Eulerian perturbation theory in non-flat universes: second-order approximation

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ABSTRACT

The problem of solving perturbatively the equations describing the evolution of self-gravitating collisionless matter in an expanding universe considerably simplifies when directly formulated in terms of the gravitational and velocity potentials: the problem can be solved exactly, rather than approximately, even for cosmological models with arbitrary density parameter $\Omega$. The Eulerian approach we present here allows to calculate the higher-order moments of the initially Gaussian density and velocity fields: in particular, we compute the gravitationally induced skewness of the density and velocity-divergence fields for any value of $\Omega$, confirming the extremely weak $\Omega$-dependence of the skewness previously obtained via Lagrangian perturbation theory. Our results show that the separability assumption of higher-order Eulerian perturbative solutions is restricted to the Einstein-de Sitter case only.

Key words: cosmology: theory - gravitation - galaxies: large-scale structure of the Universe

1 INTRODUCTION

Any model of large-scale structure formation based on the gravitational instability has to deal with the problem of the nonlinear evolution of the density and peculiar velocity fields. In the simplest of these scenarios, the mass content of the universe is treated as a cold fluid of dust. In spite of the simplicity of the dust fluid model, the analysis of the nonlinear gravitational evolution is extremely involved, the Poisson, continuity and Euler equations being highly nonlinear and nonlocal beyond the linear regime, even in their Newtonian version (Peebles 1980; see also Kofman & Pogosyan 1994, and references therein).

In the attempt of addressing fundamental cosmological issues, a big effort has been devoted to the theoretical treatment of nonlinear gravity, both in the Eulerian and Lagrangian formulation (e.g. Shandarin & Zel’dovich 1989; Sahni & Coles 1994).

The basic Eulerian approximation scheme is perturbation theory: the density contrast $\delta \equiv (\rho - \rho_b)/\rho_b$ (where $\rho$ is the density field and $\rho_b$ its mean background value) and the peculiar velocity $u$ are expanded about the background solution $\delta = 0$, $u = 0$, namely $\delta = \sum \delta^{(n)}$ and $u = \sum u^{(n)}$ with $\delta^{(n)} = O(\delta^{(1)n})$ and $u^{(n)} = O(u^{(1)n})$, where $\delta^{(1)}$ and $u^{(1)}$ correspond to the linear solutions, then the differential equations for any $\delta^{(n)}$ and $u^{(n)}$ are solved (Doroshkevich & Zel’dovich 1975; Peebles 1980; Fry 1984; Goroff et al. 1986). Thus, for example, the question of whether power can be transferred from large to small scales has been examined in the framework of the second-order approximation by Juszkiewicz (1981), Vishniac (1983), Juszkiewicz, Sonoda & Barrow (1984), Coles (1990), Suto & Sasaki (1991), Makino, Sasaki & Suto (1992), Jain & Bertschinger (1994) and Baugh & Efstathiou (1994). The weakly nonlinear deformation of the original Gaussian density and velocity distributions in terms of the gravitationally induced higher-order moments - and the corresponding correlation hierarchy - is instead analysed by Peebles (1980), Fry (1984), Goroff et al. (1986), Grinstein et al. (1987), Grinstein & Wise (1987), Bernarddeau (1992a,b), Juszkiewicz, Bouchet & Colombi (1993), Bernarddeau (1994), Catelan & Moscardini (1994a,b), Juszkiewicz et al. (1994) and Lokas et al. (1994); extensions to more general intrinsically non-Gaussian models are discussed by Luo & Schramm (1993) and Fry & Scherrer (1994). Some of these results and related issues have been also successfully investigated in complementary N-body simulations by Hansel et al. (1985), Coles & Frenk (1991), Efstathiou et al. (1990), Efstathiou, Sutherland & Maddox...
(1990), Moscardini et al. (1991), Messina et al. (1992), Weinberg & Cole (1992), Bouchet & Hernquist (1992), Lahav et al. (1993), Lucchin et al. (1994), Kofman et al. (1994) and Melott, Pellman & Shandarin (1994).

Typically, with the exception of a few papers (Bernardeau 1992a,b; 1994), this kind of investigation has been confined within the limits of the Einstein-de Sitter cosmology, mainly because the condition \( \Omega = 1 \) enormously eases the solution of the dynamical equations for \( \delta \) and \( \mathbf{u} \): apart from theoretical expectations, there is however no conclusive observational evidence that our universe is actually flat (e.g. Peebles 1991; Coles & Ellis 1994).

A powerful alternative approach is the Lagrangian description, where the individual fluid elements are followed during their motion. The most popular Lagrangian approach is the celebrated Zel’dovich approximation (Zel’dovich 1970a,b), widely used in cosmology, showing also extremely useful in reconstruction methods of initial conditions from velocity data (e.g. Nusser & Dekel 1992).

A Lagrangian perturbative approach, pioneered by Buchert (1989), Moutarde et al. (1991) and Buchert (1992), is presently object of thorough investigation. The key difference with respect to the Eulerian approach is that one searches for solutions of perturbed trajectories about the linear (initial) particle displacement \( \mathbf{S}^{(1)} : \mathbf{S} = \sum \mathbf{S}^{(n)} \) where \( \mathbf{S}^{(n)} = O(\mathbf{S}^{(1)}) \), while the continuity equation for the density field is exactly satisfied.

Solutions up to the third-order Lagrangian approximation have been obtained, in the case of an Einstein-de Sitter model by Buchert & Ehlers (1993) and Buchert (1994), and for more general Friedmann models by Bouchet et al. (1992) and Catelan (1995). Matarrese, Pantano & Saez (1994a,b) developed a relativistic Lagrangian treatment of the nonlinear dynamics of an irrotational collisionless fluid, which reduces to the standard Newtonian approach on sub-horizon scales, but is also suitable for the description of perturbations on super-horizon scales.

The higher accuracy of Lagrangian perturbative methods, as compared to other currently studied approaches [such as the frozen flow (Matarrese et al. 1992) and frozen (or linear evolution of the) potential (Brainerd, Scherrer & Villumsen 1993; Bagla & Padmanabhan 1994) approximations, is discussed by Munshi & Starobinsky (1994), Bernardeau et al. (1994b) and Munshi, Sahni & Starobinsky (1994), still in the framework of an Einstein-de Sitter cosmology. Comparisons with N-body simulations in the fully developed nonlinear regime are given in Moutarde et al. (1991), Coles, Melott & Shandarin (1993), Melott et al. (1994) and Melott, Buchert & Weiss (1995).

Finally, let us also mention some observational evidence that gravitational instability in the quasi-linear regime operates on large scales. Baumgart & Fry (1991) claim that the observed bispectrum obeys the hierarchical pattern; in Saunders et al. (1991), the distribution of QDOT-IRAS galaxies out to 140 h\(^{-1}\) Mpc (\( h \) being the Hubble constant in units of 100 km sec\(^{-1}\) Mpc\(^{-1}\)) is found positively skewed and inconsistent with the results of the standard cold dark matter model, even after a 20 h\(^{-1}\) Mpc smoothing (see also Coles & Frenk 1991; Park 1991); higher-order correlations and/or moments in the galaxy distribution have been recently also measured by Szapudi, Szalay & Boschan (1992), Meiksin, Szapudi & Szalay (1992), Bouchet et al. (1993), Gaztañaga (1992, 1994) and others, both in optically and IRAS selected catalogues; higher-order effects are detected on significantly large scales by the counts-in-cells analysis of N-body simulations of Baugh, Gaztaña & Efstathiou (1994). The general observational evidence appears to be quite consistent with gravitational instability perturbation theory applied on random-phase initial conditions.

The problem we deal with in this paper is thus of quite general interest. We present an Eulerian formalism which allows to solve in a rigorous perturbative way the equations governing a pressureless self-gravitating fluid in the expanding universe, with arbitrary density parameter \( \Omega \). The problem notably simplifies when formulated directly in terms of the peculiar gravitational and velocity potentials. In other words, we look for perturbative solutions of the Bernoulli equation rather than the Euler equation: the Bernoulli equation is the corner-stone for several nonlinear approximation techniques (Nusser et al. 1991; Matarrese et al. 1992; Gramann 1993a; Lachièze-Rey 1993; Brainerd, Scherrer & Villumsen 1993; Bagla & Padmanabhan 1994).

More in detail, we derive a particular form of the equations governing the evolution of the gravitational potential \( \varphi \) and the velocity potential \( \Phi \): the dynamics of the fields \( \varphi \) and \( \Phi \) is determined by the longitudinal component of the current \( \delta \mathbf{v} \) carried by the fluctuation \( \delta \) with peculiar velocity \( \mathbf{u} \). An advantage of this treatment is that we can easily work out the higher-order velocity contributions \( \mathbf{u}^{(n)} \) in a manifestly irrotational form in Eulerian configuration space [all the previous irrotational expressions for \( \mathbf{u}^{(n)} \) were obtained in Fourier space (see Goroff et al. 1986)]. The importance of having explicit irrotational expressions for the velocity field at any perturbative order lies in the fact that a collisionless self-gravitating fluid cannot generate vorticity (before orbit mixing) and that conservation of circulation is applicable well beyond the linear regime (see the discussion in Bertschinger 1993).

