Objectivity From Quanta Via State Information Broadcasting

R. Horodecki,1,2 J. K. Korbicz,1,2 and P. Horodecki3,2

1 Institute of Theoretical Physics and Astrophysics, University of Gdańsk, 80-952 Gdańsk, Poland
2 National Quantum Information Centre in Gdańsk, 81-824 Sopot, Poland
3 Faculty of Applied Physics and Mathematics, Gdańsk University of Technology, 80-233 Gdańsk, Poland

(Dated: December 24, 2013)

In spite of all of its successes, quantum mechanics leaves us with a central problem: How does Nature create a "foot-bridge" from fragile quanta to the objective world of everyday experience [1–3]? Here we identify within quantum mechanics a fundamental process leading to the perceived objectivity and called state information broadcasting. This is the trick that Nature uses instead of a simple cloning. We uncover it basing on minimal assumptions, without referring to any dynamical details or a concrete model. More specifically, we show how a crucial for quantum mechanics notion of non-disturbance due to Bohr [4, 5] and a natural definition of objectivity [6] lead to a canonical structure of a quantum system-environment state, reflecting objective information records about the system stored in the environment.

The emergence of objective world from quanta has been a long standing problem, already present from the very dawn of quantum mechanics [1, 2]. It is now commonly accepted that the most promising approach is the decoherence theory, based on a system-environment paradigm [3–8]: a quantum system is considered interacting with its environment. It recovers, under certain conditions, a classical-like behavior of the system alone in some preferred frame, singled out by the interaction and called a pointer basis [9], and explains it through information leakage from the system into the environment. However, decoherence theory is silent on how comes that in the classical realm information is redundant [8]: same record can exist in a large number of copies and can be independently accessed by many observers and many times. Or more basically: Since quanta cannot be cloned [10] and information redundancy is at the heart of objectivity, then what quantum process lies at the foundations of the objective classical world?

In an attempt to answer it, Zurek and collaborators have introduced quantum Darwinism model [6, 11] with a more realistic environment, composed of independent fractions. In the Everettian spirit [12], they assumed that each of these fractions effectively "measures" the system [13] and argued that after the decoherence has taken place it caries nearly complete classical information about the system. Here, we derive an objectivity-carrying structure within quantum mechanics, based on a universal approach, independent of any dynamics (like e.g. the S-matrix theory [14]): We look at the post-interaction quantum system-environments state and ask what properties should it have to reflect objectivity. Unexpectedly, the answer comes with a help of Bohr’s notion of non-disturbance, which originally used to defend the quantum [4, 17], here, ironically, defines the classical. It is obtained through the state information broadcasting process which precisely pin-points the distributed character of information and makes it essentially classical. We finally illustrate our approach on one of the emblematic examples of the decoherence theory: A dielectric sphere illuminated by photons [16, 17].

We start from the following operational definition [6]:

**Definition 1 (Objectivity)** A state of the system $S$ exists objectively if "...many observers can find out the state of $S$ independently, and without perturbing it."

The natural setting for studying it is the multiple environment paradigm [6]: the quantum system of interest $S$ interacts with multiple environments $E_1,\ldots,E_N$ (denoted collectively as $E$), also modeled as quantum systems. The environments (or their collections) are assumed to be macroscopic and are monitored by independent observers [11]. The system-environment interaction is such that it leads to a full decoherence: there exists a time scale $\tau_D$, called the decoherence time, such that asymptotically for interaction times $t \gg \tau_D$: i) there emerges in the system’s Hilbert space a unique, stable in time pointer basis $\{|i\rangle\}$; ii) the reduced state of the system $\varrho_S$ becomes stable and diagonal in the pointer basis:

$$\varrho_S \equiv \text{Tr}_{E_0S:E} \approx \sum_i p_i |i\rangle \langle i|, \quad (1)$$

where $p_i$’s are some probabilities and by $\approx$ we will always denote asymptotic equality in the deep decoherence limit $t/\tau_D \rightarrow \infty$. We assume here the full decoherence, so that the system decoheres in a basis rather than in higher-dimensional pointer superselection sectors.

In what follows we formalize Def. [11] within the standard quantum formalism as precisely as possible and investigate its consequences. First, we add a stability requirement: the observers can find out the state of $S$ without perturbing it repeatedly and arbitrary many times. It should be understood in the time-asymptotic and hence decoherence regime [11].

Next, we specify the observers. Apart from the environmental ones, we also allow for a direct observer, who can measure the system $S$ directly. Such an observer is needed as a reference, to verify that the findings of the
environmental observers are the same as if one had a direct access to the system.

The word "find out" we translate as the observers performing measurements on their subsystems. We assume von Neumann (as perfectly repeatable contrary to the generalized) measurements. By the "independence" requirement of Def. 1 there can be no correlations between them. Consequently, the global von Neumann measurement, resulting from the individual local observers' measurements, must be fully product:

$$\Pi_i^{M_i} \otimes \Pi_j^{M_j} \otimes \cdots \otimes \Pi_{jN}^{M_N},$$

where \( M_S, M_1, \ldots, M_N \) denote measurements on \( S, E_1, \ldots, E_N \) respectively and all \( \Pi_i \) are mutually orthogonal Hermitian projectors, \( \Pi_j^{M_j} \Pi_j^{M_j'} = 0 \). The observers so determine the probabilities \( p_i \) of \( |i\rangle \) in \( 1 \) (they must know the pointer basis \( \{ |i\rangle \} \), as if not, they would not know what information they get is all about).

The crucial word "perturbation" needs to be made precise. We apply here Bohr's notion of non-disturbance \( 1 \) (stronger than the Einstein-Podolsky-Rosen one \( 3 \) \( 13 \)), according to which given local measurements on the subsystems are non-disturbing if they leave the whole joint state invariant (after forgetting the results). This is a realistic mathematical idealization of a repetitive information extraction—a crucial prerequisite for objectivity.

The Bohr's non-disturbance condition together with the stability requirement and the product structure \( 14 \), implies that on each subsystem \( S, E_1, \ldots, E_N \) there exists a non-demolition von Neumann measurement, leaving the whole asymptotic state \( \varrho_{S:E}(\infty) \) of the system and the observed environment invariant (\( \infty \) denotes the \( t/\tau_D \rightarrow \infty \) asymptotic). For \( S \) it is defined by the projectors on \( |i\rangle \). For the environments we allow for higher-rank projectors \( \Pi_j^{M_j}, k = 1, \ldots, N \), not necessarily spanning the whole space, as the environments can have inner degrees of freedom not correlating to \( S \).

Consequently, the total joint probability of the results of the Bohr non-disturbing measurements is given by:

$$p_{ij_1 \ldots j_N} = \text{Tr} \left[ |i\rangle \langle i| \otimes \Pi_j^{M_j} \otimes \cdots \otimes \Pi_{jN}^{M_N} \varrho_{S:E}(\infty) \right].$$

The independent measurements will typically reveal inconsistent information about the system. Indeed, allowing for general correlations may lead to a disagreement: if one of the outcomes measures first, the ones measuring afterwards may find outcomes depending on the result of the first measurement \( 15 \). Thus, we add to Def. 1 an obvious agreement requirement: "...many observers can find out the same state of \( S \) independently...", leading to a natural conclusion:

$$p_{ij_1 \ldots j_N} \neq 0 \quad \iff \quad i_1 = j_1 = \ldots = j_N.$$  

This means that the environmental Bohr-nondisturbing measurements must be perfectly correlated with the pointer basis. Hence, after forgetting the results, the asymptotic post-measurement state \( \varrho_{S:E}^M(\infty) \) reads:

$$\varrho_{S:E}^M(\infty) = \sum_{i,j_1,\ldots, j_N} p_{ij_1 \ldots j_N} \varrho_{ij_1 \ldots j_N}^{S:E}(\infty),$$

$$= \sum_i |i\rangle \langle i| \otimes \Pi_i \varrho_{S:E}(\infty) |i\rangle \langle i| \otimes \Pi_i,$$

where \( \Pi_i = \Pi_i^{M_1} \otimes \cdots \otimes \Pi_i^{M_N} \).

Now we are ready for the crucial step: we impose the relevant form of the Bohr-nondisturbance condition:

$$\sum_i |i\rangle \langle i| \otimes \Pi_i \varrho_{S:E}(\infty) |i\rangle \langle i| \otimes \Pi_i = \varrho_{S:E}(\infty),$$

whose only solution \( 16 \) are Classical-Quantum (CQ) states \( 17 \):

$$\varrho_{S:E}(\infty) = \sum_i p_i |i\rangle \langle i| \otimes \varrho_i^E,$$

where \( p_i \) are identified with the probabilities from Eq. 1 and \( \varrho_i^E \) are some residual states in the space of all the environments with mutually orthogonal supports: \( \varrho_i^E \varrho_i^E = 0 \). Hence, \( \varrho_i^E \) are perfectly distinguishable through the assumed non-disturbing measurements \( \Pi_i \), projecting on their supports.

