A general recurrence relation for the weight-functions in Mühlbach-Neville-Aitken representations
with application to WENO interpolation and differentiation

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Abstract

In several applications, such as WENO interpolation and reconstruction [Shu C.W.: SIAM Rev. 51 (2009) 82–126], we are interested in the analytical expression of the weight-functions which allow the representation of the approximating function on a given stencil (Chebyshev-system) as the weighted combination of the corresponding approximating functions on substencils (Chebyshev-subsystems). We show that the weight-functions in such representations [Mühlbach G.: Num. Math. 31 (1978) 97–110] can be generated by a general recurrence relation based on the existence of a 1-level subdivision rule. As an example of application we apply this recurrence to the computation of the weight-functions for Lagrange interpolation [Carlini E., Ferretti R., Russo G.: SIAM J. Sci. Comp. 27 (2005) 1071–1091] for a general subdivision of the stencil \(\{x_{i-M}, \ldots, x_{i+M}\} \) of \(M+1 = M_+ + M_- \geq 1\) distinct ordered points into \(K_s + 1 \leq M := M_+ + M_- > 1\) (Neville) substencils \(\{x_{i-M+s}k_s, \ldots, x_{i+M-k_s}\}\) \((k_s \in \{0, \ldots, K_s\})\) all containing the same number of \(M - K_s + 1\) points but each shifted by 1 cell with respect to its neighbour, and give a general proof for the conditions of positivity of the weight-functions (implying convexity of the combination), extending previous results obtained for particular stencils and subdivisions [Liu Y.Y., Shu C.W., Zhang M.P.: Acta Math. Appl. Sinica 25 (2009) 503–538]. Finally, we apply the recurrence relation to the representation by combination of substencils of derivatives of arbitrary order of the Lagrange interpolating polynomial.

Keywords: Mühlbach-Neville-Aitken, weight-functions, (Lagrange) interpolation, WENO

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1. Introduction

The Neville-Aitken algorithm [1, pp. 204–209] constructs the interpolating polynomial on \(\{x_{i-M}, \ldots, x_{i+M}\}\), by recursive combination of the interpolating polynomials on substencils, with weight-functions which are also polynomials of \(x\) [1, pp. 204–209]. Carlini et al. [2], working on the Lagrange interpolating polynomial in the context of centered (central) WENO schemes [3], recognized the connexion between the Neville algorithm [1, pp. 207–208] and the determination of the optimal [3] weight-functions.

Definition 1.1 (Stencil). Let

\[
X_{i-M,i+M} := \{x_{i-M}, \ldots, x_{i+M}\} \subset \mathbb{R}
\]

\[
\begin{cases}
M_+ \in \mathbb{Z} : M := M_+ + M_- \geq 0 \\
x_{i-M} < x_{i-M+1} < \ldots < x_{i+M}, \quad \forall M > 0
\end{cases}
\]

(1)

be a set of \(M + 1\) distinct ordered real points.

\[
\square
\]

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Lemma 2.1 (General recurrence relation for weight-functions). Let \( X_{i-M,i+M} \) be a stencil (Definition 1.1) and assume \( M \geq 2 \) in (1). The \( K_s + 1 > 1 \) substencils

\[
X_{i-M_i+k_i+M_i} := \{x_{i-M_i+k_i}, \ldots, x_{i+M_i-k_i}\} \quad \begin{cases} \mathbb{N} \ni K_s \leq M - 1 := M_+ + M_- - 1 \geq 1 \\ k_i \in \{0, \ldots, K_s\} \end{cases}
\]

(2a)

each of which contains \( M - K_s + 1 \geq 2 \) distinct ordered points and which satisfy

\[
\bigcup_{k_i=0}^{K_s} X_{i-M_i+k_i+M_i-k_i} = X_{i-M_i+M_i} \tag{2b}
\]

\[
X_{i-M_i+k_i+1+M_i} - X_{i-M_i-k_i+1} = (X_{i-M_i+k_i+M_i-k_i} \setminus \{x_{i-M_i-k_i}\}) \cup \{x_{i+M_i-k_i}\} \quad \forall k_i \in \{0, \ldots, K_s - 1\} \tag{2c}
\]
correspond to the \( K_s \)-level subdivision of \( X_{i-M_i+M_i} \) to substencils of equal length, each obtained from its left neighbour by deleting the leftmost point and adding 1 point to the right (2c).

The optimal weight-functions \( \sigma_{i}^{1,2,3,4,5}(x) \) in WENO interpolation satisfy \( \sigma_{i}^{1,2,3,4,5}(x) \) (2a, 3a, 4a, 5a, 6a, 7a)

\[
p_{i}^{1,2,3,4,5}(x) = \sum_{k_i=0}^{K_s} \sigma_{i}^{1,2,3,4,5}(x) p_{i}^{1,2,3,4,5}(x) \quad \forall x \in \mathbb{R} \tag{3a}
\]

\[
\sum_{k_i=0}^{K_s} \sigma_{i}^{1,2,3,4,5}(x) = 1 \quad \forall x \in \mathbb{R} \tag{3b}
\]

where \( p_{i}^{1,2,3,4,5}(x) \) is the Lagrange interpolating polynomial (1) pp. 183–189 of the real function \( f : \mathbb{R} \rightarrow \mathbb{R} \) on the stencil \( X_{i-M_i+M_i} \). The optimal weight-functions \( \sigma_{i}^{1,2,3,4,5}(x) \) correspond to the weight-functions in Mühlbach’s theorem (5, Theorem 2.1, p. 100), where they were expressed in terms of quotients of determinants of interpolation-error functions, directly obtained by the Cramer solution (6, Proposition 5.1.1, p. 72) of error-eliminating linear systems (7, 13, p. 8489). Since WENO approaches are based on nonlinear, with respect to the function \( f(x) \), modifications of the optimal weight-functions (3), we are particularly interested in analytical explicit expressions of the weight-functions.

