A new formulation of field theory is presented, based on a pseudo-complex description. An extended group structure is introduced, implying a minimal scalar length, rendering the theory regularized a la Pauli-Villars. Cross sections are calculated for the scattering of an electron at an external Coulomb field and the Compton scattering. Deviations due to a smallest scalar length are determined. The theory also permits a modification of the minimal coupling scheme, resulting in a generalized dispersion relation. A shift of the Greisen-Zatsepin-Kuzmin-limit (GZK) of the cosmic ray spectrum is the consequence.

I. INTRODUCTION

The (Quantum) Field Theory (FT, in general) is a great success story of physics. For example, Quantum Electrodynamics describes many physical processes to an unprecedented precision, rarely known in science. The Lorentz symmetry plays an important rôle, which allows to formulate several conservation laws, like the PCT theorem. However, experiments on the cosmic ray spectrum put some doubts on this symmetry. Further high energy events were measured in Ref. [2]. Under the assumption of Lorentz symmetry, a maximal possible energy at $10^{20}$ eV is predicted, called the Greisen-Zatsepin-Kuzmin (GZK) limit (or cutoff) [6, 7]. It originates from the interaction of high energy protons with the photons of the Cosmic-Microwave-Background (CMB). At very large energies, corresponding to the GZK limit, the energy of the photon in the eigen-system of the proton is large enough to create pions. This process subtracts energy from the proton and, thus, no higher energy protons can reach earth. Of course, the assumption is important that these high energy particles are produced very far from earth, for example, in the early universe. In a recent article [8], reporting on the results of the HiRes experiment, the GZK cutoff was claimed to be observed. The results of AGASA and HiRes are put on trial in the Pierre-Augier experiment [9, 10], where final data have not been published yet.

In spite of the great success of FT, the occurrences of ultraviolet divergences is troublesome, requiring intelligent subtractions of infinities. These are due to the assumption of permitting arbitrary large momenta, i.e. small lengths. However, physics might change at the order of the Planck length or even before. Adding a smallest length ($l$) corresponds to introduce a momentum cutoff, which eliminates the infinities, though, renormalization of physical parameters, like the charge or mass, have still to be applied. A smallest length scale $l$ must have an influence on the position of the GZK cutoff. Conversely, if a shift is observed it will put a value to the $l$. As we will see in this contribution, the effect of a smallest length is larger for high energy processes. The atomic energy scale are too small. Investigating the GZK limit gives a good opportunity to look for a smallest length scale.

Up to now, the main efforts to explain the non-existence of the GZK limit concentrates on the violation of Lorentz symmetry. In accordance with our last observation, a minimal length is introduced in most models (see, e.g., the Double Special Relativity or spin-networks). The length is handled as a vector, subject to Lorentz contraction. Another way to break Lorentz invariance is to assign a velocity with respect to an universal frame, breaking rotational and, thus, also Lorentz symmetry. This is proposed in [14, 15, 16], based on a geometric approach. In [17] a Lorentz breaking interaction was considered, also containing a preferred oriented vector. In [18, 19] Lorentz breaking interactions in the Lagrange density were investigated too on a general basis.

In [20] an alternative is proposed, extending the Lorentz group to a larger one. The formalism is based on a pseudo-complex extension of the Lorentz group, where pseudo-complex numbers have to be introduced, also called hyperbolic or hypercomplex. Large part of the mathematics is described in detail in Ref. [22]. It also implies to formulate a pseudo-complex version of the field theory, which is proposed schematically in [23], however, without any calculations of physical processes. Adding a term to the Lagrangian, which simulates that the interaction happens within a finite size of space-time and not at a point (due to the occurrence of a minimal length scale $l$), changes the dispersion relation. The minimal length scale ($l$) enters and modifies the dispersion relation, giving rise to a shift in the GZK limit. However, the maximal predicted cutoff, under reasonable assumptions and independent on the choice of further structure of the additional interaction term, is by only a factor 2. The difference is proportional to $l^2$ and increases with energy. The GZK cutoff gives us the opportunity of investigating such high energy events.
not observed, at least we can obtain an upper limit on the smallest length scale $l$.

Consequently, the change in the dispersion relation is visible only at high energies, comparable to the GZK scale. At low energies, the dispersion relation is to very high approximation maintained. One may ask however, if the smallest length $l$ may also produce deviations at intermediate energies, for example, in the TeV range, accessible to experiment now. In order to be measurable, we look for differences in the cross section of a particular reaction, of the lowest power in $l$ possible.

The advantage of the proposed extended field theory is obvious: All symmetries are maintained and, thus, it permits the calculation of cross sections as we are used to. Still, an invariant length scale appears, rendering the theory regularized and reflecting the deviation of the space-time structure at distances of the order of the Planck length.

The main objective of this paper is to formulate the pseudo-complex extension of the standard field theory (SFT). For the extension we propose the name Pseudo-Complex Field Theory (PCFT). First results are reported in [20].

The structure of the paper is as follows: In section 2 the pseudo-complex numbers are introduced and it is shown how to perform calculations, like differentiation and integration. This section serves as a quick reference guide to the reader unfamiliar with the concept of pseudo-complex numbers. In section 3 the pseudo-complex Lorentz and Poincaré groups are discussed. The representations of the pseudo-complex Poincaré group are indicated. Section 4 introduces a modified variational procedure, required in order to obtain a new theory and not two separated old ones. The language is still classical. As examples, scalar and Dirac fields are discussed and an extraction procedure, on how to obtain physical observables, is constructed and at the end formally presented. Section 5 is dedicated to the symmetry properties of the PCFT. Finally, in section 6 the quantization formalism is proposed. In section 7 a couple of cross sections are calculated within the PCFT: i) The dispersion of a charged particle at a Coulomb field and ii) the Compton scattering. These, however, require higher order Feynman diagrams, which will explode the scope of the present paper. This will be investigated in a future article. The language will be within Quantum Electrodynamics and effects from the electro-weak unification will be discarded, for the moment. In section 8 we will show some relations to geometric approaches, which also contain a scalar length parameter. The results of this section will give important implications for the topic treated in Section 9, where the theory is extended such that the GZK limit is shifted. Finally, section 10 contains the conclusions and an outlook.

The paper contains at the beginning an explanatory part of a work already published [21, 22, 23, 24, 25], however, in a quite dense form. Parts, published in [21, 23, 24, 25], had to be revised and inconsistencies, physical and mathematical ones, were corrected. It also contains new contributions to the pseudo-complex formulation. The main motivation is to make this contribution self-contained and to expand the very short presentations, given in several different contributions of the pseudo-complex formulation, such that the reader appreciates the global context. The new and additional contributions can be found in the mathematical part, to the representation theory, how to extract physical observables (like cross sections), the quantization procedure and the calculation of cross sections.

II. PSEUDO-COMPLEX NUMBERS AND DERIVATIVES

The pseudo-complex numbers, also known as hyperbolic [22] or hypercomplex [26], are defined via

$$X = x_1 + I x_2 ,$$

with $I^2 = 1$. This is similar to the common complex notation except for the different behavior of $I$. An alternative presentation is to introduce

$$\sigma_\pm = \frac{1}{2} (1 \pm I)$$

with

$$\sigma^2_\pm = 1 , \sigma_+ \sigma_- = 0 .$$

The $\sigma_\pm$ form a zero divisor basis, with the zero divisor defined by $P^0 = P^0_+ \cup P^0_-$, with $P^0_\pm = \{ X = \lambda \sigma_\pm | \lambda \in R \}$. This basis is used to rewrite the pseudo-complex numbers as

$$X = X_+ \sigma_+ + X_- \sigma_- .$$
with

\[ X_\pm = x_1 \pm x_2 \quad . \]  

(5)

The set of pseudo-complex numbers is denoted by \( P = \{ X = x_1 + Ix_2 | x_1, x_2 \in \mathbb{R} \} \).

The pseudo-complex conjugate of a pseudo-complex number is

\[ X^* = x_1 - Ix_2 = X_+ \sigma_- + X_- \sigma_+ \quad . \]  

(6)

We use the notation with a star for the pseudo-complex conjugate and a bar (\( \bar{X} \)) to denote the usual complex conjugate, i.e., the pseudo-real and pseudo-imaginary part can also be complex, though, in this section we assume that they are real for didactical reasons. The norm square of a pseudo-complex number is given by

\[ |X|^2 = XX^* = x_1^2 - x_2^2 \quad . \]  

(7)

There are three different possibilities:

\[ x_1^2 - x_2^2 > 0 , \quad \text{"space like"} \]
\[ x_1^2 - x_2^2 < 0 , \quad \text{"time like"} \]
\[ x_1^2 - x_2^2 = 0 , \quad \text{"light cone"} \quad , \]  

(8)

where the notation in "...") stays for the analogy to the structure of the 1+1-dimensional Minkowski space. In each subsection, a different parametrization of the pseudo-complex sector can be applied \([21, 27]\).

i) Positive norm:

The presentation of a pseudo-complex number is very analogous to the usual complex one

\[ X = Re^{I\phi} = R(\cosh(\phi) + I\sinh(\phi)) \quad . \]  

(9)

with

\[ x_1 = R\cosh(\phi) , \quad x_2 = R\sinh(\phi) \quad . \]  

(10)

The inverse relation is given by

\[ R = \pm \sqrt{x_1^2 - x_2^2} \]
\[ \tanh(\phi) = \frac{x_2}{x_1} \quad . \]  

(11)

There are two cases: \( R > 0 \) and \( R < 0 \), corresponding to the "right" and "left" cone, respectively. Constant \( R \) corresponds to hyperboloids either on the right or left cone.

ii) Negative norm:

The only difference is an additional \( I \) in the parametrization of the pseudo-complex number, i.e.,

\[ X = RIE^{I\phi} = R(I\cosh(\phi) + \sinh(\phi)) \quad . \]  

(12)

with

\[ x_2 = R\cosh(\phi) , \quad x_1 = R\sinh(\phi) \quad . \]  

(13)

The inverse transformation is
There are two cases: $R > 0$ and $R < 0$, corresponding to the "upper" and "lower" cone, respectively. Constant $R$ corresponds to either hyperboloids on the upper or lower cone.

iii) Zero norm:

The parametrization is given by

$$X = \lambda \frac{1}{2} (1 \pm I) = \lambda \sigma_{\pm}$$

With $X^*X = 0$ it satisfies the condition for the zero norm.

As can be seen, the structure of the space is very similar to the one of the Minkowski space. In fact, the structure corresponds to the group $O(1,1)$.

A useful rule of any function $F(X)$, which is expanded into a Taylor series, can be written as

$$F(X) = F(X_+)\sigma_+ + F(X_-)\sigma_-$$

and a product of two functions $F(X)$ and $G(X)$ satisfies

$$F(X)G(X) = F(X_+)G(X_+)\sigma_+ + F(X_-)G(X_-)\sigma_- .$$

This is proved, using $\sigma_{\pm}^2 = 1$ and $\sigma_{\mp}\sigma_{\mp} = 0$ and
for arbitrary $n$ (note, that $\sigma^\pm_n = \sigma^\pm_n$ for all $n$). As an example, we have

$$e^X = e^{X_+} + e^{X_-} .$$  \hfill (19)
B. Integration

In general, we have to provide a curve \( X(t) = x_1(t) + Ix_2(t) \), with \( t \) being the curve parameter, along which we would like to perform the integration. A pseudo-complex integral can be calculated via real integrals (as for the normal complex case):

\[
\int F(X) dX = \int \frac{dr_1}{dt} + I \frac{dr_2}{dt} F(X(t)) .
\] (25)

However, no residual theorem exists. Thus, the structure of pseudo-complex numbers is very similar to the usual complex ones but not completely, due to the appearance of the zero divisor branch. This reflects the less stringent algebraic structure, i.e., that the pseudo-complex numbers are not a field but a ring.

