DIFFERENT CLASSES OF BINARY NECKLACES AND A COMBINATORIAL METHOD FOR THEIR ENUMERATIONS

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Abstract. In this paper we investigate enumeration of some classes of \(n\)-character strings and binary necklaces. Recall that binary necklaces are necklaces in two colors with length \(n\). We prove three results (Theorems 1, 1’ and 2) concerning the numbers of three classes of \(j\)-character strings (closely related to some classes of binary necklaces or Lyndon words). Using these results, we deduce Moreau’s necklace-counting function for binary aperiodic necklaces of length \(k\) \cite{12} (Theorem 3), and we prove the binary case of MacMahon’s formula from 1892 \cite{9} (also called Witt’s formula) for the number of necklaces (Theorem 4). Notice that we give proofs of Theorems 3 and 4 without use of Burnside’s lemma and Pólya enumeration theorem. Namely, the methods used in our proofs of auxiliary and main results presented in Sections 3 and 4 are combinatorial in spirit and they are based on counting method and some facts from elementary number theory.

1. Introduction

George Pólya (1887–1985) discovered a powerful general method for enumerating the number of orbits of a group on particular configurations. This method became known as the Pólya Enumeration Theorem, or PET, whose proof follows directly from Burnside’s lemma. Pólya’s theorem can be used to enumerate several objects under permutation groups. In particular, it can be used for enumeration of different classes of necklaces and bracelets.

In combinatorics, a \(k\)-ary necklace of length \(n\) is an equivalence class of \(n\)-character string over an alphabet of size \(k\), taking all rotations are equivalent. It represents a structure with \(n\) circularly connected beads of up to \(k\) different colors. A necklace of length \(n\) is primitive if its period is not a proper divisor of \(n\).

Technically, one may classify as an orbit of the action of the cyclic group of \(n\)-character strings, and a bracelet as an orbit of the dihedral group’s action.

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Namely, an \((n, k)\) - bracelet is an equivalence classs of words of length \(n\) under rotation and reflection. This enables appplication of Pólya enumeration theorem of necklaces and bracelets. An \((n, k)\) - necklace is an equivalence class of words of length \(n\) over an alphabet of size \(k\) under rotation. For example, if \(k = 2\) and the alphabet is \(\{0, 1\}\), then the following sets are examples of three binary necklaces (i.e., those with \(k = 2\)):

\[
\{0101, 1010\},
\{011011, 110110, 1011101\},
\]

and

\[
\{0110101, 110101, 0101011, 1010110, 0101101, 1011010\},
\]

The basic enumeration problem is then (Necklace Enumeration): For a given \(n\) and \(k\), how many \((n, k)\)-necklaces are there? Equivalently, we are asking how many orbits the cyclic group \(C_n\) has on the set of all words of length \(n\) over an alphabet of size \(k\). We will denote this value by \(a(n, k)\). Notice that in a group \(G\) of symmetry transformations such that only translations \((a_i \rightarrow a_{i+s})\) are allowed, \(G\) is a cyclic group \(C_n\). This case appears in \([14]\) in connection with counting necklaces made from \(n\) beads of \(k\) different kinds (translations merely rotate the necklace). It also arises in problems of coding and genetics \([6]\). The special case \(n = 12, k = 2\) occurs in finding the number of distinct musical chords (of 0, 1, \(\cdots\), or 12 notes) when inversions and transpositions to other keys are equivalences (for related calculations see \([4, \text{Section 6}]\)).

An aperiodic necklace of length \(n\) is an equivalence class of size \(n\), i.e., no distinct rotations of a necklace from such class are equal. According to Moreau’s necklace-counting function (see \([3, \text{p. 503}]\); also see \([14]\)), there are

\[
M_k(n) = \frac{1}{n} \sum_{d|n} \mu(d)k^{n/d}
\]

different \(k\)-ary aperiodic necklaces of length \(n\), where \(\mu\) is the Möbius function, where \(\mu(1) = 1\), \(\mu(n) = (-1)^r\) if \(n\) is a product of \(r\) distinct primes, and \(\mu(n) = 0\) otherwise (see the sequence A001037 in \([16]\) concerning the sequence \(\{M_2(n)\}_{n=1}^\infty\) which presents the number of binary Lyndon words). The formula (1) is called MacMahon’s formula in the book by Graham et al. \([5, \text{the formula (4.63), p. 141}]\). Notice that this formula may be derived by a simple direct argument given in \([6]\).

Each aperiodic necklace contains a single Lyndon word so that Lyndon words form representatives of aperiodic necklaces. Recall that in mathematics, in the areas of combinatorics and computer science, a Lyndon word is a nonempty string that is strictly smaller than lexicographic order than all of its rotations. More precisely, a \(k\)-ary Lyndon word of length \(n > 0\) is an \(n\)-character string over an alphabet of size \(k\), and which is the unique minimum element in the lexicographical ordering of all its rotations. Being the singularly smallest rotation implies
that a Lyndon word differs from any of its non-trivial rotations, and is therefore aperiodic (see [1]). For example (see [1]), the list of Lyndon words of length 6 on the alphabet \{0, 1\} reads

000001, 000101, 000111, 001011, 001111, 010111, 011111.

Of course, the number of Lyndon words of length \(n\) on \(k\) symbols is equal to \(M_k(n)\), where \(M_k(n)\) is given by (1).

Notice that the authors of the paper [1] investigate the historical roots of the field of combinatorics of words. They comprise applications and interpretations in algebra, geometry, and combinatorial enumeration. Combinatorics of words is a comparatively new area of discrete mathematics. It is pointed out in [1] that the collective volumes written under the pseudonym of Lothaire give an account of it (Lothaire’s first volume [7] appeared in 1983 and was reprinted with corrections in 1997 [8]).

It is also well known (see, e.g., [14, p. 162]) that the number of \((n, k)\) - necklaces is given by

\[
N_k(n) = \frac{1}{n} \sum_{d|n} \varphi(d) k^{n/d}.
\]

(2)

The formula (2) is called MacMahon’s formula in the book by Graham et al. [5] the formula (4.63), p. 141], while in Lucas’ book [3, p. 503], it is credited to M. le colonel Moreau (see the sequence A001031 in [16] concerning the sequence \(\{N_2(n)\}_{n=1}^{\infty}\) which presents the number of binary necklaces). A proof of (2) given in [5, pp. 14-141] is based on a lemma presented by Pólya [13] (see also Lemma in [14, p. 659]).

Let \(G\) be a finite group that acts on a set \(X\). For each \(g \in G\) let \(X^g\) denote the set of elements in \(X\) that are fixed by \(g\). Burnside’s lemma asserts the following formula for the number of orbits, denoted \(|X/G|\):

\[
|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.
\]

(3)

Two elements of \(X\) belong to the same “orbit” when one can be reached from the other by through the action of an element of \(G\). For example, if \(X\) is the set of colorings of a cube, and \(G\) is the set of rotations of the cube, then two elements of \(X\) belong to the same orbit precisely when one is a rotation of the other.

