New BPS Solitons in 2+1 Dimensional Noncommutative $CP^1$ Model

Hideharu Otsu  *
Faculty of Economics,
Aichi University, Toyohashi, Aichi 441-8522, Japan

Toshiro Sato †
Faculty of Policy Science, Matsusaka University,
Matsusaka, Mie 515-8511, Japan

Hitoshi Ikemori ‡
Faculty of Economics, Shiga University,
Hikone, Shiga 522-8522, Japan

Shinsaku Kitakado §
Department of Information Sciences, Meijo University,
Tempaku, Nagoya 486-8502, Japan

Abstract

Investigating the solitons in the non-commutative $CP^1$ model, we have found a new set of BPS solitons which does not have counterparts in the commutative model.

*otsu@vega.aichi-u.ac.jp
†tsato@matsusaka-u.ac.jp
‡ikemori@biwako.shiga-u.ac.jp
§kitakado@ccmfs.meijo-u.ac.jp
1 Introduction

Field theories on the non-commutative space have been extensively studied in the last few years. Particularly, BPS solitons are interesting, because they might not share the common features with those on the commutative space[1][2][3].

Solitons in the non-commutative $CP^1$ model have been studied in [4] and further developed in [5] in connection with the dynamical aspects of the theory. The non-BPS solitons, that do not exist in the commutative case, have been studied in [6]. These investigations were reviewed in [7].

In this paper, we report on a set of new BPS solitons in the non-commutative $CP^1$ model, that does not exist in the commutative limit.

We consider the $CP^1$ model on 2+1 dimensional non-commutative space-time. The space coordinates obey the commutation relation

$$[x, y] = i\theta,$$

or

$$[z, \bar{z}] = \theta > 0,$$

in terms of the complex variables, $z = \frac{1}{\sqrt{2}}(x + iy)$ and $\bar{z} = \frac{1}{\sqrt{2}}(x - iy)$. The Hilbert space can be described in terms of the energy eigenstates $|n\rangle$ of the harmonic oscillator whose creation and annihilation operators are $\bar{z}$ and $z$ respectively,

$$z |n\rangle = \sqrt{\theta} n |n - 1\rangle,$$

$$\bar{z} |n\rangle = \sqrt{\theta(n + 1)} |n + 1\rangle,$$  

The $CP^1$ lagrangian is

$$L = \text{Tr}(|D_t \Phi|^2 - |D_z \Phi|^2 - |D_{\bar{z}} \Phi|^2),$$  

where $\Phi$ is a 2-component complex vector with the constraint $\Phi^\dagger \Phi = 1$. We consider $\Phi$ to be the fundamental field and thus to be non-singular. $\text{Tr}$ denotes the trace over the Hilbert space as

$$\text{Tr O} = 2\pi \theta \sum_{n=0}^{\infty} \langle n | O | n \rangle.$$  

The covariant derivative is defined by

$$D_a \Phi = \partial_a \Phi - i \Phi A_a, \quad A_a = -i \Phi^\dagger \partial_a \Phi, \quad (a = t, z, \bar{z}),$$
where $\partial_z = -\theta^{-1} [\bar{z},]$ and $\partial_{\bar{z}} = \theta^{-1} [z,].$

For the static configurations, topological charge and static energy are given by

$$Q = \frac{1}{2\pi} \text{Tr} \left( |D_z \Phi|^2 - |D_{\bar{z}} \Phi|^2 \right),$$

and

$$E = \text{Tr} \left( |D_z \Phi|^2 + |D_{\bar{z}} \Phi|^2 \right) \geq 2\pi |Q|.$$  

The configuration which saturates the energy bound satisfies the BPS soliton equation

$$D_{\bar{z}} \Phi = 0,$$

or BPS anti-soliton equation

$$D_z \Phi = 0.$$  

It is convenient [4] to introduce the 2-component complex vector $W$ and the projection operator $P$ as

$$\Phi = W \frac{1}{\sqrt{W^\dagger W}}, \quad P = \Phi \Phi^\dagger.$$  

In terms of the projection operator, BPS soliton equations (9) and (10) are [8][9][10]

$$(1 - P) zP = 0,$$

and

$$(1 - P) \bar{z}P = 0,$$

which indicate respectively

$$zW = WV,$$

and

$$\bar{z}W = WV,$$

where $V$ is a scalar function [4]. Topological charge (7) and static energy (8) can be expressed as

$$Q = \frac{1}{2\pi} \text{Tr} \left\{ \partial_z \Phi^\dagger (1 - P) \partial_z \Phi - \partial_{\bar{z}} \Phi^\dagger (1 - P) \partial_{\bar{z}} \Phi \right\},$$

and

$$E = \text{Tr} \left\{ \partial_z \Phi^\dagger (1 - P) \partial_z \Phi + \partial_{\bar{z}} \Phi^\dagger (1 - P) \partial_{\bar{z}} \Phi \right\}.$$  

