Steering, incompatibility, and Bell inequality violations in a class of probabilistic theories

Neil Stevens† and Paul Busch‡

University of York, York YO10 5DD, UK
(Dated: December 6, 2013)

We show that connections between a degree of incompatibility of pairs of observables and the strength of violations of Bell’s inequality found in recent investigations can be extended to a general class of probabilistic physical models. It turns out that the property of universal uniform steering is possible the joint measurability of observables which are incompatible within quantum mechanics”. With this result a tight link has been established between the availability of incompatible observables and the possibility of violating a CHSH inequality. It is natural to ask whether a quantitative connection can be found between a degree of incompatibility and the strength of these violations, and whether such a connection is specific to quantum mechanics or holds in a wider class of probabilistic physical theories.

It is a well known fact that two incompatible quantum observables can be approximately measured together if some unsharpness in the measurement is allowed. A measure of the incompatibility of two observables can then be obtained by quantifying the degree of unsharpness required to obtain an approximate joint measurement. In the case of dichotomic observables this can be achieved by mixing each observable with a trivial observable (a POVM whose positive operators are multiples of the identity) and with relative weights λ, 1 − λ. The mixing weight determines the degree of unsharpness of the resulting smeared observable.

Banik et al have shown that the degree of incompatibility (they use the term complementarity) of two dichotomic observables, quantified by the largest smearing parameter, λ, for which the smeared versions are compatible, puts limitations on the maximum strength of CHSH inequality violations available in such a theory [14]. The Bell functional, B, a generalisation of what is known as the Bell operator in the quantum case, then is bounded by the parameter λopt associated with the “most incompatible” pair of observables, so that B ≤ 2/λopt.

Here we study the connection between degrees of incompatibility and CHSH inequality violation in the context of general probabilistic physical theories by way of unifying the approaches of [13] and [14]. We will see that the degree of incompatibility used by Banik et al is

I. INTRODUCTION

The Bell inequalities [1] provide constraints that certain families of joint probability distributions must satisfy to admit a common joint distribution. It is known that the satisfaction of a full set of Bell inequalities in a probabilistic system is equivalent to the existence of such a joint probability [2, 3, 4]. It was observed subsequently that joint measurability (in the sense that there exist joint probabilities of the usual quantum mechanical form for every state) entails an operator form of Bell inequalities; therefore, the Bell inequalities are satisfied whenever the observables involved in an EPR-Bell type experiment are mutually commutative [7]. In the case of unsharp observables, commutativity is not required for joint measurability and the degree of unsharpness of the observables required for joint measurability can be determined; this value is more restrictive than is needed for violations of the Bell inequalities to be eliminated in the case of the singlet state [8–11].

The connection between joint measurability and Bell inequalities – in the specific form of the CHSH inequalities [12], which apply to experiments involving runs of measurements of two pairs of dichotomic observables on a bipartite system – has been further elucidated in two interesting recent publications by Wolf et al [13] and Banik et al [14]. The former have shown that for any pair of incompatible dichotomic observables in a finite dimensional quantum system a violation of a CHSH inequality will be obtained. Hence, incompatibility is not only necessary but also sufficient for obtaining Bell inequality violations. Wolf et al [13] conclude that “if a hypothetical no-signaling theory is a refinement of quantum mechanics (but otherwise consistent with it), it cannot render useful the joint measurability of observables which are incompatible within quantum mechanics”. With this result a tight link has been established between the availability of incompatible observables and the possibility of violating a CHSH inequality. It is natural to ask whether a quantitative connection can be found between a degree of incompatibility and the strength of these violations, and whether such a connection is specific to quantum mechanics or holds in a wider class of probabilistic physical theories.

2 Such mixing procedures and their connection with goal of achieving joint measurability are investigated systematically in [14].
closely linked with an unnamed parameter used in [13] to characterise the joint measurability of two dichotomic observables. Under an additional assumption on the physical theory, namely that it supports a sufficient degree of steering, the construction used to violate the CHSH inequality generalises. This gives a sufficient condition under which the maximal violation can be saturated. This result can be rephrased by saying that probabilistic theories can be classified according to the value of the generalised Tsirelson bound, defined as the maximum value of the Bell functional, and this bound can (under said assumptions) be realised by suitable maximally incompatible observables (see Theorem 1).

Finally we illustrate the link between incompatibility and Bell violation in the class of regular polygon state spaces. It turns out that this connection appears to hold generally in the case of even-sided polygons but not, at least in the same form, for odd-sided cases.

