RAMANUJAN GRAPHS AND DIGRAPHS

ORI PARZANCHEVSKI

Abstract. Ramanujan graphs have fascinating properties and history. In this paper we
explore a parallel notion of Ramanujan digraphs, collecting relevant results from old and
recent papers, and proving some new ones. Almost-normal Ramanujan digraphs are shown
to be of special interest, as they are extreme in the sense of an Alon-Boppana theorem, and
they have remarkable combinatorial features, such as small diameter, Chernoff bound for
sampling, optimal covering time and sharp cutoff. Other topics explored are the connection
to Cayley graphs and digraphs, the spectral radius of universal covers, Alon’s conjecture for
random digraphs, and explicit constructions of almost-normal Ramanujan digraphs.

1. Introduction

A connected $k$-regular graph is called a Ramanujan graph if every eigenvalue $\lambda$ of its adjacency
matrix (see definitions below) satisfies either

$$|\lambda| = k, \quad \text{or} \quad |\lambda| \leq 2\sqrt{k - 1} \quad \text{(Ramanujan graph)}$$

While the generalized Ramanujan conjecture appears in the first constructions of such graphs
[LPS88, Mar88], the reason that led Lubotzky, Phillips and Sarnak to coin the term Ramanu-
jan graphs is that by their very definition, they present the phenomenon of extremal spectral
behavior, which Ramanujan observed in a rather different setting.

In the case of graphs, this can be stated in two ways: Ramanujan graphs spectrally mimic their
universal cover, the infinite $k$-regular tree $T_k$, whose spectrum is the interval

$$\text{Spec } (T_k) = [-2\sqrt{k - 1}, 2\sqrt{k - 1}]$$

[Kes59]; And, they are asymptotically optimal: the Alon-Boppana theorem (cf. [LPS88, Nil91,
HLW06]) states that for any $\varepsilon > 0$, there is no infinite family of $k$-regular graphs for which all
nontrivial adjacency eigenvalues satisfy $|\lambda| \leq 2\sqrt{k - 1} - \varepsilon$ (the trivial eigenvalues are by definition
$\pm k$). These two observations are closely related - in fact, any infinite family of quotients of a
common covering graph $\tilde{G}$ cannot “do better” than $\tilde{G}$ (see [Gre95, GZ99] for precise statements).

The main interest in the adjacency spectrum of a graph comes from the notion of expanders
- graphs of bounded degree whose nontrivial adjacency spectrum is of small magnitude. Such
graphs have strong connectedness properties which are extremely useful: see [Lub94, HLW06,
Lub12] for extensive surveys on properties of expanders in mathematics and computer science.

Ramanujan graphs, which stand out as the optimal expanders (from a spectral point of view),
have a rich theory and history. The purpose of this paper is to suggest that a parallel theory
should be developed for directed graphs (digraphs, for short), where by a Ramanujan digraph
we mean a $k$-regular digraph whose adjacency eigenvalues satisfy either

$$|\lambda| = k, \quad \text{or} \quad |\lambda| \leq \sqrt{k} \quad \text{(Ramanujan digraph)},$$

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where the reasons for this definition will be made clear along the paper.

The idea of “Ramanujan digraphs” arose during the work on the papers [LLP17, PS18]; While we believe that the term itself is new, several classic results can be interpreted as saying something about these graphs (see for example §3.5 and §5.2). We survey here both classic results and ones from the mentioned papers, and prove several new ones.

The paper unfolds as follows: After giving the definitions in §2 and various examples in §3, we prove that there are very few normal Ramanujan digraphs in §3.7. We then turn to almost-normal digraphs in §4, proving an Alon-Boppana type theorem, and surveying their spectral and combinatorial features, such as optimal covering, sharp cutoff, small diameter and Chernoff sampling bound. We then explore Ramanujan digraphs from the perspective of universal covers §5.1, and infinite Cayley graphs §5.2. In §5.3 we discuss an explicit construction of Ramanujan digraphs as Cayley graphs of finite groups, which is similar to the LPS construction [LPS88], but applies to any $PGL_d$ and not only to $PGL_2$. In §5.4 we touch upon zeta functions and the Riemann Hypothesis, in §5.5 we discuss Alon’s second eigenvalue conjecture for digraphs, and finally in §6 we present some questions.

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2. Definitions

Throughout the paper we denote by $G = (V_G, E_G)$ a connected $k$-regular graph on $n$ vertices, where by a graph we always mean an undirected one. Its adjacency matrix $A = A_G$, indexed by $V$, is defined by $A_{v,w} = 1$ if $v \sim w$ ($v$ and $w$ are neighbors in $G$), and 0 otherwise. Since $\sim$ is symmetric, so is $A$, hence it is self-adjoint with real spectrum. The constant function $1$ is an eigenvector of $A$ with eigenvalue $k$, and when $G$ is bipartite, say $V = L \sqcup R$, the function $1_L - 1_R$ is an eigenvector with eigenvalue $-k$. We call these eigenvalues and eigenvectors trivial, and denote by $L^2_0 = L^2_0(V)$ their orthogonal complement in $L^2(V)$, namely

$$L^2_0(V) = \begin{cases} \mathbb{1}^\perp & G \text{ is not bipartite} \\ (\mathbb{1}_L, \mathbb{1}_R)^\perp & G \text{ is bipartite.} \end{cases}$$

Observe that $A$ restricts to a self-adjoint operator on $L^2_0(V)$, and recall that for self-adjoint (and even normal) operators, the spectral radius

$$\rho(M) = \max \{|\lambda| \mid \lambda \in \text{Spec}(M)\}$$

coincides with the operator norm

$$\|M\| = \max_{v \neq 0} \frac{\|Mv\|}{\|v\|}.$$

Definition 1 ([LPS88]). A $k$-regular graph $G$ is a Ramanujan graph if

$$\rho(G) \overset{df}{=} \rho\left( A_G |_{L^2_0} \right) = \left\| A_G |_{L^2_0} \right\| \leq 2\sqrt{k-1}.$$  

Moving on to digraphs, we denote by $D$ a finite connected $k$-regular directed graph, by which we mean that each vertex has $k$ incoming and $k$ outgoing edges. Now, $A_{v,w} = 1$ whenever $v \rightarrow w$ (namely, there is an edge from $v$ to $w$) and since $A$ is no longer symmetric, its spectrum is not necessarily real. However, by regularity we still have

$$\rho(A) = \|A\|_1 = \|A\|_2 = \|A\|_\infty = k,$$

(1) On occasions we will allow loops and multiple edges, in which case $A_{v,w}$ is the number of edges between $v$ and $w$. 


