Parallel and Perpendicular Diffusion Coefficients of Energetic Charged Particles with Adiabatic Focusing

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Abstract
It is very important to understand stochastic diffusion of energetic charged particles in the nonuniform background magnetic field in plasmas of astrophysics and fusion devices. Using different methods considering an along-field adiabatic focusing effect, various authors derived a parallel diffusion coefficient $K_p$ and its correction $T$ to $K_p(0)$, where $K_p(0)$ is the parallel diffusion coefficient without an adiabatic focusing effect. In this paper, using the improved perturbation method developed by He & Schlickeiser and iteration process, we obtain a new correction $T'$ to $K_p(0)$. Furthermore, by employing the isotropic pitch-angle scattering model $D_{\mu\mu} = D(1 - \mu^2)$, we find that $T'$ has a different sign from that of $T$. In this paper, the spatial perpendicular diffusion coefficient $K_L$ with the adiabatic focusing effect is also obtained.

Key words: diffusion – magnetic fields – scattering – turbulence

1. Introduction

Energetic charged particle propagation in a magnetic turbulent field is one of the fundamental problems in astrophysics (e.g., cosmic-ray physics, astrophysical plasmas, and space weather research) and tokamak fusion devices (see, e.g., Jokipii 1966; Schlickeiser 2002; Matthaeus et al. 2003; Shalchi & Schlickeiser 2005; Shalchi et al. 2006; Qin 2007; Hauff & Jenko 2008; Shalchi 2009a, 2010; Qin & Zhang 2014). The magnetic turbulence can cause the field line wandering, or the field line random walk (Jokipii 1966; Matthaeus et al. 1995; Shalchi & Kourakis 2007; Shalchi & Qin 2010; Wang et al. 2017a), which directly affects the diffusion of charged particles. In the investigation of energetic particle transport through magnetized plasma, according to observations one usually assumes the magnetic field configuration as the superposition of a background magnetic field $B_0$ and a turbulent component $\delta B$. Because the background magnetic field breaks the symmetry of the magnetized plasma, one must distinguish particle diffusion along and across the large-scale magnetic field. However, some previous articles only consider the parallel diffusion, since it is much greater than the perpendicular one in many scenarios (see, e.g., Earl 1974, 1976; Beeck & Weiβerenz 1986; Bieber & Burger 1990; Köta 2000; Schlickeiser & Shalchi 2008; Shalchi 2009b, 2011; Litvinenko 2012a, 2012b; Shalchi & Danos 2013; Wang & Qin 2016; Wang et al. 2017b). To explore the influence of adiabatic focusing on particle transport, the perturbation method is frequently used (see, e.g., Beeck & Weiβerenz 1986; Bieber & Burger 1990; Schlickeiser & Shalchi 2008; Schlickeiser & Jenko 2010; Litvinenko & Schlickeiser 2013; He & Schlickeiser 2014). To use the perturbation method, since the anisotropic distribution function is an implicit function, by using the iteration method, one can find that the anisotropic distribution function becomes an infinite series of the spatial and temporal derivatives of the isotropic distribution function. Therefore, the governing equation of the isotropic distribution function derived from the Fokker–Planck equation contains infinite terms because of the infinite series of the anisotropic distribution function. By using the truncating method to neglect the higher-order derivative terms, the approximate correction formulas of parallel or perpendicular diffusion coefficients were obtained (see, e.g., Schlickeiser & Shalchi 2008; Schlickeiser & Jenko 2010; He & Schlickeiser 2014). However, the higher-order derivative terms probably also make the correction to the parallel and perpendicular diffusion much like the lower-order derivative ones do. The magnitude of the correction from
higher-order derivative term might not necessarily be a higher-order small quantity than the magnitude of the lower-order derivative terms. Therefore, the correction obtained by the previous authors is likely to contain significant errors. In this paper, by considering the higher-order derivative terms, we derive the parallel and perpendicular diffusion coefficients and obtain the correction formulas coming from all order derivative terms by using the improved perturbation method (He & Schlickeiser 2014) and the iteration operation. And for the weak adiabatic focusing limit we evaluate the correction to the parallel diffusion coefficient and compare it with the correction obtained in the previous papers.

The paper is organized as follows. In Section 2, by considering the adiabatic focusing effect, we derive the governing equation of the isotropic distribution function with the infinite series. In Section 2, by employing the truncation, we deduce the approximate formulas of the perpendicular and parallel diffusion coefficients, and that of the streaming term. In Section 4, the parallel and perpendicular diffusion coefficients \( \kappa_j \) and \( \kappa_z \), and the parallel streaming coefficient \( \kappa_t \) are derived, with the influence of the infinite series caused by the iteration of the anisotropic distribution function. We conclude and summarize our results in Section 5.

### 2. Equation of Isotropic Distribution Function

The starting point of this paper is the modified Fokker–Planck equation for the gyrotropic energetic charged particle distribution function, which incorporates the pitch-angle and perpendicular diffusion, and the along-field adiabatic focusing

\[
\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu} \left[ D_{\mu \mu} \frac{\partial f}{\partial \mu} - \frac{v}{2L} \left( 1 - \mu^2 \right) f \right] + D_z \Delta_f f.
\]

(1)

Here \( t \) is time, \( z \) is the distance along the background magnetic field, \( \mu = v/v \) is the pitch-angle cosine with particle speed \( v \) and its \( z \)-component \( v_z \), \( D_{\mu \mu} \) is the pitch-angle diffusion coefficient, \( D_z \) is the Fokker–Planck perpendicular diffusion coefficient, \( L(z) = -B_0(z)/dB_0(z)/dz \) is the adiabatic focusing characteristic length of the large-scale magnetic field \( B_0(z) \), and \( \Delta_a = \Delta^\|^2 + \Delta^\|^2 \Delta z^2 \) is the differential operator across the large-scale magnetic field. The source term is not included in the above equation. Because only the influence of the along-field adiabatic focusing on the parallel and perpendicular diffusion is explored in this paper, the terms related to momentum diffusion and so on are ignored in the Fokker–Planck Equation (1). The more complete form of the Fokker–Planck equation can be found in Schlickeiser (2002).

