THE ALTERNATING RUN POLYNOMIALS OF PERMUTATIONS

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Abstract. In this paper, we first consider a generalization of the David-Barton identity which relate the alternating run polynomials to Eulerian polynomials. By using context-free grammars, we then present a combinatorial interpretation of a family of \(q\)-alternating run polynomials. Furthermore, we introduce the definition of semi-\(\gamma\)-positive polynomial and we show the semi-\(\gamma\)-positivity of the alternating run polynomials of dual Stirling permutations. A connection between the up-down run polynomials of permutations and the alternating run polynomials of dual Stirling permutations is established.

Keywords: Alternating runs; Eulerian polynomials; Semi-\(\gamma\)-positivity; Stirling permutations

1. Introduction

The enumeration of permutations by number of alternating runs was first studied by André [1]. Knuth [19, Section 5.1.3] has discussed this topic in connection to sorting and searching. Over the past few decades, the study of alternating runs of permutations was initiated by David and Barton [12, 157-162].

Let \(\mathfrak{S}_n\) denote the symmetric group of all permutations of \([n] = \{1, 2, \ldots, n\}\). Let \(\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n\). An alternating run of \(\pi\) is a maximal consecutive subsequence that is increasing or decreasing (see [1, 22]). An up-down run of \(\pi\) is an alternating run of \(\pi\) endowed with a 0 in the front (see [13, 22]). Let \(\text{altrun}(\pi)\) (resp. \(\text{udrun}(\pi)\)) be the number of alternating runs (resp. up-down runs) of \(\pi\). For example, if \(\pi = 324156\), then \(\text{altrun}(\pi) = 4\), \(\text{udrun}(\pi) = 5\).

We define

\[
R_{n,k} = \# \{ \pi \in \mathfrak{S}_n : \text{altrun}(\pi) = k \},
\]
\[
T_{n,k} = \# \{ \pi \in \mathfrak{S}_n : \text{udrun}(\pi) = k \}.
\]

It is well known that these numbers satisfy the following recurrence relations

\[
R_{n+1,k} = kR_{n,k} + 2R_{n,k-1} + (n - k + 1)R_{n,k-2},
\]
\[
T_{n+1,k} = kT_{n,k} + T_{n,k-1} + (n - k + 2)T_{n,k-2},
\]

with the initial conditions \(R_{1,0} = 1\) and \(R_{1,k} = 0\) for \(k \geq 1\), \(T_{0,0} = 1\) and \(T_{0,k} = 0\) for \(k \geq 1\) (see [1, 13]). The alternating run polynomial and up-down run polynomial are respectively defined by

\[
R_n(x) = \sum_{k=0}^{n-1} R_{n,k}x^k \quad \text{and} \quad T_n(x) = \sum_{k=0}^{n} T_{n,k}x^k.
\]

A descent of \(\pi \in \mathfrak{S}_n\) is an index \(i \in [n-1]\) such that \(\pi(i) > \pi(i+1)\). Denote by \(\text{des}(\pi)\) the number of descents of \(\pi\). The classical Eulerian polynomial is defined by

\[
A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{des}(\pi) + 1}.
\]
By solving a differential equation, David and Barton [12, 157-162] established the identity:

\[ R_n(x) = \left(\frac{1 + x}{2}\right)^{n-1} (1 + w)^n A_n \left(\frac{1 - w}{1 + w}\right) \]  

(2)

for \( n \geq 2 \), where \( w = \sqrt{\frac{1 - x}{1 + x}} \). Using (2), Bóna proved that the polynomial \( R_n(x) \) has only real zeros (see [4]). Moreover, one can prove that \( R_n(x) \) has the zero \( x = -1 \) with the multiplicity \( \lfloor \frac{n}{2} \rfloor - 1 \) by using (2), which can also be obtained based on the recurrence relation of \( R_n(x) \) (see [25]). Motivated by (2), Zhuang [31] proved several identities expressing polynomials counting permutations by various descent statistics in terms of Eulerian polynomials.

Let us now recall another combinatorial interpretation of \( T_n(x) \). An alternating subsequence of \( \pi \) is a subsequence \( \pi(i_1) \cdots \pi(i_k) \) satisfying

\[ \pi(i_1) > \pi(i_2) < \pi(i_3) > \cdots > \pi(i_k), \]

where \( i_1 < i_2 < \cdots < i_k \) (see [28]). Denote by \( \text{as}(\pi) \) the number of terms of the longest alternating subsequence of \( \pi \). By definition, we see that \( \text{as}(\pi) = u\text{drun}(\pi) \). Thus

\[ T_n(x) = \sum_{\pi \in S_n} x^{\text{as}(\pi)}. \]

There has been much recent work related to the numbers \( R_{n,k} \) and \( T_{n,k} \). In [3], Bóna and Ehrenborg proved that \( R_{n,k}^2 \geq R_{n,k-1}R_{n,k+1} \). Subsequently, Bóna [4, Section 1.3.2] noted that

\[ T_n(x) = \frac{1}{2}(1 + x)R_n(x) \]

