EXISTENCE OF SOLUTIONS FOR FRACTIONAL INSTANTANEOUS AND NON-INSTANTANEOUS IMPULSIVE DIFFERENTIAL EQUATIONS WITH PERTURBATION AND DIRICHLET BOUNDARY VALUE

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ABSTRACT. A class of fractional instantaneous and non-instantaneous impulsive differential equations under Dirichlet boundary value conditions with perturbation is considered here. The existence of classical solutions is presented by using the Weierstrass theorem. An example is given to verify the validity of the obtained results.

1. Introduction. Impulse is a universal phenomenon in human social activities and nature. According to the duration of the changing process, the impulse can be divided into instantaneous impulse and non-instantaneous impulse. As the name implies, the instantaneous impulse means that the time of the sudden change process is very short relative to the whole development process and can be ignored. A non-instantaneous impulse means that the process of change is dependent on the state and lasts for a period of time that cannot be ignored. During the past years, the instantaneous impulse differential equations have received great attention, which are often used to describe abrupt change, for instance, harvesting, disasters and so on. Detailed information and applications, see e.g. [3, 6, 14, 5, 12, 15, 7, 11, 16, 13, 24, 17] and the cited references. However, not all the phenomena in real life could be described by instantaneous impulses, for example, earthquakes and tsunamis. Thus, more and more scholars began to pay attention to the study of non-instantaneous impulses which can be characterized as impulsive jumps starting at any fixed point and continuing during the finite time interval. Hernández and O'Regan [8] introduced the non-instantaneous impulse differential equations in the context of drug injection for a person. Along this line, non-instantaneous impulse differential equations gradually receive more and more attention, see for example [8, 22, 21, 20, 4, 19].

In the existing papers on non-instantaneous impulses, most only consider the evolution of the impulse in a finite time interval, and few authors take the sudden change of the state at the beginning of the above time interval into account at the same time. Taking the hemodynamic equilibrium of a person as an example, the concentration of drug in the blood will instantly increase after intravenous injection,
but as the drug is absorbed in the body, the concentration in the blood will gradually decrease, which is a gradual and continuous process. And it is more reasonable to use mathematical models with instantaneous and non-instantaneous impulse effects to describe this phenomenon. In recent years, instantaneous and non-instantaneous differential equations with impulses have been paid more attention by scholars, which take both instantaneous impulses and non-instantaneous impulses into consideration, see for instance [18, 23, 25] and the cited references. In [18], Tian and Zhang used the Ekeland’s variational principle to study the second-order differential equations, which have instantaneous and non-instantaneous impulses, and at least one classical solution was obtained. In [23], the authors considered the fractional instantaneous and non-instantaneous impulses differential equations with Dirichlet boundary value. They obtained the existence of solutions by Ekeland’s variational method. In 2020, Zhou et al. [25] studied the fractional instantaneous and non-instantaneous impulses differential equations with $p$-Laplacian term and used the critical point theory to get the existence of solutions.

However, drug absorption in the human body may be different from person to person or affected by individual behavior, which can be regarded as a perturbation, so it is necessary to take the perturbation into account. To the best of my knowledge, there are few (if any) publications concerning with Dirichlet boundary value for instantaneous and non-instantaneous impulses differential equations with perturbation. To fill that gap, in this paper, we mainly consider

\[
\begin{align*}
\frac{d}{dt}D^\alpha_T(z(t)) &= f_j(t, z(t)) + g(t)|z(t)|^{p-2}z(t), \quad t \in \bigcup_{j=0}^{k}(s_j, t_{j+1}], \\
\Delta(D^\alpha_T z)(t_j) &= I_j(z(t_j)), \\
D^\alpha_T z(t_j^+) &= D^\alpha_T z(t_j^-), \\
D^\alpha_T z(s_j) &= D^\alpha_T z(s_j^-),
\end{align*}
\]

