A Nyström method for the two dimensional Helmholtz hypersingular equation

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Abstract In this paper we propose and analyze a class of simple Nyström discretizations of the hypersingular integral equation for the Helmholtz problem on domains of the plane with smooth parametrizable boundary. The method depends on a parameter (related to the staggering of two underlying grids) and we show that two choices of this parameter produce convergent methods of order two, while all other stable methods provide methods of order one. Convergence is shown for the density (in uniform norm) and for the potential postprocessing of the solution. Some numerical experiments are given to illustrate the performance of the method.

Keywords Helmholtz equation · Boundary integral equations · Nyström methods

Mathematics Subject Classifications (2010) 65R20 · 65N38
1 Introduction

The method in a few paragraphs In this paper we propose and analyze a discretization of the hypersingular integral equation for the Helmholtz equation on a smooth parametrizable simple curve $\Gamma \subset \mathbb{R}^2$:

$$- \partial_v \int_{\Gamma} \partial_v(y) H_0^{(1)}(k|\cdot - y|) \phi(y) d\Gamma(y) = g \quad \text{on } \Gamma.$$  (1)

Here $H_0^{(1)}$ is the Hankel function of the first kind and order zero, $k$ is the wave number, $\partial_v$ is the normal derivative, and $g$ is data on $\Gamma$. The method is based on some simple ideas:

(a) The operator is first written as a bilinear integrodifferential form acting on periodic functions

$$(\psi, \phi) \mapsto \int_0^1 \int_0^1 H_0^{(1)}(k|x(s) - x(t)|) \psi'(s) \phi'(t) ds dt - k^2 \int_0^1 \int_0^1 H_0^{(1)}(k|x(s) - x(t)|) n(s) \cdot n(t) \psi(s) \phi(t) ds dt,$$

$x = (x_1, x_2)$ being a regular parametrization of $\Gamma$, and an outward pointing normal vector $n = (x_2', -x_1')$.

(b) The principal part of the bilinear form (the one with the derivatives) is formally approximated with a nonconforming Petrov-Galerkin scheme, using piecewise constant functions on two different uniform grids with the same mesh-size $h$: \{(i - \frac{1}{2})h\} and \{(j - \frac{1}{2} + \varepsilon)h\}. Since the derivatives of piecewise constant functions are linear combinations of Dirac delta distributions, that part of the bilinear form is just discretized with the matrix

$$W_{i+1,j+1} - W_{i+1,j} + W_{i,j} - W_{i,j+1},$$

where $W_{i,j} = H_0^{(1)}(k|b_i^e - b_j^e|)$, $b_i^e := x((i - \frac{1}{2} + \varepsilon)h)$, and $b_j := x((j - \frac{1}{2})h)$.

(c) The second part of the bilinear form (which has a weakly singular logarithmic singularity) is discretized with the same Petrov-Galerkin scheme, using midpoint quadrature to approximate the resulting integrals:

$$k^2 H_0^{(1)}(k|m_i^e - m_j|) n_i^e \cdot n_j \approx k^2 \int_{(i - \frac{1}{2} + \varepsilon)h}^{(i + \frac{1}{2})h} \int_{(j - \frac{1}{2})h}^{(j + \frac{1}{2})h} H_0^{(1)}(k|x(s) - x(t)|) n(s) \cdot n(t) ds dt,$$

where $m_i^e := x((i + \varepsilon)h)$, $n_i^e := h n((i + \varepsilon)h)$ and $m_j$ and $n_j$ are similarly defined.

(d) The right-hand side is tested with piecewise constant functions and then midpoint quadrature is applied to all the resulting integrals.
As can be seen from the above formulas, this method leads to a very simple
discretization of Eq. 1, requiring no assembly process, no additional numerical inte-
gration and no complicated data structures to handle the geometric data. While the
emphasis will be given to the discretization of the hypersingular operator, which is
the novel part of this work, this can be easily extended to more complicated bound-
ary integral equations where the hypersingular operator is the principal part. This
will be shown in the section of numerical experiments, and is part of what appears
in [6].

**Nyström methods in the literature**  Properly speaking, Nyström methods are applied
to integral equations of the second kind with kernels that can be evaluated every-
where. The philosophy of Nyström methods thus includes a reconstruction formula,
which fills the values of the unknown –initially only computed on the quadra-
ture points– to the entire domain of integration. In the realm of applications to the
Helmholtz equation, there is a well known method of order three which applies
to the double layer representation of the solution to the Dirichlet problem on a
smooth domain: see [17, Section 12.2] for the general ideas and an example applied
to the Laplace equation, and [27, Section 12.5] for full details. Nyström methods
can also be built for weakly singular integral equations of the second kind: in this
case, either the quadrature nodes and the collocation nodes (where the equation is
imposed) do not coincide, or the kernel is decomposed and rearranged, using inter-
polation operators applied to the unknown before some parts of the operator acts on
it. Fully discrete methods, based on a trigonometric Galerkin discretization (which
is diagonal on the principal part of the operator) and a quadrature method applied
to another part of the integral operator, are schemes that belong to the same extended
family of Nyström methods. While it is not easy to give a taxonomy of the many
existing methods, let us mention some milestones. Kress [16] created a method
for the hypersingular equation based on a decomposition of the operator, the use
of a trigonometric Galerkin scheme to deal with the principal part, and a method
used in [17] to deal with the remaining weakly singular operator. The methods
in [18] fall also in this category of fully discretized schemes based on trigono-
metric interpolation. The QBX (Quadrature by Expansion) method of Klöckner,
Barnett, Greengard, and O’Neil [14] can also be used to define a Nyström like
method.

The design of specialized quadrature rules for singular and hypersingular integrals
has been the object of interest in the community [15]. These ones can be inbuilt to
create fully discrete methods based on quadrature, for a wide class of integral opera-
tors. More sophisticated schemes have been introduced on domains with corners
[11], although analysis of this kind of schemes for non-smooth domains seems to be
an open problem.

A completely different approach can be used by taking care of different formula-
tions of the problem. Combined field representations of the solution of the Neumann
problem lead to problems where the hypersingular operator occurs, and this same
operator is present in frequency-robust representations of transmission problems.
However, regularized combined field integral equations can be used: they typically
lead to better conditioned formulations, some of which include compositions of
operators. (Note that in some cases, the integral equation is greatly simplified, but the potential representation is then complicated by the need of discretizing a weakly singular operator before making the double layer potential act.) This is the philosophy of the methods in [2] and [3]. Other regularized methods use inverses of tangential differential operators [4], and it is less clear that they can be built into Nyström type methods.

Nyström methods for equations of the first kind The use of a two-grid Nyström method for periodic logarithmic integral equations goes back to the work of Jukka Saranen, Ian Sloan and their collaborators [25, 26, 29]. It was then discovered that the values $\varepsilon = \pm 1/6$ provide superconvergent methods (of order two) and that the values $\varepsilon = \pm 1/2$ lead to unstable discretizations. The idea was further exploited in [5], showing that the methods can be used on the weakly singular equations that appear in the Helmholtz equation. The present paper shows how to transfer the same kind of ideas (and, up to a point, the same type of analysis) to the hypersingular integral equation for the Helmholtz equation. The case of the Laplace hypersingular equation is included in the present analysis.

We will show that this discretization of the hypersingular integral operator is stable in the $L^2$ norm for the underlying space of piecewise constant functions as long as $\varepsilon \neq \pm 1/2$ (the value $\varepsilon = 0$ is excluded as a possibility from the very beginning, since it leads to evaluation of the kernel functions on the singularity). We will also show that $\varepsilon = \pm 1/6$ define methods of order two and that this order is actually attained in a strong $L^\infty$ norm. The error analysis will be based on Fourier techniques [1, 25, 27] combined with an already quite extensive library of asymptotic expansions developed by two of the authors of this paper with some other collaborators [5, 8–10].

In [6] we show how to combine this discretization method for the hypersingular equation with the original method [5, 25, 26, 29] for the single layer operator and with straightforward Nyström discretization of the double layer operator and its adjoint. This results in a compatible and straightforward-to-code fully discrete Calderón calculus for the two dimensional Helmholtz equation on a finite number of disjoint smooth closed curves. This discretization set has a strong flavor of low order Finite Differences. This might make it an attractive option to build simple code for scattering problems, when the simultaneous use of several boundary integral operators is required. It is worth mentioning, that while spectral-like methods have considerably better convergence properties on $C^\infty$ boundaries, this great advantage is lost on curves of reduced regularity (see experiments in [23]), as is the case when boundaries are spline curves, often used for reconstruction in iterative methods for inverse problems. In this case, the coding simplicity of the method of this paper can make it advantageous. We do not claim, nevertheless, that this method is better than more sophisticated high order methods, while we make the point that high order spectral methods have not debunked easy-to-code central Finite Difference methods for boundary value problems. More recent work [7] has shown –currently only at the experimental level– how to take advantage of the estimates of this paper to build a class of third order discretization methods for all the integral operators for the Helmholtz equation. Consistency error
estimates for those methods are directly based on the estimates of this paper, and stability is likely to be provable using carefully tuned inf-sup estimates in the spirit of [8].

Structure of the paper In Section 2 we present the simplified model problem we will be using for the sake of presentation and analysis, as well as some more complicated and useful integral representations for the Helmholtz problems where the hypersingular integral operator appears. In Section 3 we present the method for a class of periodic hypersingular equations that include Eq. 1 after parametrization. The method is then reinterpreted as a non-conforming Petrov-Galerkin discretization with numerical quadrature. In Section 4, we introduce the functional frame for the analysis of the method, based on the theory of periodic pseudodifferential operators on periodic Sobolev spaces. In Section 5 we present the stability result of this paper in the form of an infimum-supremum condition. In Sections 6 and 7, we respectively give the consistency and convergence error estimates for the method. Section 8 contains some numerical experiments, while we have gathered in Appendix A the more technical proof of Proposition 12.

2 Hypersingular operator methods for the Helmholtz equation

We are going to present a collection of integral formulations for the exterior Neumann problem for the radiating Helmholtz equation exterior to a simple smooth curve in the plane. The extension to a finite set of non-intersecting smooth curves is straightforward. The numerical discretization will be presented for the simplest (while less interesting) model equation, given the fact that we want to emphasize the discretization of the hypersingular operator.

Let \( x : \mathbb{R} \rightarrow \mathbb{R}^2 \) be a smooth 1-periodic function such that \( |x'(s)| \neq 0 \) for all \( s \), and \( x(s) \neq x(t) \) if \( |s - t| < 1 \). The range of \( x \) is then a smooth closed curve \( \Gamma \) in the plane. Let \( n(t) \) be the outward pointing normal vector at \( x(t) \) with \( |n(t)| = |x'(t)| \). Given a periodic function \( \varphi \), we define

\[
\mathbb{R}^2 \setminus \Gamma \ni z \mapsto (DL\varphi)(z) := \frac{i}{4} \int_0^1 \nabla_y H_0^{(1)}(k|z - y|) \left| y = x(t) \right| \cdot n(t) \varphi(t) dt
\]

\[
= \frac{ik}{4} \int_0^1 H_1^{(1)}(k|z - x(t)|) \frac{(z - x(t)) \cdot n(t)}{|z - x(t)|} \varphi(t) dt,
\]

where \( H_0^{(1)} \) and \( H_1^{(1)} \) are the Hankel functions of the first kind and orders 0 and 1 respectively. The function \( U := DL\varphi \) is an outgoing solution of the Helmholtz equation

\[
\Delta U + k^2 U = 0 \quad \text{in} \, \mathbb{R}^2 \setminus \Gamma, \quad \frac{\partial U}{\partial r} - ikU = o(r^{-1/2}) \quad \text{as} \, \, r = |z| \to \infty,
\]

with the asymptotic limit (the Sommerfeld radiation condition) holding uniformly in all directions: \( U \) is the double layer potential with (parametrized) density \( \varphi \) (see
The double layer potential is discontinuous across $\Gamma'$ but its normal derivative on $\Gamma'$ coincides from both sides. If we define
\[(W\varphi)(s) := -(\nabla U(x(s)) \cdot n(s)),\]
then,
\[(W\varphi)(s) = -\frac{i}{4} \frac{d}{ds} \int_0^1 H_0^{(1)}(k|x(s) - x(t)|)\varphi'(t)dt \]
\[= -\frac{ik^2}{4} \int_0^1 H_0^{(1)}(k|x(s) - x(t)|)n(s) \cdot n(t) \varphi(t)dt. \quad (4)\]

This is just the parametrized form of a well known regularized formula for the hypersingular operator \[28, \text{Theorem 3.3.22}], \[20, \text{Exercise 9.6]. The history of the proof of this formula is rather involved \[13]. It seems that the result is known from the work of Nédélec \[21], including this and similar identities for the Laplace and Navier equations, although the first results of this kind have been traced back to Maue \[19].

