Infinite-dimensional $\ell^1$ minimization and function approximation from pointwise data

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March 10, 2015

Abstract
We consider the problem of approximating a function from finitely-many pointwise samples using $\ell^1$ minimization techniques. In the first part of this paper, we introduce an infinite-dimensional approach to this problem. Three advantages of this approach are as follows. First, it provides interpolatory approximations in the absence of noise. Second, it does not require a priori bounds on the expansion tail in order to be implemented. In particular, the truncation strategy we introduce as part of this framework is completely independent of the function being approximated. Third, it allows one to explain the crucial role weights play in the minimization, namely, that of regularizing the problem and removing so-called aliasing phenomena. In the second part of this paper we present a worst-case error analysis for this approach. We provide a general recipe for analyzing the performance of such techniques for arbitrary deterministic sets of points. Finally, we apply this recipe to show that weighted $\ell^1$ minimization with Jacobi polynomials leads to an optimal, convergent method for approximating smooth, one-dimensional functions from scattered data.

1 Introduction

Many problems in science and engineering require the approximation of a smooth function from a finite set of pointwise samples. An important example is that of uncertainty quantification, wherein such functions describe the dependence of given physical models on a set of parameters. Common features of such problems are their high dimensionality and the limited availability of data, which make accurate approximation a challenging task.

Although a classical problem in approximation theory, in the last several years there has been an increasing focus on the use of convex optimization techniques for this task [12, 14, 16, 19, 22, 23, 24, 28, 29]. As dimension increases, smooth multivariate functions are increasingly well-represented by their best $k$-term approximation in certain orthogonal expansions (e.g. multivariate Legendre polynomials). Hence the expectation is that these techniques will yield substantial gains over more standard approaches such as discrete least squares and interpolation, at least when the dimension is sufficiently high and the data points arise from appropriate sampling distributions. A number of recent studies, such as those listed above, have empirically shown this to be the case.

1.1 Current approaches

Let $f$ be a multivariate function, $\{\phi_i\}_{i \in \mathbb{N}}$ an orthonormal basis of functions (e.g. polynomials) and write $f = \sum_{i \in \mathbb{N}} x_i \phi_i$, where $x = \{x_i\}_{i \in \mathbb{N}}$ is the infinite vector of coefficients of $f$. If $\{t_n\}_{n=1}^N$ is a finite
set of points, the problem is to approximate \( x \), and therefore \( f \), from the data \( \{f(t_n)\}_{n=1}^N \). Since \( x \) is an infinite vector, in order to compute an approximation to \( f \) it is necessary to truncate in some way. In the usual formulation (see \([12, 14, 16, 19, 22, 23, 24, 28, 29]\)), one introduces a fixed \( M \geq N \) and seeks to approximate the first \( M \) coefficients \( x_1, \ldots, x_M \) of \( x \). If \( A = \{\phi_i(t_n)\}_{n=1,i=1}^{N,M} \subseteq \mathbb{C}^{N \times M} \) then the standard (weighted) \( \ell^1 \) minimization approach is as follows:

\[
\min_{z \in \mathbb{C}^M} \|z\|_{1,w} \text{ subject to } \|Az - y\| \leq \delta, \quad y = \{f(t_n)\}_{n=1}^N. \tag{1.1}
\]

Here \( \|z\|_{1,w} = \sum_{i=1}^M w_i |z_i| \) is the \( \ell^1_w \)-norm on \( \mathbb{C}^M \) with weights \( w_i > 0 \). The parameter \( \delta \) handles the truncation, and is usually chosen so that \( \{x_i\}_{i=1}^M \) is feasible for (1.1). That is,

\[
\max_{n=1,\ldots,N} \left| f(t_n) - \sum_{i=1}^M x_i \phi_i(t_n) \right| \leq \delta. \tag{1.2}
\]

In other words, the error introduced by truncating the infinite expansion to a vector of length \( M \) is viewed as noise in the data. Unfortunately, this formulation raises a number of issues, which we describe next. Overcoming these issues is the goal of this paper:

(i) In order to choose \( \delta \), one must have an \textit{a priori} estimate for the truncation error \( |f(t) - \sum_{i=1}^M x_i \phi_i(t)| \). Note that the approximation error resulting from (1.1) is highly sensitive to the choice of \( \delta \) \( [24] \). In practice, it has been proposed to address this problem using cross-validation techniques \([12, 14, 29]\). Yet such techniques can be expensive, lack theoretical support and may not result in an accurate estimation of the tail (1.2).

(ii) The approximation \( \tilde{f} \) of \( f \) obtained from (1.2) does not interpolate the data in general, i.e. \( \tilde{f}(t_n) \neq f(t_n) \), \( n = 1, \ldots, N \). Interpolatory solutions are often desirable in applications since they ensure that the approximation exactly fits the underlying function \( f \) where at the points at which \( f \) is known – in other words, the uncertainty is zero at the sampling points.

(iii) The approximation \( \tilde{f} \) can be sensitive to the choice of weights, and is prone to aliasing (also known as overfitting) if the weights are chosen inappropriately \([24]\).

(iv) Besides some specific cases where such techniques are known to perform extremely well – such as when the coefficients \( \{x_i\}_{i=1}^M \) are sparse and the data points \( \{t_n\}_{n=1}^N \) are chosen randomly according to the orthogonality measure of the basis \( \{\phi_i\}_{i \in \mathbb{N}} \) \([14, 22, 23, 24]\) – very little is known about the approximation error \( \|f - \tilde{f}\| \). In particular, for general \( f \) (not necessarily having sparse coefficients) and arbitrary \textit{deterministic} scattered points \( \{t_n\}_{n=1}^N \) the quality of the approximation \( \tilde{f} \) to \( f \) is largely unknown.

Issue (iv) has ramifications for a variety of applications where the primary limitation is the availability of data – that is, where it is time-consuming or expensive to acquire more samples – as opposed to data-rich scenarios where processing speed is the key concern (in which case classical techniques such as least-squares fitting are likely superior). If weighted \( \ell^1 \) minimization techniques are to find wide use in practice, then it is desirable to have an error bounds for non-ideal conditions (e.g. fixed, deterministic sample points) often found in applications.

### 1.2 Our contributions

The purpose of this paper is to address these issues. In \([3]\) we first propose an infinite-dimensional weighted \( \ell^1 \) minimization problem which removes the need for \textit{a priori} knowledge of magnitude
of the expansion tail. In the absence of noise in the data, its solution are exactly interpolatory, unlike solutions of (1.2). As one might expect, however, such an infinite-dimensional minimization problem cannot be solved numerically. Hence we next introduce a truncation strategy based on a user-controlled parameter $K \in \mathbb{N}$. This leads to finite-dimensional minimization problem over $C^K$, reminiscent of (1.2) but with a number of key differences. First, unlike (1.2), it requires no knowledge of the expansion tail, and second, it retains the interpolatory property of the infinite-dimensional problem. Importantly, in §6 show how to select the parameter $K$ in a manner independent of $f$, and dependent only on the basis $\{\phi_i\}_{i \in \mathbb{N}}$ and data points $\{t_n\}_{n=1}^N$.

Formulating the minimization problem in an infinite-dimensional setting also allows us to address issue (iii). In §4 we first show that unweighted $\ell^1$ minimization is largely unsuitable for the function interpolation problem, since it leads to an aliasing phenomenon. Specifically, without weights there exist infinitely many solutions of minimization problem which interpolate $f$ at the data points, but do not approximate $f$ to any accuracy away from these points. Fortunately, this problem can be completely resolved by the introduction of slowly growing weights. In effect, these weights regularize the optimization problem and ensure that such bad solutions of the unweighted problem, whilst still feasible, are no longer minimizers of the weighted problem. Through subsequent analysis we quantify how fast the weights need to grow to resolve this phenomenon, and demonstrate this result with numerical examples.

Issues (i)–(iii) are the focus of the first half of this paper (§2–6). In the second half, we consider (iv). More precisely, we pose and answer the following two questions:

(a) In the worst case, how well can one approximate a function $f$ using weighted $\ell^1$ minimization from its samples taken on an arbitrary deterministic grid of $N$ points?

(b) How does this approximation perform in comparison to existing techniques, such as least-squares fitting?

Note that in we do not assume any sparsity of the coefficients $x = \{x_i\}_{i \in \mathbb{N}}$ of $f$, although we do assume some mild decay of $x_i$ as $i \to \infty$ (otherwise the weighted $\ell^1$ problem does not make sense). We also do not assume any structure to the data: the points $\{t_n\}_{n=1}^N$ are deterministic and can be arbitrarily distributed in the domain. As is standard in scattered data approximation [26], we classify the error in terms of a simple density condition only.

Our motivation for examining (a) and (b) is the following. Least-squares fitting is a classical and widely-used technique (especially in the field of uncertainty quantification), but is well known to be intensive in the number of samples required to achieve stability and accuracy (see [9, 17, 18] and references therein). Conversely, under certain conditions – sparsity and random data points – $\ell^1$ techniques are known to give very good approximations from relatively few samples. However, in practical scenarios one may not have the luxury to choose the data points in a way to deliver the best performance of the approximation. This is the case when using legacy data, for example. Moreover, whilst functions in high dimensions tend to have sparse coefficients in polynomial bases [10, 11, 12], in low (in particular, one) dimensions polynomial coefficients usually exhibit rapid decay, but typically little sparsity. Since $\ell^1$-based techniques are computationally more intensive than classical methods such as least-squares fitting, this raises the follow question: is it still worth using $\ell^1$ techniques even when the data points are scattered and sparsity is not assured?

In §7 we present a general mathematical framework for answering these questions. We introduce a linear approximation error analysis for the infinite-dimensional weighted $\ell^1$-minimization, which allows it to be compared directly with existing techniques. In particular, we reduce (b) to a question about the behaviour of three particular quantities that depend on the data points $\{t_n\}_{n=1}^N$, the
expansion basis \( \{ \phi_i \}_{i \in \mathbb{N}} \) and the weights \( \{ w_i \}_{i \in \mathbb{N}} \). Analyzing these quantities for each specific problem setup provides an answer to (b).

To illustrate the various aspects of this framework, in the final part of this paper (§§S and §Q) we consider several examples, including one-dimensional Jacobi polynomial approximations from scattered data points. In particular, we prove the following:

**Theorem 1.1.** For \( \alpha, \beta > -1 \) let \( \{ \phi_i \}_{i \in \mathbb{N}} \) be the orthonormal Jacobi polynomial basis (2.6) on \([-1, 1]\) and let \( T = \{ t_n \}_{n=1}^N \) of \( N \) scattered points in \([-1, 1]\). Let \( h \) be the density of the points, defined by (2.2) and suppose that the truncation parameter

\[
K \geq h^{\frac{1}{2}} \xi^{-1 - \frac{1}{2}},
\]

for any \( r \in \mathbb{N} \), where \( \xi \) is the minimal separation between the points \( T \). Fix weights \( w = \{ w_i \}_{i \in \mathbb{N}} \) with \( w_i = \| \phi_i \|_{L^\infty} i^\gamma \) for some \( \gamma > 1/2 - q \), where \( q \) is as in (2.7), and let \( f = \sum_{i \in \mathbb{N}} x_i \phi_i \) with \( x \in \ell^1_w(\mathbb{N}) \) for \( \tilde{w}_i = \sqrt{r(w_i)}^2 \). Then given measurements \( y = \{ f(t_n) \}_{n=1}^N \) one can compute, via weighted \( \ell^1 \) minimization with weights \( w = \{ w_i \}_{i \in \mathbb{N}} \), an approximation \( \hat{x} \) to the coefficients \( x \) satisfying

\[
\| x - \hat{x} \| \lesssim \| x - P_M x \|_{1,w} + \| x - P_K x \|_{1,\tilde{w}},
\]

provided

\[
h \lesssim \frac{1}{M^2 \log M}.
\]

Moreover, the approximation \( \tilde{f} = \sum_{i=1}^K \hat{x}_i \phi_i \) exactly interpolates the data: \( \tilde{f}(t_n) = f(t_n), \forall n. \)

This theorem demonstrates all key aspects of our paper. First, the truncation parameter \( K \) is determined independently of \( f \) (issue (i)), and its contribution to the overall error is clarified by (1.4). In particular, if the data is roughly equally-spaced, then \( h, \xi = \mathcal{O}(1/N) \) and it suffices to take \( K = \mathcal{O}(N^{1+\frac{1}{2r}}) \) for any \( r > 0 \). Second, the approximation \( \tilde{f} \) exactly interpolates in the absence of noise (issue (ii)). Note that noise can also be dealt with within our framework (we exclude it here for ease of presentation). Third, one gets an explicit criterion for how to choose the weights (issue (iii)). Fourth, one gets an estimate (1.4) for the approximation error that only depends on the density \( h \) of the deterministic points \( T \) (issue (iv)) which can arbitrarily distributed in the domain.

As we discuss in §S the estimates (1.4) and (1.5) demonstrate not just good performance of this approach for scattered data, but in fact optimal performance (up to a log factor in \( M \)). As we explain, no stable method which is convergent as \( h \to 0 \) can exhibit an error bound of the form (1.4) with \( M \) growing faster than \( h^{-1/2} \) as \( h \to \infty \). Hence weighted \( \ell^1 \) minimization are guaranteed in the worst case to perform as well (up to log factors) as any other technique (e.g. least-squares fitting). Our numerical results support this conclusion, and in fact show that weighted \( \ell^1 \) minimization performs rather better in practice and similarly to an oracle least-squares fit.

### 1.3 Relation to previous work

A theory for reconstruction of sparse polynomial expansions from random pointwise samples was developed in a series of papers by Rauhut & Ward [23, 24]. Extension and application of this work in uncertainty quantification has been considered in [12, 14, 16, 19, 28, 29]. The use of weighted \( \ell^1 \) minimization was introduced in [19, 24, 29]. We also use weighted minimization in this paper, yet for rather different purposes. Namely, weights are chosen to regularize the minimization problem and remove the aliasing phenomenon (such a phenomenon is referred to as overfitting in [24]). Typically, this requires only very slow growth of the weights (we quantify exactly how fast later...
in the paper). Unlike other works, we do not select weights based on a priori information about decay of the polynomial coefficients. In fact, in §4 we will show that choosing weights in this way leads to inconsistent and often negligible improvements in the approximation. Moreover, from a practical standpoint, higher weights may cause issues for the numerical solvers.

The infinite-dimensional framework we introduce in this paper is inspired in part by the framework of infinite-dimensional compressed sensing in Hilbert spaces, due to A. C. Hansen and the present author [3, 4] (see also [5] for an overview). The primary difference is the need for weighted minimization in the present setup, due to the discontinuity of the sampling operator (pointwise evaluations). We note also that our worst case analysis and comparison to least-squares fitting is similar to that presented in [21] for generalized sampling in the Hilbert space setting.

The examples we use this paper consist of algebraic and trigonometric polynomials respectively. Polynomial approximations (so-called polynomial chaos expansions) are popular in areas such as uncertainty quantification [15, 27]. However, we stress that the framework and analysis of §2–7 of this paper is completely general, and can be applied to other bases. We mention several other examples in §10. Our examples are also one-dimensional. We do this so as to better elucidate the key ideas, without the notational complexities of the higher-dimensional setting. The extension to higher dimensions is nonetheless important, and we will report on the details in a future paper.

On this topic, we wish to clarify that the aim of this paper is not to propose weighted \(\ell^1\) minimization as a panacea for function approximation. In the one-dimensional setting especially there is a wealth of other techniques which are likely superior (see [6, 7, 20] and references therein). The advantages of weighted \(\ell^1\) minimization come to the fore as the dimension increases; as has been verified empirically in a number of works such as those mentioned previously. Instead, the purpose of this paper is to first propose a framework for weighted \(\ell^1\) minimization that overcomes some existing issues, and second provide a more comprehensive analysis of its approximation capabilities. We use the one-dimensional case to this end largely for illustrative purposes.

## 2 Preliminaries

Let \(D \subseteq \mathbb{R}^d\) be a domain and \(\nu(t)\) an integrable nonnegative weight function satisfying \(\int_D \nu(t) \, dt = 1\). Let \(L^2_\nu(D)\) be the space of complex-valued weighted square-integrable functions on \(D\), with norm \(\|\cdot\|_{L^2_\nu}\) and inner product \(\langle \cdot, \cdot \rangle_{L^2_\nu}\). Suppose that \(\{\phi_i\}_{i \in \mathbb{N}} \subseteq L^2_\nu(D) \cap L^\infty(D)\) is a set of functions that are orthonormal with respect to \(\nu\). In other words, \(\langle \phi_i, \phi_j \rangle_{L^2_\nu} = \delta_{ij}, \forall i, j \in \mathbb{N}\). Note that

\[
1 = \|\phi_i\|^2_{L^2_\nu} \leq \|\phi_i\|_{L^\infty}^2, \quad \forall i \in \mathbb{N},
\]

where \(\|\cdot\|_{L^\infty}\) is the uniform norm on \(D\). In particular, \(\|\phi_i\|_{L^\infty} \geq 1\) for all \(i \in \mathbb{N}\).

