ON THE INTEGRABILITY OF HILL'S EQUATION OF THE MOTION OF THE MOON

F. REIS AND B. SCÁRDUA

Abstract. We study under the standpoint of integrable complex analytic 1-forms (complex analytic foliations), a class of second order ordinary differential equations with periodic coefficients. More precisely, we study Hill's equations of motion of the moon, which are related to the dynamics of the system Sun-Earth-Moon. We associate to the complex Hill equation an integrable complex analytic one-form in dimension three. This defines a Hill foliation. The existence of first integral for a Hill foliation is then studied. The simple cases correspond to the existence of rational or Liouvillian first integrals. We then prove the existence of a Bessel type first integral in a more general case. We construct a standard two dimensional model for the foliation which we call Hill fundamental form. This plane foliation is then studied also under the standpoint of reduction of singularities and existence of first integral. For the more general case of the Hill equation, we prove for the corresponding Hill foliation, the existence of a Laurent-Fourier type formal first integral. Our approach suggests that there may be a class of plane foliations admitting Bessel type first integrals, in connection with the classification of (holonomy) groups of germs of complex diffeomorphisms associate to a certain class of second order ODEs.

Contents

1. Introduction 1
2. The complex Hill equation 2
3. Hill forms and Hill foliations 3
4. The Hill fundamental form 6
4.1. Bessel functions 6
4.2. Integrability of the Hill form 7
5. The Hill foliation 11
5.1. Non-existence of a Liouvillian first integral 11
5.2. Reduction of singularities 13
5.3. Holonomy 15
5.4. The general case 15
6. Integrrability and classical solutions 15
7. Laurent-Fourier type formal first integral for Hill foliations 17
References 18

1. Introduction

In the year of 1877, G.W. Hill published his celebrated work ([10]) “On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and moon”. In this masterpiece Hill describes the movement of the moon around the earth by considering it as a harmonic oscillator in a periodic gravitational vector field and introduced the following model
(called Hill’s equation \[1, 2\])

\begin{equation}
\label{eq:hill}
\frac{d^2u}{dz^2} + p(z)u = 0
\end{equation}

where \( p \) is a periodic function of the time \( t \) and \( u(t) \) describes the distance (position) of the moon with respect to the earth. Hill introduced and applied successfully for the first time the theory of infinite determinants. From that moment on, a great effort has been made towards the comprehension of Hill’s equation (see for instance [15]).

In this work we study the complex Hill equation. In short, this means we shall study the equation \( u''(z) + p(z)u(z) = 0 \), where \( p(z) \) is a complex analytic periodic function defined in some domain \( \Lambda \subseteq \mathbb{C} \). Since this equation is related to the problem of planetary movement and stability of the solar system, we address the following question:

**Question 1.1.** Is there any integrable structure connected to the Hill equation?

The above question may seem quite general. Indeed, one of the gains of this work is to investigate an appropriate notion of integrability in this case. Usually, when dealing with ODEs the word *integrability* means to find some kind of potential function, but this does not necessarily seem appropriate because the order of the equation is two. In this work we bring some geometrical features based in the theory of complex analytic (holomorphic) foliations. In this foliation framework the notion of integrability has essentially two interpretations. One is the existence of an integrable one-form to which the ODE is tangent (cf. Theorem \[A\]). Another is the existence of a first integral for a foliation that is tangent to the ODE (cf. Theorems \[B\] and \[D\]). We shall explore these concepts and try to shed some light into the comprehension of the Hill equation in the complex framework.

## 2. The Complex Hill Equation

In his work \[10\], G. W. Hill considered equations related to the *three-body problem*, namely earth, moon and sun, given by

\begin{align}
&\frac{d^2u}{dz^2} - 2m\frac{dv}{dz} + \frac{u}{r^3} = 3m^2u \label{eq:hill1} \\
&\frac{d^2v}{dz^2} - 2m\frac{du}{dz} + \frac{v}{r^3} = 0 \label{eq:hill2}
\end{align}

In the above equations, \( u, v \) are given rectangular coordinates of the moon having the earth as center and, \( r = \sqrt{u^2 + v^2} \). The parameter \( m \) is given by

\[
m = \frac{n'}{n - n'},
\]

where \( n' \) is the mean motion of the sun and \( n \) is the mean motion of the moon. Hill works with the estimative \( m = 0.08084893679 \). The parameter \( \chi \) is given by

\[
\chi = G\frac{M_e + M_m}{(n - n')^2},
\]

where \( M_e, M_m \) are the mass of the earth and of the moon respectively, and \( G \) is Cavendish’s gravitational constant.

G. W. Hill was able to associate to equations \((2)\) and \((3)\) the single equation,

\[
\frac{d^2u}{ds^2} + p(s)u(s) = 0
\]
where \( p \) is a periodic real valued function. The variable \( s \) is related to the time \( t \) by the formula \( s = (n - n')(t - t_0) \), where \( t_0 \) is the initial time. Most of the classical works in Hill’s equation are based on the following hypothesis:

**Hypothesis 1:** \( p(s) \) is a real integrable periodic function of period \( \pi \).

The original work of Hill and the classical reference [15] of Magnus and Winkler consider Hypothesis 1 (see for instance [9]). Despite this restriction, the solutions are allowed to have complex values.

In this work we shall start the study of the complex Hill equation

\[
u''(z) + p(z)u(z) = 0
\]

where \( p(z) \) is a complex periodic function. We shall assume that \( p(z) \) is complex analytic defined in a strip \( A \subseteq \mathbb{C} \) containing the real axis \( \Im(z) = 0 \).

Depending on the viewpoint, the complex case and the real case may be quite different, for some functions are periodic in the complex framework, but not in the real setting. This is the case of the exponential function \( p(z) = e^z, z \in \mathbb{C} \). Another important particular case is the complex Mathieu equation (2, 3)

\[
u''(z) + (a + b \cos z)u(z) = 0
\]

where \( a, b \in \mathbb{C} \) are complex numbers. For this function the main difference with the real case is the fact that the real function is bounded but this is not the case of the complex function. The complex Mathieu equation will be treated in a forthcoming work.

### 3. Hill Forms and Hill Foliations

In this section we shall study the Hill equation (5) from the point of view of integrable one-forms. We start with a more general situation of a second order complex ODE

\[
u''(z) + b(z)u = 0
\]

where \( b \) is a complex analytic function defined in an open set \( U \subseteq \mathbb{C} \). As a first step we perform the classical order reduction process where equation (7) is rewritten after the following “change of coordinates”: \( x = u, y = u', z = z \). We then obtain \( x' = u' = y, y' = u'' = -b(z)u = -b(z)x, z' = 1 \). Therefore, a natural vector field \( X : \mathbb{C}^2 \times U \rightarrow \mathbb{C}^2 \times U \) associated to equation (7) is given by \( X(x, y, z) = y\frac{\partial}{\partial x} - b(z)x\frac{\partial}{\partial y} + \frac{\partial}{\partial z} \). We shall refer to this vector field as the Hill vector field associated to the Hill equation (7).

We shall say that a complex vector field \( X \) in some open subset \( W \subseteq \mathbb{C}^3 \) is integrable in the weak sense or \( w \)-integrable for short, if it is tangent to a codimension one complex analytic foliation (possibly singular) \( \mathcal{F} \) of \( W \). The vector field is said to be \( s \)-integrable or integrable in the strong sense if it is \( w \)-integrable and the foliation \( \mathcal{F} \) can be chosen to have a first integral. For the moment we shall not decide what kind of first integral we shall be working with (polynomial, rational, meromorphic, Liouvillian, formal...).

