MODEL INDEPENDENT DETERMINATION OF
THE SHAPE FUNCTION FOR INCLUSIVE $B$ DECAYS AND
OF THE STRUCTURE FUNCTIONS IN DIS

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Abstract

We present a method to compute, by numerical simulations of lattice QCD, the
inclusive semileptonic differential decay rates of heavy hadrons and the structure func-
tions which occur in deep inelastic scattering. The method is based on first prin-
ciples and does not require any model assumption. It allows the prediction of the
differential rate in $B$ semileptonic decays for values of the recoiling hadronic mass
$W \sim \sqrt{M_B A_{QCD}}$, which is in the relevant region to extract $|V_{ub}|$ from the end-point
of the lepton spectrum in inclusive decays.
1 Introduction

In this paper we propose a method to compute, on the lattice, the shape function $f(k_+)$ which enters the calculation of the inclusive differential semileptonic decay rate of heavy hadrons. The knowledge of $f(k_+)$ is a fundamental ingredient for the extraction of $|V_{ub}|$ from the end-point of the lepton spectrum. The same function also enters the calculation of the photon spectrum in radiative $B$ decays. Furthermore, the same method can be applied to the calculation of the structure functions of Deep Inelastic Scattering (DIS). Our approach does not require any model assumption and is based on standard techniques which are used to compute the hadronic matrix elements in lattice QCD. In particular, we show that the use of the Euclidean space-time, which is unavoidable in lattice calculations, is not an obstacle to the calculation of $f(k_+)$ in the deep inelastic region.

Our main result is that, from the study of suitable combinations of lattice Green functions, we can obtain the quantity

$$G(t, \vec{Q}) = \frac{e^{-\sqrt{Q^2t}}}{2\sqrt{Q^2}} \int_0^{M_B} \tilde{f}(k_+) e^{-k_+t},$$

where $M_B$ is the mass of the decaying hadron (the $B$ meson in the example considered in this paper), $t$ is a time distance and $\vec{Q}$ a spatial momentum that we can inject in the Green functions computed in the numerical simulations. By varying $t$ and $\vec{Q}$, we can unfold the integral and extract $\tilde{f}(k_+)$. The function $\tilde{f}(k_+)$ in eq. (1) differs from the usual shape function $f(k_+)$ introduced in refs. [1, 2], as will be explained in the following. In terms of $\tilde{f}(k_+)$, the differential semileptonic decay rate can be written as

$$\frac{d\Gamma}{dE_\ell} = |V_{ub}|^2 \frac{G_F^2}{12\pi^3} E_\ell^2 \int_0^{M_B} dk_+ \tilde{f}(k_+) \Theta(m_b^* - 2E_\ell) \left[3m_b^{*2} - 4m_b^*E_\ell\right],$$

where $m_b^* = M_B - k_+$. The advantage of using $\tilde{f}(k_+)$ is that any reference to unphysical quantities, such as $\bar{\Lambda}$ or the quark mass $m_b$, disappears in the expression above and the differential rate is written in terms of hadronic quantities only.

The paper is organised as follows. In section 2 we review the main formulae for the total and differential semileptonic decay rates as they can be derived using the Operator Product Expansion (OPE); $f(k_+)$ is introduced in this section. In sec. 3 we present the basic formalism needed to explain our idea. We discuss, as a prototype, the case of the shape function which enters the differential decay rate, and then extend
the method to the computation of the structure functions. We express these quantities in terms of suitable $T$-products of local operators and discuss their continuation to the Euclidean space-time. In sec. 4, we describe the implementation of the method in lattice calculations and discuss the systematic errors which may arise in actual numerical simulations, due to the present limitations in computer resources.

## 2 Inclusive decay rates

This section contains a summary of the main results obtained in the literature for semileptonic and radiative inclusive decays. We present below the main formulae for the total and differential rates which can be derived using the OPE. This will allow us to introduce the definition of the shape function and of its moments.

The idea that inclusive decay rates of hadrons containing heavy quarks can be computed in the parton model is quite old and was used, for example, in ref. [9] to predict the charmed-hadron lifetimes. To account for bound state effects, the partonic calculation was subsequently improved by the introduction of a phenomenological model, called the “spectator model”, which has been and continues to be extensively used to extract $|V_{cb}|$ and $|V_{ub}|$ from inclusive semileptonic decays. A noble theoretical framework for the spectator model was then provided by the Wilson OPE applied to the inclusive decays of heavy hadrons. The expansion parameter is not necessarily the inverse heavy-quark mass $1/m_Q$, rather the inverse of the energy release $1/W$ of the process at hand. In a large region of the available phase space $W$ is of the order of $m_Q$. In this case, under the hypothesis of quark-hadron duality, the operator-product expansion is expected to give accurate predictions for the decay widths, expressed in terms of few non-perturbative parameters. In particular, at lowest order in $1/m_Q$, the expression of the decay widths derived with the OPE coincides with that obtained in the parton model calculation.