### 2.1 Scattered data

For \(N \in \mathbb{N}\), let \(T = \{t_n\}_{n=1}^N \subseteq \overline{D}\) be a set of \(N\) scattered data points. Our aim is to approximate functions \(f: D \to \mathbb{C}\) from the values \(\{f(t_n)\}_{n=1}^N\). To ensure an accurate approximation, we require a notion of closeness of the points \(T\). We quantify this by defining the density

\[
h = \sup_{t \in D} \min_{n=1,\ldots,N} |t - t_n|,
\]

where \(|\cdot|\) is the Euclidean distance. In our analysis later, we shall present convergence rates of the various approximations in the asymptotic regime \(h \to 0\).
Associated to the points $T$ will also be a set of values $\tau_n \geq 0$, $n = 1, \ldots, N$, which we refer to as quadrature weights. This is not to be confused with the optimization weights $w_i$ introduced later. For simplicity, we shall define these as follows

$$\tau_n = \int_{V_n} \nu(t) \, dt, \quad n = 1, \ldots, N, \quad V_n = \{ t \in D : |t - t_n| \leq |t - t_m|, \forall m \neq n \}, \tag{2.3}$$

where $V_n$ are the Voronoi cells of the points $T$. Given such quadrature weights, we define the following sesquilinear form on $L^2_p(D) \cap L^\infty(D)$:

$$\langle f, g \rangle_h = \sum_{n=1}^{N} \tau_n f(t_n) \overline{g(t_n)},$$

and write $\| \cdot \|_h = \sqrt{\langle \cdot, \cdot \rangle_h}$ for the corresponding seminorm. Note that the quadrature weights $\tau_n$ are not strictly necessary at this stage, but will play a pivotal role later in the paper.

### 2.2 Weighted spaces

For weighted minimization we need to introduce appropriate weights and weighted spaces. For the remainder of this paper, $w = \{w_i\}_{i \in \mathbb{N}}$ will be a set of positive weights satisfying

$$w_i \geq \|\phi_i\|_\infty \geq 1, \quad \forall i \in \mathbb{N}, \tag{2.4}$$

where the latter inequality is due to (2.1). Define the weighted $\ell^p$ spaces by

$$\ell^p_w(\mathbb{N}) = \left\{ x = \{x_i\}_{i \in \mathbb{N}} : \|x\|_{p,w} := \left( \sum_{i \in \mathbb{N}} (w_i)^p |x_i|^p \right)^{1/p} < \infty \right\}, \quad p > 0.$$

Note that $\|x\|_{p,w} = \|Wx\|_p$, where

$$W = \text{diag}(w_1, w_2, \ldots), \tag{2.5}$$

is the infinite diagonal matrix of weights. For the remainder of this paper, we will assume that the function we wish to recover $f = \sum_{i \in \mathbb{N}} x_i \phi_i \in L^2_w(D)$ has coefficients satisfying $x = \{x_i\}_{i \in \mathbb{N}} \in \ell^1_w(\mathbb{N})$.

### 2.3 Other notation

For $\Delta \subseteq \mathbb{N}$ we let $P_\Delta : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the projection defined by

$$(P_\Delta x)_j = x_j, \quad j \in \Delta, \quad (P_\Delta x)_j = 0, \quad j \notin \Delta.$$

If $\Delta = \{1, \ldots, K\}$ for some $K \in \mathbb{N}$, then we merely write $P_K$. We also let $\{e_j\}_{j \in \mathbb{N}}$ denote the canonical basis of $\ell^2(\mathbb{N})$, so that

$$P_\Delta(\cdot) = \sum_{j \in \Delta} \langle \cdot, e_j \rangle e_j.$$

We will allow the slight abuse of notation throughout the paper in thinking of $P_\Delta x$ as both an element of $\ell^2(\mathbb{N})$ and $\mathbb{C}^{|\Delta|}$. The intended meaning will be clear from the context.

If $x \in \mathbb{C}$, we let $\text{sign}(x) = x/|x|$ be its complex sign with the convention that $\text{sign}(0) = 0$. For $x \in \ell^\infty(\mathbb{N})$ we let $\text{sign}(x) = \{\text{sign}(x_i)\}_{i \in \mathbb{N}} \in \ell^\infty(\mathbb{N})$ be the corresponding sequence of complex signs of entries of $x$. Finally, we use the notation $a \lesssim b$ to mean that there exists a constant $C$ independent of all relevant quantities such that $a \leq Cb$. 
2.4 Examples

As mentioned in §1.3, the examples we consider in this paper consist of one-dimensional functions on bounded intervals, which we take to be $D = (-1, 1)$ without loss of generality. In §10 we discuss extensions to higher dimensions, unbounded intervals and other approximation systems.

Example 2.1 If $f$ is smooth, then it is natural to approximate it using orthogonal polynomials. In this example, we consider orthogonal polynomials with respect to the Jacobi weight

$$\nu(t) = \nu^{(\alpha, \beta)}(t) = (1 - t)'^{(1 + t)}', \quad \alpha, \beta > -1.$$ 

Following the standard notation [1, 25], we write $P_j^{(\alpha, \beta)}$ for the $j$th such polynomial, and

$$\phi_j = \left( h_j^{(\alpha, \beta)} \right)^{-1/2} P_j^{(\alpha, \beta)} - 1, \quad j \in \mathbb{N},$$

for the corresponding orthonormal polynomials, where $h_j^{(\alpha, \beta)}$ is as in (A.1). One can show that

$$\|\phi_j\|_{L^\infty} = O \left( j^{q+1/2} \right), \quad j \to \infty,$$

where $q = \max \{ \alpha, \beta \} \geq -1/2$, $\alpha, \beta \geq -1/2$.

See Appendix A (several other properties of Jacobi polynomials that will be needed later are also listed therein). Since the weights $\{w_i\} \in \mathbb{N}$ introduced in §2.2 are required to satisfy (2.4), this means that for this example they must grow at least as fast as as $j^{q+1/2}$ as $j \to \infty$.

Example 2.2 Functions that are smooth and periodic can be efficiently approximated using trigonometric polynomials. In this case, we have $\nu(t) = 1/\sqrt{2}$ and let $\{\phi_i\} \in \mathbb{N}$ be the Fourier basis

$$\phi_j(t) = e^{ij\pi t}, \quad j \in \mathbb{Z}. \quad (2.8)$$

For convenience we index over $\mathbb{Z}$ rather than $\mathbb{N}$ in this example. Note that $\|\phi_j\|_\infty = 1$ and therefore the weights $w_j$ in this example are required to satisfy $w_j \geq 1$, $\forall j \in \mathbb{Z}$.

3 Minimization problems

Define the operator $U : \ell_1^w(\mathbb{N}) \to \mathbb{C}^N$ by

$$Ux = \{\sqrt{\tau_n}g(t_n)\}_{n=1}^N, \quad where \quad g = \sum_{i \in \mathbb{N}} x_i \phi_i.$$ 

Note that this operator is a bounded operator on $\ell_1^w(\mathbb{N})$. Indeed, by (2.4),

$$\|g\|_{L^\infty} \leq \sum_{i \in \mathbb{N}} |x_i| \|\phi_i\|_\infty \leq \|x\|_{1,w} < \infty.$$ 

We shall also consider $U \in \mathbb{C}^{N \times \infty}$ as the infinite matrix with entries

$$U_{n,i} = \sqrt{\tau_n} \phi_i(t_n), \quad n = 1, \ldots, N, \quad i \in \mathbb{N}.$$ 

From now on, we make no distinction between the operator $U$ and the infinite matrix.
### 3.1 Infinite-dimensional weighted $\ell^1$ minimization

Let $f = \sum_{i \in \mathbb{N}} x_i \phi_i$ be a function we wish to recover, where $x = \{x_i\}_{i \in \mathbb{N}} \in \ell^1_\mu(\mathbb{N})$. Suppose first that we are given noiseless measurements of $f$, that is, $f(t_n)$, $n = 1, \ldots, N$, and let

$$y = \{\sqrt{\tau_n} f(t_n)\}_{n=1}^N \in \mathbb{C}^N,$$

for the vector of measurements multiplied by the quadrature weights. To recover the infinite vector $x$ of coefficients, and therefore $f$, we shall use weighted $\ell^1$ minimization. As a first approach, we formulate the following infinite-dimensional optimization problem:

$$\inf_{z \in \ell^1_\mu(\mathbb{N})} \|z\|_{1,w} \text{ subject to } Uz = y. \quad (3.1)$$

If $\hat{x} \in \ell^1_\mu(\mathbb{N})$ is a minimizer of (3.1), then the corresponding approximation $\tilde{f}$ to $f$ is given by

$$\tilde{f} = \sum_{i \in \mathbb{N}} \hat{x}_i \phi_i. \quad (3.2)$$

In general, the measurements may be noisy. Suppose we are given

$$f(t_n) + e_n, \quad n = 1, \ldots, N,$$

where $e = \{e_n\}_{n=1}^N \in \mathbb{C}^N$ is a noise vector satisfying $\|e\| \leq \eta$ for some known $\eta \geq 0$. Write

$$y = \{\sqrt{\tau_n} (f(t_n) + e_n)\}_{n=1}^N \in \mathbb{C}^N. \quad (3.3)$$

In this case, we solve the inequality-constrained optimization problem

$$\inf_{z \in \ell^1_\mu(\mathbb{N})} \|z\|_{1,w} \text{ subject to } \|Uz - y\| \leq \sqrt{\tau} \eta, \quad (3.4)$$

where $\tau = \max_{n=1,\ldots,N} \{\tau_n\}$. Note that (3.1) is just a special case of (3.4) corresponding to the case $\eta = 0$. Note also that solutions of (3.1) are interpolatory in the sense that $\tilde{f}(t_n) = f(t_n)$, $n = 1, \ldots, N$, whenever $\tilde{f}$ is given by (3.2) with $\hat{x}$ being a minimizer of (3.1). Conversely, solutions of (3.4) lead to approximations $\tilde{f}$ that are only interpolatory up to the magnitude of the noise $\eta$.

### 3.2 Truncation

Unfortunately, neither problem (3.1) or (3.4) is tractable, since they require optimizing over an infinite-dimensional space. The solution to this problem is to truncate in a suitable way. Let $K \in \mathbb{N}$ be a truncation parameter (its magnitude will be estimated later). To form the truncated problem, we replace the space $\ell^1_\mu(\mathbb{N})$ with $\mathbb{C}^K$ and truncate the $N \times \infty$ matrix $U$ to the $N \times K$ matrix $UP_K$ spanned by its first $K$ columns. Hence, we now consider the truncated problem

$$\min_{z \in \mathbb{C}^K} \|z\|_{1,w} \text{ subject to } UP_K z = y, \quad (3.5)$$

in the noiseless case, as well as its noisy analogue

$$\min_{z \in \mathbb{C}^K} \|z\|_{1,w} \text{ subject to } \|UP_K z - y\| \leq \sqrt{\tau} \eta. \quad (3.6)$$
Both of these problems are finite dimensional, and can therefore be solved using standard software. If \( \hat{x} \in \mathbb{C}^K \) is a minimizer, then the approximation to \( f \) is given by

\[
\tilde{f} = \sum_{i=1}^{K} \hat{x}_i \phi_i.
\]  

(3.7)

Note that neither (3.5) nor (3.6) modify the constraints of the infinite-dimensional problems (3.1) and (3.4). In particular, (3.5) remains interpolatory and (3.4) is interpolatory up to the noise.

With this in hand, the general idea, and really the key to this approach, is to vary \( K \) in such a way to ensure closeness of the solutions of the finite-dimensional problems (3.5) and (3.6) to those of the infinite-dimensional problems (3.1) and (3.4). Crucially, in §6 we shall show that it is possible to choose \( K \) in a completely function-independent manner, thus eliminating the need for a priori knowledge of the tail magnitude, or ad-hoc cross validation approaches to estimate it (recall §1).

**Remark 3.1** It is important that \( K \) be chosen suitably large. To see why, consider Example 2.1. If \( K = N \) then the square matrix \( UP_N \) is just the matrix of polynomial interpolation. Hence (3.5) has a unique solution and \( \tilde{f} \) is the unique polynomial interpolant of \( f \) of degree \( N - 1 \). However, for equispaced (or more generally, scattered) data this is well known to be a poor approximation to \( f \), since it suffers from Runge’s phenomenon. The approximations \( \tilde{f} \) will generally diverge and the matrix \( UP_N \) has exponentially large condition number. On the other hand, if one replaces \( UP_N \) by \( UP_K \) with \( K > N \), then provided \( K \) is sufficiently large the singular values of \( UP_K \) are all \( O(1) \) (see §6). As can be seen in Fig. 1, the resulting approximation no longer exhibits Runge’s phenomenon. Later we show that increasing \( K \) does indeed guarantee a stable and accurate approximation (see Corollary 5.2 and Theorem 6.2), and moreover the required condition on \( K \) depends only the data points and is completely independent of \( f \) (Theorem 6.5).

### 3.3 Comparison to least-squares fitting

As discussed in §1 least-squares data fitting is a classical and widely-used technique. It corresponds to the approximation \( \tilde{f} = \sum_{i=1}^{M} \hat{x}_i \phi_i \), where \( \hat{x} \) is the solution of the overdetermined least squares

\[
\min_{z \in \mathbb{C}^M} \| UP_M z - y \|.
\]  

(3.8)

An important difference between least-squares fitting and weighted \( \ell^1 \) minimization is the choice of the truncation parameter \( M \leq N \). In least-squares fitting, this parameter affects both the approximation error \( \| f - \tilde{f} \| \) and the robustness of the approximation. In practice, \( M \) must be chosen
suitably small in relation to $1/h$ to ensure a stability and robustness, whilst also being sufficiently large to give a good approximation. The issue of how to best choose $M$, which we discuss further in §7.2 for a specific instance, is nontrivial. Whilst there are many known theoretical estimates for how $M$ should scale for different function systems and datasets (see, for example, [9, 17, 18] and references therein), selecting $M$ optimally is difficult. In particular, standard theoretical guarantees usually only determine the asymptotic behaviour of $M$ with $1/h$, e.g. $M = O(1/\sqrt{h})$. Constants, if known, tend to be overly pessimistic. Moreover, this problem becomes more acute in multiple dimensions, since the ordering of the basis functions plays an increasingly important role.

Conversely, the truncation parameter $K$ in the infinite-dimensional weighted $\ell^1$ minimization formulations (3.5) and (3.6) plays a completely different role. It allows one to approximately compute solutions of the infinite-dimensional problems (3.1) and (3.4). Once $K$ is large enough so that the truncation error when passing to the finite-dimensional problems (3.5) and (3.6) is negligible, changing $K$ has no effect on the accuracy of the solution $\tilde{f}$. Moreover, once $K$ is large enough, the truncated solution space is essentially independent of the indexing of basis $\{\phi_i\}_{i \in \mathbb{N}}$.

Note also that (3.5) leads to interpolatory approximations, which is not the case for (3.8).

4 The need for weights

Before analyzing (3.1) and (3.6) in detail, we first examine the role weights play in the minimization. In particular, we shall show that it necessary for the ratios

$$w_i/\|\phi_i\|_{\infty} \to \infty, \quad i \to \infty. \quad (4.1)$$

in order for the weighted minimization problems to give good approximations to $f$. If this is not the case, then the minimization problem has multiple solutions which alias the data, leading in general to poor approximations.

In §7 we shall prove that (4.1) is sufficient to guarantee a good approximation. To demonstrate its necessary, we consider Example 2.2. Recall that $\|\phi_j\|_{\infty} = 1$ in this case.

**Proposition 4.1.** Let $D$, $\nu$ and $\{\phi_j\}_{j \in \mathbb{Z}}$ be as in (2.8). Let $T = \{t_n\}_{n=1}^{N}$ be a set of $N$ scattered data points such that $t_nP \in \mathbb{Z}$ for some $P \in \mathbb{N}$ and all $n = 1, \ldots, N$. Suppose that $\hat{x} \in \ell^1(\mathbb{Z})$ is a solution of

$$\inf_{z \in \ell^1(\mathbb{Z})} \|z\|_1 \quad \text{subject to } Uz = y, \quad (4.2)$$

where $U = \{\phi_j(t_n)\}_{n=1,j=-\infty}^{N,\infty}$ and $y \in \mathbb{C}^N$. Then every shift of $\hat{x}$ by a multiple of $2P$ is also a solution of (4.2). That is, for every $k \in \mathbb{Z}$, the element $z \in \ell^1(\mathbb{Z})$ given by

$$z_i = \hat{x}_{i-2kP}, \quad i \in \mathbb{Z}, \quad (4.3)$$

is a solution of (4.2).