where $\alpha \in (\frac{1}{p}, 1], 1 < p < 2$, $f_j, g \in C((s_j, t_{j+1}] \times \mathbb{R}, \mathbb{R})$, $I_j \in C(\mathbb{R}, \mathbb{R})$ and there exists $j \in \{1, 2, \ldots, k\}$ such that $f_j(z(t_j)) \neq 0$. $D^\alpha_T$ is the left Caputo fractional derivative and $D^\alpha_T$ is the right Riemann-Liouville fractional derivative satisfying $D^\alpha_T z(t) = -\frac{d}{dt} D^{\alpha-1}_T z(t)$. And

\[
\Delta(D^\alpha_T z)(t_j) = D^\alpha_T z(t_j^+) - D^\alpha_T z(t_j^-),
\]

where $\{t_j\}_{j \in \mathbb{Z}^+}$, $\{s_k\}_{k \in \mathbb{Z}^+}$ are two increasing sequences and satisfy $0 = s_0 < t_1 < s_1 < t_2 < \cdots < s_k < t_{k+1} = T$.

We will mainly use the Weierstrass theorem to get the existence of solutions for problem (1). Compared with the existing works, the paper has the following new sights: Firstly, we focus on the fractional instantaneous and non-instantaneous impulsive differential equations with perturbation, where $g(t)|z(t)|^{p-2}z(t)$ is the disturbing term and is sublinear; Secondly, when $p = 2$, the problem (1) is reduced to the problem in [16] and when $\alpha = 1$, problem (1) is simplified to an integer differential equation, which can be viewed as a supplement and extension of the problem in [16]. Finally, we give a definition of classical solution and then give the proof that the weak solution is also the classical solution for problem (1).

In the following, necessary notations and some Lemmas are provided in part 2. In part 3, we give the proof of the main theorem. In part 4, an example is presented to illustrate the applications of the obtained results.
2. Preliminaries.

2.1. Definitions.

Definition 2.1. Set $\mathcal{K} = [0, T]$

$$E_0^{\alpha, \gamma} = \{ z : \mathcal{K} \to \mathbb{R} | z, D_t^\alpha z \in L^\gamma(\mathcal{K}, \mathbb{R}), \ z(0) = z(T) = 0, \ \alpha \in (0, 1], \ \gamma \in (1, +\infty) \}$$

denoted with the norm

$$\|z\|_{\alpha, \gamma} = \left( \int_\mathcal{K} |z(t)|^\gamma dt + \frac{\int_\mathcal{K} ||D_t^\alpha z(t)||^\gamma dt}{\Gamma(1-\alpha)} \right)^{\frac{1}{\gamma}}.$$  

From Definition 2.1, we can see $E_0^{\alpha, \gamma}$ is defined by the closure of $C_0^{\infty}(\mathcal{K}, \mathbb{R})$. If $\gamma = 2$, we define

$$E_0^{\alpha, 2} = E_0^{\alpha}.$$

Definition 2.2. Set $\mathcal{B} = \left\{ z \in AC([0, T]) : \int_{s_j}^{t_{j+1}} \left(\int_\mathcal{B} \{ |z(t)|^2 + ||D_t^\alpha z(t)||^2 \} dt < +\infty, \ j = 0, 1, 2, \ldots, k \right) \right\},$ if $z \in \mathcal{B}$ satisfies the conditions of problem (1) and $z(0) = z(T) = 0$ holds, then $z$ is a classical solution of (1).

2.2. Propositions.

Proposition 1. Let $\alpha \in (0, 1]$, then $E_0^\alpha$ is a complete normed vector space which is separable and reflexive.

Proposition 2. Let $\alpha \in (0, 1], \ z \in AC([a, b]),$ which is the set of mappings $z : [a, b] \to \mathbb{R}$ whose components are absolutely continuous functions. Then, for $t \in [a, b]$, there exist

$$\int_a^c D_t^\alpha z(t) = a D_t^\alpha z(t) - \frac{z(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha},$$

$$\int_a^c t D_t^\alpha z(t) = t D_t^\alpha z(t) - \frac{z(b)}{\Gamma(1-\alpha)}(b-t)^{-\alpha}.$$  