II. GENERAL PROBABILISTIC MODELS

We begin by presenting the basic elements of the standard framework of probabilistic models. The framework was introduced in the 1960s by researchers in quantum foundations who used it to investigate axiomatic derivations of the Hilbert space formalism of quantum mechanics. Accessible recent introductions of the Hilbert space formalism of quantum mechanics. Operational approaches then gained renewed interest from researchers in quantum information theory, namely that it supports a sufficient degree of steering, the construction used to violate the CHSH inequality generalises. This gives a sufficient condition under which the maximal violation can be saturated. This result can be rephrased by saying that probabilistic theories can be classified according to the value of the generalised Tsirelson bound, defined as the maximum value of the Bell functional, and this bound can (under said assumptions) be realised by suitable maximally incompatible observables (see Theorem 1).

We then write Ω = Ω₁ ⊕ Ω₂, where the normalisation is given by the order unit v₁ ⊕ v₂ ∈ V₁ ⊕ V₂, but in general the positive cone is not unique [27].

Although there is much choice in general for the ordering on V₁ ⊕ V₂, there are two canonical choices, the maximal and minimal. As a minimal demand it is reasonable to expect v₁ ⊕ v₂ ≥ 0 whenever v₁, v₂ ≥ 0, therefore we make the definition

\[ (V₁ ⊕ min V₂)_+ = \left\{ \sum_{i,j} λ_{ij}v₁^{(i)} ⊕ v₂^{(j)} | λ_{ij} ∈ \mathbb{R}_+, v_k^{(i)} ∈ (V_k)_+ \right\}. \]

We can similarly make such demands on the order structure on V₁ ⊕ V₂ leading to the converse definition

\[ (V₁ ⊕ max V₂)_+ = (V₁ ⊕ min V₂)_+^*. \]

Any cone on V₁ ⊕ V₂ which lies between the maximal and minimal cones is then admissible as a viable order structure. In general the tensor product chosen is an important part in defining a theory; the only time when there is no choice (since maximal and minimal are the same) is when the local state spaces are simplexes [27].

The case where both Ω₁ and Ω₂ are quantum state spaces provides a prime example of a nonminimal, nonmaximal order structure, namely the standard quantum mechanical tensor product. By definition Ω₁ ⊗ Ω₂ contains only separable states, which form a proper subset of all bipartite states; by contrast, Ω₁ ⊗ max Ω₂ contains not only the usual quantum states, but also all normalised entanglement witnesses.

A bipartite state ω ∈ Ω₁ ⊗ Ω₂ can also be viewed as a way to prepare states in Ω₁, via the measurement of an observable on Ω₂. In this way, for each state ω, we can define the corresponding linear map \( \hat{ω} : V₂ → V₁ \) by

\[ a(\hat{ω}(b)) = ω(a, b), \quad a ∈ V₁^*, \ b ∈ V₂^*. \]

III. FUZZINESS AND JOINT MEASURABILITY

Consider a system represented by a probabilistic model, whose state space is given by the convex set Ω. Any dichotomic (or two-outcome) observable O on Ω is determined by an effect \( e = [O,+1] ∈ \mathcal{E}(Ω) \), where for any ω ∈ Ω, the probability of getting the outcome labelled by ‘+1’ in the state ω is given by \( ω(e) \), and similarly for the outcome ‘−1’ associated with the complement effect \( e' := u − e = [O,−1] \).

Two effects e and f are said to be jointly measurable if there exists g ∈ A(Ω) satisfying

\[ 0 ≤ g, \quad g ≤ e, \quad g ≤ f, \quad e + f ≤ g + u, \]

Under the assumption of tomographic locality [26], the state space of a composite system with local state spaces Ω₁ and Ω₂ naturally lives in the vector space V₁ ⊗ V₂. We then write Ω = Ω₁ ⊗ Ω₂ = (V₁ ⊗ V₂)⁺, where the normalisation is given by the order unit u₁ ⊗ u₂ ∈ V₁⁺ ⊗ V₂⁺, but in general the positive cone is not unique [27].

Although there is much choice in general for the ordering on V₁ ⊗ V₂, there are two canonical choices, the maximal and minimal. As a minimal demand it is reasonable to expect v₁ ⊕ v₂ ≥ 0 whenever v₁, v₂ ≥ 0, therefore we make the definition

\[ (V₁ ⊕ min V₂)_+ = \left\{ \sum_{i,j} λ_{ij}v₁^{(i)} ⊕ v₂^{(j)} | λ_{ij} ∈ \mathbb{R}_+, v_k^{(i)} ∈ (V_k)_+ \right\}. \]

We can similarly make such demands on the order structure on V₁ ⊕ V₂ leading to the converse definition

\[ (V₁ ⊕ max V₂)_+ = (V₁ ⊕ min V₂)_+^*. \]

Any cone on V₁ ⊕ V₂ which lies between the maximal and minimal cones is then admissible as a viable order structure. In general the tensor product chosen is an important part in defining a theory; the only time when there is no choice (since maximal and minimal are the same) is when the local state spaces are simplexes [27].