as any square matrix satisfies
\[
\|A\|_2^2 = \rho(A^*A) \leq \|A^*A\|_\infty \leq \|A\|_\infty \|A\|_\infty = \|A\|_1 \|A\|_\infty,
\]
and 1 is still a k-eigenfunction. If \(D\) is m-periodic, namely \(V_D = \bigsqcup_{j=0}^{m-1} V_j\) with every edge starting in \(V_j\) terminating in \(V_{(j+1 \mod m)}\), then \(e^{2\pi t i/m}k\) are also eigenvalues (with \(t = 1, \ldots, m-1\)), with corresponding eigenfunctions \(\sum_{j=0}^{m-1} e^{2\pi j t i/m} 1_{V_j}\). By Perron-Frobenius theory, all eigenvalues of absolute value \(k\) arise in this manner. We call these eigenvalues (including \(k\)) trivial, and denote by \(L^2_0\) the orthogonal complement of their eigenfunctions in \(L^2(V_D)\). Even though \(A\) is not self-adjoint or normal, the regularity assumptions ensures that it still restricts to \(L^2_0\), and we make the following definition:

**Definition 2.** A k-regular digraph \(D\) is a Ramanujan digraph if
\[
\rho(D) \overset{\text{def}}{=} \rho\left(A_D\big|_{L^2_0}\right) \leq \sqrt{k}.
\]
The bound \(\|A\|_{L^2_0} \leq \sqrt{k}\) does not have to hold anymore; Indeed, we will see that there are Ramanujan digraphs for which \(\|A\|_{L^2_0} = k\), which is as bad as one can have for a k-regular adjacency operator (in the undirected settings, this would mean that the graph is disconnected). For spectral analysis, the operator norm is much more important than the spectral radius, and this is what makes digraphs harder to study than graphs.

We say that a digraph \(D\) is self-adjoint, or normal, if its adjacency matrix is. In these cases we do have \(\|A_D\|_{L^2_0} = \rho(D)\), and much of the theory of expanders remains as it is for graphs (see for example [Vu08]). However, we will see in Proposition 4 that there are very few normal Ramanujan digraphs. A main novelty of [LP16], which was developed further in [LLP17], is the idea of almost-normal digraphs:

**Definition 3.** A matrix is \(r\)-normal if it is unitarily equivalent to a block-diagonal matrix with blocks of size at most \(r \times r\). A digraph is called \(r\)-normal if its adjacency matrix is \(r\)-normal, and a family of matrices (or digraphs) is said to be almost-normal if its members are \(r\)-normal for some fixed \(r < \infty\).

We shall see in §4 that for many applications, almost-normal digraphs are almost as good as normal ones.

### 3. Examples

#### 3.1. Complete digraphs.
For \(m, k \in \mathbb{N}\), we define the complete k-regular m-periodic digraph \(K_{k,m}\) by
\[
V_{K_{k,m}} = \{(x, y) \mid x \in \mathbb{Z}/m\mathbb{Z}, \ y \in [k]\}
\]
\[
E_{K_{k,m}} = \{(x, y) \to (x + 1, z) \mid x \in \mathbb{Z}/m\mathbb{Z}, \ y, z \in [k]\}.
\]
This is a normal Ramanujan digraph on \(n = km\) vertices, with \(m\) trivial eigenvalues coming from periodicity, and \((k - 1) m\) times the eigenvalue zero. This shows that one should focus on the case of bounded degree and periodicity, for otherwise infinite families of trivial examples arise.
3.2. Projective planes and hyperplanes. The Projective plane over $F_p$ is the undirected bipartite graph whose vertices represent the lines and planes in $F_p^3$, and whose edges correspond to the relation of inclusion. It is $k$-regular for $k = p + 1$, and has $n = 2(p^2 + p + 1)$ vertices. Its nontrivial spectrum is $\pm \sqrt{k - 1}$ (each repeating $p^2 + p$ times), so it is Ramanujan. In fact, it is twice better than Ramanujan, which only requires $|\lambda| \leq 2\sqrt{k - 1}$. We can therefore consider it as a digraph, with each edge appearing with both directions, and obtain a $k$-regular self-adjoint Ramanujan digraph, since the adjacency matrix remains the same.

More generally, the bipartite graph of lines against $d$-spaces in $F_{d + 1}^p$ (with respect to inclusion) has $n = 2 \cdot p^{d+1} - (p - 1) - 1$ vertices and is $k$-regular with $k = p^d - 1$. Its nontrivial eigenvalues are $\pm \sqrt{p^d - 1} = \pm \sqrt{k - \frac{p^d - 1}{p - 1}}$, so we obtain a self-adjoint Ramanujan digraph for every $d$.

3.3. Paley digraphs. For a prime $p$ with $p \equiv 3 \pmod{4}$, the Paley digraph $\mathcal{PD}(p)$ [GS71] has $V = F_p$ and

$$E = \left\{ a \to b \left| \left( \frac{k-a}{p} \right) = 1 \right. \right\},$$

where $(\cdot)$ is the Legendre symbol. It is a $k = \frac{p - 1}{2}$-regular normal digraph, with nontrivial eigenvalues $\pm \frac{\sqrt{p - 1}}{2}$ (this is a nice exercise in Legendre symbols). These are of absolute value $\sqrt{\frac{p - 1}{2}}$, so $\mathcal{PD}(p)$ is a normal Ramanujan digraph.

It turns out that examples as in §3.1-§3.3 are limited. In §3.7 we will prove:

**Proposition 4.** For any fixed $k \geq 2$ and $m \geq 1$ there are only finitely many $k$-regular $m$-periodic normal (and in particular, self-adjoint) Ramanujan digraphs.

Thus, if we wish to fix the regularity $k$ and periodicity $m$, and yet take $|V| = n$ to infinity we must move on to non-normal graphs.

3.4. Extremal directed expanders. The De Bruijn graph $\mathcal{DB}(k, s)$ is a $k$-regular aperiodic Ramanujan digraph with

- $V_{\mathcal{DB}(k, s)} = |k|^s$ (where $|k| = \{1, \ldots, k\}$, so $n = k^s$)
- $E_{\mathcal{DB}(k, s)} = \{(a_1, \ldots, a_s) \to (a_2, \ldots, a_s, t) \mid a_i, t \in [k]\}$.

