Persistently foliar composite knots

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A knot $\kappa$ in $S^3$ is persistently foliar if, for each non-trivial boundary slope, there is a co-oriented taut foliation meeting the boundary of the knot complement transversely in a foliation by curves of that slope. For rational slopes, these foliations may be capped off by disks to obtain a co-oriented taut foliation in every manifold obtained by non-trivial Dehn surgery on that knot. We show that any composite knot with a persistently foliar summand is persistently foliar and that any nontrivial connected sum of fibered knots is persistently foliar. As an application, it follows that any composite knot in which each of two summands is fibered or at least one summand is nontorus alternating or Montesinos is persistently foliar.

We note that, in constructing foliations in the complements of fibered summands, we build branched surfaces whose complementary regions agree with those of Gabai’s product disk decompositions, except for the one containing the boundary of the knot complement. It is this boundary region which provides for persistence.

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1 Introduction

Co-oriented taut foliations play an important role in the study of 3-manifolds. Recently, the search for co-oriented taut foliations in 3-manifolds has been informed by the L-space conjecture [44, 2, 33], which states that an irreducible space that is not an L-space necessarily contains a co-oriented taut foliation. Considering manifolds obtained by Dehn surgery on $S^3$, a knot $\kappa$ is called an L-space knot if some non-trivial Dehn surgery on $\kappa$ yields an L-space. A knot $\kappa$ is persistently foliar if, for each boundary slope, there is a co-oriented taut foliation meeting the boundary of the knot complement transversely in a foliation by curves of that slope. For rational slopes, these foliations may be capped off by disks to obtain a co-oriented taut foliation in every manifold obtained by Dehn surgery on that knot. In this context, we propose the L-space knot conjecture:

Conjecture 1.1 (L-space knot conjecture) A knot is persistently foliar if and only if it is not an L-space knot and has no reducible surgeries.

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Krcatovich [35] has proven that nontrivial connected sums of knots are never L-space knots. In this paper, we prove that many composite knots are also persistently foliar, as detailed in the results described below. It follows that any such knot \( \kappa \) satisfies the L-space knot conjecture, and any 3-manifold obtained by Dehn surgery along \( \kappa \) satisfies the L-space conjecture.

Let \( \kappa \) be any knot in \( S^3 \), and fix a regular neighbourhood \( N(\kappa) \) of \( \kappa \). Parametrize \( \partial N(\kappa) \) as \( S^1 \times S^1 \) so that \( \{1\} \times S^1 \) represents the meridian, and \( S^1 \times \{1\} \) represents the longitude, of \( \kappa \). A lamination of \( \partial N(\kappa) \) has slope \( m \in \mathbb{RP}^1 \) if it is isotopic to the image of lines of slope \( m \) under the universal covering map \( \mathbb{R}^2 \to S^1 \times S^1 : (s, t) \mapsto (e^{2\pi is}, e^{2\pi it}) \). The slope \( 1/0 \) is called the trivial slope.

More generally, given any oriented 3-manifold \( M \) with a single torus boundary component, which we give the standard orientation induced by the orientation of \( M \), define the set of slopes on \( \partial M \) to be the set of isotopy classes of unoriented (simple) curves on \( \partial M \). In the case that \( M \) is fibered over \( S^1 \) with fiber \( F \), we distinguish \( \partial F \) as the longitude of \( \partial M \) and denote it by \( \lambda \). In this context we define a meridian to be any curve having a single point of minimal transverse intersection with \( \lambda \). Once a distinguished meridian is chosen (see Section 3), each slope may be identified with a point in \( \mathbb{RP}^1 \). A different choice of meridian results in a parabolic shift, fixing \( 0 \) (the longitudinal slope), of the associated points of \( \mathbb{RP}^1 \); since a parabolic shift preserves the cyclic ordering, we may speak of an interval of slopes (with given endpoints and containing a given third slope in its interior) independently of this choice.

**Definition 1.2** A foliation \( \mathcal{F} \) strongly realizes a slope if \( \mathcal{F} \) intersects \( \partial N(\kappa) \) transversely in a foliation by curves of that slope.

**Remark 1.3** Note that no co-oriented taut foliation strongly realizes the meridian of a knot in \( S^3 \), since \( S^3 \) is simply connected.

We proceed as follows. First we show that connected sums behave well with respect to strong realization of slopes:

**Proposition 4.1.** Suppose \( \kappa = \kappa_1 \# \kappa_2 \) is a connected sum of knots in \( S^3 \). If the slope \( m \) along \( \kappa_1 \) is strongly realized, then so is the slope \( m \) along \( \kappa \).

**Corollary 4.2** Suppose \( \kappa = \kappa_1 \# \cdots \# \kappa_n \) is a connected sum of knots. If at least one of the \( \kappa_i \) is persistently foliar, then so is \( \kappa \).

We next show that connected sums of fibered knots are persistently foliar and therefore satisfy the L-space Knot Conjecture:
Theorem 6.1 Suppose $\kappa_1$ and $\kappa_2$ are nontrivial fibered knots in $S^3$. Any nontrivial slope on $\kappa = \kappa_1 \# \kappa_2$ is strongly realized by a co-oriented taut foliation that has a single minimal set, disjoint from $\partial N(\kappa)$. Hence $\kappa_1 \# \kappa_2$ is persistently foliar.

Combining the results above with those of [7, 8, 9], we obtain:

Corollary 6.2 Suppose $\kappa = \kappa_1 \# \cdots \# \kappa_n$ is a connected sum of knots. If at least one of the $\kappa_i$ is a nontorus alternating or Montesinos knot or a connected sum of fibered knots, then $\kappa$ is persistently foliar.

Corollary 6.3 Suppose $\kappa$ is a composite knot with a summand that is a nontorus alternating or Montesinos knot or the connected sum of two fibered knots, and $\hat{X}_\kappa$ is a manifold obtained by non-trivial Dehn surgery along $\kappa$. Then $\hat{X}_\kappa$ contains a co-oriented taut foliation; hence, $\kappa$ satisfies the L-space Knot Conjecture.

Since connected sums of fibered knots are necessarily fibered ([49]; for a geometric argument, see [13]), we can contrast the co-oriented taut foliations constructed in this paper with those constructed in [47, 48]. First, we combine some results found in [48] and restate them using the language of Honda, Kazez and Matić [29]:

Theorem 7.3 Suppose that $X$ is a fibered 3-manifold, with fiber $F$ a compact oriented surface with connected boundary, and orientation-preserving monodromy $\phi$. If there...
is a tight arc $\alpha$ so that the corresponding product disk $D(\alpha)$ has transition arcs of opposite sign, then there is a co-oriented taut foliation $F_r$ that strongly realizes slope $r$ for all slopes except $\mu$, the distinguished meridian. Furthermore, each $F_r$ extends to a co-oriented taut foliation $\hat{F}_r$ in $\hat{X}(r)$, the closed 3-manifold obtained by Dehn filling along $r$, and when $r$ intersects the meridian efficiently in at least two points, the minimal set of $\hat{F}(r)$ is genuine.

In contrast, the foliations constructed in [47, 48] are minimal up to Denjoy blowdown. Hence, when the foliations $\hat{F}_r$ have genuine minimal set, they cannot be isotopic to the foliations constructed in [47, 48]. However, it is possible that these foliations are equivalent under some coarser notion of equivalence.

**Question 1.5** Suppose $\kappa = \kappa_1 \# \kappa_2$ is a nontrivial connected sum that is fibered, and let $M$ be obtained by nontrivial Dehn surgery along $\kappa$. Let $\mathcal{F}$, and $\mathcal{F'}$ be co-oriented taut foliations in $M$, with $\mathcal{F}$ constructed as in [48] and $\mathcal{F'}$ constructed as described in this paper.

1. Are $\mathcal{F}$ and $\mathcal{F'}$ coarsely isotopic [23]?
2. Are $\mathcal{F}$ and $\mathcal{F'}$ transverse to a common smooth flow?
3. Are $\mathcal{F}$ and $\mathcal{F'}$ transverse to nowhere vanishing vector fields $v$ and $v'$, respectively, that represent a common Spin$^c$-structure $s \in$ Spin$^c(M)$ [43]?
4. If yes to (2), do weakly symplectically fillable contact structures $\xi$, $\xi'$ approximating, respectively, $\mathcal{F}$ and $\mathcal{F'}$ have common contact invariant $c(\xi) = c(\xi') \in \hat{HF}(M, s)$ [46, 30]?

The construction of co-oriented taut foliations in this paper, as well as in [7, 8, 9], involves making choices of spine and co-orientation on the branches of a spine chosen. It seems likely that different choices can lead to co-orientable taut foliations that are not isotopic (even up to reversing the co-orientation), and hence (1)–(4) of Question 1.5 apply.

Note that work of Ghiggini [26] and Ni [38, 39] (see also [31, 32]) establishes that an L-space knot is necessarily fibered. Hence, conjecturally, any non-fibered knot in $S^3$ is persistently foliar. Restricting attention to fibered knots permits us to minimize use of the theory of sutured manifolds and thus to emphasize the simplicity of the construction. In a future paper [10], we discuss more general conditions that allow for the construction of co-oriented taut foliations that strongly realize all boundary slopes except one. In particular, we make the following conjecture:

**Conjecture 1.6** Every composite knot is persistently foliar.
All constructions of co-oriented taut foliations found in this paper are adaptations of the pure arrow type construction found in [7]. Among the constructions of persistent families of co-oriented taut foliations found in [7, 8, 9, 10, 11], these are the ones closest to the sutured manifold constructions introduced by Gabai.

