GEOMETRIC CONSTRUCTION OF
CLASSES IN VAN DAELE'S K-THEORY

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Abstract. We describe explicit generators for the “real” K-theory of “real” spheres in van Daele’s picture. Pulling these generators back along suitable maps from tori to spheres produces a family of Hamiltonians used in the physics literature on topological insulators. We compute their K-theory classes geometrically, based on wrong-way functoriality of K-theory and the geometric version of bivariant K-theory, which we extend to the “real” case.

1. Introduction

Topological insulators are insulators that, nevertheless, conduct electricity on their boundaries. Even more, conducting states on the boundary are forced to be present by topological obstructions. This suggests that the boundary conducting states are quite robust under disorder. In the one-particle approximation, topological insulators may be classified by the topological K-theory of the observable C*-algebra. In translation-invariant tight-binding models, the observable algebra is isomorphic to a matrix algebra over the algebra of continuous functions on the d-torus $T^d$, where $d$ is the dimension of the material. Many interesting phenomena arise when the Hamiltonian enjoys extra symmetries that are anti-unitary or anticommute with it. Altogether, there are ten different symmetry types, and these correspond to the two complex and the eight real K-theory groups. More precisely, the torus appears through the Fourier transform, and the relevant observable algebra is the group C*-algebra of $\mathbb{Z}^d$ with real coefficients. Under Fourier transform, this becomes the real C*-algebra

$$\{ f : T^d \to \mathbb{C} : f(z) = \overline{f(z)} \text{ for all } z \in T^d \}.$$ 

The conjugation maps used here is an involution and makes the torus a “real” space. This is the same as a space with an action of the group $\mathbb{Z}/2$. In the following, we denote “real” structures as $r$. For the d-torus, we get

$$T^d = \{(x_1, \ldots, x_d, y_1, \ldots, y_d) \in \mathbb{R}^{2d} : x_j^2 + y_j^2 = 1 \text{ for } j = 1, \ldots, d\},$$

$$r_{T^d}(x_1, \ldots, x_d, y_1, \ldots, y_d) := (x_1, \ldots, x_d, -y_1, \ldots, -y_d).$$

As a result, the relevant K-theory for the study of topological phases is the “real” K-theory $KR^r(T^d)$ of $(T^d, r_{T^d})$ as defined by Atiyah [1]. This appearance in physics has renewed the interest in “real” K-theory.

This article generalises geometric bivariant K-theory as a tool for K-theory computations to the “real” case and uses this to compute the K-theory classes of certain Hamiltonians studied in the complex case already in [19]. In addition, we describe explicit generators for the “real” K-theory of spheres, extending a formula by Karoubi in [12] for the complex K-theory of even-dimensional spheres.

There have always been several different ways to describe the K-theory of a space or a C*-algebra. As noted by Kellendonk [15], the K-theory picture that is closest

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to the classification of topological insulators is van Daele’s picture. His definition
applies to a real or complex C*-algebra A with a \( \mathbb{Z}/2 \)-grading. It is based on odd,
selfadjoint unitaries in matrix algebras over A. A \( \mathbb{Z}/2 \)-grading may be interpreted
physically as a chiral symmetry, and an odd selfadjoint unitary is just the spectral
flattening of a Hamiltonian with a spectral gap at zero that respects the given
chiral symmetry. Systems without chiral symmetry may also be treated by doubling
the number of degrees of freedom to introduce an auxiliary chiral symmetry that
anticommutes with the Hamiltonian.

The starting point of this article was the discussion by Prodan and Schulz-
Baltes [19] of certain examples of Hamiltonians \( H_m \) in any dimension \( d \), namely,
\[
H_m := \frac{1}{2d} \sum_{j=1}^{d} (S_j - S_j^*) \otimes \gamma_j + \left( m + \frac{1}{2} \sum_{j=1}^{d} (S_j + S_j^*) \right) \otimes \gamma_0 \in C^*(\mathbb{Z}^d) \otimes \text{Cl}_{1,d}
\]
with Clifford generators \( \gamma_0, \ldots, \gamma_d \) and translations \( S_j \) in coordinate directions for
\( j = 1, \ldots, d \), and a mass parameter \( m \) (see [19] §2.2.4 and §2.3.3)). The selfadjoint
element \( H_m \) has a spectral gap at zero if and only if \( m \notin \{ -d, -d+2, \ldots, d-2, d \} \).
Then it defines an insulator. The top-degree Chern character of its K-theory class
and its jumps at the values in \( \{ -d, -d+2, \ldots, d-2, d \} \) are computed in the physics
literature (see [19], Equation (2.26)) and also [9,20]). Here we explain a possible
mathematical origin of \( H_m \): it is the pullback of a generator of the reduced KR-
theory of a sphere along a map \( \varphi_m : T^d \to S^{1,d} \). Then we proceed to compute the
class of \( H_m \) in “real” K-theory for all \( m \) and all dimensions \( d \).

The first step for this is to describe explicit generators for the KR-theory of
“real” spheres in van Daele’s picture. Let \( \mathbb{R}^{a,b} \) denote \( \mathbb{R}^a \times \mathbb{R}^b \) with the involution
\( \tau(x,y) := (x,-y) \) for \( x \in \mathbb{R}^a \), \( y \in \mathbb{R}^b \). Let \( S^{a,b} \subseteq \mathbb{R}^{a,b} \) be the unit sphere with
the restricted real involution. Let \( \text{Cl}_{a,b} \) denote the Clifford algebra with \( a + b \)
anticommuting, odd, selfadjoint, unitary generators \( \gamma_1, \ldots, \gamma_a+b \), and such that
\( \gamma_1, \ldots, \gamma_a \) and \( \gamma_{a+1}, \ldots, \gamma_{a+b} \) are real. Then
\[
\beta_{a,b} := \sum_{j=1}^{a+b} x_j \gamma_j \big|_{S^{a,b}}
\]
is an odd, selfadjoint, unitary, and real element of the C*-algebra \( C(S^{a,b}) \otimes \text{Cl}_{a,b} \).
Therefore, it defines a class in its van Daele K-theory, which is isomorphic to the
“real” K-theory group \( \text{KR}^{a-b-1}(S^{a,b}) \). We check that its image in the reduced “real”
K-theory is a generator in the sense that the exterior product map with it defines
an isomorphism from the “real” K-theory of a point to the reduced “real” K-theory
of \( S^{a,b} \). Our proof that \( \beta_{a,b} \) generates the “real” K-theory is based on the proof
of Bott periodicity by Kasparov [13] and Roe’s proof of the isomorphism between
KK_0(\mathbb{R}, A) and van Daele’s K-theory of A for any \( \mathbb{Z}/2 \)-graded real C*-algebra A.

The spectral flattening of the Hamiltonian \( H_m \) for \( m \in \mathbb{R} \setminus \{ -d, -d+2, \ldots, d-2, d \} \)
is an odd, selfadjoint, real unitary on the \( d \)-torus \( T^d \) with the involution by complex
conjugation in each circle factor. This may be written down as the pull-back of \( \beta_{1,d} \)
along the real map
\[
\varphi_m : T^d \to S^{1,d},
\]
\[
(x_1, \ldots, x_d, y_1, \ldots, y_d) \mapsto \frac{(x_1 + \cdots + x_d + m, y_1, y_2, \ldots, y_d)}{||(x_1 + \cdots + x_d + m, y_1, y_2, \ldots, y_d)||}
\]
Our task is to compute this pull back in the KR-theory of the torus.

There is another way to describe the generator of \( \text{KR}^{a-b-1}(S^{a,b}) \) for \( a > 0 \). Then
the two “poles”
\[
N := (1,0,\ldots,0), \quad S := (-1,0,\ldots,0),
\]
are fixed by the “real” involution on $\mathbb{S}^{a,b}$. The stereographic projection identifies $\mathbb{S}^{a,b} \setminus \{N\}$ with $\mathbb{R}^{a-1,b}$ as a “real” manifold. Thus Bott periodicity identifies the reduced $K^r$-theory of $\mathbb{S}^{0,b}$ with $K^{r-1}(\mathbb{R}^{a-1,b}) \cong KR(\mathbb{R}^n)$. The resulting map $KR^r(\mathbb{R}^n) \to KR^{r-1}(\mathbb{S}^{0,b})$ is an example of the wrong-way functoriality of $K$-theory. Namely, it is the shriek map $!i$ associated to the inclusion map $i$ of $(S)$.

Shriek maps and pull-back maps such as $!i$ and $\varphi^{-1}_m$ are among the building blocks of geometric bivariant $K$-theory. This theory also allows to compute the composites of such maps geometrically. In our case, this says that $\varphi^{-1}_m \circ !i$ is the sum of the shriek maps for all points in $\varphi^{-1}_m(S)$, equipped with appropriate orientations. Our main result, Theorem 7.6, computes the image of this in the usual direct sum decomposition of $KR^*(\mathbb{T}^d)$.

To make this computation valid in the “real” case, we show that the geometric bivariant $K$-theory as developed in [3,7] still works for $KR$-theory. These articles define geometric bivariant $K$-theory in a slightly different way than suggested originally by Connes–Skandalis [4], in order to extend it more easily to the equivariant case. A “real” involution on a space is the same as a $\mathbb{Z}$-adjoint, unitary generators.

Throughout the article, we let $R^{a,b}$ for $a, b \in \mathbb{N}$ be $R^a \times R^b$ with the involution $\tau(x, y) := (x, -y)$ for $x \in R^a$, $y \in R^b$, and we let $\mathbb{S}^{a,b}$ be the unit sphere in $\mathbb{R}^{a,b}$; so the dimension of $\mathbb{S}^{a,b}$ is $a + b - 1$. Our $R^{a,b}$ and $\mathbb{S}^{a,b}$ are denoted $R^{a,b}$ and $\mathbb{S}^{a,b}$ by Atiyah [1]; our notation is that of Kasparov [13].

A “real” structure on a $C^*$-algebra $A$ is a conjugate-linear, involutive $^*$-homomorphism $\tau: A \to A$. Then

$$A_{\mathbb{R}} := \{a \in A : \tau(a) = a\}$$

is a real $C^*$-algebra such that $A \cong A_{\mathbb{R}} \otimes \mathbb{C}$ with the involution $\tau(a \otimes \lambda) := a \otimes \overline{\lambda}$. Thus “real” $C^*$-algebras are equivalent to real $C^*$-algebras. Any commutative “real” $C^*$-algebra is isomorphic to $C_0(X)$ with the real involution $\tau(f)(x) := \overline{f(\tau(x))}$ for all $x \in X$ for a “real” locally compact space $(X, \tau)$.

Let $Cl_{a,b}$ denote the complex Clifford algebra with $a + b$ anticommuting, odd, self-adjoint, unitary generators $\gamma_1, \ldots, \gamma_{a+b}$, and such that $\gamma_1, \ldots, \gamma_a$ and $i\gamma_{a+1}, \ldots, i\gamma_{a+b}$ are real. This is a $Z/2$-graded “real” $C^*$-algebra. Let $\otimes$ denote the graded-commutative tensor product for $Z/2$-graded (“real”) $C^*$-algebras.

For $R^{1,d}$ and its subspace $S^{1,d}$ and for the corresponding Clifford algebra $Cl_{1,d}$, it is convenient to start numbering at 0. That is, the coordinates in $R^{1,d}$ are $x_0, \ldots, x_d$ and $Cl_{1,d}$ is has the anticommuting, odd, selfadjoint generators $\gamma_0, \ldots, \gamma_d$, such that $\gamma_0, i\gamma_1, \ldots, i\gamma_d$ are real.

2. Explicit $K$-theory of spheres

We are going to write down explicit generators for the reduced $K$-theory groups of spheres. This depends, of course, on the definition of $K$-theory that we are using. In terms of vector bundles given by gluing, Atiyah–Bott–Shapiro describe in [2] how to go from irreducible Clifford modules to these generators. They do treat KO-theory, but not the “real” case, which is only invented in [1]. For the description of $K_0$ through projections, Karoubi [12] has written down explicit generators for the complex $K$-theory of even-dimensional spheres. Kasparov [13] has written down...
explicit generators for his bivariant KK-theory $\text{KKR}_0(C, C_0(\mathbb{R}^{a,b}) \otimes \text{Cl}_{a,b})$; here we write $\text{KKR}$ to highlight that this means the “real” version of the theory.

We want our explicit generators in van Daele’s $K$-theory for $\mathbb{Z}/2$-graded “real” $C^*$-algebras; this version of $K$-theory is ideal for the applications to topological insulators that we have in mind (see [15]), and it is also helpful to treat “real” spheres of all dimensions simultaneously. Actually, our generators are just those of Kasparov, translated along a canonical isomorphism between Kasparov’s $KK$-theory and van Daele’s $K$-theory. This isomorphism is an auxiliary result due to Roe [21]. Roe’s isomorphism is discussed in greater detail in [3], which also relates several variants of van Daele’s $K$-theory.

2.1. Van Daele $K$-theory for graded $C^*$-algebras and Roe’s isomorphism.

The following definitions make sense both for real and complex $C^*$-algebras. We write them down in the “real” case because this is what we are going to use later. The effect of the condition $\tau(a) = a$ below is to replace a “real” $C^*$-algebra $A$ by its real subalgebra $A_\mathbb{R} := \{ a \in A : \tau(a) = a \}$.

**Definition 2.2.** Let $A$ be a unital, $\mathbb{Z}/2$-graded “real” $C^*$-algebra and $n \in \mathbb{N}_{\geq 1}$. Let

$$\mathcal{FU}_n(A) := \{ a \in \mathcal{M}_n A : a = a^*, \tau(a) = a, \ a^2 = \i, \ a \text{ odd} \},$$

equipped with the norm topology of $\mathcal{M}_n A$. Two elements in $\mathcal{FU}_n(A)$ in the same path components are called homotopic. We define the maps

$$\oplus : \mathcal{FU}_n(A) \times \mathcal{FU}_m(A) \to \mathcal{FU}_{n+m}(A), \quad (a, b) \mapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$  

Let $\mathcal{FU}_n(A) := \pi_0(\mathcal{FU}_n(A))$. We abbreviate $\mathcal{FU}(A) := \mathcal{FU}_1(A)$ and $FU(A) := \mathcal{FU}_1(A)$. We call $A$ balanced if $\mathcal{FU}(A) \neq \emptyset$.

The condition $a = a^*$ may be dropped (see [23, Proposition 2.5]). For a unitary, it means that $a^2 = 1$, forcing the spectrum to be contained in $\{ \pm 1 \}$. Then $(a + 1)/2$ is a projection. It is useless, however, because it is not even.

The operation $\oplus$ is associative and commutative up to homotopy (see [23, Proposition 2.7]). So $\bigoplus_{n=1}^{\infty} \mathcal{FU}_n(A)$ becomes an Abelian semigroup. Let $\text{GFU}(A)$ be its Grothendieck group.

**Definition 2.3.** Let $A$ be a balanced, unital, $\mathbb{Z}/2$-graded “real” $C^*$-algebra. The van Daele $K$-theory $DK(A)$ of $A$ is defined as the kernel of the homomorphism $d : \text{GFU}(A) \to \mathbb{Z}$ defined by $d_{\text{GFU}_n(A)} = n$ for all $n \in \mathbb{N}_{\geq 1}$.

Van Daele’s original definition uses an element $e \in \mathcal{FU}(A)$ with $e \sim (-e)$ to define stabilisation maps

$$\mathcal{FU}_n(A) \to \mathcal{FU}_{n+1}(A) : [x] \mapsto [x \oplus e].$$

The colimit of the resulting inductive system becomes an Abelian group under $\oplus$. This group is isomorphic to $DK(A)$ as defined above (see [3,21]).

