Skewness of the large-scale velocity divergence from non-Gaussian initial conditions

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ABSTRACT
We compute the skewness $t_3$ and the corresponding hierarchical amplitude $T_3$ of the divergence of the velocity field for arbitrary non-Gaussian initial conditions. We find that $T_3$ qualitatively resembles the corresponding hierarchical amplitude for the density field, $S_3'$, in that it contains a term proportional to the initial skewness, which decays inversely as the linear growth factor, plus a constant term which differs from the corresponding Gaussian term by a complex function of the initial three- and four-point functions. We extend the results for $S_3$ and $T_3$ with non-Gaussian initial conditions to evolved fields smoothed with a spherical top-hat window function. We show that certain linear combinations, namely $S_3 + \frac{1}{3}T_3$, $S_3 + T_3$ and $s_3 + t_3$, lead to expressions which are much simpler, for non-Gaussian initial conditions, than $S_3$ and $T_3$ (or $s_3$ and $t_3$) considered separately.

Key words: cosmology: theory – large-scale structure of Universe.

1 INTRODUCTION
The standard cosmogenesis lore attributes the formation of large-scale structure to the enhancement of primordial density fluctuations by gravity. Given that observations on scales larger than $10 h^{-1}$ Mpc show the amplitude of the rms fluctuations to be less than unity, one can successfully apply perturbative techniques to follow the evolution of the initial probability density function (PDF). However, a complete analysis of the problem requires some knowledge of the statistical nature of the initial fluctuations. Simple inflationary scenarios naturally produce initial density fluctuations characterized by a Gaussian PDF, whereas models based on cosmic strings, global texture, or inflation models with multiple scalar fields (Vilenkin 1985; Kofman 1991; Gooding et al. 1992) provide fluctuations which are initially non-Gaussian.

A great deal of analytic work has been done for Gaussian initial conditions. In particular, the hierarchical amplitudes corresponding to the skewness $S_3$ and the kurtosis $S_4$ have been known for some time (Peebles 1980; Fry 1984), and more recently Bernardeau (1992) has provided a formalism for the calculation of the full hierarchy. This work was extended to the skewness of the velocity field by Bernardeau (1994a) and Bernardeau et al. (1995), and some progress has been made toward estimating the evolution of the full PDF (Kofman et al. 1994; Bernardeau & Kofman 1995; Protogeros & Scherrer 1997). However, a firm conclusion has not yet been reached as to the nature of initial conditions that would generate, after evolution, large-scale structures with the same statistical characteristics as the ones obtained from observations such as the QDOT survey or the 1.2-Jy IRAS survey (Coles & Frenk 1991; Bouchet et al. 1993). One reason is that although the statistical description of the evolved fields has been calculated in detail for Gaussian initial conditions, as noted above, the same cannot be said for non-Gaussian fields.

For the case of non-Gaussian initial conditions, a general expression for the evolution of the skewness was derived by Fry & Scherrer (1994), and for the kurtosis by Chodorowski & Bouchet (1996). Here we extend this earlier work by examining the evolution of the divergence of the velocity field ($\theta = \nabla \cdot \mathbf{v}$) for general non-Gaussian initial conditions. Specifically, we calculate the normalized skewness $t_3 = \langle \theta^3 \rangle / \langle \theta^2 \rangle^{3/2}$, as well as the corresponding hierarchical coefficient $T_3 = \langle \theta^3 \rangle / \langle \theta^2 \rangle^2$ for arbitrary non-Gaussian initial conditions. (Throughout the paper, we use $S$ to denote statistical indicators related to the density field, and $T$ to denote the ones related to the divergence of the velocity field.) As noted above, the skewness of $\theta$ for Gaussian initial conditions has already been calculated (Bernardeau 1994a) and Bernardeau et al. (1995), and some progress has been made toward estimating the evolution of the full PDF (Kofman et al. 1994; Bernardeau & Kofman 1995; Protogeros & Scherrer 1997). However, a firm conclusion has not yet been reached as to the nature of initial conditions that would generate, after evolution, large-scale structures with the same statistical characteristics as the ones obtained from observations such as the QDOT survey or the 1.2-Jy IRAS survey (Coles & Frenk 1991; Bouchet et al. 1993). One reason is that although the statistical description of the evolved fields has been calculated in detail for Gaussian initial conditions, as noted above, the same cannot be said for non-Gaussian fields.

