Yang–Mills Theory on a Cylinder Coupled to Point Particles

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Abstract

We study a model of quantum Yang–Mills theory with a finite number of gauge invariant degrees of freedom. The gauge field has only a finite number of degrees of freedom since we assume that space–time is a two dimensional cylinder. We couple the gauge field to matter, modeled by either one or two nonrelativistic point particles. These problems can be solved without any gauge fixing, by generalizing the canonical quantization methods of Ref. [1] to the case including matter. For this, we make use of the geometry of the space of connections, which has the structure of a Principal Fiber Bundle with an infinite dimensional fiber. We are able to reduce both problems to finite dimensional, exactly solvable, quantum mechanics problems. In the case of one particle, we find that the ground state energy will diverge in the limit of infinite radius of space, consistent with confinement. In the case of two particles, this does not happen if they can form a color singlet bound state (‘meson’).
1. Introduction

Understanding nonabelian gauge theories at the quantum level is a problem of fundamental significance in particle physics as well as mathematical physics. In particle physics this is related to the as yet unresolved problem of deducing a theory of hadrons from Quantum Chromodynamics. There has been much progress in the two dimensional Quantum Chromodynamics [2] which does not have any true gauge degrees of freedom. Studying two dimensional gauge theories on spaces with nontrivial topology [3], [1], [4], [5], [6] is interesting since there are a finite number of gauge degrees of freedom. This allows us to study the dynamics of quantum gauge theories nonperturbatively in a context where the problem is exactly solvable. The insights learned from this solution should be useful in more realistic situations.

In a more mathematical direction, recall that classical gauge theories [7], [8] have a natural formulation in terms of Principal Fiber Bundles. The dynamical variable of gauge theory is a connection in a Principal Fiber bundle. The fundamental symmetry transformations of the theory, gauge transformations, form the (fiber preserving) group of automorphisms of the Principal Fiber Bundle. Matter fields coupled to the gauge field are described in terms of sections of associated bundles. In the quantum theory, we are interested not in a particular connection, but rather in the structure of the space of all connections. The group of gauge transformations act on it; two connections that differ only by a gauge transformation are to be considered equivalent. It is remarkable fact that this situation also has a natural description in terms of an infinite dimensional Principal Fiber Bundle [9], [10], [11], [12]. The space of connections is the total space and the base space is the quotient space of gauge equivalent connections. Thus the proper geometric setting for quantum gauge theories is this Principal Fiber Bundle. For example, the wavefunctions of a gauge theory are sections of an associated vector bundle. Even when the original (classical) Principal bundle is trivial this quantum bundle is nontrivial.
Choosing a section of this bundle is equivalent to choosing a gauge. The fact that there is no global section for the quantum bundle implies that there is no choice of gauge valid for all configurations: this situation is often referred to as the ‘Gribov ambiguity’ in the physics literature [13]. Moreover, on this infinite dimensional fiber bundle, there is a natural connection. The Kinetic Energy of the gauge field is the covariant Laplacian with respect to this connection. This leads to an additional coupling between the matter and gauge sectors which is not physically obvious. (This is in addition to the more physically obvious Coulomb interaction.)

In most cases of direct physical interest, this covariant Laplacian cannot yet be given a precise meaning: the divergences of quantum field theory have to be overcome. We will study a very simple class of systems which have only a finite number of (gauge invariant) degrees of freedom. Therefore it will be possible in the end to define this operator, although the steps leading up to it involve formal infinite dimensional manipulations. The gauge field has no propagating (‘wave–like’) degrees of freedom if the space–time is 1+1 dimensional. Still if there are non–contractible loops, there will be a finite number of gauge invariant degrees of freedom associated to the parallel transport (‘Wilson loop’) around these loops. The simplest case [1] is the one where space–time is a cylinder, and gauge theory without matter can then be reduced to a simple problem in quantum mechanics with a finite number of degrees of freedom. It is possible to couple this theory to matter fields, [4], [14], [15] which will have an infinite number of degrees of freedom associated to matter. This makes the problem too complicated to be solved exactly. A simpler model, more in the spirit of [1] is to couple gauge theory to sources which are nonrelativistic point particles. (One can view this as the limiting case where the matter consists of heavy particles.)

This theory has only a finite number of degrees of freedom and can be solved by reducing it to a quantum mechanical problem again. This is the approach we will follow in this paper. We believe that the detailed study of such simple systems will help the develop the
intuition and technology to be applied to more complicated problems. In fact one lesson we have learned from this exercise is that it is best to follow a manifestly gauge invariant approach, taking into account that the bundle of connections is nontrivial. We hope to apply the lessons learned from these simple models to more complicated problems such as 2+1 dimensional gauge theories.

There has been a revival of interest in exactly solvable gauge theories in 1+1 dimensions [16], [17], [18], [19], [20]. Some of this work relates this problem to topological field theories [21], [22]. There is also an interesting new approach to the quotient space of connections, [23] which we hope will be generalized to the case with matter fields. A mathematically rigorous path integral approach to two dimensional gauge theories has also been developed [24]. The exact solvability of the path integral can be understood in terms of localization [25].

Now we will summarize the contents of this paper. In section 2 we review some basic ideas on Principal Fiber Bundles. In the finite dimensional case this is standard material. We do not as yet have enough examples to develop a rigorous theory for infinite dimensional bundles. Instead, we will use the ideas familiar from the finite dimensional case in a formal way in the infinite dimensional context. In fact, one of the motivations for our work is to understand what the correct definitions should be, in a more rigorous approach. In section 3, the geometry of the bundle of connections is developed in more detail, in the special case of gauge theory on a cylinder. The simplifications of this special case are used throughout to obtain as explicit a description as possible. For example, the base space (the quotient space of connections modulo gauge transformations) is finite dimensional. The projection map of the bundle can be constructed as the solution of an ordinary differential equation. We describe a flat Riemann metric on the space of connections, invariant under gauge transformations. This induces a curved metric on the base space. The vertical vector fields on the bundle describe infinitesimal gauge transformations. We can now
define a horizontal vector to be orthogonal to all the vertical vectors. This establishes a connection on the quantum bundle. In section 4 we discuss an associated vector bundle of this principal bundle. A characterization of a section of the associated vector bundle as quasi–periodic function on the real line is given. This makes it possible to deal with sections without directly using any infinite dimensional concepts. The connection on the principal bundle defines a covariant derivative on these sections, which is found in finite form. In section 5 it is shown that the wavefunctions of the gauge–matter system are sections of an associated vector bundle. This is a direct consequence of Gauss’ law at the quantum level. Also a general expression for the hamiltonian of the coupled system is derived. In particular, it shown that the Coulomb interaction arises from the vertical part of the gauge hamiltonian. This is not immediately obvious from the purely geometrical picture. In section 6 we solve explicitly the case of gauge theory coupled to a single point particle. The solution is remarkably simple and exhibits the phenomenon of confinement: the energy of the ground state diverges as the radius of the cylinder goes to infinity. This problem has many features in common with the charge–monopole system: it deals with vector bundles over $S^3$ that are the analogues of the line bundles over $S^2$. The intermediate steps in our formulation are not manifestly translation invariant, as a preferred point has to be picked in the definition of the gauge group. (This technical point is described in greater detail in the text.) It is an important check on our theory that the final results are manifestly translation invariant. In section 7 we solve the case of two particles (one in the fundamental and the other in the conjugate–fundamental representation) coupled to the gauge field. This is a model of a meson in a cylindrical space–time. It turns out that the obvious choice of the meson wavefunctions (separately periodic in the positions of the two particles) leads to technical difficulties and a physically unreasonable answer. More consistent and physical answers are obtained by requiring periodicity only in the center of mass variable. There could be analogous subtleties in many body and field theory problems.
2. Review of Principal Fiber Bundles

We shall first define the notion of a Principal Fiber Bundle [26], [7]. Let $P$ be a manifold $G$ a Lie group acting on $P$. The action of $G$ on $P$ is defined by a map $R : G \times P \to P$

$$R(g, p) \mapsto pg^{-1},$$

(1)

where $p \in P$ and $g \in G$. We will often write $R_{g^{-1}}p = pg^{-1}$. This action is said to be free if

$$pg^{-1} = p \Rightarrow g = e$$

where $e$ is the identity element of $G$.

