Conservation laws for strings in the Abelian Sandpile Model

S. Caracciolo\(^{1(a)}\), G. Paoletti\(^{2(b)}\) and A. Sportiello\(^{1(c)}\)

\(^{1}\)Università degli Studi di Milano, Dipartimento di Fisica and INFN - Via G. Celoria 16, 20133 Milano, Italy, EU
\(^{2}\)Università di Pisa, Dipartimento di Fisica and INFN - Largo B. Pontecorvo 3, 56127 Pisa, Italy, EU

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Abstract – The Abelian Sandpile generates complex and beautiful patterns and seems to display allometry. On the plane, beyond patches, patterns periodic in both dimensions, we remark the presence of structures periodic in one dimension, that we call strings. We classify completely their constituents in terms of their principal periodic vector \(k\), that we call momentum. We derive a simple relation between the momentum of a string and its density of particles, \(E\), which is reminiscent of a dispersion relation, \(E = |k|^2\). Strings interact: they can merge and split and within these processes momentum is conserved, \(\sum a_k = 0\). We reveal the role of the modular group \(\text{SL}(2,\mathbb{Z})\) behind these laws.

Introduction. – Since the appearance of the masterpiece by D’Arcy Thompson [1], there have been many attempts to understand the complexity and variety of shapes appearing in Nature at macroscopic scales, in terms of the fundamental laws which govern the dynamics at microscopic level. Because of the second law of thermodynamics, the necessary Self-Organization can be studied only in non-equilibrium statistical mechanics.

In the context of a continuous evolution in a differential manifold, the definition of a shape implies a boundary and thus a discontinuity. This explains why catastrophe theory, the mathematical treatment of continuous action producing a discontinuous result, has been developed in strict connection to the problem of Morphogenesis [2]. More quantitative results have been obtained by the introduction of stochasticity, as for example in the diffusion-limited aggregation [3–5], where self-similar patterns with fractal scaling dimension [6] emerge which suggest a relation with scaling studies in non-equilibrium.

Cellular automata, that is, dynamical systems with discretized time, space and internal states, were originally introduced by Ulam and von Neumann in the 1940s, and then commonly used as a simplified description of phenomena like crystal growth, Navier-Stokes equations and transport processes [7]. They often exhibit intriguing patterns [8], and, in this regular discrete setting, shapes refer to sharply bounded regions in which periodic patterns appear. Despite very simple local evolution rules, very complex structures can be generated. The well-known Conway’s Game of Life can perform computations and can even emulate an universal Turing machine (see [8] also for a historical introduction on cellular automata).

In this letter, we shall concentrate on a particularly simple cellular automaton, the Abelian Sandpile Model (ASM). Originally introduced as a statistical model of Self-Organized Criticality [9], because it shows scaling laws without any fine-tuning of an external control parameter, it has been shown afterwards to possess remarkable underlying algebraic properties [10–12], and has been studied also in some deterministic approaches, exactly in connection with pattern formation [13–18]. The ASM has been shown to be able to produce allometry, that is a growth uniform and constant in all the parts of a pattern as to keep the whole shape substantially unchanged, and thus requires some coordination and communication between different parts [19]. This is at variance with diffusion-limited aggregation and other models of growing objects studied in physics literature so far, e.g. the Eden model, KPZ deposition and invasion-percolation [20–22], which are mainly models of aggregation, where growth occurs by accretion on the surface of the object, and inner parts do not evolve significantly.

In the sandpile, the regions of a configuration periodic in space, called patches, are the ingredients of pattern
Fig. 1: A string with momentum \((6, 1)\), in a background pattern with periodicities \(V = ((2, 1), (0, 2))\). String and background unit cells are shown in gray. The density in the string tile is
\[
\rho = \frac{18 \cdot 3 + 8 \cdot 2 + 4 \cdot 1 + 7 \cdot 0}{6^2 + 1^2} = 2.
\]

Fig. 2: On the left, the configuration obtained after relaxation from \(z_{\text{max}}\) plus an extra grain of sand exactly at the vertex where a defect appears. On the right, the result after removing the defect and the addition of one more grain.

