A CAP COVERING THEOREM

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A cap of spherical radius $\alpha$ on a unit $d$-sphere $S$ is the set of points within spherical distance $\alpha$ from a given point on the sphere. Let $\mathcal{F}$ be a finite set of caps lying on $S$. We prove that if no hyperplane through the center of $S$ divides $\mathcal{F}$ into two non-empty subsets without intersecting any cap in $\mathcal{F}$, then there is a cap of radius equal to the sum of radii of all caps in $\mathcal{F}$ covering all caps of $\mathcal{F}$ provided that the sum of radii is less than $\pi/2$.

This is the spherical analog of the so-called Circle Covering Theorem by Goodman and Goodman and the strengthening of Fejes Tóth’s zone conjecture proved by Jiang and the author.

1. Introduction

A finite collection $\mathcal{K}$ of convex bodies in $\mathbb{R}^d$ is called non-separable if any hyperplane intersecting $\text{conv} \bigcup \mathcal{K}$ meets a convex body of $\mathcal{K}$. The following theorem was conjectured by Erdős and proved by Goodman and Goodman [9].

Theorem 1 (A. W. Goodman, R. E. Goodman, 1945). If a finite collection of disks of radii $r_1, \ldots, r_n$ is non-separable, then it is possible to cover them by a disk of radius $r_1 + \cdots + r_n$.

Using the argument of Goodman and Goodman, it is easy to prove the high-dimensional analog of Theorem 1. Moreover, Bezdek and Lángi [6] noticed that the same argument works for a non-separable collection of homo-

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thets of a centrally symmetric convex body. Also, they investigated a similar question on covering of a non-separable collection of positive homothets of a convex body that is not necessary centrally symmetric. Later Akopyan, Balitskiy, and Grigorev [1] improved their result. The aim of the current note is to prove the spherical analog of Theorem 1; see [10, Conjecture 3].

In order to state our result, we need several definitions. Denote by $S$ the unit $d$-sphere embedded in $\mathbb{R}^{d+1}$ centered at the origin. A cap of spherical radius $\alpha$ on $S$ is defined as the set of points within spherical distance $\alpha$ from a given point on the sphere. A great sphere is the intersection of $S$ and a hyperplane passing through the origin. We call a great sphere avoiding for a collection of caps if it does not intersect any cap in this collection. A finite collection of caps is called non-separable if no avoiding great sphere divides the collection into two non-empty sets.

**Theorem 2.** Let $\mathcal{F}$ be a non-separable collection of caps of spherical radii $\alpha_1, \ldots, \alpha_n$. If $\alpha_1 + \cdots + \alpha_n < \pi/2$, then $\mathcal{F}$ can be covered by one cap of radius $\alpha_1 + \cdots + \alpha_n$.

Recall that a pair of antipodal caps can be viewed as the dual of a zone, where a zone of width $2\alpha$ on $S$ is the set of points within spherical distance $\alpha$ from a given great sphere. (The projective duality in $\mathbb{R}^{d+1}$ interchanges a line through the origin with its orthogonal hyperplane through the origin, that is, a pair of antipodal points is the dual of a great sphere.) It is worth mentioning that Theorem 2 for $\alpha_1 + \cdots + \alpha_n = \pi/2$ under some technical assumptions is a corollary of so-called Fejes Tóth’s zone conjecture [14] that is proved in [10, Theorem 1 and Corollary 3]; see also the recent work [12] of Ortega-Moreno, where the conjecture is confirmed for zones of the same width.

**Theorem 3 (Jiang, Polyanskii, 2017).** Given a collection of zones covering $S$, the sum of width of all zones in the collection is at least $\pi$.

The proof of Theorem 2 heavily relies on ideas developed in the context of studying planks [4,5,13] covering a convex body, where a plank (or slab, or strip) of width $w$ is a set of all points lying between two parallel hyperplanes in $\mathbb{R}^d$ at distance $w$. The connection between planks and zones is obvious: A zone of width $2\alpha$ is the intersection of $S$ and the centrally symmetric about the origin plank of width $2\sin\alpha$. Another key idea of our proof is considering the farthest point Voronoi diagram of the so-called Bang set (see (1)). This idea appeared in the very recent work [3] of Balitskiy, where trying to understand the proof of the theorem of Kadets [11] on the sum of inradii of convex bodies in a collection covering a unit ball, he introduced a
new concept of *multiplanks*. Since we do not use this involved concept in its greatest generality, we decided to give only a short remark in the discussion section about relation of our proof to this concept. Nevertheless, we highly recommend an interested reader to understand it: We believe that it will be very helpful in proving new results on coverings.

