THE BOUNDED ISOMETRY CONJECTURE FOR THE KODAIRA-THURSTON MANIFOLD AND 4-TORUS

ZHIGANG HAN

Abstract. The purpose of this note is to study the bounded isometry conjecture proposed by Lalonde and Polterovich [11]. In particular, we show that the conjecture holds for the Kodaira-Thurston manifold with the standard symplectic form and for the 4-torus with all linear symplectic forms.

1. Introduction and main results

Let \((M, \omega)\) be a closed symplectic manifold. There is a natural bi-invariant norm, called the Hofer norm \(\rho\), defined on the Hamiltonian diffeomorphism group \(\text{Ham}(M, \omega)\). That is, \(\rho(f)\) is the Hofer distance between the identity map \(id\) and \(f\) for all \(f \in \text{Ham}(M, \omega)\), see Section 4 for details. Lalonde and Polterovich [11] have studied the full symplectomorphism group \(\text{Symp}(M, \omega)\) within the framework of Hofer’s geometry. We first recall the notion of bounded and unbounded symplectomorphisms. Namely, for each \(\phi \in \text{Symp}(M, \omega)\), define

\[
    r(\phi) := \sup \{ \rho([\phi, f]) \mid f \in \text{Ham}(M, \omega) \},
\]

where \([\phi, f] := \phi f \phi^{-1} f^{-1}\) is the commutator of \(\phi\) and \(f\).

Definition 1.1. An element \(\phi \in \text{Symp}(M, \omega)\) is bounded if \(r(\phi) < \infty\), and is unbounded if \(r(\phi) = \infty\).

Denote by \(BI_0(M, \omega)\) the set of all bounded elements in the identity component \(\text{Symp}_0(M, \omega)\) of \(\text{Symp}(M, \omega)\). Since \(\rho\) is bi-invariant, it follows from the inequality \(\rho([\phi, f]) \leq 2\rho(\phi)\) that \(\text{Ham}(M, \omega)\) is a subgroup of \(BI_0(M, \omega)\). The converse is the following conjecture in [11].

Conjecture 1.2 (Bounded isometry conjecture). For all symplectic manifolds \((M, \omega)\), \(BI_0(M, \omega) = \text{Ham}(M, \omega)\).

This conjecture was proved in [11] for closed surfaces with area form and for arbitrary products of closed surfaces of genus greater than 0 with product symplectic form; Lalonde and Pestieau [12] confirmed it for product symplectic manifolds \(M = N \times W\) with \(N\) being any product of closed surfaces and \(W\) being any closed symplectic manifold of first real Betti number equal to zero. In this note, we give a positive answer to this conjecture for the Kodaira-Thurston manifold with the standard symplectic form and for the 4-torus with all linear symplectic forms.

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Theorem 1.3. The bounded isometry conjecture holds for the Kodaira-Thurston manifold $M$ with the standard symplectic form $\omega$.

Theorem 1.4. The bounded isometry conjecture holds for the 4-torus $(\mathbb{T}^4, \omega)$ with any linear symplectic form $\omega := \sum_{i<j} a_{ij} dx_i \wedge dx_j$.

Organization of the paper: We begin with some preparations in Section 2. Then we prove Theorem 1.3 in Section 6 and Theorem 1.4 in Section 7. We study the same conjecture for the Kodaira-Thurston manifold with all linear symplectic forms in Section 8. While we are unable to prove the conjecture in this case, some partial results are provided and the difficulties are discussed.

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2. The flux subgroup

The flux homomorphism is best defined on the universal cover $\widetilde{\text{Symp}}_0(M, \omega)$ of $\text{Symp}_0(M, \omega)$,

$$\text{flux} : \widetilde{\text{Symp}}_0(M, \omega) \to H^1(M, \mathbb{R}).$$

Let $\{\phi_t\} \in \widetilde{\text{Symp}}_0(M, \omega)$, i.e. $\phi_t$ is a smooth isotopy in $\text{Symp}_0(M, \omega)$. There exists a unique family of vector fields $X_t$ which generates the flow $\phi_t$, i.e.

$$\frac{d}{dt}\phi_t = X_t \circ \phi_t.$$

Define

$$\text{flux}(\{\phi_t\}) := \int_0^1 \iota(X_t)\omega \, dt.$$

In particular, if $\{\phi_t\}$ is the flow of the time-independent symplectic vector field $X$ on the time interval $0 \leq t \leq 1$, then

$$\text{flux}(\{\phi_t\}) = \iota(X)\omega.$$ (1)

This fact will often be used in later calculations.

The flux subgroup $\Gamma := \Gamma_\omega$ is the image

$$\text{flux}(\pi_1(\text{Symp}_0(M, \omega))) \subset H^1(M, \mathbb{R})$$

of the fundamental group of $\text{Symp}_0(M, \omega)$ under the flux homomorphism. Thus there is an induced map from $\text{Symp}_0(M, \omega)$, still denoted by flux,

$$\text{flux} : \text{Symp}_0(M, \omega) \to H^1(M, \mathbb{R})/\Gamma.$$

It is well known that this map is surjective, and its kernel is equal to $\text{Ham}(M, \omega)$. In other words, we have the following exact sequence of groups

$$0 \longrightarrow \text{Ham}(M, \omega) \longrightarrow \text{Symp}_0(M, \omega) \xrightarrow{\text{flux}} H^1(M, \mathbb{R})/\Gamma \longrightarrow 0.$$

We refer to [14] Chapter 10 for more details.
Since whether or not the flux is equal to 0 distinguishes a Hamiltonian diffeomorphism from a non-Hamiltonian symplectomorphism, one main step in our applications is to understand the flux subgroup $\Gamma$.

For this, we denote as in [6] by $C(M)$ the space of continuous maps from $M$ to $M$ with the compact open topology. Given $p \in M$, we define the evaluation map $ev_c : C(M) \to M$ by $ev_c(f) = f(p)$. Denote by $ev_s$ the restriction of $ev_c$ to $\text{Symp}_0(M,\omega)$. We will use the same notation for the induced maps on the fundamental groups. By $\tilde{ev}_s$ we denote the homomorphism from $\pi_1(\text{Symp}_0(M,\omega))$ to $H_1(M,\mathbb{Z})$, which is the composition of $ev_s$ with the Hurewitz map from $\pi_1(M)$ to $H_1(M,\mathbb{Z})$.

The following commutative diagram due to Lalonde, McDuff and Polterovich [10] plays a crucial role in the calculation of the flux subgroup $\Gamma$.