Carlotti et al. (2, 3, 4, 6, 7, 10), pp. 1074–1079] gave the expression of the polynomial weight-functions \( \sigma_{i}^{1,2,3,4,5}(x) \) for the \( (K_s = r - 1) \)-level subdivision (Definition 1.2) of \( X_{i-(r-1),i+M} \) (Definition 1.1). This result was also confirmed by Liu et al. (4, 2.2, p. 506) who further gave the analytical expression (4, 2.18), p. 511 for the polynomial weight-functions \( \sigma_{i}^{1,2,3,4,5}(x) \) for the \( (K_s = r) \)-level subdivision (Definition 1.2) of \( X_{i-r,i+M} \) (Definition 1.1). In both cases it was observed (2, 3, 4) that \( \forall x \in \mathbb{R} \) the linear weight-functions are positive (\( \geq 0 \)), so that, because of the consistency relation (3b), the combination (3a) of substencils is convex \( \forall x \in \mathbb{R} \).

The purpose of the present note is to prove (Lemma 2.1) a general recurrence relation (8, 70, p. 299) for weight-functions of an arbitrary family of functions, for which the \( (K_s = 1) \)-level subdivision (Definition 2.1) is possible. As an example of application we apply this relation to the Lagrange interpolating polynomial (Proposition 3.1), for an arbitrary level of subdivision (Definition 2.1) of a general stencil (Definition 1.1). The explicit expression of the weight-functions developed in Proposition 3.1 is used (Proposition 3.2) to study the convexity of representation (3). Then, we apply the general recurrence relation (Lemma 2.1) to determine the weight-functions for the representation of the \( n \)-derivative of the Lagrange interpolating polynomial by combination of substencils (Proposition 4.2).

2. General recurrence relation for weight-functions

We start by considering a more abstract case, where a general family of functions \( p_{M_i,M}\) depending on 2 integer indices \( M_i \in \mathbb{Z} \) : \( M_- + M_+ \geq 1 \) (which in practical applications may correspond to stencils; Definition 1.1), are equipped with a 1-level subdivision property, and develop a general recurrence relation for the weight-functions.

Lemma 2.1 (Recursive generation of weight-functions). Let \( p_{M_i,M}\) be a family of real functions

\[
p_{M_i,M} : \mathbb{R} \rightarrow \mathbb{R} \quad \forall M_i \in \mathbb{Z} \quad : \quad M := M_- + M_+ \geq 1 \quad \Rightarrow M_+ > -M_- \tag{4a}
\]
and assume that there exists an associated family of real weight-functions $\sigma_{M,M,1,0}(x)$ (also defining $\sigma_{M,M,1,1}(x) := 1 - \sigma_{M,M,1,0}(x)$)

\[
\sigma_{M,M,1,k} : \mathbb{R} \setminus S_{\mathcal{M},M,1} \to \mathbb{R}
\]

\[
\sigma_{M,M,1,0}(x) + \sigma_{M,M,1,1}(x) = 1
\]

defined everywhere in $\mathbb{R}$ except for a finite set of isolated points $S_{\mathcal{M},M,1} \subset \mathbb{R}$, which may be empty, such that

\[
p_{M,M}(x) = \sigma_{M,M,1,0}(x) p_{M,M}(x) + \sigma_{M,M,1,1}(x) p_{M,M}(x) \quad \forall x \in \mathbb{R} \\
\forall M_k \in \mathbb{Z} : M := M_ - + M_ + \geq 2
\]

Then the family of weight-functions defined recursively by

\[
\sigma_{M,M,k,k}(x) := \min(k_{k-1},k_k) \sum_{\ell_k = \max(0,k-1)}^{k_k} \sigma_{M,M,k-1,\ell_k}(x) \sigma_{M,-\ell_k,M_0-k_0,\ell_k-k_0}(x)
\]

satisfies

\[
p_{M,M}(x) = \sum_{k=0}^{K} \sigma_{M,M,k,k}(x) p_{M,M_0-k_0,M_0-k_0,k}(x) \quad \forall x \in \mathbb{R} \setminus \bigcup_{L=0}^{L_k} S_{\mathcal{M},M,-L,M_0-k_0}
\]

\[
\forall M_k \in \mathbb{Z} : M := M_ - + M_ + \geq 2 \\
\forall K_0 \leq M - 1
\]

Furthermore, for the values of $\{x, M_k, K_k\}$ satisfying the conditions of (4f)

\[
\sum_{k=0}^{K} \sigma_{M,M,k,k}(x) = 1
\]

**PROOF.** Assume $M_k \in \mathbb{Z} : M := M_ - + M_ + \geq 3 \Rightarrow (M_ - - \ell_k) + (M_ + - 1 + \ell_k) = M - 1 \geq 2 \forall \ell_k \in \mathbb{Z}$. Then, (4d) applies to both functions $p_{M,-\ell_k,M_0-1,\ell_k}$ ($\ell_k \in [0, 1]$), and we have

\[
p_{M,-\ell_k,M_0-1,\ell_k}(x) \geq \sum_{m=0}^{1} \sigma_{M,-\ell_k,M_0-1,1,\ell_k,m}(x) p_{M,-\ell_k,M_0-1,1,\ell_k,m}(x) \quad \forall x \in \mathbb{R} \setminus S_{\mathcal{M},M,-L,M_0-1,0} \\
\forall M_k \in \mathbb{Z} : M := M_ - + M_ + \geq 3 \\
\forall \ell_k \in [0, 1]
\]

where $\sigma_{M,-\ell_k,M_0-1,1,\ell_k,m}(x)$, being 1-level weight-functions, exist by (4b) (4d). Combining (4d) (5a), we have

\[
p_{M,M_0}(x) = \sum_{\ell_k=0}^{1} \sigma_{M,M_0,1,\ell_k}(x) \left( \sum_{m=0}^{1} \sigma_{M,-\ell_k,M_0-1,1,\ell_k,m}(x) p_{M,-\ell_k,M_0-1,2,\ell_k,m}(x) \right)
\]

\[
= \sum_{\ell_k=0}^{1} \sum_{m=0}^{1} \sigma_{M,M_0,1,\ell_k}(x) \sigma_{M,-\ell_k,M_0-2,1,\ell_k,m}(x) \quad (4f)
\]

