(V)$ is the negative index of $V$.

The discriminant of the quadratic space $V$ with signature $(p, q)$ is given by

$$\text{disc}(V) = (-1)^{\left\lfloor \frac{\dim V}{2} \right\rfloor} \cdot (-1)^q \in \{\pm 1\} \simeq \mathbb{R}^\times / \mathbb{R}^\times 2.$$ 

Here $\left\lfloor \frac{\dim V}{2} \right\rfloor$ is the maximal integer smaller than or equal to $\frac{\dim V}{2}$. Let the attached special orthogonal group be $\text{SO}(p, q)$. By calculation the pure inner forms of $\text{SO}(p, q)$ are given by

$$\text{SO}(p_\alpha, q_\alpha)$$

with $p_\alpha + q_\alpha = p + q$ and $p \equiv p_\alpha \mod 2$.

Among all pure inner forms of $\text{SO}(V)$, there is a particular class called quasi-split (resp. split) pure inner forms, which are pure inner forms admitting Borel subgroups (resp. maximal split torus) defined over $\mathbb{R}$.

Two admissible pairs $(W, V)$ and $(W', V')$ are called relevant if and only if

(2.2.1) $\dim W = \dim W'$, $\text{disc}(W) = \text{disc}(W')$, $\dim V = \dim V'$, $\text{disc}(W) = \text{disc}(W')$.

Fix an admissible pair $(W, V)$ with attached Gross-Prasad triple $(G, H, \xi)$. For any $\alpha \in H^1(F, H) = H^1(F, \text{SO}(W))$, there is a unique relevant admissible pair $(W_\alpha, V_\alpha = W_\alpha \oplus W_\alpha)$ with Gross-Prasad triple $(G_\alpha, H_\alpha, \xi_\alpha)$ attached to it. For any local $L$-parameter $\varphi : \mathcal{L}_F \to \mathcal{L}^\times G$, define the relevant Vogan packet (for the spherical pair $(G, H)$) as follows

(2.2.2) $\Pi_{\text{Vogan}}^{\text{rel}}(\varphi) = \bigsqcup_{\alpha \in H^1(F, H)} \Pi_{\text{G}_\alpha}^{\text{rel}}(\varphi)$.

Completed tensor product.

**Definition 2.2.1.** Following the notion in Section 2.1, for representations $\pi_V, \pi_W$ of $G_V(\mathbb{R}), G_W(\mathbb{R})$ on Fréchet spaces $V_{\pi_V}, V_{\pi_W}$, we denote by $\pi_V \boxtimes \pi_W$ the $G_V(\mathbb{R}) \times G_W(\mathbb{R})$-representation on $V_{\pi_V} \hat{\otimes} V_{\pi_W}$, the projective tensor product of $V_{\pi_V}$ and $V_{\pi_W}$ ([War12, Appendix 2.2]), with the induced group action of $G_V(\mathbb{R}) \times G_W(\mathbb{R})$.

Given $L$-parameters $\varphi_V, \varphi_W$ of $G_V, G_W$, we denote by $\varphi_V \times \varphi_W$ be the $L$-parameter of $G_V \times G_W$ defined by compositing the diagonal map $\mathcal{L}_F \to \mathcal{L}_F \times \mathcal{L}_F$ with $(\varphi_V, \varphi_W)$. Concerning the category equivalence of the Harish-Chandra modules and Casselman-Wallach representations, we have an isomorphism

$$\Pi_{\varphi_V}(G_V) \times \Pi_{\varphi_W}(G_W) \to \Pi_{\varphi_V \times \varphi_W}(G_V \times G_W)$$
this induces an isomorphism
\[ \Pi_{\varphi_V} \times \Pi_{\varphi_W} \rightarrow \Pi_{\varphi_V \times \varphi_W} \]
is an isomorphism. Therefore, in the rest of the article, when we choose an element in \( \Pi_{\varphi_V \times \varphi_W} \), we will directly write it as \( \pi_V \boxtimes \pi_W \) for \( \pi_V \in \Pi_{\varphi_V} \) and \( \pi_W \in \Pi_{\varphi_W} \).

2.3. The conjecture. In this subsection, we first review the construction of the distinguished character defined in [GP92, §10], then recall the conjecture of Gross and Prasad formulated in [GP92] [GP94]. We restrict ourselves to the cases over an archimedean local field \( F \).

Non-degenerate pairing. From [Vog93, Theorem 6.3], the following statement holds:

(2.3.1) Given a reductive group \( G \) over \( F \) and a local \( F \)-parameter \( \varphi : L_\mathbb{R} \rightarrow L_G \), let \( S_\varphi = \pi_0(S_\varphi) \) where \( S_\varphi \) is the centralizer of the image of \( \varphi \) in \( \hat{G} \) and \( S_\varphi = \pi_0(S_\varphi) \) is the corresponding connected component. After fixing a Whittaker datum for \( G \) (i.e. a quasi-split pure inner form of \( G \) with a fixed Borel subgroup over \( \mathbb{R} \), and a generic character of the corresponding unipotent radical), there is a non-degenerate pairing \( \Pi_{\varphi_V} \times S_\varphi \rightarrow \{\pm 1\} \).

In particular, for a Gross-Prasad triple \((G,H,\xi)\) attached to an admissible pair \((W,V)\), after fixing a Whittaker datum for \( G = \text{SO}(W) \times \text{SO}(V) \), there is a non-degenerate bilinear pairing, there is a non-degenerate pairing \( \Pi_{\varphi_V} \times S_\varphi \rightarrow \{\pm 1\} \) for any local \( L \)-parameter \( \varphi = \varphi_V \times \varphi_W \) of \( G \). Therefore for every \( \pi \in \Pi_{\varphi_V} \), there is a unique character \( \chi_\pi : S_\varphi \rightarrow \{\pm 1\} \).

Distinguished character. In [GP92, §10], Gross and Prasad defined a distinguished character \( \chi_\pi = \chi_V^{\varphi_V} \times \chi_W^{\varphi_W} \) of \( S_\varphi = S_{\varphi_W} \times S_{\varphi_V} \). For every element \( s \in S_{\varphi_W} \times S_{\varphi_V} \), set

(2.3.2)
\[
\begin{align*}
\chi_V^{\varphi_V}(s_V) &= \det \left(-\text{Id}_{M_V^{s_V=-1}}\right)^{\frac{\dim M_V}{2}} \cdot \det \left(-\text{Id}_{M_W}\right)^{\frac{\dim M_W^{s_W=-1}}{2}} \cdot \varepsilon \left(\frac{1}{2}, M_V^{s_V=-1} \otimes M_W, \psi\right) \\
\chi_V^{\varphi_W}(s_W) &= \det \left(-\text{Id}_{M_W^{s_W=-1}}\right)^{\frac{\dim M_W}{2}} \cdot \det \left(-\text{Id}_{M_V}\right)^{\frac{\dim M_V^{s_V=-1}}{2}} \cdot \varepsilon \left(\frac{1}{2}, M_W^{s_W=-1} \otimes M_V, \psi\right)
\end{align*}
\]

Here \( M_V \) and \( M_W \) are the space of the standard representation of \( L\text{SO}(V) \) and \( L\text{SO}(W) \). The space \( M_V^{s_V=-1} \) is the \( s_V = (-1) \)-eigenspace of \( M_V \) and \( \varepsilon(\ldots) \) is the local root number defined by Rankin-Selberg integral ( [Jac09]).
Statement of Conjecture. Now let us recall the conjecture of Gross and Prasad. Let $\pi$ be an irreducible Casselman-Wallach representation of $G(\mathbb{R})$. Set

\[(2.3.3)\quad m(\pi) = \dim \text{Hom}_{H(\mathbb{R})}(\pi, \xi).\]

Following [SZ12] [JSZ10], it is known that

\[m(\pi) \leq 1.\]

The local Gross-Prasad conjecture studies the refinement behavior of the multiplicity $m(\pi)$ in a relevant Vogan $L$-packet.

Conjecture 2. Let $(G, H, \xi)$ be a Gross-Prasad triple attached to an admissible pair $(W, V)$ over $F$. Fix a generic local $L$-parameter $\varphi$ of $G$. Then the following statements hold

1. There exists a unique member $\pi_\varphi \in \Pi_{\text{Vogan}}(\varphi)$ such that $m(\pi_\varphi) = 1$.
2. Fix the Whittaker datum for $G$ as [GP94, (6.3)]. Based on (2.3.1), the character $\chi_{\pi_\varphi} : S_\varphi \rightarrow \{\pm 1\}$ attached to the unique member $\pi_\varphi$ is equal to $\chi_\varphi$ defined in (2.3.2).

When $F = \mathbb{C}$, since $\Pi_{\text{Vogan}}^\varphi$ contains only one element. Part (1) of the conjecture implies Part (2) of the conjecture. We will prove the following theorem by constructing a non-zero element in $\text{Hom}_{H}(\pi, \xi)$ in Section 3.

Theorem 2.3.1. When $F = \mathbb{C}$, Conjecture 2 holds.

When $F = \mathbb{R}$, Luo proved Part (1) of the conjecture for tempered parameters in [Luo21] following the work of Waldspurger ([Wal10] [Wal12b]) and Beuzart-Plessis ([BP19]). The author and Luo proved Part (2) of the conjecture for tempered parameters in [CL22] by simplifying Waldspurger’s approach ([Wal10] [Wal12a] [Wal12b] [Wal12c] [Wal12d]). The main result in Section 5 is to prove Theorem 5.1.1 that implies the following theorem based on the Conjecture 2 for tempered parameters.

Theorem 2.3.2. When $F = \mathbb{R}$, Conjecture 2 holds.

3. Proof for the complex case

The codimension-one case of the Conjecture 2 over $\mathbb{C}$ was proved by J. Möllers in [Mö17]. In this section, we give proof for Conjecture 2 over $\mathbb{C}$.

The following theorem is a generalization of [Mö17, Corollary 3.4].

Theorem 3.0.1. Given a spherical pair $(G, H)$ of reductive groups over an archimedean local field $F$ such that

\[(3.0.1)\quad (1) \text{ } G \text{ is quasi-split;}\]
(2) There exists a Borel subgroup $B$ of $G$ and unitary character $\xi$ of $H(F)$ such that

$$B \cap H = 1;$$

(3) $\xi = \psi \circ \lambda_F$ for an additive character $\psi$ of $F$ and an algebraic character $\lambda : H(F) \to F$.

Then for every multiplicative character $\sigma$ of $T(F)$, we have

$$\dim \text{Hom}_H(I^G_B(\sigma), \xi) \geq 1.$$

Proof. We define a function $f$ on the open double coset $B(F) \cdot H(F)$

$$f(bh) = \delta_B^{-1/2}(b)\sigma^{-1}(b)\xi(h) \quad \forall b \in B(F), \ h \in H(F).$$

Then we can express $f$ in the form of

$$f(bh) = t^{\mu_1} \mathcal{P}^{\sigma_2} e^{is_1 \text{Re}(\lambda(h)) + s_2 \text{Im}(\lambda(h))} \quad \forall b = t \cdot n \in B(F), \ h \in H(F)$$

for certain $s_1, s_2 \in \mathbb{R}$ and $\mu_1, \mu_2 \in \text{Hom}(T, \mathbb{G}_m)$.

It is not hard to verify that for every differential operator $D$ on $B(F) \times H(F)$, the growth of $|Df|$ can be controlled by a polynomial. Therefore, $f(bh)dbdh$ defines a tempered measure on $B(F) \cdot H(F)$, which is left-($B(F), \delta_B^{1/2}\sigma$)-equivariant and right-($H(F), \xi$)-equivariant. Because $B$ is solvable, from [GSS19, Theorem B], there exists a left-($B(F), \delta_B^{1/2}\sigma$)-equivariant and right-($H(F), \xi$)-equivariant distribution on $G(F)$.

From [DC91] and the compactness of $B(F)\backslash G(F)$, there is a one-to-one correspondence between $\text{Hom}(I^G_B(\sigma), \xi)$ and the space of left-($B(F), \delta_B^{1/2}\sigma$)-equivariant and right-($H(F), \xi$)-equivariant distributions on $G(F)$. \hfill \Box

Let us return to the Gross-Prasad conjecture over $F = \mathbb{C}$. From [GP92, §11], it suffice to verify $m(\pi) \geq 1$ for every principle series representation $\pi = I^G_B(\sigma)$. From Theorem 3.0.1, it suffices to verify (3.0.1) when $(G, H, \xi)$ is the Gross-Prasad triple associated to an admissible pair $(W, V)$. Part (1) and (3) of the assumption of Theorem 2.3.1 are straightforward, so we just verify Part (2).

Set $P_V = M_V \cdot N$ be the parabolic subgroup stabilizing the totally isotropic flag $(2.1.1)$ and the Levi subgroup $M_V$ can be decomposed as $M_V = \prod_{i=1}^r \text{GL}(\mathbb{C} \cdot z_i) \times \text{SO}(V \oplus D)$. Let $\overline{P_V} = M_V \cdot \overline{N}$ be the opposite parabolic subgroup of $P_V$.

Let $(G', H', \xi')$ be the Gross-Prasad triple associated to the admissible pair $(W, W \oplus D)$. From [Möll17, §6.2.4], there exists a Borel subgroup $B'$ of $G = \text{SO}(W \oplus D) \times \text{SO}(W)$ such that $B' \cap H' = 0$. We set $B = B' \cdot \prod_{i=1}^r \text{GL}(\mathbb{C} \cdot z_i) \cdot B' \cdot (\overline{N} \times 1)$. Consider the parabolic subgroup $P = P_V \times \text{SO}(W) = M \cdot (\overline{N} \times 1)$ of $G$. Since $\prod_{i=1}^r \text{GL}(\mathbb{C} \cdot z_i)B'$ and $H'$ are subgroups of $M = M_V \times \text{SO}(W)$ such that

$$\prod_{i=1}^r \text{GL}(\mathbb{C} \cdot z_i)B' \cap H' = 1$$
we have
\[ B \cap H = \mathbb{N} \cdot \prod_{i=1}^{r} \text{GL}(\mathbb{C} \cdot z_i)B' \cap H' \cdot N = \text{GL}(\mathbb{C} \cdot z_i)B' \cap H' = 1 \]

This completes the proof for Theorem 2.3.1.

4. Representations in generic packets

In this section, we study the structure of representations in the Vogan \( L \)-packets of a special orthogonal group following [MW12, §3]. With Proposition 4.0.5, we can describe elements in Vogan \( L \)-packets of generic \( L \)-parameters.

