A Numerical Study on the Wiretap Network with a Simple Network Topology

Fan Cheng, Member, IEEE and Vincent Y. F. Tan, Member, IEEE

Abstract

In this paper, we study a security problem on Level-I/II \((n_1, n_2)\) wiretap networks, consisting of a source node S, a destination node T, and an intermediate node R. The intermediate node connects the source and the destination nodes via a set of noiseless parallel channels, with sizes \(n_1\) and \(n_2\), respectively. A message \(M\) is to be sent from S to R. The information in the network may be eavesdropped by a set of wiretappers. The wiretappers cannot communicate with one another. Each wiretapper can access a subset of channels, called a wiretap set. All the chosen wiretap sets form a wiretap pattern. A random key \(K\) is generated at S and a coding scheme on \((M, K)\) is employed to protect \(M\). We define two decoding levels at R: In Level-I, only \(M\) is required to be recovered and in Level-II, both \(M\) and \(K\) are required to be recovered. The objective is to minimize \(H(K)/H(M)\) under the perfect secrecy constraint. The first question we addressed is whether routing is optimal on this simple network. By enumerating all the wiretap patterns on the Level-I/II \((3, 3)\) networks and harnessing the power of Shannon-type inequalities, we find that gaps exist between the bounds implied by routing and the bounds implied by Shannon-type inequalities for a small fraction (< 2%) of all the wiretap patterns. The second question we investigated is what is \(\min H(K)/H(M)\) for the remaining wiretap patterns where gaps exist. We study some simple wiretap patterns and find that their Shannon bounds (i.e., the lower bound induced by Shannon-type inequalities) can be achieved by linear codes, which means routing is not sufficient even for the \((3, 3)\) network. For some complicated wiretap patterns, we study the structures of linear coding schemes under the assumption that they can achieve the corresponding Shannon bounds. The study indicates that the determination of the entropic region of 6 linear vector spaces cannot be sidestepped. Some subtle issues on the network models are discussed and interesting observations are stated.

Index Terms

Network coding, linear network coding, entropic region, cut-set bound, routing bound, Shannon bound, wiretap network.

I. INTRODUCTION

A. A Security Problem on a Simple Communication Network

In this paper, we study a security problem on a communication network (depicted in Fig. 1) with three nodes S, T, and R, where S is the source node, T is the intermediate node, and R is the destination node, respectively. There is a set of noiseless channels connecting the pairs (S, T) and (T, R), where the numbers of channels from S to T and T to R are both three.

A private message \(M\) is generated at S and is to be sent to R. As there is a collection of wiretappers that can only tap the information on a subset of the channels, a random key \(K\) which is dependent of \(M\) is also generated at node S. To protect the message \(M\), a coding scheme that depends on both \(M\) and \(K\) is employed to combat the wiretappers. This coding scheme ensures that the information read by each wiretapper is independent of \(M\). Furthermore, wiretappers cannot communicate with one another. For each wiretapper, the set of channels it accesses, called a wiretap set, is fixed before the coding scheme on \((M, K)\) is constructed. The set of all the wiretap sets is called a wiretap pattern. Now, a fundamental question arises: If we fix the size of the message, what is the minimum size of the key should be injected to protect the message? Here the “size” of a random variable is measured by its entropy. The problem we stated does not seem to be very hard since the network topology is simple and a simpler version of this problem dates back to Shannon [5]. Furthermore, there exist many results in the literature. However, as we shall show in this paper, this problem appears to be fiendishly hard. Even for such a simple network, the problem is challenging to solve completely.

B. Related Results

1) Network Coding: We leverage two important concepts in this paper, namely, routing and network coding. In most communication networks, information is transmitted in a store-and-forward manner; i.e., bits are delivered as commodities and then are routed from a node to another. The bits are unaltered on the transmission paths. Ahlswede et al. [1] proposed a network communication paradigm called network coding, where the role of the intermediate nodes is enhanced as follows. At each intermediate node, the information received on the input channels may be encoded, and may be sent on the output channels. Network coding can increase the achievable rates and even attain the capacity of the network. In [1], the classical max-flow-min-cut theorem is generalized to multicast scenario. Furthermore in [1], the authors demonstrated that network

F. Cheng is with the Department of Electrical and Computer Engineering, National University of Singapore, Singapore. Email: fcheng@nus.edu.sg.

V. Y. F. Tan is with the Department of Electrical and Computer Engineering and the Department of Mathematics, National University of Singapore, Singapore. Email: vtan@nus.edu.sg.

This paper was presented in part at the International Symposium on Information Theory, 2014 [7].
coding can outperform store-and-forward in terms of bandwidth utilization. Routing is a class of special network coding schemes. Indeed, when network coding is used, the information is coded in the network. We refer the reader to Yeung et al. for a comprehensive treatment of network coding theory.

2) Information-Theoretic Security and Wiretap Networks: Information-theoretic security was launched in Shannon’s seminal work, where the communication model is only a single channel. A key $K$ is stored at the sender and receiver before the message $M$ is sent. The sender generates $X$ from $M$ and $K$ by an encoding function. Then $X$ is sent through the channel. The receiver decodes $M$ from $X$ and $K$. The main result, called the perfect secrecy theorem, implies that the size of $K$ (measured by its entropy) is lower bounded by the size of $M$.

In wiretap networks, Cai and Yeung considered sending a private message to possibly more than one receiver through a noiseless communication network. Their model is as follows: The communication may be eavesdropped by a set of wiretappers, who cannot communicate with one another. Each of the wiretappers can access a subset of channels of the network, called a wiretap set. The wiretapper can choose an arbitrary wiretap set before communication and the choice is fixed during communication. The set of all the chosen wiretap sets is commonly referred to as a wiretap pattern, denoted by $A$. For such an $A$, the sender and the legitimate users have to design a coding scheme to combat the effect of the wiretappers. The strategy is to generate a random key $K$ to protect the message and send both the message and the key via a network coding scheme. This ensures that a wiretapper can only observe some functions of the message and the key, where the output of the functions are statistically independent of the message. On the other hand, a receiver node can recover the private message by decoding the information received from its input channels. The performance of a secure network coding scheme is measured by the sizes of the message and the key. In designing a secure network coding scheme, the aim is to maximize the size of the message and meanwhile minimize the size of the key. In [3], when $A$ consists of all subsets of channels whose sizes are at most some constant $r$, there exists a linear network code which is optimal in both the sizes of the message and the key. When $A$ is arbitrary, a cut-set bound on the ratio of the size of the key and the size of the message was obtained in Cheng and Yeung. The wiretap network model subsumes the famous perfect secrecy system studied by Shannon, and is also widely studied from many different perspectives. A comprehensive survey of the fundamental performance limits in wiretap networks can be found in Cai and Chan.

