Developable surface patches bounded by NURBS curves

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1 Introduction

In this talk we review the problem of constructing a developable surface patch bounded by two rational or NURBS (Non-Uniform Rational B-spline) curves.

NURBS curves are curves which are piecewise rational. That is, they are a generalisation of spline curves, which are piecewise polynomial curves. Similarly we define NURBS surfaces and solids. NURBS curves have been the standard in Computer Aided Design [1] for a long time through IGES and STEP specifications. They are described by a list of points called control polygon and two list of numbers: the list of weights and the list of knots.

Developable surfaces are ruled surfaces with null Gaussian curvature [2]. This implies that they can be constructed from planar surfaces by just cutting, rolling and folding, so that metric properties such as lengths and angles between curves and areas are preserved. These geometric properties are of great interest for steel and textile industry, since these are pieces designed in the plane and then combed into space.

For instance, in naval architecture sheets of steel are adapted to fit into the hull of a ship [3, 4, 5]. If these sheets are combed just with a folding machine, the costs are lower than if they require the use of heat. In textile industry cloth is planar and is cut and sewn to produce garments and the quality is improved if it is not stretched [6]. They have also been used for designing facades in architecture [7] and in automobile industry [8].

In geometric design the standard relies on the use of rational B-spline curves and surfaces (NURBS), which are described by control polygons or nets and lists of weights and knots. In the case of ruled surfaces, the control net is formed by just the control polygons of the bounding curves, since segments can be described by just their endpoints, which form their control polygon.

This problem has been addressed in several ways [9], but the key drawback is that when we require the developable surface to be NURBS and bounded by NURBS curves, the possibilities are restricted [10, 11].

In some cases, the bounding curves are planar and lie on parallel planes [12, 13] and this simplifies the problem.

For instance, one can obtain general solutions for developable NURBS surfaces bounded by NURBS curves [15, 16, 17, 18, 19].

Considering the dual space in projective geometry has been also profitable, since developable surfaces may be seen as envelopes of lines of planes [20, 21].

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For this reason our proposal is to consider developable surface patches which are not NURBS, though bounded by NURBS curves [22]. In fact, we are able to obtain every possible solution, good or bad, to this problem in our framework.

Our approach is based on performing a reparametrisation on one of the bounding curves of the surface patch and imposing that the resulting surface be developable [22]. One would expect a differential equation for the reparametrisation function. However, the condition happens to be algebraic. Another approach [23] reduces the null Gaussian curvature condition to quadratic equations.

Moreover, if the bounding curves are (piecewise) polynomial or rational of degree $n$, the developability condition is an algebraic equation of degree $2n - 2$. The degree may be lowered to $n - 1$ if the curves are (piecewise) polynomial and lie on parallel planes.

2 Methods

We start with a ruled surface parametrised by $b(t, v)$ and bounded by two parametrised curves, $c(t), d(t)$,

$$b(t, v) = (1 - v)c(t) + vd(t), \quad t, v \in [0, 1].$$

For given $c(t)$ and $d(t)$, this ruled surface will not be developable in general, but, if it is developable, this feature shall not depend on the chosen parametrisation.

The Gaussian curvature $K(t, v)$ is the quotient of the determinants of the second, $B(t, v)$, and first, $G(t, v)$, fundamental forms of the surface patch parametrised by $b(t, v)$,

$$K(t, v) = \frac{\det B(t, v)}{\det G(t, v)}.$$  

The first fundamental form $G(t, v)$ is the usual scalar product restricted to tangent vectors to the surface at the point $b(t, v)$ and is used, for instance, for calculating areas,

$$\text{Area}(S) = \int_D \sqrt{\det G(t, v)} \, dt dv,$$

but, since its determinant is positive and appears at the denominator of $K(u, v)$, we need not pay attention to it, though due to Gauss’ Theorema Egregium [2], the Gaussian curvature is an intrinsic property and may be written in terms of just the first fundamental form.

On the other hand, the second fundamental form $B(t, v)$ is the extrinsic curvature of the surface at the point $b(t, v)$ with $\nu$ as unitary normal to the surface and can be constructed with the projections of the second derivatives of the parametrisation along the unitary normal $\nu(t, v)$ to the surface at $b(t, v)$,

$$B(t, v) = \begin{pmatrix} b_{tt} \cdot \nu & b_{tv} \cdot \nu \\ b_{tt} \cdot \nu & b_{vv} \cdot \nu \end{pmatrix}_{(t,v)}.$$  

In this sense, the second fundamental form is used to compute the normal curvature at points on the surface.

For a ruled surface parametrised as in (1.1),

$$\det B(t, v) = \begin{vmatrix} (1 - v)c''(t) + vd''(t) \cdot \nu & (d'(t) - c'(t)) \cdot \nu \\ (d'(t) - c'(t)) \cdot \nu & 0 \end{vmatrix}_{(t,v)} = - ((d'(t) - c'(t)) \cdot \nu)^2,$$

we see that the Gaussian curvature is always non-positive and hence there are no elliptic points on a ruled surface, since $\det B(t, v)$ is always positive.
Figure 1: Developability is equivalent to having the same tangent plane along each ruling of a ruled surface.

Moreover, vanishing Gaussian curvature is equivalent to vanishing $(d'(t) - c'(t)) \cdot \nu$ on the points of the ruled surface.

