The double-trace spectrum of $\mathcal{N} = 4$ SYM at strong coupling

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The spectrum of IIB supergravity on AdS$_5 \times S^5$ contains a number of bound states described by long double-trace multiplets in $\mathcal{N} = 4$ super Yang-Mills theory at large ’t Hooft coupling. At large $N$ these states are degenerate and to obtain their anomalous dimensions as expansions in $1/N$ one has to solve a mixing problem. We conjecture a formula for the leading anomalous dimensions of all long double-trace operators which exhibits a large residual degeneracy whose structure we describe. Our formula can be related to conformal Casimir operators which arise in the structure of leading discontinuities of supergravity loop corrections to four-point correlators of half-BPS operators.

I. INTRODUCTION

Recently much progress has been made in understanding the structure of the spectrum of double-trace operators in $\mathcal{N} = 4$ super Yang-Mills theory at large $\lambda$ and large ’t Hooft coupling $\lambda = g^2 N$ [1]. Based on these results, OPE and bootstrap techniques have been applied in [2] to obtain closed form expressions for supergravity loop corrections of certain holographic correlators, uncovering novel and rich structure (see [3, 4] for related approaches to such loop corrections). Here we complete the picture for the double-trace spectrum and conjecture a general formula for the leading anomalous dimensions of all long double-trace operators of any twist, spin and $su(4)$ representation.

In the regime $N \to \infty$ and $\lambda \gg 1$, the theory is in correspondence with classical IIB supergravity on AdS$_5 \times S^5$ [5]. The graviton and the Kaluza-Klein multiplets are dual to protected half-BPS operators in the $[0, p, 0]$ representation of $su(4)$,

$$O_p = y^i_1 \cdots y^i_p \text{Tr} \left( \Phi_{i_1} \cdots \Phi_{i_p} \right) + \cdots$$ (1)

where $\Phi_{i_1, \ldots, 6}$ are the elementary scalar fields, the complex vector $\vec{y} \in SU(4)/S(U(2) \times U(2))$, and the ellipsis stands for $1/N$-suppressed multi-trace terms (for $p \geq 4$), whose precise nature will be described in Section II.

At leading large $N$ (for any value of $\lambda$) we may consider degenerate long double-trace superconformal primary operators of twist $\tau$, spin $l$ and $su(4)$ labels $[a, b, a]$ of the form

$$O_{pq} = O_p \partial^\tau \Box_{1/4}^{(\tau - p - q)} O_q , \quad (p \leq q) .$$ (2)

The $d$ allowed values of the pair $(p, q)$ run over a set $D_{\tau, l, a, b}^{\text{long}}$. We parametrise this set by $i, r$ as follows:

$$p = i + a + 1 + r , \quad q = i + a + 1 + b - r , \quad i = 1, \ldots , (t - 1) , \quad r = 0, \ldots , (\mu - 1) ,$$ (3)

so that $d = \mu(t - 1)$ with

$$t = (\tau - b)/2 - a , \quad \mu = \left\{ \begin{array}{ll} \frac{|b + 2|}{2} & \text{a + l even}, \\ \frac{|b + 1|}{2} & \text{a + l odd}. \end{array} \right.$$ (4)

The operators $O_{pq}$ are in long multiplets, but in the strict large $N$ limit their dimensions are protected. At order $1/N^2$ they acquire anomalous dimensions and mix amongst themselves and with other long operators. In the supergravity regime $\lambda \gg 1$, operators corresponding to massive string excitations should decouple from the spectrum leaving only those corresponding to supergravity states, e.g. the single-particle states $O_p$ and the two-particle bound states $O_{pq}$. At leading order in large $N$ the $O_{pq}$ just mix amongst themselves to produce the true scaling eigenstates, which we denote by $K_{pq}$. Mixing with higher multi-particle states will only occur at higher orders in the $1/N$ expansion. Analysis of the OPE in the tree-level supergravity regime (see Section III) leads us to the following conjecture, generalising results in [1-3].