3) Shannon- and non-Shannon-type Inequalities: The properties of Shannon information measures form a useful set of tools to investigate the properties of wiretap networks. For a set of random variables, the properties of information measures such as entropy, mutual information, conditional entropy are well known. In particular, it is well known that the above information measures are nonnegative. Information inequalities which can be implied by Shannon’s information measures are referred to as the Shannon-type inequalities; e.g., the inequality $H(X_1|X_2) \leq H(X_1)$. If only the Shannon-type inequalities are concerned, a one-to-one correspondence between information measures and set theory is established in the so-called $I$-Measure theory by Yeung [8 Ch. 3], which only involves simple set theory operations; i.e., union, intersection, complement, and set difference. Moreover, the following fundamental result [8 Ch. 14] was established for the information measures:

Let $[n] = \{1, 2, \ldots, n\}$. Any Shannon’s information measures involving random variables $X_1$, $X_2$, $\ldots$, $X_n$ can be expressed as the sum of the following two elemental forms:

(i) $H(X_i|X_{[n]-\{i\}})$, $i \in [n]$;
(ii) $I(X_i; X_j|X_K)$, where $i \neq j$ and $K \subseteq [n] - \{i, j\}$.

Noting that all the information equalities we studied are linear. If we regard these $n + \binom{n}{2}2^{n-2}$ elements as variables, then any information expression can be rewritten as a linear combination of them. This observation enables us to check the correctness of a Shannon-type inequality by transforming it into an equivalent linear program, which can be easily implemented in a computer program. ITIP [9] and Xitip [10] are two widely used software packages based on this very principle, where the latter is an upgraded version of the former. When a certain information inequality and some constraints are supplied as inputs to the software program, it will inform the user whether the information inequality is a valid Shannon-type inequality. The latest extensions of ITIP may be found in Tian and Ho et al.

It has been a long-standing problem as to whether there exist inequalities involving information measure that cannot be directly implied by the Shannon-type inequalities. This was before the seminal work by Zhang and Yeung, where the first...
non-Shannon-type inequality was proved. All the information inequalities on \( n \) random variables characterize the so-called entropic region \( \overset{*}{\Gamma}_n \). Note that the I-Measure-based method is futile for proving the non-Shannon-type inequalities. In Dougherty et al. [14], the inequality in [13] was used to reduce the capacity bound in a communication network, indicating that Shannon-type inequalities are not always sufficient in practice. The general theory of non-Shannon-type inequalities is still in its infancy and relatively little progress has been achieved. So far, the problem has been addressed when the number of random variables \( n < 4 \). When all the random variables are in one-to-one correspondences with vector spaces, the problem has been settled for \( n \leq 5 \) in Dougherty et al. [15]. For the case \( n = 6 \), Dougherty [16] showed that the number of different linear rank inequalities exceeds 1 million. The exact set of linear rank inequalities is still unknown.

4) Network Coding meets Information Inequalities: In essence, network coding problems can be perfectly represented by the aforementioned information measures. We can use random variables to denote the information transmitted on the channels. The encoding and decoding process at each node can be dealt by information equalities. The performance bounds of network coding can be expressed via information inequalities. It was proved in Chan and Grant [17] that the general secure network coding problem on multi-source and multi-sink network is as hard as determining the exact entropic region. That is, the general secure network coding problem is hard to solve. Thus, we seek bounds. We may consider only Shannon-type inequalities to obtain a bound on a concrete problem by solving the corresponding linear program, or invoking some I-Measure based softwares; e.g., ITIP or Xitip. The bound obtained by Shannon-type inequalities is referred to as the Shannon bound. However, the I-Measure based method suffers from the drawback that the computational complexity is exponential in the number of random variables, which means it works well only for few random variables. Fortunately, we may also consider the cut-set bound which is a classic tool in analyzing the performance of network coding and in some situations, the cut-set bound is tight; e.g., the bound obtained in [3].

C. The Problem We Study and The Question We Ask

Network coding trumps routing in many aspects of communication scenarios. However, routing is advantageous over network coding due to its lower complexity in encoding and decoding and it is easy to understand and analyze. In some simple networks, in lieu of network coding, routing can be shown to be sufficient. In the wiretap network, the only easily-computable bounds are the cut-set bound and Shannon bound. However, the cut-set bound is optimal for the point-to-point communication system. If we consider only routing in the network, we can transform the general network model to a point-to-point network, which means we can obtain a bound based on routing. Here we refer to it as the routing bound. For a point-to-point communication network, all these three bounds are tight; while for general networks, none of them is tight in general. The main motivation of this work is from the following fundamental question:

Can we systematically assess the tightness of these bounds in a marginally more complex network compared to a point-to-point network?

D. Main Contribution and Techniques

In this paper, we assess the optimality of routing in the wiretap network with a simple topology (Fig. 2). Recall that in [4], the cut-set bound is tight for a point-to-point communication system. Beyond the cut-set bound, no further result has been known till date and it is not clear to what extent, the routing scheme is optimal. Our network model is more complex since there is an extra encoding node. It is interesting to deduce whether routing is optimal or not. In this paper, we use ITIP/Xitip to compare the routing bound, the cut-set bound, and the Shannon bound for different wiretap patterns. We find out that for some wiretap patterns, there are gaps between the routing bounds and the Shannon bounds. We pick some examples from these wiretap patterns with gaps, and construct coding schemes to achieve the Shannon bounds. Hence, one of the main takeaways is that the routing bound is not tight even in this simple network. For some concrete wiretap patterns where gaps exist, we discuss linear coding schemes to achieve the Shannon bound. Furthermore, two different decoding levels are defined and the distinction between them is discussed.

We summarize our key contributions in this paper: Firstly, we enumerate all the possible wiretap patterns and determine the wiretap patterns where gaps between Shannon bound and routing bound exist. Secondly, for some wiretap patterns, we construct coding schemes to show that routing is not optimal. Lastly, we study some complex wiretap patterns and present linear coding scheme for these patterns. By doing so, we gain an intuitive understanding of why the Shannon bound is, in general, difficult to attain in this simple network. We make several interesting observations during the course of our numerical study. We also discuss some interesting problem inspired by the study.

