Abstract

An equational logic program is a set of directed equations or rules, which are used to compute in the obvious way (by replacing equals with “simpler” equals). We present static analysis techniques for efficient equational logic programming, some of which have been implemented in \( LR^2 \), a laboratory for developing and evaluating fast, efficient, and practical rewriting techniques. Two novel features of \( LR^2 \) are that non-left-linear rules are allowed in most contexts and it has a tabling option based on the congruence-closure based algorithm to compute normal forms. Although, the focus of this research is on the tabling approach some of the techniques are applicable to the untabled approach as well. Our presentation is in the context of \( LR^2 \), which is an interpreter, but some of the techniques apply to compilation as well.

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1 Introduction

Equational logic programming has received increasing attention in recent years and there are now fast compilers such as Elan, Maude, etc., and fast interpreters such as $LR^2$. The objective of this paper is to enhance the efficiency of interpretation (and possibly also compilation) of equational logic programming by analyzing the input program. Since we want fast and efficient interpreters, our focus is on practical and effective techniques that can be implemented very cheaply (linear or close to linear time) in terms of their time and space cost. To describe the techniques we assume the usual scenario, viz., an equational program, or a set of rules, $P$ and a term $t$ are given and the goal is to compute a normal form of $t$. We assume here that $P$ is confluent and that $t$ is well-formed but we do not assume termination. Although most of the techniques we present are of general significance, their immediate motivation is to enhance the efficiency of $LR^2$, especially its tabling mechanism, so we give a brief self-contained introduction to $LR^2$ and a description of the tabling algorithm. For more details on $LR^2$, including useful optimizations and some performance results, the reader can consult [?]. Throughout this paper, when we use the term tabling we are specifically referring to this algorithm.

1.1 $LR^2$

$LR^2$ is a laboratory for developing and evaluating fast, efficient, and practical rewriting techniques. $LR^2$ consists of an expression graph interpreter $TGR$, and an expression graph rewriter that tables the history of its reductions, called $Smaran$, based on the congruence closure approach. The input to $LR^2$ is a program representing either an orthogonal, not necessarily terminating, rewrite system, or a convergent (terminating and confluent) rewrite system, and an input expression. First-order orthogonal systems can easily express first-order functions and it is a well-known fact that parallel-outermost reduction or lazy evaluation is necessary and sufficient to compute with such systems. Observe that since convergent systems can be given to $LR^2$, hence, it allows non-left-linear rules such as $x + -x \rightarrow 0$ and $equal(x, x) \rightarrow true$, etc. Similar to algebraic specification languages like OBJ, ASF+SDF and Elan a program is composed from modules. Each module defines its own signature and rewriting rules. A module can import other modules. Expressions in $LR^2$ are written in prefix form. The language of $LR^2$ contains several built-in datatypes, viz., integers, floating-point arithmetic, booleans, characters, sets, multisets, and strings with associated operations. The set datatype supports the insertion, deletion, membership, union, etc., operations. The string datatype supports membership and indexing operations.

$LR^2$ also includes a variant detector that can determine if a new expression is an alphabetic variant of an existing expression, which is currently usable with the tabling option. If so, the appropriate variant of the result computed for the existing expression is used for further rewriting instead of starting from scratch. $LR^2$ also allows a compact form for storing lists of arithmetic progressions as these occur frequently. Instead of storing the entire list, $LR^2$ stores the initial value, the final value and the difference. $LR^2$ provides a set of commands so that it can be called by other systems for symbolic computation. This currently requires the UNIX message passing mechanism.

$LR^2$ provides a variety of options for controlling the amount of history that is stored
by the system, if the user chooses the tabling option. The default option using Smaran is to save the results of each rewrite step in a compact data structure. The language of LR^2 allows annotating specific functions with the keyword “memo”. This allows to save all the results of rewriting expressions that have the specified function symbol at the root. The Delete (history) option in LR^2 allows to delete the entire history of rewrites performed so far except for the given expression and its latest reduct after every i^th rewrite step, where i can be specified by the user. Another possibility allowed is to delete the entire history except the given expression and the latest reduct as soon as the free memory available to the user drops to a user-specified percentage of the total available memory. Options can be combined in any way to suit the application. However, the delete option overrides the other options.

Operators can also be declared associative (A) or commutative (C) or both (AC) in LR^2. However, currently only left-linear rules with A, C, or AC operators can be handled; efficient matching algorithms/heuristics for nonlinear rules with such operators are fairly involved and is currently in the testing phase. It is known that A-matching, C-matching and AC-matching of general patterns are all NP-complete even with some restrictions on occurrences of variables [?].

