Optimal Estimates for the Gradient of Harmonic Functions in the Unit Disk

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Received: 27 February 2011 / Accepted: 12 August 2011 / Published online: 28 August 2011 © Springer Basel AG 2011

Abstract Let U be the unit disk, $p \geq 1$ and let $h^p(U)$ be the Hardy space of complex harmonic functions. We find the sharp constants $C_p$ and the sharp functions $C_p = C_p(z)$ in the inequality

$$|Dw(z)| \leq C_p(1 - |z|^2)^{-1 - 1/p} \|w\|_{h^p(U)}, \ w \in h^p(U), \ z \in U,$$

in terms of Gaussian hypergeometric and Euler functions. This generalizes some results of Colonna related to the Bloch constant of harmonic mappings of the unit disk into itself and improves some classical inequalities by Macintyre and Rogosinski.

Keywords Harmonic functions · Bloch functions · Hardy spaces

Mathematics Subject Classification (2000) Primary 31A05; Secondary 42B30

1 Introduction and Statement of the Results

A harmonic function $w$ defined in the unit ball $B^n$ belongs to the harmonic Hardy class $h^p = h^p(B^n)$, $1 \leq p < \infty$ if the following growth condition is satisfied

Communicated by Mihai Putinar.

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\[
\|w\|_{h^p} := \left( \sup_{0 < r < 1} \int_S |w(r\xi)|^p d\sigma(\xi) \right)^{1/p} < \infty
\]

(1.1)

where \( S = S^{n-1} \) is the unit sphere in \( \mathbb{R}^n \) and \( \sigma \) is the unique normalized rotation invariant Borel measure on \( S \). The space \( h^\infty(\mathbb{B}^n) \) consists of all bounded harmonic functions.

It turns out that if \( w \in h^p(\mathbb{B}^n) \), then there exists the finite radial limit

\[
\lim_{r \to 1^-} w(r\xi) = f(\xi) \text{ (a.e. on } S) \nonumber
\]

and the boundary function \( f(\xi) \) belong to the space \( L^p(S) \) of \( p \)-integrable functions on the sphere.

It is well known that a harmonic function \( u \) from Hardy class can be represented as the Poisson integral

\[
u(x) = \int_S P(x, \xi) d\mu(\xi), \quad x \in \mathbb{B}^n
\]

where

\[
P(x, \xi) = \frac{1 - |x|^2}{|x - \xi|^n}, \quad x \in \mathbb{B}^n, \quad \xi \in S
\]

is the Poisson kernel and \( \mu \) is a complex Borel measure. In the case \( p > 1 \) this measure is absolutely continuous with respect to \( \sigma \) and \( d\mu(\xi) = f(\xi)d\sigma \). Moreover

\[
\|w\|_{h^p} = \|\mu\|
\]

and for \( 1 < p \leq \infty \) we have

\[
\|w\|_{h^p} = \|\mu\| = \|f\|_p,
\]

(1.2)

where we denote by \( \|\mu\| \) the total variation of the measure \( \mu \).

For previous facts we refer to the book [1, Chapter 6]. We use the classical notation \( \mathbb{U} \) and \( \mathbb{T} \) to denote the unit disk and its boundary in the complex plane \( \mathbb{C} \).

Let \( L^p(\mathbb{R}^n) \) be the space of Lebesgue integrable functions defined in \( \mathbb{R}^n \) with the norm

\[
\|f\|_p = \left( \int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p}.
\]
Let $\omega_n$ be the area of the unit sphere in $\mathbb{R}^n$. Let in addition $h^p(\mathbb{R}^n_+)$ be the Hardy space of real harmonic functions in $\mathbb{R}^n_+$, which can be represented as the Poisson integral

$$u(x) = \frac{2}{\omega_n} \int_{\mathbb{R}^n_+} \frac{x_n}{|y - x|^n} u(y') dy',$$

with boundary values in $L^p(\mathbb{R}^{n-1})$, where $y = (y', 0)$, $y' \in \mathbb{R}^{n-1}$.