E. Paper Organization

The paper is organized as follows. In Section II, the problem formulation is stated and some related results are discussed. In Section II-A, we explain how the experiments are conducted and discuss the results. From Section II-B to Section II-E, we select some wiretap patterns to further validate our claims. In Section IV, we study two hard wiretap patterns and construct linear coding schemes to achieve the Shannon bounds. We conclude the paper in Section V by summarizing our key contributions and stating directions for further work.
II. PROBLEM FORMULATION

The general problem formulation is described as follows (depicted in Fig. 2):

1. The network consists of three nodes, the source node S, the intermediate node T, and the destination node R. There are noiseless directed edges (channels) connecting the pairs (S, T) and (T, R). Denote the set of channels by $\mathcal{E}$. Let $n_1$ be the number of channels from S to T and $n_2$ be the number of channels from T to R, respectively. To simplify our discussion, we assume that the capacity of each channel is much larger than the sum of information rates.

2. At the source node S, a pair of uniformly distributed message and private key $(M, K)$ is generated, where $M \in \mathcal{M}$ is the message and $K \in \mathcal{K}$ is the private key. It is assumed that $M$ and $K$ are statistically independent; i.e.,

$$I(M; K) = 0.$$  

3. Denote the information transmitted on the channels from S to T by $X_{1,1}, X_{1,2}, \ldots, X_{1,n_1}$. Then

$$H(X_{1,1}, X_{1,2}, \ldots, X_{1,n_1} | M, K) = 0.$$  

4. Denote the information transmitted on the channels from T to R by $X_{2,1}, X_{2,2}, \ldots, X_{2,n_2}$. Then

$$H(X_{2,1}, X_{2,2}, \ldots, X_{2,n_2} | X_{1,1}, X_{1,2}, \ldots, X_{1,n_1}) = 0.$$  

5. At the destination node R, we set two different decoding levels. The first is only $M$ is needed to be decoded at R and the other is that both $M$ and $K$ are needed to be decoded at R. We refer to these two different models as Level-I $(n_1, n_2)$ network and Level-II $(n_1, n_2)$ network, respectively. Specifically, in Level-I, we have

$$H(M | X_{2,1}, X_{2,2}, \ldots, X_{2,n_2}) = 0;$$  

and in Level-II, we have

$$H(M, K | X_{2,1}, X_{2,2}, \ldots, X_{2,n_2}) = 0.$$  

6. There is a set of wiretappers who can access an arbitrary subset of $\mathcal{E}$. The choice of the set each wiretapper selected is made before communication and is fixed in the process of communication. Wiretappers cannot communicate with one another. The set of choices of the wiretappers is denote by $A \subseteq 2^\mathcal{E}$. In the sequel, $A$ is referred to as a wiretap pattern. The sender and receiver need to consider all $A \in A$, simultaneously. For a set $A \in A$, denote $(X_e, e \in A)$ by $X_A$. In this model, prefect secrecy is required. To be concrete,

$$I(M; X_A) = 0, \quad \forall A \in A.$$  

7. In this work, we are interested in minimizing $H(K)/H(M)$, given the constraints above; i.e.,

$$\min \frac{H(K)}{H(M)}.$$  

Specifically, if we fix the size of $M$, what is the minimal size of $K$ to achieve perfect secrecy.

The encoding and decoding functions at S, R, and T should abide by the conventions in network coding theory \[2\]. Moreover, the secrecy constraints \[6\] should be satisfied. The encoder at the sender S is

$$f_S : \mathcal{M} \times \mathcal{K} \rightarrow \prod_{i=1}^{n_1} X_{1,i}.$$
The encoder at node $T$ is
\[ f_T : \prod_{i=1}^{n_1} X_{1,i} \to \prod_{i=1}^{n_2} X_{2,i}. \] (9)

The decoder at node $R$ is:
\[ f_1 : \prod_{i=1}^{n_2} X_{2,i} \to M; \text{ (Level-I)} \] (10)
\[ \text{or} \]
\[ f_{11} : \prod_{i=1}^{n_2} X_{2,i} \to M \times K. \text{ (Level-II)} \] (11)

The Level-I model is a special case of the wiretap network introduced in [3]. In Level-I networks, the private key $K$ may be canceled (operated upon) at the intermediate node $T$ to potentially increase the message size whilst ensuring that the message is transmissible securely over the network. In our model, since we have removed the capacity constraints on the channels (capacities are assumed to be sufficiently large), it is interesting to understand the impact of the condition that $K$ must be recovered at the destination. In particular, we ask whether there are any wiretap patterns for which $\min H(K)/H(M)$ is changed when the decoding requirements on $K$ are different. The answer is, in general, yes.

A. Cut-set bound, Routing bound, Shannon bound

First, we state the cut-set bound for an arbitrary $A$.

**Theorem 1** ([4]). Let $W = \{e_1, e_2, ..., e_n\}$ be a cut-set and $A \subseteq 2^W$ be a wiretap pattern. Then
\[ \frac{H(K)}{H(M)} \geq \frac{1}{\max \sum_{i=1}^{n} x_i - 1}, \] (12)
where
\[ \sum_{e_i \in A} x_i \leq 1, \; \forall A \in A, \] (13)
and
\[ x_i \geq 0, \; 1 \leq i \leq n. \] (14)

This cut-set bound can be interpreted as follows. Since the focus is on the ratio between $H(K)$ and $H(M)$, we may as well set $H(K)$ to be 1. Let $x_i$ be the information rate on channel (represented by edge) $e_i$, $1 \leq i \leq n$. Assume that the symbols on the channels are mutually independent, then the constraints (13) mean that the size of the symbols in each wiretap set cannot exceed the size of the key. Furthermore, the cut-set bound is tight for a point-to-point network and its optimality can be achieved by a linear code. Hence, if we wish to know to what extent routing is optimal, we should consider a network with at least three nodes. The algorithm for computing cut-set bound is described as follows (Alg. 1):

**Algorithm 1 Algorithm for computing the cut-set bound**

1. Denote the number of edges in the cut-set by $n$ and the number of wiretap sets by $d$. Let $A$ be a $d \times n$ matrix.
2. If the $i$th wiretapper can access the $j$th edge, then $A(i,j) = 1$. Otherwise, $A(i,j) = 0$.
3. Let $1_d$ be the vector with all entries equal 1. Solve the linear program: $\max \sum_{i=1}^{n} x_i - 1$, s.t. $Ax \leq 1_d, x \geq 0$. The reciprocal of the optimal value is the cut-set bound.

