m-SYMMETRIC FUNCTIONS, NON-SYMMETRIC MACDONALD POLYNOMIALS AND POSITIVITY CONJECTURES

L. LAPOINTE

Abstract. We study the space, $R_m$, of m-symmetric functions consisting of polynomials that are symmetric in the variables $x_{m+1}, x_{m+2}, x_{m+3}, \ldots$ but have no special symmetry in the variables $x_1, \ldots, x_m$. We obtain $m$-symmetric Macdonald polynomials by $t$-symmetrizing non-symmetric Macdonald polynomials, and show that they form a basis of $R_m$. We define $m$-symmetric Schur functions through a somewhat complicated process involving their dual basis, tableaux combinatorics, and the Hecke algebra generators, and then prove some of their most elementary properties. We conjecture that the $m$-symmetric Macdonald polynomials (suitably normalized and plethystically modified) expand positively in terms of $m$-symmetric Schur functions. We obtain relations on the $(q, t)$-Koska coefficients $K_{\Omega \Lambda}(q, t)$ in the $m$-symmetric world, and show in particular that the usual $(q, t)$-Koska coefficients are special cases of the $K_{\Omega \Lambda}(q, t)$'s. Finally, we show that when $m$ is large, the positivity conjecture, modulo a certain subspace, becomes a positivity conjecture on the expansion of non-symmetric Macdonald polynomials in terms of non-symmetric Hall-Littlewood polynomials.

1. Introduction

Macdonald polynomials are a family of symmetric functions depending on two parameters $q$ and $t$ [19]. It was shown in [10] [14] that the Schur expansion of the Macdonald polynomials (suitably normalized and plethystically modified) is positive, that is, that

$$
\tilde{J}_\lambda(x; q, t) = \sum_\mu K_{\mu \lambda}(q, t) s_\mu(x)
$$

where $\tilde{J}_\lambda(x; q, t)$ stands for the plethystically modified Macdonald polynomial indexed by the partition $\lambda$.

There exists a non-symmetric version $E_\eta(x; q, t)$ of the Macdonald polynomials [8] [21] which are indexed by compositions and form a basis of the ring of polynomials in a given number $N$ of variables. Even though many combinatorial properties of the non-symmetric Macdonald polynomials are similar in nature to those of the Macdonald polynomials, there was until now no known Macdonald positivity phenomenon in the non-symmetric case (except in special cases [1] [2]). One of the main goals of this article is to develop the correct framework in which to extend the original Macdonald positivity conjectures to the non-symmetric case. In order to describe this framework, which is that of the $m$-symmetric functions, we first need to make a detour through superspace.

Generalizations to superspace of the original Macdonald positivity conjecture were provided in [5] [6]. In this setting, the Macdonald polynomials, denoted $J_A^{(\text{SUSY})}(q, t)$ and indexed by superpartitions, essentially correspond to non-symmetric Macdonald polynomials whose variables $x_1, \ldots, x_m$ (resp. $x_{m+1}, \ldots, x_N$) are antisymmetrized (resp. $t$-symmetrized), with $m$ being the fermionic sector. In short,

$$
J_A^{(\text{SUSY})}(q, t) \leftrightarrow A_{1, \ldots, m} S_{m+1, N}^{t} E_\eta(x; q, t)
$$

(1.1)

where $E_\eta(x; q, t)$ is a non-symmetric Macdonald polynomial, and where $A_{1, \ldots, m}$ (resp. $S_{m+1, N}^{t}$) stands for the antisymmetrization (resp. $t$-symmetrization) operator. The extension of the original Macdonald positivity conjecture is then that

$$
\tilde{J}_A^{(\text{SUSY})}(q, t) = \sum_\Omega K_{\Omega \Lambda}(q, t) S^{(\text{SUSY})}_\Omega
$$

with $K_{\Omega \Lambda}(q, t) \in \mathbb{N}[q, t]$

(1.2)

where $S^{(\text{SUSY})}_\Omega = j^{(\text{SUSY})}_A(0, 0)$, and where $\tilde{J}_A^{(\text{SUSY})}(q, t)$ is a plethystically modified version of $J_A^{(\text{SUSY})}(q, t)$.

For quite a while we have surmised that applying the antisymmetrization operator $A_{1, \ldots, m}$ in (1.1) was not necessary, and that the positivity conjectures could be extended to a much larger world. In this world, which

Keywords and phrases. Non-symmetric Macdonald polynomials, Hecke algebra, symmetric functions. 2020 Mathematics Subject Classification: 05E05.

Funding: this work was supported by the Fondo Nacional de Desarrollo Científico y Tecnológico de Chile (FONDECYT) Regular Grant #1210688.
we call the world of \( m \)-symmetric functions, the elements are symmetric in the variables \( x_{m+1}, x_{m+2}, \ldots \) while they have no special symmetry in the variables \( x_1, \ldots, x_m \) (the case \( m = 0 \) corresponding to the usual symmetric functions). To be more precise, we have suspected that, defining the \( m \)-symmetric Macdonald polynomials (up to a constant) as
\[
J_\Lambda(x; q, t) \propto S^\Lambda_m \cdot E_\mu(x; q, t)
\]  
(1.3)
there existed a natural basis of \( m \)-symmetric Schur functions \( s_\Omega(x; t) \) (now depending on \( t \)) such that
\[
J_\Lambda(x; q, t) = \sum_\Omega K_{\Omega \Lambda}(q, t) s_\Omega(x; t) \quad \text{with } K_{\Omega \Lambda}(q, t) \in \mathbb{N}[q, t]
\]  
(1.4)
with \( \tilde{J}_\Lambda(x; q, t) \) a plethystically modified version of \( J_\Lambda(x; q, t) \). Our main problem has been to obtain a workable definition of the \( m \)-symmetric Schur functions \( s_\Omega(x; t) \), that is, a definition allowing to demonstrate their elementary properties as well as providing a precise statement for the desired positivity conjectures for the non-symmetric Macdonald polynomials. Before describing the main ingredients of our solution to this problem, we should note that the \( m \)-symmetric Macdonald polynomials have also been considered in \cite{12, 13}, where they are called \( m \)-symmetric Hall-Littlewood polynomials.

Remarkably, it has been shown recently \cite{23} that the partially-symmetric Macdonald polynomials are in correspondence with certain \((\mathbb{C}^*)^2\)-fixed point classes of the parabolic flag Hilbert schemes \cite{7}, a result that generalizes the correspondence between Macdonald polynomials and \((\mathbb{C}^*)^2\)-fixed points of the Hilbert schemes \cite{14}.

Bases of the ring \( R_m \) of \( m \)-symmetric functions are indexed by \( m \)-partitions \( \Lambda = (a; \lambda) \) where \( a = (a_1, \ldots, a_m) \in \mathbb{Z}_{\geq 0}^m \) is a (weak) composition and where \( \lambda \) is a partition. A natural basis of \( R_m \) is provided by
\[
k_\Lambda(x; t) = H_a(x_1, \ldots, x_m; t) s_\lambda(x)
\]
where \( H_a(x_1, \ldots, x_m; t) \) is a non-symmetric Hall-Littlewood polynomial and where \( s_\lambda(x) \) is a Schur function. Let \( \langle \cdot, \cdot \rangle_m \) be the unique bilinear scalar product on \( R_m \) such that
\[
\langle k_\Lambda(x; t), k_\Omega(x; t) \rangle_m = \delta_{\Omega \Lambda} t^{\text{Inv}(a)}
\]
where \( \text{Inv}(a) \) is the number of inversions in \( a \). The reason to introduce this scalar product is that it proves more convenient to first define the dual \( m \)-symmetric Schur functions \( s^*_\Lambda(x; t) \). In the dominant case \( (\Lambda = (a; \lambda) \) with \( a_1 \geq a_2 \geq \cdots \geq a_m \), \( s^*_\Lambda(x; t) \) is simply a multi-Schur function (see Definition 11), while in the non-dominant case they can be constructed recursively using the Hecke algebra generators:
\[
T_is^*_\Lambda(x; t) = s^*_\Lambda(x; t) \quad \text{if } a_i > a_{i+1}
\]
where \( \tilde{\Lambda} = (\tilde{a}; \lambda) \) with \( \tilde{a} \) obtained from \( a \) by interchanging \( a_i \) and \( a_{i+1} \). Their dual basis (up to a power of \( t \)) with respect to the scalar product \( \langle \cdot, \cdot \rangle_m \), the sought-after \( m \)-symmetric Schur functions \( s_\Lambda(x; t) \) appearing in \cite{14}, are then the unique basis of \( R_m \) such that
\[
\langle s_\Lambda(x; t), s^*_\Omega(x; t) \rangle_m = \delta_{\Omega \Lambda} t^{\text{Inv}(a)}
\]
This definition, even though not the most direct, will prove quite fruitful. We have for instance that the Hecke algebra generator \( T_i \) is again such that
\[
T_is_\Lambda(x; t) = s_\Lambda(x; t) \quad \text{if } a_i > a_{i+1}
\]  
(1.5)
As such, it is possible to construct all \( m \)-symmetric Schur functions from those indexed by dominant \( m \)-partitions (unfortunately, we have not been able to obtain an explicit characterization of \( s_\Lambda(x; t) \) for a dominant \( \Lambda \). This was the main motivation behind the introduction of the dual \( m \)-symmetric Schur functions which, as we have seen, have a simple characterization as multi-Schur functions when \( \Lambda \) is dominant). We can also show that non-symmetric Hall-Littlewood polynomials are a sub-family of the \( m \)-symmetric Schur functions:
\[
s_{(a; \emptyset)}(x; t) = H_a(x_1, \ldots, x_m; t)
\]

Since \( R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots \), it proves natural to consider the inclusion \( i : R_m \to R_{m+1} \) and the restriction \( r : R_{m+1} \to R_m \) (which essentially consists in setting \( x_{m+1} = 0 \)). We obtain elegant formulas for the inclusion and restriction of the (dual) \( m \)-symmetric Schur functions, as well as for the inclusion and restriction of the \( m \)-symmetric Macdonald polynomials (these are probably the most technical results contained in this article). These formulas allow, among other things, to derive simple properties of the coefficients \( K_{\Omega \Lambda}(q, t) \) in \cite{14} (see Theorem 34 and

1Unfortunately, the construction of the \( m \)-symmetric Schur functions relying extensively on tableau combinatorics in the first version of this article (arXiv:2206.05177v1) was not the correct one as it yielded counter-examples to the positivity conjecture for the \( m \)-symmetric Macdonald polynomials (the smallest such instance occurring at degree 7 for \( m = 3 \)).
Proposition 57, which imply in particular that \( K_{\Omega \Lambda}(q, t) = K_{\mu \Lambda}(q, t) \) whenever \( \Omega = (0^m; \mu) \) and \( \Lambda = (0^m; \lambda) \), that is, that the usual \((q, t)\)-Kostka coefficients are special cases of the coefficients \( K_{\Omega \Lambda}(q, t) \).

Every non-symmetric Macdonald polynomial turns out to be an \( m \)-symmetric Macdonald polynomial when \( m \) is large enough (when \( m \) is as large as the length of the indexing composition to be more precise). The positivity conjecture expressed in (1.3) thus contains as a special case a generalization of the original Macdonald positivity conjecture for non-symmetric Macdonald polynomials. But a more remarkable phenomenon is at play. For any compositions \( \eta \) and \( \omega \), we can define \( K_{\omega \eta}(q, t) \) without any reference to \( m \). We can also show that for a given \( \eta \), there is only a finite number of distinct coefficients \( K_{\eta}(q, t) \), and that, as long as \( m \) is large enough, there is no loss of information in working modulo the linear span \( L_m \) of \( m \)-symmetric symmetric Schur functions \( s_{\lambda}(x; t) \) such that \( \lambda \neq \emptyset \) in \( \Lambda = (a; \lambda) \). The positivity conjecture (1.4) thus implies the following positivity conjecture for non-symmetric Macdonald polynomials (properly normalized and plethystically modified) in terms of non-symmetric Hall-Littlewood polynomials

\[
\tilde{J}_\eta(x; q, t) = \sum_{\omega \in \mathbb{Z}_{\geq 0}^m} K_{\omega \eta}(q, t) H_\omega(x_1, \ldots, x_m; t) \mod L_m, \quad \text{with } K_{\omega \eta}(q, t) \in \mathbb{N}[q, t]
\]

since, as we mentioned previously, \( s_{(\omega, \emptyset)}(x; t) = H_\omega(x_1, \ldots, x_m; t) \).

When \( q = t = 1 \), the usual \((q, t)\)-Kostka coefficient \( K_{\mu \lambda}(q, t) \) is equal to the number of standard tableaux of shape \( \mu \) (in representation-theoretic terms, this corresponds to the dimension of the irreducible representation of the symmetric group indexed by the partition \( \mu \)). We will show in a forthcoming article [10] that this property extends to the \( m \)-symmetric world. To be more precise, we will show that when \( q = t = 1 \), \( K_{\Omega \Lambda}(q, t) \) is equal to the number of standard tableaux of shape \( b \cup \mu \), where \( b \cup \mu \) is the partition obtained by reordering the entries of the concatenation of \( b \) and \( \mu \), where \( \Omega = (b; \mu) \) (in the case of compositions, \( K_{\omega \eta}(1, 1) \) is the number of standard tableaux of shape \( \omega^+ \), where \( \omega^+ \) is the partition obtained by reordering the entries of \( \omega \)). This suggests that the positivity conjectures in the \( m \)-symmetric case could still be connected to the representation theory of the symmetric group.

Our goal in introducing the positivity conjecture (1.3) was to define a much larger framework in which the extra structure could potentially lead to a combinatorial interpretation for the \((q, t)\)-Kostka coefficients. This extra structure seems for instance to include special recursions and Butler-type rules for the coefficients \( K_{\Omega \Lambda}(q, t) \) (see Conjectures 12, 13, and 14). We are thus still hopeful that this much richer framework holds the key to unraveling the mysterious combinatorics of the \((q, t)\)-Kostka coefficients.

The article, being already quite long, does not contain the \( t = 1 \) case (to be treated in [10]), or the deeper properties of the \( m \)-symmetric Macdonald polynomials (such as the orthogonality with respect to a natural scalar product, the formulas for the squared-norm and the evaluation, etc.), which are studied in [9]. We have instead focused on establishing properties of the \( K_{\Omega \Lambda}(q, t) \) coefficients, and, as such, have restricted ourselves to only presenting results that were necessary to establish such properties.

The outline of the article is the following. In Section 2 we give the necessary background on double affine Hecke algebras and non-symmetric Macdonald polynomials. It is interesting to note that we use a non-standard diagrammatic representation of compositions that, apart from providing a more elegant expression for the eigenvalues of the Cherednik operators, will prove useful when introducing the integral form of the non-symmetric Macdonald polynomials. In Section 3 we introduce the ring of \( m \)-symmetric functions. Generalizations of elementary concepts in symmetric function theory such as that of partition, Ferrers’ diagram, and dominance order are presented. Various extensions of simple bases of the ring of symmetric functions, such as that of monomials and power-sums, are also given. The \( m \)-symmetric Macdonald polynomials are defined in Section 4. This somewhat technical section culminates in a proof (using the action of certain eigenoperators involving Cherednik operators on monomials) that the \( m \)-symmetric Macdonald polynomials are unitriangularly related to the \( m \)-symmetric monomial basis, and thus form a basis of the ring of \( m \)-symmetric functions. The (dual) \( m \)-symmetric Schur functions can finally be presented in Section 5. The dual \( m \)-symmetric Schur functions are first introduced using multi-Schur functions and the Hecke algebra generators. After showing that the dual \( m \)-symmetric Schur functions form a basis of the ring of \( m \)-symmetric functions, the \( m \)-symmetric Schur function can be defined as their dual with respect to a certain scalar product. Properties of the (dual) \( m \)-symmetric Schur functions are then derived in Section 6 such as the action of the Hecke algebra generators on the (dual) \( m \)-symmetric Schur functions. Simple formulas for the inclusion and restriction of the (dual) \( m \)-symmetric Schur functions are also established. Our main conjecture, a positivity conjecture for \( m \)-symmetric Macdonald polynomials is featured in Section 7 after a notion of plethysm and the right normalization have been defined. Various relations on the Kostka coefficients \( K_{\Omega \Lambda}(q, t) \) are later given in Section 8.
These are the hardest results contained in this article (apart from actually finding the right definition of the \(m\)-symmetric Schur functions), as some of the relations depend on very technical results on \(m\)-symmetric Macdonald polynomials that are relegated to Appendix A in order not to interrupt the flow of the presentation. The case when \(m\) is large, in which case the \(m\)-symmetric Macdonald polynomials reduce to non-symmetric Macdonald polynomials, is studied in Section 10. It is shown that the positivity conjecture on \(m\)-symmetric Macdonald polynomials, modulo a subspace spanned by certain \(m\)-symmetric Schur functions, now expresses how a non-symmetric Macdonald polynomial expands into non-symmetric Hall-Littlewood polynomials. Finally, Section 10 contains conjectures on the coefficients \(K_{\Omega \Lambda}(q,t)\), including two rules reminiscent of Butler’s rule on the \((q,t)\)-Kostka coefficients.

2. Non-symmetric Macdonald polynomials

The non-symmetric Macdonald polynomials can be defined as the common eigenfunctions of the Cherednik operators \([8]\), which are operators that belong to the double affine Hecke algebra and act on the ring \(\mathbb{Q}(q,t)[x_1, \ldots, x_N]\). We now give the relevant definitions \([21, 22]\). Let the exchange operators \(\omega\), which are operators that belong to the double affine Hecke algebra and act on the ring \(\mathbb{Q}(q,t)[x_1, \ldots, x_N]\), be defined as:

\[
\omega_{\tau} \text{ is the Young diagram of } \tau \text{, such that } \omega T_i = T_{\tau} T_i \text{ for } \tau \leq \omega \text{ in the lexicographic ordering, and } \omega T_i = T_{\tau} T_i \text{ for } \tau \geq \omega.
\]

We note that \(\omega T_i = T_{i-1} \omega\) for \(i = 2, \ldots, N-1\).

We are now in position to define the Cherednik operators:

\[
Y_i = t^{N+1} T_i \cdots T_{N-1} \omega T_i^{-1} \cdots T_{i-1}^{-1},
\]

where

\[
T_j^{-1} = t^{-1} - 1 + t^{-1} T_j,
\]

which follows from the quadratic relation satisfied by the generators of the Hecke algebra. The Cherednik operators obey the following relations:

\[
T_i Y_i = Y_{i+1} T_i + (t-1) Y_i,
\]

\[
T_i Y_{i+1} = Y_i T_i - (t-1) Y_i,
\]

\[
T_i Y_j = Y_j T_i \text{ if } j \neq i, i + 1.
\]

It can be easily deduced from these relations that

\[
(Y_i + Y_{i+1}) T_i = T_i (Y_i + Y_{i+1}) \quad \text{and} \quad (Y_i Y_{i+1}) T_i = T_i (Y_i Y_{i+1}).
\]

An element \(\eta = (\eta_1, \ldots, \eta_N)\) of \(\mathbb{Z}_{\geq 0}^N\) is called a (weak) composition with \(N\) parts (or entries). It will prove convenient to represent a composition by a Young (or Ferrers) diagram. Given a composition \(\eta\) with \(N\) parts, let \(\eta^*\) be the partition obtained by reordering the entries of \(\eta\). The diagram corresponding to \(\eta^*\) is the Young diagram of \(\eta^*\) with an \(i\)-circle (a circle filled with an \(i\)) added to the right of the row of size \(\eta_i\) (if there are many rows of size
η_i, the circles are ordered from top to bottom in increasing order). For instance, given η = (0, 2, 1, 3, 2, 0, 2, 0, 0), we have

$$\eta \quad \leftrightarrow \quad \begin{array}{ccccccccc}
1 & 2 & 5 & 6 & 3 & 7 & 8 & 9 & 4
\end{array}$$

The Cherednik operators $Y_i$'s commute with each other, $[Y_i, Y_j] = 0$, and can be simultaneously diagonalized. Their eigenfunctions are the (monic) non-symmetric Macdonald polynomials (labeled by compositions). For $x = (x_1, \ldots, x_N)$, the non-symmetric Macdonald polynomial $E_{\eta}(x; q, t)$ is the unique polynomial with rational coefficients in $q$ and $t$ that is triangularly related to the monomials

$$E_{\eta}(x_1, \ldots, x_N; q, t) = x_1^{\eta_1} + \sum_{\nu < \eta} b_{\nu\eta}(q, t) x_\nu$$

and that satisfies, for all $i = 1, \ldots, N$,

$$Y_i E_{\eta} = \bar{\eta}_i E_{\eta}, \quad \text{where} \quad \bar{\eta}_i = q^{\eta_i} t^{1-r_\eta(i)}$$

with $r_\eta(i)$ standing for the row (starting from the top) in which the $i$-circle appears in the diagram of $\eta$. The order on compositions is defined as follows:

$$\nu < \eta \quad \text{iff} \quad \nu^+ < \eta^+ \quad \text{or} \quad \nu^+ = \eta^+ \quad \text{and} \quad w_\nu < w_\eta,$$

where $w_\eta$ is the unique permutation of minimal length such that $\eta = w_\eta \eta^+$ ($w_\eta$ permutes the entries of $\eta^+$), and where the order on permutations is the Bruhat order on the symmetric group ($w_\eta < w_\nu$ if $w_\eta$ has a reduced decomposition which is a proper subword of a reduced decomposition of $w_\nu$). The Cherednik operators have a triangular action on monomials \[20\], that is,

$$Y_i x^\eta = \bar{\eta}_i x^\eta + \text{smaller terms}$$

where "smaller terms" means that the remaining monomials $x^{\nu'}$ appearing in the expansion are such that $\nu < \eta$.