Just as complete digraphs, the nontrivial spectrum of $\mathcal{DB}(k, s)$ consists entirely of zeros. However, its adjacency matrix is not diagonalizable, and it has Jordan blocks of size $s$, so in particular, these do not form an almost-normal family even for a fixed $k$. The Kautz digraph is another example with similar properties.\(^{(1)}\)

In [FL92] Feng and Li show that $k$-regular $r$-periodic diagonalizable digraphs must have $\rho(D) \geq 1$ once $n > kr$. Furthermore, for any $n$ which co-prime to $k$ they give an explicit construction of a $k$-regular $r$-periodic digraph on $nr$ vertices with $\rho(D) = 1$.

De Bruijn graphs show that a direct analogue of the Alon-Boppana theorem (with respect to any positive $\varepsilon$) does not hold for digraphs in general. In §4 we will see that in the settings of almost-normal digraphs, an Alon-Boppana theorem does hold, with the bound $\sqrt{k}$.

\(^{(1)}\)For the spectrum of the symmetrization of De Bruin and Kautz digraphs, see [DT98].
3.5. Directed line graphs. In this section we assume that $G$ is a $(k + 1)$-regular graph, and we define its $k$-regular line-digraph $D_L(G)$ as follows:

$V_{D_L(G)} = \{(v, w) \mid v, w \in V_G, \ v \sim w\}$

$E_{D_L(G)} = \{(v, w) \rightarrow (w, u) \mid u \neq v\}$.

Namely, the vertices correspond to edges in $G$ with a chosen direction, and a $G$-edge is connected to another one in $D_L(G)$ if they form a non-backtracking path of length 2 in $G$. The importance of this construction is that non-backtracking walks on $G$ are encoded precisely by regular (memory-less) walks on $D_L(G)$.

![Figure 3.1. The local view of the line-digraph of a 3-regular graph (the original graph is shown in the background).](image)

By Hashimoto’s interpretation of the Ihara-Bass formula (cf. [Iha66, Sun86, Has89, Bas92, ST96, FZ99, KS00, ...]), the spectra of $G$ and $D_L(G)$ are related:

$\lambda \pm \sqrt{\lambda^2 - 4k^2} \mid \lambda \in \text{Spec } G \bigcup \{ \pm 1, \ldots, \pm 1 \}$

One can easily check that

$|\lambda| \leq 2\sqrt{(k + 1) - 1} \iff \frac{1}{2} \left( \lambda \pm \sqrt{\lambda^2 - 4k^2} \right) = \sqrt{k}$,

so that $G$ is a Ramanujan graph if and only if $D_L(G)$ is a Ramanujan digraph. Therefore, any construction of Ramanujan graphs (e.g. [LPS88, Mor94, MSS15]) can be used to construct Ramanujan digraphs.

The digraph $D_L(G)$ is not normal (as one can easily verify by applying $AA^T$ and $A^T A$ to some $\otimes$ in Figure 3.1), and it turns out that the singular values of $A$ are as bad as can be: the trivial singular value $k$ repeats $|V_G|$ times. This reflects the fact that the walk described by $A^T A$ is highly disconnected: the edges entering a fixed vertex form a connected component, since

$A^T A (v \rightarrow w) = A^T \left\{ (w \rightarrow u) \mid w \sim u \right\} = \{(u' \rightarrow w) \mid w \sim u'\}$
(this is easier to see in Figure 3.1 than algebraically). In particular, this shows that \( \| A_{L_2(V, D_L(G))} \| = k \). The breakthrough in [LP16] is the understanding that \( A \) is always 2-normal, and that this is good enough for the analysis of the random walk on \( D_L(G) \) (see §4 below).

3.6. Collision-free walks on affine buildings. In the previous example, a certain walk on the directed edges of an undirected graph \( G \) gave rise to a digraph \( D_L(G) \), which was a Ramanujan digraph whenever \( G \) was a Ramanujan graph. In [LLP17] this is generalized to higher dimension: considering some walk \( W \) on the cells of a simplicial complex \( \mathcal{X} \) (possibly oriented or ordered cells), one asks when is the digraph \( D_W(\mathcal{X}) \) which represents this walk a Ramanujan digraph.

It turns out that the key is the following property: We say that a digraph is collision-free if it has at most one (directed) path from any vertex \( v \) to any vertex \( w \). The digraph \( D_L(G) \) from §3.5 is not collision-free - indeed, a regular graph with this property must be infinite - but the line-digraph of the universal cover of \( G \), namely \( D_L(T_{k+1}) \), is indeed collision-free: Two non-backtracking walkers which start on the same directed edge on the tree will never reunite, once separated. The main theorem in [LLP17] is this:

**Theorem 5** ([LLP17]). Let \( \mathcal{X} \) be a complex whose universal cover is the affine Bruhat-Tits building \( B \), and let \( W \) be a geometric regular random walk operator. If \( W \) is collision-free on \( B \) (namely, \( D_W(B) \) is collision-free), and \( \mathcal{X} \) is a Ramanujan complex, then \( D_W(\mathcal{X}) \) is a Ramanujan digraph.

Here geometric means that the random walk commutes with the symmetries of \( B \); Properly defining the other terms in the theorem will take us to far afield, and we refer the interested reader to [LSV05a, Lub14, LLP17].

Let us give one concrete example: the geodesic edge walk on a complex goes from a directed edge \((v, w)\) to the directed edge \((w, u)\) if \( u \neq v \) (no backtracking), and in addition \{\(v, w, u\}\) is not a triangle in the complex (so the path \(v \to w \to u\) is not “homotopic” to the shorter path \(v \to u\)).

The edges of the \(d\)-dimensional Bruhat-Tits building of type \( \tilde{A}_d \) are colored by \{1, \ldots, \(d\)\} (loc. cit.), and the geodesic walk restricted to edges of color 1 forms a regular collision-free walk on the building. Thus, by the theorem above, the same walk on Ramanujan complexes of type \( \tilde{A}_d \), as constructed in [LSV05b, Fir16], gives a Ramanujan digraph\(^1\). In the case \(d = 1\), the building \( \tilde{A}_1 \) is a regular tree, its Ramanujan quotients are Ramanujan graphs, and the geodesic edge walk is simply the non-backtracking walk, so we obtain again the example from §3.5.