Clearly, the environment must be of a large dimension to have a big informational capacity, needed to carry highly redundant records about the decohered system \( S \). Since in such situation the \( S : E \) entanglement \( 20 \) is generically produced during the unitary system-environment evolution (cf. \( 21 \)), the only way to obtain a separable state \( 22 \) from an entangled one is by forgetting subsystems—some portions of the environment pass unobserved, as it is actually always the case in the reality. Thus, "many" means a sufficiently large number of subenvironments to be macroscopic, but not the whole environment. The observed fraction of \( E \) will be denoted by \( f \) or \( fE \) (depending on the context) and all the states above should be understood as \( \varrho_{S:fE}(\infty) \).

Finally, let us look at the residual states \( \varrho_i^E \) in \( 17 \). The demand of independent ability to determine the state of \( S \), already used in \( 22 \), will be completed here with a pivotal concept of strong independence: the only correlation between the environments should be the common information about the system. Thus, once one of the observers finds a particular result \( i \), the resulting conditional state should be fully product. Since the direct observer is already uncorrelated by \( 16 \), this implies that:

$$\varrho_i^E = \varrho_i^E \otimes \cdots \otimes \varrho_i^E$$

and \( \varrho_i^E_k \) must be perfectly distinguishable for each \( E_k \):

$$\varrho_i^E_k \varrho_i^E_k = 0 \quad \text{for} \quad i \neq i',$$

since by \( 16 \) for any \( k \) it holds: \( \Pi_i^{M_k} \varrho_i^E_k \Pi_i^{M_k} = \varrho_i^E_k \) and \( \Pi_i^{M_k} \Pi_i^{M_k'} = 0 \). We now formulate a broader class of independent environments, in a way paradigmatic in quantum information theory \( 22 \): the environments are independent if and only if the environmental observers may
produce the states \(\{\Psi_i\}\), exploiting only local operations (equivalent to local trace preserving maps), i.e. independent environments are those ones that simulate strong independence from the perspective of a specific resource (the class of local operations). Summarizing, under the above conditions, we have proven the following:

**Theorem 1** If a decoherence mechanism asymptotically leads to an objectively existing state of \(S\) in the sense of Def. \(\[\]\) then the asymptotic joint state of the system and the observed environment fraction (after the necessary tracing out of some of the environment) must be of a specific Classical-Classical form \([1, 12, 23, 24]\):

\[
\rho_{S:E}(\infty) = \sum_i p_i |i\rangle_S \otimes \hat{\rho}_i^{E_1} \otimes \cdots \otimes \hat{\rho}_i^{E_N},
\]

where all \(\hat{\rho}_i^{E_k}\) satisfy \([9]\).

The above structure of \(\rho_{S:E}(\infty)\), called here the *spectrum broadcast structure*, precisely reflects the objectively existing classical state of the system: for a given \(\rho_S\), state \([17]\) allows for a repetitive, local recovery of perfect copies of the probabilities \(p_i\) through the non-demolition projective measurements of the supports of \(\hat{\rho}_i^{E_k}\) (cf. \([9]\)). We stress that here \([17]\) is derived, rather than postulated on the Everettian grounds. The transition:

\[\text{initial } S : E \text{ state} \longrightarrow \text{spectrum broadcast structure } \[17\]\]

identifies a basic process, called here state information broadcasting, responsible for an emergence of the perceived objectivity. It involves broadcasting of a part of information about the system—the spectrum of its state after the decoherence, \(Sp\rho_S \equiv \{p_i\}\), into the environments and is thus similar to quantum state \([22]\) and spectrum \([24]\) broadcasting. Condition \([9]\) forces the correlations in \([17]\) to be entirely classical and thus the detailed structures of \(\hat{\rho}_i^{E_k}\) become irrelevant for the correlations.

From \([9, 17]\) it follows that under a suitable convergence:

\[
I[\rho_{S:F}(\infty)] = H_S \quad \text{for every fraction } f,
\]

where \(I[\rho_{AB}] \equiv \bar{S}_N(\rho_A) + \bar{S}_N(\rho_B) - \bar{S}_N(\rho_{AB})\) is the quantum mutual information, \(\bar{S}_N(\rho) = -\text{Tr}(\rho \log \rho)\) stands for the von Neumann entropy, and \(H_S \equiv \bar{S}_N(\rho_S(\infty)) = H(\{p_i\})\) is the entropy of the decohered state \([1]\). Condition \([11]\) postulated as a sufficient condition for objectivity in quantum Darwinism model, has a clear meaning in the classical information theory \([20]\): every fraction \(f\) carries the same information \(H_S\) about the system—the latter is redundantly encoded in the environment. However, in the quantum world its sense remains unclear \([10]\). Here, \([11]\) follows automatically from the deeper structure \([17]\).

Thm. \([1]\) states that under decoherence, state information broadcasting is a necessary condition for objectivity. Conversely, a state information broadcasting process, resulting by definition in the structure \([17]\) with the property \([9]\), implies objective existence of the classical state \(\{p_i\}\) (cf. \([27]\)). Indeed, projections on \(|i\rangle\) and on the disjoint supports of \(\hat{\rho}_i^{E_k}\) constitute the non-demolition measurements. Performing them independently, all the observers will repeatedly detect the same distribution \(\{p_i\}\), without Bohr-disturbing the joint \(S : fE\) state. Together with Thm. \([1]\) this proves the following:

**Theorem 2** In the presence of decoherence, state information broadcasting is a necessary and sufficient condition for the emergence of objectivity:

\[
\text{Decoherence } + \left( \begin{array}{c} \text{Objective Existence} \\ \text{State Information Broadcasting} \end{array} \right) \Rightarrow \left( \begin{array}{c} \text{State Information Broadcasting} \end{array} \right)
\]

Objective Existence \(\Leftrightarrow\) State Information Broadcasting.

We exemplify the above general findings on one of the central models of decoherence (see e.g. \([16, 17]\)): a dielectric sphere illuminated by photons (see Fig. 1 for the details see \([15]\)). The sphere is initially in a state without a well defined position (e.g. in \(|\psi_0^S\rangle = (|x_1\rangle + |x_2\rangle)/\sqrt{2}\)). Photons scatter elastically and slightly differently depending on where the sphere is, but this difference is vanishingly small for each individual scattering: If the observed fraction is too small, the post scattering states \(|\Psi_i^{mic}\rangle \equiv S_i |k_0\rangle\) (\(S_i\) are the scattering matrices) become identical in the thermodynamic limit: \(|\Psi_2^{mic}\rangle |\Psi_1^{mic}\rangle \equiv (\hat{r}_0 |S_2^T S_1^T r_0\rangle \underset{\text{therm.}}{\rightarrow} 1\) and the joint post scattering state approaches effectively a product:
FIG. 2: Information-theoretical phases. Schematic phase diagram showing three different phases of the illuminated sphere model, appearing in the thermodynamic and the deep decoherence limits. The horizontal axis is the observed fraction $f$ of the total photon number. The vertical axis is the asymptotic mutual information between the sphere $S$ and the fraction $\{E, I_{[\rho S; fE(\infty)]}\}$. The plot shows two phase transitions: i) at $f = 0$ from the product phase, when the observed photon fraction is too small to reveal any information on the location, to the objective, broadcasting phase $0 < f < 1$; ii) at $f = 1$ from the broadcasting to the full information phase, when all the photons are observed and maintain quantum correlation with the sphere (cf. [17]). Due to the thermodynamic limit each value of the fraction $f$ should be understood modulo a microscopic fraction, i.e. a fraction not scaling with the total number of photons.

$$(\sum_{i=1,2} p_i |\vec{x}_i⟩⟨\vec{x}_i|) \otimes |Ψ^{mic}_{i}⟩⟨Ψ^{mic}_{i}|^N_i$$, where probabilities $p_i \equiv \langle |Ψ^{S}_{i}\rangle |^2$ form the spectrum of the decohered state (cf. [1]). The photons thus force the sphere to be in a definite position $\vec{x}_i$ with the probability $p_i$, but the observed fraction carries no information about it (a product phase; Fig. 2).

However, when grouped into macroscopic fractions, the photons become almost perfectly resolving. Imagine we divide all the photons scattered up to time $t$, $N_i$, into $M$ macro-fractions of $mN_i$, $0 < m < 1$, photons (reflecting e.g. the eye’s sensitivity threshold). Then the macroscopic post scattering states $|Ψ^{mac}_{1}(t)⟩ ≡ (S_i |k_0⟩)^\otimes m N_i$, become asymptotically perfectly distinguishable:

$$|Ψ^{mac}_{1}(t)⟩|Ψ^{mac}_{2}(t)⟩ \xrightarrow{therm.} e^{-\frac{2\pi i}{\hbar} t}$$, (12)

where $\tau_D$ is the decoherence time $[16, 17]$. If we observe $fM, 0 < f < 1$, macro-fractions out of $M$, then the joint post scattering state has asymptotically the spectrum broadcast structure [17]:

$$\rho_{S; fE(0)} = \rho_{S}^0 \otimes \rho_{mac}^{1} \otimes \cdots \otimes \rho_{mac}^{M} + \tau_D \rho_{S; fE(\infty)} = \sum_{i=1,2} p_i |\vec{x}_i⟩⟨\vec{x}_i| \otimes |Ψ^{mac}_{i}⟩⟨Ψ^{mac}_{i}|^N_i$$, (13)

where $|Ψ^{mac}_{i}⟩ ≡ |Ψ^{mac}_{i}(\infty)⟩$ emerges, due to [12], as the non-disturbing environmental basis in the space of each macro-fraction. Eq. (13) identifies the state information broadcasting process: the information about the sphere’s localization, $\{p_i\}$, is redundantly transferred into the environment and becomes available in multiple copies through the measurements in $\{\{Ψ^{mac}_{i}\}\}$. The process consists of: i) decoherence [10] and ii) orthogonalization (12), and defines a broadcasting phase (Fig. 2), corresponding to the classical plateau of [17]. From Fannes-Audenaert [28] and Alicki-Fannes [29] inequalities the entropic condition (11) follows as a consequence of (13) [31]. Finally, if all the photons are observed, the post-scattering state maintains the full quantum correlation with the system $I_{[\rho S; fE(\infty)]} = I_{max}$ (a full information phase).

