Spectral locus interpolation with splines in optical instruments

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Abstract. The article deals with the problem of representation of the spectral locus curve on the CIE xy chromaticity diagram. The issue is that spectral locus is specified at discrete set of points corresponding to given set of wavelengths of monochromatic colors. Thus, the problem appears which way to calculate chromaticity coordinates for any other wavelength. Three different methods how to interpolate the spectral locus were considered. The method we proposed to build an interpolant curve is based on using of Bezier curves. This approach allows to calculate chromaticity coordinates for any arbitrary wavelength directly on the CIE xy chromaticity diagram. Obtained analytical expressions for the spectral locus curve will increase the evaluation accuracy of the luminous radiation dominant wavelength value which defines the hue and the saturation value of the emitting color and other things.

1. Introduction
In 1931 the International Commission on Illumination (CIE) established [1] the spectral tristimulus values \( \bar{x}(\lambda), \bar{y}(\lambda), \bar{z}(\lambda) \) of the constant power spectral emissions for the wavelengths \( \lambda[i] \) of the monochromatic colors, from 380 nm to 780 nm. These tristimulus values were tabulated with the step \( \Delta \lambda = 5 \) nm, \( i = 0, 1, \ldots, N_i-1 \). The total number \( N_i \) of the tabulated spectral tristimulus values \( \bar{x}(\lambda), \bar{y}(\lambda), \bar{z}(\lambda) \) is \( N_i=81 \).

According to that tabulated data for each wavelength value \( \lambda[i] \) of the monochromatic color its correspondent chromaticity coordinates \( \{x[i],y[i] \} \) can be defined directly. Although the coordinate \( z[i] \) is usually omitted in colorimetric computations because it can be easily defined with the other two coordinates: \( z[i] = 1 - x[i] - y[i] \). The chromaticity coordinates set \( \{x[i],y[i] \} \) for \( \lambda \in [380,780] \) with a step \( \Delta \lambda \) forms a point specified contour on the XY-plane, these points are nodes for interpolation procedure [2, 3] that enables to obtain the spectral locus continuous curve for the monochromatic colors which defines the color locus area. The straight line between two endpoints on the XY-plane corresponding the wavelengths 380 nm and 780 nm is called the line of purple colors (LPC).

An interpolation of a set of points \( \{x[i],y[i] \} \) must provide [4,5] a possibility to obtain the interim values of the \((x,y)\)-chromaticity coordinates of monochromatic colors with an arbitrary wavelength step which is not equal to 5 nm according to the CIE standards. Combining those values a continuous and «smooth» spectral locus graph can be obtained. A task to interpolate the tabulated spectral locus is
very important for increasing an accuracy of colorimetric computations.

2. Spectral locus interpolation approaches

The primary requirement for an interpolation method is the first derivative continuity of the interpolating function in the specified nodal points \((x[i], y[i])\) except for the endpoints \((x[0], y[0])\) and \((x[80], y[80])\). One of the effective approaches to solve the interpolation problem is the approach when a spline function is built \([6,7]\). In the colorimetry there are three different approaches to build a graph which interpolates the spectral locus (figure 1):

1) interpolation of the tabulated CIE XYZ tristimulus values — when interpolation is done with three tabulated sequences of nodal points \((\lambda[i], x[i]), (\lambda[i], y[i]), (\lambda[i], z[i])\) separately and then the \((x, y)\)-chromaticity coordinate values of the monochromatic colors are obtained;

2) interpolation of the \((x, y)\)-chromaticity coordinates — interpolation procedure is performed with two sequences of the nodal points \((\lambda[i], x[i]), (\lambda[i], y[i])\) separately;

3) the interpolation of the tabulated spectral locus nodal points \((x[i], y[i])\) directly on XY-plane to built a curve with some distinctive form features.

![Figure 1. The colorimetric curves accepted by the CIE (The CIE 1931 — solid line; the CIE 1964 — dotted line, the variable index is «10»), P – purple, B – blue, S – sky blue, G – green, Y-G – yellowish green, Y – yellow, O – orange, R – red color.)](image-url)

3. The solution of the interpolation problem for the spectral locus

To solve the interpolation problem using the first and the second approaches mentioned above the standard method of spline interpolation can be used, cubic spline interpolation technique in particular [8]. Main disadvantage of the cubic spline technique is that resulting interpolant curve usually has undesirable «oscillations».