Let \( V \) be a non-degenerate quadratic space over \( \mathbb{R} \) and \( G_V = \text{SO}(V) \). It is well-known that an \( L \)-parameter \( \varphi_V \) of \( G_V \) is generic if and only if the adjoint \( L \)-function \( L(s, \varphi_V, \text{Ad}) \) is holomorphic at \( s = 1 \) ([GP92, Conjecture 2.6] and the remark after). Based on this property, we first compute an equivalent condition for \( \varphi_V \) to be generic.

**Definition 4.0.1.** Given a generic parameters \( \varphi_V : \mathcal{W}_\mathbb{R} \to \text{L}^1\text{SO}(V) \). We denote by \( \varphi^\text{ss}_V \) the **semisimplification** of \( \varphi_V \), that is, the semisimple representation on \( M_V \) defined by compositing \( \varphi_V \) with the standard representation \( \text{std}_V : \text{L}^1\text{SO}(V) \to \text{GL}(M_V) \).

Given an \( L \)-parameter \( \varphi_V \), its semi-simplification \( \varphi^\text{ss}_V \) can be decomposed as following
\[(4.0.1) \quad \varphi^\text{ss}_V = \bigoplus \cdot \cdot \cdot \bar{\varphi}^1_{lV,i} + \bigoplus \cdot \cdot \cdot \bar{\varphi}^2_{mV,i}.\]

Here \( \varphi^1_{lV,i} (l_{V,i} \in \mathbb{Z}) \) is an one-dimensional representation of \( \mathcal{W}_\mathbb{R} = \mathbb{C} \cup \mathbb{C} \cdot j (j^2 = -1) \) defined by
\[
\varphi^1_{lV,i}(z) = 1, \quad \varphi^2_{lV,i}(z \cdot j) = (-1)^{l_{V,i}}, \quad z \in \mathbb{C},
\]
and \( \varphi^2_{mV,i} (m_{V,i} \in \mathbb{N}) \) is the two-dimensional representation of \( \mathcal{W}_\mathbb{R} \) with basis \( u, v \) satisfying
\[
\varphi^2_{mV,i}(z) u = u, \quad \varphi^2_{mV,i}(z \cdot j) u = (-1)^{m_{V,i}} v, \\
\varphi^2_{mV,i}(z) v = v, \quad \varphi^2_{mV,i}(z \cdot j) v = u.
\]

The adjoint \( L \)-function \( L(s, \varphi_V, \text{Ad}) \) is a product of factors
\[
L(s, \varphi_V, \cdot \cdot \cdot \varphi^1_{lV,i} \otimes (\cdot \cdot \cdot \varphi^1_{lV,j}), L(s, \varphi_V, \cdot \cdot \cdot \varphi^1_{lV,i} \otimes (\cdot \cdot \cdot \varphi^2_{mV,j}), L(s, \varphi_V, \cdot \cdot \cdot \varphi^2_{mV,i} \otimes (\cdot \cdot \cdot \varphi^2_{mV,j}).
\]
From [KZ82], we can compute the value of these \( L \)-functions and obtain that
\[
(1) \quad L(s, \varphi_V, \cdot \cdot \cdot \varphi^1_{lV,i} \otimes (\cdot \cdot \cdot \varphi^1_{lV,j}) \) has a pole at \( s = 1 \) if and only if
\[
\frac{1+|l_{V,i}|-|l_{V,j}|}{2} \text{ is a non-positive integer.}
\]
(2) \(L(s, \varphi_V, \cdot | s_{V,i}^{1} \varphi_{m_{V,i}}^{1} \otimes (\cdot | s_{V,j}^{2} \varphi_{m_{V,j}}^{2})^\vee)\) has a pole at \(s = 1\) if and only if 
\[1 + s_{V,i}^{1} - s_{V,j}^{2} + \frac{m_{V,j}}{2}\] 
is a non-positive integer.

(3) \(L(s, \varphi_V, \cdot | s_{V,i}^{2} \varphi_{m_{V,i}}^{2} \otimes (\cdot | s_{V,j}^{1} \varphi_{m_{V,j}}^{1})^\vee)\) has a pole at \(s = 1\) if and only if 
\[1 + s_{V,i}^{2} - s_{V,j}^{1} + \frac{m_{V,i}}{2}\] 
is a non-positive integer.

(4) \(L(s, \varphi_V, \cdot | s_{V,i}^{1} \varphi_{m_{V,i}}^{1} \otimes (\cdot | s_{V,j}^{2} \varphi_{m_{V,j}}^{2})^\vee)\) has a pole at \(s = 1\) if and only if 
\[1 + s_{V,i}^{1} - s_{V,j}^{2} + \frac{m_{V,i} + m_{V,j}}{2}\] or 
\[1 + s_{V,i}^{2} - s_{V,j}^{1} + \frac{|m_{V,i} - m_{V,j}|}{2}\] 
is a non-positive integer.

Lemma 4.0.2. An parameter \(\varphi_V\) with semisimplification \(\varphi^{\text{ss}}_V\) in (4.0.1) is generic if and only if none of

\[
\frac{1 + s_{V,i}^{1} - s_{V,j}^{1} + \frac{1 - (-1)^{|V_i| + |V_j|}}{2}}{2}, 1 + s_{V,i}^{1} - s_{V,j}^{2} + \frac{m_{V,j}}{2},
\]

\[
1 + s_{V,i}^{2} - s_{V,j}^{1} + \frac{l_{V,i}}{2}, 1 + s_{V,i}^{2} - s_{V,j}^{2} + \frac{|m_{V,i} - m_{V,j}|}{2}
\]
is a non-positive integer.

Irreducibility criteria. B. Speh and D. Vogan gave a sufficient condition for the irreducibility of limits of generalized principal series representations in [SV80, Theorem 6.19]. We apply this result to prove the irreducibility of standard models for representations in generic packets.

Definition 4.0.3. Given \(\sigma_1 \in \Pi(\text{GL}_{n_1}), \cdots, \sigma_r \in \Pi(\text{GL}_{n_r})\) and \(\pi_{V_0} \in \Pi(\text{SO}(p,q))\).

We denote by

\[\sigma_1 \times \cdots \times \sigma_r \rtimes \pi_{V_0}\]
the normalized parabolic induction

\[\Pi_{P_{n_1}, \cdots, n_r, p+q}^{\text{SO}(p+n,q+n)}(\sigma_1 \otimes \cdots \otimes \sigma_r \otimes \pi_{V_0}) \in \Pi(\text{SO}(p+n,q+n)), \ n = n_1 + \cdots + n_r.\]

Lemma 4.0.4. Fix a generic parameter \(\varphi_V = \varphi_V^{\text{GL}} \oplus \varphi_{V_0} \oplus (\varphi_V^{\text{GL}})^\vee\) of \(\text{SO}(p,q)(p > q)\), for \(\sigma \in \Pi_{\varphi_V^{\text{GL}}}\) and \(\pi_{V_0} \in \Pi_{\varphi_{V_0}^{\text{Vogan}}}^{\text{Vogan}}\), the representation \(\sigma \rtimes \pi_{V_0}\) is irreducible.

Proof. From [KZ82, Theorem 14.2], we may write the tempered representation \(\pi_{V_0}\) as a parabolic induction from a limit of discrete series representations. Then we can express \(\sigma \rtimes \pi_{V_0}\) as

\[(4.0.2) \quad \sigma_1 \times \cdots \times \sigma_l \rtimes \pi_{V_0}' \quad \sigma_i \in \Pi(\text{GL}_{n_{V,i}})\]
where \(\pi_{V_0}' \in \Pi(\text{SO}(V_0'))\) is a limit of discrete series representation and

\[\sigma_i = \cdot |s_{V,i}^{1}, \text{sgn}^i|\] or \(\sigma_i = \cdot |s_{V,i}^{2}|D_{m_{V,i}}^+\).

Following [SV80, Theorem 6.19], it suffice to check the following conditions
For every root $\alpha$ such that 

$$n_\alpha = \langle \alpha, \nu \rangle / \langle \alpha, \alpha \rangle \in \mathbb{Z},$$

(1) if $\alpha$ is a complex root ($\alpha \neq -\theta \alpha$), then $\langle \alpha, \nu \rangle \langle \theta \alpha, \nu \rangle \geq 0$;

(2) if $\alpha$ is a real root ($\alpha = -\theta \alpha$), then

$$(-1)^{n_\alpha + 1} = \epsilon_\alpha \cdot \lambda(m_\alpha)$$

Here $\lambda$ is the central character of $\sigma$, $m_\alpha$ is the image of $\rho_\alpha(-I_2)$ in $G$ for the embedding $\rho_\alpha : \text{SL}_2(\mathbb{R}) \rightarrow G(\mathbb{R})$ determined by $\alpha$ and $\epsilon_\alpha = -1$.

Then we check them using Lemma 4.0.2.

(1) For every complex root $\alpha$ such that $n_\alpha \in \mathbb{Z}$

(a) If $\alpha$ is a root of $\text{SO}(p - q)$, then $\langle \alpha, \nu \rangle \langle \theta \alpha, \nu \rangle = 0$

(b) Otherwise, $\theta \alpha = \alpha$, then $\langle \alpha, \nu \rangle \langle \theta \alpha, \nu \rangle = \langle \alpha, \nu \rangle^2 \geq 0$

(2) For every real root $\beta_{ab} = e_a - e_b$ such that $n_{\beta_{ab}} \in \mathbb{Z}$.

(a) If $E_{aa}$ is in the $\text{GL}_1$-block $\text{GL}_{n_{V,i}}$ and $E_{bb}$ is the in a $\text{GL}_1$-block $\text{GL}_{n_{V,j}}$

(in the inducing datum in (4.0.2)), then $n_{\beta_{ab}} = \frac{s_{1_{V,i}} - s_{1_{V,j}}}{2}$ is an integer, and both

$$\frac{1 + s_{1_{V,i}} - s_{1_{V,j}} + \frac{1-(-1)^{l_{V,i}+l_{V,j}}}{2}}{2}, \quad \frac{1 + s_{1_{V,i}} - s_{1_{V,j}} + \frac{1-(-1)^{l_{V,i}+l_{V,j}}}{2}}{2}$$

are not non-positive integers. If $l_{V,i} + l_{V,j}$ is odd, the sum is equal to $2$, then $s_{1_{V,i}} = s_{1_{V,j}}$ or $s_{1_{V,i}} - s_{1_{V,j}}$ is odd. If $l_{V,i} + l_{V,j}$ is even, the sum is equal to $3/2$, then $s_{1_{V,i}} - s_{1_{V,j}}$ is even.

(b) If $E_{aa}$ is in the $\text{GL}_1$-block $\text{GL}_{n_{V,i}}$ and $E_{bb}$ is the in a $\text{GL}_2$-block $\text{GL}_{n_{V,j}}$,

Lemma 4.0.2 implies

$$s_{2_{V,j}} - \frac{l_{V,j}}{2} \leq s_{1_{V,i}} \leq s_{2_{V,j}} + \frac{l_{V,j}}{2}$$

(c) If $E_{aa}$ is in the $\text{GL}_2$-block $\text{GL}_{n_{V,i}}$ and $E_{bb}$ is the in a $\text{GL}_2$-block $\text{GL}_{n_{V,j}}$,

we may assume $l_{V,j} \geq l_{V,i}$, Lemma 4.0.2 implies

$$s_{2_{V,j}} - \frac{l_{V,j}}{2} \leq s_{2_{V,i}} - \frac{l_{V,i}}{2} \leq s_{2_{V,i}} + \frac{l_{V,i}}{2} \leq s_{2_{V,j}} + \frac{l_{V,j}}{2}$$

Therefore, we checked case (b)(c) following an understanding of the parity condition in [Pra17, Theorem 2]. For case (a), the parity holds unless $l_{V,i} + l_{V,j}$ is odd and $s_{1_{V,i}} = s_{1_{V,j}}$. In this situation

$$| \cdot | s_{1_{V,i}} \cdot \text{sgn}^{l_{V,i}} \times \cdot | s_{1_{V,j}} \cdot \text{sgn}^{l_{V,j}} = | \cdot | s_{1_{V,i}} \cdot \text{sgn}^{l_{V,j}} (1 \times \text{sgn})$$
And $1 \times \text{sgn}$ is the limit of discrete series representation with parameter $\varphi_0^2$ which can be treated in cases (b)(c).

Representations in generic packets. The classification of representations of $W_R([KZ82])$ shows the following factorization into irreducible representations

$$
\varphi^*_V = \varphi^*_V \oplus \varphi_{V_0} \oplus (\varphi^*_V)^V
$$

where $\varphi_{V_0}$ is tempered and

$$
\varphi^*_V = \bigoplus_{i=1}^{l_V} |s_i \varphi_{V,i}^* | \quad \text{where } \text{Re}(s_i) > 0 \text{ for } 1 \leq i \leq l_V
$$

for discrete series $\varphi_{V,i}$ (i.e. image of $\varphi_{V,i}$ is bounded and does not lie in any proper Levi).

It is straightforward that $\varphi^*_V$ is unpaired. Let $n_{V,i} = \dim \varphi^*_{V,i}$, $n_V = \dim \varphi^*_V$ and $\sigma_{V,i}$ be the unique representation of $GL_n$ in the $L$-packet $\Pi_{\varphi^*_V}(GL_{n_{V,i}})$, then

$$
\Pi_{\varphi^*_V}(GL_n) = \{ \sigma_V \} \quad \text{where } \sigma_V = | \det |^{s_1 \sigma_{V,1}} \times \cdots \times | \det |^{s_r \sigma_{V,r}}
$$

By Lemma 4.0.4, there is an injective map

$$
\Pi_{\varphi^*_{V_0}} \to \Pi_{\varphi^*_V}
$$

$$
\pi_{V_0} \mapsto \sigma_V \times \pi_{V_0}
$$

Since $\varphi^*_V$ is unpaired, $|S_{\varphi^*_V}| = |S_{\varphi^*_V}|$ and thus $|\Pi_{\varphi^*_V}| = |\Pi_{\varphi^*_V}|$. This implies that the above map is an isomorphism and we have the following result.

**Proposition 4.0.5.** For a generic $L$-parameter $\varphi_V = \varphi^*_V \oplus \varphi_{V_0} \oplus (\varphi^*_V)^V$, every representation $\pi_V$ in $\Pi_{\varphi^*_V}$ can be expressed as $\pi_V = \sigma_V \times \pi_{V_0}$ where $\pi_{V_0} \in \Pi_{\varphi^*_V}$ and $\sigma_V$ given in (4.0.5).

Proposition 4.0.5 shows that representations in the generic packets are in the following form.

$$
\pi_V = \sigma_V \times \pi_{V_0}, \quad \sigma_V = | \det |^{s_{V,1}} \rho_{V,1} \times \cdots \times | \det |^{s_{V,r}} \rho_{V,r}.
$$

where $\text{Re}(s_{V,1}) \geq \text{Re}(s_{V,2}) \geq \cdots \geq \text{Re}(s_{V,r}) > 0$, $\pi_{V_0} \in \Pi_{\text{irr}}^{\text{temp}}(SO(V_0))$. And $\rho_{V,i} = \text{sgn}^{l_{V,i}}$ for $l_{V,i} = 0, 1$ or $\rho_{V,i} = D_{m_{V,i}}$ for $m_i \in \mathbb{N}_+$.

