SUPERTROPICAL MONOIDS:
BASICS, CANONICAL FACTORIZATION,
AND LIFTING GHOSTS TO TANGIBLES

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Abstract. Supertropical monoids are a structure slightly more general than the supertropical semirings, which have been introduced and used by the first and the third authors for refinements of tropical geometry and matrix theory in [IR1]–[IR5], and then studied by us in a systematic way in [IKR1]–[IKR3] in connection with “supervaluations”.

In the present paper we establish a category $\text{STROP}_m$ of supertropical monoids by choosing as morphisms the “transmissions”, defined in the same way as done in [IKR1] for supertropical semirings. The previously investigated category $\text{STROP}$ of supertropical semirings is a full subcategory of $\text{STROP}_m$. Moreover, there is associated to every supertropical monoid $V$ a supertropical semiring $\overline{V}$ in a canonical way.

A central problem in [IKR1]–[IKR3] has been to find for a supertropical semiring $U$ the quotient $U/E$ by a “$\text{TE}$-relation”, which is a certain kind of equivalence relation on the set $U$ compatible with multiplication (cf. [IKR2] Definition 4.5). It turns out that this quotient always exists in $\text{STROP}_m$. In the good case, that $U/E$ is a supertropical semiring, this is also the right quotient in $\text{STROP}$. Otherwise, analyzing $U/E\uparrow\downarrow E$, we obtain a mild modification of $E$ to a $\text{TE}$-relation $E'$ such that $U/E' = (U/E)^\uparrow\downarrow E'$ in $\text{STROP}$.

In this way we now can solve various problems left open in [IKR1], [IKR2] and gain further insight into the structure of transmissions and supervaluations. Via supertropical monoids we also obtain new results on totally ordered supervaluations and monotone transmissions studied in [IKR3].

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Introduction

To a large extent the algebra underpinning present day tropical geometry is based on the notion of a (commutative) bipotent semiring. Such a semiring $M$ is a totally ordered monoid under multiplication with smallest element 0, the addition being given by $x + y = \max(x, y)$, cf. [IKR1, §1] for details. In logarithmic notation, which most often is used in tropical geometry, bipotent semirings appear as totally ordered additive monoids with absorbing element $-\infty$. The primordial object here is the bipotent semifield $T = \mathbb{R} \cup \{-\infty\}$, cf. e.g. [IMS, §1.5].

In [I] the first author introduced a cover of $T = \mathbb{R} \cup \{-\infty\}$, graded by the multiplicative monoid $\mathbb{Z}_2 = \{0, 1\}$, which was dubbed the extended tropical arithmetic. Then, in [IR1] and [IR2], this structure has been amplified to the notion of a supertropical semiring. A supertropical semiring $U$ is equipped with a “ghost map” $\nu : U \to U$, which respects addition and multiplication and is idempotent, i.e., $\nu \circ \nu = \nu$. Moreover, in this semiring $a + a = \nu(a)$ for every $a \in U$ (cf. [IKR3, §3]). This rule replaces the rule $a + a = a$ taking place in the usual max-plus (or min-plus) arithmetic. We call $\nu(a)$ the “ghost” of $a$ and we term the elements of $U$, which are not ghosts, “tangible”. (The element 0 is regarded both as tangible and ghost.) $U$ then carries a multiplicative idempotent $e = e^2$ such that $\nu(a) = ea$ for every $a \in U$. The image $eU$ of the ghost map, called the ghost ideal of $U$, is itself a bipotent semiring.

Supertropical semirings allow a refinement of valuation theory to a theory of “supervaluations”, the basics of which can be found in [IKR1]-[IKR3]. Supervaluations seem to be able to provide an enriched version of tropical geometry, cf. [IKR1] §9, §11 and [IR1]. We recall the initial definitions.

An m-valuation (= monoid valuation) on a semiring $R$ is a multiplicative map $v : R \to M$ to a bipotent semiring $M$ with $v(0) = 0, v(1) = 1$, and

$$v(x + y) \leq v(x) + v(y) \quad \left[= \max(v(x), v(y))\right],$$

cf. [IKR1, §2]. We call $v$ a valuation if in addition the semiring $M$ is cancellative, by which we mean that $M \setminus \{0\}$ is closed under multiplication and is a cancellative monoid in the usual sense. If $R$ happens to be a (commutative) ring, these valuations coincide with the valuations of rings defined by Bourbaki [B] (except that we switched from additive notation there to multiplicative notation here).

Given an m-valuation $v : R \to M$ there exist multiplicative mappings $\varphi : R \to U$ into various supertropical semirings $U$ with $\varphi(0) = 0, \varphi(1) = 1$, such that $M$ is the ghost ideal of $U$ and $\nu_U \circ \varphi = v$. These are the supervaluations covering $v$, cf. [IKR1, §4].
The supervaluations lead us to the “right” class of maps between supertropical semirings \( U, V \) which we have to admit as morphisms to obtain the category STROP of supertropical semirings (formally introduced in [IKR2, §1]). These are the “transmissions”. A transmission \( \alpha : U \to V \) is a multiplicative map with \( \alpha(0) = 0, \alpha(1) = 1, \alpha(e_U) = e_V \), whose restriction \( \gamma : eU \to eV \) to the ghost ideals is a semiring homomorphism. It turned out in [IKR1, §5] that the transmissions from \( U \) to \( V \) are those maps \( \alpha : U \to V \) such that for every supervaluation \( \varphi : R \to U \) the map \( \alpha \circ \varphi : R \to V \) is again a supervaluation.

Every semiring homomorphism \( \alpha : U \to V \) is a transmission, but there also exist transmissions which do not respect addition. This causes a major difficulty for working in STROP. A large part of the papers [IKR1], [IKR2] has been devoted to constructing various equivalence relations \( E \) on a supertropical semiring \( U \) such that the map \( \pi_E : U \to U/E \), which sends each \( x \in U \) to its \( E \)-equivalence class \([x]_E\), induces the structure of a supertropical semiring on the set \( U/E \) which makes \( \pi_E \) a transmission. We call such an equivalence relation \( E \) transmissive.

It seems difficult to characterize transmissive equivalence relations in an axiomatic way. In [IKR2] Proposition 4.4 three axioms TE1–TE3 have been provided, which obviously have to hold, but then, adding a fourth axiom, we only characterized those transmissive equivalence relations on \( U \), where the ghost ideal of \( U/E \) is cancellative [IKR2, Theorem 4.5]. Dubbing the equivalence relations obeying axioms TE1–TE3 “TE-relations”, we had to leave the following problem open in general:

\( (*) \) When is a TE-relation \( E \) on a supertropical semiring \( U \) transmissive?

The problem seems to be relevant since there exist natural classes of \( m \)-valuations \( v : R \to M \), where the bipotent semiring \( M \) has no reason to be cancellative, cf. [IKR3, §1]. For \( R \) a ring such \( m \)-valuations already appeared in the work of Harrison–Vitulli [HV] and D. Zhang [Z].

The present paper gives a solution to the problem \((*)\) just described. We introduce a new category \( \text{STROP}_m \) containing the category \( \text{STROP} \) of supertropical semirings as a full subcategory. The objects of \( \text{STROP}_m \), called “supertropical monoids”, are multiplicative monoids \( U \) with an absorbing element \( 0 \), an idempotent \( e \in U \), and a total ordering on the monoid \( eU \), which makes \( eU \) a bipotent semiring.

In a supertropical monoid it is natural to speak about tangibles and ghosts in the same way as in supertropical semirings. Every supertropical semiring can be regarded as a supertropical monoid, of course. Conversely, since addition in a supertropical semiring \( U \) is determined by multiplication and the idempotent \( e \) (cf. [IKR3, Theorem 3.11]), we can turn a supertropical monoid \( U \) into a supertropical semiring in at most one way, and then say that \( U \) “is” a supertropical semiring. If \( U \) and \( V \) are supertropical monoids, the definition of a transmission \( \alpha : U \to V \), as given above for \( U \), \( V \) supertropical semirings, still makes sense, and these “transmissions” between supertropical monoids (cf. Definition \([1.5]\) are taken as the morphisms in the category \( \text{STROP}_m \).

The axioms TE1–TE3 mentioned above also make perfect sense for an equivalence relation \( E \) on a supertropical monoid \( U \) (cf. Definition \([1.7]\) below). Thus, such a relation \( E \) will again be called a TE-relation. But we have the important new fact that for a TE-relation \( E \) on a supertropical monoid \( U \) the quotient \( U/E \) always exists in the category \( \text{STROP}_m \). More precisely, the set \( U/E \) can be equipped with the structure of a supertropical monoid in a unique way, such that the map \( \pi_E \) is a transmission (cf. Theorem \([1.8]\) below).
The solution of the problem \((\ast)\) from above now reads as follows (Scholium \[1.12\] below): If \(U\) is a supertropical semiring and \(E\) is a TE-relation on \(U\), then \(E\) is transmissive iff the quotient \(U/E\) in \(\text{STROP}_m\) is a supertropical semiring.

We will provide a necessary and sufficient condition that a given supertropical monoid \(U\) is a supertropical semiring (Theorem \[1.2\] below). From this criterion it is immediate that \(U\) is such a semiring if the bipotent semiring \(U\) is cancellative, but there also are other cases, where this holds.

