Sub critical transition to turbulence in three-dimensional Kolmogorov flow

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Abstract

We study Kolmogorov flow on a three dimensional, periodic domain with aspect ratios fixed to unity. Using an energy method, we give a concise proof of the linear stability of the laminar flow profile. Since turbulent motion is observed for high enough Reynolds numbers, we expect the domain of attraction of the laminar flow to be bounded by the stable manifolds of simple invariant solutions. We show one such edge state to be an equilibrium with a spatial structure reminiscent of that found in plane Couette flow, with streamwise rolls on the largest spatial scales. When tracking the edge state, we find two branches of solutions that join in a saddle node bifurcation at a finite Reynolds number.

Keywords: Kolmogorov flow, box turbulence, sub critical transition, edge states

(Some figures may appear in colour only in the online journal)

1. Introduction

In his famous 1883 paper, Reynolds investigated the transition from ‘direct’ to ‘sinuous’ motion of water forced through a straight smooth tube (Reynolds 1883). His systematic study led to the identification of a dimensionless number, now called the Reynolds number, that essentially measures the ratio of forcing to viscous dissipation. He attempted to use this ratio to pinpoint the transition—in current terminology—from laminar to turbulent flow. His choice of studying pipe flow was a practical one. Having stated that, given their intractability,
integration of the equations of motion was ‘not promising’, he decided to study the transition experimentally in a setup that is relatively easy to realize in a laboratory. Somewhat ironically, in choosing ‘simplest possible circumstances’, he selected a transition with a highly complicated mathematical structure. To the best of our current knowledge, the laminar flow in this geometry, known as Hagen-Poiseuille flow, remains linearly stable for all flow rates. It can be unstable, however, to perturbations with finite amplitude. This type of transition, which occurs in the absence of a bifurcation of the laminar flow, is now referred to as subcritical. Although Reynolds apparently expected a linear instability to occur, having attributed to Stokes the observation that this is the general cause for the onset of sinuous motion, he carefully observed and reported the benchmarks of subcritical transition: the sinuous motion appears suddenly and with a large amplitude, and the flow rate at which it is first observed is very sensitive to the conditions of the inflowing water.

A few decades later, Kolmogorov also considered the transition problem. In a series of seminars in 1958 and 1959, he proposed to study the problem in a setting more susceptible to analysis. The seminar series was reported on by Arnol’d and Meshalkin (1960). Conditions at material boundaries, highlighted by Reynolds as particularly hard to treat analytically, are replaced by periodic boundary conditions and instead of three spatial dimensions, the problem is posed in two. A sinusoidal body force is applied on the largest spatial scale to input energy, and this force acts in one direction only.

In contrast to Reynolds, Kolmogorov explicitly mentions the possibility of a subcritical transition, noting that experimental work indicates that, with decreasing viscosity, the laminar flows usually become ‘either unstable or so weakly stable, that they are not observed in reality.’ He probably knew both situations were to occur in his model, as results on the linear stability of the laminar flow by Meshalkin and Sinai (1961) are reported in the same paper, albeit without reference. The result is different for two classes of systems: one in which the domain is elongated in the direction in which the force acts, and one in which it is elongated in the perpendicular direction. In the former case a bifurcation always takes place, i.e. it is supercritical, while in the latter the laminar flow remains linearly stable, meaning it is sub-critical. Kolmogorov hypothesized that in either case, a turbulent solution would exist with small enough viscosity.

A few years later, Iudovich (1965) proved a strong result on the subcritical case. He showed, that in this case the laminar flow is the only possible limit state, thereby disproving Kolmogorov’s hypothesis. Importantly, the proof includes the square domain. Subsequent studies of Kolmogorov flow have taken either of two paths: they consider a rectangular domain in the supercritical case, or generalize the model on a square domain, usually by increasing the wave number of the forcing. Work in the former direction has shown, that the primary bifurcation is a symmetry-breaking steady state bifurcation, while the secondary bifurcation may be of the Hopf type. Also, it was shown that the locus of the primary bifurcation approaches an infinite Reynolds number as the aspect ratio approaches unity (Okamoto and Shōji 1993). Work on the Kolmogorov flow with higher wave number forcing, meanwhile, has led to a detailed description of the possible transitions to periodic and, subsequently, turbulent flows (Chen and Price 2005).

