GLOBAL WELL-POSEDNESS OF NON-ISOTHERMAL
INHOMOGENEOUS NEMATIC LIQUID CRYSTAL FLOWS

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Abstract. In this paper, we consider the initial-boundary value problem to the non-isothermal incompressible liquid crystal system with both variable density and temperature. Global well-posedness of strong solutions is established for initial data being small perturbation around the equilibrium state. As the tools in the proof, we establish the maximal regularities of the linear Stokes equations and parabolic equations with variable coefficients and a rigid lemma for harmonic maps on bounded domains. This paper also generalizes the result in [5] to the inhomogeneous case.

1. Introduction. Liquid crystal is an intermediate state between liquid and crystal. Such materials can be obtained typically by increasing the temperature of a crystal or increasing the concentration of a liquid solvent. In particular, the nematic liquid crystals are those composed of rod-like molecules with head-to-tail symmetry. Such symmetry promises one desirable optical and magnetical properties, which are manipulatable thanks to the fluidity of the materials. Thus it has been a major interest in science research and has wide application in various industry, such as displaying devices.

The mathematical analysis of nematic liquid crystals started in 1960s by Ericksen, Leslie and their collaborators, see [11, 12, 30, 31]. In which, they proposed the so-called Oseen-Frank elastic energy density:

$$W(d, \nabla d) = k_1 (\text{div} d)^2 + k_2 (d \cdot (\nabla \times d))^2 + k_3 |d \times (\nabla \times d)|^2 + k_4 (\text{tr} (\nabla d)^2 - (\text{div} d)^2),$$

where $d$ is the unit director representing the orientation field of molecules, $k_i (i = 1, 2, 3, 4)$ are material constants. Incorporating the kinetic energy, they derive a hydrodynamic system for liquid crystals by a variational method. For more comprehensive introduction on the Ericksen-Leslie theory, we refer to [17, 42, 44].
Due to the complexity of the full Ericksen-Leslie system, Lin and Liu proposed the following extensively simplified models around 1990:

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u - \Delta u + \nabla P &= -\Delta d \cdot \nabla d, \\
\text{div } u &= 0, \quad |d| = 1, \\
\partial_t d + u \cdot \nabla d - \Delta d &= |\nabla d|^2 d,
\end{aligned}
\]

(1.1)

which is a forced incompressible Navier-Stokes equation coupled with the heat flow of harmonic maps. Here the supercritical term $|\nabla d|^2 d$ causes one of the major difficulties in mathematical analysis. Inspired by the study of harmonic maps, they replaced $|\nabla d|^2$ in the free energy by a Ginzburg-Landau approximation $\frac{1}{4\epsilon^2}(1 - |d|)$ and removed the constraint $|d| = 1$ from the system. For this approximated system, existence and partial regularity of weak solutions in both 2-D and 3-D were established in [35, 36, 37]. However, it is a challenging problem to prove the corresponding results for the original system. The main difficulty of proving global existence of weak solutions to system (1.1) is that the desired space-time $L^2$ bound on $\Delta d$ is not included in the following basic energy inequality:

\[
\frac{d}{dt} \int_{\Omega} (|u|^2 + |\nabla d|^2) \, dx + \int_{\Omega} (|\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2) \, dx \leq 0.
\]

(1.2)

For the 2-D case, by using a localized Ladyzhenskaya inequality, one can get the desired $L^2$ estimates on $\Delta d$ locally in time, in the similar way as Struwe [45] to study the heat flow of harmonic maps. Based on this observation and through different approaches, Lin-Lin-Wang [38] and Hong [22] independently proved the existence of global weak solutions for initial data $(u_0, \nabla d_0) \in L^2 \times L^2$, and such solutions are shown to be smooth away from finite many time slices. Similar results have been proved for more general system in [24, 27, 47]. The uniqueness of the above weak solutions is studied in [33, 39, 48]. Lastly, it is worth remarking that for the 2-D case, Lei-Li-Zhang proved in [29] that under a geometric condition $d_0^3 \geq \epsilon > 0$, which will be preserved along with time evolution guaranteed by the maximum principle, the weak solutions in [22, 38] are actually smooth globally.

However, the idea used for the 2-D case mentioned in the previous paragraph does not apply to the 3-D case, and the well-posedness of (1.1) in 3D remains largely open for general large initial data. Recently, under the geometric condition $d_0^3 > 0$, Lin-Wang [40] proved the existence of global weak solutions. For suitably regular initial data, Hong-Li-Xin [23] proved the local well-posedness of strong solutions to the liquid crystal system, with general Oseen-Frank free energy density, and established some blow up criteria. For small perturbed initial data around the trivial equilibrium state $(0, e)$, global existence of strong solutions in $L^p - L^q$ setting is proved in [19] by treating (1.1) as a quasilinear parabolic system. Also Wang [46] obtained the global mild solution of (1.3) for initial data $(u_0, d_0)$ belonging to possibly the largest space $BMO^{-1} \times BMO$, with small norm.

In all the works mentioned in the above, the density is assumed to be a positive constant, and the equation for the temperature is ignored. As mentioned at the beginning, by increasing the density or changing the temperature are the two typical way to forming the liquid crystals. Therefore, both physically and mathematically, to have better understanding about the behavior of the liquid crystals, one has to consider the liquid crystal system with variable density or temperature. Along this direction, there has been many studies.
If including the variable density into (1.1), i.e., considering the density-dependent simplified Ericksen-Leslie system, the corresponding mathematical analysis is more complicated than the constant density case, as the density satisfies a hyperbolic equation. For the case that the density has positive lower bound, Li [32] proved the global existence of weak solution for initial data \((\rho_0 - \bar{\rho}, u_0, \nabla d_0) \in L^q \times L^2 \times L^2\), for some \(q \in (1, \infty)\), where \(\bar{\rho}\) is a positive constant, and \(d_0^3 \geq \epsilon_0 > 0\). For the case that the initial density tends to zero at infinity, global existence of strong solution is proved in [41], for more regular initial data satisfying the same geometric condition as in [32]. As for the 3-D case, it remains largely open.

For the temperature-dependent simplified Ericksen-Leslie system, global existence of weak solutions was proved for the Ginzburg-Landau type approximated system in [13, 14] both in 2-D and 3-D. As for the original simplified system, Li-Xin [34] proved the existence of global weak solutions for the 2-D case. If the initial data is suitably regular, and if it is a small perturbation around the equilibrium state \((0, e, 1)\), global existence of strong solutions were proved by the authors [5] and Hieber-Prüss [20], by using different approaches.

There are also many works on the study of liquid crystal system in the compressible case, see, e.g., [10, 16, 18, 21, 26, 28] and the references therein.

The aim of this paper is to study the liquid crystal systems with both the effects of variable density and temperature. To this end, we consider the following system:

\[
\begin{cases}
\partial_t \rho + u \cdot \nabla \rho = 0, \\
\rho \partial_t u + \rho u \cdot \nabla u - \text{div} (\mu(\theta) D(u)) + \nabla P = -\Delta d \cdot \nabla d, \\
\partial_t d + u \cdot \nabla d = \Delta d + |\nabla d|^2 d, \\
\rho \partial_t \theta + \rho u \cdot \nabla \theta - \text{div}(\kappa \nabla \theta) = \frac{1}{2} \mu(\theta) |D(u)|^2 + |\Delta d + |\nabla d|^2 d|^2, \\
\text{div} u = 0, \quad |d| = 1.
\end{cases}
\]  

The above equations describe, respectively, the conservation of mass, linear momentum, angular momentum, internal energy, incompressibility and physical constraint on the director field. Here, we denote by \(\rho, u, d, P\) and \(\theta\) the density, velocity, director, pressure and temperature, respectively, \(D(u)\) is the Cauchy stress tensor \(D(u) = \nabla u + \nabla u^T\). The above system is simplified from those proposed in [42, 44], and such simplification can also be found in [34]. For simplicity, we will assume that the heat conducting coefficient \(\kappa\) is a positive constant, and ignore the stretching effect. Although the stretching effect is not included, this system shares some of the key mathematical difficulties with the true physical models and thus has its own interests.

We consider the system in a bounded domain \(\Omega \subset \mathbb{R}^N\) with smooth boundary, and we impose the following initial boundary conditions:

\[
(\rho, u, d, \theta)|_{t=0} = (\rho_0, u_0, d_0, \theta_0), \quad (u, B d, \partial_n \theta)|_{\partial \Omega} = (0, 0, 0),
\]  

where \(B d = \partial_n d\) or \(d - e\), \(\nu\) is the outward normal vector on \(\partial \Omega\). We always assume that \(u_0\) and \(d_0\) satisfies the necessary compatibility conditions

\[
\text{div} u_0 = 0, \quad |d_0| = 1, \quad d_0|_{\partial \Omega} = e.
\]

In addition, we assume that \(\mu(\theta)\) is continuously differentiable in \(\theta\) and satisfies

\[
0 < \underline{\mu} \leq \mu(\theta) \leq \bar{\mu} < \infty, \quad |\mu'(\theta)| \leq \bar{\mu}', \quad \forall \theta.
\]
If \((\rho, u, d, \theta)\) is a strong solution to (1.3), one easily checks that
\[
\frac{d}{dt} E(t) = 0,
\]
where \(E(t) = \int_{\Omega} (\frac{1}{2} |u|^2 + \theta + \frac{1}{2} |\nabla d|^2) \, dx.
\]

If ignoring the temperature equation, and assume that \((\rho, u, P, d)\) is a solution to the Cauchy problem of the corresponding system, then
\[
\rho^\lambda(x, t) := \rho(\lambda x, \lambda^2 t), \quad u^\lambda(x, t) := \lambda u(\lambda x, \lambda^2 t), \quad d^\lambda(x, t) := d(\lambda x, \lambda^2 t)
\]
is also a solution. This suggests us to find solutions \((\rho, u, d, P, \theta)\) to system (1.3), such that \((\rho, u, d)\) lies in the critical space \(W^{1,N} \times L^N \times W^{1,N}.
\]

We are going to establish the local or global well-posedness of strong solutions to system (1.3). By a strong solution to system (1.3) on \(\Omega \times (0, T)\), we mean that a set \((\rho, u, d, P, \theta)\) satisfies the equations in (1.3) a.e. in \(\Omega \times (0, T)\), and belongs to the function space \(E_T^{p,q,r,s}\), where \(E_T^{p,q,r,s}\) is defined as follows.

**Definition 1.1.** For \(T > 0\) and \(1 < p, q, r, s < \infty\), we denote \(E_T^{p,q,r,s}\) by the set of \((\rho, u, d, P, \theta)\) such that
\[
\begin{align*}
\rho &\in C([0, T]; W^{1,q}) \cap W^{1,\infty}(0, T; L^{\frac{2}{1-\frac{2}{p}}}), \\
u &\in C([0, T]; D_{A, r}^{1-\frac{2}{p}}(\Omega)) \cap W^{1,p}(0, T; L^r(\Omega)) \cap L^p(0, T; W^{2,r}(\Omega)), \\
d &\in C([0, T]; \dot{B}_{r,p}^{1-\frac{2}{p}}(\Omega)) \cap W^{1,p}(0, T; W^{1,\infty}(\Omega)) \cap L^p(0, T; W^{3,r}(\Omega)), \\
P &\in W^{1,p}(0, T; W^{1,\infty}(\Omega)), \quad \int_{\Omega} P \, dx = 0, \\
\theta &\in C([0, T]; \dot{B}_{q,s}^{2-\frac{2}{s}}(\Omega)) \cap W^{1,\infty}(0, T; L^q(\Omega)) \cap L^s(0, T; W^{2,q}(\Omega)).
\end{align*}
\]

We remark that the condition \(\int_{\Omega} P \, dx = 0\) in the above definition holds automatically if we replace \(P\) by
\[
P - \frac{1}{|\Omega|} \int_{\Omega} P \, dx
\]
in system (1.3). Also, the space \(D_{A, r}^{1-\frac{2}{p}}\) in Definition 1.1 stands for some fractional domain of the Stokes operator defined as the interpolation space:
\[
D_{A, r}^{1-\frac{2}{p}} := (L^r_{\sigma}, D(A_r))_{1-\frac{2}{p}},
\]
where \(D(A_r) := \{u \in W^{2,r}(\Omega), \text{div} u = 0, u|_{\partial \Omega} = 0\}\) and \(L^r_{\sigma} := \{u \in L^r(\Omega), \text{div} u = 0\}\). Moreover,
\[
D_{A, r}^{1-\frac{2}{p}} \hookrightarrow \dot{B}_{r,p}^{2(1-\frac{2}{r})} \cap L^r_{\sigma}(\Omega).
\]
The Besov space \(\dot{B}_{r,p}^{2(1-\frac{2}{r})}\) on a bounded domain with regular boundary can be regarded as the interpolation space between \(L^r\) and \(W^{2,r}\), that is,
\[
\dot{B}_{r,p}^{2(1-\frac{2}{r})} = (L^r, W^{2,r})_{1-\frac{2}{r}}.
\]

We are now ready to state our main results in this paper. The first one is the following local existence of strong solutions to system (1.3).

**Theorem 1.2.** Suppose \(\Omega \subset \mathbb{R}^N\) \((N \geq 2)\) is a bounded domain with smooth boundary. For initial data
\[
(\rho_0, u_0, d_0, \theta_0) \in W^{1,q}(\Omega) \times D_{A, r}^{1-\frac{2}{p}} \times \dot{B}_{r,p}^{3-\frac{2}{r}} \times \dot{B}_{q,s}^{2-\frac{2}{s}},
\]
(1.6)
with $0 < \rho \leq \rho_0 \leq \bar{\rho}$ on $\Omega$ and

$$1 < p < \infty, \quad 2 \leq s < \infty, \quad N < r \leq q \quad \text{and} \quad \frac{2}{p} + \frac{N}{r} < 1 + \frac{N}{2q} + 1 < 2, \quad (1.7)$$

there exists $T_0 > 0$ such that the system (1.3)-(1.4) admits a unique local strong solution $(\rho, u, d, P, \theta) \in E_{T_0}^{p,q,r,s}$.

**Remark 1.3.** Note that the index set satisfying (1.7) is not empty. One admissible choice will be $p = s = 2$ and $q = r = N + 1$.

**Remark 1.4.** Note that if one lets $p \to 1$, $q, r \to N$, the solution space for $\rho, u$ and $d$ are arbitrarily close to the critical space $W^{1,N} \times L^N \times W^{1,N}$, but the last constraint in (1.7) will fail. We believe this is caused by the lower regularity of temperature.

We also establish the global well-posedness for small perturbations of trivial equilibrium state.

**Theorem 1.5.** Under the same conditions as in Theorem 1.2, in addition, we assume that $p \leq 2s$ and $d_3^0 \geq \epsilon_0 > 0$ on $\Omega$, then there exists some $\delta_0 > 0$, such that if

$$\|(\rho_0 - 1, u_0, d_0 - e, \theta_0)\|_{W^{1,q} \times D_0^{-1,2s \times B_0,2 \times B_{s/2}}} \leq \delta_0, \quad (1.8)$$

then the system (1.3)-(1.4) admits a unique global solution $(\rho, u, P, d, \theta) \in E_T^{p,q,r,s}$, for all $T > 0$.

