STRINGY MOTIVES OF SYMMETRIC PRODUCTS

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Abstract. Given a complex smooth algebraic variety \( X \), we compute the generating function of the stringy motives of its symmetric powers as a function of motive of \( X \). In dimension two we recover the Göttsche formulas for Hilbert schemes. We use the formalism of \( \lambda \)-rings to get a particularly compact formula, which is convenient for explicit computations.

Introduction

The stringy motive \([X]_{\text{st}}\) of a log terminal complex algebraic variety \( X \) (see e.g. [1, 3]) is an invariant of \( X \) such that whenever \( X \) possesses a crepant resolution \( Y \to X \), the motive \([Y]\) coincides with \([X]_{\text{st}}\). Here motives are elements of certain extension \( A_\infty \) of the Grothendieck ring \( A \) of the category of Chow motives and the map \( X \mapsto [X] \) sending an algebraic variety to its motive is the one constructed by Gillet and Soulé [4]. The purpose of this paper is to compute the stringy motives of symmetric products of smooth manifolds.

Let us denote by \( L \) the motive \([\mathbb{A}^1]\), called the Lefschetz motive. It is known that the Grothendieck ring \( A \) has a structure of the \( \lambda \)-ring, where the \( \sigma \)-operations are defined on the level of varieties by symmetric products (see e.g. [5] for this result and basic definitions concerning \( \lambda \)-rings). In particular, \( \sigma_n(L) = L^n \). Given any \( \lambda \)-ring \( R \), we endow the ring \( R[[t]] \) with a structure of \( \lambda \)-ring by \( \sigma_n(a t^k) := \sigma_n(a) t^{nk}, a \in R \). Define the map \( \exp : R[[t]]^+ \to 1 + R[[t]]^+ \) (see e.g. [5] or [14, Appendix]) by

\[
\exp(f) := 1 + \sigma_1(f) + \sigma_2(f) + \ldots
\]

Clearly \( \exp(f + g) = \exp(f) \exp(g) \). Using the Adams operations (and assuming that \( \mathbb{Q} \subset R \)) one can also write

\[
\exp(f) = \exp \left( \sum_{k \geq 1} \frac{1}{k} \psi_k(f) \right).
\]

Our main result is the following

**Theorem 1.** Let \( X \) be a complex smooth algebraic variety of dimension \( d \geq 2 \). Then

\[
\sum_{n \geq 0} [X^{(n)}]_{\text{st}} t^n = \exp \left( \frac{[X] t}{1 - \mathbb{E}^{d/2} t} \right),
\]

where \( X^{(n)} \) denotes the symmetric product.

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If $X$ is a surface then there is a canonical crepant resolution $X^{[n]} \to X^{(n)}$, where $X^{[n]}$ denotes the Hilbert scheme of points and we get

$$\sum_{n \geq 0} [X^{[n]}] t^n = \exp \left( \frac{[X]}{1 - L^1 t} \right).$$

This formula is equivalent to the formula of Göttsche [7, 8, 10, 2, 9] but is more compact and probably more convenient for explicit computations (certainly, using some computer algebra system). In such computations one can use the above formula for $\exp$ in terms of the Adams operations as the Adams operations on the Hodge (or Poincaré) polynomials are particularly easy to describe. We discuss this together with some examples in the second section.

The idea of the proof of the theorem is to use the result of Batyrev [1] saying that the stringy motive of a finite quotient of a smooth manifold coincides with an orbifold motive. In our situation, we consider the action of the symmetric group $S_n$ on $X_n$, where the orbifold motive is very easy to compute. It should be noted that originally the Euler number of the Hilbert scheme was computed first and then it was shown that it coincides with the orbifold Euler number [11]. Thus, our proof is rather artificial but is probably the shortest one.

1. **Stringy motives**

For the definitions and basic properties of stringy motives and motivic integration we refer to [1, 3]. We will only define the ring where our motives live. Let $A$ be the Grothendieck ring of Chow motives. It is an algebra over $\mathbb{Z}[L]$ (here $L$ is considered as a variable). We define $A_n := A \otimes \mathbb{Z}[L^{\pm 1/n}]$ and $A_{\infty} := \text{colim} A_n$; thus, $A_{\infty}$ contains arbitrary rational powers of $L$. There is a non-Archimedean norm $|−| : A_{\infty} \to \mathbb{R}_{\geq 0}$ satisfying $|[X]| = 2^{\dim X}$ and $|L^s| = 2^s$. Let $\hat{A}_{\infty}$ be the completion of $A_{\infty}$ with respect to this norm. As it was mentioned in the introduction, the ring $A$ has the structure of a $\lambda$-ring; it can be extended to $\hat{A}_{\infty}$ in such a way that $\sigma_n(a L^s) = \sigma_n(a) L^{ns}$ for any $a \in A$ and $s \in \mathbb{Q}$. For any log terminal algebraic variety $X$, one can define an invariant $[X]_{st} \in \hat{A}_{\infty}$ called the stringy motive of $X$ (see [1, 3]).

