A Compact Homogeneous S-space∗

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Abstract
Under the continuum hypothesis, there is a compact homogeneous strong S-space.

1 Introduction
A space $X$ is hereditarily separable (HS) iff every subspace is separable. An $S$-space is a regular Hausdorff HS space with a non-Lindelöf subspace. A space $X$ is homogeneous iff for every $x, y \in X$ there is a homeomorphism $f$ of $X$ onto $X$ with $f(x) = y$. Under CH, several examples of $S$-spaces have been constructed, including topological groups (see [4]) and compact $S$-spaces (see [7]). It is asked in [1, 5] whether there are compact homogeneous $S$-spaces. As we shall show in Theorem 4.2, there are under CH. This cannot be done in ZFC, since there are no compact $S$-spaces under MA + ¬CH (see [12]); there are no $S$-spaces at all under PFA (see [13]).

In Section 2 we use a slightly modified version of the construction in [7, 10] to refine the topology of any given second countable space, and turn it into a first countable strong $S$-space (i.e., each of its finite powers is an $S$-space). In Section 3 we show that if the original space is compact, then there is a natural compactification of the new space which is also a first countable strong $S$-space. If in addition the original space is zero-dimensional, then the $\omega$th power of this compactification will be homogeneous by Motorov [9], proving Theorem 4.2.

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2 A Strong S-Space

If $\tau$ is a topology on $X$, we write $\tau^I$ for the corresponding product topology on $X^I$; likewise if $\tau' \subseteq \tau$ is a base we write $(\tau')^I$ for the natural corresponding base for $\tau^I$. If $E \subseteq X$, then $\text{cl}(E, \tau)$ denotes the closure of $E$ with respect to the topology $\tau$. This notation will be used when we are discussing two different topologies on the same set $X$.

The following two lemmas are well-known; the second is Lemma 7.2 in [10]:

**Lemma 2.1** If $X$ is HS and $Y$ is second countable, then $X \times Y$ is HS.

**Lemma 2.2** $X^\omega$ is HS iff $X^n$ is HS for all $n < \omega$.

The next lemma, an easy exercise, is used in the proof of Theorem 2.4:

**Lemma 2.3** If $(x, y) \in X \times Y$ and $S \subseteq X \times Y$, then $(x, y) \in \text{cl}(S)$ iff $y \in \text{cl}(\pi(S \cap (U \times Y)))$ for all neighborhoods $U$ of $x$, where $\pi : X \times Y \rightarrow Y$ is projection.

The following is proved (essentially) in [10], but our proof below may be a bit simpler:

**Theorem 2.4** Assume CH. Let $\rho$ be a second countable $T_3$ topology on $X$, where $|X| = \aleph_1$. Then there is a finer topology $\tau$ on $X$ such that $(\omega_1, \tau)$ is a first countable locally compact strong S-space.

**Proof.** WLOG, $X = \omega_1$. For $\eta < \omega_1$ we write $\rho_\eta$ for the topology of $\eta$ as a subspace of $(\omega_1, \rho)$. Applying CH, list $\bigcup_{0 < n < \omega} [\omega_1]^n \subseteq \omega$ as $\{S_\mu : \mu \in \omega_1\}$, so that each $S_\mu \subseteq \mu^{n(\mu)}$ for some $n(\mu)$ with $0 < n(\mu) < \omega$.

For $\eta \leq \omega_1$ we construct $\tau_\eta$ a topology on $\eta$ by induction on $\eta$ so as to make the following hold for all $\xi < \eta \leq \omega_1$:

1. $\tau_\xi = \tau_\eta \cap \mathcal{P}(\xi)$.
2. $\tau_\eta$ is first countable, locally compact, and $T_3$.
3. $\tau_\eta \supseteq \rho_\eta$. 

...
Note that (1) implies in particular that $\xi \in \tau_\eta$; that is, $\xi$ is open. Thus, if $\tau = \tau_{\omega_1}$, then $(\omega_1, \tau)$ is not Lindelöf. Also by (1), $\tau_\eta$ for limit $\eta$ is determined from the $\tau_\xi$ for $\xi < \eta$. So, we need only specify what happens at successor ordinals.

For $n \geq 1$ and $\xi < \omega_1$, let $Iseq(n, \xi)$ be the set of all $f \in (\omega_1)^n$ which satisfy $f(0) < f(1) < \cdots < f(n-1) = \xi$. The following condition states our requirement on $\tau_{\xi+1}$:

4. For each $\mu < \xi$ and each $f \in Iseq(n, \xi)$, where $n = n_\mu$:

$$f \in \text{cl}(S_\mu, (\tau_{\xi+1})^{n-1} \times \rho) \implies f \in \text{cl}(S_\mu, (\tau_{\xi+1})^n).$$

If $n = n_\mu = 1$, then $(\tau_{\xi+1})^{n-1} \times \rho$ just denotes $\rho$. That is, (4) requires

$$\xi \in \text{cl}(E, \rho) \implies \xi \in \text{cl}(E, \tau_{\xi+1}) \quad (\ast)$$

for all $E$ in the countable family $\{S_\mu : \mu < \xi \& n(\mu) = 1\}$. It is standard (see [7]) that one may define $\tau_{\xi+1}$ so that this holds. Now, consider (4) in the case $n = n_\mu \geq 2$. By (2), $\tau_\xi$ is second countable, so let $\tau'_\xi$ be a countable base for $\tau_\xi$. Applying Lemma 2.3, (4) will hold if whenever $U = U_0 \times \cdots \times U_{n-2} \in (\tau'_\xi)^{n-1}$ is a neighborhood of $f \upharpoonright (n-1)$,

$$\xi \in \text{cl}(\pi(S_\mu \cap (U \times (\xi + 1))), \rho) \implies \xi \in \text{cl}(\pi(S_\mu \cap (U \times (\xi + 1))), \tau_{\xi+1}),$$

where $\pi : \xi^{n-1} \times (\xi + 1) \to (\xi + 1)$ is projection. But this is just a requirement of the form (\ast) for countably many more sets $E$, so again there is no problem meeting it.

