Stability of a Volterra Integral Equation on Time Scales

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Abstract

In this paper, we study Hyers-Ulam stability for integral equation of Volterra type in time scale setting. Moreover we study the stability of the considered equation in Hyers-Ulam-Rassias sense. Our technique depends on successive approximation method, and we use time scale variant of induction principle to show that (1.1) is stable on unbounded domains in Hyers-Ulam-Rassias sense.

1 Introduction

In 1940, S. M. Ulam gave a wide range of talks at the Mathematics Club of the University of Wisconsin, in which he discussed a number of important unsolved problems. One of them was the following question:

Let $G_1$ be a group and let $G_2$ be a group endowed with a metric $d$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta,$$

for all $x, y \in G_1$, can we find a homomorphism $\theta : G_1 \to G_2$ such that

$$d(h(x), \theta(x)) < \epsilon,$$

for all $x \in G_1$?

This problem was solved by Hyers for approximately additive mappings on Banach spaces [3]. Rassias generalized, in his work [11], the result obtained by Hyers. Since then the stability of many functional, differential, integral equations have been investigated, see [4], [7], [8], and references there in.

In this paper we shall consider the non-homogeneous volterra integral equation of the first kind

$$x(t) = f(t) + \int_a^t k(t,s)x(s)\Delta s, \quad t \in I_T := [a, b]_T,$$

(1.1)
where \( f \in C_{rd}(I_T, \mathbb{R}) \), \( k \in C_{rd}(I_T \times I_T, \mathbb{R}) \) and \( x \) is the unknown function.

First, we introduce the basic definitions that will be used throughout this paper.

**Definition 1.1.** The integral equation (1.1) is said to has Hyers-Ulam stability on \( I_T \) if for any \( \varepsilon > 0 \) and each \( \psi \in C_{rd}(I_T, \mathbb{R}) \) satisfying

\[
|\psi(t) - f(t) - \int_a^t k(t,s)\psi(s)\Delta s| < \varepsilon, \quad \forall \ t \in I_T;
\]

then there exists a solution \( \varphi \) of equation (1.1) and a constant \( C \geq 0 \) such that

\[
|\varphi(t) - \psi(t)| \leq C \varepsilon, \quad \forall \ t \in I_T.
\]

The constant \( C \) is called Hyers-Ulam stability constant for equation (1.1).

**Definition 1.2.** The integral equation (1.1) is said to has Hyers-Ulam-Rassias stability, with respect to \( \omega \), on \( I_T \) if for each \( \psi \in C_{rd}(I_T, \mathbb{R}) \) satisfying

\[
|\psi(t) - f(t) - \int_a^t k(t,s)\psi(s)\Delta s| < \omega(t), \quad \forall \ t \in I_T;
\]

for some fixed \( \omega \in C_{rd}(I_T, [0, \infty)) \), then there exists a solution \( \varphi \) of equation (1.1) and a constant \( C > 0 \) such that

\[
|\varphi(t) - \psi(t)| \leq C \omega(t), \quad \forall \ t \in I_T.
\]

we shall investigate Hyers-Ulam stability and Hyers-Ulam-Rassias stability of integral equation (1.1) on both bounded and unbounded time scales intervals.

### 2 Hyers-Ulam stability

In this section we investigate Hyers-Ulam stability of equation on \( I_T := [a, b]_T \) by using iterative technique.

**Theorem 2.1.** The integral equation (1.1) has Hyers-Ulam stability on \( I_T := [a, b]_T \).

**Proof.** For given \( \varepsilon > 0 \) and each \( \psi \in C_{rd}(I_T, \mathbb{R}) \) satisfying

\[
|\psi(t) - f(t) - \int_a^t k(t,s)\psi(s)\Delta s| < \varepsilon, \quad \forall \ t \in I_T,
\]

we consider the recurrence relation

\[
\psi_n(t) := f(t) + \int_a^t k(t,s)\psi_{n-1}(s)\Delta s, \quad n = 1, 2, 3, \ldots \quad (2.1)
\]
for $t \in I_T$ with $\psi_0(t) = \psi(t)$. We prove that $\{\psi_n(t)\}_{n \in \mathbb{N}}$ converges uniformly to the unique solution of Equation (1.1) on $I_T$. We write $\psi_n(t)$ as a telescoping sum

$$\psi_n(t) = \psi_0(t) + \sum_{i=1}^{n} [\psi_i(t) - \psi_{i-1}(t)],$$

so

$$\lim_{n \to \infty} \psi_n(t) = \psi_0(t) + \sum_{i=1}^{\infty} [\psi_i(t) - \psi_{i-1}(t)], \quad \forall \ t \in I_T. \quad (2.2)$$

