Quartic isospin asymmetry energy of nuclear matter from chiral pion-nucleon dynamics

N. Kaiser
Physik Department T39, Technische Universität München, D-85747 Garching, Germany
email: nkaiser@ph.tum.de

Abstract
Based on a chiral approach to nuclear matter, we calculate the quartic term in the expansion of the equation of state of isospin-asymmetric nuclear matter. The contributions to the quartic isospin asymmetry energy $A_4(k_f)$ arising from $1\pi$-exchange and chiral $2\pi$-exchange in nuclear matter are calculated analytically together with three-body terms involving virtual $\Delta(1232)$-isobars. From these interaction terms one obtains at saturation density $\rho_0 = 0.16 \text{ fm}^{-3}$ the value $A_4(k_f) = 1.5 \text{ MeV}$, more than three times as large as the kinetic energy part. Moreover, iterated $1\pi$-exchange exhibits components for which the fourth derivative with the respect to the isospin asymmetry parameter $\delta$ becomes singular at $\delta = 0$. The genuine presence of a non-analytical term $\delta^4 \ln |\delta|$ in the expansion of the energy per particle of isospin-asymmetric nuclear matter is demonstrated by evaluating a s-wave contact interaction at second order.

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1 Introduction and summary
The determination of the equation of state of isospin-asymmetric nuclear matter has been a long-standing goal shared by both nuclear physics and astrophysics [1]. Usually one assumes a parabolic form for the energy per nucleon at zero temperature, $\bar{E}_{\omega}(\rho_p, \rho_n) = \bar{E}(\rho) + A_2(\rho) \delta^2 + O(\delta^4)$, where $\rho = \rho_p + \rho_n$ is the total nucleon density and $\delta = (\rho_n - \rho_p)/\rho$ is the isospin asymmetry related to unequal proton and neutron densities $\rho_p \neq \rho_n$. The validity of the quadratic approximation has been verified with good numerical accuracy from isospin-symmetric nuclear matter ($\delta = 0$) up to pure neutron matter ($\delta = 1$) by most of the existing nuclear many-body theories using various interactions [2]. Nonetheless, it has been shown consistently in numerous studies [3] that for some properties of neutron stars, such as the proton fraction at beta-equilibrium, the core-crust transition density and the critical density for the direct URCA process to occur, even a very small quartic isospin asymmetry energy $A_4(\rho)$ (multiplied with $\delta^4$ in the expansion of the energy per nucleon) can make a big difference.

Given the fact that all the available numerical solutions of the nuclear many-body problem confirm the validity of the quadratic approximation, the quartic isospin asymmetry $A_4(\rho)$ should be rather small. However, in the recent work by Cai and Li [4], which employs an empirically constrained isospin-dependent single-nucleon momentum-distribution and the equation of state of pure neutron matter near the unitary limit, a significant quartic isospin asymmetry energy of $A_4(\rho_0) = (7.2 \pm 2.5) \text{ MeV}$ has been found. This value amounts to about 16 times the free Fermi gas prediction, see eq.(2). On the other hand, the calculations by the Darmstadt group [5] based on chiral low-momentum interactions and many-body perturbation theory lead to a small value of $A_4(\rho_0) = (1.0 \pm 0.2) \text{ MeV}$.

The purpose of the present paper it to give a prediction for the density-dependent quartic isospin asymmetry energy $A_4(k_f)$ in the chiral approach to nuclear matter developed in refs. [6, 7].

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In this approach the long- and medium-range NN-interactions arising from multi-pion exchange are
treated explicitly and few parameters encoding the relevant short-distance dynamics are adjusted to
bulk properties of nuclear matter. Single-particle potentials \[7\], quasi-particle interactions \[8\], the
thermodynamic behavior of nuclear matter at finite temperatures \[9\] and the density-dependence
of the in-medium quark condensate \[10\] follow then as predictions in that framework (see also the
recent review article \[11\]).

