Degenerate polyexponential-Genocchi numbers and polynomials

Waseem A. Khan\textsuperscript{a,}\textsuperscript{*}, Aysha Khan\textsuperscript{b}, Idrees A. Khan\textsuperscript{c}

\textsuperscript{a}Department of Mathematics and Natural Sciences, Prince Mohammad Bin Fahd University, P.O. Box 1664, Al Khobar 31952, Saudi Arabia.

\textsuperscript{b}Department of Mathematics, College of Arts and Science-Wadi Al Dawasir, Prince Sattam Bin Abdulaziz University, Riyadh region 11991, Saudi Arabia.

\textsuperscript{c}Department of Mathematics, Faculty of Science, Integral University, Lucknow 226026, India.

Abstract

Recently, Kim et al. in [T. Kim, D. S. Kim, H. Y. Kim, L.-C. Jang, Informatica, 3 (2020), 8 pages] studied the degenerate poly-Bernoulli numbers and polynomials which are defined by using the polylogarithm function. In this paper, we study the degenerate polyexponential-Genocchi polynomials and numbers arising from polyexponential function and derive their explicit expressions and some identity involving them. In the final section, we introduce degenerate unipoly-Genocchi polynomials attached to an arithmetic function, by using polylogarithm function and investigate some identities for those polynomials.

Keywords: Polylogarithm function, degenerate poly-Bernoulli polynomials, degenerate poly-Genocchi polynomials, unipoly function.

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1. Introduction

In 1905, Hardy considered the polyexponential function \cite{9, 10} given by

\[ e(x, a|s) = \sum_{n=0}^{\infty} \frac{x^n}{(n+a)^s n!}, \quad (\Re(a) > 0). \]

For \( k \in \mathbb{Z} \), Kim and Kim \cite{24} defined the modified polyexponential function, as an inverse to the polylogarithm function by

\[ E_{ik}(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k(n-1)!}. \]  (1.1)

It is worthy to note that \( e(x, 1|k) = \frac{1}{x} E_{ik}(x) \) and \( E_{1}(x) = e^x - 1 \).
As is well known, the classical Euler and Genocchi polynomials are defined by (see [2–4, 7, 11, 15, 17, 19])

\[
\frac{2}{e^z+1} e^{uz} = \sum_{j=0}^{\infty} E_j(u) \frac{z^j}{j!}, \quad \frac{2z}{e^z+1} e^{uz} = \sum_{j=0}^{\infty} G_j(u) \frac{z^j}{j!}.
\]  

(1.2)

In the case when \( u = 0 \), \( E_j = E_j(0) \) and \( G_j = G_j(0) \) are respectively, called the Euler numbers and Genocchi numbers.

From (1.2), we see that 

\[
G_0(0) = 0, \quad E_j(u) = G_{j+1}(u) + 1, \quad (j \geq 0), \quad (\text{see [27]}). 
\]

For \( k \in \mathbb{Z} \), Kim-Kim considered the type 2 poly-Bernoulli polynomials are defined by means of the following generating function

\[
\frac{E_k(\log(1+t))}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}, \quad (\text{see [14]}). 
\]

In the case when \( x = 0 \), \( B_n^{(k)} = B_n^{(k)}(0) \) are called the type 2 poly-Bernoulli numbers.

For \( k \in \mathbb{Z} \), the polylogarithm function is defined by 

\[
\text{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}, \quad (|x| < 1), \quad (\text{see, [12, 13, 16, 19, 30]}). 
\]

Note that 

\[
\text{Li}_1(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x). 
\]

For any nonzero \( \nu \in \mathbb{R} \) (or \( \mathbb{C} \)), the degenerate exponential function is defined by 

\[
e_{\nu}(z) = (1 + \nu z)^{\frac{\nu}{z}}, \quad e_{\nu}(z) = (1 + \nu z)^{\frac{\nu}{z}}, \quad (\text{see [20–25, 27–29]}). 
\]