A different, and very natural, generalization of Kolmogorov’s model is the inclusion of the third spatial dimension. Possibly the first attempt to study the three-dimensional case, with both aspect ratios equal to unity, was made by Shebalin and Woodruff (1997). They discuss the linear stability of the laminar flow in some detail. Citing, as Reynolds did more than a century before them, the intractability of this problem, they turn to a conceptual model of ordinary differential equations to demonstrate their firm expectation that, for decreasing viscosity, an increasing number of eigenmodes will become unstable, leading to ‘essentially
the transition to turbulence described by Landau and Lifschitz. In fact, Squire’s theorem for parallel shear flows dictates that the addition of the third spatial dimension cannot render the laminar flow unstable. On the other hand, their numerical work yields convincing evidence that turbulent solutions do exist. This evidence is corroborated by Borue and Orszag (1996), who used a much higher spatial resolution, but added hyper viscosity to aid the numerics.

The inclusion of the third, ‘spanwise’ direction thus places ‘Kolmogorov’ model in the same family of elementary shear flows as pipe flow. This family also contains planar Couette and Poiseuille flow and the asymptotic suction boundary layer. What these flows have in common is that for a range of Reynolds numbers, the asymptotically stable laminar flow coexists with a turbulent state that is stable at least on a time scale much longer than the large eddy turnover time. Each of these stable, or metastable, states will have its own basin of attraction. It is possible then for these basins to intersect along a common boundary. This boundary can be called the ‘edge of turbulence’ since for each initial state, we can predict whether or not it will spawn turbulence based on its position with respect to the boundary in phase space. Since the boundary is a high dimensional object, embedded in a high dimensional space, it is generally impossible to study directly. However, it may at least locally be formed by the stable manifold of a simple invariant solution such as an equilibrium or travelling wave. Such an invariant state is then called an ‘edge state’, and computing it numerically enables us to explore aspects of the transition process. For planar Couette flow, for instance, the computation of various edge states has led to the identification of control strategies (Kawahara 2005), an explanation for the observed distribution of life times of turbulent motion (Kreilos and Eckhardt 2012), and insight in the spatial structure of minimal energy perturbations of laminar flow that induce turbulence (Cherubini and De Palma 2015).

In this paper, we commence the study of the subcritical transition process under the ‘simplest possible circumstances’, that being flow on a triply periodic domain with aspect ratios equal to unity and forcing with the smallest wave number in one direction only, through the computation of simple invariant solutions. In doing so, we hope to strike a compromise between the rigorous proofs that Kolmogorov had in mind, and ‘calculations using computers, which do not completely satisfy a mathematician.’ (Arnol’d and Meshalkin 1960).

2. 3D Kolmogorov flow

The governing equations for viscous, incompressible Kolmogorov flow are

\[
\begin{align*}
\frac{du}{dt} + u \cdot \nabla u + P_x &= - \nu \Delta u = \gamma \sin(k_f y) \\
\frac{dv}{dt} + u \cdot \nabla v + P_y &= - \nu \Delta v = 0 \\
\frac{dw}{dt} + u \cdot \nabla w + P_z &= - \nu \Delta w = 0 \\
\nabla \cdot u &= 0
\end{align*}
\]

where \( \mathbf{u} = (u, v, w) \) is the velocity of a fluid with its constant density fixed to unity, \( P \) is the pressure (divided by the constant density), \( \nu \) the kinematic viscosity and \( \gamma \) the amplitude of the forcing. We will consider this flow on a cube with linear dimension \( L = 2\pi \). Note, that for quasi-two dimensional solutions with \( w \equiv 0 \) and forcing wave number \( k_f = 1 \), the system is identical to the one posed originally by Kolmogorov (Arnol’d and Meshalkin 1960). In line with the original model, we also impose that the spatial mean of the momentum is zero.

