Coherent states of non-relativistic electron in the magnetic–solenoid field

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Abstract
In the present work we construct coherent states in the magnetic–solenoid field, which is a superposition of the Aharonov–Bohm field and a collinear uniform magnetic field. In the problem under consideration there are two kinds of coherent states, those which correspond to classical trajectories which embrace the solenoid and those which do not. The constructed coherent states reproduce exactly classical trajectories, maintain their form under the time evolution and form a complete set of functions, which can be useful in semiclassical calculations. In the absence of the solenoid field these states are reduced to the well known in the case of uniform magnetic field Malkin–Man’ko coherent states.

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1. Introduction
It is well known that the study of the Aharonov–Bohm (AB) effect began on the base of exact wavefunctions of an electron in the field of an infinitely long and infinitesimally thin solenoid [1]. Such functions allow one to analyze a nontrivial influence of the AB solenoid (ABS) on scattering of a free electron, which may give a new interpretation of electromagnetic potentials in quantum theory. Physically it is clear that in such scattering the electron is subjected to the action of the AB field for a short finite time. However, there exists a possibility of considering bound states of the electron in which it is affected by the AB field for infinite time. Such bound states exist in the so-called magnetic–solenoid field (MSF), which is a superposition of the AB

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field and a collinear uniform magnetic field. The non-relativistic and relativistic wavefunctions of an electron in the MSF were studied in [2–6]. In particular, they were used to describe the AB effect in cyclotron and synchrotron radiation [7]. We believe that such bound states of an electron in the MSF open new possibilities of studying the AB effect. One of the important questions is the construction and the study of semiclassical (coherent) states in the MSF (the importance and advantage of coherent states in quantum theory is well known [8]). Having such states in hand one can try to answer an important question: To what extent is the AB effect of pure quantum nature? In a sense constructing semiclassical states is a complementary task to the path integral construction. The latter problem is completely open in the case of a particle in the MSF. One ought to say that some attempts to construct semiclassical states in the MSF were made in [9]. However, one ought to accept that states constructed in these works have some features that do not allow one to interpret them as semiclassical and coherent states. For example, some mean values calculated in such states do not move along classical trajectories. In this work we succeeded in constructing another kind of semiclassical states in the MSF which can really be treated as coherent states. The progress is related to a nontrivial observation that in the problem under consideration there are two kinds of coherent states: those which correspond to classical trajectories which embrace the solenoid and those which do not. The constructed coherent states reproduce classical trajectories in the semiclassical limit; they maintain their form under time evolution, and form a complete set of functions, which can be useful in semiclassical calculations. In the absence of the AB field these states are reduced to the well known in the case of uniform magnetic field Malkin–Man’ko coherent states [10].

We consider the non-relativistic motion of an electron with charge \( q = -e, e > 0 \), and mass \( M \) in the MSF \( B = (B_x, B_y, B_z) \),

\[
B_x = B_y = 0, \quad B_z = B + \Phi \delta(z)\delta(r) = B + \frac{\Phi}{\pi r} \delta(r), \quad (1)
\]

which is a collinear superposition of a constant uniform magnetic field \( B \) directed along the \( z \)-axis \((B > 0)\) and the AB field (field of an infinitely long and infinitesimally thin solenoid) with a finite constant internal magnetic flux \( \Phi \). We use Cartesian coordinates \( x, y, z \) as well as cylindrical coordinates \( r, \varphi \), such that \( x = r \cos \varphi, y = r \sin \varphi \) and \( r^2 = x^2 + y^2 \). Field (1) can be described by the vector potential \( A = (A_x, A_y, A_z) \),

\[
A_x = -y \left( \frac{\Phi}{2\pi r^2} + \frac{B}{2} \right), \quad A_y = x \left( \frac{\Phi}{2\pi r^2} + \frac{B}{2} \right), \quad A_z = 0. \quad (2)
\]

The classical motion of the electron in the MSF is governed by the Hamiltonian \( H = P^2/2M, P = p - \frac{q}{c} A \), where \( p \) and \( P \) are the generalized and kinetic momentum, respectively. Trajectories that do not intersect the solenoid have the form

\[
x = x_0 + R \cos \psi, \quad y = y_0 + R \sin \psi, \quad z = \frac{p_z}{M} t + z_0; \quad \psi = \omega t + \psi_0, \quad \omega = \frac{eB}{Mc}.
\quad (3)
\]

where \( x_0, y_0, z_0, p_z, R \) and \( \psi_0 \) are integration constants. Equations (3) imply

\[
(x - x_0)^2 + (y - y_0)^2 = R^2, \quad x_0 = R_c \cos \alpha, \quad y_0 = R_c \sin \alpha,
\]

\[
r^2 = R^2 + R_c^2 + 2RR_c \cos(\psi - \alpha), \quad R_c = \sqrt{x_0^2 + y_0^2}. \quad (4)
\]

