THE HAUSDORFF DIMENSION OF THE SET OF DISSIPATIVE POINTS FOR A CANTOR-LIKE MODEL SET FOR SINGLY CUSPED PARABOLIC DYNAMICS

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Abstract. In this paper we introduce and study a certain intricate Cantor-like set \( C \) contained in unit interval. Our main result is to show that the set \( C \) itself, as well as the set of dissipative points within \( C \), both have Hausdorff dimension equal to 1. The proof uses the transience of a certain non-symmetric Cauchy-type random walk.

1. Introduction

In this paper we estimate the Hausdorff dimension of the set \( C_{\infty} \) of dissipative points within a certain Cantor-like subset \( C \) of the unit interval \([0, 1) \subset \mathbb{R}\). There are two ways to define these sets. The first is via the integration of a certain interval map \( \Phi \), where \( C \) represents the set of points with an infinite forward orbit, and \( C_{\infty} \) is equal to the basin of attraction of the set of critical points of \( \Phi \) (see Remark 1.1 at the end of this introduction). The second construction is purely in terms of fractal geometry. Let us remark that our motivation for considering the sets \( C \) and \( C_{\infty} \) stems from the investigations in [10] of the geometry of limit sets of Kleinian groups with singly cusped parabolic dynamics. For the purposes of this paper this link is irrelevant and therefore we omit the details. However, intuition coming from Kleinian groups has historically played a very important role in in the development of Real and Complex Dynamics, and this paper can be seen as adding to this tradition.

Let us begin with by giving the slightly intricate, but more down-to-earth fractal geometric construction of the sets \( C \) and \( C_{\infty} \). For this we have to define certain families of fundamental intervals by induction as follows. We start with the unit interval \([0, 1)\), and then partition the left half of \([0, 1)\) into the infinitely many intervals

\[
I_1 := \left[0, \frac{3}{\pi^2}\right), \quad \text{and} \quad I_{k+1} := \left[\frac{3}{\pi^2} \sum_{l=1}^{k} l^{-2}, \frac{3}{\pi^2} \sum_{l=1}^{k+1} l^{-2}\right), \quad \text{for} \; k \in \mathbb{N}.
\]

The family of these first-level intervals will be denoted by \( C_1 \). Note that the right half \([\frac{1}{2}, 1)\) of the unit interval, which is clearly not captured by \( C_1 \), should be interpreted as ‘the hole at the first level’. The second step is to partition each element \( I_{k_1} \in C_1 \) as follows. By starting from the left

\[
2000 \text{ Mathematics Subject Classification.} \; \text{Primary: 60D05, 14L35, 51N30.} \]

\textit{Key words and phrases.} Fractal geometry, Hausdorff dimension, Cauchy random walks, Kleinian groups.
The diameters of these intervals are mutually adjacent intervals $I_{k_1}$, and partition the left half of $I_{k_1}$ into infinitely many mutually adjacent intervals

$$I_{k_1, k_1+1, \ldots, I_{k_1, k_1+l, \ldots},}$$

where the diameters of these intervals are given by

$$|I_{k_1, k_1+l}| = \frac{3 |I_{k_1}|}{\pi^2 l^2}, \text{ for } l \in \mathbb{N}.$$ 

Similarly, by starting from the right endpoint of $I_{k_1}$ we insert into the right half of $I_{k_1}$ the $(k_1 + 1)$ mutually adjacent intervals

$$I_{k_1, k_1+1, \ldots, I_{k_1, k_1-1, \ldots, I_{k_1, 0}},}$$

with diameters given by

$$|I_{k_1, k_1-l}| = \begin{cases} 
\frac{3 |I_{k_1}|}{\pi^2 (2k_1)^2} & \text{for } l = 0 \\
\frac{3 |I_{k_1}|}{\pi^2 l^2} & \text{for } l \in \{1, \ldots, k_1\}.
\end{cases}$$

The family of these second-level intervals will be denoted by $C_2$. Note that in this way we have perforated each $I_{k_1} \in C_1$ such that there is a ‘hole’ in $I_{k_1}$ with diameter of order $|I_{k_1}|/k_1$.

We then proceed by induction as follows. Suppose that for $n \geq 2$ the $n$-th level interval $I_{k_1 \ldots k_n}$ has been constructed. The $(n + 1)$-th level intervals arising from $I_{k_1 \ldots k_n}$ are then obtained as follows. There are two cases to consider. The first case is that $k_n = 0$, and here the partition only continues in the left half of $I_{k_1 \ldots k_n-1,0}$. More precisely, in this case we start from the left endpoint of $I_{k_1 \ldots k_n-1,0}$ and partition the left half of $I_{k_1 \ldots k_n-1,0}$ into infinitely many mutually adjacent intervals

$$I_{k_1 \ldots k_{n-1},0,1, \ldots, I_{k_1 \ldots k_{n-1},0, l, \ldots},}$$

with diameters given by

$$|I_{k_1 \ldots k_{n-1},0, l}| = \frac{3 |I_{k_1 \ldots k_{n-1},0}|}{\pi^2 l^2}, \text{ for } l \in \mathbb{N}.$$ 

In the second case we have that $k_n \in \mathbb{N}$, and here we start from the left endpoint of $I_{k_1 \ldots k_n}$ and partition the left half of $I_{k_1 \ldots k_n}$ into infinitely many mutually adjacent intervals

$$I_{k_1 \ldots k_n, k_{n+1}, \ldots, I_{k_1 \ldots k_n, k_{n+1}, l, \ldots},}$$

The diameters of these intervals are

$$|I_{k_1 \ldots k_n, k_{n+1}, l}| = \frac{3 |I_{k_1 \ldots k_n, k_{n+1}}|}{\pi^2 l^2}, \text{ for } l \in \mathbb{N}. \quad (1)$$

