PERIODS OF JACOBI FORMS AND HECKE OPERATOR

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Abstract. A Hecke action on the space of periods of cusp forms, which is compatible with that on the space of cusp forms, was first computed using continued fraction [19] and an explicit algebraic formula of Hecke operators acting on the space of period functions of modular forms was derived by studying the rational period functions [8]. As an application an elementary proof of the Eichler-Selberg trace formula was derived [26]. Similar modification has been applied to period space of Maass cusp forms with spectral parameter s [21, 22, 20]. In this paper we study the space of period functions of Jacobi forms by means of Jacobi integral and give an explicit description of Hecke operator acting on this space. A Jacobi Eisenstein series $E_{2,1}(\tau, z)$ of weight 2 and index 1 is discussed as an example. Periods of Jacobi integrals are already appeared as a disguised form in the work of Zwegers to study Mordell integral coming from Lerch sums [27] and mock Jacobi forms are typical example of Jacobi integral [9].

1. Introduction

Period functions of modular forms have played an important role to understand the arithmetics on cusp forms [17]. Manin [19] studied a Hecke action on the space of period of cusp forms, which is compatible with that on the space of cusp forms, in terms of continued fractions. Later an explicit algebraic description of a Hecke operator on the space of period functions of modular forms was given by studying the rational period functions [8]. As an application, a new elementary proof of the Eichler-Selberg trace formula was derived [26]. Moreover, various modifications of period theory have been also developed.

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notion of rational period functions has been introduced and completely classified in the case of cofinite subgroups of $SL_2(\mathbb{Z})$ \[14\] [16] [15] [1]. Some period functions were attached bijectively to Maass cusp forms according to the spectral parameter $s$ and its cohomological counterpart was described (see \[18\] [2]). Similar modification to Hecke operators on the period space has also been applied to that of Maass cusp forms with spectral parameter $s$ \[21\] [22] [20].

Historically, Eichler\[10\] and Shimura\[23\] discovered an isomorphism between a space of cusp forms and Eichler cohomology group, attaching period polynomials to cusp forms. A notion of Eichler integral for arbitrary real weight with multiplier system has been introduced and shown that there is an isomorphism between the space of modular forms and Eichler cohomology group\[14\]. Along this line, recently a notion of Jacobi integral has been introduced \[7\] and shown there is also an isomorphism between the space of Jacobi cusp forms and the corresponding Eichler cohomology group\[8\]. Mock Jacobi form\[9\] is one of the typical example of Jacobi integral.

The main purpose of this paper is to give an explicit algebraic description of Hecke operator on the period functions attached to the Jacobi integrals. This is an analogous result of \[8\] to the case of Jacobi forms.

This paper is organized as follows: in section 2 we state the main results and section 3 discusses about a Jacobi Eisenstein series $E_{2,1}(\tau, z)$ as an example of Jacobi integral with period functions. Basic definitions and notations are given in section 4 and section 5 gives proofs of the main theorems by introducing various properties which are modified from the results about rational period functions \[8\].

2. Statement of Main Results

Let $f$ be an element of $J^I_{k,m}(\Gamma(1))$, that is, $f$ is a real analytic function $f : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ satisfying a certain growth condition with the following functional equation,

\[(2.1) \quad (f|_{k,m}\gamma)(\tau, z) = f(\tau, z) + P_\gamma(\tau, z), \forall \gamma \in \Gamma(1)^I,\]
with $P_\gamma \in \mathcal{P}_m$, where $\mathcal{P}_m$ is a set of holomorphic functions

$$\mathcal{P}_m = \{ g : \mathcal{H} \times \mathbb{C} \to \mathbb{C} \mid g(\tau, z) < K(\tau^\rho + v^{-\sigma}) e^{2\pi m z^2}, \text{ for some } K, \rho, \sigma > 0 \}$$

(v = Im(\tau) and $y = Im(z)$).

$P_\gamma$ in (2.1) is called a period function of $f$. If $P_\gamma(\tau, z) = 0$, $\forall \gamma \in \Gamma(1)$, $f$ is a usual Jacobi form (see [11]). It turns out that each element of the following set

$$\mathcal{P}_{\text{er} \, k, m} := \{ P : \mathcal{H} \times \mathbb{C} \to \mathbb{C} \mid \sum_{j=0}^{3} P|_{k, m} T^j = \sum_{j=0}^{5} P|_{k, m} U^j = 0 \}$$

($T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $U = ST$)

generates a system of period functions $\{P_\gamma \mid \gamma \in \Gamma(1)\}$ of $f \in \text{J}_{k, m}(\Gamma(1))$.

We also consider the following subspace:

$$\text{EJ}_{k, m}^f(\Gamma(1)) := \{ f \in \text{J}_{k, m}^f(\Gamma(1)) \mid f|_{k, m}[I, (1, 0)] = f \}.$$

It turns out that the following set

$$\mathcal{E}\mathcal{P}_{\text{er} \, k, m} := \{ P : \mathcal{H} \times \mathbb{C} \to \mathbb{C} \mid \sum_{j=0}^{3} P|_{k, m} T^j = \sum_{j=0}^{5} P|_{k, m} U^j = P - P|_{k, m}[I, (1, 0)] = 0 \}.$$

is a generating set of a system of period functions $\{P_\gamma \mid \gamma \in \Gamma(1)\}$ of $f \in \text{EJ}_{k, m}^f(\Gamma(1))$.