The following three properties of the non-symmetric Macdonald polynomials will be needed below. The first one expresses the stability of the polynomials $E_{\eta}$ with respect to the number of variables (see e.g. \[22\] eq. (3.2)):

$$E_{\eta}(x_1, \ldots, x_{N-1}, 0; q, t) = \begin{cases} E_{\eta-}(x_1, \ldots, x_{N-1}; q, t) & \text{if } \eta_N = 0, \\ 0 & \text{if } \eta_N \neq 0. \end{cases}$$

where $\eta_- = (\eta_1, \ldots, \eta_{N-1})$. The second one gives the action of the operators $T_i$ on $E_{\eta}$. It is a formula that will be fundamental for our purposes \[3\]:

$$T_i E_{\eta} = \begin{cases} \left( \frac{t^{i-1}}{1-\delta_{i,\eta}} \right) E_{\eta} + tE_{s_i \eta} & \text{if } \eta_i < \eta_{i+1}, \\ tE_{\eta} & \text{if } \eta_i = \eta_{i+1}, \\ \left( \frac{t^{i-1}}{1-\delta_{i,\eta}} \right) E_{\eta} + \frac{(1-t \delta_{i,\eta})(1-t^{-1} \delta_{i,\eta})}{(1-\delta_{i,\eta})^2} E_{s_i \eta} & \text{if } \eta_i > \eta_{i+1}, \end{cases}$$

$\delta_{i,\eta} = \bar{\eta}_i / \bar{\eta}_{i+1}$ and $s_i \eta = (\eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \eta_i, \eta_{i+2}, \ldots, \eta_N)$. The third property, together with the previous one, allows one to construct the non-symmetric Macdonald polynomials recursively. Given $\Phi_{\eta} = t^{1-N} T_{N-1} \cdots T_1 x_1$, we have that \[3\]

$$\Phi_{\eta} E_{\eta}(x; q, t) = t^{r_\eta(1)-N} E_{\Phi_\eta}(x; q, t)$$

where $\Phi_\eta = (\eta_2, \eta_3, \ldots, \eta_{N-1}, \eta_1 + 1)$. Finally, we introduce the $t$-symmetrization operator:

$$S_{m+1,N}^t = \sum_{\sigma \in S_{N-m}} T_\sigma$$

where the sum is over the permutations in the symmetric group $S_{N-m}$ and where $T_\sigma = T_{i_1+m} \cdots T_{i_k+m}$ if $\sigma = s_{i_1} \cdots s_{i_k}$ is a reduced expression. We stress that there is a shift by $m$ in the indices of the Hecke algebra generators.
Essentially, the $t$-symmetrization operator $S_{m+1,N}^t$ acts on the variables $x_{m+1}, x_{m+2}, \ldots, x_N$ while the usual $t$-symmetrization operator $S_{1,N}^t$ acts on all variables $x_1, x_2, \ldots, x_N$.

**Remark 1.** We have by (2.1) that if any polynomial $f(x_1, \ldots, x_N)$ is such that $T_i f(x_1, \ldots, x_N) = tf(x_1, \ldots, x_N)$, then $f(x_1, \ldots, x_N)$ is symmetric in the variables $x_i$ and $x_{i+1}$. As such, from (2.8), $E_\eta(x_1, \ldots, x_N)$ is symmetric in the variables $x_i$ and $x_{i+1}$ whenever $\eta_i = \eta_{i+1}$. We also have that $S_{m+1,N}^t f(x_1, \ldots, x_N)$ is symmetric in the variables $x_{m+1}, x_{m+2}, \ldots, x_N$ for any polynomial $f(x_1, \ldots, x_N)$ given that it can easily be checked that

$$T_i S_{m+1,N}^t = t S_{m+1,N}^t$$

for $i = m + 1, \ldots, N - 1$ (2.11)

The $t$-symmetrization operator satisfies the following well-known useful relations (which can be deduced from the theory of minimal coset representatives)

$$(1 + T_N + T_{N-1} + \cdots + T_{m+1} \cdots T_N) S_{m+1,N}^t = S_{m+1,N+1}^t$$

and

$$(1 + T_m + T_{m+1} + \cdots + T_{N-1} + T_{m} + T_N) S_{m+1,N}^t = S_{m,N}^t$$

We should note that, up to a constant, the usual Macdonald polynomial, $P_\lambda(x_1, \ldots, x_N; q, t)$, is obtained by $t$-symmetrizing a non-symmetric Macdonald polynomial:

$$P_\lambda(x; q, t) \propto S_{1,N}^t E_\eta(x; q, t)$$

where $\eta$ is any composition that rearranges to $\lambda$.

**3. The ring of $m$-symmetric functions**

Let $\Lambda = \mathbb{Q}(q, t)[h_1, h_2, h_3, \ldots]$ be the ring of symmetric functions in the variables $x_1, x_2, x_3, \ldots$ (the standard references on symmetric functions are [19] [25]), where

$$h_r = h_r(x_1, x_2, x_3, \ldots) = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

Bases of $\Lambda$ are indexed by partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k > 0)$ whose degree $\lambda$ is $|\lambda| = \lambda_1 + \cdots + \lambda_k$ and whose length $\ell(\lambda) = k$. Each partition $\lambda$ has an associated Young diagram with $\lambda_i$ lattice squares in the $i^{th}$ row, from top to bottom (English notation). Any lattice square $(i,j)$ in the $i$th row and $j$th column of a Young diagram is called a cell. The conjugate of $\lambda$, denoted $\lambda'$, is the reflection of $\lambda$ about the main diagonal. The partition $\lambda \cup \mu$ is the non-decreasing rearrangement of the parts of $\lambda$ and $\mu$. If the partition $\mu$ is contained in the partition $\lambda$, then the skew diagram $\lambda/\mu$ is the diagram obtained by removing the diagram corresponding to $\mu$ from the diagram of $\lambda$. The dominance order $\geq$ is defined on partitions by $\lambda \geq \mu$ when $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all $i$, and $|\lambda| = |\mu|$.

The ring $\Lambda$ can be thought as a subring of the ring of formal power series $\mathbb{Q}(q, t)[[x_1, x_2, x_3, \ldots]]$ since it consists of the elements of $\mathbb{Q}(q, t)[[x_1, x_2, x_3, \ldots]]$ that are symmetric in the variables $x_1, x_2, x_3, \ldots$ and have bounded degree. In this spirit, we will define the ring $R_m$ of $m$-symmetric functions as the subring of $\mathbb{Q}(q, t)[[x_1, x_2, x_3, \ldots]]$ made of formal power series that are symmetric in the variables $x_{m+1}, x_{m+2}, x_{m+3}, \ldots$ and have bounded degree. In other words, we have

$$R_m \simeq \mathbb{Q}(q, t)[x_1, \ldots, x_m] \otimes \Lambda_m$$

where $\Lambda_m$ is the ring of symmetric functions in the variables $x_{m+1}, x_{m+2}, x_{m+3}, \ldots$. The ring $R_m$ is graded with respect to the total degree in $x$

$$R_m = R_m^0 \oplus R_m^1 \oplus R_m^2 \oplus \cdots$$

where $R_m^d$ is the subspace of $R_m$ made of formal power series of homogeneous degree $d$. It is immediate that $R_0 = \Lambda$ is the usual ring of symmetric functions and that $R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$. Bases of $R_m$ are naturally indexed by $m$-partitions which are pairs $\Lambda = (a; \lambda)$, where $a = (a_1, \ldots, a_m) \in \mathbb{Z}_{\geq 0}^m$ is a composition with $m$ parts, and where $\lambda$ is a partition. We will call the entries of $a$ and $\lambda$ the non-symmetric and symmetric entries of $\Lambda$ respectively. In the following, unless stated otherwise, $\Lambda$ and $\Omega$ will always stand respectively for the $m$-partitions $\Lambda = (a; \lambda)$ and $\Omega = (b; \mu)$. Observe that we use a different notation for the composition $a$ with $m$ parts (which corresponds to the non-symmetric entries of $\Lambda$) than for the composition $\eta$ with $N$ parts (which will typically index a non-symmetric Macdonald polynomial).

Given a composition $a$ and a partition $\lambda$, $a \cup \lambda$ will denote the partition obtained by reordering the entries of the concatenation of $a$ and $\lambda$. The degree of an $m$-partition $\Lambda$, denoted $|\Lambda|$, is the sum of the degrees of $a$ and $\lambda$, that is, $|\Lambda| = a_1 + \cdots + a_m + \lambda_1 + \lambda_2 + \cdots$. We also define the length of $\Lambda$ as $\ell(\Lambda) = m + \ell(\lambda)$. We will say that $a$ is dominant if $a_1 \geq a_2 \geq \cdots \geq a_m$, and by extension, we will say that $\Lambda = (a; \lambda)$ is dominant if $a$ is dominant.
If \( a \) is not dominant, we let \( a^+ \) be the dominant composition obtained by reordering the entries of \( a \). We also let \( \Lambda^+ = (a^+; \lambda) \).

There is a natural way to represent an \( m \)-partition by a Young diagram. The diagram corresponding to \( \Lambda \) is the Young diagram of \( a \cup \lambda \) with an \( i \)-circle added to the right of the row of size \( a_i \) for \( i = 1, \ldots, m \) (if there are many rows of size \( a_i \), the circles are ordered from top to bottom in increasing order). For instance, given \( \Lambda = (2, 0, 2, 1; 3, 2) \), we have

\[
\begin{align*}
\Lambda & \leftrightarrow \\
& \begin{array} {ccc}
1 & 2 & \\
3 & 4 & \\
\end{array}
\end{align*}
\]

Observe that when \( m = 0 \), the diagram associated to \( \Lambda = (\lambda) \) coincides with the Young diagram associated to \( \lambda \). Also note that if \( \eta \) is a composition with \( m \) parts, then the diagram of \( \eta \) coincides with the diagram of the \( m \)-partition \( \Lambda = (a; \emptyset) \), where \( a = \eta \). We let \( \Lambda^{(0)} = a \cup \lambda \), that is, \( \Lambda^{(0)} \) is the partition obtained from the diagram of \( \Lambda \) by discarding all the circles. More generally, for \( i = 1, \ldots, m \), we let \( \Lambda^{(i)} = (a + 1^i) \cup \lambda \), where \( a + 1^i = (a_1 + 1, \ldots, a_i + 1, a_{i+1}, \ldots, a_m) \). In other words, \( \Lambda^{(i)} \) is the partition obtained from the diagram associated to \( \Lambda \) by changing all of the \( j \)-circles, for \( 1 \leq j \leq i \), into squares and discarding the remaining circles. Taking as above \( \Lambda = (2, 0, 2, 1; 3, 2) \), we have \( \Lambda^{(0)} = (3, 2, 2, 2), \Lambda^{(1)} = (3, 3, 2, 2), \Lambda^{(2)} = (3, 3, 2, 2, 1), \Lambda^{(3)} = (3, 3, 3, 2, 1) \) and \( \Lambda^{(4)} = (3, 3, 3, 2, 2, 1) \). We then define the dominance ordering on \( m \)-partitions to be such that

\[
\Lambda \geq \Omega \iff \Lambda^{(i)} \geq \Omega^{(i)} \quad \text{for all } i = 0, \ldots, m
\]

where the order on the r.h.s. is the usual dominance order on partitions. Note that the meaning of dominance order will become apparent later (see Propositions 7 and 8).

We let the \( m \)-symmetric monomial function \( m_\lambda(x) \) be defined as

\[
m_\lambda(x) := x_{a_1}^{a_1} \cdots x_m^{a_m} m_\lambda(x_{m+1}, x_{m+2}, \ldots) = x^a m_\lambda(x_{m+1}, x_{m+2}, \ldots)
\]

where \( m_\lambda(x_{m+1}, x_{m+2}, \ldots) \) is the usual monomial symmetric function in the variables \( x_{m+1}, x_{m+2}, \ldots \)

\[
m_\lambda(x_{m+1}, x_{m+2}, \ldots) = \sum_\alpha x_{m+1}^{\alpha_1} x_{m+2}^{\alpha_2} \cdots
\]

with the sum being over all derangements \( \alpha \) of \( (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}, 0, 0, \ldots) \). We emphasize that unless stated otherwise, \( x \) will always stand for the variables \( (x_1, x_2, \ldots) \) so that \( m_\lambda(x) = m_\lambda(x_1, x_2, \ldots) \). It is immediate that \( \{m_\lambda(x)\}_\lambda \) is a basis of \( R_m \), and more specifically, that

\[
\{m_\lambda(x)\}_{\lambda} \mid d = |\lambda|
\]

is a basis of \( R_m^d \). The following stronger statement holds.

**Proposition 2.** For \( \ell = 1, \ldots, m + 1 \), we have that \( \{x^a m_\lambda(x_\ell, x_{\ell+1}, \ldots)\}_{a, \lambda} \) is a basis of \( R_m \). In particular, for \( x = (x_1, x_2, x_3, \ldots) \), we have that \( \{x^a m_\lambda(x)\}_{a, \lambda} \) is a basis of \( R_m \).

**Proof.** It is obvious that \( x^a m_\lambda(x_\ell, x_{\ell+1}, \ldots) \) belongs to \( R_m \) since it is symmetric in \( x_{m+1}, x_{m+2}, x_{m+3}, \ldots \). We will show that

\[
\{x^a m_\lambda(x_\ell, x_{\ell+1}, \ldots)\}_{d = |a| + |\lambda|}
\]

is a basis of \( R_m^d \). We proceed by induction from the base case \( \ell = m + 1 \) which, as we have seen, holds. Having the right number of elements, we simply need to show that the \( x^a m_\lambda(x_\ell, x_{\ell+1}, \ldots) \)’s are linearly independent. Suppose by contradiction that \( \ell \leq m \) and that in \( R_m^d \) we have

\[
\sum_{a, \lambda} c_{a, \lambda} x^a m_\lambda(x_\ell, x_{\ell+1}, \ldots) = 0
\]

Extracting the power of \( x_\ell \) in \( x^a \), we get

\[
\sum_{a, \lambda} c_{a, \lambda} x^{a_\ell} x^b m_\lambda(x_\ell, x_{\ell+1}, \ldots) = 0
\]
where $b = (b_1, \ldots, b_m)$ is such that $b_\ell = 0$ and $b_i = a_i$ for $i \neq \ell$. Let $k$ be the lowest value of $a_\ell$ such that there exists a non-zero coefficient $c_{a, \lambda}$. Extracting $x_\ell^k$, we get that
\[ \sum_{a, \lambda} c_{a, \lambda} x_\ell^{a-k} x^b m_\lambda(x_\ell, x_{\ell+1}, x_{\ell+2}, \ldots) = 0 \]

Letting $x_\ell = 0$ in this equation, we obtain
\[ \sum_{a, \lambda: a_\ell = k} c_{a, \lambda} x^b m_\lambda(0, x_{\ell+1}, x_{\ell+2}, \ldots) = \sum_{a, \lambda: a_\ell = k} c_{a, \lambda} x^b m_\lambda(x_{\ell+1}, x_{\ell+2}, \ldots) = 0 \]

since $m_\lambda(0, x_{\ell+1}, x_{\ell+2}, \ldots) = m_\lambda(x_{\ell+1}, x_{\ell+2}, \ldots)$. Hence
\[ x_\ell \sum_{a, \lambda: a_\ell = k} c_{a, \lambda} x^b m_\lambda(x_{\ell+1}, x_{\ell+2}, \ldots) = \sum_{a, \lambda: a_\ell = k} c_{a, \lambda} x^b m_\lambda(x_{\ell+1}, x_{\ell+2}, \ldots) = 0 \]

Note that, by the minimality of $k$, there is a non-zero coefficient in the sum. This contradicts our induction hypothesis that $\{x^a m_\lambda(x_{\ell+1}, x_{\ell+2}, \ldots) | d = |a| + |\lambda| \}$ is a basis of $R_m^\ell$. 

The $m$-symmetric power sums are naturally defined as
\[ p_\lambda(x) := x_1^{a_1} \cdots x_m^{a_m} p_\lambda(x) = x^a p_\lambda(x) \]

It should be observed that the variables in $p_\lambda$, contrary to those of $m_\lambda$ in $m_\lambda(x)$, start at $x_1$ instead of $x_{m+1}$. In this expression, $p_\lambda(x)$ is the usual power-sum symmetric function
\[ p_\lambda(x) = \prod_{i=1}^{\ell(\lambda)} p_{\ell(i)}(x) \]

where $p_{\ell}(x) = x_1^{\ell} + x_2^{\ell} + \cdots$. From Proposition 2 we get immediately that $\{p_\lambda(x)\}_\lambda$ is a basis of $R_m$.

Another basis of the ring of symmetric functions is provided by the Schur functions $s_\lambda(x)$
\[ s_\lambda(x) = \det (h_{\lambda_i - j})_{1 \leq i, j \leq \ell(\lambda)} \]

where $h_i = 0$ if $i < 0$. Replacing $p_\lambda(x)$ by the Schur function $s_\lambda(x)$, we define
\[ k_\lambda(x) := x_1^{a_1} \cdots x_m^{a_m} s_\lambda(x) = x^a s_\lambda(x) \]

It is then again obvious due to Proposition 2 that $\{k_\lambda(x)\}_\lambda$ is a basis of $R_m$. We stress that this basis is not the right extension of the Schur functions in the $m$-symmetric world (hence the notation $k_\lambda$ instead of $s_\lambda$). A $t$-generalization of the $k_\lambda(x)$ basis will nevertheless prove instrumental in the construction of the $m$-symmetric Schur functions.

4. $m$-symmetric Macdonald polynomials

The $m$-symmetric Macdonald polynomials in $N$ variables are simply obtained by applying the $t$-symmetrization operator $S^t_{m+1,N}$ introduced in (2.10) to non-symmetric Macdonald polynomials. To be more specific, we associate to the $m$-partition $\Lambda = (\lambda; a_\lambda)$ the composition $\eta_{\Lambda, N} = (a_1, \ldots, a_m, 0^n, \lambda_{\ell(\lambda)}, \ldots, \lambda_1)$, where $n = N - m - \ell(\lambda) \geq 0$. The corresponding $m$-symmetric Macdonald polynomial in $N$ variables is then defined as
\[ P_\lambda(x_1, \ldots, x_N; q, t) = \frac{1}{u_{\Lambda, N}(t)} S^t_{m+1,N} E_{\eta_{\Lambda, N}}(x_1, \ldots, x_N; q, t) \]

with the normalization constant $u_{\Lambda, N}(t)$ given by
\[ u_{\Lambda, N}(t) = [n]_t^{-1} \left( \prod_{i=1}^{\ell(\lambda)} [n_{\lambda_i}(i)]_{t^{-1}} \right) t^{(N-m)(N-m-1)/2} \]

where $n_{\lambda_i}(i)$ is the number of entries in $\lambda$ that are equal to $i$, and where
\[ [k]_q! = \frac{(1-q)(1-q^2)\cdots(1-q^k)}{(1-q)^k} \]

We will see later that the normalization constant $u_{\Lambda, N}(t)$ is chosen such that the coefficient of $m_\lambda$ in $P_\lambda(x; q, t)$ is equal to 1.

