An explicit model of a polarised K3 surface of degree 8 with a symplectic action of $T_{192}$

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Abstract

The author gives a projective model of a complex polarised K3 surface via the knowledge of a finite group acting on it. This paper presents the theory used to develop an algorithm for this purpose. It relies in particular on the existence of a Gauss elimination theorem in the context of a semi-simple group algebra over an algebraically closed field of characteristic zero.

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Introduction

A K3 surface is a simply-connected, compact, complex manifold $X$ admitting a no-where vanishing holomorphic symplectic 2-form $\sigma_X \in H^{2,0}(X)$, which is unique up to scaling. Any pair $(X, L)$ consisting of a projective K3 surface $X$ and a primitive ample line bundle $L$ on $X$ is called a primitively polarised K3 surface and we define the genus of $(X, L)$ to be $\frac{1}{2}(c_1(L)^2 + 2) > 1$. Polarized K3 surfaces and their projective models have been studied by A.L. Mayer [May72] and B. Saint-Donat [SD74] who provided some examples of projective models for low genera. For instance, any double cover of $\mathbb{P}^2_{\mathbb{C}}$ branched over a sextic defines a polarized K3 surface of genus 2, and any such polarized K3 surface can be obtained in this way.

For a K3 surface $X$, we call an automorphism $f \in \text{Aut}(X)$ symplectic if $f^* \sigma_X = \sigma_X$. In [Muk88], S. Mukai shows that for a K3 surface $X$ the normal subgroup of symplectic automorphisms $G_s \subseteq \text{Aut}(X)$ is a subgroup of one among 11 maximal ones. For each of the latter groups, he provides explicit projective models of polarized K3 surfaces admitting such a group of symplectic automorphisms. In the thesis [Smi07, Chapter 1] the author constructs families of polarized K3 surfaces of genus 2 (resp. of genus 3) from the same 11 (maximal) groups classified by Mukai. These models are obtained by lifting linear actions of these groups on $\mathbb{P}^2_{\mathbb{C}}$ to $\mathbb{C}^4$ (resp. $\mathbb{P}^3_{\mathbb{C}}$ to $\mathbb{C}^4$).
and computing invariant polynomials of degree 6 (resp. degree 4). More recently, the authors in [CD22] prove that there are infinitely many polarized K3 surfaces admitting a symplectic action of the Mathieu group $M_{22}$: they produce several new explicit models as intersection of quadrics in larger projective spaces via Veronese embeddings of already known models (see [BH21, §6.1]). They also compute a model for a polarized K3 surface of genus 7 using non-trivial finite central extensions of $M_{22}$ by a cyclic group of order 4 in order to lift linear actions of $M_{22}$ on $\mathbb{P}^5_\mathbb{C}$ to $\mathbb{C}^6$.

The first goal of this paper is to develop an algorithm for computing projective models of polarized K3 surfaces given as smooth complete intersections, or s.c.i., of $t$ hyperplanes of the same degree $d$ in the projective space $\mathbb{P}^n_\mathbb{C}$, with prescribed group of symmetry $G$. This is the case for instance for general polarized K3 surfaces of genus 3 (resp. genus 5), they admit a projective model given as a smooth quartic in $\mathbb{P}^3_\mathbb{C}$ (resp. as the s.c.i. of 3 quadrics in $\mathbb{P}^4_\mathbb{C}$).

We use a similar approach as in [Smi07] and [CD22]. In Section 1 we review some results about linear representations of finite groups as well as their projective representations. The core of this paper is focused in Section 2 where we develop an algorithmic way of studying submodules of group algebra modules. We show in particular that one can parametrize, as for Grassmannian varieties in the case of vector spaces, submodules of a given dimension and given character of a larger group algebra modules. Finally, we show in Section 3 how one can use this systematic study of group algebra modules to compute defining ideals of complete intersections fixed under an action of a group on their ambient projective space. In particular, we prove the following:

**Theorem 0.1.** The polarized K3 surface $(X, L)$, with $c_1(L)^2 = 8$, corresponding to the case 77b in [BH21] admits a projective model in $\mathbb{P}^5_\mathbb{C}$ given by, for any $\lambda \in \mathbb{C}^*$,

$$S_\lambda:\begin{cases}
ix x_1 + x_2 x_3 + x_3 x_4 + \lambda x_5^2 = 0 \\
ix x_1 - x_2 x_3 - x_4 x_5 + \lambda x_5^3 = 0
\end{cases}$$

It is invariant under the linear action of $G := T_{192} \rtimes \mu_2$ on $\mathbb{P}^5_\mathbb{C}$ given by

$$\sigma_1 = \begin{pmatrix}
0 & 0 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \omega \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \sigma_2 = \frac{1}{2} \begin{pmatrix}
0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & i+1 & i-1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \sigma_3 = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$\sigma_4 = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \sigma_5 = \frac{1}{2} \begin{pmatrix}
-1 & 1 & 1 & 0 & 0 \\
1 & -1 & 1 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 \\
1 & 1 & -1 & -2 & 0 \\
0 & 0 & 0 & 0 & -2 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$  

All the algorithms written for the purpose of this paper have been implemented on the CAS Oscar [OSC22], running on Julia, and are accessible at [Mal22, Code.jl].

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## 1 Preliminaries on representation theory

In this section we recall standard results about representation theory and character theory, and we fix notations for the rest of the paper. Throughout this paper, we will work over algebraically closed fields of characteristic zero, e.g. $\mathbb{C}$, and all groups are supposed to be finite.
1.1 Linear representations and group algebra modules

The definitions and results of this subsection can be found in any classical book about representation theory of finite groups. We refer, for instance, to the notes [EGH⁺11].

Let $K$ be a algebraically closed field of characteristic zero and let $E$ be a finite group. By Maschke’s theorem [EGH⁺11, Theorem 3.1], the group algebra $KE$ is semisimple, that is all of its modules are semisimple and therefore can be decomposed as the direct sum of simple submodules. Throughout this paper, we write $KE$-modules as pair $(V, \rho)$ where $V$ is a finite dimensional $K$-vector space and $\rho$ is a $K$-linear representation of $E$ on $V$; that is $\rho$ is a homomorphism

$$\rho : G \to \text{GL}(V).$$

\textbf{Definition 1.1.} Two linear representations $\rho, \rho' : E \to \text{GL}(V)$ are called equivalent if there exists an automorphism $\mathcal{L} : V \to V$ such that, for all $g \in G$

$$\mathcal{L} \circ \rho(g) = \rho'(g) \circ \mathcal{L}.$$ 

\textbf{Definition 1.2.} Let $M = (V, \rho)$ and $M' = (V', \rho')$ be two $KE$-modules. We say that $M$ and $M'$ are equivalent, and we write $M \equiv M'$, if there exists an invertible $K$-linear map $\mathcal{L} : V \to V'$ such that $\rho^\mathcal{L}$ and $\rho'$ are equivalent representations of $E$ on $V'$, where for all $e \in E$

$$\rho^\mathcal{L}(e) := \mathcal{L} \circ \rho(e) \circ \mathcal{L}^{-1}.$$ 

We say in this paper that $M$ and $M'$ are $E$-equivariant, and we write $M \equiv_E M'$, if moreover $\rho^\mathcal{L} = \rho'$.

\textbf{Remark 1.3.} By a Krull-Schmidt theorem [EGH⁺11, Theorem 2.19], if a $KE$-module is semisimple then its decomposition into a direct sum of simple submodules is unique up to equivalence and order of the summands. Moreover, according to [Iss76, Corollary (2.5)], the equivalence classes of simple $KE$-modules correspond bijectively to the conjugacy classes of $E$.

If $M = (V, \rho)$ is a $KE$-module, then by Maschke’s theorem, one can write

$$M = \bigoplus_{i=1}^t M_i$$

(2)

where we assume that there exists $1 = i_0 < i_1 < \ldots < i_{t-1} < i_t = l + 1$ such that for all $0 \leq j \leq t - 1$

$$W_j := \bigoplus_{k \geq i_j} M_k \cong M_{i_j} \oplus M_{i_{j+1} - i_j}$$

is the complete sum of all simple submodules of $M$ which are equivalent to $M_{i_j}$. For $0 \leq j \leq t - 1$, we call $W_j$ an isotypical component of $M$ and we say that

$$M = \bigoplus_{j=0}^{t-1} W_j$$

(3)

is an isotypical decomposition of $M$ (which is unique up to equivalence and order of the summands).

