Linear orderings of combinatorial cubes

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Abstract

We show that, for every linear ordering of $[2]^n$, there is a large subcube on which the ordering is lexicographic. We use this to deduce that every long sequence contains a long monotone subsequence supported on an affine cube.

More generally, we prove an analogous result for linear orderings of $[k]^n$. We show that, for every such ordering, there is a large subcube on which the ordering agrees with one of approximately\[\frac{(k-1)!}{2^{(kn/2)}}\]orderings.

1 Statement of results

Monotone subsequences. A classic result of Erdős and Szekeres [3] asserts that every sufficiently long sequence $(a_1, a_2, \ldots)$ contains a subsequence $(a_{i_1}, \ldots, a_{i_m})$ of length $m$ that is monotone. One may wonder if one may strengthen this result by requiring that the set of indices $\{i_1, \ldots, i_m\}$ is an arithmetically structured set.

Our first result is such a strengthening. Before stating it, we recall that an affine $d$-cube is a set of the form \(\{x_0 + \varepsilon_1 x_1 + \cdots + \varepsilon_d x_d : \varepsilon_1, \ldots, \varepsilon_d \in \{0, 1\}\}\). An affine $d$-cube is proper if it contains $2^d$ distinct elements.

Theorem 1. For every $d$, there exists $m$ such that every sequence of $m$ distinct real numbers contains a monotone subsequence whose index set is a proper affine $d$-cube.

In this result one cannot replace affine cubes by arithmetic progressions. More precisely, we observe the following.

Proposition 2. There exist arbitrarily long sequences of distinct real numbers that contains no monotone subsequences whose index set is a 3-term arithmetic progression.

Proof. We construct such sequences, which we dub 3-AP-free, inductively. We start with any sequence of length 1. For the induction step, we note that if $(a_1, \ldots, a_m)$ is any 3-AP-free sequence, and $M > 2 \max |a_i|$, then $(a_1, a_1 + M, a_2, a_2 + M, \ldots, a_m, a_m + M)$ is also 3-AP-free. Indeed, suppose $\{i, j, k\}$ is a three-term arithmetic progression. If the parities of $i, j, k$ are the same, the sequence $(a_i, a_j, a_k)$ is not monotone by induction. If the parities of $i, j, k$ are different, then $(a_i, a_j, a_k)$ is not monotone by the choice of $M$. \qed

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Hales–Jewett-type result for $[2]^n$. A common way to prove Ramsey results on the integers is to deduce them from abstract statements about high-dimensional cubes. For example, in this way one deduces van der Waerden’s theorem from the Hales–Jewett theorem, and Szemerédi’s theorem from the density Hales–Jewett theorem. Our proof of Theorem 1 also follows this pattern: we deduce Theorem 1 from a Ramsey result about linear orderings of $[2]^n$.

As our most general result will apply not only to $[2]^n$, but to $[k]^n$ for any $k \geq 2$, we introduce definitions at that level of generality.

We shall think of elements of $[k]^n$ as words of length $n$ over alphabet $[k]$. A $d$-parameter word is a word $p$ over alphabet $[k] \cup \{*, 1, \ldots, *d\}$ that contains each of $*, 1, \ldots, *d$ at least once. For any word $w$ of length $d$ (possibly over a different alphabet), we let $p[w]$ be the word obtained from $p$ by replacing each $*i$ by $w_i$, for each $i$. For example, if $p = 21 *1 *2 *3$ and $w = 31$, then $p[w] = [213]133$.

If $p$ is a $d$-parameter word, then the set $\{p[w] : w \in [k]^d\}$ can be naturally regarded as a copy of $[k]^d$ inside $[k]^n$; we thus call it a (combinatorial) $d$-subcube. Two $d$-parameter words that differ in a permutation of $\{*, 1, \ldots, *d\}$ induce the same $d$-subcube. Call a $d$-parameter word $p$ canonical if the first occurrence of $*1$ precedes the first occurrence of $*2$, which in turn precedes the first occurrence of $*3$, etc. Canonical words induce a canonical bijection between $[k]^d$ and corresponding $d$-subcubes; we shall always use this bijection when identifying $d$-subcubes with $[k]^d$.