The case where both Ω₁ and Ω₂ are quantum state spaces provides a prime example of a nonminimal, nonmaximal order structure, namely the standard quantum mechanical tensor product. By definition Ω₁ ⊗ Ω₂ contains only separable states, which form a proper subset of all bipartite states; by contrast, Ω₁ ⊗ max Ω₂ contains not only the usual quantum states, but also all normalised entanglement witnesses.

A bipartite state ω ∈ Ω₁ ⊗ Ω₂ can also be viewed as a way to prepare states in Ω₁, via the measurement of an observable on Ω₂. In this way, for each state ω, we can define the corresponding linear map \( \hat{ω} : V₂ → V₁ \) by

\[ a(\hat{ω}(b)) = ω(a, b), \quad a ∈ V₁^*, \ b ∈ V₂^*. \]
where \( u \) is the order unit on \( \Omega \). This condition is equivalent to the existence of a joint observable for the dichotomic observables corresponding to \( e \) and \( f \).

Given a two-outcome observable \( A \) determined by effect \( e \), one can introduce a corresponding fuzzy observable \( A(\lambda) \) as a smearing (or fuzzy version) of \( A \), whose defining effect is given by

\[ e(\lambda) = \frac{1 + \lambda}{2} e + \frac{1 - \lambda}{2} e' = \lambda e + \frac{1 - \lambda}{2} u, \]

with smearing parameter \( \lambda \in [0, 1] \), and complement effect \( e'(\lambda) = e(\lambda) \).

Given any pair of two-outcome observables \( A_1, A_2 \), with corresponding effects \( e, f \), we can use the parameter \( \lambda \) to give a measure of how incompatible they are. First we note that for \( \lambda = \frac{1}{2} \), the choice of effect \( g = \frac{1}{2}(e + f) \) generates a joint observable for \( e \) and \( f \) since it satisfies (4), as is readily verified. Thus the set of values of \( \lambda \) which make \( e(\lambda) \) and \( f(\lambda) \) jointly measurable contains \( \frac{1}{2} \). Further, if \( e(\lambda) \) and \( f(\lambda) \) are jointly measurable, then for any \( \lambda \leq \lambda \) so are \( e(\lambda') \) and \( f(\lambda') \). Hence the set lies inside the interval \([0, \lambda_{e,f}]\), where we define \( \lambda_{e,f} \) to be the solution to the cone-linear program

\[
\begin{align*}
\text{maximise:} & \quad \lambda \\
\text{subject to:} & \quad g \leq e(\lambda) \\
& \quad g \leq f(\lambda) \\
& \quad 0 \leq g \\
& \quad e(\lambda) + f(\lambda) - u \leq g.
\end{align*}
\]

This measure of incompatibility of a pair of effects in turn leads to a measure of the degree of incompatibility of a given model by looking for the most incompatible pair:

\[ \lambda_{opt} = \inf_{e,f \in \mathcal{E}(\Omega)} \lambda_{e,f}. \]

Following a path similar to [14], we can define a different parameter \( t_{e,f} \), which will we see is closely linked with \( \lambda_{e,f} \). For a given pair of effects \( e \) and \( f \), we define \( t_{e,f} \) to be the solution to the cone-linear program:

\[
\begin{align*}
\text{minimise:} & \quad t \\
\text{subject to:} & \quad g \leq e + tu \\
& \quad g \leq f + tu \\
& \quad 0 \leq g \\
& \quad e + f - u \leq g.
\end{align*}
\]

As shown in [28], the optimal set for (8) is nonempty, so the minimum can be achieved, hence \( e \) and \( f \) are incompatible if and only if \( t_{e,f} > 0 \). Here we notice that the pair \((\lambda, g)\) being feasible for the problem (9) is equivalent to the pair \((\frac{1 - \lambda}{2\lambda}, \frac{g}{\lambda})\) being feasible for the problem (9). Combining this with the fact that the function \( \frac{1 - \lambda}{2\lambda} \) is monotonically decreasing for \( \lambda \in [0, 1] \) brings us to the promised link

\[ t_{e,f} = \frac{1 - \lambda_{e,f}}{2\lambda_{e,f}}. \]

**Examples**

In a model of discrete classical probability theory we take the state space to be the set of all probability measures on some countable set \( X \), i.e.

\[ \Omega = \left\{ (\omega_x)_{x \in X} \mid \omega_x \geq 0 \forall x \in X, \sum_x \omega_x = 1 \right\}. \]

A functional \( e \) on \( \Omega \) with action \( e(\omega) = \sum_x e_x \omega_x \) is easily seen to be positive iff \( e_x \geq 0 \) for all \( x \in X \), and the order unit satisfies \( u_x = 1 \) for all \( x \in X \).

