Acoustic resonances in two-dimensional radial sonic crystal shells

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Abstract. Radial sonic crystals (RSC) are fluidlike structures infinitely periodic along the radial direction that verify the Bloch theorem and are possible only if certain specially designed acoustic metamaterials with mass density anisotropy can be engineered (see Torrent and Sánchez-Dehesa 2009 Phys. Rev. Lett. 103 064301). A comprehensive analysis of two-dimensional (2D) RSC shells is reported here. A given shell is in fact a circular slab with a central cavity. These finite crystal structures contain Fabry–Perot-like resonances and modes strongly localized at the central cavity. Semi-analytical expressions are developed to obtain the quality factors of the different resonances, their symmetry features and their excitation properties. The results reported here are completely general and can be extended to equivalent 3D spherical shells and to their photonic counterparts.

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1. Introduction

Acoustic metamaterials are a new type of man-made structure with unusual properties. They consist of periodic arrangements of sonic scatterers embedded in a fluid or a gas and their unusual properties appear at wavelengths much larger than the lattice separation between scatterers. We have recently employed acoustic metamaterials with mass anisotropy to demonstrate that it is possible to create a new type of sonic crystal named radial sonic crystals (RSCs) [1]. These crystals are radially periodic structures that, like the regular crystals with periodicity along the Cartesian axis, verify the Bloch theorem and, consequently, sound propagation is allowed only for certain frequency bands. Some interesting applications of RSCs have been envisaged, such as dynamically orientable antennas or acoustic devices for beam forming [1].

In this paper, we present a comprehensive analysis of RSC shells, which are structures that were introduced in our previous short paper [1]. A RSC shell is, in fact, a circular slab that has a central cavity and is surrounded by a fluid or gas. It will be shown that these structures generate two types of resonant modes: one type corresponds to modes strongly localized in the cavity, while the other is defined by its weak localization in the surrounding shell. In particular, strongly localized modes or cavity modes can have huge quality factors, and this makes them suitable for developing acoustic devices in which localization is required. For example, they can be used to enhance the phonon–photon interaction in mixed phononic–photonic structures. Moreover, they can be employed as acoustic resonators of high $Q$ (quality factor), since it is shown here that acoustic cavities based on RSC shells can achieve $Q$-values much larger than those reported so far [2].

The paper is organized as follows. The theory under infinitely periodic RSC is briefly reported in section 2 for the comprehensiveness of this work. Section 3 describes the resonant properties of a particular RSC shell and analytical formulae are developed to give a physical insight into the properties associated with the two types of resonant mode. Pressure patterns of the different modes are analyzed in section 4. Finally, the present work is summarized in
section 5, where we also give a perspective on possible applications of these structures as well as an account of directions for future work.

2. Radial sonic crystals (RSCs)

The propagation of a harmonic acoustic wave with angular frequency \( \omega \), \( P(r; \omega) \), in an anisotropic fluid is given by the following wave equation [3],

\[
B \nabla (\rho^{-1} \cdot \nabla) P + \omega^2 P = 0,
\]

where \( B \) and \( \rho^{-1} \) are the bulk modulus and the reciprocal of the mass density tensor, respectively.

By assuming that the propagation takes place in 2D, the pressure field \( P \) can be factorized

\[
P(r, \theta) = \sum_q P_q(r) e^{iq\theta},
\]

where \((r, \theta)\) are the cylindrical coordinates and \( P_q(r) \) accomplishes the equation

\[
\frac{\partial}{\partial r} \left( \frac{r}{\rho_r} \frac{\partial}{\partial r} \right) P_q(r) + \left[ \frac{r}{B} \omega^2 - \frac{q^2}{r \rho_\theta} \right] P_q(r) = 0,
\]

where \( q \) are discrete numbers \( (q = 0, 1, 2, \ldots) \) and \( \rho_r \) and \( \rho_\theta \) are the components of the anisotropic 2D mass density tensor.

Due to anisotropy, coefficients in (3) are independent and, therefore, equation (3) can be made invariant under the set of translations \( r \rightarrow r + nd \), where \( n \) is an integer and \( d \) is the lattice parameter. Acoustic structures having these properties are called RSCs [1] because they are formally equivalent to crystals in solid state physics for electronic waves [4], photonic crystals for electromagnetic waves [5] or phononic crystals for acoustic and elastic waves. Their corresponding wave equations are invariant under translations along the Cartesian axis.