When routing is performed, information is transmitted from $S$ to $R$ without being modified or coded at $T$. Hence, we may assume there are $n_1 \times n_2$ paths/channels directly connecting $S$ to $R$. Since information is unchanged in each path, a wiretapper who can access channel $e$ will get the same information on all the paths that pass through $e$. Hence we need to define a new point-to-point communication system (or cut-set) and the corresponding wiretap patterns. The cut-set $W' = \{e'_{1,1}, e'_{1,2}, ..., e'_{n_1,n_2}\}$. The corresponding wiretap pattern $A'$ is constructed as follows: For each $A \in A$, there is a one-to-one corresponding $A' \in A'$ such that $A' = \{e'_{i,k}(1 \leq k \leq n_2) : e_{i,k} \in A\} \cup \{e'_{i,j}(1 \leq k \leq n_1) : e_{2,j} \in A\}$. Then the routing bound can be computed by applying the cut-set bound on $W'$ and $A'$. Since the cut-set bounds are identical in both decoding levels, routing bounds are the same in the Level-I/II $(n_1,n_2)$ networks. The algorithm is described as follows (Alg. 2).

The idea of Shannon bound is from Yeung [3 Ch. 14], where the principle is to transform the problem concerned to a linear program in the two elemental forms; i.e.,
(i) $H(X_i | X_{[n]-i})$, $i \in [n]$;
(ii) $I(X_i; X_j | X_K)$, where $i \neq j$ and $K \subseteq [n] - \{i, j\}$.
The algorithm for computing Shannon bound has been already implemented in ITIP/Xitip. When we input all the constraints, the Shannon bound will be computed.

Intuitively, we may speculate that routing is optimal for many wiretap patterns. But it is not trivial to prove its optimality. For Level-I/II (2, 2) networks, we can check that routing is optimal. For general \((n_1, n_2)\) networks, numerical searches by computer programs are preferred. For computing \(\min H(K)/H(M)\), the cut-set bound and the Shannon bound are all lower bounds and the routing bound is an upper bound. In principle, if the maximum of the two lower bounds matches and the upper bound, then \(\min H(K)/H(M)\) is determined. Otherwise, further investigations must be conducted on such wiretap patterns.

In the following sections, we choose some wiretap patterns to demonstrate the advantage of coding over routing, and the distinction between the Level-I and Level-II networks. Since the problem size is small, we just use \(X_1, \ldots, X_6\) and \(e_1, \ldots, e_6\).
to denote the random variables and edges. For ease of verification, we provide with the code written in Xitip and matlab here (https://dl.dropboxusercontent.com/u/17877098/NumericalWTN/code.txt).

From existing works of the literature, we do not know whether Shannon bound is tight and linear network coding is sufficient. We have to assume both are tight and try to design a linear coding scheme. To the best of our knowledge, there is no systematic and well-established technique and theory on how to achieve the Shannon bound. One possible method is to study the structure of the optimal solution resulting from the linear program of the Shannon bound, by which some properties about the coding scheme may be found. This approach suffers from the curse of high dimensionality of the optimal solution. Here we provide an alternative method. Noting that if we add more constraints to a linear program of the form

$$\max c^T x, \quad \text{s.t.} \quad Ax \leq b,$$

the optimal value will not be increased. If the optimal value remains the same, then we can use these additional constraints to reduce the search space for an optimal solution. If we use this technique to study the Shannon bound, some properties about the coding scheme may be obtained.

The first very simple intuition is that if we add the following constraints into the linear programs, the Shannon bounds always remain the same:

$$H(X_1, X_2, X_3) = H(X_1) + H(X_2) + H(X_3)$$
$$H(X_4, X_5, X_6) = H(X_4) + H(X_5) + H(X_6)$$

That is the random variables on the same layer are constrained to be mutually independent.

Another idea is based on the observation that for a coding scheme, a valid functional relation is described by

$$H(X_4, X_5, X_6|X_1, X_2, X_3) = 0,$$

which is rather general since the functional relationship may be more exact; e.g., $X_4$ may be a function of $X_1$ and $X_2$, and $X_3$ is not involved in the coding. The exact functional relationships will be very helpful in the construction of a linear coding scheme. The algorithm is that: For each $X_i$ ($4 \leq i \leq 6$), enumerate all the possible functional relationships with the subsets of $X_1, X_2, X_3$. Then update the functional relationship and check by ITIP/Xitip to see whether the Shannon bound is unchanged. If so, then the functional relationship is valid and does not decrease the optimal value of the linear program. Here we state an example to illustrate our approach.

In the sequel, unless otherwise stated, all the alphabets are the finite field $\mathbb{F}_q$ ($q \geq 3$).

**Example 1.** In the Level-II $(3, 3)$ network (depicted in Fig. 3), let $A = \{A_1, A_2, A_3, A_4, A_5\}$, where $A_1 = \{e_2, e_4, e_5\}$, $A_2 = \{e_5, e_3, e_6\}$, $A_3 = \{e_1, e_5, e_6\}$, $A_4 = \{e_1, e_3, e_4\}$, and $A_5 = \{e_1, e_2, e_4, e_6\}$. The routing bound is equal to 4. By the following constraints, the Shannon bound is equal to 3.

$$I(M; K) = 0$$
$$H(X_1, X_2, X_3|M, K) = 0$$
$$H(X_4, X_5, X_6|X_1, X_2, X_3) = 0$$
$$H(M, K|X_4, X_5, X_6) = 0$$
$$I(M; X_2, X_4, X_5) = 0$$
$$I(M; X_2, X_3, X_6) = 0$$
$$I(M; X_1, X_5, X_6) = 0$$
$$I(M; X_1, X_3, X_4) = 0$$
$$I(M; X_1, X_2, X_4, X_6) = 0$$

After adding the following constraints, the Shannon bound remains the same.

$$H(X_1, X_2, X_3) = H(X_1) + H(X_2) + H(X_3)$$
$$H(X_4, X_5, X_6) = H(X_4) + H(X_5) + H(X_6)$$

After checking with ITIP/Xitip, we obtain these additional functional relationships:

$$H(X_4|X_1, X_3) = 0$$
$$H(X_5|X_1, X_2, X_3) = 0$$
$$H(X_6|X_2, X_3) = 0$$

The Shannon bound will be changed if we add

$$H(X_1) = H(X_2) = H(X_3)$$
$$H(X_4) = H(X_5) = H(X_6)$$
After checking with ITIP/Xitip, we find that the following constraints can be added without changing the Shannon bound.

\[ H(X_3) = 2H(X_1) \quad (35) \]
\[ H(X_1) = H(X_2) \quad (36) \]
\[ H(X_5) = 2H(X_4) \quad (37) \]
\[ H(X_4) = H(X_6) \quad (38) \]

From these constraints, we see that the information rates on the edges from S to T and T to R are not identical any more.

The coding scheme is constructed as follows: Split \( X_3 \) as \( (X_{31}, X_{32}) \) and \( X_5 \) as \( (X_{51}, X_{52}) \).

