SHARP ESTIMATES FOR HOLOMORPHIC FUNCTIONS ON KÄHLER MANIFOLDS AND DIMENSION ESTIMATES

GANG LIU

Abstract. We generalize the classical Hadamard three circle theorem to the curved case which is also sharp. In particular, we prove two sharp monotonicity formulae for holomorphic functions. Among applications, we give the sharp dimension estimate (with rigidity) of holomorphic functions of polynomial growth when the holomorphic sectional curvature is nonnegative. When the bisectional curvature is nonnegative, this was done by Ni[26]. By using Ni’s method, Chen, Fu, Le and Zhu[2] removed the maximal volume growth condition in Ni’s paper. Note that there are complex manifolds which admit Kähler metric with positive holomorphic sectional curvature, yet do not even admits Kähler metric with nonnegative Ricci curvature. Moreover, the method is completely different and it could be adapted to the case when the holomorphic sectional curvature is only asymptotically nonnegative. This in particular includes the case when the holomorphic sectional curvature is nonnegative outside a compact set. Moreover, the degree is sharp comparing with the complex Euclidean space. Furthermore, if the size of the compact set is small (we normalize the curvature on the compact set), we can even get sharp dimension estimates (including coefficient) comparing with complex Euclidean space.

1. Introduction

In [33], Yau proposed the uniformization conjecture for complete noncompact Kähler manifolds with nonnegative holomorphic bisectional curvature. In his words, “the question is to demonstrate that every noncompact Kähler manifold with positive bisectional curvature is biholomorphic to the complex Euclidean space. If we only assume the nonnegativity of bisectional curvature, the manifold should be biholomorphic to a complex vector bundle over a compact Hermitian symmetric space.” See also [9][30]. So far there have been a lot of works on this conjecture, for example, [24][25][5]. However, Yau is conjecture is still open in general. It was further asked in [33] whether or not the ring of the holomorphic functions with polynomial growth is finitely generated, and whether or not the dimension of the spaces of holomorphic functions of polynomial growth is bounded from above by the dimension of the corresponding spaces of polynomials on \( \mathbb{C}^n \).

To bound of the dimension of holomorphic functions with polynomial growth, it suffices to bound the vanishing order by the growth rate. The following result was proved by Mok in [24]:

**Theorem 1.** Let \( M^n \) be a complete Kähler manifold with positive Ricci curvature \( n \geq 2 \), such that for some fixed point \( p \in M \),
Scalar curvature $\leq \frac{C_0}{d(p,x)^2}$ for some $C_0 > 0$;

Vol$(B(p,r)) \geq C_1 r^{2n}$ for some $C_1 > 0$.

Let $f \in O_d(M)$, then there exists a constant $C$ independent of $f$ such that the vanishing order of $f$ at $p$ satisfies

$$\text{ord}_p(f) \leq Cd.$$ 

Here $O_d(M)$ denotes holomorphic functions with polynomial growth of order $d$. See section 2 for the precise definition.

In [26], Ni proved a remarkable theorem which confirmed the conjecture of Yau on dimension of holomorphic functions by assuming the maximal volume growth condition:

**Theorem 2.** Let $M^n$ be a complete Kähler manifold with nonnegative holomorphic bisectional curvature. Assume $M$ is of maximal volume growth, then

$$\dim(O_d(M)) \leq \dim(O_d(\mathbb{C}^n))$$

for any positive integer $d$. If the equality holds for some $d$, $M$ is isometric and biholomorphic to $\mathbb{C}^n$.

Ni’s method is parabolic. The key tool is a monotonicity formula for plurisubharmonic functions:

**Theorem 3.** [26] Let $M$ be a complete Kähler manifold with nonnegative bisectional curvature. Let $v(x,t)$ be a family of plurisubharmonic functions deformed by the heat equation $(\frac{d}{dt} - \Delta)v(x,t) = 0$ such that $w(x,t) = \frac{d}{dt}v(x,t)$ is continuous for $t > 0$. Then

$$\frac{\partial}{\partial t}(tw(x,t)) \geq 0.$$ 

Theorem 3 follows from a gradient estimate of Li-Yau type [23], which resembles the trace form of Hamilton’s Li-Yau-Hamilton differential inequality [11] originally called the differential Harnack inequality, for the Ricci flow. The desired vanishing order estimate for holomorphic functions could be derived from the monotonicity formula above.

In [28], Ni and Tam proved that on a complete Kähler manifold with nonnegative bisectional curvature, if a plurisubharmonic function has exponential growth, then the plurisubharmonicity is preserved under the heat flow. Thus theorem 3 holds if we only assume $v(x,0)$ is plurisubharmonic and of exponential growth. By using same technique in [26], Chen, Fu, Le and Zhu [2] removed the maximal volume growth condition in theorem 2.

In this paper we first generalize theorem 2 to the case when the holomorphic sectional curvature is nonnegative.

**Theorem 4.** Let $M^n$ be a complete noncompact Kähler manifold with nonnegative holomorphic sectional curvature. Then for any $d > 0$, $\dim(O_d(M)) \leq \dim(O_d(\mathbb{C}^n))$. Moreover, if the equality holds for some positive integer $d$, $M$ is biholomorphic and isometric to $\mathbb{C}^n$. 
This condition in theorem 4 is much weaker than the nonnegativity of holomorphic bisectional curvature. For instance, it is not known whether the volume comparison theorem holds in this case. In the last section of this paper, we shall give examples showing that certain complex manifolds admit Kähler metric with positive holomorphic sectional curvature, but do not even admit Kähler metric with nonnegative Ricci curvature.

Theorem 4 will be deduced from the generalized Hadamard Three Circle Theorem. Recall the classical Hadamard Three Circle Theorem:

**Theorem 5.** Let \( f(z) \) be a holomorphic function on the annulus \( r_1 \leq |z| \leq r_3 \). Let \( M(r) \) be the maximum of \( |f(z)| \) on the circle \( |z| = r \). Then \( \log M(r) \) is a convex function of \( \log r \). In other words,

\[
\log \left( \frac{r_3}{r_1} \right) \log M(r_2) \leq \log \left( \frac{r_3}{r_2} \right) \log M(r_1) + \log \left( \frac{r_2}{r_1} \right) \log M(r_3)
\]

where \( r_1 < r_2 < r_3 \).

We generalize the Hadamard Three Circle theorem to Kähler manifolds with holomorphic sectional curvature lower bound. When the holomorphic sectional curvature is nonnegative, it says \( \log M(r) \) is still a convex function of \( \log r \). Note that the generalized Three Circle Theorem is new even when the bisectional curvature is nonnegative. See theorem 6 and theorem 12 for the precise statement. As corollaries, we obtain two sharp monotonicity formulae (corollary 1 and corollary 3) for holomorphic functions on complete Kähler manifolds with nonnegative holomorphic sectional curvature.

Theorem 4 is a consequence of corollary 1 and corollary 3. Since the Hadamard Three Circle Theorem (theorem 12) does not require the holomorphic sectional curvature to be nonnegative, we can extend theorem 4 to the case when the holomorphic sectional curvature is only asymptotically nonnegative and the power is still sharp. See theorem 15. This in particular includes the case when the holomorphic sectional curvature is nonnegative outside a compact set.

We also apply the Hadamard Three Circle Theorem to holomorphic maps between complete Kähler manifolds satisfying certain curvature conditions. With certain normalization, we can show that these maps form a compact set. Moreover, we can give sharp dimension estimate of the tangent space of the moduli of these maps. It is reasonable that the moduli should have some nice global structure, we shall study this in the future.

We expect more applications of the Hadamard Three Circle theorems in the future. It might be closely related with the major conjecture of Yau when the manifold has nonnegative holomorphic bisectional curvature. For example, in [14], we employ the idea in this paper to show that the ring of holomorphic functions of polynomial growth is finitely generated, provided the manifold has dimension 2 with nonnegative curvature.

The proof of the Hadamard Three Circle Theorem is surprisingly simple. It only uses the Hessian comparison theorem and the maximum principle once. In the maximum principle, we only consider the Hessian in one direction, not the Laplacian.
In the Riemannian case, there have been many articles on estimating the dimension of the harmonic functions of polynomial growth. For instance, Colding and Minicozzi [4] proved that for complete manifolds \( M^m \) with nonnegative Ricci curvature, the dimension of harmonic functions with polynomial growth is finite. In [18], Li produced an elegant short proof. However, the example of Donnelly [7] shows that the sharp inequality comparing with the Euclidean space is not true. The sharp upper bound is only obtained either when \( d = 1 \) or \( m = 1 \) in \([19][20]\) by Li and Tam. The rigidity part for \( d = 1 \) is due to Li [16] and Cheeger-Colding-Minicozzi [1]. Li and Wang [21] showed that when the sectional curvature is nonnegative and manifold has maximal volume growth, an asymptotic sharp estimate is valid. Thus there is a subtle difference between the Riemannian case and the Kähler case. One can refer to [15][17] for nice survey of these results.

Apart from the introduction, this paper is organized as follows: In section 2, we prove the Hadamard Three Circle Theorem when the holomorphic sectional curvature is nonnegative. Two monotonicity formulae are derived as corollaries. We prove theorem 4 as a byproduct. In section 3, a Liouville type theorem for plurisubharmonic functions is proved when the holomorphic sectional curvature is nonnegative. Note that when the holomorphic bisectional curvature is nonnegative, the result is due to Ni and Tam [28]. Section 4 deals with the holomorphic bundle case. The results are similar to the holomorphic function case. In section 5, we study holomorphic maps between certain complete noncompact Kähler manifolds. We prove some compactness result under some finite growth condition, also we obtain a sharp upper bound of the dimension of the moduli. Section 6 introduces the Hadamard Three Circle Theorem in the general setting. We assume that the holomorphic sectional curvature has a lower bound which might depend on the distance. In particular, we state the corresponding theorem when the holomorphic sectional curvature is no less than \(-1\) or \(1\). Then in section 7, we study holomorphic functions on complete Kähler manifolds with holomorphic sectional curvature asymptotically nonnegative. Sharp dimension estimates are obtained. In section 8, some miscellaneous results are proved. For example, we prove that in some cases, holomorphic functions with exponential growth have finite dimension. Moreover, the power is sharp. Section 9 concludes this paper with some examples showing that certain complex manifolds admit Kähler metric with positive holomorphic sectional curvature, but do not even admit Kähler metric with nonnegative Ricci curvature.

