Positive divisors on quotients of $\overline{M}_{0,n}$ and the Mori cone of $\overline{M}_{g,n}$

Claudio Fontanari

Abstract

We prove that if $m \geq n - 3$ then every $S_m$-invariant $F$-nef divisor on the moduli space of stable $n$-pointed curves of genus zero is linearly equivalent to an effective combination of boundary divisors. As an application, we determine the Mori cone of the moduli spaces of stable curves of small genus with few marked points.

1 Introduction

The birational geometry of the moduli space $\overline{M}_{g,n}$ of $n$-pointed stable curves of genus $g$ is indeed a fascinating but rather elusive subject. In particular, the problem of describing its ample cone has attracted very much attention in the last decade (see [2], [7], [6], [3], [5], [4], and [8] for a comprehensive overview).

Recall that $\overline{M}_{g,n}$ has a natural stratification by topological type, the codimension $k$ strata corresponding to curves with at least $k$ singular points.

Conjecture 1. ([6] (0.2)) A divisor on $\overline{M}_{g,n}$ is ample if and only if it has positive intersection with all one-dimensional strata.

The main result (0.3) of [6] is that Conjecture 1 holds if the same claim holds for $\overline{M}_{0,g+n}/S_g$, where $S_g$ denotes the symmetric group permuting $g$ marked points.

Question 1. ([6] (0.13)) If a divisor on $\overline{M}_{0,n}$ has non-negative intersection with all one-dimensional strata, does it follow that the divisor is linearly equivalent to an effective combination of boundary divisors?

A positive answer to Question 1 would imply Conjecture 1 "by an induction which is perhaps the simplest and most telling illustration of the power of the inductive structure of the set of all spaces $\overline{M}_{g,n}$"
Until now, Question 1 has been answered in the affirmative for $n \leq 6$, but all available approaches ([6], [3], and [4]) fail for higher $n$, leading to believe that "it seems unlikely (...) even when $n = 7$" ([5]).

It is worth stressing that the analogous question for effective divisors has a negative answer already for $n = 6$ ([10]). On the other hand, it is known that every $S_m$-invariant effective divisor on $M_{0,n}$ is an effective linear combination of boundary classes if $m \geq n - 2$ ([9]).

In the same spirit, here we present the following result:

**Theorem 1.** For every integer $n \geq 4$ and $m$ such that $n - 3 \leq m \leq n$, every $S_m$-invariant divisor on $M_{0,n}$ intersecting non-negatively all one-dimensional strata is linearly equivalent to an effective combination of boundary divisors.

This statement generalizes [2], Proposition 8 (where $n = 8$ and $m = 6$), and our argument simplifies its proof (by showing that case (ii) never occurs). As a consequence, we determine the Mori cones of moduli spaces of $n$-pointed curves of genus $g$ with small invariants $(n,g)$ as follows:

**Corollary 1.** The cone of effective curves of $M_{g,n}$ is generated by one-dimensional strata for $n = 1$, $g \leq 9$; $n = 2$, $g \leq 7$; $n = 3$, $g \leq 5$.

In the next section we address all essential preliminaries, postponing both proofs to the last section. Throughout the paper we work over the complex field $\mathbb{C}$.

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## 2 The tools

Let $\delta_S$, $S \subset P := \{1, \ldots, n\}$, and $\psi_i$, $i = 1, \ldots, n$, be the natural divisor classes on $M_{0,n}$.

**Lemma 1.** ([1], Lemma 3.3) Let $\vartheta : \overline{M}_{0,A \cup \{q\}} \to \overline{M}_{0,P}$ be the map which associates to any $A \cup \{q\}$-pointed genus zero curve the $P$-pointed genus zero curve obtained by glueing to it a fixed $A^c \cup \{r\}$-pointed genus zero curve via identification of $q$ and $r$. Then

$$
\vartheta^*(\delta_B) = \begin{cases} 
-\psi_q & \text{if } B = A \text{ or } B = A^c \\
\delta_B & \text{if } B \subset A \text{ and } B \neq A \\
\delta_{B \setminus A^c \cup \{q\}} & \text{if } B \supset A^c \text{ and } B \neq A^c \\
0 & \text{otherwise.}
\end{cases}
$$

According to the standard terminology, a divisor $D$ on $M_{0,n}$ is $F$-nef if it has non-negative intersection with all one-dimensional strata.
Lemma 2. \((6, \text{Theorem 2.1})\) Let 
\[
D = - \sum_{|S| \geq 2} b_S \delta_S + \sum_{|S| = 1} b_S \psi_S
\]
be a divisor on \(\overline{M}_{0,n}\). Then \(D\) is F-nef if and only if 
\[
b_I + b_J + b_K + b_L \geq b_{I \cup J} + b_{I \cup K} + b_{I \cup L}
\]
for every partition \(I \cup J \cup K \cup L = \{1, 2, \ldots, n\}\).