By Taylor expansion, we see 

\[
e_{\nu}(z) = \sum_{q=0}^{\infty} (\xi)_{q,\nu} \frac{z^q}{q!}, \quad (\text{see [20, 24, 29, 31]}), 
\]

where \((\xi)_{0,\nu} = 1, \quad (\xi)_{q,\nu} = \xi(\xi - \nu) \cdots (\xi - (q-1)\nu), \quad (q \geq 1).\)

Obviously 

\[
\lim_{\nu \to 0} e_{\nu}(z) = \sum_{q=0}^{\infty} \xi^q \frac{z^q}{q!} = e^{\xi z}. 
\]

In [5, 6], Carlitz considered the degenerate Bernoulli and degenerate Euler polynomials defined by 

\[
\frac{t}{(1+\lambda t)^{\frac{\nu}{z}} - 1} (1+\lambda t)^{\frac{\nu}{z}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}, 
\]

and 

\[
\frac{2}{(1+\lambda t)^{\frac{\nu}{z}} + 1} (1+\lambda t)^{\frac{\nu}{z}} = \sum_{n=0}^{\infty} E_{n,\lambda}(x) \frac{t^n}{n!}. \quad (1.3)
\]
Putting $x = 0, \beta_{n, \lambda} = \beta_{n, \lambda}(0)$ and $E_{n, \lambda} = E_{n, \lambda}(0)$ are called the degenerate Bernoulli and degenerate Euler numbers.

In [26], Kim et al. considered the the degenerate Genocchi polynomials given by

$$2z e_\lambda(z) + 1 e_\lambda(z) = \sum_{j=0}^{\infty} G_{j, \lambda}(u) \frac{z^j}{j!}.$$  (1.4)

In the case when $u = 0, G_{j, \lambda} = G_{j, \lambda}(0)$ are called the degenerate Genocchi numbers.

Very recently, Kim et al. [27] introduced the degenerate poly-Bernoulli polynomials defined by

$$E_{k, \lambda}(\log(1 + t)) e_\lambda(t) - 1 e_\lambda(t) = \sum_{n=0}^{\infty} \beta^{(k)}_{n, \lambda}(x) \frac{t^n}{n!}.$$  (1.5)

Here, $\beta^{(k)}_{n, \lambda} = \beta^{(k)}_{n, \lambda}(0)$ are called the degenerate poly-Bernoulli numbers.

The Daehee polynomials [28] are defined by

$$\log(1 + t) (1 + t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}, \quad (\text{see [19, 32]}).$$

When $x = 0, D_n = D_n(0)$ are called the Daehee numbers.

The degenerate Stirling numbers of the first kind [23] are defined by

$$\frac{1}{k!} (\log_\lambda(1 + z))^k = \sum_{j=k}^{\infty} S_{1, \lambda}(j, k) \frac{z^j}{j!}, \quad (k \geq 0).$$  (1.6)

It is notice that

$$\lim_{\lambda \to 0} S_{1, \lambda}(j, k) = S_1(j, k),$$

are calling the Stirling numbers of the first kind given by

$$\frac{1}{k!} (\log(1 + z))^k = \sum_{j=k}^{\infty} S_1(j, k) \frac{z^j}{j!}, \quad (k \geq 0), \quad (\text{see [24, 25]}).$$

The degenerate Stirling numbers of the second kind [21] are given by

$$\frac{1}{k!} (e_\lambda(z) - 1)^k = \sum_{j=k}^{\infty} S_{2, \lambda}(j, k) \frac{z^j}{j!}, \quad (k \geq 0), \quad (\text{see [1, 8]}).$$  (1.7)

Note here that

$$\lim_{\lambda \to 0} S_{2, \lambda}(j, k) = S_2(j, k),$$

standing for the Stirling numbers of the second kind given by means of the following generating function:

$$\frac{1}{k!} (e^z - 1)^k = \sum_{j=k}^{\infty} S_2(j, k) \frac{z^j}{j!}, \quad (k \geq 0), \quad (\text{see, [14, 15, 18, 20–32]}).$$
2. Degenerate polyexponential-Genocchi numbers and polynomials

