REPRESENTATION THEORY OF W-ALGEBRAS, II: RAMOND TWISTED REPRESENTATIONS

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Dedicated to Professor Akihiro Tsuchiya on the occasion of his retirement from Nagoya University

Abstract. We study the Ramond twisted representations of the affine $W$-algebra $W^k(\bar{g}, f)$ in the case that $f$ admits a good even grading. We establish the vanishing and the almost irreducibility of the corresponding BRST cohomology. This confirms some of the recent conjectures of Kac and Wakimoto [KW5]. In type $A$, our results give the characters of all irreducible ordinary Ramond twisted representations of $W^k (\mathfrak{sl}_n, f)$ for all nilpotent elements $f$ and all non-critical $k$, and prove the existence of modular invariant representations conjectured in [KW5].

1. Introduction

Let $\bar{g}$ be a complex simple Lie algebra, $f$ a nilpotent element of $\bar{g}$, $\mathfrak{g}$ the non-twisted affine Kac-Moody Lie algebra associated with $\bar{g}$. Let $W^k(\bar{g}, f)$ be the affine $W$-algebra associated with $(\bar{g}, f)$ at level $k \in \mathbb{C}$, defined by the method of the quantum BRST reduction [FF, dBT2, KRW]. The vertex algebra $W^k(\bar{g}, f)$ is in general $\mathbb{Z}_{\geq 0}$-graded [KW3]. Therefore it is natural [KW5] to consider its Ramond twisted representations. In fact it is in the Ramond twisted representations where the corresponding finite $W$-algebra $W^\text{fin}(\bar{g}, f)$ [Lyn, dBT1, P1] appears as its Zhu algebra, according to [DSK].

In the previous paper [A3] we studied the representations of $W^k(\bar{g}, f)$ in the case that $f$ is a principal nilpotent element. In the present paper we study the Ramond twisted representations of $W^k(\bar{g}, f)$ in the case that $f$ admits a good even grading. All nilpotent elements in type $A$ satisfy this condition.

There is a natural BRST (co)homology functor $H^\text{BRST}_0(?)$ from a suitable category of representations of $\mathfrak{g}$ at level $k$ to the category of Ramond twisted representations of $W^k(\bar{g}, f)$. In our case $H^\text{BRST}_0(M)$ is essentially the same BRST cohomology studied in the recent work [KW5] of Kac and Wakimoto. In the case that $f$ is principal this functor is identical to the “$-$”-reduction functor studied in [FKW, A1, A3].

The main result of this paper is the vanishing and the almost irreducibility of the BRST cohomology (Theorem 5.5.4). Though our formulation is slightly different from that of [KW5], this result proves Conjecture B of [KW5], partially. Here, recall

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1 If $f$ is an even nilpotent element then $W^k(\bar{g}, f)$ is $\mathbb{Z}_{\geq 0}$-graded and Ramond twisted representations are usual (untwisted) representations.
that a positive energy representation $M = \bigoplus_{d \in d_0 + \mathbb{Z}_{\geq 0}} M_d$, $M_d \neq 0$, of a vertex algebra $V$ is called almost irreducible if $M$ is generated by $M_{d_0}$ and there is no graded submodule of $M$ intersecting $M_{d_0}$ trivially. In particular an almost irreducible module $M$ is irreducible if and only if its “top part” $M_{d_0}$ is irreducible over the Zhu algebra of $V$.

In our case the top part of the BRST cohomology functor is identical to the Lie algebra homology functor (the Whittaker functor $[M, BK]$) from the highest weight category of $\mathfrak{g}$ to the category of $\mathcal{W}^{\text{fin}}(\mathfrak{g}, f)$-modules (see §5.4). Therefore our result reduces the study of the BRST cohomology functor to that of the Whittaker functor in the representations theory of finite $W$-algebras. Although the representation theory of finite $W$-algebras has been rapidly developing (cf. [P3, P2, Los, BGK]), not much is known about the Whittaker functor associated with $\mathcal{W}^{\text{fin}}(\mathfrak{g}, f)$ except for some special cases $M$, unless $\mathfrak{g} = \mathfrak{sl}_n$. In type $A$, Brundan and Kleshchev [BK] determined the characters of all irreducible finite-dimensional representations of $\mathcal{W}^{\text{fin}}(\mathfrak{sl}_n, f)$, by showing that the Whittaker functor sends an simple module to zero or a simple module, and any simple $\mathcal{W}^{\text{fin}}(\mathfrak{sl}_n, f)$-module is obtained in this manner. It follows that in type $A$ the almost irreducibility of the BRST cohomology actually implies the irreducibility, and furthermore, any irreducible ordinary representation of $\mathcal{W}^k(\mathfrak{sl}_n, f)$ is isomorphic to $H^k_{\text{BRST}}(\mathfrak{g}, f)$, for some irreducible highest weight representation $L(\lambda)$ of $\mathfrak{sl}_n$ with highest weight $\lambda$ (Theorem 5.7.1). Hence our result shows that the character of every irreducible ordinary Ramond twisted representation of $\mathcal{W}^k(\mathfrak{sl}_n, f)$ at any level $k \in \mathbb{C}$ is determined by that of the corresponding irreducible highest weight representation of $\mathfrak{g}$, which is known [KT] (in terms of the Kazhdan-Lusztig polynomials) provided that $k$ is not critical. This generalizes the main results of [A2, A3].

The most important representations of a vertex algebra are those irreducible ordinary representations whose normalized characters are modular invariant. Kac and Wakimoto [KW5] have recently discovered the remarkable triples $(\mathfrak{g}, f, k)$, for which the (nonzero) normalized Euler-Poincaré characters of the BRST cohomology $H^k_{\text{BRST}}(L(\lambda))$, with the coefficient in the irreducible principal admissible representations $L(\lambda)$ of $\mathfrak{g}$ at level $k$, are homomorphic functions on the complex upper half plane and span an $SL_2(\mathbb{Z})$-invariant space. Our results show in type $A$ that these Euler-Poincaré characters are indeed characters of irreducible Ramond twisted representations of $\mathcal{W}^k(\mathfrak{sl}_n, f)$, as conjectured in [KW5] (see Theorem 5.8.4).

Non-twisted representations of affine $W$-algebras are studied in our subsequent paper.

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**Notation.** Throughout this paper the ground field is the complex number \( \mathbb{C} \) and tensor products and dimensions are always meant to be as vector spaces over \( \mathbb{C} \).

2. **Preliminaries on Vertex Algebras and their Twisted Representations**

In this section we collect the necessary information on vertex algebras and their (twisted) representations. The textbook \([K2, FBZ]\) and the papers \([Li, BaK, DSK]\) are our basic references in this section.

2.1. **Fields.** Let \( V \) be a vector space. For a formal series \( a(z) \in (\text{End } V)[[z, z^{-1}]] \), we set \( a_{(n)} = \text{Res}_z z^n a(z) \), where \( \text{Res}_z \) denotes the coefficient of \( z^{-1} \).

An element \( a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1} \in (\text{End } V)[[z, z^{-1}]] \) is called a field on \( V \) if \( a_{(n)} v = 0 \) for all \( v \in V \) and \( n \gg 0 \).

The normally ordered product
\[
(a(z)b(z) := a(z)\cdot b(z) + b(z)a(z))_+
\]
of two fields \( a(z) \) and \( b(z) \) is also a field, where \( a(z)_- = \sum_{n < 0} a_{(n)} z^{-n-1} \) and \( a(z)_+ = \sum_{n \geq 0} a_{(n)} z^{-n-1} \).

Two fields \( a(z) \) and \( b(z) \) are called mutually local if
\[
(z - w)^r [a(z), b(w)] = 0 \quad \text{for } r \gg 0
\]
in \( (\text{End } V)[[z, z^{-1}, w, w^{-1}]] \).

Set
\[
\delta(z - w) = \sum_{n \in \mathbb{Z}} z^n w^{n-1} \in \mathbb{C}[[z, z^{-1}, w, w^{-1}]].
\]

The locality \((2)\) gives
\[
[a(z), b(w)] = \sum_{n \geq 0} (a_{(n)} b_{(n)} - b_{(n)} a_{(n)}) \partial_w^n \delta(z - w),
\]
where \( \partial_w^n = \partial_w^n / n! \), \( \partial_w = \frac{\partial}{\partial w} \), and
\[
a(z)_{(n)} b(w) = \text{Res}_z (z - w)^n [a(z), b(w)].
\]

2.2. **Vertex Algebras.** A vertex algebra is a vector space \( V \) equipped with the following data:

- A vector \( 1 \in V \) (vacuum vector),
- \( T \in \text{End } V \) (translation operator),
- A collection \( \{a^\alpha(z) = \sum_{n \in \mathbb{Z}} a_{(n)}^\alpha z^{-n-1}; \alpha \in A\} \) of fields on \( V \), where \( A \) is an index set (generating fields),

These data are subject to the following:

(i) \( T 1 = 0 \),
(ii) \( [T, a^\alpha(z)] = \partial_z a^\alpha(z) \) for all \( \alpha \in A \),
(iii) \( a^\alpha(z) 1 \in V[[z]] \) for all \( \alpha \in A \),
(iv) the vectors \( a_{(m_1)}^\alpha \cdots a_{(m_r)}^\alpha 1 \) with \( r \geq 0 \), \( \alpha_i \in A \) and \( m_i \in \mathbb{Z} \) span \( V \),
(v) for any \( \alpha, \beta \in A \) the fields \( a^\alpha(z) \) and \( a^\beta(z) \) mutually locally.

\[
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\]
Let $V$ be a vertex algebra. There exists a unique linear map
\begin{equation}
V \to (\text{End } V)[[z, z^{-1}]], \quad a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1}
\end{equation}
such that
(i) $Y(a, z)$ is a field on $V$ for any $a \in V$,
(ii) $Y(a, z)$ and $Y(b, z)$ are mutually local for any $a, b \in V$,
(iii) $[T, Y(a, z)] = \partial_z a(z)$ for any $a \in V$,
(iv) $Y(a, z)1 \in V[[z]]$ and $\lim_{z \to 0} Y(a, z)1 = a$ for any $a \in V$,
(v) $Y(a^{\alpha}_{(-1)}1, z) = a^{\alpha}(z)$ for any generating field $a^{\alpha}(z)$.

The map $Y(?, z)$ is called the state-field correspondence.

A Hamiltonian of a vertex algebra $V$ is a diagonalizable operator $H \in \text{End } V$ such that
\begin{equation}
[H, Y(a, z)] = Y(Ha, z) + z\partial_z Y(a, z) \quad \text{for all } a \in V.
\end{equation}
A vertex algebra with a Hamiltonian $H$ is called graded. If $a$ is an eigenvector of $H$ its eigenvalue is called the conformal weight of $a$ and denoted by $\Delta_a$. Let
\begin{equation}
V_\Delta = \{ a \in V; Ha = \Delta a \},
\end{equation}
so that $V = \bigoplus_{\Delta \in \mathbb{C}} V_\Delta$.

2.3. Twisted Representations of Vertex Algebras. Let $N \in \mathbb{N}$. An $N$-twisted field $a(z)$ on a vector space $M$ is a formal power series in $z^{1/N}$, $z^{-1/N}$ of the form
\begin{equation}
a(z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1}, \quad a(n) \in \text{End}(M)
\end{equation}
such that $a(n)m = 0$ for all $m \in M$ and $n \gg 0$.

Two $N$-twisted fields $a(z)$ and $b(z)$ on $M$ are called mutually local if they satisfy
\begin{equation}
Y(a, z)1 \in M[[z^{1/N}, z^{-1/N}], w^{1/N}, w^{-1/N}]].
\end{equation}

Let $V$ be a vertex algebra, $\sigma$ an automorphism of $V$ of order $N$. A $\sigma$-twisted representation of $V$ is a vector space $M$ equipped with a linear map from $V$ to the space of $N$-twisted fields on $M$,
\begin{equation}
V \to (\text{End } M)[[z^{1/N}, z^{-1/N}]], \quad a \mapsto Y^M(a, z) = \sum_{n \in \mathbb{Z}} a^M_n z^{-n-1},
\end{equation}
such that
\begin{equation}
Y^M(\sigma a, z) = Y^M(a, e^{2\pi i}z),
\end{equation}
\begin{equation}
Y^M(1, z) = \text{id}_M,
\end{equation}
and
\begin{equation}
\sum_{i=0}^{\infty} \binom{m}{i} a^{(r+i)}_i b^{M}_{(m+n-i)} = \sum_{i=0}^{\infty} (-1)^i \binom{r}{i} a^M_{(m+r-i)} b^M_{(n+i)} - (-1)^r b^M_{(n+r-i)} a^M_{(m+i)}
\end{equation}

\footnote{This differs from the notation in [A3].}
for \( a \in V_j, b \in V, m \in \frac{1}{N} + Z, n \in \frac{1}{N} Z, r \in Z \), where

\[
V_j = \{ \sigma(a) = (e^{\frac{2\pi i}{N} r})^{-j} a \}.
\]

The relation (10) is called the \textit{twisted Borcherds identity}.

By setting \( r = 0 \) in (10), one obtains

\[
[a^M_{(m)}, b^M_{(n)}] = \sum_{i=0}^{\infty} \binom{m}{i} (a_{(i)} b)^M_{(m+n-i)},
\]

or equivalently,

\[
[Y^M(a, z), Y^M(b, w)] = \sum_{i=0}^{\infty} Y^M(a_{(i)} b, w) \partial_w^i \delta_j(z-w)
\]

for \( a \in V_j \), where

\[
\delta_j(z-w) = z^{-j/N} w^{j/N} \delta(z-w) = \sum_{n \in j/N + Z} w^n z^{-n-1}.
\]

In particular \( Y^M(a, z) \) and \( Y^M(b, z) \) are mutually local.

The relation (11) gives (12)

\[
Y^M(a(b), w) = \text{Res}_z \sum_{k=0}^{\infty} \left( \frac{-j/N}{k} \right) z^{j/N-k} w^{-j/N} (z-w)^{n+k}[Y^M(a, z), Y^M(b, w)]
\]

for all \( n \geq 0 \). The sum in (13) is finite because of the locality. (In reality (13) holds for all \( n \in \mathbb{Z} \) in an appropriate sense, see (14)).

