Distance Functions and Generalized Means: Duality and Taxonomy

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Abstract

This paper introduces in production theory a large class of efficiency measures that can be derived from the notion of utility function. This article also establishes a relation between these distance functions and Stone-Geary utility functions. More specifically, the paper focuses on new distance function that generalizes several existing efficiency measures. The new distance function is inspired from the Atkinson inequality index and maximizes the sum of the netoutput expansions required to reach an efficient point. A generalized duality theorem is proved and a duality result linking the new distance functions and the profit function is obtained. For all feasible production vectors, it includes as special cases most of the dual correspondences previously established in the literature. Finally, we identify a large class of measures for which these duality results can be obtained without convexity.

Keywords: Utility based distance functions, generalized means, efficiency analysis, duality theory.

1 Introduction

The literature in production economics distinguishes two concepts of technical efficiency. The first concept, associated with Debreu [13] and especially Farrell [17], is related to the traditional radial efficiency measure. Focusing on input efficiency, it is defined as the minimal equiproportionate contraction in all inputs which still allows production of given outputs. Their seminal works have since then been extended to several other non-radial and directional efficiency measures.

The second concept finds its origin in the work of Koopmans [22] who gave a definition of technical efficiency which focuses on the efficient subset of the technology. A producer is technically efficient if an increase in any output or a decrease in any input requires a decrease in at least one other output, or an increase in at least one input. Along this line, for each technology for which isoquant and efficient subset diverge, there is a potential conflict between both technical efficiency concepts.

Färe and Lovell [15] define a new input efficiency measure as an arithmetic mean of its component measures (sometimes also known as a Russell efficiency measure, see also [31]). The Färe-Lovell measure overcomes some difficulties with the Debreu-Farrell’s measure that evaluates technical efficiency relative to an isoquant rather than to an efficient subset. Such a property can lead to the identification of a unit as technically efficient when it is not and is likely to produce erroneous inferences regarding technical efficiency.

Another asymmetric Färe input efficiency measure looks for the minimum over its component measures (see, e.g., [14], [16]) or [5]. An early survey of this literature is found in [16].

A generalization of Shephard’s distance function [35] has been proposed, known as the directional distance function, to analyze efficiency both in consumption and production theory. First, Luenberger [25, 26] defines the benefit function as a directional representation of preferences, thereby generalizing

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Shephard’s (34) input distance function defined in terms of a scalar output representing utility. Lu-
ennenberger 27 shows that this function transforms the question of Pareto efficiency into an optimality
principle, the so-called zero-maximum principle. In particular, he shows that an allocation is Pareto
efficient if and only if it maximizes the social benefit function and renders its value zero. Chambers
11 extends Luenberger’s method to characterize Pareto equilibria to state-contingent economies with
agents possessing incomplete preference orders. In his study of production, Luenberger also introduces
the shortage function which accounts for both input contractions and output improvements and which is
dual to the profit function. Chambers, Chung and Färe 9,10 relabel this same function as a directional
distance function and since then it is commonly known by that name. The directional distance func-
tion, in the graph oriented case, is primal to the profit function that is dual to the directional distance
function. Chavas and Cox 12 propose a graph extension of the Debreu-Farell measure that is closely
related to the directional distance function.

Along this line, Briec, Cavaignac and Kerstens 6 revisit the debate on the axiomatic properties
satisfied by various radial versus non-radial measures of technical efficiency in production. A similar
construction is proposed by 18 who relate it to the notion of slack-based distance function. These
analysis yield some new directional efficiency measures. In particular a directional Färe-Lovell measure
is proposed in 9 as well as an asymmetric directional distance function.

In a more recent paper Briec, Cavaignac and Kerstens 7 define a new, generalization of the input
proportional efficiency measures that clearly demonstrates the links between the Debreu-Farrell and
Färe-Lovell measures. Furthermore, they explore the axiomatic properties of this new input efficiency
measure, as well as the ways it can be computed using non-parametric specifications of technology.

This paper extends these contributions by introducing a wide class of technical efficiency measures
built from utility functions. The proposed construction presents an analogy with that of the Stone-Geary
model, which was developed in the context of utility theory. The Stone-Geary utility function originates
in comments made by Geary 19 on an earlier paper by Klein and Rubin 21, the scope of which
was to determine an appropriate price index in the presence of rationing. Along this line, we define a
generalized directional distance function that includes as a special limiting case the directional distance
function proposed in 9 and 10. The proposed distance function, instead of expanding the production
vector along a preassigned direction, maximizes the generalized sum of the netput quantities until an
efficient point is reached. The basic structure of this index is inspired by the inequality index introduced
by Atkinson 1 in social choice theory. Our new distance function therefore offers an encompassing
framework to the main existing distance functions and efficiency measures in the economic literature
that turn out to be just special limiting cases of the new one in the case of a feasible input-output
vector. Based on this principle, we actually show that the directional distance function can be seen as
proceeding from the maximization of a Leontief utility function. The generalized Färe-Lovell measure
also appears to be related to the CES functions following the same principle.

Along this line, a general duality result is derived, connecting the generalized directional distance
function to the profit function. We first establish a maximum norm duality theorem and extend the result
to the case where the parameter of the generalized mean is less than one. Then a dual correspondence
is developed between the generalized directional distance function and the profit function. For every
feasible production vectors, it encompasses as a special limiting case the results established in 9 and
10. Perhaps more importantly, we identify a large class of measures for which these duality results can
be obtained without convexity.

2 Assumptions on Technology and Distance Functions

2.1 Production Technology

A technology describes all the possibilities of production making it possible to transform input vectors
into output vectors. The technology can also be modelled by a set of netput vectors. By convention
inputs are negative and output positive. Let us denote T the set of all the production possibilities (or
technology set). We suppose that there are m inputs and n outputs. Therefore d = m + n and we write
For all $z = (z_1, z_2, ..., z_d)$ where $d$ is the number of commodities and $z_k$ denotes the quantity of commodity $k$ in each netput vector. We also assume that the inputs are indexed from $k = 1$ to $m$ and the outputs from $k = m + 1$ to $d = m + n$. The production set is therefore a subset of $\mathbb{R}^m \times \mathbb{R}^n$. The set valued map $T : \mathbb{R}^m \Rightarrow \mathbb{R}^n$ defined as $T(x) = \{ y : (x, y) \in T \}$ is called the input correspondence. The set valued map $T^{-1} : \mathbb{R}^n \Rightarrow \mathbb{R}^m$ defined as $T^{-1}(y) = \{ x : (x, y) \in T \}$ is called the output correspondence. Therefore, if $z := (x, y) \in T$, then $y \in T(x)$ and $x \in T^{-1}(y)$.

The following standard conditions are imposed on $T$ (see, e.g., [20] for details):

T1: $(0,0) \in T$ and $y \neq 0 \Rightarrow (0, y) \notin T$;
T2: For all $z \in T$, $\{ z' : z' \leq z \}$ is bounded;
T3: $T$ is a closed set;
T4: If $z \in T$ and $z' \leq z$ then $z' \in T$.
T5: $T$ is convex.

Apart from the traditional regularity assumptions (possibility of inaction, boundedness, and closedness), assumption T4 represents the strong or free disposability of netput.

When discussing efficiency measures and distance functions, it is important to isolate two subsets of the production set. First, we can define the weak efficient subset as follows:

$$\mathcal{W}(T) = \{ z \in T : z' > z \Rightarrow z' \notin T \}. \quad (2.1)$$

The efficient subset of a production set is defined as:

$$\mathcal{E}(T) = \{ z \in T : z' \geq z \text{ and } z' \neq z \Rightarrow z' \notin T \}. \quad (2.2)$$

Obviously, these two subsets of the production set are connected via the following inclusions: $\mathcal{E}(T) \subseteq \mathcal{W}(T) \subseteq T$.

Now, recall the definition of the directional distance function defined in [10]. The directional distance function is the map $D_T : \mathbb{R}_+^d \times \mathbb{R}_+^d \rightarrow \mathbb{R}_+ \cup \{-\infty, \infty\}$ defined by:

$$D(z; g) = \sup_{\delta \in \mathbb{R}} \{ \delta : z + \delta g \in T \}. \quad (2.3)$$

If the condition is not vacuous, that is there is a $\delta$ with $z + \delta g \in T$ and $g \neq 0$, then the sup will be achieved; otherwise it is defined to be $-\infty$. If $z \in T$ and $g = 0$, then $D_T(z, 0) = +\infty$. In words, this directional distance function indicates the maximal expansion in the direction of $g$ simultaneously in all netputs which still allow the production.

A well-known property of the directional distance function is that it exhibits the translation invariance property. Under a strong disposability assumption with $g \in \mathbb{R}_+^{d+}$, $D(z; g) = 0 \iff z \in \mathcal{W}(T)$. However, in the general case, $D(z; g) = 0$ does not imply that $z$ belongs to $\mathcal{E}(T)$. Hence, the computation of $D(z; g)$ for a given direction $g \in \mathbb{R}_+^{d}$ does not allow to conclude whether a production plan is efficient or not. In fact, $z \in \mathcal{E}(T)$ implies that $D(z; g) = 0$, but the converse is only true in the special case when $\mathcal{E}(T) = \mathcal{W}(T)$ and $g \in \mathbb{R}_+^{d+}$. For all $g \in \mathbb{R}_+^{d}$ the support of $g$ is defined as $K_g = \{ k \in [d] : g_k \neq 0 \}$. This implies that $K_g = \{ k \in [d] : g_k > 0 \}$.

Along with this approach, one can extend to the full netput space the asymmetric directional distance function defined by [6] as $AD : \mathbb{R}^d \times \mathbb{R}_+^d \rightarrow \mathbb{R}_+ \cup \{-\infty, \infty\}$,

$$AD(z; g) = \max_{k \in K_g} D(z; g_k e_k) \quad (2.4)$$

where $\{ e_k \}_{k \in [d]}$ stands for the canonical basis of $\mathbb{R}^d$. This asymmetric directional distance function takes the maximum of the dimension-wise reduction in each netput direction, which allow production. A closely related approach is proposed in [18].
By construction it is implicitly assumed that $z \in T$ to ensure a feasibility condition. This is a useful assumption to establish the main results of the paper. In the following, for all finite subsets $A$, $|A|$ stand for the cardinality of $A$.

The Färe-Lovell efficiency measure emerged from a debate on axiomatic properties of radial efficiency measures. This function $E_{FL}: \mathbb{R}^m \setminus \{0\} \times \mathbb{R}_+^n \rightarrow \mathbb{R}_+ \cup \{\infty\}$ is defined for all $z := (x, y) \in T$ as follows:

$$E_{FL}(x, y) = \inf_{\beta \in [0,1]^m} \left\{ \frac{1}{m_x} \sum_{i \in I_x} \beta_i : (\beta \odot x, y) \in T \right\}, \quad (2.5)$$

If $z \notin T$ it is defined to be $+\infty$. The symbol $\odot$ denotes the Hadamard product (element by element) of two vectors, and for all $x \in \mathbb{R}^n$ the support of $x$ is defined as $I_x = \{i : x_i \neq 0\}$ and its cardinal is $m_x$. This Färe-Lovell efficiency measure indicates the minimum average sum of dimension-wise reductions in each input dimension which maintains production of given outputs on the efficient subset of the input set.

Since the directional distance function fails to characterize the efficient subset, one can define a Färe-Lovell directional distance function as follows. The Färe-Lovell directional distance function is the map $D_{FL}: \mathbb{R}^d \times \mathbb{R}_+^d \rightarrow \mathbb{R}_+ \cup \{-\infty, +\infty\}$ defined for all $z \in T$ as:

$$D_{FL}(z; g) = \sup_{\delta \in \mathbb{R}_+^d} \left\{ \frac{1}{d_g} \sum_{k \in K_g} \delta_k : z + \delta \odot g \in T \right\}. \quad (2.6)$$

If $z \notin T$ then it is defined to be $-\infty$. If $z \in T$ and $g = 0$, then $D_{FL}(z; g) = +\infty$. This approach extends the definition proposed in along two directions. First it proposes a measure oriented in the full netput space. Second it takes into account the cardinality of the set $K_g$. This is important for some of the results further established in the paper. It is noteworthy that if $g = z$ then we retrieve the Russell proportional distance function introduced in . In particular if $g = (x, 0)$ then:

$$D_{FL}(x, y; x, 0) = 1 - E_{FL}(x, y). \quad (2.7)$$

In addition, note that if $g = 0$ then $K_g = \emptyset$ and, by construction, $D_{FL}(z; g) = D_{FL}(z; 0) = -\infty$.

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1For all $w, z \in \mathbb{R}^d$, $w \odot z = (w_1 z_1, \ldots, w_d z_d)$. 

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Figure 1: Directional Distance Function.

Figure 2: Asymmetric Directional Distance Function.
In Figure 3, \( z' \) is projected from \( z \) in the direction of \( g \). It is a weakly but not (strongly) efficient point. \( z' \) should be pushed to \( z^* \) to reach the efficient subset \( E(T) \). In Figure 4, \( x' \) is not an efficient input, while \( x^* \) is efficient. It follows that the value of the Fare-Lovell measure at point \( x^* \) is equal to 1.

2.2 Generalized Means

In the following, a suitable notion of generalized mean is considered. For all \( p \in \mathbb{R}^+ \), let \( \phi_p : \mathbb{R}^+ \rightarrow \mathbb{R} \) be the map defined by \( \phi_p(\lambda) = \lambda^p \). For all \( p \neq 0 \), the reciprocal map is \( \phi_p^{-1} := \phi_{\frac{1}{p}} \). First, it is quite straightforward to state that: (i) \( \phi_p \) is defined over \( \mathbb{R}^+ \); (ii) \( \phi_p \) is continuous over \( \mathbb{R}^+ \); and (iii) \( \phi_p \) is bijective over \( \mathbb{R}^+ \). Second, let us focus on the case \( p \in ]-\infty, 0[ \). The map \( x \mapsto x^p \) is not defined at point \( x = 0 \). Thus, it is not possible to construct a bijective endomorphism on \( \mathbb{R}^+ \).

