MAXIMAL DOMAINS OF RADIAL HARMONIC FUNCTIONS

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Abstract. We examine the maximal domain of radial harmonic functions on harmonic spaces in the context of positive, zero, and negative curvature.

1. Introduction

Throughout this paper, let \( M := (M, g) \) be a connected Riemannian manifold of dimension \( m \geq 4 \). Let \( \iota_{M,P} \) be the injectivity radius at \( P \), let \( r_{M,P}(Q) \) be the geodesic distance from \( P \) to \( Q \), and let

\[
B_{M,P} := \{ Q \in M : r_{M,P}(Q) < \iota_{M,P} \}
\]

be the geodesic ball centered at \( P \) of radius \( \iota_{M,P} \). Let \( \bar{x} \rightarrow \exp_P(x^1 e_1 + \cdots + x^m e_m) \) define geodesic coordinates centered at \( P \) where \( e_j = (e_1, \ldots, e_m) \) is a orthonormal frame for \( T_PM \) and \( \bar{x} \in \mathbb{R}^m \) satisfies \( \|\bar{x}\| < \iota_{M,P} \). The geodesic distance from \( P \) is then given by the ordinary Euclidean distance, i.e.

\[
r_{M,P}(\bar{x}) = \|\bar{x}\| = \sqrt{(x^1)^2 + \cdots + (x^m)^2}
\]

on \( B_{M,P} \).

Let \( d\bar{x} = dx^1 \cdots dx^m \) be the Euclidean measure and let \( d\text{vol}_M \) be the Riemannian measure on \( M \). If \( g_{ij} := g(\partial_{x^i}, \partial_{x^j}) \), then

\[
d\text{vol}_M = \tilde{\Theta}_{M,P} dx^1 \cdots dx^m \quad \text{where} \quad \tilde{\Theta}_{M,P} := \det(g_{ij})^{1/2}
\]

is the volume density function. Let \( S_{P}^{m-1} := \{ \tilde{\theta} \in T_PM : \|\tilde{\theta}\| = 1 \} \) be the unit sphere in \( T_PM \) and let \( S_{P}^{m-1} := (S_{P}^{m-1}, g_S) \) where \( g_S \) is the induced Euclidean metric. Introduce geodesic polar coordinates \((r, \tilde{\theta})\) to express

\[
\bar{x} = r_{M,P}(\bar{x}) \tilde{\theta}(\bar{x})
\]

for \( 0 < r_{M,P}(\bar{x}) : = \|\bar{x}\| < \iota_{M,P} \) and \( \tilde{\theta}(\bar{x}) = \|\bar{x}\|^{-1} \bar{x} \in S_{P}^{m-1} \). We may also express

\[
d\text{vol}_M = \Theta_{M,P} dr d\text{vol}_{S_{P}^{m-1}} \quad \text{for} \quad \Theta_{M,P} := r_{m-1} \tilde{\Theta}_{M,P}.
\]

We say that a smooth function \( f \), which is defined near \( P \), is radial if there exists a smooth function \( \eta_1 \) of one real variable so \( f(\bar{x}) = \eta_1(\|\bar{x}\|) \); \( f \) is smooth at \( P \) if and only if we can write \( f(\bar{x}) = \eta_2(\|\bar{x}\|^2) \) or, equivalently, \( \eta_1 \) is an even function. We say that \( M \) is central harmonic about \( P \) if \( \Theta_{M,P} \) or, equivalently, if \( \Theta_{M,P} \) is a radial function. We say that \( M \) is a harmonic space if \( M \) is central harmonic about every point.

There is a vast literature on this subject; we refer to [52 55 S11 12 13] and the references cited therein for further details. Note that if \( M \) is a harmonic space, then we can rescale the metric to replace \( g \) by \( c^2 g \) for any \( c > 0 \) to obtain another harmonic space \( M_c := (M, c^2 g) \). Similarly, we showed previously [5] that if \( M \) is central harmonic at \( P \) and if \( \psi \) is a smooth positive radial function, then the radial conformal deformation \( M_\psi := (M, \psi^2 g) \) is again central harmonic at \( P \). Thus we can construct a space which is central harmonic at a single point by taking a radial

\[\text{Volume Density Function,} \quad \text{Radial Harmonic Function,} \quad \text{Rank 1 Symmetric Space}.\]
conformal deformation of a harmonic space. There are, however, examples of spaces which are central harmonic at some point which do not arise in this fashion [9].

1.1. Radial harmonic functions. One can show that \( \mathbb{M} \) is central harmonic about \( P \) if and only if there exists a non-constant radial harmonic function \( \phi \) with domain \( B_{\mathbb{M}, P} = \{ P \} \). We will usually assume \( \mathbb{M} \) is real analytic; using analytic hypoellipticity (see, for example, the discussion in Treves [14]) this implies \( \phi \) is real analytic; we will use this fact implicitly in much of what follows. In this paper, we will be concerned with determining a maximal connected domain of \( \phi \) as often \( \phi \) has a natural extension beyond \( B_{\mathbb{M}, P} = \{ P \} \).

1.2. Examples of harmonic spaces. Let \( \rho_{ik} := g^{ik}R_{ijkl} \) be the components of the Ricci tensor and let \( \tau := g^{ik}\rho_{ik} \) be the scalar curvature where we adopt the Einstein convention and sum over repeated indices. The rank 1 symmetric spaces play a central role in the subject; they are all harmonic spaces. We give below, up to rescaling, the rank 1 symmetric spaces with \( \tau > 0 \) in Section 1.4 and the rank 1 symmetric spaces with \( \tau < 0 \) in Section 1.6. All known examples of complete harmonic spaces are real analytic.

Let \( \mathbb{M} \) be a complete simply connected harmonic space; the scalar curvature \( \tau \) is then constant. We can always rescale to replace \( \tau \) by \( c^2 \tau \) so only the sign of \( \tau \) is relevant and it is natural to study those cases separately. If \( \tau > 0 \), then \( \mathbb{M} \) is positively curved and, after a suitable rescaling of the metric, the local geometry is modeled on a positively curved rank 1 symmetric space. If \( \tau = 0 \), then \( \mathbb{M} \) is flat. If \( \tau < 0 \), the known examples are the negatively curved rank 1 symmetric spaces and the Damek–Ricci spaces defined in [6]; the Damek–Ricci spaces are homogeneous negatively curved harmonic spaces which need not be rank 1 symmetric spaces. The classification of simply connected complete harmonic spaces is not yet finished in the negatively curved setting and it is not known if there are additional negatively curved harmonic spaces.