The projections of particle trajectories on the \( xy \)-plane are circles. Particle images on the \( xy \)-plane are rotating with the synchrotron frequency \( \omega \). For an observer which is placed near the solenoid with \( z > 0 \), the rotation is anticlockwise. The particle has a constant velocity
$p_z / M$ along the $z$-axis. Since the electron freely propagates on the $z$-axis, only motion in the perpendicular plane $z = 0$ is nontrivial; this will be examined below. Denoting by $r_{\text{max}}$ the maximal possible moving off and by $r_{\text{min}}$ the minimal possible moving off, of the particle from the $z$-axis, we obtain from (4) $r_{\text{max}} = R + R_c$, $r_{\text{min}} = |R - R_c|$. It follows from (3) that

\begin{align*}
P_x &= -M\omega R \sin \psi = -M\omega (y - y_0), \\
P_y &= M\omega R \cos \psi = M\omega (x - x_0), \\
P_z^2 &= P_x^2 + P_y^2 = (M\omega R)^2.
\end{align*}

The energy $E$ of the particle rotation reads $E = P_z^2 / 2M$; then the radius $R$ can be expressed via the energy $E$ as follows:

$$R^2 = \frac{2E}{M\omega^2}. \tag{6}$$

With the help of (5) one can calculate the angular momentum projection $L_z$:

$$L_z = xp_y - yp_x = \frac{M\omega}{2} (R^2 - R_c^2) - \frac{e\Phi}{2\pi c}. \tag{7}$$

The presence of the ABS breaks the translational symmetry in the $xy$-plane, which on the classical level has only a topological effect; there appear two types of trajectories, and we label them by an index $j = 0, 1$ in what follows. On the classical level $j = 1$ corresponds to $(R^2 - R_c^2) > 0$ and $j = 0$ corresponds to $(R^2 - R_c^2) < 0$, see figure 1.

Already in classical theory it is convenient to introduce the dimensionless complex quantities $a_1$ and $a_2$ (containing $\hbar$) as follows:

\begin{align*}
a_1 &= \frac{-iP_x - P_y}{\sqrt{2\hbar M\omega}} = -\sqrt{\frac{M\omega}{2\hbar}} R e^{-i\psi}, \\
a_2 &= \frac{M\omega (x + iy) + iP_x - P_y}{\sqrt{2\hbar M\omega}} = \sqrt{\frac{M\omega}{2\hbar}} R_c e^{i\alpha}.
\end{align*}

One can see that $a_1 \exp(i\psi)$ and $a_2$ are the complex (dependent) integrals of motion. One can write that

\begin{align*}
R^2 &= \frac{2\hbar}{M\omega} a_1^* a_1, \\
R_c^2 &= \frac{2\hbar}{M\omega} a_2^* a_2, \\
x + iy &= \sqrt{\frac{2\hbar}{M\omega}} (a_2 - a_1^*). \tag{9}
\end{align*}
\[
E = \omega \hbar a_1^* a_1, \quad L_z = \hbar (a_1^* a_1 - a_2^* a_2) - \frac{e\Phi}{2\pi c}.
\]  

(10)

2. Stationary states

The quantum behavior of the electron in field (1) is determined by the Schrödinger equation with the Hamiltonian

\[
\hat{H} = \hat{H}_+ + \hat{H}_-. \quad \hat{H}_\pm = \left( \hat{p}_\pm^2 + \hat{\Psi}_1(j) \right) / 2M, \quad \hat{p}_\pm = \hat{p}_x \pm \frac{e}{c} A_x, 
\]

(11)

\[
\hat{p}_x = \hat{\rho} + \frac{e}{c} A_y, \quad \hat{p}_y = - \hbar \partial_y, \quad \hat{p}_z = - \hbar \partial_z,
\]

where \( \hat{H}_+ \) determines the nontrivial behavior on the \( xy \)-plane. It is convenient to present the magnetic flux \( \Phi \) in equation (2) as \( \Phi = (l_0 + \mu) \Phi_0 \), where \( l_0 \) is the integer, and \( 0 \leq \mu < 1 \) and \( \Phi_0 = 2\pi c h/e \) is Dirac's fundamental unit of the magnetic flux. Mantissa of the magnetic flux \( \mu \) determines, in fact, all the quantum effects due to the presence of the AB field. Stationary states of the non-relativistic electron in the MSF were first described in [2]. The corresponding radial functions were taken regularly at \( r \rightarrow 0 \); they correspond to a most natural self-adjoint extension (with a domain \( D_{\hat{H}_+} \)) of the differential symmetric operator \( \hat{H}_+ \). Considering a regularized case of a finite-radius solenoid one can demonstrate that the zero-radius limit yields such an extension, see [6]. Further, we consider only such an extension (all possible self-adjoint extensions of \( \hat{H}_+ \) were constructed in [4, 5]). The operator \( \hat{L}_z = x \hat{p}_y - y \hat{p}_x \) is self-adjoint on \( D_{\hat{H}_+} \) and commutes with the self-adjoint Hamiltonian \( \hat{H}_+ \). One can find two types \( (j = 0, 1) \) of common eigenfunctions of both operators