Points on $P$ connected by this action to the point $p \in P$ define an equivalence class called the fiber containing $p$. We require that the resulting quotient space $M \equiv P/G$ is itself a manifold with the quotient topology. Let $\pi$ denote the projection map from $P$ to $M$. Let for every open cover $\{U_\alpha\}$ of $M$ the inverse image $\pi^{-1}(U_\alpha)$ is diffeomorphic to $\{U_\alpha\} \times G$, with the diffeomorphism being given by

$$\psi_{\alpha} : \pi^{-1}(U_\alpha) \mapsto U_\alpha \times G, \psi_{\alpha}(p) = (\pi(p), \phi_{\alpha}(p))$$

where

$$\phi_{\alpha} : U_\alpha \mapsto G, \phi_{\alpha}(pg^{-1}) = \phi_{\alpha}(p)g^{-1}.$$  

Such a fiber preserving diffeomorphism is called a local trivialization. If such a local trivialization exists for every element of the open cover $\{U_\alpha\}$, then the set $\{P, \pi, M, G\}$ is called a Principal Fiber Bundle (PFB) and is denoted by $G \mapsto P \mapsto M$. $P$ and $M$ are called the total and base space respectively.

A local section of a PFB is a differentiable map $s_{\alpha} : U_\alpha \mapsto \pi^{-1}(U_\alpha)$ with the property $\pi \circ s_{\alpha} = i_M$. Local sections can be seen to be in one to one correspondence with local
trivializations of a PFB by taking
\[ \psi_{\alpha}(x, g) = s_{\alpha}(x)g^{-1}, x \in M, g \in G. \]
This also shows that if a PFB admits a global section then it is diffeomorphic to \( M \times G \).
Such PFB’s are said to be trivial.

Next we introduce the concept of an Associated Bundle. Let \( N \) be a manifold carrying an action \( \rho \) of \( G \) and let \( C = \{(p, n)\}, p \in P, n \in N \). On \( C \) define an equivalence relation \( \sim \):
\[ (p, n) \sim (pg^{-1}, \rho(g)n). \]
Let \( E = C/\sim \) and let us define a projection map \( \hat{\pi}: E \mapsto M \) given by \( \hat{\pi}(p, n) = \pi(p) \).
If we require that for any open cover \( \{U_{\alpha}\} \) of \( M \) each image \( \hat{\pi}^{-1}(U_{\alpha}) \) be an open submanifold of \( P \times N/G \), then the resulting structure is called an Associated Bundle and is denoted by \( N \mapsto E \mapsto M \).

It can be shown that every associated bundle admits a global section. Let the set of global sections of an associated bundle be denoted by \( \Gamma(E) \). We can then show that there is an isomorphism between \( \Gamma(E) \) and the set of equivariant maps \( \Omega(P) \) from \( P \) to \( N \), where
\[ \Omega(P) = \{ f : P \mapsto N | f(pg^{-1}) = \rho(g)f(p) \}. \]

Two special cases of associated bundles are of particular relevance to us. If \( N \) happens to be a vector space \( V \) and \( \rho \) a representation of \( G \) on \( V \), then the corresponding associated bundle is called an Associated Vector Bundle (AVB). They are useful for describing matter fields. If on the other hand \( N \) happens to be the group \( G \) and \( \rho \) is the adjoint action of \( G \) on itself, then the corresponding associated bundle is called the automorphism bundle. They are relevant for describing gauge transformations, to which we turn next.

Given a PFB, we can define the concept of a gauge transformation. A gauge transformation is a fiber preserving diffeomorphism \( h: P \mapsto P \) such that \( h(pg^{-1}) = h(p)g^{-1} \). The
set of such transformations denoted by \( \mathcal{G} \) is isomorphic to the set of equivariant maps from \( P \mapsto G \), given by,

\[
\Omega(P)_{\text{adj}} = \{ f : P \mapsto G | f(pg^{-1}) = gf(p)g^{-1} \}. \tag{2}
\]

The isomorphism is then given by a map

\[
\Phi : \Omega_{\text{adj}} \mapsto \mathcal{G}(P), \Phi(f)(p) = pf(p). \tag{3}
\]

By our earlier discussion \( \mathcal{G} \) is also isomorphic to the set of sections of the automorphism bundle. However, if either the PFB or the automorphism bundle is trivial then the gauge transformation reduce to maps from \( M \mapsto G \). The set of gauge transformations form a group called the gauge group.

The construction outlined so far has been primarily topological. We now turn to the description of geometric properties of the structures introduced before. First, the group action (1) on \( P \) provides a natural isomorphism, by pushforward, between the Lie algebra \( \mathcal{G} \) of \( G \) and a subspace \( V_p \) of the tangent space \( T_p \) at \( p \). \( V_p \) is called the vertical subspace at \( p \), and an element of \( V_p \) is given by

\[
V_{\lambda} = \frac{d}{dt}(pe^{-\lambda t}) \bigg|_{t=0}, \lambda \in \mathcal{G}. \tag{4}
\]

\( V_{\lambda} \) is called a vertical vector at point \( p \in P \).

At any given point \( p \) the tangent space \( T_p \) can be decomposed into the vertical and a horizontal subspace \( H_p \) such that \( T_p = V_p + H_p \). However, unlike in the case of the vertical subspace, there is no unique natural definition of the horizontal subspace. We next turn to the concept of a connection which assigns a unique horizontal subspace at each point \( p \).

A smooth assignment of the horizontal subspace \( H_p \) on all of \( P \) is called a connection if:

1. \( V_p \oplus H_p = T_p \)
2. $H_{pg^{-1}} = (R_{g^{-1}})_* H_p$ where $(R_{g^{-1}})_*$ is the pushforward induced by (1).

Another equivalent way to specify a connection is given by a Lie algebra valued one form $A \in \Lambda^1 \otimes G$ (called the connection one form or simply the connection) satisfying

1. $i_{V_\lambda}(A) = -\lambda$ for any vertical vector
2. $(R_{g^{-1}})_*(A) = ad_g(A)$ for any $g \in G$.

$H_p$ is then defined to be the kernel of $A_p$. We mention without proof that every PFB admits a connection.

In a similar vein, we next define the space of equivariant $k$-forms, denoted as $(\Lambda^k(P) \otimes W)_{eq}$. These are $k$-forms in $P$ valued in the vector space $W$ which carries a representation $\rho$ of $G$ and satisfy the relations

1. $i_{V_\lambda} \omega = 0$ for all $V_\lambda \in V_p$
2. $(R_{g^{-1}})^* \omega_{pg^{-1}} = \rho(g^{-1}) \omega_p$

The specification of a connection can be used to define a covariant derivative on the PFB;

$$D_A \omega : \Lambda^k(P) \otimes W \mapsto \Lambda^{k+1}(P) \otimes W$$

$$(D_A \omega)(u_0, u_1, ..., u_k) = d\omega(u_0^H, u_1^H, ..., u_k^H)$$

where $u_i = u_i^H + u_i^V$ gives the decomposition of a vector into horizontal and vertical parts respectively.

The covariant derivative takes a particularly simple form if we consider equivariant forms:

$$D_A \omega : (\Lambda^k(P) \otimes W)_{eq} \mapsto (\Lambda^{k+1}(P) \otimes W)_{eq}$$

$$D_A \omega = d\omega + \rho(A) \wedge \omega$$
for \( \omega \in (\Lambda^k(P) \otimes W)_{eq} \), where \( \rho \) is the representation of the Lie algebra \( G \) induced by \( \rho \) on \( W \). The rhs of the last equation also belongs to the space of equivariant forms. This means that we can take a covariant derivative again in the same way, getting

\[
D^2_A \omega = \rho(dA + \frac{1}{2}[A, A]) \wedge \omega
\]

where \( F = dA + \frac{1}{2}[A, A] \) is called the curvature two-form defined by the connection \( A \). It is possible to show that \( F = D_A A \).

Having defined the connection we next study its behaviour under gauge transformations. For this recall the definition of gauge transformations given in (2) and (3). The transformation of \( A \) is defined in terms of the pullback \( \Phi(f)^{-1} \) induced by \( \Phi(f) \) and is given by

\[
A^f \equiv \Phi(f)^{-1} A = fAf^{-1} + fdg^{-1}
\]

(5)

If the principal bundle is trivial, we can think of the connection simply as a Lie algebra–valued 1–form on \( M \). This 1–form is obtained by pullback with a global section of the principal bundle. A gauge transformation can then be thought of as a function \( g : M \to G \). It describes a change of the trivialization ( global section); its effect on the connection 1–form is

\[
A \to gAg^{-1} + gdg^{-1}.
\]

(6)

The case of interest to us will in fact be of this type.