Remark 3. We have that $T_i$ commutes with $S^t_{m+1,N}$ for $i = 1, \ldots, m - 1$. Hence, from Remark 7, if $\Lambda$ is such that $a_i = a_{i+1}$ then $P_\lambda(x_1, \ldots, x_N; q, t)$ is symmetric in the variables $x_i, x_{i+1}$. 
The $m$-symmetric Macdonald polynomials are stable with respect to the number of variables.

**Proposition 4.** Let $N$ be the number of variables and suppose that $N \geq m + \ell(\lambda)$. Then

$$P_\lambda(x_1, \ldots, x_{N-1}, 0; q, t) = \begin{cases} P_\lambda(x_1, \ldots, x_{N-1}; q, t) & \text{if } N > m + \ell(\lambda) \\ 0 & \text{if } N = m + \ell(\lambda) \end{cases}$$

**Proof.** It was shown in [22] (see formula (38) in Proposition 3, where the version of $u_{\lambda, \eta}(t)$ used therein is slightly different from the one we use here) that

$$S^t_{m+1, N} E_{\eta, N} \bigg|_{x_N = 0} = \begin{cases} \frac{u_{\lambda, \eta}(t)}{u_{\lambda, \eta}(t)} S^t_{m+1, N-1} E_{\eta, N-1} & \text{if } N > m + \ell(\lambda) \\ 0 & \text{otherwise} \end{cases}$$

The proposition then follows immediately. \[\square\]

Since the $m$-symmetric monomial functions $m_\lambda$ are also such that

$$m_\lambda(x_1, \ldots, x_{N-1}, 0) = \begin{cases} m_\lambda(x_1, \ldots, x_{N-1}) & \text{if } N > m + \ell(\lambda) \\ 0 & \text{if } N = m + \ell(\lambda) \end{cases}$$

we have immediately the following corollary.

**Corollary 5.** The expansion coefficients $d_{\Omega}(q, t)$ in

$$P_\lambda(x_1, \ldots, x_{N}; q, t) = \sum_{\Omega} d_{\Omega}(q, t)m_{\Omega}(x_1, \ldots, x_N) \quad (4.3)$$

do not depend on the number of variables $N$. Moreover, the number of terms in the r.h.s. of (4.3) does not depend on $N$ as long as $N \geq m + |\lambda|$ (the longest $m$-partition of a given degree $d$ being $(0^m; 1^d)$).

We see from Remark 6 that the $m$-symmetric Macdonald polynomials are symmetric in the variables $x_m+1, x_{m+2}, \ldots, x_N$. From the previous corollary, we can thus let $N \to \infty$ and define the $m$-symmetric Macdonald polynomials

$$P_\lambda(x; q, t) \in R_\infty,$$

for $x = (x_1, x_2, x_3, \ldots)$, as

$$P_\lambda(x; q, t) = \sum_{\Omega} d_{\Omega}(q, t)m_{\Omega}(x)$$

where the coefficients $d_{\Omega}(q, t)$ are those of (4.3).

**Remark 6.** From the comment at the end of Section 3 when $m = 0$ an $m$-Macdonald polynomial is a usual Macdonald polynomial. In the case $m = 1$, an $m$-symmetric Macdonald polynomial corresponds to the $\theta_1$ coefficient of a Macdonald polynomial in superspace with fermionic degree equal to $I$ [35].

It is obvious from [22] that $S^t_{m+1, N}$ commutes with $Y_i$, for $i = 1, \ldots, m$. It is then immediate from the definition of the $m$-symmetric Macdonald polynomials and [25] that $P_\lambda(x_1, \ldots, x_N; q, t)$ is an eigenfunction of $Y_i$ with eigenvalue $(\eta_{\lambda, N})_i$. Now, it is not hard to see that the diagram of $\Lambda$ is obtained from that of $\eta_{\Lambda, N}$ by removing the circles filled with an $m + 1, m + 2, \ldots, N$. Hence, for $i = 1, \ldots, m$, we have

$$Y_i P_\lambda(x_1, \ldots, x_N; q, t) = \varepsilon^{(i)}(q, t) P_\lambda(x_1, \ldots, x_N; q, t), \quad \text{with } \varepsilon^{(i)}(q, t) = q^a t^{1-r_{\lambda}(i)} \quad (4.4)$$

where we recall that $r_{\lambda}(i)$ is the row in which the $i$-circle appears in the diagram associated to $\Lambda$. Letting

$$D = Y_{m+1} + \cdots + Y_N - \sum_{i=m+1}^N t^{1-i}$$

it is also quite easy to check, using (2.2) and (2.3), that $D$ commutes with $S^t_{m+1, N}$. Following the argument used previously in the case of $Y_i$, this implies that $P_\lambda(x_1, \ldots, x_N; q, t)$ is an eigenfunction of $D$ with eigenvalue

$$(\eta_{\Lambda, N})_{m+1} + \cdots + (\eta_{\Lambda, N})_N - \sum_{i=m+1}^N t^{1-i}$$

Note that the $i$-circles, for $i = m + \ell(\lambda) + 1, \ldots, N$, are located in rows $m + \ell(\lambda) + 1$ up to $N$ of the diagram of $\eta_{\Lambda, N}$, which are all of length zero. From our previous observation that the diagram of $\Lambda$ is obtained from that of $\eta_{\Lambda, N}$ by removing the circles filled with an $m + 1, m + 2, \ldots, N$ we then get that

$$D P_\lambda(x_1, \ldots, x_N; q, t) = \varepsilon^{D}_\lambda(q, t) P_\lambda(x_1, \ldots, x_N; q, t), \quad \text{with } \varepsilon^{D}_\lambda(q, t) = \sum_i q^{\lambda_i(0)} t^{1-i} - \sum_{i=m+1}^{m+\ell(\lambda)} t^{1-i} \quad (4.5)$$
where the prime indicates that the sum is only over the rows of the diagram of $\Lambda$ that do not end with a circle. We stress that the eigenvalues $\varepsilon_{\Lambda}^{(i)}(q,t)$ and $\varepsilon_{\Lambda}^{(q)}(q,t)$ do not depend on the number $N$ of variables.

Before proving that the $m$-symmetric Macdonald polynomials are triangularly related to the monomials in the dominance order $\Omega \leq \Lambda$, we need the following.

**Proposition 7.** The operators $Y_i, \ldots, Y_m$ and $D$ have a triangular action on monomials (in the dominance order on $m$-partitions). To be more precise, for certain coefficients $b_{\Lambda \Omega}^{(i)}(q,t)$ and $b_{\Lambda \Omega}(q,t)$ in $\mathbb{Q}[q,t]$, we have that

$$Y_i m_{\Lambda}(x_1, \ldots, x_N) = \varepsilon_{\Lambda}^{(i)}(q,t) m_{\Lambda}(x_1, \ldots, x_N) + \sum_{\Omega < \Lambda} b_{\Lambda \Omega}^{(i)}(q,t) m_{\Omega}(x_1, \ldots, x_N)$$

and

$$D m_{\Lambda}(x_1, \ldots, x_N) = \varepsilon_{\Lambda}^{(q)}(q,t) m_{\Lambda}(x_1, \ldots, x_N) + \sum_{\Omega < \Lambda} b_{\Lambda \Omega}(q,t) m_{\Omega}(x_1, \ldots, x_N)$$

with $\varepsilon_{\Lambda}^{(i)}(q,t)$ and $\varepsilon_{\Lambda}^{(q)}(q,t)$ given in (4.4) and (4.5) respectively.

**Proof.** It is obvious that the operators $Y_i$ for $i = 1, \ldots, m$ and $D$ preserve $R_m$ since, from (2.2) and (2.6), they all commute with $T_j$ for $j = m+1, \ldots, N-1$. We thus only need to show the triangularity and that the dominant coefficients are as indicated.

It is shown in [22] (see the proof of Proposition 6 therein) that if $x^\gamma$ appears in $Y_i x^n$, then $\gamma^{(0)} \leq \eta^{(0)}$ and $\gamma^{(i)} \leq \eta^{(i)}$ for any $i$, where $\gamma^{(0)} = \gamma^+$ and $\gamma^{(i)} = (\gamma + 1^i)^+$. We therefore get that a given “smaller term” $x^\gamma$ appearing in $Y_i x^n$ is such that $\gamma^{(0)} \leq \eta^{(0)}$ and $\gamma^{(i)} \leq \eta^{(i)}$ for all $j = 1, \ldots, m$. Observe that if $x^\gamma$ is a monomial that appears in $m_{\Omega}$, then $\gamma^{(0)} = \Omega^{(0)}$ and $\gamma^{(i)} = \Omega^{(i)}$ for every $j = 1, \ldots, m$. Hence, from our previous observation,

$$Y_i m_{\Lambda} = c_{i,\Lambda}(q,t) m_{\Lambda} + \sum_{\Omega < \Lambda} b_{\Lambda \Omega}^{(i)}(q,t) m_{\Omega} \quad \text{and} \quad D m_{\Lambda} = c_{\Lambda}(q,t) m_{\Lambda} + \sum_{\Omega < \Lambda} b_{\Lambda \Omega}(q,t) m_{\Omega}$$

for certain coefficients $c_{i,\Lambda}(q,t)$, $b_{\Lambda \Omega}^{(i)}(q,t)$, $c_{\Lambda}(q,t)$, and $b_{\Lambda \Omega}(q,t)$. It remains to show that $c_{i,\Lambda}(q,t) = \varepsilon_{\Lambda}^{(i)}(q,t)$ and $c_{\Lambda}(q,t) = \varepsilon_{\Lambda}^{(q)}(q,t)$. Let $\eta$ be maximal in the order $\prec$ on compositions among all the monomials $x^n$ appearing in $m_{\Lambda}$. From (4.4), we then have that the coefficients of $x^n$ in $Y_i m_{\Lambda}$ and $D m_{\Lambda}$ are respectively equal to $\varepsilon_{\Lambda}^{(i)}(q,t)$ and $\varepsilon_{\Lambda}^{(q)}(q,t)$. By symmetry, the coefficients of $m_{\Lambda}$ are as wanted. \hfill \Box

**Proposition 8.** We have that

$$P_{\Lambda}(x; q, t) = m_{\Lambda} + \sum_{\Omega < \Lambda} d_{\Omega}(q,t) m_{\Omega}$$

Hence, the $m$-symmetric Macdonald polynomials form a basis of $R_m$.

**Proof.** We prove the result when the number of variables $N$ is finite. The infinite case will then follow from Corollary 5.

First note that the eigenvalues $\varepsilon_{\Lambda}^{(i)}(q,t)$, for $i = 1, \ldots, m$ and $\varepsilon_{\Lambda}^{(q)}(q,t)$ completely determine $\Lambda$ (the powers of $q$ in $\varepsilon_{\Lambda}^{(q)}(q,t)$ determine $\Lambda$ while the power of $q$ in $\varepsilon_{\Lambda}^{(i)}(q,t)$ determines $a_i$). Suppose that there exists a term $m_{\Omega}$ with non-zero coefficient in the monomial expansion of $P_{\Lambda}$ such that $\Omega \not\preceq \Lambda$ and let $\Gamma$ be maximal among those terms. From Proposition 7, the coefficients of $m_{\Gamma}$ in $Y_i P_{\Lambda}$ and $D P_{\Lambda}$ are respectively $\varepsilon_{\Gamma}^{(i)}$ and $\varepsilon_{\Gamma}^{(q)}$. But from (4.4) and (4.5), those coefficients are also equal to $\varepsilon_{\Lambda}^{(i)}(q,t)$ and $\varepsilon_{\Lambda}^{(q)}(q,t)$. Hence $\varepsilon_{\Gamma}^{(i)} = \varepsilon_{\Lambda}^{(i)}$ and $\varepsilon_{\Gamma}^{(q)} = \varepsilon_{\Lambda}^{(q)}$. Since, as we have seen, those eigenvalues determine $\Lambda$, we have the contradiction that $\Gamma \preceq \Lambda$.

Finally, we need to show that the coefficient of $m_{\Lambda}$ in $P_{\Lambda}$ is equal to 1. It is shown in [22] (see equation (5.35) therein) that, when $m = 0$, the coefficient of $x^\Lambda$ in $S_{1,N}^i E_{0,\Lambda}$ is equal to the coefficient of $x^\Lambda$ in $S_{1,N}^i x^{\eta,\Lambda}$, which is itself equal to $u_{\Lambda,N}(t)$. The same exact argument can be used to show that the coefficient of $x^\Lambda$ in $S_{m+1,N}^i E_{0,\Lambda}$ is equal to the coefficient of $x^\Lambda$ in $S_{m+1,N}^i x^{\eta,\Lambda}$, which is itself equal to $u_{\Lambda,N}(t)$. The coefficient of $m_{\Lambda}$ in $P_{\Lambda}$ is then equal to 1 given the normalization of $P_{\Lambda}$. \hfill \Box
5. Definition of the (dual) m-symmetric Schur functions

In the remainder of this article, we will use the notation \( s_i \Lambda = (a_1, \ldots, a_{i+1}, a_i, \ldots, a_m) \), and more generally \( s_i \Lambda = (s_i \Lambda; \Lambda) \).

Let \( H_\Lambda(x_1, \ldots, x_m; t) = E_\Lambda(x_1, \ldots, x_N; 0, t) \) be the non-symmetric Hall-Littlewood polynomial (observe that the non-symmetric Macdonald polynomial \( E_\Lambda(x_1, \ldots, x_N; q, t) \) only depends on the variables \( x_1, \ldots, x_m \) when \( q = 0 \) and the indexing composition has length at most \( m \)). For \( i = 1, \ldots, m - 1 \), the action of the Hecke operator \( T_i \) on the non-symmetric Hall-Littlewood polynomials, which can be obtained by taking the limit \( q = 0 \) in \((2.8)\), is the following:

\[
T_i H_\Lambda = \begin{cases} 
H_{s_i \Lambda} & \text{if } a_i > a_{i+1} \\
(t-1)H_\Lambda + tH_{s_i \Lambda} & \text{if } a_i < a_{i+1} \\
tH_\Lambda & \text{if } a_i = a_{i+1}
\end{cases}
\tag{5.1}
\]

We should note that the polynomial \( H_\Lambda(x_1, \ldots, x_m; t) \) can be constructed recursively from this formula since \( H_\Lambda = x_\Lambda \) when \( \Lambda \) is dominant (it is not too difficult to show that it is indeed the case using \((2.9)\) and \((5.1)\)). When \( t = 1 \), the operator \( T_i \) becomes \( K_{i,i+1} \) and it is then immediate from this construction that \( H_\Lambda(x_1, \ldots, x_m; 1) = x_\Lambda \). It is thus natural to define the following \( t \)-generalization of the \( k_\Lambda(x) \) basis:

\[
k_\Lambda(x; t) = H_\Lambda(x_1, \ldots, x_m; t)s_\Lambda(x)
\]

We stress that

\[
\{k_\Lambda(x; t) \mid \Lambda \in \mathbb{Z}_m \}
\]

is indeed a basis of \( R_m \) since we have seen that the special case \( k_\Lambda(x; 1) = k_\Lambda(x) \) is a basis of \( R_m \) (that is, the \( k_\Lambda(x; t) \)'s are linearly independent and thus form a basis of \( R_m \) by dimension considerations).

Since \( s_\Lambda(x) \) commutes with \( T_i \) for all \( i \), \((5.1)\) implies immediately the following.

**Lemma 9.** For \( i = 1, \ldots, m - 1 \), the operator \( T_i \) acts on \( k_\Lambda(x; t) \) as

\[
T_i k_\Lambda = \begin{cases} 
k_{\tilde{\Lambda}} & \text{if } a_i > a_{i+1} \\
(t-1)k_\Lambda + tk_{\tilde{\Lambda}} & \text{if } a_i < a_{i+1} \\
tk_\Lambda & \text{if } a_i = a_{i+1}
\end{cases}
\tag{5.2}
\]

where \( \tilde{\Lambda} = s_i \Lambda \).

A bilinear scalar product \( \langle \cdot, \cdot \rangle_m \) on \( R_m \) is defined by requiring that the \( k_\Lambda(x; t) \) basis be such that:

\[
\langle k_\Lambda(x; t), k_\Omega(x; t) \rangle_m = \delta_{\Lambda\Omega} t^{\text{Inv}(\Lambda)}
\tag{5.3}
\]

where

\[
\text{Inv}(\Lambda) = \# \{ i < j \mid a_i < a_j \}
\tag{5.4}
\]

is the number of inversions in \( \Lambda \).

The following proposition follows immediately from Lemma 9.

**Proposition 10.** For \( i = 1, \ldots, m - 1 \), the operator \( T_i \) is self-adjoint with respect to the scalar product \( \langle \cdot, \cdot \rangle_m \), that is,

\[
\langle T_if, g \rangle_m = \langle f, T_ig \rangle_m \quad \text{for all } f, g \in R_m
\]

**Proof.** We prove that the relation holds on the basis \( \{k_\Lambda(x; t)\}_\Lambda \). Let \( \Lambda = (\Lambda; \Lambda) \) be such that \( a_i > a_{i+1} \). Using Lemma 9 we can easily check that, for \( i = 1, \ldots, m - 1 \), we have

\[
\langle T_i k_{\Lambda}, k_\Lambda \rangle = t^{\text{Inv}(s_i \Lambda)} = t^{1 + \text{Inv}(\Lambda)} = \langle k_{\Lambda}, T_i k_{\Lambda} \rangle
\]

which also implies that \( \langle T_i k_{\Lambda}, k_\Lambda \rangle = \langle k_{\Lambda}, T_i k_{\Lambda} \rangle \). This completes the proof since the only non-zero remaining scalar products to verify are symmetric.

We are now ready to introduce the dual \( m \)-symmetric Schur functions. Once this is done, we will then be able to define the \( m \)-symmetric Schur functions by duality (see Definition 20). As mentioned in the introduction, the \( m \)-symmetric Schur functions will provide the right basis to state our positivity conjecture for the \( m \)-symmetric Macdonald polynomials (Conjecture 25). The fact, by Remark 35 that the \((q,t)\)-Kostka coefficient \( K_{\Omega \Lambda}(q,t) \) at \( q = t = 1 \) counts the number of standard tableaux of a certain shape and, by Corollary 35 that the usual \((q,t)\)-Kostka coefficients \( K_{\mu \Lambda}(q,t) \) are special cases of the coefficients \( K_{\Omega \Lambda}(q,t) \) will give further evidence that the
In the following, it will prove convenient to use the plethystic notation in which, for a symmetric function \( f \) and \( X = x_1 + x_2 + \cdots \), we let \( f[X] = f(x) = f(x_1, x_2, \ldots) \). More generally, we have using this notation that \( f[X + x_1 + \cdots + x_k] = f(x_1, \ldots, x_k, x_1, x_2, \ldots) \).

Let \( \nu \) be a partition of length \( \ell \). For a sequence of alphabets \( X_1, \ldots, X_\ell \), where \( \ell = \ell(\nu) \), the multi-Schur function \( s_\nu(X_1, \ldots, X_\ell) \) is defined as \([18]\)

\[
s_\nu(X_1, \ldots, X_\ell) = \det \left( h_{\nu_i-i+j}[X_i] \right)_{1 \leq i, j \leq \ell}
\]

Observe that when comparing with \([5.2]\), we get that \( s_\nu(X_1, \ldots, X_\ell) = s_\nu(x) \) whenever \( X = X_1 = X_2 = \cdots = X_\ell \).

**Definition 11.** The dual \( m \)-symmetric Schur functions \( s^*_\Lambda(x;t) \) are defined recursively in the following way. If \( \Lambda = (a; \lambda) \) is dominant, then

\[
s^*_\Lambda(x;t) = s_\nu(X_1, \ldots, X_\ell)
\]

where \( \nu = \Lambda^{(0)} = a \cup \lambda \), and where \( X_i \) stands for the alphabet \( X + x_1 + \cdots + x_k \) with \( k \) the number of circles weakly above row \( i \) in the diagram corresponding to \( \Lambda \). Otherwise, if \( a_i < a_{i+1} \), then

\[
s^*_\Lambda(x;t) = T_is^*_\tilde{\Lambda}(x;t)
\]

where \( \tilde{\Lambda} = s_i \Lambda \), which amounts to saying that

\[
s^*_\lambda(x;t) = T_{\sigma^{-1}}s^*_\lambda(x;t)
\]

where \( \sigma \) is the shortest permutation such that \( \sigma(a) = (a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(m)}) = a^+ \).

