Bose systems in spatially random or time-varying potentials

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Abstract

Bose systems, subject to the action of external random potentials, are considered. For describing the system properties, under the action of spatially random potentials of arbitrary strength, the stochastic mean-field approximation is employed. When the strength of disorder increases, the extended Bose-Einstein condensate fragments into spatially disconnected regions, forming a granular condensate. Increasing the strength of disorder even more transforms the granular condensate into the normal glass. The influence of time-dependent external potentials is also discussed. Fastly varying temporal potentials, to some extent, imitate the action of spatially random potentials. In particular, strong time-alternating potential can induce the appearance of a nonequilibrium granular condensate.
1 Introduction

Physics of systems with a Bose-Einstein condensate in random media has been attracting attention for many years. At the beginning, the interest was concentrated on the behavior of liquid helium in nanoporous media [1]. In recent years, the physics of dilute Bose gases has gained much interest [2–8]. Quasidisordered Bose gases have been realized experimentally by creating quasiperiodic optical lattices [9–11]. Disordered optical lattices have been realized by incorporating random impurities into an optical lattice [12]. There exists vast literature devoted to the theory of disordered optical lattices. A large number of references on the subject can be found in the recent review article [13].

A different type of disordered Bose systems corresponds to the systems without a periodic lattice potential, but which are subject to a spatially random potential. Experimentally, such random potentials are realized by means of optical speckles [14–17]. Physical properties of these Bose systems in an external random potential have been theoretically studied for the case of weak disorder [18–21] and for arbitrarily strong disorder [22,23].

The properties of uniform Bose systems and of those inside periodic lattice potentials are, of course, different [13] and may require differing theoretical methods of description. For example, the self-consistent mean-field approximation [24–29] describes well the uniform Bose systems with any strong atomic interactions. But the region of applicability of this approximation can be limited for Bose atoms in a lattice at zero temperature in the vicinity of the superfluid-insulator phase transition [30].

The properties of disordered Bose systems are also different, depending on whether external random potentials have been imposed on an initially uniform or spatially periodic system [13]. One interesting feature, common for both types of systems, is that sufficiently strong disorder can lead to the occurrence of a new phase, called Bose glass. This state of matter, occurring in disordered lattices, was suggested by Fisher et al. [31] (see also the recent articles [32,33] and the review paper [13]). The Bose glass phase is often characterized by the remaining presence of the condensate fraction ($n_0 > 0$), but with the absence of superfluidity ($n_s = 0$).

The nature of the Bose glass phase is usually described as follows. The whole system fragments into the islands of the Bose-Einstein condensate localized in the deep wells of the random potential, while other regions, surrounding these islands, are filled by the normal fluid, containing no condensate. This is why such a phase is also termed the granular condensate or the localized condensate. This phase has been observed in disordered optical lattices [34] and in nanoporous media filled by liquid helium [35]. Such a state of matter can also exist in Bose systems without lattices [13,36].

The present paper concerns the Bose systems without periodic lattices. In the absence of a random potential, such a system would be uniform, exhibiting at low temperatures Bose-Einstein condensation. The description of a Bose-condensed system in external spatially random potentials of arbitrary strength can be accomplished by means of the stochastic mean-field approximation. We emphasize that describing such random systems requires a special caution, since perturbation theory is not always applicable to them. This, e.g., concerns the weak-disorder expansion, which can become invalid for random Bose-condensed systems. For these systems, the perturbation theory with respect to the coupling parameter can also fail. Therefore the use of more refined approaches, such as the stochastic mean-field approximation, is of great importance for the correct description of random systems.
After presenting the main characteristics of Bose-condensed systems in spatial random potentials, we turn to the problem of these systems in temporal alternating potentials. We show that the action of temporal external fields can induce in a Bose system the consequences analogous to those produced by spatially random potentials. In particular, a nonequilibrium granular condensate can arise under sufficiently strong external fields. Very strong alternating fields will destroy the granular condensate, transforming the whole system into a normal turbulent fluid.

Throughout the paper, the natural system of units is employed, where $\hbar = 1$ and $k_B = 1$.

2 Bose Systems in Random Potentials

We consider a dilute Bose system composed of atoms interacting through the local potential

$$
\Phi(r) = \Phi_0 \delta(r), \quad \Phi_0 \equiv 4\pi \frac{a_s}{m},
$$

in which $a_s$ is the scattering length and $m$, atomic mass. The system is subject to a spatially random external potential $\xi(r)$. So that the energy Hamiltonian is

$$
\hat{H} = \int \psi^\dagger(r) \left[ -\frac{\nabla^2}{2m} + \xi(r) \right] \psi(r) \, dr + \frac{\Phi_0}{2} \int \psi^\dagger(r) \psi^\dagger(r) \psi(r) \psi(r) \, dr,
$$

where $\psi(r)$ is the Bose field operator.

Without the loss of generality, the random potential can be taken as zero-centered, such that its stochastic averaging gives

$$
\langle \xi(r) \rangle = 0.
$$

The stochastic correlation function

$$
R(r - r') = \langle \xi(r) \xi(r') \rangle
$$

is assumed to be real and symmetric,

$$
R^*(r) = R(-r) = R(r).
$$

The presence of a Bose-Einstein condensate implies the spontaneous gauge symmetry breaking [8], which is standardly realized through the Bogolubov shift [37] of the field operator

$$
\psi(r) \rightarrow \hat{\psi}(r) \equiv \eta(r) + \psi_1(r).
$$

Here $\eta(r)$ is the condensate wave function normalized to the number of condensed atoms,

$$
N_0 = \int |\eta(r)|^2 \, dr,
$$

and $\psi_1(r)$ is the Bose field operator of uncondensed atoms defining the number operator

$$
\hat{N}_1 = \int \psi_1^\dagger(r) \psi_1(r) \, dr.
$$
of uncondensed atoms. The field variables $\eta(r)$ and $\psi_1(r)$ are treated as two independent variables, orthogonal to each other,

$$\int \eta^*(r)\psi_1(r) \, dr = 0 \, .$$  \tag{9}

Under the Bogolubov shift (6), in order to make the theory self-consistent, it is necessary, as is proved in Refs. [24–29], to introduce the grand Hamiltonian

$$H = \hat{H} - \mu_0 N_0 - \mu_1 \hat{N}_1 \, ,$$  \tag{10}

in which $\hat{H}$ is the energy Hamiltonian (2), with the Bogolubov-shifted field operators (6), and $\mu_0$ and $\mu_1$ are the Lagrange multipliers guaranteeing the theory self-consistency.