In this paper we focus our attention to the investigation of enumerations of some classes of \(n\)-character strings and binary necklaces, i.e., for new deductions of expressions for numbers of some binary type necklaces. As noticed above, binary necklaces are necklaces in two colors with length \(n\). Observe that the authors of the paper [2] exhibit a correspondence between the binary cycles on length \(n\) and the lexicographic composition of the integer \(n\). Furthermore, in [2] the authors give an algorithm for generating all necklaces of a specific density.
The paper is organized as follows. In Section 2 we present the main results and related notions necessary for their formulations and proofs. Section 3 is devoted to the auxiliary results and related notions and notations. In Section 4 we prove Theorems 1, 1’ and 2 concerning the numbers of three classes of \( j \)-character strings (closely related to some classes of binary necklaces of Lyndon words). Furthermore, by using these results, we give proofs (of Theorems 3 and 4) of the well known formulae for two classes of binary necklaces without use of Burnside’s lemma and Pólya enumeration theorem (Burnside’s lemma is also called Burnside’s counting theorem, the Cauchy-Frobenius lemma or the orbit-counting theorem).

Notice that the formula (15) of Theorem 3 is a special (binary) case of formula (1) with \( k = 2 \). Similarly, the formula (16) of Theorem 4 is a special (binary) case of formula (2) with \( k = 2 \). As applications, we obtain some interesting congruences involving the sums of certain binomial coefficients and the function \( \mu(n) \) or \( \varphi(n) \) (Corollaries 1 and 2). In particular, we obtain two Lucas’ type congruences (Corollaries 3 and 4; see, e.g., \cite{10} and \cite{11}). The all our main results and their consequences are given in Section 2.

Methods used in all our proofs of auxiliary results presented in Section 3 are very combinatorial in spirit and they involve the applications of elementary number theory. By using these auxiliary results, in Section 4 we give proofs of our results (Theorems 1-4 and Corollaries 3 and 4).

2. The main results

Throughout this paper we suppose that \( k \geq 2 \) and \( r \geq 1 \) are fixed \( k \geq 2 \) is an arbitrary fixed integer and \( r \geq 1 \) is an integer such that \( r \leq k \). Here, as always in the sequel, we will denote by \((n,m)\) the greatest common divisor of positive integers \( n \) and \( m \), and by \( |S| \) the cardinality of a finite set \( S \). Usually, denote by \( \mu(n) \) and \( \varphi(n) \) the Möbius function and the Euler totient function, respectively. For any positive integers \( j \geq 1 \) and \( i \geq 1 \) with \( j \leq i \), denote by \( A(j,i) \) the collection of all subsets of \( A(j,i) \) consisting of those sets \( A(j,i) \) for which \( f_i(A_j) \) exists and \( f_i(A_j) \leq i \).

For an arbitrary positive integer \( s \) with \( 1 \leq s \leq i \), denote by \( A_s(j,i) \) the subset of \( A(j,i) \) consisting of those sets \( A(j,i) \) for which \( f_i(A_j) = s \). Notice that the set \( A_s(j,i) \) may be considered as a class of binary necklaces with length \( j \) whose properties are described above, i.e., for which \( f_i(A_j) = s \). Then \( R_2 = \{0,1\}, A(2,3) = \{(0,1), (0,2), (1,2)\} \). Then assuming \( A_2 = \{0,2\} \), we have \( A_2+1 = \{1,0\} \) modulo 3, \( A_2+2 = \{2,1\} \) modulo 3 and \( A_2+3 = \{0,2\} = A_2 \) modulo 3, whence it follows that \( f_3(A_2) = 3 \). If for example, \( i = j = 3 \) and
$A_3 = \{0, 2, 4\}$, then since $A_3 + 1 = \{1, 0, 2\}$ and $A_3 + 2 = \{2, 4, 0\} = A_3$ modulo 3, we find that $f_3(A_3) = 2$.

Our investigations are motivated by the following question:

**Theorem 1.** Let $A_r = \{a_1, a_2, \ldots, a_r\} \subseteq \{0, 1, 2, \ldots, k - 1\}$ be a set for which $n = f_k(A_r)$. Then $n$ divides $k$ and $d = k/n$ is a positive integer that divides $r$, that is, $k = nd$ and $r = md$ for a positive integer $m$. Furthermore, the class of $r$-character strings $A_{n}(r, k)$ consists of

$$|A_n(r, k)| = \sum_{s | (n, m)} \left( \frac{n}{s} \right) \mu(s)$$

(5)

elements, and the sum is taken over all positive divisors $s$ of the greatest common divisor $(n, m)$ of $n$ and $m$.

In particular, we have

$$|A_k(r, k)| = \sum_{s | (k, r)} \left( \frac{k}{s} \right) \mu(s).$$

(6)

**Remark 1.** Notice that the numbers $|A_n(r, k)|$ are closely related to the sequence (triangular array read by rows) A185158 in [16]. Namely, $T(n, m) = |A_n(r, k)|$ for all $n, k = 1, 2, \ldots$ (with $k = nd$ and $r = md$), where by Comments in [16], $T(n, m)$ is the number of binary Lyndon words of length $n$ containing $m$ ones (cf. the sequence/triangular array A051168 in [16]).

**Remark 2.** It follows from Theorem 1 and Example 1 at the end of Section 3 that the collection $A_n(r, k)$ is a nonempty set if and only if $d = k/n$ is an integer that divides $r$.

For a fixed positive integer $n$ that divides $k$ such that the integer $d = k/n$ divides $r$, the $r$-character strings that belong to $A_n(r, k)$ can be separated into
disjoint classes as follows. We say that the $r$-character strings $A$ and $A'$ in $A_n(r, k)$ are $k$-equivalent, writing $A \sim_k A'$, if there exists $j \in \{0, 1, 2, \ldots, k-1\}$ such that $A + j$ is equal to $A'$ modulo $k$. It is easy to see that $\sim_k$ is an equivalence relation, and that every coset (with respect to this relation) has exactly $n$ elements. More precisely, the coset $A$ represented by a set $A \in A_n(r, k)$ is equal to $\{A + j : j = 0, 1, \ldots, n-1\}$ modulo $k$. Denote by $P_n(r, k)$ the set of all these cosets. We say that each element of $P_n(r, k)$ is a $(r, k)$-period with length $n$. Thus by (5) and (6) of Theorem 1, with $k = nd$ and $r = md$, we obtain the following result.

**THEOREM 1'**. Suppose that $k = nd$ and $r = md$ for some positive integers $n$, $m$ and $d$. Then we have

$$|P_n(r, k)| = \frac{|A_n(r, k)|}{n} = \frac{1}{n} \sum_{s|(n,m)} \left(\frac{s}{n}\right) \mu(s),$$

(7)

where the sum is taken over all positive divisors $s$ of the greatest common divisor $(n, m)$ of $n$ and $m$.