**Example 12.** The diagram associated to \((2, 1; 3, 1)\) is

```
\[
\begin{array}{c}
\text{1} \\
\text{2}
\end{array}
\]
```

from which we deduce that

\[
s^*_{2,1;3,1}(x;t) = \begin{vmatrix}
h_3[X] & h_4[X] & h_5[X] & h_6[X] \\
h_1[X + x_1] & h_2[X + x_1] & h_3[X + x_1] & h_4[X + x_1] \\
0 & h_0[X + x_1 + x_2] & h_1[X + x_1 + x_2] & h_2[X + x_1 + x_2] \\
0 & 0 & h_0[X + x_1 + x_2] & h_1[X + x_1 + x_2]
\end{vmatrix}
\]

**Remark 13.** The multi-Schur functions that we use in Definition \([17]\) are essentially flagged Schur functions \([17, 26]\) in infinite alphabets instead of finite alphabets. For instance, if \( X \) were equal to \( y_1 + \cdots + y_j \), the dual \( m \)-symmetric Schur function \( s^*_{2,1;3,1}(x;t) \) would correspond in the language of \([26]\) to the flagged Schur functions \( s_{3,2,1;1}(b) \) with flags \( b_1 = 0, b_2 = 1, b_3 = 2 \) and \( b_4 = 2 \) (with the understanding that the variables \( y_1, \ldots, y_j \) would not be constrained by any of the flags).

**Remark 14.** Using the connection with the flagged Schur functions mentioned in the previous remark, we can obtain a combinatorial interpretation \([16]\) for the expansion of \( s^*_\lambda(x;t) \) in terms of the \( k_{\Omega}(x) \)'s, where we recall that \( k_{\Omega}(x) = k_{\Omega}(x;1) \). We describe this combinatorial interpretation in the case \( m = 1 \) since we will need it in Section \( \S \) In that case, is not difficult to see that Proposition \([16]\) yields

\[
s^*_{(a_1; \lambda)}(x;t) = \sum_{(b_1; \mu)} k_{(b_1; \mu)}(x) = \sum_{(b_1; \mu)} k_{(b_1; \mu)}(x;t)
\]

where the sum is over all \((b_1; \mu)\)'s such that \((\lambda \cup (a_1))/\mu \) is a horizontal \( b_1 \)-strip whose cells all lie within the first \( a_1 \) columns. Observe that the second equality holds since all \( m \)-partitions are dominant when \( m = 1 \), which implies that \( k_{\Omega}(x;t) \) does not depend on \( t \) in that case.

We first prove an elementary property of the dual \( m \)-symmetric Schur functions.

**Proposition 15.** If \( \Lambda = (0, 0, \ldots, 0; \lambda) \) then \( s^*_\Lambda(x;t) = k_{\lambda}(x;t) = s_\lambda(x) \).
Proof. If $\ell$ is the length of $\lambda$, we have in this case from Definition 11 that
\[ s^*_\lambda(x; t) = s_\lambda(x_1, \ldots, x_\ell) \]
where $X = X_1 = \cdots = X_\ell$ given that all the circles are below row $\ell$ in the diagram of $\Lambda$. It is then immediate that $s^*_\lambda(x; t) = s_\lambda(x)$ from the observation after (5.5). \hfill \Box

The following recursions due to Luis Pena [24] will prove useful. In the remainder of the article, $\lambda \setminus \lambda_j$ will stand for the partition obtained by removing the entry $\lambda_j$ from the partition $\lambda$.

**Proposition 16.** Let $\Lambda = (a; \lambda)$ be dominant. If $a_m = 0$ then
\[ s^*_\Lambda(x; t) = s^*_{\Lambda_\ell}(x; t) \]
where $\Lambda_\ell = (a_1, \ldots, a_{m-1}; \lambda)$. Otherwise,
\[ s^*_\Lambda(x; t) = s^*_\Lambda(x; t) + x_ms^*_\Lambda(x; t) \]
where $\Lambda_D = (a_1, \ldots, a_{m-1}, \lambda; (\lambda \setminus \lambda_r) \cup (a_m))$, with $\lambda_r$ the largest entry in $\lambda$ strictly smaller than $a_m$ (note that $\lambda_r$ can be equal to 0), and where $\Lambda_L = (a_1, \ldots, a_{m-1}, a_m - 1; \lambda)$.

**Example 17.** Using again $\Lambda = (2, 1; 3, 1)$, we have that
\[
\Lambda_D \leftrightarrow \begin{array}{|c|c|}
\hline
3 & 1 \\
\hline
2 & \\
\hline
\end{array}
 \quad \text{and} \quad \Lambda_L \leftrightarrow \begin{array}{|c|c|}
\hline
3 & 1 \\
\hline
2 & 2 \\
\hline
\end{array}
\]

Hence
\[ s^*_{2,1;3,1}(x; t) = s^*_2(x; t) + x_2s^*_{2,0;3,1}(x; t) \]

Proof. If $a_m = 0$ then $\Lambda(0) = \hat{\Lambda}(0)$. The result is then immediate since, in the definition of the dual $m$-symmetric Schur functions, the alphabets $X_i$ are the same for $s^*_\Lambda(x; t)$ and $s^*_\Lambda(x; t)$.

We now consider the case $a_m > 0$. Let $j$ be the row that ends with the circle $m$ in the diagram of $\Lambda$. Observe that there are no row ending with a circle below row $j$ ($\Lambda$ is dominant). Suppose that $\nu_j = \nu_{j+1} = \cdots = \nu_s$ and that $\nu_s > \nu_{s+1}$, where $\nu = \Lambda(0)$. It is known that [18]
\[ s_\nu(Y_1, \ldots, Y_{\ell}) = s_\nu(Y_1, \ldots, Y_{\ell-1}, Y_s, Y_s, Y_{s+1}, \ldots, Y_{\ell}) \quad (5.8) \]
if $Y_k$ is a subalphabet of $Y_s$ which differs from $Y_s$ by no more than $s - k$ elements for all $j \leq k \leq s$. We have that $X_i = X + x_1 + \cdots + x_m$ for all $i \geq j$ and that $X_{j-1} = X + x_1 + \cdots + x_{m-1}$. The previous equation thus yields
\[ s_\nu(X_1, \ldots, X_{\ell}) = s_\nu(X_1, \ldots, X_{j-1}, X_s, X_s, X_{s+1}, \ldots, X_{\ell}) = s_\nu(X_1, \ldots, X_{j-1}, X_{j-1}, X_{j-1}, X_{j-1}, X_{j-1}, X_s, X_s, X_{s+1}, \ldots, X_{\ell}) \]
We now use the simple identity
\[ h_i[X_s] = h_i[X_{j-1} + x_m] = h_i[X_{j-1}] + x_m h_{i-1}[X_s] \]
in row $s$ of $s_\nu(X_1, \ldots, X_{\ell})$ to deduce that
\[ s_\nu(X_1, \ldots, X_{\ell}) = s_\nu(X_1, \ldots, X_{j-1}, X_{j-1}, X_{j-1}, X_{j-1}, X_{j-1}, X_{j-1}, X_{j-1}, X_s, X_s, X_{s+1}, \ldots, X_{\ell}) + x_m s_\nu(X_1, \ldots, X_{j-1}, X_{j-1}, X_{j-1}, X_{j-1}, X_{j-1}, X_{j-1}, X_{j-1}, X_s, X_s, X_{s+1}, \ldots, X_{\ell}) \]
where $\nu = (\nu_1, \ldots, \nu_{s-1}, \nu_s - 1, \nu_{s+1}, \ldots, \nu_t)$ is a partition by hypothesis. Setting $\lambda_r = \nu_{s+1}$, it is not too difficult to see that the previous equation then corresponds to $s^*_\Lambda(x; t) = s^*_\Lambda(x; t) + x_m s^*_\Lambda(x; t)$. \hfill \Box

We now establish the triangularity of the dual $m$-symmetric Schur functions in the $k_\lambda(x; t)$ basis.

**Proposition 18.** For any $\Lambda = (a; \lambda)$, we have that
\[ s^*_\Lambda(x; t) = k_\lambda(x; t) + \sum_{\Omega} D^*_{\Omega}(t)k_{\Omega}(x; t) \quad (5.9) \]
where all the $\Omega = (b; \mu)$’s appearing in the sum are such that $|b| < |a|$. 
Proof. We proceed by induction on the size of $a$. From Proposition $\text{[15]}$ the result holds when $|a| = 0$. Now, suppose that $\Lambda$ is dominant. From Proposition $\text{[16]}$ we have that

$$
s^*_\Lambda(x; t) = s^*_{\Lambda_+}(x; t) + x_m s^*_{\Lambda_L}(x; t)
$$

(5.10)

where $\Lambda_D = (a_1, \ldots, a_{m-1}, \lambda_r; (\Lambda \setminus \lambda_r) \cup (a_m))$, with $\lambda_r$ the largest entry in $\lambda$ strictly smaller than $a_m$, and where $\Lambda_L = (a_1, \ldots, a_{m-1}, a_m - 1; \lambda)$. By definition of $\lambda_r$, we have that $|(a_1, \ldots, a_{m-1}, \lambda_r)| < |a|$. Hence by induction,

$$
s^*_{\Lambda_D}(x; t) = k_{\Lambda_D}(x; t) + \sum_{\Omega} D^*_{\Lambda_D, \Omega}(t) k_{\Omega}(x; t) = \sum_{\Gamma} D^*_{\Lambda_D, \Gamma}(t) k_{\Gamma}(x; t)
$$

(5.11)

where all the $\Gamma = (\mathbf{b}; \mu)$'s appearing in the last sum are such that $|\mathbf{b}| < |(a_1, \ldots, a_{m-1}, \lambda_r)| < |a|$. Since $|(a_1, \ldots, a_{m-1}, a_m - 1)| < |a|$, we also get by induction that

$$
s^*_{\Lambda_L}(x; t) = k_{\Lambda_L}(x; t) + \sum_{\Omega} D^*_{\Lambda_L, \Omega}(t) k_{\Omega}(x; t)
$$

(5.12)

where all the $\Omega = (\mathbf{b}; \mu)$'s appearing in the sum are such that $|\mathbf{b}| < |(a_1, \ldots, a_{m-1}, a_m - 1)|$. Since $\Lambda_L$ is dominant, we have that $k_{\Lambda_L}(x; t) = x^{\Lambda_L} s_{\Lambda}(x)$, which implies that $x_m k_{\Lambda_L}(x; t) = x^\Lambda s_{\Lambda}(x) = k_{\Lambda}(x; t)$. Hence, we get from the previous equation that

$$
x_m s^*_\Lambda(x; t) = k_{\Lambda}(x; t) + x_m \sum_{\Omega} D^*_{\Lambda_+ \Omega}(t) k_{\Omega}(x; t) = k_{\Lambda}(x; t) + \sum_{\Gamma} E^*_{\Lambda_+ \Gamma}(t) k_{\Gamma}(x; t)
$$

(5.13)

where all the $\Gamma = (\mathbf{c}; \nu)$'s appearing in the sum are such that $|\mathbf{c}| < |(a_1, \ldots, a_{m-1}, a_m - 1)| + 1 = |a|$. The result thus holds in the dominant case from (5.10), (5.11) and (5.12).

In the non-dominant case, we have from (5.7) and Lemma $\text{[9]}$ that

$$
s^*_\Lambda(x; t) = T_{\sigma^{-1}} s^*_{\Lambda^+}(x; t) = T_{\sigma^{-1}} \left( k_{\Lambda^+}(x; t) + \sum_{\Omega} D^*_{\Lambda^+ \Omega}(t) k_{\Omega}(x; t) \right) = k_{\Lambda}(x; t) + \sum_{\Gamma} D^*_{\Lambda \Gamma}(t) k_{\Gamma}(x; t)
$$

(5.14)

Since $\Lambda^+$ is dominant, we have that all the $\Omega = (\mathbf{c}; \nu)$ appearing in the first sum are such that $|\mathbf{c}| < |a|$. But this implies that all the $\Gamma = (\mathbf{c}; \nu)$'s appearing in the second sum are also such that $|\mathbf{c}| < |a|$ given that the Hecke operators $T_i$ preserve the degree of a polynomial and commute with symmetric functions.

The following important proposition is now immediate.

**Proposition 19.** The dual $m$-symmetric Schur functions form a basis of $R_m$.

**Proof.** The unitriangularity in Proposition $\text{[18]}$ immediately implies that the dual $m$-symmetric Schur functions form a basis of $R_m$. \(\square\)

Since the dual $m$-symmetric Schur functions do in fact form a basis, we can define the $m$-symmetric Schur functions, $s_{\Lambda}(x; t)$, as their dual basis (up to a power of $t$) with respect to the scalar product $\text{[5.3]}$.

**Definition 20.** The $m$-symmetric Schur functions $s_{\Lambda}(x; t)$ are defined to be such that

$$
\langle s_{\Lambda}(x; t), s^*_\Omega(x; t) \rangle_m = \delta_{\Lambda \Omega} t^{\text{inv}(\mathbf{a})}
$$

Equivalently, by duality, the $m$-symmetric Schur functions $s_{\Lambda}(x; t)$ are the unique basis of $R_m$ such that

$$
k_{\Omega}(x; t) = s_{\Omega}(x; t) + \sum_{\Lambda} D_{\Lambda \Omega}(t) s_{\Lambda}(x; t)
$$

(5.15)

where the coefficients $D_{\Lambda \Omega}(t)$ are, up to a $t$-power, the coefficients $D^*_{\Lambda \Omega}(t)$ appearing in $\text{[5.3]}$:

$$
D_{\Lambda \Omega}(t) = t^{\text{inv}(\mathbf{b})-\text{inv}(\mathbf{a})} D^*_{\Lambda \Omega}(t)
$$

(5.16)
We will first see that the action of $T_i$ on $s^*_\Lambda(x,t)$ is exactly as its action on $k_\Lambda(x;t)$ given in Lemma 5.2.

**Proposition 21.** For $i = 1, \ldots, m - 1$, the operator $T_i$ is such that

$$T_i s^*_\Lambda = \begin{cases} 
  s^*_\Lambda & \text{if } a_i > a_{i+1} \\
  (t-1)s^*_\Lambda + ts^*_\Lambda & \text{if } a_i < a_{i+1} \\
  ts^*_\Lambda & \text{if } a_i = a_{i+1}
\end{cases}$$

where $\tilde{\Lambda} = s_i\Lambda$. In particular, $s^*_\Lambda(x;t)$ is symmetric in $x_i, x_{i+1}$ if $a_i = a_{i+1}$.

**Proof.** The case $a_i > a_{i+1}$ follows immediately from (5.6). In the case $a_i < a_{i+1}$, we have from $T_i s^*_\Lambda = s^*_\Lambda$ and the quadratic relation satisfied by the generators of the Hecke algebra that

$$T_i s^*_\Lambda = T^2_i s^*_\Lambda = ((t-1)T_i + t)s^*_\Lambda$$

We thus obtain, using again $T_i s^*_\Lambda = s^*_\Lambda$, that

$$T_i s^*_\Lambda = (t-1)s^*_\Lambda + ts^*_\Lambda$$

as wanted.

The case $a_i = a_{i+1}$ is somewhat less trivial. First consider the case when $\mathbf{a}$ is dominant in which $s^*_\Lambda(x;t)$ is a multi-Schur function. From the multi-Schur property (5.8), we have that $s^*_\Lambda(x;t)$ is symmetric in $x_i$ and $x_{i+1}$ whenever $a_i = a_{i+1}$. By Remark 1 we therefore have $T_i s^*_\Lambda(x;t) = s^*_\Lambda(x;t)$ in that case.

Finally, suppose that $\mathbf{a}$ is not dominant and that $a_i = a_{i+1}$. We will proceed by induction on the number of inversions of $\mathbf{a}$, knowing that we have established the case when the number of inversions is zero. First suppose that $j$ is such that $\{j, j + 1\}$ does not intersect $\{i, i + 1\}$ and such that $s_j \mathbf{a}$ has one fewer inversion than $\mathbf{a}$. Using the formula $T_j s^*_j \Lambda = s^*_j \Lambda$ that we have already shown to hold, we then have by induction that

$$T_i s^*_\Lambda = T_i T_j(s^*_j \Lambda) = T_j(T_i s^*_j \Lambda) = tT_j s^*_j \Lambda = ts^*_\Lambda$$

Now suppose that $s_{i-1} \mathbf{a}$ has one fewer inversion than $\mathbf{a}$, and that $s_{i-1} s_{i-1} \mathbf{a}$ will have two fewer inversions than $\mathbf{a}$ since $a_i = a_{i+1}$. Hence, using this time $T_{i-1} T_i s^*_{i-1} \mathbf{a} = s^*_i \Lambda$, we have by induction that

$$T_i s^*_\Lambda = T_i T_{i-1} T_i s^*_{i-1} \mathbf{a} = T_{i-1} T_i T_{i-1} T_i s^*_{i-1} \mathbf{a} = tT_{i-1} T_i s^*_{i-1} \mathbf{a} = ts^*_\Lambda$$

where we have used the braid relations obeyed by the Hecke algebra generators. Finally, if $s_{i+1} \mathbf{a}$ has one fewer inversion than $\mathbf{a}$, then $s_i s_{i+1} \mathbf{a}$ has again two fewer inversions than $\mathbf{a}$ and we proceed as in the previous case. \hfill $\Box$

The analog of Proposition 21 holds for the $m$-symmetric Schur functions.

**Corollary 22.** For $i = 1, \ldots, m - 1$, the operator $T_i$ is such that

$$T_i s_\Lambda = \begin{cases} 
  s_\Lambda & \text{if } a_i > a_{i+1} \\
  (t-1)s_\Lambda + ts_\Lambda & \text{if } a_i < a_{i+1} \\
  ts_\Lambda & \text{if } a_i = a_{i+1}
\end{cases}$$

where $\tilde{\Lambda} = s_i\Lambda$. In particular, $s_\Lambda(x;t)$ is symmetric in $x_i, x_{i+1}$ if $a_i = a_{i+1}$.

**Proof.** Suppose that $a_i > a_{i+1}$. We will compute $T_i s_\Lambda, s^*_\Lambda$ for all $\Omega$. From Definition 20 and Proposition 10 we get that $\langle T_i s_\Lambda, s^*_\Lambda \rangle = \langle s_\Lambda, T_i s^*_\Lambda \rangle = 0$ whenever $\Omega \notin \{\Lambda, \tilde{\Lambda}\}$. When $\Omega \in \{\Lambda, \tilde{\Lambda}\}$, we have again from Definition 20 and Proposition 10 that

$$\langle T_i s_\Lambda, s^*_\Lambda \rangle = \langle s_\Lambda, T_i s^*_\Lambda \rangle = \langle s_\Lambda, s^*_\Lambda \rangle = 0$$

and that

$$\langle T_i s_\Lambda, s^*_\Lambda \rangle = \langle s_\Lambda, T_i s^*_\Lambda \rangle = \langle s_\Lambda, s^*_\Lambda \rangle = t^{1+\text{Inv}(\mathbf{a})} = t^{\text{Inv}(s_i \mathbf{a})}$$

from which we deduce that $T_i s_\Lambda = s^*_\Lambda$. The other cases can be checked similarly. \hfill $\Box$

From Proposition 13 we know explicitly $s^*_\Lambda(x;t)$ when $\Lambda = (0, 0, \ldots, 0; \lambda)$. There is also a family of $m$-symmetric Schur functions for which we have explicit expressions.

**Proposition 23.** If $\Omega = (\mathbf{b}; \emptyset)$ then $s_\Omega(x;t) = k_\Omega(x;t) = H_\mathbf{b}(x_1, \ldots, x_m;t)$. 