Fig. 3: A scattering involving pseudo-propagators with momenta \((4, 0)\), \((2, 1)\) and \((6, 1)\), on the background pattern of fig. 1 (also the symbol code is as in fig. 1).

Fig. 4: A string with momentum \((17, 4)\), in the maximally filled background. The decomposition of the unit tile into squares with diagonals, as described in eq. (2), is depicted (in orange and red, respectively for the first and second nesting level). The symbol code is as in fig. 1.

Fig. 5: The supercritical, critical and subcritical patches constructed from the propagator \(k = (3, 2) \leftarrow ((2, 1), (1, 1))\) in the \(z_{\text{max}}\) background, having densities \(\rho = 2\) and \(2 \pm 1/12\) (symbol code is as in fig. 1).

Fig. 6: The top-left \(70 \times 120\) corner in a rectangular \(175 \times 325\) domain, result of the protocol in which \(z_{\text{max}}\) is augmented with 3600 sand-grains in the center of the lattice, and then relaxed. Orange circles are points in which the condition (1) on patch incidence is satisfied. Red, pink and green circles correspond to reflection, refraction and scattering of strings. The unit tile of a \((87, 43)\) string (thus with \(|k| \sim L\)) appears.

formation. In [19], a condition on the shape of patch interfaces has been established, and proven to hold at a coarse-grained level in configurations obtained from isotropic deterministic protocols, i.e., the result of relaxation of a given isotropic strongly unstable initial state. Examples in this class are the projection of an isotropic transient state
to its recurrent representative, by adding frame identities (if the transient state is the empty state, this corresponds to determine the recurrent identity [12–16]), or adding, to an uniform configuration, a large amount of sand in a single site [16–18].

As we shall discuss, the result in [19] holds in a stronger form, which avoids the coarsening and permits to describe the emerging fine-level structures: linear interfaces and rigid domain walls with a residual one-dimensional translational invariance. These structures, that we shall call strings, are macroscopically extended in their periodic direction, while showing thickness in a full range of scales between the microscopic lattice spacing and the macroscopic volume size. We will also show how the classification of strings and patches goes up in parallel, and how the strings are the structures that mediate the allometric growth of patch structures.

The model. – While the main structural properties of the ASM can be discussed on arbitrary graphs [10], for the subject at hand here we shall need some extra ingredients (among which a local notion of translation invariance), that, for the sake of simplicity, suggest us to concentrate on the original realization on the square lattice [9], within a rectangular region $\Lambda \in \mathbb{Z}^2$.

We write $i \sim j$ if $i$ and $j$ are first neighbours. The configurations are vectors $z = \{z_i\}_{i \in \Lambda} \in \mathbb{N}^\Lambda$ ($z_i$ is the number of sand-grains at vertex $i$). Let $\bar{z} = 4$, the degree of vertices in the bulk, and say that a configuration $z$ is stable if $z_i < \bar{z}$ for all $i \in \Lambda$. Otherwise, it is unstable on a non-empty set of sites, and undergoes a relaxation process whose elementary steps are called topplings: if $i$ is unstable, we can decrease $z_i$ by $\bar{z}$, and increase $z_j$ by one, for all $j \sim i$. The sequence of topplings needed to produce a stable configuration is called an avalanche.

Avalanches always stop after a finite number of steps, which is to say that the diffusion is strictly dissipative. Indeed, the total amount of sand is preserved by topplings at sites far from the boundary of $\Lambda$, and strictly decreased by topplings at boundary sites. The stable configuration $\mathcal{R}(z)$ obtained from the relaxation of $z$, is univocally defined, as all valid stabilizing sequences of topplings only differ by permutations.

