Cohomology algebra of the orbit space of free circle group actions on lens spaces

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Abstract. Suppose that $G = S^1$ acts freely on a finitistic space $X$ whose mod $p$ cohomology ring isomorphic to that of a lens space $L^{2m-1}(p; q_1, \ldots, q_m)$. In this paper, we determine the mod $p$ cohomology ring of the orbit space $X/G$. If the characteristic class $\alpha \in H^2(X/G; \mathbb{Z}_p)$ of the $S^1$-bundle $S^1 \to X \to X/G$ is nonzero, then the mod $p$ index of the action is defined to be the largest integer $n$ such that $\alpha^n \neq 0$. We also show that the mod $p$ index of a free action of $S^1$ on a lens space $L^{2m-1}(p; q_1, \ldots, q_m)$ is $p - 1$, provided that $\alpha \neq 0$.

Key Words: Characteristic class, Finitistic space, Free action, Index, Spectral sequence.

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1. Introduction

Let $X$ be a topological space and $G$ be a topological group acting continuously on $X$. The set $\hat{x} = \{gx \mid g \in G\}$ is called the orbit of $x$. The set of all orbits $\hat{x}$, $x \in X$, is denoted by $X/G$ and assigned the quotient topology induced by the natural projection $\pi : X \to X/G$, $x \to \hat{x}$. An action of $G$ on $X$ is said to be free if $g(x) = x$, for any $x \in X \Rightarrow g = e$, the identity element of $G$. The orbit space of a free transformation group $(G, S^n)$, where $G$ is a finite group, has been studied extensively ([2], [7], [8], [10], [15]). However, a little is known if the total space $X$ is a compact manifold other than a sphere ([3], [6], [9], [14]).

The orbit space of a free involution on a real or complex projective space has been studied by the authors in [13]. We have also determined the cohomology algebra of the orbit space of free actions of $\mathbb{Z}_p$ on a generalized lens space $L^{2m-1}(p; q_1, q_2, \ldots, q_m)$ in [12]. In this note, we determine the mod $p$ cohomology algebra of orbit spaces of free actions of circle group $S^1$ on the real projective space and a lens space. Note that $S^1$ can not act freely on a 'finitistic' space having integral cohomology of a finite-dimensional complex projective space or a quaternionic projective space (Theorem 7.10 of Chapter III, [1]). We recall that a paracompact Hausdorff space is finitistic if every open covering has a finite-dimensional refinement.

Throughout this paper, $H^*(X)$ will denote the Čech cohomology of the space $X$. It is known that $H^*(\mathbb{R}P^n; \mathbb{Z}_2) = \mathbb{Z}_2[a]/\langle a^{n+1}\rangle$, where deg $a = 1$, and $H^*(L^{2m-1}(p; q_1, \ldots, q_m); \mathbb{Z}_p) = \wedge(a) \otimes \mathbb{Z}_p[b]/\langle b^m\rangle$, deg $a = 1$, $\beta(a) = b$, where $\beta$ is the Bockstein homomorphism associated with the
coefficient sequence $0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$. By $X \sim_p Y$, we shall mean that $H^*(X;\mathbb{Z}_p)$ and $H^*(Y;\mathbb{Z}_p)$ are isomorphic. We establish the following results.

**Theorem 1.** Let $G = S^1$ act freely on a finitistic space $X \sim_p L^{2m-1}$ $(p; q_1, q_2, \ldots, q_m)$, $p$ an odd prime. Then $H^*(X/G;\mathbb{Z}_p)$ is one of the following graded commutative algebras:

(i) $\mathbb{Z}_p[x, y_1, y_3, \ldots, y_{2p-3}, z]/\langle x^p, z^n, x y_q, y_q y_{q'}, A_{qq'} x^{q+q'-2p} - B_{qq'} z x^{q+q'-2p} \rangle$

where $m = np$, $\deg x = 2$, $\deg y_q = q$, $\deg z = 2p$, $A_{qq'} = 0$ when $q + q' > 2p$, $B_{qq'} = 0$ when $q + q' < 2p$ and both $A_{qq'}$ and $B_{qq'}$ are zero when $q = q'$ or $q + q' = 2p$.