**Theorem A.** A Hill vector field is always \( w \)-integrable.

For the proof of Theorem A we shall need the following technical result:

**Lemma 3.1.** Let \( X : \mathbb{C}^3 \rightarrow \mathbb{C}^3 \) be a complex vector field in \( \mathbb{C}^3 \) given by \( X(x, y, z) = f_1(x, y, z)\frac{\partial}{\partial x} + f_2(x, y, z)\frac{\partial}{\partial y} + f_3(x, y, z)\frac{\partial}{\partial z} \) where \( f_j : \mathbb{C}^3 \rightarrow \mathbb{C} \) is complex analytic \( j = 1, 2, 3 \). Let also \( \omega \) be a complex analytic 1-form given in \( \mathbb{C}^3 \) by \( \omega(x, y, z) = A(x, y, z)dx + B(x, y, z)dy + C(x, y, z)dz \), and satisfying \( \omega(X) \equiv 0 \). Then we have \( \omega \wedge dw = 0 \) provided that \( f_1f_3 B_{yy} + f_2f_3 B_{xz} + f_3f_1 B_{yz} = 0 \).
Proof. From \( \omega(X) = 0 \) we have \( Af_1 + Bf_2 + Cf_3 = 0 \). Hence,

\[(8)\quad C = -\frac{(Af_1 + Bf_2)}{f_3}.\]

The partial derivatives of \( C \) can be calculated with the aid of (8) resulting in

\[(9)\quad \frac{\partial C}{\partial x} = -\frac{f_1}{f_3} \frac{\partial A}{\partial x} - \frac{1}{f_3} \frac{\partial f_1}{\partial x} A - \frac{2}{f_3} \frac{\partial B}{\partial x} - \frac{1}{f_3} \frac{\partial f_2}{\partial x} B + \frac{f_1}{f_3} \frac{\partial f_3}{\partial x} A + \frac{2}{f_3} \frac{\partial f_3}{\partial x} B\]

and,

\[(10)\quad \frac{\partial C}{\partial y} = -\frac{f_1}{f_3} \frac{\partial A}{\partial y} - \frac{1}{f_3} \frac{\partial f_1}{\partial y} A - \frac{2}{f_3} \frac{\partial B}{\partial y} - \frac{1}{f_3} \frac{\partial f_2}{\partial y} B + \frac{f_1}{f_3} \frac{\partial f_3}{\partial y} A + \frac{2}{f_3} \frac{\partial f_3}{\partial y} B.\]

On the other hand, from the expression of \( \omega \) we have

\[
d\omega = dA \wedge dx + dB \wedge dy + dC \wedge dz = \left( \frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dx \wedge dy + \left( \frac{\partial C}{\partial x} - \frac{\partial A}{\partial z} \right) dx \wedge dz + \left( \frac{\partial C}{\partial y} - \frac{\partial B}{\partial z} \right) dy \wedge dz.
\]

Thus,

\[
\omega \wedge d\omega = A \left( \frac{\partial C}{\partial y} - \frac{\partial B}{\partial z} \right) dx \wedge dy \wedge dz + B \left( \frac{\partial C}{\partial x} - \frac{\partial A}{\partial z} \right) dy \wedge dx \wedge dz
\]

Thus,

\[
 + C \left( \frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dz \wedge dx \wedge dz
\]

Then, \( \omega \wedge d\omega = 0 \) implies \( A \frac{\partial C}{\partial y} - A \frac{\partial B}{\partial z} - B \frac{\partial C}{\partial x} + B \frac{\partial A}{\partial x} - C \frac{\partial C}{\partial x} - C \frac{\partial A}{\partial y} = 0. \) Replacing (9) and (10) in the above expression we obtain

\[
- \frac{f_1}{f_3} \frac{\partial A}{\partial y} A - \frac{1}{f_3} \frac{\partial f_1}{\partial y} A^2 - \frac{f_2}{f_3} \frac{\partial B}{\partial y} A - \frac{1}{f_3} \frac{\partial f_2}{\partial y} BA + \frac{f_1}{f_3} \frac{\partial f_3}{\partial y} A^2 + \frac{f_2}{f_3} \frac{\partial f_3}{\partial y} BA \]

\[
+ \frac{f_1}{f_3} \frac{\partial A}{\partial x} + \frac{1}{f_3} \frac{\partial f_1}{\partial x} A^2 - \frac{f_2}{f_3} \frac{\partial B}{\partial x} A - \frac{1}{f_3} \frac{\partial f_2}{\partial x} BA + \frac{f_1}{f_3} \frac{\partial f_3}{\partial x} A^2 + \frac{f_2}{f_3} \frac{\partial f_3}{\partial x} BA \]

\[
+ \frac{B}{f_3} \frac{\partial A}{\partial z} - \frac{f_1}{f_3} \frac{\partial B}{\partial z} A - \frac{f_2}{f_3} \frac{\partial B}{\partial z} + B \frac{\partial A}{\partial z} + A \frac{\partial B}{\partial z} = 0.
\]

Simplification of this equation then gives

\[
- \frac{1}{f_3} \frac{\partial f_1}{\partial y} A^2 - \frac{f_2}{f_3} \frac{\partial B}{\partial y} A - \frac{1}{f_3} \frac{\partial f_2}{\partial y} BA + \frac{f_1}{f_3} \frac{\partial f_3}{\partial y} A^2 + \frac{f_2}{f_3} \frac{\partial f_3}{\partial y} BA
\]

\[
+ \frac{f_1}{f_3} \frac{\partial A}{\partial x} + \frac{1}{f_3} \frac{\partial f_1}{\partial x} A^2 - \frac{f_2}{f_3} \frac{\partial B}{\partial x} A - \frac{1}{f_3} \frac{\partial f_2}{\partial x} BA + \frac{f_1}{f_3} \frac{\partial f_3}{\partial x} A^2 + \frac{f_2}{f_3} \frac{\partial f_3}{\partial x} BA
\]

\[
+ \frac{B}{f_3} \frac{\partial A}{\partial z} - \frac{f_1}{f_3} \frac{\partial B}{\partial z} A - \frac{f_2}{f_3} \frac{\partial B}{\partial z} + A \frac{\partial B}{\partial z} = 0.
\]
Reorganizing the terms we get
\[
\frac{f_1}{f_3} B \frac{\partial A}{\partial x} + \frac{f_2}{f_3} B \frac{\partial A}{\partial y} + B \frac{\partial A}{\partial z} = \left( \frac{1}{f_3} \frac{\partial f_1}{\partial y} - \frac{f_1}{f_3} \frac{\partial f_3}{\partial y} \right) A^2 + \left[ \frac{f_1 \partial B}{f_3} \frac{\partial B}{\partial x} + \frac{f_2 \partial B}{f_3} - \frac{\partial B}{\partial z} + B \left[ \frac{f_1 \partial f_3}{f_3} \frac{\partial f_3}{\partial x} - \frac{f_2 \partial f_3}{f_3} \frac{\partial f_3}{\partial y} - \frac{1}{f_3} \frac{\partial f_1}{\partial y} + \frac{1}{f_3} \frac{\partial f_2}{\partial y} \right] \right] A + \frac{B^2}{f_3} \left[ \frac{f_2 \partial f_3}{f_3} \frac{\partial f_3}{\partial x} - \frac{1}{f_3} \frac{\partial f_2}{\partial x} \right].
\]
Multiplication by \( f_3^2 \) ends the proof. \( \square \)

Proof of Theorem [A]. We look for an integrable complex analytic one-form \( \omega \) in \( \mathbb{C}^2 \times U \) such that \( \omega(X) = 0 \). Let us write \( \omega(x, y, z) = A(x, y, z) dx + B(x, y, z) dy + C(x, y, z) dz \) where the coefficients \( A, B, C \) are complex analytic in \( \mathbb{C}^2 \times U \). We know from Lemma 3.1 that \( \omega = A(x, y, z) dx + B(x, y, z) dy + C(x, y, z) dz \) satisfying \( \omega(X) \equiv 0 \) also satisfies \( \omega \wedge d\omega = 0 \) provided that
\[
yB \frac{\partial A}{\partial x} - [b(z)x] \frac{\partial A}{\partial y} + B \frac{\partial A}{\partial z} = A^2 + \left[ \frac{\partial B}{\partial x} - [b(z)x] \frac{\partial B}{\partial y} - \frac{\partial B}{\partial z} \right] A + b(z)B^2.
\]
and
\[
\frac{C}{y} = (-yA - b(z)xB) = yA + xp(z)B.
\]
Let us take \( B \equiv 1 \) in the last equation obtaining
\[
y \frac{\partial A}{\partial x} - [b(z)x] \frac{\partial A}{\partial y} + \frac{\partial A}{\partial z} = A^2 + b(z).
\]
Equation (12) is a semilinear first order PDE, which can be solved in real case by the classical method of characteristics. Let us try this same method in our complex framework. For this we introduce the following system:
\[
\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -b(z)x, \quad \frac{dz}{dt} = 1, \quad \frac{d\phi}{dt} = A^2 + b.
\]
Then, from this we obtain:

**Claim 3.2.** Equation (12) admits the solution \( A = \frac{y}{x} \).

In fact, for \( A = \frac{-y}{x} \) we have \( \frac{\partial A}{\partial x} = \frac{y}{x^2}, \quad \frac{\partial A}{\partial y} = -\frac{1}{x} \quad \text{and} \quad \frac{\partial A}{\partial z} = 0 \). Substituting in (12) we have
\[
y \frac{y}{x^2} - [a(z)y + b(z)x](-\frac{1}{x}) + 0 = \frac{y^2}{x^2} - a(z) \frac{y}{x} + b(z).
\]
This proves Claim 3.2.