C. Pseudo-complex Fourier integrals

In \( d \)-dimensions, the Fourier transform of a function \( F(X) \) and its inverse can be defined via

\[
F(X) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int d^d P \tilde{F}(P)e^{iP \cdot X}
\]

\[
\tilde{F}(P) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int d^d X F(X)e^{-iP \cdot X} ,
\] (26)

with both \( X \) and \( P \) in general pseudo-complex, \( P \cdot X = (P^\mu X_\mu) \) and \( n_2 \) being the number of integrations in the "time-like" sector. Here, we restrict to straight lines in either the "space-like" or "time-like" sector. Straight lines in the "space-like" sector (here, for example of the coordinate \( X \)) are parametrized as: \( X = R \exp(i\phi_0) \) (\( \phi_0 = \text{const} \)). For the integration in the "time-like" sector we have \( X = IR \exp(i\pi_0) \).

With this definition of the Fourier transform, in 1-dimension, the \( \delta \)-function is given by

\[
\tilde{\delta}(X - Y) = \frac{1}{2\pi} \int dpe^{iP(X - Y)} = I^\xi (\delta(X^+ - Y^+)\sigma_+ + \delta(X^- - Y^-)\sigma_-) ,
\] (27)

with \( \xi = 0 \) if the integration along a straight line is performed in the "space-like" sector and it is equal to 1 if the integration is performed along a line in the "time-like" sector. For a more detailed description, consult Appendix A.

III. PSEUDO-COMPLEX LORENTZ AND POINCARÉ GROUPS

Finite transformations in the pseudo-complex extension of the Lorentz group are given by \( \exp(i\omega_{\mu\nu}\Lambda^{\mu\nu}) \), where \( \omega_{\mu\nu} \) is a pseudo-complex group parameter \( (\omega_{\mu\nu}^{(1)} + I\omega_{\mu\nu}^{(2)}) \) and \( \Lambda^{\mu\nu} \) are the generators [30 31]. The finite transformation with pseudo-complex parameters form the pseudo-complex Lorentz group \( SO_P(1,3) \).

When acting on functions in the pseudo-complex coordinate variable \( X^\mu \), the representation of the generators of the Lorentz group is, using as the momentum operator \( P^\mu = \frac{1}{i}D^\mu \) \( (=P^\mu_1 + IP^\mu_2 = P^\mu_+\sigma_+ + P^\mu_-\sigma_-) \), in the two possible representations:

\[
\Lambda^{\mu\nu} = X^{\mu}P^{\nu} - X^{\nu}P^{\mu} = N^{\mu\nu}_+ \sigma_+ + N^{\mu\nu}_- \sigma_- ,
\] (28)

with

\[
\Lambda^{\mu\nu}_\pm = X^{\mu}_\pm P^{\nu}_\mp - X^{\nu}_\pm P^{\mu}_\mp .
\] (29)

The pseudo-complex Poincaré group is generated by
\[ P^\mu = iD^\mu = i \frac{D}{D\lambda_\mu} = P^\mu_+\sigma_+ + P^\mu_-\sigma_- \]
\[ \Lambda^{\mu\nu} = \Lambda^{\mu\nu}_+\sigma_+ + \Lambda^{\mu\nu}_-\sigma_- \]  
(30)

with \( P^\mu_\pm = P^\mu_1 \pm P^\mu_2 \).

As before, finite transformations of the pseudo-complex Poincaré group are given by
\[ \exp(i\omega \cdot L)\sigma_+ + \exp(i\omega \cdot L)\sigma_- \]
with \( (\omega \cdot L_\pm) = \omega_\pm L_\pm^\dagger, L_\pm \) being either \( \Lambda^{\mu\nu}_\pm \) or \( P^{\mu}_\pm \) and in general distinct pseudo-real group parameters \( \omega_\pm \) and \( \omega_\mp \). Only when \( \omega_\mp = \omega_\pm \) the group parameters \( \omega_i \) are pseudo-real and standard Lorentz group is recovered.

A Casimir of the pseudo-complex Poincaré group is
\[ P^2 = P^\mu P_\mu = \sigma_+ P^2_+ + \sigma_- P^2_- \]  
(31)

Its eigenvalue is \( M^2 = \sigma_+ M^2_+ + \sigma_- M^2_- \), i.e., a pseudo-complex mass associated to each particle. The Pauli-Ljubanski vector is given by
\[ W_\mu = -\frac{1}{2} \epsilon_{\mu\gamma\alpha\beta} P^\gamma \Lambda^{\alpha\beta} = W_\mu_+\sigma_+ + W_\mu_-\sigma_- \]  
(32)

The \( \sigma_\pm \) parts of this vector are
\[ W_{\mu\pm} = -\frac{1}{2} \epsilon_{\mu\gamma\alpha\beta} P^\gamma \Lambda^{\alpha\beta}_\pm \]  
(33)

Thus, two mass scales are associated to a particle, namely \( M_+ \) and \( M_- \), which are in general different. Its interpretation will be discussed below.

A. Interpretation of a pseudo-complex transformation

In this subsection the effect of a transformation with pseudo-imaginary group parameters is revisited. The first steps towards an extraction procedure, on how to obtain physically observable numbers, are presented. Step by step, this will be complemented to a final extraction procedure. Later, in section 4.4, all building blocks are united and a formal justification will be given.

A finite transformation of the pseudo-complex Lorentz group is expressed by
\[ \exp(i\omega_{\mu\nu} \Lambda^{\mu\nu}) = \exp(i\omega \cdot \Lambda) \]
where \( \omega_{\mu\nu} \) is pseudo-complex. In order to study the effect of a pseudo-complex transformation, it suffices to restrict to a purely pseudo-imaginary \( \omega_{\mu\nu} \rightarrow I\omega_{\mu\nu} \), where we extracted the \( I \). Thus, a finite transformation is given by
\[ \Lambda_\mu^{\nu} = \exp(iI\omega \cdot \Lambda) = \Lambda_1 \mu^{\nu} + I\Lambda_2 \mu^{\nu} \]  
(34)

were the transformation is divided into its pseudo-real and pseudo-imaginary components. The pseudo-real part can again be associated to a standard Lorentz transformation.

Now, let us consider a co-moving four-bein along the world-line of the observer. The unit vectors are denoted by \( e_\mu \). Applying to it the pseudo-complex transformation leads to new, now pseudo-complex, unit vectors \( E_\mu \), which are related to the old ones via
\[ E_\mu = \Lambda_1 \mu^{\nu} e_\nu + I\Omega_\mu^{\nu} e_\nu \]  
(35)

with
\[ \Omega_\mu^{\nu} = \frac{1}{I} (\Lambda_1^{-1})^{\lambda}_{\mu} (\Lambda_2)_{\lambda}^{\nu} \]  
(36)
It is straightforward to show that the following symmetry properties hold

\[(\Lambda_1)^{\mu\nu} = (\Lambda_1)^{\nu\mu} \quad \Omega_{\mu\nu} = -\Omega_{\nu\mu} \quad .\] (37)

Let us consider as a particular transformation the boost in direction 1, the presence of the \( I \) requires a pseudo-imaginary angle \( I\phi \) of the boost. Using \( \cosh(I\phi) = \cosh(\phi) \) and \( \sinh(I\phi) = I\sinh(\phi) \), the transformation acquires the form

\[
\Lambda = \begin{pmatrix}
\cosh(\phi) & I\sinh(\phi) & 0 & 0 \\
I\sinh(\phi) & \cosh(\phi) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} .\] (38)

With Eq. (38), this gives the relation for the relevant components

\[
\Omega_{0}^{0} = \Omega_{1}^{1} = 0 \\
\Omega_{0}^{1} = \Omega_{1}^{0} = \frac{1}{l}\tanh(\phi) ,\] (39)

where the matrix element \( \Omega_{0}^{1} \) describes the acceleration of the particle. Lowering the indices reproduces the required symmetry properties of Eq. (37). As a special case, consider \( (\Lambda_{1})^{\mu}_{\nu} = \delta^{\mu}_{\nu} \). The four-bein vectors acquire the form

\[
E_{\mu} = e_{\mu} + lI\Omega_{\mu}^{\nu}e_{\nu} \\
= e_{\mu} + lI\frac{de_{\mu}}{d\tau} ,\] (40)

where \( \tau \) is the eigen-time. The pseudo-imaginary component describes the changes of the four-bein vectors with time, i.e., for the 0-component it gives the acceleration, described by the Frenet-Serret tensor \( \Omega_{\mu}^{\nu} \), of the co-moving system along the world line of the observer. The form of \( \Omega_{\mu}^{\nu} \) implies a maximal value for the acceleration (see Eq. (39)) which is, using the limit \( \tanh(\phi) \to 1 \) for \( \phi \to \infty \),

\[
\Omega_{0}^{1} = \frac{1}{l}\tanh(\phi) \leq \frac{1}{l} .\] (41)

In conclusion, the pseudo-imaginary component of the group parameter results in the appearance of a maximal acceleration and, thus, the theory contains a minimal length scale. Which value this \( l \) acquires, cannot be decided yet. We will see that it should be of the order of the Planck length. Important to note is that \( l \) is a scalar parameter which is not subject to a Lorentz contraction.

When transformations of \( \Lambda_{kl} \), with \( k,l = 1,2,3 \) are considered, the transformed systems correspond to rotating ones [27].

Equation (40) suggests to propose as a new coordinate

\[
X^{\mu} = x^{\mu} + lI\frac{dx^{\mu}}{d\tau} \\
x^{\mu} + lIu^{\mu} ,\] (42)

with \( \tau \) the eigen-time and \( u^{\mu} \) as the four-velocity of the observer. In general, the \( X_{1}^{\mu} \) and \( X_{2}^{\mu} \) in \( X^{\mu} = X_{1}^{\mu} + IX_{2}^{\mu} \) are linear independent. Eq. (42) proposes to fix the pseudo-imaginary component, when mapped to a physical system, using this geometrical argument.

The pseudo-imaginary component of \( P^{\mu} \) allows the following interpretation: When we apply a finite transformation \( \exp(il\partial_{\mu}P^{\mu}) \) to \( X^{\mu} \), which is pseudo-imaginary as in the former case, it only affects the pseudo-imaginary component of \( X^{\mu} \), namely \( u^{\mu} \to l(u^{\mu} + \delta^{\mu}) \). This action changes the four-velocity and, thus, corresponds to an acceleration. Therefore, we can associate to the pseudo-imaginary part of the translation operator an acceleration, too. This will play an important role in sections 8 and 9, when a modified procedure is proposed on how to extract physically
obtainable numbers. In general, the two components of $P_\mu$ are linear independent and only at the end a choice of the pseudo-imaginary component is applied.

As was shown, the pseudo-complex extension includes systems which are accelerated (and rotated, when rotational transformations with pseudo-imaginary group parameter are used). However, when we want to determine physical observables, we have to do it in an inertial frame because only there the vacuum is well defined. This implies to select a subset of systems, corresponding to inertial ones, with respect to which the comparison to SFT is possible. Because the $P_2^\mu$ component is associated to acceleration, it is suggested to set $P_2^\mu$ to zero. However, adding to the Lagrange density a fixed, i.e. not linear independent, pseudo-complex component to $P_\mu$ may simulate the effect of acceleration during interaction.