Using the asymptotic behavior of Hankel functions close to the singularity, we can write
\[W := -DV_1D + V_2, \quad D\varphi := \varphi', \quad (5)\]
where
\[V_\ell\varphi := \int_0^1 V_\ell(\cdot, t)\varphi(t)dt, \quad \ell \in \{1, 2\}, \quad (6)\]
with
\[V_\ell(s, t) := \frac{i}{2\pi} A_\ell(s, t) \log(\sin^2 \pi(s - t)) + K_\ell(s, t), \quad \ell \in \{1, 2\}, \quad (7)\]
\[A_1, A_2, K_1, K_2 \in C^\infty(\mathbb{R}^2) \text{ being 1-periodic in both variables. Moreover, } A_1(s, s) \equiv i/2. \]

Let us now consider the space \[D := \{\varphi \in C^\infty(\mathbb{R}) : \varphi(1 + \cdot) = \varphi\}, \text{ of smooth 1-periodic functions. As we will see later, the injectivity condition}\]
\[\varphi \in D, \quad W\varphi = 0 \quad \implies \quad \varphi = 0, \quad (8)\]
is equivalent to invertibility of $W$ in a wide range of Sobolev spaces. This fact holds for any operator of the form \[(5)-(6)-(7), \text{ provided that } A_1 \text{ is constant on the diagonal:}\]
\[A_1(s, s) \equiv \alpha \neq 0. \quad (9)\]

The operator $W$ satisfies the injectivity condition \[(8) \text{ if and only if } -k^2 \text{ is not a Neumann eigenvalue of the Laplace operator in the interior of } \Gamma \[12, \text{Section 2.1]. In those cases, the solution of the interior-exterior Helmholtz Eq. 3 with Neumann boundary condition}\]
\[\partial_n U := (\nabla U \circ x) \cdot n = f \quad (10)\]
can be represented with the double layer ansatz (2), \( \varphi \) being a solution of

\[ W\varphi = g. \] (11)

with \( g := -f \circ x \). The exterior scattering problem by a sound hard obstacle hit by an incident wave-field \( U^{\text{inc}} \) consists of finding the solution to

\[ \Delta U + k^2 U = 0 \] in the exterior of \( \Gamma \),

\[ \frac{\partial U}{\partial r} - ikU = o(r^{-1/2}) \] as \( r \to \infty \),

\[ \partial_n U = -\partial_n U^{\text{inc}}. \] (12c)

We next briefly describe some related operator equations with the same principal part. Consider now the adjoint double layer operator

\[ (J\lambda)(s) := \frac{i}{4} \int_0^1 H_1^{(1)}(k|x(t) - x(s)|) \frac{(x(t) - x(s)) \cdot n(s)}{|x(s) - x(t)|} \lambda(t) dt. \]

The integral equation

\[ W\varphi = \frac{1}{2} \lambda + J\lambda, \quad \lambda = (\nabla U^{\text{inc}} \circ x) \cdot n, \] (13)

is uniquely solvable in the same situations where Eq. 11 is. This formulation has the advantage that \( \varphi = U \circ x \), where \( U \) is the solution of problem (12). The scattered wave field can then be represented with the formula

\[ U = DL\varphi + SL\lambda, \]

where

\[ \mathbb{R}^2 \setminus \Gamma \ni z \mapsto (SL\lambda)(z) := \frac{i}{4} \int_0^1 H_0^{(1)}(k|z - x(t)|) \lambda(t) dt \]

is the acoustic single layer potential. A combined field ansatz \( U = (DL - ikSL)\psi \) leads to the integral equation

\[ W\psi - \frac{1}{2} ik\varphi + ikJ\psi = \partial_n U^{\text{inc}}. \] (14)

This equation is always uniquely solvable; unlike problems (11) and (13), it does not resonate on Neumann eigenfrequencies. The principal part of the operator is still the hypersingular operator \( W \), and from the point of view of numerical discretization, the problem has the same difficulties as problems (11) and (13).

Using the fact that the incident field is a solution of the Helmholtz equation in the interior of \( \Gamma \), it is possible to show that a double layer ansatz \( U = DL\varphi \) satisfies the direct combined field integral equation (in the spirit of the Burton-Miller integral equation for the sound-soft problem)

\[ W\varphi - \frac{1}{2} ik\varphi + ikK\varphi = \partial_n U^{\text{inc}} - ikU^{\text{inc}} \circ x, \] (15)

where

\[ (K\varphi)(s) := \frac{i}{4} \int_0^1 H_1^{(1)}(k|x(s) - x(t)|) \frac{(x(s) - x(t)) \cdot n(t)}{|x(s) - x(t)|} \varphi(t) dt \] (16)
is the double layer operator. This equation is also uniquely solvable for all frequencies, has $W$ as principal part, and has the advantage that $\varphi = (U + U^{\text{inc}}) \circ \mathbf{x}$.

Finally, we want to mention that some, but not all, integral methods for transmission problems lead to systems of boundary integral equations where $W$ has a prominent role.

### 3 Discretization of the hypersingular operator

#### 3.1 A class of integrodifferential operators and its discretization

Consider then problem (11), where $W$ is given in the form (5), with logarithmic operators (6)–(7), with the constant diagonal condition (9), and subject to the injectivity hypothesis (8). The discretization method uses four sets of discretization points. Let $N$ be a positive integer, $h := 1/N$, and

$$
s_i = \left( i - \frac{1}{2} \right) h, \quad t_i := i h, \quad s_{i+\varepsilon} := \left( i + \varepsilon - \frac{1}{2} \right) h, \quad t_{i+\varepsilon} := (i + \varepsilon) h, \quad i \in \mathbb{Z}. \tag{17}
$$

(We will comment on $\varepsilon$ shortly.) The discretization method looks for

$$(\varphi_0, \ldots, \varphi_{N-1}) \in \mathbb{C}^N \quad \text{such that} \quad \sum_{j=0}^{N-1} W_{ij}^\varepsilon \varphi_j = h g(t_{i+\varepsilon}) \quad i = 0, \ldots, N - 1, \tag{18}
$$

where

$$W_{ij}^\varepsilon := V_1(s_{i+1+\varepsilon}, s_{j+1}) - V_1(s_{i+\varepsilon}, s_{j+1}) - V_1(s_{i+1+\varepsilon}, s_j) + V_1(s_{i+\varepsilon}, s_j)
+h^2 V_2(t_{i+\varepsilon}, t_j). \tag{19}
$$

Substitution of $\varepsilon$ by $\varepsilon + 1$ produces the same method. The option $\varepsilon \in \mathbb{Z}$ is not practicable, since it leads to evaluations of the logarithmic kernels in their diagonal singularity. The method for $\varepsilon = 1/2$ ($\varepsilon \in 1/2 + \mathbb{Z}$) will not fit in our analysis, that relies on stability properties of an $\varepsilon$–dependent discretization of logarithmic operators that is unstable for $\varepsilon = 1/2$. (We will show numerical evidence that the value $\varepsilon = 1/2$ is valid though.) All other methods will provide convergent schemes, with two superconvergent cases. Namely, we will see that for smooth enough solutions, we can prove:

$$\max_j |\varphi_j - \varphi(t_j)| = \mathcal{O}(h), \quad \text{if } \varepsilon \not\in \frac{1}{2} \mathbb{Z}
$$

(this excludes the non-practicable and unstable cases), and that

$$\max_j |\varphi_j - \varphi(t_j)| = \mathcal{O}(h^2), \quad \text{if } \varepsilon \in \pm \frac{1}{6} + \mathbb{Z}.
$$

These results will be proved as Theorems 3 and 4 respectively.
3.2 A non-conforming Petrov-Galerkin method

We next give some intuition on how to come up with the method (18)–(19). We can formally rewrite Eq. 11 in variational form

\[
\int_0^1 \psi'(s)(V_1 \varphi')(s)ds + \int_0^1 \psi(s)(V_2 \varphi)(s)ds = \int_0^1 \psi(s)g(s)ds. \tag{20}
\]

Consider now the function \( \chi_i \) that arises from 1-periodization of the characteristic function of the interval \((s_i, s_{i+1}) = (t_i - h/2, t_i + h/2)\), that is,

\[
\chi_i(1 + \cdot) = \chi_i, \quad \chi_i \big|_{(s_i, s_{i+1})} = 1, \quad \chi_i \big|_{[s_{i+1}, s_{i+N}]} \equiv 0. \tag{21}
\]

We similarly define the functions \( \chi_{i+\varepsilon} \) by periodizing the characteristic functions of the intervals \((s_{i+\varepsilon}, s_{i+1+\varepsilon}) = (t_{i+\varepsilon} - h/2, t_{i+\varepsilon} + h/2)\). The weak derivatives of these functions can be expressed through the use of Dirac delta distributions. In addition to understanding Dirac deltas as point distributions (see Section 4.2), we will admit the action of Dirac deltas on any function that is continuous around the point where the delta is concentrated. At the present stage, we only need to consider the functionals

\[
\{\delta_{s_i}, \varphi\} := \varphi(s_i), \quad \{\delta_{s_{i+\varepsilon}}, \varphi\} := \varphi(s_{i+\varepsilon}), \tag{22}
\]

acting on any 1-periodic function \( \varphi \) that is smooth in a neighborhood of \( s_i \) and \( s_{i+\varepsilon} \). Admitting formally that \( \chi_i' = \delta_{s_i} - \delta_{s_{i+1}} \) and \( \chi_{i+\varepsilon}' = \delta_{s_{i+\varepsilon}} - \delta_{s_{i+1+\varepsilon}} \), we consider a non-conforming Petrov-Galerkin discretization of Eq. 20:

\[
\varphi_h := \sum_{j=0}^{N-1} \varphi_j \chi_j \quad \text{such that} \quad \sum_{j=0}^{N} W_{ij}^{E,0} \varphi_j = \int_{t_{i+\varepsilon}-\frac{h}{2}}^{t_{i+\varepsilon}+\frac{h}{2}} g(s)ds, \quad i = 0, \ldots, N-1, \tag{23}
\]

where

\[
W_{ij}^{E,0} := V_1(s_{i+1+\varepsilon}, s_{j+1}) - V_1(s_{i+\varepsilon}, s_{j+1}) - V_1(s_{i+1+\varepsilon}, s_j) + V_1(s_{i+\varepsilon}, s_j)
\]

\[
+ \int_{t_{i+\varepsilon}-\frac{h}{2}}^{t_{i+\varepsilon}+\frac{h}{2}} \int_{t_j-\frac{h}{2}}^{t_j+\frac{h}{2}} V_2(s, t)dsdt. \tag{24}
\]

Note that

\[
\sum_{j=0}^{N-1} \left( \int_{t_{i+\varepsilon}-\frac{h}{2}}^{t_{i+\varepsilon}+\frac{h}{2}} \int_{t_j-\frac{h}{2}}^{t_j+\frac{h}{2}} V_2(s, t)dsdt \right) \varphi_j = \int_{t_{i+\varepsilon}-\frac{h}{2}}^{t_{i+\varepsilon}+\frac{h}{2}} (V_2 \varphi_h)(s)ds
\]

and that the leading term in \( W_{ij}^{E,0} \) can be understood as the action

\[
\{\delta_{s_{i+\varepsilon}} - \delta_{s_{i+1+\varepsilon}}, V_1(\delta_{s_j} - \delta_{s_{j+1}})\} = \{\chi_{i+\varepsilon}', V_1 \chi_j'\}. \tag{25}
\]

The method (18)–(19) is recovered if we use midpoint integration for all integrals in Eqs. 23–24.
Note that, as opposed to the theory of Galerkin (same trial and test spaces) methods where stability follows from strong ellipticity, there is no standard form of proving stability for (different trial and test spaces) schemes. The stability analysis has to be done on a case-by-case basis, using inf-sup conditions or direct estimates based on Fourier analysis of the principal part and perturbation arguments. The papers [1, 24] contain early examples of this form of analysis. More general results can be found in [22].

4 Functional frame

4.1 Asymptotics of hypersingular operators

We endow the space $\mathcal{D}$ of periodic $C^\infty$ complex valued functions of one variable with the metric that imposes uniform convergence of all derivatives [27, Section 5.2]. A periodic distribution is an element of $\mathcal{D}'$, the dual space of $\mathcal{D}$. Given $u \in \mathcal{D}'$, we consider its Fourier coefficients
\[ \hat{u}(m) := \langle u, \phi_m \rangle_{\mathcal{D}' \times \mathcal{D}}, \quad \phi_m(t) := \exp(2\pi i mt), \quad m \in \mathbb{Z}. \] (26)
The periodic Sobolev space of order $r \in \mathbb{R}$ is
\[ H^r := \{u \in \mathcal{D}' : \|u\|_r < \infty\}, \quad \text{where} \quad \|u\|^2_r := |\hat{u}(0)|^2 + \sum_{m \neq 0} |m|^{2r} |\hat{u}(m)|^2. \] (27)
(From here on, the symbol $\sum_{m \neq 0}$ refers to a sum over all integers except zero.) An extensive treatment of these spaces can be found in the monograph [27]. Let us just mention that $H^p \subset H^q$ for $p > q$, with dense and compact injection. Also, $H^0$ can be identified with the space of 1-periodic functions that are locally square integrable or equivalently, with the 1-periodization of $L^2(0, 1)$.

