SELECTION PROPERTIES OF THE SPLIT INTERVAL AND THE CONTINUUM HYPOTHESIS

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Abstract. We prove that every usco multimap \( \Phi : X \to Y \) from a metrizable separable space \( X \) to a GO-space \( Y \) has an \( F_\sigma \)-measurable selection. On the other hand, for the split interval \( \mathbb{I} \) and the projection \( P : \mathbb{I}^2 \to \mathbb{I}^2 \) of its square onto the unit square \( \mathbb{I}^2 \), the usco multimap \( P^{-1} : \mathbb{I}^2 \to \mathbb{I}^2 \) has a Borel (\( F_\sigma \)-measurable) selection if and only if the Continuum Hypothesis holds. This CH-example shows that known results on Borel selections of usco maps into fragmentable compact spaces cannot be extended to a wider class of compact spaces.

1. Introduction

By a multimap \( \Phi : X \to Y \) between topological spaces \( X, Y \) we understand any subset \( \Phi \subseteq X \times Y \), which can be thought as a function assigning to every point \( x \in X \) the subset \( \Phi(x) := \{ y \in Y : (x, y) \in \Phi \} \) of \( Y \). For a subset \( A \subseteq X \) we put \( \Phi[A] = \bigcup_{x \in A} \Phi(x) \). Each function \( f : X \to Y \) can be thought as a single-valued multimap \( \{(x, f(x)) : x \in X\} \subseteq X \times Y \).

For a multimap \( \Phi : X \to Y \), its inverse multimap \( \Phi^{-1} : Y \to X \) is defined by \( \Phi^{-1} := \{(y, x) : (x, y) \in \Phi\} \).

A multimap \( \Phi : X \to Y \) is called

- lower semicontinuous if for any open set \( U \subseteq Y \) the set \( \Phi^{-1}[U] \) is open in \( X \);
- upper semicontinuous if for any closed set \( F \subseteq Y \) the set \( \Phi^{-1}[F] \) is closed in \( X \);
- Borel-measurable if for any Borel set \( B \subseteq Y \) the set \( \Phi^{-1}[B] \) is Borel in \( X \);
- compact-valued if for every \( x \in X \) the subspace \( \Phi(x) \) of \( Y \) is compact and non-empty;
- usco if \( \Phi \) is upper semicontinuous and compact-valued.

It is well-known that for any surjective continuous function \( f : X \to Y \) between compact Hausdorff spaces, the inverse multimap \( f^{-1} : Y \to X \) is usco.

Let \( \Phi : X \to Y \) be a multimap between topological spaces. A function \( f : X \to Y \) is called a selection of \( \Phi \) if \( f(x) \in \Phi(x) \) for every \( x \in X \). The Axiom of Choice ensures that every multimap \( \Phi : X \to Y \) with non-empty values has a selection. The problem is to find selections possessing some additional properties like the continuity or measurability.

One of classical results in this direction is the following theorem of Kuratowski and Ryll-Nardzewski [11] (see also [15 §5.2] or [13 6.12]).

Theorem 1. Let \( X, Y \) be Polish spaces. Any Borel-measurable multimap \( \Phi : X \to Y \) with non-empty values has a Borel-measurable selection.

We recall that a function \( f : X \to Y \) between topological spaces is called Borel-measurable (resp. \( F_\sigma \)-measurable) if for every open set \( U \subseteq Y \) the preimage \( f^{-1}[U] \) is Borel (or type \( F_\sigma \)) in \( X \).

\( F_\sigma \)-Measurable selections of usco multimap with values in non-metrizable compact spaces were studied by many mathematicians [3, 4, 5, 7, 8, 6]. Positive results are known for two classes of compact spaces: fragmentable and linearly ordered.

Let us recall [2 5.0.1] (see also [14 §6]) that a topological space \( K \) is fragmentable if \( K \) has a metric \( \rho \) such that for every \( \varepsilon > 0 \) each non-empty subset \( A \subseteq K \) contains a non-empty relatively

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open set $U \subset A$ of $\rho$-diameter $< \varepsilon$. By [2, 5.1.12], each fragmentable compact Hausdorff space contains a metrizable dense $G_\delta$-subspace.

The following selection theorem can be deduced from Theorem 1' and Lemma 6 in [3].

**Theorem 2** (Hansell, Jayne, Talagrand). Any usco map $\Phi : X \to K$ from a perfectly paracompact space $X$ to a fragmentable compact space $Y$ has an $F_\sigma$-measurable selection.

A similar selection theorem holds for usco maps into countably cellular GO-spaces. A Hausdorff topological space $X$ is called a generalized ordered space (briefly, a GO-space) if $X$ admits a linear order $\leq$ such that the topology of $X$ is generated by a base consisting of open order-convex subsets of $X$. A subset $C$ of a linearly ordered space $X$ is called order-convex if for any points $x \leq y$ in $C$ the open interval $[x, y] := \{z \in X : x \leq z \leq y\}$ is contained in $X$. We say that the topology of $X$ is generated by the linear order $\leq$ if the topology of $X$ is generated by the subbase $\{(\langle a, b \rangle, a, b \rangle) : a, b \in X\}$ consisting the order-convex sets $\langle a, b \rangle := \{x \in X : x < a\}$ and $(a, b) = \{x \in X : a < x\}$.