Similarly, by starting from the right endpoint of $I_{k_1 \ldots k_n}$ we insert into the right half of $I_{k_1 \ldots k_n}$ the $(k_n + 1)$ mutually adjacent intervals

$$I_{k_1 \ldots k_n, k_{n+1}, \ldots, I_{k_1 \ldots k_n-1, k_{n+1}, \ldots, I_{k_1 \ldots k_n, 0}},}$$

with diameters given by

$$|I_{k_1 \ldots k_n, l}| = \begin{cases} 
\frac{3 |I_{k_1 \ldots k_n}|}{\pi^2 (2k_n)^2} & \text{for } l = 0 \\
\frac{3 |I_{k_1 \ldots k_n}|}{\pi^2 l^2} & \text{for } l \in \{1, \ldots, k_n\}.
\end{cases}$$

(2)
The so obtained set of intervals of the \((n + 1)\)-th level will be denoted by 
\(C_{n+1}\). That is,
\[
C_n = \{I_{k_1 \ldots k_n} : k_1 \in \mathbb{N}, k_{i+1} \in \mathbb{N}_0 \text{ for } i \in \mathbb{N}, \text{ and if } k_i = 0 \text{ then } k_{i+1} \neq 0\}.
\]
Again, note that by this we have perforated \(I_{k_1 \ldots k_n}\) such that in the first case ‘the hole’ is precisely the right half of \(I_{k_1 \ldots k_n}\), whereas in the second case the diameter of the hole is of order \(|I_{k_1 \ldots k_n}|/k_n\). Also, let us emphasize that by construction, the state 0 necessarily has to renew itself. That is, the generation following the interval \(I_{k_1 \ldots k_n - 10}\) is given by \(\{I_{k_1 \ldots k_n - 10 k_{n+1}} : k_{n+1} \in \mathbb{N}\}\). Moreover, note that the system can only be stationary at states \(k_n \in \mathbb{N}\), which means that if \(I_{k_1 \ldots k_n}\) is a given interval of some level \(n\) then \(I_{k_1 \ldots k_{n-1} k_n}\) exists if and only if \(k_n \neq 0\). Finally, note that we always assume that the intervals \(I_{k_1 \ldots k_n}\) are half open, namely closed to the left and open to the right.

With this inductive construction of the generating intervals \(I_{k_1 \ldots k_n}\) at hand, the Cantor-like set \(C\) is defined by
\[
C := \bigcap_{n \in \mathbb{N}} \bigcup_{I \in C_n} I.
\]

Next, we define the set of dissipative points in \(C\). For this we require the following canonical coding of the elements in \(C\). A finite or infinite sequence \((k_1, k_2, \ldots)\) is called admissible if \(I_{k_1 \ldots k_n} \in C_n\), for all \(n \in \mathbb{N}\). Clearly, the diameter of \(I_{k_1 \ldots k_n}\) tends to zero as \(n\) tends to \(\infty\), for every fixed infinite admissible sequence \((k_n)_{n \in \mathbb{N}}\), and therefore,
\[
\bigcap_{n=1}^{\infty} I_{k_1 \ldots k_n} \text{ is a singleton.}
\]
In particular, each \(x \in C\) is coded uniquely by an infinite admissible sequence, and this gives rise to the bijection
\[
\rho : \Sigma \to C, (k_1, k_2, \ldots) \mapsto \bigcap_{n=1}^{\infty} I_{k_1 \ldots k_n},
\]
where \(\Sigma\) refers to the set of all admissible sequences. Using this coding, the set \(C_\infty \subset C\) of dissipative points is then given by
\[
C_\infty := \left\{x \in C : x = \rho(k_1, k_2, \ldots) \text{ and } \lim_{n \to \infty} k_n = \infty\right\}.
\]
The following theorem gives the main result of this paper. Here, \(\dim_H\) refers to the Hausdorff dimension.

**Main Theorem.** For \(C\) and \(C_\infty\) as defined above, we have
\[
\dim_H (C_\infty) = \dim_H (C) = 1.
\]

**Remark 1.1.** As already mentioned at the beginning of the introduction, the sets \(C\) and \(C_\infty\) can alternatively be defined in terms of a certain interval map
\[
\Phi : \bigcup_{I_k \in C_1} I_k \to [0, 1).
\]
Namely, the map $\Phi$ is given piecewise by $\Phi|_{I_k} := \phi_k$, for each $k \in \mathbb{N}$, where the maps $\phi_k : \bigcup_{l \in C_2} I_{kl} \to I_k$ are piecewise linear in the following sense. For each $k \in \mathbb{N}$ and $l \in \mathbb{N}_0$, the map $\phi_k|_{I_{kl}}$ is a linear and

$$
\phi_k|_{I_{kl}}(I_{kl}) = \begin{cases} 
\bigcup_{m=l-1}^{\infty} I_{km} & \text{for } l \neq 0 \\
I_k \setminus I_{k0} & \text{for } l = 0. 
\end{cases}
$$