For $\pi_V$ in the form of 4.0.7, we define the following notions.

**Definition 4.0.6.** The Harish-Chandra parameter for $\pi_V$ in (4.0.7) is defined as

$$(v_1, \cdots, v_r, v_{V_0})$$
where $v_{\pi_0}$ is the Harish-Chandra parameter of the tempered representation $\pi_0$, $v_i = s_i$ when $\rho_{V,i} = \text{sgn}^{\pi_{V,i}}$, and $v_i = (s_{V,i} + \frac{m_{V,i}}{2}, s_{V,i} - \frac{m_{V,i}}{2})$ when $\rho_{V,i} = D_{m_{V,i}}$.

**Definition 4.0.7.** We define the leading index of $\pi_V$ as the largest number among $\text{Re}(s_{V,i})$. We denote it by $LI(\pi_V)$.

5. **Proof for the real case**

In this section, we complete the proof of Conjecture 2 over the real field based on the tempered cases proved in [CL22, Theorem 2.3.2]. As in [MW12], there are mainly three important steps in the proof, namely, the reduction to codimension-one cases and two inequalities to prove the multiplicity formula. The reduction to codimension-one cases can be proved following [Xue20b]. One inequality of the multiplicity formula will be proved using representation theoretic results and orbit analysis, and the other inequality will be proved using harmonic analysis.

5.1. **Main results.** Let $(G,H,\xi)$ be a Gross-Prasad triple associated to an admissible pair $(W,V)$. One can find an $(2r + 2)$-dimensional split quadratic space $Z^+$ containing the orthogonal complement $W_\perp$ in $V$. Then we associate a codimension-one Gross-Prasad triple $(G^+,H^+,\xi^+)$ to the admissible pair $(V,W^+) = (V,Z^+ \oplus W)$. The following lemma relates the multiplicities of representations of $G$ to those of $G^+$.

We expect the following result, which is the real counterpart of [MW12, Proposition 1.3].

**Conjecture 3.** Given an admissible pair $(W,V)$. Let $\pi_V \in \Pi_{CW}(SO(V))$ and $\pi_W \in \Pi_{CW}(SO(W))$ ($\pi_V, \pi_W$ may be reducible). Moreover, we can express them into parabolic induction

$$\pi_V = | \cdot |^{s_{V,1}} \rho_{V,1} \times \cdots \times | \cdot |^{s_{V,r_V}} \rho_{V,r_V} \rtimes \pi_0$$

$$\pi_W = | \cdot |^{s_{W,1}} \rho_{W,1} \times \cdots \times | \cdot |^{s_{W,r_W}} \times \rho_{W,r_W} \rtimes \pi_0$$

as in (4.0.7). Then

$$m(\pi_V \boxtimes \pi_W) = m(\pi_V \rtimes \pi_W).$$

**Theorem 5.1.1.** Conjecture 3 holds when $\pi_V$ and $\pi_W$ are in generic $L$-packets.

**Proof for Theorem 2.3.2.** Given generic parameters $\varphi_V, \varphi_W$, from Proposition 4.0.5, for every $\pi_V \boxtimes \pi_W \in \Pi^\text{Vogan}_\varphi$, we can express both $\pi_V, \pi_W$ in the form of (4.0.7). If we identify the component group $S_{\varphi_{V_0} \times \varphi_{W_0}}$ with $S_{\varphi_V \times \varphi_W}$ via the isomorphisms

$$\Pi^\text{Vogan}_{\varphi_{V_0}} \rightarrow \Pi^\text{Vogan}_{\varphi_V} \quad \Pi^\text{Vogan}_{\varphi_{W_0}} \rightarrow \Pi^\text{Vogan}_{\varphi_W}$$

given by (4.0.6), it can be verified directly that $\chi_{\varphi_{V_0} \times \varphi_{W_0}} = \chi_{\varphi_V \times \varphi_W}$. Then, with Theorem 5.1.1, we reduce Conjecture 2 to the tempered cases, which was proved in [CL22]. \qed
Then we introduce the details for the proof of Theorem 5.1.1.

**Definition 5.1.2.** Let \( s_1, s_2, \ldots, s_{r+1} \) be complex numbers. We say that \( \underline{s} = (s_1, \ldots, s_{r+1}) \) are in general position, if \( \underline{s} \in \mathbb{C}^{r+1} \) does not lie in the set of zeros of countably many polynomial functions on \( \mathbb{C}^{r+1} \).

**Lemma 5.1.3.** For \( \underline{s} = (s_1, \ldots, s_{r+1}) \in \mathbb{C}^{r+1} \) in general positions, the spherical principal series representation \( \sigma_{\underline{s}} = |\cdot|^{s_1} \times \cdots \times |\cdot|^{s_{r+1}} \) satisfy

\[
m(\pi_V \boxtimes \pi_W) = m((\sigma_{\underline{s}} \rtimes \pi_W) \boxtimes \pi_V)
\]

for every \( \pi_V \in \Pi_{CW}(SO(V)) \) and \( \pi_W \in \Pi_{CW}(SO(W)) \).

With this lemma, we find such a spherical principal series \( \sigma_{\underline{s}} \) and reduce Theorem 5.1.1 to the case for admissible pair \((V, W \oplus Z^+)\) and representations \( \sigma_{r+1} \rtimes \pi_W, \pi_V \) that can be expressed in the parabolic induction form as in (4.0.7), which is a codimension-one case.

**Proposition 5.1.4.** Given a codimension-one admissible pair \((W, V)\). Let \( \pi_V \in \Pi_{CW}(SO(V)) \) and \( \pi_W \in \Pi_{CW}(SO(W)) \) in generic packets and we can express them in the following form

\[
\pi_V = |\cdot|^{s_{V,1}} \rho_{V,1} \times \cdots \times |\cdot|^{s_{V,r_V}} \rho_{V,r_V} \rtimes \pi_{V_0}
\]

\[
\pi_W = |\cdot|^{s_{W,1}} \rho_{W,1} \times \cdots \times |\cdot|^{s_{W,r_W}} \rho_{W,r_W} \rtimes \pi_{W_0}
\]

as in (4.0.7). Then

\[
m(\pi_V \boxtimes \pi_W) = m(\pi_{V_0} \boxtimes \pi_{W_0}).
\]

We will build up the first inequality \( m(\pi_V \boxtimes \pi_W) \geq m(\pi_{V_0} \boxtimes \pi_{W_0}) \) in Section 5.4 using mathematical induction. The building blocks (induction steps) are the following proposition proved by orbit analysis and some representation theory.

**Proposition 5.1.5.** Let \( \pi_V \) in generic packet and \( \pi_W \in \Pi_{CW}(SO(W)) \).

1. When \( \dim V = \dim W + 1 \) and \( \text{Re}(s) \geq \text{LI(}\pi_V) \),

\[
m(\pi_V \boxtimes \pi_W) \geq m((|\cdot|^{-s} \rho_{V,1} \times \pi_W) \boxtimes \pi_V)
\]

2. When \( \dim V = \dim W + 3 \) and \( \text{Re}(s) \geq \text{LI(}\pi_V) \),

\[
m(\pi_V \boxtimes (|\cdot|^{-s} \rho_{V,1} \times \pi_W)) \geq m((|\cdot|^{-s+1} \rho_{W,1} \times \pi_W) \boxtimes \pi_V)
\]

To prove the second inequality \( m(\pi_V \boxtimes \pi_W) \leq m(\pi_{V_0} \boxtimes \pi_{W_0}) \), we need some building blocks. In general, we expect the following inequality.

**Conjecture 4.** Let \( \pi_V \in \Pi_{CW}(SO(V)) \), \( \pi_W \in \Pi_{CW}(SO(W)) \) and \( \sigma \) is a generic representation in \( \text{GL}(X^+) \)

\[
m(\pi_V \boxtimes \pi_W) \leq m((\sigma \times \pi_W) \boxtimes \pi_V)
\]
In Section 5.5, we prove Conjecture 4 when \( \dim V = \dim W + 1 \). In Section 5.6 we prove Conjecture 4 when \( \dim V = \dim W + 3 \), \( \sigma = |\det |^{s}D_{m} \) and \( \text{Re}(s) + \frac{m}{2} \geq \text{LC}(\pi_{V}) \) (Definition 5.6.1). With these results, we give proof for the second inequality in Section 5.6. The general cases for Conjecture 4 will be treated in my future work using more subtle harmonic analysis, which will give a complete proof for Conjecture 3.

5.2. Schwartz homology. In this section, we introduce the main tool for the proof of Lemma 5.1.3, the Schwartz homologies. We refer the readers to [CS20] and [Xue20b, §2] for more details.

Schwartz induction. To better describe the analytic theory of smooth Fréchet \( G(\mathbb{R}) \)-representations of moderate growth, we use the notions of almost linear groups ([Sun15, Definition 1.1]), Nash manifolds ([Sun15, Definition 2.1]), tempered vector bundles and Schwartz sections ([CS20, Definition 6.1]).

Let \( G \) be a linear algebraic group over \( \mathbb{R} \), then by definition, \( G(\mathbb{R}) \) can be treated as an almost linear Nash group. We work in the categories of smooth Fréchet \( G(\mathbb{R}) \)-representations of moderate growth. We denote by \( \Pi_{\text{FM}}(G) \) the objects in this category. We denote by \( \Pi_{\text{CW}}(G) \) the set of Casselman-Wallach representations of \( G \), and we use \( \Pi_{\text{irr}}^{\text{CW}}(G) \) to denote the irreducible Casselman-Wallach representations of \( G \).

**Proposition 5.2.1.** For \( \pi \in \Pi_{\text{CW}}(G) \), the projective tensor product \( \hat{\otimes} \pi \) is an exact functor in \( \Pi_{\text{FM}}(G) \).

**Proof.** From [BK14], \( \pi \) is nuclear and the proposition follows from [CHM00, Lemma A.3]. \( \square \)

Let \( H \) be an algebraic subgroup of \( G \) and \( \pi_{H} \in \Pi_{\text{FM}}(H(\mathbb{R})) \). Following [Xue20b, Section 2.1], we denote by \( H(\mathbb{R})\backslash(G(\mathbb{R}) \times \pi_{H}) \) the vector bundle over \( H(\mathbb{R})\backslash G(\mathbb{R}) \) obtained by \( G(\mathbb{R}) \times \pi_{H} \) quotient by left \( H(\mathbb{R}) \)-action \( (h, (g, v)) = (h \cdot g, \pi_{H}(h).v) \) for \( h \in H(\mathbb{R}) \), \( g \in G(\mathbb{R}) \) and \( v \in \pi_{H} \) and this vector bundle is tempered. We define the **Schwartz induction** as the functor

\[
\text{Ind}_{P}^{S,G} : \Pi_{\text{FM}}(H(\mathbb{R})) \to \Pi_{\text{FM}}(G(\mathbb{R}))
\]

\[\pi_{H} \mapsto \Gamma^{S}(H(\mathbb{R})\backslash G(\mathbb{R}), \pi_{H})\]

where \( \Gamma^{S}(H(\mathbb{R})\backslash G(\mathbb{R}), \pi_{H}) \) stands for the space of Schwartz sections over the tempered vector bundle \( H(\mathbb{R})\backslash(G(\mathbb{R}) \times \pi_{H}) \). In particular, when \( G \) is reductive, for a parabolic subgroup \( P \) of \( G \), the Schwartz induction \( \text{Ind}_{P}^{S,G} \) coincides with the smooth induction, and we denote by \( I_{P}^{G} \) the normalized induction \( \text{Ind}_{P}^{S,G}(\delta_{P}^{1/2}) \), where \( \delta_{P} \) is the modular characters of \( P(\mathbb{R}) \). We will use the following properties of Schwartz inductions.
Proposition 5.2.2.  
(1) ([CS20, Proposition 7.1]) \( \text{Ind}^{S,G}_H \) is an exact functor from \( \Pi_{\text{FM}}(H) \to \Pi_{\text{FM}}(G) \).

(2) ([CS20, Proposition 7.2]) For algebraic subgroup \( H' \) of \( H \), then
\[
\text{Ind}^{S,G}_H \circ \text{Ind}^{S,H}_{H'} = \text{Ind}^{S,G}_{H''}.
\]

(3) ([CS20, Proposition 7.4] [BK14]) \( \pi_G \in \Pi_{\text{CW}}(G) \) and \( \pi_H \in \Pi_{\text{FM}}(H) \), then
\[
\text{Ind}^{S,G}_H(\hat{\pi}_H \otimes \pi_G |_H) = \text{Ind}^{S,G}_H(\pi_H) \hat{\otimes} \pi_G.
\]

Definitions. For \( V \in \Pi_{\text{FM}}(G) \), the Schwartz homology \( H^S_i(G,V) \) is defined to be the left derived functors of the \( G(\mathbb{R}) \)-coinvariant functor \( V \mapsto V_G \), where \( V_G = V/\sum_{g \in G} (g-1)V \). In particular, \( H^S_0(G,V) = V_G \).

By definition, we have the following equation that builds a bridge between Schwartz homologies and multiplicities.

\[
(5.2.1) \quad \text{Hom}_G(H_0(G,V), \mathbb{C}) = \text{Hom}_G(V, \mathbb{C}).
\]

The following proposition is called Shapiro’s lemma for Schwartz inductions and Schwartz homologies ([CS20, Theorem 7.5]).

Proposition 5.2.3. Let \( H \) be a closed Nash subgroup of \( G \) and \( \pi_H \in \Pi_{\text{FM}}(H) \)
\[
H^S_i(G, \delta^{-1}_G \otimes \text{Ind}^{S,G}_H(\delta_H \otimes \pi_H)) = H^S_i(H, \pi_H)
\]

Graded pieces.

Definition 5.2.4. We say the Schwartz homologies of \( \pi \in \Pi_{\text{FM}}(G) \) vanish, if
\[
H^S_i(G, \pi) = 0, \quad i = 0, 1, \ldots
\]

Based on the long exact sequence of the Schwartz homologies, we have the following lemma.

Proposition 5.2.5. Given a short exact sequence
\[
0 \to \pi_1 \to \pi_2 \to \pi_3 \to 0, \quad \pi_1, \pi_2, \pi_3 \in \Pi_{\text{FM}}(G),
\]
if the Schwartz homologies of two of \( \pi \)'s vanish, those of the remaining \( \pi \) vanish.