2.1. Acoustic parameters radially periodic

The conditions that leave (3) invariant under translations of the type \( r \rightarrow r + nd \) are

\[
\frac{r + nd}{\rho_r(r + nd)} = \frac{r}{\rho_r(r)}, \quad (4a)
\]

\[
\frac{r + nd}{B(r + nd)} = \frac{r}{B(r)}, \quad (4b)
\]

\[
(r + nd)\rho_\theta(r + nd) = r \rho_\theta(r). \quad (4c)
\]

From these conditions it is easily concluded that the acoustic parameters can be cast into the form [1]

\[
\rho_r(r) = r \hat{\rho}_r(r), \quad (5a)
\]

\[
\rho_\theta^{-1}(r) = r \hat{\rho}_\theta^{-1}(r), \quad (5b)
\]

\[
B(r) = r \hat{B}(r), \quad (5c)
\]
Figure 1. Radial component of the mass density tensor as a function of radius for the case of a RSC shell. The shell consists of a defect cavity with radius $2d$ and an eight layer thick RSC shell. Cavity and background are considered as different acoustic materials.

where $\hat{\rho}_r(r)$, $\hat{\rho}_\theta(r)$ and $\hat{B}(r)$ represent periodic functions of $r$. Let us note that any periodic function is valid to obtain the invariance described in (4).

Figure 1 depicts a possible profile for $\rho_r(r)$. It has been obtained by making the product of a radially periodic function $\hat{\rho}_r$ with $r$, as shown in (5a). In fact, the RSC corresponds to the region $2 \leq r/d \leq 10$. So, this plot illustrates the case of an acoustic cavity generated by a RSC shell of eight layers; below $r/d = 2$ the mass density is $\rho_a$, and for $r/d > 10$ the background outside the shell has density $\rho_b$.

2.2. Wave propagation and band structure

Periodicity of coefficients (2.1) allows us to apply Bloch’s theorem to solve (3). Thus, the solutions $P_q(r)$ can be chosen to have the form of a plane wave times a function with the lattice periodicity [4]

$$P_q(r) = e^{iK_q r} \sum_n P_{qn} e^{iG_n r}, \quad (6)$$

where $K_q$ is the wavenumber and $G_n$ are the reciprocal lattice vectors; $G_n = 2n\pi/d$, where $n$ are integers.

The Fourier expansions of coefficients are

$$\frac{r}{\rho_r} = \sum_n f_n e^{iG_n r}, \quad (7a)$$

$$\frac{1}{r\rho_\theta} = \sum_n g_n e^{iG_n r}, \quad (7b)$$

$$\frac{r}{B} = \sum_n h_n e^{iG_n r}. \quad (7c)$$
Figure 2. Acoustic band structure for a 2D RSC. Note that each band has a well-defined $q$-symmetry. Points of equal color define modes with the same $q$-pole symmetry.

After the insertion of these expressions in (3) and after using the expansion of the field (6), the following matrix equation is obtained,

$$MP = \omega^2 NP,$$  \hspace{1cm} (8)

where the matrix elements are

$$M_{mn} = f_{m-n}(K_q + G_m)(K_q + G_n) + q^2 g_{m-n},$$  \hspace{1cm} (9a)

$$N_{mn} = h_{m-n}.$$  \hspace{1cm} (9b)

This matrix equation is a generalized eigenvalue problem that can be solved numerically by using different algorithms. Once the acoustic parameters defining the crystal are determined, the solution of the eigenvalue equation gives $\omega = \omega(K_q)$, a dispersion relation that is different for each $q$-pole, and it verifies that $\omega(K_q) = \omega(K_{-q})$ [4].

The final form for the field inside the crystal is given by a linear combination of plane waves:

$$P(r, \theta) = \sum_q e^{iK_q r} e^{iq\theta} \sum_n P_{qn} e^{iG_n}.$$  \hspace{1cm} (10)

In [1], we have considered a RSC made of two types of alternating ‘homogeneous’ layer. Let us remark that layers are not exactly ‘homogeneous’ because they need to be radially dependent. The term ‘homogeneous’ applies here because this medium is the equivalent to the system ‘ABABABA…’ employed in standard heterostructures. The difference is that, in this case, $\hat{\rho}_r$, $\hat{\rho}_\theta$ and $\hat{B}$ act as the ‘stepped’ functions.

2.3. Low frequency bandgaps

Figure 2 shows that in the low-frequency limit only the dispersion relation corresponding to modes $q = 0$ (black lines) has a linear behavior near the zero frequency. Acoustic bands
associated with higher modes, \( q > 0 \), present low-frequency bandgaps. In other words, for non-monopolar modes, the RSC behaves like a ‘radially homogeneous’ medium, that is, a system where the periodic functions \( \hat{\rho}_r, \hat{\rho}_\theta \) and \( \hat{B} \) take constant values \( \rho_r, \rho_\theta \) and \( B \), respectively. Then, in such a medium, wave equation (3) becomes

\[
\frac{\partial^2 P_q(r)}{\partial r^2} + \left[ \omega^2 \frac{\rho_r}{B} - q^2 \frac{\rho_r}{\rho_\theta} \right] P_q(r) = 0. \tag{11}
\]

The solutions of this equation are plane waves

\[
P_q(r) = e^{i k_q r}, \tag{12}
\]

where

\[
k_q^2 = \omega^2 \frac{\rho_r}{B} - q^2 \frac{\rho_r}{\rho_\theta}, \quad q = 0, 1, 2, \ldots . \tag{13}
\]

Therefore, below certain cut-off frequencies given by

\[
\omega_{q_c} = q \sqrt{\frac{B}{\rho_\theta}}, \tag{14}
\]

the wavenumbers take imaginary values and evanescent waves are the only possible solutions to (3). The cut-off frequencies define low-frequency bandgaps of the corresponding mode. This is a property characterizing the so-called ‘radially homogeneous’ medium, whose acoustic parameters (dynamical mass density and bulk modulus) linearly increase with the distance to the origin of the crystal.