In the presence of random fields, there exist two types of averaging. One type is the stochastic averaging, denoted as $\ll \ldots \gg$, which characterizes the averaging over the distribution of random potentials. And there is, as usual, the quantum statistical averaging, which for an operator $\hat{A}$ is defined as

$$\langle \hat{A} \rangle_H \equiv \text{Tr} \, \hat{\rho} \hat{A} \, ,$$  \tag{11}

where $\hat{\rho}$ is a statistical operator. The latter, for an equilibrium system, is

$$\hat{\rho} = \frac{\exp(-\beta H)}{\text{Tr} \exp(-\beta H)} \, ,$$  \tag{12}

with $\beta \equiv 1/T$ being inverse temperature. The total average of an operator $\hat{A}$,

$$\langle \hat{A} \rangle \equiv \ll \text{Tr} \, \hat{\rho} \hat{A} \gg$$  \tag{13}

includes both, the quantum and stochastic averaging procedures. The grand thermodynamic potential, corresponding to the frozen disorder, is

$$\Omega = -T \ll \ln \, \text{Tr} \, e^{-\beta H} \gg \, .$$  \tag{14}

One should also keep in mind the quantum-number conservation condition

$$\langle \psi_1(r) \rangle = 0 \, .$$  \tag{15}

To satisfy the latter, the grand Hamiltonian (10) should be complimented by one more term guaranteeing the absence in $H$ of the terms linear in $\psi_1(r)$, as is shown in Ref. [29]. Here we do not add explicitly such a linear killer, since for a uniform system or for a system uniform on average, linear in $\psi_1(r)$ terms do not arise and condition (15) is automatically satisfied [24–29].

For the zero-centered random potential, for which property (3) holds, the system can be treated as uniform on average. Then one can set

$$\eta(r) = \sqrt{\rho_0} \, (\rho_0 \equiv \frac{N_0}{V}) \, ,$$  \tag{16}

where $V$ is the system volume. The field operator of uncondensed atoms can be expanded over plane waves as

$$\psi_1(r) = \frac{1}{\sqrt{V}} \sum_{k \neq 0} a_k e^{ik \cdot r} \, .$$  \tag{17}
The field operators in the momentum representation, $a_k$, define the momentum distribution

$$n_k \equiv < a_k^\dagger a_k >$$  \hspace{1cm} (18)

and the anomalous average

$$\sigma_k \equiv < a_k a_{-k} >.$$  \hspace{1cm} (19)

The density of uncondensed atoms is

$$\rho_1 = \frac{1}{V} \sum_{k \neq 0} n_k ,$$  \hspace{1cm} (20)

and the total anomalous average is

$$\sigma_1 = \frac{1}{V} \sum_{k \neq 0} \sigma_k .$$  \hspace{1cm} (21)

With the total number of particles $N$, the system average density is

$$\rho \equiv \frac{N}{V} = \rho_0 + \rho_1 .$$  \hspace{1cm} (22)

The density of uncondensed atoms (20) consists of two terms,

$$\rho_1 = \rho_N + \rho_G ,$$  \hspace{1cm} (23)

among which $\rho_N$ is due to thermal fluctuations and interactions, while $\rho_G$ is caused by the random potential and defines the density of a glassy component or, briefly speaking, the glassy density

$$\rho_G \equiv \frac{1}{V} \int \ll |< \psi_1(r) >_H |^2 \gg \, dr .$$  \hspace{1cm} (24)

The general definition of the superfluid density [3,29] results in the expression

$$\rho_s = \rho - \frac{< \hat{P}^2 >}{3mTV} ,$$  \hspace{1cm} (25)

in which $\hat{P}$ is the momentum operator

$$\hat{P} \equiv \int \hat{\psi}(r) \ (-i\nabla) \hat{\psi}(r) \ dr = \int \psi_1(r) \ (-i\nabla) \psi_1(r) \ dr .$$

It is convenient to define the dimensionless atomic fractions. Thus, the condensate fraction

$$n_0 \equiv \frac{\rho_0}{\rho} = 1 - n_1$$  \hspace{1cm} (26)

is expressed through the uncondensed-atom fraction

$$n_1 \equiv \frac{\rho_1}{\rho} = \frac{1}{N} \sum_{k \neq 0} n_k .$$  \hspace{1cm} (27)
Equation (21) gives the anomalous fraction
\[ \sigma \equiv \frac{\sigma_1}{\rho} = \frac{1}{N} \sum_{k \neq 0} \sigma_k. \] (28)

The uncondensed-atom fraction (27) is the sum
\[ n_1 = n_N + n_G \] (29)
of the normal fraction \( n_N \equiv \rho_N / \rho \) and the glassy fraction
\[ n_G \equiv \frac{\rho_G}{\rho} = \frac{1}{N} \int \left\langle \left| <\psi_1(r) >_H \right|^2 \right\rangle \, dr. \] (30)

Finally, from Eq. (25) we have the superfluid fraction
\[ n_s = 1 - 2Q / 3T, \] (31)
in which
\[ Q \equiv \frac{\left\langle \hat{P}^2 \right\rangle}{2mN} \] (32)
is the dissipated heat per atom. With expansion (17), the glassy fraction can be written as
\[ n_G = \frac{1}{N} \sum_k \left\langle |\alpha_k|^2 \right\rangle, \] (33)
where
\[ \alpha_k \equiv <a_k>_H. \] (34)

Together with expanding over plane waves the field operator of uncondensed atoms, as in Eq. (17), let us expand the random potential as
\[ \xi(r) = \frac{1}{\sqrt{V}} \sum_k \xi_k e^{ikr}, \quad \xi_k = \frac{1}{\sqrt{V}} \int \xi(r) e^{-ikr} \, dr. \] (35)

The Fourier transformation of the correlation function (4) is
\[ R(r) = \frac{1}{V} \sum_k R_k e^{ikr}, \quad R_k = \int R(r) e^{-ikr} \, dr. \] (36)

Then, Fourier-transforming Eq. (4), we get
\[ \left\langle \xi^*_k \xi_p \right\rangle = \delta_{kp} R_k. \] (37)

Passing to the momentum representation, we accomplish in Hamiltonian (2) the Bogolubov shift (6) and substitute there the Fourier transforms (17) and (35). As a result, the grand Hamiltonian (10) acquires the form
\[ H = \sum_{n=0}^{4} H^{(n)} + H_{ext}, \] (38)
in which

\[ H^{(0)} = \left( \frac{1}{2} \rho_0 \Phi_0 - \mu_0 \right) N_0 , \quad H^{(1)} = 0 , \]

\[ H^{(2)} = \sum_{k \neq 0} \left( \frac{k^2}{2m} + 2 \rho_0 \Phi_0 - \mu_1 \right) a_k^\dagger a_k + \frac{1}{2} \sum_{k \neq 0} \rho_0 \Phi_0 \left( a_k^\dagger a_{-k}^\dagger + a_{-k} a_k \right) , \]

\[ H^{(3)} = \sqrt{\frac{\rho_0}{V}} \sum_{k,p(\neq 0)} \Phi_0 \left( a_k^\dagger a_{k+p}^\dagger a_{-p} + a_{-p}^\dagger a_{k+p}^\dagger a_k \right) , \]

\[ H^{(4)} = \frac{1}{2V} \sum_q \sum_{k,p(\neq 0)} \Phi_0 a_k^\dagger a_p^\dagger a_{k-q} a_{p+q} , \]

with the part

\[ H_{\text{ext}} = \rho_0 \xi_0 \sqrt{V} + \sqrt{\rho_0} \sum_{k \neq 0} \left( a_k^\dagger \xi_k + a_k \xi_k^* \right) + \frac{1}{\sqrt{V}} \sum_{k,p(\neq 0)} a_{k+p}^\dagger a_{k-p} \xi_{k-p} , \]

which is due to the action of the external random potential.

