DEVELOPING COMPUTER PROGRAMS FOR KNOT CLASSIFICATION

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Abstract

In this paper we summarise the work discussed in Ref. [1] and [2], in which we introduced a method helpful in solving the problem of knot classification. We also present results obtained since then.

1. INTRODUCTION

Knot Theory has attracted significant attention during recent years, both among mathematicians, and among areas of applied science such as Physics, Chemistry and Biology. In fact, a number of problems that were previously considered unrelated to each other, have been connected through applications of Knot Theory. While enormous progress has been achieved in the study of knots and of their applications, the problem of a complete classification remains still open, in spite of recent successes (Ref. [3]). In this paper we describe and discuss an algorithmic approach that could be useful in solving the problem. With the help of an algorithm which is presented in this paper, a computer program was developed, resulting to the classification of all knots with crossing number up to 11.

In Section 2 we present the main points of the algorithm, while in Section 3 we introduce a suitable notation and show how through this notation it is possible to classify knot projections. In Section 4 we show how Reidemeister moves can be used to identify projections of equivalent knots, so that ambient isotopic knots may not appear more than once at the output. In Section 5 we generate a series of “color tests” in order to demonstrate knot inequivalence; such a procedure is necessary since equivalent knots may fail to be identified through the procedure of Section 4. Finally in Section 6 we show the results obtained through this computer program. Ideally, any two knot projections should either belong to knots shown equivalent through Reidemeister moves, or to knots shown inequivalent due to different responses in one or more “color tests”. This would be the case if the computer program could run for ever; in practice the results depend on the two input parameters, one indicating the maximum crossing number considered, the other indicating the ultimate “color test” to be used. Currently this has been achieved for all knots whose crossing number does not exceed 11.

2. THE ALGORITHMIC PROCESS

In this Section we present the main steps of the algorithm. First, a suitable method to denote knot projections is introduced. Second, once the set of possible such notations has
been obtained, one needs a method to distinguish notations that correspond to actual knot
projections, from notations that do not. Third, notations that correspond to identical knot
projections must be identified. Once these steps are completed, knot projections are fully
classified. This however is not identical to classifying knots, since distinct knot projections
may correspond to ambient isotopic knots, and such knots are considered equivalent.

The next step therefore is to identify such projections. It is well known that projections
of ambient isotopic knots are related through Reidemeister moves (Ref. [4]). It is thus
necessary to know how a notation is affected by a Reidemeister move. Once this is known,
one may use such moves to identify ambient isotopic knots. In order for the program to
be finite, one may establish an upper limit to the number of Reidemeister moves to be
applied; it turns out however to be simpler to set an upper limit to the crossing number
of the knot projections involved, instead of the number of Reidemeister moves This upper
limit is one of the two input parameters used in the program.

Since however no upper limit to either this number, or the number of necessary moves
is known, there is no certainty that projections not found connected through Reidemeister
moves, will actually belong to inequivalent knots. Therefore one starts by identifying as
many equivalent knots as possible. Then one proceeds by selecting one knot from each
equivalence class, conventionally the knot appearing first, and by calculating knot charac-
teristics, in order to establish inequivalences between selected knots. As such characteristics
we shall use the so called “color tests”, which are a generalisation of the “tricolorisation”
through which the trefoil’s non-triviality may easily be shown. Each color test consists
of an \( n \times n \) matrix whose elements take values in \( \{1, 2, ..., n\} \); the strands of each knot
projections are mapped to elements of \( \{1, 2, ..., n\} \) (the \( n \) “colors”). Acceptable mappings
are the ones where the three strands meeting at each crossing, are mapped to numbers
satisfying relations determined through the \( n \times n \) matrix. Once certain constraints among
the matrix elements are satisfied, the number of acceptable mappings is invariant under
Reidemeister moves. Therefore if two projections yield different results for one or more
such color tests, they definitely belong to inequivalent knots.

Not all knots on which such “color tests” are applied, are going to yield distinct results
and thus shown inequivalent. This is due to two reasons. First, some of these knots are
actually equivalent, but due to the limitations in the Reidemeister moves considered, the
program failed to identify them. Second, even if two knots are actually inequivalent, they
may not yield distinct results due to the finite number of color tests applied. The second
input parameter indicates in fact the color tests that are applied.

Having presented the main steps of the program, we now proceed with a detailed
discussion.