In this section, we define degenerate Genocchi numbers and polynomials by using the degenerate polyexponential function which are called the degenerate polyexponential-Genocchi polynomials as follows,

\[
\frac{2\text{Ei}_k(\log(1+t))}{e^\lambda(t)+1}e^\lambda(t) = \sum_{n=0}^{\infty} G^{(k)}_{n,\lambda}(x) \frac{t^n}{n!}, \quad (k \in \mathbb{Z}).
\]

(2.1)

For \( k = 1 \), \( G^{(1)}_{n,\lambda}(x) = G_{n,\lambda}(x), (n \geq 0) \). Here, \( G^{(k)}_{n,\lambda} = G^{(k)}_{n,\lambda}(0) \) are called the degenerate polyexponential-Genocchi numbers.

From (1.2) and (2.1), we note that

\[
\lim_{\lambda \to 0} G^{(1)}_{n,\lambda}(x) = G_n(x), \quad (n \geq 0).
\]

Theorem 2.1. For \( n \geq 0 \) and \( k \in \mathbb{Z} \), we have

\[
G^{(k)}_{n,\lambda}(x) = \sum_{m=0}^{n} \sum_{l=1}^{m+1} \binom{n}{m} \frac{S_1(m+1, l)}{l^{k-1}(m+1)} G_{n-m,\lambda}(x).
\]

(2.2)

Proof. Using equations (1.1) and (2.1), we see that

\[
\text{Ei}_k(\log(1+t)) = \sum_{r=1}^{\infty} \frac{(\log(1+t))^r}{(r-1)! r^k} = \sum_{r=1}^{\infty} \frac{1}{(r-1)! r^k} \sum_{q=r}^{\infty} S_1(q, r) \frac{t^q}{q!} = \sum_{q=0}^{\infty} \sum_{r=1}^{q+1} \frac{S_1(q+1, r)}{r^{k-1}(q+1)} \frac{t^q}{q!}.
\]

(2.3)

By using equations (1.4) and (2.3), equation (2.1) is

\[
\frac{2t}{e^\lambda(t)+1}e^\lambda(t) \frac{1}{t} \text{Ei}_k(\log(1+t)) = \sum_{k=0}^{\infty} G_{k,\lambda}(x) \frac{t^k}{k!} \sum_{q=0}^{\infty} \sum_{r=1}^{q+1} \frac{S_1(q+1, r)}{r^{k-1}(q+1)} \frac{t^q}{q!},
\]

L.H.S. = \[
\sum_{n=0}^{\infty} \left( \sum_{q=0}^{n} \sum_{r=1}^{q+1} \binom{n}{q} \frac{S_1(q+1, r)}{r^{k-1}(q+1)} G_{n-q,\lambda}(x) \right) \frac{t^n}{n!}.
\]

Comparing the coefficients of \( t^n \) on both sides, we get the result (2.2).

Theorem 2.2. For \( n \geq 0 \) and \( k \in \mathbb{Z} \), we have

\[
G^{(k)}_{n,\lambda}(x) = \sum_{q=1}^{n} \binom{n}{q} \sum_{r=1}^{q} \frac{S_1(q, r)}{r^{k-1}} E_{n-q,\lambda}(x).
\]

Proof. It is proved by using (1.3) and (2.1) that

\[
\frac{2\text{Ei}_k(\log(1+t))}{e^\lambda(t)+1}e^\lambda(t) = \frac{2}{e^\lambda(t)+1} e^\lambda(t) \text{Ei}_k(\log(1+t))
\]

\[
= \frac{2}{e^\lambda(t)+1} e^\lambda(t) \sum_{r=1}^{\infty} \frac{(\log(1+t))^r}{(r-1)! r^k} t^r
\]

\[
= \frac{2}{e^\lambda(t)+1} e^\lambda(t) \sum_{r=1}^{\infty} \frac{1}{r^{k-1}} \sum_{q=r}^{\infty} S_1(q, r) \frac{t^q}{q!}.
\]
To prove this section, we first consider the following expression

\[ \sum_{n=0}^{\infty} E_{n,\lambda}(x) \frac{t^n}{n!} \sum_{q=1}^{\infty} \sum_{r=1}^{q} \frac{S_1(q,r) t^q}{r^{k-1}}. \]