Proposition 3. ([9]) Let $\alpha \in (0, 1], \ \mu, \nu \in L^2([a, b], \mathbb{R})$, then

$$\int_a^b t D_t^{\alpha-1} \mu(t) \nu(t) dt = \int_a^b \mu(t) a D_t^{\alpha-1} \nu(t) dt.$$  

Proposition 4. ([9]) Let $\alpha \in (\frac{1}{2}, 1].$ For any $z \in E_0^\alpha$, we have

$$\|D_t^\alpha z\|_{L^2} \geq -\frac{\Gamma(1+1)}{T^\alpha} \|z\|_{L^2}, \ \|D_t^\alpha z\|_{L^2} \geq \frac{\Gamma(\alpha)(2\alpha-1)}{T^{\alpha-\frac{1}{2}}} \|z\|_{\infty}, \ (2)$$

where $\|z\|_{L^2} = \left( \int_0^T |z(t)|^2 dt \right)^{\frac{1}{2}}, \ \|z\|_{\infty} = \max_{t \in [0, T]} |z(t)|.$

Proposition 5. ([9]) Let $\alpha \in (\frac{1}{2}, 1],$ then the embedding $E_0^\alpha \hookrightarrow C([0, T], \mathbb{R})$ is compact.
2.3. Lemmas.

Lemma 2.3. ([1]) (Weierstrass Theorem) Set $X$ be a complete normed vector space which is reflexive, $\Phi : X \to \mathbb{R}$ is weakly lower semicontinuous and coercive. Then $\Phi$ has a global minimum point.

Lemma 2.4. ([12]) If $\Phi : X \to (-\infty, +\infty]$ is coercive, then $\Phi$ has a minimizing sequence which is bounded.

Lemma 2.5. (Hölder inequality) $m(x), l(x)$ are two continuous and nonnegative functions on $[a, b]$, then

$$\int_a^b m(x)l(x)dx \leq \left( \int_a^b m^{p_1}(x)dx \right)^{\frac{1}{p_1}} \left( \int_a^b l^{p_2}(x)dx \right)^{\frac{1}{p_2}},$$

where $\frac{1}{p_1} + \frac{1}{p_2} = 1$, $p_1$ and $p_2$ are real numbers greater than 1.

Lemma 2.6. ([2]) Set $h(x)$ be the Lebesgue integrable function on $[a, b]$, $H(x) = \int_{[a, x]} h(x)dx + C$, $x \in [a, b]$, $C$ denotes arbitrary constant, then $H(x)$ is absolutely continuous on $[a, b]$ and $H'(x) = h(x)$, a.e.$[a, b]$.

2.4. Remark.

Remark 1. Define the norm

$$\|z\|_\alpha = \left\| \int_0^D z(t)\right\|_{L^2} = \left( \int_{\mathcal{K}} \left\| \int_0^D z(t)^2 dt \right\|^2 \right)^{\frac{1}{2}}, \forall z \in E^0_\alpha.$$

By (2), one gets that $\|z\|_{\alpha, 2}$ equals to $\|z\|_\alpha$. Thus, in the following, $\|z\|_\alpha$ denotes the norm of the space $E^0_\alpha$.

3. Main results.

Theorem 3.1. If $f_j, g, I_j$ satisfy the following assumptions:

(i) $|I_j(\eta)| \leq a_j + b_j|\eta|^{c_j}$, $\forall \eta \in \mathbb{R}$, $j = 1, 2, \cdots, k$, where $a_j, b_j > 0$ and $c_j \in [0, 1]$;
(ii) $\zeta f_j(t, \zeta) \leq \omega_j(t)\zeta^q$ for $\zeta \in \mathbb{R}, q \in [0, 2), t \in \mathcal{K}, \omega_j(t) \in C(\mathcal{K}, \mathbb{R})$;
(iii) $g \in L^{\frac{2p}{p-2}}(\mathbb{R}), 1 < p < 2$ and $g(t) > 0$ for $t \in \mathbb{R}$,

then there exists at least one classical solution of problem (1).

In order to obtain the desired results, we need the following several Lemmas to guarantee the existence of solutions for problem (1).

