Number of states with fixed angular momentum for identical fermions and bosons

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We present in this paper empirical formulas for the number of angular momentum $I$ states for three and four identical fermions or bosons. In the cases with large $I$ we prove that the number of states with the same $M$ and $n$ but different $J$ is identical if $I \geq (n - 2)J - \frac{1}{2}(n - 1)(n - 2)$ for fermions and $I \geq (n - 2)J$ for bosons, and that the number of states is also identical for the same $M$ but different $n$ and $J$ if $M \leq \min(n, 2J + 1 - n)$ for fermions and for $M \leq \min(n, 2J)$ for bosons. Here $M = I_{\text{max}} - I$, $n$ is the particle number, and $J$ refers to the angular momentum of a single-particle orbit for fermions, or the spin $L$ carried by bosons.

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The enumeration of the number of total angular momentum \( I \) states is a very common practice in nuclear structure theory. One usually obtains this number by subtracting the combinatorial number of the states with a total angular momentum projection \( M = I + 1 \) from that for \( M = I \). The combinatorial numbers of different \( M \)'s seem to be irregular, and such an enumeration procedure would be prohibitively tedious without a computer for a very large single-\( j \) shell. In textbooks \[1\] the numbers of states of a few nucleons in a single-\( j \) (\( j \) is a half integer) shell are usually tabulated for sake of convenience. There was only one algebraic formulas available for \( I = 0 \) states with 4 fermions \[2\]. It would be very interesting to have more general algebraic formulas, if possible. In another context, it was noticed \[3\] that the angular momentum 0 ground state probability of 4 fermions in a single-\( j \) shell in the presence of random interactions has a synchronous staggering with an increase of the number of \( I = 0 \) states by one. It has not been clarified yet whether there is a deep relation between these two quantities or this synchronism is just a coincidence. This “coincidence” also motivated the present study of number of states.

In this paper we give some simple formulas for the number of states, denoted as \( D(I)_j \), \( D(I)_l \), and \( D(I)_L \) for fermions in a single-\( j \) shell, fermions in a single-\( l \) shell, and bosons with spin \( L \), respectively, where \( j \) is a half integer, and \( l \) and \( L \) are integers. We are able to construct empirical formulas for all \( I \) states of three and four identical particles, and some formulas for a few lowest \( I \) states of five particles. For short we use \( D(I)_J \) to refer to all these three cases, where \( J \) can be \( j \), \( l \), or \( L \). Although these formulas are obtained empirically, we have confirmed them by computer for \( j \leq 999/2 \), \( l \leq 500 \) or \( L \leq 500 \), and for many cases with much larger \( J \)'s which were taken randomly. One therefore can use them “safely” in practice. We shall also show that the number of states is identical for the same \( n \) and \( \mathcal{M} \) (defined as \( I_{\text{max}} - I \)) but different \( J \) for states of fermions with \( I \geq (n - 2)J - \frac{1}{2}(n - 1)(n - 2) \) or for states of bosons with \( I \geq (n - 2)L \), and that the number of states with different \( n \) and \( J \) is also identical if \( \mathcal{M} \leq \text{min}(n, 2J + 1 - n) \) for fermions and \( \mathcal{M} \leq \text{min}(n, 2J) \) for bosons. In the Appendix we shall list a few formulas for the cases of five particles.

For \( n = 3 \) and \( I \leq J \), we empirically obtain

\[
D(I)_j = \left[ \frac{2I + 3}{6} \right];
\]

\[
D(I)_l = \left[ \frac{I}{3} \right] + \frac{1}{2} \left( 1 - (-)^{I+l} \right);
\]

\[
D(I)_L = \left[ \frac{I}{3} \right] + \frac{1}{2} \left( 1 + (-)^{I+L} \right), \tag{1}
\]

where \( \left[ \cdot \right] \) means to take the largest integer not exceeding the value inside.