We call a stable configuration recurrent if it can be obtained from an unstable one through an avalanche involving all sites in $\Lambda$, and transient otherwise [10]. Recurrent configurations have a structure of Abelian group [12] under the operation $z \oplus w =: \mathcal{R}(z + w)$. A configuration $z$ is called an identity if, for every $w$ recurrent, $z \oplus w = w$. The frame identity [12] is an example: it is the configuration $z$ in which $z_i$ equals the number of sides neighbouring the boundary. This is in general different from the group identity, i.e. the unique identity which is recurrent. We have only a partial knowledge of the group identity for each $\Lambda$ (see, e.g., [12,13]; recently a complete characterization has been achieved for a simplified directed lattice, the pseudo-Manhattan, or $F$-lattice [23]) while it provides an example of deterministic protocol, showing the intriguing complex patches in which we are interested.

A patch is a region filled with a pattern periodic in two directions [18]. The density $\rho$ of a patch is the average of $z_i$ within a unit tile. Structures with density $\rho > 2$, $\rho = 2$ and $\rho < 2$ are said respectively supercritical, critical and subcritical. The threshold at $\rho = 2$ relates to the fact that, for large $\Lambda$, recurrent configurations have average density $\rho(z) = \frac{1}{|\Lambda|} \sum_i z_i \geq 2 + o(1)$, and this bound is tight.

Neighbouring patches may have an interface, periodic in one dimension, along a vector which is principal for both patches. Let us suppose that in a deterministic protocol [19] we generate a region filled with polygonal patches, glued together with such a kind of interfaces. At a vertex where $\ell \geq 3$ patches meet, label cyclically with $\alpha = 1, \ldots, \ell$ these patches, call $\rho_\alpha$ the corresponding densities, and $\theta_\alpha$ the angles of the interfaces between the patch $\alpha$ and $\alpha + 1$ (subscripts $\alpha = \ell + 1 \equiv 1$). These quantities are proven to satisfy the relation

$$\sum_{\alpha=1}^\ell (\rho_{\alpha+1} - \rho_\alpha) \exp(2i\theta_\alpha) = 0, \quad (1)$$

which has non-trivial solutions only for $\ell \geq 4$ [19].

Equation (1) was initially derived as a relation amongst the coarse-grained densities of patches meeting at a point. As a result of our hierarchical classification of patches and strings, it is seen to hold exactly, in $\mathbb{Q} + i\mathbb{Q}$, in infinitely many cases, and deterministic protocols exist, realizing simultaneously an arbitrary number of these solutions.

Strings. – Call string a one-dimensional periodic defect line, with periodicity vector $k = (k_x, k_y) \in \mathbb{Z}^2$ (that we call momentum), in a background patch, periodic in both directions, and has $k$ as a periodicity vector. The background on the two sides may possibly have a periodicity offset. See fig. 1. A naïve interpretation of the outcome of deterministic protocols suggests that these strings are the fine-level defects that disturb the pattern formation, and make self-similarity hold only at a coarse-grained level. We will instead show that these structures follow strong rules, whose implications include a classification of the patches, and we will describe the design of new deterministic protocols, with pattern formation, in which defect lines are geometrically predictable, or even producing no defect lines at all.

Strings in the maximally filled background emerge also in a simple “artificial” protocol. Consider a rectangular region $\Lambda$, and the configuration $z_{\text{max}}$. Add one grain of sand at some vertex $j$. The configuration after relaxation contains an inner rectangle, of strings $(1,0)$ and $(0,1)$, equidistant from the border of $\Lambda$ and having $j$ on its perimeter, and its corners are connected to the corners of $\Lambda$ with strings $(1,1)$ and $(-1,1)$. There is one defect exactly at $j$, manifested as a single extra grain, w.r.t. the underlying periodic structure. See fig. 2. This configuration is recurrent. Now remove this extra grain (the configuration

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is now transient), and repeat the game at some new vertex $j'$, say in the region below the inner rectangle. In the configuration after relaxation appear also strings $(2,1)$ and $(-2,1)$, and once again we have a defect at $j'$. This procedure can be iterated, and, if $\Lambda$ is large enough, strings with higher and higher momenta generated. Furthermore, given that the unit tiles of momenta are classified (see the following), this protocol is completely predictable, for arbitrary $\Lambda$, through a purely geometric construction. Starting with a domain filled with any other supercritical background patch $\mathcal{B}$, the strings pertinent to $\mathcal{B}$ are obtained.

