The prisoner’s dilemma with semi-synchronous updates: evidence for a first-order phase transition

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Abstract. The emergence of cooperation in self-centered individuals has been a major puzzle in the study of evolutionary ethics. Reciprocal altruism is one of the explanations put forward and the prisoner’s dilemma has been a paradigm in this context. The emergence of cooperation was demonstrated for the prisoner’s dilemma on a lattice with synchronous update. However, the cooperation disappeared for asynchronous update and the general validity of the conclusions was questioned. Neither synchronous nor asynchronous updates are realistic for natural systems. In this paper, we make a detailed study of a more realistic system of semi-synchronous updates where $pN$ agents are updated at every time instant. We observe a transition from an all-defector state to a mixed state as a function of $p$. Our studies indicate that despite it being a transition from an absorbing state, it is a first-order transition. Furthermore, we used a damage spreading technique to demonstrate that the transition in this system could be classified as a frozen–chaotic transition.

Keywords: phase transitions into absorbing states (theory), stochastic processes (theory), applications to game theory and mathematical economics, interacting agent models
1. Introduction

For ecological and social systems, game theoretical models have been immensely successful and have emerged as standard models in certain cases. One of the interesting applications of game theory in these systems has been the explanation of the emergence of cooperative behavior. It was suggested that cooperation could be sustained if there is repeated interaction and spatial structure. The prisoner’s dilemma on a lattice has emerged as a paradigm in this context [1]. We will be analyzing a stochastic variant of this in this work.

The ‘prisoner’s dilemma’ (PD) is an interesting problem in game theory where cooperation between agents is strictly dominated by defecting and a possible equilibrium is that where everyone defects. This is despite the fact that cooperation will lead to a better average pay-off for everyone. However, cooperation can arise in an iterated version of the game. This happens mainly due to the threat of punishment for defection in the next round. If game theory in the traditional sense is applied to social systems and if we assume common knowledge and perfect rationality for each agent, cooperation is not a viable option. Refining these assumptions makes cooperation possible. In iterated games on a lattice, agents have knowledge only about their nearest neighbors and the interaction is repeated. Cooperation is indeed observed in these models. Thus it is claimed that the emergence of cooperation, which seems counterintuitive in the context of Darwinian evolution but is observed in several biological and social contexts, is explained by this model. In evolutionary games, the strategies are built in trial-and-error fashion and more successful strategies are preferred in the course of time. This procedure leads to dynamics automatically. Obviously, societies comprise several agents who interact through a complex network of acquaintances. This fact has motivated the analysis of iterated versions of PD on a variety of networks. These studies have interesting consequences in social, biological and economic systems [2]–[4]. The choice of underlying network depends on the problem being addressed. In this work, we study the PD on a two-dimensional lattice. The reason is that evolutionary games on 2D networks are a relevant and popular choice for several biological networks [5]. (We should mention that recently other spatial
structures have also received a fair share of attention [6]–[12].) An extensive survey of modeling using game theoretical models in ecology can be found in a recent book [13]. In this paper, we will focus on studying various phases in this system and the nature of the dynamic phase transitions between these phases.

In this model, the agents sitting at the nodes of a two-dimensional lattice update their strategies in a synchronous manner and pay-offs are computed using the pay-off matrix of PD. The major finding is that on introduction of spatial structure, there is emergence of cooperative behavior among selfish individuals [5]. However, this model was strongly criticized by Huberman and Glance and also by Mukherji et al [14,15]. They studied the robustness of these results with respect to stochastic fluctuations and concluded that several of the conclusions do not hold in the presence of stochastic fluctuations.

In the above model, a certain degree of asynchronicity in updating strategies is relevant and possible, and this variant is studied in detail in this work. We note that the differences between synchronous and asynchronous updates have been a topic of recent interest in the statistical physics community. Besides in game theory [18], they are studied in the contexts of Boolean networks [19,20], coupled maps [21], neural networks [22], Monte Carlo processes [23] and biological networks [24,25]. They have even been studied in the context of equilibrium models [17] such as the Ising model. In the above work, a nonequilibrium phase transition, induced by introducing semi-synchronous updating in the Ising model, is studied. In the context of game theory, there is a clear physical motivation for such a study since it is unlikely that even a strategy with slightly more pay-off is deterministically copied by everyone. A certain degree of stochasticity in updating of strategies is indeed possible in these systems. Thus there is a need for a detailed study of systems which do not evolve in a fully synchronous or asynchronous manner. Hence, we make a detailed study of the evolution for semi-synchronous update where $pN$ members update their strategy every instant in PD on a lattice.