**Proof.** Shifting the entries of $x$ does not affect its $\ell^1$ norm, therefore $\|z\|_1 = \|x\|_1$. Moreover,

$$(Uz)_n = \sum_{j \in \mathbb{Z}} z_j e^{ij\pi t_n} = \sum_{j \in \mathbb{Z}} x_j e^{ij(\xi+2kP)\pi t_n} = \sum_{j \in \mathbb{Z}} x_j e^{ij\pi t_n} = (Ux)_n = y_n, \quad n = 1, \ldots, N.$$ 

Hence $z$ is feasible for (4.2), and therefore a minimizer. \qed
Figure 2: Aliasing in $\ell^1$ minimization. The black line is the function $f(t) = \phi_0 = 1$, the red dots are the data points with $N = 11$ (left) and $N = 21$ (right) and the blue curve is the aliased solution $\phi_{10}$ (left) and $\phi_{20}$ (right). Although these solutions interpolate $f$, they do not approximate $f$ in between the data points.

The absence of quadrature weights $\tau_n$ does not change the conclusion here, since we consider the equality-constrained minimization. We could also consider the inequality-constrained problem with much the same result, but we present the equality-constrained problem to show that the phenomenon is not due in any way by the increased size of the feasible set when taking $\eta > 0$.

Taken on its own, the fact that (4.2) has multiple solutions may not be alarming. After all, infinite-dimensional convex optimization problems often do. However, in this case the effect is catastrophic. To see why, consider the problem where $f = \phi_0 = 1$ so that its coefficients are $x = e_0$. Since $y_n = f(t_n) = 1$ in this case, if $z \in \ell^1(\mathbb{Z})$ is feasible for (4.2) then

$$1 = |(Uz)_n| = \left| \sum_{j \in \mathbb{Z}} z_je^{ij\pi t_n} \right| \leq \|z\|_1.$$

Hence $x$ itself is a solution of (4.2), and by Proposition 4.1 so is every shift $z = e_{2kP}$ of $x$ by a multiple of $2P$. However, for all these solutions one has $\|x - z\| = 2$. Thus, although there is one solution of (4.2) which recovers $x$ (and therefore $f$) exactly there are also infinitely many solutions of (4.2) that give meaningless approximations to $x$.

This effect is due to the aliasing of $f$ by higher frequency Fourier modes. The shifted solutions of Proposition 4.1 correspond to the functions $\phi_{kP}$, $k \in \mathbb{Z}$, which interpolate $f$ at the data points but oscillate with frequency proportional to $kP$ in between the data points. This phenomenon is shown in Fig. 2. Of course, in the simplified scenario described here the aliasing problem could have been avoided by solving a truncated problem with truncation $K = P$. However, as discussed, this will not work in the general case when truncation with $K \gg N$ is required in order to faithfully control the tail and ensure (in the noiseless case) an interpolatory solution.

Now suppose that weights $w_i$ are added, and (4.2) is replaced by

$$\inf_{z \in \ell^1(\mathbb{Z})} \|z\|_{1,w} \quad \text{subject to } Uz = y. \quad (4.4)$$

Assume the weights $w$ satisfy $w_{-i} = w_i$, $i \in \mathbb{N}$ and $1 \leq w_0 < w_1 < w_2 < \ldots$, and consider the case of $f = \phi_0$ once more. Then none of the aliased solutions of (4.2) are solutions of (4.4), since they all have larger weighted $\ell^1$-norm: $\|e_{2kP}\| = w_{2kP} > w_0 = \|e_0\|$. Hence, adding growing weights regularizes the problem (4.4) and removes the bad, aliased solutions of (4.2). This improvement is illustrated in Fig. 3. This figure also shows that this phenomenon is not limited to the equality-constrained minimization problem.
Remark 4.2 The use of weighted minimization strategies has been previously motivated by the desire to match the decay of the true coefficients $x$ of the unknown function and thereby obtain better approximations [19, 24]. However, this is not the primary role the weights in (4.4) play. Instead, as discussed, their main role is in regularizing the problem and removing the aliasing phenomenon. To demonstrate this point, in Fig. 4 we plot the error for weighted $\ell^1$ minimization using Chebyshev polynomials for a number of different test functions and weighting strategies. As can be seen, increasing the weights does not lead to a consistent improvement across all functions, even though all functions used (beside the final one) are analytic and thus have coefficients which decay geometrically fast. Whilst weights might help in some small way by promoting smoothness, these results suggest that the effect on the approximation error is much less than the role they play in regularizing the problem. Furthermore, higher weights may well cause problems for numerical solvers, due to increasingly ill-conditioning of the $N \times K$ system matrix $UW^{-1}P_K$.

5 A general recovery result

The remainder of this paper is devoted to the analysis of the problems (3.5) and (3.6). In this section, we provide a general recovery result showing, in a standard manner, stable and robust recovery subject to the existence of an appropriate dual vector $u$. This result will allow us to firstly determine how to choose the truncation parameter $K$ (see §6), and secondly deduce a linear approximation error analysis (see §7). The final two sections, §8 and §9 we apply these general results to the examples of §2.4.

Proposition 5.1. Let $\Delta \subseteq \{1, \ldots, K\}$. Suppose that

(i) : $\|P_{\Delta}U^*UP_{\Delta} - P_{\Delta}\| \leq \alpha$,  
(ii) : $\max_{i \notin \Delta} \{\|Ue_i\|/w_i\} \leq \beta$,  

Figure 3: Recovery of the function $f(t) = \cos(\pi x) \exp(\sin(\pi x))$ (shown in black) from $N = 10$ data points (shown in red). The blue curve is the function $\hat{f}$ obtained from (weighted) $\ell^1$ minimization using the Fourier basis with weights $w_i = 1$ (left), $w_i = 1 + |i|^{1/10}$ (middle) and $w_i = 1 + |i|^{1/2}$ (right). Top row: equality-constrained minimization (3.1). Bottom row: inequality-constrained minimization (3.4) with $\eta = 10^{-2}$.  

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Proposition 5.1. Let $\Delta \subseteq \{1, \ldots, K\}$. Suppose that

(i) : $\|P_{\Delta}U^*UP_{\Delta} - P_{\Delta}\| \leq \alpha$,  
(ii) : $\max_{i \notin \Delta} \{\|Ue_i\|/w_i\} \leq \beta$,  


and that there exists a vector \( u = W^{-1}U^*u' \) for some \( u' \in \mathbb{C}^M \), such that

\[
(iii) : |W(P_\Delta u - \text{sign}(P_\Delta x))| \leq \gamma, \quad (iv) : \|P_\Delta^\perp u\|_\infty \leq \theta, \quad (v) : \|u'\| \leq L\sqrt{s},
\]

where \( s = \sum_{i \in \Delta} (w_i)^2 \), for constants \( 0 \leq \alpha, \theta < 1 \) and \( \beta, \gamma, L \geq 0 \) satisfying

\[
\frac{\sqrt{1 + \alpha \beta \gamma}}{(1 - \alpha)(1 - \theta)} < 1.
\]

Let \( \hat{x} \) be a minimizer of (3.6). Then, if \( \bar{x} \in \ell^1_w(\mathbb{N}) \) is feasible for (3.6), i.e. \( \|UP_K \bar{x} - y\| \leq \sqrt{\tau} \eta \), the error estimate

\[
\|\hat{x} - x\| \leq 2 \left( C_1 + C_2 L \sqrt{s} \right) \sqrt{\tau} \eta + 2 \left( 2\|P_\Delta^\perp x\|_{1,w} + \|x - \bar{x}\|_{1,w} \right),
\]

holds, where \( C_1 = \left( 1 + \frac{\gamma}{1 - \theta} \right) C_0 \), \( C_2 = \frac{\beta}{1 - \theta} \left( 1 + \frac{\gamma}{1 - \theta} \right) C_0 + \frac{1}{1 - \theta} \) and \( C_0 = \left( 1 - \sqrt{\frac{1 + \alpha \beta \gamma}{(1 - \alpha)(1 - \theta)}} \right)^{-1} \sqrt{\frac{1 + \alpha}{1 - \alpha}} \).

Note that the problem (3.5) can be viewed as a special case of (3.6) corresponding to the case \( \eta = 0 \). Hence this result considers only (3.6).

**Proof.** Let \( v = \hat{x} - x \). Then \( Av = P_\Delta U^*Uv - P_\Delta U^*UP_\Delta^\perp v \), where \( A \) is the restriction of \( P_\Delta U^*U\) to \( P_\Delta(\ell^1(\mathbb{N})) \). By (i), we have \( \|A^{-1}\| \leq \frac{1}{1 - \alpha} \) and

\[
\|P_\Delta U^*\|^2 = \|UP_\Delta\|^2 = \|P_\Delta U^*UP_\Delta\| \leq 1 + \alpha.
\]

Thus

\[
\|P_\Delta v\| \leq \frac{1}{1 - \alpha} \|P_\Delta U^*\| \|Uv\| + \frac{1}{1 - \alpha} \|P_\Delta U^*UP_\Delta^\perp v\| \leq \frac{\sqrt{1 + \alpha}}{1 - \alpha} \left( \|Uv\| + \|UP_\Delta^\perp v\| \right).
\]

Figure 4: Weighted \( \ell^1 \) minimization with equispaced data and Chebyshev polynomials. The error \( \|f - \tilde{f}\|_{L^\infty} \) against \( N \) is shown for the choice \( w_i = i^2 \), where \( \gamma = 0.0, 0.05, 0.1, 0.25, 0.5, 0.75, 1.0, 1.5, 2.0, 2.5, 3.0 \) (thickest to thinnest). The truncation parameter \( K = 4N \) was used. As with all numerical results in this paper, the minimization problem (3.5) was solved using the CVX optimization package.
Observe that

\[ \|U v\| = \|U \hat{x} - U x\| \leq 2\sqrt{\tau} + \delta. \tag{5.2} \]

Hence

\[ \|P_{\Delta} v\| \leq \frac{\sqrt{1 + \alpha}}{1 - \alpha} \left( (2\sqrt{\tau} + \delta) + \|UP_{\Delta} v\| \right). \]

The second term can be estimated as follows:

\[ \|UP_{\Delta} v\| \leq \sum_{i \notin \Delta} |v_i| \|U e_i\| \leq \beta \|P_{\Delta} v\|_{1,w}, \]

where the latter inequality is due to (ii). Hence we get

\[ \|P_{\Delta} v\| \leq \frac{\sqrt{1 + \alpha}}{1 - \alpha} \left( (2\sqrt{\tau} + \delta) + \beta \|P_{\Delta} v\|_{1,w} \right). \tag{5.3} \]

We shall return to this inequality later, but let us now consider \( \hat{x} \).

\[
\begin{align*}
\|\hat{x}\|_{1,w} & = \|P_{\Delta} \hat{x}\|_{1,w} + \|P_{\Delta} \hat{x}\|_{1,w} \\
& \geq \text{Re} \langle P_{\Delta} W \hat{x}, \text{sign}(P_{\Delta} x) \rangle + \|P_{\Delta} v\|_{1,w} - \|P_{\Delta} \hat{x}\|_{1,w} \\
& = \text{Re} \langle P_{\Delta} W v, \text{sign}(P_{\Delta} x) \rangle + \|P_{\Delta} x\|_{1,w} + \|P_{\Delta} v\|_{1,w} - \|P_{\Delta} x\|_{1,w} \\
& = \text{Re} \langle P_{\Delta} W v, \text{sign}(P_{\Delta} x) \rangle + \|x\|_{1,w} + \|P_{\Delta} v\|_{1,w} - 2\|P_{\Delta} x\|_{1,w}. \tag{5.4} \\
\end{align*}
\]

Now let \( \bar{x} \in \ell_{w}^1(\mathbb{N}) \) be any feasible solution for \( \bf{B0} \). Then \( \|\hat{x}\|_{1,w} \leq \|\bar{x}\|_{1,w} \) and we get

\[ \|\bar{x}\|_{1,w} \geq \text{Re} \langle P_{\Delta} W v, \text{sign}(P_{\Delta} x) \rangle + \|x\|_{1,w} + \|P_{\Delta} v\|_{1,w} - 2\|P_{\Delta} x\|_{1,w} \]

After rearranging this gives

\[ \|P_{\Delta} v\|_{1,w} \leq |\langle P_{\Delta} W v, \text{sign}(P_{\Delta} x) \rangle| + 2\|P_{\Delta} x\|_{1,w} + \|x - \bar{x}\|_{1,w}. \tag{5.5} \]

We next estimate \( |\langle P_{\Delta} W v, \text{sign}(P_{\Delta} x) \rangle| \). We have

\[ |\langle P_{\Delta} W v, \text{sign}(P_{\Delta} x) \rangle| \leq |\langle P_{\Delta} W v, \text{sign}(P_{\Delta} x) - P_{\Delta} u \rangle| + |\langle W v, u \rangle| + |\langle P_{\Delta} W v, P_{\Delta} u \rangle|. \]

Note that \( |\langle P_{\Delta} W v, \text{sign}(P_{\Delta} x) - P_{\Delta} u \rangle| \leq \gamma \|P_{\Delta} v\| \) an and also that \( \langle W v, u \rangle = \langle v, W u \rangle = \langle v, U^* u' \rangle = \langle UV, u' \rangle \). Hence, \( \bf{B2} \) and (v) give

\[ |\langle W v, u \rangle| \leq \|U v\| L\sqrt{s} \leq (2\sqrt{\tau} + \delta)L\sqrt{s}. \]

Finally, by (iv), we have \( |\langle P_{\Delta} W v, P_{\Delta} u \rangle| \leq \|P_{\Delta} u\|_{\infty} \|P_{\Delta} v\|_{1,w} \leq \theta \|P_{\Delta} v\|_{1,w} \) and therefore

\[ |\langle P_{\Delta} W v, \text{sign}(P_{\Delta} x) \rangle| \leq \gamma \|P_{\Delta} v\| + (2\sqrt{\tau} + \delta)L\sqrt{s} + \theta \|P_{\Delta} v\|_{1,w}. \]

Substituting into (5.5) and rearranging now yields

\[ (1 - \theta)\|P_{\Delta} v\|_{1,w} \leq \gamma \|P_{\Delta} v\| + (2\sqrt{\tau} + \delta)L\sqrt{s} + 2\|P_{\Delta} x\|_{1,w} + \|x - \bar{x}\|_{1,w}, \]

and applying (5.3) gives

\[ \|P_{\Delta} v\| \leq \frac{\sqrt{1 + \alpha}}{1 - \alpha} \left[ (2\sqrt{\tau} + \delta) + \beta \left( \frac{\gamma}{1 - \theta} \|P_{\Delta} v\| + \frac{2\sqrt{\tau} + \delta}{1 - \theta}L\sqrt{s} + 2\|P_{\Delta} x\|_{1,w} + \|x - \bar{x}\|_{1,w} \right) \right] . \]
Hence
\[ \| P_\Delta v \| \leq \left( 1 - \frac{\sqrt{1 + \alpha \beta \gamma}}{(1 - \alpha)(1 - \theta)} \right)^{-1} \frac{\sqrt{1 + \alpha}}{1 - \alpha} \left( 1 + \frac{\beta}{1 - \theta} L\sqrt{s} \right) (2\sqrt{\tau \eta} + \delta) \\
+ \left( 1 - \frac{\sqrt{1 + \alpha \beta \gamma q}}{(1 - \alpha)(1 - \theta)} \right)^{-1} \frac{\sqrt{1 + \alpha \beta}}{1 - \alpha} (2\| P_\Delta^iv \|_{1,w} + \| x - \bar{x} \|_{1,w}) \\
= C_0 \left( 1 + \frac{\beta}{1 - \theta} L\sqrt{s} \right) (2\sqrt{\tau \eta} + \delta) + C_0 \beta \left( 2\| P_\Delta^i x \|_{1,w} + \| x - \bar{x} \|_{1,w} \right). \]

Since \( \| P_\Delta^iv \| \leq \| P_\Delta^iv \|_{1,w} \) (recall (2.3)), we now get
\[ \| v \| \leq \| P_\Delta v \| + \| P_\Delta^iv \|_{1,w} \]
\[ \leq \left( 1 + \frac{\gamma}{1 - \theta} \right) \| P_\Delta v \| + \frac{2\sqrt{\tau \eta} + \delta}{1 - \theta} \left( \sqrt{s} \right) \left( 1 + \frac{\beta}{1 - \theta} \right) (2\| P_\Delta^i x \|_{1,w} + \| x - \bar{x} \|_{1,w}) \\
\leq \left[ \left( 1 + \frac{\gamma}{1 - \theta} \right) C_0 \left( 1 + \frac{\beta}{1 - \theta} \right) \left( \sqrt{s} \right) \left( 1 + \frac{\beta}{1 - \theta} \right) \right] \left( 2\| P_\Delta^i x \|_{1,w} + \| x - \bar{x} \|_{1,w} \right), \]

as required. \( \square \)

In practice, we shall make use of the following corollary.