3. THE CLASSIFICATION OF KNOT PROJECTIONS

Knot projections are denoted as sets of \( n \) pairs of natural numbers
\( \{(a_1, a_2), (a_3, a_4), ..., (a_{2n-1}, a_{2n})\} \), where \( n \) is the crossing number, such that
\( a_i \in \{1, 2, ..., 2n\} \) and \( i \neq j \iff a_i \neq a_j \). This set is obtained as follows. First one
chooses a starting point and an orientation. Then, as one travels around the projection,
one assigns successive natural numbers to the crossing points, starting from 1 and ending to \(2n\). Each crossing is eventually assigned two numbers, \(a_{\text{over}}\) for the overcrossing and \(a_{\text{under}}\) for the undercrossing. The set of the pairs \((a_{\text{over}}, a_{\text{under}})\) denotes the projection.

Not all possible notations yield actual knot projections, the simplest counterexample being \(\{(1,3),(2,4)\}\). One necessary condition is that odd numbers are always paired to even numbers.

This condition is not sufficient, as the counterexample \(\{(1,4),(3,6),(5,8),(7,10),(9,2)\}\) demonstrates. The necessary and sufficient condition is that any two loops obtained from an actual projection, must either share one or more line segments, or intersect at an even number of points, vertices not being counted. This condition is due to the Jordan Curve Theorem (Ref. [5]) which states that any loop on \(R^2\) or \(S^2\) which does not intersect itself, divides \(R^2\) or \(S^2\) into two disjoint pieces. In these two counterexamples, the loops 1-2-3 and 3-4-1, in the first case, and 1-2-3-4 and 5-6-7-8 in the second case, violate this rule by not sharing any common segment and intersecting at exactly one point. The maximum number of loops obtained from an \(n\) crossing knot projection is \(3^n\), since each crossing may be a vertex of the loop or may not, and if it is, there are two possible direction changes. Therefore, checking whether a notation yields an actual knot projection, is a finite process.

Once “drawable” notations have been separated from “undrawable” ones, one needs to identify notations leading to the same knot projection. For an \(n\) crossing projection, there are at most \(4n\) such projections, corresponding to \(2n\) possible starting segments and to 2 possible orientations. By altering the starting place and/or the orientation, each pair \((a_i, a_j)\) becomes \((k + \epsilon a_i, k + \epsilon a_j)\), where \(k\) indicates the change of the starting point and \(\epsilon = \pm 1\) indicates a possible change of orientation; \(\epsilon = 1\) indicates that the orientation remains the same, while \(\epsilon = -1\) indicates that the orientation has been reversed. One may thus identify all such notations and keep just one, conventionally the one appearing first. This too is a finite process, and since it is less time consuming than checking the notation’s “drawability”, the program becomes more efficient if this step precedes the previous one.

At this point the procedure of classifying two-dimensional knot projections has been completed.

4. IDENTIFYING AMBIENT ISOTOPIC KNOTS

As mentioned earlier, two projections correspond to ambient isotopic knots if and only if they can be connected through Reidemeister moves. There are three kinds of Reidemeister moves, and their pictorial forms can be found in a number of relevant books (see for example Ref. [6]). Here we present their “numerical” form, by showing how each Reidemeister move affects a notation.

A first Reidemeister move, which increases the crossing number by 1, adds a pair \((i, i + 1)\) or a pair \((i + 1, i)\) to the notation, while replacing any other number \(j\) which is larger or equal to 2, with \(j + 2\). A second Reidemeister move, which increases the crossing number by 2, adds two pairs \((i, j)\) and \((i + 1, j + 1)\), or \((i + 1, j)\) and \((i, j + 1)\), to the notation. Numbers larger or equal to \(i\) and smaller than \(j\), increase by 2; numbers larger or equal to \(j\) increase by 4. A first or second Reidemeister move which decrease the crossing
number, will have the converse effect. Finally a third Reidemeister move, which keeps the crossing number constant, replaces pairs \((i, j), (i', k)\) and \((j', k')\) with the pairs \((i, k'), (i', j')\) and \((j, k)\), where \(|i' - i| = |j' - j| = |k' - k| = 1\), while all other \(n - 3\) pairs remain the same.