For all \( p \in ]-\infty, 0[ \) we consider the function \( \phi_p \) defined by:

\[
\phi_p(\lambda) = \begin{cases} 
\lambda^p & \text{if } \lambda > 0 \\
+\infty & \text{if } \lambda \leq 0.
\end{cases}
\]  (2.8)

The reciprocal is the map \( \phi_{\frac{1}{p}} \) defined on \( \mathbb{R}^+ \cup \{\infty\} \) as:

\[
\phi_{\frac{1}{p}}(\lambda) = \begin{cases} 
\lambda^{\frac{1}{p}} & \text{if } \lambda > 0 \\
0 & \text{if } \lambda = \infty.
\end{cases}
\]  (2.9)

Let us investigate the \( \phi_p \) generalized sum analyzed in [4]. First one can introduce the binary operation \( p+ \) defined for all \( s, t \in \phi_p^{-1}(\mathbb{R}) \) as

\[
s + t = \phi_p^{-1}\left(\phi_p(s) + \phi_p(t)\right).
\]  (2.10)

This binary operation can be extended to a suitable generalized sum defined as follows. For all \( (\delta_1, \ldots, \delta_d) \in \mathbb{R}_d^+ \), and for all \( p > 0 \) the \( \phi_p \)-generalized sum is given by:

\[
\phi_p \sum_{k \in [d]} \delta_k := \phi_p^{-1}\left(\sum_{k \in [d]} \phi_p(\delta_k)\right) = \left(\sum_{k \in [d]} (\delta_k)^p\right)^\frac{1}{p}.
\]  (2.11)

If \( p < 0 \), using the symbolism \( \frac{1}{0} = +\infty \) we have, by construction:

\[
\phi_p \sum_{k \in [d]} \delta_k := \phi_p^{-1}\left(\sum_{k \in [d]} \phi_p(\delta_k)\right) = \begin{cases} 
\left(\sum_{k \in [d]} (\delta_k)^p\right)^\frac{1}{p} & \text{if } \min_k \delta_k > 0 \\
0 & \text{if } \min_k \delta_k = 0.
\end{cases}
\]  (2.12)
Briec, Dumas and Mecki [8] consider these generalized sums to propose a distance function measuring the efficiency in a social welfare context. Atkinson [1] also uses some similar formulation to introduce a large class of social welfare functional.

It is important to note that we have the following limit properties:

(i) \[ \lim_{p \to -\infty} \left( \frac{1}{|d|} \sum_{k \in [d]} \delta_k^p \right)^{1/p} = \min_{k \in [d]} \delta_k^{\frac{1}{p}}; \]

(ii) \[ \lim_{p \to 0^-} \left( \frac{1}{|d|} \sum_{k \in [d]} \delta_k^p \right)^{1/p} = \prod_{k \in [d]} \delta_k^{\frac{1}{p}}; \]

(iii) \[ \lim_{p \to +\infty} \left( \frac{1}{|d|} \sum_{k \in [d]} \delta_k^p \right)^{1/p} = \max_{k \in [d]} \delta_k^{\frac{1}{p}}. \]

The geometric structure involved by this class of generalized means is depicted in Figure 5.

![Figure 5: Curves \{(x_1, x_2): x_1 + x_2 = 2\}]
3 Utility Based Distance Functions and Generalized Means

3.1 A Utility-Based Formalism for Distance Functions and Efficiency Measures

In what follows we propose a particular type of representation of the notion of distance function. We will show later that many usual distance functions listed in the literature on efficiency analysis can be represented by this model.

The proposed approach is very close to that used to define the Stone-Geary utility function. In general the Stone-Geary utility function is constructed from the Cobb-Douglas utility function

$$W(0)(z_1, ..., z_d) = \prod_{k \in [d]} (z_k - \gamma_k)^{\alpha_k}$$

(3.1)

where $z_k$ is a consumption good and $\gamma$ and $\alpha$ parameters, with $\alpha \in \mathbb{R}_+^d$ and $\sum_{k \in [d]} \alpha_k = 1$. The Stone-Geary function is used to model problems involving subsistence levels of consumption. In these cases, a certain minimal level of some good has to be consumed. In our context a producer seeks to perform better from a minimal level of netputs. For $p \leq 1$ and $p \neq 0$ we have a CES formulation involving the function:

$$W(p)(z_1, ..., z_d) = \sum_{k \in [d]} \alpha_k \varphi_p (v_k - \gamma_k).$$

(3.2)

Inspired by this formalism, we propose a new class of distance functions. To simplify the technical exposition, for all $z \in T$, let

$$T_z = \{ u \in T : u \geq z \}$$

(3.3)

denote the set of the netput vectors that dominate $z$.

Definition 3.1.1 A distance function $D : \mathbb{R}^d \rightarrow \mathbb{R}_+ \cup \{-\infty, +\infty\}$ admits a utility-based representation if there is a map $W : \mathbb{R}_+^d \rightarrow \mathbb{R}$, such that (i) $W(0) = 0$; (ii) $W$ is upper semi-continuous; (iii) $W$ is non-decreasing; (iv) For all $z \in T$

$$D(z) = \sup_{u \in T_z} W(u - z),$$

(3.4)

where $D$ is defined to be $-\infty$ when $z \notin T$.

In what follows, we will say that a distance function satisfying the conditions of Definition 3.1.1 with respect to $W$ is a $W$-utility based distance function. Conversely, note that if $T1 - T4$ hold, it is always possible to construct a distance function $D_W$ from a map $W$ satisfying conditions (i), (ii) and (iii) in Definition 3.1.1. From $T2$ and $T3$, $T_z$ is compact for any $z \in T$. Therefore, $D_W$ is well defined and there exists some $u^* \in T_z$ such that $W(u^* - z) = D_W(z)$.

The basic intuition is that distance functions of this class seek the maximum improvement from a point $z$ to reach a boundary point $z^*$. This improvement is quantified by a function that can be identified to some kind of utility function constructed on the Stone-Geary utility scheme.

In the following, we say that $W$ is strongly increasing (strongly decreasing) if $z \neq z'$ and $z' \geq z$ implies that $W(z') > W(z)$ ($W(z') < W(z)$). We say that $W$ is weakly increasing (weakly decreasing) if $z > z'$ implies that $W(z') > W(z)$ ($W(z') < W(z)$). We say that $W$ satisfies an absorption condition if $z_k = 0$ for some $k$ implies that $W(z) = 0$. The generalized mean is an example of applications satisfying this absorption condition when $p \leq 0$ (including the multiplicative case).

Proposition 3.1.2 Suppose that $D : \mathbb{R}^d \rightarrow \mathbb{R}_+ \cup \{-\infty\}$ is a $W$-utility-based distance function, where $W : \mathbb{R}^d \rightarrow \mathbb{R}$, such that (i) $W(0) = 0$; (ii) $W$ is upper semi-continuous; (iii) $W$ is non-decreasing. Under $T1$ to $T4$, $D$ satisfies the following properties:

(a) If $W$ is strongly increasing then $D$ is strongly decreasing. Moreover $D(z) = 0$ if and only if $z \in \mathcal{E}(T)$. 

7
(b) If $W$ is weakly increasing then $D$ is weakly decreasing. Moreover if $D$ satisfies the absorption condition then $D(z) = 0$ if and only if $z \in W(T)$.

(c) If $W$ is quasi-concave and $T$ is convex then $D$ is quasi-concave over $T$.

(d) If $W$ is upper semi-continuous then $D$ is upper semi-continuous over $T$.

The following results will be useful for the remainder of the paper in which we will consider some specific sequences of functions based on the notion of generalized mean.

**Proposition 3.1.3** Let $\{D_{(p)}\}_{p \in \mathbb{N}}$ be a sequence of $W_{(p)}$-utility based distance functions where $\{W_{(p)}\}_{p \in \mathbb{N}}$ is a sequence of functions satisfying conditions (i), (ii) and (iii) in definition 3.1.1. Suppose moreover that, for each $p$, $W_{(p)}$ is continuous and for any compact subset $K$ of $\mathbb{R}^d$, the sequence $\{W_{(p)}\}_{p \in \mathbb{N}}$ uniformly converges to a function $W$. Then the map $D : \mathbb{R}^d \to \mathbb{R}_+ \cup \{-\infty\}$ defined for all $z \in T$ as:

$$D(z) = \sup_{u \in T_z} W(u - z) \quad (3.5)$$

is a $W$-utility-based distance function where $D$ is defined to be $-\infty$ when $z \notin T$. Moreover, for all $z \in T$

$$\lim_{p \to \infty} D_{(p)}(z) = D(z).$$

The following statement uses the fact that if a monotone sequence of continuous functions pointwise converges to a continuous function on a compact set, then it converges uniformly.

**Corollary 3.1.4** Let $\{D_{(p)}\}_{p \in \mathbb{N}}$ be a sequence of $W_{(p)}$-utility based distance functions where $\{W_{(p)}\}_{p \in \mathbb{N}}$ is a monotonic sequence of functions satisfying conditions (i), (ii) and (iii) in definition 3.1.1. Suppose moreover that for each $p$ $W_{(p)}$ is continuous and for any compact subset $K$ of $\mathbb{R}^d$, the sequence $\{W_{(p)}\}_{p \in \mathbb{N}}$ is pointwise convergent to a continuous function $W$ over $K$. Then the sequence $\{D_{(p)}\}_{p \in \mathbb{N}}$ converges to the map $D : \mathbb{R}^d \to \mathbb{R}_+ \cup \{-\infty\}$ defined for all $z \in T$ as:

$$D(z) = \sup_{u \in T_z} W(u - z) \quad (3.6)$$

where $D$ is defined to be $-\infty$ when $z \notin T$.

Now, for any $p \in \mathbb{R} \cup \{-\infty\}$, we will say that $W_{(p)}$ is a generalized $p$-mean utility function if $W_{(p)}$ is defined as follows for each value of $p$:

(i) $W_{(p)}(\delta) = \frac{1}{d^n} \sum_{k \in [d]} \delta_k$ if $p \in \mathbb{R} \setminus \{0\}$;

(ii) $W_{(0)}(v) = \prod_{k \in [d]} (\delta_k)^{\frac{1}{p}}$ if $p = 0$;

(iii) $W_{(-\infty)}(\delta) = \min_{k \in [d]} \delta_k$ if $p = -\infty$;

(iv) $W_{(\infty)}(\delta) = \max_{k \in [d]} \delta_k$ if $p = \infty$.

Notice that for each $p$, $W_{(p)}$ is a continuous and non-decreasing map. Therefore, for any $p$, one can construct a $W_{(p)}$-utility-based distance function as:

$$D_{W_{(p)}}(z) = \sup_{u \in T_z} W_{(p)}(u - z). \quad (3.7)$$

In particular, if $p \leq 0$ then $W_p$ satisfies the absorption condition, since $z_k = 0$ for some $k$ implies that $W_p(z) = 0$. 

8
Proposition 3.1.5 Suppose that \( p, q \in \mathbb{R} \), and let \( W_{(p)} \) and \( W_{(q)} \) be two generalized mean utility functions defined on \( \mathbb{R}^d_+ \). If \( p \geq q \), then for all \( \delta \in \mathbb{R}^d_+ \)

\[
W_{(p)}(\delta) \geq W_{(q)}(\delta).
\]

Moreover let \( \bar{p} \in \mathbb{R} \cup \{-\infty, +\infty\} \) and suppose that \( \{p_k\}_{k \in \mathbb{N}} \) is a monotone sequence of real numbers such that \( \lim_{k \to \infty} p_k = \bar{p} \) and \( \{W_{(p_k)}\}_{k \in \mathbb{N}} \) is a sequence of generalized mean utility functions that is pointwise convergent to \( W_{\bar{p}} \). Then, \( \{W_{(p_k)}\}_{k \in \mathbb{N}} \) converges uniformly on any compact subset \( K \) of \( \mathbb{R}^d_+ \). In addition:

\[
\lim_{k \to \infty} D_{W_{(p_k)}}(z) = D_{W_{(\bar{p})}}(z).
\]

The generalized directional measure boils down to an output oriented measure when \( g_k = 0 \), for all \( k \in [m] \). Along this line, it might be useful to study the utility-based formalism in the case of a contingent production set with stochastic outputs (see for instance [11] and [28]). Moreover, because of the nature of the proposed construction, it would be useful to use utility-based distance functions to determine and characterize an economic equilibrium.

3.2 From Generalized Means to Generalized Directional Distance Functions

Paralleling our earlier results, an extension of the directional distance function is proposed that generalises, for every feasible production vectors, all the aforementioned directional distance functions. This new generalized directional distance function can be defined as follows:

Definition 3.2.1 For all \( p \in \mathbb{R} \setminus \{0\} \), the generalized directional Färe-Lovell distance Function \( D_{(p)} : \mathbb{R}^d \times \mathbb{R}^d_+ \to \mathbb{R}_+ \cup \{-\infty, +\infty\} \) is defined for all \( z \in T \) by

\[
D_{(p)}(z; g) = \sup_{\delta \in \mathbb{R}^d_+} \left\{ \frac{1}{d_{g}} \sum_{k \in K} \delta_k : z + \delta \odot g \in T \right\}.
\]

If \( z \notin T \) it is defined to be \(-\infty\).