Finally, all geometric walks on quotients of a fixed building \( B \) form a family of almost-normal digraphs [LLP17, Prop. 4.5]. For the geodesic edge walk on \( \tilde{A}_d \)-Ramanujan complexes, the corresponding Ramanujan digraphs are sharply \((d+1)\)-normal [LLP17, Prop. 5.3, 5.4], and they can be made to be \(m\)-periodic for any \(m\) dividing \(d\).

3.7. Normal Ramanujan digraphs. We now turn to the proof of Proposition 4, for which we need a quantitative version of the Alon-Boppana theorem. We use the following:

**Theorem 6** ([Nil04, Thm. 1 with \(s = 2\)]). The second largest eigenvalue of a \(k\)-regular graph \( \mathcal{G} \) is at least \(2\sqrt{k-1}\) \( \cos \left(\frac{2\pi d_{\text{min}}}{d}\right)\).

\(^1\)The papers [Li04, Sar07] also construct Ramanujan complexes, but using a weaker definition of Ramanujanness - we refer the reader to the discussion of this point in [Fir16, LLP17].
Proof of Proposition 4. Let \( D \) be a \( k \)-regular normal Ramanujan digraph on \( n \) vertices, and let \( G \) be its symmetrization, namely, \( A_G = A_D + A_D^T \). Assume for now that \( D \) is aperiodic. From normality of \( A_D \) we obtain

\[
\rho(G) = \max \left\{ \lambda + \overline{\lambda} \mid \lambda \in \text{Spec } A_D \right\} \leq 2\sqrt{k},
\]

and we would like to combine this with Theorem 6. For a \( k_G \)-regular graph with \( k_G \geq 4 \), Moore’s bound [HS60] gives

\[
n \leq 1 + k_G \sum_{j=1}^{\text{diam } G} (k_G - 1)^{j-1} \leq 2 (k_G - 1)^{\text{diam } G},
\]

so that Theorem 6 implies (for \( k_G \geq 4 \))

\[
\rho(G) \geq 2 \sqrt{k_G - 1} \cos \left( \frac{2\pi}{\log k_G(n/2)} \right) \geq 2 \sqrt{k_G - 1} \left( 1 - \frac{2\pi^2}{\log^2 k_G(n/2)} \right).
\]

Our \( G \) is \( 2k \)-regular, so that (3.3) and (3.4) combine to

\[
1 - \frac{2\pi^2}{\log^2 k_G(n/2)} \leq \sqrt{\frac{k}{2k - 1}} \leq \sqrt{\frac{2}{3}},
\]

which gives

\[
n \leq 2 (2k - 1)^{10.4}.
\]

Assume now that \( D \) is \( m \)-periodic, and observe the \( k^m \)-regular digraph \( D' \) whose vertices are those of \( D \) and whose edges are the paths of length \( m \) in \( D \). Since \( A_D' = A_D^m \), the trivial eigenvalues \( e^{2\pi ji/m}k \) of \( D \) become the eigenvalue \( k^m \) in \( D' \), which has no other trivial eigenvalues. This reflects the fact that \( D' \) is a disconnected digraph with \( m \) aperiodic connected components. As \( D' \) is also normal and Ramanujan, (3.5) bounds the size of each component by \( 2 (2k^m - 1)^{10.4} \).

All together, we get

\[
n \leq 2m (2k^m - 1)^{10.4},
\]

so there are only finitely many such graphs.(1)

Remark 7. (a) In §3.2 we saw examples for 2-periodic normal Ramanujan digraphs with \( n \approx 2k^2 \), which is quite far from the bound (3.6) with \( m = 2 \). It seems interesting to ask what is the optimal bound.

(b) In [LLP17, §5.1] it is shown that for any \( i \geq 1 \) there is a walk \( W_i \) on cells of dimension \( i \) of a complex, such that if \( X \) is a Ramanujan complex of dimension \( d \) then \( D_{W_i} (X) \) are Ramanujan digraphs for \( 1 \leq i \leq d \). However, no such walk on vertices (i.e., for \( i = 0 \)) is exhibited. Proposition 4 explains why: it is well known that all geometric operators on vertices commute with each other (these are “Hecke operators” - cf. [LSV05a]). In particular such an operator commutes with its own transpose, and therefore induces normal digraphs, which cannot be Ramanujan for an infinite family by the Proposition.

4. Almost-normal digraphs

In this section we explore almost-normal digraphs, and in particular almost-normal Ramanujan digraphs. Their main feature, which goes back to [LP16, LLP17] is the behavior of powers of their adjacency matrix:

(1) An alternate way to handle periodicity is to use [Nil04, Thm. 1] with \( s = m + 1 \).
Proposition 8. Let $D$ be an $r$-normal, $k$-regular digraph with $\rho(D) = \lambda$. For any $\ell \in \mathbb{N}$,

$$\|A_D^{\ell}|_{L_2^D}\| \leq \left(\frac{\ell + r - 1}{r - 1}\right) k^{r-1} \lambda^{r-1} = O\left(\ell^{-1} \lambda^{r}\right).$$

Note that for normal digraphs $r = 1$, which gives $\|A_D^{\ell}|_{L_2^D}\| \leq \lambda^\ell$ as should be. The upshot here is that as long as the “failure of normality” is bounded, only a polynomial price is incurred. This shows why random walk on almost-normal digraphs is susceptible to spectral analysis: Let $p_t$ denote the probability distribution of the walk at time $\ell$. Assuming for simplicity that $D$ is aperiodic, so that $L_2^D = \{1\}$, the distance from equilibrium is

$$\|p_\ell - \frac{1}{n}\ell\| = \left\|\left(\frac{A}{L}\right)^\ell p_0 - \frac{1}{n}\ell\right\| = \left\|\left(\frac{A}{L}\right)^\ell (p_0 - \frac{1}{n})\right\| \leq \frac{1}{k^{\ell}} \|A^{\ell}|_{L_2^D}\| = O\left(\ell^{-1} \left(\frac{\lambda}{k}\right)^\ell\right).$$

In the case of Ramanujan digraphs $\lambda = \sqrt{k}$, and this gives an almost-optimal $L^1$-cutoff, at time $\log_k n + O(\log_k \log n)$ (see [LP16, Thm. 3.5] and [LLP17, Prop. 3.1], and [ABLS07] for related results).

