Tailored jump operators for purely dissipative quantum magnetism

Hendrik Weimer

Institut für Theoretische Physik, Leibniz Universität Hannover, Appelstr. 2, D-30167 Hannover, Germany

E-mail: hweimer@itp.uni-hannover.de

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Abstract
I propose an architecture for the realization of dissipative quantum many-body spin models. The dissipative processes are mediated by interactions with auxiliary particles and lead to a widely tunable class of correlated quantum jump operators. These findings enable the investigation of purely dissipative spin models, where coherent dynamics is entirely absent. I provide a detailed review of a recently introduced variational method to analyze such dissipative quantum many-body systems, and I discuss a specific example in terms of a purely dissipative Heisenberg model, for which I find an additional disordered phase that is not present in the corresponding ground state phase diagram.

Keywords: open quantum systems, quantum magnetism, quantum phase transitions, quantum simulation

1. Introduction
Emerging quantum technologies such as quantum simulation or quantum metrology inherently require control over nonclassical correlations such as quantum entanglement. For a long time, a central belief has been that in order to realize such technologies, it would be absolutely necessary to perfectly isolate the quantum system of interest from its environment, as otherwise there would be unavoidable decoherence destroying the nonclassical states. However, as has been demonstrated in a series of landmarking theoretical and experimental works [1–7], it is possible to use carefully engineered couplings to the environment to dissipatively prepare interesting quantum many-body states as the stationary states of the evolution of such an open quantum system.

These initial results have sparked a huge interest in the study of dissipative quantum many-body systems. Besides investigation of the aforementioned dissipative quantum state engineering [8–17], there is also a large body of works investigating their fundamental properties. In particular, identifying and understanding phase transitions in dissipative quantum systems has emerged as a central topic in this context [18–38].

The theoretical analysis of dissipative quantum many-body systems is significantly more challenging than for equilibrium systems due to the lack of an underlying concept corresponding to the partition function. Consequently, the available tools have been quite limited. For one-dimensional problems, methods related to the density matrix renormalization group have been successfully used [25, 26, 39–45], however their extension towards higher dimensional states remains an outstanding challenge. In higher dimensions, most analyses have been limited to mean-field treatments [21–23, 30, 31, 46–51], however recent results cast severe doubts on the validity of such a mean-field decoupling [43, 52–54], even in large spatial dimensions. Generic non-equilibrium methods such as the Keldysh formalism can be extended to open systems [29, 55], but are notoriously difficult to treat in the presence of strong interactions [56]. As a result, there is increasing activity to find novel computational approaches that do not suffer from these limitations [57–61]. One promising route is a variational principle for open quantum systems [62], which has given quantitatively reliable results for a variety of different systems [53, 54, 63, 64], and...
which appears to correctly describe phase transitions above the upper critical dimension [65].

The physical realizations that are being discussed in the context of dissipative quantum many-body systems are quite diverse. On the one hand, a wide range of works investigates the coupling of atoms or ions to metastable electronic excitations [19, 32, 34, 66, 67]. On the other hand, there is also a very strong activity in the context of solid state systems, e.g., in the context of exciton-polariton condensates [68–70], solid state cavity arrays [46, 71–74], or nitrogen-vacancy defect centers in diamond [75–78]. A key point in all these investigations is that the steady state of the open quantum system is not a thermal state with respect to the system Hamiltonian. One common way to realize this is to consider an explicit time-dependent driving field, e.g., an external laser, which is why such systems are sometimes referred to as being ‘driven-dissipative’. However, it is important to note that non-Markovian systems offer similar possibilities [79, 80], and hence it is possible to realize interesting dissipative quantum many-body models also without the help of external driving fields.

Nevertheless, it remains a central challenge to realize dissipative processes in a controlled way. In the context of driven-dissipative settings, the dissipation usually arises from coupling the eigenstates of the system to the vacuum of the electromagnetic field, hence the dissipation naturally acts in the eigenbasis of the system. In many cases, however, it would be desirable to engineer other dissipation channels, e.g., for the production of steady state coherence between the eigenstates. While there are proposals to realized largely arbitrary dissipation channels in the framework of universal quantum simulation architectures [3, 15, 81, 82], it is highly desirable to implement tunable dissipation without requiring such a detailed level of control over the setup.

