The infrared triangle in the context of IR safe S matrices

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In the present note we show that the recently established connections between soft theorems, large gauge transformations and memories are persistent in the infrared safe formulation of quantum field theory. They take a different and simplified form and can all be derived from the non-trivial asymptotic dynamics that is proper to any theory with massless fields. Most of the results in this paper were already presented in [1] and, with a different interpretation, in [2] and [3]. The new parts (as compared to [1]) are an improved derivation of charges for large gauge transformations in the framework of non-trivial asymptotic dynamics, the connection to the classical memory effect and an overall more accessible treatment of the topic. Since the formulation of QFT without infrared divergences is physically more appealing, the infrared safe version of the above connections should be so as well.

I. INTRODUCTION

In the last years a remarkable universal structure in classical and quantum field theories, dubbed the infrared triangle, was discovered and studied in detail, see e.g. [4–7] and references therein. A central result underlying these studies goes under the name of soft theorems - a relation between amplitudes in gauge theories with a soft particle in the final state to amplitudes without such particles. The soft theorems are a manifestation of a general problem in QFTs with massless particles: the amplitudes for soft emission are divergent and hence such theories are said to possess no S matrix. How can it be that a manifestation of the problematic infrared divergences is related (and in fact equivalent) to fundamental symmetries and even to observable effects such as memories? In order to answer this natural question one needs to first focus on the resolution of the IR problem in QFT and then see how the triangle manifests itself in the divergence free, IR safe theory. In the present paper we will do just that. Since in the IR safe theory the soft photon theorem is replaced by decoupling of soft particles, the triangle becomes in a sense even simpler than before.

Luckily, most of the work was already done (and in fact done long ago) so all that is left is to connect the dots and make the picture apparent. To keep the paper short, we will refer the reader to original papers and a recent longer paper [1] co-authored by the author himself for details.

The key point of the present paper is the well known realization that in the presence of massless particles, i.e. long range interactions, the early and late time dynamics cannot be treated as free. Instead, non-trivial asymptotic dynamics have to be found and implemented in the construction of the S matrix. Once this is done, the S matrix is free from IR divergences. As we will show, the three corners of the infrared triangle are all consequences of these asymptotic dynamics. They are

1. Soft theorem: Soft particles decouple from the IR safe S matrix

\[
\lim_{\omega \to 0} [\omega a^\lambda(\omega e_k), S] = 0
\]  

2. Large gauge transformations: The charge of a LGT consists purely of soft particles, the non-trivial asymptotic dynamics kills the hard part. It thus commutes with the S matrix and this decoupling implies antipodal matching.

3. Memories: The asymptotic value of the (zero frequency of the) massless field (scalar, photon or graviton) carries memory about the scattering process. This memory has a classical manifestation.

The picture of the triangle hence gets modified to a tetrahedron. For clarity and to be concrete, throughout the paper we will focus on QED, but the discussion can be straightforwardly carried out for other field theories with massless particles, like gravity.
II. THE TRIANGLE FROM ASYMPTOTIC DYNAMICS

The first paper describing the modification of the asymptotic dynamics in presence of long range interactions was [3]. The idea was then applied to QED in [4-11] and many subsequent papers, see also [12] for a pedagogical introduction. As established there, asymptotic dynamics in presence of long range interactions was governed by the following evolution operator (here and throughout the paper we use the same notation as in the corresponding references unless stated otherwise):

\[ U_{as}(t) = e^{-iH_0t}e^{R(t)}e^{i\Phi(t)} \]  

(2)

with

\[ R(t) = \frac{e}{(2\pi)^3} \int \frac{p^\mu}{p \cdot q} \rho(p) \left( a^\dagger_\mu(q)e^{\frac{i}{\hbar} \int_{t_0}^t \rho(x)dx} - h.c. \right) \frac{d^3q}{2\omega} d^3p \]  

(3)

\[ \Phi(t) \sim \int :\rho(p)\rho(k): \frac{p \cdot k}{(p \cdot k)^2 - m^2} \text{sign}(t) \ln(|t|) d^3p d^3k \]

The phase operator plays no role in the following discussion and we will ignore it from now on.