The boundary conditions are periodic in all directions, but nevertheless we will refer to the \( x \) and \( z \) directions as streamwise and spanwise, respectively, in analogy to planar Couette and Poiseuille flow. We will refer to the \( y \) direction as shearwise.
Mean quantities of special interest are the energy and enstrophy, defined in terms of the velocity and vorticity, $\omega = \nabla \times u$, as

$$E = \frac{1}{L^3} \int \frac{1}{2} |u^2| \, dx$$

$$Q = \frac{1}{L^3} \int \frac{1}{2} |\omega^2| \, dx$$

as well as the energy input and dissipation rates, given by

$$I = \frac{\gamma}{L^3} \int u \sin(k_{i}y) \, dx$$

$$D = 2\nu Q$$

where all integrals are over the entire periodic domain.

The laminar flow, an exact solution to equations (1)–(2), is given by

$$u^* = \frac{\gamma}{k_{i}^2 \nu} \sin(k_{i}y), \; v^* = w^* = 0, \; P^* = \text{constant}$$

and has $E^* = \gamma^2/(4k_{i}^4 \nu^2)$, $Q^* = \gamma^2/(4k_{i}^2 \nu^2)$ and $I^* = D^* = \gamma^2/(2k_{i}^2 \nu)$.

### 2.1. Reynolds numbers

Several Reynolds numbers can be defined for this system. One often used for box turbulence is based on Taylor’s micro scale:

$$Re_{\lambda} = \sqrt{\frac{\frac{10}{3} E}{\nu \sqrt{Q}}}$$

For the laminar solution, and solutions near it such as described in section 4, this Reynolds number scales as $\gamma/\nu^2$, and takes values of the order $10^3$–$10^4$ for the equilibria presented here. The Taylor micro scale is supposed to be in the inertial range of the energy spectrum. As will become clear in section 3, the near-laminar solutions do not have any inertial range in their spectrum, and therefore the Taylor micro scale may not be a useful length scale to base the Reynolds number on.

A second Reynolds number is based on the energy input rate and was used by Linkmann and Morozov (2015) in their study of box turbulence with a constant energy input rate:

$$Re_{\text{Li}} = \frac{L^{4/3} \gamma^{1/3}}{\nu}$$

This Reynolds number scales as $(\gamma/\nu^2)^{2/3}$ and is of order $10^3$ for the solutions discussed here.

A geometric Reynolds number can be defined using the box length and the amplitude of $u^*$ as velocity scale:

$$Re_{g} = \frac{L_{\gamma}}{k_{i} \nu^2}$$

For nearly laminar flow, the scaling with $\gamma$ and $\nu$ is the same as that of $Re_{\lambda}$. This definition was used, for instance, by Platt et al (1991) and Okamoto and Shôji (1993).

The Reynolds number we will use is based on the length scale $1/k_{i}$ and the velocity scale $\sqrt{L\gamma}$ and is given by
In adopting this definition, we follow the direct numerical simulations of Shebalin and Woodruff (1997), as well as dynamical systems-based work by Chandler and Kerswell (2013), which is close in spirit to our current study.

2.2. Stability of the laminar flow

Laminar flow profile (7) is a parallel shear flow to which Squire’s theorem applies. This theorem says that the first linear instability of the laminar flow, observed when increasing the geometric Reynolds number, must be independent of the spanwise coordinate, i.e. it must be quasi-two dimensional. Combined with the results of Iudovich (1965), this result tells us that, for forcing wave number \( k_f = 1 \), there is no linear instability of the laminar flow.

