Geometric Characterization of Property R

William Gordon Ritter

Harvard University Department of Physics
17 Oxford St., Cambridge, MA 02138
Email: ritter@fas.harvard.edu

ABSTRACT: Consider pairs of the form \((G, N)\), with \(G\) a group and \(N \triangleleft G\), as objects of a category \(\mathcal{PG}\). A morphism \((G_1, N_1) \rightarrow (G_2, N_2)\) will be a group homomorphism \(f : G_1 \rightarrow G_2\) such that \(f(N_1) \subset N_2\). We introduce a functor \(Q : \mathcal{PG} \rightarrow \text{Groups}\), which provides a geometric definition of Property R, since it is most naturally visualized by means of a directed graph. We compute these graphs for a number of finite groups of small order, and prove a general characterization of the graphs which occur in this way.

KEYWORDS: Group Theory, Graph Theory, Property R.
1. Introduction

For a group $G$, we denote the derived subgroup by either $G'$ or $[G,G]$. $G$ is said to have Property $R$ if $(G/N)' \neq G/N$ holds for every proper normal subgroup $N \triangleleft G$, i.e. nontrivial quotients of $G$ are never perfect. Interesting relationships between different normal subgroups can be modeled by forming graphs with the normal subgroups as points, and edges isolating the property of interest. The edge relation which characterizes Property R is given explicitly in eq. (3.1).

All groups which are either solvable, simple, or are symmetric groups satisfy Property R. The latter property is important mainly because it appears as a hypothesis in powerful simplicity theorems for groups with a Tits system or BN-pair. A well known example is the Tits Simplicity Theorem [1].

**Theorem 1 (Tits Simplicity, [1]).** Let $G$ be a group having a BN-pair. Let $Z$ be the intersection of the conjugates of $B$, let $U < B$ and let $G_1$ be the subgroup generated by the conjugates of $U$ in $G$. Assume $U \triangleleft B = UT$, $U$ has Property R, $G_1 = G_1'$, and $(W,S)$ is an irreducible Coxeter system. Then any subgroup $H \subset G$ normalized by $G_1$ is contained in $Z$ or contains $G_1$. It follows that $G_1/(G_1 \cap Z)$ is either trivial, or is nonabelian and simple.

2. The Category of Pairs

Let the objects of category $\mathcal{PG}$ be pairs of the form $(G,N)$ where $G$ is a group and $N \triangleleft G$. A morphism between two objects $(G_1,N_1) \rightarrow (G_2,N_2)$ will be a group homomorphism $f : G_1 \rightarrow G_2$ such that $f(N_1) \subset N_2$. For each such $f$, we denote the induced homomorphism by $\overline{f} : G_1/N_1 \rightarrow G_2/N_2$. In this brief note we define a functor $Q : \mathcal{PG} \rightarrow \textbf{Groups}$, and

\[ \text{See [1], IV, Sec. 1 no 9 for definitions} \]
study its fundamental properties. This functor provides a geometric definition of Property R, and is naturally visualized by means of a directed graph.

For any group \( G \), we use the notation
\[
G' = [G, G]
\]
for the derived subgroup, generated by elements of the form \( xyx^{-1}y^{-1} \). Consider the natural 1-1 correspondence given by the Lattice isomorphism theorem:
\[
\left\{ \text{subgroups of } G \text{ containing } N \right\} \longleftrightarrow \left\{ \text{subgroups of } G/N \right\}
\]
\[
H \mapsto \overline{H} = H/N \leq G/N \quad (2.1)
\]
Given an object \( (G, N) \in \text{obj}_{\mathcal{P}G} \), the commutator subgroup \([G/N, G/N]\) is a characteristic subgroup in \( G/N \). Let \( Q(G, N) \) denote the preimage of \([G/N, G/N]\) under the map \( (2.1)\).

The diagram is as follows:

\[
\begin{array}{ccc}
G & \longrightarrow & G/N \\
\downarrow & & \downarrow \\
Q(G, N) & \longrightarrow & [G/N, G/N] \\
\downarrow & & \downarrow \\
N & \longrightarrow & \{e\}
\end{array}
\]

Given a group \( G \) and a normal subgroup \( N \trianglelefteq G \), \( Q \) associates a second normal subgroup of \( G \) which is not, in general, equal to \( N \).