\[
= \sum_{k=0}^{K_0} \left( \sum_{\ell_k=0}^{\min(1,k_0)} \sum_{m=0}^{1} \sigma_{M,M_0,1,\ell_k}(x) \sigma_{M,-\ell_k,M_0-2,1,1,\ell_k,m}(x) \right) p_{M,-k_0,M_0-2,k_0}(x) \\
\forall x \in \mathbb{R} \setminus \left( S_{\mathcal{M},M_0,1} \cup S_{\mathcal{M},M_0-1,1} \cup S_{\mathcal{M},M_0-1,1} \right) \\
\forall M_k \in \mathbb{Z} : M := M_ - + M_ + \geq 3 \Rightarrow 2 \leq M - 1
\]
which proves \((4c, 4f)\), for \(K_s = 2\), because
\[
\bigcup_{L=0}^{2} \left( \bigcup_{\ell_s=0}^{L} \sigma_{M\rightarrow L, -\ell_s, 0, -\ell_s, 0} \bigcup \bigcup_{\ell_s=0}^{1} \sigma_{M\rightarrow L, -\ell_s, 0, -\ell_s, 1} \right) = \left( \sigma_{M\rightarrow L, -1, 0, -1, 0} \cup \sigma_{M\rightarrow L, -1, 0, -1, 1} \right) = \left( \sigma_{M\rightarrow L, -1, 0, -1, 0} \cup \sigma_{M\rightarrow L, -1, 0, -1, 1} \right)
\]  
(5c)

To prove \((4e, 4f)\) \(\forall K_s \in \{1, \ldots, M-1\}\), by induction, assume that \((4e, 4f)\) are valid for some \(K_s = 1 \geq 2\). Then
\[
p_{M\rightarrow M, 1}(x) = \sum_{\ell_s=0}^{1} \sum_{m_s=0}^{1} \sigma_{M\rightarrow M, 1, -\ell_s, -m_s, -1}(x)\]
(5d)

with \(\sigma_{M\rightarrow M, 1, -\ell_s, -m_s, -1}(x)\) in \((5d)\) defined by \((4e)\). Assuming \(K_s \leq M-1\) in \((5d)\), we can further subdivide each of the \(K_s\) functions \(p_{M\rightarrow M, 1}(K_s+1)(x)\) in \((5d)\), once more, and we have by \((4d)\)
\[
p_{M\rightarrow M, 1}(x) = \sum_{\ell_s=0}^{1} \sum_{m_s=0}^{1} \sigma_{M\rightarrow M, 1, -\ell_s, -m_s, -1}(x)\]
(5e)

where \(\sigma_{M\rightarrow M, 1, -\ell_s, -m_s, -1}(x)\), being 1-level weight-functions, exist by \((4b, 4d)\). Combining \((5d, 5e)\), we have
\[
p_{M\rightarrow M, 1}(x) = \sum_{\ell_s=0}^{1} \sum_{m_s=0}^{1} \sigma_{M\rightarrow M, -\ell_s, -m_s, 0}(x)\]
(5f)

\(\forall x \in \mathbb{R} \setminus \left( \bigcup_{\ell_s=1}^{K_1} \bigcup_{m_s=0}^{1} \sigma_{M\rightarrow M, 1, -\ell_s, -m_s, -1}\right) \quad \forall M_s \in \mathbb{Z} : M := M_+ + M_- \geq K_s + 1\)

This completes the proof of \((4f)\) with weight-functions \((4e)\), by induction. By \((4c)\), we have
\[
\sum_{K_s=1}^{K_1} \sum_{\ell_s=0}^{\min(K_s-1, K_1)} \sum_{m_s=0}^{1} \sigma_{M\rightarrow M, K_s, 1, -\ell_s, -m_s, -1}(x) \sigma_{M\rightarrow M, K_s, 1, -\ell_s, -m_s, -1}(x)\]
(5g)

ie the sum of the weight-functions \((4e)\) is the same \(\forall K_s \in \{1, \ldots, M-1\}\) (by induction). Since, by \((4c, 4g)\) holds for \(K_s = 1\), \((5g)\) suffices to prove \((4f)\) \(\forall K_s \in \{1, \ldots, M-1\}\), by induction. \(\square\)
3. Application to the Lagrange interpolating polynomial

By Aitken’s Lemma [1, pp. 204–205], the Lagrange interpolating polynomial satisfies the 1-level subdivision property (4b, 4c), with weight-functions which are linear polynomials, and therefore defined \( \forall x \in \mathbb{R} \), implying that \( S_{S_{x-M,i+M},K_k} = \emptyset \) in (4). Application of Lemma 2.1 to the Lagrange interpolating polynomial can be summarized in the following proposition.

**Proposition 3.1 (Weight-functions for the Lagrange interpolating polynomial)**. Assume the conditions of Definition 1.2. Then, the weight-functions \( \sigma_{x-M,i+M,K_k,k}(x) \) in the representation (3) of the Lagrange interpolating polynomial on \( S_{S_{x-M,i+M},K_k}(x; f) \), are real polynomials of degree \( K_k \) with only real roots, expressed by

\[
R_{K_k}[x] \ni \sigma_{x-M,i+M,K_k,k}(x) \coloneqq (-1)^{K_k-k} b_{i,x-M,i+M,K_k,k} \prod_{x \in X_{i-M,i+M} \setminus X_{i-M,i+M}} (x - x_m) \\
\forall x \in \mathbb{R} \quad \forall K_k \in [1, \ldots, M - 1] := M_+ + M_+ - 1
\]

where the strictly positive real numbers \( b_{i,x-M,i+M,K_k,k} \) depend on the points of the stencil \( X_{i-M,i+M} \) (Definition 1.1), and are generated by the recurrence relation

\[
R_{>0} \ni b_{i,x-M,i+M,K_k,k} \coloneqq \begin{cases} 1 & K_k = 1 \\ \\
\min(K_k-1,k) & K_k \geq 2 \\
\sum_{\ell_1 = \max(0,k-1)}^{\min(K_k-1,k)} b_{i,x-M,i+M,K_k-1,\ell_1} b_{i,x-M,i+M,K_k-1,\ell_1,\ell_2-\ell_1} \quad & \forall K_k \in [1, \ldots, M - 1] := M_+ + M_+ - 1 
\end{cases}
\]

The weight-functions (6a) satisfy the consistency condition (3b) and the recurrence relation (4c).