1.3. Radial harmonic functions. Let \( \Delta^0 \) be the Laplace-Beltrami operator on functions. If \( \mathbb{M} \) is central harmonic at \( P \) and if \( \phi \) is a radial function, then

\[
\Delta^0 \phi = - (\partial_r + \Theta_{\mathbb{M}, P}^{-1} \hat{\Theta}_{\mathbb{M}, P}) \partial_r \phi.
\]

Let \( \phi_1 = \Theta_{\mathbb{M}, P}^{-1} \) and let \( \phi_0 \) solve the ODE \( \partial_r \phi_0 = \phi_1 \). Then

\[
\Delta^0 \phi_0 = - (\partial_r + \Theta_{\mathbb{M}, P}^{-1} \hat{\Theta}_{\mathbb{M}, P}) \phi_1 = \Theta_{\mathbb{M}, P}^{-2} \hat{\Theta}_{\mathbb{M}, P} - \Theta_{\mathbb{M}, P}^{-1} \hat{\Theta}_{\mathbb{M}, P} \Theta_{\mathbb{M}, P}^{-1} = 0.
\]

Consequently, \( \phi_0 \) is a non-constant radial harmonic function; since \( \Theta_{\mathbb{M}, P} \) vanishes at \( P \), \( \phi_0 \) is singular at \( P \). If \( \phi \) is any radial solution to the equation \( \Delta^0 \phi = 0 \) on \( B_{\mathbb{M}, P} \), then we may express

\[\phi = a\phi_0 + b.\]

1.4. The rank 1 symmetric spaces with positive curvature. Let \( \mathbb{S}^m \) be the unit sphere in \( \mathbb{R}^{m+1} \), let \( \mathbb{CP}^k \) be complex projective space, let \( \mathbb{HP}^k \) be quaternionic projective space, and let \( \mathbb{OP}^2 \) be the Cayley projective plane. We give these spaces the standard metrics normalized so

| \( \mathbb{M} \) | dimension | diameter | \( \Theta_{\mathbb{M}, P} \) |
|---|---|---|---|
| \( \mathbb{S}^m \) | \( m \) | \( \pi \) | \( \sin(r)^m \) |
| \( \mathbb{CP}^k \) | 2k | \( \frac{1}{2}\pi \) | \( \sin(r)^{2k-1} \cos(r) \) |
| \( \mathbb{HP}^k \) | 4k | \( \frac{1}{2}\pi \) | \( \sin(r)^{4k-1} \cos^3(r) \) |
| \( \mathbb{OP}^2 \) | 16 | \( \frac{1}{2}\pi \) | \( \sin(r)^{15} \cos^7(r) \) |
The metric on $S^m$ is the standard metric inherited from Euclidean space, the metric on $\mathbb{C}P^k$ is the suitably normalized Fubini-Study metric, and so forth. The rank 1 symmetric spaces in positive curvature are compact 2 point homogeneous spaces with $B_{M,P} = M - C_{M,P}$ where $C_{M,P}$ is the cut-locus:

$$C_{S^m,P} = \{-P\}, \quad C_{\mathbb{C}P^k,P} = \mathbb{C}P^{k-1}, \quad C_{\mathbb{H}P^k,P} = \mathbb{H}P^{k-1}, \quad C_{\mathbb{O}P^2,P} = S^7. $$

Let $\tilde{M}$ be a complete connected harmonic space of positive curvature which is not simply connected. Let $\tilde{M}_1$ be the universal cover of $\tilde{M}$. Then $\tilde{M}_1$ is a positively curved rank 1 symmetric space. We will establish the following result in Section 2 which applies to the harmonic spaces of positive curvature but is more general since no assumption on the metric is made. Although it is well known, we shall present a proof in Section 2 based on the Gauss-Bonnet formula (see Theorem 2.2) as it motivates a number of examples that we will discuss in Section 2.

**Theorem 1.1.** Let $\tilde{M} = (\tilde{M}, \tilde{g})$ be the universal cover of a smooth connected complete Riemannian manifold $M = (M, g)$ which is not simply-connected.

1. If $\tilde{M} = S^m$ and if $m$ is even, then $\pi_1(M) = \mathbb{Z}_2$ and $M$ is not orientable.
2. If $\tilde{M} \in \{\mathbb{C}P^k, \mathbb{H}P^k\}$, then $k$ is odd, $\pi_1(M) = \mathbb{Z}_2$, and $M$ is not orientable.
3. $\tilde{M}$ is not orientable.

### 1.5. The maximal domain of $\phi_0$ in non-positive curvature.

Let $\tilde{M}$ be a complete connected Riemannian manifold and let $\pi: \tilde{M} \to \hat{M}$ be the universal cover; we assume $\hat{M}$ is not simply connected so $\pi$ is nontrivial. We assume that $\tilde{M}$ is a non-positively curved harmonic space. Since $\hat{M}$ is non-positively curved, the exponential map is a covering projection so we can identify $\tilde{M}$ with $T_P \hat{M}$ and $\pi$ with $\exp_P$ for any point $P$ of $\hat{M}$. There are no conjugate points in $\hat{M}$ or $\tilde{M}$; there is a unique geodesic segment minimizing distance between any two points of $\tilde{M}$ but there can be several such geodesic segments in $\hat{M}$. Set $\Gamma := \pi_1(M)$. Then $\Gamma$ acts on $\tilde{M}$ by isometries and we may identify $\hat{M} = \tilde{M}/\Gamma$. If $\hat{P} \in \tilde{M}$, then $\Gamma \cdot \hat{P}$ is a discrete subset of $\tilde{M}$ so we can replace inf by min to define

$$\tilde{O}_{\hat{P}} := \left\{ \hat{Q} \in \tilde{M} : d_{\tilde{g}}(\hat{P}, \hat{Q}) < \min_{\gamma \in \Gamma} d_{\tilde{g}}(\hat{P}, \gamma \hat{Q}) \right\}. $$

Since $\tilde{O}_{\hat{P}} = \gamma \tilde{O}_{\hat{P}}, \pi(\tilde{O}_{\hat{P}})$ is independent of the particular point in $\pi^{-1}(P)$ which is chosen and we may define

$$O_P := \pi(\tilde{O}_{\hat{P}}) \quad \text{for any} \quad \hat{P} \in \pi^{-1}(P).$$

Because we can use min instead of inf in defining $\tilde{O}_{\hat{P}}, \tilde{O}_{\hat{P}}$ is an open subset of $\tilde{M}$ and $\pi : \tilde{O}_{\hat{P}} \to O_P$ is a 1-1, onto, isometric map. Clearly $O_P$ is the set of all points in $\hat{M}$ so there is a unique geodesic segment from $P$ to the point in question minimizing the distance. If we replace strict inequality by equality, we obtain

$$\text{bd}(\tilde{O}_{\hat{P}}) = \left\{ \hat{Q} \in \tilde{M} : d_{\tilde{g}}(\hat{P}, \hat{Q}) = \min_{\gamma \in \Gamma} d_{\tilde{g}}(\hat{P}, \gamma \hat{Q}) \right\}. $$

Since the cut locus of $\tilde{M}$ is the set of all points where there are 2 distinct minimizing geodesic segments from $P$ to the point in question, we obtain

$$\text{bd}(O_P) = \pi \left\{ \text{bd}(\tilde{O}_{\hat{P}}) \right\} = C_P. $$

Since the cut-locus has measure 0 in $\hat{M}$ and since $M = \overline{O_P}$, we obtain

$$\text{vol}(M) = \text{vol}(O_P). \quad \text{(1.b)}$$

It is a straightforward to see that

$$l_{M,P} = \frac{1}{2} \min_{\gamma \in \Gamma} d_{\tilde{g}}(\hat{P}, \gamma \hat{P}) \quad \text{for any} \quad \hat{P} \in \pi^{-1}(P). \quad \text{(1.c)}$$
Note that $B_{M, P} \subset O_P$ and in general $O_P$ is a bigger open set. Let $\phi_M$ be a non-constant harmonic function on $\tilde{M} - \{\tilde{P}\}$ which is radial from $\tilde{P}$. Let

$$\phi_M := \phi_M \circ \left\{ \pi_{O_P} \right\}^{-1}.$$ 

The following result shows that $O_P = M - C_P$ is the maximal domain in $M$ which admits a non-constant harmonic function which is radial from $P$.

