On some explicit evaluations of multiple zeta-star values

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Abstract

In this paper, we give some explicit evaluations of multiple zeta-star values which are rational multiple of powers of $\pi^2$.

1 Main Results

The multiple zeta value (MZV) is defined by the convergent series

$$\zeta(k_1, k_2, \ldots, k_n) := \sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}},$$

where $k_1, k_2, \ldots, k_n$ are positive integers and $k_1 \geq 2$. The integers $k = k_1 + k_2 + \cdots + k_n$ and $n$ are called weight and depth respectively. Considerable amount of work on MZV’s has been done in recent years from various aspects and interests. Among them, several explicit values are known for special index sets, as will be recalled below.

In this paper, we give some evaluations of the multiple zeta-star value (MZSV), which is defined by the following series similar to the MZV:

$$\zeta^*(k_1, k_2, \ldots, k_n) := \sum_{m_1 \geq m_2 \geq \cdots \geq m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}},$$

where $k_1, k_2, \ldots, k_n$ satisfy the same condition as above. The MZSV can be expressed as a $\mathbb{Z}$-linear combination of MZV’s, and vice versa.

**Theorem A.** For positive integers $m, n$, we have

$$\zeta^*_{\overline{2m, 2m, \ldots, 2m}}_{n} = \left\{ \sum_{n_0 + \cdots + n_{m-1} = mn} (-1)^{m(n-1)} \left( \prod_{k=0}^{m-1} \frac{(2^{2n_k} - 2)B_{2n_k}}{(2n_k)!} \right) \exp \left( \frac{2\pi i}{m} \sum_{l=0}^{m-1} ln_l \right) \right\} n^{2mn}.$$
Theorem B. For positive integer \( n \), we have
\[
\zeta^*_{2n}(3,1,\ldots,3,1) = \sum_{i=0}^{n} \left\{ \frac{2}{(4i+2)!} \sum_{n_0+n_1=2(n-i)}^{n_0,n_1 \geq 0} (-1)^{n_1} \frac{(2^{2n_0} - 2)B_{2n_0}}{(2n_0)!} \frac{(2^{n_1} - 2)B_{2n_1}}{(2n_1)!} \right\} \pi^{4n}.
\]
In particular,
\[
\zeta^*_{2n}(3,1,\ldots,3,1) \in \mathbb{Q} \times \pi^{4n}.
\]

Theorem C. Let \( n \) be a positive integer, and let \( I_{2n} \) denote the set of all \( 2n+1 \) possible insertions of the number 2 in the string \( \{3,1,\ldots,3,1\} \). Then we have
\[
\sum_{\vec{s}_{2n} \in I_{2n}} \zeta^*(\vec{s}_{2n}) = \sum_{k=0}^{n} \left\{ \frac{2^{4k+3}B_{4k+2}}{(4k+2)!} \sum_{i=0}^{n-k} \frac{\alpha_{n-k-i}}{(4i+2)!} - \frac{\alpha_{n-k}}{(4k+3)!} \right\} \pi^{4n+2},
\]
where
\[
\alpha_n = \sum_{n_0+n_1=2n}^{n_0,n_1 \geq 0} (-1)^{n_1} \frac{(2^{2n_0} - 2)B_{2n_0}}{(2n_0)!} \frac{(2^{n_1} - 2)B_{2n_1}}{(2n_1)!}.
\]
In particular,
\[
\sum_{\vec{s}_{2n} \in I_{2n}} \zeta^*(\vec{s}_{2n}) \in \mathbb{Q} \times \pi^{4n+2}.
\]

For later use, we recall the corresponding results for MZV's.

Theorem 1 ([AK]). Let \( m, n \) be positive integers. Then we have
\[
\zeta(2m,2m,\ldots,2m) = C_n^{(m)} \frac{(2\pi i)^{2mn}}{(2mn)!},
\]
where \( C_n^{(m)} \) is defined by the following recurrence relations:
\[
C_0^{(m)} = 1, \quad C_n^{(m)} = \frac{1}{2n} \sum_{l=1}^{n} (-1)^l \binom{2mn}{2ml} B_{2ml} C_{n-l}^{(m)} \quad (n \geq 1),
\]
where \( B_{2n} \) are the classical Bernoulli numbers.

Theorem 2 ([BBBL1],[BBBL2]). For any positive integer \( n \), we have
\[
\zeta(3,1,\ldots,3,1) = \frac{2\pi^{4n}}{(4n+2)!}.
\]
**Theorem 3** ([BBBL1]). Let \( n \) be a positive integer, and let \( I_{2n} \) denote the set of all \( 2n + 1 \) possible insertions of the number 2 in the string \( \{3, 1, \ldots, 3, 1\} \).

Then

\[
\sum_{s_{2n} \in I_{2n}} \zeta(s_{2n}) = \frac{\pi^{4n+2}}{(4n+3)!}.
\]

**2 Algebraic setup**

We use the algebraic setup of MZV’s that was developed by Hoffman [H2]. Consider the non-commutative polynomial ring

\[ \mathcal{H} := \mathbb{Q}(x, y) \]

in two indeterminates \( x, y \). We refer to monomials in \( x \) and \( y \) as words. We also define subrings

\[ \mathcal{H}^1 := \mathbb{Q} + \mathcal{H}y \]

and

\[ \mathcal{H}^0 := \mathbb{Q} + x\mathcal{H}y. \]

For an integer \( k \geq 1 \), put \( z_k = x^{k-1}y \). Then the ring \( \mathcal{H}^1 \) is freely generated by \( z_k \) \((k = 1, 2, 3, \ldots)\). When \( k \geq 2 \), \( z_k \) is contained \( \mathcal{H}^0 \).

Now define the evaluation map \( Z : \mathcal{H}^0 \rightarrow \mathbb{R} \) by setting

\[ Z(z_{k_1}z_{k_2} \cdots z_{k_n}) = \zeta(k_1, k_2, \ldots, k_n) \]

on generators and extending it \( \mathbb{Q} \)-linearly.