For all $0 \leq j \leq t - 1$, $W_j$ defines itself a $KE$-module which we say to be isotypical of weight $\text{wgt}(W_j) := \dim_K(M_{i_j})$ (to be understood, the $K$-dimension of the underlying vector space).

We conclude this subsection by stating one of the key results we use several times in this paper.

\textbf{Theorem 1.4} (Schur’s Lemma, Proposition 1.16, Corollary 1.17, [EGH⁺11]). Let $M \cong W^\oplus t$ and $M' \cong W'^{\oplus t'}$ be two isotypical $KE$-modules. Then, under the assumption that $K$ is algebraically closed, one has

$$\text{Hom}_{KE}(M, M') = \begin{cases} M_{t'}(K) & \text{if } W \cong W' \\ 0 & \text{else} \end{cases}.$$ 

(4)

In particular, the $KE$-automorphism group of a simple $KE$-module is identified with $K^*$. 

3
1.2 Character of a representation

To any $K$-linear representation of $E$ on a vector space $V$, one can associate a so-called character. These characters encode most of the information one needs to study $KE$-modules. To read more about characters we refer to [ Isa76].

Again, let $K$ be an algebraically closed field of characteristic 0, let $E$ be a finite group, and let $M = (V, \rho)$ be a $KE$-module. We define the $K$-character $\chi_M$ of $M$ to be the mapping

$$\chi_M: E \to K, \ e \mapsto \text{Tr}(\rho(e)).$$

We say that $M$ affords $\chi_M$ and that $\chi_M$ is afforded by $M$. One notes that $\chi_p(1_E) = \dim_K(V)$ and $\chi_p$ is constant on each conjugacy class of $E$. More generally, $K$-characters of $E$ are a special case of what we call class functions on $E$, and they are all of the form $\chi_M$ for some $KE$-module $M$. We define sum and product of $K$-characters of $E$ as pointwise sum and product of their respective images in $K$. So for instance, if $\chi$ and $\chi'$ are two $K$-characters of $E$ afforded by $M$ and $M'$ respectively, then $\chi + \chi'$ is afforded by $M \oplus M'$ and vice-versa. A $K$-character $\chi$ of $E$ is said to be simple, or indecomposable, if $\chi$ cannot be written as sum of other $K$-characters of $E$.

**Proposition 1.5** (Corollary (2.5), [ Isa76]). The number of simple $K$-characters of $E$ is equal to the number of conjugacy classes of $E$ (recall that $E$ is a finite group here). In particular, simple $K$-characters of $E$ are afforded by simple $KE$-modules.

**Proposition 1.6** (Corollary (2.9), [ Isa76]). Two $KE$-modules $M$ and $M'$ are equivalent if and only if they afford the same $K$-character of $E$.

We define the dimension of a $K$-character $\chi$ of $E$ as $\chi(1_E)$. For all $n \geq 1$, we denote $\text{Irr}_K^n(E)$, the set of all $n$-dimensional simple $K$-characters of $E$ and let $\text{Irr}_K(E) := \bigcup_{n \geq 1} \text{Irr}_K^n(E)$. According to [ Isa76, Theorem (2.8)], any $K$-character $\chi$ of $E$ admits a unique decomposition

$$\chi = \sum_{\mu \in \text{Irr}_K(E)} e_\mu \mu$$

where $e_\mu \in \mathbb{Z}_{\geq 0}$ is called the multiplicity of the irreducible character $\mu$ in $\chi$. Given two $K$-characters $\chi = \sum_{\mu \in \text{Irr}_K(E)} e_\mu \mu$ and $\chi' = \sum_{\mu \in \text{Irr}_K(E)} e'_\mu \mu$ of $E$, we define their scalar product

$$\langle \chi, \chi' \rangle := \sum_{\mu \in \text{Irr}_K(E)} e_\mu e'_\mu.$$

In particular, for $\mu \in \text{Irr}_K(E)$, $\langle \mu, \mu \rangle = 1$ and $\langle \chi, \mu \rangle$ is equal to the multiplicity of $\mu$ in $\chi$. If $\chi = \sum_{\mu \in \text{Irr}_K(E)} e_\mu \mu$ and $\chi' = \sum_{\mu \in \text{Irr}_K(E)} e'_\mu \mu$ are two $K$-characters of $E$ such that $0 \leq e_\mu \leq e'_\mu$ for all $\mu \in \text{Irr}_K(E)$, then we say that $\chi$ is a constituent of $\chi'$.

We see that the decomposition of the $K$-character afforded by a $KE$-module depends only on its isotypical decomposition. In particular, we say that a $K$-character $\chi$ of $E$ is isotypical if $\chi$ is afforded by an isotypical $KE$-module.

1.3 Group actions and projective representations

In this subsection we state some relevant results about projective representations of finite groups which we use throughout this paper. We refer to [ Isa76, Chapter 11] for the readers who are not familiar with the notion of projective representations. The key point is that one can relate linear actions of a group $G$ on $\mathbb{P}^n(\mathbb{C})$ to linear actions of a possibly larger group, called a Schur cover, on $\mathbb{C}^{n+1}$.

Let $K = \mathbb{C}$ be algebraically closed of characteristic zero. Given a finite group $G$ and a finite dimensional $\mathbb{C}$-vector space $V$, we call a projective representation of $G$ on $V$ any homomorphism

$$\overline{\rho}: G \to \text{PGL}(V).$$

Such a representation is called faithful if it is injective. For any group $G$, there exists a finite abelian group $M(G)$ called the Schur multiplier of $G$ (see [ Isa76, Definition (11.12)]),
be identified with $H^2(G, \mathbb{C}^*)$, the second cohomology group of $G$ with complex coefficients. In [Sch04], I. Schur proved that for any finite group $G$, there exists a group $E$ and an exact sequence

$$1 \rightarrow H \xrightarrow{i} E \xrightarrow{\rho} G \rightarrow 1$$

such that $H = M(G) = i(H) \subseteq E'$ (the derived subgroup of $E$) and such that for any projective representation $\overline{\rho}: G \rightarrow \text{PGL}(V)$ of $G$ on a finite vector space $V$, there exists a linear representation $\rho: E \rightarrow \text{GL}(V)$ making the following diagram with exact rows commute

$$\begin{array}{ccc}
1 & \rightarrow & H \\
\downarrow & & \downarrow \\
1 & \rightarrow & \mathbb{C}^\times \\
\end{array}
\xrightarrow{\beta}
\begin{array}{ccc}
& & E \\
\downarrow & & \downarrow \pi \rho \\
& & \text{GL}(V) \\
\end{array}
\xrightarrow{\beta}
\begin{array}{ccc}
& & G \\
\downarrow & & \downarrow \pi \\
& & \text{PGL}(V) \\
\end{array}
\rightarrow
1.

(8)

Here $\beta$ is induced by the restriction of $\rho$ to $i(H)$. This result is known as Schur’s Theorem (see [Isa76, Theorem (11.17)]) and the group $E$ is referred to as a Schur cover of $G$. We moreover refer to $\rho$ as a lift of $\overline{\rho}$ and the latter as the reduction of the former. Schur’s Theorem allows us to use the results from the theory of linear representations of finite groups to work on projective representations of finite groups. In particular, given a finite group $G$, a Schur cover $E$ of $G$ and a finite dimensional complex vector space $V$, one can relate a classification of projective representations of $G$ on $V$ to a classification of linear representations of $E$ on $V$.

**Definition 1.7** ([Isa76, Definition (1.18)], [Isa76, Page 177]). Let $G$ be a finite group and let $V$ be a finite dimensional vector space. Two projective representations $\overline{\rho}, \overline{\rho}': G \rightarrow \text{PGL}(V)$ are called similar if there exists an automorphism $\mathcal{L}: V \rightarrow V$ such that, for all $g \in G$,

$$\mathcal{L} \circ \overline{\rho}(g) = \overline{\rho}'(g) \circ \mathcal{L}$$

where $\mathcal{L}: \mathbb{P}(V) \rightarrow \mathbb{P}(V)$ is induced by $\mathcal{L}$.

**Lemma 1.8** ([Isa76, Page 178]). Let $G$ be a finite group, $E$ a Schur cover of $G$ and $V$ a finite dimensional complex vector space. Assume that there are two projective representations $\overline{\rho}, \overline{\rho}': G \rightarrow \text{PGL}(V)$ lifting respectively to $\rho, \rho': E \rightarrow \text{GL}(V)$ as in Diagram 8. Then $\overline{\rho}$ and $\overline{\rho}'$ are similar if and only if there exists a homomorphism $\epsilon: E \rightarrow \mathbb{C}^\times$ such that $\rho$ and $\epsilon \rho'$ are equivalent.