Suppose we now have two effects \( e, f \in \mathcal{E}(\Omega) \). Taking \( g \) to have components \( g_x = \min\{e_x, f_x\} \), then since positivity is determined componentwise the inequalities (4) are immediately satisfied, and hence \( e \) and \( f \) are jointly measurable. Since this holds for arbitrary \( e \) and \( f \) in this case we have \( \lambda_{opt} = 1 \).

As shown in [14], in any finite dimensional Hilbert space the value of the joint measurability parameter for a pair of dichotomic observables is \( \lambda_{opt} = 1/\sqrt{2} \).

A simple non-classical, non-quantum example is that of the squit. The two dimensional state space is given by a square, denoted \( \Box \); it contains all points \((x, y, 1)\) with \(-1 \leq x + y \leq 1, -1 \leq x - y \leq 1\), and takes the shape of a square. As we will see, the squit leads to maximally incompatible effects in the sense that it leads to the smallest possible value of \( \lambda_{opt} \).

Firstly we note that for any probabilistic model \( \lambda = \frac{1}{2} \) provides a lower bound for \( \lambda_{opt} \), since \( e(\frac{1}{2}) = \frac{1}{2} e + \frac{1}{2} u \) and \( f(\frac{1}{2}) = \frac{1}{2} f + \frac{1}{2} u \) are always jointly measurable. This can be seen explicitly by setting \( g = \frac{1}{2} e + \frac{1}{2} f \), then the corresponding equations (4) are satisfied.

As a convenient parametrisation we can write a generic affine functional \( g \in \mathcal{A}(\Box) \) as a vector \( g = (a, b, c) \), with action given by the canonical inner product scaled by a factor of \( \frac{1}{2} \). In this case the order unit is given by \( u = (0, 0, 2) \). Since the positivity of a functional \( g \) on a compact convex set is equivalent to positivity on its extreme points, we can determine the structure of the set of effects by demanding that its elements \( g \) take values between 0 and 1 on the extreme points of the set of states. In the case of the squit, \( \mathcal{E}(\Box) \) is a convex polytope with defining inequalities given by

\[ u \geq g \geq 0 \iff \left\{ \begin{array}{ll}
2 \geq c + a \geq 0, & 2 \geq c + b \geq 0, \\
2 \geq c - a \geq 0, & 2 \geq c - b \geq 0.
\end{array} \right. \]

We note the extreme points: \((0, 0, 2) = u, (0, 0, 0), (1, 1, 1), (1, -1, 1), (-1, 1, 1), (-1, -1, 1)\).
In an attempt to find the lowest possible value of $\lambda_{e,f}$ we consider the case of the two orthogonal extremal effects $e = (1, 1, 1)$ and $f = (1, -1, 1)$. In order for $e^{(A)}$ and $f^{(A)}$ to be jointly measurable we need to be able to find a $g$ that satisfies all the inequalities in [3]. This entails, in particular:

$$g - e^{(A)} - f^{(A)} + u = (a - 2\lambda, b, c) \geq 0,$$

giving $2\lambda \leq a + c$;

$$e^{(A)} - g = (\lambda - a, \lambda - b, 1 - c) \geq 0,$$

giving $\lambda \leq 1 + a - c$;

$$f^{(A)} - g = (\lambda - a, -\lambda - b, 1 - c) \geq 0,$$

giving $\lambda \leq 1 - a - c$;

$$g = (a, b, c) \geq 0,$$

giving $a \leq c$.

Combining these inequalities leads to $4\lambda \leq 2 + a - c \leq 2$, so for this choice of $e$ and $f$ we must have $\lambda_{e,f} \leq \frac{1}{2}$. Given that $\frac{1}{2}$ is the lowest possible value, we conclude that in the case of the squit $\lambda_{\text{opt}} = \frac{1}{2}$.

IV. STEERING AND SATURATION OF THE GENERALISED TSIRELSON BOUND

In order to give conditions on a generalised probabilistic model under which the bound on CHSH violations can define its $A$ marginal, living in $\Omega_A$ in an analogue to the quantum mechanical partial trace:

$$\omega^A = \hat{\omega}(u_B),$$

where $u_B$ is the order unit on $B$, with a similar definition for $\omega^B$.

Following this we say that a state $\omega \in \Omega_A \otimes \Omega_B$ is steering for its $A$ marginal if for any collection of sub-normalised states that form a decomposition of that marginal, i.e., \(\{\alpha_1, ..., \alpha_n\}, \sum_i \alpha_i = \omega^A, 0 \leq u_A(\alpha_i) \leq 1\), there exists an observable $\{e_1, ..., e_n\} \subset \mathcal{E}(\Omega_B)$ with $\alpha_i = \hat{\omega}(e_i)$.

