A NILPOTENT GROUP WITHOUT LOCAL FUNCTIONAL EQUATIONS FOR PRO-ISOMORPHIC SUBGROUPS

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Abstract. The pro-isomorphic zeta function \( \zeta_\wedge^\Gamma(s) \) of a torsion-free finitely generated nilpotent group \( \Gamma \) enumerates finite index subgroups \( \Delta \leq \Gamma \) such that \( \Delta \) and \( \Gamma \) have isomorphic profinite completions. It admits an Euler product decomposition \( \zeta_\wedge^\Gamma(s) = \prod_p \zeta_\wedge^\Gamma,p(s) \).

We manufacture the first example of a torsion-free finitely generated nilpotent group \( \Gamma \) such that the local Euler factors \( \zeta_\wedge^\Gamma,p(s) \) do not satisfy functional equations. The group \( \Gamma \) has nilpotency class 4 and Hirsch length 25. It is obtained, via the Malcev correspondence, from a \( \mathbb{Z} \)-Lie lattice \( \Lambda \) with a suitable algebraic automorphism group \( \text{Aut}(\Lambda) \).

1. Introduction

Let \( \Gamma \) be a torsion-free finitely generated nilpotent group. In analogy to classical zeta functions such as the Dedekind zeta function of a number field, Grunewald, Segal and Smith introduced in [5] zeta functions counting certain finite index subgroups of \( \Gamma \).

More precisely, they defined Dirichlet generating functions

\[
\begin{align*}
\zeta_\leq^\Gamma(s) &= \sum_{n=1}^{\infty} \frac{a_\leq_n(\Gamma)}{n^s}, \quad \zeta_\supset^\Gamma(s) = \sum_{n=1}^{\infty} \frac{a_\supset_n(\Gamma)}{n^s}, \\
\zeta_\text{iso}^\Gamma(s) &= \sum_{n=1}^{\infty} \frac{a_\text{iso}_n(\Gamma)}{n^s}, \quad \zeta_\wedge^\Gamma(s) = \sum_{n=1}^{\infty} \frac{a_\wedge_n(\Gamma)}{n^s},
\end{align*}
\]

where \( a_\leq_n(\Gamma), a_\supset_n(\Gamma), a_\text{iso}_n(\Gamma) \) and \( a_\wedge_n(\Gamma) \) denote the number of subgroups \( \Delta \) of index \( n \) in \( \Gamma \) satisfying \( \Delta \leq \Gamma, \Delta \supset \Gamma, \Delta \cong \Gamma \) and \( \hat{\Delta} \cong \hat{\Gamma} \) respectively. Here \( \hat{H} \) denotes the profinite completion of a group \( H \). The Dirichlet series above are now commonly referred to as the subgroup zeta function, the normal zeta function, the isomorphic zeta function and the pro-isomorphic zeta function.

The group \( \Gamma \) being nilpotent, one easily derives Euler product decompositions

\[
\begin{align*}
\zeta_\leq^\Gamma(s) &= \prod_p \zeta_\leq^\Gamma,p(s), \quad \zeta_\supset^\Gamma(s) = \prod_p \zeta_\supset^\Gamma,p(s), \quad \zeta_\text{iso}^\Gamma(s) = \prod_p \zeta_\text{iso}^\Gamma,p(s), \quad \zeta_\wedge^\Gamma(s) = \prod_p \zeta_\wedge^\Gamma,p(s),
\end{align*}
\]

where each product extends over all rational primes \( p \) and

\[
(1.1) \quad \zeta_\wedge^\Gamma,p(s) = \sum_{k=0}^{\infty} a_\wedge^p_n(\Gamma)p^{-ks}, \quad \text{for } * \text{ one of } \leq, \supseteq, \wedge,
\]

2010 Mathematics Subject Classification. Primary 11M41; Secondary 20E07, 20F18, 20F69.

Key words and phrases. Nilpotent group, pro-isomorphic zeta function, functional equation.
is the local zeta function at a prime $p$. In contrast, the zeta function $\zeta_{\Gamma, p}^{iso}(s)$ generally does not admit a decomposition of this kind. One of the key results in [5] is that each of the local factors defined in (1.1) is in fact a rational function over $\mathbb{Q}$ in $p^{-s}$.

Over the last 20 years many important advances have been made in the study of zeta functions of nilpotent groups; for instance, see [2, 4, 10]. One of the prominent questions driving the subject has been whether the local zeta functions satisfy functional equations: is it true that

$$\zeta_{\Gamma, p}^{\ast}(s) |_{p=p-1} = (-1)^{a}p^{b-cs}\zeta_{\Gamma, p}(s)$$

for almost all $p$, where $a$, $b$, $c$ are certain integer parameters depending on $\Gamma$? In [10], Voll derived positive answers for subgroup zeta functions in general and for normal zeta functions associated to groups of nilpotency class at most 2. More precisely, he showed that the local zeta functions in question can be expressed as rational functions in $p^{-s}$ whose coefficients involve the numbers $b_{V}(p)$ of $\mathbb{F}_{p}$-rational points of certain smooth projective varieties $V$. The number $b_{V}(p)$ is obtained by writing $b_{V}(p)$ as an alternating sum of Frobenius eigenvalues and subsequently inverting these eigenvalues. It is known that normal zeta functions of nilpotent groups of class 3 may or may not satisfy local functional equations as above; see [4, Section 2.11].

In comparison to $\zeta_{\Gamma, p}^{\leq}(s)$ and $\zeta_{\Gamma, p}^{\prec}(s)$, our present picture of the local pro-isomorphic zeta function $\zeta_{\Gamma, p}(s)$ is somewhat less complete. This is perhaps due to the fact that $\zeta_{\Gamma, p}^{\wedge}(s)$ depends crucially on an intermediate object that is generally not easy to pin down, namely the algebraic automorphism group $\text{Aut}(\Lambda)$ of a $\mathbb{Z}$-Lie lattice $\Lambda$ naturally associated to $\Gamma$. Using $\text{Aut}(\Lambda)$, one can compute $\zeta_{\Gamma, p}^{\wedge}(s)$ as a $p$-adic integral similar to zeta functions of reductive algebraic groups that were studied in the 1960s by Weil, Tamagawa, Satake and Macdonald; cf. Proposition 2.2. Building upon work of Igusa [6], du Sautoy and Lubotzky [3] and Berman [1] have established functional equations for $\zeta_{\Gamma, p}^{\wedge}(s)$ subject to certain conditions on $\text{Aut}(\Lambda)$.

The purpose of this paper is to give the first example of a nilpotent group $\Gamma$ such that the local pro-isomorphic zeta functions $\zeta_{\Gamma, p}(s)$ do not satisfy functional equations in the sense discussed above.

**Theorem 1.1.** There exists a torsion-free finitely generated nilpotent group $\Gamma$, of nilpotency class 4 and Hirsch length 25, such that, for all primes $p > 3$,

$$\zeta_{\Gamma, p}^{\wedge}(s) = \frac{1 + p^{285-102s} + 2p^{286-102s} + 2p^{272-204s}}{(1-p^{285-102s})(1-p^{573-204s})}.$$  

In particular, this resolves Question 1.3 in [1]. As explained in Section 2, the proof of Theorem 1.1 reduces to the construction of a suitable nilpotent $\mathbb{Z}$-Lie lattice $\Lambda$, linked to a group $\Gamma$ via the Malcev correspondence. In Section 3 we describe a candidate for such a Lie lattice $\Lambda$, in terms of generators and relations: $\Lambda$ has nilpotency class 4 and $\mathbb{Z}$-rank 25; its construction is motivated by certain integrals presented in [1] Section 6. In Section 4 we carry out the task of pinning down the algebraic automorphism group $\text{Aut}(\Lambda)$. In Section 5 we compute $\zeta_{\Gamma, p}^{\wedge}(s)$, using our description of $\text{Aut}(\Lambda)$ and the machinery developed in [3].
2. Reduction to \( \mathbb{Z} \)-Lie lattices

Let \( \Gamma \) be a torsion-free finitely generated nilpotent group. Then the profinite completion \( \hat{\Gamma} \cong \prod_p \hat{\Gamma}_p \) is pro-nilpotent, with Sylow pro-\( p \) subgroups isomorphic to the pro-\( p \) completions \( \hat{\Gamma}_p \), and

\[
\zeta_\hat{\Gamma}(s) = \prod_p \zeta_{\hat{\Gamma}_p}(s).
\]

One of the key steps in [5] is to use the Malcev correspondence to ‘linearise’ the problem of computing the factors \( \zeta_{\hat{\Gamma}_p}(s) \) by passing from groups to Lie lattices.