It should be mentioned that the linear phase space density \( f(x, y, z, p, \mu, t) = f_0(x, y, z, p, \mu, t)/B_0(z) \), so that Equation (1) is equivalent to the standard Fokker–Planck equation

\[
\frac{\partial f_0}{\partial t} + v \frac{\partial f_0}{\partial z} = \frac{\partial}{\partial \mu} \left[ D_{\mu \mu} \frac{\partial f_0}{\partial \mu} \right] - \frac{v}{2L} \left( 1 - \mu^2 \right) \frac{\partial f_0}{\partial \mu} + D_z \Delta f_0.
\]

(2)

If the pitch-angle scattering is strong, the gyrotropic phase space distribution function quickly becomes quasiequilibrium distribution. Therefore, we can split the gyrotropic cosmic-ray phase space density \( f(x, \mu, t) \) into the dominant isotropic part \( F(x, t) \) and the subordinate anisotropic part \( g(x, \mu, t) \), following the previous articles (see, e.g., Schlickeiser et al. 2007; Schlickeiser & Shalchi 2008; He & Schlickeiser 2014)

\[
f(x, \mu, t) = F(x, t) + g(x, \mu, t)
\]

(3)

with

\[
F(x, t) = \frac{1}{2} \int_{-1}^{1} d\mu f(x, \mu, t)
\]

(4)

and

\[
\int_{-1}^{1} d\mu g(x, \mu, t) = 0.
\]

(5)

### 2.1. The Differential Equation of Anisotropic Distribution Function \( g(\mu) \)

In this subsection, we adopt the method of He & Schlickeiser (2014) to derive the differential equation of the anisotropic distribution function \( g(\mu) \).

By integrating Equation (1) over \( \mu \) from \(-1 \) to \( 1 \), we can obtain

\[
\frac{\partial F}{\partial t} + \frac{v}{2} \frac{\partial}{\partial \mu} \int_{-1}^{1} \mu f d\mu = \frac{1}{2} \Delta F \int_{-1}^{1} d\mu D_z + \frac{1}{2} \Delta \int_{-1}^{1} d\mu D_{\mu} g.
\]

(6)

Furthermore, by integrating Equation (1) over \( \mu \) from \(-1 \) to \( \mu \), the following equation can be found

\[
\frac{\partial F}{\partial t} + \frac{v}{2} \frac{\partial}{\partial \mu} \int_{-1}^{\mu} \mu f d\mu = \frac{1}{2} \Delta F \int_{-1}^{1} d\mu D_z + \frac{1}{2} \Delta \int_{-1}^{\mu} d\mu D_{\mu} g.
\]

(7)

To obtain Equations (6) and (7), the assumption \( D_{\mu \mu}(\mu = \pm 1) = 0 \) is used. Subtracting Equation (6) from (7), we can get

\[
\frac{\partial g}{\partial t} - \frac{v}{2L} \left( 1 - \mu^2 \right) g + \frac{v}{2L} \left( 1 - \mu^2 \right) \frac{\partial F}{\partial \mu} + \frac{1}{2} \Delta \int_{-1}^{\mu} d\mu D_{\mu} g = \Phi(\mu)
\]

(8)

with

\[
\Phi(\mu) = \frac{1}{D_{\mu \mu}} \left[ \left( \frac{\partial F}{\partial t} + \frac{v}{2} \int_{-1}^{\mu} \mu f d\mu \right) + \frac{1}{2} \Delta \int_{-1}^{\mu} d\mu D_{\mu} g \right]
\]

(9)

By defining the following quantity

\[
M(\mu) = \frac{1}{2} \int_{-1}^{\mu} d\nu 1 - \frac{\nu^2}{D_{\mu \mu}(\nu)}.
\]

(10)
we can obtain

\[
\frac{\partial}{\partial \mu} \left\{ g(\mu) - L \left( \frac{\partial F}{\partial z} - \frac{F}{L} \right) e^{-M(\mu)} \right\} = e^{-M(\mu)} \Phi(\mu). \tag{11}
\]

It is noted that in He & Schlickeiser (2014) the right-hand side of this equation (Equation 20 in their paper) is set to be 0.

### 2.2. The Integration Results of \( g(\mu) \)

Through integrating Equation (11) over \( \mu \), we can get the anisotropic distribution function as the following

\[
g(\mu) = L \left( \frac{\partial F}{\partial z} - \frac{F}{L} \right) \left[ 1 - \int_{-1}^{1} e^{M(\mu)} d\mu \right] e^{-M(\mu)} + e^{-M(\mu)} \left[ R(\mu) - \int_{-1}^{1} \frac{d\mu}{d\mu e^{M(\mu)}} R(\mu) \right] \tag{12}
\]

with

\[
R(\mu) = \int_{-1}^{\mu} dv e^{-M(\mu)} \Phi(\mu). \tag{13}
\]

Equation (12) contains the effect from the term on the right-hand side of Equation (11), whereas Equation (20) in He & Schlickeiser (2014) is an approximate expression.

Inserting \( R(\mu) \) (Equation (13)) into \( g(\mu) \) (Equation (12)) and considering Equation (9), we find that the anisotropic distribution function \( g(\mu) \) becomes

\[
g(\mu) = L \left( \frac{\partial F}{\partial z} - \frac{F}{L} \right) \left[ 1 - \int_{-1}^{1} e^{M(\mu)} d\mu \right] e^{-M(\mu)} + e^{-M(\mu)} \left[ \int_{-1}^{1} \frac{d\mu}{d\mu e^{M(\mu)}} R(\mu) \right] \tag{12}
\]

From the latter equation, we can find that the anisotropic distribution function \( g(\mu) \) is an implicit function. Therefore, by iterating Equation (14), i.e., applying it repeatedly, we obtain the formula of \( g(\mu) \) as

\[
g(\mu) = \sum_{m,n,p} \epsilon_{m,n,p} \frac{\partial^{m+n}}{\partial t^m \partial z^n} \Delta_F^p, \tag{15}
\]

with the coefficients \( \epsilon_{m,n,p} \). Here \( m, n, p = 0, 1, 2, 3, \ldots \).

Similarly, by inserting Equation (12) into Equation (9) and inputting Equation (9) into Equation (13), we can find that \( R(\mu) \) is also an implicit function. Therefore, iteration operation \( R(\mu) \) can also be written as

\[
R(\mu) = \sum_{m,n,p} \chi_{m,n,p} \frac{\partial^{m+n}}{\partial t^m \partial z^n} \Delta_F^p, \tag{16}
\]

with the coefficients \( \chi_{m,n,p} \) and \( m, n, p = 0, 1, 2, 3, \ldots \).

Apparently, the coefficients \( \chi_{m,n,p} \) in the latter equation are related to the coefficients \( \epsilon_{m,n,p} \) in Equation (15) since \( g(\mu) \) and \( R(\mu) \) are related according to Equations (9), (12), (13), and (14).