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2. Hadamard Three Circle Theorem, Special Case

In this section we study holomorphic functions on complete noncompact Kähler manifolds with nonnegative holomorphic sectional curvature. We prove the following theorem which is a generalization of the Hadamard three-circle theorem:

**Theorem 6.** Let \( M \) be a complete noncompact Kähler manifold with nonnegative holomorphic sectional curvature. Let \( p \in M \) and \( f \) be a holomorphic function on \( M \). Define \( r(x) = \text{dist}(x, p) \) and let \( M(r) \) be the maximum of \(|f(x)|\) for \( x \in \partial B(p, r) \). Then \( \log M(r) \) is a convex function of \( \log r \). The conclusion can also be written in the form

\[
\log \left( \frac{r_3}{r_1} \right) \log M(r_2) \leq \log \left( \frac{r_3}{r_2} \right) \log M(r_1) + \log \left( \frac{r_2}{r_1} \right) \log M(r_3)
\]

where \( r_1 < r_2 < r_3 \).

**Definition 1.** Let \( M \) be a complete noncompact Kähler manifold. Let \( O(M) \) denote the ring of holomorphic functions on \( M \). For any \( d \geq 0 \), define

\[
O_d(M) = \{ f \in O(M) | \lim_{r \to \infty} \frac{|f(x)|}{r^d} < \infty \}.
\]

Here \( r \) is the distance from a fixed point on \( M \). If \( f \in O_d(M) \), we say \( f \) is of polynomial growth with order \( d \). We also define

\[
O'_d(M) = \{ f \in O(M) | \lim_{r \to \infty} \frac{M(r)}{r^d} < \infty \}.
\]

Clearly \( O_d(M) \subseteq O'_d(M) \).

**Corollary 1 (Sharp Monotonicity I).** Under the same assumption in theorem 6, if \( f \in O'_d(M) \), then \( \frac{M(r)}{r^d} \) is nonincreasing.

**Proof:** We need to show that

\[
\frac{M(r_1)}{r_1^d} \geq \frac{M(r_2)}{r_2^d}
\]

when \( r_1 \leq r_2 \). By rescaling, we may assume \( r_1 = 1 \). From the assumption, for any \( \epsilon > 0 \), there exists a sequence \( \lambda_j \to \infty \) such that,

\[
\log M(\lambda_j) \leq \log M(1) + (d + \epsilon) \log \lambda_j.
\]

Now we take \( r_3 = \lambda_j \) sufficiently large in theorem 6

\[
M(r_2) \leq M(r_1)^{d+\epsilon}.
\]

The corollary follows if we let \( \epsilon \to 0 \). \( \square \)

**Corollary 2.** Let \( M \) be a complete noncompact Kähler manifold with nonnegative holomorphic sectional curvature, then \( O'_d(M) = O_d(M) \).

Now corollary 1 implies the sharp dimension estimate for holomorphic functions with polynomial growth. For reader’s convenience, we rewrite theorem 4 below.
Theorem 7. Let $M^n$ be a complete noncompact Kähler manifold with nonnegative holomorphic sectional curvature. Then for any $d > 0$, $\dim(O_d(M)) \leq \dim(O_d(\mathbb{C}^n))$. Moreover, if the equality holds for some positive integer $d$, $M$ is biholomorphic and isometric to $\mathbb{C}^n$.

Proof. Suppose for some $d$, the converse inequality holds, then by a linear algebra argument, for any given point $p \in M$, there exists a nonzero holomorphic function $f \in O_d(M)$ such that the vanishing order at $p$ is at least $[d] + 1$. Therefore

$$\lim_{r \to 0^+} \frac{M(r)}{r^d} = 0.$$ 

Corollary 1 says $\frac{M(r)}{r^d}$ is nonincreasing. Therefore $f \equiv 0$. This is a contradiction. We postpone the proof of the rigidity part to the end of this section. □

Remark. As we remarked in the introduction, one key point of Ni’s method to theorem 2 is that the plurisubharmonicity is preserved under the heat flow when the bisectional curvature is nonnegative. It does not seem obvious to the author whether this still holds when the holomorphic sectional curvature is nonnegative.

Corollary 3 (Sharp Monotonicity II). Under the same assumption in the theorem 6, if $f$ vanishes at $p$ with order at least $k$, then

$$M(r_1) r_2^{-k} \leq M(r_2) r_3^{-k} \leq M(r_3) r_2^{-k}$$

when $r_2 \leq r_3$. Again by rescaling, we assume $r_3 = 1$. Since $f$ vanishes at $p$ with order $k$, given any small $\epsilon > 0$, for $r$ sufficiently small,

$$\log M(r) \leq \log M(1) + (k - \epsilon) \log r.$$ 

Now take $r_1 = r$ sufficiently small in theorem 6

$$M(r_2) \leq M(r_3) r_2^{k-\epsilon}.$$ 

The corollary follows if $\epsilon \to 0$. □

Now we turn to the proof of theorem 6.

Proof. The proof follows from the maximum principle and the Hessian comparison theorem. We will use the following Hessian comparison theorem which appears in [22] by Li and Wang:
Theorem 8. Let $M$ be a complete Kähler manifold with holomorphic sectional curvature nonnegative. Let $r$ be the distance function to a point $p \in M$. Define $e_1 = \frac{1}{\sqrt{2}}(\nabla r - \sqrt{-1} J \nabla r)$. Then when $r$ is smooth, $r_{\perp r} \leq \frac{1}{2}$. Thus
\[
(\log r)_{\perp} = \frac{r_{\perp r}}{r} - \frac{r_{\| r}}{r^2} \leq 0.
\]

Since the proof is simple, we include it here for reader’s convenience.

Proof. Let $e_1$ be a unitary frame at $x \in M$ such that $r$ is smooth and $e_1 = \frac{1}{\sqrt{2}}(\nabla r - \sqrt{-1} J \nabla r)$. We also parallel transport the unitary frame along the geodesic from $p$ to $x$. Consider the Bochner formula
\[
0 = \frac{1}{2} |\nabla r|^2_{\perp} = r_{\perp r_{\perp}} + r_{\| r_{\perp}} + r_{\| r_{\parallel}} \\
\geq 2 r_{\perp r} + \frac{\partial r_{\perp}}{\partial r} + \frac{1}{2} r_{\parallel r^2_{\perp}}.
\]

Solving this inequality, with the initial condition of $r_{\perp r}$ when $r \to 0$, we obtain the proof. □

Now define
\[
F(x) = \log \left( \frac{r_3}{r} \right) \log M(r_1) + \log \left( \frac{r}{r_1} \right) \log M(r_3)
\]
for $0 < r_1 \leq r = \text{dist}(x, p) \leq r_3$. Also consider
\[
G(x) = \log \left( \frac{r_3}{r_1} \right) \log |f(x)|.
\]
It is clear that $F(x) \geq G(x)$ when $x \in \partial B_p(r_1)$ and $x \in \partial B_p(r_3)$. Suppose the $F(x) < G(x)$ somewhere inside the annulus. Let $q$ be the maximum point of $G(x) - F(x)$. Then if $q$ is not at the cut locus of $p$,
\[
\sqrt{-1} \partial \bar{\partial} (G(x) - F(x))_{x=q} \leq 0; \quad \nabla F(q) = \nabla G(q) = C \nabla r(q)
\]
where $C$ is some constant(this point will be used in the proof of theorem [3]). In particular,
\[
G_{\perp r} - F_{\perp r} \leq 0
\]
at $q$, where $e_1 = \frac{1}{\sqrt{2}}(\nabla r - \sqrt{-1} J \nabla r)$. The Poincare-Lelong equation says
\[
\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |f|^2 = [D]
\]
where $D$ is the divisor of $f$. Since $G(x) - F(x)$ have the maximum at $q$, $|f(q)| \neq 0$. Therefore
\[
G_{\parallel r} = 0.
\]
Now since $f$ is a holomorphic on $M$, we may assume $M(r_1) < M(r_3)$, otherwise $f$ is a constant. By theorem[8]
\[
F_{\perp r} \leq 0.
\]
Therefore

\[ G_{11} - F_{11} \geq 0. \]

If we can show the strict inequality, this will be a contradiction. For this, we consider a perturbation. For notational convenience, we define

\[ F_{\epsilon}(x) = a_{\epsilon} \log(r - \epsilon) + b_{\epsilon} \]

and demand that

\[ F_{\epsilon}(r_1) = M(r_1); F_{\epsilon}(r_3) = M(r_3) \]

where \( \epsilon \) is a small positive number. Clearly when \( \epsilon \to 0, F_{\epsilon} \to F \). Since \( M(r_3) > M(r_1), a_{\epsilon} > 0 \). Now we compute

\[ \left( \log(r - \epsilon) \right)_{11} = \frac{r_{11}}{r - \epsilon} - \frac{1}{2(r - \epsilon)^2} < 0 \]

by theorem 8. Therefore

\[ (G(x) - F_{\epsilon}(x))_{11} > 0 \]

which implies that it cannot assume the maximum inside the annulus if we ignore the cut locus problem. Considering the boundary data, we find that for any small \( \epsilon > 0 \),

\[ G(x) - F_{\epsilon} \leq 0 \]

inside the annulus.