For the case of non-Gaussian initial conditions, a general expression for the evolution of the skewness was derived by Fry & Scherrer (1994), and for the kurtosis by Chodorowski & Bouchet (1996). Here we extend this earlier work by examining the evolution of the divergence of the velocity field ($\theta = \nabla \cdot \mathbf{v}$) for general non-Gaussian initial conditions. Specifically, we calculate the normalized skewness $t_3 = \langle \theta^3 \rangle / \langle \theta^2 \rangle^{3/2}$, as well as the corresponding hierarchical coefficient $T_3 = \langle \theta^3 \rangle / \langle \theta^2 \rangle^2$ for arbitrary non-Gaussian initial conditions. (Throughout the paper, we use $S$ to denote statistical indicators related to the density field, and $T$ to denote the ones related to the divergence of the velocity field.) As noted above, the skewness of $\theta$ for Gaussian initial conditions has already been calculated (Bernardeau
The calculation of $T_3$ and $S_3$ is presented in the next section. As in the case of the skewness of the evolved density field, we find three contributions to $T_3$: (i) the initial, linearly evolved skewness, which decays as the inverse of the linear perturbation growth rate, (ii) a 'Gaussian' contribution, identical to the value of $T_3$ for Gaussian initial conditions, and constant in time, and (iii) a term, also constant in time, which is a complex function of the initial three- and four-point correlations. The values of $T_3$ and $S_3$ for the smoothed final density field are calculated in Section 3. We discuss our results in Section 4, and show that certain linear combinations of $T_3$ and $S_3$ give simpler expressions than each considered separately.

2 CALCULATION OF $T_3$

In what follows we adopt Peebles' (1980) notation in a line of argument paralleling that of Fry & Scherrer (1994). To simplify our derivation, we define

$$\delta = \delta^{(1)} + \delta^{(2)} + \ldots$$

$$v = v^{(1)} + v^{(2)} + \ldots$$

$$\theta = \theta^{(1)} + \theta^{(2)} + \ldots,$$

so that the skewness of $\theta$ is given by

$$\zeta_3(0) = \langle \theta^3 \rangle = \langle \delta^{(1)} \theta \rangle + 3 \langle \delta^{(1)} \delta^{(1)} \delta^{(2)} \rangle + \ldots$$

The first- and second-order solutions for $\delta$ are the well-known linear result

$$\delta^{(1)} = D_1(t) \delta(x, t_0),$$

where $D_1(t)$ is the usual growing-mode solution, and (Peebles 1980)

$$\delta^{(2)} = \frac{5}{7} \delta^{(0)} - \delta^{(0)} \Delta_{0,0} + \frac{2}{7} \Delta_{0,0} \Delta_{0,0},$$

where we have defined $\delta_0 = \delta^{(0)}$, and $\Delta_0$ is defined to be

$$\Delta_0 = \frac{1}{4\pi} \int dx' \delta_0(x') \frac{1}{|x'-x'|}.$$  \hspace{1cm} (5)

To go from these expressions to the values for $\delta^{(1)} = \nabla \cdot v^{(1)} / a$ and $\delta^{(2)} = \nabla \cdot v^{(2)} / a$, we substitute the expansions for $\delta$ and $v$ into the continuity equation,

$$\frac{\partial \delta}{\partial t} + \nabla \cdot (\delta \nabla v) = 0,$$

leading to the linear and second-order equations

$$\delta^{(1)} + \frac{1}{a} \nabla \cdot v^{(1)} = 0,$$

$$\delta^{(2)} + \frac{1}{a} \nabla \cdot v^{(2)} + \frac{1}{a} \delta^{(1)} \nabla \cdot v^{(1)} + \frac{1}{a} v^{(1)} \cdot \nabla \delta^{(1)} = 0.$$  \hspace{1cm} (8)