**Corollary 5.2.** Let \( \Delta \subseteq \{1, \ldots, K\} \). Suppose that there are constants \( 0 \leq \alpha, \theta < 1 \) such that

(a) : \( \| P_\Delta^i x \|_{1,w} \| \leq \alpha \), \quad (b) : \( \| P_\Delta^i x \|_{1,w} \| \leq \theta \),

where \( A = P_\Delta^i x \). If \( \hat{x} \) is a minimizer of (5.6) and \( s = \sum_{i \in \Delta} (w_i)^2 \), then
\[ \| \hat{x} - x \| \leq 2\sqrt{\tau \eta} + \delta \left( 1 + C_0 \sqrt{s} \right) \left( 1 + C_3 \right) \left( 2\| P_\Delta^i x \|_{1,w} + \| x - \bar{x} \|_{1,w} \right), \]

where \( \bar{x} \in \ell_w^i(N) \) is any feasible solution of (3.6) and \( C_3 = \frac{1}{1 - \theta} \left( 1 + \frac{\sqrt{1 + \alpha \beta \gamma q}}{(1 - \alpha)(1 - \theta)} \right) \).

**Proof.** We apply Proposition 5.1 with \( u = W^{-1}U^i P_\Delta^i A^{-1} WP_\Delta^i \). Note that (a) and (b) imply (i) and (iv) respectively. Also, by construction, \( P_\Delta u = P_\Delta \) and therefore (iii) holds with \( \gamma = 0 \). Now consider (ii). By definition
\[ \| U e_i \| = \| \phi_i \| N \leq \| \phi_i \| \sqrt{\sum_{n=1}^{N} \tau_n} \leq w_i, \]

where the last inequality is due to (2.3) and the fact that \( \sum_{n=1}^{N} \tau_n = 1 \) when the weights \( \tau_n \) are given by (2.3). Hence (ii) holds with \( \beta = 1 \). Finally, observe that
\[ \| u' \| = \| U P_\Delta A^{-1} WP_\Delta \| \| x \| \| A^{-1} \| \| WP_\Delta \| \| x \| \leq \frac{\sqrt{1 + \alpha \beta \gamma q}}{(1 - \alpha)(1 - \theta)} \sqrt{s}, \]

where the final inequality follows from (a). Hence (v) holds with \( L = \frac{\sqrt{1 + \alpha \beta \gamma q}}{(1 - \alpha)(1 - \theta)} \). \( \square \)
6 Handling truncation: the choice of $K$

Having obtained the error estimate (5.6), we can now discuss how to choose the parameter $K$. From (5.6), it is clear that $K$ impacts the approximation error $\|\hat{x} - x\|$ solely in the term $\|x - \bar{x}\|_{h,K}$. Hence, if $K$ is chosen such that

$$T_{h,K}(x) = \inf \{\|x - \bar{x}\|_{1,w} : \bar{x} \in \mathbb{C}^K, \|UPK\bar{x} - y\| \leq \rho\},$$

is small, where $\rho = \sqrt{\tau_N}$, then the truncation effect due to $K$ is negligible.

6.1 Main result

It is not given that (3.5) or (3.6) has a solution for arbitrary $K \geq N$, since, for example in (3.5), the vector $y$ need not lie in the range of $UPK$ unless the minimum singular value $\sigma_{\min}$ of $(UPK)^*$ is positive. Whilst this will be the case for sufficiently large $K$, our aim now is to quantify precisely how large $K$ needs to be. First, we note the following:

**Lemma 6.1.** The minimum singular value $\sigma_{\min}$ of $(UPK)^*$ satisfies

$$\sigma_{\min} \geq 1 - \sup_{g \in \mathbb{C}^N} \inf_{\|y\| = 1} \inf_{\phi \neq 0} \left\{ \frac{\|g - \phi\|_{\infty}}{\|y\| - \|g - \phi\|_{\infty}} \right\},$$

where $\Phi_K = \text{span}\{\phi_1, \ldots, \phi_K\}$ and $G_y = \{g \in L_0^2(D) \cap L^\infty(D) : g(t_n) = y_n/\sqrt{\tau_n}\}$.

**Proof.** The minimum singular value is given by $\sigma_{\min} = \inf_{y \in \mathbb{C}^N} \| (UPK)^* y \|$. Fix $y \in \mathbb{C}^N$, $\|y\| = 1$ and observe that

$$\| (UPK)^* y \| = \sup_{\|z\| = 1} |\langle y, UPKz \rangle| = \sup_{\|\phi\|_{\infty} = 1} \left|\sum_{n=1}^N \sqrt{\tau_n} y_n \overline{\phi(t_n)}\right|.$$

Let $g \in G_y$ and $\phi \in \Phi_K$. Then

$$\| (UPK)^* y \| \geq \sum_{n=1}^N \sqrt{\tau_n} y_n |\overline{\phi(t_n)}| \geq \|y\|^2 - \sum_{n=1}^N \sqrt{\tau_n} y_n |\phi(t_n) - \overline{\phi(t_n)}| \geq \|y\|^2 - \|y\| \|g - \phi\|_{L^\infty}.$$

Since $\|\phi\|_{\infty} \geq \|g\|_{\nu} - \|g - \phi\|_{\nu} \geq \|g\|_{\nu} - \|g - \phi\|_{\infty}$ the result now follows immediately.

This lemma gives an explicit criterion for the matrix $(UPK)^*$ to have full rank and thus ensure existence of a solution. Note that this trivially holds if $\Phi_K$ is interpolatory: that is, for any $\{y_n\}_{n=1}^N$ there exists a $\phi \in \Phi_K$ such that $\phi(t_n) = y_n$. This holds for all examples in this paper, e.g. algebraic or trigonometric polynomials, as well as many other cases. On the other hand, the main usefulness of this result is that it gives a lower bound for the minimal singular value of $(UPK)^*$, which in turn will lead to explicit criteria in (6.2) for how to select $K$.

Before doing this, let us now relate $\sigma_{\min}$ to the truncation error (6.1). The following is the main result of this section:

**Theorem 6.2.** Suppose that $\sigma_{\min} > 0$, where $\sigma_{\min}$ is the minimal singular value of $(UPK)^*$, and let $x \in \ell_1^1(\mathbb{N})$, where $\tilde{w} = \{\tilde{w}_i\}_{i \in \mathbb{N}}$ with $\tilde{w}_i \geq \sqrt{i}w_i^2$, $\forall i \in \mathbb{N}$. Then

$$T_{h,K}(\sqrt{\tau_N}) \leq \|x - P_Kx\|_{1,w} + 1/\sigma_{\min}\|x - P_Kx\|_{1,\tilde{w}}.$$
Proof. Since \( UP_K \) has full rank we let \( \bar{x} = P_K x + (UP_K)\dagger U(x - P_K x) \). Then

\[
\|UP_K \bar{x} - y\| = \|UP_K x + U(x - P_K x) - U x - \{\sqrt{\tau_n}e_n\}_{n=1}^N\| \leq \sqrt{\tau_1},
\]

so that \( \bar{x} \) is feasible. Moreover, \( \|x - \bar{x}\|_{1,w} \leq \|x - P_K x\|_{1,w} + \|(UP_K)\dagger U(x - P_K x)\|_{1,w} \). Note that

\[
\|(UP_K)\dagger U(x - P_K x)\|_{1,w}^2 \leq \sum_{i=1}^K (w_i)^2 \|(UP_K)\dagger U(x - P_K x)\|^2 \leq \left( \sum_{i=1}^K (w_i)^2 \right) \|U(x - P_K x)\|^2 / \sigma_{\text{min}} \leq \left( \sum_{i=1}^K (w_i)^2 \right) \|x - P_K x\|_{1,w}^2.
\]

To obtain the result, we note that

\[
\sqrt{\sum_{i=1}^K (w_i)^2 \|x - P_K x\|_{1,w}} \leq w_K \sqrt{K} \sum_{i > K} w_i |x_i| \leq \sum_{i > K} \bar{w}_i |x_i| = \|x - P_K x\|_{1,\bar{w}},
\]

as required. \( \square \)

This theorem shows that once \( K \) is chosen so that \( 1/\sigma_{\text{min}} \) is moderate in size, the effect of truncation is bounded by the decay of the coefficients \( x_i, \ i > K \). As we see later via example, this means the truncation error is effectively negligible in practice.

Remark 6.3 In practice, rather than performing an analysis of \( \sigma_{\text{min}} \), one may choose \( K \) simply by calculating the minimal singular value of \( UP_K \). Provided this is sufficiently large, then Theorem \( \ref{thm:6.2} \) guarantees that the truncation error is negligible.

Remark 6.4 A downside of this result is that it requires additional regularity of the coefficients \( x \), which now must belong to the weighted space \( \ell_{1,\bar{w}}^1 \). Removing this condition is an open problem.

6.2 Examples

Lemma \ref{lem:6.1} allows us to determine the required condition on \( K \) by analyzing the right-hand side of \( \ref{thm:6.2} \). Note that this depends completely on the points \( T \) and the basis \( \{\phi_i\}_{i \in \mathbb{N}} \). We now do this for the two examples of \ref{ex:2.4}. We first consider the Jacobi polynomial case:

Theorem 6.5. For \( \alpha, \beta > -1 \) let \( \{\phi_i\}_{i \in \mathbb{N}} \) be the orthonormal Jacobi polynomial basis \( \ref{ex:2.6} \) and let \( T = \{t_n\}_{n=1}^N \) be a set of \( N \) points in \([-1,1]\). Then for every \( r \in \mathbb{N} \) there exists a \( C_r > 0 \) such that

\[
\sigma_{\text{min}} \geq \frac{1}{\sqrt{x/(2h_N^1)}} - C_r K^{-r - 1/2} \frac{\xi^{-r - 1/2}}{\sqrt{x/(2h_N^1)} - C_r K^{-r - 1/2}},
\]

where \( \sigma_{\text{min}} \) is the minimal singular value of \( (UP_K)^* \),

\[
\xi = \frac{1}{2} \min_{n=0,\ldots,N} \{t_{n+1} - t_n\}, \quad h = \sup_{-1 \leq t \leq 1} \min_{n=1,\ldots,N} |t - t_n|,
\]

and \( t_0 = -t_1 - 2, \ t_{n+1} = 2 - t_N \). In particular, if

\[
K \geq \left( \sqrt{2} C_r (1 + \epsilon^{-1}) \right)^{\frac{1}{2}} h^{\frac{1}{2}} \xi^{-\frac{1}{2}} \theta, \quad 0 < \epsilon < 1 \text{ then } \sigma_{\text{min}} > 1 - \epsilon.
\]
We defer the proof of this result until §8.4. In the case of equispaced data, we have $h = \xi = 1/N$ and therefore it suffices to take $K \gtrsim N^{1+\frac{r}{2}}$ for any $r \in \mathbb{N}$. It follows that $K$ is almost linear in $N$. In practice, we have found that $K = 4N$ is sufficient in all examples (recall also Remark 6.3). On the other hand, if the data clusters severely, then a larger value of $K$ may be necessary.

A similar result also holds in the Fourier case:

**Theorem 6.6.** Let $\{\phi_i\}_{i \in \mathbb{Z}}$ be the orthonormal Fourier basis (2.8), $T = \{t_n\}_{n=1}^N \subseteq D$ be a set of $N$ scattered data points and suppose that $h$ is as in (2.8). Then for every $r \in \mathbb{N}$ there exists a $C_r > 0$ such that

$$\sigma_{\min} \geq 1 - \frac{C_r K^{-r} \xi^{-r-1/2}}{r^2}$$

where $\sigma_{\min}$ is the minimal singular value of $(UP_K)^*$,

$$\xi = \frac{1}{2} \min_{n=0, \ldots, N} \{t_{n+1} - t_n\}, \quad h = \sup_{|t| \leq 1} \min_{n=1, \ldots, N} |t - t_n|,$$

and $t_0 = -1$, $t_{n+1} = 1$. In particular, if

$$K \geq \left(\sqrt{2}C_r (1 + \epsilon^{-1})\right)^{\frac{1}{r}} h^\frac{r}{2} \xi^{-1-\frac{r}{2}},$$

for some $0 < \epsilon < 1$ then $0 \leq \gamma < \epsilon$.

This result is exactly the same as that for Jacobi polynomials (Theorem 6.5), except up to a minor change in the definition of $\xi$. Its proof is near identical, and hence is omitted.

### 7 Linear approximation error of weighted $\ell^1$ minimization

Besides the effects of noise and truncation, with the latter having already been addressed, Corollary 6.2 gives conditions under which $x$ is recovered up to an error depending on how well it is approximated by the coefficients $x_j$, $j \in \Delta$, for some arbitrary set $\Delta$. This depends on the conditioning of the corresponding submatrix (condition (a)) and the off-support magnitude the dual vector (condition (b)). Under a weighted sparsity condition on the coefficients $x$, and appropriate random choices of the points $T$, one may use this to prove estimates relating the number of measurements to the weighted sparsity. See [23, 24] for related works in the finite-dimensional setting.

However, as discussed in §11 sparsity is not always present in the low-dimensional setting, and the data points may not arise from such ideal distributions. In this section, we present a linear approximation error analysis for arbitrary deterministic scattered data points. This follows by setting $\Delta = \{1, \ldots, M\}$ for some $M \leq K$ and using Corollary 5.2 to address the question of how large $M$ can be chosen in relation to the density $h$ of the points $T$. As we explain in Remark 7.6, doing this will allow us to make a direct comparison with other techniques, e.g. least-squares fitting.

To make such statements, which will be asymptotic in $h \to 0$, we need an additional assumption. Let $H$ be a subspace of $L^2(D) \cap L^\infty(D)$ which is closed under multiplication and complex conjugation and such that $f \in H$ and $\{\phi_i\}_{i \in \mathbb{N}} \subseteq H$. We now assume that the points $T$ satisfy

$$\sum_{n=1}^N \tau_n g(t_n) \to \int_D g(t) \nu(t) \, dt, \quad h \to 0, \quad \forall g \in H. \quad (7.1)$$

In particular, since $H$ is closed under multiplication and complex conjugation, one has that

$$\langle f, g \rangle_h \to \langle f, g \rangle_{L^2}, \quad h \to 0, \quad \forall f, g \in H.$$
Hence the discrete inner product is equivalent to $\langle \cdot, \cdot \rangle_{L^2}$ on finite-dimensional subspaces of $H$ for sufficiently small $h$. Note that this assumption is by no means stringent. For example, if $D = (-1, 1)$ we may take $H$ to be the space of functions for which $|f(t)|^2 \nu(t)$ is Riemann integrable.