3. Resonances in a RSC shell

As with other types of crystal, it is not possible to handle an infinite RSC. So, let us consider a finite-thickness (eight layers) RSC slab containing a defect cavity of radius \( R_a = 2d \) centered at the origin of coordinates. The profile for the radial component of the inertial mass density corresponding to this structure, which is called a RSC shell, is shown in figure 1. Note that the cavity medium \( A \) is enclosed by an anisotropic shell \( S \) and both are embedded in the fluid background \( B \) existing for \( r \geq R_b \). Both of the media \( A \) and \( B \) are homogeneous and isotropic fluid-like materials with acoustic parameters \((\rho_a, B_a)\) and \((\rho_b, B_b)\), respectively. The shell is assumed to be made of a RSC satisfying (3).

Here, we apply the transfer matrix method [6, 7] to study two types of scattering problem. First, we consider the scattering of a RSC by an incident sound wave. Afterwards, we analyze the case when the exciting sound is inside the cavity shell.

In both homogeneous media, \( A \) and \( B \), the fields are given by a linear combination of the Hankel and the Bessel functions,

\[
P(r, \theta) = \sum_q \left[ C_{q \ell}^+ H_q(k_\ell r) + C_{q \ell}^- J_q(k_\ell r) \right] e^{i q \theta}, \tag{15}
\]

where \( k_\ell = \omega / c_\ell \) and \( \ell = a, b \).

For the anisotropic shell \( S \), which fills up the region \( R_a \leq r \leq R_b \), the pressure field is

\[
P(r, \theta) = \sum_q \left[ C_{q \ell}^+ \phi_q^+(r) + C_{q \ell}^- \phi_q^-(r) \right] e^{i q \theta}, \tag{16}
\]
where \( \phi^*_q \) and \( \phi^+_{q} \) are two linearly independent solutions of the wave equation. The proposed solutions in the shell are completely general and it is assumed that the acoustic parameters \( \rho_s(r) \), \( \rho_g(r) \) and \( B(r) \) are arbitrary functions of the radial coordinate \( r \). Later we will consider a particular RSC.

To determine the relationships between coefficients \( C_{i\ell q}^\pm \) for \( \ell = a, b, s \), boundary conditions must be applied at the boundaries \( R_a \) and \( R_b \). The boundary conditions are the continuity of pressure and the continuity of the radial component of particle velocity. The radial symmetry of the problem allows us to decouple the equations for the different multipolar components \( q \).

Therefore, at \( r = R_\ell \), for \( \ell = a, b \), we require that

\[
(17a)
\begin{align*}
C_{i\ell q}^+ (k_\ell R_\ell) + C_{i\ell q}^- (k_\ell R_\ell) &= C_{sq}^+ (R_\ell) + C_{sq}^- (R_\ell), \\
\frac{k_\ell R_\ell}{\rho_i} \left[ C_{i\ell q}^+ H'_q (k_\ell R_\ell) + C_{i\ell q}^- J'_q (k_\ell R_\ell) \right] &= \frac{R_\ell}{\rho_i (R_\ell)} \left[ C_{sq}^+ \partial_r \phi_+(r) + C_{sq}^- \partial_r \phi_-(r) \right]_{r=R_\ell}.
\end{align*}
\]

Note that, for convenience, the continuity equation of particle velocity has been multiplied by \( \rho_i \).

After some algebra, the matrix \( M \) can be factorized as \( M = M_a^{-1} M_b M_s \), where

\[
(19a)
M_a^{-1} = \frac{i \pi \rho_a}{2} \begin{pmatrix} k_a R_a \rho^a_{\ell} J'_q (k_a R_a) & -J_q (k_a R_a) \\ -k_a R_a \rho^a_{\ell} H'_q (k_a R_a) & H_q (k_a R_a) \end{pmatrix},
\]

\[
(19b)
M_b = \begin{pmatrix} \phi_{11} & R_b^{-1} \rho_b (R_b) \phi_{12} \\ R_a \rho^a_{\ell} (R_a) \phi_{21} & R_a^{-1} R_a \rho^a_{\ell} (R_b) \rho^b_{\ell} (R_b) \phi_{22} \end{pmatrix},
\]

\[
(19c)
M_s = \begin{pmatrix} H_q (k_b R_b) & J_q (k_b R_b) \\ k_b R_b \rho^b_{\ell} H'_q (k_b R_b) & k_b R_b \rho^b_{\ell} J'_q (k_b R_b) \end{pmatrix}.
\]

The quantities \( \phi_{ij} \) in \( M_s \) are Wronskians of the shell function \( \phi^\pm_q (r) \) and their derivatives, so that no analytical simplification can be done here. However, it is easy to demonstrate that the determinant of \( M_s \) is unity (see the appendix).