Main conjecture. Up to order $1/N^2$, the dimensions of the operators $K_{pq}$ are given by

$$\Delta_{pq} = \tau + l - \frac{2M_{r+1}^{(4)} M_{t+l+1}^{(4)}}{N^2} \frac{2M_{r}^{(4)} M_{t+l}^{(4)}}{(l + 2p - 2 - a - 12(-1/a)^{1/4})} .$$ (5)

Here $(\ldots)_6$ is the Pochhammer symbol, and we define

$$M_{t}^{(4)} \equiv (t - 1)(t + a)(t + a + b + 1)(t + 2a + b + 2) .$$ (6)

Note that for $\mu > 1$ and $t > 2$ some dimensions exhibit a residual degeneracy because they are independent of $q$. We display this property with an illustration of $D_{\tau, l, a, b}^{\text{long}}$.

The dots connected by vertical lines represent operators of common anomalous dimension.
II. HOLOGRAPHIC CORRELATORS

The correlators \( \langle O_{1}O_{2}O_{3}O_{4} \rangle \equiv \langle p_{1}p_{2}p_{3}p_{4} \rangle \) may be written as a free part plus an interacting part, 
\[
\langle p_{1}p_{2}p_{3}p_{4} \rangle = \langle p_{1}p_{2}p_{3}p_{4} \rangle_{\text{free}} + \mathcal{P} \times \mathcal{I} \times \mathcal{H}.
\] (7)

The factor \( \mathcal{P} \) carries the conformal and \( su(4) \) weights and assuming (without loss of generality) \( p_{21} \geq 0, \ p_{43} \geq 0 \) and \( p_{43} \geq p_{21} \), it takes the form 
\[
\mathcal{P} = N^{2} \sum_{p} \frac{p_{1} + p_{3} - p_{4} - p_{2} + p_{43}}{g_{12}} \frac{p_{21} + p_{34}}{g_{14}} \frac{p_{24} + p_{34}}{g_{24}} \frac{p_{1} - p_{21}}{g_{34}},
\] (8)

where \( p_{ij} = p_{i} - p_{j} \) and \( g_{ij} = (y_{i} \cdot y_{j}) / x_{ij}^{2} \). The quantities \( \mathcal{I} \) and \( \mathcal{H} \) are functions of the variables \( x, \bar{x}, y, \bar{y} \), related to the conformal and \( su(4) \) cross-ratios \( u, v, \sigma, \tau \) via 
\[
u = (1 - x)(1 - \bar{x}) = \frac{x_{12}x_{23}}{x_{13}x_{24}}, \quad \sigma = \frac{y_{12}y_{24}}{y_{13}y_{24}}, \quad \tau = (1 - y)(1 - \bar{y}) = \frac{y_{12}y_{24}}{y_{13}y_{24}}.
\]

In terms of these variables we have 
\[
\mathcal{I}(x, \bar{x}, y, \bar{y}) = (1 - x)(1 - \bar{x})/(y\bar{y})^{2}.
\]

The decomposition into free and interacting parts in (7) reflects the property of ‘partial non-renormalisation’ [4], i.e. the statement that all the dependence on the coupling appears in the function \( \mathcal{H} \). Here we consider the leading contribution to \( \mathcal{H} \) at large \( \lambda \). In the OPE of \( \langle O_{1}O_{2}O_{3} \rangle \) and \( \langle O_{3}O_{4} \rangle \), the free term contributes both a protected sector, and a long sector. Identifying the sectors is non-trivial due to possible semisimple multiplet recombination at the unitarity bound [8,9].

At leading order in the \( 1/N^{2} \) expansion, a correlator is determined by disconnected contributions to the free part. These only exist for \( \langle ppqq \rangle \) and cases related by crossing, 
\[
\langle ppqq \rangle = p_{q}p_{q} \left[ 1 + \delta_{p_{1}q_{1}} \left[ \frac{g_{12}g_{24}}{g_{12}g_{24}} \right]^{p_{1}} + \left( \frac{g_{14}g_{24}}{g_{12}g_{24}} \right)^{p_{2}} \right].
\] (11)

At the next order in \( 1/N^{2} \) in the supergravity regime, tree-level Witten diagrams contribute both the free theory connected diagrams and the first contribution to \( \mathcal{H} \).