For each positive integer $n$, define two Hecke operators $V^{\infty}_n$ and $T^{\infty}_n$ by

$$\text{(f}|_{k, m} V^{\infty}_n)(\tau, z) := n^{k-1} \sum_{a, d = n, a > 0 \atop b \text{ (mod } d)} d^{-k} f \left( \frac{a \tau + b}{d}, az \right)$$

(2.7)

$$\text{(f}|_{k, m} T^{\infty}_n)(\tau, z) := n^{k-4} \sum_{a, d = n^2, \gcd(a, b, d) = \Box \atop a, d > 0, b \text{ (mod } d), X, Y \in \mathbb{Z}/n\mathbb{Z}} (f|_{k, m}[\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}], (X, Y))(\tau, z),$$

Then

**Theorem 2.1.**\quad (1) $(f|_{k, m} V^{\infty}_n) \in \text{J}_{k, mn}^f(\Gamma(1))$ if $f \in \text{J}_{k, m}^f(\Gamma(1))$. 
(2) Let \( f|_{k,m}T = f + P_T \) and \((f|_{k,m}\mathcal{V}_n^\infty)|_{k,m}T = (f|_{k,m}\mathcal{V}_n^\infty) + \hat{P}_T\). Then
\[
\hat{P}_T = n^{\frac{k}{2} - 1}\left(P_T|_{k,m}\left[\begin{array}{cc}
\frac{1}{\sqrt{n}} & 0 \\
0 & \frac{1}{\sqrt{n}}
\end{array}\right], (0,0)\right)|_{k,m}\tilde{\mathcal{V}}_n,
\]
where
\[
\tilde{\mathcal{V}}_n = \sum_{\begin{array}{c}
ad - bc = n^2, \gcd(a, b, c, d) = \Box \\
a > c > 0, d > -b > 0 \\
X,Y \in \mathbb{Z}/n\mathbb{Z}
\end{array}} \left\{\left\lfloor\begin{array}{cc}
a & b \\
c & d
\end{array}\right\rfloor, (0,0)\right\} + \sum_{\begin{array}{c}
ad = n^2, \gcd(a, b, d) = \Box \\
-\frac{1}{2}d < b \leq \frac{1}{2}d \\
X,Y \in \mathbb{Z}/n\mathbb{Z}
\end{array}} \left\lfloor\begin{array}{cc}
a & 0 \\
c & d
\end{array}\right\rfloor, (0,0)\right\} + \sum_{\begin{array}{c}
ad = n^2, \gcd(a, c, d) = \Box \\
-\frac{1}{2}a < c \leq \frac{1}{2}a, c \neq 0 \\
X,Y \in \mathbb{Z}/n\mathbb{Z}
\end{array}} \left\lfloor\begin{array}{cc}
a & 0 \\
c & d
\end{array}\right\rfloor, (0,0)\right\}.
\]

**Theorem 2.2.**

(1) If \( f \in EJ^f_{k,m}(\Gamma(1)) \), then \( f|_{k,m}\mathcal{T}_n^\infty \in EJ^f_{k,m}(\Gamma(1)) \).

(2) Let \( f|_{k,m}T = f + P_T \) and \((f|_{k,m}\mathcal{T}_n^\infty)|_{k,m}T = (f|_{k,m}\mathcal{T}_n^\infty) + \tilde{P}_T\). Then
\[
\tilde{P}_T = n^{k-4}P_T|_{k,m}\tilde{T}_n,
\]
where
\[
\tilde{T}_n = \sum_{\begin{array}{c}
ad - bc = n^2, \gcd(a, b, c, d) = \Box \\
a > c > 0, d > -b > 0 \\
X,Y \in \mathbb{Z}/n\mathbb{Z}
\end{array}} \left\{\left\lfloor\begin{array}{cc}
a & b \\
c & d
\end{array}\right\rfloor, (X,Y)\right\} + \sum_{\begin{array}{c}
ad = n^2, \gcd(a, b, d) = \Box \\
-\frac{1}{2}d < b \leq \frac{1}{2}d \\
X,Y \in \mathbb{Z}/n\mathbb{Z}
\end{array}} \left\lfloor\begin{array}{cc}
a & 0 \\
c & d
\end{array}\right\rfloor, (X,Y)\right\} + \sum_{\begin{array}{c}
ad = n^2, \gcd(a, c, d) = \Box \\
-\frac{1}{2}a < c \leq \frac{1}{2}a, c \neq 0 \\
X,Y \in \mathbb{Z}/n\mathbb{Z}
\end{array}} \left\lfloor\begin{array}{cc}
a & 0 \\
c & d
\end{array}\right\rfloor, (X,Y)\right\}.
\]

**Remark 2.3.**

(1) There are Hecke operators acting on the space of Jacobi forms \([11]\). One needs to choose a special set of representatives to apply Hecke operators to Jacobi integral \( f \) since \( f \) is not \( \Gamma(1)^J \)-invariant.

(2) Note that \( \mathcal{T}_n^\infty \) acts only on the subspace \( EJ^f_{k,m}(\Gamma(1)) \) (see section 5 for details).

(3) It is shown that there is an isomorphism between the space of Jacobi cusp forms and Eichler cohomology group with some coefficient module(there, denoted by \( \mathcal{P}^e_{\mathcal{M}} \)) corresponding to period functions for \( f \in EJ^f_{k,m}(\Gamma(1)) \). Hence the Hecke operator \( \tilde{T}_n \) can also be regarded as an Hecke operator on the Eichler cohomology group.
3. Example of Jacobi Integral

It is well known that Eisenstein series $E_2(\tau) := 1 - 24 \sum_{n \geq 1} (\sum_{0<d|n} d) q^n$ is not a modular form on $\Gamma(1)$, but it is a modular integral (see [16]). Similarly Jacobi Eisenstein series $E_{2,1}(\tau, z)$ is not a Jacobi form on $\Gamma(1)$, but it is a Jacobi integral as we explain below.

The following Jacobi-Eisenstein series was studied in [5]:

$$E_{2,1}(\tau, z) = -12 \sum_{n,r \in \mathbb{Z}, r^2 \leq 4n} H(n-r^2)q^n\zeta^r, q = e^{2\pi i \tau}, \zeta = e^{2\pi i z},$$

where $H(n)$ be the class number of the quadratic forms of discriminant $-n$ with $H(0) = -\frac{1}{12}$.

