HEAT-KERNEL ESTIMATES FOR RANDOM WALK AMONG RANDOM CONDUCTANCES WITH HEAVY TAIL

OMAR BOUKHADRA∗

Centre de Mathématiques et Informatique (CMI),
Université de Provence;
Département de Mathématiques, Université de Constantine

Abstract. We study models of discrete-time, symmetric, \( \mathbb{Z}^d \)-valued random walks in random environments, driven by a field of i.i.d. random nearest-neighbor conductances \( \omega_{xy} \in [0, 1] \), with polynomial tail near 0 with exponent \( \gamma > 0 \). We first prove for all \( d \geq 5 \) that the return probability shows an anomalous decay (non-Gaussian) that approaches (up to sub-polynomial terms) a random constant times \( n^{-2} \) when we push the power \( \gamma \) to zero. In contrast, we prove that the heat-kernel decay is as close as we want, in a logarithmic sense, to the standard decay \( n^{-d/2} \) for large values of the parameter \( \gamma \).

keywords : Random walk, Random environments, Markov chains, Random conductances, Percolation.

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1. Introduction and results

The main purpose of this work is the derivation of heat-kernel bounds for random walks \( (X_n)_{n \in \mathbb{N}} \) among polynomial lower tail random conductances with exponent \( \gamma > 0 \), on \( \mathbb{Z}^d, d > 4 \). We show that the heat-kernel exhibits opposite behaviors, anomalous and standard, for small and large values of \( \gamma \).

Random walks in reversible random environments are driven by the transition matrix
\[
P_\omega(x, y) = \frac{\omega_{xy}}{\pi_\omega(x)}.
\]
where \( (\omega_{xy}) \) is a family of random (non-negative) conductances subject to the symmetry condition \( \omega_{xy} = \omega_{yx} \). The sum \( \pi_\omega(x) = \sum_y \omega_{xy} \) defines an invariant, reversible measure for the corresponding discrete-time Markov chain. In most situations \( \omega_{xy} \) are non-zero only for nearest neighbors on \( \mathbb{Z}^d \) and are sampled from a shift-invariant, ergodic or even i.i.d. measure \( \mathbb{Q} \).

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∗E-mail address : omar.boukhadra@cmi.univ-mrs.fr.
One general class of results is available for such random walks under the additional assumptions of uniform ellipticity,

$$\exists \alpha > 0 : \ Q(\alpha < \omega_b < 1/\alpha) = 1$$

and the boundedness of the jump distribution,

$$\exists R < \infty : |x| \geq R \Rightarrow P_\omega(0, x) = 0, \ \mathbb{Q} - a.s.$$ 

One has then the standard local-CLT like decay of the heat-kernel ($c_1, c_2$ are absolute constants), as proved by Delmotte [Del99]:

$$P_\omega^n(x, y) \leq \frac{c_1}{n^{d/2}} \exp \left\{ -c_2 \frac{|x - y|^2}{n} \right\}. \quad (1.2)$$

Once the assumption of uniform ellipticity is relaxed, matters get more complicated. The most-intensely studied example is the simple random walk on the infinite cluster of supercritical bond percolation on $\mathbb{Z}^d$, $d \geq 2$. This corresponds to $\omega_{xy} \in \{0, 1\}$ i.i.d. with $Q(\omega_b = 1) > p_c(d)$ where $p_c(d)$ is the percolation threshold (cf. [Grim99]). Here an annealed invariance principle has been obtained by De Masi, Ferrari, Goldstein and Wick [DFGW85]–[DFGW89] in the late 1980s. More recently, Mathieu and Remy [MR04] proved the on-diagonal (i.e., $x = y$) version of the heat-kernel upper bound (1.2)—a slightly weaker version of which was also obtained by Heicklen and Hoffman [HH05]—and, soon afterwards, Barlow [Ba04] proved the full upper and lower bounds on $P_\omega^n(x, y)$ of the form (1.2). (Both these results hold for $n$ exceeding some random time defined relative to the environment in the vicinity of $x$ and $y$.) Heat-kernel upper bounds were then used in the proofs of quenched invariance principles by Sidoravicius and Sznitman [SSz04] for $d \geq 4$, and for all $d \geq 2$ by Berger and Biskup [BB07] and Mathieu and Piatnitski [MPia07].

We consider in our case a family of symmetric, irreducible, nearest-neighbor Markov chains on $\mathbb{Z}^d$, $d \geq 5$, driven by a field of i.i.d. bounded random conductances $\omega_{xy} \in [0, 1]$ and subject to the symmetry condition $\omega_{xy} = \omega_{yx}$. These are constructed as follows. Let $\Omega$ be the set of functions $\omega : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}_+$ such that $\omega_{xy} > 0$ iff $x \sim y$, and $\omega_{xy} = \omega_{yx}$ ( $x \sim y$ means that $x$ and $y$ are nearest neighbors). We call elements of $\Omega$ environments.

We choose the family $\{\omega_b, b = (x, y), x \sim y, b \in \mathbb{Z}^d \times \mathbb{Z}^d\}$ i.i.d according to a law $Q$ on $(\mathbb{R}_+^\times)^\mathbb{Z}^d$ such that

$$\omega_b \leq 1 \quad \text{for all } b;$$

$$Q(\omega_b \leq a) \sim a^\gamma \quad \text{when } a \downarrow 0,$$

where $\gamma > 0$ is a parameter. Therefore, the conductances are $\mathbb{Q}$-a.s. positive.
In a recent paper, Fontes and Mathieu [FM06] studied continuous-time random walks on $\mathbb{Z}^d$ which are defined by generators $\mathcal{L}_\omega$ of the form

$$(\mathcal{L}_\omega f)(x) = \sum_{y \sim x} \omega_{xy}[f(y) - f(x)],$$

with conductances given by

$$\omega_{xy} = \omega(x) \wedge \omega(y)$$

for i.i.d. random variables $\omega(x) > 0$ satisfying (1.3). For these cases, it was found that the annealed heat-kernel, $\int dQ(\omega) P^\omega_0(X_t = 0)$, exhibits an anomalous decay, for $\gamma < d/2$. Explicitly, from [FM06], Theorem 4.3, we have

$$\int dQ(\omega) P^\omega_0(X_t = 0) = t^{-(\gamma + 1/2) + o(1)}, \quad t \to \infty. \tag{1.4}$$