In the quantum theory, we are interested in the space of all connections on a given principal bundle. Two connections are physically equivalent if they are related by a gauge transformation. It is a remarkable fact that the space of connections modulo gauge transformations is itself an infinite dimensional principal bundle. This bundle can be nontrivial even when the original ( finite dimensional) principal bundle is trivial. All the geometric ideas discussed above can be applied to this infinite dimensional case, which will be the way to get a nonperturbative formulation of \textit{quantum} Yang–Mills theories [10].
Let $\mathcal{A}$ denote the space of all connections on the PFB $P$. $\mathcal{A}$ and $\mathcal{G}$ are both infinite dimensional spaces, with $\mathcal{A}$ carrying an action of $\mathcal{G}$. We can therefore attempt to construct a PFB $\mathcal{G} \hookrightarrow \mathcal{A} \twoheadrightarrow \mathcal{A}/\mathcal{G}$. However the action of $\mathcal{G}$ on $\mathcal{A}$ may not be free. For example if $P$ admits a flat connection, the point $A = 0 \in \mathcal{A}$ is left invariant by the constant gauge transformation. Demanding that an infinitesimal gauge action be free implies that

$$A' - A = t[A, \Lambda] + td\Lambda = 0 \quad \text{iff} \quad \Lambda = 0,$$

where $\Lambda: P \mapsto \mathcal{G} \subseteq \mathcal{G}$ ($\mathcal{G}$ denotes the Lie algebra of the group $\mathcal{G}$.) The above requirement can be satisfied in two different ways:

i. Restrict connections to satisfy this property, that is $A \in \mathcal{A}^{irr}$, the space of irreducible connections.

ii. Restrict $\mathcal{G}$ to $\mathcal{G}_0 = \{ f \in \mathcal{G} | f(p_0) = e, \ \text{identity of} \mathcal{G} \}$ for all $p_0 \in \pi^{-1}(x_0)$ where $x_0$ is an arbitrary point on $M$. $\mathcal{G}_0$ is called the space of pointed gauge transformations.

We choose here to follow the second method (see next section for explanation) and consider the PFB $\mathcal{G}_0 \hookrightarrow \mathcal{A} \twoheadrightarrow \mathcal{G}_0/\mathcal{A}$.

It is now possible to show that the principal bundle $\mathcal{G}_0 \to \mathcal{A} \to \mathcal{A}/\mathcal{G}_0$ admits a natural connection. Following the earlier discussion in the case of finite dimensional PFB, we introduce the concepts of vertical and horizontal vectors. The vertical subspace consists of vectors of the type

$$V_\Lambda = \frac{d}{dt}(e^{-t\Lambda}Ae^{t\Lambda} + e^{-t\Lambda}d\Lambda)|_{t=0} = d\Lambda + [A, \Lambda]$$

where $\Lambda \in \mathcal{G}_0 : P \mapsto \mathcal{G}$.

We next assume that the original base space $M$ has a Riemann metric defined on it. This induces an inner product on the tangent space $TA$, which is the space of equivariant one forms $(\Lambda^1(P) \otimes \mathcal{G})_{eq}$. This metric is given by

$$d^2(A_1, A_2) = \|A_1 - A_2\|^2 = \int_M <A_1 - A_2, * (A_1 - A_2)>$$
where "*" is the Hodge star operator defined by the metric on $M$ and $< . >$ denotes the trace operation on the Lie algebra $G$. This metric has the property that $d(A_1^0, A_2^0) = d(A_1, A_2)$, where $A_i^0, i = 1, 2$ are the gauge transformed versions of $A_i$.

We can now define the horizontal subspace $\mathcal{H}_A$ to be one which is orthogonal to the vertical subspace with respect to this metric. Therefore, $\mathcal{H}_A = \{ \xi \in (\Lambda^1(P) \otimes G)_{eq} \text{ such that } \}$. Integrating by parts and using the fact that $\xi$ is an equivariant one-form we may rewrite this expression as,

$$D_A \ast \xi = 0 \quad \text{iff} \quad \xi \in \mathcal{H}_A.$$ 

This equation need not be true at the point in the fiber of $p_0$ since $\Lambda$ should vanish there. The solution is in general a distribution.

These ideas are developed in more detail in the next section, in the special case where $M = S^1$ and $P$ is the trivial $G$–bundle over it. Even in this case, the bundle $G_0 \to A \to A/G_0$ will be nontrivial. The base manifold will be finite dimensional; in fact $A/G_0 = G$ in this case. The ideas of associated vector bundle, connection 1–form, curvature etc. introduced in this section will be useful in formulating and solving the quantum theory of Yang–Mills fields coupled to point particles. Some of these techniques can be generalized to higher dimensions, [10] although we no longer expect the problem to be exactly solvable.

3. The Bundle of Connections

Let space–time be a cylinder $S^1 \times R$ and $G$ a connected, simply connected compact Lie group. In this case, a principal bundle with structure group $G$ over space $S^1$ will be equivalent to the trivial bundle. Then a connection on this principal bundle can be thought of as a $G$–valued 1–form $Adx$ on $S^1$. We will use a coordinate $x$ on $S^1$, with
$0 \leq x \leq 2\pi$. This 1-form is the Yang–Mills potential or ‘gauge field’. Since the form has only one component, the gauge field can also be thought of as a function $A : S^1 \to G$. Most of the time we will use this simplification. A gauge transformation $g : S^1 \to G$ acts on this as follows:

$$A \mapsto gAg^{-1} + g \frac{\partial g^{-1}}{\partial x}.$$  \hspace{1cm} (11)

Gauge fields that differ only by a gauge transformation are to be regarded as physically identical.

Let $A = S^1 G$ be the space of gauge potentials and $\mathcal{G} = S^1 G$ the group of gauge transformations. All gauge transformations are in the connected component of identity in $S^1 G$, since $\pi_0(S^1 G) = \pi_1(G) = 0$. Ideally we would like to formulate gauge theory directly on the quotient space $A/\mathcal{G}$, so that it will be manifestly gauge invariant. But there is an immediate problem with this strategy. The action of the group $\mathcal{G}$ has fixed points, so that the quotient is not a smooth manifold. The fixed points arise whenever there is a non-trivial solution to the equation

$$\frac{d}{dx} g + [A, g] = 0;$$  \hspace{1cm} (12)

i.e., whenever the connection $A$ is reducible. These singular points in $A$ can lead to unwieldy boundary conditions on the wavefunctions.

There are two ways around this difficulty:

1. consider the quotient $A^{\text{irr}} / \mathcal{G}$, of the space of irreducible connections, or,

2. consider the quotient $A / \mathcal{G}_0$ by the subgroup of ‘based’ gauge transformations, $\mathcal{G}_0 = \{g \in \mathcal{G} | g(0) = 1\}$. $\mathcal{G}_0$ acts without fixed points on $A$. This leaves the constant part (often called ‘global’ part, $\mathcal{G} / \mathcal{G}_0 = G$) of the gauge group. This can be taken care of at the end by requiring the wave functions to be equivariant under $G$.

We will follow the second method, as it seems to have a closer relation to the usual perturbative formulation of gauge theories. (The flat connection, which is the starting
point for perturbation theory, is excised in the first approach). But it has the disadvantage that translation invariance is not manifest, as a special point 0 is picked out in the definition of $G_0$. We will see that in the end, translation invariance is recovered.

There is then a principal fiber bundle $G_0 \to A \to A/G_0$. The only gauge invariant information contained in $A$ on a circle is its Wilson loop. Therefore $A/G_0$ is just $G$ and the projection map of the bundle is the Wilson loop. (It is a special property of 1 + 1-dimensional gauge theory that this quotient is finite dimensional.)

More explicitly, define the parallel transport operator $\tilde{W} : A \times S^1 \to G$

$$\frac{d\tilde{W}(A, x)}{dx} + A\tilde{W}(A, x) = 0 \quad \tilde{W}(A, 0) = 1. \quad (13)$$

Under a gauge transformation, $A \to gAg^{-1} + gdg^{-1}$, $\tilde{W}(A, x) \to g(x)\tilde{W}(A, x)g(0)^{-1}$. The Wilson loop is defined to be $W(A) = \tilde{W}(A, 2\pi)$. This is clearly invariant under the action of $G_0$. Under the action of a constant gauge transformation, $W(A) \mapsto gW(A)g^{-1}$. Since the curvature of all connections $A$ are zero, $W$ determines $A$ upto an action of $G_0$. Thus the map $W : A \to G$ is indeed the projection map of a principal fiber bundle. Once can construct local sections and transition functions for this principal bundle and verify that it is nontrivial.