### 2.3. The Governing Equation of the Isotropic Distribution Function \( F(x, t) \)

From Equation (6), we can find that in order to obtain the governing equation of the isotropic distribution function \( F(x, t) \) we have to get the expression of \( \int_{-1}^{1} d\mu \mu_g \) and \( \int_{-1}^{1} d\mu D_g \). By using Equation (12), we can obtain

\[
\int_{-1}^{1} d\mu \mu_g = -L \int_{-1}^{1} \frac{d\mu}{d\mu e^{M(\mu)}} \left( \frac{\partial F}{\partial z} - \frac{F}{L} \right) L \]

\[+ \int_{-1}^{1} \frac{d\mu}{d\mu e^{M(\mu)}} \left[ R(\mu) - \int_{-1}^{1} \frac{d\mu}{d\mu e^{M(\mu)}} R(\mu) \right] \tag{17}
\]

and

\[
\int_{-1}^{1} d\mu D_g = L \left( \frac{\partial F}{\partial z} - \frac{F}{L} \right) \int_{-1}^{1} \mu D_g \left[ 1 - \int_{-1}^{1} \frac{d\mu}{d\mu e^{M(\mu)}} \right] \]

\[+ \int_{-1}^{1} \mu D_g \left[ R(\mu) - \int_{-1}^{1} \frac{d\mu}{d\mu e^{M(\mu)}} R(\mu) \right]. \tag{18}
\]
Substituting Equations (17) and (18) into Equation (6) yields
\[
\frac{\partial F}{\partial t} - \frac{\partial}{\partial z} \left[ \nu L \int_{-1}^{1} d\mu \mu_{M(\mu)} \left( \frac{\partial F}{\partial z} - \frac{F}{L} \right) \right] \\
- \Delta F \int_{-1}^{1} d\mu D_\mu e^{M(\mu)} \\
\times \left[ 1 - \frac{2e^{M(\mu)}}{\int_{-1}^{1} d\mu M(\mu)} \right] = \Lambda(x, y, z, t) 
\]
(19)
with
\[
\Lambda(x, y, z, t) = -\frac{v}{2} \int_{-1}^{1} d\mu \mu_{M(\mu)} \left[ \frac{\partial R}{\partial z} + \frac{\int_{-1}^{1} d\mu \frac{\partial R}{\partial \mu} e^{M(\mu)}}{\int_{-1}^{1} d\mu M(\mu)} \right] \\
+ \frac{1}{2} \int_{-1}^{1} d\mu D_\mu e^{M(\mu)} \left[ \Delta R - \frac{\int_{-1}^{1} d\mu \frac{\partial R}{\partial \mu} e^{M(\mu)}}{\int_{-1}^{1} d\mu M(\mu)} \right]. 
\]
(20)
By inserting Equation (16) into the latter equation, we can obtain the following formula
\[
\Lambda(x, y, z, t) = \sum_{m,n,p} \eta_{m,n,p} \frac{\partial^{m+n}}{\partial y^m \partial z^n} \Delta^n F, 
\]
(21)
with the coefficients \(\eta_{m,n,p}\) and \(m, n, p = 0, 1, 2, 3, \ldots\) Note that some of the coefficients \(\eta_{m,n,p}\) are equal to 0.

Replacing \(\Lambda(x, y, z, t)\) in Equation (19) with the latter equation gives
\[
\frac{\partial F}{\partial t} - \frac{\partial}{\partial z} \left[ \nu L \int_{-1}^{1} d\mu \mu_{M(\mu)} \left( \frac{\partial F}{\partial z} - \frac{F}{L} \right) \right] \\
- \Delta F \int_{-1}^{1} d\mu D_\mu e^{M(\mu)} \\
- \frac{L}{2} \Delta \frac{\partial F}{\partial z} \int_{-1}^{1} d\mu D_\mu \left[ 1 - \frac{2e^{M(\mu)}}{\int_{-1}^{1} d\mu M(\mu)} \right] \\
= \sum_{m,n,p} \eta_{m,n,p} \frac{\partial^{m+n}}{\partial y^m \partial z^n} \Delta^n F. \tag{22}
\]
The latter equation is the most general resulting transport equation in this paper.

After combining similar terms for the latter equation, we can obtain
\[
\begin{align*}
(\eta_{1,0,0} + 1) \frac{\partial F}{\partial t} + & \left( \eta_{0,1,0} + \sqrt{v L} \int_{-1}^{1} d\mu \mu_{M(\mu)} \left( \frac{\partial F}{\partial z} - \frac{F}{L} \right) \right) \\
= & \left( \eta_{0,0,2} + \frac{L}{2} \int_{-1}^{1} d\mu D_\mu \right) \left[ 1 - \frac{2e^{M(\mu)}}{\int_{-1}^{1} d\mu M(\mu)} \right] \\
& + \sum_{(m,n,p) \neq A} \eta_{m,n,p} \frac{\partial^{m+n}}{\partial y^m \partial z^n} \Delta^n F.
\end{align*}
\]
(23)
Here, \(A = \{(1, 0, 0), (0, 1, 0), (0, 2, 0), (0, 0, 1), (0, 1, 1)\}.

Furthermore, Equation (23) can be simply rewritten as
\[
\begin{align*}
\kappa_{1,0,0} \frac{\partial F}{\partial t} + & \kappa_{0,1,0} \frac{\partial F}{\partial z} = \sum_{(m,n,p) \neq B} \kappa_{m,n,p} \frac{\partial^{m+n}}{\partial y^m \partial z^n} \Delta^n F \tag{24}
\end{align*}
\]
with
\[
\begin{align*}
\eta_{1,0,0} + 1, & \quad (m, n, p) = (1, 0, 0) \\
\eta_{0,1,0} + \sqrt{v L} \int_{-1}^{1} d\mu \mu_{M(\mu)}, & \quad (m, n, p) = (0, 1, 0) \\
\eta_{0,0,2} + \frac{L}{2} \int_{-1}^{1} d\mu D_\mu, & \quad (m, n, p) = (0, 0, 1) \\
\eta_{0,1,1} + \frac{L}{2} \int_{-1}^{1} d\mu D_\mu \left[ 1 - \frac{2e^{M(\mu)}}{\int_{-1}^{1} d\mu M(\mu)} \right], & \quad (m, n, p) = (0, 1, 1) \\
\eta_{m,n,p} & \quad \text{otherwise.}
\end{align*}
\]
(25)
Here, \(B = \{(1, 0, 0), (0, 1, 0)\}, \kappa_{1,0,0} \) is the coefficient of the first order time derivative term, \(\kappa_{0,1,0} \) is the coefficient of the parallel streaming term, \(\kappa_{0,0,2} \) is the parallel diffusion coefficient, \(\kappa_{0,0,1} \) is the perpendicular diffusion coefficient, and \(\kappa_{0,1,1} \) is the coefficient of the term with \(\Delta_\perp \frac{\partial F}{\partial z}\). Equation (23) and (24) are equivalent to Equation (22). They are all forms of the governing equation of the isotropic distribution function in the most general case.