(3)

for \( n \geq 2 \). Set \( \rho = \sqrt{1 - x^2} \). Stanley [28, Theorem 2.3] showed that

\[ T(x, z) = \sum_{n=0}^{\infty} T_n(x) \frac{z^n}{n!} = (1 - x) \frac{1 + \rho + 2xe^{\rho z} + (1 - \rho)e^{2\rho z}}{1 + \rho - x^2 + (1 - \rho - x^2)e^{2\rho z}}. \]

(4)

By using (3) and (4), Stanley [28] obtained explicit formulas of \( T_{n,k} \) and \( R_{n,k} \). Canfield and Wilf [6] presented an asymptotic formula for \( R_{n,k} \). In [21], another explicit formula of \( R_{n,k} \) was obtained by combining the derivative polynomials of tangent function and the following generating function obtained by Carlitz [7]:

\[ \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{n} R_{n+1,k} x^{n-k} = \frac{1 - x}{1 + x} \left(\frac{\sqrt{1 - x^2} + \sin(z\sqrt{1 - x^2})}{x - \cos(z\sqrt{1 - x^2})}\right)^2. \]

In [22], several convolution formulas of the polynomials \( R_n(x) \) and \( T_n(x) \) are obtained by using Chen’s grammars. By generalizing a reciprocity formula of Gessel, Zhuang [30] obtained generating function for permutation statistics that are expressible in terms of alternating runs. Very recently, Josuat-Vergès and Pang [18] showed that alternating runs can be used to define subalgebras of Solomon’s descent algebra.

In this paper, we continue the work initiated by David and Barton [12]. In Section 2, we consider a generalization of (2). In Section 3, we present a combinatorial interpretation of a family of \( q \)-alternating run polynomials by using Chen’s grammars. In Section 4, we show the semi-\( \gamma \)-positivity of the alternating run polynomials of dual Stirling permutations.
2. The David-Barton type identity

Let \( f(x) = \sum_{i=0}^{n} f_i x^i \) be a symmetric polynomial, i.e., \( f_i = f_{n-i} \) for any \( 0 \leq i \leq n \). Then \( f(x) \) can be expanded uniquely as

\[
f(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_k x^k (1 + x)^{n-2k},
\]

and it is said to be \( \gamma \)-positive if \( \gamma_k \geq 0 \) for \( 0 \leq k \leq \lfloor \frac{n}{2} \rfloor \) (see [15]). The \( \gamma \)-positivity provides an approach to study symmetric and unimodal polynomials and has been extensively studied (see [2, 5, 10, 20] for instance).

The first main result of our paper is the following, which shows that the David-Barton type identities often occur in combinatorics and geometry.

**Theorem 1.** Let

\[
M_n(x) = \sum_{k=0}^{\lfloor (n+\delta)/2 \rfloor} M(n, k) x^k (1 + x)^{n+\delta-2k}
\]

be a symmetric polynomial, where \( \delta \) is a fixed integer. Set \( w = \sqrt{\frac{1-x}{1+x}} \). Then

\[
N_n(x) = \left( \frac{1+x}{2} \right)^{n-\delta} (1+w)^{n+\delta} M_n \left( \frac{1-w}{1+w} \right)
\]

if and only if

\[
N_n(x) = \sum_{k=0}^{\lfloor (n+\delta)/2 \rfloor} \frac{1}{2^{k-2\delta}} M(n, k) x^k (1 + x)^{n-\delta-k}.
\]

**Proof.** Set \( \alpha = \frac{1+x}{2} \). Note that

\[
1 - w^2 = \frac{x}{\alpha},
1 - w = \frac{1 - w^2}{(1+w)^2} = \frac{1}{(1+w)^2} \frac{x}{\alpha},
1 + \frac{1 - w}{1+w} = \frac{2}{1+w}.
\]

It follows from (5) that

\[
N_n(x) = \left( \frac{1+x}{2} \right)^{n-\delta} (1+w)^{n+\delta} M_n \left( \frac{1-w}{1+w} \right)
= \alpha^{n-\delta}(1+w)^{n+\delta} \sum_k M(n, k) \frac{1}{(1+w)^{2k}} \frac{x^k}{\alpha^k} \left( \frac{2}{1+w} \right)^{n+\delta-2k}
= \sum_k M(n, k) x^k \alpha^{n-\delta-k} 2^{n+\delta-2k}
= \sum_k M(n, k) x^k \left( \frac{1+x}{2} \right)^{n-\delta-k} 2^{n+\delta-2k}
= \sum_k \frac{1}{2^{k-2\delta}} M(n, k) x^k (1 + x)^{n-\delta-k},
\]

and vice versa. This completes the proof. \( \Box \)
The reader is referred to [2] for a survey of some recent results on $\gamma$-positivity. For any $\gamma$-positive polynomial $M_n(x)$, we can define an associated polynomial $N_n(x)$ by using (6). And then we get a David-Barton type identity (5). As illustrations, in the rest of this section, we shall present two examples.

For example, Foata and Schützenberger [14] discovered that

$$A_n(x) = \sum_{k=1}^{\lceil (n+1)/2 \rceil} a(n,k) x^k (1+x)^{n+1-2k}$$

for $n \geq 1$, where the numbers $a(n,k)$ satisfy the recurrence relation

$$a(n,k) = ka(n-1,k) + (2n - 4k + 4)a(n-1,k-1),$$

with the initial conditions $a(1,1) = 1$ and $a(1,k) = 0$ for $k \neq 1$ (see [10, 26] for instance). By using the David-Barton identity (2) and Theorem 1, we immediately get the following result.