A topological space $X$ is countably cellular if every disjoint family of open sets in $X$ is at most countable. It is easy to see that each separable topological space is countably cellular. A topological space is called $F_\sigma$-perfect if every open set in $X$ is of type $F_\sigma$ in $X$ (i.e., can be represented as the countable union of closed sets). For example, every metrizable space is $F_\sigma$-perfect.

The following selection theorem will be proved in Section 2.

**Theorem 3.** Let $Y$ be a GO-space and $X$ be an ($F_\sigma$-perfect) topological space. If $X$ or $Y$ is countably cellular, then any usco map $\Phi : X \to Y$ has a Borel ($F_\sigma$-measurable) selection.

Theorem 2, 3 suggest the following problem.

**Problem 1.** Is it true that any usco map $\Phi : M \to K$ from a compact metrizable space $M$ to a compact Hausdorff space $K$ has a Borel ($F_\sigma$-measurable) selection?

In this paper we prove that this problem has negative answer under the negation of the Continuum Hypothesis (i.e., under $\omega_1 < \omega$). A suitable counterexample will be constructed using the split square $\mathbb{I}^2$, which is the square of the split interval $\mathbb{I}$.

The split interval is the linearly ordered space $\mathbb{I} = [0, 1] \times \{0, 1\}$ whose topology is generated by the lexicographic order (defined by $\langle x, i \rangle \leq \langle y, j \rangle$ if either $x < y$ or else $x = y$ and $i < j$). The split interval plays a fundamental role in the theory of separable Rosenthal compacta [14]. Let us recall that a topological space is called Rosenthal compact if it is homeomorphic to a compact subspace of the space $B_1(\mathcal{P})$ of functions of the first Baire class on a Polish space $\mathcal{P}$. It is well-known (and easy to see) that the split interval is Rosenthal compact and so is its square. By Theorem 4 of Todorčević [16], each non-metrizable Rosenthal compact space of countable spread contains a topological copy of the split interval. A topological space has countable spread if it contains no uncountable discrete subspaces.

By Theorem 3, any usco map $\Phi : X \to \mathbb{I}^2$ from an $F_\sigma$-perfect topological space $X$ has an $F_\sigma$-measurable selection. In contrast, the split square $\mathbb{I}^2$ has dramatically different selections properties. Let $p : \mathbb{I} \to \mathbb{I}$, $p : \langle x, i \rangle \mapsto x$, be the natural projection of the split interval onto the unit interval $\mathbb{I} = [0, 1]$, and

$$P : \mathbb{I}^2 \to \mathbb{I}^2, \quad P : \langle x, y \rangle \mapsto \langle p(x), p(y) \rangle,$$

be the projection of the split square $\mathbb{I}^2$ onto the unit square $\mathbb{I}^2$.

**Theorem 4.** The following conditions are equivalent:

1. the usco multimap $P^{-1} : \mathbb{I}^2 \to \mathbb{I}^2$ has a Borel-measurable selection;
2. the usco multimap $P^{-1} : \mathbb{I}^2 \to \mathbb{I}^2$ has an $F_\sigma$-measurable selection;
3. $\omega_1 = \omega$.

The implication (2) $\Rightarrow$ (1) of Theorem 4 is trivial and the implications (1) $\Rightarrow$ (3) $\Rightarrow$ (2) are proved in Lemmas 2 and 8 respectively.

Combining Theorem 4 with the Todorčević dichotomy for Rosenthal compact spaces, we obtain the following consistent characterization of metrizable compacta.
Corollary 1. Under $\omega_1 < \omega$ a Rosenthal compact space $K$ is metrizable if and only if $K$ has countable spread and each usco multimap $\Phi : I^2 \to K^2$ has a Borel-measurable selection.

Proof. The “only if” part follows from Theorem 2. To prove the “if” part, assume that a Rosenthal compact $K$ is not metrizable but has countable spread. By Theorem 4 of [10], the space $K$ contains a topological copy of the split interval $I$. We lose no generality assuming that $I \subset K$. By Theorem 2 under $\omega_1 < \omega$, the usco multimap $P^{-1} : I^2 \to I^2 \subset K^2$ does not have Borel-measurable selections. \qed

Now we pose some open problems suggested by Theorem 4.

Problem 2. Assume CH. Is it true that each usco map $\Phi : X \to I^2$ from a metrizable (separable) space $X$ has a Borel-measurable selection?

Observe that the map $p : I \to I$ is 2-to-1 and its square $P : I^2 \to I^2$ is 4-to-1. A function $f : X \to Y$ is called $n$-to-1 for some $n \in \mathbb{N}$ if $|f^{-1}(y)| \leq n$ for any $y \in Y$. By Theorem 3 of Todorčević [11], every Rosenthal compact space of countable spread admits a 2-to-1 map onto a metrizable compact space. Let us observe that the splitted square $I^2$ contains a discrete subspace of cardinality continuum and hence has uncountable spread.

Problem 3. Let $n \in \{2, 3\}$. Is there an $n$-to-1 map $f : K \to M$ from a (Rosenthal) compact space $K$ to a metrizable compact space $M$ such that the inverse multimap $f^{-1} : M \to K$ has no Borel selections?