In conclusion, based on an universal approach, independent of any dynamics or a concrete model, we have identified the primitive state information broadcasting process responsible for an emergence of the perceived objectivity. Our main result (Thm. 2) implies that the states of the form (13) are notoriously formed in Nature. In a laboratory, this can be in principle directly verified via e.g. quantum state tomography [30]. Moreover, it naturally leads to a view that in fact there may be no "quantum-to-classical transition"—what we perceive as "classical", e.g. objective information, may be merely a reflection of some specific properties of the underlying quantum states, like the spectrum broadcast structure; a view further strengthened by [31].

The emergence of redundantly encoded information in the structure of quantum states may also shed new light on the life phenomenon. Since self-replication of the DNA information is indispensable for the existence of life, it cannot be excluded that the state information broadcasting may indeed open a "classical window" for life processes within quantum mechanics [32].

[1] Bohr, N. Discussions with Einstein on Epistemological Problems in Atomic Physics, in P. A. Schilpp (Ed.), Albert Einstein: Philosopher-Scientist, Library of Living Philosophers, Evanston, Illinois (1949).
[2] Heisenberg, W. Philosophic Problems in Nuclear Science (F. C. Hayes Transl.), Faber and Faber, London (1952).
[3] Joos, E., et al., Decoherence and the Appearance of a Classical World in Quantum Theory, Springer, Berlin (2003).
[4] Bohr, N. Can Quantum-Mechanical Description of Physical Reality be Considered Complete? Phys. Rev. 48, 696 (1935).
[5] Wiseman, H. M. Quantum discord is Bohr’s notion of non-mechanical disturbance introduced to counter the
Einstein-Podolsky-Rosen argument. To appear in Ann. Phys, preprint at <http://arxiv.org/abs/1208.4964v2> (2012).

[6] Zurek, W. H. Quantum Darwinism. Nature Phys. 5, 181 (2009).

[7] Zeh, H. D. Roots and fruits of decoherence, in B. Duplantier, J.-M. Raimond, V. Rivasseau (Eds.), Quantum Decoherence, Birhäuser, Basel (2006).

[8] Schlosshauer, M. Decoherence and the Quantum-to- Classical Transition, Springer, Berlin (2007).

[9] Zurek, W. H. Pointer basis of quantum apparatus: Into what mixture does the wave packet collapse? Phys. Rev. D 24, 1516 (1981).

[10] Wootters, W. and Zurek, W. H. A Single Quantum Cannot be Cloned. Nature 299, 802 (1982).

[11] Zwalak, M. and Zurek, W. H. Complementarity of quantum discord and classically accessible information. Sci. Rep. 3, 1729 (2013).

[12] Everett, H. "Relative State" Formulation of Quantum Mechanics. Rev. Mod. Phys. 29, 454 (1957).

[13] Blume-Kohout, R. and Zurek, W. H. Quantum Darwinism: Entanglement, branches, and the emergent classicality of redundantly stored quantum information. Phys. Rev. A 73, 062310 (2006).

[14] Cao, T. Y. Conceptual Developments of 20th Century Field Theories, Cambridge University Press, Cambridge, 1997.

[15] Einstein, A., Podolsky, B., and Rosen, N. Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. 47, 777 (1935).

[16] Joos, E. and Zeh, H. D. The emergence of classical properties through interaction with the environment. Z. Phys. B 59, 223 (1985).

[17] Riedel, C. J. and Zurek, W. H. Quantum Darwinism in an everyday environment: Huge redundancy in scattered photons. Phys. Rev. Lett. 105, 020404 (2010).

[18] See Supplementary Information.

[19] Modi, K., Brodutch, A., Cable, H., Paterek, T., and Vedral, V. Quantum discord and other measures of quantum correlation. Rev. Mod. Phys. 84, 1655 (2012).

[20] Horodecki, R., Horodecki, P., Horodecki, M., and Horodecki, K. Quantum entanglement. Rev. Mod. Phys. 81, 865 (2009).

[21] Gurvits, L. and H. Barnum, H. Largest separable balls around the maximally mixed bipartite quantum state. Phys. Rev. A 66, 062311 (2002).

[22] Horodecki, P. and Horodecki, R. Distillation and Bound Entanglement. Quant. Inf. Comp. 1, 45 (2001).

[23] Piani, M., Horodecki, P., and Horodecki, R. No-Local-Broadcasting Theorem for Multipartite Quantum Correlations. Phys. Rev. Lett. 100, 090502 (2008).

[24] Korbicz, J. K., Horodecki, P., and Horodecki, R. Quantum-correlation breaking channels, broadcasting scenarios, and finite Markov chains. Phys. Rev. A 86, 042319 (2012).

[25] Barnum, H., Caves, C. M., Fuchs, C. A., Jozsa, R. and Schumacher, B. Noncommuting Mixed States Cannot Be Broadcast. Phys. Rev. Lett. 76, 2818 (1996).

[26] Cover, T. M. and Thomas, J. A. Elements of Information Theory, John Wiley and Sons, New York (1991).

[27] Zurek, W. H. Wave-packet collapse and the core quantum postulates: Discreteness of quantum jumps from unitarity, repeatability, and actionable information. Phys. Rev. A 87, 052111 (2013).

[28] Audenaert, K. M. R. A sharp continuity estimate for the von Neumann entropy. J. Phys. A: Math. Theor. 40, 8127 (2007).

[29] Alicki, R. and Fannes, M. Continuity of quantum conditional information. J. Phys. A: Math. Gen. 37, L55, (2004).

[30] Paris, M. and Řeháček, J. (Eds.), Quantum State Estimation, Lect. Notes Phys. 649, Springer, Berlin (2004).

[31] Pusey, M. F., Barrett, J., and Rudolph, T. On the reality of the quantum state. Nature Phys. 8, 476 (2012).

[32] Wigner, E. P. The Probability of the Existence of a Self-Reproducing Unit, in The Logic of Personal Knowledge: Essays Presented to Michael Polany on his Seventieth Birthday, Routledge & Kegan Paul, London (1961).

Acknowledgements We thank W. Zurek and J. Riedel for discussions and comments and M. Piani for discussions on strong independence. P.H. and R.H. acknowledge discussions with K. Horodecki, M. Horodecki, and K. Życzkowski. This research is supported by ERC Advanced Grant QOLAPS and National Science Centre project Maestro DEC-2011/02/A/ST2/00305.

Author contributions All authors contributed to all aspects of this work.

Author information Reprints and permissions information is available at www.nature.com/reprints. Correspondence and requests for materials should be addressed to R.H. (fizrh@uoguelph.ca).

I. SUPPLEMENTARY INFORMATION

II. AGREEMENT AMONG THE OBSERVERS

Let us more formally show that

\[ p_{ij\ldots jN} \neq 0 \text{ iff } i = j_1 = \ldots = j_N. \]  

(14)

considering for simplicity only two observers. If one of them measures first and gets a result \( i \), then the joint conditional state becomes \( g_{i} = (1/p_i) (\Pi_i \otimes 1) g (\Pi_i \otimes 1) \). \( p_i \equiv \text{Tr}(\Pi_i \otimes 1) \) and the subsequent measurement by the second observer will yield results \( j \) with conditional probabilities \( p_{j|i} = (1/p_i) \text{Tr}(\Pi_i \otimes \Pi_j g) \). If for some \( i \), \( p_{j|i} p_{j'|i} \neq 0 \) for \( j \neq j' \), then comparing their results after a series of measurements at some later moment, the observers will be confused as to what exactly the state the system \( S \) was: with the probability \( p_{j|i} p_{j'|i} \) the second observer will obtain different states \( j \neq j' \), while the first observer measured the same state \( i \). One would not the observers’ findings objective, unless for every \( i \) there exists only one \( j(i) \) such that \( p_{j(i)|i} \neq 0 \) (actually \( p_{j(i)|i} = 1 \), which follows from the normalization \( \sum_j p_{ij} = 1 \), so that the distributions \( p_{ij} \) are all deterministic). Reversing the measurement order and applying the same reasoning, we obtain that for every \( j \) there can exist only one \( i(j) \) such that \( p_{i(j)|j} \neq 0 \), where by the Bayes theorem \( p_{i|j} = p_{j|i} p_{i}/p_{j} \), \( p_{j} \equiv \text{Tr}(1 \otimes \Pi_j g) \). These two conditions
imply that the joint probability \( p_{ij} = \delta_{ij} \) (after an eventual renumbering). Applying the above argument to all the pairs of indices, one obtains Eq. (14).