For completeness, we define the two-person PD game in its classic form: this game describes the confrontation between two players, each of whom may choose either to cooperate (strategy $C$), or defect (strategy $D$), at any confrontation. If both players choose $C$, they get a pay-off of magnitude $R$ each; if one player chooses $D$ while the other chooses $C$, the defector player gets the biggest pay-off $T$, while the other gets $S$; if both players defect, they get pay-off $P$. In this game, the pay-off values must satisfy the inequalities $T > R > P > S$ and $2R > S + T$. For such choice of parameters, the paradox is evident. Each player is tempted to defect, but they would be worse off if both defected and the total pay-off for both together would be higher if they cooperated instead. However, the best pay-off for an individual player is obtained when they defect while the other player cooperates.

In the ecological context, Nowak and May simulated this system with the choice of parameters $R = 1$, $T = b$ (1.0 $b$ 2.0) and $S = P = 0$ [5,26]. They believe

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that most of the interesting behavior is reproduced for this choice of parameters. They studied the PD on a two-dimensional array with synchronous updating. They explored the asymptotic behavior of the game for various values of $b$. Here players interact with their local neighbors through simple deterministic rules and have no memory of the past. They found that the dynamical behavior of the system depends on the parameter $b$. For a range of values of $b$ ($1.0 < b < 2.0$), the system reaches a steady state with a non-zero fraction of cooperators. They concluded that spatial structure and repeated interactions promote cooperation in the PD.

This synchronous updating came in for heavy criticism. It was argued that global clocks in social systems are very rare and the probability of two events evolving at exactly the same time has measure zero. It was also argued that more realistic modeling would involve updating the system by individual. Thus only one player should be updated at each time step. This type of updating is called asynchronous update. As we mentioned previously, Huberman and Glance studied the PD game with the same parameters as Nowak and May used, under the asynchronous updating rule [14]. A similar argument was made by Mukherji et al [15] and Masaki and Mitsuo [27]. In this case, the system rapidly converges to the steady state where all the players become defectors. They argued that the previous results about emergence of cooperation are not generic. Nowak et al [28]–[30] replied that if they study the behavior in the entire parameter space, cooperation is observed even for asynchronous update for some choice of parameters. They found that the two updating rules are similar for a some values of $b$, but for $1.8 < b < 2.0$ the two updating rules lead to different steady states. They also argued that discrete time is appropriate for many biological situations where an interaction phase is followed by a reproduction phase. Thus synchronous update is more relevant and realistic biologically. However, neither completely synchronous nor completely asynchronous updating is realistic in natural processes and there are bound to be stochastic fluctuations in updating. There have been attempts to interpolate between these two cases [17]–[25]. In this work, we make a detailed study of PD on a 2D lattice with semi-synchronous updates from the viewpoint of statistical physics and investigate how generic the results are.

We will make a detailed study of the observed phases for semi-synchronous update and present a phase diagram. The phases of interest in this system are an all-defector state and a mixed phase with cooperators and defectors. In particular, we will study the transition between these two phases as a dynamic phase transition.

In statistical physics, a lot of effort is devoted for finding the order of transitions and critical exponents in the case of continuous phase transitions. The reason is that the critical behavior lets us distinguish between essential and not so essential details of the system. The idea of universality in the theory of phase transitions has allowed us to see how seemingly disparate models have common underlying features. Thus it is important to study the transitions in detail. All agents becoming $C$ or $D$ is an absorbing state while coexistence of $D$ and $C$ can be considered as an active phase. Hence, this system is characterized by two absorbing states, an all-defector state and an all-cooperator state. It has long been argued that all one-component systems going from an active phase to a unique absorbing state have a phase transition in the class of directed percolation (DP) if the order parameter is a scalar and there are no extra symmetries or conservation laws, and the interaction is short range [16]. Also, most systems with multiple absorbing states are found to fall in the DP class [31]. We will study the veracity of this conjecture.
Firstly we need to establish the order of the transition. The first-order (second-order) phase transitions in equilibrium systems are characterized by discontinuities in the first (second) derivatives of the free energy, e.g., the internal energy and order parameter. These singularities at the first-order phase transition are due to phase coexistence and there are no critical exponents.

For the PD game system, it was found that for a different choice of updating rule the transition to the all-defector state on variation of parameter \( b \) is in the class of directed percolation [32,33]. We will study Nowak and May’s system under the variation of the probability \( p \) of update. We observe that the transition is not continuous. We must also mention that there are several known exceptions in nonequilibrium systems which exhibit a discontinuous transition to an absorbing state [34]–[40].