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3 Preliminary Definitions

3.1 Fibered knots and product disks

A knot $\kappa$ in $S^3$ is fibered if the knot complement $X_\kappa$ is homeomorphic to

$$F \times [0, 1] / \sim,$$

where $(x, 1) \sim (\phi(x), 0)$ for some compact orientable surface $F$ and homeomorphism $\phi : F \to F$. In this case, $F$ is called a fiber of $\kappa$, and $\phi$ the monodromy map of the fibering. The homeomorphism type of $F \times [0, 1] / \sim$ is dependent only on the isotopy class of $\phi$.

We will always assume that a fibered knot $\kappa$ and its fiber $F$ are consistently oriented; namely, $F$ is oriented and $\kappa$ is isotopic to $\lambda = \partial F$ as an oriented manifold. We also assume an orientation of $S^3$, with the induced normal orientation on $F$ equal to the increasing orientation on the $[0, 1]$ factor of $F \times [0, 1]$.

We next state some results in the context of fibered 3-manifolds, rather than restricting attention to knot complements. Let $X$ denote the fibered 3-manifold

$$F \times [0, 1] / \sim,$$

where $(x, 1) \sim (\phi(x), 0)$, for some compact orientable surface $F$ and orientation preserving homeomorphism $\phi : F \to F$. We restrict attention to the case that $F$ has a
We observe the following, which will be useful in the definition of $\mu$. Two properly embedded arcs intersect efficiently if any intersections are transverse and no isotopy through properly embedded arcs (rel endpoints) reduces the number of points of intersection. A pair of properly embedded arcs $(\alpha, \beta)$ is tight if either $\alpha = \beta$ (as unoriented arcs) or if $\alpha$ and $\beta$ are non-isotopic and intersect efficiently. Given a properly embedded arc $\alpha$, we may isotope $\phi$ so that $(\alpha, \phi(\alpha))$ is a tight pair; in this case, we say that $\alpha$ is tight (with respect to $\phi$). Given a tight pair $(\alpha, \beta)$, it is clear that $(\phi(\alpha), \phi(\beta))$ is also a tight pair; furthermore, we may isotope $\phi$ so that the arcs $\alpha, \beta, \phi(\alpha)$, and $\phi(\beta)$ are pairwise tight. Indeed, any finite collection of properly embedded oriented arcs can be isotoped to be pairwise tight, and then isotoped so that the collection of arcs together with their images under $\phi$ are pairwise tight. When working with a finite collection of arcs, we will henceforth assume that these arcs and then $\phi$ have been isotoped in this way.

**Notation 3.1** Given any properly embedded arc $\alpha \in F$, with endpoints $\alpha(0)$ and $\alpha(1)$ in $\partial F$, let $D(\alpha)$ be the image of $\alpha \times [0, 1]$, and let $\delta_i(\alpha)$ be the image of $\alpha(i) \times [0, 1]$, for $i = 0, 1$, under the quotient map $F \times [0, 1] \to F \times [0, 1]/\sim$. Identify $\alpha$ with the image of $\alpha \times \{0\}$ and $\phi(\alpha)$ with the image of $\alpha \times \{1\}$; thus, $\partial D(\alpha) = \alpha \cup \phi(\alpha) \cup \delta_0(\alpha) \cup \delta_1(\alpha)$.

Now consider an oriented properly embedded tight essential arc $\alpha$. If $\alpha = \phi(\alpha)$ (as oriented arcs), set $\mu = \delta_0(\alpha)$. Suppose that $\alpha \neq \phi(\alpha)$ (as oriented arcs). The endpoints of $\delta_0(\alpha)$ cut $\lambda$ into two open arcs, $\rho_1(\alpha)$ and $\rho_2(\alpha)$ say. The simple closed curves $\mu_1(\alpha) = \delta_0(\alpha) \cup \rho_1(\alpha)$ and $\mu_2(\alpha) = \delta_0(\alpha) \cup \rho_2(\alpha)$ are meridians satisfying $|\langle \mu_1(\alpha), \mu_2(\alpha) \rangle| = 1$.

We observe the following, which will be useful in the definition of $\mu$ given below. If $\partial M$ is parametrized as $S^1 \times S^1$ so that $\{1\} \times S^1$ represents the meridian $\mu_i(\alpha)$ and $S^1 \times \{1\}$ represents $\lambda$, then, up to isotopy of this parametrization, $\delta_0(\alpha)$ maps to $\rho_2(\alpha)$ under the projection onto the second factor $S^1 \times S^1 \to S^1 : (s, t) \mapsto t$. Letting $\widehat{X}_i$ denote the 3-manifold obtained by Dehn filling along $\mu_i(\alpha)$, $i = 1, 2$, it follows that $\phi(\alpha)$ is to the left of $\alpha$ in the associated open book of $\widehat{X}_i$ if and only if $\phi(\alpha)$ is to the right of $\alpha$ in the associated open book of $\widehat{X}_j$, for $j \neq i$.

Next consider a properly embedded oriented essential arc $\beta$ such that the arcs $\alpha, \beta, \phi(\alpha)$, and $\phi(\beta)$ are pairwise nonisotopic as unoriented arcs and pairwise tight. Orient the arc...
β. Up to symmetry, including interchanging the labelings \(\rho_1\) and \(\rho_2\), there are three possibilities:

1. \(\{\mu_1(\alpha), \mu_2(\alpha)\} \cap \{\mu_1(\beta), \mu_2(\beta)\} = \{\mu_1(\alpha)\} = \{\mu_1(\beta)\}\); 
2. \(\mu_1(\alpha) \cap \mu_1(\beta) = \emptyset\) but \(\{\mu_1(\alpha), \mu_2(\alpha)\} = \{\mu_1(\beta), \mu_2(\beta)\}\); 
3. \(\mu_1(\alpha) \cap \mu_2(\beta) \neq \emptyset\) for all \(i, j = 1, 2\) (and, hence, \(\{\mu_1(\alpha), \mu_2(\alpha)\} = \{\mu_1(\beta), \mu_2(\beta)\}\)).

These are illustrated in Figure 1.

![Figure 1: Possible relative positions of the transition arcs \(\delta_0(\alpha)\) and \(\delta_0(\beta)\).](image)

If (1) holds for some choice of \((\alpha, \beta)\), let \(\mu = \mu_1(\alpha)\) be the common meridian, and set \(\delta_0(\alpha) = \mu_1(\alpha)\). In this case, \(\phi\) realizes \(\mu\) as a closed orbit and \(\mu \in \{\mu_1(\gamma), \mu_2(\gamma)\}\) for all choices of \(\gamma\). If (2) holds for two choices \((\alpha, \beta)\) and \((\alpha', \beta')\) such that \(\mu_1(\alpha) = \mu_2(\alpha')\), so \(\mu_1(\alpha) \neq \mu_1(\alpha')\), we note in passing that \(\phi|_{\partial F}\) has a periodic point of order two. If (3) holds for all \((\alpha, \beta)\), we note that \(\phi\) is isotopic to a periodic homeomorphism of order two. (If not, consider an arc \(\alpha\) such that \(\phi^{-1}(\alpha)\) is not isotopic to \(\phi(\alpha)\) and an arc \(\beta\) with \(\beta(0)\) in the component of \(\partial F \setminus \{\phi^{-1}(\alpha(0)), \phi(\alpha(0))\}\) that does not contain \(\alpha(0)\). One of \((\alpha, \beta)\) or \((\phi(\alpha), \beta)\) fails to satisfy (3)). In these cases, we defer the choice of \(\mu\) until after Corollary 3.3. Otherwise, set \(\mu = \mu_1(\alpha)\), as well.

**Theorem 3.2** [48] If (1) holds for some choice of \((\alpha, \beta)\), there are co-oriented taut foliations that strongly realize all slopes except possibly \(\mu\). Otherwise, there are co-oriented taut foliations that strongly realize all slopes in the interval of slopes containing \(\lambda\) that lie strictly between \(\mu_1(\alpha)\) and \(\mu_2(\alpha)\) for some, and hence all, choices of tight \(\alpha\).

If \(X\) is the complement of a fibered knot \(\kappa \subset S^3\), and \(\alpha\) is a tight essential arc in \(F\), then in terms of the standard parametrization of \(\partial N(\kappa)\), \(\mu_1(\alpha) = 1/n\) and \(\mu_2(\alpha) = 1/(n+1)\), for some \(n \in \mathbb{Z}\). If \(n \notin \{-1, 0\}\), then by Theorem 3.2, \(1/0\) would be strongly realized by a taut foliation, an impossibility; hence, \(1/0 \in \{\mu_1(\alpha), \mu_2(\alpha)\}\). In fact, the proof of Theorem 4.5 of [34] implies more:

**Corollary 3.3** If \(X\) is the complement of a fibered knot \(\kappa \subset S^3\), one of \(\mu_1(\alpha)\) or \(\mu_2(\alpha)\) is the standard meridian for some, and hence any, choice of tight \(\alpha\); moreover, \(\mu\) as defined as above is the standard meridian.
Finally, in the cases for which \( \mu \) has not been defined above, we define it as follows: if \( X \) is the complement of a fibered knot in \( S^3 \), choose \( \mu \) to be the standard meridian; otherwise, with reference to the observation above, choose \( \mu \in \{ \mu_1(\alpha), \mu_2(\alpha) \} \) so that \( \phi(\alpha) \) is to the right of \( \alpha \) in the \( \mu \)-Dehn filling of \( F \times [0, 1]/\sim \). Set \( \delta'_1(\alpha) = \delta_0(-\alpha) \). With these definitions we have \( \mu = \delta_i(\alpha) \cup \delta'_i(\alpha) \) for all tight \( \alpha \) and for \( i = 1, 2 \).

**Notation 3.4** Parting with previous notation, henceforth let \( \mu_i(\alpha) = \delta_i(\alpha) \cup \delta'_i(\alpha) \), \( i = 0, 1 \).