Let \( \Lambda \) be a \( \mathbb{Z} \)-Lie lattice. In analogy to the zeta functions defined for groups, the isomorphic zeta function and the pro-isomorphic zeta function of \( \Lambda \) are

\[
\zeta_\Lambda(s) = \sum_{n=1}^{\infty} \frac{a_n(\Lambda)}{n^s}, \quad \zeta_\hat{\Lambda}(s) = \sum_{n=1}^{\infty} \frac{a_n(\Lambda)}{n^s},
\]

where \( a_n(\Lambda) \) and \( a_n(\Lambda) \) denote the number of Lie sublattices \( M \) of index \( n \) in \( \Lambda \) satisfying \( \hat{M} \cong \hat{\Lambda} \) and \( \hat{\mathbb{Z}} \otimes \mathbb{Z} M \cong \hat{\mathbb{Z}} \otimes \mathbb{Z} \Lambda \) respectively. As \( \hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p \), where \( \mathbb{Z}_p \) denotes the ring of \( p \)-adic integers, the latter condition is equivalent to: \( \mathbb{Z}_p \otimes \mathbb{Z} M \cong \mathbb{Z}_p \otimes \mathbb{Z} \Lambda \) for all primes \( p \). Moreover,

\[
\zeta_\hat{\Lambda}(s) = \zeta_{\mathbb{Z}_p \otimes \mathbb{Z} \Lambda}(s) = \prod_p \zeta_{\mathbb{Z}_p \otimes \mathbb{Z} \Lambda}(s).
\]

In [5, Section 4] one finds a discussion of how starting from a group \( \Gamma \) one obtains a \( \mathbb{Z} \)-Lie lattice \( \Lambda \) such that \( \zeta_{\hat{\Gamma}_p}(s) = \zeta_{\mathbb{Z}_p \otimes \mathbb{Z} \Lambda}(s) \) for almost all primes \( p \). Our aim in this section is to complement the treatment in [5], by giving a detailed account of the transition in the opposite direction from \( \mathbb{Z} \)-Lie lattices to groups; refer to [9, Chapter 6] for an alternative approach and [7, Chapter 10] for a description of the related Malcev correspondence. We make use of the Lie correspondence between \( p \)-adic analytic pro-\( p \) groups and \( \mathbb{Z}_p \)-Lie lattices, which – similarly to the Malcev correspondence – is effected by the Hausdorff series

\[
\Phi(X,Y) = \log(\exp(X) \exp(Y)) \in \mathbb{Q} \langle \langle X, Y \rangle \rangle,
\]

a formal power series in non-commuting variables \( X, Y \), where

\[
\exp(Z) = \sum_{n=0}^{\infty} \frac{Z^n}{n!} \quad \text{and} \quad \log(Z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(Z-1)^n}{n}.
\]

By arranging terms suitably, one can write \( \Phi(X,Y) \) as a sum

\[
\Phi(X,Y) = \sum_{n=1}^{\infty} u_n(X,Y)
\]

(2.1)

of homogeneous Lie polynomials \( u_n(X,Y) \) in \( X, Y \) of total degree \( n \) with rational coefficients. For instance, modulo terms of total degree at least 5, the Hausdorff series is congruent to the Lie polynomial

\[
\Phi_4(X,Y) = X + Y + \frac{1}{2}[X,Y] - \frac{1}{12}[X,Y,X] + \frac{1}{12}[X,Y,Y] - \frac{1}{24}[X,Y,X,Y].
\]
For $c \in \mathbb{N}$, let $m(c)$ denote the least common denominator of the rational coefficients appearing in (2.1) up to total degree $c$; for instance, $m(4) = 24$.

Suppose that $\Lambda$ is a finitely generated nilpotent $\mathbb{Z}$-Lie lattice of class $c$, and let $\Phi_c(X,Y)$ denote the Lie polynomial obtained from $\Phi(X,Y)$ by truncating after terms of total degree at most $c$. Setting $m = m(c)$, we have $[m\Lambda, m\Lambda] \leq m(m\Lambda)$. Consequently, \[ \exp(m\Lambda) := (m\Lambda, *), \] defines a torsion-free finitely generated nilpotent group of class $c$. Indeed, the formal identities
\[
\Phi(\Phi(X,Y), Z) = \Phi(X, \Phi(Y, Z)), \\
\Phi(0, X) = \Phi(X, 0) = X, \quad \Phi(-X, X) = \Phi(X, -X) = 0
\]
show that $\exp(m\Lambda)$ is a group. Furthermore, the identity $\Phi(X, X') = X + X'$ for commuting variables $X, X'$ shows that on every abelian Lie sublattice $A$ of $m\Lambda$ the operation $*$ is the same as Lie addition; in particular, $\exp(m\Lambda)$ is torsion-free, and central isolated Lie sublattices correspond to central isolated subgroups. By induction on the nilpotency class $c$, one shows that $\exp(m\Lambda)$ is finitely generated nilpotent of class at most $c$ and has Hirsch length equal to the $\mathbb{Z}$-rank of $m\Lambda$. The nilpotency class of $\exp(m\Lambda)$ is equal to $c$, because each group commutator $[x_1, \ldots, x_c]_{\text{grp}}$ of length $c$ yields exactly the same element as the corresponding Lie commutator $[x_1, \ldots, x_c]_{\text{Lie}}$; this follows, by induction, from the congruence $\Phi(\Phi(-X, -Y), \Phi(X, Y)) \equiv [X, Y]$ modulo commutators of length at least 3. This establishes the first half of the following proposition.

**Proposition 2.1.** Let $\Lambda$ be a nilpotent $\mathbb{Z}$-Lie lattice of class $c$, and set $m = m(c)$. Then $\Gamma = \exp(m\Lambda)$ is a torsion-free finitely generated nilpotent group of class $c$, and for all primes $p$ with $p \nmid m$,
\[ \zeta^{\text{iso}}(s) = \zeta_{\mathbb{Z}_p \otimes \mathbb{Z}}(s). \]

**Proof.** It remains to show that the equality between the two zeta functions holds for all primes $p$ with $p \nmid m$. Let $p$ be such a prime. We observe that $\Phi_c(X,Y)$ has coefficients in $\mathbb{Z}_p$. Write $L = \mathbb{Z}_p \otimes_{\mathbb{Z}} \Lambda = \mathbb{Z}_p \otimes_{\mathbb{Z}} m\Lambda$ and $G = \exp(L) = (L, *), \,$ where $x * y = \Phi_c(x, y)$ as above. Observe that $L$ and $G$ have the same underlying sets and that $G$, with the topology inherited from $L$, is a topological group. To distinguish Lie and group commutators we write $[x, y]_{\text{Lie}}$ and $[x, y]_{\text{grp}}$ respectively.