In this paper, we only explore the properties of the first order time derivative term coefficient \(\kappa_{1,0,0}, \) the parallel streaming
coefficient $\kappa_{1,0,0}$, the parallel diffusion coefficient $\kappa_{0,2,0}$, and the perpendicular diffusion coefficient $\kappa_{0,0,1}$, which are listed as

\[
\kappa_{1,0,0} = \eta_{1,0,0} + \kappa^a_{1,0,0},
\]
\[
\kappa_{0,1,0} = \eta_{0,1,0} + \kappa^a_{0,1,0},
\]
\[
\kappa_{0,0,1} = \eta_{0,0,1} + \kappa^a_{0,0,1},
\]
\[
\kappa_{0,2,0} = \eta_{0,2,0} + \kappa^a_{0,2,0},
\]

with

\[
\kappa^a_{1,0,0} = 1,
\]
\[
\kappa^a_{0,1,0} = vL\frac{\int_{-1}^1 d\mu \mu e^{M(\mu)}}{\int_{-1}^1 d\mu e^{M(\mu)}},
\]
\[
\kappa^a_{0,0,1} = \frac{\int_{-1}^1 d\mu D_\mu e^{M(\mu)}}{\int_{-1}^1 d\mu e^{M(\mu)}},
\]
\[
\kappa^a_{0,2,0} = vL\frac{\int_{-1}^1 d\mu \mu e^{M(\mu)}}{\int_{-1}^1 d\mu e^{M(\mu)}}.
\]

Note that any of the coefficients $\eta_{1,0,0}$, $\eta_{0,1,0}$, $\eta_{0,0,1}$, and $\eta_{0,2,0}$ might be zero.

### 3. Analytical Coefficients with $\Lambda(x, y, z, t) = 0$

For the condition $\Lambda(x, y, z, t) = 0$, from Equation (20) we can find that $R(\mu) = 0$. So, from formula (12) the approximate anisotropic distribution function can be obtained

\[
g^a(\mu) = L\left(\frac{\partial F}{\partial z} - \frac{F}{L}\right)\left[1 - \frac{2e^{M(\mu)}}{\int_{-1}^1 d\mu e^{M(\mu)}}\right],
\]

which is identical to Equation (20) in He & Schlickeiser (2014). In addition, by using the condition $\Lambda(\mu) = 0$, i.e., setting $\eta_{m,n,p} = 0$ for any $m, n, p$, we can simplify Equation (23) as

\[
\frac{\partial F}{\partial t} + \kappa_1 \frac{\partial F}{\partial z} = \frac{\partial}{\partial z}\left(\kappa^a \frac{\partial F}{\partial z} + \kappa^a_1 \Delta F + \kappa^a_2 \Delta \frac{\partial F}{\partial z}\right),
\]

with

\[
\kappa^a_1 = \kappa_{1,0,0}^a = 1,
\]
\[
\kappa^a_2 = \kappa_{0,1,0}^a = \frac{\kappa^a_{1,0,0}}{L} = \frac{\int_{-1}^1 d\mu \mu e^{M(\mu)}}{\int_{-1}^1 d\mu e^{M(\mu)}},
\]
\[
\kappa^a_3 = \kappa_{0,0,1}^a = \frac{\int_{-1}^1 d\mu D_\mu e^{M(\mu)}}{\int_{-1}^1 d\mu e^{M(\mu)}},
\]
\[
\kappa^a_4 = \kappa_{0,2,0}^a = vL\frac{\int_{-1}^1 d\mu \mu e^{M(\mu)}}{\int_{-1}^1 d\mu e^{M(\mu)}}.
\]

Obviously, Equations (36)–(40) are the special forms of Equations (26)–(28) with $\Lambda(x, y, z, t) = 0$. In particular, Equation (38) shows the approximate perpendicular diffusion coefficient with the adiabatic focusing effect. Furthermore, the approximate parallel diffusion coefficient $\kappa^a_2$ (Equation (39)) is identical to the result obtained by some previous authors (see Beeck & Wibberenz 1986; Litvinenko 2012a; He & Schlickeiser 2014), and it can be written as

\[
\kappa^a_2 = \kappa_{||0} + T,
\]

where $T$ is the correction to $\kappa_{||0}$.

### 4. Analytical Coefficients with $\Lambda(x, y, z, t) \neq 0$

In general, if $\Lambda(x, y, z, t) \neq 0$, we have to obtain the coefficient formulas of the isotropic distribution function equation (see Equation (19)) with the influence of $\Lambda(\mu)$. Therefore, we need to get the terms $\partial R/\partial z$ and $\Delta R$ because of Equation (20), in consequence, we have to get the formula for $R(\mu)$. By combining Equations (9) and (13), we can get the following formula

\[
\frac{\partial R}{\partial z} = \int_{-1}^1 d\mu \frac{e^{-M(\mu)}}{D_\mu} \left[\frac{\partial^2 F}{\partial \nu^2} + \frac{\partial^2 F}{\partial t^2} - \int_{-1}^1 d\rho \frac{\partial^2 F}{\partial \rho^2} + \Delta \frac{\partial F}{\partial \rho} + \Delta \frac{\partial^2 F}{\partial \rho^2} - \int_{-1}^1 d\mu \frac{\partial^2 F}{\partial \mu^2}ight]
\]

With the latter equation, we can get the formulas of $\partial R/\partial z$ and $\Delta R$ as follows

\[
\frac{\partial R}{\partial z} = \int_{-1}^1 d\mu \frac{e^{-M(\mu)}}{D_\mu} \left[\frac{\partial^2 F}{\partial \nu^2} + \frac{\partial^2 F}{\partial t^2} - \int_{-1}^1 d\rho \frac{\partial^2 F}{\partial \rho^2} + \Delta \frac{\partial F}{\partial \rho} + \Delta \frac{\partial^2 F}{\partial \rho^2} - \int_{-1}^1 d\mu \frac{\partial^2 F}{\partial \mu^2}ight]
\]

and

\[
\Delta R = \int_{-1}^1 d\mu \frac{e^{-M(\mu)}}{D_\mu} \left[\Delta \frac{\partial F}{\partial \nu} + \Delta \frac{\partial^2 F}{\partial \nu^2} - \int_{-1}^1 d\rho \Delta \frac{\partial F}{\partial \rho} + \Delta \frac{\partial^2 F}{\partial \rho^2} - \int_{-1}^1 d\mu \Delta \frac{\partial F}{\partial \mu} + \Delta \frac{\partial^2 F}{\partial \mu^2} - \int_{-1}^1 d\rho \Delta \frac{\partial^2 F}{\partial \rho^2} + \Delta \frac{\partial^2 F}{\partial \rho^2} - \int_{-1}^1 d\mu \Delta \frac{\partial^2 F}{\partial \mu^2}ight]
\]