Remark 1. Theorem 4 provides a consistent counterexample to the problem [9] of Chris Heunen, posed on Mathoverflow.

2. Proof of Theorem 3

Theorem 3 follows from Lemmas 2 and 3 proved in this section.

First we prove one lemma, showing that our definition of a GO-space agrees with the original definition of Lutzer [12]. Probably this lemma is known but we could not find the precise reference in the literature.

Lemma 1. The linear order $\leq$ of any GO-space $X$ is a closed subset of the square $X \times X$.

Proof. Given two elements $x, y \in X$ with $x \leq y$, use the Hausdorff property of $X$ and find two disjoint order-convex neighborhoods $O_x, O_y \subset X$ of the points $x, y$, respectively. We claim that the product $O_x \times O_y$ is disjoint with the linear order $\leq$. Assuming that this is not true, find elements $x' \in O_x$, $y' \in O_y$ such that $x' \leq y'$. Taking into account that the sets $O_x, O_y$ are disjoint and order-convex, we conclude that $x' < y$ and $x < y'$. It follows from $x \leq y$ that $y < x$. Then $x' < y < x < y'$, which contradicts the assumption. This contradiction shows that the neighborhood $O_x \times O_y$ of the pair $(x, y)$ is disjoint with $\leq$ and hence $\leq$ is a closed subset of $X \times X$. \qed

Lemma 2. Any usco multimap $\Phi : X \to Y$ from an ($F_\sigma$-perfect) topological space $X$ to a countably cellular GO-space space $Y$ has a Borel ($F_\sigma$-measurable) selection.

Proof. Being a GO-space, $Y$ has a base of the topology consisting of open order-convex subsets with respect to some linear order $\leq$ on $Y$. By Lemma 1 the linear order $\leq$ is a closed subset of $Y \times Y$. Then for every $a \in Y$ the order-convex set $(\leftarrow, a] = \{y \in Y : y \leq a\}$ is closed in $Y$, which implies that each non-empty compact subset of $Y$ has the smallest element.

Then for any usco multimap $\Phi : X \to Y$ we can define a selection $f : X \to Y$ of $\Phi$ assigning to each point $x \in X$ the smallest element $f(x)$ of the non-empty compact set $\Phi(x) \subset Y$. We claim that this selection is $F_\sigma$-measurable.

A subset $U \subset Y$ is called upper if for any $u \in U$ the order-convex set $[u, \to) = \{y \in Y : u \leq y\}$ is contained in $U$.

Claim 1. For any upper open set $U \subset Y$ the preimage $f^{-1}[U]$ is open in $X$. 

Claim 2. For any closed lower set $L \subset Y$ the preimage $f^{-1}[L]$ is closed in $X$.

Proof. Observe the the complement $X \setminus L$ is an open upper set in $Y$. By Claim 1 the preimage $f^{-1}[X \setminus L]$ is open in $X$ and hence its complement $X \setminus f^{-1}[X \setminus L] = f^{-1}[L]$ is closed in $X$. □

Claim 3. For any lower set $L \subset Y$ the preimage $f^{-1}(L)$ is of type $F_\sigma$ in $X$.

Proof. If $L$ has the largest element $\lambda$, then $L = (-, \lambda]$ and $f^{-1}(L) = f^{-1}((-\lambda, \lambda])$ is closed by Claim 2. So, we assume that $L$ does not have the largest element. Then the countable cellularity of $Y$ implies that $L$ has a countable cofinal subset $C \subset L$ (which means that for every $x \in L$ there exists $y \in C$ with $x \leq y$). By Lemma 3 for every $c \in C$ the preimage $f^{-1}((-, c])$ is closed in $X$. Since $L = \bigcup_{c \in C}(-, c]$, the preimage $f^{-1}[L] = \bigcup_{c \in C}f^{-1}((-, c])$ is of type $F_\sigma$ in $X$. □

Claim 4. For any open order-convex subset $U \subset Y$ the preimage $f^{-1}[U]$ is a Borel subset of $X$ (of type $F_\sigma$ if the space $X$ is $F_\sigma$-perfect).

Proof. The order-convexity of $U$ implies that $U = \widehat{U} \cap \overrightarrow{U}$ where $\widehat{U} = \bigcup_{u \in U}(-, u]$ and $\overrightarrow{U} = \bigcup_{u \in U}[u, \rightarrow)$. Taking into account that $Y$ has a base of order-convex sets, one can show that the upper set $\overrightarrow{U}$ is open in $X$. By Claim 1 the preimage $f^{-1}[\overrightarrow{U}]$ is open in $X$ (of type $F_\sigma$ if the space $X$ is $F_\sigma$-perfect). By Claim 3 the preimage $f^{-1}[\widehat{U}]$ is of type $F_\sigma$ in $X$. Then $f^{-1}(U) = f^{-1}(\overrightarrow{U}) \cap f^{-1}[\widehat{U}]$ is Borel (of type $F_\sigma$ if $X$ is $F_\sigma$-perfect). □

Claim 5. For every open set $U \subset Y$ the preimage $f^{-1}[B]$ is Borel subset of $X$ (of type $F_\sigma$ if $X$ is $F_\sigma$-perfect).