**Theorem 1.2.** Let $\tilde{M}$ be a non-positively curved complete connected harmonic space. Adopt the notation established above.

1. $\phi_M$ is a non-constant function on $O_P - \{P\}$ which is smooth harmonic, and radial from $P$.

2. If $O$ is an open set in $M$ which contains a point of $C_P$, then $O$ does not admit a smooth non-constant harmonic function which is radial from $P$.

**Proof.** Since $\pi$ is a local isometry, $\phi_M$ is harmonic. Since $r_{\tilde{M}, \tilde{P}}(\tilde{Q}) = r_{\tilde{M}, \pi(P)}(\pi(Q))$ for $\tilde{Q} \in \tilde{O}_{\tilde{P}}$ and since $\tilde{\phi}$ is radial from $\tilde{P}$ in $\tilde{M}$, we see that $\phi$ is a radial function from $P$ on the open set $O_P$. This proves Assertion (1). Since $r_{\tilde{M}, \tilde{P}}$ is Lipschitz but not smooth on the cut-locus. Assertion (2) follows.

There are other open sets to which $\phi_0$ can be extended as a smooth harmonic function but $\phi_0$ will no longer be radial there; we refer to Remark 3.1 for details.

1.6. **The rank 1 symmetric spaces of negative curvature.** There are negative curvature duals of the spaces discussed in Section 1.4 that we shall denote by $\tilde{S}^m$ (hyperbolic space), $\tilde{C}P^k$ (complex hyperbolic space), $\tilde{H}P^4$ (quaternionic hyperbolic space), and $\tilde{O}P^2$ (Cayley hyperbolic plane). These are the rank 1 symmetric spaces of negative curvature; they are all 2-point homogeneous spaces and are geodesically complete. The curvature tensor of these spaces is obtained by reversing the sign of the curvature tensor of the corresponding positive curvature example. This fact will play an important role in the discussion of Section 4. Finally, we note that any simply-connected 2-point homogeneous space is either flat or is a rank 1 symmetric space.

If $M$ is a rank 1 symmetric space with negative curvature, then the exponential map is a global diffeomorphism so the underlying topology of all these spaces is Euclidean space; the cut locus is empty. We adopt the same normalizations as those used to normalize the positive curvature examples. We replace sin by sinh and cos by cosh in Equation (1.a) to obtain

| $M$ | Dimension | $\Theta_{M, P}$ |
|-----|-----------|-----------------|
| $\tilde{S}^m$ | $m$ | $\sinh(r)^{m-1}$ |
| $\tilde{C}P^k$ | $2k$ | $\sinh(r)^{2k-1} \cosh(r)$ |
| $\tilde{H}P^4$ | $4k$ | $\sinh(r)^{4k-1} \cosh^3(r)$ |
| $\tilde{O}P^2$ | 16 | $\sinh(r)^{15} \cosh^7(r)$ |

We say that $M$ is modeled on a homogeneous space $M_1$ if every point of $M$ has a neighborhood which is isometric to some open set in $M_1$. We will establish the following result in Section 1.7 which gives lower bounds on the volumes of compact spaces modeled on rank 1 symmetric spaces. It is not difficult to show that $O_P - \{P\}$ is of full measure in $M$ and, consequently, by Theorem 1.2 and Equation (1.b), we obtain on the volume of the maximal domain of $\phi_0$. We refer to Cahn et. al. [3] for a more general discussion of lower volume bounds on even dimensional negatively curved symmetric spaces.
Theorem 1.3. Let $M$ be a compact Riemannian manifold.

1. If the geometry of $M$ is modeled on $\tilde{S}^{2k}$, then $\text{vol}(M) \geq \frac{1}{2} \text{vol}(S^{2k})$.

2. If the geometry of $M$ is modeled on $\tilde{CP}^{k}$, then $\text{vol}(M) \geq \frac{1}{k+1} \text{vol}(CP^{k})$.

3. If the geometry of $M$ is modeled on $\tilde{HP}^{k}$, then $\text{vol}(M) \geq \frac{1}{k+1} \text{vol}(HP^{k})$.

4. If the geometry of $M$ is modeled on $\tilde{OP}^{2}$, then $\text{vol}(M) \geq \frac{1}{3} \text{vol}(OP^{2})$.

The estimates of Theorem 1.3 will arise from the Chern-Gauss-Bonnet Formula (see Theorem 2.2) and are not optimal in certain settings. We will use the Hirzebruch Signature Formula (see Theorem 2.2) to establish the following result in Section 4.3.

Theorem 1.4. Let $M$ be a compact Riemannian manifold. Let

$\varepsilon(M) := \begin{cases} 1 & \text{if } M \text{ is orientable} \\ \frac{1}{2} & \text{if } M \text{ is not orientable} \end{cases}$.

1. If the geometry of $M$ is modeled on $\tilde{CP}^{2k}$, then $\text{vol}(M) \geq \varepsilon(M) \text{vol}(CP^{2k})$.

2. If the geometry of $M$ is modeled on $\tilde{HP}^{2k}$, then $\text{vol}(M) \geq \varepsilon(M) \text{vol}(HP^{2k})$.

3. If the geometry of $M$ is modeled on $\tilde{OP}^{2}$, then $\text{vol}(M) \geq \varepsilon(M) \text{vol}(OP^{2})$.

1.7. Outline of the paper. In Section 2, we discuss the positively curved rank 1 symmetric spaces. We exhibit quite explicitly the non-constant harmonic function $\phi_0$ on the punctured disk $B_{M,P} - \{P\}$ for some low dimensional examples and complete the proof of Theorem 1.1. We discuss in some detail the domain of $\phi_0$ on a homogeneous lens space $S^{2k+1}/\mathbb{Z}_4$ and a quotient $CP^{2k}/\mathbb{Z}_2$. In Section 3, we turn our attention to flat space and discuss the rectangular torus and the infinite mobius strip. In Section 4, we exhibit the non-constant harmonic function $\phi_0$ in low dimensions for the negatively curved symmetric spaces and we complete the proof of Theorems 1.3 and 1.4.

The basepoint $P$ of the manifold will be fixed for the most part. To simplify the notation, we shall suppress the dependence in the notation on the point $P$ and set $r_M := r_{M,P}$, $\Theta_M := \Theta_{M,P}$, $\iota_M := \iota_{M,P}$ and so forth when no confusion is likely to ensue.

2. Positive curvature

Let $M$ be central harmonic at $P$. In Section 1.3, we discussed the construction of a non-constant radial harmonic function $\phi_0$ on $B_{M,P}$. In Section 2.1, we exhibit this function for the rank 1 symmetric spaces of positive curvature in low dimensions. In Section 2.2, we turn our attention to the study of non-simply connected examples $M/\Gamma$ where $M$ is a positively curved rank 1 symmetric space and $\Gamma$ is a finite group of isometries acting without fixed points to complete the proof of Theorem 1.1. In Section 2.3, we discuss spherical space forms and construct a homogeneous example $S^{2k+1}/\mathbb{Z}_4$ for which we give the maximal domain of $\phi_0$. In Section 2.4, we give an isometric fixed point free action of $\mathbb{Z}_2$ on $CP^{2k+1}$ and discuss the maximal domain of $\phi_0$ on $CP^{2k+1}/\mathbb{Z}_2$.

