$p$-adic $l$-functions and sums of powers

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Abstract. In this paper, we give an explicit $p$-adic expansion of

$$\sum_{j=1}^{np} \frac{(-1)^j}{j^r}$$

as a power series in $n$. The coefficients are values of $p$-adic $l$-function for Euler numbers.

1. Introduction

Let $p$ be a fixed prime. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will, respectively, denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$, cf. [1], [3], [6], [10]. Let $v_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $\|p\|_p = p^{-v_p(p)} = p^{-1}$. Kubota and Leopoldt proved the existence of meromorphic functions, $L_p(s, \chi)$, defined over the $p$-adic number field, that serve as $p$-adic equivalents of the Dirichlet $L$-series, cf. [8], [10]. These $p$-adic $L$-functions interpolate the values

$$L_p(1 - n, \chi) = -\frac{1}{n}(1 - \chi_n(p)p^{n-1})B_{n, \chi_n}, \quad \text{for } n \in \mathbb{N} = \{1, 2, \cdots\},$$

where $B_{n, \chi}$ denote the $n$th generalized Bernoulli numbers associated with the primitive Dirichlet character $\chi$, and $\chi_n = \chi w^{-n}$, with $w$ the Teichmüller character, see [2, 3, 5, 6, 17, 20]. In [14], L. C. Washington have proved the below interesting formula:

$$\sum_{j=1}^{np} \frac{1}{j^r} = -\sum_{k=1}^\infty \binom{-r}{k} (pn)^k L_p(r + k, w^{1-k-r}),$$

where $\binom{-r}{k}$ is binomial coefficient. In the recent many authors have studied $q$-extension of Euler numbers and Bernoulli numbers (see [1, 4, 5, 9, 12, 13]). These $q$-extensions seem to be valuable and worthwhile in the areas of mathematical physics and mathematics (see [1, 4, 6, 13, 14, 15, 17, 18, 19]). By using $q$-Volkenborn integration, Kim gave the interesting properties of $q$-Bernoulli and Euler polynomials

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[8, 9, 10, 11] and Ryoo-Kim-Agarwal have investigated the properties of the $q$-extension of Euler numbers and polynomials by using “Mathematica package”, see [14, 15]. The problems to find the sums of powers of consecutive $q$-integers were suggested. Kim and Schlosser treated the formulae for the sums of powers of consecutive $q$-integers [7,11, 16] and these formulae were used to give the $q$-extension of Washington’s $p$-adic $L$-functions and sums of powers (see [6, 10, 20]). In [11, 14], we found the interesting formulae “alternating sums of powers of consecutive integers” which are related to Euler numbers and polynomials. By using these alternating sums of powers of consecutive integers, we try to construct the $p$-adic $l$-functions and sums of powers for Euler numbers and polynomials, corresponding to Washington and Kim (see [10, 20]). The purpose of this paper is to give alternating $p$-adic harmonic series in terms of $n$ and $p$-adic $l$-function for Euler numbers.

2. A note on $l$-series associated with Euler numbers and polynomials

We begin with well known Euler polynomials $E_n(x)$.

**Definition 1.** Euler polynomials are defined by

$$\frac{2}{e^t + 1}e^{xt} = \sum_{n=1}^{\infty} \frac{E_n(x)}{n!} t^n,$$

$E_n(x)$ are called $n$-th Euler polynomials. For $x = 0$, $E_n = E_n(0)$ are called Euler numbers. By the definition of Euler polynomials, we easily see that,

$$E_l(x) = \sum_{n=0}^{l} \binom{l}{n} E_n x^{l-n} \in \mathbb{C}[x].$$

From the generating function of Euler polynomials $F(t, x) = \frac{2}{e^t + 1} e^{xt}$, we derive

$$F(t, x) = 2 e^{xt} \sum_{l=0}^{\infty} (-1)^l e^{lt}$$

$$= 2 \sum_{l=0}^{\infty} (-1)^l e^{(l+x)t}. \quad (1)$$

For $k \in \mathbb{N}$, we note that

$$\left. \frac{d^k}{dt^k} F(t, x) \right|_{t=0} = 2 \sum_{l=0}^{\infty} (-1)^l \frac{d^k}{dt^k} e^{(l+x)t} \Big|_{t=0}$$

