Upper $k$-tuple total domination in graphs

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Abstract

Let $G = (V, E)$ be a simple graph. For any integer $k \geq 1$, a subset of $V$ is called a $k$-tuple total dominating set of $G$ if every vertex in $V$ has at least $k$ neighbors in the set. The minimum cardinality of a minimal $k$-tuple total dominating set of $G$ is called the $k$-tuple total domination number of $G$. In this paper, we introduce the concept of upper $k$-tuple total domination number as the maximum cardinality of a minimal $k$-tuple total dominating set of $G$, and study the problem of finding a minimal $k$-tuple total dominating set of maximum cardinality on several classes of graphs, as well as finding general bounds and characterizations. Also, we find some results on the upper $k$-tuple total domination number of the Cartesian and cross product graphs.

Keywords: $k$-tuple total domination number, upper $k$-tuple total domination number, Cartesian and cross product graphs, hypergraph, (upper) $k$-transversal number.

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1 Introduction

All graphs considered here are finite, undirected and simple. For standard graph theory terminology not given here we refer to [26]. Let $G = (V, E)$ be a graph with the vertex set $V$ of order $n(G)$ and the edge set $E$ of size $m(G)$. The open neighborhood of a vertex $v \in V$ is $N_G(v) = \{u \in V \mid uv \in E\}$, while its cardinality is the degree of $v$ and denoted by $deg_G(v)$. The closed neighborhood of a vertex $v \in V$ is also $N[v] = N_G(v) \cup \{v\}$. The minimum and maximum degree of $G$ are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. We write $K_n$, $C_n$ and $P_n$ for a complete graph, a cycle, and a path of order $n$, respectively, while $K_{n_1,\ldots,n_p}$ denotes a complete $p$-partite graph. Also for a subset $S \subseteq V$, $G[S]$ denotes the induced subgraph of $G$ by $S$ in which $V(G[S]) = S$ and for any two vertices $x, y \in S$, $xy \in E(G[S])$ if and only if $xy \in E(G)$.

Definition 1. Let $k \geq 1$ be an integer and let $v \in S \subseteq V$. A vertex $v'$ is called a $k$-open private neighbor of $v$ with respect to $S$, or simply a $(S, k)$-opn of $v$ if $v \in N_G(v')$ and $|N_G(v') \cap S| = k$, in other words, there exists a $k$-subset $S_v \subseteq S$ containing $v$ such that $N_G(v') \cap S = S_v$. The set

$$opn_k(v; S) = \{v' \in V \mid v' \text{ is a } (S, k)\text{-opn of } v\}$$

is called the $k$-open private neighborhood set of $v$ with respect to $S$. Also, a $k$-open private neighbor of $v$ with respect to $S$ is called external or inner if the vertex is in $V - S$ or $S$, respectively.

Hypergraphs. Hypergraphs are systems of sets which are conceived as natural extensions of graphs. A hypergraph $H = (V, E)$ is a set $V$ of elements, called vertices, together with a multiset $E$ of arbitrary subsets of $V$, called edges. For integer $k \geq 1$, a $k$-uniform hypergraph is a
hypergraph in which every edge has size $k$. Every simple graph is a 2-uniform hypergraph. For a graph $G = (V, E)$, $H_G = (V, C)$ denotes the open neighborhood hypergraph of $G$ with the vertex set $V$ and edge set $C$ consisting of the open neighborhoods of vertices of $V$ in $G$.

A transversal in a hypergraph $H = (V, E)$ is a subset $S \subseteq V$ such that $|S \cap e| \geq 1$ for every edge $e \in E$; that is, the set $S$ meets every edge in $H$. The transversal number $\tau(H)$ of $H$ is the minimum size of a transversal in $H$. In a natural way, Wanless et al. generalized the concept of transversal in a Latin square to $k$-transversal [25].

**Definition 2.** [25] For any positive integer $k$, a $k$-transversal or a $k$-plex in a Latin square of order $n$ is a set of $nk$ cells, $k$ from each row, $k$ from each column, in which every symbol occurs exactly $k$ times. The maximum number of disjoint $k$-transversals in a Latin square $L$ is called its $k$-transversal number and denoted by $\tau_k(L)$. Obviously $\tau_k(L) \leq n/k$. A Latin square $L$ has a decomposition into disjoint $k$-transversals means $\tau_k(L) = n/k$.

In a similar way, we generalize the concept of transversal in a hypergraph to $k$-transversal as following:

**Definition 3.** For any integer $k \geq 1$, a $k$-transversal in a hypergraph $H = (V, E)$ is a subset $S \subseteq V$ such that $|S \cap e| \geq k$ for every edge $e \in E$; that is, every edge in $H$ contains at least $k$ vertices from the set $S$. The $k$-transversal number $\tau_k(H)$ of $H$ is the minimum cardinality of a minimal $k$-transversal in $H$, while the upper $k$-transversal number $\Upsilon_k(H)$ of $H$ is defined as the maximum cardinality of a minimal $k$-transversal in $H$.

**Domination.** Domination in graphs is now well studied in graph theory and the literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [11, 12]. A set $S \subseteq V$ is a dominating set (resp. total dominating set) of $G$ if each vertex in $V \setminus S$ (resp. $V$) is adjacent to at least one vertex of $S$. The domination number $\gamma(G)$ (resp. total domination number $\gamma_t(G)$) of $G$ is the minimum cardinality of a dominating set (resp. total dominating set) of $G$. An extension of total domination number introduced by Henning and Kazemi in [13] (the reader can study [14, 19–21] for more information).