Now, the coefficients for the two types of scattering problem of interest here can be obtained by applying the transfer matrix formalism.

First, let us consider the case when there is no sound source inside the cavity and, therefore, \( C_{aq}^+ = 0 \). The exciting field is an external wave defined by \( C_{aq}^- = A_q^0 \), and the goal is to find the coefficients \( C_{aq}^- = T_q^A A_q^0 \) and \( C_{aq}^+ = T_q^{sc} A_q^0 \) by solving

\[
(20)
\begin{pmatrix} 0 \\ T_q^{in} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} T_q^A \\ 1 \end{pmatrix}.
\]

That is,

\[
(21a)
T_q^{sc} = -\frac{M_{12}}{M_{11}},
\]

\[
(21b)
T_q^{in} = \frac{M_{22} M_{11} - M_{12} M_{21}}{M_{11}}.
\]
where the numerator of $T^{in}_q$ is the determinant of $M$, which is $|M| = |M_a||M_s||M_b| = \rho_a/\rho_b$. This yields

\begin{align}
T^{sc}_q &= -\frac{M_{12}}{M_{11}}, \quad (22a) \\
T^{in}_q &= \frac{\rho_a}{\rho_b} \frac{1}{M_{11}}. \quad (22b)
\end{align}

If the exciting field is put inside the cavity, the pressure wave is defined by the coefficients $C_{aq} = A_0$. Now, a standing wave is excited inside the cavity $C_{aq} = R_qA_0$ and an acoustic field defined by $C_{bq} = T_qA_0$ and $C_{bq} = 0$ is radiated by the cavity. This problem is solved by obtaining the coefficients $R_q$ and $T_q$ from the matrix equation

\begin{equation}
\begin{pmatrix}
1 \\
R_q
\end{pmatrix} =
\begin{pmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{pmatrix}
\begin{pmatrix}
T_q \\
0
\end{pmatrix}.
\end{equation}

(23)

Again, by solving for the coefficients,

\begin{align}
R_q &= -\frac{M_{21}}{M_{11}}, \quad (24a) \\
T_q &= \frac{1}{M_{11}}. \quad (24b)
\end{align}

For the two scattering problems analyzed above, the behavior of their corresponding coefficients is controlled by the diagonal element $M_{11}$. Moreover, $T^{in}_q = T_q$ when the cavity and the background are both the same medium, i.e. $\rho_a = \rho_b$.

If the shell is made of a RSC with $N$ periods, the matrix $M_i$ of each period is identical, due to the periodicity defining the RSC. The transfer matrix of such a system is therefore

\begin{equation}
M_i = M_a^{-1}M_s^N M_b.
\end{equation}

(25)

Since the $M_i$ matrix has modulus one, its eigenvalues are of the form $\lambda_{\pm} = e^{\pm iK_q d}$, with $K_q = K_q(\omega)$ being the dispersion relation that defines the band structure of the crystal. These properties allow us to obtain the $N$th power of the matrix $M_s$ as [7]

\begin{equation}
M_s^N = M_s \frac{\sin N K_q d}{\sin K_q d} - I \frac{\sin (N-1)K_q d}{\sin K_q d}.
\end{equation}

(26)

and the transfer matrix of the system as

\begin{equation}
M = M_a^{-1}M_sM_b^{-1} \frac{\sin N K_q d}{\sin K_q d} - M_a^{-1}M_b \frac{\sin (N-1)K_q d}{\sin K_q d}.
\end{equation}

(27)

The complex frequencies $\hat{\omega}$ solving the equation $M_{11} = 0$ completely characterize the resonant modes in the structure, i.e. their frequencies and lifetimes. However, numerical solution of such an equation is a cumbersome task and is beyond the scope of the present work. Here, we prefer to give a physical insight into the resonances $\omega$ for which the coefficients have large or small values.

Two types of resonance can be distinguished: cavity modes and Fabry–Perot-like resonances. The former are strongly localized inside the cavity shell and the latter are due to the finite thickness of the RSC making the shell.

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3.1. Cavity modes

These resonances occur at frequencies within the bandgaps of the RSC. They are characterized by their strong localization inside the cavity shell, as shown below.