**Supergravity states and free theory.** It was noticed in [10] that the connected part of \( \langle p_{1}p_{2}p_{3}p_{4} \rangle_{\text{free}} \) generated via tree-level Witten diagrams, disagrees with free theory four-point functions of single trace half-BPS operators. The resolution is that single-particle supergravity states are not dual to single trace half-BPS operators, rather they are uniquely defined as those orthogonal to all multi-trace operators. From this property we can identify multi-trace contributions to \( \text{Tr} \Phi^{p} \) for \( p \geq 4 \). The presence of multi-trace admixtures was also discussed in [11,12]. Consider for example \( O_{4} \), the condition \( \langle O_{4}O_{2}O_{2} \rangle = 0 \) determines, 
\[
O_{4} = y_{14} \ldots y_{14} \text{Tr} (\Phi_{1} \ldots \Phi_{4}) - \frac{2N^{2} - 3}{N(N^{2} + 1)} (O_{2})^{2}.
\] (12)

With this identification of \( O_{4} \) the free theory computation of (2244) agrees with that of supergravity [10]. The correct identification of the operators \( O_{p} \) is also necessary for the ‘derivative relation’ of [13] to hold, as can be directly observed for the cases \( 2 < n < m \).

More generally, connected free theory diagrams where e.g. \( O_{p_{4}} \) is joined only to \( O_{p_{4}} \) (see Fig. 1) are absent. To see this note that at twist \( p_{43} \) in the \( \langle O_{p_{4}}O_{p_{4}} \rangle \) OPE, only a half-BPS operator \( O_{p_{43}} \) of charge \( p_{43} \) could potentially be transferred. By our definition, \( O_{p_{4}} \) is orthogonal to all multi-trace operators and in particular to the double (or higher) trace operator \( [O_{p_{43}}O_{p_{43}}] \). But the vanishing two-point function \( \langle [O_{p_{43}}O_{p_{43}}] \rangle \) is just a non-singular limit of the three-point function, \( \langle O_{p_{43}}O_{p_{43}}O_{p_{43}} \rangle \), which therefore also vanishes. Hence no operator \( O_{p_{43}} \) can be exchanged and the coefficient of the above diagram must vanish. Note that this holds no matter if \( O_{p_{43}} \) is single-trace, multi-trace or a combination thereof. Obviously any topology related by a permutation to Fig. 1 also vanishes.

**Tree level dynamics.** The conjecture of (14) is a simple Mellin integral for the leading term in \( \mathcal{H} \):
\[
\mathcal{H}_{\text{RZ}} = -\frac{N_{p_{1}p_{2}p_{3}p_{4}}}{2} \int dz dw \frac{w}{z} R \left[ \frac{w}{z} \right] \Gamma_{p_{1}p_{2}p_{3}p_{4}},
\]
\[
\Gamma = \Gamma \left[ \frac{p_{1} + p_{2} - z}{2} \right] \Gamma \left( \frac{p_{1} + p_{2} - w}{2} \right) \times \Gamma \left( \frac{p_{1} + p_{2} - p_{4} - w}{2} \right),
\]
\[
\mathcal{R} = \left. \frac{u_{p_{1}p_{2}}}{v_{p_{4}p_{4}}} \sum_{i,j} a_{i,j} \tau^{i} \bar{\tau}^{j} \right| (z - \bar{z} + 2i)^{-1}.
\] (13)

In the sum \( i, j, k \geq 0 \) and we use the notation:
\[
\tilde{\mu} = p_{2} + p_{4} - 2, \quad \tilde{w} = p_{2} + p_{4} - 2,
\]
\[
\tilde{z} = \min(p_{1} + p_{2}, p_{3} + p_{4}) - 2, \quad k = M - 1 - i - j,
\]
\[
M = p_{3} - 1 + \min(0, \Lambda), \quad \Lambda = \frac{p_{1} + p_{2} - p_{4} - p_{4}}{2}.
\] (14)

Finally, the coefficients \( a_{i,j} \) are given by 
\[
a_{i,j} = \left. \frac{2^{2k}M} {(1 + [\Lambda])_{k}((1 + \frac{p_{1} + p_{2}}{2})_{i}(1 + \frac{p_{4} + p_{4}}{2})_{j})} \right|.
\] (15)

The conjecture agrees with all known supergravity computations [13] and refs. therein. The assumptions which led to [15] are spelled out in [12].

Figure 1. A free theory diagram absent from \( \langle p_{1}p_{2}p_{3}p_{4} \rangle \).
Determined $N_{p_1p_2p_3p_4}$ from the light-like limit.
The normalisation $N$ is not determined in [14]. Here we fix it using the following non-trivial statement:

$$\lim_{u,v \to 0} \frac{\langle p_1p_2p_3p_4 \rangle}{\mathcal{P}} \bigg|_{\mathcal{P}} = 0, \quad u/v \text{ fixed.}$$ \hspace{1cm} (16)

The limit $u, v \to 0$ with $(u/v)$ fixed corresponds to taking the points $x_1, x_2, x_3, x_4$ to be sequentially light-like separated.