This evolution operator gives the asymptotic current and electromagnetic field operators respectively as:

\[ J_{\mu}^{as}(x) = \int d^4p p_\mu \rho(p) \int dr \delta^4(x - pr) \]  

(4)

\[ A_{\mu}^{as}(x) = A_{\mu}^{free}(x) + \int d^4y \Delta^{ret}(x - y) J_{\mu}^{as}(y) \]  

(5)

The expressions are understood to have physical meaning only for large \(|t|\) and large separations. The above expression for \(J_{\mu}\) is simply the current of particles flying on straight lines through the origin. These equations are all that is needed to derive each corner of the triangle.

A. The soft decoupling

The definition of \(U_{as}\) results the following IR safe \(S\) matrix

\[ S_{KF} = \lim_{t_+ \to \pm \infty} e^{R(t_+)} S_D(t_+, t_-) e^{R(t_-)} \]  

\[ =: \lim_{t_+ \to \pm \infty} S_{KF}(t_+, t_-) \]  

(6)

Here \(KF\) stands for Kulish-Faddeev and \(S_D\) is the standard Dyson \(S\) matrix in the free interaction picture, i.e. \(S_D(t_+, t_-) = U_I(t_+, t_-)\).

The normal leading order soft photon theorem implies that

\[ \lim_{\omega \to 0} [\omega a^\mu(q), S_{KF}] = 0 \]  

(7)

This was already stated in [3] (though without proof). The proof involves a careful ordering of limits. First note that

\[ [a^\mu(q), S_{KF}(t_+, t_-)] = [a^\mu(q), e^{R(t_+)}] S_D(t_+, t_-) e^{R(t_-)} + e^{R(t_+)} [a^\mu(q), S_D(t_+, t_-)] e^{R(t_-)} \]

\[ + e^{R(t_+)} S_D(t_+, t_-) [a^\mu(q), e^{R(t_-)}] \]

hence to equation (7). This version of the proof was also presented in [1]. It is important to notice that the limits \(\omega \to 0\) and \(t_+ \to \pm \infty\) do not commute.

B. Charges of LGT from non-trivial dynamics

Under the assumption of free asymptotic dynamics a basis for the charges of large gauge transformations on \(\mathcal{F}^+\) in massless QED was found explicitly by Strominger et al to be (in a combination of the notation in [3] and the notation in [8]):
\[ Q_{\text{free}}^+(e_x) = \frac{1}{\sqrt{2 \epsilon}} \left( \lim_{\omega \to 0^+} \left[ \omega a_{\text{out}}(\omega e_x) + h.c. \right] \right) - 2e \int \frac{\omega p \cdot \varepsilon^+(e_x)}{p \cdot q} \rho_{\text{out}}(p) d^3p \]  
\[ =: Q_{\text{free}}^{+ \text{soft}}(e_x) + Q_{\text{free}}^{+ \text{hard}}(e_x) \]  

Here the notation means:
\[ q = \omega(1, e_x), \quad e_x = e_x(z, \bar{z}) \]  

and \( Q_{\text{free}}^+(e_x) \) is what was called
\[ Q^+(\varepsilon) \quad \text{with} \quad \varepsilon(w, \bar{w}) = \frac{1}{z(e_x) - w} \]

in [4].