Together with [Xue20b, Lemma 2.12, Proposition 2.13], we obtain

Proposition 5.2.6. Given representations \( \pi \in \Pi_{\text{FM}}(G) \), \( \pi' \in \Pi_{\text{CW}}(G) \) and a complete descending filtration of closed subspace \( \pi_\alpha \) of \( \pi \) index by countably well-order set \( I \), with graded pieces \( \pi_\alpha^+ / \pi_\alpha \), suppose the Schwartz homologies of \( \pi_\alpha^+ / \pi_\alpha \hat{\otimes} \pi' \) vanish then the Schwartz homologies of \( \pi \hat{\otimes} \pi' \) vanish.
Vanishing theorems. Let \((V_0, W)\) be a codimension-one admissible pair and \(Z_0 = X_0 \oplus Y_0\) be a non-degenerate split quadratic space such that \(X_0, Y_0\) are totally isotropic and \(\dim X_0 = 1, 2\). We take \(V = V_0 \oplus Z_0\).

We get our first vanishing theorem by comparing infinitesimal characters following [Xue20b, §3].

**Lemma 5.2.7.** Given representations \(\pi_{V_0} \in \Pi_{FM}(SO(V_0))\), \(\sigma_{X_0} \in \Pi_{CW}^{irr}(GL(X_0))\) and \(\pi_V \in \Pi_{CW}^{irr}(SO(V))\). Let \(v_\sigma = (a_1, \cdots, a_r)\) \((a_1 \geq \cdots \geq a_r \geq 0)\) be the Harish-Chandra parameter of \(\sigma_{X_0}\) and let \(v_{\pi_V} = (b_1, \cdots, b_n)\) \((b_1 \geq \cdots \geq |b_n| \geq 0)\) be the Harish-Chandra parameter of \(\pi_V\). Suppose we have \(a_1 \neq \pm b_i\) for \(1 \leq i \leq n\), then the \(\Delta_{SO(V)}\)-Schwartz homologies of \((\sigma_{X_0} \rtimes \pi_{V_0}) \otimes \pi_V\) vanish.

**Proof.** Let \(\pi_V^\vee\) be the contragredient of \(\pi_V\) and let \(\chi_{\pi_V^\vee}\) be its infinitesimal character. From [Xue20b, Corollary 2.8], it suffices to find an element \(z \in Z(G_{V,C})\) such that \(\chi_{\pi_V^\vee}(z) \neq 0\) and

\[ z.v = 0 \text{ for every } v \in \sigma_{X_0} \rtimes \pi_{W_0}. \]

Let \(h_{V,C}\) be the Cartan subalgebra of the complexified Lie algebra \(g_{V,C}\) of \(G_{V}\). With we define a polynomial \(p\) in the polynomial algebra \(P(h_{V,C})\) by

\[ p(x_1, \cdots, x_n) = \prod_{i=1}^{n} (x_i^2 - a_i^2) \]

and \(p\) is invariant under the Weyl group \(W_{G_{V,C}}\). Then we take the element \(z \in Z(g_{V,C})\) corresponding to \(p \in P(h_{V,C})^{W_{G_{V,C}}}\) under Chevalley isomorphism.

Following the proof of [Xue20b, Lemma 5.1], we have

\[ z.v = 0, \text{ for every } v \in \sigma \rtimes \pi_{V_0}. \]

From the assumption that \(a_1 \neq \pm b_i\) for \(i = 1, \cdots, n\), we have

\[ \chi_{\pi_V^\vee}(z) = p(-v_{\pi_V}) = \prod_{i=1}^{n} (-b_i + a_1)(-b_i - a_1) \neq 0. \]

\[ \square \]

**Theorem 5.2.8.** Suppose \(\pi_{V_0} \in \Pi_{FM}(SO(V_0))\) and \(\pi_V \in \Pi_{CW}^{irr}(SO(V))\). The Schwartz homologies of \((\sigma_{\underline{s}} \rtimes \pi_{V_0}) \otimes \pi_V\) vanish for \(\sigma_{\underline{s}} = | \cdot |^{s_1} \times \cdots \times | \cdot |^{s_r}\) and \(\underline{s} = (s_1, \cdots, s_r)\) in general positions.

5.3. Reduction to codimension-one case. In this section, we complete the proof for Lemma 5.1.3, which is parallel to the proof in [Xue20b, Proposition 6.1].
General idea. Let us recall the setting of Lemma 5.1.3. For an admissible pair \((W, V)\), let \((V, W^+)\) be a codimension-one admissible pair such that \(W \subset W^+\). We denote by \((G^+, H^+, \xi^+)\) the Gross-Prasad triple associated to \((V, W^+)\). In particular, \(H^+ = \text{SO}(V)\). Let \(r = \frac{\dim V - \dim W - 1}{2}\) and \(P_{W^+}\) be the parabolic subgroup of \(G_{W^+} = \text{SO}(W^+)\) stabilizing an \((r + 1)\)-dimensional totally isotropic space \(X^+ \subset W^+\).

For \(\sigma_{X^+} \in \Pi_{CW}(\text{GL}(X^+))\) and \(\pi_W \in \Pi_{CW}(G_W)\) and \(\pi_V \in \Pi_{CW}(G_V)\), we want to prove that

\[
m(\pi_V \boxtimes \pi_W) = m((\sigma_{X^+} \times \pi_W) \boxtimes \pi_V),
\]

when \(\sigma_{X^+}\) equal to spherical principal series representation \(| \cdot |^{s_1} \times \cdots \times | \cdot |^{s_{r+1}}\) for the parameter \((s_1, \ldots, s_{r+1}) \in \mathbb{C}^{r+1}\) in general positions.

We recall Definition 4.0.3 and express \(\sigma_{X^+} \times \pi_W\) as

\[
I^S_\mu(| \det |^{s_1} \sigma_{X^+} \times \pi_W) = \Gamma^S(\mathcal{X}, \mathcal{E})
\]

where

\[
(5.3.1) \quad \mathcal{E} = \mathcal{E}_{\sigma_{X^+}, \pi_W} = P_{W^+}(\mathbb{R}) \setminus (G_{W^+}(\mathbb{R}) \times (\delta^{-1/2}_{P_{W^+}} | \det |^{s_1} \sigma_{X^+} \boxtimes \pi_W)),
\]

Let \(\mathcal{X} = (P_{W^+ \backslash G_{W^+}}(\mathbb{R}) = P_{W^+}(\mathbb{R}) \backslash G_{W^+}(\mathbb{R})\) ([BT65, Theorem 4.13(a)]). The right \(\text{SO}(V)(\mathbb{R})\)-action has an open orbit \(\mathcal{U}\) and we denote by \(\mathcal{Z}\) be the complement of \(\mathcal{U}\) in \(\mathcal{X}\). Let \(\Gamma^S_2(\mathcal{X}, \mathcal{E}) = \Gamma^S(\mathcal{X}, \mathcal{E}) / \Gamma^S(\mathcal{U}, \mathcal{E})\). From Proposition 5.2.1, there is a short exact sequence

\[
(5.3.2) \quad 0 \to \Gamma^S(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V \to \Gamma^S(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V \to \Gamma^S_2(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V \to 0
\]

The strategy to prove Lemma 5.1.3 is to first show that the Schwartz homologies of \(\Gamma^S(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V\) vanish when \(\sigma_{X^+} = | \cdot |^{s_1} \times \cdots \times | \cdot |^{s_{r+1}}\) and \((s_1, \ldots, s_{r+1}) \in \mathbb{C}^{r+1}\) is in general positions. Then, from the long exact sequence, we have

\[
\text{H}_0(H^+, \Gamma^S(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V) = \text{H}_0(H^+, \Gamma^S(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V)
\]

From (5.2.1), we know

\[
(5.3.3) \quad m((\sigma_{X^+} \times \pi_W) \boxtimes \pi_V) = \dim \text{Hom}_{H^+}(\text{H}_0(H^+, \Gamma^S(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V), \mathbb{C})
\]

We will then prove Proposition 5.3.3 that implies

\[
(5.3.4) \quad m(\pi_V \boxtimes \pi_W) = \dim \text{Hom}_{H^+}(\text{H}_0(H^+, \Gamma^S(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V), \mathbb{C})
\]

when \(\sigma_{X^+} = | \cdot |^{s_1} \times \cdots \times | \cdot |^{s_{r+1}}\) and \((s_1, \ldots, s_{r+1}) \in \mathbb{C}^{r+1}\) is in general positions. Then we can conclude Lemma 5.1.3.
Orbits. We first study the structure of the right-$\text{SO}(V)$-orbits of $\mathcal{X} = P_{W^+} \setminus G_{W^+}(\mathbb{R})$.

When $\dim W^+ > 2(r+1)$, $\mathcal{X}$ consists of all $k$-dimensional totally isotropic subspaces of $V$. When $\dim W^+ = 2(r+1)$, there are exactly two maximal totally isotropic spaces and $\mathcal{X}$ is exactly one of them.

When $\dim W^+ > 2(r+1)$, there is an open $\text{SO}(V)(\mathbb{R})$-orbit $\mathcal{U}$ consisting of $(r+1)$-dimensional totally isotropic spaces that is not contained in $V$. Its complement $\mathcal{Z}$ is the space of $(r+1)$-dimensional totally isotropic spaces contained in $V$. When $\dim V = 2(r+1)$ and $X^+.g_0 \subset V$ for some $g_0 \in \text{SO}(W^+)$, $\mathcal{Z}$ has two orbits and both of them are singletons, more precisely, $[X^+.g_0]$ and $[X^+.g_0g]$ for any $g \in \text{O}(V) \setminus \text{SO}(V)$; when $\dim V = 2(r+1)$ and if $X^+.g_0 \not\subseteq V$ for all $g_0 \in \text{SO}(W^+)$, $\mathcal{Z}$ is empty; otherwise, $\mathcal{Z}$ has just one orbit. Then we can conclude that

Lemma 5.3.1.  
(1) $\mathcal{Z}$ is empty, when $\dim W^+ = 2(r+1)$, or if $\dim V = 2(r+1)$ and $X^+.g_0 \not\subseteq V$ for all $g_0 \in \text{SO}(W^+)$;
(2) $\mathcal{Z}$ has two $\text{SO}(V)$-orbits, when $\dim V \neq 2(r+1)$;
(3) $\mathcal{Z}$ has a single $\text{SO}(V)$-orbit, when $\dim V = 2(r+1)$ and $X^+.g_0 \subseteq V$ for some $g_0 \in \text{SO}(W^+)$.

Let $\mathcal{N}'_{\mathcal{Z}/\mathcal{X}}$ be the conormal bundle over $\mathcal{Z}$ ([CS20, Section 6.1]). From [CS20, Propositions 8.2, 8.3], there is a decreasing complete filtration on $\Gamma^S(\mathcal{X}, \mathcal{E})$, denoted by $\Gamma^S(\mathcal{X}, \mathcal{E})_k$, satisfying

$$\Gamma^S(\mathcal{X}, \mathcal{E}) = \lim_{\leftarrow} \Gamma^S(\mathcal{X}, \mathcal{E})_k$$

and the graded pieces are isomorphic to

$$(5.3.5) \quad \Gamma^S(\mathcal{Z}, \text{Sym}^k \mathcal{N}'_{\mathcal{Z}/\mathcal{X}} \otimes \mathcal{E}|_\mathcal{Z}), \quad k = 0, 1, \ldots$$

When $\mathcal{Z}$ is nonempty. Let $\gamma \in G_{W^+}$ be a representative of an orbit of $\mathcal{Z}$ such that $X^+.\gamma = X'$ where $X'$ is a totally isotropic subspace of $V$ satisfying $\dim X^+ = \dim X'$. Then the stabilizer group $S_\gamma$ at $[X]$ is equal to $\gamma^{-1}P_{W^+} \cap \text{SO}(V)$, which a parabolic subgroup of $G_V$ with Levi decomposition $S_\gamma = M_\gamma N_\gamma$ and the Levi subgroup $M_\gamma = \text{GL}(X') \times \text{SO}(V_0)$. The cotangent bundles and their fibers at $[X']$ are

$$T^*_{\mathcal{Z}} = \text{SO}(V) \times_{S_\gamma} S_\gamma^\perp, \quad \text{Fib}_{[X']}(T^*_{\mathcal{Z}}) = S_\gamma^\perp$$
$$T^*_{\mathcal{X}} = \text{SO}(W^+) \times_{P_{W^+}} P_{W^+}^\perp, \quad \text{Fib}_{[X']}(T^*_{\mathcal{X}}) = P_{W^+}^\perp$$

and $S_\gamma$ acts by adjoint action. Then the fiber of the conormal bundle at $[X']$

$$\text{Fib}_{[X']}(\mathcal{N}'_{\mathcal{Z}/\mathcal{X}}) = \text{Fib}_{[X']}(T^*_{\mathcal{X}})/\text{Fib}_{[X']}(T^*_{\mathcal{Z}}) = P_{W^+}^\perp/S_\gamma^\perp,$$
which is \( \dim(X') \)-dimensional. The \( G_{V_0} = \text{SO}(V_0) \) and \( N_\gamma \) act trivially and \( \text{GL}(X') \)
acts as the standard representations. Then

\[
\Gamma^S(\text{SO}(V).[X], \text{Sym}^k \mathcal{N}^Z_{/X} \otimes \mathcal{E}|_Z)
= \Gamma^\mathcal{G}_{\mathcal{S}_\gamma}([\text{Fib}_{[X]}(\text{Sym}^k \mathcal{N}^Z_{/X} \otimes \mathcal{E}|_Z))
= \Gamma^\mathcal{G}_{\mathcal{S}_\gamma}((|\det(\cdot)|^{s+\frac{1}{2}} \sigma_{X+} \otimes \text{Sym}^k \rho_{X'}^{\text{std}}) \boxtimes (\gamma_{\pi_W}|_{G_{V_0}}))
\]

Therefore,

\[
(5.3.6)
\Gamma^S(\mathcal{Z}, \text{Sym}^k \mathcal{N}^Z_{/X} \otimes \mathcal{E}|_Z) = (\Gamma^\mathcal{G}_{\mathcal{S}_\gamma}((|\det(\cdot)|^{s+\frac{1}{2}} \sigma_{X+} \otimes \text{Sym}^k \rho_{X'}^{\text{std}}) \boxtimes (\gamma_{\pi_W}|_{G_{V_0}})))^{c\in \pi_V}
\]

Here \( \rho_{X'}^{\text{std}} \) is the standard representation of \( \text{GL}(X') \) and \( c \) is the number of \( \text{SO}(V') \)-orbits in \( \mathcal{Z} \).

**Analysis on the closed orbits.** Then we prove the vanishing result we need for closed orbits.

**Proposition 5.3.2.** The \( H^+ \)-Schwartz homologies of \( \Gamma^S_Z(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V \) vanish for \( s \) in general position.