**Definition 3.5** Each \( \delta_i(\alpha) \) is a transition arc, and each \( \delta'_i(\alpha) \) is the meridian complement of \( \delta_i(\alpha) \).

In summary, parametrize \( \partial M \) as \( S^1 \times S^1 \) so that \( \{1\} \times S^1 \) represents the distinguished meridian \( \mu \), and \( S^1 \times \{1\} \) represents the longitude \( \lambda \), the isotopy class of \( \partial F \). If we consider \( D(\alpha) \), for some tight \( \alpha \), and focus on a neighbourhood of one of \( \delta_0(\alpha) \) or \( \delta_1(\alpha) \), then either \( \alpha = \phi(\alpha) \) as oriented arcs, or we see one of the models shown in Figure 2.

Reversing the orientation convention found in [47, 48], we call the first transition arc positive and the second negative. Notice that reversing the orientation of \( \kappa \) reverses the orientation on \( F \), and vice versa; so, the sign of a transition arc is independent of the initial choice of orientation on \( \kappa \). Either the endpoints of \( \alpha \) separate those of \( \phi(\alpha) \) or they do not; up to symmetry, the possibilities when \( \alpha \) is not isotopic to \( \phi(\alpha) \) (as unoriented arcs) are listed in Figure 3.

**Remark 3.6** We observe that the open book decomposition in \( \hat{X}(\mu) \) associated to \( (F, \phi) \) is right-veering (respectively, left-veering) [29] if, for every properly embedded tight oriented arc \( \alpha \in F \), either \( \alpha = \phi(\alpha) \) or the transition arc \( \delta_0(\alpha) \) is positive (respectively, negative).

![Figure 2: Positive and negative transition models.](image)

**Definition 3.7** Given any metric space \( X \) and any subset \( A \subset X \), the closed complement of \( A \) in \( X \), denoted \( X \rceil A \), is the metric completion of \( X \setminus A \).
Remark 3.8  Intuitively, $X|_{A}$ amounts to cutting $X$ open along $A$. (Although other authors have used the notation $X_{A}$, we feel the inclusion of a vertical slash evokes the notion of “cutting.”) In particular, we will consider the closed complement of a curve in a surface and of a surface or, more generally, a lamination [25], in a 3-manifold, with respect to the path metric inherited from a Riemannian metric. For example, if $F$ is a fiber of a fibered knot $\kappa$, then $M|_{F}$ is homeomorphic to $F \times [0, 1]$.

Definition 3.9  Let $M$ be a 3-manifold with nonempty boundary, and let $S$ be an oriented surface with nonempty boundary properly embedded in $M$. Label the two copies of $S$ in $(M|_{S})$ by $S_{-}$ and $S_{+}$. A product disk is an immersed disk in $M$ whose pre-image under the quotient map $M|_{S} \to M$ is properly embedded in $(M|_{S})$ with boundary consisting of two essential arcs in $\partial M$, and two essential arcs in $S$, one contained in $S_{-}$ and one in $S_{+}$. A product disk is tight if the two arcs of its boundary in $S$ are tight.

In particular, the disk $D(\alpha)$ of Notation 3.1 is a tight product disk whenever $\alpha$ is tight.
3.2 Laminations and foliations

Roughly speaking, a codimension-one foliation $\mathcal{F}$ of a 3-manifold $M$ is a disjoint union of surfaces injectively immersed in $M$ such that $(M, \mathcal{F})$ looks locally like $(\mathbb{R}^3, \mathbb{R}^2 \times \mathbb{R})$. More precisely, we have the following definition.

**Definition 3.10** Let $M$ be a closed $C^\infty$ 3-manifold. A codimension one foliation $\mathcal{F}$ of (or in) $M$ is a decomposition of $M$ into disjoint connected surfaces $L_i$, called the leaves of $\mathcal{F}$, such that $(M, \mathcal{F})$ looks locally like $(\mathbb{R}^3, \mathbb{R}^2 \times \mathbb{R})$. More precisely:

1. $\bigcup_i L_i = M$, and
2. there exists an atlas $\mathcal{A}$ on $M$ with respect to which $\mathcal{F}$ respects the following local product structure:
   - for every $p \in M$, there exists a coordinate chart $(U, (x, y, z))$ in $\mathcal{A}$ about $p$ such that $U$ is homeomorphic to $\mathbb{R}^3$ and the restriction of $\mathcal{F}$ to $U$ is the union of planes given by $z = \text{constant}$.

A foliation is co-oriented if the leaves admit co-orientations that are locally compatible.

We also consider foliations $\mathcal{F}$ in compact smooth 3-manifolds with nonempty boundary, restricting attention to the case that $\partial M$ is a nonempty union of tori and $\mathcal{F}$ intersects $\partial M$ everywhere transversely. In this case, at boundary points of $M$, $(M, \mathcal{F})$ looks locally like horizontal closed half planes $[0, \infty) \times \mathbb{R}$.

Calegari [4] proved that any foliation has an isotopy representative that is $C^\infty,0$; in particular, such that $T\mathcal{F}$ is defined and continuous, and leaves of $\mathcal{F}$ are smoothly immersed. A foliation is taut [14, 6] if for every $p \in M$ there exists a 1-submanifold that contains $p$ and is everywhere transverse to $\mathcal{F}$. The foliations constructed in this paper will have only noncompact leaves; hence they have an isotopy representative that is taut [27, 6].

Recall that a subset of $M$ is $\mathcal{F}$-saturated if it is a union of leaves of $\mathcal{F}$. A minimal set of $\mathcal{F}$ is a closed $\mathcal{F}$-saturated subset of $M$ that doesn’t properly contain a nonempty closed $\mathcal{F}$-saturated subset. The foliations constructed in this paper contain a unique minimal set, and this minimal set is disjoint from $N(\kappa)$.

A lamination $\mathcal{L}$ is a decomposition of a closed subset of $M$ into a union of injectively immersed surfaces, called the leaves of $\mathcal{L}$, such that $(M, \mathcal{L})$ looks locally like $(\mathbb{R}^3, \mathbb{R}^2 \times C)$, where $C$ is a closed subset of $\mathbb{R}$. Properly embedded compact surfaces, foliations, and $\mathcal{F}$-saturated closed subsets of $M$, such as minimal sets of foliations, are all key
examples of laminations. All laminations that arise in this paper are $\mathcal{F}$-saturated closed subsets of $M$ for some foliation $\mathcal{F}$.

A lamination strongly realizes the slope $r \in \mathbb{S}^1$ if it meets $\partial N(\kappa)$ transversely in a lamination consisting of consistently oriented curves of slope $r$. When $r$ is rational, these curves are closed, and it makes sense to talk about the manifold $\hat{M}_r$ obtained by Dehn surgery along $\kappa$ by slope $r$. If a lamination $\mathcal{L}$ strongly realizes slope $r$, then it extends to a lamination $\hat{\mathcal{L}}$ in $\hat{M}_r$ by capping off each boundary curve with disk. Moreover, if $\mathcal{L}$ is a co-oriented taut foliation, then so is $\hat{\mathcal{L}}$.

4 Any composite knot with a persistently foliar summand is persistently foliar.

Before discussing connected sums of fibered knots, we prove the useful fact that strong realization of a slope for a knot in $S^3$ extends to any connected sum with that knot.

Proposition 4.1 Suppose $\kappa = \kappa_1 \# \kappa_2$ is a connected sum of knots in $S^3$. If the slope $m$ along $\kappa_1$ is strongly realized by a co-oriented taut foliation, then so is the slope $m$ along $\kappa$.

Proof Suppose $X_{\kappa_1}$ contains a co-oriented taut foliation $\mathcal{F}_1$ that strongly realizes slope $m$. By [20], there is a co-oriented taut foliation $\mathcal{F}_2$ in $X_{\kappa_2}$ that strongly realizes the longitudinal boundary slope. We describe how to form a “connected sum” of these two foliations to produce a co-oriented taut foliation that strongly realizes slope $m$ in $X_\kappa$.

Let $P$ denote a summing sphere for this connected sum, and set $A := P \cap (S^3 \setminus \text{Int}N(\kappa)) = P \cap X_\kappa$. Observe that for each $i = 1, 2$, $\partial X_{\kappa_i}$ is the union of two annuli, one of which is $A$. Viewing $A$ as $S^1 \times I$, we may arrange that each foliation $\mathcal{F}_i$ intersects $A$ with leaves $\theta \times I$, $\theta \in S^1$. It is easy to check that, gluing $\mathcal{F}_1$ to $\mathcal{F}_2$ along $A$, we obtain a taut foliation $\mathcal{F}$ in $X_\kappa$ that realizes slope $m$. Choosing compatible co-orientations on the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ yields a co-orientation for $\mathcal{F}$. □

Corollary 4.2 Suppose $\kappa = \kappa_1 \# \cdots \# \kappa_n$ is a connected sum of knots. If at least one of the $\kappa_i$ is persistently foliar, then so is $\kappa$. □
5 Spines, Train Tracks and Branched surfaces

In this paper we construct foliations by first constructing a spine, which we then smooth to a branched surface that carries a foliation (more precisely, a lamination that extends to a foliation). We restrict attention to the case that any intersections of a spine or a branched surface with \( \partial M \) are transverse; hence the intersection of a branched surface with the boundary of a 3-manifold is a train track. The curves carried by this train track play an important role in our analysis.

A \textit{train track} is a space locally modeled on one of the spaces of Figure 4. An \textit{I-fibered neighbourhood} of a train track \( \tau \) is a regular neighborhood \( N(\tau) \) foliated (as a 2-manifold with corners) by interval fibers that intersect \( \tau \) transversely, as locally modeled by the spaces in Figure 5.

