Thermodynamic Uncertainty Relations for Bosonic Otto Engines

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We study a two-mode bosonic engine undergoing an Otto cycle. The energy exchange between the two bosonic systems is provided by a tunable unitary bilinear interaction in the mode operators modeling frequency conversion, whereas the cyclic operation is guaranteed by relaxation to two baths at different temperature after each interacting stage. We derive exact expressions for work and heat fluctuations, identities showing the interdependence among average extracted work, fluctuations and efficiency, along with thermodynamic uncertainty relations between the signal-to-noise ratio of observed work and heat and the entropy production. The full probability of the stochastic work and heat is then provided.

Nonequilibrium processes are always accompanied by irreversible entropy production [1]. When systems become smaller, as in nanoscopic heat engines [2, 3], biological or chemical systems [4–6] or nanoelectronic devices [7, 8], the fluctuations of all thermodynamical quantities as work, heat, their correlations, and entropy production itself, become very relevant. For example, a macroscopic thermal engine supplies a certain amount of work while extracting heat from a hot thermal reservoir. There can be variations in these amounts between different cycles, but typically these fluctuations are negligible. However, as the thermodynamic machine size is scaled down, the work output and heat absorbed will correspondingly be scaled down. Then, the fluctuations will become more and more significant, and it becomes useful to investigate the stochastic properties of such fluctuating quantities.

A number of fluctuation theorems has been derived [9–29] as powerful relations that characterize the behavior of small systems out of equilibrium. Fluctuation relations pose stringent constraints on the statistics of fluctuating quantities as heat and work due to the symmetries (particularly, time-reversal symmetry) characterizing the microscopic dynamics from which they emerge. Furthermore, recent relations have also been developed, so called thermodynamic uncertainty relations (TUR), where the signal-to-noise ratio of observed work and heat has been related to entropy production [29–51]. Such TURs rule for example the tradeoff between entropy production and the output power relative fluctuations, i.e. the precision of a heat machine, so that working machines operating at near-to-zero entropy production cannot be achieved without a divergence in the relative output power fluctuations.

Although independently developed, fluctuation relations and TURs have been recently connected under various approaches and assumptions [50–57]. In particular, in Ref. [54] a saturable TUR obtained from fluctuation theorems has been derived and compared with exact results pertaining to a microscopic two-qubit swap engine operating at the Otto efficiency.

In this Letter we derive thermodynamic uncertainty relations for two-mode bosonic engines, where alternately each quantum harmonic oscillator is coupled to a thermal bath allowing heat exchange and a unitary bilinear interaction determines energy exchange between the two modes by frequency conversion with tunable strength. This model is shown to achieve the Otto efficiency [54, 58–65], independently of the coupling strength and the temperature of the reservoirs. We identify the regimes where the periodic protocol works as a heat engine, a refrigerator, or a thermal accelerator, and derive its stochastic characteristics. The joint characteristic function provides all moments of work and heat, and is obtained by adopting the two-measurements scheme [29, 35, 50, 67] usually considered in the derivation of Jarzinsky equality [68], and referred to the simultaneous estimation of both work and heat. Our derivation allows to obtain an exact relation between the signal-to-noise ratio of work and heat and the average entropy production of the engine, thus showing the deep interdependence among average extracted work, fluctuations, and entropy production. From these relations we derive thermodynamic uncertainty relations that are satisfied in all the considered regimes and for any value of the strength of the bilinear interaction between the two quantum harmonic oscillators. A bound of the efficiency in terms of the average work and its fluctuations is also obtained.

The full distribution of the stochastic work and heat is then provided in closed form.