\[
\hat{H}_+ \Psi_{n,l,n_2}^{(j)}(t, r, \varphi) = \mathcal{E}_{n,l,n_2}^{(j)}(t, r, \varphi), \quad \mathcal{E}_{n,l,n_2}^{(j)} = \hbar \omega (n_1 + 1/2), 
\]

\[
\hat{L}_z \Psi_{n,l,n_2}^{(j)}(t, r, \varphi) = L_z \Psi_{n,l,n_2}^{(j)}(t, r, \varphi), \quad L_z = \hbar (l - l_0).
\]

(12)

The eigenfunctions have the form

\[
\Psi_{n,l,n_2}^{(j)}(t, r, \varphi) = \exp \left( -\frac{i}{\hbar} \mathcal{E}_{n,l} t \right) \Phi_{n,l,n_2}^{(j)}(\varphi, \rho), \quad \rho = \frac{eBr^2}{2\hbar}, \quad j = 0, 1,
\]

\[
\Phi_{n,l,n_2}^{(0)}(\varphi, \rho) = N \exp[i(l - l_0)\varphi] I_{n_2, n_1}(\rho), \quad n_1 = m, \quad n_2 = m - l - \mu, \quad -\infty < l < -1,
\]

\[
\Phi_{n,l,n_2}^{(1)}(\varphi, \rho) = N \exp[i(l - l_0)\varphi - i\pi l] I_{n_1, n_2}(\rho), \quad n_1 = m + l + \mu, \quad n_2 = m, \quad 0 < l < +\infty.
\]

Here \( l, m (m \geq 0) \) are two integers, \( I_{n, m}(\rho) \) are the Laguerre functions that are related to the Laguerre polynomials \( L_n^m(\rho) \) (see equations (8.970) and (8.972.1) from [11]) as follows:

\[
I_{m+a, m}(\rho) = \sqrt{\frac{\Gamma(m + 1)}{\Gamma(m + a + 1)}} e^{-\rho/2} \rho^{a/2} L_m^a(\rho), \quad L_n^m(\rho) = \frac{1}{m!} \rho^{-a} \frac{d^m}{d\rho^m} e^{-\rho} \rho^{m+a},
\]

(14)

and \( N \) is the normalization constant. For any real \( \alpha > -1 \) the functions \( I_{a+m, m}(\rho) \) form a complete orthonormal set on the half-line \( \rho \geq 0 \):

\[
\int_0^\infty I_{a+m, m}(\rho) I_{a+m, m}(\rho') d\rho = \delta_{\rho, \rho'}, \quad \sum_{m=0}^{\infty} I_{a+m, m}(\rho) I_{a+m, m}(\rho') = \delta(\rho - \rho').
\]

(15)

Let us define an inner product of the two functions \( f(\varphi, \rho) \) and \( g(\varphi, \rho) \) as

\[
(f, g)_\perp = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \int_0^{2\pi} d\rho f(\varphi, \rho) g(\varphi, \rho).
\]
Then eigenfunctions (13) form an orthogonal set on the $xy$-plane:

$$\langle \psi^{(j)}_{n_1,n_2}, \psi^{(j')}_{n_1',n_2'} \rangle_{\perp} = \left| N \right|^2 \delta_{n_1,n_1'} \delta_{n_2,n_2'} \delta_{j,j'}.$$  \hspace{1cm} (16)

These functions form a complete orthogonalized set on $D_{H}$. It is useful to define the self-adjoint operators $\hat{R}^2$ and $\hat{R}_c^2$ by analogy with the corresponding classical relations (6) and (7):

$$\hat{R}^2 = \frac{2\hat{H}_\perp}{M\omega^2}, \quad \hat{R}_c^2 = \hat{R}^2 - \frac{2}{M\omega} [\hat{L}_z + (l_0 + \mu) \hbar].$$  \hspace{1cm} (17)

In the semiclassical limit the sign of the mean value of the operator $\hat{R}^2 - \hat{R}_c^2$,

$$\langle \psi^{(j)}_{n_1,n_2}, (\hat{R}^2 - \hat{R}_c^2) \psi^{(j)}_{n_1,n_2} \rangle_{\perp} |N|^{-2} = \frac{2\hbar (l + \mu)}{M\omega},$$

where (12) is used, allows one to interpret the corresponding states as particle trajectories that embrace and do not embrace the solenoid. Namely, an orbit embraces the solenoid for $l \geq 0$ (type $j = 1$), and does not for $l \leq -1$ (type $j = 0$). These classifications correspond to the classical one described in the previous section, see equation (7) and figure 1. Trajectories with $l = 0$ and $l = -1$ pass most close to the solenoid.