Observe that $G^p = \{x^p \mid x \in G\}$ is equal to $G^{\{p\}} = \{x^p \mid x \in G\} = \exp(p^k L)$ for each $k \in \mathbb{N}$, because $p^k L$ is closed under $*$. Since the sublattices $p^k L$ form a base for the neighbourhoods of 0, the group $G$ is a pro-$p$ group. Similarly, one sees that $\Gamma^p = \Gamma^{\{p\}} = \exp(p^k m\Lambda)$. This implies\[ \hat{\Gamma}_p = \lim \frac{\Gamma^p}{\Gamma} = \lim \frac{\exp(m\Lambda)}{\exp(p^k m\Lambda)} \cong \lim \frac{\exp(L)}{\exp(p^k L)} \cong \exp(L) = G. \]

To finish the proof we verify that the construction $M \mapsto \exp(M)$ sets up an index-preserving one-to-one correspondence
\[ \{M \mid M \leq L \text{ a Lie sublattice, } M \cong L\} \to \{H \mid H \leq G \text{ a subgroup, } H \cong G\}. \]

Since $\Phi_c(X,Y)$ has coefficients in $\mathbb{Z}_p$, every $\mathbb{Z}_p$-Lie sublattice $M$ of $L$ gives rise to a subgroup $\exp(M)$. Moreover, the construction is functorial so that $M \cong L$ implies
\[ \exp(M) \cong \exp(L) = G. \] Conversely, suppose that \( H \) is a subgroup of \( G \) with \( H \cong G \). The group \( G \) has the properties

(i) \( G^p = G^{p^k} \) and \([G^{p^k}, G^{p^k}] \subseteq G^{p^{2k}}\) for \( k \in \mathbb{N} \),

(ii) \( G \to G, x \mapsto x^p \) is injective.

Based on these, one can recover the Lie operations on \( L \) from the group structure of \( G \) by means of

\[ x + y = \lim_{k \to \infty} (x^{p^k} \ast y^{p^k})^{p^{-k}} \quad \text{and} \quad [x, y]_{\text{Lie}} = \lim_{k \to \infty} [x^{p^k}, y^{p^k}]_{\text{grp}}^{p^{-2k}}. \]

Since \( H \cong G \), the relations (2.2) applied to \( H \) show that the set \( H \) is closed under Lie addition and the Lie bracket. Thus \( H = \exp(M) \) for a Lie sublattice \( M \) of \( L \), and again by functoriality \( M \cong L \).

Finally, we claim that, for \( x \in L \) and \( k \in \mathbb{N} \), the Lie coset \( x + p^kL \) and the group coset \( x \ast G^{p^k} \) are equal as sets. This will imply that the normalised Haar measures \( \mu_{\text{Lie}} \) on \( L \) and \( \mu_{\text{grp}} \) on \( G \) are the same so that for every Lie sublattice \( M \) of \( L \),

\[ |L : M| = \mu_{\text{Lie}}(M)^{-1} = \mu_{\text{grp}}(\exp(M))^{-1} = |G : \exp(M)|. \]

To prove the claim, note that \( x \ast G^{p^k} = \Phi_c(x, p^k L) \subseteq x + p^k L \). The reverse inclusion follows inductively: one shows that \( x + p^k \gamma_{c+1-i}(L) \subseteq x \ast G^{p^k} \) for \( i \in \{1, \ldots, c\} \).

Next we recall the notion of the algebraic automorphism group \( \text{Aut}(\Lambda) \) of a \( \mathbb{Z} \)-Lie lattice \( \Lambda \). The group \( \text{Aut}(\Lambda) \) is realised via a \( \mathbb{Z} \)-basis of \( \Lambda \) as a \( \mathbb{Q} \)-algebraic subgroup \( G \) of \( \text{GL}_d \), where \( d \) denotes the \( \mathbb{Z} \)-rank of \( \Lambda \), so that

\[ G(k) = \text{Aut}_k(k \otimes \mathbb{Z} L) \leq \text{GL}_d(k) \]

for every extension field \( k \) of \( \mathbb{Q} \). It admits a natural arithmetic structure so that

\[ G(\mathbb{Z}) = \text{Aut}(\Lambda) \quad \text{and} \quad G(\mathbb{Z}_p) = \text{Aut}(\mathbb{Z}_p \otimes \mathbb{Z} \Lambda) \]

for every prime \( p \). We state [5] Proposition 3.4.

**Proposition 2.2** (Grunewald, Segal, Smith). Let \( \Lambda \) be a nilpotent \( \mathbb{Z} \)-Lie lattice, with \( \mathbb{Z} \)-basis \( \mathcal{E} = (e_1, \ldots, e_d) \) say, and let \( G = \text{Aut}(\Lambda) \subseteq \text{GL}_d \) denote the algebraic automorphism group of \( \Lambda \), realised with respect to \( \mathcal{E} \). For each prime \( p \), let

\[ G^+_p = G(\mathbb{Q}_p) \cap \text{Mat}_d(\mathbb{Z}_p) = \text{Aut}(\mathbb{Q}_p \otimes \mathbb{Z} \Lambda) \cap \text{End}(\mathbb{Z}_p \otimes \mathbb{Z} \Lambda) \]

and let \( \mu_p \) denote the right Haar measure on the locally compact group \( G(\mathbb{Q}_p) \) such that \( \mu_p(G(\mathbb{Z}_p)) = 1 \). Then for all primes \( p \),

\[ \zeta_{\mathbb{Z}_p \otimes \mathbb{Z} \Lambda}^\text{iso}(s) = \int_{G^+_p} |\det g|_p^s d\mu_p(g). \]

In [3] Section 2] we find a treatment of \( p \)-adic integrals as in Proposition 2.2 subject to a series of simplifying assumptions. Almost all of those assumptions, relevant for our purposes, are not restrictive, in the sense that they can be realised by an appropriate equivalent representation of the automorphism group corresponding to a different choice of basis \( \mathcal{E} \); this is shown in [3] Section 4]. The integral of Proposition 2.2 is unaffected by this change of representation for almost all primes. It will turn out that the assumptions required for our application in Section 3] are automatically satisfied without the need for an equivalent representation.
We now outline the assumptions. Let $G \subseteq \text{GL}_d$ be an affine group scheme over $\mathbb{Z}$. Decompose the connected component of the identity as a semidirect product $G^o = N \rtimes H$ of the unipotent radical $N$ and a reductive group $H$. Fix a prime $p$. The first assumption is that
\[
G(\mathbb{Q}_p) = G(\mathbb{Z}_p)G^o(\mathbb{Q}_p);
\]
loosely speaking, this allows us to work with the connected group $G^o$ rather than $G$. We write $G = G^o(\mathbb{Q}_p)$, $N = N(\mathbb{Q}_p)$, $H = H(\mathbb{Q}_p)$. Assume further that $G \subseteq \text{GL}_d(\mathbb{Q}_p)$ is in block form, where $H$ is block diagonal and $N$ is block upper unitriangular in the following sense. There is a partition $d = r_1 + \ldots + r_c$ such that, setting $s_i = r_1 + \ldots + r_i - 1$ for $i \in \{1, \ldots, c\},$
\[
\begin{align*}
&\circ \text{ the vector space } V = \mathbb{Q}_p^d \text{ on which } G \text{ acts from the right decomposes into a } \\
&\text{direct sum of } H\text{-stable subspaces } U_i = \{(0, \ldots, 0)\} \times \mathbb{Q}_p^{d_i} \times \{(0, \ldots, 0)\}, \text{ where } \\
&\text{the vectors } (0, \ldots, 0) \text{ have } s_i, \text{ respectively } d - s_{i+1}, \text{ entries;} \\
&\circ \text{ setting } V_i = U_i \oplus \ldots \oplus U_c, \text{ each } V_i \text{ is } N\text{-stable and } N \text{ acts trivially on } V_i/V_{i+1}.
\end{align*}
\]