From the latter equations, we can find that the terms $\partial^2 g/\partial z^2$, $\Delta \frac{\partial g}{\partial z}$, $\Delta \frac{\partial g}{\partial t}$, and $\Delta^2 g$ need to be obtained. Then inserting the latter equations into Equation (20), we can obtain the formulas of $\eta_{1,0,0}$, $\eta_{0,1,0}$, $\eta_{0,2,0}$, and $\eta_{0,0,1}$.
4.1. The Analytical Perpendicular Diffusion Coefficient
with $\Lambda(x) \neq 0$

From Equation (23), we can find that the perpendicular diffusion coefficient is the corresponding coefficient of the term $\Delta F$, and the correction to the perpendicular diffusion coefficient from $\Lambda(x, y, z, t)$ is the coefficient of the term $\Delta F$ in $\Lambda(x, y, z, t)$. As shown in Equation (20), the term $\Lambda(x, y, z, t)$ is the function of $\partial R/\partial z$ and $\Delta R$. In the formula of $\partial R/\partial z$ (see Equation (43)), the term $\Delta F$ does not exist. In addition, from the formula of $\Delta R$ (see Equation (44)), we can find that the term including $\Delta F$ does not exist either. Therefore, $\Lambda(x, y, z, t)$ does not have the term $\Delta F$, and the correction to the perpendicular diffusion coefficient from $\Lambda(x, y, z, t)$ is zero, i.e., $\Lambda_0,\Lambda_2,\Lambda_3 = 0$. From Equation (28), we can find that the perpendicular diffusion coefficient can be written as

$$
\kappa_\perp = \kappa_\perp^a = \frac{\int_1^1 d\mu_1 \kappa_\perp^a}{\int_1^1 d\mu e^M(\mu)}.
$$

In general, the Fokker–Planck perpendicular diffusion coefficient $D_\perp$ depends on pitch-angle cosine $\mu$. By using UNLT theory (Shalchi 2010), Qin & Shalchi (2014) developed a model of $D_\perp(\mu)$ to find $D_\perp(\mu) \propto |\mu|$. But this model describes the case for uniform background magnetic field. Maybe the weakly nonlinear theory (Shalchi et al. 2004) can be used to derive the formula of $D_\perp(\mu)$ with the adiabatic focusing effect, but this procedure is too complicated. Therefore, so far, no mathematically tractable theory describing the relationship of $D_\perp$ to pitch-angle cosine $\mu$ and adiabatic focusing is obtained to explore this problem. In this paper, we only obtain the latter formula of the perpendicular diffusion coefficient with the adiabatic focusing, but do not explore it in detail.

4.2. The Analytical Parallel Diffusion Coefficient
with $\Lambda(x, y, z, t) \neq 0$

As shown in Equation (23), coefficient $\Lambda_0,\Lambda_2,\Lambda_3$ is the correction from $\Lambda(x, y, z, t)$ to the parallel diffusion coefficient. It is obvious that there is no term $\partial^2 F/\partial z^2$ in $\Lambda R$ (see Equation (44)), but there might be the term $\partial^2 F/\partial z^2$ in $\partial R/\partial z$ (see Equation (43)). Therefore, the correction from $\Lambda(x, y, z, t)$ (Equation (20)) to the parallel diffusion coefficient should only come from

$$
-\frac{\nu}{2} \int_1^1 d\mu e^M(\mu) \left[ \frac{\partial R}{\partial z} - \int_1^1 d\mu \frac{\partial R}{\partial z} e^M(\mu) \right].
$$

From Equation (43), we can find that only the terms $\partial^2 g/\partial z^2$, i.e.,

$$
\frac{\nu}{2} \int_1^1 d\mu e^M(\mu) \left[ \frac{\partial R}{\partial z} - \int_1^1 d\mu \frac{\partial R}{\partial z} e^M(\mu) \right],
$$

might have the contribution to term $\partial^2 F/\partial z^2$ in $\partial R/\partial z$. By operating $\partial^2 F/\partial z^2$ on the formula of $g(\mu)$ (see Equation (12)), we can obtain

$$
\frac{\partial^2 F}{\partial z^2} = \left( \frac{\partial^2 F}{\partial z^2} - \frac{\partial^2 F}{\partial z^2} \right) \left[ 1 - \frac{\partial^2 F}{\partial z^2} \right] \left[ 1 - \frac{\partial^2 F}{\partial z^2} \right].
$$

From Equation (42), $\partial^2 R/\partial z^2$ can be written as

$$
\frac{\partial^2 R}{\partial z^2} = \int_1^1 d\mu e^M(\mu) \frac{\partial^2 F}{\partial z^2} \left[ 1 - \frac{\partial^2 F}{\partial z^2} \right] \left[ 1 - \frac{\partial^2 F}{\partial z^2} \right].
$$

Obviously, the term $\partial^2 F/\partial z^2$ does not exist on the right-hand side of the latter equation. Therefore, we can find that the term $\partial^2 F/\partial z^2$ on the right-hand side of Equation (48) is

$$
\frac{\partial^2 F}{\partial z^2} \left[ 1 - \frac{\partial^2 F}{\partial z^2} \right] \left[ 1 - \frac{\partial^2 F}{\partial z^2} \right].
$$

By replacing $\partial^2 g/\partial z^2$ in expression (47) with expression (50), we find the term $\partial^2 F/\partial z^2$ in $\partial R/\partial z$ as

$$
\frac{\partial^2 F}{\partial z^2} \int_1^1 d\mu e^M(\mu) \left[ 1 - \frac{\partial^2 F}{\partial z^2} \right] \left[ 1 - \frac{\partial^2 F}{\partial z^2} \right].
$$

By substituting $\partial^2 g/\partial z^2$ in expression (46) with the latter expression, we can obtain the term $\partial^2 F/\partial z^2$ in $\Lambda(x, y, z, t)$ as

$$
\frac{\partial^2 F}{\partial z^2} \int_1^1 d\mu e^M(\mu) \left[ 1 - \frac{\partial^2 F}{\partial z^2} \right] \left[ 1 - \frac{\partial^2 F}{\partial z^2} \right].
$$