In particular, the number of $(r, k)$-periods with maximal length $k$ is given as

$$|P_k(r, k)| = \frac{|A_k(r, k)|}{k} = \frac{1}{k} \sum_{s|(k,r)} \left(\frac{k}{s}\right) \mu(s).$$

(8)

Now define the sum $S(r, k)$ as

$$S(r, k) = \sum_{n=1}^{k} |P_n(r, k)| = \sum_{n=1}^{k} \frac{|A_n(r, k)|}{n},$$

(9)

that is, for a fixed $r$, $S(r, k)$ is a number of all $(r, k)$-periods with arbitrary length $n$ ($1 \leq n \leq k$). Observe that by Theorem 1, $S(r, k)$ may be written as

$$S(r, k) = \sum_{n|_{(k/n)r}} \frac{|A_n(r, k)|}{n}.$$  

(10)

**THEOREM 2**. $S(r, k)$ is given as a sum

$$S(r, k) = \frac{1}{k} \sum_{s|(k,r)} \left(\frac{k}{s}\right) \varphi(s),$$

(11)

where the sum is taken over all positive divisors $s$ of the greatest common divisor $(k, r)$ of $k$ and $r$.

**REMARK 3**. It is known (see, e.g., the sequence $L_2(n, d)$ in [17]) that the number $S(r, k)$ given by (11) in Theorem 2 is the number of binary Lyndon words of length $k$ containing $r$ ones.

Notice that the numbers $|A_n(r, k)|$ are closely related to the sequence (triangular array read by rows) A185158 in [16]. Namely, $T(n, m) = |A_n(r, k)|$ for all
\( n, k = 1, 2, \ldots \) (with \( k = nd \) and \( r = md \)), where by Comments in [16], \( T(n, m) \) is the number of binary Lyndon words of length \( n \) containing \( mk \) ones (cf. the sequence/triangular array A051168 in [16]).

As the immediate consequences of (8) and (11) we get the following two congruences, respectively.

**Corollary 1.** Let \( k \geq 2 \) and \( r \geq 1 \) be integers with \( r \leq k \). Then we have

\[
\sum_{s \mid (k, r)} \left( \frac{k}{s} \right) \mu(s) \equiv 0 \pmod{k}. \tag{12}
\]

**Corollary 2.** Let \( k \geq 2 \) and \( r \geq 1 \) be positive integers with \( r \leq k \). Then we have

\[
\sum_{s \mid (k, r)} \left( \frac{k}{s} \right) \varphi(s) \equiv 0 \pmod{k}. \tag{13}
\]

Let \( R(k) \) be the sum defined as

\[
R(k) = \sum_{r=1}^{k} |P_k(r, k)| \tag{14}
\]

that is, for a fixed \( k \), \( R(k) \) is a number of all \((r, k)\)-periods \((1 \leq r \leq k)\) with arbitrary length \( n \) \((1 \leq n \leq k)\).

The following result is a special (binary) case of MacMahon's necklace-counting function (1) with \( k = 2 \).

**Theorem 3.** Let \( k \geq 2 \) be any integer. Then

\[
R(k) = M_2(k) = \frac{1}{k} \sum_{s \mid k} 2^{\frac{k}{s}} \mu(s), \tag{15}
\]

where \( M_2(k) \) is the binary case of MacMahon's necklace-counting function given by (1), where the sum on the right hand side ranges over all divisors \( s \) of \( k \).

Finally, put \( L(k) = \sum_{r=1}^{k} S(r, k) \), that is, for a fixed \( k \), \( L(k) \) is a number of all \((r, k)\)-periods \((1 \leq r \leq k)\) with any possible length \( n \) \((1 \leq n \leq k)\).

The following result is a special (binary) case of MacMahon's formula (2) with \( k = 2 \).

**Theorem 4.** Let \( k \geq 2 \) be any integer. Then

\[
L(k) = N_2(k) = \frac{1}{k} \sum_{s \mid k} 2^{\frac{k}{s}} \varphi(s), \tag{16}
\]

where \( N_2(k) \) is the binary case of MacMahon's formula (2) with \( k = 2 \), and the sum on the right is taken over all positive divisors \( s \) of \( k \).
Finally, as consequences of the congruence from Corollary 1, in Section 4 we prove the following two statements.

**Corollary 3.** (cf. [11]) Let \( n \geq 1 \) and \( m \geq 1 \) be relatively prime integers with \( m \leq n \). Then for any prime \( p \) and integer \( \alpha \geq 1 \),

\[
\binom{np^\alpha}{mp^\alpha} \equiv \binom{n}{m} \quad \text{(mod } np\text{).} \tag{17}
\]

**Corollary 4.** (see, e.g., [10] the congruence (5) on p. 6) Let \( n \geq 1 \) and \( m \geq 1 \) be any positive integers with \( m \leq n \). Then for any prime \( p \) we have

\[
\binom{np}{mp} \equiv \binom{n}{m} \quad \text{(mod } p\text{).} \tag{18}
\]

### 3. The Collections \( \mathcal{A}_s(j, i) \) and Auxiliary Results

Let \( i \) be a fixed integer greater than 1, and consider an alphabet consisting of the numbers \( 0, 1, \ldots, i - 1 \). With this alphabet form all possible \( k \)-letter words \((a_1, a_2, \ldots, a_k)\), where \( k \) is also fixed. There are evidently \( i^k \) such words in all. For our purposes, notice that the set \( R_i := \{0, 1, \ldots, i - 1\} \) is a complete residue system modulo \( i \). For a finite subset \( A \) of \( \mathbb{N}_0 := \{0, 1, 2, \ldots\} \), denote by \( \overline{A}(i) \) the (unique) subset of \( R_i \) consisting of all \( l \in R_i \) for which there is an \( a_l \in A \) such that \( a_l \equiv l \pmod{i} \). In other words, \( \overline{A}(i) \) is a set of representatives modulo \( i \) (chosen from the set \( R_i \)) of all elements which belong to \( A \). For given two finite subsets \( A \) and \( B \) of \( \mathbb{N}_0 \) we say that \( A \) equals \( B \) modulo \( i \) if \( \overline{A}(i) = \overline{B}(i) \). In this case, we shall often write \( A = B \) modulo \( i \).

For any positive integers \( i \geq 1 \) and \( j \geq 1 \) with \( j \leq i \), denote by \( \mathcal{A}(j, i) \) the collection of all subsets of \( \{0, 1, \ldots, i - 1\} \) that contain exactly \( j \) elements. Given set \( A_j = \{a_1, a_2, \ldots, a_j\} \in \mathcal{A}(j, i) \), and any positive integer \( t \), put \( A_j + t = \{a_1 + t, \ldots, a_j + t\} \). Denote by \( l = f_i(A_j) > 0 \) the smallest positive integer for which \( A_j = A_j + l \) modulo \( i \). Clearly, \( A_j = A_j + i \) modulo \( i \), whence we see that \( f_i(A_j) \) exists and \( f_i(A_j) \leq i \).

For an arbitrary positive integer \( s \) with \( 1 \leq s \leq i \), denote by \( \mathcal{A}_s(j, i) \) a subset of \( \mathcal{A}(j, i) \) consisting of those sets \( A_j \) in \( \mathcal{A}(j, i) \) for which \( f_i(A_j) = s \). It is of interest here to consider the collections \( \mathcal{A}_n(r, k) \) and \( \mathcal{A}_n(m, n) \).