Set \( b = 1, r = -2, n = 0 \) in (9). It follows that

\[
Y^M(Ta, z) = \partial_z Y^M(a, z).
\]

Suppose that \( V \) is graded by a Hamiltonian \( H \). A \( \sigma \)-twisted representation \( M \) is called \textit{graded} if there exists a diagonalizable operator \( H^M \) on \( M \) such that

\[
[H^M, a^M_{(n)}] = (Ta)_{(n+1)}^M + (Ha)_{(n)}
\]

for all \( a \in V \) and \( n \in \frac{1}{N} Z \). If \( a \) is homogeneous, (15) is equivalent to

\[
[H^M, a^M_{(n)}] = -(n - \Delta_a + 1)a^M_{(n)}.
\]

We set

\[
M_d = \{ m \in M; H^M m = dm \}
\]

for \( d \in \mathbb{C} \).

A \textit{positive energy} \( \sigma \)-twisted representation\(^6\) of \( V \) is a graded \( \sigma \)-twisted representation \( M \) of \( V \) such that there exist a finite set \( d_1, \ldots, d_r \in \mathbb{C} \) such that \( M_d = 0 \) unless \( d \in \bigcup_i d_i + \mathbb{Z}_{\geq 0} \). Let \( V\text{-}\mathfrak{mod}_\sigma \) be the category of positive energy \( \sigma \)-twisted representations of \( V \), whose morphisms are graded homomorphisms of \( \sigma \)-twisted representations.

An \textit{ordinary} \( \sigma \)-twisted representation of \( V \) is a positive energy \( \sigma \)-twisted representation of \( V \) such that \( \text{dim } M_d < \infty \) for all \( d \). Let \( V\text{-}\mathfrak{mod}_\sigma \) be the full subcategory of \( V\text{-}\mathfrak{mod}_\sigma \) consisting of ordinary \( \sigma \)-twisted representations.

\(^6\)A positive energy representations is also called an admissible representation in the literature.
When \( \sigma = \text{id}_V \), \( \sigma \)-twisted representations are just usual (non-twisted) representations. We set \( V \text{-Mod} = V \text{-Mod}_{\text{id}_V} \) and \( V \text{-mod} = V \text{-mod}_{\text{id}_V} \).

### 2.4. \( H \)-Twisted Zhu Algebras.

Let \( V \) be a vertex algebra graded by a Hamiltonian \( H \). Assume that \( V_\Delta \neq 0 \) unless \( \Delta \in \frac{1}{H} \mathbb{Z} \). Then \( \sigma_H := e^{2\pi i \text{ad} H} : V \to V \) is an automorphism of order at most \( N \).

If \( M \) is a graded \( \sigma_H \)-twisted representations of \( V \) then the number \( n - \Delta_a + 1 \) in \( (10) \) is always an integer. Set \( a_n^M = a_{(n+\Delta_a-1)}^M \), so that

\[
Y^M(a, z) = \sum_{n \in \mathbb{Z}} a_n^M z^{-n-\Delta_a}, \quad [H^M, a_n^M] = -na_n^M.
\]

Define the \( H \)-twisted Zhu algebra \( \mathbb{Z} \text{-DSK} \) \( \mathcal{Z}_H V \) by

\[
\mathcal{Z}_H V = V/V \circ V,
\]

where \( V \circ V \) is the span of the vectors

\[
a \circ b := \sum_{r \geq 0} \binom{\Delta_a}{r} a_{(r-2)} b
\]

with homogeneous vectors \( a, b \in V \). The \( \mathcal{Z}_H V \) is an associative algebra with the multiplication

\[
a * b = \sum_{r \geq 0} \binom{\Delta_a}{r} a_{(r-1)} b.
\]

Let \( M \) be an object of \( V \text{-Mod}_{\sigma_H} \). Denote by \( V_{\text{top}} \) the sum of homogeneous subspace \( V_d \) such that \( V_{d'} = 0 \) for all \( d' \in d - N \). Then \( V_{\text{top}} \) is naturally a module over \( \mathcal{Z}_H V \) by the following action:

\[
(a + V \circ V)m = a_{(\Delta_a-1)}^M m = a_0^M m.
\]

**Theorem 2.4.1 (\[\mathbb{Z} \text{-DSK}\]).** The map \( M \mapsto M_{\text{top}} \) gives a bijective correspondence between simple objects of \( V \text{-Mod}_{\sigma_H} \) and irreducible \( \mathcal{Z}_H V \text{-modules} \).

The \( M \) is said to be **almost highest weight** if (1) \( M_{\text{top}} = M_d \) for some \( d \) and (2) \( M \) is generated by \( M_{\text{top}} \) over \( V \). The \( M \) is said to be **almost co-highest weight** if (1) \( M_{\text{top}} = M_d \) for some \( d \) and (2) \( M \) contains no graded submodule intersecting \( M_{\text{top}} \) trivially. The \( M \) is called **almost irreducible** \( \text{-DSK} \) if \( M \) is both almost highest weight and almost co-highest weight. Clearly, an almost irreducible module is simple if and only if \( M_{\text{top}} \) is irreducible over \( \mathcal{Z}_H V \).

### 3. Affine W-Algebras

#### 3.1. The Setting.

Let \( \mathfrak{g} \) be a complex reductive Lie algebra, \( f \) a nilpotent element of \( \mathfrak{g} \). The corresponding affine \( W \)-algebra \( \mathcal{W}^k(\mathfrak{g}, f) \) at the level \( k \in \mathbb{C} \) is defined by the method of the quantum BRST reduction. This method was discovered by Feigin and Frenkel \([\text{FF}]\) who used it to define the \( W \)-algebra \( \mathcal{W}^k(\mathfrak{g}, f) \) associated with the principal nilpotent elements \( f \). The most general definition of \( \mathcal{W}^k(\mathfrak{g}, f) \) was given by Kac, Roan and Wakimoto \([\text{KRW}]\), and the definition of \( \mathcal{W}^k(\mathfrak{g}, f) \) given in \([\text{KRW}], [\text{KW}]\) involves another data, namely a **good grading** of \( \mathfrak{g} \) for \( f \). However, thanks to the results \([\text{BG}]\) of Brundan and Goodwin, the definition of \( \mathcal{W}^k(\mathfrak{g}, f) \) does **not** depend on the choice of a good grading for \( f \).
Throughout this paper we assume that $f$ admits a good even grading unless otherwise stated, that is, there exists a $\mathbb{Z}$-grading
\begin{equation}
\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j
\end{equation}
of $\mathfrak{g}$ such that $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{g}_0$, $f \in \mathfrak{g}_{-1}$, and $\text{ad} f : \mathfrak{g}_{\leq 0} \rightarrow \mathfrak{g}_{< 0}$ is surjective, where $\mathfrak{g}_{\leq 0} = \bigoplus_{j \leq 0} \mathfrak{g}_j$ and $\mathfrak{g}_{< 0} = \bigoplus_{j < 0} \mathfrak{g}_j$. The last condition is equivalent to that $\text{ad} f : \mathfrak{g}_{> 0} \rightarrow \mathfrak{g}_{\geq 0}$ is injective, where $\mathfrak{g}_{\geq 0} = \bigoplus_{j \geq 0} \mathfrak{g}_j$ and $\mathfrak{g}_{> 0} = \bigoplus_{j > 0} \mathfrak{g}_j$. By definition there exists an exact sequence
\begin{equation}
0 \rightarrow \mathfrak{g}^f \hookrightarrow \mathfrak{g}_{\leq 0} \xrightarrow{\text{ad} f} \mathfrak{g}_{< 0} \rightarrow 0,
\end{equation}
where $\mathfrak{g}^f$ is the centralizer of $f$ in $\mathfrak{g}$.

One can find a $\mathfrak{sl}_2$-triple $(e, h, f)$ in $\mathfrak{g}$ such that $e \in \mathfrak{g}_1$, $h \in \mathfrak{g}_0$, see Lemma 1.1 of [FK]. Below we write $h_0$ for $h$. Also, there exists a semisimple element $x_0 \in \mathfrak{g}_0$ that defines the $\mathbb{Z}$-grading, i.e.,
\begin{equation}
\mathfrak{g}_j = \{a \in \mathfrak{g} : [x_0, a] = ja\}.
\end{equation}

We fix a non-degenerate invariant inner product $(\cdot | \cdot)$ on $\mathfrak{g}$ such that $(e|f) = 1$.

Set
\begin{equation}
\tilde{\chi} = \tilde{x}_f = (f|?) \in \mathfrak{g}^*,
\end{equation}
and let $\mathcal{O}_{\tilde{\chi}}$ be the coadjoint orbit of $\tilde{\chi}$,
\begin{equation}
d_{\tilde{\chi}} = \frac{1}{2} \dim \mathcal{O}_{\tilde{\chi}}.
\end{equation}
By (22) one has
\begin{equation}
\dim \mathfrak{g}_{< 0} = \frac{1}{2} (\dim \mathfrak{g} - \dim \mathfrak{g}^f) = d_{\tilde{\chi}}.
\end{equation}

3.2. Root Data. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_0$ containing $x_0$ and $h_0$ (see above). Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\Delta$ be the set of roots of $\mathfrak{g}$. One has
\[\Delta = \bigcup_{j \in \mathbb{Z}} \Delta_j,\]
where $\Delta_j = \{\alpha \in \Delta : \langle \alpha, x_0 \rangle = j\}$. The $\Delta_0$ is the set of roots of the reductive subalgebra $\mathfrak{g}_0$. Let $\Delta_{0,+}$ be a set of positive roots of $\mathfrak{g}_0$. $\Delta_{0,-} = -\Delta_{0,+}$. Then $\Delta_+ = \Delta_{0,+} \cup \Delta_{> 0}$ is a set of positive roots of $\mathfrak{g}$, where $\Delta_{> 0} = \bigcup_{j > 0} \Delta_j$. Likewise, $\Delta_- = \Delta_{0,-} \cup \Delta_{< 0}$ is a set of negative roots of $\mathfrak{g}$, where $\Delta_{< 0} = \bigcup_{j < 0} \Delta_j$. Let $\mathfrak{n}_0 = \mathfrak{n}_{0,-} \oplus \mathfrak{h} \oplus \mathfrak{n}_{0,+}$ and $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the corresponding triangular decompositions of $\mathfrak{g}_0$ and $\mathfrak{g}$, respectively.

Let $\bar{\rho}$ be the half sum of positive roots of $\mathfrak{g}$.

Let $Q$, $Q^\vee$, $P$ and $P^\vee$ be the root lattice, the coroot lattice, the weight lattice and the coweight lattice of $\mathfrak{g}$, respectively. Denote by $W$ the Weyl group of $\mathfrak{g}$.

We fix an anti-Lie algebra involution $\mathfrak{g} \ni x \mapsto x^t \in \mathfrak{g}$ such that $e^t = f$ and $h^t = h$ (for all $h \in \mathfrak{h}$).

Set $I = \{1, \ldots, \text{rank} \mathfrak{g} \}$. Let $\{J_\alpha : \alpha \in I \cup \Delta_{\pm}\}$ be a basis of $\mathfrak{g}$ such that $J_\alpha$ with $\alpha \in \Delta$ is a root vector of root $\alpha$ and $\{J_i : i \in I\}$ is a basis of $\mathfrak{h}$. Denote by $c_{a,b}^d$ the corresponding structure constant.
3.3. Kac-Moody Lie Algebras. Let \( \mathfrak{g} \) be the Kac-Moody affinization of \( \tilde{\mathfrak{g}} \):

\[
\mathfrak{g} = \tilde{\mathfrak{g}}[t, t^{-1}] \oplus \mathbb{C} K \oplus \mathbb{C} D,
\]

where \( \tilde{\mathfrak{g}}[t, t^{-1}] = \tilde{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}] \). The commutation relations are given by

\[
[X(m), Y(n)] = [X, Y](m + n) + m \delta_{m+n,0}(X|Y)K,
\]

\[
[D, X(m)] = mX(m), \quad [K, \mathfrak{g}] = 0
\]

for \( X, Y \in \mathfrak{g}, m, n \in \mathbb{Z} \), where \( X(m) = X \otimes t^m \). The subalgebra \( \tilde{\mathfrak{g}} \otimes \mathbb{C} \subset \mathfrak{g} \) is naturally identified with \( \tilde{\mathfrak{g}} \).

The form \( (\cdot | \cdot) \) is extended from \( \tilde{\mathfrak{g}} \) to the invariant symmetric bilinear of \( \mathfrak{g} \) as follows:

\[
(X(m)|Y(n)) = (X|Y)\delta_{m+n,0}, \quad (D|K) = 1,
\]

\[
(X(m)|D) = (X(m)|K) = (D|D) = (K|K) = 0.
\]

We fix the triangular decomposition \( \mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \) as usual:

\[
\mathfrak{h} = \mathfrak{h} \oplus \mathbb{C} K \oplus \mathbb{C} D,
\]

\[
\mathfrak{n}_- = \tilde{\mathfrak{n}}_-[t^{-1}] \oplus \tilde{\mathfrak{n}}_+[t^{-1}], \quad \mathfrak{n}_+ = \tilde{\mathfrak{n}}_-[t] \oplus \tilde{\mathfrak{n}}_+[t].
\]

Let \( \mathfrak{h}^* = \mathfrak{h}^* \oplus \mathbb{C} \Lambda_0 \oplus \mathbb{C} \delta \) be the dual of \( \mathfrak{h} \). Here, \( \Lambda_0 \) and \( \delta \) are dual elements of \( K \) and \( D \), respectively. For \( \lambda \in \mathfrak{h}^* \), the number \( \langle \lambda, K \rangle \) is called the level of \( \lambda \).

Let \( \lambda \) be the restriction of \( \lambda \in \mathfrak{h}^* \) to \( \mathfrak{h}^* \). We refer to \( \lambda \) as the classical part of \( \lambda \).