For the moment, we will put the pseudo-imaginary component to zero, when extracting physical results and only later, in section 9, we will explore the consequences permitting a linear dependent pseudo-complex component.

B. Representations

One implication of the above description for fields $\Phi_r(X)$, with $r$ denoting internal degrees of freedom, is that it depends on the pseudo-complex coordinate $X$. In the zero-divisor basis this field acquires the form $\Phi_{r,+}(X)\sigma_+ + \Phi_{r,-}(X)\sigma_-$. The Casimir operator $W^2 = W_\mu W^\mu$ of the Poincaré group is proportional to $M^2 J^2 = M^2_j J^2_j \sigma_+ + M^2 J^2 \sigma_-$, with $M$ the pseudo-complex mass and $J^2 = J^2_j \sigma_+ + J^2_\sigma_- \text{ the total spin squared}$. Spin is a conserved quantity and the pseudo-complex fields have to be eigenstates of this spin-operator. Because the eigenvalue is real, the eigenvalues of $\Phi_{r,\pm}$ with respect to $J^2_\pm$ have to be the same.

The representation theory of the new field theory is completely analogous to the standard procedure [30] and it is not necessary to elaborate on it further. The same holds for the Poincaré group.

The eigenvalue $M^2$ of $P^2$ is pseudo-complex and results in two mass scales, namely $M_+$ and $M_-$. One of these scales will be associated to the physical mass $m$, as will be shown further below. The other scale will be related to $l^{-1}$, setting it equal to the Planck mass.

IV. MODIFICATION OF THE VARIATIONAL PROCEDURE

Up to now, it seems that everything can be written in parallel, i.e., one part in $\sigma_+$ and the other one in $\sigma_-$. In order to obtain a new theory, both parts have to be connected.

Given a Lagrange density, one is tempted to introduce in the pseudo-complex space an action $S = \int d^4 X \mathcal{L}$. The equations of motion are obtained by a variational procedure. However, if we require that $\delta S = 0$, we come just back to two separated actions and two separated wave equations, because we can write the action as $S = S_+ \sigma_+ + S_- \sigma_-$ and $\delta S = 0$ results in $\delta_+ S_+ = 0$ plus $\delta_- S_- = 0$. If we want to modify this, obtaining a new field theory, we are forced to extend the variational equation, such that both parts are connected. In [23, 27] the proposal is

$$\delta S \in P^0_\pm,$$

with $P^0 = P^0_+ \cup P^0_-$ and $P^0_\pm = \{X | X = \lambda \sigma_\pm \}$, i.e., the right hand side has to be in the zero divisor branch, which plays in the pseudo-complex extension the rôle of a zero (remember that the norm of $\lambda \sigma_\pm$ is zero).

Assuming now a theory with fields $\Phi_r$, we have for a 4-dimensional space $(1+3)$

$$\delta S = \int \left[ \sum_r \frac{D\mathcal{L}}{D\Phi_r} \delta \Phi_r + \sum_r \frac{D\mathcal{L}}{D(D_\mu \Phi_r)} \delta (D_\mu \Phi_r) \right] d^4 X$$

$$= \int \left[ \sum_r \frac{D\mathcal{L}}{D\Phi_r} - \sum_r D_\mu \left( \frac{D\mathcal{L}}{D(D_\mu \Phi_r)} \right) \right] \delta \Phi_r d^4 X$$

$$+ \sum_r \int D_\mu \left( \frac{D\mathcal{L}}{D(D_\mu \Phi_r)} \delta \Phi_r \right) d^4 X.$$

With $F^\mu(X) = \frac{D\mathcal{L}}{D(D_\mu \Phi_r)} \delta \Phi_r$, the last term is surface integral.
\[
\int (D_\mu F^\mu(X))d^4X = \int (D_+ \mu F^\mu_+(X))d^4X_+ \sigma_+ \\
+ \int (D_- \mu F^\mu_-(X))d^4X_- \sigma_-
\]

In standard field theory, this surface integral vanishes but here we have to permit that the numerical result is a number in the zero divisor branch \( P^0 \). This term can be added to the right hand side of the variational equation. Without loss of generality, we assume that the element of \( P^0 \) is of the form \( \sum_r A^r_+ \delta \Phi_-, \sigma_-, \) with some arbitrary \( A^r_- \).

From the variational equation we obtain

\[
D_+ \mu \left( \frac{D_+ \mathcal{L}_+}{D_+(D_+ \phi_+, \sigma_+)} \right) - \frac{D_+ \mathcal{L}_+}{D_+ \phi_+, \sigma_+} = 0
\]

\[
D_- \mu \left( \frac{D_- \mathcal{L}_-}{D_-(D_- \phi_-, \sigma_-)} \right) - \frac{D_- \mathcal{L}_-}{D_- \phi_-, \sigma_-} = 0 .
\]

Or, in a compact notation,

\[
D_\mu \left( \frac{D \mathcal{L}}{D(D_\phi_+, \sigma_+)} \right) - \frac{D \mathcal{L}}{D \phi_+, \sigma_+} \epsilon P^0_+ .
\]

Strictly speaking, this is not an equation, though, we will continue to denote it like that. The same expression is obtained when we choose \( \sum_r A^r_+ \delta \Phi_+, \sigma_+ \epsilon P^0_+ \), being different from zero. In order to obtain an equation of motion of the type \( \hat{A} = 0 \) one more step is involved as will be illustrated next, on the level of a classical field theory.

### A. Scalar Fields

The proposed Lagrange density is

\[
\mathcal{L} = \frac{1}{2} \left( D_{\mu} \Phi D^{\mu} \Phi - M^2 \Phi^2 \right) .
\]

The resulting equation of motion follows according to Eq. (47) is

\[
(P^2 - M^2) \Phi \epsilon P^0 .
\]

Multiplying by the pseudo-complex conjugate \( (P^2 - M^2)\Phi \) we arrive at

\[
(P^2_+ - M^2_+) (P^2_- - M^2_-) \Phi = 0 ,
\]

which can be seen as follows: Without loss of generality we can assume the case \( (P^2 - M^2) \Phi \epsilon P^0_. \) We have then

\[
(P^2 - M^2) \Phi = \left( (P^2_+ - M^2_+) \sigma_+ + (P^2_- - M^2_-) \sigma_- \right) \Phi
\]

This implies \( (P^2_+ - M^2_+) \Phi = 0 \), leading to \( (P^2_+ - M^2_-) \sigma_+ \Phi = (P^2_- - M^2_+) \Phi \epsilon \sigma_- = \lambda \sigma_- \), with \( \lambda \) having in general some non-zero value. Alternatively, \( (P^2 - M^2)^* (P^2 - M^2) = (P^2_+ - M^2_+) (P^2_- - M^2_-) \) (use \( \sigma_+ = \sigma_\pm \), \( \sigma_- = 0 \) and \( \sigma_\pm + \sigma_- = 1 \)), which is a pseudo-real hermitian operator whose eigenvalues are real. It can only be satisfied when \( \lambda = 0 \).

This is the connection we searched for: We only obtain an equation of motion of the form \( \hat{A} = 0 \), when both components, of the \( \sigma_+ \) and \( \sigma_- \), connected. The field equation is obtained, after having substituted \( P_\pm^\mu \) by \( p^\mu \) (see also comments at the end of section 3.1).
To obtain a solution for Eq. (50), at least one of the factors, applied to $\Phi$, has to vanish. Without loss of generality we choose the first one. This defines which part of the pseudo-complex wave function we associate with the standard physical particle. After the above introduced extraction procedure, we obtain

$$(p^2 - M_+^2) = 0 \rightarrow E^2 = p^2 + M_+^2,$$  \hspace{1cm} (52)$$

where we used $p^0 = p_0 = E$ and $p^k = -p_k$. It requires to interpret the mass scale $M_+$ as the physical mass $m$ of the particle.

Eq. (50), after setting $P_\mu^\pm = p^\mu$ and $X^\mu = x^\mu$, acquires the form

$$(p^2 - M_+^2)\varphi(x) = 0,$$  \hspace{1cm} (53)$$

with the still pseudo-complex function, equal to $\varphi_+(x)\sigma_+ + \varphi_-(x)\sigma_-$, and

$$\varphi(x) = (p^2 - M_+^2)\Phi(x).$$  \hspace{1cm} (54)$$

This gives a relation of the field $\Phi(X)$ to what we will call the physical component.

To obtain a value for the other mass scale $M_-$, we have to find a generalization of the propagator in this theory. For that reason, let us consider the propagator related to Eq. (49). Its pseudo-complex Fourier component is

$$\xi = \frac{1}{p^2 - M_+^2} = \frac{1}{p_+^2 - M_+^2}\sigma_+ + \frac{1}{p_-^2 - M_-^2}\sigma_-,$$  \hspace{1cm} (55)$$

where the factor $\xi = \xi_+\sigma_+ + \xi_-\sigma_-$ is in general pseudo-complex and has yet to be determined. We used that $\frac{A}{B} = \frac{A_+}{B_+}\sigma_+ + \frac{A_-}{B_-}\sigma_-$. Conversely, for Eq. (50), setting the pseudo-imaginary part of $P^\mu$ to zero, we expect the Fourier component

$$\left(\frac{1}{p^2 - M_+^2} - \frac{1}{p^2 - M_-^2}\right).$$  \hspace{1cm} (56)$$

In order to obtain a consistent result, we have first to set in Eq. (55) $P_+^\mu$ to $p^\mu$ (selecting an inertial frame) and taking the pseudo-real part. In a second step, we have to choose for $\xi$ the value $2I$, because $I\sigma_\pm = \pm\sigma_\pm$. Without the $I$, the wrong relative sign appears and the factor of 2 is needed, to get the correct normalization. This result obtained will be resumed in section 4.4 within a formal description of the extraction procedure.

The propagator describes the propagation of a particle characterized by two mass scales. In order to obtain the same result as SFT at low energy, the $M_-$ has to be very large. Taking the analogy to the Pauli-Villars regularization, $M_-$ should take the maximal possible value, which is $l^{-1}$.

The fact that a particle is described by two mass scales does not imply two particles, but rather the same particle with a dynamical mass, depending on the energy.

### B. Dirac Field

The proposed Lagrange density for a Dirac particle is

$$\mathcal{L} = \bar{\Psi}(\gamma_\mu P^\mu - M)\Psi.$$  \hspace{1cm} (57)$$

The equation of motion is

$$(\gamma_\mu P^\mu - M)\Psi \in \mathcal{P}^0.$$  \hspace{1cm} (58)$$

Multiplying by the pseudo-complex conjugate $(\gamma_\nu P^\nu - M)^* = (\gamma_\nu P_+^\nu - M_+)\sigma_- + (\gamma_\nu P_-^\nu - M_-)\sigma_-$, we arrive at
\[
(\gamma_\nu P_\nu^+M_+)(\gamma_\mu P_\mu^-M_-^2)\Psi = 0 .
\] (59)

Again, we have to project to the pseudo-real part of the momentum in order to compare to the result of the SFT. This leads us to assume that one of the factors in Eq. (59), applied to \(\Psi\), has to be zero. Without loss of generality we choose again the first one, which describes a particle with physical mass \(m = M_+\).