If $p$ is a $D$-parameter word of length $n$ and $p'$ is a $d$-parameter word of length $D$, then $p[p']$ is a $d$-parameter word of length $n$. Furthermore, if both $p$ and $p'$ are canonical, then so is $p[p']$. Hence, if $C_1 \supseteq C_2 \supseteq \cdots$ is a nested chain of subcubes of $[k]^n$, we may use the canonical bijection to regard $C_{i+1}$ as a subcube of $C_i$.

The canonical bijection also allows us to regard a restriction of a linear ordering on $[k]^n$ to any $d$-subcube as a linear ordering on $[k]^d$. Namely, let $\prec$ be a linear ordering on $[k]^n$ and let $p$ be a canonical $d$-parameter word of length $n$. If $w, w' \in [k]^d$, we then set $w \prec w'$ whenever $p[w] \prec p[w']$.

Given a linear ordering $\prec$ on $[k]$, the lexicographic ordering on $[k]^n$ is defined by setting $w \prec_{\text{lex}} w'$ whenever $w_i < w'_i$, where $i$ is the least index such that $w_i \neq w'_i$. Note that if $p$ is a canonical $d$-parameter word, then $w \prec_{\text{lex}} w'$ holds for $w, w' \in [k]^d$ if and only if $p[w] \prec_{\text{lex}} p[w']$. Hence, under the canonical bijection, a restriction of a lexicographic ordering to a $d$-subcube is a lexicographic ordering on $[k]^d$.

**Theorem 3.** For every $d$ there exists $n$ with the following property: for every linear ordering $\prec$ of $[2]^n$ there is a $d$-subcube $C$ of $[2]^n$ such that the restriction of $\prec$ to $C$ is the lexicographic ordering for one of the two linear orderings of $[2]$.

**Theorem 1** is an easy consequence of Theorem 3. Indeed, let $m = 3^n$ and define the projection map $\pi : [2]^n \to [m]$ by $\pi(w) \triangleq \sum_{i=1}^n w_i 3^{n-i}$ for $w = a_1, \ldots, a_m$. The sequence $a_1, \ldots, a_m$ then induces a linear ordering on $[2]^n$, where $w \prec w'$ whenever $a_{\pi(w)} < a_{\pi(w')}$. A lexicographically ordered $d$-subcube of $[2]^n$ then corresponds to a monotone subsequence of $a_1, \ldots, a_m$ whose index set is a proper affine $d$-cube.

**Hales–Jewett-type result for general $[k]^n$.** The naive generalization of Theorem 3 to the case of $[k]^n$, with $k \geq 3$, is false. As an example, define a linear ordering $\prec$ on $[3]^n$ as follows: for a word $w$ let $w_{12}$ be the word obtained from $w$ by replacing each 1 by 2, and set $w \prec w'$ if either $w_{12} \leq_{\text{lex}} w'_{12}$
or \( w_{12} = w'_{12} \) and \( w <_{\text{lex}} w' \). This ordering is different from any of the 3! lexicographic orderings, and is stable under restriction to subcubes.

To describe the class of linear orderings that generalize the lexicographic ordering for \( k \geq 3 \), we need a couple of auxiliary definitions. A Schröder tree is a rooted plane tree each of whose internal nodes has at least 2 children. A weakly decreasing Schröder tree is a Schröder tree with a binary relation \( \preceq \) on the set of internal nodes that satisfies:

1. (ST1) \( \preceq \) is a total preorder, i.e., \( \preceq \) is transitive, reflexive, and \( a \preceq b \) or \( b \preceq a \) for every two nodes \( a, b \).
2. (ST2) Every path from the root is strictly decreasing.

For a weakly decreasing Schröder tree \( T \), we denote by \( \preceq_T \) the linear ordering on the internal nodes of \( T \). If \( T \) has \( k \) leaves, and we have an linear ordering on \([k]\), we identify leaves of \( T \) with the elements of \([k]\) by labeling leaves in the increasing order. For a pair of leaves \( \{a, b\} \in \binom{[k]}{2} \) we write \([a, b]_T\) for the bottommost node of \( T \) that contains both \( a \) and \( b \).

Given a linear ordering \(<\) of \([k]\) and a weakly decreasing Schröder tree \( T \) with \( k \) leaves, we can define a linear ordering \( \preceq_T \) on \([k]^n\) as follows. Given two words \( w, w' \in [k]^n \), we consider all indices \( i \in [n] \) such that \( w_i \neq w'_i \). Among these, we pick the smallest \( i \) such that \([w_j, w'_j]_T \preceq_T [w_i, w'_i]_T \) for all \( j \in [n] \). We then declare \( w \preceq_T w' \) if \( w_i < w'_i \).