![Figure 4: Local models of a train track.](image)

A \textit{standard spine} \([5]\) is a space \( \Sigma \) locally modeled on one of the spaces of Figure 6. A standard spine with boundary has the additional local models shown in Figure 7. The \textit{critical locus} \( \Gamma \) of \( \Sigma \) is the 1-complex of points of \( \Sigma \) where the spine is not locally a manifold. The critical locus is a stratified space (graph) consisting of triple points \( \Gamma^0 \) and arcs of double points \( \Gamma^1 = \Gamma \setminus \Gamma^0 \).

\textbf{Definition 5.1} The components of \( \Sigma|_\Gamma \) are called the \textit{sectors} of \( \Sigma \).

A \textit{branched surface (with boundary)} \([50]\; \text{see also \([40, 41]\)} \) is a space \( B \) locally modeled on the spaces of Figure 8 (along with those in Figure 9); that is, \( B \) is homeomorphic
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Figure 6: Local models of a standard spine at interior points.

Figure 7: Local models of a standard spine at boundary points.

to a spine, with the additional structure of a well-defined tangent plane at each point. The branching locus $\Gamma$ of $B$ is the 1-complex of points of $B$ where $B$ is not locally a manifold; such points are called branching points. The branching locus is a stratified space (graph) consisting of triple points $\Gamma^0$ and arcs of double points $\Gamma^1 = \Gamma \setminus \Gamma^0$. The components of $B|_{\Gamma}$ are called the sectors of $B$.

Figure 8: Local model of a branched surface at interior points.

An $I$-fibered neighborhood of a branched surface $B$ in a 3-manifold $M$ is a regular neighborhood $N(B)$ foliated by interval fibers that intersect $B$ transversely, as locally modeled by the spaces in Figure 10 at interior points; if the ambient manifold $M$ has non-empty boundary, all spines and branched surfaces are assumed to be properly embedded, with $N(B) \cap \partial M$ a union (possibly empty) of I-fibers. The surface $\partial N(B) \setminus \partial M$ is a union of two subsurfaces, $\partial_v N(B)$ and $\partial_h N(B)$, where $\partial_v N(B)$, the vertical boundary, is a union of sub-arcs of I-fibers, and $\partial_h N(B)$, the horizontal boundary, is everywhere transverse to the I-fibers.
Let $\pi$ be the retraction of $N(B)$ onto the quotient space obtained by collapsing each fiber to a point. The branched surface $B$ is obtained, modulo a small isotopy, as the image of $N(B)$ under this retraction. We will freely identify $B$ with this image and the core of each component of vertical boundary with its image in $\Gamma$. Double points of the branching locus are cusps with cusp direction pointing inward from the vertical boundary if $B$ is viewed as the quotient of $N(B)$ obtained by collapsing the vertical fibers to points. Cusp directions will be indicated by arrows, as in Figures 8 and 9.

A branched surface $B$ is co-oriented if the one-dimensional foliation of $N(B)$ is oriented. When $\partial M$ is a union of tori, and co-oriented $B$ is transverse to $\partial M$, the regions

$$(\partial M \setminus \text{int} N(B), \partial_1 N(B) \cap \partial M)$$

are products, and we use the notation

$$\partial'_1 N(B) = \partial_1 N(B) \cup (\partial M \setminus \text{int} N(B)).$$

Following Gabai [14, 19, 20], we call each component of $\partial'_1 N(B)$ a suture and refer to the pair $$(M \setminus \text{int} N(B), \partial'_1 N(B))$$ as a sutured manifold. Note that each component of $\partial'_1 N(B)$ is an annulus or a torus.

A surface is said to be carried by $B$ if it is contained in $N(B)$ and is everywhere transverse to the one-dimensional foliation of $N(B)$. A surface is said to be fully carried by $B$ if it carried by $B$ and has nonempty intersection with every I-fiber of $N(B)$. A lamination $\mathcal{L}$ is carried by $B$ if each leaf of $\mathcal{L}$ is carried by $B$, and fully carried if, in addition, each I-fiber of $N(B)$ has nonempty intersection with some leaf of $\mathcal{L}$.

Similarly, a 1-manifold, or union of 1-manifolds, is said to be carried by a train track $\tau$ if it is contained in some I-fibered neighbourhood $N(\tau)$, everywhere transverse to the one-dimensional foliation of $N(\tau)$. A 1-manifold, or union of 1-manifolds, is said to be
Figure 11: Oriented spine to oriented branched surface.

fully carried by \( \tau \) if it is carried by \( \tau \) and has nonempty intersection with every I-fiber of \( N(\tau) \).

If a branched surface \( B \) is homeomorphic to a spine \( \Sigma \), we say that \( B \) is obtained by smoothing \( \Sigma \). An example is illustrated in Figure 11. We say that a choice of co-orientations on the sectors of \( \Sigma \) is compatible if there is a smoothing of \( \Sigma \) to a co-oriented branched surface \( B \) that preserves the co-orientations on sectors; in this case, we call this smoothing the smoothing determined by the co-orientations. Examples are illustrated in Figure 12 and Figure 13. Branched surfaces \( B^G(\alpha) = (F; D(\alpha)) \) as described below play a key role in this paper. We use the superscript \( G \) because Lemma 5.3 describes a special case of Gabai’s Construction 4.16 of \([20]\) applied in the context of \([17]\).

**Notation 5.2** Let \( X \) be a fibered 3-manifold, with compact fiber \( F \) and monodromy \( \phi \); assume \( \partial F \) is connected and non-empty. (For current purposes, of course, we focus on the complement \( X_\kappa \) of a fibered knot \( \kappa \).) Let \( \alpha \) be a tight arc in \( F \) such that \( \alpha \neq \phi(\alpha) \) (as unoriented arcs). Set \( \Sigma = F \cup D(\alpha) \). Any choice of orientations on the surfaces \( F \) and \( D(\alpha) \) determines a unique compatible smoothing of \( \Sigma \) to a branched surface. We denote this branched surface by \( B^G(\alpha) = (F; D(\alpha)) \). (For purposes of this notation, \( F \) and \( D(\alpha) \) are assumed oriented.)

**Lemma 5.3** The complement pair \( (X \setminus \text{int} N(B^G(\alpha)), \partial \text{int} N(B^G(\alpha))) \) is homeomorphic to \( (F|_\alpha \times I, \partial (F|_\alpha) \times I) \).

**Proof** The complement of \( \text{int} N(F) \) is homeomorphic to \( F \times [0, 1] \), and hence is a handlebody of genus twice the genus of \( F \), with \( \partial \text{int} N(F) = \partial F \times [0, 1] \). Cutting along \( D(\alpha) \) introduces two strips of vertical boundary with cores isotopic to \( \alpha \), as illustrated in Figure 14, from which it follows that the complement of \( N(B^G(\alpha)) \) is homeomorphic to \( F|_\alpha \times I \), with \( \partial \text{int} N(B^G(\alpha)) = \partial (F|_\alpha) \times [0, 1] \).
We next generalize the notation above for a particular family of splittings of the branched surface $B^G(\alpha) = \langle F; D(\alpha) \rangle$.

**Notation 5.4** Suppose $\alpha_1, \alpha_2, \ldots, \alpha_n = \alpha$ is a sequence of pairwise tight oriented arcs properly embedded in $F$, with $\alpha_i \neq \pm \alpha_{i+1} \pmod{n}$.

Let $D_i = \alpha_i \times \left[\frac{i-1}{n}, \frac{i}{n}\right]$, oriented so that $\partial D_i$ contains $\alpha_i \times \left\{\frac{i-1}{n}\right\}$ as a positively oriented subarc. Let $F_I = F \times \{\frac{i}{n}\}$.

We denote by $B^G(\alpha_1, \alpha_2, \ldots, \alpha_n) = \langle F; (D_i)_{i=1}^n \rangle$ the branched surface obtained, under the identification $(x, 1) \sim (\phi(x), 0)$, by smoothing the spine $\bigcup_{i=1}^n (F_{i-1} \cup D_i)$ with co-orientation consistent with the given orientations on the product disks $D_i$ and copies of the fiber $F_I$. Notice that $B^G(\alpha_1, \alpha_2, \ldots, \alpha_n) = \langle F; (D_i)_{i=1}^n \rangle$ is a splitting of $B^G(\alpha)$ for $n > 1$. 

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5.1 Laminar branched surfaces

A minimal set of a co-oriented taut foliation $F$ is necessarily essential, and therefore carried by an essential branched surface:

**Definition 5.5** [25] A branched surface $B$ in a closed 3-manifold $M$ is called an *essential* branched surface if it satisfies the following conditions:

1. $\partial_h N(B)$ is incompressible in $M \setminus \text{int}(N(B))$, no component of $\partial_h N(B)$ is a sphere and $M \setminus \text{int}(N(B))$ is irreducible.
2. There is no monogon in $M \setminus \text{int}(N(B))$; i.e., no disk $D \subset M \setminus \text{int}(N(B))$ with $\partial D = D \cap N(B) = \alpha \cup \beta$, where $\alpha \subset \partial_h N(B)$ is in an interval fiber of $\partial_h N(B)$ and $\beta \subset \partial_h N(B)$.
3. There is no Reeb component; i.e., $B$ does not carry a torus that bounds a solid torus in $M$.

In practice, it can be difficult to determine whether an essential branched surface fully carries a lamination. In [36, 37], Li defines the notion of *laminar*, a very useful criterion that is sufficient (although not necessary) to guarantee that an essential branched surface fully carries a lamination. We recall the necessary definitions here.
Definition 5.6 ([36, 37]) Let $B$ be a branched surface in a 3-manifold $M$. A sink disk is a disk branch sector $D$ of $B$ for which the cusp direction of each component of $\Gamma^1 \cap D$ points into $D$ (as shown in Figure 15). A half sink disk is a sink disk which has nonempty intersection with $\partial M$.