**Remark 1.6.** Here we additionally assume that $p \leq 2s$. This is due to some time independent inequalities in Corollary 3.2. Notice that the choice in Remark 1.3 is still admissible here.

As the end of this introduction, let us give some comments about the proof of our main results. Compared with the homogenous case, i.e., the case that the density is a positive constant, the density equation here produces extra difficulty due to its hyperbolic structure, and it seems that the quasilinear approach used in [20] to deal with the homogeneous case does not work to the case considered in this paper. Inspired by the work of Hu-Wang on modified nematic liquid crystal flows in [25], we linearize system (1.3) into

$$\begin{cases}
\partial_t \rho + v \cdot \nabla \rho = 0, \\
\rho \partial_t u - \text{div} (\mu(\phi) D(u)) + \nabla P = -\zeta v \cdot \nabla v - \Delta n \cdot \nabla n, \\
\partial_t d - \Delta d = -v \cdot \nabla n + |\nabla n|^2 n, \\
\rho \partial_t \theta - \Delta \theta = -\zeta v \cdot \nabla \phi + \mu(\phi)|D(v)|^2 + |\Delta n + |\nabla n|^2|^2, \\
\text{div} u = 0.
\end{cases} \quad (1.9)$$

Besides the transport equation, note that the left hand side of above system consists of Stokes equations and parabolic equations with variable density and temperature dependent viscosity. If $\rho$ is positive and bounded from below, dividing the second and fourth equation in (1.9) by $\rho$, then they reduce to the Stokes and parabolic equation with variable viscosity plus a lower order term. The solvability of Stokes equations and parabolic equations with variable coefficient is well-known in the existing literature, for example, we refer to [1, 2, 6, 43]. Here in order to close the estimates, we follow the localization argument in [7], which works on the
inhomogeneous Navier-Stokes system, to derive the similar Stokes and parabolic estimates with explicit dependence on the variable density and viscosity.

Given such estimates, it would suffice to prove the local existence of solutions. To extend such a local solution to be a global one, the main difficulty comes from the transport equation. As we shall see later in the proof, to obtain the estimates \( \|\rho\|_{L^\infty([0,T];W^{1,r})} \), one needs \( \|\nabla u\|_{L^1([0,T];L^\infty)} \). In the framework of strong solution to Navier-Stokes equations, this is ensured by the exponential decay of \( \|u(t)\|_{L^2} \) and estimates on \( \|u\|_{L^2([0,T];W^{2,2})} \). However, in our case, as suggested by (1.2), the exponential decay of \( \|u(t)\|_{L^2} \) is not obvious due to the coupling of director field. To overcome this obstacle, we impose the geometric condition on initial director and invoke the observation in [29], generalizing Lei-Li-Zhang’s results to the bounded domain case. Finally, the exponential decay of \( \|u(t)\|_{L^2} \) is recovered.

The rest of the paper is organized as follows. In Sec. 2, we collect some results on transport equation, Stokes equation and parabolic equation, and establish the estimates on variable viscosity Stokes and parabolic equation. Then in Sec. 3 and Sec. 4, local and global existence of solutions are proved, respectively. Finally in the appendix, we show the proof of estimates on variable viscosity Stokes and parabolic equation, for completeness of the paper.

2. The linear estimates.

2.1. Transport equation. The following standard estimates can be found for example in [7].

**Proposition 2.1.** Let \( \Omega \) be a Lipschitz domain of \( \mathbb{R}^N \) and \( v \in L^1(0,T;\text{Lip}) \) be a solenoidal vector field such that \( v \cdot n = 0 \) on \( \partial \Omega \). Let \( a_0 \in W^{1,q} \) with \( q \in [1, +\infty] \). Then the Cauchy problem

\[
\partial_t a + v \cdot \nabla a = 0, \quad a|_{t=0} = a_0, \tag{2.1}
\]

has a unique solution in \( L^\infty(0,T;W^{1,\infty}) \cap C([0,T];\cap_{r<\infty}W^{1,r}) \) if \( q = \infty \) and in \( C([0,T];W^{1,q}) \) if \( q < \infty \). Besides, \( \forall t \in [0,T] \), it holds that

\[
\|a\|_{C([0,t];W^{1,q})} \leq \|a_0\|_{W^{1,q}} \exp\left( \int_0^t \|\nabla v(\cdot, \tau)\|_{L^\infty} d\tau \right), \tag{2.2}
\]

\[
\text{meas.}\{x \in \Omega : \alpha \leq a(t,x) \leq \beta\} = \text{meas.}\{x \in \Omega : \alpha \leq a_0 \leq \beta\}, \alpha, \beta > 0. \tag{2.3}
\]

2.2. Linearized Stokes equation. Building on the maximal regularity of Stokes operator on bounded domain, we obtain the following estimates for the Stokes equation with variable density and the temperature-dependent viscosity:

**Proposition 2.2.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with smooth boundary, \( 1 < p, r < \infty \), \( N < q \leq r \) and \( \mu \) satisfies (1.5). Suppose \( u_0 \in L^p_\Lambda^{1-\frac{1}{p}} \), \( f \in L^p(0,T;L^r) \), \( \rho \) and \( \theta \) satisfies

\[
\rho \in C([0,T];W^{1,q}) \cap \dot{C}^{\beta_1}(0,T;L^\infty), \quad \theta \in C([0,T];W^{1,q}) \cap \dot{C}^{\beta_2}(0,T;L^\infty), \tag{2.4}
\]

for some \( \beta_1, \beta_2 > 0 \) and \( 0 < q \leq \rho(t,x) \leq \bar{\rho} \) for all \( (x,t) \in \Omega_T \). Then the system

\[
\begin{cases}
\rho \partial_t u - \text{div} (\mu(\theta)D(u)) + \nabla P = f, \\
\text{div} u = 0, \quad \int_{\Omega} P dx = 0, \\
u|_{t=0} = u_0, \quad u|_{\partial \Omega} = 0,
\end{cases} \tag{2.5}
\]
has a unique solution \((u, P)\) satisfying
\[
\begin{aligned}
\| u \|_{C([0,t]; D^{1-\frac{1}{p}, p}_T)} + \| (\partial_t u, \Delta u, u, \nabla P) \|_{L^p(0,t; L^r)} &
\leq C(1 + B_{p,\theta}(t)) e^{C_{p,\theta}(t)} \left( \| u_0 \|_{D^{1-\frac{1}{p}, p}_T} + \| f \|_{L^p(0,t; L^r)} \right), \\
\| u \|_{C([0,t]; D^{1-\frac{1}{p}, p}_T)} + \| (\partial_t u, \Delta u, u, \nabla P) \|_{L^p(0,t; L^r)} &
\leq C \left( B_{p,\theta}(t) + 1 \right) \left( \| u_0 \|_{D^{1-\frac{1}{p}, p}_T} + \| f \|_{L^p(0,t; L^r)} \right) + C_{p,\theta}(t) \| u \|_{L^p(0,t; L^r)},
\end{aligned}
\]
for any \(0 < t \leq T\), where \(C\) is independent of \(u_0\), \(f\), \(\rho\), \(\theta\) and \(T\),
\[
B_{p,\theta}(t) = \max \left\{ 1, \| \nabla \rho \|_{C([0,t]; L^q(\Omega))} + \| \nabla \theta \|_{C([0,t]; L^s(\Omega))} \right\},
\]
\[
C_{p,\theta}(t) = (1 + B_{p,\theta}(t)) \left( \| \rho \|_{C_{p_1}(0,t; L^\infty)} + \| \theta \|_{C_{p_2}(0,t; L^\infty)} \right) \frac{Q_{1,\rho} + Q_{2,\theta}}{\min(1, Q_{1,\rho} + Q_{2,\theta})},
\]
k \geq 2 is a positive constant depending only on \(p, q, r, \beta_1, \beta_2\) and \(N\).

**Remark 2.1.** Actually, \(k \to +\infty\) as \(q \to N^+\). Thus these estimates fail for the critical case \(q = N\).

Proposition 2.2 can be proved by following the arguments in [7]. We do not claim to much novelty here, but for the completeness of our presentation, the proof will be given in the appendix.

### 2.3. Linear parabolic equation

First we recall the maximal regularity for the parabolic operators (cf. Theorem 4.10.7 and Remark 4.10.9 in [4], or Chapter II.8 in [8]):

**Proposition 2.3.** Given \(1 < p < q < \infty\), and \(\Omega\) is a bounded domain with regular boundary, for the Dirichlet or Neumann problem
\[
\begin{array}{l}
\partial_t \omega - \Delta \omega = f, \\
\omega|_{t=0} = \omega_0, \\
B_\nu \omega|_{\partial \Omega} = 0,
\end{array}
\]
where \(B_\nu \omega = \omega \nu \partial_\nu \omega\), \(\nu\) is the outward normal vector on \(\partial \Omega\), \(\nu > 0 \in B_{q,p}^{s-\frac{2}{p}}\), and \(f \in L^p(0,T; W^{1,q}(\Omega))\), then system (2.8) has a unique solution \(\omega \in W^{1,p}(0,T; W^{1,q}) \cap L^p(0,T; W^{3,q})\) satisfying
\[
\| \omega \|_{C([0,T]; B_{q,p}^{s-\frac{2}{p}})} + \| \omega \|_{W^{1,p}(0,T; W^{1,q})} + \| f \|_{L^p(0,T; W^{1,q})} \leq C \left( \| \omega_0 \|_{B_{q,p}^{s-\frac{2}{p}}} + \| f \|_{L^p(0,T; W^{1,q})} \right),
\]
where \(C\) is independent of \(\omega_0\), \(f\) and \(T\).

Mimicking the proof of Proposition 2.2, we have also proved the following estimates with density-dependent parabolic operators under the Neumann condition.

**Proposition 2.4.** Suppose \(\Omega \subset \mathbb{R}^N\) is a bounded domain with regular boundary, \(1 < s < \infty\), \(q > N\), \(\omega_0 \in B_{q,s}^{2-\frac{2}{s}}\) and \(f \in L^s(0,T; L^q(\Omega))\),
\[
\forall (t,x) \in \Omega_T, \quad 0 < \rho \leq \rho(t,x) \leq \bar{\rho}, \quad \rho \in C([0,T]; W^{1,q}(\Omega)) \cap C_{\beta_1}([0,T]; L^\infty),
\]
for some \(\beta_1 > 0\), then equation
\[
\rho \partial_t \omega - \Delta \omega = f, \quad \omega|_{t=0} = \omega_0, \quad \partial_\nu \omega|_{\partial \Omega} = 0,
\]
where \(\rho \in C([0,T]; W^{1,q}(\Omega)) \cap C_{\beta_1}([0,T]; L^\infty)\).
admits a unique solution $\omega$ on $[0, T]$ satisfying for any $0 < t \leq T$,

$$
\|\omega\|_{C([0, t]; B^{s-\frac{N}{q}}_{q_1, r_1})} + \|[(\partial_t \omega, \nabla^2 \omega)]_{L_t^1(L^q)} \leq C(1 + B_p(t))^k \exp(CtC_p(t))(\|\omega_0\|_{B^{s-\frac{N}{q}}_{q_2, r_2}} + \|f\|_{L_t^1(L^q)}),
$$

where $B_p$ and $C_p$ are defined as in Proposition 2.2, and $k \geq 2$ is a positive constant depending on $s$ and $q$.

3. Existence on a small time interval. This section is devoted to proving Theorem 1.2. First, we introduce some interpolation inequalities which can also be found in [25] and [7].

Lemma 3.1. Assume $1 < p, p_*, q, r, s, s_1, q_1, r_1 < \infty (i = 1, 2)$,

$$
f \in L_t^{\infty}(B^{s_1}_{q_1, r_1}) \cap L_t^{1}(B^{s_2}_{q_2, r_2}), \quad \frac{s - s_i - \frac{N}{q_i} + \frac{N}{q}}{s_2 - s_1 - \frac{N}{q_2} + \frac{N}{q_1}} < \min \left\{1, \frac{p}{p_*}\right\},
$$

then it holds that

$$
\|f\|_{L_t^{p}(W^{s-q})} \leq C t^{\frac{1}{p} - \frac{1}{p_*}} \|f\|_{L_t^{1}(B^{s-\frac{N}{q}}_{q_1, r_1})}^{\frac{1}{\lambda}} \|f\|_{L_t^{1}(B^{s_2-\frac{N}{q_2}}_{q_2, r_2})}^{\frac{1}{1-\lambda}},
$$

for some $\lambda > 0$ and $C$ is independent of $t$ and $f$.

Proof. Noticing that

$$
B^{s_i}_{q_i, r_i} \hookrightarrow B^{s_i-\frac{N}{q_i} + \frac{N}{q}}_{q_1, r_1}, \quad \left(B^{s_1-\frac{N}{q_1} + \frac{N}{q}}_{q_1, r_1}, B^{s_2-\frac{N}{q_2} + \frac{N}{q}}_{q_2, r_2}\right)_\lambda, = B^{s_0}_{q_0, 1} \hookrightarrow W^{s, q},
$$

provided with that

$$
s_0 = \left(s_i - \frac{N}{q_1} + \frac{N}{q}\right)(1 - \lambda) + \left(s_2 - \frac{N}{q_2} + \frac{N}{q}\right)\lambda \geq s,
$$

for some $\lambda \in (0, 1)$. This is ensured by the condition on the index.

Therefore,

$$
\|f\|_{L_t^{p}(W^{s-q})} \leq C \left(\int_0^t \|f\|_{L_t^{p}(B^{s_i}_{q_i, r_i})} \|f\|_{L_t^{1}(B^{s_2}_{q_2, r_2})}^\lambda \, dt\right)^{\frac{1}{\lambda}},
$$

where we have used $p, \lambda < p$ by the condition on the index. Thus, (3.1) is proved.

The following inequalities immediately follow.

Corollary 3.1. Under the conditions of Theorem 1.2, it holds that

$$
\|\nabla f\|_{L_t^{p}(L^\infty)} \leq C T^{\frac{1}{p} - \frac{1}{p_*}} \|f\|_{L_t^{p}(B^{s_1}_{q_1, r_1})} \|f\|_{L_t^{1}(B^{s_2}_{q_2, r_2})},
$$

$$
\|\nabla f\|_{L_t^{p}(L^{2^*})} \leq C T^{\frac{1}{p} - \frac{1}{p_*} + \frac{1}{2} + \frac{N}{q}} \|f\|_{L_t^{p}(B^{s-\frac{N}{q}}_{q_1, r_1})} \|f\|_{L_t^{1}(B^{s_2-\frac{N}{q_2}}_{q_2, r_2})},
$$

$$
\|\nabla f\|_{L_t^{p}(L^{\infty})} \leq C T^{\frac{1}{p} - \frac{1}{p_*} + \frac{1}{2} + \frac{N}{q}} \|f\|_{L_t^{p}(B^{s-\frac{N}{q}}_{q_1, r_1})} \|f\|_{L_t^{1}(B^{s_2-\frac{N}{q_2}}_{q_2, r_2})},
$$

where $C$ depends only on $p, q, r, s$ and $\Omega$. 

In the proof of global existence of strong solutions, the following interpolation inequalities will play an important role.