**PROOF.** The case \( K_k = 1 \)

\[
\sigma_{x-M,i+M,i+M,K_k,k}(x) \overset{(6a)}{=} (-1)^{K_k-k} \frac{1}{x_i-M_+ - x_j-M_-} (x - x_M - k, M) \quad \forall K_k \in [0, 1]
\]

holds since it is exactly Aitken’s Lemma [1, pp. 204–205]. Since (6) hold for \( K_k = 1, \forall M_+ \in \mathbb{Z} : M_+ := M_+ + M_+ \geq 2 \) the family of Lagrange interpolating polynomials is equipped with the 1-level subdivision rule (4a–4c), and therefore satisfies the conditions of Lemma 2.1. We can therefore apply (4c) to \( \sigma_{x-M,i+M,K_k,k}(x) \). To obtain the simpler expressions (6), assume that (6a) holds for some \( K_k - 1 \geq 1 \). Then by Lemma 2.1

\[
\sigma_{x-M,i+M,i+M,K_k,k}(x) \overset{(6b)}{=} \sum_{\ell_1 = \max(0,k-1)}^{\min(K_k-1,k)} \sigma_{x-M,i+M,i+M,K_{k-1},K_{k-1}}(x) \sigma_{x-M,i+M,i+M,K_{k-1},K_{k-1},1,\ell_1} \ell_1-x_M \prod_{x \in X_{i-M,i+M} \setminus X_{i-M,i+M}} (x - x_M) \\
\times (x - x_M) \overset{(7a)}{=} (-1)^{K_k-k} \frac{1}{x_i-M_+ - x_j-M_-} (x - x_M)
\]

because \( (X_{i-M,i+M} \setminus X_{i-M,i+M}) (x_i-M_+ + \ell_1) = (X_{i-M,i+M} \setminus X_{i-M,i+M}) (x_i-M_+ + \ell_1, \ell_1, \ell_1) = (X_{i-M,i+M} \setminus X_{i-M,i+M}) (x_i-M_+ + \ell_1, \ell_1, \ell_1) \). Since (6a) (6b) are valid for \( K_k = 1 \) by Aitken’s Lemma [1, pp. 204–205], (7b) proves that they are...
also valid for $K_s = 2$, and by induction $\forall K_s \in \{1, \cdots, M-1 := M_+ + M_s - 1\}$, completing the proof. Direct computation, using (7a), proves that the consistency relation (3b) holds for $K_s = 1$, and, by Lemma 2.1 $\forall K_s \in \{1, \cdots, M-1\}$.

Finally, strict positivity of $b_{I_x,x_1,\cdots,x_r,k}$ holds for $K_s = 1$, and then by induction, using (6a), $\forall K_s \in \{0, \cdots, K_s\}$ and $\forall K_s \in \{1, \cdots, M-1 := M_+ + M_s - 1\}$. □

Because of the positivity of the numbers $b_{I_x,x_1,\cdots,x_r,k} \in \mathbb{R}_{>0}$ (6b) it is quite straightforward to study the sign of the weight-functions $\sigma_{I_x,x_1,\cdots,x_r,k}(x)$ (6a), which allows to determine the intervals on the real axis where the combination (3) of the Lagrange interpolating polynomials on the substencils is convex.

**Proposition 3.2 (Convexity in the neighbourhood of $x_1$).** Assume the conditions of Proposition 3.1. Furthermore assume that $K_s \leq \left\lceil \frac{M}{2} \right\rceil$. Then the weight-functions of the combination (3) of the Lagrange interpolating polynomials on substencils (Proposition 3.1) satisfy

$$0 \leq \sigma_{I_x,x_1,\cdots,x_r,k}(x) \leq 1 \quad \forall x \in [x_{i-M+K_s-1}, x_{i+M-K_s+1}]$$

$$\forall M_z \in \mathbb{Z} : M := M_+ + M_s \geq 2 \quad \forall K_s \in \{1, \cdots, \left\lceil \frac{M}{2} \right\rceil\}$$

(8)

$$\forall k_s \in \{0, \cdots, K_s\}$$

PROOF. Because of the consistency condition (3b), (non strict) positivity of the weight-functions $\sigma_{I_x,x_1,\cdots,x_r,k}(x)$ (6) suffices (proof by contradiction) to prove (8). Rewrite (6a) as

$$\sigma_{I_x,x_1,\cdots,x_r,k}(x) = (-1)^{K_s-k} b_{I_x,x_1,\cdots,x_r,k}$$

(9a)

Obviously we have

$$\text{sign} \left( \prod_{i=M}^{i=M+k_s-1} (x-x_i) \right) = 1 \forall k_s \in \{1, \cdots, K_s\} \quad \forall x > \max_{0 \leq k_s \leq K_s} x_{i-M+K_s-1} - x_{i-M+K_s-1}$$

(9b)

$$\text{sign} \left( \prod_{i=M}^{i=M+k_s-1} (x-x_i) \right) = (-1)^{K_s-k} \forall k_s \in \{0, \cdots, K_s-1\} \quad \forall x < \min_{0 \leq k_s < K_s} x_{i+M-K_s+k_s+1} - x_{i+M-K_s+1}$$

(9c)

Combining (9a, 9b) with the positivity of the numbers $b_{I_x,x_1,\cdots,x_r,k} \in \mathbb{R}_{>0}$ (6b), and taking into account that $(-1)^{K-s-K_s} = 1$, proves (8). Notice that the condition for the interval $[x_{i-M+K_s-1}, x_{i+M-K_s+1}]$ in (8) to contain at least 1 cell (at least 2 grid-points) is $M_+ + K_s - 1 < M_+ - K_s + 1 \iff 2K_s < M_+ + M_s - 2 \iff K_s < \left\lceil \frac{M+2}{2} \right\rceil + 1$, which explains the additional constraint on $K_s$ included in the hypotheses of Proposition 3.2. □

**Remark 3.3 (Typical stencils [2, 4]).** For $\sigma_{I_x,x_1,\cdots,x_r,k}(x)$ the positivity interval is, by (8), $[x_{i-(r-1)}+1, x_{i+(r-1)}+1] = [x_{i-1}, x_{i+2}]$ in agreement with [4, Tab. 2.1, p. 507], while for $\sigma_{I_x,x_1,\cdots,x_r,k}(x)$ the positivity interval is, by (8), $[x_{i-r+1}, x_{i+r+1}] = [x_{i-1}, x_{i+1}]$ in agreement with [4, Tab. 2.2, p. 511]. □

