Competitive Equilibria with Indivisible Goods and Generic Budgets*

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Abstract

We study competitive equilibria in the basic Fisher market model, but with indivisible goods. Such equilibria fail to exist in the simplest possible market of two players with equal budgets and a single good, yet this is a knife’s edge instance as equilibria exist once budgets are not precisely equal. Is non-existence of equilibria also a knife-edge phenomenon in complex markets with multiple goods? Our computerized search has indicated that equilibria often exist when budgets are “generic”. We prove several existence results both for the case of general preferences and for the special case of additive preferences, and relate competitive equilibria to notions of fair allocation of indivisible items.

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1 Introduction

In this paper we study competitive equilibria in markets of indivisible goods without money, when players have generic, possibly very different, budgets. Such markets correspond to many real-life applications in which resources must be allocated but money is inappropriate, including workforce scheduling (e.g., allocating shifts to shift workers), sharing scientific resources, sharing computational resources within a large company, and allocating university courses to students [15]. We have two motivations: first, to understand conditions for existence of competitive equilibria in markets of indivisible goods, and second, to study notions of fairness in division of indivisible goods where the participants may have unequal “entitlements” for the goods. We will proceed by describing each of these two motivations separately, and then present our results.

1.1 Motivation 1: Markets with Indivisible Goods

In a classic competitive market equilibrium, goods are assigned prices, each participant takes his preferred set of goods among those that are within his budget under these prices, and the market clears. By the First Welfare Theorem, the resulting allocation is Pareto efficient. The fundamental achievement of general equilibrium theory is proving the existence of such equilibria in “convex” settings, as shown in the seminal work of Arrow and Debreu [2] and in much subsequent work (see, e.g., [36, Chapters 15–20]). For concreteness and simplicity let us consider the simple one-sided model of Fisher markets [7] (see also, e.g., [40, Chapters 5–6]). Here a single seller brings \( m \) goods to the market, and \( n \) buyers each bring a certain amount of “money” to the market in the form of budgets \( b_1, \ldots, b_n \). The seller has no use for the goods, while the buyers have no use for money and only desire the goods, each buyer having a preference order among all possible bundles of goods. If the goods are infinitely divisible, then a market equilibrium is known to exist if buyers’ preferences satisfy very mild conditions [1].

But what if some goods are indivisible? In such cases it is easy to see that an equilibrium need not exist – just consider a single indivisible item sold among 2 players with the same budget of 1: if the item’s price is at most 1 then both players desire it, while if the price is strictly above 1 then neither of them does [2]. This, however, is a knife’s edge phenomenon: if the budgets are not exactly the same, say one budget is 1.0001 while the other is 1, then an equilibrium where the item has an intermediate price between the two budgets exists. For this reason, Budish [13] considered budgets that are perturbed from equality. In this paper we more generally consider the case of generic budgets, for which it turns out that it is “easier” for an equilibrium to exist.

Even with generic budgets, an equilibrium does not always exist. A case that was heavily studied is the quasi-linear one, where there is a single infinitely divisible item (which can be thought of as money that is desired by the buyers), as well as a collection of indivisible goods, and where buyers’ preferences are linear in the divisible good and monotone over the indivisible items. When budgets are large enough, this becomes exactly the standard setting of combinatorial auctions (see, e.g., [19, 3]), where gross substitutes preferences on the indivisible items are identified as those ensuring an equilibrium [31, 30]. Indeed, once there are even 2 indivisible items that are viewed as complements by one buyer and as substitutes by another, an equilibrium is no longer guaranteed to exist.

In the simple case where all items are indivisible, equilibria seem to exist much more commonly. In particular, a computerized search we ran over small instances with non-equal budgets did not produce any example for which no equilibrium was found. Similarly, a simple tâtonnement process could always find an equilibrium for larger instances with preferences taken from real-life data accumulated by the Spliddit website [29], as long as we assigned the players “generic” budgets (see Appendix D). These observations motivate seeking a finer understanding of the basic question of equilibrium existence in Fisher markets.

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1E.g., this holds when buyers have preferences with constant elasticity of substitution, such as linear or Leontief preferences, in which case efficient algorithms for finding such an equilibrium are also known [24].

2Note that players have no value for money, so a player is not happy if the price is 1 but he does not get the item.
1.2 Motivation 2: Fairness with Unequal Entitlements

How do we fairly partition a collection of goods between several players? An enormous amount of literature has been devoted to this question, both in general (see, e.g., [8]), and in computational contexts in particular (see, e.g., [10] Chapters 11–13). While there are various possible interpretations of “fairness” in this context, it is usually assumed that all claimants to the goods have an equally-strong claim, and standard notions of fairness reflect this assumption. However, it is possible that a priori some players are more “entitled” than others (see, e.g. [45] Chapters 3 and 11). There are many examples of a priori entitlement in practice. For instance, it is common in partnerships that each partner owns a different share of the holdings; how should the holdings of such a partnership be split among the partners if the partnership were to dissolve (see, e.g., [18])? As another example, consider different sub-units of a company that share computational resources (such as different cores on a cloud), and need to be assigned their “share” of the shared resources; how do we arbitrate between their differing demands for resources (see, e.g., [28])? Similar considerations may come up in splitting family heirloom [43] and in many other natural scenarios.

We thus consider the following model, focusing as above on indivisible items: We need to fairly partition \( m \) items among \( n \) players who have preferences over different bundles of the items. Each player \( i \) arrives with an inherent “entitlement share” given by a positive real number \( b_i \), where without loss of generality \( \sum_i b_i = 1 \). Clearly fairness should be defined with regards to the entitlement shares, yet it is not at all clear what would be a natural definition.

In the special case where all shares are equal, i.e., \( b_1 = \ldots = b_n = 1/n \), envy-freeness [26] is an important fairness property, but it is not immediately evident how to generalize this notion to unequal entitlements. Intuitively, a property we desire with unequal entitlements is “proportional satisfaction of claims” (cf. the high-level discussion by Broome [12, p. 95]). For example, if we have 6 identical items to share among 2 players, one with a 2/3 entitlement share and the other with a 1/3 entitlement share, then a fair allocation should give 4 items to the former and 2 items to the latter. This is related to a second important fairness notion – that of fair share [47]. If players’ preferences can be captured cardinally (i.e., they have a valuation function that assigns a non-negative real value to every possible bundle of items), then the fair share solution of weighted proportionality from the cake cutting literature can be applied: the set \( S_i \) that player \( i \) receives should be worth to him at least a \( b_i \)-fraction of the value that she assigns to all items together.

In this paper we study the adaptation of the well-known approach of fairness via competitive equilibrium with equal incomes (also known as CEEI – see, e.g., [50] or the recent work of [4]) to the case of unequal entitlement shares: Players are treated as buyers in a Fisher market for the items, whose budgets are equal to their entitlements. A competitive equilibrium allocation, in which each player gets the most preferred bundle she can afford, may be considered a fair outcome. In the special case that all players get the same budget, the outcome is a CEEI.

Of course, as we are interested in fair division of indivisible items, a competitive equilibrium is not guaranteed to exist even in the case of equal entitlement shares. But this is also the case for notions of fair division such as envy-freeness and proportionality. In some cases, the lack of existence of an allocation satisfying these notions is due to the same underlying reasons as the lack of existence of an equilibrium – a simple example is when 2 players with equal budgets desire a single item. We conclude that a more nuanced understanding of existence of generic-budget competitive equilibria in discrete Fisher markets would shed light on the existence or lack thereof of appropriate notions of fairness, and may also suggest alternative fairness remedies, especially in those markets that correspond to natural applications of fair division.
1.3 General Preferences

We start by looking at the most general setting: Each player $i$ of our $n$ players has an *ordinal* preference $\prec_i$ among subsets of the $m$ heterogeneous items. We assume a strict linear order\(^3\) and free disposal (also known as monotonicity\(^4\), i.e., that $S \prec_i T$ whenever $S \subset T$. For example, if we are allocating 3 items $\{A,B,C\}$, then a possible preference of player $i$ might be $\emptyset \prec_i \{A\} \prec_i \{B\} \prec_i \{A,B\} \prec_i \{C\} \prec_i \{B,C\} \prec_i \{A,C\} \prec_i \{A,B,C\}$. Each player has a budget $b_i > 0$, and we assume without loss of generality that $\sum_i b_i = 1$.

A *competitive equilibrium (CE)* is a price vector $p = (p_1, \ldots, p_m)$ and an allocation (partition) $(S_1, S_2, \ldots, S_n)$ of all the items (i.e., $S_i \cap S_k = \emptyset$ for $i \neq k$ and $\bigcup_i S_i = \{1, 2, \ldots, m\}$), such that each player is allocated his preferred set among all sets whose price is within his budget, i.e., $\sum_{j \in S_i} p_j \leq b_i$ and for every $T \succ_i S_i$ we have that $\sum_{j \in T} p_j > b_i$.

We proceed by identifying cases in which existence of a CE is assured.

**Theorem 1.1 (Informal)** When budgets are generic (do not satisfy a few simple equalities), a CE exists in any one of the following cases:

- At most 3 items and any number of players with any monotone preferences.
- 4 items and 2 players with any monotone preferences.
- 2 players with “leveled preferences”: they prefer a set with larger cardinality to a set with smaller cardinality, i.e., $|S| < |T|$ implies that $S \prec_i T$ (with arbitrary preferences among sets of the same cardinality).

The last result may be viewed as quite surprising since in quasi-linear settings even when there are only 2 items and 2 players, and even when all preferences are leveled (one player views the items as complements, the other as substitutes), a CE may fail to exist. We do show however that we cannot go further than Theorem 1.4.

- There exists an example with 5 items and 2 players such that for an open interval of budgets, no CE exists.

When a CE does exist, its allocation has natural fairness properties. We generalize the maximin notion of Budish\(^5\) to the case of different budgets, and define the $\ell$-out-of-$d$ *maximin share* of a player to be a bundle that is at least as good as the one he can guarantee for himself in the following protocol: the player partitions the items into $d$ parts, and he gets the worst $\ell$ parts out of the $d$ parts (others get to choose $d - \ell$ of these parts, and their choice is assumed to be worst-case). We show that:

- For every player $i$ and rational number $\ell/d \leq b_i$, a CE allocation gives player $i$ his $\ell$-out-of-$d$ maximin share\(^6\).

In light of the non-existence result for general preferences, it is natural to try and focus on special cases with better existence guarantees, and indeed we look at the following hierarchy of classes (cf. [10]): *Additive* preferences are preferences that can be represented by positive item values $v^1, v^2, \ldots, v^m$ such that $S \prec T$ if and only if $\sum_{j \in S} v^j < \sum_{j \in T} v^j$ \(^6\) A somewhat larger class of preferences is termed responsive

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\(^3\)A linear order is transitive, antisymmetric, and complete.

\(^4\)While monotonicity is a property that is often taken for granted in most fair division contexts [3] Sec. 12.1.4], there are some contexts in which it is not the case [13].

\(^5\)Note that this guarantee is stronger than the guarantee one naively gets from a cut-and-choose algorithm. For 3 items and a player with a budget of 5/13, dividing the 3 items to 13 bundles and getting the worst 5 will give him nothing, while our result ensures that in any CE he gets at least one item, since $1/3 < 5/13$.

\(^6\)Additive preferences can also represent preferences with equality of the values of two sets, but for simplicity (and unless explicitly stated otherwise) we shall assume the additive preferences are such that this does not happen.
preferences, and these satisfy that there exists an order \( \succ \) over items such that for every bundle \( S \) and items \( j, j' \notin S \) (where \( j' \) is also allowed to be \( \emptyset \) and last in the order), \( j \succ j' \iff S \cup \{ j \} \succ S \cup \{ j' \} \). It is easy to see that we get a strict hierarchy of preferences:

- Denote the sets of lexicographic preferences, additive preferences, responsive preferences, and general preferences, by LEXICOGRAPHIC, ADDITIVE, RESPONSIVE, and GENERAL, respectively. It holds that LEXICOGRAPHIC \( \subset \) ADDITIVE \( \subset \) RESPONSIVE \( \subset \) GENERAL.

One may hope to get a finer hierarchy using preferences represented by valuations from the “subadditive hierarchy” in quasi-linear markets \[32\], but we show that already the class of submodular valuations can represent arbitrary preferences:

- For every preference \( \prec \) on \( \{1, \ldots, m\} \), there exists a submodular function \( v : \{1, \ldots, m\} \rightarrow \mathbb{R}^+ \) such that \( S \prec T \) if and only if \( v(S) < v(T) \).