**Corollary 3.2.** Under the conditions of Theorem 1.5, it holds that

\[
\|\nabla f\|_{L^p_t(L^\infty)} \leq C\|f\|_{L^p_t(W^{2,r})}, \\
\|\nabla f\|_{L^2_t(L^\infty)} \leq C\|f\|_{L^2_t(B_{r,p}^{2-\frac{2}{r}})} \|f\|_{L^p_t(W^{2,r})}, \\
\|\nabla f\|_{L^q_t(L^\infty)} \leq C\|f\|_{L^q_t(B_{r,p}^{3-\frac{3}{r}})} \|f\|_{L^p_t(W^{3,r})},
\]

where \( C \) is independent of \( f \) and \( T \).

**Proof.** (3.5) immediately follows from the fact that \( W^{2,r} \hookrightarrow W^{1,\infty} \) as \( r > N \). Next, since \( p \leq 2s \), by the log-convexity of \( L^p \) norms (for example see page 27 in [3]),

\[
\|\nabla f\|_{L^p_t(L^{2s})} \leq \|\nabla f\|_{L^p_t(L^{2s})} \|\nabla f\|_{L^p_t(L^{2s})} \leq C\|f\|_{L^p_t(B_{r,p}^{2-\frac{2}{r}})} \|f\|_{L^p_t(W^{2,r})},
\]

where we have used the fact that \( B_{r,p}^{2-\frac{2}{r}} \hookrightarrow W^{1,2q} \) as \( \frac{2}{r} + \frac{N}{r} < 1 + \frac{N}{2q} \) and \( W^{2,r} \hookrightarrow W^{1,2q} \) as \( r > N \), so (3.6) is proved. By the same method, one can easily check (3.7).

Next, we begin to prove the existence of local strong solution through an iteration method. The proof will be divided into the following several steps.

**Step 1: Construction of approximate solution.** We initialize the construction of approximate solution by setting \( \rho^0 := \rho_0, u^0 := u_0, d^0 := d_0 \) and \( \theta^0 := \theta_0 \). Given \( (\rho^n, u^n, d^n, \theta^n) \in E^q_{T,p,r,\sigma} \), Proposition 2.1 enables one to define \( \rho^{n+1} \in C([0,T]; W^{1,q}) \cap W^{1,\infty}(0,T; L^\infty) \) as the unique solution of

\[
\partial_t \rho^{n+1} + u^n \cdot \nabla \rho^{n+1} = 0, \quad \rho^{n+1}|_{t=0} = \rho_0.
\]

Proposition 2.4 enables us to define \( \theta^{n+1} \in C([0,T]; B_{r,p}^{1-\frac{3}{r}}) \cap W^{1,q}_{T}(L^q) \cap L^p_T(W^{2,q}) \) as the unique global solution of the system

\[
\begin{cases}
\rho^{n+1} \partial_t \theta^{n+1} - \Delta \theta^{n+1} \\
= -\rho^{n+1} u^n \cdot \nabla \theta^n + \frac{1}{2} \mu(\theta^n) \nabla (\bar{d}^n)^2 + |\nabla \bar{d}^n|^2 (\bar{d}^n + e), \\
\theta^{n+1}|_{t=0} = \theta_0, \quad \theta^{n+1}|_{\partial\Omega} = 0,
\end{cases}
\]

where \( \bar{d}^n = d^n - e \). And by Proposition 2.3, define \( \bar{d}^{n+1} \in C([0,T]; B_{r,p}^{1-\frac{3}{r}}) \cap W^{1,q}_{T}(L^q) \cap L^p_T(W^{2,q}) \) as the unique solution of system

\[
\begin{cases}
\partial_t \bar{d}^{n+1} - \Delta \bar{d}^{n+1} \\
= -u^n \cdot \nabla \bar{d}^n + |\nabla \bar{d}^n|^2 (\bar{d}^n + e), \\
\bar{d}^{n+1}|_{t=0} = d_0 - e, \quad \bar{d}^{n+1}|_{\partial\Omega} = 0 \quad \text{or} \quad \partial_t \bar{d}|_{\partial\Omega} = 0.
\end{cases}
\]

Finally, Proposition 2.2 enables us to define \( (u^{n+1}, P^{n+1}) \in C([0,T]; D^{1-\frac{1}{p}}_{T,p}) \cap W^{1,p}_{T}(L^q) \cap L^p_T(W^{2,r}) \) as the unique global solution of

\[
\begin{cases}
\rho^{n+1} \partial_t u^{n+1} - \text{div}(\mu(\theta^{n+1}) D(u^{n+1})) + \nabla P^{n+1} \\
= -\rho^{n+1} u^n \cdot \nabla u^n - \Delta \bar{d}^n \cdot \nabla \bar{d}^n, \\
\text{div} u^{n+1} = 0, \quad \int_{\Omega} P^{n+1} \, dx = 0, \\
u^{n+1}|_{t=0} = u_0, \quad u^{n+1}|_{\partial\Omega} = 0.
\end{cases}
\]
Step 2: Uniform bounds for some small fixed time $T$. In this step, we aim at finding a positive time $T$ independent of $n$ for which $(\rho^n, u^n, \bar{df}^n, \bar{P}^n, \theta^n)_{n \in \mathbb{N}}$ is uniformly bounded in the Banach space $E^p_{T}$.

In order to keep our presentation brief, let us denote
\[
\Pi_n(t) = \|\rho^n\|_{C([0,t];W^{1,p})}, \quad \Pi_0 = \|\rho_0\|_{W^{1,p}},
\]
\[
U_n(t) := \|u^n\|_{C([0,t];D^{-\frac{1}{p}}_{A^p})} + \|u^n\|_{W^{1,p}(L^r)} + \|\bar{P}^n\|_{L^r_t(L^p)},
\]
\[
D_n(t) := \|\bar{d}^n\|_{C([0,t];B^{-\frac{1}{p}}_{r,s})} + \|\bar{d}^n\|_{W^{1,p}(L^r)} + \|\bar{\theta}^n\|_{L^p_t(L^r)},
\]
\[
\Theta_n(t) := \|\theta^n\|_{C([0,t];B^{-\frac{1}{p}}_{r,s})} + \|\theta^n\|_{W^{1,p}(L^r)},
\]
\[
U_0 := \|u_0\|_{D^{-\frac{1}{p}}_{A^p}}, \quad D_0 := \|d_0 - e\|_{B^{-\frac{1}{p}}_{r,s}}, \quad \Theta_0 := \|\theta_0\|_{B^{-\frac{1}{p}}_{r,s}},
\]
\[
E_n(t) := U_n(t) + D_n(t) + \Theta_n(t), \quad E_0 := U_0 + D_0 + \Theta_0.
\]

It is noted that in the following the constants $\eta \in [0, 1]$ and $\gamma > 0$ may vary in different inequalities and these values do not play a role in our analysis, thus from now on we do not distinguish them in notation unless otherwise claimed.

Then by Proposition 2.1, 2.2, 2.3 and 2.4, it holds that for any $0 < t < T$,
\[
\Pi_{n+1}(t) \leq \Pi_0 \exp \left( \int_0^t \|\nabla u^n(\tau, \cdot)\|_{L^\infty} \, d\tau \right) \leq \Pi_0 \exp(Ct^d U_n(t)), \quad (3.12)
\]
\[
\Theta_{n+1}(t) \leq C(1 + B_{n+1}^k(t)) \exp(CtC_{\rho^{n+1}}(1)) \left( \Theta_0 + \|\rho^{n+1}u^n \cdot \nabla \theta^n\|_{L^p_t(L^r)} \right)
\]
\[
+ \left( \frac{\|\nabla u^n\|^2}{L^p_t(L^r)} + \frac{\|\Delta \bar{d}^n + |\nabla \bar{d}^n|^2(\bar{d}^n + e)\|^2}{L^p_t(L^r)} \right), \quad (3.13)
\]
\[
D_{n+1}(t) \leq C \left( D_0 + \|u^n \cdot \nabla \bar{d}^n\|_{L^p_t(W^{1,p})} + \|\nabla \bar{d}^n| (\bar{d}^n + e)\|_{L^p_t(L^r)} \right), \quad (3.14)
\]
\[
U_{n+1}(t) \leq C(1 + B_{\rho^{n+1}, \theta^{n+1}}^k(t)) \exp(CtC_{\rho^{n+1}, \theta^{n+1}}(t)) \times \left( U_0 + \|\rho^{n+1}u^n \cdot \nabla u^n\|_{L^p_t(L^r)} + \|\Delta \bar{d}^n \nabla \bar{d}^n\|_{L^p_t(L^r)} \right), \quad (3.15)
\]
where we have used the fact that $D_{A^p}^1 \cap W^{2,p} \hookrightarrow W^{1,\infty}$, and
\[
\int_0^t \|\nabla u^n(\cdot, \tau)\|_{L^\infty} \, d\tau \leq Ct^{\frac{1}{p} - \frac{1}{p} - \frac{\eta}{p} - 1} \|u^n\|_{L^p(D_{A^p}^1)} \|u^n\|_{L^p_t(L^r)}, \quad (3.16)
\]
in the second inequality of (3.12). Next, we evaluate the terms on the RHS of (3.13)-(3.15) one by one.

Noting that for $\frac{1}{m} = \frac{1}{q} + \frac{1}{2}$,
\[
L^\infty(0, T; W^{1,q}) \cap W^{1,\infty}(0, T; L^m) \hookrightarrow C^{\beta_1}([0, T]; L^\infty), \quad \forall \beta_1 \in \left( 0, \frac{1 - \frac{N}{q}}{1 + \frac{N}{q}} \right),
\]
we have
\[
\|\rho^{n+1}\|_{C^{\beta_1}(0, T; L^\infty)} \lesssim \|\rho^{n+1}\|_{L^p_t(W^{1,q})} + \|\partial_t \rho^{n+1}\|_{L^p_t(L^m)}
\]
\[
\lesssim \Pi_0 \exp(t^d U_n(t)) + \|u^n\|_{L^p_t(L^r)} \|\nabla \rho^{n+1}\|_{L^p_t(L^r)} \quad (3.17)
\]
where we have used the fact...

therefore,

\[ B_{q,s}^{2-\frac{2}{q}} \hookrightarrow W^{1,q}, \quad W_t^{1,s}(L^q) \cap L_t^s(W^{2,q}) \hookrightarrow C_t^{\beta_2}(L^\infty), \quad \forall \beta_2 \in \left(0, 1 - \frac{N}{2q} - \frac{N}{s}\right), \]

therefore,

\[ \| \nabla \theta^{n+1} \|_{L_t^r L^s} + \| \theta^{n+1} \|_{C_t^{\beta_2} L^\infty} \leq \Theta_{n+1}(t). \]  

Combining (3.16)-(3.18), one has

\[ B_{\rho,n+1}(t) \leq \Pi_0 \exp(t^\gamma U_n(t)), \quad C_{\rho,n+1}(t) \leq (1 + U_n(t))\Pi_0^k \exp(kt^\gamma U_n(t)), \]  

\[ B_{\rho,n+1,g,n+1}(t) \leq \Pi_0 \exp(t^\gamma U_n(t)) + \Theta_{n+1}(t), \]

\[ C_{\rho,n+1,g,n+1}(t) \leq \left( (1 + (1 + U_n(t))\Pi_0) \exp(t^\gamma U_n(t)) + \Theta_{n+1}(t) \right)^k. \]

The first term on the RHS of (3.13) can be estimated as

\[ I_1 \leq C\| u^n \|_{L_t^\infty(L^q)} \| \nabla u^n \|_{L_t^r L^\infty} \]

\[ \leq C t^{\frac{1}{2}(1 - \frac{s}{2})} \| u^n \|_{L_t^\infty(D_{A_r}^{1-\frac{1}{2} - \frac{s}{2}})} \| \theta^n \|_{L_t^\infty(B_{2s}^{2 - \frac{2}{s}})} \]

\[ \leq C t^{\frac{1}{2}(1 - \frac{s}{2})} U_n(t)^2(t). \]  

\[ III_1 \leq C\| \nabla u^n \|_{L_t^r(L^r)} \| \nabla u^n \|_{L_t^r L^\infty} \]

\[ \leq C t^{\frac{1}{2}(1 - \frac{s}{2})} \| u^n \|^{2 - \eta}_{L_t^\infty(D_{A_r}^{1-\frac{1}{2}-\frac{s}{2}})} \| \nabla u^n \|_{L_t^r(W^2, r)}^\eta \]

\[ \leq C t^{\frac{1}{2}(1 - \frac{s}{2})} \| u^n \|_{L_t^r(W^2, r)}. \]  

Similarly, one can get

\[ III_2 \leq C\| \nabla \theta^n \|_{L_t^r(L^r)} \| \nabla \theta^n \|_{L_t^r L^\infty} \]

\[ \leq C t^{\frac{1}{2}(1 - \frac{s}{2})} \| \theta^n \|_{L_t^\infty(B_{2s}^{2 - \frac{2}{s}})}^2 \| \nabla \theta^n \|_{L_t^r(W^3, r)}^\eta \]

\[ \leq C t^{\frac{1}{2}(1 - \frac{s}{2})} \| \theta^n \|_{L_t^r(W^3, r)}. \]  

For \( I_2 \), it follows from (3.10) that

\[ I_2 \leq C\| \nabla u^n \|_{L_t^2(L^2)^2}^2 \]

\[ \leq C t^{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}} \| u^n \|^{2(1 - \eta)}_{L_t^\infty(D_{A_r}^{1 - \frac{1}{2} - \frac{s}{2}})} \| u^n \|_{L_t^r(W^2, r)}^2 \]

\[ \leq C t^{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}} U_n(t)^2(t). \]  

Now, we evaluate the last term as

\[ I_3 \leq C \left( \| \Delta \theta^n \|_{L_t^2(L^2)}^2 + \| \nabla \theta^n \|^2_2 \| \theta^n \|_{L_t^2(L^2)}^2 \right) \]

\[ \leq C \left( \| \Delta \theta^n \|_{L_t^2(L^2)}^2 + \| \nabla \theta^n \|^4_{L_t^4(L^4)} (\| \theta^n \|_{L_t^\infty(L^\infty)}^2 + 1) \right). \]

Since

\[ \| \Delta \theta^n \|_{L_t^2(L^2)} \leq C t^{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}} \| \nabla \theta^n \|^{1 - \eta}_{L_t^\infty(B_{2s}^{2 - \frac{2}{s}})} \| \nabla \theta^n \|_{L_t^r(W^2, r)}^\eta \]

\[ \leq C t^{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}} D_n(t), \]
where we have used inequality (3.3) and (3.4). Therefore,
\[
I_3 \leq C \xi (D_n^8(t) + D_n^4(t) + D_n^6(t)).
\] (3.26)

For the first term on the RHS of (3.14), by (3.2), one gets
\[
\begin{align*}
I_1 &\leq \|u^n \cdot \nabla \tilde{d}^n\|_{L^p(L^\infty)} + \|\nabla u^n \cdot \nabla \tilde{d}^n\|_{L^p(L^\infty)} + \|u^n \cdot \nabla^2 \tilde{d}^n\|_{L^p(L^\infty)} \\
&\leq C\left(\|u^n\|_{L^\infty(L^2)} \|
abla \tilde{d}^n\|_{L^p(L^\infty)} + \|\nabla u^n\|_{L^p(L^\infty)} \|
abla \tilde{d}^n\|_{L^p(L^\infty)}\right) \\
&\leq C t^{\frac{1}{2}} (1 + 2 ) U_n(t) D_n(t).
\end{align*}
\]

