This note will appear as an appendix to the paper of Michael Freedman, *A magnetic model with a possible Chern-Simons phase* [F]; it may, however, be read independently of [F]. The purpose of this paper is to prove the following result.

**Theorem 0.1.** For a generic value of the parameter, the Temperley-Lieb category has no non-zero, proper tensor ideal. When the parameter \( d = 2 \cos(\pi/n) \) for some \( n \geq 3 \), then the Temperley-Lieb category has exactly one non-zero, proper ideal, namely the ideal of negligible morphisms.

Our notation in the appendix differs slightly from that in [F]. We write \( t \) instead of \( -A^2 \), \( T_n \) for the Temperley-Lieb algebra with \( n \) strands, and \( TL \) for the Temperley-Lieb category. We trust that this notational variance will not cause the reader any difficulty.

1. **The Temperley-Lieb Category**

1.1. **The Generic Temperley Lieb Category.** Let \( t \) be an indeterminate over \( \mathbb{C} \), and let \( d = (t + t^{-1}) \). The *generic Temperley Lieb category* \( TL \) is a strict tensor category whose *objects* are elements of \( \mathbb{N} = \{0, 1, 2, \ldots \} \). The set of *morphisms* \( \text{Hom}(m, n) \) from \( m \) to \( n \) is a \( \mathbb{C}(t) \) vector space described as follows:

- If \( n - m \) is odd, then \( \text{Hom}(m, n) \) is the zero vector space.
- For \( n - m \) even, we first define \( (m, n) \)-TL diagrams, consisting of:
  1. a closed rectangle \( R \) in the plane with two opposite edges designated as top and bottom.
  2. \( m \) marked points (vertices) on the top edge and \( n \) marked points on the bottom edges.
  3. \((n + m)/2\) smooth curves (or “strands”) in \( R \) such that for each curve \( \gamma \), \( \partial \gamma = \gamma \cap \partial R \) consists of two of the \( n + m \) marked points, and such that the curves are pairwise non-intersecting.

Two such diagrams are *equivalent* if they induce the same pairing of the \( n + m \) marked points. \( \text{Hom}(m, n) \) is defined to be the \( \mathbb{C}(t) \) vector space with...
basis the set of equivalence classes of \((m, n)\)-TL diagrams; we will refer to equivalence classes of diagrams simply as diagrams.

The composition of morphisms is defined first on the level of diagrams. The composition \(ba\) of an \((m, n)\)-diagram \(b\) and an \((\ell, m)\)-diagram \(a\) is defined by the following steps:

1. Juxtapose the rectangles of \(a\) and \(b\), identifying the bottom edge of \(a\) (with its \(m\) marked points) with the top edge of \(b\) (with its \(m\) marked points).
2. Remove from the resulting rectangle any closed loops in its interior.
3. The product \(ba\) is \(dc\), where \(r\) is the number of closed loops removed.

The composition product evidently respects equivalence of diagrams, and extends uniquely to a bilinear product

\[
\text{Hom}(m, n) \times \text{Hom}(\ell, m) \to \text{Hom}(\ell, n),
\]

hence to a linear map

\[
\text{Hom}(m, n) \otimes \text{Hom}(\ell, m) \to \text{Hom}(\ell, n).
\]

The tensor product of objects in TL is given by \(n \otimes n' = n + n'\). The tensor product of morphisms is defined by horizontal juxtaposition. More exactly, the tensor product \(a \otimes b\) of an \((n, m)\)-TL diagram \(a\) and an \((n', m')\)-diagram \(b\) is defined by horizontal juxtaposition of the diagrams, the result being an \((n + n', m + m')\)-TL diagram.

The tensor product extends uniquely to a bilinear product

\[
\text{Hom}(m, n) \times \text{Hom}(m', n') \to \text{Hom}(m + m', n + n'),
\]

hence to a linear map

\[
\text{Hom}(m, n) \otimes \text{Hom}(m', n') \to \text{Hom}(m + m', n + n').
\]