1. Independently generate three bits of key \( K_1, K_2 \) and \( K_3 \) and one bit of message \( M \) from \( F_q \);
2. on edge \( e_1 \), \( X_1 = M + K_1 + K_2 + K_3 \);
3. on edge \( e_2 \), \( X_2 = K_1 \);
4. on edge \( e_3 \), \( X_{31} = K_2 \) and \( X_{32} = K_3 \);
5. at intermediate node \( T \), \( (M, K_1, K_2, K_3) \) can be easily recovered;
6. on edge \( e_4 \), \( X_4 = X_1 + X_{31} = M + K_1 + 2K_2 + K_3 \);
7. on edge \( e_5 \), \( X_{51} = X_1 + X_{31} + X_{32} = M + 2K_1 + 2K_2 + K_3 \) and \( X_{52} = X_1 + X_2 + X_{32} = M + 2K_1 + K_2 + 2K_3 \);
8. on edge \( e_6 \), \( X_6 = X_2 + X_{31} = K_1 + K_2 \);
9. at the destination node \( R \), \( (X_1, X_2, X_{31}, X_{32}) \) can be recovered by \( X_1 = X_{51} - X_6 \), \( X_2 = X_{51} - X_4 \), \( X_{31} = X_6 - X_{51} + X_4 \), and \( X_{32} = X_{52} + X_4 + X_6 - 2X_{51} \), which means \( (M, K_1, K_2, K_3) \) can also be recovered.

We now verify the security constraints.

- \( A_1 \): \( I(M; X_2, X_4, X_5) \)
  \[= I(M; X_2, X_1 + X_{31}, X_1 + X_2 + X_{31}, X_1 + X_2 + X_{32}) \]
  \[= I(M; X_2, X_1 + X_{31}, X_1 + X_{32}) \]
  \[= H(X_2, X_1 + X_{31}, X_1 + X_{32}) - H(X_2, X_1 + X_{31}, X_1 + X_{32}|M) \]
  \[= H(X_2, X_1 + X_{31}, X_1 + X_{32}) - H(K_1, K_1 + 2K_2 + K_3, K_1 + K_2 + 2K_3) \]
  \[= 0 \]

- \( A_2 \): \( I(M; X_2, X_3, X_6) \)
  \[= I(M; X_2, X_1 + X_{31}) \]
  \[= 0 \]

- \( A_3 \): \( I(M; X_1, X_{51}, X_6) \)
  \[= I(M; X_1, X_1 + X_2 + X_{31}, X_1 + X_2 + X_{31}, X_2 + X_{32}) \]
  \[= I(M; X_1, X_2 + X_{31}, X_2 + X_{32}) \]
  \[= 0 \]

- \( A_4 \): \( I(M; X_1, X_3, X_4) \)
  \[= I(M; X_1, X_3, X_1 + X_{31}) \]
  \[= 0 \]

- \( A_5 \): \( I(M; X_1, X_2, X_4, X_6) \)
  \[= I(M; X_1, X_2, X_1 + X_{31}, X_2 + X_{31}) \]
  \[= I(M; X_1, X_2, X_2 + X_{32}) \]
  \[= 0 \]

An interesting discovery is that, in ITIP/Xitip, if we set \( H(K) = 3H(M) \) together with (19)-(27) in the constraints, all the relationships in (28)-(32) and (53)-(58) are true. This means that if the ratio of the size of the key and the message is fixed to that prescribed by the Shannon bound, then several functional relationships between the constituent random variables in the problems can be uncovered. To the best of the authors’ knowledge, this phenomenon has not been observed in previous works.

C. Coding is necessary

In the Level-I/II \((3, 3)\) network (depicted in Fig. 3), consider the wiretap pattern \( \mathcal{A} = \{A_1, A_2, A_3, A_4\} \), where \( A_1 = \{2, 3, 5\} \), \( A_2 = \{1, 4, 5\} \), \( A_3 = \{1, 3, 6\} \), and \( A_4 = \{2, 4, 6\} \). The routing bound is equal to 3. Both of the Shannon bounds for the Level-I and Level-II settings are equal to 2. Next, we construct a code to demonstrate that 2 is optimal.

1. Independently generate two bits of key \( K_1 \) and \( K_2 \) and one bit of message \( M \) from \( F_q \);
2. on edge \( e_1 \), \( X_1 = K_1 \);
3. on edge \( e_2 \), \( X_2 = K_2 \);
4. on edge \( e_3 \), \( X_3 = M + K_1 + K_2 \);
5. at intermediate node \( T \), \( (M, K_1, K_2) \) can be easily recovered;
6. on edge \( e_4 \), \( X_4 = M + 2K_1 + 2K_2 \);
We now verify the security constraints.

Fig. 3. Level-I/II (3, 3) network

7. on edge $e_5$, $X_5 = M + K_1 + 2K_2$;
8. on edge $e_6$, $X_6 = M + 2K_1 + K_2$;
9. at the destination node $R$, $(M, K_1, K_2)$ can be recovered by $M = 2(X_5 + X_6) - 3X_4$, $K_1 = X_4 - X_5$, and $K_2 = X_4 - X_6$.

We now verify the security constraints.