Apart from the evident mathematical simplification, there are several more reasons for working directly with the gravitational and velocity potentials. Unlike the density fluctuation field, the gravitational potential is required neither to have zero mean (since only its gradients are physically meaningful), nor to satisfy the positive-mass constraint, \( \delta \geq -1 \); moreover, since it depends directly on the mass distribution, it is bias-independent. On the other hand, the velocity potential has been recently recognized as a powerful tool to test the cosmological power spectrum, becoming directly measurable from observational data (see Heavens 1993; Baugh & Efstathiou 1994).
Eulerian perturbation theory in non-flat universes

There are two major previous attempts to solve the same problem we study in this work. Martel & Freudling (1991) applied Eulerian perturbation theory, but in the attempt to get second-order solutions for the density contrast and peculiar velocity fields in non-flat models, they forced the intrinsic non-separability of the Eulerian dynamical equations. In Bouchet et al. (1992), the separability is also assumed, but for the higher-order Lagrangian perturbative ansatz; the separability hypothesis is also implicit in Gramann (1993a,b), and Lachièze-Rey (1993). Actually, the general validity of the separability ansatz in Lagrangian perturbation theory has been recently proved by Ehlers & Buchert (1995, in preparation; see also Appendix B in Catelan 1995).

As an application of our Eulerian formalism, we finally calculate the density skewness $S_{3\delta}$ induced by the gravitational evolution, starting from Gaussian initial conditions, as well as the skewness of the velocity-divergence, $S_{3\eta}$.

Before concluding this Introduction we would like to mention an important difference between the Eulerian and Lagrangian perturbative approaches, which is relevant for the present work. Eulerian perturbation theory, at any order, cannot lead to shell-crossing and consequent vorticity generation by multi-streaming, contrary to the Lagrangian perturbation theory, and, what is most important, contrary to the correct evolution which does lead to caustic formation. This is because the perturbative Eulerian velocity field, in a given position $\mathbf{x}$ and at time $t$, is obtained at any order by a set of (generally nonlocal) quantities which are functions of the position $\mathbf{x}$ itself (and time): no memory of the initial (Lagrangian) position $\mathbf{q}$ of the particle which is passing there at that time can be present; in other terms, the system does not contain particle inertia. Clear evidence of this fact is provided by the frozen flow approximation (Matarrese et al. 1992), which precisely exploits this property of the linear Eulerian velocity field (but the same would be true at higher orders). Therefore, provided the initial velocity field is sufficiently smooth, which will be always the case after suitable low-pass filtering, no caustic ever forms, as this would correspond to a discontinuity in the velocity field. Of course, one might ask whether this is a real weakness of Eulerian perturbation theory itself, compared to the Lagrangian one. Certainly, it is a limitation if one is interested in using its results to reconstruct the exact position of each particle after nonlinear evolution (e.g. by extending the method of the frozen flow approximation to higher orders). However, one is often interested in knowing only the coarse-grained features of the density and velocity fields in physical space: whether Eulerian theory at some perturbative order can reproduce such features remains in our opinion an open question. For the purpose of the present work the absence of multi-streaming is a clear advantage, for it allows the use of the velocity potential at any time during the nonlinear evolution.

The plan of the paper is as follows. In Section 2 the Hamilton-Jacobi equation and the fluid evolution equations are derived starting from the single particle Lagrangian in a generic Friedmann universe; the Hamilton-Jacobi equation allows an alternative derivation of various approximation schemes, such as the frozen flow and Zel’ dovich algorithms, which is sketched in Appendix A1, and shows an important analogy with the Bernoulli equation for the velocity potential of the fluid. In Section 3, the second-order solutions for the gravitational and velocity potentials are worked out. These results allow to calculate the gravitationally induced density and velocity-divergence skewness and their dependence on the density parameter: in Section 4 we compare these results with the corresponding ones obtained so far by means of more complicated approximation techniques or applying Lagrangian perturbation theory and transforming back to real space. Conclusions are drawn in Section 5. Technical appendices are also given.

2 DYNAMICS OF SELF-GRAVITATING COLLISIONLESS MATTER IN NON-FLAT UNIVERSES

We assume that, at the era of large-scale structure formation, the mass content of the universe is in the form of a collisionless and irrotational fluid with negligible velocity dispersion. Let $\mathbf{x}$ be Eulerian comoving spatial coordinates; physical distances are obtained by $r = a(t)x$, where $a(t)$ is the scale factor and $t$ the cosmic time. For an Einstein-de Sitter universe (density parameter $\Omega = 1$ and vanishing cosmological constant), $a(t) \propto t^{2/3}$. However, in what follows, we shall consider more general non-flat Friedmann models. In the general case a suitable time coordinate is the variable $D(t)$, in that the dynamical equations assume very simple forms: $D(t)$ is the growing mode of linear density fluctuations, whose dependence on $t$ is in general quite complicated (e.g. Peebles 1980); with whole generality $D(0) = 0$ at the initial time; in the flat case, $D(t) \propto a(t) \propto t^{2/3}$.

In this section we rederive the dynamical equations for our pressureless self-gravitating fluid directly from the single particle Lagrangian in a non-flat model. The Hamilton-Jacobi formulation of the problem is shown to be strictly related to the Bernoulli equation for the velocity potential; in terms of it the Zel’dovich and frozen flow approximations may be simply reformulated, as described in Appendix A1.

2.1 From the particles to the fluid description

Let $L_L$ be the Lagrangian for the motion of a particle with mass $m$ (e.g. Peebles 1980)

$$ L_L(\mathbf{x}, \dot{\mathbf{x}}, t) = \frac{1}{2} ma^2 \dot{x}^2 - m \phi(\mathbf{x}, t), \quad (1) $$
where $\phi$ is the gravitational potential, related through the Poisson equation to the mass fluctuations $\delta$, $\nabla^2 \phi = 4\pi G \rho_0 a^2 \delta$. The action of the particle reads $S = \int dt L_t = \int dD L_D$, where the last equality is justified by the fact that $S$ is a scalar. Here $dD = D(t) dt$; in what follows, a dot indicates the operator $d/dt$, and a prime $d/dD$. From now on, for simplicity, we shall write $L \equiv L_D \equiv D^{-1} L_t$, and $L(x', x, D) = \frac{1}{2} ma^2 D x'^2 - m D^{-1} \phi(x, D)$. An alternative form of the Lagrangian $L$ may be obtained by suitably rescaling the gravitational potential $\phi$ and the particle mass $m$. Indeed, introducing the two quantities $\varphi \equiv \left[ 2D/3e(\Omega)a^2 D^2 \right] \phi$ and $\mu(D) \equiv [aD/\sqrt{D/e(\Omega)}] m$, after some algebra one gets the final expression

$$\mathcal{L}(x', x, D) = \frac{a\mu}{2\sqrt{D/e(\Omega)}} \left\{ \frac{D}{e(\Omega)} x'^2 - 3\varphi(x, D) \right\}. \quad (2)$$

The function $e(\Omega)$ is defined in terms of the logarithmic rate of growth of mass fluctuations, $f(\Omega) \equiv d \ln D/d \ln a$, through $e(\Omega) \equiv \Omega/[f(\Omega)]^2$ (Gramann 1993a,b). Note that $e(\Omega)$ depends weakly on $\Omega$, since $e(\Omega) \approx \Omega^{-1/5}$, once $f(\Omega) \approx \Omega^{2/5}$ is used (see Peebles 1980). The Einstein-de Sitter case is easily recovered with the replacements $D \to a$, $e \to 1$ and $\mu \to 2/3 a_0^{3/2} t_0^{-1} m$ [here $t_0$ is an arbitrary reference time and $a_0 = a(t_0)$]. In terms of the potential $\varphi$ the Poisson equation reduces to

$$D \nabla^2 \varphi(x, D) = \delta(x, D). \quad (3)$$

Instead of using the velocity $x'$ as independent variable, one can alternatively introduce the conjugate momentum $p \equiv \frac{2\mu}{a} = a \sqrt{D/e(\Omega)} \mu x'$. In such a case one adopts the Hamiltonian point of view, where the Hamiltonian reads

$$\mathcal{H}(x, p, D) \equiv p \cdot x' - L(x', x, D) = \frac{p^2}{2a\mu \sqrt{D/e(\Omega)}} + \frac{3}{2} \frac{a\mu}{\sqrt{D/e(\Omega)}} \varphi(x, D). \quad (4)$$

By varying the action $S$ with respect to $x$ and $x'$ one obtains the Euler-Lagrange equations

$$x'' + \frac{3e(\Omega)}{2D}(x' + \nabla \varphi) = 0. \quad (5)$$

The three second-order Euler-Lagrange equations above are equivalent to the set of six first-order Hamilton equations, namely

$$x' = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{a\mu \sqrt{D/e}} , \quad p' = -\frac{\partial \mathcal{H}}{\partial x} = \frac{3}{2} \frac{a\mu}{\sqrt{D/e}} \nabla \varphi. \quad (6)$$

Alternatively, one can use the Hamilton-Jacobi formulation, according to which the momentum $p$ is written in terms of an action functional $S$, as $p = \nabla S$, the latter obeying the Hamilton-Jacobi equation $\frac{\partial S}{\partial D} + \mathcal{H}(x, \nabla S) = 0$. It can be, however, useful to define a scaled action functional $\Phi_S$ through $S(x, D) \equiv a \sqrt{D/e(\Omega)} \mu \Phi_S(x, D)$ so that the Hamilton-Jacobi equation takes the simple form,

$$\frac{\partial \Phi_S}{\partial D} + \frac{1}{2} (\nabla \Phi_S)^2 + \frac{3e(\Omega)}{2D} (\Phi_S + \varphi) = 0. \quad (7)$$

So far we have only dealt with the dynamics of single particles. Our problem is, however, the description of an infinite set of particles interacting only through gravity. The standard description of this system is based on the Vlasov equation (or collisionless Boltzmann equation), which governs the evolution of the (comoving) one-particle distribution function $f(x, p, D) dx dp$, namely the probability of finding one particle in the infinitesimal phase-space element $dx dp$.