Hence, substituting the values of \( A \) and \( B \) in (11), we have \( C = \frac{y^2}{x} + b(z)x \). Thus, the meromorphic 1-form \( \Omega := -\frac{y}{x} dx + dy + \left[ \frac{y^2}{x} + b(z)x \right] dz \) satisfies \( \Omega(X) = 0 \). Multiplying \( \Omega \) by its poles we obtain \( \omega = x \Omega = -ydx + xdy + [y^2 + b(z)x^2]dz \), which is a complex analytic solution to the integrability problem, proving the theorem. \( \square \)

From now on we shall consider the case where the function \( b \) is an entire periodic function \( b = p(z) \) i.e., there is a complex number \( T \in \mathbb{C} \setminus \{0\} \) such that \( p(z + T) = p(z) \). Therefore, according to Theorem [A], we can define a complex analytic foliation \( \mathcal{H} \) associated to the Hill equation (5).
Definition 3.3. We shall refer to the foliation \( \mathcal{H} \) in \( \mathbb{C}^3 \) defined by
\[
\omega_{\mathcal{H}} = -ydx + y^2 + p(z)x^2 dz = 0
\]
as Hill foliation of parameter \( p \), where \( p \) is a periodic complex analytic function of period \( T \) (real or complex).

A first question involving Hill foliations is the following:

Question 3.4. Is there any type of first integral for a Hill foliation?

Let us begin with the very basic cases.

Case \( p(z) = 0 \). In this case we have the corresponding Hill foliation
\[
-ydx + y^2 dz = 0.
\]
This foliation exhibits a rational first integral given by \( y/x + z \).

Case \( p(z) = 1 \). Here we have
\[
\omega = -ydx + x^2 dz = x^2 [d(y/x) + ((y/x)^2 + 1)dz].
\]
This form admits a Liouvillian first integral. Indeed, \( \omega/(x^2(1 + (y/x)^2) \) is closed and rational and can be integrated by logarithmic.

The interesting cases appear when we consider \( p(z) \) a non-constant periodic function. In this situation we shall ask for the existence of a first integral for the corresponding Hill foliation.

4. The Hill fundamental form

In this section we study the special case of the Hill equation where the periodic function is the exponential function, \( p(z) = e^z \). From this special case we shall obtain a fundamental form that shall play an important role in our study of the general case. We start with the differential form given above \( \omega = -ydx + y^2 + e^z x^2 dz \). Notice that
\[
\frac{1}{x^2} \omega = -ydx + x^2 dy + [y^2 + e^z]dz = x^2 (d(y/x) + (y/x)^2 + 1)dz.
\]
Let us call \( \phi = -y/x, \psi = e^z \). Then \( dz = d\psi/\psi \) and we can rewrite
\[
\frac{1}{x^2} \omega = d(y/x) + [y^2 + e^z]dz = -d\phi + [\phi^2 + \psi] d\psi = \frac{1}{\psi} [-\psi d\phi + (\phi^2 + \psi) d\psi].
\]

In a modern language, if we define the rational map \( \Pi: \mathbb{C}^3 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \) by \( \Pi(x, y, z) = (\phi, \psi) = (y/x, e^z) \) then we shall state:

Lemma 4.1. The Hill foliation \( \mathcal{H} \) on \( \mathbb{C}^3 \) given by
\[
-ydx + y^2 + e^z x^2 dz = 0
\]
is the pull-back by the rational map \( \Pi \) above of the two-dimension foliation \( \mathcal{H}_2 \) on \( \mathbb{P}^1 \times \mathbb{P}^1 \) given on affine coordinates \( (x, y) \) by
\[
-ydx + (x^2 + y)dy = 0
\]

Definition 4.2. The 1-form
\[
\Omega_2 = -ydx + (x^2 + y)dy
\]
will be called the Hill fundamental form or just Hill form on \( \mathbb{C}^2 \).

It is our belief that the Hill form given in (13) will play a key role in the study of Hill foliations. In what follows we give a more detailed study of this form. We shall prove that it does not admit first integral of Liouvillian type, but it admits a first integral which is given in terms of Bessel functions. Bessel functions are described in details in [20]. In what follows we present a short summary of their properties. Details and proofs can be found in [20].
4.1. **Bessel functions.** The Bessel functions appear as solutions to the classical *Bessel equation*

\[ z^2 u'' + zu' + (z^2 - k^2)u = 0. \]

These equations are related to problems in physics as vibrations, diffusion and electromagnetic waves. Originally defined by Bernoulli they were formally introduced by F. Bessel in 1824 while doing his astronomy research.

**Definition 4.3.** The function

\[ J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(\nu+k+1)} \left( \frac{z}{2} \right)^{2k+\nu} \]

is called *Bessel function of first type, and order \( \nu \).* Here \( \Gamma \) is the *Gamma function* defined by

\[ \Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt \]

where \( \nu \) is a number (real or complex).

**Proposition 4.4.** The first type Bessel functions satisfy the following properties for each \( \nu \in \mathbb{C} \):

\[ 2J'_\nu(z) = J_{\nu-1}(z) - J_{\nu+1}(z), \quad J_{\nu+1}(z) = \frac{2\nu}{z}J_\nu(z) - J_{\nu-1}(z); \]

Moreover, for every \( n \in \mathbb{Z} \) we have: \( J_{-n}(z) = (-1)^n J_n(z) \), where \( J' \) denotes the usual derivative with respect to \( z \).

**Remark 4.5.** The functions \( J_{-\nu}, J_\nu \) are known to be linearly independent in the case where \( \nu \not\in \mathbb{Z} \).

**Lemma 4.6.** The Bessel functions \( J_0 \) and \( J_1 \) of first type and order 0 and 1, are entire functions of the complex variable \( z \).

**Proof.** Indeed, \( J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k} (k!)^2} \). The ratio test shows that the convergence radius of this power series is \( \infty \). By its turn we have \( J_1(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{2^{2k+1} (k!(k+1)!)} \). Again the ratio test shows the convergence in the whole complex plane. \( \square \)

**Definition 4.7.** The *second type Bessel function of order \( \nu \in \mathbb{C} \) is defined by the expression

\[ Y_\nu(z) = \frac{J_\nu(z) \cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)} \]

where \( \nu \not\in \mathbb{Z} \). Here \( J_\nu, J_{-\nu} \) are first type Bessel functions as defined above. In case \( \nu = n \in \mathbb{Z} \) the corresponding second type Bessel function is defined as

\[ Y_n(z) = \lim_{\nu \to n} Y_\nu(z). \]

The above limit exists. For instance, \( Y_0(0) \) is well-defined by the well-known L’Hospital rule (\( \square \)) pg. 9).

**Remark 4.8.** Proposition 4.4 also holds for second type Bessel functions.
4.2. Integrability of the Hill form. We are now in conditions to study the integrability of the Hill form.

**Theorem B.** The Hill form

\[ \Omega_2 = -y \, dx + (x^2 + y) \, dy \]

admits a first integral \( F \) of the form

\[ F(x, y) = \frac{2x Y_0(2\sqrt{y}) - 2\sqrt{y} Y_1(2\sqrt{y})}{x J_0(2\sqrt{y}) - \sqrt{y} J_1(2\sqrt{y})} \]

where \( J_0, J_1 \) are the Bessel functions of first type and \( Y_0, Y_1 \) are the Bessel functions of second type. The function \( F \) is well-defined complex analytic in open subsets of the form \((x, y) \in \mathbb{C} \times (\mathbb{C} \setminus L)\) where \( L \) is a closed segment of line starting from the origin of \( \mathbb{C} \).