It was observed by Schrödinger that this property holds in quantum mechanics for all pure bipartite states [31], originally coining the term steering, which we generalise now, following [29]: A general probabilistic model of a system $A$ with state space $\Omega_A$ supports uniform universal steering if there is another system $B$ with state space $\Omega_B$, such that for any $\alpha \in \Omega_A$, there is a state $\omega_\alpha \in \Omega_A \otimes \Omega_B$, with $\omega^A_\alpha = \alpha$ that is steering for its $A$ marginal, and supports universal self-steering if the above is satisfied with $B = A$. The existence of steering in this manner is similar to the idea of purification to be found, for example, in [31]. Indeed any purification of a state will be steering for its marginals; however steering states being pure is not required here.

The magnitude of maximal CHSH violations is quantified in quantum mechanics by the norm of the Bell operator. We take $A_1, A_2, B_1$ and $B_2$ to be $\pm 1$-valued observables, and define following [14]

$$\mathbb{B} := \langle A_1 B_1 + A_1 B_2 + A_2 B_1 - A_2 B_2 \rangle_\omega,$$

where $A_1 := A_1^{[+1]} - A_1^{[-1]}$, etc., and $\langle X \rangle_\omega := X(\omega)$ for any affine functional $X$. We will call the map $\omega \mapsto \mathbb{B}$ the Bell functional and refer to $\sup_\omega \mathbb{B}$ as the (generalised) Tsirelson bound.

In order to see where steering enters the picture, we follow [14] to get a simple bound on the norm of $\mathbb{B}$. In order to do this we consider what effect smearing the observables of one party has by defining

$$\mathbb{B}^{(\lambda)} := \langle A_1^{(\lambda)} B_1 + A_1^{(\lambda)} B_2 + A_2^{(\lambda)} B_1 - A_2^{(\lambda)} B_2 \rangle,$$

where $A_1^{(\lambda)} = A_1^{[\lambda]} - A_1^{[-\lambda]}$ etc., with the smearing of the effects as defined as in [3]. Due to the fact that the choice of observable that is mixed to form the smearing is an unbiased trivial observable, the resulting expectation scales with the smearing parameter:

$$A_1^{(\lambda)} = \lambda A_1^{[+1]} + \frac{1 - \lambda}{2} u - \lambda A_1^{[-1]} - \frac{1 - \lambda}{2} u = \lambda A_1.$$

Now since the Bell functional is bilinear, and the same smearing parameter is being used on all functionals on the first system, the linear scaling carries over and we get $\mathbb{B}^{(\lambda)} = \lambda \mathbb{B}$.

As shown in the previous chapter, there always exists jointly measurable fuzzy versions of any pair of observables, so long as the value of the smearing parameter is small enough. Now if we take any $\lambda$ such that $A_1^{(\lambda)}$ and $A_2^{(\lambda)}$ are jointly measurable, then we know that the corresponding Bell functional satisfies the usual Bell inequality, and thus its value is bounded by $\mathbb{B}^{(\lambda)} \leq 2$. Consequently, each such value of $\lambda$ gives a bound on the Bell functional of $\mathbb{B} \leq \frac{2}{\lambda}$, and in order to obtain the lowest such upper bound we take the largest smearing parameter which still results in joint measurability, to get

$$\mathbb{B} \leq \frac{2}{\lambda_{\text{opt}}},$$

(16)

Since every probabilistic model contains observables which are jointly measurable with no smearing, and thus satisfying the usual Bell inequality, knowing the above bound for a single pair of observables will not necessarily yield information about the structure of the system itself. A more general bound however can be written down by simply taking the most incompatible pair of observables:

$$\mathbb{B} \leq \frac{2}{\lambda_{\text{opt}}}.$$
Theorem 1. In any probabilistic model of a system $A$ that supports uniform universal steering, the Tsirelson bound is given by the tight inequality that can be saturated:

$$
\mathbb{B} \leq \frac{2}{\lambda_{\text{opt}}},
$$

with $\lambda_{\text{opt}}$ defined in Eq. (7).

Proof. Suppose we have a model of a system $A$ that supports uniform universal steering, and that we have two effects $e,f \in \mathcal{E}(\Omega_A)$. The parameter introduced earlier, $\iota_{e,f}$ can now also be calculated from the dual program to (8), which can be given as [32]

$$
\text{maximise: } \mu_3(e + f - u_A) - \mu_1(e) - \mu_2(f)
$$

subject to: 

$$
(\mu_1 + \mu_2)(u_A) = 1 \quad \mu_1 + \mu_2 = \mu_3 + \mu_4 \quad 0 \leq \mu_1, \mu_2, \mu_3, \mu_4
$$

with the $\mu_i \in A(\Omega_A)^*$.