### 1.4 Additive Preferences

At this point we turn our attention to the simplest case in which we do not know if a CE exists, the case of additive preferences and 2 players. Here we assume that each player \( i \) has a set of positive item values \( v_1^i, \ldots, v_m^i \in \mathbb{R}^+ \), and values each set \( S \) at the cardinal value \( v_i(S) = \sum_{j \in S} v_j^i \), and thus his ordinal preference is determined by \( S \prec_i T \) whenever \( v_i(S) < v_i(T) \). Our results for additive preferences hold even when there are identical items, which implies that preferences are only weakly monotone, so for additive preferences we do not assume that the preference order over bundles is strict. Any CE we present assigns identical prices to identical items.

Our first result is a Second Welfare Theorem for this case:

- For every Pareto efficient allocation \( S = (S_1, S_2) \) among 2 players with additive preferences, there exist budgets \( b_1, b_2 \) and prices \( p \) such that \( (S, p) \) is a CE with these budgets and preferences.
- This is in contrast to general preferences, where there exists an example with 2 players and non-additive preferences, in which a certain Pareto efficient allocation cannot be supported by any CE with any budgets.

Our main result in this part is the identification of large classes of instances where a CE exists:

**Theorem 1.2 (Informal)** Assume a market of 2 players with additive preferences and budgets, then a CE exists in any of the following cases:

- Whenever there exists a proportional allocation \( (S_1, S_2) \), i.e., one where \( v_i(S_i) \geq b_i \cdot v_i(\{1, \ldots, m\}) \) for \( i \in \{1, 2\} \).
- The budgets of the 2 players are “almost equal”, i.e., \( b_1 > b_2 > b_1 - \epsilon \) for small enough \( \epsilon \).
- The 2 players have identical preferences and budgets are generic (do not satisfy a few simple equalities).

In all of these cases, we show that our CE has the fairness property that each player receives at least his truncated-share: as much as he gets in the best (for him) Pareto efficient allocation that still gives him at most \( b_i \) share. When no proportional allocation exists, in some sense such an allocation would be as close to proportionality as the indivisibility of the items allows.

Let us discuss the conditions of this theorem: The first shows that CEs are no rarer than the weak fairness condition of proportionality. The second condition is tied to the observation of \[13\] that a CE with almost equal budgets ensures a strong notion of “envy-freeness up to one item”, and shows that if
the goal is to reach such envy-freeness then a CE with almost equal budgets always exists. The third condition focuses on the case that would seem to be hardest, in which players are in direct competition; intuitively, it should be easier to satisfy both of them when their preferences are not the same. We leave the following as our main open problem:

**Open Problem:** Does there always exist a CE for 2 players with additive preferences and generic budgets? Does there always exist a CE for any number of players with additive preferences and generic budgets?

### 1.5 Related Work

**Discrete Fisher Markets** The model we study is a discrete Fisher Market model which is a special case of Arrow-Debreu exchange economies. Several other Arrow-Debreu exchange economies with indivisible items have been studied; these differ from our model in various aspects: The works of [18, 37, 11, 27, 39] study markets with indivisibilities and one divisible good (i.e., money carries value for the buyers). [46, 49] and subsequent works study the “house allocation” problem with unit-demand players. The combinatorial assignment model of Budish [13], which is closest to our work, allows non-monotone preferences [cf. 25] and focuses on the case of multiple units of each item. Several works on general equilibrium with indivisibilities study relaxed equilibrium notions [e.g., 28, 21, 41], or assume a continuum of traders [e.g., 35]. Deng et al. [21] study the complexity of finding a market equilibrium in an exchange economy with additive preferences. Their NP-hardness constructions rely on non-generic budgets and so do not directly apply in our model – see Appendix C.

**CEEI and Approximation** Budish [13] circumvents the non-guaranteed existence of CEEI due to indivisibilities by weakening the equilibrium concept (to one called A-CEEI). In contrast to our work, he exclusively studies budgets that are almost equal, and focuses on non-monotonic preferences and multi-unit settings. In the same model, Othman et al. [42] show PPAD-completeness of computing the guaranteed A-CEEI, and NP-completeness of finding one with better approximation. The preferences used in their hardness proofs are non-monotone, and they suggest the research direction of restricting the preferences as a way around their negative results. Recently, [4] studies CEEI for divisible items and additive preferences, characterizing it by natural axioms. In the same setting, [5] analyzes CEEI for the division of “bads” rather than goods.

**Fairness with Indivisibles and Additive Preferences** The classic fairness literature applies to divisible items. Recently there has been renewed interest in additive preferences for their succinctness and use in practice [e.g., 4]. A notion of fairness with entitlements – weighted proportionality – has been studied in the context of divisible goods (see [45, Chapters 3 and 11] for an overview; see also [9]).

There is also a stream of research on fairness with indivisible items (see [6] for a recent survey), with a particular emphasis on computational challenges (e.g., [20, 11]). The work of [33] studies envy minimization, including for additive preferences (see also [34]). Several recent works consider the fairness and approximability of maximum Nash social welfare, often for additive preferences [e.g., 17, 16]. Implementations of fair division in practice are discussed in [14] (responsive preferences), [41] (implementing [13]), [29] (Spliddit website, additive preferences) and [8] (adjusted winner algorithm, one item may possibly need to be divided).

### 1.6 Organization

After presenting the preliminaries (Section 2), we show our CE existence results for general preferences and a fairness property that CEs guarantee (Section 3). We then present our results for additive preferences.
For each allocated player, raise the price of an item in his allocated bundle until his budget is exhausted. The new prices form a CE with the original allocation since every player still gets his demanded set.

Theorem 2.3 (First Welfare Theorem) The allocation $S$ of a CE $(S, p)$ is Pareto optimal.

The next observation will be used throughout the paper.

Observation 2.4 (Budget-exhaustion is wlog) For every CE there is a CE with the same allocation where all the allocated players exhaust their budgets.

Proof: For each allocated player, raise the price of an item in his allocated bundle until his budget is exhausted. The new prices form a CE with the original allocation since every player still gets his demanded set. □
2.2 Fairness Preliminaries

According to Moulin [38] (p.166), “the two most important tests of equity” are envy-freeness and fair share. Both of these are well-defined for ordinal preferences (as well as for cardinal ones). Player \(i\) envies player \(k\) given allocation \(S\) if \(S_i \prec_i S_k\), and an allocation is envy-free (EF) if no player envies another player. This strong requirement often cannot be satisfied when items are indivisible [22]. Fair share is well-defined (for ordinal preferences) only for allocations of divisible items: it requires for each player to receive a bundle that he prefers at least as much as the bundle consisting of a \(1/n\)-fraction of every item on the market. To circumvent the issues stemming from indivisibilities, Budish [13] proposes appropriate variants of these fairness notions for indivisible items:

Definition 2.5 ([13]) An allocation \(S\) is envy-free up to one good (EF-1) if for every two players \(i\) and \(k\), for some item \(j \in S_k\) it holds that \(S_k \setminus \{j\} \prec_i S_i\).

Definition 2.6 ([13]) An allocation \(S\) gives every player his maximin share if every player receives a bundle he prefers at least as much as the bundle he can guarantee by partitioning the items into \(n\) parts and allowing the other \(n - 1\) players to chose their parts first.

Every CE with equal budgets is envy-free and gives every player his maximin share. Budish [13] shows that every CE with almost equal budgets is EF-1; a short proof appears for completeness in Appendix B. Caragiannis et al. [16] introduce a strengthening of EF-1 called EF-1* in which for every players \(i, k\) and every item \(j \in S_k\) it holds that \(S_k \setminus \{j\} \prec_i S_i\); in Appendix B we present an example showing that a CE is not necessarily EF-1*.

A strict cardinal preference of a player is defined by a valuation function \(v: 2^M \to \mathbb{R}_+\) such that for every two sets \(S, T \subseteq M\), the player prefers \(S\) over \(T (S \succ T)\) if and only if \(v(S) > v(T)\). For cardinal preferences, another central fairness notion is that of proportionality. Assume that players have “weights” \(\{b_i\}_i\) that sum up to 1. An allocation \(S\) gives player \(i\) his proportional share if player \(i\) receives at least a \(b_i\)-fraction of his value for the grand bundle of items, that is \(v_i(S_i) \geq b_i \cdot v_i(M)\). An allocation is budget-proportional (also known as weighted-proportional) if every player receives his proportional share. When all budgets are equal, such an allocation is simply called proportional.

3 General Preferences

For general preferences we are interested in understanding the existence of competitive equilibria, as well as fairness properties of such CEs. We first present the fairness notion of \(\ell\)-out-of-\(d\) maximin share and show that any CE guarantees that property. We then consider the issue of existence of a CE with general preferences. With only 2 items and any number of players with unequal budgets, there is a CE in which player 1 pays \(b_1\) for his most favorable item out of the two, and player 2 pays \(b_2\) for the remaining item. What about larger numbers of items? We first observe that there is a CE when there are 3 items and unequal budgets, and then show that even with only 2 players and 5 items, a CE is not guaranteed to exist for some open interval of budgets (in Appendix A.2 we show that such a negative result for 2 players does not hold for 4 items.) Finally, we show that the Second Welfare Theorem does not hold, that is, for some small market (with only 2 players and 5 items), there is a PO allocation such that for every budget profile, no CE exists.

3.1 Fairness of CEs with Arbitrary Budgets

We are interested in fairness properties that are guaranteed by the existence of a CE when players have arbitrary budgets (not necessarily equal or almost equal).

\[\text{For divisible items (cake-cutting), fair share guarantees that each player feels she received at least } 1/n \text{ of the cake, and envy-freeness guarantees she feels no one else received a larger piece. Fair share is also described as “probably the least controversial fairness requirement in the literature” [13]. [13] ties its justification to the “veil of ignorance” of Rawls.}\]
We define a generalized version of Budish’s maximin share guarantee (Definition 2.6) that is intended to accommodate unequal budgets. Player i’s $\ell$-out-of-$d$ maximin bundle is the bundle he can guarantee for himself in the following protocol: he partitions the items into $d$ parts, then lets the other players choose $d - \ell$ of these parts (their choice is assumed to be worst-case), and then receives the remaining $\ell$ parts. For example, an additive player who values items $(A,B,C)$ at $(1,2,3)$ has a 2-out-of-3 maximin bundle of $\{A,B\}$.

**Definition 3.1** The $\ell$-out-of-$d$ maximin bundle of player i is the bundle

$$
\max_{S_i} \left\{ \min_{T_i} \left\{ \sum_{t \in L} T_i \mid (T_1, \ldots, T_d) \text{ is a partition of the items into } d \text{ parts, } L \subseteq [d], |L| = \ell \right\} \right\},
$$

where the maximum and minimum are with respect to i’s preferences. An allocation $S$ gives player i his $\ell$-out-of-$d$ maximin share if $S_i$ is either his $\ell$-out-of-$d$ maximin bundle, or any bundle that player i prefers to it.

We use the above definition to state a fairness guarantee that holds for every CE allocation.

**Proposition 3.2** For every CE with budgets $b_1, \ldots, b_n$, for every player i and rational number $\ell/d \leq b_i$, the equilibrium allocation gives player i his $\ell$-out-of-$d$ maximin share.

Before we prove Proposition 3.2  we briefly discuss how it can be applied. First, it generalizes a result of Budish [13], who shows that for every CE with equal budgets the equilibrium allocation gives every player his 1-out-of-$n$ maximin share, and if the budgets are almost equal then every player gets his 1-out-of-$(n + 1)$ maximin share. This is because if budgets are equal then $1/n \leq b_i$ for every player i, and if it holds that $b_1 \geq b_2 \geq \cdots \geq b_n \geq \frac{n}{n+1} b_1$ then $1 - \frac{1}{n+1} \leq b_i$ for every player i. As another example of applying Proposition 3.2 consider 2 players with budgets $b_1 = 2/3$ and $b_2 = 1/3$. The proposition implies that every CE allocation gives player 1 his 2-out-of-3 share and player 2 his 1-out-of-3 share.

**Proof:** Let $(S,p)$ be a CE. Since $S$ is an allocation of all items, every item is bought by an agent and so $P := \sum_j p_j \leq \sum_i b_i = 1$. Let $(T_1, \ldots, T_d)$ be any partition of the items into $d$ parts, and observe that $1 \geq P = \sum_j p_j = p(\bigcup_{l=1}^d T_l) = \sum_{l=1}^d p(T_l)$. By the pigeonhole principle, there exists a set of $\ell$ parts whose total price is at most $\frac{\ell}{d}P \leq \frac{\ell}{d}$. Let us call this “the cheap subset”. By the assumption in Proposition 3.2 $\frac{\ell}{d} \leq b_i$. Therefore, agent i can afford the cheap subset, and by definition of CE, the bundle actually allocated to agent i must be at least as preferred by him as the cheap subset.

3.2 Existence of CE for 3 items

We start with a simple proof that a CE exists for any preferences over 3 items provided that budgets are all different.