We say that an operator $L : \mathcal{D}' \to \mathcal{D}'$ is a periodic pseudodifferential operator of order $n$, and we write for short $L \in E(n)$, when $L : H^r \to H^{r-n}$ for all $r$. It then follows from [27, Paragraph 7.6.1] that the logarithmic operators (6) can be extended to act on all periodic distributions and, consequently, so can W. Moreover, $V_1, V_2 \in E(-1)$ and $W \in E(1)$.

A first group of pseudodifferential operators that we will use extensively is that of multiplication operators. Given $a \in \mathcal{D}$ we define the operator $a \in E(0)$ by $au := a u$. The periodic Hilbert transform
\[ H_u := \sum_{m \neq 0} \text{sign}(m)\hat{u}(m)\phi_m \] (28)
is clearly a periodic pseudodifferential operator of order zero. We also consider the operators for $n \in \mathbb{Z}$:
\[ D_n u := \sum_{m \neq 0} (2\pi i m)^n \hat{u}(m)\phi_m, \quad D_n \in E(n). \] (29)
It is easy to note that \( D_1 = D \) is the differentiation operator and \( D_{-1} \) is a weak form of the following antiderivative operator

\[
(D_{-1}u)(s) = \int_0^s (u(t) - \hat{u}(0)) \, dt \quad \forall u \in H^0.
\]

In the next lemma we collect some elementary properties of these operators.

**Lemma 1** The following properties of the operators \( D_n \) defined by Eq. 29 hold:

(a) \( D_0 u = u - \hat{u}(0) \) for all \( u \).

(b) For all \( n \) and \( u \), \( D_{-n} D_n u = u - \hat{u}(0) \).

(c) For all \( n, m \), \( D_n D_m = D_{n+m} \).

(d) For all \( n \), \( D_n H = HD_n \).

(e) For all \( a \in D \), \( D_1 a = a D_1 + a' \).

The following results show how logarithmic operators and the hypersingular operators \( W \) can be represented up to operators of arbitrarily negative order as a linear combination of compositions of the simple operators given above.

**Proposition 1** Let

\[
Vu := \frac{i}{2\pi} \int_0^1 A(\cdot, t) \log(\sin^2(\pi(\cdot - t))) u(t) \, dt + \int_0^1 K(\cdot, t) u(t) \, dt,
\]

where \( A, K \in \mathcal{C}^\infty(\mathbb{R}^2) \) are 1-periodic in each variable. Then there exists a sequence \( \{a_n\}_{n \geq 1} \subset D \) such that for all \( M \),

\[
V = \sum_{n=1}^{M-1} a_n HD_{-n} + K_M, \quad K_M \in \mathcal{E}(-M).
\]

Moreover \( a_1(s) = A(s, s) \).

**Proof** See [9, Proposition A.1] or similar arguments in [27, Chapter 6].

**Proposition 2** Let \( W \) be the operator in Eq. 5. Then, there exists a sequence \( \{b_n\}_{n \geq 0} \subset D \) such that for all \( M \),

\[
W = -\alpha HD_1 + \sum_{n=0}^{M-1} b_n HD_{-n} + K_M, \quad K_M \in \mathcal{E}(-M),
\]

where \( 0 \neq \alpha \in \mathbb{C} \) is the constant in Eq. 9.

**Proof** This is just a direct consequence of Proposition 1 and Lemma 1.

**Proposition 3** Hypothesis (8) implies invertibility of \( W : H^r \to H^{r-1} \) for all \( r \).
Proof Consider the lowest order expansion of Proposition 2, namely \( W = -\alpha HD + K_0 \) with \( K_0 \in \mathcal{E}(0) \). It is clear that \(-\alpha HD : H^r \to H^{r-1}\) is Fredholm of index zero, and therefore so is \( W \).

Let now \( u \in H^r \) be such that \( Wu = 0 \). Applying that \( H^2 v = v - \hat{v}(0) \) for all \( v \), and Lemma 1 (b) and (d), it follows that

\[
0 = HD_{-1} Wu = -\alpha u + w, \quad \text{ where } w := \alpha \hat{u}(0) + HD_{-1} K_0 u \in H^{r+1}.
\]

This means that \( u \in H^s \) for all \( s \) and therefore \( u \in \mathcal{D} \). Hypothesis (8) implies then that \( u = 0 \). Therefore \( W : H^r \to H^{r-1} \) is injective and, by the Fredholm Alternative, it is invertible.

4.2 Variational formulation of the discrete method

For a fixed \( t \in \mathbb{R} \) we can define the Dirac delta distribution \( \delta_t \) by its action on elements of \( \mathcal{D} \), \( \langle \delta_t, \phi \rangle_{\mathcal{D}^\prime \times \mathcal{D}} = \phi(t) \). Using the Sobolev embedding theorem [27, Lemma 5.3.2], we can prove that \( \delta_t \in H^r \) for all \( r < -1/2 \). However, this does not allow us to apply the Dirac delta to functions that are piecewise smooth on points where they do not have jumps. If \( u \) is a 1-periodic function that is continuous in a neighborhood of \( t \), we will write \( \{\delta_t, u\} = u(t) \). Note that in general this is not a duality product \( \mathcal{D}^\prime \times \mathcal{D} \) or \( H^r \times H^r \). With this definition, we can admit the Dirac deltas \( \delta_{x_i} \) and \( \delta_{x_i + \varepsilon} \) in Eq. (22), as well as formula (25).

Let \( \mathbb{P}_0 \) be the space of constant functions. We then introduce three \( N \)-dimensional spaces:

\[
S_h := \text{span}\{\chi_i : i = 0, \ldots, N - 1\} = \{u_h \in H^0 : u_h|_{(s_i, s_{i+1})} \in \mathbb{P}_0 \ \forall i\},
\]

\[
S_{h,\varepsilon} := \text{span}\{\chi_{i+\varepsilon} : i = 0, \ldots, N - 1\} = \{u_h \in H^0 : u_h|_{(s_{i+\varepsilon}, s_{i+1+\varepsilon})} \in \mathbb{P}_0 \ \forall i\},
\]

\[
S_h^{-1} := \text{span}\{\delta_{x_i} : i = 0, \ldots, N - 1\} = \{u'_h : u_h \in S_h\} \oplus \text{span}\{d_h\},
\]

where

\[
d_h := h \sum_{j=0}^{N-1} \delta_{x_j}.
\]

Finally, we consider the discrete operators

\[
Q_h u := h \sum_{j=0}^{N-1} u(t_j) \delta_{t_j} \quad Q_{h,\varepsilon} u := h \sum_{j=0}^{N-1} u(t_{j+\varepsilon}) \delta_{t_{j+\varepsilon}},
\]

that are well defined for all periodic functions that are continuous around \( \bigcup \{t_j\} = h\mathbb{Z} \) and \( \bigcup \{t_{j+\varepsilon}\} = h(\varepsilon + \mathbb{Z}) \) respectively. In particular, we can apply \( Q_h \) to elements of \( S_h \) and \( Q_{h,\varepsilon} \) to elements of \( S_{h,\varepsilon} \).

Proposition 4 Let \( \varepsilon \notin \mathbb{Z} \) and let \( g \) be continuous in a neighborhood of \( h(\varepsilon + \mathbb{Z}) \). Then the discrete variational problem

\[
\text{find } \psi_h \in S_h \text{ such that } \{\psi'_h, V_1 \psi'_h \} + \{Q_{h,\varepsilon} \psi_h, V_2 Q_h \varphi_h\} = \{Q_{h,\varepsilon} \psi_h, g\} \quad \forall \psi_h \in S_{h,\varepsilon},
\]

is equivalent to looking for \( \varphi_h = \sum_{j=0}^{N-1} \varphi_j \chi_j \), where Eqs. 18–19 are satisfied.
Proof Given that both $S_h$ and $S_{h,\varepsilon}$ are $N$-dimensional, the problem (34) can be reduced to a $N \times N$ linear system, after choosing a basis for each of the spaces. The result follows then from several simple observations. First of all $\chi'_j = \delta_{s_{i+j}} - \delta_{s_{i+j+1}}$ and $\chi'_{i+\varepsilon} = \delta_{s_{i+\varepsilon}} - \delta_{s_{i+1+\varepsilon}}$ and $\{\delta_{s_{i+\varepsilon}}, V_1 \delta_{s_j}\} = V_1(s_{i+\varepsilon}, s_j)$. Also
\[ \{Q_{h,\varepsilon} \chi_{i+\varepsilon}, V_2 Q_h \chi_j\} = h^2 \{\delta_{s_{i+\varepsilon}}, V_2 \delta_{s_j}\} = h^2 V_2(t_{i+\varepsilon}, t_j). \]
Finally $\{Q_{h,\varepsilon} \chi_{i+\varepsilon}, g\} = h g(t_{i+\varepsilon})$. 

The non-conforming Petrov-Galerkin discretization of Eq. 20 given in Eqs. 23–24 is equivalent to the discrete variational problem
\[ \text{find } \varphi_h \in S_h \text{ such that } \{\psi'_h, V_1 \varphi'_h\} + (\psi_h, V_2 \varphi_h) = (\psi_h, g) \quad \forall \psi_h \in S_{h,\varepsilon}, \]
where $(u, v) = \int_0^1 u(t)v(t)\,dt$. In the sequel $(\cdot, \cdot)$ will be used to denote this bilinear form in $H^0$ (so that $(u, u) = \|u\|_0^2$) and its extension to a duality product $H^{-1} \times H^1$, so that for any $r \in \mathbb{R}$.
\[ \|v\|_{-r} = \sup_{0 \neq u \in H^r} \frac{|(u, v)|}{\|u\|_r} \quad v \in H^r. \]

Given $A$, we will denote by $A^*$ the adjoint with respect to this bilinear form, thus avoiding conjugation. Notice in passing, that by Eqs. 28–29, it holds
\[ D_n^* = (-1)^n D_n, \quad H^* = -H. \]

5 Stability analysis via an inf-sup condition

Consider now the bilinear form $w_h : S_{h,\varepsilon} \times S_h \rightarrow \mathbb{C}$ associated to the problem (34), namely
\[ w_h(\psi_h, \varphi_h) := \{\psi'_h, V_1 \varphi'_h\} + \{Q_{h,\varepsilon} \psi_h, V_2 Q_h \varphi_h\}. \]

Compare this with the variational form of the equation presented in Eq. 20. The aim of this section is the proof of the following result, that in particular implies that problem (34) (and by Proposition 4 also the method (18)–(19)) has a unique solution for small enough $h$.

Theorem 1 (Stability) There exist $h_0$ and $\beta_\varepsilon > 0$ such that for $h \leq h_0$,
\[ \sup_{0 \neq \psi_h \in S_{h,\varepsilon}} \frac{|w_h(\psi_h, \varphi_h)|}{\|\psi_h\|_0} \geq \beta_\varepsilon \|\varphi_h\|_0 \quad \forall \varphi_h \in S_h. \]

5.1 Stability of the non-conforming PG method

We start by considering the bilinear form associated to problem (35)
\[ w^0_h(\psi_h, \varphi_h) := \{\psi'_h, V_1 \varphi'_h\} + (\psi_h, V_2 \varphi_h), \]

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the operator $A\varphi := W D_{-1}\varphi + \hat{\varphi}(0) W 1$ and its adjoint $A^*$. Note that if $\psi_h \in S_{h,\varepsilon}$, then $A^*\psi_h$ is a smooth function except at the discontinuity points of $\psi_h$.

**Lemma 2**

$$\omega_h(\psi_h, \varphi_h) = \{\varphi'_h, A^*\psi_h\} + \hat{\varphi}_h(0)(1, A^*\psi_h) \quad \forall \psi_h \in S_{h,\varepsilon}, \varphi_h \in S_h.$$  

**Proof** A direct computation, using Eq. 37 and Lemma 1(b) shows that

$$A^*\psi = -D_{-1} W^*\psi + \hat{\varphi}_h(0) W^*_1\psi' = -V_1^*\psi' - \hat{V}_1^*\psi'(0) - D_{-1} V_2^*\psi + \hat{V}_2^*\psi(0).$$  

Noticing that $\varphi'_h, 1) = (\varphi_h, 1) = 0$, it follows from Eq. 40 and Lemma 1(b) that

$$(\varphi'_h, A^*\psi) = (\varphi'_h, V_1^*\psi'_h) + (D_{-1}\varphi'_h, V_2^*\psi)_h$$  

$$= (\varphi'_h, V_1^*\psi'_h) + (\varphi_h, V_2^*\psi_h) - \hat{\varphi}_h(0)(1, V_2^*\psi_h)$$  

$$= w_h(\psi_h, \varphi_h) - \hat{\varphi}_h(0) V_2^*\psi(0).$$

At the same time, integrating in Eq. 40, it follows that $(1, A^*\psi) = \hat{V}_2^*\psi(0)$, which finishes the proof.

**Lemma 3** $\|d_h - 1\|_1 \leq \pi h$.

**Proof** Since for all $u \in H^1$,

$$\|1 - d_h, u\|_0 = \left| \int_0^1 u(t) dt - h \sum_{j=0}^{N-1} u(s_j) \right| \leq \sum_{j=0}^{N-1} \int_{s_j - \frac{h}{2}}^{s_j + \frac{h}{2}} u(t) dt - hu(s_j)$$  

$$\leq \frac{h}{2} \sum_{j=0}^{N-1} \int_{s_j - \frac{h}{2}}^{s_j + \frac{h}{2}} |u'(t)| dt \leq \frac{h}{2} \|u'\|_0 \leq \pi h \|u\|_1,$$

the result follows from Eq. 36.