**Proposition 2.** For $n \geq 2$, we have

$$R_n(x) = \sum_{k=1}^{\lceil (n+1)/2 \rceil} \frac{1}{2^{n-2}} a(n,k) x^k (1+x)^{n+1-2k-1}.$$  

Let $\pm[n] = \{\pm 1, \pm 2, \ldots, \pm n\}$. Let $B_n$ be the hyperoctahedral group of rank $n$. Elements of $B_n$ are signed permutations of $\pm[n]$ with the property that $\pi(-i) = -\pi(i)$ for all $i \in [n]$. In the sequel, we always assume that signed permutations in $B_n$ are prepended by 0. That is, we identify a signed permutation $\pi = \pi(1) \cdots \pi(n)$ with the word $\pi(0)\pi(1) \cdots \pi(n)$, where $\pi(0) = 0$. A type $B$ descent is an index $i \in \{0, 1, \ldots, n-1\}$ such that $\pi(i) > \pi(i+1)$. Let $\text{des}^B(\pi)$ be the number of type $B$ descents of $\pi$. The type $B$ Eulerian polynomials are defined by

$$B_n(x) = \sum_{\pi \in B_n} x^{\text{des}^B(\pi)}.$$ 

It is well known that

$$B_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} b(n,k) x^k (1+x)^{n-2k},$$

where the numbers $b(n,k)$ satisfy the recurrence relation

$$b(n,k) = (1+2k)b(n-1,k) + 4(n-2k+1)b(n-1,k-1),$$  

with the initial conditions $b(1,0) = 1$ and $b(1,k) = 0$ for $k \neq 0$ (see [2, 10, 26]).

Define

$$b_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{2^k} b(n,k) x^k (1+x)^{n-k}.$$  

Then by Theorem 1, we get the following result.

**Proposition 3.** For $n \geq 1$, we have

$$b_n(x) = \left(\frac{1+x}{2}\right)^n (1+w)^n B_n \left(\frac{1-w}{1+w}\right).$$
Combining (7) and (8), we see that the polynomials \( b_n(x) \) satisfy the recurrence relation
\[
 b_{n+1}(x) = (1 + x + 2nx^2)b_n(x) + 2x(1 - x^2)b'_n(x),
\]  
with the initial conditions \( b_0(x) = 1, \ b_1(x) = 1 + x \). For \( n \geq 1 \), we define \( b_n(x) = \frac{1 + x}{x}c_n(x) \). It follows from (9) that the polynomials \( c_n(x) \) satisfy the recurrence relation
\[
 c_{n+1}(x) = (2nx^2 + 3x - 1)c_n(x) + 2x(1 - x^2)c'_n(x).
\]
Let \( \hat{B}_n = \{ \pi \in B_n | \pi(1) > 0 \} \). There is a combinatorial interpretation of \( c_n(x) \) (see [11, 29]):
\[
 c_n(x) = \sum_{\pi \in \hat{B}_n} x^{\text{altrun}(\pi)}.
\]

3. The q-alternating runs polynomials

For an alphabet \( A \), let \( \mathbb{Q}[[A]] \) be the rational commutative ring of formal power series in monomials formed from letters in \( A \). A Chen’s grammar (which is known as context-free grammar) over \( A \) is a function \( G : A \rightarrow \mathbb{Q}[[A]] \) that replaces a letter in \( A \) by an element of \( \mathbb{Q}[[A]] \), see [8, 9, 24] for details. The formal derivative \( D := D_G \) is a linear operator defined with respect to a context-free grammar \( G \). Following [9], a grammatical labeling is an assignment of the underlying elements of a combinatorial structure with variables, which is consistent with the substitution rules of a grammar.

Let us now recall two results on context-free grammars.

**Proposition 4** ([22, Theorem 6]). If \( G = \{ a \rightarrow ab, \ b \rightarrow bc, \ c \rightarrow b^2 \} \), then
\[
 D^n(a) = a \sum_{k=0}^{n} T_{n,k} b^k c^{n-k}, \quad D^n(a^2) = a^2 \sum_{k=0}^{n} R_{n+1,k} b^k c^{n-k}.
\]

**Proposition 5** ([22, Theorem 9]). If \( G = \{ a \rightarrow 2ab, \ b \rightarrow bc, \ c \rightarrow b^2 \} \), then
\[
 D^n(a) = a \sum_{k=0}^{n} R_{n+1,k} b^k c^{n-k}.
\]

Combining Leibniz’s formula and Proposition 4, we see that
\[
 R_{n+1}(x) = \sum_{k=0}^{n} \binom{n}{k} T_k(x) T_{n-k}(x).
\]
Motivated by Propositions 4 and 5 it is natural to consider the grammar
\[
 G_1 = \{ a \rightarrow qab, \ b \rightarrow bc, \ c \rightarrow b^2 \}. \quad (10)
\]