Now, we need to show that $\tau^n$ is HS for each $0 < n < \omega$. We proceed by induction, so assume that $\tau^m$ is HS for all $m < n$. Fix $A \subseteq (\omega_1)^n$; we need to show that $A$ is $\tau^n$-separable. Applying the induction hypothesis, we may assume that each $f \in A$ has all coordinates distinct. Also, since permutation of coordinates induces a homeomorphism of $(\omega_1)^n$, we may assume that each $f \in A$ is strictly increasing; that is, $f \in Iseq(n, \xi)$, where $\xi = f(n-1)$. By the induction hypothesis and Lemma 2.1, $A$ is separable in $(\tau_{\xi+1})^{n-1} \times \rho$. We can then fix $\mu$ such that $n(\mu) = n$, $S_\mu \subseteq A$, and $S_\mu$ is $(\tau_{\xi+1})^{n-1} \times \rho$-dense in $A$. Now, say $f \in A$ with $\xi = f(n-1) > \mu$. Applying (4), we have $f \in \text{cl}(S_\mu, \tau^n)$. Thus, $A \setminus \text{cl}(S_\mu, \tau^n)$ is countable, so $A$ is $\tau^n$-separable.  
\[\spadesuit\]
3 Compactification

Definition 3.1 If \( \varphi \) is a continuous map from the \( T_2 \) space \( Y \) into \( X \), then \( Y \dot{\cup} \varphi X \) denotes the disjoint union of \( X \) and \( Y \), given the topology which has as a base:

a. All open subsets of \( Y \), together with
b. All \([U, K] := U \cup (\varphi^{-1}U \setminus K)\), where \( U \) is open in \( X \) and \( K \) is compact in \( Y \).

Our main interest here is in the case where \( X \) is compact and \( Y \) is locally compact. Then, if \(|X| = 1\), we have the 1-point compactification of \( Y \), and if \( Y \) is discrete and \( \varphi \) is a bijection we have the Aleksandrov duplicate of \( X \).

Lemma 3.2 Let \( Z = Y \dot{\cup} \varphi X \), with \( X, Y \) Hausdorff:

1. \( X \) is closed in \( Z \), \( Y \) is open in \( Z \), and both \( X, Y \) inherit their original topology as subspaces of \( Z \).
2. If \( Y \) is locally compact, then \( Z \) is Hausdorff.
3. If \( X \) is compact, then \( Z \) is compact.
4. If \( X, Y \) are first countable, \( X \) is compact, \( Y \) is locally compact, and each \( \varphi^{-1}(x) \) is compact, then \( Z \) is first countable.
5. If \( X, Y \) are zero dimensional, \( X \) is compact, and \( Y \) is locally compact, then \( Z \) is zero dimensional.
6. If \( X \) is second countable and \( Y^\omega \) is HS, then \( Z^\omega \) is HS.

Proof. For (3): If \( U \) is a basic open cover of \( Z \), then there are \( n \in \omega \) and \([U_i, K_i] \in U \) for \( i < n \) such that \( \bigcup_{i<n} U_i = X \). Thus, \( \bigcup_{i<n}[U_i, K_i] \) contains all points of \( Z \) except for (possibly) the points in the compact set \( \bigcup_{i<n} K_i \subseteq Y \).

For (4): \( Z \) is compact Hausdorff and of countable pseudocountable character.

For (5): \( Z \) is compact Hausdorff and totally disconnected.

For (6): By Lemma 2.2 it is sufficient to prove that each \( Z^n \) is HS. But \( Z^n \) is a finite union of subspaces of the form \( X^j \times Y^k \), which are HS by Lemma 2.1.
4 Homogeneity

The following was proved by Dow and Pearl [3]:

**Theorem 4.1** If $Z$ is first countable and zero dimensional, then $Z^\omega$ is homogeneous.

Actually, we only need here the special case of this result where $Z$ is compact and has a dense set of isolated points; this was announced (without proof) earlier by Motorov [9].

Note that by Šapirovskii [11], any compact $HS$ space must have countable $\pi$-weight (see also [6], Theorem 7.14), so if it is also homogeneous, it must have size at most $2^{\aleph_0}$ by van Douwen [2]. Under CH this implies, by the Čech – Pospīšil Theorem, that the space must be first countable.

**Theorem 4.2** (CH) There is a (necessarily first countable) zero-dimensional compact homogeneous strong $S$-space.

**Proof.** Let $X$ be the Cantor set $2^\omega$ with its usual topology, let $Y$ be $2^\omega$ with the topology constructed in Theorem 2.4 let $\varphi$ be the identity, and let $Z = Y \cup_\varphi X$. By Lemma 3.2, $Z$, and hence also $Z^\omega$, are zero-dimensional first countable compact strong $S$-spaces; $Z^\omega$ is homogeneous by Theorem 4.1. ✩

No compact topological group can be an $S$-space or an $L$-space. However under CH there are, by [8], compact $L$-spaces which are right topological groups (i.e. they admit a group operation such that multiplication on the right by a fixed element defines a continuous map). We do not know whether there can be compact $S$-spaces which are right topological groups.
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