Let \( \Delta \) be the set of roots of \( \mathfrak{g} \), \( \Delta_+ \) the set of positive roots, \( \Delta_- = -\Delta_+ \). Then, \( \Delta = \Delta^r \sqcup \Delta^i \), where \( \Delta^r \) is the set of real roots and \( \Delta^i \) is the set of imaginary roots. Let \( \Delta^\pm = \Delta^r \sqcup \Delta^\pm \) and \( \Delta^\pm = \Delta^i \sqcup \Delta^\pm \). One has

\[
\Delta^r_+ = \{ \alpha + n\delta; \alpha \in \Delta_+, \ n \geq 0 \} \cup \{ -\alpha + n\delta; \alpha \in \Delta_+, \ n \geq 1 \}.
\]

Let \( Q \) be the root lattice, \( Q_+ = \sum_{\alpha \in \Delta_+} \mathbb{Z}_{\geq 0} \alpha \subset Q \). We define a partial ordering \( \mu \leq \lambda \) on \( \mathfrak{h}^* \) by \( -\lambda - \mu \in Q_+ \).

Let \( W \subset GL(\mathfrak{h}^*) \) be the Weyl group of \( \mathfrak{g} \) generated by the reflections \( s_\alpha \) with \( \alpha \in \Delta^r \), where \( s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \) for \( \lambda \in \mathfrak{h}^* \). The dot action of \( W \) on \( \mathfrak{h}^* \) is defined by \( w \cdot \lambda = w(\lambda + \rho) - \rho \), where \( \rho = \frac{1}{2} \rho^\vee \Lambda_0 \in \mathfrak{h}^* \) and \( \rho^\vee \) is the dual Coxeter number of \( \tilde{\mathfrak{g}} \).

For \( \lambda \in \mathfrak{h}^* \), let

\[
\Delta(\lambda) = \{ \alpha \in \Delta^r; \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z} \},
\]

\[
\Delta_+(\lambda) = \Delta(\lambda) \cap \Delta_+,
\]

\[
W(\lambda) = \{ s_\alpha; \alpha \in \Delta(\lambda) \} \subset W.
\]

One knows that \( W(\lambda) \) is a Coxeter subgroup of \( W \), and \( W(\lambda) \) is called the integral Weyl group of \( \lambda \in \mathfrak{h}^* \). Let \( \ell_\lambda : W(\lambda) \to \mathbb{Z}_{\geq 0} \) be its length function.

For an \( \mathfrak{h} \)-module \( M \) let \( M^\lambda \) be the weight space of weight \( \lambda \in \mathfrak{h}^* \):

\[
M^\lambda = \{ m \in M; am = \lambda(a)m \ \forall \alpha \in \mathfrak{h} \}.
\]

We say \( M \) admits a weight space decomposition if \( M = \bigoplus \lambda M^\lambda \) and \( \dim M^\lambda < \infty \) for all \( \lambda \). In this case we define the graded dual \( M^* \) of \( M \) by

\[
M^* = \bigoplus \lambda \text{Hom}_\mathbb{C}(M^\lambda, \mathbb{C}) \subset \text{Hom}_\mathbb{C}(M, \mathbb{C}).
\]
Also, we set\(^7\)

\begin{align*}
M_d &= \{ m \in M; -Dm = dm \}, \\
D(M) &= \bigoplus_{d \in \mathbb{C}} \text{Hom}_\mathbb{C}(M_d, \mathbb{C})
\end{align*}

for a semisimple \( \mathbb{C}D \)-module \( M \). Note that if \( M \) is a \( \mathfrak{g} \)-module then \( M_d \) is a \( \mathfrak{g} \)-submodule of \( M \) for any \( d \).

### Lemma 3.3.1.
Let \( M \) be a \( \mathfrak{h} \)-module that admits a weight space decomposition. Suppose that \( M_d \) is finite-dimensional for all \( d \). Then \( D(M) = M^* \).

#### 3.4. Universal Affine Vertex Algebras.
For \( k \in \mathbb{C} \) define the \( \mathfrak{g} \)-module \( V^k(\mathfrak{g}) \) by

\begin{equation}
V^k(\mathfrak{g}) = U(\mathfrak{g}) \otimes_{U(\widehat{\mathfrak{g}}[t] \oplus \mathbb{C}K \oplus \mathbb{C}D)} \mathbb{C}k,
\end{equation}

where \( \mathbb{C}k \) is the one-dimensional representation of \( \widehat{\mathfrak{g}}[t] \oplus \mathbb{C}K \oplus \mathbb{C}D \) on which \( \mathfrak{g} \) acts trivially and \( K \) acts as the multiplication by \( k \).

Define a field \( J(z) \) on \( V^k(\mathfrak{g}) \) for \( J \in \mathfrak{g} \) by

\begin{equation}
J(z) = \sum_{n \in \mathbb{Z}} J(n) z^{-n-1}.
\end{equation}

There is a unique vertex algebra structure on \( V^k(\mathfrak{g}) \) such that \( 1 = 1 \otimes 1 \in V^k(\mathfrak{g}) \) is the vacuum vector and \( \{ J(z); J \in \mathfrak{g} \} \) is a set of generating fields. The vertex algebra \( V^k(\mathfrak{g}) \) is called the universal affine vertex algebra associated with \( \mathfrak{g} \) at level \( k \).

#### 3.5. Clifford Vertex Algebras.
Set

\begin{equation}
L_{\mathfrak{g} > 0} = \mathfrak{g}_{> 0} \otimes \mathbb{C}[t, t^{-1}], \quad L_{\mathfrak{g} < 0} = \mathfrak{g}_{< 0} \otimes \mathbb{C}[t, t^{-1}].
\end{equation}

They are nilpotent subalgebras of \( \mathfrak{g} \).

Let \( \mathcal{C}L \) be the Clifford algebra associated with \( L_{\mathfrak{g} < 0} \oplus L_{\mathfrak{g} > 0} \) and the restriction of \( ( \_ \_ ) \) to \( L_{\mathfrak{g} < 0} \oplus L_{\mathfrak{g} > 0} \). The superalgebra \( \mathcal{C}L \) has the following generators and relations:

- generators: \( \psi_\alpha(n) \) \( (\alpha \in \bar{\Delta}_{\neq 0}, n \in \mathbb{Z}) \),
- relations: \( \psi_\alpha(n) \) is odd,
  \[ [\psi_\alpha(m), \psi_\beta(n)] = \delta_{\alpha+\beta,0} \delta_{m+n,0} \] \( (\alpha, \beta \in \bar{\Delta}_{\neq 0}, m, n \in \mathbb{Z}) \).

Here \( \bar{\Delta}_{\neq 0} = \bar{\Delta}_{< 0} \sqcup \bar{\Delta}_{> 0} \).

The algebra \( \mathcal{C}L \) contains the Grassmann algebras \( \wedge (L_{\mathfrak{g} < 0}) \) and \( \wedge (L_{\mathfrak{g} > 0}) \) as its subalgebras: \( \wedge (L_{\mathfrak{g} < 0}) = \{ \psi_\alpha(n)\alpha \in \bar{\Delta}_{< 0}, n \in \mathbb{Z} \}, \wedge (L_{\mathfrak{g} > 0}) = \{ \psi_\alpha(n)\alpha \in \bar{\Delta}_{> 0}, n \in \mathbb{Z} \} \). One has

\begin{equation}
\mathcal{C}L = \wedge (L_{\mathfrak{g} > 0}) \otimes \wedge (L_{\mathfrak{g} < 0})
\end{equation}

as linear spaces.

In view of \( \mathcal{C}L \), the adjoint action of \( \mathfrak{h} \) on \( L_{\mathfrak{g} < 0} \oplus L_{\mathfrak{g} > 0} \) induces an action of \( \mathfrak{h} \) on \( \mathcal{C}L \): \( \mathcal{C}L \cong \bigoplus_{\lambda \in \mathbb{Q}} \mathcal{C}L^\lambda \).

\(^7\)This differs from the notation in [A3].
Let $\wedge^{\infty+\bullet}(L\tilde{\mathfrak{g}}>0)$ be the irreducible representation of $\mathfrak{Cl}$ generated by the vector $1$ such that
\begin{equation}
\psi_\alpha(n)1 = 0 \quad \text{if } \alpha + n\delta \in \Delta^e_+.
\end{equation}
Then $\wedge^{\infty+\bullet}(L\tilde{\mathfrak{g}}>0) = \wedge(L\tilde{\mathfrak{g}}_< \cap n-) \otimes \wedge(L\tilde{\mathfrak{g}}>0 \cap n-)$ as linear spaces. We regard $\wedge^{\infty+\bullet}(L\tilde{\mathfrak{g}}>0)$ as an $\mathfrak{h}$-module under this identification:
\[ \wedge^{\infty+\bullet}(L\tilde{\mathfrak{g}}>0) = \bigoplus_{\lambda \in -Q_+} \wedge^{\infty+\bullet}(L\tilde{\mathfrak{g}}>0)^\lambda. \]

The space $\wedge^{\infty+\bullet}(L\tilde{\mathfrak{g}}>0)$ is $\mathbb{Z}$-graded by charge:
\begin{equation}
\wedge^{\infty+\bullet}(L\tilde{\mathfrak{g}}>0) = \bigoplus_{i \in \mathbb{Z}} \wedge^{\infty+i}(L\tilde{\mathfrak{g}}>0).
\end{equation}

The charge of $1$, $\psi_\alpha(n)$ and $\psi_{-\alpha}(n)$ with $\alpha \in \tilde{\Delta}_>$ and $n \in \mathbb{Z}$ are $0$, $-1$ and $1$, respectively. The $\mathfrak{Cl}$-module $\wedge^{\infty+\bullet}(L\tilde{\mathfrak{g}}>0)$ is often called the space of semi-infinite forms on $L\tilde{\mathfrak{g}}>0$.

There is a unique vertex (super)algebra structure on $\wedge^{\infty+\bullet}(L\tilde{\mathfrak{g}}>0)$ such that $1$ is the vacuum vector, and
\begin{align}
\psi_\alpha(z) &= \sum_{n \in \mathbb{Z}} \psi_\alpha(n)z^{-n-1} \quad \text{with } \alpha \in \tilde{\Delta}_>, \\
\psi_{-\alpha}(z) &= \sum_{n \in \mathbb{Z}} \psi_{-\alpha}(n)z^{-n} \quad \text{with } \alpha \in \tilde{\Delta}_<
\end{align}
are generating fields. The vertex algebra $\wedge^{\infty+\bullet}(L\tilde{\mathfrak{g}}>0)$ is also called the Clifford vertex algebra associated with $L\tilde{\mathfrak{g}}>0$.

3.6. **The $W$-Algebra $\mathcal{W}^k(\tilde{\mathfrak{g}}, f)$.** Because both $\mathcal{V}^k(\tilde{\mathfrak{g}})$ and $\wedge^{\infty+\bullet}(L\tilde{\mathfrak{g}}>0)$ are vertex algebras, the tensor product
\begin{equation}
\mathcal{C}^* = \mathcal{V}^k(\tilde{\mathfrak{g}}) \otimes \wedge^{\infty+\bullet}(L\tilde{\mathfrak{g}}>0)
\end{equation}
is also a vertex algebra. Set $\mathcal{C}^i = \mathcal{V}^k(\tilde{\mathfrak{g}}) \otimes \wedge^{\infty+i}(L\tilde{\mathfrak{g}}>0)$, so that
\begin{equation}
\mathcal{C}^* = \bigoplus_{i \in \mathbb{Z}} \mathcal{C}^i.
\end{equation}

Let $Q(z)$ be the odd field on $\mathcal{C}^*$ defined by
\begin{equation}
Q(z) = Q^\text{st}(z) + \chi(z),
\end{equation}
where
\begin{align}
Q^\text{st}(z) &= \sum_{\alpha \in \tilde{\Delta}_>} J_\alpha(z)\psi_{-\alpha}(z) - \frac{1}{2} \sum_{\alpha, \beta, \gamma \in \tilde{\Delta}_>} c_{\alpha, \beta, \gamma}^\gamma \psi_{-\alpha}(z)\psi_{-\beta}(z)\psi_{\gamma}(z), \\
\chi(z) &= \sum_{\alpha \in \tilde{\Delta}_>} \bar{\chi}(J_\alpha)\psi_{-\alpha}(z)
\end{align}
($\bar{\chi}$ is defined in (24)). One has
\begin{equation}
[Q(z), Q(w)] = 0,
\end{equation}
and therefore,
\begin{equation}
Q^2_{(0)} = 0.
\end{equation}
because $Q(z)$ is odd. (Recall that $Q(0) = \text{Res}_z Q(z)$, see §2.1)

Since $Q(0)C^i \subset C^{i+1}$, $(C^*, Q(0))$ is a cochain complex.

**Definition 3.6.1.** The universal affine $W$-algebra $W^k(\mathfrak{g}, f)$ associate with $(\mathfrak{g}, f)$ at level $k$ is the zeroth cohomology of the complex $(C^*, Q(0))$:

$$W^k(\mathfrak{g}, f) = H^0(C^*, Q(0)).$$

The $W$-algebra $W^k(\mathfrak{g}, f)$ inherits the vertex algebra structure from $C^*$.

### 3.7. The Hamiltonian of $W^k(\mathfrak{g}, f)$.

Set

$$H = -D - \frac{1}{2} h_0,$$

where $h_0$ is the element in the $\mathfrak{sl}_2$-triple $\{e, h_0, f\}$ fixed in §3.1. The right-hand-side is considered as an element of $\mathfrak{h}$ which acts diagonally on the complex $C^* = V^k(\hat{\mathfrak{g}}) \otimes \mathbb{F}^{+*}(L_{\hat{\mathfrak{g}}, 0})$.

One knows that $H$ defines a Hamiltonian on $C^*$ (cf. §4.9 of [K2]).

**Lemma 3.7.1.** One has $[H, Q(0)] = 0$.

**Proof.** Obviously $[H, Q(0)] = 0$. Also,

$$\alpha(h_0) = 2$$

for all $\alpha$ such that $\bar{\chi}(J_0) \neq 0$.