We illustrate now the two-mode bosonic engine under investigation, as depicted in Fig. 1. Let us fix natural units $\hbar = k_B = 1$. Each system is described by bosonic mode operators $a, a^\dagger$ and $b, b^\dagger$, respectively, with the usual commutation relation, and corresponding free Hamiltonians $H_A = \omega_A (a^\dagger a + \frac{1}{2})$ and $H_B = \omega_B (b^\dagger b + \frac{1}{2})$. Initially, the two modes $a$ and $b$ are in thermal equilibrium with their own ideal bath at temperature $T_A$ and $T_B$, respectively, and we fix $T_A > T_B$. Hence, the initial state is characterized by the tensor product of bosonic Gibbs thermal states, i.e.

$$\rho_0 = \frac{e^{-\beta_A H_A}}{Z_A} \otimes \frac{e^{-\beta_B H_B}}{Z_B},$$

with $\beta_X = 1/T_X$ and $Z_X = \text{Tr}[e^{-\beta_X H_X}]$. The two sys-
Figure 1. Two-mode bosonic Otto cycle in heat engine operation: in the first stage each quantum harmonic oscillator with frequency $\omega_A$ and $\omega_B$ is in thermal equilibrium with its respective bath at temperature $T_A$ and $T_B$, respectively, with $T_A > T_B$; in the second stage the two oscillators are isolated and let to interact by a bilinear unitary interaction ($\theta$), thus extracting work $W$; in the third stage the oscillators are let to relax to their respective thermal baths, thus absorbing heat $Q_H$ and releasing heat $Q_C$, such that the initial condition is reestablished. In the refrigeration regime all three arrows are reversed.

states are then isolated from their thermal baths and are allowed to interact via a global unitary transformation. We will consider the bilinear interaction that globally transforms the mode operators as follows

$$a' = \cos \theta a + e^{i\varphi} \sin \theta b,$$  \hspace{1cm} (2)

$$b' = \cos \theta b - e^{-i\varphi} \sin \theta a,$$  \hspace{1cm} (3)

with $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$.

The Heisenberg transformation in Eqs. (23) corresponds to a linear mixing of the modes that for $\omega_A \neq \omega_B$ describes frequency conversion, and in the Schrödinger picture is equivalent to the unitary transformation $U_\xi = \exp(\xi a^d b - \xi^* a b^d)$, with $\xi = \theta e^{i\varphi}$. We remark that $U_\xi$ incorporates the free evolutions, all interactions and classical external driving, such that the corresponding unitary for the time-reversed process is just $U_\xi^\dagger$. We also notice that an extensive study of such thermodynamical coupling, especially for general Gaussian bipartite states, has been recently put forward in Ref. [63]. In what follows the phase $\varphi$ is irrelevant, hence we pose $\varphi = 0$.

After the interaction the two harmonic oscillators are reset to their equilibrium state of Eq. (1) via full thermalization by their respective baths. The procedure can be sequentially repeated and leads to a stroke engine. We notice that for $\theta = \pi/2$ the unitary $U_{\pi/2}$ performs a swap gate which exchanges the states of the two quantum systems, analogous to the two-qubit swap engine [54, 63]. More generally, here we consider an arbitrary value of $\theta$, modeling different interaction strengths (or times). For each cycle the energy change in mode $a$ corresponds to the heat $Q_H$ released by the hot bath, i.e., $Q_H = -\Delta E_a$, and similarly with have $Q_C = -\Delta E_b$ for the heat dumped into the cold reservoir (heat is positive when it flows out of a reservoir). The work $W$ is performed ($W > 0$) or extracted ($W < 0$) during the unitary interaction, and from the first law we have $W = -Q_H - Q_C = \Delta E_a + \Delta E_b$.

We can characterize the engine by the independent random variables $W$ and $Q_H$, and study the characteristic function $\chi(\lambda, \mu)$, where $\lambda$ and $\mu$ denotes the work and heat labels such that all moments of work and heat can be obtained by the identity

$$\langle W^n Q_H^m \rangle = (-i)^{n+m} \frac{\partial^{n+m} \chi(\lambda, \mu)}{\partial \lambda^n \partial \mu^m} \big|_{\lambda=\mu=0}. $$  \hspace{1cm} (4)

The characteristic function depends on the procedure that is adopted to jointly estimate $W$ and $Q_H$. By using the two-point measurement scheme [20, 25, 66, 67], we can write the characteristic function as follows [25]

$$\chi(\lambda, \mu) = \text{Tr}[U_\theta e^{-i\mu H_A} e^{-i\lambda(H_A + H_B)} U_\theta e^{i\mu H_A} e^{-i\lambda H_A + H_B} \rho_0]. $$  \hspace{1cm} (5)