We define the harmonic product \(*\) on \( \mathcal{H}^1 \) inductively by

\[
w * 1 = 1 * w = w \]

\[
z_p w_1 * z_q w_2 = z_p(w_1 * z_q w_2) + z_q(z_p w_1 * w_2) + z_{p+q}(w_1 * w_2),
\]

for all \( p, q \geq 1 \), and any words \( w, w_1, w_2 \in \mathcal{H}^1 \), together with \( \mathbb{Q} \)-bilinearity. For instance, \( z_p * z_q = z_p z_q + z_q z_p + z_{p+q} \). This product corresponds to \( \zeta(p) \zeta(q) = \zeta(p, q) + \zeta(q, p) + \zeta(p + q) \).

The following theorem which has been proven in [H2] gives the basic algebraic properties of the \(*\)-product.

**Theorem 4** ([H2]). The harmonic product is commutative and associative.

Theorem 4 says that \( \mathcal{H}^1 \) is a \( \mathbb{Q} \)-commutative algebra with respect to the harmonic product \(*\). Then \( \mathcal{H}^0 \) is subalgebra of \( \mathcal{H}^1 \). In [H2], it has also been proved that \( Z \) is homomorphism with respect to the harmonic product \(*\):

\[ Z(w_1 * w_2) = Z(w_1)Z(w_2). \quad (w_1, w_2 \in \mathcal{H}^0) \]
We conclude this section by introducing the $Q$-linear map $S$. Let $S_1 \in Aut(\mathfrak{g})$ be defined by $S_1(1) = 1$, $S_1(x) = x$ and $S_1(y) = x + y$. Define the $Q$-linear map $S : \mathfrak{g}_1 \rightarrow \mathfrak{g}_1$ by

$$S(Fy) := S_1(F)y$$

for all words $F \in \mathfrak{g}$ and $S(1) = 1$. Then it is clear that

$$\zeta^*(k_1, k_2, \ldots, k_n) = Z(S(z_{k_1}z_{k_2}\cdots z_{k_n})).$$

For example, $\zeta^*(k_1, k_2) = \zeta(k_1 + k_2) + \zeta(k_1, k_2) = Z(S(z_{k_1}z_{k_2})). \zeta^*(k_1, k_2, k_3) = \zeta(k_1 + k_2 + k_3) + \zeta(k_1 + k_2, k_3) + \zeta(k_1, k_2 + k_3) + \zeta(k_1, k_2, k_3) = Z(S(z_{k_1}z_{k_2}z_{k_3})).$

### 3 Proof of Theorem A

We prove Theorem A by using the Laurent expansion for the cosecant function:

$$\csc x = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{(2n - 2)B_{2n} x^{2n-2}}{(2n)!}.$$

**Proof of Theorem A** Using the infinite product for the sine function, we have

$$\csc \pi x e^{\frac{\pi i}{m}} = \prod_{k=0}^{m-1} \left(1 - x^2 e^{\frac{2\pi i}{m} k}\right) \prod_{k=0}^{m-1} \left(1 - e^{\frac{2\pi i}{m} k}\right).$$

Substituting $k = 0, 1, \ldots, m - 1$ and multiplying both sides, we obtain

$$\left(\pi x e^{\frac{\pi i}{m}} \prod_{k=0}^{m-1} e^{\frac{\pi i}{m} k}\right)^{m-1} \prod_{k=0}^{m-1} \csc \pi x e^{\frac{\pi i}{m} k} = \frac{1}{\prod_{n=1}^{\infty} \left(1 - \frac{x^{2n}}{n^{2m}}\right)}.$$

The right hand side of (1) equals

$$1 + \left(\sum_{n_1>0} \frac{1}{n_1^{2m}}\right) x^{2m} + \left(\sum_{n_1\ge n_2>0} \frac{1}{n_1^{2m} n_2^{2m}}\right) x^{4m} + \cdots$$

$$= 1 + \sum_{n_1>0} \zeta^*(2m, 2m, \ldots, 2m) x^{2mn}.$$

On the other hand, the left hand side of (1) equals

$$\left(\pi x e^{\frac{\pi i}{m}} \prod_{k=0}^{m-1} e^{\frac{\pi i}{m} k}\right)^{m-1} \prod_{k=0}^{m-1} \sum_{n_k=0}^{\infty} (-1)^{n_k-1} \frac{(2n_k - 2)B_{2n_k} \pi^{2n_k-1} x^{2n_k-1} e^{\frac{\pi i}{m} k(2n_k-1)}}{(2n_k)!}\right)$$

$$= \prod_{k=0}^{m-1} \sum_{n_k=0}^{\infty} (-1)^{n_k-1} \frac{(2n_k - 2)B_{2n_k} \pi^{2n_k-2} x^{2n_k-1} e^{\frac{\pi i}{m} k n_k}}{(2n_k)!}$$
To state the main identity we need to define a sum as follows.

\[ S(z_1z_2\cdots z_n) = \prod_{j=1}^n z_j^{k_j} \sum_{\sum_{i=1}^k m_i = m} \frac{1}{m!} \left( \frac{2^{m_i} - 1}{m} \right) \]

where \( m = \sum_{i=1}^k m_i \).

Comparing coefficients of both sides, we obtain the desired identity. \( \square \)

**Corollary 5.** For positive integers \( m, n \), we have

\[ \zeta^*(2m, 2m, \ldots, 2m) \in \mathbb{Q} \times \pi^{2mn}. \]

**Proof.** The coefficient of \( \pi^{2mn} \) on the right hand side of Theorem A is invariant under the action of the galois group \( \text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \), hence belongs to \( \mathbb{Q} \). \( \square \)

**Remark.** Yasuo Ohno proves Theorem A independently. He proves this theorem in two ways, one way is to use the same method of our proof. The other is to use generating function and differential equation.

## 4 Proof of Theorem B

Theorem B will be obtained as a Corollary of a more general identity, which is stated as follows.