---

6. Properties of the (dual) $m$-symmetric Schur functions

We will first see that the action of $T_i$ on $s^*_\Lambda(x,t)$ is exactly as its action on $k_\Lambda(x;t)$ given in Lemma 5.2.
It is then straightforward to verify that 
\[ f(\Omega) = f(\hat{\Omega}^0) \] 
where \( \Omega = (b^0; \mu) \) with \( b^0 = (b_1, \ldots, b_m, 0) \). The restriction \( r : R_{m+1} \to R_m \) is defined on the other hand as
\[ r(f) = f(x_1, \ldots, x_m, 0, x_{m+2}, x_{m+3}, \ldots) \] 
\[ (x_{m+2}, x_{m+3}, \ldots) \to (x_{m+1}, x_{m+2}, \ldots) \]
It is then immediate that \( r \circ i : R_m \to R_{m+1} \) is the identity and that
\[ (i(f), g)_{m+1} = \langle f, r(g) \rangle_m \] 
for all \( f \in R_m \) and all \( g \in R_{m+1} \). Remarkably, the inclusion and restriction of the (dual) \( m \)-symmetric Schur functions satisfy very elegant formulas.

**Proposition 24.** For \( \Lambda = (\lambda_1, \ldots, \lambda_m) \), we have that
\[ r(s_{\Lambda}^*) = t^{\Inv(\lambda) - \Inv(\hat{\lambda})} s_{\hat{\Lambda}}^* \] 
where \( \hat{\lambda} = (\lambda_1, \ldots, \lambda_{m-1}; \mu \cup (a_m)) \). Equivalently, by duality and (6.2),
\[ i(s_{\Lambda}) = \sum_{\Omega} s_{\Omega} \] 
where the sum is over all \( \Omega \)'s such that \( \Omega = (\lambda_1, \ldots, \lambda_m, a_{m+1}; \mu) \) with \( \mu \cup (a_{m+1}) = \lambda \).

In other words, the \( m \)-partition \( \hat{\lambda} \) in (6.3) is obtained from \( \lambda \) by removing the \( m \)-circle, while the \( \Omega \)'s that appear in (6.4) are those whose diagram can be obtained from that of \( \lambda \) by adding an \( (m+1) \)-circle in any symmetric row (including that of size 0).

Before proving the proposition, we illustrate it with an example.

**Example 25.** The diagram of \( \Lambda = (1, 3; 4, 3, 2) \) is

![Diagram](image)

Adding a 3-circle in all possible ways gives

![Add-3-circle](image)

Hence
\[ i(s_{(1,3;4,3,2),L}) = s_{1,3,4;3,2} + s_{1,3,3;4,2} + s_{1,3,2;4,3} + s_{1,3,0;4,3} \]

Proof. Since \( T_{\sigma} \) commutes with the restriction \( r : R_m \to R_{m-1} \) when \( \sigma \in S_{m-1} \), we can suppose that \( \hat{\lambda} = (a_1, \ldots, a_{m-1}) \) is dominant. In this case, \( \lambda' = (a_1, \ldots, a_\ell, a_m, a_{\ell+1}, \ldots, a_{m-1}) \) is also dominant, where \( a_m \) is such that \( a_{\ell-1} \geq a_m > a_\ell \). Thus \( s_{\lambda'}^* = T_{m-1} \cdots T_1 s_{\lambda'}^* \)
where \( \lambda' = (\lambda'; \lambda) \). It is easy to deduce from (6.1) that
\[ r(T_{m-1}f) = t f(x_1, \ldots, x_{m-2}, 0, x_m, x_{m+1}, \ldots) \mid (x_m, x_{m+1}, \ldots) \to (x_{m-1}, x_m, \ldots) \]
on any \( f \in R_m \), and more generally, that

\[
r(T_{m-1} \cdots T_1 f) = t^{m-\ell} f(x_1, \ldots, x_{\ell-1}, 0, x_{\ell+1}, x_{\ell+2}, \ldots) \\
\left|_{(x_{\ell+1}, x_{\ell+2}, \ldots) \mapsto (x_1, x_2, \ldots)}\right.
\]

(6.6)

Hence, given that \( \Lambda' \) is dominant,

\[
r(s_{\Lambda}^*) = r(T_{m-1} \cdots T_1 s_{\Lambda}^*)
\]

is (up to a \( t \) power) obtained by setting \( x_\ell = 0 \) in the multi-Schur function corresponding to \( s_{\Lambda'}^* \), followed by relabeling the variables \( x_{\ell+1}, x_{\ell+2}, \ldots \). But this simply generates (up to a \( t \) power) the multi-Schur function associated to \( s_{\Lambda}^* \), where \( \Lambda = (a_1, \ldots, a_m; \lambda \cup (a_m)) \). Observing that \( m - \ell = \text{Inv}(a) - \text{Inv}(\hat{a}) \), (6.3) is seen to hold.

\[\square\]

**Proposition 26.** For \( \Lambda = (a_1, \ldots, a_m; \lambda) \), let \( \Lambda^0 = (a_1, \ldots, a_m; 0; \lambda) \). We have that

\[
i(s_{\Lambda}^*) = s_{\Lambda^0}^*
\]

Equivalently, by duality,

\[
r(s_{\Lambda}) = \begin{cases} 
0 & \text{if } a_m > 0 \\
\text{Inv}(\hat{\lambda}) & \text{if } a_m = 0
\end{cases}
\]

where \( \Lambda_- = (a_1, \ldots, a_{m-1}; \lambda) \).

**Proof.** Since \( T_\sigma \) commutes with the inclusion \( i \) when \( \sigma \in S_m \), we can suppose that \( \Lambda \) is dominant. In this case \( i(s_{\Lambda}^*) = s_{\Lambda^0}^* \) is immediate from Proposition [16] \[\square\]

7. Main Conjecture

Our main conjecture is a positivity conjecture for the expansion of \( m \)-symmetric Macdonald polynomials in terms of \( m \)-symmetric Schur functions akin to the original Macdonald positivity conjecture [19] (now theorem [14]). Before stating the conjecture, we need to extend the notion of plethysm to \( R_m \). Recall that the plethysm relevant to Macdonald polynomials is the linear map on the ring of symmetric functions that sends the power-sum \( p_\lambda \) to \( p_\lambda / \prod_i (1 - t^{\lambda_i}) \) [4]. The notion of plethysm that we will need will simply be the linear map \( \varphi : R_m \to R_m \) defined on the \( m \)-symmetric power-sums as

\[
\varphi(p_\lambda) = \frac{p_\lambda}{\prod_i (1 - t^{\lambda_i})}.
\]

(7.1)

We stress that the plethysm \( \varphi \) only depends on the symmetric part \( \lambda \) of \( \Lambda = (a; \lambda) \).

We now introduce the proper normalization for the integral form of the \( m \)-symmetric Macdonald polynomial. Let

\[
c_{\Lambda}(q,t) = \prod_{s \in \Lambda} (1 - q^{a(s)}t^{\ell(s) + 1})
\]

where the product is over the cells of \( \Lambda \) (excluding the circles). In the expression, \( a(s) \) and \( \ell(s) \) stand respectively for the arm-length and leg-length of \( s \). The arm-length, \( a(s) \), is equal to the number of cells in \( \Lambda \) strictly to the right of \( s \) (and in the same row). Note that if there is a circle at the end of its row, then it adds one to the arm-length of \( s \). The leg-length, \( \ell(s) \), is equal to the number of cells in \( \Lambda \) strictly below \( s \) (and in the same column). If at the bottom of its column there are \( k \) circles whose filling is smaller than the filling of the circle at the end of its row, then they add \( k \) to the value of the leg-length of \( s \). If the row does not end with a circle then none of the circles at the bottom of its column contributes to the leg-length.

**Example 27.** The values of \( a(s) \) and \( \ell(s) \) in each cell of the diagram of \( \Lambda = (2,0,0,2; 4,1,1) \) are

| 34 | 22 | 10 |
| 23 | 11 |   |
| 24 | 10 |   |
| 01 |   |   |
| 00 |   |   |
| 2  |   |   |
| 3  |   |   |

which gives

\[
c_{(2,0,0,2; 4,1,1)}(q,t) = (1 - t)(1 - qt)(1 - q^2t^3)(1 - q^3t^5)(1 - qt^2)(1 - q^2t^4)(1 - qt)(1 - q^2t^5)(1 - t^2)(1 - t)
\]
Let the integral form of the $m$-symmetric Macdonald polynomials be defined as\footnote{The integral form of the “partially symmetric Macdonald polynomials” in [12, 13] is the same as our integral form.}
\[ J_\Lambda(x; q, t) = c_\Lambda(q, t)P_\Lambda(x; q, t) \]
We are finally ready to state our positivity conjecture for the $m$-Macdonald polynomials.

**Conjecture 28.** The $m$-symmetric Macdonald polynomials in their integral form are such that
\[
\varphi(J_\Lambda(x; q, t)) = \sum_\Omega K_{\Omega \Lambda}(q, t) s_\Omega(x; t) \tag{7.2}
\]
with $K_{\Omega \Lambda}(q, t) \in \mathbb{N}[q, t]$.

It is noteworthy that, as we will see in Corollary\footnote{The integral form of the “partially symmetric Macdonald polynomials” in [12, 13] is the same as our integral form.} the usual $(q, t)$-Kostka coefficients $K_{\mu \lambda}(q, t)$ are special cases of the coefficients $K_{\Omega \Lambda}(q, t)$.

**Example 29.** We have
\[
\varphi(J_{1,0,2}) = t^2 s_{3,0;0} + qt^2 s_{0,3;0} + qt s_{0,0;3} + (qt^3 + t)s_{2,1;0} + (q^2t^2 + t)s_{2,0;1} + (q^2t^3 + t)s_{1,2;0} \\
+ (q^2t^2 + 1)s_{1,0;2} + (q^2t^2 + qt)s_{0,2;1} + (q^2t^2 + q)s_{0,1;2} + (q^2t + q)s_{0,0;2,1} \\
+ qt^2 s_{1,1;1} + qt s_{1,0;1,1} + q^2t s_{0,1;1,1} + q^2 s_{0,0;1,1,1}
\]

**Remark 30.** The symmetries $K_{\mu \lambda}(q, t) = K_{\mu' \lambda'}(t, q)$ and $K_{\mu \lambda}(q, t) = q^{n(\lambda) - n(\Lambda)}K_{\mu' \lambda'}(q^{-1}, t^{-1})$ (where $n(\lambda) = \lambda_1 + 2\lambda_2 + 3\lambda_3 + \cdots$) of the usual $(q, t)$-Kostka coefficients do not extend to the $m$-symmetric world (except in the case $m = 1$). It seems however that $K_{\Omega \Lambda}(q, t) = K_{\Omega' \Lambda'}(t, q)$ whenever $\Omega'$ and $\Lambda'$ both exist, where $\Lambda'$ is the $m$-partition whose diagram is the conjugate of the diagram of $\Lambda$.

**Remark 31.** The special case $t = 1$ will be proven in [16] using a combinatorial interpretation quite similar to the major index statistic on tableaux (note that this does not provide a formula for the $q = 1$ case since, as commented in the previous remark, the symmetry $K_{\mu \lambda}(q, t) = K_{\mu' \lambda'}(t, q)$ of the usual $(q, t)$-Kostka coefficients does not extend to the $m$-symmetric world). One important consequence of the combinatorial interpretation at $t = 1$ worth mentioning is that, as can be appreciated in Example\footnote{The integral form of the “partially symmetric Macdonald polynomials” in [12, 13] is the same as our integral form.}
\[
K_{\Omega \Lambda}(1, 1) = \# \text{ of standard tableaux of shape } \mu \cup b
\]
Finding a combinatorial interpretation for the coefficients $K_{\Omega \Lambda}(q, t)$ thus amounts to finding a statistic on standard tableaux with extra circles.

The next section will be devoted to proving properties of the $K_{\Omega \Lambda}(q, t)$ coefficients.

8. The $K_{\Omega \Lambda}(q, t)$ Coefficients

Before describing a few elementary relations satisfied by the $K_{\Omega \Lambda}(q, t)$ coefficients, we first need to establish certain results on the inclusion and restriction of $m$-symmetric Macdonald polynomials. The proofs will rely on properties of the $m$-symmetric Macdonald polynomials that are proven in Appendix\footnote{The integral form of the “partially symmetric Macdonald polynomials” in [12, 13] is the same as our integral form.}. We start with the restriction, which can be obtained in general and which is surprisingly simple in the integral form.

**Proposition 32.** For $\Lambda = (a_1, \ldots, a_m; a_{m+1}; \lambda)$, the restriction $r : R_{m+1} \to R_m$ is such that
\[
r(J_\Lambda(x; q, t)) = q^{a_{m+1}}t^\# \{a_i < a_{m+1}\}J_{\hat{\Lambda}}(x; q, t)
\]
where $\hat{\Lambda} = (a_1, \ldots, a_m; \lambda \cup (a_{m+1}))$.

**Proof.** First suppose that $a_{m+1} = 0$. Letting $N \to \infty$ in Proposition\footnote{The integral form of the “partially symmetric Macdonald polynomials” in [12, 13] is the same as our integral form.} we have in this case that
\[
J_\Lambda(x; q, t) \bigg|_{x_{m+1}=0} = J_{\hat{\Lambda}}(x_{(m+1)}; q, t)
\]
where $x_{(m+1)} = (x_1, \ldots, x_m, x_{m+2}, x_{m+3}, \ldots)$. After relabelling the variables as $(x_{m+2}, x_{m+3}, \ldots) \mapsto (x_{m+1}, x_{m+2}, \ldots)$ we obtain $r(J_\Lambda(x; q, t)) = J_{\hat{\Lambda}}(x; q, t)$ as wanted.

If $a_{m+1} \neq 0$, letting this time $N \to \infty$ in Proposition\footnote{The integral form of the “partially symmetric Macdonald polynomials” in [12, 13] is the same as our integral form.} we get that
\[
J_\Lambda(x; q, t) \bigg|_{x_{m+1}=0} = q^{a_{m+1}}t^\# \{a_i < a_{m+1}\}J_{\hat{\Lambda}}(x_{(m+1)}; q, t)
\]
which gives that $r(J_\Lambda(x; q, t)) = q^{a_{m+1}}t^\# \{a_i < a_{m+1}\}J_{\hat{\Lambda}}(x; q, t)$ after relabelling the variables. \qed
The inclusion of an $m$-symmetric Macdonald polynomial is also quite simple when the symmetric part of the indexing $m$-partition is empty.

**Proposition 33.** If $\Lambda = (a_1, \ldots, a_m; \emptyset)$, then the inclusion $i : R_m \to R_{m+1}$ is such that
\[
i(J_{\Lambda}(x; q, t)) = J_{\Lambda^0}(x; q, t)
\]
where $\Lambda^0 = (a_1, \ldots, a_m, 0; \emptyset)$.

**Proof.** We need to prove that $i(c_{\Lambda}(q, t)P_{\Lambda}(x; q, t)) = c_{\Lambda^0}(q, t)P_{\Lambda^0}(x; q, t)$. It is immediate that $c_{\Lambda^0}(q, t) = c_{\Lambda}(q, t)$ since the only difference between the two diagrams is the extra circle $m + 1$ at the bottom of the diagram of $\Lambda^0$. But this circle $m + 1$ is larger than every other circle and therefore never contributes to a leg-length. We thus have to prove that $i(P_{\Lambda}(x; q, t)) = P_{\Lambda^0}(x; q, t)$. We will show that $P_{\Lambda^0}(x; q, t)$ and $P_{\Lambda}(x; q, t)$ are equal as polynomials in $N$ variables, which will prove the proposition.

We have
\[
P_{\Lambda}(x; q, t) = \frac{1}{u_\Lambda} S_{m+1,N}^t E_{a_1,\ldots,a_m,0^{N-m}} = \frac{1}{u_\Lambda} S_{m+2,N}^t O_{m+1} E_{a_1,\ldots,a_m,0^{N-m}}
\]
where
\[
O_{m+1} = 1 + T_{m+1} + T_{m+2} + \cdots + T_{m+1} T_{m+2} \cdots T_{N-1}
\]
Since $E_{a_1,\ldots,a_m,0^{N-m}}$ is symmetric in the variables $x_{m+1}, \ldots, x_N$, we can easily check that $O_{m+1} E_{a_1,\ldots,a_m,0^{N-m}} = (1 - t^{N-m})/(1 - t) E_{a_1,\ldots,a_m,0^{N-m}}$, which implies that
\[
P_{\Lambda}(x; q, t) = \frac{1}{u_\Lambda(t)} \left(1 - \frac{t^{N-m}}{1 - t}\right) S_{m+2,N}^t = \frac{u_{\Lambda^0}(t)}{u_\Lambda(t)} \left(1 - \frac{t^{N-m}}{1 - t}\right) P_{\Lambda^0}(x; q, t)
\]
But $u_\Lambda(t)/u_{\Lambda^0}(t) = (1 - t^{N-m})/(1 - t)$ due to the extra 0 in the symmetric entries of $\Lambda$, which gives that, in $N$ variables, $P_{\Lambda^0}(x; q, t) = P_{\Lambda}(x; q, t)$.

In order to better visualize the following theorem, it will prove convenient to denote, for $\Lambda = (a; \lambda)$ and $\Omega = (b; \mu)$, the coefficient $K_{\Omega\Lambda}(q, t)$ as
\[
\left( \begin{array}{llll} a_1 & a_2 & \cdots & a_m \\ b_1 & b_2 & \cdots & b_m \end{array} \right) \lambda \mu = \left( \begin{array}{llll} a_1 & a_2 & \cdots & a_m \\ b_1 & b_2 & \cdots & b_m \end{array} \right) \lambda \mu
\]
Recall that $\lambda \setminus \lambda_j$ stand for the partition obtained by removing the entry $\lambda_j$ from the partition $\lambda$.

**Theorem 34.** The coefficients $K_{\Omega\Lambda}(q, t)$ satisfy the following relations.

1. For any entry $\lambda_j$ in $\lambda$, we have
\[
q^{|\lambda_j|} t^{|\{i | a_i < \lambda_j\}|} \left( \begin{array}{llll} a_1 & a_2 & \cdots & a_m \\ b_1 & b_2 & \cdots & b_m \end{array} \right) \lambda \mu = \left( \begin{array}{llll} a_1 & a_2 & \cdots & a_m \\ b_1 & b_2 & \cdots & b_m \end{array} \right) \lambda_j \mu
\]
2. For any entry $\mu_j$ in $\mu$, we have
\[
\left( \begin{array}{llll} a_1 & a_2 & \cdots & a_m \\ b_1 & b_2 & \cdots & b_m \end{array} \right) \emptyset \mu = \left( \begin{array}{llll} a_1 & a_2 & \cdots & a_m \\ b_1 & b_2 & \cdots & b_m \end{array} \right) \emptyset \mu_j
\]
3. If $a_{i+1} = a_i$ then
\[
\left( \begin{array}{llll} a_1 & \cdots & a_i & \cdots & a_m \\ b_1 & \cdots & b_i & \cdots & b_m \end{array} \right) \lambda \mu = \left( \begin{array}{llll} a_1 & \cdots & a_i & \cdots & a_m \\ b_1 & \cdots & b_{i+1} & \cdots & b_m \end{array} \right) \lambda \mu
\]
4. If $b_{i+1} = b_i$ and $a_i > a_{i+1}$ then
\[
t \left( \begin{array}{llll} a_1 & \cdots & a_i & \cdots & a_m \\ b_1 & \cdots & b_i & \cdots & b_m \end{array} \right) \lambda \mu = \left( \begin{array}{llll} a_1 & \cdots & a_{i+1} & \cdots & a_m \\ b_1 & \cdots & b_i & \cdots & b_m \end{array} \right) \lambda \mu
\]
Proof. We will see that (1) is an easy consequence of Propositions 20 and 32. Applying $r$ on both sides of (7.2), we get

$$r(\wp(J_{A^0}(x; q, t))) = \sum_{\Delta} K_{\Delta A^0 r}(s_{\Delta}(x; t))$$

where $A^0 = (a_1, \ldots, a_m, \lambda \setminus \lambda_1)$. Since the plethysm $\wp$ commutes with $r$, we have from Propositions 32 that

$$r(\wp(J_{A^0}(x; q, t))) = q^{\lambda_1 t \# \{i \mid a_i < \lambda_1 \}} \wp(J_{A}(x; q, t)) = q^{\lambda_1 t \# \{i \mid a_i < \lambda_1 \}} \sum_{\Omega} K_{\Omega A} s_{\Omega}(x; t)$$

(8.1)

and from Proposition 20 that

$$\sum_{\Delta} K_{\Delta A^0 r}(s_{\Delta}(x; t)) = \sum_{\Omega} K_{\Omega A^0 s_{\Omega}(x; t)}$$

(8.2)

where $\Omega^0 = (b_1, \ldots, b_m, 0; \mu)$. Equating (8.1) and (8.2), we obtain that $q^{\lambda_1 t \# \{i \mid a_i < \lambda_1 \}} K_{\Omega A}(q, t) = K_{\Omega A^0}(q, t)$ as wanted.