(ii) $\mathbb{Z}_p[z]/\langle z^m \rangle$, where $\deg z = 2$.

For free actions of circle group on a cohomology real projective space, we have

**Theorem 2.** Let $G = S^1$ act freely on a finitistic space $X \sim_2 \mathbb{R}P^{2m-1}$. Then

$$H^*(X/G;\mathbb{Z}_2) \cong \mathbb{Z}_2[z]/\langle z^m \rangle \text{ where } \deg z = 2.$$

Let $G = S^1$ act freely on a space $X$, then there is an orientable 1-sphere bundle $S^1 \hookrightarrow X \xrightarrow{\nu} X/G$, where $\nu$ denotes the orbit map. Let $\alpha \in H^2(X/G;\mathbb{Z})$ be its characteristic class. Jaworowski [4] has defined the (integral) index of a free $S^1$-action on the space $X$ to be the largest integer $n$ (if it exists) such that $\alpha^n \neq 0$. Similarly, one can define mod $p$ index of a free $S^1$-action on a space $X$. Jaworowski has shown that the (integral or
rational) $S^1$-index of $L^{2m-1}(p; q_1, q_2, \ldots, q_m)$ is $m - 1$. It follows from the Thom-Gysin sequence for bundle $S^1 \hookrightarrow X \to X/G$ that the characteristic class is nonzero only if $p > 2$ on a space $X = L^{2m-1}(p; q_1, q_2, \ldots, q_m)$. In this case, the mod $p$ $S^1$-index of $X$ is $p - 1$. It should be noted that $G = S^1$ can not act freely on $X \sim_2 \mathbb{R}P^{2m}$.

2. Preliminaries

Let $G = S^1$ act on a paracompact Hausdorff space $X$. Then there is an associated fibration $X \xrightarrow{i} X_G \xrightarrow{\pi} B_G$, where $X_G = (E_G \times X)/G$ and $E_G = S^\infty \to B_G = \mathbb{C}P^\infty$ is a universal $G$-bundle. It is known that $B_G$ is a CW-complex with $2N$-skelton $\mathbb{C}P^N$ for all $N$ and $E_G$ is a CW-complex with $2N + 1$-skelton $S^{2N+1}$. Write $E^N_G = S^{2N+1}$ and $B^N_G = \mathbb{C}P^N$. Then, $H^i(E^N_G) = 0$ for $0 < i < 2N+1$. Let $X^N_G = X \times G E^N_G$ is associated bundle over $B^N_G$ with fibre $X$. Then the equivariant projection $X \times E^N_G \to X$ induces the map $\phi : X^N_G \to X/G$. Let $G$ acts freely on $X$, then

$$\phi^* : H^i(X/G) \to H^i(X_G)$$

is an isomorphism for all $i < 2N + 1$ with coefficient group $\mathbb{Z}_p$, $p$ a prime, by Vietoris-Begle mapping theorem. By $H^i(X_G)$ we mean $H^i(X^N_G)$, $N$ is large.

To compute $H^*(X_G)$ we exploit the Leray-Serre spectral sequence of the map $\pi : X_G \to B_G$ with coefficients in $\mathbb{Z}_p$, $p$ a prime. The edge homomorphisms,

$$H^p(B_G) = E_2^{p,0} \to E_3^{p,0} \to \ldots \to E_{p+1}^{p,0} = E_{\infty}^{p,0} \subseteq H^p(X_G),$$

and
\[ H^q(X_G) \to E^0_{\infty} = E^0_{q+1} \subset \ldots \subset E^0_2 = H^q(X) \]

are the homomorphisms

\[ \pi^* : H^p(B_G) \to H^p(X_G), \quad \text{and} \quad i^* : H^q(X_G) \to H^q(X), \]

respectively. We also recall the fact that the cup product in \( E_{r+1} \) is induced from that in \( E_r \) via the isomorphism \( E_{r+1} \cong H^*(E_r) \). For the above facts, we refer to McCleary [5].