Our updating strategy is as follows. We allow every player to update their strategy with probability \( p \) in each Monte Carlo step. Under variation of this probability from \( p \to 0 \) (asynchronous) to \( p = 1 \) (synchronous) in the thermodynamic limit, we study the effect of the updating scheme on the behavior of the system. In section 2 of the paper, we establish that the transition is indeed a first-order transition. In section 3, we carry out a damage spreading analysis to study the dynamical phase diagram and critical behavior.

### 2. The model and simulation

We investigate PD on a two-dimensional lattice of size \( L \) with evolutionary dynamics. The agents on each site of lattice can choose only two strategies \( D = 0, C = 1 \). \( D \) corresponds to defecting while \( C \) corresponds to cooperating. (The defectors and cooperators have also been viewed as dead and living sites in some applications.) We employ fixed boundary conditions and assume that the agents have no memory of the past. The initial configuration consists of 30% defectors and 70% cooperators distributed randomly on the lattice. (We checked other initial conditions as well. We varied the density of defectors between 10% and 50% and found that the asymptotic stationary state did not change.) Every agent interacts with eight nearest neighbors and self-interacts. We set the parameters \( T = b \) \( (b = 1.83) \), \( R = 1 \) and \( S = P = 0 \). The pay-off matrix is

\[
\begin{array}{c|cc}
D & C & D \\
\hline
C & 3 & 0 \\
D & b & 0
\end{array}
\]

The players interact simultaneously and independently of each other. Their pay-off is the sum of the pay-offs from all nine interactions (with neighbors and the self-interaction). Generally, each player updates their strategy by imitating the strategy of most successful agent in the neighborhood. The main variation in this work is as follows: at every time step, each player updates their strategy by adopting the strategy of the most successful neighbor with probability \( p \). For asynchronous update, only one agent updates their strategy at every instant. This could be compared with evolution with \( p = 1/N \) where \( N \) is the total number of agents. On the other hand, all agents update their strategy for synchronous update and the above rule for \( p = 1 \) is the same as synchronous update which is widely studied. We vary the value of \( p \) from \( p \to 0 \) to \( p = 1 \) interpolating between asynchronous and synchronous update. The phases of interest are all-defector and mixed phases. We study the domains in the \( p-b \) plane, where almost all initial conditions lead to one of these phases.
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Figure 1. Schematic phase diagram of the semi-synchronous PD game. We plot the phases observed as a function of $p$ (probability of update for any given site) and $b$ (temptation to defect). For the values of $1.8 < b < 2.0$ the system switches from an absorbing state (all defectors) to an active state where cooperators and defectors coexist. Simulations are carried out for $L = 60$, and the initial 1000 time steps are discarded. We average over 100 different initial conditions.

We carried out a detailed investigation of this system for parameter values in the ranges $0.0 < p < 1.0$ and $1.0 < b < 2.0$. The corresponding phase diagram is shown in figure 1. For values of $b$ in the range $1.8 < b < 2.0$, the final phase depends on the value of $p$. In this work, we study the nature of the phase transition observed on varying $p$. The density of cooperators $\rho_c$ is an obvious order parameter for describing the transition since it is zero in an all-defector state and positive for a mixed state.

We compute the value of $\rho_c$ in the steady state for different values of the updating parameter $p$. We also investigate this for different lattice sizes $L$. We observe that when we change $p$ from $p \to 0$ to $p = 1$, the system switches from an all-defector absorbing state with $\rho_c = 0$ to an active phase with $\rho_c > 0$. We find clear evidence of a metastable state near the transition region. In figure 2, we plot the lifetime (average time taken by the system to reach an all-defector absorbing state) as a function of updating probability $p$ for several values of $L$. For smaller values of the lifetime ($< 10^6$), we average over 100 configurations and for larger lifetimes, we average over 10 configurations. Depending on the value of the parameter $p$ we can distinguish between three regions.

- For a long range of the values of the parameter $p < p^*$, the system rapidly converges to an absorbing all-defector state. For $p > p^*$ the time required to reach this absorbing state grows abruptly compared to the case for $p$ values smaller than $p^*$ figure 2. In this region, the lifetimes are approximately equal for all values of parameter $p$ and equal to the lifetime in the asynchronous update. Thus one could say that the behavior of the system is analogous to one obtained with purely asynchronous updates. As shown in figure 2, the value $p^*$ converges to a finite value at the thermodynamic limit.

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Figure 2. The average time required for the system to reach its absorbing state (lifetime of the coexistence state) is plotted as a function of updating probability $p$ for different lattice sizes. The lifetime changes abruptly and diverges near the critical point.