One then immediately verifies that the set $C$ is equal to the set of points which have an infinite forward orbit under $\Phi$. Moreover, the centre $c_k$ of $I_k$ is the critical point of $\phi_k$, for each $k \in \mathbb{N}$, and thus $\text{Crit}(\Phi) := \{c_k : k \in \mathbb{N}\}$ represents the countable set of critical points of $\Phi$. With $\omega_\Phi(x)$ referring to the $\omega$-limit set of an element $x \in [0, 1/2)$ with respect to $\Phi$ (that is, $\omega_\Phi(x)$ denotes the set of accumulation points of $\{\Phi^n(x) : n \in \mathbb{N}\}$), and with $\mathcal{B}(\text{Crit}(\Phi)) := \{x : \omega_\Phi(x) \subset \text{Crit}(\Phi)\}$ denoting the basin of attraction of $\text{Crit}(\Phi)$ under $\Phi$, we then have

$$
\mathcal{C}_\infty = \mathcal{B}(\text{Crit}(\Phi)).
$$

Similar types of interval maps have been studied for instance in [4] and [11] in connection with ‘wild Cantor sets’ and the search for Julia sets of positive Lebesgue measure. It seems worthwhile to point out that the ‘Martingale Argument’ of Keller (see [4], Section 4.1), which gives a criterion for the basin of attraction of a critical point to be of positive Lebesgue measure, is not applicable to the map $\Phi$ and hence does not allow to draw any conclusion for the Lebesgue measure of $\mathcal{C}_\infty$. Nevertheless, recent studies in the theory of Kleinian groups (cf. [11] [3] [5], and also [6]) have confirmed the Ahlfors Conjecture, and applying these to our situation here strongly suggests that $\mathcal{C}$ and $\mathcal{C}_\infty$ are both of 1-dimensional Lebesgue measure equal to 0. However, currently it is still a conjecture that the Lebesgue measure of $\mathcal{C}$ and $\mathcal{C}_\infty$ is equal to zero, and it would be desirable to have an elementary proof of this conjecture.

Acknowledgement: We like to thank the EPSRC for supporting a one week visit of the first author to the School of Mathematics at the University of St. Andrews. During this visit we began with the work towards this paper. Also, we like to thank the Erwin Schrödinger Institute, Vienna for supporting the workshop Ergodic Theory - Limit Theorems and Dimensions (Vienna, 17 - 21 December 2007) during which we made the finishing touches to this paper. Finally, we are grateful to Tatyana Turova for very helpful discussions concerning the random walk argument which we employ here.

2. Proof of the Main Theorem

Since $\mathcal{C}_\infty \subset \mathcal{C} \subset [0, 1)$, we have

$$
\dim_H(\mathcal{C}_\infty) \leq \dim_H(\mathcal{C}) \leq 1.
$$

Therefore, our strategy will be to construct a family of probability measures $\mu_\alpha$ on $\mathcal{C}_\infty$, for $1/2 < \alpha < 1$, such that the Hausdorff dimension $\dim_H(\mu_\alpha)$ of the measure $\mu_\alpha$ tends to 1 for $\alpha$ tending to 1. Clearly, this will then be sufficient for the proof of the Main Theorem.
2.1. The family of measures \( \mu_\alpha \).

Let \( 1/2 < \alpha \leq 1 \) be fixed. We then define a set function \( \mu_\alpha \) on the intervals \( I_{k_1 \cdots k_n} \) by induction in the following way. Define \( I_0 := (0, 1) \) and set \( I_{0k} := I_k \) for all \( k \in \mathbb{N} \). Then let

\[
\mu_\alpha (I_0) := 1,
\]

and define \( \mu_\alpha (I_{k_1 \cdots k_n k_{n+1}}) \) for each finite admissible sequence \( (k_1, \cdots, k_{n+1}) \) as follows. With \( \zeta(s) := \sum_{m=1}^{\infty} m^{-s} \) referring to the Riemann zeta function, we define for \( k_{n+1} \neq 0 \),

\[
\mu_\alpha (I_{k_1 \cdots k_n k_{n+1}}) := \begin{cases} 
\frac{\mu_\alpha (I_{k_1 \cdots k_n})}{2 \zeta(2\alpha)} \left( \frac{1}{|k_{n+1} - k_n|^{2\alpha}} + \frac{1}{(k_{n+1} + k_n)^{2\alpha}} \right) & \text{for } k_{n+1} \neq 0 \\
\frac{\mu_\alpha (I_{k_1 \cdots k_n})}{2 \zeta(2\alpha)} \frac{1}{k_n^{2\alpha}} & \text{for } k_{n+1} = 0.
\end{cases}
\]

Also, for \( k_n = k_{n+1} \) let

\[
\mu_\alpha (I_{k_1 \cdots k_n k_n}) := \frac{\mu_\alpha (I_{k_1 \cdots k_n})}{2 \zeta(2\alpha)} \frac{1}{(2k_n)^{2\alpha}}.
\]

On the first sight, this definition of the set function \( \mu_\alpha \) might appear to be slightly artificial. However, in the next section we will see that this definition reflects the transition probabilities of a certain (transient) random walk on \( \mathbb{N}_0 \), and therefore is rather canonical. Before we come to this, let us first state the following consistency property for \( \mu_\alpha \). This property can also be deduced using the random walk of Section 2.2. Nevertheless, the following gives an elementary proof of this consistency property.