By the Liouville theorem the distribution function is conserved along particle trajectories in phase-space, thus leading to the Vlasov equation, which in our case reads

$$\frac{\partial f}{\partial D} + \frac{p}{a\mu \sqrt{D/e(\Omega)}} \cdot \nabla f - \frac{3}{2} \frac{a\mu}{\sqrt{D/e(\Omega)}} \nabla \varphi \cdot \frac{\partial f}{\partial p} = 0. \quad (8)$$

This equation, together with the definition of comoving mass density

$$\rho(x, D) \equiv m \ n(x, D) \equiv \rho_0 (1 + \delta(x, D)) \equiv m \int dp \ f(x, p, D) \quad (9)$$

(here $n$ is the comoving particle number density and $\rho_0$ the mass density at $t_0$) and the Poisson equation (3), completes the description. It is however useful to attempt a fluid description of the system, which is achieved by introducing the first moment of the distribution function, namely the local streaming velocity, i.e. the mean velocity of the patch of fluid around $x$

$$u(x, D) = \frac{1}{a \mu \sqrt{D/e(\Omega)}} \int dp \ p f(x, p, D) / \int dp \ f(x, p, D). \quad (10)$$

From the Vlasov equation, we then obtain the continuity equation

$$\frac{\partial \rho}{\partial D} + \nabla \cdot (\rho \ u) = 0 \quad (11)$$

and the momentum conservation equation
By expressing the density fluctuation $δ$ and velocity field $u$ in terms of the gravitational and velocity potential, respectively, we can recast the continuity equation in the form (see also Kofman 1991)

$$
\nabla \cdot \left[ \frac{∂}{∂D} (D \nabla \varphi) + (1 + D \nabla^2 \varphi) \nabla \Phi \right] = 0 ,
$$

whence its general solution reads

$$
\frac{∂}{∂D} (D \nabla \varphi) + (1 + D \nabla^2 \varphi) \nabla \Phi = \nabla \wedge T .
$$

A standard way to single out the irrotational part of the latter equation is the following: according to Helmholtz’s theorem (e.g. Morse & Feshbach 1953), the last term in the l.h.s may be written as (up to an irrelevant harmonic function)

$$
(D \nabla^2 \varphi) \nabla \Phi = δu \equiv \nabla F + \nabla \wedge T ,
$$
in such a way that the solenoidal vector $T$ in equation (18) is automatically canceled out:

$$
\frac{∂}{∂D} (D \nabla \varphi) + \nabla \Phi + \nabla F = 0 .
$$

The function $F$ just introduced has to satisfy the equation

$$
\nabla^2 F = \nabla \cdot (δu) = δ\nabla^2 \Phi + \nabla δ \cdot \nabla \Phi .
$$

To summarize, one can write the set of equations for the cosmological potentials as follows:

\[
\begin{align*}
\frac{∂}{∂D} [D \varphi(x, D)] + \Phi(x, D) + F(x, D) &= 0 , \\
\frac{∂}{∂D} \Phi(x, D) + \frac{1}{2} [\nabla \Phi(x, D)]^2 + \frac{3e(Ω)}{2D} \left[ \Phi(x, D) + \varphi(x, D) \right] &= 0 , \\
\nabla^2 F(x, D) &= D \nabla \cdot \left[ \nabla^2 \varphi(x, D) \nabla \Phi(x, D) \right] .
\end{align*}
\]
Some remarks are appropriate. The auxiliary potential $F$ - which closes our system of scalar equations - is related to the flux of matter originated by the mass current $\mathbf{j} \equiv \delta \mathbf{u}$ carried by the density fluctuation $\delta$ moving with peculiar velocity $\mathbf{u}$: only the longitudinal component $j||$ determines the dynamics [a related quantity is introduced by Bertschinger & Hamilton (1994), although in a different context]. Furthermore, the function $F$ is, by construction, at least a second-order quantity in the fields $\delta$ and $\mathbf{u}$, because $\mathbf{j}$ is so. This fact allows us to easily linearize the equations: the first-order form of the equations (22) is

\begin{align}
\varphi^{(1)} + \Phi^{(1)} &= 0 , \\
\frac{\partial \varphi^{(1)}}{\partial D} &= 0 ,
\end{align}

(23)

so that the linear regime is described in terms of a single potential, say $\varphi$ (see Kofman 1991). Note that the second of these equations implies that the gravitational potential is time-independent, i.e. $\varphi(x, D) = \varphi^{(1)}(x)$ and thus $\Phi(x, D) = \Phi^{(1)}(x) = -\varphi^{(1)}(x)$, in the linear regime.

Only a few exact solutions of the set of equations (22) are known, mostly endowed with some symmetries, i.e. with restrictions in the initial conditions. A possible alternative strategy is to seek perturbative solutions: this program will be carried out in the next section.

3 Eulerian Perturbation Theory: Second-Order Approximation

Let us now solve the previous equations for the potentials according to the following general Eulerian perturbative ansatz:

\begin{align}
\varphi(x, D) &= \varphi^{(1)}(x) + \varphi^{(2)}(x, D) + \cdots \equiv \varphi_1(x) + D \varphi_2(x, D) + \cdots , \\
\Phi(x, D) &= \Phi^{(1)}(x) + \Phi^{(2)}(x, D) + \cdots \equiv \Phi_1(x) + D \Phi_2(x, D) + \cdots ,
\end{align}

(24)

(25)

explicitly up to second-order: of course, the expansion may be continued to higher-order terms. The functions $\varphi^{(1)}(x)$ and $\Phi^{(1)}(x)$ are the linear potentials, as already discussed. Note that in our notation, for instance, $\delta^{(1)}(x, D) = D \nabla^2 \varphi^{(1)}(x)$, but $\delta_1(x) = \nabla^2 \varphi_1(x)$, where $\delta^{(1)}(x, D) \equiv D \delta_1(x)$; instead: $u^{(1)}(x) \equiv \nabla \Phi^{(1)}(x) = \nabla \Phi_1(x) \equiv u_1(x)$. Similarly for the higher-order terms: the expansions (24) and (25) obviously generate analogous expansions for the corresponding fields $\delta$ and $\mathbf{u}$.

We assume that the general ansatz in (24) and (25) actually catches all the physical information contained in perturbation theory: in this sense we stress that, in particular, the second-order solutions $\varphi^{(2)}(x, D)$ and $\Phi^{(2)}(x, D)$ are non-separable in the variables $x$ and $D$. On the contrary, what is usually done is to assume that all the temporal information contained e.g. in $\varphi^{(2)}(x, D)$ can be factored out in the form $\varphi^{(2)}(x, D) = D \varphi_2(x)$: this is not allowed a priori, it applies in the case of the Einstein-de Sitter universe, but not in a generic Friedmann model. To be convinced of the previous statement it is enough to suppose, ab absurdo, that separability actually holds, then show that such an ansatz is not a solution of the fundamental equations; we give this proof in Appendix A2.

In terms of the perturbed quantities, our equations become

\begin{align}
D \frac{\partial}{\partial D} \varphi_2(x, D) + 2 \varphi_2(x, D) + \Phi_2(x, D) + F_2(x) &= 0 , \\
D \frac{\partial}{\partial D} \Phi_2(x, D) + \left[ 1 + \frac{3e(\Omega)}{2} \right] \Phi_2(x, D) + \frac{3e(\Omega)}{2D} \varphi_2(x, D) + \frac{1}{2} \left[ \nabla \Phi_1(x) \right]^2 &= 0 , \\
\nabla^2 F_2(x) &= \nabla \cdot \left[ \nabla^2 \varphi_1(x) \nabla \Phi_1(x) \right] = \nabla \cdot \left[ \delta_1(x) \mathbf{u}_1(x) \right] ,
\end{align}

(26)

where the dependence on the variables $x$ and $D$ is explicitly shown and, after noting that $F(x, D)$ is actually a second-order quantity, we have defined the scaled function $F_2(x) \equiv D^{-1} F^{(2)}(x, D)$; the solution of the last equation above can be immediately obtained as a convolution in Fourier space, $F_2(k) \equiv \int dx \, F_2(x) e^{i k \cdot x}$:

\begin{equation}
\bar{F}_2(k) = \frac{1}{k^2} \int \frac{d k_1 d k_2}{(2\pi)^3} \delta_D (k_1 + k_2 - k) \left[ 1 + \frac{k_1 \cdot k_2}{k^2} \right] k_1^2 \bar{\varphi}_1 (k_1) k_2^2 \bar{\varphi}_1 (k_2) ,
\end{equation}

(27)

where e.g. $k = |k|$: $F_2$ is determined by the initial condition on $\varphi_1$, while the kernel $1 + k_1 \cdot k_2/k^2$ describes the effects of the nonlinear evolution.