For each $i \in \{2, \ldots, c + 1\}$ let $N_{i-1} = N \cap \ker(\psi'_i)$, where $\psi'_i : G \to \text{Aut}(V_i/V_i)$ denotes the natural action. Let $\psi : G/N_{i-1} \to \text{Aut}(V_i/V_i)$ denote the induced map, and define
\[
(G/N_{i-1})^+ = \psi^{-1}(\psi(G/N_{i-1}) \cap \text{Mat}_{s_i}(\mathbb{Z}_p)).
\]
Assume that for every $g_0N_{i-1} \in (G/N_{i-1})^+$ there exists $g \in G^+$ such that $g_0N_{i-1} = gN_{i-1}$; this is the crucial ‘lifting condition’ [3 Assumption 2.3]. As explained in [1 p. 6], this lifting condition cannot in general be satisfied by moving to an equivalent representation. For $i \in \{2, \ldots, c\}$ there is a natural embedding of $N_{i-1}/N_i \hookrightarrow (V_i/V_{i+1})^{s_i}$ via the action of $N_{i-1}/N_i$ on $V_i/V_{i+1}$, recorded on the natural basis. The action of $H$ on $V_i/V_{i+1}$ induces, for each $h \in H$, a map
\[
\tau_i(h) : N_{i-1}/N_i \hookrightarrow (V_i/V_{i+1})^{s_i}.
\]
Define $\vartheta_{i-1} : H \to [0, 1]$ by setting
\[
\vartheta_{i-1}(h) = \mu_{N_{i-1}/N_i}(\{nN_i \in N_{i-1}/N_i \mid (nN_i)\tau_i(h) \in \text{Mat}_{s_i}(\mathbb{Z}_p)\}),
\]
where $\mu_{N_{i-1}/N_i}$ denotes the right Haar measure on $N_{i-1}/N_i$, normalised such that the set $\psi_{i+1}^{-1}(\psi_{i+1}(N_{i-1}/N_i) \cap \text{Mat}_{s_{i+1}}(\mathbb{Z}_p))$ has measure 1. Write $\mu_G$, respectively $\mu_H$, for the right Haar measure on $G$, respectively $H$, normalised such that $\mu_G(\text{Mat}_d(\mathbb{Z}_p)) = 1$ and $\mu_H(\text{Mat}_d(\mathbb{Z}_p)) = 1$. From $G = N \rtimes H$ one deduces that $\mu_G = \prod_{i=1}^c \mu_{N_{i-1}/N_i} \cdot \mu_H$. Setting $G^+ = G \cap \text{Mat}_d(\mathbb{Z}_p)$ and $H^+ = H \cap \text{Mat}_d(\mathbb{Z}_p)$, we can now state [3 Theorem 2.2].

Theorem 2.3 (du Sautoy and Lubotzky). In the set-up described above, subject to the various assumptions,
\[
\int_{G^+} |\det g|^s_p d\mu_G(g) = \int_{H^+} |\det h|^s_p \prod_{i=1}^{c-1} \vartheta_i(h) d\mu_H(h).
\]

For later use, we record a simple fact that is useful for detecting when an element of $G$, arising as in Proposition 2.2, is integral.

Lemma 2.4. Suppose that $\Lambda$ is a $\mathbb{Z}$-Lie lattice of $\mathbb{Z}$-rank $d$, generated by elements $x_1, \ldots, x_r$. Put $L = \mathbb{Z}_p \otimes_\mathbb{Z} \Lambda$, and let $g \in G = G(\mathbb{Q}_p)$, where $G = \text{Aut}(\Lambda) \subseteq \text{GL}_d$.

If $(x_1)g, \ldots, (x_r)g \in L$, then $g \in G^+ = G(\mathbb{Q}_p) \cap \text{Mat}_d(\mathbb{Z}_p)$.
Proof. Note that \( g \in G^+ \) if and only if \((z)g \in L\) for all \( z \in L \). Suppose that \((x_1)g, \ldots, (x_r)g \in L\). Let \( w \in L \) be an arbitrary commutator in \( x_1, \ldots, x_r \), i.e. \( w = [y_1, \ldots, y_n] \) with \( y_i \in \{x_1, \ldots, x_r\} \) for \( 1 \leq i \leq n \). Then \((w)g = [(y_1)g, \ldots, (y_n)g]\) is a commutator of elements of \( L \), hence lies in \( L \). By linearity, \((z)g \in L\) for all \( z \in L \), since \( x_1, \ldots, x_r \) generate \( L \) as a \( \mathbb{Z}_p \)-Lie lattice. Thus \( g \in G^+ \). □

3. The \( \mathbb{Z} \)-Lie lattice \( \Lambda \)

Let \( \mathcal{F} \) be the free nilpotent \( \mathbb{Z} \)-Lie ring of class 4 on generators \( X, Y, Z \). In the following we will primarily be dealing with Lie rings and it is convenient to use simple product notation for the Lie bracket and left-normed notation. Thus we write \( XYZ \) in place of \([X, Y, Z]\). The Hall collection process (see Appendix A) yields the following graded \( \mathbb{Z} \)-basis for \( \mathcal{F} \):

\[
X, Y, Z, \quad XY, XZ, YZ, \\
XYY, XZZ, XYZ, XZY, YZY, ZYZ.
\]

(3.1)

In particular, \( \dim_{\mathbb{Q}}(\mathbb{Q} \otimes \mathbb{Z} \mathcal{F}) = 32 \).

Let \( \mathcal{I} \) be the Lie ideal of \( \mathcal{F} \) generated by the two relations

\[
R_1 = YXXX - YZY \quad \text{and} \quad R_2 = ZXXX - ZYZ.
\]

We determine a basis for \( \mathcal{I} \). Examining terms of the form \( R_1S \) and \( R_2S \) for \( S \in \{X, Y, Z\} \), we obtain a spanning set for \( \mathcal{I} \), namely

\[
R_1, R_2, \quad YZYX, YZYY, YZYZ, \quad ZYZX, ZYZY, ZYZZ.
\]

We make repeated use of the following basic Lie identity, cf. (A.4) in Appendix A:

(3.2)

\[
PQRS = PSQR + SQPR + RSQP + SRPQ.
\]

The substitution \((P, Q, R, S) = (Y, Z, Y, Z)\) yields \( YZY = 2YZZY + ZYYZ \), hence

\[
YZY = YZZ = -ZYYZ.
\]

This shows that we may omit \( ZYYZ \) from the spanning set of \( \mathcal{I} \). We claim that the remaining seven elements

\[
R_1, R_2, \quad YZYX, ZYZX, \quad YZYY, YZYZ, ZYZZ,
\]

form a \( \mathbb{Z} \)-basis for \( \mathcal{I} \); in particular, \( \dim_{\mathbb{Q}}(\mathbb{Q} \otimes \mathbb{Z} \mathcal{I}) = 7 \). Indeed, we only need to look at the elements of weight 4. The last three of them are part of the basis (3.1), and evaluating (3.2) for \((P, Q, R, S) = (Y, Z, Y, X)\) and \((P, Q, R, S) = (Z, Y, Z, X)\) we can express the remaining two spanning elements also in terms of the basis (3.1):

\[
YZYX = -2XYZY + XZY + XYZZ, \\
ZYZX = -2XZZY + XYYZ + XZZY.
\]

(3.3)

Set \( \Lambda = \mathcal{F}/\mathcal{I} \) and write \( x, y, z \) for the images of \( X, Y, Z \) in \( \Lambda \). From our description of \( \mathcal{I} \) we conclude that \( \dim_{\mathbb{Q}}(\mathbb{Q} \otimes \mathbb{Z} \Lambda) = 25 \) and that \( \Lambda \) admits the following \( \mathbb{Z} \)-basis, obtained
by omitting the images of the underlined elements in (3.1) matching the underlined terms in the expressions above:

\begin{align*}
&x, y, z, \quad xy, xz, yz, \\
&xxy, xzz, xyz, xzy, xyx, xzx, \\
&xyy, xzz, xyz, xzy, xyy, xzy, xzy.
\end{align*}

(3.4)

For later use we record two consequences of our discussion, see (3.3):

\begin{align*}
2xyzy &= xzyy + xyyz \\
2xzyz &= xyzz + xzzz.
\end{align*}

(3.5)

Finally, we make the following observation, which can be checked by direct calculation using standard identities and (3.2); in verifying the claim, it is convenient to consider first the analogous statement for $\mathcal{F}$ and subsequently pass to the quotient $\Lambda$.