In the recent paper [7] Maz'ya and Kresin studied pointwise estimates of the gradient of real harmonic function $u$ under the assumptions that the boundary values belong to $L^p$. They obtained the following result

$$|\nabla u(x)| \leq C_p x_n^{(1-n-p)/p} \|u\|_p$$

where $C_p$ is a constant depending only on $p$ and $n$. For $p = 1$, $p = 2$ and $p = \infty$, the constant $C_p$ is concretized and it is shown the sharpness of the result. After that, in [8], they obtained similar results for the unit ball, but only for $p = 1$ and $p = 2$. Precisely, they obtained an integral representation for the sharp constant $K_p(x, l)$ in the inequality

$$|\langle \nabla u(x), l \rangle| \leq K_p(x, l)\|u\|_p, \quad 1 \leq p \leq \infty$$

and the sharp constant $K_p(x)$ in the inequality

$$|\nabla u(x)| \leq K_p(x)\|u\|_p$$

was concretized for $p = 1, 2$ and $x$ arbitrary and for $x = 0$ and all $p$.

Notice that, for $n = 2$ the results concerning the upper half-plane $H$ cannot be directly translated to the unit disk and vice-versa. Although the unit disk $\mathbb{D}$ and the upper half-plane $H$ can be mapped to one-another by means of Möbius transformations, they are not interchangeable as domains for Hardy spaces. A contribution to this difference is the fact that the unit circle has finite (one-dimensional) Lebesgue measure while the real line does not.

A complex harmonic function $w$ in a region $D$ can be expressed as $w = u + iv$ where $u$ and $v$ are real harmonic functions in $D$. For a complex harmonic function we will use sometimes the abbreviation a harmonic mapping. If $D$ is simply-connected, then there are two analytic functions $h$ and $k$ defined on $D$ such that $w = g + \overline{h}$. For a complex harmonic function $w = g + \overline{h} = u + iv$, denote by $Dw(z)$ the formal differential matrix $Dw(z) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$. Its norm is given by

$$|Dw| := \max\{|Dw(z)| : |l| = 1\}.$$

Then

$$|Dw(z)| = |g'(z)| + |h'(z)|.$$

(1.3)
Let $w$ be a harmonic function satisfying the Lipschitz condition, when regarded as a function from the hyperbolic unit disk into the complex plane $\mathbb{C}$ endowed with the Euclidean distance. The function $w$ is called Bloch with the Bloch constant

$$
\beta_w = \sup_{z \neq z'} \frac{|w(z) - w(z')|}{d_h(z, z')}
$$

Here $d_h$ is defined by

$$
\tanh \frac{d_h(z, z')}{2} = \frac{|z - z'|}{|1 - \overline{z}z'|}.
$$

It can be proved that

$$
\beta_w = \sup_{z \in U} (1 - |z|^2)|Dw(z)|.
$$

We refer to [2, Theorem 1] for the proof of (1.4). In the same paper Colonna proved that, if $w$ is a harmonic mapping of the unit disk into itself, then there holds the following sharp inequality

$$
\beta_w \leq 4 \pi.
$$

See also the book of Pavlović [12, pp. 53, 54] for a related problem.

An estimate similar to (1.5) for magnitudes of derivatives of bounded harmonic functions in the unit ball in $\mathbb{R}^3$ is obtained by Khavinson in [4].

Together with the Bloch constants, for a harmonic mapping of the unit disk onto itself consider the hyperbolic Lipschitz constant defined by

$$
\beta_{w}^{\text{hyp}} := \sup_{z \neq z'} \frac{d_h(w(z), w(z'))}{d_h(z, z')}
$$

Since $|dz| \leq |d\zeta|/(1 - |\zeta|^2)$, it follows that for $z, w \in U$ we have $d(z, w) \leq d_h(z, w)$. Thus

$$
\beta_w \leq \beta_{w}^{\text{hyp}}.
$$

If $w$ is an analytic function, then by Schwarz–Pick lemma

$$
\beta_{w}^{\text{hyp}} \leq 1,
$$

and the equality is attained for Möbius self mappings of the unit disk. More recently, it is proved in [5] that for real harmonic mappings of the unit disk onto itself there holds the following sharp inequality.
|∇w| \leq \frac{4}{\pi} \frac{1 - |w(z)|^2}{1 - |z|^2}, \quad (1.6)

and therefore $\beta_{\text{hyp}}^{w} \leq \frac{4}{\pi}$ extending thus Colonna result for real harmonic mappings. However, if we drop the assumption that $w$ is real, then $\beta_{\text{hyp}}^{w}$ can be infinite. The inequality (1.6) can be considered as a real-part theorem for an analytic function. More than one approach can be found in the book [9].

