REACTION-DIFFUSION-ADVECTION SYSTEMS WITH DISCONTINUOUS DIFFUSION AND MASS CONTROL

WILLIAM E FITZGIBBON, JEFF MORGAN, BAO QUOC TANG, AND HONG-MING YIN

Abstract. In this paper, we study unique, globally defined uniformly bounded weak solutions for a class of semilinear reaction-diffusion-advection systems. The coefficients of the differential operators and the initial data are only required to be measurable and uniformly bounded. The nonlinearities are quasi-positive and satisfy a commonly called mass control or dissipation of mass property. Moreover, we assume the intermediate sum condition of a certain order. The key feature of this work is the surprising discovery that quasi-positive systems that satisfy an intermediate sum condition automatically give rise to a new class of $L^p$-energy type functionals that allow us to obtain requisite uniform a priori bounds. Our methods are sufficiently robust to extend to different boundary conditions, or to certain quasi-linear systems. We also show that in case of mass dissipation, the solution is bounded in sup-norm uniformly in time. We illustrate the applicability of results by showing global existence and large time behavior of models arising from a spatio-temporal spread of infectious disease.

Contents

1. Introduction 2
1.1. Problem setting 2
1.2. State of the art and Motivation 3
1.3. Main results and key ideas 4
1.4. Structure of the paper 10
2. Proofs 10
2.1. Systems with control of mass: Proof of Theorem 1.1 10
2.2. Generalizations: Proof of Theorems 1.2 and 1.3 21
2.3. Other boundary conditions or quasilinear systems 25
3. Applications to a model of an infectious disease 26
4. Two technical lemmas 30
References 34

2010 Mathematics Subject Classification. 35A01, 35K57, 35K58, 35Q92.

Key words and phrases. Reaction-diffusion-advection systems; Non-smooth diffusion coefficients; Mass control; Global existence; $L^p$-energy methods.
1. Introduction

1.1. Problem setting. Let $1 \leq n \in \mathbb{N}$, and $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary $\partial \Omega$ such that $\Omega$ lies locally on one side of $\partial \Omega$. This condition is assumed throughout the current paper. Let $2 \leq m \in \mathbb{N}$. We consider the following reaction-diffusion system for vector of concentrations $u = (u_1, u_2, \ldots, u_m)$: for any $i \in \{1, \ldots, m\}$,

$$
\begin{aligned}
&\partial_t u_i - \nabla \cdot (D_i(x,t)\nabla u_i) + \nabla \cdot (B_i(x,t)u_i) = F_i(x,t,u), \quad x \in \Omega, \; t > 0, \\
u_i(x, t) = 0, \quad x \in \partial \Omega, \; t > 0, \\
u_i(x, 0) = u_{i,0}(x), \quad x \in \Omega,
\end{aligned}
$$

(1.1)

where the initial data $u_{i,0}$ is bounded and non-negative, the diffusion matrix $D_i : \Omega \times [0, \infty) \rightarrow \mathbb{R}^{n \times n}$ satisfies

$$
\lambda |\xi|^2 \leq \xi^T D_i(x,t) \xi \quad \forall (x,t) \in \Omega \times [0, \infty), \; \forall \xi \in \mathbb{R}^n, \; \forall i = 1, \ldots, m,
$$

(1.2)

for some $\lambda > 0$, and for each $T > 0$,

$$
D_i \in L^\infty(\Omega \times (0, T)) \quad \forall i = 1, \ldots, m,
$$

(1.3)

and the drift $B_i : \Omega \times [0, \infty) \rightarrow \mathbb{R}^n$ is bounded, i.e.

$$
\text{ess sup}_{x,t} |B_i| \leq \Gamma \quad \forall i = 1, \ldots, m,
$$

(1.4)

for some $\Gamma > 0$. The nonlinearities $F_i : \Omega \times \mathbb{R}_+^n \rightarrow \mathbb{R}$ satisfy the following conditions:

- (F1) for any $i = 1, \ldots, m$ and any $(x,t) \in \Omega \times \mathbb{R}_+$, $F_i(x,t,\cdot) : \mathbb{R}^m \rightarrow \mathbb{R}$ is locally Lipschitz continuous uniformly in $(x,t) \in \Omega \times (0, T)$ for any $T > 0$;

- (F2) for any $i = 1, \ldots, m$, any $(x,t) \in \Omega \times \mathbb{R}_+$, $F_i(x,t,\cdot)$ is quasi-positive, i.e. $F_i(x,t,u) \geq 0$ for all $u \in \mathbb{R}^m_+$ with $u_i = 0$, for all $i = 1, \ldots, m$;

- (F3) there exist $c_1, \ldots, c_m > 0$ and $K_1, K_2 \in \mathbb{R}$ such that

$$
\sum_{i=1}^{m} c_i F_i(x,t,u) \leq K_1 \sum_{i=1}^{m} u_i + K_2 \quad \forall (x,t,u) \in \Omega \times \mathbb{R}_+^n \times \mathbb{R}_+^m,
$$

- (F4) there exist $K_3 > 0$, $r > 0$, and a lower triangular matrix $A = (a_{ij})$ with positive diagonal entries, and non-negative entries otherwise, such that, for any $i = 1, \ldots, m$,

$$
\sum_{j=1}^{i} a_{ij} F_j(x,t,u) \leq K_3 \left( 1 + \sum_{i=1}^{m} u_i^r \right) \quad \forall (x,t,u) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+^m
$$

(we call this assumption intermediate sum of order $r$);

- (F5) the nonlinearities are bounded above by a polynomial, i.e. there exist $\ell > 0$ and $K_4 > 0$ such that

$$
F_i(x,t,u) \leq K_4 \left( 1 + \sum_{i=1}^{m} u_i^\ell \right), \quad \forall (x,t,u) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+^m, \; \forall i = 1, \ldots, m.
$$
The quasi-positivity of nonlinearities in assumption (F2) has a simple physical interpretation. If the concentration \( u_i = 0 \), it cannot be consumed in the reaction. Mathematically, this assumption implies that if the initial data is non-negative, then so is the solution, as long as it exists. At first glance, we note that (F5) implies (F4) with \( r = \ell \). However, our results only place restrictions on the size of \( r \) in (F4), while \( \ell \) in (F5) can be arbitrarily large. In addition, we note that (F4) does not imply that the vector field grows at most as a power of \( r \). It simply implies that \( F_1 \) is bounded above by a polynomial of degree \( r \), and that higher order terms in subsequent \( F_j \) have a “canceling effect”. This assumption holds naturally in many systems arising from chemistry or biology, see e.g. [21, 22]. Roughly speaking, for any reaction/interaction, a gain term for a species stems from a loss term from another species, which results in the same terms in the nonlinearities for these species but with opposite signs. This leads to the desired “canceling effect” described in assumption (F4).

Our focus is the global existence and uniform boundedness of solutions to (1.1) and its variance under the previous assumptions\(^1\). The main results, together with their proof methods and our key new ideas are detailed in the following section.

1.2. State of the art and Motivation. Reaction-diffusion systems with (F1), quasi-positivity condition (F2) and control of mass (F3) appear frequently in physical, chemical or biological models, and the study of global well-posedness for such systems has produced an extensive literature in the last four decades, see e.g. [30, 15, 11, 20, 27] and references therein. In the spatially homogeneous case, i.e. the diffusion and the advection in (1.1) are neglected, these three conditions immediately imply the global existence, uniqueness and uniform-in-time bound (in case \( K_1 = K_2 = 0 \) or \( K_1 < 0 \) in (F3)) of solutions. In case of spatially inhomogeneous systems where the diffusion is present, the situation is much more challenging. In fact, it was shown in [28] that (F2) and (F3) are not enough to prevent solutions to (1.1) from blowing up (in sup-norm) in finite time. A lot of effort has subsequently been spent on systems satisfying (F2) and (F3), and nonlinearities having polynomial growth, i.e.

\[
|f_i(u)| \lesssim 1 + |u|^r \quad \forall i = 1, \ldots, m,
\]

for some \( r \geq 1 \). In [14], it was shown under an additional assumption called entropy inequality that if \( r = 3 \) and \( n = 1 \), or \( r = 2 \) and \( n = 2 \), then the local bounded solution exists globally. This was later extended for strictly sub-quadratic growth \( r < 2 \) in all dimensions in [19, 5]. Without assuming the entropy inequality, [23, 3] showed that systems with quadratic nonlinearities, i.e. \( r = 2 \), possess global bounded solutions when \( n \leq 2 \). This was latter shown also in [29], and improved in [21] where the growth condition was replaced by a weaker intermediate sum condition (F4). The global existence of quadratic systems in higher dimensions had been open until recently when it was settled in three parallel works [32, 4, 9], in the first two works the entropy condition was still imposed, and

\(^1\)Note that except for (F1), (F2) and (F5), some of our results do not assume (F3) and (F4). Precise assumptions will be explicitly stated in each lemma or theorem.
it was removed completely in the last work. There is also work concerning weak solutions. The reader is referred to [27] for an extensive survey.

We emphasize that most, if not all, of the existing literature consider the case of constant or smooth diffusion coefficients. There is a technical reason. For constant or smooth diffusion coefficients, one can utilize the duality method (see e.g. [27, 21]) to first obtain initial a-priori estimates in $L^p(\Omega \times (0, T))$ and to extend these estimates from one component to another, and then the regularizing effect of the heat operator helps to initiate a bootstrap argument which ultimately leads to boundedness in $L^\infty(\Omega \times (0, T))$, and hence global existence. The case of inhomogeneous diffusion coefficients has been studied much less frequently, see e.g. [6] or [2]. Moreover, this work requires smoothness such as continuity or even differentiability of the diffusion coefficients. This motivates the study of the current paper where we investigate the global well-posedness of (1.1) without assuming any regularity of diffusion coefficients $D_i(x, t)$ other than their ellipticity and boundedness. We also remark that system (1.1) includes advection, which has been frequently neglected in the literature.

In many cases, advection, diffusion or reaction processes take place in highly heterogeneous domains. If the terms in the differential operators and/or the reaction terms have discontinuity, then we cannot expect well-posedness in the classical sense, but rather weak solutions in $L^p$ spaces. Concerning spatial inhomogeneity, the classical book [17] considered diffractive differential operators, which are piecewise smooth on sub regions of $\Omega$ and satisfy certain compatibility conditions that guarantee continuity of the state variables and their flux but not their gradients. Strong solutions to systems with diffractive differential operators modelling the spatio-temporal spread of diseases among animal populations distributed in heterogeneous environments are obtained in [13, 12]. What distinguishes the work at hand from previous studies is that we do not place smoothness, piecewise smoothness, or even continuity conditions on the advection and diffusion coefficients. This high degree of heterogeneity precludes strong $L^p$ solutions and forces us to consider weak solutions instead. The lack of regularity renders the aforementioned duality method inapplicable. In this paper, we overcome this issue by introducing a new family of $L^p$-energy functions which, in combination with the intermediate sum condition, provides suitable a-priori estimates to obtain global existence and uniform-in-time boundedness of solutions to (1.1). Note that the uniform-in-time bound of solutions can be applied to determine the large time behavior of the corresponding system. We believe that our results are applicable in a variety of scenarios. To illustrate, we apply our results to an infectious disease model.

1.3. Main results and key ideas. We will introduce the notion of weak solutions which is defined in the following.

**Definition 1.1.** A vector of non-negative state variables $u = (u_1, \ldots, u_m)$ is called a weak solution to (1.1) on $(0, T)$ if

$$u_i \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)), \quad F_i(u) \in L^2(0, T; L^2(\Omega)), \quad F_i(u) \in L^2(0, T; L^2(\Omega)), \quad$$
REACTION-DIFFUSION-ADVECTION SYSTEMS WITH NON-SMOOTH DIFFUSION

with \( u_i(\cdot,0) = u_{i,0}(\cdot) \) for all \( i = 1, \ldots, m \), and for any test function \( \varphi \in L^2(0,T;H^1_0(\Omega)) \) with \( \partial_t \varphi \in L^2(0,T;H^{-1}(\Omega)) \), one has

\[
\int_{\Omega} u_i(x,t)\varphi(x,t)dx \bigg|_{t=0}^{t=T} - \int_0^T \int_{\Omega} u_i \partial_t \varphi dxdt + \int_0^T \int_{\Omega} D_i(x,t) \nabla u_i \cdot \nabla \varphi dxdt = \int_0^T \int_{\Omega} u_i B_i(x,t) \cdot \nabla \varphi dxdt + \int_0^T \int_{\Omega} F_i(u) \varphi dxdt.
\]

Our first main result is the global existence and uniform boundedness of system (1.1) under, among others, the assumption on control of mass (F3).

**Theorem 1.1.** Assume (1.2), (1.3), (1.4), (F1), (F2), (F3), (F4) and (F5). Assume moreover that

\[
0 \leq r < 1 + \frac{2}{n}. \tag{1.5}
\]

Then for any non-negative, bounded initial data \( u_0 \in L^\infty(\Omega)^m \), there exists a unique global weak solution to (1.1) with \( u_i \in L^\infty_{\text{loc}}(0,\infty;L^\infty(\Omega)) \) for all \( i = 1, \ldots, m \). Moreover, if \( K_1 < 0 \) or \( K_1 = K_2 = 0 \), then the solution is bounded uniformly in time, i.e.

\[
\text{ess sup}_{t \geq 0} \| u_i(t) \|_{L^\infty(\Omega)} < +\infty, \quad \forall i = 1, \ldots, m. \tag{1.6}
\]

**Remark 1.1.**

- When \( K_1 = K_2 = 0 \), the assumption (F3) becomes the well known mass dissipation, which has been considered frequently in the literature. The case \( K_1 < 0 \) can occur in chemical reactions when there is some continuous source of reactants. An example is the Gray-Scott system.
- The uniform-in-time bound (1.6) is shown by using the fact that the \( L^1(\Omega) \)-norm of the solution is bounded uniformly in time, which is inferred from the assumption \( K_1 < 0 \) or \( K_1 = K_2 = 0 \). If the \( L^1(\Omega) \)-norm can be shown to be bounded uniformly in time by some other way, e.g. using special structures of the system at hand, then the condition \( K_1 < 0 \) or \( K_1 = K_2 = 0 \) can be relaxed.

The proof of Theorem 1.1 is based on a new \( L^p \)-energy approach. Traditionally, one seeks an energy function of (1.1) of the form

\[
\mathcal{E}[u] = \sum_{i=1}^m \int_{\Omega} h_i(u_i) dx
\]

which is decreasing or at least bounded in time. When \( h_i(z) \sim z^p \), it yields an \( L^p \)-estimate of the solution, which, for \( p \) large enough, would eventually lead to bounds in \( L^\infty \)-norm. Such an approach is, unfortunately, very likely to fail under the general assumptions (F3)-(F4), except for some very special cases. There is another approach called the duality method (see [27, 3, 21]) which has proved very efficient when dealing with systems with constant or smooth diffusion coefficients. Using this method, one gets from the mass control condition (F3) an \( L^{2+\varepsilon}(\Omega \times (0,T)) \)-estimate. This initial estimate and another duality argument are sufficient to bootstrap the regularity of the solution, using the intermediate
sum condition (F4) and the growth assumption (1.5), to eventually obtain bounds in $L^\infty(\Omega \times (0, T))$ which ensure global existence. We refer the reader to [21] for more details. However, this method seems not extendable to the case of merely bounded measurable diffusion coefficients, unless some additional regularity assumptions are imposed (see [6, 2]).