**Lemma 3.1.** Every commutator $w$ of length 4 in $x, y, z \in \Lambda$ can be expressed uniquely as a $\mathbb{Z}$-linear combination of elements of the basis (3.4) having the same weights for $x, y, z$; that is: $x, y, z$ appear in each of these basis elements with the same multiplicity as in the original $w$.

4. The algebraic automorphism group $\text{Aut}(\Lambda)$

Using the $\mathbb{Z}$-basis of $\Lambda$ listed in (3.4), the algebraic automorphism group $\text{Aut}(\Lambda)$ of $\Lambda$ can be realised as a $\mathbb{Q}$-algebraic subgroup $G$ of $\text{GL}_{25}$ so that

$$G(k) = \text{Aut}_k(k \otimes_{\mathbb{Z}} \Lambda) \leq \text{GL}_{25}(k)$$

for every extension field $k$ of $\mathbb{Q}$. Moreover,

$$G(\mathbb{Z}) = \text{Aut}(\Lambda) \quad \text{and} \quad G(\mathbb{Z}_p) = \text{Aut}(\mathbb{Z}_p \otimes_{\mathbb{Z}} \Lambda) \quad \text{for every prime} \quad p.$$  

We observe that for every extension field $k$ of $\mathbb{Q}$,

1. every $k$-automorphism of $k \otimes_{\mathbb{Z}} \Lambda$ lifts to a $k$-automorphism of $k \otimes_{\mathbb{Z}} \mathcal{F}$, and $\alpha \in \text{Aut}_k(k \otimes_{\mathbb{Z}} \mathcal{F})$ induces an automorphism $\alpha_{k \otimes \Lambda} \in \text{Aut}_{k}(k \otimes_{\mathbb{Z}} \Lambda)$ if and only if $(k \otimes_{\mathbb{Z}} \mathcal{I}) \alpha \subseteq k \otimes_{\mathbb{Z}} \mathcal{I}$;

2. every $\alpha \in \text{Aut}_k(k \otimes_{\mathbb{Z}} \mathcal{F})$ is uniquely determined by the images $X\alpha$, $Y\alpha$, $Z\alpha$; conversely any choice $X_0$, $Y_0$, $Z_0 \in k \otimes_{\mathbb{Z}} \mathcal{F}$ with $k \otimes_{\mathbb{Z}} \mathcal{F} = \text{span}_k(X_0, Y_0, Z_0) + \gamma_2(k \otimes_{\mathbb{Z}} \mathcal{F})$ yields a $k$-automorphism $\alpha$ of $k \otimes_{\mathbb{Z}} \mathcal{F}$ such that $X\alpha = X_0$, $Y\alpha = Y_0$, $Z\alpha = Z_0$.

Consequently, we identify two natural subgroups of $\text{Aut}(\Lambda)$. The affine algebraic group $T$ admits a $\mathbb{Q}$-defined faithful representation with image $\tilde{T} \leq \text{GL}_{32}$ such that, for every extension $k$ of $\mathbb{Q}$, the group $\tilde{T}(k)$ corresponds to the subgroup of $\text{Aut}_k(k \otimes_{\mathbb{Z}} \mathcal{F})$ obtained
from extending the natural action of $T(k)$ on $\text{span}_k(X, Y, Z)$ to an action on $k \otimes \mathcal{F}$:

$$
\begin{pmatrix}
\lambda & 0 & 0 \\
0 & \mu & 0 \\
0 & 0 & \nu
\end{pmatrix}
\text{acts via}
\begin{cases}
X \mapsto \lambda X \\
Y \mapsto \mu Y \\
Z \mapsto \nu Z
\end{cases}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\text{acts via}
\begin{cases}
X \mapsto X \\
Y \mapsto Z \\
Z \mapsto Y
\end{cases}
$$

We remark that the connected component of $T$ is isomorphic to the group $\varrho_{2,3}(T_2)$ presented in [1] Proposition 6.1 which defines an integral not satisfying a functional equation. Thus it is reasonable to investigate $\text{Aut}(\Lambda)$ in the hope that it will provide similar behaviour.

We also consider the $\mathbb{Q}$-defined algebraic group $M$ such that, for every extension $k$ of $\mathbb{Q}$, the group $M(k)$ consists of all $k$-automorphisms of $k \otimes \mathcal{F}$ of the form

$$
\begin{align*}
X & \mapsto X + U, & U & \in \gamma_2(k \otimes \mathcal{F}) \\
Y & \mapsto Y + vXY + \sigma YZ + V, & V & \in \gamma_3(k \otimes \mathcal{F}) \\
Z & \mapsto Z + vXZ + \tau YZ + W, & W & \in \gamma_3(k \otimes \mathcal{F}),
\end{align*}
$$

where $v, \sigma, \tau \in k$. Note that $\tilde{T}(k)$ acts faithfully on $(k \otimes \mathcal{F})/\gamma_2(k \otimes \mathcal{F})$, whereas $M(k)$ acts trivially on $(k \otimes \mathcal{F})/\gamma_2(k \otimes \mathcal{F})$. We define $S = M \rtimes \tilde{T}$ so that $S(k) = M(k) \rtimes \tilde{T}(k) \leq \text{Aut}_k(k \otimes \mathcal{F})$.

**Lemma 4.1.** The ideal $k \otimes \mathcal{I}$ is invariant under the action of $S(k)$.

**Proof.** Elements of $\tilde{T}(k)$ map $R_1$ and $R_2$ to scalar multiples of themselves. Furthermore, elements of $M(k)$ fix $R_1$ and $R_2$ modulo $\text{span}_k(R_1X, R_1Y, R_1Z, R_2X, R_2Y, R_2Z)$. Indeed, for any element $\alpha \in M(k)$ as in (4.1), using the fact that commutators of length greater than 4 are trivial, short calculations based on the identities (A.2) and (3.3) yield

$$
\begin{align*}
(R_1)\alpha & = R_1 + vYZYX + \sigma ZYZY + \tau YZYY \in k \otimes \mathcal{I}, \\
(R_2)\alpha & = R_2 + vYZXZ - \tau YZYZ - \sigma ZZYZY \in k \otimes \mathcal{I}.
\end{align*}
$$

The action of $S(k)$ on $k \otimes \Lambda$ can be described in terms of a $\mathbb{Q}$-defined morphism $\varrho: S \to \text{Aut}(\Lambda)$ to the algebraic automorphism group. Let $K = \ker(\varrho)$.

**Theorem 4.2.** The morphism $\varrho: S \to \text{Aut}(\Lambda)$ induces an isomorphism

$$
\text{Aut}(\Lambda) = \text{img}(\varrho) \cong S/K.
$$

**Proof.** Let $k$ be an extension field of $\mathbb{Q}$ and let $\alpha \in \text{Aut}_k(k \otimes \Lambda)$. We are to show that $\alpha \in (S(k))\varrho$. Clearly, $\alpha$ induces an automorphism $\overline{\alpha}$ on $(k \otimes \Lambda)/\gamma_2(k \otimes \Lambda) \cong \text{span}_k(\overline{x}, \overline{y}, \overline{z}) \cong k^3$. We write $A = (a_{ij}) \in \text{GL}_3(k)$ for the matrix representing $\overline{\alpha}$ with respect to the $k$-basis $\overline{x}, \overline{y}, \overline{z}$.