2.1. Simply-connected examples. We use Mathematica to determine $\phi_1$ and $\phi_0$ in low dimensions for the rank 1 symmetric spaces of positive curvature.
For the sphere, since \( \sin(r)^{1-m} \sim (\pi - r)^{1-m} \) as \( r \to \pi \), \( \phi_1 \) is not integrable on \((\pi - \varepsilon, \pi)\). Consequently, \( \phi_0 \) does not extend to the antipode and \( B_{M,P} - \{P\} \) is the natural domain of definition. For the remaining rank 1 symmetric spaces, since \( \phi_1 = \sin(\pi - r)^{m-1} \cos(\pi - r)^{-k} \) is not integrable on \((\frac{1}{2} \pi - \varepsilon, \frac{1}{2} \pi)\) for \( k > 0 \), \( \phi_0 \) tends to infinity as \( r \to \frac{1}{2} \pi \) and again the natural domain of \( \phi_0 \) is \( B_{M,P} - \{P\} = M - C \).

### 2.2. Non simply-connected examples: The proof of Theorem 1.1

Theorem 1.1 follows from Theorem 9.3.1 of Wolf [16]. However, to keep this discussion as self-contained as possible and to provide some examples, we shall give a different discussion based on the Chern-Gauss-Bonnet Formula for the most part; an exception being the case of \( \mathbb{H}P^{2\ell+1} \) for \( \ell > 1 \) where we do not know an elementary proof that \( \mathbb{H}P^{2\ell+1} \) does not admit a finite group acting without fixed points. We note that since we are in positive curvature, Synge’s Theorem is relevant although it does not give the full result.

Let \( M \) be a compact Riemannian manifold of dimension \( m \). If \( m = 2j \), let \( \chi(M) \) be the Euler characteristic of \( M \) and, if \( m = 4k \) and if \( M \) is orientable, let \( \text{sign}(M) \) be the signature of \( M \). We recall the following well known result:

**Lemma 2.1.**

(1) The cohomology rings of \( CP^k \), \( HP^k \), and \( OP^2 \) are truncated polynomial rings on generators \( x_i \) of degree \( i \):

\[
H^*(CP^k) = \mathbb{R}[x_2]/(x_2^{k+1} = 0) = \mathbb{R} \oplus \mathbb{R} \cdot x_2 \oplus \cdots \oplus \mathbb{R} \cdot x_2^k,
\]

\[
H^*(HP^k) = \mathbb{R}[x_4]/(x_4^{k+1} = 0) = \mathbb{R} \oplus \mathbb{R} \cdot x_4 \oplus \cdots \oplus \mathbb{R} \cdot x_4^k,
\]

\[
H^*(OP^2) = \mathbb{R}[x_4]/(x_4^2 = 0) = \mathbb{R} \oplus \mathbb{R} \cdot x_8 \oplus \mathbb{R} \cdot x_8^2.
\]

(2) The Euler characteristic and the Hirzebruch signature are given by:

| \( M \) | \( \chi(M) \) | \( \text{sign}(M) \) | \( M \) | \( \chi(M) \) | \( \text{sign}(M) \) |
|------|------|------|------|------|------|
| \( S^{4k+2} \) | 2 | - | \( S^{4k} \) | 2 | 0 |
| \( CP^{2k+1} \) | \( 2k + 2 \) | | \( CP^{2k} \) | \( 2k + 1 \) | 1 |
| \( HP^{2\ell+1} \) | \( 2k + 2 \) | | \( HP^{2\ell} \) | \( 2k + 1 \) | 1 |
| \( OP^2 \) | 3 | 1 |
Let $\text{Pf}_j$ be the Pfaffian and let $L_k$ be the Hirzebruch polynomial in the curvature of $M$. We refer to Chern \[4\] for the proof of Assertion (1) and to Hirzebruch \[10\] for the proof of Assertion (2) in the following result.

**Theorem 2.2.** Let $\mathcal{M}$ be a compact Riemannian manifold of dimension $m$. Let $\pi: \mathcal{M}_1 \to \mathcal{M}$ be a finite $\ell$-fold Riemannian cover.

1. If $m = 2k$, then $\chi(M) = \int_M \text{Pf}_k(M)$ and thus $\chi(M_1) = \ell \cdot \chi(M)$.
2. If $m = 4k$ and if $M$ is orientable, then $\text{sign}(M) = \int_M L_k(M)$ and thus $\text{sign}(M_1) = \ell \cdot \text{sign}(M)$.

Theorem \[1.1\] will follow from the following result.

**Lemma 2.3.** Let $\Gamma$ be a finite group which acts without fixed points on a compact manifold $M$ of dimension $m = 2\ell \cdot n$. Assume that

$$H^*(M) = \mathbb{R}[x_{2\ell}]/(x_{2\ell}^{n+1} = 0) = \mathbb{R} \oplus \mathbb{R} x_{2\ell} \oplus \cdots \oplus \mathbb{R} x_{2\ell}^{n}$$

(2.a)

is a truncated polynomial ring where $x_{2\ell} \in H^{2\ell}(M)$.

1. If $n$ is even, then $\Gamma$ is trivial.
2. If $n$ is odd, then either $\Gamma$ is trivial or $\Gamma = \mathbb{Z}_2$ and $\mathcal{M}/\Gamma$ is not orientable.
3. If $M \in \{\mathbb{C}P^{2k}, \mathbb{H}P^{2k}\}$ for $k \geq 1$ or $M = \mathbb{O}P^2$, then $\Gamma$ is trivial.
4. If $M \in \{\mathbb{C}P^{2k+1}, \mathbb{H}P^{2k+1}\}$, then either $\Gamma$ trivial or $\Gamma = \mathbb{Z}_2$ and $\mathcal{M}/\Gamma$ is not orientable.

**Proof.** By averaging a Riemannian metric on $M$ over the group $\Gamma$, we may assume without loss of generality that $\Gamma$ acts by isometries. Since $x_{2\ell}^j$ has even degree for all $j$, Equation (2.2) yields $\chi(M) = n + 1$. By Theorem 2.2, $\chi(M) = |\Gamma| \cdot \chi(M/\Gamma)$ and consequently $|\Gamma|$ divides $n + 1$. Let $T \in \Gamma$. We may express $T^* x_{2\ell} = \varepsilon(T) x_{2\ell}$ where $\varepsilon(T) \in \mathbb{R}$. Since $T|\Gamma| = \text{id}$ and since $|\Gamma|$ divides $n + 1$, $T^{n+1} = \text{id}$. We show $\varepsilon(T) = \pm 1$ by computing:

$$x_{2\ell} = T^* (\text{id}) x_{2\ell} = (T^*)^{n+1} x_{2\ell} = \varepsilon^{n+1}(T) x_{2\ell}.$$  

Suppose that $n$ is even. Since $\varepsilon(T)^{n+1} = 1$, this implies $\varepsilon(T) = 1$ and consequently $T^* x_{2\ell} = x_{2\ell}$ for all $j$. Since the cohomology of $M/\Gamma$ can be identified with the $\Gamma$-invariant cohomology of $M$, $H^*(M/\Gamma) = H^*(M)$ and consequently $\chi(M/\Gamma) = \chi(M)$. This implies $\Gamma$ is trivial and completes the proof of Assertion (1).