Using mathematical induction we prove the following estimate

$$|\psi_i(t) - \psi_{i-1}(t)| \leq \varepsilon M^{i-1} \frac{(t-a)^{i-1}}{(i-1)!}, \quad \forall \ t \in I_T. \quad (2.3)$$

For $i = 1$ we have

$$|\psi_1(t) - \psi(t)| < \varepsilon.$$ 

So the estimate (2.3) holds for $i = 1$. Assume that the estimate (2.3) is true for $i = n \geq 1$. We have

$$|\psi_{n+1}(t) - \psi_n(t)| \leq \int_{a}^{t} |k(t, s)||\psi_n(s) - \psi_{n-1}(s)| \Delta s \leq M \int_{a}^{t} \varepsilon M^{n-1} \frac{(s-a)^{n-1}}{(n-1)!} ds \leq \varepsilon M^n \frac{(t-a)^n}{n!},$$

hence the estimate (2.3) is valid for $i = n + 1$. This shows that the estimate (2.3) is true for all $i \geq 1$ on $I_T$.

See that

$$|\psi_i(t) - \psi_{i-1}(t)| \leq \varepsilon M^{i-1} \frac{(t-a)^{i-1}}{(i-1)!} \leq \varepsilon M^{i-1} \frac{(b-a)^{i-1}}{(i-1)!},$$

and

$$\sum_{i=1}^{\infty} \varepsilon M^{i-1} \frac{(b-a)^{i-1}}{(i-1)!} = \sum_{i=0}^{\infty} \varepsilon \frac{[(M(b-a))]^i}{i!} = \varepsilon e^{M(b-a)}.$$ 

Applying Weierstrass M-Test, we conclude that the infinite series

$$\sum_{i=1}^{\infty} [\psi_i(t) - \psi_{i-1}(t)]$$

converges uniformly on $t \in I_T$. Thus from (2.2), the sequence $\{\psi_n(t)\}_{n \in \mathbb{N}}$ converges uniformly on $I_T$ to some $\varphi(t) \in C_{r,d}(I_T, \mathbb{R})$. Next, we show that the limit
of the sequence \( \varphi(t) \) is the exact solution of (1.1). For all \( t \in I_T \) and each \( n \geq 1 \), we have

\[
\left| \int_a^t k(t,s)\psi_n(s) - \int_a^t k(t,s)\varphi(s) \Delta s \right| \leq M \int_a^t |\psi_n(s) - \varphi(s)| \Delta s.
\]

Taking the limits as \( n \to \infty \) we see that the right hand side of the above inequality tends to zero and so

\[
\lim_{n \to \infty} \int_a^t k(t,s)\psi_n(s) \Delta s = \int_a^t k(t,s)\varphi(s) \Delta s, \quad \forall t \in I_T.
\]

By letting \( n \to \infty \) on both sides of (2.4), we conclude that \( \varphi(t) \) is the exact solution of (1.1) on \( I_T \). Then there exists a number \( N \) such that \( |\psi_N(t) - \varphi(t)| \leq \varepsilon \). Thus

\[
|\psi - \varphi| \leq |\psi(t) - \psi_N(t)| + |\psi_N(t) - \varphi(t)| \leq |\psi(t) - \psi_1(t)| + |\psi_1(t) - \psi_2(t)| + \cdots + |\psi_{n-1}(t) - \psi_N(t)| + |\psi_N(t) - \varphi(t)|
\]

\[
\leq \sum_{i=1}^N |\psi_{i-1}(t) - \psi_i(t)| + |\psi_N(t) - \varphi(t)|
\]

\[
\leq \sum_{i=1}^N \varepsilon M^{i-1} \frac{(b-a)^{i-1}}{(i-1)!} + |\psi_N(t) - \varphi(t)|
\]

\[
\leq \varepsilon e^{M(b-a)} + \varepsilon = \varepsilon (1 + e^{M(b-a)}) \varepsilon \leq C \varepsilon.
\]

which completes the proof. \( \square \)