The first-order equation gives $v^{(1)} = aD_1 \Delta_{0,0}$ and $\nabla \cdot v^{(1)} = -aD_1 \delta_0(x)$, so that

$$\delta^{(1)} = -D_1 \delta_0(x),$$

while the second-order equation gives

$$\delta^{(2)} = -D_1 ^2 \left[ \frac{3}{7} \delta^{(0)} - \delta^{(0)} \Delta_{0,0} + \frac{4}{7} \Delta_{0,0} \Delta_{0,0} \right].$$

From equations (9) and (10), we obtain

$$\langle \delta^{(1)} \theta \rangle = -D_1 \langle \delta^{(0)} \rangle,$$

and

$$\langle \delta^{(2)} \theta \rangle = -D_1 ^2 \left[ \frac{3}{7} \langle \delta^{(0)} \rangle - \delta^{(0)} \Delta_{0,0} \right].$$

We now simplify the various terms in equation (12) using the results of Peebles (1980) and Fry & Scherrer (1994).

The second-, third- and fourth-order moments of the density field can be expressed in terms of the irreducible two-, three- and four-point correlation functions $\xi_{12}, \xi_{13}$ and $\eta_{1234}$:

$$\langle \delta^{(1)} \delta^{(1)} \delta^{(1)} \rangle = 3 \xi_{12} \xi_{13} \xi_{14},$$

where all the moments are functions of $|x-x'|$ due to the assumed homogeneity and isotropy of the density field distribution. Therefore,

$$\langle \delta^{(2)} \delta^{(2)} \delta^{(2)} \rangle = 3 \xi_{12} \xi_{13} \xi_{14} + \eta_{1234},$$

and

$$\langle \delta^{(3)} \delta^{(3)} \delta^{(3)} \delta^{(3)} \rangle = \frac{1}{4\pi} \int d^3 x_1 \frac{1}{|x_1-x'|} \xi_{00}(x_1) \xi_{00}(x_1-x'),$$

$$\langle \delta^{(3)} \delta^{(3)} \delta^{(3)} \delta^{(3)} \rangle = \frac{1}{3} \eta_{00}(x, x, x, x').$$  \hspace{1cm} (15)

where the zero subscripts indicate linearly evolved quantities. The first term in equation (15) yields $\xi_{12}^3(0)$, whereas the second term yields $\frac{1}{3} \eta_{00}(0)$ upon integration by parts, using the fact that $\eta(x, x, x, x) = \eta_0(0)$. Thus,

$$\langle \delta^{(2)} \delta^{(2)} \Delta_{0,0} \Delta_{0,0} \rangle = \xi_{12}^3(0) + \frac{1}{3} \eta_{00}(0).$$

Finally, we evaluate the last term in equation (12):

$$\langle \delta^{(2)} \Delta_{0,0} \Delta_{0,0} \rangle = \frac{1}{(4\pi)^2} \int d^3 x' \int d^3 x''$$

$$\frac{1}{|x-x'|} \frac{1}{|x-x''|} \langle \delta^{(2)}(x) \delta_0(x') \delta_0(x'') \rangle.$$  \hspace{1cm} (17)

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The fourth moment on the right-hand side can be expressed as
\[
\langle \delta^2_0(x) \delta_0(x') \delta_0(x'') \rangle = \xi_0(0) \xi_0(x-x') + 2 \xi_0(x-x') \xi_0(x-x'').
\tag{18}
\]

The first two terms in the double integration in equation (17) yield \(\xi_0(0)\) and \(2 \xi_0(0)\) respectively, for a total contribution of \(3 \xi_0(0)\). The last term yields \(1\langle \delta^2_0 \rangle + \eta_0(0)\), where we define \((Fry & Scherrer 1994)\)
\[
I[\eta_0] = \frac{1}{(4\pi)^2} \int d^3x' d^3x'' \eta_0(0, 0, x', x'') \frac{6P_2(\hat{x} \cdot \hat{x}')}{x'^3 x''^3}.
\tag{19}
\]
with \(P_2(\hat{x} \cdot \hat{x}')\) being the second Legendre polynomial. Therefore,
\[
\langle \delta^2_0 \Delta_{\delta,0} \Delta_{\delta,0} \rangle = \frac{5}{3} \xi_0(0) + I[\eta_0] + \frac{4}{7} \eta_0(0).
\tag{20}
\]