The present paper is organized as follows. In section 2, analytical expressions are given for the
contributions to the quartic isospin asymmetry energy \(A_4(k_f)\) as they arise from one-pion and
chiral \(2\pi\)-exchange. The three-nucleon interaction generated by \(2\pi\)-exchange and excitation of a
virtual \(\Delta(1232)\)-isobar is considered as well. These interaction contributions lead at saturation
density \(\rho_0 = 0.16\,\text{fm}^{-3}\) (or \(k_{f0} = 263\,\text{MeV}\)) to the (small) value \(A_4(k_{f0}) = 1.5\,\text{MeV}\), which amounts to
about three times the kinetic energy part. Moreover, in the course of the calculation one encounters
components of the second-order \(1\pi\)-exchange whose representation of the fourth derivative with the
respect to \(\delta\) at \(\delta = 0\) is singular. In section 3, the generic presence of a non-analytical term \(\delta^4\ln|\delta|\)
in the expansion of the energy per particle of isospin-asymmetric nuclear matter is demonstrated by
calculating in closed form the second-order contribution from a \(s\)-wave contact-interaction. Clearly,
after having established its existence, the non-analytical term \(\delta^4\ln|\delta|\) should be included in future
fits of the equation of state of (zero-temperature) isospin-asymmetric nuclear matter.

## 2 One-pion and two-pion exchange contributions

In this section we collect the expressions for the quartic isospin asymmetry \(A_4(k_f)\) as they arise from
one-pion and two-pion exchange diagrams following refs. \[6,7\]. Isospin-asymmetric (spin-saturated)
nuclear matter is characterized by different proton and neutron Fermi momenta, \(k_{p,n} = k_f(1 \mp \delta)^{1/3}\).
Expanding the energy per particle at fixed nucleon density \(\rho = 2k_f^3/3\pi^2\) in the isospin asymmetry
parameter \(\delta\) up to fourth order gives:

\[
\bar{E}_{\text{as}}(k_p, k_n) = \bar{E}(k_f) + \delta^2 A_2(k_f) + \delta^4 A_4(k_f) + \mathcal{O}(\delta^6),
\]

with \(A_2(k_f)\) the (usual) quadratic isospin asymmetry energy. We view the density-dependent expansion
coefficients \(\bar{E}(k_f), A_2(k_f)\) and \(A_4(k_f)\) as functions of the Fermi momentum \(k_f\), since they
emerge in this form directly from the calculation. The first contribution to \(A_4(k_f)\) comes from the
relativistically improved kinetic energy \(T_{\text{kin}}(p) = p^2/2M - p^4/8M^3\), and it reads:

\[
A_4(k_f)^{\text{kin}} = \frac{k_f^2}{162M} \left(1 + \frac{k_f^2}{4M^2}\right),
\]

with \(M = 939\,\text{MeV}\) the average nucleon mass. The corresponding value at nuclear matter saturation
density \(\rho_0 = 0.16\,\text{fm}^{-3}\) (or \(k_{f0} = 263\,\text{MeV}\)) is \(A_4(k_{f0})^{\text{kin}} = 0.464\,\text{MeV}\).

For the treatment of two-body interactions that depend on the momentum transfer \(|\vec{p}_1 - \vec{p}_2|\), the
following expansion formulas for integrals over two Fermi spheres are most helpful:

\[
\int \frac{d^3p_1 d^3p_2}{(2\pi)^6} F(|\vec{p}_1 - \vec{p}_2|) \left[\theta(k_p - |\vec{p}_1|) \theta(k_p - |\vec{p}_2|) + \theta(k_n - |\vec{p}_1|) \theta(k_n - |\vec{p}_2|)\right]
= \frac{2k_f^6}{3\pi^4} \int_0^1 dz \left[z^2(1-z)^2(2+z) + \frac{\delta^2 z^3}{3}\right] F(2zk_f) + \frac{\delta^4 k_f^4}{162} \left[F'(2zk_f) - 7z^4 F'(2zk_f)\right]
\]

\[
\int \frac{d^3p_1 d^3p_2}{(2\pi)^6} F(|\vec{p}_1 - \vec{p}_2|) \theta(k_p - |\vec{p}_1|) \theta(k_n - |\vec{p}_2|)
= \frac{k_f^6}{3\pi^4} \int_0^1 dz \left[z^2(1-z)^2(2+z) + \frac{\delta^2 z}{3} (z^2 - 1)\right] F(2zk_f) + \frac{\delta^4 k_f^4}{162} \left(8z^2 - 1 - 7z^4\right) F'(2zk_f)
\]