In this paper we prove the following results for the unit disk which are analogous to the results of Maz’ya and Kresin and extend the results of Colonna.

Since the case $p = 1$ is well-known, we will assume in the sequel that $p > 1$.

**Theorem 1.1** (Main theorem) Let $p > 1$ and let $q$ be its conjugate. Let $w \in h^p$ be a complex harmonic function defined in the unit disk and let $z \neq 0$. Define $n = \frac{z}{|z|}$, and $t = i \frac{z}{|z|}$.

(a) We have the following sharp inequalities

\[
|Dw(z)e^{i\tau}| \leq C_p(z, e^{i\tau})(1 - r^2)^{-1/p - 1/2} \|w\|_{hp},
\]

\[
|Dw(z)| \leq C_p(z)(1 - r^2)^{-1/p - 1/2} \|w\|_{hp},
\]

where $z = re^{i\alpha}$,

\[
C_p(z, e^{i\tau}) = \frac{1}{\pi} \left( \int_{-\pi}^{\pi} \frac{|\cos(s + \tau - \alpha)^q|}{(1 + r^2 - 2r \cos s)^{1-q}} ds \right)^{1/q}
\]

and

\[
C_p(z) = \begin{cases} 
C_p(z, n), & \text{if } p < 2; \\
C_p(z, t), & \text{if } p \geq 2.
\end{cases}
\]

Moreover

\[
\begin{cases} 
C_p(z, t) \leq C_p(z, e^{i\tau}) \leq C_p(z, n), & \text{if } p < 2; \\
C_p(z, n) \leq C_p(z, e^{i\tau}) \leq C_p(z, t), & \text{if } p \geq 2.
\end{cases}
\]

(b) For $p \geq 2$ the function $C_p(z)$ can be expressed as

\[
C_p(z) = \frac{2^{1/q}}{\pi} \left( B \left( \frac{1 + q}{2}, \frac{1}{2} \right) F \left( 1 - \frac{3q}{2}, 1 - q; \frac{q}{2}; r^2 \right) \right)^{1/q},
\]

where $B$ is the beta function and $F$ is the Gauss hypergeometric function.
Finally

\[ C_p(z) = \sup_{z \in U} C_p(z) = \begin{cases} \frac{1}{\pi} \left( \int_{-\pi}^{\pi} \frac{|\cos s|^q}{\tau_{2-2 \cos s}^{1-q}} ds \right)^{1/q}, & \text{if } 1 < p < 2; \\ \frac{1}{\pi} \left( \int_{-\pi}^{\pi} \frac{|\sin s|^q}{\tau_{2-2 \cos s}^{1-q}} ds \right)^{1/q}, & \text{if } p \geq 2. \end{cases} \]

The constant \( C_p \) is optimal for real harmonic functions as well.

**Theorem 1.2** Let \( p > 1 \) and let \( w \in h^p \), be a complex harmonic function defined in the unit disk. Then we have the following sharp inequalities

\[ |\partial w(z)|, |\bar{\partial} w(z)| \leq c_p(z)(1 - |z|^2)^{-1/p-1} \|w\|_{h^p}, \]

and

\[ |\partial w(z)|, |\bar{\partial} w(z)| \leq c_p(1 - |z|^2)^{-1/p-1} \|w\|_{h^p}, \]

where

\[ c_p(z) = (2\pi)^{1/q-1} \left( F(1 - q, 1 - q; 1; r^2) \right)^{1/q} \tag{1.12} \]

and

\[ c_p = 2^{-1+q} \pi^{-1+\frac{1}{2q}} \left( \frac{\Gamma(-1/2 + q)}{\Gamma(q)} \right)^{1/q}. \tag{1.13} \]

**Remark 1.3** (a) In particular, if in Theorem 1.1 we take \( p = 2 \), then we have the following estimate

\[ |\nabla w(z)| \leq \frac{1}{\sqrt{\pi}} \frac{(1 + |z|^2)^{1/2}}{(1 - |z|^2)^{3/2}} \|w\|_{h^2}. \tag{1.14} \]

If we assume that \( w \) is a real harmonic function, i.e. \( w = g + \bar{g} \), where \( g \) is an analytic function, then this estimate is equivalent to the real part theorem

\[ |g'(z)| \leq \frac{1}{\sqrt{\pi}} \frac{(1 + |z|^2)^{1/2}}{(1 - |z|^2)^{3/2}} \|\Re g\|_{h^2}. \tag{1.15} \]

For the proof of (1.15) we refer to [9, pp. 87, 88]. See also a higher dimensional generalization of (1.14) by Maz’ya and Kresin in the recent paper [8, Corollary 3] for \( n \geq 2 \). Also the relation (1.14) for real \( w \) can be deduced from the work of Macintyre and Rogosinski for analytic functions, see [10, p. 301].