The main idea of the $L^p$-approach in this paper is to look for an energy function consisting of mixed polynomials of order $p \in \mathbb{N}$, with well chosen coefficients, namely

$$
\mathcal{L}_p[u] = \int_{\Omega} \sum_{\deg(Q)=p} \theta_Q Q[u] dx,
$$

where $\theta_Q > 0$ depends on the monomial $Q[u]$. Thanks to the non-negativity of the solution and the choice of $\theta_Q$, $(\mathcal{L}_p[u])^{1/p}$ is an equivalent $L^p$-norm. Of course, the main challenge is to find an algorithm for choosing the coefficients $\theta_Q$ in a manner that it is compatible to both diffusion and reactions in the sense that the evolution of $\mathcal{L}_p[u]$ is well behaved. In the case at hand, we show that the intermediate sum condition (F4) allows us to choose monomials of the form

$$
Q[u] = \prod_{i=1}^{m} u_i^{\beta_i} \text{ for } \beta_i \in \mathbb{N}_0 \text{ satisfying } \sum_{i=1}^{m} \beta_i = p,
$$

and coefficients of the form

$$
\theta_Q = \frac{p!}{\beta_1! \ldots \beta_m!} \prod_{i=1}^{m} \theta_i^{\beta_i}
$$

with appropriately chosen $\theta_1, \ldots, \theta_m > 0$. We note that preliminary ideas of this $L^p$-energy approach have been used previously, see [18, 16]. A recent work [22] by the second and third authors has also used this method in the context of volume-surface systems with constant diffusion coefficients. The novelty of the present work is to significantly extend this $L^p$-approach to the case of nonsmooth diffusion (and advection) coefficients. This, in particular, requires non-trivial extensions in studying the properties of $\mathcal{L}_p[u]$ (see Lemmas 2.5 and 4.2). Moreover, we show that this method is sufficiently robust to model variants and different boundary conditions. It remains as an interesting open issue if the choice of the coefficients $\theta_Q$ is purely mathematical or due to deeper structure of the system under consideration.

It is worth noting that other conditions can also lead to a priori estimates. For example, the so-called entropy condition (see e.g. [14, 32, 10]) has been employed in a number of papers. In this case, one requires the existence of scalars $\mu_i \in \mathbb{R}$ so that

$$
\sum_{i=1}^{m} F_i(x, t, u) (\log(u_i) + \mu_i) \leq 0 \text{ for all } (x, t, u) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+^m.
$$

This condition guarantees an $L_1(\Omega)$ a priori estimate for $H(u(\cdot, t))$ where

$$
H(u) = \sum_{i=1}^{m} u_i (\log u_i - 1 + \mu_i) \text{ for all } u_i \geq 0.
$$
In this case of the entropy condition, we could obtain a priori estimates by multiplying the PDE for $u_i$ in (1.1) by $\log(u_i) + \mu_i$, integrating over $\Omega$ and summing the results. Note that this takes advantage of the positivity of the second derivative of $u_i(\log u_i - 1 + \mu_i)$. More generally, we could assume the existence of a set $M = \prod_{k=1}^m (\alpha_k, \beta_k)$ where $\alpha_k, \beta_k$ are extended real numbers such that $\alpha_k < \beta_k$ for each $i = 1, \ldots, m$, and solutions to (1.1) remain in $M$ whenever initial data lies in $M$, a function $H : M \to \mathbb{R}_+$ that is $C^2$ and has the form

$$H(u) = \sum_{i=1}^m h_i(u_i)$$

where $h_i : (\alpha_i, \beta_i) \to \mathbb{R}_+$ satisfies

$$h''_i(z) \geq 0 \quad \text{for all } z \in (\alpha_i, \beta_i),$$

$$h_i(z) \text{ is bounded implies } z \text{ is bounded},$$

$$\nabla H(u) \cdot F(x, t, u) \leq K_5 \sum_{i=1}^m h_i(u_i) + K_6 \text{ for all } (x, t, u) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+^m,$$

for some $K_5, K_6 > 0$. Then, analogous to above, we would obtain an $L_1(\Omega)$ a priori estimate for $H(u(\cdot, t))$. In addition, intermediate sum conditions could also be written in the form

$$A \begin{pmatrix} h'_1(u_1)F_1(x, t, u) \\ \vdots \\ h'_m(u_1)F_m(x, t, u) \end{pmatrix} \leq K_7 \tilde{I} \left( \sum_{i=1}^m h_i(u_i) + 1 \right)^r \text{ for all } (x, t, u) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+^m,$$

that would lead to results in the same manner as we obtain from (F3) and (F4) above. For more information, see [19, 20] in the case of constant diffusion coefficients.

Our next main result is the following theorem.

**Theorem 1.2.** Assume (1.2), (1.3), (1.4), (F1), (F2), (F5), and (H1), (H2). Assume moreover that

$$0 \leq r < 1 + \frac{2}{n}.$$

Then for any non-negative, bounded initial data $u_0 \in L^\infty(\Omega)^m$, there exists a unique global weak solution to (1.1) with $u_i \in L^\infty_{loc}(0, \infty; L^\infty(\Omega))$ for all $i = 1, \ldots, m$. Moreover, if $K_5 < 0$ or $K_5 = K_6 = 0$, then the solution is bounded uniformly in time, i.e.

$$\text{ess sup}_{t \geq 0} \|u_i(t)\|_{L^\infty(\Omega)} < +\infty, \quad \forall i = 1, \ldots, m.$$

The following theorem gives a conditional result in the sense that if, by using the specific structure of the system, one can get better a-priori estimates, then the exponent $r$ in the intermediate sum condition (F4) can be enlarged.

**Theorem 1.3.** Assume (1.2), (1.3), (1.4), (F1), (F2), (F4), (F5). Suppose that there exists either a constant $a \geq 1$ such that

$$\|u_i\|_{L^\infty(0,T; L^a(\Omega))} \leq \mathcal{F}(T), \quad \forall i = 1, \ldots, m,$$

(1.7)
or a constant \( b \geq 1 \) such that
\[
\| u_i \|_{L^b(0,T;L^b(\Omega))} \leq \mathcal{F}(T), \quad \forall i = 1, \ldots, m,
\] (1.8)
where \( \mathcal{F} : [0, \infty) \rightarrow \mathbb{R}_+ \) is a continuous function. Assume additionally that
\[
0 \leq r < \begin{cases} 
1 + 2a/n, & \text{in case of (1.7)}, \\
1 + 2b/(n+2), & \text{in case of (1.8)}.
\end{cases}
\] (1.9)

Then for any non-negative, bounded initial data \( u_0 \in L^\infty(\Omega)^m \), there exists a unique global weak solution to (1.1) with \( u_i \in L^\infty_{\text{loc}}(0,\infty;L^\infty(\Omega)) \) for all \( i = 1, \ldots, m \). Moreover, if \( \sup_{T \geq 0} \mathcal{F}(T) < +\infty \), then the solution is bounded uniformly in time, i.e.
\[
\text{ess sup}_{t \geq 0} \| u_i(t) \|_{L^\infty(\Omega)} < +\infty, \quad \forall i = 1, \ldots, m.
\]

Our approach is sufficiently robust to extend to other boundary conditions such as Neumann or Robin type, or to certain quasilinear systems. The precise results are stated in the following theorems.

**Theorem 1.4.** Consider the system
\[
\begin{aligned}
&\partial_t u_i - \nabla \cdot (D_i(x,t)\nabla u_i) = F_i(x,t,u), \quad x \in \Omega, t > 0, \\
&D_i(x,t)\nabla u_i(x,t) \cdot \nu(x) + \alpha_i u_i(x,t) = 0, \quad x \in \partial \Omega, t > 0, \\
&u_i(x,0) = u_{i,0}(x), \quad x \in \Omega,
\end{aligned}
\] (1.10)

where \( \nu(x) \) is the outward unit normal vector on \( \partial \Omega \), and \( \alpha_i \geq 0 \) for all \( i = 1, \ldots, m \).

Assume (1.2), (1.3), (F1), (F2), (F4) and (F5). Moreover, assume either (F3) or (1.7) or (1.8) with
\[
0 \leq r < \begin{cases} 
1 + 2/n, & \text{in case of (F3)}, \\
1 + 2a/n, & \text{in case of (1.7)}, \\
1 + 2b/(n+2), & \text{in case of (1.8)}.
\end{cases}
\]

Then for any non-negative, bounded initial data \( u_0 \in L^\infty(\Omega)^m \), there exists a unique global weak solution to (1.10) (see Definition 2.1) with \( u_i \in L^\infty_{\text{loc}}(0,\infty;L^\infty(\Omega)) \) for all \( i = 1, \ldots, m \). Moreover, if \( K_1 < 0 \) or \( K_1 = K_2 = 0 \) in case of (F3), or \( \sup_{T \geq 0} \mathcal{F}(T) < +\infty \) in case of (1.7) or (1.8), then the solution is bounded uniformly in time, i.e.
\[
\text{ess sup}_{t \geq 0} \| u_i(t) \|_{L^\infty(\Omega)} < +\infty, \quad \forall i = 1, \ldots, m.
\]

**Remark 1.2.**

- A direct corollary of Theorem 1.4 is the global existence and boundedness of systems with Lipschitz nonlinearities.
- The uniform in time bound of solutions in Theorem 1.4, and previous theorems, is also important in numerical analysis. This can allow one to obtain convergence of the numerical scheme uniformly in time, see e.g. [7].
Theorem 1.4 can also be extended to the case of nonlinear boundary conditions, i.e.
\[ D_i(x,t) \nabla u_i(x,t) \cdot \nu(x) + \alpha_i u_i(x,t) = G_i(u), \quad x \in \partial \Omega, \]
where the nonlinearities \( G_i \) also satisfy an intermediate sum condition. The details are left for the interested reader. We refer to [22, 31] for a related work dealing with constant diffusion coefficients.

Theorem 1.5. Consider the system
\[
\begin{aligned}
\partial_t u_i - \nabla \cdot (A_i(x,t,u) \nabla u_i) &= F_i(x,t,u), \quad x \in \Omega, t > 0, \\
u_i(x,t) &= 0, \quad x \in \partial \Omega, t > 0, \\
u_i(x,0) &= u_{i,0}(x), \quad x \in \Omega,
\end{aligned}
\]  
(1.11)

where, for each \( i = 1, \ldots, m \), \( A_i : \Omega \times (0, \infty) \times \mathbb{R}^n_+ \to \mathbb{R}^{n \times n} \) satisfies the following conditions:

(i) For a.e. \((x,t) \in \Omega \times \mathbb{R}_+\), the map \( \mathbb{R}^m \ni \omega \mapsto A_i(x,t,\omega) \) is continuous, for all \( i = 1, \ldots, m \).

(ii) There exists \( \lambda > 0 \) such that
\[ \lambda |\xi|^2 \leq \xi^\top A_i(x,t,\omega) \xi \quad \forall (x,t,\omega) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^m_+, \forall \xi \in \mathbb{R}^n, \forall i = 1, \ldots, m. \]

(iii) For each \( T > 0 \),
\[ \text{ess sup}_{(x,t,\omega) \in \Omega \times (0,T) \times \mathbb{R}^m_+} |A_i(x,t,\omega)| < +\infty. \]

Assume (F1), (F2), (F4) and (F5). Moreover, assume either (F3) or (1.7) or (1.8) with
\[ 0 \leq r < \begin{cases} 
1 + 2/n, & \text{in case of (F3)}, \\
1 + 2a/n, & \text{in case of (1.7)}, \\
1 + 2b/(n + 2), & \text{in case of (1.8)}. 
\end{cases} \]

Then for any non-negative, bounded initial data \( u_0 \in L^\infty(\Omega)^m \), there exists a global weak solution to (1.11) with \( u_i \in L^\infty_{\text{loc}}(0, \infty; L^\infty(\Omega)) \) for all \( i = 1, \ldots, m \). Moreover, if \( K_1 < 0 \) or \( K_1 = K_2 = 0 \) in case of (F3), or \( \sup_{T \geq 0} \mathcal{F}(T) < +\infty \) in case of (1.7) or (1.8), then this solution is bounded uniformly in time, i.e.
\[ \text{ess sup}_{t \geq 0} \|u_i(t)\|_{L^\infty(\Omega)} < +\infty, \quad \forall i = 1, \ldots, m. \]

In addition, if the diffusion coefficients satisfy:

(iv) for any \( T > 0 \), the mapping \( \Omega \times [0,T] \ni (x,t) \mapsto A_i(\cdot,\cdot,\omega) \) is Hölder continuous in each component,

then the weak solution obtained above is unique.
1.4. Structure of the paper. In the next section, we provide the proofs of our main results. In subsection 2.1, we prove Theorem 1.1 by first considering an approximate system where we regularize the nonlinearities to obtain global approximate weak solutions, then we derive uniform estimates, applying the key idea of the $L^p$-energy functions, and pass to the limit to obtain global existence and uniqueness of (1.1). Subsection 2.2 provides the proof of the generalized results in Theorems 1.2 and 1.3. The last subsection 2.3 gives the proof of Theorem 3.1 concerning different types of boundary conditions or quasilinear systems. Section 3 is devoted to the application of our results to an infectious disease without lifetime immunity. Finally, in Section 4 we provide technical proofs concerning the construction of the $L^p$-energy functions that give rise to our uniform bounds.

2. Proofs

2.1. Systems with control of mass: Proof of Theorem 1.1. To show the global existence of (1.1), we start with a truncated system where the nonlinearities are regularized to be bounded. It is important that the regularization preserves the properties (F1), (F2), (F3), (F4) and (F5). The next crucial step is to derive uniform estimates for truncated systems where the bound is independent of the truncation. The main idea behind our approach is the development of an $L^p$-energy function for $p \geq 2$. This allows us to obtain $L^p$-estimates for the approximate solutions for all $2 \leq p < +\infty$, which is enough to conclude the boundedness thanks to the polynomial growth (F5). The uniform-in-time bound is shown by examining the system on each cylinder $\Omega \times (\tau, \tau + 2)$ for $\tau \in \mathbb{N}$. The final step of passing to the limit is straightforward because of our obtained a-priori estimates.

2.1.1. Truncated systems. For any $\varepsilon > 0$, we consider the truncated system: for all $i = 1, \ldots, m$,

$$
\begin{aligned}
\partial_t u^\varepsilon_i - \nabla \cdot (D_i \nabla u^\varepsilon_i) + \nabla \cdot (B_i u^\varepsilon_i) &= F^\varepsilon_i(u^\varepsilon), & x \in \Omega, t > 0, \\
u_i^\varepsilon(x, t) &= 0, & x \in \partial \Omega, t > 0, \\
u_i^\varepsilon(x, 0) &= u_{i,0}^\varepsilon(x), & x \in \Omega,
\end{aligned}
$$

(2.1)

where

$$
F^\varepsilon_i(u^\varepsilon) := F_i(u^\varepsilon) \left[1 + \varepsilon \sum_{j=1}^m |F_j(u^\varepsilon)|\right]^{-1},
$$

(2.2)

and $u_{i,0}^\varepsilon \in L^\infty(\Omega)$ such that $\|u_{i,0}^\varepsilon - u_i\|_{L^\infty(\Omega)} \xrightarrow{\varepsilon \to 0} 0$. Note that since $u_{i,0} \in L^\infty(\Omega)$ one can take $u_{i,0}^\varepsilon = u_i$ to obtain the global existence of global existence. The approximation $u_{i,0}^\varepsilon$ we consider here can be relevant for numerical analysis, where the initial data is discretized and approximated by e.g. finite dimensional data.

Lemma 2.1. For any fixed $\varepsilon > 0$, there exists a global bounded, non-negative weak solution to (2.1).