Lemma 2.1 only requires the determination of weight-functions for the 1-level subdivision. It is therefore not limited to a particular family of stencils and/or subdivisions, and can be used to determine weight-functions on biased stencils, eg near the boundaries of the computational domain.
Remark 3.4 (Relation to previous work). WENO interpolation applied to the development of central WENO schemes only requires the computation of the value of the weight-functions at specific points on the stencil, and these can be computed by solving a linear system \([3]\). Carlini et al. \([2]\) pointed out that the weight-functions in representation \((5)\) are of the form \(\alpha_{k}^{\pm},\) with unknown constants \(\gamma_{k}^{\pm}\), which can be determined, in the general case, by solution of a linear triangular system. These authors \([2]\) studied in particular the stencil \(X_{r-(r-1)}\), for \(K_{r} = r - 1\), for which they obtained an analytical expression for the coefficients \(\gamma_{k}^{\pm}\), and proved convexity \(\forall x \in [x_{1}, x_{2}]\). Liu et al. \([4]\) used the same form as Carlini et al. \([2]\) for the weight-functions, computed the coefficients up to \(r = 6\), and observed that the interval of convexity is actually \([3.3]\). They also studied the stencil \(X_{r-(r+1)}\), for \(K_{r} = r\), gave an analytical expression for the coefficients \(\gamma_{k}^{\pm}\), which were computed up to \(r = 5\), and observed that the interval of convexity in this case is \([3.3]\). They also proved that the interval of convexity is actually \([3.3]\). These results are in agreement with those proven in Proposition 3.2 (Remark 3.3). Proposition 3.1 studies an arbitrary stencil \(X_{r-M+M,}\) \((\text{Definition} 3.1)\), and level of subdivision \(K_{r} \in \{1, \ldots, M - 1\}\) \((M := M_{+} + M_{-})\), and obtains an analytical recursive expression for the coefficients \(\beta_{1}^{(r-M+M,)},\) \((6a)\) in \([6a]\). In this way, we were able to give a formal proof for the interval of convexity, which was determined for general values of \(M_{+}\) and \(K_{r}\) \((\text{Remark} 3.3)\).

Remark 3.5 (Alternative formulation). The expression of the \((K_{r} = 1)\)-level weight-functions \((7a)\) can also be written in an equivalent form, using ratios of fundamental functions of Lagrange interpolation on \(X_{r-M+M,}\) \((\text{Definition} 1.1)\), and on its \((K_{r} = 1)\)-level Neville substencils \((\text{Definition} 1.2)\), \(X_{r-M+M-1}\) and \(X_{r-M+1}\). The Lagrange interpolating polynomial of a function \(f : \mathbb{R} \to \mathbb{R}\), on \(X_{r-M+M,}\), can be expressed \([1, (9.3, 9.4), \text{p. 184}]\) as

\[
\begin{align*}
\sigma_{X_{r-M+M,}}^{(1)}(x) & = \sum_{\ell = 1}^{M} \alpha_{X_{r-M+M,}}^{\ell}(x) f_{r+\ell} \quad (10a) \\
\sigma_{X_{r-M+M,}}^{(2)}(x) & = \sum_{\ell = 1}^{M} \alpha_{X_{r-M+M,}}^{\ell}(x) f_{r+\ell} \quad (10b) \\
\end{align*}
\]

where the \(M + 1\) polynomials \(\alpha_{X_{r-M+M,}}^{\ell}(x) \in \mathbb{R}_{M}[x] \) \((10b)\) are \(\neq 0_{\mathbb{R}_{M}[x]}(x)\), linearly independent, and form a basis of the space of all polynomials with real coefficients and degree \(\leq M, \mathbb{R}_{M}[x] \) \([9, \text{p. 2771}]\). It can be verified by direct computation, using definition \((10b)\), that \((7a)\) is equivalent to

\[
\begin{align*}
\sigma_{X_{r-M+M,}}^{1.1}(x) & = \sum_{\ell = 1}^{M} \alpha_{X_{r-M+M,}}^{\ell}(x) f_{r+\ell} \quad (11a) \\
\sigma_{X_{r-M+M,}}^{1.2}(x) & = \sum_{\ell = 1}^{M} \alpha_{X_{r-M+M,}}^{\ell}(x) f_{r+\ell} \quad (11b) \\
\end{align*}
\]

The interest of this alternative expression \((11)\), which is analogous to the expression of the \((K_{r} = 1)\)-level weight-functions for the representation of the Lagrange reconstructing polynomial on a homogeneous grid \([9, \text{Lemma 4.2, p. 2780}]\), is that it can be generalize for the representation of the \(n\)-derivative of the Lagrange interpolating polynomial by combination of substencils, as will be shown in \([4]\).

4. Application to the \(n\)-derivative of the Lagrange interpolating polynomial

One of the motivations that led to the formulation of Lemma 2.1 was the study of WENO reconstruction in view of the computation of \(f'(x)\), which is treated in \([9]\) results and relation to previous work are summarized in Remark 4.1. The expression of the \((K_{r} = 1)\)-level weight-functions for the Lagrange reconstructing polynomial is similar to \((11)\), upon replacing the fundamental functions of Lagrange interpolation in \((11)\) by the corresponding fundamental functions of Lagrange reconstruction \([9, (32), \text{Lemma 4.2, pp. 2780–2781}]\). It turns out that a similar relation is valid for the \(n\)-derivative of the Lagrange interpolating polynomial

\[
\begin{align*}
P_{X_{r-M+M,}}^{(n)}(x; f) & = \sum_{\ell = 1}^{M} \alpha_{X_{r-M+M,}}^{(n)}(x) f_{r+\ell} \quad (12) \\
\end{align*}
\]
Remark 4.1 (Lagrange reconstructing polynomial). The case of reconstruction of a function \( h(x) \) from its cell-averages \( f(x) \), sampled on a given stencil (Definition 1.2), is important for the construction of numerical schemes used in the solution of hyperbolic PDEs \([4]\). In the particular case of homogeneous grids \( (x_{i+1} - x_i = \Delta x = \text{const} \in \mathbb{R}_{>0} \forall i) \), reconstruction can be used for the computation of \( f'(x) \) \([3, 8]\). In the context of methods for the determination of numerical fluxes, what is needed are the values of the weight-functions at \( x_{i+\frac{1}{2}} \) (optimal weights \([3]\)), which were usually computed from the solution of a linear system \([7, (13), p. 8489]\). Recently, Arandiga et al. \([10]\) gave analytical expressions of the optimal weight functions for the Lagrange reconstructing polynomial \([8]\), obtain explicit recursive expressions for the weight-functions for the Lagrange reconstructing polynomial \([8]\), and these new results are formulated in Proposition 4.2.