In addition, in a more recent paper, Berger, Biskup, Hoffman and Kozma [BBHK08], provided universal upper bounds on the quenched heat-kernel by considering the nearest-neighbor simple random walk on $\mathbb{Z}^d$, $d \geq 2$, driven by a field of i.i.d. bounded random conductances $\omega_{xy} \in [0,1]$. The conductance law is i.i.d. subject to the condition that the probability of $\omega_{xy} > 0$ exceeds the threshold $p_c(d)$ for bond percolation on $\mathbb{Z}^d$. For environments in which the origin is connected to infinity by bonds with positive conductances, they studied the decay of the $2n$-step return probability $P^\omega_{2n}(0,0)$. They have proved that $P^\omega_{2n}(0,0)$ is bounded by a random constant times $n^{-d/2}$ in $d = 2, 3$, while it is $o(n^{-2})$ in $d \geq 5$ and $O(n^{-2} \log n)$ in $d = 4$. More precisely, from [BBHK08], Theorem 2.1, we have for almost every $\omega \in \{0 \in C_\infty\}$ ($C_\infty$ represents the set of sites that have a path to infinity along bonds with positive conductances), and for all $n \geq 1$.

$$P^\omega_{2n}(0,0) \leq C(\omega) \begin{cases} n^{-d/2}, & d = 2, 3, \\ n^{-2} \log n, & d = 4, \\ n^{-2}, & d \geq 5, \end{cases} \tag{1.5}$$

where $C(\omega)$ is a random positive variable.

On the other hand, to show that those general upper bounds (cf. (1.5)) in $d \geq 5$ represent a real phenomenon, they produced examples with anomalous decay approaching $1/n^2$, for i.i.d. laws $Q$ on bounded nearest-neighbor conductances with lower tail much heavier than polynomial and with $Q(\omega_b > 0) > p_c(d)$. We quote Theorem 2.2 from [BBHK08]:

**Theorem 1.1** (1) Let $d \geq 5$ and $\kappa > 1/d$. There exists an i.i.d. law $Q$ on bounded, nearest-neighbor conductances with $Q(\omega_b > 0) > p_c(d)$ and a random
variable $C = C(\omega)$ such that for almost every $\omega \in \{0 \in C_\infty\}$,

$$P_\omega^{2n}(0, 0) \geq C(\omega) \frac{e^{- (\log n)^k}}{n^2}, \quad n \geq 1. \quad (1.6)$$

(2) Let $d \geq 5$. For every increasing sequence $\{\lambda_n\}_{n=1}^\infty$, $\lambda_n \to \infty$, there exists an i.i.d. law $Q$ on bounded, nearest-neighbor conductances with $Q(\omega_b > 0) > p_c(d)$ and an a.s. positive random variable $C = C(\omega)$ such that for almost every $\omega \in \{0 \in C_\infty\}$,

$$P_\omega^n(0, 0) \geq \frac{C(\omega)}{\lambda_n n^2}. \quad (1.7)$$

along a subsequence that does not depend on $\omega$.

The distributions that they use in part (1) of Theorem 1.1 have a tail near zero of the general form

$$Q(\omega_{xy} < s) \approx |\log(s)|^{-\theta} \quad (1.8)$$

with $\theta > 0$.

Berger, Biskup, Hoffman and Kozma [BBHK08] called attention to the fact that the construction of an estimate of the anomalous heat-kernel decay for random walk among polynomial lower tail random conductances on $\mathbb{Z}^d$, seems to require subtle control of heat-kernel lower bounds which go beyond the estimates that can be easily pulled out from the literature. In the present paper, we give a response to this question and show that every distribution with an appropriate power-law decay near zero, can serve as such example, and that when we push the power to zero. The lower bound obtained for the return probability approaches (up to sub-polynomial terms) the upper bound supplied by [BBHK08] and that for all $d \geq 5$.

Here is our first main result whose proof is given in section 2:

**Theorem 1.2** Let $d \geq 5$. There exists a positive constant $\delta(\gamma)$ depending only on $d$ and $\gamma$ such that $Q$-a.s., there exists $C = C(\omega) < \infty$ and for all $n \geq 1$

$$P_\omega^{2n}(0, 0) \geq \frac{C}{n^{2+\delta(\gamma)}} \quad \text{and} \quad \delta(\gamma) \xrightarrow[\gamma \to 0]{} 0. \quad (1.9)$$

**Remark 1.3**

(1) The proof tells us in fact, with (1.5), that for $d \geq 5$ we have almost surely

$$-2[1 + d(2d - 1)\gamma] \leq \liminf_n \frac{\log P_\omega^{2n}(0, 0)}{\log n} \leq \limsup_n \frac{\log P_\omega^{2n}(0, 0)}{\log n} \leq -2. \quad (1.10)$$
As we were reminded by M. Biskup and T.M. Prescott, the invariance principle (CLT) (cf Theorem 2.1. in [BP07] and Theorem 1.3 in [M08]) automatically implies the “usual” lower bound on the heat-kernel under weaker conditions on the conductances. Indeed, the Markov property and reversibility of $X$ yield

$$P_0^\omega (X_{2n} = 0) \geq \frac{\pi_\omega (0)}{2d} \sum_{x \in \mathbb{C}_\infty, |x| \leq \sqrt{n}} P_0^\omega (X_n = x)^2.$$ 

Cauchy-Schwarz then gives

$$P_0^\omega (X_{2n} = 0) \geq P_0^\omega (|X_n| \leq \sqrt{n})^2 \frac{\pi_\omega (0)/2d}{|\mathbb{C}_\infty \cap [-\sqrt{n}, +\sqrt{n}]^d|}.$$ 