The theta decomposition of $E_{2,1}(\tau, z)$ is

$$E_{2,1}(\tau, z) = -12 \cdot \left( \frac{H_0(\tau)}{H_1(\tau)} \right)^t \cdot \left( \begin{array}{c} \vartheta_{1,0}(\tau, z) \\ \vartheta_{1,1}(\tau, z) \end{array} \right),$$

where

$$H_\mu(\tau) := \sum_{N \geq 0, N \equiv -\mu^2 \pmod{4}} H(N)q^N, \vartheta_{1,\mu}(\tau, z) = \sum_{r \in \mathbb{Z}} q^{\frac{r^2}{4}} \zeta^r.$$

**Remark 3.1.**

1. $E_{2,1}(\tau, z)$ was defined in [5] as the holomorphic part of

$$E_{2,1}^*(\tau, z; s) = \frac{1}{2} \sum_{c,d \in \mathbb{Z}, \gcd(c,d)=1} \sum_{\lambda \in \mathbb{Z}} e^{2\pi i (\lambda^2 - \frac{c+db+2\lambda}{c+d})} (c+d)^2 |c+d|^{2s}$$

at $s = 0$.

2. A correspondence among $E_2(\tau), E_{2,1}(\tau, z)$ and $H_2(\tau) := \sum_{n \geq 0} H(n)q^n$ was studied in [5].

3. $H_2(\tau) = H_0(4\tau) + H_1(4\tau)$ since $H(N) = 0$ unless $N \equiv 0, 3 \pmod{4}$.

The following function $F$ transforms like a modular form of weight $\frac{3}{2}$ on the group $\Gamma_0(4)$ (see [13], p 91-92):

$$F(\tau) = H_{\frac{3}{2}}(\tau) + v^{-\frac{1}{2}} \sum_{\ell=-\infty}^{\infty} \beta(4\pi \ell^2 v)q^{-\ell^2},$$
where $\beta(x) := \frac{1}{16\pi} \int_1^\infty u^{-\frac{3}{4}} e^{-\pi u} du (x \geq 0)$ and

$$
\left( v^{-\frac{1}{2}} \sum_{t \equiv 0 \pmod{2}} \beta(\pi^2 v) q^{-\frac{t^2}{4}} \right) + \left( v^{-\frac{1}{2}} \sum_{t \equiv 1 \pmod{2}} \beta(\pi^2 v) q^{-\frac{t^2}{4}} \right) = \frac{1 + i}{16\pi} \left( \int_{-\frac{1}{2}}^{i\infty} (t + \tau)^{-\frac{3}{2}} \vartheta_{1,0}(t, 0) dt \right).
$$

Consider a function

$$
\varphi(\tau, z) := F_0(\tau) \vartheta_{1,0}(\tau, z) + F_1(\tau) \vartheta_{1,1}(\tau, z) = \left( \begin{array}{c} F_0(\tau) \\ F_1(\tau) \end{array} \right) \left( \begin{array}{c} \vartheta_{1,0}(\tau, z) \\ \vartheta_{1,1}(\tau, z) \end{array} \right),
$$

with

$$
F_\mu(\tau) = H_{\mu}(\tau) + v^{-\frac{1}{2}} \sum_{t \equiv \mu \pmod{2}} \beta(\pi^2 v) q^{-\frac{t^2}{4}}, \mu = 0, 1.
$$

It is easy to check that $\varphi(\tau, z)$ transforms like a Jacobi form for $\Gamma(1)$ of weight 2 and index 1 and so $\varphi|_{2,1}T = \varphi$. On the other hand, following the computation in [13], page 92 we see

$$
\left\{ \frac{1 + i}{16\pi} \left( \int_{-\frac{1}{2}}^{i\infty} (t + \tau)^{-\frac{3}{2}} \vartheta_{1,0}(t, 0) dt \right) + \left( \vartheta_{1,0}(\tau, z) \right) \right\}_{2,1} (T - E) \right) (\tau, z)
$$

$$
= \frac{1 + i}{16} \left( \int_0^{i\infty} (\tau + w)^{-\frac{3}{2}} \vartheta_{1,0}(w, 0) dw \right)^t \left( \vartheta_{1,0}(\tau, z) \right)
$$

using the transformation formula (for example, see [6]) of theta series $\theta_{1,\mu}(\tau, z)$.

So we conclude that

$$
(E_{2,1}|_{2,1}T)(\tau, z) = E_{2,1}(\tau, z) + P_{E_{2,1},T}(\tau, z),
$$

where

$$
P_{E_{2,1},T}(\tau, z) = \frac{1 + i}{16} \left( \int_0^{i\infty} (\tau + w)^{-\frac{3}{2}} \vartheta_{1,0}(w, 0) dw \right)^t \left( \vartheta_{1,0}(\tau, z) \right)
$$

Next, for each prime $p$, let

$$
(E_2|_{2} T_{p,2}^\infty)(\tau) := p \sum_{\substack{a \equiv b \equiv 0 \\ b (mod d)}} d^{-4} E_2 \left( \frac{a \tau + b}{d} \right),
$$

$$
(H_{\frac{1}{2}}|_{2} T_{p,2}^\infty)(\tau) := \sum_{N=0,3 \pmod{4}} H(Np^2) \left( \frac{-N}{p} \right) H(N) + pH \left( \frac{N}{p^2} \right) q^N,
$$
and

\[(E_{2,1}|_{2,1} T_p^\infty)(\tau, z) := p^{-2} \sum_{\substack{a, d = p^2, a, d > 0 \\ b \equiv (a, d) \mod{d}, \gcd(a, b, d) = \Box}} \sum_{\lambda, \mu \in \mathbb{Z}/p\mathbb{Z}} (E_{2,1}|_{2,1}[[\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right), (\lambda, \mu)]])(\tau, z).\]

It can be directly checked that the following diagram commutes:

\[
\begin{array}{ccc}
E_2(\tau) & \xrightarrow{\varphi} & (E_2|_{2,2} T_p^\infty)(\tau) \\
\downarrow & & \downarrow \\
H_3(\tau) & \xrightarrow{\psi} & (H_3|_{3,2} T_p^\infty)(\tau)
\end{array}
\]

where \(\varphi(\sum_{n \geq 0} c(n) q^n) := 1 - \frac{24}{L(0, \chi)} \sum_{n \geq 1} \sum_{d \mid n} \left(\frac{D}{d}\right) c\left(\frac{n^2}{d^2} | D| \right) q^n\) with a fundamental discriminant \(D\) and \(\psi(\sum_{n \geq 0} c(n) q^n) := -12 \sum_{n \in \mathbb{Z}, r^2 \leq 4n} c(4n - r^2) q^n \zeta^r\) (see [5], Theorem 3.2).

