Solutions to the non-autonomous ABS lattice equations: Casoratians and bilinearization

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Abstract

In the paper non-autonomous H1, H2, H3δ and Q1δ equations in the ABS list are bilinearized. Their solutions are derived in Casoratian form. We also list out some Casoratian shift formulae which are used to verify Casoratian solutions.

Key words: non-autonomous ABS list, Casoratian, bilinear, soliton solutions

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1 Introduction

The “discrete integrable systems” has been a popular topic and is still drawing more and more attention. Particularly in the recent ten years it received much progress. The property of multidimensional consistency [1–3] provides an approach to investigate integrability for the discrete systems defined on an elementary quadrilateral:

\[ Q(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}; p, q) = 0, \] (1.1)

where \( p, q \) are spacing parameters. The ABS list [2] contains all quadrilateral lattice equations of the above form which are consistent-around-the-cube (CAC). In the ABS list the spacing parameters \( p, q \) can either constants or functions \( p_n \) and \( q_m \), which corresponds to autonomous case or non-autonomous case, respectively. In general an autonomous system means a differential/difference model with constant coefficients while a non-autonomous one means the model has coefficients varying with independent variables but the model can not be transformed back to an autonomous one. Obviously, the ABS lattice equations themselves are automatically non-autonomous in the sense of taking \( p = p_n, q = q_m \), and this non-autonomous case still keeps the CAC property.

Integrable non-autonomous systems have its own importance. It is known that most of discrete Painlevé equations are non-autonomous ordinary difference equations. In addition, for integrable non-autonomous forms of partial difference equations, their reductions usually lead to integrable non-autonomous mappings, which quite often are discrete Painlevé equations. To get a non-autonomous version of a partial difference equation, taking (1.1) as an example, one

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can replace constant lattice parameters \((p, q)\) by \((p_{n,m}, q_{n,m})\), but the integrability should be kept. Many criterions, such as singularity confinement, conservation laws and algebraic entropy, have been used to check integrability for the non-autonomous systems, for both ordinary and partial difference cases \([4, 7]\). Besides, it is also possible to deautonomise a discrete bilinear system if it contains spacing parameters. With suitable deautonomisation the obtained non-autonomous bilinear systems admit \(N\)-soliton solutions expressed through deformed discrete exponential functions, (see \([8, 9]\) as examples).

Recently, many solving approaches have been developed to find solutions for autonomous lattice equations in the ABS list \([10–18]\). In \([13]\) the H1, H2, H3 and Q1\(\delta\) equations in the ABS list were bilinearized and their solutions were derived in Casoratian form. In the present paper we will repeat the treatment of \([13]\) to get bilinear forms as well as solutions in Casoratian form for some non-autonomous ABS lattice equations. As we have mentioned before, the ABS lattice equations with spacing parameters \((p_n, q_m)\) are automatically non-autonomous and still CAC. Their integrable aspects are also double checked by singularity confinement and algebraic entropy approaches \([7]\).

The paper is organized as follows. Section 2 contains some basic notations for discrete systems and Casoratians and a list of non-autonomous ABS lattice equations. In Section 3 the non-autonomous H1, H2, H3\(\delta\) and Q1\(\delta\) equations are bilinearized and their solutions are derived in Casoratian form. The Appendix contains a collection of Casoratian formulae of non-autonomous case.

## 2 Preliminaries

Conventionally, we use tilde/hat notations to express the shifts in \(n/m\) directions, for example,

\[
\begin{align*}
    u &= u_{n,m}, \quad \tilde{u} = u_{n+1,m}, \quad y = u_{n-1,m}, \quad \hat{u} = u_{n,m+1}, \quad \hat{u} = u_{n,m-1}, \quad \hat{\tilde{u}} = u_{n+1,m+1}.
\end{align*}
\]

By these notations the lattice equation (1.1) is rewritten as

\[
Q(u, \tilde{u}, \hat{u}, \hat{\tilde{u}}; p, q) = 0. \tag{2.1}
\]

The non-autonomous ABS list is as follows \([3, 7]\):

\[
\begin{align*}
    \text{H1} : & \quad (u - \tilde{u})(\tilde{u} - \hat{u}) + q_m - p_n = 0, \tag{2.2a} \\
    \text{H2} : & \quad (u - \tilde{u})(\tilde{u} - \hat{u}) + (q_m - p_n)(u + \tilde{u} + \hat{u} + \hat{\tilde{u}}) + q_m^2 - p_n^2 = 0, \tag{2.2b} \\
    \text{H3}_\delta : & \quad p_n(u\tilde{u} + \tilde{u}\hat{u}) - q_m(u\tilde{u} + \tilde{u}\hat{u}) + \delta(p_n^2 - q_m^2) = 0, \tag{2.2c} \\
    \text{Q1}_\delta : & \quad p_n(u - \tilde{u})(\tilde{u} - \hat{u}) - q_m(u - \tilde{u})(\tilde{u} - \hat{u}) + \delta^2 p_n q_m (p_n - q_m) = 0, \tag{2.2d} \\
    \text{Q2} : & \quad p_n(u - \tilde{u})(\tilde{u} - \hat{u}) - q_m(u - \tilde{u})(\tilde{u} - \hat{u}) + p_n q_m (p_n - q_m)(u + \tilde{u} + \hat{u} + \hat{\tilde{u}}) \\
    & \quad - p_n q_m (p_n - q_m)(p_n^2 - p_n q_m + q_m^2) = 0, \tag{2.2e} \\
    \text{Q3}_\delta : & \quad \sin(p_n + q_m)(\tilde{u}\tilde{u} + \tilde{u}\hat{u}) - \sin(p_n(u\tilde{u} + \tilde{u}\hat{u}) - \sin q_m(u\tilde{u} + \tilde{u}\hat{u}) \\
    & \quad + \delta^2 \sin p_n \sin q_m (p_n + q_m) = 0, \tag{2.2f} \\
    \text{Q4} : & \quad \sin(p_n + q_m)(\tilde{u}\tilde{u} + \tilde{u}\hat{u}) - \sin p_n(u\tilde{u} + \tilde{u}\hat{u}) - \sin q_m(u\tilde{u} + \tilde{u}\hat{u}) \\
    & \quad + \sin p_n \sin q_m (p_n + q_m)(1 + k^2 u\tilde{u}\hat{u}) = 0, \tag{2.2g}
\end{align*}
\]
where $\delta$ is a constant, $p_n = p(n)$ and $q_m = q(m)$ are arbitrary non-zero functions of discrete variables $n$ and $m$, respectively.

Here the forms of Q3 and Q4 are in accordance with the autonomous version via the parametrization introduced by Hietarinta [19]. We omit A1$\delta$ and A2 from the above list because of the equivalence between A1$\delta$ and Q1$\delta$ by $u \rightarrow (-1)^{n+m}u$, as well as A2 and Q3$\delta=0$ by $u \rightarrow u^{(-1)^{n+m}}$.

The discrete version of Wronskian is Casoratian, which is a determinant of the Casorati matrix:

$$f = |\psi(n, m, l_1), \psi(n, m, l_2), \ldots, \psi(n, m, l_N)| = |l_1, l_2, \ldots, l_N|,$$

(2.3a)

where the basic column vector is

$$\psi(n, m, l) = (\psi_1(n, m, l), \psi_2(n, m, l), \ldots, \psi_N(n, m, l))^T,$$

(2.3b)

and the shifts are in $l$ direction. Using the standard short-hand notations [20], we list the following often-used $N$th-order Casoratians

$$|\widehat{N} - 1| = |0, 1, \ldots, N - 1|, \quad |\widehat{N} - 2, N| = |0, 1, \ldots, N - 2, N|, \quad |\widehat{N} - 1| = |0, 1, 2, \ldots, N - 1|.$$

As in [13], since in (2.3) there are three direction variables, say $n, m$ and $l$, one can introduce the operators $E^\nu (\nu = 1, 2, 3)$ by

$$E^1 \psi = \tilde{\psi} = \psi(n + 1, m, l), \quad E^2 \psi = \tilde{\psi} = \psi(n, m + 1, l), \quad E^3 \psi = \tilde{\psi} = \psi(n, m, l + 1),$$

(2.4)

then define a Casoratian w.r.t $E^\nu$-shift,

$$|\widehat{N} - 1|_{[\nu]} = |\psi, E^\nu \psi, (E^\nu)^2 \psi, \ldots, (E^\nu)^{N-1} \psi|, \quad (\nu = 1, 2, 3).$$

(2.5)

For these Casoratians we have

**Proposition 1. The Casoratians**

$$|\widehat{N} - 1|_{[3]} = |\widehat{N} - 1|_{[2]} = |\widehat{N} - 1|_{[3]},$$

(2.6)

if their column vector $\psi(n, m, l)$ satisfies the relations

$$\alpha_n \psi = \tilde{\psi} - \psi, \quad \beta_m \psi = \psi - \tilde{\psi},$$

(2.7)

where $\alpha_n$ and $\beta_m$ are arbitrary functions of discrete variables $n$ and $m$, respectively, i.e. $\alpha_n = \alpha(n), \beta_m = \beta(m)$.