By construction, if \( g = 0 \) then \( D_{(p)}(z; g) = +\infty \) for all \( z \). Obviously, from equation (2.6), we have

\[
D_{(1)}(z; g) = D_{FL}(z; g).
\]

Clearly, if \( p > 0 \), then we have:

\[
D_{(p)}(z; g) = \sup_{\delta \in \mathbb{R}^d_+} \left\{ \left( \frac{1}{d_{g}} \sum_{k \in K} \delta_k^p \right)^{1/p} : z + \delta \odot g \in T \right\}.
\]

(3.8)

Notice that, in Definition 3.2.1 we consider the cases where \( p \) takes either positive or negative values. The number \( \frac{1}{d_{g}} \) operates as a weighting scheme taking into account the dimensions of the netput space. This definition can be extended to the case \( p = 0 \) by defining a multiplicative directional Färe-Lovell distance function for all \( z \in T \) as follows: \( D_{(0)} : \mathbb{R}^d \times \mathbb{R}^d_+ \to \mathbb{R}_+ \cup \{-\infty, +\infty\} \) as

\[
D_{(0)}(z; g) = \sup_{\delta \in \mathbb{R}^d_+} \left\{ \prod_{k \in K} \delta_k^{d_k^g} : z + \delta \odot g \in T \right\}.
\]

(3.9)

If \( z \notin T \) it is defined to be \(-\infty\). This function also projects an netput vector on the efficient subset of the production set. An multiplicative analogue Färe-Lovell measure was proposed in [30].

For any nonempty subset \( K \) of \([d]\), let us denote

\[
\mathbb{R}_K = \left\{ \sum_{k \in K} w_k e_k : w_k \in \mathbb{R}, k \in K \right\}.
\]

(3.10)

For all vectors \( z \) of \( \mathbb{R}^d \), let \( z_K = \sum_{k \in K} z_k e_k \) denotes the canonical projection of \( z \) onto \( \mathbb{R}_K \). For all \( g \in \mathbb{R}^d_+ \setminus \{0\} \), it follows that if \( K = K_g \), then \( \mathbb{R}_K = \left\{ \sum_{k \in K_g} w_k e_k : w_k \in \mathbb{R}, g_k > 0 \right\} \). Moreover,
z_{K_g} = \sum_{g_k > 0} z_{k \epsilon k}$. Clearly, if $g > 0$ then $\mathbb{R}_{K_g} = \mathbb{R}^d$ and $z_{K_g} = z$. Accordingly, let $\mathbb{R}_{K,+}$ denotes the set of the nonnegative canonical projections onto $\mathbb{R}_g$.

In the following, we slightly extend the generalized $p$-mean utility functions introduced in section 3.1 in the case where $g \neq 0$.

(i) $W_{(p),g}(\delta) = \frac{1}{d_p} \sum_{k \in K_g} \frac{g_k}{g_k} \delta_k$ if $p \in \mathbb{R} \setminus \{0\}$;

(ii) $W_{(0),g}(\delta) = \prod_{k \in K_g} (\delta_k)^{\frac{1}{d_p}}$ if $p = 0$;

(iii) $W_{-\infty,g}(\delta) = \min_{k \in K_g} \delta_k$ if $p = -\infty$;

(iv) $W_{\infty,g}(\delta) = \max_{k \in K_g} \delta_k$ if $p = \infty$.

The next statement proposes a useful utility-based formulation of the directional Färe-Lovell distance function including the Stone-Geary utility function. Among other things, it relates the directional distance function to the map $\delta \mapsto \min_{k \in K_g} \delta_k$.

**Proposition 3.2.2** Under T1 to T4, for all $p \in \mathbb{R} \cup \{-\infty, +\infty\}$ and all $g \in \mathbb{R}_g^d \setminus \{0\}$, the generalized directional distance function $D_{(p)}$ has a $W_{(p),g}$-utility representation over $\mathbb{R}_{K_g,+}$:

(a) $D(z; g) = \sup_{u \in T_z} W_{-\infty,g}(u - z) = \sup_{u \in T_z} \min_{k \in K_g} \left\{ \frac{u_k - z_k}{g_k} \right\}$.

(b) $D_{(p)}(z; g) = \sup_{u \in T_z} W_{(p),g}(u - z) = \sup_{u \in T_z} \left\{ \frac{1}{d_p} \sum_{k \in K_g} \frac{u_k - z_k}{g_k} \right\}$, for all $p \in \mathbb{R} \setminus \{0\}$.

(c) $D_{(0)}(z; g) = \sup_{u \in T_z} W_{(0),g}(u - z) = \sup_{u \in T_z} \left\{ \prod_{k \in K_g} \left( \frac{u_k - z_k}{g_k} \right)^{\frac{1}{d_p}} \right\}$.

(d) $AD(z; g) = \sup_{u \in T_z} W_{\infty,g}(u - z) = \sup_{u \in T_z} \max_{k \in K_g} \left\{ \frac{u_k - z_k}{g_k} \right\}$.

It is now shown that the generalized directional Färe-Lovell distance function is associated with an optimal value of $\delta$, which allows to identify a reference netput vector. These elementary properties are important for the results established hereafter in the paper.

**Lemma 3.2.3** Under T1 to T4, for all $z \in T$, and all $g \in \mathbb{R}_g^d \setminus \{0\}$ we have the following properties:

(a) For all $p \in \mathbb{R} \setminus \{0\}$, there is some $\delta_{(p)}^* \in \mathbb{R}_{K_g,+}$ such that $D_{(p)}(z; g) = \frac{1}{d_p} \sum_{k \in K_g} \delta_{(p),k}^* \delta_k$ and $z + \delta_{(p)}^* \circ g \in \mathcal{W}(T)$. Moreover, if $p \in \mathbb{R}_{++}$, then $z + \delta_{(p)}^* \circ g \in \mathcal{E}(T)$.

(b) If $p = 0$, there is some $\delta_{(0)}^* \in \mathbb{R}_{K_g,+}$ such that $D_{(0)}(z; g) = \prod_{k \in K_g} \delta_{(0),k}^*$ and $z + \delta_{(0)}^* \circ g \in \mathcal{W}(T)$.

(c) If $p = -\infty$, then there is some $\delta_{-\infty}^* \in \mathbb{R}_{K_g,+}$ such that $D(z; g) = \min_{k \in K_g} \delta_{-\infty,k}^*$ and $z + \delta_{-\infty}^* \circ g \in \mathcal{W}(T)$.

(d) If $p = +\infty$, then there is some $\delta_{+\infty}^* \in \mathbb{R}_{K_g,+}$ such that $AD(z; g) = \max_{k \in K_g} \delta_{+\infty,k}^*$ and $z + \delta_{+\infty}^* \circ g \in \mathcal{W}(T)$.

Notice that if $p < 0$ then $z + \delta_{(p)}^* \circ g$ may not be in $\mathcal{E}(T)$. This comes from the fact that the map $\delta \mapsto \sum_{k \in [d]} \delta_k$ is not strictly increasing at point 0. 0 is an absorbing element of the generalized sum. In the following we propose an example of non-parametric technology (see [3] and [2]).
Example 3.2.4 Let $A \subset \mathbb{R}^m \times \mathbb{R}^n$ be a finite set of observed production vectors. Following [3], one can define the production technology:

$$T = \{ u \in \mathbb{R}^d : u \leq \sum_{a \in A} t_a a : \sum_{a \in A} t_a = 1, t \geq 0. \}$$

It follows that we have:

$$D_{(p)}(z; g) = \sup_{t \in \mathbb{R}^{|A|}_+} \left\{ \frac{1}{d_g^p} \sum_{k \in K_g} \phi_p \sum_{a \in A} t_a a_k - z_k \right\} : \sum_{a \in A} t_a = 1, t \geq 0 \}.$$ 

Notice that this formulation is related to the slack-based approach proposed by [18] and [37].

3.3 Axiomatic Properties

This generalized Färe-Lovell directional distance function involves a preassigned direction $g$ and it is important to characterise its axiomatic properties. This we do in the following proposition by considering a suitable refinement of the notion of efficient subset. For all $k \in [d]$ let us define the $k$-input weak efficient subset as:

$$W_k(T) = \{ z \in T : z' \geq z \text{ and } z_k > z_k' \implies z' \notin T \}. \quad (3.11)$$

This definition relaxes the standard definitions of weak efficiency by focusing on a single specific commodity $k$. Briec, Cavaignac and Kerstens [6] propose a similar idea with the notion of \textit{weak efficient subset in the direction of} $g$. The proof of the next statements is similar and thus is omitted.

The efficient subset of $T$ satisfies the above criteria for all $k \in [d]$. Therefore, it can be expressed as:

$$\mathcal{E}(T) = \bigcap_{k \in [d]} W_k(T). \quad (3.12)$$

The weak efficient subset is larger that the efficient subset and by definition:

$$W(T) = \bigcup_{k \in [d]} W_k(T). \quad (3.13)$$

For all subsets $K$ of $[d]$ let us denote:

$$\mathcal{E}_K(T) = \bigcap_{k \in K} W_k(T). \quad (3.14)$$

$\mathcal{E}_K(T)$ is termed the \textit{$K$-efficient subset}. This definition yields a weakening of the usual notion of efficient subset by focusing of some specific netputs. This subset can be characterized from the generalized Färe-Lovell directional function. Note that the following results are independent of any assumption on the returns to scale or the convexity of the technology (see [23] for recent developments on variable returns to scale).

Paralleling the definitions of the weak and strong efficient subsets in the direction of $g$, we say that $f$ is \textbf{strongly decreasing in the direction of} $g$ if $z_{K_g} \neq z_{K_g}'$ and $z' \geq z$ imply that $f(z') < f(z)$. $f$ is \textbf{weakly decreasing in the direction of} $g$, if $f$ is non increasing and if $z_{K_g}' > z_{K_g}$ implies that $f(z') < f(z)$.

Proposition 3.3.1 Under T1 to T4, for all $p \in \mathbb{R}$ and all $g \in \mathbb{R}^d_+ \setminus \{0\}$, we have:

(a) If $p > 0$, then $D_{(p)}(z; g) = 0$ if and only if $z \in \mathcal{E}_K(T)$.

(a') If $p \leq 0$, then $D_{(p)}(z; g) = 0$ if and only if $z \in W_K(T)$.

(b) For all $z' \leq z$ we have $D_{(p)}(z'; g) \geq D_{(p)}(z; g)$.

(c) For all $p > 0$, if $z' \leq z$ and $z_{K_g} \neq z_{K_g}'$, then $D_{(p)}(z'; g) > D_{(p)}(z; g)$.

(c') For all $p \leq 0$, if $z' \leq z$ and $z_{K_g} < z_{K_g}'$, then $D_{(p)}(z'; g) > D_{(p)}(z; g)$.

(d) If $L$ is a $d \times d$ positive diagonal matrice then we have the equality $D_{(p)}(z; g) = D_{(p)}(Lz; Lg)$.
Proposition 3.3.1a and 3.3.1c respectively show that if $p > 0$ then the generalized directional distance function characterizes the strong efficient subset and is strongly decreasing in the direction of $g$. Proposition 3.3.1a and 3.3.1c’ respectively show that if $p \leq 0$ then the generalized directional distance function characterizes the weak efficient subset and is weakly decreasing in the direction $g$. From Proposition 3.3.1b, the generalized directional distance function is nonincreasing in any case. Proposition 3.3.1d shows that the generalized directional distance function is independent of the units of measurement.

From Proposition 3.3.1, the directional Färe-Lovell distance function satisfies the axiomatic properties inherited from the standard case $p = 1$. However, it is well known that the directional distance function (which corresponds to the case $p = -\infty$) does not characterize the strong efficient subset. This is due to the fact that the map $\delta \mapsto \min_{k \in d} \mathcal{D}_k$ is not strictly increasing. The same situation arises with the asymmetric directional distance function derived from the map $\delta \mapsto \max_{k \in [d]} \mathcal{D}_k$ that is not strictly increasing. Therefore, its maximisation does not impose a required transfer between the netput combinations to reach the efficient subset.

This generalized directional Färe-Lovell distance function extends the properties satisfied by the input directional Färe-Lovell efficiency measure defined in [9]. It characterizes the partial efficient subset given a specific direction $g$. Notice that, in line with Lemma 3.2.3c the efficient subset is characterized by the generalized Färe-Lovell directional function only when $p > 0$. All these measures clearly belong to the class of slack-based efficiency measures considered in [18].

### 3.4 Directional, Asymmetric and Multiplicative Distance Functions as a Special Limiting Case

We now show that both the directional distance function, the directional Färe-Lovell distance function, the directional multiplicative Färe-Lovell, and the asymmetric directional distance functions can be viewed as a limiting case of the generalized directional Färe-Lovell distance function. Notice, however, that we only consider the case of feasible netput vectors.

**Proposition 3.4.1** Under T1 to T4, for all $z \in T$ and all $g \in \mathbb{R}_+^d \setminus \{0\}$, we have:

(a) $D_{(1)}(z; g) = DF_L(z; g)$.

(b) $\lim_{p \to -\infty} D_{(p)}(z; g) = D(z; g)$.

(c) $\lim_{p \to 0^+} D_{(p)}(z; g) = D_{(0)}(z; g)$.

(d) $\lim_{p \to +\infty} D_{(p)}(z; g) = AD(z; g)$.

Now we can recall the definition of the radial efficiency measure proposed by Debreu [13] and Farrell [17] defined as $E_{DF} : \mathbb{R}_+^n \times \mathbb{R}_+^m \to \mathbb{R}_+ \cup \{-\infty, \infty\}$ is defined for all $z := (x, y) \in T$ as follows:

$$E_{DF}(z) = \inf_{\lambda \in \mathbb{R}_+} \{\lambda : (\lambda x, y) \in T\}.$$  \hfill (3.15)

If the condition is not vacuous, that is there is a $\lambda$ with $(\lambda x, y) \in T$ then the sup will be achieved; otherwise it is defined to be $+\infty$. This radial efficiency measure indicates the maximal equiproportionate reduction in all inputs which still allows production of the given output vector on the isoquant of the input set.