Figure 15: A sink disk.

Sink disks and half sink disks play a key role in Li’s notion of laminar branched surface. A sink disk or half sink disk $D$ can be eliminated by splitting $D$ open along a disk in its interior; these trivial splittings must be ruled out:

Definition 5.7 ([36], [37]) Let $D_1$ and $D_2$ be the two disk components of the horizontal boundary of a $D^2 \times I$ region in $M \setminus \text{int}(N(B))$. If the projection $\pi : N(B) \to B$ restricted to the interior of $D_1 \cup D_2$ is injective, i.e, the intersection of any $I$-fiber of $N(B)$ with $\text{int}(D_1) \cup \text{int}(D_2)$ is either empty or a single point, then we say that $\pi(D_1 \cup D_2)$ forms a trivial bubble in $B$.

Definition 5.8 ([36], [37]) An essential branched surface $B$ in a compact 3-manifold $M$ is called laminar if it satisfies the following conditions:

1. $B$ has no trivial bubbles.
2. $B$ has no sink disk or half sink disk.

Theorem 5.9 ([36], [37]) Suppose $M$ is a compact and orientable 3-manifold.

(a) Every laminar branched surface in $M$ fully carries an essential lamination.

(b) Any essential lamination in $M$ that is not a lamination by planes is fully carried by a laminar branched surface.

In general, the branched surface $B^G$ of Lemma 5.3 is not laminar. However, $B^G$ admits a splitting to a laminar branched surface. Moreover, this splitting can be chosen so that the boundary train track of the resulting laminar branched surface contains the meridian as subtrack.
Definition 5.10 Let $X$ be an oriented fibered 3-manifold, with monodromy $\phi$ and compact fiber $F$; assume $\partial F$ is connected and nonempty. Let $\alpha$ be a tight arc properly embedded in $F$. A sequence $\phi(\alpha) = \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n = \alpha$ of oriented arcs properly embedded in $F$ is $\alpha$-sparse if, for all $j$,

(1) $\alpha_j \cap \alpha_{j+1} = \emptyset$ and
(2) $\alpha_j$ and $\alpha_{j+1}$ are non-isotopic.

Definition 5.11 For $i = 0, 1$, an $\alpha$-sparse sequence is $i$-end-effective, if the endpoints $\alpha_j(i), 0 \leq j \leq n$, give a monotonic sequence in the interval $\delta'_i(\alpha)$. An $\alpha$-sparse sequence is end-effective if it is both 0-end-effective and 1-end-effective.

Figure 16: Sequences that are $i$-end-effective (the example $n = 2$), as necessary in the construction of $B^G(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

Theorem 5.12 Let $X$ be an oriented fibered 3-manifold, with monodromy $\phi$ and compact fiber $F$; assume $\partial F$ is connected and nonempty. Let $\alpha$ be a tight, non-separating, oriented arc properly embedded in $F$ such that $\alpha \neq \phi(\alpha)$ (as oriented arcs). If both transition arcs are positive (respectively, negative), then there is a 0-end-effective (respectively, 1-end-effective) sequence $\phi(\alpha) = \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n = \alpha$ such that the train track $\tau = B^G(\alpha_1, \alpha_2, \ldots, \alpha_n) \cap \partial M$ contains as subtrack the meridian, with the meridional subtrack necessarily containing $\delta_0(\alpha)$ (respectively, $\delta_1(\alpha)$).

If one transition arc is positive and the other negative, then there is an end-effective sequence $\phi(\alpha) = \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n = \alpha$ such the train track $\tau = B^G(\alpha_1, \alpha_2, \ldots, \alpha_n) \cap \partial M$ contains as subtrack two disjoint copies of the meridian, with one component of the subtrack containing $\delta_0(\alpha)$ and the other containing $\delta_1(\alpha)$.

In each case, the resulting branched surface $B^G(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is necessarily laminar.
Proof  This is primarily a restatement of results found in [47, 48]. There are three possibilities, as illustrated in Figure 16.

When the transitions have a common sign and $\delta_0'(\alpha) \cap \delta_1'(\alpha) = \emptyset$, this is the main result of [47] together with Corollary 6.4 of [48]. Corollary 6.4 of [48] is easily modified to allow for the case that $\delta_0'(\alpha) \cap \delta_1'(\alpha) \neq \emptyset$; a hint is shown in Figure 17. When the transitions are of opposite sign, this is the main construction of [47] together with Corollary 6.6 of [48].

\[\text{Figure 17}\]

In particular, it follows that $B^G(\alpha_1, \alpha_2, \ldots, \alpha_n) \cap \partial M$ carries all meridians. (Recall that in this general context, a meridian is defined to be any curve having a single point of minimal transverse intersection with $\partial F$, and we use the definite article and the letter $\mu$ when referring to the distinguished meridian defined in Section 3.) When the transition arcs are of opposite sign, $B^G(\alpha_1, \alpha_2, \ldots, \alpha_n) \cap \partial M$ fully carries all meridians except $\mu$. When both transition arcs have the same sign, $B^G(\alpha_1, \alpha_2, \ldots, \alpha_n) \cap \partial M$ fully carries all meridians except the two, which we call extremal, obtained by taking the union of $\delta_i(\alpha)$ with, respectively, each of the components of $\partial F \setminus \delta_i(\alpha)$. When $\delta_i(\alpha)$ is positive (respectively, negative), the extremal meridians are $\mu$ and the simple closed curve of slope $\frac{1}{1}$ (respectively, $-\frac{1}{1}$). It follows that the train track $B^G(\alpha_1, \alpha_2, \ldots, \alpha_n) \cap \partial M$ fully carries the open interval of slopes that is bounded by these extremal meridians and contains all other meridians.
6 Connected sums of fibered knots are persistently foliar

Theorem 6.1 Suppose \( \kappa_1 \) and \( \kappa_2 \) are nontrivial fibered knots in \( S^3 \). Any nontrivial slope on \( \kappa = \kappa_1 \# \kappa_2 \) is strongly realized by a co-oriented taut foliation that has a unique minimal set, disjoint from \( \partial N(\kappa) \). Hence \( \kappa_1 \# \kappa_2 \) is persistently foliar.

Corollary 6.2 Suppose \( \kappa = \kappa_1 \# \cdots \# \kappa_n \) is a connected sum of knots. If at least one of the \( \kappa_i \) is a nontorus alternating or Montesinos knot or a connected sum of fibered knots, then \( \kappa \) is persistently foliar.

Proof All Montesinos and prime alternating knots are either persistently foliar or fibered, by the results of [7, 8, 9]. Thus the result follows immediately from Theorem 6.1 and Corollary 4.2.

Corollary 6.3 Suppose \( \kappa \) is a composite knot with a summand that is a nontorus alternating or Montesinos knot or the connected sum of two fibered knots, and \( \hat{X}_\kappa \) is a manifold obtained by non-trivial Dehn surgery along \( \kappa \). Then \( \hat{X}_\kappa \) contains a co-oriented taut foliation; hence, \( \kappa \) satisfies the L-space Knot Conjecture.

We prove Theorem 6.1 in the sections that follow. First, in Section 6.1, we describe the spine, \( \Sigma \), underlying the branched surface, \( B \), that carries the minimal set of the desired foliations. In Section 6.2 we describe compatible co-orientations on \( \Sigma \), smoothing it to obtain \( B \). In Section 6.3 we give a precise description of the complementary regions of \( B \). In Section 6.4 we prove that \( B \) carries no compact leaves, and in Section 6.5 we prove that \( B \) fully carries a lamination. Finally, in Section 6.6, we show that this lamination extends to a family of co-oriented taut foliations with unique common minimal set, carried by \( B \), that strongly realize all boundary slopes.

6.1 The spine \( \Sigma \)

Let \( \kappa = \kappa_1 \# \kappa_2 \subset S^3 \) be a connected sum, where each of the knots \( \kappa_1 \) and \( \kappa_2 \) is nontrivial and fibered, with fibers \( F_1 \) and \( F_2 \) respectively. Let \( F \) denote the band connect sum of \( F_1 \) and \( F_2 \); so \( F \) is a fiber for \( \kappa \) [13].

Let \( P \) denote a summing sphere for this connected sum, cutting \( F \) into \( F_1 \) and \( F_2 \). Set

\[
A := P \cap (S^3 \setminus \text{int}(N(\kappa))) = P \cap X_\kappa.
\]

Choose the isotopy representatives of \( \phi \) and \( P \) so that \( A|_F = D(\beta) \) for an arc \( \beta \) properly embedded in \( F \). Thus \( \phi(\beta) = \beta \) and the endpoints of \( \beta \) are fixed points of \( \phi \).
View $F$ as a compact surface properly embedded in $X_\kappa$, and view $F_1$ and $F_2$ as compact surfaces properly embedded in $X_\kappa|A$. Denote the component of $X_\kappa|A$ containing $F_i$ by $X_{\kappa_i}$; we observe that $X_{\kappa_i}$ is indeed homeomorphic to the complement of $\kappa_i$. Let $T = \partial X_\kappa$.

To simplify the exposition, we focus on the case that both $\kappa_1$ and $\kappa_2$ have right-veering monodromy. The case that they both have left-veering monodromy follows symmetrically. We address the remaining case, that $\kappa$ has monodromy that is neither right- nor left-veering, in Section 7.