III. DISCUSSION OF THE ENTROPIC OBJECTIVITY CONDITION AS A "WITNESS" FOR OBJECTIVITY

Here we show a potential problem with the entropic objectivity condition:

\[
I(\rho_{S;FE}) = H_S \quad \text{for every fraction } f,
\]

as a sufficient condition for objectivity (see e.g. Refs. [1] [2] and references therein). Although our example below is not fully conclusive, we argue that at this moment neither is the reasoning of quantum Darwinism studies.

Condition (15) has been shown to hold in several models, including the illuminated sphere [3] and spin baths (see e.g. Ref. [4]). For finite times \( t \), the equality (15) is not strict and holds within some error \( \delta(t) \), which defines the redundancy \( R_S(t) \) as the inverse of the smallest fraction of the environment \( f_E(t) \), for which \( I(\rho_{S;f_E}(\infty)) = [1 - \delta(t)]H_S \). When satisfied, (15) implies that the mutual information between the system and the environment fraction is a constant function of the fraction size \( f \) (up to an error \( \delta \) for finite times) and the plot of \( I \) against \( f \) exhibits a characteristic plateau, called the classical plateau (see e.g. Ref. [1]). The appearance of this plateau has been heuristically explained in the quantum Darwinism literature as a consequence of the redundancy: classical information about the system exists in many copies in the environment fractions and can be accessed independently and without perturbing the system by many observers, thus leading to objective existence of a state of \( S \) [1]. That would be for sure so in the classical information setting: the condition (15) is there equivalent to a perfect correlation of both systems [3], i.e. for every \( f \) the environment fraction has a full information about the system and indeed this information thus exists objectively in the sense of our definition.

But in the quantum world the situation may be different and the condition (15) alone may not be sufficient to guarantee objectivity, due to a wholistic nature of quantum correlations [3]. It is clear that the spectrum broadcast states (14) satisfy (15), but there may also be entangled states satisfying it, thus violating the spectrum broadcast form, derived as a necessary condition for objectivity. As a simple example in favour of such a statement consider the following case of two qubits, where one is the system \( S \) and the second the environment \( E \):

\[
\rho_{SE} = p \rho_1 + (1 - p) \rho_2,
\]

where \( P \equiv |\psi\rangle \langle \psi|, \quad p \neq 1/2, \quad a = \sqrt{p} \) and \( b = \sqrt{1-p} \). Then the partial state \( \rho_S = \rho_1 \rho_2 \) is diagonal in the basis \( |0\rangle, |1\rangle \) and moreover \( S_N(\rho_S) = S_N(\rho_{SE}) \equiv h(p) \) (the binary Shannon entropy [3]), so that a form of the entropic condition holds: \( I(\rho_{SE}) = S_N(\rho_S) = H_S, \quad H_S = h(p) \), but the systems are nevertheless entangled, which one verifies directly through the PPT criterion [2].

The above example is of course not conclusive, as there is only one environment, but it suggest that the functional condition (15) may indeed be not sufficient to show objectivity, as defined in the main text. The paradigmatic shift with respect to the earlier works on decoherence and quantum Darwinism models we propose here, is that the core object of the analysis should be a derived structure of the full quantum state of the system \( S \) and the observed environment \( f_E \), rather than the partial state of the system only (Decoherence Theory) or information-theoretical functions (quantum Darwinism).

IV. THE ILLUMINATED SPHERE MODEL - PURE ENVIRONMENTS

Here we present a detailed derivation of the spectrum broadcast structure:

\[
\rho_{SE}(\infty) = \sum_i p_i \rho_i |i\rangle \otimes \rho_i^{E_i} \otimes \cdots \otimes \rho_i^{E_i}
\]

in the illuminated sphere model. We first recall the basics of the model, following the usual treatment (see e.g. Refs. [3] [4] [10]). The system \( S \) is a sphere of radius \( a \) and relative permittivity \( \epsilon \), bombarded by a constant flux of photons, which constitute the multiple environments and decohere the sphere. The sphere can be located only at two positions: \( \vec{x}_1 \) or \( \vec{x}_2 \), so that effectively its state-space is that of a qubit \( \mathbb{H}_S \equiv \mathbb{C}^2 \) with a preferred orthonormal (due to the mutual exclusiveness) basis \( |\vec{x}_1\rangle, |\vec{x}_2\rangle \), which will become the pointer basis. This greatly simplifies the analysis, yet allows the essence of the effect to be observed. The sphere is sufficiently massive, compared to the energy of the radiation, so that the recoil due to the scattering can be totally neglected and photons’ energy is conserved, i.e. the scattering is elastic.

The environmental photons are assumed not energetic enough to individually resolve the sphere’s displacement \( \Delta x \equiv |\vec{x}_2 - \vec{x}_1| \):

\[
k \Delta x \ll 1,
\]

where \( h k \) is the characteristic photon momentum. Otherwise, each individual photon would be able to resolve the position of the sphere and studying multiple environments would not bring anything new. On the technical side, following the traditional approach [3] [4] [10], we describe the photons in a simplified way using box normalization: we assume that the sphere and the photons are enclosed in a large box of edge \( L \) and volume \( V = L^3 \) and photon momentum eigenstates \( |\vec{k}\rangle \) obey periodic boundary conditions. Although a more rigorous treatment was developed in Ref. [11] with well localized photon states, we choose this traditional heuristic approach as, at the expense of a mathematical rigor, it al-
lows to expose the physical situation more clearly, without unnecessary mathematical details (we remark that the findings of Ref. 11 agree with the previous works using box normalization 12). After dealing with formally divergent terms, we remove the box through the thermodynamic limit (signified by \( \equiv \)) 3, 10:

\[ V \to \infty, \quad N \to \infty, \quad \frac{N}{V} = \text{const}, \quad (19) \]

that is we expand the box and add more photons, keeping the photon density constant, as the relevant physical quantity is the radiative power, proportional to \( N/V \). The thermodynamic limit is crucial in the sense that it defines micro- and macroscopic regimes, which will turn to be qualitatively very distinct.

The detailed dynamics of each individual scattering is irrelevant—the individual scatterings are treated asymptotically in time. The interaction time \( t \) enters the model differently, thought the number of scattered photons. It may be called a “macroscopic time”. Assuming photons come from the area of \( L^2 \) at a constant rate \( N \) photons per volume per unit time, the amount of scattered photons from \( t = 0 \) to \( t \) is:

\[ N_t \equiv L^2 \frac{N}{V} ct, \quad (20) \]

where \( c \) is the speed of light. Throughout the calculations we work with a fixed time \( t \) and pass to the asymptotic limit \( t/\tau \to \infty \) (signified by \( \approx \) or \( \sim \)) at the very end.

Since multiphoton scatterings can be neglected and all the photons are treated equally (symmetric environments), the effective sphere-photons interaction up to time \( t \) is of a controlled-unitary form:

\[ U_{S:E}(t) \equiv \sum_{i=1,2} |\vec{x}_i\rangle \langle \vec{x}_i| \otimes S_i \otimes \cdots \otimes S_i \quad (21) \]

where (assuming translational invariance of the photon scattering) \( S_i \equiv S_{\vec{k}} = e^{-i\vec{k} \cdot \vec{x}} S_0 \) is the scattering matrix when the sphere is at \( \vec{x}_i \), \( S_0 \) is the scattering matrix when the sphere is at the origin, and \( \hbar \vec{k} \) is the photon momentum operator. Due to the elastic scattering, \( S_i \)’s have non-zero matrix elements only between the states \( |\vec{k}\rangle \) of the same energy \( \hbar c |\vec{k}| \). In the sector \( |1\rangle \) the interaction \( (21) \) is vanishingly small at the level of each individual photon 11: in the thermodynamic limit \( S_1 \equiv S_2 \) (in a suitable sense we clarify later), and hence \( \sum_i |\vec{x}_i\rangle \langle \vec{x}_i| \otimes S_i \equiv 1 \otimes S \). Surprisingly, this will not be true for macroscopic groups of photons. We also note that unlike in the previous treatments 3, 3, 11, already at this moment we explicitly include in the description all the photons scattered up to the fixed time \( t \). Finally, the preferred role of the basis \( |\vec{x}_i\rangle \) is already singled out now by the form of the interaction \( (21) \).

The initial, pre-scattering “in” state, is as usually assumed a full product:

\[ |\psi_{S:E}(0)\rangle \equiv |\psi_0^S \rangle \otimes (|\phi_0^{ph}\rangle)^{\otimes N_t}, \quad (22) \]

with \( |\psi_0^S\rangle \) having coherences in the preferred basis \( |\vec{x}_i\rangle \) and \( |\phi_0^{ph}\rangle \) some initial states of the photons (the environments are by assumption symmetric). Next, we introduce a crucial environment coarse-graining 1: the full environment (i.e. all the \( N_t \) photons) is divided into a number of macroscopic fractions, each containing \( mN_t \) photons, \( 0 \leq m \leq 1 \). By macroscopic we will always understand “scaling with the total number of photons \( N_t \)”. By definition, these are the environment fractions accessible to the independent observers. Such a division may seem artificial and arbitrary, as e.g. the choice of \( m \) is unspecified. However, observe that in typical situations detectors used to monitor fractions of the environment, e.g. eyes, have some minimum detection thresholds—some minimum amount of radiative energy delivered in a given time interval is needed to trigger the detection. Each macroscopic fraction \( mN_t \) is meant to reflect that detection threshold. Its concrete value (the fraction size \( m \)) is for our analysis irrelevant—it is enough that it scales with \( N_t \). This coarse-graining procedure is analogous to the one used e.g. in the description of liquids 13: each point of a liquid (a macro-fraction \( m \) here) is in reality composed of a suitable large number of microparticles (individual photons). It is also employed in mathematical approach to von Neumann measurements using, so called, macroscopic observables (see e.g. Ref. 14 and the references therein).