Writing $\mu_1 + \mu_2 = \mu$, for the $\mu_i$ that achieve the optimal value for (18), we find that $\rho \geq 0$ and $u_A(\rho) = 1$, so $\rho \in \Omega_A$. By the assumption of uniform universal steering therefore we can find a state $\omega \in \Omega_A \otimes \Omega_B$ with $\omega_A = \tilde{\omega}(u_B) = \rho$; moreover, in $\{\mu_1, \mu_2\}$ and $\{\mu_3, \mu_4\}$ we have two different decompositions of $\rho$, and we can thus find effects $\tilde{e}, \tilde{f} \in \mathcal{E}(\Omega_B)$ satisfying

$$
\tilde{\omega}(\tilde{e}) = \mu_1, \quad \tilde{\omega}(\tilde{f}) = \mu_3.
$$

To achieve the maximum CHSH violations we take $A_1, A_2, B_1$ and $B_2$ to be $\pm 1$-valued observables defined by effects $\tilde{e}', e, \tilde{f}'$ and $\tilde{f}'$ respectively; we then have

$$
A_1 = u_A - 2\tilde{f}, \quad B_1 = u_B - 2\tilde{e}, \\
A_2 = 2e - u_A, \quad B_2 = u_B + 2\tilde{f}.
$$

The value of the Bell functional can now be evaluated:

$$
\mathbb{B} = \omega(u_A - 2f, 2u_B - 2\tilde{e} - 2\tilde{f}) + \omega(2e - u_A, 2\tilde{f} - 2\tilde{e})
$$

$$
= 2\tilde{\omega}(u_B)(u_A - 2f)
$$

$$
+ 4\tilde{\omega}(\tilde{e})(f - e) + 4\tilde{\omega}(\tilde{f})(f + e - u_A)
$$

$$
= 2 + 4[(\mu_1 + \mu_2)(-f)
$$

$$
+ \mu_1(f - \mu_1(e) + \mu_3(f + e - u_A))]
$$

$$
= 2 + 4[\mu_3(e + f - u_A) - \mu_1(e) - \mu_2(f)]
$$

$$
= 2(2\iota_{e,f} + 1) = \frac{2}{\lambda_{e,f}},
$$

thus saturating the generalised Tsirelson bound as claimed.

Not every probabilistic model may possess the property of supporting uniform universal steering, and although it is a sufficient condition to obtain the conclusion of the above theorem, as the following example will show, it is not a necessary one. Indeed a model of ‘boxworld’, which contains Popescu-Rohrlich (PR) box states exhibiting the maximum possible CHSH violations, uses local state spaces that are the squits introduced earlier, and composition is given by the maximal tensor product. Despite the saturation of the generalised Tsirelson bound, such a state space does not admit uniform universal steering.

To see this, we consider a bipartite state $\omega \in \square \otimes_{\text{max}} \square$ with the corresponding map $\tilde{\omega}$. Note that from the definition of $\omega$ being a state, $\tilde{\omega}$ will automatically be a positive map sending $V_+^2$ into $V_+$. Now suppose $\omega$ is steering for its marginal $\rho$, i.e. $\tilde{\omega}(u) = \rho$, and choose a decomposition of $\rho$ into pure states: $\rho = \sum_i \alpha_i$. Since the subnormalised states in the decomposition are pure, and $\tilde{\omega}$ is positive, the inverse images $\tilde{\omega}^{-1}(\alpha_i)$ must lie on extremal rays of the cone $V_+^2$. Consider the extremal ray effect $e = (1,1,1)$ with its complement $e' = (-1,-1,1)$ (which is again extremal). With appropriate labelling of the $\alpha_i$ we can then write $\alpha_1 = \tilde{\omega}(e)$ and $\alpha_2 = \tilde{\omega}(e')$; however since we have $e + e' = u,$

$$
\alpha_1 + \alpha_2 = \tilde{\omega}(e + e') = \tilde{\omega}(u) = \rho,
$$

and hence $\rho$ can be written as a mixture of just two pure states. Since there are many points in a square that can only be written as a convex combination of a minimum of three extreme points, we conclude that such a model of ‘boxworld’ does not support universal uniform steering.

Remark 1. It is interesting to note that there is another set of conditions sufficient to obtain the conclusion of the above theorem. We say that a positive cone $V_+$ is homogeneous if the space of order automorphisms of $V$ acts transitively on the interior of $V_+$, and (weakly) self-dual if there exists a linear map $\eta : V \rightarrow V^*$ that is an isomorphism of ordered linear spaces i.e. $\eta(V_+) = V_+^*$. It is known that homogeneity follows from uniform universal steering. Conversely, if the positive cone $V_+$ generated by the state space $\Omega$ of the probabilistic model of a system $A$ is homogeneous and weakly self-dual, then uniform universal self-steering follows if the maximal tensor product is adopted. Hence the conditions of Theorem 1 are fulfilled [24] and the Tsirelson bound in the inequality $\mathbb{B} \leq 2/\lambda_{\text{opt}}$ can be saturated.