To complete the proof, we still need to handle the case when \( q_{\epsilon}(q_{\epsilon}) \) is the maximum point of \( G - F_{\epsilon} \) lies on the cut locus of \( p \). We will adopt the trick of Calabi. Consider a number \( 0 < \epsilon_1 < \epsilon \). Let \( p_1 \) be the point on the minimal geodesic connecting \( p \) and \( q_{\epsilon} \) and that \( \text{dist}(p, p_1) = \epsilon_1 \). Define

\[ \hat{r}(x) = \text{dist}(p_1, x) \]

and consider

\[ F_{\epsilon, \epsilon_1} = a_{\epsilon} \log(\hat{r} + \epsilon_1 - \epsilon) + b_{\epsilon}. \]

Then

\[ F_{\epsilon, \epsilon_1} \geq F_{\epsilon} \]

by triangle inequality and

\[ F_{\epsilon}(q_{\epsilon}) = F_{\epsilon, \epsilon_1}(q_{\epsilon}). \]

Thus \( G - F_{\epsilon, \epsilon_1} \) have maximum at \( q_{\epsilon} \). Note that

\[ (F_{\epsilon, \epsilon_1})_{11} < 0 \]

since \( \epsilon_1 < \epsilon \). Then we apply the maximum principle for \( G - F_{\epsilon, \epsilon_1} \) at \( q_{\epsilon} \) to get a contradiction. The proof of theorem 8 follows if we let \( \epsilon_1 \to 0 \) and then \( \epsilon \to 0 \). □

Now we complete the rigidity part of theorem 4. It suffices to show that \( M \) is flat (then we take the universal cover). We need to show that for any \( p \in M \) and \( X \in T_p(M) \), \( R_{X,JXJX} = 0 \). There exists a local coordinate chart \((U, z_1, z_2, ..., z_n)\) containing \( p \) such that \( X = \frac{\partial}{\partial z_1} \) and \( \frac{\partial}{\partial z_1}(p) = X - \sqrt{-1}JX \). Since \( dim(O_d(M)) = dim(O_d(C^n)) \), there exists \( f \in O_d(M) \) such that the restriction of \( f \) in \( U \) is

\[ f(z_1, z_2, ..., z_n) = z_1^d + O(r^{d+1}). \]
In particular, the vanishing order of \( f \) at \( p \) is \( d \). By corollary 1 and corollary 3,
\[
\frac{M(r)}{r^d} = \text{Constant}.
\]
By looking back to the proof of theorem 6 and recalling the definition of \( F \) and \( G \), we find that \( G(x) - F(x) \) achieves the maximum 0 on every geodesic ball \( \partial B_p(r) \). By checking the equality of the Hessian comparison (theorem 8), we find that \( R_{YJYJ} = 0 \) where \( Y = \nabla r \) at the points on \( \partial B_p(r) \) where \( |f(z)| \) achieves the maximum. Using the Taylor expansion of \( f \), we find that the limit of \( Y \) as \( r \to 0^+ \) will be on the tangent subspace of \( p \) spanned by \( X \) and \( JX \). Thus \( R_{XJXJ} = 0 \). This completes the proof of the rigidity.

3. A Liouville type theorem for plurisubharmonic functions

Recall that the classical Liouville theorem states that any bounded (or even just positive) harmonic function is constant on Euclidean space. In [32], Yau extended the classical Liouville theorem to complete noncompact Riemannian manifolds with nonnegative Ricci curvature. It was further shown by Cheng and Yau in [6] that any harmonic function with sublinear growth on a complete noncompact Riemannian manifold with nonnegative Ricci curvature must be constant.

In the Kähler category, it is easy to show that any non-constant plurisubharmonic function in \( \mathbb{C}^n \) have at least logarithmic growth. In [27], Ni showed that on a complete Kähler manifold \( M^n \) with nonnegative Ricci curvature, any plurisubharmonic function \( f \) with sublogarithmic growth has \( (\partial \bar{\partial} f)^n = 0 \). In [28], Ni and Tam proved that any plurisubharmonic function with sublogarithmic growth on Kähler manifolds with nonnegative bisectional curvature is necessarily constant. Their method combines the heat flow and [27]. We show this Liouville type theorem still holds when the holomorphic sectional curvature is nonnegative.

**Theorem 9.** Let \( M^n \) be a complete noncompact Kähler manifold with nonnegative holomorphic sectional curvature. Let \( p \in M \) and \( r(x) = \text{dist}(x, p) \). Let \( u \) be a function on \( M \) satisfying
\[
\lim_{r \to \infty} \frac{r(x)}{\log r} = 0 \quad \text{where} \quad u^+(x) = \max(u(x), 0) \quad \text{and} \quad r \quad \text{is the distance function to a point} \quad p \in M.
\]

- If \( u \) be a plurisubharmonic function (not necessarily continuous) on \( M \), then \( u \) is a constant.
- If \( u \) is smooth satisfying
  \[
  u_{i\overline{j}k\overline{l}} g^{ik} g^{jl} \geq 0,
  \]
  then \( u \) is a constant. This condition means the complex Hessian of \( u \) is only nonnegative along the gradient of \( u \).

**Remark.** If we define \( M(r) = \max u^+(x) \) and \( \lim_{r \to \infty} \frac{M(r)}{\log r} = 0 \), then theorem 9 still holds. In fact, theorem 9 holds even we assume that the holomorphic sectional curvature is only asymptotically nonnegative, see corollary 5.

**Proof.** Let \( A = \sup_{B_p(1)} u(x) \). Given any \( \epsilon > 0, 1 > \epsilon_1 > 0 \), define
\[
f_{\epsilon_1}(x) = A + \epsilon \log \frac{r - \epsilon_1}{1 - \epsilon_1}; h(x) = u(x) - f_{\epsilon_1}(x),
\]
then
\[ h_{11} > 0 \]
where \( e_1 = \frac{1}{\sqrt{2}} (\nabla r - J\nabla r) \). This means the strict mean value inequality holds on a small holomorphic disk whose tangent space is spanned by \( \nabla r, J\nabla r \) at \( x \). Note \( h(x) \leq 0 \) for \( x \in \partial B_p(1) \cup \partial B_p(R) \) for large \( R \). If \( h(x) \) achieves the maximum inside the annulus \( B(p,R) - B(p,1) \), say at \( q \), then this contradicts the strict mean value inequality. If \( q \) is on the cut locus on \( p \), then we can apply Calabi’s trick as before. The details are omitted. Therefore, on \( B(p,R) - B(p,1) \), \( h \leq 0 \).

Letting \( R \to \infty \) and \( \epsilon \to 0, \epsilon_1 \to 0 \), we obtain 
\[ u(x) \leq A. \]
But \( A \) is the maximum of \( u(x) \) in \( B(p,1) \), maximum principle says \( u(x) \equiv A \).

Now consider the case when \( u_i u_j u_k u_l g^{ij} g^{kl} \geq 0 \). Let \( A = \inf u \) on \( M \) (here \( A \) could be \( -\infty \)). Consider a decreasing sequence \( a_i \to A \). We pick \( x_i \in M \) such that \( u(x_i) \leq a_i \). Let \( b_i = a_i + \frac{1}{i} \), then there exists \( \delta_i > 0 \) such that \( u(x) \leq b_i \) for \( x \in B(x_i, \delta_i) \). Define 
\[ f_i(x) = b_i + \epsilon \log \frac{\text{dist}(x_i, x)}{\delta_i} \]
(Here \( \epsilon \) is a small positive constant). Then \( u(x) \leq f_i(x) \) for \( x \in \partial B(x_i, \delta_i) \) and \( x \in \partial B(x_i, R) \) for \( R \) sufficiently large. If \( u - f_i(x) \) achieves the maximum at \( q \) in the interior part of \( B(x_i, R) - B(x_i, \delta) \), then 
\[ \nabla u(q) = \nabla f_i(q) = C\nabla \text{dist}(x_i, x)|_{x=q}. \]
If \( q \) is on the cut locus of \( x_i \), we apply Calabi’s trick. By maximum principle as before, we find 
\[ f_i(x) \geq u(x) \]
for \( x \in M - B(x_i, \delta_i) \). Let \( \epsilon \to 0 \) and then \( i \to \infty \), we find that 
\[ u(x) \leq A. \]
Since \( A = \inf u, u \equiv A \).

4. Sharp dimension estimate on holomorphic bundle with nonpositive curvature

In this section we consider the extension of theorem 4 to holomorphic vector bundles with nonpositive curvature.

**Definition 2.** Let \( E \) be a Hermitian holomorphic vector bundle on a Kähler manifold \( M \), then we say \( E \) is nonnegative (nonpositive) in the sense of Griffith if \( \Theta(F)(\xi \otimes \nu) \geq 0 (\leq 0) \) for all nonzero indecomposable tensor \( \xi \otimes \nu \in TM \otimes F \). Here \( \Theta \) is the curvature of the Chern connection of \( E, F \) is the fibre of \( E \).
Definition 3. Let $E$ be a Hermitian holomorphic vector bundle over a Kähler manifold $M$. Let $O_M(E)$ be holomorphic sections on $E$. For any $d \geq 0$, define
\[ O_d(M, E) = \{ f \in O_M(E) | \lim_{r \to \infty} \frac{|f(x)|}{r^d} < \infty \}. \]
Here $r$ is the distance from a fixed point on $M$. If $f \in O_d(M, E)$, we say $f$ is of polynomial growth with order $d$.

Theorem 10. Let $M^n$ be a complete noncompact Kähler manifold with nonnegative holomorphic sectional curvature. If $E^m$ is a Hermitian holomorphic vector bundle of rank $m$ over $M$ such that $E$ is nonpositive in the sense of Griffith, then for any $d > 0$,
\[ \dim(O_d(M, E)) \leq \dim(O_d(\mathbb{C}^n, E')) \]
where $E'$ is the trivial flat holomorphic vector bundle of rank $m$ over $\mathbb{C}^n$. The equality holds if $(M, E)$ is holomorphic and isometric to $(\mathbb{C}^n, E')$.

Remark. This kind of theorem was proved by Ni [26] when $E$ is a nonpositive line bundle and $M$ has nonnegative bisectional curvature.

Proof. For any holomorphic section $f \in O_M(E)$, at points where $f$ does not vanish, the Poincare-Lelong equation says
\[ \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |f|^2 = -\Theta(L_f). \]
Here $L_f$ is the line bundle induced by the section $f$ at the points where $f \neq 0$, $\Theta(L_f)$ is the curvature form of $L_f$. It is well known that the curvature of a Hermitian subbundle is no greater than the curvature of the ambient bundle, e.g., [8]. Therefore,
\[ \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |f|^2 \geq 0 \]
when $f \neq 0$. In particular,
\[ \Delta(|f|^2) \geq 0, \]
thus $|f|^2$ cannot assume the maximum in the interior part of a domain. This point is important, since we need $M(r_3) > M(r_1)$ as in the proof in theorem 6. Now we can mimic the proof of theorem 6. There, to make the proof work, we only need that
\[ \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} G(x) \geq 0. \]
Therefore, we get that theorem 6 holds for $f \in O_M(E)$. In particular, corollary 1 and corollary 5 hold in this case. By a linear algebra argument, we obtain the sharp dimension estimate.