**Proof.** Suppose that $\overline{\rho}$ and $\overline{\rho}'$ are similar and let $\mathcal{L} \in \text{GL}(V)$ such that, for all $g \in G$,

$$\mathcal{L} \circ \overline{\rho}(g) \circ \mathcal{L}^{-1} = \overline{\rho}'(g).$$

Now by commutativity of Diagram 8, for all $e \in E$, one obtains that

$$\pi(\mathcal{L} \circ \rho(e) \circ \mathcal{L}^{-1}) = \mathcal{L} \circ (\pi(\rho(e))) \circ \mathcal{L}^{-1} = \overline{\rho}'(p(e)) = \pi(\rho'(e)).$$

Hence, there exists a homomorphism $\epsilon: E \rightarrow \mathbb{C}^\times$ such that $\rho$ and $\epsilon \rho'$ are equivalent.

Now, suppose there exists a map $\epsilon: E \rightarrow \mathbb{C}^\times$ and $\mathcal{L} \in \text{GL}(V)$ such that, for all $e \in E$,

$$\mathcal{L} \circ \rho(e) \circ \mathcal{L}^{-1} = \epsilon(e) \rho'(e).$$

One deduce that $\overline{\rho}$ and $\overline{\rho}'$ are similar by commutativity of Diagram 8 and surjectivity of $p$. \qed

Note that, in the context of Lemma 1.8, given a linear representation $\rho$ of $E$ on $V$, one can always define a projective representation of $G$ on $V$ which makes Diagram 8 commute, by setting $\overline{\rho} := \pi \circ \rho \circ s$ where $s$ is any section of $p$ that maps $1_G$ to $1_E$ (it can be easily shown that this definition does not depend on the choice of $s$). In other words, any linear representation of $E$ always admits a (unique) reduction to $G$. 

5
2 Parametrizing submodules of a given group algebra module

Let $K$ be an algebraically closed field of characteristic zero, $E$ a finite group and $M = (V, \rho)$ be a $KE$-module. In this section we show that the moduli space parametrizing the $KE$-submodules of $M$ is algebraic, and its irreducible components are actually rational.

2.1 Isotypical normalization and Gauss elimination

The goal of this subsection is to bring a constructive approach to the proof of the existence of a Gauss elimination theorem for isotypical $KE$-modules.

Lemma 2.1 (Key Lemma). Let $(V, \rho)$ and $(V', \rho')$ be two equivalent simple $KE$-modules. Then, up to scalar multiplication, there exists a unique $L' \in \text{GL}(V')$ such that

$$(L'V', \rho'\check{\ell}') \equiv_E (V, \rho).$$

Remark 2.2. In other words, Lemma 2.1 tells us that if two simple $KE$-modules $M$ and $M'$ are equivalent, one can always perform a base change on the underlying space of $M'$, for instance, in such a way that the action of $E$ on the respective $K$-bases of both $M$ and $M'$ is the same.

Proof. Since $(V, \rho)$ and $(V', \rho')$ are equivalent, there exists $L \in \text{Isom}_K(V, V')$ such that $\rho'$ and $\rho \check{\ell}$ are equivalent. Therefore, there exists a base change $L' \in \text{GL}(V')$ such that for all $e \in E$

$$L' \circ \rho'(e) \circ L'^{-1} = L \circ \rho(e) \circ L^{-1}.$$ 

Thus, $L$ induces an $E$-equivariance between $(L'V', \rho'\check{\ell}')$ and $(V, \rho)$. Now, if there exists another $L'' \in \text{GL}(V'')$ satisfying the same property, then for all $e \in E$

$$\rho'^{L''}(e) = \rho'^{L'}(e)$$

and therefore $\rho'^{L'' - L'} = \rho'$. This means that $L''^{-1}L'$ is a $K$-linear automorphism of $V'$ which commutes with the action of $E$ given by $\rho'$. By Schur’s lemma (Theorem 1.4), since $(V', \rho')$ is simple, one has that $L''^{-1}L' \in K^*\text{Id}_V$. $\square$

Example 1. We show here that the assumption on $K$ to be algebraically closed is crucial for the unicity up to scalar condition in Lemma 2.1 (which fails if we can’t use Schur’s lemma). Let $K = \mathbb{R}$ and let $E = \mathbb{Z}/4\mathbb{Z}$ with generator $e$. We define

$$\rho: E \to \text{GL}_2(\mathbb{R}), \ e \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$V := \text{Span}_\mathbb{R}(e_1, e_2) \text{ with } e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$ 

We have that $(V, \rho)$ is a simple 2-dimensional $\mathbb{R}E$-module and the respective actions of $E$ on the two distinct bases $\{e_1, e_2\}$ and $\{e_1 - e_2, e_1 + e_2\}$ of $V$ are the same, given by $\rho$. However, the matrix of base change between these two bases is not a multiple of the identity.

Let $M = (V, \rho)$ be a $KE$-module of $K$-dimension $n$ and fix a basis of $V$. Then for $L \in \text{GL}_n(K) \simeq \text{GL}(V)$, we denote

$$L \cdot (V, \rho) := (LV, \rho \check{\ell}).$$

We can actually extend the result from Lemma 2.1 to any isotypical $KE$-module.

Theorem 2.3 (E-equivariant block representation (EBR)). Let $M = \bigoplus_{i=1}^n M_i$ be an isotypical $KE$-module of weight $n = \dim_K(M_1)$ and fix a basis of its underlying $K$-vector space. Then there exist unique, up to scaling, $L_2, \ldots, L_t \in \text{GL}_n(K)$ such that for all $2 \leq j \leq t$, $L_j \cdot M_j \equiv_E M_1$ and

$$M \equiv M_1 \oplus \bigoplus_{j=2}^t L_j \cdot M_j.$$
In other words, if \( M = (V, \rho) = \bigoplus_{j=1}^{t} (V_j, \rho_j) \), then there exists \( L \in \text{GL}_{rt}(K) \) such that

\[
M \cong (LV, \rho^{\otimes t})
\]

and this decomposition is unique up to the action of \( \text{GL}_t(K) \) on the summand of \( M \).

**Proof.** It follows directly from Lemma 2.1 and the fact that all the summands in the isotypical decomposition of \( M \) are simple and pairwise equivalent. The unicity up to the action of \( \text{GL}_t(K) \) follows also from Schur’s lemma (Theorem 1.4) which tells us that \( \text{Aut}_{KE}(M) = \text{GL}_t(K) \). \( \square \)

**Remark 2.4.** If \( M \) is an isotypical \( KE \)-module of weight \( n \) and \( K \)-dimension \( tn \), this theorem tells us that given a simple \( KE \)-module \( W \) of \( K \)-dimension \( n \) such that

\[
M \cong W^{\otimes t}
\]

then one can find a base change conjugating the actions of \( E \) on \( M \) and \( W^{\otimes t} \). In practice, such a change of basis can be computed algorithmically (see for instance [CIK97, Theorem 2]).

An isotypical \( KE \)-module \( M \) of weight \( n \) of the form

\[
M = (V, \rho^{\otimes t})
\]

is said to be, in this paper, in \( EBR \)-form. **Theorem 2.3** tells us that given an isotypical \( KE \)-module, one can always find an equivalent isotypical \( KE \)-module in \( EBR \)-form. Before stating the main results of this subsection, we define a convenient operation for vector spaces and isotypical \( KE \)-module in \( EBR \)-form.