Chambers, Chung and Färe [9] showed that $D(x, y; x, 0) = E_{DF}(x, y)$. It follows from Proposition 3.4.1 that

$$\lim_{p \to -\infty} D_{(p)}(x, y; x, 0) = 1 - E_{DF}(x, y).$$  \hfill (3.16)

Figures 6 and 7 depict the case where $p < 0$. When $p \to -\infty$ the map $u \mapsto \frac{1}{d_y} \sum_{k \in K_y} \frac{u_k - z_k}{g_k}$ tends to some kind of netput oriented Leontief function defined

$$u \mapsto \min_{k \in K_y} \left\{ \frac{u_k - z_k}{g_k} \right\}.$$  \hfill (3.17)
Figures 6 and Figure 9 depict the case where $p > 0$. When $p \rightarrow +\infty$ the map $u \mapsto \frac{\phi_p}{d_0} \sum_{k \in K_g} \frac{u_k - z_k}{g_k}$ tends to the function defined $u \mapsto \max_{k \in K_g} \left\{ \frac{u_k - z_k}{g_k} \right\}$. Notice that for certain $p > 0$ there is no evidence that the projection point is an extreme point of $T_z$. However, this is the case when $p = \infty$. 
Figures 10 and 11 depict the case $p = 1$ and show that the projection of $z$ onto the efficient frontier may not be a kink point of $T_z$. Thus, the four types of directional distance functions discussed in Section 2 are clearly limiting cases of the new generalized directional Färe-Lovell distance function in Definition 3.2.1. Notice that in the limit cases $p \to \infty$, $p \to -\infty$ and $p \to 0$, the limit distance functions do not characterize the efficient subset $E(T)$.

Paralleling [7], one can order most of these directional distance functions. Indeed, while we clearly have $D(z; g) \leq D_{FL}(z; g) \leq D^{(0)}(z; g) \leq AD(z; g)$, the new generalized directional Färe-Lovell distance function can also be related to these existing distance functions.

**Proposition 3.4.2** Under $T_1$ to $T_4$, the generalized Färe-Lovell directional distance function can be ordered as follows for all $z \in T$ and $g \in \mathbb{R}_d^+ \setminus \{0\}$. For all $p, q \in \mathbb{R}$, if $p \geq q$ then:

$$D(z; g) \leq D_{(q)}(z; g) \leq D_{(p)}(z; g) \leq AD(z; g).$$

Notice that the sense of these inequalities is reversed in the approach proposed in [7]. Namely, it is shown that for all $q \geq p > 0$:

$$E_{DF}(z) \geq E_{(p)}(z) \geq E_{(q)}(z) \geq E_{FL}(z) \geq AF(z),$$

where $AF$ stands for the asymmetric Färe measure. Further investigations could be carried out from an axiomatic point of view, in particular by an analysis of the continuity of the various measures proposed (see [33]).

## 4 Profit Function Prices and Duality for Convex and Non-Convex Technologies

This section establishes several intermediate results connecting the generalised Färe-Lovell distance function and the profit function. First, a duality theorem is established for the maximisation of a distance over a compact subset in a normed space. In [7] the Nirenberg’s theorem (that deals with the minimisation of the distance to a convex set) is used establish a duality result relating the input generalized Färe-Lovell measure $E_{(p)}$ and the cost function. An analogue maximization theorem is proposed to relate the generalized Färe-Lovell distance function and the profit function when $p \geq 1$ (see [24] for a proof of the Nirenberg’s Theorem). Notice, however, that convexity is not required. In the case where $p < 1$ the notion of norm is no longer suitable in our context. However, a useful duality result can be derived for quasi-concave utility-based distance functions. A suitable duality result is then obtained using standard rules of Lagrangian duality. In the latter case, the formal rules involved in our duality results parallel the formal relationship existing between $\ell_p$ and $\ell_q$ norms with the condition $\frac{1}{p} + \frac{1}{q} = 1$. 
4.1 Maximum Norm Distance Functions and \( \phi_p \)-generalized Sums for \( p \geq 1 \).

In the following we consider a real normed vector space \( E \) equipped with a norm \( \| \cdot \| \). The dual space of \( E \) is the set \( E^\star \) of all the continuous linear forms defined on \( E \) (that is the topological dual of \( E \)). The dual norm of a linear form \( \varphi_y : x \mapsto \langle y, x \rangle \) is defined as

\[
\| \varphi_y \| = \sup \left\{ \frac{|\langle y, x \rangle|}{\|x\|} : x \neq 0 \right\} = \|y\|_\star.
\] (4.1)

The following result does not require the convexity of \( C \). This is due to the fact that the largest ball centered at \( z \) which contains \( C \), admits at its maximum point a supporting hyperplane which is also that of \( C \).

**Proposition 4.1.1** Let \( C \) be a compact subset of a real normed vector space \( E \). Let \( h_C : E^\star \longrightarrow \mathbb{R} \cup \{-\infty, \infty\} \) be the functional support of \( C \). Then for all \( x \in C \),

\[
\max\{\|x - z\| : x \in C\} = \sup_{y \in E^\star} \{h_C(y) - \langle y, z \rangle : \|y\|_\star = 1\}.
\]

Suppose that \( E \) is the Euclidean vector space \( \mathbb{R}^d \) endowed with the \( \ell_p \)-norm defined by \( x \mapsto (\sum_{k \in [d]} |x_k|^p)^{\frac{1}{p}} \) if \( p \in [1, \infty[ \) and as \( x \mapsto \max_{k \in [d]} |x_k| \) if \( p = \infty \). Given any vector \( y \in \mathbb{R}^d \) the dual norm of the map \( \varphi_y : x \mapsto \langle y, x \rangle = \sum_{k \in [d]} y_k x_k \) is defined as

\[
\| \varphi_y \|_p = \sup \left\{ \frac{|\langle y, x \rangle|}{\|x\|_p} : x \neq 0 \right\} = \|y\|_q,
\] (4.2)

with \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Corollary 4.1.2** Let \( C \) be a compact subset of \( \mathbb{R}^d \). Let \( h_C : \mathbb{R}^d \longrightarrow \mathbb{R} \cup \{-\infty, \infty\} \) be the functional support of \( C \). Then for all \( z \in \mathbb{R}^d \) with \( \frac{1}{p} + \frac{1}{q} = 1 \),

\[
\max\{\|x - z\|_p : x \in C\} = \sup_{y \in \mathbb{R}^d} \{h_C(y) - \langle y, z \rangle : \|y\|_q = 1\}.
\]

Let us define as \( \Pi_z : \mathbb{R}^d_+ \longrightarrow \mathbb{R} \cup \{\infty\} \) the function which yields the maximum profit for all the netput vectors of \( T_z \). Namely,

\[
\Pi_z(w) = \sup\{w.u : u \in T_z\}.
\] (4.3)

In the following we consider the situation of a \( W \)-utility-based distance function where \( W \) is a norm \( \| \cdot \| \) that is weakly increasing over the nonnegative orthant \( \mathbb{R}^d_+ \). We have the following property.

**Proposition 4.1.3** Suppose that \( T \) satisfies T1 to T4. For all \( z \in T \), if \( W = \| \cdot \| \) is weakly increasing over \( \mathbb{R}^d_+ \), then:

\[
D_{\| \cdot \|}(z) = \max\{\|u - z\| : u \in T_z\} = \sup_{w \geq 0} \{\Pi_z(w) - w.z : \|w\| = 1\}.
\]

For all \( p \in [1, \infty[ \) if \( W = \| \cdot \|_p \) then

\[
D_{\| \cdot \|_p}(z) = \max\{\|u - z\|_p : u \in T_z\} = \sup_{w \geq 0} \{\Pi_z(w) - w.z : \|w\|_q = 1\}
\]

with \( \frac{1}{p} + \frac{1}{q} = 1 \).

Not that Proposition 4.1.3 does not assume T5. It is possible to construct a utility-based distance function from any homogeneous and quasi-convex increasing function. In such a case it is, however, easy to show that it necessarily boils down to a norm.
4.2 Quasi-Concave Utility-Based Distance Functions and $\phi_p$-generalized Sums for $p < 1$

In this subsection, we establish a duality result in the case of a $W$-utility-based distance function. It should also be assumed that $W$ is homogeneous of degree 1. Let us consider the indirect utility function $W_* : \mathbb{R}^d_+ \rightarrow \mathbb{R} \cup \{+\infty\}$ defined as:

$$ W_*(w) = \sup_{z \geq 0} \{W(z) : w.z = 1\}. \quad (4.4) $$

It follows from the homogeneity of $W$ that:

$$ \alpha W_*(w) = \sup_{w \geq 0} \{W(\alpha z) : w.z = 1\} = \sup_{w \geq 0} \{W(z') : w.z' = \alpha\}. \quad (4.5) $$

A subset $C$ of $\mathbb{R}^d_+$ is comprehensive if for all $z \in C$, $0 \leq z' \leq z$ implies that $z' \in C$.

**Lemma 4.2.1** Let $C$ be a compact convex comprehensive subset of $\mathbb{R}^d_+$ that contains 0. Then for every $z \notin C$, there is a positive vector $w_z \in \mathbb{R}^d_+$ such that $w_z.z > h_C(w_z)$. Equivalently

$$ C = \bigcap_{w \in \mathbb{R}^d_+} \{z : w.z \leq h_C(w)\}. $$

Lemma 4.2.1 is useful to simplify the technical exposition of the paper and computing the indirect utility-based distance functions involved by the generalized means in the case where $p < 1$. Notice that if $w \in \mathbb{R}^d_+$, then $W_*(w) < +\infty$.

The following duality result is now established in the case where $W$ is quasi-concave.

**Proposition 4.2.2** Let $W : \mathbb{R}^d \rightarrow \mathbb{R}$, such that (i) $W(0) = 0$; (ii) $W$ is upper semi-continuous; (iii) $W$ is strongly increasing; (iv) quasi-concave. Under T1 to T5, the $W$-utility-based distance function $D_W$ satisfies the following dual property:

$$ D_W(z) = \inf_{w \geq 0} \{\Pi_z(w) - w.z : W_*(w) = 1\}. $$

The following example illustrates a situation where the dual solution is not reached under the normalization constraint $W_*(w) = 1$.

**Example 4.2.3** Let us consider the production set $T = \{(x_1, x_2) : x_1 \leq 0, x_1 + x_2 \leq 0, x_2 \leq 2\}$. Let us consider the points $z = (-3, 2)$ and $z^* = (-2, 2)$. $z^*$ is efficient and $z$ is weakly efficient. Suppose that $W(u) = (u_1^{\frac{1}{3}} + u_2^{\frac{1}{3}})^2$. In such a case $W_*(w) = (w_1^{\frac{1}{3}} + w_2^{\frac{1}{3}})^3$. $D_W(-3, 2) = (0 + 1)^2 = 1$. We have $T_z = \{(u_1, u_2) : u_1 \in [-3, -2], u_2 = 2\}$. However there does not exists a normalized supporting hyperplane that achieves the dual program having null components. If either $w = (1, 0)$ or $w = (0, 1)$, we have from the absorption property of the generalized sum $W_*(w) = +\infty$. However, let us consider for all $n \in \mathbb{N}$ the vector $w^{(n)} = (1, n)$. We have $W_*(1, n) = (1 + \frac{1}{n^3})^3$, and $\Pi(w^{(n)}) - w^{(n)}.z = -2 + 3 = 1$. Finally,

$$ D_W(-3, 2) = 1 = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n^3}\right)^3 = \lim_{n \rightarrow \infty} (\Pi(w^{(n)}) - w^{(n)}.z)W_*(w_n). $$

In the next example, there is a normalized price vector that is the solution of the dual problem.

**Example 4.2.4** Let us consider the production set $T$ defined in Example 4.2.3 with the point $z = (-3, 2)$ that is weakly efficient. Suppose that $W(u) = (u_1^{\frac{1}{3}} + u_2^{\frac{1}{3}})^2$. In such a case $W_*(w) = (w_1^{\frac{1}{3}} + w_2^{\frac{1}{3}})^3$. From the absorption property of the generalized sum, we have $D_W(-3, 2) = 0$. If $w = (0, 1)$, $\Pi_z(w) = 2$ and $W_*(0, 1) = (0 + 1)^{-3} = 1$. Moreover $\Pi(w) - w.z = 2 - 2 = 0$ and $D_W(-3, 2) = 0 = (\Pi(w) - w.z)W_*(w)$. 

16
Remark 4.2.5 In the case where \( p = 1 \), the map \( w \mapsto \sum_{k \in [d]} w_k \) is concave and convex over \( \mathbb{R}^d_+ \). Therefore Proposition 4.1.3 and 4.2.2 apply. Moreover, this map boils down to the \( \ell_1 \)-norm over \( \mathbb{R}^d_+ \).