**Definition 4.** [13] Let $k \geq 1$ be an integer and let $G$ be a graph with $\delta(G) \geq k$. A subset $S \subseteq V$ is called a $k$-tuple total dominating set, briefly $k$TDS, of $G$ if for each $x \in V$, $|N(x) \cap S| \geq k$. The minimum number of vertices of a $k$TDS of $G$ is called the $k$-tuple total domination number of $G$ and denoted by $\gamma_{x,k,t}(G)$. A $k$TDS with cardinality $\gamma_{x,k,t}(G)$ is called a min-TDS of $G$.

Finding the maximum cardinality of the set of minimal subsets of the vertices (or edges or both) of a graph with a property is one of the important problems in graph theory. According to this fact, in this paper, we initiate the study of the problem of finding a minimal $k$-tuple total dominating set of maximum cardinality in a graph. This leads to our next definition.

**Definition 5.** The upper $k$-tuple total domination number $\Gamma_{x,k,t}(G)$ of $G$ is the maximum cardinality of a minimal $k$TDS of $G$, and a minimal $k$TDS with cardinality $\Gamma_{x,k,t}(G)$ is a $\Gamma_{x,k,t}(G)$-set, or a $\Gamma_{x,k,t}$-set of $G$. Also, we say that a graph $G$ is a $\Gamma_{x,k,t}$-external graph if it has a $\Gamma_{x,k,t}$-set $S$ such that every vertex in it has an external $k$-open private neighbor with respect to $S$.

Obviously, for every graph $G$ and every positive integer $k$, $\gamma_{x,k,t}(G) \leq \Gamma_{x,k,t}(G)$, and this bound is sharp by $\gamma_{x,k,t}(K_n) = \Gamma_{x,k,t}(K_n) = k + 1$ when $1 \leq k < n$. We remark that the upper 1-tuple total domination number $\Gamma_{x,1,t}(G)$ is the well-studied upper total domination number $\Gamma_t(G)$, while the upper 2-tuple total domination number is known as the upper double total domination number. The redundancy involved in upper $k$-tuple total domination makes it useful in many applications.

In this paper, as we said before, we initiate the study of the problem of finding a minimal $k$-tuple total dominating set of maximum cardinality on several classes of graphs, as well as finding general bounds and characterizations. Also we present a Vizing-like conjecture on the upper $k$-tuple total domination number, and prove it for a family of graphs. Proving

$$\Gamma_{x,k,t}(G \times H) \geq \Gamma_{x,k,t}(G) \cdot \Gamma_{x,t,\ell}(H) \quad (\text{for any } k, \ell \geq 1)$$

is our next work in which $G \times H$ denotes the cross product of two graphs $G$ and $H$. Then we characterize graphs $G$ satisfying $\Gamma_{x,k,t}(G) = \gamma_{x,k,t}(G)$, and show that for any graph $G$ with
minimum degree at least \( k \),
1. \( \Gamma_{x,k,t}(G) = \Upsilon_k(H_G) \), and
2. \( \Gamma_{x,k,t}(G) = \gamma_{x,k,t}(G) \) if and only if \( \Upsilon_k(H_G) = \tau_k(H_G) \).

We begin our discussion with the following useful observation.

**Observation 1.** Let \( k \geq 1 \) be an integer and let \( G \) be a graph of order \( n \) with \( \delta(G) \geq k \). Then
i. \( \gamma_{x,k,t}(G) \leq \Gamma_{x,k,t}(G) \leq n \),
ii. every \( k \)\( \text{TDS} \) \( S \) of \( G \) is minimal if and only if \( \text{opt}_k(v;S) \neq \emptyset \) for every vertex \( v \in S \),
iii. all neighbors of every vertex of degree \( k \) in \( G \) belong to every \( k \)\( \text{TDS} \) of \( G \), and
iv. if \( H \) is a spanning subgraph of \( G \) which has a \( \Gamma_{x,k,t} \)-set that is also a minimal \( k \)\( \text{TDS} \) of \( G \), then \( \Gamma_{x,k,t}(H) \leq \Gamma_{x,k,t}(G) \).

Observation 1 (iii) implies the next proposition.

**Proposition 1.** For any \( k \)-regular graph \( G \), \( \Gamma_{x,k,t}(G) = n \).

The converse of Proposition 1 does not hold. For example, if \( G \) is the graph obtained by the union of two disjoint complete graphs of order \( k+1 \geq 3 \), with an edge between them, then \( G \) is not regular but \( \Gamma_{x,k,t}(G) = 2k+2 \). The next two propositions are useful for our investigations.

We recall that for any positive integer \( k \), the \( k \)-join \( G \circ_k H \) of a graph \( G \) to a graph \( H \) with \( \delta(H) \geq k \) is the graph obtained from the disjoint union of \( G \) and \( H \) by joining each vertex of \( G \) to at least \( k \) vertices of \( H \).

**Proposition 2.** [9] For any path \( P_n \) of order \( n \geq 2 \),
\[
\Gamma_t(P_n) = 2\lfloor (n+1)/3 \rfloor.
\]

**Proposition 3.** [13] Let \( G \) be a graph with \( \delta(G) \geq k \). Then \( \gamma_{x,k,t}(G) = k+1 \) if and only if \( G = K_{k+1} \) or \( G = F \circ_k K_{k+1} \) for some graph \( F \).