Combining the results of equations (14), (16) and (20), equation (12) becomes
\[
\langle \{\hat{\theta}^{(2)}\} \rangle = -D_1^2 D_t \left[ \frac{\xi_0(0) + \frac{26}{7} \eta_0(0) + \frac{12}{7} I[\eta_0]}{D_1} \right].
\tag{21}
\]

Substituting equations (11) and (21) into equation (2), we obtain our final expression for \(\xi_0(0)\) :
\[
\xi_0(0) = -D_1^2 D_t \left[ \frac{\xi_0(0) + \frac{26}{7} \eta_0(0) + \frac{12}{7} I[\eta_0]}{D_1} \right].
\tag{22}
\]

To calculate the hierarchical amplitude \(T_3\) or the normalized skewness \(t_3\), we must also calculate \(\theta(\hat{\theta}^2)\), which can be derived in a calculation similar to that for \(\xi_0\). We have
\[
\xi_0 = \langle \hat{\theta}^2 \rangle = \langle [\hat{\theta}^{(1)}]^2 \rangle + 2 \langle [\hat{\theta}^{(1)}] \hat{\theta}^{(2)} \rangle + \ldots
\tag{23}
\]
with
\[
\langle [\hat{\theta}^{(1)}]^2 \rangle = D_1^2 D_t \xi_0(0)
\tag{24}
\]
and
\[
\langle [\hat{\theta}^{(1)}] \hat{\theta}^{(2)} \rangle = D_1^2 D_t \left[ \frac{3}{7} \delta_0^2 - \delta_0 \delta_0 \Delta_{\delta,0} + \frac{4}{7} \delta_0 \Delta_{\delta,0} \right].
\tag{25}
\]

In a manner similar to our previous derivations, we obtain
\[
\langle \delta_0^3 \rangle = \xi_0(0)
\tag{26}
\]
\[
\langle \delta_0 \Delta_{\delta,0} \Delta_{\delta,0} \rangle = \frac{1}{2} \xi_0(0)
\tag{27}
\]
\[
\langle \delta_0 \Delta_{\delta,0} \Delta_{\delta,0} \rangle = I[\xi_0] + \frac{1}{3} \xi_0(0).
\tag{28}
\]

where \(I[\xi_0]\) is the integral analogous to equation (19), integrated over \(\xi_0(0, x', x'')\), rather than \(\eta_0(0, 0, x', x'')\). Therefore,
\[
\langle [\hat{\theta}^{(1)}] \hat{\theta}^{(2)} \rangle = \frac{5}{2} \xi_0(0) + \frac{8}{7} I[\xi_0]
\tag{29}
\]
\[
= \frac{5}{2} \xi_0(0) + \frac{8}{7} \xi_0(0) + \frac{12}{7} \eta_0(0) I[\eta_0].
\tag{30}
\]

Combining equations (22) and (29), with
\[
\theta = \frac{1}{H}
\]
and defining
\[
f(\Omega) = \frac{1}{H} \frac{D_1}{D_t} \Omega^{6/5},
\]
we obtain our expressions \(T_3\) and \(t_3\) for arbitrary non-Gaussian initial conditions. For the hierarchical amplitude \(T_3\) we obtain
\[
T_3 = -\frac{26}{7} \Omega^{-0.6}.
\tag{31}
\]

For comparison, the hierarchical coefficient for the density field is (Fry & Scherrer 1994)
\[
S_3 = \frac{1}{D_1} \xi_0(0) + \frac{34}{7} \eta_0(0) + \frac{6}{7} I[\eta_0] + \frac{26}{21} \xi_0(0) I[\xi_0] + \frac{8}{7} \xi_0(0) I[\xi_0].
\tag{32}
\]

We see that \(T_3\), like \(S_3\), contains three distinct contributions: a term proportional to the initial skewness which decays away as \(1/D_1(t)\), a ‘Gaussian’ piece which is constant and identical to the hierarchical amplitude in the Gaussian case, and a third contribution, also constant in time, which is a complex function of the initial skewness, the initial kurtosis and various integrals over the initial three- and four-point functions. In fact, a comparison of equations (32) and (30) indicates that the terms in the two equations are identical functions of the initial density field; only the coefficients multiplying the various terms are different. We will exploit this fact in Section 4.