**Proof.** First we write \( \Omega_2 = 0 \) as

\[ dy \, = \, \frac{y}{x^2 + y} \, dx \]

This ODE is similar to some of the models treated in [17] page 375, §13.2.3. Indeed, the change of coordinates \( y = e^t \) leads to the model

\[ x' = dx \, dt = \frac{dx \, dy}{y} \, dt = \frac{x^2 + y \, dx}{y} \, dt = \frac{x^2 + e^t \, dt}{e^t} = x^2 + e^t \]

By its turn the ODE \( x' = x^2 + e^t \) can be solved by methods similar to those in [17] 13.2.2.5 and leads us to the following:

**Claim 4.9.** The solutions of (14) are given by \( F(x, y) = c \in \mathbb{C} \) where

\[ F(x, y) = \frac{2x Y_0(2\sqrt{y}) - 2\sqrt{y} Y_1(2\sqrt{y})}{x J_0(2\sqrt{y}) - \sqrt{y} J_1(2\sqrt{y})} \]

**Proof.** We shall first calculate the partial derivatives of \( F \). The first is obtained by a standard computation

\[ \frac{\partial F}{\partial x}(x, y) = \frac{2Y_0(2\sqrt{y})(xJ_0(2\sqrt{y}) - \sqrt{y}J_1(2\sqrt{y})) - J_0(2\sqrt{y})(2xY_0(2\sqrt{y}) - 2\sqrt{y}Y_1(2\sqrt{y}))}{(xJ_0(2\sqrt{y}) - \sqrt{y}J_1(2\sqrt{y}))^2} \]

\[ \frac{\partial F}{\partial y}(x, y) = \frac{2\sqrt{y}(J_0(2\sqrt{y})Y_1(2\sqrt{y}) - J_1(2\sqrt{y})Y_0(2\sqrt{y}))(xJ_0(2\sqrt{y}) - \sqrt{y}J_1(2\sqrt{y}))}{(xJ_0(2\sqrt{y}) - \sqrt{y}J_1(2\sqrt{y}))^2} \]

The partial derivative \( \frac{\partial F}{\partial y} \):

In this case we shall make use of the properties of the Bessel functions mentioned above. Let us first analyze each term of the derivative. From Proposition 4.4 we have

\[ \frac{\partial}{\partial y}[xJ_0(2\sqrt{y})] = \frac{x}{\sqrt{y}}[J_1(2\sqrt{y}) - J_0(2\sqrt{y})] \]

\[ = \frac{x}{\sqrt{y}}[-J_1(2\sqrt{y}) - J_0(2\sqrt{y})] \]

\[ = -2 \frac{x}{\sqrt{y}} J_1(2\sqrt{y}). \]
\[
\frac{\partial}{\partial y}[\sqrt{y} J_1(2\sqrt{y})] = \frac{1}{2\sqrt{y}} J_1(2\sqrt{y}) + 2\sqrt{y} \frac{\partial}{\partial y}[J_1(2\sqrt{y})]
\]
\[
= \frac{1}{2\sqrt{y}} J_1(2\sqrt{y}) + 2\sqrt{y} \left[ \frac{1}{2\sqrt{y}} [J_0(2\sqrt{y}) - J_2(2\sqrt{y})] \right]
\]
\[
= \frac{1}{2\sqrt{y}} J_1(2\sqrt{y}) + J_0(2\sqrt{y}) - \left[ \frac{1}{2\sqrt{y}} J_1(2\sqrt{y}) - J_0(2\sqrt{y}) \right]
\]
\[
= 2J_0(2\sqrt{y}).
\]

Similarly,
\[
\frac{\partial}{\partial y}[2xY_0(2\sqrt{y})] = -2\frac{x}{\sqrt{y}} Y_1(2\sqrt{y}), \quad \frac{\partial}{\partial y}[2\sqrt{y}Y_1(2\sqrt{y})] = 2Y_0(2\sqrt{y}).
\]

Thence
\[
\frac{\partial}{\partial y}(xJ_0(2\sqrt{y}) - \sqrt{y} J_1(2\sqrt{y})) = -2\frac{x}{\sqrt{y}} J_1(2\sqrt{y}) - 2J_0(2\sqrt{y})
\]
and,
\[
\frac{\partial}{\partial y}(xY_0(2\sqrt{y}) - \sqrt{y} Y_1(2\sqrt{y})) = -2\frac{x}{\sqrt{y}} Y_1(2\sqrt{y}) - 2Y_0(2\sqrt{y}).
\]

Henceforth,
\[
\frac{\partial}{\partial y} \left[ \frac{2xY_0(2\sqrt{y}) - 2\sqrt{y}Y_1(2\sqrt{y})}{xJ_0(2\sqrt{y}) - \sqrt{y} J_1(2\sqrt{y})} \right]
\]
\[
= \frac{N}{(xJ_0(2\sqrt{y}) - \sqrt{y} J_1(2\sqrt{y}))^2}
\]
where by its turn
\[
N = \left[ -2Y_0(2\sqrt{y}) - 2\frac{x}{\sqrt{y}} Y_1(2\sqrt{y}) \right] \left[ xJ_0(2\sqrt{y}) - \sqrt{y} J_1(2\sqrt{y}) \right] - \left[ -2J_0(2\sqrt{y}) - 2\frac{x}{\sqrt{y}} J_1(2\sqrt{y}) \right] \left[ xY_0(2\sqrt{y}) - \sqrt{y} Y_1(2\sqrt{y}) \right]
\]
\[
= -2xY_0(2\sqrt{y})J_0(2\sqrt{y}) + 2\sqrt{y}Y_0(2\sqrt{y})J_1(2\sqrt{y}) - 2\frac{x^2}{\sqrt{y}} Y_1(2\sqrt{y}) J_0(2\sqrt{y})
+ 2xY_0(2\sqrt{y})J_1(2\sqrt{y}) + 2xJ_0(2\sqrt{y})Y_0(2\sqrt{y}) - 2\sqrt{y}J_0(2\sqrt{y})Y_1(2\sqrt{y})
+ 2\frac{x^2}{\sqrt{y}} J_1(2\sqrt{y}) Y_0(2\sqrt{y}) - 2xJ_1(2\sqrt{y})Y_1(2\sqrt{y})
\]
\[
= 2 \left[ \sqrt{y} + \frac{x^2}{\sqrt{y}} \right] [J_1(2\sqrt{y}) Y_0(2\sqrt{y}) - Y_1(2\sqrt{y}) J_0(2\sqrt{y})]
\]
The exterior derivative \(dF\) is then given by
\[
dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy
\]
\[
= \left[ \frac{2\sqrt{y} (J_0(2\sqrt{y}) \overline{Y}_1(2\sqrt{y}) - J_1(2\sqrt{y}) \overline{Y}_0(2\sqrt{y}))}{(xJ_0(2\sqrt{y}) - \sqrt{y} J_1(2\sqrt{y}))^2} \right] dx + \left[ \frac{N}{(xJ_0(2\sqrt{y}) - \sqrt{y} J_1(2\sqrt{y}))^2} \right] dy.
\]
Therefore,

\[
\Omega \wedge dF = \left[-ydx + (x^2 + y)dy\right] \wedge \left[\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy\right]
\]

\[
= -y \frac{\partial F}{\partial y} dx \wedge dy + (x^2 + y) \frac{\partial F}{\partial x} dy \wedge dx
\]

\[
= \left[-y \frac{\partial F}{\partial y} - (x^2 + y) \frac{\partial F}{\partial x}\right] dx \wedge dy.
\]

Now we observe that

\[
(x^2 + y) \frac{\partial F}{\partial x} = (x^2 + y) \frac{2\sqrt{y} (J_0(2\sqrt{y}) Y_0(2\sqrt{y}) - J_1(2\sqrt{y}) Y_1(2\sqrt{y}))}{(xJ_0(2\sqrt{y}) - \sqrt{y}J_1(2\sqrt{y}))^2}
\]

\[
= -2\sqrt{y}(x^2 + y) \frac{(J_1(2\sqrt{y}) Y_0(2\sqrt{y}) - J_0(2\sqrt{y}) Y_1(2\sqrt{y}))}{(xJ_0(2\sqrt{y}) - \sqrt{y}J_1(2\sqrt{y}))^2}
\]

and

\[
y \frac{\partial F}{\partial y} = 2y \left[\sqrt{y} + \frac{x^2}{\sqrt{y}}\right] \frac{J_1(2\sqrt{y}) Y_0(2\sqrt{y}) - Y_1(2\sqrt{y}) J_0(2\sqrt{y})}{(xJ_0(2\sqrt{y}) - \sqrt{y}J_1(2\sqrt{y}))^2}
\]

\[
= 2\sqrt{y}(x^2 + y) \frac{(J_1(2\sqrt{y}) Y_0(2\sqrt{y}) - J_0(2\sqrt{y}) Y_1(2\sqrt{y}))}{(xJ_0(2\sqrt{y}) - \sqrt{y}J_1(2\sqrt{y}))^2}.
\]

Then we can conclude that

\[
\Omega_2 \wedge dF = \left[-y \frac{\partial F}{\partial y} - (x^2 + y) \frac{\partial F}{\partial x}\right] dx \wedge dy
\]

\[
= 0.
\]

The claim is proved. □

The claim proves the theorem. □

We apply the above result back to the Hill foliation and prove its integrability.