The differential equations in (26) are coupled in the fields $\varphi_2(x, D)$ and $\Phi_2(x, D)$; to decouple them, we differentiate the first once, to get

\begin{equation}
D \frac{\partial^2 \varphi_2}{\partial D^2} + 3 \frac{\partial \varphi_2}{\partial D} + F_2 = 0 .
\end{equation}

(28)
By substituting into the second equation of the set (26) we obtain
\[ D^2 \frac{\partial^2 \varphi_2}{\partial D^2} + \left[ 4 + \frac{3 \varepsilon(\Omega)}{2} \right] D \frac{\partial \varphi_2}{\partial D} + \left[ 2 + \frac{3 \varepsilon(\Omega)}{2} \right] \varphi_2 = \frac{1}{2} (\nabla \varphi_1)^2 - \left[ 1 + \frac{3 \varepsilon(\Omega)}{2} \right] F_2 , \] (29)
which turns out to be an equation for the single potential \( \varphi_2 \). Now it should be noticed that, while \( \varphi_2(x, D) \) is not separable in space and time, the sum
\[ \psi(x, D) \equiv \varphi_2(x, D) + F_2(x) \] (30)
is exactly separable; in fact, after substitution in equation (29), we find
\[ D^2 \frac{\partial^2 \psi}{\partial D^2} + \left[ 4 + \frac{3 \varepsilon(\Omega)}{2} \right] D \frac{\partial \psi}{\partial D} + \left[ 2 + \frac{3 \varepsilon(\Omega)}{2} \right] \psi = C(x) , \] (31)
where
\[ C(x) \equiv \frac{1}{2} (\nabla \varphi_1(x))^2 + F_2(x) . \] (32)
Defining at this point the function
\[ A(\Omega) \equiv 2 + \frac{3}{2} \varepsilon(\Omega) = A(D) , \] (33)
we can summarize our results in terms of the following set of equations,
\[
\begin{align*}
D^2 \frac{\partial^2 \psi}{\partial D^2} + \left( 2 + A(D) \right) D \frac{\partial \psi}{\partial D} + A(D) \psi(x, D) &= C(x) , \\
\varphi_2(x, D) &= \psi(x, D) - F_2(x) , \\
\Phi_2(x, D) &= -D \frac{\partial}{\partial D} \varphi_2(x, D) - 2 \varphi_2(x, D) - F_2(x) ,
\end{align*}
\] (34)
which exactly defines the second-order potentials \( \varphi_2(x, D) \) and \( \Phi_2(x, D) \). In particular, since \( F_2(x) \) is determined by the initial conditions, the last equation of the set (34) gives the velocity potential \( \Phi_2(x, D) \) once \( \varphi_2(x, D) \) is known; the main task is to solve the first of the equations in (34); we address this problem in the next subsection.

### 3.1 Second-order perturbative solutions

We have previously shown that, although the second-order potential \( \varphi_2(x, D) \) is non-separable, the function \( \varphi_2(x, D) + F_2(x) \) is so: since the r.h.s. of the first equation in (34) does not depend on \( D \), we can assume (up to an additive constant, which can be always set to zero)
\[ \psi(x, D) \equiv B(D) C(x) , \] (35)
where the function \( B(D) \) is a function of the only variable \( D \). Therefore, the second-order potential \( \varphi_2(x, D) \) takes the form
\[ \varphi_2(x, D) = - \left[ 1 - B(D) \right] F_2(x) + \frac{1}{2} B(D) \left( \nabla \varphi_1(x) \right)^2 . \] (36)
The second-order velocity potential then reads
\[ \Phi_2(x, D) = \left[ 1 - 2B(D) - DB'(D) \right] F_2(x) - \frac{1}{2} \left( 2B(D) + DB'(D) \right) \left( \nabla \varphi_1(x) \right)^2 . \] (37)
The last relation shows that the velocity \( \mathbf{u} \) may be indeed written as the gradient of a scalar function in \( \{x\} \)-space:
\[ \mathbf{u}(x, D) = \nabla \left[ \Phi^{(1)} + \left( 1 - 2B(D) - DB'(D) \right) F^{(2)}(x, D) - \frac{1}{2} D \left( 2B(D) + DB'(D) \right) \mathbf{u}^{(1)} \right] , \] (38)
explicitly up to second-order; this expression of the velocity field is thus manifestly irrotational in physical space [see also Catelan & Moscardini (1994b); for its Fourier components see Goroff et al. (1986)].

We can now furthermore derive the final expressions of the second-order corrections \( \delta^{(2)}(x, D) \equiv D^2 \delta_2(x, D) \) and \( \nabla \cdot \mathbf{u}^{(2)}(x, D) \equiv D \nabla \cdot \mathbf{u}_2(x, D) : \)
\[ \delta^{(2)}(x, D) = \left[ 1 - B(D) \right] \delta^{(1)}(x, D)^2 - D \left( \mathbf{u}^{(1)} \cdot \nabla \right) \delta^{(1)}(x, D) + D^2 B(D) \sum_{\alpha\beta} \left( \partial_{\alpha} u^{(1)\beta} \right)^2 , \] (39)
and
\[ -D \nabla \cdot \mathbf{u}^{(2)}(x, D) = \left[ 1 - 2B(D) - DB'(D) \right] \delta^{(1)}(x, D)^2 - D \left( \mathbf{u}^{(1)} \cdot \nabla \right) \delta^{(1)}(x, D) + D^2 \left[ 2B(D) + DB'(D) \right] \sum_{\alpha \beta} \left( \partial_\alpha u^{(1)\beta} \right)^2. \] (40)

Combining these solutions, we can also derive a relation between the density contrast \( \delta \) and the divergence of the velocity field \( \nabla \cdot \mathbf{u} \), namely
\[ -D \nabla \cdot \mathbf{u} = \delta - 2D^2 \left[ DB(D) \right] \mu_2 (u_1), \] (41)
where \( \mu_2 (u_1) \) is the second-order invariant of the initial deformation tensor \( \partial_\alpha \partial_\beta \Phi_1 (x) \), which is in general different from zero in three-dimensional systems: \( 2\mu_2 (u_1) \equiv (\nabla \cdot u_1)^2 - \sum_{\alpha \beta} \left( \partial_\alpha u_1^\beta (x) \right)^2 = 2(\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3). \) The quantities \( \lambda_h, h = 1, 2, 3 \), are the eigenvalues of \( \partial_\alpha u_1^\beta (x) \). The relation (41), which quantifies the second-order deviation of \( \nabla \cdot \mathbf{u} \) from \( \delta \), may be compared with similar relations in the literature (e.g. Giavalisco et al. 1993; Mancinelli et al. 1993; Mancinelli & Yahil 1994). The importance of the shear term \( \sum_{\alpha \beta} \left( \partial_\alpha u_1^\beta (x) \right)^2 \) in the density-velocity-divergence relation, even in the case of laminar flow, is also discussed by Mancinelli & Yahil (1994).

The corresponding expressions of \( \delta^{(2)} \) and \( \mathbf{u}^{(2)} \) for the flat limit case (see Peebles 1980, where a different gravitational potential is used; Goroff et al. 1986, where a different normalization of the velocity field is chosen; also Appendix A3 for an independent derivation) read:
\[ \delta^{(2)}(x, a) = \frac{5}{7} \delta^{(1)}(x, a)^2 - a \left( \mathbf{u}^{(1)} \cdot \nabla \right) \delta^{(1)}(x, a) + \frac{2}{7} a^2 \sum_{\alpha \beta} \left( \partial_\alpha u^{(1)\beta} \right)^2, \] (42)
and
\[ -a \nabla \cdot \mathbf{u}^{(2)}(x, a) = \frac{3}{7} \delta^{(1)}(x, a)^2 - a \left( \mathbf{u}^{(1)} \cdot \nabla \right) \delta^{(1)}(x, a) + \frac{4}{7} a^2 \sum_{\alpha \beta} \left( \partial_\alpha u^{(1)\beta} \right)^2, \] (43)
where, in the Einstein-de Sitter model, \( \delta^{(2)} \) scales like \( a^2 \) and \( \mathbf{u}^{(2)} \) like \( a \), that they are separable in space and time, \( \delta^{(2)}(x, a) \equiv a^2 \delta_2 (x) \) and \( \mathbf{u}^{(2)}(x, a) \equiv a \mathbf{u}_2 (x) \).

The function \( B(D) \), defined in (35), allows to specify the (second-order) growth of the potentials \( \varphi (x, D) \) and \( \Phi (x, D) \): it is a solution of the differential equation
\[ \left\{ D^2 \frac{d^2}{dD^2} + \left[ 2 + A(D) \right] D \frac{d}{dD} + A(D) \right\} B(D) = 1. \] (44)

The initial condition (at the beginning: \( D = 0 \)) for the function \( B(D) \) may be easily found comparing equations (39) and (42): \( B(D = 0) = 2/7 \). Comparing instead equations (40) and (43), we get a relation for the derivative of \( B(D) \) at the initial time, namely \( DB'(D)|_{D=0} = 0 \).