The definition above is generalised beyond the balanced, unital case as follows. First let $A$ be a unital, $\mathbb{Z}/2$-graded “real” $C^*$-algebra. Then $A \otimes \text{Cl}_{1,1}$ is balanced because $1 \otimes \gamma_1 \in FU(A \otimes \text{Cl}_{1,1})$. If $A$ is already balanced, then there is a natural stabilisation isomorphism $DK(A) \cong DK(A \otimes \text{Cl}_{1,1})$. This justifies defining $DK(A) := DK(A \otimes \text{Cl}_{1,1})$ in general. For a $C^*$-algebra $A$ without unit, let $A^+$ be its unitisation, equipped with the canonical augmentation homomorphism $A^+ \to \mathbb{C}$. Then

$$DK(A) := \text{Ker}(DK(A^+) \to DK(\mathbb{C})).$$

This reproduces the previous definition if $A$ is unital because then $A^+ \cong A \oplus \mathbb{C}$ and $DK$ is additive. Since there is an isomorphism $DK(A) \cong \text{KKR}_0(C_{1,0}, A)$, the functor $DK$ is stable with respect to matrix algebras, Morita invariant, homotopy
invariant, and exact for all extensions of \( \mathbb{Z}/2 \)-graded \( \mathcal{C}^* \)-algebras. See also \cite{3} for direct proofs of these properties of van Daele’s K-theory using the definition above.

2.4. Roe’s isomorphism. The following proposition is due to Roe \cite{21}. The equivalent isomorphism \( \text{DK}(A) \cong \text{KKR}_0(\mathbb{C}, A) \) is proven in \cite{3}.

**Proposition 2.5** (\cite{21}). Let \( A \) be a unital, \( \mathbb{Z}/2 \)-graded “real” \( \mathcal{C}^* \)-algebra. Then \( \text{DK}(A \otimes \text{Cl}_{r,s}, A) \cong \text{KKR}_0(\mathbb{C}, A \otimes \text{Cl}_{r,s}, A) \).

This justifies defining the “real” K-theory \( \text{KR}(A) \) and its graded version for a \( \mathbb{Z}/2 \)-graded “real” \( \mathcal{C}^* \)-algebra \( A \) as

\[
\text{KR}(A) := \text{DK}(A \otimes \text{Cl}_{1,0}), \quad \text{KR}_n(A) := \text{DK}(A \otimes \text{Cl}_{1,n}),
\]

for \( n \in \mathbb{N} \). Actually, this only depends on \( n \mod 8 \), so that we may also take \( n \in \mathbb{Z}/8 \). If \( A \) is trivially graded, then \( \text{KR}(A) \) is naturally isomorphic to the ordinary \( K_0 \) of the real Banach algebra \( \mathbb{A}_R \). This is sometimes denoted \( \text{KO}_0(\mathbb{A}_R) \) to highlight that \( \mathbb{A}_R \) is only a real \( \mathcal{C}^* \)-algebra.

For a “real” locally compact space \( X \), we define its “real” K-theory \( \text{KR}(X) \) as \( \text{KR}(\mathbb{C}_0(X)) \) with the induced “real” structure. This is equivalent to the definition by Atiyah \cite{1}. For \( n \in \mathbb{Z}/8 \), we also let

\[
\text{KR}^n(X) := \text{KR}_{-n}(\mathbb{C}_0(X)).
\]

To construct explicit generators for the \( \text{KR} \)-theory of spheres, we shall need an elementary auxiliary result used in the proof in \cite{21}. Let \( \mathcal{H}_A \) be the standard graded “real” Hilbert \( A \)-module \( \ell^2(\mathbb{N}, \mathbb{C}) \otimes A \); the “real” involution is the tensor product of complex conjugation on \( \ell^2(\mathbb{N}, \mathbb{C}) \) and the given “real” involution on \( A \), and the \( \mathbb{Z}/2 \)-grading is the tensor products of the \( \mathbb{Z}/2 \)-grading on \( \ell^2(\mathbb{N}, \mathbb{C}) \) induced by the parity operator and the given one on \( A \). A cycle for \( \text{KKR}_0(\mathbb{C}, A \otimes \text{Cl}_{r,s}, A) \) is homotopic to one of the form \( (\mathcal{H}_A \otimes \text{Cl}_{r,s}, 1, F) \), where \( 1 \) denotes the left action of \( \mathbb{C} \) by scalar multiplication, and \( F \in \mathcal{B}(\mathcal{H}_A) \otimes \text{Cl}_{r,s} \) is real, odd and selfadjoint and satisfies \( F^2 - 1 \in \mathcal{K}(\mathcal{H}_A) \otimes \text{Cl}_{r,s} \). Let \( \mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}, \mathbb{C})) \), identify \( \mathcal{K}(\mathcal{H}_A) \) with \( \mathcal{K} \otimes A \cong \mathcal{K} \otimes \mathcal{A} \), and let \( \mathcal{M}(\mathcal{K} \otimes A) \) denote the stable multiplier algebra of \( A \). Then \( \mathcal{B}(\mathcal{H}_A) \otimes \text{Cl}_{r,s} \) is isomorphic to \( \mathcal{M}(\mathcal{K} \otimes A) \otimes \text{Cl}_{r,s} \). Let \( \mathcal{Q}(\mathcal{K} \otimes A) := \mathcal{M}(\mathcal{K} \otimes A) / \mathcal{K} \otimes A \). Then \( F \) is mapped to a “real”, odd, selfadjoint unitary in \( \mathcal{Q}(\mathcal{K} \otimes A) \otimes \text{Cl}_{r,s} \). Conversely, any “real”, odd, selfadjoint unitary in \( \mathcal{Q}(\mathcal{K} \otimes A) \otimes \text{Cl}_{r,s} \) lifts to an operator \( F \in \mathcal{B}(\mathcal{H}_A) \otimes \text{Cl}_{r,s} \) such that \( (\mathcal{H}_A \otimes \text{Cl}_{r,s}, 1, F) \) is a cycle for \( \text{KKR}_0(\mathbb{C}, A \otimes \text{Cl}_{r,s}, A) \). Thus we get an obvious surjective map from \( \text{FU}(\mathcal{Q}(\mathcal{K} \otimes A) \otimes \text{Cl}_{r,s}) \) onto \( \text{KKR}_0(\mathbb{C}, A \otimes \text{Cl}_{r,s}, A) \).

Since \( \mathcal{H}_A \) has infinite multiplicity, \( \mathcal{B}(\mathcal{H}_A) \otimes \text{Cl}_{r,s} \) is isomorphic to \( M_n(\mathcal{B}(\mathcal{A}_R) \otimes \text{Cl}_{r,s}) \), and the same holds for the quotient by the compact operators. Thus the construction above also defines a map

\[
\bigoplus_{n=1}^{\infty} \text{FU}_n(\mathcal{Q}(\mathcal{K} \otimes A) \otimes \text{Cl}_{r,s}) \to \text{KKR}_0(\mathbb{C}, A \otimes \text{Cl}_{r,s}).
\]

**Lemma 2.6.** The map above induces a group isomorphism \( \text{DK}(\mathcal{Q}(\mathcal{K} \otimes A) \otimes \text{Cl}_{r,s}) \cong \text{KKR}_0(\mathbb{C}, A \otimes \text{Cl}_{r,s}) \).

**Proof.** Our map is a semigroup homomorphism because the sum in DK and KKR is defined as the direct sum. As such, it induces a group homomorphism on the Grothendieck group, which we may restrict to the subgroup DK(A). There is a real, odd, selfadjoint unitary operator \( F_0 \in \mathcal{B}(\mathcal{H}_A \otimes \text{Cl}_{r,s}) \), and any such operator defines a degenerate Kasparov cycle. Such degenerate cycles represent zero in KK-theory. If \( F \in \mathcal{F}_U(\mathcal{Q}(\mathcal{K} \otimes A) \otimes \text{Cl}_{r,s}) \), then \([F] - [F_0] \in \text{DK}(A)\) is mapped to the same KK-class as \( F \). Thus the induced map on \( \text{DK}(A) \) remains surjective. Roe shows in \cite{21} that it is also injective; the main reason for this is that “homotopic” KK-cycles become “operator homotopic” after adding degenerate cycles. \( \square \)
Roe further combines the isomorphism in Lemma 2.6 with the boundary map in the long exact sequence for KK for the extension of \( C^* \)-algebras
\[
K \otimes A \otimes Cl, s \to \mathcal{M}(K \otimes A) \otimes Cl, s \to Q(K \otimes A) \otimes Cl, s
\]
to prove Proposition 2.3. The latter boundary map is an isomorphism because van Daele’s K-theory, like ordinary K-theory, vanishes for stable multiplier algebras.

Now we combine Lemma 2.6 with Kasparov’s proof of Bott periodicity (see [13, p. 545f and Theorem 7]) in bivariant KK-theory. Fix \( a, b \in \mathbb{N}; \) at some point, we will assume \( a \geq 1 \), but for now we do not need this. Let
\[
D := C_0(\mathbb{R}^{a,b}) \otimes Cl_{a,b} = C_0(\mathbb{R}^{a,b}) \otimes Cl_{a,b}.
\]
Kasparov constructs an element in KK\(_R(\mathbb{C}, D)\) and proves that it is invertible by defining an inverse and computing the Kasparov products. The definition of Kasparov’s KK-class is based on the unbounded continuous function
\[
(5) \quad \tilde{\beta}_{a,b} : \mathbb{R}^{a,b} \to Cl_{a,b}, \quad (x_1, \ldots, x_{a+b}) \mapsto \sum_{i=1}^{a+b} x_i \gamma_i.
\]

**Lemma 2.7.** \( \tilde{\beta}_{a,b} \) is selfadjoint, odd and real, and \( \tilde{\beta}_{a,b}(x)^2 = \|x\|^2 \).

**Proof.** First, \( \tilde{\beta}_{a,b} \) is selfadjoint because \( x_i^* = x_i \) and \( \gamma_i^* = \gamma_i \) for all \( i \). It is odd because all \( \gamma_i \) are odd and the grading on \( C_0(\mathbb{R}^{a,b}) \) is trivial. Finally, \( \tilde{\beta}_{a,b} \) is real because \( \gamma_i \) is real whenever \( x_i \) is real and \( \gamma_i \) is imaginary whenever \( x_i \) is imaginary. The formula \( \tilde{\beta}_{a,b}(x)^2 = \|x\|^2 \) holds because \( \gamma_i^2 = 1 \) for all \( 1 \leq i \leq a + b \) and the \( \gamma_i \) anticommute.

The bounded transform of \( \tilde{\beta}_{a,b} \) is
\[
\tilde{\beta}'_{a,b} : \mathbb{R}^{a,b} \to Cl_{a,b}, \quad x \mapsto \tilde{\beta}_{a,b}(x)/(1 + \tilde{\beta}_{a,b}(x)^2)^{1/2}.
\]
It satisfies \( 1 - (\tilde{\beta}'_{a,b})^2 = (1 + \|x\|^2)^{-1/2} \in \mathbb{K}(D) \) and is odd, selfadjoint and real because \( \tilde{\beta}_{a,b} \) is. Hence \( [\tilde{\beta}'_{a,b}] \in \text{KK}\(_R(\mathbb{C}, D)\)\). Kasparov proves that \( [\tilde{\beta}'_{a,b}] \) generates \( \text{KK}\(_R(\mathbb{C}, D)\) \cong \mathbb{Z} \).

To plug \( [\tilde{\beta}'_{a,b}] \) into the isomorphism in Lemma 2.6, we choose a degenerate cycle \( F_0 \). Adding \( F_0 \) replaces the underlying Hilbert module \( D \) of \( [\tilde{\beta}'_{a,b}] \) by \( H_D \). Thus \( \tilde{\beta}_{a,b} \oplus F_0 \) and \( F_0 \) are elements of \( \mathcal{FU}(Q(C_0(\mathbb{R}^{a,b}) \otimes Cl_{a,b})) \) and \( [\tilde{\beta}_{a,b} \oplus F_0] - [F_0] \) represents a class in \( \text{DK}(Q(C_0(\mathbb{R}^{a,b}) \otimes \mathbb{K} \otimes Cl_{a,b})) \)\). Since degenerate KK-cycles represent zero, \( [\tilde{\beta}_{a,b} \oplus F_0] - [F_0] \) is a generator for \( \text{DK}(Q(C_0(\mathbb{R}^{a,b}) \otimes \mathbb{K} \otimes Cl_{a,b})) \cong \mathbb{Z} \). Here the choice of \( F_0 \) does not matter.

We may also interpret the passage from \( \tilde{\beta}_{a,b} \) to \( \tilde{\beta}'_{a,b} \) as replacing \( \mathbb{R}^{a,b} \) by
\[
B^{a,b} := \{ x \in \mathbb{R}^{a,b} : \|x\| < 1 \},
\]
which is homeomorphic to \( \mathbb{R}^{a,b} \) through the map \( x \mapsto x/(1 + \|x\|^2)^{1/2} \). Let \( B^{a,b} \) be the closure of \( B^{a,b} \) in \( \mathbb{R}^{a,b} \). The boundary \( \partial B^{a,b} \) is the \( a + b \)-dimensional unit sphere \( S^{a,b} \subseteq \mathbb{R}^{a,b} \) with the “real” involution of \( \mathbb{R}^{a,b} \). There is a canonical embedding \( C(B^{a,b}) \otimes Cl_{a,b} \hookrightarrow C(B^{a,b}) \otimes Cl_{a,b} = \mathcal{M}(C_0(B^{a,b}) \otimes Cl_{a,b}) \). It induces a canonical embedding \( C(S^{a,b}) \otimes Cl_{a,b} \hookrightarrow \mathcal{M}(C_0(\mathbb{R}^{a,b}) \otimes Cl_{a,b})/C_0(\mathbb{R}^{a,b}) \otimes Cl_{a,b}) \).

Its image contains the image of \( \tilde{\beta}_{a,b} \) in this quotient. Namely, \( (\tilde{\beta}'_{a,b}) \) comes from \( \tilde{\beta}_{a,b} := \tilde{\beta}_{a,b}[\tilde{\beta}_{a,b}] \in \mathcal{FU}(C(S^{a,b}) \otimes Cl_{a,b}) \); this is the same \( \tilde{\beta}_{a,b} \) as in \( [2] \).

We assume from now on that \( a \geq 1 \). Then the first Clifford generator \( \gamma_1 \in Cl_{a,b} \) is an odd selfadjoint real unitary. The constant function with value \( \gamma_1 \) belongs to \( \mathcal{FU}(C(B^{a,b}) \otimes Cl_{a,b}) \) and is mapped to a degenerate KK-cycle for \( \text{KK}\(_R(\mathbb{C}, D)\) \). So we map \( F_0 \) to be a direct sum of countably many copies of \( \gamma_1 \) above.
Lemma 2.8. Let \( a \geq 1 \). The class of \([\beta_{a,b}] - [\gamma_1]\) ∈ DK(\(C(S^{a,b}) \otimes Cl_{a,b}\)) is mapped to a generator of KK\(R_0(\mathbb{C}, D)\).

*Proof.* The inclusion of \(C(S^{a,b}) \otimes Cl_{a,b}\) into \(Q(C_0(\mathbb{R}^{a,b}) \otimes \mathbb{K} \otimes Cl_{a,b})\) is not unital. The map on van Daele K-theory that it induces sends \([e] - [f]\) for \(e, f \in FU(C(S^{a,b}) \otimes Cl_{a,b})\) to \([e \oplus F_0] - [f \oplus F_0]\). For our choice of \(F_0\) above, it therefore sends \([\beta_{a,b}] - [\gamma_1]\) to \([\beta_{a,b} \oplus F_0] - [F_0] = 0\) ∈ DK(\(Q(C_0(\mathbb{R}^{a,b}) \otimes \mathbb{K} \otimes Cl_{a,b})\)). We already know that Roe’s isomorphism maps the latter to a generator of KK\(R_0(\mathbb{C}, D)\). □

Since \( a \geq 1 \), the two poles \(S\) and \(N\) in \(\mathbb{R}\) are fixed by the “real” involution on \(S^{a,b}\.

