Understanding the Linear See-saw Neutrino Mass Relations and Large Mixing Angle MSW solution

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Abstract

We study the effective ways for generating the linear see-saw neutrino mass relations and large neutrino mixing angles in two classes of grand unified SO(10) models where the texture of Dirac neutrino mass matrix is related to either the charged lepton mass matrix (case A) or the up-quark mass matrix (case B). We also briefly analyse their stability criteria and they are found to be stable under radiative corrections at low energies.

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1 Introduction

Recent results from Super-Kamiokande\cite{1,2} on solar and atmospheric neutrino oscillations indicate a strong positive hint for the existence of tiny neutrino masses. It has been inferred\cite{1} that the solution of the atmospheric neutrino anomaly requires a small squared mass difference between $\nu_\mu$ and $\nu_\tau$ with almost maximal mixing $(\Delta m_{23}^2 \sim (1.5 - 5) \times 10^{-3} eV^2$, and $\sin^2 2\theta_{23} > 0.88)$. Oscillation to sterile neutrinos is also ruled out. In case of the solar neutrino oscillation\cite{2}, the small mixing angle MSW solution, the vacuum oscillation solution and also the oscillation to sterile neutrino, have been ruled out, leaving the only possible option of the large angle MSW solution. Although the solar mixing angle is relatively large, the maximal mixing is not allowed\cite{2,3} $(\Delta m_{12}^2 \sim (2.5 - 15) \times 10^{-5} eV^2$, and $0.25 < \sin^2 2\theta_{12} < 0.65$).

The above observations lead to at least three possible interpretations of the observed $\Delta m_{ij}^2 = |m_{\nu_i}^2 - m_{\nu_j}^2|$, in terms of neutrino mass eigenvalues, viz., hierarchical, reverse hierarchical and quasi-degenerate. If the neutrino masses originate from the see-saw mechanism \cite{4}, then it is most natural to assume the existence of a physical neutrino mass hierarchy, though the other possibilities are not ruled out \cite{5}. For the hierarchical case the relation $m_{\nu_3} > m_{\nu_2} > m_{\nu_1}$ implies $\Delta m_{23}^2 = |m_{\nu_3}^2 - m_{\nu_2}^2| \approx m_{\nu_3}^2$, $\Delta m_{12}^2 = |m_{\nu_2}^2 - m_{\nu_1}^2| \approx m_{\nu_2}^2$, and this fixes the ratio of the two neutrino masses $m_{\nu_2}/m_{\nu_3} \approx \lambda^2$ where $\lambda \approx 0.22$ is the Wolfenstein parameter\cite{6}. However it does not fix the other two neutrino mass ratio $m_{\nu_1}/m_{\nu_2}$ from the observations. There are theoretical speculations\cite{7} that this ratio may range from $\lambda^2$ to $\lambda^4$, and approaches zero for massless $m_{\nu_1}$. We can see interesting features in common among the masses of all the fundamental fermions including neutrinos\cite{7}:

$$m_b : m_s : m_d = 1 : \lambda^2 : \lambda^4;$$

$$m_t : m_c : m_u = 1 : \lambda^4 : \lambda^8$$
where \( p \) is arbitrary as neutrino mass \( m_{\nu_1} \) is not yet fixed from the observations.

In the theoretical front the main ambiguities in the see-saw mechanism[4] for generating small left-handed Majorana neutrino masses, lie in the choice of the texture of the Dirac neutrino mass matrix \( M_\nu \). Grand unified \( SO(10) \) models (with or without SUSY) being employed, in principle predict the textures of the Dirac mass matrices \( M_u, M_d, M_e \) along with \( M_\nu \), and their group theoretical relations. The most general grand unified \( SO(10) \) model[8] (referred to as case A) generally predicts the relation of the Dirac mass matrices \( M_u = M_\nu \propto M_d = M_e \) where \( M_u \) stands for the mass matrix of the up-type quark sector. In another class of left-right symmetric models and their extension to SUSY \( SO(10) \) models[9] (referred to as case B) one obtains the relation \( M_\nu = M_e \tan \beta \) where \( M_e \) is the charged lepton mass matrix and \( \tan \beta = V_u/V_d \). These two cases A and B are again subject to further ambiguities arising from the texture of the right-handed Majorana mass matrix \( M_R \).