If \(S_m\) acts on \(\overline{M}_{0,n}\) by permuting the last \(m\) marked points, then we denote by \((i_R, j_S, k_T, *)\) the partition \(I \cup J \cup K \cup L = \{1, 2, \ldots, n\}\) with \(R \cup S \cup T \subseteq \{1, \ldots, n - m\}\), \(R \subseteq I\), \(S \subseteq J\), \(T \subseteq K\), and \(|I| = i\), \(|J| = j\), \(|K| = k\). We also adopt the shorthand notation \(i := i_0\).

The vector space of \(S_m\)-invariant divisors on \(\overline{M}_{0,n}\) (up to linear equivalence) is generated by the boundary divisors 
\[
B^i_T = \sum_{|S| = i \cap S = T} \delta_S
\]
with \(2 \leq i \leq n - 2\), \(T \subseteq \{1, \ldots, n - m\}\), and \(B^i_T = B^{n-i}_{\{1, \ldots, n-m\} \setminus T}\).

Lemma 3. \((9, \text{Corollary 3.2})\) If
- \(\mathcal{B}_{n,n} = \{B^i_0 : 2 \leq i \leq \lfloor n/2 \rfloor\}\)
- \(\mathcal{B}_{n,n-1} = \{B^i_T : 2 \leq i \leq \lfloor n/2 \rfloor\}\)
- \(\mathcal{B}_{n,n-2} = \{B^i_T : 2 \leq i \leq \lfloor n/2 \rfloor\} \setminus \{B^2_{i2}\}\)
- \(\mathcal{B}_{n,n-3} = \{B^i_T : 2 \leq i \leq \lfloor n/2 \rfloor\} \setminus \{B^2_{i2}, B^3_{i3}, B^3_{i23}\}\)

then \(\mathcal{B}_{n,m}\) is a basis of the vector space of \(S_m\)-invariant divisors on \(\overline{M}_{0,n}\) for \(m \geq n - 3\).

3 The proofs

Proof of Theorem\([7]\). We are going to show that if \(D\) is a \(S_m\)-invariant F-nef divisor on \(\overline{M}_{0,n}\) then 
\[
D = \sum [i]_T B^i_T
\]
where the sum runs over all \(B^i_T \in \mathcal{B}_{n,m}\) as in Lemma\([6]\) and every coefficient \([i]_T \geq 0\). We set \([n - i]_{\{1, \ldots, n-m\} \setminus T} := [i]_T\) and \([i] := [i]_0\).

If \(m = n\) and \(k \leq n - 3\) we apply Lemma\([2]\) to the partitions \((1, k, i, *)\) for \(i = 1, \ldots, n - k - 2\) and we obtain the inequalities 
\[
[k + 1] + [k + i] + [i + 1] = [k] - [i] - [n - k - i - 1] \geq 0.
\]
Summation over $i = 1, \ldots, n - k - 2$ simplifies to
\[(n - k)[k + 1] \geq (n - k - 2)[k]\]
and since $[1] = 0$ an easy induction implies that $[h] \geq 0$ for every $h \leq n - 2$.

If instead $m = n - 1$, we consider the partitions $(1, k\{1\}, i, *)$ and we get
\[(k + 1)\{1\} + [k + i]\{1\} + [i + 1] - [k]_{\{1\}} - [i] - [n - k - i - 1] \geq 0.\]

Substitution $[n - k - i - 1] = [k + i + 1]_{\{1\}}$ and summation over $i = 1, \ldots, n - k - 2$ yields
\[(n - k)[k + 1]_{\{1\}} \geq (n - k - 2)[k]_{\{1\}}.\]

Hence $[h]_{\{1\}} \geq 0$ and $[h] = [n - h]_{\{1\}} \geq 0$ for every $h$.

Let now $m = n - 2$ and $k \leq n - 3$. The same argument as above applied to the partitions $(1, k\{1,2\}, i, *)$ for $i = 1, \ldots, n - k - 2$ implies $[h]_{\{1,2\}} \geq 0$ for every $h$ by induction from $[2]_{\{1,2\}} = 0$. Moreover, $[h] = [n - h]_{\{1,2\}} \geq 0$ for every $h$. On the other hand, if we try to address also the coefficients $[h]_{\{\alpha\}}$ with $\alpha = 1, 2$ along the same lines, from the partitions $(1, k_{\{\alpha\}}, i, *)$ we obtain
\[(n - k - 1)[k + 1]_{\{\alpha\}} + [k + 1]_{\{1,2\}} \geq (n - k - 2)[k]_{\{\alpha\}}.\]

In particular, since $[2]_{\{1,2\}} = 0$ we deduce $[2]_{\{\alpha\}} = 0$.