**Theorem 6.** For positive integers \( a, b \) and nonnegative integer \( n \), we have

\[ S((z_1z_2)^n) = \sum_{i=0}^n (z_1z_2)^i * S(z_{a+b}^i), \] (2)

\[ S(z_1z_2z_3\cdots z_n)^n = \sum_{i=0}^n z_1z_2z_3\cdots z_n * S(z_{a+b}^i). \] (3)

**Proof.** By definition of \( S \), we have

\[ S(w_1w_2) = S_1(w_1)S(w_2). \] (\( w_1 \in \mathcal{F}, w_2 \in \mathcal{F} \))

Using this identity, we obtain

\[ S(z_1z_2\cdots z_n) = z_1S(z_2z_3\cdots z_n) + z_2S(z_1z_2z_3\cdots z_n) \]

\[ = z_1S(z_2z_3\cdots z_n) + z_{k_1+k_2}S(z_{k_3}\cdots z_n) + z_{k_1+k_2+k_3}S(z_{k_4}\cdots z_n) + \cdots \]

\[ = \sum_{j=1}^n z_{k_1+k_2+\cdots+k_j}S(z_{j+1}z_{j+2}\cdots z_n). \] (4)
(When \( j = n \), we regard \( S(z_{k_{j+1}} z_{k_{j+2}} \cdots z_{k_n}) \) as 1.) By using this identity, we obtain

\[
S((z_a z_b)^n) = \sum_{j=0}^{n-1} z_{(a+b)j+a} S(z_b(z_a z_b)^{n-1-j}) + \sum_{j=1}^{n} z_{(a+b)j} S((z_a z_b)^{n-j}), \tag{5}
\]

\[
S(z_b(z_a z_b)^n) = \sum_{j=0}^{n} z_{(a+b)j+b} S((z_a z_b)^{n-j}) + \sum_{j=1}^{n} z_{(a+b)j} S(z_b(z_a z_b)^{n-j}), \tag{6}
\]

\[
S(z_{a+b}) = \sum_{j=1}^{n} z_{(a+b)j} S(z_{a+b}^{n-j}). \tag{7}
\]

We prove identities (2) and (3) simultaneously by induction. The case of \( n = 0 \) is obvious. Suppose that the assertion has been proven up to \( n - 1 \).

**RHS of (2)**

\[
\begin{align*}
\text{LHS of (2)} & \equiv S(z_{a+b}^n) + \sum_{i=1}^{n-1} (z_a z_b)^i \sum_{j=1}^{n-i} z_{(a+b)j} S(z_a z_b)^{n-i-j} + (z_a z_b)^n \\
& = S(z_{a+b}^n) + \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} z_a (z_a z_b)^{i-1} \sum_{j=1}^{n-i} z_{(a+b)j} S(z_a z_b)^{n-i-j} \\\n& \quad + \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} z_{(a+b)j} S(z_a z_b)^{n-i-j} \\\n& \quad + \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} z_{(a+b)j+a} (z_a z_b)^{i-1} S(z_a z_b)^{n-i-j} + (z_a z_b)^n \\
& \equiv S(z_{a+b}^n) + z_a \sum_{i=1}^{n-1} z_b (z_a z_b)^{i-1} S(z_{a+b}^{n-i}) \\\n& \quad + \sum_{j=1}^{n-j} z_{(a+b)j} S(z_a z_b)^{n-j} \\\n& \quad + \sum_{j=1}^{n-j} z_{(a+b)j+a} (z_a z_b)^{i-1} S(z_{a+b}^{n-i-j}) + (z_a z_b)^n \\
& = S(z_{a+b}^n) + z_a \{S(z_b(z_a z_b)^{n-1}) - z_b(z_a z_b)^{n-1}\} \\
& \quad + \sum_{j=1}^{n-j} z_{(a+b)j} \{S((z_a z_b)^{n-j}) - S(z_{a+b}^{n-j})\} \\
& \quad + \sum_{j=1}^{n-j} z_{(a+b)j+a} S(z_b(z_a z_b)^{n-j-1}) + (z_a z_b)^n \\
& \quad \text{(by induction hypothesis)}
\end{align*}
\]
Hence, (2) is true for \( n \).

**RHS of (3)**

\[
\begin{align*}
\sum_{j=0}^{n-1} z_{(a+b)j} + z_a S((z_a z_b)^{n-j}) = & \sum_{j=1}^{n-1} z_{(a+b)j} S((z_a z_b)^{n-j}) + z_b (z_a z_b)^n \\
= & \sum_{i=0}^{n-1} \sum_{j=1}^{n-i} z_{(a+b)j} \left( z_a z_b \right)^{i-j} S((z_a z_b)^{n-j}) + z_b (z_a z_b)^n \\
= & \sum_{i=0}^{n-1} \sum_{j=1}^{n-i} z_{(a+b)j} \left( z_a z_b \right)^{i-j} S((z_a z_b)^{n-j}) + z_b (z_a z_b)^n \\
= & \sum_{i=0}^{n-1} \left( z_a z_b \right)^{i} S((z_a z_b)^{n-i}) + z_b (z_a z_b)^n \\
= & \sum_{j=1}^{n} z_{(a+b)j} \left( z_a z_b \right)^{n-j} S((z_a z_b)^{n-j}) + z_b (z_a z_b)^n \\
= & \sum_{j=1}^{n} z_{(a+b)j} \left( z_a z_b \right)^{n-j} + z_b (z_a z_b)^n \\
= & S((z_a z_b)^n).
\end{align*}
\]
Therefore, (2) is true for $n$. □

Proof of Theorem B. From (2), we have

$$
\zeta^\ast(3, 1, \cdots, 3, 1) = \sum_{i=0}^{n} \zeta(3, 1, \cdots, 3, 1) \zeta^\ast(4, 4, \cdots, 4).
$$

Hence, we have the assertion by Theorem 2 and Theorem A. □

5 Proof of Theorem C

As in Section 4, we prove the following identities to obtain the explicit evaluations of $\sum_{s_2 n} \zeta^\ast(s_2 n)$.