Let us digress a little to consider the special case $G = SU(2)$. As a manifold, the base $SU(2) = S^3$. Now, principal bundles on $S^3$ are labelled by $\pi_2$ of the structure group. Hence there are no such nontrivial bundles with a compact Lie group. ($\pi_2$ is zero in this case). Yet, if the structure group is the based loop group $S^1SU(2)_0$, there are nontrivial bundles over $S^3$:

$$\pi_2(G_0) = \pi_3(G) = \mathbb{Z}. \quad (14)$$

The bundle we get over $S^3$ is the one corresponding to the fundamental generator of $\pi_3(G)$. Thus the bundle we are studying is in many ways the analogue of the bundle
$U(1) \to SU(2) \to S^2$, which arises in the study of magnetic monopoles [27]. We will see that the wavefunctions of our physical system are sections of an associated vector bundle, analogous to the ‘monopole harmonics’.

Another interesting feature of our principal bundle $\mathcal{G}_0 \to \mathcal{A} \to \mathcal{G}$ is that the total space is contractible. Therefore, this is a model for the ‘Universal bundle’ [28] of the gauge group $\mathcal{G}_0$. This feature is independent of the fact that physical space is $S^1$. However it is unusual for the base manifold of a universal bundle (called the classifying space) to be finite dimensional. (The precise statement is that the classifying space is homotopic to $G$). This means that a principal bundle over any manifold $M$, with structure group $\mathcal{G}_0$ is the pullback of our bundle $\mathcal{G}_0 \to \mathcal{A} \to \mathcal{G}$ through a map $\phi : M \to \mathcal{G}$. This $\mathcal{G}_0$–bundle on $M$ is determined by the homotopy class of the map $\phi$. More generally, this suggests that the total space of any universal bundle can be thought as a space of connections. This point of view might be useful to construct gauge theories with finite dimensional or even discrete gauge groups. We will not pursue this idea here.

Vectors in $\mathcal{A}$ can be thought of as functions $\xi : S^1 \to \mathcal{G}$. A vertical vector field in $\mathcal{A}$ is one that points along the fiber:

$$\xi = \frac{d}{dx} \lambda + [A, \lambda],$$

where $\lambda$ is a map from $S^1 \to \mathcal{G}$. A connection on $\mathcal{A}$ is a splitting of the tangent space into a vertical and a ‘horizontal’ piece. We can construct a connection on $\mathcal{A}$, starting from the obvious gauge invariant Riemannian metric on $\mathcal{A}$. The distance $d(A_1, A_2)$ between any two points in $\mathcal{A}$ is given by,

$$d^2(A_1, A_2) = -\int \frac{dx}{2\pi} \text{tr}(A_1 - A_2)^2.$$

A vector field $\eta_h(A, x)$ is defined to be horizontal if it is orthogonal to any vertical vector:

$$\int \text{tr} \eta_h(A, x)(\frac{d}{dx} \lambda + [A(x), \lambda(x)])dx = 0, \text{ for all } \lambda \text{ such that } \lambda(0) = 0.$$
That is,
\[
\frac{d}{dx} \eta_h + [A(x), \eta_h(x)] = k\delta(x) \quad \text{for some constant } k. \tag{18}
\]

The delta function on the r.h.s. tells us that horizontal vectors \( \eta_h \) may have a discontinuity at the point \( x = 0 \).

Thus a horizontal vector field is a function \( \eta_h : A \times S^1 \to G \),

\[
\eta_h(A, x) = \frac{1}{\sqrt{2\pi}} \tilde{W}(A, x) \eta \tilde{W}(A, x)^{-1}. \tag{19}
\]

(The constant \( \frac{1}{\sqrt{2\pi}} \) is put in for later convenience.) The function \( \eta_h(x) \) is discontinuous at \( x = 0 \) (or \( 2\pi \)) because of the delta function in (18).

It is useful to calculate the infinitesimal change in \( \tilde{W} \), \( \tilde{W} \mapsto \tilde{W} + t\tilde{w} + O(t^2) \), when \( A \mapsto A + \frac{t}{\sqrt{2\pi}} \tilde{W}(x) \eta \tilde{W}(x)^{-1} \). We have,

\[
\frac{d}{dx} \tilde{w} + A \tilde{w} + \frac{1}{\sqrt{2\pi}} \tilde{W}(x) \eta \tilde{W}(x)^{-1} \tilde{W} = 0 \tag{20}
\]

so that

\[
\tilde{w}(A, x) = -\frac{x}{\sqrt{2\pi}} \tilde{W}(A, x) \eta. \tag{21}
\]

Also, the change in \( W(A) \) is

\[
W(A) \mapsto W(A) - t\sqrt{2\pi} W(A) \eta + O(t^2) \tag{22}
\]

Thus if \( \eta \in G \) is independent of \( A \) as well as \( x \), the above horizontal vector field descends to a left–invariant vector field on \( G \). Conversely, the left invariant vector field, \( \eta \in G \) can be lifted to a horizontal vector \( \eta_h(A, x) = \frac{1}{\sqrt{2\pi}} \tilde{W}(A, x) \eta \tilde{W}(A, x)^{-1} \) in \( A \).

4. **Associated vector bundles**

Let \( F \) be a vector space carrying a representation \( \rho \) of \( G_0 \). Then there is an associated vector bundle \( F \to A \times F / G_0 \to G \). A section of this vector bundle is a function \( \psi : A \to F \)
such that
\[ \psi(gAg^{-1} + g \frac{d}{dx} g^{-1}) = \rho(g)\psi(A), \text{ if } g(0) = 1. \] (23)

We will see that the wavefunctions of Yang–Mills theory coupled to matter satisfy this ‘equivariance condition’. Hence they are naturally identified with sections of an associated bundle.

There is a representation of $S^1G$ on $S^1V$, when $V$ carries a finite dimensional representation (also called $\rho$) of $G$, given by pointwise multiplication. Let $\mathcal{V}$ be the vector bundle associated to this representation. It will be important to find a simple description of the sections of $\mathcal{V}$, as they describe one particle wavefunctions. The ideas can then be easily generalized to multiparticle wavefunctions.

A function $\psi : \mathcal{A} \to S^1V$ is the same as a function $\psi : \mathcal{A} \times S^1 \to V$. In order to be a section of $\mathcal{V}$, it must satisfy
\[ \psi(gAg^{-1} + g \frac{d}{dx} g^{-1}, x) = \rho(g(x))\psi(A, x) \text{ for all } g \in G_0. \] (24)

This is however a complicated way of describing a section: it involves functions of an infinite number of variables. We will now find an alternative description in terms of functions of a finite number of variables, which is more useful for later calculations.

Solutions to (24) are of the form
\[ \psi(A, x) = \rho(W(A, x))\phi(W(A), x). \] (25)

This gives a second, much simpler characterization.

**Defn.** A section of $\mathcal{V}$ is a function $\phi : G \times [0, 2\pi] \to V$ satisfying the quasi–periodicity condition
\[ \phi(W, 0) = \rho(W)\phi(W, 2\pi). \] (26)

We may use this quasi–periodicity to extend $\phi$ to a function on the whole real line:
\[ \phi(W, x + 2\pi) = \rho(W^{-1})\phi(W, x),. \] (27)
Since our principal bundle has a natural connection, given a left invariant vector field \( \eta \in G \), there must be a covariant derivative operator \( \nabla_\eta \) on such sections. We can show that this is simply

\[
\nabla_\eta \phi(W, x) = \sqrt{(2\pi)} \left[ \mathcal{L}_\eta - \frac{x}{2\pi} \rho(\eta) \right] \phi(W, x),
\]

(28)

\( \mathcal{L}_\eta \) being the Lie derivative w.r.t. left invariant vector fields:

\[
\mathcal{L}_\eta \phi(W, x) = \lim_{t \to 0} \frac{\phi(W(1-t\eta), x) - \phi(W, x)}{t}.
\]

(29)

To get the above formula for \( \nabla_\eta \), recall that the covariant derivative of a section \( \psi : A \to S^1 V \) is just the Lie derivative along a horizontal vector field on \( A \). Given \( \eta \in G \), there is a corresponding horizontal vector field in \( A \), \( \eta_h(A, x) = \frac{1}{\sqrt{(2\pi)}} \tilde{W}(A, x) \eta \tilde{W}(A, x)^{-1} \).