A bipotent semiring \(M\) may be viewed as a supertropical semiring, all of whose elements are ghosts. (This is the case \(1 = e\).) Thus the category \(\text{STROP}_{m}/M\) of supertropical monoids over \(M\) may be viewed as the category of supertropical monoids \(U\) with a fixed ghost ideal \(eU = M\). Then the morphisms of \(\text{STROP}_{m}/M\) are the transmissions \(\alpha : U \to V\) with \(\alpha(x) = x\) for all \(x \in M\). We call the surjective transmissions over \(M\) fiber contractions (over \(M\)), as we did for supertropical semirings \([KR1\ §6]\). We note in passing that if \(\alpha : U \to V\) is a fiber contraction and \(U\) is a supertropical semiring, then \(V\) is again a supertropical semiring (cf. Theorem \[1.6\] below), and \(\alpha\) is a semiring homomorphism \([KR1\ Proposition 5.10.iii]\).

It turns out that for every supertropical monoid \(U\) there exists a fiber contraction \(\sigma_U : U \to \hat{U}\) with \(\hat{U}\) a supertropical semiring, called the supertropical semiring associated to \(U\), such that every fiber contraction \(\alpha : U \to V\) factors through \(\sigma_U\) (in a unique way), \(\alpha = \beta \circ \sigma_U\) with \(\beta : \hat{U} \to V\) again a fiber contraction. In more elaborate terms, \(\text{STROP}/M\) is a full reflective subcategory of \(\text{STROP}_{m}/M\), cf. \([F\ p. \ 79]\, [FS\ 1.813]\).

The reflections \(\sigma_U : U \to \hat{U}\) turn out to be useful for solving problems of universal nature for supertropical semirings and supervaluations. The strategy is, first to solve such a problem in \(\text{STROP}_{m}\), which often is easy, and then to employ reflections to obtain a solution in \(\text{STROP}\). Major instances for this are provided by Theorems \[1.6\] and \[7.9\] below.

A large part of the paper is devoted to the factorization of transmissions into appropriately defined “basic” transmissions. Let \(\alpha : U \to V\) be a surjective transmission with \(U, V\) supertropical monoids, and let \(\alpha' := \gamma : M \to N\) denote the “ghost part” of \(\alpha\), i.e., the semiring homomorphism obtained from \(\alpha\) by restriction to the ghost ideals \(M := e_vU, N := e_VV\). Then there exists an essentially unique factorization

\[
\alpha : U \xrightarrow{\lambda} U_1 \xrightarrow{\beta} V_1 \xrightarrow{\mu} V, \quad (0.1)
\]

for some supertropical monoids \(U_1\) and \(V_1\), with \(\lambda\) and \(\mu\) fiber contractions of certain types over \(M\) and \(N\) respectively and \(\beta\) a so called “strict ghost-contraction”, which means that \(\beta\) restricts to a bijection from the set \(\mathcal{T}(U_1)\) of non-zero tangible elements of \(U\) to \(\mathcal{T}(U_1)\), while \(\beta' = \gamma\), cf. Theorem \[2.10\] (Notice that \(\gamma\) has convex fibers in \(M\), since \(\gamma\) respects the orderings of \(M\) and \(N\). These convex sets are contracted by \(\gamma\) to one-point sets, hence the name “ghost-contraction”. “Strict” alludes to the property that no element of \(\mathcal{T}(U)\) is sent to a ghost in \(V\).)

From \(0.11\) we then obtain a factorization

\[
\alpha : U \xrightarrow{\lambda} U \xrightarrow{\mu} W \xrightarrow{\mu} V, \quad (0.2)
\]

which is really unique. Here \(\lambda, \beta, \mu\) are transmissions of the same types as before but are normalized to maps \(\pi_E\) given by certain TE-relations \(E\) on \(U, U, W\) respectively, which are uniquely determined by \(\alpha\), and \(\rho\) is an isomorphism over \(N = eV\). This is the “canonical
factorization” of α, appearing in the title of the paper (Definition 2.12). The transmissions λ, β, μ, ρ are called the canonical factors of α.

In §3 we make explicit how the canonical factors of a composition α2 ∘ α2 of two transmissions α1 : U1 → U2, α2 : U2 → U3 can be obtained from the canonical factors of α1 and α2.

Our primary interest is not in supertropical monoids but in supertropical semirings. In this respect the following result is useful: If U and V are supertropical semirings, then in the canonical factorization (0.2) all three supertropical monoids U, W, W are again supertropical semirings, and thus all canonical factors are morphisms in STROP (Theorem 2.10).

Besides STROP, the category STROPH deserves interest, whose objects are again the supertropical semirings but whose morphisms are only the semiring homomorphisms. (Thus STROPH is a subcategory of STROP and a full subcategory of the category of semirings.) In §5 we introduce a subcategory STROPH_m of STROP_m which turns out to be equally useful for working in STROPH, as STROP_m has proved to be useful for STROP.

The objects of STROPH_m are again the supertropical monoids, but the morphisms are suitably defined “h-transmissions”, which are designed in such a way that an h-transmission α : U → V between supertropical semirings is a semiring homomorphism, cf. Definition 5.1. Thus STROPH is a full subcategory of STROPH_m. Again it turns out that for a given bipotent semiring M the category STROPH /M is reflective in STROPH_m/M (Corollary 5.10).

If α : U → V is a surjective h-transmission, then the canonical factors of α are again h-transmissions (Theorem 5.11). It follows that, if α : U → V is a surjective semiring homomorphism, the whole canonical factorization runs in STROPH.

In §6 we study supertropical monoids which have a total ordering defined to be compatible with the supertropical monoid structure in a rather obvious way (Definition 6.1). We call them ordered supertropical monoids (= OST-monoids, for short). It turns out that the underlying supertropical monoids of an OST-monoid is a supertropical semiring (Theorem 6.4).

The “right” morphisms between OST-monoids are the transmissions compatible with the given orderings, called monotone transmissions. It turns out that every monotone transmission is a semiring homomorphism (Theorem 6.7). A major result now is that, given a monotone transmission α : U → V, all factors of the canonical factorization (0.2) of α can be interpreted as monotone transmissions. More precisely, there exist unique total orderings on U, W, W, which make U, W, W OST-monoids and all factors λ, β, μ, ρ monotone transmissions (Theorem 6.14).

In the last sections §7–§9 of the paper “m-supervaluations” play the leading role. Given an m-valuation v : R → M on a semiring R, an m-supervaluation ϕ : R → U covering v is defined in a completely analogous way as has been indicated above for a supervaluation. The only difference is that now U is a supertropical monoid instead of a supertropical semiring (Definition 7.1).

The morphisms in STROP_m are adapted to the m-supervaluations, as the morphisms in STROP were adapted to the supervaluations, to wit, a map α : U → V between supertropical monoids is a transmission iff for every m-supervaluation ϕ : R → U the map α ∘ ϕ is again an m-supervaluation. (This has not been detailed in the paper.)

In order to avoid discussions about “equivalence” of m-valuations we now tacitly assume, without essential loss of generality, that all occurring m-supervaluations ϕ : R → U are
“surjective”, i.e., \( U = \varphi(R) \cup e\varphi(R) \). Given such \( m \)-supervaluations \( \varphi : R \to U, \psi : R \to V \), we say that \( \varphi \) dominates \( \psi \), and write \( \varphi \geq \psi \), of there exists a (unique) transmission \( \alpha : U \to V \) with \( \psi = \alpha \circ \varphi \).

In \( \S 7 \) we construct the initial \( m \)-supervaluation \( \varphi_v^0 : R \to U(v)^0 \) covering a given \( m \)-supervaluation \( v : R \to M \), which means that \( \varphi_v^0 \) dominates all other \( m \)-supervaluations covering \( v \). It then is immediate that

\[
\varphi_v := \sigma_{U(v)^0} \circ \varphi_v^0 : R \to (U(v)^0)^\wedge
\]

is the initial supervaluation covering \( v \), i.e., dominates all supervaluations covering \( v \) (Theorem 7.9).

Already in [IKR1, §7] we could prove that an initial supervaluation \( \varphi_v \) covering \( v \) exists, but obtained an explicit description of \( \varphi_v \) only in the case that \( M \) is cancellative, while now we obtain an explicit description of \( \varphi_v \) in general (Scholium 7.11). (N.B. If \( M \) is cancellative, then \( \varphi_v = \varphi_v^0 \).)

More generally, if \( \varphi : R \to U \) is an \( m \)-supervaluation covering \( v \), then

\[
\hat{\varphi} := \sigma_U \circ \varphi : R \to \hat{U}
\]

is a supervaluation covering \( v \), and \( \hat{\varphi} \geq \psi \) for any other supervaluation \( \psi \) covering \( v \) with \( \varphi \geq \psi \). We call \( \varphi \) tangible, if \( \varphi(R) \subseteq T(U) \cup \{0\} \). It turns out that \( \varphi_v^0 \) is always tangible, but if \( U(v)^0 \) is not a supertropical semiring, i.e., \( \varphi_v^0 \neq \varphi_v \), then \( \varphi_v \) is not tangible, and this implies that no supervaluation covering \( v \) is tangible.