**Lemma 2.1.** For each finite admissible sequence \( (k_1, \cdots, k_n) \), we have

\[
\mu_\alpha (I_{k_1 \cdots k_n}) = \sum_{k_{n+1} \geq 0} \mu_\alpha (I_{k_1 \cdots k_n k_{n+1}}).
\]

**Proof.** For \( k_n = 0 \), we have

\[
\sum_{k_{n+1} \geq 0} \mu_\alpha (I_{k_1 \cdots k_{n-1} k_{n+1}}) = \frac{\mu_\alpha (I_{k_1 \cdots k_{n-1} 0})}{2 \zeta(2\alpha)} \sum_{l=1}^{\infty} \frac{2}{l^{2\alpha}} = \mu_\alpha (I_{k_1 \cdots k_{n-1} 0}).
\]
If \( k_n \neq 0 \), then we compute

\[
\sum_{k_{n+1} \geq 0} \mu_{\alpha} (I_{k_1 \ldots k_{n+1}})
\]

\[
= \sum_{k_{n+1} > 0} \frac{1}{2 \zeta (2\alpha)} \frac{1}{|k_{n+1} - k_n|^{2\alpha}} \mu_{\alpha} (I_{k_1 \ldots k_n}) + \frac{1}{2 \zeta (2\alpha)} \frac{1}{(k_{n+1} + k_n)^{2\alpha}} \mu_{\alpha} (I_{k_1 \ldots k_n})
\]

\[
= \mu_{\alpha} (I_{k_1 \ldots k_n}) \left( \sum_{k_{n+1} > 0} \frac{1}{|k_{n+1} - k_n|^{2\alpha}} + \sum_{k_{n+1} \neq k_n} \frac{1}{(k_{n+1} + k_n)^{2\alpha}} \right)
\]

\[
= \mu_{\alpha} (I_{k_1 \ldots k_n}) \cdot \frac{2}{2 \zeta (2\alpha)} \sum_{l=1}^{\infty} \frac{1}{l^{2\alpha}} = \mu_{\alpha} (I_{k_1 \ldots k_n}).
\]

\( \square \)

The following is an immediate consequence of the previous lemma.

**Corollary 2.2.** The measure \( \mu_{\alpha} \) is a probability measure on \( C \).

2.2. The associated random walk.

In this section we show that the measure \( \mu_{\alpha} \) can be interpreted in terms of a certain random walk. In particular, this will give that \( \mu_{\alpha} \) has the Markov property. For this, let the random variables \( \tilde{X}_n^\alpha \) be defined by the probability (with respect to \( \mu_{\alpha} \)) being in the interval \( I_{k_1 \ldots k_n k_{n+1}} \) given that in the previous step the process has been in the interval \( I_{k_1 \ldots k_n} \). That is, the random variables \( \tilde{X}_n^\alpha \) is given as follows.

- For \( k_{n+1} > 0 \) such that \( k_{n+1} \neq k_n \), let

\[
\mathbb{P} \left( \tilde{X}_{n+1}^\alpha = I_{k_1 \ldots k_n k_{n+1}} \mid \tilde{X}_n^\alpha = I_{k_1 \ldots k_n} \right) := \frac{\mu_{\alpha} (I_{k_1 \ldots k_n k_{n+1}})}{\mu_{\alpha} (I_{k_1 \ldots k_n})}
\]

\[
= \frac{1}{2 \zeta (2\alpha)} \left( \frac{1}{|k_{n+1} - k_n|^{2\alpha}} + \frac{1}{(k_{n+1} + k_n)^{2\alpha}} \right).
\]

- For \( k_{n+1} > 0 \) such that \( k_{n+1} = k_n \), let

\[
\mathbb{P} \left( \tilde{X}_{n+1}^\alpha = I_{k_1 \ldots k_n k_{n+1}} \mid \tilde{X}_n^\alpha = I_{k_1 \ldots k_n} \right) := \frac{\mu_{\alpha} (I_{k_1 \ldots k_n k_{n+1}})}{\mu_{\alpha} (I_{k_1 \ldots k_n})} = \frac{1}{2 \zeta (2\alpha)} \frac{1}{(2k_n)^{2\alpha}}.
\]

- If \( k_{n+1} = 0 \), then

\[
\mathbb{P} \left( \tilde{X}_{n+1}^\alpha = I_{k_1 \ldots k_n 0} \mid \tilde{X}_n^\alpha = I_{k_1 \ldots k_n} \right) := \frac{\mu_{\alpha} (I_{k_1 \ldots k_n 0})}{\mu_{\alpha} (I_{k_1 \ldots k_n})} = \frac{1}{2 \zeta (2\alpha)} \frac{1}{k_{n+1}^{2\alpha}}.
\]
Clearly, these conditional probabilities do not depend on \( k_1, \ldots, k_{n-1} \). Hence, we can define an associated random walk \( X_n^\alpha \) on \( \mathbb{N}_0 \) by the following transition probabilities.

- For \( l, m \in \mathbb{N}_0 \), let

\[
\mathbb{P}(X_{n+1}^\alpha = l | X_n^\alpha = m) := \begin{cases} 
\frac{1}{2\zeta(2\alpha)} \left( \frac{1}{|m-l|^{2\alpha}} + \frac{1}{(m+l)^{2\alpha}} \right) & \text{for } l \neq 0, l \neq m \\
\frac{1}{2\zeta(2\alpha)} & \text{for } l \neq 0, l = m \\
0 & \text{for } l = 0, m \neq 0 \\
0 & \text{for } l = m = 0.
\end{cases}
\]

The random walk \( X_n^\alpha \) is very closely connected to our original geometric setting, since it allows to recover the measure \( \mu_\alpha \) as follows.