For fermions with \( n = 3 \) and \( I \geq J - 1 \) or bosons with \( n = 3 \) and \( I \geq J \), the \( D(I)_J \)'s can be empirically given in a unified form:

\[
D(I)_J = \left[ \frac{I_{\text{max}} - I}{6} \right] + \delta_I, \tag{2}
\]
where

\[
\delta_I = \begin{cases} 
0 & \text{if } (I_{\text{max}} - I) \% 6 = 1 \\
1 & \text{otherwise} .
\end{cases}
\]

In this paper \(a \% b\) means to take the remainder of \(a/b\), where \(a\) is a non-negative integer and \(b\) is a natural number. For examples, \(7 \% 3 = 1\), \(27 \% 10 = 7\).

According to Eq. (1), \(D(\frac{1}{2})_j = 0\), \(D(0)_l = 1\) (or 0) if \(l\) is odd (or even), and \(D(0)_L = 1\) (or 0) if \(L\) is even (or odd). It is noted that there are overlaps of \(I\)'s covered by Eqs. (1) and (2), and that one may use either of them to obtain the number of states for these \(I\)'s.

The cases of four particles are more complicated. However, a regular staggering of \(D(I)_J\) can be easily noticed if \(I \leq 2J\). We define \(\eta_I^J = D(I)_J - D(I)_J + 3\) (even \(I\)) and \(\Delta(I) = D(I)_J - D(I + 3)_J\) (odd \(I\)). It is found that \(\eta_I^J\) changes periodically as \(J\) changes by 3 for \(I \leq 2J\). Table I shows that the \((\eta_I^J, \eta_I^{J+1}, \eta_I^{J+2})\) of four particles is actually a “trinary representation” of a natural number \((I/2 + 1)\), where \(I\) is an even number. Although the origin of this regularity is not known, one may make use of this to construct the formulas of \(D(I)_J\) for \(n = 4\) and \(I \leq 2J\). The cases of \(I \geq 2J\) will be addressed later in this paper.

For \(n = 4\) and \(I \leq 2J\) with \(I\) being even, we empirically obtain

\[
D(I)_J = \left[ \frac{\mathcal{L} - I/2}{3} \right] \times (I + 1) + C(I)m - \delta + \mathcal{D}(I),
\]

(3)

where \(m = (\mathcal{L} - I/2) \% 3\). For four fermions in a single-\(j\) shell, \(\mathcal{L} = j = \frac{1}{2}\), and the coefficients in Eq. (3) are given by

\[
C(I) = \left[ \frac{I}{6} \right] + 1 ,
\]

\[
\delta = \begin{cases} 
\delta_{m2} & \text{if } I \% 6 = 0 \\
0 & \text{otherwise}
\end{cases} ,
\]

\[
\mathcal{D}(I) = 3K(K - 1) + K + (1 + K)K + \delta_{\mathcal{K}, 4} + \delta_{\mathcal{K}, 5} ,
\]

where \(K = \left[ \frac{I + 4}{12} \right]\), and \(\mathcal{K} = \frac{(I + 4) \% 12}{2}\). For four fermions in a single-\(l\) shell, \(\mathcal{L} = l\), and the coefficients in Eq. (3) are given by

\[
C(I) = \left[ \frac{I + 4}{6} \right] ,
\]

\[
\delta = \begin{cases} 
\delta_{m2} & \text{if } (I + 4) \% 6 = 0 \\
0 & \text{otherwise}
\end{cases} ,
\]

\[
\mathcal{D}(I) = 3K(K - 1) + K + (1 + K)K + \delta_{\mathcal{K}, 4} + \delta_{\mathcal{K}, 5} ,
\]

where \(K = \left[ \frac{I + 2}{12} \right]\), and \(\mathcal{K} = \frac{(I + 2) \% 12}{2}\). For 4 bosons with spin \(L\), \(\mathcal{L} = L\), and the coefficients in Eq. (3) are given by

\[
C(I)_J = \left[ \frac{I + 4}{6} \right] ,
\]
\[ \delta = \begin{cases} \delta_{m2} & \text{if } (I + 4) \% 6 = 0 \\ 0 & \text{otherwise} \end{cases}, \]

\[ D_L(I) = 3K(K - 1) + K + (1 + K)K + \delta_{K,4} + \delta_{K,5}, \]

where \( K = \left\lfloor \frac{I + 8}{12} \right\rfloor \), and \( \mathcal{K} = \frac{(I+8)\%12}{2} \).