**Remark 3.2.** There is a one-to-one correspondence, which is Hecke equivariant, among the space of modular forms of weight \(2k - 2(\text{even})\) on \(\Gamma(1)\), the Kohnen plus space of weight \(k - \frac{1}{2}\) on \(\Gamma_0(4)\) and the space of Jacobi forms of weight \(k\) and index 1 on \(\Gamma(1)^T\). The above diagram shows that this correspondence is extended to the more general case \(E_2(\tau), H_3(\tau)\) and \(E_{2,1}(\tau, z)\).

The above commutative diagram implies that \(E_{2,1}|_{2,1} T_p^\infty = (p + 1)E_{2,1}\) since \(E_2|_{2} T_p^\infty = (p + 1)E_2\) (also see Knopp [16]). In summary we have shown the following:

**Proposition 3.3.** \((1)\) \((E_{2,1}|_{2,1} T)(\tau, z) = E_{2,1}(\tau, z) + P_{E_{2,1}}(\tau, z),\) where

\[
P_{E_{2,1}}(\tau, z) = \frac{1 + i}{16} \left(\int_0^{i\infty} (\tau + w)^{-\frac{3}{2}} \vartheta_{1,0}(w, 0) dw \right) \left(\vartheta_{1,0}(\tau, z) \vartheta_{1,1}(\tau, z) \right).
\]
(2) For each prime \( p \), let \((E_{2,1}|_{2,1} T_p^\infty|_{2,1} T)(\tau, z) = E_{2,1}(\tau, z) + \tilde{P}_{E_{2,1}, \tau}(\tau, z)\).

Then
\[
\tilde{P}_{E_{2,1}, \tau}(\tau, z) = p^{-2} P_{E_{2,1}, \tau}|_{2,1} T_p = (p + 1) \cdot P_{E_{2,1}, \tau}(\tau, z).
\]

4. Definitions and Notations

Let \( \mathcal{H} \) be the usual complex upper half plane, \( \tau \in \mathcal{H}, z \in \mathbb{C} \) and \( \tau = u + iv, z = x + iy, u, v, x, y \in \mathbb{R} \). Take \( k, m \in \mathbb{Z} \). Let
\[
\Gamma(1)^J := \Gamma(1) \ltimes \mathbb{Z}^2 = \{ [M, (\lambda, \mu)] | M \in \Gamma(1), \lambda, \mu \in \mathbb{Z} \}, (\Gamma(1) = SL_2(\mathbb{Z}))
\]
be the full Jacobi group with a group law
\[
[M_1, (\lambda_1, \mu_1)][M_2, (\lambda_2, \mu_2)] = [M_1 M_2, (\lambda, \mu)M_2 + (\lambda_2, \mu_2)].
\]

Let us introduce the following elements in \( \Gamma(1)^J \): \( S = [(\frac{1}{0}, 1), (0, 0)], T = [(\frac{0}{1}, 0), (1, 0)], T_0 = [(\frac{0}{1}, 0), (0, 0)], U = ST = [(\frac{1}{0}, 1), (1, 0)], I_2 = [I, (1, 0)], I_1 = [I, (0, 1)], E = [I, (0, 0)]. \)

In [4] it is known that \( \Gamma(1)^J \) is generated by \( S \) and \( T \). Also \( \Gamma(1)^J \) is generated by \( T \) and \( U \) and they satisfy the relations
\[
T^4 = U^6 = E,
\]
\[
U T^2 = T^2 U[I, (-1, 0)] = T^2[I, (0, -1)] U = [I, (0, 1)] T^2 U
\]
and these are the defining relations for \( \Gamma(1)^J \).

Also, let \( G^J \) be the set of triples \( [M, X, \zeta] \) \( (M \in SL_2(\mathbb{R}), X \in \mathbb{R}^2, \zeta \in \mathbb{C}, |\zeta| = 1) \). Then \( G^J \) is a group via
\[
[M, X, \zeta][M', X', \zeta'] = [MM', XM' + X', \zeta \zeta' \cdot e^{2\pi i \det \left(\frac{X''}{X'}\right)}],
\]
acting on \( \mathcal{H} \times \mathbb{C} \) as
\[
\gamma(\tau, z) = \left(\frac{a \tau + b}{c \tau + d}, \frac{z + \lambda \tau + \mu}{c \tau + d}\right), \gamma = [(\frac{a}{c}, \frac{b}{d}), (\lambda, \mu), \zeta],
\]
which defines a usual slash operator on a function \( f : \mathcal{H} \times \mathbb{C} \rightarrow \mathbb{C} \) defined by :
\[
(f|_{k,m}\gamma)(\tau, z) := j_{k,m}(\gamma, (\tau, z)) f(\gamma(\tau, z)),
\]
with \( j_{k,m}(\gamma, (\tau, z)) := \zeta^m (c \tau + d)^{-k} e^{2\pi i m(-\frac{a^2}{cd} + \lambda^2 + 2\lambda z + \lambda \mu)} \). Further let
\[
f|_{k,m}[\left(\frac{a}{c}, \frac{b}{d}\right), (X, Y)] := f|_{k,m}[\left(\frac{a}{c}, \frac{b}{d}\right), (X, Y), 1], \text{ if } ad - bc = \ell > 0 .
\]
We will omit $\zeta$ if $\zeta = 1$.

**Definition 4.1.** (Jacobi Integral)

(1) A real analytic periodic (in both variables) function $f : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ is called a *Jacobi Integral* of weight $k \in \mathbb{Z}$ and index $m \in \mathbb{Z}$ with period functions $P_\gamma$ on $\Gamma(1)^J$ if it satisfies the following relations:

(i) For all $\gamma \in \Gamma(1)^J$

\[ f|_{k,m}\gamma = f(\tau, z) + P_\gamma(\tau, z), P_\gamma \in \mathcal{P}_m. \]

(ii) It satisfies a growth condition, when $v, y \to \infty$,

\[ |f(\tau, z)|v^{-\frac{k}{2}} e^{-2\pi m y^2} \to 0. \]

The space of Jacobi integrals forms a vector space over $\mathbb{C}$ and we denote it by $J^F_{k,m}(\Gamma(1)^J)$.