Since a direct proof is more insightful, we include one based on the energy method, closely following Waleffe (2011 chapters 3 and 4). The starting point is the linearization of equations (1)–(2) about the laminar flow

\[
\begin{align*}
\mathbf{u}' + u^*\mathbf{u}' + D\mathbf{u}^* \mathbf{v}'_e + \nabla P' - \nu \Delta \mathbf{u}' &= 0 \\
\nabla \cdot \mathbf{u}' &= 0
\end{align*}
\]

where \( \mathbf{u} = u^* + \mathbf{u}', P = P^* + P', \mathbf{e}_s \) is the unit direction vector in the streamwise direction and \( D \) denotes the derivative in the shearwise direction. Taking the curl and selecting the shearwise direction, we find the Squire equation

\[
(\partial_t + u^*\partial_x - \nu \Delta) \eta + D\mathbf{u}^* v'_e = 0
\]

where \( \eta \) is the shearwise component of the deviatoric vorticity. Taking the curl twice and selecting the shearwise direction yields the Orr-Sommerfeld equation

\[
(\partial_t + u^*\partial_x - \nu \Delta) \Delta v' - D^2\mathbf{u}^* v'_e = 0
\]

We write the deviatoric shearwise vorticity and velocity as Fourier modes in the streamwise and spanwise directions

\[
\begin{align*}
\eta &= \hat{\eta}(y) e^{ik_x x + ik_z z} \\
v' &= \hat{v}(y) e^{ik_x x + ik_z z}
\end{align*}
\]

where \( k_x \) and \( k_z \) are integers. This yields

\[
[\lambda + ik_xu^* - \nu (D^2 - k_x^2 - k_z^2)] \hat{\eta} = -ik_x D\mathbf{u}^* \hat{v}
\]

\[
[\lambda + ik_xu^* - \nu (D^2 - k_x^2 - k_z^2)] (D^2 - k_x^2 - k_z^2) \hat{v} = ik_x D^2\mathbf{u}^* \hat{v}
\]

In this equation, the rate of growth of a deviation from laminar flow is the real part of \( \lambda \). We first filter out four special cases.

(i) If both \( \eta \) and \( v' \) are constant in space, the only solutions consistent with the divergence free condition are of the form \( \mathbf{u}' = (c_1 \exp(ik_y y), 0, c_2 \exp(ik_y y)) \) for arbitrary constants \( c_{1,2} \), with \( \lambda = -ik_y^2 \). For \( k_y = 0 \), these solutions correspond to Galilean boosts in the streamwise and spanwise directions, which are excluded by the condition that the mean momentum is zero.

(ii) If \( v' \) is constant in space, but \( \eta \) is not, we only need to consider Squire’s equation. Multiplying (15) by \( \hat{\eta} \), integrating over the shearwise direction and taking the real part, we find
\[ \Re(\lambda) \int_{y=0}^{2\pi} |\tilde{\eta}|^2 \, dy = -\nu \int_{y=0}^{2\pi} [(k_x^2 + k_z^2)|\tilde{\eta}|^2 + |D\tilde{\eta}|^2] \, dy < 0 \]

(iii) If \( k_z = k_z = 0 \), but \( \nu' \) or \( \eta \) depend nontrivially on the shearwise coordinate, we find from equations (15) and (16) that \( \tilde{\eta}, \tilde{\nu} \propto \exp(ik_zy) \) and \( \lambda = -\nu k_y^2 < 0 \).

(iv) If \( k_x^2 + k_z^2 > 0 \) but \( \nu' \) does not depend on the shearwise coordinate, equation (16) reduces to an algebraic equation that admits the solution \( \lambda = -\nu(k_x^2 + k_z^2) < 0 \) if and only if \( (k_x^2 + k_z^2) = k_t^2 \).

In deriving the last result, we have used the fact that \( D^2u^* = -k_t^2u^* \), which is also pivotal for the treatment of the more general case below, in which we can assume that \( (k_x^2 + k_z^2) > 0 \) and \( \nu \) depends nontrivially on \( y \).