**Theorem 4.** For every \( k \) and \( d \) there exists \( n \) with the following property: for every linear ordering \( \prec \) of \([k]^n\) there is a \( d \)-subcube \( C \) of \([k]^n\) such that the restriction of \( \prec \) to \( C \) is equal to \( \preceq_T \) for some weakly decreasing Schröder tree \( T \) with \( k \) leaves and some linear ordering of \([k]\).

For example, the ordering on \([3]^n\) above is obtained from the tree depicted on the right, under the usual ordering \( 1 < 2 < 3 \). Another ordering on \([3]^n\) can be obtained by taking a mirror image of the tree on the right. The usual lexicographic ordering is obtained by taking \( T = \emptyset \). In general, Bodini, Genitrini and Naima [1, Section 3.2] showed that the number of weakly decreasing Schröder trees with \( k \) leaves is equal to the \((k-1)\)st ordered Bell number, and hence is asymptotic to \( \frac{(k-1)!}{2(\ln 2)^k} \).

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1. A plane tree is a tree in which children of a node are ordered.
An extension of Ramsey’s theorem. Interestingly, after proving the main result in this paper (Theorem 4), we found that there is another way to prove Theorem 1, which relies on an extension of Ramsey’s theorem.

Theorem 5. For every $d$ and $r$, there exists $m$ such that for every $r$-edge-coloring of the complete graph on $[m]$ there is a monochromatic proper affine $d$-cube.

It is easy to deduce Theorem 1 from Theorem 5: Given a sequence $a_1, \ldots, a_m$ we color edge $\{i, j\}$, with $i < j$, with one of two colors according to whether $a_i < a_j$ or $a_j < a_i$. A monochromatic clique in this coloring then corresponds to a monotone subsequence.

Paper organization. The bulk of the paper is occupied by the proof of Theorem 4, which is split into two parts. We first show that, for any linear ordering $\prec$ of $[k]^n$, there is a large subcube $C$ such that the restriction of $\prec$ to $C$ enjoys a certain symmetry property, which we call uniformity. For $k = 2$, the uniform linear orderings are then easily seen to be lexicographic, which proves Theorem 3. That is done in Section 2. The case of general $k$ requires a more careful analysis of uniform linear orderings, which we carry out in Section 3. We conclude the paper with the proof of Theorem 5 in Section 4 and with some open problems in Section 5.

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2 Uniform linear orderings

Given a linear ordering $\prec$ on $[k]^n$, a restriction of $\prec$ to a $d$-subcube induces, under the canonical bijection, one of $(k^d)!$ linear orderings on $[k]^d$. We say that $\prec$ is $d$-uniform if all restrictions to $d$-subcubes induce the same linear ordering on $[k]^d$. An ordering that is $d$-uniform for all $d$ is called simply uniform.

Lemma 6. For every $k$ and $d$ there exists $n$ with the following property: for every linear ordering $\prec$ of $[k]^n$ there is a $d$-subcube $C$ of $[k]^n$ such that the restriction of $\prec$ to $C$ is uniform.

We shall deduce this from the following special case of the Graham–Rothschild theorem [4] (see also [7] for a transparent exposition and a short proof).

Lemma 7 (Graham–Rothschild theorem, the trivial group case). For every $d, D, k, t$ there exists $n = n(d, D, k, t)$ such that, for any $t$-coloring of $d$-subcubes of $[k]^n$, there is a $D$-subcube of $[k]^n$ all of whose $d$-subcubes are monochromatic.

Proof of Lemma 6. Let $\chi$ be the coloring of $d$-subcubes of $[k]^n$ in $(k^d)!$ many colors that assigns to each $d$-subcube $C$ the restriction of $\prec$ to $C$. By the Graham–Rothschild theorem, if $n$ is large enough, there is a $(2d - 1)$-subcube $C_0$ of $[k]^n$ on which $\chi$ is monochromatic. We use the canonical bijection to identify this subcube with $[k]^{2d-1}$. Let $C$ be the subcube of $C_0$ induced by the canonical word $1 \cdots 1*_{1^*} \cdots \ast_{d^*}$ (with $d - 1$ many 1’s). We claim that the restriction of $\prec$ to $C$ is uniform.