- There is a very sharp range of the value of the parameter $p^* < p < p^{**}$ where the system falls into a metastable state. The mixed state in which both cooperators and defectors are present is extremely long lived and the system falls into an all-defector state after a very long time. The average time taken by the system to reach this absorbing state increases with updating probability $p$ until it reaches a very large value as $p \rightarrow p^{**}$. For any value $p > p^{**}$, the system remains in the active phase. As $p \rightarrow p^{**}$ the system spends a longer time in the active phase before collapsing eventually to its absorbing state. We found that for the best fit the lifetime as a function of $(p^{**} - p)$ is an exponential decreasing fit (see figure 3).

- For values of $p > p^{**}$, we did indeed find that the system saturates with finite number of cooperators. We would like to assert that as the system crosses over the metastable state region toward this region the order parameter $\rho_c$ exhibits a certain jump in its value (see figure 4).

The appearance of the metastable state in this system shows some similarity to that of an equilibrium discontinuous transition. It is known that the discontinuous transition is usually accompanied by metastability.

To confirm that for any value of $p < p^{**}$ the system collapses to an absorbing state, we plot the time evolution of the density of cooperators $\rho_c(t)$ as a function of time $t$ in figure 4. We show the behavior of $\rho_c(t)$ as a function of $t$ for various values of $p$ for a system of size $L = 60$. We average over $10^3$ different initial conditions. The finite number of cooperators are observed for some time (depending on the value of $p$; the time taken to reach an absorbing state increases exponentially as $p \rightarrow p^{**}$ as mentioned previously),

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Figure 3. The semi-log plot for the lifetime of the coexistence state as a function of \((p^{**} - p)\) is plotted for \(L = 60\) and \(100\) in the metastable state. (We note that \(p^{**}\) is the value of updating probability \(p\) above which the system reaches a saturated state with coexistence of cooperators and defectors.) The solid line shows the exponential fit for the data.

followed by a collapse to an absorbing state. As \(p \to p^{**}\), the curves in figure 4 become flatter (the system needs a longer time to approach an absorbing state) until the system reaches its steady state at \(p > p^{**}\). Figure 4 shows that when the steady state of the system changes from an all-defector state to a coexistence state, there is certainly a jump in the value of the order parameter \(\rho_c\). Metastability and the long time required by the system to reach its steady state make it very difficult to locate the critical value of \(p\) above which the system reaches its stable active phase. We run the programs for very long times so that the correct estimates can be made. In the figure 5, we present the order parameter \(\rho_c\) as a function of the parameter \(p\). We used a lattice of size \(L = 100\); we averaged over \(10^2\) samples after discarding \(10^6\) transients. The figure shows the clear jump in the value of the order parameter \(\rho_c\).

This result of a clear jump in the order parameter value at the transition point coupled with the presence of a metastable state indicates that the model undergoes a first-order phase transition between the active and absorbing phases.

2.1. Analysis using the Binder cumulant

As mentioned above, the presence of long lived metastable states makes it very difficult to locate the critical point and the kind of phase transition since one could always doubt whether the jump in order parameter is genuine. Fortunately, the fourth-order reduced Binder cumulant offers a precise tool which is very sensitive to the nature of phase transitions. The fourth-order reduced Binder cumulant of the order parameter \(\rho_c\)

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Figure 4. The time evolution of the density of cooperators $\rho_c(t)$ for $L = 60$ for different values of the parameter $p$ near the steady state region. On increasing the value of $p$ from $p = 0.8925$ (bottom curve) to $p = 0.905$ (top curve) the system needs more time to reach the absorbing state. It is clear that at $p = 0.905$, the curve becomes flat, and the system reaches the active coexistence phase asymptotically. We start the simulation with 70% cooperators in all cases. There is a clear jump in the asymptotic value of $\rho_c(t)$.

Figure 5. The asymptotic value of the order parameter $\rho_c$ as a function of $p$ for lattice size $L = 100$. A clear jump in the value of the order parameter at the critical point indicates a first-order phase transition.
Figure 6. Binder cumulant $U_L$ plotted as a function of the probability $p$ for different lattice sizes. The presence of a minimum suggests that the transition is of first order.

is defined [41, 42] as

$$U_L = 1 - \frac{\langle \rho_c^4 \rangle}{3 \langle \rho_c^2 \rangle^2}. \quad (1)$$