Proof. By the definition of the topology of $Y$, each point $x \in U$ has an open order-convex neighborhood $O_x \subset U$. By the Kuratowski-Zorn Lemma, each open order-convex subset of $U$ is contained in a maximal open order convex subset of $U$. Let $C \subset B$ be the family of maximal open order-convex subsets of $U$. Observe that $U = \bigcup C$ and any distinct sets $C, D \in C$ are disjoint: otherwise the union $C \cup D$ would be an open order convex subset of $U$ and by the maximality of $C$ and $D$, $C = C \cup D = D$. Since the space $Y$ is countably cellular, the family $C$ is at most countable. By Claim 3 for every $C \in C$ the preimage $f^{-1}(C)$ is Borel (an type $F_\sigma$-set if $X$ is $F_\sigma$-perfect) and so is the countable union $f^{-1}[U] = \bigcup_{C \in C}f^{-1}_C$. □

Claim 5 completes the proof of Lemma 2.

Lemma 3. Every usco multimap $\Phi : X \rightharpoonup Y$ from a countably cellular ($F_\sigma$-perfect) topological space $X$ into a GO-space $Y$ has a Borel ($F_\sigma$-measurable) selection.

Proof. The Kuratowski-Zorn Lemma implies that the usco map $\Phi$ contains a minimal usco map $\Psi : X \rightharpoonup Y$. We claim that the image $\Psi[X] \subset Y$ is a countably cellular subspace of $Y$. Assuming the opposite, we can find an uncountable disjoint family $(U_\alpha)_{\alpha \in \omega_1}$ of non-empty open subsets in $\Psi[X]$. For every $\alpha \in \omega_1$, find $x_\alpha \in X$ such that $\Phi(x_\alpha) \cap U_\alpha \neq \emptyset$. By Lemma 3.1.2 [2], the minimality of the usco map $\Psi$ implies that $\Psi[V_\alpha] \subset U_\alpha$ for some non-empty open set $V_\alpha \subset X$. Taking into account that the family $(U_\alpha)_{\alpha \in \omega_1}$ is disjoint, we conclude that the family $(V_\alpha)_{\alpha \in \omega_1}$ is disjoint, witnessing that the space $X$ is not countably cellular. But this contradicts our assumption. This contradiction shows that the GO-subspace $\Psi[X]$ of $Y$ is countably cellular. By Lemma 2 the usco map $\Phi : X \rightharpoonup \Psi[X]$ has a Borel ($F_\sigma$-measurable) selection, which is also a selection of the usco map $\Phi$. □

Finally, let us prove one selection property of the split interval, which will be used in the proof of Lemma 8.
Lemma 4. Any selection of the multimap \( p^{-1} : \mathbb{I} \to \mathbb{I} \) is \( F_\sigma \)-measurable.

Proof. Given any open subset \( U \subset \mathbb{I} \), we need to show that \( s^{-1}[U] \) is of type \( F_\sigma \) in \( \mathbb{I} \). For every \( x \in s^{-1}[U] \), find an open order-convex set \( I_x \subset U \) containing \( s(x) \). It is well-known (see e.g. \( \mathbf{[I]} \) 3.10.C(a)] that the split interval \( \mathbb{I} \) is hereditarily Lindelöf. Consequently, there exists a countable set \( C \subset s^{-1}[U] \) such that \( \bigcup_{x \in C} I_x = \bigcup_{x \in C} s^{-1}[I_x] \) and hence \( s^{-1}[U] = \bigcup_{x \in C} s^{-1}[I_x] \). For every \( x \in C \) the order-convexity of the interval \( I_x \subset \mathbb{I} \) implies that its preimage \( s^{-1}[I_x] \) is a convex subset of \( \mathbb{I} \), containing \( x \). Since convex subsets of \( \mathbb{I} \) are of type \( F_\sigma \), the countable union \( s^{-1}[U] = \bigcup_{x \in C} s^{-1}[I_x] \) is an \( F_\sigma \)-set in \( \mathbb{I} \). \( \square \)

3. Selection properties of the split square \( \mathbb{I}^2 \) under the negation of \( \text{CH} \)

In this section we study the selection properties of the split square \( \mathbb{I}^2 \) under the negation of the Continuum Hypothesis.

By \( \langle x, y \rangle \) we denote ordered pairs of elements \( x, y \). In this way we distinguish ordered pairs from the order intervals \( \langle x, y \rangle := \{ z \in \mathbb{I} : x < z < y \} \) in linearly ordered spaces.

The split interval \( \mathbb{I} = \mathbb{I} \times \{0, 1\} \) carries the lexicographic order defined by \( \langle x, i \rangle \leq \langle y, j \rangle \) iff either \( x < y \) or \( x = y \) and \( i < j \). It is well-known that the topology generated by the lexicographic order on \( \mathbb{I} \) is compact and Hausdorff, see \( \mathbf{[I]} \) 3.10.C(b)]. By \( p : \mathbb{I} \to \mathbb{I} \), \( p : (x, i) \mapsto x \), we denote the coordinate projection and by \( P : \mathbb{I}^2 \to \mathbb{I}^2 \), \( P : (x, y) \mapsto (p(x), p(y)) \) the square of the map \( p \).

The following lemma proves the implication \((1) \Rightarrow (3)\) of Theorem \( \mathbf{[I]} \).