Examining both the free theory and interacting contributions to the LHS of (16) above, we find that it takes the form $\sum_{r=1}^{M} A_r(u/v)^{\tau}$ where

$$A_r = p_1p_2p_3p_4 \frac{(p_{21} + p_{43} + 2)}{2N^2} - N_{p_1p_2p_3p_4} R_{p_1p_2}^{p_3p_4}. \hspace{1cm} (17)$$

The first term in (17) comes from $\langle p_1p_2p_3p_4 \rangle_{\text{free/}\mathcal{P}}$ and arises from the diagrams in Fig. 2. The normalisation of each of these contributions in the planar limit can be simply obtained by counting the number of inequivalent planar embeddings. Cyclic rotation on each vertex leaves the diagram unchanged, hence the factor $p_1p_2p_3p_4$. Additionally, the diagonal propagators can be drawn inside or outside the square, giving $\frac{1}{2}(p_{21} + p_{43}) + 1$ different possibilities. The multi-trace terms in $\mathcal{P}$ do not affect the leading $N$ result for the diagram. The cases $r = 0$ or $r = M + 1$ correspond to the diagrams of Fig. 1 which are absent as discussed above.

The second contribution in (17), is obtained from $\mathcal{T} \times H_{\text{RZ}}$. Note that each term in $u/v^{\tau} R$ has the form

$$u^{i-\frac{2}{N}} v^{i-\frac{2}{N}} \sigma^{i} r^{j} \hspace{1cm} (18)$$

and upon residue integration will produce a term proportional to $(u/v)^{\tau} r^{j}$. Since $\mathcal{T} = \tau + \mathcal{O}(u, v)$, we find that the contribution to $A_r$ comes from taking the simple poles with $i = 0$ in (17). The residue is

$$R_{p_1p_2}^{p_3p_4} = |\mathcal{A}| \big(\frac{(p_{21} + p_{43})}{2}\big) \big(\frac{(p_{34} + p_{21})}{2}\big) (M - 1)!.$$

Crucially the $j$ dependence cancels between $a_{0jk}/(j!k!)$ and $\Gamma_{p_1p_2p_3p_4}$ and hence $A_r$ is in fact independent of $r$. Now the statement (16) is clearly equivalent to the statement $A_r = 0$ for all $r$. Rearranging (17) we thus obtain the result for $N_{p_1p_2p_3p_4}$,

$$N = \frac{1}{N^2} |\mathcal{A}| \big(\frac{(p_{21} + p_{43})}{2}\big) \big(\frac{(p_{34} + p_{21})}{2}\big) (M - 1)!.$$ \hspace{1cm} (20)

The result combines neatly with the coefficients $a_{ijk}$,

$$N_{p_1p_2p_3p_4} = \frac{1}{N^2} |\mathcal{A}| \big(\frac{(p_{21} + p_{43})}{2}\big) \big(\frac{(p_{34} + p_{21})}{2}\big) (M - 1)! \hspace{1cm} (21)$$

Note that the expression (20) is consistent with the results for $N_{ppq}$ and $N_{p,p+1,q,q+1}$ obtained in [11].

**III. UNMIXING EQUATIONS**

We now describe how the system of relations implied by the OPE describes an eigenvalue problem which allows us to determine the anomalous dimensions of the true double-trace eigenstates $K_{pq}$. In particular, we consider the long multiplet SCPW expansion of the correlators $\langle p_1p_2p_3p_4 \rangle$, in which the pairs $(p_1, p_2)$ and $(p_3, p_4)$ both run over the set $\mathcal{D}_{\text{long}}^{p_1, p_2}$ described in [8]. The result is a symmetric $(d \times d)$ matrix whose partial wave expansion reads

$$\langle p_1p_2p_3p_4 \rangle = \sum_{\tau, l, a, b} A_{\tau l a, b} \frac{1}{\sqrt{\tau}} \log u M^{\tau l}_{a, b} L^{\tau l}_{a, b}.$$

Terms of order $1/N^2$ which are analytic at $u = 0$, i.e. without a factor of $\log u$, have been dropped on the RHS.