3 Stochastic Mean-Field Approximation

To treat Hamiltonian (38), an approximation is needed. The third- and fourth-order terms in the operators \(a_k\), occurring in Eq. (39), can be simplified by means of the Hartree-Fock-Bogolubov approximation, as in Refs. [24–29]. However, the interaction of the random potential with atoms, described by part (40), cannot be treated in the simple mean-field procedure, since

\[ < a_k > = < \xi_k > = 0 , \]

which would kill the last term in Eq. (40). Such a case would correspond to considering asymptotically weak disorder. To treat the last term in (40), a more delicate decoupling procedure is required. For this purpose, we shall employ the stochastic mean-field approximation suggested and used earlier for other physical systems [38–43]. Following the idea of this approximation, we simplify the last term of Eq. (40) writing

\[ a_k^\dagger a_p \xi_{k-p} = \left( a_k^\dagger \alpha_p + \alpha_k^* a_p - \alpha_k^* \alpha_p \right) \xi_{k-p} , \]

where \(\alpha_k\) is the quantum average (34). Under equation (42), we have

\[ < a_k^\dagger a_p \xi_{k-p} > = \ll \alpha_k^* \alpha_p \xi_{k-p} \gg . \]

Approximations (42) and (43) allow for the consideration of arbitrarily strong disorder strengths [22, 23].

The next nontrivial procedure is the diagonalization of Hamiltonian (38) by means of the nonuniform nonlinear canonical transformation

\[ a_k = u_k b_k + v_k^* b_{-k}^\dagger + w_k \varphi_k , \]

in which \(u_k, v_k, w_k,\) and \(\varphi_k\) are to be defined so that the resulting Hamiltonian be diagonal in the operators \(b_k\), such that

\[ < b_k >_H = < b_k b_p >_H = 0 . \]
Then the transformation (44) gives

\[ \alpha_k \equiv \langle a_k \rangle_H = w_k \varphi_k . \]  

(46)

Accomplishing the diagonalization, we find

\[ u_k^2 = \frac{\omega_k + \varepsilon_k}{2\varepsilon_k} , \quad v_k^2 = \frac{\omega_k - \varepsilon_k}{2\varepsilon_k} , \quad u_k v_k = - \frac{mc^2}{2\varepsilon_k} , \quad w_k = - \frac{1}{\omega_k + mc^2} , \]  

(47)

where

\[ \omega_k \equiv \frac{k^2}{2m} + mc^2 , \quad \omega_k^2 = \varepsilon_k^2 + (mc^2)^2 , \]  

(48)

and \( \varepsilon_k \) is the Bogolubov spectrum

\[ \varepsilon_k = \sqrt{(ck)^2 + \left( \frac{k^2}{2m} \right)^2} , \]  

(49)

with the sound velocity given by the equation

\[ mc^2 = (n_0 + \sigma) \rho \Phi_0 . \]  

(50)

The random variable \( \varphi_k \) satisfies the integral equation

\[ \varphi_k = \sqrt{\rho_0} \xi_k - \frac{1}{\sqrt{V}} \sum_p \frac{\xi_{k-p} \varphi_p}{\omega_p + mc^2} \]  

(51)

of the Fredholm type. The nonuniform nonlinear transformation (44), with Eqs. (47) to (51), results in the grand Hamiltonian

\[ H = E_B + \sum_k \varepsilon_k b_k^\dagger b_k + \varphi_0 \sqrt{N_0} , \]  

(52)

in which

\[ E_B = \frac{1}{2} \sum_k (\varepsilon_k - \omega_k) - \left[ 1 - n_0(1 + \sigma) + \frac{1}{2} \left( 1 - n_1^2 \right) \right] \rho^2 \Phi_0 N . \]

With the diagonal Hamiltonian (52), it is straightforward to find the momentum distribution (18),

\[ n_k = \frac{\omega_k}{2\varepsilon_k} \coth \left( \frac{\varepsilon_k}{2T} \right) - \frac{1}{2} + \ll |\alpha_k|^2 \gg \]  

(53)

and the anomalous average

\[ \sigma_k = - \frac{mc^2}{2\varepsilon_k} \coth \left( \frac{\varepsilon_k}{2T} \right) + \ll |\alpha_k|^2 \gg . \]  

(54)

The random variable (46), in view of Eqs. (47), is

\[ \alpha_k = - \frac{\varphi_k}{\omega_k + mc^2} . \]  

(55)
\[
\ll |\alpha_k|^2 \gg = \frac{\ll |\varphi_k|^2 \gg}{(\omega_k + mc^2)^2},
\]
(56)

The fraction of uncondensed atoms (27) is represented as sum (29), in which
\[
n_N = \frac{1}{2\rho} \int \left[ \frac{\omega_k}{\varepsilon_k} \coth \left( \frac{\varepsilon_k}{2T} \right) - 1 \right] \frac{dk}{(2\pi)^3},
\]
(57)

while the glassy fraction (33) becomes
\[
n_G = \frac{1}{\rho} \int \frac{\ll |\varphi_k|^2 \gg}{(\omega_k + mc^2)^2} \frac{dk}{(2\pi)^3}.
\]
(58)

The anomalous fraction (28) is also the sum
\[
\sigma = \sigma_N + n_G,
\]
(59)

where
\[
\sigma_N = -\frac{1}{2\rho} \int \frac{mc^2}{\varepsilon_k} \coth \left( \frac{\varepsilon_k}{2T} \right) \frac{dk}{(2\pi)^3}
\]
(60)

and the \( n_G \) is the same as in Eq. (58). The terms \( n_N \) and \( \sigma_N \) are caused by thermal fluctuations and interactions, while the glassy fraction \( n_G \) is due to the action of the random field.

In the superfluid fraction (31), the dissipated heat (32) reads as
\[
Q = Q_N + Q_G,
\]
(61)

with
\[
Q_N = \frac{1}{8m\rho} \int \frac{k^2}{\sinh^2(\varepsilon_k/2T)} \frac{dk}{(2\pi)^3}
\]
(62)

being due to interactions and finite temperature, while
\[
Q_G = \frac{1}{2m\rho} \int \frac{k^2 \ll |\varphi_k|^2 \gg}{\varepsilon_k(\omega_k + mc^2)} \coth \left( \frac{\varepsilon_k}{2T} \right) \frac{dk}{(2\pi)^3}
\]
(63)

is the heat dissipated by the glassy fraction.