An interesting corollary [LLP17, Thm. 2] is that in an $r$-normal Ramanujan digraph the sphere of radius $\ell_0 = \log_k n + (2r - 1) \log_k \log n$ around any vertex $v$ covers almost all of the graph. Indeed, if the walk described by $p_\ell$ starts at $v_0$ then $\text{supp}(p_\ell) = S_\ell(v_0)$, so that

$$n - |S_\ell(v_0)| = \sum_{v \not\in S_\ell(v_0)} \frac{1}{n^2} \leq \left\|p_\ell - \frac{1}{n}\ell\right\|^2 = O\left(\frac{\ell^{2r-2}}{k^{\ell}}\right),$$

and $\ell = \ell_0$ yields $|S_{\ell_0}(v_0)| \geq n(1 - o(1))$. This in turn implies a bound of $(2 + o(1)) \log_k(n)$ on the diameter, since the $\ell_0$-spheres around any two vertices must intersect.

Yet another consequence of almost-normality is a Chernoff bound for sampling: in [PR18] we show that if $f$ is a function from the vertices to $[-1, 1]$ with sum zero, and $v_1, \ldots, v_\ell$ are the vertices visited in a random walk on an almost-normal directed expander, then

$$\text{Prob} \left[ \frac{1}{\ell} \sum_{i=1}^{\ell} f(v_i) > \gamma \right] \leq e^{-C\gamma^2 k}$$

for small enough $\gamma$, where $C$ depends on the expansion and normality. Using §3.5, this also gives a similar result for non-backtracking random walk on non-directed expanders, and via §3.6 to geodesic walks on high-dimensional expanders.

Proof of Proposition 8. By definition, $A$ is unitarily equivalent to a block-diagonal matrix with blocks of size $r \times r$. The periodic functions on $D$ correspond to “trivial” blocks of size one, and the singular values of $A^{\ell}|_{L_2^D}$ are the union of the singular values of the $\ell$-th powers of the remaining, “nontrivial” blocks. Let $B$ be a nontrivial block of size $s \times s$. By Schur decomposition, we can assume that $B$ is upper triangular, in which case the absolute values of its diagonal entries are bounded by $\lambda$. In addition, since $B$ is unitarily equivalent to the restriction of $A$ to some invariant subspace, all entries of $B$ are bounded by $\|B\|_2 \leq \|A\|_2 = k$, so that $B$ is entry-wise majorized by

$$M_{s,\lambda,k} := \begin{pmatrix} \lambda & k & \cdots & k \\ 0 & \lambda & \cdots & \vdots \\ \vdots & \ddots & \ddots & k \\ 0 & \cdots & 0 & \lambda \end{pmatrix} s.$$
It follows that $B^\ell$ is majorized by $M_{s,\lambda,k}^\ell$, hence
\[ \|B^\ell\|_2 \leq \sqrt{\|B^\ell\|_1 \|B^\ell\|_\infty} \leq \sqrt{\|M_{s,\lambda,k}^\ell\|_1 \|M_{s,\lambda,k}^\ell\|_\infty} = \|M_{s,\lambda,k}^\ell\|_1, \]
and the latter is just the sum of the first row in $M_{s,\lambda,k}^\ell$. This is maximized for $s = r$, and equals
\[ \sum_{t=0}^{r-1} \binom{r-1}{t} \binom{\ell}{t} k^t \lambda^{\ell-t} \leq \left( \frac{\ell + r - 1}{r - 1} \right) k^{r-1} \lambda^{\ell-r+1}, \]
which gives the bound in the Proposition.

It is natural to ask whether symmetrization turns directed expanders into expanders, and we suspect that this is true for almost-normal aperiodic expanders in general. We can show that this is so for the symmetrization of a high enough power:

**Proposition 9.** Let $D$ be an aperiodic $r$-normal digraph with $\rho(D) = \lambda$. If $G_\ell$ is the symmetrization of the $\ell$-th power of $D$, namely $A_{G_\ell} = A_D^\ell + (A_D^\ell)^T$, then
\[
\frac{\rho(G_{\ell-1})}{\deg G_{\ell-1}} = \frac{1}{2} + \frac{(r-1)^2}{2} \cdot \frac{\lambda}{k} + O\left( \left( \frac{\lambda}{k} \right)^2 \right), \quad \text{and} \quad \frac{\rho(G_\ell)}{\deg G_\ell} = \frac{r \lambda}{k} + O\left( \left( \frac{\lambda}{k} \right)^2 \right).
\]

**Proof.** Observe that $\deg G_\ell = 2k^\ell$. Maintaining the notations of the previous proof, we have by the same reasoning
\[
\frac{1}{\deg G_{\ell-1}} \left\| B^{\ell-1} + B^{r-1} \right\|_2 \leq \frac{1}{2k^r-1} \left\| M_{s,\lambda,k}^{\ell-1} + M_{s,\lambda,k}^{r-1} \right\|_1
\]
\[
= \frac{1}{2k^\ell} \left[ \lambda^{r-1} + \sum_{t=0}^{r-1} \binom{r-1}{t} k^t \lambda^{r-1-t} \right] = \frac{1}{2} + \frac{(r-1)^2}{2} \cdot \frac{\lambda}{k} + O\left( \left( \frac{\lambda}{k} \right)^2 \right).
\]
and the computations for $G_\ell$ are similar.

We now prove an Alon-Boppana theorem for almost-normal digraphs:

**Theorem 10.** Let $k \geq 2$ and $m \geq 1$. For any $\varepsilon > 0$, there is no infinite almost-normal family of $k$-regular $m$-periodic digraphs $D$ with $\rho(D) \leq \sqrt{k} - \varepsilon$.