**Lemma 4** For $\varepsilon \notin \frac{1}{2} \mathbb{Z}$, there exist positive constants $\beta_\varepsilon$ and $C_\varepsilon$ such that

$$\sup_{0 \neq \psi_h \in S_{h,\varepsilon}} \frac{\|\delta_h, A^*\psi_h\|}{\|\psi_h\|_0} \geq \beta_\varepsilon \|\delta_h\|_{-1}, \quad \forall \delta_h \in S_h^{-1},$$  

and

$$\|1, A^*\psi_h\| - \|d_h, A^*\psi_h\| \leq C_\varepsilon \|\psi_h\|_0 \quad \forall \psi_h \in S_{h,\varepsilon}.$$  

**Proof** Using Proposition 2 (asymptotic expansion of $W$), it is simple to see that $A = -\alpha H + K$, where $K \in E(-1)$. This places us in the hypotheses of [5, Proposition 8], which proves (41). For the second estimate, we use Eq. 37 to write $A^* = -\alpha H + K^*$ and decompose

$$(1, A^*\psi_h) - \{d_h, A^*\psi_h\} = -\alpha \left( (1, H\psi_h) - h \sum_{j=0}^{N-1} (H\psi_h)(s_j) \right) + (1 - d_h, K\psi_h).$$
The first term can be bounded using [5, Lemma 3]

\[
\left| (1, H\psi_h) - h \sum_{j=0}^{N-1} (H\psi_h)(s_j) \right| \leq C_\varepsilon h \|\psi_h\|_0 \quad \forall \psi_h \in S_{h,e}.
\]

The second one follows from Lemma 3,

\[
|(1 - d_h, K\psi_h)| \leq \|1 - d_h\|_{-1} \|K^* \psi_h\|_1 \leq \pi h \|K^*\|_{H^p \to H^q} \|\psi_h\|_0.
\]

**Lemma 5** There exist two positive constants \( c_1, c_2 \) such that

\[
c_1 \|\varphi_h^\prime + \tilde{\varphi}_h(0) d_h\|_{-1} \leq \|\varphi_h\|_0 \leq c_2 \|\varphi_h^\prime + \tilde{\varphi}_h(0) d_h\|_{-1} \quad \forall \varphi_h \in S_h.
\]

**Proof** The first bound is a simple consequence of the following inequalities

\[
\|\varphi_h^\prime\|_{-1} \leq 2\pi \|\varphi_h\|_0, \quad \|\tilde{\varphi}_h(0)\| \leq \|\varphi_h\|_0, \quad \|d_h\|_{-1} \leq 1 + \pi h
\]

(the last inequality follows from Lemma 3.) Note now that the operator \( S_h \to S_h^{-1} \) given by \( \varphi_h \mapsto \varphi_h^\prime + \tilde{\varphi}_h(0) d_h \) is injective. Its inverse is \( S_h^{-1} \ni \delta_h \mapsto D_{-1}\delta_h + \delta_h(0)(1 - D_{-1}d_h) \). Then the second inequality of the statement follows from the fact that

\[
\|D_{-1}\delta_h\| \leq \frac{1}{2\pi} \|\delta_h\|_{-1}, \quad \|\tilde{\varphi}_h(0)\| \leq \|\delta_h\|_{-1}, \quad \|1 - D_{-1}d_h\|_0 \leq 1 + \frac{1}{2\pi} \|d_h\|_{-1} \leq 1 + \frac{h}{2},
\]

where we have applied Lemma 3 again. 

**Proposition 5** There exist \( h_0 \) and \( \beta_\varepsilon > 0 \) such that for \( h \leq h_0 \),

\[
\sup_{0 \neq \varphi_h \in S_{h,e}} \frac{|w_h^0(\varphi_h, \varphi_h)|}{\|\psi_h\|_0} \geq \beta_\varepsilon \|\varphi_h\|_0 \quad \forall \varphi_h \in S_h.
\]

**Proof** By Lemma 2, we can write

\[
w_h^0(\varphi_h, \varphi_h) = \{\varphi_h^\prime + \tilde{\varphi}_h(0) d_h, A^* \varphi_h\} + \tilde{\varphi}_h(0)
\begin{pmatrix}
(1, A^* \varphi_h) - \{d_h, A^* \varphi_h\}
\end{pmatrix}.
\]

Therefore, by Lemma 4, it follows that

\[
\sup_{0 \neq \varphi_h \in S_{h,e}} \frac{|w_h^0(\varphi_h, \varphi_h)|}{\|\psi_h\|_0} \geq \sup_{0 \neq \varphi_h \in S_{h,e}} \frac{|[\varphi_h^\prime + \tilde{\varphi}_h(0) d_h, A^* \varphi_h]|}{\|\psi_h\|_0} - C_\varepsilon \|\tilde{\varphi}_h(0)\|_0
\]

\[
\geq \beta_\varepsilon \|\varphi_h^\prime + \tilde{\varphi}_h(0) d_h\|_{-1} - C_\varepsilon h \|\varphi_h\|_0
\]

\[
\geq \left( \beta_\varepsilon c_2^{-1} - C_\varepsilon h \right) \|\varphi_h\|_0 \quad \forall \varphi_h \in S_h,
\]

where we have applied Lemma 5 in the last inequality.
5.2 A perturbation argument

Consider now the quadrature error

$$E_{ij}^\varepsilon := \int_{Q_{ij}^\varepsilon} V_2(s, t) \, dx \, dt - h^2 V^2(t_{i+\varepsilon}, t_j), \quad i, j \in \mathbb{Z},$$

(43)

where $Q_{ij}^\varepsilon := (s_{i+\varepsilon}, s_{i+1+\varepsilon}) \times (s_j, s_{j+1})$. Let $E_{ij}^\varepsilon$ be the $N \times N$ matrix whose entries are the values $E_{ij}^\varepsilon$ for $i, j = 0, \ldots, N - 1$. In the sequel $|E|_p$ will denote the $p$-norm of the matrix $E$ (for $p \in \{1, 2, \infty\}$) and $|E|_{\text{Frob}}$ will denote its Frobenius norm.

**Proposition 6**

$$|\langle \psi_h, V_2 \varphi_h \rangle - \{Q_{h,\varepsilon} \psi_h, V_2 Q_{h} \varphi_h \}| \leq h^{-(N-1)} E_{ij}^\varepsilon \|\psi_h\|_0 \|\varphi_h\|_0 \quad \forall \psi_h \in S_{h,\varepsilon}, \varphi_h \in S_h.$$

**Proof** If we decompose $\psi_h = \sum_{i=0}^{N-1} \psi_i \chi_i$ and $\varphi_h = \sum_{j=0}^{N-1} \varphi_j \chi_j$, it is easy to see that

$$\langle \psi_h, V_2 \varphi_h \rangle - \{Q_{h,\varepsilon} \psi_h, V_2 Q_{h} \varphi_h \} = \sum_{i,j=0}^{N-1} \psi_i E_{ij}^\varepsilon \varphi_j.$$

The result is then straightforward noticing that $\|\psi_h\|_0 = h^{1/2} |(\psi_0, \ldots, \psi_{N-1})|$ and $\|\varphi_h\|_0 = h^{1/2} |(\varphi_0, \ldots, \varphi_{N-1})|$, where $|\cdot|$ is the Euclidean norm in $\mathbb{C}^N$.

In order to simplify some forthcoming arguments, let us restrict (without loss of generality) $\varepsilon$ to be in $[-1/2, 1/2] \setminus \{0\}$ (the restriction $\varepsilon \neq \pm 1/2$ is not needed for these arguments).

**Lemma 6** There exists $C_\varepsilon$ such that for all $h$

$$|E_{ij}^\varepsilon| \leq C_\varepsilon h^2 \log |h| \quad \forall i, j.$$

Moreover $C_\varepsilon$ diverges like $\log |\varepsilon|^{-1}$ as $|\varepsilon| \to 0$.

**Proof** Since $E_{ij}^\varepsilon = E_{ij}^\varepsilon$, we can choose $|i-j| \leq N/2$ and then

$$|s-t| \leq h + h(N/2 + |\varepsilon|) \leq \frac{3}{2} h + \frac{1}{2} \leq \frac{3}{4}, \quad (s, t) \in Q_{ij}^\varepsilon,$$

(44)

as long as $h \leq 1/6$. (This is not a restriction, since we are only missing the values $1 \leq N \leq 5$ that can be incorporated by modifying the constants in the final bound.) Also

$$|\varepsilon| h \leq |t_{i+\varepsilon} - t_j| \leq \frac{3}{4}.$$

(45)

Because of the form of the kernel function $V_2$ (see Eq. 7), in the diagonal strip $D := \{(s, t) : |s-t| \leq 3/4\}$, we can bound

$$|V_2(s, t)| \leq C_1 \log |s-t|^{-1} + C_2, \quad (s, t) \in D.$$

(46)
Therefore, by Eq. 45,
\[ |V_2(t_{i+\varepsilon}, t_j)| \leq C_1 \log |\varepsilon|^{-1} + C_1 \log h^{-1} + C_2, \quad \forall i, j. \]  
(47)

The choice of indices \(|i - j| \leq N/2\) ensures that \(Q_{ij}^e \subset D\). If \(\text{dist}(Q_{ij}, \{(s, s) : s \in \mathbb{R}\}) \geq h/2\), then by Eq. 46,
\[ \left| \int_{Q_{ij}^e} V_2(s, t)\,dxdy \right| \leq C_1 h^2 (\log 2 + \log h^{-1}) + C_2 h^2. \]  
(48)

If, on the other hand, \(\text{dist}(Q_{ij}, \{(s, s) : s \in \mathbb{R}\}) \leq h/2\), a simple geometric argument shows that \(Q_{ij}^e \subset \{(s, t) : t \in (s_j, s_{j+1}), |s - t| < c h\}\), where \(c := \sqrt{2} + 1/2 < e\). Therefore
\[ \left| \int_{Q_{ij}^e} V_2(s, t)\,dxdy \right| \leq 2C_1 h \int_0^{ch} \log u^{-1} \,du + C_2 h^2 = 2C_1 c h^2 (1 - \log c + \log h^{-1}) + C_2 h^2. \]  
(49)

We can now gather the bounds in (47), (48) and (49) rearrange terms and take upper bounds to prove the result. \(\square\)

**Lemma 7** There exists \(C\) independent of \(e\) such that for all \(h\)
\[ |E_{ij}^e| \leq C \frac{h^2}{|i - j|^2} \quad \text{for } i, j \text{ such that } 2 \leq |i - j| \leq \frac{N}{2}. \]

**Proof** If \((s, t) \in Q_{ij}^e, 2 \leq |i - j| \leq N/2, \text{ and } |\varepsilon| \leq 1/2\), then
\[ |s - t| \geq |t_{i+\varepsilon} - t_j| - |t - t_j| - |s - t_j| \geq h |i - j| - (|\varepsilon| + 1)h \geq \frac{1}{4} h |i - j|. \]  
(50)

Recall first the definition of \(D := \{(s, t) : |s - t| \leq 3/4\}\) given in the proof of Lemma 6. Taking derivatives of the kernel function \(V_2\), we can write
\[ \frac{\partial^2 V_2}{\partial s^2}(s, t) = \frac{B_1(s, t)}{|s - t|^2} + B_2(s, t), \quad \frac{\partial^2 V_2}{\partial t^2}(s, t) = \frac{B_3(s, t)}{|s - t|^2} + B_4(s, t), \]
where for \(\ell \in \{1, 2, 3, 4\}\) the functions \(B_\ell \in C^0(D)\) are bounded. Using the error bound for the midpoint formula in two variables, we can estimate
\[ |E_{ij}^e| \leq \frac{h^4}{24} \max_{(s, t) \in Q_{ij}} \left( \left| \frac{\partial^2 V_2}{\partial s^2}(s, t) \right| + \left| \frac{\partial^2 V_2}{\partial t^2}(s, t) \right| \right) \]
\[ \leq \frac{h^4}{24} \left( \|B_2\|_{L^\infty} + \|B_4\|_{L^\infty} + \|B_1\|_{L^\infty} + \|B_3\|_{L^\infty} \right) \max_{(s, t) \in Q_{ij}} |s - t|^{-2} \]
\[ \leq C_1 h^4 + C_2 \frac{h^2}{|i - j|^2} \leq \left( C_1 \frac{4}{4} + C_2 \right) \frac{h^2}{|i - j|^2}, \]
where we have applied Eq. 50 and the upper bound \(|i - j|h \leq 1/2\). \(\square\)
Lemma 8 There exists $C_\varepsilon$ such that for all $h$, $|E^\varepsilon|_2 \leq C_\varepsilon h^{3/2}$.