Recall that \( k \geq 2 \) is any fixed integer and \( r \geq 1 \) is an integer such that \( r \leq k \). In this section we give necessary conditions on integers \( k, r, n, a_1, a_2, \ldots, a_r \), to be satisfied \( f_k(A_r) = n \) for given set \( A_r = \{a_1, a_2, \ldots, a_r\} \in \mathcal{A}(r, k) \). To solve this problem, we start with the following proposition.

**Proposition 1.** For any set \( A_r = \{a_1, a_2, \ldots, a_r\} \subseteq \{0, 1, 2, \ldots, k - 1\} \), the integer \( n = f_k(A_r) \) divides \( k \).
Proof. As noticed above, \( n = f_k(A_r) \leq k \). If we suppose that the integer \( f_k(A_r) = n \) does not divide \( k \), then \( k = q_1n + r_1 \) with positive integers \( q_1 \) and \( r_1 \) such that \( 0 < r_1 \leq n - 1 \), and hence

\[
A_r = A_r + k = A_r + (q_1n + r_1) = (A_r + q_1n) + r_1 = A_r + r_1 \quad \text{modulo} \quad k.
\]

It follows that \( n = f_k(A_r) \leq r_1 < n \). This contradiction shows that \( f_k(A_r) \) divides \( n \).

\[\square\]

Let \( d \) be any divisor of \( k \), and \( k = nd \) for an integer \( n \geq 1 \). For a fixed integer \( i \) such that \( 0 \leq i \leq k - 1 \), consider the set \( C_i \) defined as

\[
C_i = \{i, i + n, i + 2n, \ldots, i + (d-1)n\}. \tag{19}
\]

Then we have the following lemma.

**Lemma 1.** Every set \( \overline{C}_i(k) \) (\( 0 \leq i \leq k - 1 \)) has exactly \( d \) elements. Moreover, \( \overline{C}_i(k) = \overline{C}_j(k) \) if and only if \( i \equiv j \) (mod \( n \)). In the case when \( i \not\equiv j \) (mod \( n \)), \( \overline{C}_i(k) \) and \( \overline{C}_j(k) \) are disjoint sets.

**Proof.** First observe that the set \( \overline{C}_i(k) \) has \( d \) elements. Namely, if \( i + d_1n \equiv i + d_2n \pmod{k} \) for some \( d_1 \) and \( d_2 \) with \( 0 \leq d_1 < d_2 \leq d - 1 \), then \( (d_2 - d_1)n \equiv 0 \pmod{dn} \). Thus \( (d_2 - d_1) \equiv 0 \pmod{d} \), and hence it must be \( d_1 = d_2 \).

Suppose that \( \overline{C}_i(k) \) and \( \overline{C}_j(k) \) have at least one common element. In other words, assume that \( i + d_1n \equiv j + d_2n \pmod{k} \) for some \( d_1 \) and \( d_2 \) with \( 0 \leq d_1, d_2 \leq d - 1 \), or equivalently, \( i - j + (d_1 - d_2)n \equiv 0 \pmod{dn} \). Therefore, we obtain \( i \equiv j \pmod{n} \), whence it follows easily that \( \overline{C}_i(k) = \overline{C}_j(k) \). \( \square \)

Given set \( A_r = \{a_1, a_2, \ldots, a_r\} \subseteq \{0, 1, 2, \ldots, k - 1\} \), put \( n = f_k(A_r) \). Then by Proposition 1, the number \( d = k/n \) is an integer. Assume that the set \( A_r \) contains \( m \) distinct representatives modulo \( n \). Choose a maximal subset \( B_m = \{a'_1, a'_2, \ldots, a'_m\} \) of \( A_r \), such that \( a'_s \not\equiv a'_q \pmod{n} \) for any integers \( s \) and \( q \) with \( 1 \leq s < q \leq m \). Then by Lemma 1,

\[
\bigcup_{j=1}^{r} \overline{C}_{a'_j}(k) = \bigcup_{j=1}^{m} \overline{C}_{a'_j}(k), \tag{20}
\]

where the sets \( \overline{C}_{a'_j}(k) \) are disjoint in pairs, that is, \( \overline{C}_{a'_s}(k) \cap \overline{C}_{a'_q}(k) \) is the empty set for any integers \( p \) and \( q \) with \( 1 \leq s < q \leq m \). Furthermore,

\[
\left| \bigcup_{j=1}^{r} \overline{C}_{a'_j}(k) \right| = \left| \bigcup_{j=1}^{m} \overline{C}_{a'_j}(k) \right| = md, \tag{21}
\]

Since by the assumption \( A_r + n = A_r \) modulo \( k \), and hence \( A_r + ln = A_r \) modulo \( k \) for all integers \( l \) with \( 0 \leq l \leq d - 1 \), we have

\[
\bigcup_{j=1}^{m} \overline{C}_{a'_j}(k) \subseteq \{a_1, a_2, \ldots, a_r\}, \tag{22}
\]
or equivalently,

\[ A'_r := \{ a'_i + jn : 1 \leq i \leq m, 0 \leq j \leq d - 1 \} \subseteq A_r. \]  

(23)

On the other hand, for any \( a_i \in A_r \), there exists \( a'_j \) with \( 1 \leq j \leq m \), such that \( n \) divides \( a_i - a'_j \). Hence, in view of the fact that \( 0 \leq a_i - a'_j \leq k - 1 \), there is an \( s \) with \( 0 \leq s \leq d - 1 \) such that \( a_i - a'_j = sn \), i.e., \( a_i = a'_j + sn \in A'_r \). Therefore, \( A_r \subseteq A'_r \), and hence, it must be \( A_r = A'_r \). It follows that \( r = |A_r| = |A'_r| = md \), and we have

\[ A_r = \{ a'_i + jn : 1 \leq i \leq m, 0 \leq j \leq d - 1 \}. \]  

(24)

The above arguments together with Proposition 1 imply the following result.

**Proposition 2.** For given set \( A_r = \{ a_1, a_2, \ldots, a_r \} \subseteq \{0, 1, 2, \ldots, k - 1\} \) put \( n = f_k(A_r) \). Then \( d = k/n \) is an integer that divides \( r \), that is, \( r = md \) for a positive integer \( m \). Moreover, the set \( A_r \) contains exactly \( m \) distinct representatives modulo \( n \). If we assume that \( a'_1, a'_2, \ldots, a'_m \) are these representatives modulo \( n \), then \( A_r \) has the form

\[ A_r = \{ a'_i + jn : 1 \leq i \leq m, 0 \leq j \leq d - 1 \}. \]  

(25)

**Remark 4.** Clearly, \( A_r = A_r + n \) modulo \( k \) for every set \( A_r \) given by (25). However, the converse of Proposition 2 is not true in the sense that generally, given positive integers \( k, r, n, m \) and \( d \) such that \( k = nd \) and \( r = md \), there are sets \( A_r \) of the form (25) for which \( f_k(A_r) < n \). To show this fact, put \( n = k = 4 \), \( d = 1 \), \( m = r = 2 \), and consider the set \( A_2 = \{0, 2\} \subset \{0, 1, 2, 3\} \). Then \( A_2 + 2 = A_2 \) modulo 4, and hence \( f_4(A_2) = 2 < 4 \).