Theorem 7. For positive integers $a$, $b$, $c$ and nonnegative integer $n$, we have

$$
\sum_{k=0}^{n} S((z_a z_b)^k z_c(z_a z_b)^{n-k}) + \sum_{k=0}^{n-1} S((z_b z_a)^k z_c(z_b z_a)^{n-1-k} z_b)
$$

$$
= 2 \sum_{k=0}^{n} z(a+b)k+c * S((z_a z_b)^{n-k})
$$

$$
- \sum_{k=0}^{n-1} S((z_a z_b)^{n-1-k}) * \left\{ \sum_{i=0}^{k} (z_a z_b)^i z_c(z_a z_b)^{k-i} + \sum_{i=0}^{k} (z_b z_a)^i z_c(z_b z_a)^{k-i} \right\}
$$

and

$$
\sum_{k=0}^{n} S(z_b(z_a z_b)^k z_c(z_a z_b)^{n-k}) + \sum_{k=0}^{n} S((z_b z_a)^k z_c(z_b z_a)^{n-k} z_b)
$$

$$
= 2 \sum_{k=0}^{n} z(a+b)k+c * S(z_b(z_a z_b)^{n-k})
$$

$$
- \sum_{k=0}^{n-1} S((z_a z_b)^{n-1-k}) * \left\{ \sum_{i=0}^{k} z_b(z_a z_b)^i z_c(z_a z_b)^{k-i} + \sum_{i=0}^{k} (z_b z_a)^i z_c(z_b z_a)^{k-i} \right\}.
$$

(We regard summations $\sum_{i=m}^{m-1} \cdots$ as 0.)

Proof. We put

$$
A_{i,j} = (z_a z_b)^i z_c(z_a z_b)^j \quad \text{and} \quad B_{i,j} = (z_b z_a)^i z_c(z_b z_a)^j z_b.
$$
Then we can rewrite (8) and (9) as follows:

\[
\sum_{k=0}^{n} S(A_{k,n-k}) + \sum_{k=0}^{n-1} S(z_a B_{k,n-1-k})
= 2 \sum_{k=0}^{n} z^{(a+b)k+c} S((z_a z_b)^{n-k})
- \sum_{k=0}^{n} S(z^{n-k}_a z^{n-k}_b) \left\{ \sum_{i=0}^{k} A_{i,k-i} + \sum_{i=0}^{k-1} z_a B_{i,k-1-i} \right\}
\]  

(10)

and

\[
\sum_{k=0}^{n} S(z_b A_{k,n-k}) + \sum_{k=0}^{n} S(B_{k,n-k})
= 2 \sum_{k=0}^{n} z^{(a+b)k+c} S((z_a z_b)^{n-k})
- \sum_{k=0}^{n} S(z^{n-k}_a z^{n-k}_b) \left\{ \sum_{i=0}^{k} z_b A_{i,k-i} + \sum_{i=0}^{k} B_{i,k-i} \right\}
\]

(11)

We prove the identities (10) and (11) simultaneously by induction. Before proceeding the proof, by using equation (4), we rewrite the quantities on the LHSs of (10) and (11).

\[
S(A_{k,n-k})
= \sum_{j=1}^{k} \left\{ z^{(a+b)(j-1)+a} S(z_b A_{k-j,n-k}) + z^{(a+b)j} S(A_{k-j,n-k}) \right\}
+ z^{(a+b)k+c} S((z_a z_b)^{n-k})
+ \sum_{j=k+1}^{n} \left\{ z^{(a+b)(j-1)+a+c} S(z_b (z_a z_b)^{n-j}) + z^{(a+b)j+c} S((z_a z_b)^{n-j}) \right\}
\]  

for \(0 \leq k \leq n\),

\[
S(z_a B_{k,n-1-k})
= \sum_{j=1}^{k} \left\{ z^{(a+b)(j-1)+a} S(B_{k-j+1,n-1-k}) + z^{(a+b)j} S(z_a B_{k-j,n-1-k}) \right\}
+ z^{(a+b)k+a} S(B_{0,n-1-k})
+ \sum_{j=k+1}^{n} \left\{ z^{(a+b)(j-1)+a+c} S((z_b z_a)^{n-j} z_b) + z^{(a+b)j+c} S((z_a z_b)^{n-j}) \right\}
\]
for $0 \leq k \leq n - 1$,

$$S(z_b A_{k,n-k})$$

$$= \sum_{j=1}^{k} \left\{ z_{(a+b)(j-1)+b} S(A_{k-j+1,n-k}) + z_{(a+b)j} S(z_b A_{k-j,n-k}) \right\}$$

$$+ z_{(a+b)k+b} S(A_{0,n-k})$$

$$+ \sum_{j=k+1}^{n} \left\{ z_{(a+b)(j-1)+b+c} S((z_a z_b)^{n-j}) + z_{(a+b)j+c} S((z_a z_b)^{n-j} z_b) \right\}$$

$$+ z_{(a+b)n+b+c}$$

for $0 \leq k \leq n$ and

$$S(B_{k,n-k})$$

$$= \sum_{j=1}^{k} \left\{ z_{(a+b)(j-1)+b} S(z_a B_{k-j,n-k}) + z_{(a+b)j} S(B_{k-j,n-k}) \right\}$$

$$+ z_{(a+b)k+c} S((z_b z_a)^{n-k} z_b)$$

$$+ \sum_{j=k+1}^{n} \left\{ z_{(a+b)(j-1)+b+c} S((z_a z_b)^{n-j}) + z_{(a+b)j+c} S((z_b z_a)^{n-j} z_b) \right\}$$

$$+ z_{(a+b)n+b+c}$$

for $0 \leq k \leq n$. Hence, we have

$$\sum_{k=0}^{n} S(A_{k,n-k}) + \sum_{k=0}^{n-1} S(z_a B_{k,n-1-k})$$

$$= \sum_{k=0}^{n} z_{(a+b)k+c} S((z_a z_b)^{n-k})$$

$$+ 2 \sum_{k=0}^{n-1} \sum_{j=k+1}^{n} \left\{ z_{(a+b)(j-1)+a+c} S((z_a z_b)^{n-j}) + z_{(a+b)j+c} S((z_a z_b)^{n-j}) \right\}$$

$$+ \sum_{j=1}^{n} z_{(a+b)j} \left\{ \sum_{k=0}^{n-j} S(A_{k,n-j-k}) + \sum_{k=0}^{n-j-1} S(z_a B_{k,n-j-1-k}) \right\}$$

$$+ \sum_{j=1}^{n} z_{(a+b)(j-1)+a} \left\{ \sum_{k=0}^{n-j} S(z_b A_{k,n-j-k}) + \sum_{k=0}^{n-j} S(B_{k,n-j-k}) \right\}$$