For the normalized skewness \(t_3\) we obtain
\[
t_3 = -s_3,0
\tag{33}
\]
\[
= \frac{26}{7} \eta_0(0) + \frac{12}{7} I[\eta_0] + \frac{5}{7} \xi_0(0) + \frac{12}{7} \eta_0(0) I[\eta_0] + \frac{14}{7} \xi_0(0) I[\xi_0] \times o_0(t).
\tag{34}
\]

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and
\[
\xi_0(0) = \langle \hat{\theta}^2 \rangle = D_1^2 D_t \left[ \frac{5}{21} \xi_0(0) + \frac{8}{7} I[\xi_0] \right].
\tag{29}
\]
which can be compared with the corresponding normalized skewness for the density (Fry & Scherrer 1994)

\[
\sigma_3 = \sigma_{3,0} + \frac{3}{7} \frac{\xi(0)}{\sigma_0} + \frac{10}{7} \frac{\xi_2(0)}{\sigma_0} - \frac{3}{7} \frac{\xi_1(0)}{\sigma_0} 
\]

In these expressions, \( \sigma_0(t) \) is the linearly evolved rms fluctuation: \( \sigma_0(t) = \sigma_0(0) \), and \( \sigma_{3,0} \) is the (constant) linearly evolved skewness: \( \sigma_{3,0} = \sigma_0(0)/\sigma_0^2(t) \). Note that \( t_3 \), unlike \( T_3 \), is independent of \( f(\Omega) \).

### 3 SMOOTHED \( T_3 \) AND \( S_3 \) RESULTS

Although the results derived in the previous section are interesting from a formal point of view, an application to the observations requires a calculation of \( T_3 \) for a field which has been smoothed with a window function. Both Fry & Scherrer (1994) and Chodorowski & Bouchet (1996) argued that the smoothed skewness and kurtosis for non-Gaussian initial conditions should qualitatively resemble the unsmoothed results. The effects of smoothing on the hierarchical amplitudes for Gaussian initial conditions have been calculated in detail for the case of spherical top-hat smoothing (Bernardeau 1994a,b), so we will follow Bernardeau’s treatment to derive an expression for the value of \( T_3 \) with non-Gaussian initial conditions and spherical top-hat smoothing. In addition, we extend the results of Fry & Scherrer (1994) by performing the same calculation to derive the smoothed value of \( S_3 \).

Consider a spherical top-hat window function with radius \( R_0 \) and Fourier transform

\[
W(kR_0) = \frac{2}{(kR_0)^3} \left[ \sin(kR_0) - kR_0 \cos(kR_0) \right].
\]

After the density and velocity-divergence fields have been smoothed with this window function, we obtain new expressions for \( S_3 \) and \( T_3 \), which we denote by \( S_3(R_0) \) and \( T_3(R_0) \). We now use the methods developed in Bernardeau (1994a) to calculate these quantities. The expressions for the smoothed \( \delta \) up to second order are (Bernardeau 1994a)

\[
\theta^{(1)}(R_0) = -f(\Omega)D_1 \int \frac{d^3k}{(2\pi)^3} \delta_k W(kR_0),
\]

\[
\theta^{(2)}(R_0) = -f(\Omega) \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \delta_{k_1} \delta_{k_2} W(|k_1 + k_2|R_0)
\times \left[ D_2 \left( P_{12} - \frac{3}{2} Q_{12} \right) + \frac{3}{4} E_{2} Q_{12} \right],
\]

where \( \delta_k \) is the Fourier transform of the initial density field, the quantities \( P_{ij}, Q_{ij} \), and \( \delta \) are defined by

\[
P_{ij} = 1 - \frac{k_i \cdot k_j}{k_j^2},
\]

\[
\delta = \frac{1}{k^2},
\]

\[
Q_{ij} = 1 - \frac{(k_i \cdot k_j)^2}{k_i^2 k_j^2},
\]

and \( D_i \) gives the time dependence of the growing mode, being the solution of the \( i \)th order time evolution equation for \( \delta \), while

\[
E_2 = D_1 \frac{d}{dD_1} (D_2 - D_1^2).
\]