Lemma 3.2. If $z \in E^0_\alpha$ is a solution to problem (1), then for any $v \in E^0_\alpha$, there holds

$$\int_{\mathcal{K}} \left( \int_0^D z(t) \right) \left( \int_0^D v(t) \right) dt = \sum_{j=0}^k \int_{t_j}^{t_{j+1}} g(t)z(t)|v|^{p-2}z(t)v(t)dt - \sum_{j=1}^k \int_{t_j}^{t_{j+1}} I_j(z(t_j))v(t_j)$$

$$+ \sum_{j=0}^k \int_{t_j}^{t_{j+1}} f_j(t, z(t))v(t)dt. \quad (3)$$

Proof. The proof is similar to that of Lemma 3.1 in [23], we therefore omit it here.

Definition 3.3. If (3) holds for any $v \in E^0_\alpha$, then $z \in E^0_\alpha$ is called a weak solution of problem (1).
Define the functional \( \Phi : E_0^\alpha \rightarrow \mathbb{R} \) associated with (1) by
\[
\Phi(z) = \frac{1}{2} \int_K |D_t^\alpha z(t)|^2 dt - \sum_{j=0}^{k} \int^{t_{j+1}}_{t_j} F_j(t, z(t)) dt - \frac{1}{p} \sum_{j=0}^{k} \int^{t_{j+1}}_{t_j} g(t)|z(t)|^p dt + \sum_{j=1}^{k} \int_{t_j}^{z(t_j)} I_j(s) ds,
\]
where \( F_j(t, z) = \int_0^z f_j(t, s) ds \). Since \( f_j, I_j \) are continuous functions, from (i) – (ii), one can get that \( \Phi \) is well defined and \( \Phi \in C^1(E_0^\alpha, \mathbb{R}) \) with derivative given by
\[
\langle \Phi'(z), v \rangle = \int_K (\hat{D}_t^\alpha z(t)) (\hat{D}_t^\alpha v(t)) dt - \sum_{j=0}^{k} \int^{t_{j+1}}_{t_j} f_j(t, z(t)) v(t) dt - \frac{1}{p} \sum_{j=0}^{k} \int^{t_{j+1}}_{t_j} g(t)|z(t)|^{p-2} z(t) v(t) dt + \sum_{j=1}^{k} I_j(z(t_j)) v(t_j), \quad \forall v \in E_0^\alpha.
\]
Clearly, if \( z \) is a critical point of \( \Phi \), then \( z \) is a weak solution of problem (1).

**Lemma 3.4.** \( z \in E_0^\alpha \) is a weak solution to problem (1) if and only if \( z \) is a classical solution to (1).

**Proof.** If \( z \) is a classical solution of (1), then \( z \) satisfies equations in (1) and impulsive conditions and Dirichlet boundary condition of problem (1), therefore one can get that \( z \) satisfies the function defined by (4), that is
\[
\frac{1}{2} \int_K |D_t^\alpha z(t)|^2 dt - \sum_{j=0}^{k} \int^{t_{j+1}}_{t_j} F_j(t, z(t)) dt - \frac{1}{p} \sum_{j=0}^{k} \int^{t_{j+1}}_{t_j} g(t)|z(t)|^p dt + \sum_{j=1}^{k} \int_{t_j}^{z(t_j)} I_j(s) ds = 0.
\]
Then, for \( v \in E_0^\alpha \), take the derivative of both sides of (5), we have
\[
\int_K (\hat{D}_t^\alpha z(t)) (\hat{D}_t^\alpha v(t)) dt - \sum_{j=0}^{k} \int^{t_{j+1}}_{t_j} f_j(t, z(t)) v(t) dt - \frac{1}{p} \sum_{j=0}^{k} \int^{t_{j+1}}_{t_j} g(t)|z(t)|^{p-2} z(t) v(t) dt + \sum_{j=1}^{k} I_j(z(t_j)) v(t_j) = 0,
\]
which implies that \( z \) is a weak solution of (1).