**Proof.** Let $D$ be an $r$-normal, aperiodic $k$-regular digraph on $n$ vertices and denote $\lambda = \rho(D)$ and $A = A_D$. Let $G$ be the graph whose adjacency matrix is $A^\ell A^\ell$, for $\ell \geq r$ which will be determined later on. Namely, $V_G = V_D$, and each edge in $G$ corresponds to a $2\ell$-path in $D$ whose first $\ell$ steps are in accordance with the directions of the edges of $D$, and the next $\ell$ steps are in discordance with them\(^{(1)}\). Since $G$ is $k^{2\ell}$-regular, (3.4) gives
\[
\rho(G) \geq 2 \sqrt{k^{2\ell} - 1} \left( 1 - \frac{2\pi^2}{\log^2 k^{2\ell-1}(n/2)} \right).
\]
On the other hand, Proposition 8 gives
\[
\rho(G) = \rho\left( A^\ell \big| L^\ell_0 \right) = \left\| A^\ell \big| L^\ell_0 \right\|^2 \leq \left( \frac{\ell + r - 1}{r - 1} \right)^2 k^{2r-2} \lambda^{2(\ell-r+1)},
\]
\(^{(1)}\)In particular, there are $k^\ell$ such closed path consisting of taking some $\ell$-path and then retracing it backwards, so that one can even take the graph whose adjacency matrix is $A^\ell A^\ell - k^\ell I$.
and together we obtain for some $C_{k,r} > 0$
\[
\lambda^{2(t-r+1)} \geq \frac{2\sqrt{k^{2r-1}}}{(t^{r-1})^2 k^{2r-2}} \left( 1 - \frac{2\pi^2}{\log^2 k^{2r-1} (n/2)} \right) \geq \frac{C_{k,r} k^{\ell-r+1}}{\ell^{2r-2}} \left( 1 - \frac{8 (\pi \ell \ln k)^2}{\ln^2 (n/2)} \right)
\]
\[
\implies \lambda \geq \sqrt{k} \cdot 2^{(l-t+r)} \frac{C_{k,r}}{\ell^{2r-2}} \left( 1 - \frac{8 (\pi \ell \ln k)^2}{\ln^2 (n/2)} \right)^{n \to \infty} \sqrt{k}.
\]
We finally choose $\ell = \sqrt{\ln (n/2)}$, obtaining
\[
\lambda \geq \sqrt{k} \cdot 2^{(l-t+r)} \frac{C_{k,r}}{\ell^{2r-2}} \left( 1 - \frac{8 (\pi \ell \ln k)^2}{\ln^2 (n/2)} \right)^{n \to \infty} \sqrt{k}.
\]
This concludes the aperiodic case, and we leave the general one to the reader. \(\square\)

5. Further exploration

5.1. Universal Objects. The universal cover of all $k$-regular graphs is the $k$-regular tree $T_k$; Ramanujan graphs are those which, save for the trivial eigenvalues, confine their spectrum to that of their forefather. It is possible to give an analogous interpretation for Ramanujan digraphs: consider the $k$-regular directed tree $T_k^\ast$, which is obtained by choosing directions for the edges in $T_k$ to create a $k$-regular digraph. The spectrum of $T_k^\ast$ was computed in [dlHRV93]:

\[
\text{Spec} \left( T_k^\ast \right) = \left\{ z \in \mathbb{C} \mid |z| \leq \sqrt{k} \right\},
\]
so indeed a $k$-regular digraph is Ramanujan if and only if its nontrivial spectrum is contained in that of its “universal directed cover” $T_k^\ast$. However, one can also consider other universal objects: for example, the line digraph $D_L (T_{k+1})$ of the $k+1$-regular tree is a $k$-regular collision-free digraph which covers all of the digraphs obtained as line graphs of $(k+1)$-regular graphs (see Figure 3.1 for $k = 2$). Its spectrum is

\[
\text{Spec} \left( D_L (T_{k+1}) \right) = \left\{ \pm 1 \right\} \cup \left\{ z \in \mathbb{C} \mid |z| = \sqrt{k} \right\},
\]

and it contains the spectrum of all Ramanujan digraphs of the form $D_L (G)$. It is also 2-normal: $L^2 (V_{D_L (T_{k+1})})$ decomposes as an orthogonal Hilbert sum of one and two-dimensional spaces, each stable under the adjacency operator. Similarly, the digraph which describes the geodesic walk on the two-dimensional buildings of type $A_2$ is 3-normal, and by computations in [KLW10] its spectrum is

\[
\text{Spec} \left( D_W (\tilde{A}_2) \right) = \left\{ z \in \mathbb{C} \mid |z| = \sqrt{k} \right\} \cup \left\{ z \in \mathbb{C} \mid |z| = \sqrt{k} \right\};
\]

one can continue to higher dimensions in this manner - see [Kan16, LLP17] for more details.

5.2. Universal Cayley graphs. For even $k$, the $k$-regular tree $T_k$ is the Cayley graph of $F_k/2$, the free group on $S = \{x_1, \ldots, x_{k/2}\}$, with respect to the generating set $S \cup S^{-1} = \{x_1, \ldots, x_{k/2}, x_1^{-1}, \ldots, x_{k/2}^{-1}\}$. In fact, for any subset $S$ of size $k/2$ in a group $G$, the following are tautologically equivalent:

1. $G$ is a free group and $S$ is a free generating set.
2. The Cayley graph $C_{ay} (G, S \cup S^{-1})$ is a tree(1).

(1) Here $\sqcup$ indicates disjoint union, so that this is always a $k$-regular graph.
The following, however, is far from a tautology:

**Theorem 11 ([Kes59])**. For \( \frac{k}{2} > 1 \), (1) and (2) above are equivalent to:

\[
(3) \quad \rho \left( A_{\text{Cay}(G,S \cup \overline{S}^{-1})} \right) = 2\sqrt{k - 1}.
\]

This does not say that \( T_k \) is the only \( k \)-regular graph with spectral radius \( 2\sqrt{k - 1} \), but rather that among Cayley graphs it is the only one. In a sense, Keten’s result says that the Ramanujan spectrum characterizes the free group. The analogue for directed graphs was revealed to be more complex in [dHRV93]. First, observe that \( T_k^- \) is the Cayley digraph of the free group with respect to the *positive* generating letters:

\[
T_k^- = \text{Cay}(F_k, \{x_1, \ldots, x_k\}).
\]

![Figure 5.1. The directed tree \( T_2^- \) as \( \text{Cay}(F_2, \{x_1, x_2\}) \).](image)

As we have said, the spectral radius of \( T_k^- \) is \( \sqrt{k} \), but it turns out that it is enough that \( S \) generate a free *semigroup* in order for this to happen:

**Theorem 12 ([dHRV93])**. Let \( S \) be a subset of size \( k \geq 2 \) in a group \( G \). If \( S \) generates a free subsemigroup of \( G \), then

\[
\rho \left( A_{\text{Cay}(G,S)} \right) = \sqrt{k},
\]

and if \( G \) has property (RD)\(^{(1)} \) then the converse holds as well.