**Proof.** By the definition of $E^\nu$ in (2.4) the relations (2.7) can be rewritten as

$$E^3 \psi = (E^1 + \alpha_n) \psi, \quad E^3 \psi = (E^2 + \beta_m) \psi,$$

(2.8)

from which one has

$$(E^3)^k = (E^1 + \alpha_n)^k = (E^1)^k + \sum_{j=1}^{k} \sum_{l_j=0}^{k-j} \prod_{i=1}^{j} \alpha_{n+l_i}(E^1)^{k-j}, \quad k = 1, 2, \ldots, N - 1,$$

(2.9a)

$$(E^3)^k = (E^2 + \beta_m)^k = (E^2)^k + \sum_{j=1}^{k} \sum_{l_j=0}^{k-j} \prod_{i=1}^{j} \beta_{m+l_i}(E^2)^{k-j}, \quad k = 1, 2, \ldots, N - 1.$$  

(2.9b)

Then, substituting them into (2.5) one can easily obtain (2.6).  

□
This Proposition will bring more flexibility for Casoratian verifications. Besides, for later convenience, we give the following Laplace expansion property [20]:

**Proposition 2.** Suppose that $B$ is a $N \times (N - 2)$ matrix, and $a, b, c, d$ are $N$th-order column vectors, then

\[
|B, a, b||B, c, d| - |B, a, c||B, b, d| + |B, a, d||B, b, c| = 0. \tag{2.10}
\]

### 3 Bilinearization and Casoratian solutions

In the following we derive bilinear forms and Casoratian solutions for the non-autonomous $H_1$, $H_2$, $H_3$, $\delta$ and $Q_{1\delta}$ models in the non-autonomous ABS list [22]. Singularity confinement might provide a possible transformation to connect discrete integrable systems with their bilinear forms, (see [21, 22] as examples), but here we will roughly use the same transformations as for the autonomous lattice equations [13]. It then turns out that these non-autonomous lattice equations can share the same bilinear forms with those autonomous ones except changing the spacing parameters accordingly.

To get Casoratian solutions one needs to use deformed discrete exponential functions and develop corresponding Casoratian shift formulae, which we have listed in Appendix.

#### 3.1 Non-autonomous $H_1$ equation

We note that the non-autonomous $H_1$ has been solved in [8] through bilinear approach and the Casoratian solutions were given, but here we give a more generalized result.

With the parametrization

\[
p_n = c - a^2_n, \quad q_m = c - b^2_m, \quad (c \text{ is an arbitrary constant}), \tag{3.1}
\]

and through the transformation

\[
u_{n,m} = \frac{g_{n,m}}{f_{n,m}} - \sum_{i=n_0}^{n-1} a_i - \sum_{j=m_0}^{m-1} b_j - \gamma, \quad (\gamma \text{ is an arbitrary constant}), \tag{3.2}
\]

the non-autonomous $H_1$ [22a] is bilinearized by

\[
\mathcal{H}_1 \equiv (\tilde{g} \tilde{f} - \tilde{g} \tilde{f}) + (a_n - b_m)(\tilde{f} \tilde{f} - \tilde{f} \tilde{f}) = 0, \tag{3.3a}
\]

\[
\mathcal{H}_2 \equiv (\tilde{g} \tilde{f} - \tilde{g} \tilde{f}) + (a_n + b_m)(\tilde{f} \tilde{f} - \tilde{f} \tilde{f}) = 0, \tag{3.3b}
\]

where in the transformation \(3.2\) \(n_0, m_0\) are arbitrary integers. The connection between [22a] and [3.3] is

\[-[\mathcal{H}_1 + (a_n - b_m)\tilde{f} \tilde{f}] [\mathcal{H}_2 + (a_n + b_m)\tilde{f} \tilde{f}] / (\tilde{f} \tilde{f} \tilde{f} \tilde{f}) + (a^2_n - b^2_m) \equiv H_1,
\]

which is the same relation as the autonomous one [13].

Solutions to the bilinear equations [3.3] can be given by

**Proposition 3.** The Casoratians

\[
f(\psi) = |\tilde{N} - 1_{[\psi]}|, \quad g(\psi) = |\tilde{N} - 2, N|_{[\psi]}, \tag{3.4}
\]
solve the non-autonomous bilinear equations \( (3.3) \), if the column vector \( \psi(n, m, l) \) symmetrically in terms of the pairs \( (n, a_n) \) and \( (m, b_m) \), satisfies the shift relations

\[
\begin{align*}
a_{n-1} \psi &= \psi - \bar{\psi}, \\
\psi &= A_{[m]} \phi, \quad b_m \bar{\phi} = \phi + \bar{\phi},
\end{align*}
\]

where \( \phi(n, m, l) \) is an auxiliary vector, the \( N \times N \) transform matrix \( A_{[m]} \) is invertible, and the subscript \([m]\) specially means \( A_{[m]} \) only depends on \( m \) but is independent of \( n, l \).

**Proof.** We prove \( \mathcal{H}_1 \) in its down-tilde-hat version \( \mathcal{H}_{\hat{1}} \):

\[
\mathcal{H}_{\hat{1}} \equiv (g \hat{f} - \hat{g}f) + (a_{n-1} - b_{m-1})(\hat{f} \hat{f} - \hat{f} f).
\]

Using the formulae given in appendix A with \( c = 0 \). In \( (3.6) \) \( f = [\overline{N - 1}]_3 \), \( \hat{f}, f, \hat{g}, g + a_{n-1} f, f \) and \( g + b_{m-1} f \) are \((A.8c), (A.6b), (A.9a), (A.6b) \) and \((A.9a) \), respectively, we have

\[
\begin{align*}
\mathcal{H}_{\hat{1}} &\equiv -(a_{n-1} - b_{m-1}) f f + f (g + a_{n-1} f) - f (g + b_{m-1} f) \\
&= -a_{n-1}^{-N+2} b_{m-1}^{-N+2} [\overline{N - 3}, \psi(N - 2), \psi(N - 1)]_{[3]} \cdot [\overline{N - 3}, \psi(N - 2)]_{[3]} \\
&\quad - [\overline{N - 3}, \psi(N - 2), \psi(N - 1)]_{[3]} \cdot [\overline{N - 3}, \psi(N - 2)]_{[3]} \\
&= 0,
\end{align*}
\]

where we have made use of Proposition 2 in which \( B = (\overline{N - 3}), (a, b, c, d) = (\psi(N - 2), \psi(N - 1), \psi(N - 2), \psi(N - 2)) \).

Next, we prove the down-tilde version of \( \mathcal{H}_2 \), which is

\[
\mathcal{H}_2 \equiv (\hat{f}g - \hat{g}f) + (a_{n-1} + b_m)(\hat{f} \hat{f} - f \hat{f}).
\]

In \( (3.7) \) we take \( f = [\overline{N - 1}]_3 \), and for \( \hat{f}, f, \hat{g}, g - b_m \hat{f}, \hat{f} \) and \( g + a_{n-1} f \) we use \((A.8b), (A.6b), (A.9b), (A.6b) \) and \((A.9a) \), with \( c = 0 \), respectively. Then we have

\[
\begin{align*}
\mathcal{H}_2 &\equiv -(a_{n-1} + b_m) f \hat{f} - f (g - b_m \hat{f}) + \hat{f} (g + a_{n-1} f) \\
&= -a_{n-1}^{-N+2} b_m^{-N+2} |A_{[m]+1} A_{[m]}^{-1}| \cdot [\overline{N - 3}, \psi(N - 2), \psi(N - 1)]_{[3]} \\
&\quad \times [\overline{N - 3}, \psi(N - 2), \hat{E}^2 \psi(N - 2)]_{[3]} \\
&\quad - [\overline{N - 3}, \psi(N - 2), \psi(N - 1)]_{[3]} \cdot [\overline{N - 3}, \psi(N - 2), \psi(N - 2)]_{[3]} \\
&\quad + [\overline{N - 3}, \psi(N - 2), \hat{E}^2 \psi(N - 2)]_{[3]} \cdot [\overline{N - 3}, \psi(N - 1), \psi(N - 2)]_{[3]} \\
&= 0,
\end{align*}
\]

by using Proposition 2 in which \( B = (\overline{N - 3}), (a, b, c, d) = (\psi(N - 2), \psi(N - 1), \psi(N - 2), \hat{E}^2 \psi(N - 2)) \). □

For the explicit forms of \( \psi \) together with the transformation matrices we can take either \((A.3)\) with \( c = 0 \) or \((A.4)\) with \( c = 0 \).

