On the $R_{fB}$ condition for the (2+m)-Einstein warped product manifolds and some almost-complex structure cases

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1. Abstract

For the studied cases in [14], the author showed that having the $f$-curvature-Base ($R_{fB}$) is equal to requiring a flat metric on the base-manifold. In [15] the authors used the condition $R_{fB}$ on (2+m)-Einstein warped product manifold, in the case $\lambda = \mu = 0$, to built a new kind of manifolds, composed by positive-dimensional manifold and negative-dimensional manifold, the so called POLJ-manifolds. This have opened up the possibility of seeking the existence of others $(n,m)$-POLJ manifolds. The aim of this paper is to extend the work done in [14] to $m$-dimensional fiber, and show if the value of $m$ can influence the result, i.e., finding base-manifolds with non-flat metric for $\text{dim}F \neq 2$ and, at the same time, check if in these cases can exists other kinds of $(2, m)$-POLJ manifolds.

In the second part of the paper, we put in relation a particular type of almost-Hermitian manifolds with POLJ-manifolds and we proceeded to give the definitions of: almost Hermitian-POLJ manifold (aH-POLJs and pseudo almost Hermitian-POLJ manifolds (paH-POLJs).

As a result, we find out that the dimension of fiber-manifold does not change the result of [14]. Moreover, we show that the $(2,m)$-almost hyperbolic Hermitian-POLJ manifolds with flat fiber and quasi-constant curvature, and $(2, m)$-quasi-Einstein almost hyperbolic Hermitian-POLJ manifolds with flat fiber and quasi-constant curvature can not exist.

Finally we add a Special Remark about the possible use of the $(n,−n)$-POLJ in super-conductor graphene theory.

keywords: $f$-curvature-Base; $R_{fB}$, Einstein warped product manifold, POLJ-manifolds, non-existence of the $(2,m)$-almost hyperbolic Hermitian-POLJ manifolds with flat fiber and quasi-constant curvature, almost Hermitian-POLJ manifold, pseudo almost Hermitian-POLJ manifold, point-like manifold.

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In recent years the study of warped product manifolds (WPM) is of great interest both for the mathematicians and physicists. Many works have been published that have studied and introduced new types of WPM, (to name a few reference see [7], [19], [4] and [3]).

Aytimur and Zgr in [1] proved some results concerning the Einstein statistical WPM, and in [15] Pigazzini et al. introduced a new type of WPM so called POLJ-manifolds, where the fiber is a manifold with negative dimension.

In [14] Pigazzini introduced a simple constraint on the base-manifold called $f$-curvature-Base $(R_{fB})$ and proposed to use it in order to simplify the equations, trying to constructing a nonRicci-flat metric on the base-manifold obtaining, as a result, that this is equivalent to the request for a flat metric. This same constraint, for Ricci-flat Einstein-WPM with Ricci-flat fiber, was used in [15] and allowed the construction of a new manifold called POLJ-manifold.

After having seen in [15] that by not setting $\dim F = 2$ (in the case $\lambda = \mu = 0$) allowed a different result from [14], this paper is in effect an extension of the work done in [14] and it is motivated by the question: "Can the fiber dimension ($\dim F$) influence the result obtained in [14] for the cases other than the one analyzed in [15]?"

In the second part of the paper we consider a particular kind of almost Hermitian manifolds with the POLJ-manifolds, defined new types of POLJ-manifolds and in a Special Remark we suggest a possible use of the $(n, -n)$-POLJs in the superconductor graphene theory.

**Definizione 2.1.** A metric which satisfies the condition $\text{Ric} = \lambda g$ for some constant $\lambda$, is said to be an Einstein metric. A manifold which admits an Einstein metric is called an Einstein manifold. (See [16]).

