Extreme events on complex networks

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We study the extreme events taking place on complex networks. The transport on networks is modelled using random walks and we compute the probability for the occurrence and recurrence of extreme events on the network. We show that the nodes with smaller number of links are more prone to extreme events than the ones with larger number of links. We obtain analytical estimates and verify them with numerical simulations. They are shown to be robust even when random walkers follow shortest path on the network. The results suggest a revision of design principles and can be used as an input for designing the nodes of a network so as to smoothly handle an extreme event.

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Extreme events (EE) taking place on the networks is a fairly common place experience. Traffic jams in roads and other transportation networks, web servers not responding due to heavy load of web requests, floods in the network of rivers, power black outs due to tripping of power grids are some of the common examples of EE on networks. Such events can be thought of as an emergent phenomena due to the transport on the networks. As EE lead to losses ranging from financial and productivity to even of life and property [1], it is important to estimate probabilities for the occurrence of EE and, if possible, incorporate them to design networks that can handle such EE.

Transport phenomena on the networks have been studied vigorously in the last several years [2, 3] though they were not focussed on the analysis of EE. However, one kind of extreme event in the form of congestion has been widely investigated [4]. For instance, a typical approach is to define rules for (a) generation and transport of traffic on the network and (b) capacity of the nodes to service them. Thus, a node will experience congestion when its capacity to service the incoming ‘packets’ has been exceeded [5]. In this framework, several results on the stability of networks, cascading failures to congestion transition etc. have been obtained. Extreme event, on the other hand, is defined as exceedences above a prescribed quantile and is not necessarily related to the handling capacity of the node in question. It arises from natural fluctuations in the traffic passing through a node and not due to constraints imposed by capacity. Thus, in rest of this paper, we discuss transport on the networks and analyse the probabilities for the occurrence of EE arising in them without having to model the dynamical processes or prescribe capacity at each of the nodes.

The transport model we adopt in this work is the random walk on complex networks [3]. Random walk is of fundamental importance in statistical physics though in real network settings many variants of random walk could be at work [6]. For instance, in the case of road traffic, the flow typically follows a fixed, often shortest, path from node $A$ to $B$ and can be loosely termed deterministic. As we show in this paper, thresholds and corresponding probabilities for the EE depend on such details as the operating principle of the network. Thus, given the operational principle of network dynamics, i.e., deterministic or probabilistic or a combination of both, can the nodes of the network be designed to have sufficient capacity to smoothly handle EE of certain magnitude? We show that we can obtain apriori estimates for the volume of transport on the nodes given the static parameters and operating principle of the network. Currently, for univariate time series, there is a widespread interest on the extreme value statistics and their properties, in particular in systems that display long memory [7]. Thus, we place our results in the context of both the random walks and EE in a network setting.

We consider a fully connected, undirected, finite network with $N$ nodes with $E$ edges. The links are described by an adjacency matrix $A$ with whose elements $A_{ij}$ are either 1 or 0 depending on whether $i$ and $j$ are connected by a link or not respectively. On this network, we have $W$ non-interacting walkers performing the standard random walk. A random walker at time $t$ sitting on $ith$ node with $K_i$ links can choose to hop to any of the neighbouring nodes with equal probability. Thus, transition probability for going from $ith$ to $jth$ node is $A_{ij} / K_i$. We can write down a master equation for the $n$–step transition probability of a walker starting from node $i$ at time $n = 0$ to node $j$ at time $n$ as,

$$P_{ij}(n + 1) = \sum_k \frac{A_{kj}}{K_k} P_{ik}(n)$$

(1)

It can be shown that the $n$–step time-evolution operator corresponding to this transition, acting on an initial distribution, leads to stationary distribution with eigenvalue unity [3] and it turns out to be

$$\lim_{n \to \infty} P_{ij}(n) = p_j = \frac{K_j}{2E}$$

(2)

The existence of stationary distribution is crucial for
defining EE. Physically, the time-independent probability in Eq. \( \frac{2}{n} \) implies that more walkers will visit a given node if it has more links.