The Fourier component of the propagator, corresponding to (58), is given by

\[
\xi_+ \frac{1}{\gamma_\mu P_\mu^-M_-} = \xi_+ \frac{1}{\gamma_\mu P_\mu^-M_+} + \xi_- \frac{1}{\gamma_\mu P_\mu^-M_-} .
\] (60)

After projection, the expected form of the propagator, according to Eq. (59), is

\[
\left(\frac{1}{\gamma_\mu P_\mu^-M_+} - \frac{1}{\gamma_\mu P_\mu^-M_-}\right) .
\] (61)

In order to be consistent with Eq. (59), requires to put \(\xi = 2I\), as in the scalar field case. Like in the former example, the final operator describes the propagation of a particle with two mass scales. In order to obtain at low energies the same result as in SFT, again the \(M_-\) has to be very large. We set \(M_-\) equal to the only mass scale left in the theory, which is \(l^{-1}\).

Note, that the theory is Pauli-Villars regularized. It is an automatic consequence of the pseudo-complex description, i.e. the introduction of a minimal invariant length.

The dispersion relation for a Dirac particle is obtained starting from Eq. (59), setting \(P_\mu^- = 0\), multiplying it from the left with \((\gamma_\nu P_\nu^-M_-)(\gamma_\mu P_\mu^-M_+)\) and using the properties of the \(\gamma^\mu\) matrices \((\frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = g_{\mu\nu})\). The final result is \((M_+\) is renamed by \(m)\)

\[
(E^2 - p^2 - m^2) = 0 .
\] (62)

As in the scalar case, we part from Eq. (59), setting \(P_\mu^\pm = p^\mu\) and obtain

\[
(\gamma_\mu p^\mu - m)\psi(x) = 0 ,
\] (63)

with

\[
\psi(x) = (\gamma_\mu p^\mu - M_-)\Psi(x) ,
\] (64)

which gives the relation between \(\Psi(X)\) to the physical projected piece.

Let’s summarize subsections 4.1 and 4.2: The procedure on how to deduce the physical component is partially outlined, which states: i) Set the pseudo-imaginary component of the linear momentum to zero. This should also be the case for the pseudo-imaginary component of the angular momentum, boost, etc.. ii) In order to get a propagator, consistent with the field equation, we have to define the Green’s function by the equation

\[
OG(X',X) = (2\pi)^4 I\delta^{(4)}(X' - X) ,
\] (65)

where \(O = (P^2 - M^2)\) for the scalar field case and \((\bar{P} - M)\) for the pseudo-complex Dirac field (we use the notation \(\bar{P} = \gamma_\mu P^\mu\)). Only then the correct sign appears, consistent with the field equation, involving the two \(\sigma_{\pm}\) components. The pseudo-complex Fourier transform leads to the propagators discussed above.

The last piece of the extraction procedure, i.e., how the fields have to be mapped, will be discussed next.
C. Extracting the physical component of a field

To a pseudo-complex field $\Phi_\nu(X) = \Phi_{+,\nu}(X_+)\sigma_+ + \Phi_{-,\nu}(X_-)\sigma_-$ a pseudo-complex mass is associated. We identified the $M_+$ as the physical mass. When the motion of a free particle is considered, the second component of the field, related to the large mass $M_-$, cannot propagate on shell, because for that at least the energy $M_-$ is required. It contributes only to internal lines of a Feynman diagram, where energy conservation is not required. Therefore, the physical component of a free in-and-out-going particle is proportional to $\Phi_{+,\nu}(X)$, where the pseudo-complex coordinate $X^\mu$ has to be subsequently substituted by $x^\mu$. This also holds for the linear momentum, when the Fourier component of the field is considered. In the case for the scalar field, the physical component is, therefore, $\varphi_{+,\nu}(x) = N'(p^2 - M^2_\nu)\Phi_{+,\nu}(x)$, with $N$ a normalization factor. Taking into account that $p^2\Phi_{+,\nu}(x)$ gives $M^2_\nu\Phi_{+,\nu}(x) = m^2\Phi_{+,\nu}(x)$ (on-shell condition), the factor $(m^2 - M^2_\nu)$ in front can be assimilated into the normalization and we can use $\Phi_{+,\nu}(x)$ as the projected physical piece of the field. The same holds for the Dirac field.

For fields describing internal lines, similar to the propagators, the physical component is given by the sum of the fields $\Phi_{\pm,\nu}$. For example, when the dispersion of a charged particle at a Coulomb field is considered, one has to take the sum of the $\sigma_\pm$ components of the pseudo-complex Coulomb field.

The discussion of this subsection leads to the last piece on how to extract physical answers: The construction of any composed object, like a $S$-matrix element which is a function of the fields and propagators, is defined by first constructing the projected pieces, i.e., extract the pseudo-real part of the propagators, take the $\sigma_+$ component of the in- and out-going fields and then compose higher order expressions. This is in analogy to Classical Electrodynamics, where the fields are expressed as complex functions, but when non-linear expressions in the fields, like the energy, are calculated, one has to decide which component to take.

D. Formal introduction to the extraction procedure

Let us first propagate the pseudo-complex scalar field $\Phi(X)$ from the space-time point $X$ to $Y$ and then project the physical part of the wave function. It can be written as

$$\Phi(Y) = i \int d^4X G(Y, X)\Phi(X)$$

$$\rightarrow \varphi_+(y) = N(p^2 - M^2_\nu)\Phi_+(y)$$

$$= N(p^2 - M^2_\nu)i \int d^4x G_+(y, x)\Phi_+(x) \ , \quad (66)$$

where $N$ is a normalization factor and $Y$ has been set to $y$. The Fourier transform of $G_+(y, x)$ is $\tilde{G}_+(p) = \frac{1}{p^2 - M^2_\nu}$. Applying a Fourier transformation also to the field $\Phi_+(x)$, denoted as $\tilde{\Phi}_+(p)$, and using the properties of the $\delta^{(4)}(p^2 - p^2)$ function, we obtain

$$\varphi_+(y) = i \int d^4p \frac{e^{ipy}}{p^2 - M^2_\nu} \tilde{\Phi}_+(p) \ . \quad (67)$$

Now, we will first project and then propagate from $x$ to $y$. Because the projected state is given by $\varphi_+(x) = N(p^2 - M^2_\nu)\Phi_+(x)$ it has to be propagated by $g(y, x)$ with the Fourier transform $\left[1/(p^2 - M^2_\nu) - 1/(p^2 - M^2_\nu)\right]$, as was suggested above. Propagating this state gives

$$i \int d^4g(y, x)\varphi_+(x) =$$

$$\left[\frac{1}{(2\pi)^3} \int d^4x \int d^4p_1 \int d^4p_2 \left(\frac{1}{p^2 - M^2_\nu} - \frac{1}{p^2 - M^2_\nu}\right) e^{ip_1 \cdot (y-x)}\varphi_+(p_2) e^{ip_2 \cdot x}\right]$$

$$\left(M^2_\nu - M^2_\nu\right)i \int d^4p \frac{e^{ipy}}{p^2 - M^2_\nu} \tilde{\Phi}_+(p) \ . \quad (68)$$

exploiting the on-shell condition for a free particle $p^2\varphi_+(p) = M^2_\nu \varphi_+(p)$, leads to the same result as in Eq. (67). Note, that we used the propagator $g(y, x)$ for $\varphi_+(x)$, while for $\Phi(X)$ it is $G(Y, X)$. Using as the physical part of the wave function the $\varphi_+(x)$, requires the physical propagator $g(y, x)$. Thus, a consistent formulation is obtained.
E. Conserved Quantities

As in the SFT, the Noether theorem can be applied. The procedure is completely analogous, except for appearance of the zero divisor.

As a particular example, let us discuss the translation in space-time, i.e.,

$$X'_\mu = X_\mu + \delta b_\mu,$$

where $\delta b_\mu$ is a constant four-vector. The variation of the Lagrange density has to be proportional at most to a divergence plus a term which is in the zero divisor branch:

$$\delta \mathcal{L} = \mathcal{L}' - \mathcal{L} = \delta b_\mu D^\mu \mathcal{L} + \xi,$$

with $\xi \in \mathbf{P}^0$.

Proceeding parallel to the SFT, using the equation of motion (47), leads to

$$D^\mu \Theta_{\mu\nu} \in \mathbf{P}^0,$$

with

$$\Theta_{\mu\nu} = -g_{\mu\nu} \mathcal{L} + \sum_r \frac{D\mathcal{L}}{D(D_\mu \Phi_r)} D_\nu \Phi_r,$$

which is the pseudo-complex energy momentum tensor. The $\Phi_r$ is some field with an index $r$ and $g_{\mu\nu}$ is the metric involved. Let us suppose that $\xi \in \mathbf{P}^0$. When we look at the $\sigma_+$ component, we get an equation $D^\mu \Theta_{\mu\nu} = 0$, which gives the usual conservation laws, setting $P^\pm = p^\mu$. Considering both components, the equation reads

$$D^\mu \Theta_{\mu\nu} \in \mathbf{P}^0.$$

For the case of a scalar field, this expression leads to the conserved linear momentum

$$P^k = -\int d^3X \Pi(X) D^k \Phi(X),$$

with $\Pi(X) = D^0 \Phi(X)$, and the Hamilton density

$$\mathcal{H} = \frac{1}{2} \left( \Pi^2 + (D^k \Phi)^2 + M^2 \right).$$

Similar steps have to be taken for the Dirac and the electro-magnetic fields and also when other symmetry operations, like rotations and phase changes, are considered.

The symmetry properties of the fields are similar to the SFT. Therefore, we will not elaborate on them further.

V. GAUGE SYMMETRY AND GAUGE BOSONS

Let us consider the case of a Dirac particle, coupled to a photon field, i.e., Pseudo-Complex Quantum Electrodynami

$$\mathcal{L} = \bar{\Psi} \left( \gamma^\mu \left( P_\mu - ig A_\mu \right) - M \right) \Psi - \frac{i}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} N^2 \sigma_+ A_\mu A^\mu,$$

with $F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu, D_\mu$ the pseudo-complex derivative and $M$ being the pseudo-complex mass of the Dirac particle. The photon has a pseudo-complex mass term given by $N^2 \sigma_+$, i.e., the physical mass $N_+$ is zero. Due to the
appearance of a mass term, one might worry about gauge invariance. However, gauge invariance is still preserved in
the pseudo-complex formulation:

The fields transform as

\[\Psi \rightarrow \exp(i\alpha(x))\Psi\]
\[A_\mu \rightarrow A_\mu + \frac{1}{g}(D_\mu \alpha(x))\]  

(77)

where \(\alpha(x)\) is the gauge angle, depending on the position in space-time. This gauge angle can be chosen the same
in both parts, pseudo-real and pseudo-imaginary. I.e., \(\alpha(x) = \eta(x)(1 + I) = (\eta(x)/2)\sigma_+\). This is justified because at
the same space-time point an observer can define the same gauge angle without violating the principle of relativity.
Therefore, \(\alpha(x)\) gives zero when applied to the mass term of the photon in the Lagrange density. No contradiction to
the principle of gauge-symmetry exists!

The formulation can be extended to higher gauge symmetries, not only \(U(1)\), as just discussed. For the case of an
\(SU(n)\) symmetry, the Lagrange density of a Dirac field coupled to a gauge boson field is given by \((a, b, c = 1, 2, \ldots , n)\)

\[L = \bar{\Psi}(\gamma^\mu (P_\mu - ig\lambda_a A^a_\mu)) - M) \Psi - \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2} N^2 \sigma_- A^a_\mu A^a_\mu\]

(78)

where \(\lambda_a\) are the generators of the \(SU(n)\) gauge group. This Lagrange density was proposed in [23], however, without
any further calculations.