**Theorem 2.** \( Q \) is a functor on the category \( \mathcal{P}G \).

**Proof of Theorem 2.** Given a morphism \( f \in \text{mor}_{\mathcal{P}G} \), \( f : (G_1, N_1) \rightarrow (G_2, N_2) \), we define an associated morphism \( Q(f) : Q(G_1, N_1) \rightarrow Q(G_2, N_2) \) by restriction:
\[
Q(f) = f|_{Q(G_1, N_1)}.
\]
Let \( i \in \{1, 2\} \) and suppose \( G_i, N_i, Q_i \) are arbitrary groups satisfying \( N_i \trianglelefteq Q_i \trianglelefteq G_i \) and \( f : (G_1, N_1) \rightarrow (G_2, N_2) \) is a morphism as above. If
\[
\overline{f}(Q_1/N_1) \subset Q_2/N_2
\]
then \( f(Q_1) \subset Q_2 \). (If \( \exists q_1 \in Q_1 \) such that \( f(q_1) \notin Q_2 \), then \( \overline{f}(q_1 N_1) = f(q_1)N_2 \notin Q_2/N_2 \).

It remains to see that
\[
f(Q(G_1, N_1)) \subset Q(G_2, N_2) \quad (2.2)
\]
This will finish the proof, since it shows that \( Q(f) \) maps into \( Q(G_2, N_2) \). To prove \( (2.2) \), note that
\[
\overline{f}(Q(G_1, N_1)/N_1) = \overline{f}((G_1/N_1)') = (\overline{f}(G_1/N_1))' \subset (G_2/N_2)' = Q(G_2, N_2)/N_2
\]
The conclusion now follows by the argument given above. \( \Box \)
3. The Forest Associated to $G$

Throughout this paper, we use the term *tree* to mean graphs with no cycles except possibly self-loops. $Q$ gives a natural way of associating a forest of trees to a group $G$. Let $T_G$ be the digraph with vertices labelled by the normal subgroups of $G$, and with edge relation $e$ defined as follows:

$$N_1 e N_2 \iff N_2 = Q(G, N_1)$$  \hspace{1cm} (3.1)

We refer to this forest as the *Property R graph* of $G$, since $G$ has Property R if and only if the vertex in $T_G$ corresponding to the whole group $G$ is isolated. An example of such a graph for a group of order 192 is the following:

![Graph Example]

Since the first homology $H_1(G)$ is the same as the abelianization $G/[G, G]$, our edge relation implies that if there is an edge $a \rightarrow b$, then $H_1(G/a) \cong G/b$. We now state our main results, and provide proofs where necessary.

**Theorem 3.** The graph defined by (3.1) has the following properties:

1. **(Inclusion in the full lattice)** Let $L$ be the full lattice of normal subgroups of $G$, with edge relation defined by inclusions. Since $N_1 \subseteq Q(G, N_1)$, it follows that $T_G$ can be viewed as a subgraph $L$ with the same vertices and fewer edges.

2. **(Self-loops)** Self-loops correspond to abelian quotients, in the following sense:

$$N = Q(G, N) \iff G/N \text{ is abelian} \iff \text{vertex } N \text{ has a self-loop}$$

Since $T_G$ has no loops other than self-loops, we can write more simply in this context

abelian quotient = loop

A quotient $G/N$ of $G$ is abelian if and only if $N$ includes $G'$, so the Property R graph provides a quick way of visualizing the normal subgroups which are between $G$ and $G'$ in the lattice.

3. **(Property R)** The graph associated to $Q$ gives a geometric definition of Property R. $G$ has Property R if and only if the vertex in $T_G$ corresponding to the whole group $G$ is isolated, because a normal subgroup $N$ with $(G/N)' = G/N$ generates a directed edge of the form $N \rightarrow G$. 
4. (Automorphism group action) There is a natural action of the automorphism group Aut(G) on the tree T_G, i.e. if N_1 \rightarrow N_2 is an edge in T_G, then f(N_1) \rightarrow f(N_2) is also an edge in T_G, for any f \in Aut(G).

Proof: Since N_2/N_1 \subseteq (G/N_1)', we may represent any coset n_2N_1 \in N_2/N_1 as g'N_1, where g' \in G'. Then f(n_1N_2) = f(g')f(N_1) \in (G/f(N_1))'. We have used the property that G' is characteristic, and hence preserved under all automorphisms. The reverse inclusion follows similarly.

