Spectral Theory for $p$-adic Banach Representations and $p$-adic Quantum Theory

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Abstract

We give an explicit definition of the normality of a $p$-adic operator, and we establish several criteria for the normality. The formulation of the normality yields a $p$-adic model of Quantum Theory. We study the relation between the normality and the reduction. By the repetition of reductions, we partially generalise Vishik’s spectral theory. In the finite dimensional case, the normality corresponds to the diagonalisability of a matrix by a unitary matrix. Therefore our work contains a certain compatibility with the diagonalisability and the reduction. For example, we show that the diagonalisation of the reduction gives a partition of unity corresponding to the reductive spectrum. It decomposes the representation space into the direct sum of subrepresentations. This decomposition is a functorial lift of the eigenspace decomposition of the reduction.

Contents

0 Introduction 2
  0.1 Comparison to the Archimedean Analysis 6
  0.2 Relation with Physics 9

1 Basic Notions 11
  1.1 Seminormed Group 12
  1.2 Seminormed Ring 13
  1.3 Seminormed Module 16
  1.4 Valuation Field 17
  1.5 Banach Space 18
  1.6 Rings of Continuous Functions 19
  1.7 Strictly Cartesian Banach Space 24
  1.8 Operator Algebra 25

2 Three Reductions 26
  2.1 Normal Reduction 26
  2.2 Spectral Reduction 27
  2.3 Matrix Reduction 28
The main themes of this paper are to give an explicit definition of the normality of a \( p \)-adic operator and to establish criteria for the normality. Roughly speaking, a normal operator is an operator admitting a certain functional calculus. The topological difference between \( \mathbb{C} \) and a non-Archimedean field yields the difficulty even in a formulation of the definition of a non-Archimedean functional calculus. In the finite dimensional case, the normality of an operator corresponds to the diagonalisability of its matrix representation.
by a unitary matrix. In order to reduce the analytic problem to the purely algebraic one, we study a relation between reductions, spectra, and functional calculi. A functional calculus of an operator on the residue field is just the diagonalisation. As results, we establish criteria for the normality for a bounded operator using the reduction technique. We simply divide it into the following two cases:

One is the case that the spectrum of the reduction is an infinite set. Then its reduction is automatically transcendental, and it admits a holomorphic functional calculus. This rare case occurs only when we consider a sufficiently large field such as \( \mathbb{C}_p \), but a base change of an operator might cause such a situation. Namely, a base change of an operator often changes the spectra, and an operator over \( \mathbb{Q}_p \) sometimes becomes such an operator after extending the base field to \( \mathbb{C}_p \). The examples in §4.2 would help a reader to understand this phenomenon.

The other one is the case that the spectrum of the reduction is a finite set. Note that many of significant operators in Number Theory have this property. For example, the spectrum of the reduction of a bounded operator over a local field is a finite set because the residue field itself is a finite set. The spectrum of an operator on a finite dimensional Banach space is also a finite set, and coincides with the set of eigenvalues of its matrix representation in the base field. In this case, a single attempt of a reduction forces a bounded operator to forget much information. However, the diagonalisation of the reduction yields a partition of unity corresponding to the reductive spectrum. It decomposes the representation space into the direct sum of subrepresentations. This decomposition is a functorial lift of the eigenspace decomposition of the reduction. Thus if the reduction is diagonalisable, then the reduction data associates new operators.

We observe that the repetition of infinitely many reductions possess enough information that recovers the operator if it admits a functional calculus. Namely, do a reduction to an operator, deform it with projections derived from the reduction data if the reduction is diagonalisable, then do a reduction again, deform it, ..., and repeat the process as many times as possible. If this repetition runs without stopping, i.e. if all the reductions appearing the repetition process are diagonalisable, then the operator turns out to be normal. Thus we give a criterion for the normality of such a bounded operator using the repeating process of reductions.

For example, in the finite dimensional case, this process yields an algorithm for the criterion of the normality of a matrix. Therefore the unitary diagonalisation of a matrix is reduced to the diagonalisation of finitely many matrices on the residue field. Unfortunately in the infinite dimensional case, the repetition process is a little difficult. If one desires the possibility of the calculation of it by hands, an operator should be assumed to admit a fractal structure in some sense. Imagine the case that the repetition does not change the type of an operator. In such a case, the repetition is successful, and the criterion works without concrete calculations. The examples in §5.4 might be helpful.

A functional calculus is the substitution of an operator to a function on its spectrum. For example, the holomorphic functional calculus is the substitution to a rigid analytic function, and the continuous functional calculus is the substitution to a continuous func-
tion. They are the extensions of the substitution to a polynomial function. In order to formulate the normality of a non-Archimedean operator, one has to consider how to justify functional calculi. One of the most basic methods is the use of the spectral measure. However, the proof of the existence of the spectral measure of an Archimedean normal operator heavily relies on the Archimedean analyses: the regularity of the resolvent operator, the harmonic analysis, the Stieltjes integral of a bounded variation function, the existence of a monotonously increasing sequence of continuous functions pointwise converging to a characteristic function, Riesz representation theorem, and so on. These elementary analyses are based on the fundamental properties of the topology of $\mathbb{R}$: the existences of the notions of convexity, positivity, arcwise-connectedness, and so on.

To tell the truth, a formulation of the spectral decomposition of a bounded analytic operator with compact spectrum has been already given by Vishik in his paper [Vis], but it just yields kind of the locally analytic functional calculus analogous to Riesz functional calculus. Assigning some suitable conditions to an operator, there are direct extensions of the result of Vishik by Richard Lance Baker in [Bak] and by Dodzi Attimu and Toka Diagana in [AD]. The direction of their works is completely different from that of ours. We would like to remove the assumptions that an operator is analytic and that its spectrum is compact. Moreover we would like to extend it to the continuous functional calculus. Furthermore we would like to generalise it to an unbounded operator. These non-trivial generalisation can not be obtained if one extends Vishik’s spectral theory directly. Indeed, the functional calculus in his work uses Shnirel’man integral, which is a non-Archimedean analogue of Cauchy integral of an analytic function introduced in [Shn]. The assumptions that an operator is analytic and that its spectrum is compact are essential there. The assumption of the boundedness is also necessary there, because a non-Archimedean Banach space is not necessarily reflective. One of the reasons why the spectral measure works well for the continuous functional calculus of an unbounded operator in the Archimedean case is because an Archimedean Hilbert space is reflective. The equality

$$h(f(A)g) = \int_{\sigma(A)} f(\lambda) d(h(E_A(\lambda)g)) \in k$$

for each $g \in A$ and $h \in V^\vee$ might determine an operator $f(A) : V \to V^{\vee\vee}$, but it is difficult to calculate the domain of it when its codomain is restricted on the weak* dense subspace $V \subset V^{\vee\vee}$. If one desires a good duality for Banach spaces, it would be better to restrict the class of base fields to spherically complete fields. Remark that an infinite dimensional strictly Cartesian Banach space is never reflective because the natural multiplication $C_{\text{bd}}(I, k) \otimes_k C_0(I, k) \to C_0(I, k)$ gives the inclusions $C_0(I, k) \subset C_{\text{bd}}(I, k) \subset C_0(I, k)^\vee$ for an infinite index set $I$. The first inclusion is obviously not surjective. Thus in order to removes the assumptions that the operator is analytic, that the spectrum is compact, and that the operator is bounded, we give up making use of the integration by Vishik’s spectral measure. Instead, we study the structures of rings of continuous functions and operator algebras. The reduction technique allows us to calculate homomorphisms between them.

Beware that the term “continuous functional calculus” is a little ambiguous in the
non-Archimedes situation in the following sense. To begin with in the Archimedean case, the base field \( \mathbb{C} \) is a proper metric space. The spectrum \( \sigma \) of a bounded operator is a bounded closed subset of \( \mathbb{C} \), and hence is compact. Therefore the rings \( C_0(\sigma, \mathbb{C}) \subset C_{bd}(\sigma, \mathbb{C}) \subset C(\sigma, \mathbb{C}) \) coincide with each other. On the other hand in the non-Archimedean case, the base field \( k \) is not necessarily proper as a metric space or even locally compact as a topological space. The spectrum \( \sigma \) of a \( k \)-valued bounded operator is just a bounded closed subset of \( k \), and hence is not compact in general. Therefore the rings \( C_0(\sigma, k) \subset C_{bd}(\sigma, k) \subset C(\sigma, k) \) may differ from each other. The ambiguity occurs in the choice of the ring for the domain of the continuous functional calculus. The first ring \( C_0(\sigma, k) \) is a non-unital Banach \( k \)-algebra, and does not contains the coordinate function \( z : \sigma \to k \) when \( \sigma \) is non-compact. The second ring \( C_{bd}(\sigma, k) \) is a Banach \( k \)-algebra contains the coordinate function \( z \) but is so large that the localisation of \( k[z] \subset C_{bd}(\sigma, k) \) by functions in \( k[z] \cap C_{bd}(\sigma, k) \) is not dense in general. The third ring \( C(\sigma, k) \) is too large and does not admits a canonical norm compatible with the supremum norm of \( C_{bd}(\sigma, k) \). The density of the localisation of \( k[z] \) is essential for the uniqueness of the continuous functional calculus, and hence we will use the other appropriate closed \( k \)-subalgebra densely containing the localisation of \( k[z] \), namely, the closure \( C_{rd}(\sigma, k) \) of the localisation of \( k[z] \) in \( C_{bd}(\sigma, k) \). The continuous functional calculus for an unbounded non-Archimedean operator is still more complicated because a unbounded closed subset of \( k \) is not \( \sigma \)-compact, while an unbounded closed subset of \( \mathbb{C} \) is countable at infinity.

In §1 we recall basic notions and conventions in the non-Archimedean analysis. This chapter contains no new result. Therefore a reader might skip it if he or she is familiar with the non-Archimedean analysis after examining Definition .

In §2 we recall the notions of three reductions and compare them with each other. The first one of them is the reduction of a Banach space with respect to the norm. It is the quotient of the closed unit ball by the open unit ball, and we call it the normal reduction. The normal reduction is not stable under the change of equivalent norms. The second one is the reduction of a commutative Banach algebra with respect to the uniform structure. It is the quotient of the subring of bounded elements by the ideal of topologically nilpotent elements, and we call it the spectral reduction. The spectral reduction is stable under the change of equivalent norms. In particular if the Banach algebra is uniform, the normal reduction coincides with the spectral reduction. The third one is the reduction of a matrix. It is the reduction of each entry, and we call it the matrix reduction. We deal not only with a finite dimensional matrix but also with an infinite dimensional matrix. For an operator, we consider the matrix reduction of its matrix representation with respect to a basis. The matrix reduction depends on the choice of a basis. In particular, for a finite dimensional matrix, its matrix reduction changes if one replace it by the matrix representation of the same linear operator with respect to another basis. In this reason, we always endow \( k^n \) with the canonical basis and the canonical norm given by the basis.

In §3 we formulate the unitarity and the normality of an operator, and we observe the relation between the normality and the diagonalisability by a unitary matrix. A unitary operator is an isometry, and a normal operator is an operator admitting a suitable func-
tional calculus. We introduce the class of a rigid continuous function for the functional calculus. Here we say a function to be rigid continuous if it is uniformly approximated by a rational function. Such a function is sometimes called a Krasner analytic function, but the term “Krasner analytic” is so widely used that ambiguity exists. The class of rigid continuous functions is more rigid than that of general continuous functions, but is much more naive than that of locally analytic functions. For example, a continuous function on a closed subset in a local field is always rigid continuous. The difference appears when we consider $C_p$. An operator $A$ admits the rigid continuous functional calculus when the Banach algebra $\mathcal{L}(A)$ generated by $A$ possesses properties similar with the Banach algebra of rigid continuous functions on the spectrum of $A$. In particular, studying the properties of such a Banach algebra helps one to predict a sufficient condition for the normality. To begin with, we observe the necessary conditions for the normality in Proposition 3.11, Proposition 3.25, and Proposition 5.18, relating to the reduction of an operator which we call the normal reduction. They contain the compatibility of the reduction and the functional calculus. Adding some conditions to them, we obtain our main results in the latter sections.

In §4, we deal with a bounded operator such that the spectrum of its normal reduction is an infinite set. We verify that $A$ always admits the holomorphic functional calculus in Theorem 4.2. Moreover, the Banach algebra $\mathcal{L}(A)$ is isometrically isomorphic to the Tate algebra, and hence the holomorphic functional calculus is the finest functional calculus whose image is contained in $\mathcal{L}(A)$. In special cases, a holomorphic functional calculus is the finest continuous functional calculus, and there is no class of more naive continuous functions that rigid analytic functions appropriate for a functional calculus. The assumption that the reductive spectrum is an infinite set is strong, but the base change of an operator with the finite reductive spectrum sometimes become such an operator. The simplest examples are the shift operator and the Toeplitz operator over $\mathbb{Q}_p$.

In §5, we deal with a bounded operator such that the spectrum of its normal reduction is a finite set. We apply to such an operator $A$ the repetition of the reduction process. In the finite dimensional case, we show that this process yields an algorithm for the criterion of the normality of a matrix in Theorem 5.7. In the infinite dimensional case, this process should be repeated in infinitely many times. If the repetition runs without stopping, we say that $A$ is naive. Suppose the case that the base field $k$ is a local field. Then since a continuous function is automatically rigid continuous, the rigid continuous functional calculus is the continuous functional calculus. We verify that $A$ admits the continuous functional calculus if and only if $A$ is naive in Theorem 5.20.

### 0.1 Comparison to the Archimedean Analysis

Here we briefly show how to define the normality of a $p$-adic operator. Before that, let us recall the normality of an operator of an Archimedean Hilbert space, and compare the properties of an Archimedean operator and a non-Archimedean operator. Specific examples of non-Archimedean operators in Appendix might help a reader to grasp the common features and the differences between them.
In the Archimedean case, the most basic examples of normal operators are a self-adjoint operator, a skew-self-adjoint operator, and a unitary operator. Conversely since the involution is diagonalisable as a $\mathbb{C}$-linear automorphism, a normal operator is uniquely decomposed into the sum $T + S$ of a self-adjoint operator $T$ and a skew-self-adjoint operator $S$ commuting with each other. A skew-self-adjoint operator is expressed as an operator of the form $iT'$ by the unique self-adjoint operator $T'$. Therefore self-adjoint operators form one of the most essential class of normal operators, and let us think about non-Archimedean counterparts of Hermitian matrices first.

A real symmetric matrix is a complex Hermitian matrix, and hence the simplest alternative might be a $p$-adic symmetric matrix. However, we have an example of a $p$-adic symmetric matrix which is not diagonalisable. See Proposition 7.3. There is another non-Archimedean analogous notion of a Hermitian matrix. The diagonalisability of an Archimedean normal matrix is easily verified by calculating the canonical inner products of its eigenvectors, because the adjoint property of the involution with respect to the inner product directly implies the complete reducibility of the linear representation associated to it. Similarly, if a non-Archimedean vector space has a good inner product, the adjoint property determines an involution. An invariant matrix of the non-Archimedean non-canonical involution behaves like a Hermitian matrix, and is diagonalisable by a unitary matrix in many cases. See Definition 7.4, Proposition 7.5, Definition 7.6, and Proposition 7.7. The last one gives the direct application to the $p$-adic Pauli matrices.

Example 0.1. Suppose $p \equiv 3 \pmod{4}$ and $\bar{k} = \overline{\mathbb{F}_p}$. Let $K/k$ be the unramified Galois extension $k[X]/(X^2 + 1)$ and $i \in K$ the image of $X \in k[X]$ by the canonical projection $k[X] \to K$. Set

\[
\sigma_x := \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \sigma_y := \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \sigma_z := \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \in M_2(K)
\]

and call them the $p$-adic counterparts of the Pauli matrices. Then the $p$-adic Pauli matrices are diagonalisable by distinct unitary matrices in $K$.

Note that even if $p \equiv 1 \pmod{4}$, the $p$-adic Pauli matrices are defined in the same way replacing $K$ by $k$ itself and fixing a square root $i$ of $-1$ in $k$, and are diagonalisable by distinct unitary matrices in $k$. However if $p = 2$, obviously the $p$-adic Pauli matrices do not work.

The definition of the normality of an operator on an Archimedean Hilbert space uses the notion of the involution. On the other hand, there seems to be no canonical involution on the non-Archimedean base field $k$, and hence a similar formulation of the normality does not works in the $p$-adic case. An involution has a strong relation to a formally real field such as a hyperreal field in the non-standard analysis, but a $p$-adic field is never formally real. The non-Archimedean properties of an order and a valuation conflict with each other in the mixed characteristic case. To tell the truth, there is a non-Archimedean ordered non-Archimedean valued field with characteristic $(0,0)$ such as Levi-Civita field, which is a formally real extension of the real Puiseux series field in
the Archimedean Novikov field, but it is not useful for a “p-adic”. Due to the absence of a canonical involution, there are no counterparts of reality and positivity in the p-adic analysis. In particular, a symmetric inner product on a k-vector space does not give a norm in general. Example 7.1 might help a reader to understand what happens. Note that we do not mean that a symmetric inner product on a vector space never gives a norm, and indeed, in rare cases such phenomena occur. There is an example in Appendix. We give an explicit definition of an inner product in a lo dimensional case in Proposition 7.2. Such a construction of an inner product which generates a norm does not work in the case of higher dimensional k-Banach spaces than 4 when k is a local field. In particular the p-adic Hilbert space 

\[ L^2(I, k, \omega) := \left\{ f := (f_i)_{i \in I} \in k^I \left| \lim_{i \in I} |\omega_i f_i|^2 = 0 \right. \right\} \]

associated to a countable index set I and a weight \( \omega = (\omega_i)_{i \in I} \in k^I \) is equipped with the canonical norm

\[ \| \cdot \|_\omega : L^2(I, k, \omega) \rightarrow [0, \infty) \]

\[ f \mapsto \|f\|_\omega := \sup_{i \in I} |\omega_i f_i|^2 \]

and the canonical symmetric inner product

\[ \langle \cdot | \cdot \rangle_\omega : \mathcal{L}^2(I, k, \omega) \otimes_k \mathcal{L}^2(I, k, \omega) \rightarrow k \]

\[ f \otimes g \mapsto \langle f | g \rangle_\omega := \sum_{i \in I} \omega_i f_i g_i \]

which are not compatible if \#I > 4.

Moreover, the absence of the canonical involution causes the difficulty of defining the normality or the self-adjointness of an operator. Remark that the parallelogram law

\[ \|f + g\| + \|f - g\| = 2(\|f\| + \|g\|) \]

never holds for a non-trivial non-Archimedean Banach space, and the law of cosines

\[ \langle f | g \rangle = \|f\| + \|g\| - \|f + g\| \]

does not work for a non-trivial non-Archimedean inner product space. Thus non-Archimedean Hilbert spaces are far from Archimedean Hilbert spaces, and there is few merit to consider a non-Archimedean inner product. We mean that it is better to forget the inner product when one considers an analogue of a normal operator. In fact, a non-Archimedean Banach space is more similar with the underlying Archimedean Banach space of an Archimedean Hilbert space than a general Archimedean Banach space in some aspects. There is a good class of non-Archimedean Banach spaces, namely, the class of strictly Cartesian Banach spaces. A strictly Cartesian Banach space is a non-Archimedean Banach space admitting an orthonormal Schauder basis. Furthermore, an isometric automorphism on a non-Archimedean Banach space behaves as a unitary transformation on
an Archimedean Hilbert space. We will simply define a non-Archimedean unitary operator as an isometric automorphism in \( \mathbb{S} \). Thus one might not stick to a \( p \)-adic Hilbert space for the formulation of the normality of an operator, and a strictly Cartesian \( p \)-adic Banach space might sufficiently stand in for it. Note that a strictly Cartesian Banach space is non-canonically isometric to the Banach space of the form \( C_0(I, k) \) for some countable discrete topological space \( I \) by definition. Since \( C_0(I, k) \) coincides with \( \mathcal{L}^2(I, k, 1) \), it admits a non-canonical inner product. We will no longer see such an inner product and will forget a structure of a Hilbert space.

Once one has defined the unitarity of an operator, the normality will be soon formulated in the finite dimensional case. A unitary matrix is a matrix whose natural representation is a unitary operator with respect to the canonical norm associated to the canonical basis. One does not have to be puzzled over the absence of the canonical involution now. The non-Archimedean counterpart of the normality of a matrix is the diagonalisability by a unitary matrix. The normality in the infinite dimensional case is a little more complicated. The normality of an Archimedean operator \( A \in \mathcal{B}_C(V) \) on an Archimedean Hilbert space \( V \) guarantees the existence of the spectral measure \( dE_A \) on the spectrum \( \sigma_{\mathcal{B}_C(V)}(A) \) of \( A \) which gives the spectral decomposition of \( A \). Suppose \( A \) is bounded and hence its spectrum is compact for convenience. The spectral measure induces the continuous functional calculus

\[
\iota_A : C(\sigma_{\mathcal{B}_C(V)}(A), \mathbb{C}) \rightarrow \mathcal{B}_C(V),
\]

namely, the unique isometric \( \mathbb{C} \)-algebra isomorphism sending the coordinate function \( z : \sigma_{\mathcal{B}_C(V)}(A) \rightarrow \mathbb{C} \) to the operator \( A \). We extract this property, and we define the normality of an operator on a \( p \)-adic Banach space as the existence of the continuous functional calculus.

### 0.2 Relation with Physics

The non-Archimedean analysis sometimes appears in the modern Physics. For example, the \( p \)-adic analysis is heavily used in the \( p \)-adic Quantum Mechanics, and the the adelic analysis is essential in Feynman’s adelic path integral. See [DD], [DR], [Dra], and [VV] for details. People are usually accustomed to the Euclidean geometry because our world is somehow built on a topological space locally isomorphic to an affine space \( \mathbb{R}^n \), but there is no reason one has to restrict all topologies which arise in physics to the Archimedean ones. A physical quantity \( T_1 \) might just be the real part of some complex physical quantity \( T_1 + iT_2 \), and the imaginary part \( T_2 \) might affect other physical quantity even though there is no method to observe it directly. The same might be true for an adelic number. There might be a physical quantity which takes values \( a = (a_\infty, a_2, a_3, a_5, \ldots, a_p, \ldots) \) in the adele

\[
A_\mathbb{Q} := \left\{ a \in \mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3 \times \mathbb{Q}_5 \times \cdots \times \mathbb{Q}_p \times \cdots \mid \# \left\{ p \in \mathbb{N} \mid p \text{ is prime}, |a_p|_p > 1 \right\} < \infty \right\}
\]

and only whose real entry \( a_\infty \in \mathbb{R} \) people can observe. In particular in Quantum Mechanics, an ideal observation always acts on the observee, and it forces to change the physical
state from the previous one according to the projection postulate. Therefore the fact that people can only observe real-valued ones does not justify the assumption of the reality of all physical quantities. Thus the $p$-adic analogue of theories in physics itself is significant, and constructions of $p$-adic models of Mechanics are meaningful. We give one of the explicit formulations of the $p$-adic Quantum Theory in Definition 3.20, Definition 3.21, Definition 3.22, and Definition 6.25.

In the Schrödinger picture of the Archimedean Quantum Theory, physical states and observables are expressed as state vectors in a Hilbert space evolving in time and constant self-adjoint operators on it. In finite dimensional cases, a self-adjoint operator and a normal operator corresponds to a Hermitian matrix and a normal matrix respectively. The reason why an observable should be expressed by a self-adjoint operator rather than a general operator in a wider class is very simple. A closable operator $T$ admits the spectral measure $dE_T$ which gives the spectral decomposition

$$T = \int_{\sigma(T)} \lambda dE_T(\lambda)$$

if and only if it is normal, and the spectrum of a normal operator is contained in $\mathbb{R} \subset \mathbb{C}$ if and only if it is self-adjoint. In finite dimensional cases, these properties are translated as the diagonalisability of a normal matrix by a unitary matrix and the reality of the eigenvalues of a Hermitian matrix respectively. An eigenvector of a fixed observable is an eigenstate, namely, a state vector in which the observed value of it is determined, and the diagonalisability by a unitary matrix guarantees that a vector state can be uniquely expressed as a linear combination of normalised orthogonal eigenstates. Thus the self-adjointness indicates the reality of observed values of an observable, the principle of the quantum superposition, and Born rule.

On the other hand, consider the non-Archimedean analogue of Quantum Theory. A non-Archimedean physical quantity should be expressed as a non-Archimedean normal operator in a certain class, and therefore the formulation of and the criterion for the normality of a non-Archimedean operator themselves are significant. A value of a non-Archimedean physical quantity does not have to be “real”, and instead one might assume the spectrum of the corresponding operator is contained in the base field $k$. We call such an operator a $k$-valued operator. Recall that a Hermitian matrix has a Jordan normal form concentrated at the diagonal with real-valued entries. Similarly a $k$-valued normal matrix has a Jordan normal form concentrated at the diagonal with $k$-valued entries. As we observed in the previous section, the non-Archimedean counterpart of a normal operator on an Archimedean Hilbert space is a normal operator on a non-Archimedean Banach space, and a self-adjoint operator on an Archimedean Hilbert space corresponds to a $k$-valued normal operator on a non-Archimedean Banach space.

Although the structure of a Hilbert space is significant when we consider Born rule in the Archimedean Quantum Theory, it seems to be doubtful that the analogue of Born rule surely makes sense in a $p$-adic model of Quantum Theory if we formulated a non-Archimedean physical quantity as an operator on a non-Archimedean Hilbert space. For
example, Born rule for an Archimedean observable $A$ with discrete intersection $U \cap \sigma_{C(V)}(A) \subset \mathbb{R}$ is given as

$$\mathcal{P}_{\psi}(A \in U) = \sum_{\lambda \in U} |\langle \psi \mid P_A(\lambda) \mid \psi \rangle| = \sum_{\lambda \in U} \|P_A(\lambda)\psi\|^2 = \left\|\sum_{\lambda \in U} P_A(\lambda)\psi\right\|^2 \in [0, \infty).$$

The analogous equalities

$$\mathcal{P}_{\psi}(A \in U) = \begin{cases} \sum_{\lambda \in U} \langle \psi \mid P_A(\lambda) \mid \psi \rangle & \in k \\ \sum_{\lambda \in U} |\langle \psi \mid P_A(\lambda) \mid \psi \rangle| & \in [0, \infty) \\ \sum_{\lambda \in U} \|P_A(\lambda)\psi\| & \in [0, \infty) \\ \sum_{\lambda \in U} \|P_A(\lambda)\psi\|^2 & \in [0, \infty) \\ \max_{\lambda \in U} \|P_A(\lambda)\psi\| & \in [0, \infty) \\ \left\|\sum_{\lambda \in U} P_A(\lambda)\psi\right\|^2 & \in [0, \infty) \end{cases}$$

for a non-Archimedean operator $A$ with discrete intersection $U \cap \sigma_{C(V)}(A) \subset k$ have distinct meanings and totally differ from each other. Furthermore when one considers the continuous spectrum, he or she has to puzzle over what kind of $p$-adic integrations defined extracting several properties of the Archimedean integration, but of course none of them possesses all the desired properties. The structure of a Hilbert space is no longer useful here, and therefore we define a $p$-adic physical quantity as a $k$-valued normal operator on a strictly Cartesian $p$-adic Banach space in Definition 3.21 and Definition 6.25. Thus there is no longer a reasonable justification for the assumption that a non-Archimedean physical quantity is expressed as an operator on a non-Archimedean Hilbert space.

Remark that the self-adjointness of an Archimedean operator guarantees the existence of the Borel functional calculus, which is stronger than the continuous functional calculus of a normal operator, but we do not use this property for the definition of a $p$-adic observable because there is few differences between the class of $p$-adic Borel-measurable functions and the class of $p$-adic continuous functions. Recall that a non-Archimedean field is totally disconnected and hence the ring of $p$-adic continuous functions on a topological space possesses enough projections. Thus a commutative uniform Banach $k$-algebra isometrically isomorphic to the ring of continuous functions on a totally disconnected compact Hausdorff space is much more similar with an Archimedean commutative von Neumann algebra than a general Archimedean commutative $C^*$-algebra. See examples and propositions in §1 such as Example 1.34 for the properties of non-Archimedean commutative $C^*$-algebras.

## 1 Basic Notions

In this chapter, we recall the basic notions and conventions for the non-Archimedean analysis. We follow the conventions in [BGR], except that we deal also with non-
commutative Banach algebras. As important examples of Banach algebras, we introduce rings of continuous functions and an operator algebra. Banach spaces of continuous functions vanishing at infinity on index sets yield the notion of a strictly Cartesian Banach space. This chapter contains nothing new, and hence a reader might skip this chapter if he or she is familiar with the non-Archimedean analysis after examining conventions in Definition [1.33] and Definition [1.40].

1.1 Seminormed Group

We observe a seminorm on an Abelian group. We mainly consider the case that a seminorm satisfies the non-Archimedean property. In fact the condition that the underlying group is Abelian is not essential. The reason why we assume it is because we always consider the case that $G$ is the underlying additive group of a ring.

Definition 1.1. Let $G$ be an Abelian group. A seminorm on $G$ is a set-theoretical map $\| \cdot \| : G \rightarrow [0, \infty)$: $g \mapsto \|g\|$ satisfying the following:

(i) $\|0\| = 0$; and

(ii) $\|g - h\| \leq \|g\| + \|h\|$ for any $g, h \in G$.

A seminorm is said to be a norm if it satisfies the additional condition

(i)' $\|g\| = 0$ if and only if $g = 0$,

which is stronger than (i). A seminorm is said to be non-Archimedean if it satisfies the additional condition

(ii)' $\|g - h\| \leq \max\{\|g\|, \|h\|\}$ for any $g, h \in G$,

which is stronger than the condition (ii).

A seminorm naturally induces a pseudo-metric. Similarly a norm (or a non-Archimedean norm) induces a metric (resp. an ultrametric). If a seminorm of a group is given, the addition and the inverse are continuous with respect to the pseudo-metric. Therefore a seminorm gives $G$ a structure of a topological group.

Example 1.2. Let $G$ be an Abelian group. The trivial norm on $G$ is the norm $\| \cdot \| : G \rightarrow [0, \infty)$ determined by the additional condition

(i)'' $\|g\| = 1$ for any $g \in G \setminus \{0\}$ and $\|0\| = 0$,

which is stronger than the conditions (i) and (ii). The trivial norm is one of the most important examples of non-Archimedean norms.
Definition 1.3. A seminormed Abelian group is data \((G, \| \cdot \|)\) of an Abelian group \(G\) with a fixed seminorm \(\| \cdot \|: G \to [0, \infty)\). The Abelian group \(G\) and the seminorm \(\| \cdot \|\) are called the underlying group and the seminorm of \((G, \| \cdot \|)\). We usually write \(G\) instead of \((G, \| \cdot \|)\) for short. A seminormed Abelian group is said to be a normed Abelian group (or non-Archimedean) if its seminorm is a norm (resp. non-Archimedean). For a normed Abelian group, we endow its underlying group with the metric topology given by the norm. A normed Abelian group is said to be complete if its underlying metric space is complete.

Definition 1.4. Let \((G, \| \cdot \|)\) be a seminormed Abelian group. For an \(r > 0\), set
\[
G(r) := \{ g \in G \mid \|g\| \leq r \}
\]
\[
G(r-) := \{ g \in G \mid \|g\| < r \} .
\]
If \((G, \| \cdot \|)\) is non-Archimedean, both \(G(r)\) and \(G(r-)\) are clopen subgroups of \(G\), and \((G(r), \| \cdot \|)\) and \((G(r-), \| \cdot \|)\) are seminormed groups.

Definition 1.5. Let \(G\) and \(H\) be seminormed Abelian groups. A group homomorphism \(f: G \to H\) is said to be bounded if \(f\) satisfies \(\sup_{g \in G(1)} \|f(g)\| < \infty\).

Remark 1.6. A bounded group homomorphism is continuous.

The ultrametric has much better properties than Euclidean metric in the analysis. For example, the convergence of a sequence is much more simplified. For a complete ultrametric space \((X, d)\), a sequence \((x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}\) converges if and only if \(\lim_{n \to \infty} d(x_n, x_{n+1}) = 0\). In particular for a complete non-Archimedean normed Abelian group \(G\), a sequence \((g_n)_{n \in \mathbb{N}} \in G^{\mathbb{N}}\) (or a series \(\sum_{n \in \mathbb{N}} g_n\)) converges if and only if \(\lim_{n \to \infty} \|g_n - g_{n+1}\| = 0\) (resp. \(\lim_{n \to \infty} \|g_n\| = 0\)). Note that this criterion for non-Archimedean convergences implies that the classes \(L^1\), \(L^2\), and \(L^\infty\) of sequences in a non-Archimedean normed Abelian group coincide with each other.

1.2 Seminormed Ring

Throughout this paper, we assume that a ring is unital and associative. We do not assume the commutativity or the inequality \(1 \neq 0\). Beware that we will assume that a field satisfies \(1 \neq 0\) in the latter section.

Definition 1.7. Let \(\mathcal{A}\) be a ring. A seminorm on \(\mathcal{A}\) is a seminorm \(\| \cdot \|: \mathcal{A} \to [0, \infty)\) of the underlying additive group of \(\mathcal{A}\) satisfying the following:

(iii) \(\|1\| \leq 1\); and

(iv) \(\|ab\| \leq \|a\| \|b\|\) for any \(a, b \in \mathcal{A}\).

Note that the condition (iii) and (iv) guarantees \(\|1\| \in \{0, 1\}\). A seminorm is said to be power-multiplicative if it satisfies the additional condition

(iv)' \(\|a^n\| = \|a\|^n\) for any \(a \in \mathcal{A}\) and \(n \in \mathbb{N}_+\),
and is said to be multiplicative if it satisfies the additional condition

\[(iv)'' \quad \|ab\| = \|a\| \|b\| \text{ for any } a, b \in \mathcal{A},\]

which is stronger than the conditions (iv) and (iv)'.

If a seminorm of a ring is given, the addition and the multiplication of a ring are continuous with respect to the induced pseudo-metric topology. Therefore a seminorm gives a ring a topology with respect to which it is a topological ring.

**Example 1.8.** For a commutative ring \(\mathcal{A}\), there are two canonical non-Archimedean seminorms, which are not necessarily distinct. One is the seminorm determined by the additional condition

\[(iii)' \quad \|a\| = 1 \text{ for any non-nilpotent element } a \in \mathcal{A} \text{ and } \|a\| = 0 \text{ for any nilpotent } a \in \mathcal{A},\]

which is stronger than the condition (iii), and call it the trivial seminorm of \(\mathcal{A}\). The other one is the norm determined by the additional condition

\[(iii)'' \quad \|a\| = 1 \text{ for any non-zero element } a \in \mathcal{A},\]

which is stronger than the condition (iii), and call it the trivial norm of \(\mathcal{A}\). The condition (i) + (iii)” is equivalent to (i)”, and hence the trivial norm of a ring coincides with the trivial norm of an Abelian group. Therefore one did not have to assume the commutativity of \(\mathcal{A}\) in the definition of the trivial norm.

The adjective “trivial” is a little ambiguous. In this paper, we only use the term for the triviality of a seminorm or a norm defined above. Instead, we call the trivial ring \(\mathcal{O}\) the zero ring. The adjectives “zero” and “non-zero” are substituted for the adjectives “trivial” and “non-trivial” in the latter sense.

**Remark 1.9.** The trivial seminorm is power-multiplicative. The trivial seminorm is multiplicative if the ring has precisely one minimal prime ideal. The trivial norm is power-multiplicative if and only if the ring is semisimple, i.e. the nilradical is the zero ideal, and then it coincides with the trivial seminorm. The trivial norm is multiplicative if and only if the ring is an integral domain or the zero ring. Moreover, a ring has a power-multiplicative norm if and only if it is semisimple, and a ring has a multiplicative norm if and only if it is an integral domain or the zero ring.

**Definition 1.10.** A seminormed ring is data \((\mathcal{A}, \| \cdot \|)\) of a ring \(\mathcal{A}\) with a fixed seminorm \(\| \cdot \| : \mathcal{A} \to [0, \infty)\). The ring \(\mathcal{A}\) and the seminorm \(\| \cdot \|\) are called the underlying ring and the seminorm of \((\mathcal{A}, \| \cdot \|)\). We usually write \(\mathcal{A}\) instead of \((\mathcal{A}, \| \cdot \|)\) for short. A seminormed ring is said to be a normed ring if its seminorm is a norm. We endow the underlying ring of a normed ring with the metric topology given by the norm. A normed ring is said to be complete if its underlying metric space is complete. A seminormed ring is said to be uniform if its seminorm is power-multiplicative. A seminormed ring is said to be a trivial seminormed ring if the underlying ring is commutative and its seminorm is the trivial seminorm of the underlying ring. A normed ring is said to be a trivial normed ring if its norm is the trivial norm of the underlying ring.
Example 1.11. The zero ring $O$ has a unique seminorm $\| \cdot \| : O \to [0, \infty)$, which sends 0 to 0. Then one has $\|1\| = \|0\| = 0$, and $(O, \| \cdot \|)$ is a trivial normed ring whose norm is non-Archimedean and multiplicative. Moreover, the property $\|1\| \in [0, 1)$ guarantees that a normed ring satisfies $\|1\| = 1$ if and only if its underlying ring is non-zero.