Proof. Since for fixed $\varepsilon > 0$, the nonlinearities $F^\varepsilon_i(u^\varepsilon)$ are Lipschitz continuous and are bounded above by $1/\varepsilon$ uniformly in $u^\varepsilon$, the global existence of a unique weak solution follows from the standard Galerkin method, see e.g. [17]. We leave the details to the
reader. It remains to show the non-negativity of $u^\varepsilon$. Denote by $u^\varepsilon_{i,+} := \max\{u^\varepsilon_i, 0\}$ and $u^\varepsilon_{i,-} := \min\{u^\varepsilon_i, 0\}$. We consider the auxiliary system of (2.1)

$$\partial_t u^\varepsilon_i - \nabla \cdot (D_i \nabla u^\varepsilon_i) + \nabla \cdot (B_i u^\varepsilon_i) = F^\varepsilon_i (u^\varepsilon_{i,+})$$

where $u^\varepsilon_+ = (u^\varepsilon_{i,+})_{i=1,...,m}$. Thanks to the uniqueness, it is sufficient to show the non-negativity for the solution of this system. By multiplying this auxiliary system by $u^\varepsilon_{i,-}$ and using the quasi-positivity assumption ($F2$) (recall that this property also holds for $F^\varepsilon_i$), we obtain

$$\frac{1}{2} \int_\Omega |u^\varepsilon_{i,-}|^2 dx + \lambda \int_\Omega |\nabla u^\varepsilon_{i,-}|^2 dx \leq \Gamma \int_\Omega |u^\varepsilon_{i,-}||\nabla u^\varepsilon_{i,-}| \leq \frac{\lambda}{2} \int_\Omega |\nabla u^\varepsilon_{i,-}|^2 dx + C \int |u^\varepsilon_{i,-}|^2 dx.$$ 

By applying the Gronwall lemma, we get $u^\varepsilon_{i,-} = 0$ a.e. in $Q_T$, since $u^\varepsilon_{i,0,-} = 0$. This shows the desired non-negativity.

2.1.2. Uniform-in-$\varepsilon$ estimates. In this subsection, we prove crucial uniform-in-$\varepsilon$ estimates for the solution to the truncated system (2.1). Moreover, we want to emphasize that all the constants in this subsection are independent of $\varepsilon$.

We start off with the estimate in $L^\infty(0,T;L^1(\Omega))$.

**Lemma 2.2.** Assume (F1), (F2) and (F3). Then for any $T > 0$, there exists a constant $M_T$ depending on $T, \Omega, \|u_{i,0}\|_{L^1(\Omega)}$ and $c_1,\ldots,c_m, K_1, K_2$ in (F3) such that

$$\sup_{t \in (0,T)} \|u^\varepsilon_i(t)\|_{L^1(\Omega)} \leq M_T \quad \forall i = 1, \ldots, m. \quad (2.3)$$

**Proof.** Formally, the desired $L^1$-bound can expected by first multiplying the equation of $u_i$ by $c_i$, summing the resultant, and then integrating the sum in $\Omega \times (0,t)$. Noticing that the solution is non-negative and vanishes on the boundary, we expect the flux $D_i \nabla u^\varepsilon_i \cdot \nu \leq 0$. Unfortunately, since we are dealing with weak solutions, this formal procedure is not justified. On the other hand, $\varphi \equiv 1$ is not admissible as a test function since it does not belong to $L^2(0,T;H^1_0(\Omega))$. We therefore have to resort to a different strategy. Let $\delta > 0$. We define the continuous (and piecewise smooth) function $h_\delta : \mathbb{R} \to \mathbb{R}$ as

$$h_\delta(s) = \begin{cases} 
1 & \text{if } s \geq \delta, \\
\frac{s}{\delta} & \text{if } 0 < s < \delta, \\
0 & \text{if } s \leq 0.
\end{cases}$$

We also use the following primitive of $h_\delta$ given by

$$H_\delta(s) = \begin{cases} 
s - \delta/2 & \text{if } s \geq \delta, \\
s^2/(2\delta) & \text{if } 0 < s < \delta, \\
0 & \text{if } s \leq 0.
\end{cases}$$
Taking $h_\delta(u_\varepsilon^i) \in L^2(0, T; H_0^1(\Omega))$ as a test function in (2.1) yields
\[
\int_0^T \left\langle \partial_t u_\varepsilon^i, h_\delta(u_\varepsilon^i) \right\rangle dt + \int_\Omega \int_0^T D_i \nabla u_\varepsilon^i \cdot \nabla (h_\delta(u_\varepsilon^i)) dx dt - \int_0^T \int_\Omega B_i u_\varepsilon^i \cdot \nabla (h_\delta(u_\varepsilon^i)) dx dt \\
= \int_0^T \int_\Omega F_i^\varepsilon(u^\varepsilon) h_\delta(u_\varepsilon^i) dx dt.
\] (2.4)

Since $\partial_t u_\varepsilon^i \in L^2(0, T; H^{-1}(\Omega))$ and $h_\delta(u_\varepsilon^i) \in L^2(0, T; H_0^1(\Omega))$ we have
\[
\int_0^T \left\langle \partial_t u_\varepsilon^i, h_\delta(u_\varepsilon^i) \right\rangle dt = \int_\Omega H_\delta(u_\varepsilon^i(\cdot, T)) dx - \int_\Omega H_\delta(u_\varepsilon^i(\cdot, 0)) dx.
\]

Due to $\nabla (h_\delta(u_\varepsilon^i)) = \chi_{\{|u_\varepsilon^i| \leq \delta\}} \delta^{-1} \nabla u_\varepsilon^i$ and (1.2) we have
\[
\int_0^T \int_\Omega D_i \nabla u_\varepsilon^i \cdot \nabla (h_\delta(u_\varepsilon^i)) dx dt \geq \lambda \int_0^T \int_\Omega \chi_{\{|u_\varepsilon^i| \leq \delta\}} \delta^{-1} |\nabla u_\varepsilon^i|^2 dx dt \geq 0,
\]
and, using (1.4),
\[
\left| \int_0^T \int_\Omega B_i u_\varepsilon^i \cdot \nabla (h_\delta(u_\varepsilon^i)) dx dt \right| \leq \delta^{-1} \int_0^T \int_\Omega |B_i| |u_\varepsilon^i| \chi_{\{|u_\varepsilon^i| \leq \delta\}} |\nabla u_\varepsilon^i| dx dt \\
\leq \Gamma \int_0^T \int_\Omega \chi_{\{|u_\varepsilon^i| \leq \delta\}} |\nabla u_\varepsilon^i| dx dt.
\]

Therefore, it follows from (2.4) that
\[
\int_\Omega H_\delta(u_\varepsilon^i(\cdot, T)) dx \leq \int_\Omega H_\delta(u_\varepsilon^i(\cdot, 0)) dx + \Delta \int_0^T \int_\Omega \chi_{\{|u_\varepsilon^i| \leq \delta\}} |\nabla u_\varepsilon^i| dx dt + \int_0^T \int_\Omega F_i^\varepsilon(u^\varepsilon) h_\delta(u_\varepsilon^i) dx dt.
\] (2.5)

Letting $\delta \to 0$, and using $u_\varepsilon^i \in L^2(0, T; H_0^1(\Omega))$ we get
\[
\limsup_{\delta \to 0} \int_0^T \int_\Omega \chi_{\{|u_\varepsilon^i| \leq \delta\}} |\nabla u_\varepsilon^i| dx dt = 0.
\] (2.6)

We will show that
\[
\lim_{\delta \to 0} \int_0^T \int_\Omega F_i^\varepsilon(u^\varepsilon) h_\delta(u_\varepsilon^i) dx dt \leq \int_0^T \int_\Omega F_i^\varepsilon(u^\varepsilon) dx dt.
\]

Indeed, we write
\[
F_i^\varepsilon(u^\varepsilon) h_\delta(u_\varepsilon^i) = F_i^\varepsilon(u^\varepsilon) \mathbf{1}_{\{u^\varepsilon > \delta\}} + \frac{1}{\delta} F_i^\varepsilon(u^\varepsilon) u_\varepsilon^i \mathbf{1}_{\{0 \leq u^\varepsilon \leq \delta\}} =: \psi^\varepsilon(x, t) + \varphi^\varepsilon(x, t).
\] (2.7)

It is clear that $\lim_{\delta \to 0} \varphi^\varepsilon(x, t) = 0$ for a.e. $(x, t) \in \Omega \times (0, T)$. Moreover, $|\varphi^\varepsilon(x, t)| \leq |F_i^\varepsilon(u^\varepsilon(x, t))| \leq 1/\varepsilon$ a.e. $(x, t)$ due to (2.2). Thus, it holds
\[
\lim_{\delta \to 0} \int_0^T \int_\Omega \varphi^\varepsilon(x, t) dx dt = 0.
\]
For \((x, t)\) such that \(u^\varepsilon(x, t) > 0\), we have \(\lim_{\delta \to 0} \psi^\delta(x, t) = F_i^\varepsilon(u^\varepsilon)\), while if \(u^\varepsilon(x, t) = 0\), then \(\lim_{\delta \to 0} \psi^\delta(x, t) = 0 \leq F_i^\varepsilon(u^\varepsilon(x, t))\) thanks to (F2) and (2.2). Hence,

\[
\lim_{\delta \to 0} \int_0^T \int_\Omega \psi^\delta(x, t) dx dt \leq \int_0^T \int_\Omega F_i^\varepsilon(u^\varepsilon) dx dt.
\]

Therefore, we can let \(\delta \to 0\) in (2.5) to obtain

\[
\int \Omega u_i^\varepsilon(\cdot, T) dx \leq \int \Omega u_i^\varepsilon(\cdot, 0) dx + \int_0^T \int_\Omega F_i^\varepsilon(u^\varepsilon) dx dt.
\]

Now, by using (F3), we obtain

\[
\sum_{i=1}^m c_i \int \Omega u_i^\varepsilon(\cdot, T) dx \leq \sum_{i=1}^m c_i \int \Omega u_i^\varepsilon(\cdot, 0) dx + \int_0^T \int_\Omega \left( K_1 \sum_{i=1}^m u_i^\varepsilon + K_2 \right) dx dt. \tag{2.8}
\]

The Gronwall lemma gives the desired \(L^1\)-estimate. \(\Box\)

**Building \(L^p\)-energy functions:** To this end, we write \(\mathbb{Z}_+^m\) as the set of all \(m\)-tuples of non-negative integers. Addition and scalar multiplication by non-negative integers of elements in \(\mathbb{Z}_+^m\) is understood in the usual manner. If \(\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{Z}_+^m\) and \(p \in \mathbb{N} \cup \{0\}\), then we define \(\beta^p = ((\beta_1)^p, \ldots, (\beta_m)^p)\). Also, if \(\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}_+^m\), then we define \(|\alpha| = \sum_{i=1}^m \alpha_i\). Finally, if \(z = (z_1, \ldots, z_m) \in \mathbb{R}_+^m\) and \(\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}_+^m\), then we define \(z^\alpha = z_{1}^{\alpha_1} \cdots z_{m}^{\alpha_m}\), where we interpret \(0^0\) to be 1. For \(p \in \mathbb{N} \cup \{0\}\), we build our \(L^p\)-energy function of the form

\[
\mathcal{L}_p[u^\varepsilon](t) = \int_\Omega \mathcal{H}_p[u^\varepsilon](t) dx \tag{2.9}
\]

where

\[
\mathcal{H}_p[u^\varepsilon](t) = \sum_{\beta \in \mathbb{Z}_+^m, |\beta| = p} \left( \frac{p!}{\beta_1! \cdots \beta_m!} \right) \theta^{\beta_2 u^\varepsilon(t)^\beta}, \tag{2.10}
\]

with

\[
\left( \frac{p!}{\beta_1! \cdots \beta_m!} \right) = \frac{p!}{\beta_1! \cdots \beta_m!}, \tag{2.11}
\]

and \(\theta = (\theta_1, \ldots, \theta_m)\) where \(\theta_1, \ldots, \theta_m\) are positive real numbers which will be determined later. For convenience, hereafter we drop the subscript \(\beta \in \mathbb{Z}_+^m\) in the sum as it should be clear. Note that if \(\theta = (1, \ldots, 1)\) in (2.10), we have

\[
\mathcal{H}_p[u^\varepsilon](t) = \sum_{|\beta| = p} \left( \frac{p!}{\beta_1! \cdots \beta_m!} \right) u^\varepsilon(t)^\beta = \left( \sum_{i=1}^m u_i^\varepsilon(t) \right)^p.
\]

Therefore, for a given \(2 \leq p \in \mathbb{N}\), \(\mathcal{H}_p[u^\varepsilon]\) is a generalization of a multinomial expansion of degree \(p\) in \(u^\varepsilon\). For \(p = 0, 1, 2\), one can write these functions explicitly as

\[
\mathcal{H}_0[u^\varepsilon](t) = 1 \quad \text{and} \quad \mathcal{H}_1[u^\varepsilon](t) = \sum_{j=1}^m \theta_j u_j^\varepsilon(t).
\]
and

$$\mathcal{H}_2[u^\varepsilon](t) = \sum_{i=1}^{m} \theta_i^4 u_i^\varepsilon(t)^2 + 2 \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \theta_i \theta_j u_i^\varepsilon(t) u_j^\varepsilon(t).$$

Thanks to the non-negativity of the solution, we have

$$\mathcal{L}_p[u^\varepsilon](t) \sim \sum_{i=1}^{m} \|u_i^\varepsilon(t)\|_{L^p(\Omega)}^p.$$

Therefore, we are going to use $\mathcal{L}_p[u^\varepsilon](t)$ to obtain a priori estimates on $u^\varepsilon$ for each $2 \leq p \in \mathbb{N}$. We will need two technical lemmas, concerning the derivative in time of $H^p$ and integration by parts, whose proofs are postponed to Section 4 in order not to interrupt the train of thought.

Next, we need the following functional inequality, which was proved in [22].

**Lemma 2.3.** [22, Lemma 2.3] Suppose $\Omega \subset \mathbb{R}^n$ such that the Gagliardo-Nirenberg inequality is satisfied and basic trace theorems apply (for instance $\Omega$ has Lipschitz boundary). Let $a \geq 1$, $p \geq 2a$ and $w : \Omega \to \mathbb{R}_+$ such that $w^{p/2} \in H^1(\Omega)$ and there exists $K \geq 0$ such that $\|w\|_{a,\Omega} \leq K$. If $0 \leq s < 2a/n$ and $\kappa > 0$, then there exists $C_\kappa \geq 0$ (depending on $p, \kappa, r, a, \Omega$, but independent of $w$) such that

$$\int_{\Omega} w^{p+s} \, dx \leq \kappa \int_{\Omega} \left( w^{p-2} |\nabla w|^2 + w^p \right) \, dx + C_\kappa. \quad (2.12)$$

The following lemma shows a consequence of the intermediate sum condition (F4) which is crucial to the construction of the $L^p$-energy function.

**Lemma 2.4.** Assume (F4). Then there exist componentwise increasing functions $g_i : \mathbb{R}^{m-i} \to \mathbb{R}_+$, for $i = 1, \ldots, m-1$ such that: if $\theta = (\theta_1, \ldots, \theta_m) \in (0, \infty)^m$ satisfies $\theta_m > 0$ and $\theta_i \geq g_i(\theta_{i+1}, \ldots, \theta_m)$ for all $i = 1, \ldots, m-1$, then

$$\sum_{i=1}^{m} \theta_i F_i^\varepsilon(x, t, u^\varepsilon) \leq K_\theta \left( 1 + \sum_{i=1}^{m} (u_i^\varepsilon)^r \right) \quad \forall (x, t, u^\varepsilon) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+^m$$

for some constant $K_\theta$ depending on $\theta$, $g_i$, and $K_3$ in (F4).

