APPLICATION OF A TWO-PARAMETER QUANTUM ALGEBRA
TO ROTATIONAL SPECTROSCOPY OF NUCLEI

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A two-parameter quantum algebra $U_{qp}(u_2)$ is briefly investigated in this paper. The basic ingredients of a model based on the $U_{qp}(u_2)$ symmetry, the $qp$-rotator model, are presented in detail. Some general tendencies arising from the application of this model to the description of rotational bands of various atomic nuclei are summarized.

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0. Introduction

In recent years, the theory of compact matrix quantum groups (or pseudogroups) [1, 2, 3, 4] and quantum algebras [5, 6, 7, 8] was applied to several fields of theoretical physics. In particular, the one-parameter quantum algebra \( U_q(\text{su}_2) \) was introduced in nuclear physics at the beginning of the 90’s. In this direction, the \( q \)-rotator models developed in [9] (see also [10]) and in [11] (see also [12, 13, 14]) are based on the \( U_q(\text{su}_2) \) symmetry. In addition, the \( \kappa \)-Poincaré rotator model [15] may be considered as a relativistic alternative to the \( q \)-rotator models.

In some preliminaries works, the authors showed the interest of using a two-parameter deformation of the Lie algebra \( u_2 \) for nuclear spectroscopy [16, 17] and molecular spectroscopy [18]. Along these lines, the junior author (R. B.) studied in his thesis [19] the two-parameter quantum algebra \( U_{qp}(u_2) \) and explored the application of such an algebra to rotational collective dynamics in nuclei.

The aim of this paper is two-fold: (i) to present the so-called \( qp \)-rotator model based on the \( U_{qp}(u_2) \) symmetry and (ii) to report the main results, derived in [19], of its application to some superdeformed nuclei in the \( A \approx 130, 150, \) and 190 mass regions as well as to rare earth and actinide deformed nuclei.

1. The \( U_{qp}(u_2) \) quantum algebra

Two-parameter deformations of the algebra \( u_2 \) were worked out by several authors (see, for instance, [20, 21, 22]). We follow here Ref. [21] where four generators \( J_\alpha \) (with \( \alpha = -, 0, +, 3 \)) span a deformation of \( u_2 \) characterized by the following commutation relations

\[
\begin{align*}
[J_0, J_\alpha] & = 0, \\
[J_3, J_\pm] & = \pm J_\pm, \\
[J_+, J_-] & = (qp)^{J_0-J_3} [2J_3]_{qp},
\end{align*}
\]

with the notation

\[
[[X]]_{qp} := \frac{q^X - p^X}{q - p}. \tag{4}
\]

The parameters \( q \) and \( p \) in (3) and (4) are \textit{a priori} complexe parameters. Hermiticity condition requirements show that they are allowed to vary on two domains: (i) \( q \in \mathbb{R} \) and \( p \in \mathbb{R} \) and (ii) \( q \in \mathbb{C} \) and \( p = \bar{q} \in \mathbb{C} \), where we exclude the values for which \( q \) and \( p \) are roots of unity. Note that by introducing the parameters \( Q \) and \( P \) defined by

\[
Q := (qp^{-1})^{\frac{1}{2}}, \quad P := (qp)^{\frac{1}{2}}, \tag{5}
\]

equation (3) can be rewritten as

\[
[J_+, J_-] = P^{2J_0-1} [2J_3]_Q, \tag{6}
\]
with

\[ [X]_Q := \frac{Q^X - Q^{-X}}{Q - Q^{-1}}. \]  

(7)

In the \(QP\)-parametrization, the domains (i) and (ii) corresponds to: (i) \((Q, P) \in \mathbb{R} \times \mathbb{R}\) and (ii) \((Q, P) \in S^1 \times \mathbb{R}\).