**Definition 2.5** (Weighted basis sum). Let \( \{V_j\}_{1 \leq j \leq t} \) be a family of \( n \)-dimensional \( K \)-vector spaces, with respective basis \( B_j := \{v_{j,1}, \ldots, v_{j,n}\} \), \( 1 \leq j \leq t \). Then, for any \( 1 \leq r \leq t \) and any matrix \( L = (a_{i,j})_{1 \leq i \leq r, 1 \leq j \leq t} \in M_{r,t}(K) \), one defines the \( L \)-weighted basis sum of \( V := V_1 \oplus \ldots \oplus V_t \), denoted \( L \ast V \), to be the \( K \)-subvector space of \( V \) spanned by the \( rn \) vectors \( (a_{1,1} B_1, \ldots, a_{r,n} B_r) \) defined by

\[
(u_{n(i-1)+1}, \ldots, u_{ni}) := \sum_{j=1}^{r} a_{i,j} B_j, \quad 1 \leq i \leq r.
\]

**Example 2.** If \( V_1 \) and \( V_2 \) are 2-dimensional with respective bases \( B_1 = \{f_1, f_2\} \) and \( B_2 = \{g_1, g_2\} \), then for any \( L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2,2}(K) \), we have

\[
L \ast (V_1 \oplus V_2) = \text{Span}_K(af_1 + bg_1, af_2 + bg_2, cf_1 + dg_1, cf_2 + dg_2) \subseteq V_1 \oplus V_2.
\]

It is not hard to see that

\[
\dim_K(L \ast (V_1 \oplus V_2)) = 2 \text{rank}_K(L).
\]

**Proposition 2.6.** Let \( M = \bigoplus_{j=1}^{t} (V_j, \rho_j) = (V, \rho^{\otimes t}) \) be an isotypical \( KE \)-module of weight \( n \) in \( EBR \)-form. Let \( 1 \leq r \leq t \) and let \( L \in M_{r,t}(K) \) be of rank \( 1 \leq k \leq r \). Then

\[
N := (L \ast V, \rho^{\otimes k})
\]

is an isotypical \( KE \)-submodule of \( M \) of weight \( n \) and \( K \)-dimension \( kn \) in \( EBR \)-form. **This construction does not depend on the choice of a representative in** \( GL_r(K) \cdot L \), **we call N the L-weighted isotypical submodule of M and we write N = L \ast M.**

**Proof.** Suppose that \( L \) is of full rank \( k = r \). Then, \( L \ast V \) is a \( K \)-subvector space of \( V \). It can be viewed as the direct sum of \( n \)-dimensional \( K \)-subvector spaces of \( V \) of the form \( R_j \ast V \), where for \( 1 \leq j \leq k \), \( R_j \) is the \( j \)-th row of \( L \) (full rank of \( L \) ensures that these \( K \)-subvector spaces are in direct sum). By definition, the action of \( E \) on each of the \( R_j \ast V \) is given by \( \rho \) (since \( M \) is in \( EBR \)-form) and therefore \( N \) defines an isotypical \( KE \)-submodule of \( M \) in \( EBR \)-form of weight \( n \) and of dimension \( n \cdot \text{rank}_K(L) = kn \). The left action of \( GL_r(K) \cong \text{Aut}_{KE}(N) \) (Schur’s Lemma) on \( L \) consists on basis manipulations and so, any other representative \( L' \) of \( GL_r(K) \cdot L \) will give rise to the same \( N \).

If \( L \) is not of full rank, one can find an element \( L' \) in \( GL_r(K) \cdot L \) whose \( r-k \) last rows are actually zero and such that \( L \ast M = L' \ast M \). Thus one applies a similar proof considering the matrix defined by the first \( k \) rows of \( L' \). \( \square \)
As in the case of $K$-vector spaces, there exists a Gauss elimination theorem for $KE$-submodules of an isotypical $KE$-module in EBR-form. This implies that weighted isotypical submodules of isotypical $KE$-modules define a standard form of $KE$-modules from which one can obtain any $KE$-submodule of an isotypical $KE$-module (not necessarily in EBR-form).

**Theorem 2.7** (Gauss elimination). Let $M = W^@t = (V, \rho^@t)$ be an isotypical $KE$-module of weight $n$ in EBR-form. Let $N$ be a $KE$-submodule of $M$. Then

1. $N$ is isotypical of weight $n$ and $K$-dimension $rn$ for some $1 \leq r \leq t$, and $r = 1$ if and only if $N$ is simple.

2. If $N$ is simple, then there exists a row vector $v = (1,a_2,\ldots,a_r) \in M_{1,t}(K)$ such that, up to reordering the summands in $M$,

$$N = v * M.$$

3. If $N$ is non simple ($r \geq 2$), then there exists a matrix $L \in M_{r,t-r}(K)$ such that, up to reordering the summands in $M$ and $N$,

$$N = (I_r|L) * M.$$

**Proof.**

1. Since $N$ is a $KE$-submodule of $M$, its character is a constituent of the character of $M$. Therefore $N$ is isotypical of weight $\dim_K(W) = n$ and dimension $rn$ with $1 \leq r \leq t$. Moreover, $r = 1$ if and only if $N$ affords the same character as $W$, i.e. $N$ is simple.

2. Since $N \cong W$, by Lemma 2.1, one can assume that $N = (U, \rho)$ for some $U \subseteq V$ of $K$-dimension $n$. The action of $E$ on $N$ and each copy of $W$ in $M$ is given by $\rho$, all the copies of $W$ are simple and in direct sum, therefore it is not hard, but tedious, to show that $U$ is a weighted basis sum of $V$, for some $L = (a_1,\ldots,a_r) \in M_{1,t}(K)$ of rank 1 (one can convince one by proving it for $t = n = 2$ and then $t = 2$; the general case follows by case distinction). By reordering the copies of $W$ in $M$ if necessary, we may assume that $a_1 \neq 0$, and denoting $v := L/a_1$, we have

$$N = v * M.$$

3. Using Lemma 2.1 and Theorem 2.3, since $N \cong W^@r$, we may assume that $N = (U, \rho^@r)$, where $U \subseteq V$ is a direct sum of $r$ $K$-subvector spaces of $V$ of $K$-dimension $n$. By applying Item 2 to each summand of $N$, we have that there exists $L \in M_{r,t}(K)$ of rank $r$ such that

$$N = L * M.$$

By performing a row reduction to $L$, we have that $L * M = (I_r|L') * M$ for some $L' \in M_{r,t-r}(K)$. \hfill \Box

**Remark 2.8.** It is good to note that in the proof of Theorem 2.7 Item 3., different choices of elements in $M_{r,t-r}(K)$ produce $E$-equivariant $KE$-submodules of $M$ (they afford the same character and the same action of $E$ on their respective bases) but which are pairwise distinct $KE$-modules.

**Corollary 2.9.** Let $M$ be an isotypical $KE$-module of weight $n$ and $K$-dimension $tn$. Then, the set of $KE$-submodules of $M$ of $K$-dimension $rn$, $1 \leq r \leq t$, can be identified with the Grassmannian variety $\text{Gr}(r,t)$.

**Proof.** Using Lemma 2.1 and Theorem 2.3, we know we can reduce to the case where $M$ is in EBR-form $W^@t$. We can see $W^@t$ as a $K$-vector space of dimension $t$, with basis given by the $t$ copies of $W$, and sum and scalar multiplication given by weighted sums. In this way, Theorem 2.7 and Remark 2.8 makes it clear that looking for $KE$-submodules of $W^@t$ of $K$-dimension $rn$ is the same as looking for $r$-dimensional $K$-subvector spaces of $(W^@t, *)$. \hfill \Box
2.2 Moduli space of submodules

In this subsection we show that using EBR-form for isotypical $KE$-modules and Gauss elimination, the space of $KE$-submodules with a given character of a $KE$-module is rational.

Recall that $M = (V, \rho)$ is a $KE$-module, and let $n := \dim_K(V)$. For all $1 \leq t \leq n$, we define $\mathcal{M}(M, t)$ to be the moduli space of $t$-dimensional $KE$-submodule of $M$. Using iteratively an argument in [MWY20, Theorem 5.11], we have that $\mathcal{M}(M, t)$ is a closed subvariety of $\text{Gr}(t, V) \subseteq \mathbb{P}^{n-1}$. In general, $\mathcal{M}(M, t)$ is not irreducible, and we give two ways to decompose it: we use the first one computationally to parametrize all $t$-dimensional submodules of $M$.

Let $\chi$ be the $K$-character of $M$. For all $1 \leq t \leq \chi(1_E)$, we define $\text{ch}_\chi(t)$ to be the set all of $t$-dimensional $K$-characters of $E$ that are a consistent of $\chi$. For $1 \leq t \leq \chi(1_E)$, each $\eta \in \text{ch}_\chi(t)$ defines an equivalence class of $t$-dimensional submodules of $M$. We denote $\mathcal{M}(M, \eta)$ the moduli space of $t$-dimensional submodules of $M$ affording $\eta$. One has that

$$\mathcal{M}(M, t) = \bigsqcup_{\eta \in \text{ch}_\chi(t)} \mathcal{M}(M, \eta).$$

The following holds:

**Theorem 2.10.** Let $1 \leq t \leq \chi(1_E)$. For all $\eta \in \text{ch}_\chi(t)$, $\mathcal{M}(M, \eta)$ is a rational closed subvariety of $\mathcal{M}(M, t)$ of dimension $\langle \eta, \chi - \eta \rangle$. In particular, $\{\mathcal{M}(M, \eta)\}_{\eta \in \text{ch}_\chi(t)}$ is the set of irreducible components of $\mathcal{M}(M, t)$.