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* Here the symmetric property between pairs \( (n, a_n) \) and \( (m, b_m) \) means, for example, once we have \( (3.5a) \), at the same time we have

\[
\begin{align*}
b_{m-1} \psi &= \psi - \bar{\psi}, \\
\psi &= A_{[n]} \omega, \quad a_n \omega = \omega + \bar{\omega}.
\end{align*}
\]
3.2 Non-autonomous H2 equation

By the parametrization (3.1) with \( c = 0 \), i.e., \( p_n = -a_n^2 \), \( q_m = -b_m^2 \), we first rewrite the non-autonomous H2 (2.2b) into

\[
\text{H2} \equiv (u - \tilde{u})(u - \tilde{u}) + (a_n^2 - b_m^2)(u + \tilde{u} + \tilde{u} - (a_n^2 + b_m^2)) = 0. \tag{3.8}
\]

Then, taking the transformation

\[
u_{n,m} = U_{n,m}^2 - 2U_{n,m} \frac{g}{f} + \frac{h + s}{f}, \tag{3.9a}
\]

with

\[U_{n,m} = \sum_{i=n_0}^{n-1} a_i + \sum_{j=m_0}^{m-1} b_j + \gamma, \quad (\gamma \text{ is an arbitrary constant}), \tag{3.9b}
\]

and

\[s - h = \alpha f, \quad (\alpha \text{ is some constant}), \tag{3.9c}
\]

one can bilinearize (3.8) by

\[
\mathcal{H}_1 \equiv (\tilde{g} \tilde{f} - \tilde{g} \tilde{f}) + (a_n + b_m)(\tilde{f} \tilde{f} - \tilde{f} \tilde{f}) = 0, \tag{3.10a}
\]

\[
\mathcal{H}_2 \equiv (\tilde{g} \tilde{f} - \tilde{g} \tilde{f}) + (a_n + b_m)(\tilde{f} \tilde{f} - \tilde{f} \tilde{f}) = 0, \tag{3.10b}
\]

\[
\mathcal{H}_3 \equiv - (a_n + b_m) \tilde{f} \tilde{g} + a_n \tilde{f} \tilde{g} + b_m \tilde{f} \tilde{g} + \tilde{f} h - \tilde{f} h = 0, \tag{3.10c}
\]

\[
\mathcal{H}_4 \equiv - (a_n + b_m) \tilde{f} \tilde{g} - a_n \tilde{f} \tilde{g} - b_m \tilde{f} \tilde{g} + \tilde{f} h - \tilde{f} h = 0, \tag{3.10d}
\]

\[
\mathcal{H}_5 \equiv b_m (\tilde{f} \tilde{g} - \tilde{f} \tilde{g}) + \tilde{f} h + \tilde{f} s - \tilde{g} \tilde{g} = 0, \tag{3.10e}
\]

where the connection is

\[\text{H2} = \sum_{i=1}^{5} \mathcal{H}_i P_i / (f \tilde{f} \tilde{f} \tilde{f}), \]

and

\[P_1 = -4(a_n + b_m)[(\tilde{U} \tilde{U} - a_n^2 + b_m^2) \tilde{f} \tilde{f} - \tilde{U} \tilde{f} \tilde{g} - (a_n - b_m) \tilde{f} \tilde{g}], \]

\[P_2 = -4[(a_n + b_m)(\tilde{U} \tilde{U} - a_n^2 + b_m^2) \tilde{f} \tilde{f} + (\tilde{U} \tilde{U} - a_n^2 + b_m^2) \tilde{f} \tilde{g} - \tilde{U} \tilde{f} \tilde{g} - (a_n - b_m) \tilde{U} \tilde{f} \tilde{g}], \]

\[P_3 = 4[(a_n + b_m) \tilde{U} \tilde{f} \tilde{f} + \tilde{U} \tilde{f} \tilde{g} + \tilde{U} \tilde{f} \tilde{g} + \tilde{f} h + \tilde{f} h], \]

\[P_4 = 4[(a_n + b_m)(\tilde{U} \tilde{f} \tilde{f} - \tilde{f} \tilde{g}) + \tilde{U}(f \tilde{g} - f \tilde{g})], \]

\[P_5 = 4(a_n^2 - b_m^2) \tilde{f} \tilde{f}, \]

where \( U = U_{n,m} \) is defined in (3.9c). This is as same as the autonomous case [13].

For solutions we have

**Proposition 4. The Casoratians**

\[f = |N - 1|_{3}, \quad g = |N - 2|_{3}, \quad h = |N - 3|_{3}, \quad N - 1, N - 1, N - 1, s = |N - 2|_{3}, N + 1|_{3}, \tag{3.12}
\]

solve the non-autonomous bilinear equations (3.10), where the basic column vector \( \psi \) satisfies the same conditions in Proposition [3].
Proof. We skip the proof for $\mathcal{H}_1$ and $\mathcal{H}_2$ as they are just the bilinear $H_1$. We prove $\mathcal{H}_5$ and shifted $\mathcal{H}_3$ and $\mathcal{H}_4$ in the following forms

\begin{align}
\mathcal{H}_3 &\equiv - (a_{n-1} + b_m) \hat{f} g + a_{n-1} \hat{f} g + b_m f \hat{g} + \hat{f} h - f \hat{h}, \\
\mathcal{H}_4 &\equiv - (a_{n-1} - b_{m-1}) \hat{f} g + f (a_{n-1} g + h) - f (b_{m-1} g + h).
\end{align}

(3.13a) (3.13b)

We still use the formulae given in appendix A with $c = 0$.

For (3.13a), $g = |\overline{N - 2}, N_{[3]}|$, and $\hat{f}$, $b_m \hat{g} - \hat{h}$, $\hat{f}$ and $a_{n-1} g + h$ are (A.8c), (A.6b), (A.9d), (A.6g) and (A.9c) respectively. Then we have

\[
\mathcal{H}_3 = - (a_{n-1} + b_m) g \hat{f} + f (b_m \hat{g} - \hat{h}) + \hat{f} (a_{n-1} g + h) = - a_{n-1} b_m [-A_{m+1} A_{m}] \cdot [N - 3, \psi(N - 2), \psi(N)]_{[3]} \times [N - 3, \psi(N - 2), \hat{E} \hat{E} \psi(N - 2)]_{[3]}.
\]

For (3.13b), $g = |\overline{N - 2}, N_{[3]}|$, and $\hat{f}$, $f$, $a_{n-1} g + h$, $\hat{f}$ and $b_m g + h$ are (A.6a), (A.6c), (A.9d), (A.6b) and (A.9a) respectively. Then we have

\[
\mathcal{H}_4 = - (a_{n-1} - b_{m-1}) f g + f (a_{n-1} g + h) - f (b_{m-1} g + h) = - a_{n-1} b_{m-1} [-A_{m+1} A_{m}] \cdot [N - 3, \psi(N - 2), \psi(N)]_{[3]} \times [N - 3, \psi(N - 2), \psi(N)]_{[3]}.
\]

where we have made use of Proposition 2 in which $B = (\overline{\psi(N - 2)}, \psi(N), \psi(N - 2), \psi(N - 2)).$

We prove $\mathcal{H}_5$ and shifted $\mathcal{H}_3$ and $\mathcal{H}_4$ in the following forms

\begin{align}
\mathcal{H}_5 &\equiv f (\hat{h} - b_m \hat{g}) - g (\hat{g} - b_m \hat{f}) + \hat{f} s = b_m^{-N+1} A_{m+1} A_{m} \cdot [N - 3, \psi(N - 2), \psi(N - 1)]_{[3]} \times [N - 3, \psi(N)]_{[3]} \times [N - 3, \psi(N - 2), \hat{E} \hat{E} \psi(N - 2)]_{[3]}
\end{align}

For the explicit forms of $\psi$ together with the transformation matrices we can take either (A.3) with $c = 0$ or (A.4) with $c = 0$. 

\[\square\]
3.3 Non-autonomous H3 equation

With the parametrization

\[ p_n = \frac{1 + \alpha_n^2}{2\alpha_n}, \quad q_m = \frac{1 + \beta_m^2}{2\beta_m}, \quad \alpha_n = -\frac{a_n - c}{a_n + c}, \quad \beta_m = -\frac{b_m - c}{b_m + c}. \]

the non-autonomous H3 equation \[2.2c\] admits two different sets of bilinear forms. One is

\[
B_1 \equiv 2cf\hat{f} + (a_n - c)\tilde{f}\hat{f} - (a_n + c)\tilde{f}\hat{f} = 0, \tag{3.14a}
\]

\[
B_2 \equiv 2cf\hat{f} + (b_m - c)\tilde{f}\hat{f} - (b_m + c)\tilde{f}\hat{f} = 0, \tag{3.14b}
\]

and the other is

\[
B'_1 \equiv (b_m + c)\tilde{f}\hat{f} + (a_n - c)\tilde{f}\hat{f} - (a_n + b_m)\tilde{f}\hat{f} = 0, \tag{3.15a}
\]

\[
B'_2 \equiv (a_n + c)f\hat{f} + (b_m - c)f\hat{f} - (a_n + b_m)f\hat{f} = 0, \tag{3.15b}
\]

\[
B'_3 \equiv (a_n - c)(b_m + c)f\hat{f} + (b_m - c)(a_n + c)f\hat{f} - 2c(a_n - b_m)f\hat{f} = 0. \tag{3.15c}
\]

Both of them share same transformation

\[
u_{n,m} = AV_{n,m}\frac{\tilde{f}_{n,m}}{f_{n,m}} + BV_{n,m}\frac{f_{n,m}}{f_{n,m}}, \quad AB = -\frac{1}{4}\delta, \tag{3.16a}
\]

where

\[
V_{n,m} = \prod_{i=n_0}^{n-1} \alpha_i \prod_{j=m_0}^{m-1} \beta_j. \tag{3.16b}
\]