We find an energy equation by multiplying equation (16) by \( \tilde{\nu} \), taking the real part and integrating over the shearwise direction:

\[ \Re(\lambda) \int_{y=0}^{2\pi} (|D\tilde{\nu}|^2 + (k_x^2 + k_z^2)|\tilde{\nu}|^2) \, dy = \frac{i k_z}{2} \int_{y=0}^{2\pi} D^2u^* (\tilde{\nu}D\tilde{\nu} - \tilde{\nu}D\tilde{\nu}) \, dy \]

\[ -\nu \int_{y=0}^{2\pi} (|D^2 - k_x^2 - k_z^2|\tilde{\nu}|^2) \, dy \]

An enstrophy equation is found by multiplying equation (16) by \( (D^2 - k_x^2 - k_z^2)\tilde{\nu} \), taking the real part and integrating over the shearwise direction, which yields

\[ \Re(\lambda) \int_{y=0}^{2\pi} (|D^2 - k_x^2 - k_z^2|\tilde{\nu}|^2) \, dy = -\frac{i k_z}{2} \int_{y=0}^{2\pi} D^3u^* (\tilde{\nu}D\tilde{\nu} - \tilde{\nu}D\tilde{\nu}) \, dy \]

\[ -\nu \int_{y=0}^{2\pi} (|D(D^2 - k_x^2 - k_z^2)\tilde{\nu}|^2 + (k_x^2 + k_z^2)(D^2 - k_x^2 - k_z^2)|\tilde{\nu}|^2) \, dy \]

Finally, we multiply the energy equation (17) by \( k_t^2 \) and subtract it from the enstrophy equation (18) to eliminate the production term. Applying integration by parts once to simplify the left hand side, we obtain

\[ \Re(\lambda) \int_{y=0}^{2\pi} [(2k_x^2 + 2k_z^2 - k_t^2)|D\tilde{\nu}|^2 + (k_x^2 + k_z^2 - k_t^2)(k_x^2 + k_z^2)|\tilde{\nu}|^2 + |D^2\tilde{\nu}|^2] \, dy \]

\[ = -\nu \int_{y=0}^{2\pi} [(k_x^2 + k_z^2 - k_t^2)(D^2 - k_x^2 - k_z^2)|\tilde{\nu}|^2 + |D(D^2 - k_x^2 - k_z^2)\tilde{\nu}|^2] \, dy \]

(19)

For perturbations with \( k_x^2 + k_z^2 \geq k_t^2 \), both integrals are positive and the growth rate \( \Re(\lambda) \) is negative.

In conclusion we can say that only deviations from the laminar flow with \( k_x^2 + k_z^2 > 0 \) and \( 0 < k_x^2 + k_z^2 < k_t^2 \) can have a positive growth rate. For the classical value \( k_t = 1 \), this means all deviations are damped.

2.3. Spatial symmetries

In addition to the Galilean boosts mentioned above, the governing equations with \( k_t = 1 \) are equivariant under a large group of symmetries, generated by

- \( T_x(\delta) \) and \( T_x(\delta) \), shifts over any distance \( \delta \) in the streamwise and spanwise directions;
- \( S_y \), reflection in the spanwise direction;
- \( S_{xys} \), simultaneous reflection in the streamwise and shearwise directions;
Sy, reflection in the shearwise direction about the plane \( y = L/4 \);

- \( S_x \), a shift over \( L/2 \) along the shearwise direction followed by a reflection in the streamwise direction.

3. Numerical simulation and equilibrium solving

The simulation code solves for the Fourier coefficients of vorticity, that satisfy

\[
\frac{d\tilde{\omega}_k}{dt} = -ik \times (u \cdot \nabla u) - \nu |k|^2 \tilde{\omega} - \gamma \cos(\gamma) e_z
\]

where we have used the semi discrete Fourier transform

\[
\tilde{\omega} = \sum_k \tilde{\omega}_k e^{ikx}
\]