### 7.1 Main result

For $h > 0$ and $M, R \in \mathbb{N}$ let us define the quantities

$$E_2(h, M) = \| P_M - P_M U^* U P_M \|, \quad E_\infty(h, M) = \| P_M - P_M U^* U P_M \|_\infty,$$

and

$$F(h, M, R) = \| P_R^1 W^{-1} U^* U P_M \|_\infty.$$  

(7.2)

For convenience, we also set $E(h, M) = \max \{ E_2(N, M), E_\infty(N, M) \}$. We first require the following three lemmas:

**Lemma 7.1.** For fixed $M$, we have $E(h, M) \to 0$ as $h \to 0$.

**Proof.** Since all norms on $\mathbb{C}^M$ are equivalent, it suffices to show that $(U^* U)_{i,j} \to \delta_{ij}$ for each $i, j = 1, \ldots, M$ as $h \to 0$. However, by (7.1) and orthogonality of the $\phi_j$, we have

$$(U^* U)_{i,j} = \langle \phi_i, \phi_j \rangle_h \to \langle \phi_i, \phi_j \rangle_{L^2} = \delta_{ij},$$

as required. \qed

**Lemma 7.2.** Suppose that the weights $w_i = z_i \| \phi_i \|_{L^\infty}$ for some $z_i \geq 1$. Then

$$F(h, M, R) \leq \sqrt{M} \frac{1 + E(N, M)}{\min_{i > R} \{ z_i \}}.$$  

**Proof.** Let $x \in P_M(\ell^2(\mathbb{N}))$, $\| x \|_\infty = 1$ be arbitrary. Then $(W^{-1} U^* U P_M x)_i = \langle g, \phi_i \rangle_h / w_i$, where $g = \sum_{j=1}^M x_j \phi_j$. By the Cauchy–Schwarz inequality,

$$\| P_R^1 W^{-1} U^* U P_M x \|_\infty \leq \sup_{i > R} \left\{ \frac{1}{w_i} \| g \|_h \| \phi_i \|_h \right\}.$$  

Now

$$\| g \|_h^2 = \langle x, P_M U^* U P_M x \rangle \leq (1 + E(h, M)) \| x \|^2 \leq M (1 + E(h, M)).$$  

Also $\| \phi_i \|_h \leq \| \phi_i \|_{L^\infty} = w_i / z_i$ and therefore

$$\| P_R^1 W^{-1} U^* U P_M \|_\infty \leq \sqrt{M} \frac{1 + E(h, M)}{\min_{i > R} \{ z_i \}},$$

as required. \qed

**Lemma 7.3.** Suppose that the weights $w_i = z_i \| \phi_i \|_{L^\infty}$, where $z_i \geq 1$ and $z_i \to \infty$ as $i \to \infty$. Then for any $0 < \epsilon < 1/2$ and any $M \in \mathbb{N}$ it is possible to find $R \in \mathbb{N}$ and $h > 0$ depending on $M$ and $\epsilon$ such that

$$E(h, M) < \epsilon, \quad E(h, R) < \epsilon \frac{\min_{M < i \leq R} \{ w_i \}}{\max_{i=1, \ldots, M} \{ w_i \}}, \quad F(h, M, R) \leq \frac{\epsilon}{\max_{i=1, \ldots, M} \{ w_i \}}.$$  

(7.4)
Proof. By Lemma 7.1 we can find an $h_1$ such that $E(h, M) < \epsilon$ for all $h \leq h_1$, thus satisfying the first condition in (7.4). Using Lemma 7.2 we note that

$$F(h, M, R) \leq \frac{\sqrt{2M}}{\min_{i > R} \{z_i\}}, \quad \forall h \leq h_1.$$  

To satisfy the third condition in (7.4), we pick $R$ sufficiently large so that

$$\min_{i > R} \{z_i\} > \frac{\sqrt{2M}}{\max_{i = 1, \ldots, M} \{w_i\}}/\epsilon.$$  

We now pick $h_2$ sufficiently small so that

$$E(h, R) < \epsilon \frac{\min_{i > M} \{w_i\}}{\max_{i = 1, \ldots, M} \{w_i\}}, \quad \forall h \leq h_2,$$

and then set $h = \min\{h_1, h_2\}$. □

We now present our main result of this section:

**Theorem 7.4.** Suppose that the weights $w_i = z_i \|\phi_i\|_{L, \infty}$, where $z_i \geq 1$ and $z_i \to \infty$ as $i \to \infty$. For $0 < \epsilon < 1/2$, let $h > 0$ and $M, R \in \mathbb{N}$ be such that (7.4) holds. Then there exists a constant $C(\epsilon)$ such that, if $\hat{x}$ is a minimizer of (3.6), we have

$$\|x - \hat{x}\| \leq C(\epsilon) \left(1 + \|P_M w\|\right) \sqrt{\varphi} + \|P_M^\perp x\|_{1, w} + T_{h, K, \sqrt{\varphi}}(x),$$

where $T_{h, K, \sqrt{\varphi}}(x)$ is as in (6.1). Moreover, $\lim_{\epsilon \to 0^+} C(\epsilon) = 4$.

Note that the weights condition here is equivalent to (2.4) and (4.1) which was argued in [1] to be necessary for the success of weighted $\ell^1$ minimization. This result shows that the same condition is also sufficient.

**Proof.** We use Corollary 5.2 with $\Delta = \{1, \ldots, M\}$. Note first that

$$\alpha \leq E(h, M) < \epsilon < 1,$$

and therefore (a) holds with $\alpha \leq \epsilon$. Now consider (b). Write

$$u = W^{-1} U^* U P_M A^{-1} W P_M \text{sign}(x),$$

so that (b) is equivalent to $\|P_M^\perp u\|_{\infty} \leq \theta$. We have

$$\|P_M^\perp u\|_{\infty} \leq \max \left\{\|P_R P_M^\perp u\|_{\infty}, \|P_M^\perp u\|_{\infty}\right\}.$$
where the first equality is due to the fact that $P_M^\perp P_M = 0$. Therefore, we obtain

$$\|P_R P_M^\perp u\|_\infty \leq \max_{1 \leq i \leq M} \{w_i\} \left( \frac{E(h, R)}{1 - E(h, M)} \right) \leq \frac{\epsilon}{1 - \epsilon}. \quad (7.9)$$

Now consider the other term in (7.8). By (7.7) and the definition of $F(h, M, R)$,

$$\|P_R^\perp u\|_\infty \leq \|P_R^\perp W^{-1} U^* U P_M\|_\infty \|A^{-1}\|_\infty \|P_M W\|_\infty \leq \max_{i=1, \ldots, M} \{w_i\} \left( \frac{1 - E(h, M)}{1 - E(h, M)} \right) F(h, M, R) \leq \frac{\epsilon}{1 - \epsilon}.$$

Combining this with (7.9) and substituting into (7.8) yields $\theta \leq \frac{\epsilon}{1 - \epsilon} < 1$. The result now follows immediately from Corollary 5.2. 

7.2 Comparison with least-squares fitting

We now examine how Theorem 7.4, which assesses the linear approximation error for weighted $\ell^1$ minimization, compares with the corresponding result for the least-squares fitting (3.8). The following result is standard (a proof is given for completeness):

**Theorem 7.5.** For $0 < \epsilon < 1$ suppose that $M \in \mathbb{N}$ and $h > 0$ are such that $E_2(h, M) \leq \epsilon < 1$, where $E_2(h, M)$ is as in (7.2). Then the problem (3.8) has a unique solution $\hat{x}$, which satisfies

$$\|x - \hat{x}\| \leq \left( 1 + \frac{1}{\sqrt{1 - \epsilon}} \right) \|x - P_M x\|_{1, w} + \frac{1}{\sqrt{1 - \epsilon}} \sqrt{\tau \eta}, \quad (7.10)$$

for any $w = \{w_i\}_{i \in \mathbb{N}}$ with $w_i \geq \|\phi_i\|_{L^\infty}$.

**Proof.** Observe that

$$\|U P_M z\|^2 = z^* P_M U^* U P_M z = \|z\|^2 - z^* (P_M - P_M U^* U P_M) z \geq (1 - E_2(h, M)) \|z\|^2.$$

Hence $U P_M$ has full rank since $E_2(h, M) \leq \epsilon < 1$ and its minimum singular value $\sigma_{\min} \geq \sqrt{1 - \epsilon}$. This implies that $\hat{x}$ is unique and is given by $\hat{x} = (U P_M)^\dagger y$. Hence

$$\hat{x} - P_M x = (U P_M)^\dagger \left( U x + \{\sqrt{\tau \eta} \epsilon_n \}_{n=1}^N \right) - P_M x = (U P_M)^\dagger \left( U (x - P_M x) + \{\sqrt{\tau \eta} \epsilon_n \}_{n=1}^N \right).$$

Therefore

$$\|\hat{x} - P_M x\| \leq \frac{1}{\sqrt{1 - \epsilon}} \left( \|U (x - P_M x)\| + \sqrt{\tau \eta} \right) \leq \frac{1}{\sqrt{1 - \epsilon}} \left( \|x - P_M x\|_{1, w} + \sqrt{\tau \eta} \right).$$

Since $\|x - \hat{x}\| \leq \|x - P_M x\| + \|\hat{x} - P_M x\| \leq \|x - P_M x\|_{1, w} + \|\hat{x} - P_M x\|$, the result now follows immediately. 

The error bound (7.10) is very similar to the linear approximation error bound (7.5) for weighted $\ell^1$ minimization. In the absence of noise and truncation error, both depend on the term $x - P_M x$, i.e. the tail of $x$ beyond its first $M$ coefficients. The main difference is in the choice of $M$, which is determined through the conditions of Theorems 7.4 and 7.5. We shall discuss this point further in the next section.
Remark 7.6 Theorems [24] and [7.3] provide a mathematical recipe for determining the worst-case behaviour of weighted $\ell^1$ minimization for scattered data. Given an orthonormal system $\{\phi_i\}_{i\in\mathbb{N}}$, an $h > 0$ and an $0 < \epsilon < 1/2$, determine:

1. the largest $M = M_1(h)$ such that $E_2(h, M) < \epsilon$,
2. the largest $M = M_2(h)$ such that $(7.3)$ holds for some appropriate $R$ and $\{w_i\}_{i\in\mathbb{N}}$.

In this case, the errors for both weighted $\ell^1$ minimization and least-squares fitting are determined by $\|x - P_M x\|_{1,w}$, where $M = M_1(h)$ for the former and $M = M_2(h)$ for the latter. Hence, if $M_1(h) \gg M_2(h)$ as $h \to 0$ it follows that both least-squares fitting and weighted $\ell^1$ minimization are guaranteed to converge at roughly the same asymptotic rate as $h \to 0$.

Since this the various quantities are dependent on the reconstruction system $\{\phi_i\}_{i\in\mathbb{N}}$ a separate analysis must be carried out in use of the prescribed recipe. The last two sections of this paper will be devoted to doing this for the examples introduced in [2.4]. Note that for weighted $\ell^1$ minimization, we also introduce a third step:

3. Given $0 < \gamma < 1$, find the smallest $K$ such that $\sigma_{\min} \geq 1 - \gamma$, where $\sigma_{\min}$ is the minimal singular value of $(UP_K)^*$.

In combination with Theorem [6.2] this will reveal how to appropriately choose the truncation parameter $K$. Empirically this can be done using the lower bound described in Lemma [6.2]. See also [6.2] for analytical results for the examples of [2.4].

8 Jacobi polynomials on the unit interval

In this section we consider Example [2.1]. For convenience, we recall the growth condition (2.7):

$$\|\phi_j\|_{L^\infty} = O\left(q^{j+1/2}\right), \quad j \to \infty,$$

where $q = \begin{cases} \max\{\alpha, \beta\}, & \alpha, \beta \geq -1/2 \\ -1/2, & \alpha, \beta < -1/2 \end{cases}$ (8.1)

8.1 Main results

Theorem 8.1. For $\alpha, \beta > -1$ let $\{\phi_i\}_{i\in\mathbb{N}}$ be the orthonormal Jacobi polynomial basis (2.7), $T = \{t_n\}_{n=1}^N \subseteq D$ be a set of scattered data points and suppose that $h$ is as in (2.4). Suppose that the weights $w_i = z_i(i^{q+1/2})$, where $q$ is given by (8.1) and $z_i \geq 1$ satisfies $z_i = O(i^\gamma)$ as $i \to \infty$ for some $\gamma > 0$. Then for each $0 < \epsilon < 1/2$ there exists a $c(\epsilon) > 0$ such that if

$$h \leq c(\epsilon) \left\{ \begin{array}{ll} \frac{1}{M^{q+1/2} \log M}, & 0 < \gamma \leq 1/2 - q \\ \frac{1}{M^{q+1/2} \log M}, & \gamma > 1/2 - q \end{array} \right. ,$$

then any minimizer $\hat{x}$ of (2.6) satisfies

$$\|x - \hat{x}\| \leq C(\epsilon) \left( M^{\gamma+q+1/2} + \|x - P_M x\|_{1,w} + T_{N,K,\sqrt{\gamma}}(x) \right),$$

for some constant $C$ depending on $\epsilon$ only, where $T_{N,K,\sqrt{\gamma}}(x)$ is as in (6.1).

Note that if $q \geq 1/2$ then the condition (8.2) reduces merely to $h \leq \frac{c(\epsilon)}{M^{1/2} \log M}$ for any choice of $\gamma > 0$. One also has a similar, albeit substantially simpler, result for least squares:
Theorem 8.2. Let \( \{\phi_i\}_{i \in \mathbb{N}} \) be the orthonormal Jacobi polynomial basis (2.6), \( T = \{t_n\}_{n=1}^N \subseteq D \) be a set of scattered data points and suppose that \( h \) is as in (2.2). Then for each \( 0 < \epsilon < 1 \) there exists a \( c(\epsilon) > 0 \) such that if
\[
    h \leq \frac{c(\epsilon)}{M^2},
\]
then the solution \( \tilde{x} \) of (3.8) exists uniquely and satisfies
\[
    \|x - \tilde{x}\| \leq \left(1 + \frac{1}{\sqrt{1-\epsilon}}\right) \|x - P_M x\|_{1,w} + \frac{1}{\sqrt{1-\epsilon}} \sqrt{r} \eta,
\]
for any \( w = \{w_i\}_{i \in \mathbb{N}} \) with \( w_i \geq \|\phi_i\|_{L^\infty} \).

These two results show that weighted \( \ell^1 \) minimization with scattered data, Jacobi polynomials and sufficiently large weights \( w_i \) is guaranteed to perform as well as least-squares fitting up to a log factor in \( h \). Specifically, both are guaranteed to recover \( x \) up to an error proportional to \( \|x - P_M x\|_{1,w} \), provided \( h \leq 1/(M^2 \log M) \) for the former and \( h \leq 1/M^2 \) for the latter. A numerical comparison verifying this conclusion is provided in (8.3).

8.2 Near-optimality of weighted \( \ell^1 \) minimization for scattered data

We first note the following theorem:

Theorem 8.3. Let \( T = \{t_n\}_{n=1}^N \) be an equispaced grid of \( N \) points in \([-1,1]\), \( E \subseteq \mathbb{C} \) be a compact set containing \([-1,1]\) in its interior and let \( B(E) \) be the Banach space of functions continuous on \( B \) and analytic in its interior, with norm \( \|f\|_B = \sup_{z \in B} |f(z)| \). Let \( F : B(E) \to L^\infty(-1,1) \) be a mapping such that for each \( f \in B(E) \), \( F(f) \) depends only on the data \( \{f(t_n)\}_{n=1}^N \), and suppose that, for constants \( C > 0 \), \( \sigma > 1 \) and \( 1/2 < \tau \leq 1 \),
\[
    \|f - F(f)\|_{L^\infty} \leq C \sigma^{-N^\tau} \|f\|_E, \quad \forall f \in B(E).
\]
If \( \|f\|_{T,\infty} = \max_{n=1,...,N} |f(t_n)| \) then there exists a constant \( \nu > 1 \) such that
\[
    \sup_{\substack{f \in B(E) \\
                    \|f\|_{T,\infty} \neq 0}} \left\{ \frac{\|F(f)\|_{L^\infty}}{\|f\|_{T,\infty}} \right\} \geq \nu^{N^{2\tau-1}}. \tag{8.4}
\]

This theorem is due to Platte, Trefethen & Kuijlaars [20] (a minor modification is made in (8.4) which is more suitable for our purposes). It states the following. For any method that achieves an error for all analytic functions in \( B(E) \) that is exponentially-decaying as \( N \to \infty \) with rate \( 2\tau - 1 \) it is possible to find a function \( f \in B(E) \) which is bounded on the set \( T \), but for which \( \|F(f)\|_{L^\infty} \) grows exponentially fast with rate \( 2\tau - 1 \). In particular, the best possible convergence rate for a robust method, i.e. one for which \( \|F(f)\|_{L^\infty} / \|f\|_{T,\infty} \leq C \) for all \( f \in B(E) \) and \( N \in \mathbb{N} \), is root-exponential in \( N \). Note that this theorem is very general: the method \( F \) can be linear or nonlinear, and \( F(f) \) only needs to be defined for extremely smooth (specifically, analytic) functions.

Now consider the cases of weighted \( \ell^1 \) minimization and least-squares fitting. If \( f \) is analytic in \( E \), then its Jacobi polynomial coefficients \( x_i \) decay geometrically fast. That is, \( |x_i| \lesssim \rho^{-i} \) as \( i \to \infty \) where \( \rho > 1 \) depends on the largest Bernstein ellipse contained in \( E \). In particular, for algebraically growing weights \( w_i \), we have
\[
    \|f - \tilde{f}\|_{L^\infty} \leq \|x - P_M x\|_{1,w} \leq C \|f\|_E (\rho')^{-M}, \quad M \to \infty,
\]
for some $\rho' > 1$. Hence the function $f$ is approximated by either technique with an error decaying geometrically in $M$. Now suppose that the data $T = \{t_n\}_{n=1}^N$ is equispaced. In this case, $h = 1/N$ and thus the scaling for least-squares fitting it is $N \gtrsim M^2$ (Theorem 8.2). Hence, as a function of $N$, the least-squares errors decays root-exponentially fast in $N$, which, according to Theorem 8.3, is the best possible convergence rate for a robust method. On the other hand, the scaling for weighted $\ell^1$ minimization (with sufficiently large weights) given by Theorem 8.1 is $N \gtrsim M^2 \log M$, and this translates into convergence that is root-exponential up to a log factor:

$$\|f - \tilde{f}\|_{L^\infty} \lesssim (\rho''')^{-\sqrt{N/\log(N)}}, \quad N \to \infty.$$ 

Hence weighted $\ell^1$ minimization, with appropriately growing weights, achieves also this optimal convergence up to a log factor.

This discussion highlights that the worst case analysis developed in §7 for weighted $\ell^1$ minimization for scattered data is not a pessimistic one, since it is sharp up to a log factor for the class of analytic functions. In other words, better accuracy can only possibly be achieved for smaller subclasses, such as functions with sparse coefficients. We will see several examples of function-dependent benefits in §8.3.