For \( n = 4 \) and \( I \leq 2J \) with \( I \) being odd, we introduce \( I = I_0 + 3 \), and obtain

\[ D(I)_j = D_j(I_0) - \left\lfloor \frac{I}{4} \right\rfloor - 1, \]
\[ D(I)_l = D_l(I_0) - \left\lfloor \frac{I + 2}{4} \right\rfloor, \]
\[ D(I)_L = D_L(I_0) - \left\lfloor \frac{I}{4} \right\rfloor - 1. \]

(4)

One easily sees that the \( D(1)_j \)'s are always zero for \( n = 4 \).

For \( n = 4 \) and \( I \geq 2J \), we define

for even \( I \): \( I = I_{\text{max}} - 2m \),
for odd \( I \): \( I = I_{\text{max}} - 3 - 2m \).

We let \( K = \left\lfloor \frac{m}{6} \right\rfloor \), \( \mathcal{K} = m \% 6 \), and obtain

\[ D(I)_j = 3K(K + 1) - K + (K + 1)(\mathcal{K} + 1) + \delta_{K0} - 1 \]

(5)

for fermions with \( I \geq 2J - \frac{1}{2}(n-1)(n-2) = 2J - 3 \) and for bosons with \( I \geq 2L = I_{\text{max}} - 2L \).

It is noted that for fermions \( D(I)_j \) of \( I = (2J - 3, 2J - 2, 2J - 1, \text{ and } 2J) \) can be obtained either by Eqs. (3) and (4) or Eq. (5). The formulas of \( D(I)_j \) for \( n = 4 \) present an even-odd staggering of the number of states: the number of states with even number of \( I \) is not smaller and mostly larger than those of their odd \( I \) neighbors. A similarity between the formulas for four fermions (in both half-integer \( j \) orbit and integer \( l \) orbit) and bosons is also easily noticed.

The situation of \( n = 5 \) is much more complex, and we are unable to construct simple and unified formulas. In the Appendix we list a few formulas for the lowest \( I \)'s.

We next point out that for fermions with \( I \geq (n - 2)J - \frac{1}{2}(n-1)(n-2) \) or for bosons with \( I \geq (n - 2)J \), \( D(I)_j \) is identical for the same \( n \) and \( \mathcal{M} \) but different \( J \). This identity is universal and exists for all three cases discussed in this paper: fermions in a single-\( j \) shell or a single-\( l \) shell, and bosons with spin \( L \). Taking \( n = 5 \) as an example, we use \( I = I_{\text{max}} - \mathcal{M} \) to get \( D(\mathcal{M})_j = D(I)_j = 1, 0, 1, 1, 2, 2, 3, 3, 5, 5, 7, 7, 10, 10, 13, 14, 17, 18, 22, 23, 28, 29, 34, 36, 42, 44, 50, 53, 60, 63, \cdots \), for \( \mathcal{M} = 0, 1, 2, \cdots, 28 \cdots \), which is independent of \( J \) and is applicable both to fermions or bosons. Below we prove this observation.
According to the standard enumeration procedure [4], suppose that there are \( p \) distinct Slater determinants \( \Psi_M(i) \) \((i = 1, \ldots, p)\) with the property \( J_2 \Psi_M(i) = M \Psi_M(i) \). If there are \( p + q \) \((q > 0)\) linearly independent Slater determinants that have the property \( J_2 \Psi_{M-1}(i) = (M-1) \Psi_{M-1}(i) \), then the \( q \) states with angular momentum \( I = M - 1 \) can be constructed. We use the convention that \( J \geq m_1 \geq m_2 \cdots \geq m_n \geq -J \) for fermions and \( L \geq m_1 \geq m_2 \cdots \geq m_n \geq -L \) for bosons. Here the sum of \( m_i \) over \( i \) is equal to \( M \) for \( \Psi_M(i) \) or \( (M-1) \) for \( \Psi_{M-1}(i) \). One sees that the number of states of different systems is the same if the hierarchies of \( m_i \) for these systems have a one-to-one correspondence.