**Definizione 2.2.** A warped product manifold is Einstein (see [14], also [12], [2]) if and only if

\[
\left\{ \begin{array}{l}
\text{Ric} - \frac{\mu}{\lambda} \nabla^2 f = \lambda g \\
\text{Ric} = \mu \bar{g} \\
\int \Delta f + (m - 1) |\nabla f|^2 + \lambda f^2 = \mu
\end{array} \right.
\]

where $\lambda$ and $\mu$ are constants, $m$ is the dimension of $F$, $\nabla^2 f$, $\Delta f$ and $\nabla f$ are, respectively, the Hessian, the Laplacian and the gradient of $f$ for $g$, with $f : (B) \to (0, \infty)$.
a smooth positive function.

Contracting first equation of (1) we get:
\[(2) \quad R_B f^2 - m f \Delta f = n f^2 \lambda \]
where \(n\) and \(R_B\) is the dimension and the scalar curvature of \(B\) respectively. By third equation, considering \(m \neq 0\) and \(m \neq 1\), we have:
\[(3) \quad m f \Delta f + m(m - 1)|\nabla f|^2 + m \lambda f^2 = m \mu \]
Now from (2) and (3) we obtain:
\[(4) \quad |\nabla f|^2 + \left[\frac{(m-n)+R_B}{m(m-1)}\right] f^2 = \frac{\mu}{(m-1)} \]

**Definizione 2.3.** Let \((M, \tilde{g}) = (B, g) \times_f (F, \tilde{g})\) be an Einstein warped-product manifold with \(\tilde{g} = g + f^2 \tilde{g}\). We define the scalar curvature of the Base-manifold \((B, g)\) as \(f\)-curvature-Base \((R_{f_B})\), if it is a multiple of the warping function \(f\) (i.e. \(R_{f_B} = cf\) for \(c\) an arbitrary constant belonging to \(\mathbb{R}\)). (See [14] as reference).

**Remarks 1:** Since a warped product manifold (WPM) implies a non-constant warping function \(f\) (otherwise it would be a simply Riemannian product-manifold, for more details see [8], [13]) and that the definition of \(POLJ\)-manifold presupposes non-zero \(R_{f_B}\), the results analyzed from now on will be considered from this point of view.

**Case 1:** Einstein warped-product manifold Ricci-flat \((\lambda = 0)\)

**Case 1a:** \(\mu = 0\)

We have already seen in [15] that the base-manifold, with non-Ricci-flat metric, exists only in the \((2, -2)\)-POLJ case.

**Case 1b:** \(\mu \neq 0\).

**Theorem 1.** Let \((M^{2+m}, \tilde{g}) = (B^2, g) \times_f (F^m, \tilde{g})\) be an Einstein warped-product manifold Ricci-flat (i.e., \(\tilde{\text{Ric}} = \lambda \tilde{g}\) with \(\lambda = 0\)), where \((B^2, g)\) is a smooth surface with non-zero \(R_{f_B}\), and \((F^m, \tilde{g})\) is a smooth Einstein-surface (i.e. \(\tilde{\text{Ric}} = \mu \tilde{g}\)).
Then \((M^{2+m}, \tilde{g})\), cannot exist.

**Proof.** In our case, we have \(n = 2\), \(\lambda = 0\) and \(R_B = R_{f_B}\) (see [14]), then (2) and (3) become:
(5) \( \Delta f - hf^2 = 0 \) \quad \text{(with} \ h = c/m. \text{)}

(6) \( f \Delta f + (m - 1) |\nabla f|^2 + \mu \)

Then (4) becomes:

(7) \( (m - 1) |\nabla f|^2 + hf^3 = \mu \)

Now, by the initial hypothesis (non-zero \( R_{\mu} \)), we assume \( h \neq 0 \) with \( f \) nonconstant and setting \( p = (m - 1) \) and \( u = -hf \), for an open set where \( u \) nonzero. Thus:

(8) \( \Delta u + u^2 = 0 \)

(9) \( u\Delta u + p|\nabla u|^2 - u^3 - h^2\mu = 0. \)

(10) \( p|\nabla u|^2 - u^3 - h^2\mu = 0. \)

For the sake of simplicity we replace the constant \( h^2\mu \) with constant \( A \).