The examples of BPS soliton are $W = (z^n, 1)^t$, with topological charge $Q = n$ and energy $E = 2\pi n$, and those of BPS anti-soliton are $W = (\bar{z}^n, 1)^t$ with topological charge $Q = -n$ and energy $E = 2\pi n$ [4]. These configurations are solitons also in the commutative theory.
2 New Solitons

We have found that the following is the BPS soliton solution of the non-commutative $CP^1$ model. The configuration of the soliton with the topological charge $Q = -n$ is

$$\Phi = \left( \frac{z^n}{\sqrt{\prod_{l=1}^{n} (\bar{z} z + l \theta)}} 0 \right), \quad (18)$$

which can also be expressed in terms of projection operator $P$ as

$$P = \left( 1 - \sum_{m=0}^{n-1} |m\rangle \langle m| 0 \right). \quad (19)$$

For $Q = n$, on the other hand, the soliton can be written as

$$\Phi = \left( \frac{1}{\sqrt{\prod_{l=1}^{n} (\bar{z} z + l \theta)}} z^n \right), \quad (20)$$

and the corresponding projection operator expression is

$$P = \left( 1 \ 0 \sum_{m=0}^{n-1} |m\rangle \langle m| \right). \quad (21)$$

We can straightforwardly confirm that (19) and (21) satisfy the BPS equations (13) and (12) respectively. The energy of these solitons are of course $E = 2\pi n$ due to the BPS property.

We calculate the topological charge of (18) and (20). For BPS anti-solitons (18), we use

$$\partial_{\bar{z}} \Phi^\dagger (1 - P) \partial_{z} \Phi = 0 \quad (22)$$

which follows from (10), and

$$\partial_{\bar{z}} \Phi^\dagger (1 - P) \partial_{z} \Phi \quad (23)$$

$$= \theta^{-2} \left[ \frac{1}{\sqrt{\prod_{l=1}^{n} (\bar{z} z + l \theta)}} \bar{z}^n, \bar{z} \right] \left( \sum_{m=0}^{n-1} |m\rangle \langle m| \right) \left( z^n, z \right) \left[ \frac{1}{\sqrt{\prod_{l=1}^{n} (\bar{z} z + l \theta)}} \right]$$

$$= \theta^{-1} n |0\rangle \langle 0|.$$  

Substituting (22) and (23) into (16), the topological charge is

$$Q = \frac{1}{2\pi} 2\pi \theta \sum_{k=0}^{\infty} \langle k| (-\theta^{-1} n |0\rangle \langle 0|) |k\rangle = -n. \quad (24)$$
Similarly, for BPS solitons (20), we use
\[ \partial_z \Phi^\dagger (1 - P) \partial_z \Phi = 0 \] (25)
which follows from (14), and
\[ \partial_z \Phi^\dagger (1 - P) \partial_z \Phi = \theta^{-2} \left[ \sum_{l=0}^{n-1} |l\rangle \langle l|, \bar{z} \right] \left( 1 - \sum_{m=0}^{n-1} |m\rangle \langle m| \right) \left[ \bar{z}, \sum_{l=0}^{n-1} |l\rangle \langle l| \right] \] (26)
\[ = \theta^{-1} n |n - 1\rangle \langle n - 1|. \]
The topological charge is \( Q = n \).

3 Discussions

First we note that these solutions do not have the commutative counterparts. In order to see this, we consider
\[ W = \left( a^{-n} \bar{z}^n \prod_{l=1}^{n} (\bar{z}z + l\theta)^{-1} \right) \] (27)
for \( Q < 0 \) and
\[ W = \left( a^{-n} \left( \prod_{l=1}^{n} (\bar{z}z + l\theta)^{-1} \right) z^n \right) \] (28)
for \( Q > 0 \), where \( a \) is a real parameter. Taking the limit \( a \to 0 \) in the non-commutative case, \( \Phi \) reduces to the non-singular configurations (18) and (20) respectively. On the other hand, in the commutative case, the same limit leads to
\[ \Phi = \begin{pmatrix} e^{-in\varphi} \\ 0 \end{pmatrix} \] (29)
and
\[ \Phi = \begin{pmatrix} e^{in\varphi} \\ 0 \end{pmatrix}, \] (30)
for \( z \neq 0 \), where \( z \equiv |z| e^{i\varphi} \). For \( z = 0 \), the limit leads to
\[ \Phi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \] (31)
in both cases. There exists a discontinuous jump at the origin. Which shows that in the commutative theory the configurations of \( CP^1 \) field \( \Phi \) corresponding to \( a \to 0 \) limit of (27), (28) are absent. Thus, we have seen that
the singular configurations in the commutative theory become non-singular in the non-commutative case which can be attributed to non-commutativity of the coordinates.