Note that \( D_{G_1}(a) = qab, \ D_{G_1}^2(a) = a(q^2b^2 + qbc) \). By induction, it is easy to verify that
\[
 D_{G_1}^n(a) = a \sum_{k=0}^{n} R_{n,k}(q) b^k c^{n-k}. \quad (11)
\]
It follows from (10) that
\[
D_{G_1}^{n+1}(a) = D_{G_1} \left( a \sum_{k=0}^{n} R_{n,k}(q) b^k c^{n-k} \right) = a \sum_{k} R_{n,k}(q) \left( k b^k c^{n-k+1} + q b^{k+1} c^{n-k} + (n-k)b^{k+2} c^{n-k-1} \right),
\]
which leads to the recurrence relation
\[
R_{n+1,k}(q) = k R_{n,k}(q) + q R_{n,k-1}(q) + (n-k)R_{n,k-2}(q).
\] (12)
The \textit{q-alternating run polynomials} are defined by
\[
R_n(x; q) = \sum_{k=0}^{n} R_{n,k}(q)x^k.
\]
In particular, \( R_n(x; 1) = T_n(x) \), \( R_n(x; 2) = R_{n+1}(x) \). The first few \( R_n(x; q) \) are given as follows:
\[
R_0(x; q) = 1, \quad R_1(x; q) = qx, \quad R_2(x; q) = q(1+qx), \quad R_3(x; q) = qx(1+3qx+x^2+q^2x^2).
\]
We define
\[
R(x, z; q) := \sum_{n=0}^{\infty} R_n(x; q) \frac{z^n}{n!}.
\]

\textbf{Proposition 6.} We have \( R(x, z; q) = T^q(x, z) \), where \( T(x, z) \) is given by (4). Therefore,
\[
\sum_{n=0}^{\infty} D_{G_1}^n(a) \frac{z^n}{n!} = a R \left( \frac{b}{c}, cz; q \right) = aT^q \left( \frac{b}{c}, cz \right).
\] (13)
Moreover, we have \( R_n(x; -q) = R_n(-x; q) \) and \( R_n(-x; -q) = R_n(x; q) \).

\textbf{Proof.} By rewriting (12) in terms of generating function \( R(x, z; q) \), we obtain
\[
(1-x^2z) \frac{\partial}{\partial z} R(x, z; q) = x(1-x^2) \frac{\partial}{\partial x} R(x, z; q) + q x R(x, z; q).
\] (14)
It is routine to check that the generating function \( T^q(x, z) \) satisfies (14). Also, this generating function gives \( T^q(0, z) = T^q(x, 0) = 1 \). Hence \( R(x, z; q) = T^q(x, z) \). It is routine to check that
\[
R(x, z; -q) = R(-x, z; q), \quad R(-x, z; -q) = R(x, z; q)
\]
which leads to the desired result. \( \square \)

We say that \( \pi \in S_n \) is a circular permutation if it has only one cycle. Let \( A = \{x_1, x_2, \ldots, x_k\} \) be a finite set of positive integers, and let \( C_A \) be the set of all circular permutations of \( A \). We will write a permutation \( w \in C_A \) by using its canonical presentation \( w = y_1 y_2 \cdots y_k \), where \( y_1 = \min A, y_i = w^{i-1}(y_1) \) for \( 2 \leq i \leq k \) and \( y_1 = w^k(y_1) \). A \textit{cycle peak} (resp. \textit{cycle double ascent, cycle double descent}) of \( w \) is an entry \( y_i, 2 \leq i \leq k \), such that \( y_{i-1} < y_i > y_{i+1} \) (resp. \( y_{i-1} < y_i < y_{i+1}, y_{i-1} > y_i > y_{i+1} \)), where we set \( y_{k+1} = \infty \). Let \( \text{cpk}(w) \) (resp. \( \text{cddasc}(w), \text{cddes}(w), \text{cyc}(w) \)) be the number of cycle peaks (resp. cycle double ascents, cycle double descents, cycles) of \( w \).

\textbf{Definition 7.} A \textit{cycle run} of a circular permutation \( w \) is an alternating run of \( w \) endowed with \( a \infty \) in the end. Let \( \text{crun}(w) \) be the number of cycle runs of \( w \).
It is clear that \( \text{crun}(w) = 2\text{cpk}(w) + 1 \). In the following discussion we always write \( \pi \in \mathfrak{S}_n \) in standard cycle decomposition: \( \pi = w_1 \cdots w_k \), where the cycles are written in increasing order of their smallest entry and each of these cycles is expressed in canonical presentation. We define

\[
\text{crun}(\pi) := \sum_{i=1}^{k} \text{crun}(w_i).
\]

In particular, \( \text{crun}((1)(2)\cdots(n)) = \sum_{i=1}^{n} \text{crun}(i) = \sum_{i=1}^{n} \text{altrun}(i\infty) = n \). We can now present the second main result.

**Theorem 8.** For \( n \geq 1 \), we have

\[
R_n(x; q) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{crun}(\pi)} q^{\text{cyc}(\pi)}.
\]

*Proof.* For \( \pi \in \mathfrak{S}_n \), we first put a \( \infty \) in the end of each cycle. We then introduce a grammatical labeling of \( \pi \) as follows:

1. Put a subscript label \( q \) at the end of each cycle of \( \pi \);
2. Put a superscript label \( a \) at the end of \( \pi \);
3. Put a superscript label \( b \) before each \( \infty \);
4. If \( \pi(i) \) is a cycle peak, then put a superscript label \( b \) before \( \pi(i) \) and a superscript label \( b \) right after \( \pi \);
5. If \( \pi(i) \) is a cycle double ascents, then put the superscript label \( c \) before \( \pi(i) \);
6. If \( \pi(i) \) is a cycle double descents, then put the superscript label \( c \) right after \( \pi(i) \).