Choose non-separating, tight, properly embedded oriented arcs $\alpha_1$ in $F_1$ and $\alpha_2$ in $F_2$ disjoint from $\beta$ and such that $\phi(\alpha_i) \neq \alpha_i$, $i = 1, 2$. Consider $D(\alpha_i) \in X_{\kappa_i}$. Set $\Sigma_0 = T \cup A \cup F \cup D(\alpha_1) \cup D(\alpha_2)$. Notice that $\Sigma_0$ is not yet a spine as two surfaces meet transversely along $\beta$. To remedy this, isotope $F_2$ so that $F_2$ remains a properly embedded surface in $X_\kappa_2$, but $F_2 \cap A$ is an isotopy representative of $\beta$ in $A$ that meets $\beta$ transversely in a single point. (We could instead have chosen this representative to be disjoint from $\beta$. Now set

$$\Sigma = T \cup A \cup F_1 \cup F_2 \cup D(\alpha_1) \cup D(\alpha_2).$$

To simplify notation, set $D_i = D(\alpha_i)$ for $i = 1, 2$.

Finally, isotope $\Sigma$ into the interior of $X_\kappa$, so that $T$ is parallel to $\partial X_\kappa$, with the annulus $A$ still contained in the summing sphere $P$.

### 6.2 The co-oriented branched surface $B$

In this section, we describe a smoothing of $\Sigma$ by fixing a compatible choice of co-orientations on the sectors of $\Sigma$.

Choose a regular neighbourhood $N(A)$ of $A$ in $\Sigma$ such that the closure of $N(A)$ is disjoint from $D_1 \cup D_2$, and fix co-orientations on the sectors of $N(A)$ as shown in Figure 18. Give $F_1$ and $F_2$ the co-orientations that agree, respectively, with the co-orientations of $F_1 \cap N(A)$ and $F_2 \cap N(A)$. Choose any co-orientations on $D_1$ and $D_2$. These induce orientations on $\alpha_1$ and $\alpha_2$. Finally, cut $T$ open along $\mu_0(\alpha_1) \cup \mu_0(\alpha_2)$ (as defined in Notation 3.4) and assign co-orientations to the resulting annuli components to agree with those of $T \cap N(A)$. We have thus described co-orientations on the sectors of $\Sigma$.

It is straightforward to check that this choice of co-orientations on the sectors of $\Sigma$ determines a compatible smoothing of $\Sigma$ to a branched surface. Call this branched surface $B$. The smoothings restricted to $N(A)$ are shown in Figure 19. Those near
\[ \mu_0(\alpha_1) \text{ and } \mu_0(\alpha_2) \text{ are shown in Figures 20 and 21, and called Type C, and those near } \mu_1(\alpha_1) \text{ and } \mu_1(\alpha_2) \text{ are shown in Figure 22, and called Type B.} \]

We note that the choice of co-orientations on the sectors of \( \Sigma \) is motivated by the theory developed in [7, 8], as is the terminology Type C (for cusp) and Type B.

It is helpful for calculations to make note of the components of \( \partial v N(B) \) that result locally from each type of smoothing near a transition arc. These are shown in (red) boldface in Figure 23.

### 6.3 The three complementary regions of B

For each \( i \), let \( g_i \) denote the genus of \( F_i \). We now describe the components of the sutured manifold \( (X_\kappa \setminus \text{int} N(B), \partial_\kappa N(B)) \), commonly referred to as the complementary regions of \( B \). Clearly there are three, one of which contains \( \partial X_\kappa \), and one lying in each \( X_\kappa \). Let \( V_1 \) and \( V_2 \) be the annuli of vertical boundary with cores \( \mu_0(\alpha_1) \) and \( \mu_0(\alpha_2) \).

**Proposition 6.4** The sutured manifold \( (X_\kappa \setminus \text{int} N(B), \partial_\kappa N(B)) \) consists of the following three (sutured manifold) components:

1. \( (\partial X_\kappa \times I, V_1 \cup V_2) \),
2. \( (F'_1 \times I, \partial F'_1 \times I) \), and
3. \( (F'_2 \times I, \partial F'_2 \times I) \),

where \( F'_i = F_i |_{\alpha_i} \) is a surface with two boundary components and genus \( g_i - 1 \).

**Proof** \( T \) is parallel to \( \partial X_\kappa \). Moreover, each of the two Type C neighbourhoods of \( B \) introduces a single meridian suture. Hence, the complementary region containing \( \partial X_\kappa \) is isomorphic to the sutured manifold described in (1).
Set $B_1^G = \langle F_1, D_1 \rangle$ and $B_2^G = \langle F_2, D_2 \rangle$. By Lemma 5.3, it suffices to show that the remaining components complementary to $\text{int} N(B)$ are isomorphic as sutured manifolds to the closed complements of $B_1^G$ and $B_2^G$, respectively. Let $Y_i$ be the component that lies in $X_{\kappa_i}$. Forgetting the sutured manifold structure of $\partial Y_i$, the compact 3-manifold $Y_i$ is a genus $2g_i - 2$ handlebody, and hence is homeomorphic to $F_i' \times I$. It suffices, therefore, to prove that this homeomorphism can be chosen so that $\partial Y_i'$ is mapped to $\partial F_i \times I$. Away from $\partial A$ and the crossings $D_i \cap T$, the core of $\partial Y_i'$ runs along $T$, parallel to $\partial F_i$. At the crossings, this core combines with the arcs $D_i \cap F_i$ (topologically) as it does in $B^G$; see Figure 23.

At $\partial A$, this core wraps partway about $\partial A$, but disjointly from $\partial F_i$. Hence $(Y_i, \partial Y_i)$ is isomorphic to $(F_i' \times I, \partial F_i' \times I)$. This is illustrated in Figure 24. \qed

**Corollary 6.5** Let $\hat{M}$ denote a closed 3-manifold obtained by Dehn filling of slope $\frac{p}{q}$ along $\kappa$. The complement $(\hat{M} \setminus \text{int} N(B), \partial \text{N}(B))$ consists of the following three components:

1. A solid torus whose meridian has minimal geometric intersection number $2|q|$ with $\mu_0(\alpha_1) \cup \mu_0(\alpha_2)$,
2. $(F_1' \times I, \partial F_1' \times I)$, and
3. $(F_2' \times I, \partial F_2' \times I)$.

---

Figure 19: The branched surface near $A$, also showing the co-orientations of $T$ near $\mu_0(\alpha_1)$ and $\mu_0(\alpha_2)$. 

Figure 20: Type C meridional smoothing, also showing the co-orientations of $T$ near $\mu_0(\alpha_1)$ and $\mu_0(\alpha_2)$.
where $F'_i$ is a surface with two boundary components and genus $g_i - 1$.

**Proof** Consider the complementary component that contains $\partial X_\kappa$. After Dehn filling $\partial X_\kappa$ with slope $p/q$, this component transforms to a solid torus with meridian intersecting each of the curves $\mu_0(\alpha_1)$ and $\mu_0(\alpha_2)$ minimally in $|\langle 1/0, p/q \rangle| = |q|$ points.

6.4 Any leaf carried by $B$ is noncompact.

**Proposition 6.6** Any surface carried by $B$ has nonempty intersection with every branch of $B$, and is noncompact. In particular, $B$ does not carry a torus.

**Proof** Let $L$ be any nonempty surface carried by $B$, and let $B_L$ be the sub-branched surface of $B$ that fully carries $L$. If $B$ fully carries $L$, then $B_L = B$. In general, $B_L$ is a union of sectors of $B$.

We first observe that $B_L$ must contain a sector of $F$ that has nonempty intersection with $T$. Suppose by way of contradiction that it does not. Since the sink directions on $D_i \cap F$ point into $F$ for each $i$, $B_L$ contains such a sector of $F$ whenever $B_L$ contains $D_i$; hence we may assume that $B_L$ contains $D_i$. But if $B_L$ does not contain $D_1$ or $D_2$, it can...
contain a sector of $F$ only when it contains every sector of $F$. Hence, we may assume that $B_L$ does not contain $D_1$, $D_2$, or any sector of $F$, and therefore does not contain any sector of $T$, since a cusp direction points from $T$ into $F$ at Type C smoothings. But it then follows that $B_L$ cannot contain a sector of $A$, and hence is empty, an impossibility.

Thus, $B_L$ contains a sector $F_0$ of $F$ that has nonempty intersection with $T$. Since $F_0$ has an arc of boundary along $T$ with outward-pointing cusp direction, $L \cap N(T)$ contains a proper embedding of a ray $[0, \infty)$ carried by a meridian of $T$; hence, $L$ is not compact. \hfill $\square$

### 6.5 $B$ fully carries a lamination $\mathcal{L}$

The branched surface $B$ might contain sink or half sink disks. However, using ideas from [48], it is straightforward to show that it can be split to a branched surface that contains no sink or half sink disk.

**Notation 6.7** Given a sequence of oriented arcs $\alpha_{i,1}, \alpha_{i,2}, \alpha_{i,3}, \ldots, \alpha_{i,n}$ embedded in $F_i$, let $D_{ij} = \alpha_{ij} \times \left[\frac{j-1}{n}, \frac{j}{n}\right]$, oriented so that $\partial D_{ij}$ contains $\alpha_{ij} \times \left\{\frac{j-1}{n}\right\}$ as a positively oriented subarc. Let $F_{ij} = F_i \times \left\{\frac{i}{n}\right\}$. 

---

---
**Proposition 6.8** The branched surface $B$ can be split open to a laminar branched surface $B'$ homeomorphic to the spine

$$T \cup A \cup \left( \bigcup_{j=1}^{n_1} F_{1,j-1} \cup D_{1,j} \right) \cup \left( \bigcup_{j=1}^{n_2} F_{2,j-1} \cup D_{2,j} \right),$$

where $\phi(\alpha_i) = \alpha_{i,0}, \alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,n_i} = \alpha_i$ is a 0-end-effective sequence in $F_i$.

The complement of $B'$ has $2n + 1$ components:

1. $\left( \partial X_{\kappa} \times I, V'_1 \cup V'_2 \right)$, where $V'_1$ and $V'_2$ are disjoint meridional annuli,
2. $n_1$ copies of $(F'_1 \times I, \partial F'_1 \times I)$, and
Type C Type B
φ(α) α φ(α)
Type C Type B
φ(α) α φ(α) α

Figure 24: The sutures of $B$ agree with those of $B^G$.