Systematic analysis of the Binder cumulant $U_L$ has been successfully used to determine the order of the phase transition for several equilibrium phase transitions [42]–[52]. For a second-order phase transition, $U_\infty = 2/3$ at the transition point in the thermodynamic limit. On the other hand, for first-order transitions, it has a minimum at the transition point. The reason is as follows. For a continuous phase transition, the distribution of values of the order parameter is always a Gaussian, the position of which keeps changing. For a first-order transition, the distribution is different. Here, we have phase coexistence and the distribution is a superposition of two Gaussians centered at values corresponding to each phase [41]–[43]. The quantity $U_L$ is not well defined numerically when $\rho_c \to 0$ which is the case here. To overcome this difficulty we follow [43, 44] and add an arbitrary fixed constant to all values of the order parameter $\rho_c$ (we fixed it to be equal 0.001 in this work), that is we rigidly shift the probability distribution of the order parameter away from zero.

We have plotted the Binder cumulant $U_L$ as a function of probability $p$ for various $L$s values in figure 6. We average over 150 configurations and $5 \times 10^6$ iterations on discarding $10^4$ transients. A clear minimum in the value $U_L$ shows that the transition is of first order. The value of probability $p$ that corresponds to a minimum of the Binder cumulant $U_L$ in the thermodynamic limit is the critical point.

It is well known that for the first-order phase transition, finite-size scaling theory predicts rounding and shifts of the critical point to be inversely proportional to the volume, $L^d$ in $d$ dimensions [41, 42, 53]. We have plotted $p_c(L)$ as a function of $L^{-2}$ in figure 7.
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Figure 7. The critical point $p_c(L)$ is plotted as a function of $L^{-2}$. The data came from the same simulation results as for figure 6. An excellent linear behavior matching with finite-size scaling predictions for the first-order transition is obtained.

found an excellent linear behavior which matches with the finite-size scaling prediction. In the thermodynamic limit we have the value $p_c = 0.8678(6)$ for the best fit.

2.2. Hysteresis

The hysteretic effect is a characteristic feature of first-order phase transitions. The magnetic systems show a first-order transition on varying the magnetic field but a second-order transition on varying temperature. Thus there is a hysteresis on variation of the magnetic field, but not on varying temperature. This is a useful tool for distinguishing first-order phase transitions from continuous phase transitions. The reason for the hysteresis in the first-order phase transition is the coexistence of two phases. However, there is one difficulty in studying hysteresis in transitions leading to an absorbing state. If the system falls into an absorbing state, it cannot come out. Thus we need to suitably modify the model. Hence, we use the spontaneous creation method (SCM) [35], [54]–[56]. The SCM overcomes this difficulty by allowing a small non-zero $\epsilon$ concentration of active sites to survive. For the second-order phase transition, this spontaneous rate of creation of active sites will destroy the phase transition. However, for a first-order phase transition a small spontaneous creation level does not change the nature of the transition. It only turns the absorbing state into a fluctuating state of average density $\epsilon$.

We carry out simulations for $L = 200$ and allow an $\epsilon = 0.0005$ fraction of active sites to survive. We vary $p$ stepwise in steps of $\Delta p = 0.005$. We record the value of the density of cooperator $\rho_c$ after $t_r$ update. We average over 100 loops and plot the density of cooperators $\rho_c$ as a function of $p$ in figure 8, for different relaxation times $t_r$. The evidence
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Figure 8. Hysteresis loops of the density of cooperators are obtained by the SCM method for lattice size $L = 200$. They are obtained for different relaxation times $t_r$ and it is clear that the loop area decreases for higher $t_r$. The loops are generated counterclockwise.

of a hysteresis effect confirms our conclusion that the transition in the above system is a first-order transition.

3. Damage spreading analysis

The damage spreading technique was suggested for the first time by Kauffman in the context of biological systems [57]. Recently, this concept attracted much attention and was applied to analyze several dynamical systems such as cellular automata [58], kinetic Ising models [59] and the surface growth model [60].

In the damage spreading technique, we follow the time evolution of two almost identical configurations. The initial condition for the second configuration is the same as that of the first configuration except for small perturbation. Now one studies the evolution of these configurations in time under the same dynamics and sees how the initial perturbation (damage) propagates. In our case, we start the simulation system from a random initial configuration. We allow the configuration to evolve until it reaches its steady state. Let us label this configuration $\sigma^A$ (the first copy). $\sigma^B$ (the second copy) is created from the saturated state of the first copy $\sigma^A$ by carrying out a small perturbation or damaging this copy. Now both the copies are evolved under the same dynamics in the following sense. We update the strategy of any agent in system A with probability $p$. Whenever we choose to update (not to update) the strategy of the agent in the first copy, we also update (do not update) the strategy of the agent in the second copy. This synchronized updating ensures that same set of random numbers is used during updating. We study the time evolutions of both of the configurations under this dynamics. Evolution of a configuration can be described by trajectories in the phase space. The question is
how the difference between these initial conditions grows or decays as a function of time. In other words, the question is whether the damage heals or spreads. If the two initially close trajectories quickly become different, the behavior is generically called chaotic. In order to measure the difference between that two systems, a useful metric is given by the Hamming distance or damage defined by \[ D(t) = \frac{1}{N} \sum_{i=1}^{N} \left| \sigma_i^A(t) - \sigma_i^B(t) \right| \]