In a given supercritical background $\mathcal{B}$, with translation vectors $V = (v_1, v_2)$, one and only one type of string of momentum $k = m_1 v_1 + m_2 v_2 = mV$ can be produced, if $\gcd(m_1, m_2) = 1$, and no strings of momentum $k$ exists for $k$ not of the form above. $V$ and $V'$ are equivalent descriptions of the background periodicity iff $V' = MV$, with $M \in SL(2, \mathbb{Z})$. Accordingly, $m' = mM^{-1}$. Indeed, sets of $m \in \mathbb{Z}^2$ with given $\gcd$’s are the only proper subsets invariant under the action of $SL(2, \mathbb{Z})$. The $gcd$ constraint arises in the classification of the elementary strings, because, when $d = \gcd(k_x, k_y) > 1$, the corresponding periodic ribbon is just constituted of $d$ parallel strings with momentum $k/d$.

The unit tile of each string, as well as of each patch, is symmetric under 180-degree rotations. In particular, momenta $k$ and $-k$ describe the same string. The tile of a string of momentum $k$ fits within a square having $k$ as one of the sides, so that each string is a row of identically filled squares. This is a non-empty statement: the tile could have required rectangular boxes of larger aspect ratio, and even an aspect ratio depending on momentum and background.

A string of momentum $k$ has an energy $E$, defined as the difference of sand-grains, in the framing unit box of side $k$, w.r.t. $\tau_{\text{max}}$. Heuristically for all the strings we have observed, and as proven for the infinite family of which we give an explicit construction later on, we have the relation $E = |k|^2$, or, in other words, the unit tile has exactly critical density, $\rho = 2$, irrespectively of the density of the surrounding background (as seen, e.g., in fig. 1). This is consistent with the fact that the networks of strings, appearing from the artificial protocol, are transient configurations, but may be turned into recurrent ones by adding only $O(1)$ sand-grains.

Two strings, respectively of momentum $p$ and $q$, can collapse in a single one of momentum $k$ (see fig. 3). In this process momentum is conserved: $p + q + k = 0$, for momenta oriented as outgoing from the collapse region. More precisely, the strings join together in such a way that the squares boxes surrounding the unit cells meet at an extended scattering vertex, a triangle of sides of lengths equal to $|k|$, $|p|$ and $|q|$, rotated by 90 degrees w.r.t. the corresponding momenta: given this geometrical construction, momentum conservation rephrases as the oriented perimeter of the triangle being a closed polygonal chain.

Local momentum conservation and the $k \leftrightarrow -k$ symmetry are reminiscent of equilibrium of tensions, in a planar network of tight material strings, from which the name “string” of these structures. On networks, this local conservation is extended to a global constraint. Choose an orthogonal frame $(x, t)$, and orient momenta in the direction of increasing $t$. Then, sections at fixed $t$ are all crossed by the same total momentum.

Rigid extended domain walls between periodic patterns, satisfying similar local and global conservations, appear in certain tiling models [24–26], which remarkably show a Yang-Baxter integrable structure, and the corresponding strings are useful interpreted as world-lines of particles in the $(x, t)$-frame. Note, however, that, at variance with these models, in the ASM we have an infinite tower of excitations for a given background, indiciated by points in the Euclid’s orchard, and infinitely many different backgrounds too.

Consider a background with periodicities $V = (v_1, v_2)$. Take a vector $k = m_1 v_1 + m_2 v_2$, with $\gcd(m_1, m_2) = 1$. Consider the segment $(0, k)$. There exists two unique points $p$ and $q$ on the sublattice, which are nearest to this segment. They are symmetric w.r.t. the point $k/2$, i.e. $p + q = k$, and, with an appropriate ordering, are such that $p \land q = v_1 \land v_2$. We write in this case $k \leftarrow (p, q)$.