Let \( g \) be the metric on \( B \) and assume that \( u \) is a nonzero (and hence necessarily positive) solution, to the above system on a simply-connected open subset \( B' \subset B \).

The equation (10) implies that \( \omega_1 = (u^3 + A)^{-\frac{1}{2}}p\frac{1}{2}du \), and this implies that we have to assume \( (u^3 + A) \) to be nonzero, \( \omega_1 \) is a 1-form with \( g \)-norm 1 on \( B' \) and hence \( g \) can be written in the form \( g = \omega_1^2 + \omega_2^2 \) for some \( \omega_2 \) which is also a unit 1-form.

Fix an orientation by requiring that \( \omega_1 \wedge \omega_2 \) is the \( g \)-area form on \( B' \), then \( \star du = (u^3 + A)^{\frac{3}{2}}p^{-\frac{1}{2}}\omega_2 \), and since \( d(\star du) = \Delta u \omega_1 \wedge \omega_2 \), it follows that:

\[
p^{-\frac{1}{2}}(u^3 + A)^{-\frac{3}{2}}u^2du \wedge \omega_2 + p^{-\frac{1}{2}}(u^3 + A)^{\frac{3}{2}}d\omega_2 =
\]

\[
d][(u^3 + A)^{\frac{3}{2}}p^{-\frac{1}{2}}\omega_2] = -u^2\omega_1 \wedge \omega_2 = -u^2(u^3 + A)^{-\frac{3}{2}}p\frac{1}{2}du \wedge \omega_2 =
\]

\[
\frac{3}{2}(u^3 + A)^{-\frac{3}{2}}u^2du \wedge \omega_2 + pu^2(u^3 + A)^{-\frac{3}{2}}du \wedge \omega_2 = -(u^3 + A)^{\frac{1}{2}}d\omega_2.
\]

Then \( (-\frac{3}{2} - p)(u^3 + A)^{-1}u^2du \wedge \omega_2 = d\omega_2 \) and we have \( d[(u^3 + A)^{\frac{3}{2}}u^{-\frac{3}{2}}\omega_2] = 0 \), i.e. \( \omega_2 = (u^3 + A)^{-\frac{3}{2}}\omega_2 \), so the metric \( g = p(u^3 + A)^{-1}du^2 + (u^3 + A)^{-1}\frac{3}{2}p dv^2 \) has a singularity in \( u^3 = -A \).

[Note 1: Here, the analysis of the singularity is substantially the same as in [14] ]

Consider an open set for which \( (u^3 + A) \) is nonzero. The Gaussian curvature is given by:

\[
K = -\frac{3}{2}(u^3 + A)^{-1}u^4p - \frac{3}{2}(u^3 + A)^{-1}u^4p^2 - (u^3 + A)^{-1}u^4p^3 + 2up^3 + 3up^2.
\]

In this case, it is easy to verify that for the initial hypothesis, where we have set \( R_B = R_{\mu} \) (i.e. \( K = -u^2h \)), we observe that \( K \) is incompatible with our analysis. In fact we have:
\[ -u_m^2 = -\frac{9}{2}(u^3 + A)^{-1}u^4p - \frac{9}{2}(u^3 + A)^{-1}u^4p^2 - (u^3 + A)^{-1}u^4p^3 + 2up^3 + 3up^2 \]

or
\[ m = (u^3 + A)^{-1}u^3(9p + 9p^2 + 2p^3) - 6p^2 - 4p^3 \]
Now remembering that \( m = p + 1 \) we have:
\[ p + 1 + 6p^2 + 4p^3 = (u^3 + A)^{-1}u^3(9p + 9p^2 + 2p^3) \]
or
\[ (p + 1 + 6p^2 + 4p^3)(u^3 + A) = u^3(9p + 9p^2 + 2p^3) \]
or
\[ (p + 1 + 6p^2 + 4p^3)u^3 + Ap + A + A6p^2 + A4p^3 = (9p + 9p^2 + 2p^3)u^3 \]
or
\[ Ap + A + A6p^2 + A4p^3 = (-2p^3 + 3p^2 + 8p - 1)u^3 \]
Since \( u \) must not be a constant, this implies:

(a) \( 4p^3 + 6p^2 + p + 1 = 0 \) and (b) \(-2p^3 + 3p^2 + 8p - 1 = 0\). But the polynomials (a) and (b) have different solutions, so (11) is satisfied only for constant \( u \) (i.e., \( f = \text{constant} \)), which is not admitted in our initial assumptions, therefore \((M^{2+m}, \bar{g})\), cannot exist. \( \square \)

**Case 2: Flat Fiber Surface (\( \mu = 0 \))**

**Case 2a:** We point out that if we consider \( \lambda = 0 \), we are obviously in the same situation treated in [15] where we introduced the \((2, -2)-POLJ\) manifold. Therefore we will not consider it further.

**Case 2b:** \( \lambda \neq 0 \).

**Theorem 2.** Let \((M^{2+m}, \bar{g}) = (B^2, g) \times_f (F^m, \bar{g})\) be an Einstein warped-product manifold, where \((B^2, g)\) is a smooth surface with non-zero \( R_{fu} \), and \((F^m, \bar{g})\) is a smooth Ricci-flat surface (i.e. \( \bar{\text{Ric}} = \mu \bar{g} \), with \( \mu = 0 \)).

Then \((M^{2+m}, \bar{g})\), cannot exist.

**Proof.** The analysis is essentially the same as seen so far, so we assume \( h \neq 0 \) and set \( u = -hf \), where \( f \) is not constant. The equations (2) and (4) become:

\[(12) \ h f^2 - \Delta f - l \lambda f = 0 \]
\[(13) \ |\nabla f|^2 + \frac{1}{p} f^2 - \frac{4}{p} f^2 + \frac{4}{p} f^3 = 0 \]
where \( h = \frac{1}{m} \), \( l = \frac{1}{m} \) and \( p = (m - 1) \).

Setting \( u = -hf \) we obtain:
(14) \( u^2 + \Delta u + Qu = 0 \)
(15) \( \nabla u^2 - Su^2 + Tu^2 - Du^3 = 0 \)
with \( Q = \lambda l, S = \frac{\lambda}{p}, T = \frac{\lambda}{p} \) and \( D = \frac{1}{p} \), where it is easy to see that \( D \neq 0, T \neq 0, S \neq 0 \) and \( S - T \neq 0 \) for \( m \neq 2 \).

By the same token as in case (1b), we obtain from (15) that \( du = (u^3D + Su^2 - Tu^2)^{1/2} \omega_1 \).
This implies that we have to assume \( (u^3D + Su^2 - Tu^2) \) to be nonzero.
Then \( \omega_1 = (u^3D + Su^2 - Tu^2)^{-1/2} du \), so \( *du = (u^3D + Su^2 - Tu^2)^{1/2} \omega_2 \).
Since \( d(*du) = \Delta u \omega_1 \wedge \omega_2 \), we obtain:

(16) \( d\omega_2 = \left(-\frac{3}{2}u^2D - Su + Tu - u^2 - Qu\right)(u^3D + Su^2 - Tu^2)^{-1} du \wedge \omega_2 \).

Thus we can write \( \omega_2 = u^{-A}(Du + S - T)^{-B} dv \) for some constant \( A \) and \( B \) and for some function \( v \).
Since for \( u = \frac{T-S}{D} \) we have a singularity and we have assumed \( (u^3D + Su^2 - Tu^2) \) to be nonzero, then we must consider \( u \neq \frac{T-S}{D} \).

[Note 2: Even here, the analysis of the singularity is substantially the same as in the previous case (i.e. 1b, it is sufficient to consider \( \frac{T-S}{D} = A \)) so both are equivalent to the studied case in [14].]