This gives $[H, \chi(0)] = 0$. \hfill \Box

From Lemma 3.7.1 it follows that $H$ defines a Hamiltonian of $W^k(\mathfrak{g}, f)$. One has

$$W^k(\mathfrak{g}, f) = \bigoplus_{\Delta \in \mathbb{Z}} W^k(\mathfrak{g}, f)_{\Delta},$$

$$W^k(\mathfrak{g}, f)_{\Delta} = \{a \in W^k(\mathfrak{g}, f); Ha = \Delta a\}.$$ 

### 3.8. Generating Fields of $W^k(\mathfrak{g}, f)$.

Set

$$\hat{J}_a(z) = \sum_{n \in \mathbb{Z}} \hat{J}_a(n) z^{-n-1} = J_a(z) - \sum_{\beta, \gamma \in \Delta_{>0}} c_{\alpha, \beta}^\gamma : \psi_{-\beta}(z) \psi_\gamma(z) :$$

for $a \in \bar{I} \sqcup \bar{\Delta}$. One has [KW3 (2.5)] on $C^*$

$$[\hat{J}_a(m), \hat{J}_b(n)] = \sum_d c_{a,b}^d \hat{J}_d(m + n) + \left((k + h)^<)(a|b) - \frac{1}{2} \kappa_{\hat{\mathfrak{g}}_0}(a, b)\right) m \delta_{m+n,0} \text{id},$$

$$[\hat{J}_a(m), \psi_\alpha(n)] = \sum_d c_{a,\alpha}^\beta \psi_\beta(m + n)$$

provided that either $a, b \in \Delta_{>0} \sqcup \bar{I}$ and $\alpha \in \Delta_{>0}$, or $a, b \in \Delta_{\leq 0} \sqcup \bar{I}$ and $\alpha \in \Delta_{<0}$, where $\kappa_{\hat{\mathfrak{g}}_0}(a, b)$ is the Killing form of $\hat{\mathfrak{g}}_0$.

Let $C^*$ be the vertex subalgebra of $C^*$ generated by the fields $\hat{J}_a(z)$ and $\psi_\beta(z)$ with $a \in \bar{I} \sqcup \Delta_{\leq 0}$ and $\beta \in \Delta_{<0}$. By (57) and (58), $C^*$ is spanned by the vectors

$$\hat{J}_{a_1}(m_1) \ldots \hat{J}_{a_r}(m_r) \psi_{\beta_1}(n_1) \ldots \psi_{\beta_s}(n_s) 1$$

with $a_i \in \bar{I} \ sqcup \Delta_{\leq 0}$, $\beta_i \in \Delta_{<0}$, $m_i, n_i \in \mathbb{Z}$. 

**Remark.**
Similarly let $\mathcal{C}^*$ be the vertex subalgebra of $\mathcal{C}^*$ generated by the fields $\hat{J}_\alpha(z)$ and $\psi_\beta(z)$ with $\alpha, \beta \in \Delta_{>0}$. Then $\mathcal{C}^*$ is spanned by the vectors 

$$\hat{J}_{\alpha_1}(m_1) \ldots \hat{J}_{\alpha_r}(m_r) \psi_{\beta_1}(n_1) \ldots \psi_{\beta_s}(n_s) 1$$

with $\alpha_i \in \Delta_{>0}$, $\beta_i \in \Delta_{>0}$, $m_i, n_i \in \mathbb{Z}$.

One has the linear isomorphism 

$$C^* \cong \mathcal{C}^* \otimes C^*_\Delta.$$

Moreover it was shown [KW3] (cf. [dBT2, FBZ]) that both $C^*_\Delta$ are subcomplexes of $C^*$, and that 

$$H^i(C^*) = \begin{cases} \mathbb{C} & (i = 0) \\ 0 & (i \neq 0) \end{cases}.$$ 

Therefore by the Künneth theorem 

$$H^\bullet(C^*) = H^\bullet(C^*_\Delta).$$

It follows that we may identify $W^k(\hat{g}, f)$ with the vertex subalgebra $H^0(C^*_\Delta)$ of $C^*$ (Note that the cohomological gradation takes only non-negative values on $C^*_\Delta$):

$$W^k(\hat{g}, f) = H^0(C^*_\Delta) \subset C^*.$$ 

Let $\hat{g}^\text{aff} = \hat{g}^f \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}1$ be the central extension of the Lie algebra $\hat{g}^f \otimes \mathbb{C}[t, t^{-1}]$ with respect to the 2-cocycle $\phi_k$, defined by $\phi_k(a, b) = (k + h\gamma)(a|b) - \frac{1}{2} K_{h}(a, b)$. Set

$$V^{\phi_k}(\hat{g}^f) = U(\hat{g}^\text{aff}) \otimes U(\hat{g}^f \otimes \mathbb{C}[t] \oplus \mathbb{C}1) \mathbb{C}$$

where $\mathbb{C}$ is the $\hat{g}^f \otimes \mathbb{C}[t] \oplus \mathbb{C}1$-module on which $\hat{g}^f \otimes \mathbb{C}[t]$ acts trivially and 1 acts as 1.

By (57) one can regard $V^{\phi_k}(\hat{g}^f)$ as a vertex subalgebra of $V^k(\hat{g})$.

**Theorem 3.8.1** ([KW3]). For any $k \in \mathbb{C}$ one has the following.

(i) It holds that $H^i(C^*_\Delta) = 0$ for all $i \neq 0$. Therefore $H^i(C^*) = 0$ for all $i \neq 0$.

(ii) There exists an exhaustive, separated filtration \( \{ F_p W^k(\hat{g}, f) \} \) of the vertex algebra $W^k(\hat{g}, f)$ such that 

$$\text{gr}^F W^k(\hat{g}, f) \cong V^{\phi_k}(\hat{g}^f)$$

as graded vertex algebras.

**Remark 3.8.2.** The filtration in Theorem 3.8.1 arises from the spectral sequence associated with the filtration of $C^*_\Delta$ defined by

$$F_p C^*_\Delta = \bigoplus_{\langle \lambda, x_0 \rangle \geq p - n} (C^*_\Delta)^{\lambda}$$

(cf. §4 of [A3]).

Because $\hat{g}^f$ is preserved by the adjoint action of $x_0$ and $h_0$, there exists a basis \{ $u_j$; $j = 1, \ldots, \dim \hat{g}^f$ \} of $\hat{g}^f$ consisting of simultaneous eigenvectors of $\text{ad} x_0$ and $\text{ad} h_0$. Let $d_j \in \frac{1}{2} \mathbb{Z}_{\geq 0}$ be the half of the eigenvalue of $\text{ad} h_0$ on $u_j$:

$$[h_0, u_j] = -2d_j u_j.$$
By Theorem [3,8,1] there exist homogeneous elements \( W^{(j)} \) of \( \mathcal{W}^k(\bar{\mathfrak{g}}, f) \) with \( j = 1, \ldots, \dim \bar{\mathfrak{g}} \) whose symbols are \( u_j(-1)1 \), and \( \mathcal{W}^k(\bar{\mathfrak{g}}, f) \) is (strongly [K2]) generated by the fields

\[
W^{(j)}(z) = Y(W^{(j)}, z)
\]

in \( \mathcal{C}^\cdot \). The vector \( W^{(j)} \) has the conformal weight \( 1 + d_j \). Thus it follows that \( \mathcal{W}^k(\bar{\mathfrak{g}}, f) \) is positively graded:

\[
\mathcal{W}^k(\bar{\mathfrak{g}}, f) = \bigoplus_{\Delta \in \frac{1}{2} \mathbb{Z}_{\geq 0}} \mathcal{W}^k(\bar{\mathfrak{g}}, f)_\Delta, \quad \mathcal{W}^k(\bar{\mathfrak{g}}, f)_0 = \mathbb{C}1.
\]

4. Ramond Twisted Representation of Affine \( W \)-Algebras

4.1. Ramond Twisted Representations of \( \mathcal{W}^k(\bar{\mathfrak{g}}, f) \). Let \( \sigma_R : \mathcal{C}^\cdot \to \mathcal{C}^\cdot \) be the automorphism of order \( \leq 2 \) defined by

\[
\sigma_R = e^{\pi i \text{ad} h_0}.
\]

By [53], \( \sigma_R \) fixes the vector \( Q = Q(-1)1 \). Therefore \( [KW3] \) \( \sigma_R \) defines an automorphism of \( \mathcal{W}^k(\bar{\mathfrak{g}}, f) \).

A \( \sigma_R \)-twisted representation of \( \mathcal{W}^k(\bar{\mathfrak{g}}, f) \) is called a Ramond twisted representation of \( \mathcal{W}^k(\bar{\mathfrak{g}}, f) \).

Note that \( \sigma_R = \sigma_H \) (see \[2.4\] and \[53\]). Therefore Ramond twisted representations are exactly the \( H \)-twisted representations.

Remark 4.1.1. If the nilpotent element \( f \) is even then \( \sigma_R \) is trivial. In this case a Ramond twisted representations are usual (non-twisted) representations.

Proposition 4.1.2. Let \( M \) be a \( \sigma_R \)-twisted representation of \( \mathcal{C}^\cdot \). Then the space

\[
\ker((Q)_{(0)}^M : M \to M) / \text{im}((Q)_{(0)}^M : M \to M)
\]

is naturally a Ramond twisted representation of \( \mathcal{W}^k(\bar{\mathfrak{g}}, f) \).

Proof. By \[11\] and \[51\], the square of \((Q)_{(0)}^M \) is equal to zero. Therefore the above space is well-defined. The rest also follows from \[11\]. \( \square \)

4.2. \( \sigma_R \)-Twisted Representations of \( \mathcal{C}^\cdot \). Set

\[
\bar{\mathfrak{g}}^\text{Dyn}_j = \{ x \in \bar{\mathfrak{g}}; [h_0, x] = 2jx \}.
\]

Then \( \bar{\mathfrak{g}} = \bigoplus_{j \in \frac{1}{2} \mathbb{Z}} \bar{\mathfrak{g}}^\text{Dyn}_j \) gives a good grading for \( f \), called the Dynkin grading [KRW].

Let

\[
\bar{\mathfrak{g}}^R = \bigoplus_{j \in \frac{1}{2} \mathbb{Z}} \bar{\mathfrak{g}}^\text{Dyn}_j \otimes \mathbb{C}t^j \oplus \mathbb{C}K \oplus \mathbb{C}D
\]

be the \( \sigma_R \)-twisted affine Lie algebra [K1], where \( \bar{\mathfrak{g}}^\text{Dyn}_j = \bigoplus_{r \equiv j \text{ (mod } 2)} \bar{\mathfrak{g}}^\text{Dyn}_r \), and the commutation relations are given by the same formula as \( \mathfrak{g} \).

We write \( J^{(n)}R \) for \( J \otimes t^n \in \bar{\mathfrak{g}}^R \). Also, to avoid confusion we write \( K^R \) and \( D^R \) for \( K \) and \( D \) in \( \bar{\mathfrak{g}}^R \), respectively.
Lemma 4.2.1. Let $M$ be a vector space. Defining a $\sigma_R$-twisted $V^k(\mathfrak{g})$-module structure on $M$ is equivalent to defining a $\mathfrak{g}^R$-module structure on $M$ of level $k$ such that $J(n)^R m = 0$ for all $m \in M$ and $n \gg 0$.

Proof. By (12), given a $\sigma_R$-twisted module structure on $M$ one has

\begin{equation}
Y^M(J(-1)1, z), Y^M(J'(-1)1, w) = Y^M([J, J'](1,1), w)\delta_j(z-w) + k(J|J') id_M \partial_w \delta_j(z-w)
\end{equation}

for $J \in \mathfrak{g}_j^{\text{Dyn}}, J' \in \mathfrak{g}$. It follows that the correspondence $J(n)^R \mapsto (J(-1)1)^M_n$ define a representation of $\mathfrak{g}^R$ on $M$ of level $k$ with $J(n)^R m = 0$ for $m \in M$ and $n \gg 0$.

Conversely, suppose that we are given a $\mathfrak{g}^R$-module structure on $M$ of level $k$ such that $J(n)^R m = 0$ for $m \in M$ and $n \gg 0$. Define a 2-twisted filed $J(z)^R$ on $M$ by

\begin{equation}
J(z)^R = \sum_{n \in j+Z} J(n)^R z^{-n-1} \quad \text{for } J \in \mathfrak{g}_j^{\text{Dyn}}.
\end{equation}

These fields satisfy the same formula as (66):

\begin{equation}
[J(z)^R, J'(w)^R] = [J, J'](w)^R \delta_j(z-w) + k(J|J') id_M \partial_w \delta_j(z-w).
\end{equation}

Let $V$ be a vertex algebra generated by $J(z)^R$ with $J \in \mathfrak{g}$ in the space of 2-twisted fields on $M$ in the sense of Li [L4]. By (68) it follows that the correspondence $J(-1)1 \mapsto J(z)^R$ defines a vertex algebra homomorphism from $V^k(\mathfrak{g})$ to $V$ (cf. [L3]). Thanks to Proposition 3.17 of [L1], this completes the proof.

Let $\mathfrak{C}^R$ be the superalgebra generated by the odd fields $\psi_\alpha(n)^R$ ($\alpha \in \Delta_{\neq 0}$, $n \in \alpha(h_0)/2 + Z$) with the relations $[\psi_\alpha(m)^R, \psi_\beta(n)^R] = \delta_{m+n,0} \alpha_{\alpha+\beta,0}$.

The proof of the following assertion is the same as that of Lemma 4.2.1

Lemma 4.2.2. Let $M$ be a $\mathfrak{C}^R$-module such that $\psi_\alpha(n)^R m = 0$ for all $m \in M$, $\alpha \in \Delta_{\neq 0}$ and $n \gg 0$. Then the formulas

\begin{align*}
Y^M(\psi_\alpha(-1)1, z) &= \psi_\alpha(z)^R = \sum_{n \in \alpha(h_0)/2+Z} \psi_\alpha(n)^R z^{-n-1} \quad (\alpha \in \Delta_{>0}), \\
Y^M(\psi_\alpha(0)1, z) &= \psi_\alpha(z)^R = \sum_{n \in \alpha(h_0)/2+Z} \psi_\alpha(n)^R z^{-n} \quad (\alpha \in \Delta_{<0})
\end{align*}

defines a $\sigma_R$-twisted $\bigwedge^3 + \bullet (L\mathfrak{g}_{>0})$-module structure on $M$.