**Claim.** For $\alpha = 1, 2$ we have
\[3[3]_{\{\alpha\}} \geq 3[2]_{\{\alpha\}} + [4]_{\{\alpha\}}\]
and if $k \geq 4$ then
\[2[k]_{\{\alpha\}} \geq \left(\frac{(k - 2)(k - 3)}{2}\right) [2] + (k - 2)[2]_{\{\alpha\}} + [k + 1]_{\{\alpha\}}.\]

The positivity of $[k]_{\{\alpha\}}$ for $k \geq 3$ follows by reverse induction from the Claim and the basis step $[n - 2]_{\{\alpha\}} = [2]_{\{1,2\}
\{\alpha\}} \geq 0$. In order to check the Claim, we consider the partition $((k - 1)_{\{\alpha\}}, 1, 1, *)$, which gives
\[2[k]_{\{\alpha\}} + [2] \geq [k - 1]_{\{\alpha\}} + [k + 1]_{\{\alpha\}}. \quad (1)\]

If $k = 3$, it is enough to take into account the partition $(1_{\{\alpha\}}, 1, 1, *)$ and subtract from (1) the corresponding inequality. If instead $k \geq 4$, we introduce the sequence of partitions $((k - 2 - i)_{\{\alpha\}}, 1, 1, *)$ for $i = 1, \ldots, k - 3$, providing a weighted sum
\[\sum_{i=1}^{k-3} i \left(2[k - 1 - i]_{\{\alpha\}} + [2] - [k - 2 - i]_{\{\alpha\}} - [k - i]_{\{\alpha\}}\right) \geq 0\]
which simplifies as

$$(k - 2)[2]_{\{\alpha\}} + \frac{(k - 2)(k - 3)}{2} [2] - [k - 1]_{\{\alpha\}} \geq 0. \tag{2}$$

By subtracting (2) from (1) we get the Claim and this completes the proof that $[h]_{\{\alpha\}} \geq 0$ for $\alpha = 1, 2$.

Finally, we turn to the case $m = n - 3$. Notice that the partition $(1\{1\}, 1\{2\}, 1\{3\}, i, *)$ yields $[2]_{\{2,3\}} \geq 0$ since both $[2]_{\{1,2\}} = 0$ and $[2]_{\{1,3\}} = 0$ and by repeating our standard argument for the partitions $(1, k_{\{1,2,3\}}, i, *)$ we obtain $[h]_{\{1,2,3\}} \geq 0$ inductively from $[3]_{\{1,2,3\}} = 0$.

In order to check the positivity of all remaining coefficients, we reduce ourselves to the previous case $m = n - 2$ by applying Lemma 1 with $P \setminus A = \{1, 2\}$, $P \setminus A = \{1, 3\}$, and $P \setminus A = \{2, 3\}$. Indeed, if $D$ is F-nef then both $\vartheta^*(D)$ and $\vartheta^*(D) + [2]_{\{2,3\}} \psi_q$ are F-nef (since $\psi_q$ is ample and $[2]_{\{2,3\}} \geq 0$). Moreover, both $\vartheta^*(D)$ on $\overline{M}_{0,P\setminus\{1\},U(q)}$ and $\vartheta^*(D) + [2]_{\{2,3\}} \psi_q$ on $\overline{M}_{0,P\setminus\{2\},U(q)}$ turn out to be expressed in the basis $B_{n,n-2}$ as in Lemma 3 (indeed, the coefficient of $B^2_{3,q}$ on $\overline{M}_{0,P\setminus\{1\},U(q)}$, of $B^2_{2,q}$ on $\overline{M}_{0,P\setminus\{1\},U(q)}$, and of $B^2_{1,q}$ on $\overline{M}_{0,P\setminus\{2\},U(q)}$ all vanish since $[3]_{\{1,2,3\}} = 0$). Hence the previous case applies and the proof is over.

Proof of Corollary 2. By [6] (0.3), it is enough to prove the same claim for for $\overline{M}_{0,g+n}/S_g$. As pointed out in [3], Proposition 6, this reduces to check that for all boundary restrictions $\nu: \overline{M}_{0,k} \to \overline{M}_{0,g+n}$ with $8 \leq k \leq g + n$ the pull-back of any F-nef divisor is an effective combination of boundary classes. In order to do so, for $k = g + n$ we directly apply Theorem 1 while if $8 \leq k < g + n$ we need also to notice that for any boundary restriction $\nu: \overline{M}_{0,g+n-1} \to \overline{M}_{0,g+n}$ the pull-back of a F-nef divisor is a F-nef divisor on $\overline{M}_{0,g+n-1}/S_{g-2}$.

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Claudio Fontanari
Politecnico di Torino
Dipartimento di Matematica
Corso Duca degli Abruzzi 24
10129 Torino (Italy)
e-mail: claudio.fontanari@polito.it