By representing the thermal states as mixture of coherent states, namely

$$\frac{e^{-\beta X}}{\pi N_C} = \frac{\beta^\frac{1}{2}}{\pi N_C} \int d^2 \gamma e^{-\frac{|\gamma|^2}{\beta}} |\gamma\rangle \langle \gamma|,$$  \hspace{1cm} (6)

with $d^2 \gamma = d\Re \gamma d\Im \gamma$ and $N_C = (e^{\beta X} - 1)^{-1}$, using the identities $e^{i\psi a^d a} |\alpha\rangle = |\alpha e^{i\psi}\rangle$ and

$$|\alpha \cos \theta + \delta \sin \theta\rangle |\delta \cos \theta - \alpha \sin \theta\rangle,$$  \hspace{1cm} (7)

we have

$$\chi(\lambda, \mu) = \frac{d^2 \alpha}{\pi N_A} \int \frac{d^2 \gamma}{\pi N_B} e^{-\frac{|\alpha|^2}{\pi N_A} - \frac{|\gamma|^2}{\pi N_B}} \times \langle \alpha \cos \theta + \gamma \sin \theta | \alpha \cos \theta + \gamma e^{i(\lambda - \mu) \omega_A - i\lambda \omega_B} \sin \theta \rangle \times \langle \gamma \cos \theta - \alpha \sin \theta | \gamma \cos \theta - \alpha e^{i(\lambda - \mu) \omega_A - i\lambda \omega_B} \sin \theta \rangle.$$  \hspace{1cm} (8)

Finally, from the identity $\langle \alpha \rangle |\gamma\rangle = \exp(-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\gamma|^2 + \bar{\alpha} \gamma)$ and lengthy but straightforward Gaussian integration we obtain

$$\chi(\lambda, \mu) = \{1 - \sin^2 \theta \times$$

$$\{(N_A + N_B + 2N_A N_B) |\cos(\mu \omega_A - \lambda (\omega_A - \omega_B)) - 1\} + i(N_A - N_B) |\sin(\mu \omega_A - \lambda (\omega_A - \omega_B))\}^{-1}. $$

We easily check the identity $\chi[|\beta_B, i(\beta_B - \beta_A)| = 1$, corresponding to the standard fluctuation theorem. Indeed, the time-reversal symmetry of the unitary operation provides the stronger identity $\chi[|i(\beta_B - \lambda, i(\beta_B - \beta_A) - \mu) = \chi(\lambda, \mu)$, corresponding to the Gallavotti-Cohen microreversibility [9, 10], and equivalent to the detailed fluctuation theorem [19, 22, 23, 27]

$$\frac{p(W, Q_H)}{p(-W, -Q_H)} = e^{(\beta_B - \beta_A) Q_H + \beta_B W}. $$  \hspace{1cm} (10)
Using Eq. \((4)\), we notice the symmetry \(\langle W^n Q_m^H \rangle = \left(\frac{\omega_A}{\omega_B - \omega_A}\right)^n \langle W^{n+m} Q_m^H \rangle = \left(\frac{\omega_B - \omega_A}{\omega_A}\right)^n \langle Q_m^H \rangle\) and, from the first law, \(\langle Q_m^H \rangle = (\omega_B / \omega_A)^n \langle Q_m^H \rangle\).

The first two moments for the work \(W\) and heat \(Q_H\) can be obtained from Eqs. \((4)\) and \((9)\), and one has

\[
\langle W \rangle = (\omega_A - \omega_B)(N_B - N_A)\sin^2\theta, \tag{11}
\]

\[
\langle Q_H \rangle = \omega_A(N_A - N_B)\sin^2\theta = \frac{\omega_A}{\omega_B - \omega_A} \langle W \rangle, \tag{12}
\]

\[
\langle W^2 \rangle = (\omega_A - \omega_B)^2[N_A + N_B + 2N_A N_B + 2(N_A N_B)^2 \sin^2\theta \sin^2\theta], \tag{13}
\]

\[
\langle Q_H^2 \rangle = \frac{\omega_A^2}{\omega_B - \omega_A} \langle W^2 \rangle, \tag{14}
\]

\[
\langle W Q_H \rangle = \frac{\omega_A}{\omega_B - \omega_A} \langle W^2 \rangle. \tag{15}
\]

We can identify three regimes of operations, namely

a) \(\omega_A > \omega_B \quad \& \quad N_A > N_B\) heat engine,

b) \(\omega_A > \omega_B \quad \& \quad N_A < N_B\) refrigerator,

c) \(\omega_A < \omega_B \quad (\Rightarrow N_A > N_B)\) thermal accelerator.

