EXISTENCE OF GORENSTEIN PROJECTIVE RESOLUTIONS

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Abstract. Gorenstein rings are important to mathematical areas as diverse as algebraic geometry, where they encode information about singularities of spaces, and homotopy theory, through the concept of model categories.

In consequence, the study of Gorenstein rings has led to the advent of a whole branch of homological algebra, known as Gorenstein homological algebra.

This paper solves one of the open problems of Gorenstein homological algebra by showing that so-called Gorenstein projective resolutions exist over quite general rings, thereby enabling the definition of a Gorenstein version of derived functors.

An application is given to the theory of Tate cohomology.

0. Introduction

Gorenstein rings are important mathematical objects originating in the work of Grothendieck and his pupils. The study of Gorenstein rings has given rise to a whole branch of homological algebra known as Gorenstein homological algebra, to which this paper is a contribution. The main point is an existence proof for Gorenstein projective resolutions; this central item has been lacking from the theory until now. However, before going into details, let me start the exposition on a classical note.

Homological algebra is one of the most versatile mathematical machines ever invented. It has an impact on most parts of mathematics — algebra, geometry, number theory... One of the central items of the theory are projective resolutions. An augmented projective resolution of a module $M$ is an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$
where the $P_i$ are projective modules. If each module $M$ over a ring $A$ has a projective resolution $P$ with $P_i = 0$ for $i \gg 0$, then $A$ is called regular. For an algebraist, one of the points of homological algebra is that it can be used to study non-regular rings in terms of their deviation from regularity; this makes it possible to use the easier regular case as a frame of reference for understanding more complicated rings.

Gorenstein homological algebra takes a parallel approach and considers Gorenstein projective resolutions. I will give the precise definition in a moment, but the point is that if each module $M$ over a ring $A$ has a Gorenstein projective resolution $G$ with $G_i = 0$ for $i \gg 0$, then $A$ is Gorenstein, that is, it has finite injective dimension as a module over itself. Gorenstein homological algebra goes back to Auslander and Bridger, see [1], and substantial contributions are due to Enochs and his coauthors, see [6], [7], [8], and several other papers.

Gorenstein homological algebra plays a role in algebraic geometry, see [9], [10], [11]; commutative ring theory, see [2], [3], [4], [13], [14], [25]; homotopy theory, see [15]; and number theory, see [26].

To describe the contents of this paper, let me make a foray into the definitions of Gorenstein homological algebra. Gorenstein projective resolutions are defined in terms of Gorenstein projective modules. These are modules of the form $G = \text{Ker}(E^1 \rightarrow E^2)$ where $E$ is a complete projective resolution, that is, an exact complex of projective modules which stays exact when one applies the functor $\text{Hom}(-, Q)$ for any projective module $Q$. An augmented Gorenstein projective resolution of a module $M$ is an exact sequence

$$
\cdots \rightarrow G_2 \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0 \quad (1)
$$

where the $G_i$ are Gorenstein projective modules, which stays exact when one applies the functor $\text{Hom}(\tilde{G}, -)$ for any Gorenstein projective module $\tilde{G}$. The complex

$$
G = \cdots \rightarrow G_2 \rightarrow G_1 \rightarrow G_0 \rightarrow 0 \rightarrow \cdots \quad (2)
$$

is then called a Gorenstein projective resolution of $M$.

For the theory to be worth anything, it is a key question whether Gorenstein projective resolutions really exist. In other words, for a given ring $A$, it is important to determine whether each $A$-module has a Gorenstein projective resolution. One could attempt to circumvent this question by dropping the requirement that the complex (1) stay exact under the functor $\text{Hom}(\tilde{G}, -)$. This makes it easy to establish existence, and such attempts have been made. However, to do so misses an important point: The purpose of requiring (1) to stay exact under
the functor Hom(\(\tilde{G}, -\)) is that this makes the Gorenstein projective resolution (2) unique up to chain homotopy, as one can easily check. This in turn means that (2) can be used to define the Gorenstein version of derived functors. Without the requirement that (1) stay exact under the functor Hom(\(\tilde{G}, -\)), any such definition fails, and the theory must remain without derived functors; a one legged life. Therefore, it is a central question whether Gorenstein projective resolutions exist.

The corresponding questions of existence of so-called Gorenstein injective and Gorenstein flat resolutions have recently been settled in the affirmative in [8] and [6], but the Gorenstein projective case has resisted the attacks of a number of authors despite partial results in papers such as [2], [7], [14], and [17]. The state of the art up to now seems to be [14, prop. 2.18]; this only gives that Gorenstein projective resolutions exist over a Gorenstein ring.

However, the present paper proves the existence of Gorenstein projective resolutions over much more general rings. This is done by showing that the resolutions exist under one simple assumption — the existence of a certain adjoint functor \(e^!\) — and by using Bousfield localization to show that this assumption holds if the ground ring has a dualizing complex. This covers many rings arising in practice. For instance, any local ring of a scheme of locally finite type over a field has a dualizing complex. Other types of rings are also covered; see remark 1.1.

In fact, it may even be the case that the functor \(e^!\) exists over any ring and hence that Gorenstein projective resolutions exist in general, but I do not know how to prove this.