**Proof.** From Proposition 5.2.6, it suffices to verify the Schwartz homologies of all graded pieces vanish.

\[
\Gamma^S(\mathcal{Z}, \text{Sym}^k \mathcal{N}^Z_{/X} \otimes \mathcal{E}|_Z) \boxtimes \pi_V = (\Gamma^\mathcal{G}_{\mathcal{S}_\gamma}((|\det(\cdot)|^{s+\frac{1}{2}} \sigma_{X+} \otimes \text{Sym}^k \rho_{X'}^{\text{std}}) \boxtimes (\gamma_{\pi_W}|_{G_{V_0}})))^{c\in \pi_V}
\]

From [Kos75, Section 1.3], the tensor product \( \sigma_{X+} \otimes \text{Sym}^k \rho \) is of finite length. Then it suffices to verify that for all irreducible components \( \pi_X \) of \( \sigma_X \otimes \text{Sym}^k \rho \). Since \( \sigma_{X+} = | \cdot |^{s_1} \times \cdots \times | \cdot |^{s_r} \), from [Kos75, Corollary 5.6], the Harish-Chandra parameters of irreducible components of \( \sigma_{X+} \otimes \text{Sym}^k \rho \) are \([s_1 + a_1, \cdots, s_l + a_l] \), where \( a_i \) are non-negative integers. Hence, from Theorem 5.2.8, for \( s = (s_1, \cdots, s_r) \) in general positions, the \( G_V \)-Schwartz homologies of \( \Gamma^S_Z(\mathcal{X}, \mathcal{E}) \boxtimes \pi_W \), that is, the \( H^+ \)-Schwartz homologies of \( \Gamma^S_Z(\mathcal{X}, \mathcal{E}) \boxtimes \pi_W \) vanish.

**Analysis on the open orbit.** Then we study \( H^S_1(H^+, \Gamma^S_Z(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V) \).

**Proposition 5.3.3.** When \( \sigma_{X+} = | \cdot |^{s_1} \times \cdots \times | \cdot |^{s_r+1} \), we have

\[
(5.3.7)
H^S_1(H^+, \Gamma^S_Z(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V) = H^S_1(H^+, \xi^{-1} \otimes (\pi_V \boxtimes \pi_W))
\]

for \( s \in \mathbb{C}^{r+1} \) in general positions.

Let us recall or introduce some notations

- Let \( d = \dim V, \ r = \frac{\dim V - \dim W - 1}{2} \), then the modular character

\[
\delta_{P_W}((m \times g_W) \times n) = |\det(m)|^{d-1-r}
\]
• Let $N_{r+1}$ be the unipotent subgroup of $GL_{r+1}$ consisting of upper-triangular unipotent matrices, and let $R_{r,1}$ be the mirabolic subgroup of $GL_{r+1}$. We denote by $N_{r,1}$ the unipotent radical of $R_{r,1}$.

• Define a character $\pi_{r+1}$ of $N_{r+1}$ by letting

$$\psi_{r+1}(n) = \psi(\sum_{i=1}^{r+1} n_{i,i+1})$$

• Let $Q_{a,b,c}$ be the intersection of the parabolic subgroup associated to the partition $(a, b, c)$ in $GL_{a+b+c}$ and the mirabolic subgroup $R_{a+b+c-1}$. We let the "Levi part" $L_{a,b,c}$ of $Q_{a,b,c}$ to be the image of $GL_a \times GL_b \times R_{c-1,1}$ diagonally embedded into $GL_{a+b+c}$. Then $Q_{a,b,c} = L_{a,b,c}U_{a,b,c}$ for the unipotent group associated to the partition $(a, b, c)$.

Let $X$ be totally isotropic space in $W^\perp$ defined in Section 2.1. We may assume $X \subset X^+$. Recall that $N$ is the unipotent radical of the parabolic subgroup $P_V$ of $G_V$ stabilizing $X$. We define $N'$, the subgroup of $N$ stabilizing $D$, then $H = (N_{r+1} \times \Delta G_W) \rtimes N'_r$.

Consider the open orbit $U = P_W \setminus P_W \times G_V = (P_W \cap G_V) \setminus G_V$. We have

$$P_W \cap G_V = (GL(X) \times 1 \times G_W) \rtimes N = G_W \rtimes (R_{r,1} \times N'_r).$$

Then, for the left-hand side of (5.3.7), we have

$$\Gamma^S_{\chi}(U, \mathcal{E}) \boxtimes \pi_V = \text{Ind}_{P_W \cap G_V}^{G_V} (\sigma_{X^+} \rtimes \pi_W | P_W \cap G_V) \boxtimes \pi_V$$

$$= \text{Ind}_{P_W \cap G_V}^{G_V} (\sigma_{X^+}|_{R_{r,1}} \boxtimes \pi_W) \boxtimes \pi_V$$

$$= \text{Ind}_{(R_{r,1} \times \Delta G_W) \rtimes N'_r}^{S, H^+} (\sigma_{X^+}|_{R_{r,1}} \boxtimes \pi_W \boxtimes \pi_V)$$

For the right-hand side of (5.3.7), we have

$$\text{Ind}^{S, H^+}_{H} (\xi^{-1} \otimes (\pi_V \boxtimes \pi_W)) = \text{Ind}_{(N_{r+1} \times \Delta SO(W)) \rtimes N'_r}^{S, H^+} (\xi^{-1} \otimes (\pi_V \boxtimes \pi_W))$$

$$= \text{Ind}_{(R_{r,1} \times \Delta G_W) \rtimes N'_r}^{S, H^+} (\text{Ind}_{N_{r+1}}^{R_{r,1}} \psi^{-1}_{r+1} \boxtimes \pi_W \boxtimes \pi_V)$$

Then we study the quotient of (5.3.9) and (5.3.10)

$$\text{Ind}_{(R_{r,1} \times \Delta G_W) \rtimes N'_r}^{S, H^+} \left( (\sigma_{X^+} |_{R_{r,1}} / \text{Ind}_{N_{r+1}}^{R_{r,1}} \psi^{-1}_{r+1}) \boxtimes \pi_W \boxtimes \pi_V \right)$$

and apply the vanishing theorems to prove Proposition 5.3.3.

Let $(s_1, \ldots, s_{r+1}) \in \mathbb{C}^{r+1}$ and $\sigma_{X^+} = | s_1 | \times \cdots \times | s_{r+1} |$, the computation in [Xue20b, Section 5.1] for the restriction of spherical principal series representations to the mirabolic subgroup $R_{r,1}$ can be generalized over the real field verbatim and we can obtain a proposition parallel to [Xue20b, Proposition 5.1].
Proposition 5.3.4. When restricted to $R_{r,1}$, the representation $\sigma_{X^+}$ has a subrepresentation $\text{Ind}_{N_{r+1}}^{R_{r,1}}(\psi_{r+1}^{-1})$. Moreover, the quotient $\sigma_{X^+}/\text{Ind}_{N_{r+1}}^{G_{r+1}}(\psi_{r+1}^{-1})$ admits an $R_{r,1}$-stable complete filtration whose graded pieces have the shape

$$\text{Ind}_{Q_{a,b,c}}^{R_{r,1}}(\tau_a \boxtimes \tau_b \boxtimes \tau_c)$$

where $a + b + c = t + 1$, $a + b \neq 0$ and $\tau_a \boxtimes \tau_b \boxtimes \tau_c$ is regarded as a $Q_{a,b,c}$ representation by trivially extended by $N_{a,b,c}$.

1. $\tau_a = \text{Ind}_{B_a}^{G_a}(\text{sgn}^{m_1} \cdot |s_{i_1} + k_1| \cdots \text{sgn}^{m_a} \cdot |s_{i_a} + k_a|)$ where $1 \leq i_1, \cdots, i_a \leq t + 1$ are integers, $l_1, \cdots, l_a \in \mathbb{Z}$ and $k_1, \cdots, k_a \in \mathbb{Z}$;
2. $\tau_b = \tau'_b \otimes \rho$ where $\tau'_b$ is a representation of the same form as $\tau_a$ and $\rho$ is a finite-dimensional representation of $GL_b(\mathbb{R})$;
3. $\tau_c = \text{Ind}_{N_c}^{R_{r,-1,1}}(\psi_{r+1}^{-1})$.

Then there is a $R_{r,1}$-equivariant embedding

$$\text{Ind}_{N_{r+1}}^{S,R_{r,1}}(\psi_{r+1}^{-1}) \hookrightarrow |\det \frac{d-1-r+c}{2} \sigma_{X^+}|$$

From the exactness of Schwartz induction and projective tensor product (Proposition 5.2.1), each graded piece in Proposition 5.3.4 gives a graded piece

$$(5.3.13) \quad \text{Ind}_{(R_{r,1} \times G_{W}) \times N'_V}^{S,H^+}(\text{Ind}_{Q_{a,b,c}}^{S,R_{r,1}}(\tau_a \boxtimes \tau_b \boxtimes \tau_c) \boxtimes \pi_W)$$

of (5.3.7). Since (5.3.13) can be expressed as the parabolic induction

$$|\det \frac{d-1-r+c}{2} (\tau_a \boxtimes \tau_b)| \times \text{Ind}_{(R_{r,1} \times G_{W}) \times N'_V}^{G_{W} \oplus D \oplus X_c}(\xi_{r+1}^{-1} \otimes \pi_W),$$

based on Lemma 5.2.7 and that fact that $a + b \geq 1$, the $H^+$-Schwartz homologies of (5.3.13) vanish for $(s_1, \cdots, s_{r+1}) \in \mathbb{C}^n$ in general position.

From Proposition 5.2.6 and long exact sequences of Schwartz homology, the map

$$H_i^S(H^+, \text{Ind}_{(R_{r,1} \times G_{W}) \times N'_V}^{S,H^+}(\text{Ind}_{N_{r+1}}^{S,R_{r,1}}(\psi_{r+1}^{-1}) \boxtimes \pi_W \boxtimes \pi_V))$$

$$\longrightarrow H_i^S(H^+, \text{Ind}_{(R_{r,1} \times G_{W}) \times N'_V}(\sigma_{X^+} \mid_{R_{r,1}} \boxtimes \pi_W \boxtimes \pi_V))$$

induced from (5.3.12) is an isomorphism when $(s_1, \cdots, s_{r+1}) \in \mathbb{C}^{r+1}$ is in general position.

Using Shapiro’s Lemma ( [CS20]) and the unimodularity of $H$ and $H^+$, we have

$$H_i^S(H, \xi^{-1} \otimes (\pi_V \boxtimes \pi_W)) = H_i^S(H^+, \text{Ind}_{H^+}^{S,H^+}(\xi^{-1} \otimes (\pi_V \boxtimes \pi_W))$$

This proves Proposition 5.3.3.

Conclusion. Following the proof at the beginning of this subsection ("the general idea" part), when $s \in \mathbb{C}^{r+1}$ is in general position, we have (5.3.3) from Proposition 5.3.2 and (5.3.4) from Proposition 5.3.3, these equations conclude the proof for Lemma 5.1.3.
5.4. Multiplicity formula: first inequality. In this section, we complete the proof for Proposition 5.1.5. The proof follows a similar approach as the previous section, the main difference is that we consider the left exact sequence of the Hom-functor instead of the long sequence of the Schwartz homology. As in the previous section, we use a vanishing result and study the Schwartz sections on the orbits.

A vanishing result. The vanishing result we use in this section is proved in [CCZ23]

Definition 5.4.1. For every \( \pi_V \in \Pi_{\text{CW}}^{\text{irr}}(G_V) \), let \( \pi_V \) is the Langlands quotient of

\[
\det |^s \rho_1 \times \cdots \times \det |^s \rho_r \times \pi_{V_0}
\]

for \( \text{Re}(s_1) \geq \cdots \geq \text{Re}(a_\rho) > 0 \) and tempered \( \rho_1, \cdots, \rho_r, \pi_{V_0} \). We let the leading index for Langlands quotient

\[ \text{LI}(\pi_V) = \text{Re}(s_1) \]

This definition is compatible with Definition 4.0.7 when the standard module (5.4.1) is irreducible. In particular, the definitions are compatible when \( \pi_V \) is in a generic packet.

Theorem 5.4.2. ([CCZ23, Lemma 5.1]) If \( \text{Re}(s) > \text{LI}(\pi_V) \), then

\[
\text{Hom}_{\Delta G_V}((\det |^s \rho \times \pi_{V_0}) \boxtimes \pi_V, \mathbb{C}) = 0, \quad \text{for } \pi_{V_0} \in \Pi_{\text{FM}}(G_{V_0}), \pi_V \in \Pi_{\text{CW}}^{\text{irr}}(G_V).
\]

Closed orbits. The analysis for the closed orbit is just parallel to that in the previous section.

We need a result for Hom-functor parallel to Proposition 5.2.6, which is a direct consequence of the left exactness of Hom and Hom(lim\( \rightarrow \alpha \pi_\alpha, \mathbb{C} \)) = lim\( \rightarrow \alpha \)Hom(\( \pi_\alpha, \mathbb{C} \)).

Lemma 5.4.3. Given representations \( \pi \in \Pi_{\text{FM}}(G_V), \pi' \in \Pi_{\text{CW}}(G_V) \) and a complete descending filtration of closed subspace \( \pi_\alpha \) of \( \pi \) index by countably well-order set \( I \), with graded pieces \( \pi_{\alpha+} / \pi_\alpha \), suppose Hom\(_{\Delta G_V}((\pi_{\alpha+} / \pi_\alpha) \boxtimes \pi', \mathbb{C}) = 0 \) vanish then Hom\(_{\Delta G_V}(\pi \boxtimes \pi', \mathbb{C}) = 0 \).

Now we introduce all the notions in Section 5.3.

Proposition 5.4.4. When \( \text{Re}(s) \geq \text{LI}(\pi_V) \), we have

\[
\text{Hom}_{\Gamma^G_{\mathcal{X}, \mathcal{E}}(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V, \mathbb{C}) = 0
\]

Proof. From Lemma 5.4.3, it suffice to prove that for the graded pieces \( \Gamma^G_{\mathcal{X}, \mathcal{E}}(\mathcal{X}, \mathcal{Z})_k \)

\[
\text{Hom}_{\Gamma^G_{\mathcal{X}}(\mathcal{X}, \mathcal{Z})_k \boxtimes \pi_V, \mathbb{C}) = 0
\]

This is equivalent to showing that

\[
\text{Hom}_{\Gamma^G_{\mathcal{X}}((\det |^s \sigma_{\mathcal{X}+} \boxtimes \text{Sym}^k(\rho_{\mathcal{X}+}^\text{std})) \boxtimes (\mathcal{I}_{\mathcal{W}} |_{G_{V_0}}))) \boxtimes \pi_V, \mathbb{C}) = 0,
\]

and it follows from Theorem 5.4.2 because

\[
\text{Re}(s) + \frac{1}{2} > \text{LI}(\pi_V).
\]
The left exactness of the Hom-functor and exact sequence (5.3.2) gives
\[ 0 \rightarrow \text{Hom}_{H^+}(\Gamma^S(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V, \mathbb{C}) \rightarrow \text{Hom}_{H^+}(\Gamma^S(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V, \mathbb{C}) \rightarrow \text{Hom}_{H^+}(\Gamma^S(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V, \mathbb{C}) \]
so Proposition 5.4.4 gives
\[ \dim \text{Hom}_{H^+}(\Gamma^S(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V, \mathbb{C}) \geq \dim \text{Hom}_{H^+}(\Gamma^S(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V, \mathbb{C}) = m((\sigma_X \times \pi_W) \boxtimes \pi_V) \]
Open orbits. To complete the proof for Proposition 5.1.5, we prove an inequality parallel to Proposition 5.3.3.