The connections are respectively

\[
H3 = \frac{-\delta^2 B^{-2}V^2_{n,m}(a_n - c)(b_m - c)P_1 + 4\delta P_2 + 16B^2V^{-2}_{n,m}(a_n + c)(b_m + c)P_3}{32(a_n^2 - c^2)(b_m^2 - c^2)f\hat{f}\tilde{f}\hat{f}},
\]

with

\[
P_1 = \hat{f}[(b_m - c)\tilde{f}\hat{B}_1 - (a_n - c)\tilde{f}\hat{B}_2] - \tilde{f}[(b_m + c)\hat{f}\tilde{B}_1 - (a_n + c)\hat{f}\tilde{B}_2],
\]

\[
P_2 = 2c[(b_m + c)(b_m - c)\hat{f}\tilde{B}_1 + f\hat{B}_1] - (a_n + c)(a_n - c)(f\hat{B}_2 + f\hat{B}_2),
\]

\[
P_3 = \tilde{f}[(b_m + c)\hat{f}\tilde{B}_1 - (a_n + c)\hat{f}\tilde{B}_2] - \hat{f}[(b_m - c)\tilde{f}\hat{B}_1 - (a_n - c)\tilde{f}\hat{B}_2],
\]

and

\[
H3 = \frac{c}{f\hat{f}\tilde{f}\hat{f}}[A^2V^2_{n,m}(a_n + c)(b_m + c)\tilde{f}\hat{f}\hat{B}'_1 - \tilde{f}\hat{f}\hat{B}'_2] + B^2V^{-2}_{n,m}(a_n - c)(b_m - c)f\hat{f}\tilde{f}\hat{f}\hat{B}'_1 - f\hat{f}\tilde{f}\hat{B}'_2
\]

\[
+ AB[(a_n + b_m)f\hat{f}\tilde{B}'_2 + f\hat{f}\tilde{B}'_2] + \frac{\tilde{f}\hat{f}\hat{B}'_1 + \tilde{f}\hat{f}\hat{B}'_1}{(a_n - c)(b_m + c)} + 2(a_n + b_m)\tilde{f}\hat{f}\hat{B}'_3]
\]

which are similar to the one in the autonomous case \[13\]. For solutions we have
Proposition 5. The Casoratians

\[ f = |\overline{N-1}|_{\nu}, \quad \nu = 1, 2, 3, \]  

(3.18)
solve non-autonomous bilinear equations (3.14) and (3.15), if the basic column vector \( \psi(n, m, l) \)
is symmetric in terms of pairs \( (n, a_n) \) and \( (m, b_m) \), and together with auxiliary vectors \( \omega(n, m, l) \)
and \( \zeta(n, m, l) \), satisfies the following shift relations

\[ (c - a_n)\psi = \psi - \widetilde{\psi}, \]  

(3.19a)
\[ \psi = A_{[n]}\omega, \quad (a_n + c)\widetilde{\omega} = \omega + \widetilde{\omega}, \]  

(3.19b)
\[ \psi = B_{[l]}\zeta, \quad (c + b_m)\overline{\zeta} = \zeta + \overline{\zeta}, \]  

(3.19c)
where \( A_{[n]} \) and \( B_{[l]} \) are \( N \times N \) transform matrices, and the matrix product \( B_{[l]}B_{[l+1]}^{-1} \) is inde-
dependent of \( l \).

Proof. We prove (3.14a) in its down-tilde version

\[ \mathcal{B}_1 \equiv 2cffe + (a_{n-1} - c)\overline{f}f - (a_{n-1} + c)\overline{f}f. \]  

(3.20)
In (3.20), for \( f, f, \overline{f}, f, \overline{f} \) and \( f \), we make use of (A.7b), (A.8i), (A.7h), (A.8h), (A.8l) and
(A.7e) respectively, and get

\[ \mathcal{B}_1 = \prod_{j=0}^{N-3} (a_{n-1} - b_{m+j})^{-1} \prod_{j=0}^{N-3} (c - b_{m+j})^{-1} \prod_{j=0}^{N-3} (c + b_{m+j})^{-1} |B_{[l+1]}B_{[l]}^{-1}| \]
\[ \times [-|\overline{N-3}|, \psi(N-2), \psi(N-2)|_2] \cdot |\overline{N-3}|, \psi(N-2), \hat{\psi}^3\psi(N-2)|_2 \]
\[ + |\overline{N-3}|, \psi(N-2), \psi(N-2)|_2 \cdot |\overline{N-3}|, \psi(N-2), \hat{\psi}^3\psi(N-2)|_2 \]
\[ - |\overline{N-3}|, \psi(N-2), \hat{\psi}^3\psi(N-2)|_2 \cdot |\overline{N-3}|, \psi(N-2), \psi(N-2)|_2 \]
\[ = 0, \]
with the help of Proposition 2 in which \( \mathbf{B} = (\overline{N-3}), \mathbf{(a, b, c, d)} = (\psi(N-2), \psi(N-2), \psi(N-2), \hat{\psi}^3\psi(N-2)) \). Here to get the coefficient \( |B_{[l+1]}B_{[l]}^{-1}| \) we request \( B_{[l]}B_{[l+1]}^{-1} \) to be independent
of \( l \), i.e. \( B_{[l]}B_{[l+1]}^{-1} = B_{[l+1]}B_{[l]}^{-1} \).

\( \mathcal{B}_2 \) holds thanks to \( \mathcal{B}_1 \) and the \( n-m \) symmetric property of \( \psi(n, m, l) \).

We prove \( \mathcal{B}_1 \) in its down-tilde shifted version, i.e.

\[ \mathcal{B}_1' \equiv (b_{m} + c)\overline{f}f + (a_{n-1} - c)\overline{f}f - (a_{n-1} + b_{m})\overline{f}f. \]  

(3.21)
In (3.21), we take \( f = |\overline{N-1}|_3 \) and \( \overline{f}, \overline{f}, \overline{f}, f, \overline{f} \) as (A.8e), (A.6e), (A.6a), (A.6a) and
(A.8l), respectively. Then we have

\[ \mathcal{B}_1' \equiv (a_{n-1} - c)^{-N+2} (b_{m} + c)^{-N+2} |A_{[m+1]}A_{[m]}^{-1}| \]
\[ \times [-|\overline{N-2}|, \psi(N-1)|_3] \cdot |\overline{N-2}|, \psi(N-1), \hat{\psi}^2\psi(N-1)|_3 \]
\[ + |\overline{N-2}|, \psi(N-1)|_3 \cdot |\overline{N-2}|, \psi(N-1), \hat{\psi}^2\psi(N-1)|_3 \]
\[-\psi(0), \tilde{N} - 2, \tilde{\hat{B}}^2 \psi(N - 1)_{[\alpha]} \cdot |\tilde{N} - 2, \psi(N - 1), \psi(N - 1)_{[\alpha]}| = 0,\]

where we have made use of Proposition 2 in which \( \mathbf{B} = (\tilde{N} - 2), (a, b, c, d) = (\psi(0), \psi(N - 1), \tilde{\hat{B}}^2 \psi(N - 1)). \)

\( \mathcal{B}'_2 \) holds thanks to \( \mathcal{B}_1 \) and the \( n-m \) symmetric property of \( \psi(n, m, l) \).

By a down-hat shift, \( \mathcal{B}_3' \) reads

\[
\mathcal{B}_3' \equiv (a_n - c)(b_{m-1} + c)f f - (a_n + c)(b_{m-1} - c)f f - 2c(a_n - b_{m-1})f f. \tag{3.23}
\]

In (3.23), for \( f, \tilde{f}, \bar{f}, \tilde{\hat{f}} \) and \( \hat{f} \) we use \([A.7d], [A.8n], [A.7c], [A.8k], [A.7l] \) and \([A.8j] \) respectively. Then we have

\[
\mathcal{B}_3' \equiv \prod_{i=1}^{N-2} (b_{m-1} - a_{n+i})^{-1} \prod_{i=1}^{N-2} (c - a_{n+i})^{-1} \prod_{i=1}^{N-2} (c + a_{n+i})^{-1}|B_{[i+1]}B_{[i]}| \times \begin{align*}
&|\tilde{N} - 2, \psi(0), \psi(N - 1)_{[\alpha]} : |\tilde{N} - 2, \psi(N - 2), \tilde{\hat{B}}^2 \psi(N - 1)_{[\alpha]}| \\
&- |\tilde{N} - 2, \psi(0), \psi(N - 1)_{[\alpha]} : |\tilde{N} - 2, \psi(N - 1), \tilde{\hat{B}}^2 \psi(N - 1)_{[\alpha]}| \\
&+ |\tilde{N} - 2, \psi(0), \tilde{\hat{B}}^2 \psi(N - 1)_{[\alpha]} : |\tilde{N} - 2, \psi(N - 1), \psi(N - 1)_{[\alpha]}| = 0,
\end{align*}
\]

where we have made use of Proposition 2 in which \( \mathbf{B} = (\tilde{N} - 2), (a, b, c, d) = (\psi(0), \psi(N - 1), \tilde{\hat{B}}^2 \psi(N - 1)). \)

For the explicit forms of \( \psi \) together with the transformation matrices we can take either \([A.3] \) or \([A.4]\).