Under gauge transformations they change to

\[\Psi \rightarrow \exp(i\alpha^a(x)\lambda_a)\Psi\]
\[A^a_\mu \lambda_a \rightarrow A^a_\mu \lambda_a + \frac{1}{g}(D_\mu \alpha^a(x)\lambda_a) + i\alpha^a(x)A^b_\mu f_{ab} \lambda_c\]  

(79)

with \(f_{ab}^c\) as the structure constants of the \(su(n)\) algebra (algebras are denoted by lower case letters).

VI. QUANTIZATION

The quantization procedure will be illustrated first for the case of a pseudo-scalar field. It is followed by the Dirac
field and finally by the electro-magnetic field.

A. Scalar Field

Note, that a pseudo-scalar field is not a scalar with respect to the pseudo-complex numbers, because it has non-
vanishing components in \(\sigma_\pm\). We refer rather to the scalar nature with respect to the usual complex numbers.

In the first step of the quantization procedure we construct a possible complete basis with respect to which we
expand the pseudo-scalar field \(\Phi(X)\). Solutions to the above field equations (49) and (50) are plane waves

\[f_P(X) = \frac{1}{(2\pi)^{3/2}\sqrt{2\omega_P}} e^{-iP\cdot X}\]

(80)

where \(X, f_P(X), P\) and \(\omega_P\) are all pseudo-complex quantities. The \(\omega_P\) reflects the dispersion relation and is given by

\[\omega_P = \sqrt{P^2 + M^2} = \sqrt{P_+^2 + M_+^2} \sigma_+ + \sqrt{P_-^2 + M_-^2} \sigma_-\]

(81)

The factor in Eq. (80) normalizes the components in each sector.
Since \( f_p(X) \) is a solution to the field equation and completeness can be shown in the same way as for pseudo-real plane waves, the next step is to expand the field \( \Phi(X) \) in terms of these pseudo-complex plane waves:

\[
\Phi(X) = \int d^3P \left[ a(p)f_p(X) + a^\dagger(p)f^*_p(X) \right].
\] (82)

For the moment, the integration is taken along a straight line (see Appendix A) in the pseudo-complex momentum space. Later, we will restrict to the pseudo-real axis only. However, the more general form has implications, discussed next. The physical consequences are not completely clear yet. Further investigation is needed.

One can deduce the commutation properties of the operators \( a \) and \( a^\dagger \). This requires to assume a particular commutation relation for equal times of the field with its canonical momentum, which is defined as

\[
\Pi(X) = \frac{DL}{D(D^0\Phi)} = D_0\Phi.
\] (83)

A general proposal for the commutation relation is

\[
[\Phi(X,X_0),\Pi(Y,X_0)] = i\hbar^2\delta^{(3)}(X - Y),
\] (84)

with \( \delta(X - Y) = \delta(X_+ - Y_+)\sigma_+ + \delta(X_- - Y_-)\sigma_- \), as introduced in the Appendix A. The natural number \( n \) has yet to be specified.

In Appendix B the inversion of the above relations is given, yielding the operators \( a \) and \( a^\dagger \) and their commutation relations:

\[
[a(P),a^\dagger(P')] = I^{n+\xi_x}\delta^{(3)}(P - P'),
\] (85)

with \( \xi_x \) related to the type of path chosen in integrations. Conversely, let us start from the last equation, assuming the given commutation relation, and deduce the commutation relation of the field with its conjugate momentum, which should give back Eq. (84). As shown in Appendix B, this requires to set \( \xi_p = \xi_x \), i.e. if the integration in \( X \) is in one sector, it has to be in the equivalent one in \( P \).

Let us suppose that \( \xi_x = 0 \) (pure straight "space-like" paths) and \( n = 0 \) or \( n = 1 \). In the first case \( (n = 0) \) we obtain the usual commutation relations, which we will adopt from now on. As we will show in the next subsection, this implies a particular definition of the propagator in terms of the fields.

Relating the results to SFT, implies setting \( P^\mu_\pm \) to \( p^\mu \), which gives for the component \( \Phi_+(x) \), now with \( X^\mu \rightarrow x^\mu \), a plane wave proportional to \( \exp(ip_{\nu}x^\nu) \). Therefore, an in- and out-going wave is described as before.

For completeness, we discuss the case with \( n = 1 \): The commutation relation of the creation and annihilation operators reduces to

\[
[a(P),a^\dagger(P')] = I\delta^{(3)}(P - P'),
\] (86)

with all other commutators equal to zero. Separating the commutator into the \( \sigma_+ \) and \( \sigma_- \) part, where the first is related to the low energy mass, also projecting to real momenta, yields

\[
[a_+(p),a^\dagger_+(p')] = +\delta^{(3)}(p - p')
\]

\[
[a_-(p),a^\dagger_-(p')] = -\delta^{(3)}(p - p').
\] (87)

The commutation relations for the \( \sigma_- \) component have the opposite sign, i.e., the part described by \( a_- \) seems to refer to a particle with unusual commutation relations.

Such structure is not new. In a formulation within Krein-spaces (see, for example [33]), finite field theories are obtained, suggesting a possible relation to our description. The field equations look very similar to those discussed in this contribution. When the particle with mass \( M_+ \) is considered as the physical one, the commutation relations are the usual ones. However, particles corresponding to the mass, \( M_- \) in our case, obey commutation relations with an opposite sign. These particles, within the Krein-space formulation, never appear as free particles but rather only in internal lines of the Feynman diagrams. Thus, choosing another path of integration in the pseudo-complex space of \( X^\mu \) and \( P^\mu \) leads possibly to a different kind of theory. Mathematically, these theories are related and it would be interesting to elaborate on it further.
1. Propagator

In section 4.1, the concept of a propagator for the scalar field was extended. Using the standard commutation relations of the fields, the creation and annihilation operators \((n = 0)\), the following definition of a propagator, consistent with our former discussion, can be given, namely

\[
I \langle 0 | \Phi(X) \Phi(Y) | 0 \rangle \quad ,
\]  
(88)

assuming now that the field and their arguments are, in general, pseudo-complex. We could have used the second choice of commutation relations \((n = 1)\) as indicated in the previous subsection. Then, there would be no factor \(I\), implying unusual (anti-)commutation relations and a different field theory. However, we prefer the standard commutation relations, because they allow the usual interpretation of the particles as bosons. The opposite requires the introduction of particles with unusual properties, as discussed above, but not observed.

Substituting the fields of Eq. (82) and using the commutator relations of the pseudo-complex creation and annihilation operators (85) gives

\[
I \int \frac{d^3P}{(2\pi)^3} \frac{e^{-ip \cdot (X-Y)}}{2\omega_P} .
\]  
(89)

For equal times \((Y_0 = X_0)\) and \(P_\pm = p\) we arrive at

\[
I \langle 0 | \Phi(X) \Phi(Y) | 0 \rangle = 
I \left\{ \frac{1}{(2\pi)^3} \int \frac{d^3P}{2\omega_+,p} e^{-ip \cdot (x-y)} \sigma_+ 
+ \frac{1}{(2\pi)^3} \int \frac{d^3P}{2\omega_-,p} e^{-ip \cdot (x-y)} \sigma_- \right\} .
\]  
(90)

This must still be projected to the pseudo-real part, which is the sum of the factor of \(\sigma_+\) and \(\sigma_-\). Due to the \(I\) as a factor and \(I\sigma_\pm = \pm \sigma_\pm\), the sign in the second term changes and we obtain the propagator of Eq. (88). This is possible having chosen the quantization, with \(n = 0\), as given above.

As can be seen, the description is consistent, using the proposed form of the propagator in Eq. (88). The advantage lies in the standard use of the commutation relations of the fields, the creation and annihilation operators and its interpretation as bosons.

B. Dirac Field

The quantization of the Dirac field has the usual form [1], using now \(E_P = \omega_P\),

\[
\Psi(X) = \sum_{\pm s} \int \frac{d^3P}{(2\pi)^3} \sqrt{\frac{M}{E_P}} \left[ b(P,s)u(P,s) e^{-iP \cdot X} 
+ d(P,s) \bar{v}(P,s) e^{iP \cdot X} \right] ,
\]

\[
\Psi^\dagger(X) = \sum_{\pm s} \int \frac{d^3P}{(2\pi)^3} \sqrt{\frac{M}{E_P}} \left[ \bar{b}(P,s) \bar{u}(P,s) e^{iP \cdot X} 
+ d(P,s) \bar{v}(P,s) e^{-iP \cdot X} \right] ,
\]  
(91)

with the exception that all functions and operators are pseudo-complex. The bar over a function indicates the normal complex conjugation. The \(s\) indicates the two possible spin directions and \(E_P = \sqrt{P^2 + M^2}\).

The anti-commutator relations at equal time are set to

\[
\{ \Psi(X, X_0), \Psi^\dagger(Y, X_0) \} = I^n \delta^{(3)}(X - Y) .
\]  
(92)
All other anti-commutators are zero. There are the two possibilities, $n = 0$ or $n = 1$. The case $n = 0$ leads to standard anti-commutation relations, while $n = 1$ leads to the commutation relations as discussed in [33]. We choose, as in the boson case, the standard anti-commutation relations.

The result is

$$\{ b(P, s), b^\dagger(P', s') \} = \delta_{ss'} \delta^{(3)}(P' - P)$$
$$\{ d(P, s), d^\dagger(P', s') \} = \delta_{ss'} \delta^{(3)}(P' - P)$$

(93)

and all other anti-commutation relations equal to zero.

Again, the propagator of the form (61) is only obtained when in terms of the fields it is defined as

$$I \langle 0 | \Psi(X') \Psi(X) | 0 \rangle.$$  

(94)

Also here, the in- and out-going states are obtained by first mapping $P^\mu_\pm \to p^\mu$ and $X^\mu_\pm \to x^\mu$. The field is then a simple in- and out-going plane wave, multiplied with the corresponding Dirac-spinor.

C. The Electro-Magnetic Field

The procedure is completely analogous to the one outlined in the two last cases. The quantized electro-magnetic field is [1]

$$A(X) = \int \frac{d^3 P}{(2\pi)^3 2\omega_P} \sum_{\lambda=1,2} e(P, \lambda) \left[ a(P, \lambda) e^{-iP \cdot X} + a^\dagger(P, \lambda) e^{iP \cdot X} \right].$$

(95)

The interpretation is analogous to the usual field theory, with the exception that the fields and variables are now pseudo-complex. The $\lambda$ indicates the two polarization directions and $e(P, \lambda)$ are the unit vectors of the polarization $\lambda$.

As in the scalar and Dirac field cases, the in- and out-going waves are proportional to $e^{i\mu} exp(\mp ip^\mu x^\mu)$. The quantization rule for equal pseudo-complex time is

$$[\Pi_i(X, X_0), A^j(Y, X_0)] = i \delta^{(tr)}_{ij} (X - Y)$$
$$= i \left[ \delta^{(tr)}_{ij} (X_+ - Y_+) \sigma_+ 
+ \delta^{(tr)}_{ij} (X_- - Y_-) \sigma_- \right],$$

(96)

with the transversal delta function on the right hand side of the equation.