After showing these results, I will give an application to the theory of Tate cohomology. This was originally defined for representations of finite groups, but I will show, again under the assumption that the adjoint functor \(e^!\) exists, that it is possible to define Tate Ext groups

\[
\widehat{\text{Ext}}^i(M, N)
\]

for any modules \(M\) and \(N\), so that classical Tate cohomology is the special case \(\widehat{\text{Ext}}^i_{kG}(k, N)\). Moreover, Tate and ordinary Ext groups will be shown to fit into a long exact sequence

\[
0 \to \text{Ext}^1_{G}(M, N) \to \text{Ext}^1(M, N) \to \widehat{\text{Ext}}^1(M, N) \to \cdots,
\]

(3)

where the \(\text{Ext}^1_{G}\) are Gorenstein Ext groups defined by

\[
\text{Ext}^i_{G}(M, N) = H^i\text{Hom}(G, N)
\]
where $G$ is a Gorenstein projective resolution of $M$. The $\Ext^i_G$ are precisely a Gorenstein version of derived functors.

A theory of Tate cohomology such as this was already accomplished in [3] and [25], but only under the assumption that $M$ had a finite Gorenstein projective resolution, hence restricting the real scope of the theory to Gorenstein rings.

The paper is organized as follows. Section 1 shows the existence of the adjoint functor $e^!$ over rings with a dualizing complex. Section 2 shows the existence of Gorenstein projective resolutions when $e^!$ exists. And section 3 defines Tate Ext groups, shows some simple properties, and shows that the Tate Ext groups fit into the exact sequence (3).

1. The adjoint functor $e^!$

This section shows the existence of a certain adjoint functor $e^!$ over rings with a dualizing complex.

Remark 1.1. Dualizing complexes are popular gadgets in homological algebra. I shall give the precise definition in setup 1.4 for noetherian commutative rings and in setup 1.4' for right-noetherian algebras over a field. But I would like already here to point out that many rings have dualizing complexes.

For instance, a noetherian local commutative ring has a dualizing complex if and only if it is a quotient of a Gorenstein noetherian local commutative ring, by the (deep) result [19, thm. 1.2]. It follows that, as mentioned in the introduction, any local ring of a scheme of locally finite type over a field has a dualizing complex. By the Cohen structure theorem, it also follows that any complete noetherian local commutative ring does.

Some important types of non-commutative noetherian algebras are also known to have dualizing complexes. For example, complete semi-local PI algebras do by [24, cor. 0.2], and filtered algebras do by [27, cor. 6.9] if their associated graded algebras are noetherian and connected, and either PI, FBN, or with enough normal elements.

Definition 1.2. If $A$ is a ring, then $E(A)$ denotes the class of complete projective resolutions of $A$-left-modules. So a complex of $A$-left-modules $E$ is in $E(A)$ if it consists of projective $A$-left-modules, is exact, and has $\Hom_A(E, Q)$ exact for each projective $A$-left-module $Q$.

Remark 1.3. I will view $E(A)$ as a full subcategory of $K(\text{Pro } A)$, the homotopy category of complexes of projective $A$-left-modules. The
inclusion functor will be denoted

\[ e_\ast : E(A) \longrightarrow K(\text{Pro } A). \]

**Setup 1.4.** Let \( A \) be a noetherian commutative ring with a dualizing complex \( D \). That is,

(i) The cohomology of \( D \) is bounded and finitely generated over \( A \).
(ii) The injective dimension \( \text{id}_A D \) is finite.
(iii) The canonical morphism \( A \longrightarrow \text{RHom}_A(D, D) \) in the derived category \( D(A) \) is an isomorphism.

**Setup 1.5.** Let \( D \xrightarrow{\cong} I \) be an injective resolution so that \( I \) is a bounded complex.

See [12, chp. V] for background on dualizing complexes.

**Remark 1.6.** Since \( A \) has a dualizing complex, it has finite Krull dimension by [12, cor. V.5.2], so by [23, Seconde partie, cor. (3.2.7)], each flat \( A \)-module has finite projective dimension.

The following lemma uses \( I \), the injective resolution of the dualizing complex \( D \).

**Lemma 1.7.** Let \( P \) be a complex of projective \( A \)-modules. Then

\[ \text{Hom}_A(P, Q) \text{ is exact for each projective } A\text{-module } Q \]

\[ \Leftrightarrow I \otimes_A P \text{ is exact.} \]

**Proof.** \( \Rightarrow \) Suppose that \( \text{Hom}(P, Q) \) is exact for each projective module \( Q \). To see that \( I \otimes P \) is an exact complex, it is enough to see that

\[ \text{Hom}(I \otimes P, J) \cong \text{Hom}(P, \text{Hom}(I, J)) \]

is exact for each injective module \( J \).

It follows from [21, thm. 1.2] that \( \text{Hom}(I, J) \) is a bounded complex of flat modules. Hence, \( \text{Hom}(I, J) \) is finitely built from flat modules in the homotopy category of complexes of \( A \)-modules, \( K(A) \), and so it is enough to see that \( \text{Hom}(P, F) \) is exact for each flat module \( F \).

Since \( F \) has finite projective dimension by remark 1.6, there is a projective resolution \( \tilde{P} \xrightarrow{\cong} F \) with \( \tilde{P} \) bounded. Since \( P \) consists of projective modules and both \( \tilde{P} \) and \( F \) are bounded, this induces a quasi-isomorphism

\[ \text{Hom}(P, \tilde{P}) \cong \text{Hom}(P, F). \]

So it is enough to see that \( \text{Hom}(P, \tilde{P}) \) is exact.
But $\tilde{P}$ is a bounded complex of projective modules, so it is finitely built from projective modules, so it is enough to see that $\text{Hom}(P,Q)$ is exact for each projective module $Q$. And this holds by assumption.

$\Leftarrow$ Suppose that $I \otimes P$ is an exact complex. I must show that $\text{Hom}(P,Q)$ is exact for each projective module $Q$.