**Corollary 4.10.** The Hill equation \( u'' + e^z u = 0 \) is \( s \)-integrable. Indeed, the Hill foliation \( \mathcal{H} : \omega = -ydx + xdy + [y^2 + e^z x^2]dz = 0 \) admits the first integral

\[
H(x, y, z) = \frac{2y Y_0(2e^{\frac{z}{e}}) - 2xe^{\frac{z}{e}} Y_1(2e^{\frac{z}{e}})}{y J_0(2e^{\frac{z}{e}}) - xe^{\frac{z}{e}} J_1(2e^{\frac{z}{e}})}.
\]

**Proof.** It follows from the preceding theorem and from the fact that the Hill foliation \( \mathcal{H} \) in \( \mathbb{C}^3 \) is obtained as the pull-back of the foliation \( \Omega_2 = 0 \) by the map \( \Pi(x, y, z) = (\frac{x}{e}, e^z) \). This last admits the first integral

\[
F(x, y) = \frac{2xy Y_0(2\sqrt{y}) - 2\sqrt{y} Y_1(2\sqrt{y})}{xJ_0(2\sqrt{y}) - \sqrt{y}J_1(2\sqrt{y})}
\]

So we may obtain a first integral \( H = \Pi^* F \) for \( \mathcal{H} \) which is of the form

\[
H = \Pi^* F(x, y) = F\left(\frac{y}{x}, e^z\right) = \frac{2y Y_0(2e^{\frac{z}{e}}) - 2xe^{\frac{z}{e}} Y_1(2e^{\frac{z}{e}})}{y J_0(2e^{\frac{z}{e}}) - xe^{\frac{z}{e}} J_1(2e^{\frac{z}{e}})}
\]

\[
= \frac{2y Y_0(2e^{\frac{z}{e}}) - 2xe^{\frac{z}{e}} Y_1(2e^{\frac{z}{e}})}{y J_0(2e^{\frac{z}{e}}) - xe^{\frac{z}{e}} J_1(2e^{\frac{z}{e}})}.
\]

□
5. The Hill foliation

As for the moment we have the following: (1) Starting with the Hill equation \( u'' + e^z u = 0 \) in \( \mathbb{C}^2 \) we can consider the vector field \( X(x, y, z) = y\partial_x - e^x \partial_y + \frac{y}{z} \partial_z \) in \( \mathbb{C}^3 \) which corresponds to the order reduction of the Hill equation. To the vector field \( X \) we associate the integrable one-form \( \omega = -ydx + xdy + [y^2 + e^z x^2]dz \) in \( \mathbb{C}^3 \) showing the strong integrability of the Hill equation \( u'' + e^z u = 0 \). Indeed, the one-form \( \frac{1}{z^2} \omega \) above is the pull-back of the Hill form \( \Omega_2 = -ydx + (x^2 + y)dy \) by the rational map \( \Pi = (-y/x, e^z) \). By its turn the Hill form \( \Omega_2 \) admits a first integral of the form \( F = \frac{2x^2 y_0 (2\sqrt{y}) - 2\sqrt{y} y_1 (2\sqrt{y})}{x^2 (2\sqrt{y}) - \sqrt{y} y_1 (2\sqrt{y})} \) given by Bessel first and second type functions.

5.1. Non-existence of a Liouvillian first integral. Our aim is to show that the Hill form \( \Omega_2 \) admits no Liouvillian first integral on \( \mathbb{CP}^2 \). Firstly, we investigate the existence of invariant algebraic curves.

**Lemma 5.1.** The only invariant algebraic curves for the Hill form \( \Omega_2 \) on \( \mathbb{CP}^2 \) are the line at infinity and the line \( \{ y = 0 \} \).

**Proof.** First of all, the lines \( \{ x = 0 \} \) and the line at infinity are invariant, by straightforward computation. Let us prove that there is no other affine invariant algebraic curve. For this observe that the leaves of \( \Omega_2 \) are transverse to the horizontal 3-dimensional cylinders \( C_r : |y| = r, r \in \mathbb{C}, r > 0 \). We may therefore investigate the existence of periodic orbits for the flow \( \mathcal{L}_r \) induced by \( \Omega_2 \) on \( C_r \). We evoke a result of H. Zoladeck improving former results of other authors:

**Proposition 5.2** (cf. [21] Proposition 4 pp. 166-167). For the system \( dz/dt = z^2 + re^{it} \) there exists a sequence \( r_j \to \infty \) of bifurcation values such that for any \( r \neq r_j \) this equation has exactly one periodic solution (of period \( 2\pi \)) and, for \( r = r_j \), the equation does not have any bounded periodic solution.

Our desired conclude is then basically a consequence of the above proposition. Indeed, it is enough to consider the two dimensional real ODE obtained from the original equation \( -ydx + (x^2 + y)dy = 0 \) by making \( y = re^{it} \) which leads to the complex ODE, \( x' = i(x^2 + re^{it}) \) corresponding to the restriction to \( C_r \).

We shall need the following result due to M. Singer:

**Theorem 5.3** ([19] Theorem 1). A polynomial differential equation \( Pdy - Qdx = 0 \) with complex coefficients admits a Liouvillian first integral iff the 1-form \( \omega = Pdy - Qdx \) admits an integrating factor of the form \( R = \exp \int Udx + Vdy \) where \( U, V \) are rational functions satisfying \( \frac{\partial U}{\partial y} = \frac{\partial V}{\partial x} \).

If we put in the above statement \( \omega = Pdy - Qdx \) and \( \eta = Udx + Vdy \) then Singer’s result says that: the existence of a Liouvillian first integral for \( \omega \) corresponds to the existence of a closed rational 1-form \( \eta \) which satisfies \( d\omega = \eta \wedge \omega \). In this case, the first integral is of the form \( F = \int \alpha \) where \( \alpha = \Omega / \exp \int \eta \).

Using now that the affine poles of a 1-form \( \eta \) as above must be invariant algebraic curves for \( \omega = Pdy - Qdx \) (cf. [6] Lemma 1) for \( \Omega \) we can prove:

*Alternatively we may consider the vertical cylinders \( V_{r} : |x| = r, |y| < \epsilon \) for \( \epsilon > 0 \) small enough. On each such cylinder \( V_{r} \) we have a induced transversely complex analytic flow \( V_{r} \) admitting a periodic orbit \( \gamma \) obtained by the intersection \( V_{r} \cap (y = 0) \). This periodic orbit has a holonomy map which is a parabolic complex diffeomorphism. By investigating the periodic points of such map we may conclude.*
Case cannot occur.

Theorem C. The Hill form
\[ \Omega_2 = -ydx + [x^2 + y]dy \]
adopts no first integral of Liouvillean type.