The deformation spanned by the operators \(J_0\)’s may be equipped with a Hopf algebraic structure \([21]\), thus producing a quantum algebra denoted as \(U_{qp}(u_2)\). Let us just mention that we can define a family of coproducts \(\Delta_{\beta}^{QP}\) via

\[
\begin{align*}
\Delta_{\beta}^{QP}(J_0) & := J_0 \otimes 1 + 1 \otimes J_0, \\
\Delta_{\beta}^{QP}(J_3) & := J_3 \otimes 1 + 1 \otimes J_3, \\
\Delta_{\beta}^{QP}(J_+) & := J_+ \otimes P^{\beta J_0} Q^{+ J_3} + P^{(2-\beta)J_0} Q^{-J_3} \otimes J_+, \\
\Delta_{\beta}^{QP}(J_-) & := J_- \otimes P^{(2-\beta)J_0} Q^{+ J_3} + P^{\beta J_0} Q^{-J_3} \otimes J_-,
\end{align*}
\]

(8) \hspace{1cm} (9) \hspace{1cm} (10) \hspace{1cm} (11)

where \(\beta\) is a real parameter. Two extreme cases have been studied, namely, the cases corresponding to \(\beta = 1\) \([21]\) and \(\beta = 2\) \([19]\). For the latter two cases, we have the properties

\[
\begin{align*}
(\Delta_{\beta}^{QP}(J_+))^\dagger & = \Delta_{1}^{QP}(J_+), & \sigma \Delta_{2}^{Q^{-1}P}(J_+), \\
(\Delta_{\beta}^{QP}(J_-))^\dagger & = \Delta_{2}^{QP}(J_-), & \sigma \Delta_{2}^{P}(J_-)
\end{align*}
\]

(12) \hspace{1cm} (13)

for the domain (i) and

\[
\begin{align*}
(\Delta_{\beta}^{QP}(J_+))^\dagger & = \Delta_{2}^{Q^{-1}P}(J_+), & \sigma \Delta_{2}^{QP}(J_+) \\
(\Delta_{\beta}^{QP}(J_-))^\dagger & = \Delta_{2}^{QP}(J_-), & \sigma \Delta_{2}^{P}(J_-)
\end{align*}
\]

(12) \hspace{1cm} (13)

for the domain (ii). In (12) and (13), \(X^\dagger\) stands for the adjoint of \(X\) and \(\sigma\) is the twist operator (such that \(\sigma(a \otimes b) = b \otimes a\)). It is to be noted that for \(p = q^{-1}\) (i.e., \(P = 1\)), the quantum algebra \(U_{qp}(u_2)\) reduces to \(U_q(\text{su}_2) \times U_1\), where \(U_q(\text{su}_2)\) is the classical one-parameter quantum algebra used in Refs. \([3, 11]\) (see also \([23, 24]\)).

The operator

\[
C_2(U_{qp}(u_2)) := \frac{1}{2}(J_+ J_- + J_- J_+) + \frac{1}{2} [2] Q P^{2 J_0} \left(\left[3\right] Q\right)^2
\]

(14)

can be shown to be an invariant operator for the algebra \(U_{qp}(u_2)\). The irreducible representations of \(U_{qp}(u_2)\) may be labelled by doublets \((j_0, j)\) with \(j_0 \in \mathbb{Z}\) and \(2j \in \mathbb{N}\). The representation \((j_0, j)\) is spanned by the set \(\{|j_0, j, m\rangle : m = -j, -j+1, \ldots, +j\}\), where the state vector \(|j_0, j, m\rangle\) is obtained from the highest weight state vector \(|j_0, j, j\rangle\) through

\[
|j_0, j, m\rangle = (qp)^{\frac{1}{2}(j_0-j)(j-m)} \sqrt{\frac{[j+m]_q!}{[2j]_q! [j-m]_q!}} \langle j_0, j, j \rangle \langle j_0, j, j \rangle^{-m} |j_0, j, j\rangle,
\]

(15)

where \([n]_q!\) is the \(qp\)-deformed factorial of \(n \in \mathbb{N}\). The eigenvalues of \(C_2(U_{qp}(u_2))\) in the irreducible representation \((j_0, j)\) assume the form

\[
(qp)^{j_0-j} [j]_q [j+1]_q = P^{2 j_0} [j]_q [j+1]_q
\]