Lemma 2.1 (LMP). Let $(M,\omega)$ be a closed symplectic manifold of dimension $2n$. Then the following diagram commutes.

$$
\begin{array}{cccc}
\pi_1(\text{Symp}_0(M,\omega)) & \xrightarrow{\tilde{ev}_s} & H_1(M,\mathbb{Z}) & \xrightarrow{\text{PD}} & H^{2n-1}(M,\mathbb{Z}) \\
\downarrow{id} & & & & \downarrow{(n-1)!\text{vol}(M)} \\
\pi_1(\text{Symp}_0(M,\omega)) & \xrightarrow{\text{flux}} & H^1(M,\mathbb{R}) & \xleftarrow{\wedge[\omega]^{n-1}} & H^{2n-1}(M,\mathbb{R}).
\end{array}
$$

3. The Kodaira-Thurston manifold

Let $G$ be the group $(\mathbb{Z}^4, \cdot)$ where

$$(m_1, n_1, k_1, \ell_1) \cdot (m_2, n_2, k_2, \ell_2) = (m_1 + m_2, n_1 + n_2, k_1 + k_2 + m_1 \ell_2, \ell_1 + \ell_2).$$

$G$ acts on $\mathbb{R}^4$ via

$$G \to \text{Diff}(\mathbb{R}^4) : (m, n, k, \ell) \mapsto \rho_{mnk\ell}$$

where

$$\rho_{mnk\ell}(s, t, x, y) = (s + m, t + n, x + k + my, y + \ell).$$

Note that $\rho_{mnk\ell}$ preserves the symplectic form $\omega = ds \wedge dt + dx \wedge dy$ on $\mathbb{R}^4$. Hence the quotient $(M := \mathbb{R}^4/G, \omega)$ is a closed symplectic manifold, known as the Kodaira-Thurston manifold, see [19]. It was the first known example of a closed symplectic manifold which admits no kähler structure, since its first betti number $b_1 = 3$, see [14] Example 3.8.

The manifold $M = \mathbb{R}^4/G$ can also be described as a torus bundle over a torus, that is $M = \mathbb{R}^2 \times_{\mathbb{Z}^2} \mathbb{T}^2$. Here $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ in the usual way, and it acts on $\mathbb{T}^2$ via

$$(m, n) \to A_{mn} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$ 

Therefore $M = \mathbb{R} \times S^1 \times \mathbb{T}^2/\sim$, where

$$(s, t, x, y) \sim (s + 1, t, x + y, y).$$

Our first task is to understand the flux subgroup $\Gamma$ of the Kodaira-Thurston manifold described above. In particular, we have
Theorem 3.1. The flux subgroup $\Gamma \subset H^1(M, \mathbb{R})$ of the Kodaira-Thurston manifold with the standard symplectic form $\omega = ds \wedge dt + dx \wedge dy$ has rank 2 over $\mathbb{Z}$. Namely, $\Gamma = \mathbb{Z}(ds, dy)$.

To prove Theorem 3.1 we need the following result on the cohomology groups of the Kodaira-Thurston manifold.

Lemma 3.2. The cohomology groups of the Kodaira-Thurston manifold $M$ described above are as follows: $H^1(M, \mathbb{R})$ is of rank 3, generated by $ds$, $dt$ and $dy$; $H^2(M, \mathbb{R})$ is of rank 4, generated by $\gamma \wedge ds$, $\gamma \wedge dy$, $ds \wedge dt$ and $dy \wedge dt$; and $H^3(M, \mathbb{R})$ is of rank 3, generated by $\gamma \wedge dy \wedge dt$, $\gamma \wedge dy \wedge ds$ and $\gamma \wedge ds \wedge dt$, where $\gamma = dx - sdy$.

Proof. This follows from an easy calculation. □

Proof of Theorem 3.1 We use the commutative diagram in Lemma 2.1. For manifolds of dimension 4, the diagram reads as

$$
\begin{array}{ccc}
\pi_1(\text{Symp}_0(M, \omega)) & \xrightarrow{\mathfrak{ev}} & H_1(M, \mathbb{Z}) \\
\downarrow{id} & & \downarrow{\text{PD}} \\
\pi_1(\text{Symp}_0(M, \omega)) & \xrightarrow{\text{flux}} & H^1(M, \mathbb{R}) \xrightarrow{\wedge[\omega]} H^3(M, \mathbb{R}).
\end{array}
$$

Denote by $C_0(M)$ the identity component of $C(M)$. It was proved in Gottlieb [2] (Theorem III.2) that for all aspherical manifolds $M$,

$$
ev_c : \pi_1(C_0(M)) \cong Z(\pi_1(M))$$

is a group isomorphism, where $Z(\pi_1(M))$ stands for the center of $\pi_1(M)$. For the Kodaira-Thurston manifold $M = \mathbb{R}^4/G$, we have $\pi_1(M) = G$. It is easy to check that $Z(\pi_1(M)) = \mathbb{Z}(\partial/\partial t, \partial/\partial x)$, and the commutator group $[\pi_1(M), \pi_1(M)] = \mathbb{Z}((\partial/\partial x)^3)$, see Example 3.8 in [14]. Thus the image of $\mathfrak{ev}_c$ in $H_1(M, \mathbb{Z})$ is contained in $Z(\pi_1(M))/[\pi_1(M), \pi_1(M)] = \mathbb{Z}(\partial/\partial t, \partial/\partial x) / \mathbb{Z}(\partial/\partial x)^3 = \mathbb{Z}(\partial/\partial t)^3$.

Note that $PD(\partial/\partial t) = -dx \wedge dy \wedge ds = -\gamma \wedge dy \wedge ds$, where $\gamma = dx - sdy$. Now look at the map $\wedge[\omega] : H^1(M, \mathbb{R}) \to H^3(M, \mathbb{R})$,

$$
ds \mapsto ds \wedge \omega = \gamma \wedge dy \wedge ds \neq 0, \\
dt \mapsto dt \wedge \omega = \gamma \wedge dy \wedge dt \neq 0, \\
dy \mapsto dy \wedge \omega = dy \wedge ds \wedge dt = 0.
$$

Here we have used the fact that the 3-form $dy \wedge ds \wedge dt = d(\gamma \wedge dt)$ is exact, so it vanishes on the cohomology level. Since $\text{vol}(M) = 1$, we conclude from the above commutative diagram that the flux subgroup $\Gamma \subset H^1(M, \mathbb{R})$ is contained in $\mathbb{Z}(ds, dy)$. An explicit construction shows that $\Gamma$ is actually equal to $\mathbb{Z}(ds, dy)$. Namely, we take two elements $\{\phi_0\}$ and $\{\psi_0\}$ in $\pi_1(\text{Symp}_0(M, \omega))$ such that

$$
\phi_0(s, t, x, y) = (s, t - \theta, x, y), 0 \leq \theta \leq 1, \\
\psi_0(s, t, x, y) = (s, t, x + \theta, y), 0 \leq \theta \leq 1.
$$

Using (1) in Section 2 one can show that flux$(\{\phi_0\}) = ds$ and flux$(\{\psi_0\}) = dy$. This completes the proof of Theorem 3.1. □
4. The Hofer norm

Let \((M, \omega)\) be a closed symplectic manifold of dimension \(2n\). Denote by \(\mathcal{A}\) the space of all normalized smooth functions on \(M\) with respect to the volume form \(\omega^n\), i.e.

\[
\mathcal{A} := \{ F \in C^\infty(M) \mid \int_M F \omega^n = 0 \}.
\]

It is well known that \(\mathcal{A}\) can be identified with the space of Hamiltonian vector fields, which is the Lie algebra \(\mathfrak{h}\) of the \(\infty\)-dimensional Lie group \(\text{Ham}(M, \omega)\).