Thus, we divide the detailed initial state of the envi-
environment \((\hat{\varrho}_0^{ph})^\otimes N_i\) into \(M = 1/m\) macroscopic fractions:

\[
\hat{\varrho}_0^{ph} \otimes \cdots \otimes \hat{\varrho}_0^{ph} = \hat{\varrho}_0^{ph} \otimes \cdots \otimes \hat{\varrho}_0^{ph} \otimes \cdots \otimes \hat{\varrho}_0^{ph} \equiv \hat{\varrho}_{mac}^{m \otimes m, N_i} \equiv \hat{\varrho}_{mac}^{m \otimes m, N_i} \equiv \hat{\varrho}_0^{mac} \equiv (\hat{\varrho}_0^{mac})^\otimes m N_i,
\]

where \(\hat{\varrho}_0^{mac} = (\hat{\varrho}_0^{ph})^\otimes m N_i\) is the initial state of each macroscopic fraction (macro-state for brevity).

After all the \(N_i\) photons have scattered, the asymptotic (in the sense of the scattering theory) "out"-state \(\hat{\varrho}_{S:E}(t) = U_{S:E}(t)\hat{\varrho}_{S:E}(0)U_{S:E}(t)\dagger\), is given from Eqs. \(21, 22, 23\) by

\[
\hat{\varrho}_{S:E}(t) = \sum_{i=1,2} \langle \vec{x}_i|\hat{\varrho}_0^S \hat{\varrho}_i |\vec{x}_i\rangle \otimes \hat{\varrho}_{i}^{mac}(t) \otimes \cdots \otimes \hat{\varrho}_{i}^{mac}(t) \tag{24}
\]

\[
+ \sum_{i \neq j} \langle \vec{x}_i|\hat{\varrho}_0^S \hat{\varrho}_j|\vec{x}_j\rangle \langle \vec{x}_j| \otimes \left( \hat{S}_1 \hat{\varrho}_0^S \hat{S}_1^\dagger \right)^\otimes \cdots \otimes \hat{S}_m \hat{\varrho}_0^S \hat{S}_m^\dagger \tag{25}
\]

where

\[
\hat{\varrho}_{i}^{mac}(t) = \left( \hat{S}_i \hat{\varrho}_0^S \hat{S}_i^\dagger \right)^\otimes m N_i, \quad i = 1, 2. \tag{26}
\]

As argued, we have to trace out some of the environment. In the current model it is important that the forgotten fraction must be macroscopic: we assume that \(f M, 0 < f < 1\) out of all \(M\) macro-fractions of Eq. \(24\) are observed, while the rest, \((1 - f)M\), is traced out. The resulting partial state reads (cf. Eqs. \(21, 22, 23\))

\[
\hat{\varrho}_{S:F}(t) = \sum_{i=1,2} \langle \vec{x}_i|\hat{\varrho}_0^S \hat{\varrho}_i |\vec{x}_i\rangle \otimes \hat{\varrho}_{i}^{mac}(t) \otimes \cdots \otimes \hat{\varrho}_{i}^{mac}(t) \tag{27}
\]

\[
+ \sum_{i \neq j} \langle \vec{x}_i|\hat{\varrho}_0^S \hat{\varrho}_j|\vec{x}_j\rangle \left( \text{Tr} \hat{S}_1 \hat{\varrho}_0^S \hat{S}_1^\dagger \right)^{(1-f)N_i} \otimes \hat{\varrho}_{i}^{mac}(t) \otimes \hat{\varrho}_{j}^{mac}(t) \otimes \left( \hat{S}_1 \hat{\varrho}_0^S \hat{S}_1^\dagger \right)^{(1-f)N_i} \tag{28}
\]

We finally demonstrate that in the soft scattering sector \(15\), the above state is asymptotically of the broadcast form \(17\) by showing that in the deep decoherence regime \(t \gg \tau_D\) two effects take place:

1. The coherent part \(\hat{\varrho}_{i}^{f,F}(t)\) given by Eq. \(28\) vanishes in the trace norm:

\[
\|\hat{\varrho}_{S:F}(t)\|_\text{tr} = \text{Tr} \left( \hat{\varrho}_{i}^{f,F}(t) \right) \approx 0. \tag{29}
\]

2. The post-scattering macroscopic states \(\hat{\varrho}_{i}^{mac}(t)\) (cf. Eq. \(26\)) become perfectly distinguishable:

\[
\hat{\varrho}_{i}^{mac}(t)\hat{\varrho}_{i}^{mac}(t) \approx 0, \tag{30}
\]

or equivalently using the generalized overlap \(17\):

\[
B[\hat{\varrho}_{1}^{mac}(t), \hat{\varrho}_{2}^{mac}(t)] = \text{Tr} \left( \hat{\varrho}_{1}^{mac}(t) \hat{\varrho}_{2}^{mac}(t) \right) \hat{\varrho}_{1}^{mac}(t) \approx 0. \tag{31}
\]

despite of the individual (microscopic) states becoming equal in the thermodynamic limit.

The first mechanism above is the usual decoherence of \(S\) by \(f E\)—the suppression of coherences in the preferred basis \(|\vec{x}_i\rangle\). Some form of quantum correlations may still survive it, since the resulting state \(24\) is generally of a Classical-Quantum (CQ) form \(14\). Those relict forms of quantum correlations are damped by the second mechanism: the asymptotic perfect distinguishability \(23\) of the post-scattering macro-states \(\hat{\varrho}_{i}^{mac}(t)\). Thus, the state \(\hat{\varrho}_{S,F}(\infty)\) becomes of the spectrum broadcast form \(17\) for the distribution:

\[
p_i = \langle \vec{x}_i|\hat{\varrho}^S \hat{\varrho}_i |\vec{x}_i\rangle, \tag{32}
\]

which by our general results gains objective existence (as a state in the sense of the classical statistical mechanics).

For the purpose of this work, we demonstrate the mechanisms \(29, 30\) and hence a formation of the broadcast state \(14\), for pure initial environments:

\[
\hat{\varrho}_{ph}^0 = |\vec{k}_0\rangle \langle \vec{k}_0|, \quad k_0 \Delta x \ll 1, \tag{33}
\]

i.e. all the photons come from the same direction and have the same momenta \(k_0\), \(k_0 \equiv |\vec{k}_0|\), satisfying \(13\). To show \(29\), observe that \(\hat{\varrho}_{S:F}(t)\), defined by Eq. \(28\), is of a simple form in the basis \(|\vec{x}_i\rangle\):

\[
\hat{\varrho}_{S:F}(t) = \left[ \begin{array}{cc} 0 & \gamma \hat{C}^\dagger \\ \gamma^* & 0 \end{array} \right], \tag{34}
\]

where \(\gamma \equiv \langle \vec{x}_1|\hat{\varrho}_0^S \hat{\varrho}_1 |\vec{x}_1\rangle \left( \text{Tr} \hat{S}_1 \hat{\varrho}_0^S \hat{S}_1^\dagger \right)^{(1-f)N_i} \), and \(C \equiv (\hat{S}_1 \hat{\varrho}_0^S \hat{S}_1^\dagger)^{(1-f)N_i} \). Since \(\hat{S}_i\)'s are unitary and \(\hat{\varrho}_0^S \geq 0, \text{Tr} \hat{\varrho}_0^S = 1\), we obtain:

\[
\|\hat{\varrho}_{S:F}(t)\|_\text{tr} = \|\hat{\varrho}_{S:F}(t)\|_\text{tr} \approx 0.
\]