Now we turn to the rigidity. Fix a point $p \in M$, in a small holomorphic chart $(U, z_1, z_2, ... z_n) \ni p$, let $e_1, e_2, ... e_m$ be a holomorphic basis for $E$ with $\nabla e_i = 0$ at $p$ for $i = 1, 2, ... m$. To show that $M$ is flat and $E$ is a flat bundle over $M$, it suffices to show that $R_{1111} = 0$ and $\Theta_{11}(e_i, \overline{e_j}) = 0$ at $p$ (since the frame is arbitrary). Here
Θ is the curvature of \( E \). Since the equality holds for some \( d > 0 \), there exists \( f \in O_d(M, E) \) such that
\[
f = z_1^d e_1 + \sum_{i=1}^{m} O(d+1) e_i.
\]
Similar to the proof of rigidity of theorem 4, we get that
\[
\frac{M(r)}{r^d} \equiv \text{Constant}
\]
where \( M(r) = \max |f(x)|, \ x \in B_p(r) \). Then by the equality case of the maximum principle as in theorem 6 (note that the Poincare-Lelong equation gives the curvature), we find that
\[
R_{YY} = 0; \Theta_\alpha \alpha(L_f)(f(x), f(x)) = 0
\]
at the points on \( \partial B_p(r) \) where \( |f(x)| \) takes the maximum. Here \( Y = \nabla r, \alpha = \frac{1}{2}(Y - \sqrt{-1} J Y) \) (note that \( f(x) \) lies in the fibre of \( E \)). Using the fact that
\[
0 \geq \Theta_\alpha \alpha(f(x), \overline{f(x)}) \geq \Theta_{\alpha \overline{\alpha}}(L_f)(f(x), \overline{f(x)}) \geq 0,
\]
we obtain
\[
\Theta_{\alpha \overline{\alpha}}(f(x), \overline{f(x)}) = 0.
\]
Letting \( r \to 0 \), we find
\[
R_{1 \overline{1}} = 0; \Theta_{1 \overline{1}}(e_1, \overline{e_1}) = 0
\]
at \( p \). We have proved that both \( M \) and \( E \) are flat. By looking at the pull back bundle of the covering map \( \mathbb{C}^n \to M \) and counting dimensions, we find that \( M \) is just \( \mathbb{C}^n \).

5. Compactness and sharp dimension estimates for holomorphic maps between certain Kähler manifolds

In this section we study holomorphic maps between certain Kähler manifolds.

**Theorem 11.** Let \( M^n \) be a complete Kähler manifold with nonnegative holomorphic sectional curvature and \( N^m \) be a simply connected Kähler manifold with nonpositive sectional curvature. Let \( o \) be a fixed point on \( M \) and define \( r(x) = \text{dist}_M(o, x) \). For any \( k \geq 0 \), let \( H_k \) be the set of holomorphic maps \( f : M \to N \) which is of polynomial growth of order \( k \). That is to say,
\[
\lim_{r \to \infty} \frac{\text{dist}_N(f(o), f(x))}{r^k} < \infty
\]
for some \( k \geq 0 \). For fixed \( c > 0, k \) and compact set \( K \subset M \), define
\[
H_{c,k,K} = \{ f \in H_k | f(o) \in K, |\frac{\partial^i f(o)}{\partial z^i}| \leq c, 1 \leq i \leq k \}.
\]
Then \( H_{c,k,K} \) is compact. Here the partial derivatives are understood in each fixed holomorphic chart in \( N \), since \( K \) is compact. Furthermore, we have the sharp estimate of the dimension of tangent space of \( H_k \):
\[
dim(TH_k) \leq \dim(O_d(\mathbb{C}^n, E'))
\]
where $E'$ is the flat trivial bundle of rank $m$ over $\mathbb{C}^n$. If the equality holds, $M$ is isometric and biholomorphic to complex Euclidean space and the pull back tangent bundle is a trivial flat holomorphic vector bundle.

**Remark.** The compactness cannot be true if we drop the bound of the derivatives of $f$ at $o$. For instance, consider the polynomial which is regarded as a holomorphic map from $\mathbb{C}$ to $\mathbb{C}$: $f_\lambda(z) = z^k + \lambda z^{k-1}$, then $f(0) = 0$. If $\lambda$ is unbounded, $f_\lambda$ cannot be compact. Note that the $k-1$th derivative is unbounded.

**Proof.** We prove the compactness first. For notational convenience, let $d(f(x), f(o)) = \text{dist}_N(f(x), f(o))$.

**Proposition 1.** $\log d(f(o), f(x))$ is plurisubharmonic.

**Proof.** Consider a unitary frame $e_i$ at $x$ and a holomorphic chart containing $f(x)$ such that $\frac{\partial}{\partial z_i} = e_i$ has unit length at $f(x)$.

\[
\frac{\partial}{\partial z_i} = \frac{d_{\partial z_i} \alpha_i}{d} + \frac{d_{\partial z_i} \beta_i}{d} - |\frac{d_{\partial z_i} \alpha_i}{d}|^2 \geq 0.
\]

Here we have used the Hessian comparison theorem $d_{\partial z_i} \alpha_i \geq \frac{1}{2} \text{Id}$; $\alpha_i = 0$; $|d_{\partial z_i} \alpha_i| \leq \frac{1}{\sqrt{n}}$.

Let $M(f(x), r)$ be the maximum of $d(f(o), f(x))$ for $x \in B(o, r)$. For simplicity, we write $M(r) = M(f(x), r)$. By similar arguments as in theorem 6 and corollary 1, we obtain

**Claim 1.** $\frac{M(r)}{r^k}$ is nonincreasing.

Now consider a sequence $f_j \in H_{c,k,K}$, we may assume $f_j(o) \to p \in N$. Consider a small holomorphic chart $U(o, z_1, z_2, ..., z_n)$ on $M$ such that for $x = (z_1, ..., z_n) \in U$,

\[
\frac{1}{2} \sqrt{\sum_{i=1}^{n} |z_i|^2} \leq |x| = \text{dist}(o, x) \leq 2 \sqrt{\sum_{i=1}^{n} |z_i|^2}.
\]

Here by scaling, we may assume $U$ contains the points $z = (z_1, z_2, ..., z_n)$ where $|z| = \sqrt{\sum_{i=1}^{n} |z_i|^2} \leq 1$.

There are two cases to consider:

Case 1:

There exists a small number $\delta > 0$ and $M > 0$ such that $\text{dist}(f_j(o), f_j(x)) \leq M$ for all $x \in B(o, \delta)$ and a sequence of $j$ going to $\infty$.

In this case, by claim 1, for any $r \geq \delta$, $M(f_j, r) \leq r^k \frac{M}{\delta^k}$.
for that sequence of \( j \). Since \( N \) is simply connected with nonpositive sectional curvature, \( N \) is a Stein manifold, which means that \( N \) could be properly holomorphically embedded in \( \mathbb{C}^{2m+1} \). Thus we can find uniform bound of the derivatives of \( f_j \) in each compact set of \( M \), since the images all lie in fixed compact sets in \( \mathbb{C}^{2m+1} \). Thus \( f_j \) has a convergent subsequence which converges to some \( f \in H_{c,k,K} \).

Case 2: Case 1 does not hold. Then for all sufficiently large \( j \), we can find \( \delta_j > 0 \) such that

\[
M(f_j(\delta_jz), |z| \leq \frac{4}{3}) = 1
\]

(here we only consider the points \( z \in U \)). Then by assumption, \( \delta_j \to 0 \). Define \( g_j(z) = f_j(\delta_jz) \). Note \( g_j \) is only defined on \( U \).

**Lemma 1.** For \( \frac{1}{3} \geq r_1 \geq r_2 > 0 \), \[
\frac{M(g_j, |z| \leq r_1)}{4^r r_1} \leq \frac{M(g_j, |z| \leq r_2)}{r_2^2}.
\]

**Proof.** We have

\[
\frac{M(g_j, |z| \leq r_1)}{2^k \delta_j r_1^k} = \frac{M(f_j(\delta_jz), |z| \leq r_1)}{2^k \delta_j r_1^k} \\
\leq \frac{M(f_j, 2\delta_j r_1)}{(2\delta_j r_1)^k} \\
\leq \frac{M(f_j, \frac{1}{2}\delta_j r_2)}{(\frac{1}{2}\delta_j r_2)^k} \\
\leq \frac{M(g_j, |z| \leq r_2)}{(\frac{1}{2}\delta_j r_2)^k} \\
= \frac{M(g_j, |z| \leq r_2)}{r_2^2}.
\]

(7)

In the middle, we have used claim 1 and 6. \( \square \)

Now \( g_j(0) \to p \) and \( M(g_j, |z| \leq \frac{1}{3}) = 1 \). By lemma 1, \( M(g_j, |z| \leq \frac{1}{3}) \) is uniformly bounded. Then we can find a subsequence such that \( g_j \to g \) uniformly for \( |z| \leq \frac{1}{3} \). Thus \( g \) is not a constant map. Uniform convergence and lemma 1 implies the following:

**Lemma 2.** For \( \frac{1}{3} \geq r_1 \geq r_2 > 0 \), \[
\frac{M(g_j, |z| \leq r_1)}{4^r r_1} \leq \frac{M(g_j, |z| \leq r_2)}{r_2^2}.
\]

Since \( \delta_j \to 0 \) and that \( |\frac{\partial f_j}{\partial z}| \leq c \) for \( 1 \leq i \leq k \), \( |\frac{\partial g_j}{\partial z}| \to 0 \) for \( 1 \leq i \leq k \), \( j \to \infty \). Since \( g_j \to g \), the partial derivatives of \( g \) at \( o \) vanishes up to order \( k \). Combining this with lemma 2 we find \( g \) to be a constant map (just let \( r_2 \to 0 \) in lemma 2). This is a contradiction.

Thus Case 2 cannot happen! The compactness of \( H_{c,k,K} \) is proved.