(52)
so the correction coefficient $\eta_{0,2,0}$ from $\Lambda(\mu)$ to the parallel diffusion coefficient can be written as

$$
\eta_{0,2,0} = \frac{v^2}{2} \left\{ \int_{-1}^1 d\mu \int_{-1}^1 d\nu \frac{e^{M(\nu)}}{D_{\mu\nu}} \right\}
\times \left[ \int_{-1}^1 d\mu \mu^2 e^{M(\mu)} \left( 1 - \frac{2e^{M(\mu)}}{\int_{-1}^1 d\mu e^{M(\mu)}} \right) \right]
- \int_{-1}^1 d\mu \int_{-1}^1 d\nu e^{M(\mu)} \left( \frac{\partial}{\partial \nu} \frac{e^{M(\nu)}}{D_{\mu\nu}} \right) \left\{ \int_{-1}^1 d\mu \mu e^{M(\mu)} \right\}. 
$$

Thus, by considering $\Lambda(\mu)$ we find a new correction $T'$ to $\eta_{0,2,0}$ as

$$
\kappa_{||} = \kappa_{||} + \eta_{0,2,0} = \kappa_{||} + T + \eta_{0,2,0}
$$

with

$$
T' = T + \eta_{0,2,0}. 
$$

4.4. The Analytical Coefficient of the First Order Time Derivative Term with $\Lambda(x, y, z, t) \neq 0$

From Equation (22), we find that $\Lambda(x, y, z, t) = 0$ is the function of $\partial R/\partial t$ and $\Delta R$. Since the term $\partial F/\partial t$ does not exist in terms $\partial R/\partial t$ and $\Delta R$, from Equation (23) we can find that the correction from $\Lambda(x, y, z, t)$ to the first order time derivative term is 0, i.e., $\eta_{1,0,0} = 0$. Therefore, from Equation (26) we can obtain

$$
\kappa_{t} = \kappa_{t} = 1. 
$$

4.5. Evaluating the Correction $T'$ for Model $D_{\mu\nu} = D(1 - \mu^2)$

In Section 4.2, the parallel diffusion coefficient for the case $\Lambda(x, y, z, t) = 0$ has been obtained as $\kappa_{||} = \kappa_{||} + T'$ with the correction $T' = T + \eta_{0,2,0}$. In Appendix A, we find that the limits of $T$ and $\eta_{0,2,0}$ all tend to 0 for the limit $L \to \infty$. That is, the correction $T'$ tends to 0 for the limit $L \to \infty$. Therefore, $\kappa_{||} \to \kappa_{||}$ when the spatially varying background magnetic field tends to the uniform one. In the following, we evaluate the correction $T'$ for the isotropic model $D_{\mu\nu} = D(1 - \mu^2)$.

As shown in He & Schlickeiser (2014), for the isotropic pitch-angle scattering model $D_{\mu\nu} = D(1 - \mu^2)$, Equation (10) can be simplified as

$$
M(\mu) = \xi(\mu + 1) 
$$

with

$$
\xi = \frac{v}{2DL}. 
$$

In Appendix B, by using the latter simple model the correction coefficient $\eta_{0,2,0}$ from $\Lambda(\mu)$ is evaluated. We can find that the magnitude of the lowest order of the correction $\eta_{0,2,0}$ is larger than that of the correction $T$. If $T$ needs to be considered, the correction coefficient $\eta_{0,2,0}$ from $\Lambda(\mu)$ cannot be ignored.

Inserting the quantities $T$ (Equation (75)), $S$ (Equation (76)), and $\eta_{0,2,0}$ (Equation (90)) into the formula of $\kappa_{||}$ (Equation (55)), we can obtain

$$
\kappa_{||} \approx \kappa_{||} \left( 1 + \frac{2}{15} \xi^2 \right). 
$$

Because $\kappa_{||} = \kappa_{||} + T'$, we can find

$$
T' \approx \frac{2}{15} \xi^2 \kappa_{||}. 
$$

The lowest order correction of $T$ obtained by the previous authors is equal to $-\xi^2 \kappa_{||}/15$ (see Beeck & Wibberenz 1986; Litvinenko 2012b; Shalchi & Danos 2013; He & Schlickeiser 2014). However, in this paper, the lowest order correction of $T'$ is equal to $2\xi^2 \kappa_{||}/15$. Therefore, at least for the isotropic model...
\[ D_{\mu} = D(1 - \mu^2) \] with \( \xi \ll 1 \), to consider the adiabatic focusing effect, the new correction \( T' \) should be used.

5. Summary and Conclusion

In this paper, by using the improved perturbation method of He \& Schlickeiser (2014) and the iteration process, we explore the influence of along-field adiabatic focusing on energetic charged particle transport. Starting from the modified linear Fokker–Planck equation with the pitch-angle scattering and perpendicular transport and adiabatic focusing effect, we obtain the governing equation of the isotropic distribution function \( F(x, t) \) with infinite terms, from which we get the coefficients of the spatial parallel and perpendicular diffusion, and the coefficient of the parallel streaming term. The parallel diffusion coefficient can be written as \( \kappa_1 = \kappa_{10} + T' \), where \( \kappa_{10} \) is the parallel diffusion coefficient for the uniform background magnetic field, and \( T' \) is the correction to the parallel diffusion coefficient. We also get \( T' = T + \eta_{0,2,0} \) with \( T \) being the correction derived in the previous papers by ignoring the higher-order derivative terms in the isotropic distribution function equation (Beecq \& Wibberenz 1986; Litvinenko 2012a; He \& Schlickeiser 2014), and \( \eta_{0,2,0} \) coming from the higher-order derivative terms obtained in this paper but ignored by the previous authors.

Moreover, for the isotropic pitch-angle scattering model \( D_{\mu} = D(1 - \mu^2) \) (\( D \) is a constant) with \( \xi \ll 1 \), we find that the magnitude of correction coefficient \( \eta_{0,2,0} \) is larger than that of the correction \( T \) obtained in the previous paper. The correction \( T' = T + \eta_{0,2,0} \) even has a different sign from \( T \). In the previous papers, the higher-order derivative terms shown by \( \Lambda(x, t) \) in the isotropic distribution function equation (Equation (22)) were neglected. However, in this paper we find that the higher-order derivative terms \( \Lambda(x, t) \) also can make correction to the parallel diffusion coefficients, i.e., the correction formula \( \eta_{0,2,0} \). Therefore, the correction \( T \) obtained in the previous papers is approximate. In addition, we find that the magnitude of \( \eta_{0,2,0} \), which is the correction of the parallel diffusion coefficient, is larger than the magnitude of \( T \) derived by the previous authors. Therefore, the higher-order derivative term in the governing equation of the isotropic distribution function cannot be arbitrarily ignored.