(3) $n_2$ copies of $(F'_2 \times I, \partial F'_2 \times I)$,
where each $F'_i$ is a surface with two boundary components and genus $g_i - 1$.

Proof Recall our assumption that all transition arcs are positive; hence Type B smoothings occur at $\delta_1$ and Type C smoothings occur at $\delta_0$ for each arc $\alpha_i, i = 1, 2$.

Applying Theorem 5.12 to $B^G(\alpha_i)$, for each $i = 1, 2$, there are 0-end-effective sequences $\phi(\alpha_i) = \alpha_{i,0}, \alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,n_i} = \alpha_i$ such that each branched surface $B^G(\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,n_i})$ is laminar, and each train track $\tau_i = B^G(\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,n_i}) \cap \partial M$, contains the meridian as a subtrack containing $\delta_0(\alpha_i)$. Denote each meridian subtrack by $\mu_{\tau_i}$. Recall that each $F_{i,j}$ is oriented consistently with $F_i, i = 1, 2$.

Setting

$$\Sigma' = T \cup A \cup \left( \bigcup_{j=1}^{n_1} F_{1,j-1} \cup D_{1,j} \right) \cup \left( \bigcup_{j=1}^{n_2} F_{2,j-1} \cup D_{2,j} \right),$$

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we describe a smoothing of $\Sigma'$ by fixing a compatible choice of co-orientations on the sectors of $\Sigma'$. Indeed, co-orientations have been fixed for all sectors except those lying in $T$. We define co-orientations in the sectors of $T$ by choosing co-orientations on the two annuli obtained by cutting $T$ open along $\mu_1 \cup \mu_2$, choosing these co-orientations to agree with the co-orientations chosen on $T \cap N(A)$.

It is straightforward to check that this choice of co-orientations on the sectors of $\Sigma'$ determines a compatible smoothing of $\Sigma'$ to a branched surface. Call this branched surface $B'$. Under this smoothing, the two meridians $\mu_1 \cup \mu_2$ become meridian cusps in the complementary region that contains $\partial X_\kappa$. This is illustrated in Figure 25. Let $V'_i$ be the annulus of vertical boundary with core $\mu_\tau_i$.

Since $B$ does not carry a torus, and $B'$ is a splitting of $B$, $B'$ does not carry a torus. Moreover, since the sequences $\alpha_{i,0}, \alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,n_i}$ are $\alpha$-sparse, neither $B^G(\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,n_i})$ has a sink disk or half sink disk; thus, $B'$ has no sink disk or half sink disk. So $B'$ is laminar.

**Corollary 6.9**  $B$ fully carries a lamination.

**Proof**  The branched surface $B'$ described in Proposition 6.8 is laminar, and hence fully carries a lamination $L$ [36]. Since $B'$ is obtained by splitting $B$, $L$ is also fully carried by $B$.

### 6.6 $L$ extends to co-oriented taut foliations that strongly realize all boundary slopes

**Proposition 6.10**  For each slope $\gamma$ (not necessarily rational), the lamination $L$ extends to a co-oriented taut foliation $F_\gamma$ that strongly realizes $\gamma$. Each $F_\gamma$ has a unique minimal set, fully carried by $B$.

**Proof**  The complementary region of $B$ that is not a product (as a sutured manifold) is the one containing $\partial X_\kappa$: $\left(\partial X_\kappa \times I, V'_1 \cup V'_2\right)$, where $V'_1$ and $V'_2$ are disjoint meridional annuli. Denote this region by $Y_\partial$. We will now show that for every nontrivial slope $\gamma$ (not necessarily rational), this region can be filled in by noncompact leaves that meet $\partial X_\kappa$ in parallel leaves of slope $\gamma$.

When $\gamma$ is rational, this region contains a properly embedded annulus $A_\gamma = \gamma \times I$. When $\gamma$ is not the meridian, any choice of co-orientation of $A_\gamma$ describes a smoothing...
of $\Sigma \cup A_\gamma$ to a branched surface in $X_\kappa$ whose complementary regions are all products (as sutured manifolds).

In general (when $\gamma$ is either rational, but not the longitude $\lambda$, or irrational), proceed instead as follows. Consider the essential annulus $A_\lambda = \lambda \times I$. Again, any choice of co-orientation of $A_\lambda$ describes a smoothing of $\Sigma_\lambda = \Sigma \cup A_\lambda$ to a branched surface in $X_\kappa$ whose complementary regions are all products. In particular, the complementary region $Y_{\partial |A_\lambda}$ is a solid torus with two longitudinal sutures (one of which is $\partial M \setminus \lambda$).

Let $D_\mu$ be the product disk for this region, isotoped so that the essential arcs $D_\mu \cap A_\lambda$ are disjoint. The two distinct choices of orientation on $D_\mu$ give rise to two smoothings of $\Sigma_\mu = \Sigma \cup D_\mu$; call the resulting branched surfaces $B_1$ and $B_2$. The isotopy representative of $D_\mu$ can be chosen so that the train tracks $B_1 \cap \partial X_\kappa$ and $B_2 \cap \partial X_\kappa$ together fully carry all nontrivial, nonlongitudinal boundary slopes. The associated measures on these train tracks describe measured laminations that are fully carried by the sub-branched surfaces (not properly embedded) with spine $A_\lambda \cup D_\mu$. See Figure 26. (Alternatively, the branched surfaces $B_1$ and $B_2$ are laminar, and hence there exist co-oriented laminations fully carried by $B_1$ or $B_2$ that strongly realize any nontrivial, nonlongitudinal slope $\gamma$ [37]. The proof of the main result of [37] reveals that these foliations can be chosen to include $L$ as a sublamination.) This argument can of course be repeated replacing $\lambda$ with any nontrivial rational slope.

Filling in the product complementary regions of the resulting lamination with parallel copies of the boundary leaves yields a co-oriented foliation $\mathcal{F}_\gamma$ that strongly realizes $\gamma$. Since $\mathcal{F}_\gamma$ has no compact leaves, it is necessarily taut. Since any leaf carried by $B$ has nonempty intersection with every branch of $B$, $\mathcal{F}_\gamma$ has exactly one minimal set. When the surgery coefficient is rational but not an integer, the minimal set of $\mathcal{F}_\gamma$ remains genuine after Dehn filling by slope $\gamma$. \hfill $\Box$

This extension of $L$ to the family of co-oriented taut foliations $\mathcal{F}_\gamma$ (and $\mathcal{F}_\gamma$) is an extension of the well known “stacking chair” construction (see, for example, Example 1.1.i in [22]). An alternate approach to moving from the lamination $L$ to co-oriented taut foliations in $\tilde{X}_\gamma$, for $\gamma$ rational, can be found as Operations 2.3.2 and 2.4.4 in [21].

We note, for the reader interested in understanding all co-oriented taut foliations in the complement of $\kappa$, that there are multiple distinct choices of compatible co-orientations on $\Sigma$ leading to branched surfaces that fully carry taut foliations.

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7 Additional constructions when the monodromy of $\kappa$ is neither right- nor left-veering.

Recall that if the monodromy of $\kappa \in S^3$ is neither right- nor left-veering, Theorem 1.4 guarantees that any nontrivial slope is strongly realized by some co-oriented taut foliation. We now introduce several new constructions of co-oriented taut foliations that give the same result, most of which differ from the foliations of [47, 48] in that they have genuine minimal set.

We note in passing that if the monodromy of $\kappa$ is neither right- nor left-veering, then it has fractional Dehn twist coefficient zero [30], or, equivalently, Gabai degeneracy $n \cdot \frac{1}{0}$ for some $n \geq 1$ [34].

Lemma 7.1 A connected sum of fibered knots in $S^3$ has right-veering (respectively,
left-veering) monodromy if and only if each of its components has right-veering (respectively, left-veering) monodromy.

Proof By induction, it suffices to consider the case of two nontrivial summands. The result follows immediately from Corollary 1.4 of [16], or, more directly, from an analysis of product disks.

Corollary 7.2 Suppose $\kappa = \kappa_1 \# \kappa_2$ is a fibered knot in $S^3$. If the monodromy is neither right- nor left-veering, then one of the following must be true:

1. at least one of $\kappa_1$ or $\kappa_2$ has monodromy that is neither right- nor left-veering, or
2. one of $\kappa_1$ or $\kappa_2$ has monodromy that is right-veering, and the other summand is left-veering.

We proceed as in the right-veering case, by first constructing a spine, and then describing a smoothing by fixing a compatible choice of co-orientations on the branches of this spine. We note for completeness that we could address each summand separately in the manner of Section 6: as in Section 6.1, let $\Sigma = T \cup A \cup F_1 \cup F_2 \cup D_1 \cup D_2$, with the orientations on $A, T, F_1, F_2, \alpha_1$, and $\alpha_2$ chosen as before in Section 6.2. The only difference in the case that $D_i$ has transition arcs of opposite sign is that, along with a local smoothing of Type C, we see a local smoothing as shown in Figures 27 and 28, which we call Type A; again, as before, the sutures of $B$ agree with those of $B^G$, as shown in Figure 29.