By (3.2), one can obtain
\[
\begin{align*}
I_2 &\leq \left\| \nabla \tilde{d}^n \right\|_{L^p(L^\infty)} + 2 \left\| \nabla \tilde{d}^n \right\|_{L^p(L^\infty)} + \left\| \nabla \tilde{d}^n \right\|_{L^p(L^\infty)} \\
&\leq C (1 + \|\tilde{d}^n\|_{L^\infty(W^{1,\infty})}) \|
abla \tilde{d}^n\|_{L^p(L^\infty)} \|
abla \tilde{d}^n\|_{L^p(W^{1,\infty})} \\
&\leq C t^{\frac{1}{2}} (1 + D_n(t)).
\end{align*}
\] (3.28)

Substituting (3.22)–(3.28) into (3.13)–(3.15), it follows that
\[
\begin{align*}
\Theta_{n+1}(t) &\leq C (1 + B_{0}^{k}(t)) \exp (C t c_{p+1}(t)) (\Theta_0 + t^{\frac{1}{2}} (E^2(t) + E^6(t))), \\
D_{n+1}(t) &\leq C (D_0 + t^{\frac{1}{2}} (E^2(t) + E^3(t))), \\
U_{n+1}(t) &\leq C (1 + B_{0}^{k}(t)) \exp (C t c_{p+1,\theta n+1}(t)) (U_0 + t^{\frac{1}{2}} E^2(t)).
\end{align*}
\] (3.31)

Now let us assume
\[
E^n(t) \leq MCE_0,
\] (3.32)
on $[0, T]$ for some $T > 0$, where $M$ is determined later. Choosing $0 < T_1 \leq T$ such that
\[
T_1 \geq M C^2 E_0 \leq \ln 2,
\] thus for any $t \in [0, T_1],
\[
\begin{align*}
\Pi_{n+1}(t) &\leq 2 \Pi_0, \\
B_{p+1}(t) &\leq (2 \Pi_0)^k, \\
C_{p+1}(t) &\leq (1 + 2 \Pi_0)^k (1 + MCE_0)^k.
\end{align*}
\]
Choosing $0 < T_2 \leq T_1$ such that
\[
C T_2 (1 + 2 \Pi_0)^k (1 + MCE_0)^k \leq \ln 2,
\] thus for any $0 < t \leq T_2,$
\[
\begin{align*}
\Theta_{n+1} &\leq C (1 + (2 \Pi_0)^k) \times (\Theta_0 + E_0) \leq 4CE_0 (1 + 2 \Pi_0)^k, \\
B_{p+1,\theta n+1} &\leq (2 \Pi_0)^k + 4CE_0 (1 + 2 \Pi_0)^k := B_0, \\
C_{p+1,\theta n+1} &\leq (1 + MCE_0 + 4CE_0)^k (1 + 2 \Pi_0)^k := C_0.
\end{align*}
\]
Choosing $0 < T_3 \leq T_2,$ such that
\[
C T_3 C_0 \leq \ln 2,
\] thus for any $t \in [0, T_3],
\[
U_{n+1}(t) \leq C (1 + B_0) \times 2 E_0 \leq 4C (1 + B_0) E_0.
\]
Choosing $0 < T_4 \leq T_3,$ such that
\[
T_4 (M C^2 E_0 + M^3 C^3 E_0^2) \leq 1,
\]
thus for any $t \in [0, T_4]$, 
\[
D_{n+1}(t) \leq C(D_0 + E_0) \leq 2CE_0.
\]
Combining these inequalities together, for any $t \in [0, T_4]$, 
\[
E_{n+1}(t) \leq CE_0 \left( 4(1 + 2\Pi_0)^k + 4(1 + B_0) + 2 \right) \leq MCE_0,
\]
(3.33)
where $M := 4(1 + 2\Pi_0)^k + 4(1 + B_0) + 2$.

Therefore, for fixed small interval $[0, T_4]$, $(\rho^n, u^n, d^n, P^n, \theta^n)$ belongs to $E_{T_4}^{p,q,r,s}$ and is uniformly bounded in the underlying norm with respect to $n$.

**Step 3: Convergence of sequence in** $E_{T_4}^{p,q,m,s}$ **for some** $T < T_4$. In this step, we claim that $(\rho^n, u^n, d^n, P^n, \theta^n)$ is a Cauchy sequence in the Banach space $E_{T_4}^{p,q,m,s}$ for sufficiently small $T \leq T_4$, where $m = \frac{n^2}{r + q}$.

Let $\delta \rho^n := \rho^{n+1} - \rho^n$, $\delta u^n := u^{n+1} - u^n$, $\delta P^n = P^{n+1} - P^n$, $\delta d^n := d^{n+1} - d^n$, $\delta \theta^n := \theta^{n+1} - \theta^n$, $\delta \mu^n = \mu(\theta^{n+1}) - \mu(\theta^n)$, denote
\[
\begin{align*}
\delta \Pi^n(t) &:= \|\delta \rho^n\|_{C([0,t];L^q(\Omega))}, \\
\delta U^n(t) &:= \|\delta u^n\|_{C([0,t];D^{1-\frac{1}{r},p}(\Omega))} + \|\delta u^n\|_{W^{1, p}(0, t; L^m(\Omega)) \cap L^p(0, t; W^{1, m}(\Omega))}, \\
\delta D^n(t) &:= \|\delta d^n\|_{C([0,t];B^{-\frac{1}{r},2}_m, \rho)} + \|\delta d^n\|_{W^{1, p}(0, t; W^{1, m}(\Omega)) \cap L^p(0, t; W^{1, m}(\Omega))}, \\
\delta \Theta^n(t) &:= \|\delta \theta^n\|_{C([0,t];B^{-\frac{1}{r},2}_m, \rho)} + \|\delta \theta^n\|_{W^{1, p}(0, t; L^{1, m}(\Omega)) \cap L^p(0, t; W^{1, m}(\Omega))}, \\
\delta E^n(t) &:= \delta U^n(t) + \delta D^n(t) + \delta \Theta^n(t).
\end{align*}
\]

Then $(\delta \rho^n, \delta u^n, \delta d^n, \delta P^n, \delta \theta^n)$ satisfies the system
\[
\begin{align*}
\left\{
\begin{array}{l}
\partial_t \delta \rho^n + u^n \cdot \nabla \delta \rho^n = -\delta u^{n-1} \cdot \nabla \rho^n, \\
\rho^{n+1} \partial_t \delta u^n - \text{div} \left( (\mu(\theta^{n+1})D(\delta u^n)) + \nabla \delta P^n = -\delta \rho^n \partial_t u^n + \text{div} \left( \delta \mu^n D(u^n) \right) \\
-\rho^n u^n \cdot \nabla u^n - \rho^n u^n \cdot \nabla \delta u^{n-1} - \rho^n \delta u^{n-1} \cdot \nabla u^{n-1} \\
-\Delta \delta u^{n-1} \cdot \nabla \delta d^{n-1} - \Delta \delta d^{n-1} \cdot \nabla d^{n-1}, \\
\partial_t \delta d^n - \Delta \delta d^n = \delta |\nabla d^n|^2 \delta d^{n-1} + \delta d^n \delta \nabla d^{n-1} d^{n-1} + \Delta \delta d^{n-1} \delta \nabla d^{n-1} d^{n-1} \\
-\delta u^{n-1} \cdot \nabla d^n - u^{n-1} \cdot \nabla \delta d^{n-1}, \\
\rho^{n+1} \delta u^n - \Delta \delta \theta^n = -\delta \rho^n \partial_t \theta^n - \rho^n u^n \cdot \nabla \theta^n - \rho^n u^n \cdot \nabla \delta \theta^{n-1} \\
-\rho^n \delta u^{n-1} \cdot \nabla \delta u^{n-1} + \delta \mu^{n-1} D(u^n) : \nabla u^n + \mu^{n-1} D(\delta u^{n-1}) : \nabla u^n \\
+ \mu^{n-1} D(u^{n-1}) : \nabla \delta u^{n-1} + (\Delta (d^n + d^{n-1}) + |\nabla d^n|^2 d^n + |\nabla d^{n-1}|^2 d^{n-1}) \\
\times (\Delta \delta P^{n-1} + \nabla d^n \delta \nabla d^{n-1} d^n + \nabla d^n \nabla d^{n-1} \delta d^{n-1} + \nabla d^{n-1} \nabla d^{n-1} d^{n-1}), \\
\text{div} \delta u^n = 0, \quad \int_{\Omega} \delta P^n \, dx = 0,
\end{array}
\right.
\end{align*}
\]
Concluding from the previous step, for any $t \in [0, T_4]$, such that 
\[
\Pi_n \leq 2\Pi_0, \quad E_n \leq MCE_0, \quad \forall \ n \in \mathbb{N}.
\]
(3.34)
Therefore, for any $t \in [0, T_4]$, 
\[
B_{\rho^n}(t), C_{\rho^n}(t), B_{\rho^n, \theta^n}(t), C_{\rho^n, \theta^n}(t) \leq C(\Pi_0, E_0), \quad \forall \ n \in \mathbb{N},
\]
\[
\|\delta \rho^n\|_{L^p_t(L^q)} \leq \|\nabla \rho^n\|_{L^p_t(L^q)} \int_0^t \|\delta u^{n-1}(\cdot, \tau)\|_{L^q} d\tau \leq C(\Pi_0) t^{2-\gamma} \delta U^{n-1},
\]
(3.35)
\[
\|\delta \theta^n\|_{C([0,t];B^{-\frac{1}{r},2}_m, \rho)} + \|\nabla \delta \theta^n, \nabla^2 \delta \theta^n\|_{L^2_t(L^m)}
\]

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Step 4: Verifying that the limit is a local strong solution. Concluding from the last step, there exists $(\rho, u, P, d, \theta)$ such that for any $0 < t < T_5$,

$$
\rho^n \to \rho \quad \text{in} \quad C([0, t]; L^3(\Omega)), \quad \partial_t \rho^n \to \partial_t \rho \quad \text{in} \quad L^\infty_t L^m(\Omega),
$$

$$
0 < \rho \leq \rho(t, x) \leq \bar{\rho} < +\infty, \quad \forall (x, t) \in \Omega_t;
$$

$$
u^n \to u \quad \text{in} \quad C([0, t]; D_A^{1-\frac{3}{p}, p} \cap W_t^{1, p} L^r \cap L^t_t(W^{2,r}),
$$

$$
\to u \quad \text{in} \quad C([0, t]; D_A^{1-\frac{3}{p}, p} \cap W_t^{1, p} L^m \cap L_t^p(W^{2,m});
$$

$$
d^n \to d \quad \text{in} \quad C([0, t]; B^{3-\frac{3}{p}, p}_{A_t} \cap W_t^{1, p} W^{1,r} \cap L_t^p(W^{3,r}),
$$

$$
\to d \quad \text{in} \quad C([0, t]; B^{3-\frac{3}{p}, p}_{A_t} \cap W_t^{1, p} W^{1,3,m} \cap L_t^p(W^{3,m});
$$

$$
\theta^n \to \theta \quad \text{in} \quad C([0, t]; B^{3-\frac{3}{p}, p}_{A_t} \cap W_t^{1, p} (L^3) \cap L_t^p(W^{2,q}),
$$

$$
\to \theta \quad \text{in} \quad C([0, t]; B^{3-\frac{3}{p}, p}_{A_t} \cap W_t^{1, p} L^m \cap L_t^p(W^{2,m}).
$$

Now we claim that every term in (3.8)–(3.11) converges to the corresponding ones in $D'(\Omega_t, L_t^1 L^m(\Omega))$, $L_t^1 W^{1,m}(\Omega)$ and $L_t^1 L^m(\Omega)$, respectively. Thus $(\rho, u, P, d, \theta)$ satisfies (1.3). To verify the claim is very straightforward, and we take the following terms as examples.
For any $\phi \in \mathcal{D}(\Omega)$, it holds that
\[
\lim_{n \to \infty} \left| \int_0^t \int_{\Omega} (u^n \cdot \nabla \rho^{n+1} - u \cdot \nabla \rho) \phi \, dx \, dt \right|
\leq \lim_{n \to \infty} \left| \int_0^t \int_{\Omega} (u^n - u) \nabla \rho^{n+1} \phi \, dx \, dt \right| + \lim_{n \to \infty} \left| \int_0^t \int_{\Omega} u \cdot \nabla(\rho^{n+1} - \rho) \phi \, dx \, dt \right|
\leq \lim_{n \to \infty} \| \nabla \rho^{n+1} \|_{L^p_t L^r} \| u^n - u \|_{L^q_t L^r} \| \phi \|_{L^p_t(L^m)}
= 0,
\]
where we have used that $u \phi \in L^1_t(L^d)$ and $W^{2,m} \hookrightarrow L^r$.

It follows from the fact $W^{1,m} \hookrightarrow L^2$ that
\[
\lim_{n \to \infty} \| \rho^{n+1} \partial_t \theta^{n+1} - \rho \partial_t \theta \|_{L^r_t(L^s)}
\leq \lim_{n \to \infty} \| \rho^n - \rho \|_{L^s_t(L^r)} \| \partial_t \theta^{n+1} \|_{L^r_t(L^s)} + \bar{\rho} \lim_{n \to \infty} \| \partial_t \theta^{n+1} - \partial_t \theta \|_{L^r_t(L^m)}
= 0,
\]
\[
\lim_{n \to \infty} \| u^n \nabla \theta^n - \rho \nabla \theta \|_{L^r_t(L^m)}
\leq \lim_{n \to \infty} \| \rho^n - \rho \|_{L^s_t(L^r)} \| u^n \|_{L^q_t(L^r)} \| \nabla \theta^n \|_{L^r_t(L^\infty)} + \bar{\rho} \lim_{n \to \infty} \| u^n \|_{L^q_t(L^r)} \| \nabla \theta^n - \nabla \theta \|_{L^r_t(L^m)}
= 0.
\]

By $B_{m,s}^{2-\frac{d}{2}} \hookrightarrow L^r$, one has
\[
\lim_{n \to \infty} \| \mu(\theta^n) \mathcal{D}(u^n) : \nabla u^n - \mu(\theta) \mathcal{D}(u) : \nabla u \|_{L^q_t(L^m)}
\leq \lim_{n \to \infty} \| \theta^n - \theta \|_{L^s_t(L^r)} \| \nabla u^n \|_{L^q_t(L^r)}^2 + \lim_{n \to \infty} \| \nabla u^n - \nabla u \|_{L^q_t(L^s)} \| \nabla u \|_{L^q_t(L^r)}
= 0.
\]

The other terms can be verified similarly, the details are left to the reader.