(12)

and

$$\sum_{k=0}^{n} S(z_b A_{k,n-k}) + \sum_{k=0}^{n} S(B_{k,n-k})$$

$$= \sum_{k=0}^{n} z_{(a+b)k+c} S((z_a z_b)^{n-k})$$
\[ + 2 \sum_{k=0}^{n-1} \sum_{j=0}^{n-k} z_{(a+b)(k+j)+b+c} S((z_{a}z_{b})^{n-j-k}) \]

\[ + 2 \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} z_{(a+b)(k+j)+c} S(z_{b}(z_{a}z_{b})^{n-j-k}) + 2z_{(a+b)n+b+c} \]

\[ + \sum_{j=1}^{n} z_{(a+b)j} \left\{ \sum_{k=0}^{n-j} S(z_{b}A_{k,n-j-k}) + \sum_{k=0}^{n-j} S(B_{k,n-j-k}) \right\} \]

\[ + \sum_{j=0}^{n} z_{(a+b)j+b} \left\{ \sum_{k=0}^{n-j} S(A_{k,n-j-k}) + \sum_{k=0}^{n-j-1} S(z_{a}B_{k,n-j-1-k}) \right\} . \tag{13} \]

Now, the case of \( n = 0 \) is obvious. Suppose that the assertion has been proven up to \( n - 1 \).

\[ \sum_{k=0}^{n} z_{(a+b)k+c} S((z_{a}z_{b})^{n-k}) \]

\[ \sum_{k=0}^{n-1} \sum_{j=0}^{n-k} z_{(a+b)(k+j)+a} S(z_{b}(z_{a}z_{b})^{n-k-j}) \]

\[ + \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} z_{(a+b)(k+j)+a+c} S(z_{b}(z_{a}z_{b})^{n-k-j}) \]

\[ = \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} z_{(a+b)k+c} z_{(a+b)(j-1)+a} S(z_{b}(z_{a}z_{b})^{n-k-j}) \]

\[ + \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} z_{(a+b)(j-1)+a+c} S(z_{b}(z_{a}z_{b})^{n-k-j}) \]

\[ + \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} z_{(a+b)k+c} z_{(a+b)j} S((z_{a}z_{b})^{n-k-j}) \]

\[ + \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} z_{(a+b)j} \left( z_{(a+b)k+c} S((z_{a}z_{b})^{n-k-j}) \right) \]

\[ + \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} z_{(a+b)(k+j)+a+c} S((z_{a}z_{b})^{n-k-j}) + z_{(a+b)n+c} \]

\[ \equiv \sum_{k=0}^{n} z_{(a+b)k+c} S((z_{a}z_{b})^{n-k}) \]
+ \sum_{j=1}^{n} z(a+b)(j-1) + \sum_{k=0}^{n-j} z(a+b)k + c \ast S(z_b(z_a z_b)^{n-k-j})
\]

\[
+ \sum_{j=1}^{n} z(a+b)j + \sum_{k=0}^{n-j} z(a+b)k + c \ast S((z_a z_b)^{n-k-j})
\]

\[
+ \sum_{k=0}^{n-1} n-k \sum_{j=1}^{n} \left\{ z(a+b)(k+j-1) + a+c S(z_b(z_a z_b)^{n-k-j}) + z(a+b)(k+j)+c S((z_a z_b)^{n-k-j}) \right\}.
\]

On the other hand,

\[
\sum_{k=0}^{n} S(z_a z_b)^n \ast \left\{ \sum_{i=0}^{k} A_i,k-i + \sum_{i=0}^{k-1} z_a B_{i,k-1-i} \right\}
\]

\[
= S(z_a z_b)^n \ast z_c
\]

\[
+ \sum_{k=1}^{n-1} S(z_a z_b)^{n-k} \ast \left\{ \sum_{i=1}^{k} A_i,k-i + \sum_{i=0}^{k-1} z_a B_{i,k-1-i} \right\}
+ \sum_{k=1}^{n-1} S(z_a z_b)^{n-k} \ast z_c(z_a z_b)^k
\]

\[
+ \sum_{i=0}^{n-1} A_{i,n-i} + \sum_{i=0}^{n-1} z_a B_{i,n-i}
\]

\[
= \sum_{k=1}^{n-1} n-k \sum_{j=1}^{n} z(a+b)j S(z_a z_b)^{n-k-j} \ast \left\{ \sum_{i=1}^{k} A_i,k-i + \sum_{i=0}^{k-1} z_a B_{i,k-1-i} \right\}
\]

\[
+ \sum_{k=0}^{n-1} n-k \sum_{j=1}^{n} \left\{ z(a+b)j S(z_a z_b)^{n-k-j} \ast \left\{ \sum_{i=1}^{k} z_b A_{i-1,k-i} + \sum_{i=0}^{k-1} B_{i,k-1-i} \right\} \right\}
\]