Lemma 5. If \( \omega_1 < \mathfrak{c} \), then the multimap \( p^{-1} : \mathbb{I}^2 \to \mathbb{I}^2 \) has no Borel selections.

Proof. To derive a contradiction, assume that the multimap \( p^{-1} \) has a Borel-measurable selection \( s : \mathbb{I}^2 \to \mathbb{I}^2 \).

For a real number \( x \in \mathbb{I} \) by \( x_0 \) and \( x_1 \) we denote the points \( \langle x, 0 \rangle \) and \( \langle x, 1 \rangle \) of the split interval \( \mathbb{I} \). Then \( \mathbb{I} = \mathbb{I}_0 \cup \mathbb{I}_1 \) where \( \mathbb{I}_i = \{ x_i : x_i \in \mathbb{I} \} \) for \( i \in \{0, 1\} \).

For any numbers \( i, j \in \{0, 1\} \) consider the set
\[
\mathbb{I}_{ij} = \{ z \in \mathbb{I}^2 : s(z) \in \mathbb{I}_i \times \mathbb{I}_j \}
\]
and observe that \( \mathbb{I}^2 = \bigcup_{i,j=0}^1 \mathbb{I}_{ij} \).

For a point \( a \in \mathbb{I} \), let \( [0, a) \) and \( (a, 1] \) be the order intervals in \( \mathbb{I} \) with respect to the lexicographic order. Observe that for any \( x \in \mathbb{I} \) we have
\[
p([0, x)) = [0, x), \quad p([0, x_1)) = [0, x], \quad p((x_0, 1]) = [x, 1], \quad p((x_1, 1]) = (x, 1].
\]

For every \( a \in (0, 2) \subset \mathbb{R} \) consider the lines
\[
L_a = \{ \langle x, y \rangle \in \mathbb{R}^2 : x + y = a \} \quad \text{and} \quad \Gamma^a = \{ \langle x, y \rangle \in \mathbb{R}^2 : y - x = a \}
\]
on the plane.

Claim 6. For every \( a \in \mathbb{R} \) the intersection \( L_a \cap Z_{00} \) is at most countable.

Proof. If for some \( a \in \mathbb{R} \) the intersection \( L_a \cap Z_{00} \) is uncountable, then we can choose a non-Borel subset \( B \subset L_a \cap Z_{00} \) of cardinality \( |B| = \omega_1 \). For every point \( \langle x, y \rangle \in B \subset Z_{00} \), the definition of the set \( Z_{00} \) ensures that \( s(\langle x, y \rangle) = \langle x_0, y_0 \rangle \) and hence the set \( U_{(x,y)} = \{ x_0, x_1 \} \times \{ 0, 1 \}, \quad U_{(x_0, y_1)} = \{ 0, x_0 \} \times \{ 0, y_0 \} \) is an open neighborhood of \( s(\langle x, y \rangle) \) in \( \mathbb{I}^2 \). Observe that \( \langle x, y \rangle \in s^{-1}(U_{(x,y)}) \subset p(U_{(x,y)}) = \{ x, y \} \times \{ 0, y \} \) and hence \( L_a \cap s^{-1}(U_{(x,y)}) = \{ \langle x, y \rangle \} \). Then for the open set \( U = \bigcup_{(x,y) \in B} U_{(x,y)} \) the preimage \( s^{-1}[U] \) is not Borel in \( \mathbb{I}^2 \) because the intersection \( s^{-1}[U] \cap L_a = B \) is not Borel. But this contradicts the Borel measurability of \( s \). \( \square \)

By analogy we can prove the following claims.

Claim 7. For every \( a \in \mathbb{R} \) the intersection \( L_a \cap Z_{11} \) is at most countable.

Claim 8. For every \( b \in \mathbb{R} \) the intersection \( \Gamma^b \cap (Z_{01} \cup Z_{10}) \) is at most countable.
Now fix any subset set $\Omega \subseteq [\frac{1}{2}, \frac{3}{2}]$ of cardinality $|\Omega| = \omega_1$. By Claims 6, 7, for every $a \in \Omega$ the intersection $L_a \cap (Z_{00} \cup Z_{11})$ is at most countable. Consequently the union

$$U = \bigcup_{a \in \Omega} L_a \cap (Z_{00} \cup Z_{11})$$

has cardinality $|U| \leq \omega_1$. Since $|U| \leq \omega_1 < \omega$, there exists a real number $b \in [\frac{1}{2}, \frac{3}{2}]$ such that the line $\Gamma^b$ does not intersect the set $U$. Since $\{b\} \cup \Omega \subseteq [\frac{1}{2}, \frac{3}{2}]$ for every $a \in \Omega$ the intersection $\Gamma^b \cap L_a \cap U^2$ is not empty. Then the set $X = \bigcup_{a \in \Omega} L_a \cap \Gamma^b \subseteq U^2$ is uncountable and $X \subset \Gamma^b \setminus U \subset \Gamma^b \cap (Z_{01} \cup Z_{10})$. But this contradicts Claim 8.

4. Selection properties of the split square $U^2$ under the Continuum Hypothesis

In this section we shall prove that under the continuum hypothesis the usco multifunction $P^{-1} : U^2 \to U^2$ has an $F_\sigma$-measurable selection.

First we introduce some terminology related to monotone functions.