For example, small cancellation theory shows that in the surface group of genus \( g \geq 2 \)

\[
S_g = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdot \ldots \cdot [a_g, b_g] \rangle,
\]

the elements \( \{a_1, b_1, \ldots, a_g, b_g\} \) generate a free semigroup. Thus, the corresponding Cayley digraph of \( S_g \) has spectral radius \( \sqrt{k} \) even though \( S_g \) is not free.

\(^{(1)}\)Property (RD), which stands for rapid decay, is satisfied both by hyperbolic groups and by groups of polynomial growth. For its definition we refer the reader to [Jol90, dHRV93].
5.3. ** Explicit constructions.** For any $k$, [MSS15] shows the existence of infinitely many $k$-regular bipartite Ramanujan graphs, and thus there exist infinitely many $k$-regular, 2-periodic, 2-normal Ramanujan digraphs, namely their line-digraphs defined in §3.5. For any prime power $k$, [LPS88, Mor94] give both aperiodic and 2-periodic $k$-regular Ramanujan digraphs, as line digraphs of explicit Cayley graphs.

Nevertheless, it is interesting to ask whether Ramanujan digraphs can be obtained as Cayley digraphs in themselves, and also which groups $G$ has a generating set $S$ such that $\text{Cay}(G, S)$ is an almost-normal Ramanujan digraph, as this gives the extremal results on random walk and diameter mentioned after the statement of Proposition 8.

For $k \in \{2, 3, 4, 5, 7, 11, 23, 59\}$, an infinite family of $k$-regular, 2-normal Ramanujan digraphs is constructed in [PS18, §5.2] as Cayley digraphs of $\text{PSL}_2(\mathbb{F}_q)$ and $\text{PGL}_2(\mathbb{F}_q)$. Each such family arises from a special arithmetic lattice in the projective unitary group $\text{PU}(2)$, which acts simply transitively on the directed edges of the tree $\mathcal{T}_{k+1}$, and whose torsion subgroup is a group of symmetries of a platonic solid. An example with $k = 4$ is shown in Figure 5.2.

In [Par18] we go much further, showing that for any prime power $q$ and any $d \geq 2$ there is an infinite family of Cayley Ramanujan digraphs on $\text{PSL}_d(\mathbb{F}_q)$ ($\ell \to \infty$), which are $S$-regular and sharply $d$-normal. This is quite different from the symmetric case: we have no reason to suspect that $\text{PSL}_d(\mathbb{F}_q)$ can be endowed with a structure of a Ramanujan Cayley graph, for $d \geq 3$. Let us sketch the main ideas: In [CS98, LSV05b] appears an arithmetic lattice $\Gamma$ in a certain division algebra, which acts simply-transitively on the vertices of the building of type $A_{d-1}$ associated with the group $\text{PSL}_d(\mathbb{F}_q((t)))$. This lattice can be enlarged to a lattice $\Gamma' \supset \Gamma$, which acts simply-transitively on the edges of color 1 in the same building. Recall from §3.6 that the geodesic walk on these edges is $k$-regular and collision-free. We take a set of generators $S \subset \Gamma'$ which induces this walk, and regard them as elements in the finite group $\ text{PSL}_d(\mathbb{F}_q')$, which is obtained as a congruence quotient of $\Gamma'$ via strong approximation. We then invoke the Jacquet–Langlands correspondence of [BR17] and the Ramanujan conjecture for function fields [Laf02] to deduce that the nontrivial spectrum of $\Gamma$ acting on the building, thus obtaining a Ramanujan digraph. Finally, sharp $d$-normality follows from [LLP17, Prop. 5.3, 5.4].

5.4. **Riemann Hypothesis.** We briefly mention the perspective of zeta functions - for a lengthier discussion see [LLP17, §6] and [KLW10, Kan16]. Ihara [Iha66] associated with a graph $\mathcal{G}$ a zeta function $\zeta_\mathcal{G}(s)$ which counts closed cycles in $\mathcal{G}$, in analogy with the Selberg zeta function of a hyperbolic surface. If $\mathcal{G}$ is $(k+1)$-regular, it is Ramanujan if and only if $\zeta_\mathcal{G}(s)$ satisfies the following “Riemann hypothesis”: every pole at $\zeta_\mathcal{G}(k^{-s})$ with $0 < \Re s < 1$ satisfies $\Re s = \frac{1}{2}$.

Indeed, Hashimoto [Has89] proved that $\zeta_\mathcal{G}(u) = \det (I - u \cdot A_{D, \gamma}(\mathcal{G}))^{-1}$, so that (3.1) and (3.2) show this equivalence (note that the trivial eigenvalues $\pm k$ of $D_{\mathcal{L}}(\mathcal{G})$ and the eigenvalues $\pm 1$ in (3.1) correspond to $s = 1$ and $s = 0$, respectively). For digraphs the story is simpler: the zeta function $Z_D(u)$ of a digraph $D$ (following [BL70, Has89, KS00]) is $Z_D(u) = \prod_{\gamma} (1 - u^{\ell(\gamma)})^{-1}$, where $\gamma$ is a primitive directed cycle of length $\ell(\gamma)$ in $\mathcal{D}$, and $[\gamma]$ is the equivalence class of its cyclic rotations. One then has $Z_D(u) = \det (I - u \cdot A_D)^{-1}$, so that by (1.1) a $k$-regular digraph $D$ is Ramanujan if and only if every pole at $Z_D(k^{-s})$ satisfies $\Re s = 1$ or $0 \leq \Re s \leq \frac{1}{2}$. The fact that we cannot rule out $s$ with $0 < \Re s < \frac{1}{2}$ is demonstrated by (5.2), for example.

5.5. **Alon’s conjecture.** One of the earliest results on graph expansion is that random regular graphs are expanders [KB67, Pin73]. In [Alo86], Alon conjectured that they are in fact almost
Ramanujan. Namely, for any $\varepsilon > 0$

$$\text{Prob} \left[ \rho(G) \leq 2\sqrt{k-1} + \varepsilon \right] \xrightarrow{n \to \infty} 1$$

where $G$ is a random $k$-regular graph on $n$ vertices.