2
The $z$-dependent weighting functions at order $\delta^2$ and $\delta^4$ have been obtained by applying several partial integrations. The contribution of the $1\pi$-exchange Fock diagram to the quartic isospin asymmetry energy reads:

$$A_4(k_f)^{(1\pi)} = \frac{g_A^4 m_\pi^4}{(36 \pi f_\pi)^3} \left\{ (4u + \frac{21}{8}u^3) \ln(1 + 4u^2) - 2u^3 - \frac{33u}{4} - u(9 + 4u^2) \frac{1}{4(1 + 4u^2)^2} \right. \right.$$  

$$\left. + \frac{m_\pi^2}{M^2} \left[ 2u^5 + 2u^3 + \frac{3u}{8} - u^3 \ln(1 + 4u^2) - \frac{u(3 + 16u^2)}{8(1 + 4u^2)^2} - \frac{3u^3}{2} \arctan 2u \right] \right\}, \quad (5)$$

with the dimensionless variable $u = k_f/m_\pi$. The second line in eq.(5) gives the relativistic $1/M^2$-correction. The occurring physical parameters are: nucleon axial-vector coupling constant $g_A = 1.3$, (neutral) pion mass $m_\pi = 135$ MeV, and pion decay constant $f_\pi = 92.4$ MeV.

Next in the chiral expansion comes the iterated (second-order) $1\pi$-exchange. With two medium insertions $\frac{1}{2} (1 + \tau_3) \theta(k_p - |\vec{p}_1|) + \frac{1}{2} (1 - \tau_3) \theta(k_n - |\vec{p}_1|)$ one gets a Hartree contribution of the form:

$$A_4(k_f)^{(H2)} = \frac{g_A^4 M m_\pi^4}{(24 \pi)^3 f_\pi^4} \left\{ 10u^3 - \frac{61u}{2} + \frac{200u^2 + 49}{6u} \ln(1 + 4u^2) - \frac{u(13 + 60u^2)}{6(1 + 4u^2)^2} - \frac{128u^2}{3} \arctan 2u \right\}, \quad (6)$$

and the corresponding Fock exchange-term reads:

$$A_4(k_f)^{(F2)} = \frac{g_A^4 M m_\pi^4}{(12 \pi)^3 f_\pi^2} \left\{ \frac{u}{8} - \frac{u^3}{3} - \frac{u}{12(1 + 2u^2)} - \frac{u}{24(1 + u^2)} \right. \right.$$  

$$\left. + u^4 \arctan u + \frac{u^2(2 + 11u^2 + 16u^4)}{6(1 + 2u^2)^2} \left[ \arctan u - \arctan 2u \right] \right. \right.$$  

$$\left. + \int_0^u dx \frac{1}{6u(1 + 2x^2)} \left[ (1 + 8x^2 + 8x^4) \arctan x - (1 + 4x^2) \arctan 2x \right] \right\}. \quad (7)$$