$$= 2 \sum_{l=0}^{\infty} (-1)^l (l+x)^k. \quad (2)$$

Therefore we can define the Euler zeta function as follows:
Definition 2. For \( s \in \mathbb{C} \), we define Euler zeta function as
\[
\zeta_E(s) = 2 \sum_{l=0}^{\infty} \frac{(-1)^l}{(l+x)^s}.
\] (3)

By using Definition 2 and (2), we obtain the following:

**Proposition 3.** For \( k \in \mathbb{N} \), we have
\[
\zeta_E(-k, x) = E_k(x).
\] (4)

For \( f(=\text{odd}) \in \mathbb{N} \),
\[
\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{2}{e^t+1} e^{xt} = 2 \sum_{l=0}^{\infty} (-1)^l e^{(l+x)t} = 2 \sum_{a=0}^{l-1} \sum_{m=0}^{\infty} (-1)^{a+lf} e^{(a+lf+x)t}
\]
\[
= \sum_{a=1}^{f} (-1)^a 2 \sum_{n=0}^{\infty} (-1)^l e^{f(l+a+xt)} = \sum_{a=1}^{f} (-1)^a \sum_{n=0}^{\infty} E_n \left( \frac{x+a}{f} \right) \frac{t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} f^n \sum_{a=1}^{f} (-1)^a E_n \left( \frac{x+a}{f} \right) \frac{t^n}{n!}.
\]

Thus we note that
\[
E_n(x) = f^n \sum_{a=1}^{f} (-1)^a E_n \left( \frac{x+a}{f} \right),
\] (5)

where \( f(=\text{odd}) \in \mathbb{N} \). This (5) is so called Distribution for Euler polynomials.

\[
2 \sum_{l=0}^{n-1} (-1)^l t^m = (-1)^{n+1} \sum_{l=0}^{m-1} E_l m^m \binom{m}{l} + ((-1)^{n+1} + 1) E_m.
\]

In particular, if \( n \) is even, then
\[
2 \sum_{l=0}^{n-1} (-1)^l t^m = - \sum_{l=0}^{m-1} E_l m^m - l.
\]

Let \( s \) be a complex variable and let \( a, F(=\text{odd}) \) be integers with \( 0 < a < F \).

\[
H(s, a|F) = \sum_{m \equiv a(F)}^{\infty} \frac{(-1)^m}{m^s} = \sum_{n=0}^{\infty} \frac{(-1)^{nF+a}}{(a+nF)^s} = (-1)^a \sum_{n=0}^{\infty} \frac{(-1)^n}{(a+nF)^s}
\]
\[
= (-1)^a \sum_{n=0}^{\infty} \frac{(-1)^n}{F^s \left( \frac{a}{F} + n \right)^s} = \frac{(-1)^s F^{-s}}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n + \frac{a}{F}}
\]
\[
= \frac{(-1)^s F^{-s}}{2} \zeta_E \left( s, \frac{a}{F} \right).
\] (6)
Note that
\[ H(-n, a|F) = (-1)^a \frac{F^n}{2} E_n \left( \frac{a}{F} \right). \]

Let \( \chi \) be the primitive Dirichlet character with conductor \( f(=\text{odd}) \in \mathbb{N} \).

\[ F_\chi(t) = 2 \sum_{n=0}^\infty e^{nt} \chi(n)(-1)^n \]
\[ = 2 \sum_{a=0}^{f-1} \sum_{n=0}^\infty e^{(a+nf)t} \chi(a+nf)(-1)^{a+nf} \]
\[ = \sum_{a=0}^{f-1} e^{at} \chi(a)(-1)^a 2 \sum_{n=0}^\infty (-1)^n e^{nf t} \]
\[ = 2 \sum_{a=0}^{f-1} e^{at} \chi(a)(-1)^a \left( \frac{1}{e^{ft}+1} \right) \]
\[ = 2 \sum_{a=0}^{f-1} \frac{e^{at} \chi(a)(-1)^a}{e^{ft}+1} \]
\[ = \sum_{n=0}^\infty E_n, \chi \frac{t^n}{n!}. \]

Thus, we can define the below generalized Euler number attached to \( \chi \).