Continuing with the calculations we have:
\( E = (u^3D + Su^2 - Tu^2)^{-1} \) and
\( G = u^{-2A}(Du + S - T)^{-2B} \)
So by Brioschi’s formula, we have that the Gaussian curvature is:
\( K = (2A^2 + A)(Du + S - T)^3 u^{(4+6A+4B)} + AD(Du + S - T)^2 u^{(6A+4B+5)} + (2ABD + 4BD)u^{(6A+4B+7)}(Du + S - T)^{(2B+3)} + (2B^2 D^2 + BD^2) u^{(6A+4B+8)}(Du + S - T)^{(2B+2)}. \)

Also in this case for the initial hypothesis, \( 2K = R_{fn} = cf \), we must have \( K = -u^m T \), which means:

(17) \( -\frac{m}{2} = (2A^2 + A)(Du + S - T)^3 u^{(3+6A+4B)} + AD(Du + S - T)^2 u^{(6A+4B+4)} + (2ABD + 4BD)u^{(6A+4B+6)}(Du + S - T)^{(2B+3)} + (2B^2 D^2 + BD^2) u^{(6A+4B+7)}(Du + S - T)^{(2B+2)}. \)

Now putting in relation the equation (16) with \( \omega_2 \), we obtain:
\[ -\frac{1}{2}u^2D-Su+Tu-u^2-Qu}{u^4D+Su^2-Tu^2} = -\frac{ADu-AS+AT-BDu}{Du^2+Su-Tu} \]

and solving the partial fractions we have:

(18) \( A = \frac{2}{Z} + 1 = \frac{m}{2-m} \), then \( m \neq 2 \) (and \( m \neq 0, m \neq 1 \) from Definition 2).

(19) \( B = \frac{3D+2}{2D} - A = \frac{m-2m^2+2}{4-2m} \), then \( m \neq 2 \) (and \( m \neq 0, m \neq 1 \) from Definition 2).

where \( Z = S - T \).

If (17) has a solution, certainly the coefficients of \( u \) with highest degree must vanish. Hence we can consider the right side of (17) composed by:

\[
\begin{align*}
P_1(u) &= (2A^2 + A)(Du + S - T)^3u^{(3+6A+4B)} \text{ with highest degree: } 6A + 4B + 6, \\
P_2(u) &= AD(Du + S - T)^2u^{(6A+4B+4)} \text{ with highest degree: } 6A + 4B + 6, \\
P_3(u) &= (2ABD + 4BD)u^{(6A+4B+6)}(Du + S - T)^{(2B+3)} \text{ with highest degree: } 6A + 6B + 9, \\
P_4(u) &= (2B^2D^2 + BD^2)u^{(6A+4B+7)}(Du + S - T)^{(2B+2)} \text{ with highest degree: } 6A + 6B + 9.
\end{align*}
\]

It is worth noticing that the highest degree of \( P_1(u) \) is equal to that of \( P_2(u) \) and the highest degree of \( P_3(u) \) is equal to that of \( P_4(u) \). But since the constants \( A \) and \( B \) can be non-integer and negative, we cannot know in advance which of the two degrees is the highest. We have 3 cases:

I) \( 6A + 6B + 9 \) is the highest degree,

II) \( 6A + 4B + 6 \) is the highest degree,

III) \( 6A + 6B + 9 = 6A + 4B + 6 \).

CASE (I):

From coefficients of \( P_3(u) \) and \( P_4(u) \) if the (17) is satisfied, we should get:

\[
2A + 2B + 5 = 0 \text{ and considering the (18) and (19) we have:} \\
\begin{align*}
(20) -4m^2 - 4m + 24 &= 0, \text{ i.e., } m = 2 \text{ that is not possible for (18) and (19), and } m = -3.
\end{align*}
\]

Now if the CASE (I) (the highest degree) vanish for \( m = -3 \), we must consider the other degrees and also they must vanish for \( m = -3 \), so we proceed to consider the degree of CASE(II).
CASE (II):
If the \((17)\) is satisfied by considering coefficients of \(P_1(u)\) and \(P_2(u)\), we get:
\[ (2A^2 + AD^3 + AD^3 = 0 \]
and since \(D\) is nonzero we can divide for \(D^3\), then:
\[ (21) \quad A = -1 \]

Considering the \((18)\): \(-2 = 0\) is not possible regardless of the value of \(m\).