**Remark 2.2.** We can find an estimate on the difference of two approximate solutions of the integral equation (1.1). Let \( \psi_1 \) and \( \psi_2 \) are two different approximate solutions to (1.1) that is for some \( \varepsilon_1, \varepsilon_2 > 0 \), and for all \( t \in I_T \)

\[
\left| \psi_1(t) - f(t) - \int_a^t k(t,s)\psi_1(s) \Delta s \right| \leq \varepsilon_1,
\]

and

\[
\left| \psi_2(t) - f(t) - \int_a^t k(t,s)\psi_2(s) \Delta s \right| \leq \varepsilon_2.
\]

So

\[
|\psi_1(t) - \psi_2(t)| \leq (\varepsilon_1 + \varepsilon_2) e_M(t,a), \quad \forall t \in I_T.
\]

If \( \psi_1 \) is an exact solution of equation (1.1), then we have \( \varepsilon_1 = 0 \).

**Proof.** Adding the two inequalities (2.4), (2.5) and making use of \(|\alpha| - |\beta| \leq |\alpha - \beta| \leq |\alpha| + |\beta|\), we get

\[
\left| \psi_1(t) - \psi_2(t) - \int_a^t k(t,s)[\psi_1(s) - \psi_2(s)] \Delta s \right| \leq \varepsilon_1 + \varepsilon_2.
\]
for all $t \in I_T$ where $\varepsilon := \varepsilon_1 + \varepsilon_2$.

Put

$$\xi(t) := |\psi_1(t) - \psi_2(t)|, \quad \forall \ t \in I_T,$$

then

$$\xi(t) \leq \varepsilon + \int_a^t |k(t,s)\xi(s)\Delta s| \leq \varepsilon + \int_a^t |k(t,s)|\xi(s)\Delta s \leq \varepsilon + \int_a^t |k(t,s)|\xi(s)\Delta s,$$

where we make an application of Grönwall’s inequality in the last step. By Theorem we have

$$\int_a^t \frac{M}{e_M(\sigma(s),a)} \Delta s = - \int_a^t \left[ \frac{1}{e_M(\sigma(s),a)} \right] \Delta s = \left( 1 - \frac{1}{e_M(t,a)} \right),$$

thus

$$\xi(t) \leq \varepsilon + \varepsilon \left| e_M(t,a) - 1 \right| = \varepsilon e_M(t,a), \quad \forall \ t \in I_T.$$
Proof. Consider the following iterative scheme

\[ \psi_n(t) := f(t) + \int_a^t k(t, s)\psi_{n-1}(s)\Delta s, \quad n = 1, 2, 3, \ldots \quad (3.1) \]

for \( t \in I_T \) with \( \psi_0(t) = \psi(t) \). By mathematical induction, it is easy to see that the following estimate

\[ |\psi_n(t) - \psi_{n-1}(t)| \leq MP^{n-1}\omega(t), \quad (3.2) \]

holds for each \( n \in \mathbb{N} \) and all \( t \in I_T \). By the same argument as in Theorem 2.1 we prove that the sequence \( \psi_n(t) \) converges uniformly on \( I_T \) to the unique solution, \( \varphi \), of the integral equation (1.1). Then there exists a positive integer \( N \) such that \( |\psi_N(t) - \varphi(t)| \leq w(t), \; t \in I_T \). Hence