(b) On the other hand, if we take \( p = \infty \), then \( C_p = \frac{4}{\pi} \) and therefore, the relation (1.8) coincides with the result of Colonna, while for real \( w \), it is a real part theorem [4] which can be expressed as
(c) Notice that $c_p < C_p < 2c_p$ if $p > 1$, and $C_1 = 2c_1 = \frac{4}{\pi}$. On the other hand, $c_{\infty} = 1$ coincides with the constant of the Schwarz lemma for analytic functions. Notice also the interesting fact that the minimum of constants $C_p$ is achieved for $p = 2$ and it is equal to $C_2 = \frac{\sqrt{2}}{\pi}$. The graphs of functions $C_p$ and $c_p$ with $1 \leq p \leq 20$, are shown in Figs. 1 and 2.

(d) From Theorem 1.1 we find out that the Khavinson–Kresin–Maz’ya hypothesis (see [8]) is not true for $n = 2$ and $2 < p < \infty$. Namely, the maximum of the absolute value of the directional derivative of a harmonic function with a fixed $L^p$-norm of its boundary values is attained at the radial direction for $p \leq 2$ and at the tangential direction for $2 < p < \infty$.

In the classical paper [10, (8.3.8)] of Macintyre and Rogosinski they obtained the inequality

\begin{equation}
|g'(z)| \leq \frac{4}{\pi} \frac{1}{1 - |z|^2} \|\Re g\|_{\infty}.
\end{equation}
\[ |f'(z)| \leq \left( 1 + \frac{|z|^2}{(p-1)^2} \right)^{1/q} (1 - |z|^2)^{-1-1/p} \|f\|_{H^p}. \tag{1.17} \]

As an immediate consequence of Theorem 1.2, we improve the inequality (1.17) as follows

**Corollary 1.4** Let \( w = f(z) \) be an analytic function from the Hardy class \( H^p(U) \). Then there holds the following inequality

\[ |f'(z)| \leq c_p(z)(1 - |z|^2)^{-1-1/p} \|f\|_{H^p}, \tag{1.18} \]

where \( c_p(z) \) is defined in (1.12).

**Remark 1.5** Corollary 1.4 is an improvement of the corresponding inequality \([10, (8.3.8)]\) because it holds

\[ (2\pi)^{1-q} F(1-q, 1-q; 1; r^2) < 1 + \frac{r^2}{(p-1)^2} \]

for all \( q > 1 \). The function \( c_2(z) = \frac{\sqrt{1+|z|^2}}{\sqrt{2\pi}} \) in (1.18) is the best possible, because for \( p = 2 \) the relation (1.18) coincides with the sharp inequality \([10, \text{p. 301, eq. (7.2.1)}]\). We expect that (1.18) is sharp for every \( p \). On the other hand, the power \(-1 - 1/p\) is optimal, see e.g. Garnett \([3, \text{p. 86}]\). The paper \([10]\) contains some sharp estimates of the form \( |f^{(k)}(z)| \leq c_p \|f\|_p \) for \( f \in H^p(U) \) and \( k \geq 1 \) but \( p \) depends on \( k \) and it seems that if \( k = 1 \) then \( p \) can be only equals to 1 or 2.

**2 Proofs**

We need the following lemmas

**Lemma 2.1** Let \( a_q(t), \ t \in [0, 2\pi], \ q \geq 1, \ 0 \leq r \leq 1, \) be a function defined by

\[ a_q(t) = \int_{-\pi}^{\pi} |\cos(s-t)|^q |r - e^{is}|^{2q-2} ds. \]