From $R_2 \equiv -YZZ$ modulo $\gamma_4(\mathcal{F})$ we deduce $(xyz)\alpha \equiv 0$ modulo $\gamma_4(k \otimes \Lambda)$. A short computation shows that, modulo $\gamma_4(k \otimes \Lambda)$,

$$

to (a_{31}x + a_{32}y + a_{33}z)(a_{21}x + a_{22}y + a_{23}z)(a_{11}x + a_{12}y + a_{13}z)$$

$$
= c_1xxy + c_2xzz + c_3xyx + c_4xzx + c_5xyz + c_6xzy,
$$

As a result, $A \equiv c_1xxy + c_2xzz + c_3xyx + c_4xzx + c_5xyz + c_6xzy$. 

\[\square\]
is invertible, we conclude that the four coefficients listed in (4.2) vanish. On the other hand, since $yzy \equiv zyx \equiv 0$ modulo $\gamma_4(k \otimes \mathbb{Z} \Lambda)$ and $yzx = xzy - xyz$. Therefore the contributions to $c_1, c_2, c_3, c_4$ are the ‘obvious ones’, as described above.

Since $xyy, xzz, xyx, xzx, xyz$ form a $k$-basis for $\gamma_3(k \otimes \mathbb{Z} \Lambda)$ modulo $\gamma_4(k \otimes \mathbb{Z} \Lambda)$, we conclude that the four coefficients listed in (4.2) vanish. On the other hand, since $A$ is invertible,

$$\text{rk} \left( \begin{array}{ccc} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right) = 2.$$  

We claim that $a_{21} = a_{31} = 0$. Otherwise $\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \neq 0$ or $\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \neq 0$. Looking at the coefficients $c_3, c_4$ of $xyx, xzx$, we conclude that $a_{31} = 0$. Looking at the coefficients $c_1, c_2$ of $xyy, xzz$, this implies that $a_{21}a_{32}^2 = a_{21}a_{33}^2 = 0$. Since $a_{32} \neq 0$ or $a_{33} \neq 0$, this shows that $a_{21} = 0$.

Multiplying $\alpha$ by a suitable element of $(\mathcal{T}(k))_0$, we may assume without loss of generality that $a_{11} = 1$ so that $A$ is of the form

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}.$$  

Next we derive consequences from the relation $(zxx)\alpha = (zyz)\alpha$, coming from the relation $R_2$. Since $\gamma_5(k \otimes \mathbb{Z} \Lambda) = 0$, we have

$$\text{(4.3)} \quad (zxx)\alpha = (a_{32}y + a_{33}z)(x + a_{12}y + a_{13}z)(x + a_{12}y + a_{13}z),$$  

and similarly

$$\text{(4.4)} \quad (zyz)\alpha = (a_{32}y + a_{33}z)(a_{22}y + a_{23}z)(a_{32}y + a_{33}z) + (b_{\bar{x}}y + c_{\bar{x}}z + d_{\bar{x}}yz)(a_{22}y + a_{23}z)(a_{32}y + a_{33}z) + (a_{32}y + a_{33}z)(b_{\bar{y}}y + c_{\bar{y}}z + d_{\bar{y}}yz)(a_{32}y + a_{33}z) + (a_{32}y + a_{33}z)(a_{22}y + a_{23}z)(b_{\bar{x}}y + c_{\bar{x}}z + d_{\bar{x}}yz),$$  

where we are writing

$$y\alpha \equiv a_{22}y + a_{23}z + b_{\bar{y}}y + c_{\bar{y}}z + d_{\bar{y}}yz$$

$$z\alpha \equiv a_{32}y + a_{33}z + b_{\bar{x}}y + c_{\bar{x}}z + d_{\bar{x}}yz$$

modulo $\gamma_3(k \otimes \mathbb{Z} \Lambda)$.

Using $yzy = -zyy = yxx$ and $yzx = -yzz = zxx$, we can express the right-hand side of (4.4) as a linear combination of commutators of length 4. This enables us to
apply Lemma 3.1. Comparing, first of all, the coefficients of \( yzx = yzy = -zyy \)
in (4.3) and (4.4), we get

\[
(4.5) \quad a_{32} \left( 1 + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \right) = 0.
\]

Similarly, comparing the coefficients of \( zxx = zyx = -yzz \) in (4.3) and (4.4), we obtain

\[
(4.6) \quad a_{33} \left( 1 - \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \right) = 0.
\]

The equations (4.5) and (4.6) yield 2 implies that

\[
(4.7) \quad \alpha = 1, \quad \gamma = 0.
\]

and similarly

\[
(4.8) \quad z(x + a_{12}y + a_{13}z)(x + a_{12}y + a_{13}z)
\]

Next we derive consequences from the relation \((yxx)\alpha = (yzy)\alpha\), coming from the relation \(R_2\). Since \(\gamma_5(k \otimes \Lambda) = 0\), we have

\[
(4.9) \quad (yxx)\alpha = (y + a_{23}z)(x + a_{12}y)(x + a_{12}y),
\]

and similarly

\[
(4.10) \quad (yzy)\alpha = (y + a_{23}z)z(y + a_{23}z)
\]

where we are writing

\[
y\alpha \equiv y + a_{23}z + b_y xy + c_z xz + d_y yz
\]

\[
z\alpha \equiv z + b_z xy + c_y xz + d_y yz
\]
modulo $\gamma_3(k \otimes_{\mathbb{Z}} \Lambda)$, as before.

The right-hand side of (4.10) can be expressed as a linear combination of commutators of length 4, so we may apply Lemma 3.1. Considering the coefficients for $zxxx = -yzz$ in (4.9) and (4.10), we get $a_{23} = -a_{23}$ and so $a_{23} = 0$. Next we consider the coefficients for $xyxy = xyyx = -yxyx = -yxxy$ in (4.9) and (4.10). Non-zero contributions only come from (4.9), and we get $a_{12}(xxyy + yxyx) = 0$, so $-2a_{12}xxyy = 0$ and hence $a_{12} = 0$. This means that $A$ is the identity matrix so that $\alpha$ acts trivially on $(k \otimes_{\mathbb{Z}} \Lambda)$.

We claim that, in fact, (4.11)$$\begin{align*}
xa &\equiv x \\
y\alpha &\equiv y + b_1xy + b_2xz \\
z\alpha &\equiv z + b_3xy.
\end{align*}$$

We claim that, in fact, $b_1 = b_2 = b_3 = 0$. Indeed, the relation $R_1$ gives

$$0 = (yxxx - yzyy)\alpha = -b_1xyyz - b_2xzzz - b_1(xy)(xy) - b_2(zy)(xz) - b_3y(xy)y.$$

From the standard relations $(yz)(xy) = xzyy - xyyz$ and $(yz)(xz) = xzzy - xzyz$, compare (A.2), we deduce that

$$0 = b_1(2xzyy - xyyz) + b_2(2xzzz - xzyz) - b_3xxyy.$$ 

Using (B.3), we obtain

$$0 = b_1xzyy + b_2(2(2xzyy - xxyz) - xzyz) - b_3xxyy$$

and

$$= b_1xzyy + b_2(3xzyy - 2xzyz) - b_3xxyy.$$

Since all four Lie products involved belong to the basis (3.4), we obtain that $b_1 = b_2 = b_3 = 0$. It follows that $\alpha$ indeed lies in $(\mathbf{M}(k))g$. \hfill \Box

5. Computation of the zeta function

Based on Theorem 4.2 and using the discussion leading up to Theorem 2.3, we proceed to compute the local pro-isomorphic zeta functions of the Lie lattice $\Lambda$ defined in Section 3. By Theorem 4.2, the connected component of the identity of the algebraic automorphism group $G = \text{Aut}(\Lambda)$ decomposes as

$$G^0 = N \rtimes H,$$

where $N = M_g$ is the unipotent radical and $H = \tilde{T}^g \varrho$ is a diagonal group. We consider $G$ as a subgroup of $\text{GL}_{25}$, via the $\mathbb{Z}$-basis $(x, y, \ldots, xzyz)$ of $\Lambda$ in (3.4).