In this paper, I discuss an architecture for the realization of largely arbitrary dissipation channels. The central strategy is to turn coherent interactions with auxiliary particles into effective dissipative elements for the reduced system under consideration. I will discuss the implementation of single-particle and two-particle operators describing dissipative quantum jumps, focusing on the case where the resulting effective dynamics is purely dissipative and coherent dynamics is entirely absent. Crucially, such purely dissipative systems are still fully quantum if the associated quantum jump operators act in different eigenbases [36, 45, 61]. As a specific example, I will focus on dissipative quantum magnetism in the form of a dissipative Heisenberg model. I will analyze the phase diagram of this model using the variational principle for open quantum systems and comment on the differences to the ground state phase diagram.

2. Tailoring quantum jump operators

Here, we are interested in realizing dissipative spin models for an ensemble of atoms or molecules loaded into an optical lattice with uniform filling. We will be investigating a case where the dissipative elements of the dynamics are mediated by a set of auxiliary atoms, see figure 1. The relevant degrees of freedom within the ensemble are also internal degrees of the constituents, e.g., hyperfine spin states. The starting point of the analysis is the microscopic quantum master equation in Lindblad form, which includes both the spin ensemble and the auxiliary atoms, given by

\[ \frac{d}{dt} \rho = -i[H, \rho] + \sum_i \mathcal{D}(c_i). \]  

(1)

where \( \rho \) is the density operator describing the state of the system, \( H \) is the Hamiltonian responsible for the coherent part of the dynamics, and the dissipative terms are given in terms of quantum jump operators \( c_i \) by

\[ \mathcal{D}(c_i) = c_i \rho c_i^\dagger - \frac{1}{2}(c_i^\dagger c_i \rho + \rho c_i^\dagger c_i). \]  

(2)

In the following, we will obtain a reduced description of the dynamics only for the atoms in the spin ensemble.

2.1. Effective operator formalism

The general setup we are considering has dissipative terms only within the auxiliary atoms. Therefore, we aim for an effective description where the dynamics of the auxiliary atoms is integrated out. To be specific, we assume that the auxiliary atoms are two-level systems with one state decaying into the other; a straightforward realization is optical pumping by laser coupling one of the two levels to a fast decaying electronic excitation, see figure 1. Within this assumption, we will employ the effective operator formalism [83] to derive an effective quantum master equation, in which the dynamics of the auxiliary atoms can be factorized out.
Within the effective operator formalism, the resulting master equation is given by

$$\dot{\rho} = -i[H_{\text{eff}}, \rho] + \sum_k D(c_k^{\text{eff}}),$$

with the effective Hamiltonian $H_{\text{eff}}$ and effective jump operators $c_k^{\text{eff}}$ being given by

$$H_{\text{eff}} = H_g - \frac{1}{2} \sum_k V_k (-\hat{A}_k^{-1} + (\hat{A}_k^{-1})^\dagger) V_k^+,$$

$$c_k^{\text{eff}} = \frac{1}{\sqrt{2}} \hat{A}_k^{-1} V_k^+ c_k,$$

Here, $H_g$ is the Hamiltonian within the ground state manifold, i.e., involving only states that are not subject to dissipation. $V_k^+$ and $V_k^-$ are interaction terms that couple an ensemble atom to the decaying state or away from it, respectively. We have also assumed that there is only one decay channel per atom, and that the decay processes occur independently from each other. Finally, $\hat{H}_k$ is the non-Hermitian Hamiltonian for the $k$th auxiliary atom, given by

$$\hat{H}_k = H_k^\dagger - \frac{1}{2} c_k^{\dagger} c_k,$$

with $H_k^\dagger$ being the part of the Hamiltonian that couples only within the manifold of states subject to dissipation. Note that although $\hat{H}_k$ is non-Hermitian, the overall effective Hamiltonian $H_{\text{eff}}$ is always Hermitian.