**Proof.** The proof follows exactly from [22, Lemma 2.2] with the observation

$$\sum_{i=1}^{m} \theta_i F_i^\varepsilon(x, t, u^\varepsilon) \leq \max \left\{ 0; \sum_{i=1}^{m} \theta_i F_i(x, t, u^\varepsilon) \right\} \quad \forall (x, t, u^\varepsilon) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+^m.$$

We are now ready to use the $L^p$-energy functions built in (2.9)-(2.10) to obtain the $L^p$-estimates of $u^\varepsilon$. 

\[\square\]
Lemma 2.5. Assume (F1), (F2), (F3) and (F4). Then for any $1 \leq p < \infty$ and any $T > 0$, there exists a constant $C_{T,p}$ depending on $T$, $p$ and other parameters such that

$$\sup_{t \in (0,T)} \| u_i^\varepsilon(t) \|_{L^p(\Omega)} \leq C_{T,p} \quad \forall i = 1, \ldots, m.$$ 

Proof. Let $u^\varepsilon$ solve (2.1), and $\mathcal{L}_p(t) := \mathcal{L}_p[u^\varepsilon](t)$ be defined in (2.9). Then

$$\mathcal{L}_p(t) = \int_\Omega \sum_{|\beta| = p - 1} \left( \begin{array}{c} p \\ \beta \end{array} \right) \theta^{\beta^2} u^\varepsilon(x,t)^\beta \sum_{k=1}^m \theta_k^{2\beta_{k+1}} \frac{\partial}{\partial t} u_k^\varepsilon(x,t) dx \quad \text{for all } \beta \neq 0.$$ 

If we apply Lemma 4.2 and integration by parts, we have

$$\int_\Omega \sum_{|\beta| = p - 1} \left( \begin{array}{c} p \\ \beta \end{array} \right) \theta^{\beta^2} u^\varepsilon(x,t)^\beta \sum_{k=1}^m \theta_k^{2\beta_{k+1}} \nabla \cdot (D_k(x,t)\nabla u_k^\varepsilon(x,t)) dx = I,$$

where

$$I = -\int_\Omega \sum_{|\beta| = p - 2} \left( \begin{array}{c} p \\ \beta \end{array} \right) \theta^{\beta^2} u^\varepsilon(x,t)^\beta \sum_{k=1}^m \theta_k^{2\beta_{k+1}} \sum_{l=1}^m C_{k,l}(\beta) (D_k \nabla u_k^\varepsilon) \cdot \nabla u_l^\varepsilon dx$$

with

$$C_{k,l}(\beta) = \begin{cases} \theta_k^{2\beta_{k+1}} \theta_l^{2\beta_{l+1}}, & k \neq l, \\ \theta_k^{4\beta_{k+4}}, & k = l. \end{cases}$$

For a given $\beta$ with $|\beta| = p - 2$, create an $mn \times mn$ matrix $B(\beta)$ made up of $m^2$ blocks $B_{k,l}(\beta)$, each of size $n \times n$, where

$$B_{k,l}(\beta) = \frac{1}{2} C_{k,l}(\beta) (D_k + D_l).$$

Note that for each $k = 1, \ldots, m$,

$$B_{k,k}(\beta) = \theta_k^{4\beta_{k+4}} D_k.$$ 

Also,

$$I = -\int_\Omega \sum_{|\beta| = p - 2} \left( \begin{array}{c} p \\ \beta \end{array} \right) \theta^{\beta^2} u^\varepsilon(x,t)^\beta \nabla u^\varepsilon(x,t)^T B(\beta) \nabla u^\varepsilon(x,t) dx,$$

where $\nabla u^\varepsilon(x,t)$ is a column vector of size $mn \times 1$, and for $j = 1, \ldots, m$, entries $n(j - 1) + 1$ to $nj$ of $\nabla u^\varepsilon(x,t)$ are $\nabla u_j^\varepsilon(x,t)$. We claim that if all of the entries in $\theta$ are sufficiently
large, then $B(\beta)$ is positive definite. In fact, it is a simple matter to show it is positive
definite if and only if the $mn \times mn$ matrix $\tilde{B}(\beta)$ made up of $n \times n$ blocks

$$
\tilde{B}_{k,l}(\beta) = \begin{cases} 
\theta_k^2 D_k, & k = l, \\
\frac{1}{2} (D_k + D_l), & k \neq l,
\end{cases}
$$
is positive definite. However, if we recall the uniform positive definiteness of the matrices
$D_k$, we can show that if $\theta_i$ is sufficient large for each $i$, then we have what we need. Consequently, returning to above, we can show there exists $\alpha_p > 0$ so that

$$
\mathcal{L}_p'(t) + \alpha_p \sum_{k=1}^m \int_\Omega |\nabla (u^\varepsilon_k)^{p/2}(x,t)|^2 dx \\
\leq \int \sum_{|\beta| = p-1} \left( \frac{p}{\beta} \right) \theta^{2\beta+1} \sum_{k=1}^m \theta_k^{2\beta+1} \left( -\nabla \cdot (B_k u^\varepsilon_k) + F^\varepsilon_k (u^\varepsilon(x,t)) \right) dx. \tag{2.13}
$$

For the first term on the right hand side of (2.13) we can use integration by parts and Hölder’s inequality to obtain

$$
- \int \sum_{|\beta| = p-1} \left( \frac{p}{\beta} \right) \theta^{2\beta+1} \sum_{k=1}^m \theta_k^{2\beta+1} \nabla \cdot (B_k u^\varepsilon_k) \\
\leq \frac{\alpha_p}{2} \sum_{k=1}^m \int_\Omega |\nabla (u^\varepsilon_k)^{p/2}(x,t)|^2 dx + C_{\theta,p} \sum_{k=1}^m \int_\Omega |u^\varepsilon_k|^p dx. \tag{2.14}
$$

We now look closely at the second term on the right hand side of (2.13), and in particular the term

$$
\sum_{k=1}^m \theta_k^{2\beta+1} F^\varepsilon_k (u^\varepsilon).
$$

Note that from (F4) and Lemma 2.4, there exist componentwise increasing functions $g_i : \mathbb{R}^{m-i} \rightarrow \mathbb{R}_+$ for $i = 1, ..., m-1$ so that if $\gamma_m > 0$ and $\gamma_i \geq g_i(\gamma_{i+1}, ..., \gamma_m)$ for $i = 1, ..., m-1$ then there exists $K_{\gamma} > 0$ so that

$$
\sum_{k=1}^m \gamma_k F^\varepsilon_k (x,t,u^\varepsilon) \leq K_{\gamma} \left( 1 + \sum_{i=1}^m (u^\varepsilon_i)^r \right) \text{ for all } (x,t,u^\varepsilon) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^m_+.
$$

So, we choose $\theta$ so that its components are sufficiently large that the previous positive definiteness condition is satisfied, and

$$
\theta_i \geq g_i(\theta_1^{2p-1}, ..., \theta_{i-1}^{2p-1}) \text{ for } i = 1, ..., m-1,
$$

where $g_i$ are functions constructed in Lemma 2.4. Then there exists $K_{\beta}$ so that for all $\beta \in \mathbb{Z}_+$ with $|\beta| = p - 1$, we have

$$
\sum_{k=1}^m \theta_k^{2\beta+1} F^\varepsilon_k (u^\varepsilon(x,t)) \leq K_{\beta} \left( 1 + \sum_{i=1}^m (u^\varepsilon_i)^r \right) \text{ for all } (x,t,u^\varepsilon) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^m_+.
It follows from this and (2.14) that there exists $C_p > 0$ so that (2.13) implies
\[
\mathcal{L}_p'(t) + \frac{\alpha_p}{2} \sum_{k=1}^{m} \int_{\Omega} |\nabla (u_k^\varepsilon)^{p/2}(x,t)|^2 \, dx \leq C_p \int_{\Omega} \sum_{k=1}^{m} \left( u_k^\varepsilon(x,t)^{p-1+r} + u_k^\varepsilon(x,t)^p + 1 \right) \, dx.
\] (2.15)

By adding both sides with $\alpha_p \sum_{k=1}^{m} \int_{\Omega} (u_k^\varepsilon)^p \, dx$ and using $(u_k^\varepsilon)^p \leq C [1 + (u_k^\varepsilon)^{p-1+r}]$ due to $r \geq 1$ we get
\[
\mathcal{L}_p'(t) + \frac{\alpha_p}{2} \sum_{k=1}^{m} \int_{\Omega} \left( |\nabla (u_k^\varepsilon)^{p/2}|^2 + |u_k^\varepsilon|^p \right) \, dx \leq C_p \left( 1 + \sum_{k=1}^{m} \int_{\Omega} (u_k^\varepsilon)^{p-1+r} \, dx \right).
\] (2.16)

Applying Lemma 2.3 with $a = 1$ and $1 \leq r < 1 + 2/n$ to the right hand side above, implies there exists $C_{T,p} > 0$, $C > 0$ and $\delta > 0$ so that
\[
\mathcal{L}_p'(t) + C \int_{\Omega} \sum_{k=1}^{m} u_k^\varepsilon(x,t)^p \, dx \leq C_{T,p},
\] (2.17)
which implies
\[
\mathcal{L}_p'(t) + \delta \mathcal{L}_p(t) \leq C_{T,p},
\] (2.18)
for some $\delta > 0$. Clearly, (2.18) allows us to obtain the estimates
\[
\sup_{t \geq 0} \mathcal{L}_p(t) \leq C_{T,p},
\]
and these in turn allow us to obtain estimates for
\[
\sup_{t \geq 0} \| u_k^\varepsilon(t) \|_{L^p(\Omega)} \leq C_{T,p} \quad \forall k = 1, \ldots, m.
\]
\[\square\]

Combining the $L^p$-estimates and the polynomial growth of the nonlinearities in (F5) we ultimately obtain the boundedness in $L^\infty$.

**Proposition 2.1.** Assume (1.2), (1.3), (1.4), (F1), (F2), (F3), (F4) and (F5). Then for any $T > 0$ the solution to the truncated system (2.1) is bounded in $L^\infty$ locally in time, i.e.
\[
\| u_i^\varepsilon \|_{L^\infty(Q_T)} \leq C_T \quad \forall i = 1, \ldots, m.
\]
for some constant $C_T$ depending on $T$ and independent of $\varepsilon > 0$.

**Proof.** From (F5),
\[
F_i^\varepsilon(u^\varepsilon) \leq F_i(u^\varepsilon) \leq G_i(u^\varepsilon) := K_4 \left[ 1 + \sum_{i=1}^{m} (u_i^\varepsilon)^t \right].
\]
Lemma 2.5 implies that for any $1 \leq p < \infty$ a constant $C_p > 0$ exists depending on $p$ but not on $T$ such that
\[
\| G_i(u^\varepsilon) \|_{L^p(Q_T)} \leq C_p \quad \forall i = 1, \ldots, m.
\]
By choosing $p > \frac{n+2}{2}$ and using the regularization of the parabolic operator $\partial_t v - \nabla \cdot (D_i \nabla v) + \nabla \cdot (B_i v)$ (see e.g. [17] or [25, Proposition 3.1]) we obtain

$$\|u_i\|_{L^\infty(Q_T)} \leq C_T \quad \forall i = 1, \ldots, m.$$  

Indeed, this follows exactly the same as in the proof of Proposition 3.1 in [25] with a slight modification concerning (A.15) therein. More precisely, with $f(s) = F_i^\varepsilon(u^\varepsilon(s))$ and $u^{(k)}(s) = (u_i^\varepsilon(s) - k)_+ \geq 0$ in [25, Eq. (A.15)], we can estimate

$$\int_\Omega f(s)u^{(k)}(s)dx = \int_\Omega F_i^\varepsilon(u^\varepsilon(s))(u_i^\varepsilon(s) - k)_+dx \leq \int_\Omega G_i(u^\varepsilon(s))(u_i^\varepsilon(s) - k)_+dx$$

and the rest of the proof of [25, Proposition 3.1] goes through. This completes the proof of Proposition 2.1.  

2.1.3. Passing to the limit - Global existence. In this Subsection, we pass to the limit $\varepsilon \to 0$ in (2.1) using the uniform estimates in Subsection 2.1.2.

**Proof of Theorem 1.1: Global existence.** From Subsection 2.1.2 we have the following uniform-in-time bound

$$\|u_i^\varepsilon\|_{L^\infty(Q_T)} \leq C_T \quad \forall i = 1, \ldots, m.$$  

Due to the polynomial growth ($F5$), it follows that

$$\|F_i(u^\varepsilon)\|_{L^\infty(Q_T)} \leq C_T \quad \forall i = 1, \ldots, m.$$  

By multiplying (2.1) by $u_i^\varepsilon$ then integrating on $Q_T$ gives

$$\frac{1}{2}\|u_i^\varepsilon(T)\|_{L^2(\Omega)}^2 + \int_0^T \int_\Omega (D_i \nabla u_i^\varepsilon) \cdot \nabla u_i^\varepsilon dxdt = \frac{1}{2}\|u_{i,0}\|_{L^2(\Omega)}^2 + \int_0^T \int_\Omega u_i^\varepsilon B_i \cdot \nabla u_i^\varepsilon dxdt + \int_0^T \int_\Omega \nabla \cdot (D_i \nabla u_i^\varepsilon) \cdot \nabla u_i^\varepsilon dxdt.$$  

The ellipticity (1.2) and Hölder’s inequality give

$$\int_0^T \int_\Omega (D_i \nabla u_i^\varepsilon) \cdot \nabla u_i^\varepsilon dxdt \geq \lambda \|u_i^\varepsilon\|_{L^2(Q_T)}^2$$

and

$$\int_0^T \int_\Omega u_i^\varepsilon B_i \cdot \nabla u_i^\varepsilon dxdt \leq \lambda \|\nabla u_i^\varepsilon\|_{L^2(Q_T)}^2 + \frac{\Gamma^2}{2\lambda} \|u_i^\varepsilon\|_{L^2(Q_T)}^2,$$

and consequently

$$\|\nabla u_i^\varepsilon\|_{L^2(Q_T)}^2 \leq C_T \quad \forall i = 1, \ldots, m.$$  

Thus,

$$\{u_i^\varepsilon\}_{\varepsilon > 0}$$  

is bounded uniformly in $\varepsilon$ in $L^\infty(Q_T) \cap L^2(0,T; H^1_0(\Omega)).$

From this, it follows easily that

$$\partial_i u_i^\varepsilon \quad \text{is bounded uniformly in } \varepsilon \text{ in } L^2(0,T; H^{-1}(\Omega)).$$

The classical Aubin-Lions lemma gives the strong convergence (up to a subsequence)

$$u_i^\varepsilon \xrightarrow{\varepsilon \to 0} u_i \quad \text{strongly in } L^2(Q_T).$$
Consequently, for any $1 \leq p < \infty$,
\[
u_i^\epsilon \overset{\epsilon \to 0}{\longrightarrow} u_i \quad \text{strongly in} \quad L^p(Q_T),
\]
thanks to the uniform $L^\infty$-bound of $u_i^\epsilon$. This is enough to pass to the limit in the weak formulation of (2.1)
\[
\begin{align*}
- \int_\Omega \varphi(\cdot, 0)u_{i,0}^\epsilon dx - \int_0^T \int_\Omega u_i^\epsilon \partial_t \varphi dx dt + \int_0^T \int_\Omega \nabla u_i^\epsilon \cdot \nabla \varphi dx dt \\
- \int_0^T \int_\Omega u_i^\epsilon B_i \cdot \nabla \varphi dx dt = \int_0^T \int_\Omega F_i^\epsilon (u^\epsilon) \varphi dx dt,
\end{align*}
\]
to obtain that $u = (u_1, \ldots, u_m)$ is a global weak solution to (1.1) and additionally
\[
\|u_i\|_{L^\infty(Q_T)} \leq C_T \quad \forall i = 1, \ldots, m.
\]
\hfill \Box

2.1.4. Uniform-in-time estimates.

**Lemma 2.6.** Assume (F1), (F2) and (F3) with either $K_1 < 0$ or $K_1 = K_2 = 0$. Then, there exists a constant $M$ independent of time such that
\[
\sup_{t \geq 0} \|u_i(t)\|_{L^1(\Omega)} \leq M \quad \forall i = 1, \ldots, m. \tag{2.19}
\]

**Proof.** By the same arguments in Lemma 2.2, we have, similarly to (2.8),
\[
\begin{align*}
\sum_{i=1}^m c_i \int_\Omega u_i(\cdot, t) dx & \leq \sum_{i=1}^m c_i \int_\Omega u_i(\cdot, s) dx + \int_s^t \int_\Omega K_1 \sum_{i=1}^m u_i(\cdot, r) dx dr + K_2|\Omega|(t-s)
\end{align*}
\]
for all $t > s \geq 0$. If $K_1 = K_2 = 0$, the desired bound (2.19) follows immediately. If $K_1 < 0$, we get for some constant $\sigma > 0$, and for all $t > s \geq 0$
\[
\sum_{i=1}^m c_i \int_\Omega u_i(t) dx + \sigma \int_s^t \left( \sum_{i=1}^m c_i \int_\Omega u_i(r) dx \right) dr \leq \sum_{i=1}^m c_i \int_\Omega u_i(s) dx + K_2|\Omega|(t-s). \tag{2.20}
\]
Define $\psi(t) = \sum_{i=1}^m c_i \int_\Omega u_i(\cdot, t) dx$ and $\varphi(s) = \int_s^t (\sum_{i=1}^m c_i \int_\Omega u_i(\cdot, r) dx) dr$. It follows from (2.20) that
\[
\varphi'(s) = -\sum_{i=1}^m c_i \int_\Omega u_i(\cdot, s) dx \leq -\psi(t) - \sigma \varphi(s) + K_2|\Omega|(t-s),
\]
which leads to
\[
(e^{\sigma s} \varphi(s))' + e^{\sigma s} \psi(t) \leq K_2|\Omega|e^{\sigma s}(t-s).
\]
Integrating with respect to $s$ on $(0, t)$, and using $\varphi(t) = 0$, we have
\[
-\varphi(0) + \psi(t) \frac{e^{\sigma t} - 1}{\sigma} \leq \frac{K_2|\Omega|}{\sigma} \left( -t + \frac{e^{\sigma t} - 1}{\sigma} \right).
\]
Since $\sigma \varphi(0) - K_2|\Omega|t \leq \sum_{i=1}^{m} c_i \int_{\Omega} u_{i0}(x)dx$ (see (2.20)) it follows that
\[
\psi(t) \leq (e^{\sigma t} - 1)^{-1} \sum_{i=1}^{m} c_i \int_{\Omega} u_{i0}(x)dx + K_2|\Omega|\sigma^{-1},
\]
which finishes the proof of Lemma 2.6.