Since at a point $b(t, v)$ on the surface we have $b_t(u, v) = (1 - v)c'(t) + d'(t)$, $b_v(t, v) = d(t) - c(t)$ as two tangent vectors to the surface, we may construct a normal vector to the surface at $b(t, v)$ as $N = b_t \times b_v$, and then having a vanishing Gaussian curvature is equivalent to having a vanishing triple product,

$$
\text{det} \left( c'(t), d'(t), d(t) - c(t) \right) = 0, \quad t \in [0, 1] \tag{1.1}
$$
at all points of the surface patch.

Our contribution to deal with this problem is based on reparameterisation of one of the bounding curves by a function $T(t)$,

$$
\tilde{b}(t, v) = (1 - v)c(t) + vd(T(t)) \tag{1.2}
$$
and require $\tilde{b}(t, v)$ to satisfy the null Gaussian curvature condition.

## 3 Results

The developability condition (1.1) applied to parametrisations such as (1.2) can be seen to be algebraic in $T(t)$, since the dependence on the derivative $T'(t)$ is factored out by the determinant,

$$
\text{det} \left( c'(t), \dot{d}(T), d(T) - c(t) \right) = 0, \tag{1.3}
$$
where the dot stands for derivation with respect to $T$.

In the case of (piecewise) polynomial or rational curves $c(t), d(t)$, further consequences may be derived:

**Theorem 1:** Let $c(t), d(T), t, T \in [0, 1]$ be rational curves of degree $n$. The parameterized ruled surface,

$$
b(t, v) = (1 - v)c(t) + vd(T(t)), \quad t, v \in [0, 1],
$$
is a developable surface if the reparameterization function $T(t)$ satisfies the algebraic equation

$$
\text{det} \left( c'(t), \dot{d}(T), d(T) - c(t) \right)_{T=T(t)} = 0,
$$
and is a real monotonically increasing function of $t$. 

3
This equation is of degree \(2n - 2\) at most. If both curves are (piecewise) polynomial and lie on parallel planes, the equation is of degree \(n - 1\) at most. We may see an example in Fig 2.

The price to pay is that solutions of this algebraic equation will not be rational or polynomial in general and \(\tilde{b}(t, v)\) will no longer be NURBS.

Since the condition on the reparametrisation is algebraic, the number of possible solutions is finite, but not all of them are geometrically acceptable.

For being a reparametrisation, \(T(t)\) must be a monotonically increasing function. Otherwise, we would have unpleasant regression areas with more than on ruling through some points of the curves (See Fig 3). This can be checked with the help of

\[
T'(t) = \frac{\det \left( c''(t), \dot{d}(T), d(T) - c(t) \right)}{\det \left( \ddot{d}(T), c'(t), d(T) - c(t) \right)} \bigg|_{T=T(t)},
\]

which we derive from the null Gaussian curvature condition.

This implies that monotonicity is granted if

\[
\sgn \left( c''(t) \cdot \nu(t) \right) = \sgn \left( \ddot{d}(T) \cdot \nu(t) \right) \bigg|_{T=T(t)},
\]

where \(\nu(t)\) is the unitary normal to the ruled surface along the segment at \(t\). This means that the normal curvatures of both curves must have the same sign for each value of \(t\).

Hence, acceptable solutions just appear if both curves are qualitatively similar regarding their curvature.

**Theorem 2:** Let \(c(t), d(T), t, T \in [0, 1]\) be parameterized curves. Let \(T(t)\) be a reparameterization function so that

\[
b(t, v) = (1 - v)c(t) + vd(T(t)), \quad t, v \in [0, 1],
\]
is a developable surface. $T(t)$ is a monotonically increasing function if and only if for all $t$,

$$\text{sgn} \left( c''(t) \cdot \nu(t) \right) = \text{sgn} \left( \frac{d(T) \cdot \nu(t)}{T(t)} \right),$$

where $\nu(t)$ is the unitary normal to the surface along the ruling at $t$.

Or equivalently, for the normal curvatures $k_{n,c}, k_{n,d}$ of both curves

$$\text{sgn} \left( k_{n,c}(t) \right) = \text{sgn} \left( k_{n,d}(T(t)) \right),$$

for all values of $t$.

In the case of parameterizations of class $C^k$ of differentiability, $T(t)$ is of class $C^{k-1}$.

### 4 Conclusions

We have produced a new approach for dealing with the problem of constructing a developable surface patch between two parametrised curves $c(t)$ and $d(t)$.

This approach is grounded on performing a reparametrisation of one of the curves and the developability condition is not a differential equation, but an algebraic equation, and provides all possible solutions to the problem.

In the case of (piecewise) polynomial or rational curves of degree $n$, the developability condition is an algebraic equation of degree $2n - 2$. Since the most usual degree in Computer Aided Design is three, this means we are dealing with a fourth degree equation, which can be handled either by numerical or analytical methods.

For (piecewise) polynomial curves of degree $n$, lying on parallel planes, the algebraic equation is of degree $n - 1$.

Requiring that the reparametrisation function be a monotonically increasing function, in order to avoid regression areas on the developable surface patch, is achieved if the sign of the normal curvatures of both bounding curves is the same at the endpoints of each ruling.

With this approach, it is easy to control the final class of differentiability of the surface in the piecewise case.

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