Topological Equivalences of E-infinity Differential Graded Algebras

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Abstract

Two DGAs are called topologically equivalent if the corresponding Eilenberg-Mac Lane ring spectra are weakly equivalent as ring spectra. Quasi-isomorphic DGAs are topologically equivalent but the converse is not necessarily true. As a counter-example, Dugger and Shipley showed that there are DGAs that are non-trivially topologically equivalent, i.e. topologically equivalent but not quasi-isomorphic.

In this work, we define $E_\infty$ topological equivalences and utilize the obstruction theories developed by Goerss, Hopkins and Miller to construct first examples of non-trivially $E_\infty$ topologically equivalent $E_\infty$ DGAs. Also, we show using these obstruction theories that for co-connective $E_\infty$ DGAs, $E_\infty$ topological equivalences and quasi-isomorphisms agree. For $E_\infty \mathbb{F}_p$-DGAs with trivial first homology, we show that an $E_\infty$ topological equivalence induces an isomorphism in homology that preserves the Dyer-Lashof operations and therefore induces an $H_\infty \mathbb{F}_p$-equivalence.

1 Introduction

Dugger and Shipley defined a new equivalence relation between associative differential graded algebras (which we call DGAs) that they call topological equivalences [DS07]. To define topological equivalences, they use the Quillen equivalence between $R$-DGAs and $HR$-algebras where $R$ denotes a discrete commutative ring [Shi07]. Two $R$-DGAs $X$ and $Y$ are said to be topologically equivalent if the corresponding $HR$-algebras $HX$ and $HY$ are weakly equivalent as $S$-algebras where $S$ denotes the sphere spectrum. Using Quillen equivalences in [Shi07], it is easy to see that topologically equivalent DGAs are Morita equivalent. Furthermore, topological equivalences appear in one of the equivalent definitions of Morita equivalences of DGAs; this is explained in Theorem 1.1 below.
By the Quillen equivalence between $R$-DGAs and $HR$-algebras, two $R$-DGAs are quasi isomorphic if and only if the corresponding $HR$-algebras are weakly equivalent as $HR$-algebras. Because the forgetful functor from $HR$-algebras to $S$-algebras preserve weak equivalences, it is clear that quasi-isomorphic DGAs are always topologically equivalent. One of the main results of [DS07] is that there are DGAs that are not quasi-isomorphic but are topologically equivalent. Such DGAs are called non-trivially topologically equivalent. On the other hand, another theorem in [DS07] states that there are no examples of non-trivial topological equivalences in $Q$-DGAs, i.e. topologically equivalent $Q$-DGAs are quasi-isomorphic. See Theorem 1.4 below. Such non-existence results are important due to their applications to Morita theory. The following theorem may be considered as one of the equivalent definitions of Morita equivalence.

**Theorem 1.1.** [DS07] For two DGAs $X$ and $Y$, the model categories of $X$-modules and $Y$-modules are Quillen equivalent if and only if $\text{Hom}_X(P,P)$ is topologically equivalent to $Y$ where $P$ is a bi-fibrant replacement of a compact generator in the derived category of $X$.

Since there are no non-trivial topological equivalences of $Q$-DGAs, one can replace the topological equivalence condition in the above theorem with weak equivalence when the two DGAs are $Q$-DGAs.

Because there is also a Quillen equivalence between $E_\infty$ $R$-DGAs and commutative $HR$-algebras [RS14], topological equivalences for $E_\infty$ DGAs can also be considered. Now we explain what we mean by topological equivalences for DGAs and $E_\infty$-DGAs. For DGAs we have the following definition of topological equivalence.

**Definition 1.1.** Two $R$-DGAs $X$ and $Y$ are **topologically equivalent** if the corresponding $HR$-algebras $HX$ and $HY$ are weakly equivalent as $S$-algebras. This is same as the definition of topological equivalences in [DS07].

The definition for topological equivalence of $E_\infty$-DGAs is the following.

**Definition 1.2.** Two $E_\infty$ $R$-DGAs $X$ and $Y$ are **$E_\infty$ topologically equivalent** if the corresponding commutative $HR$-algebras $HX$ and $HY$ are weakly equivalent as commutative $S$-algebras.

Our methods make use of the obstruction theories for ring spectra. These obstruction theories rely on the obstruction spectral sequence of Bousfield [Bou89]. The first was developed by Hopkins and Miller; it provides an obstruction spectral sequence for calculating mapping spaces of ring spectra and also an obstruction theory for showing existence of ring structures on spectra [Rez98], also see [Rob89]. Later,
Goerss and Hopkins generalized this obstruction theory for commutative ring spectra \[GH04\]. We use a particular construction of the Goerss-Hopkins obstruction theory to calculate mapping spaces of commutative ring spectra.

In this work, we construct the first examples of non-trivially \(E_\infty\) topologically equivalent \(E_\infty\) DGAs. One of these examples is in \(E_\infty \mathbb{F}_p\)-DGAs. This is particularly interesting because one of the open questions in \[DS07\] asks if there are any examples of non-trivial topological equivalences of \(k\)-DGAs for a field \(k\). Our example provides a positive answer to this question in \(E_\infty\) DGAs. Although there is an example of non-trivial \(E_\infty\) topological equivalences over \(\mathbb{F}_p\), our non-existence results for \(E_\infty\) topological equivalences hint that such examples are not common.

Before stating our non-existence results, we note that topologically equivalent DGAs have isomorphic homology rings. This is because the Quillen equivalence between \(R\)-DGAs and \(HR\)-algebras gives an isomorphism between the homology ring of an \(R\)-DGAs and the homotopy ring of the corresponding ring spectra. Therefore if \(X\) and \(Y\) are topologically equivalent DGAs, then \(H_*(X) \cong \pi_*(HX) \cong \pi_*(HY) \cong H_*(Y)\) where the isomorphisms are ring isomorphisms and the isomorphism in the middle is induced by the topological equivalence. The same is true for \(E_\infty\) topologically equivalences, but as Example 5.1 indicates, the isomorphism of homology rings may not preserve Dyer-Lashof operations. However, by Theorem 1.3 if \(X\) and \(Y\) are \(E_\infty\) topologically equivalent \(E_\infty \mathbb{F}_p\)-DGAs with trivial first homology, then the isomorphism of homology rings induced by the \(E_\infty\) topological equivalence preserves Dyer-Lashof operations, i.e. it is an isomorphism of algebras over the Dyer-Lashof algebra.

For co-connective \(E_\infty\mathbb{Z}/(m)\)-DGAs, we prove that there are no non-trivial \(E_\infty\) topological equivalences where \(m\) denotes an integer. This in particular implies that there are no non-trivial \(E_\infty\) topological equivalences in co-connective \(E_\infty\mathbb{Z}\)-DGAs.

**Theorem 1.2.** \(E_\infty\) Topologically equivalent co-connective \(E_\infty \mathbb{Z}/(m)\)-DGAs are quasi-isomorphic as \(E_\infty \mathbb{Z}/(m)\)-DGAs. That is, \(E_\infty\) topological equivalences and quasi-isomorphisms agree for co-connective \(E_\infty \mathbb{Z}/(m)\)-DGAs.

**Remark 1.1.** It is interesting to consider the following consequence of Theorem 1.2. For a co-connective commutative \(S\)-algebra \(X\), we use the \(E_\infty\) connective cover \(H\pi_0X \to X\) to obtain a map \(HZ \to X\) which gives \(X\) a commutative \(HZ\)-algebra structure. This says that there is an \(E_\infty\mathbb{Z}\)-DGA corresponding to a co-connective commutative \(S\)-algebra. By this and Theorem 1.2, we deduce that weak equivalence classes of co-connective commutative \(S\)-algebras are uniquely determined by the quasi-isomorphism classes of the corresponding \(E_\infty\mathbb{Z}\)-DGAs.
In Example 5.1 we construct \( E_\infty \mathbb{F}_p \)-DGAs that are non-trivially \( E_\infty \) topologically equivalent. Therefore it is not possible to generalize Theorem 1.2 to all \( E_\infty \mathbb{F}_p \)-DGAs. However, for \( E_\infty \mathbb{F}_p \)-DGAs with trivial first homology we have the following result.

**Theorem 1.3.** Let \( X \) and \( Y \) be \( E_\infty \mathbb{F}_p \)-DGAs with trivial first homology group. If \( X \) and \( Y \) are \( E_\infty \) topologically equivalent, then they are \( H_\infty \mathbb{F}_p \)-algebra equivalent. Furthermore, an \( S \)-algebra equivalence between \( HX \) and \( HY \) induces an isomorphism of the homology rings that preserves Dyer-Lashof operations.

We actually prove a stronger result. Theorem 7.1 states that for \( H_\infty H\mathbb{F}_p \)-algebras with trivial first homotopy, \( H_\infty \mathbb{S} \)-algebra equivalence implies \( H_\infty H\mathbb{F}_p \)-algebra equivalence.

The condition of trivial first homology follows from the fact that the dual Steenrod algebra is generated by an element of degree one as a ring with Dyer-Lashof operations. Again by Example 5.1 this condition cannot be removed from this theorem.

**Remark 1.2.** In \[Law15\], Lawson produces examples of \( H_\infty \mathbb{S} \)-algebras whose \( H_\infty \mathbb{S} \)-algebra structures do not lift to commutative \( \mathbb{S} \)-algebra structures. One of the intermediate results of \[Law15\] states that Theorem 7.1 is still true without the restriction on the first homotopy but Example 5.1 contradicts this. The examples of spectra constructed in \[Law15\] are co-connective. Therefore, Theorem 7.1 recovers the main result of \[Law15\]. We elaborate on this in Section 7.

The proof of the non-existence theorem in \[DS07\] for \( \mathbb{Q} \)-DGAs also works for \( E_\infty \mathbb{Q} \)-DGAs. We obtain the following.

**Theorem 1.4.** \( (E_\infty) \) topologically equivalent \( (E_\infty) \mathbb{Q} \)-DGAs are quasi-isomorphic. That is, \( (E_\infty) \) topological equivalences and \( (E_\infty) \) quasi-isomorphisms agree in \( (E_\infty) \mathbb{Q} \)-DGAs.

In the next section, we explain the examples of non-trivial topological equivalences given in \[DS07\] and make a correction to a mistake in their construction. Section 8 discusses the obstruction spectral sequences that we will use for calculating mapping spaces of ring spectra and Section 4 describes the dual Steenrod algebra and the Dyer-Lashof operations on it. Section 5 is devoted to our examples of non-trivial \( E_\infty \) topological equivalences. Section 6 contains the proof of Theorem 7.1 and Section 7 contains the proof of Theorem 1.3.

**Notation** As noted earlier, for a commutative ring \( R \), when we say \( R \)-DGAs we mean associative \( R \)-DGAs. Similarly for a commutative ring spectrum \( R \), \( R \)-algebras denote associative \( R \)-algebras. A smash product without a subscript \( \wedge \) denotes the smash product over the
sphere spectrum. The category of spectra we use is symmetric spectra in topological spaces with the positive model structure as in [MMSS01].

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2 Previous results on topological equivalences

Here we discuss the examples of non-trivial topological equivalences in [DS07]. The first class of examples provided in [DS07] relies on the classification of Postnikov extensions of ring spectra developed in [DS06]. Below, we point out a mistake in the construction of these examples and provide a correction which recovers the classification of quasi-isomorphism classes of Z-DGAs with homology ring $A_{F_1}(x_n)$ for $n > 0$, i.e. exterior algebra with a single generator in a positive degree.