The weight of \( \pi \) is the product of its labels. When \( n = 1, 2 \), we have

\[
\mathfrak{S}_1 = \{(1^b\infty)^a_q\}, \quad \mathfrak{S}_2 = \{(1^b\infty)_q(2^b\infty)_q^a, (1^c2^b\infty)_q^a\}.
\]

Then the weight of \( (1^b)^a_q \) is given by \( D_{\mathfrak{S}_1}(a) \), and the sum of weights of the elements in \( \mathfrak{S}_2 \) is given by \( D_{\mathfrak{S}_1}^2(a) \). Hence the result holds for \( n = 1, 2 \). Let

\[
r_n(i, j) = \{\pi \in \mathfrak{S}_n : \text{crun}(\pi) = i, \ \text{cyc}(\pi) = j\}.
\]

Suppose we get all labeled permutations in \( r_{n-1}(i, j) \), where \( n \geq 3 \). Let \( \pi' \) be obtained from \( \pi \in r_{n-1}(i, j) \) by inserting the entry \( n \). We distinguish the following four cases:

1. If we insert \( n \) as a new cycle, then \( \pi' \in r_{n-1}(i+1, j+1) \). This case corresponds to the substitution rule \( a \to qab \);
2. If we insert \( n \) before a \( \infty \), then \( \pi' \in r_{n-1}(i, j) \). This case corresponds to the substitution rule \( b \to bc \);
3. If we insert \( n \) before or right after a cycle peak, then \( \pi' \in r_{n-1}(i, j) \). This case corresponds to the substitution rule \( b \to bc \);
4. If we insert \( n \) before a cycle double ascents or right after a cycle double descents, \( \pi' \in r_{n-1}(i+2, j) \). This case corresponds to the substitution rule \( c \to b^2 \).

In each case, the insertion of \( n \) corresponds to one substitution rule in the grammar (10). It is easy to check that the action of \( D_{\mathfrak{S}_1} \) on elements of \( \mathfrak{S}_{n-1} \) generates all elements of \( \mathfrak{S}_n \). Using (11) and by induction, we present a constructive proof of (15). This completes the proof.

\( \square \)
We define
\[ R_n(x, y; q) = \sum_{\pi \in S_n} x^{\text{crun}(\pi)} y^{\text{fix}(\pi)} q^{\text{cyc}(\pi)}, \]
\[ R(x, y, z; q) = \sum_{n=0}^{\infty} R_n(x, y; q) \frac{z^n}{n!}. \]

By using the principle of inclusion-exclusion, it is routine to verify that
\[ R_n(x, y; q) = \sum_{i=0}^{n} \binom{n}{i} (qx_y - qx)^i R_{n-i}(x; q). \]

Hence
\[ R(x, y, z; q) = e^{xq_y - qx} T(x, z). \]

A permutation \( \pi \in S_n \) is a derangement if \( \pi(i) \neq i \) for any \( i \in [n] \). Let \( D_n \) denote the set of derangements in \( S_n \). Then
\[ R_n(x, 0; 1) = \sum_{\pi \in D_n} x^{\text{crun}(\pi)}. \]

**Proposition 9.** Set \( d_n(x) = R_n(x, 0; 1) \). Then the polynomials \( d_n(x) \) satisfy the recurrence
\[ d_{n+1}(x) = nx^2 d_n(x) + x(1 - x^2) d'_n(x) + nxd_{n-1}(x), \]
with the initial conditions \( d_0(x) = 1, \) \( d_1(x) = 0. \) In particular, \( d_n(-1) = -(n - 1) \) for \( n \geq 1 \).

**Proof.** Let \( d(x, z) = \sum_{n=0}^{\infty} d_n(x) \frac{z^n}{n!} \). It follow from (16) that
\[ d(x, z) = e^{-x} T(x, z). \]

By rewriting (14) in terms of generating function \( T(x, z) \), we obtain
\[ (1 - x^2) \frac{\partial}{\partial z} T(x, z) = xT(x, z) + x(1 - x^2) \frac{\partial}{\partial x} T(x, z). \]

Hence
\[ (1 - x^2) \frac{\partial}{\partial z} d(x, z) = xzd(x, z) + x(1 - x^2) \frac{\partial}{\partial x} d(x, z), \]
which yields the desired recurrence relation. \( \square \)

Let \( d_n(x) = \sum_{k=0}^{n} d_{n,k} x^k \). By using (18), it is not hard to verify that
\[ \sum_{n=0}^{\infty} d_{n,n} \frac{z^n}{n!} = \frac{e^{-x}}{\tan x + \sec x}. \]