### 3.4 Non-autonomous Q1 equation

The non-autonomous Q1 equation can have two different bilinearizations which are also similar to their autonomous cases \([10]\). First, using the parametrization

\[
p_n = \frac{r c^2}{a_n^2 - c^2}, \quad q_m = \frac{r c^2}{b_m^2 - c^2}, \tag{3.25a}
\]

where \( c \) and \( r \) are constants, and through the transformation

\[
u_{n,m} = AV_{n,m} \frac{f_{n,m}}{f_{n,m}} + BV_{n,m}^{-1} \frac{f_{n,m}}{f_{n,m}}, \quad AB = \frac{r^2 \delta^2}{16}, \tag{3.25b}
\]

where \( V_{n,m} \) is \([3.16b] \) and \( \alpha_n = \frac{a_n - c}{a_n + c}, \quad \beta_m = \frac{b_m - c}{b_m + c} \), the non-autonomous Q1 equation \([2.2d]\) can be transformed to \([3.14]\), i.e., one of bilinear form for the non-autonomous H3 equation. So its solutions consequently follow the Proposition 5. In this case, the connection is

\[
Q_1 \equiv (\alpha_n \beta_m U_{n,m}^2 A^2 P_1 + \frac{r^2}{16} \alpha_n^{-1} \beta_m^{-1} (a_n + c)^2 (b_m + c)^2 \delta^2 P_2 + \alpha_n^{-1} \beta_m^{-1} U_{n,m}^{-2} B^2 P_2) / \tilde{f} \tilde{\hat{f}} \tilde{\bar{f}},
\]
where
\[
P_1 = Y \bar{Y} - X \bar{X}, \quad X = B_1 - 2cf \bar{f}, \quad Y = B_2 - 2cf \bar{f},
\]
\[
P_2 = (a^2_n - c^2)(b_m + c)^2\left(\bar{X} \bar{X} - 4c^2 \bar{f} \bar{f} \bar{f} \bar{f}\right) + (a^2_n - c^2)(b_m - c)^2\left(X \bar{X} - 4c^2 \bar{f} \bar{f} \bar{f} \bar{f}\right)
- 4c^2(b^2_m - c^2)(X \bar{X} - 4c^2 \bar{f} \bar{f} \bar{f} \bar{f}) - (b^2_m - c^2)(a_n + c)^2\left(\bar{Y} \bar{Y} - 4c^2 \bar{f} \bar{f} \bar{f} \bar{f}\right)
- (b^2_m - c^2)(a_n - c)^2\left(Y \bar{Y} - 4c^2 \bar{f} \bar{f} \bar{f} \bar{f}\right) + 4c^2(a^2_n - c^2)(Y \bar{Y} - 4c^2 \bar{f} \bar{f} \bar{f} \bar{f}).
\]

The second bilinear form employs the transformation
\[
u_{n,m} = W_{n,m} - \left(\frac{c^2}{r} - \frac{\delta^2 r}{\delta^2}\right) \frac{g_{n,m}}{f_{n,m}},
\]
where \(c, r\) are constants,
\[
W_{n,m} = \sum_{i=n_0}^{n-1} \alpha_i + \sum_{j=m_0}^{m-1} \beta_j, \quad \alpha_n = p_n a_n, \quad \beta_m = q_m b_m, \quad p_n = \frac{c^2/r - \delta^2 r}{a^2_n - \delta^2}, \quad q_m = \frac{c^2/r - \delta^2 r}{b^2_m - \delta^2},
\]
and the bilinear form reads
\[
Q_1 \equiv (b_m - \delta) \bar{f} \bar{f} + (a_n + \delta) \bar{f} \bar{f} - (a_n + b_m) \bar{f} \bar{f} = 0,
\]
\[
Q_2 \equiv (a_n - b_m) \bar{f} \bar{f} + (b_m + \delta) \bar{f} \bar{f} - (a_n + \delta) \bar{f} \bar{f} = 0,
\]
\[
Q_3 \equiv \bar{f} \bar{f} - \bar{f} \bar{f} + (b_m - \delta) \bar{f} \bar{g} - (a_n - \delta) \bar{f} \bar{g} + (a_n - b_m) \bar{g} \bar{g} = 0,
\]
\[
Q_4 \equiv (a_n - b_m)(\bar{f} \bar{g} - \bar{f} \bar{g}) + (a_n + b_m)(\bar{g} \bar{f} - \bar{f} \bar{g}) = 0.
\]

The connection is
\[
Q1 = \frac{(c^2/r - \delta^2 r)^3}{(a^2_n - \delta^2)(b^2_m - \delta^2)(a_n - b_m)(a_n + \delta) \bar{f} \bar{f} \bar{f} \bar{f}} \sum_{i=1}^{4} Q_i P_i,
\]
where
\[
P_1 = (a_n - b_m) \left[-(a_n - b_m) \bar{f} \bar{f} \bar{g} + (a_n + b_m) \bar{g} \bar{f} \bar{f} - \bar{g} \bar{g}\right]
- (a^2_n - \delta^2) \bar{g} \bar{f} \bar{g} + (b^2_m - \delta^2) \bar{f} \bar{g} \bar{f} + (a^2_n - b^2_m) \bar{f} \bar{g} \bar{g} \bar{g},
\]
\[
P_2 = (a_n + b_m) \left[(a_n - b_m) \bar{f} \bar{f} \bar{g} + (b_m - \delta) \bar{g} \bar{f} \bar{g} - (a_n - b_m) \bar{g} \bar{f} \bar{g}\right] - (a^2_n + \delta) \bar{g} \bar{f} \bar{g} + (b^2_m + \delta) \bar{f} \bar{g} \bar{f} + (a^2_n - b^2_m) \bar{f} \bar{g} \bar{g} \bar{g},
\]
\[
P_3 = (a_n + b_m)(a_n + \delta) \left[(a_n - b_m) \bar{f} \bar{f} \bar{g} + (b_m - \delta) \bar{g} \bar{f} \bar{g} - (a_n - b_m) \bar{g} \bar{f} \bar{g}\right],
\]
\[
P_4 = (a_n + \delta) \left[-(a_n - b_m) \bar{f} \bar{f} + (a_n - \delta) (b_m - \delta) \bar{f} \bar{g} \bar{f} \bar{g}\right].
\]

For solutions to (3.27) we have

**Proposition 6. The Casoratians**

\[
f = |N - 1|_{[3]}, \quad g = |-1, N - 1|_{[3]},
\]

solve the non-autonomous bilinear equations (3.27), if their basic column vector \(\psi(n,m,l)\) has symmetric property and satisfies the shift relations (3.19) with \(c = \delta\), as well as

\[
\psi = A_{[n]} A_{[m]} \sigma, \quad (a_n + \delta) \sigma = \sigma + \sigma, \quad (b_m + \delta) \bar{\sigma} = \bar{\sigma} + \bar{\sigma},
\]

where the matrices \(A_{[n]}, A_{[m]}\) and their shifts are in an Abelian group.
Proof. The down-hat-bar shifted version of (3.27a) is
\[ Q_1 = (b_{m-1} - \delta) \hat{f} f + (a_n + \delta) \hat{f} f - (a_n + b_{m-1}) \hat{f} f, \]  
and by using (A.6d), (A.6f), (A.8d), (A.6e), (A.8a) and \( f = | - 1 \ N_3 \ \psi(N - 2)|_{[3]}, \) with \( c = \delta \) we can prove it is true. In fact, it is the same as the down hat-bar version of \( B'_2. \)

By a down-tilde shift (3.27b) is written as
\[ Q_2 = (a_{n-1} - b_m) \tilde{f} f + (b_m + \delta) \tilde{f} f - (a_{n-1} + \delta) \tilde{f} f. \]  
Thanks to Proposition [1] here we use \( f = | \ N - 1 |_{[2]} \). Then with the help of (A.6d), (A.6a), (A.7g), (A.8g) and (A.8m), with \( c = \delta \), we can rewrite (3.31) as
\[
Q_2 = \prod_{j=1}^{N-2}(\delta + b_{m+j})^{-1} \prod_{j=1}^{N-2}(a_{n-1} - b_{m+j})^{-1} |B_{[1]} B_{[2]}^{-1}| |N - 1, \ \hat{E} \psi(N - 1)|_{[2]} \times |N - 2, \ \hat{E} \psi(N - 1)|_{[2]} - |N - 2, \ \hat{E} \psi(N - 1)|_{[2]} \cdot |N - 1, \ \hat{E} \psi(N - 1)|_{[2]}
\]  
where we have made use of Proposition [2].

By a down-hat-bar shift (3.27c) is written as
\[ Q_3 = - \tilde{f} |f + (a_{n-1} - \delta) g| + \tilde{f} |f + (b_m - \delta) g| + (a_n - b_m) \tilde{f} g. \]  
With \( c = \delta \), substituting (A.10b), (A.10d), (A.10f), (A.10e), (A.10a), (A.10d) and (A.10m) into the right-hand side of (3.32), we obtain
\[ Q_3 = (a_{n-1} - \delta) Y_1 - (b_m - \delta) Y_2 + (a_n - \delta) (b_m - \delta) Y_3, \]
where
\[
Y_\mu = |E \mu \psi(-1), \ \psi(-1), \hat{N} - 2|_{[3]} + |E \mu \psi(-1), \hat{N} - 1|_{[3]} - g |E \mu \psi(-1), \hat{N} - 2|_{[3]}, \ \mu = 1, 2, \\
Y_3 = |\psi(-1), \ \psi(-1), \hat{N} - 2|_{[3]} |\psi(-1), \hat{N} - 1|_{[3]} - |\psi(-1), \psi(-1), \hat{N} - 2|_{[3]} |\psi(-1), \hat{N} - 1|_{[3]}
\]  
which are zeros in the light of Proposition [2] [13]. Thus, we have proved \( Q_3 = 0 \). To prove \( Q_4 = 0 \), we go to prove \( Q_4 = Q_3 + Q_4 = 0 \), i.e.,
\[ Q'_4 = - \tilde{f} |f - (a_n + \delta) g| - \tilde{f} |f - (b_m + \delta) g| + (a_n - b_m) \tilde{f} g = 0. \]  
In the light of (A.10b), (A.10d), (A.10f), (A.10e), (A.10a), (A.10d) and (A.10m), with \( c = \delta \) we can rewrite \( Q'_4 \) as
\[ Q'_4 = (a_n + \delta)^2 Z_1 - (b_m + \delta)^2 Z_2 - (a_n + \delta)^2 (b_m + \delta)^2 Z_3, \]
where
\[
Z_\mu = |E^\mu \psi(-1), \psi(-1), \hat{N} - 2|_{[3]} + |E^\mu \psi(-1), \hat{N} - 1|_{[3]} - g |E^\mu \psi(-1), \hat{N} - 2|_{[3]}, \ \mu = 1, 2, \\
Z_3 = |\hat{E}^1 \psi(-1), \psi(-1), \hat{N} - 2|_{[3]} |\hat{E}^2 \psi(-1), \hat{N} - 1|_{[3]} - |\hat{E}^2 \psi(-1), \psi(-1), \hat{N} - 2|_{[3]}
\]  
which are also zeros [13] in the light of Proposition [2]. Thus we finish the proof. \( \square \)