Finally, we check that $|d| = 1$ a.e. on $\Omega_t$, for any $0 < t \leq T$. In particular, consider the equations
\[
\begin{cases}
\partial_t d - \Delta d + u \cdot \nabla d = |\nabla d|^2 d, \\
d|t=0 = d_0, \quad \text{on } \Omega, \\
d = d_0, \quad \text{or } \partial_r d = 0, \quad \text{on } \partial \Omega \times [0,t).
\end{cases}
\]

Since $d \in L^\infty_t(B_{r,p}^{3-\frac{d}{2}}) \cap L^r_t(W^{3,r})$, multiplying (3.38) by $d$ and using the fact that $\Delta(|d|^2) = 2 \Delta d \cdot d + 2|\nabla d|^2$, one obtains
\[
\begin{cases}
\partial_t (|d|^2 - 1) - \Delta (|d|^2 - 1) + u \cdot \nabla (|d|^2 - 1) = 2|\nabla d|^2(|d|^2 - 1), \\
|d|^2 - 1 = 0, \quad \text{on } \Omega \times \{t = 0\}, \\
|d|^2 - 1 = 0, \quad \text{or } \partial_r (|d|^2 - 1) = 0, \quad \text{on } \partial \Omega \times [0,t).
\end{cases}
\]

Multiplying (3.39) by $|d|^2 - 1$, then integrating the resulting equation over $\Omega$, it follows that
\[
\frac{d}{dt} \int_{\Omega} (|d|^2 - 1)^2 \, dx + \int_{\Omega} |\nabla (|d|^2 - 1)|^2 \, dx = \int_{\Omega} |\nabla d|^2(|d|^2 - 1)^2 \, dx.
\]
Notice that if $p \geq 2$, $L^p_t(W^{3,r}) \hookrightarrow L^2_t(W^{1,\infty})$, and if $1 < p < 2$, $W^{1,p}_t(W^{1,r}) \cap L^p_t(W^{3,r}) \hookrightarrow L^2_t(W^{1,\infty})$. Thus, by Gronwall’s inequality, one has
\[
\int_\Omega (|d|^2 - 1)^2(t,x)dx \leq \left( \int_0^t \|\nabla d(s,\cdot)\|_{L^\infty}^2 ds \right) \int_\Omega (|d|^2 - 1)^2(0,x)dx = 0, \quad (3.41)
\]
for any $t < T_5$. This implies that $|d| = 1$ a.e. on $\Omega_{T_5}$.

Finally, the existence of local strong solution is proved. The proof of the uniqueness is standard except for the density equation. However it is well-known in existing literature, we refer to Proposition 4.2 in [7]. The details are omitted here.

4. Global existence for small perturbation. In this section we prove Theorem 1.5. As mentioned in the introduction, to extend a local strong solution to a global one, the key is to obtain uniform-in-time estimates. However, by Proposition 2.1,
\[
\|\rho\|_{C([R^+;W^{1,q})} \leq C\|\rho_0\|_{W^{1,q}} \exp \left( \int_0^{+\infty} \|\nabla u(t,\cdot)\|_{L^\infty} dt \right).
\]
The functional space we choose for $u$ here, however, is not enough to ensure $\|\nabla u\|_{L^1([R^+;W^{1,\infty})}$ due to slow decay of lower order term. Hence we recover the $L^2$ decay first, by a geometric condition on the initial director.

4.1. $L^2$ estimates for the velocity. We prove the following proposition.

**Proposition 4.1.** Under the assumptions of Theorem 1.5, suppose $(\rho, u, d, P, \theta)$ is a strong solution to system (1.3)-(1.4) on $\Omega \times [0,T)$, then for any $t < T$, it holds that
\[
\|u(t,\cdot)\|_{L^2} + \|\nabla u(t,\cdot)\|_{L^2} \leq C e^{-Ct}\left(\|u_0\|_{L^2} + \|\nabla d_0\|_{L^2}\right), \quad (4.1)
\]
where $C > 0$ only depends on the upper bound of $\|u_0\|_{L^2}$, $\|\nabla d_0\|_{L^2}$, $\epsilon_0$, $\tilde{\rho}$ and $\Omega$.

To prove the above proposition, we first establish the coercivity of the harmonic energy under the geometric restriction. This generalizes Theorem 1.5 in [29].

**Lemma 4.1.** Let $0 < \epsilon_0 < 1$ and $C_0 > 0$. There exists a positive constant $\delta_0 < 1$ depending on $\epsilon_0$ and $C_0$ such that the following holds:
If $d : \Omega \to S^2$, $\nabla d \in H^1(\Omega)$ with $\|\nabla d\|_{L^2} \leq C_0$ and $d^3 \geq \epsilon_0$ on $\Omega$, $d^3 = 1$ or $\partial_\nu d^3 = 0$ on $\partial \Omega$, where $d^3$ denotes the third component of vector $d$, then
\[
\|\nabla d\|_{L^2}^4 \leq (1 - \delta_0)\|\Delta d\|_{L^2}^2. \quad (4.2)
\]
Consequently, for such maps the associated harmonic energy is coercive, i.e.
\[
\|\Delta d + |\nabla d|^2 d\|_{L^2}^2 \geq \frac{\delta_0}{2} \left( \|\Delta d\|_{L^2}^2 + \|\nabla d\|_{L^2}^4 \right). \quad (4.3)
\]

Before proving the above lemma, we state a preparation lemma first.

**Lemma 4.2.** Let $C_0 > 0$, $\alpha > 0$, and $d_n : \Omega \to S^2$ is a sequence satisfying
\[
\|\nabla d_n\|_{L^2(\Omega)} + \|\Delta d_n\|_{L^2(\Omega)} \leq C_0 < \infty, \quad \forall n \in \mathbb{N}, \quad (4.4)
\]
\[
\|\nabla d_n\|_{L^2}^2 \geq \alpha > 0, \quad (4.5)
\]
then upon a subsequence $|\nabla d_{n_k}|^2 \to f$ strongly in $L^2(\Omega)$ and $\|f\|_{L^2(\Omega)} \geq \alpha$.

**Proof of Lemma 4.2.** Set $f_n = |\nabla d_n|^2$, then $\|f_n\|_{L^2} \geq \alpha > 0$. Moreover,
\[
\|f_n\|_{L^2} = \|\nabla d_n\|_{L^4(\Omega)}^2 \leq \|\nabla d_n\|_{L^2}^2 \|\nabla d_n\|_{L^2}^2 \leq C_0^2,
\]
\[
\nabla f_n = 2\Delta d_n \cdot \nabla d_n \in L^\frac{4}{3}(\Omega).
Thus \( f_n \in W^{1,2} \rightarrow L^q(\Omega) \), where \( 1 \leq q < 4 \), if \( N = 2 \), and \( 1 \leq q < \frac{12}{5} \), if \( N = 3 \). In summary, upon a subsequence of \( \{ f_n \} \), \( f_n \rightarrow f \in L^2(\Omega) \) strongly. Therefore, the lemma is proved. \( \Box \)

Now we are ready to prove the coercivity of the harmonic energy associated to the vector fields \( d \).

**Proof of Lemma 4.1. Case 1:** \( \| \Delta d \|_{L^2(\Omega)} = 0. \)

This implies \( \Delta d^3 = 0 \) a.e. on \( \Omega \). Thus \( d^1 \) is harmonic on \( \Omega \), which infers that \( d^3 \) attains minimum on \( \partial \Omega \). Thus \( d^3 \equiv 1 \) due to \( d^3 \leq 1 \). On the other hand, \( |d| = 1 \), thus \( d^1 = d^2 \equiv 0 \). The lemma is trivially true.

**Case 2:** \( \| \Delta d \|_{L^2(\Omega)} > 0. \)

Without loss of generality, assume \( \| \Delta d \|_{L^2(\Omega)} = 1 \), otherwise define \( \tilde{d} = \frac{d}{\| \Delta d \|_{L^2}} \).

It suffices to show that there exists \( 0 < \delta_0 < 1 \) such that

\[
\| \nabla d \|_{L^2}^2 \leq 1 - \delta_0.
\]  

(4.6)

Assume (4.6) does not hold, then there exists a sequence \( d_n : \Omega \rightarrow \mathbb{S} \) satisfying

\[
\| \Delta d_n \|_{L^2(\Omega)} = 1, \quad \| \nabla d_n \|_{L^2(\Omega)} \leq C_0, \quad d_n^3 \geq \epsilon_0 > 0, \quad \text{on} \ \Omega,
\]

\[
d^3 = 1 \quad \text{or} \quad \partial_\nu d_n^3 = 0, \quad \text{on} \ \partial \Omega \quad \text{but} \quad \| \nabla d_n \|_{L^4(\Omega)} \nrightarrow 1, \quad \text{as} \ \ n \rightarrow \infty.
\]

Let \( g_n = \Delta d_n + |\nabla d_n|^2 d_n \), then easy computation gives

\[
\| g_n \|_{L^2(\Omega)} = 1 - \| \nabla d_n \|_{L^4(\Omega)}^2 \nrightarrow 0.
\]

Then from the proof of Lemma 4.2 and the third component equation

\[
\Delta d_n^3 + |\nabla d_n|^2 d_n^3 = g_n^3, \quad \text{as} \ n \rightarrow \infty,
\]  

(4.7)

we can prove that

\[
\Delta d_n^3 \rightarrow \Delta d^3, \quad \text{weakly in} \ L^2(\Omega),
\]

\[
|\nabla d_n|^2 \rightarrow f, \quad \text{strongly in} \ L^2(\Omega),
\]

\[
g_n^3 \rightarrow 0, \quad \text{weakly in} \ L^2(\Omega).
\]

Hence

\[
\Delta d^3 + f \epsilon_0 \leq 0, \quad \text{weakly in} \ L^2(\Omega).
\]  

(4.8)

This implies \( d^3 \) is superharmonic on \( \Omega \), which yields that \( d^3 \) attains minimum on \( \partial \Omega \). Noticing that \( d^3 \rightarrow d^3 \) weakly in \( W^{2,2}(\Omega) \), one concludes that the convergence is strong in \( W^{1,2}(\Omega) \) for \( n = 2 \) and \( W^{1,6}(\Omega) \) for \( n = 3 \). Consequently, by trace theorem, \( d_n^3 \rightarrow d_\ast^3 \) or \( \partial_\nu d_n^3 \rightarrow \partial_\nu d_\ast^3 \) strongly in \( L^2(\partial \Omega) \). Therefore, \( 0 < \epsilon_0 \leq d_\ast^3 \leq 1 \) a.e. on \( \Omega \) and \( d_\ast^3 = 1 \) or \( \partial_\nu d_\ast^3 = 0 \) a.e. on \( \partial \Omega \).

Combining the above facts, one infers from the maximum principle or Hopf lemma that \( d^3 \equiv 1 \) or \( d^3 \equiv C \) on \( \Omega \), where the constant \( C \) satisfies \( \epsilon_0 \leq C \leq 1 \). So we get \( f \equiv 0 \), this contradicts to the result \( \| f \|_{L^2(\Omega)} \geq \alpha > 0 \) in Lemma 4.2. \( \Box \)

**Proof of Proposition 4.1.** First of all, multiplying the second and third equation of (1.3) by \( u \) and \( \Delta d + |\nabla d|^2 d \), respectively, summing up the resulting equalities and integrating by parts over \( \Omega \), then taking advantage of the conservation of mass, it follows that

\[
\frac{d}{dt} \int_{\Omega} (\rho |u|^2 + |\nabla d|^2) \, dx + \int_{\Omega} (|\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2) \, dx \leq 0.
\]  

(4.9)
Applying maximum principle to the third component of $d$, one can show that $\inf_{\Omega} d(t,x) \geq \epsilon_0$ for any $t > 0$. Moreover, since $L^\infty(B^{3-\frac{2}{T}}_{r,p}) \hookrightarrow L^\infty(W^{2,r}) \hookrightarrow L^\infty(H^2)$, one gets $\nabla d(t,\cdot) \in H^1(\Omega)$ for a.e. $t \in [0, T]$.

Noticing that $d(t,\cdot)|_{\partial \Omega} = e$ implies $\int_{\Omega} \nabla d(t,x) dx = 0$, it follows from Poincaré-Wirtinger inequality that

$$\|\nabla d(t,\cdot)\|_{L^2} \leq C \|\Delta d(t,\cdot)\|_{L^2} \leq C \|\Delta d + |\nabla d|^2 d(t,\cdot)\|_{L^2}, \quad (4.10)$$

where $C$ depends only on the upper bound of initial data $\|u_0\|_{L^2}$, $\|\nabla d_0\|_{L^2}$, $\epsilon_0$ and $\Omega$.

On the other hand, since $u(t,\cdot)|_{\partial \Omega} = 0$, Poincaré inequality yields that

$$\|u(t,\cdot)\|_{L^2} \leq C \|\nabla u(t,\cdot)\|_{L^2}. \quad (4.11)$$

Plugging (4.10) and (4.11) into (4.9), one reaches

$$\frac{d}{dt}(\|\sqrt{\rho}(t,\cdot)u(t,\cdot)\|_{L^2} + \|\nabla d(t,\cdot)\|_{L^2}) + C(\|u(t,\cdot)\|_{L^2} + \|\nabla d(t,\cdot)\|_{L^2}) \leq 0. \quad (4.12)$$

Multiplying both sides of (4.12) by $e^{Ct}$, and integrating the resulting equation over time, one deduces that

$$\|\sqrt{\rho}(t,\cdot)u(t,\cdot)\|_{L^2} + \|\nabla d(t,\cdot)\|_{L^2} \leq e^{-Ct}(\|\sqrt{\rho}_0 u_0\|_{L^2} + \|\nabla d_0\|_{L^2}). \quad (4.13)$$

Noting that $\rho$ is bounded from below, the proof of Proposition 4.1 is completed. ☐

**Proof of Theorem 1.5.** Suppose $T^*$ is the maximal existence time and fix $t < T^*$. Define

$$\Pi(t) = \|\rho\|_{L^\infty(W^{1,q}(\Omega))}, \quad \Pi_0 = \|\nabla \rho_0\|_{L^q(\Omega)},$$

$$U(t) := \|u\|_{C_1(B^{\frac{3-\frac{2}{T}}{r}}_{r,p})} + \|u\|_{W^{2,r}(\Omega)} + \|P\|_{L^p_t(W^{1,r})},$$

$$D(t) := \|d - e\|_{C_1(B^{2-\frac{2}{T}}_{r,p})} + \|d - e\|_{L^r_t(W^{2,r})},$$

$$\Theta(t) := \|\theta\|_{C_1(B^{2-\frac{2}{T}}_{r,p})} + \|\theta\|_{L^r_t(W^{2,r})},$$

$$E(t) := U(t) + D(t) + \Theta(t), \quad E_0 := U_0 + D_0 + \Theta_0,$$

where $U_0$, $D_0$ and $\Theta_0$ are defined as before.

Noticing that $L^2 \cap W^{2,r} \hookrightarrow W^{1,\infty}$,

$$t \int_0^t \|\nabla u(\cdot, \tau)\|_{L^\infty} d\tau \leq \int_0^t \|u(\cdot, \tau)\|_{L^2}^\phi \|u(\cdot, \tau)\|_{W^{2,r}}^\phi d\tau \leq C U_0^\phi U_0^\phi(t), \quad (4.14)$$

where we have used (4.1) and $\phi = \frac{1}{2} + \frac{1}{2} - \frac{1}{r}$ is in $(0, 1)$.