On the other hand, if \( z \in E_0^\alpha \) is a weak solution of (1), then \( z(0) = z(T) = 0 \) and (3) holds. For \( t \in [0, s_j] \cup (t_{j+1}, T] \), \( j \in [0, k] \), without loss of generality, we take \( v \in C_0^\infty(s_j, t_{j+1}] \) satisfying \( v(t) \equiv 0 \) and substitute it into (3). From Proposition 3, we can get
\[
\int^{t_{j+1}}_{t_j} \hat{D}_t^\alpha (\hat{D}_t^\alpha z(t)) v(t) dt = \int^{t_{j+1}}_{t_j} (\hat{D}_t^\alpha z(t)) (\hat{D}_t^\alpha (\hat{D}_t^\alpha v(t))) dt = \int^{t_{j+1}}_{t_j} f_j(t, z(t)) v(t) dt + \int^{t_{j+1}}_{t_j} g(t)|z(t)|^{p-2} z(t) v(t) dt < +\infty,
\]
thus,
\[ iD_T^\alpha (\delta_0 D_t^\alpha z(t)) = f_j(t, z(t)) + g(t)|z(t)|^{p-2}z(t), \quad t \in (s_j, t_{j+1}). \] (7)

Since \( z \in E_0^\alpha \subset C([0, T]) \), we get
\[ \int_{s_j}^{t_{j+1}} (|z(t)|^2 + |D_t^\alpha z(t)|^2) dt < +\infty, \quad j = 0, 1, 2, \cdots, k. \]

Combining \( z \in L^2(s_j, t_{j+1}) \), there is
\[ -iD_T^\alpha (\delta_0 D_t^\alpha z(t)) = \frac{d}{dt}(iD_T^{\alpha-1}(\delta_0 D_t^\alpha z(t))) \in L^2(s_j, t_{j+1}). \]

From Lemma 2.6, one can obtain
\[ iD_T^{\alpha-1}(\delta_0 D_t^\alpha z(t)) \in AC([s_j, t_{j+1}]), \]
then, \( iD_T^{\alpha-1}(\delta_0 D_t^\alpha z(t)) \) is left continuous at \( t_{j+1} \) and right continuous at \( s_j \). Substituting (7) into (3), one gets
\[ \int_{K} (\delta_0 D_t^\alpha z(t)) (\delta_0 D_t^\alpha v(t)) dt + \sum_{j=0}^{k} \int_{s_j}^{t_{j+1}} \frac{d}{dt}(iD_T^{\alpha-1}(\delta_0 D_t^\alpha z(t))) v(t) dt + \sum_{j=1}^{k} I_j(z(t_j)) v(t_j) \]
\[ = \sum_{j=0}^{k} iD_T^{\alpha-1}(\delta_0 D_t^\alpha z(t_{j+1}^-)) v(t_{j+1}^-) - \sum_{j=0}^{k} iD_T^{\alpha-1}(\delta_0 D_t^\alpha z(s_j^+)) v(s_j^+) \]
\[ + \sum_{j=1}^{k} \int_{t_j}^{s_j} (\delta_0 D_t^\alpha z(t)) (\delta_0 D_t^\alpha v(t)) dt + \sum_{j=1}^{k} I_j(z(t_j)) v(t_j) \]
\[ = 0. \]

For \( t \in [0, t_j] \cup (s_j, T] \), \( j = 1, 2, \cdots, k \), without loss of generality, we take \( v \in C_0^\infty(t_j, s_j) \) such that \( v(t) \equiv 0 \) and substitute it into (8). Combining (6), we get
\[ iD_T^{\alpha-1}(\delta_0 D_t^\alpha z(t_j)) = \text{Const}, \quad t \in (t_j, s_j), \]
then
\[ iD_T^{\alpha-1}(\delta_0 D_t^\alpha z(t_j^+)) = iD_T^{\alpha-1}(\delta_0 D_t^\alpha z(s_j^-)) = iD_T^{\alpha-1}(\delta_0 D_t^\alpha z(t_j)). \] (9)