**Remark 5.** If the integers \( k \) and \( r \) are relatively prime, using the same notations as in Proposition 2, this proposition implies that \( d \) divides \( (k, r) = 1 \). Hence, it must be \( d = 1 \) and \( k = n = f_k(A_r) \) for any set \( A_r = \{ a_1, a_2, \ldots, a_r \} \in A(r, k) \). It follows that \( |A_k(r, k)| = |A(r, k)| = \binom{k}{r} \) (cf. (5) and (6) of Theorem 1). This means that each set \( A_r \in A(r, k) \) belongs to certain \( (r, k) \)-period with maximal length \( k \).

The following result has an important role in the proof of Theorem 1.

**Proposition 3.** For an arbitrary common divisor \( d \geq 1 \) of \( k \) and \( r \), take \( k = nd \) and \( r = md \). Then the collections \( A_n(m, n) \) and \( A_n(r, k) \) have same cardinality, and one bijection \( h \) between these collections is given as

\[ B_m = \{ a_1, a_2, \ldots, a_m \} \mapsto A_r = \{ a_i + jn : 1 \leq i \leq m, 0 \leq j \leq d - 1 \}, \]  

(26)

where \( B_m \) is in \( A_n(m, n) \) and \( A_r \) is in \( A_n(r, k) \).

**Proof.** For a given set \( B_m = \{ a_1, a_2, \ldots, a_m \} \in A_n(m, n) \), it is easy to check that all elements of its associated set \( A_r = \{ a_i + jn : 1 \leq i \leq m, 0 \leq j \leq d - 1 \} \) are distinct modulo \( k \). Therefore, \( |A_r| = md = r \) modulo \( k \), and hence the above
It remains to show that the map $h$ is onto $A_n(r, k)$. Let $A_r \in A_n(r, k)$ be arbitrary. This means that $n = f_k(A_r) > 0$ is the smallest positive integer for which $A_r + n$ equals $A_r$ modulo $k$. It follows from Proposition 2 that the set $A_r$ contains $m$ distinct representatives modulo $n$; assume $a_{i_1}, a_{i_2}, \ldots, a_{i_m}$. Hence, by (25) we get

$$A_r = \{a_{i_l} + jn : 1 \leq l \leq m, 0 \leq j \leq d - 1\}. \quad (27)$$

Obviously, it is sufficient to show that there exists a set $B_m \in A_n(m, n)$ such that $h(B_m) = A_r$. Define $B_m = \{a_{i_1}, a_{i_2}, \ldots, a_{i_m}\}$. Clearly, $B_m$ is in $A(m, n)$, and hence it suffices to show that $f_n(B_m) = n$. Suppose that $f_n(B_m) = n_1 < n$. Then as in the proof of Proposition 1, we infer that $n_1$ divides $n$, i.e., $n = n_1 d_1$ with an integer $d_1 > 1$. Since the set $B_m + n_1$ equals $B_m$ modulo $n$, by Proposition 2, with $n_1, m, n_1 d_1$ instead of $n, r, k, d$ and $A_r$, respectively, we conclude that $d_1$ divides $m$, i.e., $m = m_1 d_1$ with $m_1 \in \mathbb{N}$. Furthermore, by Proposition 2, the set $B_m$ contains exactly $m_1$ distinct representatives modulo $n_1$, assume for example, $a_{i_1}, a_{i_2}, \ldots, a_{i_m}$. Then by (25) of Proposition 2, $B_m$ has the form

$$B_m = \{a_{i_s} + t n_1 : 1 \leq s \leq m_1, 0 \leq t \leq d_1 - 1\}, \quad (28)$$

which by (27) implies that

$$A_r = \{(a_{i_s} + t n_1) + jn : 1 \leq s \leq m_1, 0 \leq t \leq d_1 - 1, 0 \leq j \leq d - 1\}, \quad (29)$$

whence by putting $n = n_1 d_1$, we obtain

$$A_r = \{a_{i_s} + (t + j n_1) n_1 : 1 \leq s \leq m_1, 0 \leq t \leq d_1 - 1, 0 \leq j \leq d - 1\}. \quad (30)$$

Because of $k = nd = n_1 d_1 d$, it is easy by (30) to verify that the set $A_r + n_1$ is equal to the set $A_r$ modulo $k$. This implies that $f_k(A_r) \leq n_1 < n$. This contradiction with our assumption that $f_k(A_r) = n$ shows that $f_n(B_m) = n$. This means that $B_m$ is in $A_n(m, n)$, and since by (26), (27) and the definition of $B_m$, $h(B_m) = A_r$, we conclude that $h$ is a surjective map. This completes the proof. \qed

**Example 1.** We will show that $A_n(r, k)$ is a nonempty set for each positive integer $n$ satisfying conditions of Proposition 2. More precisely, for any integers $n \geq 1$ and $d \geq 1$, such that $k = nd$ and $r = md$, we will construct some elements of $A_n(r, k)$. Since by (26) of Proposition 3 it is given an one-to-one correspondence $h$ between the families $A_n(m, n)$ and $A_n(r, k) = A_n(md, nd)$, it is sufficient to consider the corresponding problem for the families $A_n(m, n)$ with $m \leq n$.

If $(n, m) = 1$, then according to Remark 5 (by replacing $k$ and $r$ with $n$ and $m$, respectively), we have $A_n(m, n) = A(m, n)$. In other words, each set $A_m = \{a_1, a_2, \ldots, a_m\} \in A(m, n)$ is in $A_n(m, n)$, and hence $|A_n(m, n)| = \binom{n}{m}$. Now we suppose that $(n, m) > 1$. First we observe that if the set $A_m$ is in $A_n(m, n)$, then it is easily seen that the set $A_{n-m} = \{0, 1, \ldots, n-1\} \setminus A_m$ is in
\[ A_n(n - m, n). \] Indeed, since \( A_m + n \) is equal to \( A_m \) modulo \( n \), then
\[
A_{n-m} + n = (\{0, 1, \ldots, n-1\} \setminus A_m) + n = \{0, 1, \ldots, n-1\} \setminus (A_m + n) = \{0, 1, \ldots, n-1\} \setminus A_m = A_{n-m} \text{ modulo } n. \quad (31)
\]

Hence, \( f_n(A_{n-m}) \leq n \). If we suppose that \( f_n(A_{n-m}) = s < n \), then as above we obtain that \( A_m + s \) is equal to \( A_m \) modulo \( n \). This contradiction with the fact that \( A_m \) is in \( A_n(m, n) \) implies that \( A_{n-m} \) is in \( A_n(n, m, n) \).