The last two sections 8–9 are motivated by our interest to put a supervaluation \( \varphi : R \to U \) covering \( v \) to use in tropical geometry. It will be relevant to apply \( \varphi \) to the coefficients of a given polynomial \( f(\lambda) \in R[\lambda] \) in a set \( \lambda \) of \( n \) variables (or a Laurent polynomial), and to study the supertropical root sets and tangible components of polynomials \( g(\lambda) \in F[\lambda] \) in \( F^n \) (cf. [IKR1] §5, §7) obtained from \( f(\lambda) \in U[\lambda] \) by passing from \( U \) to various supertropical semifields \( F \). For this purpose it will be important to have some control on the set

\[
\{a \in R \mid \varphi(a) \in M\} = \{a \in R \mid \varphi(a) = v(a)\}.
\]

Given an \( m \)-supervaluation \( \varphi : R \to Y \) covering \( v : R \to M \), we construct a tangible \( m \)-supervaluation \( \hat{\varphi} : R \to \hat{U} \), which is minimal with \( \hat{\varphi} \geq \varphi \) (Theorem 8.11). In §9 we then classify the \( m \)-supervaluations \( \psi \) with \( \varphi \leq \psi \leq \hat{\varphi} \), called the partial tangible lifts of \( \varphi \).

They are uniquely determined by their ghost value sets

\[
G(\psi) := \psi(R) \cap M,
\]

cf. Theorem 9.3. These are ideals of the semiring \( M \), and all ideals \( a \subset G(\varphi) \) occur in this way (Theorem 9.7). Unfortunately the ghost value set \( G(\psi) \) does not control the set \( \{a \in R \mid \psi(a) \in M\} \) completely. We can only state that this set is contained in \( v^{-1}(G(\psi)) \).

If \( \varphi \) is a supervaluation, then \( \hat{\varphi} := (\hat{\varphi})^\wedge \) is the supervaluation, which is a partial tangible lift of \( \varphi \) having smallest ghost value set.

**Notations.** Given sets \( X, Y \) we mean by \( Y \subset X \) that \( Y \) is a subset of \( X \), with \( Y = X \) allowed. If \( E \) is an equivalence relation on \( X \) then \( X/E \) denotes the set of \( E \)-equivalence classes in \( X \), and \( \pi_E : X \to X/E \) is the map which sends an element \( x \) of \( X \) to its \( E \)-equivalence class, which we denote by \( [x]_E \). If \( Y \subset X \), we put \( Y/E := \{[x]_E \mid x \in Y\} \).

Although this does not mean surjectivity in the usual sense, there is no danger of confusion since a supervaluation \( \varphi : R \to V \) can hardly ever be surjective as a map except in the degenerate case \( V = M \).
If $U$ is a supertropical semiring, we denote the sum $1 + 1$ in $U$ by $e$, more precisely by $e_U$ if necessary. If $x \in U$ the ghost companion $ex$ is also denoted by $\nu(x)$, and the ghost map $U \to eU$, $x \mapsto \nu(x)$, is denoted by $\nu_U$. If $\alpha : U \to V$ is a transmission, then the semiring homomorphism $eU \to eV$ obtained from $\alpha$ by restriction is denoted by $\alpha^*$ and is called the ghost part of $\alpha$. Thus $\alpha^* \circ \nu_U = \nu_V \circ \alpha$.

$\mathcal{T}(U)$ and $\mathcal{G}(U)$ denote the sets of tangible and ghost elements of $U$, respectively, cf. [IKR1, Terminology 3.7]. We put $\mathcal{T}(U)_0 := \mathcal{T}(U) \cup \{0\}$.

If $v : R \to M$ is an $m$-valuation we call the ideal $v^{-1}(0)$ of $R$ the support of $v$, and denote it by $\text{supp}(v)$.

1. Supertropical monoids

Definition 1.1. A supertropical monoid $U$ is a monoid $(U, \cdot)$ (multiplicative notation) which has an absorbing element $0 := 0_U$, i.e., $0 \cdot x = 0$ for every $x \in U$, and a distinguished central idempotent $e := e_U$ such that the following holds:

$$\forall x \in U : \quad ex = 0 \Rightarrow x = 0.$$ 

Further a total ordering, compatible with multiplication, is given on the submonoid $M := eU$ of $U$.

We then regard $M$ as a bipotent semiring in the usual way [IKR1 §1].

If $U$ is a supertropical monoid, we would like to enrich $U$ by a composition $U \times U \rightarrow U$ extending the addition on $M$, such that $U$ becomes a supertropical semiring with

$$1_U + 1_U = e_U.$$ 

We are forced to define the addition on $U$ as follows $(x, y \in U)$, cf. [IKR1 Theorem 3.11]:

$$x + y = \begin{cases} 
  y & \text{if } ex < ey, \\
  x & \text{if } ex > ey, \\
  ex & \text{if } ex = ey.
\end{cases}$$

If this addition obeys the associativity and distributivity laws, we say that the supertropical monoid $U$ “is” a semiring. In the commutative case we have the following criterion.

Theorem 1.2. A supertropical commutative monoid is a semiring iff the following holds:

$$\text{(Dis)} : \quad \forall x, y, z \in U : \quad \text{If } 0 < ex < ey, \text{ but } exz = eyz, \text{ then } yz = eyz \text{ (i.e., } yz \in eU).$$

In this case the semiring $U$ is supertropical.

Proof. Let $x, y, z \in U$ be given. Obviously, $x + y = y + x$ and $x + 0 = x$, and it is easily checked that $(x + y) + z = x + (y + z)$. It remains to investigate when we have

$$(x + y)z = xz + yz. \quad (\ast)$$

We assume without loss of generality that $ex \leq ey$. If $ex = 0$, then $x = 0$ and $(\ast)$ is true. If $ex = ey$, then $exz = eyz$, hence $x + y = ey$ and $xz + yz = eyz$. Thus $(\ast)$ is true again.

We are left with the case that $0 < ex < ey$. Then $x + y = y$ and $exz \leq eyz$. If $exz < eyz$, then $xz + yz = yz$, and $(\ast)$ is true. But if $exz = eyz$, then $xz + yz = eyz$, while $(x + y)z = yz$. Thus $(\ast)$ holds iff $yz = eyz$.

We conclude that $(\ast)$ holds for all triples $x, y, z$ iff condition (Dis) is fulfilled. □
Remark 1.3. When $U$ is not commutative, we have an analogous result. We just need to add the same condition (Dis) for the monoid $U^{\text{app}}$ obtained from $U$ by changing the multiplication $(x, y) \mapsto xy$ to $(x, y) \mapsto yx$; i.e.,

$$(\text{Dis}) : \quad \forall x, y, z \in U : \text{If } 0 < ex < ey, \text{ but } ezx = ezy, \text{ then } zy = ezy.$$

Given an element $x$ of a supertropical monoid $U$, we call $ex$ the ghost of $x$, and we denote the ghost map $U \to eU$, $x \mapsto ex$, by $\nu_U$, as we did before for $U$ a supertropical semiring.

By a (two sided) ideal $a$ of a supertropical monoid $U$ we mean a monoid ideal of $U$, i.e., a nonempty subset $a$ of $U$ with $U \cdot a \subseteq a$ and $a \cdot U \subseteq a$. Notice that in the case that $U$ is a supertropical semiring, such a set $a$ is indeed an ideal of the semiring $U$ in the usual sense, cf. [IKR2, Remark 6.21]. We call $eU$ the ghost ideal of $U$.

Many more definitions in [IKR1] and [IKR2] retain their sense if we replace the supertropical semirings by supertropical monoids, in particular the following one.

Definition 1.4. Let $U$ and $V$ be supertropical monoids. We call a map $\alpha : U \to V$ a transmission, if the following holds (cf. [IKR1, §5]):

- $TM1 : \alpha(0) = 0$,
- $TM2 : \alpha(1) = 1$,
- $TM3 : \forall x, y \in U : \alpha(xy) = \alpha(x)\alpha(y)$,
- $TM4 : \alpha(e_U) = e_V$,
- $TM5 : \forall x, y \in eU : x \leq y \Rightarrow \alpha(x) \leq \alpha(y)$.

(N.B. $\alpha$ maps $eU$ to $eV$ due to $TM3$ and $TM4$.) Notice that this means that $\alpha$ is a monoid homomorphism sending 0 to 0 and e to e, which restricts to a homomorphism $\gamma : eU \to eV$ of bipotent semirings. We then call $\gamma$ the ghost part of $\alpha$, and write $\gamma = \alpha^\nu$. We also say that $\alpha$ covers $\gamma$ (as we did in [IKR1] for $U, V$ commutative supertropical semirings). Notice that $\alpha(U)$ is a supertropical submonoid of $V$ in the obvious sense.

We introduce two sorts of “kernels” of transmissions.

Definition 1.5. Let $\alpha : U \to V$ be a transmission between supertropical monoids.

(a) The zero kernel of $\alpha$ is the set

$$\mathfrak{z}_\alpha := \{x \in U \mid \alpha(x) = 0\}.$$

(b) The ghost kernel of $\alpha$ is the set

$$\mathfrak{A}_\alpha := \{x \in U \mid \alpha(x) \in eV\}.$$

These sets are ideals of $U$, and $M \cup \mathfrak{z}_\alpha \subseteq \mathfrak{A}_\alpha$. If $U$ is a semiring, then $M \cup \mathfrak{z}_\alpha = M + \mathfrak{z}_\alpha$, (cf. [IKR2, Remark 6.21]). If $\mathfrak{A}_\alpha = M$, we say that $\alpha$ has trivial ghost kernel, and if $\mathfrak{z}_\alpha = \{0\}$, we say that $\alpha$ has trivial zero kernel.

Theorem 1.6. Let $\alpha : U \to V$ be a transmission between supertropical monoids, which is injective on the set $(eU) \setminus \{0\}$.

(i) If $\alpha$ has a trivial ghost kernel, and if $V$ is a semiring, then $U$ is a semiring.