If $\mu \neq 0$, the degeneracy of the energy spectrum is partially lifted, namely, energy levels of states (13) with $l \geq 0$ are shifted with respect to the Landau levels by $\mu \hbar \omega$, such that $E_{n_l} = \hbar \omega (m + l + \mu + 1/2)$, while energy levels of states (13) with $l \leq -1$ are still given by the Landau formula $E_{n_l} = \hbar \omega (m + 1/2)$. For $\mu = 0$ there is no impact of the ABS on the energy spectrum. Splitting of the Landau levels in the MSF is represented in figure 2.

3. Coherent states

3.1. Instantaneous coherent states on the $xy$-plane

Let us introduce the operators $\hat{a}_1, \hat{a}_2$ and $\hat{a}_1^\dagger, \hat{a}_2^\dagger$ that correspond to the classical quantities $a_1, a_2$ and $a_1^*, a_2^*$:

$$\hat{a}_1 = \frac{-i\hat{P}_x - \hat{P}_y}{\sqrt{2\hbar M\omega}}, \quad \hat{a}_2 = \frac{M\omega (x + iy) + i\hat{P}_x - \hat{P}_y}{\sqrt{2\hbar M\omega}};$$

$$\hat{a}_1^\dagger = \frac{i\hat{P}_x - \hat{P}_y}{\sqrt{2\hbar M\omega}}, \quad \hat{a}_2^\dagger = \frac{M\omega (x - iy) - i\hat{P}_x - \hat{P}_y}{\sqrt{2\hbar M\omega}}.$$  \hspace{1cm} (18)

Figure 2. Splitting of Landau levels in the MSF.
One ought to say that the momentum operators $\hat{P}_x$ and $\hat{P}_y$ are symmetric but not self-adjoint on the domain $D_{\Phi_1}$. That is why one cannot consider $\hat{a}_1^\dagger$ and $\hat{a}_2^\dagger$ as adjoint to $\hat{a}_1$ and $\hat{a}_2$. Nevertheless, they play an important auxiliary role in further constructions.

Using the properties of Laguerre functions, one can find the action of the operators $\hat{a}_1^\dagger$, $\hat{a}_1^\dagger$, $\hat{a}_2^\dagger$, $\hat{a}_2^\dagger$ on functions (13):

$$\hat{a}_1^\dagger \Phi_{n_1,n_2} (\varphi, \rho) = \sqrt{n_1} \Phi_{n_1-1,n_2} (\varphi, \rho), \quad \hat{a}_1^\dagger \Phi_{n_1,n_2} (\varphi, \rho) = \phi_{n_1+1,n_2} (\varphi, \rho),$$

$$\hat{a}_2^\dagger \Phi_{n_1,n_2} (\varphi, \rho) = \sqrt{n_2} \Phi_{n_1,n_2-1} (\varphi, \rho), \quad \hat{a}_2^\dagger \Phi_{n_1,n_2} (\varphi, \rho) = \phi_{n_1,n_2+1} (\varphi, \rho),$$

(19)

where the possible values of $n_1$ and $n_2$ depend on $m$, $l$ and $j$ according to (13) and the functions $\Phi_{n_1,n_2} (\varphi, \rho)$ are defined as follows:

$$\Phi_{n_1,n_2} (\varphi, \rho) = N \exp[i(l_0 - l - s_1 + s_2)\varphi] I_{n_1,n_2}(\varphi, \rho),$$

$$\Phi_{n_1,n_2} (\varphi, \rho) = N \exp[i(l_0 - l - s_1 + s_2)\varphi + \pi(l + s_1 - s_2)] I_{n_1,n_2}(\varphi, \rho).$$

at $s_1 = 0, \pm 1$ and $s_2 = 0, \pm 1$. There appear new functions $\Phi_{n_1,n_2} (\varphi, \rho)$ with $n_2 = m + 1 - \mu$ and $\Phi_{n_1,n_2} (\varphi, \rho)$ with $n_1 = m + \mu$, which are irregular at $r \to 0$. Such functions were not defined by equations (13). In addition, for $n_1 = 0$ or $n_2 = 0$, one has to bear in mind that

$$\hat{a}_1 \Phi_{0,l-\mu} (\varphi, \rho) = 0, \quad \hat{a}_2 \Phi_{l+\mu,0} (\varphi, \rho) = 0.$$