**Proposition 5.4.5.** When \( \sigma_{X^+} = | \cdot |^m \text{sgn}^l \) or \( \sigma_{X^+} = | \det |^l D_m \), we have
\[ \dim \text{Hom}_{H^+}(\Gamma^S(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V, \mathbb{C}) \leq m((| \cdot |^m \text{sgn}^l, \pi_W) \boxtimes \pi_V) \]
for \( \text{Re}(s) \geq \text{LI}(\pi_V) \).

When \( \sigma_{X^+} = | \cdot |^m \text{sgn}^l \), from (5.3.9), we have
\[ \Gamma^S(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V = \text{Ind}^{G_{W^D}}_{G_W}(\pi_W) \boxtimes \pi_V \]
Then Proposition 5.4.5 follows from Proposition 5.2.3 and (5.2.1).

To prove Proposition 5.4.5 when \( \sigma_{X^+} = | \det |^l D_m \), we need to introduce the following notions:
- Let \( B_2 \) be the (upper-triangular) Borel subgroup of \( \text{GL}_2 \) with Levi decomposition \( B_2 = T_2 N_2 \). Let \( K = \text{SO}_2(\mathbb{R}) \).
- We let
\[ n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad k_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]
- Let \( \mathcal{X}_2 = B_2 \backslash \text{GL}_2, U_2 = B_2 \backslash B_2 w_2 B_2 \subset \mathcal{X}_2 \) and \( Z_2 = B_2 \backslash B_2 \);
- Let \( \pi_D = \text{Ind}^{| \det(\cdot)|^m \text{sgn}^l} \cdot D_m \). By definition, \( \pi_D \) is the unique quotient of
\[ \pi_I = \text{Ind}^{\Gamma^S(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V}(\cdot \text{sgn}^l, \pi_W) \boxtimes \pi_V = \text{Ind}^{\Gamma^S(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V}(\chi_1 \chi_2 \otimes \chi_2), \]
where \( \chi_1 = \text{sgn}^m \) and \( \chi_2 = \text{sgn}^m \).
- Let \( \pi_F \) be the unique subrepresentation of \( \pi_I \). It is not hard to prove the following lemma.

**Lemma 5.4.6.** \( \pi_F \) is isomorphic to the \( n \)-dimensional \( \text{GL}_2(\mathbb{R}) \)-representation \( \chi_1 \chi_2(\text{det}(\cdot))^m \text{Sym}^n(\mathbb{C}^2) \), where \( \mathbb{C}^2 \) is the standard representation of \( \text{GL}_2(\mathbb{R}) \). All irreducible \( R_{1,1} \)-components \( \pi_F \) are
\[ \text{det}(\cdot)|^k \text{sgn}^k(\text{det}(\cdot)), \text{ for } k = 0, 1, \ldots, m-1. \]

Extension by zero gives a natural embedding of \( R_{1,1} \)-representations
\[ (5.4.3) \quad i_{UX} : \Gamma^S(\mathcal{U}, \chi_1 \chi_2 \otimes \chi_2) \rightarrow \Gamma^S(\mathcal{X}, \chi_1 \chi_2 \otimes \chi_2), \]
**Lemma 5.4.7.** The quotient $\Gamma^S(\mathcal{X}_2, \chi_1 \chi_2 \otimes \chi_2)/\Gamma^S(\mathcal{U}_2, \chi_1 \chi_2 \otimes \chi_2)$ has a decreasing complete filtration $\Gamma^S(\mathcal{X}_2, \chi_1 \chi_2 \otimes \chi_2)_k$ with graded pieces isomorphic to $R_{1, 1}$-representations

$$\chi_1 \chi_2(\det (\cdot)) \text{sgn}^k(\det (\cdot)) | \det (\cdot)|^k |_{R_{1, 1}},$$

for $k = 0, 1, \cdots$.

**Proof.** This lemma follows from [CS20, Propositions 8.2, 8.3].

We identify $\Gamma^S(\mathcal{U}_2, \chi_1 \chi_2 \otimes \chi_2)$ as $\text{Ind}_{\mathbb{R}^\times \times 1}^{S, R_{1, 1}}(\chi_2)$ using the following equation.

$$\Gamma^S(\mathcal{U}_2, \chi_1 \chi_2 \otimes \chi_2) = \Gamma^S(B_2 \setminus B_2 w_2 B_2, \chi_1 \chi_2 \otimes \chi_2)$$

$$= \Gamma^S(T_2 \setminus B_2, \chi_2 \otimes \chi_1 \chi_2)$$

$$= \Gamma^S(\mathbb{R}^\times \times 1 \setminus R_{1, 1}, \chi_2) = \text{Ind}_{\mathbb{R}^\times \times 1}^{S, R_{1, 1}}(\chi_2),$$

and then define an $R_{1, 1}$-homomorphism

$$T_d : \text{Ind}_{\mathbb{R}^\times \times 1}^{S, R_{1, 1}}(\chi_2) \rightarrow \pi_D$$

by compositing the embedding (5.4.3) and the quotient map $\pi_I$ to $\pi_F$, that is,

$$T_d : \text{Ind}_{\mathbb{R}^\times \times 1}^{S, R_{1, 1}}(\chi_2) = \Gamma^S(\mathcal{U}_2, \chi_1 \chi_2 \otimes \chi_2) \rightarrow \Gamma^S(\mathcal{X}_2, \chi_1 \chi_2 \otimes \chi_2) = \pi_I \rightarrow \pi_I/\pi_F = \pi_D.$$

**Lemma 5.4.8.** $T_d$ is injective.

**Proof.** Suppose $T_d$ is not injective then there exist $\tilde{f} \in \Gamma^S(\mathcal{U}, \chi_1 \chi_2 \otimes \chi_2)$ such that its extension by zero $\tilde{f}_G$ in $\pi_I$ is contained in $\pi_F$.

On the one hand, $f(x) = \tilde{f}(w_2 n_x)$ is a Schwartz function. For $\theta \in (0, \pi)$, we can compute $\tilde{f}$ with the decomposition

$$k_\theta = \begin{pmatrix} 1/\sin(\theta) & \cos(\theta) \\ \sin(\theta) & 1 \end{pmatrix} w_2 \begin{pmatrix} 1 & -\cot(\theta) \\ \cot(\theta) & 1 \end{pmatrix}.$$

Then we have

$$\tilde{f}_G(k_\theta) = \tilde{f}(k_\theta) = \chi_1 \chi_2(1/\sin(\theta)) \chi_2(\sin(\theta)) f(\cot(\theta)) = o(\theta^l), \quad \text{for every } l > 0.$$

Then $(\frac{d}{d\theta})^l \tilde{f}_G(k_\theta)|_{\theta=0} = 0$ for every positive integer $l$.

On the other hand, from [God74, Section 2.3], $\pi_F$ is generated by the functions

$$\varphi_{-m+1}, \varphi_{-m+3}, \cdots, \varphi_{m-1},$$

where $\varphi_l(n_x \cdot t(a, b) \cdot k_\theta) = \chi_1 \chi_2(a) \chi_2(b)^{e^{il\theta}}$.

Then $\tilde{f}_G \in \pi_F$ is a linear combination of $\varphi_k$, that is, there is a nonzero $n$-tuple $(\lambda_1, \cdots, \lambda_n) \in \mathbb{C}^n$ such that $\tilde{f}_G = \sum_{k=1}^n \lambda_k \varphi_{2k-n-1}$. Then we have

$$\frac{d}{d\theta}^l \tilde{f}_G(k_\theta)|_{\theta=0} = \sum_{k=0}^{n-1} \lambda_k ((2k - n - 1)i)^l,$$
Hence, there exists \( l \) such that \( (\frac{d}{dg})^l(\tilde{f}\mathcal{G}(k_0))|_{a=0} \neq 0 \), which leads to a contradiction. Therefore, the \( R_{1,1} \)-homomorphism \( T_d \) is injective. \( \square \)

**Lemma 5.4.9.** Coker\( (T_d) \) has a decreasing complete filtration \( \Gamma^S_S(\mathcal{X}_2, \chi_1\chi_2 \otimes \chi_2)_k \) with graded pieces isomorphic to

\[
\text{(5.4.4)}
\begin{align*}
|\det(\cdot)|^{k+s+\frac{d+m-3}{2}} \text{sgn}(\cdot)^k |_{R_{1,1}}, & \quad \text{for } k = 1, \ldots. \\
\end{align*}
\]

**Proof.** From Lemma 5.4.7, \( \Gamma^S_S(\mathcal{X}_2, \chi_1\chi_2 \otimes \chi_2) = \pi_I/\Gamma^S_S(\mathcal{U}_2, \chi_1\chi_2 \otimes \chi_2) \) has a decreasing complete filtration \( \Gamma^S_S(\mathcal{X}_2, \chi_1\chi_2 \otimes \chi_2)_k \) with graded pieces isomorphic to

\[
\text{(5.4.5)}
\begin{align*}
|\det(\cdot)|^{k} \text{sgn}(\cdot)^k \chi_1\chi_2(\det(\cdot)) |_{R_{1,1}}, & \quad \text{for } k = 0, 1, \ldots. \\
\end{align*}
\]

From Lemma 5.4.6, the finite-dimensional representation \( \pi_F \) in \( \pi_I \) has a \( R_{1,1} \)-composition series with irreducible pieces

\[
|\det(\cdot)|^{k} \text{sgn}(\cdot)^k \chi_1\chi_2(\det(\cdot)) |_{R_{1,1}}, & \quad \text{for } k = 0, 1, \ldots, m - 1. 
\]

Then the projection \( \pi_I \to \pi_I/i_{UX}(\Gamma^S_S(\mathcal{U}_2, \chi_1\chi_2 \otimes \chi_2)) \) gives an isomorphism between \( \pi_F \) and \( \pi_F = \Gamma^S_S(\mathcal{X}_2, \chi_1\chi_2 \otimes \chi_2)/\Gamma^S_S(\mathcal{X}_2, \chi_1\chi_2 \otimes \chi_2)_n \). Therefore, we have

\[
\Gamma^S_S(\mathcal{X}_2, \chi_1\chi_2 \otimes \chi_2) = \pi_F \oplus \Gamma^S_S(\mathcal{X}_2, \chi_1\chi_2 \otimes \chi_2)_m. 
\]

Therefore, \( \text{Coker}(T_d) = \pi_D/i_{UX}(\Gamma^S_S(\mathcal{U}_2, \chi_1\chi_2 \otimes \chi_2)) = (\pi_I/\Gamma^S_S(\mathcal{U}_2, \chi_1\chi_2 \otimes \chi_2))/\pi_F = \Gamma^S_S(\mathcal{X}_2, \chi_1\chi_2 \otimes \chi_2)_m \), and thus has a decreasing complete filtration with graded pieces isomorphic to

\[
\sigma_k = |\det(\cdot)|^{k} \text{sgn}(\cdot)^k \chi_2(\det(\cdot)) |_{R_{1,1}} = |\det(\cdot)|^{k+s+\frac{d+m-3}{2}} \text{sgn}(\cdot)^k |_{R_{1,1}}
\]

for positive integers \( k \).

We conclude the proof for Proposition 5.4.5 by proving the following statement.

**Proposition 5.4.10.** The map

\[
\text{Hom}_{H^+}(\text{Ind}^{S,H^+}_{(R_{1,1} \times \Delta G_W) \times N'_W}(\sigma_X |_{R_{1,1}} \otimes \pi_{W} \otimes \pi_V)), \mathbb{C})
\]

\[
\longrightarrow \text{Hom}_{H^+}(\text{Ind}^{S,H^+}_{(R_{1,1} \times \Delta G_W) \times N'_W}(\text{Ind}^{S,R_{1,1}}_{R \times 1}(\chi_2) \otimes \pi_{W} \otimes \pi_V)), \mathbb{C})
\]

induced by the embedding \( T_d \) a surjection when \( \text{Re}(s) \geq \text{LI}(\pi_V) \).

**Proof.** Notice that, for \( t = k + s + \frac{d+m-3}{2} \)

\[
\text{Ind}^{S,H^+}_{(R_{1,1} \times \Delta G_W) \times N'_W}(\sigma_k \otimes \pi_{W} \otimes \pi_V) = (|t-\frac{d-2}{2}| \text{sgn}^m \times \text{Ind}^{S,G_W \otimes R}_{G_W}(\pi_W)) \otimes \pi_V
\]

Since we assumed \( \text{Re}(s) + m/2 \geq \text{LI}(\pi_V) \) and \( k \) is a positive integer, we have

\[
(5.4.6)
\begin{align*}
t - \frac{d-2}{2} & = \text{Re}(s) + \frac{m}{2} + k - \frac{1}{2} > \text{LI}(\pi_V). \\
\end{align*}
\]
Then, from Theorem 5.4.2,
\[ \text{Hom}_{H^+}(|\cdot|^{-\frac{d+2}{2}}\text{sgn}^m \times \text{Ind}_{G_W}^{G_{W'}}(\pi_W)) \boxtimes \pi_V, C) = 0, \quad k = 1, 2, \ldots \]

From Lemma 5.4.3 and Lemma 5.4.9, we have
\[ \text{Hom}_{H^+}(\text{Ind}_{(R_1,1 \times \Delta_{W'}) \times N'_V}(\text{Coker}(T_d) \boxtimes \pi_W \boxtimes \pi_V), C) = 0. \]

From the left exact sequence of the Hom-functor, we obtain that
\[ \text{Hom}_{H^+}(\text{Ind}_{(R_1,1 \times \Delta_{W'}) \times N'_V}(\sigma_X |R_{-1} \boxtimes \pi_W \boxtimes \pi_V)), C) \rightarrow \text{Hom}_{H^+}(\text{Ind}_{(R_1,1 \times \Delta_{W'}) \times N'_V}(\text{Ind}_{R_1}^{S,1}(\chi_2) \boxtimes \pi_W \boxtimes \pi_V), C) \]
is a surjection. □

Notice that
\[ \text{Ind}_{(R_1,1 \times \Delta_{W'}) \times N'_V}(\text{Ind}_{R_1}^{S,1}(\chi_2) \boxtimes \pi_W \boxtimes \pi_V) = (|\cdot|^{-\frac{d+2}{2}}\text{sgn}^m \times \pi_W) \boxtimes \pi_V \]
so Proposition 5.4.5 follows from Proposition 5.4.10.