The process of identifying equivalent knot goes as follows. First, one obtains through the procedure of Section 3, all distinct knot projections whose crossing number does not exceed some maximum value \(N\). Then, on each projection one applies Reidemeister moves that do not increase the crossing number. Projections that cannot be connected to ones appeared before, are stored in the computer memory and are assigned two numbers, a “temporary” and a “permanent” one. Initially these numbers are equal. The permanent numbers assigned to such projections, are successive natural numbers. Projections connected to ones appeared before, are not stored in the memory, but help obtain equivalences among projections already stored. If for example some projection \(P\) is found equivalent to projections \(P_1, P_2, \ldots, P_k\) which have been assigned the permanent numbers \(p_1, p_2, \ldots, p_k\) and the temporary numbers \(t_1, t_2, \ldots, t_k\), the permanent numbers do not change, while the temporary numbers are replaced by \(\min(t_1, t_2, \ldots, t_k)\).

When all projections have been checked, only the ones stored in the memory with equal temporary and permanent numbers are going to appear at the output, since only these have not been found equivalent to preceding projections. As stated earlier, such projections may or may not be equivalent, and one thus proceeds by developing “color tests” in order to distinguish inequivalent knots.

5. ESTABLISHING “COLOR TESTS”

A simple method to show the existence of non-trivial knots is through “tricolorisation”. One maps the strands \(s_i\) of a knot projection to a number \(n_i \in \{1, 2, 3\}\), so that at each crossing, the strands involved, \(s_i, s_{i+1}\) and \(s_j\) satisfy the relation \(n_i + n_{i+1} + n_j = 0\mod 3\). If the projection is altered by a Reidemeister move, to each mapping of the old projection corresponds exactly one mapping of the new. Therefore the number of mappings is a knot invariant; if a projection \(P_1\) admits \(m_1\) mappings, while a projection \(P_2\) admits \(m_2\), and \(m_1 \neq m_2\), then \(P_1\) and \(P_2\) definitely belong to inequivalent knots. For the trefoil three such mappings are possible, each mapping the only strand to one of the elements of \(\{1, 2, 3\}\). In contrast, for the trefoil nine such mappings are possible; three map all strands to the same number, while the other six map the strands to three different numbers. Therefore the non-triviality of the trefoil is established (Ref. [7]).

Starting from this “three color test”, one may generalise to obtain more such color tests in order to distinguish equivalent knots whose responses to tricolorisation are identical. Each such color test is defined through an \(n \times n\) matrix \(M_{ij}\), so that if at some crossing the strands involved, \(s_i, s_{i+1}\) and \(s_j\) are mapped to \(n_i, n_{i+1}\) and \(n_j\), then either \(n_{i+1} = M_{n_i n_j}\), or \(n_i = M_{n_{i+1} n_j}\), depending on whether the crossing is positive or negative. Only mappings \(s_k \rightarrow n_k\), where this property is satisfied at every crossing, are considered acceptable and are counted for the corresponding knot invariant. For the “three color test” mentioned before, one may notice that \(n = 3, M_{ii} = i\), while for \(i \neq j\), \(M_{ij} = k\), where \(k \neq i\) and \(k \neq j\).
Not all possible matrices however are suitable. A matrix may only be used to define a “color test” if for any two knot projections $P$ and $P'$ differing by Reidemeister moves, to each acceptable mapping for $P$ corresponds exactly one mapping for $P'$. To ensure this property, one considers the constraints that each Reidemeister move imposes. One may easily observe that these constraints are the following.

1$^{st}$ move: $M_{ii} = i \quad \forall \quad i \in \{1, 2, ..., n\}$

2$^{nd}$ move: $M_{ij} = M_{i'j} \Leftrightarrow i = i'$

3$^{rd}$ move: $M_{ij} = k \quad \wedge \quad M_{li} = m \quad \wedge \quad M_{lj} = n \quad \Rightarrow \quad M_{nk} = M_{mj}$

In addition, an $n$-color test is not considered if there is a subset $S$ of $\{1, 2, ..., n\}$ other than the empty set and $\{1, 2, ..., n\}$ itself, such that $i \in S \Rightarrow M_{ij} \in S \quad \forall \quad j \in \{1, 2, ..., n\}$, since such a test may be reduced to simpler ones. Finally, two tests are considered identical if one may be obtained from the other by permutation, or if they are defined through matrices $M$ and $M'$ such that $M_{ij} = k \Rightarrow M'_{kj} = i$, since in such a case they are related through mirror symmetry.