We choose \(N \in S^{a,b}\) as a point at infinity and identify \(S^{a,b} \setminus \{N\} \cong \mathbb{R}^{a-1,b}\) by stereographic projection. The C*-algebra extension

\[
\begin{array}{ccc}
C_0(\mathbb{R}^{a-1,b}) & \xrightarrow{\epsilon_{N,a}} & C(S^{a,b}) \\
\downarrow & & \downarrow \\
C_0(\mathbb{R}^{a,b}) \otimes \mathbb{K} & \xrightarrow{\mathcal{M}(C_0(\mathbb{R}^{a,b}) \otimes \mathbb{K})} & Q(C_0(\mathbb{R}^{a,b}) \otimes \mathbb{K}).
\end{array}
\]

splits by taking constant functions. Since van Daele’s K-theory is split exact

\[
\text{DK}(C(S^{a,b}) \otimes Cl_{r,s}) \cong \text{DK}(C_0(\mathbb{R}^{a-1,b}) \otimes Cl_{r,s}) \oplus \text{DK}(Cl_{r,s}).
\]

Proposition 2.9. There is a Bott periodicity isomorphism

\[
\text{DK}(C_0(\mathbb{R}^{a-1,b}) \otimes Cl_{a,b}) \cong \text{KKR}_0(\mathbb{C}, C_0(\mathbb{R}^{a-1,b}) \otimes Cl_{a-1,b}) \cong \mathbb{Z}.
\]

Under the two isomorphisms above, \([\beta_{a,b}] - [\gamma_1]\) ∈ DK(\(C(S^{a,b}) \otimes Cl_{a,b}\)) is mapped to \((1, 0) ∈ \mathbb{Z} \oplus \text{DK}(Cl_{a,b})\).

*Proof.* The value of \(\beta_{a,b}\) at \(N = \gamma_1\). So evaluation at \(N\) maps \([\beta_{a,b}] - [\gamma_1]\) to zero. We compute DK(\(C_0(\mathbb{R}^{a-1,b}) \otimes Cl_{a,b}\)) through the long exact sequences for DK applied to the morphism of C*-algebra extensions

The vertical arrow \(C(B^{a,b} \setminus \{N\}) \hookrightarrow \mathcal{M}(C_0(\mathbb{R}^{a,b}) \otimes \mathbb{K})\) combines the map from bounded continuous functions on \(\mathbb{R}^{a,b}\) to multipliers of \(C_0(\mathbb{R}^{a,b})\) and the corner embedding into \(\mathbb{K}\). It induces an isomorphism on DK because the latter is homotopy invariant, stable under tensoring with \(\mathbb{K}\) and vanishes on stable multiplier algebras. The left vertical map induces an isomorphism on DK as well. Therefore, the vertical map on the quotients also induces an isomorphism

\[
\text{DK}(Q(C_0(\mathbb{R}^{a,b}) \otimes \mathbb{K})) \cong \text{KKR}_0(\mathbb{C}, D) \cong \mathbb{Z}.
\]

Lemma 2.8 shows that the image of \([\beta_{a,b}] - [\gamma_1]\) is mapped to a generator of \(\mathbb{Z}\) under these isomorphisms. □

How about generators for DK(\(C_0(\mathbb{R}^{a-1,b}) \otimes Cl_{r,s}\)) for \((r, s) \neq (a, b)\)? By Bott periodicity, it is no loss of generality to assume \(r = a + c\) and \(s = b + d\) with \(c, d \geq 0\), and this group is isomorphic to KK\(R_0(\mathbb{C}, Cl_{c,d})\). The isomorphism in Kasparov theory is given simply by exterior product. In order to compute these exterior products explicitly, we simplify the cycles for KK\(R_0(\mathbb{C}, Cl_{c,d})\). The following results may well be known, but the authors are not aware of a reference where generators for the KO-theory of a point are worked out as cycles for KK\(R_0(\mathbb{C}, Cl_{c,d})\). Of course, our results will recover the description of \(KO^*(pt)\) through Clifford modules by Atiyah–Bott–Shapiro \[I\].

Lemma 2.10. Any cycle for KK\(R_0(\mathbb{C}, Cl_{c,d})\) is homotopic to one with \(\mathbb{C}\) acting by scalar multiplication and Fredholm operator equal to 0. Isomorphism classes of such cycles are in bijection with finitely generated modules over the real subalgebra \((Cl_{c+1,d})_{\mathbb{R}}\) in \(Cl_{c+1,d}\).
Proof. Let $(\mathcal{H}, \varphi, F)$ be a cycle for $KKR_0(\mathbb{C}, Cl_{c,d})$. First, we may replace it by a homotopic one where $\mathbb{C}$ acts just by scalar multiplication. Then we may use functional calculus for $F$ to arrange that the spectrum of $F$ consists only of $\{0, 1, -1\}$. The direct summand where $F$ has spectrum $\pm 1$ is unitary and thus gives a degenerate cycle. Removing that piece gives a cycle with $F = 0$, still with $\mathbb{C}$ acting by scalar multiplication. Thus the only remaining data is the underlying $\mathbb{Z}/2$-graded “real” Hilbert $Cl_{c,d}$-module, which we still denote by $\mathcal{H}$. To give a KK-cycle with $F = 0$, the identity operator on $\mathcal{H}$ must be compact.

Next we claim that $\mathcal{H} \cong p \cdot Cl_{2,d}^{c_2}$ for some real, even projection $p \in \tilde{M}_{2n}(Cl_{c,d})$; here half of the summands in $Cl_{2,d}^{c_2}$ have the flipped grading. First, the Kasparov stabilisation theorem implies that there is a real, even unitary $\mathcal{H} \oplus (Cl_{c,d})_{\infty} \cong (Cl_{c,d})_{\infty}$, where $(Cl_{c,d})_{\infty}$ denotes the standard “real” graded Hilbert $Cl_{c,d}$-module, with the $\mathbb{Z}/2$-grading where half of the summands carry the flipped grading. This gives a real, even projection $p_0 \in B((Cl_{c,d})_{\infty})$ with $\mathcal{H} \cong p_0(Cl_{c,d})_{\infty}$. Since the identity on $\mathcal{H}$ is compact, $p_0 \in K((Cl_{c,d})_{\infty})$. Then $p_0$ is Murray–von Neumann equivalent – with even real partial isometries – to a nearby projection $p \in \tilde{M}_{2n}(Cl_{c,d})$. This implies an isomorphism $\mathcal{H} \cong p \cdot Cl_{2,d}^{c_2}$ of “real” $\mathbb{Z}/2$-graded Hilbert modules.

It is well known that any idempotent in a $C^*$-algebra is Murray–von Neumann equivalent to a projection. The proof is explicit, using the functional calculus. Therefore, if the idempotent we start with is even and real, then so are the equivalent projection and the Murray–von Neumann equivalence between the two. Therefore, the isomorphism class of $\mathcal{H}$ as a “real” graded Hilbert module is still captured by the Murray–von Neumann equivalence class of $p$ as a real, even idempotent element. Let $S$ be the ring of even, real elements in $\tilde{M}_{2}(Cl_{c,d})$. What we end up with is that the possible isomorphism classes of $\mathcal{H}$ are in bijection with Murray–von Neumann equivalence classes of idempotents in matrix algebras over $S$. These are, in turn, in bijection to isomorphism classes of finitely generated projective modules over $S$. The subalgebra of even elements of $\tilde{M}_{2}(Cl_{c,d})$ is isomorphic to the crossed product $Cl_{c,d} \rtimes \mathbb{Z}/2$ – this is an easy special case of the Green–Julg Theorem (see [1]), identifying a crossed product for an action of a compact group $G$ on a $C^*$-algebra $A$ with the fixed-point algebra of the diagonal $G$-action on $A \otimes K(L^2 G)$. The nontrivial element of $\mathbb{Z}/2$ gives an extra Clifford generator that commutes with even and anticommutes with odd elements of $Cl_{c,d}$. Thus the even part of $\tilde{M}_{2}(Cl_{c,d})$ is isomorphic to $Cl_{c+1,d}$. This implies $S \cong (Cl_{c+1,d})_\mathbb{R}$. This $\mathbb{R}$-algebra is semisimple because its complexification is a sum of matrix algebras. So all finitely generated modules over it are projective.

Remark 2.11. The lemma only describes isomorphism classes of certain special cycles for $KKR_0(\mathbb{C}, Cl_{c,d})$, it does not compute this group. We do not need this computation, but sketch it anyway. Consider a degenerate cycle for $KKR_0(\mathbb{C}, Cl_{c,d})$ that is also finitely generated. The operator $F$ on it is real and odd with $F^2 = 1$ and commutes with $Cl_{c,d}$. Multiplying $F$ with the grading gives a real, odd operator that anticommutes with the generators of $Cl_{c,d}$ and has square $–1$. This shows that a $(Cl_{c+1,d})_\mathbb{R}$-module admits an operator $F$ that makes it a degenerate cycle for $KKR_0(\mathbb{C}, Cl_{c,d})$ if and only if the module structure extends to $(Cl_{c+1,d+1})_\mathbb{R}$. Call two $(Cl_{c+1,d})_\mathbb{R}$-modules stably isomorphic if they become isomorphic after adding restrictions of $(Cl_{c+1,d+1})_\mathbb{R}$-modules to them. Since the direct sum of $(Cl_{c+1,d+1})_\mathbb{R}$-modules corresponds to the direct sum of Kasparov cycles, stably isomorphic $(Cl_{c+1,d})_\mathbb{R}$-modules give the same class in $KKR_0(\mathbb{C}, Cl_{c,d})$. Now Atiyah–Bott–Shapiro [2] have computed the stable isomorphism classes of $(Cl_{c+1,d})_\mathbb{R}$-modules and found that they give the KO-theory of the point. As a result, two $(Cl_{c+1,d})_\mathbb{R}$-modules give the same class in $KKR_0(\mathbb{C}, Cl_{c,d})$ if and only if they are stably isomorphic.
Let us look at the generators of \( \text{KKR}_0(\mathbb{C}, \text{Cl}_{a,d}) \cong \mathbb{Z}/2 \) for \( d - c \equiv 1, 2 \pmod{8} \). The simplest choice here is to take \( c = 0 \) and \( d = 1, 2 \). It follows from Lemma 2.10 that the generator of \( \text{KKR}_0(\mathbb{C}, \text{Cl}_{a,d}) \cong \mathbb{Z}/2 \) corresponds to \( (\text{Cl}_{a+1,d})^\mathbb{R} \)-module.

Since the direct sum of modules becomes the sum in KKR-theory, we may pick a simple module. For \( c = 0 \) and \( d = 1 \), we get \((\text{Cl}_{1,1})^\mathbb{R} = \mathbb{M}_2(\mathbb{R})\). Up to isomorphism, there is a unique two-dimensional simple module. Turn \( \text{Cl}_{0,1} \) into a \( \mathbb{Z}/2 \)-graded Hilbert module over itself and give it the operator \( F = 0 \). The constructions in the proof of Lemma 2.10 turn this into a two-dimensional \((\text{Cl}_{1,1})^\mathbb{R}\)-module. So \( \text{Cl}_{0,1} \) with \( F = 0 \) represents the generator of \( \text{KKR}_0(\mathbb{C}, \text{Cl}_{0,1}) \cong \mathbb{Z}/2 \).

Next, let \( c = 0 \) and \( d = 2 \). Then \((\text{Cl}_{1,2})^\mathbb{R} = \mathbb{M}_2(\mathbb{C})\). It has \( \mathbb{C}^2 \) as its unique simple module. So any module of dimension 4 over \( \mathbb{R} \) is simple. Since \( \text{Cl}_{0,2} \) as a graded Hilbert module over itself also yields a 4-dimensional module over \((\text{Cl}_{1,2})^\mathbb{R}\), the latter is simple. Thus \( \text{Cl}_{0,2} \) with \( F = 0 \) represents the generator of \( \text{KKR}_0(\mathbb{C}, \text{Cl}_{0,2}) \cong \mathbb{Z}/2 \).

Having described the generators of \( \text{KKR}_0(\mathbb{C}, \text{Cl}_{d,0}) \) for \( d = 1, 2 \) in this simple form, computing the exterior product becomes a trivial matter:

**Proposition 2.12.** Let \( d \in \{1, 2\} \). Then the isomorphism

\[
\text{DK}(\mathbb{C}(S^{a,b}) \otimes \text{Cl}_{a,b+d}) \cong \text{DK}(\mathbb{C}(\mathbb{R}^{a-1,b}) \otimes \text{Cl}_{a,b+d}) \oplus \text{DK}(\text{Cl}_{a,b+d})
\]

maps \( [\beta_{a,b}] - [\gamma_1] \) to \((1, 0)\).

**Proof.** First we recall a general formula for exterior products in a simple special case. Let \((\mathcal{E}, \varphi, F)\) be a cycle for \( \text{KKR}_0(A, B) \) for some “real” \( \mathbb{C} \)-algebras \( A, B \). Let \((\mathcal{H}, 1, 0)\) be a cycle for \( \text{KKR}_0(\mathbb{C}, \text{Cl}_{c,d})\). Then their exterior product is represented by the triple \((\mathcal{E} \otimes \mathcal{H}, \varphi \otimes 1, F \otimes 1)\); this works because \( \mathcal{H} \) has Fredholm operator 0. We now apply this to Kasparov’s Bott periodicity generator \([\beta_{a,b}']\). Then we get the \( \mathbb{Z}/2 \)-graded Hilbert \( \mathcal{C}(\mathbb{R}^{a,b}) \otimes \text{Cl}_{a+c,b+d} \)-module \( \mathcal{C}(\mathbb{R}^{a,b}) \otimes \text{Cl}_{a,b} \otimes \mathcal{H} \) with \( \mathbb{C} \) acting by scalar multiplication and with the Fredholm operator \( \beta_{a,b}' \otimes 1 \). The same arguments as above show that the isomorphism to van Daele’s K-theory transfers this Kasparov cycle to \([\beta_{a,b}'] - [\gamma_1]\) in \( \text{DK}(\mathbb{C}(S^{a,b}) \otimes \text{Cl}_{a+b} \otimes \mathcal{K}(\mathcal{H}))\). The latter is \( \mathbb{Z}/2 \)-equivariantly Morita equivalent to \( \text{DK}(\mathbb{C}(S^{a,b}) \otimes \text{Cl}_{a+c,b+d})\) because \( \mathcal{K}(\mathcal{H}) \) is \( \mathbb{Z}/2 \)-equivariantly Morita equivalent to \( \text{Cl}_{c,d} \). The isomorphism in van Daele’s K-theory induced by this Morita equivalence maps \([\beta_{a,b}] - [\gamma_1]\) to the element in \( \text{DK}(\mathbb{C}(S^{a,b}) \otimes \text{Cl}_{a+c,b+d})\) that comes from \((\mathcal{H}, 1, 0) \in \text{KKR}_0(\mathbb{C}, \text{Cl}_{c,d})\).

Now we specialise to the case where \( \mathcal{H} = \text{Cl}_{0,d} \) with the obvious structure of \( \mathbb{Z}/2 \)-graded “real” Hilbert \( \text{Cl}_{0,d} \)-module. Then \( \mathcal{K}(\mathcal{H}) \cong \text{Cl}_{0,d} \) canonically, and there is no need to invoke the Morita invariance of van Daele’s K-theory. The result is just to view \( \beta_{a,b}' \) as taking values in \( \text{Cl}_{a+b+d} \supseteq \text{Cl}_{a,b} \).

We do not describe the generator for \( \text{KKR}_0(\mathbb{C}, \text{Cl}_{0,4}) \cong \mathbb{Z} \). Since \( \mathcal{H} = \text{Cl}_{0,4} \) itself represents 0 in KKR, a nontrivial Morita equivalence is needed here.

### 3. Some canonical maps in K-theory

We have described explicit elements in the van Daele K-theory of the spheres \( S^{a,b} \) for \( a, b \in \mathbb{N} \). Topological phases are, however, described by the K-theory of the “real” torus \( T^d \), where \( d \) is the dimension of the physical system. One way to transfer K-theory classes from one space to another is the pullback functoriality. In Section 7 we are going to use this to pull our K-theory generators on \( S^{1,d} \) back to K-theory classes on \( T^d \) along certain maps \( T^d \rightarrow S^{1,d} \).

We now describe this pull back functoriality in detail. Let \( (X, \tau_X) \) and \( (Y, \tau_Y) \) be “real” locally compact spaces and let \( b: X \rightarrow Y \) be a proper continuous map that is
“real” in the sense that \( r_Y \circ b = b \circ r_X \). Such a map induces a “real” -homomorphism

\[
    b^*: C_0(Y) \to C_0(X), \quad \psi \mapsto \psi \circ b,
\]

which for \( c, d \in \mathbb{N} \) induces maps in van Daele’s K-theory

\[
    b^*: DK(C_0(Y) \otimes Cl_{c,d}) \to DK(C_0(X) \otimes Cl_{c,d}).
\]

We denote it by \( b^* \) as well because no confusion should be possible.