The periodicity condition on $f$ is equivalent to saying that $f|_{k,m}S = f|_{k,m}I_2 = f$ so that $P_S(\tau, z) = P_{I_2}(\tau, z) = 0$. A set of period functions

\[ \{P_\gamma|\gamma \in \Gamma(1)^J\} \text{ of } f \in J^F_{k,m}(\Gamma(1)) \]

satisfies the following consistency condition:

\[ P_{\gamma_1}\gamma_2 = P_{\gamma_1}|_{k,m}\gamma_2 + P_{\gamma_2}, \text{ for all } \gamma_1, \gamma_2 \in \Gamma(1)^J. \]

So using the relations $T^4 = U^6 = E$, it is easy to see that $P_T$ satisfies

\[ \sum_{j=0}^{3} P|_{k,m}T^j = \sum_{j=0}^{5} P|_{k,m}U^j = 0 \]

so that $P_T$ belongs $\mathcal{P}er_{k,m}$. In fact it is shown in [7] that $\mathcal{P}er_{k,m}$ generates the space of period functions $\{P_\gamma|\gamma \in \Gamma(1)^J\}$ of $f \in J^F_{k,m}(\Gamma(1))$.

Next we consider a subspace

\[ EJ^F_{k,m}(\Gamma(1)) = \{g \in J^F_{k,m}(\Gamma(1))|g|_{k,m}I_2 = g\} \]

By the similar method as that for $\mathcal{P}er_{k,m}$ we can show that a set of period functions $\{P_\gamma|\gamma \in \Gamma(1)^J\}$ of $g$ is spanned by $E\mathcal{P}er_{k,m}$. 

5. Hecke Operators on the space of Jacobi Integral

We start to prove Theorem 2.2 and only sketch out the proof of Theorem 2.1 briefly, since it is similar but simpler.

5.1. Proof of Theorem 2.2. It is immediate to see $f|_{k,m} \mathcal{T}_n^\infty \in E\mathcal{J}_k^J(\Gamma(1))$ if $f \in E\mathcal{J}_k^J(\Gamma(1))$ by checking the following:

$$f|_{k,m}[[a/n\ b/n\ d/n], (X, Y)] = f|_{k,m}[[1/1], (0, 0)] \times [[a/n\ b/n\ d/n], (X, Y)],$$

$$f|_{k,m}[[a/n\ b/n\ d/n], (X + n, Y)] = f|_{k,m}[I, (0, -b)] \times [I, (d, 0)] \times [[a/n\ b/n\ d/n], (X, Y)],$$

$$f|_{k,m}[[a/n\ b/n\ d/n], (X, Y + n)] = f|_{k,m}[I, (0, a)] \times [[a/n\ b/n\ d/n], (X, Y)].$$

To prove (2) in Theorem 2.2, first we need to prove a couple of propositions and lemmata. For each integer $n > 1$ let $\mathcal{M}_n^J := \{ \gamma = [\gamma_0, (X, Y)] \mid \gamma_0 \in \frac{1}{n}(\mathcal{M}_2(\mathbb{Z})/(\pm 1)) \}, X, Y \in (\mathbb{C}/n\mathbb{Z}, \det(\gamma_0) = 1 \}$. Write $M^J := \cup M^J_n$ and $R^J_n := \mathbb{Z}[\mathcal{M}_n^J]$ and $R^J_+ := \mathbb{Z}[\mathcal{M}_n^J] = \oplus_n M_n^J$, for the sets of finite integral linear combinations of elements of $M^J_n$ and $M_n^J$, respectively. Then $R^J_+$ is a (non-commutative) ring with unity and is “multiplicatively graded” in the sense that $R^J_n R^J_m \subset R^J_{nm}$ for all $m, n > 0$; in particular, each $R^J_n$ is a left and right module over the group ring $R^J_1 = \mathbb{Z}[\Gamma(1)^J]$ of $\Gamma(1)^J$. Denote by $J$ the right ideal

$$(1 + T + T^2 + T^3)R^J_1 + (1 + U + U^2 + U^3 + U^4 + U^5)R^J_1$$

of $R^J_1$. Finally denote $M^J_{n,1}$ by $\{ \gamma \in M^J_n \mid X, Y \in \mathbb{Z}/n\mathbb{Z} \}$. Then

**Proposition 5.1.** Let $\hat{T}_n^\infty$ be $n^{-k+4}T_n^\infty$.

1. For each integer $n \geq 1$, $\hat{T}_n^\infty(S - I) \equiv 0$, $\hat{T}_n^\infty(I_1 - I) \equiv 0$, $\hat{T}_n^\infty(I_2 - I) \equiv 0$ (mod $(S - I)R^J_n + (I_1 - I)R^J_n + (I_2 - I)R^J_n$) and

$$\hat{T}_n^\infty(T - I) \equiv (T - I)\hat{T}_n \pmod{(S - I)R^J_n + (I_1 - I)R^J_n + (I_2 - I)R^J_n}$$

for a certain element $\hat{T}_n \in R^J_+$.

2. This element is unique modulo $J R^J_n$.

3. If $n$ and $n'$ are positive integers coprime to $m$ then $\hat{T}_n, \hat{T}_n' \in R^J_+$ satisfy the product formula

$$\hat{T}_n \cdot \hat{T}_n' = \sum_{d \mid \gcd(n, n')} d^{2k-3}\hat{T}_{nn'} \pmod{(S - I)R^J_{nn'} + (I_1 - I)R^J_{nn'} + (I_2 - I)R^J_{nn'}.}$$
The proof of Proposition 5.1 is similar to that in [8] where the explicit Hecke operators on the space rational period functions associated to elliptic modular forms were computed.