Let $R$ be a connective commutative ring spectrum. We first explain the classification of Postnikov extensions of connective (no negative homotopy) $R$-algebras developed in [DS06]. For a connective $R$-algebra $X$, the $n$-th Postnikov section of $X$ is a map of $R$-algebras $X \to P_n X$ which induces an isomorphism on $\pi_i(X) \to \pi_i(P_nX)$ for $i \leq n$ and $\pi_i(P_nX) = 0$ for $i > n$. Given a connective $R$-algebra $Y$ with $P_{n-1} Y \simeq Y$ and a $\pi_0(Y)$-bimodule $M$, a Postnikov extension of $Y$ of type $(M, n)$ is a map of $R$-algebras $X \to Y$ which satisfies the following properties:

1. $\pi_i(X) = 0$ for $i > n$
2. $\pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i < n$
3. There is an isomorphism of $\pi_0(X)$-bimodules $\pi_n(X) \cong M$ where $\pi_0(X)$-bimodule structure structure of $M$ is obtained by the map $\pi_0(X) \to \pi_0(Y)$.

The moduli space of Postnikov extensions of $Y$ of type $(M, n)$, denoted by $\mathcal{M}_R(Y + M, n)$, is defined to be the category whose objects are Postnikov extensions of $Y$ of type $(M, n)$, and a morphism between two extensions $X_1 \to Y$ to $X_2 \to Y$ is a weak equivalence $X_1 \sim X_2$.
for which the following triangle commutes.

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\sim} & X_2 \\
\downarrow & & \downarrow \\
Y & \xleftarrow{\sim} & \text{(for the triangle to commute)}
\end{array}
\]

The main result of [DS06] is a classification of these Postnikov extensions in terms of topological Hochschild cohomology.

**Theorem 2.1.** [DS06] Assuming \( X \) is cofibrant as an \( R \)-module, the following is a bijection.

\[
\pi_0 \mathcal{M}_R(X + M, n) \cong \text{THH}_{R}^{n+2}(X, M)/\text{Aut}(M)
\]

This result is used in [DS07] to classify weak equivalence classes of \( \mathbb{Z} \)-DGAs with homology ring \( \Lambda_{\mathbb{F}_p}(x_n) \), the exterior algebra over \( \mathbb{F}_p \) with a single generator in degree \( n \), for \( n > 0 \). Any such DGA is a Postnikov extension of \( \mathbb{F}_p \) of type \((\mathbb{F}_p, n)\). By the Quillen equivalence of \( \mathbb{Z} \)-DGAs and \( H\mathbb{Z} \)-algebras, this is the same as classifying \( H\mathbb{Z} \)-algebras with homotopy ring \( \Lambda_{\mathbb{F}_p}(x_n) \) and such \( H\mathbb{Z} \)-algebras are Postnikov extensions of \( H\mathbb{F}_p \) of type \((H\mathbb{F}_p, n)\) in \( H\mathbb{Z} \)-algebras.

At this point we should note a piece of explanation that is missing in [DS07] about this classification. In [DS07] it is claimed that weak equivalence classes of \( H\mathbb{Z} \)-algebras whose homotopy ring are \( \Lambda_{\mathbb{F}_p}(x_n) \) are classified by \( \pi_0 \mathcal{M}_{H\mathbb{Z}}(H\mathbb{F}_p + H\mathbb{F}_p, n) \). However in \( \mathcal{M}_{H\mathbb{Z}}(H\mathbb{F}_p + H\mathbb{F}_p, n) \), the morphisms are weak equivalences of Postnikov extensions, i.e. for two Postnikov extensions of \( H\mathbb{F}_p \) of type \((H\mathbb{F}_p, n)\): \( X_1 \to H\mathbb{F}_p \) and \( X_2 \to H\mathbb{F}_p \), a morphism in \( \mathcal{M}_{H\mathbb{Z}}(H\mathbb{F}_p + H\mathbb{F}_p, n) \) is a weak equivalence of \( H\mathbb{Z} \)-algebras \( X_1 \cong X_2 \) for which the triangle above commutes. In the classification we are concerned with here, a morphism of two Postnikov extensions is just a weak equivalence of \( H\mathbb{Z} \)-algebras \( X_1 \cong X_2 \). In general one should not expect these two classifications to be the same. However, in this case we will show that they are actually the same. Unfortunately we cannot use simple point set arguments to prove this, even if we work with \( \mathbb{Z} \)-DGAs instead of \( H\mathbb{Z} \)-algebras, because one needs to use a cofibrant and fibrant \( H\mathbb{Z} \)-algebra model of \( H\mathbb{F}_p \) and also because we need the same result over \( \mathcal{S} \)-algebras not only for \( H\mathbb{Z} \)-algebras.

We prove that these two classifications are the same by first showing that there is a unique homotopy class of \( H\mathbb{Z} \)-algebra maps from \( X \) to \( H\mathbb{F}_p \). We prove this in the appendix by using the obstruction theory of the Hopkins-Miller theorem. Using this fact, we prove that \( \pi_0 \mathcal{M}_{H\mathbb{Z}}(H\mathbb{F}_p + H\mathbb{F}_p, n) \) actually classifies weak equivalence classes of \( H\mathbb{Z} \)-algebras with homotopy ring \( \Lambda_{\mathbb{F}_p}(x_n) \).
Proposition 2.1. The set $\pi_0 \mathcal{M}_{HZ}(H\mathbb{F}_p + H\mathbb{F}_p, n)$ is in bijective correspondence with weak equivalence classes of $HZ$-algebras with homotopy ring $\Lambda_{HZ}(x_n)$ for $n > 0$. This statement holds for $S$-algebras too. Namely, the set $\pi_0 \mathcal{M}_{S}(H\mathbb{F}_p + H\mathbb{F}_p, n)$ is in bijective correspondence with weak equivalence classes of $S$-algebras with homotopy ring $\Lambda_{S}(x_n)$ for $n > 0$.

Proof. We only prove the statement for $HZ$-algebras. The proof is similar for $S$-algebras, the only important difference is that one uses the second part of Lemma A.1 instead of the first part.

Since up to weak equivalence there is a unique $HZ$-algebra with homotopy $F_p$ concentrated at degree zero, every $HZ$-algebra with homotopy ring $\Lambda_{HZ}(x_n)$ is a Postnikov extension of $H\mathbb{F}_p$ of type $(H\mathbb{F}_p, n)$. Given two such Postnikov extensions: $\varphi_1 : X_1 \to H\mathbb{F}_p$ and $\varphi_2 : X_2 \to H\mathbb{F}_p$, if these extensions are weakly equivalent in $\mathcal{M}_{HZ}(H\mathbb{F}_p + H\mathbb{F}_p, n)$ then they are clearly weakly equivalent as $HZ$-algebras. We need to show that when these Postnikov extensions are weakly equivalent as $HZ$-algebras, they are also weakly equivalent in $\mathcal{M}_{HZ}(H\mathbb{F}_p + H\mathbb{F}_p, n)$.

Let $X_1$ and $X_2$ be weakly equivalent as $HZ$-algebras, we show that $\varphi_1$ and $\varphi_2$ are weakly equivalent in $\mathcal{M}_{HZ}(H\mathbb{F}_p + H\mathbb{F}_p, n)$. We assume that $X_1$ and $X_2$ are both fibrant and cofibrant and $HF_p$ is fibrant as $HZ$-algebras.

Because $X_1$ and $X_2$ are weakly equivalent and because $X_1$ is cofibrant and $X_2$ is fibrant, there is a weak equivalence of $HZ$-algebras $X_1 \simeq X_2$. Using this weak equivalence we define another Postnikov extension of $H\mathbb{F}_p$ of type $(H\mathbb{F}_p, n)$ which is the composite $\varphi_1 \circ \psi : X_1 \to H\mathbb{F}_p$. This Postnikov extension, $\varphi_2 \circ \psi$, is weakly equivalent to $\varphi_2$ in $\mathcal{M}_{HZ}(H\mathbb{F}_p + H\mathbb{F}_p, n)$ through $\psi$. Therefore it is sufficient to show that $\varphi_2 \circ \psi$ is weakly equivalent to $\varphi_1$ in $\mathcal{M}_{HZ}(H\mathbb{F}_p + H\mathbb{F}_p, n)$. By Lemma A.1 there is a unique homotopy class of maps from $X_1$ to $H\mathbb{F}_p$. Therefore, $\varphi_1$ and $\varphi_2 \circ \psi$ are homotopic. We have the following diagram in $HZ$-algebras which corresponds to a homotopy between these maps where $X_1 \wedge I$ is a path object of $X_1$.

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\sim} & X_1 \\
\varphi_1 \downarrow & \simeq & \downarrow \varphi_2 \circ \psi \\
X_1 \wedge I & \xrightarrow{\sim} & H\mathbb{F}_p
\end{array}
\]

The map $X_1 \wedge I \to H\mathbb{F}_p$ in the above diagram is also a Postnikov extension of type $(H\mathbb{F}_p, n)$ of $HF_p$ because it factors $\varphi_1$ by a weak equivalence. Therefore the above diagram gives a zig-zag of weak equivalences.
between \( \varphi_1 \) and \( \varphi_2 \circ \psi \) in \( \mathcal{M}_{HZ}(HF_p + HF_p, n) \). This shows that \( \varphi_1 \) and \( \varphi_2 \) are weakly equivalent in \( \mathcal{M}_{HZ}(HF_p + HF_p, n) \).

At this point, we are ready to provide the classification of weak equivalence classes of \( H\mathbb{Z} \)-algebras with homotopy ring \( \Lambda_{HF_p}(x_n) \) for \( n > 0 \) and hence, quasi-isomorphism classes of \( \mathbb{Z} \)-DGAs with homology ring \( \Lambda_{HF_p}(x_n) \) for \( n > 0 \). By Proposition 2.1 and Theorem 2.1 This is given by \( \text{THH}_{HZ}^{n+2}(HF_p, HF_p)/\text{Aut}(HF_p) \). As described in Example 3.15 of [DS07], \( \text{THH}_{HZ}^{n+2}(HF_p, HF_p) \cong \mathbb{F}_p[\sigma_2] \), a polynomial algebra with a generator in degree 2 (with cohomological grading). Calculating the quotient of \( \mathbb{F}_p[\sigma_2] \) by the multiplicative action of \( \mathbb{F}_p \), one obtains the following classification: for odd \( n > 0 \), there is a unique \( H\mathbb{Z} \)-algebra with homotopy ring \( \Lambda_{HF_p}(x_n) \) and for even \( n > 0 \), there are exactly two non-weakly equivalent \( H\mathbb{Z} \)-algebras whose homotopy ring is \( \Lambda_{HF_p}(x_n) \). For \( n = 0 \), this classification says that there are two Postnikov extensions of \( HF_p \) of type \( (HF_p, 0) \) and these are \( H\mathbb{Z}/p^2 \) and \( H\Lambda_{HF_p}(x_0) \).

Similarly, weak equivalence classes of \( \mathbb{S} \)-algebras with homotopy ring \( \Lambda_{HF_p}(x_n) \) are given by \( \text{THH}_{\mathbb{S}}^{n+2}(HF_p, HF_p)/\text{Aut}(HF_p) \). By the calculations of [Laz04], \( \text{THH}_{\mathbb{S}}^{n+2}(HF_p, HF_p) \cong \Gamma[\alpha_2] \), the divided polynomial algebra on a generator of degree 2 which is isomorphic to \( \mathbb{F}_p[x_n] \) as a module. Therefore we get a similar classification result: there are exactly two non weakly equivalent \( \mathbb{S} \)-algebras with homotopy ring \( \Lambda_{HF_p}(x_n) \) for odd \( n > 0 \) and there is only one for even \( n > 0 \).