3. Proofs

To prove our theorems, we need the following:

**Proposition.** Let \( G = S^1 \) act freely on a finitistic space \( X \) with \( H^i(X) = 0 \forall i > n \). Then \( H^i(X/G) = 0 \forall i \geq n \) with coefficient group \( \mathbb{Z}_p \), \( p \) a prime.

**Proof:** We recall that the bundle \( S^1 \to X \to X/G \) is orientable, where \( \nu : X \to X/G \) is the orbit map. Consider, The Thom-Gysin sequence

\[ \ldots \to H^i(X/G) \xrightarrow{\nu^*} H^i(X) \xrightarrow{\lambda} H^{i-1}(X/G) \xrightarrow{\mu^*} H^{i+1}(X/G) \to \ldots \]

of the bundle, where \( \mu^* \) is the multiplication by a characteristic class \( \alpha \in H^2(X/G) \). This implies that \( H^i(X/G) \xrightarrow{\mu^*} H^{i+2}(X/G) \) is an isomorphism \( \forall i \geq n \). Since \( X \) is finitistic, \( X/G \) is finitistic (Deo and Tripathi [11]). Therefore, \( H^*(X/G) \) can be defined as the direct limit of \( H^*(K(U)) \), where \( K(U) \) denotes the nerve of \( U \) and \( U \) runs over all finite dimensional open coverings of \( X/G \). Let \( \beta \in H^i(X/G) \) be arbitrary. Then, we find a finite dimensional covering \( \mathcal{V} \) of \( X/G \) and elements \( \alpha' \in H^2(K(\mathcal{V})) \), \( \beta' \in H^i(K(\mathcal{V})) \) such that \( \rho(\alpha') = \alpha \) and \( \rho(\beta') = \beta \) where
\[\rho : \sum_{U} H^i(K(U)) \rightarrow H^i(X/G)\] is the canonical map. Consequently, for \(2k + i > \dim V\), we have \((\alpha')^k \beta' = 0\) which implies that \((\mu^*)^k(\beta) = \alpha^k \beta = 0\). So, \(\beta = 0\), and the proposition follows.

Now, we prove our main theorems.

**Proof of Theorem 1.** The case \(m = 1\) is trivial. So we assume \(m > 1\). Since \(G = S^1\) acts freely on \(X\), the Leray-Serre spectral sequence of the map \(\pi : X_G \rightarrow B_G\) does not collapse at the \(E_2\)-term. As \(\pi_1(B_G)\) is trivial, the fibration \(X \rightarrow X_G \rightarrow B_G\) has a simple system of local coefficients on \(B_G\). So the spectral sequence has

\[E_2^{k,l} \cong H^k(B_G) \otimes H^l(X).\]

Let \(a \in H^1(X)\) and \(b \in H^2(X)\) be generators of the cohomology ring \(H^*(X)\) then \(a^2 = 0\) and \(b^m = 0\). Consequently, we have either \((d_2(1 \otimes a) = t \otimes 1\) and \(d_2(1 \otimes b) = 0\)\) or \((d_2(1 \otimes a) = 0\) and \(d_2(1 \otimes b) = t \otimes a\)\).