The rest of this paper is organized as follows. In Section 2, we give a description of the main algorithm for Smaran, in Section 3 we discuss techniques to enhance the efficiency of LR^2, in Section 4 we mention the techniques that have already been implemented in LR^2. We conclude with some promising directions for future research. We omit here all proofs of the results stated in this paper.

2 Smaran’s Core Algorithm

The basic algorithm at the core of Smaran was proposed in [?] for orthogonal rules and later extended in [?, ?] to non-left-linear rewrite systems under fairly general conditions. The basic algorithm in turn is an extension of the well-known congruence closure algorithm (CCA) [?, ?, ?, ?]) for ground equations.

Recall that CCA divides the set of terms into numbered equivalence classes. Membership of a term in an equivalence class is decided by its signature. The signature of a term f(t_1, . . . , t_n) is the tuple (f #[t_1] . . . #[t_n]), where #[t_i] is the number of the equivalence class containing the signature representing t_i. Equivalence class C represents term t if C contains a signature representing t. CCA operates by merging equivalence class representing terms whose equivalence follows from the given equations.

To extend CCA for normalization we make use of the concept of a distinguished signature in each equivalence class called the unreduced signature [?, ?]. Using this signature we construct a distinguished term. It has been shown in [?, ?, ?] that it is enough to examine this term to select useful rule instances, and that it is sufficient to check this term for irreducibility for rewrite systems satisfying fairly general conditions. If it exists and is irreducible, then the class containing this term has a normal form. Thus, instances of left hand sides represented by reduced signatures, do not lead to any progress towards the normal form, and any term represented by a reduced signature cannot be in normal form. Whenever a rule instance A → B can be applied to the distinguished term of a class, C, because of a match, the signature representing A is marked reduced and the class representing B (if any) is merged
with $C$. If there is no class representing $B$, then a signature representing $B$ is constructed, inserted into $C$, and marked the unreduced signature of $C$.

The algorithm starts by constructing the signature of the given term. This signature is then inserted into a class and marked the unreduced signature of the class. This class number is tracked throughout the process of normalization. Signatures of terms are constructed in the obvious bottom-up way. We illustrate this algorithm with a small example. Consider the convergent rewrite system with three rules that are numbered for convenience,

$$1: \text{fib}(x) \rightarrow f(x > 1, x), 2: f(\text{true}, x) \rightarrow \text{fib}(x - 1) + \text{fib}(x - 2), 3: f(\text{false}, x) \rightarrow 1,$$

that uses built-in arithmetic to define the fibonacci function and let the input term be $\text{fib}(2)$.

The initial set of classes is:

$$0 : \{2^*\}, 1 : \{(\text{fib } 0)^*\}$$

(the symbol ‘*’ indicates unreduced signatures). Now the match procedure is called to find a match between the unreduced signature of any class and the left-hand side of any rule. A match occurs between class 1 and rule 1. Now, the signature representing the corresponding instance of the right-hand side of rule 1 is constructed, inserted into class 1 and marked as its unreduced signature. Note that here we do not show the signatures, corresponding to the built-in datatypes, that can be evaluated directly. At this stage the classes are as follows:

$$0 : \{2^*\}, 1 : \{(\text{fib } 0), (\text{fib } 0)^*\}, 2 : \{1^*\}, 3 : \{\text{true}^*\}$$

During the next iteration class 1 matches rule 2. The right-hand side instance is $\text{fib}(1) + \text{fib}(0)$. The signature representing this is created and inserted into class 1 as its new unreduced signature. At the end of the second iteration the new/changed classes are as follows:

$$1 : \{(\text{fib } 0), (\text{fib } 0)^*, (+4.6)^*\}, 2 : \{(\text{fib } 2)^*\}, 5 : \{0^*\}, 6 : \{(\text{fib } 5)^*\}$$

After two rewrite steps, using rules 1 and 3 respectively, the term $\text{fib}(1)$ represented by class 4 reduces to 1, which is represented by class 2. Hence classes 4 and 2 are merged, say into 2, and we have:

$$0 : \{2^*\}, 1 : \{(\text{fib } 0), (\text{fib } 0)^*, (+2.6)^*\}, 2 : \{(\text{fib } 2), (\text{fib } 7, 2)^*, 1^*\}, 3 : \{\text{true}^*\}, 5 : \{0^*\}, 6 : \{(\text{fib } 5)^*\}, 7 : \{\text{false}^*\}$$