What we are really interested in here is to decide which of these non-weakly equivalent \( H\mathbb{Z} \)-algebras are weakly equivalent as \( \mathbb{S} \)-algebras. For this, one considers the the map

\[
\text{THH}_{HZ}^{n+2}(HF_p, HF_p) \to \text{THH}_{\mathbb{S}}^{n+2}(HF_p, HF_p)
\]

induced by the forgetful functor from \( H\mathbb{Z} \)-algebras to \( \mathbb{S} \)-algebras. This corresponds to a morphism of rings \( \varphi : \mathbb{F}_p[\sigma_2] \to \Gamma[\alpha_2] \). Where \( \varphi \) maps \( \sigma_2 \) to \( \alpha_2 \) because \( H\mathbb{Z}/p^2 \) and \( H\Lambda_{HF_p}(x_0) \) are non-weakly equivalent as \( \mathbb{S} \)-algebras. Since \( \alpha_2^p = 0 \) in \( \Gamma[\alpha_2] \), \( \varphi(\sigma_2^p) = 0 \). This implies that the two non-weakly equivalent \( H\mathbb{Z} \)-algebras corresponding to \( \sigma_2^p \) and 0 are weakly equivalent as \( \mathbb{S} \)-algebras. These are the first example of non-trivial topological equivalences from [DS07]. That is, there are two non-weakly equivalent \( \mathbb{Z} \)-DGAs with homology ring \( \Lambda_{HF_p}(x_{2p-2}) \) which are topologically equivalent. For \( p = 2 \), one of these DGAs is the formal one and the other one is given by the following formula

\[
\mathbb{Z}[e_1; de_1 = 2]/(e_1^4) \text{ where } |e_1| = 1.
\]
3 Obstruction theories for ring spectra

For a commutative $\mathbb{S}$-algebra $X$, a commutative $HZ$-algebra structure on $X$ is given by a map $HZ \to X$ of commutative $\mathbb{S}$-algebras. In other words, the category of commutative $HZ$-algebras is the category commutative $\mathbb{S}$-algebras under $HZ$. Therefore it is natural to consider the maps from $HZ$ to a commutative $\mathbb{S}$-algebra $X$ for the purpose of studying $E_\infty$ topological equivalences. For this, we employ an obstruction spectral sequence to calculate homotopy class of maps in commutative ring spectra.

There are several obstruction theories similar to the one we will use in the literature for calculating mapping spaces of ring spectra and showing the existence of ring structures on a given spectrum. All of these obstruction theories rely on Bousfield’s obstruction theory for cosimplicial spaces [Bou89]. The first is developed in the Hopkins-Miller theorem, see [Rez98], and it is for associative ring spectra. Hopkins and Miller use this obstruction theory to show that the Morava stabilizer group acts on the Morava $E$-theory spectrum $E_n$. Later, Goerss and Hopkins generalized this theory to commutative ring spectra [GH04].

Generalizing the obstruction theory of the Hopkins-Miller theorem to commutative ring spectra is not trivial because of the following problem. For a spectrum $X$ and a homology theory $E_*$ corresponding to another spectrum $E$, when $E_*X$ is flat over $E_*$, $E_*T(X)$ is the free associative $E_*$-algebra over $E_*X$ where $T(X)$ is the free associative ring spectrum over $X$. For commutative ring spectra, one uses the free commutative ring spectra functor $P_\mathbb{S}$ but $E_*P_\mathbb{S}(X)$ may not have a nice description, even under the above flatness assumption. However, in our case, we use $HF_p$ for $E$ and then $HF_pP_\mathbb{S}(X)$ is the free unstable algebra over the Dyer-Lashof algebra generated by $HF_p(X)$. Using this, the arguments of the Hopkins-Miller theorem go through and one obtains the following obstruction spectral sequence.

**Theorem 3.1.** [Rez98] [Fre09] [Noe15] Let $X$ be a commutative $\mathbb{S}$-algebra and let $Y$ be a commutative $HF_p$-algebra. Given a map $\phi : X \to Y$ of commutative $\mathbb{S}$-algebras, there is a second quadrant spectral sequence abutting to $\pi_{1-s}map_{\mathbb{S}-cAlg}(X,Y)$ where $\mathbb{S}$-$cAlg$ denotes commutative $\mathbb{S}$-algebras. The $E_2$ term of this spectral sequence is given by

$$E_2^{0,0} = \text{Hom}_{cAlg}(HF_{p^s}X,Y_*)$$

and for $t > 0$,

$$E_2^{s,t} = \text{Der}_{cAlg}^{s}(HF_{p^s}X,Y_*^{ST})$$

where $\text{Der}_{cAlg}^{s}(-,-)$ denotes the $s$’th Andr´e-Quillen cohomology for unstable algebras with Dyer-Lashof operations [Qui70], $Y^{ST}$ denotes the mapping spectrum from the $t$-sphere to $Y$ and $\text{Hom}_{cAlg}(HF_{p^s}X,Y_*)$ denotes morphisms preserving Dyer-Lashof operations.
Obstructions to lifting a morphism in $E^{0,0}_{2,0}$ to a morphism of commutative $S$-algebras lie in $\text{Der}^t_{R\text{-alg}}(H\mathbb{F}_{p^*}X, Y^{S^{-1}}_s)$ for $t \geq 2$.

Obstructions to up-to homotopy uniqueness of a lift lie in $\text{Der}^t_{R\text{-alg}}(H\mathbb{F}_{p^*}X, Y^S_s)$ for $t \geq 1$.

Johnson and Noel generalized the obstruction theory of the Hopkins Miller theorem to calculate mapping spaces of algebras over a general monad in a model category in [JN14]. This is called the $T$-algebra spectral sequence. The above obstruction spectral sequence may also be considered as a special case of this spectral sequence, see Proposition 2.2 of [Noe15]. Also, this is very similar to the spectral sequence studied by French in [Fre09].

4 Dyer-Lashof operations and the dual Steenrod algebra

For a commutative ring spectrum $R$ we denote the free commutative algebra functor from $R$-modules to commutative $R$-algebras by $P_R$. This functor is homotopically well behaved and induces a monad on $\text{Ho}(R\text{-mod})$ and the algebras over this monad are called $H_\infty R$-algebras. Therefore, an $E_\infty$ algebra is an $H_\infty$ algebra. The converse to this is shown to be false by counter-examples in [Noe14] and [Law15].

Dyer-Lashof operations are power operations, just like the Steenrod operations, that are constructed in a way to act on the homotopy ring of $H_\infty \mathbb{F}_p$-algebras in [BMMS86]. Equivalently, they act on the homology ring of $H_\infty \mathbb{F}_p$-DGAs. Indeed the category of $H_\infty \mathbb{F}_p$-algebras is equivalent to the category of graded commutative rings over $\mathbb{F}_p$ with Dyer-Lashof operations satisfying the allowability and $p$’th power conditions, which are called unstable algebras over the Dyer-Lashof algebra, see the discussion in section 3 of [Law15].

For each integer $s$, the Dyer-Lashof operation of degree $s$ is denoted by $Q^s$. These operations are preserved under $H_\infty \mathbb{F}_p$-algebra morphisms and hence $E_\infty \mathbb{F}_p$-algebra morphisms. The operation $Q^s$ increases the degree by $2s(p-1)$ for odd primes and by $s$ for $p = 2$. For an element $x$ in the homotopy ring of a commutative $\mathbb{F}_p$-algebra, the unstable Dyer-Lashof operations satisfy the following properties (properties for $p = 2$ are given in parenthesis).

$$Q^s x = 0 \text{ for } 2s < |x| \text{ (for } s < |x|)$$

$$Q^s x = x^p \text{ for } 2s = |x| \text{ (for } s = |x|)$$

$$Q^s 1 = 0 \text{ for } s \neq 0$$
Also, these operations satisfy the Cartan formula and the Adem relations as in Chapter III Theorem 1.1 of [BMMS86].

As mentioned earlier, for a commutative $\mathcal{S}$-algebra $X$, $HF_{pu}\mathbb{P}_S(X)$ is the free unstable algebra over the Dyer-Lashof algebra generated by $HF_{pu}X$.

**Theorem 4.1.** [Bak12] $HF_{pu}\mathbb{P}_S(X)$ is the free commutative graded $\mathbb{F}_p$-algebra generated by $Q|x_j|$ where $x_j$’s form a basis for $HF_{pu}X$ and $I = (\varepsilon_1, i_1, \varepsilon_2, ..., \varepsilon_n, i_n)$ is admissible and satisfies excess$(I) + \varepsilon_1 > |x_j|$. The definition of admissibility and excess can be found in [May71].

**Dual Steenrod Algebra.** Now we discuss the dual Steenrod algebra and the Dyer-Lashof operations on it. The dual Steenrod algebra is first described by Milnor in [Mil58] and the Dyer-Lashof operations on it are first studied in Chapter III of [BMMS86]. We also recommend [Bak15]. The dual Steenrod algebra $A_* \cong HF_{pu}HF_p$ is a free graded commutative $\mathbb{F}_p$-algebra. For $p = 2$, it is given by two different standard sets of generators

$$A_* = F_2[\xi_r \mid r \geq 1] = F_2[\zeta_r \mid r \geq 1]$$

where $|\xi_r| = |\zeta_r| = 2^r - 1$. The transpose map of the smash product applied to $HF_p \wedge HF_p$ induces an automorphism of the dual Steenrod algebra denoted by $\chi$. The reason we have two different set of generators above is to keep track of the action of $\chi$. We have $\chi(\xi_r) = \zeta_r$. The generating series $\xi(t) = t + \sum_{r<1}\xi_r t^{2r}$ and $\zeta(t) = t + \sum_{r<1}\zeta_r t^{2r}$ are composition inverses in the sense that $\zeta(\xi(t)) = t = \xi(\zeta(t))$. This in particular shows that $\xi_1 = \zeta_1$.

Since commutative $HF_p$-algebras form the category of commutative $\mathcal{S}$-algebras under $HF_p$, $HF_p \wedge HF_p$ can be given two different commutative $HF_p$-algebra structures using the maps $HF_p = HF_p \wedge S \to HF_p \wedge HF_p$ and $HF_p = S \wedge HF_p \to HF_p \wedge HF_p$. We call these maps $g_1$ and $g_2$ respectively. We denote the Dyer-Lashof operations induced on $A_*$ from the first structure map by $Q^s$ and the second structure map by $\tilde{Q}^s$.

Since the transpose map induces an isomorphism of commutative $HF_p$-algebras, $\chi$ preserves the corresponding Dyer-Lashof operations i.e. $\chi(Q^s x) = Q^s \chi(x)$. For $p = 2$, $A_*$ is generated as an algebra over the Dyer-Lashof algebra by $\xi_1$, we have

$$Q^{2^s-2} \xi_1 = \zeta_s \text{ for } s \geq 1.$$ 

By using the fact that $\chi$ preserves Dyer-Lashof operations, one obtains

$$\tilde{Q}^{2^s-2} \xi_1 = \xi_s \text{ for } s \geq 1.$$
For an odd prime $p$, the dual Steenrod algebra is given by the following.

$$\mathcal{A}_s = \mathbb{F}_p[\xi_r \mid r \geq 1] \otimes \Lambda(\tau_s \mid s \geq 0) = \mathbb{F}_p[\xi_r \mid r \geq 1] \otimes \Lambda(\tau_s \mid s \geq 0)$$

Where $|\xi_r| = |\zeta_r| = 2(p^r - 1)$ and $|\tau_s| = |\tau_s| = 2p^s - 1$. The action of the transpose map is given by $\chi(\xi_r) = \zeta_r$ and $\chi(\tau_r) = \tau_r$. We use the following formula to relate the two set of generators for the dual Steenrod Algebra, see Section 7 of [Mil58] and the proof of Lemma 4.7 in [Bak15].

$$\bar{\tau}_s + \bar{\tau}_{s-1} \xi_1^{p^{s-1}} + \bar{\tau}_{s-2} \xi_2^{p^{s-2}} + \cdots + \bar{\tau}_0 \xi_s + \tau_s = 0 \quad (1)$$

The dual Steenrod algebra is generated by $\tau_0$ as an algebra over the Dyer-Lashof algebra. We have the following formulae for the Dyer-Lashof operations for $s \geq 1$

$$Q^{(p^s - 1)/(p - 1)}\tau_0 = (-1)^s \tau_s$$

$$\beta Q^{(p^s - 1)/(p - 1)}\tau_0 = (-1)^s \zeta_s.$$

More can be found on the Dyer-Lashof operations on the dual Steenrod algebra in [Bak15] and in Chapter III of [BMMS86].