There is an alternative and original approach to interpolate the spectral locus curve represented as a trajectory which is drawn on the XY-plane by chromaticity coordinates point when it moves according to wavelength change \(\lambda[i]\) from 380 nm to 780 nm. To interpolate spectral locus directly on XY-plane, Bezier interpolation \([9,10]\) technique seems to be useful tool. Locally specified Bezier curves may be calculated and then jointed with each other in the interpolation nodes to create a joint Bezier spline for spectral locus curve. For that purpose quadratic and cubic Bezier curves are used.

An interpolating function which is used to build locus curve between wavelengths from \(\lambda[i]\) to \(\lambda[80]\).
\[ \lambda[i+1] \] on the basis of the quadratic Bezier curve must be represented as:

\[
\begin{align*}
x(\lambda) &= t_1^2(\lambda)x[i] + 2t_1(\lambda)t_2(\lambda)x_n[i] + t_2^2(\lambda)x[i+1] \\
y(\lambda) &= t_1^2(\lambda)y[i] + 2t_1(\lambda)t_2(\lambda)y_n[i] + t_2^2(\lambda)y[i+1],
\end{align*}
\]

where \(x_n[i], y_n[i], x_n[i], y_n[i] \) - control point coordinates \(Q[i]\), \(P_1[i]\) and \(P_2[i]\) on the \(XY\)-plane which define the interpolating function properties in each interval;

\[
t_1(\lambda) = (\lambda - \lambda[i])/(\lambda[i+1] - \lambda[i]); \quad t_2(\lambda) = 1 - t_1(\lambda) = (\lambda[i+1] - \lambda)/(\lambda[i+1] - \lambda[i]).
\]

An interpolating function which is used to build locus curve between wavelengths from \(\lambda[i]\) to \(\lambda[i+1]\) on the basis of the cubic Bezier curve must be represented as:

\[
\begin{align*}
x(\lambda) &= t_1^3(\lambda)x[i] + 3t_1^2(\lambda)t_2(\lambda)x_n[i] + 3t_1(\lambda)t_2^2(\lambda)x_n[i] + t_2^3(\lambda)x[i+1] \\
y(\lambda) &= t_1^3(\lambda)y[i] + 3t_1^2(\lambda)t_2(\lambda)y_n[i] + 3t_1(\lambda)t_2^2(\lambda)y_n[i] + t_2^3(\lambda)y[i+1].
\end{align*}
\]

The computation of the control point coordinates \(Q[i]\), \(P_1[i]\) and \(P_2[i]\) includes the precomputation of the angle of the tangent in each nodal point \((x[i], y[i])\), defining desired interpolating curve using the formula:

\[ \alpha[i] = angle(x[i+1] - x[i-1], y[i+1] - y[i-1]), i = 1, 2, ... , N_\lambda - 2, \]

where the function \(angle(x,y)\) computes the angle(in radians) of the directing vector which starts in the \(XY\)-plane coordinate origin and terminates in the point defined with the coordinates \((x,y)\), \(\alpha[i] \in [0, 2\pi]\). In the \(i\)-th interpolation interval the starting tangent with the inclination angle \(\alpha[i]\) to the \(X\) axis must be made through the point \((x[i], y[i])\) of the interpolation interval beginning and the terminating tangent with the inclination angle \(\alpha[i+1]\) to the \(X\) axis must be made through the point \((x[i+1], y[i+1])\) of the interpolation interval ending.

The selection method of the Bezier curve type and its construction method are shown in figure 2 using as an example two parts of a spectral locus graph.

\(\lambda[515]\) of 515 nm corresponds to the nodal point \((x[i], y[i])\) through which a starting tangent is drawn as shown in figure 2.a. The starting tangent is a line which is parallel to the line...
(shown in figure 2a, as a dotted line) which connects the point \((x[i-1], y[i-1])\) for \(\lambda[i-1]=510\) nm and the point \((x[i+1], y[i+1])\) for \(\lambda[i+1]=520\) nm. The terminating tangent which intersects the spectral locus in the point \((x[i+1], y[i+1])\) for \(\lambda[i+1]=520\) nm is a line which is parallel to the line (shown in figure 2a, as a dotted line) which connects the point \((x[i], y[i])\) for \(\lambda[i]=515\) nm and the point \((x[i+2], y[i+2])\) for \(\lambda[i+2]=525\) nm. In figure 2 the starting (ST) and terminating tangents (TT) are shown by dot and dash lines.