Formal commutators for the operators $\hat{a}_1^\dagger$, $\hat{a}_1^\dagger$, $\hat{a}_2^\dagger$, $\hat{a}_2^\dagger$ have the form

$$[\hat{a}_1, \hat{a}_1^\dagger] = 1 + f, \quad [\hat{a}_2, \hat{a}_2^\dagger] = 1 - f, \quad [\hat{a}_1, \hat{a}_2] = f, \quad [\hat{a}_1, \hat{a}_2] = 0$$

(20)

with a singular function $f = \Phi (\pi B r)^{-1} \delta(r) = 2(l_0 + \mu) \delta(\rho)$. However, one can verify with the help of (19) that this function gives zero contribution on the domain $D_{\Phi_1}$, such that on this domain $\hat{a}_1^\dagger$, $\hat{a}_2^\dagger$ and $\hat{a}_1$, $\hat{a}_2$ behave as creation and annihilation operators. The operators $\hat{R}^2$, $\hat{R}_x^2$, $\hat{R}_y^2$, $\hat{H}_\perp$, and $\hat{L}$ can be expressed in terms of the operators $\hat{a}_1^\dagger$, $\hat{a}_1$, $\hat{a}_2^\dagger$, $\hat{a}_2$ as follows:

$$\hat{R}^2 = \hbar (2\hat{N}_1 + 1), \quad \hat{R}_x^2 = \hbar (2\hat{N}_2 + 1), \quad x + iy = \sqrt{\frac{2\hbar}{\omega}} (\hat{a}_2 - \hat{a}_1^\dagger),$$

$$\hat{H}_\perp = \hbar \omega (\hat{N}_1 + 1/2), \quad \frac{1}{\hbar} \hat{L}_z = l_0 + \mu = (\hat{N}_1 - \hat{N}_2), \quad \hat{N}_1 = \hat{a}_1^\dagger \hat{a}_1, \quad s = 1, 2.$$ (21)

The functions $\Phi_{n_1,n_2} (\varphi, \rho)$ (13) can be used to construct the following useful states $\Phi_{z_1,z_2} (\varphi, \rho)$ (13) can be used to construct the following useful states

$$\Phi_{z_1,z_2} (\varphi, \rho) = \sum_l \Phi_{l,z_1,z_2} (\varphi, \rho), \Phi_{l,z_1,z_2} (\varphi, \rho) = \sum_m \frac{z_1^{n_1} z_2^{n_2}}{\sqrt{1 + n_1 + n_2}} \Phi_{n_1,n_2} (\varphi, \rho),$$

(22)

where $z_1$ and $z_2$ are the complex parameters, the possible values of $n_1$ and $n_2$ depend on $m$, $l$ and $j$ according to (13) and we set $N = 1$. We call these states instantaneous coherent states on the $xy$-plane. These states can be expressed via special functions $Y_\alpha(z_1, z_2; \rho)$.

$$Y_\alpha(z_1, z_2; \rho) = \sum_{m=0}^{\infty} \frac{z_1^m z_2^m \Theta_m(\rho)}{\Gamma(1 + m + \alpha)},$$

(23)

as follows:

$$\Phi_{0,0}^{(j)} (\varphi, \rho) = \exp[i(l_0 - l)\varphi] Y_{l-\mu}(z_1, z_2; \rho),$$

$$\Phi_{0,0}^{(j)} (\varphi, \rho) = \exp[i(l_0 - l)\varphi + \pi l] Y_{l+\mu}(z_2, z_1; \rho).$$ (24)

With the help of the well-known sum,

$$\sum_{m=0}^{\infty} \frac{z^m \Theta_m(\rho)}{\Gamma(1 + m + \alpha)} = z^{-\frac{\alpha}{2}} \exp \left( \frac{z}{2} - \frac{x}{\alpha} \right) J_\alpha(2\sqrt{xz}),$$
where \( J_\alpha(x) \) are the Bessel functions of the first kind, one can obtain the following representation for \( Y_\alpha(z_1, z_2; \rho) \):

\[
Y_\alpha(z_1, z_2; \rho) = \exp \left( z_1 z_2 - \frac{\rho}{2} \right) \left( \frac{z_2}{z_1} \right)^\alpha J_\alpha(2\sqrt{z_1 z_2 \rho}).
\]  

(25)

Then it follows from (19) that

\[
\hat{N}_\alpha(1, 2; \rho) = z_2 \partial_{\phi} \Phi_{z_1, z_2}(\rho, \rho), \quad k = 1, 2;
\]  

(26)

and

\[
a_1 \Phi_{z_1, z_2}(\rho, \rho) = z_1 \left[ \Phi_{z_1, z_2}(\rho, \rho) - (-1)^j \Phi_{z_1, z_2}^{(j-1)}(\rho, \rho) \right],
\]