5 Examples of non-trivial $E_\infty$ topological equivalences

5.1 Examples in $E_\infty \mathbb{F}_p$-DGAs

Here, we discuss the first examples of $E_\infty$ DGAs that are not quasi-isomorphic but are $E_\infty$ topologically equivalent, i.e. non-trivially $E_\infty$ topologically equivalent.

The first example we construct is in $E_\infty \mathbb{F}_p$-DGAs. By the equivalence of $E_\infty \mathbb{F}_p$-DGAs and commutative $H\mathbb{F}_p$-algebras, constructing non-trivially topologically equivalent $E_\infty \mathbb{F}_p$-DGAs is the same as constructing commutative $H\mathbb{F}_p$-algebras that are not weakly equivalent as commutative $H\mathbb{F}_p$-algebras but are weakly equivalent as commutative $S$-algebras.

As we noted earlier, commutative $H\mathbb{F}_p$-algebras form the category of commutative $S$-algebras under $H\mathbb{F}_p$. There is a model structure induced on the under-category where the weak equivalences, cofibrations and fibrations are precisely the same as for commutative $S$-algebras.

In our example, we start with a commutative $S$-algebra $X$ and induce two different commutative $H\mathbb{F}_p$-algebra structures on this object by providing two different commutative $S$-algebra maps from $H\mathbb{F}_p$ to $X$. Clearly these two commutative $H\mathbb{F}_p$-algebras are weakly equivalent.
(even isomorphic) as commutative $S$-algebras. We show that these two commutative $HF_p$-algebras are not weakly equivalent as commutative $HF_p$-algebras by showing that their homotopy rings are not isomorphic as algebras over the Dyer-Lashof algebra. By the discussion of section 3 this shows that these non-trivially $E_\infty$ topologically equivalent $E_\infty$ $F_p$-DGAs are furthermore not equivalent as $H_\infty$ $F_p$-DGAs.

**Example 5.1.** For an odd prime $p$, the $E_\infty$ $F_2$-DGAs we produce have the same homology ring given by $\Lambda_{F_2}[\tau_0, \xi_1, \tau_1]/(\tau_0 \tau_1, \tau_1 \xi_1, \tau_0 \xi_1 - \tau_1)$ where the degrees of the generators are those of the dual Steenrod algebra i.e. $|\tau_0| = 1$, $|\xi_1| = 2(p - 1)$ and $|\tau_1| = 2p - 1$. However, homology of these $E_\infty$ $F_p$-DGAs are not isomorphic as algebras over the Dyer-Lashof algebra. In one of them, $Q^1(\tau_0) = \tau_1$ and in the other, $Q^1(\tau_0) = 0$. Therefore these two $E_\infty$ $F_p$-DGAs are not equivalent as $H_\infty$ $F_p$-DGAs and therefore they are not quasi-isomorphic.

For $p = 2$ the homology ring of the two $E_\infty$ topologically equivalent $E_\infty$ $F_2$ DGAs are $\mathbb{F}_2[\xi_1]/(\xi_1^4)$ where $|\xi_1| = 1$ as in the dual Steenrod algebra. In the homology of the first $E_\infty$ $F_2$-DGA, $Q^2(\xi_1) = \xi_1^3$ and in the other one, $Q^2(\xi_1) = 0$. Again, these two $E_\infty$ $F_2$-DGAs are not quasi-isomorphic because their homology rings are not isomorphic as algebras over the Dyer-Lashof algebra and therefore they are not equivalent as $H_\infty$ $F_2$-DGAs.

First, we discuss our example for $p = 2$. Let $HF_2 \wedge HF_2 \to P_3(HF_2 \wedge HF_2)$ be the third Postnikov section of $HF_2 \wedge HF_2$ as a commutative $S$-algebra, we have $\pi_*(P_3(HF_2 \wedge HF_2)) = \mathbb{F}_2[\xi_1, \xi_2]/(\xi_1^4, \xi_2^2, \xi_1 \xi_2)$ with $|\xi_1| = 1$ and $|\xi_2| = 3$ i.e. the dual Steenrod algebra quotiented out by the ideal of elements of degree 4 and higher. By attaching a cell to $P_3(HF_2 \wedge HF_2)$ to kill the element $\xi_1^3 + \xi_2$ and then taking the third Postnikov section, we obtain another commutative $S$-algebra $X$ with $\pi_*(X) = \mathbb{F}_2[\xi_1, \xi_2]/(\xi_1^4, \xi_2^2, \xi_1 \xi_2, \xi_1^3 + \xi_2) = \mathbb{F}_2[\xi_1]/(\xi_1^4)$. The reason we kill $\xi_1^3 + \xi_2$ is because it is equal to $\xi_2$ in the dual Steenrod algebra, this follows from the generating series we discuss in Section 3. The cell attachment is explained in Lemma 5.1 below. Note that for the cell attachment, one can use the commutative $HF_p$-algebra structure on $P_3(HF_2 \wedge HF_2)$ induced by the map $HF_2 \wedge S \to HF_2 \wedge HF_2 \to P_3(HF_2 \wedge HF_2)$.

Furthermore, we have a map $P_3(HF_2 \wedge HF_2) \to X$ with the induced map on the homotopy rings being the canonical one. By pre-composing this map with the map into the Postnikov section $HF_2 \wedge HF_2 \to P_3(HF_2 \wedge HF_2)$, we obtain a map of commutative $S$-algebras $f : HF_2 \wedge HF_2 \to X$. We construct two commutative $S$-algebra maps
from $H\mathbb{F}_2$ to $X$ as shown in the diagram below.

\[
\begin{array}{c}
H\mathbb{F}_2 \cong H\mathbb{F}_2 \land S \\
g_1 \downarrow \\
H\mathbb{F}_2 \land H\mathbb{F}_2 \xrightarrow{f} X \\
g_2 \downarrow \\
S \land H\mathbb{F}_2
\end{array}
\] (2)

The maps $g_1$ and $g_2$ induce two commutative $H\mathbb{F}_2$-algebra structures on $X$. The commutative $H\mathbb{F}_2$-algebra with unit $f \circ g_1$ is denoted as $X_1$ and with unit $f \circ g_2$ as $X_2$.

As we discuss in Section 4, $H\mathbb{F}_2 \land H\mathbb{F}_2$ can be given two commutative $H\mathbb{F}_2$-algebra structures through the maps $g_1$ and $g_2$ and we call the two associated commutative $H\mathbb{F}_2$-algebras $Y_1$ and $Y_2$. The Dyer-Lashof operations on $\pi_*(Y_1)$ are denoted by $Q^*$ and the Dyer-Lashof operations on $\pi_*(Y_2)$ are denoted by $\bar{Q}^*$ in Section 4.

Because commutative $H\mathbb{F}_2$-algebra morphisms are morphisms of commutative $S$-algebras under $H\mathbb{F}_2$, from the map $f$ alone we obtain two $H\mathbb{F}_2$-algebra maps $g : Y_1 \to X_1$ and $h : Y_2 \to X_2$. These maps induce maps that preserve Dyer-Lashof operations in the homotopy rings and we use this to understand the Dyer-Lashof operations on $\pi_*(X_1)$ and $\pi_*(X_2)$. On $\pi_*(X_1)$,

\[Q^2(\xi_1) = Q^2(g_*(\xi_1)) = g_*(Q^2(\xi_1)) = g_*(\zeta_2) = \zeta_2 = \xi_1^3 + \xi_2 = 0.\]

On $\pi_*(X_2)$,

\[Q^2(\xi_1) = Q^2(h_*(\xi_1)) = h_*(\bar{Q}^2(\xi_1)) = h_*(\xi_2) = \xi_2 \neq 0.\]

Therefore, $\pi_*(X_1)$ and $\pi_*(X_2)$ are not isomorphic as algebras over the Dyer-Lashof algebra as desired.

For odd primes $p$, the construction of an example is similar. By Equation 14 in Section 4, $\tau_1 = \tau_0 \xi_1 - \tau_1$. Therefore one can attach a cell to kill $\tau_0 \xi - \tau_1$ which kills $Q^1 \tau_0$. The rest of the arguments follow similarly.

**Lemma 5.1.** Let $X$ be a connective commutative $H\mathbb{F}_p$-algebra with $\pi_0(X) = \mathbb{F}_p$ and $\pi_i(X) = 0$ for $i > n$. Given $x \in \pi_n(X)$ there is a commutative $S$-algebra $Y$ and a map of commutative $S$-algebras $X \to Y$ which induces the morphism $X_* \to X_*/(x)$ in the homotopy groups.

**Proof.** Let the $H\mathbb{F}_p$-module map $\Sigma^n H\mathbb{F}_p \to X$ represent $x$. By adjunction and by applying the $n$’th Postnikov section functor, we obtain the map $P_n(\mathbb{F}_{H\mathbb{F}_p}(\Sigma^n H\mathbb{F}_p)) \to P_n(X) \simeq X$. The homotopy ring $\pi_*(\mathbb{F}_{H\mathbb{F}_p}(\Sigma^n H\mathbb{F}_p))$ is the free unstable algebra over the Dyer-Lashof algebra over an element in degree $n$, so we have $\pi_*(P_n(\mathbb{F}_{H\mathbb{F}_p}(\Sigma^n H\mathbb{F}_p))) =$
where $|x_n| = n$. Let $Z = P_n(\mathbb{F}_p, \Sigma^n H\mathbb{F}_p))$. The required $Y$ is $P_n(H\mathbb{F}_p \wedge Z X)$ where $H\mathbb{F}_p$ is a commutative $Z$-algebra by the map $Z \to P_0Z = H\mathbb{F}_p$. Since the smash product is a pushout in commutative ring spectra, this may be considered as attaching a cell. Using the Künneth spectral sequence and the standard resolution of $\mathbb{F}_p$ over $\Lambda \mathbb{F}_p[x_n]$ which is $\Sigma^k \Lambda \mathbb{F}_p[x_n]$ at homological degree $k$, one can show that $\pi_*(H\mathbb{F}_p \wedge Z X)$ and $\mathbb{F}_p \otimes_{\Lambda \mathbb{F}_p[x_n]} X_*$ agree for degree less than $n + 1$. Therefore, $Y_* \cong \mathbb{F}_p \otimes_{\Lambda \mathbb{F}_p} X_* \cong X_*/(x)$ as desired.

5.2 Examples in $E_\infty$ $\mathbb{Z}$-DGAs

Now we discuss examples of non-trivial $E_\infty$ topological equivalences in $E_\infty$ $\mathbb{Z}$-DGAs. Example 5.3 provides examples that have a simple construction. The corresponding commutative $H\mathbb{Z}$-algebras to these $E_\infty$ $\mathbb{Z}$-DGAs have the same underlying commutative $S$-algebra $H\mathbb{Z} \wedge H\mathbb{F}_p$, but the commutative $H\mathbb{Z}$-algebra structure of the first is given by the map $H\mathbb{Z} \wedge S \to H\mathbb{Z} \wedge H\mathbb{F}_p$ and the second is given by the map $S \wedge H\mathbb{Z} \to H\mathbb{Z} \wedge H\mathbb{F}_p$. Example 5.2 below provides a general scenario where examples of non-trivially $E_\infty$ topologically equivalent $E_\infty$ $\mathbb{Z}$-DGAs occur.