First observe that by [4, thm. (3.2)], there is an isomorphism

$$Q \xrightarrow{\sim} \text{RHom}(D, D \otimes Q).$$

Of course, I can replace $D$ by $I$ to get

$$Q \xrightarrow{\sim} \text{RHom}(I, I \otimes Q).$$

(4)

Here

$$I \otimes Q \cong I \otimes Q$$

because $Q$ is projective. Moreover, $I \otimes Q$ is a bounded complex of injective modules so

$$\text{RHom}(I, I \otimes Q) \cong \text{RHom}(I, I \otimes Q) \cong \text{Hom}(I, I \otimes Q).$$

So the isomorphism (4) in the derived category is represented by the chain map

$$Q \longrightarrow \text{Hom}(I, I \otimes Q)$$

which must accordingly be a quasi-isomorphism.

Completing to a distinguished triangle in $K(A)$ gives

$$Q \longrightarrow \text{Hom}(I, I \otimes Q) \longrightarrow C \longrightarrow$$

where $C$ is exact. Here $I$ and $I \otimes Q$ are bounded, so $\text{Hom}(I, I \otimes Q)$ is bounded. As the same is true for $Q$, the mapping cone $C$ is also bounded.

Now, the distinguished triangle gives another distinguished triangle

$$\text{Hom}(P, Q) \longrightarrow \text{Hom}(P, \text{Hom}(I, I \otimes Q)) \longrightarrow \text{Hom}(P, C) \longrightarrow .$$

Here $\text{Hom}(P, C)$ is exact because $P$ is a complex of projective modules while $C$ is a bounded exact complex. So to see that $\text{Hom}(P, Q)$ is exact as desired, it is enough to see that $\text{Hom}(P, \text{Hom}(I, I \otimes Q))$ is exact.

However,

$$\text{Hom}(P, \text{Hom}(I, I \otimes Q)) \cong \text{Hom}(I \otimes P, I \otimes Q).$$

And this is exact because $I \otimes P$ is exact by assumption while $I \otimes Q$ is a bounded complex of injective modules.

\begin{lemma}
The homotopy category of complexes of projective $A$-modules, $K(\text{Pro} A)$, is a compactly generated triangulated category.
\end{lemma}
Proof. It is clear that $K(\text{Pro} A)$ is triangulated.

The ring $A$ is noetherian and hence coherent, and by remark 1.6 each flat $A$-module has finite projective dimension. So $K(\text{Pro} A)$ is compactly generated by [18, thm. 2.4]. □

Combining lemmas 1.7 and 1.8 with Bousfield localization now gives existence of the adjoint functor $e^!$.

**Proposition 1.9.** The inclusion functor $e_* : E(A) \to K(\text{Pro} A)$ has a right-adjoint $e^! : K(\text{Pro} A) \to E(A)$.

*Proof.* Consider the functor

$$k(-) = H^0((A \oplus I) \otimes_A -)$$

from $K(\text{Pro} A)$ to $\text{Ab}$, the category of abelian groups. This is clearly a homological functor respecting set indexed coproducts. Moreover,

$$k(\Sigma^i P) \cong H^i(P) \oplus H^i(I \otimes P),$$

where $\Sigma^i$ denotes $i$'th suspension, so for $P$ to satisfy $k(\Sigma^i P) = 0$ for each $i$ means

$$H^i(P) = 0$$

and

$$H^i(I \otimes P) = 0$$

for each $i$. Using lemma 1.7, this shows

$$\{ P \in K(\text{Pro} A) \mid k(\Sigma^i P) = 0 \text{ for each } i \} = E(A).$$

That is, $E(A)$ is the kernel of the homological functor $k$.

One consequence of this is that $E(A)$ is closed under set indexed coproducts. Hence [20, lem. 3.5] says that for $e_*$ to have a right-adjoint is the same as for the Verdier quotient $K(\text{Pro} A)/E(A)$ to satisfy that each Hom set is in fact a set (as opposed to a class).

Now, the category $K(\text{Pro} A)$ is compactly generated by lemma 1.8. By [22, lem. 4.5.13] with $\beta = \aleph_0$, this even implies that there is only a set of isomorphism classes of compact objects in $K(\text{Pro} A)$. Hence the version of Bousfield localization given in [17, thm. 4.1] applies to the functor $k$ on $K(\text{Pro} A)$, and gives that $K(\text{Pro} A)$ modulo the kernel of $k$ satisfies that each Hom is a set. That is, $K(\text{Pro} A)/E(A)$ satisfies that each Hom is a set, as desired. □

The methods given above also apply to non-commutative algebras. Let the following setups replace setups 1.4 and 1.5.

**Setup 1.4’.** Let $A$ be a left-coherent and right-noetherian $k$-algebra over the field $k$ for which there exists a left-noetherian $k$-algebra $B$ and
a dualizing complex \( B D_A \). That is, \( D \) is a complex of \( B \)-left-\( A \)-right-modules, and

(i) The cohomology of \( D \) is bounded and finitely generated both over \( B \) and over \( A^{\text{op}} \).

(ii) The injective dimensions \( \text{id}_B D \) and \( \text{id}_{A^{\text{op}}} D \) are finite.

(iii) The canonical morphisms

\[ A \longrightarrow \mathcal{RHom}_B(D, D) \quad \text{and} \quad B \longrightarrow \mathcal{RHom}_{A^{\text{op}}}(D, D) \]

in the derived categories \( D(A \otimes_k A^{\text{op}}) \) and \( D(B \otimes_k B^{\text{op}}) \) are isomorphisms.

**Setup 1.5’**. Let \( D \xrightarrow{\sim} I \) be an injective resolution of \( D \) over \( B \otimes_k A^{\text{op}} \). Replace \( I \) by a bounded truncation. This may ruin the property that \( I \) is an injective resolution over \( B \otimes_k A^{\text{op}} \), but because \( \text{id}_B D \) and \( \text{id}_{A^{\text{op}}} D \) are finite, I can still suppose that \( I \) consists of modules which are injective both over \( B \) and over \( A^{\text{op}} \).