Remark 1.12. A trivial normed ring is endowed with the discrete uniform topology, and hence is complete.

The correspondence from a commutative ring to the associated trivial seminormed ring determines a functor from the category of commutative rings to the category of uniform seminormed commutative rings. It is the left adjoint functor of the forgetful functor. The correspondence from a ring to the associated trivial normed ring determines a functor from the category of rings to the category of normed rings. It is the left adjoint functor of the forgetful functor.

Definition 1.13. Let $(\mathcal{A}, \| \cdot \|)$ be a normed ring. A subset $S \subset \mathcal{A}$ is said to be bounded if $S$ is a bounded subset of $\mathcal{A}$ as a metric space. An element $a \in \mathcal{A}$ is said to be bounded if the multiplicative subset $\{ a^n \mid n \in \mathbb{N} \} \subset \mathcal{A}$ generated by $a$ is bounded. Denote by $\mathcal{A}^o \subset \mathcal{A}$ the subset of bounded elements. An element $a \in \mathcal{A}$ is said to be topologically nilpotent if $\lim_{n \to \infty} a^n = 0$, or equivalently $\lim_{n \to \infty} \|a^n\| = 0$. Denote by $\mathcal{A}^{oo} \subset \mathcal{A}$ the subset of topologically nilpotent elements.

Suppose $(\mathcal{A}, \| \cdot \|)$ is a non-Archimedean seminormed ring. Then $\mathcal{A}(1) \subset \mathcal{A}$ is a clopen subring, and $\mathcal{A}(r-) \subset \mathcal{A}(1)$ are clopen two-sided ideals for any $r \in (0, 1]$. The data $(\mathcal{A}(1), \| \cdot \|_{\mathcal{A}(1)})$ is also a non-Archimedean seminormed ring. On the other hand if $(\mathcal{A}, \| \cdot \|)$ is a non-Archimedean normed commutative ring, then $\mathcal{A}^o$ is a clopen subring of $\mathcal{A}$ and $\mathcal{A}^{oo}$ is a clopen ideal of $\mathcal{A}^o$. The data $(\mathcal{A}^o, \| \cdot \|_{\mathcal{A}^o})$ is a non-Archimedean normed commutative ring. Regardless of the commutativity of $\mathcal{A}$, if $| \cdot |_{\mathcal{A}}$ is a norm, then the inclusion relations $\mathcal{A}(1) \subset \mathcal{A}^o$ and $\mathcal{A}(1-) \subset \mathcal{A}^{oo}$ hold. If in addition $\| \cdot \|_{\mathcal{A}}$ is power-multiplicative, then the equalities $\mathcal{A}(1) = \mathcal{A}^o$ and $\mathcal{A}(1-) = \mathcal{A}^{oo}$ hold.

Proposition 1.14. Let $\mathcal{A}$ be a complete normed rings. Then one has

$$1 + \mathcal{A}(1-) \subset \mathcal{A}^x$$

and $\mathcal{A}^x \subset \mathcal{A}$ is open. In addition if $\mathcal{A}$ is non-Archimedean, then the stronger inclusion

$$1 + \mathcal{A}(1-) \subset \mathcal{A}(1)^x$$

holds.

Proof. The first implication is trivial because the convergent radius of the convergent power series $(1 - z)^{-1} = \sum_{i=0}^{\infty} z^i \in \mathcal{A}(1)[[z]]$ is $1 - 1 = 0$. Take an invertible element $a \in \mathcal{A}^x$. Then for any $b \in \mathcal{A}$ with $\|b - a\| < \|a^{-1}\|^{-1}$, one has

$$b = a + (b - a) = a(1 + a^{-1}(b - a)) \in \mathcal{A}^x(1 + \mathcal{A}(1-)) = \mathcal{A}^x,$$
and hence $\mathcal{A}^\times \subset \mathcal{A}$ is open. If $\mathcal{A}$ is non-Archimedean, the ultrametric triangle inequality guarantees $1 + \mathcal{A}(1-) \subset \mathcal{A}(1)$, and it is easy to see that the inverse of an element of $1 + \mathcal{A}(1-)$ is also contained in $1 + \mathcal{A}(1)$ by the estimation of the norm of the analytic function $(1 - z)^{-1} - 1 \in \mathcal{A}(1[[z]])$. □

**Definition 1.15.** Let $\mathcal{A}$ and $\mathcal{B}$ be seminormed rings. A ring homomorphism $f : \mathcal{A} \to \mathcal{B}$ is said to be bounded if $f$ is bounded as a group homomorphism of the underlying seminormed additive groups.

### 1.3 Seminormed Module

We introduce the notions of a seminormed module and a seminormed algebra. We do not assume that an $\mathcal{A}$-algebra contains the image of $\mathcal{A}$ in the centre. However, we mainly consider the case that the base ring is a field, and therefore a reader might assume that a seminorm ring is a commutative non-Archimedean normed ring.

**Definition 1.16.** Let $(\mathcal{A}, | \cdot |_{\mathcal{A}})$ be a seminormed ring. A seminormed left $(\mathcal{A}, | \cdot |_{\mathcal{A}})$-module is data $(M, \| \cdot \|_M)$ of a left $\mathcal{A}$-module with a seminorm $\| \cdot \|_M : M \to [0, \infty)$ of the underlying additive group satisfying the following:

$(v)$ $\|am\|_M \leq |a|_{\mathcal{A}} \|m\|_M$ for any $a \in \mathcal{A}$ and $m \in M$.

In particular $(\mathcal{A}, | \cdot |_{\mathcal{A}})$ itself is a left $(\mathcal{A}, | \cdot |_{\mathcal{A}})$-module. When $\mathcal{A}$ is commutative, then we call a seminormed left $(\mathcal{A}, | \cdot |_{\mathcal{A}})$-module a seminormed $(\mathcal{A}, | \cdot |_{\mathcal{A}})$-module for short. Suppose the seminorm $| \cdot |_{\mathcal{A}}$ is multiplicative. A seminormed left $(\mathcal{A}, | \cdot |_{\mathcal{A}})$-module $(M, \| \cdot \|_M)$ is said to be faithful if it satisfies the additional condition

$(v)'$ $\|am\|_M = |a|_{\mathcal{A}} \|m\|_M$ for any $a \in \mathcal{A}$ and $m \in M$,

which is stronger than the condition $(v)$. In particular $(\mathcal{A}, | \cdot |_{\mathcal{A}})$ itself is a faithful left $(\mathcal{A}, | \cdot |_{\mathcal{A}})$-module by the assumption of the multiplicativity of the seminorm $| \cdot |_{\mathcal{A}}$.

**Example 1.17.** Let $(\mathcal{A}, | \cdot |_{\mathcal{A}})$ be a non-Archimedean seminormed ring. Then the data $(\mathcal{A}(r), | \cdot |_{\mathcal{A}(r)})$ and $(\mathcal{A}(r-), | \cdot |_{\mathcal{A}(r-)})$ are non-Archimedean seminormed left $(\mathcal{A}(1), | \cdot |_{\mathcal{A}(1)})$-modules for any $r \in (0, \infty)$.

**Example 1.18.** Let $(\mathcal{A}, | \cdot |_{\mathcal{A}})$ be a non-Archimedean norm commutative ring. Then the data $(\mathcal{A}^\infty, | \cdot |_{\mathcal{A}^\infty})$ is a non-Archimedean normed $(\mathcal{A}^\infty, | \cdot |_{\mathcal{A}^\infty})$-module.

**Definition 1.19.** Let $\mathcal{A}$ be a seminormed ring, and $M$ and $N$ be seminormed left $\mathcal{A}$-modules. An $\mathcal{A}$-module homomorphism $f : M \to N$ is said to be bounded if $f$ is bounded as a group homomorphism of the underlying seminormed additive groups.

**Definition 1.20.** Let $(\mathcal{A}, | \cdot |_{\mathcal{A}})$ be a seminormed ring and suppose the seminorm $| \cdot |_{\mathcal{A}}$ is multiplicative. A seminormed $(\mathcal{A}, | \cdot |_{\mathcal{A}})$-algebra is data $(\mathcal{B}, \| \cdot \|_\mathcal{B})$ of an $\mathcal{A}$-algebra $\mathcal{B}$ with a seminorm $\| \cdot \|_\mathcal{B} : \mathcal{B} \to [0, \infty)$ of the underlying ring with respect to which $(\mathcal{B}, \| \cdot \|_\mathcal{B})$ is a faithful seminormed left $(\mathcal{A}, | \cdot |_{\mathcal{A}})$-module. In particular $(\mathcal{A}, | \cdot |_{\mathcal{A}})$ itself
is a seminormed \((\mathfrak{A}, |·|_\mathfrak{A})\)-algebra by the assumption of the multiplicativity of the norm \(∥·∥_\mathfrak{A}\). A seminormed \((\mathfrak{A}, |·|_\mathfrak{A})\)-algebra \((\mathfrak{B}, |·|_\mathfrak{B})\) is said to be a normed \((\mathfrak{A}, |·|_\mathfrak{A})\)-algebra if its underlying seminormed ring is a normed ring.

The norm of an algebra is compatible with that of the base seminormed ring. Indeed if \((\mathfrak{B}, ∥·∥_\mathfrak{B})\) is a non-zero normed \((\mathfrak{A}, ∥·∥_\mathfrak{A})\)-algebra, the condition (v)’ implies \(|a|_\mathfrak{A} = |a|_\mathfrak{A}∥1∥_\mathfrak{B} = ∥a∥_\mathfrak{B}\) for any \(a ∈ \mathfrak{A}\).

**Definition 1.21.** Let \(\mathfrak{A}\) be a seminormed ring and suppose its seminorm is multiplicative. Let \(\mathfrak{B}\) and \(\mathfrak{B}'\) be seminormed \(\mathfrak{A}\)-algebras. An \(\mathfrak{A}\)-algebra homomorphism \(f : \mathfrak{B} → \mathfrak{B}'\) is said to be bounded if \(f\) is bounded as a group homomorphism of the underlying seminormed additive groups.

### 1.4 Valuation Field

We assume that a valuation is of rank at most 1 so that a valuation field is a normed ring. We do not assume that the valuation of a valuation field is not trivial. On the other hand, a field is assumed to be a commutative ring with \(0 ≠ 1\). In particular, the zero ring \(O\) is excluded from the definition of a field. We mainly see the norm of a valuation field rather than the additive valuation. Therefore the valuation group means the image of the norm but not the image of the additive valuation.

**Definition 1.22.** A valuation field is a normed ring whose underlying ring is a field and whose norm is non-Archimedean and multiplicative. A valuation field \((k, |·|)\) is said to be a discrete valuation field if the valuation group \(|k^×| ⊂ (0, ∞)\) is a multiplicative subgroup isomorphic to \(\mathbb{Z}\). Then we also say that the valuation of \(k\) is discrete. A valuation field \((k, |·|)\) is said to be trivial if the valuation group \(|k^×| ⊂ (0, ∞)\) is the trivial group \(\{1\}\). Then we also say that the valuation of \(k\) is trivial.

For a valuation field \(k\), the equalities \(k(1) = k^o\) and \(k(1−) = k^{oo}\) because its norm is multiplicative. The clopen subring \(k^o ⊂ k\) is a clopen integrally closed local ring, and \(k^{oo} ⊂ k^o\) is the unique maximal ideal. Call \(k^o\) the ring of integral elements or the valuation ring of \(k\), and \(k^{oo}\) the maximal ideal of \(k\).

**Example 1.23.** The associated trivial seminormed ring of a field is a trivial complete valuation field. For a valuation field \((k, |·|)\), the valuation of \(k\) is trivial if and only if \((k, |·|)\) is a trivial normed ring. The correspondence from a field to the associated trivial valuation field determines a functor from the category of fields to the category of valuation fields, which is the left adjoint functor of the forgetful functor.

Henceforth, let \(k\) denote a valuation field. We mainly deal with the case \(k\) is complete and we will assume it from §3. Since we assume that a field is not the zero ring, one has \(|k^×| ≠ \{0\}\). The valuation of \(k\) is discrete or trivial unless the valuation group \(|k^×| ⊂ (0, ∞)\) is a dense subgroup.
1.5 Banach Space

**Definition 1.24.** A seminormed $k$-module is said to be a seminormed $k$-vector space if it is non-Archimedean and faithful. A seminormed $k$-vector space is said to be a $k$-Banach space if it is a complete normed $k$-module.

**Example 1.25.** For an index set $I$, consider the $k$-vector subspace $k^\oplus I \subset \text{Map}(I, k)$ freely generated by $I$ identified with the subset of the characteristic function of each element of $I$. For the supremum norm $\| \cdot \| : k^\oplus I \rightarrow [0, \infty)$: $f \mapsto \|f\| := \max_{i \in I} |f(i)|$, the data $(k^\oplus I, \| \cdot \|)$ is a normed $k$-vector space.

**Example 1.26.** A closed $k$-vector subspace of a $k$-Banach space is again a $k$-Banach space with respect to the restriction of the norm.

In the Archimedean analysis, the class of separable Banach spaces is important. However in the non-Archimedean analysis, there might be no separable Banach space for a fixed non-Archimedean field because the base field itself might be inseparable. Instead an alternative condition of the dimension stands for the separability.

**Definition 1.27.** A $k$-Banach space is said to be of countable type if it contains dense $k$-vector subspace of countable dimension.

**Definition 1.28.** A normed $k$-algebra is said to be a Banach $k$-algebra if it is a complete non-Archimedean normed $k$-algebra. In particular a Banach $k$-algebra is a $k$-Banach space.

**Example 1.29.** A closed $k$-subalgebra $\mathcal{B}$ of a Banach $k$-algebra $\mathcal{A}$ is again a Banach $k$-algebra with respect to the restriction of the norm. If $\mathcal{A}$ is commutative (or uniform), then so is $\mathcal{B}$.

**Example 1.30.** For a complete valuation field $(k, | \cdot |_k)$, each algebraic extension $K$ of the underlying field admits a unique extension $| \cdot |_K$ of $| \cdot |_k$ as multiplicative norms, and $(K, | \cdot |_K)$ is a valuation field over $(k, | \cdot |_k)$. We always endow $K$ with the norm. If $K/k$ is a finite extension, then $(K, | \cdot |_K)$ is complete. The completion $C$ of an algebraic closure of $k$ as a metric space has the natural structure of an algebraically closed complete valuation field over $(k, | \cdot |_k)$.

**Example 1.31.** For a complete valuation field $k$ and an $r \in (0, \infty)$, the general Tate algebra

$$k_{r^{-1}T} := \left\{ F = \sum_{n \in \mathbb{N}} F_nT^n \in k[[T]] \left| \lim_{n \in \mathbb{N}} |F_n|r^n = 0 \right. \right\}$$

associated to the closed disc of radius $r$ endowed with Gauss norm

$$\| \cdot \| : k_{r^{-1}T} \rightarrow [0, \infty)$$

18
\[ F = \sum_{n \in \mathbb{N}} F_n T^n \mapsto \|F\| := \sup_{n \in \mathbb{N}} |F_n| r^n < \infty \]

is a commutative Banach \( k \)-algebra with a multiplicative norm. For \( r, s \in (0, \infty) \), the \( k \)-affinoid algebra

\[ k\{r^{-1}T, sT^{-1}\} := \left\{ F = \sum_{n \in \mathbb{Z}} F_n T^n \in k[[T, T^{-1}]] \mid \lim_{n \to \infty} \max_{n \in \mathbb{N}} |F_n| r^n, |F_n| s^{-n} = 0 \right\} \]

associated to the closed annulus of radius \((s, r)\) with Gauss norm

\[ || \cdot || : k\{r^{-1}T\} \rightarrow [0, \infty) \]

\[ F = \sum_{n \in \mathbb{Z}} F_n T^n \mapsto ||F|| := \sup_{n \in \mathbb{N}} |F_n| r^n < \infty \]

is a commutative uniform Banach \( k \)-algebra.

**Example 1.32.** For a complete valuation field \( k \) and an \( r \in (0, \infty) \), the \( k \)-subalgebra

\[ k\{(r-)^{-1}T\} := \left\{ F = \sum_{n \in \mathbb{N}} F_n T^n \in k[[T]] \mid \sup_{n \in \mathbb{N}} |F_n| r^n < \infty \right\} \]

of \( k[[T]] \) associated to the open disc of radius \( r \) endowed with Gauss norm

\[ || \cdot || : k\{(r-)^{-1}T\} \rightarrow [0, \infty) \]

\[ F = \sum_{n \in \mathbb{N}} F_n T^n \mapsto ||F|| := \sup_{n \in \mathbb{N}} |F_n| r^n < \infty \]

is a commutative Banach \( k \)-algebra with a multiplicative norm.

**Definition 1.33.** Let \( \mathcal{A} \) be a Banach \( k \)-algebra. For an element \( A \in \mathcal{A} \), denote by \( k[A] \subset \mathcal{A} \) the image of the \( k \)-algebra homomorphism \( k[T] \rightarrow \mathcal{A} : T \mapsto A \), by \( \mathcal{L}(A) \subset \mathcal{A} \) the closure of \( k[A] \), and by \( \mathcal{L}_{\mathcal{A}}(A) \subset \mathcal{A} \) the closure of the localisation of \( k[A] \) by the multiplicative subset \( k[A] \cap \mathcal{A}^\times \).

The \( k \)-subalgebra \( \mathcal{L}(A) \subset \mathcal{A} \) is a commutative Banach \( k \)-algebra. Moreover, the commutative Banach \( k \)-algebra \( \mathcal{L}(A) \) is independent of the choice of \( \mathcal{A} \). Namely, even if one replaces \( \mathcal{A} \) by a closed \( k \)-subalgebra \( \mathcal{B} \subset \mathcal{A} \) containing \( A \), the Banach \( k \)-algebra \( \mathcal{L}(A) \subset \mathcal{B} \) does not change as a \( k \)-subalgebra of \( \mathcal{A} \). Unlike \( \mathcal{L}(A) \), the commutative Banach \( k \)-algebra \( \mathcal{L}_{\mathcal{A}}(A) \) depends on the choice of not only \( A \) but also the Banach \( k \)-algebra \( \mathcal{A} \) containing \( A \).

### 1.6 Rings of Continuous Functions

We observe several classes of continuous functions. The most basic ones in non-Archimedean analysis are the class of bounded continuous functions and the class of continuous functions vanishing at infinity. We introduce another class of rigid continuous functions. We will use the class in the definition of the non-Archimedean functional calculus in §3. In this section, \( X \) denotes a topological space.
Definition 1.34. A continuous function \( f : X \to k \) is said to be bounded if \( \sup_{x \in X} |f(x)| < \infty \). Consider the \( k \)-subalgebra \( C_{bd}(X, k) \subset \text{Map}(X, k) \) of bounded continuous functions. For the supremum norm \( \| \cdot \| : C_{bd}(X, k) \to [0, \infty) : f \mapsto \|f\| := \sup_{x \in X} |f(x)| \), the data \( (C_{bd}(X, k), \| \cdot \|) \) is a commutative uniform non-Archimedean normed \( k \)-algebra. If \( k \) is complete, then \( (C_{bd}(X, k), \| \cdot \|) \) is a commutative uniform Banach \( k \)-algebra.

In the following, we denote by \( \text{Top} \) the category of topological spaces, by \( \text{TDCHTop} \) the full subcategory of totally disconnected compact Hausdorff topological spaces, and by \( i : \text{TDCHTop} \hookrightarrow \text{Top} \) the canonical embedding.

Proposition 1.35. Suppose that \( k \) is a finite field endowed with the trivial norm or a local field. There are a functor \( \text{SC}_k : \text{Top} \to \text{TDCHTop} \) and a natural transform \( \text{ev} : \text{id}_{\text{Top}} \to i \circ \text{SC}_k \) such that for each topological space \( X \), the pull-back by the continuous map \( \text{ev}(X) : X \to \text{SC}_k(X) \) gives the functorial isometric isomorphism

\[
C(\text{SC}_k(X), k) = C_{bd}(\text{SC}_k(X), k) \to C_{bd}(X, k)
\]

of Banach \( k \)-algebras.

This is a non-Archimedean analogue of the Stone-Čech compactification of \( X \) with respect to \( k \)-valued bounded continuous functions. The topological space \( \text{SC}_k(X) \) is a universal totally disconnected compact Hausdorff topological space which satisfies that any bounded continuous function \( f : X \to k \) uniquely extends to a bounded continuous function \( \text{SC}_k(f) : \text{SC}_k(X) \to k \) through the continuous map \( \text{ev}(X) : X \to \text{SC}_k(X) \).

Proof. Denote by \( \text{SC}_k(X) \) the closure of the image of the continuous map

\[
\text{ev}(X) : X \to (k^\circ)^{C_{bd}(X,k)(1)}
\]

\[
x \mapsto (f(x))_{f \in C_{bd}(X,k)(1)}.
\]

Since \( k \) is a finite field or a local field, the subset \( k^\circ \subset k \) is a totally disconnected compact Hausdorff topological space, and so is the codomain of \( \text{ev}(X) \) by Tychonoff’s theorem. Therefore the closed subset \( \text{SC}_k(X) \) is a totally disconnected compact Hausdorff topological space, and the image of \( X \) is dense in \( \text{SC}_k(X) \) by definition. The universal property immediately follows from the construction.

\( \square \)

Beware that we did not assume that \( X \) is a totally disconnected Hausdorff space, and hence bounded continuous functions do not necessarily separate the points of \( X \). Points of \( X \) which can not be separated by \( k \)-valued bounded continuous functions have the same image in \( \text{SC}_k(X) \).

Corollary 1.36. Suppose that \( k \) is a finite field endowed with the trivial norm or a local field. There is a totally disconnected compact Hausdorff topological space \( Y \) such that its cardinality of \( Y \) is at most \( \max\{2^{2^{|X|}}, 2^{|k|} \} \) and it admits an isometric isomorphism

\[
C(Y, k) = C_{bd}(Y, k) \to C_{bd}(X, k)
\]

of Banach \( k \)-algebras.
Furthermore, there is a non-Archimedean analogue of Gel’fand-Naimark theorem ([Dou] 4.29). Namely, the non-Archimedean Gel’fand transform induces the equivalence between the category of totally disconnected compact Hausdorff topological space to a certain full subcategory of commutative uniform Banach $k$-algebras ([Ber] 9.2.7). This is why one often assumes that a topological space is a totally disconnected compact Hausdorff topological space in the non-Archimedean analysis. Recall that Gel’fand-Naimark theorem guarantees that an Archimedean commutative $C^*$-algebra is functorially isometrically isomorphic to the Banach $C$-algebra of continuous functions on a compact Hausdorff space. In the non-Archimedean case, the $C^*$-condition $\|f^* f\| = \|f\|^2$ is nonsense because of the absence of a canonical involution. One of non-Archimedean analogues of commutative $C^*$-algebras is commutative uniform Banach $k$-algebras. However, there are various commutative uniform Banach $k$-algebras which are not isomorphic to the Banach $k$-algebra of certain functions on a topological space. For example, consider the commutative uniform Banach $k$-algebras $k[r^{-1}T], k[r^{-1}T, sT^{-1}], \text{and } k\{(r)^{-1}\}$. Their underlying rings are integral domains. On the other hand, the underlying ring of $C_{\text{bd}}(X, k)$ has ample idempotents, and is an integral domain if and only if it is isomorphic to $O$ or $k$. Thus the essential image of the functor $C_{\text{bd}}(\cdot, k)$ does not coincide with the category of commutative uniformly Banach $k$-algebras.

The similar extension property holds for the Alexandroff extension of a topological space. The Alexandroff extension is one of compactifications of a topological space, and coincides with what is called the one-point compactification when the topological space is locally compact and Hausdorff. In the following, $AE(X)$ denotes the Alexandroff extension of $X$.

**Definition 1.37.** A continuous function $f: X \to k$ is said to vanish at infinity if the subset $\{x \in X \mid |f(x)| \geq \epsilon\}$ is compact for any $\epsilon > 0$. Such a continuous function is automatically bounded by the maximum modulus principle, and the subset $C_0(X, k) \subset C_{\text{bd}}(X, k)$ of continuous functions vanishing at infinity is a closed ideal. For the restriction $\| \cdot \|: C_0(X, k) \to [0, \infty)$ of the supremum norm, the data $(C_0(X, k), \| \cdot \|)$ is a non-Archimedean normed $k$-vector space. It also admits the structure of a non-Archimedean uniform normed non-unital $k$-algebra with the submultiplicativity of the norm. If $k$ is complete, then $(C_0(X, k), \| \cdot \|)$ is a $k$-Banach space.

For an index set $I$, we always endow it with the discrete topology. The $k$-Banach space of the form $C_0(I, k)$ is important. The $k$-Banach space $(C_0(I, k), \| \cdot \|)$ is the completion of $(k^{0(I,k)}\|\|)$, and many of $k$-Banach spaces appearing in Number Theory are (not necessarily isometrically) isomorphic to $C_0(I, k)$ for some index set $I$. See also §1.7.

**Proposition 1.38.** The pull-back by the canonical open embedding $\iota: X \hookrightarrow AE(X)$ gives the isometric isomorphism

$$C(AE(X), k) = C_0(AE(X), k) \to C_0(X, k) \oplus k$$

$$f \mapsto ((f - f(\infty))|_X, f(\infty))$$

of commutative uniform non-Archimedean normed $k$-algebras, where $\infty \in AE(X)$ is the infinitesimal point, i.e. $AE(X) = \iota(X) \cup \{\infty\}$. The $k$-algebra structure of the left hand side is
compatible with that of the right hand side as the unitisation of a non-unital commutative
$k$-algebra.

Proof. For a continuous function $f : \text{AE}(X) \to k$, the restriction $(f - f(\infty))|_X \in C(\text{AE}(X), k)$
satisfies $|\cdot| \circ (f - f(\infty))|_X \in C_0(\text{AE}(X), \mathbb{R})$ by the definition of the Alexandroff extension. It follows $(f - f(\infty))|_X \in C_0(\text{AE}(X), k)$.

Conversely for an $f \in C_0(X, k)$, let $\tilde{f} : \text{AE}(X) \to k$ denote the map given by setting $\tilde{f}(\iota(x)) := f(x)$ for each $x \in X$ and $\tilde{f}(\infty) := 0$. Then $\tilde{f}$ is continuous by the definition of the Alexandroff extension. The map $(f, a) \mapsto \tilde{f} + a$ is the inverse map of the given $k$-algebra homomorphism. The isometry is immediately follows from the fact that the supremum norm is compatible with the partition of the underlying topological space. □

The Alexandroff extension of a totally disconnected locally compact Hausdorff topological space, being the one-point compactification, is a totally disconnected compact Hausdorff topological space. Unlike $\text{SC}_k(X)$, the compactification $\text{AE}(X)$ is not necessarily totally disconnected if $X$ is not a totally disconnected locally compact Hausdorff topological space. In the non-Archimedean analysis, a compactification is preferred to be a totally disconnected compact Hausdorff topological space. Such a good topological space can be constructed taking the quotient of $\text{AE}(X)$ by the equivalence relation given by the separability by continuous functions vanishing at infinity. The quotient topological space is a totally disconnected compact Hausdorff topological space endowed with a canonical continuous map from $X$ with respect to which it has the extension property for a $k$-valued continuous function vanishing at infinity.

**Corollary 1.39.** There is a totally disconnected compact Hausdorff topological space $Y$
such that its cardinality is at most $\#X + 1$ and it admits an isometric isomorphism

$$C(Y, k) \to C_0(X, k) \oplus k$$

of normed $k$-algebras, where the right hand side is endowed with the normed $k$-algebra structure as the unitisation of a non-unital commutative normed $k$-algebra.

Let $\sigma \subset k$ be a non-empty bounded closed subset. The coordinate function $z : \sigma \hookrightarrow k$
is a bounded continuous function by the definition of the supremum norm. The $k$-subalgebra $k[z] \subset C_{bd}(\sigma, k)$ is not dense in general, and hence a bounded $k$-algebra homomorphism from $\mathcal{L}(z) \subset C_{bd}(\sigma, k)$ might extend to two distinct bounded $k$-algebra homomorphism on $C_{bd}(\sigma, k)$. We desire the uniqueness of a functional calculus, and hence we consider a wider class of continuous functions than $\mathcal{L}(z)$.

**Definition 1.40.** For a non-empty bounded closed subset $\sigma \subset k$, set

$$C_{\text{rig}}(\sigma, k) := \mathcal{L}_{C_{bd}(\sigma, k)}(z) \subset C_{bd}(\sigma, k).$$

Call an element of $C_{\text{rig}}(\sigma, k)$ a rigid continuous function on $\sigma \subset k$. 22
There is a notion of the rigidity of a continuous function called “Krasner analytic”. Suppose \( k \) is algebraically closed. People often say a continuous function \( f \) is Krasner analytic if it is defined on an open subset \( U \subseteq k \) and if it admits a sequence of rational functions with poles outside \( U \) which uniformly converges to \( f \) on \( U \). People sometimes say a continuous function \( f \) is Krasner analytic if it is defined on an open subset \( U \subseteq k \) and if for any \( \epsilon > 0 \), it admits a sequence of rational functions with poles in the \( \epsilon \)-neighbourhood \( X_{\epsilon} \) of \( k \setminus U \) which uniformly converges to \( f \) on \( k \setminus X_{\epsilon} \subseteq U \). It is similar to the convergence on compact subsets. The conditions that the domain \( U \subseteq k \) is open, and/or that the base field \( k \) is algebraically closed might be removed. If \( U \subseteq k \) is not open, people sometimes say a continuous function \( f \) is Krasner analytic if \( U \) admits a covering by discs on which the restrictions of \( f \) are expressed by convergent power series. People sometimes use the term “rigid analytic” instead of “Krasner analytic”, but a rigid analytic function usually means a function in an affinoid algebra. Thus the term “Krasner analytic” is a little ambiguous, and hence we use the new term “rigid continuous” here.

**Example 1.41.** There are two specific examples.

(i) \( k := \mathbb{Q}_p \), \( \sigma := \mathbb{Z}_p = \mathbb{Q}_p^\circ \subset \mathbb{Q}_p \)

\[ \mathcal{L}(z) = C_{\text{rig}}(\mathbb{Z}_p, \mathbb{Q}_p) = C_{\text{bd}}(\mathbb{Z}_p, \mathbb{Q}_p) = C(\mathbb{Q}_p, \mathbb{Q}_p). \]

For more general cases, see Proposition 1.44

(ii) \( k := \mathbb{C}_p \), \( \sigma := \mathbb{C}_p^\circ \subset \mathbb{C}_p \)

\[ \mathcal{L}(z) = \mathbb{C}_p[z] = C_{\text{rig}}(\mathbb{C}_p^\circ, \mathbb{C}_p) \subseteq \mathbb{C}_p \{ z^\lambda \mid n \in \mathbb{N}_+ \} \subseteq C_{\text{bd}}(\mathbb{C}_p^\circ, \mathbb{C}_p) \subseteq C(\mathbb{C}_p^\circ, \mathbb{C}_p), \]

where \( (z^{1/n})_{n \in \mathbb{N}_+} \in C_{\text{bd}}(\mathbb{C}_p^\circ, \mathbb{C}_p)^{\mathbb{N}_+} \) is a fixed compatible system of continuous \( n \)-th roots \( z^{1/n} : \mathbb{C}_p^\circ \to \mathbb{C}_p \) of the coordinate function \( z : \mathbb{C}_p^\circ \to \mathbb{C}_p \). This is a special case of Theorem 1.2

**Proposition 1.42.** In the situation in Definition 1.40, for a polynomial \( P \in k[T] \) of with no zeros on \( \sigma \), the polynomial function \( P(z) : \sigma \to k \) is invertible in \( C_{\text{rig}}(\sigma, k) \).

**Proof.** A function \( f \in C_{\text{bd}}(\sigma, k) \) is invertible in \( C_{\text{bd}}(\sigma, k) \) if and only if its absolute value function \( |f| : \sigma \to [0, \infty) \) has a positive lower bound on \( \sigma \). Take a polynomial \( P \in k[T] \) with no zeros on \( \sigma \). Take an algebraic closure of \( k \) and let \( C/k \) be the completion of it with respect to the unique extension of the valuation. Denote by \( S \subseteq C \setminus \sigma \) the finite set of the zeros of \( P \). Since \( S \) is compact and \( \sigma \) is closed, the distance \( d \) between \( S \) and \( \sigma \) is positive definite. Expressing \( P = a \prod_{s \in S}(T-s)^{n_s} \) by a coefficient \( a \in k^\times \) and multiplicities \( (n_s)_{s \in S} \in \mathbb{N}_+^S \), one has \( |P(\lambda)| = |a| \prod_{s \in S} |\lambda - s|^{n_s} \geq |a|d^n \) for any \( \lambda \in \sigma \), where \( n := \sum_{s \in S} n_s \). Therefore \( \inf_{s \in S} |P(\lambda)| \geq |a|d^n > 0 \), and hence \( |P(z)| : \sigma \to [0, \infty) \) admits a positive lower bound \( |a|d^n \).

**Proposition 1.43.** In the situation in Definition 1.40 the equalities \( C_{\text{rig}}(\sigma, k)^\times = C_{\text{rig}}(\sigma, k) \cap C_{\text{bd}}(\sigma, k)^\times \) and \( \sigma_{\text{c_{rig}(\sigma, k)}}(z) = \sigma \) hold.
Proof. For the rings, the inclusion \( \subset \) is obvious. Take a rigid continuous function \( f: \sigma \to k \) invertible in \( C_{\text{bd}}(\sigma, k) \). By the definition of \( C_{\text{rig}}(\sigma, k) \), for any \( \epsilon \in (0, 1) \), there is a continuous function \( g: \sigma \to k \) contained in the localisation of \( k[z] \subset C_{\text{bd}}(\sigma, k) \) such that \( \|f - g\| < \epsilon \min(\|f^{-1}\|^{-1}, \|f^{-1}\|^{-2}) \). For any point \( \lambda \in \sigma \), one has \( \|f(\lambda)\| = \|f^{-1}(\lambda)\|^{-1} \geq \|f^{-1}\|^{-1} > \|f - g\| \geq |f(\lambda) - g(\lambda)| \), and hence \( |g(\lambda)| = |f(\lambda)| > 0 \). Therefore \( |g(\lambda)|^{-1} = |f(\lambda)|^{-1} \). Since \( g(\lambda)^{-1} = g^{-1}(\lambda) \), and it follows that \( g \) is invertible in \( C_{\text{bd}}(\sigma, k) \). Take polynomials \( P, Q \in k[T] \) with \( Q(z) \in C_{\text{bd}}(\sigma, k)^x \) and \( g = Q(z)^{-1} P(z) \). Since \( Q(z), g \in C_{\text{bd}}(\sigma, k)^x \), \( Q(z) \) and \( g \) have no zeros on \( \sigma \), and hence so does \( P(z) = gQ(z) \). Therefore \( P(z) \) is invertible in \( C_{\text{bd}}(\sigma, k) \) by Proposition \[1.42\] and hence \( g^{-1} = P(z)^{-1} Q(z) \) is contained in the localisation of \( k[z] \). Moreover one obtains

\[
\|f^{-1} - g^{-1}\| \leq \|f^{-1}\| \|g^{-1}\| |g - f| = \|f^{-1}\|^2 \|f - g\| < \epsilon,
\]

and thus \( f \in C_{\text{rig}}(\sigma, k) \).

For the spectra, the inclusion \( \supset \) is trivial because \( \sigma \) coincides with the image of \( z \). The opposite inclusion \( \subset \) directly follows by the definition of \( C_{\text{rig}}(\sigma, k) \). \( \square \)

**Proposition 1.44.** In the situation in Definition \[7.40\] if \( \sigma \subset k \) is compact, then \( k[z] \) is dense in \( C_{\text{rig}}(\sigma, k) \) and the equalities \( C_0(\sigma, k) = \mathcal{L}(z) = C_{\text{rig}}(\sigma, k) = C_{\text{bd}}(\sigma, k) = C(\sigma, k) \) holds.