Note that signatures containing the class number 4 have been updated to contain class number 2. After two more rewrite steps, the term $\text{fib}(0)$ represented by class 6 reduces to 1. Hence classes 6 and 2 are merged, say into 2, and we get:

$$0 : \{2^*\}, 1 : \{(\text{fib } 0), (\text{fib } 0)^*, (+2.2)^*\}, 2 : \{(\text{fib } 2), (\text{fib } 5), (\text{fib } 7, 2)^*, 1^*\}, 3 : \{\text{true}^*\}, 5 : \{0^*\}, 7 : \{\text{false}^*\}$$

Now, the unreduced signature of class 1 can be evaluated to yield the term 2, which is in class 0. Hence, classes 1 and 0 are merged, say into 0, and we get:

$$1 : \{2^*, (\text{fib } 0)^*, (\text{fib } 0)^*, (+2.2)^*\}, 2 : \{(\text{fib } 2), (\text{fib } 5), (\text{fib } 7, 2)^*, 1^*\}, 3 : \{\text{true}^*\}, 5 : \{0^*\}, 7 : \{\text{false}^*\}$$

No more matches are found. Therefore, the algorithm checks for the existence of a normal form for the given term. Class 1 contains the unreduced signature 2, which is irreducible. Thus the normal form of $\text{fib}(2)$ is 2. Note that if the normal form of $\text{fib}(\text{fib}(2))$ is needed, no more computations are needed since this term is also represented by the signature $(\text{fib } 0)$ in class 1. It is simply extracted from class 1. The normal form of this term is also 2. On the other hand, an interpreter that does not store history would compute $\text{fib}(2)$ twice to normalize this term. This compact data structure helps exploit the advantages of storing history and can also speed up normalization.
3 When does tabling help?

In this section, we present techniques that analyze the program $P$ and term $t$ to determine whether tabling could be helpful in computing the normal form of $t$. Recall that tabling can help in two important ways:

1. It can improve the termination characteristics, i.e., tabled rewriting can halt in cases where untabled would fail to halt because there is no normal form. For example, consider the rule $a \rightarrow f(a)$.

2. It can improve the efficiency of the computation. Each rule instance is applied at most once and its results are stored. This is extremely useful for: (a) problems that recursively solve lots of overlapping subproblems, e.g., dynamic programming problems, (b) proving theorems of the form $A = B$, and determining whether a critical-pair $(A, B)$ generated during KB-completion [?] is trivial. (It is possible that $A$ and $B$ were reduced to a common term albeit at different times, since this common term is remembered in tabling, it will terminate immediately whereas untabled methods may go on forever.), and (c) incremental computation.

We now develop two sets of necessary conditions that a program $P$ must satisfy so that tabling can potentially be useful for $P$, one set for each way in which tabling can help.

3.1 When does tabling improve termination?

First we give necessary conditions on $P$ alone for tabling to help termination. Next, we will analyze both $P$ and $t$ to develop even stronger necessary conditions. As usual, we partition the constants and function symbols of $P$ into two classes: constructors and defined symbols.

**Definition 1** A symbol is called defined if it is the outermost symbol of the left-hand side of some rule in $P$, otherwise it is a constructor.

For example, $fib$ is a defined symbol in the program given above, whereas $true$ is a constructor. Note that constructors include the constants from the predefined types such as bool, float, int, char, etc., (we know that there can be no rules with integer, float, char and bool constants as left-hand sides). The standard operators over these predefined types are treated as constructor functions in this section.

**Definition 2** Let $f$ be a defined symbol of $P$ and let $g$ be any function symbol or constant of $P$. We say that $f$ needs $g$ if $g$ appears in the right-hand side of any rule that has $f$ as its defined symbol.

Now we define the needs graph of $P$ as the directed graph $G = (V, E)$, where $V$ contains a vertex for each function symbol or constant of $P$ and $E$ contains a directed edge from the vertex for $f$ to the vertex for $g$ precisely when $f$ needs $g$. Note that this graph may have self-loops, i.e., cycles with just one edge.
**Proposition 1 (Necessary Condition)** A necessary condition for tabling to improve termination of the computation of normal form of $t$ with respect to $P$ is the existence of a cycle in the needs graph of $P$.

This condition can be checked in time that is linear in the sum of the lengths of the right-hand sides of $P$ using a standard search algorithm. We only introduce vertices in the graph that are necessary, i.e., they represent either defined symbols with at least one edge or a symbol that is adjacent to a defined symbol. The reader may wonder if it is possible to have a defined symbol with no originating edge. This happens when all the rules defining this symbol are collapsing, i.e., the right-hand side is just a variable.