CASE (III):
The equality in the CASE (III) implies \(B = -\frac{3}{2}\) and for the \((19)\) this means:

\[ (22) \quad -2m^2 - 2m + 8 = 0, \]
that has no solution for the integer values of \(m\).

We showed that \((17)\) could be satisfied only for some constant value of function \(u\) (i.e. \(f\) constant), which is not admitted in our initial assumptions. Then, also in this case, \((M^{2+m}, \bar{g})\), cannot exist.

3. Almost-complex structure cases

Before starting we remember that (See [6]): - A Hermitian manifold \(M\) is a complex manifold with a smoothly varying Hermitian inner product (Hermitian metric) on each (holomorphic) tangent space. One can also define a Hermitian manifold as a real manifold with a Riemannian metric that preserves a complex structure. A complex structure is essentially an almost complex structure with an integrability condition.

By dropping this condition, we get an almost Hermitian manifold. An almost complex structure \(J\) on \(M\) is a linear complex structure (that is, a linear map which squares to \(-1)\) on each tangent space of the manifold, which varies smoothly on the manifold. In other words, we have a smooth tensor field \(J\) of degree \((1, 1)\) such that \(J^2 = 1\) when regarded as a vector bundle isomorphism \(J : TM \rightarrow TM\) on the tangent bundle. A manifold equipped with an almost complex structure is called an almost complex manifold.

- A Hermitian metric \(h\) on a complex manifold \(M\), can expressed as \(h = g + i\omega\), where \(g\) is a Riemannian metric, where its imaginary part \(\omega\) is a nondegenerate antisymmetric bilinear form.
- A Hermitian metric \(h\) on an (almost) complex manifold \(M\) defines a Riemannian metric \(g\) on the underlying smooth manifold and \(g\) is defined to be the real part of \(h\), i.e., \(g = \frac{1}{2}(h + \bar{h})\).
- A Kählerian metric is a Hermitian metric $h = g + i\omega$ on a complex manifold $M$ whose fundamental form $\omega$ is closed, i.e., $d\omega = 0$.

**Definition 4:** The tangent chain complex of an algebro-geometric object is meant to behave as the (sheaf of) sections of the tangent bundle. At least in the generality of derived algebraic stacks $X$, the tangent chain complex is equivalently (up to a shift in chain degree) the module of sections of the infinitesimal disk bundle of $X$ (for further information see [17] and [9]).

[Note 3: Defining an almost-complex structure $J$ on the tangent chain complex of the derived differential manifold (i.e., chain degree different from 0) is the same thing as that of the usual tangent bundles of the Riemannian manifolds.]

**Definition 5:** The tangent complex functor, is a functor from the category of derived smooth manifolds to the category of sheaves of chain complexes. It sends the morphism $\varphi : F \to M$ (where $F$ is the differential derived fiber of a POLJ-manifold $M$) to a morphism of sheaves of chain complexes $T\varphi : TF \to \varphi^*TM$.

In particular, if $W$ is a vector field on $F$, i.e., a section of $TF$, then $T\varphi(W)$ is a section of $\varphi^*TM$. (See for example [10]-[11]).

4. **Almost Hermitian-POLJ manifolds (AH-POLJs)**

In this section, we will consider the POLJ-manifold as an almost Hermitian manifold, so we give the following definition:

**Definition 6:** A POLJ-manifold compatibly equipped with almost complex structure $J$ is called almost Hermitian-POLJ manifold.