Then

\[ \max_{0 \leq t \leq 2\pi} a_q(t) = \begin{cases} a_q(\frac{\pi}{2}), & \text{if } q \leq 2; \\ a_q(0), & \text{if } q > 2 \end{cases} \]

and

\[ \min_{0 \leq t \leq 2\pi} a_q(t) = \begin{cases} a_q(0), & \text{if } q \leq 2; \\ a_q(\frac{\pi}{2}), & \text{if } q > 2. \end{cases} \]
Proof Since the case $q = 1$ is trivial, assume that $q > 1$ and

$$a(t) := a_q(t) = \int_{-\pi}^\pi |\cos(t - s)|^q (1 + r^2 - 2r \cos s)^{q-1} ds.$$ 

Note that $a$ is $\pi$-periodic. Because sub-integral expression is $2\pi$-periodic with respect to $s$, we obtain

$$a(t) = \int_0^{2\pi} |\cos s|^q (1 + r^2 - 2r \cos(t + s))^{q-1} ds,$$

and therefore

$$a'(t) = 2(q - 1)r \int_0^{2\pi} |\cos s|^q \sin(t + s)(1 + r^2 - 2r \cos(t + s))^{q-2} ds.$$ 

Again by using the periodicity of sub-integral expression, we find that

$$a'(t) = 2(q - 1)r \int_0^{2\pi} |\cos(t - s)|^q \sin s(1 + r^2 - 2r \cos s)^{q-2} ds.$$ 

Next we need the following transformations

$$a'(t) = 2(q - 1)r \int_0^\pi |\cos(t - s)|^q \sin s (1 + r^2 - 2r \cos s)^{q-2} ds$$

$$+ 2(q - 1)r \int_{\pi}^{2\pi} |\cos(t - s - \pi)|^q \sin(s + \pi)(1 + r^2 - 2r \cos(s + \pi))^{q-2} ds$$

$$= 2(q - 1)r \int_0^{2\pi} |\cos(t - s)|^q \sin s Q(r, s - \pi/2) ds$$

$$= 2(q - 1)r \int_{-\pi/2}^{\pi/2} |\sin(t - s)|^q \cos s Q(r, s) ds,$$

where

$$Q(r, s) = (1 + r^2 + 2r \sin s)^{q-2} - (1 + r^2 - 2r \sin s)^{q-2}.$$
Thus, the derivative is
\[
a'(t) = 2r(q - 1) \int_{-\pi/2}^{\pi/2} h(t, s) \cos s \, ds,
\]
where
\[
h(t, s) = |\sin(t - s)|^q Q(r, s).
\]
Also \(a'(t)\) is \(\pi\)-periodic and
\[
a'(0) = a'(\pi/2) = 0.
\]
Further, we have
\[
h(t, s) + h(t, -s) = (|\sin(t - s)|^q - |\sin(t + s)|^q) Q(r, s).
\]
If \(1 < q < 2\), then for \(0 < t < \pi/2\) we have
\[
h(t, s) + h(t, -s) > 0, \quad 0 < s < \pi/2
\]
and for \(\pi/2 < t < \pi\)
\[
h(t, s) + h(t, -s) > 0, \quad 0 < s < \pi/2.
\]
We claim that
\[
a'(t) = 2r(q - 1) \int_0^{\pi/2} (h(t, s) + h(t, -s)) \cos s \, ds > 0, \quad 0 < t < \pi/2
\]
and
\[
a'(t) = 2r(q - 1) \int_0^{\pi/2} (h(t, s) + h(t, -s)) \cos s \, ds < 0, \quad \pi/2 < t < \pi.
\]
This means that the minimum of \(a\) is achieved in 0 and the maximum in \(\pi/2\).

Similarly, it can be treated the case \(q > 2\). For \(q = 2\) the function \(a(t)\) is a constant.

The proof of Lemma 2.1 is completed. \(\Box\)

**Lemma 2.2** Let \(\lambda \geq 0\), \(0 \leq r \leq 1\) and \(q \geq 1\). For all \(t\) there exists \(t' \in [0, 2\pi]\) such that
\[
\int_0^{2\pi} |\cos(s - t)|^\lambda |r - e^{is}|^{2q-2} \, ds \leq \int_0^{2\pi} |\cos(s - t')|^\lambda |1 - e^{is}|^{2q-2} \, ds.
\]

**Proof** In order to prove Lemma 2.2, we need the following result.
Proposition 2.3 [6, Lemma 3.2] Let $U \subset \mathbb{C}$ be the open unit disk and let $(A, \mu)$ be a measured space with $\mu(A) < \infty$. Let $f(z, \omega)$ be a holomorphic function for $z \in \mathbb{U}$ and measurable for $\omega \in A$. Let $b > 0$ and assume in addition that there exists an integrable function $\chi \in L^{\max\{b, 2\}}(A, d\mu)$ such that

$$|f(0, \omega)| + |f'(z, \omega)| \leq \chi(\omega), \quad (2.1)$$

for $(z, \omega) \in \mathbb{U} \times A$, where by $f'(z, \omega)$ we mean the complex derivative of $f$ with respect to $z$. Then the function

$$\phi(z) = \log \int_A |f(z, \omega)|^b d\mu(\omega)$$

is subharmonic in $\mathbb{U}$.