For each \(n \in \mathbb{N}_0\), \(T_n := \text{End}(n)\) is a \(\mathbb{C}(t)\)-algebra, with the composition product. The identity \(1_n\) of \(T(n)\) is the diagram with \(n\) vertical (non-crossing) strands. We have canonical embeddings of \(T_n\) into \(T_{n+m}\) given by
$x \mapsto x \otimes 1_m$. If $m > n$ with $m - n$ even, there also exist obvious embeddings of $\text{Hom}(n, m)$ and $\text{Hom}(m, n)$ into $T_m$ as follows: If $\cap$ and $\cup$ denote the morphisms in $\text{Hom}(0, 2)$ and $\text{Hom}(2, 0)$, then we have linear embeddings

$$a \in \text{Hom}(n, m) \mapsto a \otimes \cap(m-n)/2 \in T_m$$

and

$$b \in \text{Hom}(m, n) \mapsto b \otimes \cup(m-n)/2 \in T_m.$$

Note that these maps have left inverses which are given by premultiplication by an element of $\text{Hom}(n, m)$ in the first case, and postmultiplication by an element of $\text{Hom}(m, n)$ in the second. Namely,

$$a = d^{-(m-n)/2} (a \otimes \cap(m-n)/2) \circ (1_n \otimes \cap(m-n)/2)$$

and

$$b = d^{-(m-n)/2} (1_n \otimes \cap(m-n)/2) \circ (b \otimes \cap(m-n)/2).$$

By an ideal $J$ in TL we shall mean a vector subspace of $\bigoplus_{n,m} \text{Hom}(n, m)$ which is closed under composition and tensor product with arbitrary morphisms. That is, if $a, b$ are composable morphisms, and one of them is in $J$, then the composition $ab$ is in $J$; and if $a, b$ are any morphisms, and one of them is in $J$, then the tensor product $a \otimes b$ is in $J$.

Note that any ideal is closed under the embeddings described just above, and under their left inverses.

1.2. Specializations and evaluable morphisms. For any $\tau \in \mathbb{C}$, we define the specialization $\text{TL}(\tau)$ of the Temperley Lieb category at $\tau$, which is obtained by replacing the indeterminant $t$ by $\tau$. More exactly, the objects of $\text{TL}(\tau)$ are again elements of $\mathbb{N}_0$, the set of morphisms $\text{Hom}(m, n)(\tau)$ is the $\mathbb{C}$–vector space with basis the set of $(m, n)$–TL diagrams, and the composition rule is as before, except that $d$ is replaced by $d(\tau) = (\tau + \tau^{-1})$. Tensor products are defined as before. $T_n(\tau) := \text{End}(n)$ is a complex algebra, and $x \mapsto x \otimes 1_m$ defines a canonical embedding of $T_n(\tau)$ into $T_{n+m}(\tau)$. One also has embeddings $\text{Hom}(m, n) \to T_n$ and $\text{Hom}(n, m) \to T_n$, when $m < n$, as before. An ideal in $\text{TL}(\tau)$ again means a subspace of $\bigoplus_{n,m} \text{Hom}(n, m)$ which is closed under composition and tensor product with arbitrary morphisms.

Let $\mathbb{C}(t)_{\tau}$ be the ring of rational functions without pole at $\tau$. The set of evaluable morphisms in $\text{Hom}(m, n)$ is the $\mathbb{C}(t)_{\tau}$–span of the basis of $(n, m)$–TL diagrams. Note that the composition and tensor product of evaluable morphisms are evaluable. We have an evaluation map from the set of evaluable morphisms to morphisms of $\text{TL}(\tau)$ defined by

$$a = \sum s_j(t)a_j \mapsto a(\tau) = \sum s_j(\tau)a_j,$$

where the $s_j$ are in $\mathbb{C}(t)_{\tau}$, and the $a_j$ are TL-diagrams. We write $x \mapsto x(\tau)$ for the evaluation map. The evaluation map is a homomorphism for the composition and tensor products. In particular, one has a $\mathbb{C}$–algebra homomorphism from the algebra $T^*_n$ of evaluable endomorphisms of $n$ to the algebra $T_n(\tau)$ of endomorphisms of $n$ in $\text{TL}(\tau)$. 
The principle of constancy of dimension is an important tool for analyzing the specialized categories $\text{TL} (\tau)$. We state it in the form which we need here:

**Proposition 1.1.** Let $e \in T_n$ and $f \in T_m$ be evaluable idempotents in the generic Temperley Lieb category. Let $A$ be the $\mathbb{C}(t)$–span in $\text{Hom}(m,n)$ of a certain set of $(m,n)$–TL diagrams, and let $A(\tau)$ be the $\mathbb{C}$–span in $\text{Hom}(m,n)(\tau)$ of the same set of diagrams. Then

$$\dim_{\mathbb{C}(t)} eAf = \dim_{\mathbb{C}} e(\tau)A(\tau)f(\tau).$$

**Proof.** Let $X$ denote the set of TL diagrams spanning $A$. Clearly

$$\dim_{\mathbb{C}(t)} A = \dim_{\mathbb{C}} A(\tau) = |X|.$$ 

Choose a basis of $e(\tau)A(\tau)f(\tau)$ of the form $\{e(\tau)xf(\tau) : x \in X_0\}$, where $X_0$ is some subset of $X$. If the set $\{exf : x \in X_0\}$, were linearly dependent over $\mathbb{C}(t)$, then it would be linearly dependent over $\mathbb{C}[t]$, and evaluating at $\tau$ would give a linear dependence of $\{e(\tau)xf(\tau) : x \in X_0\}$ over $\mathbb{C}$. It follows that

$$\dim_{\mathbb{C}(t)} eAf \geq \dim_{\mathbb{C}} e(\tau)A(\tau)f(\tau).$$

But one has similar inequalities with $e$ replaced by $1 - e$ and/or $f$ replaced by $1 - f$. If any of the inequalities were strict, then adding them would give $\dim_{\mathbb{C}(t)} A > \dim_{\mathbb{C}} A(\tau)$, a contradiction.

1.3. **The Markov trace.** The Markov trace $\text{Tr} = \text{Tr}_n$ is defined on $T_n$ (or on $T_n(\tau)$) by the following picture, which represents an element in $\text{End}_0 \cong \mathbb{C}(t)$ (resp. $\text{End}(0) \cong \mathbb{C}$).

![Diagram of the Markov trace](image)

Figure 2. The categorical trace of an element $a \in T_n$.

On an $(n,n)$–TL diagram $a \in T_n$, the trace is evaluated geometrically by closing up the diagram as in the figure, and counting the number $c(a)$ of components (closed loops); then $\text{Tr}(a) = d(c(a))$.

It will be useful to give the following inductive description of closing up a diagram. We define a map $\varepsilon_n : T_{n+1} \to T_n$ (known as a conditional expectation in operator algebras) by only closing up the last strand; algebraically
it can be defined by

\[ a \in T_{n+1} \mapsto (1_n \otimes \cup) \circ (a \otimes 1) \circ (1_n \otimes \cap). \]

If \( k > n \), the map \( \varepsilon_{n,k} \) is defined by \( \varepsilon_{n,k} = \varepsilon_n \circ \varepsilon_{n+1} \cdots \circ \varepsilon_{k-1} \). It follows from the definitions that \( \text{Tr}(a) = \varepsilon_{0,n} \) for \( a \in T_n \).

It is well-known that \( \text{Tr} \) is indeed a functional satisfying \( \text{Tr}(ab) = \text{Tr}(ba) \); one easily checks that this equality is even true if \( a \in \text{Hom}(n,m) \) and \( b \in \text{Hom}(m,n) \). We need the following well-known fact:

**Lemma 1.2.** Let \( f \in T_{n+m} \) and let \( p \in T_n \) such that \( (p \otimes 1_m)f(p \otimes 1_m) = f \), where \( p \) is a minimal idempotent in \( T_n \). Then \( \varepsilon_{n,n+m}(f) = \gamma p \), where \( \gamma = \text{Tr}_{n+m}(f)/\text{Tr}_n(p) \)

**Proof.** It follows from the definitions that

\[ p\varepsilon_{n,n+m}(f)p = \varepsilon_{n,n+m}(p \otimes 1_m)f(p \otimes 1_m)) = \varepsilon_{n,n+m}(f). \]

As \( p \) is a minimal idempotent in \( T_n \), \( \varepsilon_{n,n+m}(f) = \gamma p \), for some scalar \( \gamma \). Moreover, by our definition of trace, we have \( \text{Tr}_{n+m}(f) = \text{Tr}_n(\varepsilon_{n,n+m}(f)) = \gamma \text{Tr}_n(p) \). This determines the value of \( \gamma \).

The negligible morphisms \( \text{Neg}(n,m) \) are defined to be all elements \( a \in \text{Hom}(n,m) \) for which \( \text{Tr}(ab) = 0 \) for all \( b \in \text{Hom}(m,n) \). It is well-known that the negligible morphisms form an ideal in \( TL \).