The \(L_\infty\) norm on \(\mathcal{A}\)

\[
\| F \|_\infty = \max F - \min F
\]
gives rise to the Hofer metric \(d\) on \(\text{Ham}(M, \omega)\) in the following way: we define the Hofer length of a smooth Hamiltonian path \(\alpha : [0, 1] \to \text{Ham}(M, \omega)\) as

\[
\text{length}(\alpha) := \int_0^1 \| \dot{\alpha}(t) \|_\infty dt = \int_0^1 \| F_\alpha(t) \|_\infty dt,
\]

where \(F_\alpha(x) = F(t, x)\) is the time-dependent Hamiltonian function generating the path \(\alpha\). The Hofer distance \(d\) between two Hamiltonian diffeomorphisms \(f, g\) is defined by

\[
d(f, g) := \inf \{ \text{length}(\alpha) \},
\]

where the infimum is taken over all Hamiltonian paths \(\alpha\) connecting \(f\) and \(g\). The Hofer norm \(\rho(f)\) is the Hofer distance between the identity map \(id\) and \(f\), i.e.

\[
\rho(f) := d(id, f).
\]

It is easy to check that \(d\) is bi-invariant in the sense that

\[
d(fh, gh) = d(hf, hg) = d(f, g)
\]

for all \(f, g, h \in \text{Ham}(M, \omega)\). The fact that \(d\) is nondegenerate is highly nontrivial. This was proved by Hofer [5] for the case of \(\mathbb{R}^{2n}\), then generalized by Polterovich [17] to some larger class of symplectic manifolds, and finally proved in the full generality by Lalonde and McDuff [8] using the following energy-capacity inequality

\[
e(S) \geq \frac{1}{2} \text{capacity}(S)
\]

for a subset \(S\) of \(M\). Here the capacity of \(S\) is equal to \(\pi r^2\) when \(S\) is a symplectically embedded ball of radius \(r\), and is defined in general as the supremum of the capacities of all symplectically embedded balls in \(S\). The displacement energy \(e(S)\) is defined to be the infimum of the Hofer norms of all \(f \in \text{Ham}(M, \omega)\) such that \(f(S) \cap S = \emptyset\).

Note that the energy-capacity inequality provides a lower bound for the Hofer norm. Namely, we have

\[
f(S) \cap S = \emptyset, \text{capacity}(S) > c \implies \rho(f) > c/2.
\]

This fact will be crucial in our proof of Theorem 1.3.

Recall in Definition 1.1 that an element \(\phi \in \text{Symp}(M, \omega)\) is called unbounded if

\[
r(\phi) := \sup \{ \rho([\phi, f]) \mid f \in \text{Ham}(M, \omega) \} = \infty.
\]

As a vector space, the Lie algebra is by definition the tangent space to the Lie group at the identity. The tangent spaces to the Lie group at other points are identified with the Lie algebra with the help of right shifts of the group.
Note that all Hamiltonian diffeomorphisms are bounded since \( r(g) \leq 2\rho(g) < \infty \) for all \( g \in \text{Ham}(M, \omega) \), where \( \rho(g) \) is the Hofer norm of \( g \). According to Proposition 1.2. A in [11], \( r \) satisfies the triangle inequality \( r(\phi \psi) \leq r(\phi) + r(\psi) \). Since \( \text{Ham}(M, \omega) \) is the kernel of the flux homomorphism, two symplectomorphisms \( \phi \) and \( \psi \) have the same flux if and only if they differ by a Hamiltonian diffeomorphism. Combining these facts, we have the following

**Observation A.** [11] In order to prove \( BI_0(M, \omega) = \text{Ham}(M, \omega) \), it suffices to show that for each nonzero value \( v \in H^1(M, \mathbb{R})/\Gamma \), there exists some unbounded element \( \phi \in \text{Symp}_0(M, \omega) \) with flux(\( \phi \)) = \( v \).

5. The admissible lift

To prove an element \( \phi \in \text{Symp}_0(M, \omega) \) is unbounded, one has to show that \( \rho([\phi, f]) \) can be arbitrarily large by choosing different \( f \in \text{Ham}(M, \omega) \). Hence the energy-capacity inequality will not work directly for closed manifolds since the capacity of the manifold itself is finite. To go around this difficulty, we recall the notion of admissible lifts which was first introduced by Lalonde and Polterovich [11]. We shall point out that our definition is slightly different from theirs, but the two definitions are equivalent.

Let \( \pi : (\widetilde{M}, \widetilde{\omega}) \rightarrow (M, \omega) \) be a symplectic covering map, i.e. a covering map \( \pi \) between two symplectic manifolds such that \( \widetilde{\omega} = \pi^* \omega \).

**Definition 5.1.** For every \( g \in \text{Ham}(M, \omega) \), assume \( g \) is the time-1 map of the Hamiltonian flow generated by time-dependent Hamiltonian function \( H_t \). An admissible lift \( \tilde{g} \in \text{Ham}(\widetilde{M}, \widetilde{\omega}) \) of \( g \) with respect to \( \pi \) is defined to be the time-1 map of the Hamiltonian flow generated by \( H_t \circ \pi \).

**Lemma 5.2** (existence and uniqueness of admissible lifts). For all \( g \in \text{Ham}(M, \omega) \), such an admissible lift \( \tilde{g} \in \text{Ham}(\widetilde{M}, \widetilde{\omega}) \) exists and is unique.

**Proof.** The existence follows from the definition. For the uniqueness, it suffices to show that the admissible lift \( \tilde{g} \) of \( g \) is independent of the choice of the Hamiltonian function \( H_t \).

Note that the choice of \( H_t \) is equivalent to the choice of the Hamiltonian isotopy \( g_t \) connecting \( \text{id} \) to \( g \). For every point \( p \in M \), let

\[
\tilde{e}v_p : \pi_1(\text{Ham}(M, \omega), \text{id}) \rightarrow \pi_1(M, p)
\]

be the map induced by the evaluation map \( ev_p : \text{Ham}(M, \omega) \rightarrow M \) which takes \( g \) to \( g(p) \). It follows from Floer theory that for all symplectic manifolds \( (M, \omega) \), the induced map \( \tilde{e}v_p \) is trivial, see Chapter 11 [14] for instance. This deep result implies that for any two different paths \( g^1_t \) and \( g^2_t \) in \( \text{Ham}(M, \omega) \) connecting \( \text{id} \) to \( g \), \( g^1_t(p) \) and \( g^2_t(p) \) must be homotopic paths in \( M \). Therefore, for every point \( \tilde{p} \in \tilde{M} \), the image \( \tilde{g} \tilde{(p)} \) of \( \tilde{p} \) under \( \tilde{g} \), being the endpoint of the lift of the path \( g_t(p) \), is independent of the choice of the Hamiltonian isotopy \( g_t \). This proves the uniqueness of admissible lifts.

For our purposes, we consider the universal cover \( \tilde{M} \) of \( M \). Note that \( \tilde{M} \) is not necessarily compact, and the admissible lift \( \tilde{g} \) of \( g \in \text{Ham}(M, \omega) \) is not necessarily compactly supported in \( \tilde{M} \). Instead, it belongs to \( \text{Ham}_b(\tilde{M}, \tilde{\omega}) \) of time-1 maps of
bounded Hamiltonians $\tilde{M} \times [0, 1] \to \mathbb{R}$. The Hofer norm is still well defined and the same energy-capacity inequality still holds for this setting. This idea is due to Lalonde and Polterovich [11]. We shall spell out some details here for the sake of clarity.