\[
|\psi - \varphi| \leq |\psi(t) - \psi_N(t)| + |\psi_N(t) - \varphi(t)|
\leq |\psi(t) - \psi_1(t)| + |\psi_1(t) - \psi_2(t)| + \cdots + |\psi_{n-1}(t) - \psi_N(t)| + |\psi_N(t) - \varphi(t)|
\leq \sum_{k=1}^N |\psi_{k-1}(t) - \psi_k(t)| + |\psi_N(t) - \varphi(t)|
\leq \sum_{k=1}^N MP^{k-1}\omega(t) + |\psi_N(t) - \varphi(t)|
\leq \sum_{k=1}^N MP^{k-1}\omega(t) + \omega(t)
\leq \sum_{k=1}^\infty MP^{k-1}\omega(t) + \omega(t)
\leq M \cdot \frac{1}{1 - P}\omega(t) + \omega(t) = \left(1 + \frac{M}{1 - P}\right)\omega(t),
\]

which shows that (1.1) has Hyers-Ulam-Rassias stability on \( I_T \).

\[ \Box \]

**Theorem 3.2.** Assume that for a family of statements \( A(t), \; t \in [t_0, \infty)_T \) the following conditions holds

1. \( A(t_0) \) is true.

2. for each right-scattered \( t \in [t_0, \infty)_T \) we have \( A(t) \Rightarrow A(\sigma(t)) \).

3. for each right-dense \( t \in [t_0, \infty)_T \) there is a neighborhood \( U \) such that \( A(t) \Rightarrow A(s) \) for all \( s \in U, \; s > t \).

4. for each left-dense \( t \in [t_0, \infty)_T \) one has \( A(s) \) for all \( s \) with \( s < t \Rightarrow A(t) \).

Then \( A(t) \) is true for all \( t \in [t_0, \infty)_T \).

Next, we prove that the integral equation (1.1) has Hyers-Ulam-Rassias on unbounded domains.
Theorem 3.3. Consider the integral equation (1.1) with \( I_T := [a, \infty)_\tau \). Let \( f \in C_{rd}([a, \infty)_\tau, \mathbb{R}) \) and \( k(t, \cdot) \in C_{rd}([a, \infty)_\tau, \mathbb{R}) \) for some fixed \( t \in [a, \infty)_\tau \). Assume \( \psi \in C_{rd}(I_T, \mathbb{R}) \) satisfying

\[
\left| \psi(t) - f(t) - \int_a^t k(t, s) \psi(s) \Delta s \right| < \omega(t), \quad t \in I_T; \tag{3.3}
\]

where \( \omega \in C_{rd}([a, \infty)_\tau, \mathbb{R}_+) \) with the property

\[
\int_a^t \omega(\tau) \Delta \tau \leq \lambda \omega(t), \quad \forall \, t \in [a, \infty)_\tau. \tag{3.4}
\]

for \( \lambda \in (0, 1) \). Then the integral equation (1.1) has Hyers-Ulam-Rassias stability, with respect to \( \omega \), on \([a, \infty)_\tau \).

Proof. We apply the time scale mathematical induction in \([a, \infty)_\tau \) on the following statements

I. \( A(a) \) is trivially true.

II. Let \( r \) be a right scattered point and that \( A(r) \) holds. That means equation (1.1) has Hyers-Ulam-Rassias stability, with respect to \( \omega \), on \([a, r)_\tau \), i.e. for each \( \psi : [a, r)_\tau \rightarrow \mathbb{R} \) satisfying

\[
\left| \psi(t) - f(t) - \int_a^t k(t, s) \psi(s) \Delta s \right| < \omega(t), \quad t \in [a, r)_\tau;
\]

where \( \omega \in C_{rd}([a, r)_\tau, \mathbb{R}_+) \), then there exist a unique solution to equation (1.1) \( \phi_r : [a, r)_\tau \rightarrow \mathbb{R} \) such that

\[
|\phi_r(t) - \psi(t)| \leq C_1 \omega(t), \quad t \in [a, r)_\tau.
\]

We want to prove that \( A(\sigma(r)) \) is true. Assume that the function \( \psi \) satisfies

\[
\left| \psi(t) - f(t) - \int_r^{\sigma(r)} k(t, s) \psi(s) \Delta s \right| < \omega(t), \quad t \in [r, \sigma(r)]_\tau.
\]