Proof. According to the above discussion it is enough to prove that there is no closed rational 1-form \( \eta \) satisfying

\[ d\Omega_2 = \eta \wedge \Omega_2. \tag{16} \]

Let us write \( \eta = A(x, y)dx + B(x, y)dy \). Then
\[ d\Omega_2 = -dy \wedge dx + 2xdx \wedge dy = (1 + 2x)dx \wedge dy \]
and,
\[ \eta \wedge \Omega_2 = [A dx + B dy] \wedge [-y dx + [x^2 + y] dy] = ([x^2 + y]A + yB) dx \wedge dy. \]

Hence, from (16) we get
\[ (1 + 2x) = (x^2 + y)A + yB. \tag{17} \]

By Lemma 5.1 the affine poles of \( \eta \) are contained in the line \( \{ y = 0 \} \). Since \( \eta \) is closed, by the Integration lemma (14) we can write
\[ \eta = \lambda \frac{dy}{y} + d\left( \frac{Q}{y^n} \right). \]

for some \( \lambda \in \mathbb{C} \) and some irreducible polynomial \( Q(x, y) \) of degree \( n \in \mathbb{N} \). Hence,
\[ \eta = \lambda \frac{dy}{y} + d\left( \frac{Q}{y^n} \right) = \lambda \frac{dy}{y} + \frac{1}{y^n} \left( \frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial y} dy \right) + Q \left( -\frac{n}{y^{n+1}} \right) dy \]
\[ = \frac{1}{y^n} \frac{\partial Q}{\partial x} dx + \left( \frac{\lambda}{y} + \frac{1}{y^n} \frac{\partial Q}{\partial y} - \frac{nQ}{y^{n+1}} \right) dy. \]

It follows from (17) that
\[ 1 + 2x = \frac{x^2 + y}{y^n} \frac{\partial Q}{\partial x} + \lambda + y \left( \frac{1}{y^n} \frac{\partial Q}{\partial y} - \frac{nQ}{y^{n+1}} \right). \]

Then \( \lambda = 1 \). By writing \( Q(x, y) = \sum_{i=0}^{n} p_i(y)x^i \) we get
\[ 2x = \frac{x^2 + y}{y^n} \sum_{i \geq 1} ip_i(y)x^{i-1} + \frac{1}{y^{n+1}} \sum_{i \geq 0} p_i'(y)x^i - \frac{n}{y^n} \sum_{i \geq 0} p_i(y)x^i. \tag{18} \]

Case \( n = 0 \): Equation (18) is given by \( x^2 p_1(y) + yp_1(y) + yp_0'(y) = 2x \). Analyzing the coefficients of \( x^2 \) we get \( p_1 \equiv 0 \). Then, equation (18) would write as \( yp_0'(y) = 2x \). Therefore, this case cannot occur.

Case \( n = 1 \): Equation (18) is given by \( \frac{p_1(y)}{y} x^2 + p_1(y) + 2\frac{p_2(y)}{y} x^3 + 2p_2(y)x + p_0'(y) + p_1'(y)x = 2x \).
Analyzing the coefficients of \( x^2 \) and \( x^3 \) we conclude that \( p_1, p_2 \equiv 0 \). This implies \( p_0'(y) = 2x \). Again this case cannot occur.

Case \( n = 2 \): Equation (18) is given by \( \frac{x^2 + y}{y^2} [p_1(y) + 2xp_2(y) + 3x^2p_3(y)] + \frac{1}{y} [p_0'(y) + xp_1'(y) + x^2p_2'(y)] - \frac{2}{y^2} [p_0(y) + xp_1(y) + x^2p_2(y)] = 2x \).
Analyzing each power of \( x \) we obtain
\[ x^0 : \frac{2}{y^2} p_1(y) + \frac{1}{y} p_0'(y) - \frac{2}{y} p_0(y) = 0. \]
We consider the quadratic blow-up at the origin 0 and refer to [5, 12, 18]. Let us check which models of singularities arise after a number of blow-ups.

To extract more information about the Hill form. For more details in this technique we refer to the next section.

Case $n = 3$ : Equation (18) is given by $x^3 + \frac{y}{y^n} [p_1(y) + 2xp_2(y) + \cdots + (n-1)x^{n-2}p_{n-1}(y) + nx^{n-1}p_n(y) + (n+1)x^n p_{n+1}(y)]$.

The above equations imply that $p_1, p_2, p_3 \equiv 0$. Then, we obtain $\frac{1}{y} p'_0(y) - \frac{2}{y^2} p_0(y) = 2x$. Hence this case does not occur.

Case $n > 2$ : Equation (18) writes

\[ \frac{x^2}{y} [p_1(y) + 2xp_2(y) + \cdots + (n-1)x^{n-2}p_{n-1}(y) + nx^{n-1}p_n(y) + (n+1)x^n p_{n+1}(y)] \]

Again, the analysis of each power of $x$ gives

The above equations imply that $p_1, p_2, p_3, p_4 \equiv 0$. Then, we obtain $\frac{1}{y} p'_0(y) - \frac{3}{y^2} p_0(y) = 2x$. Therefore this case is not possible.

Thus, the general case $n > 2$ is also excluded. This ends the proof.

5.2. Reduction of singularities. In what follows we make use of the blow-up technique in order to extract more information about the Hill form. For more details in this technique we refer to [5, 12, 18]. Let us check which models of singularities arise after a number of blow-ups. We consider the quadratic blow-up at the origin $0 \in \mathbb{C}^2$ as pattern in what follows.

**Proposition 5.4.** Let

\[ \Omega_2 = -ydx + (x^2 + y)dy \]
be the Hill form. Then,

(i) The first blow-up originates two singularities in the exceptional divisor. One singularity is of Siegel type of the form

\[ -ydx + (1 - x + x^2y)dy = 0. \]

The second singularity is nilpotent and given by:

\[ (-y + yx + y^2)dx + (x^2 + yx)dy = 0. \]

(ii) After performing \( n > 1 \) blow-ups starting from the second singularity and always repeating the process on the corresponding nilpotent singularity we have: an exceptional divisor \( \mathbb{E} = \bigcup_{j=1}^{n} \mathbb{E}_j \) with a nilpotent singularity at the intersection of the strict transform of the \( x \)-axis and \( \mathbb{E}_n \), which is given by:

\[ (-y + nyx + nx^{n-1}y^2)dx + (x^2 + yx^n)dy. \]

Moreover, at each corner \( \mathbb{E}_j \cap \mathbb{E}_{j+1} \), we have a Siegel type singularity given by:

\[ (-y + jxy^2 + jx^{j-1}y^{j+1})dx + (-x + (j + 1)x^2y + (j + 1)x^jy^j)dy. \]

The proof of Proposition 5.4 is a straightforward computation with blow-ups and we chose to omit it. For mode details in the subject of reduction of singularities we refer to [5] or [18].

Next we give the explicit first integrals for the models arising in the reduction of singularities of the Hill form as described in Proposition 5.4.

**Proposition 5.5.**

(i) Let \( n \geq 1 \) be given. The singularity

\[ (-y + nyx + nx^{n-1}y^2)dx + (x^2 + yx^n)dy = 0 \]

admits a first integral

\[ F_n(x, y) = \frac{2xyY_0(2x^2y^{\frac{1}{2}}) - 2x^2y^{\frac{1}{2}}Y_1(2x^2y^{\frac{1}{2}})}{xJ_0(2x^2y^{\frac{1}{2}}) - x^2y^{\frac{1}{2}}J_1(2x^2y^{\frac{1}{2}})}. \]

(ii) Let \( j \in \{1, \ldots, n-1\} \) be given. The Siegel type singularity

\[ (-y + jxy^2 + jx^{j-1}y^{j+1})dx + (-x + (j + 1)x^2y + (j + 1)x^jy^j)dy \]

admits the first integral

\[ G_j(x, y) = \frac{2xyY_0(2x^2y^{\frac{j+1}{2}}) - 2x^2y^{\frac{j+1}{2}}Y_1(2x^2y^{\frac{j+1}{2}})}{xyJ_0(2x^2y^{\frac{j+1}{2}}) - x^2y^{\frac{j+1}{2}}J_1(2x^2y^{\frac{j+1}{2}})}. \]

**Proof.** Let \( \pi_n(x, y) = (x, x^n y) \), \( \Omega_2 = -ydx + (y + x^2)dy \) then \( \pi_n^* \Omega = x^n[(-y + nyx + nx^{n-1}y^2)dx + (x^2 + yx^n)dy] \). Now we apply Theorem 13 and obtain (i). Now, let us define \( \pi(x, y) = (xy, y) \). Let

\[ \xi_j = (-y + jyx + jx^{j-1}y^2)dx + (x^2 + yx^j)dy. \]

Then \( \pi^* \xi_j = y[(-y + jxy^2 + jx^{j-1}y^{j+1})dx + (-x + (j + 1)x^2y + (j + 1)x^jy^j)dy] \). From this pull-back together with (i) we get (ii).