**Proof.** Let $\eta \in \text{ch}_\chi(t)$ and let $\eta = \sum_{\mu \in \text{Irr}_K(E)} e_\mu \mu$ be the isotypical decomposition of $\eta$. Then, for a $KE$-module $N$ in $\mathcal{M}(M, \eta)$, $N$ is a direct sum of isotypical $KE$-submodules of $M$ affording $\mu$ respectively $e_\mu \mu$ for $\mu \in \text{Irr}_K(E)$ such that $e_\mu \neq 0$. Let

$$M = \bigoplus_{\mu \in \text{Irr}_K(E)} W_\mu^{e_\mu}$$

be an isotypical decomposition of $M$ (where $W_\mu$ affords $\mu$ and some of the $f_\mu$’s can be zero). For all $\mu \in \text{Irr}_K(E)$, one has $0 \leq e_\mu \leq f_\mu$, and the isotypical component $N_\mu^{e_\mu}$ of $N$ affording $e_\mu \mu$ is a $KE$-submodule of $W_\mu^{e_\mu}$, i.e.

$$N_\mu^{e_\mu} \in \mathcal{M}(W_\mu^{e_\mu}, e_\mu \mu).$$

By Corollary 2.9, $\mathcal{M}(W_\mu^{e_\mu}, e_\mu \mu)$ is isomorphic to the Grassmannian variety $\text{Gr}(e_\mu, f_\mu)$, which is of dimension $e_\mu(f_\mu - e_\mu) = (e_\mu, f_\mu - e_\mu)$. Therefore, noticing that

$$\mathcal{M}(M, \eta) = \prod_{\mu \in \text{Irr}_K(E), e_\mu \neq 0} \mathcal{M}(W_\mu^{e_\mu}, e_\mu \mu)$$

one deduces that $\mathcal{M}(M, \eta)$ is of the wanted form. Rationality follows from the rationality of Grassmannian varieties (or directly from Theorem 2.7), which is preserved under product. $\mathcal{M}(M, \eta)$ is an irreducible closed subvariety of $\mathcal{M}(M, t)$ of dimension

$$\sum_{\mu \in \text{Irr}_K(E)} (e_\mu, f_\mu - e_\mu) = \langle \eta, \chi - \eta \rangle.$$

Theorem 2.10 offers a feasible way to parametrising $t$-dimensional submodules of a given $KE$-module $M$. Indeed, one can compute an EBR-form of $M$ with its base change, and for all $t$-dimensional constituent $\eta$ of the character $\chi$ of $M$, one may use Theorem 2.7, Corollary 2.9 and Theorem 2.10 to construct a concrete parametrisation of $\mathcal{M}(M, \eta)$. 




### 2.3 Determinantal character

In this subsection, we give another decomposition of $\mathcal{M}(M, t)$ from Section 2.2, by looking at the determinantal characters of the $t$-dimensional $K\mathcal{E}$-submodules of $M$.

Let $1 \leq t \leq \chi(1, E)$, where we recall that $\chi$ is the character afforded by a fixed $K\mathcal{E}$-module $M = (V, \rho)$ of dimension $n$. Any element of the $t$-th exterior $\Lambda^t V$ of $V$ is called a $t$-tensor of $V$ and those of the form $v_1 \wedge \ldots \wedge v_t$ are called completely decomposable or pure. Any element of $\Lambda^t V$ can be written as a finite sum of pure $t$-tensors. There is moreover an induced action of $E$ on $\Lambda^t V$ given by, for all $e \in E$ and for any pure $t$-tensor $v_1 \wedge \ldots \wedge v_t$ of $V$,

$$ e \cdot (v_1 \wedge \ldots \wedge v_t) := (\rho(e)v_1) \wedge \ldots \wedge (\rho(e)v_t). \quad (11) $$

We denote $\Lambda^t \rho$ the previous representation of $E$ on $\Lambda^t V$ and $\Lambda^t M := (\Lambda^t V, \Lambda^t \rho)$ the corresponding $K\mathcal{E}$-module. We call it the $t$-antisymmetric part of $M$.

**Proposition 2.11.** Let $M$ be a $K\mathcal{E}$-module of finite complex dimension $n$. Then $\mathcal{M}(M, t)$ is non-empty if and only if $\Lambda^t M$ admits a 1-dimensional $K\mathcal{E}$-submodule whose underlying $K$-vector space is spanned by a pure $t$-tensor.

**Proof.** This follows from Eq. (11) and from the fact that a pure tensor $v_1 \wedge \ldots \wedge v_m$ is non zero if and only if its components $v_1, \ldots, v_m$ are linearly independent in $V$. \qed

Let $\Lambda^t M = U_{\mu \in \text{Irr}_K(E)} \Lambda^t \rho_{\mu}$ be an isotypical decomposition of $\Lambda^t M$. Therefore, $\Lambda^t M$ has a 1-dimensional $K\mathcal{E}$-submodule if and only there exists $\mu \in \text{Irr}_K^1(E)$ such that $g_\mu \neq 0$. For a linear (i.e. 1-dimensional) $K$-character $\mu$ of $E$ with $g_\mu \neq 0$, the action of $E$ on each summand of $U_{\mu}^{\Lambda^t}$ is given by $\mu$. Therefore, since $\mu$ is a homomorphism

$$ U_{\mu}^{\Lambda^t} = \bigcap_{e \in E} \text{Eigenspace} \left( (\Lambda^t \rho)(e), \mu(e) \right). \quad (12) $$

This proves the following.

**Theorem 2.12 (Theorem 4, [Ser77]).** With the same notation as in Proposition 2.11, $\mathcal{M}(M, t)$ is non empty if, and only if, there exists $\mu \in \text{Irr}_K^1(E)$ such that $g_\mu \neq 0$ and $U_{\mu}^{\Lambda^t}$ contains a pure tensor.

Denote by $\chi$ the $K$-character of $E$ afforded by $M$. For any $\eta \in \text{ch}_\chi(t)$, we call the determinantal character of $\eta$, denoted $\det(\chi)$, the character afforded by the $t$-antisymmetric part of any $K\mathcal{E}$-module affording $\eta$. This is a 1-dimensional character, constituent of the character $\Lambda^t \chi$ afforded by $\Lambda^t M$. Note that two distinct constituents $\eta, \eta' \in \text{ch}_\chi(t)$ of $\chi$ can have the same determinantal character. For any linear $K$-character $\mu$ of $E$, we denote $\mathcal{M}(M, t, \mu)$ the moduli space of $t$-dimensional $K\mathcal{E}$-submodules of $M$ having determinantal character equal to $\mu$. Then, we have the decompositions

$$ \mathcal{M}(M, t) = \bigsqcup_{\mu \in \text{Irr}_K^1(E)} \mathcal{M}(M, t, \mu) $$

and for all $\mu \in \text{Irr}_K^1(E)$

$$ \mathcal{M}(M, t, \mu) = \bigsqcup_{\eta \in \text{ch}_\chi(t), \det(\eta) = \mu} \mathcal{M}(M, \eta). $$

In the rest of this section, we show that for all $\mu \in \text{Irr}_K^1(E)$ such that the set $\{ \eta \in \text{ch}_\chi(t) \mid \det(\eta) = \mu \}$ is non empty, $\mathcal{M}(M, t, \mu)$ is an algebraic subvariety of $\mathcal{M}(M, t)$ by explaining how one can compute its defining ideal. For this, we make explicit how to find pure tensors in an isotypical component of weight 1 of $\Lambda^t M$.

**Proposition 2.13 (Page 64, [Har92]).** Let $V$ be a $K$-vector space of dimension $n$ and let $1 \leq t \leq n - 1$. Then for any non-zero $w \in \Lambda^t V$, $w$ is a pure $t$-tensor if and only if the linear map

$$ \varphi: \ V \to \Lambda^{t+1} V \quad v \mapsto v \wedge w $$

has kernel of dimension $t$. 

