Quantifying the levitation picture of extended states in lattice models

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The behavior of extended states is quantitatively analyzed for two dimensional lattice models. A levitation picture is established for both white-noise and correlated disorder potentials. In a continuum limit window of the lattice models we find simple quantitative expressions for the extended states levitation, suggesting an underlying universal behavior. On the other hand, these results point out that the Quantum Hall phase diagrams may be disorder dependent. PACS number(s) 73.43.Nq, 72.15.Rn, 71.30.+h

I. INTRODUCTION

In the past few years increasing attention has been payed to the challenge of connecting two important limits for non interacting two-dimensional (2D) disordered systems: the existence of extended states at the center of broadened Landau bands in the integer Quantum Hall (QH) regime and the prevailing view of localization of all electronic states, according to the scaling theory of localization. The first attempt at unifying both limits appeared in the form of a conjecture, proposed independently by Laughlin and Khmelnitskii: the extended states at the center of the Landau bands should float up (or levitate) in energy above the Fermi level with decreasing magnetic field or increasing disorder.

A landmark in the history of this problem was the proposal of the Global Phase Diagram (GPD) of the integer QH effect, which is based on the levitation conjecture. Initially performed experiments could verify transitions from the first QH plateau, $\nu = 1$ (or $\nu = 2$ in the case of non polarized spin systems), to the insulator state, according to the Diagram prediction. Several recent experiments, however, show evidences of direct transitions extending states has been investigated by perturbative approaches [8–10], as well as, by several numerical works based on lattice models [11–18]. The perturbative approaches identify weak levitation regimes in the strong magnetic field limit. The scenario of numerical works is controversial, including a non-float-up picture, where the extended states are supposed to disappear at finite $B$ or disorder strength [11,12]. Other works show evidences of incipient floating up for white-noise disorder [13,14]. Some recent works have considered lattice models with correlated disorders [17,18] arguing that correlations would extended the floating up process to lower magnetic fields, smoothing out lattice effects. Nevertheless, a full microscopic understanding of the levitation is still lacking and, moreover, a consensual quantitative description of how this effect takes place is not available.

The aim of this work is to go a step forward in the direction of such quantitative description. The problem is treated within the framework of a 2D tight-binding lattice, and both white-noise and correlated disorder potentials are investigated. With a careful emulation of the continuum limit of the lattice model, the dependence of the extent states levitation on the disorder potential landscape, magnetic flux and Landau level index, could be mapped out, for an energy range within the corresponding Landau bands, but far beyond perturbative limits. We could find an universal quantitative relation describing the levitation of extended states as a function of the relevant parameters of the problem.

II. LATTICE MODEL CALCULATION

For sake of completeness we briefly describe the model Hamiltonian for a square lattice of $s$-like orbitals, with nearest-neighbor interactions only:

$$H = \sum_i \varepsilon_i c_i^\dagger c_i + \sum_{<i,j>} V(e^{i\phi_{ij}} c_i^\dagger c_j + e^{-i\phi_{ij}} c_j^\dagger c_i) \quad (1)$$

where $c_i$ is the fermionic operator on site $i$. The magnetic field is introduced by means of the phase $\phi_{ij} = 2\pi(e/h) \int_0^L A \cdot dl$ in the hopping parameter $V = 1$. In the Landau gauge, $\phi_{ij} = 0$ along the $x$ direction and $\phi_{ij} = \pm 2\pi(x/a)\Phi/\Phi_0$ along the $y$ direction, with $\Phi/\Phi_0 = Ba^2 e/h$ (a is the lattice constant). Disorder is introduced by assigning random fluctuations to the orbital energy $\varepsilon_i$. Two different kinds of disorder are considered: in the white-noise case these energies are uncorrelated, taking $\varepsilon_i \leq |W/2|$. In the correlated disorder model, a gaussian correlation $\varepsilon_i = \frac{1}{\sqrt{2\pi} \lambda} \sum_j \varepsilon_j e^{-|R_i - R_j|^2/\lambda^2}$ is assumed, with correlation length $\lambda$ and $\varepsilon_j \leq |W/2|$. We consider unit cells of $40\times40$ sites with periodic boundary conditions.

The degree of localization of the states is evaluated by means of the Participation Ratio (PR):

$$PR = 1/(N \sum_{i=1}^N |\Psi_i|^4) \quad (2)$$

where $N$ is the number of lattice sites and $\Psi_i$ is the amplitude of the normalized wavefunction on site $i$. Extended...
states can be identified by well resolved peaks in the PR as a function of energy.