**Proof of Proposition 5.1**

(1) For the assertion \( \hat{T}_n^\infty (S - I) \equiv 0 \) we compute as in [8]. To claim \( \hat{T}_n^\infty (I_1 - I) \equiv 0 \) check that

\[
\sum_{ad=n^2, \gcd(a,b,d)=\Box} \left[ \left( \begin{array}{cc} a/n & b/n \\ d/n \\ 0 \end{array} \right), (X, Y) \right] \left[ \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), (0, 1) \right]
\]

\[
= \sum_{ad=n^2, \gcd(a,b,d)=\Box} \left[ \left( \begin{array}{cc} a/n & b/n \\ d/n \\ 0 \end{array} \right), (X, Y + 1) \right]
\]

\[
\equiv \sum_{ad=n^2, \gcd(a,b,d)=\Box} \left[ \left( \begin{array}{cc} a/n & b/n \\ d/n \\ 0 \end{array} \right), (X, Y) \right].
\]

For the assertion \( \hat{T}_n^\infty (I_2 - 1) \equiv 0 \) check that

\[
\sum_{ad=n^2, \gcd(a,b,d)=\Box} \left[ \left( \begin{array}{cc} a/n & b/n \\ d/n \\ 0 \end{array} \right), (X, Y) \right] \left[ \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), (1, 0) \right]
\]

\[
= \sum_{ad=n^2, \gcd(a,b,d)=\Box} \left[ \left( \begin{array}{cc} a/n & b/n \\ d/n \\ 0 \end{array} \right), (X + 1, Y) \right]
\]

\[
\equiv \sum_{ad=n^2, \gcd(a,b,d)=\Box} \left[ \left( \begin{array}{cc} a/n & b/n \\ d/n \\ 0 \end{array} \right), (X, Y) \right].
\]

To claim \( \hat{T}_n^\infty (T - I) \equiv (T - I) \hat{T}_n \pmod{(S - I)R_n^J + (I_1 - I)R_n^J + (I_2 - I)R_n^J} \) we need the following lemma.

**Lemma 5.2.** Take any \( \gamma \in \Gamma(1)^J \). Then

\[ \gamma - I \in (T - I)R_1^J + (S - I)R_1^J. \]

**Proof of Lemma 5.2** This lemma follows by induction on the word length. Assume this holds for some \( \gamma \in \Gamma(1)^J \). Note that \( T\gamma - I = (T - I)\gamma + (\gamma - I) \), \( S\gamma - I = (S - I)\gamma + (\gamma - I) \), \( S^{-1}\gamma - I = (S - I)(-S^{-1}\gamma) + (\gamma - I) \) also belong to \( (T - I)R_1^J + (S - I)R_1^J \). Since \( S, T \) generates \( \Gamma(1)^J \), lemma follows
by induction on the word length. \qed

Now write $\hat{T}_n^\infty$ as $\sum M_i$. Note that
\[
[[\left(\frac{a/n}{0} \ b/n \ \ 0 \ \ d/n\right), (X, Y)]T = \left[\left(\frac{b/(b,d)}{d/(b,d)} \ \ \beta \right), (0, 0)\right][\left(\frac{(b,d)/n}{0} \ -a\delta/n \ \ n/(b,d)\right), (Y + 1, -X)]
\]
for some $\beta, \delta \in \mathbb{Z}$ such that $\frac{b}{(b,d)}\delta - \frac{d}{(b,d)}\beta = 1$. Hence for each index $i$ we can choose index $i'$ such that $M_i T = \gamma_i M_{i'}$ for some $\gamma_i \in \Gamma(1)$ so that the set of $i'$s is equal to that of $i$. Then $\hat{T}_n^\infty(T - 1) = \sum \gamma_i M_{i'} - M_i = \sum \gamma_i (\gamma_i - 1) M_{i'}$, and this belongs to $(T - 1)\mathcal{R}_n^J + (S - 1)\mathcal{R}_n^J$ by Lemma 5.2.

(2) To characterize the elements of $\mathcal{J}_n^J$ consider an “acyclicity” condition, which was introduced in [8] in the case of the modular group: suppose $V$ is an abelian group on which $\Gamma(1) \mathcal{J}$ acts on the left. Then $V$ is a left $\mathcal{R}_n^J$-module. For $X \in \mathcal{R}_n^J$ let $\text{Ker}(X) := \{v \in V \mid Xv = 0\}$, $\text{Im}(X) := \{Xv \mid v \in V\}$. We call $V$ acyclic if
\[
\text{Ker}(1 + T + T^2 + T^3) \cap \text{Ker}(I + U + \cdots + U^5) = \{0\},
\]
\[
\text{Ker}(I - T) = \text{Im}(I + T + T^2 + T^3),
\]
and
\[
\text{Ker}(I - U) = \text{Im}(I + U + U^2 + U^3 + U^4 + U^5).
\]

Then the following holds:

**Lemma 5.3.** $\mathcal{R}_n^J$ is an acyclic $\mathcal{R}_n^J$-module for all $n$.

**Proof of Lemma 5.3** First we claim that $\text{Ker}(1 + T + T^2 + T^3) \cap \text{Ker}(I + U + \cdots + U^5) = \{0\}$: let $X = \sum n_\gamma (n_\gamma \in \mathbb{Z}, \gamma \in \mathcal{M}_n)$ be an element of $\mathcal{R}_n^J$. Suppose that $X \in \text{Ker}(1 + T + T^2 + T^3) \cap \text{Ker}(I + U + \cdots + U^5)$. Take any $r(\tau) = \frac{1}{\tau - a}$, where $a \in \mathbb{C}$ is not rational or quadratic and let $q(\tau, z) := (r|_{k,m}X)(\tau, z)$. Then $q(\tau, z)$ behaves somewhat like the rational period functions in [8], i.e., it has finite number of singularities as a function of $\tau$ (when $z$ is fixed), and these singularities are rational or real quadratic. But it can be seen that for some $z_0$, $q(\tau, z_0)$ has a singularity at some point $\tau = M^{-1}a$ with $n_M \neq 0$, contradicting to the fact that $q(\tau, z_0)$ have singularities only at rational or quadratic irrational points. On the other hand, if $X$ is left invariant under
Then \( n_M = n_{TM} = n_{T^2M} = n_{T^3M} \) for all \( M \), and since \( M, TM, T^2M, \) and \( T^3M \) are distinct, this means that \( X \) can be written as an integral linear combination of elements \( M + TM + T^2M + T^3M = (1 + T + T^2 + T^3)M \in R_N^f \). Similarly, \( X = UX \) implies \( n_M = n_{UM} = \cdots = n_{U^5M} \) for all \( M \) and hence \( X \in (1 + U + \cdots + U^5)R_n^f \). This proves the second hypothesis in the definition of acyclicity.