\[
\mathbb{P}(X_0^\alpha = k_1, \ldots, X_n^\alpha = k_n) = \mathbb{P}(X_n^\alpha = k_n | X_{n-1}^\alpha = k_{n-1}) \cdot \mathbb{P}(X_{n-1}^\alpha = k_{n-1} | X_{n-2}^\alpha = k_{n-2}) \cdot \ldots \cdot \mathbb{P}(X_1^\alpha = k_1 | X_0^\alpha = 0)
\]

\[
= \frac{\mu_\alpha(I_{k_1-k_{n-1}})}{\mu_\alpha(I_{k_1-k_{n-1}})} \frac{\mu_\alpha(I_{k_1-k_{n-1}})}{\mu_\alpha(I_{k_1-k_{n-1}})} \ldots \frac{\mu_\alpha(I_{k_1-k_{n-1}})}{\mu_\alpha(I_{k_1-k_{n-1}})} = \mu_\alpha(I_{k_1-k_{n}}).
\]

The aim now is to show that the random walk \( X_n^\alpha \) is transient. This will then allow us to deduce that \( \mu_\alpha \) is non-trivial on \( \mathcal{C}_\infty \).

**Theorem 2.3.** For each \( 1/2 < \alpha < 1 \), the random walk \( X_n^\alpha \) on \( \mathbb{N}_0 \) is transient. That is, we have \( \mathbb{P} \)-almost surely,

\[
\lim_{n \to \infty} X_n^\alpha = \infty.
\]

**Proof.** Let \( 1 < \beta < 2 \) be fixed, and consider the Cauchy-type random walk \( Y_n^\beta \) on \( \mathbb{Z} \), given by the transition probabilities

\[
\mathbb{P}(Y_{n+1}^\beta = m + l | Y_n^\beta = m) := \frac{1}{2\zeta(\beta)} \left( \frac{1}{|l|^{\beta}} \right) \text{ for } n \in \mathbb{N}, m \in \mathbb{Z} \text{ and } l \in \mathbb{Z} \setminus \{0\}.
\]

It is well known that \( Y_n^\beta \) is symmetric, and that \( Y_n^\beta \) is transient if and only if \( \beta < 2 \). Let \( \tilde{Y}_n^\beta \) denote the random walk which arises from \( Y_n^\beta \) in the following way. Let \( l_1, \ldots, l_n \) be the sequence of jumps of \( Y_n^\beta \) after the first \( n \) steps. With \( r_1 \) referring to the first time at which \( Y_k^\beta \) crosses 0, we set \( \tilde{Y}_{r_1}^\beta := |Y_{r_1}^\beta| \).

Subsequently, we apply the jumps \( |l_{r_1}| \) up to \( |l_{r_2}| \), where \( r_2 \) refers to the next time \( Y_{r_2}^\beta \) crosses 0. After that, we apply the jumps \( |l_{r_2}| \) up to \( |l_{r_3}| \), where \( r_3 \) denotes the next time at which the process \( Y_k^\beta \) first crosses 0 again.

More precisely, with \( r_i \) referring to the \( i \)-th time the process \( Y_k^\beta \) crosses 0, we let \( \tilde{Y}_{r_i}^\beta := |Y_{r_i}^\beta| \), and between each two consecutive crossings \( r_i \) and \( r_{i+1} \) we define the jumps of \( \tilde{Y}_{r_i}^\beta \) to be \( |l_{r_i+1}|, \ldots, |l_{r_i+1}| \). Note that the random walk we have just described is equal to the random walk \( |Y_n^\beta| \). Also, note that since \( Y_n^\beta \) is a symmetric random walk, the above modification of the sample path does not alter its probability. Therefore, it immediately follows that the transience of \( Y_n^\beta \) implies that \( \tilde{Y}_n^\beta \) is transient. Using the fact that by symmetry of \( Y_n^\beta \) we have \( \mathbb{P}(Y_n^\beta = m) = \mathbb{P}(Y_n^\beta = -m) \), we now compute
the transition probabilities of the random walk $\tilde{Y}_n^\beta$ on $\mathbb{N}_0$ as follows. For 
$l, m \in \mathbb{N}_0$ such that $l \neq 0$, and $m \neq l$, we have
$$
\mathbb{P}(\tilde{Y}_{n+1}^\beta = l \mid \tilde{Y}_n^\beta = m) = \frac{\mathbb{P}(|Y_{n+1}^\beta| = l \mid |Y_n^\beta| = m)}{\mathbb{P}(Y_n^\beta = m \text{ or } Y_n^\beta = -m)}
\quad = \frac{\mathbb{P}(|Y_{n+1}^\beta| = l, Y_n^\beta = m) + \mathbb{P}(|Y_{n+1}^\beta| = l, Y_n^\beta = -m)}{2\mathbb{P}(Y_n^\beta = m)}
\quad = \frac{\mathbb{P}(|Y_{n+1}^\beta| = l, Y_n^\beta = m)}{\mathbb{P}(Y_n^\beta = m)}
\quad = \frac{1}{2\zeta(\beta)} \left( \frac{1}{|m-l|^\beta} + \frac{1}{|m+l|^\beta} \right).
$$

Similarly, we obtain for $l = m \neq 0$,
$$
\mathbb{P}(\tilde{Y}_{n+1}^\beta = m \mid \tilde{Y}_n^\beta = m) = \frac{1}{2\zeta(\beta) (2m)^\beta},
$$
and for $l = 0$ and $m \neq l$,
$$
\mathbb{P}(\tilde{Y}_{n+1}^\beta = 0 \mid \tilde{Y}_n^\beta = m) = \frac{1}{2\zeta(\beta) m^\beta}.
$$

Finally, note that we immediately have
$$
\mathbb{P}(\tilde{Y}_{n+1}^\beta = 0 \mid \tilde{Y}_n^\beta = 0) = 0.
$$

This shows that the transition probabilities of $\tilde{Y}_n^\beta$ coincide with the ones of $X_n^{\beta/2}$. Therefore, since $\tilde{Y}_n^\beta$ is transient, it follows that $X_n^{\beta/2}$ is transient. This finishes the proof of the theorem.