Bloch wavenumbers within the bandgaps are complex numbers, $K_q \approx i \hat{K}_q$, and as a consequence the periodic functions of $NK_q d$ in (3) become hyperbolic functions of $N \hat{K}_q d$. Therefore, the dominant term in the expression for $M_{11}$ is proportional to $\sin NK_q d \approx \frac{1}{2} e^{N \hat{K}_q d}$.

If we consider, for example, that the sound source is placed inside the cavity shell, the field radiated out of the shell is given by (24b), that is,

$$T_q = 1/M_{11} \approx e^{-N \hat{K}_q d}. \quad (28)$$

This exponentially decreasing behavior with increasing $N$ has been previously characterized in other types of crystal, such as photonic or phononic crystals. For example, the attenuation by a phononic crystal slab has been demonstrated to be exponentially decreasing with the number of crystal periods [8].

However, it is also possible to demonstrate that, in the bandgap region, $T_q$ have strong transmission peaks, which are associated with the existence of cavity modes. Thus, after some algebra, the $M_{11}$ element can be cast as

$$M_{11} = \frac{1r_0}{2} \left[ X_q H_q(k_b R_b) + Y_q k_b R_b \rho_a^{-1} H'_q(k_b R_b) \right], \quad (29)$$

where

$$X_q = M_{11}^N k_a \rho_a^{-1} R_a J'_q(k_a R_a) - J_q(k_a R_a) M_{21}^N \quad (30a)$$

and

$$Y_q = M_{12}^N k_a \rho_a^{-1} R_a J'_q(k_a R_a) - J_q(k_a R_a) M_{22}^N. \quad (30b)$$

When $X_q = 0$, we have

$$Y_q = -J_q(k_a R_a) / M_{11}^N \approx e^{-N \hat{K}_q d}, \quad (31)$$

which means that now $M_{11}$ behaves inversely as before; that is, the transmission coefficient increases exponentially with the number of layers $N$,

$$T_q = 1/M_{11} \approx e^{N \hat{K}_q d}. \quad (32)$$

A similar behavior is found after analyzing the case $Y_q = 0$.

The resonance frequency can be obtained from the condition $X_q = 0$. The following condition must hold,

$$k_a \rho_a^{-1} R_a J'_q(k_a R_a) / J_q(k_a R_a) = M_{21}^N R_a \rho_a^{-1} (R_a) \frac{\phi_{21}}{\phi_{11}}, \quad (33)$$

where the quantities $\phi_{ij}$ have already been introduced as factors in the elements of $M_i$ (see (19b)). Since we are working at bandgap frequencies, the ratio $\phi_{21}/\phi_{11}$ can be approximated by

$$\frac{\phi_{21}}{\phi_{11}} \approx iK_q = -\text{Im}[K_q(\omega)]. \quad (34)$$
Figure 3. Localized modes and Fabry–Perot resonances for a finite RSC consisting of eight periods and an inner cavity of radius $R_a = 2d$ (see figure 1). Top panels: graphic resolution of equation (20) for modes 0 and 1, respectively. The dots define the frequency positions at which the mode cavities are predicted. The shadowed zones define the acoustic bandgaps shown in figure 2. Bottom panels: the corresponding scattering coefficients, $T_0$ and $T_1$, which have peaks at the frequencies approximately determined by (20) (see text).

Therefore, the equation for the frequency $\omega_0$ of the cavity modes is rapidly found,

$$\omega_0 J_q'(\omega_0 R_a/c_a) = -\frac{\rho_a c_a}{\rho_r(R_a)} \text{Im}[K_q(\omega_0)] J_q(\omega_0 R_a/c_a), \quad (35)$$

which can be solved graphically.

Figures 3 and 4 show the graphical solution of (35), where the left-hand-side term is depicted with black dot symbols, while the red lines represent the right-hand-side term. The color dots at the crossing points give the (approximate) frequencies of cavity modes. The values are further confirmed by the exact numerical solution of the coefficients $R_q$ and $T_q$, which are depicted in the bottom panels of figures 3 and 4. Note how, at frequencies within the bandgaps (shadowed zones in the top panels), the coefficients show peaks that practically coincide with the points obtained from the graphical solution of (35). The peaks in the passband regions (white zones) correspond to Fabry–Perot-like modes, which are discussed in the following subsection. Let us note that coefficients $R_q$ and $T_q$ in figures 3 and 4 have been calculated with a resolution of $10^{-5}$ along the frequency axis and, therefore, their values at the peaks are not necessarily converged. For an exact determination of peak heights, lower frequency steps should be chosen till the convergence is obtained.
Figure 4. The same as figure 3 but for modes $q = 2$ and $q = 3$, respectively. Large-frequency bandgaps (shadowed regions) now appear below a certain frequency cut-off (see text). Fabry–Perot-like resonances show up in the narrow passband frequencies (white regions).