With the $L^1$-bound (uniformly in time), we are ready to show the uniform-in-time boundedness of the solution.

**Proof of Theorem 1.1: Uniform-in-time boundedness.** We first show that for any $1 \leq p < \infty$, there exists a constant $C_p > 0$ such that
\[
\sup_{t \geq 0} \|u_i(t)\|_{L^p(\Omega)} \leq C_p \quad \forall i = 1, \ldots, m. \tag{2.21}
\]
By using the $L^p$-energy function $L_p[u]$ defined in (2.9) and the computations similar to Lemma 2.5, we obtain (2.16) which we recall here
\[
L_p^\prime(t) + \frac{c_p}{2} \sum_{k=1}^{m} \int_{\Omega} (|\nabla u_k|^{p/2} + |u_k|^p) \, dx \leq C_p \left( 1 + \sum_{k=1}^{m} \int_{\Omega} u_k^{p-1+r} \, dx \right).
\]
We now apply Lemma 2.3 to the right hand side, bearing in mind that the $L^1$-bound is uniform in time (thanks to Lemma 2.6), to get
\[
L_p^\prime(t) + C \int_{\Omega} \sum_{k=1}^{m} u_k(x,t)^p \, dx \leq C
\]
for some constant $\sigma > 0$. Gronwall’s lemma yields
\[
\sup_{t \geq 0} L_p(t) \leq C,
\]
which implies (2.21). To finally see that the solution is bounded uniformly in time in sup norm, we use a smooth time-truncated function $\psi : \mathbb{R} \to [0,1]$ with $\psi(s) = 0$ for $s \leq 0$ and $\psi(s) = 1$ for $s \geq 1$, $0 \leq \psi' \leq C$ and its shifted version $\psi_T(\cdot) = \psi(\cdot - \tau)$ for any $\tau \in \mathbb{N}$. Let $\tau \in \mathbb{N}$ be arbitrary. It is straightforward to show that since $u = (u_i)_{i=1, \ldots, m}$ is a weak solution to (1.1), the function $\psi_T u = (\psi_T u_i)_{i=1, \ldots, m}$ is a weak solution to the following
\[
\begin{cases}
\partial_t (\psi_T u_i) - \nabla \cdot (D_i \nabla (\psi_T u_i)) + \nabla \cdot (B_i(\psi_T u_i)) = \psi_T' u_i + \psi_T F_i(x, t, u), & x \in \Omega, t \in (\tau, \tau + 2) \\
(\psi_T u_i)(x,t) = 0, & x \in \partial\Omega, t \in (\tau, \tau + 2) \\
(\psi_T u_i)(x, \tau) = 0, & x \in \Omega.
\end{cases}
\]
Thanks to (2.21) and the polynomial growth (F5), we have
\[
\psi_T' u_i + \psi_T F_i(x, t, u) \leq G_i(x, t, u) := C \left( 1 + \sum_{k=1}^{m} u_k^\ell \right).
\]
Thanks to (2.21) for any \(1 \leq p < \infty\) a constant \(C_p > 0\) exists such that

\[
\|G_i(x, t, u)\|_{L^p(\Omega \times (\tau, \tau+2))} \leq C_p \quad \forall i = 1, \ldots, m.
\]

Therefore, by the smooth effect of parabolic operator \(\partial_t v - \nabla \cdot (D_i \nabla v) + \nabla \cdot (B_i v)\) (see e.g. [25, Proposition 3.1] or [17]), similar to Proposition 2.1, we get

\[
\|\psi_t u_i\|_{L^\infty(\Omega \times (\tau, \tau+2))} \leq C \quad \forall i = 1, \ldots, m,
\]

where \(C\) is a constant independent of \(\tau \in \mathbb{N}\). Thanks to \(\psi_t \geq 0\) and \(\psi|_{(\tau+1, \tau+2)} \equiv 1\), we obtain finally the uniform-in-time bound

\[
\sup_{t \geq 0} \|u_i(t)\|_{L^\infty(\Omega)} \leq C \quad \forall i = 1, \ldots, m.
\]

\[\square\]

2.2. Generalizations: Proof of Theorems 1.2 and 1.3.

**Proof of Theorem 1.2.** We give a formal proof as its rigor can be easily obtained through approximation. Since \(h_k(\cdot)\) is convex, we have

\[
\nabla \cdot (D_i \nabla (h_i(u_i)))) = \nabla \cdot (D_i h'_i(u_i) \nabla u_i) = h'_i(u_i) \nabla \cdot (D_i \nabla u_i) + h''_i(u_i) D_i |\nabla u_i|^2 
\geq h'_i(u_i) \nabla \cdot (D_i \nabla u_i).
\]

Therefore, by defining \(v_i := h_i(u_i) \geq 0\) we have

\[
\partial_t v_i - \nabla \cdot (D_i \nabla v_i) \leq G_i(x, t, u) := h'_i(u_i) F_i(x, t, u), \quad x \in \Omega, \ t > 0,
\]

with initial data \(v_i(x, 0) = h_i(u_i, 0(x))\) and homogeneous Dirichlet boundary condition \(v_i(x, t) = 0\) for \(x \in \partial \Omega\) and \(t > 0\). Thanks to (H1) and we have

\[
\sum_{i=1}^{m} G_i(x, t, u) \leq K_5 \sum_{i=1}^{m} v_i + K_6
\]

and

\[
A \left( \begin{array}{c}
G_1(x, t, u) \\
\vdots \\
G_m(x, t, u)
\end{array} \right) \leq K_7 \prod_{i=1}^{m} \left( \sum_{i=1}^{m} v_i + 1 \right)^r.
\]

We can now reapply the methods in the proof of Theorem 1.1, keeping in mind that though \(v_i\) satisfies (2.22) with an inequality all calculations are still available thanks to its non-negativity, to obtain \(v_i \in L^\infty_{\text{loc}}(0, \infty; L^\infty(\Omega))\), and in case \(K_5 < 0\) or \(K_5 = K_6 = 0\) in (H1),

\[
\text{ess sup}_{t \geq 0} \|v_i(t)\|_{L^\infty(\Omega)} < +\infty, \quad \forall i = 1, \ldots, m.
\]

Thanks to the assumption (H1), the global existence and boundedness of (1.1) follow immediately. \[\square\]
Proof of Theorem 1.3. For the global existence, we only need to show that for any \(1 \leq p < \infty\) and any \(T > 0\), there exists \(C_{T,p} > 0\) such that

\[
\|u_i\|_{L^p(Q_T)} \leq C_{T,p} \quad \forall i = 1, \ldots, m. \tag{2.23}
\]

The rest follows exactly as in Proposition 2.1. To show (2.23), we utilize the \(L^p\)-energy functions \(\mathcal{L}_p(t)\) constructed in Lemma 2.5. Repeat the arguments in the proof of Lemma 2.5 until (2.16) we end up with

\[
\mathcal{L}_p'(t) + \frac{\alpha_p}{2} \sum_{k=1}^{m} \int_{\Omega} \left( |\nabla u_k^{p/2}|^2 + |u_k|^p \right) dx \leq C_p \left( 1 + \sum_{k=1}^{m} \int_{\Omega} u_k^{p-1+r} dx \right). \tag{2.24}
\]

In case (1.7) holds, we apply Lemma 2.3 to estimate

\[
\int_{\Omega} u_k^{p-1+r} dx \leq \frac{\alpha_p}{2} \int_{\Omega} \left( |\nabla u_k^{p/2}|^2 + |u_k|^p \right) dx + C_T
\]

where \(C_T\) depends on \(\mathcal{F}(T)\). Inserting this into (2.24) yields

\[
\mathcal{L}_p'(t) + \delta \mathcal{L}_p(t) \leq C_{p,T},
\]

which implies (2.23), thanks to Gronwall’s lemma.

In case (1.8) holds, we integrate (2.24) in time to obtain

\[
\sup_{t \in (0,T)} \mathcal{L}_p(t) + \frac{\alpha_p}{2} \sum_{k=1}^{m} \int_{0}^{T} \int_{\Omega} \left( |\nabla u_k^{p/2}|^2 + |u_k|^p \right) dx dt
\]

\[
\leq \mathcal{L}_p(0) + C_p T + C_p \sum_{k=1}^{m} \int_{0}^{T} \int_{\Omega} u_k^{p-1+r} dx dt. \tag{2.25}
\]

Denote by \(y_k := u_k^{p/2}\). The left hand side of (2.25) can be estimated below by

\[
\text{LHS of (2.25)} \geq C \sum_{k=1}^{m} \left( \|y_k\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|y_k\|_{L^2(0,T;H^1(\Omega))}^2 \right). \tag{2.26}
\]

For the right hand side of (2.25), we first consider

\[
\theta > p - 1 + r \tag{2.27}
\]

as a constant to be determined later. Of course we are only interested in the case when \(p - 1 + r > b\), otherwise the right hand side of (2.25) is bounded thanks to (1.8). By the interpolation inequality we have

\[
\int_{0}^{T} \int_{\Omega} u_k^{p-1+r} dx dt = \|u_k\|_{L^{p-1+r}(Q_T)}^{p-1+r} \leq \|u_k\|_{L^\theta(Q_T)}^{\theta(p-1+r)} \|u_k\|_{L^\theta(Q_T)}^{(1-\theta)(p-1+r)}, \tag{2.28}
\]

where \(\theta \in (0, 1)\) satisfies

\[
\frac{1}{p - 1 + r} = \frac{\theta}{b} + \frac{1 - \theta}{\vartheta},
\]
which implies
\[(1 - \theta)(p - 1 + r) = \frac{\vartheta(p - 1 + r - b)}{\vartheta - b}.\]

Using this, and taking into account (1.8), (2.28) implies
\[
\int_0^T \int_\Omega u_k^{p-1+r} \, dx \, dt \leq \mathcal{F}(T)^{\theta(p-1+r)} \left\| u_k \right\|_{L^p(Q_T)}^{\frac{\vartheta(p-1+r-b)}{\vartheta - b}}.
\]
\[
= \mathcal{F}(T)^{\theta(p-1+r)} \left( \int_0^T \int_\Omega u_k^\vartheta \, dx \, dt \right)^{\frac{p-1+r-b}{\vartheta - b}}.
\]
\[
= C_T \left( \int_0^T \int_\Omega y_k^{\frac{2\vartheta}{p}} \, dx \, dt \right)^{\frac{p-1+r-b}{\vartheta - b}}.
\]

For \( p \) large enough, we choose \( \vartheta \) close enough to (but bigger than) \( p - 1 + r \) so that \( H^1(\Omega) \hookrightarrow L^{\frac{2\vartheta}{p}}(\Omega) \). That means \( \vartheta \) is arbitrary for \( n \leq 2 \) and
\[
\frac{2\vartheta}{p} \leq \frac{2n}{n - 2} \Leftrightarrow \vartheta \leq \frac{pn}{n - 2} \quad \text{for} \quad n \geq 3.
\]

Thus, we can use the Gagliardo-Nirenberg’s inequality to estimate
\[
\int_{\Omega} y_k^{\frac{2\vartheta}{p}} \, dx \leq \left\| y_k \right\|_{L^{\frac{2\vartheta}{p}}(\Omega)}^{\frac{2\vartheta}{p}} \leq C \left\| y_k \right\|_{H^1(\Omega)}^{\alpha \frac{2\vartheta}{p}} \left\| y_k \right\|_{L^2(\Omega)}^{(1 - \alpha) \frac{2\vartheta}{p}}
\]
where \( \alpha \in (0, 1) \) satisfies
\[
\frac{p}{2\vartheta} = \left( \frac{1}{2} - \frac{1}{n} \right) \alpha + \frac{1 - \alpha}{2}.
\]

From this
\[
\alpha \cdot \frac{2\vartheta}{p} = \frac{n(\vartheta - p)}{p} \quad \text{and} \quad (1 - \alpha) \cdot \frac{2\vartheta}{p} = \frac{np - (n - 2)\vartheta}{p}.
\]

Therefore, we obtain from (2.31) that
\[
\int_{\Omega} y_k^{\frac{2\vartheta}{p}} \, dx \leq C \left\| y_k \right\|_{H^1(\Omega)}^{\frac{n(\vartheta - p)}{p}} \left\| y_k \right\|_{L^2(\Omega)}^{\frac{np - (n - 2)\vartheta}{p}}.
\]

It follows that
\[
\int_0^T \int_{\Omega} y_k^{\frac{2\vartheta}{p}} \, dx \, dt \leq C \int_0^T \left\| y_k \right\|_{H^1(\Omega)}^{\frac{n(\vartheta - p)}{p}} \left\| y_k \right\|_{L^2(\Omega)}^{\frac{np - (n - 2)\vartheta}{p}} \, dt
\]
\[
\leq C \left\| y_k \right\|_{L^{\infty}(0,T;L^2(\Omega))}^{\frac{np - (n - 2)\vartheta}{p}} \int_0^T \left\| y_k \right\|_{H^1(\Omega)}^{\frac{n(\vartheta - p)}{p}} \, dt.
\]

We choose
\[
\frac{n(\vartheta - p)}{p} \leq 2 \Leftrightarrow \vartheta \leq \frac{p(n + 2)}{n},
\]
\[
(2.32)
\]
which is possible since \( p - 1 + r < p + \frac{2p}{n} \) for \( p \) large enough. Thus, by Hölder’s inequality,
\[
\int_0^T \int_{\Omega} y_k^{p-1} \, dx \, dt \leq C_T \|y_k\|_{L^\infty(0,T;L^2(\Omega))} \|y_k\|_{L^p(0,T;H^1(\Omega))}^{n/(\vartheta - p)}.
\]
Inserting this into (2.29) yields
\[
\int_0^T \int_{\Omega} u_k^{p-1+r} \, dx \, dt \leq C_T \left( \|y_k\|_{L^\infty(0,T;L^2(\Omega))} \|y_k\|_{L^p(0,T;H^1(\Omega))} \right) \left( \frac{p-1+r-b}{\vartheta-b} \right)^{2}.
\]  
We check that we can choose \( \vartheta \) such that
\[
\frac{p-1+r-b}{\vartheta-b} \cdot \left( \frac{np-n(\vartheta-2)}{p} + \frac{n(\vartheta-p)}{p} \right) < 2.
\]  
Indeed, this is equivalent to
\[
\frac{p-1+r-b}{\vartheta-b} < 2 \iff (p-1+r-b)\vartheta < (\vartheta-b)p
\]
\[
\iff \vartheta > \frac{bp}{1+b-r}.
\]  
We now check that we can choose \( \vartheta \) which satisfies all the conditions (2.27), (2.30), (2.32) and (2.35). This is fulfilled provided
\[
\frac{bp}{1+b-r} < \frac{p(n+2)}{n},
\]
which is equivalent to \( r < 1 + \frac{2b}{n+2} \), and this is exactly our assumption (1.9). Now by (2.34), we can use Young’s inequality of the form
\[
x^{\lambda_1}y^{\lambda_2} \leq \varepsilon(x^2 + y^2) + C_\varepsilon \quad \text{for} \quad \lambda_1 + \lambda_2 < 2,
\]
to estimate (2.33) further as
\[
\int_0^T \int_{\Omega} u_k^{p-1+r} \, dx \, dt \leq \varepsilon \left( \|y_k\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|y_k\|_{L^p(0,T;H^1(\Omega))}^2 \right) + C_{T,\varepsilon}.
\]
Thus
\[
\text{RHS of (2.25)} \leq \mathcal{L}_p(0) + \varepsilon \sum_{k=1}^m \left( \|y_k\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|y_k\|_{L^p(0,T;H^1(\Omega))}^2 \right)^2 + C_{T,\varepsilon}.
\]
Combining this with (2.26) gives us the desired estimate (2.23).