**Proof.** The equality \( C_{\text{bd}}(\sigma, k) = C(\sigma, k) \) holds by the maximal modulus principle. It suffices to show that \( k[z] \) is dense in \( C(\sigma, k) \). This is the well-known \( p \)-adic analogue of Stone-Weierstrass theorem, which is originally verified by Kaplan/ky in \[Kap\] as a generalisation of Dieudonné’s theorem in \[Die\]. For other sophisticated proofs, see \[Mur1\] 1.7 or \[Ber\] 9.2.6. \( \square \)

### 1.7 Strictly Cartesian Banach Space

In this section, assume that \( k \) is complete and let \( V \) denote a \( k \)-Banach space. We define a strictly Cartesian \( k \)-Banach space using \( C_0(I, k) \). Note that the original notion of being strictly Cartesian is defined for a normed \( k \)-vector space. For example, see \[BGR\] 2.5.2/1.

**Definition 1.45.** A discrete subset \( I \subset V \) is said to be an orthonormal Schauder basis of \( V \) if the canonical \( k \)-linear homomorphism \( k^{\text{ad}} \to V \) given by the inclusion \( I \hookrightarrow V \) uniquely extends to the isometric isomorphism

\[
C_0(I, k) \to V.
\]

If \( V \) is finite-dimensional, then simply call \( I \) an orthonormal basis.

Do not confuse \( C_0(I, k) \) with \( C_{\text{bd}}(I, k) \). The subset \( I \subset C_{\text{bd}}(I, k) \), identified with the characteristic functions of singletons in \( I \), is not an orthonormal Schauder basis unless \( I \) is a finite set. Beware that an orthonormal basis of a finite-dimensional \( k \)-Banach space is a basis of the underlying \( k \)-vector space but the converse does not hold in general. There are criteria for the orthonormality in \[BGR\] 2.5.1/3 and \[BGR\] 2.5.2/2.

24
**Definition 1.46.** A $k$-Banach space is said to be strictly Cartesian if it admits an orthonormal Schauder basis. It is equivalent to the condition that a $k$-Banach space is isometrically isomorphic to the $k$-Banach space $C_0(I, k)$ for some discrete topological space $I$.

**Example 1.47.** Any $k$-vector subspace of the $n$-dimensional $k$-Banach space $k^n$ is strictly Cartesian with respect to the restriction of the norm of $k^n$ given by its canonical basis for any $n \in \mathbb{N}$.

**Example 1.48.** The $\mathbb{Q}_p$-Banach spaces $C(\mathbb{Z}_p, \mathbb{Q}_p)$, $C_0(\mathbb{Q}_p, \mathbb{Q}_p)$, and $\mathbb{C}_p$ are strictly Cartesian $\mathbb{Q}_p$-Banach spaces of infinite dimension.

In general, any subspace of a finite-dimensional strictly Cartesian Banach space is again strictly Cartesian by [BGR] 2.5.1/4. The notion of being strictly Cartesian depends on the choice of the norm. If one equips $k^n$ with another equivalent norm, then $k^n$ might not be strictly Cartesian. It is remarkable that a Banach space of countable type always admits an equivalent norm with respect to which it admits an orthonormal Schauder basis by [BGR] 2.7.2/8. Such a countability can be removed if the valuation of $k$ is discrete. See the paragraph in the end of §2.1.

### 1.8 Operator Algebra

**Definition 1.49.** Let $V$ be a seminormed $k$-vector spaces. An operator on $V$ is a $k$-linear homomorphism $f : W \to V$ from a $k$-vector subspace. Call $W$ the domain of $f$, and $f$ is said to be defined on $W$. In particular if $W = V$, then $f$ is said to be everywhere defined. An operator is said to be bounded if it is everywhere defined and is a bounded endomorphism. Denote by $\mathcal{B}_k(V) \subset \text{End}_k(V)$ the subset of bounded operators.

The subset $\mathcal{B}_k(V) \subset \text{End}_k(V)$ is a $k$-subalgebra, and $(\mathcal{B}_k(V), \| \cdot \|)$ is a seminormed $k$-algebra for the operator seminorm

$$
\| \cdot \| : \mathcal{B}_k(V) \to [0, \infty),
A \mapsto \sup_{v \in V} \|Av\| < \infty.
$$

Call $(\mathcal{B}_k(V), \| \cdot \|)$ the (bounded) operator algebra on $V$. In addition if $V$ is a normed $k$-vector space, then $(\mathcal{B}_k(V), \| \cdot \|)$ is a normed $k$-algebra, and call $\| \cdot \|$ the operator norm. Further more if $V$ is a $k$-Banach space, then $(\mathcal{B}_k(V), \| \cdot \|)$ is a Banach $k$-algebra.

**Remark 1.50.** Suppose $k$ is non-trivial. An everywhere-defined operator $f : V \to V$ is bounded if and only if $f$ is continuous. Significant bounded operators, such derivations and unbounded physical quantities, are basically defined on a proper subspace, and one often means “everywhere-defined” by “bounded”.

**Definition 1.51.** A $k$-Banach representation of a Banach $k$-algebra $\mathcal{A}$ is a bounded $k$-algebra homomorphism $\mathcal{A} \to \mathcal{B}_k(V)$ for a $k$-Banach space $V$. 

25
**Definition 1.52.** A $k$-Banach representation $\mathcal{A} \to \mathcal{B}_k(V)$ is said to be of countable type if $V$ is of countable type.

**Definition 1.53.** Suppose $k$ is complete. A $k$-Banach representation $\mathcal{A} \to \mathcal{B}_k(V)$ is said to be normalisable if $V$ is strictly Cartesian and of countable type.

For a discrete group $G$, the convolution product on $k^G = k[G]$ is continuous with respect to the supremum norm, and hence it induces a unital $k$-algebra structure of $C_0(G, k)$. Giving a $k$-Banach representation $C_0(G, k) \to \mathcal{B}_k(V)$ is equivalent to giving a Banach representation $G \times V \to V$. On the other hand, suppose that $G$ is a totally disconnected locally compact Hausdorff topological group. The group algebra $k[G]$ is not contained in $C_0(G, k)$, and there is no $k$-valued Haar measure on $G$ if $G$ contains a pro-$p$ component. Therefore there is no canonical convolution product on $C_0(G, k)$, and a Banach representation $G \times V \to V$ can not be interpreted as a Banach representation of $C_0(G, k)$. Alternatively, Peter Schneider and Jeremy Teitelbaum verified in [ST2] that the Iwasawa algebra $k[[G]]$ works well for a profinite group $G$.

### 2 Three Reductions

There are several basic ways of reduction. Recall that one of the main theme of this paper is the use of the reduction for the functional calculus of an operator. We always endow spaces obtained by reductions with the discrete uniform topology. For example, the reductions of a Banach algebra is a Banach algebra over the residue field with respect to the trivial norm.

#### 2.1 Normal Reduction

**Definition 2.1** (Normal reduction). For a non-Archimedean seminormed group $G$, set $\overline{G} := G(1)/G(1-)$, and call it the normal reduction of $G$. For an element $g \in G(1)$, denote by $\overline{g} \in \overline{G}$ its image and call it the normal reduction of $g$.

The normal reduction of a non-Archimedean seminormed ring is the normal reduction of the underlying non-Archimedean seminormed group endowed with the quotient ring structure. In particular call $\overline{k}$ the residue field of $k$. The normal reduction of a seminormed $k$-vector space $V$ (or a non-Archimedean seminormed $k$-algebra $\mathcal{A}$) is the normal reduction of the underlying non-Archimedean seminormed group (resp. ring). It admits the natural structure of a $\overline{k}$-vector space (resp. a $\overline{k}$-algebra). Beware that the normal reduction heavily depends on the seminorm, and changes if one replaces the seminorm by an equivalent one. A replacement of an equivalent norm of corresponds to a change of a lattice $V(1) \subset V$ (resp. the integral model $\mathcal{A}'(1) \subset \mathcal{A}$) over $k^\circ$.

**Remark 2.2.** The residue field $\overline{C}$ of $C$ is an algebraic closure of the residue field $\overline{k}$ of $k$.

**Remark 2.3.** If $X$ is a compact topological space, the maximum modulus principle implies that the canonical $\overline{k}$-algebra homomorphism $\overline{C(X, k)} \to C(X, \overline{k})$ is an isomorphism.
Hereafter $p \in \mathbb{N}$ denotes the characteristic of the residue field $\overline{k}$. We mainly consider the case $p > 0$.

**Definition 2.4.** A valuation field $k$ is said to be a local field if $k$ is a complete discrete valuation field and if the residue field $\overline{k}$ is a finite field.

Let $\mathcal{A}$ and $\mathcal{B}$ be non-Archimedean seminormed rings. A ring homomorphism $f: \mathcal{A} \to \mathcal{B}$ which is a contraction induces a bounded ring homomorphism $\mathcal{A}(1) \to \mathcal{B}(1)$ and a ring homomorphism $\mathcal{A} \to \mathcal{B}$. In other words, the correspondences $\mathcal{A} \to \mathcal{A}(1)$ and $\mathcal{A} \to \mathcal{B}$ are functorial with respect to ring homomorphisms which are contractions. A bounded homomorphism which is not a contraction does not work here.

**Proposition 2.5.** Let $\mathcal{A}$ be a complete non-Archimedean normed ring. The surjective ring homomorphism $\Pi: \mathcal{A}(1) \twoheadrightarrow \mathcal{A}$ induces the surjective group homomorphism $\Pi: \mathcal{A}(1)^\times \twoheadrightarrow \mathcal{A}^\times$. Moreover one has $\Pi^{-1}(\mathcal{A}^\times) = \mathcal{A}(1)^\times$.

**Proof.** For an element $a \in \mathcal{A}(1)$, suppose $a^{-1} \in \mathcal{A}$. Take a lift $b \in \mathcal{A}(1)$ of the inverse $a^{-1} \in \overline{\mathcal{A}}$. Since $ab - 1 = a^{-1} - 1 = 0 \in \overline{\mathcal{A}}$, one has $\|ab - 1\| < 1$, and hence $ab \in 1 + \mathcal{A}(1-)$, and $\mathcal{A}(1)^\times$ by Proposition 1.14. Thus $a \in \mathcal{A}(1)^\times$. □

The normal reduction is useful when one examines whether a given system is an orthonormal Schauder basis. Suppose $k$ is complete. For a finite dimensional strictly Cartesian $k$-Banach space $V$ such as $k^n$, a subset $I \subset V(1)$ is an orthonormal basis of $V$ if and only if the restriction $I \to \overline{V}: i \mapsto i$ of the reduction map is injective and its image $\overline{I}$ is a basis of the $\overline{k}$-vector space $\overline{V}$ by [BGR] 2.5.1/3. Suppose in addition that the valuation of $k$ is discrete. For a $k$-Banach space $V$ with $\|V\| = |k|$, a subset $I \subset V(1)$ is an orthonormal Schauder basis of $V$ if and only if the restriction $I \to \overline{V}: i \mapsto i$ of the reduction map is injective and its image $\overline{I}$ is a basis of the $\overline{k}$-vector space $\overline{V}$, and in particular $V$ is automatically strictly Cartesian. For example, see [BGR] 2.5.3/11 and 2.4.4/2, [FP] 1.2.2/(1), or [Mon] IV.3 for details.

### 2.2 Spectral Reduction

**Definition 2.6 (Spectral reduction).** For a commutative non-Archimedean normed ring $\mathcal{A}$, set $\mathcal{A}^{\text{red}} := \mathcal{A}^\circ \mid \mathcal{A}^{\text{oo}}$, and call it the spectral reduction of $\mathcal{A}$. For an element $A \in \mathcal{A}^\circ$, denote by $A^{\text{red}} \in \mathcal{A}^{\text{red}}$ its image and call it the spectral reduction of $A$.

If $\mathcal{A}$ is uniform, then the equality $\mathcal{A}^{\text{red}} = \overline{\mathcal{A}}$ holds because $\mathcal{A}^\circ = \mathcal{A}(1)$ and $\mathcal{A}^{\text{oo}} = \mathcal{A}(1-)$. In particular, one has $k^{\text{red}} = \overline{k}$. Unlike the normal reduction, the spectral reduction depends only on the uniform topology, and hence does not change if one replaces the norm by an equivalent one.

Beware that the spectral reduction works only when the underlying ring is commutative. However, it is still important also for a non-commutative normed ring such as an operator algebra $\mathcal{B}(V)$, because the closed $k$-subalgebra $\mathcal{L}(A) \subset \mathcal{B}(V)$ is commutative for any $A \in \mathcal{B}(V)$. 27
For normed rings $A$ and $B$, a bounded ring homomorphism $f: A \to B$ sends
a bounded element to a bounded element, and a topologically nilpotent element to
a topologically nilpotent element. Therefore in particular if $A$ and $B$ is commutative and
non-Archimedean, then $f$ induces a bounded ring homomorphism $A^\circ \to B^\circ$ and a ring
homomorphism $A_{\text{red}} \to B_{\text{red}}$. In other words, the correspondences $A \mapsto A^\circ$ and
$A \mapsto A_{\text{red}}$ are functorial with respect to bounded homomorphisms.

Let $A$ be a commutative non-Archimedean normed $k$-algebra. The spectral reduction
$A_{\text{red}}$ of $A$ is the spectral reduction of the underlying commutative non-Archimedean
normed ring. It admits the natural structure of a $k$-algebra by the functoriality.

The spectral reduction might seem to be easy to handle when we deal with commu-
tative Banach algebras. However, it is hard to calculate $A^\circ$ and $A_{\text{red}}$ when $A$ is not
uniform, and hence the spectral reduction is applied mainly to uniform Banach algebras.

### 2.3 Matrix Reduction

In this section, suppose $k$ is complete for simplicity. Roughly speaking, the matrix reduc-
tion an operator is the reduction of the entries of its matrix representation. Beware that
in the infinite dimensional case, such a reduction makes sense and is well-defined only
when the Banach space is strictly Cartesian.

**Definition 2.7** (Matrix reduction). Let $I$ be an index set. The matrix reduction of the
operator algebra $B_k(C_0(I,k))$ is the $k$-algebra $\text{End}_k(\mathbb{K}^\infty)$ endowed with the canonical
$k^\circ$-algebra homomorphism $\Pi_f: B_k(C_0(I,k))(1) \to \text{End}_k(\mathbb{K}^\infty)$ sending an operator $A \in
B_k(C_0(I,k))(1)$ to a $k$-linear endomorphism $\Pi_f(A): \mathbb{K}^\infty \to \mathbb{K}^\infty$ whose matrix represen-
tation $\Pi_f(M) \in (\mathbb{K}^\infty)^I \subset \mathbb{K}^{I \times I}$ with respect to the canonical basis $I \subset \mathbb{K}^\infty$ is given as
the matrix whose entries are the reduction of the entries of the matrix representation
$M \in C_0(I,k)^I \subset \mathbb{K}^{I \times I}$ of $A$ with respect to the canonical orthonormal Schauder basis
$I \subset C_0(I,k)$. The $k$-linear map $\Pi_f(A)$ is called the matrix reduction of $A$.

Note that in the construction above, each column of $\Pi_f(M) \in \mathbb{K}^{I \times I}$ surely belongs to
$\mathbb{K}^{\infty} \subset \mathbb{K}^I$, i.e. its all but finite entries are 0, because each column of $A \in k^{I \times I}$ belongs
to $C_0(I,k)$, i.e. its entries are tending to 0. In the finite dimensional case, the matrix
reduction coincides with the classical reduction of a matrix.

**Example 2.8.** For an $n \in \mathbb{N}$, we identify $M_n(k)$ as $B_k(k^n)$ in the natural way, where
$k^n$ is endowed with the norm given by the canonical basis. The matrix reduction of an
$M \in M_n(k^n)$ is the matrix $\Pi_n(M) \in M_n(\overline{k}) = \text{End}_k(k^n)$ whose entries are the
reductions of the entries.

In particular, the matrix reduction of a matrix coincides with the normal reduction
with respect to the pull-back of the operator norm by the identification $M_n(k) = B_k(k^n)$.
Although the normal reduction of $B_k(k^n)$ heavily depends on the norm of $k^n$, the iso-
 morphism class of the matrix reduction $M_n(\overline{k})$ is independent of the choice of a norm
of $k^n$ with respect to which $k^n$ is strictly Cartesian. However the matrix reduction map
\( M_n(\kappa)(1) \to M_n(\overline{\kappa}) \) depends on the norm of \( \kappa^n \). A replacement of the norm by an equivalent norm of with respect to which \( \kappa^n \) is strictly Cartesian causes a change of the integral model \( M_n(\kappa^n)(1) \subset M_n(\kappa^n) \).

**Proposition 2.9.** The identity \( \mathcal{B}_k(C_0(I, k)) \to \mathcal{B}_k(C_0(I, k)) \) induces the canonical \( \overline{\kappa} \)-algebra epimorphism \( \Pi_I: \mathcal{B}_k(C_0(I, k)) \to \text{End}_k(\overline{\kappa}^{op}) \). When \( I \) is a finite set, then \( \Pi_I \) is an isomorphism. When \( I \) is an infinite set, then \( \Pi_I \) is an isomorphism if and only is the valuation of \( \kappa \) is discrete or trivial.

**Proof.** When \( I \) is a finite set, or when the valuation of \( \kappa \) is discrete or trivial, then the operator norm of a bounded operator \( A \) on \( C_0(I, k) \) is archived by the absolute value of an entry of its canonical matrix representation \( M \). Therefore if \( A \in \mathcal{B}_k(C_0(I, k))(1) \) and \( \Pi_I(M) = 0 \), then one has \( \| A \| < 1 \). \( \square \)

**Corollary 2.10.** Suppose that \( I \) is a finite set, or that \( I \) is an infinite set and the valuation of \( \kappa \) is discrete or trivial. Then the canonical \( \overline{\kappa} \)-algebra homomorphisms

\[
\overline{C}_0(I, k) \to C_0(I, \overline{k}) = \overline{\kappa}^{op}
\]

\[
\overline{C}_{bd}(I, k) \to C(I, \overline{k}) = \overline{k}^{\prime}
\]

are isomorphisms.

**Proof.** The \( \overline{\kappa} \)-algebra homomorphism \( \overline{C}_0(I, k) \to C_0(I, \overline{k}) = \overline{\kappa}^{op} \) is an isomorphism because the maximum modulus principle holds for a compact-supported continuous function. Identify \( C_{bd}(I, k) \) and \( C(I, \overline{k}) = \overline{k}^{\prime} \) as the diagonals of \( \mathcal{B}_k(C_0(I, k)) \) and \( \mathcal{B}_k(C_0(I, k)) = \text{End}_k(\overline{\kappa}^{op}) \) respectively. \( \square \)

**Definition 2.11.** Let \( V \) be a strictly Cartesian \( k \)-Banach space. For an orthonormal Schauder basis \( I \subset V \), we consider the isometric isomorphism

\([i]: V \to C_0(I, k)\)

of \( k \)-Banach spaces, the isometric isomorphism

\( \mathcal{B}_k([i]): \mathcal{B}_k(V) \to \mathcal{B}_k(C_0(I, k))\)

of Banach \( k \)-algebras, the isomorphism

\([\tau]: \overline{V} \to \overline{\kappa}^{op}\)

of \( \overline{k} \)-vector spaces, and the isomorphism

\( \text{End}_k([\tau]): \text{End}_k(\overline{V}) \to \text{End}_k(\overline{\kappa}^{op})\)

of \( \overline{k} \)-algebras. The matrix reduction of \( \mathcal{B}_k(V) \) is the \( \overline{k} \)-algebra \( \text{End}_k(\overline{V}) \) endowed with the canonical \( \kappa^e \)-algebra homomorphism

\( \Pi_V: \mathcal{B}_k(V)(1) \to \mathcal{B}_k(C_0(I, k))(1) \to \text{End}_k(\overline{\kappa}^{op}) \to \text{End}_k(\overline{V}) \).

For an operator \( A \in \mathcal{B}_k(V)(1) \), call \( \Pi_V(A) \in \text{End}_k(\overline{V}) \) the matrix reduction of \( A \).
The homomorphism $\Pi_V$ is independent of the choice of an orthonormal Schauder basis $I \subset V$. Indeed, the matrix reduction of the matrix representation of a change of orthonormal Schauder bases coincides with the matrix representation of the change of reductive bases. In the case $V = C_0(I, k)$, one has $\Pi_V = \Pi_I$ identifying $\overline{V} = \overline{k}^{\text{ad}}$ by the natural isomorphism given by the representation by the canonical basis $I \subset V$. In this case, we denote simply by $\overline{A}$ the matrix reduction $\Pi_V(A) = \Pi_I(A)$ of an operator $A$ with matrix representation $M$ and by $\overline{M}$ the matrix representation of $\overline{A}$.

**Definition 2.12.** Let $V$ be a strictly Cartesian $k$-Banach space admitting an orthonormal Schauder basis. An operator $A \in \mathcal{B}(V)$ is said to be reductively scalar if its matrix reduction $\Pi_V(A) \in \text{End}_k(\overline{V})$ is scalar, i.e. contained in $\overline{k}$, and is said to be reductively transcendental if its matrix reduction $\Pi_V(A) \in \text{End}_k(\overline{V})$ is not integral over $\overline{k}$.

### 3 Functional Calculus and Unitary Diagonalsability

Henceforth, we assume that the base field $k$ is a complete valuation field, $V$ denotes a $k$-Banach space, and $\mathcal{A}$ denotes a Banach $k$-algebra.

#### 3.1 Unitary Operator

We introduce the notion of the unitarity of an non-Archimedean operator. Unlike the unitarity of an Archimedean operator on a Hilbert space, we formulate the unitarity as a property of a non-Archimedean operator on a Banach space. It is because $k$ has no canonical involution.

**Definition 3.1.** An operator $A \in \mathcal{B}(V)$ is said to be unitary if $A: V \to V$ is an isometric isomorphism.

We usually give $k^n$ the canonical basis and the canonical structure of the $k$-Banach space, and identify the operator algebra $\mathcal{B}(k^n)$ with the matrix algebra $M_n(k)$.

**Proposition 3.2.** For a matrix $U \in M_n(k)$, the following are equivalent:

(i) $U$ is unitary;

(ii) $\|U\| = 1$ and $|\det U| = 1$;

(iii) $U \in M_n(k^n)$ and $\overline{U} \in \text{GL}_n(\overline{k})$;

(iv) The columns of $U$ form an orthonormal basis of $k^n$;

(v) $U \in \text{GL}_n(k^n)$;

(vi) $U \in \text{GL}_n(k)$ and $U^{-1}$ is unitary; and

(vii) $^tU$ is unitary.
Proof. Suppose (i) holds. Since $U$ is an isometric isomorphism on $k^n$, one has $\|U\| = \|U^{-1}\| = 1$ by the definition of the operator norm. It follows all eigenvalues of $U$ in an algebraic closure of $k$ have norm no greater than 1, and the same holds for $U^{-1}$. It follows all eigenvalues of $U$ has norm 1, and the same holds for $U^{-1}$. It follows all eigenvalues of $U$ has norm 1, and it follows $|\det U| = 1$. Therefore (i) implies (ii).

Suppose (ii) holds. Then one has $\det \overline{U} = \overline{\det U} \neq 0$, and it follows $\overline{U} \in \text{GL}_n(k)$. Therefore (ii) implies (iii).

The conditions (iii) and (iv) are equivalent. Indeed for a matrix $U \in M_n(k^o)$, the columns of $\overline{U}$ form a basis of $k^n$ if and only if the columns of $U$ form an orthonormal basis of $k^n$ by [BGR] 2.5.1/4.

Suppose (iii) holds. Then Proposition 2.5 and Proposition 2.9 guarantees that $U \in \text{GL}_n(k^o)$. Therefore (iii) implies (v).

Suppose (v) holds. The condition $U \in \text{GL}_n(k^o)$ guarantees $U, U^{-1} \in M_n(k^o)$ and hence $0 < \|U^{\pm 1}\| \leq 1$. For a vector $v \in k^n$, one has

$$\|v\| \leq \frac{\|v\|}{\|U^{-1}\|} \leq \|U^{-1}v\| \leq \|U^{-1}\| \|v\| \leq \|v\|,$$

and hence $\|U^{-1}v\| = \|v\|$. Therefore (v) implies (vi).

The condition (vi) obviously implies (i), and by the equivalence of (i) and (ii), it is easy to see that (i) and (vii) are equivalent. \[\square\]

If a matrix $U \in M_n(k)$ satisfies $^tUU = U^tU \in (k^o)^\times$, then $U$ is unitary because it satisfies the condition (v). There is a unitary matrix which does not satisfy the condition $^tUU = U^tU \in (k^o)^\times$. For example, the matrices

$$U := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, U^{-1} := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

are unitary, but one has

$$^tUU = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \not\in (k^o)^\times$$

Note that a real matrix (or a complex matrix) gives an isometric transformation if and only if it is an orthonormal matrix (resp. a unitary matrix in the complex sense). The reason why such an example as above exists is because of the non-Archimedean property of the norm. The upper right entry 1 does not affect the norm of the image.

Remark 3.3. As we mentioned, the notion of the unitarity of a matrix deeply depends on the choice of the structure of a k-Banach space on $k^n$. One usually equips it with the canonical norm associated with the canonical basis, and the change of the basis yields the change of the norm, which is equivalent to the canonical one. For an invertible matrix $U \in \text{GL}_n(k)$ whose eigenvalues are contained in $(k^o)^\times$, if one endows $k^n$ with the norm $\| \cdot \|_b$ associated with a basis $b \subset k^n$ which gives a Jordan normal form of $U$, then $U$ represents a unitary operator on the k-Banach space $(k^n, \| \cdot \|_b)$. However if the matrix $B$ associated with the basis $b$ is not unitary, then the canonical norm differs from $\| \cdot \|_b$, and $U$ is not unitary with respect to it.
3.2 Spectrum of an Operator

We introduce the notion of the spectrum of an element of a Banach algebra. This is the analogue of the classical Archimedean spectrum of a bounded operator on a Banach space, and we will deal with the spectrum of an unbounded operator in $\mathbb{C}$.

**Definition 3.4.** Let $R$ be a ring and $S$ an $R$-algebra. If $R \neq O$ and $S = O$, set $\sigma_S(0) := \{0\} \subset R$. Otherwise, for an element $A \in S$, set $\sigma_S(A) := \{ \lambda \in R \mid A - \lambda \notin S^\times \}$. Call $\sigma_S(A)$ the spectrum of $A$ in $S$, and also call an element of $\sigma_S(A)$ a spectrum of $A$ in $S$. Similarly call the complement of $\sigma_S(A)$ the resolvent of $A$ in $S$ and its element a resolvent of $A$ in $S$.

In particular, consider the case that $R$ is a complete valuation field $k$ and $S$ is a Banach $k$-algebra $\mathcal{A}$. For an element of $A \in \mathcal{A}$, the spectrum of $A$ in $\mathcal{A}$ is defined as the spectrum $\sigma_\mathcal{A}(A) \subset k$ of $A$ in the underlying $k$-algebra of $\mathcal{A}$. If $\mathcal{A}$ is a matrix algebra $M_n(k)$, a matrix $M \in M_n(k)$ is integral over $k$ but its characteristic polynomial is not necessarily decomposed in $k$ when $k$ is not algebraically closed. Therefore the spectrum $\sigma_{M_n(k)}(M)$ might be empty. The infinite dimensional case is still more complicated. In the Archimedean case, the spectrum of an element of a Banach $\mathbb{C}$-algebra is non-empty. The non-emptiness is derived from Liouville’s theorem for the resolvent function, which is an analytic function on a $\mathbb{C}$-Banach space, and it directly implies Gel’fand-Mazur theorem. On the other hand in the non-Archimedean case, it is well-known that a complete valuation field always has a counter-example for Gel’fand-Mazur theorem. The same proof is no longer valid because the resolvent function is just locally analytic but is not rigid analytic. Liouville’s Theorem surely holds for a rigid analytic function, but does not for a general locally analytic function. Indeed, there are ample locally constant functions. The main target of our spectral theory is the operator algebra on a strictly Cartesian Banach space of countable type, but there is an example of a non-Archimedean operator with empty spectrum even for such a good Banach space over an algebraically closed field.

**Example 3.5.** Denote by $\bar{\mathbb{F}}_p$ the residue field $\mathbb{F}_p$ of $\mathbb{C}_p$, and by $[\cdot] : \mathbb{F}_p \rightarrow \mathbb{C}_p$ the canonical Teichmüller embedding sending a root of unity $a \in \mathbb{F}_p^\times$ to the root of unity $[a] \in \mathbb{C}_p^\times$ with $[a] = a$. Set $I := \mathbb{N} \cup (\bar{\mathbb{F}}_p \times \mathbb{N})$, and then $I$ is a countable discrete topological space. For a strictly Cartesian $\mathbb{C}_p$-Banach space $C_0(I, \mathbb{C}_p)$, denote by $1_z \in C_0(I, \mathbb{C}_p)$ the image of $z \in I$ by the canonical embedding $I \hookrightarrow C_0(I, \mathbb{C}_p)$. Then the collection $I$ identified with the subset $\{ 1_z \mid z \in I \} \subset C_0(I, \mathbb{C}_p)$ forms an orthonormal Schauder basis of $C_0(I, \mathbb{C}_p)$. Consider the operator

$$A : \mathbb{C}_p^{\otimes I} \rightarrow C_0(I, \mathbb{C}_p)$$

$$1_z \mapsto \begin{cases} 1_{a+1} & (z = n \in \mathbb{N} \subset I) \\ 1_{(0,n-1)} & (z = (0,n) \in \{0\} \times \mathbb{N}_+ \subset I) \\ 1 + [a]1_{a,0} & (z = (a,0) \in \bar{\mathbb{F}}_p \times \{0\} \subset I) \\ 1_{(a,n-1)} + [a]1_{(a,n)} & (z = (a,n) \in \bar{\mathbb{F}}_p \times \mathbb{N}_+ \subset I) \end{cases}$$

32
Proposition 3.6. For a ring $R$, let $S$ and $T$ be $R$-algebras. For an $R$-algebra homomorphism $\phi: S \to T$ and an element $A \in S$, one has $\sigma_{\phi,R}(C_0(I, C_p))(A) \subset C_p$. 

For an element $\lambda \in C_p$, consider the operator $B: C_p^d \to C_0(I, C_p)$, 

$$B: C_p^d \to C_0(I, C_p)$$

$$1_{\lambda} \mapsto \begin{cases} \sum_{i=0}^{n-1} \lambda^{n-i-1} 1_i + \lambda^n \sum_{i=0}^{\infty} (\lambda-\lambda[i])^i 1_{(\lambda,i)} & (z = n \in \mathbb{N} \subset I) \\ \sum_{i=n+1}^{\infty} (\lambda-\lambda[i])^i 1_{(\lambda,i)} & (z = (\lambda, n) \in \lambda \times \mathbb{N} \subset I) \\ (|\lambda|-a)^{-n} 1_{(\lambda,0)} - \sum_{i=0}^{n-1} \sum_{j=i}^{\infty} \frac{b(j)(|\lambda|-a)^{j-1}(a)^j}{(j-n)!} 1_{(a,i)} & (z = (a, n) \in \lambda(\mathbb{F}_p \setminus \lambda) \times \mathbb{N} \subset I) \end{cases}$$

densely defined on $C_0(I, C_p)$. By the estimation of the norms of entries of the canonical matrix representation, $A$ is uniquely extended to an everywhere defined bounded operator, and one has $\sigma_{\phi,R}(C_0(I, C_p))(A) \subset C_p$.

An operator with empty spectrum is exceptional in this paper because such an operator never admits a functional calculus. We mainly deal with an operator with spectrum one has.

**Proof.** It immediately follows from the inclusion $\phi(S^\times) \subset T^\times$. 

In particular, the inclusion $\sigma_{\phi}(A) \subset \sigma_{\phi(1)}(A)$ for an element $A \in \mathcal{A}(1)$ is important. Here $\mathcal{A}(1)$ is regarded as a $k^\circ$-algebra. Note that unlike an Archimedean $C^*$-algebra, the spectrum $\sigma_{\phi}(A)$ depends not only on $A$ but also $\mathcal{A}$. It is because the proof of the independence of the choice of $\mathcal{A}$ in the Archimedean case heavily uses the topological property of the real line $\mathbb{R}$ and the complex plain $\mathbb{C}$. See [Dou] 4.28.

**Example 3.7.**

$$\sigma_{k[T]}(T) = k^\circ$$

$$\sigma_{k[T,T^{-1}]}(T) = (k^\circ)^{\times} = k^\circ \setminus k_{00}^\circ.$$ 

*In particular, the spectra of them are not stable under the base change.*

**Proposition 3.8.** For any element $A \in \mathcal{A}$, its spectrum $\sigma_{\mathcal{A}}(A) \subset k$ is a bounded closed subset.

**Proof.** For an element $a \in k$ with $|a| > \|A\|$, one has $\|a^{-1}A\| < 1$ and hence

$$A - a = -a(1 - a^{-1}A) \in k^\times(1 + \mathcal{A}(1-)) \subset \mathcal{A}^{\times}.$$
by Proposition 1.14. Therefore \( \sigma_{A}(A) \) is contained in \( \{ a \in k \mid |a| \leq \|A\| \} \) and is bounded.

Take a resolvent \( a \in k \setminus \sigma_{A}(A) \). For an element \( b \in k \) with \( |b - a| < \|(A - a)^{-1}\| \), one has \( \|(b - a)(A - a)^{-1}\| < 1 \) and hence
\[
A - b = (A - a) - (b - a) = (A - a)(1 - (b - a)(A - a)^{-1}) \in \mathcal{A}^\times(1 + \mathcal{A}(1-)) \subset \mathcal{A}^\times
\]
by Proposition 1.14. Therefore \( \sigma_{A}(A) \subset k \) is closed. \( \square \)

**Corollary 3.9.** Suppose \( k \) is a finite field endowed with the trivial norm or a local field. For any element \( A \in \mathcal{A} \), its spectrum \( \sigma_{A}(A) \subset k \) is compact.

**Proof.** The ring \( k^\circ \) of integral elements is compact and any closed disc of radius in \( |k^\times| \subset (0, \infty) \) is isomorphic to \( k^\circ \). Any bounded closed subset is contained in such a closed disc, and hence is compact. \( \square \)

### 3.3 Reduction of the Spectrum

We observe a certain compatibility of spectra and reductions. For an element \( A \in \mathcal{A}(1) \) the reduction of the spectrum of an element \( A \) in \( \mathcal{A} \) does not coincides with the spectrum of the reduction \( \overline{A} \) in \( \mathcal{A} \) in general. However, the spectrum of \( A \) in \( \mathcal{A}(1) \) is sufficiently large.

**Proposition 3.10.** For an element \( A \in \mathcal{A}(1) \), one has
\[
\sigma_{\mathcal{A}(1)}(A) = \bigcup_{\lambda \in \sigma_{\mathcal{A}}(A)} \overline{\lambda}.
\]

Beware that an element \( \overline{\lambda} \) of the quotient set \( \overline{\mathcal{A}} = \mathcal{A}(1)/\mathcal{A}(1-) \) is a coset, which is a subset of \( \mathcal{A}(1) \), in the sense of the set-theoretical tautology.

**Proof.** The assertion is the direct implication from Proposition 2.5. \( \square \)

**Proposition 3.11.** Let \( \sigma \subset k \) be a non-empty compact subset with \( \sigma \neq \{0\} \). For any element \( a \in k^\times \) with \( |a| = \sup_{\lambda \in \sigma} |\lambda| > 0 \), the canonical homeomorphism \( \sigma \rightarrow a^{-1}\sigma \subset k^\circ \) and the canonical projection \( k^\times \rightarrow \overline{k} \) induces a surjective map \( \sigma_{C(\sigma,k)}(z) \rightarrow \sigma_{C(\sigma,k)}(a^{-1}z) \).

Note that the compactness of the non-empty set \( \sigma \) guarantees that there is an element \( a \in \sigma \) with \( |a| = \sup_{\lambda \in \sigma} |\lambda| > 0 \).