\[ \square \]

**Lemma 5.4.** If \( V \) is an acyclic \( \Gamma \)-module and \( v \in V \), then

\[ (I-T)v \in (I-S)V \iff v \in (1+T+T^2+T^3)V + (I+U+U^2+U^3+U^4+U^5)V = JV. \]

**Proof of Lemma 5.4** The direction “\( \Leftarrow \)” is true for any \( \Gamma(1) \)-module, since \( v = (1 + T + T^2)w + (I + U + U^2 + U^3 + U^4 + U^5)y \) implies

\[ (I-T)v = (1 - S^{-1}U)(I + U + \cdots + U^5)y = (S - 1)S^{-1}(I + U + \cdots + U^5)y. \]

Conversely, assume that \( V \) is acyclic and \( (1-T)v = (1-S)w \) for some \( w \in V \). Then

\[ (1-T)(v - (1+T+T^2)w) = (1-T)v - (1-T^3)w = (T^3 - S)w = (1-U)T^3w. \]

This element must vanish since \( \text{Im}(1-T) \cap \text{Im}(1-U) \subset \text{Ker}(1+T+T^2+T^3) \cap \text{Ker}(I+U+\cdots+U^5) = \{0\} \). But then \( v - (1+T+T^2)w \in \text{Im}(1+T+T^2+T^3) \) and \( T^3w \in \text{Im}(I+U+\cdots+U^5) \) by the second hypothesis in the definition of acyclicity, so \( v = (v - (1+T+T^2)w) + (1+T+T^2+T^3)w - T^3w \in (1+T+T^2+T^3)V + (I+U+\cdots+U^5)V. \)

\[ \square \]

Lemmas 5.3 and 5.4 give a characterization of \( JR_n^f \) as \( \{X \in R_n^f | (1-T)X \in (1-S)R_n^f \} \).

(2) The uniqueness of \( \tilde{T}_n \) modulo \( JR_n^f \) follows immediately from this characterization and the definition of \( \tilde{T}_n \).
(3) Finally,
\[
(T - 1)(\mathcal{T}_n \cdot \mathcal{T}_n' - \sum_{d \mid \gcd(n,n')} d^{2k-3} \mathcal{T}_n'_{\infty})
\equiv \mathcal{T}_n^\infty(T - 1)\mathcal{T}_n' - \sum_{d \mid \gcd(n,n')} d^{2k-3}(T - 1)\mathcal{T}_n'_{\infty}
\]
\[
\equiv \mathcal{T}_n^\infty[T - (S - 1)X_n' - (I_1 - 1)Y_n' - (I_2 - 1)Z_n'] - \sum_{d \mid \gcd(n,n')} d^{2k-3} \mathcal{T}_n'_{\infty}(T - 1)
\]
\[
\equiv (\mathcal{T}_n^\infty \cdot \mathcal{T}_n'^\infty - \sum_{d \mid \gcd(n,n')} d^{2k-3} \mathcal{T}_n'^\infty)\mathcal{T}_n'_{\infty} \equiv 0 \pmod{(S - 1)\mathcal{R}_n^J + (I_1 - 1)\mathcal{R}_n^J + (I_2 - 1)\mathcal{R}_n^J},
\]
and
\[
\mathcal{T}_n^\infty \cdot \mathcal{T}_n'^\infty - \sum_{d \mid \gcd(n,n')} d^{2k-3} \mathcal{T}_n'^\infty \equiv 0 \pmod{(S - 1)\mathcal{R}_n^J + (I_1 - 1)\mathcal{R}_n^J + (I_2 - 1)\mathcal{R}_n^J}
\]
by the usual calculation for the commutation properties of Hecke operators.

This completes the proof of Proposition 5.1.

Now we are ready to prove Theorem 2.2-(2):

5.2. Proof of Theorem 2.2-(2). For any \((X,Y) \in \mathbb{Z}^2 / n\mathbb{Z}^2\), the maps
\[
A = [(a \ b \ c \ d), (X,Y)] \rightarrow I_{1}^{-1}TS^{-[\frac{n}{d}]}A = \left[\left(-\frac{c}{a+c} \frac{d}{a} \right), (X,Y)\right]
\]
\[
B = [(a \ b \ c \ d), (X,Y)] \rightarrow I_{1}^{-1}S^{[\frac{n}{d}]}TB = \left[\left(-\frac{c}{a} \frac{d}{b} \right), (X,Y)\right],
\]
where \([\ ]\) denotes the integral part, are to give inverse bijections between the sets
\[
\mathcal{A}_n := \{[(a \ b \ c \ d), (X,Y)] \in \mathcal{M}_{n,1}^J \mid a > c > 0, d > -b \geq 0, b = 0 \Rightarrow a \geq 2c, \gcd(a, b, c, d) = \Box\}
\]
and
\[
\mathcal{B}_n = \{[(a \ b \ c \ d), (X,Y)] \in \mathcal{M}_{n,1}^J \mid a > -c \geq 0, d > b > 0, c = 0 \Rightarrow d \geq 2b, \gcd(a, b, c, d) = \Box\}.
\]