Note that deriving the above formulas we have used the form of the Lagrange multiplier
\[
\mu_0 = (1 + n_1 + \sigma)\rho \Phi_0,
\]
(64)

obtained by minimizing the grand potential (14) with respect to the number of condensed atoms \( N_0 \), and the multiplier
\[
\mu_1 = (1 + n_1 - \sigma)\rho \Phi_0,
\]
(65)

found from the condition of the condensate existence \([13]\), which is equivalent to the requirement that the spectrum of collective excitations be gapless.
4 Failure of Weak-Disorder Perturbation Theory

Uniform Bose-condensed systems under the action of random potentials are usually considered in the case of asymptotically weak interactions and weak disorder. Exceptions are the articles [22,23], where the stochastic mean-field approximation [38–43] was employed allowing for the description of systems with arbitrarily strong atomic interactions and arbitrarily strong disorder. It is necessary to stress that the perturbation theory with respect to the disorder strength may be inapplicable to the Bose systems in random potentials. In the present section, we show that the weak-disorder perturbation expansion may lead to incorrect results.

Let us consider the last term

\[ H_{\text{ran}} \equiv \varphi_0 \sqrt{N_0} \]  

in Hamiltonian (52), explicitly describing the influence of the random potential. The related contribution to the internal energy is

\[ E_{\text{ran}} \equiv \langle H_{\text{ran}} \rangle , \]

which results in

\[ E_{\text{ran}} = \langle \varphi_0 \rangle \sqrt{N_0} = \ll \varphi_0 \gg \sqrt{N_0} . \]

Suppose, one considers weak disorder and assumes that the weak-disorder perturbation theory should be valid. Then, the first term in Eq. (51) can be treated as the zero-order approximation

\[ \varphi_k^{(0)} = \sqrt{\rho_0} \xi_k . \]

Iterating with Eq. (69) the second term of Eq. (51) gives the first-order approximation

\[ \varphi_k^{(1)} = \sqrt{\rho_0} \xi_k - \sqrt{N_0} \frac{\sum_p \xi_{k-p} \xi_p}{\omega_p + mc^2} . \]

Employing approximation (70) makes it easy to derive all results obtained by other researchers using the weak-disorder perturbation theory (see Ref. [22]). For example, the internal energy (68), due to random fields, becomes

\[ E_{\text{ran}}^{(1)} = - \frac{N_0}{V} \sum_p \ll |\xi_p|^2 \gg \frac{\omega_p + mc^2}{\omega_p + mc^2} . \]

Using here the correlation formula (37) and passing to integration yields

\[ E_{\text{ran}}^{(1)} = - \int \frac{N_0 R_p}{(\omega_p + mc^2)} \frac{dp}{(2\pi)^3} . \]

It is exactly this expression (72) that has been rederived by all authors who have used the weak-disorder perturbation theory. Equation (72) tells us that random fields diminish the internal energy.

But the result is very different, if no perturbation theory has been involved. According to Eqs. (34) and (41), we have

\[ \ll a_k \gg = 0 . \]
Then, from Eq. (46), it follows that

$$\ll \varphi_k \gg = 0 .$$

(74)

Consequently, the random-field contribution to the internal energy (68) is exactly zero,

$$E_{\text{ran}} \equiv < H_{\text{ran}} > = 0 .$$

(75)

This conclusion sounds reasonable, if one remembers that the zero-centered potential, with the zero mean (3), is considered. Thus, the perturbation-theory formula (72) is in contradiction with the exact expression (75), hence, Eq. (72) does not seem to be correct. The same concerns other formulas that have been derived by means of the weak-disorder expansion. Such formulas do not seem to be reliable. The reason for the weak-disorder perturbation theory failure is the noncommutativity, in some cases, of the operations of expanding over the random fields and of averaging over these fields.

## 5 Random Potential with Local Correlations

The consideration of the previous sections is general, being applicable to any spatially random potentials. To specify the problem, it is necessary to concretize the type of the random potential. When the correlation length of the correlation function (4) is sufficiently short, at least much shorter than the healing length, then the random potential can be modelled by the Gaussian white noise, with the local correlation function

$$R(r) = R_0 \delta(r) .$$

(76)

Then the Fourier transform (36) gives \( R_k = R_0 \) and Eq. (37) becomes

$$\ll \xi_k^* \xi_p \gg = \delta_{kp} R_0 .$$

(77)

For numerical investigation, it is convenient to introduce dimensionless variables. The interaction strength is characterized by the gas parameter

$$\gamma \equiv \rho^{1/3} a_s .$$

(78)

The dimensionless temperature is defined as

$$t \equiv \frac{mT}{\rho^{2/3}} .$$

(79)

For the dimensionless sound velocity, we have

$$s \equiv \frac{mc}{\rho^{1/3}} .$$

(80)

Disorder is known to possess the property of localizing atomic motion inside the regions of the characteristic localization length, which can be estimated as the Larkin length [44]

$$l_{\text{loc}} \equiv \frac{4\pi}{im^2 R_0} .$$

(81)
The strength of disorder can be described by the disorder parameter

\[ \zeta \equiv \frac{a}{l_{\text{loc}}} \left( \rho^{1/3} a = 1 \right), \tag{82} \]

where \( a \) is the mean interatomic distance.

If the random potential \( \xi(r) \) is limited by a finite amplitude \( V_R \), such that

\[ |\xi(r)| \leq V_R \]

for all \( r \) pertaining to the considered system, and if the correlation function (4) has a finite correlation length \( l_R \), then the parameter \( R_0 \) in Eqs. (76) and (77) can be represented as

\[ R_0 = V_R^2 l_R^3. \tag{83} \]

In that case, the localization length (81) becomes

\[ l_{\text{loc}} = \frac{4\pi}{\tau m^2 V_R^2 l_R^3}, \tag{84} \]

which is close to the expression given in Ref. [17]. Returning to the local correlation function (76) implies that \( l_R \to 0 \) and \( V_R \to \infty \), so that product (83) be finite.

When disorder is asymptotically weak, so that \( R_0 \to 0 \), then \( l_{\text{loc}} \) extends to the size of the whole system. The latter then represents the standard Bose-condensed system with extended condensate. With growing disorder, when the localization length becomes much smaller than the system linear size \( L \), but when \( l_{\text{loc}} \) is yet much larger than the mean interatomic distance \( a \), that is, when

\[ a \ll l_{\text{loc}} \ll L, \tag{85} \]

then the Bose-Einstein condensate fragments into multiple pieces separated by the normal phase with no gauge symmetry breaking [34–36, 45–47]. This type of matter is termed the Bose glass or the granular condensate. Finally, when disorder is so strong that \( l_{\text{loc}} \sim a \), then no condensate is possible, since atoms are localized separately, each of them being trapped in a deep randomly located well of the random potential. The latter phase forms the normal glassy matter. Such a phase was observed [14] in a strong random potential created by laser speckles. At zero temperature, the phase portrait, in the variables of the gas parameter (78) and disorder parameter (82), should look as in Fig. 1. The granular condensate starts appearing when \( l_{\text{loc}} \sim (10 - 100)a \), hence \( \zeta \sim 0.01 - 0.1 \). And the granular condensate transforms into the normal glass when \( l_{\text{loc}} \sim a \), so that \( \zeta \sim 1 \). The phase transition between the extended condensate and granular condensate is continuous, while that between the granular condensate and the normal glass is of first order.