4.1. **The Non-existence of the (2, m)-quasi-Einstein almost hyperbolic Hermitian-POLJ manifold with flat fiber and quasi-constant curvature.**

In [18], Shukla analized interesting kind of manifolds so called almost hyperbolic Hermitian manifold with quasi-constant curvature and quasi-Einstein almost hyperbolic Hermitian manifold with quasi-constant curvature and defined its scalar curvature as $R = n[(n-1)a-2b]$ and $R = \left[\frac{na}{(n-1)} - \frac{(n-1)b}{(n-2)}\right]$ respectively, where $n$ is the dimension of the
manifold and $a$, $b$ are scalars given by $a = \frac{(2n-3)b}{n(n-2)}$.

**Theorem 3.** The $(2,m)$-almost hyperbolic Hermitian-POLJ manifolds with flat fiber and quasi-constant curvature, and $(2,m)$-quasi-Einstein almost hyperbolic Hermitian-POLJ manifolds with flat fiber and quasi-constant curvature, cannot exist.

**Proof.** From Definition 6 (in compatible with [18]), we consider an almost Hermitian POLJ-manifold $M$. In this sense, a complex structure is an endomorphism of the tangent bundle $J : TM \to TM$ such that $J^2 = -1$ (it plays the role of the multiplication by $\sqrt{-1}$). Saying that a Riemannian metric $g$ preserves the complex structure amounts to saying that $g(Jv, Jv) = g(v, v)$ for all tangent vector $v$.

Mind that this forces the (real) dimension of $M$ to be even. Moreover, given any $J$, the complexification of the (real) tangent bundle of $M$ is written as $TM \otimes \mathbb{C} := T_{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M$, where $T^{(1,0)}M$ is the vector bundle associated to the eigenvalue $i$ of $J_{\mathbb{C}}$ (where $J_{\mathbb{C}}(v \otimes z) = J(v) \otimes z$) and $T^{(0,1)}M$ is the vector bundle associated to the eigenvalue $-i$ of $J$. Thus there is a $\mathbb{C}$-linear isomorphism between $(TM, J)$ and $T^{(1,0)}M$ as complex bundles.

After these considerations, our analysis will be the same one done so far. In fact, the scalar curvatures $R = n[(n - 1)a - 2b]$ and $R = \left[\frac{na}{(n-1)} - \frac{(n-1)b}{(n-2)}\right]$ are both constant and if we replace these with the constant $n\lambda$ (i.e., $R = n\lambda$) we are in the same case of the Theorem 2 and hence the proof.

\[\square\]

4.2. The pseudo almost Hermitian-POLJ manifolds (paH-POLJs).

In this section we want to consider a kind of POLJ-manifold $M$ where its dimension is zero and we want extend "a sort" of concept of almost-Hermitian which we call pseudo almost Hermitian-POLJ manifold.

As we have seen in [15] the POLJ-manifolds can "appear" as point-like manifolds and this can be interpreted, not as a "point", but as a manifold with "hidden dimensions", such as to make it appear as a point.

For this reason we want to extend the concept of almost-Hermitian to this case, but since the dimension is zero, we must confine it to its "hidden dimensions", i.e., base-manifold and fiber-manifold. For this reason we called it pseudo almost-Hermitian.
Definizione 4.1. A Riemannian base (B) and a differential derived fiber (F) (of a POLJ-manifold M), if compatibly equipped with an almost complex structure J (i.e., an endomorphism of the tangent bundle J : TB → TB (and J : TF → TF) such that J^2 = −1 and also their respective (pseudo)Riemannian metrics g (and ĝ) satisfy: g(Jv, Jv) = g(v, v) (and ĝ(Jw, Jw) = ĝ(w, w)) for all tangent vectors v (and w), respectively). We call (B) and (F): almost-Hermitian base and almost-Hermitian derived fiber, respectively.

Important Note: We recall that with derived differential manifold we consider only the derived manifolds as smooth Riemannian manifolds by adding a vector bundle of obstructions, then the usual Riemannian geometry works for the underlying Riemannian manifold as for "ordinary” manifolds. In this way we can, therefore, consider ourselves to be able to work with the (pseudo-)Riemannian geometry on our derived fiber-manifold F.