Set $U_k(\mathfrak{g}) = U(\mathfrak{g})/(K - k1)$. Let $M$ be a $U_k(\mathfrak{g}) \otimes \mathfrak{C}^R$-module such that $J(n)^R m = \psi_\alpha(n)^R m = 0$ for $n \gg 0$, $m \in M$, $J \in \mathfrak{g}$ and $\alpha \in \Delta_{\neq 0}$. Then by Lemmas 4.2.1 and 4.2.2, $M$ can be naturally considered as a $\sigma_R$-twisted representation of $\mathfrak{C}^R$. By Proposition 4.1.2, the space $\ker(Q)_0^M/\im(Q)_0^M$ is a Ramond twisted representation of $W^k(\mathfrak{g}, f)$. One has

\begin{equation}
(Q)_0^M = (Q^{at})_0^M + \chi_0^M,
\end{equation}
where \((Q_{\text{st}})^{(M)}_{(0)}\) and \(\chi^{(M)}_{(0)}\) are explicitly expressed as follows:

\[
(Q^M_{\text{st}})_{(0)} = \sum_{\alpha \in \Delta_\geq 0} J_\alpha(n)^M \psi_{-\alpha}(-n)^M
\]

\[
- \frac{1}{2} \sum_{k, \beta, \gamma \in \Delta_\geq 0, \varepsilon \in \Delta} c_{\alpha, \gamma}^\gamma \psi_{-\alpha}(-k)^M \psi_{-\beta}(-l)^M \psi_{\gamma}(k + l)^M,
\]

\[
\chi^M_{(0)} = \sum_{\alpha \in \Delta_0} \chi(J_\alpha) \psi_{-\alpha}(1)^M.
\]

4.3. Identification with Non-Twisted Representations. The superalgebra \(U(\mathfrak{g}^R) \otimes \mathfrak{cl}^R\) is isomorphic to \(U(\mathfrak{g}) \otimes \mathfrak{cl}\) [KW5]: the isomorphism is given by:

\[
\tilde{t}_{-\frac{1}{2}h_0} : J_\alpha(n)^R \mapsto J_\alpha(n + \alpha(h_0)/2) \quad (\alpha \in \Delta),
\]

\[
J_i(n)^R \mapsto J_i(n) \quad (i \in \tilde{I}, n \neq 0),
\]

\[
J_i(0)^R \mapsto J_i(0) + \frac{1}{2}(h_0 | J_i) K
\]

\[
K^R \mapsto K,
\]

\[
D^R \mapsto D - \frac{1}{2} h_0(0),
\]

\[
\psi_{\alpha}(n)^R \mapsto \psi_{\alpha}(n + \alpha(h_0)/2) \quad (\alpha \in \Delta_\neq 0, n \in \mathbb{Z}),
\]

Set \(U_k(\mathfrak{g}) = U(\mathfrak{g})/\langle K - k \rangle\) for \(k \in \mathbb{C}\). Let \(\widehat{\omega}_0 \in \text{Aut}(U_k(\mathfrak{g}) \otimes \mathfrak{cl})\) be a lift of the longest element \(\omega_0\) of the Weyl group \(\tilde{W}\). Set

\[
(70) \quad \tilde{g}_0 = \widehat{\omega}_0 \tilde{t}_{-\frac{1}{2}h_0}.
\]

Then \(\tilde{g}_0\) defines an isomorphism \(U_k(\mathfrak{g}^R) \otimes \mathfrak{cl}^R \tilde{\mapsto} U_k(\mathfrak{g}) \otimes \mathfrak{cl}\).

Let \(M\) be a (non-twisted) positive energy representation of \(V^k(\mathfrak{g})\). Then the space \(M \otimes \bigwedge^{\ast +}(L_{\mathfrak{g}^R_0})\) can be regarded as a \(\sigma_R\)-twisted representation of \(\mathfrak{cl}\), by the action

\[
(71) \quad u \cdot m = \tilde{g}_0(u)m
\]

for \(m \in M \otimes \bigwedge^{\ast +}(L_{\mathfrak{g}^R_0})\) and \(u \in U_k(\mathfrak{g}^R) \otimes \mathfrak{cl}^R\). Note that in this case the differential \((Q)^{(M)}_{\text{st}} \bigwedge^{\ast +}(L_{\mathfrak{g}^R_0})\) is homotopic to

\[
Q_- = Q_{\text{st}}^- + \chi_-,
\]

where

\[
(72) \quad Q_{\text{st}}^- = \sum_{\alpha \in \Delta_\leq 0} J_\alpha(-n) \psi_{-\alpha}(n)
\]

\[
- \frac{1}{2} \sum_{\alpha, \beta, \gamma \in \Delta_\leq 0} c_{\alpha, \beta}^\gamma \psi_{-\alpha}(-k) \psi_{-\beta}(-l) \psi_{\gamma}(k + l),
\]

\[
(73) \quad \chi_- = \sum_{\alpha \in \Delta_0} \chi(J_\alpha) \psi_{-\alpha}(0).
\]
One has
\[ Q_-(M \otimes \bigwedge^{\geq i} (L\mathfrak{g}_{>0})) \subset M \otimes \bigwedge^{\geq i-1} (L\mathfrak{g}_{>0}). \]

It follows that by Proposition 4.1.2 the homology space
\[ H_{\bullet}^{BRST}(M) := \bigoplus_{d \in d_0 + Z_{>0}} H_{\bullet}^{BRST}(M)_d, \]

and therefore,
\[ H_{\bullet}^{BRST}(M)_{top} = H_{\bullet}^{BRST}(M)_{d_0}, \]

provided that \( H_{\bullet}^{BRST}(M)_{d_0} \neq 0. \)

In this case \( H_{\bullet}^{BRST}(M)_{top} \) is easily described as follows: Identify the Grassmann algebra \( \bigwedge^\bullet (\bar{n}_-) \) of \( \bar{n}_- \) with the subalgebra of \( \mathfrak{c} \) generated by \( \psi_\alpha(0) \) with \( \alpha \in \Delta_- \).

Then \( \bigwedge^\bullet (\bar{n}_-) \) is also identified with the subspace \( \bigwedge^{\geq i} (L\mathfrak{g}_{>0})_{top} \) of \( \bigwedge^{\geq i} (L\mathfrak{g}_{>0}). \)

One has
\[ H_{0}^{BRST}(M)_{top} = H_0(M_{top} \otimes \bigwedge^\bullet (\bar{n}_-), Q_-). \]

One sees that the operator \( Q_- \) acts on \( M_{top} \otimes \bigwedge^\bullet (\bar{n}_-) \) as
\[ Q_- = \sum_{\alpha \in \Delta_{<0}} (J_\alpha(0) - \chi(J_{-\alpha}) \psi_{-\alpha}(0)) \]
\[ -\frac{1}{2} \sum_{\alpha, \beta, \gamma \in \Delta_{<0}} c_{\alpha, \beta}^\gamma \psi_{-\alpha}(0) \psi_{-\beta}(0) \psi_\gamma(0). \]

From this formula it follows that the complex \( (M_{top} \otimes \bigwedge^\bullet (\bar{n}_-), Q_-) \) is identical to the Chevalley-Eilenberg complex which defines the Lie algebra \( \mathfrak{g}_{<0} \)-homology \( H_{\bullet}^{Lie}(\mathfrak{g}_{<0}, M_{top} \otimes \mathbb{C}_{\bar{\chi}_-}) \) with the coefficient in the \( \mathfrak{g}_{<0} \)-module \( M \otimes \mathbb{C}_{\bar{\chi}_-} \), where \( \mathbb{C}_{\bar{\chi}_-} = U(\mathfrak{g}_{<0})/\ker \bar{\chi}_- \) and \( \bar{\chi}_- \) is the character of \( \mathfrak{g}_{<0} \) defined by
\[ \chi_- (J_\alpha) = \chi(J_{-\alpha}). \]

Thus one has
\[ H_{\bullet}^{BRST}(M)_{top} = H_{\bullet}^{Lie}(\bar{n}_-, M_{top} \otimes \mathbb{C}_{\bar{\chi}_-}). \]

This in particular means that \( H_{\bullet}^{Lie}(\bar{n}_-, M_{top} \otimes \mathbb{C}_{\bar{\chi}_-}) \) is a module over \( Zh(H^k(\mathfrak{g}, f)) \).
Recall [DSK] that

\[(79)\quad Zh_H(\mathcal{W}(\bar{g}, f)) \cong \mathcal{W}(\bar{g}, f),\]

where \(\mathcal{W}(\bar{g}, f)\) is the finite \(W\)-algebra associated with \((\bar{g}, f)\). The finite \(W\)-algebra \(\mathcal{W}(\bar{g}, f)\) may be defined by means of the quantum BRST reduction [KS] [DHK]:

Let \(\mathcal{C}\) be the Clifford algebra associated with \(\bar{g}_{<0} \oplus \bar{g}_{>0}\) and \((\, \mid \, )_{\bar{g}_{<0} \oplus \bar{g}_{>0}}\). We identify \(\mathcal{C}\) with the subalgebra of \(\mathcal{C}\) generated by \(\psi_\alpha = \psi_\alpha(0)\) with \(\Delta_{\alpha} \neq 0\). One has the subalgebra \(U(\bar{g}) \otimes \mathcal{C}\) in \(U(\bar{g}) \otimes \mathcal{C}\), and \(\mathcal{Q}_-\) can be considered as an odd element of \(U(\bar{g}) \otimes \mathcal{C}\). One has \((\mathcal{Q}_-)^2 = 0\), and thus

\[\text{(ad} \mathcal{Q}_-)^2 = 0.\]

Therefore \((U(\bar{g}) \otimes \mathcal{C}, \text{ad} \mathcal{Q}_-)\) is a chain complex (with respect the grading by charge). The corresponding homology

\[\text{(80) } H_*(U(\bar{g}) \otimes \mathcal{C}) = H_*(U(\bar{g}) \otimes \mathcal{C}, \text{ad} \mathcal{Q}_-)\]

is naturally a \(\mathbb{Z}\)-graded superalgebra.

**Theorem 4.4.1** ([DHK], cf. Theorem 2.4.2 of [A3]).

(i) It holds that \(H_*(U(\bar{g}) \otimes \mathcal{C}) = 0\) for all \(i \neq 0\).

(ii) There is an algebra isomorphism \(H_0(U(\bar{g}) \otimes \mathcal{C}) \cong \mathcal{W}(\bar{g}, f)\).

For a \(\mathcal{g}\)-module \(M\), \(M \otimes \Lambda(\bar{g}_{<0})\) is naturally a \(U(\bar{g}) \otimes \mathcal{C}\)-module. Therefore the algebra \(H_0(U(\bar{g}) \otimes \mathcal{C})\) naturally acts on \(H_*^{\text{Lie}}(\mathcal{g}, M \otimes \mathcal{C}_-)\). As in the same manner as [A3], it follows that the action of \(\text{Zh}_H(\mathcal{W}(\bar{g}, f))\) on \(H_*^{\text{BRST}}(M)_{\text{top}}\) coincides with the action of \(H_0(U(\bar{g}) \otimes \mathcal{C})\) on the space \(H_*^{\text{Lie}}(\mathcal{g}, M \otimes \mathcal{C}_-)\), via the isomorphisms (78) and (ii) of Theorem 4.4.1.

5. Representation Theory of Affine \(W\)-Algebras via the BRST Cohomology Functor

### 5.1. The Vanishing of the Lie Algebra Homology

Recall the notation from §3.1 and §3.2.

Let \(L(\lambda)\) be the irreducible highest weight representation of \(\mathcal{g}\) with highest weight \(\lambda \in \bar{h}^*\).

Let \(\mathcal{O}_0(\mathcal{g})\) be the full subcategory of the category of finitely generated \(\mathcal{g}\)-modules consisting of objects \(M\) such that (1) \(\dim U(\bar{n}_+)m < \infty\) for all \(m \in M\), (2) \(\bar{h}\) acts semisimply on \(M\), (3) \(M\) is a direct sum of finite-dimensional \(\mathcal{g}_0\)-modules.

Set

\[(81)\quad \bar{P}_0^+ = \{ \lambda \in \bar{h}^*; \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in \Delta_{0, +}\}.\]

For \(\lambda \in \bar{P}_0^+\) put \(M_0(\lambda) = U(\bar{g}) \otimes U(\bar{g}_{\geq 0})L(\lambda)\), where \(L(\lambda)\) is the irreducible finite-dimensional representation of \(\mathcal{g}_0\) with highest weight \(\lambda\), considered as a \(\mathcal{g}_{\geq 0}\)-module on which \(\mathcal{g}_{>0}\) acts trivially. The \(M_0(\lambda)\) has \(L(\lambda)\) as its unique simple quotient. Every simple object of \(\mathcal{O}_0(\mathcal{g})\) is isomorphic to exactly one of the \(L(\lambda)\) with \(\lambda \in \bar{P}_0^+\).

For a finitely generated \(\mathcal{g}\)-module \(M\) let \(\text{Dim } M\) be the Gelfand-Kirillov dimension of \(M\). By (26), one has

\[\text{(82) } \text{Dim } M \leq d_{\bar{h}}\]

for all \(M \in \mathcal{O}_0(\mathcal{g})\).
Set
\[
H^\text{Lie}_c(M) = H^\text{Lie}_c(\mathfrak{g}_{<0}, M \otimes \mathbb{C}_{\leq -}).
\]

One sees that $H^\text{Lie}_c(M)$ is finite-dimensional for any object $M$ of $\mathcal{O}_0(\mathfrak{g})$ as in Lemma 2.5.1 of [A3]. From [A3] it follows that the correspondence $M \mapsto H^\text{Lie}_c(M)$ defines a functor from $\mathcal{O}_0(\mathfrak{g})$ to $\text{Fin}(\text{W}^\text{fin}(\mathfrak{g}, f))$, the category of finite-dimensional representations of $\text{W}^\text{fin}(\mathfrak{g}, f)$.

**Theorem 5.1.1** (Matumoto [M]).

(i) The functor $H^\text{Lie}_c(?) : \mathcal{O}_0(\mathfrak{g}) \to \text{Fin}(\text{W}^\text{fin}(\mathfrak{g}, f))$ is exact.

(ii) Let $M$ be an object of $\mathcal{O}_0(\mathfrak{g})$. One has $H^\text{Lie}_c(M) \neq 0$ if and only if $\dim M = d_{\chi}$.