A relation to the $U(2)$ sigma model discussed in Ref. [9] is interesting. If we define $U = 1 - 2P$ using the projection operator $P$ of (11), $U$ satisfies $U^U = U^2 = 1$ and as such is a $U(2)$ field. Furthermore, the BPS equation in $U(2)$ sigma model are expressed as (12) and (13) [9]. From this viewpoint, our solutions (19) and (21) could be considered as embedding of abelian solutions into $U(2)$ sigma model. Actually, our projector $P$ for $CP^1$ solitons is unitarily equivalent to

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} P_n & 0 \\ 0 & 0 \end{pmatrix}$$

(32)

for $Q = -n$ and

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & P_n \end{pmatrix},$$

(33)

for $Q = n$, where $P_n$ is the projector for abelian solitons of ref. [9]. Accordingly, the computations of energy for our solitons could be reduced to those for the embedded abelian solitons. Which makes it easy to verify $E = 2\pi n$ by means of BPS equations.

Finally, a few words on the properties of the solutions (18) and (20). It follows from (11), that with the scalar function $\Delta$ that commutes with $WW$, the transformation $W \rightarrow W\Delta$ does not change $\Phi$. In order to take the $a \rightarrow 0$ limit in (27) (28), we can use this invariance and rewrite these as

$$W = \left( a^n \prod_{l=1}^{N} (\bar{z}z + l\theta) \right),$$

(34)

and

$$W = \left( a^n \bar{z}z + n-1 \sum_{m=0}^{N-1} |m\rangle \langle m| \right),$$

(35)

where

$$\Delta = a^n \prod_{l=1}^{N} (\bar{z}z + l\theta)$$

(36)

for (27) and

$$\Delta = a^n \bar{z}z + n-1 \sum_{m=0}^{N-1} |m\rangle \langle m|$$

(37)

for (28). Taking the limit of $a \rightarrow 0$, the BPS soliton solutions (27) (28) can be expressed in terms of $W$ as

$$W = \begin{pmatrix} \bar{z}^n \\ 0 \end{pmatrix},$$

(38)
and
\[ W = \left( \sum_{m=0}^{n-1} |m\rangle \langle m| \right). \] (39)

The configurations (38) and (39) satisfy the BPS equations (15) and (14) with
\[ V = \bar{z} \] (40)
and
\[ V = z - \sqrt{n\theta} |n - 1\rangle \langle n|, \] (41)
respectively.

To summarize, we have considered solitons in the non-commutative \( CP^1 \) model and have found new solitons that do not have their counterparts in the commutative theory. Further properties of these solutions will be examined using more general approaches[11][12][13] in a future publication.

References

[1] J. A. Harvey, Komaba lectures on noncommutative solitons and D-branes, [hep-th/0102076].

[2] N. Nekrasov and A. Schwarz, Instantons on noncommutative \( R^2 \) and (2,0) superconformal six dimensional theory, Commun. Math. Phys. 198 (1998) 689–703, [hep-th/9802068].

[3] R. Gopakumar, S. Minwalla, and A. Strominger, Noncommutative solitons, JHEP 05 (2000) 020, [hep-th/0003160].

[4] B.-H. Lee, K.-M. Lee, and H. S. Yang, The \( CP^n \) model on noncommutative plane, Phys. Lett. B498 (2001) 277–284, [hep-th/0007140].

[5] K. Furuta, T. Inami, H. Nakajima, and M. Yamamoto, Low-energy dynamics of noncommutative \( CP^1 \) solitons in 2+1 dimensions, Phys. Lett. B537 (2002) 165–172, [hep-th/0203125].

[6] K. Furuta, T. Inami, H. Nakajima, and M. Yamamoto, Non-BPS solutions of the noncommutative \( CP^1 \) model in (2+1)-dimensions, JHEP 08 (2002) 009, [hep-th/0207166].

[7] J. Murugan and R. Adams, Comments on noncommutative sigma models, JHEP 12 (2002) 073, [hep-th/0211171].
[8] L. Hadasz, U. Lindstrom, M. Rocek, and R. von Unge, Noncommutative multisolitons: Moduli spaces, quantization, finite theta effects and stability, JHEP 06 (2001) 040, [hep-th/0104017].

[9] O. Lechtenfeld and A. D. Popov, Noncommutative multi-solitons in 2+1 dimensions, JHEP 11 (2001) 040, [hep-th/0106213].

[10] R. Gopakumar, M. Headrick, and M. Spradlin, On noncommutative multi-solitons, Commun. Math. Phys. 233 (2003) 355–381, [hep-th/0103256].

[11] K. Hashimoto, Fluxons and exact BPS solitons in non-commutative gauge theory, JHEP 12 (2000) 023, [hep-th/0010251].

[12] M. Hamanaka and S. Terashima, On exact noncommutative BPS solitons, JHEP 03 (2001) 034, [hep-th/0010221].

[13] M. Hamanaka, ADAHM/Nahm construction of localized solitons in noncommutative gauge theories, Phys. Rev. D65 (2002) 085022, [hep-th/0109070].