A subset $f \subseteq U^2$ is called a

1. a function if for any $\langle x_1, y_1\rangle, \langle x_2, y_2\rangle \in f$ the equality $x_1 = x_2$ implies $y_1 = y_2$;
2. strictly increasing if for any $\langle x_1, y_1\rangle, \langle x_2, y_2\rangle \in f$ the strict inequality $x_1 < x_2$ implies $y_1 < y_2$;
3. strictly decreasing if for any $\langle x_1, y_1\rangle, \langle x_2, y_2\rangle \in f$ the inequality $x_1 < x_2$ implies $y_1 > y_2$;
4. strictly monotone if $f$ is strictly increasing or strictly decreasing.

Lemma 6. Each strictly increasing function $f \subseteq U^2$ is a subset of a Borel strictly increasing function $\bar{f} \subseteq U^2$.

Proof. It follows that the strictly increasing function $f$ is a strictly increasing bijective function between the sets $pr_1[f] = \{x \in U : \exists y \in U \langle x, y \rangle \in f\}$ and $pr_2[f] = \{y \in U : \exists x \in U \langle x, y \rangle \in f\}$. It is well-known that monotone functions of one real variable have at most countably many discontinuity points. Consequently, the sets of discontinuity points of the strictly monotone functions $f$ and $f^{-1}$ are at most countable. This allows us to find a countable set $D_f \subset f$ such that the set $f \setminus D_f$ coincides with the graph of some increasing homeomorphism between subsets of $U$. Replacing $D_f$ by a larger countable set, we can assume that $D_f = f \cap (pr_1[D_f] \times pr_2[D_f])$, where $pr_1, pr_2 : U^2 \to U$ are coordinate projections. By the Lavrentiev Theorem [10, 3.9], the homeomorphism $f \setminus D_f$ extends to a (strictly increasing) homeomorphism $h \subseteq U^2$ between $G_\delta$-subsets of $U^2$ such that $f \setminus D_f$ is dense in $h$. It is easy to check that the Borel subset $\bar{f} = (h \setminus (pr_1[D_f] \times pr_2[D_f])) \cup D_f$ is a strictly increasing function extending $f$.

By analogy we can prove

Lemma 7. Each strictly decreasing function $f \subseteq U^2$ is a subset of a Borel strictly decreasing function $\bar{f} \subseteq U^2$.

Now we are ready to prove the main result of this section.

Lemma 8. Under $\omega_1 = \omega$ the multifunction $P^{-1} : U^2 \to U^2$ has an $F_\sigma$-measurable selection.

Proof. Let $M$ be the set of all infinite strictly monotone Borel functions $f \subseteq U^2$. Since $\omega_1 = \omega$, the set $M$ can be written as $M = \{f_\alpha\}_{\alpha < \omega_1}$. It is clear $\bigcup_{\alpha < \omega_1} f_\alpha = U^2$. So, for any point $z \in U^2$ we can find the smallest ordinal $\alpha_z < \omega_1$ such that $z \in f_{\alpha_z}$. Consider the sets

$$L := \{z \in U^2 : f_{\alpha_z} \text{ is strictly increasing}\} \quad \text{and} \quad \Gamma := \{z \in U^2 : f_{\alpha_z} \text{ is strictly decreasing}\} = U^2 \setminus L.$$

Define a selection $s : U^2 \to U^2$ of the multifunction $P^{-1} : U^2 \to U^2$ letting

$$s(\langle x, y \rangle) = \begin{cases} \langle x_1, y_1 \rangle & \text{if } \langle x, y \rangle \in L, \\ \langle x_1, y_0 \rangle & \text{if } \langle x, y \rangle \in \Gamma, \end{cases}$$

for $\langle x, y \rangle \in U^2$. 


We claim that the function \( s : \mathbb{I}^2 \to \mathbb{I}^2 \) is \( F_\sigma \)-measurable. Given any open set \( U \subset \mathbb{I}^2 \), we should prove that its preimage \( s^{-1}[U] \) of type \( F_\sigma \) in \( \mathbb{I}^2 \). Consider the open subset \( V := U \cap (0,1)^2 \subset \mathbb{I}^2 \) of \( U \). Using Lemma 4, it can be shown that the set \( s^{-1}[U \setminus V] \subset \mathbb{I}^2 \setminus (0,1)^2 \) is of type \( F_\sigma \) in \( \mathbb{I}^2 \).

Therefore, it remains to show that the preimage \( s^{-1}[V] \) is of type \( F_\sigma \) in \( \mathbb{I}^2 \).

Let \( \mathbb{Q} := \{ \frac{n}{m} : n, m \in \mathbb{N}, n < m \} \) be the set of rational numbers in the interval \((0, 1)\).