Now we turn to the dimension estimate. We need to establish the following lemma:
Lemma 3. Let $M^n$ be a complete Kähler manifold and $N^m$ be a simply connected Kähler manifold with nonpositive sectional curvature. Let $f, g$ be holomorphic maps from $M$ to $N$. Then $\log \text{dist}_N(f(x), g(x))$ is a plurisubharmonic function on $M$.

Proof. We may assume that at a point $x$, $f(x) \neq g(x)$. Let $\gamma$ be the geodesic connecting $f(x)$ and $g(x)$ and $e_\alpha(\alpha = 1, 2, \ldots, 2m)$ be an orthonormal frame on $\gamma$ which is also parallel. Let $e_1$ be tangential to the $\gamma$. We also assume that $Je_{2s-1} = e_{2s}$ for $s = 1, 2, \ldots, m$. Define $h_s = \frac{1}{\sqrt{2}}(e_{2s-1} - \sqrt{-1}e_{2s})$. Near $x \in M$, we consider a normal coordinate $(U, z_1, \ldots, z_n)$.

\[
\frac{\partial^2 \log d(f(x), g(x))}{\partial z_j \partial \bar{z}_j} = \left( \frac{d(f(x), g(x))}{d(f(x), g(x))} \right)_j = \left( \frac{(d(f(x), g(x)))_j}{d(f(x), g(x))} \right)_j = \left( \frac{(d(f(x), g(x)))_j}{d(f(x), g(x))} \right)^2.
\]

We need the second variation of arc length. For completeness, we include the calculations. Let $F(\lambda, t) : [0, 1] \times (-\varepsilon, \varepsilon) \to N$ be a smooth map. Let $X = \frac{\partial}{\partial \lambda}$. We demand that for fixed $t$, $F(\lambda, t)$ is a geodesic in $N$ and $\nabla_X X = 0$. For each $t$, let $L(t)$ be the arc length $1 \int_0^1 \sqrt{\langle X, X \rangle} d\lambda$. To simplify the notation, we write $t = \frac{\partial}{\partial m}$. First variation at $t = 0$:

\[
\frac{dL(t)}{dt} = \int_0^1 \frac{\langle \nabla \langle X, X \rangle \rangle}{\sqrt{\langle X, X \rangle}} d\lambda = \int_0^1 \frac{\langle \nabla_X t, X \rangle}{\sqrt{\langle X, X \rangle}} d\lambda = \frac{\langle t, X \rangle}{L(0)}|_{t=0}.
\]
Second variation at $t = 0$:

\[
\frac{d^2 L(t)}{dt^2} = \frac{d}{dt} \left( \int_0^1 \frac{\langle \nabla_t \nabla t, X \rangle}{\sqrt{\langle X, X \rangle}} d\lambda \right)
\]

\[
= \int_0^1 \frac{\langle \nabla_t \nabla X, X \rangle + |\nabla_t X|^2}{\sqrt{\langle X, X \rangle}} - \frac{|\langle \nabla_t X, X \rangle|^2}{\langle X, X \rangle} d\lambda
\]

\[
= \int_0^1 \frac{R_{XX} + \langle \nabla X, \nabla_t X \rangle + |\nabla_{\perp} X|^2}{\sqrt{\langle X, X \rangle}} d\lambda
\]

\[
\int_0^1 |\nabla_{\perp} X|^2 d\lambda + \langle \nabla t, X \rangle |_{\lambda = 1} \geq \frac{1}{L(0)}.
\]

Here $\perp$ is the projection orthogonal to $X$.

Let $\gamma_1, \gamma_2$ be two normal geodesics on $M$ starting from $x$, with initial tangent vectors $Re\frac{\partial}{\partial z}$ and $Im\frac{\partial}{\partial z}$. Set $F_1(\lambda, t), F_2(\lambda, t)$ be two maps satisfying lemma 3 and boundary conditions $F_j(0, t) = f(\gamma_j(t)), F_j(1, t) = g(\gamma_j(t)), j = 1, 2$. Let $L_j$ be the length function for $F_j$.

Lemma 4. $\nabla_{\perp} t_1 + \nabla_{\perp} t_2 = 0$ for $\lambda = 0$ or $\lambda = 1$. Here $t_j = \frac{\partial}{\partial z_j}, j = 1, 2$.

Proof. For $\lambda = 0$, $\nabla_{\perp} t_1 + \nabla_{\perp} t_2 = \nabla f, \frac{\partial}{\partial z_j} = f, \frac{\partial}{\partial z_j}$, since $f$ is holomorphic. Similarly, the result holds for $\lambda = 1$. □

Let

\[
f_s \frac{\partial}{\partial z_s} = \sum_{s=1}^m (a_s(e_{2s-1} - \sqrt{-1} e_{2s}) + b_s(e_{2s} + \sqrt{-1} e_{2s-1}));
\]

\[
g_s \frac{\partial}{\partial z_s} = \sum_{s=1}^m c_s(e_{2s-1} - \sqrt{-1} e_{2s}) + d_s(e_{2s} + \sqrt{-1} e_{2s-1})).
\]

Also let

\[
\frac{\partial}{\partial t_1} |_{t_1 = 0} = \sum_{s=1}^m (u_s e_{2s-1} + v_s e_{2s}); \quad \frac{\partial}{\partial t_2} |_{t_2 = 0} = \sum_{s=1}^m (\bar{u}_s e_{2s-1} + \bar{v}_s e_{2s}).
\]

Note that

\[
v_1(0) = b_1, \bar{v}_1(0) = -a_1, v_1(1) = d_1, \bar{v}_1(1) = -c_1.
\]
By (10) and Lemma 4, 

\[
\frac{d^2L_1(t_1)}{dt_1^2} + \frac{d^2L_2(t_2)}{dt_2^2} \\
\geq \frac{1}{L(0)} \int_0^1 |v_1'(|\lambda|)|^2 + |v_1'(|\lambda|)|^2 + \sum_{s=2}^m |(u_2'(|\lambda|)|^2 + |u_2'(|\lambda|)|^2 + |v_2'(|\lambda|)|^2 + |\bar{v}'_2(|\lambda|)|^2) d\lambda \\
\geq \frac{1}{L(0)} \int_0^1 |v_1'(|\lambda|)|^2 + |v_1'(|\lambda|)|^2 d\lambda \\
\geq \frac{(b_1 - d_1)^2 + (a_1 - c_1)^2}{L(0)}.
\]

(11)

In the last step, we have used Cauchy-Schwarz inequality. By (9), 

\[ \frac{dL_1(t)}{dt} = c_1 - a_1, \quad \frac{dL_2(t)}{dt} = b_1 - d_1. \]

The lemma follows if we plug these (8). \[\square\]

Let \( f, g \in H_k \), then \( \text{dist}_N(f(x), g(x)) \leq 3C r^k \) when \( r \) is sufficiently large. Lemma 3 says \( \log \text{dist}_N(f(x), g(x)) \) is plurisubharmonic. Then by similar arguments as in corollary 1, we find that 

\[
\text{dist}_N(f(x), g(x)) \leq A r^k
\]

for all \( r \geq 1 \) where \( A = \max_{r \leq 1} \text{dist}_N(f(x), g(x)) \). Consider a sequence of holomorphic maps \( f_i \in H_k \) converging to \( f \) uniformly in each compact set of \( M \), then 

\[
A_i = \max_{r \leq 1} \text{dist}_N(f_i(x), f(x)) \to 0
\]

as \( i \to \infty \). After taking a subsequence, the limit of the “difference” between \( f_i \) and \( f \), with a suitable normalization, will be a holomorphic section \( \Gamma \in f^*T^{1,0}N \). Since \( N \) has nonpositive sectional curvature, \( f^*T^{1,0}N \) will be nonpositive in the sense of Griffith. By (12), \( \Gamma \in O_k(M, f^*T^{1,0}N) \). By theorem 10, we find the sharp dimension estimate of \( TH_k \). The rigidity also follows. \[\square\]

Remark. Holomorphic maps with polynomial growth between complex Euclidean spaces are the prototype of theorem 11. It is also interesting to compare it with various Schwarz lemmas by different authors, e.g. Yau’s Schwarz lemma [34] and Royden’s Schwarz lemma [29]. One feature is that for Schwarz lemmas, the target manifold should have some curvature bounded from above by a negative constant. In theorem 11, we are dealing with the borderline case, i.e. the curvature of the target is nonpositive.
6. Hadmard Three Circle Theorem, General Case

The following theorem is the general version of the Hadmard Three Circle theorem in the curved case.

**Theorem 12.** Let $M^n$ be a complete noncompact Kähler manifold, $p \in M$. Let $r(x) = \text{dist}(p, x)$, $e_1 = \frac{1}{\sqrt{2}}(\nabla r - \sqrt{-1}J\nabla r)$. Let the function $g(r)$ satisfies $R_{1111} \geq g(r)$ for all $r \geq 0$. Note that we can find the upper bound of $r_{11}$ by Hessian comparison theorem. Thus there exists a function $h(r)$ satisfying

- $\lim_{r \to 0^+} \frac{h(r)}{r} = 1$;
- $h'(r) > 0$ for all $r > 0$;
- $h(r)_{11} \equiv \frac{1}{2}h'' + h'r_{11} = 0$.

Then for $f \in \mathcal{O}(M)$, $\log M(r)$ is convex in terms of $h(r)$. In particular, if $f \in \mathcal{O}(M)$ vanishes at $p$ of order $d$, $\frac{M(r)}{r^d}$ is nondecreasing. Thus, given certain growth control of the holomorphic function, we can bound the vanishing order at $p$. This gives the dimension estimate.

**Remark.** Theorem [12] is sharp in general. In fact, one can just consider the unitary symmetric metric on $\mathbb{C}^n$ or the unit ball in $\mathbb{C}^n$ with the prescribed holomorphic sectional curvature on the radial direction. Then any homogeneous polynomial holomorphic function satisfies the equality, e.g. $z_1^2 - 2z_1z_2, z_3^3 + 5z_2z_3$.