Clearly, the above condition is not sufficient for two different kinds of reasons: (i) for instance, $t$ could be a normal form and $P$ can have many cycles, but obviously tabling is of no help, (ii) $P$ may have cycles but still represent a terminating system (e.g., the fibonacci system above). Trying to decide termination of $P$ is in general undecidable and even when it is decidable it can be prohibitively expensive. Therefore, we only try to ameliorate the first source of insufficiency by analyzing $t$ as well.

**Proposition 2 (Necessary Condition)** A necessary condition for tabling to improve termination of the computation of normal form of $t$ with respect to $P$ is the existence of a cycle in the needs graph of $P$ containing a vertex that is reachable by a directed path from a vertex representing a function symbol or constant that appears also in $t$.

This condition can be checked by marking the vertices that appear on some cycle in the needs graph and then running a modified search algorithm that looks for marked vertices.

The needs graph defined above is a refinement of the call graph from classical compilation techniques, where procedure parameters are also taken into account. Variations of these graphs have appeared in other contexts in rewriting (e.g., for narrowing a related but different graph is used in [?]), functional and logic programming (e.g., for termination). However, generally speaking, there are differences between such graphs in existing literature (that we are aware of) and our approach. Many of the applications here appear to be new as well.

### 3.2 When does tabling help efficiency?

Now we focus on the second benefit of tabling, i.e., the possibility that the computation of the normal form of $t$ would need the same reduction step more than once. Since the tabling approach here shares common subexpressions to the extent possible, repeated subterms in $t$ or its reducts do not necessarily lead to repeated reduction steps.

**Proposition 3 (Necessary Condition)** A necessary condition for tabling to improve the efficiency of the sequential computation of normal form of $t$ with respect to $P$ is the existence of a node with in-degree at least one and outdegree at least one in the needs graph of $P$.

This condition is easily checked in time that is linear in the sum of right-hand sides of $P$. Again, this condition is not sufficient for two kinds of reasons. Informally, they are: (i) $t$ has no “need” for this vertex, or (ii) $P$ does not “allow” to use this vertex from different terms. Again, it is quite complex to decide (ii). Hence, we only try to ameliorate the first by analyzing $t$. 
Proposition 4 (Necessary Condition) A necessary condition for tabling to improve the efficiency of the sequential computation of normal form of $t$ with respect to $P$ is the existence of a node with in-degree more than one and outdegree at least one in the needs graph of $P$ that is reachable via directed paths from two different symbols of $t$.

To check this condition we can use a standard search algorithm. Note that we are looking for a very “short” chain in the needs graph, i.e., we are looking for the possibility that just one reduction step can be saved. The length of the desired chain can be adjusted based on the ratio of the time it takes to do a reduction step versus the time it takes to lookup the results of a reduction step in the table, which grows with the table size.

3.3 Techniques for Efficient Reduction and Tabling

For technical reasons, the standard operators over the predefined types are not treated as constructor functions in this section, even though we do not have any rules in the system for them.

3.3.1 Optimizing Reduction

We define the following class of signatures called the don’t-reduce signatures that represent a subclass of normal forms that are formed from only constructors, or constructor terms.

**Definition 3** (a) Base case: All constructor constants are don’t-reduce signatures. (b) Inductive case: If classes $c_1, ..., c_n$ contain only don’t-reduce signatures as unreduced signatures, then $f(c_1, ..., c_n)$ is also a don’t-reduce signature if $f$ is a constructor.

Now note that no rule can match a don’t-reduce signature. This means that when we do leftmost-outermost reduction and we reach a class that contains an unreduced signature which is a don’t-reduce signature, then we need not go down (below this class) recursively to find any matches. Actually, if a class contains a normal form, then, of course, we need not explore the structure below to find any matches. However, to detect normal forms one must match rules against the unreduced signature of a class, whereas the detection of don’t-reduce signatures is easier since it can be done in a bottom-up manner without any matching. Moreover, our situation is even more complicated with respect to detection of normal forms, since we have built-in arithmetic and other types. It is easy to construct a scenario where no rules match an unreduced signature $s$ because $s$ depends on a class which contains an unevaluated arithmetic signature as unreduced signature and once it is evaluated rules may or may not match. Although we state this in terms of signatures here, it is clear that this idea applies to the untabled approach as well by marking subterms of terms that are constructor terms.

3.3.2 Optimizing Matching

By analyzing the term $t$ together with the program $P$ we can eliminate unnecessary rules, i.e. those rules that are never needed in the computation of the normal form of $t$. For this we find all the defined symbols of $P$ that are reachable from all the symbols in $t$ using a standard
search algorithm on the needs graph of $P$. We then include only rules for the reachable defined symbols in building efficient data structures for matching, e.g., discrimination nets. In compilation this corresponds to not generating code for the unnecessary rules and could speed up the compilation process itself (of course, the set of terms to be normalized must be known in advance and specified before compilation). The generated code uses less space.