Now the invariance principle implies that $P_0^\omega (|X_n| \leq \sqrt{n})^2$ has a positive limit as $n \to \infty$ and the Spatial Ergodic Theorem shows that $|\mathbb{C}_\infty \cap [-\sqrt{n}, +\sqrt{n}]^d|$ grows proportionally to $n^{d/2}$. Hence we get

$$P_0^\omega (X_{2n} = 0) \geq \frac{C(\omega)}{n^{d/2}}, \quad n \geq 1,$$

with $C(\omega) > 0$ a.s. on the set $\{0 \in \mathbb{C}_\infty\}$. Note that, in $d = 2, 3$, this complements nicely the “universal” upper bounds derived in [BBHK08]. In $d = 4$, the decay is at most $n^{-2} \log n$ and at least $n^{-2}$.

The result of Fontes and Mathieu (1.4) (cf. [FM06], Theorem 4.3) encourages us to believe that the quenched heat-kernel has a standard decay when $\gamma \geq d/2$, but the construction seems to require subtle control of heat-kernel upper bounds. In the second result of this paper whose proof is given in section 3, we prove, for all $d \geq 5$, that the heat-kernel decay is as close as we want, in a logarithmic sense, to the standard decay $n^{-d/2}$ for large values of the parameter $\gamma$. For the cases where $d = 2, 3$, we have a standard decay of the quenched return probability under weaker conditions on the conductances (see Remark 1.3).

**Theorem 1.4** Let $d \geq 5$. There exists a positive constant $\delta(\gamma)$ depending only on $d$ and $\gamma$ such that $\mathbb{Q}$-a.s.,

$$\limsup_{n \to +\infty} \sup_{x \in \mathbb{Z}^d} \frac{\log P_0^\omega (0, x)}{\log n} \leq -\frac{d}{2} + \delta(\gamma) \quad \text{and} \quad \delta(\gamma) \xrightarrow[\gamma \to +\infty]{} 0. \quad (1.11)$$

In what follows, we refer to $P_\omega^x (\cdot)$ as the *quenched* law of the random walk $X = (X_n)_{n \geq 0}$ on $((\mathbb{Z}^d)^\mathbb{N}, \mathcal{G})$ with transitions given in (1.1) in the environment $\omega$, where $\mathcal{G}$ is the $\sigma$-algebra generated by cylinder functions, and let $\mathbb{P} := \mathbb{Q} \otimes P_0^\omega$ be
the so-called *annealed* semi-direct product measure law defined by

\[ P(F \times G) = \int_F Q(d\omega) P_0^\omega(G), \quad F \in \mathcal{F}, G \in \mathcal{G}. \]

where \( \mathcal{F} \) denote the Borel \( \sigma \)-algebra on \( \Omega \) (which is the same as the \( \sigma \)-algebra generated by cylinder functions).

2. Anomalous heat-kernel decay

In this section we provide the proof of Theorem 1.2.

We consider a family of bounded nearest-neighbor conductances \((\omega_b) \in \Omega = [0,1]^{B^d}\) where \( b \) ranges over the set \( B^d \) of unordered pairs of nearest neighbors in \( \mathbb{Z}^d \).

The law \( Q \) of the \( \omega \)'s will be i.i.d. subject to the conditions given in (1.3).

We prove this lower bound by following a different approach of the one adopted by Berger, Biskup, Hoffman and Kozma [BBHK08] to prove (1.6–1.7). In fact, they prove that in a box of side length \( \ell_n \) there exists a configuration where a strong bond with conductance of order 1, is separated from other sites by bonds of strength \( 1/n \), and (at least) one of these “weak” bonds is connected to the origin by a “strong” path not leaving the box. Then the probability that the walk is back to the origin at time \( n \) is bounded below by the probability that the walk goes directly towards the above pattern (this costs \( e^{O(\ell_n)} \) of probability) then crosses the weak bond (which costs \( 1/n \)), spends time \( n - 2\ell_n \) on the strong bond (which costs only \( O(1) \) of probability), then crosses a weak bond again (another factor of \( 1/n \)) and then heads towards the origin to get there on time (another \( e^{O(\ell_n)} \) term). The cost of this strategy is \( O(1)e^{O(\ell_n)}n^{-2} \) so if \( \ell_n = o(\log n) \) then we get leading order \( n^{-2} \).

Our method for proving Theorem 1.2 is, in fact, simple - we note that due to the reversibility of the walk and with a good use of Cauchy-Schwartz, one does not need to condition on the exact path of the walk, but rather show that the walker has a relatively large probability of staying within a small box around the origin.

Our objective will consist in showing that for almost every \( \omega \), the probability that the random walk when started at the origin is at time \( n \) inside the box \( B_{n^\delta} = [-3n^\delta, 3n^\delta]^d \), is greater than \( c/n \) (where \( c \) is a constant and \( \delta = \delta(\gamma) \downarrow 0 \)). Hence we will get \( P_\omega^{2n}(0,0)/\pi(0) \geq c/n^{2+6\delta} \) by virtue of the following inequality which, for almost every environment \( \omega \), derives from the reversibility of \( X \), Cauchy-Schwarz inequality and (1.3):

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\[
\frac{P^{2n}(0, 0)}{\pi_\omega(0)} \geq \sum_{y \in B_{n^\delta}} \frac{P^n(0, y)^2}{\pi_\omega(y)} \\
\geq \left( \sum_{y \in B_{n^\delta}} P^n(0, y) \right) \frac{1}{\pi_\omega(B_{n^\delta})} \\
\geq \frac{P_0^\omega(X_n \in B_{n^\delta})^2}{\#B_{n^\delta}}. \tag{2.1}
\]

In order to do this, our strategy is to show that the random walk meets a trap, with positive probability, before getting out from \([-3n^\delta, 3n^\delta]^d\), where, by definition, a trap is an edge of conductance of order 1 that can be reached only by crossing an edge of order 1/n. The random walk, being imprisoned in the trap inside the box \([-3n^\delta, 3n^\delta]^d\), will not get out from this box before time \(n\) with positive probability. Then the Markov property yields \(P_0^\omega(X_n \in [-3n^\delta, 3n^\delta]^d) \geq c/n\). Thus, we will be brought to follow the walk until it finds a specific configuration in the environment.