The decoherence factor \(\text{Tr} \hat{S}_1 \hat{\varrho}_0^S \hat{S}_1^\dagger\) for the pure case \(35\) has been extensively studied before (see. e.g. Refs. \(3, 4, 6\)). Let us briefly recall the main results. Under the condition \(13\) and using the classical cross section of a dielectric sphere in the dipole approximation \(k_0 \Delta x \ll 1\), one obtains in the box normalization:

\[
\langle \vec{k}_0| \hat{S}_1 \hat{\varrho}_0^S \hat{S}_1^\dagger \rangle = 1 + \frac{8 \pi \Delta x k_0^5 \Delta \theta}{3 L^2} \cos \Theta - \frac{2 \pi \Delta x^2 k_0^5 \Delta \theta}{15 L^2} (3 + 11 \cos^2 \Theta) + O \left( \frac{(k_0 \Delta x)^3}{L^2} \right) \tag{37}
\]

where \(\Theta\) is the angle between the incoming direction \(\vec{k}_0\) and the displacement vector \(\Delta \vec{x} \equiv \vec{x}_2 - \vec{x}_1\) and \(\Delta \vec{x} \equiv a(\epsilon - \epsilon_0) \Delta x \).
Eqs. (36,39) imply that \( ||\Psi^\text{mic}_1(t)\|^2 \approx (19) \) and the deep decoherence Eq. (43) more and more distinguishable in the thermodynamic (represented by the big solid slabs on the right) become by

\[
\text{FIG. 4: Orthogonalization of macroscopic states. At the microscopic level, the individual post-scattering states } |\Psi^\text{mic}_1(t)\rangle = S_i |\vec{k}_0\rangle, \text{ corresponding to the sphere being at } \vec{x}_i, \text{ (represented by the small solid slabs on the left) become identical in the thermodynamic limit (cf. Eq. (17)) and hence completely indistinguishable. They carry vanishingly small amount of information about the sphere’s localization, which is due to the assumed weak coupling between the sphere and each individual environmental photon } \Omega. \text{ On the other hand, the collective states of macroscopic fractions } |\Psi^\text{mac}(t)\rangle = (S_i |\vec{k}_0\rangle)^\otimes m_{N_i} \text{ (represented by the big solid slabs on the right) become by Eq. (36) more and more distinguishable in the thermodynamic (119) and the deep decoherence } t \gg \tau_D \text{ limits. Together with the decoherence mechanism (29) this leads to a formation of the spectrum broadcast state (17) with pure environmental states, and hence to the objective existence of the (classical) state of the sphere.}
\]

\[
1/(\epsilon + 2)^{1/3}. \text{ This implies:}
\]

\[
\left[ \text{Tr}_{S_0} e^{\Delta t} S_i |\vec{k}_0\rangle |\vec{k}_0\rangle \right]^{(1-f)N_i} \approx \left[ 1 - \frac{2\pi \Delta x^2 k_0^6 \delta^6}{15L^2} (3 + 11 \cos^2 \Theta) \right]^{(1-f)x} e^{-\frac{(1-f)D}{\Delta t}}. \tag{38}
\]

In the second line above we used Eq. (37) up to the leading order in \( 1/L \); in the last line we removed the box normalization through the thermodynamical limit (119) and thus obtained the decoherence time (2, 119):

\[
\tau_D^{-1} = \frac{2\pi N}{15V} \Delta x^2 \Delta k_0^6 \delta^6 (3 + 11 \cos^2 \Theta). \tag{40}
\]

Eqs. (36, 39) imply that \( ||\rho_S^{\text{SF}E}(t)||_{1,x} \leq 2e^{-(1-f)t/\tau_D} ||\vec{x}_1 \rangle |\vec{k}_0^\Theta \langle \vec{x}_2| \rangle \), since the sequence \( (1 + x/N)^N \) is monotonically increasing. As a result, whenever we forget a macroscopic fraction of the environment \( f < 1 \), the resulting coherent part \( \tilde{\rho}_S^{\text{SF}E}(t) \) decays in the trace norm exponentially, with the characteristic time \( \tau_D/(1-f) \). This completes the first step (29).

The asymptotic orthogonalization (39) is also straightforward to show in the case of pure environments. The post-scattering states of the environment macro-

fractions, Eq. (29), are all pure:

\[
\rho_i^\text{mac}(t) = \left( S_i |\vec{k}_0\rangle \langle \vec{k}_0| S_i^\dagger \right)^\otimes m_{N_i} \equiv |\Psi^\text{mac}(t)| \langle \Psi^\text{mac}(t)|, \tag{41}
\]

so it is enough to consider their overlap:

\[
|\langle \Psi^\text{mac}(t)| \Psi^\text{mac}(t)\rangle| = \left| \langle \vec{k}_0| S_2^\dagger S_1 \vec{k}_0 \rangle \right|^2 e^{-\frac{2m \Delta}{\tau_D} x t}. \tag{42}
\]

Thus, for \( t \gg \tau_D \) the states of the macro-fractions \( \Psi^\text{mac}(t) \) asymptotically orthogonalize and moreover on the same timescale \( \tau_D \) as the decay of the coherent part described by Eq. (13) (note that \( 0 < m, f \leq 1 \) so the timescales from Eqs. (39, 43) do not differ considerably). This shows the asymptotic formation of the broadcast state (17) with pure encoding states \( \tilde{\rho}_i \):

\[
q_{S-fE}(t) = \tilde{\rho}_0^S \otimes \rho_0^\text{mac} \otimes \cdots \otimes \rho_0^\text{mac} \xrightarrow{t \gg \tau_D} \text{therm.} \otimes \rho_{S-fE}(\infty) = \sum_{i=1,2} p_i |\vec{x}_i\rangle \langle \vec{x}_i| \otimes |\text{mac}\rangle \otimes \cdots \otimes |\text{mac}\rangle \otimes |\text{mac}\rangle. \tag{44}
\]

where \( p_i \) is given by Eq. (32) and \( |\text{mac}\rangle \equiv |\Psi^\text{mac}(\infty)\rangle \) emerges as the non-disturbing environmental basis in the space of each macro-fraction, spanning a two-dimensional subspace, which carries the correlation between the macro-fraction and the sphere (this basis depends on the initial state \( |\vec{k}_0\rangle \)). Thus, the correlations become effectively among the qubits. The full process (14) is a combination of the measurement of the system in the pointer basis \( |\vec{x}_i\rangle \) and spectrum broadcasting of the result, described by a CC-type channel (17):

\[
A_{\infty}^{S-fE} (\tilde{\rho}_0^S) = \sum_{i} |\vec{x}_i\rangle \langle \vec{x}_i| \otimes |\text{mac}\rangle \otimes \cdots \otimes |\text{mac}\rangle \otimes |\text{mac}\rangle. \tag{45}
\]

Entropic objectivity condition and the classical plateau follow now form Eq. (14):

\[
I[\rho_{S-fE}(t)] \approx H_S, \tag{46}
\]

because of the conditions (29, 31) (see the next Section for the details). Thus the mutual information becomes asymptotically independent of the fraction \( f \) (as long as it is macroscopic).

In quantum Darwinism simulations for finite, fixed times \( t \) (see e.g. Refs. 3, 10), one can observe that the formation of the plateau is stronger driven by increasing the time rather than the macro-fraction \( f \) (keeping all other parameters equal). This can be straightforwardly explained by looking at the Eqs. (39, 43): the fractions \( f, m \) are by definition at most 1, and hence have little effect on the decay of the exponential factors, while \( t \) can be arbitrarily greater than \( \tau_D \), thus accelerating the formation of the broadcast state (14).
There is a very distinct difference in the macro- and microscopic behavior of the environment, already alluded to in Refs. 3, 10. From Eq. 47, it follows that within the sector \( \Omega \) the post-scattering states of individual photons (micro-states) \( |\Psi_{\text{mic}}\rangle \equiv S_1|\tilde{k}_0\rangle \), become identical in the thermodynamic limit and hence encode no information about the sphere’s localization:

\[
|\Psi_{\text{mic}}\rangle \equiv |\Psi_{\text{mic}}\rangle_2|\Psi_{\text{mic}}\rangle_1 \xrightarrow{\text{therm.}} |1\rangle.
\]  

(47)

This is not surprising due to the condition 18. On the other hand, and despite of it, by Eq. 19 macroscopic groups of photons are able to resolve the sphere’s position and in the asymptotic limit resolve it perfectly. It leads to the appearance of the different information-theoretical phases in the model, which we now describe. We stress that the macro-fraction \( m \) can be arbitrarily small (which only prolongs the orthogonalization time, cf. Eq. 21), but must scale with the total number of photons \( N_L \). Indeed, for a microscopic, i.e. not scaling with \( N_L \) fraction \( \mu \) the limit 17 still holds: \( (|k_0\rangle S_1|\tilde{k}_0\rangle \xrightarrow{\text{therm.}} |1\rangle) \). Thus, if the assumed portion of the environment is microscopic, the asymptotic post-scattering state is in fact a product one:

\[
\bar{\rho}_{S:FE}(0) = \begin{pmatrix} \rho_0 \otimes (\rho_0^{\text{mic}})_{\mu} & 0 \\ 0 & 0 \end{pmatrix} \xrightarrow{\text{therm.}} \bar{\rho}_{S:FE}(\infty) = \sum_{i=1,2} p_i|\tilde{x}_i\rangle\langle \tilde{x}_i| \otimes (S_i|\tilde{k}_0\rangle \xrightarrow{\text{therm.}} S_i|\tilde{k}_0\rangle S_i^\dagger)_{\mu} = \sum_{i=1,2} p_i|\tilde{x}_i\rangle\langle \tilde{x}_i| \otimes |\Psi_{\text{mic}}\rangle(\Psi_{\text{mic}}\rangle_{\mu},
\]  

(48)

where \( |\Psi_{\text{mic}}\rangle \equiv S_1|\tilde{k}_0\rangle \approx S_2|\tilde{k}_0\rangle \) because of Eq. 17 (and \( \approx \) denotes equality in the thermodynamic limit 19). This is the product phase, in which \( I[\bar{\rho}_{S:FE}(\infty)] = 0 \).