Alon’s conjecture was eventually proved by Friedman [Fri08], and other proofs followed [FK14, Bor15]. While working on the paper [LLP17], the author conjectured that random regular digraphs are almost Ramanujan as well, in the sense that

$$\text{Prob} \left[ \rho(D) \leq \sqrt{k} + \varepsilon \right] \xrightarrow{n \to \infty} 1$$

where $D$ is a random $k$-regular digraph on $n$ vertices,

for any $\varepsilon > 0$; and furthermore, that they behave as almost-normal digraphs, in the sense that the operator norm of their powers is well behaved as in Proposition 8. In joint work with Doron Puder we tried to extend the methods from [Pud15] to prove this conjecture, and made partial progress which is described below. This project was disrupted by the appearance of a solution on the arXiv:

**Theorem 13** ([Cos17, Thm. 1 with $\delta = \Delta = k$]). Statement (5.4) is true.

Since our methods are quite different from the ones in [Cos17], and might lead to other results (such as understanding of the adjacency-powers), we sketch them here.

In the seminal paper [BS87] the value of $\rho(G)$ for a random $k$-regular graph on $n$ vertices is bounded in the following manner: for even $\ell$, $\rho(G)^\ell \leq \text{tr} \left( A^\ell \right) - k^\ell$, and $\text{tr} \left( A^\ell \right)$ equals the number of closed paths of length $\ell$ in $G$. In the permutation model for $G$ (see [BS87, Wor99, Pud15]) each path of length $\ell$ corresponds to a starting vertex, and a word $\omega$ of length $\ell$ in $S = \{x_1^{\pm 1}, \ldots, x_{k/2}^{\pm 1}\}$. If $\omega$ is trivial as an element of $F_{k/2}$, this path is completely backtracking in every instance of $G$, and in particular closed. Denoting $p_\omega = \text{Prob} \left( \text{a path in } G \text{ which starts at } v \text{ and is labeled by } \omega \text{ ends at } v \right)$ --
\[ \frac{1}{n}, \] one obtains
\[ \mathbb{E} \left( \rho(G)^{\ell} \right) \leq \mathbb{E} \left( \text{tr} \left( A^{\ell} \right) \right) - k^{\ell} = n \sum_{\omega \in S^\ell} p_\omega, \]

and each trivial \( \omega \) contributes \( p_\omega = 1 - \frac{1}{n} \). In [BS87] it is shown that \( p_\omega \) is small for words which are not trivial or proper powers in \( \mathbb{F}_{k/2} \), and the number of trivial and power words is bounded, giving a bound on \( \mathbb{E}(\rho(G)^{\ell}) \). An appropriate choice of \( \ell \) then implies \( \rho(G) \leq 3k^{3/4} \) a.a.s. as \( n \to \infty \). In [Pud14, PP15] it is shown that \( p_\omega \) depends on the so-called primitivity rank of \( \omega \), and in [Pud15] this is made qualitatively precise, and words of each primitivity rank are counted, leading to \( \rho(G) \leq 2\sqrt{k-1} + 1 \) a.a.s.

Next, let \( D \) be a \( k \)-regular digraph on \( n \) vertices, so that \( A_D \) is simply the sum of \( k \) independent \( n \times n \) permutation matrices. We cannot use \( \text{tr} \left( A^{\ell} \right) \) directly to bound \( \rho(G) \), since \( A \) is not normal any more. Instead, denoting \( A_0 = A|_{L^2} \) we use Gelfand’s formula:
\[ 2\ell \sqrt{\rho \left( A_0^{2\ell} A_0^0 \right)} = \sqrt{\| A_0^0 \|} \rho \left( A_0 \right), \]

and to bound \( \rho \left( A_0^{2\ell} A_0^0 \right) \) we study \( \text{tr} \left( \left( A^{2\ell} A^{\ell} \right)^{\ell} \right) \). The entries of \( \left( A^{2\ell} A^{\ell} \right)^{\ell} \) correspond to “\( \ell \)-alternating words” of length \( 2\ell t \): words in \( S = \{ x^{\pm}_1, \ldots, x^{\pm}_k \} \) which are composed of alternating sequences of \( \ell \) negative letters followed by \( \ell \) positive ones. Given a starting vertex, each such word translates to a path in \( D \), where negative letters indicate crossing a directed edge in the “wrong” direction. Again \( p_\omega \) is the probability that this path is closed, so that
\[ \rho \left( \left( A_0^{2\ell} A_0^0 \right)^{\ell} \right) \leq \text{tr} \left( \left( A^{2\ell} A^{\ell} \right)^{\ell} \right) - k^{2\ell} = n \cdot \sum_{\omega \in (S^\ell \times S^\ell)^{\ell}} p_\omega. \]

Now, \( \mathbb{E} \left( \text{tr} \left( \left( A^{2\ell} A^{\ell} \right)^{\ell} \right) \right) \) can be bounded similarly to [Pud15], this time by counting \( \ell \)-alternating words of each primitivity rank, and choosing both \( \ell \) and \( t \) carefully. We discovered that already from \( \ell = 2 \) one obtains the bound \( \rho(D) \leq \sqrt{2k + \epsilon} \) a.a.s., and we expect that as \( \ell \) goes to infinity one should recover (5.4) up to an additive constant. As remarked above, this analysis goes through the spectral norm of \( A^{\ell} \), so it might lead to other interesting results on \( D \).

6. Questions

(1) A non-regular graph \( G \) is said to be Ramanujan if its nontrivial spectrum is contained in the \( L^2 \)-spectrum of its universal cover (which is a non-regular tree). This definition is justified both by the extended Alon-Boppana theorem [Gre95, GŻ99] and by the behavior of random covers [Fri03, Pud15, FK14, BDH18]. What is the appropriate definition of a non-regular Ramanujan digraph?

(2) Can standard results on expanders (such as the Cheeger inequalities and the expander mixing lemma) be extended to almost-normal directed expanders?

(3) Does symmetrization turns a family of almost-normal directed expanders into a family of expanders?

(4) Are there infinite almost-normal families of non-periodic \( k \)-regular Ramanujan digraphs for \( k \) which is not a prime power?

(5) Almost-normality is an “algebraic” phenomenon: it originates from representation theory in [LLP17], and from the special structure of line-digraphs in [LP16]. There seems to be no reason that random models will have this property, or that it will be stable under perturbations. Is there a more flexible definition of almost-normality, which still gives a theorem in the spirit of Proposition 8?
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EINSTEIN INSTITUTE OF MATHEMATICS
HEBREW UNIVERSITY, JERUSALEM 91904, ISRAEL.
parzan@math.huji.ac.il.