In the maximally filled background with $V = I$ this construction is simplified. For each $k$ with $\gcd(k_x, k_y) = 1$ and $k_x, k_y > 0$ there exists a unique ordered pair of momenta $p$ and $q$, with non-negative components, such that $p \land q = k$ and the matrix $\left( \begin{smallmatrix} p_x & p_y \\ q_x & q_y \end{smallmatrix} \right)$ is in $SL(2, \mathbb{Z})$. For example, $(17, 4) \leftarrow ((13, 3), (4, 1))$.

We have found a simple algorithm to derive hierarchically the string textures. The tile for $k \leftarrow (p, q)$ is essentially composed of four interlaced tiles, two for $p$ and two for $q$, adjacent to the four vertices of the $k$ box. The consistency issue for the overlapping region, of side $\sim |p - q|$, is solved by the fact that, for $k \leftarrow (p, q)$, then $p \leftarrow (p - q, q)$ if $|p| > |q|$, and $q \leftarrow (p, q - p)$ otherwise, again by a property of $SL(2, \mathbb{Z})$. The reader can see in fig. 3 the similarity of the tile of a composed string with the ones of its component. By iterating this procedure, from only the knowledge of the strings of minimum momentum, we can understand the textures of an infinite family of strings (that we conjecture being the full catalog). In particular, this conjecture being true, it would make the protocol for producing networks of strings completely predictable.

A careful analysis shows that, if the unit tiles of momenta $p, q$ and $(p - q)$ have density $\rho = 2$, then also the one associated to $k$ has density 2. So, given that $\rho = 2$ for the basic momenta $(1, 0)$ and $(1, 1)$, the dispersion relation $E(k) = |k|^2$ follows by induction for this full family.

$SL(2, \mathbb{Z})$ is generated by $\left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$ and $T = \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right)$, and $S^2 = -I$, so, for each $k$, we can write

$$k \leftarrow \left( \begin{array}{c} p_x \\ q_x \\ p_y \\ q_y \end{array} \right) = \pm T^\ell \prod_{t=1}^\ell S^{c_t},$$

for some set $(c_1, \ldots, c_\ell) \in \mathbb{Z}^\ell$, and a minimal $\ell$. It results from the recursive construction above that the tile corresponding to $k$ is composed of a square of tiles, with side
for which the outer-most square has side 3 + 1.

Following the analysis of [19], we introduce the graph- now, besides interfaces between patches, incident strings.

of must be of the form .

in a single background, as in fig. 3, and the case of two

scattering patches plus three incident strings

break into new types of patches, in a Sierpiński fashion,

with a new nesting level added for each factor .

Comparing configurations in which the grains of sand

are added one by one, it emerges that the allometric

specialization of (5) can be read as a Snell’s law for

ASM strings. For the case of three strings on a common

background , we get

\[ \sum_{\alpha=1}^{\ell} \tilde{E}^{(\alpha)} \left| k^{(\alpha)} \right|^2 k^{(\alpha)} = 0, \]

which shows that momentum conservation implies a

dispersion relation of the form \( E^{(\alpha)} = c_B |k^{(\alpha)}|^2 \), and vice versa.

Emergence of allometry. – The classification of the

strings precludes to a classification of the patches. To any

string of momentum , decomposed as \( k \leftarrow (p, q) \), we can

associate three patches, respectively supercritical, criti-

cal and subcritical, through a geometrical construction,

involving \( p \) and \( q \), sketched in an example in fig. 5.

Consider a momentum \( k \leftarrow (p, q) \). The recursive

construction of unit tiles shows that a single string of

momentum \( k' = mp + q \), for \( m \) a large integer, looks as a

strip-shaped patch of -period width, of the tile of the

critical patch associated to \( p \), crossed by a string,

whose tile is related to \( q \), that reflects twice per period

\( k' \).

Similarly, the scattering of \( mp + q \) into \( q \) and \( m \) parallel \( p \)

strings, looks as the string ultimately leaving the critical

patch through a refraction event (cf. fig. 6). This is yet

another aspect of the interplay between one-dimensional

strings and two-dimensional patches, for classification

purposes.