The relation between any expressions on \( \mathcal{F}^+ \) derived us-

| \[ Q^+(e_x) = \frac{1}{\sqrt{2 \epsilon}} \left( \lim_{\omega \to 0^+} \left[ \omega a_{\text{out}}(\omega e_x) + h.c. \right] \right) 2e \int \frac{\omega p \cdot \varepsilon^+(e_x)}{p \cdot q} \rho_{\text{out}}(p) d^3p - 2e \int \frac{\omega p \cdot \varepsilon^+(e_x)}{p \cdot q} \rho_{\text{out}}(p) d^3p \]  

In other words, the non-trivial asymptotic dynamics has

killed the hard part of the charge that creates LGT and

we have
\[ Q^+(e_x) = \frac{1}{\sqrt{2 \epsilon}} \left( \lim_{\omega \to 0^+} \left[ \omega a_{\text{out}}(\omega e_x) + h.c. \right] \right) \]

A similar calculation applies to \( \mathcal{F}^- \) and results in
\[ Q^-(e_x) = \frac{1}{\sqrt{2 \epsilon}} \left( \lim_{\omega \to 0^-} \left[ \omega a_{\text{out}}(\omega e_x) + h.c. \right] \right), \]

where \( z, \bar{z} \) are now coordinates on \( \mathcal{F}^- \). Similar

statements (but with different derivations and interpre-

tations) can be found in [3]. The vacuum in QED is
degenerate, two vacua are related to each other by LGT

or equivalently by soft photons and the soft photon,

being the Goldstone boson of a spontaneously broken

global symmetry, decouples from the S matrix. The

picture becomes very unified and simple.

We find it necessary to comment on the meaning of equa-
tions [3] and [4] in order to avoid possible confusion. From

the fact that the charges of LGT do not contain a hard

part does not follow that they commute with the charged

fields. After all, if they did, they would not create large
gauge transformations anymore. There are two ways to

see that expressions \( Q^+ \) and \( Q^- \) still generate LGT de-

spite their purely soft appearance. The simplest way is to

note that when expressed in terms of asymptotic field op-

erators, the charges are given by the expressions (3.2) and

(5.6) in [3] (with the appropriate \( \varepsilon \), as in [1]) and hence

they obviously generate the correct transformations. An

alternative way would be to compute the action of \( Q^\pm \)
on the asymptotic expressions for field operators of charged

particles. This will be done in the appendix.

C. From decoupling to symmetry and matching

Due to the decoupling of soft modes from the IR safe

S matrix, [1] and the fact that charges of LGT contain

purely soft modes, it is clear that LGT constitute sym-

metries of the S matrix. The antipodal matching from

\( \mathcal{F}^- \) to \( \mathcal{F}^+ \) is obtained using that for any in-operator \( O_{in} \)
\[ S^\dagger O_{in} S = O_{out} \]  

and from the decoupling
\[ [S, Q^- (e_x)] = 0 \]

which results in the antipodal matching
\[ Q^- (e_x(z, \bar{z})) = Q^+(e_x(z, \bar{z})). \]  

See also [1] for an extended discussion of this topic.
D. From asymptotic dynamics to memory

The connection of the topics treated above to classical
memory is surprisingly simple and is most easily un-
derstood by comparing the works of Zwanzinger 10 and
Rohrlich 12 to the recent work by Tolish, Wald et al.
13,14. The crux is in the equations 4–5 which appear in
both approaches. A classical scattering process from
particles with incoming momenta \( \{ p_i^{\text{in}} \} \) to \( \{ p_j^{\text{out}} \} \) can be
reconstructed within QFT as having the system in the
state \( \{|p_i^{\text{in}}\rangle \} \) in the far past and in \( \{|p_j^{\text{out}}\rangle \} \) in the far future.
Assuming the interaction is to happen instantaneously
at time \( t = 0 \), as it was done in 13,14, it is appropriate
to use the asymptotic expressions 4–5 for all times \( t \neq 0 \).
The momenta eigenstates \( \{|p_i^{\text{in/out}}\rangle \} \) are eigenstates of
the asymptotic current, hence assuming that the state
\( \{|p_i^{\text{in}}\rangle \} \) evolves into the state \( \{|p_j^{\text{out}}\rangle \} \) at time \( t = 0 \) and
computing the expectation value of \( J_{\mu}^{\text{as}}(x) \) results in the
current given by formulas (21) and (22) in [14]. Then
the expectation value of the asymptotic electromagnetic potential \( A_{\mu}^{\text{as}}(x) \) coincides with formula (24) in [14],
which leads to electromagnetic memory.