Lattice models have to be used very carefully: one has to know how lattice and size effects may hinder valid conclusions for the continuum limit (which should be described by the effective mass approximation), where the actual physical situation takes place. This can only be warranted for low magnetic flux values through the lattice unit cell and for the lowest few Landau levels, which constitutes, indeed, the continuum limit of the Hofstadter spectrum [21]. However, at low magnetic fluxes, it is expected that lattice effects start to manifest on the localization character of the states for sufficiently strong disorder: states with negative Chern numbers moving down from the band center eventually annihilate extended states related to the lowest Landau levels [18]. In this way, we consider only the results for the lowest Landau bands for a correlated disorder potential [22].

### III. RESULTS AND DISCUSSION

#### A. Density of states and localization of states

The Participation Ratio is a very intuitive quantity for discriminating the degree of localization of electronic states [14]. In the context where quantizing magnetic fields are present, the PR has shown to be a good tool in the identification of the extended states positions, compared to other approaches like the Thouless number [21]. For well separated Landau bands (small disorder strength or high magnetic fields), we have observed the peaks in the PR occurring exactly at the center of each band. However, decreasing the magnetic flux or increasing the disorder, deviations of the PR peaks to the high energy side of the bands, could be progressively observed, characterizing the floating up of delocalized states. Fig. 1 is an example of our calculations, showing the density of states (DOS) and the PR for the lowest two broadened Landau bands, for a correlated disorder potential in which $W/V = 4.0$ and $\lambda = 1.5a$. The energy floating up of the $1^{st}$ Landau band extended states is indicated by $\delta E_0$. It’s important to note here that $\delta E$ deviations are already observed before the Landau bands superpose. For all calculations presented throughout this work, the DOS and the PR are averages over 100 disorder realizations.

A wide range of disorder and flux parameters has been analyzed, for white-noise disorder and for various correlation lengths in the described correlated disorder model. In a previous work [22], we have discussed the main differences in the DOS and in the qualitative character of the localization of states, as a function of modifications in the potential landscape. For white-noise disorder, the Landau bands show a constant broadening and are fairly Gaussian-like. In Fig. 1, the broadening (taken at half height) $\Gamma_0$, is defined for the lowest band as an illustration. It also can be seen that as soon as a finite correlation length is introduced, $\Gamma$ shrinks with increasing Landau level index [23]. Other important feature observed is that as the potential is smoothed, the line-shape of the DOS changes, becoming perfectly fitted by Gaussian curves (the sum of dotted gaussian fits coincides with the calculated DOS in the scale of Fig. 1). This observed dependence may contribute to resolve the discrepancies reported in the literature [21] about the exact form of the Landau level DOS.

#### B. Levitation in the lowest landau level

The results obtained from varying the disorder intensities, for fixed magnetic fluxes ($\Phi/\Phi_0 = 1/20$), and following the levitation of extended states of the $1^{st}$ Landau band are summarized in Fig. 2. The normalized shifts, $\delta E_0/\hbar \omega_c$, are plotted as a function of $\hbar \omega_c / \Gamma_0$, the ratio between the energy separation of Landau levels and the band width. It should be noticed that $\Gamma$ is a linear function of the disorder amplitude $W/V$, the actual input in the simulations. Each curve in Fig. 2 corresponds to a different disorder potential: circles, fitted by the darkest line, are for the white-noise case; the other lines are for different correlation length cases. These re-
sults represent our first quantification of the levitation process: in all cases shown, the fittings are of the form 
\[ \delta E_0/\hbar \omega_c = \alpha (\hbar \omega_c/\Gamma_0)^{-2} \]
The white-noise case is a clear upper limit for the levitation of the lowest extended state, with \( \alpha_{wn} = 0.37 \). Increasing the disorder correlation length decreases the energy shift and \( \alpha \) becomes linearly dependent on the length scale \( l_B/\lambda \), where \( l_B = \sqrt{\hbar/eB} \) is the magnetic length (inset of Fig. 2). This linear behavior begins to saturate for \( l_B/\lambda > 2 \), since in this limit there is no distinction between the correlated and white noise potential landscapes concerning the extended states.