There are other ways to transfer K-theory classes between spaces. A famous example is the Atiyah–Singer index map for a family of elliptic differential operators. There are two ways to compute this index map, one analytic and one topological.

We are going to use the topological approach. One of its ingredients is functoriality for open inclusions: let \( U \subseteq X \) be an open subset with \( r_X(U) = U \), so that the “real” structure \( r_X \) restricts to one on \( U \). Then extension by zero defines a “real” -homomorphism

\[
    \iota_U: C_0(U) \to C_0(X),
\]

which for \( c, d \in \mathbb{N} \) induces maps in van Daele’s K-theory

\[
    \iota_{U,*}: DK(C_0(U) \otimes Cl_{c,d}) \to DK(C_0(X) \otimes Cl_{c,d}).
\]

The other ingredient of the topological index map is the Thom isomorphism. Let \( E \to X \) be a \( \mathbb{Z}/2 \)-equivariant vector bundle. Let \( Cl_E \) be the Clifford algebra bundle of \( E \); this is a locally trivial bundle of finite-dimensional \( \mathbb{Z}/2 \)-graded, real \( C^* \)-algebras over \( X \). Each fibre \( C_0(E_x) \otimes Cl(E_x) \) carries a canonical Kasparov generator \( \tilde{\beta}_E^* \). Letting \( x \) vary, these combine to a class

\[
    \tilde{\beta}_E^* \in \text{KK}R_0(C_0(X), C_0(\mathbb{Z}/2) \otimes Cl_E).
\]

This class is also invertible and produces a raw form of the Thom isomorphism.

**Definition 3.1.** A KR-orientation on a \( \mathbb{Z}/2 \)-equivariant vector bundle \( E \to X \) is a \( \mathbb{Z}/2 \)-graded, “real”, \( C_0(X) \)-linear Morita equivalence between \( C_0(X, Cl_E) \) and \( C_0(X) \otimes Cl_{a,b} \) for some \( a, b \in \mathbb{N} \). That is, it is a full Hilbert bimodule \( M \) over \( C_0(X, Cl_E) \) and \( C_0(X) \otimes Cl_{a,b} \) with a compatible “real” structure and \( \mathbb{Z}/2 \)-action and with the extra property that the two actions of \( C_0(X) \) by multiplication on the left and right are the same. A vector bundle is called KR-orientable if it has such a KR-orientation. We call \( a - b \) mod \( 8 \) \( \mathbb{Z}/8 \) the KR-dimension \( \dim_{KR} E \) of \( E \).

Morita equivalence for “real” \( \mathbb{Z}/2 \)-graded \( C^* \)-algebras with an action of a locally compact groupoid is explored by Moutuou [17]. Our definition is the special case where the groupoid is the space \( X \) with only identity arrows. The idea to describe K-theory orientations through Morita equivalence goes back to Plymen [18].

Two Clifford algebras \( Cl_{a,b} \) and \( Cl_{a',b'} \) are Morita equivalent as “real” \( \mathbb{Z}/2 \)-graded \( C^* \)-algebras if and only if \( a - b \equiv a' - b' \) mod \( 8 \); this is part of the computation of the Brauer group of the point in [17, Appendix A]. This shows that the KR-dimension is well defined and that it suffices to consider, say, the cases \( b = 0, a \in \{0, 1, \ldots, 7\} \) in Definition 3.1. A trivial vector bundle \( X \times \mathbb{R}^{a,b} \to X \) is, of course, KR-oriented because \( Cl_{X \times \mathbb{R}^{a,b}} = X \times Cl_{a,b} \). Its KR-dimension is \( a - b \) mod \( 8 \).

If \( a = b = 0 \), then a KR-orientation is a \( \mathbb{Z}/2 \)-graded, “real” Morita equivalence between \( C_0(X, Cl_E) \) and \( C_0(X) \). This is equivalent to a \( \mathbb{Z}/2 \)-graded “real” Hilbert \( C_0(X) \)-module \( S \) with a \( C_0(X) \)-linear isomorphism \( k(S) \cong C_0(X, Cl_E) \). This forces \( S \) to be the space of \( C_0 \)-sections of a complex vector bundle over \( X \), equipped with a \( \mathbb{Z}/2 \)-grading and a “real” involution. Then the isomorphism \( k(S) \cong C_0(X, Cl_E) \) means that the fibre \( S_x \) of this vector bundle carries an irreducible representation of \( Cl_{E_x} \). Thus \( S \) is a “real” spinor bundle for \( E \). Allowing \( a, b \neq 0 \) gives appropriate analogues of the spinor bundle for “real” vector bundles of all dimensions.
The KR-orientation induces a KKR-equivalence because KKR is Morita invariant. Together with the KKR-equivalence in \(\mathbb{T}e \in \text{KKR}_0(C_0(X), C_0([E]) \otimes \text{Cl}_{a,b}) \cong \text{KKR}^{\dim_{\text{KR}} E}(C_0(X), C_0([E])),\) called the Thom isomorphism class. It induces Thom isomorphisms in K-theory

\[ \text{KR}^n(X) \cong \text{KR}^{n+\dim_{\text{KR}} E}(E) \quad \text{for } n \in \mathbb{Z}/8. \]

**Example 3.2.** Let \(E \to X\) be a complex vector bundles with a “real” structure. A Thom isomorphism for \(E\) is defined in [1] Theorem 2.4. It is analogous to the Thom isomorphism for complex vector bundles in ordinary complex K-theory. Atiyah’s Thom isomorphism may also be defined using a KR-orientation of dimension 0 as defined above. Namely, the sum of the complex exterior powers of \(E\) provides a spinor bundle for \(E\), which also carries a canonical “real” structure to become a KR-orientation of KR-dimension 0 as in Definition 3.1.

**Lemma 3.3.** Let \(V_1, V_2\) be two \(\mathbb{Z}/2\)-equivariant vector bundles over \(X\). If two of the vector bundles \(V_1, V_2, V_1 \oplus V_2\) are KR-oriented, then this induces a canonical KR-orientation on the third one, such that

\[ \dim_{\text{KR}} V_1 + \dim_{\text{KR}} V_2 = \dim_{\text{KR}} (V_1 \oplus V_2). \]

**Proof.** A \(C_0(X)\)-linear Morita equivalence is the same as a continuous bundle over \(X\) whose fibres are Morita equivalences. Such bundles may be tensored together on \(X\). With the graded tensor product over \(X\), we get \(\text{Cl}_{V_1 \oplus V_2} \cong \text{Cl}_{V_1} \otimes \text{Cl}_{V_2}\). Therefore, KR-orientations for \(V_1\) and \(V_2\) induce one for \(V_1 \oplus V_2\). Assume, conversely, that \(V_1 \oplus V_2\) and \(V_2\) are KR-oriented. Then the bundle \(\text{Cl}_{V_2}\) is Morita equivalent to the trivial bundle with fibre \(\text{Cl}_{0,b}\) for some \(a, b \in \mathbb{N}\). So \(\text{Cl}_{V_2} \otimes \text{Cl}_{b,a}\) is Morita equivalent to a trivial bundle of matrix algebras, making it Morita equivalent to the trivial rank-1 bundle \(X \times \mathbb{C}\). Therefore, \(\text{Cl}_{V_1}\) is Morita equivalent to \(\text{Cl}_{V_2} \otimes \text{Cl}_{b,a} \cong \text{Cl}_{V_1 \oplus V_2} \otimes \text{Cl}_{b,a}\). Since \(V_1 \oplus V_2\) is KR-oriented, this is also Morita equivalent to some trivial Clifford algebra bundle. This gives a KR-orientation on \(V_1\).

The following proposition clarifies how many KR-orientations a KR-orientable vector bundle admits.

**Proposition 3.4.** Two KR-orientations for the same \(\mathbb{Z}/2\)-equivariant vector bundle become isomorphic after tensoring one of them with a \(\mathbb{Z}/2\)-graded “real” complex line bundle \(L \to X\), which is determined uniquely.

**Proof.** Let \(M_1\) and \(M_2\) be two \(C_0(X)\)-linear “real” graded Morita equivalences between \(C_0(X, \text{Cl}_E)\) and \(C_0(X) \otimes \text{Cl}_{a,b}\). Then the composite of \(M_1\) and the inverse of \(M_2\) is a \(C_0(X)\)-linear “real” graded Morita self-equivalence of \(C_0(X) \otimes \text{Cl}_{a,b}\). Taking the exterior product with \(\text{Cl}_{a,b}\) is clearly bijective on isomorphism classes of Morita self-equivalences if \(a = b\) because then \(\text{Cl}_{a,b}\) is a matrix algebra. If \(a \neq b\), then it is also bijective on isomorphism classes because when we tensor first with \(\text{Cl}_{a,b}\) and then with \(\text{Cl}_{b,a}\), we tensor with \(\text{Cl}_{a+b,a+b}\), which is already known to be bijective on isomorphism classes. Therefore, we may replace the self-equivalence of \(C_0(X) \otimes \text{Cl}_{a,b}\) by one of \(C_0(X)\). Such a Morita self-equivalence of \(C_0(X)\) is well known to be just a complex line bundle \(L\). In our case, the line bundle must also carry a “real” involution and a \(\mathbb{Z}/2\)-grading, of course. Unravelling the bijections on isomorphism classes, we see that \(M_2\) is isomorphic to \(L \otimes M_1\).

**Remark 3.5.** Assume that \(X\) is connected. Let \(L \to X\) be a “real” complex line bundle. Then there are two ways to define a \(\mathbb{Z}/2\)-grading on \(L\): we may declare all of \(L\) to have parity 0 or 1. When we tensor a Morita equivalence \(M\) with a line bundle of negative parity, we flip the \(\mathbb{Z}/2\)-grading on \(M\). Clearly, we may always
flip the \( \mathbb{Z}/2 \)-grading on a Morita equivalence and get another Morita equivalence. This operation is called orientation-reversal.

4. Representable KR-theory and KR-theory with supports

We are going to construct a geometric bivariant KR-theory for spaces with a “real” involution. We follow [6,7], where equivariant geometric bivariant \( K \)-theory is developed for spaces with a groupoid action. While it is mentioned there that the theory also works for KO-theory, the more general KR-theory is not mentioned there explicitly. Nevertheless, the theory developed there applies because KR-theory comes from a \( \mathbb{Z}/2 \)-equivariant cohomology theory. This is checked in some detail in [10]. The proof is similar to the proof of the same result for \( \mathbb{Z}/2 \)-equivariant KO-theory, which only differs in that the group \( \mathbb{Z}/2 \) acts linearly instead of conjugate-linearly. Therefore, we do not repeat the proof here.

We clarify, however, how to define the relevant cohomology theory because we will use this later anyway. KR-theory, like ordinary \( K \)-theory, is not a cohomology theory because it is only functorial for proper continuous maps. The cohomology theory from which \( K \)-theory comes is called representable \( K \)-theory. Adapting the approach in [5] for groupoid-equivariant \( K \)-theory to the “real” case, we define the representable analogue of KR-theory as

\[
KR^X_a(X) := KKR^X_a(C_0(X), C_0(X));
\]

the right hand side means the \( C_0(X) \)-linear “real” version of Kasparov theory, which Kasparov [14] denotes by \( RKK \).

The following result is shown in [10] and is what is needed to apply the machinery developed in [6,7] to KR-theory:

**Theorem 4.1** (10). Representable KR-theory is a multiplicative \( \mathbb{Z}/2 \)-equivariant cohomology theory, and KR-oriented vector bundles are oriented for it.

Our notation for representable KR-theory alludes to a further generalisation, namely, the KR-theory \( KR^*_Z(X) \) of a space \( X \) with \( Z \)-compact support, given a map \( b: X \to Z \). By analogy to [5], we define this by

\[
KR^*_Z(X) := KKR^*_Z(C_0(Z), C_0(X)).
\]

If \( Z = \text{pt} \), then \( KR^*_Z = KR \) is the KR-theory as defined above. Representable KR-theory is the special case where \( Z = X \) and \( b \) is the identity map. More generally, if \( b \) is proper, then \( KR^*_Z(X) = KR^*_X(X) \) is the representable KR-theory of \( X \). In particular, if \( X \) is compact, then \( KR^*_Z(X) = KR^*_X(X) \) for all \( b: X \to Z \).

Let \( X \) be a \( \mathbb{Z}/2 \)-manifold. Then any “real” complex vector bundle over \( X \) defines a class in \( KR^*_X(X) \). With our Kasparov theory definition, this is the Hilbert \( C_0(X) \)-module of sections of \( X \) with \( C_0(X) \) acting by pointwise multiplication also on the left. This only defines a class in the usual \( KR^0(X) \) if \( X \) is compact. If \( X \) is compact, we also know that \( KR^0(X) = KR^0_X(X) \) is the Grothendieck group of the monoid of such “real” vector bundles.

For a finite-dimensional CW-complex \( X \), it is well known that its representable \( K \)-theory and KO-theory are the Grothendieck groups of the monoids of complex and real vector bundles over \( X \). The analogous result for \( \mathbb{Z}/2 \)-equivariant \( K \)-theory is false, however, as shown by the counterexample in [16], Example 3.11]. This counterexample for \( K^\mathbb{Z}/2 \), however, does not work like this for KR-theory. Therefore, it is possible that the representable KR-theory of a finite-dimensional \( \mathbb{Z}/2 \)-CW-complex is always isomorphic to the Grothendieck group of the monoid of “real” complex vector bundles over \( X \). We have not investigated this question.

More generally, let us add a \( \mathbb{Z}/2 \)-map \( b: X \to Z \). Choose two “real” complex vector bundles \( E_{\pm} \) over \( X \) together with an isomorphism \( \varphi: E_+|_{X\setminus A} \cong E_-|_{X\setminus A} \).
for an open subset \( A \subseteq X \) whose closure is \( Z \)-compact in the sense that \( b_{\overline{A}} : \overline{A} \to Z \) is proper. We may, of course, equip \( E_{\pm} \) with inner products. We may arrange these so that \( \varphi \) is unitary. Using the Tietze Extension Theorem, we may extend \( \varphi \) to a continuous section \( \tilde{\varphi} \) of norm 1 of the vector bundle \( \text{Hom}(E_+, E_-) \) on all of \( X \). Then we define a cycle for \( \text{KKR}_Z^0(C_0(Z), C_0(X)) \) as follows. The Hilbert module consists of the space of sections of \( E_+ \oplus E_- \) with the \( \mathbb{Z}/2 \)-grading induced by this decomposition and with the “real” structures induced by the “real” structures on \( E_{\pm} \).

A function \( h \in C_0(Z) \) acts on this by pointwise multiplication with \( h \circ \varphi \). The Fredholm operator is pointwise multiplication with \( \tilde{\varphi} \). This is indeed a cycle for \( \text{KKR}_Z^0(C_0(Z), C_0(X)) \) because \( \tilde{\varphi} \) is unitary outside a \( Z \)-compact subset.

With the definition in (9), it becomes obvious that a class \( \xi \in \text{KR}_Z^0(X) \) yields an element of \( [\xi] \in \text{KKR}_{-n}(C_0(Z), C_0(X)) \), which induces maps \( \xi : \text{KR}^a(Z) \to \text{KR}^{a+n}(X) \) for \( a \in \mathbb{Z}/2 \). This generalises both the pull back functoriality for continuous proper maps \( b : X \to Z \) and the map on \( \text{KR}^*(X) \) that multiplies with a vector bundle on \( X \).

### 5. Wrong-Way Functoriality of KR-theory

In the following, we shall specialise some of the theory in [6,7] and simplify it a bit for our more limited purposes. First, we take the groupoid denoted \( \mathcal{G} \) in [6,7] to be the group \( \mathbb{Z}/2 \). This is because a “real” structure is the same as a \( \mathbb{Z}/2 \)-action. So the object space of \( \mathcal{G} \), which is denoted \( Z \) in [6,7], is just the one-point space \( \text{pt} \).

Thus fibre products over \( Z \) become ordinary products.

Secondly, we work in the smooth setting, that is, with smooth manifolds without boundary, and with smooth maps only. We also assume smooth manifolds to be finite-dimensional, which is automatic if they are connected. We briefly call a finite-dimensional smooth manifold with a smooth \( \mathbb{Z}/2 \)-action a \( \mathbb{Z}/2 \)-manifold.