Proof We first decompose the matrix $E^\varepsilon = E^\varepsilon_{\text{trid}} + E^\varepsilon_{\text{off}}$, where $E^\varepsilon_{\text{trid}}$ gathers all tridiagonal terms (modulo $N$) of $E^\varepsilon$

$$E^\varepsilon_{\text{trid},ij} := \begin{cases} E^\varepsilon_{ij}, & |i - j| \leq 1 \text{ or } |i - j| = N - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Using Lemma 6 and the fact that $E^\varepsilon_{\text{trid}}$ has only three non-vanishing elements in each row and column, it is easy to estimate $|E^\varepsilon_{\text{trid}}|_1 + |E^\varepsilon_{\text{trid}}|_\infty \leq 3C_\varepsilon h^2 |\log h|$. Therefore, by the Riesz-Thorin theorem

$$|E^\varepsilon_{\text{trid}}|_2 \leq |E^\varepsilon_{\text{trid}}|_1^{1/2} |E^\varepsilon_{\text{trid}}|_\infty^{1/2} \leq 3C_\varepsilon h^2 |\log h|.$$  (51)

On the other hand, we can estimate the off-diagonal terms using Lemma 7 (recall that we can move indices so that $|i - j| \leq N/2$)

$$|E^\varepsilon_{\text{off}}|_2 \leq |E^\varepsilon_{\text{off}}|_{\text{Frob}} \leq \sum_{i=0}^{N-1} \sum_{2 \leq |i - j| \leq N/2} |E^\varepsilon_{ij}|^2 \leq C^2 h^4 \sum_{i=0}^{N-1} \sum_{2 \leq |i - j| \leq N/2} \frac{1}{|i - j|^4} = h^3 2C^2 \sum_{k=2}^\infty \frac{1}{k^4}.$$  (52)

Gathering Eqs. 51 and 52, the result follows.

Proof (Proof of Theorem 1.) Note that

$$w_h(\psi_h, \varphi_h) = w^\circ_h(\psi_h, \varphi_h) + \{Q_{h,\varepsilon} \psi_h, V_2 Q_h \varphi_h\} - (\psi_h, V_2 \varphi_h).$$

As a direct consequence of Proposition 6 and Lemma 8, we can bound

$$|(\psi_h, V_2 \varphi_h) - \{Q_{h,\varepsilon} \psi_h, V_2 Q_h \varphi_h\}| \leq C_\varepsilon h^{1/2} \|\psi_h\|_0 \|\varphi_h\|_0 \quad \forall \psi_h \in S_{h,\varepsilon}, \varphi_h \in S_h.$$  (53)

The proof is thus a simple consequence of this bound and Proposition 5.

6 Consistency error analysis

The analysis of the consistency error is based in the careful use of estimates for quadrature error and the combination of asymptotic expansions of discrete and continuous operators. We start this section with some technical results that will be needed in the sequel.
6.1 Estimates for quadrature error

Lemma 9 The following bounds hold for all $h$ and all $\varepsilon$:

(a) $|(Q_{h,\varepsilon}\psi_h, u) - (\psi_h, u)| \leq \frac{1}{2} (\pi h)^2 \|\psi_h\|_0 \|u\|_2$ for all $\psi_h \in S_{h,\varepsilon}$ and $u \in H^2$.
(b) $|(Q_{h,\varepsilon}\psi_h, u)| \leq \|\psi_h\|_0 (\|u\|_0 + \pi h \|u\|_1)$ for all $\psi_h \in S_{h,\varepsilon}$ and $u \in H^1$.

Proof Using Taylor expansions, it is easy to prove the following well-known bound for the midpoint formula

$$\left| \int_{c-h/2}^{c+h/2} u(t) dt - hu(c) \right| \leq \frac{h^2}{8} \int_{c-h/2}^{c+h/2} |u''(t)| dt,$$

from where

$$|(Q_{h,\varepsilon}\psi_h, u) - (\psi_h, u)| = \left| \sum_{j=0}^{N-1} \psi_j \left( \int_{t_j+\varepsilon-h/2}^{t_j+\varepsilon+h/2} u(t) dt - hu(t_j+\varepsilon) \right) \right|$$

$$\leq \frac{h^2}{8} \int_0^1 |\psi_h(t)| |u''(t)| dt \leq \frac{h^2}{8} \|\psi_h\|_0 (2\pi)^2 \|u\|_2.$$

This proves (a). To prove (b) we proceed similarly, showing first that

$$|(Q_{h,\varepsilon}\psi_h, u) - (\psi_h, u)| \leq \pi h \|\psi_h\|_0 \|u\|_1 \quad \forall \psi_h \in S_{h,\varepsilon}, u \in H^1,$$

and then applying the inverse triangle inequality.

□

Lemma 10 There exists $C_\varepsilon$ such that

$$|(Q_{h,\varepsilon}\psi_h, V_2 Q_h u)| \leq C_\varepsilon \|\psi_h\|_0 (\|u\|_0 + h \|u\|_1) \quad \forall \psi_h \in S_{h,\varepsilon}, u \in H^1.$$

Proof Let $u_h := \sum_{j=0}^{N-1} u(t_j) \chi_j \in S_h$ be the midpoint interpolant of $u$ onto $S_h$. A direct estimate shows that

$$\|u_h\|_0 \leq \|u\|_0 + \|u - u_h\|_0 \leq \|u\|_0 + \frac{\pi h}{\sqrt{2}} \|u\|_1.$$

On the other hand, since $Q_h u = Q_h u_h$, it follows from Eq. 53 that

$$|(Q_{h,\varepsilon}\psi_h, V_2 Q_h u)| = \|(Q_{h,\varepsilon}\psi_h, V_2 Q_h u_h)\| \leq (\psi_h, V_2 u_h) + C_\varepsilon h^{1/2} \|\psi_h\|_0 \|u_h\|_0 \leq (\|V_2\|_{H^0 \rightarrow H^0} + C_\varepsilon h^{1/2}) \|\psi_h\|_0 \|u_h\|_0.$$

Applying Eq. 55, the result follows.

□

6.2 Discrete operators and expansions

The truncation operator for the Fourier series

$$F_h u := \sum_{m \in \Lambda_N} \hat{u}(m) \phi_m \quad \text{where } \Lambda_N := \{m : -N/2 < m \leq N/2\}$$
gives optimal approximation properties in all Sobolev norms \([27, \text{Theorem 8.2.1}]\)
\[
\| F_h u - u \|_s \leq (\sqrt{2}h)^{r-s} \| u \|_r \quad \forall r \geq s.
\]  
(56)

We can also define a discretization operator onto \(S_h\) based on matching the central Fourier coefficients
\[
D_h u \in S_h \quad \text{such that} \quad \hat{D}_h u(m) = \hat{u}(m) \quad \forall m \in \Lambda_N.
\]

This operator is based on a class of spline-trigonometric projectors introduced in \([1]\). Here we will use it as introduced in \([10]\). The following property
\[
Q_{h,1/2} F_h D = D D_h
\]  
(57)
is a consequence of \([5, \text{Lemma 5}]\).

Consider the \(1\)-periodic functions \(B_\ell\) such that \((-1)^\ell \ell! B_\ell\) restricted to \((0, 1)\) is equal to the Bernoulli polynomial of degree \(\ell\) for all \(\ell\). Consider also \(C_\ell := H B_\ell\). By comparing their Fourier coefficients \([5, \text{Section 3}]\), it is easy to prove that
\[
C_1(t) = -\frac{1}{2\pi i} \log(4 \sin^2(\pi t))
\]  
and therefore \(C_1(\pm \frac{1}{6}) = 0\).  
(58)

Note that \(\pm 1/6 + \mathbb{Z}\) are the only zeros of \(C_1\).

**Proposition 7** Let \(a_1, a_2 \in \mathcal{D}\) and
\[
V := a_1 HD_{-1} + a_2 HD_{-2} + K_3 \quad \text{where} \ K_3 \in \mathcal{E}(-3).
\]

Let then \(L_1 := a_1\) and \(L_2 := a_1 D - a_2\), and consider the operators
\[
R_h u := Vu - VQ_h F_h u + hC_1(\cdot / h)L_1 F_h u,
\]
\[
T_h u := Vu - VQ_{h,1/2} F_h u + hC_1(\cdot / h + 1/2)L_1 F_h u + h^2 C_2(\cdot / h + 1/2)L_2 F_h u.
\]

Then
\[
\| R_h u \|_0 + h \| R_h u \|_1 \leq Ch^2 \| u \|_1 \quad \forall u \in H^1,
\]  
(59)
\[
\| T_h u \|_0 + h \| T_h u \|_1 \leq Ch^3 \| u \|_2 \quad \forall u \in H^2.
\]  
(60)

**Proof** It is a direct consequence of \([5, \text{Proposition 16}]\). \(\Box\)

**Proposition 8** Let \(E_h u := u - D_h u + hB_1(\cdot / h + 1/2) F_h u'\). Then
\[
\| E_h u \|_0 + h \| E_h u \|_1 \leq Ch^2 \| u \|_2 \quad \forall u \in H^2.
\]

**Proof** The bound for \(\| E_h u \|_0\) is given in \([9, \text{Proposition 1}]\). The \(H^1\) bound can be obtained with similar arguments (see the proof of \([5, \text{Proposition 16}]\)). \(\Box\)

### 6.3 Consistency error

In the definition of the bilinear form \((38)\), we only admitted discrete arguments. In this section we will admit a continuous second argument. The definition is equally valid.
Proposition 9 Let \( \varphi_h \) be the solution of Eq. \( 34 \) with right-hand side \( g = W \varphi \). Then
\[
|w_h(\psi_h, \varphi_h - \varphi) + hC_1(\varepsilon)(\psi_h, a_0 \varphi)| \leq Ch^2\|\psi_h\|_0\|\varphi\|_3 \quad \forall \psi_h \in S_{h, \varepsilon},
\]
where \( a(s) := A_2(s, s) \).

Proof Note first that by definition of \( \varphi_h \)
\[
w_h(\psi_h, \varphi_h - \varphi) = (Q_{h, \varepsilon} \psi_h, W_1 \varphi) - (\psi_h', V_1 \varphi') - \{Q_{h, \varepsilon} \psi_h, V_2 Q_h \varphi\}
\[
 \quad = -(Q_{h, \varepsilon} \psi_h, DV_1 \varphi) + (\psi_h', DV_1 \varphi)
\[
 \quad + \{Q_{h, \varepsilon} \psi_h, V_2 Q_h (F_h \varphi - \varphi)\} + \{Q_{h, \varepsilon} \psi_h, V_2 (\varphi - Q_h F_h \varphi)\}.
\]

In order to estimate \( T_1 \), we apply Lemma 9(a) with \( u = DV_1 \varphi \) and note that \( DV_1 D \in \mathcal{E}(1) \), to obtain
\[
|T_1| \leq \frac{1}{2} h^2 \pi^2 \|\psi_h\|_0\|DV_1 \varphi\|_2 \leq Ch^2\|\psi_h\|_0\|\varphi\|_3. \quad (62)
\]

To estimate \( T_2 \), we apply Lemma 10 and the approximation properties of \( F_h \) \((56)\), so that
\[
|T_2| \leq \|\psi_h\|_0(\|F_h \varphi - \varphi\|_0 + \pi h \|F_h \varphi - \varphi\|_1) \leq Ch^2\|\psi_h\|_0\|\varphi\|_2. \quad (63)
\]

To bound \( T_3 \) we will apply Proposition 7 to the operator \( V_2 \) (see Proposition 1). It is simple to verify that
\[
\{Q_{h, \varepsilon} \psi_h, C_1(\cdot / h) u\} = C_1(\varepsilon, (Q_{h, \varepsilon} \psi_h, u) \quad \forall \psi_h \in S_{h, \varepsilon}.
\]

Then, by Proposition 7, and denoting \( a(s) := A_2(s, s) \),
\[
T_3 = -hC_1(\varepsilon)(Q_{h, \varepsilon} \psi_h, a_0 F_h \varphi) + (Q_{h, \varepsilon} \psi_h, R_h \varphi)
\[
= -hC_1(\varepsilon)(\psi_h, a_0 \varphi)
\[
 + hC_1(\varepsilon)(\psi_h - Q_{h, \varepsilon} \psi_h, a_0 \varphi) + hC_1(\varepsilon)(Q_{h, \varepsilon} \psi_h, a_0 (\varphi - F_h \varphi)) + (Q_{h, \varepsilon} \psi_h, R_h \varphi).
\]

Applying Lemma 9(a) with \( u = a_0 \varphi \) we can easily bound
\[
|T_{31}| \leq Ch^3\|\psi_h\|_0\|\varphi\|_2, \quad (64)
\]
while Lemma 9(b) applied to \( u = a_0 (\varphi - F_h \varphi) \) yields
\[
|T_{32}| \leq h\|\psi_h\|_0(\|a_0 (\varphi - F_h \varphi)\|_0 + h\|a_0 (\varphi - F_h \varphi)\|_1) \leq Ch^3\|\psi_h\|_0\|\varphi\|_2. \quad (65)
\]
Finally, we apply Lemma 9(b) again, using the bound for $R_h \varphi$ provided by Eq. 59, which yields

$$|T_{33}| \leq \| \psi_h \|_0(\|R_h \varphi\|_0 + \pi h \|R_h \varphi\|_1) \leq Ch^2 \| \psi_h \|_0 \| \varphi \|_1. \tag{66}$$