In the following, we will be interested in the case where the effective dynamics is purely dissipative. To this end, we consider situations where $H_g = 0$, which can be realized by ensuring that there are no interactions within the spin ensemble, while possible effective terms corresponding to local operators can be canceled using suitable local potentials. The contribution to the effective Hamiltonian of the auxiliary atoms vanishes in case the non-Hermitian Hamiltonian $H_{\text{eff}}$ is purely imaginary; this occurs when the two states of the auxiliary atom have the same energy in a suitable rotating frame.

2.2. Single-particle jump operators

As a first step, let us discuss the requirements for the realization of arbitrary single-particle jump operators. To be specific, we will exemplify the proposed method by considering the realization of the jump operator $\ket{-}\bra{+}$, with $\ket{\pm} = (\ket{\uparrow} \pm \ket{\downarrow})/\sqrt{2}$, as this jump operator is not acting within the eigenbasis of the spin ensemble and therefore represents a particularly challenging case.

Without loss of generality, we can assume that the spin up state of the auxiliary atom is decaying into the spin down state, hence the microscopic jump operator is given by

$$c_k = \sqrt{\gamma} \sigma_k^z = \sqrt{\gamma} [\sigma_k^z - i\sigma_k^x]/2$$

in terms of the Pauli matrices $\sigma_k^x$, $\sigma_k^y$, and $\sigma_k^z$, with $\gamma$ being the associated decay rate. Then, we will be interested in the case where the interaction Hamiltonian between the $i$th atom in the spin ensemble and the $k$th auxiliary atoms is given by

$$H_{\text{ik}} = E_0 \ket{-}\bra{+} \sigma_k^x + \text{H.c.} = E_0/2 [\sigma_i^z \sigma_k^x + \sigma_i^x \sigma_k^z],$$

with $E_0$ being an energy scale related to the concrete physical implementation of the interaction. Such tunable spin interactions can be efficiently realized in a wide range of physical systems, most notably Rydberg-dressed atoms [84, 85] and rotational excitations of ultracold polar molecules [86–88].

Within the context of the effective operator formalism, we find that the term containing $\sigma_k^x$ corresponds to $V_k^+$, while its Hermitian conjugate is identical to $V_k^-$. Then, we find for the effective jump operator

$$c_i^{\text{eff}} = E_0 \frac{1}{\sqrt{2}} \ket{\uparrow} \bra{\uparrow} \pm \ket{\downarrow} \bra{\downarrow}.$$

In leading order in $E_0/\gamma$, the auxiliary atoms are confined to the spin down state and hence the auxiliary atom can be factorized out from the dynamics, leaving only the desired jump operator $\ket{-}\bra{+}$.

2.3. Multi-particle jump operators

Now, let us turn to the realization of two-body jump operators. In the same way as the realization of single-particle jump operators requires two-body interactions including one of the auxiliary atoms, two-body jump operators require the presence of three-body interactions. Such three-body terms can arise in a variety of different contexts, e.g.: (i) canceling the two-body interaction leaves three-body interactions as the leading interaction term [89], (ii) Strong blockade effects in Rydberg-dressed atoms lead to higher-order interaction terms [90]. (iii) Time-dependent driving of a system can lead to three-body interactions in the rotating frame of the driving [91]. We would like to point out that adapting the first approach to the field of Rydberg atoms appears to be particularly promising. By a combination of a static electric field and microwave driving close to a resonance between different Rydberg states, it is possible to cancel the two-body interaction on neighboring sites in an optical lattice. The remaining interaction term is then dominated by a three-body interaction, whose strength is generically on the same order as the van der Waals interaction in the absence of the external fields. As many Rydberg blockade experiments have already demonstrated and characterized the relevance of van der Waals interaction [92], this method appears a very promising route to realize three-body interaction potentials. Combination with Rydberg-dressing proposals involving multiple hyperfine ground states [84, 85] will then allow for the implementation of arbitrary three-body spin interactions. However, in the following, we will not make detailed assumptions about the underlying physical mechanism to realize the three-body interaction terms, allowing to retain a general treatment of the emerging dissipative many-body dynamics.