The behavior of coefficients $T_q$ near the resonances suggests that their moduli have the following functional form:

$$|T_q(\omega)|^2 = \frac{1}{A(\omega - \omega_0)^2 e^{N\hat{k}_q d} + B e^{-N\hat{k}_q d}}. \quad (36)$$

The quality factor is defined by $Q = \omega_0/\Delta \omega$, where $\Delta \omega$ is the full-width at half-maximum (FWHM) of the corresponding peak. In other words, $\Delta \omega = \omega_+ - \omega_-$, where $\omega_\pm$ are the frequencies at which $|T_q(\omega_\pm)|^2 = |T_q(\omega_0)|^2/2$. It is easy to show that

$$\omega_\pm = \omega_0 \pm \sqrt{\frac{B}{A}} \quad (37)$$

and therefore

$$Q = 2\sqrt{\frac{A}{B}} e^{N\hat{k}_q d}. \quad (38)$$

This formula predicts that the $Q$-factor of a cavity mode grows up exponentially with the number of layers $N$ building the RSC shell. This result is also known in the field of photonic crystals, where it was numerically demonstrated, for example, that the $Q$-factor of a cylindrical defect surrounded by a finite metallic 2D photonic crystal exponentially increases with the number of layers surrounding the cavity defect [9].

To demonstrate the validity of (38), we have calculated $Q$ as a function of $N$ for three different cavity modes. It must be pointed out that $Q$ is obtained directly from the peaks in $T_q$ and, therefore, a very accurate determination of the peak’s intensity at the resonant frequencies
is needed. This requirement is accomplished by using extremely small frequency steps until convergence is obtained. As an example, figure 5 shows the calculation of $T_1$ and $T_2$ by using frequency steps of $10^{-7}$ and $10^{-12}$, respectively. These spectra have already been depicted in figures 3 and 4 by using much larger frequency steps. Now, the intensities of peaks in $T_q$ spectra are much higher than in figures 3 and 4 since they are converged.

Figure 6 presents the $Q$-factor calculated for several values of $N$. We have analyzed one dipolar mode and two quadrupolar modes. The dipolar mode, which has the frequency 0.1973, corresponds to the resonant peak having the lowest frequency appearing in figure 5(a). The two quadrupolar modes have frequencies 0.3167 and 0.5538, respectively, and are shown in figure 5(b). It is observed that the predicted exponential dependence is confirmed by the numerical simulations and provides the order of magnitude of $Q$ for these new structures. These structures used as acoustic resonators could provide $Q$ values much larger than those already obtained by perfect cylinders or spherical resonators [2].

3.2. Fabry–Perot resonances

Fabry–Perot-like resonances are generated by the oscillating terms $\sin N K_q d$ and $\sin(N - 1) K_q d$ in (3), where $Nd$ defines the RSC thickness, $Nd = R_b - R_a$. These resonances are confined inside the RSC region. They are equivalent to the Fabry–Perot modes already characterized for sonic crystal slabs [10]. A typical feature of these resonances is that they are frequency equidistant in the regions where the dispersion relation is linear. This feature is clearly observed in the profile of the coefficients $R_0$ and $T_0$ shown in figure 3, where the peaks in the lower part of the first frequency band are equally spaced.
Figure 6. The quality factor of three different modes localized in the cavity of a RSC shell made of \(N\) periods. The mode with the frequency 0.1973 has dipolar symmetry \((q = 1)\), while modes with frequencies 0.3167 and 0.5538 both have quadrupolar symmetry \((q = 2)\).

The Fabry–Perot frequency oscillations, which are produced by the periodic functions of \(K_q(R_b - R_a)\) in (27), are coupled with the oscillating behavior of the quasi-periodic functions with argument \(k_a R_a\) embedded in \(M_a\). Therefore, two parametric regimes can be distinguished. First, the regime of thick shells occurs when \(R_b \gg R_a\) and, consequently, resonances localized in the RSC are dominants and the cavity modes will be scarcely shown and will appear as additional peaks on top of the oscillating profile of \(R_q\) and \(T_q\). The second regime is defined by the condition \(R_b \approx R_a\) and corresponds to the case of thin shells. In this case, the oscillating profile will be dominated by peaks associated with cavity modes and, therefore, Fabry–Perot oscillations, if any, will appear in between.

Fabry–Perot modes have \(Q\)-factors much lower than cavity modes. For example, for the dipolar mode located at the frequency 0.3649 (see the arrow in figure 5(a)), we obtain \(Q = 380\). The modes with quadrupolar symmetry located at 0.5936 and 0.6122 (see the arrows in figure 5(b)) have \(Q\) values of 4151 and 3364, respectively. Lower \(Q\)-values mean less localization and, consequently, a long evanescent tail that could be employed for their possible excitation by external sources located at intermediate distances. It is interesting to point out that the existence of a great number of resonances combined with their easy excitation could be employed to design acoustic devices for broadband detection of external sound sources, as was suggested in [1].