The definition of dualizing complexes over non-commutative algebras is due to [27, def. 1.1].

With setups 1.4 and 1.5 replaced by setups 1.4’ and 1.5’, let me inspect the rest of this section. As the ground ring \( A \) is now non-commutative, I must replace “module” by “left-module” throughout. Remark 1.6 also needs to be replaced by the following.

**Remark 1.6’**. Under setup 1.4’, each flat \( A \)-left-module has finite projective dimension by [16].

After this, the proof of lemma 1.7 goes through if one keeps track of left and right structures throughout, and the proofs of lemma 1.8 and proposition 1.9 also still work.

Hence I can sum up the results of this section in the following theorem.

**Theorem 1.10**. Consider either of the following two situations.

(i) \( A \) is a noetherian commutative ring with a dualizing complex (see setup 1.4).

(ii) \( A \) is a left-coherent and right-noetherian \( k \)-algebra over the field \( k \) for which there exists a left-noetherian \( k \)-algebra \( B \) and a dualizing complex \( B D_A \) (see setup 1.4’).

Then the inclusion functor

\[ e_* : E(A) \longrightarrow K(\text{Pro} A) \]
has a right-adjoint
\[ e^! : K(\text{Pro} A) \to E(A). \]

2. GORENSTEIN PROJECTIVE RESOLUTIONS

This section shows the existence of Gorenstein projective resolutions when the adjoint functor \( e^! \) exists.

**Setup 2.1.** For the rest of this paper, \( A \) is a ring for which the inclusion functor

\[ e_* : E(A) \to K(\text{Pro} A) \]

has a right-adjoint

\[ e^! : K(\text{Pro} A) \to E(A). \]

**Remark 2.2.** The existence of the right-adjoint \( e^! \) is precisely the hypothesis under which the constructions of this paper work.

The functor \( e^! \) exists over fairly general rings; see theorem 1.10 and remark 1.1. As mentioned in the introduction, it may even be the case that the functor \( e^! \) exists over any ring, but I do not know how to prove that.

**Remark 2.3.** If \( P \) is a complex of projective \( A \)-left-modules, then \( e^! P \) can be thought of as the best approximation to \( P \) by a complete projective resolution.

Elaborating on this, if \( M \) is an \( A \)-left-module with projective resolution \( P \), then \( e^! P \) can be thought of as the best approximation to \( M \) by a complete projective resolution. This point will be made more precise in lemma 3.6.

**Construction 2.4.** If \( P \) is a complex of \( A \)-left-modules, then for each \( i \) there is a chain map

\[
\begin{array}{cccccccc}
\cdots & \to & 0 & \xrightarrow{id} & P^i & \xrightarrow{\partial^i} & P^{i+1} & \xrightarrow{\partial^{i+1}} & \cdots \\
\downarrow{id} & & \downarrow{\partial^i} & & \downarrow{\partial^{i+1}} & & \downarrow{\partial^{i+2}} & \\
\cdots & \to & P^{i-1} & \xrightarrow{\partial^{i-1}} & P^i & \xrightarrow{\partial^i} & P^{i+1} & \xrightarrow{\partial^{i+1}} & \cdots
\end{array}
\]

where the upper complex is null homotopic.

If \( T \xrightarrow{t} P \) is now a chain map, then I can add the upper complex to \( T \) and thereby change \( t \) so that the \( i \)'th component \( T^i \xrightarrow{t^i} P^i \) becomes surjective. Doing so does not change the isomorphism class of \( t \) in \( K(A) \), the homotopy category of complexes of \( A \)-left-modules.
Construction 2.5. If $M$ is an $A$-left-module, then let $P$ be a projective resolution concentrated in non-positive cohomological degrees and consider the counit morphism $e_*e^! P \xrightarrow{\epsilon_P} P$ in $\text{K}(\text{Pro} A)$. By applying construction 2.4 in each degree, I can assume that $\epsilon_P$ is represented by a surjective chain map, so setting

$$F = e_*e^! P,$$

there is a short exact sequence of complexes

$$0 \to K \to F \to P \to 0.$$  

Note that since both $F$ and $P$ consist of projective modules, the sequence is semi-split (that is, split in each degree) and $K$ also consists of projective modules.

Lemma 2.6. Consider the complex $K$ from construction 2.5. Then

$$\text{Hom}_{\text{K}(\text{Pro} A)}(E, K) = 0$$

for $E$ in $E(A)$.

Proof. The chain map $F \to P$ represents the counit morphism

$$e_*e^! P \xrightarrow{\epsilon_P} P$$

which leads to a commutative diagram

$$\begin{array}{ccc}
\text{Hom}_{E(A)}(E, e^! P) & \xrightarrow{e_*(-)} & \text{Hom}_{\text{K}(\text{Pro} A)}(e_*E, e_*e^! P) \\
& & \downarrow \text{Hom}(e_*E, \epsilon_P) \\
& & \text{Hom}_{\text{K}(\text{Pro} A)}(e_*E, P),
\end{array}$$

where the diagonal map is the adjunction isomorphism while the horizontal map is an isomorphism because $e_*$ is the inclusion functor of a full subcategory. The vertical map must therefore also be an isomorphism. That is,

$$\text{Hom}_{\text{K}(\text{Pro} A)}(E, F) \to \text{Hom}_{\text{K}(\text{Pro} A)}(E, P) \tag{5}$$

is an isomorphism.