(16)
and clearly depend on the two parameters $q$ and $p$ (or, alternatively, $Q$ and $P$). We have calculated the Clebsch-Gordan coefficients $(j_{01}, j_{02}, j_1, j_2, m_1, m_2)_{q,p}$ corresponding to the coproducts $\Delta_1$ (with $\beta = 1$) and $\Delta_2$ (with $\beta = 2$). As a result, we have

\begin{align*}
(j_{01}, j_{02}, j_1, j_2, m_1, m_2)_{q,p} &= \delta_{j_0, j_{01}} + \delta_{j_2, j_{02}} (j_1, j_2, m_1, m_2 | j, m)_Q, \\
(j_{01}, j_{02}, j_1, j_2, m_1, m_2)_{q,p} &= \delta_{j_0, j_{01}} + \delta_{j_2, j_{02}} (j_1, j_2, m_1, m_2 | j, m)_Q \text{P}^{j_{01}(j-m)-j_0(j_1-m_1)},
\end{align*}

where $(j_1, j_2, m_1, m_2 | j, m)_Q$ is the one-parameter Clebsch-Gordan coefficient for the algebra $U_q(\text{su}_2)$ (see [23] and [24]).

2. The $qp$-rotator model

Let us now list the basic hypotheses of the $qp$-rotator model for describing rotational bands of nuclei. (The model is also of interest for diatomic molecules.) As a first hypothesis, we take the Hamiltonian

\[ H := \frac{1}{2J} C_2(U_q(u_2)) + E_0, \]

where the constants $E_0$ and $J$ stand for the bandhead energy and the effective moment of inertia of a given nucleus, respectively. Such an Hamiltonian exhibits the $U_q(u_2)$ symmetry. As a second hypothesis, the physical state vectors are chosen as basis vectors for the irreducible representation $(I, I)$ (i.e., $j_0 = j = I$, where $I$ is the angular momentum of the nucleus under consideration). Therefore, the eigenvalues of $H$ turn out to be

\[ E(I)_{q,p} = \frac{1}{2J} [I]_{q,p} [I + 1]_{q,p} + E_0 = \frac{1}{2J} P^{2I-1} [I]_Q [I + 1]_Q + E_0. \]

Finally, the third hypothesis concerns the calculation of the $B(E2)$ electric-quadrupole transition probability for the $\gamma$-transition $(K : I + 2) \rightarrow (K : I)$ between the levels $I + 2$ and $I$ of the $K$ band. We assume that the $BE(2)$ reduced transition probability is defined by

\[ B(E2; K I_1 \rightarrow K I_2)_{\beta,q} := \frac{5}{16\pi} Q^2_0 \left| (I_1, I_2 - I_1, I_1, 2, K, 0 | I_2, I_2, K)_{\beta,q} \right|^2, \]

for the $U_q(u_2)$ symmetry, where $Q_0$ is the intrinsic electric-quadrupole moment. Equations (16) and (17) lead to

\[ B(E2; K I + 2 \rightarrow K I)_{1,q} = B(E2; K I + 2 \rightarrow K I)_Q \]

and

\[ B(E2; K I + 2 \rightarrow K I)_{2,q} = P^{4K} B(E2; K I + 2 \rightarrow K I)_Q \]

for $\beta = 1$ and $\beta = 2$, respectively, where $B(E2; K I + 2 \rightarrow K I)_Q$ is the $B(E2)$ reduced transition probability for the $q$-rotator model developed in Refs. [11, 12, 13, 14] on the basis of the $U_q(\text{su}_2)$ symmetry.
3. Physical results and conclusions

As a first test of the \( qp \)-rotator model presented in Section 2, we fitted \( \gamma \)-transitions on experimental data for rotational bands of superdeformed (SD) and deformed (D) nuclei. We have chosen two ranges of variation, compatible with the above mentioned domains (i) and (ii), for the parameters \( q \) and \( p \). They correspond to the parametrizations:

\[
(i) \quad q = e^{a+b}, \quad p = e^{a-b} \quad \text{and} \quad (ii) \quad q = e^{a+ib}, \quad p = e^{a-ib}.
\]