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Proof. The result follows from the observation that for a non-zero pure tensor \( w = v_1 \wedge \ldots \wedge v_t \), \( v_1, \ldots, v_t \) are \( K \)-linearly independent and any \( v \) such that \( v \wedge w = 0 \) must lie in the \( t \)-dimensional \( K \)-vector space spanned by the \( v_i \)'s. In particular, one remarks that for a general non-zero \( t \)-tensor \( w \in \Lambda^t V \), the kernel of \( \varphi \) is of dimension at most \( t \), with equality if and only if \( w \) is pure. \( \square \)

Remark 2.14. We chose here to state Proposition 2.13 regarding the dimension of the kernel of \( \varphi \), but one can state a similar result regarding the rank of \( \varphi \), using the Rank-nullity theorem. Indeed, we see with the last comment of the proof that the dimension of the kernel of \( \varphi \), for any tensor in \( \Lambda^t V \), will never exceed \( t \). Therefore, for any non-zero tensor in \( \Lambda^t V \), \( \varphi \) has rank at least \( n - t \), with equality if the tensor is pure. In particular, one only needs to check whether the rank is at most \( n - t \).

We state also a Corollary to Proposition 2.13. For this, we consider the following: fix a volume form given by the natural pairing

\[
\Lambda^t V \times \Lambda^{n-t} V \to \Lambda^n V \xrightarrow{\varphi} K
\]

(13)

To define \( \varphi \), fix a basis \( \{v_1, \ldots, v_n\} \) of \( V \) and for \( w'' \in \Lambda^n V \) non zero, via permutation of the factors, one can write uniquely \( w'' = \varphi(w''')v_1 \wedge \ldots \wedge v_n \) (and \( \varphi(0) = 0 \)). Eq. (13) provides an isomorphism between \( \Lambda^t V \) and \( (\Lambda^{n-t} V)\star \). Moreover, there exists an isomorphism between \( (\Lambda^{n-t} V)\star \) and \( \Lambda^{n-t} V^\star \), given on the elements of the basis by \( (v_1 \wedge \ldots \wedge v_{n-t})\star = v_1^* \wedge \ldots \wedge v_{n-t}^* \). Therefore, to each non-zero \( w \in \Lambda^t V \) one can associate a non-zero element \( w^* \in \Lambda^{n-t} V^\star \) (which depends on the choice of the volume form). Via those descriptions, one sees that \( w \) is a pure \( t \)-tensor if and only if \( w^* \) is a pure \((n-t)\)-tensor.

**Corollary 2.15.** Let \( V \) be a complex vector space of dimension \( n \) and let \( 1 \leq t \leq n - 1 \). Then for any non-zero \( w \in \Lambda^t V \), \( w \) is a pure tensor if and only if the linear map

\[
V^* \to \Lambda^{n-t+1} V^*
\]

\[
w^* \mapsto w^* \wedge v^*
\]

has rank at most \( t \) (since it is at least \( t \), by duality with Remark 2.14). Here \( w^* \) is obtained by fixing a volume form on \( \Lambda^n V \) (see Eq. (13)).

It is good to keep in mind both Proposition 2.13 and its Corollary 2.15 for computational aspects. Indeed, one has to choose to work either with \( V \) or its dual depending on \( t \) and \( n \).

Now, if \( W = (U, \mu^{(t)}) \) is an isotypical component of weight 1 of \( \Lambda^t M = (\Lambda^t V, \Lambda^t \rho) \), identifying \( \mathbb{P}(\Lambda^n V) \) with \( \mathbb{P}^{(t)}_K \) as finite dimensional \( K \)-vector spaces, we have that the set of pure tensors \( U' \) in \( U \) is actually the same as

\[
U' = \mathbb{P}(U) \cap \text{Gr}(t, V).
\]

Here we implicitly identify the Grassmannian variety \( \text{Gr}(t, V) \) with its image via

\[
\text{Gr}(t, V) \xrightarrow{\iota^t} \mathbb{P}^{(t)}_K \supset \mathbb{P}(\Lambda^n V)
\]

where \( \iota \) is the usual Plucker embedding. Using Proposition 2.13, we can describe the ideal defining the projective variety \( U' \) in \( \mathbb{P}^{(t)}_K \). In fact, denoting \( (u_1, \ldots, u_l) \) a basis of \( U \), for any non-zero \( u \in U \), there exist scalars \( y_1, \ldots, y_l \in K \), not all zero, such that

\[
u = \sum_{i=1}^l y_i u_i.
\]

Therefore, \( U' \) is defined as the set of tuples \( (y_i)_{1 \leq i \leq l} \in K^l \setminus \{0\} \) for which the map

\[
\varphi(y_1, \ldots, y_l) \colon V \to \Lambda V
\]
associated to $u = \sum_{i=1}^{t} y_i u_i$ has rank $n-t$. Such a set can be obtained by constructing the polynomial matrix $P$ with entries in $K[y_1, \ldots, y_t]$ corresponding to the linear map

$$\varphi(y_1, \ldots, y_t): V \to (\bigwedge V)[y_1, \ldots, y_t], \ v \mapsto v \wedge (\sum_{i=1}^{t} y_i u_i)$$

and computing the ideal $I$ generated by all $(n-t+1)$-minors of $P$: in this situation, one can show that $U' = V(I)$.

### 3 Finding complete intersections with prescribed symmetry

In this section, we show how one can use the method explained in the previous sections to find defining ideal of complete intersections that are fixed by a linear action of a finite group on their ambient complex projective space.

Let $X$ be a complex projective variety, sitting in $P^n_C$, given as a complete intersection of $t$ hyperplanes of the same degree $d$. Let $G$ be a finite group and suppose that $G$ acts linearly on $P^n_C$ while fixing $X$. We see that $X$ is therefore given by a regular section of $\mathcal{O}_{P^n_C}(d)^{\#t}$ whose vanishing locus is invariant under the linear action of $G$ on $P^n_C$. Let $E$ be a Schur cover of $G$ by $H \cong M(G)$ (see Section 1.3). We define $\overline{\rho}: G \to \text{PGL}_{n+1}(\mathbb{C})$ the projective representation of $G$ associated to its linear action on $P^n_C$, and we let $\rho: E \to \text{GL}_{n+1}(\mathbb{C})$ be a lift of $\overline{\rho}$ making the following commutative diagram with exact rows commute

$$\begin{array}{cccccc}
1 & \to & H & \to & E & \to & G & \to & 1 \\
& & \beta \downarrow & & \rho \downarrow & & \pi \\
1 & \to & C^\times & \xrightarrow{\text{ad}_v} & \text{GL}(\mathbb{C}^{n+1}) & \to & \text{PGL}(\mathbb{C}^{n+1}) & \to & 1
\end{array} \quad (14)
$$

We write $X = V(s)$ where $s$ is a regular section of $\mathcal{O}_{P^n_C}(d)^{\#t} (n,d,t \in \mathbb{Z}_{\geq 0})$ with $G$-invariant vanishing locus: we say that $s$ is a (regular) $(G,n,d,t)$-section.

### 3.1 From invariant ideals to group algebra modules

In this subsection we transform the problem of finding $(G,n,d,t)$-sections to finding $t$-dimensional $CE$-submodules of $R_d$, the $d$-homogeneous part of the polynomial algebra associated to $\mathbb{C}^{n+1}$.

Let us fix $(G,n,d,t)$-section $s$ and a projective representation $\overline{\rho}: G \to \text{PGL}(\mathbb{C}^{n+1})$: we are in the context of Diagram 14. The ideal $I$ defining the vanishing locus $V(s)$ of $s$ is homogeneous and generated by $t$ homogeneous polynomials $f_1, \ldots, f_t \in C[x_0, \ldots, x_n]$ of common degree $d$. We denote by $R_s := \oplus_{h \geq 0} C[x_0, \ldots, x_n]$ the $\mathbb{Z}$-graded $C$-algebra of polynomials in $n+1$ variables. Considering Diagram 14, the action of $E$ on $C^{n+1}$ defined by $\rho: E \to \text{GL}(\mathbb{C}^{n+1})$ naturally induces, for all $h \geq 0$, a linear action on $R_h$. It is given by: for any $h \geq 0$, $P \in R_h$, $e \in E$ and $x \in \mathbb{C}^{n+1}$,

$$(e \cdot P)(x) := P(\rho(e)^{-1}(x)). \quad (15)$$

It is a well-defined action, because the action of $E$ on $\mathbb{C}^{n+1}$ is linear, which we denote by $\rho_h$. Collecting these actions for all $h \geq 0$ gives $(R_s, \rho_s)$ the structure of a $CE$-algebra: $R_s$ is a $\mathbb{Z}$-graded $C$-algebra and all of its homogeneous components $R_h (h \geq 0)$, equipped with the action $\rho_h$, are $CE$-modules.