Since the dependence of \( A(D) \) on \( D \) is definitively non-trivial, we transform to a new variable \( \tau \) which, e.g. in the case \( \Omega < 1 \), is defined by (see e.g. Peebles 1980)
\[ \tau \equiv \frac{1}{\Omega} - 1, \] (45)
where \( 0 \leq \tau < \infty \): \( \tau = 0 \) corresponds to the Big Bang (where one asymptotically approaches the Einstein-de Sitter model). The previous differential equation can now be written as
\[ \left\{ J(\tau) \right\} \frac{d^2}{d\tau^2} + \left[ 1 + J'(\tau) + A(\tau) \right] J(\tau) \frac{d}{d\tau} + A(\tau) \right\} B(\tau) = 1, \] (46)
and the initial conditions keep the same form. Here and in what follows, a prime denotes differentiation with respect to \( \tau \). To maintain the notation concise we have defined
\[ J(\tau) \equiv \frac{D(\tau)}{D'(\tau)} = \tau f(\tau)^{-1}, \] (47)
and \( J(\tau) \to \tau \) in the limit \( \tau \to 0 \). The growing mode \( D(\tau) \) reads (e.g. Peebles 1980):
\[ D(\tau) = 1 + \frac{3}{\tau} + \frac{3\sqrt{1 + \tau}}{\tau^{3/2}} L(\tau); \] (48)
with \( L(\tau) \equiv \ln \left( \sqrt{1 + \tau} - \sqrt{\tau} \right) \); \( D(\tau) \to 2\tau/5 \) in the limit \( \tau \to 0 \). The function \( A(\tau) \) is now simply given by
\[ A(\tau) = 2 + \frac{3}{2} \left( 1 + \tau \right) f(\tau)^2, \] (49)
where finally
\[ f(\tau) = -\frac{3}{2} \left[ \frac{3\sqrt{\tau(\tau+1)} + (3 + 2\tau)L(\tau)}{3 + \tau \sqrt{\tau(\tau+1)} + (3 + 3\tau)L(\tau)} \right] \] (50)
and \( f(0) = 1 \).

The analytical solution of Eq.(46) for the open case, with growing mode initial conditions, \( B(\tau)|_{\tau=0} = 2/7 \) and \( \tau B'(\tau)|_{\tau=0} = 0 \), can be guessed by noting that there exists a close relation between our Eq.(46) and Eq.(7) in Bouchet et al. (1992), such that \( B = \frac{1}{2} \left[ 1 + \frac{E(\tau)}{\tau(\tau)^2} \right] \), where \( E(\tau) \) is also reported in eq.(70) below. Indeed, the function
\[ B(\tau) = \frac{1}{2} - \frac{1}{4D(\tau)^2} - \frac{9}{4\tau D(\tau)^2} \left\{ 1 + \sqrt{\frac{1 + \tau}{\tau}} L(\tau) + \frac{1}{2} \left[ \sqrt{\frac{1 + \tau}{\tau}} + L(\tau) \right]^2 \right\} \] (51)
solves the differential equation (46) with the appropriate initial conditions. This result also shows the equivalence between the Eulerian perturbation theory, which is in general characterized by non-separable perturbative solutions, and the Lagrangian description, whose higher-order solutions are intrinsically separable (see the discussion in Appendix B of Catelan 1995 for the second-order case; the separability of higher-order Lagrangian modes has been recently demonstrated by Ehlers & Buchert 1995, in preparation; Buchert, private communication).

Incidentally we give here also the corresponding expressions directly in terms of the variable \( \Omega \), which can alternatively be chosen as time variable: in the order,
\[ J(\Omega) = -\Omega(1 - \Omega) f(\Omega)^{-1} \] (52)
\[ D(\Omega) = \frac{1}{(1 - \Omega)^{3/2}} \left[ (1 + 2\Omega)^{1/2} \Omega + 3\Omega L(\Omega) \right] \] (53)
\[ f(\Omega) = -\frac{3\Omega}{2} \left\{ \frac{3\sqrt{\Omega - \Omega} + (2 + \Omega)L(\Omega)}{1 + 2\Omega)^{1/2} \Omega + 3\Omega L(\Omega)} \right\} \] (54)
where \( L(\Omega) \equiv \ln \left[ 1 - \sqrt{1 - \Omega}/\sqrt{\Omega} \right] \) and \( L(1) = 0 \). There are two approximations of the function \( f(\Omega) \) usually found in the literature: \( f(\Omega) \approx \Omega^{3/5} \) (Peebles 1980) and \( f(\Omega) \approx \Omega^{4/7} \) (Fry 1985). Also, we give all the formulae for the closed model (\( \Omega > 1 \)) in Appendix A4.

4 APPLICATIONS: THE GRAVITATIONAL SKEWNESS

The previous results may be used to compute the gravitationally induced skewness of the density fluctuation, \( \delta \), and divergence of the velocity field, \( \eta \equiv -D \nabla \cdot \mathbf{u} \). We assume that the primordial potential \( \varphi \) is Gaussian distributed; then, for the density contrast the first non-zero contribution to the skewness is
\[ \langle \delta^3 \rangle = 3\langle \delta^{(1)} \delta^{(2)} \rangle + O(\delta^{(1)}^6) \] (55)
It is well known that this kind of calculations is better performed in Fourier space, namely we need to compute the term \( \langle \delta^{(1)}(k_1, D) \delta^{(1)}(k_2, D) \delta^{(2)}(k_3, D) \rangle \). From equation (39), one finally gets
\[ \delta_2(k, D) = \int \frac{dk_1 dk_2}{(2\pi)^3} \delta_D(k_1 + k_2 - k) J_s^{(2)}(k_1, k_2; D) \delta_1(k_1) \delta_1(k_2) \] (56)
where the symmetrized kernel \( J_s^{(2)} \) is defined by the relation
\[ J_s^{(2)}(k_1, k_2; D) \equiv \left[ 1 - B(D) \right] + \frac{1}{2} \left( \frac{k_1}{k_2} + \frac{k_2}{k_1} \right) \frac{k_1 \cdot k_2}{k_1 k_2} + B(D) \left( \frac{k_1 \cdot k_2}{k_1 k_2} \right)^2 \] (57)
The gravitational skewness of the unfiltered density field may thus be written in the form
\[ \langle \delta^3 \rangle = 6 \int \frac{dk dk'}{(2\pi)^3} J_s^{(2)}(k, k'; D) P(k) P(k') \] (58)
where \( P(k) \) is the primordial power spectrum of the density field. Performing the integration, we obtain the following expression for the density skewness parameter \( S_{3s} \):
\[ S_{3s}(\tau) \equiv \frac{\langle \delta^3 \rangle}{\langle \delta^2 \rangle^2} = \frac{3\langle \delta^{(1)} \delta^{(2)} \rangle}{\langle \delta^{(1)} \rangle^2} = 6 - 4B(\tau) \] (59)
having emphasized the dependence on, e.g., the variable \( \tau \). Note that, at the beginning \( B(0) = 2/7 \), and therefore \( S_3(0) = 34/7 \), which is the well-known value of the unfiltered density skewness in the Einstein-de Sitter model (Peebles 1980).

Similar calculations can be done for the velocity-divergence field \( \eta \). It turns out that

\[
\tilde{\eta}_2(k, D) = \int \frac{dk_1dk_2}{(2\pi)^3} \delta_D (k_1 + k_2 - k) \, K^{(2)}(k_1, k_2; D) \, \hat{\delta}_1(k_1) \, \hat{\delta}_1(k_2),
\]

where the kernel \( K^{(2)} \) is defined by

\[
K^{(2)}(k_1, k_2; D) \equiv \left[ 1 - 2B(D) - DB'(D) \right] + \frac{1}{2} \left( \frac{k_1}{k_2} + \frac{k_2}{k_1} \right) \frac{k_1 \cdot k_2}{k_1 k_2} + \left[ 2B(D) + DB'(D) \right] \left( \frac{k_1 \cdot k_2}{k_1 k_2} \right)^2.
\]

Then

\[
\langle \eta^3 \rangle = 6 \int \frac{dkdk'}{(2\pi)^6} \, K^{(2)}(k, k'; D) \, P(k) \, P(k'),
\]

from which

\[
S_{3\eta}(\tau) \equiv \frac{\langle \eta^3 \rangle}{\langle \eta^2 \rangle^2} = 6 - 8B(\tau) - 4J(\tau)B'(\tau).
\]

Note that, for the Einstein-de Sitter model, we recover the standard result \( S_{3\eta}(0) = 26/7 \) (see Bernardeau 1994).

### 4.1 Previous determinations of the skewness

Both the results (59) and (63) can be directly compared with the corresponding results in the literature.

#### 4.1.1 Martel & Freudling (1991)

In Martel & Freudling (1991), the second-order Eulerian equation for the density field is reduced to a separable version of it, by replacing \( \Omega^2 \rightarrow \Omega \) in one of the terms. Martel and Freudling call this “almost–separability” of the perturbation equation for the density field, and the consequent approximation “MP approximation” (see also Martel 1995). In practice, this corresponds to assume that the expansion of the density contrast is of the factorized form

\[
\delta(x, \tau) = D(\tau) \delta_1(x) \left[ 1 + \kappa(\tau) \mathcal{C}(x) + \cdots \right],
\]

up to the second-order term. The function \( \mathcal{C}(x) \) is given by the expression

\[
\mathcal{C}(x) = \frac{5}{2} \delta_1(x) - \frac{7}{8\pi \delta_1(x)} \nabla \delta_1(x) \cdot \nabla \chi(x) + \frac{1}{16\pi^2 \delta_1(x)} (\partial_\alpha \partial_\beta \chi(x))^2,
\]

where \( \chi \equiv \int dx' \, \delta_1(x')/|x' - x| \); furthermore, the function \( \kappa(\tau) \) is a solution of the differential equation

\[
\tau^2 \kappa''(\tau) + 2 \left( \frac{\Omega^3/5 + 1 - \Omega}{\tau} \right) \tau \kappa' = (\tau + 1)^{-1} D(\tau),
\]

with \( \kappa(0) = \kappa'(0) = 0 \) and \( \kappa(\tau) \to 2D(\tau)/7 \) in the limit \( \tau \to 0 \). It turns out that the density skewness is given by the expression

\[
S_{3\kappa}(\tau) = 17 \frac{\kappa(\tau)}{D(\tau)}.
\]

In the mentioned references no explicit estimate of \( S_{3\eta} \) is reported.