Inequalities (64)–(66) imply that

$$|T_3 + hC_1(\varepsilon)(\psi_h, a \varphi)| \leq Ch^2 \| \psi_h \|_0 \| \varphi \|_2. \tag{67}$$

Carrying Eqs. 62, 63, and 67 to Eq. 61 the result follows. \hfill \square

**Proposition 10** Let $\alpha$ be the constant in Eq. 9. Then

$$|w_h(\psi_h, \varphi - D_h \varphi) + h \alpha C_1(\varepsilon)(\psi_h', \varphi')| \leq Ch^2 \| \psi_h \|_0 \| \varphi \|_3 \quad \forall \psi_h \in S_{h, \varepsilon}.$$  

**Proof** Using Eq. 57, we can write

$$w_h(\psi_h, \varphi - D_h \varphi) = \{\psi_h', V_1(\varphi' - Q_{h,1/2} F_h \varphi')\} + \{Q_{h,\varepsilon} \psi_h, V_2 Q_h(\varphi - D_h \varphi)\}. \tag{68}$$

To estimate $T_4$ we will use Proposition 7 applied to $V_1$ (see Proposition 1). An easy computation shows that

$$\{\psi_h', C_\varepsilon(\cdot / h + 1/2) u\} = C_\varepsilon(\psi_h', u) \quad \forall \psi_h \in S_{h, \varepsilon}. \tag{69}$$

Let $L_1$ and $L_2$ be the differential operators associated to the expansion of $V_1$ in Proposition 7 (note that $L_1 u = \alpha u$). By Eq. 69 and Proposition 7 it follows that

$$T_4 = -hC_1(\varepsilon)(\psi_h, L_1 \varphi') \tag{70}$$

Using Eq. 56 we can easily bound

$$|T_{41}| = |hC_1(\varepsilon)(\psi_h, DL_1(\varphi' - F_h \varphi'))| \leq Ch \| \psi_h \|_0 \| \varphi' - F_h \varphi' \|_1 \leq C'h^2 \| \psi_h \|_0 \| \varphi \|_3 \tag{71}$$

and

$$|T_{42}| = |h^2C_2(\varepsilon)(\psi_h, DL_2 F_h \varphi')| \leq Ch^2 \| \psi_h \|_0 \| \varphi \|_3. \tag{72}$$

Similarly, using the bound for $T_h$ given by Eq. 60, we estimate

$$|T_{43}| \leq \| \psi_h \|_0 \| DT_h \varphi' \|_0 \leq Ch^2 \| \psi_h \|_0 \| \varphi \|_3. \tag{73}$$

Taking Eqs. 71–73 to Eq. 70 we have proved that

$$|T_4 + h \alpha C_1(\varepsilon)(\psi_h', \varphi')| \leq Ch^2 \| \psi_h \|_0 \| \varphi \|_3. \tag{74}$$
We next estimate the term $T_5$ in Eq. 68. Note that

$$Q_h(B_1(\cdot/h + 1/2)u) = B^1_1(1/2)Q_h u = 0,$$

(75)

by the coincidence of $B_1$ with the Bernoulli polynomial of first degree in $(0, 1)$. Therefore, using Proposition 8 and Lemma 10(b) it follows that

$$|T_5| = |\{Q_h, \psi_h, V_2 Q_h E_h \phi\}| \leq C_\varepsilon \|\psi_h\|_0 (\|E_h \phi\|_0 + h \|Q_h E_h \phi\|_1) \leq C_\varepsilon h^2 \|\psi_h\|_0 \|\phi\|_2.$$  

(76)

The collection of Eqs. 68, 74 and 76 proves the result. □

**Corollary 1** Let $\psi_h$ be the solution of Eq. 34 with right-hand side $g = W \phi$. Then

$$|w_h(\psi_h, \varphi_h - D_h \phi) + h C_4(\varepsilon)(\psi_h, a \phi - \alpha \phi'')| \leq C h^2 \|\psi_h\|_0 \|\varphi\|_3 \quad \forall \psi_h \in S_{h,\varepsilon},$$

and

$$|w_h(\psi_h, \varphi_h - D_h \phi) + h C_4(\varepsilon)(Q_h, \psi_h, a \phi - \alpha \phi'')| \leq C h^2 \|\psi_h\|_0 \|\varphi\|_3 \quad \forall \psi_h \in S_{h,\varepsilon},$$

(77)

(78)

where $a(s) = A_2(s, s)$ and $\alpha$ is the constant in Eq. 9.

**Proof** The first bound is a straightforward consequence of Propositions 9 and 10. The bound (78) can be derived from the first and Eq. 54, although it has already been implicitly given in the proofs above. □

**Proposition 11 (Zero order asymptotics)** The following reduced estimate holds:

$$|w_h(\psi_h, \varphi_h - D_h \phi)| \leq C h \|\psi_h\|_0 \|\varphi\|_2 \quad \forall \psi_h \in S_{h,\varepsilon}.$$  

(79)

**Proof** If we go back to the notation of the proofs of Propositions 9 and 10, it is clear from Eqs. 63, 67, and 76 that

$$|T_2| + |T_3| + |T_5| \leq C h \|\psi_h\|_0 \|\varphi\|_2 \quad \forall \psi_h \in S_{h,\varepsilon}^{-1}.$$  

Using Eq. 54 instead of Lemma 9(a), it is also simple to bound

$$|T_1| \leq C h \|\psi_h\|_0 \|\varphi\|_2 \quad \forall \psi_h \in S_{h,\varepsilon}^{-1}.$$  

For the operator $T_{h^2}u := V u - V Q_{h,1/2} F_h u + h C_1(\cdot/h + 1/2) L_1 F_h u$, we can bound $\|T_{h^2}u\|_0 + h \|T_{h^2}u\|_1 \leq C h^2 \|u\|_1$ [5, Proposition 16]. Using this bound instead of Proposition 7, we can prove that

$$|T_4| \leq C h \|\psi_h\|_0 \|\varphi\|_2 \quad \forall \psi_h \in S_{h,\varepsilon}^{-1}.$$  

This finishes the proof. □
In order to set up clearly the precise formulas of the second term in the asymptotic expansion of \( w_h(\psi_h, \varphi_h - D_h\varphi) \), we need to consider the first two terms in the expansions of \( V_1 \) and \( V_2 \) given by Proposition 1:

\[
V_2 = aHD_{-1} + bHD_{-2} + K_3, \quad K_3 \in E(-3),
\]

\[
V_1 = \alpha HD_{-1} + cHD_{-2} + J_3, \quad J_3 \in E(-3).
\]

**Proposition 12 (Second order asymptotics)** Let \( \varphi_h \) be the solution of Eq. 34 with right-hand side \( g = W\varphi \). Let

\[
P_1^{\varepsilon} := C_1(\varepsilon)\left(\alpha D^2 - a\right),
\]

\[
P_2^{\varepsilon} := C_2(\varepsilon)\left(D(\alpha D - c)D - (aD - b)\right) + \frac{1}{24}(D^3V_1D + V_2D^2).
\]

Then

\[
\left| w_h(\psi_h, \varphi_h - D_h\varphi) - \sum_{\ell=1}^{2} \frac{2}{h^\ell} (Q_{h,\varepsilon} \psi_h, P_\ell^{\varepsilon} \varphi) \right| \leq Ch^3 \| \psi_h \|_0 \| \varphi \|_4 \quad \forall \psi_h \in S_{h,\varepsilon}.
\]

**Proof** See Appendix A. \( \square \)

### 7 Convergence theorems

The results of Sections 5 and 6 give a first simple \( H^0 \) estimate of the convergence of the method, showing that when \( \varepsilon = \pm 1/6 \), the solution superconverges to the projection \( D_h\varphi \). We first recall that [9, Formula (5)]

\[
\| D_h \varphi - \varphi \|_s \leq C h^{r-s} \| \varphi \|_r, \quad s \leq r \leq 1, \quad s < 1/2.
\]

**Theorem 2** Let \( \varphi_h \) be the solution of Eq. 34 with right-hand side \( g = W\varphi \) and \( \varepsilon \notin \frac{1}{2} \mathbb{Z} \). Then

\[
\| \varphi_h - \varphi \|_0 \leq C_\varepsilon h \| \varphi \|_2.
\]

Moreover,

\[
\| \varphi_h - D_h\varphi \|_0 \leq Ch^2 \| \varphi \|_3 \quad \text{if} \ \varepsilon \in \{-1/6, 1/6\}.
\]

**Proof** Using Theorem 1, and Eq. 79, we can prove that

\[
\beta_\varepsilon \| \varphi_h - D_h\varphi \|_0 \leq \sup_{0 \neq \psi_h \in S_{h,\varepsilon}} \frac{|w_h(\psi_h, \varphi_h - D_h\varphi)|}{\| \psi_h \|} \leq Ch \| \varphi \|_2.
\]

Applying Eq. 80, this proves Eq. 81. The superconvergence bound (82) follows from Corollary 1 (note that \( C_1(\pm 1/6) = 0 \)) and Theorem 1. \( \square \)
The superconvergence estimate can be first exploited with a postprocessing of the solution: given $v$ smooth enough we approximate

$$
\int_0^1 \varphi(t)v(t)dt \approx h \sum_{j=0}^{N-1} \varphi_j v(t_j) = (Q_h \varphi_h, v).
$$

This includes the fully discrete double layer potential (85) to approximate (2).

**Corollary 2** Let $\varphi_h$ be the solution of Eq. 34 with right-hand side $g = W\varphi$, and $\varepsilon \in \{-1/6, 1/6\}$. Then

$$
|(Q_h \varphi_h, v) - (\varphi, v)| \leq C h^2 \|\varphi\|_3 \|v\|_2 \quad \forall v \in H^2.
$$

**Proof** Using Lemma 9(a) (with $\varepsilon = 0$), we can easily bound

$$
|(Q_h \varphi_h, v) - (\varphi, v)| \leq |(Q_h \varphi_h - \varphi_h, v)| + |(\varphi_h - D_h \varphi_h, v)| + |(D_h \varphi - \varphi, v)|
$$

$$
\leq C_1 h^2 \|\varphi_h\|_0 \|v\|_2 + \|\varphi_h - D_h \varphi_h\|_0 \|v\|_0 + \|D_h \varphi - \varphi\|_1 \|v\|_1
$$

$$
\leq C_2 h^2 (\|\varphi\|_2 \|v\|_2 + \|\varphi\|_3 \|v\|_0 + \|\varphi\|_1 \|v\|_1),
$$

by Eqs. 81, 82, and 80.

We next introduce the interpolation operator

$$
I_h u := \sum_{j=0}^{N-1} u(t_j) \chi_j.
$$

The Sobolev embedding theorem [27, Lemma 5.3.2] and Proposition 8 show that

$$
\max_j |u(t_j) - (D_h u)(t_j) - h B_1(t_j/h + 1/2)(F_h u')(t_j)| \leq \|E_h u\|_{L^\infty} \leq C \|E_h u\|_1 \leq C h \|u\|_2.
$$

However, $B_1(t_j/h + 1/2) = B_1(1/2) = 0$ and therefore

$$
\|I_h u - D_h u\|_{L^\infty} \leq C \|u\|_2 \quad \text{and} \quad \|D_h u\|_{L^\infty} \leq C \|u\|_2. \quad (83)
$$

**Theorem 3** Let $\varphi_h$ be the solution of Eq. 34 with right-hand side $g = W\varphi$ and $\varepsilon \notin \frac{1}{2} \mathbb{Z}$. Then

$$
\max_j |\varphi_j - \varphi(t_j)| = \|\varphi_h - I_h \varphi\|_{L^\infty} \leq C h \|\varphi\|_3.
$$

**Proof** We rely on the first order asymptotic formula of Corollary 1. Let $C_h$ be the solution operator associated to Eq. 34, namely $C_h \varphi = \varphi_h$. Let $\xi := C_1(\varepsilon) W^{-1}(a \varphi'' - \alpha \varphi)$. Then Eq. 78 shows that

$$
|w_h(\psi_h, \varphi_h - D_h \varphi) - h(Q_{h,\varepsilon} \psi_h, W \xi)| \leq C h^2 \|\psi_h\|_0 \|\varphi\|_3 \quad \forall \psi_h \in S_{h,\varepsilon}^{-1},
$$

which can also be written as

$$
|w_h(\psi_h, C_h \varphi - D_h \varphi - h C_h \xi)| \leq C h^2 \|\psi_h\|_0 \|\varphi\|_3 \quad \forall \psi_h \in S_{h,\varepsilon}^{-1}.
$$
By the inf-sup condition in Theorem 1, it follows that
\[ \| C_h \varphi - D_h \varphi - h C_h \xi \|_0 \leq C h^2 \| \varphi \|_3, \]
and therefore, by Eq. 81 applied to \( \xi \),
\[ \| C_h \varphi - D_h \varphi - h D_h \xi \|_0 \leq h \| C_h \xi - D_h \xi \|_0 + C h^2 \| \varphi \|_3 \leq C h^2 (\| \xi \|_2 + \| \varphi \|_3) \]
\[ \leq C'h^2 \| \varphi \|_3, \tag{84} \]
since \( \| \xi \|_2 \leq C \| \varphi \|_3 \). Note that for piecewise constant functions on a uniform grid of meshsize \( h \) we can estimate \( \| \rho_h \|_{L^\infty} \leq h^{-1/2} \| \rho \|_0 \). Thus,
\[ \| C_h \varphi - I_h \varphi \|_{L^\infty} \leq h^{-1/2} \| C_h \varphi - D_h \varphi - h D_h \xi \|_0 + \| D_h \varphi - I_h \varphi \|_{L^\infty} + h \| D_h \xi \|_{L^\infty} \]
\[ \leq C h^{3/2} \| \varphi \|_3 + C h \| \varphi \|_2 + C h \| \xi \|_2, \]
by Eq. 83. This proves the result. \( \square \)