- $A_1$: $I(X_2, X_3, X_5; M)$
  
  $= H(X_2, X_3, X_5) - H(X_2, X_3, X_5|M)$
  
  $= H(K_2, M + K_1 + K_2, M + K_1 + 2K_2) - H(K_2, M + K_1 + K_2, M + K_1 + 2K_2|M)$
  
  $= H(K_2, M + K_1) - H(K_2, K_1|M)$
  
  $= 0$

- $A_2$: $I(X_1, X_4, X_5; M)$
  
  $= H(X_1, X_4, X_5) - H(X_1, X_4, X_5|M)$
  
  $= H(K_1, M + 2K_1 + 2K_2, M + K_1 + 2K_2) - H(K_1, M + 2K_1 + 2K_2, M + K_1 + 2K_2|M)$
  
  $= H(K_1, M + 2K_2) - H(K_1, K_2|M)$
  
  $= 0$

- $A_3$: $I(X_1, X_3, X_6; M)$
  
  $= H(X_1, X_3, X_6) - H(X_1, X_3, X_6|M)$
  
  $= H(K_1, M + K_1 + K_2, M + 2K_1 + K_2) - H(K_1, M + K_1 + K_2, M + 2K_1 + K_2|M)$
  
  $= H(K_1, M + K_2) - H(K_1, K_2|M)$
  
  $= 0$

- $A_4$: $I(X_2, X_4, X_6; M)$
  
  $= H(X_2, X_4, X_6) - H(X_2, X_4, X_6|M)$
  
  $= H(K_2, M + 2K_1 + 2K_2, M + 2K_1 + K_2) - H(K_2, M + 2K_1 + 2K_2, M + 2K_1 + K_2|M)$
  
  $= H(K_2, M + 2K_1) - H(K_2, K_1|M)$
  
  $= 0$

D. Distinction between Level-I and Level-II Networks

In this example, we show that for the same network topology and wiretap pattern, the routing bound is tight for the Level-II network while it is loose for the Level-I network. Consider the wiretap pattern $A = \{A_1, A_2, A_3, A_4\}$, where $A_1 = \{1, 4\}$, $A_2 = \{2, 3, 4\}$, $A_3 = \{1, 2, 5, 6\}$, and $A_4 = \{3, 5, 6\}$. The routing bound is 3. The Shannon bound in Level-I network is 2 and in Level-II network is 3. Next, we construct a code to show that 2 is optimal for the Level-I network.

1. Independently generate two bits of key $K_1$ and $K_2$ and one bit of message $M$ from $\mathbb{F}_q$;
2. on edge $e_1$, $X_1 = M + K_1$;
3. on edge $e_2$, $X_2 = K_2$;
4. on edge $e_3$, $X_3 = K_1$;
5. it is easy to see at intermediate node $T$, $(M, K_1, K_2)$ can be recovered.
6. on edge $e_4$, $X_4 = K_1 + K_2$;
7. on edge $e_5$, $X_5 = M + K_1 + K_2$;
8. on edge $e_6$, transmit nothing;
9. at the destination node $R$, $M$ can be recovered by $M = X_5 - X_4$, $K_1$ and $K_2$ cannot be recovered.

We now verify the security constraints.
Fig. 4. Level-I/II (3, 2) network

- $A_1$: $I(X_1, X_4; M)$
  
  \[
  = H(X_1, X_4) - H(X_1, X_4|M) \\
  = H(M + K_1, K_1 + K_2) - H(M + K_1, K_1 + K_2|M) \\
  = H(M + K_1, K_1 + K_2) - H(K_1, K_2|M) \]
  
  \[
  = 0
  \]

- $A_2$: $I(X_2, X_3, X_4; M)$
  
  \[
  = I(K_2, K_1, K_1 + K_2; M) \\
  = 0
  \]

- $A_3$: $I(X_1, X_2, X_5, X_6; M)$
  
  \[
  = H(X_1, X_2, X_5, X_6) - H(X_1, X_2, X_5, X_6|M) \\
  = H(M + K_1, K_2, M + K_1 + K_2) - H(M + K_1, K_2, M + K_1 + K_2|M) \\
  = H(M + K_1, K_2) - H(K_1, K_2|M) \]
  
  \[
  = 0
  \]

- $A_4$: $I(X_3, X_5, X_6; M)$
  
  \[
  = H(X_3, X_5, X_6) - H(X_3, X_5, X_6|M) \\
  = H(K_1, M + K_1 + K_2) - H(K_1, M + K_1 + K_2|M) \\
  = H(K_1, M + K_2) - H(K_1, K_2|M) \]
  
  \[
  = 0
  \]

E. Sub-optimality of routing in Level-II/III $(n_1, 2)$ and $(2, n_2)$ networks

The Level-II $(n_1, n_2)$ network is equivalent to the Level-II $(n_2, n_1)$ network. For Level-I networks, the situation is more subtle. For Level-II/III $(2, n_2)$ networks, experiments can show that routing is optimal when $n_2 \leq 3$. The problem is still open for the case $n_2 \geq 4$. In the following, we show that there exists a Level-I (3, 2) network (depicted in Fig. 4), where routing is not optimal. Consider the following wiretap pattern $A = \{A_1, A_2, A_3, A_4\}$, where $A_1 = \{1, 2, 4\}$, $A_2 = \{3, 4\}$, $A_3 = \{2, 5\}$, and $A_4 = \{1, 3, 5\}$. The routing bound is equal to 3. The Shannon bound for Level-I is equal to 2 and for Level-II is equal to 3. Next, we construct a code to show that 2 is optimal for the Level-I network.

1. Independently generate two bits of key $K_1$ and $K_2$ and one bit of message $M$ from $\mathbb{F}_2$;
2. on edge $e_1$, $X_1 = K_1$;
3. on edge $e_2$, $X_2 = M + K_2$;
4. on edge $e_3$, $X_3 = K_2$;
5. it is easy to see at intermediate node $T$, $(M, K_1, K_2)$ can be recovered.
6. on edge $e_4$, $X_4 = M + K_1 + K_2$;
7. on edge $e_5$, $X_5 = K_1 + K_2$;
8. at the destination node $R$, $M$ can be recovered by $M = X_4 - X_5$, $K_1$ and $K_2$ cannot be recovered.

We now verify the security constraints.

- $A_1$: $I(X_1, X_2, X_4; M)$
  
  \[
  = H(X_1, X_2, X_4) - H(X_1, X_2, X_4|M) \\
  = H(K_1, M + K_2, M + K_1 + K_2) - H(K_1, M + K_2, M + K_1 + K_2|M) \\
  = H(K_1, M + K_2) - H(K_1, K_2|M) \]
  
  \[
  = 0
  \]

- $A_2$: $I(X_3, X_4; M)$
  
  \[
  = H(X_3, X_4) - H(X_3, X_4|M) \\
  = H(K_2, M + K_1 + K_2) - H(K_2, M + K_1 + K_2|M) \]
= H(K_2, M + K_1) - H(K_2, K_1|M)
= 0

- A_3: \(I(X_2, X_5; M)\)
  \[= H(X_2, X_5) - H(X_2, X_5|M)\]
  \[= H(M + K_2, K_1 + K_2) - H(M + K_2, K_1 + K_2|M)\]
  \[= H(M + K_2, K_1 + K_2) - H(K_2, K_1|M)\]
  \[= 0\]

- A_4: \(I(X_1, X_3, X_5; M)\)
  \[= I(K_1, K_2, K_1 + K_2; M)\]
  \[= 0\]

### IV. Two Hard Examples

In the previous sections, we have already elucidated the differences between Level-I and Level-II wiretap networks. In the sequel, we focus on the Level-II wiretap networks since the problem may become simpler when \(K\) is required to be recovered. An observation from the experiments in the previous section is that gaps exist only if \(4 \leq |A| \leq 12\). For \(|A| = 4, \ldots, 12\), the number of wiretap patterns where gaps exist is 18, 252, 1494, 4842, 9144, 9648, 5400, 1494, and 180, respectively.