Because every projective object of $\mathcal{O}_0(\mathfrak{g})$ is free over $U(\mathfrak{g}_{<0})$, the following assertion follows from (i) of Theorem 5.1.1 in the same manner as Theorem 2.5.6 of [A3].

**Theorem 5.1.2.** One has $H^\text{Lie}_c(M) = 0$ for all $i \neq 0$ and for all $M \in \mathcal{O}_0(\mathfrak{g})$.

### 5.2. Representations of Finite $W$-Algebras in Type $A$

In [BK], Brundan and Kleshchev gave a complete description of irreducible finite-dimensional representations of $\text{W}^\text{fin}(\mathfrak{g}, f)$ in type $A$, as we recall below:

Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. As usual, we write $\Delta = \{\alpha_{i,j} ; 1 \leq i, j \leq n\}$, $\Delta_+ = \{\alpha_{i,j} ; 1 \leq i < j \leq n\}$.

Let $Y_f$ be the partition $(p_1 \leq p_2 \leq \cdots \leq p_r)$ of $n$ corresponding to the nilpotent element $f$. Following [BK], we identify $Y_f$ with the Young diagram with $p_i$ boxes in the $i$th row, and number the boxes of $Y_f$ by $1, 2, \ldots, n$ down columns from left to right. The corresponding good grading is defined so that

\[
\bar{\Delta} = \{\alpha_{i,j} ; \text{the $i$th and the $j$th boxes belong to the same column}\}
\]  
(see [EK] [BK] for details). Let

\[
\bar{\Delta}_f = \{\alpha \in \bar{\Delta} ; \alpha(h) = 0 \ \forall h \in \mathfrak{h}\},
\]

\[
\bar{\Delta}_+= \bar{\Delta}_f \cap \Delta_+.
\]

It is easy to see that

\[
\bar{\Delta}_f = \{\alpha_{i,j} \in \bar{\Delta} ; \text{the $i$th and the $j$th boxes belong to the same row}\}.
\]

Let

\[
W_f = \{w \in W ; w(h) = h \ \forall h \in \mathfrak{h}\}.
\]

Then $W_f$ is the subgroup of $W = \mathfrak{S}_n$ generated by $s_{\alpha}$ with $\alpha \in \bar{\Delta}_f$.

**Theorem 5.2.1** (Brundan and Kleshchev [BK], $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$).

(i) For $\lambda \in \check{P}_0^+$, $H^\text{Lie}_c(L(\lambda)) \neq 0$ if and only if $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{N}$ for all $\alpha \in \check{\Delta}_f$. In this case $H^\text{Lie}_c(L(\lambda))$ is irreducible. Further, any irreducible finite-dimensional representation of $\text{W}^\text{fin}(\mathfrak{g}, f)$ arises in this way.

(ii) Nonzero $H^\text{Lie}_c(L(\lambda))$ and $H^\text{Lie}_c(L(\mu))$, with $\lambda, \mu \in \check{P}_0^+$, are isomorphic if and only if $\mu + \rho \in W_f(L(\lambda + \rho))$. 

5.3. The Category $\mathcal{O}_{0,k}$ of $\mathfrak{g}$-Modules. Recall the notation from §3.3.

For $\lambda \in \mathfrak{h}^*$ let $L(\lambda)$ be the irreducible representation of $\mathfrak{g}$ with highest weight $\lambda$.

Let $\mathcal{O}_{0,k}$ be the full subcategory of the category of left $\mathfrak{g}$-modules consisting of objects $M$ such that the following hold:

- $K$ acts as the multiplication by $k$ on $M$;
- $M$ admits a weight space decomposition;
- there exists a finite subset $\{\mu_1, \ldots, \mu_n\}$ of $\mathfrak{h}_k^*$ such that $M = \bigoplus_{\mu \in \bigcup_i \mu_i - Q_+} M^\mu$;
- for each $d \in \mathbb{C}$, $M_d$ is an object of $\mathcal{O}_0(\mathfrak{g})$ as $\mathfrak{g}$-modules.

Set
\begin{equation}
P_0^+ = \{ \lambda \in \mathfrak{h}_k^*; \lambda \in \hat{P}_0^+, \langle \lambda, K \rangle = k \}.
\end{equation}

For $\lambda \in P_0^+$, let
\begin{equation}
M_0(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h}[t] \oplus \mathbb{C}K \oplus \mathbb{C}D)} M_0(\lambda),
\end{equation}
where $\hat{M}_0(\lambda)$ is considered as a $\mathfrak{g}[t] \oplus \mathbb{C}K \oplus \mathbb{C}D$-module on which $\mathfrak{g}[t]t$ acts trivially, and $K$ and $D$ act the multiplication by $\langle \lambda, K \rangle$ and $\langle \lambda, D \rangle$, respectively. The $M_0(\lambda)$ is an object of $\mathcal{O}_{0,k}$, and has $L(\lambda)$ as its unique simple quotient. Every irreducible object of $\mathcal{O}_k$ is isomorphic to exactly one of the $L(\lambda)$ with $\lambda \in P_0^+$.

The correspondence $M \mapsto M^*$ defines a duality functor on $\mathcal{O}_{0,k}$. Here, $\mathfrak{g}$ acts on $M^*$ by
\begin{equation}
(Xf)(v) = f(X^t v)
\end{equation}
where $X \mapsto X^t$ is the anti-automorphism of $\mathfrak{g}$ define by $K^t = K$, $D^t = D$ and $J(n)^t = J^t(-n)$ for $J \in \mathfrak{g}$, $n \in \mathbb{Z}$.

Clearly, $(M^*)^* = M$ for $M \in \mathcal{O}_{0,k}$. It follows that $L(\lambda)^* = L(\lambda)$.

Let $\mathcal{O}_{0,k}^{\Delta}$ be the full subcategory of $\mathcal{O}_{0,k}$ consisting of objects $M$ that admit a finite filtration $M = M_0 \supset M_1 \supset \cdots \supset M_r = 0$ such that each successive subquotient $M_l/M_{l+1}$ is isomorphic to some generalized Verma module $M_0(\lambda_i)$ with $\lambda_i \in P_0^+$. Dually, let $\mathcal{O}_{0,k}^{\nabla}$ be the full subcategory of $\mathcal{O}_k$ consisting of objects $M$ such that $M^* \in \bigoplus j \mathcal{O}_{0,k}^{\Delta}$.

For $\lambda \in P_0^+$, let $\mathcal{O}_{0,k}^{\leq \lambda}$ be the Serre full subcategory of $\mathcal{O}_{0,k}$ consisting of objects $M$ such that $M = \bigoplus_{\mu \leq \lambda} M^\mu$. It is well-known (see e.g., [S]) that every $L(\mu)$ that lies in $\mathcal{O}_{0,k}^{\leq \lambda}$ admits the indecomposable projective cover $P_{\leq \lambda}(\mu)$ in $\mathcal{O}_{0,k}^{\leq \lambda}$, and hence, every finitely generated object in $\mathcal{O}_{0,k}^{\leq \lambda}$ is an image of a projective object of the form $\bigoplus_{i=1}^r P_{\leq \lambda}(\mu_i)$. The $P_{\leq \lambda}(\mu)$ is an object of $\mathcal{O}_{0,k}^{\Delta}$. Dually, $I_{\leq \lambda}(\mu) = P_{\leq \lambda}(\mu)^*$ is the injective envelope of $L(\mu)$ in $\mathcal{O}_{0,k}^{\leq \lambda}$.

5.4. The “Top” Part of the BRST Cohomology. Let $M$ be an object of $\mathcal{O}_{0,k}$.

Clearly, $M_{\text{top}}$ is a $\mathfrak{g}$-submodule of $M$. By Theorem 5.1.2, $H^i_{\text{BRST}}(M_{\text{top}}) = 0$ for all $i > 0$, and $H^0_{\text{BRST}}(M_{\text{top}})$ is a finite-dimensional $\mathcal{W}^\text{fin}(\mathfrak{g}, f)$-module.

The following assertion follows from Theorem 5.1.2.
**Lemma 5.4.1.** Let $M$ be an object of $\mathcal{O}_{0,k}$. Assume that $H_{\Lambda}^{\text{Lie}}(M_{\text{top}}) \neq 0$. Then one has the following isomorphism of $\mathcal{W}^{\text{fin}}(\mathfrak{g}, f)$-modules:

$$H_i^{\text{BRST}}(M)_{\text{top}} \cong \begin{cases} 
H_0^{\text{Lie}}(M_{\text{top}}) & \text{for } i = 0 \\
0 & \text{for } i \neq 0.
\end{cases}$$

The following assertion follows from Theorems 5.1.1, 5.1.2 and Lemma 5.4.1.

**Proposition 5.4.2.** One has

$$H_i^{\text{BRST}}(M_0(\lambda))_{\text{top}} \cong \begin{cases} 
H_0^{\text{Lie}}(\tilde{M}_0(\tilde{\lambda})) & \text{for } i = 0 \\
0 & \text{for } i \neq 0,
\end{cases}$$

and if $\dim \tilde{L}(\tilde{\lambda}) = d_{\tilde{\chi}}$, then

$$H_i^{\text{BRST}}(L(\lambda))_{\text{top}} \cong \begin{cases} 
H_0^{\text{Lie}}(\tilde{L}(\tilde{\lambda})) & \text{for } i = 0 \\
0 & \text{for } i \neq 0.
\end{cases}$$

**5.5. The Vanishing and the Almost Irreducibility.**

**Theorem 5.5.1.** Let $M$ be an object of $\mathcal{O}_{0,k}$. Then $H_i^{\text{BRST}}(M)_d$ is finite-dimensional for all $d$. If $M$ is an object of $\mathcal{O}_{0,k}^{\leq \lambda}$ then $H_i^{\text{BRST}}(M)_d = 0$ unless $|i| \leq d - \langle \lambda, D \rangle$.

**Proof.** By Theorem 5.1.2 one has $H_i^{\text{Lie}}(M|_{\mathfrak{g}}) = 0$ for all $i \neq 0$. Therefore by considering the Hochschild-Serre spectral sequence for $\mathfrak{g}_{<0} \subset L\mathfrak{g}_{<0}$, the assertion follows in the same manner as Theorem 7.4.2 of [A3].

Theorem 5.5.1 in particular implies that $H_i^{\text{BRST}}(M)$ is an ordinary representations for all $M \in \mathcal{O}_{0,k}$. It follows that one has the functor

$$\mathcal{O}_{0,k} \to \mathcal{W}^{k}(\mathfrak{g}, f)-\text{mod}_{\sigma_k}, \quad M \to H_0^{\text{BRST}}(M).$$

**Theorem 5.5.2 ([KW3]).** For $\lambda \in P_{0,+}^k$ one has the following:

(i) $H_i^{\text{BRST}}(M_0(\lambda)) = 0$ for all $i \neq 0$.

(ii) $H_0^{\text{BRST}}(M_0(\lambda))$ is almost highest weight.

(The proof of Theorem 5.5.2 is the same as that of Theorem 3.8.1)

**Theorem 5.5.3.** For $\lambda \in P_{0,+}^k$ one has the following:

(i) $H_i^{\text{BRST}}(M_0(\lambda)^{\ast}) = 0$ for all $i \neq 0$.

(ii) $H_0^{\text{BRST}}(M_0(\lambda)^{\ast})$ is almost co-highest weight.

The proof of Theorem 5.5.3 is given in Section 6.

Though our formulation is slightly different from that of [KW5], the following assertion essentially confirms Conjecture B of [KW3], partially (cf. Theorems 5.7.1, 5.8.1 and 5.8.3 below).

**Theorem 5.5.4 (The main result).** Let $k$ be any complex number.

(i) Let $M$ be an object of $\mathcal{O}_{0,k}$. Then $H_i^{\text{BRST}}(M) = 0$ for all $i \neq 0$. In particular the functor $H_0^{\text{BRST}}(\cdot) : \mathcal{O}_{0,k} \to \mathcal{W}^{k}(\mathfrak{g}, f)-\text{mod}_{\sigma_k}$ is exact.

(ii) For $\lambda \in P_{0,+}^k$, $H_0^{\text{BRST}}(L(\lambda))$ is zero or almost irreducible. Further, one has $H_0^{\text{BRST}}(L(\lambda)) \neq 0$ if and only if $\dim \tilde{L}(\tilde{\lambda}) = d_{\chi}$. 
**Theorem 5.6.1.**

For $H$ and hence those of Theorems 7.6.1 and 7.6.3 of [A3].

We give only the sketch of the proof because it is essentially the same as that of Theorems 6.1 and 6.3 of [A3].

By the Euler-Poincaré principle one has [FKW, KRW, KW5]

(92)

with Theorem 5.5.1 gives the vanishing of $H^i_{\text{BRST}}(M)$ for all $i > 0$ and all $M \in \mathcal{O}_{0,k}$. Likewise, Theorem 5.5.3 (i) gives $H^i_{\text{BRST}}(M) = 0$ for all $i < 0$ and all $M \in \mathcal{O}_{0,k}$. This shows (i). (ii) follows from (i) using Theorem 5.1.1 (ii), Theorem 5.5.2 (ii) and Theorem 5.5.3 (ii).

**Corollary 5.5.5.** Let $\lambda \in P_{0,+}^k$ with $k \in \mathbb{C}$. The representation $H^0_{\text{BRST}}(L(\lambda))$ is irreducible over $\mathcal{W}^k(\hat{g},f)$ if and only if $H^0_{\text{Lie}}(L(\lambda))$ is irreducible over $\mathcal{W}^{\text{fin}}(\hat{g},f)$.

**Proof.** The assertion follows immediately from Proposition 5.4.2 and Theorem 5.5.4 (ii). □

### 5.6. The Character of $H^0_{\text{BRST}}(L(\lambda))$

Let $\text{ch} L(\lambda)$ be the character of $L(\lambda)$: $\text{ch} L(\lambda) = \sum\limits_{\mu} e^\mu \text{dim} L(\lambda)^\mu$. One has

$$\text{ch} L(\lambda) = \sum\limits_{\mu \in \mathfrak{h}^*} c_{\lambda,\mu} e^\mu \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\text{dim} g_\alpha}$$

with some $c_{\lambda,\mu} \in \mathbb{Z}$. The coefficient $c_{\lambda,\mu}$ is known by Kashiwara and Tanisaki [KT] (in terms of the Kazhdan-Lusztig polynomials) provided that $k$ is not critical (for any simple summand of $H$).