At the limit of \( t \to 0 \), \( D_2 \) satisfies the relation

\[
D_2(t) \approx \frac{34}{21} D_1^2(t).
\]

Furthermore, the two-, three- and four-point functions of the random field \( \delta_k \) are expressed as

\[
\langle \delta_k \delta_k \rangle = \xi_{kk},
\]

\[
\langle \delta_k \delta_k \delta_k \rangle = \xi_{kkk},
\]

\[
\langle \delta_k \delta_k \delta_k \delta_k \rangle = \xi_{kkk} + \xi_{kk} \xi_{kk} + \xi_{kk} \xi_{kk} + \eta_{kkkk},
\]

where the two-point function is related to the power spectrum \( P(k) \) of the \( \xi_{kk} \) field as \( \xi_{kk} = \delta_k(k + \kappa, k')P(k) \). Using these relations and given that \( \langle \theta^{(2)} \rangle = \langle \theta^{(1)} \theta^{(1)} \rangle + 2 \langle \theta^{(1)} \theta^{(2)} \rangle + \ldots \), we obtain

\[
\langle \theta^2(R_0) \rangle = f(\Omega)^2 D_1^2 \sigma^2(R_0) + 2f(\Omega)^2 D_1^2 I_1[\xi_{kk}],
\]

where we have used the definitions

\[
\sigma^2(R_0) = \int \frac{d^3k}{(2\pi)^3} W(kR_0)^2 P(k),
\]

and

\[
I_1[\xi_{kk}] = \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{d^3k_3}{(2\pi)^3} W_1 W_2 W_3 \xi_{kkk},
\]

\[
\times \left[ \left( P_{32} - \frac{3}{2} Q_{32} \right) + \frac{3}{4} E_{2} Q_{32} \right],
\]

where \( W_k = W(kR_0) \), and \( W_0 = W(|k_1 + k_2|R_0) \). In a similar fashion, we obtain

\[
\langle \theta^{(1)} \delta \rangle = -f(\Omega) D_1^2 I_2[\xi_{kk}],
\]

where we define

\[
I_2[\xi_{kk}] = \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{d^3k_3}{(2\pi)^3} W_1 W_2 W_3 \xi_{kkk}.
\]

Finally, as shown in Appendix A, we obtain

\[
\langle \theta^{(1)} \theta^{(2)} \rangle = -f(\Omega)^2 D_1^2 \sigma^2(R_0) \frac{26}{21} + \gamma f(\Omega)^2 D_1^2 I_1[\xi_{kk}],
\]

where we define

\[
I_1[\xi_{kk}] = \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{d^3k_3}{(2\pi)^3} \frac{d^3k_4}{(2\pi)^3} W_1 W_2 W_3 \xi_{kkk}.
\]
\begin{equation}
\times \left[ P_{s4} - \frac{3}{2} Q_{s4} \right] + \frac{3}{4} E_{q} D_{q} Q_{s4}
\end{equation}

Since \( \langle \theta^3 \rangle = \langle \theta(1)^3 \rangle + 3 \langle \theta^2 \rangle^2 \theta(1) + \ldots \), we obtain

\begin{equation}
\langle \theta^3 (R_0) \rangle = -f(\Omega)^3 D_{q}^2 I_3 [s_{43}] - f(\Omega)^3 D_{q}^4 \sigma^4 (R_0)
\end{equation}

\begin{equation}
\times \left( \frac{26}{7} + \gamma \right) - \frac{3}{f(\Omega)^3} D_{q} I_\eta [s_{43}],
\end{equation}

where \( \gamma \) is defined as

\begin{equation}
\frac{d \log [\sigma^2 (R_0)]}{\gamma} = \frac{d \log (R_0)}{\gamma}.
\end{equation}

