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MAPPING THEOREMS

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Riemannian surfaces are conformally equivalent to the sphere, plane or disk, in the case where the Beltrami differential is zero by the Uniformization Theorem of Poincaré-Klein-Koebe, i.e. Hilbert’s XXII problem (which actually asks for higher dimensions too), and more generally when the Beltrami is bounded, by Ahlfors-Bers [1] who use $K$—quasiconformal (QC) mappings, i.e. homeomorphisms mapping small balls to ellipsoids of bounded eccentricity $K$. In higher dimensions Liouville shows that the conformal mappings are essentially trivial [23] and so one seeks quasi-orthogonal co-ordinates instead. Lavrentieff (1938) introduced quasiconformal mappings of space for PDE. A geometric-analytic theory, initiated by Löwner (1959), was sustained by the school of Gehring and Väisälä as well as Soviet mathematicians such as Reshetnyak and Zoric, with notable contributions by Tukia and Rickman (not to mention Ahlfors, Carleson, Donaldson and Sullivan). In 1965 Gehring and Väisälä [9] formulated what Ahlfors in his review called the main problem: find a characterisation of the $K$—quasiconformal images of the unit ball, i.e. QC-balls. For two dimensions this is the Riemann Mapping Theorem, proved by Koebe in 1907 (since by Ahlfors-Bers [1] the QC version is equivalent). In their plenary addresses to the International Congress both Ahlfors(1978) and Gehring(1986) gave this as the main open problem of the theory.

Our insight comes from reflections\footnote{sense reversing idempotent homeomorphisms of $\mathbb{R}^3$ onto itself}. Now according to Smith [28] any reflection $F$ of the sphere $\hat{\mathbb{R}}^3$ has a set $T$ of fixed points forming a topological sphere with complement being two disjoint domains $D, D'$ (called the “complementary domains”). Smith’s conjecture that $F$ is topologically conjugate to a Euclidean reflection was disproved by Bing [4] by constructing a “wild reflection”, i.e. the complementary domains are not simply connected\footnote{“The Smith Conjecture” program gave ”yes” for diffeomorphisms [26]}. As Bing’s construction of “wild reflections” was rather indirect he asked [5] for an explicit example. We constructed an explicit example [14] which although biholder is not quasiconformal. It was expected (see Heinonen and Semmes [22], and communications from Sullivan) that there exist wild $K$—quasiconformal reflections, however in [16] we prove:

**THEOREM 1** (QC Smith conjecture) QC reflections are tame.
Remarks: Although $F$ is topologically conjugate to a Euclidean reflection, the conjugation need not be QC (unlike two dimensions).

A fundamental concept in our theory is renormalization. For example consider an injection $F: \hat{\mathbb{R}}^m \to \hat{\mathbb{R}}^n$, for $n \geq m$. Its family of renormalisations would be $\hat{F} = N \circ F \circ L$ where $L$ is any conformal automorphism of $\hat{\mathbb{R}}^m$ and $N$ chosen so that say $(0,0,0)$ and $(0,0,1)$ are fixed, i.e. $\hat{F}$ is a normalised rescaling. For $n = m$ this gives a well known characterization of QC mappings $F$: the family of renormalizations of a QC map is a precompact family in the space of homeomorphisms, i.e. normalized rescalings of quasiconformal mappings have subsequences converging uniformly to another QC map. For $n > m$ we get the so called quasisymmetric (QS) mappings which like QC maps have the property that triangles are roughly preserved. We shall also renormalize sets $S \subset \hat{\mathbb{R}}^3$, in this case the family is $\tilde{S} = L(S)$ where the conformal mapping fixes a point of $S$. For example the famous quasicircles of Ahlfors are Jordan curves $\Gamma \subset \hat{\mathbb{R}}^2$ whose family of renormalizations is precompact in the space of Jordan curves (where the Hausdorff metric between sets is used). By Ahlfors these are parameterized by QS maps of $\hat{\mathbb{R}}$, i.e. “quasisymmetric 1-spheres”. In general a quasisymmetric m-sphere is a QS embedding of $\hat{\mathbb{R}}^m$ in $\hat{\mathbb{R}}^n$. Concentrating on $\hat{\mathbb{R}}^3$ we are concerned with $m = 2$ the so called QS spheres. In general any object with renormalizations being precompact in some metric will be called uniform.

The fixed set of a $K$-QC reflection is called a quasireflector. By Ahlfors [2] the quasireflectors of $\hat{\mathbb{R}}^2$ are the QS circles. We find that in $\mathbb{R}^3$ however there is a difference between topological spheres which are uniform as sets and those which have uniform parameterizations.