## 2 Cycles and complete multipartite graphs

In this section, we calculate the upper \( k \)-tuple total domination number of a cycle and a complete multipartite graph. Proposition 1 implies \( \Gamma_{x,2,1}(C_n) = n \). The next proposition calculates \( \Gamma_t(C_n) \).

**Proposition 4.** For any cycle \( C_n \) of order \( n \geq 3 \),
\[
\Gamma_t(C_n) = \begin{cases} 2\lfloor n/3 \rfloor + 1 & \text{if } n \equiv 2 \pmod{3}, \\
2\lfloor n/4 \rfloor & \text{otherwise}. \end{cases}
\]

**Proof.** Let \( V(C_n) = \{1, 2, \ldots, n\} \), and let \( ij \in E(C_n) \) if and only if \( j \equiv i+1 \pmod{n} \). Let \( S \) be a \( \Gamma_t(C_n) \)-set. If at least one vertex of any two consecutive vertices belongs to \( S \), then \( n \equiv 0 \pmod{3} \). Since, otherwise, \( S \) will contain at least three consecutive vertices of \( V(C_n) \), which contracts the minimality of \( S \). Hence \( |S| = 2\lfloor n/3 \rfloor \), when \( n \equiv 0 \pmod{3} \). Now, assume there exist two consecutive vertices, say \( ij \) and \( (i+1) \), out of \( S \). Then \( S \) is also a minimal \( k \)\( \text{TDS} \) in the path \( P_n = C_n - \{e\} \) in which \( e = 1n \in E(C_n) \). This implies
\[
|S| = \Gamma_t(C_n) \leq \Gamma_t(P_n) = 2\lfloor (n+1)/3 \rfloor \quad (\text{by Proposition 2}).
\]

Now since \( \{3i+1, 3i+2\} \leq 0 \leq i \leq \lfloor n/3 \rfloor \) is a minimal \( k \)\( \text{TDS} \) in \( C_n \) with cardinality \( \Gamma_t(P_n) \) when \( n \not\equiv 2 \pmod{3} \), we obtain \( \Gamma_t(C_n) = 2\lfloor n/3 \rfloor + 1 = 2\lfloor n/4 \rfloor \). Now let \( n \equiv 2 \pmod{3} \) and let \( S \) be a minimal \( k \)\( \text{TDS} \) of \( C_n \) with cardinality \( \Gamma_t(P_n) \). Then there exist seven consecutive vertices, say \( 1, 2, \ldots, 7 \), such that \( S \cap \{1, 2, \ldots, 7\} = \{i\} \) which \( i = 2 \) or \( 4 \). Since \( S - \{i+1\} \) is a \( k \)\( \text{TDS} \) of \( C_n \), we obtain \( |S| < \Gamma_t(P_n) \), and so \( \Gamma_t(C_n) \leq \Gamma_t(P_n) - 1 \). Now since \( \{3i+1, 3i+2\} \leq 0 \leq i \leq \lfloor n/3 \rfloor \) is a minimal \( k \)\( \text{TDS} \) of \( C_n \) with cardinality \( \Gamma_t(P_n) - 1 = 2\lfloor n/3 \rfloor + 1 \), we obtain \( \Gamma_t(C_n) = 2\lfloor n/4 \rfloor + 1 \).

**Theorem 1.** Let \( G = K_{n_1,n_2,\ldots,n_p} \) be a complete \( p \)-partite graph with \( \delta(G) \geq k \geq 1 \) in which \( n_1 \leq n_2 \leq \ldots \leq n_p \). Then
\[
\Gamma_{x,k,t}(G) = k + \max\{x \mid (\ell-1)x = k \text{ and } x \leq \min\{k, n_{p-\ell+1}, \ldots, n_p\} \}.\]
Proof. Let $S$ be a minimal $k$TDS of $G = K_{n_1,n_2,...,n_p}$ and let $V = X_1 \cup X_2 \cup ... \cup X_p$ be the partition of the vertex set of $G$ to the $p$ independent sets $X_1, X_2, \cdots, X_p$ in which $|X_i| = n_i$ for each $i$ and $n_1 \leq n_2 \leq ... \leq n_p$. Let $I = \{i_j | j = 1, ..., \ell\}$ be an index subset of $\{1, 2, ..., p\}$ for some $2 \leq \ell \leq p$ such that $S \cap X_{i_j} \neq \emptyset$ if and only if $i_j \in I$. Also assume $S \cap X_{i_j} = x_{i_j}$ for each $i_j \in I$, and $x_{i_1} \leq x_{i_2} \leq ... \leq x_{i_\ell}$. The minimality of $S$ implies $x_{i_j} \leq k$ for each $i_j \in I$, and there exists a $(\ell - 1)$-subset $L \subseteq I$ such that $\sum_{i_j \in L} x_{i_j} = k$. Then, by the minimality of $S$, $\sum_{i_j \in L} x_{i_j} = k$ for every $(\ell - 1)$-subset $L \subseteq I$, and so $x_{i_1} = x_{i_2} = ... = x_{i_\ell}$. Let $x := x_{i_1} = x_{i_2} = ... = x_{i_\ell}$. Then $x_{i_1} + x_{i_2} + ... + x_{i_\ell} = \ell x = k + \ell - 1 \leq \Gamma_{x,k,t}(G)$ where $x \leq \min\{k, n_{i_1}, ..., n_{i_\ell}\}$, and so

$$\Gamma_{x,k,t}(G) = \begin{cases} k + \max\{x | (\ell - 1)x = k \text{ and } x \leq \min\{k, n_{i_1}, ..., n_{i_\ell}\}\} \\ k + \max\{x | (\ell - 1)x = k \text{ and } x \leq \min\{k, n_{p-\ell+1}, ..., n_p\}\}\end{cases}.$$ 

\[\square\]

Corollary 1. Let $G = K_{n_1,n_2,...,n_p}$ be a complete $p$-partite graph. For any integer $k \geq 1$ if $|\{ i | n_i \geq k\}| \geq 2$, then $\Gamma_{x,k,t}(G) = 2k$.