**Remark 8.4** Although Theorem 8.3 applies only to equispaced data, it can also be formulated for general scattered data. Loosely speaking, unless the data clusters quadratically at the endpoints $x = \pm 1$, the same conclusions apply. A work discussion this is in preparation [2].

### 8.3 Numerical examples

In Fig. 5 and 6 we give a results for Legendre polynomials approximations from equispaced and jittered data. Although it has only been proved above that weighted $\ell^1$ minimization performs as
well (up to a log factor) as least-squares fitting, these results show that it in fact exhibits rather better performance, similar to that of the best possible least-squares fit. Note that this ‘oracle’ least squares cannot be implemented in practice since the aspect ratio \( M \) is calculated by minimizing the approximation error. In essence, weighted \( \ell^1 \) minimization appears to be able to find this sweet spot without any knowledge of the function beyond the given data. We see also that weighted \( \ell^1 \) minimization performs better for some functions than for others, i.e. it delivers function-dependent benefits in line with earlier discussions regarding sparsity.

8.4 Proofs

The proof of Theorem 8.1 relies on the following three lemmas, which provide estimates for the quantities \( E_2(h, M), E_\infty(h, M) \) and \( F(h, M, R) \) respectively.

Lemma 8.5. For \( \alpha, \beta > -1 \) let \( \{\phi_i\}_{i \in \mathbb{N}} \) be the orthonormal Jacobi polynomial basis (2.6), \( T = \{t_n\}_{n=1}^N \subseteq D \) be a set of scattered data points and suppose that \( h \) is as in (2.2). If \( hM^2 \leq 1 \) then

\[
E_2(h, M) \lesssim \sqrt{hM},
\]

where \( E_2(h, M) \) is as in (7.2).

Proof. By self-adjointness,

\[
\|P_M - P_M U^* U P_M\| = \sup_{x \in P_M(\ell^2(\mathbb{N}))} \|\langle P_M - P_M U^* U P_M \rangle x, x \|.
\]

Let \( x \in P_M(\ell^2(\mathbb{N})), \|x\| = 1 \) be arbitrary and set \( g = \sum_{j=1}^M x_j \phi_j \in P_{M-1} \) so that \( \|g\|_{L^2(\alpha, \beta)} = 1 \). Let \( \{V_n\}_{n=1}^N \) be the Voronoi cells of the points \( \{t_n\}_{n=1}^N \) and set \( \chi(t) = \sum_{n=1}^N g(t_n) I_{V_n}(t) \). Then, by the definition (2.3) of the weights \( \tau_n \), we have

\[
\|\chi\|_{L^2(\alpha, \beta)}^2 = \sum_{n=1}^N \tau_n |g(t_n)|^2,
\]

Figure 6: The same as Fig. 5 except for randomly jittered data.
and therefore
\[ |\langle (P_M - P_M U^* U P_M) x, x \rangle| = \left| 1 - \sum_{n=1}^N \tau_n |g(t_n)|^2 \right| = \left| \|g\|_L^2_{\nu(\alpha, \beta)} - \|\chi\|_L^2_{\nu(\alpha, \beta)} \right|. \]

Hence
\[ |\langle (P_M - P_M U^* U P_M) x, x \rangle| \leq \|g - \chi\|_{L^2_{\nu(\alpha, \beta)}} \left( 2\|g\|_{L^2_{\nu(\alpha, \beta)}} + \|g - \chi\|_{L^2_{\nu(\alpha, \beta)}} \right), \]  
and so it suffices to show that
\[ \|g - \chi\|_{L^2_{\nu(\alpha, \beta)}} \lesssim \sqrt{h} M \|g\|_{L^2_{\nu(\alpha, \beta)}}. \]

We have
\[ \|g - \chi\|_{L^2_{\nu(\alpha, \beta)}}^2 = \sum_{n=1}^N \int_{V_n} |g(t) - g(t_n)|^2 \nu^{(\alpha, \beta)}(t) \, dt = \sum_{n=1}^N I_n. \]

Let \( n_0 \) be the unique number such that \( 0 \in V_{n_0} \). Then we write this as
\[ \|g - \chi\|_{L^2_{\nu(\alpha, \beta)}}^2 = \sum_{n=n_0+1}^N I_n + \sum_{n=1}^{n_0-1} I_n + I_{n_0} = S_1 + S_{-1} + S_0. \]

We shall address each term separately. Consider \( S_1 \). Since \( \nu^{(\alpha, \beta)}(t) \lesssim (1 - t)^\alpha \) on \([0,1]\), we have
\[ I_n \lesssim \int_{V_n} |g(t) - g(t_n)|^2 (1 - t)^\alpha \, dt, \quad n = n_0, \ldots, N. \]

We now consider three cases: (i) \(-1 < \alpha < 0\), (ii) \( \alpha = 0 \) and (iii) \( \alpha > 0 \). Consider case (i). Then
\[ I_n \lesssim \int_{V_n} \left( \int_{V_n} |g'(s)| \, ds \right)^2 (1 - t)^\alpha \, dt \lesssim \int_{V_n} (1 - t)^\alpha \, dt \int_{V_n} (1 - t)^{-\alpha-1} \, dt \int_{V_n} |g'(t)|^2 (1 - t)^{\alpha+1} \, dt. \]

By construction, \( V_n \) is of width at most \( 2h \). Hence, after a short calculation we get that
\[ I_n \lesssim h \int_{V_n} |g'(t)|^2 (1 - t)^{\alpha+1} \, dt, \quad -1 < \alpha < 0. \]

Now consider case (ii). We have
\[ I_n \lesssim \int_{V_n} \left( \int_{V_n} |g'(s)| \, ds \right)^2 \left( \int_{V_n} |g(t)|^2 \, dt \right) \lesssim h^2 \int_{V_n} |g'(t)|^2 \, dt, \quad \alpha = 0. \]

Final, consider case (iii). By similar arguments, we get that
\[ I_n \lesssim \left( \int_{V_n} (1 - t)^\alpha \int_{t_n}^t (1 - s)^{-\alpha-1} \, ds \, dt \right) \int_{V_n} |g'(t)|^2 (1 - t)^{\alpha+1} \, dt. \]

Write \( V_n = (a, b) \) where \( 0 \leq a \leq t_n \) and \( t_n \leq b \leq 1 \). Then
\[ \int_{V_n} (1 - t)^\alpha \int_{t_n}^t (1 - s)^{-\alpha-1} \, ds \, dt = \frac{1}{\alpha} \int_a^b (1 - t)^\alpha \left( (1 - t)^{-\alpha} - (1 - t_n)^{-\alpha} \right) \, dt \]
\[ = \frac{1}{\alpha} \left[ (b - a) + \frac{1}{\alpha + 1} \frac{(1 - b)^{\alpha+1} - (1 - a)^{\alpha+1}}{(1 - t_n)^\alpha} \right] \]
Note that $1 - b \leq 1 - t_n$ and $1 - a \geq 1 - t_n$. Therefore
\[
\int_{V_n} (1 - t)^\alpha \int_{t_n}^t (1 - s)^{-\alpha - 1} ds \, dt \leq \frac{1}{\alpha} \left[ b - a - \frac{b - a}{\alpha + 1} \right] \lesssim h.
\]
Hence
\[
I_n \lesssim h \int_{V_n} |g'(t)|^2 (1 - t)^{\alpha + 1} \, dt, \quad \alpha > 0.
\]
With these estimates to hand, we now deduce the following bound for the term $S_1$ in (8.8):
\[
S_1 \lesssim \begin{cases} h^2 \|g\|_{L^2(0,1)}^2 & \alpha = 0 \\
\|g\|_{L^2(\alpha + 1, \beta + 1)}^2 & \alpha \neq 0 \end{cases}, \tag{8.9}
\]
This follows from the fact that the $V_n$ form a partition of $(-1, 1)$, the definition of $n_0$ and the fact that $(1 + t)^{\beta + 1} \geq 1$ for $t \in [0, 1]$. Identical arguments give a similar result for $S_{-1}$:
\[
S_{-1} \lesssim \begin{cases} h^2 \|g\|_{L^2(-1,0)}^2 & \beta = 0 \\
\|g\|_{L^2(\alpha + 1, \beta + 1)}^2 & \beta \neq 0 \end{cases}. \tag{8.10}
\]
Finally, consider $S_0$. Let $V_{n_0} = J_{-1} \cup J_1$, where $J_1 \subseteq [0, 1]$ and $J_{-1} \subseteq [-1, 0]$. If $hM^2 \lesssim 1$ we may assume that $h \leq 1/2$ so that $|1 - t| \geq 1/2$ and $|1 + t| \geq 1/2$ for $t \in V_{n_0}$. Then we have
\[
J_{\pm 1} \lesssim \left( \int_{J_{\pm 1}} dt \right)^2 \int_{J_{\pm 1}} |g'(t)|^2 \, dt \lesssim h^2 \int_{J_{\pm 1}} |g'(t)|^2 \, dt \lesssim h^2 \int_{J_{\pm 1}} |g'(t)|^2 \nu(\gamma, \delta)(t) \, dt,
\]
for any $\gamma, \delta > -1$. It now follows that $J_{\pm 1}$ satisfy exactly the same bounds as (8.9) and (8.10) for $S_{\pm 1}$. Therefore, in order to estimate the left-hand side of (8.8) it suffices from now on to consider only $S_{\pm 1}$. For this, we shall use Markov’s inequality:
\[
\|g\|_{L^2(I)}^2 \lesssim M^2 / |I| \|g\|_{L^2(I)}, \quad \forall g \in \mathbb{P}_M, \tag{8.11}
\]
where $I$ is an arbitrary bounded interval, as well as the following Markov-type inequality:
\[
\|g\|_{L^2(\nu, \alpha + 1, \beta + 1)}^2 \lesssim M \|g\|_{L^2(\nu, \alpha, \beta)}^2, \quad \forall g \in \mathbb{P}_M. \tag{8.12}
\]
Markov’s inequality (8.11) is well-known. A proof of (8.12) is given in Appendix A.

There are now four cases: (a) $\alpha = \beta = 0$, (b) $\alpha = 0$, $\beta \neq 0$, (c) $\alpha \neq 0$, $\beta = 0$ and (d) $\alpha \neq 0$, $\beta \neq 0$. Consider case (a). Then by (8.8), (8.9), (8.10) and (8.11) we find that
\[
\|g - \chi\|_{L^2}^2 \lesssim h^2 M^4 \|g\|_{L^2}^2.
\]
Since $hM^2 \lesssim 1$, this now gives (8.7) for case (a). Now consider case (b). By (8.8), (8.9), (8.10), (8.11) and (8.12) we get that
\[
\|g - \chi\|_{L^2(\nu, \beta)}^2 \lesssim h^2 M^4 \|g\|_{L^2(\nu, 0, 1)}^2 + hM^2 \|g\|_{L^2(\nu, 0, \beta)}^2 \lesssim hM^2 \|g\|_{L^2(\nu, 0, \beta)}^2.
\]
Here in the final inequality we use the facts that $hM^2 \lesssim 1$ and $(1 + t)^\beta \geq 1$ for $t \in [0, 1]$. Thus we get (8.7) in this case as well. Case (c) is near-identical to case (b). To complete the proof, we consider case (d). Using (8.8), (8.9), (8.10) and (8.12), we get
\[
\|g - \chi\|_{L^2(\nu, \alpha, \beta)}^2 \lesssim hM^2 \|g\|_{L^2(\nu, 0, \beta)}^2,
\]
which yields (8.7). \qed
Lemma 8.6. For \( \alpha, \beta > -1 \) let \( \{\phi_i\}_{i \in \mathbb{N}} \) be the orthonormal Jacobi polynomial basis \( \{2, 0\} \), \( T = \{t_n\}_{n=1}^N \subseteq D \) be a set of scattered data points and suppose that \( h \) is as in \( (2.2) \). If \( h M^2 \leq 1 \) then

\[
E_\infty(h, M) \lesssim h M^2 \log M,
\]

where \( E_\infty(h, M) \) is as in \( (7.2) \).

Proof. Let \( x \in P_M(\ell^2(\mathbb{N})) \), \( \|x\|_\infty = 1 \) and set \( g = \sum_{j=1}^M x_j \phi_j \in \mathbb{P}_{M-1} \) as in the previous proof. Then

\[
\|(P_M - P_M U^* U P_M) x\|_\infty = \max_{i=1, \ldots, M} \left| \langle g, \phi_i \rangle_{L^2_{\nu(\alpha, \beta)}} - \langle g, \phi_i \rangle_h \right|.
\] (8.13)

Observe that

\[
\left| \langle g, \phi_i \rangle_{L^2_{\nu(\alpha, \beta)}} - \langle g, \phi_i \rangle_h \right| = \left| \sum_{n=1}^N \int_{V_n} (g(t) \phi_i(t) - g(t_n) \phi_i(t_n)) \nu(\alpha, \beta)(t) \, dt \right|
\]
\[
\leq \sum_{n=1}^N \int_{V_n} \left| \int_{t_n}^t (g(s) \phi_i(s))' \, ds \right| \nu(\alpha, \beta)(t) \, dt
\]
\[
\lesssim \sum_{n=1}^N \int_{V_n} \nu(\alpha, \beta)(t) \, dt \int_{V_n} \left| (g(s) \phi_i(s))' \right| \, ds.
\]

Since \( \|x\|_\infty = 1 \), we have \( \|(g(s) \phi_i(s))'\| \leq \sum_{j=1}^M \left| (\phi_i(s) \phi_j(s))' \right| \) and we now substitute into (8.13) to deduce that

\[
\|(P_M - P_M U^* U P_M) x\|_\infty \lesssim \max_{i=1, \ldots, M} \sum_{j=1}^M \sum_{n=1}^N \int_{V_n} \nu(\alpha, \beta)(t) \, dt \int_{V_n} \left| (\phi_i(s) \phi_j(s))' \right| \, ds
\]
\[
= \max_{i=1, \ldots, M} \sum_{j=1}^M \sum_{n=1}^N I_n = \max_{i=1, \ldots, M} \sum_{j=1}^M (S_1 + S_{-1} + S_0),
\] (8.14)

where, as in the previous lemma, \( S_1 \) corresponds to \([0, 1]\), \( S_{-1} \) corresponds to \([-1, 0]\) and \( S_0 \) corresponds to the term \( I_{n_0} \) where \( 0 \in V_{n_0} \).

Consider the term \( S_1 \). By (A.2) and (A.8), we have

\[
|\phi_i^{(k)}(t)| \lesssim \min \left\{ (\sqrt{1 - t^2})^{-\alpha - k - 1/2}, t^{2k + \alpha + 1/2} \right\}, \quad 0 \leq t \leq 1.
\]

Since \( 1 \leq i, j \leq M \), we have that

\[
S_1 \lesssim \sum_{n=n_0+1}^N \int_{V_n} (1 - t)^\alpha \, dt \int_{V_n} \min \left\{ (\sqrt{1 - t^2})^{-2\alpha - 2}, M^{2\alpha + 3} \right\} \, dt.
\]