In view of the above natural correspondence between the families \( A_n(m, n) \) and \( A_n(n-m, n) \), and the fact that \( (n, m) > 1 \), we may suppose that \( 2 \leq m \leq \lfloor \frac{n}{2} \rfloor \). For such a \( m \) define
\[
V = \{ v \in \mathbb{N} : v \leq 2m - 2, (v, n) > 1 \}. \quad (32)
\]
Since \( v = m \leq 2m - 2 \) and by the assumption, \( (n, m) > 1 \), it follows that \( V \) is a nonempty set. Let \( u = \max\{v \in V \} \) and
\[
S = \{ s_1s_2 : s_1 \in \mathbb{N}, s_2 \in \mathbb{N}, s_1 < \frac{n}{u}, s_2 \leq n, (s_2, n) = 1 \}. \quad (33)
\]
We will prove that \( B_s = \{ s, 2s, \ldots, ms \} \in A_n(m, n) \) for each \( s \in S \). In particular, we have \( \{1, 2, \ldots, m\} \in A_n(m, n) \). First show that \( B_s \) has exactly \( m \) different elements modulo \( n \). Assume that for a fixed \( s \in S \), the integers \( l_1s \) and \( l_2s \) are in \( B_s \) with \( 1 \leq l_1 - l_2 \leq m - 1 \) such that \( n \mid (l_1 - l_2)s \). We write \( s = s_1s_2 \) with integers \( s_1 \) and \( s_2 \) as described by (33). Then since \( (s_2, n) = 1 \), it must be \( n \mid (l_1 - l_2)s_1 \). Now, since \( 1 \leq s_1 < \frac{n}{u} < n \), it follows that \( l_1 - l_2 = l_3s_3 \) such that \( (s_3, n) = 1 \) and \( (l_3, n) > 1 \). So \( n \mid l_3s_1 \), and because of \( l_3 \leq m - 1 \) and \( (l_3, n) > 1 \), we have \( l_3 \leq u \) with \( u \) defined above. Thus \( 1 \leq l_3s_1 = u \cdot \frac{n}{u} = n \), and therefore, \( l_3s_1 \not\equiv 0(\text{mod } n) \). This contradiction shows that \( |S| = m \) modulo \( n \).

It remains to show that \( B_s \) is not equal to \( B_s + t \) modulo \( n \) for any integer \( t \) with \( 1 \leq t \leq n - 1 \). Indeed, if \( B_s = B_s + t \) modulo \( n \) for some \( t \) with \( 1 \leq t \leq n - 1 \), then there is an integer \( p \) with \( 1 \leq p \leq m - 1 \) such that \( s + t = (p + 1)s \). Therefore, \( B_s + t = \{ (p + 1)s, (p + 2)s, \ldots, (p + m)s \} \) modulo \( n \), and hence we have
\[
\{ s, 2s, \ldots, ms \} = \{ (p + 1)s, (p + 2)s, \ldots, (p + m)s \} \text{ modulo } n. \quad (34)
\]
Thus there is an integer \( q \) with \( p + 1 \leq q \leq p + m \leq 2m - 1 \) such that \( n \mid qs - s \).

It follows that \( n \mid (q - 1)s_1s_2 \), whence since \( (s_2, n) = 1 \), we have \( n \mid (q - 1)s_1 \).

Since \( s_1 \leq n - 1 \), it follows that \( (q - 1, n) = 1 \), i.e., \( q - 1 = q_1q_2 \) for integers \( q_1 \) and \( q_2 \) with \( 1 \leq q_1 \leq q - 1 \leq 2m - 2 \) such that \( (q_1, n) > 1 \) and \( (q_2, n) = 1 \). This yields that \( q_1 \leq u \), and \( n \mid q_1s_1 \), which is impossible since \( 1 \leq q_1s_1 < u \cdot \frac{n}{u} = n \).

This contradiction implies that \( B_s \) is in \( A_n(m, n) \) for each \( s \in S \).

4. Proofs of Theorems 1–4 and Corollaries 3 and 4

We give here a combinatorial proof of Theorem 1 which is based on auxiliary results obtained in Section 3 and on property of function \( \tau(a, t) \) defined as follows.
DEFINITION 1. For any integers \( a > 1 \) and \( t \geq 1 \), denote by \( \tau(a, t) \) the number of \( t \)-tuples \((a_1, a_2, \ldots, a_t)\) of integers such that \( a_j \geq 2 \) for all \( j = 1, 2, \ldots, t \), and \( a = a_1 a_2 \cdots a_t \). Obviously, \( \tau(a, t) = 0 \) for all \( t \geq a \).

In the proof of Theorem 1, we use the following property of the function \( \tau(a, t) \).

**Lemma 2.** For each integer \( a > 1 \), we have

\[
\sum_{t=1}^{a-1} (-1)^t \tau(a, t) = \mu(a),
\]

(35)

where \( \mu(a) \) is the Möbius function.

**Proof.** We derive the proof by induction on \( a \geq 2 \). Since \( -\tau(2, 1) = -1 = \mu(2) \), we see that (35) is true for \( a = 2 \). Suppose that \( a > 2 \) and that (35) is satisfied for all integers less than \( a \).

Obviously, there holds \( \tau(a, 1) = 1 \) for all \( a > 1 \). Letting that the first coordinate \( q = a_1 \) of \( t \)-tuples \((a_1, a_2, \ldots, a_t)\) of integers satisfying \( a_j \geq 2 \) for all \( j = 1, 2, \ldots, t \), and \( a = a_1 a_2 \cdots a_t \), is taken over all divisors of \( a \), by Definition 1 of \( \tau(a, t) \), we have

\[
\tau(a, t) = \sum_{q\mid a \atop q > 1} \tau \left( \frac{a}{q}, t-1 \right).
\]

(36)

Now by using the induction hypothesis, (36) and the basic property of the Möbius function (see, e.g., [1, (32) on p. 181]) given by

\[
\sum_{s\mid a} \mu(s) = \begin{cases} 
1 & \text{if } a = 1 \\
0 & \text{if } a > 1,
\end{cases}
\]

(37)
we get
\[
\sum_{t=1}^{a-1} (-1)^t \tau(a, t) = \sum_{t=2}^{a-1} (-1)^t \sum_{q \mid a \land 1 < q < a} \tau\left(\frac{a}{q}, t-1\right) - \tau(a, 1)
\]
\[
= - \sum_{t=2}^{a-1} \sum_{q \mid a \land 1 < q < a} (-1)^{t-1} \tau\left(\frac{a}{q}, t-1\right) - 1
\]
\[
= - \sum_{q \mid a \land 1 < q < a} \sum_{t=1}^{a-1} (-1)^t \tau\left(\frac{a}{q}, t\right) - 1
\]
\[
= - \sum_{q \mid a \land 1 < q < a} \mu\left(\frac{a}{q}\right) - 1 \quad \text{(by the hypothesis for } a < q )
\]
\[
= - \sum_{q \mid a \land 1 < q < a} \mu\left(\frac{a}{q}\right) + \mu(a) = - \sum_{s \mid a} \mu(s) + \mu(a) = \mu(a).
\]

This completes the proof. \qed

We are now ready to prove the main result.

Proof of Theorem 1. Note that the first assertion of Theorem 1 is contained in Proposition 2. It remains to prove the equality (5).