For the explicit forms of \( \psi \) together with the transformation matrices we can take either (A.3) with \( c = \delta \) or (A.4) with \( c = \delta \).
4 Conclusions

We have derived bilinear forms and Casoratian solutions for the non-autonomous H1, H2, H3, and Q1 models in the non-autonomous ABS list. The transformations that we used to fulfill bilinearization are quite similar to those used in the autonomous cases. Besides, the bilinear forms and Casoratian structures of solutions are also similar to the autonomous cases. In addition, in Appendix we listed Casoratian shift formulae for non-autonomous case.

It would be interesting to have a look at the non-autonomous deformation in terms of the bilinearizations, bilinear equations and Casoratian vectors. With comparison we can sum up the following deformations from autonomous case to non-autonomous case:

- Line parameters: \( (a, b) \) \( \rightarrow \) \( (a_n, b_m) \),
- Linear function: \( an + bm \) \( \rightarrow \) \( \sum_{i=n_0}^{n-1} a_i + \sum_{j=m_0}^{m-1} b_j \),
- Discrete exponential function: \( \left( \frac{a+k}{a-k} \right)^n \left( \frac{b+k}{b-k} \right)^m \) \( \rightarrow \) \( \prod_{i=n_0}^{n-1} \left( \frac{a_i+k}{a_i-k} \right) \prod_{j=m_0}^{m-1} \left( \frac{b_j+k}{b_j-k} \right) \).

We note that many discrete bilinear equations can be deautonomised using these deformations (see [8]). Besides, actually, here we have seen that these deformations can well keep the correspondence of autonomous and non-autonomous ABS lattice equations.

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A Casoratian shift formulae

We derive shift formulae for the Casoratians

\[
\begin{align*}
    f &= |\mathcal{N}-1|_{\nu}, &
    g &= |\mathcal{N}-2, \mathcal{N}|_{\nu}, &
    h &= |\mathcal{N}-2, \mathcal{N}|_{\nu}, &
    s &= |\mathcal{N}-3, \mathcal{N}-1, \mathcal{N}|_{\nu}, \mathcal{N}+1|_{\nu},
\end{align*}
\]

(A.1)

for \( \nu = 1, 2, 3, \ldots \), where the basic column vector \( \psi(n, m, l) \) satisfies the following relation

\[
(a_{n-1} - c)\psi = \psi - \overline{\psi}, \quad (b_{m-1} - c)\psi = \psi - \overline{\psi},
\]

(A.2a)

and its auxiliary vectors \( \omega(n, m, l), \phi(n, m, l), \zeta(n, m, l) \) and \( \sigma(n, m, l) \) satisfy

\[
\begin{align*}
    \psi &= A_{[n]}\omega, &
    (a_n + c)\omega &= \omega + \overline{\omega},
    \quad &
    (A.2b) \\
    \psi &= A_{[m]}\phi, &
    (a_n + c)\phi &= \phi + \overline{\phi},
    \quad &
    (A.2c) \\
    \psi &= B_{[l]}\zeta, &
    (c + b_m)\zeta &= \zeta + \overline{\zeta},
    \quad &
    (A.2d) \\
    \psi &= A_{[n]}A_{[m]}\sigma, &
    (a_n + c)\sigma &= \sigma + \overline{\sigma}, \quad (b_m + c)\sigma &= \sigma + \overline{\sigma},
    \quad &
    (A.2e)
\end{align*}
\]

where \( c \) is a constant, \( A_{[n]}, A_{[m]} \) and \( B_{[l]} \) only depends on \( n, m \) and \( l \), respectively, \( A_{[n]}, A_{[m]} \) and their shifts are in an Abelian group, and the matrix product \( B_{[l+1]}B_{[l]}^{-1} \) is independent of \( l \).

We give two explicit forms for \( \psi \) satisfying the above criterion (A.2). One is

\[
\psi(n, m, l) = \psi^+(n, m, l) + \psi^-(n, m, l),
\]

(A.3a)
\[ \psi^\pm(n, m, l) = (\psi_1^\pm(n, m, l), \psi_2^\pm(n, m, l), \ldots, \psi_N^\pm(n, m, l))^T, \]  
(A.3b)

with
\[ \psi_r^\pm(n, m, l) = \rho_r^\pm(c \pm k_r)^l \prod_{i=n_0}^{n-1} (a_i \pm k_r) \prod_{j=m_0}^{m-1} (b_j \pm k_r), \quad r = 1, 2, \ldots, N, \]  
(A.3c)

where \( \rho_r^\pm \) and \( k_r \) are constants. The available transform matrices are
\[ A_{[n]} = \text{Diag}(A_{[n]}(k_1, n), \ldots, A_{[n]}(k_N, N)), \quad A_{[n]}(k_r, n) = \prod_{i=n_0}^{n-1} (a_i^2 - k_r^2), \]  
(A.3d)
\[ A_{[m]} = \text{Diag}(A_{[m]}(k_1, m), \ldots, A_{[m]}(k_N, m)), \quad A_{[m]}(k_r, m) = \prod_{j=m_0}^{m-1} (b_j^2 - k_r^2), \]  
(A.3e)
\[ B_{[l]} = \text{Diag}(B_{[l]}(k_1, l), \ldots, B_{[l]}(k_N, l)), \quad B_{[l]}(k_r, l) = (c^2 - k_r^2)^l, \]  
(A.3f)

and the auxiliary vectors \( \omega, \phi, \zeta, \sigma \) are correspondingly defined by these transform matrices and \( \psi \) through \([A.2b],[A.2c]\). Another explicit form of \( \psi \) is
\[ \psi(n, m, l) = A_+ \psi^+(n, m, l) + A_- \psi^-(n, m, l), \]  
(A.4a)
\[ \psi^\pm(n, m, l) = (\psi_1^\pm(n, m, l), \psi_2^\pm(n, m, l), \ldots, \psi_N^\pm(n, m, l))^T, \]  
(A.4b)

with
\[ \psi_r^\pm(n, m, l) = \frac{1}{(r - 1)!} \partial_r^{r-1} \{ \rho_1^\pm(c \pm k_1)^l \prod_{i=n_0}^{n-1} (a_i \pm k_1) \prod_{j=m_0}^{m-1} (b_j \pm k_1) \}, \quad r = 1, 2, \ldots, N, \]  
(A.4c)

where \( A_\pm \) are two arbitrary non-singular lower triangular Toeplitz matrix(see [23]). The available transform matrices of this case are
\[ A_{[n]} = (a_{s,i}(k_1))_{N \times N}, \quad a_{s,i}(k_1) = \begin{cases} \frac{\partial_{s-i}^k}{(s-i)} \prod_{i=n_0}^{n-1} (a_i^2 - k_r^2), & s \geq i, \quad s, i = 1, \ldots, N, \\ 0, & s < i, \end{cases} \]  
(A.4d)
\[ A_{[m]} = (a_{s,j}(k_1))_{N \times N}, \quad a_{s,j}(k_1) = \begin{cases} \frac{\partial_{s-j}^k}{(s-j)} \prod_{j=m_0}^{m-1} (b_j^2 - k_r^2), & s \geq j, \quad s, j = 1, \ldots, N, \\ 0, & s < j, \end{cases} \]  
(A.4e)
\[ B_{[l]} = (b_{s,j}(k_1))_{N \times N}, \quad b_{s,j}(k_1) = \begin{cases} \frac{\partial_{s-j}^k}{(s-j)} (c^2 - k_r^2)^l, & s \geq j, \quad s, j = 1, \ldots, N, \\ 0, & s < j, \end{cases} \]  
(A.4f)

The auxiliary vectors \( \omega, \phi, \zeta, \sigma \) are also correspondingly defined by these transform matrices and \( \psi \) through \([A.2b],[A.2c]\). Since \( A_{[n]}, A_{[m]}, B_{[l]} \) are non-singular lower triangular Toeplitz matrices which compose an Abelian group (see [23]), the commutative property holds automatically and the matrix product \( B_{[l+1]}B_{[l]}^{-1} \) is independent of \( l \) (see [13]).