4. **Semi-\( \gamma \)-positive polynomials**

Let \( g(x) = \sum_{i=0}^{2n} g_i x^i \) be a symmetric polynomial. Note that
\[ g(x) = \sum_{i=0}^{n} \gamma_i x^i (1 + x)^{2(n-i)} \]
\[ = \sum_{i=0}^{n} \gamma_i x^i (1 + 2x + x^2)^{n-i} \]
\[ = \sum_{i=0}^{n} \sum_{\ell=0}^{n-i} \binom{n-i}{\ell} 2^\ell \gamma_i x^i + \ell (1 + x^2)^{n-i-\ell}. \]
Hence \( g(x) \) can be expanded as
\[
g(x) = \sum_{k=0}^{n} \lambda_k x^k (1 + x^2)^{n-k}.
\]

It is clear that if \( \gamma_i \geq 0 \) for all \( 0 \leq i \leq n \), then \( \lambda_k \geq 0 \) for all \( 0 \leq k \leq n \). Furthermore, we have
\[
g(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \lambda_{2k} x^{2k} (1 + x^2)^{n-2k} + \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \lambda_{2k+1} x^{2k+1} (1 + x^2)^{n-2k-1}
\]
\[
= g_1(x^2) + xg_2(x^2).
\]

Similarly, if \( h(x) = \sum_{i=0}^{2n+1} h_i x^i \) a symmetric polynomial, then we have
\[
h(x) = \sum_{i=0}^{n} \beta_i x^i (1 + x)^{2n+1-2i}
\]
\[
= (1 + x) \sum_{i=0}^{n} \sum_{\ell=0}^{n-i} \binom{n-i}{\ell} 2^\ell \beta_i x^{i+\ell} (1 + x^2)^{n-i-\ell}.
\]

Hence \( h(x) \) can be expanded as
\[
h(x) = (1 + x) \sum_{k=0}^{n} \mu_k x^k (1 + x^2)^{n-k}.
\]

**Definition 10.** If \( f(x) = (1 + x)^\nu \sum_{k=0}^{n} \lambda_k x^k (1 + x^2)^{n-k} \) and \( \lambda_k \geq 0 \) for all \( 0 \leq k \leq n \), then we say that \( f(x) \) is semi-\( \gamma \)-positive, where \( \nu = 0 \) or \( \nu = 1 \).

It should be noted that a semi-\( \gamma \)-positive polynomial is not always \( \gamma \)-positive. From the above discussion it follows that we have the following result.

**Proposition 11.** If \( f(x) = (1 + x)^\nu \left( f_1(x^2) + x f_2(x^2) \right) \) is a semi-\( \gamma \)-positive polynomial, then both \( f_1(x) \) and \( f_2(x) \) are \( \gamma \)-positive.

In the following, we shall show the semi-\( \gamma \)-positivity of the alternating run polynomials of dual Stirling permutations. Following \[16\], a **Stirling permutation** of order \( n \) is a permutation of the multiset \( \{1,1,\ldots,n,n\} \) such that for each \( i \), \( 1 \leq i \leq n \), all entries between the two occurrences of \( i \) are larger than \( i \). There has been much recent work on Stirling permutations, see \[17, 24\] and references therein.

Denote by \( \mathcal{Q}_n \) the set of **Stirling permutations** of order \( n \). Let \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_{2n} \in \mathcal{Q}_n \). Let \( \Phi \) be the injection which maps each first occurrence of entry \( j \) in \( \sigma \) to \( 2j \) and the second \( j \) to \( 2j - 1 \), where \( j \in [n] \). For example, \( \Phi(21331) = 432651 \). Let \( \Phi(\mathcal{Q}_n) = \{ \pi \mid \sigma \in \mathcal{Q}_n, \Phi(\sigma) = \pi \} \) be the set of **dual Stirling permutations** of order \( n \). Clearly, \( \Phi(\mathcal{Q}_n) \) is a subset of \( \mathcal{S}_{2n} \). For \( \pi \in \Phi(\mathcal{Q}_n) \), the entry \( 2j \) is to the left of \( 2j - 1 \), and all entries in \( \pi \) between \( 2j \) and \( 2j - 1 \) are larger than \( 2j \), where \( 1 \leq j \leq n \). Noted that \( \pi \in \Phi(\mathcal{Q}_n) \) always ends with a descending run. The alternating runs polynomials of dual Stirling permutations are defined by
\[
F_n(x) = \sum_{\sigma \in \Phi(\mathcal{Q}_n)} x^{\text{allrun} (\sigma)} = \sum_{k=1}^{2n-1} F_{n,k} x^k.
\]
According to [23], the numbers $F_{n,k}$ satisfy the recurrence relation

$$F_{n+1,k} = k F_{n,k} + F_{n,k-1} + (2n - k + 2) F_{n,k-2},$$  \hspace{1cm} (19)

with the initial conditions $F_{0,0} = 1$, $F_{1,1} = 1$ and $F_{n,0} = 0$ for $n \geq 1$. It follows from (19) that

$$F_{n+1}(x) = (x + 2nx^2) F_n(x) + x(1 - x^2) F'_n(x).$$

The first few $F_n(x)$ are given as follows:

$$F_1(x) = x,$$

$$F_2(x) = x + x^2 + x^3,$$

$$F_3(x) = x + 3x^2 + 7x^3 + 3x^4 + x^5,$$

$$F_4(x) = x + 7x^2 + 29x^3 + 31x^4 + 29x^5 + 7x^6 + x^7.$$

Let

$$r(x) = \frac{\sqrt{1 + x}}{1 - x}.$$

By induction, it is to verify that

$$\left( x \frac{d}{dx} \right)^{2n} r(x) = \frac{r(x) F_{2n}(x)}{(1 - x^2)^{2n}},$$

$$\left( x \frac{d}{dx} \right)^{2n+1} r(x) = \frac{F_{2n+1}(x)}{r(x)(1 - x^2)^{2n+1}}.$$