As an example, let us study the implementation of a correlated jump operator of the form $\ket{\psi_k} \langle \psi_i \rangle$, where the states $\ket{\psi_k}$ are the Bell states

$$\ket{\psi_k} = \frac{1}{\sqrt{2}} (\ket{\uparrow \downarrow} \pm \ket{\downarrow \uparrow}).$$

As this jump operator pumps the system towards the
maximally entangled state $|\psi_i\rangle$, it can be seen as a building block for the efficient generation of entangled quantum states using dissipation. In close analogy to the preceding subsection, we can realize such a jump operator with the help of the auxiliary atoms. Specifically, we find this jump operator being realized when the interaction Hamiltonian $H_{ijk}$ between two atoms of the spin ensemble $i$ and $j$ and one auxiliary atom $k$ takes the form

$$H_{ijk} = E_0|\psi_i\rangle\langle \psi_i|\sigma_i^+ + H.c. = E_0/4 [\sigma_i^+\sigma_j^+\sigma_k^- + \sigma_i^+\sigma_j^-\sigma_k^+ - \sigma_i^-\sigma_j^+\sigma_k^- + \sigma_i^-\sigma_j^-\sigma_k^+] .$$

(10)

Calculation of the effective jump operators then leads to the desired result.

3. Variational analysis of purely dissipative Heisenberg models

We will now turn to the analysis of purely dissipative spin models that can be realized using the techniques laid out in the previous section. However, before turning to a detailed analysis of the models, let us go through a detailed introduction to the variational principle presented in [62], which will serve as the basis for our investigation.

3.1. Introduction to the variational principle for dissipative quantum many-body systems

In general, finding the exact solution to the steady state of the quantum master equation is an exponentially hard problem. Therefore, we will aim for an approximate solution that can be calculated efficiently. To this end, we perform a variational parametrization of the density matrix. As the first step, we take a variational basis in terms of tensor products of single atom density matrices. As a further simplification, we assume that all single atom density matrices are identical. Then, we have

$$\rho = \rho_0 \otimes \rho_0 \otimes \cdots$$

(11)

$$\rho_0 = \frac{1}{2} (1 + \alpha_x \sigma_x + \alpha_y \sigma_y + \alpha_z \sigma_z).$$

(12)

Here, $\alpha_x, \alpha_y,$ and $\alpha_z$ are our (real) variational parameters, which we will vary to find the best possible approximation to the steady state. Consequently, we have boiled down the full quantum many-body problem down to the computation of three variational parameters. Note that we have already incorporated the requirement of the trace being identical to one into the construction of $\rho_0$, as the Pauli matrices are traceless. However, the constraint that the density matrix must be non-negative leads to the condition $\alpha_x^2 + \alpha_y^2 + \alpha_z^2 \leq 1$, which has to be accounted for within the numerical optimization.

As the underlying variational principle, we want to minimize a suitable norm $\|d\rho/dt\|$, as the norm going to zero indicates that we have found the true steady state. Here, the suitable norm is given by the trace norm, i.e., $\text{Tr} \|d\rho/dt\|$ [62]. However, computing the variational norm is still an exponentially hard task, requiring the diagonalization of a matrix whose dimension increases exponentially with the size of the system, even for our rather simple ansatz containing only few variational parameters. Therefore, we minimize a slightly different variational functional that provides an upper bound. In the case of our ansatz, this upper bound can be given as

$$\left\| \frac{d}{dt}\rho \right\| \leq \sum \left\| \frac{d}{dt}\rho^{(ij)} \right\| ,$$

(13)

where $d\rho^{(ij)}/dt$ is the time derivative of the reduced system consisting of only two atoms [62]. For system sizes where a direct comparison can be made, taking the upper bound instead of the full problem leads to very similar results [53]. Crucially, the reduced time derivative is given by a $4 \times 4$ matrix, which can be easily diagonalized. In the case of a homogeneous system, all terms of the sum are identical, meaning that we have to minimize the norm of a single $4 \times 4$ matrix.