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### 2. Covering zones instead of caps

From now on, we consider only open centrally symmetric about the origin planks, and thus we omit the phase ‘open centrally symmetric about the origin’ most of the time. For a plank $P$, denote by $w(P)$ a vector $w$ such that $P = \{ x \in \mathbb{R}^d : |\langle x, w \rangle| < \langle w, w \rangle \}$. For a zone $Z$, set $w(Z) = w(P)$, where $P$ is the open plank such that $Z$ coincides with the closure of $S \cap P$.

The main tool of the current paper is the following dual reformulation of [10, Lemma 4].

**Lemma 4.** Let $Z_1, \ldots, Z_n \subset S$ be zones of width $2\alpha_1, \ldots, 2\alpha_n$, respectively, such that $\alpha = \alpha_1 + \cdots + \alpha_n \leq \pi/2$. Set $w_i := w(Z_i)$. If $w = \sum_{i=1}^n w_i$ satisfies the following inequalities

$$|w| \geq \sin \alpha \text{ and } |w - w_i| \leq \sin(\alpha - \alpha_i) \text{ for all } i \in [n],$$

then there is a zone of width $2\alpha$ covering $Z_1, \ldots, Z_n$.

We first show that the following theorem (Theorem 5) implies Theorem 2. We remark that Theorem 2 also easily implies Theorem 5.

**Theorem 5.** Let $Z_1, \ldots, Z_n \subset S$ be zones of width $2\alpha_1, \ldots, 2\alpha_n$, respectively, such that $\alpha_1 + \cdots + \alpha_n < \pi/2$. If $S \setminus \bigcup_{i=1}^n Z_i$ has at most one pair of two antipodal connected components, then the zones $Z_1, \ldots, Z_n$ can be covered by a zone of width $2\alpha_1 + \cdots + 2\alpha_n$.

**Theorem 5 implies Theorem 2.** Suppose that spherical caps $D_1, \ldots, D_n$ satisfy the conditions of Theorem 2. Let $D'_i$ be an open cap concentric with $D_i$ of spherical radius $\pi/2 - \alpha_i$. By projective duality, the center of an open hemisphere lies in $D'_i$ if and only if this hemisphere covers $D_i$. Since no avoiding great sphere divides $\{D_1, \ldots, D_n\}$ into two non-empty sets, by projective
duality, we get
\[
\bigcap_{i=1}^{n} \varepsilon_i D'_i = \emptyset \text{ unless all } \varepsilon_i \in \{\pm 1\} \text{ are the same.}
\]

Hence, the zones \( S \setminus (D'_1 \cup (-D'_1)) \), \ldots, \( S \setminus (D'_n \cup (-D'_n)) \) satisfy the conditions of Theorem 5. Therefore, they can be covered by a zone \( Z \) of width \( 2\alpha_1 + \cdots + 2\alpha_n \). Since \( S \setminus Z \) is the union of two antipodal open caps \( D' \) and \( -D' \) of radii \( \pi/2 - (\alpha_1 + \cdots + \alpha_n) \), without loss of generality we obtain
\[
D' \subseteq \left( \bigcap_{i=1}^{n} D'_i \right).
\]

By projective duality, the closed cap concentric with \( D' \) of radius \( \alpha_1 + \cdots + \alpha_n \) covers caps \( D_1, \ldots, D_n \).

Having shown that Theorem 5 implies Theorem 2, we now proceed with the proof of Theorem 5.

**Proof of Theorem 5.** Denote by \( P_i \) the open plank such that the intersection of its closure with \( S \) is \( Z_i \). Set \( w_i = w(P_i) \) for all \( i \in [n] \). Without loss of generality let us assume that \( w = w_1 + \cdots + w_n \) has the maximum norm among vectors of the Bang set

\[
L = \left\{ \sum_{i=1}^{n} \pm w_i \right\}.
\]

First, let us show that we can assume \( |w| \leq \sin(\alpha_1 + \cdots + \alpha_n) \). Indeed, suppose that \( |w| > \sin(\alpha_1 + \cdots + \alpha_n) \). Thus the family of all subsets \( J \subset [n] \) such that
\[
\left| \sum_{i \in J} w_i \right| > \sin \left( \sum_{i \in J} \alpha_i \right)
\]
is non-empty. Choose among them a minimal subset \( I \). Since \( |w_i| = \sin \alpha_i \), we have \( |I| > 1 \), and so we can apply Lemma 4 to \( I \) and cover zones \( Z_i, i \in I \), by the zone \( Z \). Replacing the zones \( Z_i, i \in I \), by \( Z \), we obtain a new collection of zones with the same sum of width covering the original collection of zones. Put \( S \setminus Z_i = D_i \cup (-D_i) \) and \( S \setminus Z = D \cup (-D) \), where \( D_i \) and \( D \) are open caps
such that $D ⊂ \bigcap_{i ∈ I} D_i$. By the conditions of the theorem, we can assume that

$$\bigcap_{i=1}^{n} \varepsilon_i D_i = \emptyset$$

unless all $\varepsilon_i$ are the same.