Strings and patches interplay also at the level of dynam-

ics in deterministic protocols, and the strings are the

key ingredient to clarify allometry and pattern formation

for the ASM. Take for example a centrally seeded sand-

pile, in a maximally filled rectangular region of aspect

ratio . Near to the corners we see configurations as the

one in fig. 6. The triangoloid structure is asymptotically

a Sierpiński-like fractal, containing infinitely many different

patches, and where the theoretical formula (1) has in-

finitely many distinct realizations, while the strip connect-

ning the corner to the center of the domain is a string of

momentum \( k \sim m(1, 2) \), for \( m \) a large integer.

If one takes aspect ratio , with \( r \) an integer, one gets

\( r - 1 \) distinct triangoloids on the corners, arranged in a

parabolic shape, and connected to the center by a string of

momentum \( k \sim m(1, r) \). Further families of triangoloids,

also arranged in parabolas, may be generated through

different protocols, that we do not describe here.

If one enlarges the domain, with fine-tuned aspect ratio

and fixed amount of sand, the center and corner part of

the configuration remain stable, and what only changes

is the number of periods for the string appearing in the

diagonal ribbon. If instead both the size is scaled by \( \lambda \),

and the amount of sand is scaled by \( \lambda^2 \), the whole picture

is allometrically scaled by \( \lambda \) (thus the momentum of the

central string as well), and the triangoloids on the corners

break into new types of patches, in a Sierpiński fashion,

with a new nesting level added for each factor \( \lambda = 3 \).

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\[ |c_1| + 1 \] and a special diagonal (if \( c_1 > 0 \), or anti-diagonal, 

if \( c_1 < 0 \)), these tiles being shaped as squares crossed 

by a diagonal, or antidiagonal, and so on up to \( \ell \) nestings. 

For example, for \( k = (17, 4) \leftarrow \left( \frac{13}{3} \right) \), the fact that \( (13, 3) \geq 3 \cdot (4, 1) \) reflects in the exponent 3 for the product 

\( \left( \frac{13}{3} \right) = -T^3 ST^4 S \), and the decomposition of the unit tile for \( k \) into squares with special diagonals, shown in fig. 4, 

for which the outer-most square has side 3 + 1.

Conservation laws. – Let us go back to the problem of \( \ell \) interfaces which meet at a given corner, but allow 

now, besides interfaces between patches, incident strings. 

Following the analysis of [19], we introduce the graph- 

vector \( T = \{ T_i \} \), where \( T_i \) is the number of topplings at 
i in the relaxation of the starting configuration, and study 

its characteristics in a region that, in the starting configura-

tion, was uniformly filled with a patch. However, now we 

allow for topping distributions which are piecewise both 

quadratic and linear (the linear term was not required in 
the context of [19], as subleading in the coarsening).

For any relevant direction \( \alpha \), allow for a patch interface, 
or a string, or both. Call \( \tilde{E}^{(\alpha)} \) the difference for unit length 
(not for period), in the total number of grains of sand w.r.t. 
\( \tau_{\text{max}} \), due to presence of a string, i.e. \( \tilde{E}^{(\alpha)} = E^{(\alpha)}/|k^{(\alpha)}| \), or the contribution from a non-zero impact parameter in 
the interface. It can be shown, by reasonings as in [19], that 
the difference between the extrapolated topping profile 
for two contiguous patches, at a polar coordinate \((r, \theta)\), 
must be of the form

\[ T_{r, \theta}^{(\alpha+1)} - T_{r, \theta}^{(\alpha)} = \frac{r^2}{2} (\rho_{\alpha+1} - \rho_\alpha) \sin^2(\theta - \theta_\alpha) \]

\[ + r \tilde{E}^{(\alpha)} \sin(\theta - \theta_\alpha) + O(1). \]

Then, by summing over \( \alpha \) and matching separately the 
quadric and linear terms, we conclude that, for each \( \theta \),

\[ \sum_{\alpha=1}^{\ell} (\rho_{\alpha+1} - \rho_\alpha) \sin^2(\theta - \theta_\alpha) = 0, \]

\[ \sum_{\alpha=1}^{\ell} \tilde{E}^{(\alpha)} \sin(\theta - \theta_\alpha) = 0, \]

so that, besides the anticipated eq. (1) for patches alone, 
that was deduced in [19], we obtain

\[ \sum_{\alpha=1}^{\ell} \tilde{E}^{(\alpha)} \exp(i\theta_\alpha) = 0, \]

which describes the string and interface-offset contribu-
tions.