**Remark.** The above theorem (Hadmard Three Circle theorem) also works for holomorphic sections on Hermitian holomorphic vector bundle with nonpositive curvature. If the curvature of the bundle has small positive curvature, then the sharp monotonicity holds. Note that in this case, it is crucial that $h' > 0$. This is only possible when the curvature of the bundle is not too large (we have to solve $(h(r))_{11} \leq -2\pi\Theta_{11}(a, \overline{a})$. Here $\Theta$ is the curvature of the bundle; $e_n$ is of type $(1, 0)$ in the fibre of the bundle with unit length). The arguments are similar.

**Proof:** The argument is similar to theorem [6]. We only prove the monotonicity formula here. The proof of Three circle theorem is the same. Let $0 < r_1 \leq r_2$. Consider

$$F_\epsilon(x) = \log |f(x)| - \log M(r_2) - (d - \epsilon)(h(r) - h(r_2)),$$

here $\epsilon > 0$. Then $F_\epsilon(x) \leq 0$ for $r(x) = r_2$ or $r < \delta$ where $\delta < r_1$ is very small. We apply the maximum principle for $\delta \leq r \leq r_2$ which is the same as in the proof of theorem [6]. For the perturbation function as in theorem [6] take

$$h_{\epsilon_1} = \log(e^h - \epsilon_1),$$

then one sees that

$$(h_{\epsilon_1})_{11} < 0.$$

Write

$$F_{\epsilon, \epsilon_1}(x) = \log |f(x)| - \log M(r_2) - (d - \epsilon)(h_{\epsilon_1}(r) - h_{\epsilon_1}(r_2)),$$

here $\epsilon_1 \ll \delta$ so that $F_{\epsilon, \epsilon_1}(x) \leq 0$ for $r = \delta$. We apply the maximum principle for $F_{\epsilon, \epsilon_1}$ on $B(p, r_2) - B(p, \delta)$. If the maximum point $q_{\epsilon_1}$ is in the interior part and not on the cut locus of $p$, $(F_{\epsilon, \epsilon_1})_{11}(q_{\epsilon_1}) \leq 0$. By direct computation, $(F_{\epsilon, \epsilon_1})_{11} > 0$, this
is a contradiction. If $q_{\epsilon_1}$ is on the cut locus of $p$, we employ Calabi’s trick again. Namely, define $\hat{r} = \text{dist}(p_1, x)$ where $p_1$ is the point on the minimal geodesic connecting $p$ and $q_{\epsilon_1}$ such that $\text{dist}(p, p_1) = \epsilon_2$. Define
\[
F_{\epsilon, \epsilon_1, \epsilon_2}(x) = \log |f(x)| - \log M(r_2) - (d - \epsilon)(h_{\epsilon_1}(\hat{r} + \epsilon_2) - h_{\epsilon_1}(r_2)) \leq F_{\epsilon, \epsilon}(x)
\]
on $B(p, r_2) - B(p, \delta)$. Here we have used the triangle inequality and that $h' > 0$. Moreover, $F_{\epsilon, \epsilon_1, \epsilon_2}(q_{\epsilon_1}) = F_{\epsilon, \epsilon_1}(q_{\epsilon_1})$, therefore $q_{\epsilon_1}$ is the maximum of $F_{\epsilon, \epsilon_1, \epsilon_2}$. Thus
\[
(F_{\epsilon, \epsilon_1, \epsilon_2})_T \leq 0.
\]
For fixed $\epsilon_1 > 0$, if $\epsilon_2$ is small enough, then one can show that
\[
(F_{\epsilon, \epsilon_1, \epsilon_2})_T > 0
\]
at $q_{\epsilon_1}$ (this follows from the continuity of the lower bound of the holomorphic sectional curvature). This is a contradiction.

Then
\[
F_{\epsilon, \epsilon_1, \epsilon_2}(x) \leq 0
\]
for $\delta \leq r \leq r_2$. Letting $\epsilon_2 \to 0$, $\epsilon_1 \to 0$, $\epsilon \to 0$ and putting $r(x) = r_1$, we find
\[
\log M(r_1) - \log M(r_2) - d(h(r_1) - h(r_2)) \leq 0.
\]
This is equivalent to
\[
\frac{M(r_1)}{e^{h(r_1)d}} \leq \frac{M(r_2)}{e^{h(r_2)d}}.
\]

We next apply the theorem above to the case when the holomorphic sectional curvature has lower bound 1 or $-1$. The proof follows from a simple computation.

**Theorem 13.** Let $M^n$ be a complete noncompact Kähler manifold such that the holomorphic sectional curvature is no less than $-1$. Let $p \in M$ and $r(x) = \text{dist}(p, x)$. Then for $f \in O(M)$, $M(r)$ is convex in terms of $\log \frac{1}{r + 1}$. In particular, $\frac{M(r)}{(r + 1)^d}$ is nondecreasing where $d$ is the vanishing order of $f$ at $p$.

**Theorem 14.** Let $M$ be a complete (compact) Kähler manifold with holomorphic sectional curvature no less than 1, $p \in M$. Let $f$ be a holomorphic function defined on $B_p(r_0) \subset M$. Then $M(r)$ is convex in terms of $\log(\tan \frac{r}{2})$. In particular, if $f$ vanishes at $p$ with order $d$, $\frac{M(r)}{(\tan \frac{r}{2})^d}$ is nondecreasing.

**7. Complete Kähler manifolds with holomorphic sectional curvature asymptotically nonnegative**

In this section we apply the Hadamard Three Circle Theorem to Kähler manifolds with holomorphic sectional curvature asymptotically nonnegative.

**Theorem 15.** Let $M^n$ be a complete noncompact Kähler manifold, let $p \in M$ and $r(x) = \text{dist}(p, x)$. Suppose there exists a constant $\epsilon, A > 0$ such that for any $\epsilon_1 \in T^{1.0}M$ with unit length, $R_{\epsilon_1} \geq -\frac{1}{(\epsilon + 1)^d}$, then there exists $C = C(A, \epsilon) > 0$ such that for any $d \geq 1$,
\[
dim(O_d(M)) \leq Cd^\epsilon.
\]
Thus the power of $d$ is sharp comparing with the complex Euclidean space. If $d \leq e^{-\frac{3A}{e}}$, $\dim(O_d(M)) = 1$. Finally, if $\frac{A}{e} \leq \frac{1}{3d}$, we have the sharp dimension estimate $\dim(O_d(M)) \leq \dim(O_d(\mathbb{C}^n))$.

**Remark.** The above theorem is not true for $\epsilon \leq 0$. In fact, in this case it is possible that the dimension of bounded holomorphic functions is infinite. For example, on page 226 in [31], it is remarked that on the unit disk, the metric $ds^2 = \frac{1}{1-|z|^2} dz \otimes d\bar{z}$ ($m \geq 3$) has quadratic holomorphic sectional curvature decay, but bounded holomorphic functions have infinite dimension.

**Proof.** We may assume $\epsilon < \frac{1}{2}$. We shall analyze along a minimal geodesic. Recall in theorem 8 that $2r^2_1 + \frac{\partial r_1}{\partial r} + \frac{1}{2}R_{1111} \leq 0$.

**Claim 2.** For any $r > 0$, $r_1 \leq p(r) = \frac{1}{2} + \frac{A}{(r+1)^{1+\epsilon}}$.

**Proof.** Let $g(r) = r_1 - \frac{1}{2r}$. Then $g$ satisfies

$$2g^2 + \frac{2g}{r} + g' + \frac{1}{2}R_{1111} \leq 0.$$

It is easy to see that $r_1 \leq p(r)$ when $r$ is sufficiently small. If the claim is not true, let $r_0$ be the first time that the inequality is violated. Then $g(r_0) = \frac{A}{(r_0+1)^{1+\epsilon}}$ and $g'(r_0) \geq \frac{-A(1+\epsilon)}{(r_0+1)^{2+\epsilon}}$.

$$g'(r_0) \leq -2g^2 - \frac{2g}{r_0} - \frac{1}{2}R_{1111} \leq \frac{3A}{2(r_0 + 1)^{2+\epsilon}}$$

$$\leq \frac{-A(1 + \epsilon)}{(r_0 + 1)^{2+\epsilon}} \leq g'(r_0).$$

This is a contradiction. \hfill $\square$

Now let $h(r) \in C[0, \infty)$ be the solution to the equation in theorem 12 with $r_1 = \frac{1}{2} + \frac{A}{(r+1)^{1+\epsilon}}$. We find $h'(r) = \frac{e^{-\frac{3A}{r}}}{r} e^{-\frac{2A}{r}}$. Therefore,

$$h(r) = \int_1^r \frac{e^{-\frac{2A}{t}}}{t} e^{-\frac{2A}{r}} dt + C$$

(14)

$$\geq \int_1^r \frac{1 + \frac{2A}{t(r+1)^\epsilon}}{t} e^{-\frac{2A}{r}} dt + C$$

$$\geq e^{-\frac{3A}{r}} \ln r + C$$
for $r \geq 1$. By theorem [12] if $f \in \mathcal{O}(M)$ and $f$ vanishes at $p$ with order $d$, $\frac{M(r)}{d}$ is nondecreasing. Thus

$$M(R) \geq e^{h(R)rd} \lim_{r \to 0} \frac{M(r)}{e^{h(R)rd}} \geq e^{Cd^d} R^d e^{\frac{2d}{d+1}}.$$

By linear algebra,

$$\dim(\mathcal{O}_d(M)) \leq \dim(\mathcal{O}_{de^d}((\mathbb{C}^n))) = C(A, \epsilon)d^n.$$

If $d \leq e^{-\frac{2d}{d+1}}$,

$$\dim(\mathcal{O}_d(M)) \leq \dim(\mathcal{O}_{de^d}((\mathbb{C}^n))) \leq \dim(\mathcal{O}_{e^d}((\mathbb{C}^n))) = 1.$$

Finally, if $\frac{d}{d+1} \leq \frac{1}{2d}$, $e^\frac{2d}{d+1} d < d + 1$, we have the sharp dimension estimate

$$\dim(\mathcal{O}_d(M)) \leq \dim(\mathcal{O}_{de^d}((\mathbb{C}^n))) = \dim(\mathcal{O}_d((\mathbb{C}^n))).$$

□

By similar arguments, we have the following:

**Corollary 4.** Let $M^n$ be a complete noncompact Kähler manifold such that the holomorphic sectional curvature is nonnegative outside a compact set $K$, then there exists $C = C(K) > 0$ such that for any $d \geq 1$,

$$\dim(\mathcal{O}_d(M)) \leq Cd^n.$$

Thus the power is sharp. Furthermore, if we rescale $M$ such that the holomorphic sectional curvature is no less than $-1$ and if $\text{diam}(K) \leq \frac{1}{2d}$ for some $d \geq 1$, then we have the sharp dimension estimate

$$\dim(\mathcal{O}_d(M)) \leq \dim(\mathcal{O}_d((\mathbb{C}^n))).$$

Finally, there exists a small number $\epsilon = \epsilon(K) > 0$ such that

$$\dim(\mathcal{O}_\epsilon(M)) = 1,$$

i.e, it is spanned by constants.