Liu et al. \([4]\), also study the representation of the first two derivatives \( (n \in [1, 2]) \) of the Lagrange interpolating polynomial for the particular homogeneous stencils and sub-stencils studied in the reconstruction case (Remark 4.1). In the present work, we show that the \( (K_n = 1) \)-level subdivision (Definition 1.2) weight-functions can be explicitly determined for the \( n \)-derivative of the Lagrange interpolating polynomial \([12]\). Using Lemma 2.1 we define the weight-functions for the representation of the \( n \)-derivative of the Lagrange interpolating polynomial \([12]\) on \( x_{i-M+i} \), \( (\text{Definition} \ 1.1) \), by combination of the \( n \)-derivative of the Lagrange interpolating polynomials on the \( K_n \)-level sub-stencils (Definition 1.2), requiring that \( n \leq M - K_n \), so that the \( n \)-derivative be \( \neq 0_{x_{i-M+i} \rightarrow x_i}(x) \) on the sub-stencils. The result is formulated in the following Proposition.

Proposition 4.2 (Weight-functions for the \( n \)-derivative of the Lagrange interpolating polynomial). Assume the conditions of Proposition 5.1. Then, \( \forall M_n \in \mathbb{Z} : M := M + M_n \geq 2, \forall K_n \leq M - 1, \forall n \leq M - K_n \), the \( n \)-derivative with respect to \( x \) of the Lagrange interpolating polynomial on \( x_{i-M+i} \), \( (\text{Definition} \ 1.1) \) can be represented, almost everywhere, by combination of the \( n \)-derivative of the Lagrange interpolating polynomials on the \( K_n \)-level sub-stencils (Definition 1.2) of \( x_{i-M+i} \), as

\[
p_{j,n}^{(n)}(x; f) = \sum_{K_n=0}^{K_n} \sigma_{l_0,\ldots,l_{K_n}}^{\text{Lag},x_{i-M+i}}(x) \quad \forall x \in \mathbb{R} \setminus S_{l_0,\ldots,l_{K_n}}^{(n)}
\]

where the rational weight-functions \( \sigma_{l_0,\ldots,l_{K_n}}^{\text{Lag},x_{i-M+i}}(x) \) are defined recursively by

\[
\sigma_{l_0,\ldots,l_{K_n}}^{\text{Lag},x_{i-M+i}}(x) :=
\begin{cases}
\sigma_{l_0,\ldots,l_{K_n}}^{\text{Lag},x_{i-M+i}}(x) & K_n = 1 \\
\sigma_{l_0,\ldots,l_{K_n}}^{\text{Lag},x_{i-M+i}}(x) & K_n \geq 2 \\
\min(K_n - 1, K_n) \sum_{\ell_0=\max(0,l_0-1)}^{K_n} \sigma_{l_0,\ldots,l_{K_n},K_n}^{\text{Lag},x_{i-M+i}}(\cdot) \sigma_{l_0,\ldots,l_{K_n},K_n}^{\text{Lag},x_{i-M+i}}(\cdot) & K_n \geq 2
\end{cases}
\]

for all \( l_0, \ldots, l_{K_n} \) \( \forall k_n \in [0, \ldots, K_n] \) \( \forall K_n \in [1, \ldots, M - 1] \) \( \forall n \in [0, \ldots, M - K_n] \)

and satisfy the consistency condition

\[
\sum_{k_n=0}^{K_n} \sigma_{l_0,\ldots,l_{K_n}}^{\text{Lag},x_{i-M+i}}(x) = 1 \quad \forall x \in \mathbb{R} \setminus S_{l_0,\ldots,l_{K_n}}^{(n)}
\]

\(^1\) on general inhomogeneous grids reconstruction does not provide \( f'(x) \), only numerical fluxes for the discretization of the PDE \([3]\)
The set of poles of the rational weight-functions $S_{l_i, \xi_1, \ldots, \xi_M \mid K_s}$ satisfies

\[
S_{l_i, \xi_1, \ldots, \xi_M, \mid K_s} \subseteq \left\{ x \in \mathbb{R} : \prod_{i=0}^{K_s-1} \prod_{\ell=0}^{L-1} \alpha_{l_i, \xi_1, \ldots, \xi_M, \mid M}^{(a)}(x) = 0 \right\}
\]  

\[
\forall K_s \in \{1, \ldots, M-1\}
\]  

(13d)

PROOF. Because of Lemma 2.1 it suffices to prove that Proposition 4.2 is valid for $K_s = 1$. Consider first the consistency relation (15a) for the $(K_s = 1)$-level weight-functions defined by (13a).

\[
\sigma_{l_i, \xi_1, \ldots, \xi_M, \mid K_s = 1}(x) = \frac{\alpha_{l_i, \xi_1, \ldots, \xi_M, \mid M, \mid M}^{(a)}(x)}{\alpha_{l_i, \xi_1, \ldots, \xi_M, \mid M+1, \mid M}^{(a)}(x)} \quad \forall n \in \{0, \ldots, M-1\}
\]  

(14a)

\[
\sigma_{l_i, \xi_1, \ldots, \xi_M, \mid K_s = 1}(x) = \frac{\alpha_{l_i, \xi_1, \ldots, \xi_M, \mid M+1, \mid M}^{(a)}(x)}{\alpha_{l_i, \xi_1, \ldots, \xi_M, \mid M, \mid M}^{(a)}(x)} \quad \forall n \in \{0, \ldots, M-1\}
\]  

(14b)