The granular condensate is a phase that cannot already be treated as uniform on average, as it is done for the extended condensate. The granular condensate is a principally nonuniform condensate, which requires a separate consideration taking into account the spatial nonuniformity. Assuming that the system is uniform on average does not distinguish between the extended and granular condensates, but this description covers all the region, where any condensate is possible. However, the phase transition between the system with a condensate and the normal glass can be described. Keeping this in mind, we follow the consideration of the previous sections.
First, we need to solve the integral equation (51). A good approximate solution to this equation is

\[ \varphi_k = \frac{\sqrt{\rho_0} \xi_k}{1 + \frac{1}{\sqrt{V}} \sum_p \xi_p \omega_p + mc^2}. \]  

This random variable enters the above equations, such as Eqs. (53), (54), (55), (58), and (63), in the form \( \ll |\varphi_k|^2 \gg \). To find the latter, we employ the self-similar approximation theory [48–50] in the variant involving the self-similar factor approximants [51–54]. As a result, we obtain

\[ \ll |\varphi_k|^2 \gg = \frac{n_0 R_0 s^{3/7}}{a^3(s - \zeta)^{3/7}}. \]  

Then the glassy fraction (58) is

\[ n_G = \frac{n_0 \zeta}{s^{4/7}(s - \zeta)^{3/7}}. \]  

Note that the disorder parameter (82) naturally appears in Eqs. (87) and (88). Therefore these equations can be considered as defining the disorder parameter (82) as such.

In dimensionless variables (78) to (80), the dimensional equation (50) for the sound velocity becomes

\[ s^2 = 4\pi \gamma (1 - n_1 + \sigma). \]  

Due to Eqs. (29) and (59), this can be rewritten as

\[ s^2 = 4\pi \gamma (1 - n_N + \sigma_N). \]  

For the normal fraction (57), we have

\[ n_N = \frac{s^3}{3\pi^2} \left\{ 1 + \frac{3}{2\sqrt{2}} \int_0^\infty \left( \sqrt{1 + x^2} - 1 \right)^{1/2} \left[ \coth \left( \frac{s^2 x}{2t} \right) - 1 \right] dx \right\}. \]  

According to expressions (22) and (23), the atomic fractions are normalized as

\[ n_0 + n_N + n_G = 1. \]  

Using this, the glassy fraction (88) can be represented as

\[ n_G = \frac{(1 - n_N)\zeta}{\zeta + 7s^{4/7}(s - \zeta)^{3/7}}. \]  

The superfluid fraction (31) contains the dissipated heat (61). The part (63) of the dissipated heat, due to the dissipation on the glassy fraction, in the case of the white noise, contains Eq. (87). Then, integral (63) diverges, but can be regularized [22]. So that, finally, for the superfluid fraction, we find

\[ n_s = 1 - \frac{4}{3} n_G - \frac{s^5}{6\sqrt{2} \pi^2 t} \int_0^\infty \frac{x \left( \sqrt{1 + x^2} - 1 \right)^{3/2} dx}{\sqrt{1 + x^2} \sinh^2(s^2 x/2t)}. \]  

The anomalous fraction (60), because of the local interaction (1), also diverges and requires to be regularized [13,24–29,55]. In this regularization, the asymptotic properties of
function (60), with respect to temperature, should remain correctly defined. Thus, in the
limit of zero temperature, using the dimensional regularization, we have [13,24,27–29]

$$\sigma_N \simeq \frac{2s^2}{\pi^2} \sqrt{\pi \gamma n_0} \quad (t \to 0). \quad (95)$$

While, when temperature tends to the critical point

$$t_c = 3.312498, \quad (96)$$

then the correct limit of Eq. (60) is

$$\sigma_N \simeq -\frac{s t}{2\pi} \quad (t \to t_c). \quad (97)$$

The critical point (96) is the same as for the ideal Bose gas, as it has to be for a mean-field
approximation [29].

It is important to stress that taking account of the anomalous averages is principally
necessary. It would be absolutely incorrect to use the Shohn trick [56] by omitting the
anomalous averages, as one often does. This is mathematically wrong, since the anomalous
averages at low temperature can be larger than the normal fraction $n_N$, and they are of
the same order at finite temperatures below the critical point [23,57]. Keeping what is
of the same order, but neglecting what can even be larger, cannot be called a reasonable
approximation. It is also straightforward to show [24,29] that omitting the anomalous average
renders the consideration not self-consistent and the system unstable. The origin of the
resulting inconsistency is very easy to understand [29]. The existence of the anomalous
average is due to the gauge symmetry breaking. The latter is also the cause of the Bose-
Einstein condensate existence. Hence both, the anomalous average and the condensate,
either exist together or do not arise at all. If one wishes to omit the anomalous average,
then, to be self-consistent, one must neglect the condensate existence. Or, when the latter
is assumed, one has to retain the anomalous average as well. The omission of the anomalous
average, in addition to breaking the system stability and making the thermodynamics not
self-consistent, also distorts the phase transition order, provoking a first-order transition.
The latter is, of course, incorrect, since the Bose-Einstein condensation is the second-order
phase transition, irrespectively to the interaction strength [29].

The correct asymptotic behavior of the anomalous average in the vicinity of the critical
point $t_c$, as in Eq. (97), guarantees the second-order phase transition for any value of the gas
parameter [13,28,29]. This can be demonstrated by direct numerical calculations [13,28] and
also by expanding the quantities of interest in powers of the relative temperature deviation

$$\tau \equiv \left| \frac{t - t_c}{t_c} \right| \to 0 \quad (98)$$

in the vicinity of $t_c$. Then we obtain the dimensionless sound velocity

$$s \simeq \frac{3\pi}{t_c} \tau + \frac{9\pi}{t_c} \left(1 - \frac{2\pi}{\gamma t_c^2}\right) \tau^2, \quad (99)$$

the condensate fraction

$$n_0 \simeq \frac{3}{2} \tau - \frac{3}{8} \tau^2. \quad (100)$$
the anomalous average
\[ \sigma_N \simeq -\frac{3}{2} \tau + \frac{3}{8} \left( 1 + \frac{6\pi}{\gamma t_c^2} \right) \tau^2, \]  \hspace{1cm} (101)
and the superfluid fraction
\[ n_s \simeq \frac{3}{2} \tau - 1.741 \tau^2. \]  \hspace{1cm} (102)

From these asymptotic expressions, the second-order phase transition is evident.