**Definition 4.** Let \( \chi \) be the Dirichlet character with conductor \( f(=\text{odd}) \in \mathbb{N} \). Then we define the generalized Euler numbers attached to \( \chi \) as follows;

\[ \frac{2 \sum_{a=0}^{f-1} e^{at} \chi(a)(-1)^a}{e^{ft}+1} = \sum_{n=0}^\infty E_{n, \chi} \frac{t^n}{n!}. \]

\( E_{n, \chi} \) will be called the \( n \)-th generalized Euler numbers attach to \( \chi \).

From the Definition 4, we derive the below formula:

\[ \sum_{n=0}^\infty E_{n, \chi} \frac{t^n}{n!} = \frac{2 \sum_{a=0}^{f-1} e^{at} \chi(a)(-1)^a}{e^{ft}+1} = \sum_{a=0}^{f-1} \chi(a)(-1)^a \left( \frac{2}{e^{ft}+1} e^{at} \right) \]
\[ = \sum_{a=0}^{f-1} \chi(a)(-1)^a \sum_{n=0}^\infty E_n \left( \frac{a}{f} \right) \frac{t^n}{n!} = \sum_{n=0}^\infty \sum_{a=0}^{f-1} \chi(a)(-1)^a E_n \left( \frac{a}{f} \right) \frac{t^n}{n!}. \]

By comparing the coefficients on both sides, we easily see that

\[ E_{n, \chi} = f^n \sum_{a=0}^{f-1} \chi(a)(-1)^a E_n \left( \frac{a}{f} \right). \]
**Definition 5.** For \( s \in \mathbb{C} \), we define Dirichlet’s \( l \)-function as follows:

\[
l(s, \chi) = 2 \sum_{n=1}^{\infty} \frac{\chi(n)(-1)^n}{n^s}.
\]

Note that

\[
F_{\chi}(t) = 2 \sum_{n=1}^{\infty} e^{nt} \chi(n)(-1)^n = \sum_{n=0}^{\infty} E_{n,\chi} \frac{t^n}{n!}.
\]

For \( k \in \mathbb{N} \),

\[
E_{k,\chi} = \left. \frac{d^k}{dt^k} F_{\chi}(t) \right|_{t=0} = 2 \sum_{n=1}^{\infty} \chi(n)(-1)^n \left. \frac{d^k}{dt^k} e^{nt} \right|_{t=0} = 2 \sum_{n=1}^{\infty} \chi(n)(-1)^n n^k.
\]

By Definition 5 and (10), we easily see that \( l(-k, \chi) = E_{k,\chi} \), where \( k \in \mathbb{N} \). Therefore we obtain the following:

**Proposition 6.** Let \( k \) be the positive integer. Then we have

\[
l(-k, \chi) = E_{k,\chi}.
\]

Let \( \chi \) be the Dirichlet character with conductor \( f(=\text{odd}) \in \mathbb{N} \). Then we note that

\[
l(s, \chi) = 2 \sum_{a=1}^{f} \chi(a) H(s, a|f)
\]

In Eq. (12), we give a value of \( l(s, x) \) at negative integer:

\[
l(-n, \chi) = 2 \sum_{a=1}^{f} \chi(a) H(-\eta, a|f)
\]

\[
= 2 \sum_{a=1}^{f} \chi(a)(-1)^n \frac{f^n}{2} E_n \left( \frac{a}{f} \right)
\]

\[
= f^n \sum_{a=0}^{f-1} \chi(a)(-1)^a E_n \left( \frac{a}{f} \right)
\]

\[
= E_{n,\chi}.
\]

The function \( H(s, a|F) \) will be called partial zeta function which interpolates Euler polynomials at negative integers. The values of \( l(s, \chi) \) at negative integers are algebraic, hence may be regarded as lying in an extension of \( \mathbb{Q}_p \). We therefore
look for a $p$-adic function which agrees with $l(s, \chi)$ at negative integers in later.