**DEFINITION 1** We say that a flat sphere $T$ is uniform if its family of “renormalizations” $\tilde{T}$ is precompact in the space of flat spheres with respect to the Hausdorff metric between compact sets.$^3$

As the renormalizations of a QC reflections is a precompact family from Theorem 1 we deduce that the fixed set is uniform, the converse is

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$^3$Equivalently, by Bing’s criterion for flat spheres, at all scales $T$ can be squeezed between “uniform” polyhedra from the complementary domains
**THEOREM 2** (see [17]) \( T \) is the fixed set of a QC reflection iff it is a uniform sphere.

Remarks: By work of Tukia and Väisälä this also solves the problem of characterising the fixed set of bilipschitz reflections (a question first considered by Poincaré (1898), see Jones [24]).

Another fundamental question was to characterize “quasispheres”: the image of the unit sphere under a QC mapping \( F : \mathbb{R}^3 \to \mathbb{R}^3 \). The two dimensional analog again is due to Ahlfors who showed that “quasicircle” was necessary and sufficient. Of course in \( \mathbb{R}^3 \) any quasisphere is a QS sphere, but the converse is false as QS spheres can be wild. Nor is every uniform sphere also QS. In fact the uniform sphere \( \gamma \times \mathbb{R}^1 \) for fractal quasicircle \( \gamma \) is not quasisymmetric. In other words a topological sphere can be uniform as a set but without any uniform parametrization.

**DEFINITION 2**: A topological sphere \( T \subset \mathbb{R}^3 \) is regular if it a uniform sphere with quasisymmetric parametrization.

This provides the characterization required:

**THEOREM 3** (see [18]) \( T \subset \mathbb{R}^3 \) is the image of \( S^2 \) under a QC mapping of \( \mathbb{R}^3 \) iff \( T \) is a regular sphere.

Remarks: As a special case we have the Ahlfors problem of extending a quasisymmetric mapping \( F : S^2 \to S^2 \) to a quasiconformal mapping of \( \mathbb{R}^3 \), see L. Ahlfors [3], L. Carleson [8] and (for higher dimensions) Tukia [30]. Indeed we find various equivalent conditions, e.g.

**COROLLARY 1** Quasisphere \( \Leftrightarrow \) uniform and “Löwner”.

Remarks: “Löwner” (in the terminology of [6]) means that uniform annuli of \( T \) have uniformly bounded 2-capacity. In fact being a QS sphere is quite delicate. In [6] it is proved that \( T \) is QS sphere iff it is “locally linearly connected”, “doubling” and “Löwner”. As our regular spheres already satisfy the first two conditions the corollary is an immediate consequence of Theorem 2. Essentially all of this means that \( T \) must be “geometrically uniform”.
The QC Riemann Mapping Theorem is a one sided version of this. Now in two dimensions the characterization is that \( D \) is simply connected (with non empty boundary). One difference between two and three dimensions is that any topological 2-ball in \( \mathbb{R}^2 \) is “uniformly simply connected” (USC), i.e. its renormalizations form a precompact family in the space of topological disks (wrt weak convergence). However in \( \mathbb{R}^3 \) a sequence of renormalizations of a topological ball could converge to a torus. Now a QC-ball is USC. However by the previous discussion we expect extra boundary conditions. Gehring \cite{12} showed the complement \( \mathbb{R}^3 - D \) is “linearly locally connected” (LLC).

We must now refer to the prime-ends introduced by Caratheodory and generalized higher dimensions by Zoric, see \cite{...}. These are the “ends” cut off by compacta with diameters converging to zero. Now any simply connected proper subdomain of \( \mathbb{R}^2 \) has boundary which is a topological circle (in the prime-end metric). This is not true in \( \mathbb{R}^2 \) even for topological balls. However Zoric showed that any QC mapping of the unit ball extends to a homeomorphism of the prime-end boundary \( \partial D \) which is thus a topological sphere. More generally we find that a USC-LLC domain has prime-end boundary \( \partial \hat{D} \) parametrised by a homeomorphism of \( S^2 \). Let \( B \) be the space of prime-end boundary maps of USC-LLC domains using the natural prime-end metric.

**DEFINITION 3** We say that a USC domain \( D \) is a “regular ball” if it has parametization \( H : S^2 \to \partial \hat{D} \) whose renormalizations are precompact in \( B \).