**Remark.** The inequality $\dim(\mathcal{O}_d(M)) \leq \dim(\mathcal{O}_d((\mathbb{C}^n)))$ is not true if we only assume $M$ has nonnegative holomorphic sectional curvature outside a compact set. One can easily construct the example in complex one dimensional case (the rotationally symmetric case on $\mathbb{C}$).

**Corollary 5.** Theorem [9] is valid under the assumption of theorem [13]

**Proof.** In the proof of theorem [9] we just replace $\log r$ by $h(r)$. Then everything works as the same. □

**Remark.** We can also generalize theorem [10] and [11] to these cases. The results are similar.

**Corollary 6.** Under the same assumption as in theorem [13] any holomorphic map from $M$ to $N$ must be constant map. Here $N$ is a complete simply connected Kähler manifold with nonpositive sectional curvature and that the sectional curvature satisfies $\text{sec} \leq -\frac{K}{d^2}$ for all large $d$. Here $d$ is the distance function to a point $q$ on $N$, $K$ is a positive constant.
Proof. We shall construct a bounded plurisubharmonic function on $N$, the pull it back to $M$. Let $f$ be a holomorphic map from $M$ to $N$ and $p \in f(M)$. It is easy to see that there exists a constant $C > 0$ such that the sectional curvature on $N$ satisfies $\sec \leq -\frac{C}{r^2}$ for $r \geq r_0$. Here $r(x) = \text{dist}(x, p)$, $r_0$ is a positive constant. Recall the comparison theorem in [31] by Siu and Yau (page 227):

**Theorem 16.** Let $M_1$ and $M_2$ be two simply connected, complete Riemannian manifolds of real dimension $n$ with nonpositive sectional curvature. Let $\gamma_i : [0, a] \to M$ be a geodesic parametrized by its arc-length $t$ and let $r_i$ be the distance function on $M_i$ measured from the point $\gamma_i(0)(i = 1, 2)$. Suppose that for some $a > 0$ and for every $0 < t < a$ the sectional curvature of $M_1$ at any 2-plane in the tangent space of $M_1$ at $\gamma(t)$ is greater than or equal to the sectional curvature of $M_2$ at any 2-plane in the tangent space of $M_2$ at $\gamma_2(t)$. Let $X_i$ be any unit tangent vector of $M_i$ at $\gamma_i(a)(i = 1, 2)$ perpendicular to the radial direction. Then $H(r_1)(X_1, X_1) \leq H(r_2)(X_2, X_2)$.

**Remark.** For theorem 16, we can weaken the condition that $M_1$ has nonpositive sectional curvature. It suffices to assume that $M_j$ infinite injective radius at $\gamma_j(0)$ and the sectional curvature on $\gamma_j(t)$ is nonpositive between $\nabla r_j$ and any other direction.

Let $M_2 = N$. We shall construct a complete simply connected Kähler manifold $M_1$ of unitary symmetric metric with respect to a point $o$ satisfying

\[
\begin{align*}
\sec &= 0 \\
r(x) &= \text{dist}(x, o) \leq r_0 \\
0 &\geq \sec(\nabla r, X) \geq -\frac{C}{r^2}
\end{align*}
\]  

(15)

Here $X$ is a unit tangent vector orthogonal to $\nabla r$. Then $M_1, M_2$ will satisfy the remark of theorem 16. $\gamma_1$ is the geodesic emanating from $o$, $\gamma_2$ is the geodesic emanating from $p$.)

The example is given as follows: First for complex dimension 1, a unitary symmetric Kähler metric with respect to $o$, along the geodesic emanating from $o$, we have that

\[
\frac{\partial r_{1\bar{1}}}{\partial r} + 2r_{1\bar{1}}^2 + \frac{1}{2}R_{1\bar{1}1\bar{1}} = 0.
\]

(16)

Here $r$ is the distance function to $o$, $e_1 = \frac{1}{\sqrt{2}}(\nabla r - \sqrt{-1}J\nabla r)$. We may assume that the metric is given by $\omega = \partial \overline{\partial} p(d)$ in the two dimensional case where $d$ is the Euclidean distance. In this case, the manifold is conformal to the unit disk in $\mathbb{C}$ or $\mathbb{C}$ itself. Then we extend the function $p$ to the unit ball in $\mathbb{C}^n$ or $\mathbb{C}$ and define the Kähler form $\omega = \partial \overline{\partial} p(d)$. Let $g$ denote the induced metric. For any point $x$ which is not the origin, let $e_2$ be a $(1, 0)$ type tangent vector at $x$ which is orthogonal to $e_1$, i.e., $g_{1\bar{2}} = 0$. For such a unitary symmetric metric, by equation (4.1) and (4.2) in [13],

\[
\frac{\partial r_{2\bar{2}}}{\partial r} + r_{2\bar{2}}^2 + \frac{1}{2}R_{1\bar{1}2\bar{2}} = 0;
\]

(17)
(18) \[
\frac{\partial r_{22}}{\partial r} = 2r_{22}r_{11} - 2r_{22}^2
\]

First consider a \(C^2\) function \(\lambda(r)\) given by

\[
\begin{align*}
\lambda(r) &= 2 & 0 \leq r \leq 2r_0 \\
\frac{-5\epsilon}{r_0} &\leq \lambda'(r) \leq 0 & 2r_0 \leq r \leq 4r_0 \\
\lambda &= 2 - \epsilon & r \geq 4r_0.
\end{align*}
\]

Here \(0 < \epsilon < 1\) is a small constant which will be determined later. Now we would like

(20) \[r_{11} = \frac{1}{\lambda r},\]

then we determine \(R_{1111}\) and \(R_{1122}\) by (17), (16) and (18). Since \(\lambda \leq 2\) and \(\lambda' \leq 0\), we find that

(21) \[R_{1111}^2r^2 = 2(\lambda + \rho \lambda' - 2) \leq 0.\]

(22) \[
\begin{align*}
R_{1111} &= 0 & r \leq 2r_0 \\
R_{1111} &\geq -\frac{\epsilon + \frac{5\epsilon}{r_0^2}}{4\lambda r^2} & 2r_0 \leq r \leq 4r_0 \\
R_{1111} &= -\frac{\epsilon}{(2-\epsilon)r^2} & r \geq 4r_0.
\end{align*}
\]

Claim 3. \(2r_{11} - r_{22} \geq 0.\)

Proof. By (21), \(R_{1111} \leq 0,\) (16) says \(\frac{\partial r_{11}}{\partial r} + 2r_{22}^2 \geq 0.\) Using (18), we find

(23) \[
\frac{\partial}{\partial r} \left( 2r_{11} - r_{22} \right) \geq -4r_{11}^2 - 2r_{11}r_{22}^2 + 2r_{22}^2 \\
\quad = -2(2r_{11} - r_{22})(r_{11} + r_{22}).
\]

Since \(\lim_{r \to 0} (2r_{11} - r_{22}) = 0,\) the claim follows from a simple ode argument. \(\Box\)

From (17), (18),

(24) \[
\frac{1}{2}R_{1122} = -\left( \frac{\partial r_{22}}{\partial r} + r_{22}^2 \right) = -r_{22}(2r_{11} - r_{22}) \leq 0.
\]

In the last step, we have used the fact that \(r_{22} \geq 0\) which directly follows from (18). It is easy to see that for \(0 \leq r \leq 2r_0,\)

(25) \[R_{1122} = 0.\]

By (20),

(26) \[
\frac{1}{2r} \leq r_{11} \leq \frac{1}{r}.
\]

(18) and a simple ode analysis imply that for all \(r \geq 0,\)

(27) \[
\frac{1}{r} \leq r_{22} \leq \frac{2}{r}.
\]
Let \( u(r) = 2r_1 - r_2 \geq 0 \). Then by (27), (26), (13), (16), (22) and (28)
\[
\begin{align*}
 u' &= -2u(r_1 + r_2) - R_{1112} \\
 &= -2u(r_1 + r_2) - \frac{3u}{r} + \frac{100\epsilon}{r^2}.
\end{align*}
\]
Note if \( u(2r_0) = 0 \). Therefore \( r^3u' \leq 100\epsilon r \). Therefore 
\[
 u(r) \leq \frac{50\epsilon(r^2 - 4r_0^2)}{r^3} \leq \frac{50\epsilon}{r}.
\]
By (24),
\[
 R_{1112} \geq -\frac{100\epsilon}{r^2}.
\]
By (22), (25), (24), (29), if we take \( \epsilon = \frac{C}{200} \), \( M_1 \) satisfies condition (15).

Now we construct a bounded plurisubharmonic function on \( N \) with the form 
\[
 h(x) = F(r(x)).
\]
Define a \( C^\infty \) function \( u \) satisfying
\[
 \begin{cases}
   \ u(r) = 0 & 0 \leq r \leq 4r_0 \\
   \ u' \geq -\frac{2}{(2-\epsilon) r} & 4r_0 \leq r \leq 8r_0 \\
   \ u' = -\frac{2}{(2-\epsilon) r} & r \geq 8r_0.
\end{cases}
\]
Define
\[
 F(r) = \int_0^r e^{u(t)} dt.
\]
It is easy to see that when \( r \geq 8r_0, \)
\[
 F(r) = A - Br^{-\frac{2}{(2-\epsilon) r}}
\]
where \( A, B \) are positive constants. In particular, \( F \) is uniformly bounded for all \( r \geq 0 \). By theorem 16, we can check that \( F(r) \) is a bounded plurisubharmonic function on \( N \). Pulling back to \( M \), we obtain a bounded plurisubharmonic function \( g \) on \( M \). By corollary 5, \( g \) is a constant. Since \( p \in f(M) \), \( g \equiv F(p) = 0 \). Since \( p \) is the unique minimum point of \( F(r) \), \( f(M) = p \).

\[\square\]

8. Miscellaneous Results

**Theorem 17.** Let \( M^n \) be a complete noncompact Kähler manifold with nonnegative holomorphic sectional curvature. Suppose the holomorphic sectional curvature is positive at one point, then there exists \( \epsilon > 0 \) depending only on \( M \) such that for any integer \( d \geq 1 \), \( \dim(\mathcal{O}_d(M)) \leq \dim(\mathcal{O}(1-\epsilon)d(\mathbb{C}^n)) \).