However, there is a simpler and more general construction which does not depend on having a connected sum, given in the following proposition:

Theorem 7.3 Suppose that $X$ is a fibered 3-manifold, with fiber $F$ a compact oriented surface with connected boundary, and orientation-preserving monodromy $\phi$. If there is a tight arc $\alpha$ so that the corresponding product disk $D(\alpha)$ has transition arcs of opposite sign, then there is a co-oriented taut foliation $\mathcal{F}_\gamma$ that strongly realizes slope $\gamma$ for all slopes except $\mu$, the distinguished meridian. The foliation $\mathcal{F}_\gamma$ has a unique minimal set, and this minimal set is genuine and disjoint from $\partial M$. Furthermore, each $\mathcal{F}_\gamma$ extends to a co-oriented taut foliation $\tilde{\mathcal{F}}_\gamma$ in $\tilde{X}(\gamma)$, the closed 3-manifold obtained by Dehn filling along $\gamma$, and when $\gamma$ intersects the meridian efficiently in at least two points, the minimal set of $\tilde{\mathcal{F}}_\gamma$ is genuine as well.

Proof Set $D = D(\alpha)$, $\mu_0 = \mu_0(\alpha)$, and $\mu_1 = \mu_1(\alpha)$. Choose a small annular neighbourhood of $\partial F$ in $F$, and let $F_0$ denote the complement of this annulus in $F$. We
may assume $\phi$ restricts to a homeomorphism of $F_0$; so $(F_0 \times I)/\phi$ is a codimension zero submanifold of $X$. Let $T$ denote the torus boundary of this submanifold.

Now consider the spine $T \cup F_0 \cup D$. Fix an arbitrary co-orientation on $F_0$, and choose the co-orientation on $D$ that results in the smoothing in the interior of $F_0$ that is indicated in Figure 30.

Now choose co-orientations on the components of $T_{|\mu_0 \cup \mu_1}$ in a neighbourhood of the spine about each transition so that a meridian cusp is introduced at each, as modelled in Figures 20 and 21. Since the transition arcs are of opposite sign, there is a compatible choice of co-orientation on the components of $T_{|\mu_0 \cup \mu_1}$ that agrees with these local choices. As illustrated in Figure 30, the complementary region of the resulting branched surface that does not contain $\partial X_\kappa$ is isomorphic as a sutured manifold to $B^G(F_0, D)$.

We thus obtain a branched surface $B$ with one complementary region homeomorphic to a $(F'_0 \times I, \partial F'_0 \times I)$, where $F'_0 = F_0|_{\alpha}$, and one complementary region homomorphic to $(\partial X_\kappa \times I, V_0 \cup V_1)$, where $V_0$ and $V_1$ are disjoint meridional annuli with cores $\mu_0$ and $\mu_1$, respectively. It is therefore essential. Apply the arguments of Section 6 to the splitting of $B$ guaranteed by Theorem 5.12 to see that $B$ splits to a laminar branched surface. The desired conclusions now follow as in our previous constructions. \qed
Figure 28: $B$ in a neighbourhood of a positive Type A transition.

References

[1] S. Boyer and A. Clay, *Foliations, orders, representations, L-spaces and graph manifolds*, Adv. Math. **310** (2017), 159–234.

[2] S. Boyer, C. Gordon and L. Watson, *On L-spaces and left-orderable fundamental groups*, Math. Ann. **356** (2013), no. 4, 1213–1245.

[3] G. Burde and H. Zieschang, *Knots*, De Gruyter Studies in Mathematics, 5. Walter de Gruyter and Co., Berlin, 2003.

[4] D. Calegari, *Leafwise smoothing laminations*, Algebr. Geom. Topol. **1** (2001), 579–585.

[5] B. Casler, *An imbedding theorem for connected 3-manifolds with boundary*, Proc. A.M.S. **16** (1965), 559–566.

[6] V. Colin, W. H. Kazez and R. Roberts, *Taut foliations*, 2016, ArXiv:1605.02007 (to appear in Comm. Anal. Geom.).

[7] C. Delman and R. Roberts, *Nontorus alternating knots are persistently foliar*, preprint.

[8] C. Delman and R. Roberts, *Persistently foliar Montesinos knots*, preprint.

[9] C. Delman and R. Roberts, *Montesinos knots satisfy the L-space knot conjecture*, preprint.

[10] C. Delman and R. Roberts, *Taut double-diamond replacements*, preprint.

[11] C. Delman and R. Roberts, *Modifying branched surfaces at the boundary*, in preparation.
Figure 29: The sutures of $B$ agree with those of $B^G$.

Figure 30: Introducing two meridian cusps.

[12] J. Etnyre and J. Van Horn-Morris, *Fibered Transverse Knots and the Bennequin Bound*, IMRN 2011 (2011), 1483–1509.

[13] D. Gabai, *The Murasugi sum is a natural geometric operation*, Contemp. Math. 20 (1983), 131-143.

[14] D. Gabai, *Foliations and the topology of 3-manifolds*, J. Differential Geom. 18 (1983), no. 3, 445–503.

[15] D. Gabai, *Foliations and genera of links*, Topology 23 (1984), 381–394.

[16] D. Gabai, *The Murasugi sum is a natural geometric operation II*, Contemp. Math. 44 (1985), 93–100.

[17] D. Gabai, *Detecting fibered links in $S^3$*, Comment. Math. Helvetici 61 (1986), 519–555.

[18] D. Gabai, *Genera of the arborescent links*, Memoirs of the AMS 59 (339) (1986), 1–98.

[19] D. Gabai, *Foliations and the topology of 3-manifolds. II*, J. Differential Geom. 26 (1987), no. 3, 461–478.

[20] D. Gabai, *Foliations and the topology of 3-manifolds. III*, J. Differential Geom. 26 (1987), no. 3, 479–536.

[21] D. Gabai, *Taut foliations and suspensions of $S^3$*, (1992).

[22] D. Gabai, *Problems in foliations and laminations*, Geometric Topology (ed. W. Kazez), Proceedings of the 1993 Georgia International Topology Conference; 2 (1997), 1–33.
[23] D. Gabai, *Essential laminations and Kneser normal form*, J. Diff. Geom. **53** (1999), 517–574.

[24] D. Gabai and W. Kazez, *Homotopy, Isotopy and Genuine Laminations of 3-Manifolds*, Geometric Topology, Vol 1, (W H Kazez Ed.) AMS/IP (1997) 123–138.

[25] D. Gabai and U. Oertel, *Essential Laminations in 3-Manifolds*, Ann. Math. **130**, (1989), 41–73.

[26] P. Ghiggini, *Knot Floer homology detects genus-one fibred knots*, Amer. J. Math. **130** (2008), no. 5, 1151–1169.

[27] A. Haefliger, *Variétés feuilletées*, Ann. Scuola Norm. Sup. Pisa (3) **16** (1962), 367–397.

[28] M. Hirasawa and K. Murasugi, *Genera and fibredness of Montesinos knots*, Pac. J. Math **225(1)** (2006), 53–83.

[29] K. Honda, W. Kazez and G. Matić, *Right-veering diffeomorphisms of compact surfaces with boundary*, Invent. Math. **169** (2007), 427–449.

[30] K. Honda, W. Kazez, G. Matić, *The contact invariant in sutured Floer homology*, Invent. Math. **176** (2009), no. 3, 637–676.

[31] A. Juhász, *Floer homology and surface decompositions*, Geom. Topol. **12** (2008), 299–350.

[32] A. Juhász, *The sutured Floer homology polytope*, Geom. Topol. **14** (2010), 1303–1354.

[33] A. Juhász, *A survey of Heegaard Floer homology*, New Ideas in Low Dimensional Topology, World Scientific (2014), 23–296.

[34] W. H. Kazez and R. Roberts, *Fractional Dehn twists in knot theory and contact topology*, Algebr. Geom. Topol. **13(6)**, (2013), 3603–3637.

[35] D. Krcatovich, *The reduced knot Floer complex*, Topol. Appl. **194** (2015) 171-201.

[36] T. Li, *Laminar branched surfaces in 3-manifolds*, Geom. Top. **6** (2002), 153–194.

[37] T. Li, *Boundary train tracks of laminar branched surfaces*, Topology and geometry of manifolds (Athens, GA, 2001), 269â€“285, Proc. Sympos. Pure Math., 71, Amer. Math. Soc., Providence, RI, 2003.

[38] Y. Ni, *Knot Floer homology detects fibred knots*, Invent. Math. **170** (2007), 577–608.

[39] Y. Ni, *Erratum: Knot Floer homology detects fibred knots*, Invent. Math. **170** (2009), no. 1, 235–238.

[40] U. Oertel, *Incompressible branched surfaces*, Invent. Math. **76** (1984), 385–410.

[41] U. Oertel, *Measured laminations in 3-manifolds*, Trans. A.M.S. **305** (1988), 531–573.

[42] P. Ozsváth and Z. Szabó, *Holomorphic disks and genus bounds*, Geom. Topol. **8** (2004), 311–334.

[43] P. Ozsváth and Z. Szabó, *Holomorphic disks and topological invariants for closed 3-manifolds*, Ann. Math. **159(3)**, (2004), 1027–1158.
[44] P. Ozsváth and Z. Szabó, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. of Math. (2), 159 (3) (2004), 1159–1245.

[45] P. Ozsváth and Z. Szabó, *On Heegaard diagrams and holomorphic disks*, European Congress of Mathematics, Eur. Math. Soc., Zürich, 2005, 769–781.

[46] P. Ozsváth and Z. Szabó, *Heegaard Floer homology and contact structures*, Duke Math. J. 129(1) (2005), 39–61.

[47] R. Roberts, *Taut foliations in punctured surface bundles. I*, Proc. London Math. Soc. (3) 82 (2001), no. 3, 747–768.

[48] R. Roberts, *Taut foliations in punctured surface bundles, II*, Proc. London Math. Soc. (3) 83 (2001), no. 2, 443–471.

[49] J. Stallings, *Constructions of fibered knots and links*, Proc. Symp. Pure Math. AMS 27 (1975), 315-319.

[50] R. Williams, *Expanding attractors*, Inst. Hautes Études Sci. Publ. Math. 43 (1974), 169–203.

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