First, we will need to prove one lemma. Let \(B_N = [-3N, 3N]^d\) be the box centered at the origin and of radius \(3N\) and define \(\partial B_N\) to be its inner boundary, that is, the set of vertices in \(B_N\) which are adjacent to some vertex not in \(B_N\). We have \(#B_N \leq (7N)^d\). Let \(H_0 = 0\) and define \(H_N, N \geq 1\), to be the hitting time of \(\partial B_N\), i.e.

\[H_N = \inf\{n \geq 0 : X_n \in \partial B_N\}.
\]

The box \(B_N\) being finite for \(N\) fixed, we have then \(H_N < \infty\) a.s., \(\forall N \geq 1\).

Let \(\hat{e}_i, i = 1, \ldots, d\), denote the canonical unit vectors in \(\mathbb{Z}^d\), and let \(x \in \mathbb{Z}^d\), with \(x := (x_1, \ldots, x_d)\). Define \(i_0 := \max\{i : |x_i| \geq |x_j|, \forall j \neq i\}\) and let \(\epsilon(x) : \mathbb{Z}^d \to \{-1, 1\}\) be the function such that

\[\epsilon(x) = \begin{cases} +1 & \text{if } x_{i_0} \geq 0 \\ -1 & \text{if } x_{i_0} < 0 \end{cases}
\]

Now, let \(\alpha, \xi\) be positive constants such that \(Q(\omega_b \geq \xi) > 0\). Define \(A_N(x)\) to be the event that the configuration near \(x, y = x + \epsilon(x)\hat{e}_{i_0}\) and \(z = x + 2\epsilon(x)\hat{e}_{i_0}\) is as follows:

1. \(\frac{1}{2}N^{-\alpha} < \omega_{xy} \leq N^{-\alpha}\).
2. \(\omega_{yz} \geq \xi\).
3. every other bond emanating out of \(y\) or \(z\) has \(\omega_b \leq N^{-\alpha}\).
The event $\mathcal{A}_N(x)$ so constructed involves a collection of $4d - 1$ bonds that will be denoted by $\mathcal{C}(x)$, i.e.

$$\mathcal{C}(x) := \{[x, y], [y, z], [y, y'], [z, z'], [z, z_0']; y = x + \epsilon(x)\hat{e}_{i_0}, z = x + 2\epsilon(x)\hat{e}_{i_0},$$

$$y' = y \pm \hat{e}_i, z' = z \pm \hat{e}_i, \forall i \neq i_0, z'_0 = z + \epsilon(x)\hat{e}_{i_0}\}$$

Let us note that if $x \in \partial B_N$, for some $N \geq 1$, the collection $\mathcal{C}(x)$ is outside the box $B_N$ and if $y \in \partial B_K$, for $K \neq N$, we have $\mathcal{C}(x) \cap \mathcal{C}(y) = \emptyset$. If the bonds of the collection $\mathcal{C}(x)$ satisfy the conditions of the event $\mathcal{A}_N(x)$, we agree to call it a trap that we will denote by $\mathfrak{P}_N$.

The lemma says then that:

**Lemma 2.1** The family $\{\mathcal{A}_N^k = \mathcal{A}_N(X_{H_k})\}_{k=0}^{N-1}$ is $\mathbb{P}$-independent for each $N$.

**Proof.** The occurrence of the event $\mathcal{A}_N(X_{H_k})$ means that the random walk $X$ has met a trap $\mathfrak{P}_N$ situated outside of the box $B_k$ when it has hit for the first time the boundary of the box $B_k$.

Let $q_N$ be the $\mathbb{Q}$-probability of having the configuration of the trap $\mathfrak{P}_N$. We have $q_N = \mathbb{Q}(\mathcal{A}_N(x)) = \mathbb{P}[\mathcal{A}_N(X_{H_k})], \forall x \in \partial B_k$ and $\forall k \leq N - 1$. Indeed, by virtue of the i.i.d. character of the conductances and the Markov property, when the random walk hits the boundary of $B_k$ for the first time at some element $x$, the probability that the collection $\mathcal{C}(x)$ constitutes a trap, i.e., satisfies the conditions of the event $\mathcal{A}_N(x)$, depends only on the edges of the collection $\mathcal{C}(x)$, which have not been visited before.

Let $k_1 < k_2 \leq N - 1$ and $x \in \partial B_{k_2}$, we have then

$$\mathbb{P} \left[ \mathcal{A}_N^{k_1}, X_{H_{k_2}} = x, \mathcal{A}_N^{k_2} \right] = \mathbb{P} \left[ \left\{ \mathcal{A}_N^{k_1}, X_{H_{k_2}} = x \right\} \cap \mathcal{A}_N(x) \right]$$

$$= \mathbb{P} \left[ \mathcal{A}_N^{k_1}, X_{H_{k_2}} = x \right] \mathbb{P} \left[ \mathcal{A}_N(x) \right]$$

$$= q_N \mathbb{P} \left[ \mathcal{A}_N^{k_1}, X_{H_{k_2}} = x \right],$$

since the events $\{\mathcal{A}_N^{k_1}, X_{H_{k_2}} = x\}$ and $\mathcal{A}_N(x)$ depend respectively on the conductances of the bonds of $B_{k_2}$ and the conductances of the bonds of the collection $\mathcal{C}(x)$ which is situated outside the box $B_{k_2}$ when $x \in \partial B_{k_2}$.