In the most general case, the contribution to the operator $d\rho^{(ij)}/dt$ consists of three different parts

$$\frac{d}{dt}\rho^{(ij)} = \rho^{(ij)}_{\text{loc}} + \rho^{(ij)}_{\text{int}} + \rho^{(ij)}_{\text{MF}} .$$

(14)

The first term, $\rho^{(ij)}_{\text{loc}}$, refers to terms that act only within the local Hilbert space of a single site. The second term, $\rho^{(ij)}_{\text{int}}$ refers to two-particle operators describing coherent interactions and correlated jump operators that act only on particles $i$ and $j$. These first two terms are identical to the ones found in the microscopic quantum master equation. The third term, $\rho^{(ij)}_{\text{MF}}$ is slightly more subtle and corresponds to a mean-field treatment of the sites surrounding $i$ and $j$, given by

$$\rho^{(ij)}_{\text{MF}} = \sum_k \text{Tr} \{ [H_{ik} + H_{jk}, \rho^{(ij)}_{\text{loc}}] + \mathcal{D}(c_{ik}) + \mathcal{D}(c_{jk}) \},$$

(15)

where $\rho^{(ij)} = \rho^{(i)} \otimes \rho^{(j)}$ is the reduced density operator for three sites. If we compose the interaction term $H_{ik}$ into operators of the form $H_{ik} = A_i \otimes B_{ik}$, we can write their contribution to $\rho^{(ij)}_{\text{MF}}$ as

$$\text{Tr} \{ [H_{ik}, \rho^{(ij)}_{\text{loc}}] \} = \text{Tr} \{ B_i \rho_{ik} [A_i \otimes 1], \rho^{(ij)}_{\text{loc}} \},$$

(16)

which describes a mean-field decoupling of the interaction [93]. Similar expressions can also be found in the case of correlated jump operators acting on multiple sites at the same time.

The sum over $k$ in equation (15) runs over all sites adjacent to $i$ and $j$ and therefore contains $2(z-1)$ terms, where $z$ is the coordination number of the underlying lattice. Crucially, this results in the mean-field term being dominant in the limit of large $z$. If we want to capture, for instance, the changes in phase diagrams with changing $z$, it is therefore convenient to renormalize all coupling constants $\lambda$ corresponding to two-particle operators (both coherent and dissipative) according to $\lambda = \lambda’/z - 1$. Then, the mean-field term remains constant in the large $z$ limit. However, this does not apply to the second term $\rho^{(ij)}_{\text{int}}$, which decays like $1/(z-1)$ in this limit. As this term captures the buildup of correlations, this means that correlations become irrelevant in the limit of large $z$ and the product state solution becomes exact.
The variational approach has proven to be especially successful for the analysis of dissipative phase transitions [53, 62, 65]. The unique advantage is the possibility to interpret the variational norm in close correspondence to the free energy functional for equilibrium systems. Then, in close analogy to Landau theory for equilibrium transitions, one can formally perform a series expansion of the variational norm in the order parameter $\phi$ of the dissipative phase transition. A common situation is that due to symmetry reasons, the odd powers of the expansion vanish and one is left with a $\phi^4$ theory according to

$$
\|\bar{\mu}(\delta)\| = u_0 + u_2 \phi^2 + u_4 \phi^4 + O(\phi^6).
$$

(17)

Such a $\phi^4$ theory has a phase transition at a critical value of $u_2 = 0$, going from a disordered phase with $\phi = 0$ for $u_2 > 0$ to an ordered phase with $\phi = 0$ for $u_2 < 0$. Close to the phase transition, the order parameter behaves according to $\phi \sim u_2^{1/2}$, from which we can identify the critical exponent $\beta = 1/2$. Moreover, we can establish the regime of validity of our product state approach from the known critical properties of $\phi^4$ theories. Above the upper critical dimension $d_u = 4$, the critical behavior is correctly described by the theory. Between the lower critical dimension $d_l = 1$ and $d_u$, critical exponents will be incorrect, but basic properties of the phase diagram remain valid.

### 3.2. Application to purely dissipative Heisenberg models

The Heisenberg model is a paradigmatic model for the study of quantum magnetism. Here, we will be interested in a particular variant of a spin 1/2 model, where the coupling constant along two axes is identical, which is also known as the XXZ model. Then, the Hamiltonian of the equilibrium model is given by

$$
H = -J\sum_{\langle ij \rangle} \sigma_x^i \sigma_x^j + \sigma_y^i \sigma_y^j + (1 - \lambda) \sigma_z^i \sigma_z^j.
$$

(18)

Constructing a dissipative analog of this model can be done in different ways, but in general, the dissipative model should exhibit the same symmetries as the equilibrium one. This means that for $\lambda = 0$, the model exhibits an $SU(2)$ symmetry corresponding to global rotation around any axis, while for $\lambda = 0$ this symmetry is partially lifted and only a $U(1)$ symmetry corresponding to rotations around the $z$ axis remains. Here, we have chosen the interaction in the $XY$ plane to be ferromagnetic, however, an underlying symmetry along the $z$ axis results in the antiferromagnetic case having the same spectrum [94].