The pseudo-scalar mass of this field has a zero $\sigma_+$ component, which is related to the zero physical rest mass at low energy. The $\sigma_-$ component has to be large, as argued also in the case of a bosonic and fermionic field. Again, the only mass scale left, equal to $l^{-1}$, is taken for the $\sigma_-$ component, denoted by $N$. It is reflected in the dispersion relations $\omega_P = \omega_+ + \omega_- + p \sigma_+$ and $\omega_+ = \sqrt{P^2 + N^2}$. This choice leads, with the additional $I$ in the definition of the pseudo-complex propagator, to

$$\frac{1}{P^2_+ \sigma_+} - \frac{1}{P^2_+ - N^2 \sigma_-}.$$  

(97)

Setting $P^\mu_\pm = p^\mu$ and extracting the pseudo-real part, leads to

$$\frac{1}{p^2} - \frac{1}{p^2 - N^2}.$$  

(98)
A consequence of (98) is an effective mass of the photon as a function in energy. We set the propagator in (98) equal to $1/(p^2 - m(\omega)^2)$, with $m(\omega)$ being an effective rest mass at a fixed energy $\omega$. Solving for $m(\omega)$ yields $p^2/N$.

At energies in the GeV range, the $m(\omega)$ is of the order of $10^{-20}$ GeV, far too low to be measured. Thus, the photon appears to have no mass. At energies of $10^{11}$ GeV, the scale of the GZK limit, this mass rises to about 500 GeV. It sounds large, but it has to be compared with the energy scale, giving a ratio of about $5 \times 10^{-9}$.

This has a measurable effect on the dispersion relation. The energy of the photon is given by

$$\omega^2 = k^2 + m(\omega)^2,$$

(100)

where we used $(\hbar = 1) E = \omega$ and $p = k$. Solving for $\omega$, using Eq. (99) leads in lowest order in $l$ to ($N = l^{-1}$)

$$\frac{\omega}{k} = 1 - \frac{1}{2}(lk)^2,$$

(101)

which shows the deviation from the light velocity. For energies at the GZK scale ($\omega = 10^{11}$ GeV) and using $l = 5 \times 10^{-20}$ GeV$^{-1}$, the second term on the right hand side acquires the value of $2.5 \times 10^{-18}$, still quite low. For energies of the order of 50 TeV = 50000 GeV the effect is even smaller, about $10^{-34}$. In [34] upper limits on the correction to the speed of light were deduced for energies in the TeV range, using experimental observations. The stringent limit, obtained for the case of Compton scattering of photons in the TeV range, is $< 10^{-16}$.

For a free propagating photon, the effect of the mass $N$ via the vacuum polarization can be assimilated in a renormalization of the charge, as shown in [1]. It results in a modification, due to the mass scale $l^{-1}$, to the dependence on the energy scale in the running coupling constant. Thus, renormalization is still necessary, although it is not related to erase infinities any more.

There is an interesting interpretation of the zero component of the vector potential. Using the propagator and searching for the Fourier transform, gives for the $\sigma_+$ part a simple Coulomb potential $-1/r$, while the $\sigma_-$ part describes a propagating particle with mass $N$, i.e., it results in a Yukawa potential $e^{-Nr}$. Projecting to the pseudo-real part, with $r_\pm = r$, gives

$$A_0 \sim -\frac{1}{r} \left(1 - e^{-Nr} \right).$$

(102)

For large $r$ it is essentially the Coulomb potential. However, for $r$ of the order of $\frac{l}{N} \sim l$ a deviation appears. For $r \to 0$ we get $A_0 \to 0$, which is a large number.

VII. CALCULATION OF SOME CROSS SECTIONS

We will determine two different cross sections: a) the dispersion of a charged particle in an external Coulomb field and b) the Compton scattering. The steps are analogous to the ones described in [1]. The two cross sections chosen, differ to lowest order in the internal lines, which is a photon in the first case and a fermion in the latter.

We use the proposed projection method on how to extract numerical results. It requires the construction of the building blocks of the $S$-matrix elements, i.e., the $\sigma_+$ component of the fields and the pseudo-real part of the propagators, and then compose the $S$-matrix element. When a field appears in internal lines, it is treated similar to the propagators, i.e., the sum of the $\sigma_\pm$ components have to be taken. The cross section is proportional to the square of the $S$-matrix element.

We take into account only electro-magnetic interactions. The united electro-weak field theory should be used, because the interesting deviations will probably occur at very large energies. This would, however, explode the scope of the present contribution. To get more realistic cross sections at ultra-high energies, we refer to a later publication.
A. Scattering of a charged Particle at a Coulomb Potential

We proceed in a completely analogous way as in Ref. [1]. The transition matrix element is given by

\[ S_{fi} = -ie \int d^3X \tilde{\Psi}_f(X) \gamma^\mu A_\mu(X) \Psi_i(X) , \]  

(103)

where the indices \( i \) and \( f \) refer to the initial and final state respectively. The fields in Eq. (103) are substituted according to the rules formerly established.

The in- and out-coming field are given by

\[ \Psi_i(x) = \sqrt{\frac{m}{E_i V}} u(p_i, s_i) e^{-ip_i \cdot x} \]
\[ \Psi_f(x) = \sqrt{\frac{m}{E_f V}} \bar{u}(p_f, s_f) e^{-ip_f \cdot x} , \]  

(104)

with \( E_{i/f} = \sqrt{p_{i/f}^2 + m^2} \). The Coulomb field describes the mediating photons and one has to take the pseudo-real component of

\[ A_0(X) = 2 \left[ -\frac{Ze}{4\pi |X_+|} \sigma_+ + \frac{Ze}{4\pi |X_-|} e^{-N|X_-|} \sigma_- \right] . \]  

(105)

Determining the partial cross section involves integration over the coordinates, which we assume to be along the pseudo-real axis, i.e., \( |X| = r \), and \( P_\pm = p \).

Taking the pseudo-real part of (105), leads to the transition matrix element

\[ S_{fi} = iZe^2 \frac{1}{V} \sqrt{\frac{m^2}{E_f E_i}} \frac{\bar{u}(p_f, s_f) \gamma^0 u(p_i, s_i)}{|q|^2} 2\pi \delta(E_f - E_i) \]
\[ -iZe^2 \frac{1}{V} \sqrt{\frac{m^2}{E_f E_i}} \frac{\bar{u}(p_f, s_f) \gamma^0 u(p_i, s_i)}{|q|^2 + N^2} 2\pi \delta(E_f - E_i) , \]  

(106)

with \( q = p_f - p_i \). The mass \( N \), is the \( \sigma_- \) component of the photon’s pseudo-complex mass.

One finally arrives at the expression

\[ \frac{d\sigma}{d\Omega} = 4Z^2\alpha^2 m^2 \left( \frac{1}{|q|^2} - \frac{1}{|q|^2 + N^2} \right)^2 \frac{1}{|\bar{u}(p_f, s_f) \gamma^0 u(p_i, s_i)|^2} . \]  

(107)

Using the mean value of the cross section for different spin orientations [1] and the kinematic relations \( E_i = E_f = E \) (elastic scattering) \( (p_i \cdot p_f) = E^2 - p^2 \cos\theta \), we arrive at

\[ \frac{d\bar{\sigma}}{d\Omega} = \frac{Z^2\alpha^2}{4p^2 \beta^2 (\sin^2 \frac{\theta}{2})^4} \left( 1 - \beta^2 \sin^2 \frac{\theta}{2} \right) \]
\[ \left[ 1 - \frac{p^2 \sin^2 \frac{\theta}{2}}{N^2 + 4p^2 \sin^2 \frac{\theta}{2}} \right]^2 . \]  

(108)

The bar over the \( \sigma \) indicates the summation over the spin directions of the in- and out-coming particle. The factor in front of \( "[..]^{2n} \) is the Mott formula for the scattering of an electron at a Coulomb potential of a nucleus.

Considering that \( N = \frac{1}{4} \), we get to lowest order in \( l \).
\[
\frac{d\bar{\sigma}}{d\Omega} \approx \frac{d\bar{\sigma}}{d\Omega}_{\text{Mott}} \left[ 1 - 8g^2 p^2 \sin^2 \frac{\theta}{2} \right].
\]  

(109)

The largest correction is at back scattering (\(\theta = \pi\)). However, even for linear momenta near the GKZ cutoff (\(p \approx 10\,11\) GeV), the corrections are of the order of \(10^{-16}\) \((l \approx 5 \times 10^{-20}\text{GeV}^{-1}\), corresponding to the Planck length), beyond any hope to be measured in near future. At momenta in the TeV range, the situation is even more hopeless. The corrections would be of the order of \(10^{-31} - 10^{-32}\).

### B. Compton Scattering

The calculation of the cross section proceeds in the same way as explained in [1]. Traces of \(\gamma\)-matrices appear which are of the form (we use the Dirac notation \(A = \gamma_m A^m\))

\[
B_{\sigma_1,\sigma_2} = \text{Tr} \left[ \frac{\hat{p}_f + m}{2m} \Gamma_{\sigma_1} \frac{\hat{p}_i + m}{2m} \bar{\Gamma}_{\sigma_2} \right],
\]

(110)

with \(\sigma_k = \pm\) and

\[
\Gamma_{\pm} = \frac{\hat{\epsilon} \cdot \hat{\epsilon}'}{2p_i \cdot k + (m^2 - M_\pm^2)}\]

\[
+ \frac{\hat{\epsilon} \cdot \hat{k} + M_\pm \hat{\epsilon}'}{2p_i \cdot k + (m^2 - M_\pm^2)}
\]

(111)

and \(\bar{\Gamma} = \gamma^0 \Gamma \gamma^0\). We use \(M_+ = m\) and \(M_- = \frac{1}{\lambda}\). For the plus sign we get the usual expression. The two possibilities of \(\Gamma_\pm\) appear because to the propagator \(1/(\hat{p} - m)\) of the SFT one has to add the second term \(-1/(\hat{p} - M_-)\).

When the minus index is used, we can exploit the large value of \(M_- > p_i, p_f, k\) and approximate \(\Gamma_-\) through

\[
\Gamma_- \approx -\frac{2(\epsilon \cdot \epsilon')}{M_-}.
\]

(112)

We arrive finally at the expression for the total cross section, having made the usual relations between \(p_i\) and \(p_f\) [1] and evaluate the cross section in the laboratory frame, with \(p_i = (m,0,0,0)\). We obtain

\[
\frac{d\bar{\sigma}}{d\Omega} (\lambda', \lambda) \approx \frac{1}{4m^2} \alpha^2 \omega^2 \left\{ \frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 4(\epsilon \cdot \epsilon')^2 - 2 \right\}
\]

\[
-4 \frac{m}{M_-} \left\{ \frac{1}{m} (\epsilon \cdot \epsilon')(\epsilon \cdot k')(\epsilon' \cdot \epsilon) \left( \frac{1}{\omega} - \frac{1}{\omega'} \right) \right.
\]

\[
\left. + (\epsilon \cdot \epsilon')^2 \left( \frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 2 \right) \right\}.
\]

(113)

Summing over the initial polarizations \((\lambda, \lambda')\) of the photons we arrive at

\[
\frac{d\bar{\sigma}}{d\Omega_k} \approx \frac{\alpha^2 \omega^2}{2m^2 \omega} \left( \frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2 \theta \right)
\]

\[
\left\{ 1 - \frac{m}{M_-} \left[ \frac{|\omega'|^2 \cos \theta \sin^2 \theta + (1 + \cos^2 \theta)(\omega' + \omega + 2)}{|\omega' + \omega - \sin^2 \theta |} \right] \right\}.
\]

(114)

As can be seen, the correction is proportional to \(\frac{m}{M_-} = ml\). The deviations are increased when heavy particles, like \(W^\pm\) and \(Z\) bosons are involved. Choosing \(\theta\) in the forward or backward scattering and using a particle of mass
where $u$ now the reciprocity theorem of Born. which treats momentum and coordinate on an equal footing. This was already observed by M. Born [36, 37], called He argued for this combination of position and momentum in the same line element due to the uncertainty relation momentum

\[ \approx \]

electron.

a shift of the GZK limit. The language will be held simple in order to avoid unnecessary complex notations. Related to our approach. It also will give a hint on how to extend the pseudo-complex field theory such that it permits processes which include dominantly electron internal lines.