Denote by $(N, \sigma)$ a noncompact symplectic manifold without boundary. We do not often consider the group $\text{Ham}(N, \sigma)$ of all Hamiltonian diffeomorphisms with arbitrary support. One reason in our context is that it would not be possible to define the Hofer norm on $\text{Ham}(N, \sigma)$ using the $L_{\infty}$ norm on the space $\mathcal{A}$ of all Hamiltonian functions with arbitrary support, since not all elements in $\mathcal{A}$ have finite $L_{\infty}$ norms.

One may consider the group $\text{Ham}^c(N, \sigma)$ of Hamiltonian diffeomorphisms with compact support. The Hofer norm $\rho$ is well defined on $\text{Ham}^c(N, \sigma)$, and the energy-capacity inequality

$$e_c(S) \geq \frac{1}{2}\text{capacity}(S)$$

is valid as usual, where

$$e_c(S) := \inf \{ \rho(f) \mid f \in \text{Ham}^c(N, \sigma), f(S) \cap S = \emptyset \}.$$

As we have already pointed out, however, this setting is not sufficient for our purposes since the admissible lift is usually not compactly supported. Hence we need to consider the larger group $\text{Ham}_b(N, \sigma)$ of Hamiltonian diffeomorphisms which are time-1 maps of bounded Hamiltonians $H : N \times [0, 1] \to \mathbb{R}$. Note that one can not use an arbitrary bounded Hamiltonians $H$, since the Hamiltonian flow generated by $H$ need not be integrable. Instead, we only restrict to those bounded Hamiltonians whose flows are integrable.

The Hofer norm can be defined on $\text{Ham}_b(N, \sigma)$ exactly the same way as in Section 4. For a subset $S$ of $N$, define also the bounded displacement energy $e_b(S)$ as

$$e_b(S) := \inf \{ \rho(f) \mid f \in \text{Ham}_b(N, \sigma), f(S) \cap S = \emptyset \}.$$

Note that $\text{Ham}_c(N, \sigma) \subset \text{Ham}_b(N, \sigma)$ implies $e_b(S) \leq e_c(S)$ for any subset $S \subset N$. In fact, for any compact subset $S$, we have

$$e_b(S) = e_c(S).$$

To prove the other inequality, note that if $f \in \text{Ham}_b(N, \sigma)$ displaces a compact subset $S$ from itself, one can easily construct some cut-off $f_{\text{cut}} \in \text{Ham}_c(N, \sigma)$ of $f$ which still displaces $S$ from itself, and the Hofer norm satisfies $\rho(f) \geq \rho(f_{\text{cut}})$. Taking the infimum implies $e_b(S) \geq e_c(S)$.

The above argument implies that the energy-capacity inequality still holds for the bounded displacement energy. That is

$$e_b(S) \geq \frac{1}{2}\text{capacity}(S).$$

Now back to our discussion about the admissible lift. Note that the admissible lift $\tilde{g}$ of $g \in \text{Ham}(M, \omega)$ belongs to $\text{Ham}_b(\tilde{M}, \tilde{\omega})$. And it follows from the definition of the admissible lift that

$$\rho(g) \geq \rho(\tilde{g})$$

for all $g \in \text{Ham}(M, \omega)$ and the admissible lift $\tilde{g} \in \text{Ham}_b(\tilde{M}, \tilde{\omega})$ of $g$. Here the two $\rho$’s are the Hofer norms on $\text{Ham}(M, \omega)$ and $\text{Ham}_b(\tilde{M}, \tilde{\omega})$ respectively. Combining the above discussions, we have
Observation B. \[11\] To construct $g \in \text{Ham}(M, \omega)$ of arbitrarily large Hofer norm, it suffices to make sure that the unique admissible lift $\tilde{g} \in \text{Ham}_b(\tilde{M}, \tilde{\omega})$ of $g$ displaces from itself a symplectic ball in $\tilde{M}$ of arbitrarily large capacity.

6. Proof of Theorem 1.3

In this section, we prove Theorem 1.3. Recall that $(M, \omega)$ is the Kodaira-Thurston manifold with the standard symplectic form $\omega = ds \wedge dt + dx \wedge dy$. Recall also that $H^1(M, \mathbb{R}) = \mathbb{R}\langle ds, dy, dt \rangle$ and the flux subgroup $\Gamma = \mathbb{Z}\langle ds, dy \rangle$ by Lemma 3.2 and Theorem 3.1. In view of Observation A, to prove $BI_0^1(M, \omega) = \text{Ham}(M, \omega)$, it suffices to show that for every nonzero element $v \in H^1(M, \mathbb{R})/\Gamma = \mathbb{R}/\mathbb{Z}\langle ds, dy \rangle \oplus \mathbb{R}\langle dt \rangle$, there exists some unbounded symplectomorphism with flux equal to $v$. We begin with an explicit construction of symplectomorphisms with given fluxes.

**Lemma 6.1.** Let $v$ be an element in $H^1(M, \mathbb{R})/\Gamma = \mathbb{R}/\mathbb{Z}\langle ds, dy \rangle \oplus \mathbb{R}\langle dt \rangle$, say $v = \alpha ds + \beta dy + \epsilon dt$ where $\alpha, \beta \in \mathbb{R}/\mathbb{Z}$ and $\epsilon \in \mathbb{R}$. Then there exists an element $\phi_{\alpha\beta\epsilon} \in \text{Symp}_0^0(M, \omega)$ with $\text{flux}(\phi_{\alpha\beta\epsilon}) = v$. Namely, $\phi_{\alpha\beta\epsilon}(s, t, x, y) = (s + c, t - \alpha, x + \beta, y)$.

**Proof.** First $\phi_{\alpha\beta\epsilon}$ is well-defined. For instance, since $(s, t, x, y)$ and $(s + 1, t, x + \beta, y)$ represent the same point on $M$, one has to show that $\phi_{\alpha\beta\epsilon}(s, t, x, y) \sim \phi_{\alpha\beta\epsilon}(s + 1, t, x, y, y)$. This is true since $\phi_{\alpha\beta\epsilon}(s, t, x, y) = (s + c, t - \alpha, x + \beta, y)$, and $\phi_{\alpha\beta\epsilon}(s + 1, t, x + \beta, y) = (s + 1 + c, t - \alpha, x + y + \beta, y)$. It is easy to see that $\phi_{\alpha\beta\epsilon}$ preserves $\omega$, and the obvious isotopy from $id$ to $\phi_{\alpha\beta\epsilon}$ implies that $\phi_{\alpha\beta\epsilon} \in \text{Symp}_0^0(M, \omega)$. The calculation for $\text{flux}(\phi_{\alpha\beta\epsilon}) = v$ is straightforward using (1) in Section 2. \[\Box\]

The following theorem due to Lalonde and Polterovich \[11\] is an important criteria for unbounded symplectomorphisms.

**Theorem 6.2** (Theorem 1.4.A \[11\]). Let $L \subset M$ be a closed Lagrangian submanifold admitting a Riemannian metric with non-positive sectional curvature, and whose inclusion in $M$ induces an injection on fundamental groups. Let $\phi$ be an element in $\text{Symp}_0(M, \omega)$ such that $\phi(L) \cap L = \emptyset$. Then $\phi$ is unbounded.

For the proof, one passes to the universal cover $\tilde{M}$ of $M$. The hypothesis implies that the lift of a neighbourhood $U$ of $L$ has infinite capacity. One then constructs a Hamiltonian isotopy $f_\tau$ supported in $U$ so that the admissible lift $[\phi, f_\tau]$ of the commutator $[\phi, f_\tau]$ will displace a symplectic ball of arbitrarily large capacity as $\tau$ goes to infinity. This implies $\phi$ is unbounded according to Observation B. See \[11\] for details.

