Colorings with neighborhood parity condition

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Abstract

In this short paper, we introduce a new vertex coloring whose motivation comes from our series on odd edge-colorings of graphs. A proper vertex coloring \(\varphi\) of graph \(G\) is said to be odd if for each non-isolated vertex \(x \in V(G)\) there exists a color \(c\) such that \(\varphi^{-1}(c) \cap N(x)\) is odd-sized. We prove that every simple planar graph admits an odd 9-coloring, and conjecture that 5 colors always suffice.

Keywords: planar graph, neighborhood, proper coloring, odd coloring.

1 Introduction

All considered graphs in this paper are simple, finite and undirected. We follow [3] for all terminology and notation not defined here. A \(k\)-(vertex-)coloring of a graph \(G\) is an assignment \(\varphi : V(G) \rightarrow \{1, \ldots, k\}\). A coloring \(\varphi\) is said to be proper if every color class is an independent subset of the vertex set of \(G\). A hypergraph \(\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))\) is a generalization of a graph, its (hyper-)edges are subsets of \(V(\mathcal{H})\) of arbitrary positive size. There are various notions of (vertex-)coloring of hypergraphs, which when restricted to graphs coincide with proper graph coloring. One such notion was introduced by Even at al. [9] (in a geometric setting) in connection with frequency assignment problems for cellular networks, as follows. A coloring of a hypergraph \(\mathcal{H}\) is conflict-free (CF) if for every edge \(e \in E(\mathcal{H})\) there is a color \(c\) that occurs exactly once on the vertices of \(e\). The CF chromatic number of \(\mathcal{H}\) is the minimum \(k\) for which \(\mathcal{H}\) admits a CF \(k\)-coloring. For graphs, Cheilaris [5] studied the CF coloring with respect to neighborhoods, that is, the coloring in which for every non-isolated vertex \(x\) there is a color that occurs exactly once in the (open) neighborhood \(N(x)\), and proved the upper bound \(2\sqrt{n}\) for the CF chromatic number of a graph of order \(n\). For more on CF colorings see, e.g., [8, 11, 13, 16, 20].

A similar but considerably less studied notion (concerning a weaker requirement for the occurrence of a color) was introduced by Cheilaris et al. [7] as follows. An odd coloring of hypergraph \(\mathcal{H}\) is a coloring such that for every edge \(e \in E(\mathcal{H})\) there is a color \(c\) with an odd number of vertices of \(e\) colored by \(c\). Particular features of the same notion (under the
name weak-parity coloring) have been considered by Fabrici and Göring \cite{10} (in regard to face-hypergraphs of planar graphs) and also by Bunde et al. \cite{4} (in regard to coloring of graphs with respect to paths, i.e., path-hypergraphs). For various edge colorings of graphs with parity condition required at the vertices we refer the reader to \cite{1, 6, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21}.

In this paper we study certain aspects of odd colorings for graphs with respect to (open) neighborhoods, that is, the colorings of graph $G$ such that for every non-isolated vertex $x$ there is a color that occurs an odd number of times in the neighborhood $N_G(x)$. Our focus is on colorings that are at the same time proper, and we mainly confine to planar graphs.

Let us denote by $\chi_o(G)$ the minimum number of colors in any proper coloring of a given graph $G$ that is odd with respect to neighborhoods, call this the odd chromatic number of $G$. Note that the obvious inequality $\chi(G) \leq \chi_o(G)$ may be strict; e.g., $\chi(C_4) = 2$ whereas $\chi_o(C_4) = 4$. Similarly, $\chi(C_5) = 3$ whereas $\chi_o(C_5) = 5$. In fact, the difference $\chi_o(G) - \chi(G)$ can acquire arbitrarily large values. Indeed, let $G$ be obtained from $K_n$ ($n \geq 2$) by subdividing each edge once. Since $G$ is bipartite, $\chi(G) = 2$. On the other hand, it is readily seen that $\chi_o(G) \geq n$. Note in passing another distinction between the chromatic number and the odd chromatic number. The former graph parameter is monotonic in regard to the ‘subgraph relation’, that is, if $H \subseteq G$ then $\chi(H) \leq \chi(G)$. This nice monotonicity feature does not hold for the odd chromatic index in general. For example, $C_4$ is a subgraph of the kite $K_4 - e$, but nevertheless we have $\chi_o(C_4) = 4 > 3 = \chi_o(K_4 - e)$.