**Proposition 3.1.** Let $K$ be a field, let $E$ be a group and let $(R_s, \rho_s)$ be a $\mathbb{Z}$-graded $KE$-algebra. Let $I$ be a homogeneous ideal of $R_s$ being finitely generated by $t$ homogeneous elements $r_1, \ldots, r_t \in R_s$ of respective degrees $d_1, \ldots, d_t$ (possibly non pairwise distinct). We denote by $I_h := I \cap R_h$ the $h$-homogeneous part of $I$. Then, $I$ is invariant for the given action of $E$ on $R_s$ if and only if $(I_{d_i}, \rho_{d_i})$ is a $KE$-submodule of $(R_{d_i}, \rho_{d_i})$ for all $i = 1, \ldots, t$ (here we use the same notation for the restriction of $\rho_h$ to $I_h$, $h \geq 0$).
First, remark that \( I = \bigoplus_{h \in \mathbb{Z}} I_h = \sum_{i=1}^{t} I_h_i \) as \( R_0 \)-modules since \( I \) is generated by the \( t \) homogeneous elements \( r_1, \ldots, r_t \). Therefore, we see that if \( E \cdot I = I \) (i.e. \( I \) is \( E \)-invariant) then for all \( i = 1, \ldots, t \), \( E \cdot I_h_i = E \cdot (I \cap R_h_i) \subseteq I_h_i \), because \( R_h_i \) is fixed under the action of \( E \). Therefore, \((I_h_i, \rho_h_i)\) is a \( CE \)-submodule of \((R_h_i, \rho_h_i)\), for all \( i = 1, \ldots, t \).

Now suppose that for all \( i = 1, \ldots, t \), \( I_h_i \) is \( E \)-invariant. Since \( I \) is generated by \( \bigcup_{i=1}^{t} I_h_i \) as a \( R_\ast \)-module and \((R_\ast, \rho_\ast)\) is a \( KE \)-module, then \( I \) is \( E \)-invariant. \( \square \)

Recall that \( X = V(s) \) is the complete intersection in \( \mathbb{P}^n \) defined by \( s \). Considering Diagram 14 with \( R_d \) instead of \( C^{n+1} \), we know that \( \rho_d \) reduces to a unique projective representation of \( G \) on \( R_d \). By commutativity of Diagram 14, one sees that \((I_d, \rho_d)\) is a \( CE \)-submodule of \((R_d, \rho_d)\) if and only if \( P(I_d) \) is invariant under the induced action of \( G \) on \( P(R_d) \). Moreover, the fact that \( X \) is fixed under the action of \( G \) on \( \mathbb{P}^n \) is equivalent to have \( P(I_d) \) invariant under the induced action of \( G \) on \( P(R_d) \). Therefore, according to Proposition 3.1, having \( X \) invariant under the action of \( G \) on \( \mathbb{P}^n \) is equivalent to have \( I \) invariant under the induced action of \( E \) on \( R_\ast \). This also means that if \( V \) is the \( C \)-span of \( f_1, \ldots, f_t \) in \( R_d \), then \( I \) is the ideal of \( R \) generated by \( V \) and \( X \) being \( E \)-invariant is equivalent to have \((V, \rho_d)\) being a \( CE \)-submodule of \((R_d, \rho_d)\).

Therefore, in order to explicitly determine the \( f_i \)'s, one can equivalently search for a linear representation \( \rho \) of \( E \) on \( C^{n+1} \) that reduces to a faithful projective representation of \( G \) (see Subsection 3.2), and a \( CE \)-submodule \( W \) of \((R_d, \rho_d)\) whose underlying \( C \)-vector space is spanned by \( t \) homogeneous polynomials of common total degree \( d \) (see Section 2).

### 3.2 Classification of projectively faithful representations

In Definition 1.7 we define an equivalence relation on the linear representations of \( E \). In this subsection, we define another equivalence relation, coarser than the previous one. More precisely, we classify linear representations of \( E \) having faithful reduction, up to similarity of their respective reductions to \( G \).

Let \( E \) and \( G \) as before, and let \( V \) be a finite dimensional \( C \)-vector space (e.g. \( V = C^{n+1} \)).

**Definition 3.2.** A linear representation \( \rho : E \rightarrow GL(V) \) is said to be projectively faithful if \( \text{im}(\pi \circ \rho) \) is isomorphic to \( G \).

By commutativity of Diagram 14 and by surjectivity of \( p \), we see that any representation \( \rho \) of \( E \) on \( V \) is projectively faithful if and only if its reduction \( \overline{\rho} : G \rightarrow PGL(V) \) is faithful. Therefore, in order to find a \( CE \)-module \( W \) whose underlying vector space generates the defining ideal \( I \) of \( X \) (see Section 3.1), we start by classifying the projectively faithful representations of \( E \) on \( C^{n+1} \).

**Definition 3.3** (Definition (2.26), [ Isa76 ]). Let \( \chi \) be the \( C \)-character of \( E \) afforded by a \( CE \)-module \((V, \rho)\). Then, the center of the character \( \chi \) is defined to be

\[
Z(\chi) := \left\{ e \in E \mid \frac{\chi(e)}{\chi(1)} \text{ is a root of unity} \right\}.
\]

**Proposition 3.4.** With the notations of Definition 3.3 and Diagram 14, \( Z(\chi) = \ker(\pi \circ \rho) \).

**Proof.** According to [ Isa76, Lemma (2.27)],

\[
Z(\chi) = \{ e \in E \mid \rho(e) \in C^\ast Id_V \}
\]

so the first inclusion \( Z(\chi) \subseteq \ker(\pi \circ \rho) \) holds. Now, let \( e \in E \) such that \( \pi(\rho(e)) = 1_{PGL(V)} \). In particular, \( \rho(e) \in \ker(\pi) = C^\ast Id_V \). Therefore, according to Eq. (16), \( e \in Z(\chi) \). \( \square \)

**Corollary 3.5.** With the notation of Diagram 14 and Proposition 3.4, \( \overline{\rho} \) is faithful if, and only if, \( E/Z(\chi) \cong G \).

Using Corollary 3.5, we are now able to decide whether a linear representation of \( E \) is projectively faithful or not. In order to classify them, we first consider the following: if \( \overline{\rho} \) and \( \overline{\rho}' \) are two
similar projective representations of $G$ on $V$, we denote by $\mathcal{L} \in \text{Aut}(V)$ an automorphism of $V$ such that, for all $g \in G$,
\[
\overline{\gamma} \circ \overline{\gamma}(g) \circ \overline{\gamma}^{-1} = \overline{\gamma}(g).
\] (17)

Then, a $\mathbb{C}$-subvector space $W \subset V$ satisfies that $\mathbb{P}(W)$ is invariant under the action of $G$ on $\mathbb{P}(V)$ given by $\overline{\gamma}$ if and only if $\mathbb{P}(\mathcal{L}W)$ is invariant under the action of $G$ on $\mathbb{P}(V)$ given by $\overline{\gamma}$. This means that the projective varieties whose defining ideals are respectively generated by $W$ and $\mathcal{L}W$ are $G$-equivariantly isomorphic. Using Lemma 1.8, we can thus classify projectively faithful representations of $E$ on $\mathbb{C}^{n+1}$ by equivalence modulo $\text{Irr}_E^1(E)$. This concludes the classification: among all the linear representations of $E$ on $\mathbb{C}^{n+1}$, we choose only a finite number of them corresponding to representatives of classes in
\[
\{\text{projectively faithful representations of } E \text{ on } \mathbb{C}^{n+1} \} / \{ \rho \sim \rho' \text{ iff } \exists \epsilon \in \text{Irr}_E^1(E) \text{ s.t. } \chi_\rho = \epsilon \chi_\rho' \} \quad (18)
\]

We ensure that there are only finitely many such classes since the number of equivalence classes of linear representations of $E$ on $\mathbb{C}^{n+1}$ is actually finite (see [Isa76, Corollary (2.5)]).

### 3.3 Application to the case of K3 surfaces

In this subsection, we apply the previous theory to compute projective models of K3 surfaces given as complete intersections of hyperplanes of the same degree.