where \( N \) is the number of sites of the system. The quantity \( D(t) \) measures the fraction of sites of configuration \( \sigma^A \) which have different strategies to those from configuration \( \sigma^B \). In the thermodynamic limit, \( D(t) \) may go to zero if the damage heals completely while it may tend to a positive value if it does not heal. For a chaotic system, we expect the damage to grow in time and reach a certain asymptotic value. However, in the so-called frozen phase, \( D(t) \) will go to zero [62]–[64].

Nowak and May, in their original paper, do not exactly make a systematic study of damage spreading, as is done in the present work. However, they empirically simulate different configurations and call a phase chaotic if a slightly perturbed initial condition leads to a very different state. They state that the steady state of PD with synchronous update is chaotic for the parameter range \( 1.8 < b < 2.0 \) [5]. This is very similar to the definition of a chaotic phase in damage spreading studies. In this work, we will make a systematic study of damage spreading in PD as a function of updating probability \( p \).

We would like to mention that similar studies were carried out for the stochastic game of life (SGL) by Monetti and Albano. They found a first-order transition as a function of stochasticity and a rich dynamic critical behavior in the system [65]. Since our system is similar, we expect an analogous dynamic behavior in our system.

### 3.1. Analysis of damage spreading

We followed the same updating role and initial values as were described in the previous section. We start our simulation with 70% cooperators and 30% defectors distributed randomly on the sites of a square lattice. As mentioned before, we update the strategies with probability \( p \) and wait until the system reaches its steady state and we label this configuration as \( \sigma^A \). The second configuration \( \sigma^B \) is created with small damage in the central sites of configuration \( \sigma^A \). The evolution of damage spreading is computed as a function of time for different update probabilities. We have plotted \( D(t) \) as a function of \( t \) in the active phase for various values of \( p \) in figure 9(a). On the log–log scale the damage \( D(t) \) evolves linearly as a function of \( t \) until it saturates to a finite value of damage after some time. In the figure 9(b), we have plotted the average asymptotic value of damage \( \langle D(\infty) \rangle \) for different values of the parameter \( p \) for a lattice of size \( L = 200 \). It is clear that below a certain threshold probability, the damage reaches zero asymptotically.

This threshold probability is very close to \( p_c \) for the transition mentioned above. So, we conclude that the dynamical behavior of the phase transition in this system, between the absorbing all-defector state and the mixed state, is intimately connected to the frozen–chaotic transition in damage spreading.

We also study the effect of finite lattice size on damage spreading. In figure 10(a), we plot the damage \( D(t) \) as a function of \( t \) for various values of lattice size \( L \) for \( p = 1 \). The
behavior obtained is very similar to the one observed in SGL [65]. It is evident that the slopes and saturation values of all curves are independent of lattice size $L$. However, the time required for the system to reach the plateau $\tau(L)$ increases with $L$. The behavior can be summed up by the scaling form [65]

$$D(x) \propto \left\{ \begin{array}{ll}
x^\alpha & x \leq 1 \\
\text{const} & x > 1
\end{array} \right.$$

(3)

where $x = t/\tau(L)$ and $\alpha$ is an exponent. In figure 10(b), we show the scaled data. It is clear that the data for four different lattice sizes collapse to a single curve using the above
Figure 11. The average mean square distance for epidemic spreading $R^2(t)$ is plotted as a function of $t$ for lattice size $L = 199$ for different values of $p$.

Table 1. Values of the exponents $\alpha, \beta$ and $d_f$ at different values of the parameter $p$.

| $p$  | $\alpha$ | $\beta$ | $d_f$ |
|-----|----------|----------|-------|
| 1.0 | 1.96(9)  | 2.03(7)  | 1.93(1) |
| 0.98| 1.88(7)  | 2.01(5)  | 1.87(1) |
| 0.96| 1.85(8)  | 1.98(9)  | 1.86(8) |
| 0.92| 1.64(2)  | 1.90(1)  | 1.72(6) |
| 0.90| 1.52(8)  | 1.82(2)  | 1.67(1) |
| 0.88| 0.97(8)  | 1.59(4)  | 1.22(1) |

scaling form. We calculate the value of $\alpha$ for the different values of the parameter $p$ in table 1.