Now, the short exact sequence from construction 2.5 is semi-split and therefore gives a distinguished triangle

$$K \to F \to P \to$$

in $\text{K}(\text{Pro} A)$. Hence there is a long exact sequence consisting of pieces

$$\text{Hom}_{\text{K}(\text{Pro} A)}(\Sigma^i E, K) \to \text{Hom}_{\text{K}(\text{Pro} A)}(\Sigma^i E, F) \to \text{Hom}_{\text{K}(\text{Pro} A)}(\Sigma^i E, P).$$
Since $\Sigma^i E$ is in $\mathcal{E}(A)$ for each $i$, the second homomorphism here is of the type from equation (5), so is an isomorphism for each $i$. This implies $\text{Hom}_{K(\text{Pro } A)}(E, K) = 0$ as desired. □

**Remark 2.7.** For the following lemma, recall that a Gorenstein projective $A$-left-module is a module of the form $G = \text{Ker}(E^1 \to E^2)$ where $E$ is in $\mathcal{E}(A)$ (cf. definition 1.2).

**Lemma 2.8.** Consider the complex $K$ from construction 2.5. Suppose that the sequence
\[
\cdots \to K^{i-2} \to K^{i-1} \xrightarrow{k} N \to 0
\]
obtained from $K$ is exact.

Let $\tilde{G}$ be Gorenstein projective and let $\tilde{G} \xrightarrow{\tilde{g}} N$ be a homomorphism. Then $\tilde{g}$ lifts through $k$,

\[
\begin{array}{c}
\tilde{G} \\
\downarrow \tilde{g} \\
K^{i-1} \\
\downarrow k \\
N.
\end{array}
\]

**Proof.** By (de)suspending, I can clearly pick a complex $E$ in $\mathcal{E}(A)$ with $\tilde{G} = \text{Ker}(E^i \to E^{i+1})$, and it is not hard to see that there is a chain map $E \xrightarrow{e} K$ which fits together with $\tilde{G} \xrightarrow{\tilde{g}} N$ in a commutative diagram

\[
\begin{array}{cccccc}
\cdots & \to & E^{i-2} & \to & E^{i-1} & \to & E^i & \to & E^{i+1} & \to & \cdots \\
| & & | & & | & & | & & | & & | \\
\downarrow e^{i-2} & & \downarrow e^{i-1} & & \downarrow e^i & & \downarrow e^{i+1} & & \\
\cdots & \to & K^{i-2} & \to & K^{i-1} & \to & K^i & \to & K^{i+1} & \to & \cdots \\
| & & | & & | & & | & & | \\
\downarrow e & & \downarrow \tilde{g} & & \downarrow \varepsilon & & \downarrow \varepsilon & & \\
\cdots & \to & N & \to & \cdots
\end{array}
\]

Since lemma 2.6 says $\text{Hom}_{K(\text{Pro } A)}(E, K) = 0$ for $E$ in $\mathcal{E}(A)$, the chain map $e$ must be null homotopic. Let $\varepsilon$ be a null homotopy with $e = \varepsilon \partial E + \partial K \varepsilon$, consisting of components $E^j \xrightarrow{\varepsilon^j} K^{j-1}$. Then it is straightforward to prove
\[
k \circ (e^i \ell) = \tilde{g},
\]
so \( \tilde{G} \xrightarrow{\delta} N \) has been lifted through \( K^{-1} \xrightarrow{k} N \) as desired. \( \square \)

**Remark 2.9.** For the next theorem, recall that an augmented Gorenstein projective resolution of an \( A \)-left-module \( M \) is an exact sequence

\[
\cdots \rightarrow G_2 \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0
\]

where the \( G_i \) are Gorenstein projective modules, which stays exact when one applies the functor \( \text{Hom}(\tilde{G}, -) \) for any Gorenstein projective module \( \tilde{G} \). The complex

\[
G = \cdots \rightarrow G_2 \rightarrow G_1 \rightarrow G_0 \rightarrow 0 \rightarrow \cdots
\]

is then called a Gorenstein projective resolution of \( M \).

**Remark 2.10.** Recall construction 2.5. The complex \( F \) is in \( E(A) \). In particular it is exact, and therefore the cohomology long exact sequence shows

\[
H^i K = \begin{cases} M & \text{for } i = 1, \\ 0 & \text{for } i \neq 1. \end{cases}
\]

Hence there is an exact sequence

\[
\cdots \rightarrow K^{-2} \rightarrow K^{-1} \rightarrow K^0 \rightarrow \text{Ker} \partial^1_K \rightarrow M \rightarrow 0.
\]

**Theorem 2.11.** Let \( M \) be an \( A \)-left-module. Then the exact sequence from remark 2.10 is an augmented Gorenstein projective resolution of \( M \).

**Proof.** The modules \( K^0, K^{-1}, \ldots \) are projective and hence Gorenstein projective.

As for \( \text{Ker} \partial^1_K \), observe that in the short exact sequence from construction 2.5, the complex \( P \) is concentrated in non-positive cohomological degrees, so the modules \( P^1 \) and \( P^2 \) are zero. So in degrees 1 and 2, the short exact sequence gives

\[
\begin{array}{ccc}
K^2 & \xrightarrow{\cong} & F^2 \\
\downarrow \partial^1_K & & \downarrow \partial^1_F \\
K^1 & \xrightarrow{\cong} & F^1.
\end{array}
\]

Hence \( \text{Ker} \partial^1_K \cong \text{Ker} \partial^1_F \), and \( \text{Ker} \partial^1_K \) is Gorenstein projective because \( F \) is in \( E(A) \).

To complete the proof, I must show that the exact sequence from remark 2.10,

\[
\cdots \rightarrow K^{-2} \rightarrow K^{-1} \rightarrow K^0 \rightarrow \text{Ker} \partial^1_K \rightarrow M \rightarrow 0,
\]
remains exact when one applies the functor \( \text{Hom}(\tilde{G}, -) \) for any Gorenstein projective module \( \tilde{G} \).