Pauli-blocking effects at second order are included through diagrams with three (isospin-asymmetric) medium insertions [6]. We consider here only the factorizable Fock contribution for which the energy denominator gets canceled by factors from the momentum-dependent $\pi N$-vertices (see eqs.(11,26) in ref.[6]). Its contribution to the quartic isospin asymmetry energy can be represented as a one-parameter integral, $A_4(k_f)^{(fac)} = g_A^4 M m_\pi^4 (12 \pi f_\pi)^{-4} f_0^2 dx I(x, u)$, where the lengthy integrand $I(x, u)$ involves the function $\ln[1 + (u + x)^2] - \ln[1 + (u - x)^2]$ and its square. The corresponding value at saturation density is $A_4(k_0)^{(fac)} = -1.35$ MeV, thus counterbalancing most of the Fock term $A_4(k_0)^{(F2)} = 1.70$ MeV without Pauli-blocking written in eq.(7). For the non-factorizable pieces the representation of the fourth derivative with respect to $\delta$ at $\delta = 0$ includes singularities of the form $(u - x)^{-\nu}, \nu = 1, 2$. When subtracting these singular terms from the integrand only very small numerical values are obtained for the non-factorizable Hartree contribution. In the case of the quadratic isospin asymmetry energy $A_2(k_0)$ one finds that the non-factorizable pieces (see eqs.(24,26) in ref.[6]) tend to cancel each other almost completely as: $(-11.6 + 12.0)\text{ MeV}$. Therefore one can expect that the omission of the non-factorizable pieces does not change much the final result for the quartic isospin asymmetry energy $A_4(k_f)$. However, the observation that the iterated $1\pi$-exchange has components with a singular representation of their fourth derivative with respect to $\delta$ at $\delta = 0$, indicates that the expansion in eq.(1) becomes non-analytic beyond the quadratic order $\delta^2$. This feature is demonstrated in section 3 by calculating in closed form the second-order contribution from a s-wave contact-interaction.

We continue with the contribution of the irreducible $2\pi$-exchange to the quartic isospin asymmetry energy. Using a twice-subtracted dispersion relation for the $2\pi$-exchange NN-potential in momentum-space and the master formulas in eqs.(3,4), one obtains:

$$A_4(k_f)^{(2\pi)} = \frac{1}{81 \pi^3} \int_{2m_\pi}^\infty d\mu \left\{ \text{Im}(V_C + 2\mu^2 V_T) \left[ \frac{7\mu k_f}{4} - \frac{2k_f^5}{3\mu^3} - \frac{\mu k_f^3(7\mu^2 + 36k_f^2)}{2(\mu^2 + 4k_f^2)^2} \right] \right\} \int_2^\infty dM \frac{1}{M^2} \left\{ 1 - \frac{M^2}{m_\pi^2} \right\}.$$

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virtual excitation of a ∆(1232)-isobar \[7\]. The corresponding three-body Hartree contribution reads:

\[
-\frac{7\mu^3}{16k_f} \ln \left(1 + \frac{4k_f^2}{\mu^2}\right) + \text{Im}(W_C + 2\mu^2W_T)\left[\frac{2k_f^5}{\mu^3} + \frac{k_f^3}{\mu} + \frac{21\mu k_f}{4}\right] - \frac{\mu k_f^3(7\mu^2 + 36k_f^2)}{2(\mu^2 + 4k_f^2)^2} - \frac{\mu}{16k_f}(21\mu^2 + 32k_f^2) \ln \left(1 + \frac{4k_f^2}{\mu^2}\right)
\]

where \(\text{Im}C,T\) and \(\text{Im}W_{C,T}\) are the spectral functions of the isoscalar and isovector central and tensor NN-amplitudes, respectively. These imaginary parts are composed of the functions \(\sqrt{\mu^2 - 4m_n^2}\) and \(\text{arctan}(\sqrt{\mu^2 - 4m_n^2}/2\Delta)\), with \(\Delta = 293\) MeV the delta-nucleon mass splitting. Note that due to the implemented subtractions the \(k_f\)-expansion of \(A_4(k_f)\) in eq. (8) starts with the power \(k_f^5\). A short-distance contribution proportional to \(k_f^4\) is supplemented by the subtraction constants:

\[
A_4(k_f) = \frac{10k_f^5}{(3M)^4} \left(\frac{2B_5}{3} - B_{n,5}\right),
\]

with the parameters \(B_5 = 0\) and \(B_{n,5} = -3.58\) adjusted in ref.\[7\] to the empirical nuclear matter saturation point and quadratic isospin asymmetry energy \(A_2(k_f) = 34\) MeV.