In the conventional quadratic see-saw mechanism[4] the tiny left-handed Majorana neutrino mass eigenvalues \( m_{\nu_i} \) vary as Dirac mass eigenvalues \( m_i^2 \) when the right-handed Majorana neutrino masses are assumed to be degenerate \( M_{N_i} = M_N \). This is true for pure diagonal cases[10], and the neutrino mass ratios are simply obtained as \( m_{\nu_2}/m_{\nu_3} = (m_e^2/m_\tau^2) \sim \lambda^8 \), and \( m_{\nu_2}^2/m_{\nu_3}^2 \sim \lambda^4 \) for two types of \( SO(10) \) GUT models in cases A and B respectively. Such quadratic relations to Dirac masses fail to conform with the observations[11]. Alternatively, if the eigenvalues of \( M_{N_i} \) follow the same hierarchy as \( m_i \) in case of non-degenerate right-handed neutrino mass, then it could be possible that \( m_{\nu_i} \) vary linearly as \( m_i \) (hence the name linear seesaw formula)[10]. This can be understood from the fact that \( m_i = h_i V_u \) and \( M_{N_i} = f_i V_R \), and one may take \( f_i/h_i = (1,1,1) \) for \( i = 1,2,3 \), at the symmetry breaking scale.
$M_R$ where the see-saw mechanism is operative[11]. The neutrino mass ratio $m_{\nu 2}/m_{\nu 3}$ are given as $(m_e/m_t) \sim \lambda^4$ in case A, and $(m_{\mu}/m_{\tau}) \sim \lambda^2$ in case B respectively. It is quite clear that the linear see-saw mass relation with charged leptons in case B agrees quite well with the neutrino mass relation of the MSW solution whereas the linear see-saw relation with up-quarks in case A predicts too low neutrino mass ratio[11]. This indicates the ratio $f_i/h_i$ needs to be modified for the case A. These shortcomings had remained for long time until recently Babu and Barr[12] addressed this problem in a class of $SO(10)$ models where $M_R$ has a hierarchy similar to $M_\nu$, and the texture of $M_\nu$ is related to the texture of $M_u$ through a multiplicative factor due to Clebsch coefficients in analogy with Georgi-Jarlskog mechanism[13] in $SU(5)$ GUT. Such modification could rescue [12] the linear see-saw neutrino mass relations in case A, thus keeping the linear see-saw mass relations for both cases A and B at equal footing. However such analysis is true for nearly diagonal textures, and the lepton mixing parameters are inherently absent[12]. The desired lepton mixing angles ($\theta_{12}, \theta_{23}$) for both solar and atmospheric neutrino oscillations can in principle be generated through a number of ways[14]. In case of the linear see-saw formula with up-quarks (in case A) it is easier to get large $\theta_{23}$ and small $\theta_{12}$ as in ref.[12]. This is due to the fact that the large atmospheric mixing and small solar mixing can be imparted from the texture of charged lepton mass matrix, but it is difficult to generate large MSW solar mixing angle from charged lepton sector.