The Lie derivative of \( \psi \) along this horizontal vector field is

\[
\mathcal{L}_{\eta_h} \psi(A, x) = \lim_{t \to 0} \frac{\tilde{W}(A + t\eta_h, x) \phi(W(A + t\eta_h), x) - \tilde{W}(A, x) \phi(A, x)}{t}.
\]

(30)

We can calculate the r.h.s explicitly and then extract the covariant derivative in terms of \( \phi \):

\[
\nabla_\eta \phi(W, x) = \rho(\tilde{W}(A, x)^{-1}) \mathcal{L}_{\eta_h} \psi(A, x).
\]

(31)

We pause to make a technical remark. The space \( A \) has been implicitly assumed to be consisting smooth functions \( A : S^1 \to G \). This means that a one–form in \( A \) would be a distribution on \( S^1 \) valued in \( G \). In particular it could be a discontinuous function on \( S^1 \). The connection on the bundle \( G_0 \to A \to G \) can be thought of as a one–form on \( A \). It turns out that it has a discontinuity as a function on \( S^1 \). This also means that the covariant derivative of a quasi–periodic function is no longer quasi–periodic. This is not an inconsistency.

Given \( \eta_1, \eta_2 \in G \) the curvature of this connection can be calculated to be

\[
\Omega_{\eta_1, \eta_2}(x) = [\nabla_{\eta_1}, \nabla_{\eta_2}] - \nabla_{[\eta_1, \eta_2]} = \left( \frac{x}{\sqrt{(2\pi)}} - 1 \right) \frac{x}{\sqrt{(2\pi)}} \rho([\eta_1, \eta_2]).
\]

(32)
Viewed as a function on $S^1$, this is discontinuous; that is allowed for the technical reason mentioned above.

Under a gauge transformation $g \in G$ (possibly with $g(0) \neq 1$), the section $\phi(W,x)$ transforms as follows:

$$\phi(W,x) \mapsto g(0)\phi(g(0)^{-1}Wg(0)).$$

(33)

It is straightforward to check that this is consistent with the quasi–periodicity condition.

The operator $d_A : \psi(A, x) \mapsto \frac{d}{dx}\psi(A, x) + A(x)\psi(x)$ maps a section $\psi : A \times S^1 \to V$ of the associated bundle to another section. If we think of sections instead as functions $\phi(W, x)$ satisfying the quasi–periodicity condition, this operator becomes just $\frac{d}{dx}$. This is part of the simplicity of thinking in terms of $\phi$ rather than $\psi$. We can also describe the one–particle wave–functions of the systems in the momentum basis, as functions $\tilde{\phi} : G \times R \to V$, with

$$e^{2\pi i k} \tilde{\phi}(W, k) = W^{-1} \tilde{\phi}(W, k)$$

(34)

in which case the covariant derivative will be

$$\nabla_\eta = \sqrt{(2\pi)}[L_\eta - \frac{\rho(\eta)}{2\pi i} \frac{\partial}{\partial k}].$$

(35)

5. The Quantization of Gauge Fields Coupled to Matter

The action for Yang–Mills theory coupled to matter can be chosen to be

$$S = \int \text{tr} E(\partial_t A - \partial_x A_0 + [A_0, A])dxdt + \frac{1}{2} \int \text{tr} E^2dxdt - i \int \text{tr} \rho(x)A_0(x)dx + S_m(A, \chi),$$

(36)

$S_m$ being the action for the matter variables $\chi$ and $\rho(x)$ the charge density of matter. The configuration space of the matter will be called $Q$. Classically, the variation with respect to $A_0$ leads to the constraint equation (Gauss’ law):

$$\frac{d}{dx}E + [A, E] - i\rho(x) = 0.$$
In fact $A_0$ is just the Lagrange multiplier that enforces these constraints. The constraints are first class in the sense of Dirac. The action also implies that $A$ and $E$ are canonically conjugate to each other. Furthermore, we can read off the hamiltonian
\begin{equation}
H = - \int dx tr E^2 + H_m(A, \chi). \tag{38}
\end{equation}
Here $H_m$ is the hamiltonian of the matter fields.

In the quantum theory, wave functions are functions $\psi : A \times Q \to C$. Denote by $\mathcal{F}$ the space of functions on $Q$ (the space of matter wavefunctions). Then, $\psi$ can be thought of as a function on $A$ valued in $\mathcal{F}$, $\psi : A \to \mathcal{F}$. The electric field, being canonically conjugate to $A$, is represented by the operator $E(x) = \frac{\delta}{i \delta A(x)}$. More precisely, upon smoothing out with the (Lie algebra valued) function $\xi$,
\begin{equation}
\int dx tr \xi(x) E(x) \psi(A, \chi) = -i \mathcal{L}_\xi \psi(A) = -i \lim_{t \to 0} \frac{\psi(A + t \xi) - \psi(A)}{t}. \tag{39}
\end{equation}

Also, upon quantizing the matter field, the charge density $\rho(x)$ is given by an operator. This operator must provide a representation of the Lie algebra of the gauge group on the Hilbert space of matter wavefunctions $\mathcal{F}$. That is, if $\rho(\lambda) = \int dx tr \lambda(x) \rho(x)$,
\begin{equation}
[\rho(\lambda), \rho(\lambda')] = \rho([\lambda, \lambda']). \tag{40}
\end{equation}
Being first class, the constraint equation becomes a differential equation on the allowed (‘physical’) wave functions. If $\lambda \in S^1 G$, we can write it as
\begin{equation}
- \int dx tr d_A \lambda E \psi(A, \chi) = i \int tr \rho(x) \lambda(x) \psi(A, \chi). \tag{41}
\end{equation}
Thus, under an infinitesimal gauge transformation,
\begin{equation}
\mathcal{L}_{d_A \lambda} \psi = \rho(\lambda) \psi. \tag{42}
\end{equation}
Since our gauge group $G$ is connected, this may be integrated to a constraint under finite gauge transformations:
\begin{equation}
\psi(g A g^{-1} + g d g^{-1}) = \rho(g) \psi(A). \tag{43}
\end{equation}
(To avoid proliferation of symbols, we use $\rho$ to denote the representation of the group as well as the Lie algebra.) If we restrict to gauge transformations that satisfy $g(0) = 1$, this is just the ‘equivariance’ condition that $\psi$ be a section of the associated vector bundle $A \times \mathcal{F}/G$. They have to satisfy in addition an equivariance condition under the ‘global’ (constant) gauge transformations $G/G_0 = G$.

$$\psi(gAg^{-1}) = \rho(g)\psi(A)$$ (44)

More generally, given any vector bundle $\mathcal{F} \to \mathcal{T} \to \mathcal{G}$ associated to the principal bundle $G_0 \to A \to G$, we have a theory of matter coupled to Yang–Mills theory. Any representation of the gauge group will provide such an associated bundle. * Actually all that is necessary is a 1–cocycle of the group. It is thus possible to have more general ‘matter fields’ whose transformation law is not just a representation of $G_0$, but instead depends on $A$. We will only study here the case of bundles arising from some obvious representations.

We now need to understand the meaning of the hamiltonian in this language. The matter part of the hamiltonian is an operator on each fiber and requires no further comment. (We will work out some special cases later.) We will show now that the term $\int dx \, tr E^2$ is the sum of a covariant Laplacian in $G$ and the Coulomb energy. The topologically nontrivial part is of course in the covariant Laplacian [12]. It is interesting that in our view the Coulomb energy arises from the ‘kinetic energy’ $\int dx \, tr E^2(x)$ of the Yang–Mills field.

We know that

$$\int tr\eta(x)E(x)\psi(A)dx = E_\eta\psi(A) = -i \lim_{t \to 0} \frac{\psi(A + t\eta) - \psi(A)}{t}. \quad (45)$$

* There could be additional restrictions for the theory to be physically reasonable. For example, the hamiltonian should be represented by a self–adjoint operator that is bounded below.
If $\eta$ is a horizontal vector in $A$, this is the same as the covariant derivative (upto a factor of $-i$):
\[
\int dx \text{tr}\eta(x)E(x)\psi = -i\nabla_\eta \psi \quad \text{if } \eta \text{ is horizontal.} \quad (46)
\]

On the other hand, if $\eta = d_A\lambda$ is a vertical vector, then the component along this vertical direction is given by the constraint on the wavefunctions:
\[
E_{d_A\lambda}\psi(A,\chi) = -i \int dx \lambda(x)\rho(x)\psi(A,\chi). \quad (47)
\]

Let $\eta_a$ be a complete set of horizontal vectors and $d_A\lambda_m$ a complete set of vertical vectors, orthonormal with respect to the metric on $A$:
\[
\int \text{tr}\eta_a(x)\eta_b(x) = -\delta_{ab}, \quad \int \text{tr}d_A\lambda_m(x)d_A\lambda_n(x) = -\delta_{mn}. \quad (48)
\]

Then we can define the operator $\int \text{tr}E^2(x)$ to be
\[
\int \text{tr}E^2(x)dx = -\sum_a E^2_{\eta a} - \sum_m E^2_{d_A\lambda m}. \quad (49)
\]

The functions $\lambda_m$ must vanish at the origin in order to be in $G_0$. Furthermore, they might have discontinuous first derivatives at that point.