The Four Color Theorem \cite{1, 19} asserts the tight upper bound $\chi(G) \leq 4$ for the chromatic number of any planar graph $G$. One naturally starts wondering about an analogous bound for the odd chromatic number of all planar graphs. Since $\chi_o(C_5) = 5$, four colors no longer suffice. It is our belief that five colors always suffice.

**Conjecture 1.1.** For every planar graph $G$ it holds that $\chi_o(G) \leq 5$.

The main purpose of this paper is to provide first support to Conjecture \cite{11} by proving the following.

**Theorem 1.2.** For every planar graph $G$ it holds that $\chi_o(G) \leq 9$.

## 2 Proof of Theorem 1.2

Throughout, we will refer to a coloring of this kind as to a nice coloring, that is, a nice coloring is a proper weak-odd coloring that uses at most 9 colors. Arguing by contradiction, let $G$ be a counter-example of minimum order $n = n(G)$. Clearly, $G$ is 2-edge-connected and has $n \geq 10$ vertices. We proceed to exhibit several structural constraints of $G$.

**Claim 1.** The minimum degree $\delta(G)$ equals 5.

Since $G$ is a connected planar simple graph with $n > 2$ vertices, the inequality $\delta(G) \leq 5$ is a consequence of Euler’s formula. Consider a vertex $v$ of degree $d_G(v) = \delta(G)$. If $d_G(v) = 1$ or 3, then take a nice coloring of $G - v$. By forbidding at most six colors at $v$ (namely, at most three colors used for $v$’s neighbors and at most three additional colors in regard to weak-oddness concerning the neighborhoods in $G - v$ of these vertices), the coloring extends to a nice coloring of $G$. 

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Suppose next that \( d_G(v) = 2 \), and say \( N_G(v) = \{x, y\} \). Construct \( G' \) by removing \( v \) from \( G \) plus connecting \( x \) and \( y \) if they are not already adjacent. By minimality, \( G' \) admits a nice coloring \( c \). Say \( c(x) = 1 \) and \( c(y) = 2 \), and let color(s) \( 1' \) and \( 2' \), respectively, have odd number of occurrences in \( N_{G'}(x) \) and \( N_{G'}(y) \). If there are more possibilities for \( 1' \), then choose \( 1' \neq 2' \); do similarly for \( 2' \) in regard to \( 1' \). Extend the coloring to \( G \) by using for \( v \) a color different from \( 1, 2, 1', 2' \). The properness of the coloring is clearly preserved. As for the weak-oddness concerning neighborhoods, \( v \) is fine because \( 1 \neq 2 \). If \( 1' \neq 2' \) then \( x \) is fine since \( 1' \) remains to be odd on \( N_G(x) \). Contrarily, if \( 1' = 2' \) then \( c(v) \) is odd on \( N_G(x) \). Similarly, the vertex \( y \) is also fine.

Finally, suppose \( d_G(v) = 4 \) and let \( w \in N_G(v) \). Remove \( v \) and connect \( w \) by an edge to any other non-adjacent neighbor of \( v \). The obtained graph \( G'' \) is simple and planar. Indeed, it can be equivalently obtained from \( G \) by contracting the edge \( vw \) (if parallel edges arise through possible mutual neighbors of \( v \) and \( w \), then for each such adjacency a single edge is kept and its copies are deleted). Notice that under any nice coloring of \( G'' \), the color of \( w \) occurs exactly once in \( N_G(v) \). Therefore, any such coloring extends to a nice coloring of \( G \) by forbidding at most eight colors at \( v \).

We refer to any vertex of degree \( d \) as to a \( d \)-vertex. Similarly, a vertex of degree at least \( d \) is a \( d^+ \)-vertex. Analogous terminology applies to faces in regard to a planar embedding of \( G \).