**Proposition 3.3** For $n$ players with any monotone preferences and 3 items, if all 4 largest budgets are different then a CE exists.

**Proof:** It is clear that in any CE, any player that is not one of the 3 players with largest budgets will not get any item. So in any CE we construct, those players will get nothing. Assume that the 3 largest budgets are $b_1 > b_2 > b_3$.

- If $b_1 > 3b_2$, then player 1 gets all 3 items, each for a price of $b_1/3 > b_2$.

\[13\]Since $b_1 \geq 1/n$, otherwise the budgets would not sum up to 1.

\[14\]The claim also holds if there are only 2 players (treating $b_i$ as 0 for any $i \geq 3$) or 3 players (same for $i \geq 4$).
• If $3b_2 \geq b_1 > 2b_2$, then player 1 gets the bundle of size 2 that he most prefers, and pays $b_1/2 > b_2$ for each item, and player 2 gets the remaining item for price of $b_2$.

• If $2b_2 \geq b_1 > b_2 + b_3$, then if the pair of items that player 1 prefers most does not contain player 2’s most preferred item, give player 1 this pair and charge him $b_1/2$ for each item, and give player 2 the remaining item for price of $b_2$. Otherwise, give player 2 his second most preferred item for a price of $b_2$, give the other 2 items to player 1 and charge him $b_2 + \epsilon > b_2$ for player 2’s most favorable item, and $b_1 - b_2 - \epsilon > b_3$ for the other item.

• If $b_1 = b_2 + b_3$, then if there is an item that player 1 prefers over all pairs of items, he gets the item and pays his budget, while each other player – in the order of their budgets – picks his most favorable item and pays his budget. If there is no such item, player 2 gets his most favorable item for price $b_2$, and player 1 get the remaining 2 items, each for a price of $b_1/2 < b_2$.

• If $b_2 + b_3 > b_1 > b_2$, then each player in the order of budgets picks his most favorable item out of the remaining items, paying his budget.

It is easy to verify that each of these price vectors indeed form a CE for the given budgets. ∎

### 3.3 Non-existence of CE for general preferences

While for 3 items and any number of players with different budgets a CE exists, we next shows that with 5 items, even with 2 players a CE does not exist, for an open interval of budget pairs. For the same market, the Second Welfare Theorem fails to hold.

**Proposition 3.4** There exist monotone strict preferences for 2 players and 5 items, such that for any budgets $b_1 > b_2$ such that $(4/3)b_2 > b_1$, no CE exists. Moreover, for these preference there exists a PO allocation such that for any budgets $b_1, b_2$ no CE exists.

**Proof:** Let us call the two players Alice and Bob, Alice has budget $b_1$ and Bob has budget $b_2$ such that $(4/3)b_2 > b_1 > b_2$. There are 5 items: $A, B, C, D, E$. We next present cardinal preferences for Alice and Bob, the reader should consider the implied ordinal preferences. For the first two items, Alice and Bob have different perspective: Alice values item $A$ at 10, and item $B$ at 20, and the pair at 700. Bob has value of 500 for $A$ and 501 for $B$, and a value of 502 for both. Alice and Bob have additive preferences over the three items $C, D$ and $E$, Both Alice and Bob values them at 201, 202 and 203, respectively. For both players, the value of a set is additive across the subset of the pair \{$A, B$\} and the three remaining items, for example, if Alice gets \{$A, B, C, D$\} her value is $700 + 201 + 202 = 1103$.

To prove the claim we show that no CE exists. By Theorem 2.3 any allocation that is part of a CE is Pareto optimal, so we only need to consider PO allocations. Note that in any PO allocation each of the 5 items is allocated to one of the two players.

We first observe that as Alice has more money, it is not possible that Bob gets both $A$ and $B$, since Alice can afford Bob’s bundle, and she prefers \{$A, B$\} over \{$C, D, E$\} (and any subset of it).

Alice cannot get \{$A, B$\} or any superset of it. This is because in such a CE one of the items $A$ or $B$ will have price at most $b_1/2 < b_2$, and Bob prefers that item over any pair of items from his set, and there is such a pair that costs at least $(2/3)b_2 > b_1/2$, thus Bob will deviate.

We are left to consider the case that Alice gets one of $A$ and $B$, and Bob gets the other. If Alice also gets all three items in \{$C, D, E$\}, the allocation is not PO as it is dominated by giving Alice the set \{$A, B$\} and Bob the rest. By Theorem 2.3 any allocation that is part of a CE is Pareto optimal, so we can disregard such allocations. If Alice gets at most one of the items in \{$C, D, E$\}, she would rather exchange bundles with Bob, and can afford it, so it is not a CE.

We are left with the case in which Alice gets one of $A$ or $B$, and two items from the set \{$C, D, E$\}. Let $X_1 \in \{$A, B$\}$ and $Y_1, Y_2 \in \{$C, D, E$\}$ with $Y_1 \neq Y_2$ be the items that Alice got, and denote the
items that Bob got by $X_2 \in \{A, B\}$ and $Y_3 \in \{C, D, E\}$. As both Alice and Bob prefer Alice’s bundle \{X_1, Y_1, Y_2\} over Bob’s bundle \{X_2, Y_3\}, to support that allocation in a CE it must be the case that $b_1 > b_2$. Assume we have a CE, by Observation 2.4 we can assume that both Alice and Bob exhaust their budgets, that is, $b_1 = p_{X_1} + p_{Y_1} + p_{Y_2}$ and $b_2 = p_{X_2} + p_{Y_3}$. In such a CE, since Alice prefers replacing $X_1$ by $Y_3$ and replacing \{Y_1, Y_2\} by $X_2$, it must be that $p_{X_1} < p_{Y_3}$ and $p_{Y_1} + p_{Y_2} < p_{X_2}$. This implies that $b_1 = p_{X_1} + p_{Y_1} + p_{Y_2} < p_{X_2} + p_{Y_3} = b_2$, a contradiction. Finally, we note that any such allocation is PO as both Alice and Bob prefer $B$ over $A$, and both have the same order over items $C, D$ and $E$, and thus there is no exchange of items that improves the situation for both players simultaneously. Thus, for any such allocation that is PO, and for any budgets $b_1, b_2$ no CE exists.

Since for the above preference there exists a PO allocation such that for any budgets $b_1, b_2$ no CE exists, the Second Welfare Theorem does not hold for general preferences.

**Corollary 3.5** The Second Welfare Theorem does not hold for monotone strict ordinal preferences. That is, for some markets (with only 2 players and 5 items), there is a PO allocation such that for every budgets, no CE exists.

As even for 3 players, a CE may fail to exist for general ordinal preferences, in the next section we consider 2 players with additive cardinal preferences. Afterwards, in Section 5 we take a step back and look at other classes of ordinal and cardinal preferences.

## 4 Two Players with Additive Preferences

We have seen that even with only 2 players, a CE does not exist for monotone strict preferences in general. We thus focus on additive preferences and study the case of 2 players with different budgets. Our positive results hold even when there are identical items, which implies that preferences are only weakly-monotone, so in this section we only assume that an additive preference assigns positive value to each item, but we do not assume that the preference order over bundles is strict. We note that all the CEs we present in our positive results assign identical prices to identical items.

We start by presenting a characterization of CEs for 2 players with weakly-monotone cardinal preferences, and then use it to prove the Second Welfare Theorem for additive preferences. We then present sufficient conditions for existence of a CE for given budgets. Throughout we assume wlog that “everything is normalized”, i.e., $b_1 + b_2 = 1$ and $v_1(M) = v_2(M) = 1$. Our main technical result in this section is Theorem 4.12, and we illustrate its proof via Figures 2 to 5.

### 4.1 A Characterization of CEs for 2 Players with Cardinal Preferences

We present necessary and sufficient conditions for a budget-exhausting pricing and a PO allocation to form a CE when preferences are cardinal. We later use this characterization to prove the Second Welfare Theorem for additive preferences. Fix cardinal preferences $v_1, v_2$. For player $i$ with preference $v_i$ and for two disjoint sets $X, Y$, the marginal value of $X$ given $Y$ is $v_i(X \mid Y) = v_i(X \cup Y) - v_i(Y)$. Recall that budget exhaustion is wlog (Observation 2.4).

**Proposition 4.1 (Characterization)** For 2 players with weakly-monotone cardinal preferences, consider a PO allocation $S = (S_1, S_2)$ in which both players are allocated, and budget-exhausting item prices $p$. Then $(S, p)$ forms a CE if and only if for every $i, k \in \{1, 2\}, i \neq k$, and for every two bundles $S \subseteq S_i, T \subseteq S_k$,

$$v_i(S \mid S_i \setminus S) > v_i(T \mid S_i \setminus S) \text{ and } v_k(S \mid S_k \setminus T) > v_k(T \mid S_k \setminus T) \implies p(S) > p(T). \quad (1)$$

**Proof:** Assume first that $(S, p)$ is a CE. Assume for contradiction that Condition \[1\] is violated. Wlog assume that this is the case for $S \subseteq S_1$ and $T \subseteq S_2$, i.e., it holds that $v_1(S \mid S_1 \setminus S) > v_1(T \mid
since player 2’s budget is exhausted, this means there must be bundles $S \setminus S_2 \setminus T$ while $p(S) \leq p(T)$. Then player 2 prefers to swap $T$ for $S$ and has enough budget to do so, in contradiction to the fact that he gets his demanded set in the CE.

For the other direction, consider a pair $(S, p)$ such that $p(S) > p(T)$ for every $S, T$ as in the proposition statement. Assume for contradiction and wlog that player 2 is not allocated his demanded set. Then since player 2’s budget is exhausted, this means there must be bundles $S \subseteq S_1$ and $T \subseteq S_2$ such that $v_2(S \mid S_2 \setminus T) > v_2(T \mid S_2 \setminus T)$ and $p(S) \leq p(T)$. Therefore $v_1(S \mid S_1 \setminus S) \leq v_1(T \mid S_1 \setminus S)$, and so by swapping $S, T$ in the allocation we arrive at a new allocation strictly preferred player 2, and no worse for player 1, in contradiction to the Pareto optimality of $S$.

\[ \square \]

### 4.2 Second Welfare Theorem

We have seen that for general preferences the Second Welfare Theorem does not hold even with 2 players; we next show that for additive preferences it does. That is, given additive preferences and a PO allocation, we can find budgets and prices such that the allocation forms a CE with these prices. The prices will be derived from the additive preferences by a weighted linear combination, as follows.

**Definition 4.2** Consider 2 additive preferences $v_1, v_2$, and parameters $\alpha, \beta \in \mathbb{R}_+$ such that $\max\{\alpha, \beta\} > 0$. The combination pricing $p$ with parameters $\alpha, \beta$ is an item pricing that assigns every item $j$ the price $p_j = \alpha v_1(\{j\}) + \beta v_2(\{j\})$.

Observe that by additivity of $v_1, v_2$ in Definition 4.2, the combination pricing with parameters $\alpha, \beta$ assigns every bundle $S$ the price $p(S) = \alpha v_1(S) + \beta v_2(S)$. Recall that $p$ is budget-exhausting with respect to an allocation $S = (S_1, S_2)$ if $p(S_1) = b_1$ and $p(S_2) = b_2$. Note that a combination pricing assigns identical prices to identical items (items $j, j'$ are identical if for every agent $i$ it holds that $v_i(\{j\}) = v_i(\{j'\})$).

**Lemma 4.3 (Budget-exhausting combination pricing)** Consider 2 players with additive preferences and budgets $b_1 \geq b_2 > 0$ (possibly equal). For every PO allocation $S$, if there exists a budget-exhausting combination pricing $p$ then $(S, p)$ is a CE.

**Proof:** First notice that by existence of a budget-exhausting pricing $p$, we know that both players must be allocated in $S$. By Proposition 4.1, to prove the lemma it is sufficient to show Condition (1), which for additive preferences can simply be written as: for every $i, k \in \{1, 2\}, i \neq k$, and for every two bundles $S \subseteq S_i, T \subseteq S_k$,

$$v_i(S) > v_i(T) \quad \text{and} \quad v_k(S) > v_k(T) \implies p(S) > p(T).$$

Indeed, for every $S, T$ such that $v_1(S) > v_1(T)$ and $v_2(S) > v_2(T)$, it holds that $p(S) = \alpha v_1(S) + \beta v_2(S) > \alpha v_1(T) + \beta v_2(T) = p(T)$, thus Condition (1) holds for any $i \neq k \in \{1, 2\}$. So $(S, p)$ is a CE by Proposition 4.1.

We can now derive the Second Welfare Theorem from the lemma.