We use the continuity equation, \( k \cdot \tilde{u}_k = 0 \), to select two scalar fields to solve for. The nonlinear term is computed by a standard, FFT-based spectral method and aliasing interactions are removed by the phase-shift method of Patterson and Orszag (1971). The spatial resolutions is fixed to 64\(^3\) grid points and the highest wave number Fourier component computed is \( k_{\text{max}} = 30 \). Keeping only Fourier coefficients inside this de-aliasing radius for two components of vorticity, the dynamical system resulting from this spectral truncation has \( N = 230, 240 \) real-valued degrees of freedom. In figure 1 typical three-dimensional energy spectra are shown for nearly laminar and turbulent flow. The former is clearly well-resolved, while the resolution is marginal for the latter. Note, that the spectrum of turbulence flow we find is very similar to that of Shebalin and Woodruff (1997). The FFTs are computed in parallel, and nearly linear scaling can be achieved up to 64 cores. The system is time-stepped with a fourth-order accurate explicit Runge–Kutta-Gill scheme with a fixed time step of \( 1.8 \times 10^{-3} \) in units \( T = 1/\sqrt{k^2 L^2} \), the time scale that \( Re \) is based on. The time it takes to advance the system over one unit \( T \) is typically about 30 CPU minutes on modern processors.
The computation of edge states proceeds in three steps. Each of these steps, and the
algorithms used, have become fairly standard in the study of invariant solutions to the
Navier–Stokes equation over the past three decades. Therefore, we will not discuss them in
detail, but rather mention the main points and key papers.

The first step is to identify a long-lived turbulent state. We first computed a turbulent
state at a high Reynolds number, and then gradually increased the viscosity, allowing the flow
to equilibrate in every step. Around \( Re = 100 \) we start to observe a rapid laminarisation of the
flow, while at \( Re = 170 \), the turbulence is sustained for a few hundred units \( T \).

In the second step, we fix the viscosity to the latter value and then bisect between the
turbulence state and the laminar state, following the recipe of Itano and Toh (2001). This
process is illustrated in figure 1(left). We pick initial conditions on a line in phase space
connecting the turbulent to the laminar state, and find a critical value of the bisection para-
meter for which the fluid lingers on the boundary of the domain of attraction of the latter.

In the third step, we inspect the behaviour on the boundary and identify transient
approaches to equilibrium states. Such transient approaches are then used as initial guesses for
Newton iteration. We denote the discretised and spectrally truncated vorticity equation by
\( n = Xf \), for a vector \( \mathbb{X} \) of unknowns, each corresponding to the real or imaginary
part of one of the Fourier coefficients of vorticity in equation (20). The solution of this system
of ordinary differential equations is denoted by \( \phi(X, t, \nu) \). We then look for a solution to
\[
\phi(X, p, \nu) - T_x(\delta_x)T_z(\delta_z)X = 0
\]

The action of the operators of translation in the streamwise and spanwise directions on the
vector of unknowns is readily found from the Fourier transform (21). A solution to this
system of equation is a travelling wave with speeds \( \delta_x/p \) and \( \delta_z/p \) in the streamwise and
spanwise directions, respectively. Since the number of unknowns is \( N + 3 \), corresponding to
the elements of \( X, \delta_x \) and \( \delta_z \) and \( \nu \), we need to add three equations in order to
find an isolated solution. Assuming we have a known solution \( (X^0, \delta_x^0, \delta_z^0, \nu^0) \) and an initial guess for the next
solution, \( (X^{(n)}, \delta_x^{(n)}, \delta_z^{(n)}, \nu^{(n)}) \), we impose that
\[
(X - X^{(n)}) \cdot \frac{dT_x(\delta_x)}{d\delta_x} X^{(n)} \bigg|_{\delta_x=0} + \delta_x - \delta_x^{(n)} = 0
\]
\[
(X - X^{(n)}) \cdot \frac{dT_z(\delta_z)}{d\delta_z} X^{(n)} \bigg|_{\delta_z=0} + \delta_z - \delta_z^{(n)} = 0
\]
\[
(X - X^{(n)}) \cdot (X^{(n)} - X_j) + (\delta_x - \delta_x^{(n)})(\delta_x^{(n)} - \delta_x') + (\delta_z - \delta_z^{(n)})(\delta_z^{(n)} - \delta_z')
+ (\nu - \nu^{(n)})(\nu^{(n)} - \nu_j) = 0
\]