For the uniform-in-time bounds, we use similar arguments to the proof of Theorem 1.1 study (1.1) (multiplied by a time-truncated function \( \psi_\tau \)) on each interval \((\tau, \tau + 2)\) to ultimately obtain
\[
\sup_{\tau \in \mathbb{N}} \|u_i\|_{L^\infty(\Omega \times (\tau,\tau+2))} < +\infty \quad \forall i = 1, \ldots, m.
\]
We leave the details for the interested reader. \( \square \)
2.3. Other boundary conditions or quasilinear systems. First, we state the definition of weak solutions to (1.10) as it is different from that of (1.1).

**Definition 2.1.** A vector of non-negative concentrations \( u = (u_1, \ldots, u_m) \) is called a weak solution to (1.10) on \((0, T)\) if

\[
  u_i \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \quad F_i(u) \in L^2(0, T; L^2(\Omega)),
\]

with \( u_i(\cdot, 0) = u_{i,0}(\cdot) \) for all \( i = 1, \ldots, m \), and for any test function \( \varphi \in L^2(0, T; H^1(\Omega)) \) with \( \partial_t \varphi \in L^2(0, T; H^{-1}(\Omega)) \), it holds that

\[
  \int_\Omega u_i(x, t) \varphi(x, t) \, dx \bigg|_{t=0}^{t=T} - \int_0^T \int_\Omega u_i \partial_t \varphi \, dx dt + \int_0^T \int_\Omega D_i(x, t) \nabla u_i \cdot \nabla \varphi \, dx dt \\
  + \alpha_i \int_0^T \int_{\partial \Omega} u_i \varphi \, d\mathcal{H}^{n-1} \, dt = \int_0^T \int_\Omega F_i(x, t, u) \varphi \, dx dt.
\]

(2.36)

**Proof of Theorem 1.4.** The proof of this theorem is similar to that of Theorems 1.1 and 1.3, except for the fact that the \( L^1 \)-norm can be obtained in a different, and easier, way. Since \( \varphi \equiv 1 \) is an admissible test function we get from (2.36) and (F3) that

\[
  \sum_{i=1}^m \int_\Omega c_i u_i(\cdot, t) \, dx \bigg|_{t=0}^{t=T} + \sum_{i=1}^m c_i \alpha_i \int_0^T \int_{\partial \Omega} u_i \, d\mathcal{H}^{n-1} \, dt \\
  \leq K_1 \int_0^T \int_\Omega u_i \, dx dt + K_2 |\Omega| T.
\]

The \( L^1 \)-bound, which is uniform in time in case \( K_1 < 0 \) or \( K_1 = K_2 = 0 \), then follows immediately from the Gronwall’s inequality. \( \square \)

**Proof of Theorem 1.5.** The definition of weak solutions of (1.11) is similar to Definition 1.1. Note that the Galerkin method in the proof of Lemma 2.1 is not applicable due to the nonlinear dependence of \( A_i(x, t, u) \) on \( u \). We resort to a different strategy.

For fixed \( \varepsilon > 0 \), we consider the truncated system: for all \( i = 1, \ldots, m \),

\[
  \begin{cases}
    \partial_t u_i^\varepsilon - \nabla \cdot (A_i^\varepsilon(x, t, u^\varepsilon) \nabla u_i^\varepsilon) = F_i^\varepsilon(x, t, u^\varepsilon), & x \in \Omega, t > 0, \\
    u_i(x, t) = 0, & x \in \partial \Omega, t > 0, \\
    u_i(x, 0) = u_{i,0}(x), & x \in \Omega,
  \end{cases}
\]

(2.37)

where for all \( i = 1, \ldots, m \),

- \( F_i^\varepsilon(x, t, u^\varepsilon) = F_i(x, t, u^\varepsilon) \left[ 1 + \varepsilon \sum_{j=1}^m |F_j(x, t, u^\varepsilon)| \right]^{-1} \),
- \( u_{i,0} \in C^2(\Omega) \cap C(\bar{\Omega}) \) and \( \lim_{\varepsilon \to 0} \| u_{i,0}^\varepsilon - u_{i,0} \|_{L^\infty(\Omega)} = 0 \),
- \( \Omega \times \mathbb{R}_+ \times \mathbb{R}^m \ni (x, t, \omega) \mapsto A_i^\varepsilon(x, t, \omega) \) is Hölder continuous in each component,

\[
  \frac{\lambda}{2} |\xi|^2 \leq \xi^T A_i^\varepsilon(x, t, \omega) \xi \quad \forall (x, t, \omega) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^m, \forall \xi \in \mathbb{R}^n.
\]
we obtain that

\[ A^\epsilon_i(x, t, \omega) \to A_i(x, t, \omega) \quad \forall \omega \in \mathbb{R}^m, \text{ a.e.}(x, t) \in \Omega \times \mathbb{R}_+ . \]

Thanks to these regularization, the existence of a weak solution \( u^\epsilon = (u^\epsilon_i)_{i=1,...,m} \) to (2.37) follows from classical results, see e.g. [26]. Moreover, this solution is unique [24]. Thanks to this uniqueness, we can also get the non-negativity of \( u^\epsilon \) using similar computations to that of Lemma 2.1. Following the arguments of Lemma 2.2, Lemma 2.5, Proposition 2.1, and Theorem 1.3 we obtain that

\[ \|u^\epsilon_i\|_{L^\infty(Q_T)} \leq C_T \quad \forall i = 1, \ldots, m. \]

Note that this is possible since we used only the uniform ellipticity of \( A^\epsilon_i \) to derive this bound, which is uniformly in \( \epsilon > 0 \). From (1.1), we obtain \( \{u^\epsilon_i\}_{\epsilon > 0} \) is bounded in \( L^2(0, T; H^1_0(\Omega)) \) and \( \{\partial_t u^\epsilon_i\}_{\epsilon > 0} \) is bounded in \( L^2(0, T; H^{-1}(\Omega)) \). The Aubin-Lions lemma gives strong in \( L^2(Q_T) \), and consequently point-wise, convergence \( u^\epsilon_i \to u_i \) (up to a subsequence) for all \( i = 1, \ldots, m \). Passing to the limit \( \epsilon \to 0 \) for the weak formulation of (2.37), we only have to show that

\[ \int_0^T \int_\Omega A^\epsilon_i(x, t, u^\epsilon) \nabla u^\epsilon_i \cdot \nabla \varphi dx dt \xrightarrow{\epsilon \to 0} \int_0^T \int_\Omega A_i(x, t, u) \nabla u_i \cdot \nabla \varphi dx dt . \]

This is guaranteed thanks to the boundedness of \( A^\epsilon_i, A_i(x, t, u^\epsilon) \to A_i(x, t, u) \) almost everywhere, and \( u^\epsilon_i \to u_i \) weakly in \( L^2(0, T; H^1_0(\Omega)) \). The uniform-in-time bound in sup-norm can be obtained in similar ways as in previous theorems, so we omit the details.

If \( A_i \) is merely bounded in \( (x, t) \), it seems not possible to show uniqueness of weak solutions. Under the stronger assumption (iv) that, \( A(\cdot, \cdot, \omega) \) is Hölder continuous on \( \Omega \times \mathbb{R}_+ \), one can show that the gradient \( \nabla u_i \) is sup-norm bounded, and the uniqueness follows. We refer the interested reader to [24] for more details. \( \square \)

3. APPLICATIONS TO A MODEL OF AN INFECTIOUS DISEASE

We apply our theoretical results to a model for the spread of infectious disease within a host population that occupies a highly heterogeneous habitat. Related problems have been considered in many works in the literature, see e.g. [13, 1, 8, 33, 34] and the references therein. Here we extend the model considered in [34]. We consider a population confined in a bounded domain \( \Omega \subset \mathbb{R}^n, n \geq 1 \). The population is compartmentalized into the susceptible class \( S \), the infected class \( I \), and the recovered class \( R \), whose densities are denoted by \( s(x, t), i(x, t) \) and \( r(x, t) \), respectively. Moreover, we denote by \( b(x, t) \) the density of the pathogen associated with the disease. We consider the following system

\[
\begin{cases}
\partial_t s = \nabla \cdot (D_S(x, t) \nabla s) - \sigma_I(x)s - \sigma_B(x)sb + \gamma r, & x \in \Omega, \ t > 0, \\
\partial_t i = \nabla \cdot (D_I(x, t) \nabla i) + \sigma_I(x)s + \sigma_B(x)sb - (\lambda + \alpha)i, & x \in \Omega, \ t > 0, \\
\partial_t r = \nabla \cdot (D_R(x, t) \nabla r) + \lambda i - \gamma r, & x \in \Omega, \ t > 0, \\
\partial_t b = \nabla \cdot (D_B(x, t) \nabla b + c(x, t)b) + \phi(x)i - \delta b, & x \in \Omega, \ t > 0, 
\end{cases}
\]  

(3.1)

where we assume
• The diffusivities $D_S(x,t), D_I(x,t), D_R(x,t)$ and $D_B(x,t)$ are uniformly bounded on $\Omega$ and there exists a $d_*>0$ so that
\[
0 < d_* \leq \min_{x \in \Omega, t>0} \{ D_S(x,t), D_I(x,t), D_R(x,t), D_B(x,t) \}.
\] (3.2)

• The exists a $c^*>0$ so that
\[
\max_{1 \leq j \leq n} \sup_{t>0} \| c_j(t) \|_{L^\infty(\Omega)} < c^*.
\] (3.3)

• There exist $\sigma_*$ and $\sigma^*$ such that
\[
0 < \sigma_* \leq \min_{x \in \Omega} \{ \sigma_I(x), \sigma_B(x) \} \leq \max_{x \in \Omega} \{ \sigma_I(x), \sigma_B(x) \} \leq \sigma^*.
\] (3.4)

The system (3.1) is subject to no-flux boundary condition for $x \in \partial \Omega$, $t>0$,
\[
D_S \nabla s \cdot \nu = D_I \nabla i \cdot \nu = D_R \nabla r \cdot \nu = (D_B \nabla b + c(x,t)b) \cdot \nu = 0,
\]
and bounded, non-negative initial data for $x \in \Omega$,
\[
s(x,0) = s_0(x), \quad i(x,0) = i_0(x), \quad r(x,0) = r_0(x), \quad b(x,0) = b_0(x).
\]

For more discussion about the modeling of (3.1), we refer the reader to [34].

We are in position to show the global existence, boundedness and asymptotic behavior of solutions to (3.1). It is emphasized that the uniform-in-time boundedness plays an important role in establishing the large time behavior of the solution.

**Theorem 3.1.** Assume (3.2), (3.3), (3.4), positive parameters $\gamma, \lambda, \alpha, \delta > 0$, $\phi(x) \geq 0$ a.e. $x \in \Omega$ and
\[
\| \phi \|_{L^\infty(\Omega)} \leq \alpha.
\] (3.5)

For any bounded, non-negative initial data, there exists a unique global bounded weak solution to (1.1) which is bounded uniformly in time, i.e.
\[
\sup_{t \geq 0} \{ \| s(t) \|_{L^\infty(\Omega)}, \| i(t) \|_{L^\infty(\Omega)}, \| r(t) \|_{L^\infty(\Omega)}, \| b(t) \|_{L^\infty(\Omega)} \} < +\infty.
\] (3.6)

Moreover, we have the following asymptotic behavior for any $1 < p < \infty$,
\[
\lim_{t \to \infty} \left( \| i(t) \|_{L^p(\Omega)} + \| r(t) \|_{L^p(\Omega)} + \| b(t) \|_{L^p(\Omega)} + \| s(t) - s_\infty \|_{L^p(\Omega)} \right) = 0
\] (3.7)

where
\[
0 < s_\infty = \int_{\Omega} (s_0 + i_0 + r_0) dx - \alpha \int_{\Omega} \int_{\Omega} i(x,z) dxdz < +\infty.
\]

One can interpret the convergence of $s$ as meaning that the final susceptible population equals to the difference of the initial total population $\int_{\Omega} (s_0 + i_0 + r_0) dx$ and the total mortality over time $\alpha \int_0^\infty \int_{\Omega} i(x,z) dxdz$, which is expected from realistic situations.

**Proof.** Denote by $u = (s, i, r, b)$ the vector of unknowns, and $F_s, F_i, F_r$ and $F_b$ the nonlinearities in the equations of $s, i, r$ and $b$, respectively. Thanks to (3.5), we have $F_s(u) \leq \gamma r$, $F_s(u) + F_i(u) \leq \gamma r$, $F_s(u) + F_i(u) + F_r(u) \leq 0$, and $F_s(u) + F_i(u) + F_r(u) + F_b(u) \leq 0$. The global existence and uniform boundedness (3.6) of a unique weak solution to (3.1) then
follow from Theorem 1.4 with slight modifications (due to the different boundary condition of \( b \)).