Let us recall how this works for the inclusive semileptonic processes $\bar{B} \rightarrow X_{c,u}\ell\bar{\nu}_\ell$. Using the OPE, the total semileptonic decay rate is given by

$$\Gamma(\bar{B} \rightarrow X_{c,u}\ell\bar{\nu}_\ell) = |V_{(c,u)b}|^2 \frac{G_F^2 m_b^5}{192 \pi^3} \left[ \left( 1 + \frac{\lambda_1}{2 m_b^2} \right) C_0(x_{c,u}) - \frac{9 \lambda_2}{2 m_b^2} C_1(x_{c,u}) + \ldots \right],$$

where we have neglected QCD radiative corrections and phase space effects due to the mass of the final charged lepton. The dots stand for higher-order perturbative and/or power corrections. We have omitted to write explicitly the terms of $O(1/m_b^6)$ because they are expected to give negligible contributions to $\Gamma(\bar{B} \rightarrow X_{c,u}\ell\bar{\nu}_\ell)$. $C_{0,1}(x_{c,u})$ are known phase-space factors which depend on the ratios of the final-quark masses to $m_b$, etc.
\[ x_{c,u} = m_{c,u}^2/m_b^2 \] \[ \text{In eq. (3), the term which coincides with the parton model result} \]
\[ \Gamma_{PM}(\bar{B} \to X_{c,u} \ell \bar{\nu}_\ell) = |V_{(c,u)b}|^2 \frac{G_F^2 m_b^5}{192\pi^3} C_0(x_{c,u}) \] \[ \text{(4)} \]
\[ \text{corresponds to the insertion of the leading, dimension-three operator } \bar{b}b \text{ appearing in the OPE. In the heavy quark effective theory (HQET) at fixed velocity } v, \text{ this is a conserved operator } (\bar{b}b \to \bar{h}_v h_v + \ldots) \text{ satisfying the following normalisation condition} \]
\[ \langle \bar{B}(v)|\bar{h}_v h_v|\bar{B}(v)\rangle = 1. \] \[ \text{(5)} \]

The hadron state \(|\bar{B}(v)\rangle\) is normalised to \(v^0\) instead of the usual relativistic normalisation of \(2E\) because this is more convenient for the heavy quark expansion. The corrections of \(O(1/m_b)\) vanish since the only possible dimension-four operator, \(\bar{b}iD\bar{b}\), can be reduced to \(\bar{b}b\) by using the equations of motion. The kinetic energy of the heavy quark in the \(B\) meson, \(\lambda_1\), and the chromomagnetic moment of the heavy quark, \(\lambda_2\), correspond to matrix elements of local, dimension-five operators appearing in the OPE \[8\]. Written in terms of the fields of the HQET, these operators are given by
\[ \lambda_1 = \langle \bar{B}(v)|\bar{h}_v (iD)^2 h_v|\bar{B}(v)\rangle \] \[ \lambda_2 = \langle \bar{B}(v)|\bar{h}_v \sigma_{\mu\nu} iD^\mu iD^\nu h_v|\bar{B}(v)\rangle. \] \[ \text{(6)} \]
Thus, up to \(O(1/m_b^3)\) the problem is reduced to the calculation of the matrix elements of the two local operators appearing in eq. (6). For these matrix elements several theoretical estimates exist \[1\]. A recent review of these estimates can be found, for example, in ref. \[12\].

Unfortunately, in order to determine \(|V_{ub}|\) from the lepton spectrum, one has to study the differential distribution and perform cuts to suppress decays involving charmed particles in the final state. For the differential distribution, the relevant scale of the OPE is the squared momentum of the hadronic system recoiling against the lepton pair, \(Q_b = m_b v - q\), where \(q\) is the momentum of the lepton system. By approaching the end-point of the electron spectrum \(W = \sqrt{Q_b^2}\) becomes small.

In the region where \(v \cdot Q_b\) is of \(O(M_B)\) and \(W \sim \sqrt{M_B \Lambda_{QCD}}\) we can still use the parton model by introducing a “shape function”, analogous to the distribution function of the spectator model \[11\], corresponding to a modified OPE. This is analogous to deep inelastic scattering, where one can introduce non-perturbative distribution functions

\[ \text{1 We are not concerned here with the precise definition of } \lambda_{1,2} \text{ when QCD corrections are taken into account.} \]
for the light-cone component of the quark momenta. In terms of the shape function $f(k_+)$ the differential distribution is given by

$$
\frac{d\Gamma}{dE_\ell} \equiv \int_{-m_b}^\Lambda dk_+ f(k_+) \frac{d\Gamma_{PM}}{dE_\ell}(m_b^*, E_\ell)
$$

$$
= |V_{ub}|^2 \frac{G_F^2}{12\pi^3} E_\ell^2 \int_{-m_b}^\Lambda dk_+ f(k_+) \Theta(m_b^* - 2E_\ell) \left[3m_b^{*2} - 4m_b^2E_\ell \right],
$$

(7)

where $m_b^* = m_b + k_+$ and

$$
f(k_+) = \langle \bar{B}(v) | \bar{h}_v \delta(k_+ - iD_+) h_v | \bar{B}(v) \rangle.
$$

(8)

$D_+ = n \cdot D$, with $n^\mu = Q_b^\mu / (v \cdot Q_b)$. Note that in the region where $W \sim \sqrt{M_B \Lambda_{QCD}}$, $n^2 \sim \Lambda_{QCD} / M_B \sim 0$.