**Proof.** Since
\[
\sigma_{C(\sigma,k)}(z) = \sigma_{C_{k}(\sigma,k)}(z) = \sigma
\]
and
\[
\sigma_{C(\sigma,k)}(a^{-1}z) = \sigma_{C_{k}(\sigma,k)}(a^{-1}z) = \sigma_{C_{k}(a^{-1}\sigma,k)}(z') = a^{-1}\sigma
\]

34
for the coordinate function $z': a^{-1}\sigma \leftrightarrow k$ by Proposition [1.44] we may and do assume $a = 1$ replacing $\sigma$ by $a^{-1}\sigma$. Note that $C(\sigma, k)(1) = C(\sigma, k'^*)$ and $C(\sigma, k) = C(\sigma, \overline{k})$. Since $\overline{k}$ is discrete, the Banach $\overline{k}$-algebra $C(\sigma, \overline{k})$ is the $\overline{k}$-subalgebra of $k'^* \circ$ consisting of locally constant functions. The inverse of a locally constant function invertible in $k'^*$ is again locally constant. Since the image of $\overline{z}: \sigma \rightarrow \overline{k}$ is the image $\overline{\sigma} \subset \overline{k}$ of $\sigma$ by the canonical projection $k^o \rightarrow \overline{k}$, one has $\overline{\sigma(\overline{\alpha}(\overline{z}))} = \overline{\sigma}$, which was what we wanted. 

\[ \square \]

### 3.4 Rigid Continuous Functional Calculus

We define the normality of an operator by the admissibility of a functional calculus by the class of rigid continuous functions introduced in Definition [1.40].

**Definition 3.12.** For a non-empty bounded closed subset $\sigma \subset k$, a clopen subset $S \subset \sigma$ is said to be $\sigma$-measurable or simply a measurable subset of $\sigma$ if the characteristic function $1_S : \sigma \rightarrow k$ of $S$ is contained in $C_{rig}(\sigma, k)$. Denote by $\Omega_\sigma \subset 2^\sigma$ the collection of measurable subsets of $\sigma$.

**Example 3.13.** If $\sigma \subset k$ is compact, every clopen subset $S \subset \sigma$ is $\sigma$-measurable by Proposition [1.44].

**Definition 3.14.** For an element $A \in \mathcal{A}$, the rigid continuous functional calculus of $A$ in $\mathcal{A}$ is an isometric $k$-algebra homomorphism

$\iota_A : C_{rig}(\sigma_{\mathcal{A}}(A), k) \rightarrow \mathcal{A}$

sending the coordinate function $z: \sigma \rightarrow k$ to $A$. In the case that $C_{rig}(\sigma_{\mathcal{A}}(A), k) = C(\sigma_{\mathcal{A}}(A), k)$, call it the continuous functional calculus of $A$.

The rigid continuous functional calculus is unique because the localisation of $k[z] \subset C_{rig}(\sigma_{\mathcal{A}}(A), k)$ is dense. The uniqueness holds even if we replace the condition “isometric” to “homeomorphic onto the image”. The rigid continuous functional calculus is deeply related to unitary operators by the condition “isometric”.

**Definition 3.15.** For an element $A \in \mathcal{A}$ admitting the rigid continuous functional calculus in $\mathcal{A}$, set $\Omega_A := \Omega_{\sigma_{\mathcal{A}}}(A)$. Call a subset $S \subset k$ belonging to $\Omega_A$ an $A$-measurable set.

The condition that $C_{rig}(\sigma_{\mathcal{A}}(A), k) = C(\sigma_{\mathcal{A}}(A), k)$ is too strong when $k$ is not a local field, and hence we mainly consider the rigid continuous functional calculus instead of the continuous functional calculus. If $A$ admits the rigid continuous functional calculus in $\mathcal{A}$, then the spectrum $\sigma_{\mathcal{A}}(A) \subset k$ is non-empty because $C_{rig}(0, k) = 0$ and $\sigma_{\mathcal{A}}(0) = 0 \subset k$. Remark that the image of the rigid continuous functional calculus, if exists, is a closed $k$-subalgebra of the commutative Banach $k$-algebra $\mathcal{L}_{\mathcal{A}}(A)$. Moreover if $\sigma_{\mathcal{A}}(A)$ is compact, then $C_{rig}(\sigma_{\mathcal{A}}(A), k) = C(\sigma_{\mathcal{A}}(A), k)$ and the continuous functional calculus induces the isometric isomorphism $C_{rig}(\sigma_{\mathcal{A}}(A), k) \cong_k \mathcal{L}(A)$ by Proposition [1.44]. First we deal with the rigid continuous functional calculus of a bounded normal operator. The rigid continuous functional calculus of an unbounded normal operator will appear in §6.
Proposition 3.16. If an element $A \in \mathcal{A}$ admits the rigid continuous functional calculus in $\mathcal{A}$, then the equality
\[
\|A\| = \sup_{\lambda \in \sigma_{\mathcal{A}}(A)} |\lambda|
\]
holds.

Proof. The right hand side is the supremum norm of the coordinate function $z: \sigma \hookrightarrow k$, and the equality holds because the reductive continuous functional calculus is an isometry. □

Definition 3.17. A bounded operator $A$ on a $k$-Banach space $V$ is said to be a bounded $k$-valued normal operator if $A$ admits the rigid continuous functional calculus in $\mathcal{B}_k(V)$.

Definition 3.18. A matrix $M \in M_n(k)$ is said to be a $k$-valued normal matrix if $M$ corresponds to a bounded $k$-valued normal operator on $k^n$ as the matrix representation with respect to the canonical basis.

A matrix $M \in M_n(k)$ is a $k$-valued normal matrix if and only if $M$ is diagonalisable by a unitary matrix in $k$. As we mentioned, the notion of the unitarity of an operator on $k^n$ makes sense only when one fixes the norm of $k^n$. If one replaces the canonical basis by another one, the normality of an operator changes. It is remarkable that a matrix is diagonalisable in $k$ if and only if it represents a bounded $k$-valued normal operator on $k^n$ with respect to some norm of $k^n$.

Definition 3.19. A bounded operator $A$ on a $k$-Banach space $V$ is said to be a bounded normal operator if there is a finite extension $K/k$ such that the image of $A \otimes 1 \in \mathcal{B}_k(V) \otimes_k K$ by the bounded $K$-algebra isomorphism $\mathcal{B}_k(V) \otimes_k K \to \mathcal{B}_K(V \otimes_k K)$ is a bounded $K$-valued normal operator.

Here the complete tensor product $\hat{\otimes}$ means the completion of the tensor product with respect to the non-Archimedean tensor norm. Since we only consider a finite extension $K/k$, the completion process can be skipped.

3.5 Passage to $p$-adic Quantum Theory

Once the normality of a bounded operator is defined, one obtains a formulation of $p$-adic Quantum theory with bounded $p$-adic physical quantities. For the unbounded generalisation, see Definition 6.25.

Definition 3.20 ($p$-adic physical state). A $p$-adic state vector over $k$ is a non-zero vector $|\phi\rangle$ in a strictly Cartesian $k$-Banach space $V$. When there is no ambiguity of the base field $k$, just say $|\phi\rangle$ a $p$-adic vector state in $V$. A $p$-adic physical state represented by $|\phi\rangle \in V$ is the lay $[|\phi\rangle]$ in the projective space $\mathbb{P}(V) := (V \setminus \{0\})/k^\times$.

Definition 3.21 (bounded $p$-adic physical quantity). A bounded $p$-adic physical quantity over $k$ is a bounded $k$-valued normal operator on a strictly Cartesian $k$-Banach space.
**Definition 3.22** (Born rule). For a bounded p-adic physical quantity $A$ over $k$ on a strictly Cartesian $k$-Banach space $V$, define the probability function $P(A \in \cdot)$ in the following way:

$$P(A \in \cdot): \mathcal{F}(V) \times \Omega_A \rightarrow [0, 1]$$

$$([[\phi]], S) \mapsto P_{[[\phi]]}(A \in S) := \|k_A(1_S) | \phi \rangle\|.$$

where $| \phi \rangle \in [[| \phi \rangle]$ is a representative with the normalisation condition $\| | \phi \rangle\| = 1$. The probability function is well-defined. For a p-adic physical state $[[| \phi \rangle]$ and an $A$-measurable clopen subset $S \subset \sigma_{\mathcal{B}_k(V)}(A)$, call $P_{[[\phi]]}(A \in S)$ the probability that the observed value of $A$ in $[[| \phi \rangle]$ is contained in $S$.

The probability is non-Archimedean, and hence is not additive. For a p-adic physical state $[[| \phi \rangle]$ and $A$-measurable clopen subsets $S, S' \in \Omega_A$, one has the following properties:

(i) $P_{[[\phi]]}(A \in \emptyset) = 0, P_{[[\phi]]}(A \in \sigma_{\mathcal{B}_k(V)}(A)) = 1$;

(ii) Even if $P_{[[\phi]]}(A \in S) = 1$ and $S \cap S' = \emptyset$, the equality $P_{[[\phi]]}(A \in S') = 0$ does not hold in general;

(iii) The non-Archimedean property

$$P_{[[\phi]]}(A \in S \cup S') \leq \max \left\{ P_{[[\phi]]}(A \in S), P_{[[\phi]]}(A \in S') \right\}$$

holds;

(iv) If $S \subset S'$, the monotonousness

$$P_{[[\phi]]}(A \in S) \leq P_{[[\phi]]}(A \in S')$$

holds; and

(v) If $S \cap S' = \emptyset$, the non-Archimedean orthogonality

$$P_{[[\phi]]}(A \in S \sqcup S') = \max \left\{ P_{[[\phi]]}(A \in S), P_{[[\phi]]}(A \in S') \right\}$$

holds.

For proofs of the properties (iii)-(v), see Proposition 5.11 and Proposition 5.12.

### 3.6 Reductive Functional Calculus

In order to establish an algorithm for a criterion of the unitarity of an operator, we study the functional calculus of the reduction. There are two cases concerning the reduction. One is the case that the reduction has the finite spectrum, and the other one is the case that the reduction has the infinite spectrum. We will deal with the infinite reductive spectrum case in §4 and with the finite reductive spectrum case in §5.
Definition 3.23. An element \( A \in \mathcal{A} \) is said to admit the reductive functional calculus in \( \mathcal{A} \) if \( \|A\| \in |k^\times| \) and if for an element \( a \in k^\times \) with \( \|A\| = |a| \) the normal reduction \( a^{-1}A \) of \( a^{-1}A \in \mathcal{A} \) of \( a^{-1}A \in \mathcal{A} \) admits the continuous functional calculus \( \iota_{a^{-1}A} : C(\sigma_{\mathcal{A}}(a^{-1}A), \overline{k}) \to \overline{A} \).

The residue field \( \overline{k} \) is a trivial valuation field. When \( \sigma_{\mathcal{A}}(a^{-1}A) \subset \overline{k} \) is a finite subset, one has \( C_{rig}(\sigma_{\mathcal{A}}(a^{-1}A), \overline{k}) = C(\sigma_{\mathcal{A}}(a^{-1}A), \overline{k}) \cong (\overline{k})^{\sigma_{\mathcal{A}}(a^{-1}A)} \), and hence in the reductive rigid continuous functional calculus is always the reductive functional calculus. We mainly consider the reductive functional calculus of an operator such that the spectrum of the reduction is a finite set, and hence one might assume that the reductive functional calculus is just the rigid continuous functional calculus of the reduction.

Example 3.24. A matrix \( M \in M_n(k) \) admits the reductive functional calculus in \( M_n(k) \) if and only if \( M \neq 0 \) and for an element \( a \in k^\times \) with \( \|M\| = |a| \) the matrix reduction \( a^{-1}M \in M_n(k)^- = M_n(\overline{k}) \) is diagonalisable in \( \overline{k} \).

Proposition 3.25. If a non-zero element \( A \in \mathcal{A}\setminus\{0\} \) has the compact spectrum and admits the continuous functional calculus in \( \mathcal{A} \), then \( A \) admits the reductive functional calculus in \( \mathcal{A} \).

Proof. Since \( A \neq 0 \) admits the rigid continuous functional calculus in \( \mathcal{A} \), one has \( \|A\| = |k^\times| \) by Proposition 3.16 and by the compactness of \( \sigma := \sigma_{\mathcal{A}}(A) \). Replacing \( A \) by \( a^{-1}A \) for an element \( a \in k^\times \) with \( \|A\| = |a| \), we may and do assume \( \|A\| = 1 \). We follow the conventions in the proof of Proposition 3.11. To begin with, we verify \( \sigma_{\mathcal{A}}(A) = \overline{\sigma} \). Since \( \sigma_{\mathcal{A}}(A) \subset \sigma_{\mathcal{A}}(1)(A) \) by definition, the canonical projection \( k^\circ \to \overline{k} \) induces the reduction map \( \sigma \to \sigma_{\mathcal{A}}(\overline{A}) \) by Proposition 3.5, and hence one has \( \overline{\sigma} \subset \sigma_{\mathcal{A}}(\overline{A}) \). Assume the reduction map \( \sigma \to \sigma_{\mathcal{A}}(\overline{A}) \) is not surjective, and take a reductive spectrum \( \Lambda \in \sigma_{\mathcal{A}}(\overline{A}) \) which is not contained in the image \( \overline{\sigma} \). Take a representative \( \lambda \in \Lambda \). Due to the choice of \( \Lambda \), one has \( \|\lambda' - \lambda\| = 1 \) for any \( \lambda' \in \sigma \), and hence \( (z - \lambda)^{-1} \in C(\sigma, k)(1)^\times \). It follows \( z - \lambda \in C(\sigma, \overline{k}^\times) \) by Proposition 3.5 and it contradicts the condition \( \Lambda \in \sigma_{\mathcal{A}}(\overline{A}) \).

Thus one has \( \sigma_{\mathcal{A}}(\overline{A}) = \overline{\sigma} \).

Since the continuous functional calculus \( \iota_{\Lambda} : C(\sigma, k) \to \mathcal{A} \) is an isometry, it induces the injective ring homomorphism \( \iota_{\Lambda} : C(\sigma, k) \to \mathcal{A} \). For an element \( f = (f_\lambda)_{\lambda \in \sigma} \in k^\sigma \), the composition

\[
c(f) : \sigma \to \overline{\sigma} \xrightarrow{f} \overline{k}
\]
sending \( \lambda \in \sigma_{\mathcal{A}}(A) \) to \( f_\lambda \) is continuous. One obtains a \( \overline{k} \)-algebra homomorphism

\[
\iota_{\overline{k}} : C(\sigma_{\mathcal{A}}(\overline{A}), \overline{k}) = k^\sigma \to C(\sigma_{\mathcal{A}}(A), \overline{k}) \cong \overline{k} C(\sigma_{\mathcal{A}}(A), \overline{k}) \xrightarrow{\iota_{\Lambda}} \mathcal{A},
\]

which is obviously injective and sends the coordinate function \( \overline{z} : \sigma_{\mathcal{A}}(\overline{A}) \to \overline{k} \) to \( \iota_{\Lambda}(z) = \overline{A} \in \mathcal{A} \), where the isomorphism \( C(\sigma_{\mathcal{A}}(A), \overline{k}) \cong \overline{k} C(\sigma_{\mathcal{A}}(A), \overline{k}) \) is given in Remark 3.3. This is the reductive functional calculus of \( A \) in \( \mathcal{A} \). \( \square \)
Thus the normality implies the reductive normality. Then a natural question arises: When/Whether does the reductive normality imply the normality? It contains the question when/whether the diagonalisation of a matrix can be constructed by the diagonalisation of its reduction. We will answer the question, and the answer yields a criterion for the normality. See Theorem 4.2, Theorem 5.7, and Theorem 5.20.

3.7 Holomorphic Functional Calculus

We define a holomorphic functional calculus of an operator. In the Archimedean case, the regularity “holomorphic” of a complex function means that the function admits the Taylor development at each point in the domain, but such a regularity is very weak in the non-Archimedean case because a non-Archimedean field $k$ is totally disconnected. Therefore we use here a stronger regularity: “rigid analytic”. Denote by $\mathbb{A}^1_k$ the affine line as Berkovich’s analytic space. It is the analytification of the scheme-theoretic affine line $\text{Spec}(k[T])$. A rational domain of $\mathbb{A}^1_k$ a closed subset $V \subset \mathbb{A}^1_k$ which is contained in a sufficiently large closed disc of $\mathbb{A}^1_k$ as a rational domain. For a closed subset $\sigma \subset k$, a rational neighbourhood of $\sigma$ is a rational domain of $\mathbb{A}^1_k$ containing the image of $\sigma$ by the identification $k \cong \mathbb{A}^1_k(k)$. We use a rational neighbourhood of the spectrum of an operator for a holomorphic functional calculus. The rationality is essential for the uniqueness of the holomorphic functional calculus because an affinoid domain is rational if and only if the image of the suitable localisation of the global section is dense in the corresponding affinoid algebra. For conventions for Berkovich’s spectrum, see [Ber].

**Definition 3.26.** For an element $A \in \mathcal{A}$, a holomorphic functional calculus of $A$ in $\mathcal{A}$ is a bounded $k$-algebra homomorphism

$$\iota_A : H^0(V, O_{\mathbb{A}^1_k}) \to \mathcal{A}$$

sending the coordinate function $z : V \hookrightarrow k$ to $A$ for a rational neighbourhood $V \subset \mathbb{A}^1_k$ of $\sigma_{\mathcal{A}}(A)$.

For a fixed rational neighbourhood of $\sigma_{\mathcal{A}}(A)$, the holomorphic functional calculus is unique because the localisation of $k[T]$ is dense in the corresponding affinoid domain. The definition of a holomorphic functional calculus depends on the choice of a rational neighbourhood $V \subset \mathbb{A}^1_k$ of $\sigma_{\mathcal{A}}(A)$. It is obvious that the restriction of a rigid analytic function on an affinoid domain to a narrower affinoid domain is again rigid analytic, and hence one hopes that the rational neighbourhood in the definition of a holomorphic functional calculus is taken to be as small as possible.

4 Reductively Infinite Operator

To begin with, we deal with a reductively infinite operator. A reductively infinite operator has the non-compact spectrum, and hence Vishik’s non-Archimedean analogue of Riesz functional calculus in [Vis] does not work here.
4.1 Structure Theorem

We verify that a reductively infinite operator generates the closed subalgebra isomorphic to the Tate algebra, and hence the holomorphic functional calculus always works.

**Definition 4.1.** Let $\mathcal{A}$ be a Banach $k$-algebra. An element $A \in \mathcal{A}$ is said to be reductively infinite if $\|A\| \in k^\times$ and if the spectrum $\sigma_{\mathcal{A}}(a^{-1}A) \subset \bar{k}$ of the normal reduction $a^{-1}A \in \mathcal{A}$ is an infinite set for an element $a \in k^\times$ with $\|A\| = |a|$. The cardinality of $\sigma_{\mathcal{A}}(a^{-1}A)$ is independent of the choice of $a \in k^\times$ with $\|A\| = |a|$. Indeed, for elements $a, b \in k^\times$ with $\|A\| = |a| = |b|$, the multiplication by $ab^{-1} \in \bar{k}$ induces the bijection $\sigma_{\mathcal{A}}(a^{-1}A) \to \sigma_{\mathcal{A}}(b^{-1}A)$.

**Theorem 4.2.** Let $\mathcal{A}$ be a Banach $k$-algebra and $A \in \mathcal{A}$ a reductively infinite element. Then the $k$-algebra homomorphism $k[T] \to \mathcal{A}$: $T \mapsto A$ induces the isometric isomorphism $i_A: k[T] \to \mathcal{L}(A)$ sending $T$ to $A$.

**Proof.** To begin with, we prove that $\|F(A)\| = 1$ for a polynomial $F \in k^\times[T] \subset k[T]$ with $\|F\| = 1$ with respect to the Gauss norm $\|\cdot\|$ of $k[T]$. Since $F \in k^\times[T]$, one has $\|F(A)\| \leq 1$. The equality $\|F\| = 1$ implies that the reduction $\bar{F} \in k[T] = \bar{k}[T]$ is non-trivial. The condition that $A$ is reductively infinite guarantees that $\bar{F}(A) = \bar{F}(\bar{A}) \neq 0 \in \bar{k}$. Indeed, assume that $\bar{F}(A) = 0$. The non-trivial polynomial $\bar{F} \in \bar{k}[T]$ has at most finitely many zeros, and hence there is an element $a \in \sigma_{\mathcal{A}}(\bar{A}) \subset \bar{k}$ such that $\bar{F}(a) \neq 0 \in \bar{k}$.

Set $\overline{G} := \frac{\bar{F}(T) - \bar{F}(a)}{T - a} \in \bar{k}[T]$. One has

$$1 = \left(\bar{F}(a)^{-1}(\bar{F}(a))\right)^{-1} = \bar{F}(a)^{-1}(\bar{F}(a) - \bar{F}(a)) = -\bar{F}(a)^{-1}\overline{G(\bar{A} - a)} \in \mathcal{A} \otimes_{\bar{k}} \bar{k},$$

and therefore $\bar{A} - a \in \mathcal{A}^\times$. It contradicts the condition that $a \in \sigma_{\mathcal{A}}(\bar{A})$. Thus $\bar{F}(\bar{A}) = \bar{F}(\bar{A}) \neq 0 \in \mathcal{A}$. It follows that $\|F(A)\| = 1$.

Now take a polynomial $F \in k[T]$ and suppose $\|F(A)\| \leq 1$. Assume $F \notin k^\times[T]$. Then there is an element $a \in k \setminus k^\circ$ such that $\|F\| = |a|$. One has $\|a^{-1}F\| = |a|^{-1}\|F\| = 1$ and hence $a^{-1}F \in k^\times[T]$. On the other hand, the inequality $\|(a^{-1}F)(A)\| = |a|^{-1}\|F(A)\| \leq |a|^{-1} < 1$ implies that $a^{-1}F(A) = 0 \in \mathcal{A}$, and it contradicts the fact we proved above. Thus $F \in k^\times[T]$.

Finally take a non-trivial polynomial $F \in k[T]$. For an element $a \in k^\times$ with $\|F\| = |a|$, one has $\|(a^{-1}F)(A)\| = 1$ and therefore $\|F(A)\| = |a|\|(a^{-1}F)(A)\| = |a| = \|F\|$. It follows that $k[T] \to \mathcal{A}$: $T \mapsto A$ is an isometry with respect to the Gauss norm $\|\cdot\|$ of $k[T]$, and hence it induces the isometry $i_A: k[T] \to \mathcal{A}$. Since $k[T] \subset k[T]$ is a dense $k$-subalgebra, the image of $i_A$ is contained in $\mathcal{L}(A)$ as a dense subspace. In general, the image of a closed subset by an isometry between complete metric spaces is complete, and hence it is again closed. Therefore the image of $i_A$ coincides with $\mathcal{L}(A) \subset \mathcal{A}$. One has obtained the isometric isomorphism $i_A: k[T] \to \mathcal{L}(A)$. \qed
Corollary 4.3. A reductively infinite element $A \in \mathcal{A}$ admits the holomorphic functional calculus $i_A : H^0(||A||D^1_k, \mathcal{G}_a) \rightarrow \mathcal{A}$, where $||A||D^1_k \subset A^1_k$ is the strictly $k$-affinoid domain $\mathcal{M}_k(k(||A||^{-1}T)) \subset A^1_k$.

Corollary 4.4. For a discrete index set $I$, a reductively transcendental operator $A \in \mathcal{B}_k(C_0(I, k))$ admits the holomorphic functional calculus $i_A : H^0(||A||D^1_k, \mathcal{G}_a) \rightarrow \mathcal{B}_k(C_0(I, k))$.

Corollary 4.5. Let $\rho : \mathcal{A} \rightarrow \mathcal{B}_k(V)$ be a normalisable $k$-Banach representation. For an element $A \in \mathcal{A}$, if $||A|| \leq ||\rho(A)||$ and if $\rho(A)$ is reductively transcendental operator on $V$, then $A$ admits the holomorphic functional calculus $i_A : H^0(||A||D^1_k, \mathcal{G}_a) \rightarrow \mathcal{A}$.

Thus a reductively infinite operator automatically admits a holomorphic functional calculus. Suppose that $k = \mathbb{C}_p$ and $\sigma_{\mathcal{B}_k(V)}(A) = \mathbb{C}^o_p$ for an operator $A \in \mathcal{B}_{\mathbb{C}_p}(V)$. As we referred in Example 4.1, it immediately follows from Theorem 4.2 that $\mathbb{C}_p[z] = \mathbb{C}_{rig}(\mathbb{C}_p, \mathbb{C}_p)$. Therefore in this case, the holomorphic functional calculus $i_A : H^0(D^1_{\mathbb{C}_p}, \mathcal{G}_a) \rightarrow \mathcal{A}$ is the rigid continuous functional calculus, and hence is the finest functional calculus. If one needs to extend the holomorphic functional, he or she should consider addition topological data such as the topology of pointwise convergences of $C_{bd}(\mathbb{C}_p, \mathbb{C}_p)$ and the strong operator topology of $\mathcal{B}_k(V)$. The subsequent two examples might help a reader to grasp ways of extensions.

4.2 Shift Operator and Toeplitz Operator

The following two examples might help a reader to understand the holomorphic functional calculus. We first assume that the valuation of $k$ is discrete and that $\bar{k}$ is an infinite field, but the assumptions are easily removed by a base change argument.

Example 4.6. Suppose that the valuation of $k$ is discrete and that $\bar{k}$ is an infinite field. Let $A \in \mathcal{A} := \mathcal{B}_k(C_0(\mathbb{Z}, k))$ be the shift operator $(Af)(z) = f(z+1)$. Its matrix representation $M$ with respect to the canonical orthonormal Schauder basis is

$$M = \begin{pmatrix} \ldots & \ldots & \ldots & \ldots & \cdot \ldots \\ \cdot 0 1 0 0 & \ldots \\ \cdot 0 0 1 0 & \ldots \\ \cdot 0 0 0 1 & \ldots \\ \cdot 0 0 0 0 & \ldots \end{pmatrix},$$

where the diagonal component is the line with 0’s right lower than the line with 1’s. It is obvious that $A$ is unitary, and hence one has $||A|| = 1$ and $\sigma_{\mathcal{A}}(A) \subset (\bar{k})^\times$. Its normal reduction $\overline{A} \in \mathcal{A}$ coincides with the matrix reduction $\overline{A} \in \mathcal{B}_k(\overline{\mathbb{C}_p})$ by Proposition 2.9 and the matrix reduction is the shift operator $\overline{A}e_n = e_{n-1}$, where $(e_n)_{n \in \mathbb{Z}}$ is the canonical basis of $\overline{\mathbb{C}_p}$. The spectrum of the algebraic shift operator $\overline{A}$ coincides with $\bar{k}^\times$, and hence one has $\sigma_{\mathcal{A}}(A) = (\bar{k})^\times$ by Proposition 2.5. Therefore since $\bar{k}$ is an infinite field, then $A$
and \( A^{-1} \) are reductively infinite. Applying the holomorphic functional calculus to both \( A \) and \( A^{-1} \), one obtains the holomorphic functional calculus

\[
\tau_A : k[T, T^{-1}] \to \mathcal{A}
\]

\[
F = \sum_{n \in \mathbb{Z}} F_n T^n \mapsto F(A) = \sum_{n \in \mathbb{Z}} F_n A^n.
\]

For a rigid analytic function \( F \in k[T, T^{-1}] \), the matrix representation \( F(M) \) of the substitution \( \tau_A(F) = F(A) \) is

\[
F(M) = \begin{pmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots \\
\cdots & F_0 & F_1 & F_2 & F_3 & \cdots \\
\cdots & F_{-1} & F_0 & F_1 & F_2 & \cdots \\
\cdots & F_{-2} & F_{-1} & F_0 & F_1 & \cdots \\
\cdots & F_{-3} & F_{-2} & F_{-1} & F_0 & \cdots \\
\ddots & \vdots & \vdots & \vdots & \vdots 
\end{pmatrix}
\]

and this result is analogous to the theory of the Fourier development of a function on \([0, 1]\).

**Example 4.7.** Suppose that the valuation of \( k \) is discrete and that \( \overline{k} \) is an infinite field. Let \( A \in \mathcal{A} := \mathcal{B}_k(\mathcal{C}_0(\mathbb{N}, k)) \) be the Toeplitz operator \((Af)(z) = f(z + 1)\). Its matrix representation \( M \) with respect to the canonical orthonormal Schauder basis is

\[
M = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 1 & \cdots \\
0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots 
\end{pmatrix},
\]

where the order of \( \mathbb{N} \) is given along the positive direction. It is obvious that \( ||A|| = 1 \) and \( \sigma_{ad}(A) \subset k^\circ \). Its normal reduction \( \overline{A} \in \mathcal{A} \) coincides with the matrix reduction \( \overline{A} \in \mathcal{B}_k(k^{\overline{\mathbb{N}}}) \) by Proposition 2.9, and the matrix reduction is the same Toeplitz operator

\[
\overline{A} : e_0 \mapsto 0, e_n \mapsto e_{n-1}, \forall n \in \mathbb{N}^+,
\]

where \((e_n)_{n \in \mathbb{N}}\) is the canonical basis of \( k^{\overline{\mathbb{N}}} \). The spectrum of the algebraic Toeplitz operator \( \overline{A} \) coincides with \( \overline{k} \), and hence one has \( \sigma_{ad}(A) = k^\circ \) by Proposition 2.5. Therefore since \( \overline{k} \) is an infinite field, then \( A \) is reductively infinite. Applying the holomorphic functional calculus to \( A \), one obtains the holomorphic functional calculus

\[
F(M) = \begin{pmatrix}
F_0 & F_1 & F_2 & F_3 & \cdots \\
0 & F_0 & F_1 & F_2 & \cdots \\
0 & 0 & F_0 & F_1 & \cdots \\
0 & 0 & 0 & F_0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots 
\end{pmatrix}
\]

42
and it is easy to see that the functional calculus is continuously extended to the rigid analytic functions $k^\times[[T]] \otimes_k k$ on the open unit disc with respect to the Frechet topology of $k^\times[[T]] \otimes_k k$ and the strong operator topology of $A$.

Note that both examples do not need the assumptions of the discreteness of the valuation of $k$ and the infiniteness of $\mathcal{T}$ in fact, because the restriction of the scalar works after the base change by a suitable extension of complete valuation fields. Such a base change technique is valid for analytic or algebraic functions, but not for general continuous functions. It is remarkable that when the residue field is an infinite field, the holomorphic functional calculi are in fact the rigid continuous functional calculi.

5 Reductively Finite Operator

A reductively finite operator possesses few information in its normal reduction and its matrix reduction. For example, there is an operator of operator norm 1 such that its matrix reduction is zero, as we mentioned at Proposition 2.9. Such a difficulty occurs only when we consider a reductively finite operator on an infinite dimensional Banach space. Subtracting a suitable scalar before reducing, we avoid the exceptional case.

5.1 Reduction Algorithm for Matrices

To begin with, we observe the reduction of a matrix. The diagonalisation is the decomposition of the representation space into the direct sum of eigenspaces as vector spaces. The unitary diagonalisation corresponds to the decomposition into the direct sum as Banach spaces. The diagonalisation of the reduction yields a certain decomposition of the representation space as a Banach space. It is a lift of the decomposition of the reduction of the representation space.

Definition 5.1. Let $\mathcal{A}$ be a Banach $k$-algebra. An element $A \in \mathcal{A}$ is said to be reductively finite if $\|A\| \in |k^\times|$ and if the spectrum $\sigma_{\mathcal{A}}(a^{-1}A) \subset \mathcal{A}$ of the normal reduction $a^{-1}A \in \mathcal{A}$ is a finite set for an element $a \in k^\times$ with $\|A\| = |a|$.

Example 5.2. For an extension $K/k$ of non-Archimedean fields, a non-zero matrix $M \in M_n(K)$ is reductively finite if and only if the norm of each entry is contained in $|k|$. If $K/k$ is not algebraic extension such as the complete fractional field of $k\{T\}$ and the non-strict $k$-affinoid field $k\{r^{-1}T, rT^{-1}\}$ for $r \in (0, \infty)\setminus |k^\times|$, the spectrum of the normal reduction $M$ might be empty even if $\|M\| \in |k^\times|$.

Definition 5.3. A matrix $M \in M_n(k)$ is said to be $k$-valued if each eigenvalue of $M$ is contained in $k$.

Definition 5.4. For a $k$-valued matrix $M \in M_n(k)$, define the family $\Sigma(M)$ in the following way: Since $M$ is $k$-valued, the spectrum $\sigma_{M_0(k)}(M)$ is non-empty. For elements $\lambda_1, \lambda_2 \in \sigma_{M_0(k)}(M)$, we write $\lambda_1 \sim \lambda_2$ if $|\lambda_1 - \lambda_2| < \|M\|$. The binary relation $\sim$
is obviously an equivalence relation on the ultrametric space \( \sigma_{M,(k)}(M) \subset k \), and set \( \Sigma(M) := \sigma_{M,(k)}(M) / \sim \).

For a matrix, the criterion of the unitarity in Proposition 5.2 guarantees the compatibility of the reductions and the spectra. See also Proposition 3.11.

**Proposition 5.5.** For a non-zero \( k \)-valued matrix \( M \in M_n(k) \setminus \{0\} \) and \( a \in k^\times \) with \( ||M|| = |a| \), the canonical homeomorphism \( \sigma_{M,(k)}(M) \rightarrow \sigma_{M,(k)}(a^{-1}M) \subset k^\times \) and the canonical projection \( k^\times \rightarrow k \) induces a bijective map \( \Sigma(M) \rightarrow \sigma_{M,(k)}(a^{-1}M) \).

**Proof.** Replacing \( M \) by \( a^{-1}M \) for a suitable element \( a \in k^\times \), we may and do assume \( ||M|| = 1 \) without loss of generality. For a \( \lambda \in \sigma_{M,(k)}(M) \subset k^\times \), one has \( M - \lambda \notin \text{GL}_n(k) \) and in particular \( M - \lambda \notin \text{GL}_n(k^\times) \). It follows \( \lambda \in \sigma_{M,(k)}(M) \) by Proposition 2.5. Therefore the projection \( k^\times \rightarrow k \) induces an injective map \( \Sigma(M) \rightarrow \sigma_{M,(k)}(M) \), which is obviously injective by the definition of \( \sim \). In order to prove the surjectivity, assume there is a \( \Lambda \in \sigma_{M,(k)}(M) \) such that \( \Lambda \) is not contained in the image of \( \Sigma(M) \), i.e., \( \Lambda \cap \sigma_{M,(k)}(M) = \emptyset \) regarding \( \Lambda \) as a subset of \( k^\times \) by the set-theoretical tautology. Take any representative \( \lambda \in \Lambda \). Since \( \Lambda \cap \sigma_{M,(k)}(M) = \emptyset \), one has \( M - \lambda \in \text{GL}_n(k) \). Let \( P \in k[T] \) be the characteristic polynomial of \( M - \lambda \). Since \( M \) is \( k \)-valued, the zeros of \( P \) coincides with \( \sigma_{M,(k)}(M - \lambda) \). The condition \( \Lambda \cap \sigma_{M,(k)}(M) = \emptyset \) guarantees that \( (k^\times)^\times \cap \sigma_{M,(k)}(M - \lambda) = \emptyset \) and hence \( |\text{det}(M - \lambda)| = |P(0)| = 1 \). It follows \( M - \lambda \in \text{GL}_n(k^\times) \) by Proposition 3.2 and therefore \( M - \lambda = \overline{M - \lambda} \in \text{GL}_n(k) \) by Proposition 2.5. It contradicts the assumption \( \Lambda \in \sigma_{M,(k)}(M) \), and hence the map \( \Sigma(M) \rightarrow \sigma_{M,(k)}(M) \) is surjective. \( \Box \)

A reductively scalar matrix is trivially reductively diagonalisable, but there are trivial examples of reductive scalar matrices which are not diagonalisable. To begin with, we prepare the method of removing reductively scalar matrices.

**Lemma 5.6.** For a matrix \( M = (M_{i,j})_{i,j=1}^n \in M_n(k) \), the matrix \( M - M_{1,1} \) is reductively scalar if and only if \( M \) is scalar.

**Proof.** It immediately follows from the fact that the \((1, 1)\)-entry of \( M - M_{1,1} \) is 0. \( \Box \)

Avoiding the case that a matrix is reductively scalar before running the matrix reduction by the method above, one obtains a criterion of unitary diagonalisibility of it.

**Theorem 5.7.** A \( k \)-valued non-scalar matrix \( M = (M_{i,j})_{i,j=1}^n \in M_n(k) \) is unitarily diagonalisable if and only if \( M - M_{1,1} \) is reductively diagonalisable and the matrix representation of the restriction of the operator \( M \) on the \( M \)-stable subspace \( \ker \prod_{\lambda \in \Lambda}(M - M_{1,1} - \lambda) \subset k^n \) with respect to an orthonormal basis is unitarily diagonalisable for any \( \Lambda \in \Sigma(M - M_{1,1}) \).