Suppose that $n$ is odd and $\Gamma$ is non-trivial. The map $T \to \varepsilon(T)$ is a group homomorphism from $\Gamma$ to $\mathbb{Z}_2 = \{ \pm 1 \}$. Let $G_0 = \ker(\varepsilon)$ and let $M_0 := M/G_0$. The same argument given to prove Assertion (1) now shows $M_0 = M$ and hence $\ker(\varepsilon)$ is trivial. This implies $\Gamma = \mathbb{Z}_2$ and that if $T \neq \text{id}$, $T^* x_{2\ell} = -x_{2\ell}$. We then have $T^* (x_{2\ell}^n) = -x_{2\ell}^n$ so $T$ reverses the orientation and $M/\Gamma$ is not orientable. This completes the proof of Assertion (2). Assertions (3) and (4) then follow from Assertion (1) and Lemma 2.1. \[2.1\]

2.3. **Spherical space forms.** We shall say that $\mathcal{M}$ is a spherical space form if $\mathcal{M}$ has constant sectional curvature $+1$ or, equivalently, if the universal cover of $\mathcal{M}$ is the sphere $S^m$ for $m \geq 2$.

Let $\Gamma \subset O(m + 1)$ be a non-trivial finite subgroup of the orthogonal group acting by isometries on $S^m$ without fixed points. If $m$ is even, then $\chi(S^m) = 2$ so $|\Gamma|$ divides 2 by Theorem 2.2 and hence $|\Gamma| \geq 2$ since $\Gamma$ is assumed non-trivial. Let $\gamma$ be the non-trivial element of $\Gamma$. Since $\gamma^2 = \text{id}$ and $\gamma$ is fixed point free, $\gamma = -\text{id}$ and $\mathcal{M} = \mathbb{R}P^m$. This is a homogeneous space, so the point in question is irrelevant. The diameter of $\mathbb{R}P^m$ is $\frac{\pi}{2}$, the cut-locus is $\mathbb{R}P^{m-1}$, $\iota_{\mathbb{R}P^m} = \frac{\pi}{2}$, and $B_{\mathbb{R}P^m}$ is the maximal domain of definition for $\phi_0$. 


If $m = 2n + 1$, the situation is very different. The possible groups and group actions which can arise have been classified by Wolf [16]. There is one example which will play an important role in our future development.

**Example 2.4.** If we identify $\mathbb{R}^{2k+2} = \mathbb{C}^{k+1}$ and $\mathbb{Z}_4 = \{\pm 1, \pm i\}$, then complex multiplication defines a fixed point free isometric action of $\mathbb{Z}_4$ on $S^{2k+1}$ and we let $M = S^{2k+1}/\mathbb{Z}_4$. Alternatively, let $T$ generate $\mathbb{Z}_4$. We can define the action in purely real coordinates by setting

$$T(x_1, x_2, \ldots, x_{2k+2}) = (-x_2, x_1, \ldots, -x_{2k+2}, x_{2k+1}).$$

(2.4)

We then have a sequence of 2-fold Riemannian covering projections $S^{2k+1} \to \mathbb{R}P^{2k+1}$ and $\mathbb{R}P^{2k+1} \to M$ so we can equivalently regard $M = \mathbb{R}P^{2k+1}/\mathbb{Z}_2$. The unitary group acts transitively on $S^{2k+1}$ by isometries and commutes with this action of $\mathbb{Z}_4$ so $\mathbb{R}P^{2k+1}$ and $M$ are homogeneous spaces. We take as basepoint $P = (1, 0, \ldots, 0)$ in $\mathbb{R}^{2k+2}$. Let $\xi$ be a unit vector which is perpendicular to $P$. $\sigma(t) := \cos(t)P + \sin(t)\xi$ is a unit speed geodesic from $P$ with $\sigma(0) = \xi$. It now follows that

$$r_{\xi}^{2k+1, P}(x^1, \ldots, x^{2k+2}) = \arccos(x^1) \quad \text{so}$$

$$B_{\xi, P} = \{\vec{x} \in S^m : x^1 > x^2\} \cap \{\vec{x} \in S^m : x^1 > -x^2\};$$

the condition $x^1 > 0$ imposed by $T^2$ is then immediate. This is the intersection of the two hemispheres centered at $\left(\frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}, 0, \ldots, 0\right)$; $\nu_{\xi, P} = \arccos(\frac{1}{\sqrt{2}}) = \frac{\pi}{4}$. We present a picture below (where the $x^1$ axis is vertical) of what results when we take a slice by setting $x^4 = \cdots = x^{2k+2} = 0$; this picture motivates the terminology lens spaces. The 90 degree rotation in the $(x^1, x^2)$ plane identifies one great circle in boundary of the fundamental domain with the other. The fundamental domain is in light blue; the ball of radius $\frac{\pi}{4}$ about the north pole is in dark blue.

**Picture 2.1**

**Remark 2.5.** Let $M = S^{2k+1}/\Gamma$ where $\Gamma$ is a finite subgroup of the orthogonal group $O(2k+2)$. If $M$ is homogeneous, then, up to conjugation, either $\pi_1(M)$ is the group of units of order $q$ in $\mathbb{C}$ acting on $S^{2k+1} \subset \mathbb{C}^{k+1}$ by complex multiplication or $m \equiv 3 \mod 4$ and the group in question is one of 4 exceptional groups; the spherical space forms are rarely homogeneous. We refer to Theorem 11.6.1 of Wolf [15] for details. Similarly, we refer to Wolf [15] Corollary 10.6 to see that if $M$ is a connected homogeneous manifold of constant negative sectional curvature, then $M$ is hyperbolic space. Thus the examples $\mathbb{S}^m/\Gamma$ are never homogeneous.

2.4. **An isometric fixed point free action of $\mathbb{Z}_2$ on $\mathbb{C}P^{2k+1}$.** Lemma 2.3 is a theorem in topology; no assumption on the metric is assumed. It leaves open the question of finding fixed point free actions of $\mathbb{Z}_2$ on $\mathbb{C}P^{2k+1}$ and $\mathbb{H}P^{2k+1}$ for $k \geq 1$.

In the following extended example, we construct an isometric fixed point free action of $\mathbb{Z}_2$ on $\mathbb{C}P^{2k+1}$ by generalizing the construction of Example 2.4. We note that Theorem 9.3.1 of Wolf [16] shows that there is no fixed point free action isometric action of $\mathbb{Z}_2$ on $\mathbb{H}P^{2k+1}$ for $k \geq 1$; we know of no elementary proof of this fact.