Now we can obtain the distribution of random walkers on a given node. We ask for the probability \( f(w) \) that there are \( w \) walkers on a given node having degree \( K \). Since the random walkers are independent and non-interacting, the probability of encountering \( w \) walkers at a given node is \( p^w \) while rest of \( W - w \) walkers are distributed on all the other nodes. This turns out to be binomial distribution given by

\[
f(w) = \binom{W}{w} p^w (1 - p)^{W-w}.
\] (3)

Now, the mean and variance for a given node can be explicitly written down as

\[
\langle f \rangle = \frac{WK}{2E}, \quad \sigma^2 = W \frac{K}{2E} \left( 1 - \frac{K}{2E} \right). \tag{4}
\]

Quite as expected, the mean and the variance depends on the degree of the node for fixed \( W \) and \( E \). Note that \( K/2E << 1 \) and the relation between the mean and the variance for walkers passing through node can be written as \( \sigma \approx \langle f \rangle^{1/2} \). This reproduces the relation proposed in Ref. \[10\], later shown to have limited validity.

One natural extension of the result in Eq. \( \frac{2}{n} \) is to account for fluctuations in the number of walkers. We assume that the total number of walkers is a random variable uniformly distributed in the interval \([W - \Delta, W + \Delta]\). Then the probability of finding \( w \) walkers becomes

\[
f^\Delta(w) = \sum_{j=0}^{2\Delta} \frac{1}{2\Delta + 1} \binom{\tilde{W} + j}{w} p^w (1 - p)^{\tilde{W} + j - w}, \tag{5}\]

where \( \tilde{W} = W - \Delta \). The mean and variance of this distribution can be obtained as,

\[
\langle f^\Delta \rangle = \langle f \rangle, \quad \sigma^2_{\Delta} = \langle f^\Delta \rangle \left[ 1 + \langle f^\Delta \rangle \left\{ \frac{\Delta^2}{3W^2} + \frac{\Delta}{3W^2} - \frac{1}{W} \right\} \right]. \tag{6}
\]

In the spirit of extreme value statistics, an extreme event is one whose probability of occurrence is small, typically associated with the tail of the probability distribution function. In the network setting, we will apply the same principle to each of the nodes. Based on Eqns. \( \frac{2}{n} \) and \( \frac{1}{n} \) we will designate an event to be extreme if more than \( q \) walkers traverse a given node at any time instant. Notice that necessarily the cut-off \( q \) will have to depend on the node (or rather, the traffic flowing through the node) in question. Applying uniform threshold independent of the node will lead to some nodes always experiencing an extreme event while some others never encountering any extreme event at all. Hence we define the threshold for extreme event to be \( q = \langle f \rangle + m\sigma \), where \( m \) is any real number. Then, the probability for extreme event can be obtained as

\[
F(K) = \sum_{j=0}^{2\Delta} \frac{1}{2\Delta + 1} \sum_{k=|q|+1}^{\tilde{W} - j} \binom{\tilde{W} + j}{k} p^k (1 - p)^{\tilde{W} - j - k}, \tag{8}\]

where \( \lfloor u \rfloor \) is the floor function defined as the largest integer not greater than \( u \).

It does not seem possible to write this summation in closed form. However, for the special case when \( \Delta = 0 \) Eq. \( \frac{8}{n} \) simplifies to

\[
F(K) = \sum_{k=|q|+1}^{W} f(K) = I_p(|q| + 1, w - |q|) \tag{9}\]

where \( I_p(\ldots) \) is the regularized incomplete Beta function \[11\]. For a given choice of network parameter \( E \) and number of walkers \( W \), the extreme event probability at any node depends only on its degree. In Fig.\[1\] we show \( F(K) \) as a function of degree \( K \) superimposed on the results obtained from random walk simulations. The agreement between Eq. \[1\] and the simulated results is quite good. Further, each point in the figure represents an average over all the nodes with the same degree. We emphasise that the oscillations seen in Fig.\[1\] are inherent in the analytical and numerical results and not due to insufficient ensemble averaging.

An important feature of this result is that the nodes with smaller degree \( K < 20 \) reveal, on an average, higher probability for the occurrence of EE as compared to the nodes with higher degree, say, \( K > 100 \). By careful choice of parameters, the probability \( F(K) \) can differ
suggests that it might be scaling with respect to the recurrence of EE at all the nodes. The graph in Fig 2(a) would be extreme and we see an equal probability of occurrence. In the limiting case of larger number of EE and hence higher probability of occurrence, the threshold for EE decreases and this leads to a fluctuating quantity. It must be pointed out that the choice of threshold for extreme event probability in Eq 9 scales with the choice of parameter that characterises the recurrence distribution, as a function of degree $K$.