The average mean square distance $R^2(t)$ over which the initial damage spreads from the center of the lattice toward the boundary is also calculated in the following manner. We start the simulation at $t = 0$ with initial damage of one site at the center of the lattice. The lattice size is fixed at $L = 199$ and the mean square distance $R^2(t)$ is computed as a function of time $t$. We carry out the simulations for various values of parameter $p$. We average over 100 configurations and the results are plotted in figure 11. It is found that $R^2(t)$ and $D(t)$ exhibit similar behavior. Thus the following scaling ansatz should hold [65]:

$$R^2(x) \propto \begin{cases} x^\beta & x \leq 1 \\ \text{const} & x > 1 \end{cases}$$

where $\beta$ is an exponent. The value of exponent $\beta$ is tabulated as a function of $p$ in table 1.
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Figure 12. (a) Snapshot of damaged lattice sites at $t = 100$ for $p = 1.0$. (b) Snapshot of damaged lattice sites at $t = 100$ for $p = 0.88$. The initial damage is of one site at the center of a $400 \times 400$ lattice.

The number of damaged sites is related to the spatial extent of the damage through the fractal dimension $d_f$ of the damaged cloud, so

$$D(t) \propto R^{d_f}(t).$$

(5)

Then from equations (3) and (4) we get that

$$2\alpha = \beta d_f.$$  

(6)

We tabulate values of $d_f$ in table 1 using equation (6). While the fractal dimension $d_f$ is found very close to 2 for synchronous update ($p = 1$), it deviates appreciably for smaller values of $p$. For the value of $p$ near the critical point $p_c$, the fractal dimension $d_f$ is smaller. We find that the damage spreading near the critical point is certainly a fractal object. On the other hand, the snapshot in figure 12 shows that for synchronous update ($p = 1$), the damage spreading is approximately compact. Thus the damage spreading is fractal near the critical point and compact for synchronous update.

4. Conclusions

We have studied the PD on a 2D lattice with an update rule which interpolates between asynchronous and synchronous update as a function of the parameter $p$. Here each agent updates their strategy with probability $p$ at each time step. We observe that this system crosses from a mixed (active) phase to an all-defector (absorbing) phase when we vary the parameter $p$. We studied the time evolution of this system and found that this system exhibits a long lived metastable state near the critical point. The order parameter of this system shows a clear jump at the critical point. We carry out detailed quantitative analysis to show that the above transition is a first-order transition. We confirm this result by studying the average lifetime of the metastable state, the Binder cumulant and the hysteresis effect.

The damage spreading technique is a useful tool for studying the sensitivity of the system dynamics to the initial condition. A damage spreading analysis of semi-synchronous update leads to the conclusion that the active phase is chaotic and the transition in this system is the same as the frozen–chaotic transition. The damage spreading inside the active phase (far from the critical point) is compact. However, near the critical point the damage spreading is fractal.
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References