Finally, if we consider the extreme region where $W \sim \Lambda_{QCD}$, the partonic picture breaks down. In this region the rate is dominated by few single states or resonances and the appropriate description can only be obtained by predicting the rate as a sum over the lowest exclusive channels.

At the lowest order in $1/m_b$, the integration of $d\Gamma/dE_\ell$ over $E_\ell$ in eq. (7) gives the total rate. It also gives some of the $O(1/m_b^2)$ corrections appearing in eq. (3). For the following discussion, it is convenient to introduce the moments of $f(k_+)$, defined as

$$
\mathcal{M}_n = \int_{-m_b}^\Lambda dk_+ k_+^n f(k_+).
$$

(9)

The first few moments are known: the leading contribution, given by $\mathcal{M}_0 = 1$, is the normalisation of $f(k_+)$ and is fixed by the normalisation of the conserved scalar density operator of eq. (3); the second moment $\mathcal{M}_1$, related to the correction of $O(1/m_b)$, vanishes because there are no operators contributing at this order; the correction of $O(1/m_b^2)$ is given by $\mathcal{M}_2 = -\lambda_1/3$. The contribution of the heavy-quark chromomagnetic moment is not included in the shape function.

In the case of the integrated rate, the contributions coming from higher moments of the shape functions are related to operators of dimension larger than five. For this reason, they are suppressed by higher powers in $1/m_b$ and can be safely ignored. For the differential distribution of eq. (7) near the end-point, instead, all moments $\mathcal{M}_n$ become of the same order in $1/m_b$ and cannot be neglected. The knowledge of the full shape function $f(k_+)$, and not only of the first few moments, is then needed in this region. This is analogous to what happens for the structure functions in DIS processes in the region where the Bjorken variable $x \to 1$.

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2 A more extensive discussion of this point will be presented in section 3.
So far, several models for the shape function have been proposed [1, 2, 11, 13]. In particular, one can show that the old ACCMM model is equivalent, up to a process-dependent redefinition of the heavy quark mass [1], to a specific choice of the shape function. It remains true, however, that the extraction of $|V_{ub}|$ from the experimental measurements of the end-point of the lepton spectrum using these models is “model dependent” and consequently affected by a systematic error which is difficult to estimate.

3 The shape function $\tilde{f}(k_+)$

In this section, we show that it is possible to determine the full shape function, and not only a few moments of it (as for example $\lambda_1$ [14]), by numerical simulations on the lattice. The shape function can be obtained from the time and momentum dependence of suitable Euclidean Green functions of the same type as those which are currently used on the lattice to compute hadronic matrix elements of local operators. We show that, in the inelastic region where the knowledge of the full shape function is necessary, this can be done in spite of the fact that lattice simulations are performed in the Euclidean space-time. This is new since, as discussed below and in ref. [15], the use of the Euclidean space-time, instead of the Minkowsky one, seems to prevent the possibility of computing inclusive quantities on the lattice [4]. We also show that, by using the same approach, it is possible to compute the structure functions of DIS, and not only few moments of it, as done so far [17–18].

A further advantage of our proposal is the following. Higher moments of the shape function (and those of the structure functions) are related to matrix elements of local operators of higher dimensions which are, in general, afflicted by power divergences when a hard cutoff is used (or renormalon ambiguities in dimensional regularisations). This makes the definition of the renormalised operators, which have finite matrix elements when the cutoff is removed, very problematic [19]. With our method, instead, no renormalisation is needed as we get directly the shape function from the lattice correlation functions, as explained in sec. 4.

We now give all the details of the derivation of our method.