**Proof for the first inequality.** Then we make use of Proposition 5.1.5 to prove the inequality (5.4.7)
\[ m(\pi_V \boxtimes \pi_W) \leq m(\pi_{V_0} \boxtimes \pi_{W_0}) \]
in Proposition 5.1.4.

**Proof.** We express \( \pi_V = \sigma_V \times \pi_{V_0}, \pi_W = \sigma_W \times \pi_{W_0} \) in the form of (4.0.7) and prove the inequality by mathematical induction on
\[ N(\sigma_V, \sigma_W) = \sum_{\text{Re}(s_{V,i}) \neq 0} n_{V,i} + \sum_{\text{Re}(s_{W,i}) \neq 0} n_{W,i}. \]
Here \( s_{V,i}, s_{W,i}, n_{V,i}, n_{W,i} \) are defined as in (4.0.7).

If \( N(\sigma_V, \sigma_W) = 0 \), both \( \pi_V \) and \( \pi_W \) are tempered, then the inequality follows from the Conjecture 2 for tempered parameters, which was proved in [CL22].

In other cases, we may assume
\[ \text{Re}(s_{V,1}) \geq \text{Re}(s_{V,2}) \geq \cdots \geq \text{Re}(s_{V,l}) > 0, \quad \text{Re}(s_{W,1}) \geq \text{Re}(s_{W,2}) \geq \cdots \geq \text{Re}(s_{W,l}) > 0. \]

Suppose the proposition holds when \( N(\sigma_V, \sigma_W) = k \), then when \( N(\sigma_V, \sigma_W) = k+1 \), we consider the following cases.

Case 1: If \( l_V \neq 0 \) and \( \text{Re}(s_{V,1}) \geq \text{Re}(s_{W,1}) \), then let \( \tilde{\sigma}_V = |\det(\cdot)|^{s_{V,2}}\sigma_{V,2} \times \cdots \times |\det(\cdot)|^{s_{V,l}}\sigma_{V,l}. \)
(a) If \( n_{V,1} = 1 \), from Proposition 5.1.5(1) we have
\[
m((\sigma_V \times \pi_{V_0}) \boxtimes (\sigma_W \times \pi_{W_0})) \leq m((\sigma_W \times \pi_{W_0}) \boxtimes (\tilde{\sigma}_V \times \pi_{V_0})),
\]
(b) If \( n_{V,1} = 2 \), let \( \tilde{\sigma}_V = | \cdot |^{s_{V,1} + m_{V,1}^{W}} \operatorname{sgn}^{m_{V,1} + 1} \times \tilde{\sigma}_V \), and from Proposition 5.1.5(2), we have
\[
m((\sigma_V \times \pi_{V_0}) \boxtimes (\sigma_W \times \pi_{W_0})) \leq m((\sigma_W \times \pi_{W_0}) \boxtimes (\tilde{\sigma}_V \times \pi_{V_0})),
\]
Since \( N(\tilde{\sigma}_V, \sigma_V), N(\tilde{\sigma}_V, \sigma_W) \leq N(\sigma_V, \sigma_W) - 1 = k \), we have
\[
m((\sigma_W \times \pi_{W_0}) \boxtimes (\tilde{\sigma}_V \times \pi_{V_0})) \leq m(\pi_{V_0} \boxtimes \pi_{W_0}), \quad m((\sigma_W \times \pi_{W_0}) \boxtimes (\tilde{\sigma}_V \times \pi_{V_0})) \leq m(\pi_{V_0} \boxtimes \pi_{W_0})
\]
Therefore, we have
\[
m((\sigma_V \times \pi_{V_0}) \boxtimes (\sigma_W \times \pi_{W_0})) \leq m(\pi_{V_0} \boxtimes \pi_{W_0}),
\]
Case 2: If \( l_V = 0 \) or \( \operatorname{Re}(s_{V,1}) < \operatorname{Re}(s_{W,1}) \), then we switch the order of \( V, W \) to reduce to Case 1. More explicitly, we take \( \sigma^{(s')}_W = | \cdot |^{s'} \times \sigma_{W_0} \), there is a \( s' \in i\mathbb{R} \), such that
\[
m((\sigma_V \times \pi_{V_0}) \boxtimes (\sigma_W \times \pi_{W_0})) = m((\sigma^{(s')}_W \times \pi_{W_0}) \boxtimes (\sigma_V \times \pi_{V_0}))
\]
From [SV80, Theorem 1.1] and Langlands classification, we may assume \( \sigma^{(s')}_W \times \pi_{W_0} \) is irreducible. Then the pair \( (\sigma^{(s')}_W, \sigma_V) \) belongs to Case 1 and \( N(\sigma^{(s')}_W, \sigma_V) = N(\sigma_V, \sigma_W) = k + 1 \), so
\[
m((\sigma^{(s')}_W \times \pi_{W_0}) \boxtimes (\sigma_V \times \pi_{V_0})) \leq m(\pi_{V_0} \boxtimes \pi_{W_0}).
\]
Therefore, we have
\[
m((\sigma_V \times \pi_{V_0}) \boxtimes (\sigma_W \times \pi_{W_0})) \leq m(\pi_{V_0} \boxtimes \pi_{W_0}).
\]
Then the proposition follows from mathematical induction on \( N(\sigma_V, \sigma_W) \). □

5.5. Multiplicity formula: second inequality. In this section, we complete the proof for the second inequality of Proposition 5.1.4.

A construction. We first prove Proposition 4 by construction. Recall that, for an admissible pair \( (W, V) \), we can construct a codimension-one admissible pair \( (V', W') \) by taking \( W' = W \oplus (X^+ \oplus Y^+) \) for certain totally isotropic spaces \( X^+ \) and \( Y^+ \). Let \( G^+ = \text{SO}(W^+) \times \text{SO}(V), H^+ = \Delta \text{SO}(V), P^+ \) is the parabolic subgroup \( P_{W^+} \times \text{SO}(V) \), where \( P_{W^+} \) is the parabolic subgroup of \( \text{SO}(W^+) \) stabilizing \( X^+ \).

From the multiplicity-one theorem ( [SZ12], \( m(\pi_V \boxtimes \pi_W) \leq 1 \), so it suffices to prove the following proposition.

Proposition 5.5.1. When \( r = 0 \) and \( m(\pi_V \boxtimes \pi_W) \neq 0 \) and \( \sigma_{X^+} \) is a generic representation of \( \text{GL}(X^+) \), then one can construct a nonzero element in \( \text{Hom}_{G_w}((\sigma_{X^+} \times \pi_W) \boxtimes \pi_V, \mathbb{C}) \).
The main idea for proving this proposition is from the following theorem.

**Theorem 5.5.2.** ([GSS19, Proposition 4.9]) For a Casselman-Wallach representation \( \sigma^+ \) of \( P^+ \), suppose

1. The complement \( G^+(\mathbb{R}) - H^+(\mathbb{R})P^+(\mathbb{R}) \) is the zero set of a polynomial \( f^+ \) on \( G^+(\mathbb{R}) \) that is left \( H^+ \)-invariant and right \((P^+, \psi_{P^+})\)-equivariant for an algebraic character \( \psi_{P^+} \) of \( P^+ \).
2. \( H^+(\mathbb{R}) \) has finitely many orbits on the flag of a minimal parabolic subgroup of \( G^+(\mathbb{R}) \).
3. \( \sigma^+ \) admits a nonzero \((P^+ \cap H^+, \delta_{P^+ \cap H^+} \cdot \delta_{H^+}^{-1})\)-equivariant continuous linear functional, where \( \delta_{P^+ \cap H^+} \) and \( \delta_{H^+} \) are the modular characters of \( P^+ \cap H^+ \) and \( H^+ \) respectively.

then \( \text{Ind}_{P^+}^{G^+}(\sigma^+) \) admits a nonzero \( H^+ \)-invariant functional.

**Proof for Proposition 5.5.1.** We denote by \( \sigma^+ \) the representation of \( P^+ \) induced from the representation \( \sigma_X \cdot \boxtimes_{\pi_W} \boxtimes \pi_V \) on the Levi component \( M^+ = (GL(X^+) \times SO(W)) \times SO(V) \) of \( P^+ \). It suffices to verify the conditions in the theorem.

1. For every \( (g_W, g_V) \in G^+(\mathbb{R}), g \in G^+(\mathbb{R}) - H^+(\mathbb{R})P^+(\mathbb{R}) \) if and only if \( g_W X \subset V \), equivalently, the \((n + 1) \times (n + 1 + r)\)-matrix
   
   \[ A_g = \left[ g_V.v_1, \cdots, g_V.v_n, g_W^{-1}, g_W^{1}, \cdots, g_W^{-1}, g_W^{1} \right] \]

   is of rank \( n \). We let
   
   \[ f(g) = \text{det}(A_g A_g^t), \]

   then \( f \) is left-\( H^+ \)-invariant and right-\((P^+, \psi_{P^+})\)-equivariant, where \( \psi_{P^+}(p_{W^+}, g_V) = \text{det}(g_X)^2 \) for \( p_{W^+} = (g_X, g_W) \cdot n_{W^+} \in P_{W^+} \) and \( g_V \in G_V \).
2. Since \( G^+/H^+ \) is an absolutely spherical variety (Section 3), the Borel subgroup has finitely many orbits, so the complexification of the minimal parabolic also has finitely many orbits. Then condition (2) is a direct consequence of the finiteness of the first Galois cohomology for groups over local fields.
3. Using the notion in (5.3.8), \( P^+ \cap H^+ = \Delta(P_{W^+} \cap G_V) \), where \( P_{W^+} \cap G_V = \Delta(R_{r+1} \times SO(W)) \rtimes N_V' \). Since \( \delta_{P^+ \cap H^+}(\Delta(g_X, g_W) \rtimes n_V) = |\text{det}(g_X)|^{\dim W + 1} \), it suffices to give a functional in

   \[ \text{Hom}_{\Delta(P_{W^+} \cap G_V)}((|\text{det}(\cdot)|^{\dim W + 1} \sigma_X \cdot |_{R_{r+1}} \boxtimes \pi_W) \boxtimes \pi_V, \mathbb{C}) \]

   When \( r = 0 \), \( P^+ \cap H^+ = \Delta SO(W) = H \), so

   \[ \text{Hom}_{\Delta(P_{W^+} \cap G_V)}((|\text{det}(\cdot)|^{\dim W + 1} \sigma_X \cdot |_{R_{r+1}} \boxtimes \pi_W) \boxtimes \pi_V, \mathbb{C}) = \text{Hom}_{H}(\pi_V \boxtimes \pi_W, \mathbb{C}) \neq 0 \]

\( \square \)
5.6. **An expedient: a trick using Schwartz homology.** In this section, we use the same trick as in [Xue20b] to give complete proof for the second inequality. In other sections, the condition we essentially use is that both \( \pi_V, \pi_W \) are in the form of (4.0.7). In this section, we need to use the fact that they lie in the generic packet. We first introduce a notion called the leading coefficient that is different from the leading index defined in Definition 4.0.7.

**Definition 5.6.1.** For a representation \( \pi_V \) in the form of (4.0.7), we define leading coefficient of \( \pi_V \) as the largest number among \( \text{Re}(s_{V,i}) + \frac{m_{V,i}}{2} \) (we take \( m_{V,i} = 0 \) when \( \rho_{V,i} = \text{sgn}^{v_i} \)). We denote it by \( \text{LC}(\pi_V) \).

**Theorem 5.6.2.** Suppose \( \sigma = |\text{det}|sD_m, \pi_{V_0} \in \Pi_{FM}(SO(V_0)) \) and \( \pi_V \) is in a generic packet. The Schwartz homologies of \((\sigma \rtimes \pi_{V_0}) \hat{\otimes} \pi_V \) vanish when \( \text{Re}(s) + \frac{m}{2} \geq \text{LC}(\pi_V) \).

**Proof.** From Lemma 4.0.2, it suffices to show that \( s + \frac{m}{2} \) does not equal any coefficient in the Harish-Chandra parameter of \( \pi_V \). From Proposition 4.0.5 and [KZ82], we can express \( \pi_V = \sigma_V \rtimes \pi_{V_0} \) where \( \pi_{V_0} \) is a limit of discrete series representation, \( \text{HC}(\pi_V) = (\text{HC}(\sigma_V), \text{HC}(\pi_{V_0})) \).

Since \( \text{Re}(s) + \frac{m}{2} \geq \text{LC}(\pi_V) \), \( s + \frac{m}{2} \) does not equal to any coefficient in \( \text{HC}(\sigma_V) \). From Lemma 4.0.2, both \( 1 + s + (-s) \) and \( 1 + (-s) - s \) are not positive integers, so \( s \) is not equal to any half-integers, so \( s + \frac{m}{2} \) does not equal to any coefficient in \( \text{HC}(\pi_{V_0}) \). □

Based on this vanishing theorem, one can write the analysis of Section 5.4 in the language of Schwartz homologies as Section 5.3, what can be obtained is the following multiplicity formula.

**Proposition 5.6.3.** When \( \sigma = |\text{det}|sD_m, \pi_W \in \Pi^{irr}_{CW}(SO(W)) \) and \( \pi_{V} \) is in a generic packet,

\[
m(\pi_V \boxtimes (|s + \frac{m}{2} \text{sgn}^{m+1} \rtimes \pi_W)) = m(|\text{det}sD_m \rtimes \pi_W) \boxtimes \pi_V)
\]

for \( \text{Re}(s) + \frac{m}{2} \geq \text{LC}(\pi_V) \).

**Proof.** After writing the analysis in Section 5.4, in the language of Schwartz homology. It suffices to verify (5.4.2)(5.4.6) in this setup.

When \( \text{Re}(s) + \frac{m}{2} \geq \text{LC}(\pi_V), \text{Re}(s) + \frac{m}{2} + \frac{1}{2} > \text{LC}(\pi_V) \), then we can apply Theorem 5.6.2 to prove the vanishing of the \( H^+ \)-Schwartz homologies of \( \Gamma^S_Z(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V \) as in Proposition 5.3.2.