Definizione 4.2. We called M a pseudo almost Hermitian-POLJ manifold if M is a point-like manifold with almost-Hermitian base and/or almost-Hermitian derived fiber.

Example: A simple example of paH-POLJ can be done by reconsidering the (2, −2)-POLJ studied in [15]. It is well known that for every integer n, the flat space $R^{2n}$ admits an almost complex structure. So if we consider the derived differential fiber-manifold (F) (its metric represents $R^{2n}$ in negative dimensions), we can define the almost-complex structure (1 ≤ i, j ≤ 2) : $J_{ij} = -\delta_{i,j-1}$ on it and get an almost-Hermitian derived fiber. From Definition 8 we obtained a (2,-2)-paH-POLJ.

5. Special Remarks about (n, −n)-POLJ manifolds in Superconductors Graphene Model

First of all we recall and highlight that the purpose of the POLJ-manifolds is precisely to present the point-like manifolds from a mathematical point of view, and introduce a type of manifold with a new kind of hidden dimensions.

In [5], Capozziello et al. introduced the concept of the "point-like manifold” building superconductors with graphene, in particular they argue that superconductor graphene can be produced by molecules organized in point-like structures where sheets are constituted by $(N + 1)$-dimensional manifold. Particles like electrons, photons and effective gravitons are string modes moving on this manifold. In fact, according to string theory, bosonic and fermionic fields like electrons, photons and gravitons are particular states or modes of strings. In their important work, they show that at the beginning, there
are point-like polygonal manifolds (with zero spatial dimension) in space which strings attaching them, where all interactions between strings on one manifold are the same and are concentrated on one point which manifold is located on it. They also attaching to show that by joining these manifolds, 1-dimensional polygonal manifolds are emerged on which gauge fields and gravitons live and so, these manifolds glued to each other build higher dimensional polygonal manifolds with various orders of gauge fields and curvatures.

In this context, we think that the \((n, -n)\)-POLJ manifolds can play an important role. In fact \((n, -n)\)-POLJ appears as a point (point-like), because in general, from the global point of view, it is a point (positive and negative dimensions cancel each other out and it amounts to zero dimension), but in special, it is composed by two manifolds. In fact if we chose, for example, the trajectories where the negative dimensions (-n) were not considerate, i.e., they were constant and fixed in zero (i.e. in Origin), we could obtain the considered n-base manifold. So if we move along the sections where one or more dimensions are not considered, then we can obtain submanifolds with nonzero dimension, but as a whole it is a point; in [15] we studied the \((2, -2)\)-POLJ manifold case.

Back to the graphene superconductors model, our \((n, -n)\)-POLJ manifold consists of two manifolds with nonzero dimensions (one with n-dimension and one with -n-dimension, where these two manifolds can be thought as a result of intersection between other manifolds). Then we can consider these two manifolds as contained in a "\(n\)-dimensional BULK", but their warped product (which generates the \((n, -n)\)-POLJ) will create the point-like polygonal manifold.

The \((n, -n)\)-POLJs can be considered as possible mathematical interpretation of point-like manifolds, because they render, for the first time, this abstract concept as a coherent mathematical object.

6. Conclusions

We have observed that, the dimension of fiber-manifold does not influence the result with respect to what obtained in [14].

Not even the construction of a POLJ-manifold is made possible in the studied cases (i.e., 1b and 2b), and for the Ricci-flat case \((\lambda = 0)\), the only POLJ-manifold, living up to the case is for the studied case in [15], i.e. for \(\mu = 0\).

Besides some remarks on almost-complex spacial cases, we showed that the \((2, m)\)-quasi-Einstein almost hyperbolic Hermitian-POLJ manifold with quasi-constant curvature and the \((2, m)\)-almost hyperbolic Hermitian-POLJ manifold with quasi-constant curvature
cannot exist.

New kinds of manifolds have been introduced which we called: the $aH$-POLJs and the $paH$-POLJs.

In conclusion an important application is observed for the $(n, -n)$-POLJ manifolds in the context of the Graphene Superconductors Model.

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