![FIG. 2. Extended states shift vs. \( \hbar \omega_c/\Gamma_0 \). Circles and dark line are for the white-noise case. The other symbols, fitted by light continuous lines are for correlated disorder cases, progressively deviating from the white noise case: \( \lambda = 1.0a \), \( \lambda = 1.5a \), \( \lambda = 2.0a \) and \( \lambda = 3.0a \), respectively. The inset shows the dependence of the levitation with \( l_B/\lambda \) (see text).](image)

For the white-noise case, the equivalence between decreasing the magnetic field and increasing disorder on the floating up of the extended states has been established \[22\]: results for different magnetic fluxes and different disorder amplitudes collapse on the same curve. For finite correlation lengths, however, this equivalence does not hold anymore, since a new length scale, \( l_B/\lambda \), is introduced. In this way, when \( W/V \) is varied for other flux values, different \( \alpha \) from curve fittings are obtained.

Having this length scale, \( l_B/\lambda \), in mind, a general relation for the floating up of the extended states can be recovered for all magnetic fields and all correlation lengths. As can be seen in Fig. 3, the extended state energy shifts collapse on:

\[ \frac{\delta E_0}{\hbar \omega_c} = \beta \frac{l_B}{\lambda} \left( \frac{\Gamma_0}{\hbar \omega_c} \right)^2 \]  
(3)

where \( \beta \approx 0.17 \). These results indicate that the levitation of the first extended state follows a well defined and universal behavior. It is expected that for small correlation lengths (\( l_B/\lambda > 2 \)) a white-noise like potential landscape is recovered. Indeed, eq.(3) holds for all smoothened disorder potentials until \( \beta l_B/\lambda = \alpha_{wn} \) (i.e, \( l_B/\lambda \approx 2.1 \)), and for \( l_B/\lambda > 2 \) extended states show maximum levitation, according to \( \delta E_0/\hbar \omega_c = \alpha_{wn} (\Gamma_0/\hbar \omega_c)^2 \). A slight saturation of the levitation can be observed for \( \lambda = 3a \) in Fig. 3, indicating that size effects may become important for longer correlation lengths.

![FIG. 3. The overall dependence of \( \delta E_0/\hbar \omega_c \) varying the disorder amplitude and magnetic fluxes, for disorder systems with different correlation lengths. Squares are for \( \lambda = 1.0a \); diamonds for \( \lambda = 1.5a \); black circles for \( \lambda = 2.0a \); and stars for \( \lambda = 3.0a \). All these results are for \( \Phi/\Phi_0 = 0.05 \). Open triangles are for \( \lambda = 2.0a \) and \( \Phi/\Phi_0 = 0.025 \).](image)

**C. Levitation in higher Landau levels**

The evolution of the extended states related to higher Landau level could also be followed, within the same procedure described for the lowest one. A particular example is shown in Fig. 4, where the energy shifts for the lowest and the second extended states were calculated for five different disorder amplitudes, \( W/V \). It should be noticed that for any given disorder \( \Gamma_1 < \Gamma_0 \), but even taking this into account for the analysis, the levitation is always less pronounced in the second Landau level. It is not shown here, but we verified a levitation even smaller for the third band, and so forth. This evidence that the levitation is reduced as \( N \) increases, is in opposition to the original levitation conjecture \[24\] and to perturbative calculations \[9\].

Although the curve for levitation in the \( N = 1 \) level is clearly separated from that for \( N = 0 \) in Fig. 4, a similar dependence of the energy shift with the energy scale ratio has been found: \( \delta E_1/\hbar \omega_c = \alpha_1 (\hbar \omega_c/\Gamma_1)^{-2} \). However, to obtain a general expression like eq.(3) valid for \( N > 0 \), we have found that a new quantity has to be considered, namely the ratio between the widths of different Landau
bands: $\Gamma_N/\Gamma_0 \leq 1$.

Considering only the lowest and the second extended state, the ratio $\delta E_1/\delta E_0$ as a function of $\lambda/l_B$ is represented in the inset of Fig. 4. An upper limit of equally intense levitation occurs for the white-noise case and a clear minimum for this ratio is seen at a finite correlation length. The evolution of $\delta E_1/\delta E_0$ is qualitatively similar to the $\Gamma_1/\Gamma_0$ variation with the correlation length, as discussed by Ando and Uemura [23]. In fact it is verified that $\delta E_1/\delta E_0 \propto (\Gamma_1/\Gamma_0)^2$. Having these results in mind, the energy shifts of the extended states, a generalization of eq.(3) is possible, valid now for any Landau level index $N$:

$$\frac{\delta E_N}{h\omega_c} = \beta \frac{l_B}{\lambda} \left( \frac{\Gamma_N}{h\omega_c} \right)^2 \left( \frac{\Gamma_N}{\Gamma_0} \right)^2. \quad (4)$$

This is illustrated in Fig. 5, where energy shifts for the second and third Landau levels are included. Considering the ratio $(\Gamma_1/\Gamma_0)^2$, these shifts for higher Landau level index all collapse on the same line obtained for the lowest one (represented by the the dashed line, for comparison).