In this paper we study the generation of the effective linear see-saw neutrino mass relations using generalised textures of $M_R$ and $M_\nu$ in both cases A and B, and also generate large solar and maximal atmospheric mixings. We then discuss the stability criteria of the linear see-saw neutrino mass relations and mixing angles under quantum corrections at low energies.
2 Models for the effective linear see-saw mass relation

The left-handed Majorana neutrino mass matrix is given by the see-saw formula \[4\]
\[ m^{\nu}_{LL} = -M_{\nu}M^{-1}_{RR}M^{T}_{\nu} \]  
and the MNS mixing matrix\[15\] by
\[ V_{MNS} = V_{eL}V^{\dagger}_{\nu L} \]
where $V_{\nu L}$ and $V_{eL}$ are defined through the diagonalisation $m^{diag}_{LL} = V_{\nu L}m^{\nu}_{LL}V^{\dagger}_{\nu L}$, and $M^{diag}_{e} = V_{eL}M_{e}V^{\dagger}_{eR}$ respectively.

We first discuss case B where there is a class of SUSY $SO(10)$ model\[9\] which predicts the relation
\[ M_{\nu} = M_{e} \tan \beta \]  
In the basis where the charged lepton mass matrix is diagonal $M_{e} = \text{Diag}(m_{e}, m_{\mu}, m_{\tau})$, the MNS mixing matrix in Eq.(2) is entirely from the texture of $M_{R}$ only through the see-saw formula (1). The light Majorana neutrino mass matrix in Eq.(1) is then given by
\[ m^{\nu}_{LL} = -\tan^{2} \beta M_{e}M^{-1}_{RR}M_{e} \]  
Using the texture of the right-handed neutrino mass matrix\[9\]
\[ M_{R} \sim \begin{pmatrix} \eta & \delta & 0 \\ \delta & \epsilon^{2} & b\epsilon \\ 0 & b\epsilon & 1 \end{pmatrix} v_{R} \]  
the light Majorana neutrino mass matrix in Eq.(4) becomes\[9\]
\[ m^{\nu}_{LL} \sim \begin{pmatrix} -\frac{\epsilon^{2}}{\eta}(1-b^{2})m_{e} & \frac{\delta}{\eta}m_{\mu} & -\frac{\delta}{\eta}b\epsilon m_{\tau} \\ \frac{\delta}{\eta}m_{\mu} & -m_{\mu}^{2} & b\epsilon m_{\mu}m_{\tau} \\ -\frac{\delta}{\eta}b\epsilon m_{\tau} & b\epsilon m_{\mu}m_{\tau} & -(\epsilon^{2}-\frac{\delta^{2}}{\eta})m_{\tau}^{2} \end{pmatrix} \]  
This can generate small angle MSW solution\[9\] with the proper choice of parameters in $M_{R}$ in Eq.(5). Here we are interested to generate large mixing angle MSW solution
and maximal atmospheric mixing along with the right order of linear see-saw neutrino mass ratio. We take the values of the following parameters in Eq.(6):

$$\frac{\delta}{\eta} = -\frac{1}{\lambda}; b = -(1 + k),$$
$$\epsilon = \lambda^2; \delta = \lambda^6, \eta = -\lambda^7$$

and obtain the neutrino mass matrix (6) of the form,

$$m^\nu_{LL} \sim \begin{pmatrix} 2k\lambda^5 & \lambda & (1 + k)\lambda \\ \lambda & 1 & 1 + k \\ (1 + k)\lambda & 1 + k & 1 + \lambda \end{pmatrix}$$

For a specific choice of the value of \(k = 0.18\), Eq.(7) gives the neutrino mass ratio \(m^\nu_2/m^\nu_3 = 0.0341 \sim \lambda^2\), and the following MNS matrix,

$$V_{MNS} = V^\dagger_{\nu L} = \begin{pmatrix} 0.9210 & -0.3623 & 0.1436 \\ -0.3607 & -0.6530 & 0.6660 \\ 0.1475 & 0.6651 & 0.7320 \end{pmatrix}$$

which in turn gives large angle solar and maximal atmospheric mixing parameters (\(\sin^2 2\theta_{12} = 0.464\) and \(\sin^2 2\theta_{23} = 0.991\)) respectively. The neutrino mass ratio can still be increased with the choice of higher value of \(k = 0.2\).