#### 4.1.2 Bouclier et al. (1992)

The dynamical equations for the displacement \( \mathbf{S}(q, \tau) \) from the Lagrangian (initial) positions \( q \) of the fluid elements are solved in Bouclier et al. (1992) according to the perturbative expansion

\[
\mathbf{S}(q, \tau) = D(\tau) \mathbf{S}^{(1)}(q) + E(\tau) \mathbf{S}^{(2)}(q) + \cdots,
\]

up to the second-order correction [see also Catelan (1995) for the notation]. After (perturbatively) inverting the relation \( \mathbf{x} = q + \mathbf{S}(q, \tau) \), to get the Eulerian location of the mass elements, one obtains the following expression for the second-order density contrast \( \delta^{(2)} \):
where the function $E(\tau)$ is given by the relation

$$E(\tau) = -\frac{1}{2} + \frac{9}{2\tau} \left\{ 1 + \sqrt{\frac{1 + \tau}{\tau}} L(\tau) + \frac{1}{2} \left[ \sqrt{\frac{1 + \tau}{\tau}} + \frac{L(\tau)}{\tau} \right]^2 \right\},$$

(70)

which, in the limit $\tau \to 0$, yields $E \to -3D^2/7$. The formal analogy between Eq.(69) and our expression Eq.(39), suggests that the two results coincide provided $B = \frac{1}{2} \left[ 1 + \frac{E(\tau)}{\tau^2} \right]$, which is indeed the case, as noticed above. The density skewness results

$$S_{3\delta}(\tau) = 4 - 2 \frac{E(\tau)}{D(\tau)^2},$$

(71)

while the skewness of the velocity-divergence reads (Bernardeau et al. 1994a):

$$S_{3\theta}(\tau) = 2 - 2 \frac{E'(\tau)}{D(\tau)D'(\tau)}.$$  

(72)

The divergence of the velocity field $\theta$ in Bernardeau et al. (1994a) is defined in such a way that $\theta = -f(\Omega)\eta$ and thus

$$S_{3\theta} = -f(\Omega)^{-1}S_{3\eta}.$$  

4.1.3 Bernardeau (1994)

The Eulerian perturbative expansion here is based on the spherical model; it reads

$$\delta(x,\tau) = D(\tau)\delta_1(x) + \frac{1}{2!}D_2(\tau)\delta_1(x)^2 + \cdots ,$$

(73)

explicitly up to the second-order contributions. The function $D_2(\tau)$ is a solution of the differential equation

$$\tau^2D''_2 + \frac{\tau(3 + 4\tau)}{2(1 + \tau)}D'_2 - \frac{3}{2(1 + \tau)}D_2 = \frac{3}{1 + \tau}D^2 + \frac{8}{3}\tau^2D'^2,$$

(74)

and, in the limit $\tau \to 0$, $D_2 \to 3\bar{D}^2/21$. The density skewness is

$$S_{3\delta}(\tau) = 3 \frac{D_2(\tau)}{D(\tau)^2},$$

(75)

and the velocity-divergence one is

$$S_{3\theta}(\tau) = -6 + 3 \frac{D'_2(\tau)}{D(\tau)D'(\tau)}.$$  

(76)

Again, the divergence of the velocity field $\theta$ in Bernardeau (1994) is such that $\theta = -f(\Omega)\eta$. These Eulerian results for the skewness are identical to those obtained with Lagrangian methods by Bouchet et al. (1992): the expressions of $S_{3\delta}$ and $S_{3\theta}$, respectively in equations (71) and (72) and equations (75) and (76), coincide, in that the relation $D_2 = \frac{1}{4}(2D^2 - E)$ actually holds.

From this brief review we understand that, including ours, there are in the literature four independent calculations of $S_{3\delta}(\Omega)$ and three of $S_{3\theta}(\Omega)$. In particular, comparing equation (59) with equations (67) and (71), the two different expressions of $B(\tau)$ have been used: 

i) $B_1(\tau) = \frac{1}{2} \left[ 6 - 17 \frac{E(\tau)}{\tau^2} \right]$ is the equivalent of our function $B(\tau)$ in Martel & Freudling (1991); 

ii) $B_2(\tau) = \frac{1}{2} \left[ 1 + \frac{E(\tau)}{D(\tau)^2} \right]$, in Bouchet et al. (1992), coinciding with our $B(\tau)$, solution of Eq.(46). The disagreement with the Martel & Freudling results is summarized in Fig. 1.

It should however be noticed that, in the observationally relevant $\Omega$ range, the two determinations of the skewness differ only by a few percents, while the deviations become more conspicuous for low $\Omega$ values.

5 SUMMARY AND CONCLUSIONS

In this paper we developed an Eulerian perturbative formalism which allows to solve the Newtonian equations governing the dynamics of a cold irrotational fluid in a generic Friedmann universe, with vanishing cosmological constant. In particular, the problem is notably simplified once described in terms of the gravitational and velocity potentials, respectively $\varphi$ and $\Phi$. The general set of scalar equations for these potentials was obtained and it was shown that, in order to close this set one needs to introduce an auxiliary scalar potential related to the current $j = \delta u$, carried by the density contrast $\delta$, moving with peculiar velocity $u$.

Once the fundamental equations (22), which are nonlinear and nonlocal in the potentials, were obtained, the next step
was to solve them for arbitrary density parameter $\Omega$. We solved them according to the general Eulerian perturbative ansatz given in (24) and (25): apart from the approximation intrinsic in any perturbative approach, no further approximation was introduced. In particular, we focused in seeking the second-order perturbative solutions $\varphi^{(2)}$ and $\Phi^{(2)}$, whose explicit expressions are reported in equations (36) and (37), which we demonstrated not to be separable with respect to their space and time dependence; the separability assumption is correct only in the case of the Einstein-de Sitter universe, but not for more general Friedmann models.

As a first application of our results, we have computed the second-order deviation of the velocity-divergence, $\nabla \cdot \mathbf{u}$, from the density contrast, $\delta$. We finally calculated the skewness parameters $S_{3\delta}$ and $S_{3\eta}$ of the density fluctuation and velocity-divergence fields, respectively; these third-order moments quantify the asymmetric deformation of the original Gaussian density and velocity-divergence distributions, induced by gravity during the (weakly) nonlinear evolution. The weak dependence of $S_{3\delta}$ and $S_{3\eta}$ on the density parameter $\Omega$, first demonstrated applying Lagrangian perturbation theory, was confirmed.

Our results also clarify that the Eulerian perturbation theory, which is characterized by non-separable perturbative higher-order solutions (in the case of a general non flat Friedmann model), is equivalent to the corresponding Lagrangian description, whose perturbative solutions of the dynamical fluid equations are intrinsically separable with respect to the space and time coordinates.

The present work provides a first attempt of building up a complete higher-order Eulerian perturbation theory; most previous papers only dealt with the construction of the density fluctuation and velocity-divergence fields [a remarkable exception being the work by Munshi & Starobinsky (1994), which however, besides adopting a different formalism is restricted to the flat case]. This kind of treatment should allow a direct comparison of the Eulerian approach with the Lagrangian one, which might lead to understanding many non-trivial aspects of the nonlinear dynamics of self-gravitating collisionless matter in the expanding universe.

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REFERENCES

Bagla J.S., Padmanabhan T., 1994, MNRAS, 266, 227
Baugh C.M., Efstathiou G., 1994, MNRAS, in press
Baugh C.M., Gaztañaga E., Efstathiou G., 1994, MNRAS, submitted
Baumgart D.J., Fry J.N., 1991, ApJ, 375, 25
Bernardeau F., 1992a, ApJ, 390, L61
Bernardeau F., 1992b, ApJ, 392, 1
Bernardeau F., 1994, ApJ, 433, 1
Bernardeau F., Juszkiewicz R., Dekel A., Bouchet F.R., 1994a, MNRAS, submitted
Eulerian perturbation theory in non-flat universes

Bernardeau F., Singh T.P., Banerjee B., Chitre S.M., 1994b, MNRAS, 269, 947

Bertschinger E., 1993, in Cosmic Velocity Fields, eds. Bouchet F.R., Lachièze-Rey M., Editions Frontière, p.137

Bertschinger E., Hamilton A., 1994, ApJ, 435, 1

Brainerd T.G., Scherrer R.J., Villumsen J.V., 1993, ApJ, 418, 570

Bouchet F.R., Hernquist L., 1992, ApJ, 400, 25

Bouchet F.R., Juszkiewicz R., Colombi S., Pellat R., 1992, ApJ, 394, L5

Bouchet F.R., Strauss M., Davis M., Fisher K.B., Yahil A., Huchra J.P. 1993, ApJ, 417, 36