First, let \( i \leq 0 \) be an integer and let \( \tilde{G} \xrightarrow{\tilde{g}} K^i \) be a homomorphism whose composition with the subsequent homomorphism in the exact sequence is zero. I must show that \( \tilde{g} \) lifts through \( K^{i-1} \xrightarrow{} K^i \).

I can view \( \tilde{g} \) as a homomorphism \( \tilde{G} \xrightarrow{\tilde{g}} \text{Ker } \partial_K^i \), and must then show that \( \tilde{g} \) lifts through the canonical homomorphism \( K^{i-1} \xrightarrow{} \text{Ker } \partial_K^i \). But this follows from lemma 2.8 applied to

\[
\cdots \xrightarrow{} K^{i-2} \xrightarrow{} K^{i-1} \xrightarrow{} \text{Ker } \partial_K^i \xrightarrow{} 0.
\]

Secondly, let \( \tilde{G} \xrightarrow{\tilde{g}} \text{Ker } \partial_K^1 \) be a homomorphism whose composition with the subsequent homomorphism in the exact sequence, \( \text{Ker } \partial_K^1 \xrightarrow{} M \), is zero. I must show that \( \tilde{g} \) lifts through \( K^0 \xrightarrow{} \text{Ker } \partial_K^1 \).

I can view \( \tilde{g} \) as a homomorphism \( \tilde{G} \xrightarrow{\tilde{g}} \text{Im } \partial_K^0 \), and must then show that \( \tilde{g} \) lifts through the canonical homomorphism \( K^0 \xrightarrow{} \text{Im } \partial_K^0 \). But this follows from lemma 2.8 applied to

\[
\cdots \xrightarrow{} K^{-1} \xrightarrow{} K^0 \xrightarrow{} \text{Im } \partial_K^0 \xrightarrow{} 0.
\]

Thirdly, let \( \tilde{G} \xrightarrow{\tilde{g}} M \) be a homomorphism. I must show that \( \tilde{g} \) lifts through \( \text{Ker } \partial_K^1 \xrightarrow{} M \). However, from the data given I can construct a commutative diagram

\[
\begin{array}{ccc}
\cdots & \xrightarrow{} & K^0 \\
& \downarrow{\partial_K} & \downarrow{\partial_K} \\
& K^1 & \xrightarrow{} K^2 & \cdots \\
& \downarrow{\text{Ker } \partial_K^1} & \downarrow{\text{Coker } \partial_K^0} & \\
& \text{Coker } \partial_K^0 & \downarrow{\gamma} & \\
& M &
\end{array}
\]

and by applying lemma 2.8 to

\[
\cdots \xrightarrow{} K^0 \xrightarrow{} K^1 \xrightarrow{} \text{Coker } \partial_K^0 \xrightarrow{} 0
\]

I find that \( \tilde{G} \xrightarrow{j\tilde{g}} \text{Coker } \partial_K^0 \) lifts through \( K^1 \xrightarrow{} \text{Coker } \partial_K^0 \). It is a small diagram exercise to see that hence, \( \tilde{G} \xrightarrow{j\tilde{g}} M \) lifts through \( \text{Ker } \partial_K^1 \xrightarrow{} M \) as desired. \( \square \)

Let me close the section with the following easy consequence.

**Remark 2.12.** Recall that a Gorenstein projective precover of an \( A \)-left-module \( M \) is a homomorphism \( G \xrightarrow{\tilde{g}} M \) where \( G \) is a Gorenstein
projective module, so that if $\tilde{G}$ is any Gorenstein projective module with a homomorphism $\tilde{G} \xrightarrow{\tilde{g}} M$, then $\tilde{g}$ lifts through $g$,

\[
\begin{array}{c}
\tilde{G} \\
\text{g}
\end{array}
\xrightarrow{\tilde{g}}
\begin{array}{c}
G \\
\text{g}
\end{array}
\xrightarrow{g}
M.
\]

**Theorem 2.13.** Each $A$-left-module has a Gorenstein projective precover.

**Proof.** It follows from theorem 2.11 that the homomorphism

$\text{Ker } \partial_1^K \rightarrow M$

is a Gorenstein projective precover. □

3. Tate Ext groups

This section defines Tate Ext groups, and goes on to show some simple properties: A short exact sequence in either variable gives rise to a long exact sequence of Tate Ext groups; when the Tate Ext groups from [3] and [25] are defined, they agree with the ones defined in this paper; and classical Tate cohomology is the special case $\hat{\text{Ext}}_i^{kG}(k, N)$ of the Tate Ext groups. Finally, it is proved that the Tate Ext groups fit into the long exact sequence (3) from the introduction.

**Remark 3.1.** It is classical that the category of $A$-left-modules $\text{Mod}(A)$ is equivalent to the full subcategory of $K(\text{Pro} A)$ consisting of projective resolutions of $A$-left-modules. Let

$\text{res} : \text{Mod}(A) \rightarrow K(\text{Pro} A)$

be a functor implementing the equivalence.

**Definition 3.2.** If $M$ and $N$ are $A$-left-modules, then the $i$'th Tate Ext group is

$\hat{\text{Ext}}_i(M, N) = H^i\text{Hom}_A(c^i \text{res } M, N)$.