As already announced before, Theorem 2.3 has the following important implication.

**Corollary 2.4.** For every $1/2 < \alpha < 1$, we have
$$
\mu_\alpha(C_\infty) = 1.
$$

**Remark 2.5.** Note that the proof of Theorem 2.3 relies heavily on the fact that $1/2 < \alpha < 1$. Namely, for instance for $\alpha = 1$ the associated random walk is recurrent, and consequently the measure $\mu_\alpha$ vanishes on $C_\infty$.

### 2.3. Approximating the essential support of $\mu_\alpha$.

In order to prepare our estimate of the lower pointwise dimension of $\mu_\alpha$, we need a further approximation of the essential support of this measure. We will see that $\mu_\alpha$-almost surely the diameters of the coding intervals of an element of $C_\infty$ do not shrink too fast. For this we define, for $\gamma \in \mathbb{R}$,

$$
C_\infty^\gamma := \left\{ x \in C_\infty : x = \rho(k_1, k_2, \ldots) \text{ such that } \limsup_{n \to \infty} \frac{|k_{n+1} - k_n|}{n^\gamma} \leq 1 \right\}.
$$

**Lemma 2.6.** For each $1/2 < \alpha < 1$ and $\gamma > 1/(2\alpha - 1)$, we have
$$
\mu_\alpha(C_\infty^\gamma) = 1.
$$
Proof. The proof is an easy consequence of the Borel-Cantelli lemma. Indeed, first note that for \( \beta = 2\alpha \) and \( k \in \mathbb{N} \) we have
\[
\mathbb{P}(|Y_n^\beta - Y_{n-1}^\beta| \geq k) \geq \mathbb{P}(|\tilde{Y}_n^\beta - \tilde{Y}_{n-1}^\beta| \geq k).
\]
The latter is an immediate consequence of the fact that the random walk \( Y_n^\beta \) has the same distribution as \( \tilde{Y}_n^\beta \), but without reflections at 0. Hence, it suffices to prove the lemma for the symmetric random walk \( Y_n^{2\alpha} \). For this we define
\[
p_n^\gamma := \mathbb{P}(|Y_n^{2\alpha} - Y_{n-1}^{2\alpha}| \geq n^\gamma).
\]
We then have, for each \( n \in \mathbb{N} \) and with \( c(\beta) > 0 \) referring to some universal constant,
\[
p_n^\gamma = 2 \sum_{k=n^{\gamma}}^{\infty} \frac{1}{k^{\beta}} \leq c(\beta) \frac{1}{n^{\gamma(\beta-1)}}.
\]
Since the series \( \sum_{n=1}^{\infty} p_n^\gamma \) converges for \( \gamma > 1/(\beta - 1) \), the Borel-Cantelli Lemma implies that \( \mathbb{P} \)-almost surely there are at most finitely many \( n \) which satisfy the inequality
\[
|Y_n^{2\alpha} - Y_{n-1}^{2\alpha}| \geq n^\gamma.
\]
This shows that for \( \mu_\alpha \)-almost every \( x = \rho(k_1, k_2, \ldots) \in C^\gamma \) we have
\[
\limsup_{n \to \infty} \frac{|k_{n+1} - k_n|}{n^\gamma} \leq 1.
\]
\[\square\]

2.4. The lower pointwise dimension on fundamental intervals.

The main result of this section will be the following estimate for the lower pointwise dimension of the measure \( \mu_\alpha \) restricted to the fundamental intervals \( I_{k_1 \ldots k_n} \).

**Proposition 2.7.** For each \( \epsilon > 0 \) there exists \( 1/2 < \alpha < 1 \) and \( \gamma > 1/(2\alpha - 1) \) such that for every \( x = \rho(k_1, k_2, \ldots) \in C^\gamma \),
\[
\liminf_{n \to \infty} \frac{\log \mu_\alpha(I_{k_1 \ldots k_n})}{\log |I_{k_1 \ldots k_n}|} \geq \alpha - \epsilon.
\]

Furthermore, in here we have that \( \alpha \) tends to 1 for \( \epsilon \) tending to 0.

**Proof.** Since we are interested in the asymptotic behaviour of \( I_{k_1 \ldots k_n} \) for points \( x = \rho(k_1, k_2, \ldots) \in C^\gamma \), Corollary [2.4] and Lemma [2.6] imply that we can assume without loss of generality that \( k_n > 0 \) and \( |k_{n+1} - k_n| \leq n^\gamma \), for all \( n \in \mathbb{N} \). Furthermore, for ease of exposition we only consider sequences which do not contain repetitions. That is, we assume that \( k_n \neq k_{n+1} \), for all \( n \in \mathbb{N} \). The case with repetitions can be dealt with in similar way and is
left to the reader. Using the definition of $\mu_\alpha$, we then have