\[
+ \sum_{k=1}^{n-1} n-k \sum_{j=1}^{n} z(a+b)j + a \left\{ S(z_a z_b)^{n-k-j} \ast \left\{ \sum_{i=1}^{k} z_b A_{i-1,k-i} + \sum_{i=0}^{k-1} B_{i,k-1-i} \right\} \right\}
\]
\[
\sum_{k=0}^{n-1} \sum_{j=1}^{n-k} z_{(a+b)j} \left( S(z_{a+b}^{n-k-j}) * z_c(z_a z_b)^k \right) \\
+ z_c \left( z_{(a+b)j} S(z_{a+b}^{n-k-j}) * (z_a z_b)^k \right) + z_{(a+b)j+c} \left( S(z_{a+b}^{n-k-j}) * (z_a z_b)^k \right) \\
+ \sum_{i=0}^{n} A_{i,n-i} + \sum_{i=0}^{n-1} z_a B_{i,n-1-i} \\
= \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} z_{(a+b)j} \left\{ S(z_{a+b}^{n-k-j}) * \left( \sum_{i=1}^{k} A_{i,k-i} + \sum_{i=0}^{k-1} z_a B_{i,k-1-i} \right) \right\} \\
+ \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} z_{(a+b)j} \left( S(z_{a+b}^{n-k-j}) * z_c(z_a z_b)^k \right) \\
+ \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} \left\{ z_c \left( z_{(a+b)j} S(z_{a+b}^{n-k-j}) * (z_a z_b)^k \right) + z_{(a+b)j+c} \left( S(z_{a+b}^{n-k-j}) * (z_a z_b)^k \right) \right\} \\
+ \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} A_{(a+b)j} + \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} \left\{ S(z_{a+b}^{n-k-j}) * \left( \sum_{i=1}^{k} z_b A_{i-1,k-i} + \sum_{i=0}^{k-1} B_{i,k-1-i} \right) \right\} \\
+ \sum_{i=0}^{n} A_{i,n-i} + \sum_{i=0}^{n-1} z_a B_{i,n-1-i} \\
= \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} z_{(a+b)j} \left\{ S(z_{a+b}^{n-k-j}) * \left( \sum_{i=0}^{k} A_{i,k-i} + \sum_{i=0}^{k-1} z_a B_{i,k-1-i} \right) \right\} \\
+ \sum_{j=1}^{n} z(a+b)j \left( S(z_{a+b}^{n-j}) * z_c \right) \\
+ z_c \sum_{k=0}^{n-1} S(z_{a+b}^{n-k}) * (z_a z_b)^k + \sum_{j=1}^{n} z_{(a+b)j+c} \sum_{k=0}^{n-j} S(z_{a+b}^{n-k-j}) * (z_a z_b)^k \\
+ \sum_{k=1}^{n-1} \sum_{j=0}^{n-k} z_{(a+b)j+a} \left\{ S(z_{a+b}^{n-k-j}) * \left( \sum_{i=1}^{k} z_b A_{i-1,k-i} + \sum_{i=0}^{k-1} B_{i,k-1-i} \right) \right\} \\
+ \sum_{i=0}^{n} A_{i,n-i} + \sum_{i=0}^{n-1} z_a B_{i,n-1-i} \\
= \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} z_{(a+b)j} \left\{ S(z_{a+b}^{n-k-j}) * \left( \sum_{i=0}^{k} A_{i,k-i} + \sum_{i=0}^{k-1} z_a B_{i,k-1-i} \right) \right\}
\]
\[ + z_c S((z_a z_b)^n) - z_c (z_a z_b)^n + \sum_{j=1}^{n} z_{(a+b)j + c} S((z_a z_b)^{n-k}) \]

\[ + \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} z_{(a+b)(j-1) + a} \left\{ S(z_{a+b}^{n-k-j}) \ast \left( \sum_{i=0}^{k-1} z_{bA_{i,k-1-i}} + \sum_{i=0}^{k-1} B_{i,k-1-i} \right) \right\} \]

\[ + \sum_{i=0}^{n-1} A_{i,n-i} + \sum_{i=0}^{n-1} z_a B_{i,n-1-i} \]

\[ = \sum_{k=0}^{n} \sum_{j=1}^{n-k} z_{(a+b)j} \left\{ S(z_{a+b}^{n-k-j}) \ast \left( \sum_{i=0}^{k} A_{i,k-i} + \sum_{i=0}^{k} z_{aB_{i,k-i}} \right) \right\} \]

\[ + \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} z_{(a+b)j + c} S((z_a z_b)^{n-j}) \]

\[ + \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} z_{(a+b)(j-1) + a} \left\{ S(z_{a+b}^{n-k-j}) \ast \left( \sum_{i=0}^{k} z_{bA_{i,k-i}} + \sum_{i=0}^{k} B_{i,k-i} \right) \right\} \]

Therefore, we have

\[ 2 \sum_{k=0}^{n} z_{(a+b)k + c} S((z_a z_b)^{n-k}) \]

\[- \sum_{k=0}^{n} S(z_{a+b}^{n-k}) \ast \left\{ \sum_{i=0}^{k} A_{i,k-i} + \sum_{i=0}^{k} z_{aB_{i,k-1-i}} \right\} \]

\[ = \sum_{k=0}^{n} z_{(a+b)k + c} S((z_a z_b)^{n-k}) \]

\[ + 2 \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} \left\{ z_{(a+b)(k+j-1) + a + c} S(z_b (z_a z_b)^{n-k-j}) + z_{(a+b)(k+j) + c} S((z_a z_b)^{n-k-j}) \right\} \]
Hence (10) is true for \( n = (n_1, n_2) \).

By induction hypothesis

\[
\sum_{k=0}^{n-1} \sum_{j=k+1}^{n} z(a+b)(j-1)+a \left\{ \sum_{k=0}^{n-j} S(A_{k,n-j-k}) + \sum_{k=0}^{n-j} S(z_b B_{k,n-j-1-k}) \right\} + \sum_{j=1}^{n} z(a+b)(j-1)+a \left\{ \sum_{k=0}^{n-j} S(z_b A_{k,n-j-k}) + \sum_{k=0}^{n-j} S(B_{k,n-j-k}) \right\} = \sum_{k=0}^{n} z(a+b)(k+c) S((z_a z_b)^{n-k})
\]

(11) is true for \( n \). Next we prove (11) for \( n \).

\[
\sum_{k=0}^{n} z(a+b)(k+c) * S(z_b(z_a z_b)^{n-k})
\]