From the above discussion, we get
\[
\int \text{tr}E(x)^2dx = \sum_a \nabla^2_{\eta a} + \sum_m \int dxdy\text{tr}[\rho(x)\lambda_m(x)]\text{tr}[\rho(y)\lambda_m(y)]. \quad (50)
\]

Now it is important to recall that $\lambda_m$ are orthonormal with respect to an inner product that depends on $A$. Thus $\sum \lambda_m(x) \otimes \lambda_m(y) = G_A(x,y)$ also depends on $A$. We can compute $G_A$ by expressing $\lambda_m$ in terms of the eigenfunctions of the operator $d_A^2$:
\[
d^2_A\mu_m = -a_m\mu_m, \quad \int \text{tr}\mu_m(x)\mu_n(x)dx = -\delta_{mn} \quad \sum_m \mu_m(x) \otimes \mu_m(y) = -\delta(x,y). \quad (51)
\]

The eigenvalues $a_m$ are positive. (The zero eigenvalue exists only if there is a constant eigenvector; but that is excluded by the condition $\lambda_m(0) = 0$.) Now we see that $\lambda_m =$
\( \frac{1}{\sqrt{\lambda_m}} \mu_m \) for \( a_m \neq 0 \). Also, \( \sum_m \lambda_m(x) \otimes \lambda(y) = G_A(x, y) \) is the Green’s function of the one-dimensional Laplace operator: \( d^2G_A(x, y) = \delta(x, y) \). (The boundary condition \( G_A(x, 0) = G_A(0, y) = 0 = G_A(2\pi, y) = 0 = G_A(x, 2\pi) = 0 \) has to be imposed on \( G_A \)).

We can in fact find \( G_A \) more explicitly. By its definition, \( G_A(x, y) \) is a matrix in the adjoint representation. That is, under a gauge transformation, it transforms as

\[
G_{gAg}^{-1} + gdg^{-1}(x, y) = \text{Ad}(g(x))G_A(x, y)\text{Ad}(g(y))^T.
\] (52)

Using the definition of \( \tilde{W} \), it is possible to check that

\[
G_A(x, y) = \text{Ad}(\tilde{W}(A, x))G_0(x, y)\text{Ad}(\tilde{W}(A, y))^T
\] (53)

where

\[
G_0(x, y) = \frac{1}{2|x - y|} + \frac{xy}{2\pi} - \frac{1}{2}(x + y).
\] (54)

Thus the total hamiltonian of the gauge–matter system can be written as

\[
H = -\nabla^2 + \int \text{tr}\rho(x)\rho(y)G_A(x, y)dxdy + H_m(A).
\] (55)

We see that matter couples to the Yang–Mills field in two different ways: through the Coulomb term and the covariant derivative. If we had not used the proper geometric language, we might have missed the first term.

It is useful to write the Coulomb energy more explicitly. If we introduce an orthonormal basis \( \eta_a \) in \( G \) labelled by \( a = 1 \cdots \dim G \), we can write

\[
\int dxdy\text{tr}\rho(x)G_A(x, y)\rho(y) = \int dxdy\rho^a(x)\rho^b(y)G^{ab}_A(x, y).
\] (56)

The \( \rho^a(x) \) are operators on matter wavefunctions satisfying

\[
[\rho^a(x), \rho^b(y)] = f^{abc}\rho^c(x)\delta(x, y).
\] (57)
6. Yang–Mills field coupled to a single particle

Now we consider the simplest possible kind of matter: a nonrelativistic point particle of mass \( \mu \), coupled to the gauge field. The wavefunction of the matter field can be thought of as a function \( f : S^1 \to V \), \( V \) being finite dimensional and carrying a unitary irreducible representation \( \rho \) of \( G \).* The wavefunctions of the Yang–Mills–matter system will be functions \( \psi : A \times S^1 \to V \) such that

\[
\psi(gAg^{-1} + gdg^{-1}, x) = \rho(g(x))\psi(A, x).
\] (58)

We already know that these can be written as

\[
\psi(A, x) = \rho(\tilde{W}(A, x))\phi(W, x)
\] (59)

where the \( \phi : G \times \mathbb{R} \to V \) satisfy the constraints

\[
\phi(W, x + 2\pi) = \rho(W^{-1})\phi(W, x)
\]

\[
\phi(gWg^{-1}, x) = \rho(g)\phi(W, x) \text{ for } g \in G.
\]

The first one is the condition for \( \phi \) to define a section of the associated vector bundle with fiber \( F \) over \( G \). The second is the equivariance condition under the ‘global’ or constant part of the gauge transformations.

Now it is clear right away that not all representations \( \rho \) are allowed. For example, if \( G = SU(N) \) and \( V = \mathbb{C}^N \), the second equation becomes

\[
\phi(gWg^{-1}, x) = g\psi(W, x).
\] (60)

The only solution is \( \phi = 0 \). To see this, consider the special case \( g = W \):

\[
\phi(W, x) = W\phi(W, x)
\] (61)

* Strictly speaking, the wavefunctions need to be periodic only up to a phase, in \( S^1 \). Such \( \theta \)-parameters will be ignored largely, as our present aim is to construct the simplest quantum theory, not the most general one.
and recall that $W$ is an arbitrary element of $SU(N)$ (which may not have eigenvalue 1.)

This has a simple meaning. The condition says that the wave–function must be invariant under the constant transformations, if the transformations of the gauge field and matter are both taken into account. Now, the gauge field, described by $W$, transforms under the adjoint representation. There is no way to form a singlet by combining a power of the adjoint representation and the fundamental one. We see that in order to have a nontrivial solution to the constraints, the matter representation must have the center of $G$ in its kernel. (For $SU(2)$, these are the integer spin representations; for $SU(3)$, these are the representations of zero triality.)

Let us now return to studying the constraints. The second equation with $g = W^{-1}$ implies that $\phi(W, x) = \rho(W^{-1})\phi(W, x)$. This means we can simplify the first constraint:

$$
\phi(W, x + 2\pi) = \phi(W, x)
$$
$$
\phi(gWg^{-1}, x) = \rho(g)\phi(W, x).
$$

Thus the wavefunctions are just equivariant periodic functions $\phi : G \times S^1 \to G$.

The results of the last section can be used to simplify the hamiltonian. In general,

$$
H = -\nabla^2 + \int dxdy tr\rho(x)G(x, y)\rho(y) + H_m(A).
$$

(62)

In our case the matter hamiltonian is $-\frac{1}{2\mu}(\frac{d}{dx} + \text{ad } A)^2$ when acting on $\psi$. After changing variables to $\phi$, it is just

$$
H_m\phi = -\frac{1}{2\mu} \frac{d^2\phi}{dx^2}.
$$

(63)

From earlier arguments,

$$
-\nabla^2\phi(W, x) = 2\pi \sum_a [L_{\eta_a} - \frac{x}{2\pi} \rho(\eta_a)]^2 \phi
$$

(64)

where $\eta$ are an orthonormal basis in $G$ and $L_{\eta_i}$ is the corresponding left–invariant vector field on $G$. 

25
This leaves the Coulomb energy. The charge density operator \( \rho(x) \) is defined by,

\[
\rho(\lambda)\phi(W, x) = \int \text{tr}\lambda(x)\rho(x)dx\phi(W, x) = \rho(\lambda(x))\phi(W, x)
\]  

(65)

If we use the orthonormal basis \( \eta_a \) in \( G \), the charge density is the operator

\[
\rho^a(y)\phi(W, x) = \rho(\eta_a)\delta(y - x)\phi(W, x)
\]

(66)

That is, charge density is concentrated at the position of the particle. The Coulomb energy \( V \) becomes,

\[
V\Phi(W, x) = \int \text{tr}\rho^a(y)\rho^b(z)G_{AB}^{ab}(y, z)dydz\phi(W, x) = \rho(\eta_a)\rho(\eta_b)G_{AB}^{ab}(x, x)\phi(W, x).
\]

(67)

This just describes the self–energy of the particle. Now, in matrix notation,

\[
G_A(x, y) = \text{Ad}(\tilde{W}(A, x))\text{Ad}(\tilde{W}(A, y))^T G_0(x, y).
\]

(68)

The first two factors cancel each other when \( x = y \), since the adjoint representation is orthogonal. Thus

\[
G_{AB}^{ab}(x, x) = \delta^{ab}G_0(x, x),
\]

(69)

and

\[
V\phi(W, x) = G_0(x, x)\rho(\eta_a)\rho(\eta_a)\phi(W, x) = C_2(\rho)G_0(x, x)\phi(W, x)
\]

(70)

where \( C_2(\rho) \) is the quadratic Casimir of the representation \( \rho \).