Using these results, it is easy to show then that

\begin{equation}
T_3 (R_0) = -\frac{1}{f(\Omega)} \left( \frac{26}{7} + \gamma \right)
\end{equation}

\begin{equation}
= -\frac{1}{\Omega^6} \left( \frac{26}{7} - (n + 3) \right),
\end{equation}

in agreement with Bernard (1994a), where the last relation holds for a power spectrum \( P(k) \propto k^n \).

Using the same line of argument, it is easy to show that for the smoothed density field, the mean values of \( \delta^2 \) and \( \delta^3 \) are

\begin{equation}
\langle \delta^3 (R_0) \rangle = D_{\Omega}^2 \sigma^2 (R_0) + 2D_{\Omega}^3 \{ I_3 [s_{43}] + K [s_{43}] \},
\end{equation}

and

\begin{equation}
\langle \delta^5 (R_0) \rangle = D_{\Omega}^2 I_3 [s_{43}] + D_{\Omega}^4 \sigma^4 (R_0) \frac{34}{7} + \gamma
\end{equation}

\begin{equation}
+ 3D_{\Omega}^4 \{ I_\eta [s_{43}] + K [s_{43}] \},
\end{equation}

Then the skewness of the density field smoothed with a spherical top-hat window function for non-Gaussian initial conditions is

\begin{equation}
S_3 (R_0) = \frac{I_3 [s_{43}]}{D_{\Omega} \sigma^4 (R_0)}
\end{equation}

\begin{equation}
+ \left( \frac{34}{7} + \gamma \right) + \frac{3}{\sigma^4 (R_0)} \{ I_\eta [s_{43}] + K [s_{43}] \}
\end{equation}

\begin{equation}
- \frac{4}{\sigma^4 (R_0)} \{ I_3 [s_{43}] + I_3 [s_{43}] K [s_{43}] \},
\end{equation}

where we use the definitions

\begin{equation}
S_3 (R_0) = \frac{34}{7} + \gamma,
\end{equation}

\begin{equation}
= \frac{34}{7} - (n + 3),
\end{equation}

for a power spectrum \( P(k) \propto k^n \), consistent with Bernard (1994a).

4 DISCUSSION

A calculation of this sort can be used in two different ways: to gain further insight into the process of gravitational clustering at a fundamental level, and to compare with observations to try to determine if the initial perturbations were non-Gaussian. For the latter purpose, only our smoothed results are useful, although the unsmoothed results may be of interest for the former.

As noted earlier, the hierarchical amplitude for the velocity divergence \( T_3 \) resembles qualitatively the corresponding expression for the density field, \( S_3 \). We end up with one term, proportional to the initial skewness, which decays as \( 1/D_{\Omega} \), while the term which is constant in time can be broken down into a ‘Gaussian’ piece, equal to the contribution for Gaussian initial conditions, and a ‘non-Gaussian’ piece, which depends in a complex manner on the three- and four-point functions of the initial density field.

A problem with applying the results of either Section 2 or Section 3 is their complexity in comparison with their Gaussian counterparts. Even the unsmoothed results are non-local, depending on integrals over the initial distribution functions. However, it is possible to simplify these results somewhat by examining combinations of \( S_3 \) and \( T_3 \).

Consider first the unsmoothed case for \( \Omega = 1 \). In this case, the non-local terms arise from the last term in equation (12). These can be eliminated by evaluating \( S_3 + 1/2 T_3 \), for which we obtain

\begin{equation}
S_3 + \frac{1}{2} T_3 = \frac{1}{2D_{\Omega} \sigma^2 (R_0)} + 3 + \frac{\eta_2 (0)}{\pm (0)} - \frac{1}{\pm (0)}.
\end{equation}

This expression is a function only of the initial skewness and kurtosis of the non-Gaussian density field.