In the quantum probabilistic model, the tensor product is not maximal but still uniform universal steering holds. The classical model (trivially) satisfies the conditions of weak self-duality and homogeneity, and the tensor product is maximal. The squit is weakly self-dual but does not satisfy uniform universal steering, so that homogeneity fails; but it allows enough self-steering so that the maximal Bell-Tsirelson bound of 4 can be realised.

V. GENERALISED TSIRELSON BOUNDS FOR POLYGON STATE SPACES

Work in [33] suggests that there is a spectrum of values for the generalised Tsirelson bound in the case of 2-dimensional polygon state spaces (given as the convex
hulls of regular polygons). It is shown there that for a system composed of two identical polygon state spaces with an odd number of vertices, the maximally entangled state does not lead to a violation of the standard Tsirelson bound of $2\sqrt{2}$, whereas in the case of an even number of vertices this bound can be exceeded. This suggests that among the class of polygon state spaces, the generalised Tsirelson bound can be either smaller or greater than the standard Tsirelson bound.

**Remark 2.** We note that of the polygon state spaces, the only cases in which homogeneity holds are the $n = 3$ triangle, and the $n \to \infty$ circle. Hence in general uniform universal steering is not available, however it may still be possible to saturate the generalised Tsirelson bound in some cases, but in other this may not be possible.

As shown in [33], in the case of ‘boxworld’, where each local state space is a square, the maximally entangled state is a PR box; it takes the maximum possible value for the Bell functional of 4. This agrees with the result that the squit does indeed lead to the maximum amount of incompatibility, and shows that in this case the generalised Tsirelson bound can be saturated. We have been able to show that this conclusion holds also in regular polygon state spaces where the number of vertices is a multiple of 8. We expect this result to extend to all even-sided cases. This strengthens the expectation, expressed in [32], that the in these cases the Tsirelson bound is saturated with the maximally entangled state.

Moving to the $n = 5$ case makes things a lot more interesting however. To see this we follow the notation in [33] and define the family of state spaces $\Omega_n$ to be the convex hull of the points

$$\omega_i = \left( \frac{r_n \cos(\frac{2\pi i}{n})}{r_n \sin(\frac{2\pi i}{n})}, 1 \right), \quad i = 1, ..., n$$

with $r_n = \sqrt{\sec(\frac{\pi}{n})}$.

The qualitative difference between the state spaces of odd and even sided polygons first appears in the structure of the set of effects. For the case of even $n$, along with $0$ and $u$, there are $n$ extremal effects:

$$e_i = \frac{1}{2} \left( \frac{r_n \cos(\frac{(2i-1)\pi}{n})}{r_n \sin(\frac{(2i-1)\pi}{n})}, 1 \right), \quad i = 1, ..., n$$

and in this case all the $e_i$ lie on extremal rays of the cone $V^+_U$. This important fact occurs since for each of the $e_i$ we can find another effect $e_j$, also extremal, which is its complement, i.e. $e_j = e'_i = u - e_i$, namely for $j = i + \frac{n}{2} \mod n$. For the case of odd $n$, a seemingly similar expression arises for the ray extremal effects:

$$e_i = \frac{1}{1 + r^2_n} \left( \frac{r_n \cos(\frac{2\pi i}{n})}{r_n \sin(\frac{2\pi i}{n})}, 1 \right), \quad i = 1, ..., n$$

On this occasion however, the compliments of the $e_i$ are given by

$$e'_i = u - e_i = \frac{1}{1 + r^2_n} \left( -r_n \cos(\frac{2\pi i}{n}), -r_n \cos(\frac{2\pi i}{n}) \right), \quad i = 1, ..., n$$

which do not coincide with the $e_i$, and thus there are $2n$ non-trivial extreme points of $\mathcal{E}(\Omega_n)$.

Now we can pose the question of what the value is for $\lambda_{opt}$ when the state space is $\Omega_5$, and whether is it possible to achieve the corresponding Bell value $B = 2/\lambda_{opt}$. Since each extreme two valued observable is determined by a ray effect, the largest value of incompatibility will come from one of the possible pairs of the $e_i$. However due to the symmetry of the state space, the affine transformation of rotating by $\pi/3$ serves only to cyclically permute the indices of the $e_i$ modulo 5. This means that there are only two possible values of $\lambda_{e_1,e_2}$, those for nearest neighbors, and those for next nearest neighbors. Calculation shows that these values are, for example

$$\lambda_{e_1,e_2} = \frac{3 + 2\sqrt{5}}{11} \approx 0.67928,$$

$$\lambda_{e_1,e_3} = \frac{8 + 3\sqrt{5}}{19} \approx 0.77416.$$
the optimal value of the optimisation program

$$\begin{align*}
\text{maximise:} & \quad \lambda \\
\text{subject to:} & \quad g \leq e^{(\lambda, p)} \\
& \quad g \leq f^{(\lambda, q)} \\
& \quad 0 \leq g \\
& \quad e^{(\lambda, p)} + f^{(\lambda, q)} - u \leq g \\
& \quad 0 \leq p, q \leq 1.
\end{align*}$$