(12) \[
\sum_{k=0}^{n} z(a+b)(k+c) * \left\{ \sum_{j=0}^{n-k} S((z_a z_b)^{n-k-j}) + \sum_{j=1}^{n} z(a+b)S(z_b(z_a z_b)^{n-k-j}) \right\}
\]

\[
= \sum_{k=0}^{n-1} \sum_{j=0}^{n-k} z(a+b)(k+c) * (z(a+b)j+bS((z_a z_b)^{n-k-j}) + z(a+b)jS(z_b(z_a z_b)^{n-k-j})
\]

\[
+ \sum_{k=0}^{n-1} z(a+b)(n-k) * z_b + \sum_{k=0}^{n-1} \sum_{j=0}^{n-k} z(a+b)(j+k+c) * (z(a+b)j+bS((z_a z_b)^{n-k-j})
\]

\[
+ \sum_{k=0}^{n-1} \sum_{j=0}^{n-k} z(a+b)(j+k+c) * (z(a+b)j+bS((z_a z_b)^{n-k-j})
\]

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On the other hand,

\[ + \sum_{k=0}^{n-1} \sum_{j=0}^{n-k} z_{(a+b)k+c} z_{(a+b)j} S(z_b(z_a z_b)^{n-k-j}) \]

\[ + \sum_{k=0}^{n-1} \sum_{j=0}^{n-k} z_{(a+b)j} (z_{(a+b)k+c} * S(z_b(z_a z_b)^{n-k-j})) \]

\[ + \sum_{k=0}^{n-1} \sum_{j=0}^{n-k} z_{(a+b)(k+j)+c} S(z_b(z_a z_b)^{n-k-j}) \]

\[ + z_{(a+b)n+c} z_b + z_b z_{(a+b)n+c} + z_{(a+b)n+b+c} \]

\[ \sum_{k=0}^{n} z_{(a+b)k+c} S(z_b(z_a z_b)^{n-k}) \]

\[ + \sum_{j=0}^{n-j} z_{(a+b)j+b} \sum_{k=0}^{n-j} z_{(a+b)k+c} * S((z_a z_b)^{n-k-j}) \]

\[ + \sum_{j=1}^{n} z_{(a+b)j} \sum_{k=0}^{n-j} z_{(a+b)k+c} * S(z_b(z_a z_b)^{n-k-j}) \]

\[ + \sum_{k=0}^{n-1} \sum_{j=0}^{n-k} z_{(a+b)(k+j)+b+c} S((z_a z_b)^{n-k-j}) \]

\[ + \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} z_{(a+b)(k+j)+c} S(z_b(z_a z_b)^{n-k-j}) \]

\[ + z_{(a+b)n+b+c}. \]

On the other hand,

\[ \sum_{k=0}^{n} S(z_{a+b}^{n-k}) * \left\{ \sum_{i=0}^{k} z_b A_{i,k-i} + \sum_{i=0}^{k} B_{i,k-i} \right\} \]

\[ = S(z_{a+b}^n) * (z_b z_c + z_c z_b) \]

\[ + \sum_{k=1}^{n-1} S(z_{a+b}^{n-k}) * \left\{ \sum_{i=0}^{k} z_b A_{i,k-i} + \sum_{i=1}^{k} B_{i,k-i} \right\} + \sum_{k=1}^{n-1} S(z_{a+b}^{n-k}) * z_c (z_b z_a)^k z_b \]

\[ + \sum_{i=0}^{n} z_b A_{i,n-i} + \sum_{i=0}^{n} B_{i,n-i} \]

\[ \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} S(z_{a+b}^{n-k-j}) * \left\{ \sum_{i=0}^{k} z_b A_{i,k-i} + \sum_{i=1}^{k} B_{i,k-i} \right\} \]