Recall [KW3, KW5] that the “Cartan subalgebra” of $\mathcal{W}^k(\hat{g},f)$ is given by

$$t = \mathfrak{i} \oplus \mathbb{C} D, \quad \text{where } \mathfrak{i} = \mathfrak{h}^{\mathfrak{f}}.$$ 

Because it commutes with $Q_{-}$, $t$ acts on the space $H^0_{\text{BRST}}(M)$.

Let

$$\text{ch} H^0_{\text{BRST}}(L(\lambda)) = \sum\limits_{\xi \in t^\mathfrak{p}} e^\xi \text{dim} H^0_{\text{BRST}}(L(\lambda))_{\xi},$$

where $H^0_{\text{BRST}}(L(\lambda))_{\xi} = \{ c \in H^0_{\text{BRST}}(L(\lambda)); tc = \xi(t)c \ \forall t \in t \}$.

Set

$$\chi_{H^0_{\text{BRST}}(L(\lambda))} = \sum\limits_{i = -\infty}^{\infty} (-1)^i \text{ch} H^i_{\text{BRST}}(L(\lambda)).$$

By the Euler-Poincaré principle one has [FKW, KRW, KW5]

(93)

$$\chi_{H^0_{\text{BRST}}(L(\lambda))} = \prod_{j \geq 1} (1 - e^{-j\delta(1)} \text{rank} g \prod_{\alpha \in \Delta_{0,+}^e} (1 - e^{-\alpha(1)})^\alpha),$$

where $\Delta_{0,+}^e = \{ \alpha \in \Delta_{0,+}^e; \alpha \in \Delta_0 \}$.

The following assertion follows immediately from Theorem 5.5.4

**Theorem 5.6.1.** For $\lambda \in P_{0,+}^k$ one has

$$\text{ch} H^0_{\text{BRST}}(L(\lambda)) = \chi_{H^0_{\text{BRST}}(L(\lambda))}.$$
5.7. Type A Case. In type A, the following assertion follows immediately from (79), Theorems 4.4.1, 5.2.1 and 5.5.4 (in the notation of (94)).

Theorem 5.7.1 (g = sl_n). Let k be any complex number.

(i) One has $H^1_{BRST}(M) = 0$ for all $i \neq 0$ and all $M \in \mathcal{O}_{0,k}$.

(ii) For $\lambda \in P_{0,+}, H^0_{BRST}(L(\lambda)) \neq 0$ if and only if $\langle \lambda + \bar{\rho},\alpha^\vee \rangle \not\in \mathbb{N}$ for all $\alpha \in \bar{\Delta}_+$. In this case $H^0_{BRST}(L(\lambda))$ is irreducible. Further, any irreducible ordinary Ramond twisted representation of $W^k(sl_n,f)$ arises in this way.

(iii) Nonzero $H^0_{BRST}(L(\lambda))$ and $H^0_{BRST}(L(\mu))$ with $\lambda,\mu \in P_{0,+}$ are isomorphic if and only of $\bar{\mu} + \bar{\rho} \in \bar{W}(\lambda + \bar{\rho})$.

Theorems 5.6.1 and 5.7.1 determine the characters of all irreducible ordinary Ramond twisted representations of $W^k(sl_n,f)$ for all nilpotent elements $f$ at all non-critical levels $k$.

Remark 5.7.2. If $\bar{g}$ is not of type $A$, it not true that nonzero $H^0_{BRST}(L(\lambda))$ is always irreducible, see Theorem 3.6.3 of [M]. However it is likely that $H^0_{BRST}(L(\lambda))$ is a direct sum of irreducible modules.

5.8. Modular Invariant Representations. In this section we assume that $\bar{g}$ is simple.

Let $P^k_r$ be the set of principal admissible weights [KW2, KW3] of $\bar{g}$ of level $k$. For $\lambda \in P^k_r$ one has [KW1]

\begin{equation}
\text{ch} L(\lambda) = \sum_{w \in W(\lambda)} (-1)^{f_\lambda(w)} \frac{e^{w\circ \lambda}}{\prod_{j \geq 1}(1 - e^{-j\alpha})^{\text{rank}\, g} \prod_{\alpha \in \Delta_+}(1 - e^{-\alpha})}.
\end{equation}

Let $\bar{\Delta}(\lambda) = \Delta(\lambda) \cap \bar{\Delta}$, and let $\bar{W}(\lambda) \subset \bar{W}$ be the integral Weyl group of $\bar{\lambda} \in \bar{h}^*$ generated by $s_{w}$ with $\alpha \in \bar{\Delta}(\lambda)$. The formula (94) in particular implies that

\begin{equation}
\text{ch} \bar{L}(\bar{\lambda}) = \sum_{w \in \bar{W}(\bar{\lambda})} (-1)^{f_\lambda(w)} \frac{e^{w\circ \lambda}}{\prod_{\alpha \in \bar{\Delta}_+}(1 - e^{-\alpha})}.
\end{equation}

We remark that an element $\lambda$ of $P^k_r$ does not necessarily belong to $P^k_{0,+}$. However the Euler-Poincaré character $\chi_{H^0_{BRST}(L(\lambda))}$ makes sense for all $\lambda \in P^k_r$ [KW5], and coincides with the right-hand-side of (95). Thus it has the form

\begin{equation}
\chi_{H^0_{BRST}(L(\lambda))} = e^{(\lambda,D)\delta^0} \sum_{j \in \mathbb{Z}_{\geq 0}} e^{-j\delta^0} \varphi_{\lambda,j}
\end{equation}

with

\begin{equation}
\varphi_{\lambda,0} = \frac{\sum_{w \in W(\lambda)} (-1)^{f_\lambda(w)} e^{w\circ \lambda} \prod_{\alpha \in \Delta_+}(1 - e^{-\alpha})}{\prod_{\alpha \in \Delta_+}(1 - e^{-\alpha})}.
\end{equation}

Note that $\varphi_{\lambda,0}$ is the Euler-Poincaré character of $H^0_{BRST}(L(\lambda))$.

\textbf{Remark 5.8.1.} In the case of $f$ is a principal nilpotent element the characters of all irreducible positive energy representations of $W^k(g,f)$ was previously determined in [A3] (for all $\bar{g}$ and all $k \in \mathbb{C}$). Also, in the case $f$ is a minimal nilpotent element the characters of all irreducible (non-twisted) positive energy representations of $W^k(g,f)$ was previously determined in [A3] (for all $\bar{g}$ and all non-critical $k$).
Proposition 5.8.3. Let By Theorem 2.3 of [KW5],
Proof. For such a (101) \( z \in \mathfrak{g} \).
Further, again by (102), it follows that this action of \( \bar{\mathfrak{g}} \) is irreducible.
Remark 5.8.2. Let \( \lambda \in \mathcal{M}_k \). Then \( \mathcal{H}^\mathcal{B}_{\mathcal{R}S}^\mathcal{B}(L(\lambda)) \) is irreducible.
Proof. By Corollary 5.5.5 it is sufficient to show that \( \mathcal{H}^\mathcal{B}_{\mathcal{R}S}(L(\bar{\lambda})) \) is irreducible over \( \mathcal{W}^\mathcal{B}(\bar{\mathfrak{g}}, f) \).
By Corollary 2.2 of [KW5] (or its proof) one has
\( |\bar{\Delta}(\lambda)| = |\Delta_0|. \)
In our setting \( \Delta_0 \subseteq \Delta_{1/2} \) in [KW5] is identified with \( \bar{\Delta}_0 \), see [BG]. Because \( \lambda \in \mathcal{P} \), \( \bar{\Delta}_0 \subset \Delta_+(\lambda) \), and hence \( \Delta(\lambda) = \Delta_0 \). This implies
\( \bar{\Delta}(\lambda) = \bar{\Delta}_0 \). Thanks to Theorem 3.4.4 of [M], this gives the irreducibility of \( \mathcal{H}^\mathcal{B}_{\mathcal{R}S}(L(\lambda)) \).

### Remark 5.8.2
Let \( \lambda \in \mathcal{P}_0^k. \) From Theorem 5.5.3 it follows that \( \chi_{\mathcal{H}^\mathcal{B}_{\mathcal{R}S}(L(\lambda))} \) is almost convergent if and only if \( \dim \bar{L}(\lambda) = d_\lambda \).

Recall [KW5] that the pair \( (k, f) \) is called exceptional if the Euler-Poincare character \( \chi_{\mathcal{H}^\mathcal{B}_{\mathcal{R}S}(L(\lambda))} \) is almost convergent for some \( \lambda \in \mathcal{P}_k \), and is either zero or almost convergent for all \( \lambda \in \mathcal{P}_k \).
The exceptional pairs are classified in [KW5] in type A: Each admissible number \( \mathcal{P}_n \) of \( \mathfrak{sl}_n \) is written as
\( k + n = \frac{p}{q}, \quad p \geq n, \quad q \geq 1, \quad (p, q) = 1. \)

For such a \( k \) the pair \( (k, f) \) is exceptional if and only if \( f \) is the nilpotent element corresponding to the partition \( (s, q, q, \ldots, q) \) \( s \equiv n \pmod{q}, \quad 0 \leq s < q). \)
The following assertion was implicitly proved\(^9\) in [KW5].

### Proposition 5.8.3
Let \( (k, f) \) be an exceptional pair for \( \mathfrak{sl}_n \). There is a bijection
\( \bar{\mathcal{W}}^f \times \mathcal{M}_k \sim \bar{\mathcal{M}}_k, \quad (w, \lambda) \mapsto w \circ \lambda. \)

Proof. By Theorem 2.3 of [KW5],
\( \bar{\mathcal{M}}_k = \{ \lambda \in \mathcal{P}_r^k; \bar{\Delta}(\lambda) \subset \bar{\Delta} \setminus \bar{\Delta}_f \}. \)
Let \( \lambda \in \bar{\mathcal{M}}_k, \ w \in \bar{\mathcal{W}}^f. \) Since \( \mathcal{W}^r \cap w^{-1}(\mathcal{W}^r) \subset \bar{\Delta}_f, \) (102) gives \( \mathcal{W}^r(\lambda) \cap w^{-1}(\mathcal{W}^r) = \emptyset, \) or equivalently, \( \bar{\Delta}(\lambda) \subset \bar{\Delta}_f \) because
\( \bar{\Delta}(\lambda) \subset \bar{\Delta}_f \), \( \forall w \in \bar{\mathcal{W}}^f, \)
the element \( w \circ \lambda \) belongs to \( \bar{\mathcal{M}}_k. \) Therefore the shifted action of \( \bar{\mathcal{W}}^f \) preserves \( \bar{\mathcal{M}}_k. \) Further, again by (102), it follows that this action of \( \bar{\mathcal{W}}^f \) on \( \bar{\mathcal{M}}_k \) is faithful, and that \( \mathcal{M}_k \cap (\bar{\mathcal{W}}^f \circ \lambda) = \{ \lambda \} \) for all \( \lambda \in \mathcal{M}_k. \)

\(^9\)In the case that \( f \) is a principal nilpotent element (= the case that \( q \geq n, \bar{\Delta}_0 = \emptyset \) and \( \bar{\Delta}_f = \bar{\Delta} \)) Proposition 5.5.3 was proved in [FRW].
Next let \( k \) be as in (101). By Lemma 3.1 of \([KW5]\) one has
\[
\text{rank } \Delta(\lambda) \geq \min(n - q, 0) = \text{rank } \Delta_0, \quad \forall \lambda \in P_r^k.
\]
According to (the proofs of) Propositions 3.2 and 3.3 of \([KW5]\), the rank of any root subsystem in \( \Delta \setminus \Delta_f \) is equal to or smaller than \( \text{rank } \Delta_0 \), and is equal to \( \text{rank } \Delta_0 \) if and only if it is \( W_f \)-conjugate to \( \Delta_0 \). Thus for \( \lambda \in M_k \) there exists \( w \in W_f \) such that \( \Delta(\lambda) = w(\Delta_0) \), and thus \( w^{-1} \circ \lambda \in M_k \). This completes the proof. \( \square \)

According to \([KW5]\), Theorem 6.1.2 and Proposition 5.8.3 give the following assertion.

**Theorem 5.8.4** (Conjectured by Kac and Wakimoto \([KW5]\)). Let \((k, f)\) be an exceptional pair for \( \mathfrak{sl}_n \). The linear span of the normalized characters of irreducible ordinary Ramond twisted representations \( H^\text{BRST}_{(\lambda, \psi)}(L(\lambda)) \) of \( W^k(\mathfrak{sl}_n, f) \), with \( \lambda \in M_k \), are closed under the natural action of \( SL_2(\mathbb{Z}) \).

6. Proof of Theorem 5.5.3

The proof of Theorem 5.5.3 is essentially the repetition of the argument of §7 of \([A3]\). Therefore we give only the sketch of the proof.

6.1. Step 1. Let
\[
C^*(M_0(\lambda)) := M_0(\lambda) \otimes \bigwedge^{\circ +} (L_{\mathfrak{g}_0} > 0).
\]
As in §8.2 of \([A3]\), we identify \( M_0(\lambda)^* \otimes \bigwedge^{\circ +} (L_{\mathfrak{g}_0} > 0) \) with \( C^*(M_0(\lambda))^* \) \((^* \text{ is defined in } (30))\):
\[
H^\text{BRST}_{(\lambda, \psi)}(M_0(\lambda)^*) = H_*(C^*(M_0(\lambda))^*), \quad Q_-. \]

The differential \( Q_- \) acts on \( C^*(M_0(\lambda))^* \) by
\[
(Q_- \phi)(c) = \phi(Q_+ c)
\]
for \( \phi \in C^*(M_0(\lambda))^* \), \( v \in C^*(M_0(\lambda)) \), where
\[
Q_+ = (Q^\text{ad}_+)_0 + \chi'_+, \quad \chi'_+ = \sum_{\alpha \in \Delta_A^+} \chi(\alpha)\psi_{-\alpha}(0).
\]

Below we regard \( C^*(M_0(\lambda)) \) as a \( \sigma_R \)-twisted representation of \( C^* \) by the action
\[
X(n)^R \mapsto \tilde{L}_{-\frac{1}{2}h_0}(X(n)) \quad \psi_{\alpha}(n)^R \mapsto \tilde{L}_{-\frac{1}{2}h_0}(\psi_{\alpha}(n))
\]
(see Remark 6.1.2 below). Then
\[
(Q_{(-1)}1_{(0)}(M_0(\lambda))) = Q_+
\]
(in the notation of (220)).