Solving this updated problem in the case of the pentagon again gives the optimal value on e.g. $e_1$ and $e_2$, with

$$\lambda_{opt} = \frac{5 + \sqrt{5}}{10} \approx 0.72361,$$

which occurs for the values $p = q = 1$.

This is indeed a different value from earlier, but still we have that $\frac{2}{\lambda_{opt}} \neq \frac{6}{\sqrt{5}}$, however in this case, the unbiased nature of the observables mixed in means such a simple link is no longer expected, and indeed we can see that there is a link to the Bell value on the maximally entangled state as follows. As in the previous, we can define a smeared version of the Bell functional, where the smearing is all done on the functionals of one party:

$$\mathcal{E}^{(\lambda, 1)} = \langle A_1^{(\lambda, 1)} B_1 + A_1^{(\lambda, 1)} B_2 + A_2^{(\lambda, 1)} B_1 - A_2^{(\lambda, 1)} B_2 \rangle,$$

but now instead of having the nice linear scaling in $\lambda$, we gain an extra expectation term $\mathcal{E}^{(\lambda, 1)} = \lambda \mathcal{E} + 2(1 - \lambda) \langle B_1 \rangle$, and again under the assumption that $\lambda$ is small enough to ensure joint measurability, and then taking the largest such value we can write the inequality

$$\mathcal{E} \leq \frac{2[1 - (1 - \lambda_{opt}) \langle B_1 \rangle]}{\lambda_{opt}}.$$

The link to the maximally entangled state on two pentagons now comes from noting that the expectation of any observable $B_1$ defined by an extreme effect on the maximally entangled state is $\langle B_1 \rangle = \frac{2 - \sqrt{3}}{2}$. This means that if evaluated in the maximally entangled state, the inequality in (24), for the value of $\lambda_{opt}$ given above, is indeed saturated.

VI. CONCLUSION

By combining and developing ideas from the works of Wolf et al. [13] and Banik et al. [14], we have shown that probabilistic models can be classified according to their associated value of the generalised Tsirelson bound, which specifies the maximum possible violation of CHSH inequalities. We have given conditions (defined and studied in [29]), that probabilistic models may or may not satisfy, under which the maximal CHSH violations are attained for appropriate choices of maximally incompatible dichotomic observables. Here the degree of the incompatibility of two observables is defined by the minimum amount of smearing of these observables necessary to turn them into jointly measurable observables.

The authors of [13] concluded that observables that are incompatible in quantum mechanics remain incompatible in any probabilistic model that serves as an extension of quantum mechanics. Here we have shown that this conclusion applies to extensions of any probabilistic model that allows for sufficient steering.

As an illustration of the general results we have considered the squit system which underlies the PR box model, and have identified the pair of maximally incompatible extremal effects of the squit that give rise to the saturation of the largest possible value (i.e., 4) of the Tsirelson bound. In addition, we have obtained partial confirmation of the conjectured maximality of the Bell functional if evaluated on the maximally entangled state in the class of regular polygon state spaces considered in [33].

In the case of the pentagon state space we discovered that the connection between incompatibility and Bell violation is not always of the simple form envisaged originally and used through most of this paper; this suggests that the definitive universal expression of this connection remains yet to be found.

The methods used here are taken from amongst some of the standard tools of quantum measurement and information theory used in [13] and [14], and we have shown that they apply equally well in a wide class of probabilistic models. This insight may prove valuable in future investigations into the characterisation of quantum mechanics among all probabilistic models.

ACKNOWLEDGEMENTS

We wish to thank Takayuki Miyadera for the suggestion to consider the generalisation of the optimisation problem for the incompatibility parameter that allows mixing with trivial observables that are not necessarily unbiased. We are also grateful to Manik Banik and Alexander Wilce for helpful comments on a draft version of the paper. N.S. gratefully acknowledges support through the award of an Annie Currie Williamson PhD Bursary at the University of York.

[1] J.S. Bell. On the Einstein Podolsky Rosen paradox. *Physics*, 1:195–200, 1964.

[2] Arthur Fine. Hidden variables, joint probability, and the Bell inequalities. *Phys. Rev. Lett.*, 48:291–295, Feb 1982.