**Proof of Theorem 1.3.** In view of Observation A, it suffices to show that the symplectomorphisms $\phi_{\alpha\beta\epsilon}$ constructed in Lemma 6.1 are unbounded in all cases,
as long as the flux \( \mathbf{v} = \alpha ds + \beta dy + c dt \) does not vanish. We argue case by case. In the first two cases, this is a direct consequence of Theorem 6.2.

**Case 1.** \( \alpha \neq 0 \in \mathbb{R}/\mathbb{Z} \).

Let \( L \subset M \) be the subset of \( M \) defined by
\[
L := \{(s, t, x, y) \in M \mid t = 0, y = 0\}.
\]
It is easy to check that \( L \) is a Lagrangian torus satisfying the hypothesis of Theorem 6.2, and \( \phi_{\alpha\beta c} \) displaces \( L \) from itself. Thus \( \phi_{\alpha\beta c} \) is unbounded.

**Case 2.** \( \alpha = 0 \in \mathbb{R}/\mathbb{Z}, \beta \neq 0 \in \mathbb{R}/\mathbb{Z} \) and \( c = 0 \in \mathbb{R} \).

In this case, \( \phi_{\beta} := \phi_{\alpha\beta c} \) maps \( (s, t, x, y) \) to \( (s, t, x + \beta, y) \). As in the first case, \( \phi_{\beta} \) displaces from itself a Lagrangian torus \( L \) of \( M \) defined by
\[
L := \{(s, t, x, y) \in M \mid s = 0, x = 0\}.
\]
We again get \( \phi_{\beta} \) is unbounded in view of Theorem 6.2.

**Case 3.** \( \alpha = 0 \in \mathbb{R}/\mathbb{Z} \) and \( c \neq 0 \in \mathbb{R} \).

We write \( \phi_{\beta c} = \phi_{\alpha\beta c} \) for \( \phi_{\alpha\beta c} \) in this case,
\[
\phi_{\beta c} := \phi_{\alpha\beta c} : (s, t, x, y) \mapsto (s + c, t, x + \beta, y).
\]
Consider two different situations, one of which is simple, while the other is more complicated.

**3A.** \( \alpha = 0 \in \mathbb{R}/\mathbb{Z} \) and \( c \notin \mathbb{Z} \).

As in case 1 and 2, \( \phi_{\beta c} \) is unbounded as it displaces from itself
\[
L := \{(s, t, x, y) \in M \mid s = 0, x = 0\}.
\]

**3B.** \( \alpha = 0 \in \mathbb{R}/\mathbb{Z} \) and \( c \in \mathbb{Z}\setminus\{0\} \).

Note that \( (s + c, t, x + \beta, y) \sim (s, t, x + \beta - cy, y) \). So the map \( \phi_{\beta c} : M \to M \) can also be expressed as
\[
\phi_{\beta c}(s, t, x, y) = (s, t, x + \beta - cy, y).
\]

In contrast to all previous cases where we used the same argument, here we are facing a difficulty. The trouble is that in this case we are unable to find a Lagrangian torus of \( M \) which is disjoined from itself by the map \( \phi_{\beta c} \). Thus the above argument breaks down.

To resolve this difficulty, we take \( f_{\tau} \) to be the Hamiltonian isotopy whose support is in the subset
\[
U := \{(s, t, x, y) \in M \mid |s| < \epsilon, |x| < \epsilon\}
\]
of \( M \). We require \( f_{\tau} \) to flow only along \( y \) and \( t \) direction in \( U \) and its restriction to
\[
V := \{(s, t, x, y) \in M \mid |s| < \epsilon/2, |x| < \epsilon/2\}
\]
is defined by
\[
f_{\tau}(s, t, x, y) = (s, t, x - \tau).\]

In the discussion below, \( [f, g] := fgf^{-1}g^{-1} \) stands for the commutator of \( f \) and \( g \). Our goal is to show that the unique admissible lift \( \tilde{\phi}_{\beta c}, f_{\tau} \) of \( [\phi_{\beta c}, f_{\tau}] \) still displaces from itself a subset of \( \mathbb{R}^4 \) of arbitrarily large capacity when \( \tau \) goes to infinity. For this, we need the following
Lemma 6.3. Let $\phi \in \text{Symp}_0(M, \omega)$, and $f_\tau$ be a Hamiltonian isotopy of $M$. Let $\tilde{\phi} : \tilde{M} \to \tilde{M}$ be any lift of $\phi$, and $[\phi, f_\tau]$ and $\tilde{f}_\tau$ be the unique admissible lift of $[\phi, f_\tau]$ and $f_\tau$ respectively. Then $[\phi, f_\tau] = [\tilde{\phi}, \tilde{f}_\tau]$.

Proof. Note that $f_\tau$ is Hamiltonian implies $[\phi, f_\tau]$ is Hamiltonian. So both admissible lifts $[\phi, f_\tau]$ and $\tilde{f}_\tau$ make sense. To simplify notation, denote $A_\tau := [\phi, f_\tau]$ and $B_\tau := [\tilde{\phi}, \tilde{f}_\tau]$. We want to show $A_\tau = B_\tau$, which is equivalent to $A_\tau B_\tau^{-1} = id$. Since $A_\tau$ and $B_\tau$ are both lifts of $[\phi, f_\tau]$, $A_\tau B_\tau^{-1}$ is the deck transformation of the covering map $\pi : \tilde{M} \to M$. Now $A_0 B_0 = id$, and $\tau \to A_\tau B_\tau^{-1}$ is a continuously parametrized path into the discrete set of all deck transformations. Thus $A_\tau B_\tau^{-1} = id$ for all $\tau$. $\square$

Now back to the proof of Theorem 4.3. To prove $\phi_{\beta c}$ is unbounded, we need to show that the commutator $[\phi_{\beta c}, f_\tau]$ has arbitrarily large Hofer norm when $\tau$ goes to infinity. Let $V_0 \subset \mathbb{R}^4$ be the subset of $\mathbb{R}^4$ defined by

$$V_0 := \{(s, t, x, y) \in \mathbb{R}^4 \mid |s| < \epsilon/2, t \in \mathbb{R}, |x| < \epsilon/2, 0 < y < \tau/2\}.$$ 

Since $V_0$ has arbitrarily large capacity as $\tau$ goes to infinity, according to Observation B, it suffices to show that the admissible lift $[\phi_{\beta c}, f_\tau]$ displaces $V_0$ from itself.

For this, denote by $\tilde{\phi}_{\beta c} : \mathbb{R}^4 \to \mathbb{R}^4$ the preferred lift of the map $\phi_{\beta c}$ such that $\tilde{\phi}_{\beta c}(s, t, x, y) = (s, t, x + \beta - cy, y)$. By the above lemma, it suffices to show that $[\tilde{\phi}_{\beta c}, \tilde{f}_\tau](V_0) \cap V_0 = \emptyset$, which is equivalent to $\tilde{\phi}_{\beta c}^{-1} \tilde{f}_\tau^{-1}(V_0) \cap \tilde{f}_\tau^{-1} \tilde{\phi}_{\beta c}^{-1}(V_0) = \emptyset$.