\[
\frac{\log \mu_\alpha(I_{k_1\ldots k_n \kappa_{n+1}})}{\log |I_{k_1\ldots k_n \kappa_{n+1}}|} = \frac{-\log(2\zeta(2\alpha))}{\log |I_{k_1\ldots k_n \kappa_{n+1}}|} + \log \left[ \frac{1}{|k_{n+1} - k_n|^{2\alpha}} + \frac{1}{(k_n + k_{n+1})^{2\alpha}} \right] \frac{\mu_\alpha(I_{k_1\ldots k_n})}{\log |I_{k_1\ldots k_n \kappa_{n+1}}|}
\]

\[
\geq -\log(2\zeta(2\alpha)) + \log \left[ \frac{2}{|k_{n+1} - k_n|^{2\alpha}} \mu_\alpha(I_{k_1\ldots k_n}) \right] \frac{\mu_\alpha(I_{k_1\ldots k_n})}{\log |I_{k_1\ldots k_n \kappa_{n+1}}|}
\]

\[
= -\log(2\zeta(2\alpha)) + \frac{\log \left[ 2^{2\alpha} \zeta(2\alpha)^{\alpha} |I_{k_1\ldots k_{n+1}}|^{\alpha} \mu_\alpha(I_{k_1\ldots k_{n+1}}) \right]}{\log |I_{k_1\ldots k_n \kappa_{n+1}}|}
\]

\[
= \alpha + \frac{\log \left[ 2^{2\alpha} \zeta(2\alpha)^{\alpha} |I_{k_1\ldots k_{n+1}}|^{\alpha} \mu_\alpha(I_{k_1\ldots k_{n+1}}) \right]}{\log |I_{k_1\ldots k_n \kappa_{n+1}}|}.
\]

Since

\[
|I_{k_1\ldots k_{n+1}}| < \left( \frac{1}{2\zeta(2)} \right)^n,
\]

it follows for each $\kappa > 0$ and for all $n$ sufficiently large,

\[
\frac{\log 2^{2\alpha} \zeta(2\alpha)^{\alpha}}{\log |I_{k_1\ldots k_n \kappa_{n+1}}|} > -\kappa.
\]

Clearly, we even have that the limit of the latter expression is equal to 0. This settles the second term in the final line in the above calculation. The third term is more subtle, and for this we proceed as follows. Using (1) and (2), we derive with the convention $k_0 \equiv 0$,

\[
|I_{k_1\ldots k_n}|^\alpha = \prod_{i=0}^{n-1} \left( \frac{1}{2^{2\alpha} \zeta(2\alpha)^{\alpha} |k_{i+1} - k_i|^{2\alpha}} \right).
\]

Similarly, using the recursive definition of $\mu_\alpha$, we obtain

\[
\mu_\alpha(I_{k_1\ldots k_n}) = \prod_{i=0}^{n-1} \left( \frac{1}{2^{2\alpha} \zeta(2\alpha)} \left[ \frac{1}{|k_{i+1} - k_i|^{2\alpha}} + \frac{1}{(k_{i+1} + k_i)^{2\alpha}} \right] \right).
\]
Before we start with the discussion of the second case, first note that since

\[ \text{Furthermore, since } \alpha \text{ tends to 1 as } \kappa \text{ tends to 0.} \]

Before we start with the discussion of the second case, first note that since

\[ 0 < \left| \frac{k_{i+1} - k_i}{k_{i+1} + k_i} \right|^{2\alpha} \leq 1, \text{ we clearly always have} \]

\[ \frac{1}{n} \sum_{i=0}^{n-1} \log \left( 1 + \left| \frac{k_{i+1} - k_i}{k_{i+1} + k_i} \right|^{2\alpha} \right) \leq \log 2. \quad (5) \]

Furthermore, since \( k_n \) tends to infinity, there exists \( j(\kappa) \in \mathbb{N} \) such that