(ii) If $\alpha$ is surjective, and if $U$ is a semiring, then $V$ is a semiring.
Proof. Again we prove this for commutative monoids, leaving the obvious modifications in the noncommutative case to the interested reader.

We use the criterion for a supertropical monoid to be a semiring given in Theorem 1.2.

(i): Let \( x, y, z \in U \) with \( 0 < ex < ey \) and \( exz = eyz \). Then
\[
\alpha(x) = \alpha(ex) < \alpha(ey) = e\alpha(y),
\]
since \( \alpha \) is injective on \( (eU)\setminus\{0\} \), and
\[
\alpha(x)\alpha(z) = e\alpha(y)\alpha(z).
\]
Since \( V \) is a semiring, we deduce that \( \alpha(yz) = \alpha(y)\alpha(z) \in eV \). Since \( \alpha \) has a trivial ghost kernel, it follows that \( yz \in eU \), as desired.

(ii): Let \( x, y, z \in U \) with
\[
0 < e\alpha(x) < e\alpha(y) \quad \text{and} \quad e\alpha(x)\alpha(z) = e\alpha(y)\alpha(z).
\]
We are done if we verify that \( \alpha(y)\alpha(z) \in eV \). We have \( 0 < \alpha(ex) < \alpha(ey) \) and \( \alpha(exz) = \alpha(eyz) \). The inequalities imply \( 0 < ex < ey \).

Case I: \( \alpha(eyz) = 0 \). Now \( \alpha(eyz) = 0 \), hence \( \alpha(yz) = 0 \), hence \( \alpha(y)\alpha(z) = 0 \).

Case II: \( \alpha(eyz) \neq 0 \). Now \( exz \neq 0 \) and \( eyz \neq 0 \). Since \( \alpha \) is injective on \( (eU)\setminus\{0\} \), it follows that \( exz = eyz \). Since \( 0 < ex < ey \) and \( U \) is a semiring, we conclude that \( yz \in eU \), hence \( \alpha(y)\alpha(z) = \alpha(yz) \in eV \).

Thus \( \alpha(y)\alpha(z) \in eV \) in both cases. \( \Box \)

Definition 1.7. If \( U \) is a supertropical monoid, we call an equivalence relation \( E \) on the set \( U \) a TE-relation, if the following holds (cf. [IKR2, §4]):

\begin{align*}
\text{TE1:} & \quad E \text{ is multiplicative, i.e., } \forall x, y, z \in E : \quad x \sim_E y \Rightarrow xz \sim_E yz, \quad zx \sim_E zy \\
\text{TE2:} & \quad \text{The equivalence relation } E/M \text{ is order compatible, i.e.:} \\
& \quad \text{If } x_1, x_2, x_3, x_4 \in M \text{ and } x_1 \leq x_2, \ x_3 \leq x_4, \ x_1 \sim_E x_4, \ x_2 \sim_E x_3, \\
& \quad \text{then } x_1 \sim_E x_2. \text{ (Hence all } x_i \text{ are } E\text{-equivalent.)} \\
\text{TE3:} & \quad \text{If } x \in U \text{ and } ex \sim_E 0, \text{ then } x \sim_E 0.
\end{align*}

We have the following almost trivial but important fact.

Theorem 1.8. Let \( U \) be a supertropical monoid and \( E \) a TE-relation on \( U \). Then the set \( U/E \) of equivalence classes carries a unique structure of a supertropical monoid such that the map
\[
\pi_E : U \to U/E, \quad x \mapsto [x]_E,
\]
is a transmission.

Proof. This is just some universal algebra. We are forced to define the multiplication on the set \( \overline{U} := U/E \) by the rule \((x, y) \in U \)
\[
[x]_E \cdot [y]_E = [xy]_E.
\]
This makes sense since the equivalence relation \( E \) is multiplicative. Now \( \overline{U} \) is a monoid with absorbing element \( 0_{\overline{U}} := [0_U]_E \). We are further forced to take as distinguished idempotent on \( \overline{U} \) the element \( e_{\overline{U}} := [e_U]_E \). Clearly
\[
e_{\overline{U}} \overline{U} = M/E := \{[x]_E \mid x \in M\}.
\]
Finally, we are forced to choose on the submonoid \( M/E \) of \( U/E \) the total ordering given by \((x, y \in M)\)
\[
[x]_E \leq [y]_E \iff x \leq y.
\]
This total ordering is well-defined since the restriction \( E|M \) of \( E \) to \( M \) is order compatible (cf. [IKR2, §2]).

It is now evident that \( \bar{U} \) has become a supertropical monoid and \( \pi_E \) a transmission.

\[ \square \]

Remark 1.9. Conversely, if \( \alpha : U \to V \) is a transmission from \( U \) to a supertropical monoid \( V \), then the equivalence relation \( E(\alpha) \) is TE, and the map \( [x]_{E(\alpha)} \mapsto \alpha(x) \) is an isomorphism from the supertropical monoid \( U/E(\alpha) \) onto the (supertropical) submonoid \( \alpha(U) \) of \( V \).

Example 1.10. Let \( U \) be a supertropical monoid and \( M := eU \). As in the case of supertropical commutative semirings [IKR1, §6] we define an MFCE-relation on \( U \) as an equivalence relation \( E \) on \( U \), which is multiplicative, and is fiber conserving, i.e., \( x \sim_E y \Rightarrow ex = ey \).

Then we have an obvious identification \( M/E = M \), and \( E \) is a TE-relation.

The functorial properties of transmissions between supertropical semirings stated in [IKR1, Proposition 6.1] remain true if we admit instead supertropical monoids, and can be proved in exactly the same way. Thus we get:

Proposition 1.11. Let \( \alpha : U \to V \) and \( \beta : V \to W \) be maps between supertropical monoids.

(i) If \( \alpha \) and \( \beta \) are transmissions, then \( \beta \alpha \) is a transmission.

(ii) If \( \alpha \) and \( \beta \alpha \) are transmissions and \( \alpha \) is surjective, then \( \beta \) is a transmission.

Starting from now we assume that all occurring supertropical monoids are commutative. But we mention that all major results to follow can be established also for noncommutative monoids with obvious modifications of the proofs (in a similar way as indicated in Remark [IKR3]). This will save space and hopefully help the reader not to get distracted from the central ideas of the paper. At the time being, the commutative case suffices for the applications we have in mind (cf. the Introduction).

We define the category of supertropical monoids \( \text{STROP}_m \) as follows: the objects of \( \text{STROP}_m \) are the (commutative) supertropical monoids, and the morphisms are the transmissions between them. \( \text{STROP}_m \) contains the category \( \text{STROP} \) of supertropical semirings as a full subcategory.

Scholium 1.12. Let \( U \) be a supertropical semiring and \( E \) a TE-relation on \( U \). Then the map \( \pi_E : U \to U/E \) from \( U \) to the supertropical monoid \( U/E \) is a morphism in \( \text{STROP}_m \). Since \( \text{STROP} \) is full in \( \text{STROP}_m \), it follows that \( \pi_E \) is a morphism in \( \text{STROP}_m \) iff the supertropical monoid \( U/E \) is a semiring. This means in terms of [IKR2, §4], that the TE-relation \( E \) is transmissive iff the supertropical monoid \( U/E \) is a semiring.

We define initial transmissions and pushout transmissions in \( \text{STROP}_m \) as we defined such transmissions in [IKR2, §1] in the category \( \text{STROP} \). Just repeat [IKR2, Definition 1.2] and [IKR2, Definition 1.3], respectively, replacing everywhere the word “supertropical semiring” by “supertropical monoid”.

The pleasant news now is that in \( \text{STROP}_m \) the pushout transmission exists for any supertropical monoid \( U \) and surjective homomorphisms \( \gamma \) from \( M := eU \) to a bipotent semiring \( N \), and that it has the same explicit description as given in [IKR2, Theorem 1.11] and [IKR2, Example 4.9] (in the category \( \text{STROP} \)) if \( N \) is cancellative.
More precisely the following holds and can be proved by the same arguments as used in [IKR2, Example 4.9] and the proofs of [IKR2, Theorems 1.11 and 4.14].

**Theorem 1.13.** Let \( U \) be a supertropical monoid and \( \gamma \) a homomorphism from \( M := eU \) onto a (bipotent) semiring \( \mathbb{N} \). We obtain a TE-relation \( F(U, \gamma) \) on \( U \) by decreeing for \( x, y \in U \):

\[
x \sim_{F(U, \gamma)} y \iff \begin{cases} 
\text{either} & x = y, \\
\text{or} & x = ey, \ y = ey, \ \gamma(ex) = \gamma(ey), \\
\text{or} & \gamma(ex) = \gamma(ey) = 0.
\end{cases}
\]

The map

\[
\pi_{F(U, \gamma)} : U \rightarrow U/F(U, \gamma)
\]

is a pushout transmission in \( \text{STROP}_m \) covering \( \gamma \). \{Here we identify \( M/F(U, \gamma) = \mathbb{N} \) in the obvious way.\}

In particular every initial transmission in \( \text{STROP}_m \) is a pushout transmission in \( \text{STROP}_m \).

**Scholium 1.14.** We consider the special case that \( U \) is a supertropical semiring. If the supertropical monoid \( U/F(U, \gamma) \) happens to be a semiring, then it is clear that \( \pi_{F(U, \gamma)} \) is a pushout in the category \( \text{STROP} \). Thus, following [IKR2, Notation 1.7], we now have

\[
F(U, \gamma) = E(U, \gamma), \quad \pi_{F(U, \gamma)} = \alpha_{U, \gamma}.
\]

But if \( U/F(U, \gamma) \) is not a semiring then the relation \( F(U, \gamma) \) is different from \( E(U, \gamma) \).