Example 5.2. For an odd prime $p$, let $X$ be an $E_\infty \mathbb{F}_p$-$DGA$ that satisfies the following two conditions.

1. $H_i X = 0$ for $i > 2p^2 - 4$
2. $H_i X \neq 0$ for either $i = 2p - 1$ or $i = 2p - 2$

For such an $X$, there exists an $E_\infty \mathbb{Z}$-$DGA$ that is $E_\infty$ topologically equivalent to $X$ but not quasi-isomorphic to $X$.

For $p = 2$, there is a similar result. Let $X$ be an $E_\infty \mathbb{F}_2$-algebra that satisfies the following two conditions.

1. $H_i X = 0$ for $i > 4$
2. $H_i X \neq 0$ for $i = 2$

In this case, there exists an $E_\infty \mathbb{Z}$-$DGA$ that is $E_\infty$ topologically equivalent to $X$ but not quasi-isomorphic to $X$.

Indeed, in both cases we show that the $E_\infty \mathbb{Z}$-$DGA$ we construct is not quasi-isomorphic to any $E_\infty \mathbb{F}_p$-$DGA$.

It is clear that $E_\infty \mathbb{F}_p$-$DGAs$ that satisfy the conditions above exist. One can start with a graded commutative ring that satisfies the above conditions and use the corresponding formal commutative $\mathbb{F}_p$-$DGA$.

If one uses the formal commutative $\mathbb{F}_2$-$DGA$ with homology the exterior algebra with a generator in degree 2 for $X$, our examples, after
forgetting to associative Z-DGAs, are the main example of non-trivially topologically equivalent associative Z-DGAs in [DS07]. We already discussed this example in Section 2. Moreover, our methods provide a generalization of this example. We start with an X as described in the $p = 2$ case above and let Y be the $E_\infty$ Z-DGA we produce, then X and Y are not quasi-isomorphic as associative Z-DGAs although they are topologically equivalent i.e. X and Y are non-trivially topologically equivalent as associative DGAs.

To produce our example, we start with an $X$ as described in the $p = 2$ case above and let $Y$ be the $E_\infty$ Z-DGA we produce, then $X$ and $Y$ are not quasi-isomorphic as associative Z-DGAs although they are topologically equivalent i.e. $X$ and $Y$ are non-trivially topologically equivalent as associative DGAs.

We describe our example when $p$ is an odd prime and $H^{2p-1}_X \neq 0$. The case $H^{2p-2}_X \neq 0$ is similar. We explain the $p = 2$ example at the end.

For the obstruction spectral sequence of Theorem 3.1, we use the map $H_Z \to H_F^p \varphi_X U(X)$ as a base-point where the map $\varphi_X$ is the $H_F^p$ structure morphism of X. Using this base point and by setting up the obstruction spectral sequence to calculate the commutative $S$-algebra maps from $H_Z$ to $X$, we obtain that obstructions to lifting a morphism of unstable algebras over the Dyer-Lashof algebra in $E_0^{0,0} = \text{Hom}_{R-\text{alg}}(H_F^p HZ, X_*)$ to a commutative $S$-algebra map from $H_Z$ to $U(X)$ lie in the cohomology groups

$$\text{Def}_{R-\text{alg}}(H_F^p HZ, X_*^{s-1})$$

for $t \geq 2$.

In Lemma 5.2 below we show that these groups that contain the obstructions are trivial. Therefore every map in $\text{Hom}_{R-\text{alg}}(H_F^p HZ, X_*)$ lifts to a commutative $S$-algebra map.

The canonical morphism $H_F^p HZ \to H_F^p H_F^p$ is an injection and the image of this morphism is the free commutative algebra generated by $\zeta_i$ and $\tau_i$ for $i \geq 1$ when $p$ is odd. For $p = 2$, the image is generated by $\xi_i^1$ and $\zeta_i$ for $i \geq 1$. Since the above inclusion comes from a map of commutative $H_F^p$-algebras, it preserves Dyer-Lashof operations. This says that the Dyer-Lashof operations on $H_F^p HZ$ are those of $H_F^p H_F^p$.

To construct the commutative $S$-algebra map $\varphi_Y : HZ \to U(X)$ that defines Y as wanted, we start with any morphism $f$ in $E_2^{0,0} = \text{Hom}_{R-\text{alg}}(H_F^p HZ, X_*)$ that maps $\tau_1$ to a nonzero element in $H_{2p-1} X$
and use the lift of this map to a commutative $S$-algebra map. Here, $\varphi_Y$ being a lift of $f$ means that the map

$$HF_p \wedge HZ \xrightarrow{id \wedge \varphi_Y} HF_p \wedge U(X) \rightarrow U(X)$$

induces $f$ in homotopy where the second map is given by the $HF_p$ module structure map of $X$.

Now we show that the $E_\infty \mathbb{Z}$-DGAs corresponding to $X$ and $Y$ are not quasi-isomorphic, i.e., $X$ and $Y$ are not weakly equivalent as commutative $HZ$-algebras. Assume that they are weakly equivalent over $HZ$. We start with an $X$ that is cofibrant and fibrant as a commutative $HF_p$-algebra. Therefore, $Y$ is also fibrant as a commutative $HZ$-algebra because the underlying commutative $S$-algebra of $Y$ is the underlying commutative $S$-algebra of $X$. Our $X$ is also cofibrant as a commutative $HZ$-algebra since we use a cofibrant $HF_p$ so that the initial map $HZ \hookrightarrow HF_p \hookrightarrow X$ is a composition of two cofibrations. Recall that cofibrations of commutative $HF_p$-algebras are those of commutative $HZ$-algebras. Therefore, we have a weak equivalence of commutative $HZ$-algebras $\psi : X \simeq Y$. That is, we have the following commuting diagram in commutative $S$-algebras.

$$\begin{array}{ccc}
HZ & \xrightarrow{\varphi_Y} & HF_p \\
\downarrow & & \downarrow \varphi_X \\
U(X) = U(Y) & \xleftarrow{\psi} & U(X)
\end{array}$$

(3)

From the above diagram, by applying the $HF_p$ homology functor we obtain the following diagram.

$$\begin{array}{ccc}
HF_{p_*}HZ & \xrightarrow{HF_{p_*}\varphi_Y} & HF_{p_*}HF_p \\
\downarrow & & \downarrow HF_{p_*}\varphi_X \\
HF_{p_*}X & \xleftarrow{HF_{p_*}\psi} & HF_{p_*}X
\end{array}$$

(4)

Therefore, all the morphisms in this triangle preserve Dyer-Lashof operations. The bottom left arrow is induced by the $HF_p$-module structure map of $X$. This is a morphism of commutative $HF_p$-algebras, therefore the bottom left vertical arrow also preserves Dyer-Lashof opera-
tions operations. In conclusion, all the arrows in this diagram preserve Dyer-Lashof operations.

The composition of the vertical arrows on the left gives the map $f$ as chosen above. Therefore, $\tau_1$ in $HF_\mathfrak{p}_sHZ$ is mapped to a non-zero element in $X_*$ by the composition of the vertical arrows. Because the triangle above commutes, if we travel $\tau_1$ through the diagonal arrow to $HF_\mathfrak{p}_sHF_p$ and then to $X_*$, we see that $\tau_1$ in $HF_\mathfrak{p}_sHF_p$ must also be mapped to a non-zero element in $X_*$. Because $\pi_1X$ is trivial, $\tau_0$ in $HF_\mathfrak{p}_sHF_p$ is mapped to zero in $X_*$. However this, and the fact that $\tau_1 = -\beta Q^1\tau_0$ imply that $\tau_1$ in $HF_\mathfrak{p}_sHF_p$ is mapped to zero in $X_*$. This contradicts $f(\tau_1) \neq 0$. Therefore, $Y$ and $X$ are not weakly equivalent as commutative $HZ$-algebras.

For the case $H_{2p-2}X \neq 0$, we use an $f$ in $\text{Hom}_{\mathcal{R}\text{-alg}}(HF_\mathfrak{p}_sHZ, X_*)$ that maps $\zeta_1$ to a non-zero element in $\pi_{2p-2}X$. Since $Q^1\tau_0 = -\zeta_1$, the rest of the argument follows similarly.

For $p = 2$, we start with an $f$ in $\text{Hom}_{\mathcal{R}\text{-alg}}(HF_2HZ, X_*)$ that maps $\xi_1^2$ in $HF_2HZ$ to a non-zero element in $\pi_2X$. Note that $\xi_1^2$ in $HF_2HZ$ is a free algebra generator and $\xi_1 \not\in HF_2HZ$. The arguments are similar but we do not use Dyer-Lashof operations in this case. Again considering Diagram 4, $\xi_1^2$ in $HF_2HZ$ is mapped to a non-zero element in $\pi_2X$ but since $\pi_1X = 0$, $\xi_1$ is mapped to zero and this is a contradiction. Since we haven’t used Dyer-Lashof operations, we may consider the underlying associating $HZ$-algebras of $X$ and $Y$ and the above arguments still work, i.e. the associative $Z$-DGAs corresponding to $X$ and $Y$ are non-trivially topologically equivalent.

What is left to prove is the following lemma which says that the obstructions in the above setting for lifting $f$ to a map of commutative $S$-algebras are zero.

**Lemma 5.2.** In the setting of Example 5.2

\[ \text{Der}_r^{t}(HF_\mathfrak{p}_sHZ, X_*^{S_{t-1}}) = 0 \text{ for } t \geq 2 \]

**Proof.** We describe the odd prime case, the proof is similar for $p = 2$. Let $F_R(\zeta_1, \tau_1)$ denote the free unstable algebra over the Dyer-Lashof algebra generated by two elements whose degrees are the degrees of the corresponding generators in the dual Steenrod algebra. The free unstable algebra over the Dyer-Lashof algebra is described in Theorem 4.1. The lowest degree generator of $F_R(\zeta_1, \tau_1)$ after $\zeta_1$ and $\tau_1$ is $\beta Q^p\zeta_1$ with degree $2p^2 - 3$. Note that $HF_\mathfrak{p}_sHZ$ has no free algebra generators at this degree, showing that $HF_\mathfrak{p}_sHZ$ cannot be the free unstable algebra over the Dyer-Lashof algebra generated by $\zeta_1$ and $\tau_1$. However, also note that $HF_\mathfrak{p}_sHZ$ agrees with $F_R(\zeta_1, \tau_1)$ up to degree $2p^2 - 4$. Therefore, the morphism $F_R(\zeta_1, \tau_1) \rightarrow HF_\mathfrak{p}_sHZ$ which preserves Dyer-Lashof operations maps $\zeta_1$ to $\zeta_1$ and $\tau_1$ to $\tau_1$ is an isomorphism up to degree $2p^2 - 4$. 

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Furthermore, considering the free simplicial resolution of these objects as unstable algebras over the Dyer-Lashof algebra, we get a morphism

$$F_R^{*+1}(F_R(\zeta_1, \tau_1)) \to F_R^{*+1}(HF_{p_*}HZ)$$

describing simplicial unstable algebras over the Dyer-Lashof algebra. Note that up to degree \( n > 0 \), \( F_R(M) \) only depends on the part of the vector space \( M \) up to degree \( n \). Therefore, the morphism above is an isomorphism up to degree \( 2p^2 - 4 \) at each simplicial degree.