In a similar way, the next theorem can be proved.

Theorem 2. Let $G = K_{n_1,n_2,...,n_p}$ be a complete $p$-partite graph with $\delta(G) \geq k \geq 1$ in which $n_1 \leq n_2 \leq ... \leq n_p$. Then

$$\gamma_{x,k,t}(G) \leq k + \min\{x|(\ell - 1)x = k \text{ and } x \leq \min\{k, n_1, ..., n_\ell\}\}.$$ 

3 Two upper bounds

In this section, we present two upper bounds for the upper $k$-tuple total domination number of a graph. The first is in terms of $k$, the order and the minimum degree of the graph, and the second is in terms of the upper $k$-tuple total domination number of the graph for some $\ell < k$.

Theorem 3. If $G$ is a graph of order $n$ with $\delta \geq k + 1 \geq 2$, then $\Gamma_{x,k,t}(G) \leq n - \delta + k$, and this bound is sharp.

Proof. Let $G$ be a graph of order $n$ with $\delta \geq k + 1 \geq 2$ and let $S$ be a $\Gamma_{x,k,t}(G)$-set. Then for every $v \in S$ there exist a $k$-subset $S_v \subseteq S$ and a vertex $v' \in V(G)$ such that $N_G(v') \cap S = S_v$ by Observation 1 (ii). If $v' \in S$, then $N_G(v') - S_v \subseteq V(G) - S$, and so

$$\delta - k \leq deg(v') - k \leq n - |S| = n - \Gamma_{x,k,t}(G),$$

which implies $\Gamma_{x,k,t}(G) \leq n - \delta + k$. If $v' \not\in S$, then $v'$ is not adjacent to at least $|S| - k$ vertices of $S - S_v$, and so

$$\delta \leq deg(v') \leq n - |S| + k - 1 = n - \Gamma_{x,k,t}(G) + k - 1,$$

which implies $\Gamma_{x,k,t}(G) < n - \delta + k$.

The sharpness of this bound can be seen as following: Let $\delta \geq k + 1 \geq 2$ be integers. Consider $b$ vertex-disjoint complete graphs $K_{k+1}$ where $b \geq \left\lceil \frac{k}{\delta} \right\rceil$, and let $H_b = K_{k+1} + ... + K_{k+1}$ be the union of $b$ vertex-disjoint complete graphs $K_{k+1}$. Also consider an empty graph $T$ with $\delta - k$ vertices. Let $G_b = H_b \lor T$ be the join of $H_b$ and $T$, which is the union of $H_b$ and $T$ such that every vertex of $H_b$ is adjacent to all vertices in $T$. Then $G_b$ is a connected graph of order $n = b(k + 1) + \delta - k$ with minimum degree $\delta$. Since $V(H_b)$ is a minimal $k$TDS of $G_b$, we obtain $\Gamma_{x,k,t}(G_b) \geq n - \delta + k$, and consequently $\Gamma_{x,k,t}(G_b) = n - \delta + k$. \[\square\]

Theorem 4. Let $G$ be a graph with $\delta \geq k \geq 1$. Let $L = \cap_{v \in S_v} S_v$ be a set of cardinality $\ell$ in which $S$ is a $\Gamma_{x,k,t}(G)$-set and $S_v$ is the set given in Definition 1. If $\ell < k$, then

$$\Gamma_{x,k,t}(G) \leq \Gamma_{x,(k-\ell),t}(G) + \ell.$$ 

Proof. Let $S$ be a $\Gamma_{x,k,t}(G)$-set and let $L = \cap_{v \in S_v} S_v$ be a set of cardinality $\ell$ in which $S_v$ is the set given in Definition 1 and $\ell < k$. Since $S - L$ is a minimal $(k - \ell)$TDS of $G$, we obtain

$$\Gamma_{x,(k-\ell),t}(G) \geq |S - L| = |S| - \ell = \Gamma_{x,k,t}(G) - \ell.\[\square\]$$
4 Cartesian product and a Vizing-like conjecture

The Cartesian product $G \square H$ of two graphs $G$ and $H$ is a graph with the vertex set $V(G) \times V(H)$ and two vertices $(g_1, h_1)$ and $(g_2, h_2)$ are adjacent if and only if either $g_1 = g_2$ and $(h_1, h_2) \in E(H)$, or $h_1 = h_2$ and $(g_1, g_2) \in E(G)$. For more information on product graphs see [23]. The Cartesian product $K_n \square K_m$ is known as the $n \times m$ rook’s graph, as edges represent possible moves by a rook on an $n \times m$ chess board. For example see Figure 1.

![Figure 1: The $3 \times 4$ rook’s graph, i.e., $K_3 \square K_4$.](image)

Now for integers $n \geq m \geq k+1 \geq 3$ we consider the $n \times m$ rook’s graph $K_n \square K_m$ with the vertex set $V = \{(i, j) \mid 1 \leq i \leq n, 1 \leq j \leq m\}$. Since the set $\{(i, j) \mid 1 \leq i \leq n, 1 \leq j \leq k\}$ is a minimal $k$TDS of $K_n \square K_m$, we have the following proposition.