**Case I.** If \(d_2(1 \otimes a) = 0\) and \(d_2(1 \otimes b) = t \otimes a\), then we have \(d_2(1 \otimes b^q) = qt \otimes ab^{q-1}\) and \(d_2(1 \otimes ab^q) = 0\) for \(1 \leq q < m\). So \(0 = d_2[(1 \otimes b^{m-1}) \cup (1 \otimes b)] = mt \otimes ab^{m-1}\). This forces \(p|m\). Suppose that \(m = np\). Now,

\[d_2 : E_2^{k,l} \rightarrow E_2^{k+2,l-1}\]

is an isomorphism if \(l\) is even and \(2p\) does not divide \(l\); and the trivial homomorphism if \(l\) is odd or \(2p\) divides \(l\). So \(E_3^{k,l} \cong E_2^{k,l} \cong \mathbb{Z}_p\) for even \(k\) and \(l = 2qp\) or \(2(q + 1)p - 1\), \(0 \leq q < n\); \(k = 0, l\) is odd and \(2p\) does not divide \(l\); and \(E_3^{k,l} = 0\), otherwise. Clearly, all the differentials
\(d_3, d_4, \ldots, d_{2p-1}\) are trivial. Obviously,

\[d_{2p} : E_{2p}^{k,2qp} \rightarrow E_{2p}^{k+2p,2(q-1)p+1}\]

are the trivial homomorphisms for \(q = 1, 2, \ldots, n - 1\). If

\[d_{2p} : E_{2p}^{0,2p-1} \rightarrow E_{2p}^{2p,0}\]

is also trivial, then

\[d_{2p} : E_{2p}^{k,2qp-1} \rightarrow E_{2p}^{k+2p,2(q-1)p}\]

is the trivial homomorphism for \(q = 2, \ldots, n - 1\), because every element of
\(E_{2p}^{k,2qp-1}\) (even \(k\)) can be written as the product of an element of \(E_{2p}^{k,2(q-1)p}\)
by \([1 \otimes ab^{p-1}] \in E_{2p}^{0,2p-1}\). It follows that \(d_r = 0, \forall r > 2p\) so that \(E_{\infty} = E_3\).
This contradicts the fact that \(H^i(X_G) = 0\) for all \(i \geq 2m - 1\). Therefore,

\[d_{2p} : E_{2p}^{0,2p-1} \rightarrow E_{2p}^{2p,0}\]

must be non-trivial. Assume that \(d_{2p}([1 \otimes ab^{p-1}]) = [t^p \otimes 1]\). Then

\[d_{2p} : E_{2p}^{k,2qp-1} \rightarrow E_{2p}^{k+2p,2(q-1)p}\]

is an isomorphism for all \(k\) and \(1 \leq q \leq n\). Now, it is clear that
\(E_{\infty} = E_{2p+1}\). Also, \(E_{2p+1}^{k,l} \cong \mathbb{Z}_p\) for ((even) \(k < 2p\), \(l = 2qp\), \((0 \leq q < n)\))
and ((\(k = 0\), \(l\) is odd and 2p does not divide \(l\)). Thus

\[H^j(X_G) = \begin{cases} 0, & j = 2qp - 1(1 \leq q \leq n) \text{ or } j > 2np - 2 \\ \mathbb{Z}_p & \text{otherwise.} \end{cases} \]

The elements \(1 \otimes b^q \in E_2^{0,2p}\) and \(1 \otimes ab^{(h-1)/2} \in E_2^{0,h}\), for \(h = 1, 3, \ldots, 2p-3\)
are permanent cocycles. So they determine \(z \in E_\infty^{0,2p}\) and \(y_q \in E_\infty^{0,q}\),
\(q = 1, 3, \ldots, 2p - 3\), respectively. Obviously, \(i^*(z) = b^q\), \(z^n = 0\) and
Let $x = \pi^*(t) \in E_{2,0}^2$. Then $x^p = 0$. It follows that the total complex $\text{Tot} \ E_{\infty}^{*,*}$ is the graded commutative algebra

$$\text{Tot} E_{\infty}^{*,*} = \frac{\mathbb{Z}_p[x, y_1, y_3, \ldots, y_{2p-3}, z]}{(x^p, y_q y_q', x y_q, z^n)}$$

where $q, q' = 1, 3, \ldots, 2p - 3$.