Thus we see that the hamiltonian becomes

\[
H\phi(W, x) = \{-2\pi\sum_a[\mathcal{L}_{\eta_a} - \frac{x}{2\pi}\rho(\eta_a)]^2 + C_2G_0(x, x) - \frac{1}{2\mu}\frac{d^2}{dx^2}\phi(W, x).
\]

(71)

We must find the eigenvectors of this \( H \) subject to the constraints above.

The first two terms in the hamiltonian are not individually translation invariant; yet the sum is invariant on the subspace of functions satisfying the constraints. So we should
be able to eliminate all explicit $x$ dependence from the Hamiltonian, using the constraints. The infinitesimal form of the constraint

$$\phi(gWg^{-1}, x) = \rho(g)\phi(W, x)$$

(72)

is

$$(L_a + R_a)\phi(W, x) = \rho(\eta_a)\phi(W, x).$$

(73)

Here, $L_a = \mathcal{L}_{\eta_a}$ are the left invariant vector fields and $R_a$ the right invariant vector fields on $G$. Note also that the operators $L_a^2$ and $R_a^2$ are the same on $G$, since they are just different ways of expressing the Laplace operator on $G$. Now we can simplify the first two terms in the Hamiltonian using

$$L_a\rho(\eta_a)\phi = L_a(L_a + R_a)\phi = \frac{1}{2}(L_a + R_a)^2\phi = \frac{1}{2}C_2(\rho)\phi$$

(74)

The third step uses $L_a^2 = R_a^2$. The $x$ dependent terms now cancel out. Thus the Hamiltonian simplifies to just

$$H\phi(W, x) = -[2\pi L_a^2 + \frac{1}{2\mu} \frac{d^2}{dx^2}]\phi(W, x).$$

(75)

The constraints

$$\phi(W, x + 2\pi) = \phi(W, x) \quad \phi(gWg^{-1}, x) = \rho(g)\phi(W, x)$$

(76)

are also quite simple to solve.

This is to be compared to the corresponding result for pure Yang–Mills theory:

$$H\phi(W) = -2\pi L_a^2\phi(W)$$

(77)

and

$$\phi(gWg^{-1}) = \phi(W).$$

(78)
This is just the special case where \( \rho \) is the trivial representation and the particle therefore decouples from the gauge field. The eigenfunctions were character functions of the various irreducible representations; the eigenvalues then are the corresponding quadratic Casimirs.

The parameters of our problem are the gauge coupling constant \( \alpha \), the radius \( R \) of the circle and the mass \( m \) of the particle. We can use units in which \( \hbar = 1 \). Since we deal with nonrelativistic particles, the velocity of light never enters the theory: \( c \neq 1 \). Thus there are two dimensions, (say) length and time. The dimensions of the physical quantities are

\[
[A] = L^{-1}, [E] = 1, \quad [H] = T^{-1}, \quad [\alpha] = T^{-1}L^{-1}, \quad [m] =TL^{-2};
\]  
(79)

In the above we have used units with \( \alpha = R = 1 \), and \( \mu = m\alpha R^3 \) is the only dimensionless parameter of the theory. If we express the hamiltonian explicitly in terms of the physical parameters, we get

\[
H\phi(W, x) = -[\alpha RL_a^2 + \frac{1}{2m} \frac{d^2}{dx^2}]\phi(W, x).
\]  
(80)

Let us consider the solution when the gauge group is \( SU(2) \) and the representation \( \rho \) of the matter field has dimension \( 2j + 1 \). If \( j \) is half an odd integer, there is no solution to the constraint

\[
\phi(gWg^{-1}, x) = \rho(g)\phi(W, x)
\]  
(81)
at all (except \( \phi = 0 \)). This is seen by considering \( g = -1 \), and noting that \( \rho(g) = -1 \) for such even dimensional representations. If \( j \) is an integer, there is exactly one solution for each half-integer or integer \( l \) such that \( 2l \geq j \). To see this, note that there is a representation \( R \) of \( SU(2) \times SU(2) \times SU(2) \) on the space of functions \( \phi : SU(2) \rightarrow C^{2j+1} \):

\[
\phi(W) \rightarrow \rho_j(g_1)\phi(g_2^{-1}Wg_3).
\]  
(82)
The solutions to our condition, \( \phi(W) = \rho(g)\phi(g^{-1}Wg) \) are invariant under the diagonal \( SU(2) \subset SU(2) \times SU(2) \times SU(2) \) subgroup. The representation \( R \) of \( SU(2)^3 \) above can
be expanded in terms of irreducible ones:

\[ R = \bigoplus_{l=0,\frac{1}{2},1,\ldots} (j, l, l) \]  

(83)

and can be reduced to the following representation of \( SU(2) \times SU(2) \) by combining the actions of the last two:

\[ \bigoplus_{l=0,\frac{1}{2},1,\ldots} [(j, 2l) \oplus (j, 2l - 1) \oplus (j, 2l - 2) \cdots, \oplus (j, 0)]. \]  

(84)

In order that there be an invariant when the last two \( SU(2) \) are combined, there should be a term in the square bracket of the form \((j, j)\). In that case there will be exactly one such invariant vector. Thus to each \( l = 0, \frac{1}{2}, 1, \cdots \) satisfying \( 2l \geq j \) there is one solution to the constraint.

Let us call this solution \( \psi_l(W) \). Then the eigenfunctions of the Hamiltonian are \( \psi_l(W)e^{ipx} \) and the eigenvalues are given by

\[ H\psi_l(W)e^{ipx} = [\alpha R^2 \pi l(l + 1) + \frac{p^2}{2m}]\psi_l e^{ipx}. \]  

(85)

Here \( l = \frac{j}{2}, \frac{j}{2} + 1, \cdots \) and \( p \) is an integer.

In particular, the ground state corresponds to \( p = 0, l = \frac{j}{2} \). Thus even the ground state has energy of order \( \alpha R \). This situation has a simple physical meaning: to form a singlet under the global symmetry, the particle has to ‘borrow’ some color from the gluon sector. This costs an amount of color proportional to \( \alpha R \). In the limit where the radius of the circle is sent to infinity keeping \( \alpha \) fixed, the energy of any particle carrying color will diverge linearly. This is just the expression of confinement in our simple situation. There is also a close analogy with the energy levels of the charge–monopole system \[27\].

7. Bound State of Two Particles

We will now consider the case of two point particles on a circle interacting through the Yang–Mills field. If the two particles combine to form a color non–singlet, the ground
state energy of the system will be of order $R$. This is because the center of mass variable of the system by itself will behave like the example considered above. Some color would be borrowed from the gluon sector to form a singlet, but that would increase the energy of the gluon sector by an amount of order $R$. Thus in the limit of large $R$, the low lying states of the system will be in the sector where the two particle bound state is in the color singlet state (if that is possible). We will study in detail the case of two particles in complex conjugate fundamental representations of $G = SU(N)$, which allows for both possibilities. We might think of this a particle (‘quark’) and its antiparticle, interacting through the Yang–Mills field. Generalization to other cases is possible, but we do not expect any essential changes.