In addition, we obtain the formula of the perpendicular diffusion coefficient \( \eta_1 \). Since there is no appropriate theory describing the relationship of the Fokker–Planck perpendicular diffusion coefficient \( D_\theta \) to pitch-angle cosine \( \mu \) and adiabatic focusing effect, we do not explore it in detail. Furthermore, we find that, from the terms ignored by the previous authors in the governing equation of \( F \), the corrections to the coefficients of the spatial perpendicular diffusion, the parallel streaming, and the first order time derivative term are equal to 0.

It is noted that \( D_{\mu} \) used in computing \( \kappa_1 \) in this paper as well as in previous ones does not include the adiabatic focusing effect. However, \( D_{\mu} \) is corrected by the adiabatic focusing effect, which is represented by the adiabatic focusing characteristic length \( L \), i.e., \( D_{\mu} = D_{\mu}(\mu, L) \) (Tautz et al. 2014). In fact, pitch-angle diffusion coefficient \( D_{\mu} \) and perpendicular diffusion coefficient \( D_\theta \) are related to each other (e.g., Shalchi 2009a). Therefore, \( D_\theta \) should also be corrected by the adiabatic focusing effect, i.e., \( D_\theta = D_\theta(\mu, L) \). So far, no mathematically tractable theory describing the relationship of \( D_{\mu} \) and \( D_\theta \) to pitch-angle cosine \( \mu \) and adiabatic focusing can be used to explore this problem. In addition, the correction formula \( \eta_{0,2,0} \) (Equation (53)) is too complicated and it is very difficult to compute the correction. Our purpose is to show that in order to explore the correction effect the higher-order derivative terms in the governing equation cannot be neglected.

Compared with the method used in this paper, Legendre polynomial expansions to solve the z-integrated Fokker–Planck equation is a more systematic approach. But the recursive relation of different order coefficients of the expansion series cannot be obtained. In addition, we have not found a method to transform the z-integrated Fokker–Planck equation into another form, from which a one to one relationship of coefficients of the expansion series can be obtained. In the future we will continue to investigate this problem.

By using different truncating methods and transformations, from the modified Fokker–Planck equation, one can obtain the diffusion equation, the telegraph equation, and other equations. The telegraph equation is also very important. As another project, we are exploring the telegraph equation derived from the modified Fokker–Planck equation by employing the method of He \& Schlickeiser (2014).

In the future, to obtain more accurate analytical formulas of the spatial parallel and perpendicular diffusion coefficients, we plan to get the mathematical tractable formulas of \( D_{\mu}(L, \mu) \) and \( D_\theta(L, \mu) \). In addition, we plan to numerically compute the correction \( T' \) and compare it with \( T \) for different turbulence models and conditions. In addition, the analysis of the different length scales and timescales in the problem is also important. We will explore this problem by using scale analysis and dimensional analysis in the future.

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Appendix A

Parallel Diffusion Coefficient \( \kappa_1 \) for the Limit \( L \to \infty \)

From Equations (55) and (56), the parallel diffusion coefficient \( \kappa_1 \) with adiabatic focusing effect can be written as \( \kappa_1 = \kappa_{10} + T' = \kappa_{10} + T + \eta_{0,2,0} \) with the parallel diffusion coefficient for uniform background field \( \kappa_{10} \), the correction \( T \) obtained in previous papers, and the correction formula \( \eta_{0,2,0} \) derived in this paper. If adiabatic focusing characteristic length \( L \) tends to infinity, i.e., \( \xi \) tends to zero, the parallel diffusion coefficient \( \kappa_1 \) should tend to the uniform field parallel diffusion coefficient \( \kappa_0 \), i.e., \( \lim_{L \to \infty} T' = 0 \). Therefore, \( \lim_{L \to \infty} \eta_{0,2,0} = 0 \) and \( \lim_{L \to \infty} T = 0 \). In the following, we prove the above limits.

The correction coefficient \( \eta_{0,2,0} \) (see Equation (53)) is the function of quantity \( M(\mu) \) (see Equation (10)). So that we have to explore \( M(\mu) \) for the limit \( L \to \infty \), i.e., \( \lim_{L \to \infty} M(\mu) \). If the integral \( \int_0^1 dv (1 - \nu^2) / D_{\mu}(\nu) \) in \( M(\mu) \) (see Equation (10)) is convergent, we can find \( \lim_{L \to \infty} M(\mu) = 0 \). Accordingly, the limit
\[
\lim_{L \to \infty} e^{M(\mu)} = 1,
\]
\[
\lim_{L \to \infty} e^{-M(\mu)} = 1,
\]
can be obtained. Therefore, we can find the following relation
\[
\lim_{L \to \infty} \int_0^1 d\mu e^{M(\mu)} = 0.
\]
Similarly, the following formula can be found

\[ \lim_{L \to \infty} \int_{-1}^{1} d\nu \left( 1 - \frac{2e^{M(\nu)}}{\int_{-1}^{1} d\mu e^{M(\mu)}} \right) = 0. \]  

(67)

By inserting Equations (66) and (67) into Equation (53), we can easily find that \( \lim_{L \to \infty} \eta_{0,2,0} = 0. \)

Second, we investigate \( \lim_{L \to \infty} \kappa_{\parallel 0} \). We already find that \( \lim_{L \to \infty} M(\mu) = 0 \) in the above paragraph. Using \( \lim_{L \to \infty} e^{M(\nu)} = 1 + \lim_{L \to \infty} M(\mu) \), from Equation (39) the following equation can be obtained

\[ \lim_{L \to \infty} \kappa_{\parallel 0} = \nu \frac{L}{2} \lim_{L \to \infty} \int_{-1}^{1} d\mu M(\mu) \left( \frac{1 - \mu^2}{D_{\mu\mu}(\mu)} \right). \]  

(68)

By using integration in parts and the definition of \( M(\mu) \) (see Equation (10)), we can obtain

\[ \int_{-1}^{1} d\mu M(\mu) = \frac{V}{4L} \int_{-1}^{1} d\mu \left( 1 - \mu^2 \right)^2 / D_{\mu\mu}(\mu). \]  

(69)