Applying Proposition 2.1 to the first equation in system (1.3), one has

$$\|\nabla \rho\|_{L^\infty(\infty, \Omega)} \leq \|\nabla \rho_0\|_{L^q} \exp \left(\int_0^t \|\nabla u(\cdot, \tau)\|_{L^\infty} d\tau\right) \leq \Pi_0 e^{C U_0^{1-\phi} U_0^\phi(t)}. \quad (4.15)$$

Therefore,

$$B_\rho(t) \leq (\Pi_0 e^{C U_0^{1-\phi} U_0^\phi(t)})^{\frac{1}{1-\phi}}, \quad (4.16)$$

$$C_\rho(t) \leq (1 + \Pi_0 e^{C U_0^{1-\phi} U_0^\phi(t)})^k (1 + U(t)) \Pi_0 e^{C U_0^{1-\phi} U_0^\phi(t)} \frac{1}{1-\phi}. \quad (4.17)$$
Applying Proposition 2.4 to the temperature equation,
\[
\Theta(t) \leq C(1 + \Pi_0 e^{C_2 U_0^{1-\Phi} U^\phi(t)}) \kappa \left( \Theta^0 + \|u \cdot \nabla \theta\|_{L^\infty_t(L^2)} + \bar{\mu} \|\nabla u\|_{L^2_t(L^\infty)} \right) + \|\Delta d + |\nabla d|^2 d|_{L^\infty_t(L^2)} + \left( (1 + U(t)) \Pi_0 e^{C_2 U_0^{1-\Phi} U^\phi(t)} \right) \|\Theta\|_{L^\infty_t(L^2)}.
\]
where we have used Corollary 3.2. Assume that for some \(T > 0\),
\[
E(t) \leq C M E_0 (1 + \Pi_0)^k, \quad \forall \ t \in [0, T],
\]
where \(M\) is independent of \(E_0, \Pi_0, t\) and will be determined later. If
\[
\begin{cases}
C^{1+\Phi} U_0^{1-\Phi} E_0^\Phi (1 + \Pi_0)^k \Phi \leq \ln 2,
M^2 C^2 E_0 (1 + \Pi_0)^{2k} \leq 1,
M^2 C^6 \leq (1 + \Pi_0)^6k \leq 1,
CM E_0 (1 + \Pi_0)^k \leq 2^{\beta_1} - 1,
(2\Pi_0) \leq MC (1 + \Pi_0)^k \leq 1,
\end{cases}
\]
then
\[
\Theta(t) \leq 5C E_0 (1 + 2\Pi_0)^k \leq 5 \times 2^k C E_0 (1 + \Pi_0)^k. \tag{4.21}
\]
If in addition,
\[
5 \times 2^k C E_0 (1 + \Pi_0)^k \leq 1, \tag{4.22}
\]
then
\[
B_{\rho, \phi}(t) \leq (1 + 2\Pi_0) \frac{\pi}{\rho}, \quad C_{\rho, \phi}(t) \leq (1 + 2\Pi_0)^k. \tag{4.23}
\]
Hence, for any \(t \in [0, T]\), it follows from (2.7) and Corollary 3.2 that
\[
U(t) \leq C^k (1 + \Pi_0)^k (U_0 + E^2(t) + U_0) \leq 3 \times 2^k C E_0 (1 + \Pi_0)^k, \tag{4.24}
\]
where we have used
\[
\|u\|_{L^\infty_t L^2} \leq \epsilon \|u\|_{L^\infty_t L_{2,\infty}} + C(\epsilon) \|u\|_{L^\infty_t L^2} \leq \epsilon \|u\|_{L^\infty_t L_{2,\infty}} + C(\epsilon) U_0.
\]
Finally, for any \(t \in [0, T]\), it follows from (2.9) and Corollary 3.2 that
\[
D(t) \leq C (D_0 + E^2(t) + E^3(t)) \leq 3CE_0. \tag{4.25}
\]
Therefore,
\[
E(t) \leq C E_0 (1 + \Pi_0)^k (5 \times 2^k + 3 \times 2^k + 3) := \frac{M}{2} C E_0 (1 + \Pi_0)^k, \tag{4.26}
\]
where \(M = 2(5 \times 2^k + 3 \times 2^k + 3)\). Then by an induction argument, \(\Pi(t)\) and \(E(t)\) is uniformly bounded for any \(t > 0\), and the solution exists globally. The smallness condition of the initial data is determined by (4.20) and (4.22). \(\square\)
5. Appendix.

5.1. Proof of Proposition 2.2. Recall the maximal regularity for the linear Stokes operator (cf. Theorem 3.2 [7]):

**Theorem 5.1.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with smooth boundary and \( 1 < p, r < \infty \). Assume that \( u_0 \in D^{1-\frac{1}{p}}_{A_r} \), \( f \in L^p(0, T; L^r) \) and \( \mu \) is constant. Then the system

\[
\begin{aligned}
\partial_t u - \mu \Delta u + \nabla P &= f, \\
\text{div} u &= 0, \quad \int_\Omega P \, dx = 0, \\
u|_{t=0} &= u_0, \quad u|_{\partial \Omega} = 0,
\end{aligned}
\]

has a unique global solution \((u, P)\) satisfying

\[
\mu^{1-\frac{1}{p}} \|u\|_{L^\infty(0, T; D^{1-\frac{1}{p}}_{A_r})} + \| (\partial_t u, \mu \Delta u, u, \nabla P) \|_{L^p(0, T; L^r)} 
\leq C \left( \mu^{1-\frac{1}{p}} \|u_0\|_{D^{1-\frac{1}{p}}_{A_r}} + \| f \|_{L^p(0, T; L^r)} \right),
\]

for all \( T \geq 0 \), with \( C = C(p, r, \sigma(\Omega)) \), where \( \sigma(\Omega) \) stands for the open set

\[
\sigma(\Omega) = \left\{ \frac{x}{\delta(\Omega)} \big| x \in \Omega \right\},
\]

with \( \delta(\Omega) \) denoting the diameter of \( \Omega \).

The basic idea to prove Proposition 2.2 is that if \( \rho \) and \( \theta \) are close to constants \( \overline{\rho} = \inf_{x \in \Omega} \rho(x) \) and \( \overline{\theta} = \inf_{x \in \Omega} \theta(x) \), respectively, then Theorem 5.1 provides us with the desired estimates. More precisely, one can rewrite the system as

\[
\begin{aligned}
\rho \partial_t u - (\mu(\theta) - \mu(\overline{\theta})) \Delta u + \mu'(\overline{\theta}) \nabla \theta \cdot \nabla u + (\rho - \overline{\rho}) \partial_t u, \\
\text{div} u &= 0, \quad \int_\Omega P \, dx = 0, \\
u|_{t=0} &= u_0, \quad u|_{\partial \Omega} = 0.
\end{aligned}
\]

Now if \( \| \rho - \overline{\rho} \|_{L^\infty} \) and \( \| (\mu(\theta) - \mu(\overline{\theta})) \Delta u \|_{L^r(\Omega)} \) are small enough, the terms \( \| (\mu(\theta) - \mu(\overline{\theta})) \Delta u \|_{L^r(\Omega)} \) and \( \| (\rho - \overline{\rho}) \partial_t u \|_{L^\infty} \) may be absorbed by the LHS of the inequality given in Theorem 5.1.

The proof of Proposition 2.2 is organized as follows. First, we restrict ourselves to the case of null initial data, i.e., \( u_0 \equiv 0 \), and prove the \textit{a priori} estimates for \((u, P)\) under the assumption that \( \theta \) and \( \rho \) are independent of time. Next, we prove the similar estimates for time-dependent temperature and density. These estimates would suffice to prove Proposition 2.2 for the null initial data case. Finally, we derive the desired estimates for the general initial data \( u_0 \in D^{1-\frac{1}{p}}_{A_r} \).

5.1.1. Existence of solution for null initial data. We divide the proof into the following three steps.

(a) \textit{A priori} estimates with time-independent temperature and density

**Theorem 5.2.** Suppose \( p, q, r, \mu, f \) and \( \Omega \subset \mathbb{R}^N \) satisfy the assumptions in Proposition 2.2, \( u_0 = 0 \), \( \rho = \rho(x) \in W^{1,q}(\Omega) \), \( 0 < \rho \leq \overline{\rho} \) and \( \theta = \theta(x) \in W^{1,q} \). If \((u, P)\) is a smooth solution to system \((2.5)\) on \( \overline{\Omega} \times [0, T) \), then for any \( t < T \), it holds that

\[
\| (\partial_t u, \Delta u, \nabla P) \|_{L^r_t(L^r)} \leq C(B^k_{\rho, \theta} + 1) \| f \|_{L^r_t(L^r)} + B^k_{\rho, \theta} \| u \|_{L^r_t(L^r)},
\]

where \( B^k_{\rho, \theta} \) and \( C \) are constants.
where
\( B_{p, \theta} = \| \nabla \rho \|^\frac{q}{q-1} + \| \nabla \theta \|^\frac{q}{q}, \)

\( C \) is independent of \( f, \theta \) and \( t, k \geq 2 \) depends only on \( p, q, r \) and \( N \).

Proof. Rewriting (2.5) as (5.1) and applying Theorem 5.1, we obtain

\[
\|u\|_{C([0, T]; D_{L^\infty}^{\frac{1}{r} - \frac{1}{p}})} + \|(\partial_t u, \Delta u, \nabla P)\|_{L^p(0, T; L^r)} \\
\leq C\left(\|f\|_{L^p(0, T; L^r)} + \|\theta - \bar{\theta}\|_{L^\infty} \|\Delta u\|_{L^p(0, T; L^r)} \\
+ \|\nabla u \nabla \theta\|_{L^p(0, T; L^r)} + \|\rho - \bar{\rho}\|_{L^\infty} \|\partial_t u\|_{L^p(0, T; L^r)}\right),
\]

where \( C \) depends only on \( p, q, r, \Omega, \bar{\mu}, \bar{\rho}, \bar{\mu} \) and \( \bar{\mu}' \). From now on, we will keep this dependence of \( C \) in silence unless otherwise claimed, and the value of \( C \) may change from line to line.

By Gagliardo-Nirenberg interpolation, Poincaré-Wirtinger inequality and Young’s inequality, we arrive at

\[
|\nabla u \nabla \theta|_{L^q(\Omega)} \leq \epsilon \|\Delta u\|_{L^q(\Omega)} + \epsilon \frac{N+q}{q} \|\nabla \theta\|_{L^q(\Omega)} \|u\|_{L^q(\Omega)}^q,
\]

for any \( \epsilon > 0 \).

On the other hand, since \( q > N \), \( \dot{W}^{1,q}(\Omega) \hookrightarrow \dot{C}^{1,q}(\Omega) \) for \( \alpha = 1 - \frac{N}{q} \in (0, 1) \), we have

\[
\|\theta - \bar{\theta}\|_{L^\infty(\Omega)} \leq C^\alpha(\Omega) \|\nabla \theta\|_{L^q(\Omega)}, \quad \|\rho - \bar{\rho}\|_{L^\infty(\Omega)} \leq C^\alpha(\Omega) \|\nabla \rho\|_{L^q(\Omega)}.
\]

If \( C^\alpha(\Omega) \|\nabla \theta\|_{L^q(\Omega)} + \|\nabla \rho\|_{L^q(\Omega)} \leq \frac{1}{4} \), then the corresponding terms are absorbed by the LHS of (5.3). Choosing \( \epsilon = \frac{1}{4C} \) and substituting (5.4) and (5.5) into (5.3), we obtain

\[
\|u\|_{L^p_{\infty}(\dot{D}^{\frac{1}{r} - \frac{1}{p}})} + \|(\partial_t u, \Delta u, \nabla P)\|_{L^p(\Omega)} \\
\leq C\left(\|f\|_{L^p(\Omega)} + \|\nabla \theta\|_{L^q(\Omega)} \|u\|_{L^q(\Omega)}ight).
\]

Otherwise if \( C^\alpha(\Omega) \|\nabla \theta\|_{W^{1,q}(\Omega)} + \|\rho\|_{W^{1,q}(\Omega)} > \frac{1}{4} \), we perform the space localization to adjust \( \delta(\Omega) \).

Consider the following subordinate partition of \( \Omega \):

\( \{\Omega_k\}_{k=1}^K \) is an open covering of \( \Omega \) such that the multiplicity of the covering is \( m \), and for \( 1 \leq k \leq K \), it holds that

\( \delta(\Omega_k) \leq \lambda \in (0, \delta(\Omega)) \),

the value of \( \lambda \) will be determined later. \( \{\phi_k\}_{k=1}^K \) is a family of characteristic function such that

\( 0 \leq \phi_k \leq 1, \phi_k \in C_0^2(\Omega_k), \quad \sum_{k=1}^K \phi_k(x) = 1, \quad \forall x \in \Omega, \)

\( \|\nabla^\alpha \phi_k\|_{L^\infty(\Omega_k)} \leq C\lambda^{-|\alpha|}, \quad |\alpha| \leq 2. \)

The number \( K \) of the covering is of order \( \delta(\Omega)^{-1} \), and the number \( K' \) of domains \( \Omega_k \) intersecting with \( \partial \Omega \) is of order \( \delta(\Omega)^{-1} \).