Finally, we consider the long-range three-nucleon interaction generated by \(2\pi\)-exchange and virtual excitation of a ∆(1232)-isobar \[7\]. The corresponding three-body Hartree contribution reads:

\[
A_4(k_f)^{(\Delta)} = \frac{g_A^4m^6\pi}{\Delta(6\pi f_x)^4} \left\{ \left(\frac{16u^2}{3} + \frac{21}{4}\right) \ln(1 + 4u^2) - \frac{4u^4}{3} - \frac{41u^2}{3} - \frac{2u^2(11 + 99u^2 + 236u^4)}{3(1 + 4u^2)^3} \right\},
\]

while the associated three-body Fock term can be represented as \(g_A^4m^6\pi^2(12\pi f_x)^{-4}\Delta^{-1} f_0^u dx J(x, u)\), where the lengthy integrand \(J(x, u)\) involves the functions \(\text{arctan}(u + x) + \text{arctan}(u - x)\) and \(\ln[1 + (u + x)^2] - \ln[1 + (u - x)^2]\). Note that the three-body contact-term proportional to \(\zeta\) introduced additionally in ref.\[7\] does not contribute to the quartic isospin asymmetry energy \(A_4(k_f)\).

Summing up all the calculated contributions, one obtains the result for the density-dependent quartic isospin asymmetry energy \(A_4(k_f)\) of nuclear matter as shown in Fig. 1 in the density region \(0 < \rho < 2\rho_0 = 0.32\) fm\(^{-3}\). The predicted value at saturation density \(\rho_0 = 0.16\) fm\(^{-3}\) is \(A_4(k_f) = 1.49\) MeV. It amounts to 3.2 times the free Fermi-gas part \(A_4(k_f) = 0.464\) MeV. Note that

![Figure 1: Quartic isospin asymmetry \(A_4(k_f)\) as a function of the nucleon density \(\rho = 2k_f^3/3\pi^2\).](image-url)
interaction contributions to $A_4(k_f)$ start (at least) with the power $k_f^3$. The density-dependence of the full line in Fig.1 is to a good approximation $\rho^{5/4}$. Actually, it should be noted that the present calculation of the quartic isospin asymmetry energy $A_4(k_f)$ is performed in a framework where empirical constraints from bulk properties of nuclear matter are satisfied. This does not apply to the recent work in ref.[4], where a large value of $A_4(k_{f0}) = (7.2 \pm 2.5)$ MeV has been found.

3 S-wave contact interaction to second order

The analysis of the Pauli-blocking corrections to the second-order (iterated) $1\pi$-exchange has indicated that non-analytical terms may occur in the $\delta$-expansion of the energy per particle of isospin-asymmetric nuclear matter beyond the quadratic order. In the extreme case there could be a cubic term $|\delta|^3$, which is after all even under the exchange of protons and neutrons: $\delta \rightarrow -\delta$. In order to clarify the situation, we consider a s-wave contact interaction:

$$V_{ct} = \frac{\pi}{M} [a_s + 3a_t + (a_t - a_s) \vec{\sigma}_1 \cdot \vec{\sigma}_2],$$

and examine it in second-order many-body perturbation theory. For this simple interaction, the occurring integrals over three Fermi spheres with (at most) two different radii $k_p, k_n$ can be solved in closed analytical form. The pertinent function to express the result in the isospin-asymmetric configuration of interest is:

$$35 \int_0^1 dz (z - z^4) \left\{ 2xz + (x^2 - z^2) \ln \frac{x + z}{|x - z|} \right\}$$

$$= \frac{x}{2} (15 + 33x^2 - 4x^4) + \frac{1}{4} (42x^2 - 15 - 35x^4) \ln \frac{x + 1}{|x - 1|} + 2x^7 \ln \frac{x^2}{|x^2 - 1|},$$