**Theorem 4** Let \( \varphi_h \) be the solution of Eq. 34 with right-hand side \( g = W \varphi \) and \( \varepsilon \in \{-1/6, 1/6\} \). Then
\[ \max_j | \varphi_j - \varphi(t_j) | = \| \varphi_h - I_h \varphi \|_{L^\infty} \leq C h^2 \| \varphi \|_4. \]

**Proof** The proof of this estimate is very similar to that of Theorem 3. We need to rely on the second order asymptotics of the error (Proposition 12) to reveal the first non-vanishing term in the asymptotic error expansion when \( \varepsilon \in \{-1/6, 1/6\} \). In addition to this, using Proposition 14 and the Sobolev imbedding theorem, it is easy to show that the estimate (83) can be improved to \( \| I_h u - D_h u \|_{L^\infty} \leq C h^2 \| u \|_3 \). An inverse inequality, the stability estimate (Theorem 1) and Theorem 2, can be used to show that
\[ \| \varphi_h - D_h \varphi - h^2 D_h \gamma \|_{L^\infty} \leq C h^{5/2} \| \varphi \|_4, \]
where \( \gamma := W^{-1} P_2^e \varphi \), and \( P_2^e \in E(2) \) is given in Proposition 12. All remaining details are omitted. \( \square \)

**8 Numerical experiments**

We will now illustrate some of the previous convergence estimates with a simple example. We take two ellipses, one centered at \((0, 0)\) with semiaxes 1 and 2, and a second one centered at \((4, 5)\) with semiaxes 2 and 1. We look for solutions of Eq. 3 (radiating solutions of the Helmholtz equation) in the exterior domain that lies outside both ellipses, with Neumann conditions on the boundaries (see Section 2). As exact solution we take \( U(z) := H_0^{(1)}(k|z - z_0|) \), where \( z_0 := (0.1, 0.2) \) is a point inside the first of the obstacles. We have taken \( k = 1 \) in Examples 1 through 4.
**Experiment #1 (indirect method)** After parametrization of the ellipses, a double layer potential (2) is defined on each of the curves. They are then used to set up a $2 \times 2$ system of parametrized boundary integral equations, with diagonal terms of the form (5) and integral operators with smooth kernels as off–diagonal terms. We solve the system and plug the resulting densities in a fully discrete double layer potential:

$$U_h(z) := \frac{i k}{4h} \sum_{j=0}^{N-1} H_1^{(1)}(k|z - x(t_j)|) \frac{(z - x(t_j)) \cdot n(t_j)}{|z - x(t_j)|} \varphi_j.$$  \hspace{1cm} (85)

We compute the errors:

$$e_h := \max_{z \in O} |U(z) - U_h(z)|,$$

where $O := \{(4, 4), (5, 5.5), (6, 7), (7, 7.6), (6.8, 6)\}$, that is, we observe the difference of the exact and discrete solutions at five external points. We expect $e_h = \mathcal{O}(h)$ (this follows from Theorem 2) and $e_h = \mathcal{O}(h^2)$ when $\varepsilon \in \{-1/6, 1/6\}$ (Corollary 2). The results are shown in Table 1. To see how the superconvergent values of $\varepsilon$ are reflected in the error, we plot the error $e_h$ as a function of $\varepsilon$ for a fixed value of $N$ in Fig. 1.

**Experiment #2 (Richardson extrapolation)** With the same geometric configuration, exact solution, and numerical scheme as in the superconvergent case ($\varepsilon = 1/6$), we apply Richardson extrapolation to propose the potential

$$U_h^* := U_{h/2} + \frac{1}{3}(U_{h/2} - U_h),$$

| $N$ | error   | e.c.r  |
|-----|---------|--------|
| 10  | 4.3005e-02 |        |
| 20  | 1.9193e-02 | 1.1640 |
| 40  | 9.0917e-03 | 1.0779 |
| 80  | 4.4279e-03 | 1.0379 |
| 160 | 2.1852e-03 | 1.0189 |
| 320 | 1.0855e-03 | 1.0094 |
| 640 | 5.4097e-04 | 1.0047 |

| $N$ | error   | e.c.r  |
|-----|---------|--------|
| 10  | 9.7262e-03 |        |
| 20  | 2.5602e-03 | 1.8995 |
| 40  | 6.2157e-04 | 2.0645 |
| 80  | 1.5443e-04 | 2.0090 |
| 160 | 3.8588e-05 | 2.0007 |
| 320 | 9.6507e-06 | 1.9995 |
| 640 | 2.4135e-06 | 1.9995 |

The variable $N$ is the number of points on each of the curves. The top table corresponds to $\varepsilon = 1/3$ (order one) and the bottom table to $\varepsilon = 1/6$.  

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Fig. 1 Error as a function of $\varepsilon$ in Experiment #1. The superconvergent methods can be clearly seen as kinks in the error graph (corresponding to the first term in the asymptotic expansion of the error going through a zero). The methods becomes unstable as $\varepsilon \to Z$. Although our analysis does not cover this case, it is clear from the graph that $\varepsilon = 1/2$ continues smoothly the error graph as an improved approximation of the solution. The result of Proposition 12 points clearly to the existence of an asymptotic expansion of the error, very much in the style of those obtained for operator equations of zero or negative order in [5]. The numerical results displayed in Table 2 support our suppositions since clearly $e_h := \max_{z \in \Omega} |U(z) - U_h^*(z)| = \mathcal{O}(h^3)$.

| $N$  | error       | e.c.r  |
|------|-------------|--------|
| 20   | 4.3437e-06  |        |
| 40   | 7.4235e-08  | 5.8707 |
| 80   | 5.6231e-09  | 3.7227 |
| 160  | 6.6107e-10  | 3.0885 |
| 320  | 8.1033e-11  | 3.0282 |
| 640  | 1.0052e-11  | 3.0110 |

Errors are computed in several external observation points. The result reported with $N = 20$ uses a grid of 10 points as $h-$grid and a refined grid of 20 points as $h/2-$grid.
Table 3  Errors and estimated convergence rates for Experiment #3

| N   | boundary error | e.c.r  |
|-----|----------------|--------|
| 10  | 4.8555e-01     |        |
| 20  | 1.3426e-01     | 1.8546 |
| 40  | 5.4891e-02     | 1.2904 |
| 80  | 2.4641e-02     | 1.1555 |
| 160 | 1.1792e-02     | 1.0632 |
| 320 | 5.7677e-03     | 1.0318 |
| 640 | 2.8527e-03     | 1.0157 |

The top table corresponds to $\varepsilon = 1/3$ (order one) and the bottom table to $\varepsilon = 1/6$. The table shows errors $\|\psi_h - I_h\psi\|_{L^\infty}$.

Experiment #3 (direct method)  We now apply a direct boundary integral equation method for the same exterior Neumann problem as in the previous experiments. This leads to a $2 \times 2$ system with the same matrix of operators as in the previous formulation, but the adjoint double layer operator appears in the right-hand side of the system (13). This operator is simply discretized with midpoint formulas on each of the intervals: see [8] for a similar treatment in systems related to the single layer potential. With this formulation, the unknown is the parametrized form of the trace of the exact solution $\psi = U \circ x$ and we can thus compare $L^\infty$ errors (Theorems 3 and 4). We measure maximum absolute value of errors for $\varphi$ on the points $t_j$. The results are reported in Table 3.

Experiment #4 (condition numbers)  In this experiment, we pick the matrix of the previous examples and compute its spectral condition number. We then show how

Table 4  Condition numbers for the matrix W of Experiments #1 and #2 and for the matrix VW, with V given by Eq. 86

| N   | cond VW | cond W  |
|-----|---------|---------|
| 10  | 6.9548  | 5.7212  |
| 20  | 6.5994  | 11.7992 |
| 40  | 6.5349  | 23.7403 |
| 80  | 6.5196  | 47.5489 |
| 160 | 6.5159  | 95.1320 |
| 320 | 6.5150  | 190.2811|
| 640 | 6.5148  | 380.5709|
Table 5  Errors for Double Layer formulation in Experiment #5 for several wave-numbers $k$ and discretization points $N$

| $N \setminus k$ | 4     | 8     | 16    | 32    | 64    | 128   | 256   |
|-----------------|-------|-------|-------|-------|-------|-------|-------|
| 40              | 1.1450e-01 | 1.8237e-01 | 1.8700e-01 | 1.4588e-01 | 6.0793e-02 | 9.3351e-02 | 1.3088e-01 |
| 80              | 3.5120e-02 | 9.4205e-02 | 4.6639e-01 | 2.3802e-01 | 1.2502e-01 | 1.3428e-01 | 1.2381e-01 |
| 160             | 9.4806e-03 | 2.2454e-02 | 3.2514e-02 | 7.7855e-02 | 8.7453e-02 | 3.2517e-02 | 1.3763e-01 |
| 320             | 2.4138e-03 | 4.8222e-03 | 1.5854e-02 | 2.2473e-02 | 4.1011e-02 | 1.0971e-01 | 8.3771e-02 |
| 640             | 6.0619e-04 | 1.1480e-03 | 4.3486e-03 | 8.3657e-03 | 1.2624e-02 | 1.9410e-02 | 7.3018e-02 |
| 1280            | 1.5172e-04 | 2.8294e-04 | 1.0753e-03 | 2.4665e-03 | 3.5540e-03 | 1.3566e-02 | 5.0767e-02 |
| 2560            | 3.7943e-05 | 7.0424e-05 | 2.6822e-04 | 6.2952e-04 | 9.1453e-04 | 6.1168e-03 | 2.6895e-02 |
| 5120            | 9.4867e-06 | 1.7579e-05 | 6.7141e-05 | 1.5764e-04 | 2.3076e-04 | 2.0423e-03 | 1.4318e-02 |
a Calderón preconditioner based on premultiplying the matrix \( W_{ij} \) by the matrix \([5, 8]\)

\[
V_{ij} = H_{0}^{(1)}(k|\mathbf{x}(t_i) - \mathbf{x}(t_{j+\epsilon})|)
\]

reduces the condition number of the resulting linear system to what is basically a constant \( h \)-independent condition number (see Table 4).

**Experiment #5** The aim of this last experiment is to demonstrate the performance of the method as \( k \) becomes large, specifically how \( N \) has to be increased to ensure the convergence of the method. A well established rule of thumb in engineering states that about 10 nodes per wavelength have to be used in order to capture the oscillatory behavior of the solution. To check if our method follows a similar restriction, namely that the number of nodes has to be proportional to \( k \), we consider a new problem with an ellipse of semiaxes 2 and 1 centered at \((0.1, 0.2)\). On the exterior of the ellipse, we solve the Neumann problem for the Helmholtz equation when the exact solution is given by \( U(x) = i/4H_{1}^{0}(k|x|) \). We compute the error of the potentials \(|U - U_h|\) at \((1.1, 4)\) and \((-1, 0)\) as an indicator of the convergence of our method.

First we consider the Double Layer formulation (2–11). Since the perimeter of the ellipse is approximately 9.68, the distance between two consecutive nodes on the curve is about \( 10/N \). On the other hand, the wavelength of the exact solution is, for \( k \) large enough, approximately \( 2\pi/k \). Thus, one can expect that about \( 100k/2\pi \approx 16k \) points should have to be used to enter into the stable convergence zone. The numerical results are presented in Tables 5 and 6. They show that about \( N \approx h^{-1} \approx 40k \) points per wavelength are needed to observe second order convergence. This number is slightly higher than what the rule of thumb would predict, but equally growing linearly with \( k \).