One can also make a direct connection to memories with-
out referring to the work by Tolish et al. As described in
13 and 14, the classical EM memory (in the absence of
charged massless particles) can be reconstructed from
\[
\Delta W = \lim_{t \to \infty} r A_\mu(u = \infty, r, e_x) - r A_\mu(u = -\infty, r, e_x)
\]
where we rewrote equation (46) from 13 in terms of \( \mathscr{S}^{\pm} \) quantities, just as it was done in 14. Now in QFT one
must replace \( A_\mu \) by \( \langle A_\mu \rangle \). In the far past and future the
time evolution is given by the formulas 4 and 5. Thus, one
can find the memory of a scattering process by computing
\[
\lim_{u \to \infty} \langle p_j^{\text{out}} \mid r A_\mu^{\text{as}}(u, r, e_x) \mid p_j^{\text{out}} \rangle \quad (19)
\]
and
\[
\lim_{u \to -\infty} \langle p_j^{\text{in}} \mid r A_\mu^{\text{as}}(u, r, e_x) \mid p_j^{\text{in}} \rangle \quad (20)
\]
and using equations 4 and 5 or equivalently directly 2 and
the usual identities for coherent states. In that context
see also 12,17 and references therein. This results pre-
cisely in the formula (48) from 14. In this context see
also 19 where memories were studied from a different
point of view.

E. Memories and large gauge transformations

In classical electromagnetism the angular components of
the field \( A_\mu \) are pure gauge (i.e. derivatives of a scalar)
at \( u \to \pm \infty \) for any scattering and hence the electromag-
netic memory is found from the difference of two pure
gauge configurations at \( \mathscr{S}^{\pm} \) - a large gauge transforma-
tion. This can be immediately read off the formula (3.3)
in 6 which coincides with the fixed angle LGT charge
(3.4) of 5. At the same time we found that soft pho-
tons and hence the charges of LGT decouple from the \( S \)
matrix. Note that these statements are not in conflict
with each other - also in the previous treatments charges
for LGT decoupled from the (IR divergent) \( S \) matrix.
However, since in the IR safe scenario the probability to
excite soft photons is zero - just like the probability to
tunnel from one vacuum to another in any field theory
with a continuous vacuum degeneracy - one cannot say
that the physically real memory effect is directly related
to an emission of soft particles. The soft factor from
the scattering amplitudes determines the memory of a
scattering process because it enters the dressing operator
\( R(t) \) and this in turn is the only important quantity for
the evaluation of 19 and 20. This completes the infrared
tetrahedron (figure 1) in the context of IR safe theories.

F. A note on other field theories

Although we have focused on electrodynamics in this pa-
per, most of the derivations and concepts hold for any
theory with massless particles. For example, the deriva-
tion of the infrared triangle in gravity is a straightforward
(though technically more challenging) analogy. However,
the triangle seems incomplete for the case of massless
scalars with the missing corner being large gauge trans-
formations. All other corners are still there for scalars -
the asymptotic dynamics is nontrivial. It might be
possible to uncover large gauge transformations for scalar
fields by using their duality to gauge fields, see 20.