**Proposition 5.** For any integers $n \geq m \geq k+1 \geq 3$, $\Gamma_{\times,k,t}(K_n \square K_m) \geq kn$.

As we will see in Proposition 6 for $n = m = k+1 \geq 3$, we guess equality holds in Proposition 5.

**Proposition 6.** For any integer $k \geq 2$, $\Gamma_{\times,k,t}(K_{k+1} \square K_{k+1}) = k(k+1)$.

**Proof.** Let $V(K_{k+1} \square K_{k+1}) = \{(i, j) \mid 1 \leq i, j \leq k+1\}$ in which $k \geq 2$. We know $\Gamma_{\times,k,t}(K_{k+1} \square K_{k+1}) \geq k(k+1)$ by Proposition 5. Now let

$$S = \bigcup_{1 \leq i \leq k+1} S_i^r = \bigcup_{1 \leq j \leq k+1} S_j^c$$

be a minimal $k$TDS of $K_{k+1} \square K_{k+1}$ with cardinality more than $k(k+1)$ in which

$$S_i^r = S \cap \{(i, j) \mid 1 \leq j \leq k+1\},$$

$$S_j^c = S \cap \{(i, j) \mid 1 \leq i \leq k+1\}.$$

Then $|S_i^r| \geq k$ and $|S_j^c| \geq k$ for each $i$ and each $j$, and also $|S_i^r| = k+1$ and $|S_j^c| = k+1$ for some $t$ and some $\ell$. Now since $S - \{(t, \ell)\}$ is a $k$TDS of $K_{k+1} \square K_{k+1}$ which contradicts the minimality of $S$, we obtain $\Gamma_{\times,k,t}(K_{k+1} \square K_{k+1}) = k(k+1)$. See Figure 2 for an example.

![Figure 2: The dark vertices highlight a minimal 3TDS of $K_4 \square K_4$ with maximum cardinality.](image)

In 1963, more formally in 1968, Vizing [24] made an elegant conjecture that has subsequently become one the most famous open problems in domination theory.
Conjecture 1 (Vizing’s Conjecture). For any graphs $G$ and $H$,

$$\gamma(G) \cdot \gamma(H) \leq \gamma(G \square H).$$

Over more than fifty years (see [1] and references therein), Vizing’s Conjecture has been shown to hold for certain restricted classes of graphs, and furthermore, upper and lower bounds on the inequality have gradually tightened. Additionally, research has explored inequalities (including Vizing-like inequalities) for different forms of domination [12]. A significant breakthrough occurred in 2000, when Clark and Suen [7] proved that

$$\gamma(G) \cdot \gamma(H) \leq 2\gamma(G \square H)$$

which led to the discovery of a Vizing-like inequality for total domination [15, 16], i.e.,

$$\gamma_t(G) \cdot \gamma_t(H) \leq 2\gamma_t(G \square H),$$

as well as for paired [4, 5, 17], and fractional domination [8], and the $\{k\}$-domination function (integer domination) [3, 6, 18], and total $\{k\}$-domination function [18]. In 1996, Nowakowski and Rall in [22] made the following Vizing-like conjecture for the upper domination of Cartesian products of graphs.

Conjecture 2 (Nowakowski-Rall’s Conjecture). For any graphs $G$ and $H$,

$$\Gamma(G) \cdot \Gamma(H) \leq \Gamma(G \square H).$$

A beautiful proof of the Nowakowski-Rall’s Conjecture was recently found by Brešar [2]. Also Paul Dorbec et al. in [10] proved that for any graphs $G$ and $H$ with no isolated vertices,

$$\Gamma_t(G) \cdot \Gamma_t(H) \leq 2\Gamma_t(G \square H),$$

We guess (2) can be extended as follows:

Conjecture 3. (Vizing-like conjecture for upper $k$-tuple total domination)
For any integer $k \geq 2$ and any graphs $G$ and $H$ with minimum degrees at least $k$,

$$\Gamma_{x,k,t}(G) \cdot \Gamma_{x,k,t}(H) \leq \frac{k+1}{k} \cdot \Gamma_{x,k,t}(G \square H).$$

Let $G_1, G_2, \ldots, G_n$ and $H_1, H_2, \ldots, H_m$ be respectively the all connected components of two graphs $G$ and $H$ which have minimum degrees at least $k \geq 2$. Then $G \square H$ is a disconnected graph with the connected components $G_i \square H_j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. By the truth of Conjecture 3 for connected graphs, since

$$\Gamma_{x,k,t}(G \square H) = \sum_{i=1}^{n} \sum_{j=1}^{m} \Gamma_{x,k,t}(G_i \square H_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{k}{k+1} \cdot \Gamma_{x,k,t}(G_i) \cdot \Gamma_{x,k,t}(H_j) = \frac{k}{k+1} \cdot \Gamma_{x,k,t}(G) \cdot \Gamma_{x,k,t}(H)$$

we may conclude that Conjecture 3 is true for disconnected graphs. Proposition 6 shows the bound in Conjecture 3, if true, is best possible. Theorem 6, which is obtained by Theorem 5, shows that Conjecture 3 is true for a family of graphs.