## 2. The Structure of the Temperley Lieb Algebras

### 2.1. The generic Temperley Lieb algebras

Recall that a Young diagram \( \lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k] \) is a left justified array of boxes with \( \lambda_i \) boxes in the \( i \)-th row and \( \lambda_i \geq \lambda_{i+1} \) for all \( i \). For example,

\[ [5,3] = \begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array} \]

All Young diagrams in this paper will have at most two rows. For \( \lambda \) a Young diagram with \( n \) boxes, a Young tableau of shape \( \lambda \) is a filling of \( \lambda \) with the numbers 1 through \( n \) so that the numbers increase in each row and column. The number of Young tableaux of shape \( \lambda \) is denoted by \( f_\lambda \).

The generic Temperley Lieb algebras \( T_n \) are known ([J]) to decompose as direct sums of full matrix algebras over the field \( \mathbb{C}(t) \), \( T_n = \bigoplus_\lambda T_\lambda \), where the sum is over all Young diagrams \( \lambda \) with \( n \) boxes (and with no more than two rows), and \( T_\lambda \) is isomorphic to an \( f_\lambda \)-by-\( f_\lambda \) matrix algebra.

When \( \lambda \) and \( \mu \) are Young diagrams of size \( n \) and \( n+1 \), one has a (non-unital) homomorphism of \( T_\lambda \) into \( T_\mu \) given by \( x \mapsto (x \otimes 1)z_\mu \), where \( z_\mu \) denotes the minimal central idempotent in \( T_{n+1} \) such that \( T_\mu = T_{n+1}z_\mu \). Let \( g_{\lambda,\mu} \) denote the rank of \( (e \otimes 1)z_\mu \), where \( e \) is any minimal idempotent in \( T_\lambda \). It is known that \( g_{\lambda,\mu} = 1 \) in case \( \mu \) is obtained from \( \lambda \) by adding one box, and \( g_{\lambda,\mu} = 0 \) otherwise.

One can describe the embedding of \( T_n \) into \( T_{n+1} \) by a Bratteli diagram (or induction-restriction diagram), which is a bipartite graph with vertices labelled by two-row Young diagrams of size \( n \) and \( n+1 \) (corresponding to
the simple components of $T_n$ and $T_{n+1}$) and with $g_{\lambda,\mu}$ edges joining the vertices labelled by $\lambda$ and $\mu$. That is $\lambda$ and $\mu$ are joined by an edge precisely when $\mu$ is obtained from $\lambda$ by adding one box. The sequence of embeddings $T_0 \to T_1 \to T_2 \to \ldots$ is described by a multilevel Bratteli diagram, as shown in Figure 3.

A tableau of shape $\lambda$ may be identified with an increasing sequence of Young diagrams beginning with the empty diagram and ending at $\lambda$; namely the $j$-th diagram in the sequence is the subdiagram of $\lambda$ containing the numbers $1, 2, \ldots, j$. Such a sequence may also be interpreted as a path on the Bratteli diagram of Figure 3, beginning at the empty diagram and ending at $\lambda$.

2.2. Path idempotents. One can define a family of minimal idempotents $p_t$ in $T_n$, labelled by paths $t$ of length $n$ on the Bratteli diagram (or equivalently, by Young tableaux of size $n$), with the following properties:

1. $p_tp_s = 0$ if $t, s$ are different paths both of length $n$.
2. $z_\lambda = \sum \{p_t : t \text{ ends at } \lambda\}$.
3. $p_t \otimes 1 = \sum \{p_s : s \text{ has length } n+1 \text{ and extends } t\}$

Let $t$ be a path of length $n$ and shape $\lambda$ and let $\mu$ be a Young diagram of size $n + m$. It follows that $(p_t \otimes 1_m)z_\mu \neq 0$ precisely when there is a path

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{bratteli_diagram.png}
\caption{Bratteli diagram for the sequence $(T_n)$}
\end{figure}
on the Bratteli diagram from \( \lambda \) to \( \mu \). It has been shown in [J] that (in our notations) \( \text{Tr}(p_t) = [\lambda_1 - \lambda_2 + 1] \), where \([m] = (t^m - t^{-m})/(t - t^{-1})\) for any integer \( m \), and where \( \lambda \) is the endpoint of the path \( t \). Observe that we get the same value for diagrams \( \lambda \) and \( \mu \) which are in the same column in the Bratteli diagram.