In the three regimes we have \(a) \langle W \rangle < 0, \langle Q_H \rangle > 0; \quad b) \langle W \rangle > 0, \langle Q_H \rangle < 0; \quad c) \langle W \rangle > 0, \langle Q_H \rangle > 0\); respectively.

In terms of the temperature of the reservoirs, it is useful to observe that

\[
\beta_A \omega_A \leq \beta_B \omega_B \quad \Leftrightarrow \quad N_A \geq N_B, \tag{16}
\]

and thus the three regimes are equivalently identified by

a) \(T_A < \frac{T_B}{\omega_A} < 1; \quad \beta_B \omega_B = \frac{T_B}{T_A} \quad \Leftrightarrow \quad \frac{T_B}{T_A} = \eta_C\), corresponding to the Otto cycle efficiency. The Carnot efficiency \(\eta_C\) is achieved only for \(\omega_A/\omega_B = T_A/T_B\) (i.e., for \(N_A = N_B\) with zero output power). Analogously, the coefficient of performance (COP) for the refrigerator is given by

\[
\zeta = \frac{\langle Q_H \rangle}{\langle -W \rangle} = \frac{\omega_B}{\omega_A - \omega_B} \leq \frac{T_B}{T_A} = \frac{T_A}{T_B} \quad \Rightarrow \quad \zeta_C. \tag{17}
\]

Notice that both the efficiency and the COP are independent of \(\theta\) and the temperature of the reservoirs. Since \([U_a, a^\dagger a + b^\dagger b] = 0\) one has \(\Delta E_b = -\frac{\omega_B}{\omega_A} \Delta E_a\), and hence the entropy production \(\langle \Sigma \rangle\) can be written as follows

\[
\langle \Sigma \rangle = \beta_A \Delta E_a + \beta_B \Delta E_b = \frac{\beta_A \omega_A - \beta_B \omega_B}{\omega_A - \omega_B} \langle W \rangle,
\]

\[
= \ln \left(\frac{N_A + 1}{N_A(N_A + 1)}\right) \langle N_B - N_A \rangle \sin^2\theta. \tag{17}
\]

From Eq. \((16)\), as expected, one always has \(\langle \Sigma \rangle \geq 0\). Work, heat and entropy production are depicted in Fig. 2 for fixed parameters \(\omega_A = 1, \beta_A = 1,\) and \(\beta_B = 2,\) with \(\theta = \pi/2\).

For the heat engine, using the identity \(\frac{\beta_A \omega_A - \beta_B \omega_B}{\omega_A - \omega_B} = -\frac{T_B}{T_A} \frac{\log \eta_C}{\eta - 1}\), one obtains the following relation between average extracted work, entropy production and efficiency

\[
\langle \Sigma \rangle = \frac{(-W)}{T_B} \left(\frac{\eta_C}{\eta} - 1\right). \tag{18}
\]

Analogously, for the refrigerator one obtains \(\langle \Sigma \rangle = \frac{\langle Q_H \rangle}{T_A} \left(\frac{\eta_C}{\eta} - 1\right)\). Using Eqs. \((11-15)\) we can evaluate the variance for \(W\) and \(Q_H\), and their covariance as follows

\[
\text{var}(W) = (\omega_A - \omega_B)^2, \tag{19}
\]

\[
\text{var}(Q_H) = \frac{\omega_A^2}{(\omega_A - \omega_B)^2} \text{var}(W), \tag{20}
\]

\[
\text{cov}(W, Q_H) = \frac{\omega_A}{(\omega_A - \omega_B)} \text{var}(W). \tag{21}
\]