If \( U \) is a supertropical semiring and \( a \) is an ideal of \( U \) we introduced in [IKR2, §5] the saturum \( \text{sat}_{U, a} \) and the equivalence relation \( E(a) = E_U(a) \), and obtained there descriptions of these objects, which do not mention addition but only employ multiplication and the idempotent \( e \) ([IKR2, Corollary 5.5, Theorem 5.4]).

We now use these descriptions to define \( \text{sat}_{U, a} \) and \( E_U(a) \) if \( U \) is only a supertropical monoid.

**Definition 1.15.** Let \( a \) be an ideal of the supertropical monoid \( U \).

(a) The saturum \( \text{sat}_{U, a} \) of \( a \) is the set of all \( x \in U \) with \( ex \leq ea \) for some \( a \in a \). We call \( a \) saturated if \( \text{sat}_{U, a} = a \).

(b) The equivalence relation \( E := E(a) := E_U(a) \) is defined as follows:

\[
x \sim_E y \iff \begin{cases} 
\text{either} & x = y \\
\text{or} & x \in \text{sat}_{U, a}, \ y \in \text{sat}_{U, a}.
\end{cases}
\]

As in [IKR2, §5] the following fact can be verified in an easy straightforward way.

**Proposition 1.16.**

(i) \( \text{sat}_U(a) \) is again a monoid ideal of \( U \).

(ii) The saturated ideals \( a \) correspond uniquely with the ideals \( c \) of \( M \) which are lower sets of \( M \), via

\[
c = a \cap M = ea \quad \text{and} \quad a = \{x \in U | ex \in c\}.
\]

(iii) \( E_U(a) \) is a TE-relation on \( U \).

(iv) If \( b \) is a second ideal of \( U \) then

\[
E_U(a) = E_U(b) \iff \text{sat}_U(a) = \text{sat}_U(b).
\]

\[2\] Notice that a homomorphic image of a bipotent semiring is again bipotent.
Let $U$ be a supertropical monoid and $\gamma : M \to M'$ a surjective homomorphism from $M := eU$ to a (bipotent) semiring $M'$. Further, let $\mathfrak{A}$ be an ideal of $U$ containing $M$ and the saturated ideal $\mathfrak{A}_\gamma := \{ x \in U \mid \gamma(e x) = 0 \}$.

(i) The equivalence relation $E(U, \mathfrak{A}, \gamma)$ on $U$, given by $(x_1, x_2 \in U)$

\[
x_1 \sim_E x_2 \iff \text{either } x_1 = x_2 \text{ or } x_1 \in \mathfrak{A}, \ x_2 \in \mathfrak{A}, \ \gamma(e x_1) = \gamma(e x_2),
\]

is a TE-relation.

(ii) The transmission $\pi_E : U \to U/E$ has the ghost kernel $\mathfrak{A}$. The ghost part $(\pi_E)^\nu$ is the map $\gamma : M \to M'$. {Here we identify the ghost ideal $M/E$ of $U/E$ with $M'$ in the obvious way.}

(iii) Assume that a transmission $\beta : U \to V$ to a supertropical monoid $V$ is given with ghost kernel $\mathfrak{A}_\beta \supset \mathfrak{A}$, further a homomorphism $\delta : M' \to eV$ is given such that $\delta \gamma = \beta^\nu$. Then there exists a unique transmission $\eta : U/E \to V$ with $\eta^\nu = \delta$ and $\eta \alpha = \beta$.

(cf. the diagram following [IKR2, Problem 1.1].)

Remark 1.18. It can be readily verified that

\[ E(U, M \cup \mathfrak{A}_\gamma, \gamma) = F(U, \gamma). \]

Thus the present theorem is a generalization of Theorem 1.13.

For any ideal $\mathfrak{A} \supset M$ of $U$ we define

\[ E(U, \mathfrak{A}) := E(U, \mathfrak{A}, \text{id}_M), \]

as we did in [IKR2, §6] for $U$ a supertropical semiring. In this special case Theorem 1.17 reads as follows.

Corollary 1.19. $E(U, \mathfrak{A})$ is a TE-relation on $U$. A transmission $\alpha : U \to V$ (with $V$ a supertropical monoid) factors through $\pi_{E(U, \mathfrak{A})}$ iff $\mathfrak{A} \subset \mathfrak{A}_\alpha$. 

2. Canonical factorization of a transmission

Given a surjective transmission \( \alpha : U \to V \) between supertropical semirings we start out to write \( \alpha \) as a composition of transmissions of simple nature in a somewhat canonical way. More precisely, we will do this first in the category \( \text{STROP}_m \) of supertropical monoids. Afterward we will prove that, if \( U \) and \( V \) are semirings, this “canonical factorization” remains valid in the smaller category \( \text{STROP} \) of supertropical semirings, which has our primary interest.

Let \( U \) and \( V \) be supertropical monoids, and \( M := eU, N := eV \) their ghost ideals. We first exhibit the “transmissions of simple nature” we have in mind. These are the ideal compressions, tangible fiber contractions, and strict ghost contractions to be defined now.

**Definition 2.1.** As in the case that \( U \) and \( V \) are supertropical semirings (cf. [IKR3 §6]) we say that a surjective transmission \( \alpha : U \to V \) is a **fiber contraction** if the ghost part \( \alpha' = \gamma : M \to N \) is an isomorphism. We say that \( \alpha \) is a **fiber contraction over** \( M \), if \( N = M \) and \( \gamma = \text{id}_M \).

Notice that \( \alpha \) is a fiber contraction iff the equivalence relation \( E(\alpha) \) is an MFCE-relation, hence \( \alpha = \rho \circ \pi_E \) with \( E \) an MFCE-relation and \( \rho \) an isomorphism. Then \( \alpha \) is a fiber contraction over \( M \) iff \( M = N \) and \( \rho \) is an isomorphism over \( M \).

**Definition 2.2.** We call a surjective transmission \( \alpha : U \to V \) an **ideal compression**, if \( \alpha \) is a fiber contraction over \( M \) which maps \( U\backslash A_\alpha \) bijectively on to \( V\backslash N = T(M) \). (Recall that \( A_\alpha \) denotes the ghost kernel of \( \alpha \).)

This means that \( \alpha = \rho \circ \pi_{E(U,A)} \) with \( A \) an ideal of \( U \) containing \( M \) and \( \rho \) an isomorphism from \( \overline{U} := U/E(U,A) \) onto \( V \) over \( M \). We have \( A = A_\alpha \).

**Definition 2.3.** We call a transmission \( \alpha \) **tangible** if

\[ \alpha(T(U)) \subset T(V) \cup \{0\}, \]

and **strictly tangible** if

\[ \alpha(T(U)) \subset T(V). \]

In other terms, \( \alpha \) is tangible iff \( A_\alpha = M \cup 0_\alpha \), and \( \alpha \) is strictly tangible iff \( A_\alpha = M \).

What does this means in the case that \( \alpha \) is a fiber contraction? Clearly, a tangible fiber contraction \( \alpha : U \to V \) is strictly tangible. If \( E \) is an MFCE-relation on \( U \), then \( \pi_E : U \to U/E \) is tangible iff \( E \) is **ghost separating** (cf. [IKR2 Definition 6.19]), in other terms, iff \( E \) is finer than the equivalence relation \( E_t := E_{t,U} \) on \( U \) which has the equivalence classes \( \{a \in T(U) \mid ex = a\}, a \in M \backslash \{0\} \), and the one-point equivalence classes \( \{y\}, y \in M \) (cf. [IKR1 Example 6.4.v]).

**Definition 2.4.** We call the MFCE-relations \( E \) on \( U \) with \( E \subset E_t \) **tangible MFCE-relations**.

In this terminology the tangible fiber contractions \( \alpha : U \to V \) over \( M := eU \) are the products

\[ \alpha = \rho \circ \pi_T \]

with \( T \) a tangible MFCE-relation on \( U \) and \( \rho \) an isomorphism over \( M \).

**Definition 2.5.** We call a transmission \( \alpha : U \to V \) a **ghost contraction**, if \( \alpha' \) is a homomorphism from \( M \) onto \( N \), and if \( \alpha \) maps \( U \backslash (M \cup 0_\alpha) \) bijectively onto \( T(V) = V \backslash N \).
This means that

$$\alpha = \rho \circ \pi_{F(U, \gamma)}$$

with $\gamma : M \to N$ a surjective homomorphism, namely $\gamma = \alpha'$, and $\rho$ an isomorphism over $N$ from $U / F(U, \gamma)$ to $V$. Thus $\alpha$ is a ghost contraction iff $\alpha$ is a surjective pushout transmission in STROP$_m$. \{The equivalence relation $F(U, \gamma)$ had been introduced in Theorem 1.13\}

**Definition 2.6.** In the situation of Definition 2.3 and Definition 2.5, respectively, we also say abusively that $V$ is an ideal compression (resp. a ghost contraction) of $U$.

**Definition 2.7.** We call a ghost contraction $\alpha : U \to V$ **strict**, if $\alpha^{-1}(0) \subset M$. This means that $\alpha$ is also a strict tangible transmission.

Notice that every ghost contraction $\alpha : U \to V$ with $\gamma^{-1}(0) = \{0\}$, $\gamma = \alpha'$, is strict, and that $\gamma^{-1}(0) = \{0\}$ iff $\mathfrak{z}_\alpha = \{0\}$. Of course, there exist other strict ghost contractions. The maps $\pi_{F(U, \gamma)}$, where $\gamma : M \to N$ is a homomorphism with $\gamma^{-1}(0) \neq \{0\}$, but where $U$ has no tangibles with ghost companion in $\gamma^{-1}(0)$, are main examples for this.