Note that the restriction of $\tilde{f}_\tau$ to $\tilde{V}$ is defined by $\tilde{f}_\tau(s, t, x, y) = (s, t, x, y - \tau)$.

We have $\tilde{f}_\tau^{-1}(V_0) = \{(s, t, x, y) \in \mathbb{R}^4 \mid |s| < \epsilon/2, t \in \mathbb{R}, |x| < \epsilon/2, y \in \mathbb{R}\}$.

Hence $\tilde{\phi}_{\beta c}^{-1} \tilde{f}_\tau^{-1}(V_0) = \{|s| < \epsilon/2, t \in \mathbb{R}, |x + \beta - cy| < \epsilon/2, \tau < y < 3\tau/2\}$.

On the other hand, $\tilde{\phi}_{\beta c}^{-1}(V_0) = \{|s| < \epsilon/2, t \in \mathbb{R}, |x + \beta - cy| < \epsilon/2, 0 < y < \tau/2\}$.

Note that in the set $\tilde{\phi}_{\beta c}^{-1} \tilde{f}_\tau^{-1}(V_0)$ we have $|x| > |cy| - |\beta| - \epsilon/2 > |c|\tau - |\beta| - \epsilon/2$,

and in $\tilde{\phi}_{\beta c}^{-1}(V_0)$ we have $|x| < |cy| + |\beta| + \epsilon/2 < |c|\tau/2 + |\beta| + \epsilon/2$. 

Thus for sufficiently large \( \tau \), these two sets do not share the same values in \( x \) coordinates. Since the flow \( \tilde{f}_\tau^{-1} \) only changes the \( y \) and \( t \)-coordinates when restricted to \( \tilde{\phi}_{\beta c}(V_0) \), we conclude

\[
\tilde{\phi}_{\beta c}^{-1}(V_0) \cap \tilde{f}_\tau^{-1} \tilde{\phi}_{\beta c}^{-1}(V_0) = \emptyset.
\]

As we have already mentioned above, this implies \( \phi_{\beta c} \) is unbounded in case 3B, which completes the proof of Theorem 1.3.

\[\square\]

7. Proof of Theorem 1.4

We have already mentioned in Section 1 that the bounded isometry conjecture holds for the torus with the standard symplectic form. In this section we prove Theorem 1.4 which states that the conjecture holds for the 4-torus with any linear symplectic form.

**Remark 7.1.** The 2-form \( \omega = \sum_{i<j} a_{ij} dx_i \wedge dx_j \) on \( T^4 \) is symplectic, i.e. nondegenerate if and only if \( a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} \neq 0 \).

For each \( 1 \leq i \leq 4 \), let \( \{ \phi_i' \} \in \pi_1(\text{Symp}_0(T^4, \omega)) \) be the loop of rotations of \( T^4 \) along \( x_i \) direction. Let \( \xi_i \in H^3(T^4, \mathbb{R}) \) be the image of \( \{ \phi_i' \} \) under the flux homomorphism. Using (1) in Section 2 one easily gets

\[
\xi_i := \text{flux}(\{ \phi_i' \}) = \sum_{j=1}^4 a_{ij} dx_j.
\]

Here we take the convention that \( a_{ij} = -a_{ji} \). In particular, \( a_{ii} = 0 \).

**Lemma 7.2.** For the 4-torus with the linear symplectic form \( \omega := \sum_{i<j} a_{ij} dx_i \wedge dx_j \), the flux subgroup \( \Gamma \subset H^3(T^4, \mathbb{R}) \) is generated by the above \( \xi_i \)’s over \( \mathbb{Z} \). That is, \( \Gamma = \mathbb{Z}(\xi_1, \xi_2, \xi_3, \xi_4) \).

**Proof.** According to Lemma 2.1 we have the following commutative diagram for the manifold \((T^4, \omega)\).

\[
\begin{align*}
\pi_1(\text{Symp}_0(T^4, \omega)) & \xrightarrow{\text{id}} H_1(T^4, \mathbb{Z}) \xrightarrow{\text{PD}} H^3(T^4, \mathbb{Z}) \\
\downarrow{\text{id}} & \downarrow{\text{vol}(T^4)} \\
\pi_1(\text{Symp}_0(T^4, \omega)) & \xrightarrow{\text{flux}} H^1(T^4, \mathbb{R}) \xrightarrow{\wedge [\omega]} H^3(T^4, \mathbb{R}).
\end{align*}
\]

Note that \( \tilde{v}_s \) is surjective, and \( \wedge [\omega] : H^1(T^4, \mathbb{R}) \to H^3(T^4, \mathbb{R}) \) is an isomorphism. Note also that \( \text{vol}(T^4) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} \). It follows from a similar argument as in the proof of Theorem 3.1 that \( \xi_i(1 \leq i \leq 4) \) span the flux subgroup \( \Gamma \) over \( \mathbb{Z} \).

Now let \( \phi \in \text{Symp}_0(T^4, \omega) \) such that

\[
\phi(x_1, x_2, x_3, x_4) = (x_1 + \alpha_1, x_2 + \alpha_2, x_3 + \alpha_3, x_4 + \alpha_4)
\]

where \( \alpha_i \in \mathbb{R}/\mathbb{Z} \) for \( 1 \leq i \leq 4 \). Then

\[
\text{flux}(\phi) = \sum_{i=1}^4 \alpha_i \xi_i.
\]
Recall that in view of Observation A in Section 4, to prove Theorem 1.4, it suffices to show \( \phi \) is unbounded as long as at least one \( \alpha_i \in \mathbb{R}/\mathbb{Z} \) is nonzero. One may attempt to apply Theorem 6.2 by showing \( \phi \) disjoins some Lagrangian torus \( L \subset T^4 \) from itself. For a general symplectic form \( \omega \), however, there may not exist any such Lagrangian torus in \( T^4 \). Nevertheless, we can still prove \( \phi \) is unbounded using the following

Lemma 7.3. Let \((M, \omega)\) be an aspherical symplectic manifold. Let \( f_\tau \in \text{Ham}(M, \omega) \) be the flow generated by an autonomous Hamiltonian which has no nonconstant contractible orbits. Then the Hofer norm \( \rho(f_\tau) \) goes to infinity as \( \tau \) goes to infinity.

This result can be found in Oh [16], Schwarz [18] and Kerman-Lalonde [7]. The main idea of the argument is that the Hofer norm is bounded from below by the spectral norm, while the spectral norm of such \( f_\tau \) grows linearly with respect to \( \tau \).

Proof of Theorem 1.4. Let \( \phi \in \text{Symp}_0(T^4, \omega) \) such that
\[
\phi(x_1, x_2, x_3, x_4) = (x_1 + \alpha_1, x_2 + \alpha_2, x_3 + \alpha_3, x_4 + \alpha_4).
\]
As discussed above, it suffices to show \( \phi \) is unbounded when at least one \( \alpha_i \in \mathbb{R}/\mathbb{Z} \) is nonzero. Assume \( \alpha_1 \neq 0 \) without loss of generality. Thus \( \phi(U) \cap U = \emptyset \) where \( U \subset T^4 \) is defined by
\[
U := \{(x_1, x_2, x_3, x_4) \in T^4 \mid |x_1| < \epsilon \}.
\]
for sufficiently small \( \epsilon \).