Next we discuss the generation of linear see-saw mass relation in case A. In this class of grand unified SO(10) models[11], all the five Yukawa matrices \(Y_f\) where \(f = u, d, e, \nu, R\) are predicted by the theory. One can have the relation, \(Y_\nu = Y_u \propto Y_d = Y_e\), subject to modification due to group theoretical Clebsch coefficients[13]. We assume these relations are true in SUSY SO(10) as well. The textures of these Yukawa matrices take the following forms[11]:

$$Y_u \sim \begin{pmatrix} 0 & \lambda^6 & 0 \\ \lambda^6 & \lambda^4 & 0 \\ 0 & 0 & 1 \end{pmatrix}, Y_d \sim \begin{pmatrix} 0 & 0 & a\lambda^3 \\ a\lambda^3 & b\lambda/3 & 0 \\ 0 & c & 1 \end{pmatrix}, Y_e \sim \begin{pmatrix} 0 & a\lambda^3 & 0 \\ 0 & -b\lambda & c \\ a\lambda^3 & 0 & 1 \end{pmatrix}$$

$$Y_\nu \sim \frac{3}{8} \begin{pmatrix} 0 & \lambda^6 & 0 \\ \lambda^6 & -8\lambda^4 & 0 \\ 0 & 0 & 1 \end{pmatrix}, Y_R \sim \begin{pmatrix} 0 & \lambda^6 & 0 \\ \lambda^6 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
The light see-saw Majorana mass matrix (1) is given by
\[
m'_{LL} \sim \frac{9}{64} \begin{pmatrix}
  0 & \lambda^6 & 0 \\
  \lambda^6 & -16\lambda^4 & 0 \\
  0 & 0 & 1
\end{pmatrix}
\] (10)

The choice of the coefficients \((a, b, c) = (0.5, 0.4, 0.8)\) in Eq.(9), gives good fits to charged lepton mass hierarchy \((m_e/m_\mu = 0.004, m_\mu/m_\tau = 0.054)\), and \(m'_{LL}\) in Eq.(10) is nearly diagonal, and gives the linear see-saw neutrino mass ratios \(m_{\nu 2}/m_{\nu 3} = 16(m_c/m_t) \sim \lambda^2\) and \(m_{\nu 1}/m_{\nu 2} = \frac{1}{(16)^2}(m_u/m_c) \sim \lambda^8\). In this model large atmospheric mixing angle is derived from charged lepton sector \((\sin^2 2\theta_{23} = 0.953)\). However the model predicts SMA MSW solution \((\sin^2 2\theta_{12} = 0.013)\). In order to generate large solar mixing angle we have to modify the texture of the charged lepton texture in Eq.(9). Following ref.[3] one can construct the charged lepton texture which can give large solar mixing angle and maximal atmospheric angle. The most general charged lepton mass matrix in terms of the mass eigenvalues can be constructed using the relation, \(M_e = V_{eL}^T M_{e\text{diag}} V_{eL}\) where we can have the input \(V_{eL}[3]\),
\[
V_{eL} = \begin{pmatrix}
  -0.93 & 0.37 & 0 \\
  -0.28 & -0.70 & 0.66 \\
  0.24 & 0.61 & 0.75
\end{pmatrix}
\] (11)

Together with \(V_{eL}\) extracted from \(m'_{LL}\) in Eq.(10), one gets the MNS mixing matrix
\[
V_{MNS} = \begin{pmatrix}
  -0.929 & 0.373 & 0 \\
  -0.282 & -0.699 & 0.660 \\
  0.242 & 0.609 & 0.750
\end{pmatrix},
\] (12)

which gives \(\sin^2 2\theta_{23} = 0.984\) and \(\sin^2 2\theta_{12} = 0.472\) leading to maximal atmospheric mixing and large solar mixings respectively. The above two examples, each for cases A and B, show the consistency of the effective linear see-saw model with the large neutrino mixings for solar and atmospheric neutrino oscillations.