**Corollary 4.4 (Second Welfare Theorem)** Consider 2 players with additive preferences. For every PO allocation $S$, there exist budgets $b_1, b_2$ and prices $p$ for which $(S, p)$ is a CE when the players have budgets $b_1, b_2$. Moreover, if both players are allocated in $S$, then any combination pricing $p$ defines budgets $b_1 = p(S_1), b_2 = p(S_2)$ such that $(S, p)$ is a CE with $b_1, b_2$.

**Proof:** If in $S$ all items are allocated to one of the players, wlog player 1, then we have a CE be pricing every item at $\rho > 0$ and setting $b_1 = mp$ and $b_2 < \rho$. Otherwise, fixing any combination pricing $p$ (e.g., with parameters $\alpha = \beta = 1$) and setting $b_i = p(S_i)$ for every player $i$ completes the proof by Lemma 4.3.

\[ \square \]
4.3 Existence of a CE for Given Budgets

The Second Welfare Theorem (Corollary 4.4) ensures that for 2 players, for any given additive preferences and PO allocation there are budgets for which a CE exists. We next ask a different question: given additive preferences and budgets, does a CE exist for any PO allocation? We present several results, each providing sufficient conditions for existence. We first make one remark ruling out a natural candidate for a simple equilibrium allocation.

Remark 4.5 (A Failed Candidate for an Equilibrium Allocation) Given the preferences and budgets, finding a CE requires that we find a PO allocation as well as item prices. Which PO allocation might be a good candidate for an equilibrium allocation? One might guess that a PO allocation with maximum Nash social welfare will be a CE allocation \[\text{cf. } 14\]. An allocation \(S = (S_1, S_2)\) maximizes the Nash social welfare (weighted by budgets) if it maximizes \(\prod_i (v_i(S_i))^{b_i}\), or equivalently, maximizes \(\sum_i b_i \log v_i(S_i)\). We observe that the allocation that maximizes the Nash social welfare might not be supported in a CE, even in a simple market with only 2 items in which a CE with a different allocation exists (the proof appears in Appendix A).

Observation 4.6 There exists a market of 2 players with additive preferences over 2 items and unequal budgets, such that the unique PO allocation that maximizes the weighted Nash social welfare is not supported in a CE, while a different allocation is supported in a CE.

4.3.1 All Players Get Their Budget-Proportional Share

In a budget-proportional allocation, each player “believes” that he gets at least his fair share with respect to the budgets. In an anti-proportional allocation, each player believes that he gets (weakly) less than his fair share.

Definition 4.7 (Budget-proportional) We say that player \(i\) with cardinal preference \(v_i\) and budget \(b_i\) gets his budget-proportional share in allocation \(S\) if \(v_i(S_i) \geq b_i v_i(M)\) (if \(v_i\) is normalized this simplifies to \(v_i(S_i) \geq b_i\)). We say that player \(i\) gets at most his budget-proportional share if \(v_i(S_i) \leq b_i v_i(M)\). An allocation is budget-proportional if every player gets his budget-proportional share. An allocation is anti-proportional if every player gets at most his budget-proportional share.

It is clear that a budget-proportional allocation does not always exist (say, if there is a single item). We next show that if such an allocation does exist then a CE exists as well: every budget-proportional allocation is dominated by a budget-proportional PO allocation, and we show that every such PO allocation is supported in a CE. By the same argument, anti-proportional PO allocations are also supported in a CE. Observation A.2 shows that there are markets in which the allocation of every CE is anti-proportional.

Proposition 4.8 For 2 players with additive preferences \(v_1, v_2\) and any budgets \(b_1, b_2\) (possibly equal), any PO allocation that is budget-proportional is supported in a CE, and for that CE, identical items have identical prices. Thus, if there exists a budget-proportional allocation then a CE exists. Additionally, any PO allocation that is anti-proportional is supported in a CE.

Proof: Consider any PO allocation that dominates the budget-proportional allocation. Clearly, that PO allocation is also budget-proportional. Denote that allocation by \(S = (S_1, S_2)\). Recall that \(\log b_1 + b_2 = 1\) and \(v_1(M) = v_2(M) = 1\). Since \(v_1(S_1) \geq b_1\) and \(v_2(S_2) \geq b_2\), note that \(v_1(S_2) = 1 - v_1(S_1) \leq 1 - b_1 = b_2\) and \(v_2(S_1) = 1 - v_2(S_2) \leq 1 - b_2 = b_1\).

\[\text{Notice that the allocation that maximizes the Nash social welfare does not change when scaling the budgets or the valuations.}\]
If \(v_1(S_1) + v_2(S_2) = 1\) (and thus \(b_1 = v_1(S_1)\) and \(b_2 = v_2(S_2)\)), we set \(\alpha = 1\) and \(\beta = 0\). Otherwise, \(v_1(S_1) + v_2(S_2) - 1 \neq 0\) and we set

\[
\alpha = \frac{v_2(S_2) - b_2}{v_1(S_1) + v_2(S_2) - 1}, \quad \beta = 1 - \alpha = \frac{v_1(S_1) - b_1}{v_1(S_1) + v_2(S_2) - 1}.
\]

Observe that

\[
\alpha v_1(S_1) + \beta v_2(S_1) = \frac{v_2(S_2) - b_2}{v_1(S_1) + v_2(S_2) - 1} v_1(S_1) + \frac{v_1(S_1) - b_1}{v_1(S_1) + v_2(S_2) - 1} (1 - v_2(S_2)) = b_1
\]

and that similarly \(\alpha v_1(S_2) + \beta v_2(S_2) = b_2\). Since the allocation is budget-proportional, it holds that \(\alpha, \beta \geq 0\). In each of the two cases we thus have a combination pricing \(p\) with parameters \(\alpha, \beta\) such that \(p(S_1) = b_1\) and \(p(S_2) = b_2\), and thus by Lemma 4.3, \((S, p)\) is a CE.

Additionally, for every anti-proportional PO allocation in which one player gets at most his budget-proportional share and the other gets less then his share, the same pair of parameters \(\alpha, \beta\) defined in (2) will be non-negative and so will give a budget-exhausting combination pricing (and thus a CE).

Note that being supported in a CE does not in itself “refine” the set of budget-proportional (or anti-proportional) PO allocations, since any such allocation is supported.

### 4.3.2 Sufficient Conditions for CEs that are not Budget-Proportional

We have seen that if there is a budget-proportional (or anti-proportional PO) allocation then there exists a CE (Proposition 4.8). Next, we look for conditions that will guarantee existence of a CE when such an allocation does not exist. We use fairness-related definitions (Definitions 4.9 and 4.10) to present general sufficient conditions for a CE to exist (Theorem 4.12), and in Section 4.3.3 we show that these conditions are met when the two players have almost equal budgets, and that they are also met when the players have the same preferences and generic budgets.

For the remainder of Section 4.3.2 we assume that a budget-proportional or anti-proportional PO allocation does not exist; Figure 1 presents this pictorially and introduces some of the notation we will use in the rest of Section 4.

**A Fairness Guarantee** Interestingly, the equilibrium allocation that we exhibit has an attractive fairness property. When it is not possible to give each player his budget-proportional share in a CE, the next best thing a player can hope for is to get at least the maximal share he can obtain in any PO allocation in which he gets at most his budget-proportional share (due to the indivisible nature of the items this is not necessarily his budget-proportional share). We call this the player’s “truncated” share. The CE we present in Theorem 4.12 gives every player his truncated share. To formalize this we introduce the following notation: let \(\text{PO} = \text{PO}(v_1, v_2)\) be the set of PO allocations for preferences \(v_1, v_2\).

**Definition 4.9 (Truncated share)** Let \(b_i^- = \max_{S \in \text{PO} | v_i(S) \leq b_i} \{v_i(S_i)\}\) be the maximal share player \(i\) can obtain in any PO allocation in which he gets at most his budget-proportional share. Denote by \(S_i = \hat{S}^i(b_i)\) the maximizing PO allocation, i.e., \(b_i^- = v_i(S_i)\)\(^{15}\). An allocation \(S\) gives player \(i\) his downwards-truncated budget-proportional share, or truncated share in short, if \(v_i(S_i) \geq b_i^-\).

Analogously to Definition 4.9 we also define the “augmented” share of a player.

\(^{15}\)The allocation \(\hat{S}^i(b_i)\) is well-defined as it is possible to give nothing to player \(i\), so the maximum is taken over a non-empty set of allocations.
Figure 1: This figure presents the situation we analyze in Section 4.3.2 and some notation we use. It shows the value of an allocation for player 1 on the $v_1$-axis and the value of an allocation for player 2 on the $v_2$-axis. Every allocation $S$ can be represented by the point $(v_1(S_1), v_2(S_2))$ on the $(v_1 \times v_2)$-plane. The players’ budgets $b_1, b_2$ are shown on the same axes. The closure of the solid blue area (at or above $b_2$ and at or to the right of $b_1$) includes all allocations that are budget-proportional, and is empty by assumption. The closure of the red dotted area represents anti-proportional allocations and has no PO allocations by assumption. Let $A, B$ be two PO allocations. By Pareto optimality, the blue striped areas to the right and above $A$ and $B$ are both empty (the only allocation in their closures are $A$ and $B$). The figure also marks rectangles $T_1, T_2, X, Y, Z$ which will play a role in our arguments.

Definition 4.10 (Augmented share) Let $b_i^+ = \min_{S \in \tilde{PO}|v_i(S_i) \geq b_i} \{v_i(S_i)\}$ be the minimal share player $i$ can obtain in any PO allocation in which he gets at least his budget-proportional share. Denote by $\hat{S}_i = \hat{S}_i(b_i)$ the minimizing PO allocation, i.e., $b_i^+ = v_i(\hat{S}_i)$. An allocation $S$ gives player $i$ his augmented share if $v_i(S_i) \geq b_i^+$.

We remark that an equilibrium allocation does not guarantee that each player gets his truncated share, as the next example shows.

Example 4.1 Consider 2 players with symmetric additive preferences who both value items $(A, B, C, D)$ at $(7.9, 1, 5, 2)$. Player 1 has budget $b_1 = 1/2 + \epsilon$ and player 2 has budget $b_2 = 1/2 - \epsilon$ for some sufficiently small $\epsilon$. Since the preferences are symmetric, every allocation is Pareto optimal, in particular the allocation $(\{B, C, D\}, \{A\})$ where player 2 gets a share of $\frac{7.9}{15.9} < b_2$. This allocation together with prices $p = \left(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{2} + \epsilon, \frac{1}{2} - \epsilon\right)$ is a CE. However, the allocation $(\{A, B\}, \{C, D\})$ is also an equilibrium allocation, despite the fact that player 2’s share drops to $\frac{7}{15.9}$. Corresponding equilibrium prices are $p = \left(\frac{1+\epsilon}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} - \epsilon\right)$.

The following lemma shows a simple but important fact about allocations $\hat{S}_i, \hat{S}_k, \check{S}_i, \check{S}_k$ (the allocations that give players $i, k$ their truncated or augmented shares, respectively). Namely, it turns out that $\check{S}_i = \check{S}_k$ and $\check{S}_k = \check{S}_i$.

Lemma 4.11 Consider 2 players $i \neq k \in \{1, 2\}$ with additive preferences and any budgets. Assume there are no budget-proportional allocations nor PO anti-proportional allocations. Then the PO allocation $\hat{S}_i$ coincides with the PO allocation $\check{S}_k$. That is, $\check{S}_i$ obtains share $b_i^+$ for player $i$ and share $b_k^+$ for player $k$.  

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By definition of $b$ to the left of $B$ allocations except for $X$ a point in the interior of anti-proportional allocations (the closure of the dotted red area is empty from PO allocations). Yet such left of $b$ receiving value above $A$ Observe that indeed in allocation $A$ player 1 is receiving value above $b_1$ and in allocation $B$ player 2 is receiving value above $b_2$. In Figure 1 we have seen that the closure of the blue striped area does not include any allocation but $A$ and $B$. We now argue that $A = S^1 = S^2$ (showing $A = S^1 = S^2$ is similar).

By definition of $B = S^2$, it is the lowest PO allocation at or above $b_2$. Any PO allocation at or to the left of $b_1$ that is to the right of $B$ must be in the interior of the gray rectangle $X$, as there are no PO anti-proportional allocations (the closure of the dotted red area is empty from PO allocations). Yet such a point in the interior of $X$ is not only to the right of $B$, it is also below it. This means that it is closer than $B$ to $b_2$ from above, yielding a contradiction. The closed rectangle $X$ must therefore be empty of PO allocations except for $B$ (and thus must also be empty of non-PO allocations). But the point corresponding to allocation $\hat{S}^1$ must fall within closure of rectangle $X$ by definition (it is the rightmost PO point at or to the left of $b_1$, and it cannot lie to the left of $B$). Thus $\hat{S}^1 = B$, completing the proof.