Define the mapping \( \varphi_{\sigma(r)} : [a, \sigma(r)]_\tau \rightarrow \mathbb{R} \) such that

\[
\varphi_{\sigma(r)}(t) = \begin{cases} 
\varphi_r(t), & t \in [a, r)_\tau; \\
f(\sigma(r)) + \mu(r)k(\sigma(r), r)\varphi_r(r), & t = \sigma(r).
\end{cases}
\]

It is clear that \( \varphi_{\sigma(r)} \) is a solution of (1.1) on \([a, \sigma(r)]_\tau \). Moreover, on we have

\[
|\varphi_{\sigma(r)}(t) - \psi(t)| = \begin{cases} 
|\varphi_r(t) - \psi(t)|, & t \in [a, r)_\tau; \\
|f(\sigma(r)) + \mu(r)k(\sigma(r), r)\varphi_r(r) - \psi(\sigma(r))|, & t = \sigma(r).
\end{cases}
\]

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See that
\[
|\varphi_{\sigma(r)}(\sigma(r)) - \psi(\sigma(r))| = |f(\sigma(r)) + \mu(r)k(\sigma(r), r)\varphi_r(r) - \mu(r)k(\sigma(r), r)\psi(r) + \mu(r)k(\sigma(r), r)\psi(r) - \psi(\sigma(r))| \\
\leq |f(\sigma(r)) + \mu(r)k(\sigma(r), r)\psi(r) - \psi(\sigma(r))| + |\mu(r)k(\sigma(r), r)||\varphi_r(r) - \psi(r)| \\
\leq \omega(\sigma(r)) + MC_1 \mu(r)\omega(r).
\]

So we have
\[
|\varphi_{\sigma(r)}(t) - \psi(t)| \leq \begin{cases} 
C_1\omega(t), & t \in [a, r]_{\mathbb{T}}; \\
\omega(\sigma(r)) + MC_1 \mu(r)\omega(r), & t = \sigma(r).
\end{cases}
\]

**III.** Let \( r \in [a, \infty)_{\mathbb{T}} \) be right-dense and \( U_r \) be a neighborhood of \( r \). Assume \( A(r) \) is true, i.e. for each \( \psi : [a, r]_{\mathbb{T}} \to \mathbb{R} \) satisfying
\[
|\psi(t) - f(t) - \int_a^t k(t, s)\psi(s)\Delta s| < \omega(t), \quad \text{for } t \in [a, r]_{\mathbb{T}},
\]
where \( \omega \in C_{rd}([a, r]_{\mathbb{T}}, \mathbb{R}_+) \), then there exist a unique solution to equation (1.1) \( \varphi_r : [a, r]_{\mathbb{T}} \to \mathbb{R} \) such that
\[
|\varphi_r(t) - \psi(t)| \leq C_1 \omega(t), \quad \text{for } t \in [a, r]_{\mathbb{T}}.
\]

We show that \( A(\tau) \) is true for all \( \tau \in U_r \cap (r, \infty)_{\mathbb{T}} \). For \( \tau > r \) assume that the function \( \psi \) satisfies
\[
|\psi(t) - f(t) - \int_r^\tau k(t, s)\psi(s)\Delta s| < \omega(t), \quad \text{for } t \in [r, \tau]_{\mathbb{T}}.
\]

By Theorem for each \( \tau \in U_r, \tau > r \), the integral equation
\[
x(t) = f(t) + \int_r^\tau k(t, s)x(s)\Delta s, \quad \text{for } t \in [r, \tau]_{\mathbb{T}},
\]
has exactly one solution \( \varphi_\tau(\cdot) \). Therefore the mapping \( \xi_\tau : [a, \tau]_{\mathbb{T}} \to \mathbb{R} \) defined by
\[
\xi_s(t) = \begin{cases} 
\varphi_r(t), & t \in [a, r]_{\mathbb{T}}; \\
\varphi_\tau(t), & t \in [r, \tau]_{\mathbb{T}}.
\end{cases}
\]
is a solution of the integral equation
\[
x(t) = f(t) + \int_a^\tau k(t, s)x(s)\Delta s, \quad \text{for } t \in [a, \tau]_{\mathbb{T}}.
\]