Let us introduce a class of functions that may be useful. Denote by \( A(\mathcal{F}) \) the algebra of functions generated by compositions of first and second type (one variable) Bessel functions and complex valued algebraic functions of two complex variables. Also denote by \( K(\mathcal{F}) \) the field of fractions of \( A(\mathcal{F}) \). Finally, we have the following definition:

**Definition 5.6.** We shall say that a function \( F(x, y) \) of two complex variables is generated by Bessel functions if it belongs to the field \( K(\mathcal{F}) \).
Question 5.7. Let \( A(x, y)dx + B(x, y)dy = 0 \) be a germ of Siegel type singularity at the origin \( 0 \in \mathbb{C}^2 \). Under what conditions can we assure the existence of a first integral generated by Bessel functions?

Notice that a function \( F(x, y) \in K(J) \) always defines a uniform complex analytic function in some dense open subset of the complex plane \( \mathbb{C}^2 \).

5.3. Holonomy. If we consider the Hill form \( \Omega_2 = -ydx + (x^2 + y)dy \) then we have a singularity at the origin \( 0 \in \mathbb{C}^2 \). This germ of foliation admits a separatrix given by \( \Gamma : (y = 0) \). We may ask about the holonomy map of this separatrix. If we consider a simple loop \( \gamma : x(t) = r_0 e^{it}, \ 0 \leq t \leq 2\pi \) in \( \Gamma \) then we may consider the holonomy map of \( \Gamma \) with respect to the transverse section \( \Sigma : (x = r_0) \) as follows: let \( q_0 \) be the point \( (r_0, 0) \). The holonomy map \( h : (\Sigma, q_0) \to (\Sigma, q_0) \) is given by \( h(y_0) = y(2\pi, y_0) \) where \( y(t, y_0) \) is the solution of the ODE

\[
\frac{dy}{dt} = \frac{y}{r_0 e^{2it} + y} rie^t
\]

that satisfies \( y(0, y_0) = y_0 \). The first integral \( F(x, y) = \frac{2x y_0 (2\sqrt{y} - 2\sqrt{y} \gamma (2\sqrt{y}))}{x y_0 (2\sqrt{y} - 2\sqrt{y} \gamma (2\sqrt{y}))} \) may be used to calculate (study) the holonomy map \( h \), for it satisfies the relation \( F(r_0, h(y)) = F(r_0, y) \). This is an expression involving Bessel type functions.

All this suggests that there may be a theory of foliations admitting Bessel type functions as first integrals, in terms of their holonomy groups.

5.4. The general case. Now we turn our attention to the general case of the Hill equation: \( u'' + p(z)u = 0 \) where \( p(z) \) is a periodic complex analytic function defined in the complex plane. We may assume that \( p \) is not constant and of period \( 2\pi i \). By ordinary covering spaces theory we can write \( p(z) = f(e^z) \) for some complex analytic function \( f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \). The corresponding Hill foliation is then given by \( \omega = -ydx + xdy + [y^2 + p(z)x^2]dz = -ydx + xdy + [y^2 + f(e^z)x^2]dz \).

We may rewrite

\[
\omega/x^2 = (-ydx + xdy)/x^2 + [(y/x)^2 + f(e^z)]dz
\]

If we put \( \varphi = -y/x \) and \( \psi = e^z \) then we have \( dz = d\psi/\psi \) and then

\[
\omega/x^2 = -d\varphi + (\varphi^2 + f(\psi))d\psi/\psi
\]

This shows that the foliation is the pull-back by the map \( (\phi, \psi) \) of the two dimensional model

\[
\theta_2 = -ydx + (x^2 + f(y))dy = 0.
\]

6. Integrability and classical solutions

G. W. Hill has constructed periodic solutions for (5) in the form of trigonometric series

\[
u(z) = \sum_{k=0}^{\infty} A_{2k+1} \cos((2k + 1)z), \quad v(z) = \sum_{k=0}^{\infty} B_{2k+1} \sin((2k + 1)z).
\]

where the coefficients \( A_{2k+1}, B_{2k+1} \) are represented by power series with respect to \( m \). Hill has not studied their convergence though. The first attempt to estimate the convergence radius was done by Lyapunov [14].

Let us explore some of these ideas in our framework. We start by the following useful result.

Theorem 6.1. Let \( w_1, w_2 \) be two linearly independent solutions of the Hill equation \( u'' + p(z)u = 0 \) defined in a strip \( A \subseteq \mathbb{C} \). Then

\[
H(x, y, z) = \frac{xw'_1(z) - yw_1(z)}{xw'_2(z) - yw_2(z)}
\]

is a first integral for the Hill foliation

\[
\omega_H = -ydx + xdy + [y^2 + p(z)x^2]dz = 0
\]
admits the following first integral

Proof. First observe that $H$ is not constant. Indeed, $H = \frac{\gamma}{x} \in \mathbb{C}$ implies $x(w'_1(z) - cw'_2(z)) = y(w_1(z) - cw_2(z))$ and then $w_1(z) = cw_2(z)$ which is a contradiction since $w_1$ and $w_2$ are linearly independent. Now we assume that there is a (real analytic nondegenerate) curve $\Gamma$ of indefinite points for $H$. Then we have $\frac{w'_1}{w'_2} = \frac{y}{x} = \frac{w'_1}{w'_2}$ in this curve. This implies that $\ln w_1 = \ln w_2 + c$ for some constant and therefore $w_1 = kw_2$ for some constant $k$ along this curve. Since $w_1$ and $w_2$ are one variable complex analytic functions this implies that $w_1 = kw_2$ identically, yielding another contradiction. This shows that the set of indefinite points of $H$ is discrete. Since we are in dimension 3 this shows that $H$ is indeed free of indefinite points (Hartogs’ theorem for instance). Let us now prove that $H$ is actually a first integral for $\mathcal{H} : -ydx + xdy + [y^2 + p(z)x^2]dz = 0$. We start with the model in dimension two, i.e., the Hill form

$$\Omega_H = -dx + [x^2 + p(y)]dy.$$ 

We put

$$F(x, y) = \frac{w'_1(y) + xw_1(y)}{w'_2(y) + xw_2(y)}.$$ 

Now we compute the partial derivatives of $F$ with respect to $x$ and $y$:

$$\frac{\partial F(x, y)}{\partial x} = \frac{w_1(w'_2 + xw_2) - w_2(w'_1 + xw_1)}{(w'_2 + xw_2)^2}, \quad \frac{\partial F(x, z)}{\partial x} = \frac{w_1 w'_2 - w_2 w'_1}{(w'_2 + xw_2)^2}.$$ 

Thus,

$$\frac{\partial F(x, z)}{\partial x} = \frac{w'_1 w'_2 + xw'_1 w_2 - w_2 w'_1}{(w'_2 + xw_2)^2} = \frac{(w''_1 + xw'_1)(w'_2 + xw_2) - (w''_2 + xw'_2)(w'_1 + xw_1)}{(w'_2 + xw_2)^2} - \frac{(w'_1 + xw_1)(w'_2 + xw_2)}{(w'_2 + xw_2)^2}$$

$$= \frac{(-p w_2 + xw'_1)(w'_2 + xw_2) - (-p w_1 + xw'_2)(w'_1 + xw_1)}{(w'_2 + xw_2)^2}.$$ 

Therefore, $dF \wedge \Omega_H = \left(\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy\right) \wedge (-dx + (p + x^2)dy) = \left[(p + x^2)\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\right]dx \wedge dy = 0.$

This shows that $F$ is a first integral for $\Omega_H$. Now we use the pull-back, i.e., define $H = \Pi^*F$, where $\Pi(x, y, z) = (-\frac{y}{x}, z)$, obtaining then $dH \wedge \omega_H = d(\Pi^*F) \wedge \Pi^*\Omega_H = \Pi^*(dF \wedge \Omega_H) = 0.$ Explicitly we have

$$H(x, y, z) = \frac{w'_1(z) - \frac{y}{x}w_1(z)}{w'_2(z) - \frac{y}{x}w_2(z)} = \frac{yw_1(z)}{xw_2(z) - yw_2(z)}$$

is a first integral for $\omega_H$. \hfill \Box

As a straightforward interesting consequence of the above theorem is the Bessel-Hill case below:

**Corollary 6.2 (Bessel-Hill form).** The Bessel-Hill foliation

$$\omega_H = -ydx + xdy + [y^2 - (r^2 - k^2e^2 z)x^2]dz$$

admits the following first integral

$$H(x, y, z) = \frac{xY'_r(ke^z) - yY_r(ke^z)}{xJ'_r(ke^z) - yJ_r(ke^z)}.$$ 

Proof. Indeed, this follows from the above theorem by observing that $J_r(ke^z), Y_r(ke^z)$ are (well-known classical) solutions of the Bessel equation $u'' + (r^2 - k^2e^2 z)u = 0.$ \hfill \Box
Remark 6.3. We shall point-out that the introduction of the perturbation $r_2 - k^2 e^{2z}$ instead of the function $e^{2z}$ in the Hill equation, generates an important change in the first integral. Indeed, the constant is directly connected to the order of the Bessel functions in the first integral. For instance, when $r = 1/2$, we have $J_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \sin z$ which is a Liouvillian function and the first integral is Liouvillian as well.