Let \( n_0 + 1 \leq n^* < N \) be arbitrary (its value will be chosen later) and split this sum into two according to \( n^* \). Then

\[
S_1 \lesssim M \sum_{n=n_0+1}^{n^*} \int_{V_n} (1 - t)^\alpha \, dt \int_{V_n} (\sqrt{1 - t^2})^{-2\alpha - 2} \, dt + M^{2\alpha + 3} \sum_{n=n^*+1}^N \int_{V_n} (1 - t)^\alpha \, dt \int_{V_n} 1 \, dt
\]
\[
= MS_{-1}^1 + M^{2\alpha + 3} S_1^+.
\] (8.15)
We consider $S_1^+$ separately. For $S_1^+$, we have
\[
S_1^+ \lesssim h \int_{z^*}^1 (1-t)\alpha \, dt \lesssim h(1-z^*)^{\alpha+1},
\]
where $z^*$ is the right endpoint of $V_{n^*}$. Now consider $S_1^-$:
\[
S_1^- \lesssim h \sum_{n=n_0+1}^{n^*} \int_{V_n} (1-t)^\alpha \, dt \sup_{t \in V_n} (1-t)^{-\alpha-1} = h \sum_{n=n_0+1}^{n^*} \int_{V_n} (1-t)^\alpha (1-z_n)^{-\alpha-1} \, dt.
\]
where $z_n$ is the right endpoint of $V_n$. However, if $t \in V_n$ is arbitrary then $z_n \leq t + h$. Thus
\[
(1-z_n)^{-\alpha-1} \leq (1-t-h)^{-\alpha-1},
\]
and this gives
\[
S_1^- \lesssim h \int_0^{z^*} (1-t)^\alpha (1-t-h)^{-\alpha-1} \, dt \leq h \int_{1-z^*}^1 s^{-1}(1-h/s)^{-\alpha-1} \, ds
\]
Suppose now that $n^*$ is chosen so that
\[
1 - 2/M^2 - 2h \leq z^* \leq 1 - 2/M^2,
\]
(recall that the Voronoi cells are of width at most $2h$, hence such a choice is possible). Then, since $hM^2 \leq 1$, we find that $1 - h/s \gtrsim 1$ for $1 - z^* \leq s \leq 1$. Therefore, if (8.17) holds we have
\[
S_1^- \lesssim h \int_{1-z^*}^1 s^{-1} \, ds \lesssim h \log(1-z^*) \lesssim h \log(M).
\]
Combining this with (8.16) and (8.17), and using the fact that $hM^2 \leq 1$ once more, we now get that $S_1^+ \lesssim h(M^{-2} + h)^{\alpha+1} \lesssim hM^{-2\alpha-2}$. Substituting this and (8.18) back into (8.15) now gives
\[
S_1 \lesssim S_1^- \lesssim hM \log(M) + hM \lesssim hM \log(M),
\]
which completes the estimate for $S_1$. The estimate for $S_{-1}$ is near-identical, except that we use (A.2) as well as (A.8) since $S_1$ sums over integrals contained in the negative portion of the interval. Hence we get
\[
S_{-1} \lesssim hM \log(M),
\]
for this term as well. Next we need to estimate
\[
S_0 = \int_{V_{n_0}} \nu(\alpha,\beta)(t) \, dt \int_{V_{n_0}} |(\phi_i(s)\phi_j(s))'| \, ds
\]
Since $V_{n_0} \subseteq [-2h,2h]$, we have that $\nu(\alpha,\beta)(t) \lesssim 1$ for $t \in V_{n_0}$. Also, by (A.8) and (A.9), we have
\[
|(\phi_i(s)\phi_j(s))'| \lesssim M, \quad s \in V_{n_0}.
\]
Hence we get $S_0 \lesssim hM$. Combining this with (8.19) and (8.20) and substituting into (8.14) now gives
\[
\| (P_M - P_M U^* U P_M)x \|_\infty \lesssim \sum_{j=1}^M hM \log M = hM^2 \log M,
\]
from which the result follows immediately.
Lemma 8.7. For $\alpha, \beta > -1$ let $\{\phi_i\}_{i\in\mathbb{N}}$ be the orthonormal Jacobi polynomial basis \([2.6]\), $T = \{t_n\}_{n=1}^N \subseteq D$ be a set of $N$ scattered data points and suppose that $h$ is as in \([2.2]\). Suppose that $hM^2 \leq 1$. If the weights $w_i \gtrsim i^{q+1/2}$, where $q$ is as in \([8.7]\), then the quantity $F(h, M, R)$ defined by \([7.3]\) satisfies

$$F(h, M, R) \lesssim \sqrt{M} \sup_{i > R} \left\{ \frac{i^{q+1/2}}{w_i} \right\},$$

(8.21)

Moreover, if the weights $w_i \gtrsim i \log i$ and $R \geq M$ then

$$F(h, M, R) \lesssim hM \sup_{i > R} \left\{ \frac{i \log i}{w_i} \right\}.$$  

(8.22)

Proof. As before, let $x \in P_M(\ell^2(\mathbb{N}))$, $\|x\|_\infty = 1$ and set $g = \sum_{j=1}^M x_j \phi_j \in P_{M-1}$. Then

$$\|P_R W^{-1} U^* U P_M x\|_\infty = \sup_{i > R} \left| \frac{\langle g, \phi_i \rangle_h}{w_i} \right| \leq \|g\|_h \sup_{i > R} \left\{ \frac{\|\phi_i\|_h}{w_i} \right\}.$$  

Note that $\|\phi_i\|_h \leq \|\phi_i\|_{L^\infty} \lesssim i^{q+1/2}$ by \([2.7]\) and also

$$\|g\|_h^2 = \langle P_M U^* U P_M x, x \rangle \leq (1 + \|P_M - P_M U^* U P_M\|) \|x\|^2 \lesssim M,$$

where in the final inequality we use Lemma 8.5 and the fact that $\|x\| \leq \sqrt{M} \|x\|_\infty = \sqrt{M}$. This now gives \([8.21]\).

For \([8.22]\) we use orthogonality and the fact that $R \geq M$ to get

$$\|P_R W^{-1} U^* U P_M x\|_\infty = \sup_{i > R} \left| \frac{\langle g, \phi_i \rangle_h}{w_i} \right| = \sup_{i > R} \left\{ \frac{1}{w_i} \left| \langle g, \phi_i \rangle_{L^2} - \langle g, \phi_i \rangle_h \right| \right\},$$

(8.23)

We now proceed in a similar manner to the proof of Lemma 8.6. First, since $\|x\|_\infty = 1$ we have

$$\left| \langle g, \phi_i \rangle_{L^2} - \langle g, \phi_i \rangle_h \right| \leq \sum_{j=1}^M \sum_{n=1}^N \int_{V_n} \nu^{(\alpha, \beta)}(t) \int_{V_n} \left| (\phi_i(s) \phi_j(t))' \right| \, ds.$$  

We now argue in a similar way, using the fact that $j \leq M \leq R \leq i$. This gives

$$\left| \langle g, \phi_i \rangle_{L^2} - \langle g, \phi_i \rangle_h \right| \lesssim \sum_{j=1}^M \frac{h i \log(i)}{w_i} \lesssim M h i \log(i).$$

Substituting back into \([8.23]\) now gives the required result. $\square$

We are now ready to prove Theorem 8.1.

Proof of Theorem 8.1. We use Theorem 7.4 and the estimates of Lemmas 8.5, 8.7. Note that

$$\min_{M < i \leq R} \{w_i\}, \max_{i=1, \ldots, M} \{w_i\} = O\left(M^{\gamma + q + 1/2}\right), \quad M \to \infty.$$  

Hence, we require $h$, $M$ and $R \geq M$ such that

$$E(h, M) < \epsilon, \quad E(h, R) \lesssim \epsilon, \quad F(h, M, R) \lesssim \epsilon M^{-\gamma - q - 1/2}.$$  

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Since $R \geq M$, Lemmas 8.5 and 8.6 give that the first two conditions are satisfied provided

$$h \lesssim \frac{\epsilon}{R^2 \log R}.$$  

We now consider $F(h, M, R)$. Suppose first that $0 < \gamma \leq 1/2 - q$. Then Lemma 8.7 gives that $F(N, M, R) \lesssim \epsilon M^{-\gamma - q - 1/2}$ provided $R \gtrsim \epsilon^{-1/2} M^{1+(q+1)/\gamma}$. Hence this results in the condition

$$h \lesssim \frac{c(\epsilon)}{M^{2+2(q+1)/\gamma} \log M},$$  

which gives the result for $0 < \gamma \leq 1/2 - q$. Now suppose $\gamma > 1/2 - q$. Then $w_i \gtrsim i^{\gamma + q + 1/2} \gtrsim i \log i$ and therefore Lemma 8.7 gives $F(N, M, R) < \epsilon M^{-\gamma - q - 1/2}$ whenever

$$h \frac{\log R}{R^{\gamma + q - 1/2}} < \epsilon M^{-3/2 - \gamma - q}.$$  

Suppose that $R = cM$ for some $c$. Then the above condition holds, provided

$$h \lesssim \frac{\epsilon}{M^{2+2(q+1)/\gamma} \log M}.$$  

Moreover, substituting $R = cM$ into (8.24) gives precisely the same condition on $h$ in terms of $M$. Hence the result follows.

The proof of Theorem 8.2 is straightforward:

**Proof of Theorem 8.2.** We use Theorem 7.5 in combination with Lemma 8.5.

We may now present the proof of Theorem 6.5:

**Proof of Theorem 6.5.** We shall use Lemma 6.1. Let $y \in \mathbb{C}^N$, $\|y\| = 1$ be given. Let $\chi \in C_c^\infty(-1, 1)$ be a smooth compactly-supported function in $(-1, 1)$ with the properties

$$\|\chi\| = 1, \quad \chi(0) = 1, \quad 0 \leq \chi(x) \leq 1, \quad x \in (-1, 1).$$

Define

$$g(t) = \sum_{n=1}^{N} \frac{y_n}{\sqrt{\tau_n}} g_n(t), \quad g_n(t) = \chi \left( \frac{x - t_n}{\xi_n} \right),$$

where $\xi_n = \text{dist}(t_n, \partial V_n)$ is the distance of the point $t_n$ from the boundary of its Voronoi cell (in this one-dimensional setting, $V_n$ is an interval and its boundary is the set of the two endpoints). Observe that

$$\xi_n = \frac{1}{2} \min \{t_{n+1} - t_n, t_n - t_{n-1} \}, \quad n = 1, \ldots, N,$$

and therefore $\xi_n \geq \xi$ for each $n = 1, \ldots, N$. By construction $\text{supp}(g_n) \subseteq V_n$, $n = 1, \ldots, N$, and therefore $\text{supp}(g_n) \cap \text{supp}(g_m) = 0, n \neq m$. It follows that $g \in G_y$. Since $g \in C^\infty[-1, 1]$ a standard result in polynomial approximation gives that

$$\inf_{\phi \in \Phi_K, \phi \neq 0} \|g - \phi\|_\infty \leq C_r K^{-r} \|g^{(r)}\|_\infty, \quad K \geq r,$$

for some constant $C_r > 0$ independent of $K$ and $g$ (see (5.4.16))). Observe that

$$\|g^{(r)}\|_\infty = \max_{n=1,\ldots,N} \left( \frac{\|y_n\| \|g_n^{(r)}\|_\infty}{\sqrt{\tau_n}} \right) \leq \|y\| \max_{n=1,\ldots,N} \left( \frac{\xi_n}{\sqrt{\tau_n}} \right) \lesssim \xi^{-r - 1/2},$$

for some constant $C_r > 0$ independent of $K$ and $g$. Observe that
where in the last inequality we use that fact that $\xi_n \leq \tau_n$. Hence
\[
\inf_{\phi \in \Phi_K \atop \phi \neq 0} \|g - \phi\|_\infty \leq C_r K^{-r} \xi^{-r-1/2}, \quad K \geq r.
\] (8.25)

In order to apply (6.2), it remains to estimate $\|g\|_\nu = \|g\|$. Since the $g_n$’s have disjoint supports, we have
\[
\|g\|^2 = \sum_{n=1}^N \frac{\|y_n\|^2}{\tau_n} \|g_n\|^2 = \sum_{n=1}^N \frac{\|y_n\|^2}{\tau_n} \xi_n \geq \min_{n=1, \ldots, N} \{\xi_n/\tau_n\} \geq \xi_N/\max_{n=1, \ldots, N} \tau_n.
\]
Observe that $\tau_n \leq 2h$. Therefore, $\|g\| \geq \sqrt{\xi/(2h)}$. Substituting this and (8.25) into (6.2) now gives the result.

Finally, we now note that Theorem 9.1 follows immediately from Theorems 6.5 and 8.1.

9 Trigonometric polynomials on bounded intervals

We now consider Example 2.2. Note that in this case we define the projections $P_N : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ by $P_Nx = \{x_0, x_1, \ldots, x_{N-1}, 0, 0, \ldots\}$. Our main result is as follows:

**Theorem 9.1.** Let $\{\phi_i\}_{i \in \mathbb{Z}}$ be the Fourier basis (2.8), $T = \{t_n\}_{n=1}^N \subseteq D$ be a set of $N$ scattered data points and suppose that $h$ is as in (2.2). Suppose that the weights $w_i \geq 1$ satisfy $w_i = O(|i|^{-\gamma})$ as $|i| \to \infty$ for some $\gamma > 0$. Then for each $0 < \epsilon < 1/2$ there exists a $c(\epsilon) > 0$ such that if
\[
h \leq c(\epsilon) \begin{cases} M^{-3/2-3/(4\gamma)} & 0 < \gamma < 1 \\ M^{-3/2} & \gamma \geq 1 \end{cases},
\]
then any minimizer $\hat{x}$ of (3.6) satisfies
\[
\|x - \hat{x}\| \leq C(\epsilon) \left( M^{\gamma+1/2} \left( \sqrt{\tau_\eta + \delta} \right) + \|P_M^1\|_1 w + T_{N,K\sqrt{\tau_\eta}}(x) \right).
\]
for some constant $C$ depending on $\epsilon$ only, where $T_{N,K\sqrt{\tau_\eta}}(x)$ is as in (6.1).

For least-squares fitting, we have the following:

**Theorem 9.2.** Let $\{\phi_i\}_{i \in \mathbb{Z}}$ be the orthonormal Fourier basis (2.8), $T = \{t_n\}_{n=1}^N \subseteq D$ be a set of $N$ scattered data points and suppose that $h$ is as in (2.2). Then for each $0 < \epsilon < 1$ there exists a $c(\epsilon) > 0$ such that if
\[
h \leq \frac{c(\epsilon)}{M},
\]
then the solution $\hat{x}$ of (3.8) exists uniquely and satisfies
\[
\|x - \hat{x}\| \leq \left(1 + \frac{1}{\sqrt{1 - \epsilon}}\right) \|x - P_Mx\|_1 w + \frac{1}{\sqrt{1 - \epsilon}} \sqrt{\tau_\eta},
\]
for any $w = \{w_i\}_{i \in \mathbb{N}}$ with $w_i \geq 1$.

Unlike the case of Jacobi polynomials, these results give a worse recovery guarantee for weighted $\ell^1$ minimization than that of least-squares fitting. However, we do not believe the scaling $h \lesssim M^{-3/2}$ is sharp, and instead we conjecture that the true scaling is $h \lesssim (M \log M)^{-1}$. Proving this conjecture is an open problem. We remark in passing that this conjecture holds in the special case where the data is equispaced (we omit the proof for brevity’s sake). 

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9.1 Numerical examples

In Fig. 7 we give a comparison of the two techniques for jittered data. In align with the discussion above, these results suggest the scaling predicted by Theorem 9.1 is not optimal. In fact, weighted $\ell^1$ minimization performs better than least-squares fitting (including the oracle case) in all the examples. We suspect this strong performance is due in part to the presence of some sparsity in the functions considered, and the fact that jittered points are near-optimal points for the recovery of sparse trigonometric polynomials [13].

9.2 Proofs

We first require the following lemma:

Lemma 9.3. Let $\{\phi_i\}_{i\in\mathbb{Z}}$ be the orthonormal Fourier basis (2.8), $T = \{t_n\}_{n=1}^N \subseteq D$ be a set of $N$ scattered data points and suppose that $h$ is as in (2.2). Suppose that $hM \leq 1$. If $E_2(h, M)$ and $E_\infty(h, M)$ are as in (7.2) and (7.2) respectively, then

$E_2(h, M) \lesssim hM, \quad E_\infty(h, M) \lesssim hM^{3/2}.$

Proof. Consider $E_2(h, M)$ first. As in the proof of Lemma 8.5 let $x \in P_M(\ell^2(\mathbb{Z}))$, $\|x\| = 1$ be arbitrary and set $g = \sum_{j=-M}^{M-1} x_j \phi_j$ so that $\|g\|_{L^2} = 1$. Arguing in an identical manner, we see that it suffices to show that

$$\sum_{n=1}^N \int_{V_n} |g(t) - g(t_n)|^2 \, dt \lesssim h^2 M^2. \quad (9.2)$$

Observe that $|g(t) - g(t_n)|^2 \leq h \int_{V_n} \|g'(s)^2\| \, ds$ and therefore $\sum_{n=1}^N \int_{V_n} |g(t) - g(t_n)|^2 \, dt \lesssim h^2 \|g'\|_{L^2}^2$. To get (9.2) we recall Bernstein’s inequality $\|g'\|_{L^2} \lesssim M \|g\|_{L^2}$ for trigonometric polynomials.
For $E_\infty(h, M)$ we let $x \in P_M(\ell^2(\mathbb{Z}))$ with $\|x\|_\infty = 1$ and defined $g$ as before. As in the proof of Lemma 8.6 it suffices to estimate

$$\max_{i=\pm M, \ldots, M-1} |\langle g, \phi_i \rangle_{L^2} - \langle g, \phi_i \rangle_h|.$$  

Arguing in the standard way, we see that

$$\max_{i=\pm M, \ldots, M-1} |\langle g, \phi_i \rangle_{L^2} - \langle g, \phi_i \rangle_h| \lesssim h \|g \phi_i\|_{L^1} \leq h \|g \phi_i\|_{L^2}$$

Since $g \phi_i$ is a trigonometric polynomial of degree at most $2M$ Bernstein’s inequality gives

$$\max_{i=\pm M, \ldots, M-1} |\langle g, \phi_i \rangle_{L^2} - \langle g, \phi_i \rangle_h| \lesssim h M \|x\|_2 \leq h M^{3/2},$$

where in the final inequality we use the Cauchy–Schwarz inequality the fact that $\|x\|_\infty = 1$. This now gives the estimate for $E_\infty(h, M)$.  