4. Modes’ pressure fields

Pressure maps are used in this section to discuss in brief the salient features of the different resonant modes that can be excited in a RSC shell. As typical examples, we consider modes with dipolar \((q = 1)\) and quadrupolar \((q = 2)\) symmetry to illustrate their behavior under different excitation conditions. Cavity modes and Fabry–Perot-like modes are shown. As was pointed out, the main feature distinguishing the cavity modes from the Fabry–Perot-like modes is the difficulty of being excited by using external sound sources.
Figure 7. Pressure maps (amplitude) of the cavity mode with frequency 0.1973 (in reduced units) and dipolar symmetry existing in the RSC shell made of ten layers (see also figure 3). (a) Mode excited by a punctual sound source (not depicted) put at position \((x, y) = (0, 1)\) inside the cavity. (b) Mode excited by an external sound wave with a plane wavefront. White circles define the borders of the RSC shell.

Figures 7(a) and (b) depict a cavity mode with dipolar symmetry localized in a RSC shell made of ten layers. The maps are obtained under the two different excitation conditions reported in section 3: (a) by using a punctual sound source placed outside the cavity center and (b) by using an external sound with a plane wavefront. In both cases, the exciting sound has the same frequency as the excited mode. In both cases, the cavity mode is excited and strong localization inside the cavity is clearly observed. The black circle in figure 7(a) defines the region not calculated by our computer code due to the presence of the exciting source.

Figures 8 and 9 present the pressure maps of the two quadrupolar modes whose \(Q\)-factors are analyzed in figure 6. Figure 8 has been obtained by using an inner punctual source, while figure 9 was obtained by using the scattering of an external sound wave interacting resonantly with the cavity mode. In figure 9, the RSC shell thickness is eight layers since the modes are more easily excited from outside. The oscillations observed in figure 9(b) outside the shell are produced by the external sound coming from the left side of the \(x\)-axis. Like the cavity mode with dipolar symmetry, the quadrupolar modes are also strongly localized inside the cavity shell.

Finally, Fabry–Perot modes with dipolar and quadrupolar symmetry are shown in figures 10–12 under two different excitation conditions. Note that these modes are contained in the RSC shell, which in all cases has ten layers. Also note, in figure 10(b), that the dipole is excited along the direction of the impinging wave, the \(x\)-axis. This property could be used as a sonic device to determine the direction of external sound sources and even to determine their exact positions by using an additional RSC shell [1]. With regard to quadrupolar modes, a feature to note is related to the additional number of oscillations that present the mode with frequency 0.6135 in comparison with that having lower frequency (0.5936). This difference is related to the higher number of oscillations that the periodic functions with larger argument \(K_q(R_a - R_b)\) have.
Figure 8. Pressure maps (amplitude) of the two cavity modes existing in the RSC shell with quadrupolar symmetry ($q = 2$). Their frequencies are 0.3159 (a) and 0.5564 (b) in reduced units. They are excited by a punctual sound source (not depicted) put at position $(x, y) = (0, 1)$ inside the cavity. White circles define the borders of the RSC shell.

Figure 9. Pressure maps (amplitude) calculated for the scattering of a sound plane wave impinging a RSC shell made of eight layers. (a) External sound with the frequency 0.3159 (in reduced units), which is resonant with the first quadrupole cavity mode in figure 5(b). (b) External sound with the frequency 0.5564 (in reduced units), which is resonant with the second quadrupole cavity mode in figure 5(b).

5. Summary

We have studied the resonances existing in 2D RSC shells, which are acoustic structures consisting of a circular cavity surrounded by a RSC slab. Two types of resonant mode have been characterized. Cavity-like modes have frequencies within the acoustic bandgaps of the
Figure 10. Pressure maps (amplitude) of the Fabry–Perot mode of dipole symmetry \((q = 1)\) that appears at the frequency 0.3648 (see figure 5). (a) Excitation by a punctual sound source (not depicted) put inside the central cavity; at \((x, y) = (0, d)\). (b) Excitation by an external sound with a plane wavefront. Black circles define the borders of the RSC shell.

Figure 11. Pressure maps (amplitude) of the first two Fabry–Perot-like modes existing in the RSC shell with quadrupole symmetry \((q = 2)\). Their frequencies are 0.5936 (a) and 0.6135 (b), respectively. They are excited by a punctual sound source (not depicted) put inside the central cavity; at \((x, y) = (0, d)\). White circles define the borders of the RSC shell.

infinitely periodic RSC and are characterized by their strong localization inside the cavity shell. Fabry–Perot-like modes appear in frequency regions where RSCs have acoustic passbands. These modes are equivalent to those characterized in sonic crystal slabs [10] and are confined in the shell region, i.e. between the cavity and the embedded background.
A main advantage of the RSC is the actual possibility of making acoustic band engineering to select the mode symmetry of resonances to be excited in RSC shells. Symmetry selection could be performed either in the cavity modes or in the Fabry–Perot modes. This mode symmetry selection could be used, for example, in designing devices for tailoring the radiation field generated by omnidirectional sound sources. Moreover, since the Fabry–Perot resonances are selectively excited along the direction of the external sound source, their dipolar modes could be used to produce acoustic devices capable of determining the exact position of external sound sources emitting at the frequency range where these modes are isolated. Moreover, RSC shells are excellent candidates for designing devices with functionality similar to the cochlea by following a recent proposal by Bazan et al [11], who employed circular cavities based on phononic crystals with such a purpose. In this regard, RSC shells have the advantage of their large tailoring possibilities.