The way we construct the dissipative Heisenberg model is to consider two sets of jump operators. The first set of jump operators pumps the system into a dark state where the $SU(2)$ symmetry is spontaneously broken. In the ferromagnetic case, these jump operators have the form

$$
c_3^{(\phi)} = |\downarrow \downarrow \rangle \langle \psi_1 | \downarrow \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

(19)

$$
c_2^{(\phi)} = |\downarrow \downarrow \rangle \langle \psi_2 | \downarrow \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

(20)

$$
c_3^{(\phi)} = |\psi_4 \rangle \langle \psi_4 | \downarrow \rangle = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

(21)

with $|\psi_4 \rangle$ defined as in equation (9). This choice of jump operators ensures that the quantum master equation is $SU(2)$-symmetric, equally pumping the system into the three possible dark states of the ferromagnetic Heisenberg interaction. This is achieved by pumping the antisymmetric state $|\psi_4 \rangle$ into all symmetric states with equal strength. It is straightforward to check that the dark states of this set of jump operators are ferromagnetic states with all spins pointing in the same direction, but the direction itself can lie anywhere on the Bloch sphere. Finally, it is worth mentioning that it is also possible to construct the appropriate jump operators to implement the antiferromagnetic model. However, in the antiferromagnetic case the fluctuations created by the dissipation are so strong that they always drive the system towards an unpolarized state.

The second set of jump operators will now lift the $SU(2)$ symmetry in the same way the symmetry is lifted in the equilibrium model. Here, we consider the set of jump operators given by

$$
c_1^{(\phi)} = |\uparrow \uparrow \rangle \langle \psi_5 | \uparrow \rangle = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

(22)

$$
c_2^{(\phi)} = \sqrt{\lambda} |\uparrow \uparrow \rangle \langle \uparrow \uparrow | \downarrow \rangle = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

(23)

$$
c_3^{(\phi)} = \sqrt{\lambda} |\uparrow \uparrow \rangle \langle \downarrow \downarrow | \uparrow \rangle = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

(24)

$$
c_4^{(\phi)} = \sqrt{\lambda} |\uparrow \uparrow \rangle \langle \downarrow \downarrow | \downarrow \rangle = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},
$$

(25)

Crucially, these jump operators have two antiferromagnetic dark states, $|\uparrow \downarrow \rangle$ and $|\downarrow \uparrow \rangle$. It is straightforward to decompose all jump operators into the required elementary interaction terms as discussed in section 2.3. The complete model is therefore given by the quantum master equation

$$
\frac{d}{dt} \rho = \sum_{\langle ij \rangle} \sum_{\mu=1}^{7} \frac{c_{\mu}^{(\phi)} c_{\mu}^{(\phi) \dagger}}{2} \rho c_{\mu}^{(\phi) \dagger} + c_{\mu}^{(\phi)} c_{\mu}^{(\phi)} + \rho c_{\mu}^{(\phi) \dagger} c_{\mu}^{(\phi)}.
$$

(26)
We will now perform the variational analysis of the dissipative Heisenberg model. Most importantly, there is no Hamiltonian in the problem, hence all the quantum features of the model arise from the non-commuting quantum jump operators. Additionally, all jump operators involve two-site operators, i.e., the $\rho_{\text{loc}}$ term in equation (14) is zero.

We will perform the variational analysis in two parts. First, we investigate the behavior for small values of $\lambda$, where we expect the existence of an XY phase, in which the $U(1)$ symmetry is spontaneously broken. Second, we look at the limit of large $\lambda$, where the model is expected to realize an Ising antiferromagnet with broken $Z_2$ symmetry. To be explicit, we always focus on three-dimensional models on a cubic lattice, as in higher dimensions the product state ansatz of equation (11) becomes more reliable [65]. The minimization of the variational norm is done numerically using standard nonlinear optimization techniques.