Notice that for both the heat engine and the refrigerator the sign of \(\text{cov}(W, Q_H)\) is negative. On the other hand, for the thermal accelerator where external work is consumed to increase the heat flow from hot to cold reservoir the covariance is positive. For the inverse signal-to-noise ratios one obtains

\[
\frac{\text{var}(W)}{(\langle W \rangle)^2} = \frac{\text{var}(Q_H)}{(\langle Q_H \rangle)^2} = \frac{\text{cov}(W, Q_H)}{(\langle W \rangle)^2} = 1 + \frac{N_A + N_B + 2N_A N_B}{(N_A - N_B)^2 \sin^2\theta} \tag{22}
\]

These ratios are minimized versus \(\theta\) for \(\theta = \frac{\pi}{2}\), for which also the entropy production \(\langle \Sigma \rangle\) achieves the maximum. Notice also that operating at zero entropy production (i.e. for \(N_A \rightarrow N_B\), thus approaching the Carnot efficiency) will produce a divergence in Eq. \((22)\). By combining Eqs. \((17)\) and \((22)\), independently of \(\theta\) we obtain the following exact relation

\[
\frac{\text{var}(W)}{(\langle W \rangle)^2} = 1 + \frac{1}{N_A(N_B + 1)} \frac{N_A + N_B + 2N_A N_B}{\langle \Sigma \rangle} \times \ln \left(\frac{N_A + 1}{N_A(N_B + 1)}\right). \tag{23}
\]

Then, reducing the noise-to-signal ratio associated to work extraction (or cooling performance) comes at a price of increased entropy production. This equation can be
conveniently rewritten as
\[
\frac{\text{var}(W)}{(W)^2} = 1 + \frac{h(x)}{(\Sigma)} \left( \beta_A \omega_A - \beta_B \omega_B \right),
\]
where \( h(x) = x \coth(x/2) \). Since \( h(x) \geq 2 \), the following thermodynamic uncertainty relation is always satisfied:
\[
\frac{\text{var}(W)}{(W)^2} \geq 1 + \frac{2}{(\Sigma)}.
\]
In Fig. 3 we plot the work variance and compare it with the bound obtained by Eq. (25), for fixed parameters \( \omega_A = 1, \beta_A = 1, \) and \( \beta_B = 2 \). Different from the two-qubit case studied in Ref. [54], we do not observe violation of the bound in Eq. (25). Indeed, the tightest saturable bound from Ref. [54]
\[
\frac{\text{var}(W)}{(W)^2} \geq f(\langle \Sigma \rangle),
\]
where \( f(x) = \cosh^2[g(x/2)] \) and \( g(x) \) denotes the inverse function of \( x \tanh(x) \), becomes quite loose for the present bosonic engine for \( \omega_B \ll \omega_A \) (see Fig. 3).

From Eqs. (18) and (25) we can write a relation between the average extracted work, fluctuations and efficiency
\[
\langle -W \rangle \leq \frac{\text{var}(W)}{2T_B} \left( \frac{\eta_c}{\eta} - 1 \right).
\]
This can also be written as a bound on the efficiency, determined by the average work and fluctuations, namely
\[
\eta \leq \frac{\eta_c}{1 + 2T_B\langle -W \rangle / \text{var}(W)}.
\]
In fact, this can also be understood by noting that the characteristic function has periodicity \( \frac{2\pi}{|\omega_A - \omega_B|} \) and \( \frac{2\pi}{\omega_A} \) in the variables \( \lambda \) and \( \mu \). The joint probability for work and heat is then given by
\[
p[W = m(\omega_A - \omega_B), Q_H = n\omega_A] = \frac{\omega_A|\omega_A - \omega_B|}{(2\pi)^2} \int_{-\frac{\omega_A - \omega_B}{2\pi}}^{\omega_A - \omega_B} d\lambda \int_{-\frac{\omega_A - \omega_B}{2\pi}}^{\omega_A - \omega_B} d\mu \chi(\lambda, \mu) e^{-i\lambda m(\omega_A - \omega_B) - imn\omega_A}
\]
where
\[
p[Q_H = n\omega_A] = \frac{1}{2\pi} \int_{0}^{2\pi} \{ 1 - \sin^2 \theta \{(N_A + N_B + 2N_A N_B\} - \mu \} \} e^{-i\mu n} d\mu.
\]
The solution of the integral in Eq. (30) is given in the Supplemental Material [72], thus providing the full distribution of the stochastic work and heat in closed form. This also allows to verify the detailed fluctuation theorem of Eq. (10).