Statement (2) is statement (1) with $\lambda_j = 0$. The proof is the same as the proof of statement (1).

For (3), let $A = (a_1, \ldots, a_m; 0)$. Using Proposition 33 and the commutativity of $i$ and $\wp$, we have

$$\sum_{\Gamma} K_{\Gamma A}(q, t)i(s_{\Gamma}(x; t)) = i(\wp(J_{A}(x; q, t))) = \wp(J_{A^0}(x; q, t)) = \sum_{\Omega} K_{\Omega A^0}(q, t)s_{\Omega}(x; t)$$

We thus get from (6.4) that

$$\sum_{\Delta} K_{\Gamma A}(q, t)s_{\Delta}(x; t) = \sum_{\Omega} K_{\Omega A^0}(q, t)s_{\Omega}(x; t)$$

where the sum is over all $\Delta$’s that can be obtained from $\Gamma$ by adding a circle $m + 1$ on a symmetric entry. We thus have that $K_{\Omega A^0}(q, t) = K_{\Gamma A}(q, t)$ if $\Omega$ can be obtained from $\Gamma$ by adding a circle $m + 1$ on a symmetric entry (including a zero entry). But this is exactly the statement of (3) when the entry $\mu_i$ is not equal to zero.

In (4), $J_A(x; q, t)$ is in this case symmetric in $x_i$ and $x_{i+1}$ from Remark 3 and so is $\wp(J_A(x; q, t))$. Hence

$$T_i(\wp(J_A(x; q, t))) = t\wp(J_A(x; q, t)) = t \left( \sum_{\Omega} K_{\Omega A}(q, t)s_{\Omega}(x; t) \right)$$

(8.3)

On the other hand,

$$T_i(\wp(J_A(x; q, t))) = T_i \left( \sum_{\Omega} K_{\Omega A}(q, t)s_{\Omega}(x; t) \right) = \sum_{\Omega} K_{\Omega A}(q, t)T_i s_{\Omega}(x; t)$$

(8.4)

Suppose without loss of generality that $\Omega = (b; \mu)$ is such that $b_i > b_{i+1}$ and let $\Omega = s_{\Omega}$. We then have from Corollary 22 that $T_i s_{\Omega} = s_{\Omega}$ and $T_i s_{\Omega} = (t - 1)s_{\Omega} + ts_{\Omega}$. Thus, comparing the terms involving $s_{\Omega}$ and $s_{\Omega}$ in (8.3) and (8.4), we obtain

$$tK_{\Omega A}(q, t)s_{\Omega}(x; t) + tK_{\tilde{\Omega} A}(q, t)s_{\tilde{\Omega}}(x; t) = K_{\Omega A}(q, t)s_{\Omega}(x; t) + K_{\tilde{\Omega} A}(q, t)(t - 1)s_{\tilde{\Omega}}(x; t) + tK_{\tilde{\Omega} A}(q, t)s_{\Omega}(x; t)$$

which, by linear independence of $s_{\Omega}$ and $s_{\tilde{\Omega}}$, holds iff $K_{\tilde{\Omega} A}(q, t) = K_{\Omega A}(q, t)$.

Finally, we consider (5). Let $\tilde{A} = s_i A$. We have that $c_{A}(q, t)$ and $c_{A}(q, t)$ only differ by the circles $i$ and $i + 1$ that are switched. Because $i$ and $i + 1$ are consecutive numbers, all the cells in $A$ and $\tilde{A}$ will contribute the same value to $c_{A}(q, t)$ and $c_{A}(q, t)$ except the cell at the intersection of the two circles. This cell will contribute a factor of $(1 - q^{a t + 1})$ in $c_{A}(q, t)$ while in $c_{A}(q, t)$ it will contribute a factor of $(1 - q^{a t + 2})$ (given that the circle $i$ is below the circle $i + 1$ in $\tilde{A}$), where the arm-length $a$ and the leg-length $\ell$ are relative to $A$. Using (2.5) with $x = (x_1, \ldots, x_N)$, we also have in our case that

$$T_i P_A(x; q, t) = \left( \frac{t - 1}{1 - q^{-a t + 1}} \right) P_A(x; q, t) + \left( \frac{1 - q^{a t + 2}q^{a t + 1}}{1 - q^{a t + 2}} \right) P_A(x; q, t)$$

since $\bar{n}_i/\bar{n}_{i+1} = q^{a t + 1}$ and $u_{A, N}(t) = u_{\tilde{A}, N}(t)$. Therefore, taking $c_{A}(q, t)$ and $c_{A}(q, t)$ into account, we get

$$T_i J_A(x; q, t) = \left( \frac{t - 1}{1 - q^{-a t + 1}} \right) J_A(x; q, t) + \left( \frac{1 - q^{a t + 2}q^{a t + 1}}{1 - q^{a t + 2}} \right) c_{A}(q, t) J_A(x; q, t)$$

$$= \left( \frac{t - 1}{1 - q^{-a t + 1}} \right) J_A(x; q, t) + \left( \frac{1 - q^{a t + 1}}{1 - q^{a t + 1}} \right) J_{\tilde{A}}(x; q, t)$$

(8.5)
from our previous observation on \(c_\lambda(q, t)\) and \(c_\Lambda(q, t)\). After applying the plethysm \(\wp\) (which commutes with \(T_i\)), we obtain

\[
T_i \wp(J_\Lambda(x; q, t)) = \left(\frac{t - 1}{1 - q^{-a_i}t^{-1}}\right) \wp(J_\Lambda(x; q, t)) + \left(\frac{1 - q^{a_i}t}{1 - q^{-a_i}t^{1}}\right) \wp(J_\Lambda(x; q, t))
\]

Focusing on the terms involving \(s_\Omega\) on both sides of the equation then yields

\[
K_{\Omega}(q, t)T_i s_\Omega = \left(\frac{t - 1}{1 - q^{-a_i}t^{-1}}\right) K_{\Omega}(q, t) + \left(\frac{1 - q^{a_i}t}{1 - q^{-a_i}t^{1}}\right) K_{\Omega}(q, t)
\]

where we used the fact that \(s_\Omega\) cannot appear in any \(T_i s_\Omega\) such that \(\Gamma \neq \Omega\). We have from Corollary \(22\) that \(T_i s_\Omega = ts_\Omega\) since \(b_i = b_{i+1}\), which implies that

\[
tK_{\Omega}(q, t) = \left(\frac{t - 1}{1 - q^{-a_i}t^{-1}}\right) K_{\Omega}(q, t) + \left(\frac{1 - q^{a_i}t}{1 - q^{-a_i}t^{1}}\right) K_{\Omega}(q, t)
\]

Combining the coefficients of \(K_{\Omega}(q, t)\), we obtain

\[
t \left(\frac{1 - q^{a_i}t}{1 - q^{-a_i}t^{1}}\right) K_{\Omega}(q, t) = \left(\frac{1 - q^{a_i}t}{1 - q^{-a_i}t^{1}}\right) K_{\Omega}(q, t)
\]

which readily implies our claim. \(\Box\)

We get from Theorem \(33\) that the usual \((q, t)\)-Kostka coefficients \(K_{\mu\lambda}(q, t)\) are special cases of the coefficients \(K_{\Omega}(q, t)\).

**Corollary 35.** If \(\Lambda = (0^m; \lambda)\) and \(\Omega = (0^m; \mu)\) then

\[
K_{\Omega}(q, t) = K_{\mu\lambda}(q, t)
\]

where \(K_{\mu\lambda}(q, t)\) is the usual Macdonald \((q, t)\)-Kostka coefficient.

Let

\[
\Psi_N = (1 - t)(1 + T_{N-1} + T_{N-2}T_{N-1} + \cdots + T_m \cdots T_{N-1})\Phi_q
\]

where we recall that \(\Phi_q\) was defined in (2.9). The following theorem, which is the analog of (2.9) in the \(m\)-symmetric world, is proved in Appendix [A]

**Theorem 36.** For any \(N\) and any \(m \geq 1\), we have that

\[
\Psi_N J_\Lambda(x_1, \ldots, x_N; q, t) = t^{-\#(2 \leq j \leq m | a_j \leq a_1)} J_{\Lambda\Box}(x_1, \ldots, x_N; q, t)
\]

where \(\Lambda \Box = ((a_2, \ldots, a_m); \lambda \cup (a_1 + 1))\). In particular, if \(m = 1\), we have

\[
\Psi_N J_{(a_1; \lambda)}(x_1, \ldots, x_N; q, t) = J_{\lambda\Box(a_1+1)}(x_1, \ldots, x_N; q, t)
\]

We can now establish the next proposition, which provides a decomposition of the usual \((q, t)\)-Kostka coefficients in terms of \((q, t)\)-coefficients at \(m = 1\) of lower degree.

**Proposition 37.** For any entry \(\lambda_j\) in \(\lambda\), the usual \((q, t)\)-Kostka coefficient \(K_{\mu\lambda}(q, t)\) is such that

\[
K_{\mu\lambda}(q, t) = \left(\begin{array}{c}
\lambda \\
\mu
\end{array}\right) = \sum_{\mu_1} \left(\begin{array}{c}
\lambda_j - 1 \\
\mu - 1
\end{array}\right)
\]

where the sum is over all distinct parts \(\mu_i\) in \(\mu\).

The positivity conjecture for \(m\)-symmetric Macdonald polynomials at \(m = 1\) is thus a refinement of the usual Macdonald positivity.

**Example 38.** Taking \(\lambda = (3, 3)\) and \(\mu = (3, 2, 1)\), the decomposition (which is unique in this case since \(\lambda\) has only one distinct entry) gives

\[
K_{(3,2,1),(3,3)}(q, t) = K_{(2,2,1),(2,3)}(q, t) + K_{(1,3,1),(2,3)}(q, t) + K_{(0,3,2),(2,3)}(q, t)
\]

A computer calculation tells us that

\[
K_{(2,2,1),(2,3)}(q, t) = q^5t^3 + q^3t^2 + q^2t^2 + q^2t + qt,
\]

\[
K_{(1,3,1),(2,3)}(q, t) = q^5t^3 + q^4t^2 + q^3t^2 + q^3t + q^2t + q
\]

and

\[
K_{(0,3,2),(2,3)}(q, t) = q^5t^2 + q^4t^2 + q^4t + q^3t + q^2
\]

which is consistent with the value of \(K_{(3,2,1),(3,3)}(q, t)\) found in [19]. Observe that the diagram of \((2, 3)\) is obtained from that of \(\lambda = (3, 3)\) by transforming a box into a circle (indexed by a 1). Similarly, the diagram of \((2, 2, 1), (1,3,1)\) and \((0,3,2)\) are those that can be obtained from that of \(\mu = (3, 2, 1)\) by transforming a box into a circle.
Proof. We work in $N$ variables. Consider the operator $\Psi_N = \varphi \circ \Psi_N \circ \varphi^{-1}$. From Theorem 38 we have that $\Psi_N \circ \varphi(J_{\alpha_1:1}) = \varphi(J_{\lambda_1:1})$. In order to obtain our decomposition, we thus need to obtain the action of $\Psi_N$ on the $m$-symmetric Schur functions. Before obtaining that action, we will first show that $\Psi_N$ is such that

$$\Psi_N(k_{(b_1;\mu)}(x; t)) = h_{b_1+1}(x)s_\mu(x)$$  \hspace{1cm} (8.7)

From $k_{(b_1;\mu)}(x; t) = x_1^{b_1}s_\mu$, we get

$$\varphi^{-1}(k_{(b_1;\mu)}(x; t)) = x_1^{b_1}\varphi^{-1}(s_\mu)$$

Since $\varphi^{-1}(s_\mu)$ commutes with $\Psi_N$, we then have that

$$\Psi_N \circ \varphi^{-1}(k_{(b_1;\mu)}(x; t)) = \varphi^{-1}(s_\mu)\Psi_N(x_1^{b_1})$$

Now, in the case $m = 1$, the operator $\Psi_N$ is such that

$$\Psi_N(x_1^{b_1}) = (1-t)(1 + T_{N-1} + T_{N-2} + \cdots + T_1)P_{b_1+1}(x_1, \ldots, x_N; t)$$

where $P_{b_1+1}(x_1, \ldots, x_N; t)$ is the (symmetric) Hall-Littlewood polynomial indexed by a single entry [19]. In that case, it is known [1,19] that

$$(1-t)P_{b_1+1}(x_1, \ldots, x_N; t) = \varphi^{-1}(h_{b_1+1})$$

from which we obtain that $\Psi_N(x_1^{b_1}) = \varphi^{-1}(h_{b_1+1})$. The map $\varphi^{-1}$ being a homomorphism, it then follows that

$$\Psi_N \circ \varphi^{-1}(k_{(b_1;\mu)}(x; t)) = \varphi^{-1}(s_\mu)\Psi_N(x_1^{b_1}) = \varphi^{-1}(h_{b_1+1}s_\mu)$$

which implies (8.7).

We will now show that

$$\tilde{\Psi}_N(s_{\lambda_1:1}) = s_{\lambda_1 \cup (a_1+1)}$$  \hspace{1cm} (8.8)

In the $m = 1$ case, every $m$-partition is dominant. We thus obtain from Remark 14 using the duality (see (5.13) and 5.14) that

$$k_{(b_1;\mu)}(x; t) = \sum_{(a_1;\lambda)} s_{(a_1;\lambda)}(x; t)$$

where the sum is over all $(a_1;\lambda)$’s such that $(\lambda \cup (1))/\mu$ is a horizontal $b_1$-strip whose entries all lie within the first $a_1$ columns. Therefore, it follows that

$$\tilde{\Psi}_N(k_{(b_1;\mu)}) = h_{b_1+1}s_\mu = \sum_\nu s_\nu = \sum_{(a_1;\lambda)} \tilde{\Psi}_N(s_{(a_1;\lambda)})$$

where the first sum is over all $\nu$’s such that $\nu/\mu$ is a horizontal $(b_1+1)$-strip. Since the $k_{(b_1;\mu)}$’s form a basis of $R_1$, it suffices to prove that $\tilde{\Psi}_N(s_{(a_1;\lambda)}) = s_{\lambda \cup (a_1+1)}$ implies that $\tilde{\Psi}_N(k_{(b_1;\mu)}) = h_{b_1+1}s_\mu$. We will see that it is indeed the case. We have

$$\sum_{(a_1;\lambda)} \tilde{\Psi}_N(s_{(a_1;\lambda)}) = \sum_{(a_1;\lambda)} s_{\lambda \cup (a_1+1)}$$

where the sum in the second sum can be thought as being over all $(a_1;\lambda)$’s such that $(\lambda \cup (a_1+1))/\mu$ is a horizontal $(b_1+1)$-strip whose entries all lie within the first $a_1 + 1$ columns (and including a cell in column $a_1 + 1$). This implies that the map $\nu \mapsto \lambda \cup (a_1 + 1)$, where $a_1 + 1$ is the rightmost column in $\nu/\mu$, is an obvious bijection between the $\nu$’s in $s_\nu$ and the $(a_1;\lambda)$’s in $s_{\lambda \cup (a_1+1)}$. Hence $\tilde{\Psi}_N(s_{(a_1;\lambda)}) = s_{\lambda \cup (a_1+1)}$ implies $\tilde{\Psi}_N(k_{(b_1;\mu)}) = h_{b_1+1}s_\mu$ and (8.8) holds.

Applying $\tilde{\Psi}_N$ on both sides of

$$\varphi(J_{(a_1;\lambda)}) = \sum_{(b_1;\mu)} K_{(b_1;\mu)(a_1;\lambda)}(q, t)s_{(b_1;\mu)}$$

we thus get from Theorem 38 that

$$\varphi(J_{\lambda \cup (a_1+1)}) = \sum_{\nu} K_{\nu \cup (a_1+1)}(q, t)s_\nu = \sum_{(b_1;\mu)} K_{(b_1;\mu)(a_1;\lambda)}(q, t)s_{\mu \cup (b_1+1)}$$

Equating the coefficients of $s_\nu$ on both sides, the proposition then follows straightforwardly. \qed
Suppose that \( \eta = (\eta_1, \ldots, \eta_\ell, 0^{N-\ell}) \) is a composition of length \( \ell \). Then
\[
T_i E_\eta(x; q, t) = t E_\eta(x; q, t)
\]
for all \( \ell < i < N \) since \( \eta_i = \eta_{i+1} = 0 \) in that case. Hence, \( S^r_m E_\eta(x; q, t) \) is equal to \( E_\eta(x; q, t) \) times a constant whenever \( m \geq \ell \). We thus get that if \( \Lambda = (\eta_1, \ldots, \eta_\ell, 0^{m-\ell}; \emptyset) \) then
\[
P_\Lambda(x; q, t) = E_\eta(x; q, t)
\]
since the coefficient of \( x^0 \) is equal to 1 in both functions. That is, when \( m \) is large enough, a non-symmetric Macdonald polynomial is also an \( m \)-Macdonald polynomial. In this case, the integral form of the non-symmetric Macdonald polynomial, \( J_\eta(x; q, t) = c_{(\eta_1, \ldots, \eta_\ell, \emptyset)}(q, t) E_\eta(x; q, t) \), has the following expansion when considered as an \( m \)-symmetric function:
\[
\varphi(J_\eta(x; q, t)) = \sum_{\Omega} K_{\Omega(\eta_1, \ldots, \eta_\ell, 0^{m-\ell}; \emptyset)}(q, t) s_\Omega(x; t)
\]
As we have seen in Theorem 34 by increasing the value of \( m \), one can move to the non-symmetric side the entries on the symmetric side of \( \Lambda \) and \( \Omega \). We can also use this idea in the other direction. Let \( \omega \) be a composition of arbitrary length \( \ell' \). Taking \( m \geq \max(\ell, \ell') \), we can define \( K_{\omega \eta}(q, t) \) as
\[
K_{\omega \eta}(q, t) = \left( \begin{array}{cccccc} \eta_1 & \cdots & \eta_\ell & 0 & \cdots & 0 \\ \omega_1 & \cdots & \omega_{\ell+1} & \omega_{\ell'+1} & \cdots & \omega_{\ell'} \\ \end{array} \right) = K_{\omega^\prime(\eta_1, \ldots, \eta_\ell; \emptyset)}(q, t)
\]
where the number of columns on the non-symmetric side is \( m \) (we have supposed that \( \ell' > \ell \) only for the sake of illustrating the general behavior). Note that this definition does not depend on \( m \) because we can add any number of columns of \( 0 \)'s to the right without changing the value of \( K_{\omega \eta}(q, t) \) by Theorem 34(2). Now, from Theorem 34(2)(3)(4), we also have that
\[
K_{\omega \eta}(q, t) = \left( \begin{array}{cccccc} \eta_1 & \cdots & \eta_\ell & 0 & \cdots & 0 \\ \omega_1 & \cdots & \omega_{\ell+1} & \omega_{\ell'+1} & \cdots & \omega_{\ell'} \\ \nu_1 & \cdots & \nu_{\ell(\nu)} & 0 & \cdots & 0 \\ \end{array} \right) = K_{\omega(\eta_1, \ldots, \eta_\ell; \emptyset)}(q, t)
\]
where \( \nu = (\omega_{\ell+1}, \ldots, \omega_{\ell'})^+ \) is the partition obtained by reordering the entries of \( (\omega_1, \ldots, \omega_{\ell'}) \), and where \( \Omega^\omega = (\omega_1, \ldots, \omega_{\ell'}; \nu) \). For a given \( \eta \), there are thus only a finite number of distinct coefficients \( K_{\omega \eta}(q, t) \) (corresponding to the number of \( \ell \)-partitions of degree \( |\eta| \)). We also have that if \( m \geq \ell + |\eta| \), then every \( K_{\Omega(\eta_1, \ldots, \eta_\ell; \emptyset)} \) is equal to a given \( K_{\omega \eta}(q, t) \) such that \( \ell(\omega) \leq m \). As such, all the information is contained in the \( K_{\omega \eta}(q, t) \)'s such that \( \ell(\omega) \leq \ell + |\eta| \).