From Proposition 4.1.3

\[
D_{\| \cdot \|_1}(z) = \sup_{w \geq 0} \left\{ \Pi(w) - w.z : \max_{k \in [d]} |w_k| = 1 \right\}. \tag{4.6}
\]

A solution of the dual problem is a price that defines a supporting hyperplane of the \( \ell_1 \)-ball \( B_1(z, \alpha \) with \( \alpha = D_{\| \cdot \|_1}(z) \). Due to the geometric structure of the \( \ell_1 \)-ball we have \( w^* = \mathbb{1}_d \). However, if \( W = \| \cdot \|_1 \), then for every price vector \( w > 0 \), we have \( W_*(w) = \max_{k \in [d]} w_k^{-1} \). Therefore

\[
D_W(z) = \inf_{w \geq 0} \left\{ \Pi(w) - w.z : \max_{k \in [d]} w_k^{-1} = 1 \right\} = \inf_{w \geq 0} \left\{ \Pi(w) - w.z : \min_{k \in [d]} w_k = 1 \right\}. \tag{4.7}
\]

It follows that the normalization condition \( \min_{k \in [d]} w_k = 1 \) ensures that \( \Pi(w) - w.z \) is minimum for \( w^* = \mathbb{1}_d \) and we find the same result \( D_W(z) = D_{\| \cdot \|_1}(z) \) with two different dual programs.

In the following we consider the case where \( p < 1 \). In such a case the map \( v \mapsto \sum_{k \in [d]} v_k \) is not immediately related to the standard notion of norm. The next result is useful to connect the generalized Färe-Lovell distance function to the profit function. Along this line, a duality result is derived from Proposition 4.2.2.

Lemma 4.2.6 Let \( a, b \in \mathbb{R}^d_+ \) and \( c > 0 \) be a nonnegative real number.
(a) For all \( p < 1 \) with \( p \neq 0 \), let us consider the problem

\[
\sup_{v \in \mathbb{R}^d_+} \left\{ \sum_{k \in [d]} a_k v_k : \langle b, v \rangle = c, v \in \mathbb{R}^d_+ \right\}. \tag{4.8}
\]

Then there is a solution \( v^*_k \in \mathbb{R}^d_+ \) with

\[
v^*_k = c \left[ \sum_{k \in [d]} (a_k^{-1} b_k)^{\frac{1}{p}} \right]^{-1} \left( a_k^{-p} b_k \right)^{\frac{1}{p-1}}.
\]

Moreover

\[
\left( \sum_{k \in [d]} a_k v^*_k \right)^{\frac{1}{q}} = c \left( \sum_{k \in [d]} (a_k^{-1} b_k)^{q} \right)^{-\frac{1}{q}},
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).
(b) Suppose that \( t \in \mathbb{R}^d_+ \) and let us consider the problem

\[
\sup_{v \in \mathbb{R}^d_+} \left\{ \prod_k (a_k v_k)^{t_k} : \langle b, v \rangle = c, v \in \mathbb{R}^d_+ \right\}.
\]

Then there is a solution \( v^*_k \) with \( v^*_k = ct_k \left( \sum_{k \in [d]} t_k \right)^{-1} b_k^{-1} \). The value of the objective function is then:

\[
\prod_{k \in [d]} (a_k v^*_k)^{t_k} = \prod_{k \in [d]} \left[ c a_k t_k \left( \sum_{k \in [d]} t_k \right)^{-1} b_k^{-1} \right]^{t_k}.
\]

It follows that we have the following situations. If \( W(v) = \sum_{k \in [d]} a_k v_k \) for all \( W \in \mathbb{R}^d_+ \) and \( v \in \mathbb{R}^d_+ \), then, for all \( p < 1 \), with \( p \neq 0 \), we have:

\[
W_*(w) = \left[ \sum_{k \in [d]} a_k^{-1} w_k \right]^{-1}, \tag{4.9}
\]
where $\frac{1}{p} + \frac{1}{q} = 1$. If $W(v) = \prod_{k \in [d]} v_k^{a_k}$ for all $v \in \mathbb{R}^d$ and $\sum_{k \in [d]} t_k = 1$, $W$ is homogenous and:

$$W^*(w) = \prod_{k \in [d]} a_k^{-1} \left[ t_k^{-1} w_k \right]^{t_k}.$$  \hfill (4.10)

The next result is then an immediate consequence of Proposition 4.2.2.

**Proposition 4.2.7** Suppose that the production set satisfies $T_1 - T_5$ and let $a \in \mathbb{R}^d_{++}$.

(a) Suppose that $W(\delta) = \sum_{k \in [d]} a_k \delta_k$ for all $v \in \mathbb{R}^d_+$. Then, for all $p \in \mathbb{R} \setminus \{0\}$, we have:

$$D_W(z) = \inf \left\{ \Pi_z(w) - w.z : \sum_{k \in [d]} a_k^{-1} w_k = 1, w \in \mathbb{R}^d_+ \right\},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

(b) If $\sum_{k \in [d]} t_k = 1$, and $W(\delta) = \prod_{k \in [d]} \delta_k^{a_k}$ for all $v \in \mathbb{R}^d$:

$$D_W(z) = \inf \left\{ \Pi_z(w) - w.z : \sum_{k \in [d]} a_k^{-1} \left[ t_k^{-1} w_k \right]^{t_k} = 1, w \in \mathbb{R}^d_+ \right\}.$$ 

### 4.3 Profit Function and Duality: Generalized Means and Directional Cases

In the following, a dual formulation of the directional Färe-Lovell distance function in terms of the profit function is provided. This we do by showing that, in the case where $p \geq 1$, the generalized directional Färe-Lovell distance function can be interpreted as the maximisation of a norm over a suitable restriction of the production set. Applying the above maximum norm theorem, a duality result is obtained. Along this line, a dual formulation of the directional Färe-Lovell distance function is proposed and it is shown that the dual properties of the asymmetric directional distance function appear as a limiting case.

Notice that one can equivalently write

$$D(p)(z; g) = \begin{cases} \sup_{v \in T_z} \left( \sum_{k \in [d]} \frac{1}{d} \left( \frac{|u_k - z_k|}{g_k} \right)^p \right)^{\frac{1}{p}} & \text{if } z \in T \\ +\infty & \text{otherwise.} \end{cases} \hfill (4.11)$$

In the case where $p \geq 1$, it is interesting to see that one can provide a dual interpretation based upon the maximum norm theorem.

For example, if $g \in \mathbb{R}^d_{++}$, then the map

$$\|\cdot\|_{g^{-1}, p} : \delta \mapsto \left( \frac{1}{d} \sum_{k \in [d]} \frac{\delta_k |u_k|}{g_k} \right)^{\frac{1}{p}}$$

defines a weighted norm on $\mathbb{R}^d$ and we have for all $u \in \mathbb{R}^d$ and all $p \in [1, \infty[$:

$$\|\delta\|_{g^{-1}, p} = \left( \sum_{k \in [d]} \frac{1}{d} \frac{\delta_k |u_k|}{g_k} \right)^{\frac{1}{p}}.$$  \hfill (4.13)

If $p = \infty$ then

$$\|\delta\|_{g^{-1}, \infty} = \lim_{p \to \infty} \|\delta\|_{g^{-1}, p} = \max_{k \in [d]} \frac{|u_k|}{g_k}.$$  \hfill (4.14)
By definition, it follows that
\[ D_p(z; g) = \sup \{ \|u - z\|_{g^{-1},p} : u \in T_z \}. \] (4.15)

The dual norm is defined by
\[ \|w\|_{g,q} = \left( \sum_{k \in [d]} d_k^p |g_k w_k|^q \right)^{\frac{1}{q}} = d^{-\frac{1}{q}} \left( \sum_{k \in [d]} |g_k w_k|^q \right)^{\frac{1}{q}} \] (4.16)

with \( \frac{1}{p} + \frac{1}{q} = 1 \), where by definition \( \|w\|_{g,q} = \sup \{|w,u| : \|u\|_{g^{-1},p} = 1\} \). In the above cases we have considered the situation where \( g > 0 \). In the following we consider the general case. Let \( \Pi_{z,g} : \mathbb{R}_+^d \to \mathbb{R} \cup \{+\infty\} \) be the map defined as:
\[ \Pi_{z,g}(w) = \sup \left\{ w.u : u \in T_z, u_k = z_k, k \notin K_g \right\}. \] (4.17)

This profit function is limited to optimizing the netputs by fixing them at their initial levels when they are associated with a zero component of the direction \( g \).

If \( p < 1 \), we consider the quasi-concave utility functions:
\[ W_p(g)(\delta) = \sum_{k \in [d]} d_k^p \delta_k g_k^{p \neq 0} \text{ and } W_{(0),g}(\delta) = \prod_{k \in [d]} (\frac{\delta_k}{g_k})^{\frac{1}{q}} \] (4.18)

The indirect utility functions are:
\[ W_{*,(p),g}(w) = \left( \sum_{k \in [d]} d_k^p g_k w_k^q \right)^{\frac{1}{q}} \text{ and } W_{*,(0),g}(w) = d \prod_{k \in K_g} (g_k w_k)^{\frac{1}{q}} \] (4.19)

with \( \frac{1}{p} + \frac{1}{q} = 1 \). The next results are obtained by applying our earlier results to the subspace \( \mathbb{R}_{K_g} \) whose dimension is \( d_{K_g} \).

**Proposition 4.3.1** Under T1 to T4, for all \( z \in T \) and all \( g \in \mathbb{R}_+^d \setminus \{0\} \), we have the following properties:
(a) For all \( p \geq 1 \)
\[ D_p(z; g) = \sup_{u \geq 0} \left\{ \Pi_{z,g}(w) - w.z : d^{-\frac{1}{q}} \left( \sum_{k \in K_g} (g_k w_k)^q \right)^{\frac{1}{q}} = 1 \right\} \]
with \( \frac{1}{p} + \frac{1}{q} = 1 \).
(b) If \( p = 1 \),
\[ D_{FL}(z; g) = \sup_{u \geq 0} \left\{ \Pi_{z,g}(w) - w.z : d \max_{k \in K_g} w_k g_k = 1 \right\}. \]
(c) If \( p = \infty \)
\[ AD(z; g) = \sup_{u \geq 0} \left\{ \Pi_{z,g}(w) - w.z : w.g = 1 \right\}. \]
Suppose now that convexity holds (T5), we have:
(d) If \( p = 0 \)
\[ D_{(0)}(z; g) = \inf_{u \geq 0} \left\{ \Pi_{z}(w) - w.z : d \prod_{k \in K_g} (g_k w_k)^{\frac{1}{q}} = 1 \right\}. \]
If \( p < 1 \) and \( p \neq 0 \)

\[
D_{(p)}(z; g) = \inf_{w \geq 0} \left\{ \Pi_z(w) - w.z : d_g^{\frac{q}{p}} \sum_{k \in K_g} \frac{\phi_g}{g_k} w_k = 1 \right\}.
\]

If \( p = -\infty \)

\[
D(z; g) = \inf_{w \geq 0} \left\{ \Pi_z(w) - w.z : w.g = 1 \right\}.
\]

Figure 12: Duality in the case \( p \in ]1, +\infty[ \)

Figure 13: Duality in the case \( p \in ]-\infty, 1[ \)

The duality results established in Proposition 4.3.1.(a), 4.3.1.(b) 4.3.1.(d) have a similar interpretation. We consider, for any netput vector a suitable profit function restricted to the dominating production vectors.

As a result, the calculation of the generalized Färe-Lovell distance function boils down to determining a normalized dual price that minimizes the difference between the profit function and the value of a given netput. The differences between all these measures are related to the normalization constraints to which the dual prices are subject. If \( p \in [1, \infty[ \), then the problem of computing the generalized directional Färe-Lovell distance function boils down to maximizing a smooth \( \ell_p \) norm that selects an efficient point on the frontier. In the directional Färe-Lovell case, the normalization condition imposed on shadow prices allows to modify each netput price in coordinate directions to reach an efficient point. This is not the case with the directional distance function because of its additive nature that imposes a projection in the direction of \( g \) of any netput vector. Notice that the difference between the asymmetric and the standard directional cases (the duality result was established in [10]) comes from the fact that in the former case one consider a maximization of the difference between the profit function and the profit computed at point \( z \) while in the later it is a minimization. In addition, note that the shadow prices \( w^\star \) that are solutions to the dual problem achieve the maximum of the profit function at point \( z^\star = z + D(z; g)g \). It follows that \( \Pi_z(w^\star) = \Pi(w^\star) \), and we retrieve the standard duality result. More importantly, we have established in Proposition 4.3.1.(b) that \( \lim_{p \to -\infty} D_{(p)}(z; g) = D(z; g) \). When \( p \to -\infty \), since \( q = \frac{p}{p-1} \), we have \( q \to 1 \). Considering the normalization constraint in Proposition 4.3.1.(e), we retrieve the standard normalization condition established in [10]:

\[
\lim_{q \to 1} \left( d_g^{\frac{q}{p}} \sum_{k \in K_g} \frac{\phi_g}{g_k} w_k \right) = w.g = 1.
\]  

Note that the differences in the formalism of the duality results parallel the taxonomy proposed in [33]. In this paper, structural differences appear between Fare-Lovell and directional measures, which refer, respectively, to the cases \( p \geq 1 \) and \( p < 1 \).
5 Concluding Comments

We have introduced a new generalized directional Färe-Lovell distance function. This distance function extends to the full netput space the generalized Färe-Lovell input efficiency measure recently proposed in [7]. Paralleling the results established in [7], we have established that one can obtain the directional distance function [10] when \( p \) tends to minus infinity, the Färe-Lovell directional distance function [6] for \( p = 1 \), the asymmetric directional distance function when \( p \) tends to minus infinity, and finally a directional version of the multiplicative Färe-Lovell measure when \( p = 0 \).