Let $X$ be a complex smooth algebraic variety of dimension $d$, $G$ be a finite group acting effectively on $X$ and suppose that there exists a geometric quotient $X/G$. We assume that the ramification locus, that is $\cup_{g \neq 1} X^g$, has codimension at least 2 in $X$ and that $X^g$ is connected for any $g \in G$. Let us define the orbifold motive of the pair $(X, G)$ according to [1]. Given a vector space $V$ of dimension $d$ and a linear operator $T$ on $V$ of finite order, let $(e^{2\pi i \alpha_1}, \ldots, e^{2\pi i \alpha_d})$ with $0 \leq \alpha_k < 1$ be the eigenvalues of $T$. Define the weight

$$\text{wt}(T) := \sum_{k=1}^d \alpha_k \in \mathbb{Q}.$$ 

For any $g \in G$, define the weight $\text{wt}(g)$ to be the weight of the action of $g$ on $T_x X$, for some $x \in X^g$. It does not depend on the choice of the point $x \in X^g$. Define then the orbifold motive

$$[X/G]_{\text{orb}} := \sum_{[g] \subseteq G} [X^g/C(g)] \cdot L^{\text{wt}(g)},$$
where \([g]\) runs over the conjugacy classes of \(G\) (or rather their representatives) and \(C(g) \subset G\) is the centralizer of \(g\).

**Theorem 1.1** (see [1, Theorem 7.5]). It holds \([X/G]_{st} = [X/G]_{orb}\).

We apply this theorem in the case of the symmetric group \(S_n\) acting in a natural way on \(X^n\), where \(X\) is smooth. For any partition \(\lambda = (\lambda_1, \lambda_2, \ldots) = (1^{a_1} \cdots r^{a_r})\), we define \(X^\lambda := X^{(a_1)} \times \cdots \times X^{(a_r)}\). We define the length of \(\lambda\) to be \(l(\lambda) := a_1 + \cdots + a_r\) and the weight of \(\lambda\) to be \(|\lambda| := \sum_{i \geq 1} \lambda_i = \sum_{k \geq 1} ka_k\).

**Proposition 1.2.** Let \(X\) be a complex smooth algebraic variety of dimension \(d \geq 2\). Then, for any \(n \geq 1\),

\[ [X^{(n)}]_{st} = \sum_{|\lambda|=n} [X^{(\lambda)}] \cdot L^{d(n-l(\lambda))}, \]

where the sum runs over all partitions \(\lambda\) of \(n\).

**Proof.** Consider the natural action of \(S_n\) on \(Y = X^n\). The conjugacy classes of \(S^n\) are parameterized by the partitions of \(n\), where a partition \(\lambda = (1^{a_1} \cdots r^{a_r})\) corresponds to the permutations \(g \in S_n\) that have \(a_1\) 1-cycles, \(a_2\) 2-cycles etc. The manifold \(Y^g\) is isomorphic to \(X^{a_1} \times \cdots \times X^{a_r}\) and has codimension at least \(d \geq 2\) in \(Y\) whenever \(\lambda \neq (1^n)\), i.e., \(g \neq 1\). The centralizer of \(g\) consists of those elements of \(S_n\) that permute the cycles of \(g\). It follows that the variety \(X^g/C(g)\) is isomorphic to \(X^{(a_1)} \times \cdots \times X^{(a_n)} = X^{(\lambda)}\). We note that the cycle \((12\ldots k)\) acting by permutation on \(\mathbb{C}^k\) has an eigenvalue \(\xi = e^{2\pi i j/k}\) with an eigenvector \((\xi, \xi^2, \ldots, \xi^k)\) for any \(j = 0, \ldots, k-1\). This implies that the weight of the permutation equals \((1 + \cdots + (k-1))/k = (k-1)/2\). It follows that the cycle \((12\ldots k)\) acting on \((\mathbb{C}^d)^k\) by permutation has weight \(d(k-1)/2\). Finally, an element \(g\) of type \(\lambda = (1^{a_1} \cdots r^{a_r})\) in \(S_n\) has weight

\[ \text{wt}(g) = \sum_{k=1}^r \frac{d(k-1)}{2} a_k = \frac{d}{2} (n - l(\lambda)). \]

Now the statement of the proposition follows from the previous theorem. \(\square\)

We are ready to prove the main result of the paper.

**Proof of Theorem** Using the previous proposition we get

\[ \sum_{n \geq 0} [X^{(n)}]_{st} t^n = \sum_{\lambda} [X^{(\lambda)}] \cdot \mathbb{L}^{d(l(\lambda)-l(\lambda))} t^{l(\lambda)} \]

\[ = \sum_{\lambda=(1^{a_1} \cdots r^{a_r})} \prod_{k=1}^r ([X^{(a_k)}] \cdot \mathbb{L}^{d(ka_k-a_k)} t^{ka_k}) \]

\[ = \sum_{\lambda=(1^{a_1} \cdots r^{a_r})} \prod_{k=1}^r \sigma_{a_k}([X] \cdot \mathbb{L}^{d(k-1)} t^k) \]

\[ = \prod_{k \geq 1} \prod_{a \geq 0} \sigma_a([X] \cdot \mathbb{L}^{d(k-1)} t^k) = \prod_{k \geq 1} \exp([X] \cdot \mathbb{L}^{d(k-1)} t^k) \]

\[ = \exp(\sum_{k \geq 0} [X] t (\mathbb{L}^{d/2} t)^k) = \exp \left( \frac{[X] t}{1 - \mathbb{L}^{d/2} t} \right). \]