Let \( C_+(\lambda) \) the \( C^*_+ \)-submodule of \( C^*(M_0(\lambda)) \) spanned by the vectors
\[
\tilde{J}_{\alpha_1}(m_1) \ldots \tilde{J}_{\alpha_r}(m_r)\psi_{\beta_1}(n_1) \ldots \psi_{\beta_s}(n_s) v_\lambda
\]
with \( \alpha_1 \in \Delta_{\leq 0} \setminus \hat{f} \) and \( \beta_i \in \Delta_{< 0} \), where \( v_\lambda \) is the highest weight vector of \( C^*(M_0(\lambda)) \).

As in \([3, 8]\), it follows that \( C^*_+(\lambda) \) is an subcomplex of \( C^*(M_0(\lambda)) \).

\[\text{10}\] However the rationality of the simple quotient of \( W^k(\mathfrak{sl}_n, f) \) still remains to be an open problem.
The graded dual space $C^\bullet_+(\lambda)^*$ of $C^\bullet_+(\lambda)$ is a quotient complex of $C^\bullet(M_0(\lambda))^*$. Thus there is a natural map
\begin{equation}
H^\bullet_{\text{BRST}}(M_0(\lambda)^*) \to H^\bullet(C^\bullet_+(\lambda)^*). \tag{112}
\end{equation}
The space $C^\bullet_+(\lambda)^*$ is a $C^\bullet_+$-submodule of $M_0(\lambda)^* \otimes \bigwedge^\infty (L_{\bar{g}}_{>0})$ with respect to the action $[111]$. Hence by $[111]$ it follows that $[112]$ is a homomorphism of Ramond twisted representations of $W^k(\bar{g}, f)$.

One has the following assertion (cf. Proposition 8.3.4 of $[A3]$):

**Proposition 6.1.1.** The map $[112]$ gives the isomorphism
\[ H^\bullet_{\text{BRST}}(M_0(\lambda)^*) \cong H^\bullet(C^\bullet_+(\lambda)^*) \]
of $W^k(\bar{g}, f)$-modules.

**Remark 6.1.2.** Using the action $[109]$ one can define a $\sigma_R$-twisted $C^\bullet$-module structure on $M_0(\lambda)^* \otimes \bigwedge^\infty (L_{\bar{g}}_{>0})$ by the formula
\[ (X(n)\phi)(c) = \phi(X^n(-n)c) \quad \text{with} \; X \in \bar{g}. \]
This is not the same as action $[111]$, but as easily seen $H^\bullet_{\text{BRST}}(M_0(\lambda)^*)$ is almost co-highest weight if and only if it is so with respect to this new action of $W^k(\bar{g}, f)$.

**6.2. Step 2.** One has
\[ C^\bullet_+(\lambda) = \bigoplus_{d \in -\langle \lambda, D \rangle + \mathbb{Z}_{\geq 0}} C^\bullet_+(\lambda)_d, \quad \dim C^\bullet_+(\lambda)_d = \infty. \]
Note that the subspace $C^\bullet_+(\lambda)^\text{top} = C^\bullet_+(\lambda)_{-\langle \lambda, D \rangle}$ is the subcomplex of $(C^\bullet_+(\lambda), Q_+)$ spanned by the vectors
\begin{equation}
\hat{J}_{a_1}(0) \ldots \hat{J}_{a_r}(0)\psi_{\beta_1}(0) \ldots \psi_{\beta_r}(0)v_\lambda
\end{equation}
with $a_i \in \bar{\Delta}_{\geq 0} \sqcup \bar{I}$, $\beta_i \in \bar{\Delta}_{<0}$, and hence,
\begin{equation}
C^\bullet_+(\lambda)^\text{top} = M_0(\hat{\lambda}) \otimes \bigwedge^\bullet (\bar{g}_{>0}). \tag{114}
\end{equation}
One has the weight space decomposition
\[ C^\bullet_+(\lambda)^\text{top} = \bigoplus_{\mu \in \mathbb{H}^+, \langle \lambda - \mu, x_0 \rangle \geq 0} C^\bullet_+(\lambda)^\mu_{\text{top}}. \]
Define a decreasing filtration
\begin{equation}
C^\bullet_+(\lambda)^\text{top} = F^0 C^\bullet_+(\lambda)^\text{top} \supset F^1 C^\bullet_+(\lambda)^\text{top} \supset \ldots
\end{equation}
of $C^\bullet_+(\lambda)^\text{top}$ by
\begin{equation}
F^p C^\bullet_+(\lambda)^\text{top} = \bigoplus_{\mu \in \mathbb{H}^+, \langle \lambda - \mu, x_0 \rangle \geq p} C^\bullet_+(\lambda)^\mu_{\text{top}}, \tag{115}
\end{equation}
Then
\begin{equation}
(Q^\bullet_+(0) \cdot F^p C^\bullet_+(\lambda)^\text{top} \subset F^p C^\bullet_+(\lambda)^\text{top}, \tag{116}
\end{equation}
\begin{equation}
\chi_+ \cdot F^p C^\bullet_+(\lambda)^\text{top} \subset F^{p+1} C^\bullet_+(\lambda)^\text{top}. \tag{117}
\end{equation}
Let $F^p C^*_+(\lambda)$ be the subspace of $C^*_+(\lambda)$ generated by $C^*_+(\lambda)_{\text{top}}$ over $C^*_+$. One has
\begin{align}
C^*_+(\lambda) &= F^0 C^*_+(\lambda) \supset F^1 C^*_+(\lambda) \supset \ldots, \\
\bigcap_p F^p C^*_+(\lambda) &= 0,
\end{align}
(120) $Q_p F^p C^*_+(\lambda) \subset F^p C^*_+(\lambda)$,
(121) $a(\lambda)_n : F^p C^*_+(\lambda) \subset F^p C^*_+(\lambda)$ \quad (a \in \mathbb{C}^+, n \in \mathbb{Z})
(\text{cf. Proposition 8.5.3 of } \text{[A3]})

Let $(\vee E^p,q,d_r)$ be the corresponding spectral sequence:
\begin{align}
\vee E^p_0 &= F^p C^*_+(\lambda)/F^{p+1} C^*_+(\lambda), \\
\vee E^p_1 &= H^{p+q}(\vee E^p_0).
\end{align}
We do not claim that this spectral sequence converges to $H^\bullet(C^*_+(\lambda))$. We will show in Proposition 6.4.2 below that $\vee E_r$ converges to the dual $D(H^\bullet_{\text{BRST}}(M_0(\lambda)^*))$ of $H^\bullet_{\text{BRST}}(M_0(\lambda)^*)$.

6.3. **Step 3.** Set
\begin{align}
F_p C^*_+(\lambda)^* &= (C^*_+(\lambda)/F^p C^*_+(\lambda))^* \subset C^*_+(\lambda)^*.
\end{align}

Then $\{F_p C^*_+(\lambda)^*\}$ defines an exhaustive, increasing filtration of the chain complex $\{C^*_+(\lambda)^*\}$ which is obviously bounded below (cf. Lemma 8.5.4 and Proposition 8.5.5 of [A3]). It follows that one has the corresponding converging spectral sequence
\begin{align}
E^r_r &\Rightarrow H^\bullet(C^*_+(\lambda)^*) = H^\bullet_{\text{BRST}}(M_0(\lambda)^*).
\end{align}

Let $\{F_p H^\bullet_{\text{BRST}}(M_0(\lambda)^*)\}$ be the corresponding increasing filtration of $H^\bullet_{\text{BRST}}(M_0(\lambda)^*)$.

Because the filtration is compatible with the action of the Hamiltonian $-D$, each $E^r_{p,q}$ decomposes into eigenspaces of $-D$ as complexes:
\begin{align}
E^r_{p,q} &= \bigoplus_{d \in (-\lambda,D) + \mathbb{Z}_{\geq 0}} (E^r_{p,q})_d.
\end{align}

It follows that
\begin{align}
E^\infty_{p,q} &= \bigoplus_{d \in (-\lambda,D) + \mathbb{Z}_{\geq 0}} (E^\infty_{p,q})_d,
\end{align}
and each $(E^r)_d$ converges to $(E^\infty)_d$. In particular one has
\begin{align}
\bigoplus_{p+q=n} (E^\infty_{p,q})_{\text{top}} &= \begin{cases} 
\text{gr}_F H^\bullet_{\text{BRST}}(M_0(\lambda)^*)_{\text{top}} & \text{if } p+q = 0, \\
0 & \text{if } p+q \neq 0.
\end{cases}
\end{align}

by Proposition 6.4.2.

Also by 121 this filtration is compatible with the $\sigma_R$-twisted action of $\mathcal{W}^k(\mathfrak{g},f)$. Hence each $E^r_{p,q}$ is a Ramond twisted representation of $\mathcal{W}^k(\mathfrak{g},f)$, and the differential $d^r$ is a morphism in $\mathcal{W}^k(\mathfrak{g},f)\cdot \mathbf{Mod}_{\sigma_R}$. Therefore $\{F_p H^\bullet_{\text{BRST}}(M_0(\lambda)^*)\}$ is a filtration of Ramond twisted representations of $\mathcal{W}^k(\mathfrak{g},f)$, and the corresponding graded space
\begin{align}
\text{gr}_F H^\bullet_{\text{BRST}}(M_0(\lambda)^*) = \bigoplus_{p+q=0} E^\infty_{p,q}
\end{align}
is also an object of $\mathcal{W}^k(\mathfrak{g},f)\cdot \mathbf{Mod}_{\sigma_R}$. 
6.4. Step 4. Consider the subcomplex
\[(\wedge E_0^p)^\top = (\wedge E_0^p)_{(\lambda, D)} = F^p C^p_+ (\lambda)\top / F^{p+1} C^p_+ (\lambda)\top \cong \bigoplus_{(\lambda - \mu, x_0) = p} C^p_+ (\lambda)\top \]
of \(\wedge E_0^p\). By (117) one has
\[(\wedge E_0^p)^\top \cong \bigoplus_{(\lambda - \mu, x_0) = p} (C^p_+ (\lambda)\top, (Q^p_+)_{(0)})\]
as complexes.

By definition \(\wedge E_0^p\top\) is spanned by the vectors
\[(\bar{J}_1(m_1) \cdots \bar{J}_{\lambda}(m_r) \psi_{\beta_1}(n_1) \cdots \psi_{\beta_i}(n_s)c)\]
with \(c \in (\wedge E_0^p)^\top\), and \(m_i, n_i < 0\). It follows that each \(D\)-eigenspace \((\wedge E_0^p)^\top\) is finite-dimensional. Thus by Lemma 3.3.1
\[(131) E_{p, q}^0 = (\wedge E_0^{p-1, q+1})^\ast = D(\wedge E_0^{p-1, q+1}).\]
The following assertion follows immediately from (131).

**Proposition 6.4.1.** One has \(E_{p, q}^1 = D(\wedge E_1^{p-1, q+1})\), or equivalently, \(\wedge E_0^{p, q} = D(E_1^{p+1, q-1})\).

The following assertion follows from Proposition 6.4.1 by the inductive argument.

**Proposition 6.4.2.** The spectral sequence \(\wedge E^r\) converges to \(D(E_\infty)\).

The proof of the following assertion is the same as that of Theorem 3.8.1

**Proposition 6.4.3.** One has \(\wedge E_0^{p, q} = 0\) for \(p + q \neq 0\) and there is a linear isomorphism
\[U(\hat{g}^f [t^{-1}] t^{-1}) \otimes (\wedge E_1^{p-1, q})^\top \cong \wedge E_1^{p-1, q} \]
of the form
\[(132) u_{r_1}(-n_1) \cdots u_{r_1}(-n_r) \otimes v \mapsto W_{-n_1}^{(r_1)} \cdots W_{-n_r}^{(r_1)} v\]
with a fixed PBW basis \(\{u_{r_1}(-n_1) \cdots u_{r_1}(-n_r)\}\) of \(U(\hat{g}^f \otimes \mathbb{C}[t^{-1}] t^{-1})\).

Thanks to Proposition 6.4.3 the following assertion follows by induction.

**Proposition 6.4.4.** There exist isomorphisms of chain complexes
\[(\wedge E_r^{p, q}, d_r) \cong (U(\hat{g}^f [t^{-1}] t^{-1}) \otimes (\wedge E_r^{p, q})^\top, 1 \otimes d')\]
of the form (132) with \(v \in (\wedge E_r^{p, q})^\top\) for all \(r \geq 1\). Therefore one has the linear isomorphism
\[\wedge E_r^{p, q} \cong U(\hat{g}^f [t^{-1}] t^{-1}) \otimes (\wedge E_r^{p, q})^\top \]
of the form (132) with \(v \in (\wedge E_r^{p, q})^\top\).

By (128) and Proposition 6.4.1 one has
\[(\wedge E_r^{p, q})^\top = D(E_r^{p+1, q-1}) = 0 \text{ if } p + q \neq 0.\]
By Proposition 6.4.4 this gives \(\wedge E_{\infty}^{p, q} = 0\) if \(p + q \neq 0\), or equivalently,
\[(133) E_{\infty}^{p, q} = 0 \text{ if } p + q \neq 0.\]
This gives that \( H^{\text{BRST}}_n(M_0(\lambda)^*) = 0 \) for all \( n \neq 0 \).

Also, from Proposition 6.4.4 it follows that each \( \vee E_{p,-p} \) is almost highest weight. Therefore \( E_{\infty, p} = \text{gr}_p H^{\text{BRST}}_0(M_0(\lambda)^*) = D(E_{\infty-1,-p+1}) \) is almost co-highest weight with \( (E_{\infty, -p})_{\text{op}} = (E_{\infty, -p})_{\langle \lambda, D \rangle} \) (see Remark 6.1.2). Hence \( H^{\text{BRST}}_0(M_0(\lambda)^*) \) is also co-highest weight.

This completes the proof of (ii) of Theorem 5.5.3. \( \square \)

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