Table 1 summarizes how to obtain the main directional distance function as limiting cases for different values of the parameter \( p \) in the generalized directional distance function:

| Value of \( p \) | Distance function |
|-----------------|-------------------|
| \(-\infty\)     | Directional Distance Function [10] |
| 0               | Multiplicative Directional Färe-Lovell |
| 1               | Directional Färe-Lovell [6] |
| \(+\infty\)     | Asymmetric Directional distance function [6] |

Table 1: Summary of Main Results

Table 2 provides a taxonomy of the duality results relating the generalized directional Färe-Lovell measure and the profit function for any value of \( p \). In addition it indicates which dual optimization criterion is considered (maximisation or minimisation) in any case.

| Value of \( p \) | Optimization Criterion | Price Normalisation | Convexity |
|-----------------|------------------------|---------------------|-----------|
| \(-\infty\)     | Minimization           | \( w \cdot g = 1 \) | Yes       |
| \( p < 0 \)     | Minimization           | \( d_g \sum_{k \in K_g} g_k w_k = 1 \) | Yes       |
| \( p = 0 \)     | Minimization           | \( d_g \prod_{k \in K_g} (g_k w_k)^{\frac{1}{p}} = 1 \) | Yes       |
| \( p \in [0,1] \)| Minimization           | \( d_g \sum_{k \in K_g} g_k w_k = 1 \) | Yes       |
| \( p = 1 \)     | Maximization           | \( \max_{k \in K_g} w_k g_k = 1 \) | No        |
| \( p > 1 \)     | Maximisation           | \( d_g \left( \sum_{k \in K_g} (g_k w_k)^q \right)^\frac{1}{q} = 1 \) | No        |
| \( p = +\infty \)| Maximization           | \( w \cdot g = 1 \) | No        |

Table 2: Duality Results

Finally, Table 3 summarizes the results in the norm and utility-based cases.

| Utility-Based | Optimization Criterion | Price Normalisation | Convexity |
|---------------|------------------------|---------------------|-----------|
| \( \| \cdot \|_\text{\normalsize{2}} \) | Maximization           | \( \| w \|_\text{\normalsize{2}} = 1 \) | No        |
| \( W \)-Quasiconcave | Minimization           | \( W(w) = 1 \) | Yes       |

Table 3: Duality Results

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that from the absorption condition, if that \( z / W \) then this implies that \( z / W \). However, by hypothesis, if \( u > z \) then we cannot find any \( u \) that strongly dominates \( z \). Thus \( z \) is efficient. To prove the reciprocal suppose that \( D(z) > 0 \). In such a case there exists some \( u \in T_z \) with \( W(u - z) > 0 \). However, by hypothesis, if \( u = z \), then \( W(u - z) = 0 \). Hence, we deduce that \( u \neq z \) and this implies that \( z \in \mathcal{E}(T) \), which proves the reciprocal. (b) Suppose that \( v > z \). There exists \( v^* \in T_z \) such that \( W(v^* - v) = D(v) \). It follows that \( v^* - v \leq v^* - z \). Therefore, since \( W \) is weakly increasing: \( D(v) = W(v^* - v) < W(v^* - z) \leq \sup \{ W(u - z) : u \in T_z \} = D(z) \), hence \( D(v) < D(z) \). Consequently, if \( D(z) = 0 \) then we cannot find any \( u \) that strongly dominates \( z \). Thus \( z \) is efficient. To prove the reciprocal suppose that \( D(z) > 0 \). In such a case there exists some \( u \in T_z \) with \( W(u - z) > 0 \). However, from the absorption condition, if \( u_k - z_k = 0 \) for some \( k \), \( W(u - z) = 0 \). Hence, we deduce that \( u \neq z \) and this implies that \( z \notin W(T) \), which proves the reciprocal. (c) Let \( z, v \in T \). Since \( W \) is upper semi-continuous there exist \( z^*, v^* \in T \) such that \( D(z) = W(z^* - z) \) and \( D(v) = W(v^* - v) \). This for all \( \theta \in [0, 1] \), \( \theta z + (1 - \theta)v \in T \) and \( \theta z^* + (1 - \theta)v^* \in T \). Since \( W \) is quasi-concave

\[
W(\theta z^* + (1 - \theta)v^* - (\theta z + (1 - \theta)v)) \geq \min \{ W(z^* - z), W(v^* - v) \} \\
\geq \min \{ D(z), D(v) \}.
\]
However:

\[
W(\theta z^* + (1 - \theta)v^* - (\theta z + (1 - \theta)v)) \leq \sup_{w \in T_\varepsilon(\theta z + (1 - \theta)v)} W(u - (\theta z + (1 - \theta)v)) \\
\leq D(\theta z + (1 - \theta)v).
\]

Hence \(D(\theta z + (1 - \theta)v) \geq \min\{D(z), D(v)\}\) which ends the proof.

(d) Suppose that \(D\) is not upper semi-continuous and let us show a contradiction. Let \(z \in T\) and let us consider a sequence \(\{z_n\}_{n \in \mathbb{N}}\) such that \(\lim_{n \to \infty} z_n = z\). Suppose that \(\limsup_{n \to \infty} D(z_n) > D(z)\).

This implies that there is an increasing subsequence \(\{n_k\}_{k \in \mathbb{N}}\) such that \(\lim_{k \to \infty} D(z_{n_k}) > D(z)\). Since \(W\) is upper semi-continuous, for any \(k\) there exists \(z^*_{n_k}\) such that \(W(z^*_{n_k} - z_{n_k}) = D(z_{n_k})\). Suppose that \(z_k = \inf_{k} z_{n_k}\). By construction \(z_{n_k} \in T_{z_k}\) for every \(n\). However, \(T_{z_k}\) is a compact subset of \(T\) and for all \(n_k\), \(z^*_{n_k}\) is upper semi-continuous, it therefore has a limit \(z^* \in T_{z_k}\). Moreover, for any \(q, z^*_{n_k} \geq z^*_{n_k}\), therefore \(z^* \geq z\). Since \(W\) is upper semi-continuous, it follows that:

\[
\lim_{q \to \infty} D(z_{n_k}) = \lim_{q \to \infty} W(z^*_{n_k} - z_{n_k}) \leq \limsup_{n \to \infty} W(z^*_n - z_n) \leq W(z^* - z) \leq \sup_{u \in T_z} W(u - z) \leq D(z).
\]

However, this is a contradiction, which proves (d).

**Proof of Proposition 3.1.3** Since the convergence is uniform over any compact subset \(K\) of \(\mathbb{R}^d\), \(W\) is continuous over \(\mathbb{R}^d\). Therefore, there is some \(u^* \in T_z\) such that \(W(u^* - z) = \sup_{u \in T_z} W(u - z)\). Since \(\lim_{p \to \infty} W_p(u^* - z) = W(u^* - z)\), it follows that

\[
\lim_{p \to \infty} D_p(z) = \lim_{p \to \infty} \liminf_{p \to \infty} D_p(u(z) - z) \geq \lim_{p \to \infty} \limsup_{p \to \infty} W(u) = W(u - z).
\]

Conversely, for each \(p\), since \(W_p\) is continuous this implies that there is some \(u(p) \in T_z\) such that \(\lim_{u \in T_z} W_p(u) = W_p(u(p) - z)\). Therefore

\[
\limsup_{p \to \infty} D_p(z) = \limsup_{p \to \infty} \sup_{u \in T_z} W_p(u) = \limsup_{p \to \infty} W_p(u(p)) - z).
\]

Since \(T_z\) is a compact subset of \(\mathbb{R}^d\), it follows that there exists an increasing sequence \(\{p_k\}_{k \in \mathbb{N}}\) and some \(\bar{u} \in T_z\) such that \(\lim_{k \to \infty} u^*_{p_k} = \bar{u}\) and

\[
\lim_{p \to \infty} D_p(z) = \lim_{p \to \infty} W_p(u(p) - z) = \lim_{k \to \infty} W_p(u(p_k) - z).
\]

However, using the triangular inequality, we have:

\[
|W(p_k)(u(p_k) - z) - W(\bar{u} - z)| \leq |W(p_k)(u(p_k) - z) - W(u(p_k) - z)| + |W(u(p_k) - z) - W(\bar{u} - z).touch
\leq \sup_{u \in T_z} |W(p_k)(u - u)| + |W(u(p_k) - z) - W(\bar{u} - z)|.
\]

Consequently, \(\limsup_{p \to \infty} D(p)(z) = \lim_{k \to \infty} W(p_k)(u(p_k) - z) = W(\bar{u} - z) \leq \limsup_{u \in T_z} W(u - z)\), which implies that

\[
\limsup_{p \to \infty} D(p)(z) \leq \lim_{p \to \infty} W(u - z) \leq \limsup_{p \to \infty} D(p)(z).
\]

Finally, conditions (i) and (ii) in Definition 3.1.1 are satisfied from the continuity of \(W\). Since for all \(p\), \(W_p\) is non-decreasing condition (iii) immediately follows.

**Proof of Corollary 3.1.4** Since \(\{W_p\}_{p \in \mathbb{N}}\) is a monotonic sequence of functions, from the Dini’s theorem, we deduce that \(\{W_p\}_{p \in \mathbb{N}}\) uniformly converges to \(W\) for any compact subset \(K\) of \(\mathbb{R}^d\). Hence, the result follows from Proposition 3.1.3. \(\square\)
Proof of Proposition 3.1.5: First we prove that if $\delta \in \mathbb{R}_+^d$ and then the map $p \mapsto (\frac{1}{d} \sum_{i \in [d]} \delta_i^p)^\frac{1}{p}$ is increasing in $p$. Suppose that $p > q > 0$ and let us consider the map $\phi_\frac{1}{q}$. Since $q < p$ it follows that $\phi_\frac{1}{p}$ is concave. From the concavity of $\phi_\frac{1}{p}$ over $\mathbb{R}_+^d$:

$$
\phi_\frac{1}{p} \left( \frac{1}{d} \sum_{i \in [d]} \delta_i^p \right) \geq \frac{1}{d} \sum_{i \in [d]} \phi_\frac{1}{p} (\delta_i^p) = \frac{1}{d} \sum_{i \in [d]} \delta_i^q.
$$

Now, since the map $\phi_\frac{1}{p}$ is increasing and since $\phi_\frac{1}{p} \circ \phi_\frac{1}{q} = \phi_\frac{1}{q}$ we deduce that

$$
\left( \frac{1}{d} \sum_{i \in [d]} \delta_i^q \right)^\frac{1}{q} \geq \left( \frac{1}{d} \sum_{i \in [d]} \delta_i^p \right)^\frac{1}{p}.
$$

Suppose now that $0 > p > q$. If $\delta_i = 0$ for some $i$, $\frac{1}{d} \sum_{i \in [d]} \frac{1}{d^p} \delta_i = 0$ and the result is immediate. Suppose that $\delta_i > 0$ for all $i$. Since $p > q$ and $\frac{p}{q} = |\frac{p}{q}| < 1$, we deduce that:

$$
\phi_\frac{1}{p} \left( \frac{1}{d} \sum_{i \in [d]} \delta_i^q \right) \geq \frac{1}{d} \sum_{i \in [d]} \phi_\frac{1}{q} (\delta_i^q) = \frac{1}{d} \sum_{i \in [d]} \delta_i^p.
$$

Since the map $\lambda \mapsto \lambda^\frac{1}{p}$ is decreasing:

$$
\left( \frac{1}{d} \sum_{i \in [d]} \delta_i^q \right)^\frac{1}{q} \leq \left( \frac{1}{d} \sum_{i \in [d]} \delta_i^p \right)^\frac{1}{p},
$$

which ends the first part of the proof. Now since for all $\delta \in \mathbb{R}_+^d$, $\lim_{p \to 0} \frac{1}{d^p} \sum_{i \in [d]} \delta_i = \prod \delta_i$, we deduce that for all $q < 0$, $\frac{1}{d^q} \sum_{i \in [d]} \delta_i \leq \prod \delta_i^\frac{1}{q}$. Finally, from the concavity of the logarithm function, we deduce that for all $p > 0$, and all $\delta \in \mathbb{R}_+^d$, $\prod \delta_i^\frac{1}{q} \leq \sum \delta_i \frac{1}{d^p} \delta_i$ since $\prod \delta_i^\frac{1}{q} = 0$ if there is some $\delta_i = 0$.