The first two conditions are that the direction in which we search for the new solution is
perpendicular to the generators of translations in the streamwise and spanwise directions. The
last condition is that the search direction is perpendicular to the tangent to the curve of
solutions. The first time we use an initial guess obtained from bisection, no previously
computed solution is available, and we replace the last phase condition by \( \nu - \nu^{(0)} = 0 \), i.e.
we keep the viscosity constant.

The resulting system of \( N + 3 \) nonlinear, coupled equations is solved by means of
Newton iteration. The linear system that must be solved for each Newton update step is, in
turn, solved by a Krylov subspace method. The resulting combination of iterative methods is
referred to as Newton-Krylov iteration and was first introduced by Sánchez et al (2004). If the
initial guess is far from the solution, as will generally be the case if it has been filtered from a
time series, the Newton iterates may not converge. In that case, we employ the Newton-hook
step, which greatly increases the radius of convergence, at the cost of increasing the number of necessary iterations. In implementing the Newton-hook step in conjunction with the Krylov subspace method, we follow Viswanath (2007). This paper also contains a description of the phase constraints related to the translational symmetries.

Once the residual of the system of equations \((22) - (23)\), normalised by the norm of the solution vector \((X, \delta_x, \delta_z, \nu)\), has dropped below \(10^{-5}\), we accept the new point on the solution curve, i.e. we set \(X, \delta_x, \delta_z, \nu\) = \(X_{i+1}, \delta_x^{i+1}, \delta_z^{i+1}, \nu_{i+1}\). A new initial guess is then generated from known solutions by parameter continuation, i.e. changing only \(\nu\), or by extrapolation, and the process of solving the nonlinear system is repeated. For each equilibrium solution, we compute the four eigenvalues with the largest real part by means of Arnoldi iteration (see, e.g. Gollub and van Loan 1996, chapter 9), which is also based on Krylov subspace iteration and uses the same ingredients as the Newton-Krylov iteration. This algorithm is stopped when the relative convergence of the eigenvalues is below \(10^{-4}\).

Sánchez et al (2004) explain the convergence of the Krylov subspace iteration, which depends on the integration time \(p\). A longer integration time generally leads to faster convergence. On the other hand, the CPU time taken by each Krylov iteration, which involves time-stepping the vorticity equation and its linearization, increases linearly with \(p\). In this study, we fixed \(p = 3.5T\). The number of Krylov subspace iterations necessary to maintain quadratic convergence of the Newton iteration varies from about 30, on the top branch in figure 2, to about 400 on the bottom branch. Since each equilibrium takes two to four Newton iterations to converge, the computation time for the points in this diagram varies from about 40 to 500 CPU hours, and the total computation time is about five CPU years excluding the eigenvalue computation, which adds half a CPU year.

![Figure 2](image-url)

**Figure 2.** Continuation of the equilibrium state in the Reynolds number. On the vertical axis, the energy input rate is shown, normalized by its value in laminar flow. Linear stability is indicated by colour as follows: edge states, with a single positive eigenvalue, are shown in red, stable states in green and states with at least two unstable eigenvalues in blue. The solid dots indicate the solutions computed during the continuation. The solid triangle denotes a saddle-node bifurcation and the solid square a Hopf bifurcation. Labels a and b correspond to the physical space portraits in figure 3. Note, that the top branch of equilibria is most similar to laminar flow. To avoid confusion with the conventional nomenclature of ‘upper’ and ‘lower’ branch, we will refer to the solutions branches as ‘top’ and ‘bottom’. Continuation of the bottom branch is ongoing.
4. An edge state

Starting from the time series shown in figure 1, the Newton-Krylov-hook method reveals the presence of an equilibrium state close to the laminar flow. It has no drift in the streamwise or spanwise direction, and has only a single unstable eigenvalue. A continuation of this state in the Reynolds number is shown in figure 2. As the Reynolds number increases, its bulk properties, such as the energy input rate plotted in the figure, approach that of the laminar flow. Following the curve in the other direction, we find a saddle-node bifurcation around $Re = 142.5$. The equilibria on the bottom branch are completely stable up to $Re = 145.3$, where a Hopf bifurcation occurs. Beyond this point, several more Hopf bifurcations are crossed and the number of unstable eigenvalues increases rapidly.