Buchert T., 1989, A&A, 223, 9

Buchert T., 1992, MNRAS, 254, 729

Buchert T., 1994, MNRAS, 267, 811

Buchert T., Ehlers J., 1993, MNRAS, 264, 375

Catelan P., 1995, MNRAS, in press

Catelan P., Moscardini L., 1994a, ApJ, 426, 14

Catelan P., Moscardini L., 1994b, ApJ, 436, 5

Coles P., 1990, MNRAS, 243, 171

Coles P., Ellis G., 1994, Nature, 370, 609

Coles P., Frenk C.S., 1991, MNRAS, 253, 727

Coles P., Melott A.L., Shandarin S., 1993, MNRAS, 260, 765

Doroshkevich A.G., 1973, Astrophys. Letters, 14, 11

Doroshkevich A.G., Zel’dovich Ya.B., 1975, Ap Space Sci, 35, 55

Efstathiou G., Kaiser N., Saunders W., Lawrence A., Rowan- Robinson M., Ellis R.S., Frenk C.S., 1990, MNRAS, 247, 10P

Efstathiou G., Sutherland W.J., Maddox S.J., 1990, Nature, 348, 705

Fry J.N., 1984, ApJ, 279, 499

Fry J.N., 1985, Phys. Lett., 158B, 211

Fry J.N., Scherrer R.J., 1994, ApJ, 429, 36

Gaztañaga E., 1992, ApJ, 398, L17

Gaztañaga E., 1994, MNRAS, 286, 913

Giavalisco M., Mancinelli B., Mancinelli P.J., Yahil A., 1993, ApJ, 411, 9

Goroff M.H., Grinstein B., Rey S.-J., Wise M.B., 1986, ApJ, 311, 6

Gramann M., 1993a, ApJ, 405, 449

Gramann M., 1993b, ApJ, 405, L47

Grinstein B., Politzer H.D., Rey S.-J., Wise M., 1987, ApJ, 314, 431

Hansel D., Bouchet F.R., Pellat R., Ramani A., 1985, Phys. Rev. Lett., 55, 437

Heavens A.F., 1993, in Cosmic Velocity Fields, eds. Bouchet F.R., Lachièze-Rey M., Editions Frontière, p.341

Jain B., Bertschinger E., 1994, ApJ, 431, 495

Juszkiewicz R., 1981, MNRAS, 197, 931

Juszkiewicz R., Bouchet F.R., Colombi S., 1993, ApJ, 412, L9

Juszkiewicz R., Sonoda D.H., Barrow J.D., 1984, MNRAS, 209, 139

Juszkiewicz R., Weinberg D.H., Amsterdamski P., Chodorowski M., Bouchet F.R. 1994, ApJ, to appear

Kofman L., 1991, in Primordial Nucleosynthesis and Evolution of the Early Universe, ed. Sato K., Kluwer Academic, Dordrecht

Kofman L., Bertschinger E., Gelb M.J., Nusser A., Dekel A., 1994, ApJ, 420, 44

Kofman L., Pogosyan D., 1994, ApJ, submitted

Lachièze-Rey M., 1993, ApJ, 408, 403

Lahav O., Tsoi M., Inagaki S., Suto Y., 1993, ApJ, 402, 387

Lokas E.L., Juszkiewicz R., Weinberg D.H., Bouchet F.R., 1994, MNRAS, submitted

Lucchin F., Matarrese S., Melott A.L., Moscardini L., 1994, ApJ, 422, 430

Luo X.C., Schramm D.N., 1993, ApJ, 408, 33

Makino N., Sasaki M., Suto Y., 1992, Phys. Rev., D46, 585

Mancinelli P.J., Yahil A., 1994, preprint

Mancinelli P.J., Yahil A., Ganon G., Dekel A., 1993, in Cosmic Velocity Fields, eds. Bouchet F.R., Lachièze-Rey M., Editions Frontière, p.215

Martel H., 1995, ApJ, in press

Martel H., Freudling W., 1991, ApJ, 371, 1

Matarrese S., Lucchin F., Moscardini L, Saez D., 1992, MNRAS, 259, 437

Matarrese S., Pantano O., Saez D., 1994a, Phys. Rev. Lett., 72, 320

Matarrese S., Pantano O., Saez D., 1994b, MNRAS, 271, 513

Meiksin A., Szapudi I., Szalay A.S., 1992, ApJ, 394, 87

Melott A.L., Buchert T., Weiss A., 1995, A&A, 294, 345.

Melott A.L., Lucchin F., Matarrese S., Moscardini L., 1994, ApJ, 422, 430

Melott A.L., Pellman T., Shandarin S., 1994, MNRAS, 269, 626

Messina A., Lucchin F., Matarrese S., Moscardini L, 1992, Astroparticle Phys., 1, 99.

Morse P.M., Feshbach H., 1953, Methods of Theoretical Physics, McGraw-Hill (Vol. 1, Chapter 1)

Moscardini L., Matarrese S., Lucchin F., Messina A., 1991, MNRAS, 248, 424

Moutarde F., Alimi J.-M., Bouchet F.R., Pellat R., Ramani A., 1991, ApJ, 382, 377

Munshi D., Sahni V., Starobinsky A.A., 1994, ApJ, submitted

Munshi D., Starobinsky A.A., 1994, ApJ, 428, 433

Nusser T.A., Dekel A., 1992, ApJ, 391, 443
APPENDIX A1: APPROXIMATE SOLUTIONS OF THE HAMILTON-JACOBI EQUATION

A standard method to deal with the solution of the Hamilton-Jacobi equation (7) is through the simple ansatz

\[ \Phi_S = \mathcal{W}(x) - \epsilon D, \]  

(A1)

where \( \mathcal{W} \) is Hamilton’s principal function, and the constant \( \epsilon \) plays the role of particle ‘energy’ per unit mass. In our case, however, this ansatz cannot work, since the peculiar gravitational potential, self-consistently determined from the Poisson equation (3), generally contains a non-trivial time dependence. Nevertheless, in most approximation schemes aimed at moving particles, the physical peculiar gravitational potential is practically replaced by some external, or ‘mock’, potential, e.g. derived from perturbation theory, which allows to simply close the fluid dynamical equations, without fully solving for the self-gravitation of the moving particles. Examples of this are: i) the Zel’dovich approximation (Zel’dovich 1970a,b), where one makes the ansatz \( \varphi(x, D) = -\Phi(x, D) \), and ii) the frozen flow approximation (Matarrese et al. 1992); where one can write \( \varphi(x, D) = \varphi^{(1)}(x) - D \frac{\partial}{\partial D} \left( \nabla \varphi^{(1)}(x) \right)^2 \) [extending to \( \Omega \neq 1 \) the expression in (Matarrese et al. 1992)]. Second-order type solutions can also be put in a similar form, but we will not deal with this issue here.

Let us now analyze the two cases above in more detail.

A1.1 Zel’dovich approximation

The action in this case has to satisfy the standard Hamilton-Jacobi equation for inertial motion,

\[ \frac{\partial \Phi_S}{\partial D} + \frac{1}{2} (\nabla \Phi_S)^2 = 0, \]  

(A2)

which can be immediately solved in the form (A1) with \( \mathcal{W}(x) = -\varphi^{(1)}(x) - \nabla \varphi^{(1)}(x) (x - q) \) and \( \epsilon = \frac{1}{2} (\nabla \varphi^{(1)}(x))^2 \). One then obtains the standard formula (e.g. Kofman 1991)

\[ \Phi_S(x, D) = -\varphi^{(1)}(x) + \frac{(x - q)^2}{2D}, \]  

(A3)

with \( x(q, D) = q - D \nabla \varphi^{(1)}(q) \).

A1.2 Frozen flow approximation

In this case, after replacing the appropriate expression for \( \varphi \) in the Hamilton-Jacobi equation, we immediately obtain

\[ \Phi_S(x, D) = -\varphi^{(1)}(x), \]  

(A4)

corresponding to vanishing particle ‘energy’, \( \epsilon = 0 \). This conclusion is consistent with a simple picture of this approximation according to which the fluid elements behave as test particles, which just trace the initial stream-lines.

The advantage of formulating these two approximations in terms of the Hamilton-Jacobi approach is that, once the solution is known, both mass and momentum conservation are exactly satisfied at all stages of the nonlinear evolution.
APPENDIX A2: FAILURE OF THE SEPARABILITY ANSATZ IN NON-FLAT MODELS

In this appendix we explicitly demonstrate that, in the case of a generic Friedmann universe, any separable perturbative approximation is not a solution of the dynamical equations for the cosmological potentials \( \varphi \) and \( \Phi \). To avoid an exceeding proliferation of mathematical symbols, we use in this appendix the same notation of the main text.