**Remark 3.3.** As pointed out in remark 2.3, the complex $c^i \text{res } M$ can be thought of as the best approximation to $M$ by a complete projective resolution. So taking Hom into $N$ and taking cohomology is the obvious way to get Tate Ext groups.
Proposition 3.4. Let
\[ 0 \to M' \to M \to M'' \to 0 \]
and
\[ 0 \to N' \to N \to N'' \to 0 \]
be short exact sequences of $A$-left-modules. Then there are natural long exact sequences
\[ \cdots \to \widehat{\text{Ext}}^i(M'', N) \to \widehat{\text{Ext}}^i(M, N) \to \widehat{\text{Ext}}^i(M', N) \to \cdots \]
and
\[ \cdots \to \widehat{\text{Ext}}^i(M, N') \to \widehat{\text{Ext}}^i(M, N) \to \widehat{\text{Ext}}^i(M, N'') \to \cdots . \]

Proof. It is well known that the first short exact sequence in the proposition results in a distinguished triangle in $K(\text{Pro} A)$,
\[ \text{res} M' \to \text{res} M \to \text{res} M'' \to . \]
Since $e_*$ is a triangulated functor, so is its adjoint $e^!$, so there is also a distinguished triangle in $E(A)$,
\[ e^! \text{res} M' \to e^! \text{res} M \to e^! \text{res} M'' \to . \]
This again results in a distinguished triangle
\[ \text{Hom}_A(e^! \text{res} M'', N) \to \text{Hom}_A(e^! \text{res} M, N) \to \text{Hom}_A(e^! \text{res} M', N) \to \]
whose cohomology long exact sequence is the first long exact sequence in the proposition.

The complex $e^! \text{res} M$ is in $E(A)$ so consists of projective modules, so the second short exact sequence in the proposition gives a short exact sequence of complexes
\[ 0 \to \text{Hom}_A(e^! \text{res} M', N') \to \text{Hom}_A(e^! \text{res} M, N) \to \text{Hom}_A(e^! \text{res} M, N'') \to 0 \]
whose cohomology long exact sequence is the second long exact sequence in the proposition. \qed

Remark 3.5. If $M$ and $N$ are $A$-left-modules, then the earlier definition of Tate Ext groups given in [3] and [25] is
\[ \widehat{\text{Ext}}^i(M, N) = H^i \text{Hom}_A(T, N) \]
where $T$ is a complete projective resolution of $M$. This means that $T$ is in $E(A)$ and sits in a diagram of chain maps
\[ T \xrightarrow{t} P \to M \]
(6)
where $P \to M$ is a projective resolution and where $T^i \xrightarrow{t^i} P^i$ is bijective for $i \ll 0$. 

Note that not all $A$-left-modules have complete projective resolutions. In fact, the ones that do are exactly the ones which have finite Gorenstein projective dimension by \cite[thm. 3.4]{25}.

**Lemma 3.6.** Let $M$ be an $A$-left-module which has a projective resolution $P$ and a complete projective resolution $T$. Then

$$e^i P \cong T$$

in $K(\text{Pro} A)$.

**Proof.** All projective resolutions of $M$ are isomorphic in $K(\text{Pro} A)$, so I may as well prove the lemma for the specific projective resolution $P$ from equation (6).

By applying construction 2.4 to the chain map $T \xrightarrow{t} P$ in cohomological degrees larger than some number, I can assume that $t$ is surjective. Hence there is a short exact sequence of complexes

$$0 \to K \to T \xrightarrow{t} P \to 0. \tag{7}$$

Since both $T$ and $P$ consist of projective modules, the sequence is semi-split and $K$ also consists of projective modules. Moreover, by assumption, $T^i \xrightarrow{t^i} P^i$ is bijective for $i \ll 0$, so $K^i = 0$ for $i \ll 0$. So $K$ is a left-bounded complex of projective modules.

Now let $E$ be in $E(A)$. In particular, $\text{Hom}_A(E, Q)$ is exact when $Q$ is a projective module. It is classical that $\text{Hom}_A(E, K)$ is then also exact, because $K$ is a left-bounded complex of projective modules. Indeed, this follows by an argument analogous to the one which shows that if $X$ is an exact complex and $I$ is a left-bounded complex of injective modules, then $\text{Hom}_A(X, I)$ is exact.

Since the sequence (7) is semi-split, it stays exact under the functor $\text{Hom}_A(E, -)$. So there is a short exact sequence of complexes

$$0 \to \text{Hom}_A(E, K) \to \text{Hom}_A(E, T) \to \text{Hom}_A(E, P) \to 0.$$ 

Since $\text{Hom}_A(E, K)$ is exact, the cohomology long exact sequence shows that there is an isomorphism

$$H^0 \text{Hom}_A(E, T) \cong H^0 \text{Hom}_A(E, P)$$

which is natural in $E$. That is, there is a natural isomorphism

$$\text{Hom}_{K(\text{Pro} A)}(E, T) \cong \text{Hom}_{K(\text{Pro} A)}(E, P)$$

which can also be written

$$\text{Hom}_{E(A)}(E, T) \cong \text{Hom}_{K(\text{Pro} A)}(E, P)$$

because $E$ and $T$ are in $E(A)$. 
On the other hand, I also have a natural isomorphism
\[ \text{Hom}_{K(\text{Pro}_A)}(E, P) = \text{Hom}_{K(\text{Pro}_A)}(e^*E, P) \cong \text{Hom}_{E(A)}(E, e^!P). \]
Combining the last two equations gives a natural isomorphism
\[ \text{Hom}_{E(A)}(E, T) \cong \text{Hom}_{E(A)}(E, e^!P), \]
proving \( T \cong e^!P \) as desired. \( \square \)

**Proposition 3.7.** Let \( M \) be an \( A \)-left-module which has a complete projective resolution \( T \). Then the Tate Ext groups of this paper (see definition 3.2) coincide with the Tate Ext groups which were defined in [3] and [25] (see remark 3.5).