We deduce that for all $p, q$, if $p \geq q$, then $\frac{1}{d^q} \sum_{i \in [d]} \delta_i \leq \frac{1}{d^p} \sum_{i \in [d]} \delta_i$. Now, by construction we have:

$$
\lim_{k \to \infty} W(p_k) = W(p).
$$

Since $\{p_k\}_{k \in \mathbb{N}}$ is a monotone sequence, the sequence $\{W(p_k)\}_{k \in \mathbb{N}}$ is monotone and it follows from Corollary 3.1.4 that:

$$
\lim_{k \to \infty} D W(p_k) = D W(p). \quad \Box
$$

Proof of Proposition 3.2.2. (a) Equivalently we can write:

$$
D(z; g) = \sup_{\delta} \left\{ \delta : z + \delta g \leq u \in T_z \right\} = \sup_{\delta} \left\{ \delta : \delta g \leq u - z, u \in T_z \right\}.
$$

Hence

$$
D(z; g) = \sup_{\delta} \left\{ \delta : \delta \leq \min_{k \in K_g} \frac{u_k - z_k}{g_k}, u \in T_z \right\}.
$$

The first statement immediately follows from T2. (b) Now, note that the map $\delta \mapsto \sum_{k \in [d]} \delta_k$ is non-decreasing on $\mathbb{R}^d$. One can equivalently write:

$$
D(p)(z; g) = \sup_{\delta \geq 0} \left\{ \frac{1}{d^p} \sum_{k \in K_g} \delta_k : z + \delta g \leq u, u \in T_z \right\} = \sup_{\delta \geq 0} \left\{ \frac{1}{d^q} \sum_{k \in K_g} \delta_k : \delta g \leq u - z, u \in T_z \right\}.
$$
Therefore:
\[ D(p)(z; g) = \sup_{\delta \geq 0} \left\{ \frac{1}{d_g} \sum_{k \in K_g} \phi_{\delta_k} \delta_k : \delta_k \leq \frac{u_k - z_k}{g_k}, k \in K_g, u \in T_z \right\}, \]
which yields the result from T2. The proof of (c) is obtained similarly. (d) By definition, we have:
\[ AD(z; g) = \max \sup_{k \in K_g, u \in T_z} \left\{ z + \delta g_k e_k \leq u \right\}. \]
Since T2 holds, we have:
\[ AD(z; g) = \max \sup_{k \in K_g, u \in T_z} \left\{ z_k + \delta g_k \leq u_k \right\}. \]
We deduce that
\[ AD(z; g) = \max \sup_{k \in K_g, u \in T_z} \left\{ \frac{u_k - z_k}{g_k} \right\} = \sup_{u \in T_z} \max_{k \in K_g} \left\{ \frac{u_k - z_k}{g_k} \right\}. \]

**Proof of Lemma 3.2.3.** (a) Let \( \Delta = \{ \delta \in \mathbb{R}_{K_g,+} : z + \delta g \in T \} \). Suppose that \( g \in \mathbb{R}^d_+ \setminus \{0\} \). Since \( T \) is closed and since \( T_2 \) holds, \( \Delta \) is closed and bounded, therefore it is a compact subset of \( \mathbb{R}_{K_g,+} \). From [7], if \( p \neq 0 \), the map \( \delta \mapsto \frac{1}{d_g} \sum_{k \in K_g} \phi_{\delta_k} \delta_k \) is continuous over \( \mathbb{R}^d_+ \). By definition,
\[ D(p)(z; g) = \sup \left\{ \frac{1}{d_g} \sum_{k \in K_g} \phi_{\delta_k} \delta_k : \delta \in \Delta \right\}. \]
Therefore, since \( \Delta \) is compact, there is some \( \delta(p) \), such that
\[ D(p)(z; g) = \frac{1}{d_g} \sum_{k \in K_g} \phi_{\delta_k}(p) \delta_k \]. Moreover \( z + \delta(p) g \in W(T) \). Since for all \( p > 0 \) the generalized mean is increasing at point 0, the result follows. (b) The proof is similar if \( p = 0 \), since the map \( \delta \mapsto \prod_{k \in K_g} \phi_{\delta_k}^{\frac{1}{p}} \) is weakly increasing. Finally, (c) and (d) respectively follow from the fact that the functions min and max are weakly increasing. □

**Proof of Proposition 3.3.1.** (a) For all \( p \in \mathbb{R}^d_+ \) let the map
\[ \delta \mapsto W_p(\delta) = \frac{1}{d_g} \sum_{k \in K_g} \phi_{\delta_k}. \]
If \( g \neq 0 \), then this map is well-defined. If \( p > 0 \), then \( \phi_{\delta} \) and its reciprocal \( \phi_{\delta_k} \) are increasing. It follows that \( W_p \) is increasing for all \( p > 0 \). Now, remark that for all \( \delta \in \mathbb{R}^d_+ \), \( W_p(\delta) \geq 0 \), which implies that \( D(p)(z; g) \geq 0 \). Suppose that \( z \notin E_{K_g}(T) \). In such a case we can find some \( k \in K_g \) and some \( u \in T \) with \( u \geq z \) and \( u_k > z_k \). Therefore, there exists \( \delta \in \mathbb{R}^d_+ \) such that \( \delta_k > 0 \) and \( z_k < z_k + \delta_k g_k \leq u_k \). Let us consider \( \delta \in \mathbb{R}^d_+ \) defined for all \( j \in [d] \) as:
\[ \delta_j = \begin{cases} \delta_k & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases} \]
We have \( z + \delta \circ g \leq u \), and since \( u \in T \) and \( W_p(\delta) = \delta_k > 0 \), this implies that \( D(p)(z; g) > 0 \). Therefore, if \( D(p)(z; g) = 0 \) then one cannot find any \( u \in T \) such that \( u \geq z \) and \( u_k \neq z_k \) where \( u_K_g \) denotes the canonical projection on \( \mathbb{R}_g \). Hence, \( z \in E_{K_g}(T) \).

Conversely, if \( z \in E_{K_g}(T) \), then one cannot find any \( \delta \in \mathbb{R}^d_{K_g,+} \) with \( \delta_k > 0 \) for some \( k \in K_g \) and \( z + \delta g \in T \). This implies \( D(p)(z; g) = 0 \). This proves (a). (a') Suppose that \( z \notin W_{K_g}(T) \). In such a case we can find some \( u \in T \) with \( u \geq z \) and \( u_k > z_k \) for all \( k \in K_g \). Therefore, there exists \( \delta \in \mathbb{R}^d_{K_g} \) such that \( \delta_k > 0 \) and \( z_k < z_k + \delta_k g_k \leq u_k \) for all \( k \in K_g \). However, in such a case we have \( W_p(\delta) > 0 \). This
implies that $D(p)(z; g) > 0$. Therefore, if $D(p)(z; g) = 0$ then $z \in W_{K_g}(T)$. Conversely, note that if $p < 0$, then $\phi_p$ and its reciprocal $\phi_{-p}$ are nonincreasing and the composition of two nonincreasing functions is nondecreasing. Since $W_p(\delta) > 0$ if and only if $\delta_k > 0$ for all $k \in K_g$, we deduce the result. (b) We first note that if $u \leq z$ then, from the free disposal assumption,

$$\{ \delta : u + \delta \subseteq g \in T \} \supset \{ \delta : z + \delta \subseteq g \in T \}.$$  

Hence, weak monotonicity follows. (c) From (b) we have $D(p)(z; g) \leq D(p)(z'; g)$. Suppose that

$$D(p)(z; g) = W_p(\delta) \quad \text{and} \quad D(p)(z'; g) = W_p(\delta').$$

Let us denote $u = z + \delta \subseteq g$ and $u' = z + \delta' \subseteq g$. Since $z'_K > z_k$ and $g_k > 0$ for all $k \in K_g$, it follows that $u' \geq u$ with $u' \neq u$. Hence, one can find some $\epsilon > 0$ and some $k \in K_g$ such that $\delta'_k < \delta_k$. Hence $W_p(\delta') = D(p)(z'; g) < D(p)(z; g)$. (c') Suppose that

$$D(p)(z; g) = W_p(\delta) \quad \text{and} \quad D(p)(z'; g) = W_p(\delta').$$

Let us denote $u = z + \delta \subseteq g$ and $u' = z + \delta' \subseteq g$. Since $z'_K > z_k$ and $g_k > 0$ for all $k \in K_g$, it follows that $u'_k > u_k$ for all $k \in K_g$. Hence, one can find some $\epsilon \in \mathbb{R}^+_1$ such that $\delta' + \epsilon = \delta$. Since $M$ is weakly increasing, $W_p(\delta') = D(p)(z'; g) < D(p)(z; g)$. (d) is immediate. □

**Proof of Proposition 3.4.1** (a) is immediate. Let us prove (b), (c) and (d). We have for all $u \in T_z$:

\[
\begin{align*}
\lim_{p \to -\infty} W_{(p),g}(u-z) &= W_{-\infty,g}(u-z) = \min_{k \in K_g} \left\{ \frac{u_k - z_k}{g_k} \right\}; \\
\lim_{p \to 0^-} W_{(p),g}(u-z) &= W_{(0),g}(u-z) = \prod_{k \in K_g} \left( \frac{u_k - z_k}{g_k} \right)_{\mathbb{R}_+}; \\
\lim_{p \to +\infty} W_{(p),g}(u-z) &= W_{+\infty,g}(u-z) = \max_{k \in K_g} \left\{ \frac{u_k - z_k}{g_k} \right\}.
\end{align*}
\]

For any monotone sequence of real numbers $\{p_k\}_{k \in \mathbb{N}}$ and all $u \in T_z$, the sequence $\{W_{(p_k),g}(u-z)\}_{k \in \mathbb{N}}$ is monotone. Suppose that $\{p_k\}_{k \in \mathbb{N}}$ is monotone and converges to some $\bar{p} \in \mathbb{R} \cup \{-\infty, +\infty\}$. Since map $W_{(0)}$, $W_{-\infty}$ and $W_{+\infty}$ are continuous, the convergence is uniform and

\[
\lim_{k \to \infty} D_{(p_k),g}(z; g) = \lim_{k \to \infty} \sup_{u \in T_z} W_{(p_k),g}(u-z) = \sup_{u \in T_z} W_{(\bar{p}),g}(u-z) = D_{(\bar{p})}(z; g).
\]

Since this is true for every monotone sequences $\{p_k\}_{k \in \mathbb{N}}$, we deduce the result. □

**Proof of Proposition 3.4.2** For all real numbers $p, q$, if $p \geq q$ then $W_{(p),g} \geq W_{(q),g}$ which implies that $D_{(q)}(z; g) \leq D_{(p)}(z; g)$. Since

\[
\lim_{p \to -\infty} D_{(p)}(z; g) = D(z; g) \quad \text{and} \quad \lim_{p \to +\infty} D_{(p)}(z; g) = AD(z; g)
\]

we deduce the result. □

**Proof of Proposition 4.4.1** Since $C$ is compact and since the map $x \mapsto \|x-z\|$ is continuous, there exists some $\bar{x} \in \partial C$ such that

$$\|\bar{x} - z\| = \max\{\|x-z\| : x \in C\} = \bar{d}.$$  

It follows that the $B(z, \bar{d})$ contains $C$. Moreover, there exists some $x^* \in E_*$ such that

$$\langle x^*, \bar{x} \rangle = \sup\{\langle x^*, x \rangle : x \in B(z, \bar{d})\} = \sup\{\langle x^*, x \rangle : x \in C\} = h_C(x^*).$$

27
However $B(z, \bar{d}) = B(0, \bar{d}) + z$ and it follows that

$$\sup \{ \langle x^*, x \rangle : x \in B(z, \bar{d}) \} = \bar{d} \| x^* \|_* + \langle x^*, z \rangle = h_C(x^*).$$

Since $x^*$ can be normalized in the the dual space, we can find some $y^* \in E_*$ such that

$$\bar{d} = h_C(y^*) - \langle y^*, z \rangle.$$

For all $y \in E_*$ with $\|y\|_* = 1$ there is some $\bar{x} \in \partial C$ such that $h_C(y) = \langle y, \bar{x} \rangle$. However since $\|\bar{x} - z\| \leq \|\bar{x} - z\|$, it follows that $h_C(y) \leq h_C(y^*)$. Hence,

$$\bar{d} = \sup \{ h_C(y) - \langle y, z \rangle : y \in \mathbb{R}^d, \|y\|_* = 1 \}. \quad \square$$

**Proof of Proposition 4.1.3** From Proposition 4.1.1 we have:

$$\max \{ \|u - z\| : u \in T_z \} = \sup_{w \in \mathbb{R}^d} \{ h_{T_z}(w) - w.z : \|w\|_* = 1 \}.$$

Moreover, there exists a point $u^* \in \mathbb{R}^d$ such that $\|u^* - z\| = D_{\|\|}(z)$. Moreover, there exists some $w^* \in \mathbb{R}^d$ such that:

$$w^*.u^* = h_{T_z}(w^*).$$

However, since $\|\cdot\|$ is weakly increasing over $\mathbb{R}^d_+$, it follows that $T_z \cap (u^* + \mathbb{R}^d_+) = \emptyset$. Consequently $\{ u \in \mathbb{R}^d : w^*.u = h_{T_z}(w^*) \} \cap \mathbb{R}^d_+ = \emptyset$. From the Farkas Lemma, this implies that $w^* \geq 0$. Therefore:

$$h_{T_z}(w) = \Pi_z(w^*)$$

which proves the first part of the result. The second part immediately follows. \square

**Proof of Lemma 4.2.1** First note that since $C$ is a comprehensive and convex subset of $\mathbb{R}^d_+$ that contains 0:

$$C = \bigcap_{w \in \mathbb{R}^d_+} \{ z : w.x \leq h_C(w) \}.$$

Suppose that there is some $z \in \mathbb{R}^d_+$ such that $z \notin C$ and $z \in \bigcap_{w \in \mathbb{R}^d_+} \{ z : w.x \leq h_C(w) \}$. Suppose that $z \notin C$, and let us prove a contradiction. In such a case, there is some $\bar{w} \in \mathbb{R}^d_+$ such that $\bar{w}.z > h(\bar{w})$.