A more useful combination, from the point of view of the observations, can be derived from the smoothed hierarchical amplitudes given in the previous section. For the case \( \Omega = 1 \), if we simply take the sum of \( S_3 (R_0) \) and \( T_3 (R_0) \), we obtain

\begin{equation}
S_3 (R_0) + T_3 (R_0) = \frac{8}{7} + \frac{3K [s_{43}]}{\sigma^4 (R_0)} - \frac{4I_3 [s_{43}] K [s_{43}]}{\sigma^4 (R_0)}.
\end{equation}

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This quantity has several interesting properties. For the case of Gaussian initial conditions, it reduces to

\[ S^2_R(R_0) + T^2_R(R_0) = \frac{8}{7}, \]  

(59)

which is independent of the initial power spectrum. This result does not extend to higher-order amplitudes; e.g., \( S_4 - T_4 \) does depend on the initial power spectrum. For non-Gaussian initial conditions, the time-dependent term produced by the initial skewness has vanished, so equation (58) gives a much clearer estimate of the deviation from Gaussian hierarchical clustering for non-Gaussian initial conditions; any deviation from \( 8/7 \) indicates the presence of non-Gaussian initial conditions or \( \Omega \neq 1 \).

One of the main reasons for investigating the behaviour of \( T_3(R_0) \) is its sensitivity to different values of \( \Omega \) (Bernardeau et al. 1995, 1997). Unfortunately, this works to our disadvantage in equation (58): it is not possible to disentangle the effects of \( \Omega \neq 1 \) from the effects of non-Gaussian initial conditions. A more useful quantity if one is interested in the statistics of the initial conditions is \( s_3(R_0) + t_3(R_0) \), since \( t_3 \) is independent of \( f(\Omega) \). A straightforward calculation similar to our derivation of \( S_3(R_0) + T_3(R_0) \) leads to

\[ s_3(R_0) + t_3(R_0) = \left[ \frac{8}{7} + \frac{3K_{[3,12]} \sigma^4(R_0)}{\sigma^4(R_0)} \right] D_1 \sigma^4(R_0), \]  

(60)

a result which holds for any value of \( \Omega \). Thus, equation (60) provides a clear distinction between the evolution of Gaussian and non-Gaussian initial conditions.

The application of hierarchical amplitudes to the case of non-Gaussian initial conditions continues to be difficult, due primarily to the much greater complexity of the results. However, our calculations provide some simpler expressions which represent a step in the right direction.

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APPENDIX A

In Section 2 we used the integrals

\[ \frac{1}{(4\pi)^2} \int d^3x \, d^3y \, \xi(x' - x^0) \frac{6P_2(k^0 \cdot \hat{k}^0)}{x^0 x^y} \xi(0), \]  

(A1)

\[ \frac{1}{(4\pi)^2} \int d^3x \, d^3y \, \xi(x' - x^0) \left( \frac{1}{|x - x^0|} \right)_a \left( \frac{1}{|x - x^y|} \right)_a = \xi(0), \]  

(A2)

which are easily derived using an integration by parts and the relation

\[ \nabla \nabla \cdot \frac{3\xi \cdot \delta_j - \delta_{ij}}{x^i} \frac{4\pi}{3} \delta_{ij}(x). \]  

(A3)

In Section 3 we used the results given by Bernardeau's equations (A26) and (A27) (Bernardeau 1994a), specifically,

\[ \int \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} P_{12} W_{1} W_{2} P(k_1) P(k_2) = \sigma^4(R_0) \left[ 1 + \frac{1}{6} \frac{R_0}{\sigma^2(R_0)} \frac{d\sigma^2(R_0)}{dR_0} \right], \]  

(A4)

\[ \int \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} Q_{12} W_{1} W_{2} P(k_1) P(k_2) = \frac{2}{3} \sigma^4(R_0), \]  

(A5)

and the fact that in the equation for \( \langle \delta^{(1)} \bar{\delta}^{(2)} \rangle \), integrals containing the factors \( P_{ij}(\delta(k_1) \delta(k_2)) \) and \( Q_{ij}(\delta(k_1) \delta(k_2)) \) vanish.