Using the notations introduced in Section 3, if \( k = nd \) and \( r = md \) for an integer \( d \geq 1 \), then by Proposition 2 we have
\[
|A_n(md, nd)| = |A(n, m)|.
\]
(39)

If integers \( k \) and \( r \) are relatively prime, then it must be \( d = 1 \), \( r = m \), and thus \( n = k = f_k(A_r) \) for any set \( A_r = \{a_1, a_2, \ldots, a_r\} \subset \{0, 1, 2, \ldots, k-1\} \). Therefore,
\[
|A_n(r, k)| = |A(r, k)| = \binom{k}{r} = \binom{n}{m} = \mu(1) = \binom{n}{m},
\]
(40)
whence follows (5).

Now suppose that \( (k, r) > 1 \). To determine \( |A_n(r, k)| \), where \( k = nd \) and \( r = md \) with \( d \geq 1 \), denote
\[
\overline{A}_n(m, n) = A(m, n) \setminus A_n(m, n).
\]
(41)
Then since \( |A(m, n)| = \binom{n}{m} \), we have
\[
|A_n(m, n)| = \binom{n}{m} - |\overline{A}_n(m, n)|.
\]
(42)

Moreover, \( \overline{A}_n(m, n) = \bigcup_{1 \leq s < n} A_s(m, n) \), and by Proposition 2, \( A_s(m, n) \) is a nonempty set if and only if \( s \) divides \( n \) and \( \frac{d}{s} \) divides \( m \). When this is the
case, by (39), with \( \frac{n}{d_1}, \frac{m}{d_1} \) and \( d_1 \) instead of \( n, m \) and \( d \), respectively, we obtain

\[
|A_{\frac{m}{d_1}}(m, n)| = \left| A_{\frac{m}{d_1}} \left( \frac{m}{d_1}, \frac{n}{d_1} \right) \right|.
\]

(43)

Therefore, we obtain

\[
\mathcal{A}_n(m, n) = \sum_{d_1 \mid (m, n)} \bigcup_{d_1 > 1} A_{\frac{m}{d_1}} \left( \frac{m}{d_1}, \frac{n}{d_1} \right),
\]

(44)

whence it follows that

\[
|\mathcal{A}_n(m, n)| = \sum_{d_1 \mid (m, n)} \left| A_{\frac{m}{d_1}} \left( \frac{m}{d_1}, \frac{n}{d_1} \right) \right|.
\]

(45)

Thus by (42) and (45), we get

\[
|A_n(m, n)| = \binom{n}{m} - \sum_{d_1 \mid (m, n)} \left| A_{\frac{m}{d_1}} \left( \frac{m}{d_1}, \frac{n}{d_1} \right) \right|.
\]

(46)

Applying (46) on the all terms of the sum on the right hand side of (46), with \( \frac{m}{d_1} \) instead of \( m \) and \( \frac{n}{d_1} \) instead of \( n \), and iterating the same procedure at most
\[ m - 1 \text{ times, we have} \]
\[
|\mathcal{A}_n(m, n)| = \binom{n}{m} - \sum_{d_1 \mid (m, n), d_1 > 1} \left| \frac{A_{\frac{n}{d_1}}(m, \frac{n}{d_1})}{\binom{m}{d_1}} \right|
\]
\[
= \binom{n}{m} - \sum_{d_1 \mid (m, n)} \left( \frac{n}{d_1 m} \right) + \sum_{d_1 \mid (m, n)} \left( \frac{n}{d_1 m} \right) - \frac{A_{\frac{n}{d_1}, d_2}(m, d_1, \frac{n}{d_1})}{d_2 > 1}
\]
\[
= \binom{n}{m} - \sum_{d_1 \mid (m, n)} \left( \frac{n}{d_1 m} \right) + \sum_{d_1 \mid (m, n)} \left( \frac{n}{d_1 m} \right) + \sum_{d_1 \mid (m, n)} \left( \frac{n}{d_1 m} \right)
\]
\[
= \ldots
\]
\[
= \binom{n}{m}
\]
\[
+ m - 1 \sum_{j=1}^{m-1} (-1)^j \left( \sum_{d_1 \mid (m, n), d_1 > 1} \left( \frac{n}{d_1 m} \right) \sum_{d_2 \mid (\frac{n}{d_1}, d_1)} \left( \frac{n}{d_2 d_1} \right) \ldots \sum_{d_j \mid (\frac{n}{d_1 \ldots d_{j-1}}, d_1 \ldots d_{j-1})} \left( \frac{n}{d_1 d_2 \ldots d_j} \right) \right)
\]

Hence for a fixed divisor \( s > 1 \) of \((n, m)\) with \( s = d_1 d_2 \cdots d_j \) for some \( j \geq 1 \) and the integers \( d_1 > 1, d_2 > 1, \ldots, d_j > 1 \), the factor premultiplying the binomial coefficient \( \left( \frac{n}{d_1 m} \right) \) in the last sum of (47) is equal to
\[
\sum_{j=1}^{m-1} (-1)^j \tau(s, j),
\]
which is by Lemma 2 equal to \( \mu(s) \). Therefore, by (47), we obtain
\[
|\mathcal{A}_n(m, n)| = \sum_{s \mid (m, n)} \left( \frac{n}{s m} \right) \mu(s).
\]
This by (39) implies (5), and this completes the proof of Theorem 1. \( \square \)

We will need the following result for the proof of Theorem 2.
Lemma 3. [1, (9) on p. 240] For any integer \( q > 1 \), we have
\[
\sum_{s|q} \frac{\mu(s)}{s} = \frac{\varphi(q)}{q},
\]
where \( \varphi(q) \) is the Euler totient function.

Proof of Theorem 2. By (7) of Theorem 1', Theorem 1 and (50) of Lemma 3, we have
\[
S(r, k) = \sum_{n=1}^{k} |P_n(r, k)| = \sum_{d|(k, r)} |P_{\frac{k}{d}}(r, k)|
\]
\[
= \sum_{d|(k, r)} d |A_{\frac{k}{d}}(r, k)| = \frac{1}{k} \sum_{d|(k, r)} d \sum_{s|\left(\frac{k}{d}, r\right)} \left(\frac{k}{ds}\right) \mu(s)
\]
\[
= \frac{1}{k} \sum_{q|(k, r)} q \left(\frac{k}{q}\right) \sum_{s|q} \mu(s) = \frac{1}{k} \sum_{q|(k, r)} \left(\frac{k}{q}\right) \varphi(q),
\]
as required. \( \square \)

Proof of Theorem 3. To determine \( R(k) = \sum_{r=1}^{k} |P_k(r, k)| \), by (8) of Theorem 1', and using the well known property \( \sum \mu(d)_{\text{diln}} = 1 \) if \( n = 1 \), and \( \sum \mu(d)_{\text{diln}} = 0 \) if \( n > 1 \) (see, e.g., [15]), for each \( k > 1 \) we find that
\[
R(k) = \frac{1}{k} \sum_{r=1}^{k} \sum_{s|k} \left(\frac{k}{s}\right) \mu(s)
\]
\[
= \frac{1}{k} \sum_{s|k} \left(\frac{k}{s}\right) + \left(\frac{k}{2s}\right) + \ldots + \left(\frac{k}{k}\right) \mu(s)
\]
\[
= \frac{1}{k} \sum_{s|k} \sum_{i=1}^{k} \left(\frac{k}{s}\right) \mu(s)
\]
\[
= \frac{1}{k} \sum_{s|k} \left(2^{\frac{k}{s}} - 1\right) \mu(s)
\]
\[
= \frac{1}{k} \sum_{s|k} 2^{\frac{k}{s}} - \frac{1}{k} \sum_{s|k} \mu(s)
\]
\[
= \frac{1}{k} \sum_{s|k} 2^{\frac{k}{s}},
\]
as desired. \( \square \)
Proof of Theorem 4. The proof follows in the same manner as that of Theorem 3 with $\varphi(s)$ instead of $\mu(s)$, by using the well known property $\sum_{d|n} \varphi(d) = n$ established by Gauss (see, e.g., [15]), and hence may be omitted.