\(\square\)
Remark 1.3. In the same way as above one can show also
\[ \sum_{n \geq 0} \frac{[X^{(n)}]_{st} t^n}{L^{dn/2}} = \text{Exp} \left( \frac{[X]}{L^{d/2}} \frac{t}{1 - t} \right). \]

2. Examples

We endow the ring \( \mathbb{Z}[x_1, \ldots, x_r] \) with the structure of a \( \lambda \)-ring by \( \sigma_n(x^\alpha) = x_1^{n\alpha}, \) where \( \alpha \in \mathbb{Z}_+^r \). Most conveniently, this \( \lambda \)-structure can be written using the Adams operations
\[ \psi_n(f(x_1, \ldots, x_r)) = f(x_1^n, \ldots, x_r^n), \quad f \in \mathbb{Z}[x_1, \ldots, x_n]. \]

In order to avoid any problems with the Adams operations in what follows, we tensor our \( \lambda \)-rings with \( \mathbb{Q} \) without mentioning that and so we always assume that our \( \lambda \)-rings contain \( \mathbb{Q} \).

There are three basic realizations of the ring of motives \( A \). The Euler number \( e : A \to \mathbb{Z} \), the virtual Poincaré polynomial \( P : A \to \mathbb{Z}[v] \), and the virtual Hodge polynomial (also called Hodge-Deligne polynomial) \( E : A \to \mathbb{Z}[u, v] \) (see e.g. [4]). The Euler number of a variety can be defined as the Euler characteristic of the complex of cohomologies with compact support. To define the virtual Hodge polynomial or the virtual Poincaré polynomial, one uses the mixed Hodge structure on the cohomologies with compact support. In the case of the Lefschetz motive one has \( e(L) = 1, P(L) = v^2, E(L) = uv \). The relations between the above three realizations are
\[ e(X) = P(X; 1), \quad P(X; v) = E(X; v, v). \]

An important property of the above realizations is that they are morphisms of \( \lambda \)-rings. In the case of the Poincaré polynomial this corresponds to the results of Macdonald [12]. In the case of the Hodge polynomial see e.g. [15]. All three realizations can be extended to \( A_\infty \) after extending accordingly the rings \( \mathbb{Z}, \mathbb{Z}[v], \) and \( \mathbb{Z}[u, v] \). In this way one can also define the stringy Euler number, the stringy Poincaré function, and the stringy Hodge function (usually called stringy E-function). After all these remarks we can rewrite Theorem 1 substituting there any of three described realizations. For example, given a complex smooth algebraic variety \( X \) of dimension \( d \geq 2 \), it holds
\[ \sum_{n \geq 0} P(X^{(n)}_{st}) t^n = \text{Exp} \left( \frac{P(X) t}{1 - v^{dt}} \right). \]

Example 2.1. For the Euler numbers we get
\[ \sum_{n \geq 0} e(X^{(n)}_{st}) t^n = \text{Exp} \left( \frac{e(X) t}{1 - t} \right) = \prod_{k \geq 1} \text{Exp}(t^k e(X)) = \prod_{k \geq 1} (1 - t^k)^{-e(X)}. \]

For surfaces, this is the Göttsche formula for the Euler numbers of Hilbert schemes.

As it has already been mentioned, the \( \lambda \)-structure on the ring of polynomials is written most conveniently using the Adams operations. In view of the above formulas, it is important to know the formula for \( \text{Exp} \) in terms of the Adams operations. There is a formal identity between symmetric functions (see e.g. [13 2.10])
\[ \sum_{n \geq 0} h_n t^n = \exp(\sum_{k \geq 1} \frac{1}{k} p_k t^k) \]
which implies
\[
\text{Exp}(f) = \sum_{n \geq 0} \sigma_n(f) = \exp\left( \sum_{k \geq 1} \frac{1}{k} \psi_k(f) \right).
\]

**Example 2.2.** Let us find \( P(X^{[2]}), \) where \( X \) is a K3-surface. It is known that \( P(X; v) = v^4 + 22v^2 + 1. \) Applying the main theorem to the stringy Poincaré functions we get, noticing that the Adams operations are ring homomorphisms
\[
1 + P(X)t + P(X^{[2]})t^2 + o(t^2) = \text{Exp}(P(X)t(1 + v^2t)) + o(t^2)
\]
\[
= \exp\left( P(X)(t + v^2t^2) + \frac{1}{2} \psi_2(P(X))t^2 \right) + o(t^2).
\]

This implies
\[
P(X^{[2]}) = v^2 P(X) + \frac{1}{2} \psi_2(P(X)) + \frac{1}{2} P(X)^2 = v^8 + 23v^6 + 276v^4 + 23v^2 + 1.
\]

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