When \( \text{Re}(s) + \frac{m}{2} \geq \text{LC}(\pi_V), \text{Re}(s) + \frac{m}{2} + k - \frac{1}{2} > \text{LC}(\pi_V) \) for any positive integer \( k \), then we can apply Theorem 5.6.2 to prove

\[
H^S(H^+, \Gamma^S_Z(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V) = H^S(H^+, \pi_V \boxtimes |s + \frac{m}{2} \rtimes \pi_W)
\]

as in Proposition 5.3.3 using Lemma 5.4.9. Here \((G^+, H^+, \xi^+)\) is a codimension-one Gross-Prasad triple associated to \((W \oplus H, V)\), where \( H \) is a hyperbolic plane.
Then the multiplicity formula follows from the proof in Section 5.3 verbatim. \(\square\)

Then we make use of Proposition 5.5.1 and Proposition 5.6.3 to prove the inequality (5.6.1)

\[ m(\pi_V \boxtimes \pi_W) \geq m(\pi_{V_0} \boxtimes \pi_{W_0}) \]

in Proposition 5.1.4 using a mathematical induction similar to that in Section 5.4.

**Proof.** We express \(\pi_V = \sigma_V \rtimes \pi_{V_0}, \pi_W = \sigma_W \rtimes \pi_{W_0}\) in the form of (4.0.7) and prove the inequality by mathematical induction on

\[ N(\sigma_V, \sigma_W) = \sum_{\text{Re}(s_{V,i}) \neq 0} n_{V,i} + \sum_{\text{Re}(s_{W,i}) \neq 0} n_{W,i}. \]

Here \(s_{V,i}, s_{W,i}, n_{V,i}, n_{W,i}\) are defined as in (4.0.7).

If \(N(\sigma_V, \sigma_W) = 0\), both \(\pi_V\) and \(\pi_W\) are tempered, then the inequality follows from the Conjecture 2 for tempered parameters, which was proved in [CL22].

In other cases, we may assume

\[ \text{Re}(s_{V,1}) + \frac{m_{V,1}}{2} \geq \text{Re}(s_{V,2}) + \frac{m_{V,2}}{2} \geq \cdots \geq \text{Re}(s_{V,l}) + \frac{m_{V,l}}{2} > 0 \]

\[ \text{Re}(s_{W,1}) + \frac{m_{W,1}}{2} \geq \text{Re}(s_{W,2}) + \frac{m_{W,2}}{2} \geq \cdots \geq \text{Re}(s_{W,l}) + \frac{m_{W,l}}{2} > 0. \]

Suppose the proposition holds when \(N(\sigma_V, \sigma_W) \leq k\), then when \(N(\sigma_V, \sigma_W) = k+1\), we consider the following cases.

Case 1: If \(l_V \neq 0\) and \(\text{Re}(s_{V,1}) + \frac{m_{V,1}}{2} \geq \text{Re}(s_{W,1}) + \frac{m_{W,1}}{2}\), then let \(\tilde{\sigma}_V = |\text{det}(\cdot)|^{s_{V,2}}\sigma_V \rtimes \cdots \rtimes |\text{det}(\cdot)|^{s_{V,l}}\sigma_{V,l} \).

(a) If \(n_{V,1} = 1\), from Proposition 5.5.1 we have

\[ m(\langle \sigma_V \rtimes \pi_{V_0} \rangle \boxtimes (\sigma_W \rtimes \pi_{W_0})) \geq m(\langle \sigma_W \rtimes \pi_{W_0} \rangle \boxtimes (\sigma_V \rtimes \pi_{V_0})), \]

(b) If \(n_{V,1} = 2\), let \(\tilde{\sigma}_V = |\cdot|^{s_{V,1} + \frac{m_{V,1}}{2}}\text{sgn}^{m_{V,1} + 1} \times \sigma_V\), and from Proposition 5.6.3, we have

\[ m(\langle \sigma_V \rtimes \pi_{V_0} \rangle \boxtimes (\sigma_W \rtimes \pi_{W_0})) = m(\langle \sigma_W \rtimes \pi_{W_0} \rangle \boxtimes (\tilde{\sigma}_V \rtimes \pi_{V_0})), \]

Since \(N(\tilde{\sigma}_V, \sigma_W), N(\tilde{\sigma}_V, \sigma_W) \leq N(\sigma_V, \sigma_W) - 1 = k\), we have

\[ m(\langle \sigma_W \rtimes \pi_{W_0} \rangle \boxtimes (\tilde{\sigma}_V \rtimes \pi_{V_0})) \geq m(\pi_{V_0} \boxtimes \pi_{W_0}), \quad m(\langle \sigma_W \rtimes \pi_{W_0} \rangle \boxtimes (\tilde{\sigma}_V \rtimes \pi_{V_0})) \geq m(\pi_{V_0} \boxtimes \pi_{W_0}) \]

Therefore, we have

\[ m(\langle \sigma_V \rtimes \pi_{V_0} \rangle \boxtimes (\sigma_W \rtimes \pi_{W_0})) \geq m(\pi_{V_0} \boxtimes \pi_{W_0}), \]

Case 2: If \(l_V = 0\) or \(\text{Re}(s_{V,1}) + \frac{m_{V,1}}{2} < \text{Re}(s_{W,1}) + \frac{m_{W,1}}{2}\), then we switch the order of \(V, W\) to reduce to Case 1. More explicitly, we take \(\sigma_W^{(s')} = |\cdot|^{s'} \rtimes \sigma_{W_0}\). From Lemma 5.1.3, there is a \(s' \in i\mathbb{R}\) such that

\[ m(\langle \sigma_V \rtimes \pi_{V_0} \rangle \boxtimes (\sigma_W \rtimes \pi_{W_0})) = m(\langle \sigma_W^{(s')} \rtimes \pi_{W_0} \rangle \boxtimes (\sigma_V \rtimes \pi_{V_0})). \]
From [SV80, Theorem 1.1] and Langlands classification, we may assume \( \sigma^{(s')} \rtimes \pi_{W_0} \) is irreducible and thus in generic packet. Since the pair \((\sigma^{(s')}, \sigma_V)\) belongs to Case 1 and \(N(\sigma^{(s')} \rtimes \pi_{W_0}, \sigma_V) = k + 1\), so

\[
m((\sigma^{(s')} \rtimes \pi_{W_0}) \boxtimes (\sigma_V \rtimes \pi_{V_0})) \geq m(\pi_{V_0} \boxtimes \pi_{W_0}).
\]

Therefore, we have

\[
m((\sigma_V \rtimes \pi_{V_0}) \boxtimes (\sigma_W \rtimes \pi_{W_0})) \geq m(\pi_{V_0} \boxtimes \pi_{W_0}).
\]

Then the proposition follows from mathematical induction on \( N(\sigma_V, \sigma_W) \). \(\square\)

References

[AGRS10] Avraham Aizenbud, Dmitry Gourevitch, Stephen Rallis, and Gérard Schiffmann. Multiplicity one theorems. *Annals of Mathematics*, pages 1407–1434, 2010.

[Ato18] Hiraku Atobe. The local theta correspondence and the local Gan–Gross–Prasad conjecture for the symplectic-metaplectic case. *Mathematische Annalen*, 371(1):225–295, 2018.

[BK14] Joseph Bernstein and Bernhard Krötz. Smooth Fréchet globalizations of Harish-Chandra modules. *Israel Journal of Mathematics*, 199(1):45–111, 2014.

[BP14] Raphaël Beuzart-Plessis. Expression d’un facteur epsilon de paire par une formule intégrale. *Canadian Journal of Mathematics*, 66(5):993–1049, 2014.

[BP16] Raphaël Beuzart-Plessis. La conjecture locale de Gross-Prasad pour les représentations tempérées des groupes unitaires, 2016.

[BP19] Raphaël Beuzart-Plessis. A local trace formula for the Gan-Gross-Prasad conjecture for unitary groups: the archimedean case. *Asterisque*, 2019.

[BP22] Raphaël Beuzart-Plessis. Relative trace formulae and the gan-gross-prasad conjectures. *Proceedings of the 2022 ICM*, 2022.

[BT65] Armand Borel and Jacques Tits. Groupes réductifs. *Publications Mathématiques de l’IHÉS*, 27:55–151, 1965.

[Cas89] William Casselman. Canonical extensions of Harish-Chandra modules to representations of \( g \). *Canadian Journal of Mathematics*, 41(3):385–438, 1989.

[CCZ23] Cheng Chen, Rui Chen, and Jiangliang Zou. Fourier-Jacobi on reals: basic cases. *In preparation*, 2023.

[CHM00] William Casselman, Henryk Hecht, and Dragan Milicic. Bruhat filtrations and Whittaker vectors for real groups. In *Proceedings of Symposia in Pure Mathematics*, volume 68, pages 151–190. Providence, RI; American Mathematical Society; 1998, 2000.

[CL22] Cheng Chen and Zhilin Luo. The local Gross-Prasad conjecture over \( \mathbb{R} \): Epsilon dichotomy. *arXiv preprint arXiv:2204.01212*, 2022.

[CS20] Yangyang Chen and Binyong Sun. Schwartz homologies of representations of almost linear Nash groups. *Journal of Functional Analysis*, page 108817, 2020.

[DC91] Fokko Du Cloux. Sur les représentations différentiables des groupes de Lie algébriques. In *Annales scientifiques de l’École normale supérieure*, volume 24, pages 257–318, 1991.

[GGP12] Wee Teck Gan, Benedict H Gross, and Dipendra Prasad. Symplectic local root numbers, central critical \( l \)-values, and restriction problems in the representation theory of classical groups. *Astérisque*, 346(1):1–109, 2012.
[GI16] Wee Teck Gan and Atsushi Ichino. The Gross–Prasad conjecture and local theta correspondence. *Inventiones mathematicae*, 206(3):705–799, 2016.

[God74] Roger Godement. Notes on Jacquet–Langlands theory. *Matematika*, 18(2):28–78, 1974.

[GP92] Benedict H Gross and Dipendra Prasad. On the decomposition of a representation of $SO_n$ when restricted to $SO_{n-1}$. *Canadian Journal of Mathematics*, 44(5):974–1002, 1992.

[GP94] Benedict H Gross and Dipendra Prasad. On irreducible representations of $SO(2n+1) \times SO(2m)$. *Canadian Journal of Mathematics*, 46(5):930–950, 1994.

[GSS19] Dmitry Gourevitch, Siddhartha Sahi, and Eitan Sayag. Analytic continuation of equivariant distributions. *International Mathematics Research Notices*, 2019(23):7160–7192, 2019.

[Jac09] Hervé Jacquet. Archimedean Rankin-Selberg integrals. *Contemporary Mathematics*, 14:57, 2009.

[JSZ22] Dihua Jiang, Dongwen Liu, and Lei Zhang. Arithmetic wavefront sets and generic $l$-packets. *arXiv preprint arXiv:2207.04700*, 2022.

[JSZ10] Dihua Jiang, Binyong Sun, and Chen-Bo Zhu. Uniqueness of Bessel models: the archimedean case. *Geometric and Functional Analysis*, 20(3):690–709, 2010.

[JZ20] Dihua Jiang and Lei Zhang. Arthur parameters and cuspidal automorphic modules of classical groups. *Annals of Mathematics*, 191(3):739–827, 2020.

[Kna94] AW Knapp. Local Langlands correspondence: the archimedean case. In *Proc. Sympos. Pure Math*, volume 55, Part 2, pages 393–410, 1994.

[Kos75] Bertram Kostant. On the tensor product of a finite and an infinite dimensional representation. *Journal of Functional Analysis*, 20(4):257–285, 1975.

[KZ82] Anthony W Knapp and Gregg J Zuckerman. Classification of irreducible tempered representations of semisimple groups. *Annals of Mathematics*, pages 389–455, 1982.

[Lan70] Robert P Langlands. Problems in the theory of automorphic forms to solomon bochner in gratitude. In *Lectures in modern analysis and applications III*, pages 18–61. Springer, 1970.

[Lan73] RP Langlands. On thé classification of irreducible représentations of real algebraic groups. *preprint (sic), IAS, Princeton*, 1973.

[Luo21] Zhilin Luo. A local trace formula for the local gross-prasad conjecture for special orthogonal groups. *thesis*, 2021.

[Möll17] Jan Möllers. Symmetry breaking operators for strongly spherical reductive pairs and the Gross-Prasad conjecture for complex orthogonal groups. *arXiv preprint arXiv:1705.06109*, 2017.

[MW12] Colette Mœglin and Jean-Loup Waldspurger. La conjecture locale de Gross-Prasad pour les groupes spéciaux orthogonaux: le cas général. *Astérisque*, 347:167–216, 2012.

[Pra17] Dipendra Prasad. Reducible principal series representations, and langlands parameters for real groups. *arXiv preprint arXiv:1705.01445*, 2017.

[Sun15] Binyong Sun. Almost linear Nash groups. *Chinese Annals of Mathematics, Series B*, 36(3):355–400, 2015.

[SV80] Birgit Speh and David A Vogan. Reducibility of generalized principal series representations. *Acta Mathematica*, 145:227–299, 1980.

[SZ12] Binyong Sun and Chen-Bo Zhu. Multiplicity one theorems: the archimedean case. *Annals of Mathematics*, pages 23–44, 2012.
[Vog93] David A Vogan. The local langlands conjecture. *Contemporary Mathematics*, 145:305–305, 1993.

[Wal94] Nolan R Wallach. Real reductive groups II. *Bull. Amer. Math. Soc.*, 30:157–158, 1994.

[Wal10] J-L Waldspurger. Une formule intégrale reliée à la conjecture locale de Gross–Prasad. *Compositio Mathematica*, 146(5):1180–1290, 2010.

[Wal12a] Jean-Loup Waldspurger. Calcul d’une valeur d’un facteur ε par une formule intégrale par. *Astérisque*, 347:1–102, 2012.

[Wal12b] Jean-Loup Waldspurger. La conjecture locale de Gross-Prasad pour les représentations tempérées des groupes spéciaux orthogonaux. *Astérisque*, No. 347:103–165, 2012.

[Wal12c] Jean-Loup Waldspurger. Une formule intégrale reliée à la conjecture locale de gross-prasad, 2e partie: extension aux représentations tempérées. *Astérisque*, 346:171–312, 2012.

[Wal12d] Jean-Loup Waldspurger. Une variante d’un résultat de Aizenbud, Gourevitch, Rallis et Schiffmann. *Astérisque*, no. 346:313–318, 2012.

[War12] Garth Warner. *Harmonic analysis on semi-simple Lie groups I*, volume 188. Springer Science & Business Media, 2012.

[Xue] Hang Xue. Fourier–jacobi models for real unitary groups.

[Xue20a] Hang Xue. Bessel models for real unitary groups: the tempered case. *preprint*, 1(5):7, 2020.

[Xue20b] Hang Xue. Bessel models for unitary groups and Schwartz homology. *preprint*, 2020.

School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA

Email address: chen5968@umn.edu