3. A note on $p$-adic $l$-function

We define $\langle x \rangle = x^{w(x)}$, where $w(x)$ is the Teichmüller character. When $F (= \text{odd})$ is a multiple of $p$ and $(a, p) = 1$, we define

$$H_p(s, a|F) = (-1)^a \frac{\langle a \rangle}{2} \sum_{j=0}^{\infty} \left( \frac{-s}{j} \right) \left( \frac{F}{a} \right)^j E_j,$$

for $s \in \mathbb{Z}_p$.

It is easy to see that

$$H_p(-n, a|F) = \frac{(-1)^a \langle a \rangle^n}{2} \sum_{j=0}^{n} \binom{n}{j} \left( \frac{F}{a} \right)^j E_j$$

$$= \frac{(-1)^a}{2} F^n w^{-n}(a) \sum_{j=0}^{n} \binom{n}{j} \left( \frac{a}{F} \right)^{n-j} E_j$$

$$= \frac{(-1)^a}{2} F^n w^{-n}(a) E_n(a)$$

$$= w^{-n}(a) H(-n, a|F),$$

for all positive integers. Now we consider $p$-adic interpolation function for Euler numbers as follows;

$$l_p(s, \chi) = 2 \sum_{n=1}^{F} \chi(a) H_p(s, a|F)$$

for $s \in \mathbb{Z}_p$.

Let $n$ be natural number. Then we have

$$l_p(-n, \chi) = 2 \sum_{n=1}^{F} \chi(a) H_p(-n, a|F)$$

$$= E_n, \chi w^{-n} - p^n \chi w^{-n}(p) E_n, \chi w^{-n}$$

$$= (1 - p^n \chi w^{-n}(p)) E_n, \chi w^{-n}.$$

In fact, we have the formula

$$l_p(s, \chi) = \sum_{a=1}^{F} (-1)^a \langle a \rangle^{-s} \chi(a) \sum_{j=0}^{\infty} \left( \frac{-s}{j} \right) \left( \frac{F}{a} \right)^j E_j,$$

for $s \in \mathbb{Z}_p$. 

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This is a $p$-adic analytic function and has the following properties for $\chi = w^t$.

\[ l_p(-n, w^t) = (1 - p^n)E_n, \quad \text{where } n \equiv t \pmod{p - 1}, \]

\[ l_p(s, w^t) \in \mathbb{Z}_p \quad \text{for all } s \in \mathbb{Z}_p, \quad \text{when } t \equiv 0 \pmod{p - 1}. \]

If $t \equiv 0 \pmod{p - 1}$, then $l_p(s_1, w^t) \equiv l_p(s_2, w^t) \pmod{p}$ for all $s_1, s_2 \in \mathbb{Z}_p$,

\[ l_p(k, w^t) \equiv l_p(k + p, w^t) \pmod{p}. \]

It is easy to see that

\[ \frac{1}{r + k - 1} \binom{-r}{k} \left( \binom{1 - r - k}{j} \right) = \frac{-1}{j + k} \binom{-r}{k + j - 1} \binom{k + j}{j}, \]

for all positive integers with $r, j, k$ with $j, k \geq 0$, $j + k > 0$ and $r \neq 1 - k$.