Theorem 5. For any two $\Gamma_{x,k,t}$-external graphs $G$ and $H$ with minimum degree at least $k \geq 2$,

$$\Gamma_{x,k,t}(G \square H) \geq \max\{\Gamma_{x,k,t}(G) \cdot |V(H)|, \Gamma_{x,k,t}(H) \cdot |V(G)|\}.$$

Proof. Let $G$ and $H$ be two $\Gamma_{x,k,t}$-external graphs with minimum degree at least $k \geq 2$, and let $\Gamma_{x,k,t}(G) \cdot |V(H)| = \max\{\Gamma_{x,k,t}(G) \cdot |V(H)|, \Gamma_{x,k,t}(H) \cdot |V(G)|\}$. Assume $D_G$ is a $\Gamma_{x,k,t}(G)$-set in which every vertex of it has an external $(D_G, k)$-opn. Obviously, $D = D_G \times V(H)$ is a $k$TDS of $G \square H$. To show that $D$ is minimal, it is sufficient to prove

$$\text{opm}_k((v, w); D) \neq \emptyset$$

for any vertex $(v, w) \in D$. 

Let \( v' \in \text{opn}_k(v; D_G) \cap (V(G) - D_G) \). Then \( N_G(v') \cap D_G = \{v, v_1, v_2, ..., v_{k-1}\} \) for some vertices \( v_1, v_2, ..., v_{k-1} \in D_G \), and so
\[
N_G \Box_H ((v', w)) \cap D = \left((N_G(v') \times \{w\}) \cup \{v'\} \times N_H(w)\right) \cap D
= \left((N_G(v') \cap D_G) \times \{w\} \cup (\emptyset \times N_H(w)\right)
= \{(v, w), (v_1, w), (v_2, w), ..., (v_{k-1}, w)\},
\]
which implies \( \text{opn}_k((v, w); D) \neq \emptyset \) for every vertex \((v, w) \in D\). Hence
\[
\Gamma_{x,k,t}(G \Box_H H) \geq |D|\geq \Gamma_{x,k,t}(G) \cdot |V(H)|
= \max\{\Gamma_{x,k,t}(G) \cdot |V(H)|, \Gamma_{x,k,t}(H) \cdot |V(G)|\}.
\]

**Theorem 6.** Let \( G \) be a \( \Gamma_{x,k,t} \)-external graph with \( \delta(G) \geq k \geq 2 \). Then for any graph \( H \) with \( \delta(H) \geq k \),
\[
\Gamma_{x,k,t}(G \Box H) \geq \Gamma_{x,k,t}(G) \cdot \Gamma_{x,k,t}(H).
\]

The proof of Theorem 5 with Proposition 1 and Theorem 3 imply next theorem.

**Theorem 7.** Let \( G \) be a \( \Gamma_{x,k,t} \)-external graph, and let \( H \) be an arbitrary graph. Then the following statements hold.

i. If \( \delta(H) \geq k + 1 \), then \( \Gamma_{x,k,t}(G \Box H) \geq \Gamma_{x,k,t}(G)(\Gamma_{x,k,t}(H) + \delta(H) - k) \).

ii. If \( H \) is \( k \)-regular, then \( \Gamma_{x,k,t}(G \Box H) \geq \Gamma_{x,k,t}(G) \cdot \Gamma_{x,k,t}(H) \).

iii. If \( H \) is not \( k \)-regular and \( \delta(H) = k \), then \( \Gamma_{x,k,t}(G \Box H) \geq \Gamma_{x,k,t}(G)(\Gamma_{x,k,t}(H) + 1) \).

## 5 Cross product graphs

In this section, we study the upper \( k \)-tuple total domination number of the cross product of two graphs. First we recall that the cross product (also known as the direct product, tensor product, categorical product, and conjunction in the literature) \( G \times H \) has \( V(G) \times V(H) \) as vertex set and two vertices \((g_1, h_1)\) and \((g_2, h_2)\) are adjacent if and only if \((g_1, g_2) \in E(G)\) and \((h_1, h_2) \in E(H)\). For example see Figure 3.

![Figure 3: The \( K_3 \times K_3 \).](image)

**Theorem 8.** If \( G \) and \( H \) are graphs satisfying \( \delta(G) \geq k \geq 1 \) and \( \delta(H) \geq \ell \geq 1 \), then
\[
\Gamma_{x,k,\ell}(G \times H) \geq \Gamma_{x,k,\ell}(G) \cdot \Gamma_{x,\ell,\ell}(H).
\]

**Proof.** Let \( D_G \) and \( D_H \) be two \( \Gamma_{x,k,\ell} \)-sets of \( G \) and \( H \), respectively. For a vertex \((u, v) \in V(G \times H)\), let \( D_{G,u} = D_G \cap N_G(u) \) and \( D_{H,v} = D_H \cap N_H(v) \). Since \( D_G \) is a \( k \)TDS of \( G \) and \( D_H \) is an \( \ell \)TDS of \( H \), we have \( |D_{G,u}| \geq k \) and \( |D_{H,v}| \geq \ell \), and so \( |D_{G,u} \times D_{H,v}| \geq k \ell \). Now by knowing
\[
D_{G,u} \times D_{H,v} \subseteq N_G(u) \times N_H(v)
= N_{G \times H}((u, v)),
\]
we conclude the Cartesian product \( D_G \times D_H \) of \( D_G \) and \( D_H \) is a \( k \ell \)TDS of \( G \times H \). To prove the minimality of \( D_G \times D_H \) let \((a, b) \in D_G \times D_H \). Then \( a \in D_G \) and \( b \in D_H \) and the minimality of \( D_G \) and \( D_H \) imply
\[
N_G(a') \cap D_G = S_a \text{ for some vertex } a' \in V(G) \text{ and some } k \text{-subset } S_a \subseteq D_G, \text{ and}
\]
$N_H(b') \cap D_H = S_b$ for some vertex $b' \in V(H)$ and some $\ell$-subset $S_b \subseteq D_H$, and so

$$N_{G \times H}((a', b')) \cap (D_G \times D_H) = S_a \times S_b$$

for the vertex $(a', b') \in V(G \times H)$ and the $k\ell$-subset $S_a \times S_b$. Hence $D_G \times D_H$ is a minimal $k\ell$-TDS of $G \times H$, and so

$$\Gamma_{x,k\ell,t}(G \times H) \geq |D_G \times D_H| = |D_G| \times |D_H| = \Gamma_{x,k\ell,t}(G) \cdot \Gamma_{x,k\ell,t}(H).$$

\[\square\]