**Definition 2.8.** If $\gamma : M \to N$ is a surjective homomorphism for $M = eU$ to a semiring $N$, we put

$$a_{U, \gamma} := \{x \in U \mid \gamma(ex) = 0\},$$

an ideal already used in Theorem 1.17.

In this notation the ghost contraction $\pi_{F(U, \gamma)}$ is strict iff $a_{U, \gamma} \subset M$.

**Theorem 2.9.** Let $\alpha : U \to V$ be a surjective transmission between supertropical monoids and let $\gamma : M \to N$ denote the homomorphism between the ghost ideals $M := eU$, $N := eV$ obtained from $\alpha$ by restriction, $\gamma = \alpha'$.

(i) There exists a factorization

$$\alpha = \mu \circ \beta \circ \lambda$$

with $\lambda$ and ideal compression of $U$, $\beta$ a strict ghost contraction, and $\mu$ a tangible fiber contraction over $N := eV$.

(ii) The factorization is essentially unique. More precisely, if $\alpha = \mu' \circ \beta' \circ \lambda'$ is a second such factorization of $\alpha$, then there exist isomorphisms $\rho$ over $M$ and $\sigma$ over $N$ (of supertropical monoids) such that

$$\lambda' = \rho \lambda, \quad \mu' = \mu \sigma^{-1}, \quad \beta' = \sigma \beta \rho^{-1}.$$  

(iii) In particular we can choose

$$\lambda = \pi_{E(U, \mathfrak{A})} : U \longrightarrow \overline{U} := U / E(U, \mathfrak{A})$$

with $\mathfrak{A} := \mathfrak{A}_\alpha$, the ghost kernel of $\alpha$,

$$\beta = \pi_{F(U, \gamma)} : \overline{U} \longrightarrow W := \overline{U} / F(\overline{U}, \gamma)$$  

and $\mu : W \to V$ the resulting tangible fiber contraction over $N$ such that $\alpha = \mu \beta \lambda$ (see proof below).

**Proof.**  

a) Let $\gamma := \alpha' : M \to N$, $\mathfrak{A} := \mathfrak{A}_\alpha$, and $\overline{U} := U / E(U, \mathfrak{A})$. Then $\alpha$ factors through $\lambda := \pi_{E(U, \mathfrak{A})}$ in a unique way,

$$\alpha : U \xrightarrow{\lambda} \overline{U} \xrightarrow{\pi} V,$$
with $\overline{a}$ a surjective transmission having trivial ghost kernel. This is clear from [IKR2, Proposition 6.20], adapted to the category of supertropical monoids.

b) We have $(\overline{a})^\nu = \gamma$. Let $\beta = \pi_F(U, \gamma)$. By Theorem 1.13 we know that $\beta$ is an initial transmission in the category $\text{STROP}_m$ (even a pushout). Thus we have a unique transmission

$$\mu : W := U/F(U, \gamma) \rightarrow V$$

such that $\overline{a} = \mu \circ \beta$, hence $\alpha = \mu \circ \beta \circ \lambda$. From $(\overline{a})^\nu = \gamma = \mu^\nu \circ \beta^\nu$ and $\beta^\nu = \gamma$ it follows that $\mu^\nu$ is the identity of $N$.

Since $\overline{a}$ has trivial ghost kernel and $\beta$ is surjective, both $\beta$ and $\mu$ have trivial ghost kernels. We conclude that $\beta$ is a strict ghost contraction and $\mu$ is a tangible fiber contraction over $N$. Parts i) and iii) of the theorem are proven.

c) Retaining the transmissions $\lambda, \beta, \mu$ which we have defined above, we turn to the claim of uniqueness in part ii) of the theorem. Let $\alpha = \mu' \circ \beta' \circ \lambda'$ another factorization of $\alpha$ of the kind considered here. Both $\beta'$ and $\mu'$ have trivial ghost kernel. Thus the ideal compression $\lambda'$ has the same ghost kernel $\mathfrak{A}$ as $\alpha$. We conclude that

$$\lambda' = \rho \pi_{E(U, \mathfrak{A})} = \rho \lambda$$

with some isomorphism $\rho$ over $M$.

From $\alpha = (\mu' \beta' \rho) \lambda$ we then conclude that $\mu' \beta' \rho = \overline{a}$. Now $\beta' \rho$ is a strict ghost contraction covering $\gamma$, since $\beta'$ is such a ghost contraction and $\rho$ covers $\text{id}_M$. It follows that

$$\beta' \rho = \sigma \pi_F(U, \gamma) = \sigma \beta$$

with some isomorphism $\sigma$ over $N$, and hence $\beta' = \sigma \beta \rho^{-1}$. We finally obtain

$$\alpha = \mu' \sigma \beta \lambda = \mu \beta \lambda,$$

and then $\mu' \sigma = \mu$. $\square$

**Theorem 2.10.** Let $\alpha : U \rightarrow V$ be a surjective transmission between supertropical semirings, and assume that

$$\alpha : U \xrightarrow{\lambda} U_1 \xrightarrow{\beta} V_1 \xrightarrow{\mu} V,$$

is a factorization of $\alpha$ as described in Theorem 2.9.i (in the category $\text{STROP}_m$). Then both $U_1$ and $V_1$ are supertropical semirings, hence all three factors $\lambda, \beta, \mu$ are morphisms in $\text{STROP}$.

**Proof.** $\lambda$ and $\mu$ are surjective and $\lambda^\nu = \text{id}_M$, $\mu^\nu = \text{id}_N$. Moreover $\mu$ has trivial ghost kernel. Thus $V_1$ is a semiring by Theorem 1.6.i, and $U_1$ is a semiring by Theorem 1.6.ii. $\square$

**Corollary 2.11.** Let $\alpha : U \rightarrow V$ be a surjective transmission between supertropical semirings covering $\overline{a} = \gamma : M \rightarrow N$. Then for the supertropical semiring

$$\overline{U} := U/E(U, \mathfrak{A}_\alpha)$$

the transmission

$$\pi_F(U, \gamma) : \overline{U} \rightarrow \overline{U}/F(\overline{U}, \gamma)$$

is pushout in the category $\text{STROP}$. In other terms (cf. [IKR2, Notation 1.7])

$$F(\overline{U}, \gamma) = E(\overline{U}, \gamma).$$

**Proof.** Theorem 2.10 tells us that $\overline{U}/F(\overline{U}, \gamma)$ is a supertropical semiring. We know from 11 that $\pi_F(U, \gamma)$ is pushout in $\text{STROP}_m$. A fortiori this transmission is pushout in $\text{STROP}$. $\square$
\textbf{Definition 2.12.} Let $\alpha : U \to V$ be a surjective transmission covering $\alpha'' = \gamma : M \to N$. We know by Theorem 2.9 that there exists a unique factorization
\[ \alpha = \rho \circ \pi_T \circ \pi_{F(U,\gamma)} \circ \pi_{E(U,\gamma)} \tag{\ast} \]
with $A$ an ideal of $U$ containing $M \cup \beta_n$, $U := U/E(U, A)$, $T$ a tangible MFCE-relation on $W := U/F(U, \gamma)$, and $\rho$ an isomorphism from $W/T$ to $U$ over $N$. Here $A$, $T$, and hence $\rho$ are uniquely determined by $\alpha$. We call (\ast) the \textbf{canonical factorization} of $\alpha$, and $\pi_{F(U,\gamma)}$, $\pi_{E(U,\gamma)}$, $\pi_T$, $\rho$ the \textbf{canonical factors} of $\alpha$.

If one of these maps is the identity map, we feel justified to omit it in the list of the canonical factors of $\alpha$.

We discuss some simple cases of canonical factorizations.

\textbf{Scholium 2.13.} (The case of $M = N$.) Assume that $U$ and $V$ are supertropical monoids with $eU = eV = M$. Let $\alpha : U \to V$ be a fiber contraction over $M$ with ghost kernel $A = A_\alpha$.

(i) $\alpha$ has the factorization $\alpha = \mu \circ \lambda$ with $\lambda = \pi_{E(U,\alpha)}$ and
\[ \mu : U := U/E(U, A) \to V \]
a (strict) tangible fiber contraction over $M$. This is clear from Theorem 2.9 or directly from the universal property of $\pi_{E(U,\alpha)}$, cf. Corollary 1.19. Then $\mu = \rho \circ \pi_T$ with $T$ a tangible MFCE-relation on $U$. Thus $\alpha$ has the canonical factors $\pi_{E(U,\alpha)}$, $\pi_T$, and $\rho$.

(ii) If $U$ is a semiring then both $U$ and $V$ are semirings, as follows directly from Theorem 1.6.(ii).

\textbf{Example 2.14.} (Factorization of a ghost contraction.) Assume $\alpha : U \to V$ is a ghost contraction covering $\alpha'' = \gamma : M \to N$. Let $a$ denote the zero kernel of $\alpha$, $a := \beta_\alpha$.

(i) $\alpha$ has the factorization $\alpha = \beta \circ \lambda$ with $\lambda = \pi_{E(U,\alpha)}$ and
\[ \beta : U := U/E(U, M \cup a) \to V \]
a strict ghost contraction. We have
\[ \beta = \rho \circ \pi_{F(U, \gamma)} \]
with $\rho$ an isomorphism from $U/F(U, \gamma)$ to $V$ over $N$. Thus $\alpha$ has the canonical factors $\pi_{E(U,\alpha)}$, $\pi_{F(U, \gamma)}$, and $\rho$.