We employ the set-up of [3 Section 2] as summarised in Section 2. Fix a prime $p$ and observe that the condition $G(Q_p) = G(Z_p)G^0(Q_p)$ is satisfied. We write

$$G = G^0(Q_p), \quad N = N(Q_p), \quad H = H(Q_p).$$

The natural $G$-module $V = \text{span}_{Q_p} \langle x, y, \ldots, xzyz \rangle \cong Q_p^{25}$ contains $L = Z_p \otimes_{\mathbb{Z}} \Lambda$ as a sublattice: $L$ consists of the integral elements of $V$. Moreover, $V$ decomposes into a
direct sum $V = U_1 \oplus U_2 \oplus U_3 \oplus U_4$ of $H$-stable subspaces

\begin{align*}
U_1 &= \langle x, y, z \rangle & \text{of dimension } r_1 = 3, \\
U_2 &= \langle xy, xz, yz \rangle & \text{of dimension } r_2 = 3, \\
U_3 &= \langle xyy, xzz, xyz, xzy, xyx, xzx \rangle & \text{of dimension } r_3 = 6, \\
U_4 &= \langle xyy, xzzz, \ldots, xzyz \rangle & \text{of dimension } r_4 = 13.
\end{align*}

Indeed, the elements of $H$ are precisely the diagonal matrices of the form

\begin{equation}
\begin{pmatrix}
1 & 0 & 0 & \alpha_1 & \alpha_2 & \alpha_3 & \delta_{1,7} & \delta_{1,8} & \cdots & \delta_{1,25} \\
0 & 1 & 0 & \nu & 0 & \sigma & \delta_{2,7} & \delta_{2,8} & \cdots & \delta_{2,25} \\
0 & 0 & 1 & 0 & \nu & \tau & \delta_{3,7} & \delta_{3,8} & \cdots & \delta_{3,25}
\end{pmatrix},
\end{equation}

where $a, b, c \in \mathbb{Q}_p^\times$ such that $a^3 = bc$. Furthermore, for every element $n \in N \subseteq \mathrm{GL}_{25}(\mathbb{Q}_p)$, the $3 \times 25$ matrix comprising the first three rows of $n$ has the form

\begin{equation}
\begin{pmatrix}
1 & 0 & 0 & \alpha_1 & \alpha_2 & \alpha_3 & \delta_{1,7} & \delta_{1,8} & \cdots & \delta_{1,25} \\
0 & 1 & 0 & \nu & 0 & \sigma & \delta_{2,7} & \delta_{2,8} & \cdots & \delta_{2,25} \\
0 & 0 & 1 & 0 & \nu & \tau & \delta_{3,7} & \delta_{3,8} & \cdots & \delta_{3,25}
\end{pmatrix},
\end{equation}

with $\alpha_1, \alpha_2, \alpha_3, \nu, \sigma, \tau, \delta_{i,j} \in \mathbb{Q}_p$ for $(i, j) \in \{1, 2, 3\} \times \{7, 8, \ldots, 25\}$. Observe that the entries in other rows are uniquely determined by those in the first three rows, due to our choice of basis.

For $i \in \{1, \ldots, 5\}$ we put $V_i = \gamma_i(\mathbb{Q}_p \otimes \mathbb{Z} L)$, so $V_i = U_i \oplus \cdots \oplus U_4$; in particular $V_5 = 0$. For $i \in \{2, \ldots, 5\}$ let $N_{i-1}$ be the kernel of the action of $N$ on $V/V_i$, so in particular $N_1 = N$ and $N_4 = 1$. Recall from Section 2 that the natural representation of $G/N_{i-1}$ in $\mathrm{Aut}(V/V_i)$ is used to define the integral elements $(G/N_{i-1})^\times$ of $G/N_{i-1}$. In order to apply Theorem 2.3 we need to confirm that the ‘lifting condition’ Assumption 2.3 is satisfied. As stated in Section 2, this condition asserts that for every $g_0 N_{i-1} \in (G/N_{i-1})^\times$ there exists $g \in G^+$ such that $g_0 N_{i-1} = g N_{i-1}$. Let $g_0 = n_0 h$, with $n_0 \in N$ and $h \in H$, such that $g_0 N_{i-1} \in (G/N_{i-1})^\times$. Referring to the description of general elements of $N$ in (5.2), choose $n \in N$ to have the same entries as $n_0$ in the first three rows, except that those parameters $\delta_{1,j}, \delta_{2,j}, \delta_{3,j}$ of $n$ effecting contributions from $V_i$ are taken to be zero. Putting $g = nh$, we conclude that $g_0 N_{i-1} = g N_{i-1}$ and that all entries of $g$ in the first three rows are integral, hence $(x)g, (y)g, (z)g \in L$. By Lemma 2.4 $g \in G^+$, hence the lifting condition is indeed satisfied.

Using Theorem 2.3 it is now straightforward to calculate the integral we are interested in. We first calculate $\vartheta_{i-1}(h)$ for $i \in \{2, 3, 4\}$ and $h \in H$, as defined in Section 2. Recall that the free parameters in a matrix $n \in N$ all come from the top three rows. Moreover, Lemma 2.4 shows that to test whether $(n N_i) \tau_i(h) \in \mathrm{Mat}_{n_i \times i}(\mathbb{Z}_p)$ for $n N_i \in N_{i-1}/N_i$ it is necessary and sufficient to check the corresponding entries in the first three rows; see Section 2 for an explanation of how $N_{i-1}/N_i$ is identified with a $\mathbb{Q}_p$-vector space.

For $h = \text{diag}(a, b, c, \ldots) \in H$, we read off from (5.1) that

\begin{align*}
\vartheta_1(h) &= |a^3 b^c c_p|^{-1} \min\{|b|_{p}^{-1}, |c|_{p}^{-1}\}, & \vartheta_2(h) &= |a^{24} b^{15} c_{p}^{45}|_{p}^{-1}, & \vartheta_3(h) &= |a^{66} b^{45} c_{p}^{45}|_{p}^{-1}
\end{align*}

and $\det(h) = a^{33} b^{23} c^{23}$. We define the auxiliary set

\begin{equation}
\mathcal{X} = \{ (a, b) \in \mathbb{Z}_p^2 \mid a, b \neq 0 \text{ and } |b|_p \geq |a|_p^3 \} \subseteq (\mathbb{Q}_p^\times)^2.
\end{equation}
Using Theorem 2.3, this yields

$$\int_{G^+} |\det g_p^*| d\mu_G(g) = \int_{H^+} |\det h_p^*| \prod_{i=1}^3 \partial_i(h) d\mu_H(h)$$

$$= \int_{h=\text{diag}(a,b,c,\ldots)\in H^+} |a|_p^{-93+33s} |b|_p^{-64+23s} |c|_p^{-64+23s} \min\{|b|_p^{-1}, |c|_p^{-1}\} \, d\mu_H(h)$$

$$= \int_{(a,b)\in X_{c=a^3b^{-1}}} |bc|_p^{-31+11s} |b|_p^{-64+23s} |c|_p^{-64+23s} \min\{|b|_p^{-1}, |c|_p^{-1}\} \, d\mu_{Q_p^\times}^s(a) \, d\mu_{Q_p^\times}^s(b)$$

$$= \int_{(a,b)\in X_{c=a^3b^{-1}}} |bc|_p^{-95+34s} \min\{|b|_p^{-1}, |c|_p^{-1}\} \, d\mu_{Q_p^\times}^s(a) \, d\mu_{Q_p^\times}^s(b)$$

$$= \sum_{i,j \geq 0 \atop 3|(i+j)} p^{(i+j)(95-34s)} \min\{p^i, p^j\} \mu_{Q_p^\times}^s(p^{(i+j)/3}Z_p^s) \mu_{Q_p^\times}^s(p^iZ_p^s)$$

$$= \sum_{i,j \geq 0 \atop 3|(i+j)} \min\{p^i, p^j\} X^{i+j} \bigg|_{X=p^{95-34s}} ,$$

where $\mu_{Q_p^\times}$ denotes the Haar measure on the multiplicative group $Q_p^\times$, normalised so that $\mu_{Q_p^\times}(Z_p^s) = 1$. Now

$$\sum_{i,j \geq 0 \atop 3|(i+j)} \min\{p^i, p^j\} X^{i+j} = \sum_{m,n \geq 0} \min\{p^{3m}, p^{3n}\} X^{3m+3n}$$

$$+ \sum_{m,n \geq 0} \min\{p^{3m+1}, p^{3n+2}\} X^{3m+3n+3} + \sum_{m,n \geq 0} \min\{p^{3m+2}, p^{3n+1}\} X^{3m+3n+3}$$

$$= \sum_{m,n \geq 0} \min\{p^{3m}, p^{3n}\} X^{3m+3n} + 2pX^3 \sum_{m,n \geq 0} \min\{p^{3m}, p^{3n+1}\} X^{3m+3n} .$$