Let $\vec{z} = (z^1, \ldots, z^{2k+2}) \in S^{4k+3} \subset \mathbb{C}^{2k+2}$ and let $\langle \vec{z} \rangle$ denote homogeneous coordinates on $\mathbb{C}P^{2k+1}$ where we identify $\langle \vec{z} \rangle$ with $\langle \mu \vec{z} \rangle$ for $\mu \in S^1 \subset \mathbb{C}$. We can
construct a fixed point free isometry $T$ of order 4 on $S^{4k+3} \subset \mathbb{C}^{2k+1}$ by generalizing Equation (2.b) to define

$$T(\bar{z}) = (-\bar{z}_2, \bar{z}_1, \ldots, -\bar{z}_{2k+2}, \bar{z}_{2k+1}).$$

Since $T(\bar{z}) = cT(\bar{z})$, $T$ descends to $\mathbb{C}P^{2k+1}$ and we can define

$$\langle T \rangle(\langle \bar{z} \rangle) = \langle T\bar{z} \rangle.$$

Since $T^2(\bar{z}) = -\bar{z}$, $\langle T \rangle^2(\langle \bar{z} \rangle) = \langle \bar{z} \rangle$ and $(T)$ is idempotent. If we assume $T$ has a fixed point projectively, then $z_1 = -cz_2$ and $z_2 = cz_1 = -|c|^2z_2$ for $c \neq 0$. Thus $z_1 = z_2 = 0$ and we conclude inductively $\langle \bar{z} \rangle = (\bar{0})$ which is, of course not possible. We take, therefore, $\bar{M} = \mathbb{C}P^{2k+1}/\mathbb{Z}_2$ as our exemplar. Alternatively, we may identify $\mathbb{C}P^{2k+2}$ with $\mathbb{H}^{k+1}$. We then have $T(\bar{z}) = j \cdot \bar{z} \bar{f}$ so the quaternions make their appearance. The right action by the symplectic group acts transitively on the unit sphere in $\mathbb{H}^{k+1}$ and commutes with left multiplication by the quaternions. Consequently, $\bar{M}$ is a homogeneous space.

We investigate the distance functions on $S^{4k+3}$ and on $\mathbb{C}P^{2k+1}$. We take as basepoint the north pole $P = (1, 0, \ldots, 0)$; the particular point in question is irrelevant as we are dealing with homogeneous spaces. If $\bar{\xi} \in P^\perp$ with $\|\bar{\xi}\| = 1$, then the curve

$$\sigma(t) := \cos(t)P + \sin(t)\bar{\xi} \text{ for } t \in (-\pi, \pi)$$

is a unit speed geodesic from $P$ with initial direction $\xi$. Thus

$$d_{\mathbb{C}P^{k+1}}(P, \sigma(t)) = t = \arccos(|x^1|) \text{ so } r_{\mathbb{C}P^{k+1}, P}(\bar{z}) = \arccos(\Re(z^1)),$$

$$r_{\mathbb{C}P^{k+1}, P}(\bar{z}) = \arccos(|z^1|).$$

Let $\pi: \mathbb{C}P^{2k+1} \to \bar{M} := \mathbb{C}P^{2k+1}/\mathbb{Z}_2$ be the natural projection. We then have

$$r_{\mathbb{C}P^{k+1}, P}(\pi(\bar{z})) = \arccos(\min(|z^1|, |z^2|)).$$

This shows that $\bar{m}_{M,P} = \frac{\pi}{4}$ and the maximal domain of $r_{\bar{m}}$ is given by the open set

$$\{(t, z^2, \ldots, z^k) : \|z^2\| < t \} \subset \{(t, z^2, \ldots, z^k) : t^2 + \|z^2, \ldots, z^k\|^2 = 1\}.$$

The requirement that the first component is positive normalizes the point of projective space and the requirement that the action of $\mathbb{Z}_2$ is suitably normalized is reflected in the requirement $|z^2| < t$. We present a picture of the slice obtained by taking $z^3 = \cdots = z^k = 0$; the $t$ axis is up; it is a 45 degree cone, shown in dark blue, about the $z$-axis where the boundary circle is identified by a 90 degree rotation; it is $\mathbb{R}^2$ topologically speaking. The full sphere is shown in gray but is not part of $\bar{M}$.

Previously, we let $\mathbb{C}P^{2k+1} = S^{4k+3}/S^1$. We now regard

$$\mathbb{C}P^{2k+1} = \{\mathbb{C}^{2k+2} - \{0\}\}/\{\mathcal{C} - \{0\}\}$$

and introduce corresponding homogeneous coordinates $\langle \bar{z} \rangle$ for $\bar{z} \in \mathbb{C}P^{2k+2} - \{0\}$. If $k = 0$, then $\mathbb{C}P^1 = S^2 = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere and the map is

![Picture 2.2](image-url)
\langle 1, w^2 \rangle \rightarrow \langle -w^2, 1 \rangle = \langle 1, -\frac{1}{w^2} \rangle \text{ away from } (1, 0). \text{ This is anti-holomorphic and hence reverses the orientation. More generally, we have}

\langle T \{ \langle 1, w^2, \ldots, w^{2k+2} \rangle \} = \left\langle 1, -\frac{1}{w^2}, \frac{w^4}{w^2}, -\frac{w^3}{w^2}, \ldots \right\rangle \text{ for } w^2 \neq 0. \text{ This is an anti-holomorphic function of } 2k+1 \text{ complex variables and hence reverses the orientation as expected.}

**Remark 2.6.** Theorem 9.3.1 of Wolf [16] shows that any other compact Riemannian manifold which is not simply-connected and which is modeled on \( \mathbb{CP}^{2k+1} \) for \( k > 0 \) is in fact isometric to the example described above.

### 3. Flat space

If \( M \) is a simply-connected complete harmonic space with scalar curvature \( \tau = 0 \), then \( \mathbb{M} \) is flat and \( \mathbb{M} = (\mathbb{R}^m, g_\tau) \) is isometric Euclidean space with the standard flat metric. This is a homogeneous space and \( \Theta_{M,P} = \infty \) for any \( P \). We have

\[ \Theta_{M,P} = r^{m-1}, \quad \phi_1 = r^{1-m}, \quad \phi_0 = \begin{cases} \log(r) & \text{if } m = 2 \\ (2 - m)^{-1} r^{2-m} & \text{if } m > 2 \end{cases}. \]

More generally, if \( M \) is a complete harmonic space with \( \tau = 0 \), then the universal cover of \( M \) is \( \mathbb{R}^m \). Let \( \Gamma := \pi_1(M) \) and let \( \pi : \mathbb{R}^m \rightarrow \mathbb{R}^m/\Gamma = \mathbb{M} \) be the universal cover map. If \( M \) is orientable, then \( \mathbb{M} = \mathbb{R}^k \times \{ \mathbb{R}^{m-k}/\Gamma \} \) where \( \Gamma \) is a co-compact lattice in \( \mathbb{R}^{m-k} \) that defines a torus of dimension \( m - k \). More generally, \( M \) is a generalized Klein bottle. We illustrate the results of Theorem 1.2 in Section 2.2 by examining the square torus, and in Section 3.2 by studying the open Mobius strip.

#### 3.1. The square torus

Let \( \mathbb{Z}^2 \) be the integer lattice in \( \mathbb{R}^2 \) and let \( \mathbb{M} := \mathbb{R}^2/\mathbb{Z}^2; \) this is a homogeneous space so without loss of generality we may assume \( \bar{P} = 0 \) and \( P = 0 \). We then have \( \Theta_{M,0} = \frac{1}{2} \) and \( \bar{O}_0 = (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2}) \) which we identify with \( O_0 \) in \( M \). The cut-locus is the image of the boundary of the closed square \([ -\frac{1}{2}, \frac{1}{2} ] \times [ -\frac{1}{2}, \frac{1}{2} ] \) and is topologically a figure 8. We picture the situation below.