The matrix $A^{\tau l}_{a, b}$ in (22) is determined by disconnected free theory and is diagonal due to the form of the disconnected contributions [11]. The matrix $M^{\tau l}_{a, b}$ is ob-
tained from the discontinuity around \( u = 0 \) of \( \mathcal{H}_{RZ} \). For completeness, we recall the explicit expression 10, 17 of a long supermultiplet of twist \( \tau \), spin \( l \) and \( su(4) \) rep \( \mathcal{R} = [n - m, 2m + p_{43}, m - n] \),

\[
\mathbb{L}^{\tau, l}_{\mathcal{R}}(y) = \mathcal{P}(x, \bar{x}, y, \bar{y}) \frac{\mathcal{Y}_{nm}(y, \bar{y}) \mathcal{B}^{\tau, l}(x, \bar{x})}{u^{2l + 4} + p_{43}}.
\]

(23)

This structure is the simplest among the determinantal superconformal blocks [9], since it factorises into an ordinary conformal block \( \mathcal{B}^{\tau, l}(x, \bar{x}) \) [18],

\[
\mathcal{B}^{\tau, l}(x, \bar{x}) = (-t)^{x + l - 1} f_{x+l}(x) f_{x-1}(\bar{x}) - (x \leftrightarrow \bar{x}),
\]

(24)

and an \( su(4) \) block \( \mathcal{Y}_{nm}(y) \) [19],

\[
\mathcal{Y}_{nm}(y, \bar{y}) = -\mathcal{P}_{n+1}(y)\mathcal{P}_m(y) - \mathcal{P}_n(y)\mathcal{P}_{n+1}(\bar{y}),
\]

(25)

where \( \mathcal{Y}_{nm}(y, \bar{y}) \) is a Jacobi polynomial.

The matrices \( \mathcal{A} \) and \( \mathcal{M} \) contain CFT data for the operators \( K_{pq} \):

\[
\mathcal{A}_{a,b}^{\tau, l} = \mathcal{C}_{\tau, l, a, b} \mathcal{C}^T_{\tau, l, a, b},
\]

\[
\mathcal{M}_{a,b}^{\tau, l} = \mathcal{C}_{\tau, l, a, b} \cdot \eta \mathcal{C}^T_{\tau, l, a, b}.
\]

(26)

Here the \((d \times d)\) matrix \( \mathcal{C} \), indexed by pairs \((p_1, p_2)\) and \((q_1, q_2)\) running over \( D_{\tau, l, a, b} \), is given by

\[
\mathcal{C} = \left\{ \left( \mathcal{O}_{p_1}, \mathcal{O}_{p_2} \right) K_{q_1, q_2} \right\},
\]

(27)

and \( \eta = \text{diag}(\eta_{pq}) \) is a \((d \times d)\) diagonal matrix where \( \eta_{pq} \) is (half) the anomalous dimension of the operator \( K_{pq} \) for \((p, q) \in D_{\tau, l, a, b} \).

\[
\Delta_{pq} = \tau + l + \frac{2}{N} \eta_{pq} + O(1/N^4).
\]

(28)

The eigenvalue problem [20] is well defined as a consequence of the equality:

\[
\left\{ \text{# independent entries of } \mathcal{A} \& \mathcal{M} \right\} = \left\{ \text{# of } \left( \mathcal{O}_{p_1}, \mathcal{O}_{p_2} \right) K_{pq} \right\} + \eta_{pq} \}
\]

(29)

Let us comment on the structure of the matrices \( \mathcal{A} \) and \( \mathcal{M} \). The SCPW expansion of disconnected free theory has the following compact expression:

\[
\mathcal{A}_{a,b}^{\tau, l} = \text{diag} \left( \mathcal{F}_{1+a+i+r,b-2r,a,l+a+r} \right)_{1 \leq i \leq (r-1)},
\]

where the function \( \mathcal{F} \) is given by

\[
\mathcal{F}_{p,h,m,a,s} = \frac{p+h(1+h+a)(1+a)(2m+2+h+a)(1+(1+2s+2h)(p-1-m)(p-2-m-a)(p+m+a))}{(p+m+a)(p+m+h+a)(p+m+a)(p+m+a)} \times \frac{(s+m)(s+m+1+a)(s+1+a)(s-m-a)}{(s+m+1+a)(m+1+a)}. \]

\[
\Pi_{s} = \Pi_{s+1},
\]