For fermions in a single-\( j \) or a single-\( l \) shell, the hierarchical structure of number of states obtained by successively operating \( J_- \) on \( \Psi_{M_{max}} \) (where \( m_1 = m_2 = \cdots = m_n = L \)) successively until one comes to the case in which only \( m_n \) is decreased to become \( -J \) while other \( m_i \) are unchanged. In other words, before the process of the \( J_- \) operation is iterated \( 2J + 2 - n \) times (Here \( J = j \) or \( l \)), the hierarchy is the same for all \( \Psi_M \)'s obtained by operating \( J_- \) on \( \Psi_{M_{max}} \) successively: If there were not such a requirement that \( J \geq m_1 \geq m_2 \cdots \geq m_n \geq -J \), the above hierarchical structure for fermions would be always the same for all \( I \) states. The minimum angular momentum \( I \) which keeps the identical hierarchy for \( n \) fermions in a single-\( J \) shell is thus given by:

\[
(J) + (J-1) + (J-2) + \cdots (J-n+2) + (-J) = (n-2)J - \frac{1}{2}(n-1)(n-2),
\]

where \( J = j \) or \( l \).

Similarly, for bosons with spin \( L \), one keeps the hierarchical structure of number of states while one operates \( J_- \) on \( \Psi_{M_{max}} \) (where \( m_1 = m_2 = \cdots = m_n = L \)) successively until one comes to the case in which only \( m_n \) is \( -L \) and others unchanged. Thus the minimum angular momentum \( I \) which keeps the identical hierarchy for \( n \) bosons with spin \( L \) is given by:

\[
L + L + \cdots + L = (n-2)L.
\]

It is noted that the Pauli blocking produces a smaller \( I_{max} \) which equals to \( nJ - \frac{1}{2}n(n-1) \) while the \( I_{max} \) of bosons is \( nJ \). Despite of this difference, there is a one-to-one correspondence between \( |m_1, m_2, \cdots, m_n| \) of bosons and that of fermions in the process of successive operation of \( J_- \) on \( \Psi_{M_{max}} \). Thus \( D(M)J \) for both fermions in a single-\( j \) or a single-\( l \) shell and that of bosons with spin \( L \) are also the same for the same \( n \) but different \( J \), if \( I \geq (n-2)J - \frac{1}{2}(n-1)(n-2) \) for fermions and \( I \geq (n-2)L \) for bosons.

We finally address the above discussion in an alternative procedure: One lets \( I = I_{max} - M \), and uses \( \mathcal{P}(M) \) to label the number of partitions of \( M = i_1 + i_2 + \cdots + i_n \) with \( i_1 \geq i_2 \geq \cdots \geq 0 \). Defining \( \mathcal{P}(0) = 1 \), one easily finds that \( D(I)J = D(M)J = \mathcal{P}(M) - \mathcal{P}(M-1) \), unless \( I < (n-2)J - \frac{1}{2}(n-1)(n-2) \) for fermions or \( I < (n-2)L \) for bosons. This procedure not only proves the identity discussed above but also reveals a relevant fact: when \( M \leq \min(n, 2J+1-n) \) for fermions or \( M \leq \min(n, 2J) \) for bosons,
$D(I)_J = D(M)_J$ is the same for fermions and bosons, because the hierarchy of $P(i)$ of the systems is identical for different $n$ and $J$. For example, $D(M = 5)_{J_j = 31/2}$ of 5 fermions is equal to $D(M = 5)_{\ell = 30}$ of 10 bosons. The $D(M)_{J}$ series are: 1, 0, 1, 2, 2, 4, 4, 7, 8, 12, 14, 21, 24, 34, 41, 55, 66, 88, 105, 137, 165, 210, 253, 320, for $M = 0, 1, \cdots 24$, respectively.