Thus, we note that

\[ \frac{1}{r + k - 1} \binom{-r}{k} \left( \binom{1 - r - k}{j} \right) = \frac{1}{r - 1} \binom{-r + 1}{k + j} \binom{k + j}{j}. \]

Hence, we have

\[ \frac{r}{r + k} \binom{-r + 1}{k} \binom{-r - k}{j} = \binom{-r}{k + j} \binom{k + j}{j}, \]

where $k, j$ are positive integers. Let $\mathcal{F}(=\text{odd})$ be positive integers. Then

\[
\sum_{l=0}^{n-1} \frac{(-1)^{F+l+a}}{(Fl + a)^r} = \sum_{l=0}^{n-1} (-1)^{F+l+a} r^{-r} \sum_{s=0}^{\infty} \binom{-r}{s} \left( \frac{F}{a} \right)^s \\
= \sum_{m=0}^{\infty} \left( \frac{-r}{m} \right) a^{-r} \left( \frac{F}{a} \right)^m (-1)^a \sum_{l=0}^{n-1} (-1)^l l^m \\
= \sum_{m=0}^{\infty} \left( \frac{-r}{m} \right) a^{-r} \left( \frac{F}{a} \right)^m (-1)^a \frac{(-1)^{n+1}}{2} \sum_{l=0}^{m-1} E_l n^{m-l} \binom{m}{l} + ((-1)^{m-1} + 1)E_m. 
\]
when $n$ is even integer

\[
\sum_{l=0}^{n-1} \frac{(-1)^l}{(Fl+a)^r} = a^{-r}(-1)^a \sum_{m=0}^{\infty} \left( \frac{-r}{m} \right) \left( \frac{F}{a} \right)^m \left\{ \frac{(-1)^{m+1}}{2} \sum_{l=0}^{m+1} E_l n^{m-l} \binom{m}{l} \right\} = -a^{-r}(-1)^a \sum_{s=0}^{\infty} \left( \frac{-r}{s} \right) \left( \frac{F}{a} \right)^s \sum_{l=0}^{s+1} E_l n^{s-l} \binom{s}{l} \]

\[
\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left( \frac{-r}{k+l} \right) w^{-r}(a) \left( \frac{a}{F} \right)^{-k-l} n^{k+l} \frac{(-1)^a}{2} (a)^{-r} \sum_{l=0}^{s+1} E_l n^{s-l} \binom{s}{l} \]

\[
\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{r}{r+k} \left( \frac{-r-1}{k} \right) \left( \frac{-r-k}{l} \right) a^{-r} \left( \frac{a}{F} \right)^{-k-l} n^{k+l} \frac{(-1)^a}{2} (a)^{-r} \sum_{l=0}^{\infty} \left( \frac{-r-k}{l} \right) E_l \left( \frac{F}{a} \right) \]

\[
-\sum_{k=0}^{\infty} \frac{r}{r+k} \left( \frac{-r-1}{k} \right) w^{-k-r}(a)(nF)^k H_p(r+k, a|F). \]

For $F = p$, $r$, $n (= even) \in \mathbb{N}$, we see that

\[
2 \sum_{j=1}^{np} \frac{(-1)^j}{j^p} = 2 \sum_{a=1}^{p-1} \sum_{l=0}^{n-1} \frac{(-1)^{a+l}}{(a+pl)^r} \]

\[
-2 \sum_{a=1}^{p-1} \sum_{k=0}^{\infty} \frac{r}{r+k} \left( \frac{-r-1}{k} \right) w^{-k-r}(a)(pn)^k H_p(r+k, a|p) \]

\[
-\sum_{k=0}^{r} \frac{r}{r+k} \left( \frac{-r-1}{k} \right) (pn)^k \sum_{a=1}^{p-1} w^{-k-r}(a)H_p(r+k, a|p) \]

\[
-\sum_{k=0}^{r} \frac{r}{r+k} \left( \frac{-r-1}{k} \right) (pn)^k H_p(r+k, w^{-k-r}) \]

\[
-\sum_{k=0}^{\infty} \frac{r}{r+k} \left( \frac{-r-1}{k} \right) (pn)^k H_p(r+k, w^{-k-r}). \]

Therefore we obtain the following:
Theorem 6. Let $p$ be an odd prime and let $n (=\text{even}), r \in \mathbb{N}$. Then we have

$$2 \sum_{j=1}^{\frac{np}{j \equiv 1 \mod p}} \frac{(-1)^j}{j^r} = -\sum_{k=0}^{\infty} \frac{r}{r+k} \binom{-r-1}{k} (pn)^k l_p(r+k, w^{-k-r}).$$

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