Note that by [BGR] 2.5.1/5, any subspace of \( k^n \) is a strictly Cartesian normed \( k \)-vector space and admits an orthonormal basis. By Lemma 5.6 if \( M \) is a non-scalar unitarily diagonalisable matrix, one has \( \#\Sigma(M - M_{1,1}) > 1 \) and the subspace \( \ker \prod_{\lambda \in \Lambda}(M - M_{1,1} - \lambda) \subset k^n \) is a proper subspace for each \( \Lambda \in \Sigma(M - M_{1,1}) \). The
restriction of $M$ on the $M$-stable subspace corresponding to a $\Lambda \in \Sigma(M-M_{1,1})$ is again $k$-valued because its minimal polynomial has roots only in $\Lambda \subset k$. Hence Theorem 5.7 surely gives an algorithm for a criterion of unitary diagonalisability.

Proof. For each family $\Lambda \in \Sigma(M-M_{1,1})$, denote by $V_{\Lambda}$ the $M$-stable subspace $\ker \prod_{\lambda \in \Lambda} (M-M_{1,1} - \lambda)^{n} \subset k^{n}$. Take an element $a \in k^{\times}$ with $\|M-M_{1,1}\| = |a|$. To begin with, suppose $M$ is unitarily diagonalisable. Take a unitary matrix $U \in \text{GL}_{n}(k^{\times})$ such that $U^{-1}TU$ is a non-scalar diagonal matrix $\text{diag}(\lambda_{1}, \ldots, \lambda_{n})$. One has

$$
\max_{i=1}^{n} |\lambda_{i} - M_{1,1}| = \|\text{diag}(\lambda_{1} - M_{1,1}, \ldots, \lambda_{n} - M_{1,1})\| = \|U^{-1}(M-M_{1,1})U\| = \|M-M_{1,1}\| = |a|.
$$

It follows

$$
U^{-1} \frac{M-M_{1,1}}{a} U = \text{diag} \left( \frac{\lambda_{1} - M_{1,1}}{a}, \ldots, \frac{\lambda_{n} - M_{1,1}}{a} \right)
$$

$$
\therefore \quad U^{-1} \frac{M-M_{1,1}}{a} U = U^{-1} \frac{M-M_{1,1}}{a} U = \text{diag} \left( \frac{\lambda_{1} - M_{1,1}}{a}, \ldots, \frac{\lambda_{n} - M_{1,1}}{a} \right)
$$

and $M-M_{1,1}$ is reductively diagonalisable. Take a family $\Lambda \in \Sigma(M-M_{1,1})$. Now columns of the unitary matrix $U^{-1}$ form an orthonormal basis of $k^{n}$ consisting of eigenvectors of $M$. Let $u_{1}, \ldots, u_{m} \in k^{n}$ be the distinct eigenvectors belonging to the eigenvalues contained in $\Lambda$. Since $M$ is diagonalisable, one has $V_{\Lambda} = \ker \prod_{\lambda \in \Lambda} (M-M_{1,1} - \lambda)$ and hence $u_{1}, \ldots, u_{m}$ is an orthonormal basis of $V_{\Lambda}$. The matrix representation of the restriction of the operator $M$ on $V_{\Lambda}$ with respect to the orthonormal basis $u_{1}, \ldots, u_{m}$ is the $m$-dimensional diagonal matrix whose entries are the corresponding eigenvalues contained in $\Lambda$, and hence is unitarily diagonalisable.

On the other hand, suppose $M-M_{1,1}$ is reductively diagonalisable and the matrix representation of the restriction of the operator $M$ on the $V_{\Lambda}$ with respect to an orthonormal basis is unitarily diagonalisable for any $\Lambda \in \Sigma(M-M_{1,1})$. For each $\Lambda \in \Sigma(M-M_{1,1})$, take a non-zero vector $v_{\Lambda} \in V_{\Lambda} \setminus \{0\}$. We verify that the vectors $v_{\Lambda}$ form an orthogonal system. Take a linear combination

$$
v = \sum_{\Lambda \in \Sigma(M-M_{1,1})} b_{\Lambda} v_{\Lambda}
$$

for a $(b_{\Lambda}) \in k^{\Sigma(M-M_{1,1})}$, and assume that $\|v\| < \max_{\Lambda \in \Sigma(M-M_{1,1})} |b_{\Lambda}| \|v_{\Lambda}\|$. In particular it is a non-trivial linear combination. Fix elements $c_{\Lambda} \in k^{\times}$ with $|c_{\Lambda}| = \|v_{\Lambda}\|$ for each $\Lambda \in \Sigma(M-M_{1,1})$. Multiplying an appropriate scalar, we may and do assume $b_{\Lambda} c_{\Lambda} \in k^{n}$ for any $\Lambda \in \Sigma(M-M_{1,1})$ and $|b_{\Lambda} c_{\Lambda}| = 1$ for some $\Lambda \in \Sigma(M-M_{1,1})$. Set $v'_{\Lambda} := c_{\Lambda}^{-1} v_{\Lambda}$ for each $\Lambda \in \Sigma(M-M_{1,1})$. Then one has

$$
\|v\| < \max_{\Lambda \in \Sigma(M-M_{1,1})} |b_{\Lambda}| \|v_{\Lambda}\| = \max_{\Lambda \in \Sigma(M-M_{1,1})} |b_{\Lambda} c_{\Lambda}| = 1
$$

45
and

\[ 0 = \overline{v} = \sum_{\lambda \in \Sigma(M-M_{1,1})} b_{\lambda} v_{\lambda} = \sum_{\lambda \in \Sigma(M-M_{1,1})} (b_{\lambda} c_{\lambda}) v'_{\lambda} = \sum_{\lambda \in \Sigma(M-M_{1,1})} b_{\lambda} c_{\lambda} v'_{\lambda} \in \overline{k}. \]

Now the equality \(|a| = \|M - M_{1,1}\|\) implies \(|a^{-1}| \leq 1\) for any \(\lambda \in \Lambda\) and \(\Lambda \in \Sigma(M-M_{1,1})\).

Since \(M - M_{1,1}\) is reductively diagonalisable, the reduction \(a^{-1}(M - M_{1,1}) \in M_n(\overline{k})\) is diagonalisable, and the minimal polynomial of \(a^{-1}(M - M_{1,1})\) is reduced. On the other hand, one has

\[ 0 = \left( \prod_{\lambda \in \Lambda} \left( \frac{M - M_{1,1}}{a} - \frac{\lambda}{a} \right) \right) v'_{\lambda} = \left( \prod_{\lambda \in \Lambda} \left( \frac{M - M_{1,1}}{a} - \frac{\lambda}{a} \right) \right) v_{\lambda} = \left( \frac{M - M_{1,1}}{a} - \frac{\lambda}{a} \right)^{\# \Lambda} v_{\lambda} \]

for a representative \(\lambda_{\Lambda} \in \Lambda\), and it follows

\[ \left( \frac{M - M_{1,1}}{a} - \frac{\lambda_{\Lambda}}{a} \right) v_{\lambda} = 0. \]

Therefore \(v'_{\lambda} \in \overline{k}^n\) is an eigenvector of \(a^{-1}(M - M_{1,1}) \in M_n(\overline{k})\) belonging to the eigenvalue \(a^{-1}\lambda_{\Lambda} \in \overline{k}\). Note that \(a^{-1}\lambda_{\Lambda}\) is independent of the choice of the representative \(\lambda_{\Lambda} \in \Lambda\) by the definition of \(\Sigma(M-M_{1,1})\). Let \(l := \#\Sigma(M-M_{1,1})\). Take a well-order \(\lambda_1 < \cdots < \lambda_l\) on \(\Sigma(M-M_{1,1})\), and rearranging the order, we may and do assume that there is an integer \(0 \leq m \leq l\) such that \(|b_{\lambda_1} c_{\lambda_1}| = \cdots = |b_{\lambda_m} c_{\lambda_m}| = 1\) and \(1 > |b_{\lambda_{m+1}} c_{\lambda_{m+1}}| \geq \cdots \geq |b_{\lambda_l} c_{\lambda_l}|\).

The assumption that there is some \(\Lambda \in \Sigma(M-M_{1,1})\) such that \(|b_{\lambda} c_{\lambda}| = 1\) guarantees \(m \geq 1\). For each \(i = 0, \ldots, m - 1\), one has

\[ \sum_{j=1}^{m} \left( \frac{\lambda_{\Lambda_j}}{a} \right) b_{\lambda_j} c_{\lambda_j} v'_{\lambda_j} = \sum_{j=1}^{m} \left( \frac{M - M_{1,1}}{a} \right) b_{\lambda_j} c_{\lambda_j} v'_{\lambda_j} = \left( \frac{M - M_{1,1}}{a} \right) \left( \sum_{j=1}^{m} b_{\lambda_j} c_{\lambda_j} v'_{\lambda_j} \right) = \left( \frac{M - M_{1,1}}{a} \right) \sum_{j=1}^{m} b_{\lambda_j} v_{\lambda_j} = \left( \frac{M - M_{1,1}}{a} \right) \sum_{j=1}^{l} b_{\lambda_j} v_{\lambda_j} = 0, \]

and hence

\[ \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} v'_{\lambda_1} & v'_{\lambda_2} & \cdots & v'_{\lambda_m} \end{pmatrix} \begin{pmatrix} b_{\lambda_1} c_{\lambda_1} & b_{\lambda_1} c_{\lambda_1} a^{-1} \lambda_{\Lambda_1} & \cdots & b_{\lambda_1} c_{\lambda_1} a^{-1} \lambda_{\Lambda_1}^{m-1} \\ b_{\lambda_2} c_{\lambda_2} & b_{\lambda_2} c_{\lambda_2} a^{-1} \lambda_{\Lambda_2} & \cdots & b_{\lambda_2} c_{\lambda_2} a^{-1} \lambda_{\Lambda_2}^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{\lambda_m} c_{\lambda_m} & b_{\lambda_m} c_{\lambda_m} a^{-1} \lambda_{\Lambda_m} & \cdots & b_{\lambda_m} c_{\lambda_m} a^{-1} \lambda_{\Lambda_m}^{m-1} \end{pmatrix}. \]
valued matrix

an orthogonal system. Therefore presentation of the restriction of the operator is unitarily diagonalisable guarantees that the subspace are distinct by the definition of \( \lambda \).

Take a linear combination consisting of vectors with norm 1, and hence it su\( \lambda \). We conclude that \( ||v|| = \max_{\Lambda \in \Sigma(M-M,1),1} |b_\lambda| ||v_\lambda|| \) and \( v_\lambda,\ldots,v_\lambda \) form an orthogonal system. Therefore \( V_\lambda \)'s are orthogonal.

Now considering the decomposition of \( k^\alpha \) into the generalised eigenspaces of the \( k \)-valued matrix \( M-M,1,1 \), one obtains

\[
k^\alpha = \bigoplus_{\Lambda \in \Sigma(M-M,1,1)} V_\lambda.
\]

Set \( d_\lambda \) := \( \text{dim}_k V_\lambda \) for each \( \Lambda \in \Sigma(M-M,1,1) \). The assumption that the matrix representation of the restriction of the operator \( M \) on \( V_\lambda \) with respect to an orthogonal basis is unitarily diagonalisable guarantees that the subspace \( V_\lambda \subset k^\alpha \) admits an orthonormal basis \( e_{\Lambda,1,\ldots,e_{\Lambda,1,1,1}} \) consisting of eigenvectors of \( M \) for each \( \Lambda \in \Sigma(M-M,1,1) \). We verify that the system \( e_{\Lambda,1,1,\ldots,e_{\Lambda,1,1,1,1,1}} \) forms an orthonormal basis of \( k^\alpha \). Obviously it is a basis consisting of vectors with norm 1, and hence it suffices to show that it is an orthogonal system. Take a linear combination

\[
v = \sum_{j=1}^l \sum_{i=1}^{d_{\lambda_j}} b_{\lambda_j,i} e_{\lambda_j,i}
\]

for \( (b_{\lambda,j},j) \in \prod_{j=1}^l k^{d_{\lambda_j}} \). Since \( \sum_{i=1}^{d_{\lambda_j}} b_{\lambda_j,i} e_{\lambda_j,i} \in V_\lambda \) for each \( 1 \leq j \leq l \), the orthogonality of \( V_\lambda \)'s implies

\[
||v|| = \max_{j=1}^l \left\| \sum_{i=1}^{d_{\lambda_j}} b_{\lambda_j,i} e_{\lambda_j,i} \right\|,
\]

and the orthonormality of \( e_{\Lambda,1,1,\ldots,e_{\Lambda,1,1,1,1,1}} \) guarantees

\[
||v|| = \max_{j=1}^l \left\| \sum_{i=1}^{d_{\lambda_j}} b_{\lambda_j,i} e_{\lambda_j,i} \right\| = \max_{j=1}^l \max_{i=1}^{d_{\lambda_j}} |b_{\lambda_j,i}|.
\]

Thus the system \( e_{\Lambda,1,1,\ldots,e_{\Lambda,1,1,1,1,1}} \) forms an orthonormal basis of \( k^\alpha \). Since \( k^\alpha \) admits an orthonormal basis consisting of eigenvectors of \( M \), the matrix \( M \) is unitarily diagonalisable.
5.2 Lift of the Reductive Eigenspace Decomposition

The result for a matrix is easily extended to a criterion of the admissibility of the rigid continuous functional calculus of a reductively finite operator on an infinite dimensional Banach space. As the reductive diagonalisation of a matrix gives the decomposition of the representation space which is a lift of the eigenspace decomposition of the reductive representation space, the reductive functional calculus yields a partition of unity which is a lift of the system of the images of characteristic functions by the reductive functional calculus.

**Proposition 5.8.** Let $\mathcal{A}$ be a Banach $k$-algebra. If a reductively finite element $A \in \mathcal{A}$ with $\|A\| = 1$ admits the reductive functional calculus, then there is a canonical system $(P_{A,\Lambda})_{\Lambda \in \mathcal{A}} \in \mathcal{A}$ of idempotents satisfying the following:

(i) $\sum_{\Lambda \in \mathcal{A}} P_{A,\Lambda}(A) = 1$.

(ii) $P_{A,\Lambda}P_{A,\Lambda'} = 0$ for any $\Lambda, \Lambda' \in k$ with $\Lambda = \Lambda'$;

(iii) $\|P_{A,\Lambda}\| = \begin{cases} 1 & (\Lambda \in \sigma_{\mathcal{A}}(A)) \\ 0 & (\Lambda \in k \setminus \sigma_{\mathcal{A}}(A)) \end{cases}$;

(iv) $P_{A,\Lambda}A = AP_{A,\Lambda}$ for any $\Lambda \in k$; and

(v)

\[
\sigma_{\mathcal{A}}(P_{A,\Lambda}A) = \begin{cases} (\Lambda \cap \sigma_{\mathcal{A}}(A)) \cup \{0\} & (\sigma_{\mathcal{A}}(A) \neq \{\Lambda\}) \\ \sigma_{\mathcal{A}}(A) & (\sigma_{\mathcal{A}}(A) = \{\Lambda\}) \end{cases}
\]

\[
\sigma_{(P_{A,\Lambda})}(P_{A,\Lambda}A) = \begin{cases} \Lambda \cap \sigma_{\mathcal{A}}(A) & (\Lambda \in \sigma_{\mathcal{A}}(A)) \\ 0 & (\Lambda \in k \setminus \sigma_{\mathcal{A}}(A)) \end{cases}
\]

where $(P_{A,\Lambda}) \subset \mathcal{A}$ is the closure of the subset $P_{A,\Lambda} \subset A$, for any $\Lambda \in \sigma_{\mathcal{A}}(A)$.

In other words, there is a canonical partition of unity corresponding to the decomposition of the spectrum by the discrete reductive spectrum.

Note that for a $\Lambda \in \sigma_{\mathcal{A}}(A)$, the subset $(P_{A,\Lambda}) \subset \mathcal{A}$ is not a $k$-subalgebra of $\mathcal{A}$ because it does not contain $1 \in \mathcal{A}$ when $\#\sigma_{\mathcal{A}}(A) > 1$, but is a unital Banach $k$-algebra whose identity is $P_{A,\Lambda}$. For an element $B \in \mathcal{A}$ commuting with $P_{A,\Lambda}$, one has $P_{A,\Lambda}BP_{A,\Lambda} = P_{A,\Lambda}B = P_{A,\Lambda}B$ and hence we use the expression $P_{A,\Lambda}B$ rather than $P_{A,\Lambda}BP_{A,\Lambda}$ even when we regard it as an element of $P_{A,\Lambda} \subset (P_{A,\Lambda})$. In particular the structure homomorphism $k \to (P_{A,\Lambda})$ sends an element $a \in k$ to $P_{A,\Lambda}a \in (P_{A,\Lambda})$. Beware that $P_{A,\Lambda} \in \mathcal{A}$ is not a central idempotent in general, and hence the canonical projection $\mathcal{A} \to (P_{A,\Lambda})$: $B \mapsto P_{A,\Lambda}BP_{A,\Lambda}$ is not necessarily a $k$-algebra homomorphism. Therefore the inclusion $\sigma_{(P_{A,\Lambda})}(P_{A,\Lambda}A) \subset \sigma_{\mathcal{A}}(P_{A,\Lambda}A)$ is not obvious.
Proof. Set \( P_{\Lambda} := 0 \in \mathcal{A} \) for each \( \Lambda \in \mathbb{k}(\sigma_{\mathcal{A}}(\Lambda)) \). Since \( A \) admits the reductive functional calculus, the reductive spectrum \( \sigma_{\mathcal{A}}(\Lambda) \subset \mathbb{k} \) is not empty. If \( \#\sigma_{\mathcal{A}}(\Lambda) = 1 \), set \( P_{\Lambda} := 1 \in \mathcal{A} \) for the unique element \( \Lambda \in \sigma_{\mathcal{A}}(\Lambda) \). Then the conditions (i)-(v) are trivial. Note that \( \mathcal{A} \) is non-zero because \( \|A\| = 1 \) and hence \( \|1\| = 1 \). Now we consider the case \( \#\sigma_{\mathcal{A}}(\Lambda) > 1 \).

Since \( A \) is reductively finite, one has \( n := \#\sigma_{\mathcal{A}}(\Lambda) < \infty \). Take a \( \Lambda \in \sigma_{\mathcal{A}}(\Lambda) \). Let \( \Lambda = \Lambda_1, \ldots, \Lambda_n \in \mathbb{k} \) be the spectra of \( A \). Since \( A \) admits the reductive functional calculus, one has the canonical injective \( \mathbb{k} \)-algebra homomorphism \( \iota_\Lambda : \mathbb{k}(\sigma_{\mathcal{A}}(\Lambda)) \hookrightarrow \mathbb{A} \) sending the coordinate function \( (\Lambda_i)_{\Lambda_i \in \sigma_{\mathcal{A}}(\Lambda)} \in \mathbb{k}(\sigma_{\mathcal{A}}(\Lambda)) \) to \( \mathbb{A} \). Recall that the image of \( \iota_\Lambda \) is contained in \( \mathcal{L}(\Lambda) = \mathcal{L}(\Lambda) \) by Proposition \( \mathcal{L} \), and that \( \mathcal{L}(\Lambda) = \mathbb{k}[\Lambda] \subset \mathcal{L}(\Lambda) \). Take a lift \( P \in \mathcal{L}(A)(1) \) of the image \( \iota_\Lambda(1_\Lambda) \in \mathbb{A} \) of the characteristic function \( 1_\Lambda \in \mathbb{k}(\sigma_{\mathcal{A}}(\Lambda)) \) of the singleton \( \{ \Lambda \} \subset \sigma_{\mathcal{A}}(\Lambda) \). The equality \( 1_\Lambda^2 = 1_\Lambda \) guarantees that \( \|P^2 - P\| < 1 \). Set \( x_0 := P \). Inductively define the sequence \( (x_j)_{j \in \mathbb{N}} \) in \( \mathcal{L}(A)(1)^{\mathbb{N}} \) by the recurrence relation \( x_{j+1} = 3x_j^2 - 2x_j^3 \) for any \( j \in \mathbb{N} \). Then for an \( j \in \mathbb{N} \), one has

\[
\begin{align*}
x_{j+1}^2 - x_j^2 - x_{j+1} &= (3x_j^2 - 2x_j^3)^2 - (3x_j^2 - 2x_j^3) = 4x_j^6 - 12x_j^5 + 9x_j^4 + 2x_j^3 - 3x_j^2 \\
 &= (x_j^2 - x_j)(4x_j^4 - 8x_j^3 + x_j^2 + 3x_j) = (x_j^2 - x_j)(x_j^2 - x_j)(4x_j^2 - 4x_j - 3) \\
 &= -(x_j^2 - x_j)^2 (3 - 4(x_j^2 - x_j))
\end{align*}
\]

and hence

\[
\|x_{j+1}^2 - x_j^2\| \leq \|x_j^2 - x_j\|^2 \leq \|P^2 - P\|^{2^{j+1}}.
\]

It follows that \( \lim_{j \to \infty} \|x_j^2 - x_j\| = 0 \). Moreover, the equalities

\[
\begin{align*}
x_{j+1} - x_j &= 3x_j^2 - 2x_j^3 - x_j = -x_j(x_j - 1)(2x_j - 1) \\
\|x_{j+1} - x_j\| \leq \|x_j^2 - x_j\| \|2x_j - 1\| \leq \|x_j^2 - x_j\|
\end{align*}
\]

imply that the sequence \( (x_j)_{j \in \mathbb{N}} \in \mathcal{A}(1)^{\mathbb{N}} \) converges to an element \( P_{\Lambda} \in \mathcal{L}(A)(1) \). By the continuity of the norm, the multiplication, and the addition, one has

\[
\|P_{\Lambda}^2 - P_{\Lambda}\| = \lim_{j \to \infty} \|x_j^2 - x_j\| = 0
\]

and therefore \( P_{\Lambda}^2 = P_{\Lambda} \) is an idempotent. Moreover \( P_{\Lambda} \) is non-zero. Indeed, for an \( j \in \mathbb{N} \), it is easy to see that

\[
\|x_{j+1}\| = \|3x_j^2 - 2x_j^3\| = \|x_j + (x_j^2 - x_j)(-2x_j + 1)\| = \|x_j\| = \cdots = \|P\| = 1
\]

and hence

\[
\|P_{\Lambda}\| = \lim_{j \to \infty} \|x_j\| = 1
\]

49
by the continuity of the norm. Note that

\[ \|P_\Lambda - P\| = \left\| \sum_{j=0}^{\infty} (x_{j+1} - x_j) \right\| \leq \sup_{j \in \mathbb{N}} \|x_{j+1} - x_j\| = \|x_1 - x_0\| \]

\[ = \left\| (3P^2 - 2P^3) - P \right\| = \| P(P - 1)(2P - 1) \| \leq \| P^2 - P \| < 1, \]

and hence \( \overline{P_\Lambda} = \overline{P} = \iota_\omega(1_\Lambda). \) The idempotent lift \( P_\Lambda \) is independent of the choice of a lift \( P \). Indeed, suppose \( P_\Lambda, P'_\Lambda \in \mathcal{L}(A) \) are two idempotent lifts of \( \iota_\omega(1_\Lambda) \). Since \( \mathcal{L}(A) \) is commutative, one has

\[ (P_\Lambda - P'_\Lambda)^4 = P_\Lambda^4 - 4P_\Lambda^3P'_\Lambda + 6P_\Lambda^2P'_\Lambda^2 - 4P_\Lambda P'_\Lambda^3 + P'_\Lambda^4 \]

\[ = P_\Lambda - 2P_\Lambda^3P'_\Lambda + P'_\Lambda = (P_\Lambda - P'_\Lambda)^2. \]

It follows

\[ \|(P_\Lambda - P'_\Lambda)^2\| \leq \|(P_\Lambda - P'_\Lambda)^4\| \leq \|(P_\Lambda - P'_\Lambda)^2\|^2 < 1 \]

and

\[ \|(P_\Lambda - P'_\Lambda)^2\| = \|(P_\Lambda - P'_\Lambda)^4\| \leq \|(P_\Lambda - P'_\Lambda)^2\|^2. \]

Therefore one obtains \( \|(P_\Lambda - P'_\Lambda)^2\| = 0. \) Furthermore, one acquires

\[ (1 - P_\Lambda)(1 - P'_\Lambda) = (1 - P_\Lambda) - P_\Lambda^3P'_\Lambda = (1 - P_\Lambda) - P_\Lambda^2 + P_\Lambda P'_\Lambda \]

\[ = (1 - P_\Lambda) + (P'_\Lambda - P_\Lambda)P_\Lambda = (1 - P_\Lambda) + ((3P'_\Lambda - P_\Lambda)P_\Lambda + P_\Lambda) - (2(P'_\Lambda - P_\Lambda)P_\Lambda + P_\Lambda) \]

\[ = (1 - P_\Lambda) + ((P_\Lambda - P_\Lambda) + P_\Lambda)^2 = (1 - P_\Lambda) - (P_\Lambda^3 - P'_\Lambda^2) \]

\[ = 1 - P_\Lambda \]

and similarly \( (1 - P_\Lambda)(1 - P'_\Lambda) = 1 - P'_\Lambda. \) It implies \( P_\Lambda = P'_\Lambda. \)

We have constructed the system \( (P_\Lambda)_{\Lambda \in \mathcal{A}} \in \mathcal{L}(A) \) of idempotents. By the argument above, the condition (iii) holds. Since \( \mathcal{L}(A) \) is a commutative Banach \( k \)-algebra, containing \( A, \) the condition (iv) holds. For any \( \Lambda, \Lambda' \in \sigma_\omega(\mathcal{A}) \) with \( \Lambda \neq \Lambda', \) since \( 1_\Lambda 1_{\Lambda'} = 0 \in \mathcal{A}, \) one has

\[ P_\Lambda P_{\Lambda'} = P_{\Lambda'} P_\Lambda = \iota_\omega(1_\Lambda)\iota_\omega(1_{\Lambda'}) = \iota_\omega(1_\Lambda 1_{\Lambda'}) = 0 \]

and hence \( \|P_\Lambda P_{\Lambda'}\| < 1. \) On the other hand, since \( \mathcal{L}(A) \) is commutative, the product \( P_\Lambda P_{\Lambda'} \) of idempotents is again an idempotent. It follows

\[ \|P_\Lambda P_{\Lambda'}\| = \|(P_\Lambda P_{\Lambda'})^2\| \leq \|P_\Lambda P_{\Lambda'}\|^2, \]

and \( \|P_\Lambda P_{\Lambda'}\| = 0. \) Therefore the condition (ii) holds. Moreover, since

\[ \sum_{\Lambda \in \mathcal{A}} P_\Lambda = \sum_{\Lambda \in \mathcal{A}} \overline{P_\Lambda} = \sum_{\Lambda \in \mathcal{A}} \iota_\omega(1_\Lambda) = \iota_\omega(1) = 1 \neq 0 \in \mathcal{A}, \]

50
one has \( \|1 - \sum_{\Lambda \in \mathcal{A}} P_\Lambda\| < 1 \). The commutativity of \( \mathcal{L}(A) \) and the equality \( P_\Lambda P_{\Lambda'} = 0 \) for each \( \Lambda, \Lambda' \in \tilde{k} \) with \( \Lambda \neq \Lambda' \) guarantee that the difference \( 1 - \sum_{\Lambda \in \mathcal{A}} P_\Lambda \) is again an idempotent, and hence coincides with 0 by the same argument as the independence of \( P_\Lambda \) of the choice of a lift of \( \lambda(1, \Lambda) \) for \( \Lambda \in \sigma_{A}(A) \). Therefore (i) holds. We verify the property (v). Take a \( \Lambda \in \sigma_{A}(A) \), and let \( \Lambda = \overline{\Lambda}_1, \ldots , \overline{\Lambda}_n \in \tilde{k} \) be the spectra of \( A \) again.

First, one has \( \sigma_{A}(P_\Lambda A) \subset k^\circ \) and \( \sigma_{(P_\Lambda)}(P_\Lambda A) \subset k^\circ \) because \( \|P_\Lambda A\| \leq \|A\| = 1 \). Since \( \#(\sigma_{A}(A)) > 1 \), there is another element \( \Lambda' \in \sigma_{A}(A) \). The equality \( P_{\Lambda'} P_\Lambda = 0 \) with \( \Lambda' \neq 0 \) guarantees that \( P_\Lambda \notin A^\infty \), and hence \( P_\Lambda A \notin A^\infty \). Therefore \( 0 \in \sigma_{A}(P_\Lambda A) \).

Secondly, take a resolvent \( \lambda \in k^\circ \setminus \sigma_{A}(A) \). One has

\[
(P_\Lambda A - P_\Lambda \lambda)(P_\Lambda(A - \lambda)^{-1}P_\Lambda) = P_\Lambda^3 = P_\Lambda \in (P_\Lambda)
\]

and

\[
(P_\Lambda(A - \lambda)^{-1}P_\Lambda)(P_\Lambda A - P_\Lambda \lambda) = P_\Lambda^3 = P_\Lambda \in (P_\Lambda)
\]

because \( P_\Lambda \) and \( A \) commutes. It follows that \( P_\Lambda A - P_\Lambda \lambda \in (P_\Lambda)^\times \) and \( \lambda \in k^\circ \setminus \sigma_{(P_\Lambda)}(P_\Lambda A) \). In addition if \( \lambda \neq 0 \), one obtains

\[
(P_\Lambda A - \lambda)(P_\Lambda(A - \lambda)^{-1}P_\Lambda + (1 - P_\Lambda)\lambda^{-1})
\]

\[
= (P_\Lambda(A - \lambda) + (1 - P_\Lambda)\lambda)(P_\Lambda(A - \lambda)^{-1}P_\Lambda + (1 - P_\Lambda)\lambda^{-1})
\]

\[
= P_\Lambda^3 + P_\Lambda(A - \lambda)\lambda^{-1} + (1 - P_\Lambda)P_{\lambda}A(A - \lambda)^{-1}P_\Lambda A + (1 - P_\Lambda)^2
\]

\[
= P_\Lambda + 0 + 0 + (1 - P_\Lambda) = 1,
\]

and

\[
(P_\Lambda(A - \lambda)^{-1}P_\Lambda + (1 - P_\Lambda)\lambda^{-1})(P_\Lambda A - \lambda)
\]

\[
= (P_\Lambda(A - \lambda) + (1 - P_\Lambda)\lambda)(P_\Lambda(A - \lambda)^{-1}P_\Lambda + (1 - P_\Lambda)\lambda)
\]

\[
= P_\Lambda^3 + P_\Lambda(A - \lambda)^{-1}P_\Lambda (1 - P_\Lambda)\lambda + (1 - P_\Lambda)P_{\lambda}A^{-1}(A - \lambda) + (1 - P_\Lambda)^2
\]

\[
= P_\Lambda + 0 + 0 + (1 - P_\Lambda) = 1.
\]

It implies that \( P_\Lambda A - \lambda \in A^\infty \) and \( \lambda \in k^\circ \setminus \sigma_{A}(P_\Lambda A) \). Therefore \( \sigma_{(P_\Lambda)}(P_\Lambda A) \subset \sigma_{A}(A) \) and \( \sigma_{A}(P_\Lambda A) \subset \sigma_{A}(A) \).

Thirdly, take an element \( \lambda \in k^\circ \setminus \Lambda \). Since \( \overline{\Lambda} \neq \Lambda \), one has

\[
P_\Lambda A - P_\Lambda \lambda = \iota(\Lambda)(\tilde{A} - 1_A) = \iota(\Lambda)(1_A \Lambda - 1_A \lambda) = \iota(1_A)(\Lambda - \lambda)
\]

\[
\in \quad P_\Lambda A^\times \subset (P_\Lambda)^\times,
\]

and hence \( P_\Lambda A - P_\Lambda \lambda \in (P_\Lambda)^\times \) by Proposition 2.5. It follows that \( \lambda \in k^\circ \setminus \sigma_{(P_\Lambda)}(P_\Lambda A) \). Let \( B \in (P_\Lambda)^\times \) be the inverse of \( P_\Lambda A - P_\Lambda \lambda \) in \( (P_\Lambda) \). Since \( P_\Lambda(1 - P_\Lambda) = (1 - P_\Lambda)P_\Lambda = 0 \) and \( B \in (P_\Lambda) \), one obtains \( B(1 - P_\Lambda) = (1 - P_\Lambda)B = 0 \) by the continuity of the multiplication. In addition if \( \lambda \neq 0 \), one has

\[
(P_\Lambda A - \lambda)(B - (1 - B)\lambda^{-1}) = (B - (1 - B)\lambda^{-1})(P_\Lambda A - \lambda) = 1,
\]

51
and hence \( P_A - \lambda \in \mathcal{A}^\times \). It follows that \( \lambda \in k^\times \setminus \mathcal{A}(P_A A) \). Therefore \( \mathcal{A}(P_A A) \subset \Lambda \) and \( \mathcal{A}(P_A A) \).

Finally, take a \( \lambda \in \Lambda \cap \mathcal{A}(A) \). Assume that \( \lambda \notin \mathcal{A}(P_A A) \). Then the argument above guarantees \( \lambda \neq 0 \) and the inverse \( (P_A - \lambda)^{-1} = (P_A - \lambda)^{-1} \) of \( P_A - \lambda = P_A - \lambda \) exists in \( \mathcal{A} \). By the calculation in the third case, one has \( \lambda \in k^\times \setminus \mathcal{A}(P_A A) \) and hence the inverse \( (P_A - \lambda)^{-1} \in \mathcal{A} \) of \( P_A - \lambda \) exists for any \( i = 2, \ldots, n \). The equalities

\[
(A - \lambda) \left( \sum_{i=1}^n P_i(P_A - \lambda)^{-1} P_i \right) = \left( \sum_{i=1}^n P_i(A - \lambda) \right) \left( \sum_{i=1}^n P_i(P_A - \lambda)^{-1} P_i \right)
\]

\[
= \left( \sum_{i=1}^n P_i(P_A - \lambda) \right) \left( \sum_{i=1}^n P_i(P_A - \lambda)^{-1} P_i \right)
\]

\[
= \sum_{i=1}^n \sum_{j=1}^n P_i P_j (P_A - \lambda)(P_A - \lambda)^{-1} P_i = \sum_{i=1}^n P_i^2 = \sum_{i=1}^n P_i = 1
\]

and

\[
\left( \sum_{i=1}^n P_i(P_A - \lambda)^{-1} P_i \right) (A - \lambda) = \left( \sum_{i=1}^n P_i(P_A - \lambda)^{-1} P_i \right) \left( \sum_{i=1}^n P_i(A - \lambda) \right)
\]

\[
= \left( \sum_{i=1}^n P_i(P_A - \lambda)^{-1} P_i \right) \left( \sum_{i=1}^n P_i(P_A - \lambda) \right)
\]

\[
= \sum_{i=1}^n \sum_{j=1}^n P_i(P_A - \lambda)^{-1} P_i P_j (P_A - \lambda) = \sum_{i=1}^n P_i^2 = \sum_{i=1}^n P_i = 1
\]

imply \( A - \lambda \in \mathcal{A}^\times \) and it contradicts the condition \( \lambda \in \Lambda \cap \mathcal{A}(A) \) \( \subset \mathcal{A}(A) \). It follows that \( \lambda \in \mathcal{A}(P_A A) \). Similarly assume that \( \lambda \notin \mathcal{A}(P_A A) \). Then the inverse \( B_1 \in (P_A) \) of \( P_A A - P_A \lambda = P_A A - P_A \lambda \) in \( (P_A) = (P_A) \) exists. By the calculation in the third case, one has \( \lambda \in k^\times \setminus \mathcal{A}(P_A A) \) and hence the inverse \( B_1 \in (P_A) \) of \( P_A A - P_A \lambda \) in \( (P_A) \) exists for any \( i = 2, \ldots, n \). Namely, for each \( i = 1, \ldots, n \), one has

\[
(P_A - P_A \lambda) B_i = B_i (P_A A - P_A \lambda) = P_A
\]

Since \( P_A \) and \( A \) commutes, one has

\[
(P_A - P_A \lambda) P_A B_i P_A = P_A (P_A A - P_A \lambda) B_i P_A = P_A B_i P_A = P_A
\]

and

\[
P_A B_i P_A (P_A - P_A \lambda) = P_A B_i (P_A A - P_A \lambda) P_A = P_A B_i P_A = P_A
\]

It follows \( P_A B_i P_A = B_i \) by the uniqueness of the inverse in the unital ring \( (P_A) \). Now the equalities

\[
(A - \lambda) \left( \sum_{i=1}^n B_i \right) = \left( \sum_{i=1}^n P_i(A - \lambda) \right) \left( \sum_{i=1}^n P_i B_i P_A \right) = \sum_{i=1}^n \sum_{j=1}^n P_i P_j (A - \lambda) B_j P_A
\]

52
Example 5.10. \( P(A - \lambda)B_iP_{\lambda_i} = P^2_i(A - \lambda)B_iP_{\lambda_i} = P_{\lambda_i} = 1 \) and

\[
\left( \sum_{i=1}^{n} B_i \right) (A - \lambda) = \left( \sum_{i=1}^{n} P_{\lambda_i}B_iP_{\lambda_i} \right) (A - \lambda) = \sum_{i=1}^{n} P_{\lambda_i}B_iP_{\lambda_i}(A - \lambda)
\]

\[
= \sum_{i=1}^{n} P_{\lambda_i}B_iP_{\lambda_i}^2(A - \lambda) = P_{\lambda_i}B_i(P_{\lambda_i}A - P_{\lambda_i}^2) = P_{\lambda_i}^2 = P_{\lambda_i} = 1
\]

imply that \( A - \lambda \in \mathcal{A}^\times \), and it contradicts the condition \( \lambda \in \Lambda \cap \sigma_{\mathcal{A}}(A) \subset \sigma_{\mathcal{A}}(A) \). It follows that \( \lambda \in \sigma(P_A(A)) \). Therefore \( \Lambda \cap \sigma_{\mathcal{A}}(A) \subset \sigma_{\mathcal{A}}(P_A(A)) \) and \( \Lambda \cap \sigma_{\mathcal{A}}(A) \subset \sigma(P_A(A)) \).