L.H.S = \( \sum_{n=1}^{\infty} \left( \sum_{q=1}^{n} \sum_{r=1}^{q} \frac{S_1(q,r) t^q}{r^{k-1}} - E_{n-q,\lambda}(x) \right) \frac{t^n}{n!}. \)

Therefore, by (2.1) and above equation, we complete the proof.

For the next theorem, we need the following well-known from ([7]) that

\[
\left( \frac{t}{\log(1+t)} \right)^r (1+t)^{-1} = \sum_{n=0}^{\infty} B_{n}^{(n-r+1)}(x) \frac{t^n}{n!}, \quad (r \in \mathbb{C}),
\]  

(2.4)

where \( B_{n}^{(r)}(x) \) are called the higher-order Bernoulli polynomials which are given by the generating function

\[
\left( \frac{t}{e^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} B_{n}^{(r)}(x) \frac{t^n}{n!}.
\]

**Theorem 2.3.** For \( n \geq 0, k \in \mathbb{Z}, \) we have

\[
G_{n,\lambda}^{(k)} = \sum_{m=0}^{n} \binom{n}{m} n_{m_1,\ldots,m_{k-1} = m} G_{n-m,\lambda}(x) \times \left( \sum_{m_{k-1}} \frac{B_{m_1}^{(m_1)}(0)}{m_1+1} \frac{B_{m_2}^{(m_2)}(0)}{m_1+m_2+1} \cdots \frac{B_{m_{k-1}}^{(m_{k-1})}(0)}{m_1+m_2+\cdots+m_{k-1}+1} \right).
\]

**Proof.** To prove this section, we first consider the following expression

\[
\frac{d}{dx} E_{i_k}(\log(1+x)) = \frac{d}{dx} \sum_{n=1}^{\infty} \frac{(\log(1+x))^n}{(n+1)! n^k} = \frac{1}{(1+x) \log(1+x)} \sum_{n=1}^{\infty} \frac{(\log(1+x))^n}{(n+1)! n^{k-1}} = \frac{1}{(1+x) \log(1+x)} E_{i_{k-1}}(\log(1+x)).
\]

(2.5)

From (2.5), \( k \geq 2, \) we have

\[
E_{i_k}(\log(1+x)) = \int_{0}^{x} \frac{1}{(1+t) \log(1+t)} E_{i_{k-1}}(\log(1+t)) dt
\]

\[ = \int_{0}^{x} \frac{1}{(1+t) \log(1+t)} \int_{0}^{t} \frac{1}{(1+t) \log(1+t)} \int_{0}^{t} \frac{1}{(1+t) \log(1+t)} dt \cdots dt \]

\[ \times E_{i_{1}}(\log(1+x)) dt \cdots dt \]

\[ = \int_{0}^{x} \frac{1}{(1+t) \log(1+t)} \int_{0}^{t} \frac{1}{(1+t) \log(1+t)} \int_{0}^{t} \frac{1}{(1+t) \log(1+t)} dt \cdots dt. \]

(2.6)

By (2.1), (2.6), and (2.4), we get

\[
\sum_{n=0}^{\infty} G_{n,\lambda}^{(k)} \frac{x^n}{n!} = \frac{2E_{i_k}(\log(1+x))}{e_{\lambda}(x) + 1} = \frac{2}{e_{\lambda}(x) + 1}
\]
Thus, we complete the proof.

On setting $k = 2$, Theorem 2.3 gives the following result.