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3For exclusive processes involving more than one particle in the final state, the problem is usually referred to as the Maiani-Testa no-go theorem [16].
The spin-averaged, differential semileptonic decay rate is given by

$$\frac{d\Gamma}{dq^2 dE_\ell dE_{\nu\ell}} = |V_{(c,u)b}|^2 \frac{G_F^2}{2\pi^3} \left[ W_1 q^2 + W_2 \left( 2E_\ell E_{\nu\ell} - \frac{1}{2}q^2 \right) + W_3 q^2 \left( E_\ell - E_{\nu\ell} \right) \right].$$

(10)

with the Lorentz-invariant functions $W_i$ defined by the expansion of the hadronic tensor

$$W^{\mu\nu} = (2\pi)^3 \sum_X \delta^4(p_B - q - p_X)(\bar{B}(v)|J^\mu \dagger(x)X|J^\nu|\bar{B}(v)), $$

(11)

which can be written as

$$W^{\mu\nu} = -g^{\mu\nu}W_1 + \epsilon^{\alpha\mu\beta\nu} q_\alpha v_\beta W_3 + q^\mu q^\nu W_2 + (q^\mu v^\nu + q^\nu v^\mu) T_5. $$

(12)

In eq. (11), the weak current is defined as $J^\mu \equiv (\bar{q}(x)\gamma^\mu b(x)$, where $q(x)$ represents the field of the light final quark and $\gamma^\mu_L = \gamma^\mu(1 - \gamma_5)/2$. Using the optical theorem, the form factors $W_i$ are given by

$$W_i = -\frac{1}{\pi}\text{Im}T_i,$$

(13)

where the $T_i$ are defined by the forward matrix element of the $T$-product of the two weak currents

$$T^{\mu\nu} = -i \int d^4x \ e^{-iq \cdot x} \langle \bar{B}(v)|T \left( \bar{J}^\mu(x)J^\nu(0) \right)|\bar{B}(v)\rangle = -g^{\mu\nu}T_1 + \epsilon^{\alpha\mu\beta\nu} q_\alpha v_\beta T_3 + q^\mu q^\nu T_4 + (q^\mu v^\nu + q^\nu v^\mu) T_5. $$

(14)

Thus the problem is reduced to the calculation of the forward matrix element of the $T$-product defined above.

One could think that, with sufficient computer resources to work with very small values lattice spacing $a$ such that

$$\Lambda_{QCD} \ll m_b \ll \frac{1}{a},$$

(15)

it would be possible to compute directly $W^{\mu\nu}$ on the lattice using the $T$-product of eq. (14). This is in general illusory. The problem arises from the fact that numerical simulations are necessarily performed in the Euclidean space-time [15]. Let us illustrate where the problem comes from.

In the Minkowsky case, for $t > 0$, one has

$$W^{\mu\nu} = \frac{1}{\pi}\text{Im} \left[ i \int d^4x \ e^{-iq \cdot x} \langle \bar{B}(v)|J^\mu \dagger(x) J^\nu(0)|\bar{B}(v)\rangle \right]$$

6
Green functions.

and that it is possible to extract the wave function by studying suitable Euclidean space-time of the $\Sigma$ where $T_{\mu\nu}$ is given by the integral over the time of an oscillating exponential. In the Euclidean case, instead, eq. (16) becomes

$$W_{\mu\nu} \sim \sum_X \int dt \; e^{i(p_B^\mu - q^\mu - E_X)t} \frac{(2\pi)^3}{\pi} \delta^3(\vec{p}_B - \vec{q} - \vec{p}_X) \langle \vec{B}(v)|J_{\mu \nu}^{\dagger}(0)|X\rangle \langle X|J_{\nu}(0)|\vec{B}(v)\rangle.$$ \hspace{1cm} (16)

For $t \leq 0$, the cut corresponds to an intermediate state with two $b$-quarks and a $c$, which is not the process we are interested in. The $\delta$ function related to energy conservation is given by the integral over the time of an oscillating exponential. In the Euclidean case, instead, eq. (16) becomes

$$W_{\mu\nu} \sim \sum_X \int dt \; e^{i(p_B^\mu - q^\mu - E_X)t} \frac{(2\pi)^3}{\pi} \delta^3(\vec{p}_B - \vec{q} - \vec{p}_X) \langle \vec{B}(v)|J_{\mu \nu}^{\dagger}(0)|X\rangle \langle X|J_{\nu}(0)|\vec{B}(v)\rangle.$$ \hspace{1cm} (17)

Thus, at large time distances, the integral over $t$ is dominated by the states with the smallest energy $E_X$, and not by those which satisfy the energy conservation condition $E_X = p_B^0 - q^0$. This is similar to what happens in the case of exclusive decays which was discussed in ref. [16]. The same problem is also encountered in other analytic approaches which study inclusive cross-sections or decay rates in the Euclidean [20].

We now show that, in the deep inelastic limit, the analytic continuation to the Euclidean space-time of the $T$-product of the currents in eq. (14) becomes very simple and that it is possible to extract the wave function by studying suitable Euclidean Green functions.