For wiretap patterns in which the routing bounds are not optimal, we have constructed linear coding schemes to achieve the Shannon bounds by hand. This may be tedious and non-systematic. It is thus of great interest to see whether we can apply similar techniques to all the remaining unknown cases. When \(|A| = 4\), there are only 18 wiretap patterns, where all the routing bounds are equal to 1/3 and Shannon bounds are equal to 1/2. We can check that all these Shannon bounds can be achieved by linear coding schemes by the method in Sec. III-B (like the coding scheme in Sec. III-C). For wiretap patterns where \(|A| > 4\) there are instances for which optimal linear coding schemes are not easy to construct by hand. Next, we state some wiretap patterns to demonstrate the difficulty of these specific instances.

**Example 2.** Let \(A = \{A_1, A_2, A_3, A_4, A_5\}\), where \(A_1 = \{e_2, e_4, e_5\}\), \(A_2 = \{e_2, e_3, e_6\}\), \(A_3 = \{e_1, e_4, e_6\}\), \(A_4 = \{e_1, e_3, e_5\}\), and \(A_5 = \{e_1, e_2, e_4\}\). The routing bound is equal to 3. By the following constraints, the Shannon bound is equal to 7/3.

\[
3H(K) \geq 7H(M) \tag{39}
\]
\[
I(M; K) = 0 \tag{40}
\]
\[
H(X_1, X_2, X_3|M, K) = 0 \tag{41}
\]
\[
H(X_4, X_5, X_6|X_1, X_2, X_3) = 0 \tag{42}
\]
\[
H(M, K|X_4, X_5, X_6) = 0 \tag{43}
\]
\[
I(M; X_2, X_4, X_5) = 0 \tag{44}
\]
\[
I(M; X_2, X_3, X_6) = 0 \tag{45}
\]
\[
I(M; X_1, X_4, X_6) = 0 \tag{46}
\]
\[
I(M; X_1, X_3, X_5) = 0 \tag{47}
\]
\[
I(M; X_1, X_2, X_4) = 0 \tag{48}
\]

The following constraints can be added without changing the Shannon bound.

\[
H(X_1, X_2, X_3) = H(X_1) + H(X_2) + H(X_3) \tag{49}
\]
\[
H(X_4, X_5, X_6) = H(X_4) + H(X_5) + H(X_6) \tag{50}
\]
\[
H(X_1) = H(X_2) \tag{51}
\]
\[
3H(X_3) = 4H(X_1) \tag{52}
\]
\[
H(X_5) = H(X_6) \tag{53}
\]
\[
2H(X_4) = H(X_5) \tag{54}
\]
\[
H(X_4|X_1, X_2, X_3) = 0 \tag{55}
\]
\[
H(X_5|X_1, X_3) = 0 \tag{56}
\]
\[
H(X_6|X_2, X_3) = 0 \tag{57}
\]

Since \(H(K)/H(M) = 7/3\), we need to construct a linear code on a vector with size 10 and the bit rates on each edges from \(S\) to \(T\) are 3, 3, and 4 and from \(T\) to \(R\) are 2, 4, and 4, respectively. That is tantamount to find a \(10 \times 10\) matrix over \(\mathbb{F}_q\). Except brute force search, we have no other choice at the moment. Since the space of the feasible solutions is huge, the linear coding scheme for this example is unknown.

Next, an even more complicated wiretap pattern with 12 wiretap sets is studied.
Example 3. Let $A = \{A_1, A_2, ..., A_{12}\}$, where $A_1 = \{e_3, e_5, e_6\}$, $A_2 = \{e_3, e_4, e_6\}$, $A_3 = \{e_3, e_4, e_5\}$, $A_4 = \{e_2, e_5, e_6\}$, $A_5 = \{e_2, e_4, e_6\}$, $A_6 = \{e_2, e_3, e_6\}$, $A_7 = \{e_2, e_3, e_5\}$, $A_8 = \{e_2, e_3, e_4\}$, $A_9 = \{e_1, e_5, e_6\}$, $A_{10} = \{e_1, e_3, e_6\}$, $A_{11} = \{e_1, e_3, e_5\}$, and $A_{12} = \{e_1, e_2, e_4, e_5\}$. The routing bound is equal to 4. The Shannon bound is equal to 19/5 by the following constraints.

\[
5H(K) \geq 19H(M) \quad (58)
\]

\[
I(M; K) = 0 \quad (59)
\]

\[
H(X_1, X_2, X_3|M, K) = 0 \quad (60)
\]

\[
H(X_4, X_5, X_6|X_1, X_2, X_3) = 0 \quad (61)
\]

\[
H(M, K|X_4, X_5, X_6) = 0 \quad (62)
\]

\[
I(M; X_3, X_5, X_6) = 0 \quad (63)
\]

\[
I(M; X_3, X_4, X_6) = 0 \quad (64)
\]

\[
I(M; X_3, X_4, X_3) = 0 \quad (65)
\]

\[
I(M; X_2, X_5, X_6) = 0 \quad (66)
\]

\[
I(M; X_2, X_4, X_6) = 0 \quad (67)
\]

\[
I(M; X_2, X_3, X_6) = 0 \quad (68)
\]

\[
I(M; X_2, X_3, X_5) = 0 \quad (69)
\]

\[
I(M; X_2, X_3, X_4) = 0 \quad (70)
\]

\[
I(M; X_1, X_5, X_6) = 0 \quad (71)
\]

\[
I(M; X_1, X_3, X_6) = 0 \quad (72)
\]

\[
I(M; X_1, X_3, X_4) = 0 \quad (73)
\]

\[
I(M; X_1, X_2, X_4, X_5) = 0 \quad (74)
\]

We can show via ITIP/Xitip and hence prove that the Shannon bound is unchanged after adding the following constraints.

\[
H(X_1, X_2, X_3) = H(X_1) + H(X_2) + H(X_3) \quad (75)
\]

\[
H(X_4, X_5, X_6) = H(X_4) + H(X_5) + H(X_6) \quad (76)
\]

\[
7H(X_1) = 9H(X_2) \quad (77)
\]

\[
8H(X_1) = 9H(X_3) \quad (78)
\]

\[
3H(X_4) = 4H(X_5) \quad (79)
\]

\[
5H(X_4) = 4H(X_6) \quad (80)
\]

\[
H(X_4|X_1, X_2, X_3) = 0 \quad (81)
\]

\[
H(X_5|X_1, X_2, X_3) = 0 \quad (82)
\]

\[
H(X_6|X_1, X_2, X_3) = 0 \quad (83)
\]

Therefore, we need to define a linear coding scheme on a vector with size 24 such that $H(K) = 19$ and $H(M) = 5$. Moreover, the bit rates on the layers from $S$ to $T$ are 9, 7, and 8, and from $T$ to $R$ are 8, 6, and 10, respectively. The functional relationships cannot be further improved. We need to split $X_1, ..., X_6$ first, then by ITIP/Xitip, we can obtain more exact functional relationships. Due to the number of random variables (i.e., 24), ITIP/Xitip cannot afford such a huge computation. The linear coding schemes are unknown.