**Corollary 2.4.** For $n \geq 0$, we have

$$G_{n, \lambda}^{(2)} = \sum_{l=0}^{n} \left(\begin{array}{c} n \\ l \end{array}\right) B_{l}^{(1)}(0) \frac{n}{l+1} G_{n-l, \lambda}.$$ 

**Theorem 2.5.** Let $k \geq 1$ and $m \in \mathbb{N} \cup \{0\}$, $s \in \mathbb{C}$, we have

$$\chi_{k, \nu}(-m) = (-1)^{m} G_{m, \nu}^{(k)}.$$ 

**Proof.** Let $k \geq 1$ be an integer. For $s \in \mathbb{C}$, we define the function $\chi_{k}(s)$ as

$$\chi_{k, \nu}(s) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} \frac{z^{s-1}}{e_{\nu}(z) + 1} E_{k}(\log(1+z)) \, dz. \quad (2.7)$$

From (2.7), we note that

$$\chi_{k, \nu}(s) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} \frac{z^{s-1}}{e_{\nu}(z) + 1} E_{k}(\log(1+z)) \, dz$$

$$= \frac{1}{\Gamma(s)} \int_{0}^{1} \frac{z^{s-1}}{e_{\nu}(z) + 1} E_{k}(\log(1+z)) \, dz + \frac{1}{\Gamma(s)} \int_{1}^{\infty} \frac{z^{s-1}}{e_{\nu}(z) + 1} E_{k}(\log(1+z)) \, dz. \quad (2.8)$$

The second integral converges absolutely for any $s \in \mathbb{C}$ and hence, the second term on the right hand side vanishes at non-positive integers. That is,

$$\lim_{s \to -m} \left| \frac{1}{\Gamma(s)} \int_{1}^{\infty} \frac{z^{s-1}}{e_{\nu}(z) + 1} E_{k}(\log(1+z)) \, dz \right| \leq \frac{1}{\Gamma(-m)} M = 0. \quad (2.9)$$

On the other hand, for $\Re(s) > 0$, the first integral in (2.9) can be written as

$$\frac{1}{\Gamma(s)} \sum_{r=0}^{\infty} \frac{G_{r, \lambda}^{(k)}}{r!} \frac{1}{s+r},$$

which defines an entire function of $s$. Thus, we may include that $\chi_{k}(s)$ can be continued to an entire function of $s$. 

Further, from (2.8) and (2.9), we obtain

\[
\chi_k(-m) = \lim_{s \to -m} \frac{1}{\Gamma(s)} \int_0^1 \frac{z^{s-1}}{e^z + 1} E_k(\log(1 + z)) \, dz = \lim_{s \to -m} \frac{1}{\Gamma(s)} \sum_{r=0}^\infty \frac{G_r^{(k)}}{r!} \cdot \frac{1}{s + r \cdot r!} = \cdots + 0 + \cdots + \lim_{s \to -m} \frac{1}{\Gamma(s)} \frac{G_m^{(k)}}{m!} + 0 + \cdots
\]

Thus, we complete the proof of this theorem. \(\square\)

**Theorem 2.6.** For \(n \geq 0\) and \(k \in \mathbb{Z}\), we have

\[
G_{m,\lambda}^{(k)} S_2(n, m) = \sum_{i=0}^n \sum_{r=0}^i \left( \binom{n}{i} \binom{i}{r} G_{j,\lambda} S_2(r, j) B_{r-i} \right) \frac{1}{(n-i+1)^k}.
\]

**Proof.** By replacing \(t\) by \(e^t - 1\) in (2.1), we get

\[
\frac{2}{e^\lambda(e^t - 1) + 1} E_k(t) = 2 \sum_{m=0}^\infty \frac{G_m^{(k)}}{m!} (e^t - 1)^m = \sum_{m=0}^\infty \frac{G_m^{(k)}}{m!} \sum_{n=m}^\infty S_2(n, m) \frac{t^n}{n!} = \sum_{n=0}^\infty \left( \sum_{m=0}^n \frac{G_m^{(k)}}{m!} \right) \frac{t^n}{n!}.
\]