\[ \log \frac{\kappa^2 k_i^2}{4} > \frac{2 \log 2}{\kappa^2}, \text{ for all } i \geq j(\kappa). \quad (6) \]
Let us now come to the second case. That is, we now assume that for some \( n \in \mathbb{N} \) we have
\[
\frac{1}{n} \sum_{i=0}^{n-1} \log \left( 1 + \left[ \frac{|k_{i+1} - k_i|}{k_{i+1} + k_i} \right]^{2\alpha} \right) \geq \kappa.
\]
Since \( x > \log(1+x) \) for all \( x > 0 \), we then have
\[
\frac{1}{n} \sum_{i=0}^{n-1} \left[ \frac{|k_{i+1} - k_i|}{k_{i+1} + k_i} \right]^{2\alpha} \geq \kappa.
\]
Let us make the following two observations. Firstly, using the fact that
\[
0 < \left[ \frac{|k_{i+1} - k_i|}{k_{i+1} + k_i} \right]^{2\alpha} \leq 1,
\]
we can apply Chebyshev’s Inequality, which gives that for \( n \) sufficiently large,
\[
\text{card } I_n > \kappa n,
\]
where
\[
I_n := \left\{ i \in [j(\kappa), n] : \left[ \frac{|k_{i+1} - k_i|}{k_{i+1} + k_i} \right]^{2\alpha} \geq \frac{\kappa}{2} \right\}.
\]
Secondly, note that for \( 1/2 < \alpha < 1 \) the following implication holds.
If \( \left[ \frac{|k_{i+1} - k_i|}{k_{i+1} + k_i} \right]^{2\alpha} \geq \frac{\kappa}{2} \), then \( |k_{i+1} - k_i| > \frac{\kappa}{2} k_i \).
Combining these two observations with (6), we then compute
\[
\frac{1}{n} \sum_{i=0}^{n-1} \log |k_{i+1} - k_i|^{2\alpha} \geq \frac{\alpha}{n} \sum_{i \in I_n} \log |k_{i+1} - k_i|^2 \geq \frac{\alpha}{n} \sum_{i \in I_n} \log \frac{\kappa^2 k_i^2}{4}
\]
\[
\geq \frac{\alpha}{n} \sum_{i \in I_n} \frac{2 \log 2}{\kappa^2} > \frac{\log 2}{\kappa^2 n} \text{card } I_n > \frac{\log 2}{\kappa}.
\]
Inserting this into our estimate above and using (5), it follows
\[
\log(2^\zeta(2\alpha)) - \alpha \log(2^\zeta(2)) - \frac{1}{n} \sum_{i=0}^{n-1} \log \left( 1 + \left[ \frac{|k_{i+1} - k_i|}{k_{i+1} + k_i} \right]^{2\alpha} \right)
\]
\[
\geq \frac{n+1}{n} \log(2^\zeta(2)) + \frac{1}{n} \sum_{i=0}^{n-1} \log |k_{i+1} - k_i|^{2\alpha}
\]
\[
\geq \frac{- \log 2}{\log(2^\zeta(2)) + \log \frac{2}{\kappa}} = \frac{\kappa}{\kappa} \log(2^\zeta(2)) + \log \frac{2}{\kappa}
\]
\[
\geq -\kappa.
\]
This finishes the second case.
Combining the latter results with (4), and putting \( \epsilon := 3\kappa \), we have now shown that
\[
\liminf_{n \to \infty} \frac{\log \mu_\alpha(I_{k_1 \ldots k_n})}{|I_{k_1 \ldots k_n}|} \geq -\epsilon,
\]
and hence,
\[
\liminf_{n \to \infty} \frac{\log \mu_\alpha(I_{k_1 \ldots k_n})}{\log |I_{k_1 \ldots k_n}|} \geq \alpha - \epsilon,
\]
Since \( \alpha \) tends to 1 for \( \epsilon \) tending to 0, the proof is complete.
2.5. The proof of the Main Theorem.

Recall that the lower pointwise dimension $d_{\nu}(x)$ of a Borel measure $\nu$ on $\mathbb{R}$ at a point $x \in \mathbb{R}$ is given by

$$d_{\nu}(x) := \liminf_{r \to 0} \frac{\log \nu(B(x, r))}{\log r},$$

where $B(x, r)$ refers to the interval centred at $x$ with diameter equal to $2r$.

The idea is to apply the well-known Mass Distribution Principle of Frostman [8] and Billingsley [2] (see also e.g. [7]).

In order to be able to apply the Mass Distribution Principle, we still require the following straightforward generalization of Furstenberg’s Lemma [9].

**Lemma 2.8.** Let $\nu$ be a Borel measure on $\mathbb{R}$, and let $(r_n)$ be a sequence of positive numbers for which $\lim_{n \to \infty} r_n = 0$ and $\lim_{n \to \infty} (\log r_{n+1}/\log r_n) = 1$. We then have for every $x \in \mathbb{R}$,

$$d_{\nu}(x) = \liminf_{n \to \infty} \frac{\log \nu(B(x, r_n))}{\log r_n}.$$

**Proof.** For $r > 0$ we define $n = n(r) := \max\{k \in \mathbb{N} : r_k \geq r\}$. The assertion of the lemma is then an immediate consequence of the following simple calculation.

$$\frac{\log \nu(B(x, r))}{\log r} \geq \frac{\log \nu(B(x, r_n))}{\log r_{n+1}} = \frac{\log r_n}{\log r_{n+1}} \frac{\log \nu(B(x, r_n))}{\log r_n}.$$

□

**Proof of the Main Theorem.** Let $\epsilon > 0$ be given, and then fix $1/2 < \alpha < 1$ and $\gamma > 1/(2\alpha - 1)$ as in Proposition 2.7. By Lemma 2.8 we have that in order to find a lower bound for $\dim_H(\mu_\alpha)$ it is sufficient to give an estimate for $d_{\mu_\alpha}(x)$ from below, for each $x = \rho(k_1, k_2, \ldots) \in C_\gamma$. For this note that Proposition 2.7 implies

$$\liminf_{n \to \infty} \frac{\log \mu_\alpha(I_{k_1\ldots k_n})}{\log |I_{k_1\ldots k_n}|} \geq \alpha - \epsilon.$$

In order to deduce the desired lower bound for $d_{\mu_\alpha}(x)$, we then use [3] and the definition of $C_\infty$, which gives for $r_n := |I_{k_1\ldots k_n}|$,

$$\lim_{n \to \infty} \frac{\log r_{n+1}}{\log r_n} = 1 + \lim_{n \to \infty} \frac{\log |I_{k_1\ldots k_n}|}{\log |I_{k_1\ldots k_n}|} \log n \log(2\zeta(2)) = 1.$$

Therefore, Lemma 2.8 implies that for each $x \in C_\infty$,

$$d_{\mu_\alpha}(x) \geq \alpha - \epsilon.$$

Combining this with Corollary 2.2, Corollary 2.3 and Lemma 2.8 we have by the Mass Distribution Principle,

$$\dim_H(C) \geq \dim_H(C_\infty) \geq \dim_H(C_\infty^\gamma) \geq \dim_H(\mu_\alpha) \geq \alpha - \epsilon.$$

Finally, note that by Proposition 2.7 we have that $\alpha$ tends to 1 for $\epsilon$ tending to 0. This completes the proof of the theorem.
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