Trying to interpolate the anomalous average between the asymptotic limits (95) and (97), we can reorganize Eq. (60) to the identical form
\[ \sigma_N = -\frac{1}{2\rho} \int \frac{mc^2}{\varepsilon_k} \frac{d\varepsilon_k}{(2\pi)^3} - \frac{1}{2\rho} \int \frac{mc^2}{\varepsilon_k} \left[ \coth \left( \frac{\varepsilon_k}{2T} \right) - 1 \right] \frac{d\varepsilon_k}{(2\pi)^3}. \]  \hspace{1cm} (103)

Employing for the first term of Eq. (103) the dimensional regularization and passing to the dimensionless quantities (78) to (80), we get [24,27–29]
\[ \sigma_N = \frac{2s^2}{\pi^2} \sqrt{\pi\gamma n_0} - \sqrt{2} s^3 \int_0^\infty \frac{\left( \sqrt{1 + x^2} - 1 \right)^{1/2}}{\sqrt{1 + x^2}} \left[ \coth \left( \frac{s^2x}{2t} \right) - 1 \right] dx. \]  \hspace{1cm} (104)

This expression gives the correct low-temperature limit (95). However, the critical behavior (97) becomes disturbed. This disturbance, as numerical calculations [28] show, is not essential for the gas parameter in the range $0 < \gamma \leq 0.3$, for which the phase transition remains continuous. For larger $\gamma$, the use of expression (104) would result in the discontinuity at the critical point $t_c$. Therefore, for $\gamma > 0.3$, form (104) can be used below the critical region, while close to $T_c$, it has to be replaced by the correct limit (97) guaranteeing the continuous phase transition [28].

At zero temperature, the normal fraction (91) and the superfluid fraction (94) reduce to
\[ n_N = \frac{s^3}{3\pi^2}, \quad n_s = 1 - \frac{4}{3} n_G. \]  \hspace{1cm} (105)

These equations, together with Eqs. (90), (92), (93), and (95), have been analysed in detail [22,23], demonstrating a first-order phase transition on the line $\zeta = \zeta(\gamma)$ between the system with a condensate and the normal glass with no condensate.

### 6 Simple Model with Quenched Disorder

In Sec. 4, we showed that the weak-disorder perturbation theory fails in the description of the Bose-condensed system in spatially random potentials. Therefore, for correctly describing such systems, more refined approaches are necessary, for instance, as the stochastic mean-field approximation of Sec. 3, which allows for the consideration of arbitrarily strong disorder.

Here we show that, in the presence of disorder, perturbation theory with respect to atomic interactions can also fail. We demonstrate this by a simple model with quenched disorder, which allows for an explicit illustration of the point where the weak-coupling perturbation theory fails.
Let us consider the model Hamiltonian

\[ H = (1 + \xi)\varphi^2 + g\varphi^4 \quad (106) \]

of the so-called zero-dimensional $\varphi^4$-theory in the presence of quenched disorder. Here the variable $\varphi \in (-\infty, \infty)$ imitates the Bose condensate function. The coupling parameter $g > 0$ characterizes the interaction strength. And the random variable $\xi$ describes an external random potential, with a distribution $p(\xi)$. As is usual, we assume that the random potential is zero-centered, such that

\[ \langle \xi \rangle = \int_{-\infty}^{\infty} \xi p(\xi) \, d\xi = 0 \quad (107) \]

Its dispersion is given by

\[ \Delta^2 \equiv \langle \xi^2 \rangle = \int_{-\infty}^{\infty} \xi^2 p(\xi) \, d\xi \quad (108) \]

The often used examples of the random-variable distribution are the Gaussian distribution

\[ p_G(\xi) = \frac{1}{\sqrt{2\pi} \xi_0} \exp \left( -\frac{\xi^2}{2\xi_0^2} \right) \quad (109) \]

and the uniform distribution

\[ p_U(\xi) = \frac{1}{2D} \Theta(D - |\xi|) \quad (110) \]

Their dispersions, respectively, are

\[ \Delta^2_G = \xi_0^2, \quad \Delta^2_U = \frac{D^2}{3} \]

The partition function of the system in the frozen random field is

\[ Z(g, \xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-H} \, d\varphi \quad (111) \]

The free energy of the system with quenched disorder writes as

\[ f(g) = -\int_{-\infty}^{\infty} p(\xi) \ln Z(g, \xi) \, d\xi \quad (112) \]

In the limit of no disorder, when $\Delta^2 \rightarrow 0$, then, in both cases (109) and (110), one has $p(\xi) \rightarrow \delta(\xi)$. As a result,

\[ f(g) \rightarrow -\ln Z(g, 0) \quad (\Delta^2 \rightarrow 0) \]

with

\[ Z(g, 0) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left( -\varphi^2 - g\varphi^4 \right) \, d\varphi. \]

The case of no disorder allows for the use of the weak-coupling perturbation theory, provided the latter is complimented by a resummation procedure [58,59].
However, for finite disorder, the situation is more complicated. The partition function (111), with Hamiltonian (106), reads as

\[ Z(g, \xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left\{ -(1 + \xi)\varphi^2 - g\varphi^4 \right\} \, d\varphi. \]  

(113)

Expanding here the integrand in powers of \( g \) and integrating gives the series

\[ Z(g, \xi) = \sum_{n=0}^{\infty} z_n(\xi)g^n \]  

(114)

with the coefficients

\[ z_n(\xi) = \frac{(-1)^n \Gamma(2n + 1/2)}{\sqrt{\pi} \, \Gamma(n + 1)(1 + \xi)^{2n+1/2}}. \]  

(115)

Substituting series (114) into the free energy (112) shows that the latter diverges for all \( n = 0, 1, 2, \ldots \), in the case of the Gaussian distribution, because of the pole at \( \xi = -1 \) in coefficient (115). In the case of the uniform distribution (110), integral (112) is finite only for weak disorder, for which \( D < 1 \) and \( \Delta_U^2 < 1/3 \). But for any stronger disorder, with \( D \geq 1 \), the free energy is not defined, since Eq. (112) diverges. This shows that perturbation theory with respect to the coupling parameter, generally speaking, fails for the disordered system.

In order to develop a perturbation theory, some special tricks are necessary. For example, the disorder strength and the interaction strength could be treated not as independent quantities, but as being related through the ratio

\[ \lambda \equiv \frac{\xi^2}{2g}, \]  

(116)

which is assumed to be fixed [60]. Then the free energy (112), for the Gaussian distribution (109), takes the form

\[ f(g, \lambda) = -\frac{1}{\sqrt{4\pi g\lambda}} \int_{-\infty}^{\infty} \exp \left( -\frac{\xi^2}{4g\lambda} \right) \ln Z(g, \xi) \, d\xi. \]  

(117)

This expression can be expanded in powers of \( g \), yielding in the \( k \)-order

\[ f_k(g, \lambda) = \sum_{n=1}^{k} a_n(\lambda)g^n. \]  

(118)

The coefficients \( a_n(\lambda) \) are not easy to define, and for \( 0 \leq \lambda \leq 1 \) they were found to be represented as the sum

\[ a_n(\lambda) = b_n(\lambda) + c_n(\lambda), \]  