Let

\( \rho_k = \min_{x \in \Omega_k \cap \Omega} \rho(x) > 0, \quad \theta_k = \min_{x \in \Omega_k \cap \Omega} \theta(x). \)

\(^1\)The multiplicity of an covering means at most how many subsets intersect with each other, this quantity only depends of the space dimension \( N \).
Now define $u_k = u\phi_k$, $P_k = P\phi_k$, $f_k = f\phi_k$ and $\mu_k = \mu(\theta_k)$. Then $(u_k, P_k, f_k)$ satisfies the following system

$$
\begin{aligned}
\partial_t u_k &- \mu_k \Delta u_k + \nabla (\frac{P_k}{\rho_k}) = \frac{f_k}{\rho_k} + \frac{\mu_k - \mu(\theta_k)}{\rho_k} \partial_t u\phi_k - \frac{\mu_k - \mu(\theta_k)}{\rho_k} \Delta u\phi_k \nabla \phi_k , \\
- \mu_k \frac{1}{\rho_k} \partial_t u\phi_k - 2 \mu_k \partial_t \nabla f \cdot \nabla \phi_k \mu + \frac{\mu(\theta)}{\rho_k} D(u) \nabla \theta \phi_k + \frac{P_k}{\rho_k} \nabla \phi_k , \\
\text{div } u_k & = u \cdot \nabla \phi_k , \quad \int_{\Omega} P_k \, dx - \int_{\Omega} P\phi_k \, dx = 0 , \\
u_k |_{t=0} & = 0 , \quad u_k |_{\partial \Omega} = 0 .
\end{aligned}
$$

(5.7)

Notice that $u_k$ is not divergence-free and the localization procedure produces some additional lower order terms. To obtain the estimates of the above system, we use a theorem proved by Danchin:

**Theorem 5.3 (Theorem 3.6 in [7]).** Let $\Omega$ be a $C^{2+\varepsilon}$ bounded domain of $\mathbb{R}^N$ and $1 < p, r < \infty$. Let $\Omega' \subset \Omega$ be open and star-shaped with respect to small ball of diameter $d > 0$. Let $\tau \in L^p(0,T;W^{1,r})$ satisfy $\tau(0,\cdot) \equiv 0$,

$$
\int_{\Omega} \tau \, dx = 0 , \quad \partial_t \tau = \tau_0 + \text{div } R , \quad \forall \ t \in (0,T) , \quad \text{supp } \tau_0(\cdot,\cdot) \cap \text{supp } R(t,\cdot) \subset \Omega' ,
$$

with $R$ and $\tau_0$ in $L^p(0,T;L^r(\Omega))$ and $R \cdot n$ in $L^p(0,T;L^r(\partial\Omega))$. Let $v_0 \in D_{A_r}^{1-\frac{1}{p},p}$, $f \in L^p(0,T;L^r(\Omega))$ and $\mu$ is a constant. Then the following system

$$
\begin{aligned}
\partial_t v - \mu \Delta v + \nabla P & = f , \\
\text{div } v & = \tau , \quad \int_{\Omega} P \, dx = 0 , \\
v |_{t=0} & = v_0 , \quad v |_{\partial \Omega} = 0 ,
\end{aligned}
$$

has a unique solution $(v, P)$ on $\Omega \times [0,T)$ such that the following estimate holds true with $C = C(r,p,N,\sigma(\Omega))$:

$$
\| (\partial_t v, \mu \nabla^2 v, \nabla P ) \|_{L^p_r((\Omega'))} \leq C \left( \mu^{1-\frac{1}{p}} \| v_0 \|_{D_{A_r}^{1-\frac{1}{p},p}} + \| f \|_{L^p_r((\Omega'))} + \| R \|_{L^p_r((\Omega'))} \right) \\
+ \mu \| \nabla \tau \|_{L^p_r((\Omega'))} + \delta(\Omega') \| \tau_0 \|_{L^p_r((\Omega'))} \\
+ \delta(\Omega') \| R \cdot n \|_{L^p_r((\partial \Omega))} ,
$$

With Theorem 5.3 in hand, we let $\tau = u \cdot \nabla \phi_k$, then $\tau(0,\cdot) = u(0,\cdot) \nabla \phi_k \equiv 0$, $\int_{\Omega} \tau \, dx = \int_{\Omega} \text{div } u_k \, dx = 0$. Moreover,

$$
\partial_t \tau = \partial_t u \cdot \nabla \phi_k
$$

$$
= \frac{f}{\rho} \nabla \phi_k + P \partial_t \left( \frac{\partial \phi_k}{\rho} \right) - \mu(\theta) D_{ij}(u) \partial_j \left( \frac{\partial \phi_k}{\rho} \right) + \partial_j \left( \mu(\theta) D_{ij}(u) \frac{\partial \phi_k}{\rho} - \frac{P}{\rho} \partial_j \phi_k \right) ,
$$

and $\text{supp } \tau_0(t,\cdot) \cap \text{supp } R(t,\cdot) \subset \Omega_k$. Hence by Theorem 5.3, there exists a unique solution $(u_k, P_k)$ to (5.7) satisfying

$$
\| (\partial_t u_k, \nabla^2 u_k, \nabla P_k ) \|_{L^p_r(0,T;L^r(\Omega))} \lesssim \| g_k \|_{L^p_r((\Omega'))} + \| R \|_{L^p_r((\Omega'))} + \lambda \| \tau_0 \|_{L^p_r((\Omega'))} \\
+ \lambda^\frac{1}{2} \| R \cdot n \|_{L^p_r((\partial \Omega))} + \| \nabla \tau \|_{L^p_r((\Omega'))} ,
$$

(5.8)
where $g_k$ is the right hand side of the first equation of (5.7). Next, we evaluate the terms on the RHS of (5.8) one by one.

$$
\|g_k\|_{L^p_q(\Omega)} \lesssim \|f\|_{L^p_q(\Omega)} + \lambda^\alpha (\|\nabla \theta\|_{L^r} \|\Delta u\|_{L^p_q(\Omega)})
+ \|\nabla \rho\|_{L^r} \|D u\|_{L^p_q(\Omega)}) + \lambda^{-1} \|\nabla u\|_{L^p_q(\Omega)}
+ \lambda^{-2} \|u\|_{L^p_q(\Omega)} + \|\nabla u \nabla \theta\|_{L^p_q(\Omega_k)}
+ \lambda^{-1} \|P\|_{L^p_q(\Omega_k)}.
$$

(5.9)

Similarly, $\nabla \tau$ and $R$ are bounded by RHS of (5.9).

$$
\lambda \|\tau_0\|_{L^p_q(\Omega)} \lesssim \|f\|_{L^p_q(\Omega)} + \lambda^{-1} \|P\|_{L^p_q(\Omega)} + \lambda^{-1} \|\nabla u\|_{L^p_q(\Omega)}
+ \|\nabla \log \rho\|_{L^p_q(\Omega)} + \|\nabla \log \rho\|_{L^p_q(\Omega)}
+ \lambda^{-1} \|P\|_{L^p_q(\Omega)} + \|\nabla u \nabla \theta\|_{L^p_q(\Omega)}.
$$

(5.10)

Substituting (5.9)-(5.11) into (5.8), summing over $k$, noting that

$$
\forall z \in L^p_q(\Omega), \quad \sum_{k=1}^K \|z\|_{L^p_q(\Omega)} \leq m^\frac{p}{q} K^{\max(0,1-\frac{p}{q})}\|z\|_{L^p_q(\Omega)},
$$

one finally reaches

$$
\|(-\partial u, \Delta u, \nabla P)\|_{L^p_q(\Omega)} \lesssim \lambda^{-N'} \left(\|f\|_{L^p_q(\Omega)} + \lambda^{-1} \|\nabla u\|_{L^p_q(\Omega)}
+ \|\nabla \nabla \log \rho\|_{L^p_q(\Omega)} + \|\nabla \nabla \log \rho\|_{L^p_q(\Omega)}
+ \lambda^{-1} \|P\|_{L^p_q(\Omega)} + \|\nabla u \nabla \theta\|_{L^p_q(\Omega)}
+ \lambda^{-1} \lambda^{-\frac{N-1}{2}} \left(\|\nabla u\|_{L^p_q(\Omega)} + \|\nabla \theta\|_{L^p_q(\Omega)}\right),
$$

(5.12)

where $\zeta = \max\{0, \frac{1}{p} - \frac{1}{r}\}$ and we have chosen $\lambda = \kappa \left(\|\nabla \rho\|_{L^q}^{\frac{q}{q'}} + \|\nabla \theta\|_{L^q}^{\frac{q}{q'}}\right)$ for some $\kappa$ sufficiently small.

Standard interpolation inequalities enable us to further simplify the RHS of (5.12). By Gagliardo-Nirenberg and Young’s inequality, it follows that

$$
\|\nabla u\|_{L^r(\Omega)} \leq C_r (\eta_1 u + \eta_2 \|\nabla^2 u\|_{L^r(\Omega)}), \quad \forall \eta_1 > 0,
$$

(5.13)

$$
\|\nabla \nabla \log \rho\|_{L^r(\Omega)} \leq \epsilon \|\nabla \nabla P\|_{L^r} + Ce^{-\frac{\lambda}{\sqrt{\lambda}}} \|\nabla \nabla \log \rho\|_{L^r} \|P\|_{L^r}, \quad \forall \epsilon > 0,
$$

(5.14)

$$
\|\nabla \nabla \log \rho\|_{L^r(\Omega)} \leq \epsilon \|\nabla^2 u\|_{L^r} + Ce^{-\frac{\lambda}{\sqrt{\lambda}}} \|\nabla \nabla \log \rho\|_{L^r} \|u\|_{L^r}, \quad \forall \epsilon > 0,
$$

(5.15)

where we have used $q \geq r > N$ in (5.14) and (5.15). And according to the trace theorem (page 63 in [15]), one deduces that

$$
\|P\|_{L^r(\partial \Omega)} \leq C \left(\eta_1 \|P\|_{L^r(\Omega)} + \eta_2 \|\nabla P\|_{L^r(\Omega)}\right), \quad \forall \eta_2 > 0,
$$

(5.16)

$$
\|\nabla u\|_{L^r(\partial \Omega)} \leq C \left(\eta_2^{-\frac{1}{4}} \|u\|_{L^r(\Omega)} + \eta_2 \|\nabla^2 u\|_{L^r(\Omega)}\right), \quad \forall \eta_2 > 0.
$$

(5.17)

Again by (5.4),

$$
\|\nabla \nabla \theta\|_{L^r(\Omega)} \leq C \left(\eta_3 \|\nabla^2 u\|_{L^r(\Omega)} + \eta_3 \|\nabla \theta\|_{L^r(\Omega)} \|u\|_{L^r(\Omega)}\right), \quad \forall \eta_3 > 0.
$$

(5.18)

Choosing $\epsilon = \kappa \lambda^N \zeta$, $\eta_1 = \kappa \lambda^{N+1}$, $\eta_2 = \kappa \lambda^{N-1} \epsilon^{r'}$ and $\eta_3 = \kappa \lambda^N \zeta$, with $\kappa << 1$, the terms $\|\nabla P\|_{L^p_q(\Omega)}$ and $\|\nabla^2 u\|_{L^p_q(\Omega)}$ can be absorbed by the LHS.
Choosing suitable $\Lambda$. Consequently, substituting (5.13)-(5.18) into (5.12), we reach
\[
\| (\partial_t u, \Delta u, \nabla P) \|_{L^q_t(L^r(\Omega))} \lesssim \lambda^{-N \zeta} \| f \|_{L^q_t(L^r(\Omega))} + (\lambda^{-k_1} + \lambda^{-k_2}) (\| u \|_{L^q_t(L^r(\Omega))} + \| P \|_{L^q_t(L^r(\Omega))}),
\]
(5.19)
where $N \zeta + 2 \leq k_1 \leq k_2$ and $k_1, k_2$ go to $+\infty$ as $q \to N$.

It remains to show the pressure estimates in terms of $f$ and $u$ to complete the proof.

To this end, we evaluate $P$ by a duality argument
\[
\| P \|_{L^r(\Omega)} = \sup_{\| h \|_{L^{r'}(\Omega)} \leq 1} \int_\Omega P h \, dx.
\]
Let
\[
div(\rho^{-1} \nabla v) = h, \quad \partial_r v|_{\partial \Omega} = 0.
\]
(5.20)
Since $q \geq r'$, then according to Proposition C.1 in [7], we have
\[
\| \nabla v \|_{L^{r'}(\Omega)} \leq C(\Omega) \| h \|_{L^{r'}(\Omega)} \left( 1 + \| \nabla \log \rho \|_{L^2(\Omega)}^{\frac{s}{q}} \right) \leq C(B_{\rho, \theta} + 1) \| h \|_{L^{r'}},
\]
(5.21)
\[
\| \nabla^2 v \|_{L^{r'}(\Omega)} \leq C(\Omega) \| h \|_{L^{r'}(\Omega)} \left( 1 + \| \nabla \log \rho \|_{L^2(\Omega)}^{\frac{s}{q}} \right) \leq C(B_{\rho, \theta}^2 + 1) \| h \|_{L^{r'}}.
\]
(5.22)
Hence,
\[
\int_\Omega P h \, dx = \int_\Omega P \Delta v \, dx = -\int_\Omega \nabla \cdot (\nabla v) \, dx
\]
\[
\lesssim \| f \|_{L^r(\Omega)} \| \nabla v \|_{L^{r'}(\Omega)} + \| \nabla u \|_{L^{r'}(\partial \Omega)} \| \nabla v \|_{L^{r'}(\partial \Omega)}
\]
\[
\quad + \| \nabla u \|_{L^r(\Omega)} \| \Delta v \|_{L^{r'}(\Omega)} + \| \nabla u \|_{L^{r'}(\Omega)} \| \nabla \log \rho \|_{L^s(\Omega)},
\]
where we have used (5.21).

By interpolation and Young’s inequality, it holds that
\[
\| \nabla v \|_{L^{r'}(\partial \Omega)} \lesssim \| \nabla u \|_{L^r(\Omega)} \| \nabla v \|_{L^{s}}, \quad \frac{1}{s} = \frac{1}{q'} - \frac{1}{r'},
\]
(5.23)
\[
\| \nabla v \|_{L^s} \lesssim \| \nabla v \|_{L^{r'}(\Omega)}^{1 - \frac{s}{r'}} \| \nabla^2 v \|_{L^{s'}}^{\frac{s}{r'}} \lesssim (1 + B_{\rho, \theta}^{1 + \frac{s}{q'}}) \| h \|_{L^{r'}},
\]
(5.24)
\[
\| \nabla u \|_{L^s(\Omega)} \lesssim \epsilon_1^{-1} \| u \|_{L^r(\Omega)} + \epsilon_1 \| \nabla^2 u \|_{L^r(\Omega)}, \quad \forall \epsilon_1 > 0.
\]
(5.25)
Finally, one can use trace theorem to simplify the boundary terms as
\[
\| \nabla v \|_{L^{r'}(\partial \Omega)} \lesssim \| \nabla v \|_{L^{r'}(\Omega)}^\frac{1}{r} \left( \| \nabla^2 v \|_{L^{r'}(\Omega)} + \| \nabla v \|_{L^{r'}(\Omega)}^{\frac{1}{\rho}} \right)
\]
\[
\lesssim (1 + B_{\rho, \theta}^{1 + \frac{1}{q'}}) \| h \|_{L^{r'}(\Omega)},
\]
(5.26)
\[
\| \nabla u \|_{L^r(\partial \Omega)} \lesssim \epsilon_2^{-1 - \frac{1}{s}} \| u \|_{L^r(\Omega)} + \epsilon_2 \| \nabla^2 u \|_{L^r(\Omega)}, \quad \forall \epsilon_2 > 0.
\]
(5.27)
Choosing suitable $\epsilon_{1,2}$, we obtain the pressure estimates:
\[
\| P \|_{L^{r}(\Omega)} \lesssim (B_{\rho, \theta}^{\epsilon_1} + B_{\rho, \theta}^{\epsilon_2}) \| f \|_{L^{r}(\Omega)} + \| u \|_{L^r(\Omega)}.
\]
(5.28)
Plugging (5.28) into (5.19), one obtains the desired inequality. Therefore, the proof of Theorem 5.2 is completed.

(b) A priori estimates with time-dependent temperature and density

The main result is the following.
Theorem 5.4. Suppose $p$, $q$, $r$, $\Omega$, $f$ and $\mu$ satisfy the assumptions in Proposition 2.2, $u_0 = 0$ and the temperature satisfies
\begin{align}
\rho &\in C^\beta_2([0, T]; L^\infty(\Omega)) \cap L^\infty(0, T; W^{1, q}(\Omega)), \\
\theta &\in C^\beta_2([0, T]; L^\infty(\Omega)) \cap L^\infty(0, T; W^{1, q}(\Omega)),
\end{align}
for some $\beta_{1, 2} \in (0, 1)$. If $(u, P)$ is a smooth solution to (2.5) on $\Omega \times [0, T)$, then it holds that
\begin{align}
&\left\| (\partial_t u, \Delta u, \nabla P) \right\|_{L^p_\tau(L^r(\Omega))} \\
&\quad \leq C \left( (B^{k}_{p, \theta}(T) + 1) \left\| f \right\|_{L^p_\tau(L^r(\Omega))} + C_{p, \theta}(T) \left\| u \right\|_{L^p_\tau(L^r(\Omega))} \right),
\end{align}
where $k \geq 2$, $B_{p, \theta}(t)$, $C_{p, \theta}(t)$ and $C$ are defined as in Proposition 2.2.