Taking (9) into (8) yields
\[ \sum_{j=1}^{k} (iD_T^{\alpha-1}(\delta_0 D_t^\alpha z(t_{j+1}^-)) - iD_T^{\alpha-1}(\delta_0 D_t^\alpha z(t_{j}^+)) + I_j(z(t_j))) v(t_j) \]
\[ + \sum_{j=1}^{k} (iD_T^{\alpha-1}(\delta_0 D_t^\alpha z(t_{j+1}^-)) - iD_T^{\alpha-1}(\delta_0 D_t^\alpha z(s_j^+))) v(s_j) \]
\[ = 0. \]

Then, one gets
\[ iD_T^{\alpha-1}(\delta_0 D_t^\alpha z(t_{j+1}^+)) - iD_T^{\alpha-1}(\delta_0 D_t^\alpha z(t_{j}^-)) = I_j(z(t_j)), \]
and
\[ iD_T^{\alpha-1}(\delta_0 D_t^\alpha z(t_{j}^+)) = iD_T^{\alpha-1}(\delta_0 D_t^\alpha z(s_j^+)) = iD_T^{\alpha-1}(\delta_0 D_t^\alpha z(s_j^-)). \]
Therefore, \( z \) is a classical solution of (1). \( \square \)

**Proof of Theorem 3.1.** From (ii), it holds

\[
F_j(t, \zeta) \leq \frac{\omega_j(t)}{q} \zeta^q, \quad 0 \leq q < 2.
\] (10)

Combining the Hölder inequality, Sobolev inequality and (iii), there exists constant \( C > 0 \) such that

\[
\int_{s_j}^{t_j+1} g(t)|z(t)|^p dt \leq \left( \int_{s_j}^{t_j+1} g(t)^{\frac{q}{2\alpha}} dt \right)^{\frac{2\alpha}{p}} \left( \int_{s_j}^{t_j+1} |z(t)|^2 dt \right)^{\frac{q}{2}} \leq C\|z\|_{L^2}^p.
\] (11)

Then from (i), (10), (11) and proposition 4, we have

\[
\Phi(z) = \frac{1}{2}\|z\|_\alpha^2 - \sum_{j=0}^{k} \int_{s_j}^{t_j+1} F_j(t, z(t)) dt - \frac{1}{p} \sum_{j=0}^{k} \int_{s_j}^{t_j+1} g(t)|z(t)|^p dt + \sum_{j=1}^{k} \int_0^{z(t_j)} I_j(s) ds
\]
\[
\geq \frac{1}{2}\|z\|_\alpha^2 - \|z\|_\infty^q \left( \sum_{j=0}^{k} \int_{s_j}^{t_j+1} \frac{\omega_j(t)}{q} dt \right) - \left( \sum_{j=1}^{k} a_j \|z\|_\infty + \sum_{j=1}^{k} b_j \|z\|_{c_j+1} \right) C\|z\|_{L^2}^p
\]
\[
\geq \frac{1}{2}\|z\|_\alpha^2 - \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^2} \|z\|_{\alpha}^q \left( \sum_{j=0}^{k} \int_{s_j}^{t_j+1} \frac{\omega_j(t)}{q} dt \right) - \frac{C T^{\alpha}}{\Gamma(\alpha+1)} \|z\|_{\alpha}^p - \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^2} \|z\|_{\alpha} \left( \sum_{j=1}^{k} a_j \right) - \frac{b_j}{c_j+1} \left( \sum_{j=1}^{k} \left( \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^2} \right)^{c_j+1} \right) \|z\|_{c_j+1}.
\] (12)

Since \( \omega_j(t) \) is continuous, then it is integrable on \((s_j, t_{j+1})\). Thus, one has

\( \Phi(z) \rightarrow +\infty, \) as \( \|z\|_\alpha \rightarrow \infty. \)

From Lemma 2.4, \( \Phi \) has a minimizing sequence, without loss of generality, we denote it by \( \{z_i\}_{i=1}^\infty \subset E_0^\alpha \), we can get \( \{z_i\} \) is bounded from (12) and up to sequence, there exists \( z \in E_0^\alpha \) such that \( z_i \rightharpoonup z \) in \( E_0^\alpha \). Thus one obtains

\( \|z\|_{\alpha} \leq \lim\inf_{i \rightarrow \infty} \|z_i\|_{\alpha}. \)