In (1), the first non-trivial value for \( \ell \) is 4 [19]. In our

generalization, 4 is the minimal value for the number of 
patches plus the number of strings, and thus includes new 
opportunities: a scattering event, with three incident strings 
in a single background, as in fig. 3, and the case of two 
strings and two patches, producing diagrams reminiscent 
of total reflection and refraction in optics, so that the
growth of the Sierpiński structure is mediated by a monotonic translation of the string network along the domain. This fact emerges in a clearer way in a deterministic protocol which has, everywhere in the domain, totally predictable defect lines, and generates the very same Sierpiński structures described above. Consider a domain of size \((4L - 1) \times (2L - 1)\), filled with a rhombus of parallel strings of momentum \((2, 1)\), such as for \(L = 5\)

\[
\begin{array}{cccccccccc}
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
2 & 0 & 2 & 3 & 2 & 0 & 2 & 3 & 2 & 0 \\
2 & 0 & 2 & 3 & 2 & 0 & 2 & 3 & 2 & 0 \\
2 & 0 & 2 & 3 & 2 & 0 & 2 & 3 & 2 & 0 \\
2 & 0 & 2 & 3 & 2 & 0 & 2 & 3 & 2 & 0 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
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3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
\end{array}
\]

This configuration is transient and it projects into a recurrent configuration by adding a certain number of frame identities. The two cases \(L_y = 4L + 1\) and \(L_y = 4L + 3\) generate the very same configuration, up to, in the second case, a cross of strings in a simple background, and require the same number \(\ell(\ell + 1)/2\) of frame identities. Concentrate on the case \(L_y = 4L + 1\). We obtain a Sierpiński structure with the following characteristics: transient and recurrent patches alternates. Subcritical patches correspond to the “small filled triangles” in the Sierpiński topology, and have all the same number \(A \in \{1, 2, 3\}\) of tiles per side (if we had \(A \geq 4\), a portion of the subcritical patch would be a forbidden subconfiguration, and the whole configuration would be transient). Supercritical patches correspond to the “big empty triangles” in the Sierpiński topology, and have associated nesting level \(\nu\) (the central empty triangle has nesting level \(\nu = 1\), and the level increases as the triangles get smaller, up to the maximum level \(N\)). A patch of level \(\nu\) is a polygonal shape with 3 convex angles and \(3(2N-\nu-1)\) concave angles. All patches at level \(\nu\) are crossed by the same number \(b_\nu \in \{0, 1, 2\}\) of strings, starting from the convex angles, and meeting in a scattering vertex in the center of the patch (again, if we had \(b_\nu \geq 3\), in the region near to the scattering vertex there would be a forbidden subconfiguration, and the whole configuration would be transient).

The exact relation between \(\ell\) and the structure \((N, A; b_1, \ldots , b_N)\) is given by the unique writing of \(\ell\) as

\[
\ell = 3^N A + \frac{3^N - 1}{2} + \sum_{\nu=1}^{N} 3^{\nu - 1} b_\nu. \tag{7}
\]

Thus, the special values of \(\ell\) at which we have only patches, with as many periods as possible \((A = 3\) in subcritical patches) and no strings at all, are the ones of the form

\[
\ell = \frac{7 \cdot 3^N - 1}{2} = 3, 10, 31, 94, \ldots , \tag{8}
\]

i.e., domains of sizes \(27 \times 13\), \(83 \times 41\), \(251 \times 125\), and so on.

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