Doing the expansion in the \( m \)-symmetric world (with \( m \geq \ell + |\eta| \)), we obtain
\[
\varphi(J_\eta(x; q, t)) = \sum_{\omega \in \mathbb{Z}^m_{\geq 0}} K_{\omega \eta}(q, t) s_{(\omega_1, \ldots, \omega_m; \emptyset)}(x; t) \pmod{\mathcal{L}_m}
\]
(9.1)
where \( \mathcal{L}_m \) is the linear span of \( m \)-symmetric Schur functions indexed by \( m \)-partitions whose symmetric side is not empty. From the discussion above, there is no loss of information in working modulo \( \mathcal{L}_m \), as the full \( m \)-symmetric expansion can be recovered from the r.h.s. in (9.1). From Proposition 34 we also get \( s_{(\omega_1, \ldots, \omega_m; \emptyset)}(x; t) = H_\omega(x; t) \). We thus have the following positivity conjecture on non-symmetric Macdonald polynomials.

**Conjecture 39.** Let \( \eta \) be a composition of length \( \ell \). Then, for \( m \geq \ell \), we have that
\[
\varphi(J_\eta(x; q, t)) = \sum_{\omega \in \mathbb{Z}^m_{\geq 0}} K_{\omega \eta}(q, t) H_\omega(x; t) \pmod{\mathcal{L}_m}
\]
with \( K_{\omega \eta}(q, t) \in \mathbb{N}[q,t] \).

**Remark 40.** It is not clear whether there is a connection between the coefficients \( K_{\omega \eta}(q, t) \) and the composition Kostka functions \( K_{\lambda \mu}(q, t) \) of [13]. For instance if \( \omega = \lambda = (2, 2) \) and \( \eta = \mu = (3, 1) \), we get \( K_{\omega \eta}(q, t) = qt + q^2t + q^4t^2 \) while the composition Kostka function is \( K_{\lambda \mu}(q, t) = t + qt + qt^2 \) (see (12.13) in [13]), which is equal in this case to the usual \( (q,t) \)-Kostka coefficient. But if we replace \( \eta = (3, 1) \) by \( \eta = (0, 0, 3, 1) \), we get \( K_{\omega \eta}(q, t) = q^4(t + qt + q^2t^2) \) which is \( K_{\lambda \mu}(q, t) \) up to a \( q \)-power, suggesting that there might be a non-trivial connection between the two types of coefficients.
Remark 41. As discussed above, there is no loss of information in working modulo $\mathcal{L}_m$ when $m \geq \ell + |\eta|$. Working modulo $\mathcal{L}_m$ also has the advantage of allowing to compute the coefficients $\varphi_{\omega}(q,t)$ without having to do the full expansion in the $m$-symmetric Schur function basis. Indeed, we can first expand $\varphi(J_q(x; q,t))$ in the $k_\Omega(x; t)$ basis and then use the relation

$$k_\Omega(x; t) = \sum_{a \in \mathbb{Z}^{\geq 0}_0} D_{\langle a, \emptyset \rangle}\Omega(t) H_a(x; t) \ mod \ \mathcal{L}_m$$

(9.2)

which is obtained from [5,13] by simply killing the terms involving $m$-symmetric Schur functions indexed by multipartitions whose symmetric part is not empty (observe that Proposition 23 allows us to then change the remaining $s_{\langle a, \emptyset \rangle}(x; t)$'s into $H_a(x; t)$'s).

10. Conjectures on the coefficients $K_{\Omega\Lambda}(q,t)$

It does not appear that a perfect analog of Proposition 37 exists when $m > 0$. We can however formulate the following conjecture which states that a certain portion of the coefficient $K_{\Omega\Lambda}(q,t)$ can be recovered from coefficients of lower degree and larger $m$.

Conjecture 42. For any entry $\lambda_i$ in $\lambda$, we have that

$$t\#\{ k | \lambda_i \leq a_k \} \left( \begin{array}{cccc} a_1 & \cdots & a_m \\ b_1 & \cdots & b_m \end{array} \right) \left( \begin{array}{cc} \lambda & \mu \\ \mu_i \end{array} \right) - \sum_{\mu_i} t\#\{ k | \mu_i \leq b_k \} \left( \begin{array}{cccc} \lambda_i - 1 & a_1 & \cdots & a_m \\ \mu_i - 1 & b_1 & \cdots & b_m \end{array} \right) \in \mathbb{N}[q,t]$$

where the sum is over all distinct entries $\mu_i$ in $\mu$.

The next conjecture tells us how Theorem 53(4) can be somehow extended when $a_i \neq a_{i+1}$.

Conjecture 43. For a fixed $i < m$, let $\tilde{\Omega} = s_i\Omega$ and

$$A_i = \frac{\varepsilon^{(i)}(q,t)}{\varepsilon^{(i+1)}(q,t)} = q^{a_i-a_{i+1}} t^{r_{\lambda}(i)-r_{\lambda}(i)}$$

Then, if $b_i > b_{i+1}$, we have that

$$\frac{A_i t^{-1} K_{\Omega\Lambda}(q,t) - K_{\tilde{\Omega}\Lambda}(q,t)}{A_i t^{-1} - 1} \in \mathbb{N}[q,t] \ and \ \frac{K_{\Omega\Lambda}(q,t) - K_{\tilde{\Omega}\Lambda}(q,t)}{1 - A_i t^{-1}} \in \mathbb{N}[q,t]$$

(10.1)

Note that if $b_i < b_{i+1}$, we only need to switch $\Omega$ and $\tilde{\Omega}$ in the relations.

Assuming that Conjecture 28 holds, the rule essentially states that in order to obtain $K_{\tilde{\Omega}\Lambda}(q,t)$ from $K_{\Omega\Lambda}(q,t)$, one simply needs to multiply each monomial in $K_{\Omega\Lambda}(q,t)$ by either $A_i t^{-1}$ or 1. Equivalently, in a given combinatorial interpretation using standard tableaux of the coefficients $K_{\Omega\Lambda}(q,t)$, the statistic associated to a given standard tableau would either gain $A_i t^{-1}$ or remain invariant when going from $\Lambda$ to $\tilde{\Lambda}$. This phenomenon is very reminiscent of Butler’s rule on $(q,t)$-Kostka coefficients which relates the coefficients $K_{\mu\lambda}(q,t)$ and $K_{\mu\omega}(q,t)$, where $\nu$ is a partition that can be obtained from $\lambda$ by moving exactly one box from a given position to another.

The next conjecture is similar to the previous one. It relates this time the entries $K_{\Omega\Lambda}(q,t)$ and $K_{\tilde{\Omega}\Lambda}(q,t)$.

Conjecture 44. For a fixed $i < m$, let $\tilde{\Lambda} = s_i\Lambda$ and $\tilde{\Omega} = s_i\Omega$.

(1) If $a_i > a_{i+1}$ and $b_i > b_{i+1}$, we have that

$$\frac{t K_{\Omega\Lambda}(q,t) - K_{\tilde{\Omega}\Lambda}(q,t)}{t - 1} \in \mathbb{N}[q,t] \ and \ \frac{K_{\Omega\Lambda}(q,t) - K_{\tilde{\Omega}\Lambda}(q,t)}{1 - t} \in \mathbb{N}[q,t]$$

(10.2)

(II) If $a_i < a_{i+1}$ and $b_i < b_{i+1}$, we have that

$$\frac{t^2 K_{\Omega\Lambda}(q,t) - K_{\tilde{\Omega}\Lambda}(q,t)}{t - 1} \in \mathbb{N}[q,t] \ and \ \frac{t K_{\Omega\Lambda}(q,t) - K_{\tilde{\Omega}\Lambda}(q,t)}{1 - t} \in \mathbb{N}[q,t]$$

(10.3)

Note again that if $a_i < a_{i+1}$, we simply need to interchange the roles of $\Lambda$ and $\tilde{\Lambda}$ in the relations.

Acknowledgments. We thank P. Mathieu, J. Morse and L. Pena for their useful comments. We also thank D. Orr for letting us know about the work contained in [12,13], and we are grateful to B. Goodberry for confirming that the integral form in [12] was the same as ours.
APPENDIX A. SOME ADDITIONAL PROPERTIES OF THE $m$-SYMMETRIC MACDONALD POLYNOMIALS

We prove in this appendix a few properties of the $m$-Symmetric Macdonald polynomials whose proofs were too technical to include in the main presentation. From now on, we will always work with a finite number $N$ of variables. As such $x_i$ will denote the variables $(x_1, \ldots, x_N)$. Recall that $x_{(i)}$ stands for $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$.

The following lemma on non-symmetric Macdonald polynomials is probably already known. The version given in [11] had stronger conditions but the proof is the same. We reproduce it for completeness.

Lemma 45. Let $\eta = (\eta_1, \ldots, \eta_N)$ be a composition such that $\eta_i = 0$ and $\eta_j \neq 0$ whenever $j > i$. Then

$$E_\eta(x)\bigg|_{x_i=0} = E_{\eta(i)}(x_{(i)})$$

(A.1)

and

$$E_\eta(x_1, \ldots, x_N)\bigg|_{x_j=0} = 0 \quad \text{if } j > i.$$  

(A.2)

where $\eta(i) = (\eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \ldots, \eta_N)$ stands for the composition $\eta$ without its $i$-th entry.

Proof. We proceed by induction. We know from (2.7) that

$$E_\eta(x)\bigg|_{x_N=0} = E_{\eta(N)}(x_N) \quad \text{when } i = N$$

(A.3)

while

$$E_\eta(x)\bigg|_{x_N=0} = 0 \quad \text{when } i < N \text{ and } j = N$$

(A.4)

Now, suppose by induction that

$$E_\eta(x)\bigg|_{x_k=0} = E_{\eta(k)}(x_{(k)}) \quad \text{when } i = k$$

(A.5)

while

$$E_\eta(x)\bigg|_{x_k=0} = 0 \quad \text{when } i < k \text{ and } j = k$$

(A.6)

We thus need to prove that (A.5) and (A.6) still hold when $k$ is replaced by $k - 1$. We first consider the case (A.5). Suppose that $i = k - 1$. Since $\eta_{k-1} = 0$ and $\eta_k \neq 0$ by hypothesis, we have from (2.8) that

$$T_{k-1}E_\eta(x) = c_\eta E_\eta(x) + tE_{s_{k-1}\eta}(x),$$

(A.7)

where $c_\eta$ is an irrelevant constant. By definition of $T_{k-1}$, we also obtain

$$T_{k-1}E_\eta(x) = tE_\eta(x) + \frac{tx_{k-1} - x_k}{x_{k-1} - x_k} (E_\eta(x_1, \ldots, x_{k-2}, x_k, x_{k-1}, x_{k+1}, \ldots, x_N) - E_\eta(x)).$$

(A.8)

Since $i = k - 1$, we have by hypothesis that $E_\eta(x)\bigg|_{x_k=0} = 0$ and that

$$E_{s_{k-1}\eta}(x)\bigg|_{x_k=0} = E_{\eta(k-1)}(x_{(k)}).$$

(A.9)

Equating the r.h.s. of (A.7) and (A.8), we thus get after letting $x_k = 0$ that

$$tE_\eta(x_1, \ldots, x_{k-2}, 0, x_{k-1}, x_{k+1}, \ldots, x_N) = tE_{\eta(k-1)}(x_{(k)})$$

(A.10)

which, after the change of variables $x_{k-1} \mapsto x_k$, is equivalent to

$$E_\eta(x)\bigg|_{x_{k-1}=0} = E_{\eta(k-1)}(x_{(k-1)}).$$

(A.11)

Hence (A.5) holds when $k$ is replaced by $k - 1$.

We now prove that (A.6) holds when $k$ is replaced by $k - 1$. Since $i < j = k - 1$, we have by hypothesis that $\eta_{k-1} \neq 0$ and $\eta_k \neq 0$. We thus obtain by induction that

$$E_\eta(x)\bigg|_{x_k=0} = 0 \quad \text{and} \quad E_{s_{k-1}\eta}(x)\bigg|_{x_k=0} = 0.$$  

(A.12)

Therefore, after again equating the r.h.s. of (A.7) and (A.8) and letting $x_k = 0$, we get

$$tE_\eta(x_1, \ldots, x_{k-2}, 0, x_{k-1}, x_{k+1}, \ldots, x_N) = 0.$$  

(A.13)

But, after the change of variables $x_{k-1} \mapsto x_k$, this is amounts to $E_\eta(x)\bigg|_{x_{k-1}=0} = 0$. \(\square\)
Lemma 46. If \( m + 1 < i < N \) and \( \eta_i > \eta_{i+1} \), then
\[
S_{m+1,N}^t E_{\eta} = \frac{(1 - tA_i)}{t(1 - A_i)} S_{m+1,N}^t E_{\eta_i} \tag{A.14}
\]
where \( A_i = \eta_i/\eta_{i+1} \).

Proof. For \( i > m + 1 \), we have \( S_{m+1,N}^t T_i = tS_{m+1}^t \) (this is similar to (2.11)). Hence, from (2.3), we get
\[
tS_{m+1,N}^t E_{\eta} = S_{m+1,N}^t T_i E_{\eta} = S_{m+1,N}^t \left( \frac{(t - 1)}{(1 - A_i^{-1})} E_{\eta} + \frac{(1 - tA_i)(1 - t^{-1}A_i)}{(1 - A_i)^2} E_{\eta_i} \right) \tag{A.15}
\]
Comparing the l.h.s and the r.h.s of (A.15), we then obtain
\[
t\left( 1 - t^{-1}A_i \right) S_{m+1,N}^t E_{\eta} = \frac{(1 - tA_i)(1 - t^{-1}A_i)}{(1 - A_i)^2} S_{m+1,N}^t E_{\eta_i}
\]
which readily implies our claim. \( \square \)

Lemma 47. Suppose that \( \eta_{m+2} = \eta_{m+3} = \cdots = \eta_{m+n+1} = 0 \) (with \( n \geq 1 \)) are the only zero entries in \( \eta = (\eta_1, \ldots, \eta_N) \) to the right of \( \eta_{m+1} > 0 \). Then
\[
S_{m+2,N}^t E_{\eta}(x)\bigg|_{x_{m+1} = 0} = \frac{A_{m+1}(1 - t^n)}{t(1 - t^{-1}A_{m+1})} S_{m+2,N}^t E_{\eta(m+2)}(x_{m+1}) \tag{A.16}
\]
where \( A_{m+1} = \eta_{m+1}/\eta_{m+2} \).

Proof. From (2.8), we get
\[
T_{m+1} E_{\eta}\bigg|_{x_{m+1} = 0} = \frac{(t - 1)}{(1 - A_{m+1})} E_{\eta}\bigg|_{x_{m+1} = 0} + \frac{(1 - tA_{m+1})(1 - t^{-1}A_{m+1})}{(1 - A_{m+1})^2} E_{s_{m+1}\eta}\bigg|_{x_{m+1} = 0}
\]
But we also have directly using (2.1) that
\[
T_{m+1} E_{\eta}\bigg|_{x_{m+1} = 0} = (t - 1) E_{\eta}\bigg|_{x_{m+1} = 0} + E_{\eta}(x_{m+1})
\]
The previous two equations then imply that
\[
\frac{(1 - t)}{(1 - A_{m+1})} E_{\eta}\bigg|_{x_{m+1} = 0} = E_{\eta}(x_{m+1}) - \frac{(1 - tA_{m+1})(1 - t^{-1}A_{m+1})}{(1 - A_{m+1})^2} E_{s_{m+1}\eta}\bigg|_{x_{m+1} = 0} \tag{A.17}
\]
First, suppose that \( n = 1 \) (that is, that there is only one zero entry to the right of \( \eta_{m+1} \)). From Lemma 46 we have
\[
E_{\eta}(x_{m+1}) = E_{\eta(m+2)}(x_{m+1}) \quad \text{and} \quad E_{s_{m+1}\eta}\bigg|_{x_{m+1} = 0} = E_{\eta(m+2)}(x_{m+1})
\]
Using those equalities in (A.17) gives after some simplifications
\[
\frac{(1 - t)}{(1 - A_{m+1})} E_{\eta}\bigg|_{x_{m+1} = 0} = \frac{A_{m+1}(1 - t)^2}{t(1 - A_{m+1})^2} E_{\eta(m+2)}(x_{m+1}) \tag{A.18}
\]
and it is then immediate that (A.16) holds when \( n = 1 \). When \( n > 1 \), we need to work with \( S_{m+1}^t \) on the left from the start to prove our claim. Using (A.14) again and again (and realizing that each time \( \eta_{m+1} \) is switched with the zero to its right, the corresponding factor \( A_{m+1} \) is multiplied by \( t \)), we have
\[
S_{m+2,N}^t E_{s_{m+1}\eta}\bigg|_{x_{m+1} = 0} = \frac{(1 - t^2A_{m+1}) \cdots (1 - t^nA_{m+1})}{t^n(1 - tA_{m+1}) \cdots (1 - t^{n-1}A_{m+1})} S_{m+2,N}^t E_{s_{m+\cdots s_{m+1}\eta}}\bigg|_{x_{m+1} = 0} \tag{A.19}
\]
Lemma 16 gives
\[
E_{s_{m+\cdots s_{m+1}\eta}}\bigg|_{x_{m+n} = 0} = E_{s_{m+\cdots s_{m+1}\eta}(m+n)}(x_{m+n}) = E_{s_{m+\cdots s_{m+1}\eta}}(x_{m+n})
\]
which implies that
\[
E_{s_{m+\cdots s_{m+1}\eta}}\bigg|_{x_{m+1} = 0} = E_{s_{m+\cdots s_{m+1}\eta}(m+2)}(x_{m+1})
\]
since \( E_{s_{m+\cdots s_{m+1}\eta}} \) is symmetric in the variables \( x_{m+1}, \ldots, x_{m+n} \) by Remark 1. We thus obtain
\[
S_{m+2,N}^t E_{s_{m+1}\eta}\bigg|_{x_{m+1} = 0} = \frac{(1 - t^2A_{m+1}) \cdots (1 - t^nA_{m+1})}{t^n(1 - tA_{m+1}) \cdots (1 - t^{n-1}A_{m+1})} S_{m+2,N}^t E_{s_{m+\cdots s_{m+1}\eta}(m+2)}(x_{m+1})
\]
Now using (A.14) in the other direction (with a shift in $t$ due to the loss of a zero to the left), we get

$$S_{m+2,N}^t E_{s_{m+1}} \bigg|_{x_m+1=0} = \frac{(1 - A_{m+1}) (1 - t^n A_{m+1})}{(1 - t A_{m+1}) (1 - t^{n-1} A_{m+1})} S_{m+2,N}^t E_0 (x_{m+1})$$

On the other hand, using again Lemma 45 yields

$$E_{\eta} \bigg|_{x_{m+n+1}=0} = E_{\eta(m+n)} (x_{m+n+1})$$

which implies this time that

$$E_{\eta} \bigg|_{x_{m+2}=0} = E_{\eta(m+2)} (x_{m+2})$$

given that $E_{\eta}$ is symmetric in the variables $x_{m+2}$ up to $x_{m+n+1}$ and that $\eta(m+n+1) = \eta(m+2)$. Hence

$$E_{\eta} \bigg|_{(x_{m+1}, x_{m+2}) \to (x_{m+2}, 0)} = E_{\eta(m+2)} (x_{m+2}) \bigg|_{(x_{m+1}, x_{m+2}) \to (x_{m+2})} = E_{\eta(m+2)} (x_{m+1})$$

Equation (A.17) then becomes

$$\frac{(1 - t)}{(1 - A_{m+1})} S_{m+2,N}^t E_{\eta} \bigg|_{x_{m+1}=0} = \left( 1 - \frac{(1 - t^n A_{m+1}) (1 - t^{n-1} A_{m+1})}{(1 - t A_{m+1}) (1 - t^{n-1} A_{m+1})} \right) S_{m+2,N}^t E_{\eta(m+2)} (x_{m+1})$$

which, after some straightforward manipulations, yields

$$\frac{(1 - t)}{(1 - A_{m+1})} S_{m+2,N}^t E_{\eta} \bigg|_{x_{m+1}=0} = \frac{A_{m+1} (1 - t^n) (1 - t)}{t (1 - t^{n-1} A_{m+1}) (1 - A_{m+1})} S_{m+2,N}^t E_{\eta(m+2)} (x_{m+1})$$

The claim (A.16) then follows after performing the obvious cancellations. \hfill \Box

**Proposition 48.** For $\Lambda = (a_1, \ldots, a_m; \lambda)$ and $\Lambda^0 = (a_1, \ldots, a_m, 0; \lambda)$, we have that

$$c_{\Lambda^0} (q, t) P_{\Lambda^0} (x; q, t) \bigg|_{x_m+1=0} = c_{\Lambda} (q, t) P_{\Lambda} (x_{m+1}; q, t)$$

where the number of variables $N$ in $x = (x_1, \ldots, x_N)$ is such that $N > \ell(\Lambda)$.