By straightforward computation using the expression (10b) of the fundamental functions of Lagrange interpolation, we have

\[
\alpha_{l_i, \xi_1, \ldots, \xi_M, \mid M, \mid M}^{(a)}(x) = \prod_{k=-M+1}^{M-1} \frac{x - x_{i+k}}{x_i - M_k} = \prod_{k=-M+1}^{M-1} \frac{x_{i+k} - x_i}{x_i - M_k} \prod_{k=-M+1}^{M-1} x_i - M_k
\]  

(15a)

and taking into account that the product in the last line of (15a) is independent of $x$, we have by differentiation

\[
\alpha_{l_i, \xi_1, \ldots, \xi_M, \mid M, \mid M}^{(a)}(x) = \left( \prod_{k=-M+1}^{M-1} \frac{x_{i+k} - x_i}{x_i - M_k} \right) \left( \prod_{k=-M+1}^{M-1} \frac{x_i - M_k}{x_i - M_k} \right) \alpha_{l_i, \xi_1, \ldots, \xi_M, \mid M, \mid M}^{(a)}(x) \quad \forall n \in \{0, \ldots, M-1\}
\]  

(15b)

where $n = 0$ implies no differentiation, i.e. (15a). Since the product in (15b) is independent of $n$, we also have

\[
\alpha_{l_i, \xi_1, \ldots, \xi_M, \mid M, \mid M}^{(a)}(x) = \prod_{k=-M+1}^{M-1} \frac{x_{i+k} - x_i}{x_i - M_k} \alpha_{l_i, \xi_1, \ldots, \xi_M, \mid M, \mid M}^{(a)}(x) \quad \forall n \in \{1, \ldots, M-1\}
\]  

(15c)

We know by Aitken’s Lemma [11. pp. 204–205] that the $K_s = 1$ consistency relations hold for $n = 0$, and by direct computation from the expression (10b) of the fundamental functions of Lagrange interpolation, we have (7a) [11]

\[
\frac{\alpha_{l_i, \xi_1, \ldots, \xi_M, \mid M, \mid M}^{(a)}(x)}{\alpha_{l_i, \xi_1, \ldots, \xi_M, \mid M+1, \mid M}^{(a)}(x)} = \frac{x - x_{i+M}}{x_i - M_k - x_i}
\]  

(15d)

\[
\frac{\alpha_{l_i, \xi_1, \ldots, \xi_M, \mid M+1, \mid M}^{(a)}(x)}{\alpha_{l_i, \xi_1, \ldots, \xi_M, \mid M, \mid M}^{(a)}(x)} = \frac{x - x_i}{x_i - M_k - x_i - M_k}
\]  

(15e)

which give by successive differentiation (proof by induction)

\[
\alpha_{l_i, \xi_1, \ldots, \xi_M, \mid M, \mid M}^{(a)}(x) \quad \frac{\alpha_{l_i, \xi_1, \ldots, \xi_M, \mid M, \mid M}^{(a)}(x) - \alpha_{l_i, \xi_1, \ldots, \xi_M, \mid M+1, \mid M}^{(a)}(x)}{\alpha_{l_i, \xi_1, \ldots, \xi_M, \mid M, \mid M}^{(a)}(x) - \alpha_{l_i, \xi_1, \ldots, \xi_M, \mid M+1, \mid M}^{(a)}(x)} \quad \alpha_{l_i, \xi_1, \ldots, \xi_M, \mid M, \mid M}^{(a)}(x)
\]  

(15f)

\[
\alpha_{l_i, \xi_1, \ldots, \xi_M, \mid M+1, \mid M}^{(a)}(x) \quad \frac{\alpha_{l_i, \xi_1, \ldots, \xi_M, \mid M+1, \mid M}^{(a)}(x) - \alpha_{l_i, \xi_1, \ldots, \xi_M, \mid M, \mid M}^{(a)}(x)}{\alpha_{l_i, \xi_1, \ldots, \xi_M, \mid M+1, \mid M}^{(a)}(x) - \alpha_{l_i, \xi_1, \ldots, \xi_M, \mid M, \mid M}^{(a)}(x)} \quad \alpha_{l_i, \xi_1, \ldots, \xi_M, \mid M+1, \mid M}^{(a)}(x)
\]  

(15g)
Using the above relations (14, 15), we readily have

\[ \sigma_{I_n, x_1, \ldots, x_M + 1}(x) = \sigma_{I_n, x_1, \ldots, x_M, 1}(x) \]

\[ = \frac{x - x_i + M_i}{x_i + M_i - x + M_{i-1}} + \frac{x - x_i - M_i}{x_i + M_i - x - M_{i-1}} \]

\[ = 1 \quad \forall n \in \{1, \ldots, M - 1\} \] (16)

By (16), we have proven the consistency condition (13c), for \( K_n = 1 \), \( \forall n \in \{0, \ldots, M - 1\} \), the case \( n = 0 \) following from Proposition [11]. Obviously the weight-functions (14) are defined almost everywhere, except at the roots of the denominator, which because of (15b), are defined by (13d). The \( \equiv \) relation in (15d) is used, because there may be cancelation of poles by polynomial division, as in the interpolation case \( (n = 0) \), where there are no singularities [6].

To prove the representation (13a), for \( K_n = 1 \), we start from the remainder theorem of the Lagrange interpolating polynomial, which states [11, Theorem 9.2, p. 187] that for any real function \( f : \mathbb{R} \to \mathbb{R} \) of class \( C^{M+1} \)

\[ \forall f \in C^{M+1}(\mathbb{R}) \quad \forall x \in \mathbb{R} \quad \exists \{x; x_{n+1}, \ldots, x_M; i, \} \in [\min(x, x_{n+1}), \max(x, x_M)] : \]

\[ p_{I_n, x_{n+1}, \ldots, x_M}(x; f) = f(x) - \frac{1}{(M+1)!} \left( \prod_{k=1-M}^{M} (x - x_{i+k}) \right) f^{(M+1)}(i) \] (17a)