To summarize, we found in this paper that there are simple structures in the number of states of three and four identical particles, which enabled us to construct empirical formulas for $n = 3$ and 4. For $n = 5$ we presented formulas for a few lowest $I$ states. We have confirmed the validity of these empirical formulas throughout $j \leq 999/2$ and $l \leq 500$ which are large enough for practical use.

We also proved that the hierarchy of states obtained by operating $J_-$ on $\Psi_{MM_{\text{max}}}$ for fermions and bosons is the same if $I \geq nJ - \frac{1}{2}(n-1)(n-2)$ for fermions or $I \geq (n-2)J$ for bosons, which gives identical series of number of states for fermions in a single-$j$ shell or bosons with spin $l$ with the same particle numbers. The number of states is also identical for the same $M$ but different $n$ and $J$ if $M \leq \min(n, 2J + 1 - n)$ for fermions and for $M \leq \min(n, 2J)$ for bosons. These facts have eluded from observation in the long history of the enumeration procedure for the dimension $D(I)_J$.

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Table I  For the cases with $n = 4$, $\eta_I^J$, $\eta_I^{J+1}$ and $\eta_I^{J+2}$ of angular momenta $I$ and $I + 3$ ($I$ is even) states change periodically at an interval $\Delta_J = 3$ when $I \leq 2J$. One thus easily constructs formulas for $D(I)_J$’s for states with $I \leq 2J$.

| $I$ | $I + 3$ | $\eta_I^J$ | $\eta_I^{J+1}$ | $\eta_I^{J+2}$ |
|-----|---------|-------------|-----------------|-----------------|
| 0   | 3       | 1           | 0               | 0               |
| 2   | 5       | 1           | 1               | 0               |
| 4   | 7       | 1           | 1               | 1               |
| 6   | 9       | 2           | 1               | 1               |
| 8   | 11      | 2           | 2               | 1               |
| 10  | 13      | 2           | 2               | 2               |
| 12  | 15      | 3           | 2               | 2               |
| 14  | 17      | 3           | 3               | 2               |
| 16  | 19      | 3           | 3               | 3               |
| ... | ...     | ...         | ...             | ...             |
Appendix  

number of states for five particles lying in a few lowest \( I \) states.

First, we come to the case with five fermions in a single-\( j \) shell. We define \( A(j, j_0, d) = \lfloor \frac{j - j_0}{d} \rfloor \), and \( B(j, j_0, d) = (j - j_0) \% d \), and obtain

\[
D(\frac{1}{2})_j = 6A^2(j, \frac{9}{2}, 12) + 3A(j, \frac{9}{2}, 12) \\
+ \left( A(j, \frac{9}{2}, 12) + 1 \right) \left( B(j, \frac{9}{2}, 12) + 1 \right) + \delta_B(j, \frac{9}{2}, 12),
\]

where

\[
\delta_B(j, \frac{9}{2}, 12) = \begin{cases} 
- B(j, \frac{9}{2}, 12) & \text{if } B(j, \frac{9}{2}, 12) \leq 2 \\
- 2 & \text{if } 3 \leq B(j, \frac{9}{2}, 12) \leq 4 \\
- 3 & \text{otherwise}
\end{cases}
\]