**Claim 2.** If \( v \) is a \( 5 \)-vertex of \( G \), then it has at most one neighbor of odd degree.

Suppose each of two vertices \( u, w \in N_G(v) \) has an odd degree in \( G \). Consider a nice coloring \( c \) of \( G - v \). We intend to extend \( c \) to \( G \). Since \( N_G(u) \) and \( N_G(w) \) are odd-sized, no color is blocked at \( v \) in regard to oddness in the neighborhoods of \( u \) and \( w \). Moreover, as \( N_G(v) \) is odd-sized as well, the oddness of a color in this particular neighborhood is guaranteed. Therefore, by forbidding at most eight colors at \( v \) (all of \( c(N_G(v)) \)) and at most 3 additional colors concerning oddness in neighborhoods of the vertices forming \( N_G(v) \setminus \{u, w\} \), the coloring \( c \) extends to a nice coloring of \( G \).

Since the graph \( G \) is connected, for an arbitrary planar embedding Euler’s formula gives

\[
|V(G)| - |E(G)| + |F(G)| = 2,
\]

where \( F(G) \) is the set of faces. Our proof is based on the discharging technique. We assign initial charges to the vertices and faces according to the left-hand side of the following equality (which immediately follows from Euler’s formula):

\[
\sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in F(G)} (2d(f) - 6) = -12.
\]

Thus, any vertex \( v \) receives charge \( d(v) - 6 \), and any face \( f \) obtains charge \( 2d(f) - 6 \). As \( G \) is simple, for every face \( f \) it holds that \( d(f) \geq 3 \), implying that its initial charge is non-negative. By Claim 1, the only vertices that have negative initial charge are the 5-vertices of \( G \) (each is assigned with charge \(-1\)).

**Claim 3.** Any \( d \)-face \( f \in F(G) \) is incident with at most \( \left\lfloor \frac{2d}{5} \right\rfloor \) 5-vertices.

By Claim 2, on a facial walk of \( f \) no three consecutive vertices are of degree 5. From this, the stated upper bound for the number of 5-vertices incident with \( f \) follows immediately. □

We use the following discharging rules (cf. Figure 1):
(R1) Every 4*-face \( f \) sends charge 1 to any incident 5-vertex \( v \).

(R2) Every 8*-vertex \( u \) sends charge to any adjacent 5-vertex \( v \) if both faces incident with the edge \( uv \) are triangles, as follows: if one of these triangles has another 5-vertex (besides \( v \)), then the sent charge equals \( \frac{1}{3} \); otherwise, the sent charge equals \( \frac{1}{2} \).

(R3) Every 4*-face \( f \) sends through every incident edge \( uw \) with \( d(u) = d(w) = 6 \) charge \( \frac{1}{2} \) to a 5-vertex \( v \) if \( uvw \) is a triangular face and \( f \) is not a 4-face incident with two 5-vertices.

![Figure 1](image_url)

**Figure 1:** In (R2), the numbers standing beside non-labeled vertices indicate their degrees. In (R3), \( f \) is not a 4-face incident with two 5-vertices and \( uvw \) is an adjacent triangular face.

**Claim 4.** No 8*-vertex becomes negatively charged by applying (R2).

Suppose there is a \( d \)-vertex \( u \) with \( d \geq 8 \) that becomes negatively charged by applying Rule 2. Recall that its initial charge was \( d - 6 \). Consider a circular ordering of \( N_G(u) \) in regard to the embedment of \( G \). Note that no three consecutive neighbors \( v_i, v_{i+1}, v_{i+2} \in N_G(u) \) have received charge from \( u \) during the discharging process. Therefore, \( u \) gave charge to at most \( \left\lfloor \frac{2d}{3} \right\rfloor \) of its neighbors, and at most \( \frac{1}{2} \) of charge per neighbor. Consequently, \( d - 6 < \frac{2d}{3} \cdot \frac{1}{2} \). Equivalently, \( d < 9 \). So \( u \) is an 8-vertex. However, it is easily seen that an 8-vertex gives away at most \( \frac{11}{6} \cdot \frac{2}{3} + \frac{2}{3} + \frac{1}{2} \) of its initial charge 2, a contradiction.