Let \( H \) be a time-independent Hamiltonian function of \( T^4 \) supported in \( U \). Denote by \( f_\tau \) the (autonomous) Hamiltonian flow generated by \( H \). Since \( \phi(U) \cap U = \emptyset \), we know that \([\phi, f_\tau] := \phi f_\tau \phi^{-1} f_\tau^{-1}\) is also an autonomous Hamiltonian flow supported in the union of two disjoint sets \( U \cup \phi(U) \). If we further require that \( H \) depend only on the first coordinate \( x_1 \), using the fact that \( \omega \) is a linear symplectic form, we conclude that \([\phi, f_\tau]\) has no nonconstant contractible orbits. Thus it follows from Lemma 7.3 that the Hofer norm \( \rho([\phi, f_\tau]) \) goes to infinity as \( \tau \) goes to infinity. Hence \( \phi \) is unbounded in the sense of Definition 1.1. \( \square \)

8. The Kodaira-Thurston Manifold with Linear Symplectic Forms

So far we have studied bounded isometries for the Kodaira-Thurston manifold with the standard symplectic form and for the 4-torus with all linear symplectic forms. In particular, we have shown that the bounded isometry conjecture holds in both cases. In this section we will study the same question for the Kodaira-Thurston manifold with all linear symplectic forms.

Question 8.1. Does the bounded isometry conjecture hold for the Kodaira-Thurston manifold with all linear symplectic forms?

We expect the answer to be positive. Although we are not able to give a complete proof yet at this time, we shall provide some partial results below. We begin by describing the linear symplectic forms on the Kodaira-Thurston manifold \( M \). Recall that it follows from Lemma 3.2 that \( H^2(M, \mathbb{R}) \) is of rank 4, generated by \( \gamma \wedge ds, \gamma \wedge dy, ds \wedge dt \), and \( dy \wedge dt \) where \( \gamma = dx - sdy \). We consider linear 2-forms
\[
\omega_{abc} := a \gamma \wedge ds + b \gamma \wedge dy + c ds \wedge dt + f dy \wedge dt.
\]
Note that $\omega_{abef}$ is a symplectic form if and only if $be - af \neq 0$. In particular, the standard symplectic form corresponds to $b = c = 1$ and $a = f = 0$. The following lemma on the flux subgroup generalizes Theorem 3.1.

**Lemma 8.2.** The flux subgroup $\Gamma \subset H^1(M, \mathbb{R})$ of the Kodaira-Thurston manifold with the linear symplectic form $\omega_{abef}$ has rank 2 over $\mathbb{Z}$. More precisely, we have $\Gamma = \mathbb{Z}(eds + fdy, ads + bdy)$.

**Proof.** The proof follows the same lines as that of Theorem 3.1. According to Lemma 2.1, we have the following commutative diagram.

$$
\begin{array}{ccc}
\pi_1(\text{Symp}_0(M, \omega_{abef})) & \xrightarrow{\text{ev}} & H^1(M, \mathbb{Z}) \\
\downarrow \text{id} & & \downarrow \text{PD} \quad \downarrow \text{vol}(\mathcal{M}) \\
\pi_1(\text{Symp}_0(M, \omega_{abef})) & \xrightarrow{\text{flux}} & H^1(M, \mathbb{R}) \xrightarrow{\wedge [\omega_{abef}]} H^3(M, \mathbb{R}).
\end{array}
$$

As in the proof of Theorem 3.1, the image of $ev_s$ in $H_1(M, \mathbb{Z})$ is contained in $\mathbb{Z}([\gamma])$. Note that $PD([\gamma]) = -\gamma \wedge dy \wedge ds$, where $\gamma = dx - sdy$. Now look at the map $\gamma \wedge \omega_{abef} : H^1(M, \mathbb{R}) \to H^3(M, \mathbb{R})$,

$$
ds \mapsto ds \wedge \omega_{abef} = b\gamma \wedge dy \wedge ds - fdy \wedge ds \wedge dt = b\gamma \wedge dy \wedge ds,
$$

$$
dt \mapsto dt \wedge \omega_{abef} = a\gamma \wedge ds \wedge dt + b\gamma \wedge dy \wedge dt,
$$

$$
dy \mapsto dy \wedge \omega_{abef} = -a\gamma \wedge dy \wedge ds + edy \wedge ds \wedge dt = -a\gamma \wedge dy \wedge ds.
$$

Here we have used the fact that the 3-form $dy \wedge ds \wedge dt = d(\gamma \wedge dt)$ is exact, so it vanishes on the cohomology level. Since $\text{vol}(\mathcal{M}) = be - af \neq 0$, we conclude by tracing the diagram that the flux subgroup $\Gamma \subset H^1(M, \mathbb{R})$ is contained in $\mathbb{Z}(eds + fdy, ads + bdy)$. Note that the fact $be - af \neq 0$ implies that $eds + fdy$ and $ads + bdy$ are linearly independent. An explicit construction shows that $\Gamma$ is actually equal to $\mathbb{Z}(eds + fdy, ads + bdy)$. Namely, we take two elements $\{\phi_0\}$ and $\{\psi_0\}$ in $\pi_1(\text{Symp}_0(M, \omega_{abef}))$ such that

$$
\phi_0(s, t, x, y) = (s, t - \theta, x, y), 0 \leq \theta \leq 1,
$$

$$
\psi_0(s, t, x, y) = (s, t, x + \theta, y), 0 \leq \theta \leq 1.
$$

A straightforward calculation using 1 in Section 2 shows that $\text{flux}(\{\phi_0\}) = eds + fdy$ and $\text{flux}(\{\psi_0\}) = ads + bdy$. $\square$

As in Lemma 6.1, we explicitly construct below symplectomorphisms with given fluxes.

**Lemma 8.3.** Let $v$ be an element in

$$
H^1(M, \mathbb{R})/\Gamma = \mathbb{R}/\mathbb{Z}(eds + fdy, ads + bdy) \oplus \mathbb{R}(dt),
$$

say

$$
v = \alpha(eds + fdy) + \beta(ads + bdy) + c(be - af)dt
$$

where $\alpha, \beta \in \mathbb{R}/\mathbb{Z}$ and $c \in \mathbb{R}$. Then there exists $\phi_{\alpha\beta c} \in \text{Symp}_0(M, \omega_{abef})$ with $\text{flux}(\phi_{\alpha\beta c}) = v$. Namely,

$$
\phi_{\alpha\beta c}(s, t, x, y) = (s + bc, t - \alpha, x + \beta - acs, y - ac).
$$
Proof. First $\phi_{\alpha \beta c}$ is well-defined. For instance, since $(s, t, x, y)$ and $(s+1, t, x+y, y)$ represent the same point in $M$, one has to show that

$$\phi_{\alpha \beta c}(s, t, x, y) \sim \phi_{\alpha \beta c}(s+1, t, x+y, y).$$

This is true since

$$\phi_{\alpha \beta c}(s, t, x, y) = (s + bc, t - \alpha, x + \beta - acs, y - ac)$$

and

$$\phi_{\alpha \beta c}(s+1, t, x+y, y) = (s + 1 + bc, t - \alpha, x + y + \beta - ac(s+1), y - ac)$$

also represent the same point. One can check that $\phi_{\alpha \beta c}^* \omega_{abef} = \omega_{abef}$, and the obvious isotopy from $id$ to $\phi_{\alpha \beta c}$ implies that $\phi_{\alpha \beta c} \in \text{Symp}_0(M, \omega_{abef})$.