$T_{\mu\nu}$ can be written as

$$T_{\mu\nu} = -i \int d^4x \; e^{-iq \cdot x} \langle \vec{B}(v)|\bar{b}(x)\gamma_{L}^{\mu}S(x-0)\gamma_{L}^{\nu}b(0)|\vec{B}(v)\rangle$$

$$= -i \int d^4x \; e^{i(p_B \cdot q - E_X)x} \langle \vec{B}(v)|\bar{b}_v(x)\gamma_{L}^{\mu}S(x-0)\gamma_{L}^{\nu}b_v(0)|\vec{B}(v)\rangle$$

$$= -i \int d^4x \; \langle \vec{B}(v)|\bar{b}_v(x)\gamma_{L}^{\mu}S_Q(x-0)\gamma_{L}^{\nu}b_v(0)|\vec{B}(v)\rangle,$$ \hspace{1cm} (18)

where $S(x-0)$ is the final light-quark propagator; $b(x) \equiv e^{-ip_B \cdot x}b_v(x)$ and $S_Q(x-0) = e^{iQ \cdot x}S(x-0)$, with $Q = p_B - q = MBv - q$. Our definition of the heavy quark field $b_v(x)$, written in terms of the physical momentum of the meson $p_B$ instead of the quark momentum $p_b$, is legitimate since the shape function is only defined at the lowest order of the heavy quark expansion. The reason for this choice will be explained below. It is possible to expand $S_Q(x-0)$ as follows (for clarity we neglect the mass of the light quark)

$$S_Q(x-0) = \left( \frac{i}{Q + iD^2 + i\epsilon} \right)_{x-0} = \left( \frac{Q + iD}{Q^2 + 2iQ \cdot D - D^2 - 1/2D^2 + i\epsilon} \right)_{x-0}.$$
\[ T_{\mu\nu} = \frac{1}{2} \left( Q^{\mu} v^{\nu} + Q^{\nu} v^{\mu} - g^{\mu\nu} Q \cdot v + i \epsilon^{\mu\nu\alpha\beta} Q_\alpha v_\beta \right) \int_0^{M_B} dk_+ \frac{\tilde{f}(k_+)}{k_+^2 - 2v \cdot Q k_+ + i\epsilon}, \] (20)

where the shape function \( \tilde{f}(k_+) \) is defined through the relations

\[ (-1)^n \langle \bar{B}(v)|\bar{b}_0 \gamma^\mu (iD^{n_1}) \cdots (iD^{n_\mu}) b_0|\bar{B}(v) \rangle = M_n v^{n_1} \cdots v^{n_\mu} + B_n \delta^{\mu n_1} v^{n_2} \cdots v^{n_\mu} + \cdots. \] (21)

The moments \( \mathcal{M}_n \) in the equation above are given by

\[ \mathcal{M}_n = \int_0^{M_B} dk_+ k_+^n \tilde{f}(k_+) \] (22)

With our choice of \( Q \), written in terms of the \( B \)-meson momentum, and up to a trivial change in sign, the \( k_+ \) of eq. (17) is changed into \(-k_+ + \Lambda\), leading to eq. (2). Note that with our definition of \( k_+ \), \( m_b^* = M_B - k_+ \), which is in our opinion more physical: the differential distribution is now expressed in terms of hadronic quantities only, without any reference to unphysical quantities such as the quark mass or \( \Lambda \).

The moments \( \mathcal{M}_n \sim \Lambda_{QCD}^n \) have been defined in eq. (22) and give rise to terms proportional to \( \mathcal{M}_n(v \cdot Q/Q^2)^n \sim (\Lambda_{QCD} M_B/W^2)^n \). Thus, their contribution to the rate is suppressed as \( (\Lambda_{QCD}/M_B)^n \) when \( W \sim M_B \), whereas it becomes of \( \mathcal{O}(1) \) in the region where \( W \sim \sqrt{M_B \Lambda_{QCD}} \). The contributions proportional to \( B_n \) are subleading in \( 1/M_B \) with respect to the corresponding \( \mathcal{M}_n \) in all the physical region of interest, i.e. for all values of \( W \), including \( W \sim \sqrt{M_B \Lambda_{QCD}} \) (but not in the elastic region where \( W \sim \Lambda_{QCD} \)). Note that in order to derive eq. (20), we have never used the HQET, but only the OPE, by separating the large frequency modes (\( \sim W \)) from the low energy ones (\( \sim \Lambda_{QCD} \)) and expanding in powers of \( \Lambda_{QCD}/W \). For simplicity, without loss of generality, we consider in the following the case of a \( B \) meson at rest, namely \( v = (1,0) \).

We now consider, for \( t > 0 \), the Fourier transform of \( T_{\mu\nu} \) defined as

\[ T_{\mu\nu}(t, \vec{Q}) = \int \frac{dQ_0}{2\pi} e^{-iQ_0 t} T_{\mu\nu}. \] (23)

In terms of the \( T \)-product of the currents, we have

\[ T_{\mu\nu}(t, \vec{Q}) = -i \int d^3x \, e^{-i\vec{Q} \cdot \vec{x}} \langle \bar{B}(\vec{p}_{B} = 0)|J_{\nu}^{\mu}(\vec{x}, t)J_{\nu}^{\mu}(0)|\bar{B}(\vec{p}_{B} = 0) \rangle \]

\[ = -i e^{-iM_B t} \int d^3x \, e^{-i\vec{Q} \cdot \vec{x}} \langle \bar{B}(\vec{p}_{B} = 0)|J_{\nu}^{\mu}(\vec{x}, t)J_{\nu}^{\mu}(0)|\bar{B}(\vec{p}_{B} = 0) \rangle, \] (24)
where $J_\mu^v(x) = \bar{q}(x)\gamma_\mu^v b_v(x)$. The factor $e^{-iM_bt}$ appearing in eq. (24) cancels the corresponding term in the three-dimensional Fourier transform of the correlator, so that the r.h.s. of this equation goes as $e^{-iQ_0t}$, which is the expected behaviour for a hadronic system with energy $Q_0$.