**Lemma 12** ([23]). If

$$G_2 = \{ x \rightarrow xyz, y \rightarrow yz^2, z \rightarrow y^2z \},$$  \hspace{1cm} (20)

then we have

$$D^n_{G_2}(x) = x \sum_{\sigma \in \Phi(\mathcal{Q}_n)} y^{\text{altrun}(\sigma)} z^{2n - \text{altrun}(\sigma)} = x \sum_{k=0}^{2n-1} F_{n,k} y^k z^{2n-k}.$$  \hspace{1cm} (21)

We now recall another combinatorial interpretation of $F_n(x)$. An occurrence of an ascent-plateau of $\sigma \in \mathcal{Q}_n$ is an index $i$ such that $\sigma_{i-1} < \sigma_i = \sigma_{i+1}$, where $i \in \{2, 3, \ldots, 2n - 1\}$. An occurrence of a left ascent-plateau is an index $i$ such that $\sigma_{i-1} < \sigma_i = \sigma_{i+1}$, where $i \in \{1, 2, \ldots, 2n - 1\}$ and $\sigma_0 = 0$. Let $\text{ap}(\sigma)$ and $\text{la}(\sigma)$ be the numbers of ascent-plateaus and left ascent-plateaus of $\sigma$, respectively. The number of flag ascent-plateaus of $\sigma$ is defined by

$$\text{fap}(\sigma) = \begin{cases} 2\text{ap}(\sigma) + 1, & \text{if } \sigma_1 = \sigma_2; \\ 2\text{ap}(\sigma), & \text{otherwise}. \end{cases}$$

Clearly, $\text{fap}(\sigma) = \text{ap}(\sigma) + \text{la}(\sigma)$. Following [24, Section 3], we have

$$D^n_{G_2}(x) = x \sum_{\sigma \in \mathcal{Q}_n} y^{\text{fap}(\sigma)} z^{2n - \text{fap}(\sigma)}.$$

Thus,

$$F_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^\text{fap}(\sigma).$$

In fact, it is easy to verify that $\text{fap}(\sigma) = \text{altrun}(\Phi(\sigma))$ for any $\sigma \in \mathcal{Q}_n$. 

Proposition 13. For $n \geq 1$, we have

$$F_n(x) = \sum_{k=1}^{n} \gamma_{n,k} x^k (1 + x)^{2n-2k},$$

where the numbers $\gamma_{n,k}$ satisfy the recurrence relation

$$\gamma_{n+1,k} = k\gamma_{n,k} + (2n - 4k + 5)\gamma_{n,k-1}, \quad (22)$$

with the initial conditions $\gamma_{1,1} = 1$ and $\gamma_{1,k} = 0$ for $k \neq 1$. In particular,

$$\gamma_{n+1,n+1} = (-1)^n (2n - 1)!! \text{ for } n \geq 1.$$

Proof. We first consider a change of the grammar (20). Set $a = yz$ and $b = y + z$. Then we have $D(x) = xa, D(a) = a(b^2 - 2a), D(b) = ab$. If

$$G_3 = \{ x \to xa, a \to a(b^2 - 2a), b \to ab \},$$

then by induction, we see that there exist integers $\gamma_{n,k}$ such that

$$D^n_{G_3}(x) = x \sum_{k=0}^{n} \gamma_{n,k} a^k b^{2n-2k}. \quad (23)$$

Note that

$$D^{n+1}_{G_3}(x) = D_{G_3} \left( x \sum_{k=1}^{n} \gamma_{n,k} a^k b^{2n-2k} \right) = x \sum_{k} \gamma_{n,k} a^k b^{2n-2k} \left( a + kb^2 - 2ka + (2n - 2k)a \right)$$

By comparing the coefficients of $a^k b^{2n-2k+2}$, we immediately get (22). Moreover, it is clear that $\gamma_{n,0} = 0$ for $n \geq 1$. By using (23), upon taking $a = yz$ and $b = y + z$, we get

$$D^n_{G_2}(x) = x \sum_{k=0}^{n} \gamma_{n,k} (yz)^k (y + z)^{2n-2k}. \quad (24)$$

Then comparing (24) with (21), we see that $F_n(x) = \sum_{k=1}^{n} \gamma_{n,k} x^k (1 + x)^{2n-2k}$ for $n \geq 1$. By using (22), we obtain

$$\gamma_{n+1,n+1} = -(2n - 1)\gamma_{n,n},$$

which yields the desired explicit formula. \[\square\]

For $n \geq 1$, let $\gamma_n(x) = \sum_{k=1}^{n} \gamma_{n,k} x^k$. It follows from (22) that

$$\gamma_{n+1}(x) = (2n + 1)x\gamma_n(x) + x(1 - 4x)\gamma'_n(x).$$

The first few $\gamma_n(x)$ are $\gamma_0(x) = 1, \gamma_1(x) = x, \gamma_2(x) = x - x^2, \gamma_3(x) = x - x^2 + 3x^3$. From Proposition 13 we see that for any positive even integer $n$, the polynomial $F_n(x)$ is not $\gamma$-positive.