\[
D(\frac{3}{2})_j = 3A^2(j, \frac{11}{2}, 6) + 4A(j, \frac{11}{2}, 6) \\
+ \left( A(j, \frac{11}{2}, 6) + 1 \right) \left( B(j, \frac{11}{2}, 6) + 1 \right) + \delta_B(j, \frac{11}{2}, 6) + 1,
\]

where

\[
\delta_B(j, \frac{11}{2}, 6) = \begin{cases} 
1 & \text{if } B(j, \frac{11}{2}, 6) \geq 4 \\
0 & \text{otherwise}
\end{cases}
\]

\[
D(\frac{5}{2})_j = 2A^2(j, \frac{9}{2}, 4) + 3A(j, \frac{9}{2}, 4) \\
+ \left( A(j, \frac{9}{2}, 4) + 1 \right) \left( B(j, \frac{9}{2}, 4) + 1 \right) + \delta_B(j, \frac{9}{2}, 4) + 1,
\]

where

\[
\delta_B(j, \frac{9}{2}, 4) = \begin{cases} 
1 & \text{if } B(j, \frac{9}{2}, 4) \geq 2 \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
D(\frac{7}{2})_j = 6A^2(j, \frac{3}{2}, 6) + \left( 2A(j, \frac{3}{2}, 6) + 1 \right) \left( B(j, \frac{3}{2}, 6) + 1 \right) + \delta_B(j, \frac{3}{2}, 6) - 1,
\]

where

\[
\delta_B(j, \frac{3}{2}, 6) = \begin{cases} 
-1 & \text{if } 1 \leq B(j, \frac{3}{2}, 6) \leq 3 \\
0 & \text{otherwise}
\end{cases}
\]
It is noted that $D(\frac{3}{2})_j = 1$ for $n = 5$ when $j = \frac{9}{2}$, which is not covered in the formula (9). An interesting behavior is that there exists an approximate relation for 5 fermions in a single-$j$ shell: $D(I)_j \sim (I + \frac{1}{2})D(\frac{3}{2})_j$ when $I < j$.

Next, we come to five fermions in a single-$l$ shell. For $I = 0$, we define

$$k = \begin{cases} 
\frac{(l - 2)}{2} & \text{if } l\%2 = 0 \\
\frac{(l - 11)}{2} & \text{if } l\%2 = 1 
\end{cases}, \quad K = [k/6], \quad \mathcal{K} = k \% 6,$$

and obtain

$$D(0)_l = 3K (K + 1) - K + (K + 1) (K + 1) + \delta_{\mathcal{K}0} - 1; \quad (12)$$

for $I = 1$, we define

$$k = \begin{cases} 
\frac{(l - 1)}{2} & \text{if } L\%2 = 1 \\
\frac{(L - 4)}{2} & \text{if } L\%2 = 0 
\end{cases}, \quad K = [k/2], \quad \mathcal{K} = k \% 2 ,$$

and obtain

$$D(1)_l = K (K + 1) + \mathcal{K}(K + 1). \quad (13)$$

We finally come to the case of five bosons with spin $L$. For $I = 0$, we define

$$k = \begin{cases} 
\frac{L}{2} & \text{if } L\%2 = 0 \\
\frac{(L - 9)}{2} & \text{if } L\%2 = 1 
\end{cases}, \quad K = [k/6], \quad \mathcal{K} = k \% 6,$$

and obtain

$$D(0)_L = 3K (K + 1) - K + (K + 1) (K + 1) + \delta_{\mathcal{K}0} - 1; \quad (14)$$

for $I = 1$, we define

$$k = \begin{cases} 
\frac{L}{2} & \text{if } L\%2 = 1 \\
\frac{(L - 3)}{2} & \text{if } L\%2 = 0 
\end{cases}, \quad K = [k/2], \quad \mathcal{K} = (k \% 2) ,$$

and obtain

$$D(1)_L = (K + 1) (K + \mathcal{K} + 1). \quad (15)$$

The formulas for larger $I$'s with $n = 5$ are more complicated and are not addressed in this paper.