We conclude \( \sigma_{\mathcal{A}}(P_A(A)) = (\Lambda \cap \sigma_{\mathcal{A}}(A)) \cup \{0\} \) and \( \sigma_{P_A}(P_A(A)) = \Lambda \cap \sigma_{\mathcal{A}}(A) \). Thus the condition (v) holds. \( \square \)

The definition of the system \( (P_A(A))_{\Lambda \in \mathcal{T}} \) seems to depend not only on \( A \) but also on \( \mathcal{A} \) because it uses the reductive spectrum \( \sigma_{\mathcal{A}}(A) \subset \mathcal{T} \). However, it is independent of \( \mathcal{A} \) containing \( A \). The condition that \( A \) admits the reductive functional calculus in \( \mathcal{A} \) guarantees the stability of the reductive spectrum \( \sigma_{\mathcal{A}}(A) \) under changes of \( \mathcal{A} \). Beware that \( (P_A(A))_{\Lambda \in \mathcal{T}} \) is obtained as a system in \( \mathcal{L}(A) \subset \mathcal{A} \) but the Banach \( k \)-algebra \( \mathcal{L}(A) \) does not necessarily holds the condition that \( A \) admits the reductive functional calculus in \( \mathcal{L}(A) \).

**Definition 5.9.** A system \( (P_i)_{i \in I} \in \mathcal{A}^I \) of idempotents indexed by an index set \( I \) is said to be a partition of unity if \#\{ \( i \in I \mid P_i \neq 0 \) \} < \( \infty \), \( \sum_{i \in I} P_i = 1 \), and \( \|P_i\| \in \{0, 1\} \) for any \( i \in I \).

**Example 5.10.** The system \( (P_A(A))_{\Lambda \in \mathcal{T}} \) in Proposition 5.8 is a partition of unity.

**Proposition 5.11.** A partition \( (P_i)_{i \in I} \in \mathcal{A}^I \) of unity with \( P_i \neq 0 \) for any \( i \in I \) is a system of orthonormal projections, i.e. for any \( (c_i)_{i \in I} \in \mathbb{C}^I \), one has

\[
\left\| \sum_{i \in I} c_i P_i \right\| = \max_{i \in I} |c_i|.
\]

**Proof.** Set \( A := \sum_{i \in I} c_i P_i \). The inequality \( \|A\| \leq \max_{i \in I} |c_i| \) follows from the equalities \( \|P_i\| = 1 \) for any \( i \in I \). Take an index \( i \in I \) with \( |c_i| = \max_{r \in I} |c_r| \). Then the equality

\[
(A - c_i)P_i = \sum_{r \in I} (c_r - c_i)P_r P_i = 0
\]

guarantees that \( A - c_i \notin \mathcal{A}^\times \), and hence \( c_i \in \sigma_{\mathcal{A}}(A) \). Therefore one has \( \|A\| \geq |c_i| = \max_{r \in I} |c_r| \). \( \square \)
Proposition 5.12. Let \( V \) be a \( k \)-Banach space. A partition \((P_i)_{i \in I} \in \mathcal{B}(V)^\dagger\) of unity with \( P_i \neq 0 \) for any \( i \in I \) satisfies the orthogonal property, i.e., for any \( v \in V \), one has

\[
\|v\| = \max_{i \in I} \|P_i v\|.
\]

In other words, it gives the decomposition

\[
V = \bigoplus_{i \in I} P_i(V) = \bigoplus_{i \in I} \ker(1 - P_i)
\]
as a \( k \)-Banach space.

Proof. The inequality \( \leq \) is obvious by the definition of a partition of unity, and the opposite inequality \( \geq \) follows from the condition \( \|P_i\| = 1 \).

Proposition 5.13. In the situation in Proposition 5.8, \( A \) admits the rigid continuous functional calculus in \( \mathcal{A} \) if and only if \( P_{A,\Lambda}A \) admits the rigid continuous functional calculus in \( (P_{A,\Lambda}) \subset \mathcal{A} \) for any \( \Lambda \in \overline{k} \).

Proof. To begin with, suppose \( A \) admits the rigid continuous functional calculus

\[
\iota_A : C_{\text{rig}}(\sigma(\mathcal{A}), k) \to \mathcal{A}.
\]

Since the partition \((P_{A,\Lambda})_{\Lambda \in \overline{k}}\) of unity is a system of idempotents in \( \mathcal{L}^{\dagger}(A) \), it is contained in the image of \( \iota_A \) by the isometry of \( \iota_A \). For any \( \Lambda \in \overline{k} \setminus \sigma(\mathcal{A}) \), the equalities \( P_{A,\Lambda} = 0 \), \( \sigma(P_{A,\Lambda}A) = \sigma_0(0) = \emptyset \) \( C_{\text{rig}}(\sigma_i(P_{A,\Lambda}A), k) = C(0, k) = 0 \) guarantee that \( P_{A,\Lambda}A \) admits the rigid continuous functional calculus in \((P_{A,\Lambda})\). Let \((E_{\Lambda})_{\Lambda \in \overline{k}}\) be the partition of unity in \( C_{\text{rig}}(\sigma(\mathcal{A}), k) \) which is the pre-image of \((P_{A,\Lambda})_{\Lambda \in \overline{k}}\), and \((1_{\bar{\Lambda}})_{\Lambda \in \overline{k}}\) the partition of unity in \( C_{\text{bd}}(\sigma(\mathcal{A}), k) \) consisting of the characteristic functions \( 1_{\bar{\Lambda}} \) of \( \Lambda \cap \sigma(\mathcal{A}) \subset \sigma(\mathcal{A}) \) for a \( \Lambda \in \overline{k} \). The isometry of \( \iota_A \) guarantees that \( \|E_{\Lambda}\| = \|P_{A,\Lambda}\| = 1 \) for any \( \Lambda \in \sigma(\mathcal{A}) \). By the uniqueness of the rigid continuous functional calculus, it is compatible with the normal reduction, and hence one has \( E_{\Lambda} = 1_{\bar{\Lambda}} = 1 \) by the construction of \( P_{A,\Lambda} \) for any \( \Lambda \in \sigma(\mathcal{A}) \). Therefore \( \|E_{\Lambda} - 1_{\bar{\Lambda}}\| < 1 \). Since \( C_{\text{bd}}(\sigma(\mathcal{A}), k) \) is commutative, the difference \( E_{\Lambda} - 1_{\bar{\Lambda}} \) of idempotents with the same reduction is 0 by the same calculation as that in the proof of Proposition 5.8. It follows that \((E_{\Lambda})_{\Lambda \in \overline{k}}\) is the partition of unity in \( C_{\text{rig}}(\sigma(\mathcal{A}), k) \) consisting of the characteristic functions of \( \Lambda \cap \sigma(\mathcal{A}) \subset \sigma(\mathcal{A}) \) for \( \Lambda \in \overline{k} \). It is obvious that the restriction \( C_{\text{rig}}(\sigma(\mathcal{A}), k) \to C_{\text{rig}}(\Lambda \cap \sigma(\mathcal{A}), k) \) induces the isometric isomorphism \((E_{\Lambda}) \to C_{\text{rig}}(\Lambda \cap \sigma(\mathcal{A}), k) \) of commutative unital uniform Banach \( k \)-algebras, and its inverse is the restriction \( C_{\text{rig}}(\Lambda \cap \sigma(\mathcal{A}), k) \to (E_{\Lambda}) \) of the non-unital \( k \)-algebra homomorphism \( C_{\text{rig}}(\Lambda \cap \sigma(\mathcal{A}), k) \to C_{\text{rig}}(\sigma(\mathcal{A}), k) : f \mapsto E_{\Lambda} f \). Therefore one obtains the rigid continuous functional calculus

\[
\iota_{P_{A,\Lambda}} : C_{\text{rig}}(\sigma(P_{A,\Lambda}A), k) = C_{\text{rig}}(\Lambda \cap \sigma(\mathcal{A}), k) \to (E_{\Lambda}) \xrightarrow{\iota_A} (P_{A,\Lambda}).
\]

On the other hand, suppose \( P_{A,\Lambda}A \) admits the rigid continuous functional calculus

\[
\iota_{P_{A,\Lambda}} : C_{\text{rig}}(\sigma(P_{A,\Lambda}A), k) \to (P_{A,\Lambda})
\]
for each $\Lambda \in \sigma^\infty_{\mathcal{A}}(A)$. The inclusions $(P_{A,A}) \hookrightarrow \mathcal{A}$ induce a bounded $k$-linear homomorphism

$$
\phi: \prod_{\Lambda \in k} (P_{A,A}) \cong_k \prod_{\Lambda \in \sigma^\infty_{\mathcal{A}}(A)} (P_{A,A}) \rightarrow \mathcal{A}
$$

$$
F = (F_{\Lambda})_{\Lambda \in k} \mapsto \phi(F) := \sum_{\Lambda \in k} F_{\Lambda}.
$$

It is obvious that $\phi$ sends $(P_{A,A})_{\Lambda \in k}$ to $A$ because $(P_{A,A})_{\Lambda \in k}$ is a partition of unity. Note that for elements $\Lambda, \Lambda' \in k$ with $\Lambda \neq \Lambda'$, one has $BB' = 0$ for any $(B, B') \in (P_{A,A}) \times (P_{A,A'})$. Indeed, it is trivial when $B = 0$ or $B' = 0$. If $B \neq 0$ and $B' \neq 0$, then for any $\varepsilon \in (0, 1)$, take elements $(P_{A,A}DP_{A,A}, P_{A,A}D'P_{A,A'}) \in P_{A,A} \mathcal{A} \times P_{A,A} \mathcal{A}$ such that $\|B - P_{A,A}DP_{A,A}\| < \min\{\varepsilon\|B\|^{-1}, \|B\|\}$ and $\|B' - P_{A,A}D'P_{A,A'}\| < \min\{\varepsilon\|B\|^{-1}, \|B'\|\}$. Then one obtains

$$
\|BB'\| = \|(B - P_{A,A}DP_{A,A})(B' - P_{A,A}D'P_{A,A'}) + (B - P_{A,A}DP_{A,A})P_{A,A}D'P_{A,A'} + P_{A,A}DP_{A,A}(B' - P_{A,A}D'P_{A,A'})\|
$$

$$
\leq \max\{\varepsilon^2, \varepsilon, \varepsilon, 0\} = \varepsilon
$$

and hence $\|BB'\| = 0$. Therefore for elements $F, F' \in \prod_{\Lambda \in k}(P_{A,A})$, one has

$$
\phi(FF') = \phi((F_{\Lambda}F'_{\Lambda})_{\Lambda \in k}) = \sum_{\Lambda \in k} F_{\Lambda}F'_{\Lambda} = \sum_{\Lambda \in k} \sum_{\Lambda' \in k} F_{\Lambda}F'_{\Lambda'} = \left(\sum_{\Lambda \in k} F_{\Lambda}\right)\left(\sum_{\Lambda' \in k} F'_{\Lambda'}\right)
$$

$$
= \phi(F)\phi(F'),
$$

and hence $\phi$ is multiplicative. Moreover the sum $\sum_{\Lambda \in k} P_{A,A}$ is the identity $1 \in \mathcal{A}$ and it means that $\phi$ is a $k$-algebra homomorphism. Furthermore it is injective because it has the right inverse map

$$
\psi: \mathcal{A} \rightarrow \prod_{\Lambda \in k}(P_{A,A})
$$

$$
B \mapsto (P_{A,A}BP_{A,A})_{\Lambda \in k}.
$$

Beware that $\psi$ is neither the left inverse map of $\phi$ nor a $k$-algebra homomorphism in general. Now for an element $F \in \prod_{\Lambda \in k}(P_{A,A})$, one has

$$
\|\phi(F)\| = \left\|\sum_{\Lambda \in k} F_{\Lambda}\right\| \leq \max_{\Lambda \in k}\|F_{\Lambda}\| = \|F\|
$$

and

$$
\|F\| = \|\psi(\phi(F))\| = \max_{\Lambda \in k}\|P_{A,A}\phi(F)P_{A,A}\| \leq \max_{\Lambda \in k}\|P_{A,A}\|\|\phi(F)\|\|P_{A,A}\| = \|\phi(F)\|,
$$

55
and it follows that $\phi$ is an isometry.

Thus the composition

$$C_{\text{rig}}(\sigma_{\mathcal{A}}(A), k) \cong_k \prod_{\Lambda \in \mathcal{K}} C_{\text{rig}}(\Lambda \cap \sigma_{\mathcal{A}}(A), k) = \prod_{\Lambda \in \mathcal{K}} C_{\text{rig}}(\sigma_{(P_{A,\Lambda})}(P_{A,\Lambda}A), k)$$

$$\prod_{\Lambda \in \mathcal{K}}(P_{A,\Lambda}) \rightarrow \mathcal{A}$$

is an isometric $k$-algebra homomorphism sending the coordinate function $z: \sigma_{\mathcal{A}}(A) \rightarrow k$ to $A \in \mathcal{A}$, and is the rigid continuous functional calculus of $A$ in $\mathcal{A}$. \hfill \Box

5.3 Repetitive Reduction Technique

Now we obtain a direct analogue of Theorem 5.7. We mainly consider the case that $\mathcal{A}$ is an operator algebra and $A$ is an operator. Different from the matrix case, the following does not give a criterion for the normality of $A$ because the induction on the dimension works only in the finite dimensional case. In this section, suppose that $k$ is a local field and $||\mathcal{A}|| \subset |k|$. Fix a uniformiser $\pi \in k$.

Definition 5.14. For a non-zero element $A \in \mathcal{A}\{0\}$, denote by $v(A)$ the logarithm $-\log|\pi|\|A\| \in \mathbb{Z}$.

The additive norm $v(A)$ is independent of the choice of the uniformiser $\pi \in k$. On the other hand, the arguments below deeply depend on $\pi$.

Definition 5.15. Set $\mathcal{N}_{-1}(\mathcal{A}) \coloneqq \mathcal{A}$, and define the maps $P_0$ and $r_0$ in the following way:

$$P_0: \mathcal{N}_{-1}(\mathcal{A}) \rightarrow \mathcal{A} \quad A \mapsto 1$$

$$r_0: \mathcal{N}_{-1}(\mathcal{A}) \rightarrow \mathcal{A} \quad A \mapsto P_0(A)A = A.$$

An element $A \in \mathcal{A}$ is said to be naive of level 0 at 0 in $\mathcal{A}$ if one of the following holds:

(i) $A = 0$; or

(ii) $A$ is a reductively finite element admitting the reductive continuous functional calculus in $\mathcal{A}$, and the canonical projection $\sigma_{\mathcal{A}}(a^{-1}A) \rightarrow \sigma_{\mathcal{A}}(a^{-1}A)$ is surjective for any $a \in k^*$ with $|a| = ||A||$.

Denote by $\mathcal{N}_0(\mathcal{A}) \subset \mathcal{A}$ the subset of naive elements of level 0 at 0 in $\mathcal{A}$, and define the maps $P(\cdot, \mathcal{A}) = P_1$ and $r_1$ in the following way:

$$P(\cdot, \mathcal{A}) = P_1: \mathcal{N}_0(\mathcal{A}) \rightarrow \mathcal{A}$$

56
Proof. Let \( N_0(\mathcal{A}) := \{0\} \) (If (i) holds.)

\[
A \mapsto P(A, \mathcal{A}) = P_1(A) := \begin{cases} 
0 & \text{(If (i) holds.)} \\
\Pi_{\iota(A,0)} & \text{(If (ii) holds.)}
\end{cases}
\]

\[r_1 : N_0(\mathcal{A}) \to \mathcal{A}
A \mapsto P_1(A)A = P(A, \mathcal{A})A.
\]

Inductively for each \( i \in \mathbb{N} \), set \( N_{i+1}(\mathcal{A}) := r_1^{-1}(N_i(\mathcal{A})) \subset N_i(\mathcal{A}) \), and define the maps \( P_{i+2} \) and \( r_{i+2} \) in the following way:

\[
P_{i+2} : N_{i+1}(\mathcal{A}) \to (P_{i+1}(A)) \subset \mathcal{A}
A \mapsto P(P_{i+1}(A)A, (P_{i+1}(A))P_{i+1}(A))
\]

\[
r_{i+2} : N_{i+1}(\mathcal{A}) \to \mathcal{A}
A \mapsto r_{i+1}(r_1(A))
\]

Then one has \( r_{i+1}(A) = P_{i+1}(A)A \) and \( P_{i+2}(A)P_{i+1}(A) = P_{i+1}(A)P_{i+2}(A) = P_{i+2}(A) \) for any \( i \in \mathbb{N} \) and \( A \in N_i(\mathcal{A}) \). An element \( A \in \mathcal{A} \) is said to be naive of level \( i \) at a point \( \lambda \in \sigma_{\sigma}(A) \) in \( \mathcal{A} \) if \( A - \lambda \in N_i(\mathcal{A}) \) for an \( i \in \mathbb{N} \). An element \( A \in \mathcal{A} \) is said to be naive at a point \( \lambda \in \sigma_{\sigma}(A) \) in \( \mathcal{A} \) if \( A \) is naive of level \( i \) at a point \( \lambda \in \sigma_{\sigma}(A) \) for any \( i \in \mathbb{N} \) in \( \mathcal{A} \). An element \( A \in \mathcal{A} \) is said to be naive in \( \mathcal{A} \) if \( A \) is naive at any point \( \lambda \in \sigma_{\sigma}(A) \) in \( \mathcal{A} \).

**Proposition 5.16.** In the situation in Proposition 5.8, for an element \( a \in k^\times \), one has \( \|A - a\| < 1 \) if and only if \( \sigma_{\mathcal{A}}(\overline{A}) = [\overline{a}] \). Moreover then one has \( A \in \mathcal{A}(1)^\times \) and \( a \in (k^{irr})^\times \).

**Proof.** Suppose \( \|A - a\| < 1 \). Then obviously \( \overline{A - a} = \overline{A} - \overline{a} = 0 \in \mathcal{A} \), and therefore \( \overline{a} \in \sigma_{\mathcal{A}}(\overline{A}) \). For an element \( b \in k^\times \) with \( \overline{b} \neq \overline{a} \), one has

\[
A - b = (a - b) + (A - a) = (a - b)(1 + (a - b)^{-1}(A - a)) \in k(1^\times(1 + \mathcal{A}(1^-)) \subset \mathcal{A}(1)^\times
\]

by Proposition 1.14 and hence \( \overline{b} \notin \sigma_{\mathcal{A}}(\overline{A}) \) by Proposition 2.5. It follows \( \sigma_{\mathcal{A}}(\overline{A}) = [\overline{a}] \). The inequality \( \|A - a\| < 1 \) implies \( |a| = 1 \), and hence \( 0 \notin \sigma_{\mathcal{A}}(\overline{A}) \). Therefore \( A \in \mathcal{A}(1)^\times \) by Proposition 2.5.

On the other hand, suppose \( \sigma_{\mathcal{A}}(\overline{A}) = [\overline{a}] \). Since \( A \) admits the reductive functional calculus, one has \( \overline{A} = \overline{a} \). Therefore \( \|A - a\| < 1 \) and \( |a| = \|A - (A - a)\| = 1 \).

**Lemma 5.17.** In the situation in Proposition 5.8, \( P_{A,0}A = 0 \) if and only if \( k^{\infty} \cap \sigma_{\mathcal{A}}(A) \subset \{0\} \).

**Proof.** The direct implication is obvious by the definition of \( P_{A,0} \). If \( P_{A,0}A = 0 \), one has \( k^{\infty} \cap \sigma_{\mathcal{A}}(A) \subset \sigma_{\mathcal{A}}(P_{A,0}A) \subset \{0\} \) applying the condition (v) for \( k^{\infty} = 0 \in \mathbb{F} \).

**Lemma 5.18.** For a non-empty compact subset \( \sigma \subset k \), one has \( \|C(\sigma, k)\| = |k| \) and the coordinate function \( z : \sigma \to k \) in the Banach \( k \)-algebra \( C(\sigma, k) \) is naive in \( C(\sigma, k) \).

**Proof.** The first implication is trivial by the maximal modulus principle of a continuous function on a compact topological space. For an \( i \in \mathbb{N} \) and a \( \lambda \in \sigma_{C(\sigma, k)}(z) = \sigma \), we verify that \( z \) is naive of level \( i \) at a point \( \lambda \in \sigma \). One has

\[
\sigma_{C(\sigma, k)}(z - \lambda) = \sigma - \lambda := \{ \lambda' - \lambda \mid \lambda' \in \sigma \},
\]

57
and the parallel transform $\sigma \mapsto \sigma - \lambda : z \mapsto z - \lambda$ induces the isometric isomorphism $C(\sigma, k) \to C(\sigma - \lambda, k)$ sending $z - \lambda$ to the coordinate function $z' : \sigma - \lambda \mapsto k$. Therefore replacing $\sigma$ by $\sigma - \lambda$, we may and do assume $\lambda = 0 \in \sigma$. If $z \equiv 0$, i.e. $\sigma = \{0\}$, then $z$ is naive at $0$ by definition. Otherwise, replacing $\sigma$ by $a^{-1}\sigma$ for an element $a \in k^\times$ with $\|z\| = |a|$, we assume $\|z\| = 1$. Denote by $\overline{\sigma} \subset \overline{k}$ the image of $\sigma$ by the canonical projection $k^o \to \overline{k}$, and note that $\sigma_{C(\sigma, k)}(\overline{z}) = \overline{\sigma}$ by the proof of Proposition 3.25.

It is straightforward from Proposition 3.25 that $z$ is naive of level $0$ at $0$ in $C(\sigma, k)$, and the partition $(P_{\pm, 0})_{A \in \mathcal{A}}$ of unity is the system of the characteristic functions of pre-images of the reduction $\sigma \mapsto k^o \to \overline{k}$. The multiplicative $k$-Banach subspace $(P_{\pm, 0}) \subset C(\sigma, k)$ is identified with the Banach $k$-algebra $C(\sigma \cap k^\infty, k)$ by the restriction, and the function $r_1(z) = P_{\pm, 0} z \in C(\sigma, k)$ corresponds to the coordinate function of the non-empty compact subset $\sigma \cap k^\infty \subset k$ through this identification. Therefore the induction on the level $i$ works, and hence $z$ is naive at $0$ in $C(\sigma, k)$. □

**Lemma 5.19.** The inequality $||r_{i+1}(A)|| \leq |\pi|^{i+1}||A||$ holds for an $A \in \mathcal{A}(\mathcal{A})$ and an $i \in \mathbb{N}$.

**Proof.** It suffices to verify the inequality in the case $||A|| = 1$ and $i = 0$. In particular $A \neq 0$ and hence $A$ is a reductively finite element admitting the reductive continuous functional calculus $\xi : (\overline{k})^{\sigma_{\overline{k}}(\overline{A})} \to \overline{k}$. If $0 \in \overline{k}\setminus\sigma_{\overline{k}}(\overline{A})$, then one has $||r_1(A)|| = ||P_{A,0}|| = ||0|| = 0 < |\pi|$. Suppose $0 \in \sigma_{\overline{k}}(\overline{A})$. Denote by $z \in k^{\sigma_{\overline{k}}(\overline{A})}$ the coordinate function $z : \sigma_{\overline{k}}(\overline{A}) \mapsto \overline{k}$ and by $1_0 \in k^{\sigma_{\overline{k}}(\overline{A})}$ the characteristic function of $0 \in \overline{k}$. By the construction of $P_{A,0}$, one has

$$P_{A,0} A = P_{\overline{A},\overline{0}} \overline{A} = \xi(1_0)\xi(z) = \xi(1_0z) = \xi(0) = 0,$$

and hence $||r_1(A)|| = ||P_{A,0}|| < 1$. Since $\pi$ is a uniformiser of $k$, it implies $||r_1(A)|| \leq |\pi|$. □

**Theorem 5.20.** An element $A \in \mathcal{A}$ admits the continuous functional calculus in $\mathcal{A}$ if and only if $A$ is naive in $\mathcal{A}$.

**Proof.** If $A$ admits the continuous functional calculus in $\mathcal{A}$, then $A$ is naive by Lemma 5.18. Suppose $A$ is naive in $\mathcal{A}$. Take a locally constant function $c \in C(\sigma_{\mathcal{A}}(A), k)$. Since $\sigma_{\mathcal{A}}(A)$ is compact, there is an $r > 0$ such that $c$ is constant on each disc of radius $r$ in $\sigma_{\mathcal{A}}(A)$. Here we mean by a disc of radius $r$ (or $r-$) a subset of $\sigma_{\mathcal{A}}(A)$ of the form $\{ b \in \sigma_{\mathcal{A}}(A) \mid |b - a| \leq r \}$ (resp. $\{ b \in \sigma_{\mathcal{A}}(A) \mid |b - a| < r \}$) for some $a \in \sigma_{\mathcal{A}}(A)$. Replacing $r$ by a smaller number, we may and do assume $r \in \{k^\infty\}$. For each $\lambda \in \sigma_{\mathcal{A}}(A)$, since $A$ is naive at $\lambda$, there is an $i \in \mathbb{N}$ such that $||r_i(A - \lambda)|| \leq r$ by Lemma 5.19 and denote by $i_1 \in \mathbb{N}$ the least integer among such integers. Closed discs centred at some $\lambda \in \sigma_{\mathcal{A}}(A)$ of radius $||r_{i_1}(A - \lambda)||$ covers $\sigma_{\mathcal{A}}(A)$, and $||r_{i_1}(A - \lambda)|| = 0$ only when $\lambda$ is isolated in $\sigma_{\mathcal{A}}(A)$ by the construction of $r_{i_1}$ and Lemma 5.17. Therefore the closed discs have a finite disjoint subcovering because a closed disc of positive radius is open in an ultrametric space. Let $\lambda_1, \ldots, \lambda_n \in \sigma_{\mathcal{A}}(A)$ be the centres of the closed discs in the
subcovering. Define the substitution $c(A)$ as the sum $\sum_{j=1}^n c(\lambda_j)P_{ij}(A - \lambda_j) \in \mathcal{A}$. By Proposition 5.11 one has

$$\|c(A)\| = \max_{j=1}^n |c(\lambda_j)| = |c|.$$  

Now the image $c(A)$ is independent of the choice of $r > 0$ and the covering of $\sigma_{c(A)}$ by closed discs of radii no greater than $r$. Indeed, the sum of the partition of unity is 1, and hence the image $c(A)$ is stable under the refinement of the covering by the condition of the compatibility of the spectra and the reductions in the definition of the naivety.

Given two locally constant functions $c, c' : \sigma_{c(A)} \to k$, take coverings for the definition of $c(A)$ and $c'(A)$. Refining them, suppose they are a common covering. Then the equalities $(c + c')(A) = c(A) + c'(A)$ and $(cc')(A) = c(A)c'(A)$ are obvious. Moreover, for a constant map $c \in k \subset C(\sigma_{c(A)}, k)$, the image of $c(A)$ coincides with $c \in k$ by definition: the covering is taken by the singleton of the total space $\sigma_{c(A)}$. Thus the substitution map $c \mapsto c(A)$ is a $k$-algebra homomorphism from the $k$-subalgebra $c(\sigma_{c(A)}, k) \subset C(\sigma_{c(A)}, k)$ of locally constant functions to $\mathcal{A}$, and is an isometry with respect to the restriction of the supremum norm of $C(\sigma_{c(A)}, k)$.

Since $\sigma_{c(A)} \subset k$ is a totally disconnected compact Hausdorff topological space, the $k$-subalgebra $c(\sigma_{c(A)}, k) \subset C(\sigma_{c(A)}, k)$ is dense, and therefore it can be uniquely extended to be the isometry $\iota_A : C(\sigma_{c(A)}, k) \to \mathcal{A}$. It is obvious that it sends a sequence of locally constant functions converging to the coordinate function $z : \sigma_{c(A)} \to k$ to a sequence converging to $A \in \mathcal{A}$ because the approximation of $A$ by the linear combination of the projections works in modulo $\mathcal{A}$ because the approximation of $A$ by the linear combination of the projections works in modulo $\mathcal{A}$. We conclude that $\iota_A$ is the continuous functional calculus. \hfill $\Box$

### 5.4 Fractal Operator

The criterion for the normality in Theorem 5.20 contains the process of the inductive calculus with infinite steps, and hence the practical use of the criterion needs some conditions of an operator which give it kind of the structure of a fractal. See the following example:

**Example 5.21.** Suppose $k$ is a local field over $\mathbb{Q}_p$. Set $\mathcal{A} := \mathcal{B}(\mathcal{O}(\mathbb{Z}, k))$, and consider the elements $A_1, A_2 \in \mathcal{A}$ whose matrix representations with respect to the canonical orthonormal Schauder basis of $\mathcal{O}(\mathbb{Z}, k)$ are

$$M_1 = \begin{pmatrix}
-2 & p & p^2 & p^3 & p^4 & \cdots \\
0 & -1 & p & p^2 & p^3 & \cdots \\
0 & 0 & 0 & p & p^2 & \cdots \\
0 & 0 & 0 & 1 & p & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},$$

$$M_2 = \begin{pmatrix}
p & p^2 & p^3 & \cdots \\
0 & p^2 & p^3 & \cdots \\
0 & 0 & p & \cdots \\
0 & 0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$
where the centres of the matrix representations above are the \((0,0)\)-entry. For \(i = 1, 2\), it is obvious that \(\|A_i\| \leq 1\) and \(\mathbb{Z} \subset \sigma_{\omega}(A_i)\). Moreover one has \(\|A_i\| = 1\) and \(\mathbb{Z}_p = \sigma_{\omega}(A_i)\). Its normal reduction coincides with the matrix reduction by Proposition 2.9 and the reduction trivially admits the continuous functional calculus because the matrix representation of the reduction is a diagonal matrix. The reductive spectrum \(\sigma_{\omega}(A_i)\) coincides with \(\mathbb{F}_p \subset \overline{\mathbb{k}}\). For the partition \((P_{A_i, \Lambda})_{\Lambda \in \mathbb{F}_p}\), it is easy to see that the triads \((A_j, \omega, C_0(\mathbb{Z}, k))\) and \((P_{A_i, A_j}, (P_{A_j, \mathbb{Z}}), C_0(j + p\mathbb{Z}, k))\) of the \(p\)-adic Banach representations admits similar matrix representations for each \(j = 0, \ldots, p - 1\), and hence the induction process in Theorem 5.20 works. Indeed it admits the continuous functional calculus \(\iota_{A_i} : C(\mathbb{Z}_p, k) \to \omega\), and for a continuous map \(F : \mathbb{Z}_p \to k\), the matrix representations \(F(M_1)\) and \(F(M_2)\) of the substitutions \(\iota_{A_1}(F) = F(A_1)\) and \(\iota_{A_2}(F) = F(A_2)\) are

\[
F(M_1) = \begin{pmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & F(-2) & \partial F(-2)p & \partial F(-2)p^2 & \partial F(-2)p^3 & \partial F(-2)p^4 & \vdots \\
\vdots & 0 & F(-1) & \partial F(-1)p & \partial F(-1)p^2 & \partial F(-1)p^3 & \vdots \\
\vdots & 0 & 0 & F(0) & \partial F(0)p & \partial F(0)p^2 & \vdots \\
\vdots & 0 & 0 & 0 & F(1) & \partial F(1)p & \vdots \\
\vdots & 0 & 0 & 0 & 0 & F(2) & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots 
\end{pmatrix},
\]

\[
F(M_2) = \begin{pmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & F(-2) & \partial F(-2)p & \frac{\partial^2 F(-2)}{2!}p^2 & \frac{\partial^3 F(-2)}{3!}p^3 & \frac{\partial^4 F(-2)}{4!}p^4 & \vdots \\
\vdots & 0 & F(-1) & \partial F(-1)p & \frac{\partial^2 F(-1)}{2!}p^2 & \frac{\partial^3 F(-1)}{3!}p^3 & \vdots \\
\vdots & 0 & 0 & F(0) & \frac{\partial^2 F(0)}{2!}p^2 & \frac{\partial^3 F(0)}{3!}p^3 & \vdots \\
\vdots & 0 & 0 & 0 & F(1) & \frac{\partial^2 F(1)}{2!}p^2 & \vdots \\
\vdots & 0 & 0 & 0 & 0 & F(2) & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots 
\end{pmatrix},
\]

where we set \(\partial G(n) := G(n) - G(n - 1)\) for each \(n \in \mathbb{Z}_p\) and each \(G \in C(\mathbb{Z}_p, k)\). They are analogous to the holomorphic functional calculus of the shift operator in Example 4.6. It is remarkable that the difference \((n!)^{-1}\partial^n F\) appears in continuous functional calculi, while the differential coefficient \(F_n = (n!)^{-1}\partial^n F/dT^n\) does in holomorphic functional calculi. The differential operator \(F(z) \mapsto F'(z)\) is continuous in the class of rigid analytic functions \(F \in k[z]\), and the difference operator \(F(z) \mapsto \partial F(z)\) is continuous in the naive class of continuous functions \(F \in C(\mathbb{Z}_p, k)\).
6 Unbounded Normal Operator

Finally we formulate the rigid continuous functional calculus for an unbounded operator and give an explicit definition of a non-Archimedean physical quantity. Many of important Archimedean physical quantities, such as the momentum $P = d/dx$ and the Hamiltonian $E = H$, are unbounded. Note that an Archimedean physical quantity is expressed by a closed self-adjoint operator, and hence it admits the rigid continuous functional calculus using the spectral measure even if it is unbounded. Therefore a non-Archimedean physical quantity should also be expressed by an operator admitting the rigid continuous functional calculus in some sense.

6.1 Extended Rigid Continuous Functional Calculus

To begin with, we introduce the notion of the spectrum of an unbounded operator. In this section, let $V$ denote a $k$-Banach space, $W \subset V$ a $k$-vector subspace, $\sigma \subset k$ a non-empty closed subset, and $z: \sigma \leftrightarrow k$ the coordinate function.

**Definition 6.1.** For an operator $A: W \rightarrow V$, denote by $\rho(A) \subset k$ the subset of points $t \in k$ such that the operator $A - t: W \rightarrow V$ is injective and its inverse map $(A - t)^{-1}: V \rightarrow W \leftrightarrow V$ is bounded. Call it the $k$-rational resolvent of $A$. Set $\sigma(A) := k \setminus \rho(A)$, and call it the $k$-rational spectrum of $A$. If there is no ambiguity on $k$, just call it the spectrum of $A$ for short.

**Proposition 6.2.** For an operator $A: W \rightarrow V$, the spectrum $\sigma(A) \subset k$ is closed. If $A$ is an everywhere defined bounded operator, i.e. $A \in \mathcal{B}_k(V)$, then $\sigma(A) = \sigma_{\mathcal{B}_k(V)}(A)$ and it is a bounded closed subset.

**Proof.** The first assertion follows from the equality

$$(A - t)^{-1} = (A - t_0)^{-1}(1 - (t - t_0)(A - t_0)^{-1}) = (A - t_0)^{-1}\sum_{n=0}^{\infty}(t - t_0)^n(A - t_0)^{-n} \in \mathcal{B}_k(V)$$

for any $t_0 \in \rho(A)$ and $t \in k$ with $|t - t_0| < \|A - t_0\|$. The second assertion is trivial by definition. $\square$

**Definition 6.3.** For a totally disconnected metric space $X$, a continuous function $f: X \rightarrow k$ is said to be tempered if the restriction $f|_S: S \rightarrow k$ is bounded for any bounded clopen subset $S \subset X$. The subset $C_{\text{tem}}(X, k) \subset C(X, k)$ of tempered continuous functions coincides with the intersection of the pre-image of $C_{\text{bd}}(S, k) \subset C(S, k)$ by the restriction map $C(X, k) \rightarrow C(S, k)$ for all bounded clopen subsets $S \subset X$, and hence is a $k$-subalgebra. We endow $C_{\text{tem}}(X, k)$ with the topology of uniform convergences on each bounded clopen subset, i.e. a net $(f_i)_{i \in I}$ in $C_{\text{tem}}(X, k)$ indexed by a directed set $I$ converges to $f \in C_{\text{tem}}(X, k)$ if and only if for each bounded clopen subset $S \subset X$, the restriction $(f_i|_S)_{i \in I}$ converges to $f|_S$ in $C_{\text{bd}}(S, k)$.