Corollary 2.4 Assume together with the assumptions of the previous proposition that $z \mapsto f(z, \omega)$ is a continuous map up to the boundary $T$. Then we have the following inequality

$$\phi(z) \leq \max_{\tau \in [0, 2\pi)} \phi(e^{i\tau}) = \phi(e^{i\tau'}).$$

In order to apply Corollary 2.4, we take

$$d\mu(s) = |\cos(s - t)|^\lambda ds, \quad f(z, s) = z - e^{is} \text{ and } b = 2q - 2$$

and observe that

$$\max_{\tau} \int_0^{2\pi} |\cos(s - t)|^\lambda |e^{i\tau} - e^{is}|^{2q-2} ds = \int_0^{2\pi} |\cos(s - t)|^\lambda |e^{i\tau'} - e^{is}|^{2q-2} ds = \int_0^{2\pi} |\cos(s - t')|^\lambda |1 - e^{is}|^{2q-2} ds.$$ 

This finishes the proof of Lemma 2.2. \qed

The Poisson kernel for the disk can be expressed as

$$P(z, e^\theta) = \frac{1 - |z|^2}{|z - e^{i\theta}|^2} = -\left(1 + \frac{e^{-i\theta}}{\overline{z} - e^{-i\theta}} + \frac{e^{i\theta}}{z - e^{i\theta}}\right).$$
Then we have
\[ \text{grad}(P) = (P_x, P_y) = P_x + i P_y = 2 \partial P = \frac{2e^{-i\theta}}{(z - e^{-i\theta})^2}, \]
\[ \partial P = \frac{e^{i\theta}}{(z - e^{i\theta})^2} \]
and
\[ \bar{\partial} P = \frac{e^{-i\theta}}{(\bar{z} - e^{-i\theta})^2}. \]

**Proof of Theorem 1.1** (a) Let \( l = e^{i\tau}. \) Then for \( p > 1 \)

\[ Dw(z)l = \frac{1}{2\pi} \int_{0}^{2\pi} \langle \text{grad}(P), l \rangle f(e^{i\theta}) d\theta = \frac{1}{\pi} \int_{0}^{2\pi} \Re \frac{e^{-i(\theta+\tau)}}{(\bar{z} - e^{-i\theta})^2} f(e^{i\theta}) d\theta. \]  \( (2.2) \)

By applying (2.2) and Hölder inequality we obtain
\[ |Dw(z)l| \leq \frac{1}{\pi} \left( \int_{0}^{2\pi} \Re \frac{e^{-i(\theta+\tau)}}{(\bar{z} - e^{-i\theta})^2} d\theta \right)^{1/q} \left( \int_{0}^{2\pi} |f(e^{i\theta})|^p d\theta \right)^{1/p}. \]

We should consider the integral
\[ I_q = \int_{0}^{2\pi} \Re \frac{e^{-i(\theta+\tau)}}{(\bar{z} - e^{-i\theta})^2} d\theta. \]

First of all
\[ I_q = \int_{0}^{2\pi} \Re \frac{e^{i(\theta+\tau)}}{(z - e^{i\theta})^2} d\theta = \int_{0}^{2\pi} \Re \frac{e^{i(\theta+\tau-\alpha)}}{(r - e^{i\theta})^2} d\theta. \]

Take the substitution
\[ e^{i\theta} = \frac{r - e^{is}}{1 - re^{is}}. \]
Then
\[ de^{i\theta} = \frac{1 - r^2}{(1 - re^{i\theta})^2} de^{is}, \]
and thus
\[ d\theta = \frac{1 - r^2}{(1 - re^{i\theta})^2} e^{is} \frac{1 - re^{i\theta}}{r - e^{i\theta}} ds = \frac{1 - r^2}{1 + r^2 - 2r \cos s} ds. \]