Inserting Equations (10) and (69) into Equation (68) yields

\[ \lim_{L \to \infty} \kappa_{\parallel 0} = \frac{V^2}{4L} \lim_{L \to \infty} \int_{-1}^{1} d\mu \left( 1 - \mu^2 \right)^2 / D_{\mu\mu}(\mu). \]

(70)

with

\[ U = \frac{V}{2} \int_{-1}^{1} d\mu \int_{-1}^{1} d\nu (1 - \nu^2) / D_{\mu\nu}(\nu). \]

(71)

If \( U \) is finite, the following can be obtained

\[ \lim_{L \to \infty} \frac{U}{L} = 0, \]

(72)

and Equation (70) becomes

\[ \lim_{L \to \infty} \kappa_{\parallel 0} = \frac{V^2}{8} \int_{-1}^{1} d\mu \left( 1 - \mu^2 \right)^2 / D_{\mu\mu}(\mu). \]

(73)

The latter formula is identical to the parallel diffusion coefficient \( \kappa_{\parallel 0} \) for uniform background magnetic field (Jokipii 1966; Hasselmann & Wilberenz 1968; Earl 1974; Shalchi 2006). So, \( \lim_{L \to \infty} U = 0 \). Therefore, we find \( \lim_{L \to \infty} \kappa_{\parallel 0} = \kappa_{\parallel 0}, \) i.e., \( \lim_{L \to \infty} T = 0. \)

\section*{Appendix B}

The Correction Coefficient \( \eta_{0,2,0} \) from \( \Lambda(\mu) \) for the Isotropic Model \( D_{\mu\mu} = D(1 - \mu^2) \)

Here, by employing the isotropic model \( D_{\mu\mu} = D(1 - \mu^2) \), we approximately evaluate \( \eta_{0,2,0} \). After inserting Equation (60) into Equation (53) we can get

\[ \eta_{0,2,0} = \frac{\nu^2}{2} \left\{ \int_{-1}^{1} d\mu e^{\xi \mu} \int_{-1}^{1} d\nu e^{-\xi \nu} \frac{D_{\mu\mu}}{D_{\mu\mu}} \left[ \frac{\nu^2 - 1}{2} - \int_{-1}^{1} d\mu e^{\xi \mu} \right] \right\}. \]

(74)

The correction \( T \) to the parallel diffusion coefficient \( \kappa_{\parallel 0} \) has been obtained in the previous papers (see Beeck & Wilberenz 1986; Litvinenko 2012b; Shalchi & Danos 2013; He & Schlickeiser 2014) as

\[ T = \kappa_{\parallel 0} S \]

(75)

with

\[ S = \frac{1}{15} \xi^2 + \frac{2}{315} \xi^4 + \cdots. \]

(76)

To proceed, by employing the latter formula, we can get

\[ \int_{-1}^{1} d\mu e^{\xi \mu} = \frac{1 + S}{3} \xi. \]

(77)

By inputting Equation (77) into Equation (74), we can obtain

\[ \eta_{0,2,0} = \frac{\nu^2}{2} \left[ \frac{1 + S}{3} \xi Y_1 \right]. \]

(78)

with

\[ Y_1 = \int_{-1}^{1} d\mu e^{\xi \mu} \int_{-1}^{1} d\nu Z(\nu), \]

(79)

\[ Y_2 = \int_{-1}^{1} d\mu e^{\xi \mu} \int_{-1}^{1} d\nu Z(\nu), \]

(80)

\[ Z(\mu) = \frac{e^{-\xi \mu}}{D_{\mu\mu}} \left[ \frac{\nu^2 - 1}{2} - \int_{-1}^{1} d\mu e^{\xi \mu} + \frac{1 + S}{3} \xi \right]. \]

(81)

By integration in parts for Equations (79) and (80), we can obtain

\[ Y_1 = \frac{\nu}{\xi} \int_{-1}^{1} d\mu Z(\mu) - \frac{1}{\xi} \int_{-1}^{1} d\mu e^{\xi \mu} Z(\mu), \]

(82)

\[ Y_2 = \frac{e^{\xi \mu}}{\xi} \int_{-1}^{1} d\mu Z(\mu) - \frac{1}{\xi} \int_{-1}^{1} d\mu e^{\xi \mu} Z(\mu) - \frac{1}{\xi} Y_1. \]

(83)
After inserting Equations (82) and (83) into Equation (78), we can get

\[
\eta_{0,2,0} = \frac{v^2}{2} \left[ e^\xi \left( \frac{1}{\xi} - \frac{1}{\xi^2} - \frac{1}{3} \right) \int_{-1}^{1} d\mu Z(\mu) \\
- \frac{1}{\xi} \int_{-1}^{1} d\mu \mu e^{\xi} Z(\mu) \\
+ \left( \frac{1}{\xi^2} + \frac{1 + S}{3} \right) \int_{-1}^{1} d\mu \mu e^{\xi} Z(\mu) \right].
\] (84)

For \( \xi \ll 1 \), by using the following equations

\[
e^\xi = 1 + \xi + \frac{1}{2} \xi^2 + \frac{1}{6} \xi^3 + \frac{1}{24} \xi^4 + \ldots,
\] (85)

\[
e^{\mu \xi} = 1 + \mu \xi + \frac{1}{2} (\mu \xi)^2 + \frac{1}{6} (\mu \xi)^3 + \frac{1}{24} (\mu \xi)^4 + \ldots,
\] (86)

we find that Equation (84) becomes

\[
\eta_{0,2,0} \approx \frac{v^2}{2} \left[ \frac{1}{2} \int_{-1}^{1} d\mu Z(\mu)(1 - \mu^2) \\
+ \frac{\xi}{3} \int_{-1}^{1} d\mu Z(\mu) \mu(1 - \mu^2) \\
+ \frac{\xi^2}{24} \int_{-1}^{1} d\mu Z(\mu)(1 - \mu^2)(3\mu^2 - 1) \right].
\] (87)

Similarly, employing Equations (85) and (86) with \( \xi \ll 1 \), formula (81) can be simplified as

\[
Z(\mu) \approx \frac{1}{D_{\mu}(\mu)} \left[ -\frac{\mu^3}{3} + \left( \frac{\mu^2}{12} - \frac{\mu^4}{8} \right) + \frac{\mu^4}{3} \xi^2 \right].
\] (88)

By combining Equations (87) and (88), we can find

\[
\eta_{0,2,0} \approx \frac{v^2}{30D} \xi^2.
\] (89)

In addition, because of formula \( \kappa_{||0} = v^2/(6D) \) (see, e.g., Shalchi 2009a), the latter equation becomes

\[
\eta_{0,2,0} \approx \frac{1}{5} \xi^2 \kappa_{||0}.
\] (90)