Since

$$\bigcap_{i ∈ [n] \setminus I} \varepsilon_i D_i \cap \varepsilon D \subseteq \left( \bigcap_{i ∈ [n] \setminus I} \varepsilon_i D_i \right) \cap \left( \bigcap_{i ∈ I} \varepsilon D_i \right),$$

we get that the new collection of zones satisfies the conditions of the theorem. Therefore, we can reduce the number of zones and assume that $|w| ≤ \sin(\alpha_1 + \cdots + \alpha_n)$.

Next, we use the following version of so-called Bang's Lemma. It seems that it was first published in a similar form by Fenchel [8].

**Lemma 6 (Bang’s Lemma).** If $t$ has maximum norm among elements of $t + x - L$ for $x = \sum_{i=1}^{n} \varepsilon_i w_i ∈ L$, then

$$t ∈ M_x := \bigcap_{i=1}^{n} \left\{ y ∈ \mathbb{R}^d : \langle y, -\varepsilon_i w_i \rangle ≥ \langle w_i, w_i \rangle \right\} \subseteq \mathbb{R}^d \setminus \bigcup_{i=1}^{n} P_i.$$

**Proof.** Suppose that $t ∈ \{ y ∈ \mathbb{R}^d : \langle y, -\varepsilon_i w_i \rangle < \langle w_i, w_i \rangle \}$ for some $i ∈ [n]$. Then the vector $t + 2\varepsilon_i w_i ∈ t + x - L$ is longer than $t$ (see Figure 1), a contradiction.

![Figure 1. Proof of Lemma 6](image-url)
Let $B$ be the unit open ball with center at the origin. For $x \in L$, consider the set

$$A_x = \{ t \in B : \text{t has maximum norm among of elements of } t + x - L \}.$$ 

Denote by $\text{pr}: \mathbb{R}^d \setminus \{0\} \to S$ the central projection onto $S$, where $0$ is the origin. By Lemma 6, we have $\text{pr}(A_x) \subseteq \text{pr}(M_x \cap B) \subseteq \text{pr}(B \cup \bigcup_{i=1}^n P_i)$. Since $\text{pr}(M_x \cap B)$ is an open set, we obtain $\text{pr}(A_x)$ is a subset of $S \setminus \bigcup_{i=1}^n Z_i$. (Recall that $Z_i$ is the closure of $S \cap P_i$.)

We claim that

\begin{equation}
A_w = \{ t \in B : \langle t, -w \rangle \geq \langle w, w \rangle \} = \{ t \in B : \|t\| \geq \|t + 2w\| \}.
\end{equation}

Before proving (2), let us show how to finish the proof of the theorem. Since $|w| \leq \sin(\alpha_1 + \cdots + \alpha_n) < 1$, the set $\text{pr}(A_w)$ is an open cap $X$ of radius at least $\pi/2 - (\alpha_1 + \cdots + \alpha_n)$ lying in $S \setminus \bigcup_{i=1}^n Z_i$; see Figure 2. Therefore, the zone $Z := S \setminus (X \cup (-X))$ of width at most $2\alpha_1 + \cdots + 2\alpha_n$ covers $Z_1, \ldots, Z_n$. So, to finish the proof, it is enough to show (2).

![Figure 2. Proof of Theorem 5](image)

First, we prove that

\begin{equation}
A_x = \emptyset \text{ for all } x \in L \setminus \{\pm w\}.
\end{equation}

Indeed, using the fact that any two sets $M_x$ for $x \in L$ are strictly separated by one of the planks $P_i$, any two sets $\text{pr}(A_x)$ for $x \in L$ are strictly separated subsets of $S \setminus \bigcup_{i=1}^n Z_i$ consisting of at most two connected antipodal regions. Therefore, among sets $A_x$ for $x \in L$ there are at most two non-empty sets. Since $w \in A_{-w}$ and $-w \in A_w$, we conclude (3).
Next, we show that

\[(4) \quad \text{if } y - w, y + w \in B, \text{ then } y - x \in B \text{ for all } x \in L.\]

Suppose the contrary: For some \(x \in L \setminus \{\pm w\}\), the point \(y - x\) does not lie in \(B\). Choosing a proper \(0 < \lambda < 1\) and using convexity of \(B\), we get that \(\lambda y - x \in B\) for all \(x \in L\) and \(\lambda y - x' \in A_{x'}\) for some \(x' \in L \setminus \{\pm w\}\), a contradiction with (3).