Then $i^*(y_q) = ab^{(q-1)/2}$, $y_q^2 = 0$ and $y_q y_{2p-q} = 0$. It follows that

$$H^*(X_G) = \frac{\mathbb{Z}_p[x, y_1, y_3, \ldots, y_{2p-3}, z]}{(x^p, z^n, x y_q, y_q y_q' - A_{qq'} x^{\frac{q+q'}{2}} - B_{qq'} x^{\frac{q+q'-2p}{2}})}$$

where $m = np$, $A_{qq'} = 0$ when $q + q' > 2p$, $B_{qq'} = 0$ when $q + q' < 2p$ and both $A_{qq'}$ and $B_{qq'}$ are zero when $q = q'$ or $q + q' = 2p$, $\deg x = 2$, $\deg z = 2p$, $\deg y_q = q$.

**Case II.** If $d_2(1 \otimes a) = t \otimes 1$ and $d_2(1 \otimes b) = 0$, then

$$d_2 : E_{2}^{k,l} \to E_{2}^{k+2,l-1}$$

is an isomorphism for $k$ even and $l$ odd and the trivial homomorphism for remaining values of $k$ and $l$. Obviously, $E_3^{k,l} \cong \mathbb{Z}_p$ for $k = 0$ and $l = 0, 2, 4, \ldots, 2m - 2$. So that $E_\infty = E_3$. Therefore, we have

$$E_\infty^{k,l} = \begin{cases} \mathbb{Z}_p, & k = 0 \text{ and } l = 0, 2, 4, \ldots, 2m - 2 \\ 0 & \text{otherwise.} \end{cases}$$

The element $1 \otimes b \in E_2^{0,2}$ is a permanent cocycle and determines an element $z \in E_{0,2}^{0,2}$. We have $i^*(z) = b$ and $z^m = 0$. Therefore, the total complex $\text{Tot} E_{\infty}^{*,*}$ is the graded commutative algebra

$$\text{Tot} E_{\infty}^{*,*} = \mathbb{Z}_p[z]/(z^m), \quad \deg z = 2.$$
It shows that $E_{\infty}^{0,l} = H^l(X_G) \forall l$ and hence

$$H^*(X_G) = \mathbb{Z}_p[z]/(z^m), \quad \deg z = 2.$$  

Since the action of $G$ on $X$ is free, the mod $p$ cohomology rings of $X_G$ and $X/G$ are isomorphic. This completes the proof.

**Proof of Theorem 2.** As above we have

$$E_2^{kl} \cong H^k(B_G) \otimes H^l(X).$$

Let $a \in H^1(X)$ be the generator of the cohomology ring $H^*(X)$. If $d_2(1 \otimes a) = 0$, then $d_2(1 \otimes a^q) = 0$, by the multiplicative structure of spectral sequence. Consequently, the spectral sequence degenerates which contradicts our hypothesis. Therefore, we must have $d_2(1 \otimes a) = t \otimes 1$. It is easily seen that

$$d_2 : E_2^{k,l} \to E_2^{k+2,l-1}$$

is an isomorphism for $k$ even and $l$ odd; and the trivial homomorphism otherwise. So

$$E_{\infty}^{k,l} \cong \begin{cases} \mathbb{Z}_2, & k = 0 \text{ and } l = 0, 2, 4, \ldots, 2m - 2 \\ 0, & \text{otherwise} \end{cases}$$

It follows that $H^*(X_G)$ and Tot $E_{\infty}^{*,*}$ are the same as the graded commutative algebra. The case $m = 1$ is obvious, so assume that $m > 1$.