**Remark 5.1.** Swan’s Theorem says that any vector bundle over a paracompact space of finite covering dimension is a direct summand in a trivial vector bundle. This basic result may fail for groupoid-equivariant K-theory, even when the spaces are compact and the groupoid is a bundle of Lie groups (see [6] Example 2.7). This creates the need to speak of “subtrivial” equivariant vector bundles in the general setting considered in [6,7]. However, for the finite group \( \mathbb{Z}/2 \), any \( \mathbb{Z}/2 \)-equivariant vector bundle over a \( \mathbb{Z}/2 \)-manifold is subtrivial by [6] Theorem 3.11]. We use this occasion to point out that the hypotheses in that theorem are wrong: it should be assumed that the space \( Y \) and not \( X \) is finite-dimensional. Any smooth finite-dimensional manifold \( X \) with smooth \( \mathbb{Z}/2 \)-action has a structure of finite-dimensional \( \mathbb{Z}/2 \)-CW-complex. In the following, we may therefore drop the adjective “subtrivial” as long as we restrict attention to \( \mathbb{Z}/2 \)-manifolds.

**Remark 5.2.** Since \( \mathbb{Z}/2 \) is a finite group, its regular representation is a “full vector bundle” over \( \text{pt} \). Such a vector bundle is needed for several results in [6,7], and it comes for free in our case. Any linear representation of \( \mathbb{Z}/2 \) is isomorphic to a direct sum of copies of the two characters of \( \mathbb{Z}/2 \). That is, it is isomorphic to \( \mathbb{R}^a,b \) where the generator of \( \mathbb{Z}/2 \) acts by the “real” involution on that space.

After these preliminary remarks, we construct wrong-way functoriality or shriek maps for KR-oriented smooth maps. This is based on factorising smooth maps in a certain way. The factorisation is called a normally nonsingular map in [6,7]. Under the extra assumptions that we impose, any smooth map has such a factorisation and it is unique up to “smooth equivalence”; this implies that the shriek map is independent of the factorisation. So the factorisation becomes irrelevant in the special case that we consider here. This fails already for smooth maps between smooth manifolds with boundary (see [6] Example 4.7). So the theory of normally nonsingular maps
is needed to treat this more general class of spaces. Our applications, however, concern only manifolds without boundary. Therefore, we will define KR-oriented correspondences only in this special case to simplify the theory. Nevertheless, we include the basic definition of a normally nonsingular map to clarify what would be needed to extend the theory to more general spaces than \( \mathbb{Z}/2 \)-manifolds.

**Definition 5.3.** Let \( X \) and \( Y \) be \( \mathbb{Z}/2 \)-manifolds. A (smooth) normally nonsingular \( \mathbb{Z}/2 \)-map from \( X \) to \( Y \) consists of the following data:

- \( V \), a \( \mathbb{Z}/2 \)-equivariant \( \mathbb{R} \)-vector bundle over \( X \);
- \( E = \mathbb{R}^{a,b} \) for some \( a, b \in \mathbb{N} \), an \( \mathbb{R} \)-linear representation of the group \( \mathbb{Z}/2 \), which we treat as a \( \mathbb{Z}/2 \)-equivariant vector bundle over \( pt \);
- \( \hat{f} : |V| \to Y \times \mathbb{R}^{a,b} \), a \( \mathbb{Z}/2 \)-equivariant diffeomorphism between the total space of \( V \) and an open subset of \( Y \times \mathbb{R}^{a,b} = Y \times pt \{ E \} \).

Here \( |V| \) is the total space of the vector bundle \( V \). Let \( \zeta_V : X \to |V| \) be the zero section of \( V \) and let \( \pi_V : Y \times \mathbb{R}^{a,b} \to Y \) be the coordinate projection. The trace of \( (V, E, \hat{f}) \) is the dotted composite map

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\zeta_V} & & \downarrow{\pi_V} \\
|V| & \xrightarrow{\hat{f}} & Y \times \mathbb{R}^{a,b}.
\end{array}
\]

(10)

The following definition describes two ways to change a normally nonsingular map. The important feature is that they do not change the resulting shriek map.

**Definition 5.4.** Let \( (V, E, \hat{f}) \) be a normally nonsingular map and let \( E_0 \) be another linear representation of \( \mathbb{Z}/2 \). Then \( (V \oplus (X \times E_0), E \oplus E_0, \hat{f} \times \text{id}_{E_0}) \) is another normally nonsingular map, called a lifting of \( (V, E, \hat{f}) \). A normally nonsingular map from \( X \times [0,1] \) to \( Y \times [0,1] \) where the map \( \hat{f} \) is a map over \( [0,1] \) is called an isotopy between the two normally nonsingular maps that arise by restricting to the end points of \([0,1]\). Two smooth normally nonsingular maps are called smoothly equivalent if they have liftings that are isotopic.

**Proposition 5.5** ([6] Theorems 3.25 and 4.36). Any smooth \( \mathbb{Z}/2 \)-map is the trace of a normally nonsingular map, which is unique up to smooth equivalence. Two smooth \( \mathbb{Z}/2 \)-maps are smoothly homotopic if and only if their lifts to normally nonsingular maps are smoothly equivalent.

We recall how to lift a smooth \( \mathbb{Z}/2 \)-map \( f : X \to Y \) to a normally nonsingular map. There is a \( \mathbb{Z}/2 \)-equivariant, proper embedding \( i : X \to \mathbb{R}^{a,b} \) for some \( a, b \in \mathbb{N} \). Let \( E := \mathbb{R}^{a,b} \. The map \( (f, i) : X \to Y \times \mathbb{R}^{a,b} \) is still a \( \mathbb{Z}/2 \)-equivariant, proper embedding. Let \( V \) be its normal bundle. By the Tubular Neighbourhood Theorem, the total space of \( V \) is \( \mathbb{Z}/2 \)-equivariantly diffeomorphic to a neighbourhood of the image of \((f, i)\). This gives the open embedding \( \hat{f} \) for a normally nonsingular map.

**Definition 5.6.** Let \( f : X \to Y \) be a smooth \( \mathbb{Z}/2 \)-map. A stable normal bundle for \( f \) is a vector bundle \( V \to X \) such that \( V \oplus TX \) is isomorphic to \( f^*(TY) \oplus (X \times \mathbb{R}^{a,b}) \) for some \( a, b \in \mathbb{N} \). A KR-orientation for \( f \) is a stable normal bundle \( V \) for \( f \) together with a KR-orientation of \( V \). Its KR-dimension is the difference of the KR-dimension of \( V \) and \( a - b \mod 8 \), the KR-dimension of the trivial bundle \( X \times \mathbb{R}^{a,b} \to X \).

**Lemma 5.7.** Let \( (V, E, \hat{f}) \) be a normally nonsingular \( \mathbb{Z}/2 \)-map and let \( f : X \to Y \) be its trace. Then a KR-orientation for \( f \) is equivalent to a KR-orientation of the vector bundle \( V \).
Proof. The bundle $V$ is a stable normal bundle for $f$. Therefore, a KKR-orientation of $V$ induces one for $f$. Any two stable normal bundles of $f$ are stably isomorphic. Thus Lemma 4.3 implies that a KKR-orientation for one stable normal bundle of $f$ induces KKR-orientations on all other stable normal bundles of $f$. 

**Definition 5.8.** A KKR-orientation on a $\mathbb{Z}/2$-manifold $X$ is a KKR-orientation on its tangent vector bundle $TX$. The KKR-dimension $\dim_{\text{KR}} X$ is defined as $\dim_{\text{KR}} TX$.

If $X$ and $Y$ are two KR-oriented $\mathbb{Z}/2$-manifolds, then any smooth map $f: X \to Y$ inherits a KKR-orientation by Lemma 5.3 and its KKR-dimension is

$$\dim_{\text{KR}} f = \dim_{\text{KR}} Y - \dim_{\text{KR}} X.$$

**Example 5.9.** We are going to describe KKR-orientations on the $\mathbb{Z}/2$-manifolds $\mathbb{R}^{a,b}$, $S^{a,b}$, and $T^d$. The tangent bundle of $\mathbb{R}^{a,b}$ is the trivial bundle with fibre $\mathbb{R}^{0,d}$. So its Clifford algebra bundle is already isomorphic to the trivial Clifford algebra bundle with fibre $C_{a,b}$.

The covering $\mathbb{R} \to T$, $t \mapsto \exp(it)$, becomes a “real” covering $\mathbb{R}^{0,1} \to T$. This induces an isomorphism between the tangent bundle of $T$ and the trivial bundle with fibre $\mathbb{R}^{0,1}$. Taking a $d$-fold product, we get an isomorphism from the tangent bundle of $T^d$ to the trivial bundle with fibre $\mathbb{R}^{0,d}$. This induces an isomorphism between the Clifford algebra bundle of $T^d$ and the trivial bundle $T^d \times C_{a,b}$.

The outward pointing radial vector field $\partial/\partial r$ on the unit sphere spans the normal bundle of the inclusion map $S^{a,b} \to \mathbb{R}^{a,b}$. Since the involution on $\mathbb{R}^{a,b}$ is linear, it preserves this vector field, that is, the normal bundle is $\mathbb{Z}/2$-equivariantly isomorphic to the trivial bundle with fibre $\mathbb{R}^{1,0}$. So $T\mathbb{R}^{a,b} \oplus (S^{a,b} \times \mathbb{R}^{1,0}) \cong S^{a,b} \times \mathbb{R}^{a,b}$. Then Lemma 3.3 gives a KKR-orientation on $T \mathbb{R}^{a,b}$. We also find

$$\dim_{\text{KR}}(\mathbb{R}^{a,b}) = a - b, \quad \dim_{\text{KR}}(T^d) = -d, \quad \dim_{\text{KR}}(S^{a,b}) = a - b - 1,$$

where the three numbers are understood as elements of $\mathbb{Z}/8$. Notice that the above argument for $S^{a,b}$ also works for $a = 0$.

Let $(V, E, \hat{f})$ be a KR-oriented, normally nonsingular $\mathbb{Z}/2$-map and let $f: X \to Y$ be its trace. Then there are Thom isomorphisms both for the vector bundle $V \to X$ and the trivial vector bundle $Y \times \mathbb{R}^{a,b} \to Y$. The open inclusion $\hat{f}$ identifies $C_0([V])$ with an ideal in $C_0(Y \times \mathbb{R}^{a,b})$. Thus each of the solid maps in (10) induces a map in KKR-theory. Taking the composite gives induced maps

$$f_1: \text{KR}^{n}(X) \to \text{KR}^{n+\dim_{\text{KR}} f}(Y) \quad \text{for } n \in \mathbb{Z}/8.$$

Since the Thom isomorphisms come from KKR-classes, the map $f_1$ is the Kasparov product with an element in $\text{KKR}_{\text{dim}_{\text{KR}} f}(C_0(X), C_0(Y))$, which we will also denote by $f_1$. It turns out that $f_1$ is preserved under lifting and isotopy and is functorial in the sense that

$$f_1 \circ g = (f \circ g)_1,$$

for two composable $\mathbb{Z}/2$-maps $f, g$ (see [6]). In particular, by Proposition 5.5, $f_1$ only depends on the smooth map $f$ and its KKR-orientation, justifying the notation $f_1$. The map $f_1$ is called the shriek map of $f$, and the map that sends $f$ to $f_1$ is the wrong-way functoriality of KKR-theory.

**Example 5.10.** Let $E \to X$ be a KR-oriented vector bundle. Then the zero section $\zeta_E: X \to E$ and the bundle projection $\pi_E: E \to X$ are KR-oriented in a canonical way, such that their shriek maps are the Thom isomorphism and its inverse $\text{KR}^{n}(X) \leftrightarrow \text{KR}^{n+\dim_{\text{KR}} E}(E)$.

**Example 5.11.** Let $X$ be a $\mathbb{Z}/2$-manifold. Let $TX$ be its tangent bundle. As a complex manifold, it carries a canonical KR-orientation of KR-dimension 0 (see Example 3.2). This induces a KKR-orientation on the constant map $f: TX \to \text{pt}$.
to the one-point space. The shriek map $f_!: \text{KR}^0(TX) \to \text{KR}^0(\text{pt}) = \mathbb{Z}$ is the Atiyah–Singer topological index map on $X$.

**Remark 5.12.** If $f$ is a smooth submersion, then $f_!$ has an analytic variant $f_{!, \text{an}}$, given by the class in Kasparov theory of the family of Dirac operators along the fibres of $f$. This analytic version is equal to the topological one, that is, $f_{!, \text{an}} = f_!$ holds in $KK_{-\dim X}(\mathcal{C}_0(X), \mathcal{C}_0(Y))$. The proof is the same as for [7] Theorem 6.1, which deals with the analogous statement in KK, that is, when we forget the “real” structures. This is equivalent to the families version of the Atiyah–Singer index theorem for the family of Dirac operators along the fibres of the submersion $f$.

We are particularly interested in the shriek maps for an inclusion $f : \text{pt} \to X$, where pt denotes the one-point space with the trivial “real” structure. So $f(\text{pt}) \in X$ must be a fixed point of the “real” involution on $X$. The map $f_!$ is KR-orientable because any vector bundle over a point is trivial and thus KR-orientable. A KR-orientation for $X$ chooses a canonical KR-orientation $f_!$. Let us specialise to the case where $X = \mathbb{R}^{a,b}$ with $a > 1$, so that $S$ and $N$ are fixed points. Let $S : \text{pt} \to \mathbb{R}^{a,b}$ be the inclusion of the north pole. Give $\mathbb{R}^{a,b}$ the KR-orientation from Example 5.9. Then we get a canonical map

$$S_! : \mathbb{Z} \cong \text{KR}^0(\text{pt}) \to \text{KR}^{a-b-1}(\mathbb{S}^{a,b}) \cong DK(C(\mathbb{S}^{a,b}) \otimes \text{Cl}_{a,b}).$$

**Lemma 5.13.** There is $\chi \in \{\pm 1\}$ such that the map $S_!$ sends the generator $1 \in \mathbb{Z}$ to $\chi : ([\beta_{a,b}] - [\gamma_1]).$

**Proof.** The stereographic projection at the north pole induces a $\mathbb{Z}/2$-equivariant diffeomorphism $\mathbb{S}^{a,b} \setminus \{N\} \cong \mathbb{R}^{a-1,b}$. We could use this as a tubular neighbourhood for the inclusion of the south pole into $\mathbb{S}^{a,b}$. Therefore, $S_!$ factors through an isomorphism onto the direct summand $\text{KR}^{a-b-1}(\mathbb{S}^{a,b} \setminus \{N\}) \cong \mathbb{Z}$. Since $[\beta_{a,b}] - [\gamma_1]$ generates this summand, $S_!$ must send the generator $1 \in \mathbb{Z}$ to $\pm([\beta_{a,b}] - [\gamma_1]).$ □

The sign $\chi$ depends on the choices of KR-orientations and signs in boundary maps, and we do not compute it. It will appear in several formulas below.

The power of geometric bivariant K-theory is that there often is a simple way to compute composite maps $b_! \circ f_!$ for a KR-oriented map $f : X \to Y$ and a proper continuous map $b : Z \to Y$. We will use this to compute the pullback of $[\beta_{a,d}]$ to the van Daele K-theory of $\mathbb{T}^d$ using only geometric considerations. This circumvents rather messy Chern character computations in the physics literature (see [9] [20]), and it also works in the real case. The following proposition makes precise when and how we may compute $b_! \circ f_!$.