where the variable $x > 0$ is set to a ratio of Fermi momenta, $[(1 + \delta)/(1 - \delta)]^{1/3} \leq 1$. Note that the function defined in eq.(12) has at $x = 1$ the value $22 - 4\ln 2$. Combining the second-order Hartree and Fock diagrams generated by $V_{ct}$ according to their spin- and isospin-factors and performing the expansion in powers of $\delta$, one obtains the following result for the energy per particle:

$$\bar{E}_{as}(k_p, k_n)^{(2nd)} = \frac{k_f^2}{5\pi^2 M} \left\{ \frac{3}{4}(a_s^2 + a_t^2)(11 - 2\ln 2) + \frac{4\delta^2}{3} [a_s^2(3 - \ln 2) - a_t^2(2 +\ln 2)] \right\}$$

$$+ \frac{\delta^4}{81} \left\{ a_s^2 \left( 10\ln \frac{|\delta|}{3} + 2\ln 2 - \frac{41}{6} \right) + a_t^2 \left( 30\ln \frac{|\delta|}{3} + 2\ln 2 + \frac{31}{2} \right) \right\} + O(\delta^6).$$

The crucial and novel feature which becomes evident from this expression is the presence of the non-analytical logarithmic term $\delta^4 \ln(\delta)/3$. Interestingly, the corresponding coefficient is three times as large in the spin-triplet channel as in the spin-singlet channel. For comparison the first-order contribution of the s-wave contact interaction $V_{ct}$ reads, $\bar{E}_{as}(k_p, k_n)^{(1st)} = k_f^2 [a_s - a_t + \delta^2(a_t - a_s/3)])/2\pi M$, without any higher powers of $\delta$. Note that the sign-convention for the scattering lengths $a_s, a_t$ is chosen here such that positive values correspond to attraction. As a check we have rederived the same results at first and second order by using the alternative (and equivalent) form of the s-wave contact interaction $V_{ct} = \pi |3a_s + a_t + (a_s - a_t) \vec{\tau}_1 \cdot \vec{\tau}_2|/M$.

In Fig.2 the dependence of the second-order energy per particle $\bar{E}_{as}(k_p, k_n)^{(2nd)}$ on the isospin asymmetry parameter $\delta$ is shown for three different choices of the s-wave scattering lengths: $a_s = a_t$, $a_t = 0$ and $a_s = 0$. In each case the full line shows the exact result and the (nearby) dashed line gives the expansion in powers of $\delta$ truncated at fourth order according to eq.(13). One observes that these expansions reproduce the full $\delta$-dependence very well over the whole range $-1 \leq \delta \leq 1$. Note also that the prefactor $k_f^2 a_s^2/5\pi^2 M$ of dimension energy has been scaled out in Fig.2.
Figure 2: Dependence of the 2nd order energy per particle $\bar{E}_{as}(k_p, k_n)^{(2nd)}$ on the isospin asymmetry $\delta$. Three different choices for the scattering lengths, $a_s = a_t$, $a_s = 0$ and $a_s = 0$, are considered.

If one performs for the second-order energy density the fourth derivative with respect to $\delta$ at $\delta = 0$ under the integral, then one encounters integrands with singularities of the form $(1 - z)^{-\nu}$, $\nu = 1, 2$. The origin of these singularities, or in the proper treatment the non-analytical term $\delta^4 \ln(|\delta|/3)$, lies in the energy denominator of second-order diagrams. For an infinite (normal) many-fermion system the energy spectrum has a vanishing gap between bound states in the Fermi sea and excited states in the continuum. Such a gap-less energy spectrum causes a singularity, respectively a non-analyticity, if small asymmetries of the Fermi levels of two components are analyzed with too high resolution.

In summary, we have demonstrated that the non-analytical term $\delta^4 \ln(|\delta|/3)$ will be generically present in calculations of isospin-asymmetric nuclear matter that go beyond the mean-field Hartree-Fock level. Therefore, a term $\delta^4 \ln |\delta|$ should be included in future fits of the equation of state of (zero-temperature) isospin-asymmetric nuclear matter and its role should be further examined.

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