![Picture 3.1]

**Remark 3.1.** If \( O(\delta) = (-\delta, 1 - \delta) \times (-\delta, 1 - \delta) \) for \( 0 < \delta < 1 \), then we have that \( \phi_0 := \log(x^2 + y^2) \) is a harmonic real analytic function on \( O(\delta) \) which is radial near 0 and which agrees with \( \phi_0 \) near the origin. It is radial on all of \( O(\delta) \) if and only if \( \delta = \frac{1}{2} \).

#### 3.2. The open Klein bottle

There is a unique real line bundle \( \mathcal{O} \) over the circle which is not orientable. We can realize this as \([ -\frac{1}{2}, \frac{1}{2} ] \times \mathbb{R} / \sim \) where we identify the point \((-\frac{1}{2}, y)\) with the point \((\frac{1}{2}, -y)\). Alternatively, let \( T(x, y) = (x + 1, -y) \) define an action of \( \mathbb{Z} \) on \( \mathbb{R}^2 \). Then we may identify \( \mathcal{O} \) with \( \mathbb{R}^2/\mathbb{Z} \). Let \( P = (x, y) \in \mathbb{R}^2 \).
The translations \((x, y) \to (x + a, y)\) and the reflection \((x, y) \to (x, -y)\) commute with the action of \(\mathbb{Z}\) so \(\mathcal{M}\) is homogeneous in the \(x\) direction and also the reflection \((x, y) \to (x, -y)\). Thus we may assume that \(P = (0, a)\) for \(a \geq 0\). By Equation (3.a),

\[
\tau_{\mathcal{M},(0,a)} = \frac{1}{2} \min_{a \neq 0} \left\{ \left\| (0, a) - T^{n}(0, a) \right\| \right\} = \frac{1}{2} \min \left\{ \min_{k \neq 0} \left\{ \left\| (0, a) - (2k, a) \right\| \right\}, \min_{k \neq 0} \left\{ \left\| (0, a) - (2k + 1, -a) \right\| \right\} \right\} \quad (3.a)
\]

In particular, \(\mathcal{M}\) is not homogeneous since \(\tau_{\mathcal{M},P}\) is not constant.

We now determine the cut locus. Suppose first \(P = (0, 0)\). Let \(\mathcal{O} := (-5, 5) \times \mathbb{R}\). Then \(\pi\) is a diffeomorphism from \(\mathcal{O}\) to an open neighborhood of the origin in \(\mathcal{M}\) and the argument given above for the torus shows \(r_{\mathcal{M},P}(x, y) = (x^2 + y^2)^{1/2}\) on \(\mathcal{O}\). The cut locus is \(\pi\{\pm \frac{1}{2} \times \mathbb{R}\}\) and \(r\) is not smooth there; \(\mathcal{O}\) is a maximal domain for the non-constant radial harmonic function \(\phi_1\). We refer to Picture 3.2 below.

On the other hand, if \(a \neq 0\), the situation is a bit different. We consider the set of points \((x, y)\) so that \(\left\| (x, y) - (0, a) \right\| < \left\| T^{n}(x, y) - (0, a) \right\|\) for all \(n \neq 0\), i.e. \((x, y)\) is the closest point to \((0, a)\) among all possible pre-images of \((x, y)\). We compute:

\[
\left\| T^{n}(x, y) - (0, a) \right\|^2 - \left\| (x, y) - (0, a) \right\|^2 = \begin{cases} n^2 + 2nx & \text{if } n \text{ is even} \\ n^2 + 2nx + 4ay & \text{if } n \text{ is odd} \end{cases}
\]

If \(n = \pm 2\), then we obtain \(4 \pm 4x > 0\) and hence \(|x| < 1\). We impose this condition by restricting to \(\mathcal{O} = (-1, 1) \times \mathbb{R}\). We then have \(n^2 + 2nx > 0\) if \(n \neq 0\). And, furthermore, if \(n\) is odd, then \(n^2 + 2nx + 4ay\) attains its minimum when \(n = \pm 1\). Thus we must impose the conditions:

\[
1 + 2x + 4ay > 0 \quad 1 - 2x + 4ay > 0
\]

Thus the fundamental region on which \(r\) is smooth is the region

\[-1 < x < 1, \quad 1 + 2x + 4ay > 0, \quad 1 - 2x + 4ay > 0\]

Suppose \(a > 0\). To create the final region, the line from \((-1, \frac{1}{4}a)\) to \((-1, \infty)\) is glued to the line from \((1, \frac{1}{4}a)\) to \((1, \infty)\) using \(T^2\), and the directed line segment from \((-1, \frac{1}{4}a)\) to \((0, -\frac{1}{4}a)\) is glued to the line segment \((0, -\frac{1}{4}a)\) to \((1, \frac{1}{4}a)\) using \(T\). The pattern is similar if \(a < 0\). As \(a \to 0\), the two lines \(1 \pm 2x + 4ay = 0\) converge to the lines \(x = \pm \frac{1}{2}\). The closest points to the center on the 4 line segments are at \((a, \pm 1)\) and \((0, \pm \frac{1}{2})\). Thus, as noted in Equation (3.a), the injectivity radius is \(\tau_{\mathcal{M},P} = \min(1, \sqrt{a^2 + \frac{1}{4}})\); \((0, 0)\) has the smallest injectivity radius of \(\frac{1}{2}\) and the points \((0, a)\) for \(a \geq \sqrt{2}\) have the largest injectivity radius of 1.

![Picture 3.2](image)

It is worth noting that there are other maximal domains for \(\phi_1\) other than those provided by removing the cut locus. Let \(P = (0, 0)\) and let \(\mathcal{O}_2 := (-.25, .75) \times \mathbb{R}\). Then \(\Phi_1 = \log(x^2 + y^2)\) is harmonic on \(\mathcal{O}_2\). More generally, if \(\mathcal{U}\) is any connected open subset of \(\mathbb{R}^2\) containing \((0, a)\) on which \(\pi\) is 1-1, then \(\pi(\mathcal{U})\) is a domain on
We obtain: 1. Simply connected examples. The solutions \( \phi_0 \) and \( \phi_1 \) of Table 4.1 must be adjusted to replace the corresponding trigonometric functions by their hyperbolic analogues with appropriate sign changes to obtain the corresponding solutions for the negatively curved rank 1 symmetric spaces; they are defined on all of \( \mathbb{M} - \{ P \} \). We obtain:

| \( \mathbb{M} \) | \( \phi_1 \) | \( \phi_0 \) |
|---|---|---|
| \( \mathbb{S}^2 \) | \( \frac{1}{\sinh(r)} \) | log \( \left( \tanh \left( \frac{r}{2} \right) \right) \) |
| \( \mathbb{S}^3 \) | \( \frac{1}{\sinh(r)^2} \) | - coth \( r \) |
| \( \mathbb{S}^4 \) | \( \frac{1}{\sinh(r)} \) | \(-\frac{1}{2} \text{csch}^2 \left( \frac{r}{2} \right) - \frac{1}{2} \text{sech}^2 \left( \frac{r}{2} \right) - \frac{1}{2} \log \left( \tanh \left( \frac{r}{2} \right) \right) \) |
| \( \mathbb{S}^5 \) | \( \frac{1}{\sinh(r)} \) | \( \frac{2}{3} \coth(r) - \frac{1}{3} \coth(r) \text{csch}^2(r) \) |
| \( \mathbb{CP}^2 \) | \( \sinh(r) \frac{1}{\cosh(r)} \) | \(-\frac{1}{2} \text{csch}^2(r) - \log(\tanh(r)) \) |
| \( \mathbb{CP}^3 \) | \( \sinh(r) \frac{1}{\cosh(r)} \) | \(-\frac{1}{4} \text{csch}^4(r) + \frac{1}{2} \text{csch}^2(r) + \log(\tanh(r)) \) |
| \( \mathbb{CP}^4 \) | \( \sinh(r) \frac{1}{\cosh(r)} \) | \(-\frac{1}{6} \text{csch}^6(r) + \frac{1}{4} \text{csch}^4(r) - \frac{1}{2} \text{csch}^2(r) - \log(\tanh(r)) \) |
| \( \mathbb{HP}^2 \) | \( \sinh(r) \frac{1}{\cosh(r)^3} \) | \( \frac{1}{2} \left( -\frac{1}{2} \text{csch}^6(r) + \text{csch}^4(r) - 3 \text{csch}^2(r) - \text{sech}^2(r) - 8 \log(\tanh(r)) \right) \) |
| \( \mathbb{HP}^3 \) | \( \sinh(r) \frac{1}{\cosh(r)^3} \) | \(-\frac{1}{12} \text{csch}^{10}(r) + \frac{1}{2} \text{csch}^6(r) - \frac{1}{2} \text{csch}^6(r) + \text{csch}^4(r) - \frac{5}{2} \text{csch}^2(r) - \frac{1}{2} \text{sech}^2(r) - 6 \log(\tanh(r)) \) |
| \( \mathbb{HP}^4 \) | \( \sinh(r) \frac{1}{\cosh(r)^3} \) | \( \frac{1}{2} \left( -\frac{1}{2} \text{csch}^{14}(r) + \frac{1}{3} \text{csch}^{12}(r) - \text{csch}^{10}(r) + \frac{5}{2} \text{csch}^8(r) - \frac{25}{6} \text{csch}^6(r) + 14 \text{csch}^4(r) - 42 \text{csch}^2(r) - \frac{1}{2} \text{sech}^2(r) - 2 \text{sech}^2(r) - 12 \log(\tanh(r)) \right) \) |
| \( \mathbb{OP}^2 \) | \( \sinh(r) \frac{1}{\cosh(r)^3} \) | \(-\frac{1}{16} \text{csch}^{14}(r) + \frac{1}{3} \text{csch}^{12}(r) - \text{csch}^{10}(r) + \frac{5}{2} \text{csch}^8(r) - \frac{25}{6} \text{csch}^6(r) + 14 \text{csch}^4(r) - 42 \text{csch}^2(r) - \frac{1}{2} \text{sech}^2(r) - 2 \text{sech}^2(r) - 12 \log(\tanh(r)) \) |

4.2. The proof of Theorem [1,3]. Let \( \mathbb{M} \) be a compact Riemannian manifold of dimension \( m = 4\ell \) which is modeled on a rank 1 symmetric space \( \mathbb{M}_- \) of negative curvature. Let \( \mathbb{M}_+ \) be the dual rank 1 symmetric space of positive curvature. The argument of Section [2] using the Chern-Gauss-Bonnet Formula to obtain information on \( |\Gamma| \) in the positive curvature setting can be turned around to obtain information on \( \text{Vol}(\mathbb{M}) \). Let \( \text{Pf}_\ell \) be the Pfaffian; this is a polynomial of degree \( \ell \) in the curvature tensor so \( \text{Pf}_\ell(\mathbb{M}) = \text{Pf}_\ell(\mathbb{M}_-) = (-1)^\ell \text{Pf}_\ell(\mathbb{M}_+) \); these are constant since the spaces in question are locally homogeneous. The Chern-Gauss-Bonnet Formula then shows:

\[
\chi(M_+) = \text{Pf}_\ell(M_+) \cdot \text{vol}(M_+)
\]

\[
\chi(M) = \text{Pf}_\ell(M) \cdot \text{vol}(M) = (-1)^\ell \text{Pf}_\ell(M_+) \cdot \text{vol}(M_+) \cdot \frac{\text{vol}(M)}{\text{vol}(M_+)}
\]

\[
= (-1)^\ell \chi(M_+) \cdot \frac{\text{vol}(M)}{\text{vol}(M_+)}. \]

Since \( \chi(M_+) > 0 \), this implies \( \chi(M) \neq 0 \). Consequently

\[
1 \leq |\chi(M)| = \chi(M) \cdot \frac{\text{vol}(M)}{\text{vol}(M_+)} \quad \text{so} \quad \text{vol}(M) \geq \frac{\text{vol}(M_+)}{\chi(M_+)}.
\]
Theorem 1.3 now follows since, by Lemma 2.1,
\[ \chi(S^m) = 2, \quad \chi(CP^k) = k + 1, \quad \chi(HP^k) = k + 1, \quad \text{and} \quad \chi(OP^2) = 3. \]

4.3. The proof of Theorem 1.4. Let \( M \) and \( M_{\pm} \) be as in Section 4.2. Suppose that \( m = 4j \) is divisible by 4. The estimates of Theorem 1.3 arise from the Chern-Gauss-Bonnet Formula and are not optimal in certain settings. By using the Hirzebruch Signature Formula, we can obtain better estimates. Let \( M_{\pm} \in \{ \tilde{CP}^{2k}, \tilde{HP}^{2k}, \tilde{OP}^{2k} \} \) so \( M_+ \in \{ CP^{2k}, HP^{2k}, OP^{2k} \} \).

Suppose first that \( M \) is orientable. The Hirzebruch polynomial \( L_j \) is a constant multiple of the oriented volume form. Since \( L_j \) is quadratic in the curvature tensor, we have that \( L_j(M) = L_j(M_+) = L_j(M_-) \) where we adjust the orientations to ensure this equality. By Lemma 2.1 \( \text{sign}(M_+) = 1 \), Therefore,
\[
\text{sign}(M) = L_j(M) \cdot \text{vol}(M) = L_j(M) \cdot \text{vol}(M_+) \cdot \frac{\text{vol}(M)}{\text{vol}(M_+)}
\]
This shows \( \text{sign}(M) \) is positive and hence \( \text{sign}(M) \geq 1 \) so \( \text{vol}(M) \geq \text{vol}(M_+) \).

If \( M \) is not orientable, we let \( M_0 \) be the orientable double cover and estimate
\[
\frac{1}{2} \text{vol}(M) = \text{vol}(M_0) \geq \text{vol}(M_+). \]

Dedication

On 11 March 2004, 10 bombs exploded on 4 trains near the Atocha Station in Madrid killing 191 and injuring more than 1800; 18 Islamic fundamentalists and 3 Spanish accomplices were convicted of the bombings which was one of Europe’s deadliest terrorist attacks in the years since World War II. Subsequently, one of us (Gilkey) and his coauthors dedicated a paper [7] writing “En memoria de todas las víctimas inocentes. Todos éramos en ese tren.” This paper is being written during one of the worst outbreaks of war in Europe since World War II. We dedicate this paper, writing in a similar vein to show our solidarity with the innocent victims in Ukraine, that: Ми всi в Українi (“we are all in Ukraine”).

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