We now turn to the large time behavior. By summing the equations of \( s, i, r \) and taking \( \varphi \equiv 1 \) as a test function, we have

\[
\int_{\Omega} (s(x, t) + i(x, t) + r(x, t)) + \alpha \int_{0}^{t} \int_{\Omega} i(x, z)dx dz = \int_{\Omega} (s_{0}(x) + i_{0}(x) + r_{0}(x))dx. \quad (3.8)
\]

Due to the non-negativity of \( i \), it follows that \( \int_{0}^{\infty} \| i(z) \|_{L^{1}(\Omega)}dz < +\infty \), which implies

\[
\lim_{t \to \infty} \int_{t}^{t+1} \| i(z) \|_{L^{1}(\Omega)}dz = 0.
\]

By integrating the equations of \( r \) and \( b \) on \( \Omega \times (0, t) \), we obtain easily

\[
\lim_{t \to \infty} \int_{t}^{t+1} \| r(z) \|_{L^{1}(\Omega)}dz = 0 \quad \text{and} \quad \lim_{t \to \infty} \int_{t}^{t+1} \| b(z) \|_{L^{1}(\Omega)}dz = 0.
\]

Thanks to the bound in \( L^{\infty}(\Omega) \)-norm (3.6) and the inequality \( \| f \|_{L^{p}(\Omega)} \leq \| f \|_{L^{p-1}(\Omega)} \| f \|_{L^{1}(\Omega)} \) we get for any \( p > 1 \)

\[
\lim_{t \to \infty} \int_{t}^{t+1} \left( \| i(z) \|_{L^{p}(\Omega)} + \| r(z) \|_{L^{p}(\Omega)} + \| b(z) \|_{L^{p}(\Omega)} \right) dz = 0. \quad (3.9)
\]

We now show the decay of \( b \) in \( L^{p}(\Omega) \)-norm for \( p > 1 \). For \( \tau \in \mathbb{N} \), recall the time cut-off function \( \varphi_{\tau} \in C_{c}^{\infty}(\mathbb{R}) \) satisfying \( \varphi_{\tau}(\cdot) = 0 \) on \((-\infty, \tau]\) and \( \varphi_{\tau}(\cdot) = 1 \) on \([\tau, +\infty)\). It's clear that \( \varphi_{\tau}u = (\varphi_{\tau}s, \varphi_{\tau}i, \varphi_{\tau}r, \varphi_{\tau}b) \) is also a weak solution to (3.1), which satisfies \( \varphi_{\tau}y(x, \tau) = 0 \) for \( y \in \{ s, i, r, b \} \). In particular, by taking \( p\varphi_{\tau}b^{p-1} \) as a test function for the equation of \( b \) we obtain

\[
\| (\varphi_{\tau}b)(\tau + 1) \|_{L^{p}(\Omega)}^{p} + p(p - 1) \int_{\tau}^{\tau+1} \int_{\Omega} \varphi_{\tau}^{p}b^{p-2}D_{B}(x, z)\nabla b(x, z) \cdot \nabla b(x, z)dx dz
\]

\[
- \delta p \int_{\tau}^{\tau+1} \int_{\Omega} \varphi_{\tau}^{p}b(x, z)^p dx dz = -p \int_{\tau}^{\tau+1} \int_{\Omega} c(x, z)\varphi_{\tau}^{p}b^{p-1}\nabla bdx dz
\]

\[
+ p \int_{\tau}^{\tau+1} \int_{\Omega} \varphi_{\tau}^{p}\varphi_{\tau}^{'}b^{p} dx dz + p \int_{\tau}^{\tau+1} \int_{\Omega} \varphi_{\tau}^{p}\varphi_{\tau}^{'}b^{p} dx dz.
\]

By Hölder’s and Young’s inequalities, we have

\[
\left| p \int_{\tau}^{\tau+1} \int_{\Omega} c\varphi_{\tau}^{p}b^{p-1}\nabla bdx dz \right| \leq p(p - 1)\frac{d_{s}}{2} \int_{\tau}^{\tau+1} \int_{\Omega} \varphi_{\tau}^{p}b^{p-2}|\nabla b|^{2}dx dz + C \int_{\tau}^{\tau+1} \int_{\Omega} |b|^{p}dx dz.
\]

Inserting this into (3.10), using (3.2) and Hölder’s inequality yield

\[
\| b(\tau + 1) \|_{L^{p}(\Omega)}^{p} \leq C \int_{\tau}^{\tau+1} \left( \| i(z) \|_{L^{p}(\Omega)}^{p} + \| b(z) \|_{L^{p}(\Omega)}^{p} \right) dz.
\]
where $C$ is independent of $\tau$. Thus, it follows from (3.9) that $\lim_{t \to \infty} \| b(t) \|_{L^p(\Omega)} = 0$. By using similar arguments, $\lim_{t \to \infty} \| r(t) \|_{L^p(\Omega)} = 0$. For $i$, we have

$$
\| i(\tau + 1) \|^p_{L^p(\Omega)} \leq C \int_\tau^{\tau + 1} \int_\Omega \left( \| i(z) \|^p + | s(z) | \| i(z) \|^p + | s(z) | \| b(z) \| \| i(z) \|^{p-1} \right) dx dz
$$

$$
\leq C \left( 1 + \sup_{t \geq 0} \| s(t) \|_{L^\infty(\Omega)} \right) \int_\tau^{\tau + 1} \left( \| i(z) \|^p_{L^p(\Omega)} + \| b(z) \|^p_{L^p(\Omega)} \right) dz
$$

which implies $\lim_{t \to \infty} \| i(t) \|_{L^\infty(\Omega)} = 0$. It remains to show the asymptotic of $s$. Denote by $\bar{s}(t) = \frac{1}{\Omega} \int_\Omega s(x, t) dx$. Direct calculations give

$$
\frac{1}{2} \frac{d}{dt} \| s - \bar{s} \|^2_{L^2(\Omega)} \leq -d_s \int_\Omega | \nabla s |^2 dx + \| s - \bar{s} \|_{L^2(\Omega)} \| \sigma_i s i + \sigma_B s b - \gamma \bar{s}_t \|_{L^2(\Omega)}.
$$

By using Young's inequality, it yields

$$
\| s - \bar{s} \|^2_{L^2(\Omega)} \| \sigma_i s i + \sigma_B s b - \gamma \bar{s}_t \|^2_{L^2(\Omega)} \leq \frac{d_s C_{PW}}{2} \| s - \bar{s} \|^2_{L^2(\Omega)} + \frac{1}{2d_s C_{PW}} \| \sigma_i s i + \sigma_B s b - \gamma \bar{s}_t \|^2_{L^2(\Omega)}.
$$

Thus, by the Poincaré-Wirtinger inequality $\int_\Omega | \nabla s |^2 dx \geq C_{PW} \| s - \bar{s} \|^2_{L^2(\Omega)}$, we have

$$
\frac{d}{dt} \| s - \bar{s} \|^2_{L^2(\Omega)} + d_s C_{PW} \| s - \bar{s} \|^2_{L^2(\Omega)} \leq C \| \sigma_i s i + \sigma_B s b - \gamma \bar{s}_t \|^2_{L^2(\Omega)}.
$$

From the uniform bounds of $\sigma_i, \sigma_B, s$, and the decay of $i$ and $b$, we have

$$
\lim_{t \to \infty} \| \sigma_i s i + \sigma_B s b \|_{L^2(\Omega)} \leq \lim_{t \to \infty} \| s(t) \|_{L^\infty(\Omega)} \left( \| i(t) \|_{L^2(\Omega)} + \| b(t) \|_{L^2(\Omega)} \right) = 0.
$$

From the equation of $s$,

$$
| \bar{s}_t | \leq (\sigma^* + \gamma) \left( \| s \|_{L^\infty(\Omega)} (\| i \|_{L^1(\Omega)} + \| b \|_{L^1(\Omega)}) + \gamma \| r \|_{L^1(\Omega)} \right),
$$

and thus

$$
\| \bar{s}_t (z) \|^2_{L^2(\Omega)} = | \Omega | | \bar{s}_t (z) |^2 \xrightarrow{z \to \infty} 0.
$$

From (3.12) and (3.13), (3.11) implies that

$$
\lim_{t \to \infty} \| s(t) - \bar{s}(t) \|^2_{L^2(\Omega)} = 0.
$$

Finally, by letting $t \to \infty$ in (3.8), it follows

$$
\lim_{t \to \infty} \bar{s}(t) = \lim_{t \to \infty} \int_\Omega s(x, t) dx = \int_\Omega (s_0 + i_0 + r_0) dx - \alpha \int_0^\infty \int_\Omega i(x, z) dx dz = s_\infty,
$$

which, in combination with (3.14) implies

$$
\lim_{t \to \infty} \| s(t) - s_\infty \|^2_{L^2(\Omega)} = 0.
$$

The uniform bound of $s$ in $L^\infty(\Omega)$ and interpolation inequality imply the behavior (3.7). \qed
4. TWO TECHNICAL LEMMAS

Lemma 4.1. Suppose \( m_1 \in \mathbb{N}, \theta = (\theta_1, \ldots, \theta_m), \) where \( \theta_1, \ldots, \theta_m \) are positive real numbers, \( \beta \in \mathbb{Z}_+^m, \) and \( \mathcal{H}_p[u] \) is defined in (2.10). Then

\[
\frac{\partial}{\partial t} \mathcal{H}_0[u](t) = 0, \quad \frac{\partial}{\partial t} \mathcal{H}_1[u](t) = \sum_{j=1}^m \theta_j \frac{\partial}{\partial t} u_j(t),
\]

and for \( p \in \mathbb{N} \) such that \( p \geq 2, \)

\[
\frac{\partial}{\partial t} \mathcal{H}_p[u](t) = \sum_{|\beta| = p-1} (\frac{p}{\beta}) \theta^{\beta} u(t)^{\beta} \sum_{j=1}^m \theta_j^{\beta_j+1} \frac{\partial}{\partial t} u_j(t).
\]

Proof. The results for \( \mathcal{H}_0[u](t) \) and \( \mathcal{H}_1[u](t) \) are trivial. The same is true for the case when \( m = 1. \) Suppose \( p \geq 2 \) and \( m_1 \geq 2. \) We proceed by induction on the value \( m, \) and assume \( k \in \mathbb{N} \) such that the result is true for \( m = k. \) Suppose \( m = k + 1 \) and denote \( \tilde{\beta} = (\beta_2, \ldots, \beta_m) \) and \( \tilde{u} = (u_2, \ldots, u_m). \)

Then we can rewrite \( \mathcal{H}_p[u] \) as

\[
\mathcal{H}_p[u] = \sum_{\beta_1 = 0}^p \frac{1}{\beta_1!} \theta_1^{\beta_1} u_1^{\beta_1} \sum_{|\tilde{\beta}| = p - \beta_1} \left( \frac{p}{\beta} \right) \tilde{\theta}^{\tilde{\beta}_2} \tilde{u}^{\tilde{\beta}}.
\]

Consequently,

\[
\frac{\partial}{\partial t} \mathcal{H}_p[u] = \sum_{\beta_1 = 1}^p \frac{1}{\beta_1!} \theta_1^{\beta_2} \theta_1^{\beta_1} u_1^{\beta_1-1} \frac{\partial}{\partial t} u_1 \sum_{|\tilde{\beta}| = p - \beta_1} \left( \frac{p}{\beta} \right) \tilde{\theta}^{\tilde{\beta}_2} \tilde{u}^{\tilde{\beta}}
\]

\[
+ \sum_{\beta_1 = 0}^p \frac{1}{\beta_1!} \theta_1^{\beta_2} u_1^{\beta_1} \frac{\partial}{\partial t} \left( \sum_{|\tilde{\beta}| = p - \beta_1} \left( \frac{p}{\beta} \right) \tilde{\theta}^{\tilde{\beta}_2} \tilde{u}^{\tilde{\beta}} \right)
\]

\[
= \sum_{\beta_1 = 1}^p \frac{1}{\beta_1!} \theta_1^{\beta_2} \theta_1^{\beta_1} u_1^{\beta_1-1} \frac{\partial}{\partial t} u_1 \sum_{|\tilde{\beta}| = p - \beta_1} \left( \frac{p}{\beta} \right) \tilde{\theta}^{\tilde{\beta}_2} \tilde{u}^{\tilde{\beta}}
\]

\[
+ \sum_{\beta_1 = 0}^{p-1} \frac{1}{\beta_1!} \theta_1^{\beta_2} u_1^{\beta_1} \frac{p!}{(p - \beta_1)!} \frac{\partial}{\partial t} H_{p-\beta_1}[\tilde{u}].
\]

Now, from our induction hypothesis,

\[
\frac{\partial}{\partial t} \mathcal{H}_{p-\beta_1}[\tilde{u}] = \sum_{|\beta| = p - \beta_1 - 1} \left( \frac{p - \beta_1}{\beta} \right) \tilde{\theta}^{\tilde{\beta}_2} \tilde{u}^{\tilde{\beta}} \sum_{j=1}^{m-1} \theta_j^{\beta_j+1} \frac{\partial}{\partial t} \tilde{u}_j.
\]
Therefore, substituting (4.3) into (4.2), and noting that $\tilde{u}_j = u_{j+1}$ and $\tilde{\theta}_j = \theta_{j+1}$, gives
\[
\frac{\partial}{\partial t} \mathcal{H}_p[u] = \sum_{\beta_1 = 0}^{p} \frac{1}{\beta_1!} \theta_1^{\beta_2} \beta_1 u_1^{\beta_1-1} \frac{\partial}{\partial t} u_1 \sum_{|\beta| = p-\beta_1} \left( \begin{array}{c} p \\ \beta \end{array} \right) \tilde{\theta}^{\beta_2} \tilde{u}^{\beta} \\
+ \sum_{\beta_1 = 0}^{p-1} \frac{1}{\beta_1!} \theta_1^{\beta_2} \beta_1 \frac{p!}{(p-\beta_1)!} \sum_{|\beta| = p-\beta_1-1} \left( \begin{array}{c} p-\beta_1 \\ \beta \end{array} \right) \tilde{\theta}^{\beta_2} \tilde{u}^{\beta} \\
= \sum_{\beta_1 = 0}^{p} \frac{1}{\beta_1!} \theta_1^{\beta_2} \beta_1 u_1^{\beta_1} \frac{\partial}{\partial t} u_1 \sum_{|\beta| = p-\beta_1} \left( \begin{array}{c} p \\ \beta \end{array} \right) \tilde{\theta}^{\beta_2} \tilde{u}^{\beta} \\
+ \sum_{\beta_1 = 0}^{p-1} \frac{1}{\beta_1!} \theta_1^{\beta_2} \beta_1 \frac{p!}{(p-\beta_1)!} \sum_{|\beta| = p-\beta_1-1} \left( \begin{array}{c} p-\beta_1 \\ \beta \end{array} \right) \tilde{\theta}^{\beta_2} \tilde{u}^{\beta} \\
= \sum_{|\beta| = p-1} \left( \begin{array}{c} p \\ \beta \end{array} \right) \theta^{\beta_2} u^{\beta_1+1} \frac{\partial}{\partial t} u_1 + \sum_{|\beta| = p-1} \left( \begin{array}{c} p \\ \beta \end{array} \right) \theta^{\beta_2} u^{\beta} \sum_{j=2}^{m} \tilde{\theta}_j^{\beta_2} \frac{\partial}{\partial t} u_j.
\]
(4.4)

We also need a second technical result. We have recently proved this in the far simpler case when $(a_{i,j}^k) = d_k I_{n \times n}$ in [22]. The general case is handled below.

**Lemma 4.2.** Suppose $m \in \mathbb{N}$, $\theta = (\theta_1, \ldots, \theta_m)$, where $\theta_1, \ldots, \theta_m$ are positive real numbers, and let $\mathcal{H}_p[u]$ be defined in (2.10). If $p \in \mathbb{N}$ such that $p \geq 2$, then
\[
\sum_{|\beta| = p-1} \left( \begin{array}{c} p \\ \beta \end{array} \right) \theta^{\beta_2} \sum_{k=1}^{m} \theta_k^{\beta_k+1} (A_k \nabla u_k) \cdot \nabla u = \sum_{|\beta| = p-2} \left( \begin{array}{c} p \\ \beta \end{array} \right) \theta^{\beta_2} \sum_{k=1}^{m} \sum_{r=1}^{m} C_{k,r}(\beta) (A_k \nabla u_k) \cdot \nabla u_r,
\]
where
\[
C_{k,r}(\beta) = \begin{cases} 
\theta_k^{\beta_2} \theta_r^{\beta_r+1}, & k \neq r, \\
\theta_k^{4\beta_k+4}, & k = r.
\end{cases}
\]

**Proof.** The result is easily verified when $m_1 = 1$, regardless of the choice of $p$, and for $p = 2$, regardless of the choice of $m_1$. Suppose $p \geq 2$ and $m_1 \geq 2$. We proceed by induction on the value $m_1$, and assume $k \in \mathbb{N}$ such that the result is true for $m_1 = k$. Suppose $m_1 = k + 1$ and (as in the proof of Lemma 4.1) denote
\[
\tilde{\beta} = (\beta_2, \ldots, \beta_{m_1}) \text{ and } \tilde{u} = (u_2, \ldots, u_{m_1}).
\]
Also, whenever $w \in \mathbb{R}^n$ and $k \in \{1, \ldots, n\}$, we denote entry $k$ in $w$ by $w_k$. Then
\[
\sum_{|\beta| = p-1} \left( \begin{array}{c} p \\ \beta \end{array} \right) \theta^{\beta_2} \sum_{i=1}^{m_1} \theta_i^{\beta_i+1} (A_i \nabla u_i) \cdot \nabla u_i
\]
\[
= \sum_{\beta_1=0}^{p-1} \sum_{|\beta|=p-\beta_1-1} \left( \frac{p}{\beta} \right) \theta_1^{\beta_1} \tilde{\theta}^{\tilde{\beta}} \left[ \theta_i^{2\beta_1+1} (A_i \nabla u_1) \cdot \nabla u^\beta + \sum_{i=1}^{m_1-1} \tilde{\theta}_i^{2\tilde{\beta}_1+1} (A_{i+1} \nabla \tilde{u}_i) \cdot \nabla (u_i^{\beta_1} \tilde{u}^{\tilde{\beta}}) \right].
\]