(30)

The SCPW of matrix elements in \( \mathcal{M}_{a,b}^{\tau, l} \) has the form

\[
\frac{(t+1+i+a+r+p_{43}+p_{43})!(t+1+i+a+r+p_{43})!}{(2l+4+4p_{43})!} \mathcal{P}_{d}(l)
\]

(31)

where \( \mathcal{P}_{d}(l) \) is a polynomial in \( l \) of degree \( d = \min(p_1 + p_2, p_3, p_4) - (p_3 - p_2) + 4 \), and \( r \) labels \((p_3, p_4)\). We determine this polynomial case-by-case, and solve the eigenvalue problem following [13]. We have verified that our conjecture [3] holds systematically in the \( su(4) \) channels \([a, b, a] \) with \( 0 \leq a \leq 3, 0 \leq b \leq 6 \) up to twist 24 for both even and odd spins. In particular, we have been able to perform non-trivial tests on the pattern of residual degeneracies. It would be fascinating to understand how higher order corrections might lift the pattern of residual degeneracies observed at order \( 1/N^2 \).

IV. CASIMIR OPERATORS

Quadratic and quartic conformal Casimir operators have played a useful role in understanding and simplifying the structure of correlators 3, 5, 20. Here we extend the analysis of 3 to all \( su(4) \) channels \([a, b, a] \) of any correlator \((p_1, p_2, p_3, p_4)\). The quadratic and quartic Casimirs are given by

\[
D_{2}^{p_1, p_2} = D_{1}^{p_1, p_2} + 2 \frac{x \bar{x}}{x - \bar{x}} \left( (1 - x) \partial_x - (1 - \bar{x}) \partial_{\bar{x}} \right),
\]

\[
D_{4}^{p_1, p_2} = \left( \frac{x \bar{x}}{x - \bar{x}} \right)^2 D_{1}^{p_1, p_2} \left( \frac{x \bar{x}}{x - \bar{x}} \right)^{-2} D_{1}^{p_1, p_2}.
\]

(32)

where \( D_{2}^{p_1, p_2} = D_{1}^{p_1, p_2} \pm 2 b_{p_1, p_2} \) and

\[
D_{2}^{p_1, p_2} = x^2 \partial_x (1 - x) \partial_x - (p_1 + p_2) x^2 \partial_x - p_1 p_2 x^2.
\]

(33)

The labels \( p_r \) are given by \( p_1 = -\frac{1}{2} p_{21}, p_2 = \frac{1}{2} p_{34} \). The eigenvalues of \( D_2 \) and \( D_4 \) on \( B(2^3 \mid 3) \) are

\[
\lambda_2 (\tau, l) = \frac{1}{2} (l(l + 2) + (\tau + l)(\tau + l + 4))
\]

\[
\lambda_4 (\tau, l) = l(l + 2)(\tau + l + 1)(\tau + l + 3).
\]

(34)

Consider the combination of Casimirs

\[
\Delta^{(8)} = -\frac{1}{8} \left( D_{4}^{p_1, p_2} - (D_{2}^{p_1, p_2})^2 + g_{1}^{a,b} D_{1}^{p_1,p_2} - g_{2}^{a,b} \right) \times \left( D_{4}^{p_1, p_2} - (D_{2}^{p_1, p_2})^2 + g_{3}^{a,b} D_{2}^{p_1, p_2} - g_{4}^{a,b} \right)
\]

(35)

with the coefficients \( g_{i}^{a,b} \) given by

\[
g_{1}^{a,b} = (b + 2a)^2 + 6(b + 2a) + 6,
\]

\[
g_{2}^{a,b} = \frac{1}{4}(b + 2a)(b + 2a + 2)(b + 2a + 4)(b + 2a + 6),
\]

\[
g_{3}^{a,b} = (b^2 + 2b - 2),
\]

\[
g_{4}^{a,b} = \frac{1}{4}(b - 2)(b + 2)(b + 4).
\]

(36)

The operator \( \Delta^{(8)} \) has the property that its eigenvalue on the conformal blocks reproduces exactly the numerator of the anomalous dimensions given in equation [5], i.e.

\[
\Delta^{(8)} B(2^3 \mid 3) = -2 M_4^{(4)} M_{t+1}^{(4)} B(2^3 \mid 3).
\]