(27)

\[
a_2 \Phi_{z_1, z_2}(\rho, \rho) = z_2 \left[ \Phi_{z_1, z_2}^{(j)}(\rho, \rho) + (-1)^j \Phi_{z_1, z_2}^{(j)}(\rho, \rho) \right].
\]

Then, using equations (6.615) from [11], we obtain

\[
\left( \Phi_{z_1, z_2}^{(j)}, \Phi_{z_1, z_2}^{(j)} \right) = \delta_{jj} \mathcal{R}^{(j)},
\]

\[
\mathcal{R}(0) = Q_{1-\mu} \left( \sqrt{z_1^2 z_2^2}, \sqrt{z_1^2 z_2^2} \right), \quad \mathcal{R}(1) = Q_\mu \left( \sqrt{z_1^2 z_2^2}, \sqrt{z_1^2 z_2^2} \right).
\]

(28)

\[
Q_\alpha(u, v) = Q_{1-\mu} \left( \frac{u}{v/\mu} \right) I_\alpha(2uv), \quad Q_{\mu}^*(u, v) = \sum_{l=1}^{\infty} \left( \frac{v}{u} \right)^{\alpha+l} I_{\alpha+l}(2uv),
\]

where \( I_\alpha(u) \) are the modified Bessel functions of the first kind. We define the mean value of an operator \( \tilde{F} \) in the form

\[
\langle \tilde{F} \rangle_{z_1, z_2} = \left( \Phi_{z_1, z_2}, \Phi_{z_1, z_2} \right) \left( \Phi_{z_1, z_2}, \Phi_{z_1, z_2} \right)^{-1}.
\]

Using (26), one can calculate the mean values of \( \hat{N}_\alpha \):

\[
\langle \hat{N}_\alpha \rangle_{z_1, z_2} = z_2 \partial_{\phi} \ln \mathcal{R}^{(j)} \bigg|_{z_1 = z_2}, \quad s = 1, 2.
\]

(29)

This allows one to connect the mean values of \( \hat{R}^2 \) and \( \hat{R}_c^2 \) with the parameters \( z_1 \) and \( z_2 \). We expect in the semiclassical limit that \( \langle \hat{N}_\alpha \rangle_{z_1, z_2} \approx |z_1|^2 \). At the same time length scales defined by the means \( \langle \hat{R}^2 \rangle_{z_1, z_2}, \langle \hat{R}_c^2 \rangle_{z_1, z_2} \) have to be large enough, which implies \( |z_1|^2 \gg 1 \). We expect that the sign of the difference \( \langle \hat{R}^2 \rangle_{z_1, z_2} - \langle \hat{R}_c^2 \rangle_{z_1, z_2} \) is related to the trajectory type if the difference is sufficiently large, such that for states with \( j = 0 \) we have \( |z_1|^2 < |z_2|^2 \), and for states with \( j = 1 \), we have \( |z_1|^2 > |z_2|^2 \). We note that in both cases the corresponding functions \( Q_\alpha(u, v) \) are calculated at \( |v| > |u| \gg 1 \).

There exist all the derivatives \( \partial_{\phi}[(v/u)^{\alpha+l} I_{\alpha+l}(2uv)] \); the series \( Q_\alpha^*(u, v) \) converges and the series of derivatives \( \sum_{l=1}^{\infty} \partial_{\phi}[(v/u)^{\alpha+l} I_{\alpha+l}(2uv)] \) converges uniformly on the half-line, \( 0 \leq Re < \infty \). Thus, one arrives at a differential equation with respect to \( Q_\alpha^*(u, v) \):

\[
\frac{dQ_\alpha^*(u, v)}{dv} = 2v[(v/u)^{\alpha} I_\alpha(2uv)] + Q_{\mu}^*(u, v).
\]

To evaluate asymptotics, we represent its solution as follows:

\[
Q_\alpha^*(u, v) = e^{v^2 + v^2} \left[ 1 - T(u, v) \right], \quad T(u, v) = 2e^{-v^2} \int_v^{\infty} e^{-\tilde{v}^2} \left( \frac{\tilde{v}}{u} \right)^{\alpha} I_\alpha(2u\tilde{v}) d\tilde{v},
\]

(30)

where formula (6.631.4) from [11] is used. Then

\[
\hat{Q}_\alpha(u, v) = e^{v^2 + v^2} \hat{Q}_\alpha(u, v), \quad \hat{Q}_{\mu}(u, v) = \left[ 1 - T(u, v) + e^{-v^2 - v^2}(v/u)^{\alpha} I_\alpha(2uv) \right].
\]

(31)

Thus, the mean values (29) have the form

\[
\langle \hat{N}_\alpha \rangle_{z_1, z_2} = |z_1|^2 + z_2 \partial_{\phi} \ln \mathcal{R}^{(j)} \bigg|_{z_1 = z_2}, \quad s = 1, 2,
\]