C. Lamb Shift and magnetic Moment of the electron

We also looked at possible changes in the Lamb shift and the magnetic moment of the electron. After having applied the charge and mass renormalization [1], the main corrections come from the internal photon line and it is proportional to $q^2 l^2 = q^2 * 25 * 10^{-40}$, with $q$ being the momentum exchange. Because of the smallness of $l$ and $q$, the corrections are far less than the current accuracy of about $10^{-11}$ in, e.g., the anomalous magnetic moment of the electron.

Thus, the appearance of a smallest length scale in the pseudo-complex field theory does not result in measurable effects, considering standard high precision experiments. The only hope to see a difference is the possible observation of the GZK cutoff.

VIII. RELATION TO GEOMETRIC APPROACHES

It is illustrative to show a connection to geometrical formulations in flat space-time, especially those which are related to our approach. It also will give a hint on how to extend the pseudo-complex field theory such that it permits a shift of the GZK limit. The language will be held simple in order to avoid unnecessary complex notations.

Caianiello showed in (1981) [35] the existence of a maximal acceleration, by combining the metric in the coordinate and momentum space. This metric is very similar to the one in Eq. (117) below, for the pseudo-complex description. He argued for this combination of position and momentum in the same line element due to the uncertainty relation which treats momentum and coordinate on an equal footing. This was already observed by M. Born [36, 37], called now the reciprocity theorem of Born.

To show more details, let us define an orbit in the space-time manifold by

\[ X^\mu = x^\mu + lI u^\mu, \]

\[ \text{(115)} \]

where $u^\mu = \frac{dx^\mu}{d\tau}$ is the four-velocity, $l$ the invariant length scale and $\tau$ the proper time. The $l$ appears for dimensional reasons. It is a generalized notation also encountered and justified in section 3.1, including, besides the position of the observer, the information about his four velocity. In Eq. (115), the observer is not only characterized by its position $x^\mu$ in the Minkowski space, but also by its four-velocity, which defines a 4-bein along the world line he realizes in the Minkowski space. Eq. (115) is a possibility to unite in one coordinate the position $x^\mu$ of the observer with the co-tangent space, given by the 4-bein, defined through $u^\mu = \frac{dx^\mu}{d\tau}$. The geometrical implications are much more involved, related to the fiber bundle description on which we will not elaborate here.

In a similar way the four velocity $w^\mu = \frac{dx^\mu}{d\tau}$ and the four momentum $p^\mu = \gamma m a^\mu$ are modified to

\[ U^\mu = u^\mu + lI a^\mu, \]

\[ P^\mu = p^\mu + lI f^\mu, \]

\[ \text{(116)} \]

with $a^\mu = \frac{du^\mu}{d\tau}$ as the four-acceleration and $f^\mu$ as the four-force. $U^\mu$ is obtained through the derivation of $X^\mu$ with respect to the eigen-time.

The scalar product with respect to the $dX^\mu$, defines a new line element, given by [21, 27]

\[ d\omega^2 = \eta (dX, dX) = dX^\mu dX_\mu = dx^\mu dx_\mu + l^2 du^\mu du_\mu + lI(2dx^\mu dx_\mu) . \]

\[ \text{(117)} \]

The $d\omega$ is also considered as a generalized proper time. Now, $x^\mu u_\mu = x^\mu \frac{dx^\mu}{d\tau} = 0$. Therefore, it can be rewritten as (with $u^\mu = \frac{dx^\mu}{d\tau}$, $a^\mu = \frac{du^\mu}{d\tau}$ and $d\tau^2 = dx^\mu dx_\mu$),
\[ d\omega^2 = d\tau^2 (1 - l^2 g(u,a)) \]
\[ g(u,a) = -a^\mu a_\mu = -a_0^2 + a_1^2. \]  

(118)

Using \( l^2 g(u,a) = \frac{g^2}{G_0} \), with \( G_0 = \frac{1}{l^2} \), and requiring the positive definiteness of the metric \( (d\omega^2 > 0) \), we arrive at the maximal acceleration \( G_0 \).

The new proper time \( d\omega \) is related to the standard eigen-time \( d\tau \) via

\[ d\omega = \sqrt{1 - l^2 g^2} \, d\tau. \]  

(119)

The factor in front of \( d\tau \) reflects an additional \( \gamma \) factor, added to the one in special relativity.

In contributions based on a geometric description \( [23, 38, 39] \), one usually defines the \( d\omega^2 \) as

\[ d\omega^2 = dx^\mu dx_\mu + l^2 du^a du_a \]  

(120)

alone. This metric is invariant under transformations of \( O(2, 6) \) (the measure contains two time and 6 space components: \( dx^0, du^0 \) and \( dx^4, du^4 \)). Comparing this with the pseudo-complex metric, the difference is in the term \( 2I(dx^\mu du_\mu) = 0 \). This might be irrelevant. However, as stated in \( [27] \), its omission leads to a contradiction to the Tachibana-Okumura no-go theorem \([40]\). It states that when the space-time manifold has an almost complex structure, as the theories published in \( [38, 39] \) have, a parallel transport does not leave invariant this structure. In contrast, when the line element is chosen as in \([117]\), the space-time manifold has an almost product structure and the Tachibana-Okumura theorem is satisfied. However, in \( [38, 39] \) the symplectic structure, which produces the almost complex structure, is essential in order to maintain the commutation relations between the coordinates and momenta. This indicates that there are still important points not understood, which have to be studied yet.

In \( [38, 41] \) the representation theory is discussed, allowing only canonical, symplectic transformations, \( Sp(4) \). This restriction is necessary in order to maintain the commutation relation of the coordinates with the momenta invariant. The intersection is the group \( U(1,3) \simeq O(2,6) \cup Sp(4) \). Including translations, Low arrives at what he denominates as the group \( CR(1,3) \simeq U(1,3) \oplus H(1,3) \), where \( \oplus \) refers to the semi-direct product and \( H(1,3) \) to the Heisenberg group in 1+3-dimensions. For details, consult the references \( [38, 41] \).

Beil \( [39, 42, 43, 44] \) included the electromagnetic gauge potential into the metric, showed the connection to a maximal acceleration and claims a relation to Finslerian metrics. The approach by Beil, however, is put on doubt \([45, 46, 47, 48]\), due to several inconsistencies, like identifying the energy with the Lagrangian at one point and mixing notations in \([39]\).

There are several other geometrical approaches, where the relation to ours is not so clear up to now: Brandt \( [46, 47, 48] \), developed a geometric formulation, including gauge transformations. All gauge fields are included in the metric.

We also mention different geometrical approaches, based on the pseudo-complexification of geometry. To our knowledge, the first person, who proposed this extension is A. Crumeyrolle. In \([22]\) pseudo-complex numbers where introduced (called by him hyperbolic numbers) and the coordinates \( x^\mu \) of space-time were complexified hyperbolically. A couple further contributions appeared \( [49, 50] \) and other authors continued on this line \( [51, 52, 53, 54] \). The theory presented has some mathematical overlap to our formulation but the basic physical description and philosophy are different.

As a last example of the geometric approach we mention Refs. \( [14, 15, 16] \). They introduce a preferred velocity, thus, breaking rotational invariance explicitly. The Poincaré group breaks down to a subgroup with 8 generators. Lorentz invariance is explicitly broken and it is proven to be related to a Finslerian metric. Three different solutions of possible metrics are discussed, corresponding to a space-, time- and light-like velocity vector. How this is related to the pseudo-complex description is not clear yet. Only in the former mentioned approaches a relation is presented for a flat space-time manifold.

Conclusion: In flat space-time there is a correspondence to some geometric approaches previously discussed in the literature. They will give a hint on how to extend the extraction to physically observable numbers.

IX. EXTENSION OF THE PSEUDO-COMPLEX FIELD THEORY

The last section contains a hint on how possibly to extend the pseudo-complex field theory. It is related to a modification of the extraction procedure. Up to now, the \( P^\mu \) has two linear independent components, namely the
pseudo-real $P^{\mu}_1$ and the pseudo-imaginary $P^{\mu}_2$ one. In the last section we saw, however, that one can interpret the pseudo-imaginary component in a consistent way as a force, acting on the particle along its world line. This can be seen as a projection to a subspace in the pseudo-complex space of $P^\mu$, with the constriction of $P^{\mu}_2 = \ell f^\mu$. Therefore, instead of setting the pseudo-imaginary component to zero, it is substituted by $\ell f^\mu$. This is equivalent to add to the Lagrange density an additional term, reflecting the effect of the particle’s acceleration during the interaction. But it is more: This interaction originates as a part of the pseudo-complex linear momentum and, thus, represents an extension of the minimal coupling scheme to the pseudo-complex formulation:

$$P_\mu \rightarrow p_\mu + I f_\mu \ .$$  \hfill (121)

We can then proceed in the same way as done in \cite{20}, where the first results of the PCFT, related to the shift of the GZK limit, were presented. The equation of motion for a Dirac particle changes to

$$\left(\gamma^\mu (p_\mu + I f_\mu) - M\right) \Psi \in \mathcal{P}^0 \ ,$$  \hfill (122)

with $\mathcal{P}^0 = \mathcal{P}^0_+ \cup \mathcal{P}^0_-$, is the set of zero divisors. The $f_\mu$ may contain a dependence on the photon field, but not necessarily.

Using $P^{\mu}_+ = p^{\mu} \pm \ell f^\mu$, multiplying by the pseudo-complex conjugate of the operator gives $(P_+ - M_+)(P_- - M_-)\Psi = 0$, and subsequently multiplying by $(\gamma_\mu [p^\mu - I f_\mu] + M_-)$ $(\gamma_\mu [p^\mu + I f_\mu] + M_+)$ and using the properties of the $\gamma^\mu$ matrices, we arrive at the equation

$$(P_{+\mu} - M^2_+)(P_{-\mu} - M^2_-)\Psi = 0 \ .$$  \hfill (123)

Selecting the first factor, using $P_{+\mu} P^{\mu}_+ = E^2 - p^2 + l^2 f_\mu f^{\mu} + l(p_\mu f^{\mu} + f_\mu p^{\mu})$, we arrive at the dispersion relation

$$E^2 = p^2 + (l f)^2 + l(p f + f p) + M^2_+ ,$$  \hfill (124)

with $f^2 = -f_\mu f^{\mu} > 0$. and $p f = -p_\mu f^{\mu}$, $f p = -f_\mu p^{\mu}$. When $f^\mu$ is a force, it is proportional to $\frac{d\psi}{dt}$ and, thus, $p f = f p = 0$.