The wavelength \(\lambda[i]\) of 390 nm corresponds to the nodal point \((x[i], y[i])\) in figure 2b through which the starting tangent is drawn. The starting tangent is a line which is parallel to the line (shown in figure 2b, as a dotted line) which connects the point \((x[i-1], y[i-1])\) for \(\lambda[i-1]=385\) nm and the point \((x[i+1], y[i+1])\) for \(\lambda[i+1]=395\) nm. The terminating tangent which intersects the spectral locus in the point \((x[i+1], y[i+1])\) for \(\lambda[i+1]=395\) nm is a line which is parallel to the line (shown in figure 2b, as a dotted line) passing through two points \((x[i], y[i])\) for \(\lambda[i]=390\) nm and \((x[i+2], y[i+2])\) for \(\lambda[i+2]=400\) nm.

To select the Bezier curve type the supporting line inclination angle must be calculated between the nodal points \(\{x[i], y[i]\}\) and \(\{x[i+1], y[i+1]\}\):

\[
\beta[i] = \angle(x[i+1] - x[i], y[i+1] - y[i]), i = 0, 1, \ldots, N_i - 2. 
\]

(4)

For the spectral locus ending nodal points in the expressions (3) and (4) the edge conditions must be assumed: \(\alpha[0]=\beta[0]\), \(\alpha[N_i-1]=\beta[N_i-2]\). In that way in each of the \(i\)-th interpolation interval which correspond to the wavelength \(\lambda[i]\) three parameters must be assigned: the angle \(\alpha[i]\) of the starting tangent inclination to the desired curve in the interval beginning; the angle \(\beta[i]\) of the supporting line (SL) inclination between the nodal points \(\{x[i], y[i]\}\) and \(\{x[i+1], y[i+1]\}\); the angle \(\alpha[i+1]\) of the terminating tangent inclination to the desired curve in the interval ending.

The reciprocal ratios of the parameters \(\alpha[i]\), \(\beta[i]\), \(\alpha[i+1]\) determine the choice of the Bezier curve type to interpolate a part of the curve in a particular interval: a quadratic Bezier curve with a single control point \(Q[i]\); a cubic Bezier curve with two control points \(P_1[i]\) and \(P_2[i]\).

The control point \(P_1[i]\) lies on the starting tangent at a distance of \(L_1[i]\) from the nodal point with coordinates \((x[i], y[i])\). The control point \(P_2[i]\) lies on the terminating tangent at a distance of \(L_1[i]\) from the point \((x[i+1], y[i+1])\). The control points in figure 2 designated with a mark \(\bullet\), the nodal points — with a mark \(\circ\). The control points coordinates \(P_1[i]\) and \(P_2[i]\) for cubic curve (2) must be preliminarily calculated with the formulas:

\[
x_{\eta}[i] = x[i] + L_1[i] \cos(\alpha[i]), \quad y_{\eta}[i] = y[i] + L_1[i] \sin(\alpha[i]), 
\]

(5)

\[
x_{\eta}[i] = x[i+1] - L_1[i] \cos(\alpha[i+1]), \quad y_{\eta}[i] = y[i+1] - L_1[i] \sin(\alpha[i+1]). 
\]

(6)

When the control points \(P_1[i]\) and \(P_2[i]\) are found on the different sides from the supporting line which is drawn through the nodal points \((x[i], y[i])\) and \((x[i+1], y[i+1])\) the preliminary control points must be finally set in the positions (5), (6) and in this interval a cubic Bezier curve must be drawn. In other case, when preliminarily calculated control points \(P_1[i]\) and \(P_2[i]\) are found on the same side (both in the upper part, in the lower part, in the left part or in the right part) from the supporting line, then calculating of a cubic Bezier curve become pointless. In this case one control point \(Q[i]\) must be assigned instead of two preliminarily calculated. This new point is the point where the starting and terminating tangents are crossed. As the result, a quadratic Bezier curve must be build.

The control point coordinates \(Q[i]\) which are used to interpolate spectral locus by a quadratic curves (2) must be calculated with the following formula:

\[
x_{\varphi}[i] = \left\lfloor \frac{b[i+1] - b[i]}{\tan(\alpha[i]) - \tan(\alpha[i+1])} \right\rfloor \tan(\alpha[i]) \neq \tan(\alpha[i+1]), 
\]

(7)

\[
y_{\varphi}[i] = \tan(\alpha[i])x_{\varphi}[i] + b[i], \quad b[i] = y[i] - \tan(\alpha[i])x[i]. 
\]

(8)