We have
\[
|\xi_s(t) - \psi(t)| = \begin{cases} 
|\varphi_r(t) - \psi(t)|, & t \in [a, r]_{\mathbb{T}}; \\
|\varphi_\tau(t) - \psi(t)|, & t \in [r, s]_{\mathbb{T}}.
\end{cases}
\]
For \( t \in [r, s]_\mathbb{T} \), see that
\[
|\varphi_s(t) - \psi(t)| = \left| f(t) + \int_r^t k(t, \tau)\varphi_s(\tau) \Delta \tau - \psi(t) + \int_r^t k(t, \tau)\psi(\tau) \Delta \tau \right|
\leq |f(t) + \int_r^t k(t, \tau)\psi(\tau) \Delta \tau - \psi(t)| + \int_r^t |k(t, \tau)||\varphi_s(\tau) - \psi(\tau)| \Delta \tau \\
\leq C_1 \omega(t) + M \int_r^t \omega(\tau) \Delta \tau \\
\leq C_1 \omega(t) + MP\omega(t) = (C_1 + MP)\omega(t).
\]

**IV.** Let \( r \in (a, \infty)_\mathbb{T} \) be left-dense such that \( A(s) \) is true for all \( s < r \). We prove that \( A(r) \) by the same argument as in (III). By the induction principle the statement \( A(t) \) holds for all \( t \in [a, \infty)_\mathbb{T} \), that means the integral equation \((1.1)\) has Hyers Ulam Rassias stability on \( t \in [a, \infty)_\mathbb{T} \).

Now we give an example to show that Hyers Ulam stability of volterra Integral equation \((1.1)\) not necessarily holds on unbounded interval for general time scale.

**Example 3.4.** The integral dynamic equation
\[
x(t) = 1 + 5 \int_0^t x(s) \Delta s, \quad t \in [0, \infty)_\mathbb{T},
\]
has exactly one solution \( x(t) = e_5(t, 0) \), also we have \( x(t) = 0 \) as approximate solution. From Bernoulli’s inequality \([?]\), we have
\[
e_5(t, 0) \geq 1 + 5(t - 0),
\]
then we get
\[
\sup_{t \in [0, \infty)} |e_5(t, 0) - 0| \geq \sup_{t \in [0, \infty)} (1 + 5t) = \infty.
\]

Hence, there is no Hyers Ulam stability constant.

**References**

[1] C. C. Tisdell, A. H. Zaidi, *Successive approximations to solutions of dynamic equations on time scales*, 16(1), 2009, 61–87.

[2] V. Lakshmikantham, S. Sivasundaram, B. Kaymakalan, *Dynamic Equation on Measure Chains*, Springer, 1996.

[3] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA 27(4), (1941), 222.
[4] M. Gachpazan, O. Baghani, *Hyers-Ulam stability of Volterra integral equation*, Int. J. Nonlinear Anal. Appl. 1(2), (2010), 1925.

[5] C. Corduneanu, *Integral equations and applications*, Cambridge: Cambridge University Press, 1991.

[6] L. P. Castro, A. Ramos, *Hyers-Ulam-Rassias stability for a class of nonlinear Volterra integral equations*, Banach J. Math. Anal., 3(1), 2009, 36-43.

[7] S. András, A. R. Mészáros, *Ulam-Hyers stability of dynamic equations on time scales via Picard operators*, Appl. Math. Comput., 219(2013), 4853 - 4864.

[8] S. M. Jung, *A fixed point approach to the stability of a Volterra integral equation*, Fixed Point Theory Appl., 2007, Article ID 57064, 9 pages.

[9] D. R. Anderson, *Hyers-Ulam stability of second-order linear dynamic equations on time scales*, arXiv preprint, arXiv:1008.3726, (2010).

[10] E. Akin-Bohner, M. Bohner, F. Akin, *Pachpatte inequalities on time scales*, Journal of Inequalities in Pure and Applied Mathematics, 6(1), 1-23.

[11] T. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proceedings of the American Mathematical Society, 72(2)(1978), 297300.