Consider the subsets \( L_V := L \cap s^{-1}(V) \) and \( \Gamma_V := \Gamma \cap s^{-1}(V) \). For every \( (x, y) \in L_V \) we have \( s((x, y)) = (x_1, y_1) \in V \) and by the definition of the topology of the split interval, we can find rational numbers \( a(x, y), b(x, y) \in \mathbb{Q} \) such that \( x < a(x, y), y < b(x, y) \) and \( s((x, y)) = (x_1, y_1) \in [x_1, a(x, y)_0) \times [y_1, b(x, y)_0) \subset V \). Then

\[
[x, a(x, y)) \times [b(x, y), y) = s^{-1}[[x_1, a(x, y)_0) \times [y_1, b(x, y)_0) \subset s^{-1}[V].
\]

On the other hand, for every \( (x, y) \in \Gamma_V \) there are rational numbers \( a(x, y), b(x, y) \in \mathbb{Q} \) such that \( x < a(x, y), b(x, y) < y \) and \( s((x, y)) = (x_1, y_0) = [x_1, a(x, y)_0) \times \{b(x, y)_0\} \subset V \). In this case

\[
[x, a(x, y)) \times (b(x, y), y) = s^{-1}[[x_1, a(x, y)_0) \times \{b(x, y)_0\}) \subset s^{-1}[V].
\]

It follows that

\[
s^{-1}[V] = \left( \bigcup_{(x, y) \in L_V} [x, a(x, y)) \times [y, b(x, y)) \right) \cup \left( \bigcup_{(x, y) \in \Gamma_V} [x, a(x, y)) \times (b(x, y), y) \right).
\]

This equality and the following claim imply that the set \( s^{-1}[V] \) is of type \( F_\sigma \) in \( \mathbb{I}^2 \).

**Claim 9.** There are countable subsets \( L' \subset L_V \) and \( \Gamma' \subset \Gamma_V \) such that

\[
\bigcup_{(x, y) \in L_V} [x, a(x, y)) \times [y, b(x, y)) = \bigcup_{(x, y) \in L'} [x, a(x, y)) \times [y, b(x, y))
\]

and

\[
\bigcup_{(x, y) \in \Gamma_V} [x, a(x, y)) \times (b(x, y), y) = \bigcup_{(x, y) \in \Gamma'} [x, a(x, y)) \times (b(x, y), y).
\]

We shall show how to find the countable set \( L' \subset L_V \). The countable set \( \Gamma' \subset \Gamma_V \) can be found by analogy.

For rational numbers \( r, q \in \mathbb{Q} \), consider the set

\[
L_{r,q} = \{ (x, y) \in L_V : a(x, y) = r, b(x, y) = q \}
\]

and observe that \( L_V = \bigcup_{r,q \in \mathbb{Q}} L_{r,q} \).

**Claim 10.** For any rational numbers \( r, q \in \mathbb{Q} \) there exists a countable subset \( L'_{r,q} \subset L_{r,q} \) such that

\[
\bigcup_{(x, y) \in L_{r,q}} [x, r) \times [y, q) = \bigcup_{(x, y) \in L'_{r,q}} [x, r) \times [y, q).
\]

**Proof.** For every rational numbers \( r' \leq r \) and \( q' \leq q \), consider the numbers

\[
y(r') := \inf\{ y : (x, y) \in L_{r,q}, x < r' \}
\]

and \( y(q') := \inf\{ x : (x, y) \in L_{r,q}, y < q' \} \).

Choose countable subsets \( L'_{r,q} \subset L_{r,q} \) such that

\[
y(r') = \inf\{ y : (x, y) \in L'_{r,q}, x < r' \}
\]

and \( y(q') = \inf\{ x : (x, y) \in L'_{r,q}, y < q' \} \)

and

\[
y(q') = \min\{ y : (x, y) \in L_{r,q}, x < r' \}
\]

and \( y(q') = \min\{ x : (x, y) \in L_{r,q}, y < q' \} \).

Consider the countable subset

\[
L''_{r,q} := \bigcup\{ L'_{r,q} \cup L_{r,q} : r', q' \in \mathbb{Q}, r' < r, q' < q \}
\]

of \( L_{r,q} \).
Claim 11. \( \bigcup_{(x, y) \in L_{r, q}} ([x, r) \times [y, q)) \setminus \{ (x, y) \} \subset \bigcup_{(x, y) \in L'_{r, q}} [x, r) \times [y, q). \)

**Proof.** Fix any pairs \((x, y) \in L_{r, q}\) and \((x', y') \in ([x, r) \times [y, q)) \setminus \{ (x, y) \}.\) Three cases are possible:

1. \(x < x' < r\) and \(y < y' < q;\)
2. \(x = x'\) and \(y < y' < q;\)
3. \(x < x' < r\) and \(y = y'.\)

In the first case there exist rational numbers \(r', q'\) such that \(x < r' < x < r\) and \(y < q' < y' < q.\)

The definition of \(x(q')\) ensures that \(x(q') \leq x < x'.\) By the choice of the family \(L_{r, q}^{0};\) there exists \((x'', y'') \in L_{r, q}^{0} \subset L_{r, q}\) such that \(x'' < x' < r\) and \(y'' < q' < y' < q.\) Then \((x', y') \in [x'', r) \times [y'', q).\)

Next, assume that \(x = x'\) and \(y < y' < q.\) In this case we can choose a rational number \(q'\) such that \(y < q' < y'.\) It follows that \(x(q') = x = x'.\) If \(x(q') < x',\) then by the definition of the family \(L_{r, q}^{0};\) there exists \((x'', y'') \in L_{r, q}^{0} \subset L_{r, q}\) such that \(x'' < x' < r\) and \(y'' < q' < y' < q.\) Then \((x', y') \in [x'', r) \times [y'', q).\)