3 Stability under quantum corrections

We discuss in brief the stability criteria[16] of the linear see-saw mass relations under quantum corrections[17][18]. The stability conditions imply here that the ratio of
two neutrino masses \( m_{\nu 2}/m_{\nu 3} \) and also the two mixing angles \( \theta_{12} \) and \( \theta_{23} \) do not change much when one moves from the lowest right-handed neutrino mass threshold \( M_{R1} \sim 10^{13} GeV \) scale down to top-quark mass scale. This has been found to be true for mixing angles in hierarchical case [16][19]. In the diagonal charged lepton basis with \( m_{LL}^{\text{diag}} = \text{Diag}(m_{\nu 1}, m_{\nu 2}, m_{\nu 3}) \) and lepton mixing matrix (MNS) in Eq.(2),

\[
V_{\text{MNS}} = \begin{pmatrix}
V_{e1} & V_{e2} & V_{e3} \\
V_{\mu 1} & V_{\mu 2} & V_{\mu 3} \\
V_{\tau 1} & V_{\tau 2} & V_{\tau 3}
\end{pmatrix}
\] (13)

the RG equation for the neutrino mass eigenvalues \( m_{\nu a} \) for MSSM is worked out as [20] \((t = \ln(\mu))\)

\[
d\frac{dt}{16\pi^2} m_{\nu a} = \frac{1}{16\pi^2} \sum_{b = e, \mu, \tau} (K + 2h_b^2 V_{ba}) m_{\nu a}, a = 1, 2, 3
\] (14)

where

\[
K = [ -\frac{6}{5} g_1^2 - 6g_2^2 + 6 Tr M_U^2 ]
\] (15)

Neglecting \( h_\mu^2 \) and \( h_e^2 \) in Eq.(14), and integrating from the lowest right-handed neutrino mass scale \( t_{R1} = \ln(M_{R1}) \) down to top-quark mass scale \( t_0 = \ln(m_t) \), we get the mass ratio \((\text{for } a = 2, 3)\)

\[
R_{23}(t_0)/R_{23}(t_{R1}) \approx \exp[2 \Delta V_{\tau32}^2 I_\tau]
\] (16)

where \( R_{23} = m_{\nu 2}/m_{\nu 3} \) and \( I_\tau = \frac{1}{16\pi^2} \int_{t_{R1}}^{t_0} h_\tau^2 dt \). In getting Eq.(16) we have taken \( \Delta V_{\tau32}^2 = (V_{\tau3}^2 - V_{\tau2}^2) \geq 0 \) approximately constant in the entire range of integration. For large \( \tan \beta \) region one can take roughly \( I_\tau \sim 0.15, V_{\tau3} = 0.8 \) and \( V_{\tau2} = 0.6 \) as in Eq.(12), then the increase in \( R_{23} \) while running from \( M_{R1} \) scale down to at \( m_t \) scale is nearly 10%, thus maintaining the neutrino mass hierarchy even at low energies[19]. This is a desirable result and helps in attaining best-fit value at low energies. The stability of the atmospheric mixing parameter \( S_{at} = \sin^2 2\theta_{23} \) is clear from the evolution equation[20]

\[
16\pi^2 \frac{d}{dt} \sin^2 2\theta_{23} = -2\sin^2 2\theta_{23} \cos^2 \theta_{23} (h_\tau - h_\mu) \frac{m_{\nu 3} + m_{\nu 2}}{m_{\nu 3} - m_{\nu 2}}. \] (17)
Since linear see-saw neutrino mass relation guarantees hierarchical relation $m_{\nu 3} > m_{\nu 2}$, the parameter $S_{at}$ increases at low energies as long as $V_{\tau 2}$ approaches $V_{\tau 3}$ and this helps in reaching maximal value at low energies. The same analysis holds true for the solar mixing as well[19].