It remains to show that $\text{flux}(\phi_{\alpha \beta c}) = v$. Note that $\phi_{\alpha \beta c}$ is the time-1 map of the flow generated by the time-independent symplectic vector field

$$X := bc \frac{\partial}{\partial s} - \alpha \frac{\partial}{\partial t} + (\beta - acs) \frac{\partial}{\partial x} - ac \frac{\partial}{\partial y}.$$

Using (11) in Section 2, we have

$$\text{flux}(\phi_{\alpha \beta c}) = \iota(X) \omega_{abef}$$

$$= \iota(bc \frac{\partial}{\partial s} - \alpha \frac{\partial}{\partial t} + (\beta - acs) \frac{\partial}{\partial x} - ac \frac{\partial}{\partial y}) \omega_{abef}$$

$$= -abc(dx - sdy) + bced + ceds + \alpha f dy$$

$$+ a(\beta - acs)ds + b(\beta - ac)dy + a^2 csds + abcdx - ac f dt$$

$$= \alpha(eds + f dy) + \beta(ads + bdy) + c(be - af)dt$$

$$= v.$$

$\square$

To answer Question 8.1, one has to check whether $\phi_{\alpha \beta c}$ constructed in Lemma 8.3 is always unbounded whenever its flux $v$ is nonzero in $H^1(M, \mathbb{R})/\Gamma$. This is in general a very hard question. In the remaining of this section, we will give a proof for some known cases. For the unknown cases, we will try to point out what difficulty is involved.

**Case 1:** $\alpha \neq 0 \in \mathbb{R}/\mathbb{Z}$. In this case we will prove $\phi_{\alpha \beta c}$ is always unbounded. Note that $\phi_{\alpha \beta c}(U) \cap U = \emptyset$ where $U \subset M$ is defined by

$$U := \{(s, t, x, y) \in M \mid |t| < \epsilon\}$$

for sufficiently small $\epsilon$. We will apply Lemma 7.3 as in the proof of Theorem 1.4. Recall that the only thing we need to do is to construct time-independent Hamiltonian $H$ supported in $U$ whose flow has no nonconstant contractible orbits. This follows from a tedious but straightforward calculation which asserts that

$$\iota(X) \omega_{abef} = dt$$

where

$$X := \frac{1}{bc - af}(-as \frac{\partial}{\partial x} - a \frac{\partial}{\partial y} + b \frac{\partial}{\partial s}).$$

Note that this is actually a special case of the construction in Lemma 8.3. And the fact that $X$ is a well defined vector field on $M$ follows from the equivalence relation $(s, t, x, y) \sim (s + 1, t, x+y, y)$. Since $a$ and $b$ can not be both zero, if we further
require $H$ to depend only on the $t$-coordinates, we know that the Hamiltonian flow generated by $H$ will have no nonconstant contractible orbits. Therefore $\phi_{\alpha\beta c}$ is always unbounded in this case.

**Case 2:** $\alpha = 0 \in \mathbb{R}/\mathbb{Z}$ and $c \neq 0 \in \mathbb{R}$. First we assume $ac$ and $bc$ are not both integers. Note that this is always the case when the ratio $a : b$ is irrational. Under this assumption, $\phi_{\beta c} := \phi_{\alpha\beta c}$ is unbounded in view of Theorem 6.2 as it disjoins a Lagrangian torus

$$L := \{(s, t, x, y) \in M \mid s = 0, y = 0\}.$$ If the ratio $a : b$ is rational, then there exists $c \neq 0$ such that both $ac$ and $bc$ are integers. In this case, using the equivalence relation $(s, t, x, y) \sim (s + 1, t, x + y, y)$, we can write the map

$$\phi_{\beta c} : (s, t, x, y) \mapsto (s + bc, t, x + \beta - acs, y - ac)$$ as

$$\phi_{\beta c} : (s, t, x, y) \mapsto (s, t, x + \beta - acs - bcy, y).$$ It is natural to attempt the admissible lift argument as in Case 3B of Theorem 1.3 for the standard Kodaira-Thurston manifold. One would try to construct a Hamiltonian isotopy $\tilde{f}_\tau$ on $\mathbb{R}^4$ supported in

$$\tilde{U} := \{(s, t, x, y) \in \mathbb{R}^4 \mid \|es + fy\| < \epsilon, |x| < \epsilon\}$$ which flows only along $s$ and $y$ directions, and whose restriction to

$$\tilde{V} := \{(s, t, x, y) \in \mathbb{R}^4 \mid \|es + fy\| < \epsilon/2, |x| < \epsilon/2\}$$ is defined by

$$\tilde{f}_\tau(s, t, x, y) = (s + f\tau, t, x, y - e\tau).$$ Note that the above construction allows us to show that the lift

$$\tilde{\phi}_{\beta c} : (s, t, x, y) \mapsto (s, t, x + \beta - acs - bcy, y)$$ of $\phi_{\beta c}$ is unbounded on the universal cover level. For this, one would argue as in Case 3B of Theorem 1.3 that the commutator $[\tilde{\phi}_{\beta c}, \tilde{f}_\tau]$ displaces some subset $V_0 \subset \mathbb{R}^4$ of arbitrarily large capacity with respect to the symplectic form $\tilde{\omega}_{abef} := \pi^*\omega_{abef}$. Namely,

$$V_0 := \{|es + fy| < \epsilon/2, t \in \mathbb{R}, |x| < \epsilon/2, 0 < as + by < |be - af|\tau/2\}.$$ The problem here is that $\tilde{f}_\tau$ does not descend to a Hamiltonian isotopy on $M$. Note that in proving $\phi_{\beta c}$ itself is unbounded, it is crucial to have such a Hamiltonian isotopy on $M$, not just on the universal cover $\mathbb{R}^4$. Hence this case is still unsolved.

**Case 3:** $\alpha = 0 \in \mathbb{R}/\mathbb{Z}$, $c = 0 \in \mathbb{R}$ and $\beta \neq 0 \in \mathbb{R}/\mathbb{Z}$. In this case, the map

$$\phi_{\beta} := \phi_{\alpha\beta c}$$ has the simple form

$$\phi_{\beta} : (s, t, x, y) \mapsto (s, t, x + \beta, y).$$ We do not know in general how to prove $\phi_{\beta}$ is unbounded for this seemingly easy case. The difficulty in applying Theorem 6.2 is that the obvious torus

$$L := \{(s, t, x, y) \in M \mid s = 0, x = 0\}$$ displaced by $\phi_{\beta}$ is not necessarily Lagrangian with respect to all symplectic forms $\omega_{abef}$. If we assume $f = 0$, then $L$ is actually a Lagrangian torus, and $\phi_{\beta}$ will be unbounded in view of Theorem 6.2. 
Note also that Lemma 7.3 does not work here either since our situation here is different from Case 1 above. The main reason is that

\[ U := \{(s, t, x, y) \in M \mid |x| < \epsilon\} \]

is not a well defined set in \( M \). Thus one can no longer apply Lemma 7.3 by constructing a time-independent Hamiltonian \( H \) supported in \( U \) whose flow has no nonconstant contractible orbits.

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Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003-9305, USA

E-mail address: han@math.umass.edu