**Remark.** Similar results were proved by Chen, Fu, Le and Zhu [2].

**Proof.** Like the case when the curvature has a negative lower bound, we can estimate \( h(r) \) in theorem 12. For large \( r \), one can show that \( h(r) \geq (1 + \delta) \log r + C \) (here \( \delta > 0, C \) are constants depending only on \( M \)). The details will be omitted. The theorem follows from theorem 12 and linear algebra.

\[\square\]
Below we show that if the holomorphic sectional curvature is positive near infinity and does not decay too fast, then the dimension of holomorphic functions with exponential growth is finite.

**Definition 4.** Let $M$ be a complete noncompact Kähler manifold and $p \in M$. Let $r(x) = \text{dist}(x, p)$. For any $A > 0, d \geq 1$, define $E_{A,d}(M) = \{ f \in O(M)| \lim_{r \to \infty} \frac{|f(x)|}{r^d} < \infty \}$. For simplicity, we denote $E_{1,d}(M)$ by $E_d(M)$.

**Theorem 18.** Let $M^n$ be a complete noncompact Kähler manifold and $p \in M$. Let $r(x) = \text{dist}(x, p)$. Suppose the holomorphic sectional curvature satisfies $R_{i\bar{j}k\bar{l}} \geq \frac{C}{r^2}$ for all sufficiently large $r$. Here $e_1 = \frac{1}{\sqrt{2}}(\nabla r - \sqrt{-1}J\nabla r)$; $C$ is a constant satisfying $0 < C \leq \frac{1}{4}$. Let $a > \frac{1}{4}$ be the positive number satisfying $2a^2 - a + \frac{C}{2} = 0, A = 1 - 2a$. Then for any $d \geq 1$,

$$\dim(E_{A,d}(M)) \leq C_1 d^a,$$

here $C_1$ is a constant depending only on $M$. In particular, $O_d(M) \equiv \text{constant for any } d \geq 0$. Finally, the power in (31) is sharp.

**Proof.** Since $\frac{\partial r_1}{\partial r} + 2r_1 \frac{\partial}{\partial r} + \frac{C}{r^2} \leq 0$, by the assumption,

$$\frac{\partial r_1}{\partial r} + 2r_1 \frac{\partial}{\partial r} + \frac{C}{r^2} \leq 0$$

for all large $r$. Below we analyze along a minimal geodesic from $p$.

**Claim 4.** Let $a > \frac{1}{4}, b < \frac{1}{4}$ be the positive solutions to $2a^2 - a + \frac{C}{2} = 0$ for $0 < C \leq \frac{1}{4}$. Then there exists a constant $B > 0$ depending only on $M$ such that

$$r_1 \leq \frac{aB^k - b}{r(B^k - 1)}$$

for large $r$. Here $k = 2a - 2b > 0$.

**Proof.** It is easy to see $a < \frac{1}{4}$. Let $g(r) = rr_1$. Then we can plug in (32) to find

$$rg' - g + 2g^2 + \frac{C}{2} = rg' + 2(g - a)(g - b) \leq 0$$

for $r \geq r_0$ (here $r_0$ is a large number). We may assume $g(r_0) \geq a$, otherwise $g(r) \leq a$ for all $r \geq r_0$ and the conclusion is obvious. Solving inequality (33), we find

$$\frac{1}{a - b} \ln \frac{g - a}{g - b} \leq -2 \ln r + B_1.$$ 

Here $B_1$ is a constant depending only on $g(r_0), r_0, a, b$. Let $B = e^{-B_1}$, then a simplification of (34) gives the claim.

Now we can solve the differential equation $h(r)_1 = 0$ with $\lim_{r \to 0} \frac{h(r)}{r} = 1$. It is not hard to see that for $r$ very large,

$$h(r) \geq c_1 r^{1-2a} + c_2$$

SHARP ESTIMATES 25
with some constants $c_1, c_2(c_1 > 0)$. By theorem [12] given any $f \in O_M$, if the vanishing order at $p$ is no less than $d$, then $\frac{|f(x)|}{e^{hr}}$ is nondecreasing. The lower bound of $h(r)$ and a linear algebra argument give the dimension estimate. If $f \in O_d(M)$ for some $d \geq 0$, then for any integer $k \geq 1$, $f^k \in E_{A,d}(M)$. If $f$ is not a constant, $\{f^k\}$ will be linearly independent. This contradicts that $\text{dim}(E_{A,d}(M))$ is finite.

Finally, one can easily construct an example in complex dimension 1 case (unitary symmetric) to verify the sharpness of the power in (31).

$\square$

Remark. For $C = \frac{1}{4}$, one can also get sharp estimates by solving differential equations. However, the expression is more messy. For $C > \frac{1}{4}$, $M$ is automatically compact. Finally observe that $C$ in theorem 18 is a rescale invariant.

In the next theorem the sharp model is the generalized cigar soliton. This is the unitary symmetric Kähler metric on $\mathbb{C}^n$ defined by $\partial \bar{\partial} p(r)(r)$ (r is the Euclidean distance on $\mathbb{C}^n$). Here $p(r)$ is a function such that $\partial \bar{\partial} p(r)$ defines the cigar soliton on $\mathbb{C}$. For details of the cigar soliton, see [12] and [2].

**Theorem 19.** Let $M$ be a complete noncompact Kähler manifold and $p \in M$. Let $r(x) = \text{dist}(x, p)$. Suppose the holomorphic sectional curvature satisfies $R_{\bar{\nabla} \nabla} \geq \frac{8}{(e^r + e^{-r})^2}$ where $e_1 = \frac{1}{\sqrt{2}}(\nabla r - \sqrt{-1} J \nabla r)$, then we have the sharp dimension estimate

$$\text{dim}(E_d(M)) \leq \text{dim}(E_d(N)) = \text{dim}(O_d(\mathbb{C}^n))$$

where $N$ is the generalized cigar soliton.

Proof. Define $h(r) = \log \frac{e^r - e^{-r}}{2}$. We show that $(h(r))_{\bar{\nabla} \nabla} \leq 0$. It suffices to verify $r_{\bar{\nabla} \nabla} \leq \frac{2}{e^{2r} - e^{-2r}}$. By theorem [8]

$$2r^2_{\bar{\nabla} \nabla} + \frac{\partial r_{\bar{\nabla} \nabla}}{\partial r} + \frac{4}{(e^r + e^{-r})^2} \leq 0.$$

Then the conclusion follows from a simple ode argument. Let $f \in O(M)$ which vanishes with order $k$ at $p$. By theorem [12] $M(f)_{\bar{\nabla} \nabla}$ is nondecreasing. This implies that if $f \in E_d(M)$, the vanishing order of $f$ at $p$ is no greater than $d$. By linear algebra, the sharp dimension inequality holds. $\square$

9. Examples

In this section we give some examples showing that certain Kähler manifolds admit complete Kähler metric with positive holomorphic sectional curvature, yet do not admit complete Kähler metric with nonnegative Ricci curvature. Recall the Hirzebruch surface $\mathbb{F}_n$ is $\mathbb{P}(O_{\mathbb{P}^1}(n) \oplus O_{\mathbb{P}^1})$ (here $n \geq 0$).

**Proposition 2.** For any natural number $n$ and positive integer $m$, complex manifolds $\mathbb{F}_n \times \mathbb{C}^m$ admits complete Kähler metric with positive holomorphic sectional curvature. For infinitely many $n$, those manifolds do not admit complete Kähler metric with nonnegative Ricci curvature.
Remark. There should be more (nontrivial) complex manifolds which admit complete Kähler metric with positive holomorphic sectional curvature, yet do not admit Kähler metric with nonnegative Ricci curvature. We are not going to pursue this in the paper.

Proof. Recall in [10], Hitchin showed that for any integer $n \geq 0$, $\mathbb{F}_n$ admits a Kähler metric with positive holomorphic sectional curvature. Since $\mathbb{C}^m$ admits complete Kähler metric with positive holomorphic bisectional curvature, the product metric $\mathbb{F}_n \times \mathbb{C}^m$ has positive holomorphic sectional curvature.

Now assume $M = \mathbb{F}_n \times \mathbb{C}^m$ admits a Kähler metric with nonnegative Ricci curvature. We have

$$TM|_{\mathbb{F}_n} = TF_n \oplus N$$

where $N$ is the normal bundle over $\mathbb{F}_n$ which is trivial (here we are talking about the holomorphic tangent bundle). Therefore $c_1(T\mathbb{F}_n) \geq 0$. According to [8], for $\mathbb{F}_n$, the canonical divisor

$$K = -2E_0 + (n-2)C$$

where $E_0$ is the zero-section of $\mathbb{F}_n$ and $C$ is the fibre. Consider section $(\sigma, 0) \in O_{\mathbb{P}^1}(n) \oplus O_{\mathbb{P}^1}$, where $\sigma$ is any section on $O_{\mathbb{P}^1}(n)$. Away from the zeros of $\sigma$, $(\sigma, 0)$ gives a curve in $\mathbb{F}_n$; let $E_\infty$ be the closure of this curve. We have $E_0 \cdot E_0 = n; E_0 \cdot C = 1; E_\infty \sim E_0 - nC$. Since $K \cdot E_\infty \leq 0$, $n \leq 2$. This concludes the proof of the proposition.

□

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Department of Mathematics, University of California, Berkeley, Berkeley, CA 94720
E-mail address: liuxx895@math.umn.edu