Now any QC-ball is a regular ball. The converse is the first of our three versions of the QC mapping theorem:

**THEOREM 4** Any regular ball is a QC-ball.

Previously Väisälä \cite{32} had the best criterion. He gave necessary and sufficient conditions for cylindrical domains \( D = A \times \mathbb{R}^1 \subset \mathbb{R}^3 \) to be the QC image of the ball. Väisälä gave several equivalent conditions but the one we quote is that the (prime-end) boundary \( \partial \hat{D} \) must be the image of \( S^1 \times \mathbb{R}^1 \) under parametrization quasisymmetric with respect to the the inner-length:

\[
\nu(X, Y) = \inf \{ \text{dia}(\alpha) : X, Y \in \text{connected } \alpha \subset D \},
\]

i.e \( \partial \hat{D} \) is \( \nu \)-quasisymmetric.
Indeed a general concept of quasisymmetry is the actual way we proceed to Theorem 4. We find that USC domains with LLC boundary is a Gromov-Uniform-Tree (GUT), see [19]. This is analogous to Gromov’s theory of Hyperbolic Trees which arose in the study of discrete groups, for an exposition see [13]. Without giving a formal definition, a GUT is uniformly approximated from within by uniform trees of uniform polyhedra. Then there is a metric $\gamma$ on $D$ which extends to another ideal boundary, the so-called Gromov boundary which turns out to be equivalent to the prime-end boundary $\partial \hat{D}$. The metric is defined by the weights $c^{-n}$ where $n$ is the number of steps on disjoint uniform polyhedra from a fixed special point. In fact (analogous to the case of Gromov Hyperbolic Spaces) QC mappings extend to mappings of the Gromov Boundary quasisymmetric in the Gromov metric. Therefore Theorem 4 is actually proved by [19]:

**THEOREM 5** QC-ball $\Leftrightarrow$ USC domain bounded by a $\gamma$-QS sphere.

Remarks: One way is relatively easy. The converse problem is to extend the QS mapping to a QC mapping of the interiors.

The $\nu$–metric is more explicit than the gromov metric, so we generalise the Väisälä result to the following sufficient condition:

**COROLLARY 2** Any USC domain bounded by a $\nu$-QS sphere is a QC-ball.

Remarks: One could also use the the ordinary (euclidean) QS maps to get a weaker sufficient condition.

Bonk and Kleiner [7] gave three conditions for a metric space to be QS equivalent to $S^2$: “doubling”, LLC and “Lowner”. The first two already hold for USC-LLC domains. The “Löwner” criteria is: uniform annuli of $\partial \hat{D}$ have bounded 2-capacity $\text{cap}$, i.e. for any annulus

$$A = \{X : r < \gamma(X, Y) < 2r\}$$

we have $a < \text{cap}(A) < b$, for absolute constants $a, b$. This is a thickness condition. Note that 2-capacity would then be measured in the fairly implicit gromov metric. Actually Bonk and Kliener define 2-capacity in general metric spaces via approximating circle packings. Likewise we approximate
the boundary by PL surfaces and obtain 2-capacity on the boundary. However we show that one may use the usual capacities (and call the boundary “Löwner”)

Our “three point condition” is:

**THEOREM 6** A domain $D$ is a QC-ball iff

1. $D$ is USC
2. $\mathbb{R}^3 - D$ is LLC
3. $\partial D$ is Löwner

Remarks: All three conditions are necessary. For example a quasireflector satisfies the first two but is not in general QS equivalent to $\mathbb{S}^2$.

Sullivan and Thurston proved $^4$ that a domain inscribed by round balls is a QC-ball. For example the “soap bubble” domain, obtained by first adding to some initial ball $B_0$ some packing of its boundary by disjoint balls $B_n$, then doing the same to the $B_n - B_0$ and continuing outward so that all new balls are disjoint. (An infinite cylinder is another example). However there are QC-balls which cannot be inscribed by round balls. Using quasiballs instead we have the concept of quasi-inscribing. Suppose the boundaries of disjoint K-quasiballs $\Omega_j$ meet on common faces bounded by curves $\gamma(i,j)$ where the K-QS maps $H_j : \partial \Omega_j \rightarrow \mathbb{S}^2$ match up, i.e. $H_j|_{\gamma(i,j)} = H_i|_{\gamma(i,j)}$.

**THEOREM 7** Suppose the union $D$ of the $\Omega_j$ and the common faces bounded by the $\gamma(i,j)$ gives a USC domain then $D$ is a $K'$-QC ball.

Example: The so called “Manhattan” domains $D$ are formed by adjoining to the half-space $x_3 < 0$ the (maybe infinite) cylinders of the form:

$$A_j \times \{0 \leq x_3 < h_j \leq \infty\}$$

whose bases $A_j$ are disjoint squares$^5$. Then any Manhattan domain is a $K$-QC ball, even the one built over a square grid.

$^4$we thank Sullivan for telling us about this
$^5$actually any Väisälä type cylinders with bounded QC constants will do
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