Conversely, if we have access to the full environment, ignoring perhaps only a microscopic fraction \( \mu \), the arguments leading to Eqs. 20, 23 do not work anymore, since from Eq. 17:

\[
|\text{Tr} S_1 \rho^{\text{ph}} S_2^\dagger|_{\mu} \xrightarrow{\text{therm.}} |1\rangle,
\]  

(50)

and thus there is no decoherence nor orthogonalization. The post-scattering state contains then the full quantum information about the system due to the unsuppressed system-environment entanglement produced by the controlled-unitary interaction 21. As a result, the mutual information attains in the thermodynamical limit its maximum value \( I_{\text{max}} = 2H_S \) (for a pure \( \rho_0^{\text{ph}} \), since the interaction is of a controlled unitary form 21) and this defines the full information phase. We note that the rise of \( I_{\text{FE}} \) above \( H_S \) certifies the presence of entanglement 18. The intermediate phase described by Eq. 44 is the broadcasting phase.

The quantity experiencing discontinuous jumps is the mutual information between the system \( S \) and the observed environment \( FE \), and the parameter which drives the phase transitions is the fraction size \( f \). As discussed above, each value of \( f \) has to be understood modulo a micro-fraction. The appearance of the phase diagram is a reflection of both the thermodynamic and the deep decoherence limits and its form is in agreement with the previously obtained results (see e.g. Refs. 3, 10).

A. Derivation of the entropic objectivity condition in the illuminated sphere model

Here we present an independent derivation of the entropic objectivity condition

\[
I[\bar{\rho}_{S:FE}(t)] \approx H_S,
\]  

(51)

for the illuminated sphere model. Although illustrated on a concrete model, our derivation is indeed more general—instead of a direct, asymptotic calculation of the mutual information \( I[\bar{\rho}_{S:FE}(t)] \) in the model (cf. Refs. 3, 10), we will show that Eq. 51 follows from the mechanisms of: i) decoherence, Eq. 23, and ii) distinguishability, Eq. 31, once they are proven. In light of our findings, this puts a clear physical meaning to Eq. 51—it is a consequence of the state information broadcasting. Most of the proof is for general, mixed states.

Let the post-interaction \( S : FE \) state for a fixed, finite box \( L \) and time \( t \) be \( \bar{\rho}_{S:FE}(L, t) \). It is given by Eqs. 27, 28 and now we explicitly indicate the dependence on \( L \) in the notation. Then:

\[
H_S - I[\bar{\rho}_{S:FE}(L, t)] \leq I[\bar{\rho}_{S:FE}(L, t)] - I[\bar{\rho}_{S:FE}(L, t)]_{\mu} + H_S - I[\bar{\rho}_{S:FE}(L, t)]_{\mu}.
\]  

(52)

where \( \bar{\rho}_{S:FE}(L, t) \) is the decohered part of \( \bar{\rho}_{S:FE}(L, t) \), given by Eq. 27. We first bound the difference 22, decomposing the mutual information using conditional information \( S_N(\bar{\rho}_{S:FE}(L, t)) = S_N(\bar{\rho}_{S:FE}) - S_N(\bar{\rho}_{FE}) \):

\[
I[\bar{\rho}_{S:FE}] = S_N(\bar{\rho}_S) - S_N(\bar{\rho}_{S:FE}),
\]  

(54)

so that:

\[
I[\bar{\rho}_{S:FE}(L, t)] - I[\bar{\rho}_{S:FE}(L, t)]_{\mu} \leq S_N[\rho_S(L, t)] - S_N[\bar{\rho}_{S:FE}(L, t)]_{\mu} + S_N[\bar{\rho}_{S:FE}(L, t)]_{\mu} - S_N[\bar{\rho}_{S:FE}(L, t)]_{\mu}.
\]  

(55)

From Eq. 15, the total \( S : FE \) Hilbert space is finite-dimensional for a finite \( L, t \), there are \( N_L = fL^2(N/V)ct \) photons (cf. Eq. 20) and the number of modes of each photon is approximately \((4\pi/3)/L(2\pi\Delta x)^3 \). Hence, the total dimension is \( 2 \times L^2 f(N/V)ct \times (1/6\pi^2)(L/\Delta x)^3 < \)
\( \overline{s_{nl}} \) and we can use the Fannes-Audenaert [19] and the
Alicki-Fannes [20] inequalities to bound (55) and (56) respectively (cf. Ref. [2]). For (59) we obtain:
\[
\left| S_{SN} \left[ \rho_{S}(L,t) \right] - S_{SN} \left[ \rho_{S}^{i=j}(L,t) \right] \right| \\
\leq \frac{1}{2} \epsilon_{E}(L,t) \log(d_{S} - 1) + h \left( \frac{\epsilon_{E}(L,t)}{2} \right),
\]
(57)
where \( \epsilon_{E}(t) = -\epsilon \log \epsilon - (1 - \epsilon) \log(1 - \epsilon) \) is the binary Shannon entropy and:
\[
\epsilon_{E}(L,t) = \| \rho_{S}(L,t) - \rho_{S}^{i=j}(L,t) \|_{tr} \leq 2|c|_{2}[1 - \frac{1}{c^{2}D L^{2}} \left( \frac{N}{V} \right)^{-1}]^{L^{2} \epsilon_{E}^{2}},
\]
(59)
with \( c_{2} \equiv \langle \bar{f}_{1}^{S} | \rho_{S}^{S} \rangle \bar{f}_{2} \rangle \), where we have used the reasoning [34, 39], but with \( f = 0 \). For (56) the same reasoning and the Alicki-Fannes inequality give:
\[
\left| S_{SN} \left[ \rho_{S,fE}(L,t) \| \rho_{E}(L,t) \|_{tr} \right] - S_{SN} \left[ \rho_{S,fE}^{i=j}(L,t) \| \rho_{E}^{i=j}(L,t) \right] \right| \\
\leq 4\epsilon_{E}(L,t) \log d_{S} + 2h[\epsilon_{E}(L,t)],
\]
(60)
with:
\[
\epsilon_{E}(L,t) = \| \rho_{S,fE}(L,t) - \rho_{S,fE}^{i=j}(L,t) \|_{tr} \leq \| \rho_{S,fE}(L,t) \|_{tr} \equiv 2|c|_{2}[1 - \frac{1}{c^{2}D L^{2}} \left( \frac{N}{V} \right)^{-1}]^{L^{2} \epsilon_{E}^{2}},
\]
(62)
Above \( L, t \) are big enough so that \( \epsilon_{E}(L,t), \epsilon_{E}(L,t) < 1 \). Eqs. (55, 56) give an upper bound on the difference (52) in terms of the decoherence speed (20).

To bound the "orthogonalization" part (55) (see Ref. 2 for a related analysis), we note that since \( \rho_{S,fE}^{i=j}(L,t) \) is a CQ-state (cf. Eq. (27)), its mutual information is given by the Holevo quantity [21]:
\[
I \left[ \rho_{S,fE}^{i=j}(L,t) \right] = \chi \left( \rho_{i}, \rho_{i}^{mac}(t) \otimes fM \right),
\]
(64)
where \( \rho_{i} \) is given by Eq. (32). From the Holevo Theorem it is bounded by [21]:
\[
I_{\text{max}}(t) \leq \chi \left( \rho_{i}, \rho_{i}^{mac}(t) \otimes fM \right) \leq H \left( \{ \rho_{i} \} \right) = H_{S},
\]
(65)
where \( I_{\text{max}}(t) \equiv \max_{T} I \left[ \rho_{i}, \rho_{i}^{mac}(t) \right] \) is the fixed time maximal mutual information, extractable through generalized measurements \( \{ \mathcal{E}_{j} \} \) on the ensemble \( \{ \rho_{i}, \rho_{i}^{mac}(t) \otimes fM \} \), and the conditional probabilities read:
\[
\rho_{ij}(t) = \text{Tr} \left[ \mathcal{E}_{j} \rho_{i}^{mac}(t) \otimes fM \right]
\]
(66)
(here and below \( i \) labels the states, while \( j \) the measurement outcomes). We now relate \( I_{\text{max}}(t) \) to the generalized overlap \( B \left[ \rho_{i}^{mac}(t) \otimes fM, \rho_{j}^{mac}(t) \otimes fM \right] \) (cf. Eq. (31)), which we have calculated for pure states in Eq. (12, 13).