We calculate these two pieces separately:

$$\sum_{m,n \geq 0} \min\{p^{3m}, p^{3n}\} X^{3m+3n} = 2 \sum_{m,k \geq 0} p^{3m} X^{6m+3k} - \sum_{m \geq 0} p^{3m} X^{6m}$$

$$= \frac{1 + X^3}{(1 - X^3)(1 - p^3X^6)}$$

and

$$2pX^3 \sum_{m,n \geq 0} \min\{p^{3m}, p^{3n+1}\} X^{3m+3n} = 2pX^3 \left( \sum_{n \geq m \geq 0} p^{3m} X^{3m+3n} + \sum_{m > n \geq 0} p^{3n+1} X^{3m+3n} \right)$$

$$= 2pX^3 \left( \sum_{m,k \geq 0} p^{3m} X^{6m+3k} + \sum_{n \geq 0} \sum_{k > 0} p^{3n+1} X^{6n+3k} \right)$$

$$= \frac{2pX^3 + 2p^2X^6}{(1 - X^3)(1 - p^3X^6)} .$$
Summing the two expressions, we arrive at
\[
\int_{G^+} |\det g|^s dg d\mu_G(g) = \left. \frac{1 + X^3 + 2pX^3 + 2p^2X^6}{(1 - X^3)(1 - p^3X^6)} \right|_{X = p^{95 - 34s}}^X = \frac{1 + p^{285 - 102s} + 2p^{286 - 102s} + 2p^{572 - 204s}}{(1 - p^{285 - 102s})(1 - p^{573 - 204s})}.
\]

Applying Propositions 2.1 and 2.2, we obtain Theorem 1.1.

**APPENDIX A. A BASIS FOR THE FREE NILPOTENT LIE ALGEBRA OF CLASS 4 ON 3 GENERATORS**

Let \( F \) be the free nilpotent \( \mathbb{Z} \)-Lie ring of class \( c \) on \( n \) generators \( X_1, \ldots, X_n \). We use simple product notation for the Lie bracket and left-normed notation. The Hall collection process yields an ordered \( \mathbb{Z} \)-basis for \( F \); cf. [8, Ch. 4]. The elements \( X_1, \ldots, X_n \) are basic elements of weight 1, and we order them as \( X_1 < \ldots < X_n \). Basic elements of higher weight \( w \geq 2 \) are defined inductively as follows. If \( C_1 \) and \( C_2 \) are basic elements of weights \( w_1 \) and \( w_2 \) such that \( w = w_1 + w_2 \), then \( B = C_1C_2 \) is a basic element of weight \( w \) provided that (i) \( C_1 > C_2 \) and (ii) if \( C_1 = D_1D_2 \) for basic elements \( D_1, D_2 \), then \( D_2 \leq C_2 \). If \( B \) is a basic element of weight \( w \), then \( C < B \) for any basic element \( C \) of weight less than \( w \). Moreover, if \( C_1, C_2, C_3, C_4 \) are basic elements such that \( B_1 = C_1C_2 \) and \( B_2 = C_3C_4 \) are basic elements of weight \( w \), then \( B_1 < B_2 \) if one of the following holds: (i) \( C_1 < C_3 \), (ii) \( C_1 = C_3 \) and \( C_2 < C_4 \). It is well known that the basic elements of weight up to \( c \) provide a \( \mathbb{Z} \)-basis for \( F \). In fact, for each \( i \in \{1, \ldots, c\} \), the basic elements of weight \( i \) induce a \( \mathbb{Z} \)-basis for the abelian Lie lattice \( \gamma_i(F)/\gamma_{i+1}(F) \).

We are interested in the case \( c = 4 \) and \( n = 3 \). Writing \( X = X_1, Y = X_2 \) and \( Z = X_3 \), we obtain 32 basic elements of weight up to 4. They are, in the described order,

\[
\begin{align*}
X, Y, Z, & \quad YX, ZX, ZY, \\
YXX, YXY, YXZ, & \quad ZXX, ZXY, ZZX, ZYY, ZYZ, \\
(ZX)(YX), & \quad (ZY)(YX), (ZY)(ZX), \\
YXXX, YXXY, YXXZ, & \quad YXXX, ZYYZ, ZYZZ.
\end{align*}
\]

(A.1)

For our computations it is slightly easier to work with left-normed products as basis elements. Using the relations

\[
\begin{align*}
(ZX)(YX) &= \underline{YXXZ} - YXZX \\
(ZY)(YX) &= XYZY - XYZY, \\
(ZY)(ZX) &= XZZY - XYZY,
\end{align*}
\]

(A.2)

where we have underlined terms already occurring up to a sign change in (A.1), one sees that it is permissible to replace the three basis elements which are not left-normed, i.e., \((ZX)(YX), (ZY)(YX)\) and \((ZY)(ZX)\), by \(XYZZ, XYZY\) and \(XZZY\). Reordering the resulting basis, we arrive at the basis displayed in (3.1).
The relations (A.2) are obtained as follows. For elements $T, U, V, W$ of any Lie ring, the Jacobi identity – applied to $T, U, VW$ – yields

\[
(TU)(VW) = (T(VW))U + T(U(VW)) = VWUT - VWTU.
\]

Suitable substitutions for $T, U, V, W$ now provide the relations for $X, Y, Z$.

Applying (A.3) and the Jacobi identity, we derive another useful identity for elements $P, Q, R, S$ of any Lie ring, namely

\[
PQRS = PQSR + (PQ)(RS)
= (PQS)R + RSQP - RSPQ
= ((PS)Q + P(QS))R + RSQP - RSPQ
= PSQR + SQPR + RSQP + SRPQ.
\]

This is the identity (3.2) stated earlier in the paper.

References

[1] M. N. Berman, Uniformity and functional equations for local zeta functions of $\mathbb{R}$-split algebraic groups, Amer. J. Math. 133 (2011), 1–27.
[2] M. P. F. du Sautoy and F. Grunewald, Analytic properties of zeta functions and subgroup growth, Ann. of Math. 152 (2000), 793–833.
[3] M. P. F. du Sautoy and A. Lubotzky, Functional equations and uniformity for local zeta functions of nilpotent groups, Amer. J. Math. 118 (1996), 39–90.
[4] M. P. F. du Sautoy and L. Woodward, Zeta functions of groups and rings, Lecture Notes in Mathematics 1925, Springer Verlag, 2008.
[5] F. J. Grunewald, D. Segal and G. Smith, Subgroups of finite index in nilpotent groups, Invent. Math. 93 (1988), 185–223.
[6] J-I. Igusa, Universal $p$-adic zeta functions and their functional equations, Amer. J. Math. 111 (1989), 671–716.
[7] E. I. Khukhro, $p$-Automorphisms of finite $p$-groups, London Math. Soc. Lecture Note Series 246, Cambridge University Press, Cambridge, 1998.
[8] C. Reutenauer, Free Lie algebras, London Math. Soc. Monographs New Series 7, Claredon Press, Oxford, 1993.
[9] D. Segal, Polycyclic groups, Cambridge Tracts in Mathematics 82, Cambridge University Press, Cambridge, 1983.
[10] C. Voll, Functional equations for zeta functions of groups and rings, Ann. of Math. 172 (2010), 1181–1218.

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