On the other hand, we easily find that
\[ \Re \frac{e^{i(\theta + \tau - \alpha)}}{(r - e^{i\theta})^2} = \frac{(1 + r^2 - 2r \cos s) \cos(s + \tau - \alpha)}{(1 - r^2)^2}. \]

Therefore, finally we have the relation
\[ \int_0^{2\pi} \left| \Re \frac{e^{-i(\theta + \tau)}}{(z - e^{-i\theta})^2} \right|^q d\theta = (1 - |z|^2)^{1-2q} \int_{-\pi}^{\pi} \frac{|\cos(s + \tau - \alpha)|^q}{(1 + r^2 - 2r \cos s)^{1-q}} ds, \quad (2.3) \]
which together with the first relation gives
\[ |Dw(z)| \leq C_p(z, l)(1 - |z|^2)^{-1-1/p} \|w\|_{h^p}. \]

Now by using Lemma 2.1, we conclude that
\[ C_p(z) = \begin{cases} C_p(z, n), & \text{if } p < 2; \\ C_p(z, t), & \text{if } p \geq 2, \end{cases} \]
which coincides with (1.9). This implies (1.8). Lemma 2.1 implies at once (1.10).

(b) By using the following formula
\[ \int_0^{\pi} \frac{\sin^{\mu-1} t}{(1 + r^2 - 2r \cos t)^v} dt = B \left( \frac{\mu}{2}, \frac{1}{2} \right) F \left( \frac{\nu}{2}, \frac{1 - \mu}{2}; \frac{1 + \mu}{2}, r^2 \right) \quad (2.4) \]
(see, e.g., Prudnikov et al. [11, 2.5.16(43)]), where \( B(u, v) \) is the Beta-function, and \( F(a, b; c; x) \) is the hypergeometric Gauss function; taking \( \mu = q + 1 \) and \( v = 1 - q \), because \( |\cos(s + \tau - \alpha)|^q = |\sin s|^q \), for \( \tau = \alpha + \frac{\pi}{2} \), we obtain (1.11).
(c) By Lemma 2.2 and Lemma 2.1, we obtain
\[ C_p(z, l) \leq C_p(1, l') \leq C_p \]
for some \( l' \) with \( |l'| = 1 \), and we have the second conclusion of the main theorem.

Let us now show that the constant \( C_p \) is sharp. We will show the sharpness of the result for \( p \leq 2 \). A similar analysis works for \( p > 2 \). Let \( 0 < \rho < 1 \) and take
\[ e^{is} = \frac{\rho - e^{it}}{1 - \rho e^{it}}, \]
i.e.,
\[ e^{it} = \frac{\rho - e^{is}}{1 - \rho e^{is}}. \]

Define
\[ f_{\rho}(e^{it}) = (1 - \rho^2)^{-1/p} |\cos s (1 - \cos s)|^{q-1} \text{sign}(\cos s) \]
and take
\[ w_{\rho} = P[f_{\rho}]. \]

Then
\[ dt = \frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos s} ds, \]
\[ \Re e^{it} \frac{e^{it}}{(r - e^{it})^2} = \frac{(1 + r^2 - 2r \cos s) \cos(s)}{(1 - r^2)^2}, \]
and
\[ 2\pi \int_0^{|f_{\rho}(e^{it})|^p dt = 2\pi \int_0^{|f_{\rho}(e^{it})|^p \frac{1}{1 + \rho^2 - 2\rho \cos s} ds \]
\[ = 2\pi \int_0^{2\pi} |\cos s (1 - \cos s)|^{q} \frac{1}{1 + \rho^2 - 2\rho \cos s} ds. \]

Thus
\[ \lim_{\rho \to 1} \| f_{\rho} \|_p^p = \int_0^{2\pi} |f_{\rho}(e^{it})|^p dt = \frac{\pi^q}{2q} C_q^p. \quad (2.5) \]
Taking $r = \rho$, we obtain

\[
(1 - \rho^2)^{1+1/p} |Dw_\rho(\rho)| = \frac{(1 - \rho^2)^{1+1/p}}{\pi} \int_0^{2\pi} \Re \frac{e^{it}}{(\rho - e^{it})^2} f_\rho(e^{it}) dt
\]

\[
= \frac{(1 - \rho^2)^{1+1/p}}{\pi} \int_0^{2\pi} \frac{(1 + \rho^2 - 2\rho \cos s) \cos(s)}{(1 - \rho^2)^2} \left(1 - \rho^2\right)^{-1/p} \times |\cos s(1 - \cos s)|^{q-1} \text{sign}(\cos s) \frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos s} ds
\]

\[
= \frac{1}{\pi} \int_0^{2\pi} |\cos s|^q (1 - \cos s)^{q-1} ds
\]

\[
= \frac{\pi^{q-1}}{2q-1} C_p^q
\]