It would be reasonable to assume that wavefunction of the two particle system (excluding the Yang–Mills degrees of freedom) transforms under an action of $G$ as
\[
\psi(x, y) \rightarrow g(x)\psi(x, y)g(y)^{-1}.
\]
(86)

At first it seems reasonable also that the wavefunction should be separately periodic in the coordinates $x$ and $y$. However, that leads to a physically unreasonable answer: the ground state energy will be order $\alpha R$ when the radius $R$ of the cylinder is large. Therefore we will only impose periodicity under the simultaneous shift of $x$ and $y$ by $2\pi$ (in units where $R = 1$). If we shift $x$ alone by $2\pi$ keeping $y$ fixed, the particle would have to pass through the position of the antiparticle. In one–dimensional situations, it is possible that discontinuities in the wavefunction arise as result of this. The allowed boundary conditions (at the points where the positions of the two particles coincide) is related to the self–adjoint extension of the hamiltonian. Thus we will impose
\[
\psi(x + 2\pi, y + 2\pi) = \psi(x, y).
\]
(87)

Additional boundary conditions will become clear only after we know the hamiltonian in explicit form.
We can regard the matter wavefunctions as functions $\psi: (R \times R)/2\pi Z \to V \times V^*$, where the vector space $V$ carries the fundamental representation of $G$. The wavefunctions of the matter–gauge system is then the space of sections of the corresponding associated vector bundle. These sections can be thought of as functions $\Psi: A \times \left((R \times R)/2\pi Z\right) \to V \otimes V^*$ satisfying the equivariance condition

$$\Psi(gAg^{-1} + gdg^{-1}, x, y) = g(x)\Psi(A, x, y)g(y)^{-1}. \tag{88}$$

To be a section of the associated vector bundle, it is sufficient that this condition be satisfied for $g$ with $g(0) = 1$. However, physical wavefunctions must in fact satisfy this for all $g$, even those that do not become the identity at the origin. We can solve this equation as before by the ansatz

$$\Psi(A, x, y) = \tilde{W}(A, x)\Phi(W, x, y)\tilde{W}(A, y)^{-1}. \tag{89}$$

Now the equivariance is automatic if $g(0) = 1$. $\Psi$ will be equivariant under the full gauge group if $\Phi$ satisfies the constraint

$$\Phi(gWg^{-1}, x, y) = g\Phi(W, x, y)g^{-1}. \tag{90}$$

Also, the periodicity condition becomes, in terms of $\Phi$,

$$\Phi(W, x + 2\pi, y + 2\pi) = W^{-1}\Phi(W, x, y)W. \tag{91}$$

The hamiltonian operator acting on $\Psi$ is,

$$H\Psi = -\int \text{tr} E^2(z)\Psi dz - \frac{1}{2m}\left[\frac{\partial}{\partial x} + A(x)\right]^2\Psi - \frac{1}{2m}\left[\frac{\partial}{\partial y} + A^*(y)\right]^2\Psi. \tag{92}$$

We have assumed that the two particles have the same mass. There is no essential difference if they are not equal.
We can now decompose the first term into a horizontal and a vertical part as before. The derivative along the horizontal direction becomes,

\[ E_{\eta_a} \Psi(A, x, y) = i\sqrt{(2\pi)} \tilde{W}(x) [\mathcal{L}_{\eta_a} - \frac{x}{2\pi} \eta_{aL} + \frac{y}{2\pi} \eta_{aR}] \Phi \tilde{W}(y)^{-1}. \] (93)

It is again possible to express the contribution of the vertical derivatives to the Hamiltonian in terms of the Green’s functions. It becomes the sum of the self–energies and the interaction energy of the particle and the antiparticle. In more detail,

\[ \sum_m E_{d,\lambda_m}^2 \Psi = \sum_m [-\lambda_m(x)\lambda_m(x)\Psi - \Psi \lambda_m^\dagger(y)\lambda_m^\dagger(y) - 2\lambda_m(x)\Psi \lambda_m(y)^\dagger]. \] (94)

We already know how to simplify the first two terms. The last term may be simplified using the identity

\[ \sum_m \lambda_m(x)_k^i \lambda_m(y)_j^{\dagger l} = G_0(x, y)[P_j^i(x, y)P_k^l(x, y) - \frac{1}{N}\delta_k^i \delta_m^l] \] (95)

where \( P_j^i(x, y) = (\tilde{W}(x)\tilde{W}(y)^{-1})_j^i \). This term describes the interaction energy of the particle–antiparticle pair.

Now it is possible to write the Hamiltonian in terms of \( \Phi \):

\[ H \Phi = -2\pi \sum_a [\mathcal{L}_{\eta_a} - \frac{x}{2\pi} \eta_{aL} + \frac{y}{2\pi} \eta_{aR}]^2 \Phi + C_2[G_0(x, x) + G_0(y, y)]\Phi + 2G_0(x, y)[\mathrm{tr}\Phi - \frac{1}{N}\Phi] - \frac{1}{2m} \frac{\partial^2}{\partial x^2} \Phi - \frac{1}{2m} \frac{\partial^2}{\partial y^2} \Phi. \]

The wavefunction \( \Phi \) must satisfy the condition of equivariance under the constant gauge transformations:

\[ \Phi(gWg^{-1}, x, y) = g\Phi(W, x, y)g^{-1}. \] (96)

Furthermore, it should satisfy the periodicity condition

\[ \Phi(W, x + 2\pi, y + 2\pi) = W^{-1}\Phi(W, x, y)W. \] (97)
On such functions, the above hamiltonian is in fact translation invariant, although the separate terms are not. This provides a nontrivial consistency check of our formalism: there should be no dependence on the choice of the origin once the condition of equivariance under the constant gauge transformations are imposed.

We can now exploit translation invariance and change variables to center of mass and relative coordinates $Z = x + y$ and $z = y - x$. The result of a somewhat long calculation is,

$$H\Phi = -2\pi R_a^2 \Phi + z R_a (\eta_a L + \eta_a R) \Phi + |z|[c_1 \mathcal{P}_0 \Phi - c_2 \Phi] - \frac{1}{m} \frac{\partial^2}{\partial z^2} \Phi - \frac{1}{4m} \frac{\partial^2}{\partial Z^2} \Phi. \quad (98)$$

The constants $c_1$ and $c_2$ depend on the particular group $G$. For $G = SU(2)$, $c_1 = 1$ and $c_2 = -\frac{1}{2}$. Here $\mathcal{P}_0$ denotes the projection to the trivial representation.

We need to rewrite the constraints in terms of the new variables, $z$ and $Z$. The equivariance under the full gauge group is the same, but the periodicity, is given by

$$W\Phi(W, Z + 4\pi, z)W^{-1} = \Phi(W, Z, z). \quad (99)$$

This suggests the ansatz

$$\Phi(W, Z, z) = \chi_P(W, z)e^{iPZ}. \quad (100)$$

If we impose the periodicity constraint we get the following eqn,

$$W\chi_P(W, z)W^{-1} = e^{-4iP\pi}\chi_P(W, z). \quad (101)$$

From the equivariance under the constant (global) gauge transformations, we know that

$$\chi_P(gWg^{-1}, z) = g\chi_P(W, z)g^{-1}$$

and if we take $g = W$, we get

$$\chi_P(W, z) = W\chi_P(W, z)W^{-1}.$$
If we now use this in the periodicity condition, we find that $2P$ is an integer.

Now one may look for solutions to the constraint arising from the constant gauge transformations. In the case $G = SU(2)$, the general solution is

$$
\chi_P(W, z) = f_P(W, z) + Wg_P(W, z) \quad (102)
$$

where $f$ and $g$ are complex valued central functions of $W$:

$$
f_P(gWg^{-1}, z) = f_P(W, z) \quad g_P(gWg^{-1}, z) = g_P(W, z). \quad (103)
$$

The simplest eigenfunctions of the Hamiltonian satisfying the constraint are those independent of $W$: $\chi_P(W, z) = f(z)$. This corresponds to the gauge field being in its ground state. In fact, if the radius of the cylinder is $R$, any other state will have an energy of order $\alpha R$. At least when $\alpha R$ is large, the low lying states of the system will consist of wavefunctions independent of $W$. In this case, $\Phi(W, z, Z) = f(z)e^{iPZ}$ and the Schrodinger equation reduces to Airy’s equation:

$$
\frac{3}{2}|z|f(z) - \frac{1}{m} \frac{d^2}{dz^2}f(z) = E'f(z). \quad (104)
$$

The total energy is $E' + \frac{P^2}{m}$. The physical meaning is clear: the quarks have combined to form a color singlet meson, and the gluon sector is left in its ground state. The energy of such a state will approach a constant value as $R \to \infty$.

The internal energy $E'$ is determined by the boundary conditions on the wavefunctions. The relative coordinate $z$ takes values in the range $[-2\pi, 2\pi]$. The self–adjointness of the Hamiltonian will require that the wavefunctions in its domain satisfy

$$
\frac{\partial}{\partial z}\Phi(W, z, Z) + \kappa\Phi(W, z, Z) = 0 \text{ for } z = -2\pi, 2\pi. \quad (105)
$$

We can also require $\Phi$ and its derivative to be continuous at $z = 0$. The constant $\kappa$ is not determined by the classical theory and must be picked such that the limit of infinite radius of the cylinder makes sense.
The next interesting case to study would be the bound state of \( N \) quarks forming a baryon in the case where the structure group is \( SU(N) \). In the limit \( N \to \infty \) this ought to tend to a soliton. It is interesting to study how this soliton co-exists with the gauge excitations. We will not pursue this idea here.

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