Let the functor

$$\text{Der}_{R\text{-alg}}(-, X_s^{s_{t-1}}) = \text{Map}_{R\text{-alg}}(-, X_s^{s_{t-1}})$$

denote the degree preserving derivations that preserve Dyer-Lashof operations. Since \( X_s^{s_{t-1}} \) is concentrated in degree \( 2p^2 - 4 \) and below, this functor depends only on the input up to degree \( 2p^2 - 4 \). Therefore the morphism of simplicial sets above induces an isomorphism

$$\text{Der}_{R\text{-alg}}(F_R^{*+1}(HF_{p_*}HZ), X_s^{s_{t-1}}) \cong \text{Der}_{R\text{-alg}}(F_R^{*+1}(F_R(\zeta_1, \tau_1)), X_s^{s_{t-1}}).$$

In cohomology, this induces the isomorphism

$$\text{Der}_{R\text{-alg}}^t(HF_{p_*}HZ, X_s^{s_{t-1}}) \cong \text{Der}_{R\text{-alg}}^t(F_R(\zeta_1, \tau_1), X_s^{s_{t-1}}) = 0 \text{ for } t > 0.$$  

The last equality follows because André-Quillen cohomology of a free object is trivial above degree zero.

\textbf{Example 5.3.} We use the construction of Example 5.5 in \cite{DS07} to produce non-trivially \( E_\infty \) topologically equivalent \( E_\infty \) \( \mathbb{Z} \)-DGAs. We start with \( H\mathbb{Z} \wedge HF_p \) and produce two commutative \( H\mathbb{Z} \)-algebras by using the structure map \( H\mathbb{Z} \wedge S \to H\mathbb{Z} \wedge HF_p \) and \( S \wedge H\mathbb{Z} \to H\mathbb{Z} \wedge HF_p \). Denote these commutative \( H\mathbb{Z} \)-algebras by \( X \) and \( Y \) respectively. By construction, the underlying commutative \( S \)-algebras of \( X \) and \( Y \) are the same. Therefore, the \( E_\infty \) \( \mathbb{Z} \)-DGAs corresponding to \( X \) and \( Y \) are \( E_\infty \) topologically equivalent. Hence, we only need to show that these \( E_\infty \) \( \mathbb{Z} \)-DGAs are not quasi-isomorphic, i.e. we need to show that \( X \) and \( Y \) are not weakly equivalent as commutative \( H\mathbb{Z} \)-algebras.

In \cite{DS07}, it is shown that for \( p = 2 \), \( X \) and \( Y \) are non-trivially topologically equivalent as associative DGAs. We generalize this to odd primes in the \( E_\infty \) case.

Assume that \( X \) and \( Y \) are weakly equivalent as commutative \( H\mathbb{Z} \)-algebras. Taking cofibrant and fibrant replacements, we assume that there is a weak equivalence of commutative \( H\mathbb{Z} \)-algebras \( X \xrightarrow{\sim} Y \). This means that there is the following diagram in commutative \( S \)-algebras.

\begin{equation}
\begin{array}{ccc}
HZ & \xrightarrow{\varphi_X} & U(X) \\
\xleftarrow{\psi} & & \xrightarrow{\psi} \\
& \xrightarrow{\varphi_Y} & U(Y)
\end{array}
\end{equation}
Here, $U$ denotes the forgetful functor to commutative $\mathcal{S}$-algebras and $\varphi_X$ and $\varphi_Y$ denote the commutative $H\mathbb{Z}$-algebra structure maps of $X$ and $Y$ respectively. Note that by the Künneth spectral sequence,

$$H\mathbb{F}_pU(X) \cong H\mathbb{F}_{ps}(H\mathbb{Z} \wedge H\mathbb{F}_p) \cong H\mathbb{F}_{ps}H\mathbb{Z} \otimes_{\mathbb{F}_p} H\mathbb{F}_{ps}H\mathbb{F}_p.$$  

Taking the $H\mathbb{F}_p$ homology of the diagram above, we obtain the following.

$$\begin{array}{ccc}
H\mathbb{F}_{ps}H\mathbb{Z} \otimes_{\mathbb{F}_p} H\mathbb{F}_{ps}H\mathbb{F}_p & \xrightarrow{H\mathbb{F}_{ps}\varphi_X} & H\mathbb{F}_{ps}H\mathbb{Z} \\
& & \xrightarrow{H\mathbb{F}_{ps}\varphi_Y} \\
\cong & \xrightarrow{H\mathbb{F}_{ps}\psi} & H\mathbb{F}_{ps}H\mathbb{Z} \otimes_{\mathbb{F}_p} H\mathbb{F}_{ps}H\mathbb{F}_p \\
\end{array}$$

With this identification, $H\mathbb{F}_{ps}\varphi_X(x) = x \otimes 1$. Similarly, we have $H\mathbb{F}_{ps}U(Y) \cong H\mathbb{F}_{ps}H\mathbb{Z} \otimes_{\mathbb{F}_p} H\mathbb{F}_{ps}H\mathbb{F}_p$ and $H\mathbb{F}_{ps}\varphi_Y(x) = 1 \otimes x$. As noted earlier, the canonical map $H\mathbb{F}_{ps}H\mathbb{Z} \to H\mathbb{F}_{ps}H\mathbb{F}_p$ is an inclusion and $H\mathbb{F}_{ps}H\mathbb{Z}$ is a free commutative ring generated by the same generators as $H\mathbb{F}_{ps}H\mathbb{F}_p$ except $\tau_0$. The only degree one element in $H\mathbb{F}_{ps}H\mathbb{Z} \otimes_{\mathbb{F}_p} H\mathbb{F}_{ps}H\mathbb{F}_p$ is $1 \otimes \tau_0$ and since $H\mathbb{F}_{ps}\psi$ is an isomorphism, this is mapped by $H\mathbb{F}_{ps}\psi$ to $1 \otimes \tau_0$. Since $Q^1(1 \otimes \tau_0) = 1 \otimes \tau_1$, $H\mathbb{F}_{ps}\psi(1 \otimes \tau_1) = 1 \otimes \tau_1$. But the commutativity of the triangle forces $H\mathbb{F}_{ps}\psi(\tau_1 \otimes 1) = 1 \otimes \tau_1$ and this contradicts the injectivity of $H\mathbb{F}_{ps}\psi$. Therefore $X$ and $Y$ are not equivalent as commutative $H\mathbb{Z}$-algebras. The argument for $p = 2$ is similar.

6 Proof of Theorem 1.2

To prove Theorem 1.2, we need to show that two co-connective commutative $H\mathbb{Z}/(m)$-algebras that are weakly equivalent as commutative $\mathcal{S}$-algebras are also weakly equivalent as commutative $H\mathbb{Z}/(m)$-algebras for any $m \in \mathbb{Z}$.

Since the category of commutative $H\mathbb{Z}/(m)$-algebras is the category of commutative $\mathcal{S}$-algebras under $H\mathbb{Z}/(m)$, for our purpose, it is natural to consider the homotopy class of commutative $\mathcal{S}$-algebra maps from $H\mathbb{Z}/(m)$ to a co-connective $H\mathbb{Z}/(m)$-algebra $X$. We omit the forgetful functor to commutative $\mathcal{S}$-algebras and denote this by $\pi_0\text{map}_{\mathcal{S}-\text{cAlg}}(H\mathbb{Z}/(m), X)$. We show in Proposition 6.1 that there is a unique homotopy class of maps in $\pi_0\text{map}_{\mathcal{S}-\text{cAlg}}(H\mathbb{Z}/(m), X)$. The proof of Theorem 1.2 is based on this fact.

Proof of Theorem 1.2 Let $X$ and $Y$ be co-connective commutative $H\mathbb{Z}/(m)$-algebras that are weakly equivalent as commutative $\mathcal{S}$-algebras. We assume $X$ and $Y$ are cofibrant and fibrant as commutative $H\mathbb{Z}/(m)$-algebras. Recall that cofibrations, fibrations and weak equivalences
of commutative $HZ/(m)$-algebras are precisely those of commutative $S$-algebras. Therefore $X$ and $Y$ are also fibrant as commutative $S$-algebras. Furthermore, they are also cofibrant commutative $S$-algebras because the initial map $S \to U(X)$ factors as a composition of two commutative $S$-algebra cofibrations as $S \hookrightarrow HZ/(m) \hookrightarrow U(X)$, where $U$ denotes the forgetful functor to commutative $S$-algebras.

Since $X$ and $Y$ are weakly equivalent as commutative $S$-algebras and they are cofibrant and fibrant as commutative $S$-algebras, there is a commutative $S$-algebra weak equivalence $U(X) \sim \to U(Y)$. Let $\varphi_X : HZ/(m) \to U(X)$ and $\varphi_Y : HZ/(m) \to U(Y)$ denote the commutative $S$-algebra maps that are the $HZ/(m)$ structure maps of $X$ and $Y$ respectively. Since $\psi$ is only a commutative $S$-algebra map, it may not preserve the $HZ/(m)$ structure, i.e. $\psi \circ \varphi_X$ is not necessarily equal to $\varphi_Y$.

Let $Y'$ be the commutative $HZ/(m)$-algebra whose underlying commutative $S$-algebra is $U(Y)$ and whose $HZ/(m)$ structure map is $\psi \circ \varphi_X$. With this $HZ/(m)$ structure of $Y'$, $\psi$ becomes a weak equivalence of commutative $HZ/(m)$-algebras from $X$ to $Y'$. Therefore it is sufficient to show that $Y'$ and $Y$ are weakly equivalent as $HZ/(m)$-algebras.

By Proposition 6.1, $\pi_0 \map_{S-cAlg}(HZ/(m), Y) = \{\ast\}$. Therefore, $\varphi_Y$ and $\psi \circ \varphi_X$, the structure maps of $Y$ and $Y'$ respectively, are homotopic. A homotopy between $\varphi_Y$ and $\psi \circ \varphi_X$ is given by the following diagram.

\[ \begin{array}{ccc}
HZ/(m) & \xrightarrow{f} & U(Y) \\
\downarrow \varphi_X & & \downarrow \varphi_Y \\
U(Y) & \xrightarrow{p_1} & U(Y) \\
\downarrow \approx & & \downarrow \approx \\
U(Y) & \xrightarrow{p_2} & U(Y) = U(Y') \\
\end{array} \]

Here, $U(Y)^I$ denotes a path object of $U(Y)$. This is a diagram in commutative $S$-algebras. However, if we give $U(Y)^I$ a commutative $HZ/(m)$-algebra structure using $f$ and call this commutative $HF_{p^r}$-algebra $Z$, $p_1$ becomes a weak equivalence of commutative $HZ/(m)$-algebras from $Z$ to $Y$ and $p_2$ becomes a weak equivalence of commutative $HZ/(m)$-algebras from $Z$ to $Y'$. Therefore, $Y$ and $Y'$ are weakly equivalent commutative $HZ/(m)$-algebras and so are $Y$ and $X$.

What is left to prove is the following proposition.

**Proposition 6.1.** For a co-connective commutative $HZ/(m)$-algebra $X$, the mapping space $\map_{S-cAlg}(HZ/(m), X)$ is contractible.
We use the obstruction spectral sequence to show that all the homotopy groups of this mapping space are trivial. However, since we work over a general \( \mathbb{Z}/(m) \) where \( m \) may not be a prime, we do not have a description of the \( E_2 \) page of the spectral sequence as in Theorem 3.1. It turns out that it is sufficient to consider the \( E_1 \) page only. To construct the obstruction spectral sequence, one starts with the bar resolution of \( \mathbb{Z}/(m) \) with respect to the commutative \( \mathcal{S} \)-algebra monad \( \mathbb{P}_S \). We denote this augmented simplicial commutative \( \mathcal{S} \)-algebra resolution by \( \mathbb{P}_S^{+1} \mathbb{Z}/(m) \to \mathbb{Z}/(m) \). Applying the contravariant functor map \( \mathcal{S}-cAlg(\mathbb{P}_S^{+1} \mathbb{Z}/(m), X) \) to this resolution one obtains a co-simplicial resolution of the mapping space map \( \mathcal{S}-cAlg(\mathbb{P}_S^{+1} \mathbb{Z}/(m), X) \), see Section 14 of [Rez98]. Here, we assume \( \mathbb{Z}/(m) \) is cofibrant and \( X \) is fibrant as commutative \( \mathcal{S} \)-algebras.