**Proposition 5.14.** Let $X$ and $Y$ be $\mathbb{Z}/2$-manifolds. Let $f : X \to Y$ be a KR-oriented smooth $\mathbb{Z}/2$-map and let $b : Z \to Y$ be a proper smooth $\mathbb{Z}/2$-map. Assume that $b$ and $f$ are transverse. Then the fibre product $X \times_Y Z$ is a smooth $\mathbb{Z}/2$-manifold. The projection $\pi_X : X \times_Y Z \to X$ is proper, and $\pi_Z : X \times_Y Z \to Z$ inherits from $f$ a KR-orientation of the same KR-dimension. The following diagram commutes:

$$\begin{array}{ccc}
\text{KR}^n(X) & \xrightarrow{f_!} & \text{KR}^{n+\dim_{KR} f}(Y) \\
\pi_X^! & \downarrow & \downarrow b^!
\text{KR}^n(X \times_Y Z) & \xrightarrow{(\pi_Z)_!} & \text{KR}^{n+\dim_{KR} f}(Z).
\end{array}$$

**Proof.** This is a special case of [7] Theorem 2.32] about composing KR-oriented correspondences. We will discuss a more general result later. Here we only describe the KR-orientation that the projection $\pi_Z$ inherits. The transversality assumption implies that there is an exact sequence of vector bundles over $X \times_Y Z$ as follows:

$$T(X \times_Y Z) \to \pi_X^!(TX) \oplus \pi_Z^!(TZ) \to \pi^!(TY).$$
Since any such extension splits $\mathbb{Z}/2$-equivariantly, it follows that
\[
\pi^*(TY) \oplus T(X \times_Y Z) \cong \pi_X^*(TX) \oplus \pi_Z^*(TZ).
\]

Since $f$ is KR-oriented, there are $a, b \in \mathbb{N}$ and a KR-oriented vector bundle $V \to X$ such that $f^*(TY) \oplus (X \times \mathbb{R}^{a,b}) \cong TX \oplus V$. Then $\pi^*(TY) \oplus (X \times_Y Z \times \mathbb{R}^{a,b}) \cong \pi_X^*(TX) \oplus \pi_Z^*(TV) \oplus T(X \times_Y Z)$.

Since the $\mathbb{Z}/2$-equivariant vector bundle $\pi_X^*(TX)$ is subtrivial, we may add another $\mathbb{Z}/2$-equivariant vector bundle to arrive at an isomorphism
\[
\pi_Z^*(TV) \oplus (X \times_Y Z \times \mathbb{R}^{a',b'+b''}) \cong (X \times_Y Z \times \mathbb{R}^{a',b'}) \oplus \pi_X^*(TV) \oplus T(X \times_Y Z).
\]

Thus $(X \times_Y Z \times \mathbb{R}^{a',b'}) \oplus \pi_X^*(TV)$ is a stable normal bundle for $\pi_Z$. It inherits a KR-orientation from the obvious KR-orientation on the trivial bundle and the given KR-orientation of $V$ by Lemma 3.3.

We will later need the special case when $f$ is the inclusion of a point $y_0 \in Y$, so that $X = pt$. A KR-orientation of $f$ is the same as a KR-orientation on the tangent space $T_{y_0}Y$. In this case, the coordinate projection $\pi_Z$ is a bijection between $X \times_Y Z$ and the preimage $b^{-1}(y_0)$ of $y_0$ in $Z$. Transversality means in this case that the differential of $b$ is surjective in all points of $b^{-1}(y_0)$. Then $b^{-1}(y_0)$ is a smooth submanifold. It is also compact because $b$ is proper. The normal bundle of the inclusion of $b^{-1}(y_0)$ into $Z$ is canonically isomorphic to the trivial bundle with fibre $T_{y_0}Y$. This gives the induced KR-orientation of $\pi_Z$.

The commuting diagram (12) is best understood in the setting of geometric bivariant KR-theory. This theory describes $\text{KKR}_+(C_0(X), C_0(Y))$ for two $\mathbb{Z}/2$-manifolds $X$ and $Y$ in a geometric way. Its main ingredients are

\[ b_* \in \text{KKR}_0(C_0(Y), C_0(Z)), \quad f^! \in \text{KKR}_{-\dim_{\text{Kr}}(f)}(C_0(X), C_0(Y)) \]

for proper smooth maps $b: Z \to Y$ and KR-oriented smooth maps $f: X \to Y$. The details are explained in the next section.

6. Geometric bivariant KR-theory

We now define KR-oriented correspondences between two $\mathbb{Z}/2$-manifolds $X$ and $Y$ as in [7]. These produce a geometric version of bivariant “real” Kasparov theory. The definition of a correspondence in [7] differs slightly from the original definition of correspondences by Connes and Skandalis in [4, §III] by allowing maps $b$ that fail to be proper. This greatly simplifies the proof that the geometric and analytic bivariant KR-theories agree. We shall use smooth maps, whereas the definition in [7] uses normally nonsingular maps. This makes no difference for $\mathbb{Z}/2$-manifolds because of Proposition 5.5.

Definition 6.1. Let $X$ and $Y$ be $\mathbb{Z}/2$-manifolds. A (smooth) KR-oriented correspondence from $X$ to $Y$ is a quadruple $(M, b, f, \xi)$, where

- $M$ is a $\mathbb{Z}/2$-manifold,
- $b: M \to X$ is a smooth $\mathbb{Z}/2$-map,
- $f: M \to Y$ is a KR-oriented smooth $\mathbb{Z}/2$-map, and
- $\xi \in \text{KR}_X^3(M)$ for some $a \in \mathbb{Z}/8$; here $X$-compact support in $M$ refers to the map $b: M \to X$.
The KR-dimension of the correspondence is defined as $a + \dim_{KR}(f)$. We often depict a correspondence as

\[
(M, \xi) \quad \xleftarrow{b} X \xrightarrow{f} Y.
\]

The letters $f$ and $b$ in the definition stand for “forwards” and “backwards”.

**Example 6.2.** A proper, smooth $\mathbb{Z}/2$-map $b: Y \to X$ yields the following correspondence from $X$ to $Y$:

\[
(X, 1) \quad \xleftarrow{id_X} X \xrightarrow{id_Y} Y.
\]

Here $id_Y$ is the identity map with its canonical KR-orientation, which is indeed a unit arrow in the category of KR-oriented smooth maps, and $1 \in KR^0_X(Y)$ comes from the trivial “real” vector bundle of rank 1.

**Example 6.3.** A KR-oriented smooth $\mathbb{Z}/2$-map $f: X \to Y$ yields a KR-oriented correspondence from $X$ to $Y$:

\[
(X, 1) \quad \xleftarrow{id_X} X \xrightarrow{id_Y} Y.
\]

**Example 6.4.** Any class $\xi$ in the representable KR-theory $KR^\ast_X(X)$ of $X$ yields a correspondence from $X$ to itself:

\[
(X, \xi) \quad \xleftarrow{id_X} X \xrightarrow{id_Y} X.
\]

We shall mainly consider correspondences where the map $b$ is proper and $\xi$ is the unit element in $KR^0_X(X)$ as in Example 6.2.

A KR-oriented correspondence $(M, b, f, \xi)$ induces an element

\[
f \circ [\xi] \in KK^{p-a+\dim_{KR}(f)}(C_0(X), C_0(Y)),
\]

which induces maps

\[
f \circ \xi: KR^p(X) \to KR^{p+a+\dim_{KR}(f)}(Y)
\]

for all $p \in \mathbb{Z}/8$.

For instance, the correspondence in Example 6.2 gives the class of the $^*$-homomorphism $b^*: C_0(X) \to C_0(Y)$ in $KK_0(C_0(X), C_0(Y))$; the correspondence in Example 6.3 gives $f$; the correspondence in Example 6.4 gives the image of $\xi$ in $KK_0(C_0(X), C_0(X))$ under the forgetful map.

The geometric bivariant KR-theory $\widetilde{KR}^\ast(X, Y)$ is defined as the set of equivalence classes of KR-oriented correspondences, where “equivalence” means the equivalence relation generated by bordism (see [7, Definition 2.7]) and Thom modification (see [7, Definition 2.8]), which replaces $M$ in a correspondence by the total space of a KR-oriented vector bundle over it. We shall not use the precise form of these relations below and therefore do not repeat them here. It is shown in [7] that the disjoint union of correspondences makes $\widetilde{KR}^\ast(X, Y)$ an Abelian group.

**Theorem 6.5.** Let $n \in \mathbb{Z}/8$ and let $X$ and $Y$ be $\mathbb{Z}/2$-manifolds. The map above from KR-oriented correspondences to Kasparov theory defines an isomorphism

\[
\widetilde{KR}^n(X, Y) \cong KK^{-n}(C_0(X), C_0(Y)).
\]
Proof. This is shown by following the proof in [7] for equivariant KK-theory. The same arguments as in the proof of [7] Theorem 4.2 show that the map is well defined and a functor for the composition of KR-oriented correspondences defined in [7]. Next, [7] Theorem 2.25 for the \( \mathbb{Z}/2 \)-equivariant cohomology theory KR shows that the map in the theorem is bijective if \( X = pt \). The bivariant case is reduced to this easy case using Poincaré duality. Any \( \mathbb{Z}/2 \)-manifold \( X \) admits a “symmetric dual” in KR-theory because of [7] Theorem 3.17. This is another \( \mathbb{Z}/2 \)-manifold \( P \) with a smooth \( \mathbb{Z}/2 \)-map \( P \to X \) such that there are duality isomorphisms \( \mathbb{K}R^n(X, Y) \cong \mathbb{K}R^{n+a}(Y \times P) \) in geometric bivariant KR-theory. As in [7] Theorem 4.2, the geometric bivariant KR-theory classes that give this duality also give an isomorphism \( \mathbb{K}R_{-n}(C_0(X), C_0(Y)) \cong \mathbb{K}R^{n+a}(Y \times P) \). \( \square \)

Thus KR-oriented correspondences provide a purely geometric way to describe Kasparov cycles between the \( C^* \)-algebras of functions on \( \mathbb{Z}/2 \)-manifolds. An important feature is that the Kasparov product may be computed geometrically under an extra transversality assumption:

**Theorem 6.6.** Let two composable smooth KR-oriented correspondences be given, as described by the solid arrows in the following diagram:

\[
\begin{array}{ccc}
X & \overset{b}{\longrightarrow} & (M_1 \times_Y M_2, \xi) \\
& \overset{f_1}{\downarrow} & \downarrow \pi_1 \\
(M_1, \xi_1) & \longrightarrow & (M_2, \xi_2) \\
& \overset{f_2}{\downarrow} & \downarrow \pi_2 \\
& \overset{f}{\longrightarrow} & (M, \xi) \\
& \overset{b_1}{\downarrow} & \downarrow b_2 \\
Y & \longrightarrow & Z
\end{array}
\]

Assume that the smooth maps \( f_1 \) and \( b_2 \) are transverse, that is, if \( m_1 \in M_1 \), \( m_2 \in M_2 \) satisfy \( y := f_1(m_1) = b_2(m_2) \), then \( Df_1(T_{m_1}M_1) + Db_2(T_{m_2}M_2) = T_yY \). Then the following hold. First, \( M_1 \times_Y M_2 \) is a smooth \( \mathbb{Z}/2 \)-manifold. Secondly, the exterior tensor product \( \xi := \pi_1^*(\xi_1) \otimes_Y \pi_2^*(\xi_2) \) has \( X \)-compact support with respect to \( b := b_1 \circ \pi_1 \), that is, it belongs to \( \mathbb{K}R_X^{*}(M_1 \times_Y M_2) \). Thirdly, the composite map \( f := f_2 \circ \pi_2 \) inherits a KR-orientation from \( f_1 \) and \( f_2 \), whose KR-dimension is \( \dim_{\mathbb{K}R} f = \dim_{\mathbb{K}R} f_1 + \dim_{\mathbb{K}R} f_2 \). Finally, the resulting KR-oriented correspondence \( (M_1 \times_Y M_2, b, f, \xi) \) is the composite of the two given KR-oriented correspondences, that is, its image in Kasparov theory is the Kasparov product of the Kasparov theory images of \( (M_1, b_1, f_1, \xi_1) \) and \( (M_2, b_2, f_2, \xi_2) \).

Proof. In [7], the composition of correspondences is first defined in a special case. Any KR-oriented correspondence is equivalent to one where the forward map is the restriction of the coordinate projection \( Y \times \mathbb{R}^{\omega,b} \to Y \) to an open subset \( M \subseteq Y \times \mathbb{R}^{\omega,b} \) (see [4] Theorem 2.24). Then \( f \) is a submersion and hence transverse to any smooth map. The composition of “special” correspondences gives geometric bivariant KR-theory a category structure, and the canonical map to Kasparov theory is a functor. With this preparation, the claim in our theorem mostly follows from [7] Theorem 2.32 and Example 2.31. That theorem says that the “intersection product”, which is described in the statement of the theorem, is equivalent to the composite in geometric bivariant KR-theory. The example in [4] shows that the usual transversality notion from differential geometry implies the transversality assumption that is assumed in the theorem in [7] (which makes sense for correspondences with a normally nonsingular forward map). \( \square \)

Proposition 5.14 is the special case of Theorem 6.6 where \( b_1 \) and \( f_2 \) are identity maps, \( b_2 \) is proper, and \( \xi_1 \) and \( \xi_2 \) are the units in representable KR-theory.

**Remark 6.7.** The exterior product of two KR-oriented correspondences is defined by simply taking the product of all spaces and maps and the exterior product of
the KR-theory classes involved. This defines a symmetric monoidal structure on
groupoids of Hamiltonians that have been considered in the mathematical physics
literature. They are given by the formula

\[ H_m := \frac{1}{2i} \sum_{j=1}^{d} (S_j - S_j^*) \otimes \gamma_j + \left( m + \frac{1}{2} \sum_{j=1}^{d} (S_j + S_j^*) \right) \otimes \gamma_0 \in C^*(\mathbb{Z}^d) \otimes \mathcal{C}_1.d \]

for \( m \in \mathbb{R} \) (see [19, §2.4 and §2.3.3]). Here \( S_j \in C^*(\mathbb{Z}^d) \) is the unitary for the
states of the system. Throughout this section, we start numbering Clifford generators
for \( \mathcal{C}_1.d \) at 0 and not at 1 as before. We give \( C^*(\mathbb{Z}^d) \) the trivial \( \mathbb{Z}/2 \)-grading and
the “real” involution where the generators \( S_i \) are real. Then the element \( H_m \)
is selfadjoint, odd, and real because \( \gamma_0, \gamma_1, \ldots, \gamma_d \) are real by our conventions. The
Fourier transform identifies the commutative \( C^*-\)algebra \( C^*(\mathbb{Z}^d) \) with the \( C^*-\)algebra
of continuous functions on the torus \( \mathbb{T}^d \). Our “real” structure on \( C^*(\mathbb{Z}^d) \) transforms
under this isomorphism to the “real” structure on \( \mathbb{T}^d \) in (1).

Define \( \tilde{\beta}_{1,d} : \mathbb{R}^{1,d} \to \mathcal{C}_1.d \) as in (5) and define
\[ \tilde{\varphi}_m : \mathbb{T}^d \to \mathbb{R}^{1,d}, \quad (x, y) \mapsto (x_1 + \cdots + x_d + m, y_1, \ldots, y_d). \]

This map is “real”. Since \( S_i \) and \( S_i^* \) have the Fourier transforms \( x_i \pm iy_i \), the Fourier transfrom of \( H_m \) in \( C(\mathbb{T}^d, \mathcal{C}_1.d) \) is equal to \( \tilde{\beta}_{1,d} \circ \tilde{\varphi}_m \).

**Lemma 7.1.** If \( m \notin \{-d, -d + 2, \ldots, d - 2, d\} \), then \( \tilde{\varphi}_m(\mathbb{T}^d) \subseteq \mathbb{R}^{1,d} \setminus 0 \).

**Proof.** Assume \( \tilde{\varphi}_m(x, y) = 0 \). Then \( x_1 + \cdots + x_d + m = 0 \) and \( y_1 = y_2 = \cdots = y_d = 0 \). The latter forces \( x_i = \pm 1 \) for \( i = 1, \ldots, d \), and then \( m = -\sum_{i=1}^{d} x_i \) must belong to \( \{-d, -d + 2, \ldots, d - 2, d\} \).

From now on, we assume \( m \notin \{-d, -d + 2, \ldots, d + 2, d\} \). By Lemma 7.1, this is equivalent to \( H_m \) being invertible, which is needed to define its topological phase.