Proof of Corollary 3. We proceed by induction on $\alpha \geq 1$. If $\alpha = 1$, then since $(n, m) = 1$, $\mu(1) = 1$ and $\mu(p) = -1$, (12) of Corollary 1 with $np$ and $mp$ instead of $k$ and $r$, respectively, immediately implies that

$$\left(\frac{np}{mp}\right) - \left(\frac{n}{m}\right) \equiv 0 \pmod{np}. \tag{53}$$

Now suppose that $\alpha \geq 2$ and (17) holds for all positive integers $\beta < \alpha$. Then by using the fact that $\mu(p^\beta) = 0$ for each $\beta > 1$, (12) gives

$$\left(\frac{np^\alpha}{mp^\alpha}\right) \equiv \left(\frac{np^{\alpha-1}}{mp^{\alpha-1}}\right) \pmod{np}. \tag{54}$$

The above congruence together with the induction hypothesis $\left(\frac{np^{\alpha-1}}{mp^{\alpha-1}}\right) \equiv \left(\frac{n}{m}\right)$ (mod $np$) yields (17). This completes the induction proof.

Proof of Corollary 4. We deduce the proof by induction on $\sigma = n + m \geq 2$. If $\sigma = 2$, that is $n = m = 1$, (18) is obvious. Suppose that $\sigma > 2$ and that the congruence (18) is satisfied for any $n$ and $m$ such that $n + m < \sigma$.

Assume that $n'$ and $m'$ be positive integers such that $n' + m' = \sigma$. If $n'$ and $m'$ are relatively prime, then (18) is in fact (17) of Corollary 1 with $n = n'p$, $m = m'p$ and $\alpha = 1$. Now suppose that $(n', m') = d > 1$. If $d = p^\alpha$ with $\alpha \geq 1$, i.e., $n' = n''p^\alpha$ and $m' = m''p^\alpha$ with $(n'', m'') = 1$, then (12) implies that

$$\left(\frac{n'p}{m'p}\right) \equiv \left(\frac{n''p^{\alpha+1}}{m''p^{\alpha+1}}\right) \equiv \left(\frac{n''}{m''}\right) \pmod{p}. \tag{55}$$

If there exists a prime $q \neq p$ that divides $(n', m') = d$, then applying the induction hypothesis on integers $n'' = n'/q$ and $m'' = m'/q$, for any divisor $s$ of $(n''/q, m''/q)$, we get

$$\left(\frac{n''}{q}\right) \equiv \left(\frac{n''}{q}\right) \pmod{p}. \tag{56}$$

By (12) of Corollary 1, we have

$$\sum_{s|(n'p, m'p)} \left(\frac{n''}{s}\right) \mu(s) \equiv 0 \pmod{p}. \tag{57}$$

Since $\mu(s'p) = 0$ if $p \mid s'$, and each divisor $s'$ of $(n', m')$ with $s' \neq 0$(mod $p$) can be uniquely associated to the divisor $s'p$ of $(n'p, m'p)$ with $\mu(s'p) = -\mu(s')$, the
above congruence can be written as

\[
\binom{n'}{m'} - \binom{n'}{m'} + \sum_{1 < s' \mid (n', m')} \left( \binom{n'}{s'} \binom{m'}{s'} - \binom{n'}{s'} \binom{m'}{s'} \right) \mu(s') \equiv 0 \pmod{p}. \tag{58}
\]

Since each term into parantheses is by the hypothesis divisible by \( p \), we obtain

\[
\binom{n'}{m'} \equiv \binom{n'}{m'} \pmod{p}. \tag{59}
\]

This finishes the induction proof. \( \square \)

**References**

[1] J. Berstel and D. Perrin, “The origins of combinatorics on words,” *European Journal of Combinatorics*, vol. 28, pp. 996–1022, 2007.

[2] H. Fredricksen and I.J. Kessler, “An algorithm for generating necklaces of beads in two colors,” *Discrete Mathematics*, vol. 61, nos. 2–3, pp. 181–188, 1986.

[3] E. Lucas, *Théorie des Nombres*, Gauthier-Villars, 1891, reprinted by Albert Blanchard, 1961.

[4] E.N. Gilbert and J. Riordan, “Symmetry types of periodic sequences,” *Illinois Journal of Mathematics*, vol. 5, no. 4, pp. 657–665, 1961.

[5] R. L. Graham, D.E. Knuth and O. Patashnik, *Concrete Mathematics*, Addison Wesley, Second Edition, 1994.

[6] S.W. Golomb, Basil Gordon and L.R. Welch, “Comma-free codes,” *Canadian Journal of Mathematics*, vol. 10, no. 2, pp. 202–209, 1958.

[7] M. Lothaire, *Combinatorics on Words*, in: Encyclopedia of Mathematics and its Applications, vol. 17, Addison-Wesley, Reading, Mass., 1983.

[8] M. Lothaire, *Combinatorics on Words*, Cambridge University Press, Cambridge, 1997. Corrected reprint of the 1983 original.

[9] P.A. MacMahon, “Applications of a theory of permutations in circular procession to the theory of numbers,” *Proc. London Math. Soc.*, vol. 23, pp. 305-313, 1892.

[10] R. Meštrović, “Lucas’ theorem: its generalizations, extensions and applications (1878–2014),” preprint arXiv:1409.382v1 [math.NT], 2014.

[11] R. Meštrović, “A note on the congruence \( \binom{n}{md} \equiv \binom{n}{m} \pmod{q} \),” vol. 116, no. 1, pp. 75–7, 2009.

[12] C. Moreau, “Sur les permutations circulaires distincts”, *Nov. Ann. Math.*, vol. 11, pp. 309–314, 1872.

[13] G. Pólya, “Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen,” *Acta Math.*, vol. 68, pp. 145–253, 1937.

[14] J. Riordan, *An introduction to combinatorial analysis*, New York, Wiley, 1958.

[15] H.N. Shapiro, *Introduction to the Theory of Numbers*, John Wiley & Sons, New York, Inc., 1983.

[16] N.J.A. Sloane, *On-Line Encyclopedia of Integer Sequences*, published electronically at [www.research.att.com/~njas/sequences/](http://www.research.att.com/~njas/sequences/).

[17] [http://theory.cs.uvic.ca/inf/neck/NecklaceInfo.html](http://theory.cs.uvic.ca/inf/neck/NecklaceInfo.html)