Indeed, let $C'$ be an arbitrary $d'$-subcube $C'$ of $C$. We can then complete $C'$ to a $d$-subcube $C_0'$ of $C_0$ in such a way that $C'$ is the subcube of $C_0'$ induced by the canonical word $1 \cdots 1*_{1^*} \cdots \ast_{d^*}$. In
this way every \(d'\)-subcube \(C'\) is identified with the same \(d'\)-subcube of \([k]^d\), and so is ordered in the same way. \(\square\)

If \(\prec\) is uniform, with slight abuse of notation, we think of \(\prec\) as a linear ordering on each \([k]^d\) for \(d = 1, 2, \ldots\). In particular, \(\prec\) induces an ordering on \([k]\), which we denote \(<\). The following implies Theorem 3.

**Proposition 8.** Let \(\prec\) be a uniform linear ordering on \([k]^n\), for \(n \geq 3\). Suppose \(a, b \in [k]\) satisfy \(a < b\). Then the restriction of \(\prec\) to \(\{a, b\}^n\) is the lexicographic ordering for the restriction of \(\prec\) to \(\{a, b\}\).

**Proof.** We claim that \(ab \prec ba\). Indeed, if \(ba \prec ab\), then we reach a contradiction by considering the sequence of inequalities

\[aab \prec bab \prec aba \prec aab,\]

where the first inequality follows because \(a \prec b\), whereas the last two follow because \(ba \prec ab\).

To show that the restriction of \(\prec\) to \(\{a, b\}^n\) coincides with the lexicographic ordering on \(\{a, b\}^n\), it suffices to show \(w \prec w'\) whenever \(w'\) is the successor of \(w\) in the lexicographic ordering. If \(w = a^n\), then \(w' = a^{n-1}b\), and \(w \prec w'\) because \(a \prec b\). Otherwise \(w = w_0ab\) and \(w' = w_0ba\) for some nonnegative integer \(t\) and a word \(w_0\). In that case \(w \prec w'\) because \(ab^t \prec ba^t\) follows from \(ab \prec ba\). \(\square\)

## 3 The proof of Theorem 4

By Lemma 6, it suffices to show that every uniform linear ordering is of the form \(\prec_T\) for some weakly decreasing Schröder tree \(T\) and some linear ordering \(\prec\) on \([k]\). By permuting the elements of \([k]\) if necessary, we may assume that the restriction of the uniform linear ordering to \([k]\) coincides with the usual ordering on \(\mathbb{N}\). Hence, in this section \(\prec\) denotes the usual ordering on \([k]\).

Call a subinterval of \([k]\) *nontrivial* if it is of length at least 2. Given a linear ordering \(\prec\) on \([k]^n\), we define a binary relation \(\preceq\) on nontrivial subintervals of \([k]\) as follows. For any \(a, b, c, d \in [k]\) with \(a < b\) and \(c < d\), we write \([a, b] \preceq [c, d]\) if \(cb \prec da\). If both \([a, b] \preceq [c, d]\) and \([c, d] \preceq [a, b]\) hold, then we write \([a, b] \approx [c, d]\).

Note that if \(\preceq = \preceq_T\), then \([a, b] \preceq [c, d]\) holds if and only if \([a, b]_T \preceq_T [c, d]_T\) holds. Furthermore, if \(\preceq = \preceq_T\), then \(\preceq\) satisfies the following three properties:

1. Transitivity: \([a, b] \preceq [c, d]\) and \([c, d] \preceq [e, f]\) together imply \([a, b] \preceq [e, f]\).
2. Comparability: \([a, b] \preceq [c, d]\) or \([c, d] \preceq [a, b]\).
3. Ultrametric property: if \(a < b < c\), then \([a, c] \approx \max([a, b], [b, c])\).

Call any relation \(\preceq\) on nontrivial subintervals of \([k]\) *tree-like* if it satisfies these three properties. Given a uniform linear ordering on \([k]^n\), we first show that \(\preceq\) is tree-like, and then use \(\preceq\) to build a weakly decreasing Schröder tree. That is done in the next two lemmas. Then in Lemma 11, we show that the ordering induced by the resulting tree (almost) coincides with the original ordering on \([k]^n\).