\[ + \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} z_{(a+b)j} S(z_{a+b}^{n-k-j}) * z_c (z_b z_a)^k z_b + \sum_{j=1}^{n} z_{(a+b)j} S(z_{a+b}^{n-j}) * z_b z_c \]
\[+ \sum_{i=0}^{n} z_b A_{i,n-i} + \sum_{i=0}^{n} B_{i,n-i} = \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} z_{(a+b)j} \left\{ S(z_{a+b}^{n-k-j}) \ast \left( \sum_{i=0}^{k} z_b A_{i,k-i} + \sum_{i=1}^{k} B_{i,k-i} \right) \right\} + \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} z_b \left( z_{(a+b)j} S(z_{a+b}^{n-k-j}) \left( \sum_{i=0}^{k} A_{i,k-i} + \sum_{i=1}^{k} z_a B_{i-1,k-i} \right) \right) + \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} z_{(a+b)j+b} \left( S(z_{a+b}^{n-k-j}) \ast \left( \sum_{i=0}^{k} A_{i,k-i} + \sum_{i=1}^{k} z_a B_{i-1,k-i} \right) \right) + \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} z_{(a+b)j} \left( S(z_{a+b}^{n-k-j}) \ast z_c (z_b z_a)^k z_b \right) + \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} z_c (z_{(a+b)j} S(z_{a+b}^{n-k-j}) \ast (z_b z_a)^k z_b) + \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} z_{(a+b)j} \left( S(z_{a+b}^{n-k-j}) \ast \left( \sum_{i=0}^{k} z_b A_{i,k-i} + \sum_{i=1}^{k} B_{i,k-i} \right) \right) + \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} z_b \left( z_{(a+b)j} S(z_{a+b}^{n-k-j}) \ast z_c (z_b z_a)^k z_b \right) + \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} z_{(a+b)j+b} \left( S(z_{a+b}^{n-k-j}) \ast \left( \sum_{i=0}^{k} A_{i,k-i} + \sum_{i=1}^{k} z_a B_{i-1,k-i} \right) \right) + \sum_{j=1}^{n} \left\{ z_b \left( z_{(a+b)j} S(z_{a+b}^{n-j}) \ast z_c \right) + z_{(a+b)j+b} \left( S(z_{a+b}^{n-j}) \ast z_c \right) \right\} \]
\[ 
\sum_{k=0}^{n-1} \sum_{j=1}^{n-k} z_c \left( z_{a+b} j S(z_{a+b}^{n-k-j}) \ast (z_b z_a)^k z_b \right) \\
+ \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} z_{a+b} j + c \left( S(z_{a+b}^{n-k-j}) \ast (z_b z_a)^0 z_b \right) \\
+ \sum_{i=0}^{n} z_b A_{i,n-i} + \sum_{i=0}^{n} B_{i,n-i} \\
= \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} \left( S(z_{a+b}^{n-k-j}) \ast \left( \sum_{i=0}^{k} z_b A_{i,k-i} + \sum_{i=0}^{k} B_{i,k-i} \right) \right) \\
+ \sum_{j=1}^{n} \left( z_{a+b} j \left( S(z_{a+b}^{n-j}) \ast z_c z_b \right) + z_{a+b} j \left( S(z_{a+b}^{n-j}) \ast z_b z_c \right) \right) \\
+ \sum_{k=1}^{n-1} \sum_{j=0}^{n-k} z_{a+b} j + c \left( S(z_{a+b}^{n-k-j}) \ast (z_b z_a)^k z_b \right) \\
+ \sum_{i=0}^{n} z_b A_{i,n-i} + \sum_{i=0}^{n} B_{i,n-i} \\
= \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} \left( S(z_{a+b}^{n-k-j}) \ast \left( \sum_{i=0}^{k} z_b A_{i,k-i} + \sum_{i=0}^{k} B_{i,k-i} \right) \right) \\
+ \sum_{k=0}^{n-1} \sum_{j=0}^{n-k} z_{a+b} j + c \left( S(z_{a+b}^{n-j-k}) \ast \left( \sum_{i=0}^{n-j} z_b z_a \right)^k - z_c (z_b z_a)^n z_b \right) \\
+ \sum_{i=0}^{n} z_b A_{i,n-i} + \sum_{i=0}^{n} B_{i,n-i} \\
= \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} \left( S(z_{a+b}^{n-k-j}) \ast \left( \sum_{i=0}^{k} z_b A_{i,k-i} + \sum_{i=0}^{k} B_{i,k-i} \right) \right) \\
+ \sum_{k=0}^{n-1} \sum_{j=0}^{n-k} z_{a+b} j + c \left( S(z_{a+b}^{n-k-j}) \ast \left( \sum_{i=0}^{k} A_{i,k-i} + \sum_{i=0}^{k} B_{i,k-i} \right) \right) \\
+ \sum_{i=0}^{n} z_b A_{i,n-i} + \sum_{i=0}^{n} B_{i,n-i} \\
\]
Therefore, we have

\[
\begin{align*}
2 \sum_{k=0}^{n} z(a+b)_{k+c} S(z_b(z_a z_b)^{n-k}) & - \sum_{k=0}^{n} S(z_{a+b}^{n-k}) \left( \sum_{i=0}^{k} z_b A_{i,k-i} + \sum_{i=0}^{k} B_{i,k-i} \right) \\
= \sum_{k=0}^{n} z(a+b)_{k+c} S(z_b(z_a z_b)^{n-k}) & + 2 \sum_{k=0}^{n} z(a+b)_{k+j} + b+c S((z_a z_b)^{n-k-j}) \\
& + 2 \sum_{k=0}^{n} \sum_{j=0}^{n} z(a+b)_{(k+j)+c} S(z_b(z_a z_b)^{n-k-j}) + 2 z(a+b)_{n+b+c} \\
& + \sum_{j=1}^{n} z(a+b)_{j} \left\{ 2 \sum_{k=0}^{n-j} z(a+b)_{k+c} S((z_a z_b)^{n-j-k}) \\
& - \sum_{k=0}^{n-j} S((z_{a+b})^{n-j-k}) \left( \sum_{i=0}^{k} z_b A_{i,k-i} + \sum_{i=0}^{k} B_{i,k-i} \right) \right\} \\
& + \sum_{j=0}^{n} z(a+b)_{j+b} \left\{ 2 \sum_{k=0}^{n-j} z(a+b)_{k+c} S((z_a z_b)^{n-j-k}) \\
& - \sum_{k=0}^{n-j} S((z_{a+b})^{n-j-k}) \left( \sum_{i=0}^{k} A_{i,k-i} + \sum_{i=1}^{k} z_a B_{i-1,k-i} \right) \right\} \\
= \sum_{k=0}^{n} z(a+b)_{k+c} S(z_b(z_a z_b)^{n-k})
\end{align*}
\]
\[+ 2 \sum_{k=0}^{n-1} \sum_{j=0}^{n-k} z_{(a+b)(k+j)+b+c} S((z_a z_b)^{n-k-j})\]
\[+ 2 \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} z_{(a+b)(k+j)+c} S(z_b (z_a z_b)^{n-k-j}) + 2z_{(a+b)n+b+c}\]
\[+ \sum_{j=1}^{n} z_{(a+b)j} \left\{ \sum_{k=0}^{n-j} S(z_b A_{k,n-j-k}) + \sum_{k=0}^{n-j} S(B_{k,n-j-k}) \right\}\]
\[+ \sum_{j=0}^{n} z_{(a+b)j+b} \left\{ \sum_{k=0}^{n-j} S(A_{k,n-j-k}) + \sum_{k=0}^{n-j-1} S(z_a B_{k,n-j-1-k}) \right\}\]
(by induction hypothesis and (10) for \(n)\]
\[= \sum_{k=0}^{n} S(z_b A_{k,n-k}) + \sum_{k=0}^{n} S(B_{k,n-k}).\]

**Proof of Theorem C.** By (8), we obtain
\[\sum_{\vec{s}_2 \in I_{2n}} \zeta^* (\vec{s}_{2n}) = 2 \sum_{k=0}^{n} \zeta(4k + 2) \zeta^*(3,1,\ldots,3,1) - \sum_{k=0}^{n} \zeta^*(4,\ldots,4) \sum_{\vec{s}_2 \in I_{2k}} \zeta(\vec{s}_{2k}).\]
Hence, we have the assertion by Theorem F Theorem H Theorem A and Theorem B.

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**References**

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