As noted before, for $\lambda = 0$ any ferromagnetic state with all spins pointing in the same direction is an exact dark state of the quantum master equation. For any finite value of $\lambda > 0$, however, we find that the variational minimum is realized for $\langle \sigma_z \rangle = 0$, i.e., the ferromagnetic order is confined to the XY plane on the Bloch sphere, thus confirming the expectation of a $U(1)$ phase. The spontaneous magnetization thus simply follows from the length of the spin vector on the Bloch sphere. Investigating the decay of the ferromagnetic order, we find that the $U(1)$ phase breaks down at $\lambda_{c1} = 1/2$, undergoing a continuous transition, see figure 2. Interestingly, in contrast to the equilibrium case [95, 96], we do not find that the XY phase is immediately connected to the antiferromagnet, but instead the transition is to a disordered phase. From the variational analysis, it appears that this novel intermediate phase is a paramagnet, however, ordered phases involving a nonlocal order parameter cannot be ruled out at this point.

I will now turn to the variational analysis for large values of $\lambda$. In the limit $\lambda \to \infty$, the system has two dark states corresponding to the antiferromagnetic ordering of the spins.

![Figure 2](image.png)

**Figure 2.** Critical behavior of the dissipative phase transition close to $\lambda_{c1} = 1/2$ between the XY and disordered phase. The fit represents a critical scaling of the order parameter according to $m \sim (\lambda_{c1} - \lambda)^{\nu/2}$.

![Figure 3](image.png)

**Figure 3.** Critical behavior of the dissipative phase transition close to $\lambda_{c2} = 3/2$ between the Ising antiferromagnet and the disordered phase. The fit represents a critical scaling of the order parameter indicating staggered magnetization according to $m_s \sim (\lambda - \lambda_{c2})^{\nu/2}$.

To incorporate antiferromagnetic ordering into the variational approach it is necessary to consider variational states of the form

$$\rho = \rho_A \otimes \rho_B \otimes \rho_A \otimes \rho_B \otimes \cdots.$$  \hspace{1cm} (27)

The amount of antiferromagnetic ordering is captured in the staggered magnetization $m_s$, which is reduced for finite values of $\lambda$. Again, we find a continuous phase transition at $\lambda_{c2} = 3/2$ to the same disordered phase, see figure 3. For both continuous transitions, the effective critical theory appears to be a $\phi^4$ theory, resulting in the Landau critical exponent of $\beta = 1/2$. The full phase diagram of the dissipative Heisenberg model, including a comparison to the equilibrium phase diagram, is shown in figure 4.

![Figure 4](image.png)

**Figure 4.** Phase diagram for the Heisenberg model. (a) Ground state phase diagram exhibiting a single transition from an XY phase to an Ising antiferromagnet. (b) Phase diagram of the dissipative model showing the presence of an additional disordered phase between the XY ferromagnet and the Ising antiferromagnet.
unit vectors on the Bloch sphere. However, in open systems it is well possible that going from pure states to mixed states leads to a better approximation of the steady state. In this case, the length of the Bloch vector is reduced until it vanishes completely in a disordered phase. This is especially likely if the system is far away from an exact dark state solution, which is the case around $\lambda \approx 1$ for the dissipative Heisenberg model. However, in contrast to the situation in [64], the fully unpolarized state is never an exact steady state of the quantum master equation, leaving the possibility for more exotic ordering mechanisms to be realized within the dissipative Heisenberg model.

4. Summary

I have presented a generic way to realize a large class of dissipative quantum many-body spin models, including purely dissipative model without any coherent dynamics. By performing a variational analysis of a purely dissipative Heisenberg model, I have shown that the steady state phase diagram exhibits an intermediate disordered phase that is not present in the ground state phase diagram. These findings underscore that the behavior of dissipative quantum many-body systems is potentially richer than in equilibrium, and that the recently introduced variational principle is very well suited to treat such dissipative quantum many-body problems. In future developments, it will be possible to adapt the variational principle to nonlocal variational parameters in order to analyze whether the intermediate phase is characterized by an exotic hidden order parameter.

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