Using (3) and (4), we conclude that if the points \(y - w\) and \(y + w\) lie in \(B\), then \(y - w \in A_w\) or \(y + w \in A_{-w}\). Therefore, we obtain (2).

\[\]

3. Discussion

First, we discuss the connection of our proof with the concept of multiplank.

**Remark.** Using the terminology [3, Definition 2.1] of Balitskiy, it is easy to see that the set

\[P_L := \mathbb{R}^d \setminus \bigcup_{x \in L} \{t \in \mathbb{R}^d : t \text{ has maximum norm among of elements of } t + x - L\}\]

is the open *multiplank* of the set \(L\) covering \(\bigcup_{i=1}^n P_i\); see [3, Proposition 4.1]. In some sense our proof of Theorem 2 relies on the fact \(P_L \cap B = P \cap B\), where \(P\) is the plank with \(w(P) = w\); see the stratification of a multiplank in the general case in [3, Theorem 4.5].

We recall the following problem resembling Theorem 2. It was proved for \(\alpha \geq \pi/2\) in [7, Theorem 6.1] but still open for \(\alpha < \pi/2\); see also [2, Problem 6.2].

**Conjecture 7.** If a cap of spherical radius \(\alpha\) is covered with a collection of convex spherical domains, then the sum inradii of all domains in the collection is at least \(\alpha\).

We finish with the following conjecture generalizing Theorems 3 and 5 posed by Maxim Didid.

**Conjecture 8.** Let \(Z_1, \ldots, Z_n \subset S\) be zones of width \(\beta_1, \ldots, \beta_n\), respectively. If \(S \setminus \bigcup_{i=1}^n Z_i\) consists of convex connected components \(L_1, \ldots, L_{2m}\) with inradius \(\gamma_1, \ldots, \gamma_{2m}\), respectively, then \(\beta_1 + \cdots + \beta_n + \gamma_1 + \cdots + \gamma_{2m} \geq \pi\).
References

[1] A. Akopyan, A. Balitskiy and M. Grigorev: On the Circle Covering Theorem by A. W. Goodman and R. E. Goodman, *Discrete & Computational Geometry* **59** (2018), 1001–1009.

[2] A. Akopyan and R. Karasev: Kadets-Type Theorems for Partitions of a Convex Body, *Discrete & Computational Geometry* **48** (2012), 766–776.

[3] A. Balitskiy: A multi-plank generalization of the Bang and Kadets inequalities. *Israel Journal of Mathematics*, (2021). https://doi.org/10.1007/s11856-021-2210-5

[4] K. Ball: The plank problem for symmetric bodies, *Inventiones mathematicae* **104** (1991), 535–543.

[5] T. Bang: A Solution of the “Plank Problem”, *Proceedings of the American Mathematical Society* **2** (1951), 990–994.

[6] K. Bezdek and Zs. Lángi: On Non-separable Families of Positive Homothetic Convex Bodies, *Discrete & Computational Geometry* **56** (2016), 802–813.

[7] K. Bezdek and R. Schneider: Covering large balls with convex sets in spherical space, *Beitrage zur Algebra und Geometrie* **51** (2010), 229–235.

[8] W. Fenchel: On Th. Bang’s solution of the plank problem, *Matematisk Tidsskrift. B* (1951), 49–51.

[9] A. W. Goodman and R. E. Goodman: A Circle Covering Theorem, *The American Mathematical Monthly* **52** (1945), 494–498.

[10] Z. Jiang and A. Polyanskii: Proof of László Fejes Tóth’s zone conjecture, *Geometric and Functional Analysis* **27** (2017), 1367–1377.

[11] V. Kadets: Coverings by convex bodies and inscribed balls, *Proceedings of the American Mathematical Society* **133** (2005), 1491–1495.

[12] O. Ortega-Moreno: An optimal plank theorem, *Proceedings of the American Mathematical Society* **149** (2021), 1225–1237.

[13] A. Tarski: Uwagi o stopniu równoważności wielokątów [Remarks on the degree of equivalence of polygons], *Parametr* **2** (1932), 310–314.

[14] L. Fejes Tóth: Exploring a Planet, *The American Mathematical Monthly* **80** (1973), 1043–1044.

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