The idempotents \( p_t \) were defined by recursive formulas in [W2], generalizing the formulas for the Jones-Wenzl idempotents in [W1].

### 2.3. Specializations at non-roots of unity.

When \( \tau \) is not a proper root of unity, the Temperley Lieb algebras \( T_n(\tau) \) are semi-simple complex algebras with the “same” structure as generic Temperley Lieb algebras. That is, \( T_n(\tau) = \bigoplus_{\lambda} T_\lambda(\tau) \), where \( T_\lambda(\tau) \) is isomorphic to an \( f_\lambda \)-by-\( f_\lambda \) matrix algebra over \( \mathbb{C} \). The embeddings \( T_n(\tau) \to T_{n+1}(\tau) \) are described by the Bratteli diagram as before. The idempotents \( p_t \), and the minimal central idempotents \( z_\lambda \), in the generic algebras \( T_n \), are evaluable at \( \tau \), and the evaluations \( p_t(\tau) \), resp. \( z_\lambda(\tau) \), satisfy analogous properties.

### 2.4. Specializations at roots of unity and evaluable idempotents.

We require some terminology for discussing the case where \( \tau \) is a root of unity. Let \( q = \tau^2 \), and suppose that \( q \) is a primitive \( \ell \)-th root of unity. We say that a Young diagram \( \lambda \) is critical if \( w(\lambda) := \lambda_1 - \lambda_2 + 1 \) is divisible by \( \ell \). The \( m \)-th critical line on the Bratteli diagram for the generic Temperley Lieb algebra is the line containing the diagrams \( \lambda \) with \( w(\lambda) = m\ell \). See Figure 4.

Say that two non-critical diagrams \( \lambda \) and \( \mu \) with the same number of boxes are reflections of one another in the \( m \)-th critical line if \( \lambda \neq \mu \) and \( |w(\lambda) - m\ell| = |w(\mu) - m\ell| < \ell \). (For example, with \( \ell = 3 \), [2, 2] and [4] are reflections in the first critical line \( w(\lambda) = 3 \).)

For \( \tau \) a proper root of unity, the formulas for path idempotents in [W1] and [W2] generally contain poles at \( \tau \), i.e. the idempotents are not evaluable. However, suitable sums of path idempotents are evaluable.

Suppose \( w(\lambda) \leq \ell \) and \( t \) is a path of shape \( \lambda \) which stays strictly to the left of the first critical line (in case \( w(\lambda) < \ell \), or hits the first critical line for the first time at \( \lambda \) (in case \( w(\lambda) = \ell \)); then \( p_t \) is evaluable at \( \tau \), and furthermore \( \text{Tr}(p_t) = [w(\lambda)]_\tau = (\tau^{w(\lambda)} - \tau^{-w(\lambda)})/(\tau - \tau^{-1}) \).

For each critical diagram \( \lambda \) of size \( n \), the minimal central idempotent \( z_\lambda \) in \( T_n \) is evaluable at \( \tau \). Furthermore, for each non-critical diagram \( \lambda \) of size \( n \), an evaluable idempotent \( z^L_\lambda = \sum p_t \in T_n \) was defined in [GW] as follows: The summation goes over all paths \( t \) ending in \( \lambda \) for which the last critical line hit by \( t \) is the one nearest to \( \lambda \) to the left and over the paths obtained from such \( t \) by reflecting its part after the last critical line in the critical line (see Figure 5).

These idempotents have the following properties (which were shown in [GW]):

1. \( \{ z_\lambda(\tau) : \lambda \text{ critical } \} \cup \{ z^L_\mu(\tau) : \mu \text{ non-critical } \} \) is a partition of unity in \( T_n(\tau) \); that is, the idempotents are mutually orthogonal and sum to the identity.
Proposition 2.1. The ideal of negligible morphisms in $\text{TL}(\tau)$ is generated by the idempotent $p_{[\ell-1]}(\tau) \in T_{\ell-1}(\tau)$. 

Figure 4. Critical lines

2. $z_\lambda(\tau)$ is a minimal central idempotent in $T_n(\tau)$ if $\lambda$ is critical, and $z^L_\lambda(\tau)$ is minimal central modulo the nilradical of $T_n$ if $\lambda$ is not critical (see [GW], Theorem 2.2 and Theorem 2.3).