In the following we list some Casoratian shift formulae. For convenience, we define operators \( \hat{E}^\nu, \nu = 1, 2, 3, \) as follows
\[ \hat{E}^1 \psi = A_{[n]}A^{-1}_{[n+1]}E^1 \psi, \quad \hat{E}^2 \psi = A_{[m]}A^{-1}_{[m+1]}E^2 \psi, \quad \hat{E}^3 \psi = B_{[l]}B^{-1}_{[l+1]}E^3 \psi. \]  
(A.5)

These Casoratian shift formulae are
\[ (a_{n-1} - c)^{N-1} f_{[\nu]} = \left( \frac{N-2}{2}, \psi(N-1) \right)_{[\nu]}, \]  
(A.6a)
\[(a_{n-1} - c)^{N-2} f_{[3]} = -|\hat{N} - 2, \psi(N - 2)|_{[3]}, \quad \text{(A.6b)}\]

\[(b_m + c)^{N-1} \hat{f}_{[3]} = |A_{[m+1]} A_{[n]}^{-1}| \cdot |\hat{N} - 2, \hat{E}^2 \psi(N - 1)|_{[3]}, \quad \text{(A.6c)}\]

\[(a_n + c)^{N-2} \hat{f}_{[3]} = |A_{[n+1]} A_{[n]}^{-1}| \cdot |\hat{N} - 2, \hat{E}^1 \psi(N - 2)|_{[3]}, \quad \text{(A.6d)}\]

\[(b_{m-1} - c)^{N-2} \hat{f}_{[3]} = -|\hat{N} - 2, \psi(N - 2)|_{[3]}, \quad \text{(A.6e)}\]

\[(b_{m-1} - c)^{N-1} \hat{f}_{[3]} = |-1, \hat{N} - 3, \psi(N - 2)|_{[3]}, \quad \text{(A.6f)}\]

\[(b_m + c)^{N-2} \hat{f}_{[3]} = |A_{[m+1]} A_{[n]}^{-1}| \cdot |\hat{N} - 2, \hat{E}^2 \psi(N - 2)|_{[3]}, \quad \text{(A.6g)}\]

\[(b_m + c)^{N-2} \hat{f}_{[3]} = |A_{[m+1]} A_{[n]}^{-1}| \cdot |\hat{N} - 1, \hat{E}^2 \psi(N - 1)|_{[3]}, \quad \text{(A.6h)}\]

\[
\prod_{j=0}^{N-2} (a_{n-1} - b_{m+j}) f_{[2]} = |\hat{N} - 2, \psi(N - 1)|_{[2]}, \quad \text{(A.7a)}
\]

\[
\prod_{j=0}^{N-3} (a_{n-1} - b_{m+j}) f_{[2]} = -|\hat{N} - 2, \psi(N - 2)|_{[2]}, \quad \text{(A.7b)}
\]

\[
\prod_{j=0}^{N-3} (c - b_{m+j}) f_{[2]} = -|\hat{N} - 2, \psi(N - 2)|_{[2]}, \quad \text{(A.7c)}
\]

\[
\prod_{i=0}^{N-2} (c - a_{n+i}) f_{[1]} = |\hat{N} - 2, \psi(N - 1)|_{[1]}, \quad \text{(A.7d)}
\]

\[
\prod_{i=0}^{N-2} (c + a_{n+i}) \hat{f}_{[1]} = |B_{[l+1]} B_{[l]}^{-1}| \cdot |\hat{N} - 2, \hat{E}^3 \psi(N - 1)|_{[1]}, \quad \text{(A.7e)}
\]

\[
\prod_{j=0}^{N-3} (c + a_{n+i}) \hat{f}_{[1]} = |B_{[l+1]} B_{[l]}^{-1}| \cdot |\hat{N} - 2, \hat{E}^3 \psi(N - 2)|_{[1]}, \quad \text{(A.7f)}
\]

\[
\prod_{j=0}^{N-2} (c + b_{m+j}) \hat{f}_{[2]} = |B_{[l+1]} B_{[l]}^{-1}| \cdot |\hat{N} - 2, \hat{E}^3 \psi(N - 1)|_{[2]}, \quad \text{(A.7g)}
\]

\[
\prod_{j=0}^{N-3} (c + b_{m+j}) \hat{f}_{[2]} = |B_{[l+1]} B_{[l]}^{-1}| \cdot |\hat{N} - 2, \hat{E}^3 \psi(N - 2)|_{[2]}, \quad \text{(A.7h)}
\]

\[
\prod_{j=0}^{N-2} (c + b_{m+j}) \hat{f}_{[2]} = |B_{[l+1]} B_{[l]}^{-1}| \cdot |\hat{N} - 1, \hat{E}^3 \psi(N - 1)|_{[2]}, \quad \text{(A.7i)}
\]

\[
\prod_{j=0}^{N-2} (b_{m-1} - a_{n+i}) f_{[1]} = |\hat{N} - 2, \psi(N - 1)|_{[1]}, \quad \text{(A.7j)}
\]

\[(b_{m-1} + a_n)(a_n + c)^{N-2} (b_{m-1} - c)^{N-2} \hat{f}_{[3]} = |A_{[n+1]} A_{[n]}^{-1}| \cdot |\hat{N} - 3, \psi(N - 2), \hat{E}^1 \psi(N - 2)|_{[3]}, \quad \text{(A.8a)}
\]
\[(a_{n-1} + b_m)(b_m + c)^{N-2}(a_{n-1} - c)^{N-2} f_{[2]} = |A_{[m+1]} A_{[m]}^{-1}| \cdot |\hat{N} - 3, \psi(N - 2), \hat{E}^2 \psi(N - 2)|_{[3]}, \quad (A.8b)\]
\[(a_{n-1} - b_{m-1})(a_{n-1} - c)^{N-2}(b_{m-1} - c)^{N-2} f_{[3]} = |\hat{N} - 3, \psi(N - 2), \psi(N - 2)|_{[3]}, \quad (A.8c)\]
\[(a_n + c)^{N-2} f_{[3]} = |A_{[n+1]} A_{[n]}^{-1}| \cdot \hat{E}^1 \psi(N - 2)|_{[3]}, \quad (A.8d)\]
\[(a_{n-1} - c)^{N-2} \tilde{f}_{[3]} = -\hat{N} - 1, \psi(N - 1)|_{[3]}, \quad (A.8e)\]
\[(a_{n-1} + b_m)(b_m + c)^{N-2}(a_{n-1} - c)^{N-2} \tilde{f}_{[3]} = |A_{[m+1]} A_{[m]}^{-1}| \cdot |\hat{N} - 2, \psi(N - 1), \hat{E}^2 \psi(N - 1)|_{[3]}, \quad (A.8f)\]
\[\prod_{j=1}^{N-2} (a_{n-1} - b_{m+j}) \tilde{f}_{[2]} = -\hat{N} - 1, \psi(N - 1)|_{[2]}, \quad (A.8g)\]
\[(a_{n-1} - c) \prod_{j=0}^{N-3} (c - b_{m+j}) \prod_{j=0}^{N-3} (a_{n-1} - b_{m+j}) f_{[2]} = |\hat{N} - 3, \psi(N - 2), \psi(N - 2)|_{[2]}, \quad (A.8h)\]
\[2c \prod_{j=0}^{N-3} (c + b_{m+j}) \prod_{j=0}^{N-3} (c - b_{m+j}) f_{[2]} = |B_{[l+1]} B_{[l]}^{-1}| \cdot |\hat{N} - 3, \psi(N - 2), \hat{E}^3 \psi(N - 2)|_{[2]}, \quad (A.8i)\]
\[2c \prod_{i=1}^{N-2} (c + a_{n+i}) \prod_{i=1}^{N-2} (c - a_{n+i}) \tilde{f}_{[1]} = |B_{[l+1]} B_{[l]}^{-1}| \cdot |\hat{N} - 2, \psi(N - 1), \hat{E}^3 \psi(N - 1)|_{[1]}, \quad (A.8j)\]
\[(b_{m-1} - c) \prod_{i=1}^{N-3} (b_{m-1} - a_{n+i}) \prod_{i=1}^{N-2} (c - a_{n+i}) \tilde{f}_{[1]} = -|\hat{N} - 2, \psi(N - 1), \psi(N - 1)|_{[1]}, \quad (A.8k)\]
\[(b_{m-1} - c) \prod_{j=0}^{N-3} (c + b_{m+j}) \prod_{j=0}^{N-3} (a_{n-1} - b_{m+j}) \tilde{f}_{[2]} = |B_{[l+1]} B_{[l]}^{-1}| \cdot |\hat{N} - 3, \psi(N - 2), \hat{E}^3 \psi(N - 2)|_{[2]}, \quad (A.8l)\]
\[(b_{m-1} + c) \prod_{j=1}^{N-2} (c + b_{m+j}) \prod_{j=1}^{N-2} (a_{n-1} - b_{m+j}) \tilde{f}_{[2]} = |B_{[l+1]} B_{[l]}^{-1}| \cdot |\hat{N} - 2, \psi(N - 1), \hat{E}^3 \psi(N - 1)|_{[2]}, \quad (A.8m)\]
\[\prod_{i=1}^{N-2} (c + a_{n+i}) \prod_{i=1}^{N-2} (b_{m-1} - a_{n+i}) \tilde{f}_{[1]} = |B_{[l+1]} B_{[l]}^{-1}| \cdot |\hat{N} - 2, \psi(N - 1), \hat{E}^3 \psi(N - 1)|_{[1]}, \quad (A.8n)\]
\[(a_{n-1} - c)^{N-2} g_{[3]} + (a_{n-1} - c) f_{[3]} = -|\hat{N} - 3, \psi(N - 1), \psi(N - 2)|_{[3]}, \quad (A.9a)\]
\[(b_{m} + c)^{N-2} g_{[3]} - (b_{m} + c) f_{[3]} = |A_{[m+1]} A_{[m]}^{-1}| \cdot |\hat{N} - 3, \psi(N - 1), \hat{E}^2 \psi(N - 2)|_{[3]}, \quad (A.9b)\]
\[(a_{n-1} - c)^{N-2} h_{[3]} + (a_{n-1} - c) g_{[3]} = -|\hat{N} - 3, \psi(N), \psi(N - 2)|_{[3]}, \quad (A.9c)\]
\[(b_{m} + c)^{N-2} h_{[3]} - (b_{m} + c) g_{[3]} = |A_{[m+1]} A_{[m]}^{-1}| \cdot |\hat{N} - 3, \psi(N), \hat{E}^2 \psi(N - 2)|_{[3]}, \quad (A.9d)\]
\[(b_{m-1} - c)^{N-2} g_{[3]} + (b_{m-1} - c) f_{[3]} = -|\hat{N} - 3, \psi(N-1), \psi(N-2)|_{[3]}, \quad (A.9e)\]
\[(b_{m-1} - c)^{N-2} [(b_{m-1} - c) g_{[3]} + h_{[3]}] = -|\hat{N} - 3, \psi(N), \psi(N - 2)|_{[3]}, \quad (A.9f)\]
\[ f_{[3]} = f_{[3]} - (b_{m-1} - c)|\psi(-1)\overline{N-2}_{[3]}, \quad (A.10a) \]
\[ \overline{g}_{[3]} = f_{[3]} - (b_{m-1} - c)g_{[3]} + (b_{m-1} - c)^2|\psi(-1)\overline{N-1}_{[3]}, \quad (A.10b) \]
\[ g_{[3]} = |\psi(-1)\overline{N-2}_{[3]} - (b_{m-1} - c)|\psi(-1), \quad (A.10c) \]
\[ f_{[3]} = f_{[3]} - (a_{n-1} - c)|\psi(-1), \overline{N-2}_{[3]}, \quad (A.10d) \]
\[ \overline{g}_{[3]} = f_{[3]} - (a_{n-1} - c)g + (a_{n-1} - c)^2|\psi(-1), \overline{N-1}_{[3]}], \quad (A.10e) \]
\[ (-1)^N \overline{f}_{[3]} = |A_{[n+1]}A_{[n]}^{-1}|f_{[3]} - (a_n + c)|\hat{E}^1\psi(-1), \overline{N-2}_{[3]}, \quad (A.10f) \]
\[ (-1)^N \overline{g}_{[3]} = |A_{[n+1]}A_{[n]}^{-1}|g_{[3]} + (a_n + c)|\hat{E}^1\psi(-1), \overline{N-1}_{[3]}, \quad (A.10g) \]
\[ (-1)^N \overline{f}_{[3]} = |A_{[m+1]}A_{[m]}^{-1}|f_{[3]} - (b_m + c)|\hat{E}^2\psi(-1)\overline{N-2}_{[3]}, \quad (A.10h) \]
\[ (-1)^N \overline{g}_{[3]} = |A_{[m+1]}A_{[m]}^{-1}|g_{[3]} - (b_m + c)^2|\hat{E}^2\psi(-1), \overline{N-1}_{[3]}, \quad (A.10i) \]
\[ (-1)^N \overline{f}_{[3]} = |A_{[m+1]}A_{[m]}^{-1}|f_{[3]} - (b_m + c)|\hat{E}^2\psi(-1), \overline{N-1}_{[3]}, \quad (A.10j) \]
\[ (-1)^N \overline{g}_{[3]} = |A_{[m+1]}A_{[m]}^{-1}|g_{[3]} + (b_m + c)g - (b_m + c)^2|\hat{E}^2\psi(-1), \overline{N-1}_{[3]}, \quad (A.10k) \]
\[ (a_{n-1} - b_m)\overline{f}_{[3]} - (a_{n-1} - c)^2|\psi(-1), \overline{N-1}_{[3]} - (b_m + c)|\hat{E}^2\psi(-1), \overline{N-2}_{[3]}, \quad (A.10l) \]
\[ (a_{n-1} - b_m)\overline{g}_{[3]} - (a_{n-1} - c)^2|\psi(-1), \overline{N-1}_{[3]} + (b_m + c)^2|\hat{E}^2\psi(-1), \overline{N-2}_{[3]}, \quad (A.10m) \]
\[ (a_{n-1} - b_m)\overline{f}_{[3]} = (a_{n-1} - b_m)f_{[3]} - (a_{n-1} - c)^2|\psi(-1), \overline{N-1}_{[3]} + (b_m + c)^2|\hat{E}^2\psi(-1), \overline{N-2}_{[3]}, \quad (A.10n) \]