**Proof:** We present the proof in Figure 2.

**Sufficient Conditions** Fix additive preferences $v_1, v_2$ and a player $i$ with budget $b_i$. Let $k$ be the other player whose budget is $b_k$. Theorem 4.12 presents two conditions that together are sufficient for a CE to exist. One of these conditions is genericity of the budgets, defined as not belonging to some finite set of budget pairs $R_i(v_1, v_2)$. The other condition is that some “rectangle of allocations” $T_i$ is empty.

Before stating and proving Theorem 4.12, we formally define $R_i(v_1, v_2)$ and $T_i(b_i, v_1, v_2)$. While somewhat technical, the definition of $R_i(v_1, v_2)$ becomes clearer when observing its role in the proof of Theorem 4.12 whereas $T_i = T_i(b_i, v_1, v_2)$ is depicted in Figure 1 (as well as Figures 3 and 4).

- Let $d = \lvert \text{PO} \rvert$. Order all allocations in $\text{PO}$ by player $i$’s preference, such that his $r$-th least preferred PO allocation is at index $r \in [d]$. Denote this allocation by $S(r)$, so that $S(r + 1) \succ_i S(r)$ for every $r \in [d - 1]$. The budget pair in which player $i$’s budget is $b_i$ (and the other player’s budget is $1 - b_i$) belongs to $R_i(v_1, v_2)$ if there exists an index $r$ such that $\frac{b_i}{v_i(S(r+1)_i)} = \frac{1-b_i}{1-v_i(S(r))}$. Note that there can be at most $d \leq 2^m$ budget pairs in $R_i(v_1, v_2)$.

- Let $T_i = T_i(b_i, v_1, v_2)$ be the set of allocations $\mathcal{S}$ satisfying $v_i(\hat{S}_i^1) < v_i(S_i) < v_i(\hat{S}_i^k)$ and $0 < v_k(S_k) < v_k(\hat{S}_k^k)$ (where $\hat{S}_i = \hat{S}^i(b_i)$, $\hat{S}_k = \hat{S}^k(b_k)$ are as defined in Definition 4.9). $T_1$ and $T_2$ are illustrated in Figure 1 for allocations $A = S^1$ and $B = S^2$. 

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Figure 2: This figure presents the proof of Lemma 4.11, using the notation of Figure 1, for the case of $k = 1, i = 2$ (the case of $k = 2, i = 1$ is similar). Let $A = \hat{S}^1$ and $B = \hat{S}^2$ (both PO allocations). Observe that indeed in allocation $A$ player 1 is receiving value above $b_1$ and in allocation $B$ player 2 is receiving value above $b_2$. In Figure 1 we have seen that the closure of the blue striped area does not include any allocation but $A$ and $B$. We now argue that $B = \hat{S}^2 = \hat{S}^1$ (showing $A = \hat{S}^1 = \hat{S}^2$ is similar).

Theorem 4.12 presents two conditions that together are sufficient for a CE to exist. Before stating and proving Theorem 4.12, we formally define $R_i(v_1, v_2)$ and $T_i(b_i, v_1, v_2)$. While somewhat technical, the definition of $R_i(v_1, v_2)$ becomes clearer when observing its role in the proof of Theorem 4.12 whereas $T_i = T_i(b_i, v_1, v_2)$ is depicted in Figure 1 (as well as Figures 3 and 4).
We are now ready to state and prove this section’s main technical result. We present sufficient conditions for the existence of a CE. Every such CE is supported by item prices that are identical for identical items, as these prices are simply a weighted combination with non-negative coefficients of the players’ values for the items.

**Theorem 4.12** Consider 2 players with additive preferences $v_1, v_2$ and budgets $b_1 > b_2 = 1 - b_1$. Assume there are no budget-proportional allocations nor PO anti-proportional allocations. If for some player $i$, $(b_1, b_2) \notin R_i(v_1, v_2)$ and the set $T_i = T_i(b_i, v_1, v_2)$ is empty, then a CE exists. Moreover, in this CE every player gets his truncated share, and identical items have identical prices.

**Proof:** Let $i$ be the player for which the conditions of Theorem 4.12 hold. By Lemma 4.11, both PO allocations $\mathcal{S}^1$ and $\mathcal{S}^2$ give both players their truncated share. To prove the theorem it is thus sufficient to show that at least one of these allocations is supported in a CE. We next show that indeed, for some $\gamma \in (0, 1)$, at least one of these two allocations is supported by item prices of the form $p_j = \gamma v_i(\{j\})$ for every item $j$.

Let $\gamma \in (0, 1)$. We first characterize the set of allocations that are within the budget of each player when prices are set to $p_j = \gamma v_i(\{j\})$ for every item $j$ (i.e., prices are a linearly scaled down version of player $i$’s valuation). Player $i$ can afford any allocation $\mathcal{S}$ such that $\gamma v_i(\mathcal{S}_i) \leq b_i$. Player $k$ can afford any allocation $\mathcal{S}$ such that $\gamma v_i(\mathcal{S}_k) \leq b_k$, or equivalently $\gamma (1 - v_i(\mathcal{S}_i)) \leq 1 - b_i$ (using that both valuations and budgets are normalized, that is, $b_1 + b_2 = v_1(M) = v_2(M)$). We illustrate this for $\gamma = 1$ in Figure 3.

Now define

$$\gamma_i = \max \left\{ \frac{b_i}{v_i(\mathcal{S}_i^1)}, \frac{1 - b_i}{1 - v_i(\mathcal{S}_k^1)} \right\} = \max \left\{ \frac{b_i}{b_k}, \frac{b_k}{v_i(\mathcal{S}_k^1)} \right\} < 1,$$

and note that $\gamma_i$ is well-defined and less than 1. The assumption that the pair of budgets does not belong to $R_i(v_1, v_2)$ implies that the maximum is obtained by only one of the terms, which is crucial in proving that both players demand the same allocation at prices $p_j = \gamma_i v_i(\{j\})$ for every item $j$. The proof follows by analyzing these two cases, as illustrated in Figures 4 and 5 respectively:

- **Case 1:** $\gamma_i = \frac{b_i}{b_k}$.

  We show that $\mathcal{S}^i$ is supported by item prices $p_j = \gamma_i v_i(\{j\})$. For every allocation $\mathcal{S} = (\mathcal{S}_i, \mathcal{S}_k)$ he can afford satisfies $v_i(\mathcal{S}_i) \leq b_i/\gamma_i$ and this holds with equality for $\mathcal{S}^i_i$. For player $k$, every allocation $\mathcal{S} = (\mathcal{S}_i, \mathcal{S}_k)$ he can afford satisfies $\frac{b_k}{v_i(\mathcal{S}_k)} v_i(\mathcal{S}_k) \leq 1 - b_i$. Since we are in the case that $\frac{b_k}{v_i(\mathcal{S}_k)} > \frac{1 - b_i}{1 - v_i(\mathcal{S}_i)}$ we derive:

$$\frac{b_k}{v_i(\mathcal{S}_k)} (1 - v_i(\mathcal{S}_k)) = \frac{b_k}{v_i(\mathcal{S}_k)} v_i(\mathcal{S}_k) \leq 1 - b_i < \frac{b_i}{v_i(\mathcal{S}_i)} (1 - v_i(\mathcal{S}_i)),$$

or equivalently $v_i(\mathcal{S}_i) > v_i(\mathcal{S}_k)$. We claim that player $k$’s most preferred allocation that satisfies this is $\mathcal{S}^k$: By Lemma 4.11, as it holds that $\mathcal{S}^i = \mathcal{S}^k$, that is $\mathcal{S}^i$ is also the PO allocation in which player $k$ gets at most his share $b_k$, and his share is maximal. Since there is no allocation $\mathcal{S}$ in which $v_i(\mathcal{S}_i) > v_i(\mathcal{S}_k)$ and $v_k(\mathcal{S}_k) > v_k(\mathcal{S}_k)$, the claim follows.

- **Case 2:** $\gamma_i = \frac{b_k}{v_i(\mathcal{S}_k)}$.

  We show that $\mathcal{S}^k$ is supported by item prices $p_j = \gamma_i v_i(\{j\})$. For player $k$ any allocation $\mathcal{S} = (\mathcal{S}_i, \mathcal{S}_k)$ such that he can afford $\mathcal{S}_k$ satisfies $v_i(\mathcal{S}_k) \leq \frac{b_k}{\gamma_i}$ and this holds as equality for $\mathcal{S}_k^k$. For player $i$, any allocation $\mathcal{S} = (\mathcal{S}_i, \mathcal{S}_k)$ such that he can afford $\mathcal{S}_i$ satisfies $v_i(\mathcal{S}_i) \leq \frac{b_k}{\gamma_i} < \frac{b_k}{v_i(\mathcal{S}_k)} = v_i(\mathcal{S}^i_i)$, as $\frac{b_k}{v_i(\mathcal{S}_k)} < \frac{1 - b_i}{1 - v_i(\mathcal{S}_i)} = \gamma_i$. Since $T_i$ is empty, it cannot be the case that $v_i(\mathcal{S}_i) < v_i(\mathcal{S}_k) < v_i(\mathcal{S}_i^k)$. Thus the most preferred allocation that $i$ can afford gives $i$ at most $v_i(\mathcal{S}_i)$ and this indeed what he gets at allocation $\mathcal{S}^k$, thus $\mathcal{S}_i^k$ is demanded by $i$. 

\[
\square
\]
Figure 3: This figure illustrates the first part of the proof of Theorem 4.12, using the notation of Figures 1 and 2. It shows the allocations that each player can afford given his budget when prices are \( p = v_1 \) (i.e., according to player 1’s valuation). Allocations in the yellow rectangle (at or to the left of \( b_1 \)) have value at most \( b_1 \) for player 1, and thus also price at most \( b_1 \), so player 1 can afford them. Allocations in the red rectangle (at or to the right of \( b_1 \)) have value at least \( b_1 \) for player 1, and thus player 1 values player 2’s allocation at most at \( 1 - b_1 = b_2 \) (by normalization), so the price is at most \( b_2 \) and affordable for player 2. This blue striped area marks allocation with value for player 1 that is above his value for \( B \), and value for player 2 that is above his value for \( A \). The area has no allocation at all, as it is subset of the union of the following areas: the blue areas from Figure 1 without allocations below \( B \) and to the left of \( A \); the interiors of \( X \) and \( Y \) that are empty by the proof of Lemma 4.11 in Figure 2; and the interior of \( Z \) that must be empty, as an allocation there must be dominated by some PO allocation in the areas we just argued are empty, or by an anti-proportional PO allocation (which does not exist by assumption). Therefore, at these prices, if rectangle \( T_1 \) is empty then player 1 demands the allocation \( B = \hat{S}^2 \) (the rightmost allocation within the yellow area – his budget), while player 2 demands the allocation \( A = \hat{S}^1 \) (the highest allocation within the red area – his budget).

4.3.3 CE Existence for Symmetric Players and for Almost Equal Budgets

Symmetric Players A direct corollary of Theorem 4.12 is that if every allocation is PO then a CE exists:

**Proposition 4.13** Consider 2 players with additive preferences and any budgets \( b_1 > b_2 \) such that \( (b_1, b_2) \notin R(v_1, v_2) \). If every allocation is PO then there exists a CE, and in that CE, identical items have identical prices.

**Proof:** If there exists a budget-proportional or anti-proportional PO allocation, there exists a CE by Proposition 4.8. Otherwise the conditions of Theorem 4.12 hold for both players: any allocation in \( T_1 \) is dominated by \( \hat{S}^2 \), and any allocation in \( T_2 \) is dominated by \( \hat{S}^1 \), but there are no Pareto dominated allocations, so these two sets must be empty. By Theorem 4.12 there exists a CE in which every player gets at least his truncated share.

In the symmetric case where both players share the same additive preference, we have a “constant-sum game” and thus every partition is PO. In addition, an anti-proportional allocation in which some
Figure 4: This figure illustrates Case 1 in the proof of Theorem 4.12, using the notation of Figures 1 and 2, for player $i = 1$. Prices are $p_j = \gamma_1 v_1 \{j\}$ for every item $j$, where $\gamma_1 = b_1/b^1_\bot$. The yellow and red rectangles are the allocations that players 1 and 2 can afford, respectively. Both players can afford more allocations than when $\gamma = 1$ (cf. Figure 3). The overlap (the closure of the orange rectangle) contains the allocations that both players can afford. The value of $\gamma_1$ is such that player 1 can exactly afford allocation $\mathcal{A}$, which is clearly demanded by him at these prices (the rightmost allocation within his budget). We show in the proof that player 2 cannot yet afford allocation $\mathcal{B}$ and any other allocation that gives him the same value, and so allocation $\mathcal{A}$ is in his demand (the highest allocation within his budget, using that the blue striped area above $\mathcal{A}$ that is within his budget, is empty). Thus $(\mathcal{A}, p)$ is a CE.

player gets less than his truncated share cannot exist. The next corollary follows from the proof of Proposition 4.13.