Let us note that the results obtained for 2D shells can be extrapolated to their 3D counterparts, in which the cavity defect is a spherical hole. The high $Q$ values that these structures will probably achieve make them suitable to fabricate acoustic resonators with possible applications in sensing and metrology [2].

With a few modifications, the results described here can be extended to the realm of electromagnetic waves, where radial photonic crystals (RPC) have also been proposed by us [1]. In this context, photonic cavities based on RPC shells are expected to create cavity modes having huge $Q$-factors, a property that can be used to fabricate low-threshold microcavity laser lighting with a selected symmetry. Also, they could be used to design fibers in which their confinement effects are obtained by true complete bandgaps associated with the crystal properties of these anisotropic structures and not by omnidirectional bandgaps produced by alternating layers of two homogeneous dielectric materials. The properties of these cavities are under investigation and the results will be published elsewhere.
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Appendix. Demonstration of $|M_s| = 1$

Let us define the matrix $\hat{M}_s(r)$ as

$$\hat{M}_s(r) = \begin{pmatrix} \phi_\ell^+(r) & \phi_\ell^-(r) \\ r\rho_\ell^{-1}(r)\partial_r \phi_\ell^+(r) & r\rho_\ell^{-1}(r)\partial_r \phi_\ell^-(r) \end{pmatrix}.$$  \hfill (A.1)

The matrix $M_s$ can be written as a function of $\hat{M}_s(r)$ as

$$M_s = \hat{M}_s^{-1}(R_b)\hat{M}_s(R_a).$$  \hfill (A.2)

Therefore, the determinant of $M_s$ is

$$|M_s| = \frac{|\hat{M}_s(R_a)|}{|\hat{M}_s(R_b)|}.$$  \hfill (A.3)

By applying (3) it can be shown that the derivative of the determinant $|\hat{M}_s(r)|$ is zero and, therefore, this quantity is not a function of $r$, that is

$$\partial_r |\hat{M}_s(r)| = \partial_r \left[ \phi_\ell^+(r)r\rho_\ell^{-1}(r)\partial_r \phi_\ell^+(r) \right] - \partial_r \left[ \phi_\ell^-(r)r\rho_\ell^{-1}(r)\partial_r \phi_\ell^-(r) \right] = 0,$$  \hfill (A.4)

and therefore

$$|M_s| = 1.$$  \hfill (A.5)

References

[1] Torrent D and Sánchez-Dehesa J 2009 Radial wave crystals: radially periodic structures from anisotropic metamaterials for engineering acoustic or electromagnetic waves Phys. Rev. Lett. B 103 064301

[2] Miklos A, Hess P and Bozoki Z 2001 Application of acoustic resonators in photoacoustic trace gas analysis and metrology Rev. Sci. Instrum. 72 1937

[3] Torrent D and Sánchez-Dehesa J 2009 Sound scattering by anisotropic metafluids based on two-dimensional sonic crystals Phys. Rev. B 79 174104

[4] Ascroft N W and Mermin D 1976 Solid State Physics (Holt: Rinehart and Winston)

[5] Joannopoulos J D, Meade R D and Winn J N 1995 Photonic Crystals: Molding the Flow of Light (Princeton, NJ: Princeton University Press)

[6] Born M and Wolf E 1964 Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light (Oxford: Pergamon)

[7] Bendicksson J M, Dowling J P and Scalora M 1996 Analytic expressions for the electromagnetic mode density in finite, one-dimensional, photonic band-gap structures Phys. Rev. E 53 4107
[8] Goffaux C, Maseri F, Vasseur J O, Djafari-Rouhani and Lambin Ph 2003 Measurements and calculations of the sound attenuation by a phononic band gap structure suitable for an insulating partition application Appl. Phys. Lett. 83 281

[9] Ochiai T and Sánchez-Dehesa J 2002 Localized defect modes in finite metallic two dimensional photonic crystals Phys. Rev. B 65 245111

[10] Sanchis L, Hkansson A, F, Cervera F and Sánchez-Dehesa J 2003 Acoustic interferometers based on two-dimensional arrays of rigid crystals in air Phys. Rev. B 67 035422

[11] Bazan A, Torres M, Montero de Espinosa F R, Quintero-Torres R and Aragon J L 2007 Conformal mapping of ultrasonic crystals: confining ultrasound and cochlearlike waveguiding Appl. Phys. Lett. 90 094101