HIGH DEGREE GRAPHS CONTAIN LARGE-STAR FACTORS

NOGA ALON* and NICHOLAS WORMALD†

Dedicated to László Lovász, for his 60th birthday

We show that any finite simple graph with minimum degree \( d \) contains a spanning star forest in which every connected component is of size at least \( \Omega((d/ \log d)^{1/3}) \). This settles a problem of Havet, Klazar, Kratochvil, Kratsch and Liedloff.

**Dedication**

This paper is dedicated to Laci Lovász, for his 60th birthday. It settles a problem presented by Jan Kratochvil at the open problems session of the meeting Building Bridges, which took place in Budapest in August 2008, celebrating this birthday. The Lovász Local Lemma is applied extensively throughout the proof. This work is therefore a typical example illustrating the immense influence of Laci, who not only provided the community with powerful tools and techniques, but also stimulated research by his books, lectures and organization of conferences.

1. INTRODUCTION

All graphs considered here are finite and simple. A *star* is a tree with one vertex, the *center*, adjacent to all the others, which are *leaves*. A *star factor* of a graph \( G \) is a spanning forest of \( G \) in which every connected component is a star. It is easy to see that any graph with positive minimum degree contains a star factor in which every component is a star with at least one

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edge. A conjecture of Havet et al. [9], communicated to us by Jan Kratochvil [10], asserts that if the minimum degree is large then one can ensure that all stars are large. More precisely, they conjectured that there is a function $g(d)$ that tends to infinity as $d$ tends to infinity, so that every graph with minimum degree $d$ contains a star factor in which every star contains at least $g(d)$ edges. Our main result shows that this is indeed the case, for a function $g(d)$ that grows moderately quickly with $d$, as follows.

**Theorem 1.1.** There exists an absolute positive constant $c$ so that for all $d \geq 2$, every graph with minimum degree $d$ contains a star factor in which every star has at least $cd^{1/3}/(\log d)^{1/3}$ edges.

The motivation for the conjecture of Havet et al. arises in the running time analysis of a recent exact exponential time algorithm for the so called $L(2,1)$-labeling problem of graphs. See [9] for more details.

As preparation for the proof of the main result, we prove the following simpler statement.

**Theorem 1.2.** There exists an absolute positive constant $c'$ such that for all $d \geq 2$, every $d$-regular graph contains a star factor in which every star has at least $c'd/\log d$ edges. This is optimal, up to the value of the constant $c'$.

Throughout the paper we make no attempt to optimize the absolute constants. To simplify the presentation we omit all floor and ceiling signs whenever these are not crucial. We may and will assume, whenever this is needed, that the minimum degree $d$ considered is sufficiently large. It is easy to find, in any graph with all vertices of degree at least 1, a star factor with stars of at least two vertices each, so the theorems then follow for all $d \geq 2$. All logarithms are in the natural base, unless otherwise specified.

Our notation is standard. In particular, for a graph $G = (V,E)$ and a vertex $v \in V$, we let $N_G(v)$ denote the set of all neighbors of $v$ in the graph $G$, and let $d_G(v) = |N_G(v)|$ denote the degree of $v$ in $G$. For $X \subset V$, $N_G(X) = \cup_{x \in X} N_G(x)$ is the set of all neighbors of the members of $X$.

The rest of this short paper is organized as follows. In Section 2 we present the simple proof of Theorem 1.2, and in Section 3 the proof of the main result. Section 4 contains some concluding remarks and open problems.