**Lemma 9.** Suppose \(n \geq 3\). If a linear ordering \(\prec\) on \([k]^n\) is uniform, then \(\preceq\) is tree-like.
Proof. Transitivity: By the assumption we have \( cb \prec da \) and \( ed \prec fc \). From this it follows that \( edb \prec fcb \prec fda \), which implies that \( eb \prec fa \) by uniformity.

Comparability: Suppose \( [a, b] \not= [c, d] \), and so \( da \prec cb \). Also, Proposition 8 tells us that \( \{c, d\} \) is ordered lexicographically, implying that \( cbd \prec dcb \). Hence, \( dad \prec cbd \prec dbc \), which is to say \( ad \prec bc \).

Ultrametric property: We first show that \( \max([a, b], [b, c]) \preceq [a, c] \). We have \( ab \prec ac \prec ca \), and so \( [a, b] \preceq [a, c] \). Similarly, \( ac \prec bc \prec cb \), and so \( [b, c] \preceq [a, c] \).

We next show that \( [a, c] \preceq \max([a, b], [b, c]) \). Suppose \( [a, c] \not= [a, b] \). Then \( ba \prec ac \), and so \( bca \preceq acc \preceq cca \), and hence \( [a, c] \preceq [b, c] \).

Lemma 10. For every tree-like ordering \( \preceq \) on \( \binom{[k]}{2} \) there exists a weakly decreasing Schröder tree \( T \) such that \( \preceq = \preceq_T \).

In a weakly decreasing Schröder tree \( T \) we may have \( [a, b]_T \approx [c, d]_T \) for two reasons: either because \( [a, b]_T = [c, d]_T \), or neither of \( [a, b]_T \) and \( [c, d]_T \) is a descendant of one another, and they happen to be equal in the \( \preceq_T \) preorder. Therefore, to build a tree out of a tree-like ordering, we need to distinguish these two situations. We achieve this by identifying the node with the widest interval that generates it.

Proof of Lemma 10. For a nontrivial subinterval \( [a, b] \) of \( [k] \), let \( a' \) be the least element of \( [k] \) such that \( [a', b] \approx [a, b] \). Likewise, let \( b' \) be the largest element of \( [k] \) such that \( [a, b'] \approx [a, b] \). Define \( [a, b]_{\text{def}} = [a', b'] \).

Let \( \mathcal{N} = \{ [a, b] : a \preceq b \} \). We shall take \( \mathcal{N} \) be the set of nodes of our tree. To show that we indeed obtain a tree, we must prove that every two intervals from \( \mathcal{N} \) are either disjoint or one of them contains the other. To do this we first show two basic properties of the map \( [a, b] \mapsto [a, b] \).

Claim 1: \( [a, b] \approx [a, b] \) for every \( [a, b] \in \binom{[k]}{2} \). Indeed, suppose \( [a, b] = [a', b'] \). If either \( a = a' \) or \( b = b' \), then the claim follows. Say \( a' < a < b < b' \). From the ultrametric property for the triple \( a' < a < b' \) we deduce that either \( [a', b'] \approx [a, b'] \) or \( [a', b'] \approx [a', b] \). In the former case \( [a, b] \approx [a, b'] \approx [a', b'] = [a, b] \), proving the claim. In the latter case, two applications of the ultrametric property yield \( [a', b'] \approx [a', a] \approx [a', b] \approx [a', b'] \), and so \( [a', b'] \approx [a', [a, b]] \) as well.

Claim 2: \( [a, b]_{\text{def}} = [a, b]_{\text{def}} \) for every \( [a, b] \in \binom{[k]}{2} \). Indeed, suppose \( [a, b] = [a', b'] \) and \( [a, b] = [a''', b'''] \). Then by two applications of Claim 1 it follows that \( [a'', b] \approx [a'''', b'''] \approx [a', b'] \approx [a, b] \), which, by the minimality of \( a' \), implies that \( a'' = a' \). Similarly, \( b'' = b' \).