7. Laurent-Fourier type formal first integral for Hill foliations

We turn again our attention to the basic complex Hill equation $u''(z) + p(z)u(z) = 0$ where $p$ is a complex analytic periodic function defined in a strip $A \subseteq \mathbb{C}$, containing the real axis $\Im(z) = 0$. We shall apply Floquet theory in order to obtain some formal solutions. Order reduction of the problem is given by $x = u, y = u'$ and then we obtain the linear first order problem $X' = AX$ where $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 1 \\ -p(z) & 0 \end{pmatrix}$. The characteristic equation of the matrix $A$ is $\lambda^2 + p(z) = 0$ which has symmetric roots. Applying then the classical Floquet theory we conclude that a solution of the Hill equation is of the form

$$\varphi_1(z) = c_1 e^{\mu z} P(z)$$

where $c_1, \mu$ are constants ($\mu$ is a characteristic exponent) and $P(z)$ is a periodic complex valued analytic function (see for instance [11] page 244). Moreover, another solution to (5), linearly independent with respect to $\varphi_1$ is given by

$$\varphi_2(z) = c_1 e^{-\mu z} P(-z)$$

An immediate corollary of Theorem 6.1 is that we have a first integral of the Hill foliation $H : \omega_H = -ydx + xdy + [y^2 + p(z)x^2] = 0$, of parameter $p$ as follows

$$H_p(x, y, z) = \frac{x \varphi_1'(z) - y \varphi_1(z)}{x \varphi_2'(z) - y \varphi_2(z)}.$$ 

Replacing then (23) e (24) in the above expression we obtain:

Corollary 7.1. The Hill foliation of parameter $p(z)$, periodic and complex analytic in the strip $A \subseteq \mathbb{C}$, admits a first integral of the form:

$$H_p(x, y, z) = e^{2\mu z} \frac{x \mu P(z) + x P'(z) - y P(z)}{-x \mu P(-z) + x P'(-z) - y P(-z)},$$

where $P(z)$ is a periodic complex valued analytic function and $\mu$ is a constant.

It is well-known that a periodic complex analytic function $P(z)$ defined in a strip $A \subseteq \mathbb{C}$ containing $\Im(z) = 0$ admits a Fourier series expansion. Indeed, if the period is of $P$ is $2\pi i$, we have $P(z) = \sum_{k=-\infty}^{\infty} a_k e^{kz}$. Replacing the series of $P'(z), P(-z)$ and $P'(-z)$ in the expression of $H_p(x, y, z)$ above we obtain ($\Sigma_k$ stands for $\sum_{k=-\infty}^{\infty}$):

Theorem D. Let $p(z)$ be a complex analytic function periodic in a strip $A \subseteq \mathbb{C}$ containing the real axis $\Im(z) = 0$. Then the corresponding Hill foliation $-ydx + xdy + [y^2 + p(z)x^2] = 0$ admits a Laurent-Fourier type formal first integral given by an expression

$$H_p(x, y, z) = e^{2\mu z} \frac{x \mu \sum_k a_k e^{kz} + x \sum_k k a_k e^{kz} - y \sum_k k a_k e^{kz}}{-x \mu \sum_k a_k e^{-kz} - x \sum_k k a_k e^{-kz} - y \sum_k a_k e^{-kz}}.$$ 

Remark 7.2. The term formal in the above statement has to be clarified. Indeed, we are talking about quotients of formal Laurent series (i.e., quotients of series with an infinite number of terms with positive and negative exponents). The convergence of such series must be a subject of a deeper discussion. Anyway, once we have made the above convention, we shall refer
to the expression $H_p$ as \textit{formal first integral for the Hill foliation}. The “change of coordinates” \( \Pi(x,y,z) = (u,v) \) where \( u = -\frac{y}{x} \) and \( v = e^z \) gives the following Laurent-type formal first integral:

\[
H = \Pi^* F(u,v) = v^{2\mu} \frac{\mu \sum_k a_k v^k + \sum_k k a_k v^{k+1} + u \sum_k a_k v^k}{-\mu \sum_k a_k v^{-k} - \sum_k k a_k v^{-k} + u \sum_k a_k v^{-k}}.
\]  

(26)

REFERENCES

[1] V.I. Arnold; Geometrical Theory of Differential Equations, Springer-Verlag, New York (1983).
[2] V.I. Arnold; Ordinary Differential Equations, Springer Textbook, Springer Science & Business Media, 1992.
[3] V.I. Arnold; \textit{Remarks on the perturbation theory for problems of Mathieu type.} Uspekhi Mat. Nauk, 38:4(232) (1983), 189–203; Russian Math. Surveys, 38:4 (1983), 215–233
[4] C. Camacho, A. Lins Neto and P. Sad; \textit{Foliations with algebraic limit sets}; Ann. of Math. 136 (1992), 429–446.
[5] C. Camacho and P. Sad; \textit{Invariant Varieties Through Singularities of Holomorphic Vector Fields}; Annals of Mathematics Second Series, Vol. 115, no. 3 (1982), 579–595
[6] C. Camacho, B. Scárdua; \textit{Holomorphic foliations with Liouvillian first integrals}, Ergodic Theory and Dynamical Systems (2001), 21, pp.717-756.
[7] G. Floquet; \textit{Sur les équations différentielles linéaires à coefficients périodiques}, Annales Scientifiques de L’É.N.S (12) (1883), 47–88
[8] T. W. Gamelin, Complex Analysis, Springer 2001.
[9] H. Hochstadt; \textit{Results, old and new, in the Theory of Hill’s equation}, Transactions of the New York Academy of Sciences, 1964. https://doi.org/10.1111/j.2164-0947.1964.tb02962.x
[10] G.W. Hill; \textit{On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and moon}, Cambridge, Mass., Press of J. Wilson and son, 1877.
[11] E. Hille; Ordinary Differential Equation in the Complex Domain, 1894, John Wiley and Sons, Inc.
[12] Yu. Ilyashenko & S. Yakovenko. Lectures on analytic differential equations. Graduate Studies in Mathematics. American Mathematical Society (December 27, 2007).
[13] B.G Korenev; Bessel functions and their applications, Taylor and Francis LTD, London, UK, 2002.
[14] A. M. Lyapunov; \textit{On the series proposed by Hill for representation of Moon’s motion}, Academy of Sciences of the USSR, 1 (1956), 418–446.
[15] W. Magnus, S. Winkler; Hill’s Equation, Interscience Tracts in Pure and Applied Mathematics, 1966.
[16] N.W. McLachlan; Theory and application of Mathieu functions, 1947 Oxford, UK.
[17] A. D. Polyanin, V. F. Zaitsev; Handbook of Ordinary Differential Equations: Exact Solutions, Methods, and Problems. Chapman and Hall CRC, (2018).
[18] A. Seidenberg; \textit{Reduction of singularities of the differential equation Ady = Bdx}, Amer. J. Math. 90 (1968), 248–269.
[19] M.F. Singer; \textit{Liouvillian first integrals of differential equations}, Transactions of the American Mathematical Society 333 (2) (1992) 673–688.
[20] G.N. Watson; \textit{A Treatise on the Theory of Bessel Functions}, Cambridge University Press (1922).
[21] H. Zoladeck; \textit{The Method of Holomorphic Foliations in Planar Periodic Systems: The Case of Riccati Equations}, Journal of Differential Equations Volume 165, Issue 1, 20 July 2000, Pages 143-173.