**Proof.** The extra circle $m+1$ at the bottom of $\Lambda^0$ does not affect any of the hook-lengths, from which we get that $c_{\Lambda^0} (q, t) = c_{\Lambda} (q, t)$. It is also easy to see that $u_{\Lambda^0,N} (t) = u_{\Lambda,N-1} (t)$. From the definition of the $m$-symmetric Macdonald polynomials, we thus have left to prove that

$$S_{m+2,N}^t E_{\eta_{\Lambda^0,N}} (x; q, t) \bigg|_{x_{m+1}=0} = S_{m+2,N}^t E_{\eta_{\Lambda,N-1}} (x_{m+1}; q, t)$$

Suppose that in $\eta_{\Lambda^0,N}$ there are zeroes in entries $m+1$ up to $m+n$. From Lemma 45 we obtain that

$$E_{\eta_{\Lambda^0,N}} (x; q, t) \bigg|_{x_{m+n}=0} = E_{\eta_{\Lambda,N-1}} (x_{m+n}; q, t)$$

which implies that

$$E_{\eta_{\Lambda^0,N}} (x; q, t) \bigg|_{x_{m+1}=0} = E_{\eta_{\Lambda,N-1}} (x_{m+1}; q, t)$$

since $E_{\eta_{\Lambda^0,N}} (x; q, t)$ is symmetric in the variables $x_{m+1}, \ldots, x_{m+n}$ by Remark 1. The proposition then follows given that $S_{m+2,N}^t$ does not depend on $x_{m+1}$. \hfill \Box

**Proposition 49.** For $\Lambda = (a_1, \ldots, a_m; \lambda)$ and $\Lambda^0 = (a_1, \ldots, a_m, \lambda \setminus \lambda_j)$, we have that

$$c_{\Lambda^0} (q, t) P_{\Lambda^0} (x; q, t) \bigg|_{x_{m+1}=0} = q^{\lambda_j \cdot \# \{ i \mid a_i < \lambda_j \}} c_{\Lambda} (q, t) P_{\Lambda} (x_{m+1}; q, t)$$

where the number of variables $N$ in $x = (x_1, \ldots, x_N)$ is such that $N > \ell(\Lambda)$.

**Proof.** We have that

$$\frac{c_{\Lambda^0} (q, t)}{c_{\Lambda} (q, t)} = \prod_{s \in \lambda_j} \frac{1 - q^a \lambda (s) + 1 + q^a \lambda (s) + 1 + q^a \lambda (s)}{1 - q^a \lambda (s) + 1 + q^a \lambda (s)}$$

(A.19)

where the product is over all boxes $s$ in the row corresponding to $\lambda_j$ in the diagram of $\Lambda$ (the highest row of size $\lambda_j$ that does not end with a circle in the diagram of $\Lambda$), and where $c_{\Lambda} (s)$ is equal to the number of circles in the column of $s$ (the new circle $m+1$ at the end of the row of size $\lambda_j$ will always be larger than any circle in the column of $s$).
We also have that the normalization $u_{\Lambda,N}$ in (4.2) is such that
\[
\frac{u_{\Lambda,N-1}(t)}{u_{\Lambda,N}(t)} = \frac{(1 - t^{-\mu})}{(1 - t^{-n})} = \frac{(1 - t^{\mu})}{(1 - t^{n})}
\] (A.20)

Let $\nu = \lambda \setminus \lambda_j$ and let $\hat{\lambda}$ be the partition obtained from $\lambda$ by removing all the parts larger than $\lambda_j$. Using (A.10), we obtain that
\[
S_{m+2,N}^t E_{\eta,\lambda,N}(x; q, t) \bigg|_{x_{m+1} = q} = \frac{A_{m+1}(1 - t^n)}{(1 - t^{n-1})} S_{m+2,N}^t E_{a_1, \ldots, a_m, \lambda_j, 0^{n-1}, \nu, \ell, \ell_1(x_{m+1}); q, t}
\]

We now need to use (A.14) again and again to move $\lambda_j$ to the right to obtain the composition
\[
(a_1, \ldots, a_m, 0^{n-1}, \lambda_j, \nu, \ell) = \eta_{\Lambda,N-1}
\]

Let $f_{\lambda}(q, t)$ be the constant that appears in this process, that is,
\[
S_{m+2,N}^t E_{a_1, \ldots, a_m, \lambda_j, 0^{n-1}, \nu, \ell}(x_{m+1}; q, t) = f_{\lambda}(q, t) S_{m+2,N}^t E_{a_1, \ldots, a_m, \lambda_j, 0^{n-1}, \nu, \ell}(x_{m+1}; q, t)
\]

Taking (A.19) and (A.20) into consideration and using $A_{m+1} = q^{\lambda_j} t^{\# \{i : a_i < \lambda_j\} + \ell(\hat{\lambda})}$, the proposition will follow from the definition of the $m$-symmetic Macdonald polynomials if we can prove that
\[
\left( \prod_{s \in \lambda_j} \frac{1 - q^{a_s(s) + 1} t^{\lambda(s) + 1 + o_s(s)}}{1 - q^{a_s(s) + 1} t^{\lambda(s) + 1 + o_s(s)}} \right) \left( \frac{1 - q^{\lambda_j} t^{\# + \ell(\hat{\lambda}) + n - 1}}{1 - q^{\lambda_j} t^{\# + \ell(\hat{\lambda}) + n - 1}} \right) f_{\lambda}(q, t) = q^{\lambda_j} t^{\#}
\]

where we use the shorthand $\#$ for $\# \{i : a_i < \lambda_j\}$. This amounts to showing that
\[
f_{\lambda}(q, t) = \left( \prod_{s \in \lambda_j} \frac{1 - q^{a_s(s) + 1} t^{\lambda(s) + 1 + o_s(s)}}{1 - q^{a_s(s) + 1} t^{\lambda(s) + 1 + o_s(s)}} \right) \left( \frac{1 - q^{\lambda_j} t^{\# + \ell(\hat{\lambda}) + n - 1}}{1 - q^{\lambda_j} t^{\# + \ell(\hat{\lambda}) + n - 1}} \right) = 1
\]

We proceed by induction on the number of exchanges that need to be done in order to move $\lambda_j$ to the right.

If there are no exchanges to do, we need to show that $f_{\lambda}(q, t) = 1$. In this case, we have $n = 1$ and $\lambda = (\lambda_j^{(\lambda_j)})$. Moreover, in the diagram of $\Lambda$, all the rows that are shorter than $\lambda_j$ end with a circle. Hence, if there is a cell $s'$ immediately to the right of $s$ in the row of $\lambda_j$, we get
\[
1 - q^{a_s(s) + 1} t^{\lambda(s) + 1 + o_s(s)} = 1
\]
The first product in $f_{\lambda}(q, t)$ thus simplifies to
\[
\prod_{s \in \lambda_j} \frac{1 - q^{a_s(s) + 1} t^{\lambda(s) + 1 + o_s(s)}}{1 - q^{a_s(s) + 1} t^{\lambda(s) + 1 + o_s(s)}} = 1
\]

From (A.21), this yields that $f_{\lambda}(q, t) = 1$ since $n = 1$ and $a_s(\lambda_j) = \ell(\hat{\lambda})$.

We now consider the general case. First suppose that $n = 1$ (the case in which there are no more 0’s with which to do do an exchange). The first exchange that needs to be done is with a circle in a row of size $\lambda_{\ell} = \lambda_{\ell}(\lambda)$. From (A.14), it gives rise to the term
\[
1 - q^{\lambda_j - \lambda_{\ell}(\lambda) + 2 + o_s(s')} t^{(1 - q^{\lambda_j - \lambda_{\ell}(\lambda) + 2 + o_s(s')})}
\]

where $s'$ is the cell in the row of $\lambda_j$ and column $\lambda_{\ell} - \lambda_{\ell} + 1$. The remaining exchanges are equal by induction to $f_{\Omega}(q, t)$, where $\Omega$ is $\Lambda$ with the row corresponding to $\lambda_{\ell}$ now having a circle. Considering that $n = 1$, we obtain
\[
f_{\Omega}(q, t) = \left( \prod_{s \in \lambda_1} \frac{1 - q^{a_s(s) + 1} t^{\lambda(s) + 1 + o_s(s)}}{1 - q^{a_s(s) + 1} t^{\lambda(s) + 1 + o_s(s)}} \right) \left( \frac{1 - q^{\lambda_j} t^{\# + \ell(\hat{\lambda})}}{1 - q^{\lambda_j} t^{\# + \ell(\hat{\lambda})}} \right)
\]

where we used the fact that, when compared to $\Lambda$, $\#$ increased by 1 while $\ell(\lambda)$ decreased by one. We thus have to show that the product of (A.22) and (A.23) gives (A.21) in the case $n = 1$. After performing easy cancellations, this amounts to showing that
\[
1 - q^{\lambda_j - \lambda_{\ell}(\lambda) + 2 + o_s(s')} \left( \prod_{s \in \lambda_1} \frac{1 - q^{a_s(s) + 1} t^{\lambda(s) + 1 + o_s(s)}}{1 - q^{a_s(s) + 1} t^{\lambda(s) + 1 + o_s(s)}} \right) = \left( \prod_{s \in \lambda_1} \frac{1 - q^{a_s(s) + 1} t^{\lambda(s) + 1 + o_s(s)}}{1 - q^{a_s(s) + 1} t^{\lambda(s) + 1 + o_s(s)}} \right)
\]
The only difference in the products over the cells $s$ is when $s = s'$, where $s'$ lies in the column of the extra circle in $\Omega$ (in column $\lambda_j - \lambda_j + 1$). Hence, we need to show that
\[
\frac{1 - q^{\lambda_j - \lambda_j + 1} t^{|s|} + o(s')}{1 - q^{\lambda_j - \lambda_j + 1} t^{|s|} + o(s')} = \frac{1 - q^{a(s')} t^{|s|} + o(s')}{1 - q^{a(s')} t^{|s|} + o(s')}
\]
Using $a(s') = a(s') = \lambda_j - \lambda_j - 1$, $l_s(s') = l_s(s')$, and $o(s') = o(s') + 1$, the result is seen to hold.

Finally, in the case $n > 1$, the proof is exactly as in the previous case except that $s'$ is now in the first column, $\lambda_j$ is replaced by 0, and the quantities ($1 - q^{\lambda_j t^{|\lambda|} + o(\lambda)}$) and $t^{\ell(\lambda) - n(\lambda_j)}$ are replaced by ($1 - q^{\lambda_j t^{|\lambda|} + o(\lambda)}$) and $t^{\ell(\lambda) - n(\lambda_j) + n - 1}$ respectively.

\[\square\]

We will finish this appendix with a proof of Theorem 36. Recall that its main statement says that for any $N$ and any $m \geq 1$, we have that
\[
\Psi_N J_A(x_1, \ldots, x_N; q, t) = t^{-\#(2 \leq j \leq m | a_j \leq a_1)} J_A\square(x_1, \ldots, x_N; q, t)
\]
where $\Lambda\square = \{(a_2, \ldots, a_m) ; \lambda \cup (a_1 + 1)\}$.

**Proof of Theorem 36** We first establish that
\[
\Psi_N S_{m+1, N} = (1 - t) S_{m, N} \Phi_q
\]
for $m \geq 1$. From the braid relations, it is easy to see that
\[
(T_{N-2} \cdots T_1)T_1 = T_{i-1}(T_{N-2} \cdots T_1)
\]
for all $i = 2, \ldots, N - 1$. Hence, for $m \geq 1$, we obtain
\[
\Phi_q S_{m+1, N} = S_{m, N-1} \Phi_q
\]
where $\Phi_q$ is defined in (2.9). It is then immediate from (2.12) that (A.24) holds.

Now, given that by definition
\[
J_A = \frac{c_A(q, t)}{u_A(N^t)} S_{m+1, N} E_{\Lambda, N}
\]
we have to show that
\[
\Psi_N S_{m+1, N} E_{\eta_A, N} = t^{-\#(2 \leq j \leq m | a_j \leq a_1)} \frac{c_A(q, t)}{c_A(q, t)} \frac{u_A(N^t)}{u_{\Lambda\square, N}(t)} S_{m, N} E_{\eta_A, N}
\]
When the circle indexed by a 1 in the diagram of $\Lambda$ becomes a square, it will only change the contribution in $c_A(q, t)$ of the cells in the column of the circle indexed by a 1 that lie in rows that do not end with a circle (otherwise the cell indexed by a 1 already contributes to the leg-length). Hence, also considering the change in the symmetric rows of length $a_1 + 1$, we obtain
\[
\frac{c_A(q, t)}{c_A(q, t)} = \left( \prod_{s \in \text{col}(1)} \frac{1 - q^{a(s)} t^{|s|} + 2}{1 - q^{a(s)} t^{|s|} + 1} \right) \left( 1 - t^{n(\lambda(a_1 + 1)) + 1} \right)
\]
where the product is over the cells in the column of the circle indexed by 1 that lie in rows that are larger than $a_1 + 1$ and that do not end with a circle. We also have
\[
\frac{u_A(N^t)}{u_{\Lambda\square, N}(t)} = \left( \frac{1 - t^{-1}}{1 - t^{-n(\lambda(a_1 + 1)) - 1}} \right) \frac{t^{(N-m)(N-m-1)/2}}{t^{(N-m-1)(N-m)/2}} \left( 1 - t \right) \frac{t^{n(\lambda(a_1 + 1)) + m - N}}{1 - t^{n(\lambda(a_1 + 1)) + 1}}
\]
On the other hand, we obtain from (2.9) and (5.6) that
\[
\Psi_N S_{m+1, N} E_{\eta_A, N} = (1 - t)^{t^{n(\lambda(a_1 + 1)) + 1}} S_{m, N} E_{\Phi(\eta_A, N)}
\]
Using Lemma 46 again and again to reorder the last entry of $\Phi(\eta_A, N)$, we obtain
\[
S_{m, N} E_{\Phi(\eta_A, N)} = t^{-\#(j | \lambda_j > a_1 + 1)} \left( \prod_{s \in \text{col}(1)} \frac{1 - q^{a(s)} t^{|s|} + 2}{1 - q^{a(s)} t^{|s|} + 1} \right) S_{m, N} E_{\eta_A, N}
\]
where the product is again over the cells in the column of the circle indexed by 1 that lie in rows that are larger than \(a_1 + 1\) and that do not end with a circle. Therefore, \(\text{A.25}\) becomes

\[
\Psi_N S_{m+1,N} E_{\eta_1,N} = (1 - \ell) r_N(1) - N - \#\{ j | \lambda_j > a_1 + 1\} \prod_{s \in \text{col}(1)} \frac{1 - q^{\rho(s) - 1}}{1 - q^{\rho(s)}} S_{m,N} E_{\eta_2,N}.
\]

Using \(\text{A.26}\) and \(\text{A.27}\), we then see from the previous equation that \(\text{A.25}\) will hold if

\[
r_N(1) = \ell(\Lambda) - \#\{2 \leq j \leq m | a_j \leq a_1\} - \#\{ j | \lambda_j \leq a_1\} + \#\{ j | \lambda_j > a_1 + 1\} + n_\Lambda(a_1 + 1).
\]

But this easily follows after realizing that \(r_N(1) = \ell(\Lambda) - \#\{2 \leq j \leq m | a_j \leq a_1\} - \#\{ j | \lambda_j \leq a_1\}\) and that \(\ell(\Lambda) = m + \#\{ j | \lambda_j \leq a_1\} + \#\{ j | \lambda_j > a_1 + 1\} + n_\Lambda(a_1 + 1)\).

\[
\square
\]

References

[1] P. Alexandersson and M. Sawhney, Properties of non-symmetric Macdonald polynomials at \(q = 0\) and \(q = 1\), Ann. Comb. 23 (2019), 219–239.

[2] S. Assaf, Non-symmetric Macdonald polynomials and a refinement of Kostka-Foulkes integrals, Trans. Amer. Math. Soc. 370 (2018), 8777–8796.

[3] T. H. Baker and P. J. Forrester A \(q\)-analogue of the type A Dunkl Operator and Integral Kernel, Int. Math. Res. Not. 14 (1997), 667–686.

[4] F. Bergeron, Algebraic combinatorics and coinvariants spaces, CMS Treatise in Mathematics, A.K. Peters Publishers, 2009.

[5] O. Blondeau-Fournier, P. Desrosiers, L. Lapointe and P. Mathieu, Macdonald polynomials in superspace: conjectural definition and positivity conjectures, Lett. Math. Phys. 101, 27–47 (2012).

[6] O. Blondeau-Fournier, P. Desrosiers, L. Lapointe and P. Mathieu, Macdonald polynomials in superspace as eigenfunctions of commuting operators, Journal of Combinatorics 3, no. 3, 495–562 (2012).

[7] E. Carlsson, G. Gorsky and A. Mellit, The \(h,q,t\) algebra and parabolic flag Hilbert schemes, Math. Ann. 376 (2020), 1303–1336.

[8] I. Cherednik, Non-symmetric Macdonald polynomials, Int. Math. Res. Notices 10 (1995) 483-515.

[9] M. Concha and L. Lapointe, The \(m\)-symmetric Macdonald polynomials, arXiv:2311.12625.

[10] A.M. Garsia and M. Haiman, A graded representation model for Macdonald polynomials, Ann. Comb. 99 (1982), 447–450.

[11] L. Lapointe and L. Pena, The norm and the evaluation of the Macdonald polynomials in superspace, European J. Combin. 33, 103018, 30pp (2020).

[12] B.G. Goodberry, Partially-symmetric Macdonald polynomials, Ph.D. Thesis, Virginia Polytechnic Institute and State University, 2022.

[13] B. Goodberry, Type A partially symmetric Macdonald polynomials, arXiv:2311.12216.

[14] M. Haiman, Hilbert schemes, polygraphs, and the Macdonald positivity conjecture, J. Amer. Math. Soc. 14 (2001), 941–1006.

[15] F. Knop, Composition Kostka functions in Algebraic groups and homogeneous spaces, volume 19 of Tata Inst. Fund. Res. Stud. Math., Mumbai, 2007.

[16] L. Lapointe and L. Pena, in preparation.

[17] A. Lascoux and M.-P. Schützenberger, Polynômes de Schubert, C.R. Acad. Sc. Paris 295 (1982), 447–450.

[18] I. G. Macdonald, Notes on Schubert polynomials, Publications du Laboratoire de Combinatoire et d’Informatique Mathématique, vol. 6, LACIM, 1991.

[19] I. G. Macdonald, Symmetric functions and Hall polynomials, 2nd edition, Clarendon Press, Oxford, 1995.

[20] I. G. Macdonald, Affine Hecke algebras and orthogonal polynomials, Séminaire Bourbaki 1994-95, exposé 797, p. 189-207.

[21] I. G. Macdonald, Affine Hecke algebras and orthogonal polynomials, Cambridge University Press, 2003.

[22] D. Marshall, Symmetric and nonsymmetric Macdonald polynomials, Ann. Comb. 3 (1999), 385–415.

[23] D. Orr and M.B. Weising, Stable-limit partially symmetric Macdonald functions and parabolic flag Hilbert schemes, arXiv:2310.13642.

[24] L. Pena, Ph.D. Thesis, Universidad de Talca, in preparation.

[25] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge University Press, (1999).

[26] M. Waxs, Flagged Schur functions, Schubert polynomials, and symmetrizing operators, J. Combin. Theory Ser. A 40(2) (1985), 276–289.