From (2.5) it follows that

\[
\lim_{\rho \to 1} \frac{(1 - \rho^2)^{1+1/p} |Dw_\rho(\rho)|}{\|f_\rho\|_p} = C_p.
\]

This shows that the constant $C_p$ is sharp. \qed

**Proof of Theorem 1.2** First of all

\[
\partial w = \int_0^{2\pi} \frac{e^{it}}{(z - e^{it})^2} f(e^{it}) d\theta.
\]

By applying Hölder inequality, we have

\[
|\partial w| \leq \frac{1}{2\pi} \left( \int_0^{2\pi} \frac{1}{|z - e^{it}|^{2q}} d\theta \right)^{1/q} \left( \int_0^{2\pi} |f(e^{it})|^p d\theta \right)^{1/p}
\]

\[
= (1 - |z|^2)^{1/q-2} \frac{1}{2\pi} \left( \int_0^{2\pi} \frac{(1 - |z|^2)^{2q-1}}{|z - e^{it}|^{2q}} d\theta \right)^{1/q} \left( \int_0^{2\pi} |f(e^{it})|^p d\theta \right)^{1/p}.
\]

It remains to estimate the integral

\[
J_q = \int_0^{2\pi} \frac{(1 - |z|^2)^{2q-1}}{|z - e^{it}|^{2q}} d\theta = \int_0^{2\pi} \frac{(1 - r^2)^{2q-1}}{|r - e^{it}|^{2q}} d\theta.
\]
By making use again of the change
\[ e^{i\theta} = \frac{r - e^{is}}{1 - re^{is}}, \]
we obtain
\[ d\theta = \frac{1 - r^2}{|1 - re^{is}|^2} ds \]
and
\[ r - e^{i\theta} = \frac{(1 - r^2)e^{is}}{1 - re^{is}}. \]

Therefore, by using Lemma 2.2 for \( \lambda = 0 \), we obtain
\[ J_q = \frac{2}{\pi} \int_0^{2\pi} \frac{(1 - r^2)^{2q-1}}{|r - e^{i\theta}|^{2q}} d\theta = (1 - r^2)^{1-q} \int_0^{2\pi} |1 - re^{is}|^{2q-2} d\theta \]
\[ = (1 - r^2)^{1-q} \int_0^{2\pi} |1 + r^2 - 2r \cos s|^{q-1} d\theta \]
\[ \leq 2^{q-1} (1 - r^2)^{1-q} \int_0^{2\pi} |1 - \cos s|^{q-1} d\theta. \]

Thus
\[ |\partial w| \leq c_p (1 - |z|^2)^{-1-1/p} \| f \|_{L^p(\Gamma)}, \]
where
\[ c_p = 2^{-1+q} \frac{\Gamma(-1/2 + q)}{\pi^{1+q} \Gamma(q)} \left( \frac{\Gamma(-1/2 + q)}{\Gamma(q)} \right)^{1/q}. \]

This proves (1.13). By formula (2.4), for \( \mu = 1, \nu = 1 - q \) we have
\[ \int_0^{2\pi} |1 + r^2 - 2r \cos s|^{q-1} d\theta = 2 \int_0^{\pi} |1 + r^2 - 2r \cos s|^{q-1} ds \]
\[ = 2\pi F \left( 1 - q, 1 - q; 1, r^2 \right). \]
This implies (1.12). The sharpness of constant $c_p$ can be verified by taking
\[ f^\pm_\rho(e^{it}) = (1 - \rho^2)^{-1/p}|\cos s(1 - \cos s)|^{q-1}e^{\pm is} \]
and following the proof of sharpness of $C_p$. \hfill \Box

**Acknowledgments** After we wrote the first version of this paper, we had useful discussion about this subject with Professor Vladimir Maz’ya. We thank Professor Romeo Meštrović for some language remarks.

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