This leads to a modification of the threshold momentum, for the production of pions in a collision of a proton with a photon from the CMB, \cite{20}

$$p_{1,\mathrm{thr}} \approx \frac{(m_2 + \tilde{m}_3)^2 - \tilde{m}_1^2}{4\omega} \approx \frac{(m_2 + m_3)^2 - m_1^2}{4\omega}$$

$$+ \frac{l^2}{4\omega} \left[ (m_2 + m_3) \left( \frac{f_2^2}{m_2} + \frac{f_3^2}{m_3} \right) - f_1^2 \right] .$$  \hfill (125)

The analysis showed that, if $f^\mu$ is interpreted as a Lorentz force, the maximal shift of the GZK is at most by a factor of two. Equation (125) is the result of a ”back on the envelope calculation”, with the energy parameter $\omega$ of the photon from the CMB. It suffices to estimate the shift, but in order to obtain the shape of the cosmic ray spectrum, a complete determination involves the folding with the thermal spectrum of the CMB \cite{56}.

### X. CONCLUSIONS

In this contribution we presented a pseudo-complex formulation of Quantum field theory, suggested schematically in Refs. \cite{23, 27}, however, without further calculations. The pseudo-complex field theory (PCFT) shows important properties, like i) it contains a maximal acceleration, implying a scalar minimal length parameter, due to which ii) it is Pauli-Villars regularized, iii) maintains the concept of gauge invariance and iv) for each particle two mass scales appear, where one is associated to the usual physical mass and the other one to a mass of Planck scale, as argued in the main text.

The appearance of a smallest length scale in the theory is by itself interesting, asking: What are its influences on possible observable deviations to SFT? Where and how we have to search for it?

A new variational procedure had to be used, leading to the two mass scales, associated to each particle. The quantization process was discussed and shown how to define, in a consistent manner, propagators. Two distinct
quantization formulations were investigated. The first one leads to standard (anti-)commutation relations, while the second one has an opposite sign in the (anti-)commutation relations of the fields in the $\sigma_-$ component, leading possibly to a different field theory. The deep physical consequences of choosing one or the other are not clear in detail. We indicated that they lead to some equivalent results, like the field equations, suggesting a connection.

An extraction procedure has been formulated for obtaining physical observable numbers, which are pseudo-real. The cross sections for the scattering of a charged particle at an external Coulomb field and the Compton scattering where determined and deviations to SFT calculated. As one result, differences to SFT are most probably detected, when processes with fermion internal lines are considered. In this case, the deviations are proportional to the minimal length scale $l$, while for photon internal lines the deviation is proportional to $l^2$. The largest correction is of the order of $10^{-18}$. These results show that the introduction of a minimal length scale does not modify sensibly the old results of SFT at the energies applied up to now.

The effect of $N$ on the effective photon mass was also discussed, leading to the dispersion relation $\omega \approx k \left(1 - \frac{1}{2}(lk)^2\right)$. At energies of the GZK scale, the corrections are of the order of $10^{-18}$. At $TeV$ range, these corrections reduce to the order of $10^{-34}$, far too low to be observed in near future. The actual experimental upper limits of the correction to the light velocity is of the order of $10^{-16}$, for the Compton scattering of photons at $50 TeV$ [34].

Finally, we discussed a modification of the theory, which allows a shift of the GZK limit. First results were published in [20]. The relation of the present theory to several geometric approaches was discussed, showing that there is an overlap, but also differences appear which have still to be understood. It is important to note that in the pseudo-complex field theory, as presented here, a clear method of quantization is available and on how to extract cross sections. Discussing the geometrical relation we obtained hints on how to extended the minimal coupling scheme.

Important problems ly ahead: One has to include the unified electro-weak field theory, because the interesting processes happen at high energy, where effects of the $W$ and $Z$ bosons are of significance. It will be necessary to calculate the dispersion of a proton when interacting with a photon of the CMB, producing a pion and $e^\pm$ pairs, in order to obtain the cross section at ultra high energies. The Auger experiment measures this cross section. Indeed, the shift of the GZK limit is presently the only existing experimental hint for a new microscopic structure, a smallest length scale. Another interesting line of investigation is to search for an inclusion of Gravity in the pseudo-complex formulation, i.e., General Relativity. The effects of a smallest length scale have to be investigated. For example, it would be interesting to consider the modified Scharzschild metric, giving clues on how $l$ affects the structure of a black hole.

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APPENDIX A: FOURIER TRANSFORMS AND THE PSEUDO-COMPLEX DELTA FUNCTION

Let us start with the pseudo-complex $\delta$-function: It is defined as

$$\hat{\delta}(X - Y) = \frac{1}{(2\pi)} \int dP e^{iP(X - Y)} .$$  \hspace{1cm} (A1)

A straight line in the "space-like" sector is given by $X = R e^{i\phi_0}$, with $\phi_0 = const$. The ratio of the pseudo-imaginary and pseudo-real part is

$$X_1 = R \cosh(\phi_0)$$
$$X_2 = R \sinh(\phi_0)$$
$$\frac{X_2}{X_1} = \tanh(\phi_0) .$$  \hspace{1cm} (A2)

The integration over this straight line gives
\[ \hat{\delta}(X - Y) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int dR e^{i\phi_0} e^{R e^{i\phi_0}(X - Y)} \]
\[ = \frac{e^{I\phi_0}}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{+\infty} dR \left\{ e^{Re^{i\phi_0}(X_+ - Y_+)} \sigma_+ \\
+ e^{Re^{-i\phi_0}(X_- - Y_-)} \sigma_- \right\} \]
\[ = e^{I\phi_0} \left( \delta(e^{\phi_0}(X_+ - Y_+)) \sigma_+ \\
+ \delta(e^{-\phi_0}(X_- - Y_-)) \sigma_- \right) , \tag{A3} \]

where we used that \( e^{I\phi_0} = e^{\phi_0} \sigma_+ + e^{-\phi_0} \sigma_- \). Using this relation again, \( \sigma_\pm = 0 \) and \( \sigma_+ \sigma_- = 0 \), and the property \( \delta(aX) = \frac{1}{|a|} \delta(X) \), we arrive at

\[ \delta(X_+ - Y_+) \sigma_+ + \delta(X_- - Y_-) \sigma_- , \tag{A4} \]

which acts like the usual delta function, i.e.

\[ \int f(X) \delta(X - Y) \, dX = \int f(X_+) \delta(X_+ - Y) \sigma_+ \\
+ \int f(X_-) \delta(X_- - Y) \sigma_- \\
= f(Y) \sigma_+ + f(Y) \sigma_- \\
= f(Y) . \tag{A5} \]

For an integration on a “time-like” straight line, for \( X \) the parametrization \( IRe^{x_0(I\phi_0)} \) has to be used. The final result is

\[ I \left[ \delta(X_+ - Y_+) \sigma_+ + \delta(X_- - Y_-) \sigma_- \right] , \tag{A6} \]

which acts almost like the usual delta function, i.e.

\[ \int f(X) \delta(X - Y) \, dX = \int f(X_+) \delta(X_+ - Y) \sigma_+ \\
- \int f(X_-) \delta(X_- - Y) \sigma_- \\
= f(Y_+) \sigma_+ - f(Y_-) \sigma_- \\
= I f(Y) . \tag{A7} \]

It does not give the function \( f(Y) \) but \( If(Y) \).

Let us now return to the Fourier transform in 1-dimension. We define it as

\[ \tilde{F}(P) = \frac{I^x}{(2\pi)^{\frac{1}{2}}} \int dX F(Y) e^{-iPY} . \tag{A8} \]

Let us calculate

\[ \frac{1}{(2\pi)^{\frac{1}{2}}} \int dP \tilde{F}(P) e^{iPX} . \tag{A9} \]

Substituting \( \tilde{F}(P) \), we arrive at
The expression in the parenthesis gives \( I^\xi \delta(X - Y) \), with \( \xi = 0 \) for an integration along a straight line in the "space-like" sector and \( \xi = 1 \) for the integration along a straight line in the "time-like" sector. The final result is \( I^\xi F(X) \), showing that the inverse transformation is given by

\[
F(X) = \frac{I^\xi}{(2\pi)^{n_2/2}} \int dP \tilde{f}(P)e^{iPX} .
\]  

(A11)

For a multi-dimensional integral, the factor in front changes to \( I^{n_2/2}(2\pi)^{d} \), where \( n_2 \) gives the number of times one integrates in a "time-like" sector and \( d \) is the dimension of the integral. In order to be consistent, one is only allowed to integrate along straight lines. Arbitrary curves lead to integrals which can not be related to simple delta-functions.

Wether the extended definition of the \( \delta \)-function to the pseudo-complex space plays an important rôle has still to be verified.

**APPENDIX B: COMMUTATION RELATIONS OF \( a \) AND \( a^\dagger \)**

With a bar above a pseudo-complex variable we indicate usual complex conjugation. With this we have

\[
\int \tilde{f}_P(X)\Phi(X)d^3X = \int d^3P' a(p') \int \tilde{f}_P(X)f_{P'}(X)d^3X \\
+ \int d^3P' a^\dagger(p') \int \tilde{f}_P(X)f_{P'}(X)d^3X .
\]

This involves integrals of the type \( \int e^{i(P-P')X}d^3X \). In Appendix A we saw that these integrals are related to a generalized \( \tilde{\delta}(X - Y) \) function.

There is, therefore, an ambiguity over which curve we have to integrate. This is the same problem we face for the integration over \( X \). In order to keep a certain liberty, we multiply such an integral with \( I^\xi \), where \( \xi \) has yet to be specified. It is zero when the integration is taken along a straight line in the "space-like" sector and it is 1 when the integration path is along a straight line in the "time-like" sector. The same holds for an integration in the coordinate space, where the index \( P \) is substituted by \( X \).

Finally we arrive at

\[
\int \tilde{f}_P(X,x_0)\Phi(X,X_0) = I^\xi \left\{ \frac{1}{2\omega_p}a(P) + \frac{1}{2\omega_p}a^\dagger(-P)e^{2i\omega_pX_0} \right\} .
\]

(B2)

Using the conjugate momentum of the field, we get also

\[
\int \tilde{f}_P(X,x_0)D_0\Phi(X,X_0) = I^\xi \left\{ -\frac{i}{2}a(P) + \frac{i}{2}a^\dagger(-P)e^{2i\omega_pX_0} \right\} .
\]

(B3)

The two equations lead to

\[
a(P) = iI^\xi \int d^3X \tilde{f}_P(X,X_0) \leftrightarrow D_0\Phi(X,X_0)
\]

\[
a^\dagger(P) = -iI^\xi \int d^3X f_P(X,X_0) \leftrightarrow D_0\Phi(X,X_0) ,
\]

(B4)

with \( A(X_0) \leftrightarrow D_0B(X_0) = A(X_0)(D_0B(X_0)) - (D_0A(X_0))B(X_0) \).

The commutator of the operators \( a \) and \( a^\dagger \) is, using Eq. (31),

\[
[a(P),a^\dagger(P')] = i^{n+\xi} \delta^{(3)}(P - P') ,
\]

(B5)
where one integration in \( X \) had to be performed.

It depends now very much on the path of integration, which quantization property we obtain of the operators \( a \) and \( a^\dagger \). This might have also consequences on the physical theory, which are not clear yet.

Conversely, starting from the commutation relation (B5) of the creation and annihilation operators, the one for the field and its conjugate momentum is

\[
[\Phi(X, X_0), \Pi(Y, X_0)] = i\tau^{+\xi_x + \xi_p}\delta^{(3)}(X - Y),
\]

which requires to set \( \xi_x \pm \xi_p \) to even values, for reasons of consistency to Eq. (84). This implies that either both are zero or both are 1. It is not clear what an integration along the pseudo-imaginary axis of the momentum means, i.e., the case \( \xi_x = \xi_p = 1 \).

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