**Corollary 4.14** Consider 2 players with additive preferences and any budgets $b_1 > b_2$ such that $(b_1, b_2) \notin R(v_1, v_2)$. If both players share the same preference (i.e., are symmetric), then there exists a CE. Moreover, in this CE every player gets at least his truncated share, and identical items have identical prices.

**Almost Equal Budgets** With a single item and equal budgets a CE does not exist, and so CEs with equal budgets cannot always be used to fairly split items between two players. One can consider breaking the symmetry that causes the problem by using unequal but still very close budgets. We show this is indeed sufficient to ensure existence of a CE for additive preferences over any number of items.

**Proposition 4.15** For 2 players with additive preferences, there exists an $\epsilon > 0$ such that for every budgets $b_1 > b_2 = 1 - b_1$ such that $b_1 - b_2 \leq \epsilon$, there exists a CE. Moreover, in this CE every player gets at least his truncated share, and identical items have identical prices.

**Proof:** If there is an allocation that gives each player a value of exactly $1/2$, then there is a PO allocation that gives each player at least $1/2$. Such an allocation is budget-proportional for $b_1 = b_2 = 1/2$, and thus by Proposition 4.8 a CE $(\mathcal{S}, p)$ exists for that allocation, and prices are identical for identical items. For $\epsilon > 0$ that is small enough, let $b_1 = (1/2 + \epsilon)/(1 + \epsilon) > 1/2$ and $b_2 = 1 - b_1$, that is, we slightly increase the budget of player 1 while normalizing the sum $b_1 + b_2$ to 1. We claim that $(\mathcal{S}, p)$ is also a CE with these budgets. Indeed, as prices have not changed, player 2 gets his demand, and player 1, while his budget is
since $v_1$ argument holds for $A$ between 1 and the largest value to player 1 that is below $1/\epsilon$. For sufficiently small $\epsilon > 0$ that is slightly larger, cannot afford any set that is more expensive than his set $S_1$ when we pick $\epsilon > 0$ that is smaller than the difference in prices of any two bundles of non-identical price.

We next assume that no allocation gives each player value of exactly $1/2$. Assume first there is an allocation that gives both players strictly more than $1/2$, and consider any PO allocation that dominates it. In this case, for sufficiently small $\epsilon$, for any budgets $b_1 > b_2 = 1 - b_1 \geq b_1 - \epsilon$ the PO allocation becomes budget-proportional, and so the result follows from Proposition 4.8. Note that if there is an allocation that gives both players strictly less than $1/2$ then the allocation in which the two players swap their bundles gives both players more than $1/2$, and we are back to the previous case. So from now on we assume that every allocation give strictly more than $1/2$ to one player, and strictly less than $1/2$ to the other player. For sufficiently small $\epsilon$, for any budgets $b_1 > b_2 = 1 - b_1 \geq b_1 - \epsilon$ such an allocation is neither budget-proportional, nor anti-proportional.

Recall from Definition 4.9 that $\hat{S}^1(b_1)$ and $\hat{S}^2(b_2)$ are PO allocations that give players 1 and 2 their truncated shares with respect to budgets $b_1, b_2$. As the set of PO allocations is finite, we can find $\epsilon > 0$ such that there is no PO allocation $S$ such that $1/2 - 2\epsilon < v_1(S) < 1/2 + 2\epsilon$. For such small $\epsilon$, consider budgets $b_1 > b_2 = 1 - b_1 \geq b_1 - \epsilon$. Using the notation of Figure 2, let $A = \hat{S}^2(b_2)$ and $B = \hat{S}^1(b_1)$. We first claim that $A = \hat{S}^2(1/2)$ and $B = \hat{S}^1(1/2)$. This is so as $B$ is the PO allocation that gives the largest value to player 1 that is below $1/2 - 2\epsilon$, but there are no PO allocations that give player 1 value between $1/2 - 2\epsilon$ and $1/2$, and thus it also gives the largest value to player 1 that is below $1/2$. A similar argument holds for $A$ and the truncated share of player 2. Notice now that by our above assumption, since $v_2(A_2) < 1/2$ then $v_1(A_1) > 1/2$, and since $v_1(B_1) < 1/2$ then $v_2(B_2) > 1/2$.

We show that $A$ and $B$ are “symmetric” in the sense that one is obtained from the other by swapping bundles among the players (and so $v_1(A_1) = v_1(B_2) = 1 - v_1(B_1)$ and $v_2(A_2) = 1 - v_2(B_2)$, as can be seen in Figure 6). The proof of the symmetry claim appears in Figure 6 which also shows that $T_1, T_2$ as...
Figure 6: This figure proves the main claims in Proposition 4.15, using the notation of Figures 1 and 2. The budgets $b_1, b_2$ are almost equal, i.e., both are very close to $1/2$. For every allocation $S$ there is a “symmetric” allocation $\tilde{S}$ obtained by swapping the allocated bundles, which corresponds to a $180^\circ$ rotation around the point $(1/2, 1/2)$. Assume for contradiction that $A$ is not symmetric to $B$, that is $B \neq \tilde{A} = B'$. Then it is also the case that $A \neq B = A'$. Notice that for sufficiently small $\epsilon$, one of $A', B'$ must be located in the interior of the axes-parallel rectangle $Q$ as illustrated in the figure. But since $A = \tilde{S}^2(b_2)$ gives player 2 his truncated share closest to $b_2 \approx 1/2$ from below, $A' \neq A$ cannot be located as in the figure. Similarly, $B' \neq B$ cannot be located in the interior of $Q$ as in the figure, a contradiction. We have thus established that $A, B$ are symmetric. We now show that the closure of $T_1$ must be empty, except for $A$: If that were not the case – say, $T_1$ contained an allocation $C \neq A$ – then its symmetric allocation $\tilde{C}$ would Pareto dominate $B$ (due to the symmetry of $A, B$), a contradiction. $T_2$ only contains the allocation $B$ by a similar argument using the Pareto optimality of $A$.

defined for Theorem 4.12 are empty. We now decrease $\epsilon$ to some positive number that is sufficiently small so that $(b_1, b_2) \notin R_i(v_1, v_2)$ (such $\epsilon > 0$ exists as $R_i(v_1, v_2)$ is finite). We can now invoke Theorem 4.12 to complete the proof of the proposition.

5 Classes of Preferences

Proposition 3.4 shows that for general ordinal preferences, a CE does not necessarily exist, even under generic budgets and even with only 2 players and 5 items. Since a CE does not exist for general ordinal preferences even when budgets are generic, in this section we focus on understanding more restricted families of preferences. Some of these families will be cardinal, derived from some cardinal valuation function. We say that a strict ordinal preference $\prec$ is represented by a (cardinal) valuation $v$ if for every $S, T$, $v(S) < v(T) \iff S \prec T$.

In the market we used in the proof of Proposition 3.4, Alice’s preference exhibits strong complementarity between items, and one might suspect that this is the reason for the non-existence of a CE. We show this is not the case, by showing that every monotone and strict ordinal preference can be represented by a submodular cardinal preference. We then consider the classes of lexicographic and responsive preferences, and observe a strict hierarchy among these and additive preferences.

We conclude with a positive result: we consider leveled preferences and show that for two players with generic budgets, a CE always exists.
5.1 Every Preference has a Submodular Representation

We show that every monotone and strict preference can be represented by a submodular preference. Let \( \text{SUBMODULAR} \) the set of strict and monotone preferences that can be represented by a submodular valuation function.

**Proposition 5.1 (Submodular is general)** Any monotone and strict preference is represented by a submodular preference. That is, \( \text{SUBMODULAR} = \text{GENERAL} \).

**Proof:** Consider an ordinal monotone strict preference \( \emptyset < S_1 < S_2 < \ldots < S_k \) (where \( k = 2^m - 1 \)). Define \( v(S_i) = 1 - 2^{-i} \) and observe that this cardinal preference represents the ordinal preference. To see that \( v \) is submodular, consider the marginal value of some item \( j \) relative to some set \( S_i \): \( v(S_i \cup \{ j \}) - v(S_i) \) is at least \( 2^{-i}/2 \) (with equality iff the set \( S_i \cup \{ j \} \) appears in place \( i + 1 \)) but is always less than \( 2^{-i} \) (since all values are less than 1). It follows that if \( S_i \) is contained in \( S_{i'} \) (so because of monotonicity \( i' \geq i + 1 \)), the marginal value of any item with respect to \( S_i \) is at least \( 2^{-i}/2 = 2^{-(i+1)} \), and is greater than the marginal value of any item with respect to \( S_{i'} \), which is less than \( 2^{-i} \leq 2^{-(i+1)} \). \( \square \)

5.2 Lexicographic and Responsive Preferences

We next consider two additional families of preferences, lexicographic and responsive, and present a simple observation regarding the hierarchy of preferences, showing that \( \text{LEXICOGRAPHIC} \subset \text{ADDITIVE} \subset \text{RESPONSIVE} \subset \text{GENERAL} \).

**Definition 5.2 (Lexicographic)** A preference \( \succ_i \) is lexicographic (in \( \text{LEXICOGRAPHIC} \)) if there exists an order over items such that a bundle \( S \) is preferred over \( T \) if and only if the first item in the order that does not belong to both bundles belongs to \( S \).

**Definition 5.3 (Responsive)** A preference \( \succ_i \) is responsive (in \( \text{RESPONSIVE} \)) if there exists an order \( \succ \) over items such that for every bundle \( S \) and items \( j, j' \notin S \) (where \( j' \) is also allowed to be \( \emptyset \) and last in the order), \( j > j' \iff S \cup \{ j \} \succ_i S \cup \{ j' \} \).

Our next observation establishes the relationship between the different preference classes.

**Observation 5.4** \( \text{LEXICOGRAPHIC} \subset \text{ADDITIVE} \subset \text{RESPONSIVE} \subset \text{GENERAL} \).

**Proof:** Clearly not every additive preference is lexicographic. Consider a lexicographic preference where the lexicographic ordering of the items is wlog \( 1 < \cdots < m \). Define the following additive preference \( v \): item \( j \) has value \( 2^{j-1} \). Observe that \( v \) is lexicographic as any item’s value is larger than the total value of all earlier items, and it clearly represents the same preference.

Clearly, every additive preference is responsive. Consider now a responsive preference \( \succ \), responding to a linear order among the items (say \( a > b > c > d > e \)). This gives a partial order over bundles of the same cardinality (captured by say \( ab > ac > bc > ad > bd > cd > ae > be > ce > de \) or \( ab > ac > bc > ad > bd > ae > cd > be > ce > de \) for bundles of size 2, in this case), but does not give an order among bundles of different cardinalities. The order among such bundles must only respect monotonicity. Therefore we may assume that both \( da > dbc \) and \( ebc > ea \) – this will violate neither responsiveness nor monotonicity. But the resulting preference is not additive: assume for contradiction there is a representative additive preference \( v \), then \( da > dbc \) implies \( v(a) > v(bc) \) but \( ebc > ea \) implies the opposite, contradiction. \( \square \)

We observe that for lexicographic preferences, running serial dictatorship in order of budgets (each player picking his most favorable item out of the remaining items), and charging each player his budget, results in a CE.\(^{16}\)

\(^{16}\)Note that the same procedure works also for unit-demand preferences, even though they are not strict.
Observation 5.5 For lexicographic preferences, if no two players have the same budget than a CE exists.

We leave open the issue of CE existence for responsive preferences.

5.3 Leveled Preferences

A player has a leveled preference if he prefers a larger bundle over any smaller bundle.

Definition 5.6 (Leveled) A preference is leveled (in LEVELED) if every bundle of \( t \) items is preferable to every bundle of \( t - 1 \) items for every \( t \geq 1 \) (but among bundles of the same cardinality there can be any preference ordering).

We remark that LEVELED is incomparable to LEXICOGRAPHIC, ADDITIVE and RESPONSIVE (neither contains the other). The construction in Observation 5.4 that separates RESPONSIVE and ADDITIVE presents a responsive preference that is not a leveled preference. To see the other direction, observe that the following leveled preference over 3 items is not responsive:

\[
abc \succ bc \succ ab \succ ac \succ c \succ b \succ a \succ \emptyset.
\]

This is because \( ab \succ ac \) but \( c \succ b \). Moreover, it is clear that not every leveled preference is lexicographic, and not every lexicographic preference is leveled.

Next we consider leveled preference with 2 players and generic budgets, and show that a CE exists. This is in contrast to the quasi-linear case in which a CE might not exist even with leveled preferences of two players over two items.

Proposition 5.7 Consider \( m \) items. For 2 players with leveled preferences and budgets \( b_1 > b_2 = 1 - b_1 \) such that \( mb_1 \) is not an integer, a CE exists.