By analogy, in the third case \((x < x' < r\) and \(y = y')\) we can find a pair \((x'', y'') \in L_{r, q}^{0}\) such that \((x', y') \in [x'', r) \times [y'', q).\)

Claim 11 implies that the set

\[ D_{r, q} = \left( \bigcup_{(x, y) \in L_{r, q}} [x, r) \times [y, q) \right) \setminus \left( \bigcup_{(x, y) \in L'_{r, q}} [x, r) \times [y, q) \right) \]

is contained in \(L_{r, q}.\)

**Claim 12.** The set \(D_{r, q}\) is a strictly decreasing function.

**Proof.** First we show that \(D_{r, q}\) is a function. Assuming that \(D_{r, q}\) is not a function, we can find two pairs \((x, y), (x', y') \in D_{r, q}\) with \(y < y'.\) Applying Claim 11 we conclude that

\[ (x, y') \in ([x, r) \times [y, q)) \setminus \{ (x, y) \} \subset \bigcup_{(x''', y''')} [x''', r) \times [y''', q) \]

and hence \((x, y') \notin D_{r, q},\) which contradicts the choice of the pair \((x, y').\) This contradiction shows that \(D_{r, q}\) is a function.

Assuming that \(D_{r, q}\) is not strictly decreasing, we can find pairs \((x, y), (x', y') \in D_{r, q}\) such that \(x < x'\) and \(y < y'.\) Applying Claim 11 we conclude that

\[ (x', y') \in ([x, r) \times [y, q)) \setminus \{ (x, y) \} \subset \bigcup_{(x''', y''')} [x''', r) \times [y''', q) \]

and hence \((x', y') \notin D_{r, q},\) which contradicts the choice of the pair \((x, y').\) This contradiction shows that \(D_{r, q}\) is strictly decreasing.

**Claim 13.** The set \(D_{r, q}\) is at most countable.

**Proof.** To derive a contradiction, assume that \(D_{r, q}\) is uncountable. By Lemma 7 the strictly decreasing function \(D_{r, q}\) is contained in some Borel strictly decreasing function, which is equal to \(f_\alpha\) for some ordinal \(\alpha < \omega_1.\) Since the intersection of a strictly increasing function and a strictly decreasing function contains at most one point, the set

\[ D'_{r, q} = \bigcup \{ D_{r, q} \cap f_\beta : \beta \leq \alpha, f_\beta \text{ is strictly increasing} \} \]

is at most countable. We claim that \(D_{r, q} = D'_{r, q} \cup f_\alpha.\) To derive a contradiction, assume that \(D_{r, q} \setminus D'_{r, q}\) contains some pair \(z = (x, y).\) It follows from \(z \in D_{r, q} \subset f_\alpha\) that \(\alpha < \omega.\) Since \(z \notin D'_{r, q},\) the strictly monotone function \(f_\alpha\) is not strictly increasing and hence \(f_\alpha\) is strictly decreasing.
Then the definition of the set \( L \) guarantees that \( z \notin L \), which contradicts the inclusion \( z \in D_{r,q} \subset L \).

Now consider the countable subset \( L'_{r,q} := L''_{r,q} \cup D_{r,q} \) of \( L_{r,q} \) and observe that

\[
\bigcup_{\langle x,y \rangle \in L_{r,q}} ([x, r) \times [y, q)) \subset \bigcup_{\langle x,y \rangle \in L'_{r,q}} [x, r) \times [y, q))
\]

This completes the proof of Claim 10.

**Claim 14.** There exists a countable subset \( L' \subset L_V \) such that

\[
\bigcup_{\langle x,y \rangle \in L'} ([x, a(x, y)) \times [y, b(x, y)) = \bigcup_{\langle x,y \rangle \in L_V} ([x, a(x, y)) \times [y, b(x, y)).
\]

**Proof.** By Claim 10 for any rational numbers \( r, q \in \mathbb{Q} \) there exists a countable subset \( L'_{r,q} \subset L_{r,q} \) such that

\[
\bigcup_{\langle x,y \rangle \in L'_{r,q}} ([x, a(x, y)) \times [y, b(a, y)]) = \bigcup_{\langle x,y \rangle \in L'_{r,q}} ([x, r) \times [y, q)) =
\]

\[
\bigcup_{\langle x,y \rangle \in L_{r,q}} ([x, r) \times [y, q)) = \bigcup_{\langle x,y \rangle \in L_{r,q}} ([x, a(x, y)) \times [y, b(x, y))).
\]

Since \( L_V = \bigcup_{r,q \in \mathbb{Q}} L_{r,q} \), the countable set \( L' := \bigcup_{r,q \in \mathbb{Q}} L'_{r,q} \) has the required property. \( \square \)

By analogy with Claim 14 we can prove

**Claim 15.** There exists a countable subset \( \Gamma' \subset \Gamma_V \) such that

\[
\bigcup_{\langle x,y \rangle \in \Gamma'} ([x, a(x, y)) \times (b(x, y), y)) = \bigcup_{\langle x,y \rangle \in \Gamma_V} ([x, a(x, y)) \times (b(x, y), y)).
\]

Claims 14 and 15 complete the proof of Claim 9 and the proof of Lemma 8. \( \square \)

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