Lemma 9.4. Let $\{\phi_i\}_{i \in \mathbb{Z}}$ be the orthonormal Fourier basis (2.8), $T = \{t_n\}_{n=1}^N \subseteq D$ be a set of $N$ scattered data points and suppose that $h$ is as in (2.2). Suppose that $hM \leq 1$. Then the quantity $F(h, M, R)$ defined by (7.3) satisfies

$$F(h, M, R) \lesssim \sqrt{M} \sup_{i>R} \{1/w_i\}.$$  

Moreover, if the weights $w_i \gtrsim i$ and $R \geq M$, then

$$F(h, M, R) \lesssim h \sqrt{M} \sup_{i>R} \{i/w_i\}.$$  

Proof. We argue as in the proof of Lemma 8.7. In the first case, since the functions $\phi_i$ are uniformly bounded we have $\|P_R W^{-1} U^* U P_M x\|_\infty \leq \|g\|_h \sup_{i>R} \{1/w_i\}$, where $g = \sum_{j=-M}^{M-1} x_j \phi_j$ and $\|x\|_\infty = 1$. By the same argument, we find that $\|g\|_h \lesssim \|g\|_{L^2} \lesssim \sqrt{M}$, which gives the first result.

Now consider the second. With $x$ and $g$ as before, we have

$$\|P_R W^{-1} U^* U P_M x\|_\infty = \sup_{i>R} \left\{ \frac{1}{w_i} |\langle g, \phi_i \rangle_{L^2} - \langle g, \phi_i \rangle_h| \right\}.$$  

As in the previous lemma, we note that $|\langle g, \phi_i \rangle_{L^2} - \langle g, \phi_i \rangle_h| \leq h \|g \phi_i\|_{L^2}$, and therefore by Bernstein’s inequality $|\langle g, \phi_i \rangle_{L^2} - \langle g, \phi_i \rangle_h| \lesssim h i \|x\|_2 \lesssim h i \sqrt{M}$. This gives the second result. 

Proof of Theorem 9.1. With this lemma in hand, the proof is identical in manner to that of Theorem 8.1. We omit the details.

10 Conclusions

We have presented an infinite-dimensional framework for weighted $\ell^1$ minimization. Its advantages are that it does not require a priori knowledge of the expansion tail in order to be implemented and in the absence of noise it leads to genuinely interpolatory approximations. We have discussed the role weights play in the minimization in resolving the aliasing phenomenon, as opposed to promoting smoothness, and provided an explicit way to choose the truncation parameter. In the second half this paper we performed a worse-case linear error analysis for this framework valid for
arbitrary scattered data, and used it to show near-optimal performance of weighted $\ell^1$ minimization with Jacobi polynomial bases.

There are many topics for future research. Two immediate open problems are to obtain a better scaling than (9.1) in the trigonometric polynomial case, and to find ways to estimate the truncation error $T_{h,K,\rho}(x)$ that do not require additional regularity of $x$ (see Theorem 6.2 and the discussion that follows). Besides these, a question of singular importance is the extension of the analysis of §8 to the higher-dimensional setting, and to unbounded intervals (using Laguerre and Hermite polynomials, for example). This will be reported on in future papers. Other higher-dimensional problems should also be investigated, such as approximations in spherical harmonics (see [22] for some work in this direction).

Another topic is the issue of weights. The results of this paper suggest that weights aid the approximation by resolving the aliasing phenomenon and not necessarily by matching the decay of the expansion coefficients. In particular, slowly growing weights seem sufficient, at least in the one-dimensional setting. Nevertheless, finding the optimal choice of weights is an important problem, especially in higher dimensions, and one currently under investigation. We also note the possibility of using reweighted $\ell^1$ minimization, where weights iteratively updated to get a better estimation of the support set of $x$ [19, 29]. We expect this technique can be combined with our framework.

As mentioned, this paper is not about sparsity and random sampling but rather the worst-case, non-sparse analysis of these techniques. Another problem for future work is to develop sparse recovery guarantees for the infinite-dimensional minimization problems (3.5)–(3.6), similar to those introduced in [14, 23, 24] for the finite-dimensional problem (1.1). A promising way to do this would be to adapt the analysis of [3] to the present setting.

\section*{Acknowledgements}

A preliminary version of this work was presented during the Research Cluster on “Computational Challenges in Sparse and Redundant Representations” at ICERM in November 2014. The author would like to thank all the participants for the useful discussions and feedback received during the programme. He would also like to thank Alireza Doostan, Anders Hansen, Rodrigo Platte, Aditya Viswanathan, Rached Ward and Dongbin Xiu.

\appendix

\section*{A Jacobi polynomials}

Following the notation introduced in Example 2.1, given $\alpha, \beta > -1$ let $P_{j}^{(\alpha,\beta)}$ be the Jacobi polynomial of degree $j$. Such polynomials are orthogonal on $D = (-1,1)$ with respect to $\nu^{(\alpha,\beta)}(t) = (1-t)^{\alpha}(1+t)^{\beta}$, with

$$\langle P_{j}^{(\alpha,\beta)}, P_{k}^{(\alpha,\beta)} \rangle_{L^{2}_{\nu^{(\alpha,\beta)}}} = \delta_{j,k}h_{j}^{(\alpha,\beta)},$$

where

$$h_{j}^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}}{2j+\alpha+\beta+1} \frac{\Gamma(j+\alpha+1)\Gamma(j+\beta+1)}{j!\Gamma(j+\alpha+\beta+1)}. \quad (A.1)$$

They have the normalization

$$P_{j}^{(\alpha,\beta)}(1) = \binom{j+\alpha}{j}.$$
The corresponding orthonormal polynomials are defined by \( \phi_j(t) = \left(h_j^{(\alpha, \beta)} \right)^{-1/2} P_j^{(\alpha, \beta)}(t), \ j \in \mathbb{N}. \) Note first that
\[
\lambda_j^{(\alpha, \beta)} \sim \frac{2^{\alpha+\beta}}{j}, \quad j \to \infty.
\] (A.2)
and also that
\[
P_j^{(\alpha, \beta)}(1) \sim \frac{j^\alpha}{\Gamma(\alpha + 1)}.
\]
The polynomials \( P_j^{(\alpha, \beta)} \) satisfy the differential equation
\[
\left( \nu_{\alpha+1, \beta+1} \left( P_j^{(\alpha, \beta)} \right) \right)' + \lambda_j^{(\alpha, \beta)} \nu^{(\alpha, \beta)} P_j^{(\alpha, \beta)} = 0,
\] (A.3)
where \( \lambda_j^{(\alpha, \beta)} = j(j + \alpha + \beta + 1). \) In particular, the derivatives \( (P_j^{(\alpha, \beta)})' \) are orthogonal with respect to \( \nu_{\alpha+1, \beta+1}. \) Therefore, it follows that
\[
(P_j^{(\alpha, \beta)})' = \sqrt{\frac{\nu_{\alpha+1, \beta+1}}{\lambda_j^{(\alpha, \beta)}}} P_j^{(\alpha, \beta)}.
\] (A.4)

**Lemma A.1.** Let \( \alpha, \beta > -1. \) Then \( \|p'\|_{L^2_{\nu_{\alpha+1, \beta+1}}} \leq \lambda_M^{(\alpha, \beta)} \|p\|_{L^2_{\nu_{\alpha, \beta}}}, \forall p \in \mathbb{P}_M. \)

**Proof.** We can write \( p \) in terms of Jacobi polynomials \( P_j^{(\alpha, \beta)}: \)
\[
p(t) = \sum_{j=0}^{M} \frac{x_j}{h_j^{(\alpha, \beta)}} P_j^{(\alpha, \beta)}(t), \quad x_j = \langle p, P_j^{(\alpha, \beta)} \rangle_{L^2_{\nu_{\alpha, \beta}}},
\]
Observe that
\[
\|p\|^2_{L^2_{\nu_{\alpha, \beta}}} = \sum_{j=0}^{M} \frac{|x_j|^2}{h_j^{(\alpha, \beta)}}
\] (A.5)
Similarly,
\[
p'(t) = \sum_{j=0}^{M-1} \frac{y_j}{h_j^{(\alpha+1, \beta+1)}} P_j^{(\alpha+1, \beta+1)}(t), \quad y_j = \langle p', P_j^{(\alpha+1, \beta+1)} \rangle_{L^2_{\nu_{\alpha+1, \beta+1}}},
\] and
\[
\|p'\|^2_{L^2_{\nu_{\alpha+1, \beta+1}}} = \sum_{j=0}^{M-1} \frac{|y_j|^2}{h_j^{(\alpha+1, \beta+1)}}.
\] (A.6)
Consider \( x_j. \) By the differential equation (A.3) and the fact that \( \nu_{\alpha+1, \beta+1}(\pm 1) = 0, \) we have
\[
x_j = \int_{-1}^{1} p(t) P_j^{(\alpha, \beta)}(t) \nu^{(\alpha, \beta)}(t) \, dt = -\frac{1}{\lambda_j^{(\alpha, \beta)}} \int_{-1}^{1} p(t) \left( \nu_{\alpha+1, \beta+1}(t) \left( P_j^{(\alpha, \beta)}(t) \right)' \right)' \, dt
\]
\[
= \frac{1}{\lambda_j^{(\alpha, \beta)}} \int_{-1}^{1} p'(t) \left( P_j^{(\alpha, \beta)}(t) \right)' \nu_{\alpha+1, \beta+1}(t) \, dt.
\]

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Hence, by (A.4),

\[
x_j = \sqrt{\frac{h_j^{(\alpha,\beta)}}{\lambda_j^{(\alpha,\beta)} h_{j-1}^{(\alpha+1,\beta+1)}}} \langle p', P_{j-1}^{(\alpha,\beta)} \rangle_{L_2^{(\alpha+1,\beta+1)}} = \sqrt{\frac{h_j^{(\alpha,\beta)}}{\lambda_j^{(\alpha,\beta)} h_{j-1}^{(\alpha+1,\beta+1)}} y_{j-1}.
\]

Using (A.5) and (A.6) we now get that

\[
\|p\|_{L_2^{(\alpha,\beta)}}^2 \geq \sum_{j=1}^M \frac{|y_{j-1}|^2}{\lambda_j^{(\alpha,\beta)} h_{j-1}^{(\alpha+1,\beta+1)}} \geq \frac{1}{\lambda_M^{(\alpha,\beta)}} \|p'\|_{L_2^{(\alpha+1,\beta+1)}}^2,
\]

as required.

We use several results concerning the asymptotic behaviour of Jacobi polynomials. The first is as follows (see [25, Thm. 7.32.1]):

\[
\|P_j^{(\alpha,\beta)}\|_{L_\infty} = \mathcal{O}(j^q), \quad n \to \infty, \quad q = \left\{ \begin{array}{ll}
\max\{\alpha, \beta\} & \alpha, \beta \geq -1/2 \\
-1/2 & \alpha, \beta < -1/2 .
\end{array} \right.
\]

Hence, using (A.2) we find that the normalized functions \(\phi_j\) defined by (2.6) satisfy

\[
\|\phi_j\|_{L_\infty} = \mathcal{O}(j^{q+1/2}), \quad j \to \infty,
\]

which gives (2.7). We also note the following local estimates for Jacobi polynomials. If \(k = 0, 1, 2, \ldots\) and \(c > 0\) is a fixed constant then

\[
\left| \frac{d^k P_j^{(\alpha,\beta)}(t)}{dt^k} \right|_{t=\cos \theta} = \left\{ \begin{array}{ll}
\theta^{-\alpha-k-1/2} \mathcal{O}(j^{k-1/2}) & c_j^{-1} \leq \theta \leq \pi/2 \\
\mathcal{O}(j^{2k+\alpha}) & 0 \leq \theta \leq c_j^{-1},
\end{array} \right.
\]

as \(j \to \infty\). See [25, Thm. 7.32.4]. This estimate bounds the Jacobi polynomial and its derivatives for \(0 \leq t \leq 1\). For negative \(t\), we may use the relation

\[
P_j^{(\alpha,\beta)}(-t) = (-1)^j P_j^{(\beta,\alpha)}(t).
\]

Hence behaviour of \(P_j^{(\alpha,\beta)}(t)\) and its derivatives for \(t < 0\) is given by (A.8) with \(\alpha\) replaced by \(\beta\).

References

[1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions*. Dover, 1974.

[2] B. Adcock. Necessary and sufficient sampling conditions for approximations in polynomial spaces. *In preparation*, 2015.

[3] B. Adcock and A. C. Hansen. Generalized sampling and infinite-dimensional compressed sensing. *Technical report NA2011/02, DAMTP, University of Cambridge*, 2011.

[4] B. Adcock and A. C. Hansen. A generalized sampling theorem for stable reconstructions in arbitrary bases. *J. Fourier Anal. Appl.*, 18(4):685–716, 2012.

[5] B. Adcock, A. C. Hansen, B. Roman, and G. Teschke. Generalized sampling: stable reconstructions, inverse problems and compressed sensing over the continuum. *Advances in Imaging and Electron Physics*, 182:187–279, 2014.
[6] B. Adcock, D. Huybrechs, and J. Martín-Vaquero. On the numerical stability of Fourier extensions. Found. Comput. Math., 14(4):635–687, 2014.

[7] J. P. Boyd and J. R. Ong. Exponentially-convergent strategies for defeating the Runge phenomenon for the approximation of non-periodic functions. I. Single-interval schemes. Commun. Comput. Phys., 5(2–4):484–497, 2009.

[8] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang. Spectral methods: Fundamentals in Single Domains. Springer, 2006.

[9] A. Cohen, M. A. Davenport, and D. Leviatan. On the stability and accuracy of least squares approximations. Found. Comput. Math., 13:819–834, 2013.

[10] A. Cohen, R. A. DeVore, and C. Schwab. Convergence rates of best $N$-term Galerkin approximations for a class of elliptic sPDEs. Found. Comput. Math., 10:615–646, 2010.

[11] A. Cohen, R. A. DeVore, and C. Schwab. Analytic regularity and polynomial approximation of parametric and stochastic elliptic PDE’s. Analysis and Applications, 9:11–47, 2011.

[12] A. Doostan and H. Owhadi. A non-adapted sparse approximation of PDEs with stochastic inputs. J. Comput. Phys., 230(8):3015–3034, 2011.

[13] S. Foucart and H. Rauhut. A Mathematical Introduction to Compressive Sensing. Birkhauser, 2013.

[14] J. Hampton and A. Doostan. Compressive sampling of polynomial chaos expansions: Convergence analysis and sampling strategies. arXiv:1408.4157, 2014.

[15] O. P. Le Maître and O. M. Knio. Spectral Methods for Uncertainty Quantification. Springer, 2010.

[16] L. Mathelin and K. A. Gallivan. A compressed sensing approach for partial differential equations with random input data. Commun. Comput. Phys., 12(4):919–954, 2012.

[17] G. Migliorati and F. Nobile. Analysis of discrete least squares on multivariate polynomial spaces with evaluations in low-discrepancy point sets analysis of discrete least squares on multivariate polynomial spaces with evaluations in low-discrepancy point sets. Preprint, 2014.

[18] G. Migliorati, F. Nobile, E. von Schwerin, and R. Tempone. Analysis of the discrete $L^2$ projection on polynomial spaces with random evaluations. Found. Comput. Math., 14:419–456, 2014.

[19] J. Peng, J. Hampton, and A. Doostan. A weighted $\ell_1$-minimization approach for sparse polynomial chaos expansions. J. Comput. Phys., 267:92–111, 2014.

[20] R. Platte, L. N. Trefethen, and A. Kuijlaars. Impossibility of fast stable approximation of analytic functions from equispaced samples. SIAM Rev., 53(2):308–318, 2011.

[21] C. Poon. A stable and consistent approach to generalized sampling. Preprint, 2013.

[22] H. Rauhut and R. Ward. Sparse recovery for spherical harmonic expansions. In Proceedings of the 9th International Conference on Sampling Theory and Applications, 2011.

[23] H. Rauhut and R. Ward. Sparse Legendre expansions via l1-minimization. J. Approx. Theory, 164(5):517–533, 2012.

[24] H. Rauhut and R. Ward. Interpolation via weighted $\ell_1$ minimization. arXiv:1308.0759, 2013.

[25] G. Szegő. Orthogonal Polynomials. American Mathematical Society, Providence, RI, 1975.

[26] H. Wendland. Scattered Data Approximation. Cambridge University Press, 2004.

[27] D. Xiu and G. E. Karniadakis. The Wiener–Askey polynomial chaos for stochastic differential equations. SIAM J. Sci. Comput., 24(2):619–644, 2002.

[28] L. Yan, L. Guo, and D. Xiu. Stochastic collocation algorithms using $\ell_1$-minimization. Int. J. Uncertain. Quantif., 2(3):279–293, 2012.

[29] X. Yang and G. E. Karniadakis. Reweighted $\ell_1$ minimization method for stochastic elliptic differential equations. J. Comput. Phys., 248:87–108, 2013.