In fact, these formulae can be proved in a similar way as in [13][17][18]. We need to use the following relations which are derived from (A.2):

\[ (a_{n-1} - b_m)\psi = \psi - \psi, \quad (b_m - a_n)\psi = \psi - \psi, \quad (A.11a) \]
\[ (a_n + c)\psi = A_{[n+1]}A_{[n]}^{-1}\psi + \overline{\psi}, \quad (b_m - a_n)\overline{\psi} = \overline{\psi} + A_{[n+1]}A_{[n]}^{-1}\psi, \quad (A.11b) \]
\[ (c + b_m)\overline{\psi} = B_{[n+1]}B_{[n]}^{-1}\psi + \overline{\psi}, \quad (a_{n-1} + c)\overline{\psi} = \overline{\psi} + B_{[n+1]}B_{[n]}^{-1}\psi, \quad (A.11c) \]
\[ (b_m + c)\overline{\psi} = A_{[m+1]}A_{[m]}^{-1}\psi + \overline{\psi}, \quad (a_n + b_m)\overline{\psi} = \overline{\psi} + A_{[m+1]}A_{[m]}^{-1}\psi. \quad (A.11d) \]

In the following as examples we only prove (A.7a) and (A.8a). Let us prove (A.7a). For \( \overline{f}_{[2]} \), using the relation (A.11a) we first have

\[ (a_{n-1} - b_m)\overline{f}_{[2]} = |(a_{n-1} - b_m)\psi(0), \psi(1), \ldots, \psi(N-1)|_{[2]} \]
\[ = |\psi(0), \psi(1), \ldots, \psi(N-1)|_{[2]} \]
Then, for the second column,

\[(a_{n-1} - b_{m+1})(a_{n-1} - b_m)f_{[2]} = \psi(0), (a_{n-1} - b_{m+1})\psi(1), \ldots, \psi(N-1) |_{[2]}\]

\[= \psi(0), \psi(1), \ldots, \psi(N-1) |_{[2]},\]

Repeating this procedure we reach to

\[\prod_{j=0}^{N-2} (a_{n-1} - b_{m+j})f_{[2]} = \hat{N} - 2, \psi(N-1) |_{[2]},\]

i.e., (A.7c). Next, let us prove (A.8i). Based on (A.7c) and using (A.11c), we first have

\[(c + b_m) \prod_{j=0}^{N-3} (c - b_{m+j})f_{[2]} = -[(c + b_m)\psi(0), \psi(1), \ldots, \psi(N-2), \psi(N-2) |_{[2]}\]

\[= -[B_{[l+1]}B_{[l]}^{-1}\psi(0), B_{[l+1]}B_{[l]}^{-1}\psi(1), \ldots, \psi(N-2), \psi(N-2) |_{[2]}\]

For the second column, we have

\[(c + b_{m+1})(c + b_m) \prod_{j=0}^{N-3} (c - b_{m+j})f_{[2]}\]

\[= -[B_{[l+1]}B_{[l]}^{-1}\psi(0), (c + b_{m+1})\psi(1), \ldots, \psi(N-2), \psi(N-2) |_{[2]}\]

\[= -[B_{[l+1]}B_{[l]}^{-1}\psi(0), B_{[l+1]}B_{[l]}^{-1}\psi(1), \ldots, \psi(N-2), \psi(N-2) |_{[2]}\]

Repeating this procedure, we reach to

\[\prod_{j=0}^{N-3} (c + b_{m+j}) \prod_{j=0}^{N-3} (c - b_{m+j})f_{[2]}\]

\[= -[B_{[l+1]}B_{[l]}^{-1}\hat{N} - 3, B_{[l]}B_{[l+1]}^{-1}\hat{N} - 3, B_{[l]}B_{[l+1]}^{-1}\psi(N-2), \hat{E}^3\psi(N-2) |_{[2]}\]

where we have rewritten \(B_{[l]}B_{[l+1]}^{-1}\psi(N-2)\) by \(\hat{E}^3\psi(N-2) |_{[2]}\). Then, still using (A.11c) and the fact that \(B_{[l+1]}B_{[l]}^{-1}\) is independent of \(l\), we have

\[2c \prod_{j=0}^{N-3} (c + b_{m+j}) \prod_{j=0}^{N-3} (c - b_{m+j})f_{[2]}\]

\[= [B_{[l+1]}B_{[l]}^{-1}\hat{N} - 3, 2c(\hat{E}^3\psi(N-2) |_{[2]}\]

\[= [B_{[l+1]}B_{[l]}^{-1}\hat{N} - 3, B_{[l]}B_{[l+1]}^{-1}B_{[l]}B_{[l-1]}^{-1}\psi(N-2), \hat{E}^3\psi(N-2) |_{[2]}\]

\[= [B_{[l+1]}B_{[l]}^{-1}\hat{N} - 3, \psi(N-2), \hat{E}^3\psi(N-2) |_{[2]}\]

which is (A.8i).
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