(32)

\[
\mathcal{R}(0) = \hat{Q}_{1-\mu} \left( \sqrt{z_1^2 z_2^2}, \sqrt{z_1^2 z_2^2} \right), \quad \mathcal{R}(1) = \hat{Q}_\mu \left( \sqrt{z_1^2 z_2^2}, \sqrt{z_1^2 z_2^2} \right).
\]
Using asymptotics of the function \( I_a(2\nu) \) one can verify that if \( |v| > |u| \gg 1 \), then \(|z_1|^2 \gg z_0^2\), in \( R^{(j)} |z_1=\alpha_2 \), in (32). For semiclassical states corresponding to the orbits placed far enough from the solenoid, i.e. for \(||z_1|^2 - |z_2|^2| \gg 1\), the contribution \( z_0^2 \), in \( R^{(j)} |z_1=\alpha_2 \), is small as \( \exp(-||z_1|^2 - |z_2|^2|) \). Finally, we obtain

\[
|z_1|^2 \approx \frac{M \omega}{2\hbar} (R^2)_{(j)}, \quad |z_2|^2 \approx \frac{M \omega}{2\hbar} (R^2)_{(j)}, \quad |z_3|^2 \gg 1.
\]

With the help of (27), one can find

\[
(a_1)_{(0)} = z_1 \Delta_{1,\mu}(|z_1|, |z_2|), \quad (a_2)_{(0)} = z_2, \quad (a_1)_{(1)} = z_1,
\]

\[
(a_2)_{(1)} = z_2 \Delta_{\mu}(|z_2|, |z_1|), \quad \Delta_\mu(u, v) = \frac{Q_\mu(u, v)}{Q_\mu(u, v)}.
\]

such that these means match with equations (32) in the classical limit.

### 3.2. Time-dependent coherent states

Consider the Schrödinger equation with the complete three-dimensional Hamiltonian \( \hat{H} \) (11) and corresponding solutions \( \Psi(t, r) \), with a given momentum \( p_z \):

\[
\Psi(t, r) = N \exp \left\{ -\frac{i}{\hbar} \left[ \left( \frac{p_z^2}{2M} + \frac{\hbar \omega}{2} \right) t - p_z z \right] \right\} \Phi(t, \varphi, \rho),
\]

where \( N \) is the normalization constant. The functions \( \Phi(t, \varphi, \rho) \) obey the following equation:

\[
i\hbar \partial_t \Phi(t, \varphi, \rho) = \omega \hat{N}_1 \Phi(t, \varphi, \rho).
\]

One can obey (35) setting \( \Phi(t, \varphi, \rho) = \Phi^{(j)}_{|z_1, z_2|} (\varphi, \rho) |z_1=\alpha_2(t), \) where \( z_1(t) \) is a complex function of time \( t \). Then

\[
i\hbar \partial_{z_1} \Psi^{(j)}_{|z_1, z_2|} = i\hbar \partial_{\alpha_2} \Phi^{(j)}_{|z_1, z_2|}, \quad z_1 = d\alpha_2/\partial t.
\]

Substituting (36) into (35), we find \( i\hbar = \omega \alpha_2 \), where (26) is used. It is convenient to write a solution for \( z_1(t) \) as follows:

\[
\psi(z_1) = -|z_1| \exp(-i\psi), \quad \psi = \omega t + \psi_0,
\]

where \( |z_1| \) is a given constant. Thus, the functions

\[
\Psi_{|z_1, z_2|}^{(j)} (t, r) = N \exp \left\{ -\frac{i}{\hbar} \left[ \left( \frac{p_z^2}{2M} + \frac{\hbar \omega}{2} \right) t - p_z z \right] \right\} \Phi^{(j)}_{|z_1, z_2|} (\varphi, \rho)
\]

are solutions of the Schrödinger equation. At the same time they have special properties that allow us to treat them as coherent (and under certain conditions as semiclassical) states.

Let us consider the mean values \( \langle x \rangle_{(j)} \) and \( \langle y \rangle_{(j)} \) of the coordinates with respect to the states \( \Psi_{|z_1, z_2|}^{(j)} \). To this end it is enough to find the mean value \( \langle x + iy \rangle_{(j)} \). With the help of (21) we obtain

\[
\langle x + iy \rangle_{(j)} = \sqrt{\frac{2\hbar}{M \omega}} (a_2)_{(j)} - (a_1)_{(j)}^2.
\]

Taking into account equations (34) and (37), one can see that a point with coordinates \( \langle x \rangle_{(j)} \) and \( \langle y \rangle_{(j)} \) is moving along a circle on the \( xy \)-plane with the cyclotron frequency \( \omega \), i.e. its trajectory has the classical form. The same equations allow one to find a radius \( \langle R \rangle_{(j)} \) of such a circle and the distance \( \langle R \rangle_{(j)} \) between its center and the origin,

\[
\langle R \rangle_{(0)} = \sqrt{\frac{2\hbar}{M \omega}} |z_1| \Delta_{1,\mu}(|z_1|, |z_2|), \quad \langle R \rangle_{(1)} = \sqrt{\frac{2\hbar}{M \omega}} \Delta_{\mu}(|z_2|, |z_1|).
\]
Figure 3. Spread of particle position around two types of semiclassical orbits at $R \approx R_c$, where $R_c = (R_{c1})$ and $R = (R_0)$.