The element $1 \otimes a^2 \in E_2^{0,2}$ is a permanent cocycle and determines an element $z \in E_{\infty}^{0,2} = H^2(X_G)$. We have $i^*(z) = a^2$ and $z^m = 0$. Therefore,
the total complex $\text{Tot} \ E_{\infty}^{\ast\ast}$ is the graded commutative algebra.

$$\text{Tot} E_{\infty}^{\ast\ast} = \mathbb{Z}_2[z]/(z^m), \text{ where } \deg z = 2.$$ 

Thus $H^\ast(X_G) = \mathbb{Z}_2[z]/(z^m)$, where $\deg z = 2$. This completes the proof.

4. Examples

Consider the $(2m - 1)$ sphere $S^{2m-1} \subset \mathbb{C} \times \ldots \times \mathbb{C}$ ($m$ times). The map $(\xi_1, \ldots, \xi_m) \rightarrow (z\xi_1, \ldots, z\xi_m)$, where $z \in S^1$, defines a free action of $G = S^1$ on $S^{2m-1}$ with the orbit space $S^{2m-1}/S^1$ the complex projective space. Let $N = \langle z \rangle$, where $z = e^{2\pi i/p}$, then the orbit space $S^{2m-1}/N$ is the lens space $L^{2m-1}(p; 1, \ldots, 1)$ (resp. real projective space $\mathbb{R}P^{2m-1}$ for $p = 2$).

It follows that there is a free action of $S^1 = G/N$ on a lens space with the complex projective space as the orbit space. This realizes the second case of Theorem 1 and Theorem 2. We note that characteristic class of the bundle $S^1 \hookrightarrow L^{2m-1}(p; q_1, q_2, \ldots, q_m) \rightarrow L^{2m-1}(p; q_1, q_2, \ldots, q_m)/G$ over $\mathbb{Z}_p$ is zero. So, mod $p$ index of this action is not defined.

References

[1] Bredon, G.E., *Introduction to compact transformation groups*, Academic Press, 1972.

[2] Browder, W. and Livesay, G.R., *Fixed point free involutions on homotopy spheres*, Bull. Amer. Math. Soc 73(1967), 242 -245.

[3] Dotzel, R.M., Singh, Tej B. and Tripathi, S.P., *The cohomology rings of the orbit spaces of free transformation groups of the product of two
spheres, Proc. Amer. Math. Soc, 129(2000), 921-230.

[4] Jan Jaworowski, *The index of free circle actions in lens spaces*, Topology and its applications, 123(2002), 125-129.

[5] McCleary, J., *Users guide to spectral Sequences*, Publish or Perish, 1985.

[6] Myers, R., *Free involutions on lens spaces*, Topology, 20(1981),313-318

[7] Rice, P.M., *Free actions of $\mathbb{Z}_4$ on $S^3$*, Duke Math. J., 36(1969), 749-751

[8] Ritter, Gerhard, X., *Free $\mathbb{Z}_8$ Actions on $S^3$*, Trans. Amer. Math. Soc, 181(1973),195-212

[9] ———, *Free actions of cyclic groups of order $2^n$ on $S^1 \times S^2$*, Proc. Amer. Math. Soc. 46(1974), 137-140.

[10] Rubinstein H., *Free Actions of some Finite Groups on $S^3$*, Mathematische Annalen, 240(1979), 165-175.

[11] Satya Deo and Tripathi, H. S., *Compact Lie Group Actions on Finitistic Spaces*, Topology, 4(1982), 393-399.

[12] Singh, H. K. and Singh, Tej B., *On the cohomology of orbit space of free $\mathbb{Z}_p$-actions on lens space*, Proc. Indian Acad. Sci. (Math.Sci) 117(2007), 287-292

[13] Singh, H. K. and Singh, Tej B., *Fixed point free involutions on cohomology projective spaces*, Indian Journal of Pure and Applied Mathematics, to appear in June 2008.
[14] Tao. Y., *On fixed point free involutions on $S^1 \times S^2$*, Osaka J. Math., 14(1962) 145-152.

[15] Wolf, J., *Spaces of constant curvature*, McGraw-Hill, 1967.