Note that for \(1 \leq \beta_1 \leq p-2\) and \(|\tilde{\beta}| = p - \beta_1 - 1\)
\[
\nabla (u_1^{\beta_1} \tilde{u}^{\tilde{\beta}}) = \beta_1 u_1^{\beta_1 - 1} \tilde{u}^{\tilde{\beta}} \nabla u_1 + \sum_{j=1, \tilde{\beta}_j \neq 0}^{m_1-1} \tilde{\beta}_j u_1^{\beta_1} \tilde{u}^{\tilde{\beta} - e_j} \nabla \tilde{u}_j.
\]

Therefore, from (4.5) and (4.6), we have
\[
\sum_{|\beta|=p-1} \left( \frac{p}{\beta} \right) \theta_1^{\beta_1} \tilde{\theta}^{\tilde{\beta}} \left[ \theta_i^{2\beta_1+1} (A_i \nabla u_1) \cdot \nabla u^\beta + \sum_{i=1}^{m_1-1} \tilde{\theta}_i^{2\tilde{\beta}_1+1} (A_{i+1} \nabla \tilde{u}_i) \cdot \tilde{\beta}_j u_1^{\beta_1} \tilde{u}^{\tilde{\beta} - e_j} \nabla \tilde{u}_j \right] = I + II,
\]

where
\[
I = \sum_{\beta_1=0}^{p-1} \sum_{|\beta|=p-\beta_1-1} \left( \frac{p}{\beta} \right) \theta_1^{\beta_1} \tilde{\theta}^{\tilde{\beta}} \left[ \theta_i^{2\beta_1+1} (A_i \nabla u_1) \cdot \nabla u^\beta + \sum_{i=1, \tilde{\beta}_i \neq 0}^{m_1-1} \tilde{\theta}_i^{2\tilde{\beta}_1+1} (A_{i+1} \nabla \tilde{u}_i) \cdot \tilde{\beta}_j u_1^{\beta_1} \tilde{u}^{\tilde{\beta} - e_j} \nabla \tilde{u}_j \right]
\]

and
\[
II = \sum_{\beta_1=0}^{p-2} \sum_{|\beta|=p-\beta_1-1} \left( \frac{p}{\beta} \right) \theta_1^{\beta_1} \tilde{\theta}^{\tilde{\beta}} \left[ \theta_i^{2\beta_1+1} (A_i \nabla u_1) \cdot \nabla u^\beta + \sum_{i=1, \tilde{\beta}_i \neq 0}^{m_1-1} \tilde{\theta}_i^{2\tilde{\beta}_1+1} (A_{i+1} \nabla \tilde{u}_i) \cdot \tilde{\beta}_j u_1^{\beta_1} \tilde{u}^{\tilde{\beta} - e_j} \nabla \tilde{u}_j \right]
\]

Above, \(e_j\) denotes row \(j\) of the \((m_1 - 1) \times (m_1 - 1)\) identity matrix. We start with the analysis of II. We can rewrite
\[
II = \sum_{\beta_1=0}^{p-2} \frac{1}{\beta_1!} \theta_1^{\beta_1} u_1^{\beta_1} \sum_{|\beta|=p-\beta_1-1} \left( \frac{p}{\beta} \right) \theta_1^{\beta_1} \tilde{\theta}^{\tilde{\beta}} \sum_{i=1, \tilde{\beta}_i \neq 0}^{m_1-1} \tilde{\theta}_i^{2\tilde{\beta}_1+1} (A_{i+1} \nabla \tilde{u}_i) \cdot \tilde{\beta}_j u_1^{\beta_1} \tilde{u}^{\tilde{\beta} - e_j} \nabla \tilde{u}_j
\]
\[
= \sum_{\beta_1=0}^{p-2} \frac{1}{\beta_1} \theta_1^{\beta_1} u_1^{\beta_1} \sum_{|\beta|=p-\beta_1-1} \left( \frac{p}{\beta} \right) \theta_1^{\beta_1} \tilde{\theta}^{\tilde{\beta}} \sum_{i=1, \tilde{\beta}_i \neq 0}^{m_1-1} \tilde{\theta}_i^{2\tilde{\beta}_1+1} (A_{i+1} \nabla \tilde{u}_i) \cdot \nabla \tilde{u}^{\tilde{\beta}}
\]
\[
= \sum_{\beta_1=0}^{p-2} \frac{1}{\beta_1} \theta_1^{\beta_1} u_1^{\beta_1} \frac{p!}{(p - \beta_1)!} \sum_{|\beta|=p-\beta_1-1} \left( \frac{p - \beta_1}{\beta} \right) \theta_1^{\beta_1} \tilde{\theta}^{\tilde{\beta}} \sum_{i=1, \tilde{\beta}_i \neq 0}^{m_1-1} \tilde{\theta}_i^{2\tilde{\beta}_1+1} (A_{i+1} \nabla \tilde{u}_i) \cdot \nabla \tilde{u}^{\tilde{\beta}}
\]
\[
= \sum_{\beta_1=0}^{p-2} \frac{1}{\beta_1} \theta_1^{\beta_1} u_1^{\beta_1} \frac{p!}{(p - \beta_1)!} \sum_{|\beta|=p-\beta_1-2} \left( \frac{p - \beta_1}{\beta} \right) \theta_1^{\beta_1} \tilde{\theta}^{\tilde{\beta}} \sum_{i=1, j=1, \tilde{\beta}_i \neq 0}^{m_1-1} C_{i+1,j+1} (A_{i+1} \nabla \tilde{u}_i) \cdot \nabla \tilde{u}_j
\[ I = \sum_{|\beta| = p - 2} \left( \frac{p}{|\beta|} \right)^{\beta^2} u^\beta \sum_{i,j=2}^{m_1} C_{i,j} (A_i \nabla u_i) \cdot \nabla u_j, \] (4.10)

where the last step follows from the induction hypothesis. Now let’s investigate \( I \). We begin by expanding the \( \nabla u^\beta \) term to find

\[
\begin{align*}
I &= \sum_{\beta_1 = 0}^{p-1} \sum_{|\beta| = p - 1 - \beta_1} \left( \frac{p}{\beta} \right)^{\beta^2} \theta_1^{\beta_1+1} (A_1 \nabla u_1) \cdot \left( \beta_1 u_1^{\beta_1-1} \nabla \beta \nabla u_1 + \sum_{i=1}^{m-1} \beta_i u_1^{\beta_1-1} u^{\beta - e_i} \nabla u_i \right)
\end{align*}
\]

\[ + \sum_{i=1, \beta_1 \neq 0}^{m-1} \theta_i^{\beta_i+1} (A_{i+1} \nabla u_i) \cdot \beta_1 u_1^{\beta_1-1} u^{\beta} \nabla u_1 \right] =: I_{1,1} + \sum_{i=2}^{m_1} I_{i,1}, \] (4.11)

where

\[
I_{1,1} = \sum_{\beta_1 = 0}^{p-1} \sum_{|\beta| = p - 1 - \beta_1} \left( \frac{p}{\beta} \right)^{\beta^2} \theta_1^{\beta_1+1} \beta_1 u_1^{\beta_1-1} \nabla \beta \nabla u_1 \right)
\]

\[ = \sum_{\beta_1 = 0}^{p-2} \sum_{|\beta| = p - 1 - \beta_1} \left( \frac{p}{\beta} \right)^{\beta^2} \theta_1^{\beta_1+4} (A_1 \nabla u_1) \right)
\]

and for \( i \in \{2, \ldots, m_1 \}, \)

\[
I_{i,1} = \sum_{\beta_1 = 0}^{p-1} \sum_{|\beta| = p - 1 - \beta_1} \left( \frac{p}{\beta} \right)^{\beta^2} \theta_1^{\beta_1+4} (A_1 \nabla u_1) \right)
\]

\[ = \sum_{\beta_1 = 0}^{p-2} \sum_{|\beta| = p - 1 - \beta_1} \left( \frac{p}{\beta} \right)^{\beta^2} \theta_1^{\beta_1+4} (A_1 \nabla u_1) \right)
\]

\[ = \sum_{|\beta| = p - 2} \left( \frac{p}{|\beta|} \right)^{\beta^2} u^\beta \sum_{l=1}^{n} \theta_1^{\beta_1+1} \theta_1^{\beta_1+1} (A_1 \nabla u_1) \right)
\]
REACTION-DIFFUSION-ADVECTION SYSTEMS WITH NON-SMOOTH DIFFUSION

\[ + \sum_{|\beta|=p-2} \left( \frac{p}{\beta} \right) \theta^{\beta_2} u^{\beta} \sum_{l=1}^{n} \theta_{1}^{2\beta_{1}+1} \theta_{i}^{2\beta_{i}+1} (A_{i} \nabla u_{i})_{l} \frac{\partial u_{l}}{\partial x_{l}}. \]  

(4.13)

The result follows by combining (4.7), (4.8), (4.10), (4.11), (4.12) and (4.13). \[ \square \]

Acknowledgements: We would like to thank the referees for their comments and suggestions, which improve the presentation of this paper. The second author is partially supported by NAWI Graz.

References

[1] M Bendahmane, M Langlais, and M Saad. Existence of solutions for reaction-diffusion systems with \( L^1 \) data. Advances in Differential Equations, 7(6):743–768, 2002.
[2] Dieter Bothe, Andre Fischer, Michel Pierre, and Guillaume Rolland. Global wellposedness for a class of reaction–advection–anisotropic-diffusion systems. Journal of Evolution Equations, 17(1):101–130, 2017.
[3] José A Canizo, Laurent Desvillettes, and Klemens Fellner. Improved duality estimates and applications to reaction-diffusion equations. Communications in Partial Differential Equations, 39(6):1185–1204, 2014.
[4] M Cristina Caputo, Thierry Goudon, and Alexis F Vasseur. Solutions of the 4-species quadratic reaction-diffusion system are bounded and \( C^\infty \)-smooth in any space dimension. Analysis & PDE, 12(7):1773–1804, 2019.
[5] M Cristina Caputo and Alexis Vasseur. Global regularity of solutions to systems of reaction–diffusion with sub-quadratic growth in any dimension. Communications in Partial Differential Equations, 34(10):1228–1250, 2009.
[6] Laurent Desvillettes, Klemens Fellner, Michel Pierre, and Julien Vovelle. Global existence for quadratic systems of reaction-diffusion. Advanced Nonlinear Studies, 7(3):491–511, 2007.
[7] Herbert Egger, Klemens Fellner, J-F Pietschmann, and Bao Quoc Tang. Analysis and numerical solution of coupled volume-surface reaction-diffusion systems with application to cell biology. Applied Mathematics and Computation, 336:351–367, 2018.
[8] Marisa C Eisenberg, Zhisheng Shuai, Joseph H Tien, and P Van den Driessche. A cholera model in a patchy environment with water and human movement. Mathematical Biosciences, 246(1):105–112, 2013.
[9] Klemens Fellner, Jeff Morgan, and Bao Quoc Tang. Global classical solutions to quadratic systems with mass control in arbitrary dimensions. Annales de l’Institut Henri Poincaré C, Analyse non linéaire, 37(2):281–307, 2020.
[10] Julian Fischer. Global existence of renormalized solutions to entropy-dissipating reaction–diffusion systems. Archive for Rational Mechanics and Analysis, 218(1):553–587, 2015.
[11] WE Fitzgibbon, SL Hollis, and JJ Morgan. Stability and lyapunov functions for reaction-diffusion systems. SIAM Journal on Mathematical Analysis, 28(3):595–610, 1997.
[12] WE Fitzgibbon, M Langlais, and JJ Morgan. A reaction-diffusion system modeling direct and indirect transmission of diseases. Discrete & Continuous Dynamical Systems-B, 4(4):893, 2004.
[13] William E Fitzgibbon, Michel Langlais, and Jeffrey J Morgan. A mathematical model of the spread of feline leukemia virus (felv) through a highly heterogeneous spatial domain. SIAM journal on mathematical analysis, 33(3):570–588, 2001.
[14] Thierry Goudon and Alexis Vasseur. Regularity analysis for systems of reaction-diffusion equations. In Annales scientifiques de l’Ecole normale supérieure, volume 43, pages 117–142, 2010.
[15] Selwyn L Hollis, Robert H Martin, Jr, and Michel Pierre. Global existence and boundedness in reaction-diffusion systems. SIAM Journal on Mathematical Analysis, 18(3):744–761, 1987.
[16] Said Kouachi. Existence of global solutions to reaction-diffusion systems via a lyapunov functional. *Electronic Journal of Differential Equations (EJDE)* [electronic only], 2001:Paper–No, 2001.

[17] Olga A Ladyženskaja, Vsevolod Alekseevich Solonnikov, and Nina N Uralceva. *Linear and quasi-linear equations of parabolic type*, volume 23. American Mathematical Soc., 1988.

[18] Simon Malham and Jack X Xin. Global solutions to a reactive boussinesq system with front data on an infinite domain. *Communications in mathematical physics*, 193(2):287–316, 1998.

[19] Jeff Morgan. Global existence for semilinear parabolic systems. *SIAM journal on mathematical analysis*, 20(5):1128–1144, 1989.

[20] Jeff Morgan. Boundedness and decay results for reaction-diffusion systems. *SIAM Journal on Mathematical Analysis*, 21(5):1172–1189, 1990.

[21] Jeff Morgan and Bao Quoc Tang. Boundedness for reaction–diffusion systems with lyapunov functions and intermediate sum conditions. *Nonlinearity*, 33(7):3105, 2020.

[22] Jeff Morgan and Bao Quoc Tang. Global well-posedness for volume-surface reaction-diffusion systems. *arXiv preprint arXiv:2101.07982*, 2021.

[23] Jeff Morgan and Sheila Waggonner. Global existence for a class of quasilinear reaction-diffusion systems. *Communications in Applied Analysis*, 8(2):153–166, 2004.

[24] Robin Nittka. Quasilinear elliptic and parabolic robin problems on lipschitz domains. *Nonlinear Differential Equations and Applications NoDEA*, 20(3):1125–1155, 2013.

[25] Robin Nittka. Inhomogeneous parabolic neumann problems. *Czechoslovak Mathematical Journal*, 64(3):703–742, 2014.

[26] Dian Palagachev and Lubomira G Softova. Quasilinear divergence form parabolic equations in reifenberg flat domains. *Discrete & Continuous Dynamical Systems-A*, 31(4):1397, 2011.

[27] Michel Pierre. Global existence in reaction-diffusion systems with control of mass: a survey. *Milan Journal of Mathematics*, 78(2):417–455, 2010.

[28] Michel Pierre and Didier Schmitt. Blowup in reaction-diffusion systems with dissipation of mass. *SIAM review*, 42(1):93–106, 2000.

[29] Michel Pierre, Takashi Suzuki, and Yoshio Yamada. Dissipative reaction diffusion systems with quadratic growth. *Indiana University Mathematics Journal*, 2017.

[30] Franz Rothe. Global-solutions of reaction-diffusion systems. *Lecture Notes in Mathematics*, 1072:1–214, 1984.

[31] Vandana Sharma. Global existence and uniform estimates of solutions to reaction diffusion systems with mass transport type boundary conditions. *Communications on Pure & Applied Analysis*, 20(3):955, 2021.

[32] Philippe Souplet. Global existence for reaction–diffusion systems with dissipation of mass and quadratic growth. *Journal of Evolution Equations*, 18(4):1713–1720, 2018.

[33] Kazuo Yamazaki and Xueying Wang. Global stability and uniform persistence of the reaction-convection-diffusion cholera epidemic model. *Mathematical Biosciences and Engineering*, 14:559–579, 2017.

[34] Hong-Ming Yin. On a reaction-diffusion system modeling infectious diseases without life-time immunity. *European Journal Applied Mathematics*, in press, 2020.