Proof: Denote by \( p^* = 1/m \) the “average fair” item price. Now denote by \( k_1 = \lfloor mb_1 \rfloor \) and \( k_2 = \lfloor mb_2 \rfloor \) the “target proportional” number of items we want to allocate to the two players. Notice that \( k_1 + k_2 = m - 1 \) by our assumption that \( m \cdot b_1 \) is not an integer.

Notice that \( b_1/k_1 > p^* > b_1/(k_1 + 1) \) (and similarly \( b_2/k_2 > p^* > b_2/(k_2 + 1) \). Now let us see which of \( b_1/(k_1 + 1) \) and \( b_2/(k_2 + 1) \) is higher – assume wlog that \( b_1/(k_1 + 1) > b_2/(k_2 + 1) \). In this case we will give player 2 his most preferred set of \( k_2 \) items (among all sets of \( k_2 \) items) and give player 1 the other \( k_1 + 1 \) items. Each of player 1’s items will be priced at \( b_1/(k_1 + 1) \) and each of player 2’s items will be priced at \( b_2/k_2 \).

Now let us see if any player wants to deviate: Player 1 gets \( k_1 + 1 \) items; since the preference is leveled he may only want to deviate to another set with (at least) \( k_1 + 1 \) items but all such sets are above his budget – he is currently exhausting his budget with items that are priced \( b_1/(k_1 + 1) \), but all other items cost \( b_2/k_2 > p^* > b_1/(k_1 + 1) \) so he cannot get a set of size \( k_1 + 1 \) that contains even a single other item. Player 2 gets his most preferred set of \( k_2 \) items, so he may only want to deviate to larger sets but all item prices are at least \( b_1/(k_1 + 1) > b_2/(k_2 + 1) \) so he cannot afford any set of \( k_2 \) items.

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A Missing Proofs

A.1 Missing Proofs from Section 2

Proof: [Theorem 2.3] Assume for contradiction there exists an alternative allocation $S'$, in which for every player $i$ such that $S_i \neq S'_i$, $S_i \preceq_i S'_i$. Consider the total payment $\sum_i p(S'_i)$ for the alternative allocation given the CE prices $p$. By market clearance, $\sum_i p(S'_i) \leq \sum_i p(S_i)$. Therefore there must exist a player $i$ for whom $S_i \neq S'_i$ but $p(S'_i) \leq p(S_i)$. Thus $S_i$ is not player $i$’s demand, contradiction. \qed

A.2 Missing Proofs from Section 3

Proposition A.1 For 2 players with general monotone preferences over 4 items and budgets $b_1 > b_2$ such that $b_1 \notin \{4b_2, 3b_2, 3b_2/2\}$, a CE exists.

Proof: Let the items be $\{A, B, C, D\}$. We partition the space of budgets by the ratio of $b_1$ and $b_2$, as follows:

- If $b_1 > 4b_2$: Price every item at $b_1/4$ and give all items to player 1.
- If $4b_2 > b_1 > 3b_2$: Give player 1 the bundle of size 3 that he most prefers, and price each item at $b_1/3$. Give the leftover item to player 2 at price $b_2$. 


• If \( 3b_2 > b_1 > 3b_2/2 \): If player 1 prefers any triplet of items to any pair, then give player 2 the item that he most prefers at price \( b_2 \), and give player 1 the remaining triplet at price \( b_1/3 \) per item. Otherwise assume without loss of generality that \( \{A, B\} \) is player 1’s most preferred pair. If player 1 prefers \( \{A, B\} \) only to one triplet, without loss of generality \( \{B, C, D\} \), then there are three cases:

  - Subcase (i): Player 2’s favorite item is not \( A \). Give player 2 the item that he most prefers at price \( b_2 \), and give player 1 the remaining triplet at price \( b_1/3 \) per item.
  - Subcase (ii): Player 2 prefers \( \{C, D\} \) to \( \{B\} \). If \( b_1 > 2b_2 \), give player 1 the pair \( \{A, B\} \) and player 2 the pair \( \{C, D\} \). Set the prices to be \( (b_1 - b_2 + \epsilon, b_2 - \epsilon, b_2/b_2/2, b_2/2) \). Otherwise if \( b_1 \leq 2b_2 \) set the prices to be \( (b_2 + \epsilon, b_1 - b_2 - \epsilon, b_2/b_2/2, b_2/2) \).
  - Subcase (iii): Player 1 prefers \( \{A, C, D\} \) and player 2 the item \( B \). Set the prices to be \( (b_2 + \epsilon, b_2, \frac{b_1 - b_2 - \epsilon}{2}, \frac{b_1 - b_2 - \epsilon}{2}) \).

The remaining case is that player 1 prefers \( \{A, B\} \) to both \( \{A, C, D\} \) and \( \{B, C, D\} \). If \( b_1 > 2b_2 \), then give player 1 the pair \( \{A, B\} \) and give player 2 the pair \( \{C, D\} \). Set the prices to be \( (b_1/2, b_1/2, b_2/2, b_2/2) \). Otherwise \( 2b_2 \geq b_1 \) and there are two cases:

  - Subcase (i): Player 2 prefers \( \{C, D\} \) to \( \{A\} \) or to \( \{B\} \); assume without loss of generality the latter. Then give player 1 the pair \( \{A, B\} \) and give player 2 the pair \( \{C, D\} \). Set the prices to be \( (b_2 + \epsilon, b_1 - b_2 - \epsilon, b_2/b_2/2, b_2/2) \).
  - Subcase (ii): Player 2 prefers \( \{A\} \) and \( \{B\} \) to \( \{C, D\} \). Assume without loss of generality that player 1 prefers \( \{A, C, D\} \) to \( \{B, C, D\} \). Give player 1 the triplet \( \{A, C, D\} \) and give player 2 the item \( B \). Set the prices to be \( (b_2 + \epsilon, b_2, \frac{b_1 - b_2 - \epsilon}{2}, \frac{b_1 - b_2 - \epsilon}{2}) \).

We finally give the proof for the case that \( 3b_2/2 > b_1 > b_2 \). Without loss of generality let \( A \) be the single item most preferred by player 2. Let \( \delta = b_1 - b_2 \). The proof is by case analysis:

• Case 1: The complement to a pair that player 1 prefers over \( \{B, C, D\} \) appears before \( A \) in player 2’s preference ordering. Give player 1 the most preferred such pair, which must include \( A \) by monotonicity, and give player 2 its complement.

  - Subcase (i): Player 1 gets his most preferred pair. Set the prices to be \( b_1/2, b_1/2 \) for player 1’s items, and \( b_2/2, b_2/2 \) for player 2’s items.
  - Subcase (ii): Player 1 gets his second most preferred pair, without loss of generality \( \{A, C\} \) where the most preferred pair is \( \{A, B\} \). This means that \( \{C, D\} \) appears after \( \{A\} \) in player 2’s preference ordering. Set the prices to be \( (b_1 - b_2/2, b_2/2 + \epsilon, b_2/2, b_2/2 - \epsilon) \), where \( \epsilon < \delta \).
  - Subcase (iii): Player 1 gets his third most preferred pair, without loss of generality \( \{A, D\} \) where the first and second most preferred pairs are \( \{A, B\}, \{A, C\} \). This means that \( \{C, D\} \) and \( \{B, D\} \) appear after \( \{A\} \) in player 2’s preference ordering. Set the prices to be \( (b_1/2 + \delta, b_2/2, b_2/2, b_1/2 - \delta) \).

• Case 2: Player 1 prefers \( \{A\} \) over \( \{B, C, D\} \). Give player 1 the item \( A \) and give player 2 the bundle \( \{B, C, D\} \). Set the prices to be \( (b_1, b_2/3, b_2/3, b_2/3) \).

• Case 3: Give player 1 the bundle \( \{B, C, D\} \) and give player 2 the item \( A \).

  - Subcase (i): Player 1 has one pair preferred over \( \{B, C, D\} \), without loss of generality \( \{A, B\} \). This means that \( \{C, D\} \) appears after \( \{A\} \) in player 2’s preference ordering. Set the prices to be \( (b_2, b_1 - 2\epsilon, \epsilon, \epsilon) \), where \( \delta/2 < \epsilon < \delta \).
  - Subcase (ii): Player 1 has two pairs preferred over \( \{B, C, D\} \), without loss of generality \( \{A, B\} \) and \( \{A, C\} \). This means that \( \{C, D\} \) and \( \{B, D\} \) appear after \( \{A\} \) in player 2’s preference ordering. Set the prices to be \( (b_2, \frac{b_1 - \epsilon}{2}, \frac{b_1 - \epsilon}{2}, \epsilon) \), where \( \epsilon < \delta \).
Subcase (iii): Player 1 has three pairs preferred over \{B, C, D\}, these pairs are \{A, B\}, \{A, C\} and \{A, D\}. This means that \{C, D\}, \{B, D\} and \{B, C\} appear after \{A\} in player 2’s preference ordering. Set the prices to be \((b_2, b_1/3, b_1/3, b_1/3)\).

\[\square\]

### A.3 Missing Proofs from Section 4

**Proof:** [Observation 4.6] Consider 2 players Alice and Bob with budgets 101, 100, and 2 items \(A, B\) valued by Alice \(v_1(A) = 5, v_1(B) = 4\) and by Bob \(v_2(A) = 1000, v_2(B) = 1\). Clearly the only CE gives item \(A\) to Alice and item \(B\) to Bob (since Bob cannot prevent Alice from taking item \(A\) even if he pays his full budget for it), but the only allocation maximizing the Nash social welfare (i.e., maximizing \(101 \log v_1(S_1) + 100 \log v_2(S_2)\)) gives item \(A\) to Bob and item \(B\) to Alice.

\[\square\]

**Observation A.2** There exists a market of 2 items and 2 players with unequal budgets for which every CE allocation is anti-proportional (and a CE exists).

**Proof:** Let \(b_1 = 5/8, b_2 = 3/8, v_1(A) = 100, v_1(B) = 101, v_2(A) = 1, v_2(B) = 1000\). Since \(b_1 > b_2\), in every CE player 1 gets his preferred item, item \(B\). Moreover, he cannot get both items, as \(b_2 > b_1/2\) so player 2 can always afford at least one item. So in every CE, player 1 gets item \(B\) and player 2 gets item \(A\), and their shares are \(100/201 < 5/8\) and \(1/1001 < 3/8\), respectively. Equilibrium prices that support this allocation are \(p(A) = 3/8, p(B) = 5/8\).

\[\square\]

### B Additional Fairness Guarantees

In this appendix we briefly discuss the fairness notions of EF-1, EF-1* and justified envy freeness. We begin with the following result, included here for completeness.

**Proposition B.1** (Budish [13]) For every competitive equilibrium \((S, p)\) with almost equal budgets \(b_1 \geq b_2 \geq \cdots \geq b_n \geq \frac{m-1}{m} b_1\), the equilibrium allocation is EF-1.

**Proof:** Fix any two players \(i, k\). We show that there is some item \(j^* \in S_k\) such that \(S_i \succ_i S_k \setminus \{j^*\}\). By Observation 2.4 we may assume without loss of generality that the budget of player \(k\) is exhausted. So there exists some item \(j^* \in S_k\) such that its price \(p_{j^*}\) is at least \(b_k / |S_k| \geq b_k / m\). The price of \(S_k \setminus \{j^*\}\) is therefore at most \(\frac{m-1}{m} b_k \leq b_n \leq b_i\), and so \(i\) can afford \(S_k \setminus \{j^*\}\). Since \(S\) is a CE allocation and bundle \(S_k \setminus \{j^*\}\) is within \(i\)’s budget, \(S_i \succ_i S_k \setminus \{j^*\}\) as needed.

\[\square\]

A requirement stronger than EF-1 and weaker than EF is the following:

**Definition B.2** (Caragiannis et al. [16], Definition 4.4) An allocation \(S\) is EF-1* if for every two players \(i\) and \(k\), for every item \(j \in S_k\) it holds that \(S_k \setminus \{j\} \prec_i S_i\).

An EF-1* allocation always exists for two players (the cut-and-choose procedure from cake-cutting results in such an allocation), but it is an open question whether it always exists in general [16]. We demonstrate (by an example with non-strict preferences) that an EF-1* allocation is not necessarily EF even when an EF allocation exists in the market:

**Example B.1** Consider 3 symmetric additive players, 22 “small” items worth 1 each, and 2 “large” items worth 7 each. An EF allocation is two bundles of 1 large item and 5 small items each, and one bundle of 12 small items. An EF-1* allocation that is not EF is one bundle of 2 large items, and two bundles of 11 small items each.
While Proposition \ref{prop:ce-ef1} shows that CE implies EF-1 for almost equal budgets, we next demonstrate that CE does not imply the stronger property EF-1*.