**Corollary 2.** If $G$ and $H$ are graphs satisfying $\delta(G) \geq \delta(H) \geq k \geq 1$, then

$$\Gamma_{x,k\ell,t}(G \times H) \geq \max\{\Gamma_{x,k\ell,t}(G) \cdot \Gamma_t(H), \Gamma_{x,k\ell,t}(H) \cdot \Gamma_t(G)\}.$$

Next proposition shows that the bound given in Theorem 8 is tight.

**Proposition 7.** For any integers $1 \leq k \leq n - 1$, $\Gamma_{x,k\ell,t}(K_n \times K_2) = 2k + 2$.

**Proof.** For integers $1 \leq k \leq n - 1$ let $K_n \times K_2$ be the cross product of $K_n$ and $K_2$ with $V(K_n \times K_2) = V_1 \cup V_2$ in which $V_i = \{1, 2, \ldots, n\} \times \{i\}$ for $i = 1, 2$. For a minimal $k\ell$-TDS $S$ of $K_n \times K_2$ with maximum cardinality, let $S_i = S \cap V_i$ for $i = 1, 2$, and $|S_1| \geq |S_2|$. Obviously $|S_i| \geq k$ for each $i$, and the minimality of $S$ implies $|S_2| \leq k + 1$. Furthermore since $S$ has maximum cardinality, $|S_2| = k + 1$. If $|S_1| > k + 1$, then for any vertex $v \in S_1 - S_2$ the set $S - \{v\}$ is a $k\ell$-TDS of $K_n \times K_2$ in which $S_2' = \{(a, 1) | (a, 2) \in S_2\}$, a contradiction. Hence $|S_1| = |S_2| = k + 1$, and so $\Gamma_{x,k\ell,t}(K_n \times K_2) \leq 2k + 2$. Now equality can be obtained by Corollary 2. Figure 4 shows a minimal 2TDS of $K_4 \times K_2$ with maximum cardinality.

![Figure 4: The dark vertices highlight a minimal 2TDS of $K_4 \times K_2$ with maximum cardinality.](image)

As a natural question we may ask the next question.

**Question 1.** For any integers $n, m \geq 2$ such that $\max\{n, m\} \geq k + 1$, whether

$$\Gamma_{x,k\ell,t}(K_n \times K_m) = 2k + 2?$$

Now we present a lower bound for the upper $k$-tuple total domination number of the cross product of two complete multipartite graphs.

**Proposition 8.** Let $G \times H$ be the cross product of two complete multipartite graphs $G = K_{t_1, t_2, \ldots, t_m}$ and $H = K_{s_1, s_2, \ldots, s_n}$ with $\delta(G \times H) \geq k$. If

$$\sum_{1 \leq \ell \leq n} t_\ell s_\ell \geq \sum_{1 \leq \ell \leq n} t_j s_\ell \geq 2k \text{ for some } 1 \leq i \neq j \leq m, \text{ or}$$

$$\sum_{1 \leq \ell \leq m} s_\ell t_\ell \geq \sum_{1 \leq \ell \leq m} s_r t_\ell \geq 2k \text{ for some } 1 \leq \ell \neq r \leq m,$$

then $\Gamma_{x,k\ell,t}(G \times H) \geq 4k$.

**Proof.** Let $G = K_{t_1, t_2, \ldots, t_m}$ be a complete $m$-partite graph which has the partition $V(G) = X_1 \cup X_2 \cup \ldots \cup X_m$ to the disjoint independent sets $X_1, X_2, \ldots, X_m$ in which $|X_i| = t_i$ for each $i$. Similarly, let $H = K_{s_1, s_2, \ldots, s_n}$ be a complete $n$-partite graph which has the partition $V(H) = Y_1 \cup Y_2 \cup \ldots \cup Y_n$ to the disjoint independent sets $Y_1, Y_2, \ldots, Y_n$ in which $|Y_i| = s_i$ for each $i$. If $\Gamma_{x,k\ell,t}(G \times H) < 4k$, then there exists a $k\ell$-TDS $D_G \times D_H$ of $G \times H$ that does not contain any of the vertices in $S_1 \times S_2$.

As a natural question we may ask the next question.

**Question 1.** For any integers $n, m \geq 2$ such that $\max\{n, m\} \geq k + 1$, whether

$$\Gamma_{x,k\ell,t}(K_n \times K_m) = 2k + 2?$$

Now we present a lower bound for the upper $k$-tuple total domination number of the cross product of two complete multipartite graphs.
each $i$. Then $V(G \times H) = \bigcup_{1 \leq i \leq m, 1 \leq j \leq n} (X_i \times Y_j)$ is the partition of the vertex set of $G \times H$ to the independent sets $X_i \times Y_j$. Without loss of generality, we may assume $m \geq n \geq 2$ and

$$\sum_{1 \leq i \leq n} t_{1st} \geq \sum_{1 \leq i \leq n} t_{2st} \geq 2k.$$ 

For $1 \leq i \leq r$, let $k_i = \min\{t_{1si}, t_{2si}, t_{1si+r}, t_{2si+r}\}$ be a positive integer such that $k = k_1 + \cdots + k_r$. Now we choose a subset $S$ of $V(G \times H)$ such that $|S \cap (X_i \times Y_j)| = k_i$ for each $i$. It can be easily seen that $S$ is a minimal $k$-TDS of $G \times H$, and so $\Gamma_{x,k,t}(G \times H) \geq 4k$.