For $t \geq 0$, by closing the contour of the integration over $Q_0$ below the real axis, it is straightforward to find

$$T^{\mu\nu}(t, \vec{Q}) = -\frac{i}{2} \int_0^{M_B} dk_+ \tilde{f}(k_+) \left( \bar{Q}^\mu \delta^\nu_0 + \bar{Q}^\nu \delta^\mu_0 - g^{\mu\nu} \bar{Q}_0 - i\epsilon^{0\mu\nu\alpha} \bar{Q}_\alpha \right) \times \frac{e^{-\left(k_+ + \sqrt{\vec{Q}^2}\right)t}}{2\sqrt{\vec{Q}^2}}, \tag{25}$$

where $\bar{Q} \equiv (Q_0^+, \vec{Q})$, with

$$Q_0^+ = k_+ + \sqrt{k_+^2 + \vec{Q}^2} \sim k_+ + \sqrt{\vec{Q}^2}. \tag{26}$$

Using eq. (23) it is very easy to make the analytic continuation to the Euclidean space-time

$$W^{\mu\nu}(t, \vec{Q}) = -\frac{1}{\pi} \text{Im} T_E^{\mu\nu}(t, \vec{Q}) \tag{27}$$

$$= \frac{1}{2\pi} \int_0^{M_B} dk_+ \tilde{f}(k_+) e^{-\left(k_+ + \sqrt{\vec{Q}^2}\right)t} \times \left( \bar{Q}^\mu \delta^\nu_0 + \bar{Q}^\nu \delta^\mu_0 - g^{\mu\nu} \bar{Q}_0 - i\epsilon^{0\mu\nu\alpha} \bar{Q}_\alpha \right).$$

By a suitable choice of the Lorentz components $\mu$ and $\nu$ of the currents and of the spatial momentum $\vec{Q}$, one can isolate either an integral of the form given in eq. (1) or the following one

$$G(t, \vec{Q}) = \frac{e^{-\sqrt{\vec{Q}^2}t}}{2\sqrt{\vec{Q}^2}} \int_0^{M_B} dk_+ \tilde{f}(k_+) e^{-k_+t} Q_0^+. \tag{28}$$

By studying the time dependence of $G(t, \vec{Q})$ at several values of $\vec{Q}$, we can unfold both the integral above and the one in eq. (1) and extract the shape function.

One may be surprised that the analytic continuation to the Euclidean space-time is so simple, in spite of the argument made at the beginning of the section. Indeed, also in the Minkowsky case, in order to apply the OPE, one has first to expand the $T$-product of the currents for large space-like momenta, which is equivalent to work in the Euclidean, and then continue the resulting expression to the kinematic region of interest. If this is possible, which implicitly corresponds to the assumption of quark-hadron duality, then the expansion has the very simple form of eqs. (21) and (25).
In these equations, the singularities are those of a free particle propagator, the Wick rotation of which is straightforward and leads to the result given in eq. (27). The contamination of intermediate states with small invariant masses, which would dominate at large time distances, is suppressed by injecting in the correlation functions a large spatial momentum $\vec{Q}$, with $\sqrt{\vec{Q}^2} \gg \Lambda_{QCD}$.

We now show that the same method can be used to compute the structure functions of deep inelastic scattering. The starting formula is, as before, the $T$-product of two currents. We have

$$T^{\mu\nu} = -i \int d^4x \ e^{-i\vec{q} \cdot \vec{x}} \langle N| T\left(J^\mu(x) J^\nu(0)\right) |N\rangle,$$  \hfill (29)

where $|N\rangle$ represents a generic hadronic state and $q$ is the momentum of the external vector boson with space-like momentum $q^2 \leq 0$. In this case, we have only $q$ as large momentum in the game. Thus we may write

$$T^{\mu\nu} = -i \int d^4x \ e^{-i\vec{q} \cdot \vec{x}} \langle N| \bar{\psi}(x)\Gamma^\mu S(x-0)\Gamma^\nu \psi(0)|N\rangle,$$  \hfill (30)

where $\Gamma^\mu = \gamma^\mu$ or $\Gamma^\mu = \gamma^\mu_L$ for electromagnetic or weak charged currents respectively and $S_q(x-0) = e^{-i\vec{q} \cdot \vec{x}}S(x-0)$. By expanding $S_q(x-0)$ in powers of $-q^2$, as done before for the inclusive decay rate, we may define a distribution function also in this case. Thus the same formulae apply to the case in which the large momentum is space-like, as in DIS, or time-like, as in inclusive $B$ decays.