Thus

$$\mathbb{P} \left[ \mathcal{A}_N^{k_1} \mathcal{A}_N^{k_2} \right] = \sum_{x \in \partial B_{k_2}} \mathbb{P} \left[ \mathcal{A}_N^{k_1}, X_{H_{k_2}} = x, \mathcal{A}_N^{k_2} \right]$$

$$= q_N \sum_{x \in \partial B_{k_2}} \mathbb{P} \left[ \mathcal{A}_N^{k_1}, X_{H_{k_2}} = x \right]$$

$$= q_N \mathbb{P} \left[ \mathcal{A}_N^{k_1} \right] = q_N^2.$$
With some adaptations, this reasoning remains true in the case of more than two events $A^k_N$.

We come now to the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Let $d \geq 5$ and $\gamma > 0$. Set $\alpha = \frac{1-\epsilon}{(4d-2)\gamma}$ for arbitrary positive constant $\epsilon < 1$ (the constant $\alpha$ is the same used in the definition of the event $A_N(x)$). As seen before (cf. (2.1)), for almost every environment $\omega$, the reversibility of $X$, Cauchy-Schwarz inequality and (1.3) give

$$P_{\omega}(0,0) \geq \frac{|B_n^{1/\alpha}|}{\pi_{\omega}(0)} P_0(\omega_0(X_n \in B_n^{1/\alpha}))^2 \leq \frac{c}{N^{1-\epsilon}}.$$  \hspace{1cm} (2.2)

By the assumption (1.3) on the conductances and the definition of the event $A_N(x)$, the probability of having the configuration of the trap $P_N$ is greater than $cN^{-\epsilon}$ (where $c$ is a constant that we use henceforth as a generic constant). Indeed, when $N$ is large enough, we have

$$q_N = \mathbb{Q}\left(\frac{1}{2} N^{1-\epsilon} < \omega_{xy} \leq N^{-\epsilon}\right) \mathbb{Q}(\omega_{yz} \geq \xi) [\mathbb{Q}(\omega_b \leq N^{-\epsilon})]^{4d-3} \geq \frac{c}{N^{1-\epsilon}}.$$  \hspace{1cm} (2.3)

Consider now the following event

$$\Lambda_N := \bigcup_{k=0}^{N-1} A^k_N.$$  \hspace{1cm} (2.4)

The event $\Lambda_N$ so defined may be interpreted as follows: at least, one among the $N$ disjoint collections $\mathcal{C}(X_{H_k}), k \leq N - 1$, constitutes a trap $\mathcal{P}_N$. The events $A^k_N$ being independent by lemma 2.1, we have

$$\mathbb{P}[\Lambda_N^c] \leq (1 - cN^{-\epsilon})^N \leq \exp\left\{ N \log \left(1 - cN^{-\epsilon}\right) \right\} \leq \exp\left\{ -cN^\epsilon \right\}. \hspace{1cm} (2.3)$$

Chebyshev inequality and (2.3) then give

$$\sum_{N=1}^{\infty} \mathbb{Q}\{\omega : P_0^c(\Lambda_N^c) \geq 1/2\} \leq 2 \sum_{N=1}^{\infty} \mathbb{P}[\Lambda_N^c] < +\infty. \hspace{1cm} (2.4)$$

It results by Borel-Cantelli lemma that for almost every $\omega$, there exists $N_0 \geq 1$ such that for each $N \geq N_0$, the event $A_N(x)$ occurs inside the box $B_N$ with positive probability (greater than 1/2) on the path of $X$, for some $x \in B_{N-1}$. For almost every $\omega$, one may say that $X$ meets with positive probability a trap $\mathcal{P}_N$ at some site $x \in B_{N-1}$ before getting outside of $B_N$. 


Suppose that \( N \geq N_0 \) and let \( n \) be such that \( N^\alpha \leq n < (N+1)^\alpha \). Define
\[
D_N := \begin{cases} 
\inf\{k \leq N - 1 : \mathcal{A}_N^k \text{ occurs}\} & \text{if } \Lambda_N \text{ occurs} \\
+\infty & \text{otherwise},
\end{cases}
\]
to be the rank of the first among the \( N \) collections \( \mathcal{C}(X_{H_k}), k \leq N - 1 \), that constitutes a trap \( \mathfrak{P}_N \). If \( D_N = k \), the random variable \( D_N \) so defined depends only on the steps of \( X \) up to time \( H_k \). Thus, if \( D_N = k \), we have \( X_{H_k} \in B_{N-1} \) and \( \mathcal{C}(X_{H_k}) \) constitutes a trap \( \mathfrak{P}_N \). So, if we set \( X_{H_k} = x \), the bond \([x, y]\) (of the trap \( \mathfrak{P}_N \)) will have then a conductance of order \( N^{-\alpha} \). In this case, the probability for the random walk, when started at \( X_{H_k} = x \), to cross the bond \([x, y]\) is by the property (1) of the definition of the event \( \mathcal{A}_N(x) \) above greater than
\[
\frac{(1/2)N^{-\alpha}}{\pi_\omega(x)} \geq \frac{1/2}{2dN^\alpha} = \frac{1}{4dN^\alpha}. \tag{2.5}
\]
Here we use the fact that \( \pi_\omega(x) \leq 2d \) by virtue of (1.3). This implies by the Markov property and by (2.5) that
\[
P_0^\omega(X_n \in B_N|D_N \leq N - 1)
\]
\[
= \sum_{k=0}^{N-1} \sum_{x \in B_k} P_0^\omega(X_n \in B_N, D_N = k, X_{H_k} = x) \frac{P_0^\omega(D_N \leq N - 1)}{P_0^\omega(D_N \leq N - 1)}
\]
\[
\geq \sum_{k=0}^{N-1} \sum_{x \in B_k} P_0^\omega(H_N \geq n, D_N = k, X_{H_k} = x) \frac{P_0^\omega(D_N \leq N - 1)}{P_0^\omega(D_N \leq N - 1)} P_x^\omega(H_N \geq n)
\]
\[
\geq \sum_{k=0}^{N-1} \sum_{x \in B_k} P_0^\omega(D_N = k, X_{H_k} = x) \frac{P_0^\omega(D_N \leq N - 1)}{P_0^\omega(D_N \leq N - 1)} P_y^\omega(H_N \geq n) P_x^\omega(X_1 = y)
\]
\[
\geq \frac{1}{4dN^\alpha} \sum_{k=0}^{N-1} \sum_{x \in B_k} P_0^\omega(D_N = k, X_{H_k} = x) \frac{P_0^\omega(D_N \leq N - 1)}{P_0^\omega(D_N \leq N - 1)} P_y^\omega(H_N \geq n)
\]
\[
\geq \frac{1}{4d} \sum_{k=0}^{N-1} \sum_{x \in B_k} P_0^\omega(D_N = k, X_{H_k} = x) \frac{P_0^\omega(D_N \leq N - 1)}{P_0^\omega(D_N \leq N - 1)} P_y^\omega(H_N \geq n).
\]