However, in the general case, the quantities $(R_{c1})$ and $(R_{c0})$ do not coincide with the corresponding quantities

$$\sqrt{(R^2)_{(j)}} = \sqrt{\frac{\hbar}{M\omega}} \sqrt{2(N_1_{(j)})+1}, \quad \sqrt{(R_{c}^2)_{(j)}} = \sqrt{\frac{\hbar}{M\omega}} \sqrt{2(N_2_{(j)})+1},$$

which are expressed in terms of the mean values of the operators $\hat{H}_\perp$ and $\hat{L}_z$ according to (21), see also (29).

It follows from equation (34) that $\Delta_{1-\mu}(|z_1|,|z_2|) < 1$ and $\Delta_{\mu}(|z_2|,|z_1|) < 1$. This allows us to give the following interpretation for two types of states with $j = 0, 1$. States with $j = 1$ correspond to orbits that embrace the ABS (which corresponds to $|z_1|^2 \gtrsim |z_2|^2$ in the semiclassical limit). For such orbits $(R_{c1}) < R_c$, where the quantity $R_c = \sqrt{2\hbar/M\omega}|z_2|$ is interpreted by us as a distance between ABS and the orbit center as a consequence of equation (33). At the same time, the mean radius of the orbit coincides with the classical radius $R = \sqrt{2\hbar/M\omega}|z_1|$. The interpretation of $R$ as the classical radius follows from equation (33). States with $j = 0$ correspond to orbits that do not embrace the ABS (which corresponds to $|z_1|^2 \lesssim |z_2|^2$ in the semiclassical limit). For such orbits $(R_{c0}) = R_c$ and $(R_{0}) < R$.

By using formulas (31), (32) and (34) one can calculate the variances for $\hat{R}^2, \hat{R}_{c}^2$ and $x + y$ with respect to the coherent states in the semiclassical limit. With this result one can see that these variances are relatively small for the semiclassical orbits situated far enough from the solenoid, i.e. for $||z_1| - |z_2|| \gg 1$. In this case the coherent states are highly concentrated around the classical orbits. In the most interesting case when a semiclassical orbit is situated near the solenoid, such that the condition $||z_1| - |z_2|| \ll 1$ holds, the variance for $x + y$ increases significantly while the variances for $\hat{R}^2, \hat{R}_{c}^2$ remain relatively small. In this case $R \approx R_c$, however, one has $(R_{c1}) < R$ and $(R_{0}) < R$, as of course it must be for such semiclassical orbits. Having in mind that the standard deviation of $x + y$, $\delta R$, is relatively large at $R \approx R_c$, such that $\delta R \gg |R - (R_{c1})|, |(R_{0}) - R_c|$, we illustrate the typical spread of particle position around two types of semiclassical orbits at $R \approx R_c$ on figure 3.

Thus, for $\mu \neq 0$, classical relations between the parameters of particle trajectory in the constant magnetic field, such that relations between the circle parameters $R$ (related to particle
energy) and $R_c$ (related to particle angular momentum) are affected in the presence of the ABS. Such relations are not affected by the presence of the ABS for $\mu = 0$, and, even for $\mu \neq 0$, in the classical limit (in the leading approximation for sufficiently large radii) discussed above.

Thus, in contrast to the problem in the constant uniform magnetic field (and in contrast to any problem with a quadratic Hamiltonian) in the MSF, we meet a completely new situation. Here time-dependent coherent states can be constructed (which is a completely nontrivial fact due to the nonquadratic nature of the Hamiltonian in the MSF). The respective mean values move along classical trajectories; however, classical relations between physical quantities imply additional semiclassical restrictions. Not all coherent states correspond to a semiclassical approximation, which is natural for nonquadratic Hamiltonians.

Finally, we ought to mention that only linear combinations of the form
\[
\Psi(c_0, c_1; t, r) = c_0 \Psi_{CS}^{(0)}(t, r) + c_1 \Psi_{CS}^{(1)}(t, r),
\]
with $c_0$ and $c_1$ arbitrary and $c_0 c_1 \neq 0$ were considered earlier as coherent states in [9]. Mean values of the operators $\hat{a}_1$ and $\hat{a}_2$ in such mixed states do not coincide with the classical expressions (8).

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