(ii) We further have the factorization
\[ \beta = \overline{\beta} \circ \pi_{E(a)} \]
with $\overline{a} := \lambda(a)$ and
\[ \overline{\beta} : U/E(\overline{a}) = U/E(a) \to V, \]
which is a strict ghost contraction with zero kernel $\{0\}$. Notice that the ideal $a = \beta_\alpha$ is saturated in $U$ and $\overline{a}$ is saturated in $U$.

\textbf{Example 2.15.} (The transmissions $\pi_{E(U,\alpha)}$.)

Let $U$ be a supertropical monoid and let $\gamma : M \to N$ be a surjective homomorphism from $M = eU$ to a semiring $N$. Further let $A$ be an ideal of $U$ containing $M \cup a_{U,\gamma}$. 

Thus in all cases except the second we have
\[ M_\alpha = \pi_{F(U, \mathcal{A})} \circ \pi_{E(U, \mathcal{A})} \]
with \( \overline{U} := U/E(U, \mathcal{A}) \). Indeed we have \( E(U, \mathcal{A}, \gamma)/E(U, \mathcal{A}) = F(\overline{U}, \gamma) \), as has been stated in [IKR2, Theorem 6.22].

(ii) The ghost contractions \( \alpha : U \to V \) covering \( \gamma \) are precisely the maps

\[ \alpha = \rho \circ \pi_{E(U, \mathcal{A}, \gamma)} \]
with \( \mathcal{A} = M \cup \mathcal{A}_{U, \gamma} \) and \( \rho \) an isomorphism over \( N \), as is clear from the above and Example 3.3.

3. The canonical factors of a product of two basic transmissions

Definition 3.1. Let \( \alpha : U \to V \) be a surjective transmission between supertropical monoids, and let \( \gamma := \alpha^\circ : M \to N \) denote the ghost part of \( \alpha \). We call \( \alpha \) a basic transmission, if \( \alpha \) is of one of the following 4 types.

**Type 1**: \( \alpha = \pi_{E(U, \mathcal{A})} \) with \( \mathcal{A} \) an ideal of \( U \) containing \( M \).

**Type 2**: \( \alpha = \pi_{F(U, \gamma)} \) and \( \alpha^{-1}(0) = \gamma^{-1}(0) \).

**Type 3**: \( \alpha = \pi_T \) with \( T \) a tangible MFCE-relation on \( U \).

**Type 4**: \( \alpha = \rho \) with \( \rho \) an isomorphism over \( M \).

In short, the basic transmissions are the factors occurring in the canonical factorizations of transmissions (cf. Definition 2.12).

Problem 3.2. Given basic transmissions \( \alpha : U \to V \) of type \( i \) and \( \beta : V \to W \) of type \( j \) with \( 1 \leq j \leq i \leq 4 \), find the canonical factorization of \( \beta \alpha \) explicitly.

It would be easy to find these canonical factorizations up to an undetermined isomorphism \( \rho \) as first factor (cf. Definition 2.12) by running through parts a) and b) of the proof of Theorem 2.10. But we want a completely explicit description of all factors. For this we will rely on realizations of the quotient monoids \( U/E(U, \mathcal{A}) \), \( U/F(U, \gamma) \), \( U/T \) arising up in Definition 3.1 such that the basic transmissions \( \pi_{E(U, \mathcal{A})} \), \( \pi_{F(U, \gamma)} \), \( \pi_T \) have a particularly well amenable appearance.

Conventions 3.3. Let \( U \) be a supertropical monoid, \( \mathcal{A} \) an ideal of \( U \) containing \( M := eU \), furthermore \( \gamma : M \to N \) a surjective homomorphism to a (bipotent) semiring \( N \) with \( \mathcal{A}_{U, \gamma} \subset M \) (cf. Definition 2.3), and \( T \) a tangible MFCE-relation on \( U \).

(a) We write \( \mathcal{A} := M \cup S \) with \( S \) a subset of \( T(U) \) such that \( S \cdot T(U) \subset S \cup M \). Justified by [IKR2, Theorem 6.16], adapted to the monoid setting, we declare that \( U/E(U, \mathcal{A}) \) is the subset

\[ U\setminus S = (T(U)\setminus S) \cup M \]
of \( U \), and, for any \( x \in M \)

\[ \pi_{E(U, \mathcal{A})}(x) = \begin{cases} x & \text{if } x \in U\setminus S, \\ e & \text{if } x \in S. \end{cases} \]
For \( x, y \in U \setminus S \) the product \( x \odot y \) in \( U/E(U, \mathfrak{A}) \) is given by

\[
    x \odot y = \begin{cases} 
        xy & \text{if } xy \not\in S, \\
        ey & \text{if } xy \in S.
    \end{cases}
\]

(b) We identify \( \mathcal{T}(U/F(U, \gamma)) \) with \( \mathcal{T}(U) \) such that \( \llbracket x \rrbracket_{F(U, \gamma)} = x \) for \( x \in \mathcal{T}(U) \). Now \( \mathcal{T}(U/F(U, \gamma)) = T(U) \cup N \) and, for \( x \in U \),

\[
    \pi_{F(U, \gamma)}(x) = \begin{cases} 
        x & \text{if } x \in \mathcal{T}(U), \\
        \gamma(x) & \text{if } x \in M.
    \end{cases}
\]

If \( x, y \in \mathcal{T}(U) \), the product \( x \odot y \) in \( U/F(U, \gamma) \) is given by

\[
    x \odot y = \begin{cases} 
        xy & \text{if } xy \in \mathcal{T}(U), \\
        \gamma(xy) & \text{if } xy \in M.
    \end{cases}
\]

(c) For \( x \in M \) we identify \( x \) with \( [x]_T \) (as we usually did for MFCE-relations before, but notice that now \( [x]_T = \{x\} \)). We have \( U/T = \mathcal{T}(U)/T \cup M \), and, for \( x \in U \),

\[
    \pi_{F(U, \gamma)}(x) = \begin{cases} 
        [x]_T & \text{if } x \in \mathcal{T}(U), \\
        x & \text{if } x \in M.
    \end{cases}
\]

If \( x, y \in \mathcal{T}(U) \) the product \( [x]_T \odot [y]_T \) in \( U/T \) is given by

\[
    [x]_T \odot [y]_T = \begin{cases} 
        [xy]_T & \text{if } xy \in \mathcal{T}(U), \\
        xy & \text{if } xy \in M.
    \end{cases}
\]

We also need more terminology on equivalence relations.

**Definition 3.4.**

(i) If \( \eta : X \rightarrow Y \) is a map between sets and \( F \) is an equivalence relation on \( Y \), then \( \eta^{-1}(F) \) denotes the equivalence relation on \( X \) given by

\[
    x_1 \sim_{\eta^{-1}(F)} x_2 \iff \eta(x_1) \sim_F \eta(x_2).
\]

Thus \( \pi_{\eta^{-1}(F)} = \pi_F \circ \eta \).

(ii) We further have a unique map

\[
    \bar{\eta} : X/\eta^{-1}(F) \rightarrow Y/F
\]

such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\pi_{\eta^{-1}(F)}} & X/\eta^{-1}(F) \\
\downarrow{\eta} & & \downarrow{\bar{\eta}} \\
Y & \xrightarrow{\pi_F} & Y/F
\end{array}
\]

commutes. We denote this map \( \bar{\eta} \) by \( \eta^F \), and then have the formula

\[
    \pi_F \circ \eta = \eta^F \circ \pi_{\eta^{-1}(F)}.
\]
Let \( E \) be again a transmission. We have the formula

\[
\rho
\]

Lemma 3.5.

Let \( V \rightarrow U \) be a transmission between supertropical monoids, and let \( E \) be a TE-relation on \( U \). Then \( \eta^{-1}(E) \) is a TE-relation on \( V \), and the induced map

\[
\eta^E : V/\eta^{-1}(E) \rightarrow U/E
\]

is again a transmission. We have the formula

\[
\pi_E \circ \eta = \pi^E \circ \eta^{-1}(E).
\]

This lemma already gives us the solution of Problem 3.2 for \( \alpha \) basic of type 4.

Proposition 3.6. Let \( U \) and \( V \) be supertropical monoids with \( eU = eV =: M \), and let \( \rho : V \rightarrow U \) be a tangible fiber contraction over \( M \) (e.g. \( \rho \) is an isomorphism over \( M \)).

(a) If \( T \) is a tangible MFCE-relation on \( U \), then \( \rho^{-1}(T) \) is a tangible MFCE-relation on \( V \) and

\[
\pi_T \circ \rho = \rho' \circ \pi_{\rho^{-1}(T)}
\]

with \( \rho' : V/\rho^{-1}(T) \rightarrow U/T \) the obvious homomorphism over \( M \) induced by \( \rho \), namely

\[
\rho' = \rho^T.
\]

(b) If \( \gamma : M \rightarrow N \) is a surjective homomorphism from \( M \) to a semiring \( N \) with

\[
a_{V,\gamma} := \{ x \in V \mid \gamma(ex) = 0 \} \subset M,
\]

hence also \( a_{V,\gamma} \subset M \), then

\[
\pi_{F(U,\gamma)} \circ \rho = \rho' \circ \pi_{F(U,\gamma)}
\]

with

\[
\rho' := \rho^{F(U,\gamma)} : V/F(V,\gamma) \rightarrow U/F(U,\gamma).
\]

\( \rho' \) is a tangible fiber contraction over \( N \) with the following explicit description:

Writing \( V/F(U,\gamma) = T(V) \uplus N \) and \( U/F(U,\gamma) = T(U) \uplus N \) (cf. Convention \[3.3b\]), we have \( \rho'(x) = \rho(x) \) if \( x \in T(U) \) and \( \rho'(x) = x \) if \( x \in N \).