Let $K(\bar{w}) = \{ k \in [d] : \bar{w}_k > 0 \}$. Let $\{ w^{(n)} \}_{n \geq 1}$ be a sequence of positive vectors such that:

$$w^{(n)}_k = \begin{cases} \bar{w}_k & \text{if } k \in K(\bar{w}) \\ \frac{1}{n} & \text{if } k \notin K(\bar{w}). \end{cases}$$

By hypothesis, for all $n$, $w^{(n)}.z \leq h_C(w_n)$ and $\bar{w}.z > h_C(\bar{w})$. However, the latter condition implies that there is some $\epsilon > 0$:

$$h(\bar{w}) + \epsilon < \bar{w}.z = \sum_{k \in K(\bar{w})} \bar{w}_k z_k \leq \sum_{k \in K(\bar{w})} \bar{w}_k z_k + \frac{1}{n} \sum_{k \notin K(\bar{w})} z_k \leq h(w^{(n)}).$$

Now, note that since $C$ is compact there is a nonnegative $M$ such that $\sup \{ \sum_{k \in [d]} z_k : z \in C \} \leq M$. Moreover

$$h(w^{(n)}) = \sup_{w \in C} \{ \sum_{k \in K(\bar{w})} \bar{w}_k u_k + \frac{1}{n} \sum_{k \notin K(\bar{w})} u_k \} \leq \sup_{w \in C} \{ \sum_{k \in K(\bar{w})} \bar{w}_k u_k \} + \frac{1}{n} \sup_{w \in C} \{ \sum_{k \notin K(\bar{w})} u_k \} \leq h_C(\bar{w}) + \frac{M}{n}.$$
Therefore

\[ h(\bar{w}) + \epsilon \leq h_C(\bar{w}) + \frac{M}{n}. \]

However, there exists some \( n \) sufficiently large such that \( \frac{M}{n} < \epsilon \) that is a contradiction. Consequently \( z \in C \), which ends the proof. \( \square \)

**Proof of Proposition 4.2.2.** For all \( z \in \mathbb{R}^d_+ \), since \( T_z \) is a compact set, there is some \( \alpha \geq 0 \) such that:

\[ \sup_u \left\{ W(u - z) : u \in T_z \right\} = \alpha. \]

It follows that for all \( \epsilon > 0 \):

\[ \left\{ u : W(u - z) > \alpha + \epsilon \right\} \cap T_z = \emptyset. \]

Since \( T_z \) satisfies \( T_2 \) and \( W \) is upper-semicontinuous and nondecreasing map, it follows that

\[ \sup_u \left\{ W(u - z) : u \in T_z \right\} = \sup_u \left\{ W(u - z) : u \in (T_z - \mathbb{R}^d_+) \cap \mathbb{R}^d_+ \right\}. \]

Moreover

\[ T_z = \bigcap_{w \in \mathbb{R}^d_+} \{ u \in \mathbb{R}^d_+ : w.u \leq \Pi_z(w) \}. \]

From the compactness of \( T_z \), for all non-negative vector \( u' \not\in T_z \), there exists some \( w' \in \mathbb{R}^d_+ \) such that \( w'.u' > \Pi_z(w') \). It follows that

\[ T_z = \bigcap_{w \in \mathbb{R}^d_+} \{ u \in \mathbb{R}^d_+ : w.u \leq \Pi_z(w) \}. \]

Since the map \( W \) is quasi-concave, it follows that \( \{ u : W(u - z) \geq \alpha + \epsilon \} \) is convex and from the upper-continuity of \( W \), it is closed. Hence for every \( \epsilon > 0 \), there exists some \( w_\epsilon \in \mathbb{R}^d_+ \) such that

\[ \{ u : W(u - z) > \alpha + \epsilon \} \subset \{ u : w_\epsilon.u \geq \Pi_z(w_\epsilon) \} \]

and

\[ T_z \subset \{ u : w_\epsilon.u \leq \Pi_z(w_\epsilon) \}. \]

Consequently,

\[ \sup_u \left\{ W(u - z) : w_\epsilon.u \leq \Pi_z(w_\epsilon) \right\} \leq \alpha + \epsilon. \]

Since this is true for every \( \epsilon > 0 \), we deduce from Lemma 4.2.6 that:

\[ \sup_u \left\{ W(u - z) : u \in T_z \right\} = \inf_{w \in \mathbb{R}^d_+} \sup_u \left\{ W(u - z) : w.u \leq \Pi_z(w) \right\} \]

\[ = \inf_{w \in \mathbb{R}^d_+} \sup_v \left\{ W(v) : w \cdot v = \Pi_z(w) - w.z \right\} \]

\[ = \inf_{w \in \mathbb{R}^d_+} \left\{ (\Pi_z(w) - w.z)W_*(w) \right\} \]

\[ = \inf_{w \in \mathbb{R}^d_+} \left\{ \Pi_z(w) - w.z : W_*(w) = 1 \right\}. \]

Now suppose that \( w \) is a normalized vector in \( \mathbb{R}^d_+ \) that may have some null components. Since \( T_z \leq \{ u : w.z \leq \Pi_z(w) \} \) we deduce that \( D_W(z) \leq \sup_u \left\{ W(u - z) : u \leq \Pi_z(w) \right\} = (\Pi_z(w) - w.z)W_*(w) = \Pi_z(w) - w.z. \) Therefore:

\[ D_W(z) = \inf_{w \in \mathbb{R}^d_+} \left\{ \Pi_z(w) - w.z : W_*(w) = 1 \right\}. \] \( \square \)
Proof of Lemma 4.2.6: (a) Since \( a, b \in \mathbb{R}_{++}^d \) and \( c > 0 \), one can find some \( v \in \mathbb{R}_{++}^d \) such that \( \sum_{k \in [m]} a_k v_k > 0 \). Hence, since \( v_k = 0 \) for some \( k \) implies that \( \sum_{k \in [m]} a_k v_k = 0 \), any solution of the optimization problem belongs to \( \mathbb{R}_{++}^d \). Therefore, the problem can equivalently be written:

\[
\sup_v \left\{ \sum_{k \in [d]} (a_k v_k)^p : \langle b, v \rangle = c \right\}.
\]

The objective function is quasi-concave and the Lagrangian is defined as:

\[
L(v, \lambda) = \sum_{k \in [d]} (a_k v_k)^p - \lambda (\langle b, v \rangle - c).
\]

The first order condition yields:

\[
\frac{\partial L(v, \lambda)}{\partial x_k} = a_k^p \frac{p}{vk^{1-p}} - \lambda b_k = 0.
\]

Therefore, for all \( k \)

\[
v_k = \left( \frac{p}{\lambda b_k} \right)^{\frac{1}{1-p}} a_k^{\frac{p}{1-p}}.
\]

Hence, from \( \frac{\partial L(v, \lambda)}{\partial \lambda} = 0 \), we deduce that

\[
\langle b, v \rangle = \sum_{k \in [d]} b_k v_k = \sum_{k \in [d]} b_k \left( \frac{p}{\lambda b_k} \right)^{\frac{1}{1-p}} a_k^{\frac{p}{1-p}} = \sum_{k \in [d]} \left( \frac{p a_k b_k^p}{\lambda} \right)^{\frac{1}{1-p}} = c.
\]

Hence

\[
\left( \frac{p}{\lambda} \right)^{\frac{1}{1-p}} \sum_{k \in [d]} (a_k^{-1} b_k)^{\frac{p}{1-p}} = c \quad \text{and} \quad \lambda = p \left( \frac{1}{c} \sum_{k \in [d]} (a_k^{-1} b_k)^{\frac{p}{1-p}} \right)^{1-p}.
\]

Therefore for all \( i \)

\[
v_k = \left( \frac{p a_k}{p \left( \frac{1}{c} \sum_{k \in [d]} (a_k^{-1} b_k)^{\frac{p}{1-p}} \right)^{1-p}} \right)^{\frac{1}{1-p}} = c \left[ \sum_{k \in [d]} (a_k^{-1} b_k)^{\frac{p}{1-p}} \right]^{-1} (a_k p b_k^{-1})^{\frac{1}{1-p}}.
\]

The value of the objective function is then:

\[
\left( \sum_{k \in [d]} (a_k v_k)^p \right)^{\frac{1}{p}} = c \left[ \sum_{k \in [d]} (a_k^{-1} b_k)^{\frac{p}{1-p}} \right]^{-1} \left( \sum_{k \in [d]} a_k p (a_k p b_k^{-1})^{\frac{p}{1-p}} \right)^{\frac{1}{p}}
\]

\[
= c \left[ \sum_{k \in [d]} (a_k b_k^{-1})^{\frac{p}{1-p}} \right]^{-1} \left( \sum_{k \in [d]} (a_k b_k^{-1})^{\frac{p}{1-p}} \right)^{\frac{1}{p}}
\]

\[
= c \left( \sum_{k \in [d]} (a_k^{-1} b_k)^{\frac{p}{1-p}} \right)^{-1}.\]

(b) The result is standard and the proof is therefore omitted. □

Proof of Proposition 4.3.1: (a) Let us denote \( \mathbb{R}_{K_g} = \{ \sum_{k \in K_g} z_k e_k : z_k \in \mathbb{R}, k \in K_g \} \). \( \mathbb{R}_{K_g} \) is a vector subspace of \( \mathbb{R}^d \) of dimension \( d_g \). The norm

\[
\| \cdot \|_{p^{-1}, p} : u \mapsto \left( \sum_{k \in K_g} \frac{1}{d_g} \sum_{k \in K_g} \left| \frac{u_k}{g_k} \right|^p \right)^{\frac{1}{p}}
\]

(5.1)
defines a weighted norm on \( \mathbb{R}_{K_g} \) and a topology on this subspace. The dual norm is then defined by

\[
\|u\|_{g,q} = \left( \sum_{k \in K_g} d_g \frac{1}{g_k} |g_k u_k|^q \right)^{\frac{1}{q}}
\]

with \( \frac{1}{p} + \frac{1}{q} = 1 \). We have shown that:

\[
D_{(p)}(z;g) = \sup_{u \in \mathbb{R}} \left\{ \sum_{k \in K_g} \frac{\phi_p}{q g_k} : u \in T \right\}.
\]

For the sake of simplicity, let us denote \( z' = \sum_{k \in K_g} z_k e_k \) and \( u' = \sum_{k \in K_g} u_k e_k \) the respective projections of \( z \) and \( u \) onto \( \mathbb{R}_{K_g} \). Moreover, let \( T'_z = \{ \sum_{k \in K_g} u_k e_k : u \in T \} \) be the projection of \( T_z \) onto \( \mathbb{R}_{K_g} \). By hypothesis \( u \geq z \) implies that \( u' \geq z' \). Since \( T \) satisfies \( T_2 \), if \( u \geq z \) and \( u \in T \), then the vector \( v = u' + (z - z') \) is in \( T'_z \). Hence \( v - z = u' - z' \in \mathbb{R}_{K_g} \). This implies that \( v_k = z_k \) for all \( k \notin K_g \). Furthermore, \( u' \in T'_z \) since \( T_2 \) holds:

\[
\sup_{u \geq z} \left\{ \sum_{k \in K_g} \frac{u_k - z_k}{g_k} : u \in T \right\} = \sup_{v \geq z} \left\{ \sum_{k \in K_g} \frac{v_k - z_k}{g_k} : u \in T \right\} = \sup_{u'} \left\{ \|z' - u'\|_{g^{-1},p} : u' \in T'_z \right\}.
\]

Equivalently:

\[
D_{(p)}(z;g) = \sup_{u'} \left\{ d_{g^{-1},p}(u', z') : u' \in T'_z \right\}.
\]

For all \( w' \in \mathbb{R}_{K_g} \) let us denote

\[
h'_{T_z}(w') = \sup_{v'} \{ \langle w', v' \rangle : v' \in T'_z \}.
\]

\( h'_{T_z} \) is the functional support of \( T'_z \) with respect to \( \mathbb{R}_{K_g} \). From Proposition 4.1.1

\[
\max\{d_{g^{-1},p}(u', z') : u' \in T'_z\} = \sup_{w' \in \mathbb{R}_{K_g}} \{ h'_{T_z}(w') - \langle w', z' \rangle : \|w'\|_{g,q} = 1 \},
\]

However, for all \( w, u, z \in \mathbb{R}^d \)

\[
h'_{T_z}(w') - \langle w', z' \rangle = \Pi_{z,g}(w) - w.z \quad \text{and} \quad \|w'\|_{g,q} = d_g \frac{1}{q} \left( \sum_{k \in K_g} |g_k w_k|^q \right)^{\frac{1}{q}}.
\]

We deduce (a).

(b) If \( p = 1 \) then

\[
\|w\|_{g^{-1},1} = \frac{1}{d_g} \sum_{k \in K_g} |u_k|.
\]

The dual norm is

\[
\|w\|_{g,\infty} = d_g \max_{k \in K_g} |g_k w_k|.
\]

(c) If \( p = \infty \), we have

\[
\|u\|_{g^{-1},\infty} = \max_{k \in K_g} \frac{|u_k|}{g_k}.
\]
Since \( q = 1 \), the dual norm is
\[
\|u\|_{g,1} = \sum_{k \in K_g} |g_k u_k|.
\]

From Proposition 4.1.1, we deduce the result. (d) We have
\[
D_{(0)}(z; g) = \sup_{u \in T_z} \prod_{k \in K_g} \left( \frac{u_k - z_k}{g_k} \right)^{\frac{1}{d_g}}.
\]

The set \( \{ \sum_{k \in K_g} (u_k - z_k) e_k : u \in T_z \} \) is a convex and comprehensive subset of \( \mathbb{R}_{K_g,+} \). Therefore, we deduce the result applying Proposition 4.2.7b. We have, using the notations of (a)
\[
D_{(0)}(z; g) = \inf_{w' \geq 0} \left\{ \Pi_{z,g}(w') - w'.z' : d_g \prod_{k \in K_g} (g_k w'_k)^{\frac{1}{d_g}} = 1 \right\}.
\]

However, setting \( W_{(0)}(v) = \prod_{k \in K_g} \left( \frac{v}{g_k} \right)^{\frac{1}{d_g}} \), we have for every price vector \( w \in \mathbb{R}^{d}_{+} \)
\[
\sup_v \{ W_{(0)}(v) : w.v \leq \Pi_{z,K_g}(w) \} \leq \sup_v \{ W_{(0)}(v) : w.v \leq \Pi_{z}(w) \}.
\]

Moreover, from the Hanh-Banach extension theorem, one can find some \( \bar{w} \in \mathbb{R}^{d} \) such that \( \Pi_{z,g}(\bar{w}^*) = \Pi_{z}(\bar{w}) \). In such a case:
\[
\Pi_{z}(\bar{w}) - \bar{w}.z = \Pi_{z,g}(\bar{w}') - \bar{w}'.z',
\]
and we deduce (d). (e). Similarly, the result is a consequence of consequence of Proposition 4.2.7a. (f) is the standard result established in [10]. \( \square \)