*Proof.* Lemma 3.6 gives that the projective resolution \( \text{res } M \) of \( M \) satisfies \( e^! \text{res } M \cong T \). Combining this with the formulae in definition 3.2 and remark 3.5 proves the proposition. \( \square \)

**Proposition 3.8.** Let \( k \) be a field, \( G \) a finite group, and \( N \) a finite dimensional \( k \)-linear representation of \( G \). Then the Tate Ext group
\[ \hat{\text{Ext}}^i_{kG}(k, N) \]
of this paper is defined and isomorphic to the \( i \)'th classical Tate cohomology group of \( N \).

*Proof.* The group algebra \( kG \) is a finite dimensional \( k \)-algebra. It is clearly left-coherent and right-noetherian, and since it is in fact self injective, it is clear that \( kG kG kG \) is a dualizing complex (cf. setup 1.4').

Hence \( e^! \) exists over \( kG \) by theorem 1.10, and so the Tate Ext groups of this paper are defined over \( kG \).

The Tate Ext groups \( \hat{\text{Ext}}^i_{kG}(k, N) \) from [3] and [25] are also defined, and the \( i \)'th one is isomorphic to the \( i \)'th classical Tate cohomology group of \( N \) according to [3, exam. 5.1] with \( k \) in place of \( \mathbb{Z} \).

But the Tate Ext groups of this paper and the ones from [3] and [25] are isomorphic by proposition 3.7, so the present proposition follows. \( \square \)

**Definition 3.9.** If \( M \) and \( N \) are \( A \)-left-modules, then the \( i \)'th Gorenstein Ext group \( \text{Ext}^i_G(M, N) \) is
\[ \text{Ext}^i_G(M, N) = H^i \text{Hom}_A(G, N) \]
where \( G \) is a Gorenstein projective resolution of \( M \) (cf. remark 2.9).

**Remark 3.10.** The resolution \( G \) exists by theorem 2.11. Note that \( \text{Ext}^i_G(\cdot, \cdot) \) is a well defined bifunctor; see [3] or [13] for this and other properties.
Construction 3.11. Consider the short exact sequence from construction 2.5,
\[ 0 \to K \to F \to P \to 0, \]
where \( P \) is a projective resolution of the \( A \)-left-module \( M \) and where \( F = e_*e^!P \). Truncating the complexes \( K \) and \( F \) gives a new short exact sequence of complexes,
\[
\begin{array}{ccccccc}
\vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 \\
0 & \text{Ker} \partial_K^1 & \text{Ker} \partial_F^1 & 0 & 0 \\
0 & K^0 & F^0 & P^0 & 0 \\
0 & K^{-1} & F^{-1} & P^{-1} & 0 \\
\vdots & \vdots & \vdots & \vdots \\
\end{array}
\]
which I will denote
\[ 0 \to K' \to F' \to P \to 0. \tag{8} \]

Theorem 3.12. Let \( M \) and \( N \) be \( A \)-left-modules. Then there is a long exact sequence
\[
\begin{array}{cccccccc}
0 & \to & \text{Ext}^1_G(M, N) & \to & \text{Ext}^1(M, N) & \to & \hat{\text{Ext}}^1(M, N) \\
& \to & \cdots \\
& \to & \text{Ext}^1_G(M, N) & \to & \text{Ext}^1(M, N) & \to & \hat{\text{Ext}}^1(M, N) & \to & \cdots,
\end{array}
\]
natural in \( M \) and \( N \).

Proof. Consider the short exact sequence (8) from construction 3.11. The complex \( P \) is a projective resolution of \( M \) and in order to make everything natural in \( M \), I can clearly suppose
\[ P = \text{res} M \]
where \( \text{res} M \) is a projective resolution depending functorially on \( M \). Since \( P = \text{res} M \) consists of projective modules, the short exact sequence (8) is semi-split and therefore stays exact under the functor \( \text{Hom}_A(-, N) \). So there is a short exact sequence of complexes
\[
0 \to \text{Hom}_A(\text{res} M, N) \to \text{Hom}_A(F', N) \to \text{Hom}_A(K', N) \to 0. \quad (9)
\]

Since \( \text{res} M \) is a projective resolution of \( M \), I have
\[
\text{H}^i \text{Hom}_A(\text{res} M, N) = \text{Ext}^i(M, N)
\]
for each \( i \).

The complex
\[
F = e_\pi e_i^! P = e_i^! \text{res} M
\]
is in \( E(A) \) so it is exact, so
\[
F' = \cdots \to F^{-1} \to F^0 \to \text{Ker} \partial^1_F \to 0 \to \cdots
\]
is also exact, and hence \( \text{H}^0 \text{Hom}_A(F', N) = 0 \). On the other hand, the form of \( F' \) makes it clear that
\[
\text{H}^i \text{Hom}_A(F', N) = \text{H}^i \text{Hom}_A(F, N) = \text{H}^i \text{Hom}_A(e_i^! \text{res} M, N)
\]
for \( i \geq 1 \).

Finally, theorem 2.11 says that
\[
K' = \cdots \to K^{-1} \to K^0 \to \text{Ker} \partial^1_K \to 0 \to \cdots
\]
is a Gorenstein projective resolution of \( M \), shifted one step to the right. Hence
\[
\text{H}^i \text{Hom}_A(K', N) = \text{Ext}^{i+1}_G(M, N)
\]
for \( i \geq -1 \).

So looking at the cohomology long exact sequence of (9), starting with \( \text{H}^0 \text{Hom}_A(F', N) = 0 \), gives
\[
0 \to \text{Ext}^1_G(M, N) \to \text{Ext}^1(M, N) \to \text{Ext}^{1}_1(M, N) \to \cdots
\]
as desired. \( \square \)

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