As pointed out before, many types of flow on the network, such as the information packets flowing through the network of routers and traffic on roads, use more intelligent routing algorithms [12] rather than performing a random walk. In order to check the robustness of results in Eq. 8 we implemented the random walk simulation with the constraint that the traffic from node $i$ of EE,

$$\frac{F_m(K)}{K^{1-S_m}} = \text{constant}$$  \hspace{1cm} (10)$$

where $F_m(K)$ represents extreme event probability for threshold value $q$ with parameter $m$. In this, $S_m$ is the slope of the curves $F_m(K)$ in the Fig. 2(a). In Fig. 2(b), we show the effect of scaling for several choices of $q$. Using Eq. 10 on the simulated data for $\Delta = 0$, we find that all the curves for the probability of EE collapse into one curve to a good approximation.

In the study of EE, distribution of their return intervals is an important quantity of interest. This carries the signature of the temporal correlations among the EE and is useful for hazard estimation in many areas. We focus on the return intervals for a given node of the network. Since the random walkers are non-interacting, the events on the node are uncorrelated. Then, the recurrence time distribution is given by $P(\tau) = e^{-\tau/\langle\tau\rangle}$, where the mean recurrence time is $\langle\tau\rangle = 1/F(K)$. In the inset of Fig. 3 we show the recurrence time distribution obtained from random walk simulations for three nodes which have different degrees. In semi-log plot, they reveal an excellent agreement with the analytical distribution $P(\tau)$ (shown as solid line). The main graph of Fig. 3 shows the mean recurrence time $\langle\tau\rangle$, the only parameter that characterises the recurrence distribution, as a function of the degree and it agrees with the analytical result.

As pointed out before, many types of flow on the network, such as the information packets flowing through the network of routers and traffic on roads, use more intelligent routing algorithms [12] rather than performing a random walk. In order to check the robustness of results in Eq. 8 we implemented the random walk simulation with the constraint that the traffic from node $i$
to \( j \) takes the shortest path on the network. If multiple shortest paths are available to go from node \( i \) to \( j \), the algorithm chooses any one of them with equal probability. Thus, in this setting, for every random choice of source-destination pair the paths are laid out by the algorithm and randomness arises only when multiplicity of shortest paths are available. In this sense, this can be thought of as walk with a large deterministic component. We used the shortest path algorithm developed in Ref. \cite{8}. The simulation results, shown in Fig. 4 as solid circles, are qualitatively similar to the trend displayed in Fig. 1. In this scenario of predominantly deterministic dynamics due to shortest paths constraint, it is conceivable that the degree of a node does not determine the flux passing through it. This role is played by the centrality of the node with respect to the shortest paths in the network and this is quantified by the betweenness centrality \( b \) of a given node \( K \). Based on this qualitative argument, the results in Fig. 4 can be understood if we replace Eq. 2 with \( p = \beta b / B \) where \( B \) is normalisation factor that depends on the sum of betweenness centrality of all the nodes on the network. From the numerical simulations, we obtain \( \beta \approx 0.94 \). Using this \( p \) in Eq. 2 we can go through the same set of arguments as before and obtain \( \langle f \rangle, \sigma^2, q \) and the probability for occurrence of EE \( F_{up}(b) \) analytically. In Fig 3(a) \( F_{sp}(b) \) is shown as solid line. In Fig 3(b) the same data for \( F_{sp}(b) \) is shown as a function of \( K \) for easier comparison with Fig 1. Thus, even with the shortest path algorithm thrown in, the extreme event probabilities are higher for the nodes with lower degree \( (K < 20) \) than for the ones with higher degree \( (K > 100) \).

Finally we comment on how these results can be applied as a basis to design nodes of a network. The central result in this paper in Eq. 3 allows us to \textit{apriori} estimate the extreme event probabilities. These estimates depend on whether operating principle of dynamics can be modelled as a purely random walk or on the basis of shortest paths. If the idea is to avoid congestion or any other problems arising due to EE of certain magnitude, then these estimates can be used as an input to the design principles for the nodes. For instance, for the road traffic that operates broadly on the shortest path principle the probabilities can be used as an input to design principles (such as higher capacity to nodes) that will avoid bottlenecks arising from EE of a given magnitude.

In scale-free networks, low degree nodes, which are more prone to experience the EE, form the bulk. But design principles and practice generally focus on the hubs. The results in this work suggest that a revision from such an approach is necessary. A careful design for the capacity of low degree nodes needs to be given equal importance. It must be emphasised that incorporating such extreme event estimates in design principles will only help in better preparedness to meet the expected extreme event. The extreme events discussed here being due to inherent fluctuations will nevertheless take place and cannot be avoided.

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