From the form of Eq. (29), similarly to the case of the two-qubit swap engine [63], one recognizes that the efficiency is indeed a self-averaging quantity. In fact, in principle the efficiency \( \eta = \langle -W \rangle (\langle Q_H \rangle) \) is different from the expectation of the stochastic efficiency \( \eta_s = \langle -W/Q_H \rangle \). However, here we have for all moments \( \langle -W/Q_H \rangle^n = \langle -W/R \rangle^n = 1 - \frac{\omega_B}{\omega_A} \), namely there are no efficiency fluctuations.

In conclusion we have derived exact expressions for work and heat fluctuations pertaining to a two-mode bosonic Otto engine, where two quantum harmonic oscillators are alternately subject to a tunable unitary bilinear interaction allowing energy exchange and to thermal relaxation to their own reservoirs. We have identified the regimes when the protocol works as a heat engine, a refrigerator, or a thermal accelerator, and derived the characteristic function for work and heat. We have obtained thermodynamic uncertainty relations that shows the interdependence among average extracted work, fluctuations and entropy production, which hold in all range of coupling strength between the two quantum harmonic oscillators. Finally, we have provided the full joint distribution of the stochastic work and heat. Our results confirm the general meaning of TUR’s, namely that reducing the noise-to-signal ratio associated with a given current comes at a price of increased entropy production. In terms of efficiency, our results shows that its enhancement requires either sacrifice of the output work or increase of the fluctuations, thus reducing the engine constancy. The connection between fluctuation theorems and thermodynamic uncertainty relations represents a
significant advance in our understanding of nonequilibrium phenomena, and is relevant for the design of quantum thermodynamic machines, by posing strict bounds that relate work, heat, fluctuations, efficiency, and reliability.

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SUPPLEMENTAL MATERIAL

Probability for the stochastic work and heat

In order to obtain the probability for the stochastic work and heat we need to perform the integration in Eq. (30) of the main text. The integral can be solved by using the residue theorem, after posing $z = e^{i\mu}$ and integrating on the complex plane along the unit circle $\gamma$. We observe that

$$p(Q_H = n\omega_A) = \frac{1}{2\pi i} \oint_{\gamma} \left\{ 1 - \sin^2 \theta \{(N_A + N_B + 2N_A N_B)(z + z^{-1})/2 - 1 \} + i(N_A - N_B)(z - z^{-1})/(2i) \right\}^{-1} z^{-n} \frac{dz}{iz}.$$

For $n \leq 0$ the poles are easily evaluated as

$$z_{\pm} = \frac{1 + \sin^2 \theta(N_A + N_B + 2N_A N_B) \pm \sqrt{1 + 2\sin^2 \theta(N_A + N_B + 2N_A N_B) + (N_A - N_B)^2 \sin^4 \theta}}{2\sin^2 \theta N_A(N_B + 1)}.$$

We observe that

$$z_+ > \frac{1 + \sin^2 \theta[N_A + N_B + 2N_A N_B] + |N_A - N_B|}{2\sin^2 \theta N_A(N_B + 1)}.$$

Then, for $N_A \geq N_B$ clearly one has $z_+ > 1$. For $N_A < N_B$, one also has

$$z_+ > \frac{1 + 2\sin^2 \theta N_B(N_A + 1)}{2\sin^2 \theta N_A(N_B + 1)} > 1,$$

since $N_B > N_A > 0 \iff N_B(N_A + 1) > N_A(N_B + 1)$. Hence, the pole $z_+$ lies outside the unitary circle.