Subsequently, a computer program was developed that recorded the matrices that yield distinct valid color tests. The running time grew exponentially with the number of colors; to obtain all color tests for up to 11 colors the time needed was a few days, while for 12 colors it would exceed one month. The number of color tests per number of colors came out to be as follows.

| Number of Colors | Number of Tests |
|------------------|----------------|
| 1                | 1              |
| 2                | 0              |
| 3                | 1              |
| 4                | 1              |
| 5                | 2              |
| 6                | 2              |
| 7                | 3              |
| 8                | 2              |
| 9                | 6              |
| 10               | 1              |
| 11               | 5              |

As shown in Section 6, these tests are not sufficient for distinguishing knots of high crossing numbers, and the method of establishing tests by explicitly checking every possible matrix is not efficient enough. Instead, one obtains an infinite number of tests by generalising from the tests already established. One such class of tests is defined through matrices $M_{(i,j)} = (k + 1)j - ki \mod n$, where the Greatest Common Divisor $\text{GCD}(k,n)=\text{GCD}(k+1,n)=1$. The existence of non-trivial mappings depends on the determinant of the linear homogeneous system that is defined through the equations satisfied at each crossing. This determinant is the $\text{Alexander-Conway}$ polynomial (Ref.[8]). One may thus calculate and compare the Alexander-Conway polynomials of various knots,
and apply additional color tests only for knots whose Alexander-Conway polynomials are identical.

A second class of tests associates the “colors” to group elements $g_i$, and is defined through the matrix $M(g_i, g_j) = g_j g_i g_j^{-1}$ (Ref. [9]). In particular, one may use as groups the permutation groups $S_n$; each conjugacy class, defined through a partition $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ of $n$, ($\lambda_1 + \lambda_2 + \ldots + \lambda_k = n$), defines a valid color test.

6. COMPUTER RESULTS

The maximum value of the crossing number of the projections studied, was set equal to $N = 14$. In order for the program to run, the CPU time needed was 8 days, and the memory required was about 10 MBytes. The number of knots that were not connected through Reidemeister moves, came out as follows.

| Number of Crossings | Number of Knots |
|---------------------|-----------------|
| 0                   | 1               |
| 1                   | 0               |
| 2                   | 0               |
| 3                   | 1               |
| 4                   | 1               |
| 5                   | 2               |
| 6                   | 3               |
| 7                   | 7               |
| 8                   | 21              |
| 9                   | 49              |
| 10                  | 165             |
| 11                  | 552             |
| 12                  | 2191            |
| 13                  | 29781           |

Due to memory constraints, 14 crossing knots were not recorded. As pointed out earlier, these numbers are mere upper limits, since it is certain that many of them although equivalent, may only be connected through Reidemeister moves involving more than 14 crossings. One thus proceeds by applying the color tests in order to obtain topologically inequivalent knots.

When the color tests listed in the table of Section 5 were applied, which are all the color tests involving at most 11 colors, all knots with crossing number up to 7 were shown inequivalent. This was not the case however with knots whose crossing number is 8, and therefore this method is good enough for only the first 15 knots.

When the Alexander-Conway polynomials were calculated, the results were slightly better; all 36 knots whose crossing number does not exceed 8, possess distinct Alexander-Conway polynomials. When knots with crossing number 9 are also considered, one faces the first cases of inequivalent knots with identical Alexander-Conway polynomials.
We later applied color tests derived from permutation groups, as discussed at the end of Section 5. Permutation groups up to $S_5$ were sufficient to demonstrate the inequivalence of all knots that possess identical Alexander polynomials and whose crossing number does not exceed 10. For crossing number 11, one has to go up to $S_7$, until all 802 knots with crossing number not exceeding 11 were shown inequivalent. For a complete list of all these knots and the characteristics through which these knots were distinguished, the reader is referred to Ref. 10.

For crossing numbers 12 and 13 it is almost certain that equivalent knots do exist, which would require the study of projections with crossing number higher than 14. The basis of this assumption is the fact that 11 crossing knots may only be distinguished once 14 crossing projections are studied; if the maximum value is set equal to 13 crossings, then 3 pairs of 11 crossing ambient isotopic knots cannot be identified.

At this point we have derived a full list of all knots whose crossing number does not exceed 11. In principle the method discussed could lead to extending this list to an arbitrary high crossing number; the CPU time and computer memory however rise very rapidly with the crossing number.

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