Under our assumption on \( m \), there is a well defined “real” function
\[ \varphi_m : \mathbb{T}^d \to S^{1,d}, \quad \varphi_m(z) := \frac{1}{\|\tilde{\varphi}_m(z)\|} \cdot \tilde{\varphi}_m(z) \]
as in (3). Since \( \tilde{\beta}_{1,d}(x)^2 = \|x\|^2 \), the “spectral flattening” of \( H_m \) with spectrum \( \{\pm 1\} \)
is the operator whose Fourier transform in \( C(\mathbb{T}^d, \mathcal{C}_1.d) \) is \( \tilde{\beta}_{1,d} \circ \varphi_m \). Since \( \varphi_m \) takes values in \( S^{1,d} \) by construction, this is just \( \beta_{1,d} \circ \varphi_m \) with the real, odd selfadjoint unitary \( \beta_{1,d} \in C(S^{1,d}) \otimes \mathcal{C}_1.d \) in (2). The composite \( \beta_{1,d} \circ \varphi_m \in \mathcal{FU}(C(\mathbb{T}^d) \otimes \mathcal{C}_1.d) \) is its pullback along \( \varphi_m \).

To get a class in van Daele’s K-theory, we must consider a formal difference
\( [\beta_{1,d} \circ \varphi_m - f] \) for some \( f \in \mathcal{FU}(C(\mathbb{T}^d) \otimes \mathcal{C}_1.d) \). Physically, \( f \) describes the
topological phase that we choose to call “trivial”. An obvious choice in our case is
\( f = \gamma_0 \), the constant function on \( \mathbb{T}^d \) with value \( \gamma_0 \). Another obvious choice would
be \( -\gamma_0 \). In the complex case, these two are homotopic. In the “real” case, however,
these two choices turn out to have different classes in KR-theory for \( d \leq 2 \). So the
sign choice here actually matters.
Lemma 7.2. Up to the sign $\chi$ from Lemma $5.13$ the class $[\beta_{1,d} \circ \varphi_m] - [\gamma_0]$ in $\text{DK}(C(T^d) \otimes Cl_{1,d}) \cong \text{KR}^{-d}(T^d)$ is the composite of the KR-oriented correspondences

$$S_i := \left( \begin{array}{c} (pt, 1) \\ pt \end{array} \right) \xrightarrow{S_{1,i}} S_{1,d}, \quad \varphi_m^* := \left( \begin{array}{c} (T^d, 1) \\ S_{1,d} \end{array} \right).$$

Proof. The correspondence $S_i$ represents the class $\chi \cdot ([\beta_{1,d} - [\gamma_0])]$ by Lemma $5.13$. Here $\gamma_0$ denotes the constant function on $S_{1,d}$ with value $\gamma_0$, and we changed our numbering of Clifford generators to start at $0$. Composing with the correspondence denoted $\varphi_m^*$ pulls this back along the map $\varphi_m$. This gives $\chi \cdot ([\beta_{1,d} \circ \varphi_m] - [\gamma_0])$, where now $\gamma_0$ denotes the constant function on $T^d$ with value $\gamma_0$. □

Lemma 7.3. The two correspondences in Lemma 7.2 are transverse. So their composite is their intersection product. The fibre product $T^d \times_{S_{1,d}} pt$ in the intersection product is diffeomorphic to the finite subset $\varphi_m^{-1}(S) \subset T^d$. For $z \in \varphi_m^{-1}(S)$, let $\text{sign}(z)$ be the number of $-1$ among the coordinates of $z$. Then

$$[\beta_{1,d} \circ \varphi_m] - [\gamma_0] = \chi \sum_{z \in \varphi_m^{-1}(S)} (-1)^{\text{sign}(z)} z_1$$

holds in $\text{DK}(C(T^d) \otimes Cl_{1,d}) \cong \text{KR}^{-d}(T^d)$.

Proof. We must show that the differential of $\varphi_m$ is a surjective map onto $T_S S_{1,d}$ at all points $(x, y) \in T^d$ with $\varphi_m(x, y) = S$. The tangent space $T_S S_{1,d}$ is the subspace $\{0\} \times \mathbb{R}^d$, spanned by the basis vectors $e_1, \ldots, e_d$. If $\varphi_m(x, y) = S$, then $y_1 = y_2 = \cdots = y_d = 0$ follows as in the proof of Lemma 7.2. At these points, the tangent space of $T^d$ is spanned by the vectors in the directions $y_1, \ldots, y_d$. The differential of $\tilde{\varphi}_m$ maps these to the vectors $e_1, \ldots, e_d$. On the preimage of $S$, the differential of the radial projection map $\mathbb{R}^1 \setminus \{0\} \to S_{1,d}$, $z \mapsto z/||z||$, just multiplies $e_j$ for $j = 1, \ldots, d$ with a positive constant, so that the images still span $T_S S_{1,d}$. Thus $\varphi_m : T^d \to S_{1,d}$ is transverse to $S : pt \to S_{1,d}$.

The canonical map $\pi_T : T^d \times_{S_{1,d}} pt \to T^d$ is a diffeomorphism onto the closed submanifold $\varphi_m^{-1}(S) \subset T^d$ by the definition of the fibre product. Since the differential of $\varphi_m$ is bijective at all points in the preimage of $S$, this preimage is discrete. Since $T^d$ is compact, it must be finite. In fact, we may compute it easily: it consists of all points $(x, 0) = (x_1, x_2, 0, \ldots, 0) \in T^d$ with $\sum_{j=1}^{d} x_j + m < 0$. Here $(x, 0) \in T^d$ if and only if $x_j \in \{\pm 1\}$ for $j = 1, \ldots, d$.

Let $z = (x, y) \in T^d$ satisfy $v(z) = z$. Then $y = 0$ and hence $x_i \in \{\pm 1\}$ for $i = 1, \ldots, d$. These points satisfy $\varphi_m(z) \in \{N, S\}$ for the north and south pole in $\{4\}$, simply because $\varphi_m$ is “real” and $N, S$ are the only points fixed by the involution on $S_{1,d}$. Give $T^d$ the KR-orientation described in Example 5.9. This induces a KR-orientation on the fibre $T_{x,z} \mathbb{R}^d$. Since the vector field generated by the exponential function points upwards at $(1, 0) \in T^1$ and downwards at $(-1, 0) \in T^1$, the projection to the $y$-coordinate $T_{1,0} \mathbb{R}^1 \to \mathbb{R}^0$ preserves the orientation at $+1$ and reverses it at $-1$. Therefore, the projection to the $y$-coordinate $T_{z} \mathbb{R}^d \to \mathbb{R}^0$ multiplies the orientation with the sign $(-1)^{\text{sign}(z)}$. The sum in geometric bivariant KR-theory is the disjoint union of correspondences. Therefore, the discrete set $\varphi_m^{-1}(S)$ in the composite correspondence contributes the sum of $(-1)^{\text{sign}(z)} z_1$ over all $z \in \varphi_m^{-1}(S)$. Lemma 5.2 identifies this sum with $\chi \cdot ([\beta_{1,d} \circ \varphi_m] - [\gamma_0])$. □

If $d - 2 < m < d$, then $\varphi_m^{-1}(S)$ has only one element, namely, the single point $(-1, -1, \ldots, -1, 0, \ldots, 0) \in T^d$ with sign $(-1)^d$. Thus we get

$$[\beta_{1,d} \circ \varphi_m] - [\gamma_0] = \chi \cdot (-1)^d \cdot (-1, -1, \ldots, -1, 0, \ldots, 0).$$

(13)
Up to a sign, this is the generator of $\text{KR}^{-d}(\mathbb{T}^d)$ that is mapped to a generator of the KR-theory of the Roe $C^*$-algebra of $\mathbb{Z}^d$ (see [8]). It is argued in [8] why this generator describes strong topological phases. For other values of $m$, we would like to simplify the formula in Lemma 7.3 further by comparing the point inclusions $z_1; pt \to \mathbb{T}^d$ for different $z \in \mathbb{T}^d$ with $r(z) = z$. If we work in complex K-theory, then all point inclusions are homotopic and therefore give equivalent correspondences. This fails, however, in the “real” case. We need some preparation to explain this.

**Lemma 7.4.** If $m < -d$, then $H_m$ is homotopic to $-\gamma_0$ in $\mathcal{FU}(C(\mathbb{T}^d) \otimes \text{Cl}_{1,d})$.

**Proof.** The Hamiltonians $H_s$ for $s \in (-\infty, m]$ give a homotopy of real, odd, selfadjoint unitaries $[H_s]^{-1}H_s$ between $[H_m]^{-1}H_m$ and

\[
\lim_{s \to -\infty} [H_s]^{-1}H_s = -\gamma_0.
\]

The same argument shows that $H_m$ is homotopic to $\gamma_0$ for $m > d$. This is consistent with Lemma 7.3, which says that $[H_m] - [\gamma_0] = 0$ in $\text{KR}^{-d}(\mathbb{T}^d)$ for $m > d$.

Combining Lemmas 7.3 and 7.4 gives

\[
[-\gamma_0] - [\gamma_0] = \chi \sum_{z \in \mathbb{T}^d, r(z) = z} (-1)^{\text{sign}(z)} z_1
\]

because if $m < -d$, then $\varphi_m$ maps all points in $\mathbb{T}^d$ with $r(z) = z$ to $S$. In particular, for $d = 1$, this says that

\[(1,0) = (-1,0) + \chi \cdot ([-\gamma_0] - [\gamma_0]).\]

The difference $[-\gamma_0] - [\gamma_0]$ is represented by constant functions and thus is in the image of $\text{DK}(\text{Cl}_{1,1}) \cong \text{KO}^{-1}(pt) \cong \mathbb{Z}/2$ in $\text{KR}^{-1}(\mathbb{T}^1)$. Since this group is 2-torsion, we may drop the sign $\chi$ in (14).

**Lemma 7.5.** The isomorphism above maps $[-\gamma_0] - [\gamma_0]$ to the nontrivial element $\mu \in \text{KO}^{-1}(pt) \cong \mathbb{Z}/2$.

**Proof.** Identify $M_n(\text{Cl}_{1,1}) \cong \hat{M}_{2n}$. The odd selfadjoint real unitaries in $M_n(\text{Cl}_{1,1})$ are identified with the $2n \times 2n$-matrices

\[
\begin{pmatrix}
0 & U \\
U^* & 0
\end{pmatrix}
\]

for an orthogonal $n \times n$-matrix $U$. Two such matrices are homotopic among odd selfadjoint unitaries if and only if the orthogonal matrices are in the same connected component of the orthogonal group or, equivalently, have the same determinant. Here the elements $\pm \gamma_0$ correspond to $\pm 1 \in O(1) = \{\pm 1\}$, which lie in the two different connected components.

We recall how to describe the KR-theory of tori. There is a split extension of “real” $C^*$-algebras

\[
C_0(\mathbb{R}^{0,1}) \to C(\mathbb{T}^1) \to \mathbb{C}.
\]

It induces a KKR-equivalence $C(\mathbb{T}^d) \cong C(\mathbb{R}^{0,1}) \oplus \mathbb{C}$. Since $C(\mathbb{T}^d)$ is the tensor product of $d$ copies of $C(\mathbb{T}^1)$ and the tensor product bifunctor is additive on KKR, it follows that $C(\mathbb{T}^d)$ is KKR-equivalent to a direct sum of tensor products $A_1 \otimes \cdots \otimes A_d$, where each $A_j$ is either $C_0(\mathbb{R}^{0,1})$ or $\mathbb{C}$. We label such a summand by the set $I \subseteq \{1, \ldots, d\}$ of those factors that are $\mathbb{C}$.

**Theorem 7.6.** Let $m \in (-d + 2n, -d + 2n + 2)$ for some $n \in \{0, \ldots, d - 1\}$. Let $\chi$ be the sign from Lemma 5.13. The image of $[H_m] - [\gamma_0]$ in $\text{KR}^{-d}(\mathbb{T}^d)$ in the summand $\text{KR}^{-2d}(\mathbb{R}^{0,1})^{d^{-1}I}$ corresponding to $I \subseteq \{1, \ldots, d\}$ is computed as follows:

- If $I = \emptyset$, the image in $\text{KR}^{-d}(\mathbb{R}^{0,1}) \cong \mathbb{Z}$ is $(-1)^d \chi \cdot (d^{-1})$;
• if $|I| = 1$ and $n \geq 1$, the image in $\text{KR}^{-d}(\mathbb{R}_0,d-1) \cong \mathbb{Z}/2$ is $\begin{pmatrix} d-2 \\ n-1 \end{pmatrix}$ mod 2;

• if $|I| = 2$ and $n \geq 2$, the image in $\text{KR}^{-d}(\mathbb{R}_0,d-2) \cong \mathbb{Z}/2$ is $\begin{pmatrix} d-3 \\ n-2 \end{pmatrix}$ mod 2;

it is zero in the other cases, that is, for $|I| \geq 3$ or $n < |I|$.

Proof. We first consider the image of $(x,0)_I$ in the summand labeled by $I$. The shriek map $(x_1,\ldots,x_d,0,\ldots,0)_I$ is the exterior product of $(x_i,0)_I$ for $i = 1,\ldots,d$ (see Remark 6.7). In the decomposition of $\text{KR}^{-1}(\mathbb{T}^1) \cong \text{KR}^{-1}(\mathbb{R}_0,1) \oplus \text{KO}^{-1}(pt) \cong \mathbb{Z} \oplus \mathbb{Z}/2$, $(-1,0)_I$ becomes $(pt,0)$ with the standard generator $pt$ of $\text{KR}^{-1}(\mathbb{R}_0,1) \cong \mathbb{Z}$. Equation (14) shows that $(1,0)_I = (pt,\mu)$ with the nontrivial element $\mu \in \text{KO}^{-1}(pt) \cong \mathbb{Z}/2$. So $(x,0)_I$ is the exterior product of $d$ factors that are $(pt,0)$ if $x_i = -1$ and $(pt,\mu)$ if $x_i = 1$. The component in the summand labeled by $I$ is zero unless $x_i = +1$ for all $i \in I$. If $x_i = +1$ for all $i \in I$, then we get the exterior product of $pt \in \text{KR}^{-1}(\mathbb{R}_0,1)$ for all $i \notin I$ and $\mu \in \text{KO}^{-1}(pt)$ for all $i \in I$. The exterior product $\mu \otimes \mu \in \text{KO}^{-2}(pt \times pt) \cong \mathbb{Z}/2$ is known to be the nontrivial element (this also follows from the discussion above Proposition 2.12), whereas $\mu \otimes \mu \otimes \mu$ and hence also all higher exterior products of $\mu$ vanish because $\text{KO}^{-3}(pt) = 0$. Thus the image of $(x,0)_I$ is zero for all summands with $|I| \geq 3$ or $x_i = -1$ for some $i \in I$, and the standard generator of $\text{KR}^{-d}(\mathbb{R}_0,d-1)$ if $|I| \leq 2$ and $x_i = +1$ for all $i \in I$.

Now we sum up $\sum_{I \in \mathcal{I}} (-1)^{\text{sign}(x,0)_I}(x,0)_I$ over all $(x,0) \in \mathbb{T}^d$ with $\varphi_m(x,0) = 0$ or, equivalently, $\sum_{i=1}^d x_i + m < 0$. The latter means that the number of $+1$ among the coordinates $x_i$ is at most $n$. There are $C_n^d$ points $(x,0)$ with $x_i = +1$ for exactly $j$ indices $i$. For all of them, the image of $(x,0)_I$ in the direct summand $\text{KR}^{-d}(\mathbb{R}_0,d) \cong \mathbb{Z}$ for $I = \emptyset$ is the same standard generator. Hence the image of $[H_m] - [\gamma_0]$ in this direct summand is

\[( -1)^d \chi \cdot \sum_{j=0}^n (-1)^{n-j} \binom{d}{j} = ( -1)^d \chi \cdot \binom{d-1}{n} \]

because of (13). Now let $|I| = 1$, so that $I = \{i_0\}$ for some $i_0 \in \{1,\ldots,d\}$. The corresponding direct summand $\text{KR}^{-d}(\mathbb{R}_0,d-1) \cong \mathbb{Z}/2$ only sees $(x,0)_I \in \varphi_m^{-1}(S)$ with $x_{i_0} = 1$. There are $C_n^{d-1}$ points with $x_{i_0} = 1$ and $x_i = +1$ for exactly $j$ indices $i = 1,\ldots,d$. Thus the overall contribution in this summand isomorphic to $\mathbb{Z}/2$ vanishes if $n = 0$ and otherwise is equal to the class mod 2 of

\[ \sum_{j=1}^n (-1)^{n-j} \binom{d-1}{j-1} = - \sum_{j=0}^{n-1} (-1)^{n-j} \binom{d-1}{j} = \binom{d-2}{n-1}; \]

we could leave out all signs because $+1 = -1$ in $\mathbb{Z}/2$. Similarly, for $I \subseteq \{1,\ldots,d\}$ with $|I| = 2$, we get $C_n^{d-2}$ if $n \geq 2$ and 0 if $n \leq 1$.