![Graph](https://example.com/graph.png)

**FIG. 4.** $\delta E_N/h\omega_c$ as a function of $h\omega_c/\Gamma_N$ for the first (circles) and second (squares) extended states for a correlation length $\lambda = 1.5a$ and $\Phi/\Phi_0 = 0.05$. Inset: ratio between the levitation of the second extended state and the lowest one, $\delta E_1/\delta E_0$, for the different calculated cases.

The importance of the present quantification of the levitation also lies in the fact that comparisons with other than numerical approaches, start to become possible. The levitation in energy of the extended states is defined by a length scale, $l_B/\lambda$, and an energy scale, $\Gamma/h\omega_c$. In this way, concerning the length scale, the weak levitation limit result of Fogler [10] reminds our eq.(4). On the other hand, the present energy scale dependence can be identified in the perturbative approach of Haldane and Yang [3] and even in the original levitation conjecture [3]. However, a strict comparison is still not possible. As we have seen, $\Gamma$ defines the broadening of the Landau band and is disorder and Landau level index dependent. A clear connection to the single particle relaxation time used in ref. [1] is therefore not available. The other approaches [3,10] rely on the scattering time, $\tau$. Although both quantities, $\Gamma$ and $\tau$, are related, they are not equivalent [23,25].

The continuum limit of the lattice model connects to the effective mass approximation by a tight-binding parameterization emulating the effective mass of an electron: $m^* = h^2/(2|V|^2)$. This parameterization points out the possibility of direct comparison with experimental results by further increasing the system size. Such comparisons should take into account independent measurements of the DOS [24], since the present work reveals a correlation between the general features of the DOS and the qualitative character of the levitation of extended states.

![Graph](https://example.com/graph.png)

**FIG. 5.** General scaling law, describing the overall dependence of $\delta E_N/h\omega_c$ on the relevant parameters of the problem. Results for $N=1$ and $N=2$ Landau levels. Dashed line is the relation obtained for the lowest level. Second Landau level ($N=1$): squares are for $\lambda = 1.0a$; diamonds for $\lambda = 1.5a$ and black circles for $\lambda = 2.0a$. Third Landau level ($N=2$): open circles are for $\lambda = 1.0a$ and stars for $\lambda = 1.5a$.

**IV. FINAL REMARKS**

In conclusion, we verified that the relevant parameters to map the behavior of the levitation are: first, the energy scales ratio, $\Gamma/h\omega_c$; and secondly, the length scales ratio $l_B/\lambda$; while dependence with the Landau level index can be scaled by the Landau band widths ratio. The important aspect presented here is that the levitation can be described by a simple scaling expression, eq.(4), valid for a wide parameter region, where we can assure that a lattice model emulates the continuum model. For the system sizes considered, this continuum window spans from low disorder or high magnetic field (consistent with the
emulation proposed) to $\hbar \omega_c/\Gamma \approx 1$. The highest value calculated in this region is $\delta E_0/\hbar \omega \approx 0.4$, still slightly below the crossing to the second Landau band that occurs at $\delta E_0/\hbar \omega = 0.5$, but far beyond a weak levitation limit [10].

The dependence of the levitation with the Landau index of the extended state could lead to important consequences on the QH phase diagrams. The GPD [5] requires that $\delta E_{N+1}/\delta E_N > 1$. Although the present results are for a levitation regime still within the same Landau band, we find $\delta E_1/\delta E_0 \approx 1$ as an upper limit (in the white-noise case), while for finite correlation lengths (smoothened disorder potentials) $\delta E_1/\delta E_0 < 1$, consistent with a phase diagram where direct transition from the Hall insulator to $\nu > 1$ are allowed. Hence, the exact form of the QH phase diagram could be disorder model dependent. The present results give therefore a clear guidance for future work: the extension of the scaling to lower magnetic fields and longer correlation lengths (larger systems), keeping within the continuum limit of the lattice model.

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