If the trap \( \mathfrak{P}_N \) retains enough the random walk \( X \), we will have \( H_N \geq n \), when it starts at \( y \) (always the same \( y = x + \epsilon(x) \hat{e}_0 \) of the collection \( \mathcal{C}(x) \)). Let
\[
E_N := \bigcup_{j=0}^{n-1} \{X_j \text{ steps outside of the trap } \mathfrak{P}_N\}.
\]
and we say “

\[ X_j \text{ steps outside of the trap } P_N \]”, when \( X_{j+1} = y \pm \hat{e}_i , \forall i \neq i_0 \), or \( X_{j+1} = x \) (resp. \( X_{j+1} = z \pm \hat{e}_{i_0} \)) if \( X_j = y \) (resp. if \( X_j = z \)).

The complement of \( E_N \) is in fact the event that \( X \) does not leave the trap during its first \( n \) jumps, i.e. \( X \) jumps \( n \) times, starting at \( y \), in turn on \( z \) and \( y \), which, according to the configuration of the trap, costs for each jump a probability greater than

\[
\frac{\xi}{\xi + (2d - 1)N^{-\alpha}}.
\]

Then, we have by the Markov property

\[
P^\omega_y(H_N \geq n) \geq P^\omega_y(E^*_N) \geq \left( \frac{\xi}{\xi + (2d - 1)N^{-\alpha}} \right)^n,
\]

and since by the choice of \( N^\alpha \leq n < (N + 1)^\alpha \)

\[
\left( \frac{\xi}{\xi + (2d - 1)N^{-\alpha}} \right)^n \to e^{-(2d-1)/\xi},
\]

it follows for all \( N \) large enough that

\[
P^\omega_y(H_N \geq n) \geq e^{-(2d-1)/\xi}.
\]

So, putting this in (2.6), we obtain

\[
P^\omega_0(X_n \in B_N | D_N \leq N - 1) \geq \frac{e^{-(2d-1)/\xi}}{8dn} \sum_{k=0}^{N-1} \sum_{x \in B_{N-1}} \frac{P^\omega_0(D_N = k, X_{H_k} = x)}{P^\omega_0(D_N \leq N - 1)}
\]

\[
\geq \frac{e^{-(2d-1)/\xi}}{8dn}.
\]

Now, according to (2.4), we have \( P^\omega_0(D_N \leq N - 1) \geq \frac{1}{2} \). Then we deduce

\[
P^\omega_0(X_n \in B_N) \geq P^\omega_0(X_n \in B_N | D_N \leq N - 1) P^\omega_0(D_N \leq N - 1) \geq \frac{e^{-(2d-1)/\xi}}{16dn}.
\]

A fortiori, we have

\[
P^\omega_0(X_n \in B_{n^{1/\alpha}}) \geq P^\omega_0(X_n \in B_N) \geq \frac{e^{-(2d-1)/\xi}}{16dn}.
\]

Thus, for all \( N \geq N_0 \), by replacing the last inequality in (2.2), we obtain

\[
P^\omega_{2n}(0, 0) \geq \frac{\pi(0) \left( e^{-(2d-1)/\xi} / 16dn \right)^2 7^{-d}}{n^{2 + \delta(\gamma)}}.
\]

where \( \delta(\gamma) := d(4d - 2)\gamma / (1 - \epsilon) \). When we let \( \epsilon \to 0 \), we get (1.10). \(\square\)
3. Standard heat-kernel decay

We give here the proof of Theorem 1.4.

Let us first give some definitions and fix some notations besides those seen before.

Consider a Markov chain on a countable state-space $V$ with transition probability denoted by $P(x, y)$ and invariant measure denoted by $\pi$. Define $Q(x, y) = \pi(x)P(x, y)$ and for each $S_1, S_2 \subset V$, let

$$Q(S_1, S_2) = \sum_{x \in S_1} \sum_{y \in S_2} Q(x, y).$$

For each $S \subset V$ with $\pi(S) \in (0, \infty)$ we define

$$\Phi_S = \frac{Q(S, S^c)}{\pi(S)}$$

and use it to define the isoperimetric profile

$$\Phi(r) = \inf \{ \Phi_S : \pi(S) \leq r \}.$$

(Here $\pi(S)$ is the measure of $S$.) It is easy to check that we may restrict the infimum to sets $S$ that are connected in the graph structure induced on $V$ by $P$.