On the other hand, we see that

\[
\frac{2}{e^\lambda(e^t - 1) + 1} E_k(t) = \frac{2(e^t - 1)}{e^\lambda(e^t - 1) + 1} \frac{1}{e^t - 1} \sum_{l=0}^\infty \frac{t^l}{(l-1)!l!} = \sum_{j=0}^\infty \frac{G_{j,\lambda}}{j!} (e^t - 1)^j \sum_{i=1}^\infty \frac{B_i}{i!} \sum_{l=0}^\infty \frac{t^l}{(l+1)^k l!} = \sum_{r=0}^\infty \sum_{j=0}^r \frac{G_{j,\lambda} S_2(r, j)}{r! \cdot \sum_{l=0}^r \frac{B_i}{i!} \sum_{l=0}^\infty \frac{t^l}{(l+1)^k l!}} \frac{t^r}{r!} \sum_{i=0}^\infty \frac{t^i}{i!} \sum_{l=0}^\infty \frac{t^l}{(l+1)^k l!}
\]

\[
\text{L.H.S} = \sum_{m=0}^\infty \left( \sum_{i=0}^n \sum_{r=0}^i \left( \binom{n}{i} \binom{i}{r} G_{j,\lambda} S_2(r, j) B_{r-i} \right) \frac{1}{(n-i+1)^k} \right) \frac{t^n}{n!}.
\]

When equating (2.10) and (2.11) gives desired proof. \(\square\)
3. Degenerate unipoly-Genocchi polynomials and numbers

In this section, we define the degenerate unipoly-Genocchi polynomials by using the unipoly function and derive some multifarious properties.

Let \( p \) be any arithmetic function which is a real or complex valued function defined on the set of positive integers \( \mathbb{N} \). Kim and Kim [24] defined the unipoly function attached to polynomials \( p(x) \) by

\[
u_k(x|p) = \sum_{n=1}^{\infty} \frac{p(n)}{n^k} x^n, \quad (k \in \mathbb{Z}).
\]

Moreover,

\[
u_k(x|1) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} = \text{Li}_k(x), \quad \text{(see [12])},
\]

is the ordinary polylogarithm function.

Now, we define the degenerate unipoly-Genocchi polynomials attached to polynomials \( p(x) \) by

\[
\frac{2}{e_\lambda(t) + 1} u_k (\log(1+t)|p) e_\lambda^k(t) = \sum_{n=0}^{\infty} G_{n,\lambda,p}^{(k)}(x) \frac{t^n}{n!}.
\] (3.1)

In case when \( x = 0 \), \( G_{n,\lambda,p}^{(k)} = G_{n,\lambda,p}^{(k)}(0) \) are called the degenerate unipoly-Genocchi numbers attached to \( p \). If we take \( p(n) = \frac{1}{\Gamma(n)} \), then we have

\[
\sum_{n=0}^{\infty} G_{n,\lambda}^{(1)}(x) \frac{t^n}{n!} = \frac{2}{e_\lambda(t) + 1} e_\lambda^k(t) u_k \left( \log(1+t) \right) \frac{1}{n!} = \frac{2}{e_\lambda(t) + 1} e_\lambda^k(t) \sum_{m=1}^{\infty} \frac{(\log(1+t))^m}{m^k(m-1)!}.
\]

In particular, for \( k = 1 \), we obtain

\[
\sum_{n=0}^{\infty} G_{n,\lambda}^{(1)}(x) \frac{t^n}{n!} = \frac{2}{e_\lambda(t) + 1} e_\lambda^1(t) \sum_{m=1}^{\infty} \frac{(\log(1+t))^m}{m^k(m-1)!} = \frac{2t}{e_\lambda(t) + 1} e_\lambda^1(t).
\] (3.2)