**Claim 5.** No 4*-face becomes negatively charged by applying (R1) and (R3).

Suppose there is a \( d \)-face \( f \) that becomes negatively charged by applying Rules 1 and 3. Recall that its initial charge was \( 2d - 6 \). For the purposes of this proof, it is useful to think of the charge sent by \( f \) according to Rule 3 as follows. In case of an ‘isolated’ adjacent triangular face, the charge \( \frac{1}{2} \) coming from \( f \) to this face first splits evenly and goes to the pair of 6-vertices shared with \( f \), and only afterwards reaches the targeted 5-vertex. In case of two ‘consecutive’ triangular faces that are adjacent to \( f \), the total charge \( 1 = \frac{1}{2} + \frac{1}{2} \) first goes to their common 6-vertex, and only afterwards splits evenly and reaches the targeted 5-vertices. With this
perspective in mind, consider a facial walk of $f$ and notice that during the discharging it gives away to each incident vertex charge of either $0, \frac{1}{2}$, or $1$; moreover, no three consecutive vertices receive from $f$ charge $1$. Hence, on at most $\left\lfloor \frac{2d}{3} \right\rfloor$ occasions $f$ gives charge $1$, and on the other $\left\lceil \frac{d}{3} \right\rceil$ occasions $f$ gives charge at most $\frac{1}{2}$. Consequently, $2d - 6 < \frac{2d}{3} + \frac{4d}{3}$. Equivalently, $d < \frac{21}{7}$. So $f$ is a $4$-face. However, it is easily seen that a $4$-face gives away at most $2$ of its initial charge (which also equals $2$), a contradiction.

Since the total charge remains negative (it equals $-12$), from Claims 4 and 5 it follows that there is a $5$-vertex $v$ which remains negatively charged even after applying the discharging rules. By (R1) and (R2), we have the following.

**Claim 6.** The $5$-vertex $v$ has only $3$-faces around it, and it is a neighbor of at most two $8^+$-vertices. Moreover, if $v$ neighbors exactly two $8^+$-vertices, then these three vertices have another $5$-vertex as a common neighbor.

Let $v_1, v_2, v_3, v_4, v_5$ be the neighbors of $v$ in a circular order regarding the considered plane embedding, i.e., such that $v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1$ are the five $3$-faces incident with $v$.

**Claim 7.** For some $j = 1, 2, \ldots, 5$, the vertices $v_j, v_{j+2}$ are not adjacent and their only common neighbors are $v, v_{j+1}$.

Arguing by contradiction, suppose that every pair of vertices $v_j, v_{j+2}$ are either adjacent or have a common neighbor $\neq v, v_{j+1}$. This readily implies the existence of a vertex $x \neq v$ that is adjacent to all five vertices $v_1, v_2, v_3, v_4, v_5$. Now take a nice coloring $c$ of $G - v$, and let $s = |c(N_G(v))|$. In view of the $5$-cycle $v_1v_2v_3v_4v_5$, it holds that $s \in \{3, 4, 5\}$. In case $s = 3$, the coloring $c$ extends to a nice coloring of $G$ since at most $8$ colors are forbidden at $v$. These are the three colors used on $N_G(v)$ and at most five additional colors in regard to ‘oddness’ in $N_{G-v}(v_i)$ for $i = 1, 2, \ldots, 5$. Hence $s \in \{4, 5\}$, and from this we are able to further deduce that there are precisely $s$ vertices $v_i (i = 1, 2, \ldots, 5)$ such that $c(v_{i-1}) \neq c(v_{i+1})$.