3. For $\lambda$ and $\mu$ non-critical, $z^L_\lambda(\tau)T_n(\tau)z^L_\mu(\tau) \neq 0$ only if $\lambda = \mu$, or if there is exactly one critical line between $\lambda$ and $\mu$ which reflects $\lambda$ to $\mu$. If in this case $\mu$ is to the left of $\lambda$, $z^L_\lambda T_n z^L_\mu \subseteq T_\mu$ (in the generic Temperley Lieb algebra).

4. Let $z^{reg}_n = \sum p_t$, where the summation goes over all paths $t$ which stay strictly to the left of the first critical line, and let $z^{nil}_n = 1 - z^{reg}_n$. Then both $z^{reg}_n$ and $z^{nil}_n$ are evaluable; this is a direct consequence of the fact that $z^{reg}_n = \sum_{\lambda} z^L_\lambda$, where the summation goes over diagrams $\lambda$ with $n$ boxes with width $w(\lambda) < \ell$. 

Proposition 2.1. The ideal of negligible morphisms in $\text{TL}(\tau)$ is generated by the idempotent $p_{[\ell-1]}(\tau) \in T_{\ell-1}(\tau)$. 


Figure 5. A path and its reflected path.

Proof. Let us first show that $z_{n \ell}^{nil}(\tau)$ is in the ideal generated by $p_{[\ell-1]}(\tau)$ for all $n$. This is clear for $n < \ell$, as $z_{\ell-1}^{nil} = p_{[\ell-1]}$ and $z_n^{nil} = 0$ for $n < \ell - 1$.

Moreover, $z_n^{nil}$ is a central idempotent in the maximum semisimple quotient of $T_n$, whose minimal central idempotents are the $z_{\lambda}^{L}$ with $w(\lambda) \geq \ell$. One checks pictorially that $p_{[\ell-1]}z_{\lambda}^{L} \neq 0$ for any such $\lambda$ (i.e. the path to $[\ell-1]$ can be extended to a path $t$ for which $p_t$ is a summand of $z_{\lambda}^{L}$). This proves our assertion in the maximum semisimple quotient of $T_n$; it is well-known that in this case also the idempotent itself must be in the ideal generated by $p_{[\ell-1]}$. In particular, $\text{Hom}(n, m)z_{m}^{nil}(\tau) + z_n^{nil}(\tau)\text{Hom}(n, m)$ is also contained in this ideal.

By [GW], Theorem 2.2 (c), for $\lambda$ a Young diagram of size $n$, with $w(\lambda) < \ell$, $z_{\lambda}^{L}T_nz_{\lambda}^{L}(\tau)$ is a full matrix algebra, which moreover contains a minimal
idempotent \( p_t \) of trace \( \text{Tr}(p_t) = [w(\lambda)]_\tau \neq 0 \). Therefore
\[
z^L_\lambda T_n z^L_\lambda (\tau) \cap \text{Neg}(n, n) = (0).
\]
Furthermore,
\[
z^\text{reg}_n T_n z^\text{reg}_n (\tau) = \sum z^L_\lambda T_n z^L_\lambda (\tau),
\]
by Fact 4 above, so
\[
z^\text{reg}_n T_n z^\text{reg}_n (\tau) \cap \text{Neg}(n, n) = (0)
\]
as well. Now for \( x \in \text{Neg}(n, n) \), one has \( z^\text{reg}_n (\tau)x z^\text{reg}_n (\tau) = 0 \), so
\[
x \in T_n(\tau)z^\text{nil}_n (\tau) + z^\text{nil}_n (\tau)T_n(\tau).
\]

We have shown that \( \text{Neg}(n, n) \) is contained in the ideal of \( \text{TL}(\tau) \) generated by \( p_{[l-1]} \), for all \( n \). That the same is true for \( \text{Neg}(m, n) \) with \( n \neq m \) follows from using the embeddings, and their left inverses, described at the end of Section 1.1.

\[\square\]

3. Ideals

**Proposition 3.1.** Any proper ideal in \( \text{TL} \) (or in \( \text{TL}(\tau) \)) is contained in the ideal of negligible morphisms.

**Proof.** Let \( a \in \text{Hom}(m, n) \). For all \( b \in \text{Hom}(n, m) \), \( \text{tr}(ba) \) is in the intersection of the ideal generated by \( a \) with the scalars \( \text{End}(0) \). If \( a \) is not negligible, then the ideal generated by \( a \) contains an non-zero scalar, and therefore contains all morphisms.