(119)

whose terms for \( n \gg 1 \) are [60]

\[ b_n(\lambda) = (-1)^{n+1} \frac{4^n(n - 1)!}{\sqrt{2\pi}} (1 - \lambda)^{n-1/2}, \]

\[ c_n(\lambda) = \frac{4^n n! \lambda^n}{\sqrt{\pi} \, n^{3/2}} \exp \left( -\gamma \sqrt{n + \alpha} \right) \cos \left( \mu \sqrt{n + \beta} \right), \]
where 
\[ \gamma = \frac{2.959237}{\sqrt{\lambda}}, \quad \mu = \frac{3.184867}{\sqrt{\lambda}}, \quad \alpha = \frac{\ln 2}{4\lambda}, \quad \beta = \frac{3\pi}{4\lambda}. \]

Although the found coefficients \( a_n = a_n(\lambda) \) are valid only for \( n \gg 1 \), formally using them for all \( n \geq 1 \), and fixing \( \lambda = 0.1 \), gives
\[ a_1 = 0.854049, \quad a_2 = -3.07481, \quad a_3 = 22.1387, \]
\[ a_4 = -239.098, \quad a_5 = 3443, \quad a_6 = -61974.1, \]
and so on. These coefficients grow very fast, making series (118) strongly divergent. The series were shown \([60]\) to be so widely divergent that they could not be Borel summed.

But the divergent series (118) can be summed employing the self-similar approximation theory \([48–50]\). We have accomplished the resummation procedure by using the self-similar factor approximants \([51–54]\). The latter have the form

\[ f_k^*(g, \lambda) = a_1(\lambda)g \prod_{i=1}^{N_k}(1 + A_i g)^{n_i}, \quad (120) \]

where
\[ N_k = \begin{cases} k/2, & k = 2, 4, \ldots \\ (k + 1)/2, & k = 3, 5, \ldots \end{cases} \]

The coefficients \( A_i \) and \( n_i \) are defined by re-expanding form (120) and equating it to series (118), which yields
\[ \sum_{i=1}^{N_k} n_i A^n_i = B_n \quad (n = 1, 2, \ldots, k), \]
\[ B_n = \frac{(-1)^{n-1}}{(n-1)!} \lim_{g \to 0} \frac{d^n}{dg^n} \ln \left[ f_k(g, \lambda) \right], \]
and \( A_1 = 1 \) for the odd \( N_k = (k + 1)/2 \). For the fixed \( \lambda = 0.1 \) and different \( g \), we calculated the factor approximants (120) up to the order \( k = 10 \). For \( g = 0.1 \), we get \( f_{10}^* = 0.0662 \), with the relative error 22% as compared to the exact value 0.0542. For \( g = 1 \), we have \( f_{10}^* = 0.229 \), with an error 19%, as compared to the exact value 0.251. And for \( g = 10 \), we find \( f_{10}^* = 0.712 \), deviating from the exact value 0.642 by 11%. These approximants should be considered as quite good, if one remembers that expressions (119) have been defined only for \( n \gg 1 \), but used for all \( n = 0, 1, 2, \ldots, 10 \). Therefore the resulting errors rather characterize the inaccuracy of these coefficients (119), but not merely the accuracy of the factor approximants (120).

The main message of this Section is that, under disorder, the perturbation theory with respect to atomic interactions can become inapplicable. For the Gaussian distribution of the disorder potential, this perturbation theory fails for any strength of disorder, and for the uniform disorder distribution, the perturbation theory becomes invalid at sufficiently strong disorder. To realize the weak-coupling perturbation theory, it is necessary to invoke some tricks, like fixing a relation between the disorder strength and the interaction strength. However, resorting to such tricks looks too artificial, since in reality the disorder and interaction strength are independent characteristics.
Creation of Nonequilibrium Granular Condensate

In the previous Sections, we have considered equilibrium systems with Bose-Einstein condensate, which, by applying spatially random fields, could be transformed into a Bose-glass type phase with a granular condensate. In the present Section, we show that a Bose system, subject to the action of an external time-alternating potential enjoys the properties to some extent analogous to those of a system in a spatially random potential. In particular, in such a nonequilibrium system a nonequilibrium granular condensate can be created.

Let us consider a Hamiltonian

\[ H(t) = \hat{H} + \hat{V}(t) , \tag{121} \]

in which the first term

\[ \hat{H} = \int \psi^\dagger(\mathbf{r}) \left[ -\frac{\nabla^2}{2m} + U(\mathbf{r}) \right] \psi(\mathbf{r}) \, d\mathbf{r} + \frac{\Phi_0}{2} \int \psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r})\psi(\mathbf{r})\psi(\mathbf{r}) \, d\mathbf{r} \tag{122} \]

describes Bose atoms in a trapping potential \( U(\mathbf{r}) \), and the second term

\[ \hat{V}(t) = \int \psi^\dagger(\mathbf{r})V(\mathbf{r}, t)\psi(\mathbf{r}) \, d\mathbf{r} \tag{123} \]

corresponds to an external alternating potential. Again, for brevity we do not write explicitly the time dependence of the field operators \( \psi(\mathbf{r}) = \psi(\mathbf{r}, t) \).

The external potential \( V(\mathbf{r}, t) \) is assumed to vary in time so that its characteristic variation time \( t_{\text{var}} \) is much larger than the local-equilibrium time \( t_{\text{loc}} \), but much shorter than the observation time,

\[ t_{\text{loc}} \ll t_{\text{var}} \ll t_{\text{obs}} . \tag{124} \]

As an estimate of the local-equilibrium time, we can take the values of the parameters typical of the dilute trapped gases \( ^{87}\text{Rb} \) and \( ^{23}\text{Na} \), say, \( m \sim 10^{-22} \text{g}, \rho \sim 10^{15} \text{cm}^{-3}, \) and \( a_s \sim 10^{-7} \text{cm} \). The local-equilibrium time is defined \([3]\) as

\[ t_{\text{loc}} = \frac{m}{\rho a_s} . \tag{125} \]

With the above parameters, this gives \( t_{\text{loc}} \sim 10^{-3} \text{s} \).

Under condition (124), it is possible to define the local-equilibrium free energy

\[ F(t) = -T(t) \ln \text{Tr} e^{-\beta(t)H(t)} , \tag{126} \]

in which \( T(t) \equiv 1/\beta(t) \) is the local-equilibrium temperature. Since the observation time is much longer than the local-equilibrium time (125), it is appropriate to introduce the average free energy

\[ F = \frac{1}{t_{\text{var}}} \int_{0}^{t_{\text{var}}} F(t) \, dt , \tag{127} \]

averaging the local-equilibrium free energy (126) over the characteristic variation time of the alternating potential \( V(\mathbf{r}, t) \).