Let us therefore suppose that a possible class of solutions of the fundamental equations for the cosmological potentials may be written in the form

\[
\varphi(x, D) = \varphi^{(1)}(x) + \varphi^{(2)}(x, D) + \cdots \equiv \varphi_1(x) + D g_\varphi(D) \varphi_2(x) + \cdots, \\
\Phi(x, D) = \Phi^{(1)}(x) + \Phi^{(2)}(x, D) + \cdots \equiv \Phi_1(x) + D g_\Phi(D) \Phi_2(x) + \cdots,
\]

where the functions \( \varphi_2(x) \) and \( \Phi_2(x) \) are assumed to be independent of \( D \), while the functions \( g_\varphi(D) \) and \( g_\Phi(D) \) are assumed to depend only on \( D \). The calculations performed in Section 2 can then be repeated up to Eqs.(36) and (37), which we rewrite in the form

\[
g_\varphi(D) \varphi_2(x) = - \left[ 1 - B(D) \right] F_2(x) + \frac{1}{2} B(D) \left[ \nabla \varphi_1(x) \right]^2,
\]

and

\[
g_\Phi(D) \Phi_2(x) = \left[ 1 - 2B(D) - DB'(D) \right] F_2(x) - \frac{1}{2} \left[ 2B(D) + DB'(D) \right] \left[ \nabla \varphi_1(x) \right]^2,
\]

where the two space-dependent functions \( \varphi_2(x) \) and \( \Phi_2(x) \) can be easily determined from the \( \Omega \to 1 \) limit (see Eqs.(A14) and (A15) below). It is then clear that the two functions \( g_\varphi(D) \) and \( g_\Phi(D) \) cannot be space-independent, as they should, unless, either one restricts to the Einstein-de Sitter case, in which case \( g_\varphi(D) \equiv g_\Phi(D) \equiv 1 \), or one imposes suitable restrictions on the functional form of the initial potential \( \varphi_1(x) \). Specifically, it is easy to show that the separability ansatz (A5), (A6) for the gravitational and velocity potentials requires the validity of the constraint

\[
F_2(x) \equiv K \left[ \nabla \varphi_1(x) \right]^2,
\]

with \( K \) a suitable constant. It is interesting to note that a sufficient condition for the validity of this constraint is that the second-order invariant of the initial deformation tensor vanishes, \( \mu_2(u_1) = 0 \) (this is e.g. the case of one-dimensional perturbations, which are described by the Zel’dovich solution). This can be easily shown by replacing the relation (A9) into the last of Eqs.(26).

APPENDIX A3: SECOND-ORDER EULERIAN PERTURBATION THEORY IN AN EINSTEIN-DE SITTER MODEL

In this appendix, we want to apply our formalism to the case of an Einstein-de Sitter universe, without cosmological constant. In particular, and as an example of how our formalism works, we want to derive the expressions of \( \varphi^{(2)}(x, a) \) and \( \Phi^{(2)}(x, a) \), then the second-order corrections \( \delta^{(2)}(x, a) \equiv a^2 \delta_2(x) \) and \( u^{(2)}(x, a) \equiv a u_2(x) \) already known in the literature. Again, we use in this appendix the same notation of the main text.

The fundamental equations for the cosmological potentials \( \varphi \) and \( \Phi \) are given by

\[
\begin{aligned}
\frac{\partial \Phi}{\partial a} + \frac{1}{2} (\nabla \Phi)^2 + \frac{3}{2a} (\Phi + \varphi) &= 0, \\
\frac{\partial}{\partial a} (a \varphi) + \Phi + F &= 0, \\
\nabla^2 F &= a \nabla \cdot \left[ \nabla^2 \varphi \nabla \Phi \right].
\end{aligned}
\]

As in Section 3, to find approximate perturbative solutions of these equations, we expand \( \varphi, \Phi \) and \( F \) according to the relations

\[
\varphi(x, a) = \varphi^{(1)}(x) + \varphi^{(2)}(x, a) + \cdots \equiv \varphi_1(x) + a \varphi_2(x) + \cdots, \\
\Phi(x, a) = \Phi^{(1)}(x) + \Phi^{(2)}(x, a) + \cdots \equiv \Phi_1(x) + a \Phi_2(x) + \cdots,
\]

and \( F(x, a) = F^{(2)}(x, a) \equiv a F_2(x) \); we remind that \( \delta^{(1)}(x, a) = a \nabla^2 \varphi^{(1)}(x) \), while \( \delta_1(x) = \nabla \varphi_1(x) \), where \( \delta^{(1)}(x, a) = a \delta_1(x) \); \( \varphi_1(x) = -\Phi_1(x) \). Remind that \( F \) is by definition at least a second-order quantity. Finally, we stress that the general expansions (24) and (25) have become separable in space and time in the flat case. In terms of the perturbed quantities one has
From the first two equations it is possible to obtain the expressions for \( \varphi_2 \) and \( \Phi_2 \):

\[
\varphi_2 = -\frac{5}{7} F_2 + \frac{1}{7} (\nabla \varphi_1)^2 ,
\]

\[
\Phi_2 = \frac{3}{7} F_2 - \frac{2}{7} (\nabla \varphi_1)^2 .
\]

We stress again that the velocity field \( \mathbf{u} \) is manifestly irrotational in physical space, namely

\[
\mathbf{u}(x,t) = \nabla \left[ \Phi^{(1)}(x) + \frac{3}{7} F^{(2)}(x,t) \right] .
\]

The final expressions for \( \delta \) and \( \nabla \cdot \mathbf{u} \) are

\[
\delta = \delta^{(1)} + \nabla^2 \left[ -\frac{5}{7} a F^{(2)} + \frac{1}{7} a^2 \mathbf{u}^{(1)} \right]
\]

\[
= \delta^{(1)}(x,t) + \frac{5}{7} \delta^{(1)}(x,t)^2 - a (\mathbf{u}^{(1)} \cdot \nabla) \delta^{(1)}(x,t) + \frac{2}{7} a^2 \sum_{\alpha \beta} \left( \partial_\alpha u_\beta \right)^2 ,
\]

and

\[
- a \nabla \cdot \mathbf{u} = \delta^{(1)} + \nabla^2 \left[ -\frac{3}{7} a F^{(2)} + \frac{2}{7} a^2 \mathbf{u}^{(1)} \right]
\]

\[
= \delta^{(1)}(x,t) + \frac{3}{7} \delta^{(1)}(x,t)^2 - a (\mathbf{u}^{(1)} \cdot \nabla) \delta^{(1)}(x,t) + \frac{4}{7} a^2 \sum_{\alpha \beta} \left( \partial_\alpha u_\beta \right)^2 ,
\]

which gives equations (42) and (43) of the main text; they can be also compared with equation (18.8) in Peebles (1980), where, however, a slightly different definition of gravitational potential is used, with equations (19a,b) in Wise (1988) and with equations (8) and (10) in Catelan & Moscardini (1994b).

**APPENDIX A4: SECOND-ORDER EULERIAN PERTURBATION THEORY IN A CLOSED FRIEDMANN MODEL**

The \( \Omega > 1 \) case may be treated starting from equation (44) and transforming to the new variable (see e.g. Peebles 1980)

\[
\tau = 1 - \frac{1}{\Omega} ,
\]

where now \( 0 \leq \tau < 1 \), the lower limit corresponding to the Big Bang, the upper limit to the instant of maximum expansion: in this appendix we consider only the expanding phase.

The equation to be integrated is formally identical to equation (46) and the initial conditions are the same. Finally, all the non-trivial expressions valid for a closed model are

\[
J(\tau) \equiv \frac{D(\tau)}{D'(\tau)} = \tau f(\tau)^{-1} ,
\]

and \( J(\tau) \to \tau \) in the limit \( \tau \to 0 \), like in the open models; furthermore

\[
D(\tau) = -1 + \frac{3}{\tau} - \frac{3\sqrt{1-\tau}}{\tau^{3/2}} M(\tau) ,
\]

with \( M(\tau) \equiv \arctan(\sqrt{\tau/(1-\tau)}); D(\tau) = 2\tau/5 \) in the limit \( \tau \to 0 \), as in the case \( \Omega < 1 \). The function \( A(\tau) \) is now given by

\[
A(\tau) = 2 + \frac{3}{2} \left[ (1-\tau) f(\tau)^2 \right]^{-1} ,
\]

and

\[
f(\tau) = -\frac{3}{2} \left[ \frac{3\sqrt{\tau(1-\tau)} - (3 - 2\tau) M(\tau)}{(3-\tau)\sqrt{\tau(1-\tau)} - (3 - 3\tau) M(\tau)} \right] ,
\]

with \( f(0) = 1 \). We then have:
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\begin{equation}
B(\tau) = \frac{1}{2} - \frac{1}{4D(\tau)^2} + \frac{9}{4\pi D(\tau)^2} \left\{ 1 - \sqrt{\frac{1 - \tau}{\tau}} M(\tau) - \frac{1}{2} \left[ \sqrt{\frac{1 - \tau}{\tau}} - \frac{M(\tau)}{\tau} \right]^2 \right\}.
\end{equation}

(A24)

If, alternatively, one prefers to work directly in terms of the time-variable $\Omega$, the formulae which have to be used are, respectively:

\begin{align}
J(\Omega) &\equiv \frac{D(\Omega)}{D'(\Omega)} = \Omega(\Omega - 1) f(\Omega)^{-1}, \tag{A25} \\
D(\Omega) &\equiv \frac{1}{(\Omega - 1)^{3/2}} \left[ (1 + 2\Omega)\sqrt{\Omega - 1} - 3\Omega M(\Omega) \right], \tag{A26} \\
f(\Omega) &\equiv -\frac{3\Omega}{2} \left[ \frac{3\sqrt{\Omega - 1} - (2 + \Omega) M(\Omega)}{(1 + 2\Omega)\sqrt{\Omega - 1} - 3\Omega M(\Omega)} \right], \tag{A27}
\end{align}

where $M(\Omega) \equiv \arctan \sqrt{\Omega - 1}$ and $M(1) = 0$. 
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