(37)
The operator $\Delta^{(8)}$ greatly simplifies the sums which compute the leading discontinuities of a correlator to any loop order. In a large $N$ expansion we have

$$\mathcal{H} = \sum_{k \geq 1} \frac{1}{N^{2k}} \sum_{r=0}^{k-1} \frac{1}{r!} (\log u)^r \sum_{m \leq n} Y_{mn} \mathcal{H}_{r,n,m}^{(k)}. \quad (38)$$

Then the leading discontinuity $\mathcal{H}_{k,n,m}^{(k)}$ in an $su(4)$ channel with $a = n - m$ and $b = 2m - p_43$ is given by

$$\mathcal{H}_{k,n,m}^{(k)} = \sum_{\tau, l, a, b, q_1, q_2} \left( \frac{\tau_{l,a,b}}{q_1 q_2} \right)^k C_{q_1 q_2} \frac{B(q_{l+1})}{u^{2+q_{l+1}}}, \quad (39)$$

with $C_{q_1 q_2} = \langle O_{p_1} O_{p_2} K_{q_1 q_2} \rangle \langle O_{p_3} O_{p_4} K_{q_1 q_2} \rangle$. Since the numerator of the anomalous dimensions does not depend on $(q_1, q_2)$, we may pull out $(k-1)$ factors of $\Delta^{(8)}$ and remove $(k-1)$ powers of the numerator from the anomalous dimension. These reduced sums are considerably simpler. Indeed the resummed result for general $k$ is of a similar complexity as the $k = 1$ case (the $\log u$ coefficient of the tree-level supergravity result). One can then recover the full leading discontinuity by applying $\Delta^{(8)}(k-1)$ times to the resummed expression.

For concreteness, let us consider the simplest example: $p_1 = 2$, for which we have $\rho_1, \rho_2 = 0$ and the only $su(4)$ channel for long multiplets is the singlet $a, b = 0$. The $(\log u)^2$ term of the $(2222)$ correlator was computed at one loop in \[2\] (and recently reproduced using $\Delta^{(8)}$ in \[5\]). With the aid of $\Delta^{(8)}$ one can produce a closed formula for the highest transcendental weight part ($w_1$) of the leading $(\log u)^k$ discontinuity for any loop order:

$$\mathcal{H}_{k,n,m}^{(k)}_{\text{top}} = \frac{1}{u^2} (\Delta^{(8)})^{k-1} \left[ \frac{G_k(x, \bar{x}) - v^7 G(x', \bar{x}')}{(x - \bar{x})^7} \right].$$

$$G(x, \bar{x}) = a_k(x, \bar{x}) \sum_{a_i = 0,1} [H_{a_1, a_2, a_3, a_4}(x) - (x \leftrightarrow \bar{x})]. \quad (40)$$

Here $x' = \frac{x + \bar{x}}{2}$ and $H_{a_1, a_2, a_3, a_4}(x)$ are harmonic polynomials of weight $k$ \[22\]. Finally, the coefficient polynomial for the case $(2222)$ is given by

$$a_k(x, \bar{x}) = -2^{7-3k} 3^{1-k} u^4 \left[ 2^4 \left( \hat{u} + v \right) \left( \hat{u}^2 + 8 \hat{u}v + (v + 6) \right) \right. \left. - 6 \left( \hat{u}^3 + 7 \hat{u}^2 v + 3 \hat{u}v + 2 - (v - 4)v^2 \right) \right. \left. + 3 \hat{u}^2 - k \left( \hat{u}^3 - 3 \hat{u}v^2 + 3 \hat{u} + 2 \hat{u}v + v^3 \right) \right], \quad (41)$$

with $\hat{u} = u - 1$. Similar results have been obtained for the correlators $(2233)$, $(2323)$, and $(3333)$, for which the quantum numbers $(\rho_1, \rho_2)$ and $(a, b)$ of $\Delta^{(8)}$ are non-trivial.

We believe that the results on the anomalous dimensions \[3\] together with the Casimir operators \[35\] will aid in the construction of one-loop supergravity (i.e. order $1/N^4$) contributions to all correlators $(p_1 p_2 p_3 p_4)$. It would be fascinating to see if the methods described in \[22\] can be used to make contact with such supergravity loop corrections and the spectrum results described here.

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