Suppose the alternating potential has the form

\[ V(\mathbf{r}, t) = V(\mathbf{r})f(t) , \tag{128} \]
in which the temporal function $f(t)$ varies in the interval

\[ f_{\text{min}} \leq f(t) \leq f_{\text{max}} \, . \] (129)

Let us define the function $t = t(\xi)$ by the equation

\[ f(t(\xi)) = \xi \, . \] (130)

By introducing the effective Hamiltonian

\[ H_{\text{eff}}(\xi) \equiv H(t(\xi)) \, , \] (131)

we get

\[ H_{\text{eff}}(\xi) = \hat{H} + \int \psi(\mathbf{r})\xi V(\mathbf{r})\psi(\mathbf{r}) \, d\mathbf{r} \, . \] (132)

Also, we define the effective temperature

\[ T_{\text{eff}}(\xi) \equiv T(t(\xi)) \equiv \frac{1}{\beta_{\text{eff}}(\xi)} \, . \] (133)

The quantity

\[ p(\xi) \equiv \frac{1}{t_{\text{var}}} \left| \frac{dt(\xi)}{d\xi} \right| \] (134)

plays the role of the distribution of the variable $\xi$. With these notations, the free energy (127) transforms into

\[ F = - \int_{f_{\text{min}}}^{f_{\text{max}}} T_{\text{eff}}(\xi) \ln \text{Tr} \exp \{-\beta_{\text{eff}}(\xi)H_{\text{eff}}(\xi)\} \, p(\xi) \, d\xi \, . \] (135)

Let $\xi^*$ be a value in the interval $f_{\text{min}} \leq \xi^* \leq f_{\text{max}}$, and let us define

\[ T^* \equiv T_{\text{eff}}(\xi^*) \equiv \frac{1}{\beta^*} \, . \] (136)

Then, by the theorem of mean, the free energy (135) can be approximately represented as

\[ F \simeq -T^* \int_{f_{\text{min}}}^{f_{\text{max}}} \ln \text{Tr} \exp \{-\beta^*H_{\text{eff}}(\xi)\} \, p(\xi) \, d\xi \, . \] (137)

This expression looks analogously to the free energy of a system in an external random potential $\xi V(\mathbf{r})$ that enters Eq. (132).

To exemplify explicitly the form of distribution (134), let us take the time-dependent factor of the alternating potential (128) as

\[ f(t) = \cos(\omega t) \, . \] (138)

The characteristic variation time here is, clearly, the period $t_{\text{var}} = 2\pi/\omega$. The variation range of function (138), as defined in Eq. (129), is given by $f_{\text{min}} = -1$ and $f_{\text{max}} = 1$. The function $t(\xi)$ is found from Eq. (130), yielding

\[ t(\xi) = \frac{1}{\omega} \arccos \xi \, . \] (139)
From here, for distribution (134), we get

\[ p(\xi) = \frac{1}{\pi \sqrt{1 - \xi^2}} \quad (-1 \leq \xi \leq 1). \]  

(140)

The variable \( \xi \), with distribution (140), is zero-centered, since

\[ \int_{-1}^{1} \xi p(\xi) \, d\xi = 0, \]

and its dispersion is

\[ \Delta^2(\xi) = \int_{-1}^{1} \xi^2 p(\xi) \, d\xi = \frac{1}{2}. \]

In this way, when considering the time-averaged behavior of a Bose system, subject to the action of an external alternating potential, we come to the picture that is similar to the description of a Bose system in an external spatially random potential. The consequences, therefore, should also be similar. The overall behavior of trapped atoms, in the presence of an alternating potential \( V(r, t) \), should be as follows.

Let \( V_0 \) be the strength of the alternating potential \( V(r, t) \sim V_0 \). The latter plays the role of \( V_R \) in the localization length (84). If the whole trap is perturbed by the alternating potential, then the characteristic trap length \( l_0 = 1/\sqrt{m\omega_0} \), where \( \omega_0 \) is the trap frequency, plays the role of \( l_R \) in Eq. (84). Hence, for the localization length (84), we have

\[ l_{\text{loc}} \sim \left( \frac{\omega_0}{V_0} \right)^2 l_0. \]

When the amplitude of the alternating potential is small, such that \( V_0 \ll \omega_0 \), then there is an extended condensate filling the trap. If the alternating potential oscillates with a frequency far detuned from any transition frequency for trapped atoms, then the extended condensate is just a perturbed ground-state condensate. But if the frequency of the oscillating potential is in resonance with the transition frequency between the ground-state condensate and an excited topological mode, then the extended condensate is formed by the fluctuating ground-state condensate mode and an excited coherent mode [61–64].

Increasing the amplitude \( V_0 \) of the alternating potential perturbs the Bose-condensed system stronger. When \( V_0 \) reaches the trap frequency \( \omega_0 \), the localization length becomes comparable with the effective trap length \( l_0 \). At this moment, the granular condensate starts being formed. Since the applied external field is time dependent, the created granular condensate is nonequilibrium. This means that the disconnected regions with the condensate, with time, change their shapes, disappear and appear again, so that on average there always exist several such regions of the granular condensate.

The nonequilibrium granular condensate can exist for the amplitude of the alternating potential in the range

\[ \omega_0 \leq V_0 \leq \omega_0 \sqrt{\frac{l_0}{a}}. \]

If the amplitude is so large that \( V_0 \sim \omega_0 \sqrt{\frac{l_0}{a}} \), then \( l_{\text{loc}} \sim a \), and the granular condensate is completely destroyed. Then the whole system is a strongly nonequilibrium normal matter, containing no condensate, but being rather in a normal turbulent state.
The phase portrait of the nonequilibrium system, at zero temperature, qualitatively corresponds to that of the equilibrium random system, as is shown in Fig. 1, with the following analogies between the systems:

- equilibrium extended condensate ↔ nonequilibrium extended condensate,
- equilibrium granular condensate ↔ nonequilibrium granular condensate,
- equilibrium normal glass ↔ turbulent normal fluid.

The principal difference between the equilibrium and nonequilibrium granular condensates is as follows. The equilibrium granular condensate represents a system with randomly distributed in space regions of the condensate, surrounded by the normal phase with no condensate, these regions being stationary in time, their location in space being fixed. The nonequilibrium granular condensate, at each instant of time, is analogous to its equilibrium counterpart. However, the regions with the condensate are not fixed in time. But, as time varies, the condensate regions change their shapes and locations. They can appear in new spatial locations, but disappear in others. What is fixed for a nonequilibrium granular condensate is the average (over space and time) concentration of condensed atoms, so that the average condensate fraction \( n_0 \) remains a well defined order parameter. At each instant of time, the system with the nonequilibrium granular condensate is a kind of a snapshot, which is analogous to the equilibrium granular condensate. Therefore, if trapped atoms are released from the trap, the time-of-flight observations for a nonequilibrium system will be similar to those for an equilibrium system.

We may note that the nature of bosons, considered above, can be any. These could be usual bosonic atoms [2–5]. Or these could be composite bosonic molecules formed of fermions [3,6,5].

The discussed analogies between the random Bose-condensed systems and the nonequilibrium Bose systems can be tested experimentally. Such experiments are now in progress in the Institute of Physics of São Carlos, University of São Paulo, Brazil. Preliminary results confirm these analogies. But a detailed exposition of experiments will be done in separate publications.

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Figure 1: Qualitative phase portrait of a Bose system in a spatially random potential at zero temperature on the plane of the gas parameter $\gamma$ and disorder parameter $\zeta$. 