### 3.3.3 Optimizing dependency lists in Tabling

To identify the signatures that must be updated in the event a class is unioned into another class, a dependency list is usually associated with each class. It contains all the signatures that “depend” on this class. Our next optimizations are intended to reduce the size of these lists, since processing them can get quite expensive.

For this purpose, we define a class of signatures called the don’t-add signatures, which is a subclass of the class of don’t-reduce signatures.

**Definition 4**

(a) **Base case:** All constructor constants are don’t-add signatures.  
(b) **Inductive case:** If classes $c_1, \ldots, c_n$ contain only don’t-add signatures, then $f(c_1, \ldots, c_n)$ is also a don’t-add signature if $f$ is a constructor.

Note the difference in case (b) between don’t-add vs. don’t-reduce signatures. All constants are, of course, never added to any dependency list since they do not depend any class. We need (a) for the inductive step.

Now we have the following rule for don’t-add signatures: do not add a don’t-add signature to dependency lists. However, there is a practical difficulty in implementing this rule, since it is quite possible that $f(c_1, \ldots, c_n)$ is a don’t-add signature, but because of the insertion of a signature into class $c_i$, class $c_i$ contains a signature which is an “add” signature and hence $f(c_1, \ldots, c_n)$ is also now an “add” signature. It may be possible to do this efficiently, but it will require time and also the programming may be somewhat complex.

Therefore, we identify a subclass of don’t-add signatures, called never-add signatures.

### 3.3.4 Never-add signatures

In the following we assume that the rewrite system is also non-collapsing, i.e., does not have a rule whose right-hand side is a variable.

While we are parsing the rules, we can also look for: (i) if any constructor constant is the right-hand side of some rule (note ”is” the rhs and not ”is in” the rhs) and (ii) if any standard operators over the predefined types appear in the right-hand side of some rule.

Now, we define the class of never-add signatures.

**Definition 5**

(a) **Base case:** Every user-defined constructor constant that does not appear as the rhs of any rule is a never-add constant. If $t$ is a constant from one of predefined types in (ii) above, then $t$ is a never-add constant only if there is no rule containing an operator that takes an argument of the type of $t$ in the right-hand side and the term to be normalised also does not contain any operator that takes an argument of type $t$. (b) **Inductive case:** If classes $c_1, \ldots, c_n$ contain only never-add signatures and $f$ is a constructor that does not appear as the outermost symbol of the right-hand side of any rule, then $f(c_1, \ldots, c_n)$ is a never-add signature.
Never-add signatures are a proper subclass of don’t-add signatures, which are easily identified in a bottom-up manner and they can never change on the union of two classes or on the insertion of a signature in a class containing a never-add signature. So in Smaran we never add a never-add signature to the dependency list of any class.

If the rewrite system can contain collapsing rules, then it is not sufficient to check that no constructor constant and no constructor function appears as the outermost symbol of any rule. Consider, the following rewrite system for example:

\[
\begin{align*}
from(x, y) & \rightarrow if(y > 0, cons(x, from(x + 1, y - 1)), nil) \\
if(true, x, y) & \rightarrow x \\
if(false, x, y) & \rightarrow y
\end{align*}
\]

In this system both nil and cons do not appear as the outermost symbol of any rule but there are terms \( T \) that reduce to terms of the form nil and cons(...). Hence, it is possible that because of a union one of the terms from \( T \) changes and so on. In fact, it is in general undecidable whether given a term \( t \) and a rewrite system \( R \) whether \( t \) reduces to a constructor term.

4 Implementation in \( LR^2 \)

Some of the techniques discussed above have been implemented in \( LR^2 \) and have proven their effectiveness. The techniques that have been implemented are the identification of don’t-reduce and never-add signatures. We are currently in the process of implementing the needs-graph analysis to help the user with choosing the tabling or the non-tabling option in \( LR^2 \).

5 Discussion and Future Work

In this paper, we have presented some easily computed static analysis techniques to enhance the efficiency of tabled and untabled computations with equational logic programs. These techniques have implications for interpreters (and sometimes compilers) and some are applicable in a wider context, e.g., functional or logic programming. The analysis described here is a first approximation since it ignores the arguments of functions symbols. A useful direction for future research is to use constraint-based analysis with the needs graph to take into account the arguments as well. Care must be taken to ensure that the benefits of the analyses outweigh the costs. Other avenues are transformation of the program including partial evaluation.