Two physical space visualizations of equilibrium solutions are shown in figure 3. Their structure is similar to that of some nearly laminar equilibria found in channel flow. On top of the streamwise and spanwise independent shear flow, induced by the body force, streamwise vortices are formed and the isosurfaces of vorticity exhibit a weak, sinusoidal variation in the spanwise direction. If one imagines walls to be present at the maximum and minimum of streamwise velocity, the patterns resemble the streaky solutions computed by Waleffe (2003) with free-slip boundary conditions.

It can be confirmed by an inspection of the Fourier coefficients of vorticity that this family of equilibria is, up to round-off and truncation error, invariant under symmetries $S_{x}$, $S_{z}$, and $T_{x}(L/2) \circ S_{x}$. The latter symmetry is found to hold approximately in turbulent minimal plane Couette flow, and is sometimes imposed on that flow from the outset, as by Kawahara and Kida (2001). A second symmetry they imposed is $T_{x}(L/2) \circ S_{x}$. While our equilibrium only depends weakly on the streamwise direction, it does not appear to have any discrete translational symmetry in this direction.
5. Discussion

The observation that Kolmogorov flow on a three-dimensional domain with aspect ratios fixed to unity follows a subcritical transition to turbulence is perhaps surprising in the light of previous results on spatially periodic flows. The laminar state in ABC flow, for instance, was shown to turn unstable in a supercritical Hopf bifurcation (Ashwin and Podvigina 2003) and the same instability occurs in Kida-Pelz flow (Kida et al 1989, van Veen 2005). Similarly, the primary instability of laminar Kolmogorov flow with a forcing wave number greater than one will be that of the corresponding flow in two dimensions, either pitchfork or Hopf.

On the other hand, recent results by Linkmann and Morozov (2015) indicate that a forcing mechanism often used in large-scale simulations of homogeneous isotropic turbulence may also generate turbulence in the presence of a stable laminar flow. This forcing mechanism keeps the energy input rate constant in time by scaling a number of low wave number Fourier components in every time step. Because of the difference in the nature of the forcing, a direct comparison is difficult, but it seems their results have a much lower Reynolds number than those presented here. Their Reynolds number based on the energy input rate, $Re_I$, varies in the range 54–98 while in our flow it takes values from about $Re_I = 800$, in turbulent flow, to $Re_I = 2500$, in near laminar flow. Another qualitative difference is the fact that the laminar flow found in their system depends on the initial condition as the forcing depends on the instantaneous flow field.

We hope, that a further study of the edge state will shed new light on Kolmogorov's original question about the nature of laminar to turbulent transition in spatially periodic flows. We are currently computing time-periodic solutions, such as those branching off the current equilibrium in various Hopf bifurcations. Initial brute-force computations indicate that turbulence can be sustained on a long time scale for Reynolds numbers as low as $Re = 40$, hinting at the existence of additional edge states. The dynamics of—possibly transient—turbulence can be investigated by computing periodic solutions far removed from the laminar state, along the lines of Chandler and Kerswell (2013). In addition, it will be interesting to compare the results to those obtained in minimal plane Couette flow to see what aspects of the dynamics of transition are directly related to the presence of material boundaries. Finally, we expect that three-dimensional Kolmogorov flow will prove a fertile test ground for new algorithms for the computation of invariant solutions since the absence of material boundaries, and the associated polynomial basis functions, renders the simulation algorithm comparatively simple.

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