## 4 Conclusion

To summarise, we have studied the ways to generate effective linear see-saw neutrino mass ratios in two classes of $SO(10)$ models (cases A and B) where the texture of Dirac neutrino mass matrix is related to either up-quark or charged lepton mass matrix. In both cases we obtained LMA MSW solution and maximal atmospheric mixing while preserving linear see-saw neutrino mass relation. It will be useful to examine more examples in both cases for generating LMA MSW solution. The linear hierarchical neutrino mass ratio and the mixing angles in these models are found to be stable under radiative corrections. Such hierarchical linear relation is important in finding a common dynamical scheme for generating and understanding possible relations among the masses of all fundamental fermions in nature.

## References

[1] H.Sobel, talk presented at the XIX International Conference on Neutrino Physics and Astrophysics, Sudbury, Canada, June 16-21, 2000.

[2] Y.Suzuki, talk presented at the XIX International Conference on Neutrino Physics and Astrophysics, Sudbury, Canada, June 16-21, 2000.

[3] B.R.Desai, U.Sarkar, A.R.Vaucher, [hep-ph/0007344](http://arxiv.org/abs/hep-ph/0007344) and further references therein.
[4] M.Gell-Mann, P.Ramond and R.Slansky in Sanibel Talk, CALT-68-709, Feb 1979, and in Supergravity (North Holland, Amsterdam 1979);
T.Yanagida in Proc.of the Workshop on Unified Theory and Baryon Number of the Universe, KEK, Japan, 1979;
R.N.Mohapatra and G.Senjanovic, Phys.Rev.Lett.44(1980)912; Phys. Rev. D23 (1981) 165.

[5] S.F.King and N.Nimai Singh, hep-ph/0007243, and further references therein.

[6] L.Wolfenstein, Phys. Rev. Lett 51(1983) 1945.

[7] P.M.Fishbane and P.Kaus, J.Phys.G: Nucl.Part.Phys.26(2000)295.

[8] K.S.Babu and R.N.Mohapatra, Phys.Rev.Lett.70(1993)2845.

[9] K.S.Babu, B.Dutta and R.N.Mohapatra, Phys. Rev. D60 (1999) 9500; Phys. Lett. B458 (1999) 93.

[10] S.A.Bludman, D.C.Kennedy and P.G.Langacker, Nucl. Phys. B374 (1992) 373;
M.K.Parida and M.Rani, Phys. Lett. Phys. lett. B377 (1996) 89; M.K.Parida and N.Nimai Singh, Phys. Rev. D59 (1998) 032002.

[11] N.Nimai Singh and S.Biramani Singh, Pramana J. Phys. 54 (2000) 235;
S. Biramani Singh and N. Nimai Singh, J. Phys. G: Nucl. Part. Phys. 25 (1999) 1009.

[12] K.S.Babu and S.M.Barr, hep-ph/0004118.

[13] H.Georgi and C.Jarlskog, Phys.Lett.B86(1979)297.

[14] S.M.Barr and I.Dorsner, hep-ph/0003058.

[15] Z.Maki, M.Nakagawa and S.Sakata, Prog.Theor.Phys.28(1962)870.
[16] N. Haba and N. Okamura, Eur. Phys. J. C14 (2000) 347; J. A. Casas, J. R. Espinosa, A. Ibarra and I. Navarro, Nucl. Phys. B556 (1999) 3; J. Ellis, S. Lola, Phys. Lett. B458 (1999) 310; K. R. S. Balaji, A. S. Dinghe, R. N. Mohapatra, M. K. Parida, Phys. Rev. Lett. 84 (2000) 5034.

[17] K. S. Babu, C. N. Leung and J. Pantaleone, Phys. Lett. B319 (1993) 319.

[18] P. H. Chankowski and Z. Pluciennik, Phys. Lett. B316 (1993) 312.

[19] S. F. King and N. Nimai Singh, hep-ph/0006229.

[20] P. H. Chankowski, W. Krolakowski and S. Pokorski, Phys. Lett. B473 (2000) 109.