The residue for the first-order pole $z_-$ is given by

$$\text{Res} \left( \frac{z^n}{[1 + \sin^2 \theta(N_A + N_B + 2N_A N_B)]z - \sin^2 \theta[N_A(N_B + 1)z^2 + N_B(N_A + 1)]}, z_- \right)$$

$$= \frac{1}{[1 + \sin^2 \theta(N_A + N_B + 2N_A N_B)] - 2\sin^2 \theta N_A(N_B + 1)} \bigg|_{z=z_-}$$

$$= \frac{1}{\sqrt{1 + 2\sin^2 \theta(N_A + N_B + 2N_A N_B) + (N_A - N_B)^2 \sin^4 \theta}} \times \left( \frac{1 + \sin^2 \theta(N_A + N_B + 2N_A N_B) - \sqrt{1 + 2\sin^2 \theta(N_A + N_B + 2N_A N_B) + (N_A - N_B)^2 \sin^4 \theta}}{2\sin^2 \theta N_A(N_B + 1)} \right)^{|n|}.$$

For $n > 0$, we also have a $n$-order pole in $z = 0$. However, we can recast the integration as for the case $n < 0$ by the change of variable $\mu \to -\mu$, which is then equivalent to exchange $N_A$ with $N_B$. Hence, the probability for the
stochastic work and heat is given by

\[ p[Q_H = n\omega_A] = p[W = -n(\omega_A - \omega_B)] = \frac{1}{\sqrt{1 + 2\sin^2 \theta(N_A + N_B + 2N_AN_B)(N_A - N_B)^2\sin^4 \theta}} \]

\[ \times \left\{ \left( \frac{1 + \sin^2 \theta(N_A + N_B + 2N_AN_B)}{2\sin^2 \theta N_{(N_A + 1)}} \right)^{n} \right\} \quad \text{for } n \geq 0 , \]

\[ \left\{ \left( \frac{1 + \sin^2 \theta(N_A + N_B + 2N_AN_B)}{2\sin^2 \theta N_{(N_B + 1)}} \right)^{|n|} \right\} \quad \text{for } n < 0 . \]

In the case of the swap engine \( \theta = \frac{\pi}{2} \), one can directly derive the analytic expression for \( p[Q_H = n\omega_A] \) as follows

\[ p[Q_H = n\omega_A] = p[W = -n(\omega_A - \omega_B)] = \sum_{l,s=0}^{\infty} \text{Tr}(|l\rangle \langle l| \otimes I_B)U_{\pi/2}(|s\rangle \otimes \rho_{N_B})U_{\pi/2}^\dagger(|s\rangle \otimes \rho_{N_A})|s\rangle \delta_{n,s-l} \]

\[ = \sum_{l,s=0}^{\infty} \frac{1}{N_A + 1} \left( \frac{N_A}{N_A + 1} \right)^{s} \frac{1}{N_B + 1} \left( \frac{N_B}{N_B + 1} \right)^{l} \delta_{n,s-l} = \left\{ \left( \frac{N_A}{N_A + 1} \right)^{n} \right\} \quad \text{for } n \geq 0 , \]

\[ \left\{ \left( \frac{N_B}{N_B + 1} \right)^{|n|} \right\} \quad \text{for } n < 0 . \]

consistent with Eq. (A.36) for \( \theta = \frac{\pi}{2} \).

In Fig. 4 we report the work distribution for \( N_A = 8 \) and \( N_B = 2 \), pertaining to two different values of strength interaction, i.e. \( \theta = \pi/2 \) and \( \theta = \pi/4 \).

![Figure 4](image)

Figure 4. Distribution of work in \( \omega_A - \omega_B \) units, for \( N_A = 8 \) and \( N_B = 2 \), for interaction strength \( \theta = \pi/2 \) (left) and \( \theta = \pi/4 \) (right). By exchanging \( n \to -n \), the same histograms represent the distribution of heat released by the hotter reservoir in \( \omega_A \) units [see Eq. (A.37) and (A.36)].

The closed form for the probability of Eq. (A.36) allows one to explicitly verify the detailed fluctuation theorem, namely Eq. (10) of the main text, as follows

\[ \frac{p[W = -n(\omega_A - \omega_B), Q_H = n\omega_A]}{p[W = n(\omega_A - \omega_B), Q_H = -n\omega_A]} = \frac{p[Q_H = n\omega_A]}{p[Q_H = -n\omega_A]} = \left[ \frac{N_A(N_B+1)}{N_B(N_A+1)} \right]^n \]

\[ = e^{(\beta_B\omega_B - \beta_A\omega_A)n} = e^{(\beta_B - \beta_A)n}(\omega_A - \omega_B) = e^{(\beta_B - \beta_A)Q_H + \beta_B W} . \]