Because \( X \) is a commutative \( \mathbb{Z}/(m) \)-algebra, there is a map of commutative \( \mathcal{S} \)-algebras \( \mathbb{Z}/(m) \to U(X) \) that serves as a base point for this spectral sequence. By [Bou89] [BK72], there is a second quadrant spectral sequence whose first page is given by

\[
E_1^{s,t} = \pi_t \text{map}_{\mathcal{S}-cAlg}(\mathbb{P}_S^{+1} \mathbb{Z}/(m), X),
\]

where the homotopy groups are calculated at the given base point. This spectral sequence abuts to

\[
\pi_{t-s} \text{map}_{\mathcal{S}-cAlg}(\mathbb{Z}/(m), X).
\]

**Proof.** We will show that in the above spectral sequence, \( E_1^{s,t} = 0 \) for \( t > 0 \) and \( E_2^{0,0} = 0 \). This is sufficient to show that the homotopy groups of the mapping space are trivial.

We start by showing that \( E_2^{0,0} = 0 \). We have the following isomorphisms for \( E_1^{0,0} \) that we explain below.

\[
E_1^{0,0} = \pi_0 \text{map}_{\mathcal{S}-cAlg}(\mathbb{P}_S \mathbb{Z}/(m), X) \\
\cong \pi_0 \text{map}_{\mathcal{S}-mod}(\mathbb{Z}/(m), X) \\
\cong \text{Hom}_{\mathcal{S}-mod}(\mathbb{Z}/(m), X) \\
\cong \text{Hom}_{\mathcal{S}-mod}(\mathbb{Z}/(m), X_0)
\]

The first isomorphism follows by adjunction. For the second isomorphism, we use the universal coefficient spectral sequence of theorem 4.5 in Chapter IV of [EKMM07] with respect to the homology theory \( \mathbb{H}X \). This works because \( X \) is an \( \mathbb{H}X \)-module by forgetting the \( \mathbb{H}X/(m) \)-module structure through the map \( \mathbb{H}X \to \mathbb{H}X/(m) \). Calculations of the spectral sequence are done purely by degree considerations and using the fact that \( \mathbb{Z} \) has global dimension 1. Also, note that using the Tor spectral sequence of Theorem 4.1 in Chapter IV of [EKMM07], it can be shown that \( \mathbb{H}X, \mathbb{H}X/(m) \) is connective and \( \mathbb{H}X_0 \mathbb{H}X/(m) \cong \mathbb{Z}/(m) \). The third isomorphism follows by this because \( X \) is co-connective.
The $E_2^{0,0}$ term is the equalizer of the diagram

$$\pi_0 \text{map}_{\text{S-CAlg}}(\mathbb{P}_S(\mathbb{H}^Z/(m)), X) \rightrightarrows \pi_0 \text{map}_{\text{S-CAlg}}(\mathbb{P}_S^2(\mathbb{H}^Z/(m)), X).$$

Which is the equalizer of

$$\pi_0 \text{map}_{\text{S-mod}}(\mathbb{H}^Z/(m), X) \rightrightarrows \pi_0 \text{map}_{\text{S-mod}}(\mathbb{P}_S(\mathbb{H}^Z/(m)), X).$$

This means that a morphism $\varphi : \mathbb{H}^Z/(m) \to X$ of $\mathbb{S}$-modules in $E_0^{0,0}$ survives to $E_2^{0,0}$ if the induced morphisms $\mathbb{P}_S(\mathbb{H}^Z/(m)) \to \mathbb{H}^Z/(m) \xrightarrow{\varphi} X$ and $\mathbb{P}_S(\mathbb{H}^Z/(m)) \xrightarrow{\mathbb{P}_S(\varphi)} \mathbb{P}_S(X) \to X$ agree up-to homotopy. This, in particular, says that such a map should preserve the identity element in the homotopy groups. Since $\mathbb{Z}/(m)$ is generated as an abelian group by 1, there is only one such morphism in $E_1^{0,0} \cong \text{Hom}_{\mathbb{Z}-\text{mod}}(\mathbb{Z}/(m), X_0)$. Furthermore, we know that this morphism lifts to the $E_2^{0,0}$ term because it is represented by a morphism which is an actual commutative $\mathbb{S}$-algebra map $\mathbb{H}^Z/(m) \to X$ which is our base point. In conclusion, $E_2^{0,0} = \text{pt}$.

Now we will show that $E_1^{s,t} = 0$ for $t > 0$. Again by adjunction, we have

$$E_1^{s,t} = \pi_t \text{map}_{\text{S-CAlg}}(\mathbb{P}_S^{s+1}(\mathbb{H}^Z/(m)), X) \cong \pi_t \text{map}_{\text{S-mod}}(\mathbb{P}_S^s(\mathbb{H}^Z/(m)), X).$$

There is a spectral sequence for calculating homotopy groups of the homotopy orbit of a spectrum with an action of a group $G$.

$$H_p(G, \pi_q Y) \Rightarrow \pi_{p+q} Y_{hG}$$

Using this spectral sequence it is clear that the homotopy orbit spectrum of a connective spectrum is connective. Therefore $\mathbb{P}_S(Y)$ is connective when $Y$ is because $\mathbb{P}_S(Y)$ is wedges of homotopy orbits of $Y^\wedge n$ with respect to the action of the symmetric group. To show that $\pi_t \text{map}_{\text{S-mod}}(\mathbb{P}_S^s(\mathbb{H}^Z/(m)), X) \cong 0$, we again use the universal coefficient spectral sequence with respect to the homology theory $H_*$. Since $X$ is co-connective and $\mathbb{P}_S^s(\mathbb{H}^Z/(m))$ is connective, degree considerations are sufficient to show that the relevant terms in the universal coefficient spectral sequence are zero.

\[\square\]

7 Proof of Theorem 1.3

In this section, we prove the following theorem.

**Theorem 7.1.** Let $X$ and $Y$ be $H_\infty HF_p$-algebras with trivial first homotopy groups. If $X$ and $Y$ are equivalent as $H_\infty S$-algebras, then they are equivalent as $H_\infty HF_p$-algebras.
This is a slightly stronger result than Theorem 1.3. If two $E_\infty$ DGAs are $E_\infty$ topologically equivalent, then the corresponding ring spectra are commutative $\mathbb{S}$-algebra equivalent and therefore $H_\infty \mathbb{S}$-algebra equivalent. Therefore, Theorem 1.3 is a corollary of Theorem 7.1.

Remark 7.1. As mentioned in Remark 1.2, one of the intermediate results of [Law15], Proposition 5, states that Theorem 7.1 is still true for $H_\infty H\mathbb{F}_p$-algebras with non-trivial first homology and this contradicts Example 5.1. The proof of Proposition 5 of [Law15] ends by stating that the canonical map

$$[\mathbb{P}_{H\mathbb{F}_p}(M), M]_{H\mathbb{F}_p\text{-mod}} \to [\mathbb{P}_\mathbb{S}(M), M]_{\mathbb{S}\text{-mod}}$$

between homotopy class of maps in $H\mathbb{F}_p$-modules to $\mathbb{S}$-modules is injective where $M$ is an $H\mathbb{F}_p$-module. This says that $H_\infty H\mathbb{F}_p$-algebra structure maps forget injectively to $H_\infty \mathbb{S}$-algebra structure maps. However, this does not imply the desired result since one needs to consider $H_\infty H\mathbb{F}_p$-equivalences and $H_\infty \mathbb{S}$-equivalences between different $H_\infty H\mathbb{F}_p$-algebra and $H_\infty \mathbb{S}$-algebra structures on $M$.

In the proof of Theorem 7.1, we use the following facts about $H_\infty$ algebras which can be derived using the results of [BMMS86]. In the items below, $X$ denotes an $H_\infty H\mathbb{F}_p$-algebra.

1. A morphism of $H_\infty \mathbb{S}$-algebras induces a map of rings in the homotopy groups.

2. The structure map $\mu_X : H\mathbb{F}_p \wedge X \to X$ induced by the $H\mathbb{F}_p$-module structure on $X$ is a map of $H_\infty H\mathbb{F}_p$-algebras. Therefore this map preserves Dyer-Lashof operations on the homotopy ring.

3. There is an equivalence $H\mathbb{F}_p \wedge X \cong (H\mathbb{F}_p \wedge H\mathbb{F}_p) \wedge H\mathbb{F}_p X$. Using this, we obtain the identification $\pi_\ast(H\mathbb{F}_p \wedge X) \cong \mathbb{A}_\ast \otimes_{\mathbb{F}_p} X_\ast$. Note that the Dyer-Lashof operations on $\mathbb{A}_\ast \otimes_{\mathbb{F}_p} X_\ast$ are not those of the tensor product because the $H\mathbb{F}_p$ structure on $H\mathbb{F}_p \wedge X$ is given by multiplication with the $H\mathbb{F}_p$ factor on the left. With this identification, $\mu_{X_\ast}$ is given by $\mu_{X_\ast}(a \otimes x) = ax$ if $a \in \mathbb{A}_0 = \mathbb{F}_p$ and $\mu_{X_\ast}(a \otimes x) = 0$ if $a \in \mathbb{A}_i$ for $i > 0$.

4. The unit map $\eta_X : \mathbb{S} \wedge X \to H\mathbb{F}_p \wedge X$ satisfies $\mu_X \circ \eta_X = \text{id}$. However, $\eta_X$ is only a map of $H_\infty \mathbb{S}$-algebras and it may not preserve Dyer-Lashof operations in the homotopy ring. By the identification of $\pi_\ast(H\mathbb{F}_p \wedge X)$ above, the morphism induced by $\eta_X$ on the homotopy ring is given by $\eta_{X_\ast}(x) = 1 \otimes x$.

Proof of Theorem 7.1. Let $\varphi : X \to Y$ be an equivalence of $H_\infty \mathbb{S}$-algebras. We will show that $\varphi$ induces an equivalence of $H_\infty H\mathbb{F}_p$-algebras by showing that $\varphi$ preserves Dyer-Lashof operations. This is
sufficient because an $H_\infty HFP_\infty$-algebra equivalence type is determined by the isomorphism class of its homotopy ring as an algebra over the Dyer-Lashof algebra.

We have the following diagram.

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi} & Y \\
\downarrow{\eta_X} & & \downarrow{\eta_Y} \\
HFP_\infty \wedge X & \xrightarrow{id \wedge \varphi} & HFP_\infty \wedge Y \\
\downarrow{\mu_X} & & \downarrow{\mu_Y} \\
X & \xrightarrow{\mu} & Y
\end{array}
\] (9)

Applying the homotopy functor to this diagram produces the following.

\[
\begin{array}{ccc}
X_* & \xrightarrow{\varphi_*} & Y_* \\
\downarrow{\eta_{X_*}} & & \downarrow{\eta_{Y_*}} \\
A_* \otimes_{FP_\infty} X_* & \xrightarrow{\psi} & A_* \otimes_{FP_\infty} Y_* \\
\downarrow{\mu_{X_*}} & & \downarrow{\mu_{Y_*}} \\
X_* & \xrightarrow{\mu_*} & Y_*
\end{array}
\] (10)

The middle horizontal morphism $\psi$ is the morphism induced on the homotopy groups by $id \wedge \varphi$. Because we do not assume $\varphi$ to be a map of $H_\infty HFP_\infty$-algebras, $\psi$ may not be induced by two morphisms on the tensor factors. However, $\psi$ preserves Dyer-Lashof operations because it is the morphism in $HFP_\infty$ homology induced by $\varphi$.