We think that the finding some complete multipartite graphs $G$ and $H$ with $\Gamma_{x,k,t}(G \times H) = 4k$ is a good problem to work.

6 Upper $k$-transversal in hypergraphs

In this section, we show that the problem of finding upper $k$-tuple total dominating sets in graphs can be translated to the problem of finding upper $k$-transversal in hypergraphs. We recall that $H_G$ denotes the open neighborhood hypergraph of a graph $G$.

**Theorem 9.** If $G$ is a graph with $\delta(G) \geq k \geq 1$, then $\Gamma_{x,k,t}(G) = \Upsilon_k(H_G)$.

**Proof.** Since every $k$TDS of $G$ contains at least $k$ vertices from the open neighborhood of each vertex in $G$, we conclude every $k$TDS of $G$ is a $k$-transversal in $H_G$. On the other hand, we know every $k$-transversal of $H_G$ contains at least $k$ vertices from the open neighborhood of each vertex of $G$, and so is a $k$TDS of $G$. This shows that we have proved that a vertex subset $S$ is a $k$TDS of $G$ if and only if it is a $k$-transversal in $H_G$, and so $\Gamma_{x,k,t}(G) = \Upsilon_k(H_G)$.

The authors in [13] proved the problem of finding $k$-tuple total dominating sets in graphs can be translated to the problem of finding $k$-transversal in hypergraphs, that is, for every integer $k \geq 1$ and every graph $G$ with minimum degree $k$, $\gamma_{x,k,t}(G) = \tau_k(H_G)$. This fact and Theorem 9 imply the next theorem.

**Theorem 10.** For any graph $G$ with $\delta(G) \geq k \geq 1$,

$$\Gamma_{x,k,t}(G) = \gamma_{x,k,t}(G) \text{ if and only if } \Upsilon_k(H_G) = \tau_k(H_G).$$

As we saw before, Proposition 3 characterize graphs $G$ satisfying $\gamma_{x,k,t}(G) = k + 1$. The next theorem characterizes graphs $G$ satisfying $\gamma_{x,k,t}(G) = m$ for each $m \geq k + 2 \geq 3$. We note that in the next three theorems, $K_m'$ denotes a simple graph of order $m$ which has minimum degree at least $k$.

**Theorem 11.** Let $G$ be a graph with $\delta(G) \geq k \geq 1$, and let $m \geq k + 2$ be an integer. Then $\gamma_{x,k,t}(G) = m$ if and only if $G = K_m'$ or $G = F \circ_k K_m'$ in which $m$ is minimum in

$$T = \{t \mid G = F' \circ_k K'_t \text{ for some graphs } F' \text{ and } K'_t\},$$

and $F = G - K_m'$.

**Proof.** Let $G$ be a graph with $\delta(G) \geq k \geq 1$, and let $S$ be a min-$k$TDS of $G = (V, E)$ with cardinality $m \geq k + 2$. Then $G[S] = K_m'$ for some graph $K_m'$ (because every vertex has at least $k$ neighbors in $S$). If $|V| = m$, then $G = K_m'$. Otherwise, let $F = G[V - S]$. Since every vertex in $V - S$ has at least $k$ neighbors in $S$, we conclude $G = F \circ_k K_m'$, and by the definition of the $k$-tuple total domination number, $m$ is minimum in $T$.

Conversely, let $G = K_m'$ or $G = F \circ_k K_m'$, in which $m$ is minimum in $T$, and let $F = G - K_m'$. Then $\gamma_{x,k,t}(G) = m$ because $V(K_m')$ is a $k$TDS with cardinality $m$. Now if $\gamma_{x,k,t}(G) = m'$ for some $m' < m$, then, by the previous discussion, $G = F' \circ_k K_{m'}'$ for some graph $F'$ and some graph $K_{m'}'$, which contradicts the minimality of $m$. This implies $\gamma_{x,k,t}(G) = m$.

Proposition 3 and Theorem 11 imply the next theorem.
Theorem 12. For any graph $G$ with $\delta(G) \geq k \geq 1$, $\Gamma_{k.t}(G) = \gamma_{k.t}(G)$ if and only if $G = K'_m$ or $G = F \circ_k K'_m$ in which $m$ is minimum in
\[ T = \{ t \mid t \geq m + 1, G = F' \circ_k K'_t \text{ for some graphs } F' \text{ and } K'_t \}, \]
and $F = G - K'_m$.

Now by Theorems 10 and 12, we conclude:

Theorem 13. For any integer $k \geq 1$ and any hypergraph $H$, $\Upsilon_k(H) = \tau_k(H)$ if and only if $H = H_G$, in which $G$ is $K'_m$ or $F \circ_k K'_m$ for some graph $K'_m$ and $m$ is minimum in
\[ T = \{ t \mid t \geq m + 1, G = F' \circ_k K'_t \text{ for some graphs } F' \text{ and } K'_t \}, \]
and $F = G - K'_m$.

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