**4 Implementation of the method in numerical simulations**

In this section, we briefly explain the method to extract $\tilde{f}(k_+)$ from the Euclidean lattice correlation functions and discuss the feasibility of our approach. Without loss of generality, we work with a $B$ meson at rest. The extension of our method to a $B$ meson with an arbitrary velocity $v$ is straightforward.

$W^{\mu\nu}(t, \vec{Q})$ can be readily obtained from the ratio

$$W^{\mu\nu}(t, \vec{Q}) = \lim_{t_f, t_i \to \infty} \frac{W^{\mu\nu}_{t_f, t_i}(t, \vec{Q})}{S_{t_f, t_i}} e^{-M_B t},$$  \hfill (31)

where

$$W^{\mu\nu}_{t_f, t_i}(t, \vec{Q}) = \frac{1}{\pi} \int d^3x \ e^{-i\vec{Q} \cdot \vec{x}} \langle 0| \Phi_{p_B = 0}^{\dagger}(t_f) J^\mu(x, t) J^\nu(0) |\Phi_{p_B = 0}(t_i) 0\rangle,$$  \hfill (32)
\[ S_{t_f, t_i} = \langle 0 | \Phi_B^\dagger(t_f) \Phi_B(t_i) | 0 \rangle. \]  

(33)

\( \Phi_B(t) \) is the \( B \) interpolating field with definite spatial momentum \( \vec{p}_B \)

\[ \Phi_B(t) = \int d^3 x e^{i \vec{p}_B \cdot \vec{x}} \Phi_B(\vec{x}, t). \]  

(34)

It is very easy to demonstrate eq. (31). In the Euclidean, using the transfer matrix formalism, we have

\[ \Phi_B(\vec{x}, t) = e^{\hat{H}t} \Phi_B(\vec{x}) e^{-\hat{H}t}, \]  

(35)

so that the correlation functions have an exponential dependence on the energy of the external states. This implies that, in the limit \( t_i, t_f \to \infty \), the lightest boson state, corresponding to a \( \bar{B} \) meson, dominates the correlation functions (32) and (33), since all higher-energy states are exponentially suppressed. In this limit, with \( t > 0 \), we then obtain

\[ W_{t_f, t_i}^{\mu\nu}(t, \vec{Q}) \to \frac{1}{\pi} \langle 0 | \Phi_B^\dagger(0) | \bar{B}(\vec{p}_B = 0) \rangle \times \left[ \int d^3 x e^{-i \vec{Q} \cdot \vec{x}} \langle B | J_\mu^\dagger(\vec{x}, t) J_\nu(0) | \bar{B} \rangle \right] \times \langle \bar{B}(\vec{p}_B = 0) | \Phi_B(0) | 0 \rangle e^{-M_B(t_i + t_f)}. \]  

(36)

and

\[ S_{t_f, t_i} \to \langle 0 | \Phi_B^\dagger(0) | \bar{B}(\vec{p}_B = 0) \rangle \langle \bar{B}(\vec{p}_B = 0) | \Phi_B(0) | 0 \rangle e^{-M_B(t_i + t_f)}. \]  

(37)

Thus we have shown that for \( t_i, t_f \to \infty \) the ratio in eq. (31) directly gives the required quantity and that the knowledge of the coupling of the interpolating field \( \Phi_B \) to the physical meson state is not necessary.

We now discuss the feasibility of the method in actual numerical simulations. If one uses the complete light quark propagator, a systematic effect in the extraction of \( \tilde{f}(k_+) \) in the relevant region, \( k_+ \sim \bar{\Lambda} \), is induced by contributions of states with a small invariant mass, which dominate the Euclidean correlation functions when \( t \to \infty \), see eq. (17). By injecting a large momentum \( \vec{Q} \) in the correlation functions, the difference between the energy \( E_X \) of these states and the energy \( Q_0^+ + Q_0^\perp - \sqrt{Q^2} \sim \bar{\Lambda} \). Moreover the contribution of these states is suppressed at least as \( \bar{\Lambda}^2/M_B^2 \). In order to compute \( \tilde{f}(k_+) \), the condition

\[ \mathcal{F} = \frac{\bar{\Lambda}^2}{M_B^2} e^{\bar{\Lambda} t} \ll 1 \]  

(38)

must then be satisfied. Thus eq. (38) gives a condition on the maximum time distance which can be used to extract \( \tilde{f}(k_+) \). We can formulate the condition on the maximum allowed time, in units of the lattice spacing, as follows

\[ N_t = \frac{t}{a} \ll \frac{1}{\bar{\Lambda} a} \ln \left( \frac{M_B^2}{\bar{\Lambda}^2} \right). \]  