III. COMPARISON WITH THE TRIANGLE IN
THE IR DIVERGING THEORY

Just as in the above references, the related literature on
the infrared triangle usually considers the asymptotic
dynamics for field operators to be free. In such a setup
the antipodal matching of LGT is equivalent to the
usual soft theorems which encode infrared divergences

\footnote{That can be seen either from direct computation or from
the form of \( J_{\mu}^{as} \) and a comparison to the classical treatment of 18
or also from the treatment of 14.}

\footnote{This idea was communicated to me by Gia Dvali}
of the quantum theories. Thus, it might appear that the divergences of amplitudes follow from fundamental symmetries and hence cannot be circumvented. This would be particularly inconvenient, since usually in the very same treatments the notion of the S matrix is used - but it is precisely the IR divergences that lead to the non-existence of such an operator. The connection to memory is then done through analogues of formula (5.6) of [5] where the ratio of two amplitudes is interpreted as the asymptotic expectation value of a field operator. To the author’s knowledge there is no a priori derivation of this relation.

In the IR safe scenario the picture is significantly clearer. An IR safe S-matrix exists and soft particles decouple from it. LGT are generated by purely soft quanta and their antipodal matching is equivalent to the soft decoupling. The memory effect is found directly from studying asymptotic expectation values for field operators for a determined scattering process, no formulas like the above mentioned (5.6) of [5] are needed.

IV. CONCLUSIONS AND OUTLOOK

In this paper we have presented the infrared triangle within the IR safe theory where it is enhanced to a tetrahedron. All corners and the connections between them have been highlighted. A natural next step would be to include subleading (in the frequency of the soft particles) results. In order to do that, subleading terms in the dressing operators $R(t)$ would be needed to be found. A first step in this direction was taken in [1], but the construction there was rather ad hoc. A more natural expansion in frequency would be preferable. Furthermore, the IR safe S matrix has not been made into a practical tool yet, in particular, there is no simple diagrammar for it (although some attempts and calculations have been carried out - see [17], [21] and references therein). It would also be very interesting to find the manifestation of the soft decoupling for other space-times different from Minkowski, in particular, for space-times with horizons where due to infinite redshift the infrared structure is much richer. We plan to address some of these issues in future work.

V. APPENDIX

It is clear that the charges [13] and [14] generate the correct gauge transformations on the asymptotic operator $A^a_\mu$ but less obvious that they also generate the right transformation for matter fields. In order to explicitly demonstrate that on an example, we look at the asymptotics of a massless scalar field $\phi$ on $\mathcal{I}^+$. Using the saddle point approximation for large $r$ (see [6] and the appendix of [1]) and the asymptotic time evolution\(^4\) one obtains

$$\phi(u, r \gg 1, e_x) \sim \frac{1}{r} e^{R(e_x, t)} \int e^{i a_p \cdot p_{\text{out}}(p, e_x)} d\mu$$

(21)

where we only write out the particle but not the anti-particle part and we have defined

$$R(e_x, t) := \frac{e}{(2\pi)^3} \int \frac{p^\mu}{p \cdot q} \rho(p) \left(a^+_\mu(q)e^{\frac{i \omega_{p\rho}}{\omega}} - h.c.\right) \frac{d^3q}{2\omega}$$

(22)

with

$$p_\mu = p_0(1, e_x)$$

(23)

so that the above expression is indeed independent of $p_0$. Then using the commutation relation

$$\lim_{\omega \to 0} \left[\omega a_\pm(\omega e_y), e^{R(e_x, t)}\right] = \frac{p \cdot e_\pm(\omega e_y)}{p \cdot (1, e_y)} e^{R(e_x, t)}$$

(24)

one obtains that on $\mathcal{I}^+$

$$[Q^+_y(\phi(u, e_x)] = -\varepsilon(e_y)\phi(u, e_x)$$

(25)

as it must be. Here we again used the notation as indicated in [14].

VI. ACKNOWLEDGMENTS

I would like to thank Cesar Gomez for many useful discussions and for bringing several important references to my attention. This work was supported by the ERC Advanced Grant “UV-completion through Bose-Einstein Condensation” (Grant No. 339169).

\(^3\) This makes immediate physical sense since the time scale on which soft particles interact is infinity.

\(^4\) Note that from the soft decoupling follows that the ratio of amplitudes in such expressions vanishes.
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