To prove Theorem 1.4, we combine basically two facts. On the one hand, we use Theorem 2 of Morris and Peres [MorPer05] that we summarize here: Suppose that $P(x, x) \geq \sigma$ for some $\sigma \in (0, 1/2]$ and all $x \in V$. Let $\epsilon > 0$ and $x, y \in V$. Then

$$P^n(x, y) \leq \epsilon \pi(y)$$

for all $n$ such that

$$n \geq 1 + \frac{(1 - \sigma)^2}{\sigma^2} \int_{4\pi(x) \wedge \pi(y)}^{1/\epsilon} \frac{4}{u\Phi(u)^2} \, du.$$

Let $B_{N+1} = \{-(N + 1), N + 1\}^d$ and $B_{N+1}$ denote the set of nearest-neighbor bonds of $B_{N+1}$, i.e., $B_{N+1} = \{b = (x, y) : x, y \in B_{N+1}, x \sim y \}$. Call $\mathbb{Z}_e^d$ the set of even points of $\mathbb{Z}^d$, i.e., the points $x := (x_1, \ldots, x_d)$ such that $|\sum_{i=1}^d x_i| = 2k$, with $k \in \mathbb{N}$ ($0 \in \mathbb{N}$), and equip it with the graph structure defined by: two points $x, y \in \mathbb{Z}_e^d \subset \mathbb{Z}^d$ are neighbors when they are separated in $\mathbb{Z}^d$ by two steps, i.e.

$$\sum_{i=1}^d |x_i - y_i| = 2.$$

We operate the following modification on the environment $\omega$ by defining $\tilde{\omega}_b = 1$ on every bond $b \notin B_{N+1}$ and $\tilde{\omega}_b = \omega_b$ otherwise. Then, we will adapt the machinery above to the following setting

$$V = \mathbb{Z}_e^d, \quad P = P_\omega^2 \quad \text{and} \quad \pi = \pi_\omega,$$
with the objects in (3.1–3.3) denoted by \( Q_\omega, \Phi_S(\omega) \) and \( \Phi_\omega(r) \). So, the random walk associated with \( P_\omega^2 \) moves on the even points.

On the other hand, we need to know the following standard fact that gives a lower bound of the conductances of the box \( B_N \). For a proof, see [FM06], Lemma 3.6.

**Lemma 3.1** Under assumption (1.3),

\[
\lim_{N \to +\infty} \log \inf_{b \in B_N} \omega_b \frac{\log N}{\log N} = -\frac{d}{\gamma}, \quad Q - a.s. \tag{3.7}
\]

Thus, for arbitrary \( \mu > 0 \), we can write \( Q - a.s. \), for all \( N \) large enough

\[
\inf_{b \in B_{N+1}} \omega_b \geq N^{-\left(\frac{d}{\gamma} + \mu\right)}. \tag{3.8}
\]

Our next step involves extraction of appropriate bounds on surface and volume terms.

**Lemma 3.2** Let \( d \geq 2 \) and set \( \alpha(N) := N^{-\left(\frac{d}{\gamma} + \mu\right)} \), for arbitrary \( \mu > 0 \). Then, for a.e. \( \omega \), there exists a constant \( c > 0 \) such that the following holds: For \( N \) large enough and any finite connected \( \Lambda \subset \mathbb{Z}_d^d \), we have

\[
Q_\omega(\Lambda, \mathbb{Z}_d^d \setminus \Lambda) \geq c\alpha(N)^2 \pi_\omega(\Lambda)^{\frac{d-1}{2d}}. \tag{3.9}
\]

The proof of lemma 3.2 will be a consequence of the following well-known fact of isoperimetric inequalities on \( \mathbb{Z}^d \) (see [Woe00], Chapter I, § 4). For any connected \( \Lambda \subset \mathbb{Z}^d \), let \( \partial \Lambda \) denote the set of edges between \( \Lambda \) and \( \mathbb{Z}^d \setminus \Lambda \). Then, there exists a constant \( \kappa \) such that

\[
|\partial \Lambda| \geq \kappa |\Lambda|^{\frac{d-1}{d}} \tag{3.10}
\]

for every finite connected \( \Lambda \subset \mathbb{Z}^d \). This remains true for \( \mathbb{Z}_e^d \).

**Proof of lemma 3.2.** For some arbitrary \( \mu > 0 \), set \( \alpha := \alpha(N) = N^{-\left(\frac{d}{\gamma} + \mu\right)} \) and let \( N \gg 1 \). For any finite connected \( \Lambda \subset \mathbb{Z}_e^d \), we claim that

\[
Q_\omega(\Lambda, \mathbb{Z}_e^d \setminus \Lambda) \geq \alpha^2 \frac{2d}{\partial \Lambda} \tag{3.11}
\]

and

\[
\pi_\omega(\Lambda) \leq 2d |\Lambda|. \tag{3.12}
\]

Then, Lemma 3.1 gives a.s. \( \inf_{b \in B_N} \omega(b) > \alpha \) and by virtue of (3.10), we have \( |\partial \Lambda| \geq \kappa |\Lambda|^{\frac{d-1}{d}} \), then (3.9) will follow from (3.11–3.12).

It remains to prove (3.11–3.12). The bound (3.12) is implied by \( \pi_\omega(x) \leq 2d \). For (3.11), since \( P_\omega^2 \) represents two steps of a random walk, we get a lower bound on \( Q_\omega(\Lambda, \mathbb{Z}_e^d \setminus \Lambda) \) by picking a site \( x \in \Lambda \) which has a neighbor \( y \in \mathbb{Z}^d \) that has a
neighbor $z \in \mathbb{Z}^d$ on the outer boundary of $\Lambda$. By Lemma 3.1, if $x$ or $z \in B_{N+1}$, the relevant contribution is bounded by

$$\pi_\omega(x) P_\omega(x, z) \geq \pi_\omega(x) \frac{\tilde{\omega}_{xy}}{\pi_\omega(x)} \frac{\tilde{\omega}_{yz}}{\pi_\omega(y)} \geq \frac{\alpha^2}{2d}.$$  \hspace{1cm} (3.13)

For the case where $x, z \notin \mathbb{Z}^d \cap B_{N+1}$, clearly the left-hand side of (3.13) is bounded by $1/(2d) > \alpha^2/(2d)$. Once $\Lambda$ has at least two elements, we can do this for $(y, z)$ ranging over all bonds in $\partial \Lambda$, so summing over $(y, z)$ we get (3.11). \hfill \Box

Now we get what we need to estimate the decay of $P^{2n}_\omega(0, 0)$.