From these two examples, we observe that we do not have efficient methods to construct linear coding schemes to achieve their respective Shannon bounds in general. The only method as of now is to use a brute force search over all the possible linear network coding schemes over $\mathbb{F}_q$. The computational complexity of such a search is $q^{n^2}$, where $n$ is the size of the vector and $q$ is the field size. In light of the number of the remaining wiretap patterns, it is intractable and computationally prohibitive if we simply rely on the state-of-the-art resources. Even if we can find a solution by brute force search, such a solution fails to provide us with an intuitive understanding of the structure of the coding scheme. Thus, one of the take-home messages in this paper is that information theorists need to construct new techniques and concepts to systematically achieve the Shannon bound using linear coding schemes. Without such techniques, it is difficult to make progress. To assist the readers of this paper to investigate the problem further, we have provided the data at the following link:

https://sites.google.com/site/chengfancuhk/NumericalWiretapNetwork
V. Conclusion

In this paper, we have defined Level-I/II \((n_1, n_2)\) wiretap networks and have numerically computed bounds on \(H(K)/H(M)\). The performances of the routing bound on various wiretap patterns has been compared to the cut-set bound and the Shannon bound. Examples are constructed to show that under both decoding levels, routing is not sufficient even for a simple \((3,3)\) network. Our numerical calculations also demonstrate the differences between Level-I and Level-II networks.

Systematic coding schemes of achieving the Shannon bound are few and far between. One of the stumbling blocks for us to achieve the Shannon bound is our lack of a complete understanding of information inequalities. By studying the tightness of the Shannon bound and their associated coding schemes on some networks with simple topologies, we may gain new insights that may help us to further understand \(H(K)/H(M)\) for general networks. For general Level-I/II \((n_1, n_2)\) networks, we numerically compute all three bounds for all the wiretap patterns for the \((n_1 = 3, n_2 = 3)\) case. We also list some wiretap patterns in which the determination of the optimal \(H(K)/H(M)\) is challenging. The examples stated in Sec. [IV] indicate that the determination of the entropic region of 6 linear vector spaces \(L_6\) cannot be sidestepped, since the solution of some complicated wiretap patterns may depend on \(L_6\). Optimistically speaking, we may conjecture that the Shannon bound is tight in Level-I/II \((n_1, n_2)\) networks and the bounds can be achieved by linear coding schemes. Resolving this conjecture is of great interest in information theory. To do so requires us to enhance both the mathematical theory behind information-theoretic security and to aid in numerical investigations, more sophisticated software will be useful.

Acknowledgements

The authors are supported by NUS grants R-263-000-A98-750/133 and an NUS Young Investigator Award R-263-000-B37-133. This work was also partially funded by a grant from the University Grants Committee of the Hong Kong Special Administrative Region (Project No. AoE/E-02/08) and Shenzhen Key Laboratory of Network Coding Key Technology and Application, Shenzhen, China (ZSDY20120619151314964).

References

[1] R. Ahlswede, N. Cai, S.-Y. R. Li, and R. W. Yeung, “Network information flow,” IEEE Trans. Inform. Theory, IT-46: 1204-1216, 2000.
[2] R. W. Yeung, S.-Y. R. Li, N. Cai, and Z. Zhang, “Network coding Theory,” Foundations and Trends in Comm. and Info. Theory, vol. 2, nos. 4 and 5, 241-381, 2005.
[3] N. Cai and R. W. Yeung, “Secure Network Coding on a Wiretap Network,” IEEE Trans. Inform. Theory, 57(1): 424-435, Jan. 2011.
[4] F. Cheng and R. W. Yeung, “Performance Bounds in Secure Network Coding,” IEEE International Symposium on Network Coding (NetCod), Jul. 2011.
[5] C. E. Shannon, “Communication theory of secrecy systems,” Bell Sys. Tech. Journal 28, pp. 656-715, 1949.
[6] N. Cai and T. Chan, “Theory of secure network coding,” Proc. IEEE, vol. 99, no. 3, pp. 421-437, Mar. 2011.
[7] F. Cheng, “Optimality of Routing on the Wiretap Network with Simple Network Topology,” 2014 IEEE International Symposium on Information Theory, (ISIT 2014).
[8] R. W. Yeung, Information Theory and Network Coding, Springer, 2008.
[9] R. W. Yeung and Y.-O. Yan, ITIP (Information Theoretic Inequality Prover), http://user-www.ie.cuhk.edu.hk/~ITIP
[10] R. Pulikoonattu, E. Perron and S. Diggavi, Xitip, http://xitip.epfl.ch
[11] C. Tian, “Characterizing the rate region of the (4, 3, 3) exact-repair regenerating codes,” IEEE Journal on Selected Areas in Communications, vol. 32, no. 5, pp. 967-975, May 2014.
[12] S.-W. Ho, C. W. Tan, and R. W. Yeung, “Proving and Disproving Information Inequalities,” 2014 IEEE International Symposium on Information Theory, (ISIT 2014).
[13] Z. Zhang and R. W. Yeung, “On characterization of entropy function via information inequalities,” IEEE Trans. Inform. Theory, vol. 44, no. 4, pp. 1440-1452, Jul. 1998.
[14] R. Dougherty, C. Freiling, and K. Zeger, “Matroids, networks, and non-Shannon information inequalities,” IEEE Trans. Inform. Theory, vol. 53, no. 6, Jun. 2007.
[15] R. Dougherty, C. Freiling, and K. Zeger, “Linear rank inequalities on five or more variables,” arXiv 0910.0284, 2010.
[16] R. Dougherty, “Computations of linear rank inequalities on six variables,” 2014 IEEE International Symposium on Information Theory, (ISIT 2014).
[17] T. H. Chan and A. Grant, “Network Coding Capacity Regions via Entropy Functions,” IEEE Trans. Inform. Theory, vol. 60, no. 9, pp. 5347-5374, Sept. 2014.
[18] N. J. A. Sloane, “The On-Line Encyclopedia of Integer Sequences: A014466,” http://oeis.org/A014466