The top square in Diagram 10 commutes because it is induced by the commutative square in Diagram 9. Although the bottom square is not induced by a commuting square, we show that it also commutes. For this purpose we need to know more about the Dyer-Lashof operations on $A_* \otimes_{FP_\infty} X_*$. We have the following map

\[
HFP_\infty \wedge HFP_\infty \cong (HFP_\infty \wedge HFP_\infty) \wedge_{HFP_\infty} HFP_\infty \rightarrow (HFP_\infty \wedge HFP_\infty) \wedge_{HFP_\infty} X \cong HFP_\infty \wedge X
\]

induced by the map of $H_\infty HFP_\infty$-algebras $HFP_\infty \rightarrow X$. This is a map of $H_\infty HFP_\infty$-algebras when the $HFP_\infty$ multiplication on $HFP_\infty \wedge HFP_\infty$ is given by that of the $HFP_\infty$ factor on the left. Therefore the morphism $A_* \rightarrow A_* \otimes_{FP_\infty} X_*$ induces on the homotopy groups preserves Dyer-Lashof operations. This says that on $A_* \otimes_{FP_\infty} \{1\} \subseteq A_* \otimes_{FP_\infty} X_*$, Dyer-Lashof operations are given by the ones on the dual Steenrod algebra i.e. $Q^\ast(a \otimes 1) = (Q^\ast a) \otimes 1$.

Now we show that the bottom square in Diagram 10 commutes. We first show this for elements of the form $a \otimes x \in A_* \otimes_{FP_\infty} X_*$ with $|a| > 0$. By the description of $\mu_{X_*}$ in the paragraph before this proof,
we have $\mu_X^*(a \otimes x) = 0$ and therefore $\varphi_* \circ \mu_X^*(a \otimes x) = 0$. Therefore our goal is to show that $\mu_Y^* \circ \psi(a \otimes x) = 0$. Let $\tau_0$ denote the degree 1 element in $A_*$ that generates it as an algebra over the Dyer-Lashof algebra (this element is called $\xi_1$ for $p = 2$). Because $\pi_1(Y) = 0$, $\mu_Y^* \circ \psi(\tau_0 \otimes 1) = 0$. Since $\mu_Y^* \circ \psi$ is a morphism of rings that preserves Dyer-Lashof operations, $\mu_Y^* \circ \psi(a \otimes 1) = 0$ whenever $|a| > 0$. Therefore when $|a| > 0$,

$$\mu_Y^* \circ \psi(a \otimes x) = (\mu_Y^* \circ \psi(a \otimes 1)) \cdot (\mu_Y^* \circ \psi(1 \otimes x)) = 0.$$  

After this, we just need to show that the bottom square in Diagram 10 commutes for elements in $A_* \otimes_{\mathbb{F}_p} X_*$ of the form $a \otimes x$ where $a \in A_0 = \mathbb{F}_p$. Clearly, it is sufficient to work only with the elements of the form $1 \otimes x$. By the description of $\mu_X^*$ in the paragraph before this proof, we have $\varphi_* \circ \mu_X^*(1 \otimes x) = \varphi_*(x)$. Therefore, our goal is to show that $\mu_Y^* \circ \psi(1 \otimes x) = \varphi_*(x)$. Because the top square in Diagram 10 commutes, we deduce that

$$\psi(1 \otimes x) = \psi(\eta_{X^*}(x)) = \eta_{Y^*}(\varphi_*(x)) = 1 \otimes \varphi_*(x). \quad (11)$$

Using this, we obtain what we wanted to show:

$$\mu_Y^* \circ \psi(1 \otimes x) = \mu_Y^* (1 \otimes \varphi_*(x)) = \varphi_*(x).$$

At this point, we know that the bottom square in Diagram 10 commutes and we are ready to show that $\varphi_*$ preserves Dyer-Lashof operations. Given $x \in X_*$, we have

$$\varphi_*(Q^* x) = \varphi_*(Q^* \mu_X^*(1 \otimes x)) = \varphi_* \circ \mu_X^*(Q^*(1 \otimes x)) = \mu_Y^* \circ \psi(Q^*(1 \otimes x)).$$

Therefore we need to show that $\mu_Y^* \circ \psi(Q^*(1 \otimes x)) = Q^* \varphi_*(x)$. This is given by the following chain of equalities

$$\mu_Y^* \circ \psi(Q^*(1 \otimes x)) = Q^* \mu_Y^* (\psi(1 \otimes x)) = Q^* \mu_Y^* (1 \otimes \varphi_*(x)) = Q^* \varphi_*(x).$$

The first equality follows because both $\psi$ and $\mu_Y^*$ preserve Dyer-Lashof operations and the second equality follows by Equation 11. Since $\varphi_*$ preserves Dyer-Lashof operations, it induces an isomorphism between $X_*$ and $Y_*$ as algebras over the Dyer-Lashof algebra and therefore $X$ and $Y$ are equivalent as $H_\infty H\mathbb{F}_p$-algebras.

\[\square\]

### Appendix

Let $X$ denote an $H\mathbb{Z}$-algebra or an $S$-algebra with homotopy ring $\pi_*(X) \cong \Lambda_{\mathbb{F}_p}(x_n)$ for $|x_n| > 0$. We prove the following.
Lemma A.1. Let $X$ be an $H\mathbb{Z}$-algebra as above, we have

$$\pi_0 \text{map}_{H\mathbb{Z}\text{-alg}}(X, H\mathbb{F}_p) = \text{pt}.$$ 

For an $S$-algebra $X$ as above, we have

$$\pi_0 \text{map}_{S\text{-alg}}(X, H\mathbb{F}_p) = \text{pt}.$$ 

The Hopkins-Miller obstruction theory states that obstructions to lifting a morphism in $E_2^{0,0} = \text{Hom}_{\mathbb{F}_p\text{-alg}}(\pi_\ast H\mathbb{F}_p \wedge_{H\mathbb{Z}} X, \mathbb{F}_p)$ to a map of $H\mathbb{Z}$-algebras lie in André-Quillen cohomology for associative algebras, $E_2^{t,t-1} = \text{Der}^t(\pi_\ast H\mathbb{F}_p \wedge_{H\mathbb{Z}} X, \Omega^{t-1}\mathbb{F}_p)$ for $t \geq 2$ and obstructions to homotopy unqiueness of the lift lie in $E_2^{t,t} = \text{Der}^t(\pi_\ast H\mathbb{F}_p \wedge_{H\mathbb{Z}} X, \Omega^t\mathbb{F}_p)$ for $t \geq 1$. By the Künneth spectral sequence, $\pi_0 H\mathbb{F}_p \wedge_{H\mathbb{Z}} X = \mathbb{F}_p$. Therefore there is only a single map in $\text{Hom}_{\mathbb{F}_p\text{-alg}}(\pi_\ast H\mathbb{F}_p \wedge_{H\mathbb{Z}} X, \mathbb{F}_p)$ because these morphisms preserve the identity and the grading. To show that the obstructions to existence and uniqueness are zero, first, note that by the Künneth spectral sequence, $\pi_\ast H\mathbb{F}_p \wedge_{H\mathbb{Z}} X$ is connected and $\Omega^t\mathbb{F}_p$ is in negative degrees for $t > 0$. Therefore, $\text{Der}(F^{*+1}(\pi_\ast H\mathbb{F}_p \wedge_{H\mathbb{Z}} X), \Omega^t\mathbb{F}_p) = 0$ for $t > 0$ where $F$ denotes the free associative algebra functor. Therefore, the cohomology of this co-simplicial abelian group is also zero. This proves the desired result. The argument for homotopy class of maps in $S$-algebras is similar, the only difference is that one uses $\pi_\ast H\mathbb{F}_p \wedge S X$ instead of $\pi_\ast H\mathbb{F}_p \wedge_{H\mathbb{Z}} X$.

References

[Bak12] Andrew Baker. Calculating with topological André-Quillen theory, I: Homotopical properties of universal derivations and free commutative $S$-algebras. arXiv preprint arXiv:1208.1868, 2012.

[Bak15] Andrew Baker. Power operations and coactions in highly commutative homology theories. Publications of the Research Institute for Mathematical Sciences, 51(2):237–272, 2015.

[BK72] Aldridge Knight Bousfield and Daniel Marimus Kan. Homotopy limits, completions and localizations, volume 304 of Lecture Notes in Mathematics. Springer Berlin Heidelberg, 1972.

[BMMS86] RR Bruner, JP May, JE McClure, and M Steinberger. $H_\infty$ ring spectra and their applications, volume 1176 of Lecture Notes in Mathematics. Springer-Verlag Berlin Heidelberg, 1986.
[Bou89] AK Bousfield. Homotopy spectral sequences and obstructions. Israel Journal of Mathematics, 66(1-3):54–104, 1989.

[DS06] Daniel Dugger and Brooke Shipley. Postnikov extensions of ring spectra. Algebr. Geom. Topol., 6:1785–1829, 2006.

[DS07] Daniel Dugger and Brooke Shipley. Topological equivalences for differential graded algebras. Advances in Mathematics, 212(1):37–61, 2007.

[EKMM07] Anthony D Elmendorf, Igor Kriz, Michael A Mandell, and J Peter May. Rings, modules, and algebras in stable homotopy theory, volume 47 of Mathematical Surveys and Monographs. American Mathematical Soc., Providence, RI, 2007. With an appendix by M. Cole.

[Fre09] Jennifer French. A comparison of spectral sequences computing unstable homotopy groups of $p$-complete, nilpotent spaces. arXiv preprint arXiv:0909.4597, 2009.

[GH04] P. G. Goerss and M. J. Hopkins. Moduli spaces of commutative ring spectra. In Structured ring spectra, volume 315 of London Math. Soc. Lecture Note Ser., pages 151–200. Cambridge Univ. Press, Cambridge, 2004.

[JN14] Niles Johnson and Justin Noel. Lifting homotopy $T$-algebra maps to strict maps. Advances in Mathematics, 264:593–645, 2014.

[Law15] Tyler Lawson. A note on $H_\infty$ structures. Proceedings of the American Mathematical Society, 143(7):3177–3181, 2015.

[Laz04] Andrey Lazarev. Cohomology theories for highly structured ring spectra. Structured ring spectra, 315:201–231, 2004.

[May71] J Peter May. Homology operations on infinite loop spaces. In Algebraic topology, volume XXII of Proc. Sympos. Pure Math, pages 171–185. Amer. Math. Soc., 1971.

[Mil58] John Milnor. The Steenrod algebra and its dual. Annals of Mathematics, 67(1):150–171, 1958.

[MMSS01] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley. Model categories of diagram spectra. Proc. London Math. Soc. (3), 82(2):441–512, 2001.

[Noe14] Justin Noel. $H_\infty \neq E_\infty$. Contemporary Mathematics, 617:237–240, 2014.
[Noe15] Justin Noel. The T-algebra spectral sequence: Comparisons and applications. *Algebr. Geom. Topol.*, 14(6):3395–3417, 2015.

[Qui70] Daniel Quillen. On the (co-) homology of commutative rings. In *Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968)*, pages 65–87. Amer. Math. Soc., Providence, R.I., 1970.

[Rez98] Charles Rezk. Notes on the Hopkins-Miller theorem. *Contemporary Mathematics*, 220:313–366, 1998.

[Rob89] CA Robinson. Obstruction theory and the strict associativity of Morava K-theories. *London Mathematical Society Lecture Notes*, 139:143–152, 1989.

[RS14] Birgit Richter and Brooke Shipley. An algebraic model for commutative $H_\mathbb{Z}$-algebras. *arXiv preprint arXiv:1411.7238*, 2014. to appear in Algebr. Geom. Topol.

[Shi07] Brooke Shipley. $H_\mathbb{Z}$-algebra spectra are differential graded algebras. *American Journal of Mathematics*, 129(2):351–379, 2007.