The control points position (5), (6) and (7), (8) on the starting and terminating tangents enables to
combine the Bezier curves in the nodal points to ensure the continuity of the interpolating function first derivative. The decision function \( n[i] \) which enables a preferred selection of the quadratic (2) or the cubic (3) interpolated curve in the \( i \)-th interpolation interval may be represented as:

\[
n[i] = \begin{cases} 
(\beta[i]) (x_{n} [i] - x[i]) + y[i] - y_{n} [i] & \text{if } n[i] > 0 \\
(\beta[i]) (x_{n} [i] - x[i]) + y[i] - y_{n} [i] & \text{if } n[i] \leq 0 
\end{cases}
\]  

(9)

If \( n[i] > 0 \) a quadratic Bezier curve must be selected and if \( n[i] \leq 0 \) — a cubic Bezier curve must be selected. As a result an interpolating function system which defines the spectral locus on the \( XY \)-plane must be represented as:

\[
\begin{align*}
x(\lambda) &= \begin{cases} 
t_2^2 (\lambda) x[i] + 2 t_1 (\lambda) t_2 (\lambda) x_{n} [i] + t_1^2 (\lambda) x[i+1], & n[i] > 0 \\
t_2^2 (\lambda) x[i] + 3 t_1 (\lambda) t_2 (\lambda) x_{n} [i] + 3 t_1^2 (\lambda) t_2 (\lambda) x_{n} [i], & n[i] \leq 0 
\end{cases} \\
y(\lambda) &= \begin{cases} 
t_2^2 (\lambda) y[i] + 2 t_1 (\lambda) t_2 (\lambda) y_{n} [i] + t_1^2 (\lambda) y[i+1], & n[i] > 0 \\
t_2^2 (\lambda) y[i] + 3 t_1 (\lambda) t_2 (\lambda) y_{n} [i] + 3 t_1^2 (\lambda) t_2 (\lambda) y_{n} [i], & n[i] \leq 0 
\end{cases}
\end{align*}
\]  

(10)

Definition of the distances \( L_{0}[i] \) and \( L_{2}[i] \) may impact significantly the interpolating function incurrence in the intervals with a cubic Bezier curve is formed. The spectral locus interpolation problem in the given method can be solved with the following expressions: \( L_{0}[i] = L[i-1] + L[i] / (L[i-1] + L[i]) \), \( L_{2}[i] = L[i-1] + L[i] + L[i+1], i = 0, 1, \ldots, N_i - 2 \).

The value \( L[i] \) is a Euclidian distance between the points \((x[i], y[i])\) and \((x[i+1], y[i+1])\) on the \( XY \)-plane. For the starting and terminating spectral locus interpolation intervals the following distance is used: \( L[0] = L[N_i - 2] = L[N_i - 2] = L[N_i - 2] / 2 \).

In figure 2 the nodal points are connected with straight solid lines and that connection method gives a possibility to evaluate visually the proposed interpolation method efficiency in comparison with the method of linear interpolation on \( XY \)-plane. The spectral locus interpolating curves obtained with three different methods are shown in figure 3 at a larger scale in order to reveal some features of each method of interpolation.

**Figure 3.** The spectral locus interpolation results: a) in the band 380 nm - 420 nm, b) in the band 510 nm - 525 nm (the dotted curve is obtained with the method of CIE XYZ tristimulus values interpolation by cubic splines, the pointed curve — the chromaticity coordinates data interpolation by cubic splines, the solid curve — the proposed interpolation method on \( XY \)-plane using Bezier splines).
The most important point of interest is an analysis of spectral intervals where the position of the spectral locus nodal points tabulated in the CIE standard corresponds to the curve parts of the function maximum curvature: at the curve beginning in the spectral band from 380 nm to 420 nm; in the curve part which corresponds to the wavelength from 515 nm to 525 nm. The analysis of figure 3 shows that the application of the spline interpolation method in the tristimulus values data (method 1) and the chromaticity coordinates data (method 2) causes «space oscillations» in the interpolating function at the beginning of the visible spectral band wavelength 380 nm – 415 nm. For the spectral interval 515–525 nm where curvature variations of the spectral locus are significantly less, all three methods seem to be nearly equivalent.

4. Conclusion
As a result for the spectral locus interpolation, the authors propose the system (10) which defines analytical expressions for interpolating function allowing to calculate \((x, y)\)-chromaticity coordinates for monochromatic colors with any given wavelength \(\lambda\) in the visible band 380 nm – 780 nm.

The solution to the spectral locus interpolation problem is obviously useful for increasing the accuracy of the quantitative colorimetric computations. In particular, the analytical expressions for the spectral locus representation will increase the evaluation accuracy of the luminous radiation dominant wavelength value which defines the hue and the saturation value of the emitting color and other things.

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