We are now ready to prove that every pair of intervals in \( \mathcal{N} \) is either disjoint or comparable. Let \( [a, b] \) and \( [c, d] \) be any two intervals from \( \mathcal{N} \), and suppose that they are not disjoint. Say \( c < b \) (the case \( a \leq d \) is analogous, and can be reduced to this case by swapping the roles of the two intervals). If we also have \( c \leq a \), then the interval \( [c, d] \) contains \( [a, b] \). So, assume that \( a < c \). By Claim 2 we may assume that \( a \) is the minimal \( a' \) such that \( [a', b] \approx [a, b] \), and similarly for \( b, c, d \). From the minimality of \( c \) we infer that \( [a, d] \not= [c, d] \). By the ultrametric property applied to the triple \( a < c < d \), it follows that \( [a, d] \approx [a, c] \). By another application of the ultrametric property,
time to $a < c < b$, we infer that $[a, c] \preceq [a, b]$, and so $[a, d] \preceq [a, b]$. The ultrametric property of $a < b < d$ then implies that $[a, d] \simeq [a, b]$, and so $b \geq d$ by the maximality of $b$. Hence, $[c, d]$ is contained in $[a, b]$.

It follows that intervals in $\mathcal{N}$ naturally form a tree under the containment relation. The tree is plane, with intervals ordered in the natural way. We add leaves to the tree by declaring that leaf $a$ is a descendant of all intervals that contain $a$. For tree nodes $[a, b], [c, d] \in \mathcal{N}$, we order them $[a, b] \preceq_T [c, d]$ if and only if $[a, b] \preceq [c, d]$. The ultrametric property then ensures that every path from the root is decreasing. Denote the resulting weakly decreasing Schröder tree by $T$.

Recall that $[a, b]_T$ is the bottommost node of $T$ containing $a$ and $b$. Since $a, b \in [a, b]$, it follows that $[a, b]_T \subseteq [a, b]$. On the other hand, $[a, b] \subset [a, b]_T$, which implies that $[a, b] \subseteq [a, b]_T = [a, b]_T$. So, $[a, b]_T = [a, b]$ for every $\{a, b\} \in \binom{[k]}{2}$ from which $\preceq = \preceq_T$ follows. \hfill $\square$

The last ingredient in the proof of Theorem 4 is the next result.

**Lemma 11.** If $T$ is a weakly decreasing Schröder tree with $k$ leaves, and $\triangleleft$ is a uniform linear ordering on $[k]^n$ such that $\preceq = \preceq_T$, then $\triangleleft$ is equal to $\triangleleft_T$ on every $(n-1)$-subcube of $[k]^n$.

Note that, for the reason that will become clear from the proof, we do not assert that $\triangleleft = \triangleleft_T$. Theorem 4 nonetheless follows as we may restrict to a subcube of one dimension smaller.

**Proof.** It suffices to show that $w \triangleleft w'$ whenever $w \triangleleft_T w'$ and $w, w'$ differ in $t \leq n-1$ positions. The proof is by induction on $t$. The case $t = 1$ holds because our assumption that the ordering on $[k]$ is the same for $\triangleleft$ and $\triangleleft_T$. The case $t = 2$ holds because $\preceq = \preceq_T$. So, assume that $t \geq 3$.

For ease of notation we identify the $t$-subcube of $[k]^n$ containing both $w$ and $w'$ with $[k]^t$. This way, $w$ and $w'$ differ in every position. Let $i$ be the smallest natural number such that $[w_j, w'_j]_T \preceq_T [w_i, w'_i]$ for all $j \in [n]$. Note that $w_i < w'_i$ because $w \triangleleft_T w'$.

The symbol $w_i$ breaks $w$ into three parts, the prefix, the symbol $w_i$ itself, and the suffix. The prefix and the suffix cannot be both empty. Suppose first that the prefix is non-empty; we then write $w$ and $w'$ as

$$w = \check{w} w_{i-1} w_i \check{w},$$

$$w' = \check{w}' w'_{i-1} w'_i \check{w}'$$

for some words $\check{w}, \check{w}' \in [k]^{i-2}$ and $\check{w}, \check{w}' \in [k]^{n-i}$. If $w_{i-1} \triangleleft w'_{i-1}$, then $w \triangleleft \check{w} w_{i-1} \check{w}$ and since $\check{w} w_{i-1} \check{w} \triangleleft w'$ by the induction hypothesis, the inequality $w \triangleleft w'$ follows. So, we may assume that $w'_{i-1} \triangleleft w_{i-1}$.