5. (Edge stabilizers) An arbitrary edge of T_G may be written N \rightarrow Q(G, N). The stabilizer of this edge under the action of the group Aut(G) on the forest T_G is the same as the stabilizer of N under the action of Aut(G) on subgroups of G.

4. Characterization of Property R Graphs

The n-star graph, denoted S_n, is a tree on n + 1 nodes with one node having vertex degree n and the others having degree 1.

We also define S_n to be the star graph S_n, modified by the addition of a self-loop at the center. It is easy to see that Property R graphs, modulo self-loops, are disjoint unions of stars.

Theorem 4. The Property R graph of any group G is a union of disjoint copies of S_j for various values of j \in \mathbb{N}, and isolated points. If we define S_{-1} to be an isolated point, then the isomorphism type of the graph is encoded in one list of numbers, \{j : S_j \subset T, j \geq -1\}, counted with multiplicity.

Proof of Theorem 4. Suppose to the contrary that there is a nontrivial configuration of the form a \rightarrow b \rightarrow c where a, b, c are distinct proper normal subgroups. The arrow a \rightarrow b means that b/a \cong (G/a)''. Then we have

\[
\frac{G/b}{b/a} \cong \frac{G/a}{(G/a)''} = H_1(G/a)
\]

which is abelian. Therefore (G/b)'' \cong \{e\} and so any arrow from b must be a self-loop. In particular b = c. □

Our computations lead us to conjecture the converse of Theorem 4:

Conjecture 1. For any configuration G of stars S_j and isolated points as in Lemma 4, there exist some group G such that T_G = G.
5. Computation and Specific Examples

For small groups, the Property R graphs $T_G$ studied in the previous section can be computed explicitly and visualized through simple and elegant computer algebra methods. We include a table of such computations for finite groups of small order. These calculations were performed using GAP for the group theory, and Mathematica for the combinatorics and graph visualization. The GAP program analyzes the lattice of normal subgroups, computes the necessary quotients and derived groups, and generates graphs in a format which is acceptable input for Combinatorica, a standard package for Mathematica since version 4.

The heart of the group theory program is the following four lines:

```plaintext
NS := NormalSubgroups(G);
for i in [ 1 .. Length(NS) ] do
  phi := NaturalHomomorphismByNormalSubgroupNC(G, NS[i]);
  j := Position(NS, PreImage(phi, DerivedSubgroup(Image(phi))));
```

The GAP program at this point writes instructions for Combinatorica to place a directed edge from vertex $i$ to vertex $j$ in the graph it is building.

To denote some specific examples, we introduce the notation that if $G, H$ are two graphs, then the product $GH$ denotes the disjoint union. In particular, $G^n = \bigsqcup_{i=1}^n G$.

Eq. (5.1) shows the result of some computations for general linear groups of low order.

\[
\begin{array}{ccc}
GL(2, 7) & (S_0S_2)^2 & GL(3, 2) \ S_1 \\
GL(2, 13) & (S_0S_1S_2)^3 & GL(3, 3) \ (S_1)^2 \\
GL(2, 17) & S_0S_2(S_1)^3 & GL(3, 4) \ S_0S_2 \\
GL(2, 19) & (S_0S_2)^3 & GL(3, 5) \ (S_1)^3
\end{array}
\]

In addition, we have the following isomorphisms between graphs,

\[ T_{GL(2,23)} = T_{GL(3,7)} = T_{GL(2,7)} = (S_0S_2)^2 \]

The Property R graphs of simple groups, such as $PSL$ over a finite field, for obvious reasons all take the form $\bullet \rightarrow \bigcirc$, which is denoted $S_1$. It is not an interesting property for simple groups.

Figure 1 and Figure 2 give the Property R diagrams for $2 \times 2$ and $3 \times 3$ matrix groups over finite fields $GF(q^n)$ for the smallest interesting values of $q$ and $n$. \(^2\)

References

[1] Bourbaki, Nicolas *Elements de mathematique*. (French) [Elements of mathematics] *Groupes et algebres de Lie*. Chapitres 4, 5 et 6. [Lie groups and Lie algebras. Chapters 4, 5 and 6] Masson, Paris (1981)

\(^2\)The author wishes to thank the Harvard Math department for some processor time on a powerful Sun computer, which was used for the calculations in Figure 1 and Figure 2.
Figure 1: $GL(2,q^n)$
Figure 2: $GL(3, q^n)$