61
The $k$-subalgebra $k[z] \subset C(\sigma, k)$ generated by $z$ is contained in $C_{\text{tem}}(\sigma, k)$ because $z|_S \in C_{\text{bd}}(S, k)$ for any bounded clopen subset $S \subset \sigma$. For good metric spaces, the ring of tempered continuous functions is easily calculated.

**Example 6.4.** If a totally disconnected metric space $X$ is bounded, then one has $C_{\text{tem}}(X, k) = C_{\text{bd}}(X, k)$ by definition.

**Example 6.5.** For a local field $K$ such as $\mathbb{Q}_l$ for a prime number $l \in \mathbb{N}$, one has $C_{\text{tem}}(K, k) \cong_k C(K, k)$ because each bounded closed subset of $K$ is compact. The induced topology of the right hand side coincides with the compact uniform convergence topology.

**Lemma 6.6.** For a continuous function $f \in k[z]$ invertible in $C(\sigma, k)$, its inverse $f^{-1} \in C(\sigma, k)$ is bounded, and in particular tempered.

*Proof.* Take an algebraic closure $K$ of $k$. Since $k$ is complete subspace of the metric space $K$, $k$ is closed in $K$. For a continuous function $f \in k[z]$ invertible in $C(\sigma, k)$, take a polynomial $F \in k[T]$ with $F(z) = f$, and let $Z \subset K$ be the set of zeros of $F$ in $K$. If $F \in k$, then $f$ is a constant function and hence its inverse $f^{-1}$ is bounded. Therefore consider the case $F \not\in k$. In particular, $Z \subset K$ is a non-empty finite subspace of the metric space $K$, and the distance $d \in [0, \infty)$ between $Z$ and $\sigma \subset k \subset K$ is positive definite. Therefore for a $\lambda \in \sigma$, one has $|f^{-1}(\lambda)| = |F(z)^{-1}(\lambda)| \leq d^{-\deg F}$, and hence $f^{-1}$ is bounded. $\Box$

The inclusion $C_{\text{bd}}(\sigma, k) \hookrightarrow C_{\text{tem}}(\sigma, k)$ is continuous because the restriction map $C_{\text{bd}}(\sigma, k) \to C_{\text{bd}}(S, k)$ is a contraction for any bounded clopen subset $S \subset \sigma$. Remark that the topology of uniform convergences on $\sigma$ and the topology of uniform convergences on bounded clopen subsets $S \subset \sigma$ differ when $\sigma$ is not bounded. Therefore the inclusion above is not a homeomorphism onto the image.

**Definition 6.7.** For a bounded clopen subset $S \subset \sigma$, define a multiplicative $k$-linear map $1^*_S : C_{\text{bd}}(S, k) \to C_{\text{bd}}(\sigma, k)$ as a map sending a bounded continuous function $f : S \to k$ to the extension of $f$ by $0$ outside $S$, which is continuous because $S \subset \sigma$ is clopen.

For a bounded clopen subset $S \subset \sigma$, the extension map $1^*_S : C_{\text{bd}}(S, k) \to C_{\text{bd}}(\sigma, k)$ is the right inverse map of the restriction map $C_{\text{bd}}(\sigma, k) \to C_{\text{bd}}(S, k)$. Since $1^*_S$ is a contraction, it is continuous.

**Definition 6.8.** Denote by $C_{\text{rig}}(\sigma, k) \subset C_{\text{tem}}(\sigma, k)$ the closure of the localization of $k[z] \subset C_{\text{tem}}(\sigma, k)$ by functions in $k[z] \cap C_{\text{tem}}(\sigma, k)^\times$.

The rigidity is an obvious generalisation of that of a continuous function on a bounded closed subset in Definition rigid continuous function. We extends the notion of the measurability of a clopen subset in Definition $[3, 12]$

**Definition 6.9.** For a non-empty closed subset $\sigma \subset k$, a bounded clopen subset $S \subset \sigma$ is said to be $\sigma$-measurable or simply a measurable subset of $\sigma$ if the characteristic function $1^*_S : \sigma \to k$ of $S$ is contained in $C_{\text{rig}}(\sigma, k)$. Denote by $\Omega_\sigma \subset 2^\sigma$ the collection of measurable subsets of $\sigma$. 62
Definition 6.10. For a non-empty closed subset $\sigma \subset k$, a measurable filtration of $\sigma$ is an increasing sequence $S_0 \subset S_1 \subset \cdots \subset \sigma$ of $\sigma$-measurable sets such that $\bigcup_{n \in \mathbb{N}} S_n = \sigma$ and there is an $n \in \mathbb{N}$ with $S \subset S_n$ for any bounded subset $S \subset \sigma$.

Definition 6.11. A non-empty closed subset $\sigma \subset k$ is said to be analysable if it admits a measurable filtration.

Example 6.12. Every bounded closed subset $\sigma \subset k$ is analysable. Indeed, it admits the canonical measurable filtration $S_0 = S_1 = \cdots = \sigma$.

Example 6.13. If $k$ is a local field, then the total space $k$ is analysable. It is because for any bounded clopen subsets $S, T \subset k$ with $S \subset T$, $T$ is compact and hence the characteristic function $1_S$ of $S$ is uniformly approximated by polynomial functions uniformly on $T$ by Proposition 1.44.

Beware that a refinement of a measurable filtration is not necessarily a measurable filtration, because the class of rigid continuous functions does not possess enough idempotents while that of bounded continuous functions does. By a similar reason, for a bounded clopen measurable subset $S \subset \sigma$, the image of $\text{C}_{\text{rig}}(S, k)$ by the extension map $1_S^* : \text{C}_{\text{rig}}(S, k) \to \text{C}_{\text{bd}}(\sigma, k)$ seems not to be contained in $\text{C}_{\text{rig}}(\sigma, k)$. A function in $\text{C}_{\text{rig}}(S, k)$ can be approximated by a rational function with poles outside $S$, while a function in $(1_S^*)^{-1}(\text{C}_{\text{rig}}(\sigma, k)) \subset \text{C}_{\text{bd}}(S, k)$ can be only approximated by a rational function with poles outside $\sigma$. Therefore the inclusion $(1_S^*)^{-1}(\text{C}_{\text{rig}}(\sigma, k)) \subset \text{C}_{\text{rig}}(S, k)$ holds. Note that if a rational function is always approximated by a polynomial function, then any function in $\text{C}_{\text{rig}}(S, k)$ is approximated by polynomials uniformly on $S$, and hence $(1_S^*)^{-1}(\text{C}_{\text{rig}}(\sigma, k)) = \text{C}_{\text{rig}}(S, k)$. Such a situation occurs when $k$ is a local field by Proposition 1.44.

Definition 6.14. An operator $A : W \to V$ is said to be closed if $W \subset V$ is dense and if its graph $\Gamma_A := \{ (w, Aw) \mid w \in W \} \subset W \times V$ is closed. Denote by $\text{Cl}_k(V)$ the set of closed operators on $V$.

The class of closed operators is one of practical generalisations of the class of everywhere defined bounded operators. For example, an everywhere defined bounded operator is closed and therefore one has $\mathcal{B}_k(V) \subset \text{Cl}_k(V)$. Conversely, the closed graph theorem holds also for the non-Archimedean situation. Namely, if the valuation of $k$ is non-trivial, then everywhere defined closed operator $A : V \to V$ is bounded by [BGR] 2.8.1. Now we define the rigid continuous functional calculus for a closed operator.

Definition 6.15. An operator $A : W \to V$ is said to admits the extended rigid continuous functional calculus if the spectrum $\sigma(A) \subset k$ is analysable, and there is a map $\iota_A : \text{C}_{\text{rig}}(\sigma(A), k) \to \text{Cl}_k(V)$ satisfying the following:
Definition 6.16. Let \( A : W \to V \) be a closed operator admitting the extended rigid continuous functional calculus \( \iota_A \). Set \( \Omega_A := \Omega_{\sigma(A)} \). Call a subset \( S \subset k \) belonging to \( \Omega_A \) an \( A \)-measurable set.

An everywhere defined bounded operator \( A : V \to V \) admits the extended rigid continuous functional calculus as a closed operator if and only if it admits the rigid continuous functional calculus as an everywhere defined bounded operator by the following Proposition.

Proposition 6.17. Let \( \mathcal{A} \) be a Banach \( k \)-algebra. For an element \( A \in \mathcal{A} \) admitting the rigid continuous functional calculus \( \iota_A \) in \( \mathcal{A} \) and an \( A \)-measurable set \( S \in \Omega_A \), there is a unique isometric \( k \)-algebra homomorphism

\[
\iota_{A,S} : (1_S^*)^{-1}(C_{rig}(\sigma, k)) \to (\iota_A(1_S))
\]
with respect to the restriction of the norm of $C_{bd}(S, k)$, such that the diagram

\[
\begin{array}{ccc}
(1_S)^{-1}(C_{rig}(\sigma, k)) & \xrightarrow{\iota_A} & (\iota_A(1_S)) \\
\downarrow \quad \iota_S & & \downarrow \\
C_{rig}(\sigma, k) & \xrightarrow{\iota_A} & \mathfrak{A}
\end{array}
\]

of normed $k$-vector spaces commutes, where the second column is the inclusion.

**Proof.** Since $\iota_A$ is an isometry, it sends the closed ideal $(1_S)$ to the closed two-sided ideal $(\iota_A(1_S)) \subset C_{rig}(\sigma, k)$. The ideal $(1_S)$ is isometrically identified with $(1_S)^{-1}(C_{rig}(\sigma, k))$ through the restriction map and the extension map $1_S$ as normed $k$-vector spaces, and hence one obtains a desired isometry $(1_S)^{-1}(C_{rig}(\sigma, k)) \rightarrow (\iota_A(1_S))$. The uniqueness follows from the injectivity of the two columns. \hfill \square

### 6.2 Uniqueness of the Functional Calculus

We prepare a little for a proof of the uniqueness of the extended rigid continuous functional calculus. The condition (iii) implies that for any $f \in C_{rig}(\sigma(A), k)$ and any measurable filtration $S_0 \subset S_1 \subset \cdots \subset \sigma$, the sequence $(\iota_A(1_{S_n}f))_{n \in \mathbb{N}} \in \mathcal{B}(V)^{\mathbb{N}}$ converges to $\iota_A(f) \in \text{Cl}_k(V)$ with respect to the strong operator topology. The strong convergence makes sense if the intersection of the domains of $\iota_A(f)$ and $\iota_A(1_{S_n}f)$’s has enough vectors. The following two lemmas guarantee that the intersection is dense in $V$.

**Lemma 6.18.** Let $A : W \rightarrow V$ be a closed operator admitting an extended rigid continuous functional calculus $\iota_A$. The union $\tilde{W} \subset W$ of the images of the projections $\iota_A(1_S)$ is dense in $V$, where $S$ runs through $\Omega_A$.

**Proof.** Take a measurable filtration $S_0 \subset S_1 \subset \cdots \subset \sigma(A)$. For any $v \in V$, one has

$$
\lim_{n \to \infty} \iota_A(1_{S_n})v = \iota_A(1)v = v
$$

by the conditions (i) and (iii). Therefore $\tilde{W} \subset V$ is dense. \hfill \square

**Lemma 6.19.** Let $A : W \rightarrow V$ be a closed operator admitting an extended rigid continuous functional calculus $\iota_A$. For any $f \in C_{rig}(\sigma(A), k)$ the domain of the closed operator $\iota_A(f)$ contains $\tilde{W}$.

**Proof.** Let $S_0 \subset S_1 \subset \cdots \subset \sigma(A)$ be a measurable filtration. For any $w \in \tilde{W}$, take a measurable subset $S \in \Omega_A$ with $\iota_A(1_S)w = w$. By the definition of a measurable filtration, there is an $N \in \mathbb{N}$ such that $\iota_A(1_{S_N})w = \iota_A(1_S)w = w$. By the conditions (i) and (iv), $\iota_A(1_{S_N}f) = \iota_{A,S_N}(1_{S_N}f)\iota_A(1_{S_N})$ is an everywhere defined bounded operator for any $n \in \mathbb{N}$. One has

$$
\lim_{n,m \to \infty} \|\iota_A(1_{S_n}f) - \iota_A(1_{S_m}f)w\| = \lim_{n,m \to \infty} \|\iota_A(1_{S_n}f)\iota_A(1_{S_N})w - \iota_A(1_{S_n}f)\iota_A(1_{S_N})w\|
$$

65
by the conditions (i) and (iii), and hence the sequence \((\tau_A(1_{S_m})w)_{m \in \mathbb{N}}\) converges to an element \(v \in V\) by the completeness of \(V\). Since \(\tau_A(f)\) is densely defined and a metric space \(V\) is automatically first countable, there is a sequence \((w_n)_{n \in \mathbb{N}}\) in the domain of \(\tau_A(f)\) converging to \(w\). One obtains

\[
\lim_{m \to \infty} \lim_{n \to \infty} (\tau_A(1_{S_m})w_n, \tau_A(1_{S_m})w_n) = \lim_{m \to \infty} (\tau_A(1_{S_m})w_n, \tau_A(1_{S_m})w_n) = \lim_{m \to \infty} (\tau_A(1_{S_m})w_n, \tau_A(1_{S_m})w_n) = \lim_{m \to \infty} (\tau_A(1_{S_m})w, \tau_A(1_{S_m})w_n) = \lim_{m \to \infty} (\tau_A(1_{S_m})w, \tau_A(1_{S_m})w) = (w, v)
\]

by the conditions (i), (ii), and (iii) and by the continuity of \(\tau_A(1_{S_m})\) and \(\tau_A(1_{S_m})f\) for each \(m \in \mathbb{N}\). Therefore \(w\) is contained in the domain of \(\tau_A(f)\) by the closeness of \(\tau_A(f)\). \(\square\)

We did not assume that the extended functional calculus sends the coordinate function \(z: \sigma(A) \to k\) to \(A\) in the definition. However, such a desired property automatically holds by other conditions.

**Lemma 6.20.** Let \(A: W \to V\) be a closed operator admitting the extended rigid continuous functional calculus \(\tau_A\). For any polynomial \(F \in k[T]\), \(F(A)\) is a densely defined closable operator with the closure \(\overline{F(A)}\), and \(\tau_A(F(z)) = F(\overline{A})\).

**Proof.** Since \(V_S\) is \(A\)-stable for each \(S \in \Omega_A\), so is \(\overline{W}\). Therefore \(A\) induces an endomorphism \(A|_{\overline{W}}: \overline{W} \to \overline{W}\), and hence the domain of \(F(A)\) contains the dense subspace \(\overline{W}\) in \(V\) for any \(F \in k[T]\). By the condition (i), it suffices to show that \(\tau_A(z) = A\). Let \(S_0 \subset S_1 \subset \cdots \subset \sigma(A)\) be a measurable filtration of \(\sigma(A)\). For any \(w \in \overline{W}\), the conditions (iii) and (iv) and Lemma 6.19 guarantee that

\[
\tau_A(z)w = \lim_{n \to \infty} \tau_A(1_{S_n}z)w = \lim_{n \to \infty} \tau_{A,S_n}(1_{S_n}z)(\tau_A(1_{S_n})w) = \lim_{n \to \infty} (\tau_{S,A}(1_{S_n})A)(\tau_A(1_{S_n})w)
\]

and hence \(\tau_A(z) = A\) because \(\overline{W} \subset W\) is dense. \(\square\)

In particular by the condition (i), one has \(\tau_A((z - s)^{-1}) = (A - s)^{-1} \in \mathcal{B}_k(V)\) for any resolvent \(s \in \rho(A)\). In the spectral decomposition of an Archimedean self-adjoint operator, the uniqueness of the continuous functional calculus immediately follows from such equalities for resolvents because a self-adjoint operator has enough points in the resolvent. There are many important cases that the resolvent of a non-Archimedean operator is closed, and hence the proof of the uniqueness is a little complicated. By Lemma 6.18 and Lemma 6.19, the image \(\tau_A(f) \in \mathcal{B}_k(V)\) is uniquely determined by the restriction \(\tau_{1_{S},A}(1_{S})f \in \mathcal{B}_k(V_S)\) for each \(S \in \Omega_A\). Moreover, \(\tau_{S,A}(1_{S}f)\) is uniquely determined by
continuous functional calculus bounded operators independent of the choice of \( \iota_A \). Thus if \( \iota_A(1_S) \) can be described by \( S \) and \( A \), the extended rigid continuous function turns out to be unique. Beware that we have not verified that the common dense domain \( \hat{W} \subset V \) is independent of the choice of \( \iota_A \), and one has to avoid the direct use of \( \hat{W} \) in the proof.

**Proposition 6.21.** Let \( A : W \to V \) be a closed operator admitting an extended rigid continuous functional calculus \( \iota_A \). For an \( A \)-measurable set \( S \in \Omega_A \), the projection \( \iota_A(1_S) \in \mathcal{B}(V) \) admits an expression as a strongly convergent limit of everywhere defined bounded operators independent of the choice of \( \iota_A \).

**Proof.** Denote by \( V_{A^n} \subset V \) the intersection of domains of \( A^n \) for \( n \in \mathbb{N} \). Since \( V_{A^n} \) contains \( \hat{W} \), it is a dense subspace of \( V \). Moreover \( V_{A^n} \) is independent of \( \iota_A \) because it can be determined by the collection \{ \( A^n | n \in \mathbb{N} \) \}. For any \( \epsilon > 0 \) and any bounded clopen subset \( T \subset \sigma \), take an element \( f \in C_{rig}(\sigma, k) \) in the localisation of \( k[z] \) by \( k[z] \cap C_{rig}(\sigma, k) \) with \( |f(\lambda) − 1_S(\lambda)| < \epsilon \) for any \( \lambda \in T \). By the definition of the localisation of \( k[z] \) by \( k[z] \cap C_{rig}(\sigma, k) \), there are polynomials \( P, Q \in k[T] \) such that \( Q \) has no zeros on \( \sigma \) and \( f(\lambda) = Q(\lambda)^{-1}P(\lambda) \) for any \( \lambda \in \sigma \). In particular, the polynomials \( P, Q \) satisfies \( |1_S(\lambda) − Q(\lambda)^{-1}P(\lambda)| < \epsilon \) for any \( \lambda \in T \). Denote by \( I \subset 2^\sigma \) the directed set of bounded clopen subsets of \( \sigma \) containing \( S \). By the axiom of choice, one obtains a map \( (0, \infty) \times I \to k[T]^2 : (n, T) \mapsto (P_{n,T}, Q_{n,T}) \) such that the net \( \{ (n, T) \mapsto C_{rig}(\sigma, k) : (n, T) \mapsto Q_{n,T}^{-1}P_{n,T} \} \) converges to \( 1_S \). By the condition (v), the domain of the operator \( Q_{n,T}(A)^{-1}P_{n,T}(A) \) contains \( V_{A^n} \) for any \( (n, T) \in \mathbb{N} \times I \). By the condition (iii), one has

\[
\iota_A(1_S)_v = \iota_A \left( \lim_{(n,T)\to\infty} Q_{n,T}^{-1}P_{n,T} \right)_v = \lim_{(n,T)\to\infty} Q_{n,T}(A)^{-1}P_{n,T}(A)_v
\]

for any \( v \in V_{A^n} \). The right hand side and the dense subspace \( V_{A^n} \subset V \) are independent of the choice of \( \iota_A \). Thus \( \iota_A(1_S) \) admits an expression independent of the choice of \( \iota_A \). \( \square \)

**Corollary 6.22.** Let \( A : W \to V \) be a closed operator. Then the extended rigid continuous functional calculus \( \iota_A \) of \( A \) is unique, and the common dense domain \( \hat{W} \subset V \) is independent of the choice of \( \iota_A \).

**Definition 6.23.** An operator \( A : W \to V \) is said to be a \( k \)-valued normal operator if it is a closed operator admitting the extended rigid continuous functional calculus.

**Definition 6.24.** An operator \( A : W \to V \) is said to be a normal operator if there is a finite extension \( K/k \) such that the base extension \( A \otimes_k 1 : W \otimes_k K \to V \otimes_k K \) is a \( K \)-valued normal operator.

**Definition 6.25** (unbounded physical quantity). A \( p \)-adic physical quantity over \( k \) is a \( k \)-valued normal operator on a strictly Cartesian \( k \)-Banach space of countable type. In particular, a bounded \( p \)-adic physical quantity over \( k \) is a \( p \)-adic physical quantity.
6.3 Galois Descent of the Normality

Recall that the resolvent of a normal operator might be empty. An Archimedean self-adjoint operator has the sufficiently large resolvent because its spectrum is contained in a thin subset $\mathbb{R} \subset \mathbb{C}$. In the non-Archimedean case, there is no canonical subfield of $k$ corresponding to $\mathbb{R} \subset \mathbb{C}$, but conversely $k$ admits a non-trivial finite extension $K/k$ if $k$ is not an algebraically closed. Here is an observation toward a base change of a normal operator.

**Definition 6.26.** An operator $A: W \to V$ is said to be a potentially $k$-valued normal operator if there is a finite Galois extension $K/k$ with the Galois group $\text{Gal}(K/k)$ such that the base change the following hold:

(i) The base change $A \otimes_k 1: W \otimes_k K \to V \hat{\otimes}_k K$ is a $K$-valued normal operator;

(ii) $\sigma(A \otimes_k 1) = \sigma(A)$;

(iii) The extended rigid continuous functional calculus $\iota_A \otimes_k 1$ is $\text{Gal}(K/k)$-equivariant.

Here the Galois group $\text{Gal}(K/k)$ isometrically acts on $C_{bd}(S, K)$ for any bounded clopen subset $S \subset \sigma(A)$, and therefore the subring $C_{rig}(\sigma(A), K) \subset C_{ten}(\sigma, K)$ is $\text{Gal}(K/k)$-stable. The action of $\text{Gal}(K/k)$ on $V \otimes_k K$ is the tensor representation with respect to the trivial action on $V$. For an operator $B: W' \to V \hat{\otimes}_k K$, we consider the operator

$$gB: g(W') \to V \hat{\otimes}_k K$$

$$w \mapsto g(B(g^{-1}w))$$

for each $g \in \text{Gal}(K/k)$. Since the action on $K$ is an isometric representation, so is the action on $V \hat{\otimes}_k K$. Since $K$ is $k$-linearly homeomorphic to a strictly Cartesian $k$-Banach space by [BGR] 2.8.2/2, the embedding $V \hookrightarrow V \hat{\otimes}_k K$ is an isomorphism onto the image. In particular the image of $V$ in $V \hat{\otimes}_k K$ is complete and hence closed. Therefore the restriction of a $\text{Gal}(K/k)$-equivariant closed operator on $V \hat{\otimes}_k K$ on $V$ is again closed. Therefore taking the $\text{Gal}(K/k)$-invariants, $\iota_A \otimes_k 1$ induces a well-defined map

$$\iota_A: C_{rig}(\sigma(A), k) \to \text{Cl}_k(V).$$

Thus one obtains the following criterion:

**Proposition 6.27.** An operator $A: W \to V$ on a strictly Cartesian $k$-Banach space $V$ of countable type is a $p$-adic physical quantity if and only if it is a potentially $k$-valued normal operator.

Note that this proposition implies that $\iota_A$ is independent of the choice of a finite Galois extension $K$ by the uniqueness of the extended rigid continuous functional calculus. This criterion allows one to reduce it to the case that the resolvent is sufficiently large.

**Proof.** The non-trivial property is the local isometry of $\iota_A$. Suppose $K/k$ is a finite Galois extension with the appropriate conditions. Since $V$ admits an orthogonal Schauder basis, the embedding $V \hookrightarrow V \hat{\otimes}_k K$ is an isometry. Therefore the induced locally bounded map $\iota_A$ is a local isometry. Since $\iota_A \otimes_k 1$ satisfies the other conditions in the definition of the extended rigid continuous functional calculus, so does $\iota_A$. $\square$
6.4 Operator with Proper Spectrum

We have finished the formulation of the non-Archimedean analogue of the rigid continuous functional calculus of an unbounded operator. The conditions in the definition of the extended rigid continuous functional calculus are hard to verify in general if an operator is not bounded, and hence we introduce a good class of operators where the conditions are much clearer.

Definition 6.28. A metric space $X$ is said to be proper if any bounded closed subset is compact.

A proper metric space is countable at infinity as a topological space because it admits an exhaustion by closed balls of increasing radii. In particular it is $\sigma$-compact and hence Lindelöf. Moreover, it is locally compact Hausdorff. A closed subset of a proper metric space is again proper. For example, closed subsets of $k$ is proper if $k = \mathbb{R}$ or $\mathbb{C}$, if $k$ is a finite field endowed with the trivial norm, or if $k$ is a local field. In particular, a spectrum of an operator on a Banach space over such a field $k$ is proper. Conversely if a subspace $S$ of a complete metric space $X$ is proper, then $S$ is closed in $X$. Indeed if a sequence in $S$ converges to a point $x \in X$, the closed unit ball $D \subset X$ centred at $x$ contains a compact subset $S \cap D$. The point $x$ is the limit point of the subsequence contained in $S \cap D$, and hence contained in $S \cap D$ by the completeness of the compact metric space $S \cap D$.

Lemma 6.29. If a closed subset $\sigma \subset k$ is a proper metric space, then any bounded clopen subset $S \subset \sigma$ is $\sigma$-measurable.

Proof. Let $I$ be the directed set of bounded clopen subsets of $\sigma$ containing $S$. The family $I$ contains $S$, and for any bounded clopen subset $T \subset \sigma$, $T \cup S$ is contained in $I$. It follows that $I$ is a non-empty family final in the collection of bounded clopen subsets of $\sigma$. For pairs $(T, n), (T', n') \in I \times \mathbb{N}$, we write $(T, n) \leq (T', n')$ if $T \subset T'$ and $n \leq n'$, and then the binary relation $\leq$ on $I \times \mathbb{N}$ is a directed order. For each $(T, n) \in I \times \mathbb{N}$, $T$ is compact by the properness of $\sigma$. Therefore there is some continuous function $f : T \to k$ contained in the $k$-subalgebra $k[1_{T\sigma}] \subset C(T, \sigma)$ generated by the coordinate function $1_{T\sigma} : T \to k$ such that $\Vert 1_{S} - f \Vert < p^{-n}$ by Proposition 1.44. Since the restriction map $k[z] \to k[1_{T\sigma}] : z \mapsto 1_{T\sigma}$ is surjective, take an element $f_{T, n} \in k[z]$ whose restriction is $f$. By the axiom of choice, one obtains a directed system $(f_{T, n})_{(T, n) \in I \times \mathbb{N}} \in k[z]^{I \times \mathbb{N}} \subset C_{\text{tem}}(\sigma, k)^{I \times \mathbb{N}}$ such that $\Vert 1_{S} - f_{T, n} \Vert < p^{-n}$ for any $(T, n) \in I \times \mathbb{N}$. The system converges to $1_{S}$ in $C_{\text{tem}}(\sigma, k)$ by the definition of the topology of $C_{\text{tem}}(\sigma, k)$, and hence one has $1_{S} \in C_{\text{rig}}(\sigma, k)$.

Proposition 6.30. Every non-empty proper metric subspace $\sigma \subset k$ admits a measurable filtration. In particular if $k$ is a finite field endowed with the trivial norm or a local field, then every non-empty closed subset $\sigma \subset k$ is analysable.

Proof. By Lemma 6.29, every closed ball of $\sigma$ is $\sigma$-measurable, and hence $\sigma$ admits a measurable filtration consisting of closed balls. The second assertion directly follows from the fact that a finite field and a local field is a proper metric space.
For a closed operator with a proper spectrum, we reinterpret Definition \[6.15]\. This interpretation helps one to examine whether a closed operator on a finite field and a local field admits the extended rigid continuous functional calculus or not.

**Lemma 6.31.** For a non-empty proper metric subspace \( \sigma \subset k \) and a bounded clopen subset \( S \subset \sigma \), the equality \((1_S^\sigma)^{-1}(C_{\text{rig}}(\sigma, k)) = C_{\text{rig}}(S, k)\) holds.

**Proof.** Every function \( f \in C_{\text{rig}}(S, k) \) can be approximated by polynomial functions \((f_n)_{n \in \mathbb{N}}\) uniformly on \( S \) by Proposition \[1.44]\. The characteristic function \( 1_S \) can be approximated by rational functions \((q_m)_{m \in \mathbb{N}}\) with poles outside \( \sigma \) by Lemma \[6.29]\. Denote by \( I \subset 2^\sigma \) the directed set of bounded clopen subsets of \( \sigma \) containing \( S \). For each \( n \in \mathbb{N} \) and \( T \in I \), since \( T \) is compact by the properness of \( \sigma \), there is an \( M \in \mathbb{N} \) such that \([f_n(\lambda)1_S(\lambda) - f_n(\lambda)q_m(\lambda)] \leq 2^{-n}\) for any \( m \geq M \) and \( \lambda \in T \). Denote by \( M_{n,T} \in \mathbb{N} \) the minimum of such an \( M \). Then \((f_nq_{M_{n,T}})_{(n,T) \in I \times I} \) is a net consisting of rational functions with poles outside \( \sigma \) converging to the extension \( 1_Sf \in C(\sigma, k) \). Therefore \( 1_Sf \in C_{\text{rig}}(\sigma, k) \), and hence \( f \in (1_S^\sigma)^{-1}(C_{\text{rig}}(\sigma, k)) \). \( \square \)

**Theorem 6.32.** Suppose \( k \) is a finite field endowed with the trivial norm or a local field. A closed operator \( A : W \rightarrow V \) admits the extended rigid continuous functional calculus if and only if there are projections \( P_{A,n} \in \mathcal{B}_k(V) \) for \( n \in \mathbb{N} \) satisfying the following:

(i) \( \|P_{A,n}\| \in \{0, 1\} \) for any \( n \in \mathbb{N} \);

(ii) The range \( V_n \) of \( P_{A,n} \) is contained in \( W \) for any \( n \in \mathbb{N} \);

(iii) The closure of \( P_{A,n}A : W \rightarrow V \) coincides with \( AP_{A,n} : V \rightarrow V \) for any \( n \in \mathbb{N} \);

(iv) The restriction \( A_n \) of the everywhere defined bounded operator \( AP_{A,n} \) on \( V_n \) admits the continuous functional calculus \( t_{A_n} \) in \( \mathcal{B}_k(V_n) \) for any \( n \in \mathbb{N} \);

(v) The spectrum \( \sigma_{\mathcal{B}_k(V_n)}(A_n) \) coincides with the compact subset \( \sigma(A) \cap k(p^{-n}) \) for any \( n \in \mathbb{N} \);

(vi) \( P_{A,n}P_{A,n+1} = P_{A,n+1}P_{A,n} = P_{A,n} \) for any \( n \in \mathbb{N} \);

(vii) The system \((P_{A,n})_{n \in \mathbb{N}} \in \mathcal{B}_k(V)^\mathbb{N}\) converges to 1 in the strong operator topology; and

(viii) The diagram

\[
\begin{array}{ccc}
C_{\text{rig}}(\sigma(A) \cap k(p^{-(n+1)}), k) & \xrightarrow{t_{A,n+1}} & \mathcal{B}_k(V_{n+1}) \\
\downarrow \quad \downarrow & & \downarrow \\
C_{\text{rig}}(\sigma(A) \cap k(p^{-n}), k) & \xrightarrow{t_{A,n}} & \mathcal{B}_k(V_n)
\end{array}
\]

of \( k \)-Banach spaces commutes for any \( n \in \mathbb{N} \), where the second column is the bounded \( k \)-linear homomorphism \( \mathcal{B}_k(V_{n+1}) \rightarrow \mathcal{B}_k(V_n) : B \mapsto P_{A,n}BP_{A,n} \).
Note that the condition (i) indicates that $P_{A_n}$ is an orthonormal projection by the proof of Proposition 5.12. The condition (ii) guarantees $AP_{A_n}$ is everywhere defined, and the condition (iii) adds the information that $AP_{A_n}$ is an everywhere defined closed operator. By the closed mapping theorem, $AP_{A_n}$ is an everywhere defined bounded operator. Moreover, the condition (iii) implies that $V_n$ is $A$-stable, and hence $AP_{A_n}$ induces an everywhere defined bounded operator $A_n$ on $V_n$. The condition (vi) ensures that $V_n \subset V_{n+1}$ and the identity $P_n \in B(V_n)$ is naturally identified with the projection $P_n = P_{n+1}P_n P_{n+1} \in B_k(V_{n+1})$.

**Proof.** If $A$ admits the extended rigid continuous functional calculus $\iota_A$, then the data $P_{A,n} : = \iota_A (1_{\sigma(A) \cap (p^{-1})})$ and $iota_{A,n} : = \iota_A, \sigma(A) \cap (p^{-1}) = A_n$ satisfy the conditions by Lemma 6.31. Suppose $A$ admits the system $(P_{A,n})_{n \in \mathbb{N}}$ of projections satisfying the conditions. By the condition (vii), the union $\mathring{W} := \bigcup V_n \subset V$ is a dense $k$-vector subspace. Take elements $f \in C_{\mathbb{R}}(\sigma(A), k)$ and $w \in \mathring{W}$. Let $n \in \mathbb{N}$ be an integer with $w \in V_n$. Set

$$\iota_A (f) w := P_{A,n} \iota_{A,n} (f) |_{\sigma(A) \cap (p^{-1})}) P_{A,n} w \in P_{A,n} B_k (V_n) P_{A,n} w \subset V \subset V.$$

By the condition (viii), $\iota_A (f) w \in V$ is independent of the choice of $n \in \mathbb{N}$. This determines a densely defined operator $\iota_A (f) w : \mathring{W} \to V : w \mapsto \iota_A (f) w$. We verify that $\iota_A (f) w$ is closable, and we will denote by $\iota_A (f)$ the closure.

To begin with, consider the case that $f = 1_{\sigma(A) \cap (p^{-1})}$ for some $n \in \mathbb{N}$. Since $\iota_A (1_{\sigma(A) \cap (p^{-1})}) = \iota_A (1) = P_{A,n} \in B_k (V_n)$, one has $\iota_A (1_{\sigma(A) \cap (p^{-1})}) |_{\mathring{W}} = P_{A,n} |_{\mathring{W}}$. Therefore $\iota_A (1_{\sigma(A) \cap (p^{-1})}) |_{\mathring{W}}$ is closable and its closure is $P_{A,n}$.