**Proof of Theorem 1.4.** Let $d \geq 5$, $\gamma > 8d$ and choose $\mu > 0$ such that $\mu < 1 \frac{8}{\gamma - d}$. Let $n = \lceil N/2 \rceil$, $N \gg 1$, and consider the random walk on $\tilde{\omega}$.

We will derive a bound on $\Phi^{(\tilde{\omega})}_\Lambda$ for connected $\Lambda \subset \mathbb{Z}^d$. Henceforth $c$ denotes a generic constant. Observe that (3.9) implies

$$\Phi^{(\tilde{\omega})}_\Lambda \geq c \alpha^2 \pi_\omega(\Lambda)^{-1/d}.$$  \hspace{1cm} (3.14)

Then, we conclude that

$$\Phi^{(\tilde{\omega})}_\Lambda(r) \geq c \alpha^2 r^{-1/d}.$$  \hspace{1cm} (3.15)

The relevant integral is thus bounded by

$$\frac{(1 - \sigma)^2}{\sigma^2} \int_{[\pi(0) \setminus \pi(x)]} \frac{4}{u \sqrt{\Phi_\omega(u)^2}} du \leq c \alpha^{-4} \sigma^{-2} \varepsilon^{-2/d}$$  \hspace{1cm} (3.16)

for some constant $c > 0$. Setting $\varepsilon$ proportional to $n^\frac{4d^2}{\gamma - 2d - 4}$, and noting $\sigma \geq \alpha^2/(2d)$, the right-hand side is less than $n$ and by setting $\delta(\gamma) = 4d^2/\gamma$, we will get

$$P^{2n}_\omega(0, x) \leq \frac{c}{n^{\frac{\delta(\gamma)}{2} - 4d}} \quad \forall x \in \mathbb{Z}^d.$$  \hspace{1cm} (3.17)

As the random walk will not leave the box $B_N$ by time $2n$, we can replace $\tilde{\omega}$ by $\omega$ in (3.17), and since $P^{2n}_\omega(0, x) = 0$ for each $x \notin B_N$, then after letting $\mu \to 0$, we get

$$\limsup_{n \to +\infty} \sup_{x \in \mathbb{Z}^d} \frac{\log P^{2n}_\omega(0, x)}{\log n} \leq - \frac{d}{2} + \delta(\gamma).$$

This proves the claim for even $n$; for odd $n$ we just concatenate this with a single step of the random walk. \hfill \Box

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REFERENCES

[Ba04] Barlow, M.T. (2004). Random walks on supercritical percolation clusters. Ann. Probab., Vol. 32, no. 4, 3024–3084.

[BB07] Berger, N. and Biskup, M. (2007). Quenched invariance principle for simple random walk on percolation clusters. Probab. Theory Rel. Fields, Vol. 137, no. 1–2, 83–120.

[BBHK08] Berger, N., Biskup, M., Hoffman, C. E. and Kozma, G. (2008). Anomalous heat-kernel decay for random walk among bounded random conductances. Ann. Inst. Henri Poincaré Probab. Statist., Vol. 44, no. 2, 374–392.

[BP07] Biskup, M. and Prescott, T.M. (2007). Functional CLT For Random Walk Among Bounded Random Conductances. Electron. J. Probab., Vol. 12, no. 49, 1323–1348.

[De99] Delmotte, T. (1999). Parabolic Harnack inequality and estimates of Markov chains on graphs. Rev. Mat. Iberoamericana, Vol. 15, no. 1, 181-232.

[DFGW85] De Masi, A., Ferrari, P.A., Goldstein, S. and Wick, W.D. (1985). Invariance principle for reversible Markov processes with application to diffusion in the percolation regime. In :Particle Systems, Random Media and Large Deviations (Brunswick, Maine), pp. 71–85, Contemp. Math., Vol. 41, Amer. Math. Soc., Providence, RI.

[DFGW89] De Masi, A., Ferrari, P.A., Goldstein, S. and Wick, W.D. (1989). An invariance principle for reversible Markov processes. Applications to random motions in random environments. Journal of Statistical Physics, Vol. 55, no. 3–4, 787–855.

[FM06] Fontes, L.R.G. and Mathieu, P. (2006). On symmetric random walks with random conductances on $\mathbb{Z}^d$. Probab. Theory Rel. Fields, Vol. 134, no. 4, 565–602.

[Gri99] Grimmett, G.R. (1999). Percolation (Second edition), Grundlehren der Mathematischen Wissenschaften, vol. 321. Springer-Verlag, Berlin.

[HH05] Heicklen, D., and Hoffman, C. (2005). Return probabilities of a simple random walk on percolation clusters. Electron. J. Probab., Vol. 10, no. 8, 250–302 (electronic).

[MP08] Mathieu, P. (2008). Quenched invariance principles for random walks with random conductances. Journal of Statistical Physics, Vol. 130, no. 5, 1025-1046.

[MorPer05] Morris, B. and Peres, Y. (2005). Evolving sets, mixing and heat kernel bounds. Probab. Theory Rel. Fields, Vol. 133, no. 2, 245–266.

[MPi07] Mathieu, P. and Piatnitski, A.L. (2007). Quenched invariance principles for random walks on percolation clusters. Proceedings A of the Royal Society, Vol. 463, 2287-2307.

[MR04] Mathieu, P. and Remy, E. (2004). Isoperimetry and heat kernel decay on percolation clusters. Ann. Probab. Vol. 32, no. 1A, 100–128.

[SSz04] Sidoravicius, V. and Sznitman, A.-S. (2004). Quenched invariance principles for random walks on percolation or among random conductances. Probab. Theory Rel. Fields, Vol. 129, no. 2, 219–244.

[Woe00] Woess, W. (2000). Random walks on infinite graphs and groups. Cambridge tracts in Mathematics (138), Cambridge university press.

CMI, 39 rue F. Joliot-Curie 13453 Marseille cedex 13, France.
DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE CONSTANTINE, BP 325, ROUTE AIN EL BEY, 25017, CONSTANTINE, AlgÉRIE.