Unless the coloring $c$ extends to $G$, each of the nine available colors is blocked at the vertex $v$, either due to properness or to ‘oddness’. Consequently, there are at least $(9 - s)$ vertices among the $v_i$'s each of which blocks at $v$ a separate color $\neq c(v_1), c(v_2), c(v_3), c(v_4), c(v_5)$ in regard to ‘oddness’ in its neighborhood within $G - v$. Clearly, any such $v_i$ is of even degree in $G$ and any color from $\{c(v_{i-1}), c(v_{i+1})\}$ appears an even number of times in $N_{G-v}(v_i)$. Moreover, for at least four such vertices $v_i$, say $v_1, v_2, v_3, v_4$, it also holds that $c(v_{i-1}) \neq c(v_{i+1})$. Noting that the color $c(x)$ occurs in every of the four neighborhoods $N_{G-v}(v_i)$ for $i = 1, \ldots, 4$, it follows that $c(x)$ has an odd number of appearances in such a neighborhood for at most one vertex $v_i$. Therefore, at least three $v_i$'s are of degree $\geq \frac{2 + 2 + 2 + 1}{2} = 7$ in $G - v$. (Namely, each of the colors $c(v_{i-1}), c(v_{i+1}), c(x)$ yields a summand of $2$, and the summand of $1$ refers to the color blocked by $v_i$ due to ‘oddness’ in $N_{G-v}(v_i)$). However, this yields at least three $8^+$-neighbors of $v$ in $G$, contradicting Claim 6.

Let $v_j, v_{j+2}$ fulfill Claim 7. Consider the graph $(G - v)/\{v_j, v_{j+2}\}$ obtained from $G - v$ by identifying $v_j$ and $v_{j+2}$, and then deleting one of the arising two parallel edges (between the new vertex $\{v_j, v_{j+2}\}$ and the vertex $v_{j+1}$). Take a nice coloring of $(G - v)/\{v_j, v_{j+2}\}$, and look into the inherited coloring $c$ of $G - v$.

**Claim 8.** All $v_i$'s are of even degree in $G$, and the coloring $c$ is nice.
Since $c$ does not extend to a nice coloring of $G$, by reasoning in the same manner as at the beginning of the proof of Claim 7, we are able to conclude that $|c(N_G(v))| \geq 4$. On the other hand, by the construction of $c$ we clearly have $|c(N_G(v))| < 5$. So it must be that exactly four colors appear in $N_G(v)$. Moreover, as $c$ is non-extendable, each of the vertices $v_1, v_2, v_3, v_4, v_5$ blocks at $v$ a separate color $\notin c(N_G(v))$ in regard to already (and uniquely) fulfilled ‘oddness’ in its neighborhood in $G - v$. However, this readily implies that each $v_i$ is of even degree in $G$, and also that $c$ is a nice coloring of $G - v$. 

The proof of Claim 8 certifies that, upon permuting colors, the following color distribution occurs under $c$: the colors 1, 2, 1, 3, 4 are used on $v_1, v_2, v_3, v_4, v_5$, respectively, and each of the colors 5, 6, 7, 8, 9 happens to be the only color with an odd number of appearances on a separate neighborhood $N_{G-v}(v_i)$ (cf. Figure 2(a)). Moreover, as colors 5, 6, 7, 8, 9 are ‘exclusive’ in regard to ‘oddness’ on $N_{G-v}(v_1), N_{G-v}(v_2), N_{G-v}(v_3), N_{G-v}(v_4), N_{G-v}(v_5)$, respectively, any of the colors 1, 2, 3, 4 has an even (possibly 0) number of occurrences in each set $N_{G-v}(v_i)$ ($i = 1, \ldots, 5$). Consequently, the situation depicted in Figure 2(b) is present, that is, at four of the $v_i$’s (namely, $v_1, v_2, v_4, v_5$) we are guaranteed two more ‘free hanging’ colors per vertex.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Local color distribution and guaranteed ‘free hanging’ colors.}
\end{figure}

Let $f_i$ ($i = 1, \ldots, 5$) be the face incident with $v_iv_{i+1}$ but not with $v$ (cf. Figure 3). Note that $f_1, f_2, f_3, f_4, f_5$ need not be pairwise distinct. We say that $f_i$ is convenient if it is a 4-face and $v_i, v_{i+1}$ are 6-vertices whose only common neighbor is $v$.

**Claim 9.** If all of $v_1, v_2, v_3, v_4, v_5$ are 6-vertices in $G$, then at least two members of the list $f_1, f_2, f_3, f_4, f_5$ are convenient.