The formula in Theorem 7.6 is compatible with the Chern character computation in [19, Equation (2.26)]. The latter, however, only gives partial information about the K-theory class, even in the complex case, because it only concerns the top-dimensional part of the Chern character.

As a result, we find that the Hamiltonian $H_m$ for $d-2 < m < d$ represents a generator of the direct summand $\text{KR}^{-d}(\mathbb{R}_0,d) \cong \mathbb{Z}$ in $\text{KR}^{-d}(\mathbb{T}^d)$. We get different KR-classes by stacking insulators in lower dimension along some direction. In our framework, this means that we consider a “coordinate” projection $\varphi : \mathbb{T}^d \twoheadrightarrow \mathbb{T}^k$ for some $0 \leq k \leq d$ and some $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, which only keeps the coordinates $x_{i_j}, y_{i_j}$ for $j = 1,\ldots,k$. Then we may pull back the generator of $\text{KR}^{-k}(\mathbb{R}_0,k) \subseteq \text{KR}^{-k}(\mathbb{T}^k)$ of the form $H_m$ along the map $\varphi$ to a class in $\text{KR}^{-k}(\mathbb{T}^d)$;
We represent thus we may implement with the symmetry type depends only on grading and the "real" involution are implemented by a unitary operator and thus for \( k \) then \( \Theta : \mathbb{R}^k \otimes \mathbb{R}^k \to \mathbb{R}^k \) and the even subalgebra of \( Cl_k \) identified with the subalgebra of \( Cl_k \) for \( k \), this covers all the summands \( KR^{-k}(\mathbb{R}^{0,k}) \) in \( KR^*(\mathbb{T}^d) \). Proposition 2.12 gives explicit generators for the summands isomorphic to \( \mathbb{Z}/2 \) as well: simply view \( H_m \) in (15) as taking values in \( Cl_{k,d} \) for \( d = 1, 2 \). We do not describe generators for the remaining summands of the form \( KR^{-k+4}(\mathbb{R}^{0,k}) \cong \mathbb{Z} \).

8. Transfer to Hilbert space

Now we transfer the Clifford algebra-valued function \( H_m \) to an operator on a Hilbert space with extra symmetries depending on the Clifford algebra (see also [15]). We represent \( C(\mathbb{T}^d) \) on \( \ell^2(\mathbb{Z}^d) \) in the usual way. Then \( C(\mathbb{T}^d) \otimes Cl_{a,b} \) is represented on \( \ell^2(\mathbb{Z}^d) \otimes C^k \) for some \( k \in \mathbb{N} \), with some extra symmetries acting on \( C^k \). Here \( k \in \mathbb{N} \) and the symmetry type depend on \( a, b \in \mathbb{N} \), and so there is a number of cases to consider. We let \( j := b - a + 1 \mod 8 \), so that \( DK(C(\mathbb{T}^d) \otimes Cl_{a,b}) = KR^{-j}(\mathbb{T}^d) \). The symmetry type depends only on \( j \in \mathbb{Z}/8 \) because Clifford algebras with the same \( j \) are Morita equivalent as graded "real" algebras. Therefore, it will suffice to look at one representative Clifford algebra for each \( j \in \mathbb{Z}/8 \).

First assume that \( j \) is even or, equivalently, \( b - a \) is odd. Then there is an isomorphism \( Cl_{a,b} \cong M_{2^k} \mathbb{C} \oplus M_{2^k} \mathbb{C} \) for \( k = (a + b - 1)/2 \) such that the \( \mathbb{Z}/2 \)-grading automorphism \( \alpha \) merely flips the two summands \( M_{2^k} \mathbb{C} \). Thus selfadjoint, odd elements of \( Cl_{a,b} \) become elements of the form \( (x, -x) \in M_{2^k} \mathbb{C} \oplus M_{2^k} \mathbb{C} \) for some \( x \in M_{2^k} \mathbb{C} \). As a ring automorphism, the "real" involution \( r \) on \( Cl_{a,b} \) permutes the two direct summands \( M_{2^k} \mathbb{C} \). It induces either the trivial or the nontrivial permutation. We first assume that the permutation is trivial. Then \( r \) restricts to the same "real" involution on both summands \( M_{2^k} \mathbb{C} \) because it commutes with \( \alpha \). This involution is implemented as conjugation by \( \Theta \) for an antunitary operator \( \Theta : C^k \to C^k \). The pair \( (x, -x) \) for \( x \in M_{2^k} \mathbb{C} \) is fixed by \( r \) if and only if \( x \) commutes with \( \Theta \). Thus we identify "real" selfadjoint odd unitaries in \( Cl_{a,b} \) with selfadjoint unitaries in \( M_{2^k} \mathbb{C} \) that commute with \( \Theta \). In other words, we are dealing with systems with a time-reversal symmetry \( \Theta \).

Since \( \Theta \) induces an antilinear involution, \( \Theta^2 = \pm 1 \). If \( \Theta^2 = +1 \), then the real subalgebra \( (M_{2^k} \mathbb{C})_R \) fixed by \( r \) is \( M_{2^k} \mathbb{R} \) and \( (Cl_{a,b})_R \cong M_{2^k} \mathbb{R} \otimes M_{2^k} \mathbb{R} \); this happens for \( Cl_{1,0} \) and thus for \( j \equiv 0 \mod 8 \). If \( \Theta^2 = -1 \), then \( k \geq 1 \) and \( (M_{2^k} \mathbb{C})_R \cong M_{2^k-1} \mathbb{H} \) and \( (Cl_{a,b})_R \cong M_{2^k-1} \mathbb{H} \oplus M_{2^k-1} \mathbb{H} \) for the quaternions \( \mathbb{H} \); this happens for \( Cl_{0,3} \) and thus for \( j \equiv 4 \mod 8 \).

Next, we assume that \( r \) flips the two summands \( M_{2^k} \mathbb{C} \). Then \( \Theta \) maps each direct summand into itself and induces the same real involution on both summands \( M_{2^k} \mathbb{C} \). Thus we may implement \( r \circ \alpha \) by an antunitary operator \( \Theta : C^k \to C^k \) as above. The difference is that a pair \( (x, -x) \) is real if and only if \( \Theta \) anticommutates with \( x \). Thus \( \Theta \) is now a particle-hole symmetry.

Since \( r \) flips the two summands, the real subalgebra \( (Cl_{a,b})_R \) is isomorphic to \( M_{2^k} \mathbb{C} \), identified with the subalgebra of \( (x, \Theta^{-1}x) \in M_{2^k} \mathbb{C} \oplus M_{2^k} \mathbb{C} \) for \( x \in M_{2^k} \mathbb{C} \). Once again, there are the two possibilities \( \Theta^2 = \pm 1 \). If \( \Theta^2 = +1 \), then the even subalgebra of \( (Cl_{a,b})_R \) is \( M_{2^k} \mathbb{R} \); this happens for \( Cl_{0,1} \) and thus for \( j \equiv 2 \mod 8 \). If \( \Theta^2 = -1 \), then \( k \geq 1 \) and the even subalgebra of \( (Cl_{a,b})_R \) is \( M_{2^k-1} \mathbb{H} \); this happens for \( Cl_{0,3} \) and thus for \( j \equiv 6 \mod 8 \).

Now assume that \( a - b \) is even. Then \( Cl_{a,b} \cong M_{2^k} \mathbb{C} \) for some \( k \in \mathbb{N} \). The \( \mathbb{Z}/2 \)-grading and the "real" involution are implemented by a unitary operator \( \Xi : C^k \to C^k \) and an antunitary operator \( \Theta : C^k \to C^k \). A real, selfadjoint odd unitary in \( Cl_{a,b} \)
then becomes a selfadjoint unitary in $\mathbb{M}_{2k}\mathbb{C}$ that anticommutes with $\Xi$ and commutes with $\Theta$. Thus it has $\Theta$ as a time-reversal and $\Xi$ as a chiral symmetry. Then $\Theta\Xi$ is a particle-hole symmetry. Multiplying $\Xi$ with a scalar, we may arrange that $\Xi^2 = 1$. Since $\Theta$ induces an antunitary involution, $\Theta^2 = \pm 1$. Since $\tau$ commutes with the grading, $\Xi^{-1}\Theta\Xi = \pm \Theta$. This is equivalent to $(\Theta\Xi)^2 = \pm \Theta^2$. So there are four possibilities for the sign. If $\Theta^2 = +1$, then $(\mathcal{C}_{\mathbb{R}}a,b) \cong \mathbb{M}_{2\mathbb{R}}\mathbb{R}$. If $\Theta^2 = -1$, then $k \geq 1$ and $(\mathcal{C}_{a,b})_0 \cong \mathbb{M}_{2k-1}\mathbb{H}$. The even subalgebra of $\mathcal{C}_{a,b}$ is $\mathbb{M}_{2k-1}\mathbb{C} \oplus \mathbb{M}_{2k-1}\mathbb{C}$ if $k \geq 1$. If $\Xi^{-1}\Theta\Xi = \Theta$, then the real involution restricted to the even part preserves the two direct summands $\mathbb{M}_{2k-1}\mathbb{C}$, so that the even real subalgebra of $\mathcal{C}_{a,b}$ is a direct sum of two simple algebras. If $\Xi^{-1}\Theta\Xi = -\Theta$, however, then the real involution restricted to the even part flips the two direct summands, so that the even real subalgebra of $\mathcal{C}_{a,b}$ is simple. An inspection now shows the following:

- $\Theta^2 = +1, \Xi^{-1}\Theta\Xi = +\Theta$, and $(\Xi\Theta)^2 = +1$ for $\mathcal{C}_{1,1}$ with $j \equiv 1 \mod 8$;
- $\Theta^2 = +1, \Xi^{-1}\Theta\Xi = -\Theta$, and $(\Xi\Theta)^2 = -1$ for $\mathcal{C}_{2,0}$ with $j \equiv 7 \mod 8$;
- $\Theta^2 = -1, \Xi^{-1}\Theta\Xi = -\Theta$, and $(\Xi\Theta)^2 = +1$ for $\mathcal{C}_{0,2}$ with $j \equiv 3 \mod 8$;
- $\Theta^2 = -1, \Xi^{-1}\Theta\Xi = +\Theta$, and $(\Xi\Theta)^2 = -1$ for $\mathcal{C}_{0,4}$ with $j \equiv 5 \mod 8$.

The correspondence between $j$ and the symmetry types is the same as in [22, Table 1].

**References**

[1] Michael F. Atiyah, *K-theory and reality*, Quart. J. Math. Oxford Ser. (2) 17 (1966), 367–386, doi: 10.1093/qmath/17.1.367 [MR 0206940]

[2] Michael F. Atiyah, Raoul Bott, and A. Shapiro, *Clifford modules*, Topology 3 (1964), no. suppl. suppl. 1, F–38, doi: 10.1016/0040-9383(64)90003-5 [MR 167985]

[3] Chris Bourne, Johannes Kellendonk, and Adam Rennie, *The Cayley transform in complex, real and graded K-theory*, Internat. J. Math. 31 (2020), no. 9, 2050074, 50, doi: 10.1142/S0129167X20500743 [MR 4140811]

[4] Alain Connes and Georges Skandalis, *The longitudinal index theorem for foliations*, Publ. Res. Inst. Math. Sci. 20 (1984), no. 6, 1139–1183, doi: 10.2977/prims/1195180375 [MR 775126]

[5] Heath Emerson and Ralf Meyer, *Equivariant representable K-theory*, J. Topol. 2 (2009), no. 1, 121–156, doi: 10.1112/jtopol/jtp003 [MR 2499449]

[6] Collin Mark Joseph, *Geometric construction of Hamiltonians*, Master’s thesis, Universität Göttingen, 2022.

[7] Pierre Julg, *K-Théorie équivariante et produits croisés*, C. R. Acad. Sc. Paris Sér. I Math. 292 (1981), no. 13, 629–632 [MR 625361]

[8] Max Karoubi, *Lectures on K-theory*, Cohomology of groups and algebraic K-theory (Hangzhou, China, July 1), 2010, pp. 217–268.

[9] Gennadi G. Kasparov, *The operator K-functor and extensions of C*-algebras*, Izv. Akad. Nauk SSSR Ser. Mat. 44 (1980), no. 3, 571–636, 719, available at http://mi.mathnet.ru/izv1739 English transl., Math. USSR-Izv. 16 (1981), no. 3, 513–572, doi: 10.1070/IM1981v016n03ABEH001320 [MR 582160]

[10] Collin Mark Joseph, *Geometric construction of Hamiltonians*, Master’s thesis, Universität Göttingen, 2022.

[11] Pierre Julg, *K-Theory équivariante et produits croisés*, C. R. Acad. Sc. Paris Sér. I Math. 292 (1981), no. 13, 629–632 [MR 625361]

[12] Max Karoubi, *Lectures on K-theory*, Cohomology of groups and algebraic K-theory (Hangzhou, China, July 1), 2010, pp. 217–268.

[13] Gennadi G. Kasparov, *The operator K-functor and extensions of C*-algebras*, Izv. Akad. Nauk SSSR Ser. Mat. 44 (1980), no. 3, 571–636, 719, available at http://mi.mathnet.ru/izv1739 English transl., Math. USSR-Izv. 16 (1981), no. 3, 513–572, doi: 10.1070/IM1981v016n03ABEH001320 [MR 582160]

[14] Collin Mark Joseph, *Geometric construction of Hamiltonians*, Master’s thesis, Universität Göttingen, 2022.

[15] Johannes Kellendonk, *On the C*-algebraic approach to topological phases for insulators*, Ann. Henri Poincaré 18 (2017), no. 7, 2251–2300, doi: 10.1007/s00023-017-0583-3 [MR 3665214]

[16] Wolfgang Lück and Bob Oliver, *The completion theorem in K-theory for proper actions of a discrete group*, Topology 40 (2001), no. 3, 585–616, doi: 10.1016/S0040-9383(99)00077-4 [MR 1838997]

[17] El-haioum M. Moutuou, *Graded Brauer groups of a groupoid with involution*, J. Funct. Anal. 266 (2014), no. 5, 2689–2739, doi: 10.1016/j.jfa.2013.12.019 [MR 3158706]
[18] Roger Plymen, *Strong Morita equivalence, spinors and symplectic spinors*, J. Operator Theory 16 (1986), no. 2, 305–324, available at [http://jot.theta.ro/jot/archive/1986-016-002/1986-016-002-008.html](http://jot.theta.ro/jot/archive/1986-016-002/1986-016-002-008.html) MR 860349

[19] Emil Prodan and Hermann Schulz-Baldes, *Bulk and boundary invariants for complex topological insulators*, Mathematical Physics Studies, Springer, 2016. From $K$-theory to physics. doi: 10.1007/978-3-319-29351-6 MR 3468838

[20] Xiao-Liang Qi, Taylor L. Hughes, and Shou-Cheng Zhang, *Topological field theory of time-reversal invariant insulators*, Phys. Rev. B 78 (2008), 195424, doi: 10.1103/PhysRevB.78.195424

[21] John Roe, *Paschke duality for real and graded $C^*$-algebras*, Q. J. Math. 55 (2004), no. 3, 325–331, doi: 10.1093/qmath/hsh055 MR 2082096

[22] Hermann Schulz-Baldes, *Topological insulators from the perspective of non-commutative geometry and index theory*, Jahresber. Dtsch. Math.-Ver. 118 (2016), no. 4, 247–273, doi: 10.1365/s13291-016-0142-5 MR 3554424

[23] Allons Van Daele, *K-theory for graded Banach algebras. I*, Quart. J. Math. Oxford Ser. (2) 39 (1988), no. 154, 185–199, doi: 10.1093/qmath/39.2.185 MR 947500

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