(39)
By working with an arbitrarily small lattice spacing $a$, the condition (39) can always be satisfied for large values of $N_t$ and $\tilde{f}(k_+)$ can then be computed with negligible uncertainty. In practice, the value of $a$ is limited by computer resources and we have to worry whether we have a sufficient number of points in time to unfold the shape function. To give an example, with a lattice spacing $a^{-1} = 6$ GeV, a value which will probably be reached with the 100-gigaflips/teraflops machines already available or in construction, a heavy meson mass $M \sim 2$ GeV and $\bar{\Lambda} = 0.3$ GeV, one finds $N_t \sim 75$, which is certainly large enough. Note that $f(k_+)$ cannot be studied for $k_+ \gg \bar{\Lambda}$ because, in practice, the correlation function is dominated by subleading terms in $1/M_B$ at all accessible time distances. This region, however, can be studied perturbatively.

In order to avoid the limitations on $N_t$, a possibility is, of course, to perform the numerical simulations using the approximate propagator of eq. (19) (we omit the $Q$ in the numerator because this gives rise to trivial kinematic factors)

$$\tilde{S}_Q = \frac{1}{Q^2 + 2iQ \cdot D + i\epsilon} = \left( \frac{1}{v \cdot Q} \right) \frac{1}{Q^2/v \cdot Q + 2iD_+ + i\epsilon}. \quad (40)$$

It is straightforward to show that $\tilde{S}_Q(x)$ can be written as

$$\tilde{S}_Q(x) \equiv \frac{e^{i(v \cdot Q)x_+/2}}{v \cdot Q} S_{LEET}(x), \quad (41)$$

where $x_+ = n \cdot x$ and $S_{LEET}(x)$ is the light-cone propagator of the Large Energy Effective Theory (LEET) [23], which satisfies the equation

$$2iD_+ S_{LEET}(x) = \delta^4(x). \quad (42)$$

Thus, the extraction of $\tilde{f}(k_+)$ from $\tilde{S}_Q(x)$ is equivalent to the use of the LEET. In our case, we are allowed to use this approximation since it has recently been shown that the LEET is applicable to inclusive processes [24], in spite of the difficulties that it may have for exclusive decays [25]. Note that the calculation of the physical shape function using the LEET propagator, which is more singular than the propagator of the full theory, requires a further logarithmic renormalisation of $W^{\mu\nu}(t, \bar{Q})$ [28], which can be computed in lattice perturbation theory. The ultraviolet divergences of the LEET correspond perturbatively to infrared divergences in the full theory [29]. In the latter case the infrared divergences are automatically regularised by the non-perturbative contributions in the physical matrix elements of the $T$-product of the two currents and no renormalisation is required.

It is not clear to us whether the use of $S_{LEET}(x)$ will be convenient in practice, since this propagator is much more singular than the full one. For this reason we expect
that the correlation functions computed in numerical simulations using $S_{LEET}(x)$ will be affected by larger statistical fluctuations.

We stress that in all the formulae derived in this paper, we never used the fields of the HQET and, indeed, the same formalism also applies to the calculation of the structure functions in DIS. The reason is that the shape function is defined from the OPE in powers of $1/W$ (at lowest order in $1/m_b$) and one has only to worry about those terms which become of $O(1)$ at the end-point of the lepton spectrum. Thus, the use of the fields of the HQET would only affect terms of higher order in $1/m_b$ on which we do not have control anyway. In the calculation of the relevant correlation functions, one may use the heavy-quark propagator of the lattice HQET for the b quark. We recall, however, that calculations done using the lattice HQET are afflicted by considerable difficulties, which are absent in the full theory [26, 27]. Thus, we believe that the best strategy is to compute the shape function in the full theory, both for the light and the heavy quarks, at several values of the heavy quark mass, and extrapolate the results to the $B$ case. This strategy has been already successful in the calculation of the heavy meson decay constants, of the form factors for exclusive semileptonic decays, and of the $B^0-\bar{B}^0$ mixing $B$-parameters [26, 27].

5 Conclusions

In this paper, we have shown that it is possible to compute, by numerical simulations of lattice QCD, the shape function which describes the lepton spectrum in semileptonic $B$ decays and the photon spectrum in radiative $B$ decays. The same approach can be used for the calculation of the structure functions in DIS. The method is based on first principles and does not require any model assumption. Moreover it avoids the calculation of the matrix elements of higher dimensional operators, which are plagued by power divergences and the renormalisation of which is very difficult to achieve [17]. Indeed, no renormalisation is needed and the shape function can be extracted directly from suitable ratios of lattice correlation functions. We have also proposed a redefinition of the shape function which avoids any reference to $\Lambda$ or to the b-quark mass, and allows us to write the differential rate in terms of hadronic quantities only.

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