Therefore by (3.1) and (3.2), we have

\[
G_{n,\lambda,\mp}^{(1)}(x) = G_{n,\lambda}(x), \quad (n \geq 0).
\]

**Theorem 3.1.** For \( n \geq 0 \) and \( k \in \mathbb{Z} \), we have

\[
G_{n,\lambda,p}^{(k)}(x) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \frac{p(m+1)(m+1)!}{(m+1)^k} \frac{S_1(l+1,m+1)}{l+1} G_{n-l,\lambda}(x).
\]

Moreover,

\[
G_{n,\lambda,p}^{(k)}(x) = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \frac{1}{(m+1)^{k-1}} \frac{S_1(l+1,m+1)}{l+1} G_{n-l,\lambda}(x).
\]

**Proof.** From (3.1), we have

\[
\sum_{n=0}^{\infty} G_{n,\lambda,p}^{(k)}(x) \frac{t^n}{n!} = \frac{2}{e_\lambda(t) + 1} e_\lambda^k(t) u_k (\log(1+t)|p)
\]

\[
= \frac{2}{e_\lambda(t) + 1} e_\lambda^k(t) \sum_{m=1}^{\infty} \frac{p(m)}{m^k} (\log(1+t))^m
\]
Proof. By comparing the coefficients of the same powers in $t$ of above equation and (2.1), we obtain the desired result. 

**Theorem 3.2.** For $n \geq 0$ and $k \in \mathbb{Z}$, we have

$$G_{n,\lambda,p}^{(k)}(x) = \sum_{m=0}^{n} \binom{n}{m} G_{m,\lambda,p}^{(k)}(x)_{n-m,\lambda}.$$  

**Proof.** Recalling from (3.1) that

$$\sum_{n=0}^{\infty} G_{n,\lambda,p}^{(k)}(x) \frac{t^n}{n!} = \frac{2}{e_{\lambda}(t)+1} \frac{u_k}{p} (\log(1+t)) \frac{e_{\lambda}(t)}{t}. $$

By comparing the coefficients of $\frac{t^n}{n!}$, we complete the proof. 

**Theorem 3.3.** For $n \geq 0$ and $k \in \mathbb{Z}$, we have

$$G_{n,\lambda,p}^{(k)} = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n-l}{j} \binom{l}{1} D_{n-j-1} \frac{p(m+1)m!}{(m+1)^k} S_1(l, m).$$  

**Proof.** It is proved by using (3.1) that

$$\sum_{n=0}^{\infty} G_{n,\lambda,p}^{(k)} \frac{t^n}{n!} = \frac{2}{e_{\lambda}(t)+1} \frac{u_k}{p} (\log(1+t))^m$$

$$= \frac{2}{e_{\lambda}(t)+1} \frac{p(m)}{m^k} (\log(1+t))^m$$

$$= \frac{2t}{e_{\lambda}(t)+1} \frac{\log(1+t)}{t} \sum_{m=0}^{\infty} \frac{p(m+1)m! (\log(1+t))^m}{m!}.$$
Thus, we complete the proof of this theorem.

4. Conclusion

In this paper, we studied the degenerate polyexponential-Genocchi numbers and polynomials and derived explicit expressions and some identity involving them. In more detail, we obtained an expression of the degenerate polyexponential-Genocchi polynomials in terms of the degenerate Bernoulli polynomials and Stirling numbers of the first kind. We also deduced an expression of the degenerate polyexponential-Genocchi numbers in terms of the degenerate Bernoulli numbers and values of higher-order Bernoulli numbers. Also, we derived an identity involving the degenerate polyexponential-Bernoulli numbers, degenerate Stirling numbers of the second kind, and the degenerate Genocchi numbers. In the last section, we defined degenerate unipoly-Genocchi polynomials attached to arithmetic function by using the modified polyexponential function and obtained the identity degenerate unipoly-Genocchi numbers and polynomials in terms of Stirling numbers of the first kind and Daehee numbers and Bernoulli numbers.

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