It suffices to observe the following: if a given $f_i$ is a 3-face, then the vertices $v_i$ and $v_{i+1}$ must share a free hanging color. Since all the $v_i$’s are 6-vertices, it follows that $v_3$ and $v_4$ do not have a common free hanging color. Consequently, $f_3$ is convenient. Quite similarly, $f_5$ is convenient.

Following the same line of reasoning, we also conclude the following:

**Claim 10.** If precisely one of $v_1, v_2, v_3, v_4, v_5$ is an 8-face in $G$, then some $f_i$ is convenient.

Indeed, we may assume that neither $v_1$ nor $v_5$ is the 8-neighbor of $v$. Then $v_1, v_5$ are 6-vertices without a common free hanging color. Consequently, $f_5$ is convenient. 

\[ \square \]
Our next (and final) claim assures that every convenient face $f_i$ sends charge $\frac{1}{2}$ to $v$, in accordance with (R3).

**Claim 11.** No convenient face is a quadrangle $ABCD$ where $A, B$ are 5-vertices and $C, D$ are 6-vertices.

Arguing by contradiction, we suppose that the situation depicted in Figure 4(a) is present. Without loss of generality, we may assume that $B$ and $D$ are non-adjacent and $N_G(B) \cap N_G(D) = \{A, C\}$. Indeed, if each of the pairs of vertices $A, C$ and $B, D$ are either adjacent or share a common neighbor outside the set $\{A, B, C, D\}$, then by planarity, the situation depicted in Figure 4(b) occurs. However, then $C$ and $D$ have a common neighbor $\neq v$, which contradicts that the considered face is convenient.

![Figure 3](image-url)

**Figure 3:** The faces adjacent with the local triangulation around $v$.

![Figure 4](image-url)

**Figure 4:** A convenient face next to the 5-vertex $v$ that happens to be a quadrangle $ABCD$, where $A, B$ are 5-vertices and $C, D$ are 6-vertices.

With our assumption for the pair $B, D$, we look at the graph $G' = (G - \{v, A, C\})/\{B, D\}$, that is, $G'$ is obtained from $G - \{v, A, C\}$ by identifying $B$ and $D$. Since $G'$ is planar and smaller than $G$, it admits a nice coloring. Consider the inherited partial coloring $c$ of $G$. Note
that vertices $B$ and $D$ are colored the same under $c$, and the only uncolored vertices are $v, A$ and $C$. We extend $c$ to a nice coloring of $G$ as follows.

First we color $v$, and for the time being we don’t care about preserving oddness in $N_{G-A-C}(D)$. Apart from the color $c(D)$ and possibly a second color in regard to unique oddness in $N_{G-\{v,A\}}(C)$, at most $6 = 3 \cdot 2$ more colors are forbidden at $v$ (by its three neighbors $\neq C, D$ and oddness in their respective neighborhoods within $G - \{v, A, C\}$). Thus there is a color which is available for $v$.

Next we color $C$. There are at most $8 = 1 + 1 + 3 \cdot 2$ colors forbidden at $C$ (namely, the color $c(B) = c(D)$, the color $c(v)$ and possibly six more colors concerning the other three neighbors of $C$ and oddness in their respective neighborhoods within $G - A$). So there is a color available for $C$. Once $C$ is assigned with an available color, note that there is a color with an odd number of appearances in $N_{G-A}(D)$ (because $d_{G-A}(D) = 5$).

Finally, we color $A$. Since this is a 5-vertex, it suffices to choose a color for $A$ so that the properness of the coloring and the oddness in $N_{G-A}(D)$ are preserved. As $B$ and $D$ are colored the same, and $B$ is a 5-vertex, the number of forbidden colors at $A$ is at most $8 = 1 + 1 + 3 \cdot 2$ (the color $c(B) = c(D)$, possibly a second color in regard to unique oddness in $N_{G-A}(D)$, and six more colors concerning the other three neighbors of $A$ and oddness in their respective neighborhoods within $G - A$). We conclude that $c$ indeed extends to a nice coloring of $G$, a contradiction. □

From Claims 9-11 it follows that the vertex $v$ receives charge at least 1 during the discharging process, hence it cannot remain to be negatively charged. This contradiction concludes our proof.

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