**Corollary 3.2.** The categories \( \text{TL} \) and \( \text{TL}(\tau) \) for \( \tau \) not a proper root of unity have no non-zero proper ideals.

**Proof.** There are no non-zero negligible morphisms in \( \text{TL} \) and in \( \text{TL}(\tau) \) for \( \tau \) not a proper root of unity.

**Theorem 3.3.** Suppose that \( \tau \) is a proper root of unity. Then the negligible morphisms form the unique non-zero proper ideal in \( \text{TL}(\tau) \).

**Proof.** Let \( J \) be a non-zero proper ideal in \( \text{TL}(\tau) \). By the embeddings discussed at the end of Section 1.1, we can assume \( J \cap T_n \neq 0 \) for some \( n \).

Now let \( a \) be a non-zero element of \( J \cap T_n(\tau) \). Since \( \{z_\lambda(\tau)\} \cup \{z^L_\mu(\tau)\} \) is a partition of unity in \( T_n(\tau) \), one of the following conditions hold:

(a) \( b = az_\mu(\tau) \neq 0 \) for some critical diagram \( \mu \).

(b) \( b = z^L_\mu(\tau)az^L_\mu(\tau) \neq 0 \) for some non-critical diagram \( \mu \).

(c) \( b = z^L_\lambda(\tau)az^L_\lambda(\tau) \neq 0 \) for some pair \( \lambda, \lambda' \) of non-critical diagrams which are reflections of one another in a critical line. In this case, let \( \mu \) denote the leftmost of the two diagrams \( \lambda, \lambda' \).

In each of the three cases, one has \( b \in e(\tau)T_n(\tau)f(\tau) \), where \( e, f \) are evaluable idempotents in \( T_n \) such that \( eT_nf \subseteq T_\mu \). Let \( \alpha \) be a Young diagram...
on the first critical line of size \( n + m \), such that there exists a path on the generic Bratteli diagram connecting \( \mu \) and \( \alpha \). Then one has

\[
\dim \mathbb{C} z_\alpha(\tau)(e(\tau) \otimes 1_m)(T_n(\tau) \otimes \mathbb{C} 1_m)(f(\tau) \otimes 1_m)
= \dim \mathbb{C}(t) z_\alpha(e \otimes \text{id}_m)(T_n \otimes \mathbb{C}(t) 1_m)(f \otimes 1_m)
= \dim \mathbb{C}(t) cT_n f = \dim \mathbb{C} e(\tau)T_n(\tau)f(\tau)
\]

where the first and last equalities result from the principle of constancy of dimension, and the second equality is because \( x \mapsto z_\alpha(x \otimes 1_m) \) is injective from \( T_\mu \) to \( T_\alpha \). But then it follows that \( x \mapsto z_\alpha(\tau)(x \otimes 1_m) \) is injective on \( e(\tau)T_n(\tau)f(\tau) \). In particular \( (b \otimes 1_m)z_\alpha \) is a non-zero element of \( J \cap T_\alpha \). Hence there exists \( c \in T_\alpha \) such that \( f = c(b \otimes 1_m)z_\alpha \) is an idempotent. After conjugating (and multiplying with \( p_{[\ell-1]} \otimes 1_m \), if necessary), we can assume \( f \) to be a subidempotent of \( p_{[\ell-1]} \otimes 1_m \). But then \( \varepsilon_{\ell-1+m,\ell-1}(f) \) is a multiple of \( p_{[\ell-1]} \), by Lemma 1.2, with the multiple equal to the rank of \( f \) in \( T_\alpha \). This, together with Prop. 2.1, finishes the proof.

It is easily seen that TL has a subcategory TL\textsuperscript{ev} whose objects consist of even numbers of points, and with the same morphisms between sets of even points as for TL. The evaluation TL\textsuperscript{ev}(\tau) is defined in complete analogy to TL(\tau).

**Corollary 3.4.** If \( \tau^2 \) is a proper root of unity of degree \( \ell \) with \( \ell \) odd, the negligible morphisms form the unique non-zero proper ideal in TL\textsuperscript{ev}.

**Proof.** If \( \ell \) is odd, \( p_{[\ell-1]} \) is a morphism in TL\textsuperscript{ev}. The proof of the last theorem goes through word for work (one only needs to make sure that one stays within TL\textsuperscript{ev}, which is easy to check). \( \Box \)

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