Next, we solve the same problem but using the combined potential (14) instead. In Tables 7 and 8 we show the error as well as the estimated convergence order. Again we observe that taking \( N \) proportional to \( k \) ensures a steady convergence of the method. It is however noticeable that the convergence in this case is more stable and it starts earlier than in the other experiment (the restriction \( N \approx 20k \) is suggested by the numerical results).

| \( k \)       | \( k = 4 \) | \( k = 8 \) | \( k = 16 \) | \( k = 32 \) | \( k = 64 \) | \( k = 128 \) | \( k = 256 \) |
|------------|------|------|------|------|------|------|------|
| 1.7050     | 0.9530 | 1.3185 | −0.7063 | −1.0402 | −0.5245 | 0.0811 |
| 1.8892     | 2.0688 | 3.8424 | 1.6122 | 0.5158 | 2.0460 | −0.1527 |
| 1.9737     | 2.2192 | 1.0362 | 1.7926 | 1.0925 | −1.7544 | 0.7163 |
| 1.9935     | 2.0706 | 1.8662 | 1.4256 | 1.6998 | 2.4988 | 0.1982 |
| 1.9984     | 2.0206 | 2.0158 | 1.7620 | 1.8287 | 0.5168 | 0.5244 |
| 1.9995     | 2.0064 | 2.0033 | 1.9701 | 1.9583 | 1.1491 | 0.9166 |
| 1.9999     | 2.0022 | 1.9982 | 1.9976 | 1.9866 | 1.5826 | 0.9095 |
| $N \backslash k$ | 4 | 8 | 16 | 32 | 64 | 128 | 256 |
|--------------|---|---|----|----|----|-----|-----|
| 40           | 6.2746e-03 | 1.4193e-02 | 2.0585e-01 | 1.1614e-01 | 2.1205e-01 | 9.7028e-02 | 1.2984e-01 |
| 80           | 1.2602e-03 | 1.8432e-03 | 1.7137e-03 | 1.6324e-01 | 5.9806e-02 | 2.5977e-01 | 1.6631e+00 |
| 160          | 3.1311e-04 | 4.6298e-04 | 7.4709e-04 | 2.0872e-03 | 4.1720e-02 | 5.7107e-02 | 9.8900e-02 |
| 320          | 7.7698e-05 | 1.1361e-04 | 2.3348e-04 | 3.7603e-04 | 1.9252e-03 | 3.9573e-02 | 8.7337e-02 |
| 640          | 1.9333e-05 | 2.8005e-05 | 6.3102e-05 | 7.8491e-05 | 3.7306e-04 | 7.1196e-04 | 6.5937e-03 |
| 1280         | 4.8207e-06 | 6.9426e-06 | 1.6281e-05 | 1.9685e-05 | 8.8021e-05 | 1.5008e-04 | 7.1460e-04 |
| 2560         | 1.2035e-06 | 1.7278e-06 | 4.1280e-06 | 4.9382e-06 | 2.1621e-05 | 3.6105e-05 | 1.0749e-04 |
| 5120         | 3.0067e-07 | 4.3092e-07 | 1.0389e-06 | 1.2372e-06 | 5.3660e-06 | 8.9494e-06 | 2.5859e-05 |
Table 8 Estimated convergence rate for the errors presented in Table 7

| $k$ | $k = 4$ | $k = 4$ | $k = 16$ | $k = 32$ | $k = 64$ | $k = 128$ | $k = 256$ |
|-----|---------|---------|---------|---------|---------|---------|---------|
|     | 2.3159  | 2.9449  | 6.9083  | -0.4911 | 1.8260  | -1.4208 | -3.6791 |
| 2.0089 | 1.9932  | 1.1978  | 6.2893  | 0.5196  | 2.1855  | 4.0718  |         |
| 2.0107 | 2.0269  | 1.6780  | 2.4726  | 4.4377  | 0.5292  | 0.1794  |         |
| 2.0068 | 2.0203  | 1.8875  | 2.2602  | 2.3675  | 5.7966  | 3.7274  |         |
| 2.0038 | 2.0121  | 1.9545  | 1.9954  | 2.0835  | 2.2461  | 3.2059  |         |
| 2.0020 | 2.0065  | 1.9797  | 1.9950  | 2.0254  | 2.0555  | 2.7329  |         |
| 2.0010 | 2.0034  | 1.9904  | 1.9969  | 2.0105  | 2.0123  | 2.0555  |         |

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Appendix A: Second order asymptotics

This section contains the proof of Proposition 12. Note that this result is required for the proof of $L^\infty$ convergence of the superconvergent methods. In order to prove Proposition 12 we have to go one term further in the different asymptotic expansions that were used in the proofs of Propositions 9 and 10.

A. 1 Technical background

Lemma 11 There exists $C$ such that for all $\varepsilon$ and $h$

$$\left| \{Q_{h,\varepsilon} \psi_h, u\} - (\psi_h, u) + \frac{h^2}{24} \{Q_{h,\varepsilon} \psi_h, u''\} \right| \leq Ch^3 \|\psi_h\|_0 \|u\|_3 \quad \forall \psi_h \in S_{h,\varepsilon}, u \in H^3.$$

Proof It is based on the same ideas as the proof of Lemma 9, using the inequality

$$\left| \int_{c-\frac{h}{2}}^{c+\frac{h}{2}} u(t) dt - hu(c) - \frac{h^3}{24} u''(c) \right| \leq Ch^3 \int_{c-\frac{h}{2}}^{c+\frac{h}{2}} |u^{(3)}(t)| dt,$$

as starting point. \qed

Proposition 13 Let $a_1, a_2, a_3 \in D$ and consider an operator

$$V := a_1 HD_{-1} + a_2 HD_{-2} + a_3 HD_{-3} + K_4 \quad \text{where } K_4 \in \mathcal{E}(-4).$$
Let then $L_1 := a_1$, $L_2 := a_1D - a_2$, $L_3 := a_1D^2 - 2a_2D + a_3$, and consider the operators

$$R_h^# u := Vu - VQ_hF_hu + \sum_{\ell=1}^{2} h^\ell C_\ell (\cdot / h)L_\ell F_hu,$$

$$T_h^# u := Vu - VQ_{h,1/2}F_hu + \sum_{\ell=1}^{3} h^\ell C_\ell (\cdot / h + 1/2)L_\ell F_hu.$$ 

Then

$$\|R_h^# u\|_0 + h\|R_h^# u\|_1 \leq Ch^3\|u\|_2 \quad \forall u \in H^2, \quad (87)$$

$$\|T_h^# u\|_0 + h\|T_h^# u\|_1 \leq Ch^4\|u\|_3 \quad \forall u \in H^3. \quad (88)$$

**Proof** It is a direct consequence of [5, Proposition 16].

**Proposition 14** Let $E_h^# u := u - D_hu + \sum_{\ell=1}^{2} h^\ell B_\ell (\cdot / h + 1/2)F_hu^{(\ell)}$. Then

$$\|E_h^# u\|_0 + h\|E_h^# u\|_1 \leq Ch^3\|u\|_3 \quad \forall u \in H^3. \quad (89)$$

**Proof** See [9, Proposition 1] and the proof of [5, Proposition 16].

8.1 Proof of Proposition 12

Following Eq. 61 and 68, we consider the decomposition of the consistency error in five terms

$$w_h(\psi_h, \varphi_h - D_h \varphi) = w_h(\psi_h, \varphi_h - \varphi) + w_h(\psi_h, \varphi - D_h \varphi) = (T_1 + T_2 + T_3) + (T_4 + T_5). \quad (89)$$

To bound $T_1$ we use Lemma 11 with $u = DV_1D$:

$$|T_1 - \frac{h^2}{24}\{Q_{h,\varepsilon}\psi_h, D^3V_1D\varphi\} \leq Ch^3\|\psi_h\|_0\|\varphi\|_4. \quad (90)$$

Proceeding as in Eq. 63 we can bound

$$|T_2| \leq Ch^3\|\psi_h\|_0\|\varphi\|_3. \quad (91)$$

To expand $T_3$ we use Proposition 13 applied to $V_2$ and Eq. 56 to obtain

$$T_3 = -hC_1(\varepsilon)(Q_{h,\varepsilon}\psi_h, a\varphi) - h^2C_5(\varepsilon)(Q_{h,\varepsilon}\psi_h, (aD - b)\varphi)$$

$$+ hC_1(\varepsilon)(Q_{h,\varepsilon}\psi_h, (a\varphi - F_h\varphi)) + h^2C_5(\varepsilon)(Q_{h,\varepsilon}\psi_h, (aD - b)(\varphi - F_h\varphi))$$

$$+ (Q_{h,\varepsilon}\psi_h, R_h^{\#}\varphi).$$

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We now use Lemma 9(b) and Eq. 56 to bound $|T_{3a}| + |T_{3b}| \leq C h^3 \| \psi_h \|_0 \| \varphi \|_2$, as well as Lemma 9(b) and Eq. 87 to bound $|T_{3c}| \leq C h^3 \| \psi_h \|_0 \| \varphi \|_2$. Therefore

$$|T_3 + hC_1(\varepsilon)(Q_{h,\varepsilon} \psi_h, a\varphi) + h^2 C_2(\varepsilon)(Q_{h,\varepsilon} \psi_h, (aD - b)\varphi)| \leq C h^3 \| \psi_h \|_0 \| \varphi \|_2.$$  \hfill (92)

To expand $T_4$ we use Proposition 13 applied to $V_1$. Note that $L_1 = \alpha I$, $L_2 = \alpha D - c$ and $L_3 \in E(2)$. Because of Eq. 69, we can write

$$T_4 = \sum_{\ell = 1}^{2} h^\ell C_\ell(\varepsilon)(Q_{h,\varepsilon} \psi_h, DL_\ell D \varphi)$$

$$+ \sum_{\ell = 1}^{2} C_\ell(\varepsilon) \left( h^\ell (\psi_h - Q_{h,\varepsilon} \psi_h, DL_\ell D \varphi) + h^\ell (\psi_h', DL_\ell (\varphi' - F_h \varphi')) \right)$$

$$\quad - h^3 C_3(\varepsilon)(\psi_h', L_3 F_h \varphi') + (\psi_h', T_h \# \varphi').$$

Using Lemma 9(a) and Eq. 54 we can bound

$$|T_{4a}^1| + |T_{4a}^2| \leq C h^3 \| \psi_h \|_0 (\| DL_1 D \varphi \|_2 + \| DL_2 D \varphi \|_1) \leq C' h^3 \| \psi_h \|_0 \| \varphi \|_4.$$

By Eq. 56 (and using the commutation $DF_h = F_h D$ to simplify some expressions) we next bound

$$|T_{4b}^1| + |T_{4b}^2| \leq \| \psi_h \|_0 \left( h \| DL_1 D (\varphi - F_h \varphi) \|_0 + h^2 \| DL_2 D (\varphi - F_h \varphi) \|_0 \right)$$

$$\leq C \| \psi_h \|_0 \left( h \| \varphi - F_h \varphi \|_2 + h^2 \| \varphi - F_h \varphi \|_3 \right) \leq C' h^3 \| \psi_h \|_0 \| \varphi \|_4.$$

Similarly

$$|T_{4c}| \leq h^3 |C_3(\varepsilon)| \| \psi_h \|_0 \| DL_3 DF_h \varphi \|_0 \leq C h^3 \| \psi_h \|_0 \| \varphi \|_4.$$

Finally, by Eq. 88

$$|T_{4d}| \leq \| \psi_h \|_0 \| DT_h^{\#} \varphi' \|_0 \leq C h^3 \| \psi_h \|_0 \| \varphi \|_4.$$

Collecting all these bounds we have just proved that

$$|T_4 - \sum_{\ell = 1}^{2} h^\ell C_\ell(\varepsilon)(Q_{h,\varepsilon} \psi_h, DL_\ell D \varphi)| \leq C h^3 \| \psi_h \|_0 \| \varphi \|_4.$$ \hfill (93)

We are only left to deal with $T_5$. Using Proposition 14, the argument in Eq. 75, and the fact that $B_2(1/2) = -1/24$, we can write

$$Q_h(\varphi - D_h \varphi) = -h B_1(1/2) Q_h F_h \varphi' - h^2 B_2(1/2) Q_h F_h \varphi'' + Q_h E_h^{\#}$$

$$= \frac{h^2}{24} Q_h F_h \varphi'' + Q_h E_h^{\#} \varphi.$$
Therefore,

\[
T_5 = \frac{h^2}{24} \left[ Q_{h,\epsilon} \psi_h, V_2 Q_h F_h \varphi'' \right] + \left[ Q_{h,\epsilon} \psi_h, V_2 Q_h E_h \varphi \right]
\]

\[
= \frac{h^2}{24} \left( Q_{h,\epsilon} \psi_h, V_2 \varphi'' \right)
\]

\[
+ \frac{h^3}{24} C_1(e) \left( Q_{h,\epsilon} \psi_h, a F_h \varphi'' \right) - \frac{h^2}{24} \left( Q_{h,\epsilon} \psi_h, R_h \varphi'' \right) + \left[ Q_{h,\epsilon} \psi_h, V_2 Q_h E_h \varphi \right],
\]

where we have applied Proposition 7. By Lemma 9(b) and Eq. 59 we can bound \( |T_{5a}| + |T_{5b}| \leq C h^3 \| \psi_h \|_0 \| \varphi \|_3 \), while by Lemma 10 and Proposition 14, we can bound \( |T_{5c}| \leq C h^3 \| \psi_h \|_0 \| \varphi \|_3 \). Therefore

\[
|T_5 - \frac{h^2}{24} \left( Q_{h,\epsilon} \psi_h, V_2 \varphi'' \right) | \leq C h^3 \| \psi_h \|_0 \| \varphi \|_3.
\]

The result is the combination of Eqs. 89–94.

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