The relevant energy formulas used in our fitting procedures are

\[
E(I)_{qp} \equiv E(a, b; I)_{(i)} = \frac{1}{2J} e^{(2I-1)a} \frac{\sinh(Ib) \sinh[(I+1)b]}{\sinh^2 b} + E_0
\]

and

\[
E(I)_{qp} \equiv E(a, b; I)_{(ii)} = \frac{1}{2J} e^{(2I-1)a} \frac{\sin(Ib) \sin[(I+1)b]}{\sin^2 b} + E_0
\]

for the cases (i) and (ii), respectively. The fitted values of the \( qp \)-rotator parameters \( (J, a, \text{and } b) \) were obtained by minimizing the standard deviation

\[
\chi := \sqrt{\frac{1}{n-m} \sum_I \left[ \frac{E_{\gamma}^\text{th}(I) - E_{\gamma}^\text{ex}(I)}{\Delta E_{\gamma}^\text{ex}(I)} \right]^2},
\]

where \( n \) is the number of fitted \( \gamma \)-transitions, \( m \) the number of fitting parameters, and \( \Delta E_{\gamma}^\text{ex}(I) \) the experimental error for the \( \gamma \)-transition from the \( E(I)_{qp} \) level to the \( E(I-2)_{qp} \) level.

As a second important test of our model, we calculated the theoretical dynamical moment of inertia \( J_{th}^{(2)} \) defined by

\[
J_{th}^{(2)}(I) := \left( \frac{d^2E}{dx^2} \right)^{-1}, \quad E \equiv E(I)_{qp}, \quad x \equiv x(I) := \sqrt{I(I+1)}.
\]

For each considered nucleus, the moments of inertia \( J_{th}^{(2)} \) were calculated, as a function of the angular momentum \( I \), from the values obtained for the fitting parameters \( (J, a, \text{and } b) \) and compared to the experimental ones.

The two tests were performed on 20 SD bands in the \( A \approx 130, 150, \) and 190 mass regions and on 29 D bands in the rare earth and actinide mass regions. For the purpose of comparison, the same tests were achieved through the use of the \( q \)-rotator models of Refs. [9] and [11] and of the \( \kappa \)-Poincaré model of Ref. [15]. The main results of our analyses can be summed up as follows.

(i) The results obtained from the \( qp \)-rotator model are better, both for the (fitted) \( \gamma \)-transitions and the (calculated) dynamical moments of inertia, than the ones derived from the \( q \)-rotator models and the \( \kappa \)-Poincaré model (see [13] for an exhaustive study).
(ii) The best results for the qp-rotator model are obtained with the first domain of variation of the parameters $q$ and $p$ (i.e., with (25)) in the $A \approx 130$ and 150 mass regions and with the second domain (i.e., with (26)) in the $A \approx 190$, rare earth, and actinide mass regions. It is to be noted that the latter fact parallels the experimental situation according to which the dynamical moments of inertia decrease (respectively, increase) for the $A \approx 130$ and 150 SD bands (respectively, for the $A \approx 190$ SD bands and for the rare earth and actinide D bands). Furthermore, it is also to be noted that the dynamical moments of inertia derived from the $q$-rotator models and the $\kappa$-Poincaré model are not in good agreement with the corresponding experimental values in the $A \approx 130$ and 150 mass regions.

(iii) The ranges of variation of the qp-rotator parameters $a$ and $b$ depend on the bands under consideration. Indeed, the parameter $b$ (respectively $a$) is of the order $10^{-3}$ (respectively, $10^{-4}$) for the $A \approx 190$ SD bands and of the order $10^{-2}$ (respectively, $10^{-3}$) for the rare earth and actinide D bands. This is in accordance with the experimental fact that the increasing of the dynamical moment of inertia is more important in the D bands than in the $A \approx 190$ SD bands. In other words, the two parameters $a$ and $b$ of the qp-rotator model can be interpreted as inertial parameters that describe the softness of the D and SD nuclei.

(iv) To close this paper, let us emphasize that the second deformation parameter, viz., $p$, of the quantum algebra $\mathcal{U}_{qp}(u_2)$ plays an important rôle via the factor $e^a$. The parameter $a$ ($a$ is 10 times lower than $b$) acts as a correction of the softness mainly described by the parameter $b$. This is especially evident when comparing our results with the ones given by the $q$-rotator model of Ref. [11]. Finally, the fact that the qp-rotator model yields better results than the $\kappa$-Poincaré model for rare earth and actinide nuclei shows that our model works also well for heavy nuclei and thus take into account, in a phenomenological way, some relativistic effects.

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