In the paper [BH21], S. Brandhorst and K. Hashimoto study pairs $(S, G)$ consisting of a complex polarised K3 surface $S$ and a (maximal) finite subgroup $G \leq \text{Aut}(S)$ of automorphisms of $S$ such that the proper subgroup $G_s$ of symplectic automorphisms is among the 11 maximal symplectic subgroups classified by Mukai ([Muk88]). Such pairs $(X, G)$ come with a canonical polarization $L$. They gather such pairs into 42 different isomorphism classes and they exhibit 25 cases for which an explicit projective model is known. In particular, all of the pairs $(S, G)$ for $S$ of genus $\leq 6$ have been treated except one of genus 5 (case 77b). Following the notation in [BH21] (except the polarization that we denote "$L$" here and not "$L$"), we enter in Table 1 some information about this isomorphism class of K3 surfaces.

| case | $G_s$ | $\Lambda_{K3}^{G_s}$ | SO($\Lambda_{K3}^{G_s}$) | $G/G_s$ | $c_1(L)^2$ | $G$ |
|------|------|----------------|-----------------|--------|---------|-----|
| 77b  | $T_{192}$ | \begin{pmatrix} 4 & 0 & 0 \\ 0 & 8 & 4 \\ 0 & 4 & 8 \end{pmatrix} | $D_6$ | $\mu_2$ | 8 | $T_{192} \times \mu_2$ GAP Id [384,5602] |

Table 1: Specification for the triple $(S, G, G_s)$ considered in this paper

According to a remark in [SD74, Page 615], either the polarization $L$ is hyperelliptic and $|L|$ defines a degree 2 map onto a curve of degree 4 in $\mathbb{P}_C^2$, or it is not hyperelliptic and $S$ is isomorphic to a smooth complete intersection (s.c.i.) of 3 quadrics in $\mathbb{P}_C^5$ (see [May72, Page 9]). Once such a polarization is known, it is algorithmically possible to check whether it is hyperelliptic or not:

**Theorem 3.6** ([SD74, Theorem 5.2]). Let $|L|$ be a complete linear system on a K3 surface $S$ without fixed components and such that $c_1(L)^2 \geq 4$. Then $L$ is hyperelliptic only if one of the following holds

- There exists an irreducible curve $E \in \text{NS}(S)$ of genus 1 and such that $c_1(L) \cdot E = 2$;
- There exists an irreducible curve $B \in \text{NS}(S)$ of genus 2 and such that $c_1(L) = 2B$.

According to the recent database of S. Brandhorst and T. Hofmann [BH22], this pair $(S, G)$ corresponds to the case "77.2.1.3", and using in complement an algorithm of I. Shimada (see [Shi15, Algorithm 2.2]) one may show that the surface $S$ has a polarization $L$ with $L^2 = 8$ and which does not satisfy any of the two conditions in Theorem 3.6. Therefore, we may take $L$ to be not hyperelliptic and $|L|$ defines an isomorphism $\varphi_{\mathcal{L}1}$ from $S$ to a s.c.i. of type $(2,2,2)$ in $\mathbb{P}_C^5$. Since the group $G$ acts faithfully on $S$ and it preserves $L$, $G$ acts faithfully on $\mathbb{P}_C^5 \cong |L|^\vee$ and the
image of $\varphi|_{U}$. This image can be seen as the vanishing locus of a smooth section $s$ of the vector bundle $\mathcal{O}_{P^{2}}(2)^{\oplus 3}$, i.e. a smooth $(G, 5, 2, 3)$-section.

Let $G$ be a group of Id [384,5602] (see the Small Group Library [BEO22]). Using GAP [GAP21], one can show that this group has Schur multiplier $M(G)$ isomorphic to $C_{2}^{2}$, and therefore any Schur cover of $G$ has order 3072. We fix such a Schur cover $E$, using for instance the GAP method SCHURCOVER: the following steps may differ depending on the choice of the Schur cover, yet the final result shall remain true. $E$ has 12 classes of projectively faithful representations on $F^{6}$, where $F := \mathbb{Q}[x]/(x^{24} - 1) = \mathbb{Q}(z)$ with 24 being the exponent of $E$. Here we choose $F$ instead of $\mathbb{C}$ for computational reasons: according to [Isa76, Corollary (9.15)], $F$ is a splitting field for $E$, so we are allowed to restrict to $F$ (the results remain true over $\mathbb{C}$). In what follows, we denote $i := z^{6}$ and $\omega := z^{3}$.

Let $M$ be the $FE$-module $(F^{6}, \rho)$, where $\rho$ is given by the $\sigma_i$'s in Theorem 0.1. $\rho$ is projectively faithful, and $S^{2}M^{\vee} := (R_{2}, \rho_{2})$ is a 21-dimensional $FE$-module (where $(R_{s}, \rho_{s})$ is defined as in Section 3.1). Let $\chi$ be the $F$-character of $S^{2}M^{\vee}$. One has that $ch_{\chi}(3) = \{\mu\}$ where $\mu \in \text{Ir}_{F}^{1}(E)$ and $(\chi, \mu) = 2$. Therefore, we have that $M(S^{2}M^{\vee}, 3)$ is irreducible of dimension 1, equal to $M(S^{2}M^{\vee}, \mu)$. Let $W$ be the isotypical component of $S^{2}M^{\vee}$ affording $2\mu$. An EBR-form of $W$ has bases for its summands given by

$$
\begin{pmatrix}
ix_{0}x_{1} + x_{0}x_{2} + x_{1}x_{3} + ix_{2}x_{3} \\
ix_{0}x_{1} - x_{0}x_{2} - x_{1}x_{3} + ix_{2}x_{3} \\
-x_{0}x_{3} - x_{1}x_{2} \\
-x_{0}x_{3} - x_{1}x_{2}
\end{pmatrix},
\begin{pmatrix}
x_{4}^{2} \\
x_{5}^{2}
\end{pmatrix}
$$

where $(x_{0}, \ldots, x_{6})$ is a basis for the dual space of $F^{6}$. Let $v = (a, b) \in M_{1,2}(F)$. It is easy to see that if $ab = 0$, then the ideal generated by the basis of $v \ast W$ does not define a smooth variety. Assume that $a = 1$ and $b = \lambda \in F^{\ast}$. Then, the underlying vector space of $v \ast W$ generates the ideal defining the variety $S_{\lambda}$ given in Theorem 0.1. For all $\lambda \in F^{\ast}$, $S_{\lambda}$ is by construction a K3 surface, and for distinct non-zero $\lambda_{1} \neq \lambda_{2}$, it is clear that $S_{\lambda_{1}}$ and $S_{\lambda_{2}}$ are $G$-equivariantly isomorphic. In this case, we say that $\{S_{\lambda}\}_{\lambda \in F^{\ast}}$ is a 1-dimensional $G$-isotrivial family. Finally, to ensure that $S_{\lambda}$ ($\lambda \in F^{\ast}$) corresponds to the case 77b in [BH21], we need that the subgroup $G_{s}$ of automorphisms of $G$ acting symplectically on $S_{\lambda}$ is isomorphic to the group $T_{192}$. We use the following Lemma:

**Lemma 3.7** (Lemma (2.1), [Muk88]). Let $S$ be a smooth complete intersection of hypersurfaces $H_{i} = V(f_{i})$ ($1 \leq i \leq k$) in $\mathbb{P}^{N}$. Assume that $\sum_{i=1}^{k}\deg(f_{i}) = N + 1$ and let $\sigma \in \text{GL}_{N+1}(\mathbb{C})$ be a linear transformation preserving $V_{S} := \text{Vect}_{\mathbb{C}}((f_{i})_{i=1,\ldots,k})$. $\sigma$ induces an automorphism of $S$ which we denote $\varphi$. Let $\omega \in H^{2}(S, \mathcal{O}_{S})$ be no-where vanishing.

1. If, for all $1 \leq i \leq k$, there exists $a_{i} \in \mathbb{C}^{\ast}$ such that $f_{i}^{\sigma} = a_{i}f_{i}$ then

$$
\varphi^{\ast}\omega = \frac{\det(\sigma)}{\prod_{i=1}^{k}a_{i}}\omega.
$$

2. If $\sigma$ is of finite order and induces a linear transformation on $\bar{\sigma}$ on $V_{S}$ then

$$
\varphi^{\ast}\omega = \frac{\det(\sigma)}{\det(\bar{\sigma})}\omega.
$$

**Lemma 3.7** offers a practical, and computationally feasible, way to compute $G_{s}$. Here, since $6 = 5+1$ we can apply to the group generated by the $\sigma_i$'s on $S_{\lambda}$. One finds that $G_{s} \cong T_{192}$ with, in particular, $\sigma_{i}$ acting symplectically on $S_{\lambda}$ for $i = 1, 2, 3, 4$ and $\sigma_{5}$ being a non-symplectic involution.

**Remark 3.8.** The methods exposed in this paper have been used to compute projective models of 17 other (isotrivial families of) K3 surfaces from the database in [BH22]. They are available at [Mul22, models/] in the format of an Oscar-readable database.

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