Since for any polynomial of degree \( \leq M \) we have

\[ \forall q \in \mathbb{R}_M[x] \implies \left\{ \begin{array}{l} q^{(M)}(x) = \text{coeff}[x^M, q(x)] M! \\ q^{(M+1)}(x) = 0 \end{array} \right\} \quad \forall x \in \mathbb{R} \] (17b)

we can write because of (17a)

\[ p_{I_n, x_{n+1}, \ldots, x_M}(x; q) = q(x) - \prod_{k=1-M}^{M} (x - x_{i+k}) \] (17c)

\[ p_{I_n, x_{n+1}, \ldots, x_M}(x; q) = q(x) \quad \forall q \in \mathbb{R}_M[x] : \text{coeff}[x^M, q(x)] = 1 \] (17d)

\[ p_{I_n, x_{n+1}, \ldots, x_M}(x; q) = q(x) - \prod_{k=1-M+1}^{M} (x - x_{i+k}) \] (17e)

Using the expression (10b) of the fundamental functions of Lagrange interpolation in (17c, 17e) and differentiating, we have, \( \forall n \in \mathbb{N}_0 \),

\[ p_{I_n, x_1, \ldots, x_M, -1}(x; q) = q^{(0)}(x) - \prod_{k=1-M}^{M-1} (x_{i+k} - x_{i+k}) \] (17f)

\[ p_{I_n, x_1, \ldots, x_M}(x; q) = q^{(0)}(x) \quad \forall q \in \mathbb{R}_M[x] : \text{coeff}[x^M, q(x)] = 1 \] (17g)

\[ p_{I_n, x_{n+1}, \ldots, x_M}(x; q) = q^{(0)}(x) - \prod_{k=1-M+1}^{M} (x_{i+k} - x_{i+k}) \] (17h)
Combining \((14\text{b, 17b})\), and using \((15b)\), we have
\[
\begin{align*}
\text{(14b, 17b)} & \\
& = \left(\sigma_{k_0,\ldots,k_{M-1},1,0}^\prime (x) + \sigma_{k_0,\ldots,k_{M-1},1,1}^\prime (x)\right) q^{(n)}(x) \\
& = q^{(n)}(x) \quad \forall q \in \mathbb{R}[x] : \text{coeff}[x^M, q(x)] = 1 \quad \forall n \in [1, \ldots, M-1]
\end{align*}
\]

Applying \((17)\), successively, to the polynomials
\[
\mathbb{R}[x] \ni \prod_{m=-M}^{M} (x - x_{m;k}) = 0 \quad \forall x \in \{x_{-M,\ldots,-1}, x_{1,k}\} \setminus \{x_{k}\} \quad \forall k \in \{-M,\ldots,M\}
\]

yields
\[
\begin{align*}
\sigma_{k_0,\ldots,k_{M-1},1,0}^\prime (x) & = \sigma_{k_0,\ldots,k_{M-1},1,1}^\prime (x) \\
\sigma_{k_0,\ldots,k_{M-1},i+M}^\prime (x) & = \sigma_{k_0,\ldots,k_{M-1},i+1,1}^\prime (x) \\
\sigma_{k_0,\ldots,k_{M-1},i+\ell}^\prime (x) & = \sigma_{k_0,\ldots,k_{M-1},i+1,1}^\prime (x) \quad \forall \ell \in \{-M+1,\ldots,M\} - \{1\}
\end{align*}
\]

Replacing the expression \((12)\) for the \(n\)-derivative of Lagrange interpolating polynomials in \((13a)\), proves, by \((17)\), \((17n)\), \((13a)\) \(\forall f : \mathbb{R} \rightarrow \mathbb{R}\), for \(K = 1\). Lemma \(2.1\) completes the proof.

\begin{remark}
\text{(Relation of Proposition 2.2 to previous work.) To the author’s knowledge, the case \(n \geq 3\) has not been studied before. For \(n \in \{1,2\}\), the simple analytical recursive expression \((13b)\) for the weight-functions agrees with the expressions obtained by Liu et al. \((4)\), for the case of a homogeneous grid, using symbolic calculation for \(r \in \{2,\ldots,7\}\), for the \((K = r-1)\)-level subdivision of the stencils \(X_{i-r,\ldots,i} := [-r+\frac{1}{r}, \ldots, -\frac{1}{r}]\) \((4)\), Tab. 3.2, p. 516, for \(n = 1\), and Tab. 3.8, p. 520, for \(n = 2\) and for the \((K = r)\)-level subdivision of the stencils \(X_{i-r,\ldots,i} := [-r+\frac{1}{r}, \ldots, r+\frac{1}{r}]\) \((4)\), Tab. 3.5, p. 518, for \(n = 1\), and Tab. 3.9, p. 521, for \(n = 2\). Proposition \(2.2\) formally proves the existence and analytical expression of the weight-functions, for the \(n\)-derivative \((\forall n \in [0,\ldots,M-K])\), for a general \(K\)-level subdivision \((K_e \in \{1,\ldots,M-1\})\), for an arbitrary stencil of \(M+1\) distinct ordered points (Definition \(1.1\)), on an inhomogeneous grid.}
\end{remark}
5. Conclusions

Every system of functions \( f_i \), depending on 2 integer parameters, which is equipped with an associated system of weight-functions satisfying a 1-level subdivision property also satisfies \( K_s \)-level subdivision relations (Lemma 2.1), with weight-functions generated by the recurrence, which can be interpreted as an inverted generalized Neville algorithm [1, pp. 207–208].

As an application of Lemma 2.1 we developed simple explicit expressions for \( K_s \)-level weight-functions of the Lagrange interpolating polynomial (Proposition 3.1) on a general stencil in an inhomogeneous grid, which allow explicit determination of the interval of positivity of the weight-functions (Proposition 3.2) generalizing previous results [2, 4]. By (8) the length of the positivity interval is \( M_+ - M_- - 2K_s + 2 \) cells.

We further investigated the existence of \( K_s \)-level weight-functions for the \( n \)-derivative of the Lagrange interpolating polynomial \( (n \in \{1, \ldots, M - K_s\}) \). Having proved simple analytical expressions for the 1-level weight-functions, Lemma 2.1 was applied to develop an analytical recursive expression for the \( K_s \)-level weight-functions of the \( n \)-derivative \( (n \in \{1, \ldots, M - K_s\}) \). These results are valid for general inhomogeneous grids.

Other potential applications of Lemma 2.1 include WENO integration [4], the development of WENO schemes for biased near-boundary stencils, and other than Lagrange types of polynomial interpolation [3].

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