Using the method of Ref. [13], slightly modified to unequal a priori probabilities \( p_{i} \), we obtain for an arbitrary measurement \( \mathcal{E} \):
\[
I \left( \rho_{ij}(p_{i}) \right) = I \left( \rho_{ij}(p_{j}) \right) = H \left( \{ p_{i} \} \right) - \sum_{j=1,2} \rho_{ij}(p_{j}) h \left( \rho_{ij}(p_{j}) \right)
\]
(67)
\[
\geq H \left( \{ p_{i} \} \right) - 2 \sum_{j=1,2} \rho_{ij}(p_{j}) \sqrt{\rho_{ij}(p_{j}) \left( 1 - \rho_{ij}(p_{j}) \right)}
\]
(68)
\[
= H \left( \{ p_{i} \} \right) - 2 \sqrt{p_{1}p_{2}} \sum_{j=1,2} \rho_{ij}(p_{j}) \rho_{ij}(p_{j}),
\]
(69)
where we have first used Bayes Theorem \( \rho_{ij}(p_{j}) = (p_{i}/\rho_{ij}(p_{j})) \), \( \rho_{ij}(p_{j}) \equiv \sum_{i=1,2} \rho_{ij}(p_{i}) = \text{Tr}(\mathcal{E}_{j} \sum_{i} \rho_{i}) \), then the fact that we have only two states: \( \rho_{ij}^{2} = 1 - \rho_{ij}^{1} \), so that \( H(\rho_{ij}^{2}) = h(\rho_{ij}^{1}) \), and finally \( p(\rho_{ij}) \leq 2 \sqrt{p_{1}(1 - p_{1})} \). On the other hand, \( B(\rho_{ij}, \rho_{j}) = \min_{\sum_{i} \rho_{ij}^{1}, \rho_{ij}^{2}} \sum_{i} \rho_{ij}^{1} \) [13]. Denoting the optimal measurement by \( \mathcal{E}^{B}(t) \) and recognizing that \( H(\{ p_{i} \}) = H_{S} \), we obtain:
\[
I(\rho_{ij}(p_{i})) \geq I \left[ \rho_{S,fE}(L,t) \right] \leq 2 \sqrt{p_{1}p_{2}} B \left[ \rho_{1}^{mac}(t), \rho_{2}^{mac}(t) \right],
\]
(70)

Inserting the above into the bounds (55) gives the desired upper bound on the difference (52):
\[
\left| H_{S} - I \left[ \rho_{S,fE}(L,t) \right] \right| \leq 2 \sqrt{p_{1}p_{2}} B \left[ \rho_{1}^{mac}(t), \rho_{2}^{mac}(t) \right],
\]
(73)

where the generalized overlap is given by Eqs. (12, 13):
\[
B \left[ \rho_{1}^{mac}(t), \rho_{2}^{mac}(t) \right] = \left| \langle \psi_{2}^{mac}(t) | \psi_{1}^{mac}(t) \rangle \right| \approx
\]
\[
1 - \frac{1}{c^{2}D L^{2}} \left( \frac{N}{V} \right)^{-1} \right]^{L^{2} \epsilon_{E}^{2}}.
\]
(74)

Gathering all the above facts together finally leads to a bound on \( |H_{S} - I \left[ \rho_{S,fE}(L,t) \right]| \) in terms of the speed of i) decoherence (20) and ii) distinguishability [31]:
\[
|H_{S} - I \left[ \rho_{S,fE}(L,t) \right]| \leq \frac{\epsilon_{E}(L,t)}{2} + 2h \left[ \epsilon_{E}(L,t) \right] +
\]
\[
4 \epsilon_{E}(L,t) \log 2 + 2 \sqrt{p_{1}p_{2}} B \left[ \rho_{1}^{mac}(t), \rho_{2}^{mac}(t) \right],
\]
(76)
where \( \epsilon_{E}(L,t), \epsilon_{E}(L,t), B \left[ \rho_{1}^{mac}(t), \rho_{2}^{mac}(t) \right] \) are given by Eqs. (31), (33), and (74) respectively. Choosing \( L, t \) big enough so that \( \epsilon_{E}(L,t), \epsilon_{E}(L,t) \leq 1/2 \) (when the binary entropy \( h(\cdot) \) is monotonically increasing), we remove the unphysical box and obtain an estimate on the
speed of convergence of $I [\varrho_{S,E}(L,t)]$ to $H_S$:

$$\lim_{L \to \infty} |H_S - I [\varrho_{S,E}(L,t)]| \leq \hbar \left( |c_{12}| e^{-\frac{\epsilon E t}{\hbar}} \right)$$  \hfill (77)

$$+ 2\hbar \left( 2|c_{12}| e^{-\frac{\epsilon E t}{\hbar d}} \right) + 8|c_{12}| e^{-\frac{\epsilon E t}{\hbar d t}} \log 2$$  \hfill (78)

$$+ 2\sqrt{p_1 p_2} e^{-\frac{\epsilon E t}{\hbar d t}}.$$  \hfill (79)

This finishes the derivation of the condition (61).

We note that the result (81,76) is in fact a general statement, valid in any model where: i) the system $S$ is effectively a qubit; ii) the system-environment interaction is of an environment-symmetric controlled-unitary type:

**Lemma 3** Let a two-dimensional quantum system $S$ interact with $N$ identical environments, each described by a $d$-dimensional Hilbert space, through a controlled-unitary interaction:

$$U(t) \equiv \sum_{i=1,2} |i\rangle \langle i| \otimes U_i(t) \otimes^N.$$  \hfill (80)

Let the initial state be $\varrho_{S,E}(0) = \varrho_0^S \otimes (\varrho_0^E)^{\otimes N}$ and $\varrho_{S,E}(t) \equiv U(t) \varrho_{S,E}(0) U(t)\dagger$. Then for any $0 < f < 1$ and $t$ big enough:

$$|H(\{p_i\}) - I [\varrho_{S,E}(t)]| \leq \hbar \left[ \frac{\epsilon_E(t)}{2} \right] + 2\hbar [\epsilon_E(t)] +$$  \hfill (81)

$$4\epsilon_E(t) \log 2 + 2\sqrt{p_1 p_2} B [\varrho_1(t), \varrho_2(t)]^{\otimes N},$$  \hfill (82)

where:

$$p_i \equiv \langle i\rangle \langle 0|, \varrho_i(t) \equiv U_i(t) \varrho_0^E U_i(t)\dagger,$$  \hfill (83)

$$\epsilon_E(t) \equiv ||\varrho_S(t) - \varrho_{S,i}^{\otimes N}||_{tr},$$  \hfill (84)

$$\epsilon_E(t) \equiv ||\varrho_{S,E}(t) - \varrho_{S,E,i}^{\otimes N}||_{tr}.$$  \hfill (85)

[1] Zurek, W. H. Quantum Darwinism. Nature Phys. 5, 181 (2009).

[2] Zwolak, M. and Zurek, W. H. Complementarity of quantum discord and classically accessible information. Sci. Rep. 3, 1729 (2013).

[3] Riedel, C. J. and Zurek, W. H. Quantum Darwinism in an everyday environment: Huge redundancy in scattered photons. Phys. Rev. Lett. 105, 020404 (2010).

[4] Zwolak, M., Quan, H. T., and Zurek, W. H. Redundant imprinting of information in nonideal environments: Objective reality via a noisy channel. Phys. Rev. A 81, 062110 (2010).

[5] Cover, T. M. and Thomas, J. A. Elements of Information Theory, John Wiley and Sons, New York (1991).

[6] Horodecki, M. and Oppenheim, J., and Winter, A. Partial quantum information. Nature 436, 673 (2005).

[7] Peres, A. Separability Criterion for Density Matrices. Phys. Rev. Lett. 77, 1413 (1996); Horodecki, M., Horodecki, P., and Horodecki, R. Separability of mixed states: necessary and sufficient conditions. Phys. Lett. A 223, 1 (1996).

[8] Joos, E. and Zeh, H. D. The emergence of classical properties through interaction with the environment. Z. Phys. B 59, 223 (1985).

[9] Gallis, M. R. and Fleming, G. N. Environmental and spontaneous localization. Phys. Rev. A 42, 38 (1990).

[10] Riedel, C. J. and Zurek, W. H. Redundant information from thermal illumination: quantum Darwinism in scattered photons. New J. Phys. 13, 073038 (2011).

[11] Hornberger, K. and Sipe, J. E. Collisional decoherence reexamined. Phys. Rev. A 68, 012105 (2003).

[12] Adler, S. L. Normalization of collisional decoherence: squaring the delta function, and an independent cross-check. J. Phys. A 39, 14067 (2006).

[13] Landau, L. D. and Lifshitz, E. M. Fluid Mechanics, Course of theoretical physics, Vol. 6, J. B. Sykes and W. H. Reid Transl., Pergamon Press, Oxford (1987).

[14] Sewell, G. On the mathematical structure of quantum measurement theory. Rep. Math. Phys. 56, 271 (2005).

[15] Fuchs, C. A. and van de Graaf, J. Cryptographic Distiguishability Measures for Quantum Mechanical States. IEEE Trans. on Inf. Theor. 45, 1216 (1999).

[16] The fact that CQ and QC states carry some form of non-classical correlations has been shown e.g. through the no-local-broadcasting theorem: Piani, M., Horodecki, P., and Horodecki, R. No-Local-Broadcasting Theorem for Multipartite Quantum Correlations. Phys. Rev. Lett. 100, 090502 (2008), or through entanglement activation in Piani, M. et al. All Nonclassical Correlations Can Be Activated into Distillable Entanglement. Phys. Rev. Lett. 106, 220403 (2011).

[17] Korbie, J. K., Horodecki, P., and Horodecki, R. Quantum-correlation breaking channels, broadcasting scenarios, and finite Markov chains. Phys. Rev. A 86, 042319 (2012).

[18] Horodecki, R. and Horodecki, P. Quantum redundancies and local realism. Phys. Lett. A 194, 147 (1994).

[19] Audenaert, K. M. R. A sharp continuity estimate for the von Neumann entropy. J. Phys. A: Math. Theor. 40, 8127 (2007).

[20] Alicki, R. and Fannes, M. Continuity of quantum conditional information. J. Phys. A: Math. Gen. 37, L55, (2004).

[21] Holevo, A. S. Some estimates of the information transmitted by quantum communication channels. Problm. Infor. Transm. 9, 177 (1973).