Now consider the general case. Take any $v \in V$ and $(w_n)_{n \in \mathbb{N}}, (w'_n)_{n \in \mathbb{N}} \in \mathring{W}^\mathbb{N}$ such that $\lim_{n \to \infty} w_n = \lim_{n \to \infty} w'_n = v$ and both of $\lim_{n \to \infty} \iota_A (f) w_n$ and $\lim_{n \to \infty} \iota_A (f) w'_n$ exist. It suffices to show that $\lim_{n \to \infty} \iota_A (f) w_n = \lim_{n \to \infty} \iota_A (f) w'_n$. For any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $\| \iota_A (f) w_n - \iota_A (f) w'_n \| < \epsilon$ and $\| \iota_A (f) w_n - \iota_A (f) w'_n \| < \epsilon$ for any $n, m \geq N$. Take an $l \in \mathbb{N}$. By the conditions (i), the system $(P_{A,l}, 1 - P_{A,l})$ forms a partition of unity. By the construction of $\iota_A (f) w$ and the calculation above, one has $\iota_A (1_{\sigma(A) \cap (p^{-1})}) |_{\mathring{W}} \subseteq P_{A,l} \iota_A (f) w \subseteq P_{A,l} B_k (V) P_{A,l}$ and $\iota_A (1_{\sigma(A) \cap (p^{-1})}) |_{\mathring{W}} \subseteq P_{A,l} \iota_A (f) w \subseteq P_{A,l} B_k (V) P_{A,l}$. Moreover since $V_l \subset \mathring{W}$, the composition $\iota_A (f) w P_{A,l}$ is everywhere defined, and coincides with the bounded operator $P_{A,l} \iota_A (1_{\sigma(A) \cap (p^{-1})}) |_{\mathring{W}} \subseteq P_{A,l} B_k (V) P_{A,l} = P_{A,l} B_k (V) P_{A,l} \subseteq P_{A,l} B_k (V)$. By the construction, it follows that $\iota_A (1_{\sigma(A) \cap (p^{-1})}) |_{\mathring{W}}$ is closable and its closure is the everywhere defined bounded operator. Since $(P_{A,l}, 1 - P_{A,l})$ is a partition of unity again, one obtains

$$\| \iota_A (1_{\sigma(A) \cap (p^{-1})}) u \| = \| P_{A,l} \iota_A (f) u \| \leq \| \iota_A (f) u \|$$

for any $u \in \mathring{W}$. Hence one acquires $\| \iota_A (1_{\sigma(A) \cap (p^{-1})}) f w_n - \iota_A (1_{\sigma(A) \cap (p^{-1})}) f w'_n \| < \epsilon$ and $\| \iota_A (1_{\sigma(A) \cap (p^{-1})}) f w_n - \iota_A (1_{\sigma(A) \cap (p^{-1})}) f w'_n \| < \epsilon$ for any $n, m \geq N$. Taking the limit $m \to \infty$, one gains that $\| \iota_A (1_{\sigma(A) \cap (p^{-1})}) f w_n - \iota_A (1_{\sigma(A) \cap (p^{-1})}) f w'_n \| < \epsilon$ and $\| \iota_A (1_{\sigma(A) \cap (p^{-1})}) f w_n - \iota_A (1_{\sigma(A) \cap (p^{-1})}) f w'_n \| < \epsilon$ for any $n \in \mathbb{N}$ because $\iota_A (1_{\sigma(A) \cap (p^{-1})})$ is a closable operator with an everywhere defined bounded closure. It implies that $\| \iota_A (1_{\sigma(A) \cap (p^{-1})}) f w_n - \iota_A (1_{\sigma(A) \cap (p^{-1})}) f w'_n \| < \epsilon$. Taking the limit $l \to \infty$, one derives $\| \iota_A (f) w_n - \iota_A (f) w'_n \| < \epsilon$ from the condition.
(vii). Therefore one concludes $\lim_{n \to \infty} \ell_A(f)w_n = \lim_{n \to \infty} \ell_A(f)w'_n$. Thus $\ell_A(f)|_{\overline{W}}$ is closable, and denote by $\ell_A(f) \in Cl_k(V)$ the closure. We has constructed the desired map $\ell_A : C_{rig}(\sigma(A), k) \to Cl_k(V)$. 

By the proof above, the domain of the closed operator $\ell_A(f)$ is described as

$$V_f := \left\{ v \in V \Big| \lim_{n,m \to \infty} \|\ell_A(1_{\sigma(A)}(\pi^{-1})^n)f)P_{A,n}v - \ell_A(1_{\sigma(A)}(\pi^{-1})^m)P_{A,m}v\| = 0 \right\}.$$ 

This expression resembles that of the Borel functional calculus of an Archimedean self-adjoint operator.

### 6.5 Position and Momentum

We finish the main part of this paper by observing normal operators on the strictly Cartesian $\mathbb{Q}_p$-Banach space $C_0(\mathbb{Q}_p, \mathbb{Q}_p)$ of countable type. Let $x$ and $\pi$ be the operators on $C_0(\mathbb{Q}_p, \mathbb{Q}_p)$ defined as the multiplication of the coordinate function $z : \mathbb{Q}_p \to \mathbb{Q}_p$ and the action of the differential $dz/dz$ respectively. Since $C_0(\mathbb{Q}_p, \mathbb{Q}_p)$ admits an orthonormal Schauder basis consisting of characteristic functions on bounded clopen subsets, they are densely defined. Indeed, a characteristic function of a bounded clopen subset is contained in the domains of $x$ and $\pi$. Moreover, it is obvious that they are closed and the spectra of them coincide with $\mathbb{Q}_p$. The position operator $x$ and the momentum operator $\pi$ are $p$-adic physical quantities, and we observe the extended rigid continuous functional calculus of them. For each $f \in C_{rig}(\mathbb{Q}_p, \mathbb{Q}_p)$, denote by $V_f \subset C_0(\mathbb{Q}_p, \mathbb{Q}_p)$ the dense subspace consisting of functions $g$ such that $fg : \mathbb{Q}_p \to \mathbb{Q}_p$ is contained in $C_0(\mathbb{Q}_p, \mathbb{Q}_p)$. The extended rigid continuous functional calculus of $x$ is given by setting

$$\ell_x(f) : V_f \to C_0(\mathbb{Q}_p, \mathbb{Q}_p)$$

$$g \mapsto fg.$$ 

On the other hand, the description of $\ell_\pi$ is much complicated because $\pi$ corresponds to the Fourier transform of $x$. For each $q \in \mathbb{Q}_p$, denote by $W_f \subset C_0(\mathbb{Q}_p, \mathbb{Q}_p)$ the subspace of continuously differentiable functions $g$ with $g' = qg$. For example, $W_0$ coincides with the dense subspace of locally constant functions. The exponential map multiplied by a characteristic function of a closed ball $D$ centred at 0 of suitable radius belongs to $W_1$. Replacing $D$ by a smaller ball, the $q$-th power of such a function makes sense, and is an element of $W_q$. Conversely, take an invertible function $e_q$ contained in $W_q$. For any $f \in W_q$, one has

$$\frac{d(fe_q^{-1})}{dz} = \frac{df}{dz}e_q^{-1} - fe_q^{-1}\frac{de_q}{dz} = qf e_q^{-1} - qf e_q^{-1} = 0,$$

and hence $fe_q^{-1} \in W_0$. It follows that $W_q$ is a free $W_0$-module of rank 1. Note that by the equality $(d/dz)|_{W_q} = q$, a function in $W_q$ is automatically a $C^\infty$-class function. By an easy calculation with a Vandermonde matrix, the sum $\sum_{q \in \mathbb{Q}_p} W_q \subset C_0(\mathbb{Q}_p, \mathbb{Q}_p)$ is the direct
sum. Set \( W := \bigoplus_{q \in \mathbb{Q}_p} W_q \subset C_0(\mathbb{Q}_p, \mathbb{Q}_p) \). The restriction of the extended rigid continuous functional calculus of \( \pi \) on \( W \) is given by setting
\[
\iota_x(\pi)|_W : W \to C_0(\mathbb{Q}_p, \mathbb{Q}_p)
\]
\[
\sum_{q \in \mathbb{Q}_p} g_q \mapsto \sum_{q \in \mathbb{Q}_p} qg_q.
\]
We expect that the functional calculi of \( x, \pi, (1 + x)\pi \), and others are deeply related to other fields in Number Theory such as the Fourier transform, Iwasawa theory, and dilogarithms. The injective map
\[
W \hookrightarrow \prod_{q \in \mathbb{Q}_p} W_q = \prod_{q \in \mathbb{Q}_p} W_0e_q \cong_k \text{Map}(\mathbb{Q}_p, W_0)
\]
\[
g = \sum_{q \in \mathbb{Q}_p} g_q \mapsto \left( \int_{\mathbb{Q}_p} g_qe_q^{-1}dq : q' \mapsto g_q(q')e_q(q')^{-1} \right)
\]
is an analogue of the Fourier transform. The normed \( k \)-algebra \( W_0 \) corresponds to the Archimedean base field \( \mathbb{C} \), and the non-canonical basis \((e_q(z))_{q \in \mathbb{Q}_p}\) corresponds to the canonical basis \((\exp(2\pi i\zeta z))_{\zeta \in \mathbb{C}}\).

7 Appendix

Here we put examples and propositions referred in §[3]. We follow the conventions in §[3]

7.1 Degeneracy of the Euclidean Inner Product

Unlike \( \mathbb{R} \), a valuation field has an example of a symmetric inner product which admits the trivial matrix representation and which is not degenerate.

Example 7.1. For an \( n \in \mathbb{N} \) with \( n > 1 \), we endow the \( \mathbb{Q}_5 \)-vector space \( \mathbb{Q}_5^n \) with the canonical symmetric \( \mathbb{Q}_5 \)-bilinear inner product \( \langle \cdot | \cdot \rangle : \mathbb{Q}_5^n \otimes \mathbb{Q}_5^n \to \mathbb{Q}_5 : \phi \otimes \psi \mapsto \langle \phi | \psi \rangle \) with respect to the canonical basis, and consider the map \( \| \cdot \| : \mathbb{Q}_5^n \to [0, \infty) \) given by setting
\[
\|f\| := \sqrt{\langle f | f \rangle} = \sqrt{|f_1^2 + \cdots + f_n^2|}
\]
for each \( f = (f_1, \ldots, f_n) \in \mathbb{Q}_5^n \). Then \((\mathbb{Q}_5^n, \| \cdot \| )\) is not a normed \( \mathbb{Q}_5 \)-vector space. Indeed, the map \( \| \cdot \| \) does not satisfy the triangle inequality and there is a non-zero element \( f \in \mathbb{Q}_5^n \) with \( \|f\| \). For example, consider the element \( i = \cdots 431212 \in \mathbb{Q}_5 \) with \( i^2 = \cdots 4444 = -1 \) and the vectors
\[
f := \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ g := \begin{pmatrix} 0 \\ i \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ f \pm g = \begin{pmatrix} 1 + i \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{Q}_5^n.
\]
Then they satisfy
\[ \| f \pm g \| = \sqrt{|1 + (-1) + 0 + \cdots + 0|} = 0 \]
and
\[ \|(f + g) + (f - g)\| = \|2f\| = |2| = 1 > 0 = \|f + g\| + \|f - g\|. \]

7.2 Example of a Non-degenerate Symmetric Inner Product
Unlike \( \mathbb{C} \), a discrete valuation field always has an example of a non-degenerate symmetric inner product on a low dimensional vector space. Like \( \mathbb{R} \), such an inner product gives a sufficient condition for the diagonalisability.

**Proposition 7.2.** If \( k \) is a discrete valuation field with a uniformiser \( \pi \in k \), the symmetric inner product \( \langle \cdot | \cdot \rangle : k^2 \otimes_k k^2 \to k \) given by setting
\[
\begin{align*}
\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} | \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle &= 1, \\
\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} | \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle &= 0, \\
\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} | \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rangle &= \pi
\end{align*}
\]
determines the norm
\[ \| \cdot \| : k^2 \to [0, \infty), \quad f \mapsto \|f\| := \sqrt{\langle f | f \rangle}, \]
which is equivalent to the norm of the \( k \)-Banach space \( k^2 \) associated with the canonical basis.

**Proof.** For an element \( (a, b) \in k^2 \), one has
\[ \|(a, b)\| = \sqrt{|a^2 + \pi b^2|} = \max\{|a|, |\pi|^{1/2}|b|\} \]
and hence \((k^2, \|\cdot\|)\) is a non-strict Cartesian \( k \)-Banach space admitting an orthogonal basis \((1, 0), (0, 1) \in k^2\). \( \square \)

**Proposition 7.3.** If \( p \equiv 1 \pmod{4} \), or if \( p \equiv 3 \pmod{4} \) and \( \overline{k} \neq \mathbb{F}_p \), then there is a symmetric matrix \( M \in M_2(k) \) such that \( M \) is not diagonalisable in an algebraic closure of \( k \). On the other hand if \( p \equiv 3 \pmod{4} \) and \( \overline{k} = \mathbb{F}_p \), a symmetric matrix \( M \in M_2(k) \) is diagonalisable by a unitary matrix in an unramified extension of \( k \) of dimension at most 2.
Proof. To begin with, suppose \( p \equiv 1 \mod 4 \), or suppose \( p \equiv 3 \mod 4 \) and \( \overline{k} \neq \mathbb{F}_p \).
Then there is a square root \( i \) of \(-1\) in \( k \). Set \( M := \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \in M_2(k) \).

Then one has \( M^2 = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \).

but \( M \neq 0 \). Therefore the minimal polynomial of \( M \) is not reduced and \( M \) is not diagonalisable in an algebraic closure of \( k \).

On the other hand, suppose \( p \equiv 3 \mod 4 \) and \( \overline{k} = \mathbb{F}_p \). Take a symmetric matrix \( M \in M_2(k) \), and set
\[
M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}
\]
for \( a, b, c \in k \). If \( a = c \) and \( b = 0 \), then \( M \) is a scalar matrix and hence is trivially diagonalisable by a unitary matrix. Suppose either \( a \neq c \) or \( b \neq 0 \). The characteristic polynomial of \( M \) is \( P = T^2 - (a + c)T + ac - b^2 \in k[T] \), and its roots are
\[
T = \frac{a + c \pm \sqrt{(a + c)^2 - 4(ac - b^2)}}{2} = \frac{a + c \pm \sqrt{(a - c)^2 + (2b)^2}}{2} \in K := k\left(\sqrt{(a - c)^2 + (2b)^2}\right).
\]

Assume \((a - c)^2 + (2b)^2 = 0\). Since either \( a \neq c \) or \( b \neq 0 \), one has both \( a \neq c \) and \( b \neq 0 \). It follows
\[
\left(\frac{2b}{a - c}\right)^2 = -1,
\]
but it contradicts the fact \( \#(\overline{k}^\times) = \#k - 1 = \#\mathbb{F}_p - 1 = p - 1 \equiv 2 \mod 4 \). Therefore \((a - c)^2 + (2b)^2 \neq 0\), and \( P \) has two distinct roots. Thus \( M \) is diagonalisable in \( K \). The eigenvectors are
\[
e_1 = \begin{pmatrix} -2b \\ (a - c) + \sqrt{(a - c)^2 + (2b)^2} \end{pmatrix}, 
\]
\[
e_2 = \begin{pmatrix} -2b \\ (a - c) - \sqrt{(a - c)^2 + (2b)^2} \end{pmatrix}.
\]

If \( 0 < |a - c| \leq |b| \), then \( \|(2b)^{-1}e_1\| = \|(2b)^{-1}e_2\| = 1 \). The reductions \((2b)^{-1}e_1, (2b)^{-1}e_2 \in \overline{K}^2\) are obviously linearly independent because
\[
(2b)^{-1}e_1 - (2b)^{-1}e_2 = \begin{pmatrix} a-c \\ 2b + \sqrt{(a-c)^2 + (2b)^2} \end{pmatrix} - \begin{pmatrix} a-c \\ 2b - \sqrt{(a-c)^2 + (2b)^2} \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{(a-c)^2 + (2b)^2} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

Therefore \( b^{-1}e_1, b^{-1}e_2 \) are orthonormal by [BGR] 2.5.1/3. If \( |a - c| > |b| > 0 \), then rearranging the signature of \( \sqrt{(a - c)^2 + (2b)^2} \) so that \( |b| = \sqrt{(a-c)^2 + (2b)^2} - (a-c) < \)}
\[ \sqrt{(a-c)^2 + (2b)^2} + (a-c) = |a-c|, \text{ one has } |(a-c)^{-1} e_1| = |b^{-1} e_2| = 1. \] Indeed it is easy to show that

\[
(a-c)^{-1} e_1 = \begin{pmatrix}
0 \\
1 + \frac{\sqrt{(a-c)^2 + (2b)^2}}{a-c}
\end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in k^2,
\]

\[
(2b)^{-1} e_2 = \begin{pmatrix}
-1 \\
-\frac{2b}{(a-c) + \sqrt{(a-c)^2 + (2b)^2}}
\end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \in k^2.
\]

Obviously the reductions \((a-c)^{-1} e_1, (2b)^{-1} e_2\) are linearly independent, and hence \((a-c)^{-1} e_1, (2b)^{-1} e_2\) are orthonormal. We conclude that \(M\) has an orthonormal basis consists of eigenvectors, and hence is diagonalisable by a unitary matrix by Proposition 3.2. \(\square\)

### 7.3 Examples of Non-canonical Involutions

We observe examples of non-canonical involutions on the matrix algebra. Like the canonical involution on a Hilbert space over \(\mathbb{C}\), several non-canonical involutions give sufficient conditions for the diagonalisability.

**Definition 7.4.** Suppose \(k\) is a discrete valuation field with mixed characteristic \((0, p)\), and fix a uniformiser \(\pi \in k\). For a matrix \(M \in M_2(k)\), set

\[ M^* := \begin{pmatrix} 1 & 0 \\ 0 & \pi^{-1} \end{pmatrix} M \begin{pmatrix} 1 & 0 \\ 0 & \pi^{-1} \end{pmatrix} \in M_2(k). \]

Obviously one has \(M^{**} = M\), and hence it determines a \(k\)-linear involution \(*: M_2(k) \to M_2(k)\). Since the characteristic of \(k\) is not 2, the \(k\)-linear representation \(*\) is diagonalisable. A matrix \(M \in M_2(k)\) is said to be \(*\)-symmetric if it is \(M^* = M\), and to be \(*\)-antisymmetric if \(M^* = -M\).

**Proposition 7.5.** In the situation in Definition 7.4, the non-Archimedean involution \(*\) has the adjoint property with respect to the symmetric inner product \(\langle \cdot | \cdot \rangle\) defined in Proposition 7.2. Suppose \(p \neq 2\). Consider a matrix

\[ M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \in M_2(k) \]

which is \(*\)-symmetric or \(*\)-antisymmetric. The matrix \(M\) is diagonalisable in an algebraic closure of \(k\) if and only if \(|M_1 - M_4| \geq |M_2|\). In this case, \(M\) is diagonalisable by a unitary matrix in a totally ramified extension of \(k\) of dimension at most 2.

**Proof.** The adjoint property is obvious. Take a \(*\)-symmetric matrix \(M \in M_2(k)\). The equality \(M = M^*\) implies that \(M\) is of the form

\[ M = \begin{pmatrix} a & b \\ \pi b & c \end{pmatrix} \]
for \(a, b, c \in k\). If \(b = 0\), then \(|M_1 - M_4| = |a - c| \geq 0 = |M_2|\) and trivially \(M\) is diagonalisable by a unitary matrix. Hence we may and do assume \(b \neq 0\). The discriminant of its characteristic polynomial is \((a - c)^2 + \pi(2b)^2\), and is never 0 by the calculation in the proof of Proposition 7.2. The same holds for \(M - \lambda\) for each \(\lambda \in k\), and hence \(M\) is diagonalisable in the totally ramified extension \(k(\sqrt{(a - c)^2 + \pi(2b)^2})/k\). Here we rearrange the signature of the square root \(\sqrt{(a - c)^2 + \pi(2b)^2}\) in the following way: If \(|a - c| \geq |2b| > 0\), set

\[
\sqrt{(a - c)^2 + \pi(2b)^2} := (a - c) \left(1 + \pi \left(\frac{2b}{a - c}\right)^2\right)^{1/2} \in k = k(\sqrt{(a - c)^2 + \pi(2b)^2}),
\]

and if \(|a - c| \leq |2b\pi| < |2b|\), set

\[
\sqrt{(a - c)^2 + \pi(2b)^2} := 2b\sqrt{\pi} \left(1 + \left(\frac{a - c}{2b\pi}\right)^2\right)^{1/2} \in k = k(\sqrt{(a - c)^2 + \pi(2b)^2})
\]

for a fixed square root \(\sqrt{\pi}\) of \(\pi\). Note that the convergent radius of the analytic function \((1 + z)^{1/2}\) is 1 because \(p \neq 2\). The eigenvalues of \(M\) are

\[
\lambda_{\pm} := \frac{a + c}{2} \pm \frac{\sqrt{(a - c)^2 + \pi(2b)^2}}{2},
\]

and the corresponding eigenvectors are

\[
f_+ := \begin{pmatrix} -2b \\
(a - c) - \sqrt{(a - c)^2 + \pi(2b)^2} \end{pmatrix}, 
\]

\[
f_- := \begin{pmatrix} -2b \\
(a - c) + \sqrt{(a - c)^2 + \pi(2b)^2} \end{pmatrix} \in k^2.
\]

We repeat the totally same process in the proof of Proposition 7.3. To begin with, suppose \(|a - c| \geq |2b| > 0\). Then \(|M_1 - M_4| \geq |M_2|\). Take the integer \(n \in \mathbb{N}\) with \(|(a - c)^n| = |2b|\) and set \(u := (a - c)^{-1}\pi^{-n/2} \in k^\times\). One has

\[
f_+ = \begin{pmatrix} -2b \\
(a - c) - \sqrt{(a - c)^2 + \pi(2b)^2} \end{pmatrix} = \begin{pmatrix} -2b \\
\frac{(a-c)^2-(a-c)^2+\pi(2b)^2}{(a-c)+\sqrt{(a-c)^2+\pi(2b)^2}} \end{pmatrix} = 2b \begin{pmatrix} -1 \\
\frac{u^n}{1+\sqrt{1+u^2\pi^{2n+1}}} \end{pmatrix},
\]

\[
f_- = \begin{pmatrix} -2b \\
(a - c) + \sqrt{(a - c)^2 + \pi(2b)^2} \end{pmatrix} = (a - c) \begin{pmatrix} -un^n \\
1 + \sqrt{1+u^2\pi^{2n+1}} \end{pmatrix}.
\]

Therefore \(||(2b)^{-1}f_+|| = ||(a - c)^{-1}f_-|| = 1\) and

\[
(2b)^{-1}f_+ = \begin{pmatrix} -1 \\
\frac{u^n}{1+\sqrt{1+u^2\pi^{2n+1}}} \end{pmatrix} = \begin{pmatrix} -1 \\
0 \end{pmatrix} \neq \begin{pmatrix} 0 \\
0 \end{pmatrix} \in k^2,
\]

\[
(a - c)^{-1}f_- = \begin{pmatrix} -un^n \\
1 + \sqrt{1+u^2\pi^{2n+1}} \end{pmatrix} = \begin{pmatrix} 0 \\
2 \end{pmatrix} \neq \begin{pmatrix} 0 \\
0 \end{pmatrix} \in k^2.
\]
It follows that \((2b)^{-1}f_+\) and \((a-c)^{-1}f_+\) are linearly independent, and hence \((2b)^{-1}f_+\) and \((a-c)^{-1}f_+\) are orthonormal by [BGR] 2.5.1/3. Since \(M\) admits an orthonormal basis consisting of eigenvectors, \(M\) is diagonalisable by a unitary matrix in a suitable field.

On the other hand, suppose \(|a-c| \leq |2b\pi| < |2b|\), and set \(v := (2b\pi)^{-1}(a-c)\) Then \(|M_1-M_4| < |M_2|\) and one has

\[
\begin{align*}
f_+ &= \left(\frac{-2b}{(a-c)-\sqrt{(a-c)^2+\pi(2b)^2}}\right) = 2b\left(\frac{-1}{\nu\pi-(1+\nu^2\pi\sqrt{\pi}}\right), \\
f_- &= \left(\frac{-2b}{(a-c)+\sqrt{(a-c)^2+(2b)^2}\pi}\right) = 2b\left(\frac{-1}{\nu\pi+(1+\nu^2\pi\sqrt{\pi}}\right).
\end{align*}
\]

Therefore \(\|(2b)^{-1}f_+\| = 1\) and

\[
\begin{align*}
(2b)^{-1}f_+ &= \left(\frac{-1}{\nu\pi-(1+\nu^2\pi\sqrt{\pi}}\right) = \left(\frac{-1}{0}\right) \in K^2, \\
(2b)^{-1}f_- &= \left(\frac{-1}{\nu\pi+(1+\nu^2\pi\sqrt{\pi}}\right) = \left(\frac{-1}{0}\right) = (2b)^{-1}f_+ \in K^2.
\end{align*}
\]

It follows

\[
\|f_+ - f_-\| = |2b| \|(2b)^{-1}f_+ - (2b)^{-1}f_-\| < |2b| = \max\{\|f_+\|, \|f_-\|\}
\]

and hence \(f_+\) and \(f_-\) are not orthogonal. Since \(M\) does not admit an orthogonal basis consisting of eigenvectors, \(M\) is not diagonalisable by a unitary matrix. \(\Box\)

The involution above is non-canonical because we used the adjoint property of a non-canonical symmetric inner product \(\langle \cdot | \cdot \rangle\). The same construction of a non-canonical involution works for other non-canonical inner products. Here is an example.

**Definition 7.6.** Suppose \(p \equiv 3 \pmod{4}\) and \(\overline{k} = \mathbb{F}_p\). Let \(K/k\) be the unramified Galois extension \(k[X]/(X^2+1)\) and \(i \in K\) the image of \(X \in k[X]\) the canonical projection \(k[X] \rightarrow K\). Denote by \(g: K \rightarrow K\) the unique non-trivial element. For a matrix \(M \in M_2(K)\), set

\[
M^* \coloneqq i M^t \in M_2(K),
\]

where the action of \(g\) on \(M_2(K)\) is given through the action on the entries. Obviously one has \(M^{**} = M\), and hence it determines a \(k\)-linear involution \(*: M_2(K) \rightarrow M_2(K)\). Since the characteristic of \(k\) is not 2, the \(k\)-linear representation \(*\) is diagonalisable over \(k\). A matrix \(M \in M_2(K)\) is said to be \(*\)-symmetric if \(M^* = M\), and to be \(*\)-antisymmetric if \(M^* = -M\).

**Proposition 7.7.** In the situation above, a matrix \(M \in M_2(K)\) which is \(*\)-symmetric or \(*\)-antisymmetric is diagonalisable by a unitary matrix in an unramified extension of \(K\) of dimension at most 2.

78
Proof. Take a matrix $M \in M_2(K)$ which is $\ast$-symmetric or $\ast$-antisymmetric. If $M$ is $\ast$-antisymmetric, then $iM$ is $\ast$-symmetric, and hence we may and do assume that $M$ is $\ast$-symmetric. The equality $M^* = M$ implies that $M$ is a matrix of the form

$$M = \begin{pmatrix} a & b \\ b^g & c \end{pmatrix}$$

for $a, c \in k, b \in K$. If $b = 0$, then trivially $M$ is diagonalisable by the identity matrix, and hence we may and do assume $b \neq 0$. The discriminant of the characteristic polynomial of $M$ is $(a - c)^2 + 4N_{K/k}(b)$, where $N_{K/k}$ is the norm map $K \to k$: $f \mapsto f^g f \in k$. One has the commutative diagram

$$
\begin{array}{c}
0 \longrightarrow \overline{K} \times (1 + K^{\infty}) \longrightarrow K^\times \longrightarrow |K^\times| \longrightarrow 0 \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
0 \longrightarrow \overline{k} \times (1 + k^{\infty}) \longrightarrow k^\times \longrightarrow |k^\times| \longrightarrow 0
\end{array}
$$

whose columns are exact sequences given by the multiplicative valuation $|\cdot| : K^\times \to \mathbb{R}^\times$ and the Teichmüller embedding $\overline{K}^\times \to K^\times$ and whose second rows are given by the norm maps. Note that even if the valuation of $k$ is not discrete and even if one does not fixes a lift of the Frobenius, the Teichmüller embedding works in a unique way because the residue fields are finite. They consists of 0 and roots of unity. The elementary Number Theory and the assumption $\overline{k} = \mathbb{F}_p$ imply the image of the norm map $\overline{K}^\times \to \overline{k}^\times$ is the subgroup $(\mathbb{F}_p^\times)^2 \subset \mathbb{F}_p^\times$, and the restriction $1 + K^{\infty} \to 1 + k^{\infty}$ of the norm map is surjective because the convergent radius of the square root function $(1 + z)^{1/2}$ is 1. Therefore the image of the first row is $(\overline{k}^\times)^2 \times 1 + k^{\infty}$. Beware that in this proof, for a multiplicative Abelian group $G$, we denote by $G^2$ the subgroup of $G$ consisting of square elements but not the direct product $G \times G$. The image of the third row is $|K^\times|^2 = |k^\times|^2 \subset |k^\times|$ because the extension $K/k$ is unramified, and hence the image $N_{K/k}(K^\times)$ of the second row satisfies that the sequence

$$
0 \longrightarrow (\overline{k}^\times)^2 \times (1 + k^{\infty}) \longrightarrow N_{K/k}(K^\times) \longrightarrow |k^\times|^2 \longrightarrow 0
$$

is exact by the snake lemma. Considering the commutative diagram

$$
\begin{array}{c}
0 \longrightarrow (\overline{k}^\times \times (1 + k^{\infty}))^2 \longrightarrow (k^\times)^2 \longrightarrow |k^\times|^2 \longrightarrow 0 \\
\| \quad \quad \downarrow \quad \quad \| \\
0 \longrightarrow (\overline{k}^\times)^2 \times (1 + k^{\infty}) \longrightarrow N_{K/k}(K^\times) \longrightarrow |k^\times|^2 \longrightarrow 0,
\end{array}
$$

one has $N_{K/k}(K^\times) = (k^\times)^2$. In particular, the discriminant $(a - c)^2 + 4N_{K/k}(b)$ is contained in $(k^\times)^2 + (k^\times)^2 := \{ f_1^2 + f_2^2 \mid f_1, f_2 \in k \} \subset k$, and the subset does not contains $0 \in k$ by the proof of Proposition 7.3. It follows that the discriminant is never 0. The same
holds for $M - \lambda$ for any $\lambda \in k$, and hence $M$ is diagonalisable in the unramified extension $K(\sqrt{(a - c)^2 + 4N_{K/k}(b)})/K$. The eigenvalues of $M$ are

$$
\lambda_k := \frac{a + c}{2} \pm \frac{\sqrt{(a - c)^2 + 4N_{K/k}(b)}}{2}
$$

and the corresponding eigenvectors are

$$
f_+ := (a - c) - \sqrt{(a - c)^2 + 4N_{K/k}(b)} \quad , \quad f_- := (a - c) + \sqrt{(a - c)^2 + 4N_{K/k}(b)} \quad \in k^2.
$$

We repeat the totally same process in the proofs of Proposition 7.3 and Proposition 7.5. First, suppose $|a - c| > |2b| > 0$. Let $n \in \mathbb{N}$ be the integer with $|(a - c)\pi^{n+1}| = |2b|$ and set $u := ((a - c)^2 - \pi^{-2n-2}4N_{K/k}(b)) \in k^x$ and $v := (a - c)^{-1}\pi^{-n-1}2b \in k^x$. Here we rearrange the signature of the square root $\sqrt{(a - c)^2 + 4N_{K/k}(b)}$ in the following way:

$$\sqrt{(a - c)^2 + 4N_{K/k}(b)} := (a - c)\left(1 + uu\pi^{2n+2}\right)^{1/2} \quad \in k = k\left(\sqrt{(a - c)^2 + \pi(2b)^2}\right).
$$

One has

$$
f_+ = \begin{pmatrix}
-2b
\frac{2b}{(a - c) + \sqrt{(a - c)^2 + 4N_{K/k}(b)}}
\end{pmatrix} = \begin{pmatrix}
-2b
\frac{2b}{(a - c)(1 + \sqrt{1 + uu\pi^{2n+2}})}
\end{pmatrix},
$$

$$
f_- = \begin{pmatrix}
-2b
\frac{2b}{(a - c) - \sqrt{(a - c)^2 + 4N_{K/k}(b)}}
\end{pmatrix} = \begin{pmatrix}
-2b
\frac{2b}{(a - c)(1 - \sqrt{1 + uu\pi^{2n+2}})}
\end{pmatrix}.
$$

Therefore $\|((a - c)\pi^{n+1})^{-1}f_+\| = \|((a - c)^{-1})f_-\| = 1$ and

$$
\|((a - c)\pi^{n+1})^{-1}f_+\| = \begin{pmatrix}
\frac{-v}{1 + uu\pi^{2n+2}}
\frac{-v\pi^{n+1}}{1 + uu\pi^{2n+2}}
\end{pmatrix} = \begin{pmatrix}
-\frac{\pi}{0}
-\frac{0}{0}
\end{pmatrix} \neq \begin{pmatrix}
0
0
\end{pmatrix} \in k^2,
$$

$$
\|(a - c)^{-1}f_-\| = \begin{pmatrix}
\frac{-v\pi^{n+1}}{1 + uu\pi^{2n+2}}
\frac{-v\pi^{n+1}}{1 + uu\pi^{2n+2}}
\end{pmatrix} = \begin{pmatrix}
0
0
\end{pmatrix} \neq \begin{pmatrix}
0
0
\end{pmatrix} \in k^2.
$$

It follows that $((a - c)\pi^{n+1})^{-1}f_+$ and $(a - c)^{-1}f_-$ are linearly independent and hence $(a - c)\pi^{n+1})^{-1}f_+$ and $(a - c)^{-1}f_-$ are orthonormal by [BGR] 2.5.1/3. Since $M$ admits an orthonormal basis consisting of eigenvectors, $M$ is diagonalisable by a unitary matrix in a suitable field.

Secondly, suppose $0 < |a - c| < |2b|$. Let $n \in \mathbb{N}$ be the integer with $|a - c| = |2b\pi^{n+1}|$ and set $u := b^{-2}N_{K/k}(b) \in k^x$ and $v := (2b\pi^{n+1})^{-1}((a - c) \in k^x$. One has

$$
f_+ = \begin{pmatrix}
-2b
\frac{2b}{(a - c) - \sqrt{(a - c)^2 + 4N_{K/k}(b)}}
\end{pmatrix} = 2b\begin{pmatrix}
-1
\frac{-1}{v\pi^{n+1} - uu\pi^{n+1}}
\end{pmatrix},
$$

80
\[ f_+ = \begin{pmatrix} -2b \\ a - c + \sqrt{(a - c)^2 + 4N_{K/k}(b)} \end{pmatrix} = 2b \begin{pmatrix} -1 \\ v_{\pi^{n+1}} + u + v_{\pi^{n+1}} \end{pmatrix}. \]

Therefore \( \|(2b)^{-1}f_+\| = \|(2b)^{-1}f_-\| = 1 \) and

\[
(2b)^{-1}f_+ = \begin{pmatrix} -1/v_{\pi^{n+1}} - \sqrt{u + v_{\pi^{n+1}}} \\ \sqrt{u + v_{\pi^{n+1}}^{-1}} \end{pmatrix} = \begin{pmatrix} -1 \\ -\sqrt{u} \end{pmatrix} \neq \begin{pmatrix} -1 \\ 0 \end{pmatrix} \in \mathbb{k}^2,
\]

\[
(2b)^{-1}f_- = \begin{pmatrix} -1/v_{\pi^{n+1}} + \sqrt{u + v_{\pi^{n+1}}} \\ \sqrt{u + v_{\pi^{n+1}}^{-1}} \end{pmatrix} = \begin{pmatrix} -1 \\ \sqrt{u} \end{pmatrix} \neq \begin{pmatrix} -1 \\ 0 \end{pmatrix} \in \mathbb{k}^2.
\]

It follows that \((2b)^{-1}f_+\) and \((2b)^{-1}f_-\) are linearly independent and hence \((2b)^{-1}f_+\) and \((2b)^{-1}f_-\) are orthonormal by [BGR] 2.5.1/3. Since \(M\) admits an orthonormal basis consisting of eigenvectors, \(M\) is diagonalisable by a unitary matrix in a suitable field.

Finally, suppose \(|a - c| = |2b| > 0\). Set \(u := (a - c)^{-1}2b\) and \(v := (a - c)^{-1}\sqrt{(a - c)^2 + 4N_{K/k}(b)}\). One has

\[
f_+ = \begin{pmatrix} -2b \\ a - c - \sqrt{(a - c)^2 + 4N_{K/k}(b)} \end{pmatrix} = (a - c) \begin{pmatrix} -u \\ 1 - v \end{pmatrix},
\]

\[
f_- = \begin{pmatrix} -2b \\ a - c + \sqrt{(a - c)^2 + 4N_{K/k}(b)} \end{pmatrix} = (a - c) \begin{pmatrix} -u \\ 1 + v \end{pmatrix}.
\]

Therefore \(\|(a - c)^{-1}f_+\| = \|(a - c)^{-1}f_-\| = 1\) and

\[
(a - c)^{-1}f_+ = \begin{pmatrix} -u \\ 1 - v \end{pmatrix} = \begin{pmatrix} -\tilde{u} \\ 1 - \tilde{v} \end{pmatrix} \in \mathbb{k}^2,
\]

\[
(a - c)^{-1}f_- = \begin{pmatrix} -u \\ 1 + v \end{pmatrix} = \begin{pmatrix} -\tilde{u} \\ 1 + \tilde{v} \end{pmatrix} \in \mathbb{k}^2
\]

with \(\tilde{u} \neq 0\) and \(\tilde{v} \neq 0\). It follows that \((a - c)^{-1}f_+\) and \((a - c)^{-1}f_-\) are linearly independent and hence \((a - c)^{-1}f_+\) and \((a - c)^{-1}f_-\) are orthonormal by [BGR] 2.5.1/3. Since \(M\) admits an orthonormal basis consisting of eigenvectors, \(M\) is diagonalisable by a unitary matrix in a suitable field. \(\square\)

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83