[1] Axelrod R, 1997 The Complexity of Cooperation: Agent-Based Model of Competition and Collaboration (Princeton: Princeton University Press)
[2] Nowak M A and Sigmund K, Evolutionary dynamics of biological games, 2004 Science 303 793
[3] Turner P E and Chao L, Prisoner’s dilemma in an RNA virus, 1999 Nature 398 441
[4] Turner P E and Chao L, Escape from Prisoner’s dilemma in RNA phage Φ 6, 2003 Am. Nat. 161 497
[5] Nowak M A and May R M, Evolutionary games and spatial chaos, 1992 Nature 359 826
[6] Szabó G and Fáth G, Evolutionary games on graphs, 2006 Phys. Rep. 446 97
[7] Abramson G and Kuperman M, Social games in a social network, 2001 Phys. Rev. E 63 030901
[8] Szabó G, Vukov J and Szolnoki A, Phase diagrams for an evolutionary prisoner’s dilemma game on two-dimensional lattices, 2005 Phys. Rev. E 72 047107
[9] Vukov J and Szabó G, Evolutionary prisoner’s dilemma game on hierarchical lattices, 2005 Phys. Rev. E 71 036133
[10] Szabó G and Toke C, Evolutionary prisoner’s dilemma game on a square lattice, 1998 Phys. Rev. E 58 69
[11] Kim B J, Trusina A, Holme P, Minnhagen P, Chung J S and Choi M Y, Dynamic instabilities induced by asymmetric influence: Prisoner’s dilemma game in small-world networks, 2002 Phys. Rev. E 66 021907
[12] Vukov J, Szabó G and Szolnoki A, Cooperation in the noisy case: Prisoner’s dilemma game on two types of regular random graphs, 2006 Phys. Rev. E 73 067103
[13] Nowak M A, 2006 Evolutionary Dynamics: Exploring the Equations of Life (Cambridge: Harvard University Press)
[14] Huberman B A and Glance N S, Evolutionary games and computer simulations, 1993 Proc. Nat. Acad. Sci. 90 7716
[15] Mukherji A, Rajan V and Slagle J R, Robustness of cooperation, 1996 Nature 379 125
[16] Hinrichsen H, Nonequilibrium critical phenomena and phase transition into absorbing states, 2000 Adv. Phys. 49 815 and references therein
[17] Radicchi F, Vilone D and Meyer-Ortmanns H, Phase transition between synchronous and asynchronous updating algorithms, 2007 J. Stat. Phys. 129 593
[18] Blok H J and Bergersen B, Synchronous versus asynchronous updating in the game of Life, 1999 Phys. Rev. E 59 3876
[19] Klemm K and Bornholdt S, Robust gene regulation: deterministic dynamics from asynchronous networks with delay, 2003 arXiv:q-bio/0309013
[20] Greil F and Drossel B, Dynamics of critical Kauffman networks under asynchronous stochastic update, 2005 Phys. Rev. Lett. 95 048701
[21] Wolfram S, A new kind of science (Wolfram Media, 2002)
[22] Hopfield J J, Neural networks and physical systems with emergent collective computational abilities, 1982 Proc. Nat. Acad. Sci. 79 2554
[23] Choi M Y and Huberman B A, The nature of time in Monte Carlo processes, 1984 Phys. Rev. B 29 2796
[24] Klemm K and Bornholdt S, Topology of biological networks and reliability of information processing, 2005 Proc. Nat. Acad. Sci. 102 18414
[25] Klemm K and Bornholdt S, Stable and unstable attractors in Boolean networks, 2005 Phys. Rev. E 72 055101
[26] Nowak M A and May R M, The spatial dilemma of evolution, 1993 Int. J. Bifurcation Chaos 3 35
[27] Tomochi M and Kono M, Spatial prisoner’s dilemma games with dynamic payoff matrices, 2002 Phys. Rev. E 65 026112
[28] Nowak M A, Bonhoeffer S and May R M, More spatial games, 1994 Int. J. Bifurcation Chaos 4 33
[29] Nowak M A, Bonhoeffer S and May R M, Robustness of cooperation, 1996 Nature 379 126
[30] Nowak M A, Bonhoeffer S and May R M, Spatial games and the maintenance of cooperation, 1994 Proc. Nat. Acad. Sci. 91 4877
[31] Marques M C and Mendes J F F, A parity conserving dimer model with infinitely many absorbing states, 1999 Eur. Phys. J. B 12 123
[32] Hauert C and Szabó G, Game theory and physics, 2005 Am. J. Phys. 73 405

doi:10.1088/1742-5468/2009/07/P07023
The prisoner's dilemma with semi-synchronous updates

Vojta T and Schreiber M, Differences between regular and random order of updates in damage-spreading simulations, 1998 Phys. Rev. E 58 7998

Nobre F D, Mariz A M and Sousa E S, Spreading of damage: an unexpected disagreement between the sequential and parallel updating in Monte Carlo simulations, 1992 Phys. Rev. Lett. 69 13

Wang F and Suzuki S, Time-evolution of damage in the Ising model, 1996 Physica A 223 34

[60] Kim Y and Lee C K, Dynamical self-affinity of damage spreading in surface growth models, 2000 Phys. Rev. E 62 3376

[61] Herrmann H J, 1992 The Monte Carlo Method in Condensed Matter Physics ed K Binder (Berlin: Springer)

[62] Albano E V, Dynamics of damage spreading in irreversible reaction processes, 1994 Phys. Rev. Lett. 72 108

[63] Albano E V, Damage spreading in the Ziff–Gulari–Barshad model, 1994 Phys. Rev. E 50 1129

[64] Rieger H, Schadschneider A and Schreckenberg M, Re-entrant behaviour in the Domany–Kinzel cellular automaton, 1994 J. Phys. A: Math. Gen. 27 L423

[65] Monetti R A and Albano E V, Critical edge between frozen extinction and chaotic life, 1995 Phys. Rev. E 52 5825

doi:10.1088/1742-5468/2009/07/P07023