We can now present the third main result of this paper.

THE ALTERNATING RUN POLYNOMIALS OF PERMUTATIONS
Theorem 14. The polynomial \( F_n(x) \) is semi-\( \gamma \)-positive. More precisely, we have
\[
F_n(x) = \sum_{k=0}^{n} f_{n,k} x^k (1 + x^2)^{n-k},
\]
where the numbers \( f_{n,k} \) satisfy the recurrence relation
\[
f_{n+1,k} = kf_{n,k} + f_{n,k-1} + 4(n-k+2)f_{n,k-2},
\]
(25)
with the initial conditions \( f_{0,0} = 1 \) and \( f_{n,0} = 0 \) for \( n \geq 1 \). Let \( f_n(x) = \sum_{k=0}^{n} f_{n,k} x^k \). Then
\[
f(x, z) = \sum_{n=0}^{\infty} f_n(x) \frac{z^n}{n!} = \sqrt{T(2x, z)},
\]
(26)
where \( T(x, z) \) is given by (4).

Proof. We first consider the grammar (20). Note that
\[
D(x) = xyz, \quad D(yz) = yz(y^2 + z^2), \quad D(y^2 + z^2) = 4y^2 z^2.
\]
Set \( u = yz \) and \( v = y^2 + z^2 \). Then we have \( D(x) = xu, \ D(u) = uv \) and \( D(v) = 4u^2 \). If
\[
G_4 = \{ x \to xu, \ u \to uv, \ v \to 4u^2 \},
\]
(27)
then by induction we see that there exist nonnegative integers \( f_{n,k} \) such that
\[
D^\circ_{G_4} G_4(x) = x \sum_{k=0}^{n} f_{n,k} u^k v^{n-k}.
\]
(28)
Note that
\[
D^{n+1}_{G_4} = D_{G_4} \left( x \sum_{k=1}^{n} f_{n,k} u^k v^{n-k} \right)
\]
\[= x \sum_{k} f_{n,k} \left( u^{k+1} v^{n-k} + ku^k v^{n-k+1} + 4(n-k)u^{k+2} v^{n-k-1} \right).\]
By comparing the coefficients of \( u^{k+1} v^{n+1-k} \) we get (25). Moreover, it follows from (27) that \( f_{0,0} = 1 \) and \( f_{n,0} = 0 \) for \( n \geq 1 \). By using (28), upon taking \( u = yz \) and \( v = y^2 + z^2 \), we get
\[
D^n_{G_2} G_4(x) = x \sum_{k=0}^{n} f_{n,k} (yz)^k (y^2 + z^2)^{n-k}.
\]
(29)
By comparing (29) with (21), we get
\[
F_n(x) = \sum_{k=0}^{n} f_{n,k} x^k (1 + x^2)^{n-k}.
\]
(30)
We now consider a change of the grammar (10). Set \( q = \frac{1}{2}, \ a = x, \ b = 2u, \ c = v \). Then
\[
D(x) = xu, \ D(u) = uv, \ D(v) = 4u^2,
\]
which are the substitution rules in the grammar (27). Hence it follows from (13) that
\[
\sum_{n=0}^{\infty} D^n_{G_2} G_4(x) \frac{z^n}{n!} = x \sum_{n=0}^{\infty} \sum_{k=0}^{n} f_{n,k} u^k v^{n-k} \frac{z^n}{n!} = x R \left( \frac{2u}{v}, vz; \frac{1}{2} \right),
\]
which leads to \( f(x, z) = R(2x, z; 1/2) = \sqrt{T(2x, z)} \). This completes the proof. \( \square \)
Combining (26) and (30), we immediately get the following result.

**Corollary 15.** We have

\[ F(x, z) = \sum_{n=0}^{\infty} F_n(x) \frac{z^n}{n!} = \sqrt{T \left( \frac{2x}{1 + x^2}, (1 + x^2)z \right)}. \]

It would be interesting to present a combinatorial interpretation of Corollary 15. By using (26), it is not hard to verify that

\[ \sum_{n=0}^{\infty} f_{n,n} \frac{x^n}{n!} = \frac{1 + \tan x}{1 - \tan x}. \]

It should be noted that the numbers \( f_{n,n} \) appear as A012259 in [27].

5. **Concluding remarks**

This paper gives a survey of some results related to alternating runs of permutations. We present a method to construct David-Barton type identities, and based on the survey [2], one can derive several David-Barton type identities. Moreover, we introduce the definition of semi-\( \gamma \)-positive polynomial. The \( \gamma \)-positivity of a polynomial \( f(x) \) is a sufficient (not necessary) condition for the semi-\( \gamma \)-positivity of \( f(x) \). In particular, we show that the alternating run polynomials of dual Stirling permutations are semi-\( \gamma \)-positive.

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