**Observation B.3** The equilibrium allocation of a CE from almost equal budgets is not necessarily EF-1*, even for 2 symmetric players with an additive preference over 4 items.

**Proof:** The proof follows from Example \ref{example:ce-ef1}. Recall that the allocation \(\{a, b\}, \{c, d\}\) is a CE allocation, but it is not EF-1*: player 2 envies player 1 even if he gives up item \(b\).

Borrowing from the matching literature, we define justified envy as the envy of a player with a higher budget towards a player with a lower budget.

**Definition B.4** An allocation \(S\) is justified envy free given budgets \(b_1 \geq \cdots \geq b_n\) if for every two players \(i > k\), player \(i\) does not envy player \(k\). An allocation \(S\) is justified envy free for coalitions if for every player \(i\) and set of players \(K\) such that \(i \notin K\) and \(b_i \geq \sum_{k \in K} b_k\), player \(i\) does not envy \(K\), i.e., \(\bigcup_{k \in K} S_k \prec_i S_i\).

**Observation B.5** Every CE allocation is justified envy free for coalitions.

**Proof:** Assume that \(b_i \geq \sum_{k \in K} b_k\). Since the total price \(\sum_{k \in K} p(S_k)\) is at most \(\sum_{k \in K} b_k\), player \(i\) can afford the bundle \(\bigcup_{k \in K} S_k\). Because \(S\) is a CE allocation, it must hold that \(\bigcup_{k \in K} S_k \prec_i S_i\).

### C Some Computational Aspects

Most computational problems associated with markets with discrete goods seem hard. Let us list the basic ones and what can be said about their computational complexity. Below we will be using the following elements:

- We will always have \(m\) items and \(n\) players with strict additive preferences (see Remark \ref{remark:strict-additive}). The input and output size will always be polynomial in \(n\) and \(m\).
- An additive preference is given by a vector of positive rational numbers \(v = (v^1, v^2, \ldots, v^m)\).
- A budget is given by a positive rational number \(b > 0\).
- A set of items \(S\) is given by its characteristic vector.
- An allocation \(S = (S_1, S_2, \ldots, S_n)\) is given by listing the sets in it.
- A price vector is given as a list of positive rational numbers \(p = (p_1, p_2, \ldots, p_m)\).
- A perturbation is given by a positive rational number \(\epsilon > 0\).

We next list the problems and their status.

**Problem 1:** Is a given set a demand set? **Input:** A preference \(v\), a budget \(b\), prices \(p\), a set \(S\). **Question:** Is \(S\) the demanded set of the given preference and budget under these prices. **Status:** co-NP-complete (even for \(n = 2\)) by an easy reduction from Knapsack.

**Problem 2:** Demand Oracle. **Input:** A preference \(v\), a budget \(b\), prices \(p\). **Output:** Find \(S\) that is the demanded set of the given preference and budget under these prices. **Status:** NP-complete under Turing reductions (even for \(n = 2\)). (The status is stated for Turing reductions since this is not a decision problem.) Hardness can be seen e.g. from the hardness of Problem 4 below.
Problem 3: Is a given allocation PO? Input: Preferences $v_1, v_2, \ldots, v_n$, and an allocation $S = (S_1, S_2, \ldots, S_n)$. Question: Is the given allocation PO for these preferences? Status: co-NP-complete even for $n = 2$. We can reduce from the Partition problem: We are given a list of positive integer numbers that sum to $K$ and asked whether they can be partitioned into two sets each whose sum is exactly $K/2$. We treat each of these numbers as an item where both players have this value for the item (we call these “regular” items), and add two new items whose values for the first player are $K/2 - \epsilon$ and $K/2 + 2\epsilon$, respectively, while for the second player they are the opposite, $K/2 + 2\epsilon$ and $K/2 - \epsilon$, respectively ($\epsilon$ is chosen to be sufficiently small). Consider the allocation giving all the regular items to the first player and the two new items to the second player. It is Pareto-optimal if and only if there is no way in which the regular items can be split into two halves with the same sums (in which case each player would get one of these halves as well as the new item that he likes).

Problem 4: Is a pair of given allocation and prices a CE? Input: Preferences $v_1, v_2, \ldots, v_n$, budgets $b_1, b_2, \ldots, b_n$, an allocation $S = (S_1, S_2, \ldots, S_n)$, and prices $p$. Question: Do the given allocation and price vector form a CE for these preferences? Status: co-NP-complete even for $n = 2$. Consider the construction of Problem 3, and add to it budgets of $K + 2\epsilon$ for the first player and $K + 4\epsilon$ for the second player, and prices where each regular item’s price is exactly its value and the new items are priced at $K/2 + 2\epsilon$ each. Then this is an equilibrium if and only if there is no partition into two equal halves.

Problem 5: Can a given allocation be a CE allocation? Input: Preferences $v_1, v_2, \ldots, v_n$, budgets $b_1, b_2, \ldots, b_n$ and an allocation $S = (S_1, S_2, \ldots, S_n)$. Question: Are there prices $p$ that with the given allocation give a CE for these preferences? Status: co-NP-complete, even for $n = 2$. Note that it is co-NP-complete and not NP-complete, despite what the syntactic form may suggest. To show that it is in co-NP notice that the existence of equilibrium prices for the given allocation can be captured by a linear program whose inequalities are: (a) $S_i$ is in budget for player $i$; (b) every set $T$ that is prefered by $i$ to $S_i$ is out of budget for $i$. While there are exponentially many such inequalities, if the LP is infeasible then this is demonstrated already by $n + 1$ inequalities whose specification is the required proof. Hardness follows from the construction in Problem 4, which provides prices for the Pareto-optimal allocation used in the reduction of Problem 3, while certainly no such prices exist otherwise.

Problem 6: Do given preferences have a CE? Input: Preferences $v_1, v_2, \ldots, v_n$ and budgets $b_1, b_2, \ldots, b_n$, $b_1 = b_2 = 1/2$ by a reduction similar to that in Problem 3 (see [21], Remark C.1 and Appendix C.1). Status: NP-hard even for $n = 2$.

Problem 7: Do given preferences with generic budgets have a CE? Input: Preferences $v_1, v_2, \ldots, v_n$, budgets $b_1, b_2, \ldots, b_n$, perturbation $\epsilon$. Question: Is there a CE for these preferences with budgets perturbed by at most $\epsilon$? Status: Open. In particular, it may be the case that the answer is always “yes” (or at least that it is always “yes” for $n = 2$).

Remark C.1 In the above reductions from hard problems to Fisher markets, the resulting preferences can be non-strict. However we can add small perturbations to transform them into strict preferences without affecting the arguments. Consider for example the reduction from the Partition problem to Problem 6 for Fisher markets with two players and $b_1 = b_2 = 1/2$. Let $v$ be the preference for the items resulting from letting the value of each item be the corresponding integer. Add small perturbations that result in two different preferences $v_1, v_2$ for the items as follows: For every two bundles $S \neq T$ such that $v(S) = v(T)$, $v_1(S) > v_1(T)$ and $v_2(T) > v_2(S)$, and for every two bundles $S' \neq T'$ such that $v(S') > v(T')$, $v_1(S') > v_1(T')$ and $v_2(S') > v_2(T')$ (this can be achieved e.g. by small perturbations that result in two opposite lexicographic tie-breaking rules). It is not hard to verify that the reduction still holds, i.e., there is a CE in the market (with prices equal to the unperturbed integers) if and only if there is a partition into equal halves.
C.1 The NP-Hardness Result of Deng et al. [21]

We briefly review the NP-hardness result of [21] for Problem 6 above, noting the differences from our model. The NP-hardness result has two parts: For 2 players, the reduction is from the Partition problem (as sketched in Remark C.1), and for \( n \) players, the reduction is from the Exact Cover by 3-Sets problem (and establishes hardness of determining existence of even an approximate CE). One superficial difference is that the result is stated for Arrow-Debreu exchange economies rather than for Fisher markets, however it immediately applies to Fisher markets (as noted also in [14]) by simply removing the sellers (for this reason we say the reduction from Partition results in 2 rather than 3 players – we are counting only the buyers). Another superficial difference is non-strictness of the preferences (see Remark C.1). The main difference is non-genericity of the budgets, due to which the NP-hardness result is not directly applicable to our model. For 2 players we have already noted that, in the Fisher market resulting from the reduction, any perturbation of the equal budgets leads to equilibrium existence. We do not know whether this is also the case for \( n \) players, and leave it as an open question.

D Computerized Search for Equilibria

We attempted to find CEs, or find a market with no CEs, for both general and additive preferences. The instances examined were either randomly generated by sampling from distributions, or instances taken from real-world data from Spliddit. Our instances were small enough to exhaustively iterate over all PO allocations and search for CE prices using linear programming (LP). We also used another type of algorithm – tatonnement based – to search for CEs. While our initial procedure for picking random instances was naïve, below we describe our most refined procedure that is based on our current theoretical understanding, focusing the search on domains in which we have no proof of CE existence.

D.1 Randomly Sampled Preferences

Setup Our first computerized search ran on instances with between 4 and 8 items and 2 players with randomly generated preferences\footnote{The instances with 4 items were generated as a “sanity check”, as we know from Proposition A.1 that a CE exists for these instances.}. We generated both random additive preferences, where the values for the items were drawn from the uniform or Pareto distributions and then normalized to sum up to 1, as well as random monotone preferences. To generate the monotone preferences we randomly picked an order for all singletons, then randomly placed all pairs among the singletons while maintaining monotonicity, then placed all triplets and so on.

In choosing budgets for the random additive instances, our goal was to avoid instances for which we know from Proposition 4.8 or Theorem 4.12 that a CE exists. We thus iterated over consecutive pairs of allocations on the Pareto optimal frontier, and for each such pair tested several budgets that “crossed” in between those allocations. To illustrate this, recall Figure 2 in which the budgets “cross” between \( A \) and \( B \). This choice ruled out the existence of budget-proportional allocations. We used additional such considerations to carefully chose the budgets in order to rule out all “easy cases”. For random non-additive instances, we simply used several choices of arbitrary non-equal budgets.

For each of the resulting instances we conducted an exhaustive search for an equilibrium: we iterated over all possible PO allocations, and for each one of them we used CVX with the LP solver MOSEK 7 to look for equilibrium prices (see the linear program described in Appendix C, Problem 5). Note that although the problem is possibly computationally hard, our instances were small enough that they could be completely solved by the solver in a matter of seconds. We verified the equilibria found by the LP solver by implementing a demand oracle. The run time for 10,000 instances of 4 items was several minutes, and run time increased noticeably as the number of items increased.
Results  In all instances with additive preferences that we tested, we found and verified an equilibrium. As for general preferences, in all cases with 4 items we found an equilibrium (as expected), and even for 5 items we needed to go over several hundred instances before we found one that does not have an equilibrium. Instances with general preferences that do not have an equilibrium seemed to become more rare as the number of items increased.

D.2 Spliddit Data with Additive Preferences

Setup  We ran our second computerized search on instances of Spliddit data, specifically, 803 instances created so far through Spliddit’s “divide goods” application that were kindly provided to us by the Spliddit team [cf. [16], Sec. 4.3]. In every Spliddit instance, every player divides a pool of 1000 points among the instance’s indivisible items in order to indicate his values for the items; the resulting preference is additive in these values.

We implemented a simple tâtonnement process: Prices start at 0, all players are asked for their demand at these prices, and then the price of over-demanded items is increased by 1, and the price of undemanded items is decreased by 1. Prices thus remain integral throughout the process, and since our budgets are reasonably-sized integers the process is likely to converge reasonably quickly (we do not allow it to run for more than 20,000 iterations). The running time was typically well under a minute, usually no more than a second or two. One issue that deserves mention (and possibly further research) is how to update the prices when more than a single item is over- or under-demanded. Our first attempts either updated only a single such item’s price in every iteration, or updated all such items’ prices – both variants converged to a CE fairly often. We improved upon this by randomly deciding after each price update whether or not to continue updating prices in the current iteration.

Results  We focused on the non-demo instances with between 5 and 10 items (since we know from Proposition 4.1 that a CE exists for every instance with at most 4 items). There were 14 such instances available in the data, with between 3 and 9 players each. As Spliddit assumes that players have equal entitlements, we started by giving all players equal budgets of 100, in which case an equilibrium was found for only 5 out of the 14 instances. When we added small perturbations to make the budgets only almost equal (resulting in the budget vector \((100, 103, 106, \ldots)\)), we found a CE in all 14 instances. The same was true for other small perturbations that we tried (resulting in budget vectors like \((100, 101, 104, 109, \ldots)\)). We also tried several other budget vectors with budgets that are far from equal (such as \((100, 151, 202, \ldots)\) or \((100, 200, 300, \ldots)\)), and CEs were always found for these as well.