Note that
\[
\sum_{ad - bc = n^2, \gcd(a, b, c, d) = \Box \atop a > c > 0, d > -b > 0 \atop X,Y \in \mathbb{Z}/n\mathbb{Z}} \{[(a \ b \ c \ d), (X,Y)] - T[(\frac{a}{-c} \frac{b}{d}), (X,Y)]\}\]
\[
\equiv \sum_{ad = n^2, \gcd(a, b, d) = \square} \left[ \begin{pmatrix} 0 & -d \\ a & b \end{pmatrix}, (X, Y) \right] - \sum_{ad = n^2, \gcd(a, c, d) = \square} \left[ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, (X, Y) \right].
\]

Conjugating this equation by \(\alpha := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, (0, 0)\) changes the sign of all the off-diagonal coefficients of each \(2 \times 2\) matrices and each \(X\)'s, and preserves the property \("\equiv\", since \(\alpha \beta \alpha^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, (0, 0)\), \(\alpha I_1 \alpha^{-1} = I_1\), and \(\alpha I_2 \alpha^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (-1, 0)\). Also since \(X\) can be chosen freely in \(\mathbb{Z} / n \mathbb{Z}\), note that we may change \(-X\) to \(X\). Adding the result to the original equation, we get

\[
(I - T) \sum_{ad - bc = n^2, \gcd(a, b, c, d) = \square} \left[ \begin{pmatrix} (a b) & (a - b) \\ \frac{a c}{d} & d \end{pmatrix} \right], (X, Y)]\]

Hence,

\[
(I - T) \hat{T}_n \equiv \sum_{ad = n^2, \gcd(a, b, d) = \square} \left[ \begin{pmatrix} (a b) & (a - b) \\ \frac{a c}{d} & d \end{pmatrix} \right], (X, Y)]\]

The first sum on the right is \(\equiv \hat{T}_n^\infty (I - T)\), while the second equals to

\[
\sum_{ad = n^2, \gcd(a, d) = \square} \left[ \begin{pmatrix} 0 & -d \\ a & \frac{a - d}{2} \end{pmatrix} \right] - \left( \begin{pmatrix} d & 0 \\ \frac{a - d}{2} & a \end{pmatrix} \right), (X, Y)].
\]
\( = \sum_{\alpha \beta = \frac{\pi^2}{4}, \gcd(\alpha, \beta) = \square} (S^2 - I)((\frac{2\alpha}{-\alpha} \beta), (X, Y)) \equiv 0. \)

5.3. **Proof of Theorem 2.1-(1).** To see \( f|_{k,m}V_n^\infty \in J_{k,m}(\Gamma(1)) \) if \( f \in J_{k,m}(\Gamma(1)) \) note that

\[
\left( (f|_{k,m}V_n^\infty)|_{k, mn} \left[ \left[ \left( \frac{\alpha}{\gamma} \beta \right), (\lambda, \mu) \right] \right](\tau, z) \right)
= n^{b-1} \left\{ \sum_{\substack{a \equiv -n \pmod{d} \ b \equiv 0 \pmod{d} \ \ b \ (\text{mod} \ d) \ a \neq -n \}} \left( f|_{k,m} \left[ \left( \frac{1}{\sqrt{n}} 0 \right), (0, 0) \right] \right)|_{k, mn} \left[ \left[ \left( \frac{a}{c} \beta \sqrt{n} d/\sqrt{n} b/\sqrt{n} \right), (0, 0) \right] \left[ \left( \frac{\alpha}{\gamma} \beta \right), (\sqrt{n} \lambda, \sqrt{n} \mu) \right] \right] \right\},
\]

and

\[
\left( (f|_{k,m} \left[ \left( \frac{\alpha}{\gamma} \beta \right), (\lambda, \mu) \right] \right)|_{k, mn} V_n^\infty)(\tau, z)
= n^{b-1} \left\{ \sum_{\substack{a \equiv -n \pmod{d} \ b \equiv 0 \pmod{d} \ \ b \ (\text{mod} \ d) \ a \neq -n \}} \left( f|_{k,m} \left[ \left( \frac{1}{\sqrt{n}} 0 \right), (0, 0) \right] \right)|_{k, mn} \left[ \left[ \left( \frac{a}{c} \beta \sqrt{n} d/\sqrt{n} b/\sqrt{n} \right), (0, 0) \right] \left[ \left( \frac{\alpha}{\gamma} \beta \right), (\sqrt{n} \lambda, \sqrt{n} \mu) \right] \right] \right\}.
\]

So applying the following equalities

\[
\sum_{\substack{a \equiv 1 \pmod{d} \ b \equiv 0 \pmod{d} \ \ b \ (\text{mod} \ d) \ a \neq 1 \}} \left[ \left( \frac{a}{\sqrt{T}} b/\sqrt{T} \right), (0, 0) \right] \left[ \left( \frac{1}{1 1} \right), (0, 0) \right] = \sum_{\substack{a \equiv 1 \pmod{d} \ b \equiv 0 \pmod{d} \ \ b \ (\text{mod} \ d) \ a \neq 1 \}} \left[ \left( \frac{1}{1 1} \right), (0, 0) \right] \left[ \left( \frac{a}{\sqrt{T}} b/\sqrt{T} \right), (0, 0) \right],
\]

\[
\sum_{\substack{a \equiv 1 \pmod{d} \ b \equiv 0 \pmod{d} \ \ b \ (\text{mod} \ d) \ a \neq 1 \}} \left[ \left( \frac{a}{\sqrt{T}} b/\sqrt{T} \right), (0, 0) \right] \left[ \left( \frac{1}{0 1} \right), (0, \sqrt{T}) \right] = \sum_{\substack{a \equiv 1 \pmod{d} \ b \equiv 0 \pmod{d} \ \ b \ (\text{mod} \ d) \ a \neq 1 \}} \left[ \left( \frac{1}{0 1} \right), (0, a) \right] \left[ \left( \frac{a}{\sqrt{T}} b/\sqrt{T} \right), (0, 0) \right]
\]

to \( \left( f|_{k,m} \left[ \left( \frac{1}{\sqrt{n}} 0 \right), (0, 0) \right] \right)(\tau, z) \) we conclude our claim.

5.4. **Proof of Theorem 2.1-(2).** By following similar way to the proof of Theorem 2.2-(2) we find an explicit formula for \( \tilde{V}_n \) and the detailed proof will be skipped.
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