Proof. First, rewrite (2.5) as the following system
\begin{equation}
\begin{cases}
\rho_0 \partial_t u - \text{div} \left( (\mu(\theta)_0) D(u) \right) + \nabla P = f + (\rho_0 - \rho) \partial_t u - \text{div} \left( (\mu(\theta) - \mu(\theta)_0) D(u) \right) \\
\text{div} u = 0, \quad \int_\Omega P \, dx = 0, \\
\left. u \right|_{t=0} = 0, \quad u|_{\partial \Omega} = 0. 
\end{cases}
\end{equation}

Applying Theorem 5.2 to system (5.32), we have for any $t < T$
\begin{align}
&\left\| (\partial_t u, \nabla^2 u, \nabla P) \right\|_{L^p_\tau(L^r(\Omega))} \lesssim (1 + B^{k}_{p, \theta}(t)) \left( \left\| f \right\|_{L^p_\tau(L^r(\Omega))} \\
&\quad + \left\| (\mu(\theta) - \mu(\theta)_0) \Delta u \right\|_{L^p_\tau(L^r(\Omega))} + \left( \left\| \mu'(\theta) \nabla \theta - \mu'(\theta)_0 \nabla \theta_0 \right\|_{L^p_\tau(L^r(\Omega))} \\
&\quad + \left\| (\rho_0 - \rho) \partial_t u \right\|_{L^p_\tau(L^r(\Omega))} + \left\| u \right\|_{L^p_\tau(L^r(\Omega))} \right) \\
&\quad + \left\| (\rho_0 - \rho) \partial_t u \right\|_{L^p_\tau(L^r(\Omega))} + \left\| u \right\|_{L^p_\tau(L^r(\Omega))}.
\end{align}

Noticing that
\begin{align}
&\left\| (\mu(\theta)) \nabla \theta - \mu'(\theta)_0 \nabla \theta_0 \nabla u \right\|_{L^p_\tau(L^r(\Omega))} \\
&\quad \lesssim \epsilon \left\| \nabla^2 u \right\|_{L^p_\tau(L^r(\Omega))} + \epsilon^{\frac{2s}{2s-2}} \left( \left\| \nabla \theta \right\|_{L^s(\Omega)} + \left\| \nabla \theta_0 \right\|_{L^s(\Omega)} \right)^{\frac{2s}{2s-2}} \left\| u \right\|_{L^p_\tau(L^r(\Omega))}.
\end{align}

Thus choose $\epsilon = k(B^{k_{1}}_{p, \theta_0} + B^{k_{2}}_{p, \theta_0})^{-1}$ with $k << 1$ such that $\nabla^2 u$ can be absorbed by the LHS of (5.33). On the other hand,
\begin{align}
&\left\| (\rho - \rho_0) \partial_t u \right\|_{L^p_\tau(L^r(\Omega))} \lesssim t^{\beta_1} \left\| \rho \right\|_{C^{\beta_1}(0, t; L^\infty(\Omega))} \left\| \partial_t u \right\|_{L^p_\tau(L^r(\Omega))}, \\
&\left\| (\mu(\theta) - \mu(\theta)_0) \Delta u \right\|_{L^p_\tau(L^r(\Omega))} \lesssim t^{\beta_2} \left\| \theta \right\|_{C^{\beta_2}(0, t; L^\infty(\Omega))} \left\| \Delta u \right\|_{L^p_\tau(L^r(\Omega))}.
\end{align}

Substituting the above two inequalities into (5.33), we obtain
\begin{align}
&\left\| (\partial_t u, \Delta u, \nabla P) \right\|_{L^p_\tau(L^r(\Omega))} \\
&\quad \lesssim (1 + B^{k}_{p, \theta}(t)) \left( \left\| f \right\|_{L^p_\tau(L^r(\Omega))} + t^{\beta_2} \left\| \theta \right\|_{C^{\beta_2}(0, t; L^\infty(\Omega))} \left\| \Delta u \right\|_{L^p_\tau(L^r(\Omega))} \\
&\quad + t^{\beta_1} \left\| \rho \right\|_{C^{\beta_1}(L^\infty(\Omega))} \left\| \partial_t u \right\|_{L^p_\tau(L^r(\Omega))} + \left\| u \right\|_{L^p_\tau(L^r(\Omega))} \right) \\
&\quad + t^{\beta_1} \left\| \rho \right\|_{C^{\beta_1}(L^\infty(\Omega))} \left\| \partial_t u \right\|_{L^p_\tau(L^r(\Omega))} + \left\| u \right\|_{L^p_\tau(L^r(\Omega))}.
\end{align}

If $\left( t^{\beta_1} \left\| \rho \right\|_{C^{\beta_1}(L^\infty(\Omega))} + t^{\beta_2} \left\| \theta \right\|_{C^{\beta_2}(L^\infty(\Omega))} \right) (1 + B^{k}_{p, \theta}(T)) \leq \frac{1}{2\epsilon}$, then the second term on the RHS of (5.34) can be absorbed by the LHS, which gives the desired estimates.

Otherwise, we perform time localization to adjust the time interval. Specifically, choosing
\begin{align}
\tau = \min \left\{ T, \kappa \left( 1 + B^{k}_{p, \theta}(T) \right) \left( \left\| \rho \right\|_{C^{\beta_1}(L^\infty(\Omega))} + \left\| \theta \right\|_{C^{\beta_2}(L^\infty(\Omega))} \right)^{-\frac{1}{\min(\beta_1, \beta_2)}} \right\}, \quad \kappa << 1.
\end{align}
Then for any \( t \in [0, \tau] \), it holds that
\[
\| (\partial_t u, \Delta u, \nabla P) \|_{L^p(\frac{1}{2}, t; L^r(\Omega))} \lesssim (1 + B_{\rho, \theta}^k(t))(\| f \|_{L^p(\frac{1}{2}, t; L^r(\Omega))} + \| u \|_{L^p(\frac{1}{2}, t; L^r(\Omega))}).
\] (5.35)

Next, we try to extend the above estimates to \([0, T]\).

To this end, we perform a partition on time interval as the following:
Suppose \( \{\psi_k\}_{k \in \mathbb{N}} \) is a partition of unity of \([0, T]\) such that
\[
supp \psi_0 \subset [0, \tau], \quad \psi_0 \equiv 1 \text{ on } [0, \frac{\tau}{2}],
\]
\[
supp \psi_k \subset \left( \frac{k}{2}, \frac{k}{2} + \tau \right) \text{ for } k \geq 1, \quad \| \partial_t \psi_k \|_{L^\infty} \leq C t^{-1},
\]
\[
\sum_{k=0}^K \psi_k(t) = 1, \forall t \in [0, T], \quad \frac{K}{2} \tau \leq T \leq \frac{K + 1}{2} \tau.
\]

Denote \( u_k = u \psi_k, P_k = P \psi_k, f_k = f \psi_k \), then \((u_k, P_k, f_k)\) satisfy the following system
\[
\rho \partial_t u_k - \text{div} (\mu(\theta) \nabla u_k) + \nabla P_k = f_k + \rho u \partial_t \psi_k,
\]
\[
\text{div} u_k = 0, \quad \int_\Omega P_k \, dx = 0,
\]
\[
u_k|_{t=\frac{k}{2} \tau} = 0, \quad u_k|_{\partial \Omega} = 0.
\] (5.36)

Let \( I_k = \left[ \frac{k}{2} \tau, \frac{k}{2} + \tau \right], k = 0, \cdots, K - 1, I_K = \left[ \frac{K}{2} \tau, T \right] \).

For any \( t \in I_k \), it follows from (5.35) that
\[
\| (\partial_t u_k, \Delta u_k, \nabla P_k) \|_{L^p(\frac{1}{2}, t; L^r(\Omega))} \lesssim \left( B_{\rho, \theta}^k(T) + B_{\rho, \theta}^k(T) \right)
\times \left( \| f_k \|_{L^p(\frac{1}{2}, t; L^r(\Omega))} + \| u \partial_t \psi_k \|_{L^p(\frac{1}{2}, t; L^r(\Omega))} + \| u_k \|_{L^p(\frac{1}{2}, t; L^r(\Omega))} \right).
\] (5.37)

Notice that
\[
\| u \partial_t \psi_k \|_{L^p(\frac{1}{2}, t; L^r(\Omega))} \lesssim \tau^{-1} \| u \|_{L^p(\frac{1}{2}, t; L^r(\Omega))}
\lesssim (1 + B_{\rho, \theta}^k(T)) \left( \| \rho \|_{C^\alpha_{\rho, \theta}(L^\infty)} + \| \theta \|_{C^\alpha_{\rho, \theta}(L^\infty)} \right)^{-1} \| u \|_{L^p(\frac{1}{2}, t; L^r(\Omega))}.
\]

By the definition of \( C_{\rho, \theta} \), the theorem is proved.

(\(c\)) Existence and uniqueness of solution to (2.5) with null initial data

The existence and uniqueness to (2.5) immediately follow from the following \textit{a priori} estimates.

\textbf{Theorem 5.5.} Suppose all the assumptions in Proposition 2.2 are true and \( u_0 = 0 \), then the system (2.5) has a unique strong solution \((u, P)\) satisfying
\[
\| (\partial_t u, \Delta u, \nabla P) \|_{L^p(\frac{1}{2}, t; L^r(\Omega))} \leq C \left( 1 + B_{\rho, \theta}^k(t) \right) \exp(C t C_{\rho, \theta}(t)) \| f \|_{L^p(\frac{1}{2}, t; L^r(\Omega))},
\] (5.38)

for any \( t < T \), where \( k, B_{\rho, \theta}(t), C_{\rho, \theta}(t) \) and \( C \) are defined as in Proposition 2.2.

\textbf{Proof.} Suppose \((u, P)\) is a smooth solution, indeed, we have
\[
\frac{d}{dt} \| u \|_{L^r(\Omega)} \leq \| \partial_t u \|_{L^r(\Omega)}.
\] (5.39)
According to Theorem 5.4, for all \( \epsilon > 0 \)

\[
\|u(t, \cdot)\|_{L^p(\Omega)} = \int_0^t \|u(\tau, \cdot)\|_{L^p(\Omega)} \frac{d}{dt} u(\tau, \cdot) d\tau \\
\leq (p-1) \epsilon \int_0^t \|u(\tau, \cdot)\|_{L^p(\Omega)} d\tau + \epsilon^{1-p} \int_0^t \|\partial_t u\|_{L^p(\Omega)} d\tau \tag{5.40}
\]

Now, let us denote

\[
\alpha(t) = \epsilon + \epsilon^{1-p}C_{\rho, \theta}(t), \quad \beta(t) = \epsilon^{1-p}(1 + B^k_{\rho, \theta}(t))p \|f\|_{L^p_t(L^r(\Omega))}, \\
F(t) = \|u\|_{L^p_t(L^r(\Omega))}, \quad \gamma(t) = \int_0^t \beta(s) ds,
\]

then from (5.40), it follows that

\[
F(t) \leq C \int_0^t \alpha(s)F(s)ds + C \gamma(t). \tag{5.41}
\]

Therefore, Gronwall’s lemma implies that

\[
F(t) \leq C^{1-p}t(1 + B^k_{\rho, \theta}(t))p \|f\|_{L^p_t(L^r(\Omega))} \exp(C \epsilon t + C \epsilon^{1-p}tC_{\rho, \theta}(t)). \tag{5.42}
\]

Choosing \( \epsilon = C_{\rho, \theta}(t) \), then (5.42) turns into

\[
\int_0^t \|u(\tau, \cdot)\|_{L^p(\Omega)} d\tau \leq C^{1-p}t(1 + B^k_{\rho, \theta}(t))p \|f\|_{L^p_t(L^r(\Omega))} \exp(C \epsilon t + C \epsilon^{1-p}tC_{\rho, \theta}(t)), \tag{5.43}
\]

where we have used \( tC_{\rho, \theta}(t) \leq \exp(tC_{\rho, \theta}(t)) \).

Inserting (5.43) into (5.31), we finally obtain (5.38). \( \square \)

5.1.2. General initial data. In this section, we generalize the previous result to the case of general initial data. Consider the following two systems:

\[
\begin{cases}
\rho \partial_t \omega - \mu(\theta) \Delta \omega + \nabla \Pi = f, \\
\text{div } \omega = 0, \quad \int_\Omega \Pi \, dx = 0, \\
\omega|_{t=0} = u_0, \quad \omega|_{\partial \Omega} = 0,
\end{cases}
\begin{cases}
\rho \partial_t v - \text{div } (\mu(\theta) \nabla v) + \nabla Q = \text{div } (\mu(\theta) - \mu(\theta)) \nabla \omega) + (\rho - \rho) \partial_t \omega, \\
\text{div } v = 0, \quad \int_\Omega Q \, dx = 0, \\
v|_{t=0} = 0, \quad v|_{\partial \Omega} = 0,
\end{cases}
\]

where \( \theta = \inf_{x \in \Omega} \theta_0 \) and \( \rho = \inf_{x \in \Omega} \rho_0 \). It is easy to verify that \( u = v + \omega \) and \( P = \Pi + Q \) satisfy system (2.5) if \( \omega \) and \( v \) satisfy the corresponding system.

Theorem 5.1 implies that there exists a unique solution \((\omega, \Pi)\) to the first system of (5.44) satisfying

\[
\|\text{div}(\omega, \Delta \omega, \nabla \Pi)\|_{L^p_t(L^r(\Omega))} + \|\omega\|_{C^{1,1/2}_r(D^{1/2-p}_{A_r})} \lesssim \|f\|_{L^p_t(L^r(\Omega))} + \|u_0\|_{D^{1/2-p}_{A_r}}. \tag{5.45}
\]
On the other hand, Theorem 5.5 implies that there exists a unique solution \((v, Q)\) to the second system of (5.44) such that
\[
\|(\partial_t v, \Delta v, \nabla Q)\|_{L^p_t(L^r(\Omega))} + \|v\|_{C([0,t]; D_1^{\frac{1}{r} - 1}p,pAr)}
\lesssim \left(1 + B_k^{t}(t)\right) \exp(C t C \rho, \theta(t)) \left(\|f\|_{L^p_t(L^r(\Omega))} + \|u_0\|_{D_1^{\frac{1}{r} - 1}p,pAr}\right),
\]
where we have used (5.45).

Adding up (5.45) and (5.46) yields that
\[
\|u\|_{C([0,t]; D_1^{\frac{1}{r} - 1}p,pAr)} + \|\Delta u, \nabla P\|_{L^p_t(L^r(\Omega))}
\lesssim \left(1 + B_k^{t}(t)\right) \exp(C t C \rho, \theta(t)) \left(\|u_0\|_{D_1^{\frac{1}{r} - 1}p,pAr} + \|f\|_{L^p_t(L^r(\Omega))}\right),
\]
Thus, (2.6) is proved.

Estimate (2.7) can be established in the same manner, so this completes the proof of Proposition 2.2.

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