Because $t \leq n-1$ and $\triangleleft$ is uniform, the inequality $w \triangleleft w'$ will follow once we show that $\check{w} \triangleleft \check{w}'$, where

$$\check{w} \overset{\text{def}}{=} \check{w} w_{i-1} w_i \check{w},$$

$$\check{w}' \overset{\text{def}}{=} \check{w}' w'_{i-1} w'_i \check{w}'$$

The definition of $i$ implies that $[w_i, w'_i]_T \npreceq_T [w'_{i-1}, w_{i-1}]$. Hence, $[w_i, w'_i] \npreceq [w'_{i-1}, w_{i-1}]$, which is to say $w_{i-1} \triangleleft w'_i$. Therefore, $\check{w} \triangleleft \check{w}'$. On the other hand, $\check{w} w_{i-1} w_i \check{w} \triangleleft \check{w}'$.
follows from the uniformity of \( \prec \) and the induction hypothesis applied to the words \( \bar{w}w_i\bar{w} \) and \( \bar{w}'w_i'\bar{w}' \). Together these imply \( \bar{w} \prec \bar{w}' \).

If the prefix of \( w \) (before \( w_i \)) is empty, we write

\[
  w = w_iw_{i+1}\bar{w}, \\
  w' = w'_iw'_{i+1}\bar{w}',
\]

and define

\[
  \bar{w} = w_iw_{i+1}\bar{w}, \\
  \bar{w}' = w'_iw'_{i+1}\bar{w}'.
\]

Because \( [w'_{i+1}, w_{i+1}] \approx_T [w_i, w'_i] \), we have \( [w'_{i+1}, w_{i+1}] \approx [w_i, w'_i] \), and so \( w_iw_{i+1} \prec w'_iw'_{i+1} \).

Since the induction hypothesis tells us that \( w_i\bar{w} \prec w'_i\bar{w}' \), we have \( \bar{w} \prec \bar{w}' \), and so \( w \prec w' \) in this case as well.

4 Extension of Ramsey's theorem

In this section, we prove Theorem 5.

Let \( n = n(2d, 2, 2) \) be as in the Graham–Rothschild theorem, and set \( m = 3^n \). Let \( \chi : \binom{[m]}{2} \to [r] \) be an \( r \)-coloring of the edges of \( K_m \). Define the projection map \( \pi : [2]^n \to [m] \) by \( \pi(w) \overset{\text{def}}{=} \sum_{i=1}^n w_i3^{n-i} \).

The coloring \( \chi \) of \( \binom{[m]}{2} \) then induces a coloring \( \chi' \) of \( \binom{[2]^n}{2} \) via \( \chi'(w, w') \overset{\text{def}}{=} \chi(\pi(w), \pi(w')) \).

We can then define a 2-coloring of 2-subcubes of \( [2]^n \) as follows. Let \( C \) be any 2-subcube. We identify it with \( [2]^2d \) with the aid of the canonical bijection. Then \( \chi''(C) \) is equal to the \( \chi' \)-color of the edge \( \{01, 10\} \).

By the Graham–Rothschild theorem, there is a 2\( d \)-subcube \( C \) on which \( \chi'' \) is monochromatic. Call pair of words \( w, w' \) incomparable if there exist both \( i \in [n] \) such that \( (w_i, w'_i) = (0, 1) \) and \( j \in [n] \) such that \( (w_j, w'_j) = (1, 0) \). Since \( \chi'' \) is monochromatic, every two incomparable words in \( C \) are of the same color.

Identify \( C \) with \( [2]^{2d} \), and consider the set

\[
  S \overset{\text{def}}{=} \{ w \in [2]^{2d} : w_{2i-1} \neq w_{2i} \text{ for all } i = 1, 2, \ldots, d \}.
\]

Though \( S \) is not a \( d \)-subcube, its image under the map \( \pi \) is an affine \( d \)-cube. Since every two words in \( S \) are incomparable, it follows that \( \pi(S) \) is monochromatic.

5 Open problems

- Conlon and Kamčev [2] showed that for every \( r \)-coloring of \( [3]^n \) there are monochromatic lines whose wildcard set is a union of at most \( r \) intervals (see also [6, 5] for a strengthening for even \( r \)). We do not know if one can find a combinatorial line whose wildcard set is an arithmetic progression.

- In this paper we made no effort to obtain good quantitative bounds. The right dependence of \( m \) on \( d \) in Theorem 1 is probably doubly exponential.
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