From Proposition 5, \( \{z_i\}_{i=1}^\infty \) is convergent uniformly to \( z \) in \( C(K, \mathbb{R}) \). Since \( F \) is continuous, then

\( F_j(t, z_i(t)) \rightharpoonup F_j(t, z(t)), \)

in particular, we can get that

\( \int_{s_j}^{t_{j+1}} F_j(t, z_i(t)) dt \rightharpoonup \int_{s_j}^{t_{j+1}} F_j(t, z(t)) dt. \)
Set $m = \inf_{z \in E^a_0} \Phi(z)$, then

$$m \leq \Phi(z)$$

$$= \frac{1}{2} \|z\|_\alpha^2 - \frac{1}{p} \sum_{j=0}^{k} \int_{s_j}^{t_{j+1}} F_j(t, z(t)) dt$$

$$- \frac{1}{p} \sum_{j=0}^{k} \int_{s_j}^{t_{j+1}} g(t)|z(t)|^p dt + \sum_{j=1}^{k} \int_0^{z(t_j)} I_j(s) ds$$

$$\leq \lim_{i \to \infty} \left( \frac{1}{2} \|z\|_\alpha^2 - \frac{1}{p} \sum_{j=0}^{k} \int_{s_j}^{t_{j+1}} F_j(t, z_i(t)) dt - \frac{1}{p} \sum_{j=0}^{k} \int_{s_j}^{t_{j+1}} g(t)|z_i(t)|^p dt \right)$$

$$+ \sum_{j=1}^{k} \int_0^{z_i(t_j)} I_j(s) ds$$

$$= \lim_{i \to \infty} \inf \Phi(z_i)$$

$$= m,$$

which shows that $\Phi(z) = m$. Thus, $z$ is a global minimum for $\Phi$ on $E^a_0$ which is a critical point of $\Phi$. Let $\bar{z} \in E^a_0$ be a critical point of $\Phi$, since $\Phi_j(z(t_j)) \neq 0$, $j = 1, 2, \cdots, k$, one can get $\bar{z} \notin Const$. Therefore, problem (1) has a non-trivial classical solution. \qed

4. Examples. Let $\alpha = \frac{5}{6}$, $p = \frac{8}{7}$, $T = 1$. Consider

$$\begin{cases}
  D_t^\frac{2}{3} \left( \left( D_t^\frac{1}{2} \right)^2 z(t) \right) = z^{\frac{2}{3}} + |z(t)|^{\frac{2}{3}}, & t \in \bigcup_{j=0}^{k} (s_j, t_{j+1}], \\
  \Delta (D_t^\frac{1}{2} \left( \left( D_t^\frac{1}{2} \right)^2 z \right)) (t_j) = \frac{1}{2} \sin |z(t)|^{\frac{2}{3}}, & j = 1, 2, \cdots, k, \\
  D_t^\frac{1}{2} \left( \left( D_t^\frac{1}{2} \right)^2 z \right) (t_j^+), & t \in \bigcup_{j=1}^{k} (t_j, s_j], \\
  D_t^\frac{1}{2} \left( \left( D_t^\frac{1}{2} \right)^2 z \right)(s_j^+), & j = 1, 2, \cdots, k, \\
  z(0) = z(T) = 0,
\end{cases}$$

where $g(t) = 1$, $f_j(t, z(t)) = z^{\frac{2}{3}}$, $I_j(z(t_j)) = \frac{1}{2} \sin |z(t)|^{\frac{2}{3}}$ with $a_j = 0$, $b_j = 2$, $c_j = \frac{2}{3}$, $\omega_j(t) = 2$, $q = \frac{8}{7}$. Obviously, all the assumptions in Theorem 3.1 are satisfied, therefore, problem (13) has at least one classical solution.

5. Conclusion. We have considered the Dirichlet boundary value problem for instantaneous and non-instantaneous impulsive equations with perturbation. The Weierstrass theorem has been used to ensure the existence of solutions. In the future, we will consider whether the same result can be obtained when the nonlinear term $f$ satisfies linear or superlinear growth condition.
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