(c) Let \( \mathfrak{A} \) be an ideal of \( U \) containing \( M \) and let \( \mathfrak{B} := \rho^{-1}(\mathfrak{A}) \), which is an ideal of \( V \) containing \( M \). Then

\[
\pi_{E(U,\mathfrak{A})} \circ \rho = \rho' \circ \pi_{E(V,\mathfrak{B})},
\]

with

\[
\rho' := \rho^{E(V,\mathfrak{B})} : V/E(V,\mathfrak{B}) \rightarrow U/E(U,\mathfrak{A}).
\]

\( \rho' \) is a tangible fiber contraction over \( N \), which has the following explicit description:

We write \( \mathfrak{A} = M \uplus S \) with \( S \subset T(U) \), and have \( \mathfrak{B} = M \uplus \rho^{-1}(S) \) with \( \rho^{-1}(S) \subset T(V) \). By Convention \[3.3a\]

\[
U/E(U,\mathfrak{A}) = (T(U) \setminus S) \uplus M,
\]

\[
V/E(V,\mathfrak{B}) = (T(V) \setminus \rho^{-1}(S)) \uplus M.
\]
The map \( \rho' \) is obtained from \( \rho \) by restriction to these subsets of \( U \) and \( V \). \{N.B. It is easy to check directly that \( \rho' \) respects multiplication.\}

(d) If \( \rho \) is an isomorphism, then in all three cases \( \rho' \) is again an isomorphism. Thus, if \( \beta : U \to W \) is a basic transmission of type \( i = 1, 2, 3 \) and \( \rho : V \to U \) is basic of type \( 4 \), then \( \beta \rho = \rho' \gamma \) with \( \beta', \rho' \) again of type \( i \) and \( 4 \) respectively.

Proof. Straightforward by use of Lemma \ref{lemma-3.5}. \( \square \)

Remark 3.7. If in Proposition \ref{prop-3.6} we dismiss the assumption that \( a_{V,\gamma} \subset M \), we have the same result, but with a slightly more complicated description of the tangible fiber contraction \( \rho' \) as follows: We now have natural identifications

\[
V/F(V, \gamma) = (T(V)\setminus a_{V,\gamma}) \cup N, \\
U/F(U, \gamma) = (T(U)\setminus a_{U,\gamma}) \cup N,
\]

and then

\[
\rho'(x) = \begin{cases}
\rho(x) & \text{if } x \in T(V)\setminus a_{V,\gamma} \\
x & \text{if } x \in N.
\end{cases}
\]

The following three propositions contain the solution of Problem \ref{problem-3.2} in the remaining cases \( i \leq j \leq 3 \). The stated canonical factorizations can always quickly be verified by inserting an element \( x \) of \( T(U) \) and comparing both sides. \{For \( x \in eU \) equality is always evident.\}

Often more conceptional proofs are also possible. With one exception we do not give the details.

Proposition 3.8. \( \text{(The case } i = j.\text{)} \)

Let \( U \) be a supertropical monoid and \( M := eU \).

(a) Assume that \( \mathfrak{A} = M \uplus S \) is an ideal of \( U \) containing \( M \) and \( \mathfrak{B} = M \uplus S' \) is an ideal of \( \overline{U} := U/E(U, \mathfrak{A}) = U\setminus S \) containing \( M \). \{Thus \( S \) and \( S' \) are disjoint subsets of \( T(U) \).\} Then

\[
\mathfrak{B} := \pi^{-1}_{E(U, \mathfrak{A})}(\mathfrak{B}) = M \uplus S \uplus S'
\]

is an ideal of \( U \) and

\[
\pi_{E(U, \mathfrak{B})} \circ \pi_{E(U, \mathfrak{A})} = \pi_{E(U, \mathfrak{B})}.
\]

(b) Let \( \gamma : M \to N \) and \( \delta : N \to L \) be surjective homomorphisms of bipotent semirings with \( a_{U,\gamma} \subset M \) and \( a_{V,\delta} \subset N \), where \( V := U/F(U, \gamma) \). Then

\[
\pi_{F(V, \delta)} \circ \pi_{F(U, \gamma)} = \pi_{F(U, \delta)}.
\]

(c) If \( T \) is a tangible MFCE-relation on \( U \) and \( T' \) is a tangible MFCE-relation on \( U/T \), then \( T' \circ T \) is again a tangible MFCE-relation and

\[
\pi_{T'} \circ \pi_T = \pi_{T' \circ T}.
\]

Proposition 3.9. \( \text{(The case } \alpha = \pi_T.\text{)} \)

Let \( U \) be a supertropical monoid, \( T \) a tangible MFCE-relation on \( U \), and

\[
\overline{U} := U/T = (T(U)/T) \uplus M.
\]

(a) If \( \gamma \) is a surjective homomorphism from \( M := eU \) to a semiring \( N \), and \( a_{U,\gamma} \subset M \), then

\[
\pi_{F(U, \gamma)} \circ \pi_T = \pi_{T'} \circ \pi_{F(U, \gamma)}.
\]
where $T'$ is the tangible MFCE-relation on $V := U/F(U, \gamma) = T(U) \cup N$ defined as follows. For any $x, y \in V$

$$x \sim_{T'} y \iff \begin{cases} 
\text{either} & x, y \in T(U) \text{ and } x \sim_T y, \\
\text{or} & x = y \in N.
\end{cases}$$

(b) Let $\mathcal{B}$ be an ideal of $U$ containing $M$, hence $\mathcal{B} = M \cup \mathcal{S}$ with $\mathcal{S}$ a subset of $T(U)/T$. We put $S := \pi_T^{-1}(\mathcal{S}) \subset T(U)$. Then

$$\pi_{E(U, \mathcal{S})} \circ \pi_T = \pi_{T'} \circ \pi_{E(U, \mathcal{B})},$$

with $\mathcal{A} := \pi_T^{-1}(\mathcal{B}) = M \cup S$, and $T'$ a tangible MFCE-relation on $V := U/E(U, \mathcal{A}) = (T(U)/S) \cup M$.

$T'$ is obtained from $T$ by restriction to the subset $U \setminus S$ of $U$. {Notice that $T(U)/S$ is a union of $T$-equivalence classes.}

**Proposition 3.10.** (The remaining case $j = 1, i = 2$.)

Let $U$ be a supertropical monoid and $\gamma$ a homomorphism from $M := eU$ onto a semiring $N$ with $a_{U, \gamma} \subset M$. Let $V := U/F(U, \gamma) = T(U) \cup N$.

(i) The ideals $\mathcal{A} \supset M$ of $U$ correspond uniquely with the ideals $\mathcal{B} \supset N$ of $V$ via $\mathcal{A} = \beta^{-1}(\mathcal{B})$, $\mathcal{B} = \beta(\mathcal{A})$, where $\beta := \pi_{F(U, \gamma)}$. We then have $\mathcal{A} = M \cup A$, $\mathcal{B} = N \cup S$ with the same set $S \subset T(U) = T(V)$, and $E(U, \mathcal{A}) = \beta^{-1}(E(V, \mathcal{B}))$. Finally

(ii) $\pi_{E(V, \mathcal{B})} \circ \pi_{F(U, \gamma)} = \pi_{T(U)} \circ \pi_{E(U, \mathcal{A})}$

with $\mathcal{U} := U/E(U, \mathcal{A}) = (T(U)/S) \cup M$.

**Proof.** i): The point is that for $S$ a subset of $T(U)$ we have $S \cdot T(U) \subset M$ in $U$ iff $S \cdot T(U) \subset N$ in $V$, since $\beta^{-1}(N) = M$. (Recall that we identified $T(U) = T(V)$.)

ii): Again just insert a given $x \in T(U)$ in both sides of the equation and compare. $\square$

**Summary 3.11.** If $\alpha : U \to V$ and $\beta : V \to W$ are basic transmissions, $\alpha$ of type $i$ and $\beta$ of type $j \leq i$, cf. Definition 3.11, then in case $i = j$ the transmission $\beta \alpha$ is again basic of type $i$, and otherwise $\beta \alpha = \alpha' \beta'$ with $\alpha'$ basic of type $i$ and $\beta'$ basic of type $j$, and the new basic transmissions can be determined from $\alpha$ and $\beta$ in an explicit way.

If $\alpha : U \to V$ and $\beta : V \to W$ are any transmissions with known canonical factors, then the canonical factorization of $\beta \alpha$ can be determined explicitly in at most $4 + 3 + 2 + 1 = 10$ steps.

4. THE SEMIRING ASSOCIATED TO A SUPERTROPICAL MONOID;
INITIAL TRANSMISSIONS

Let $U$ be a supertropical monoid and $M := eU$ its ghost ideal. We start out to convert $U$ into a supertropical semiring in a somewhat canonical way.

If $S$ is any subset of $U$, then the set $(US) \cup M$ is the smallest ideal of $U$ containing both $S$ and $M$. For convenience we introduce the notation

$$E(U, S) := E(U, US \cup M).$$

Corollary 3.19 tells us the meaning of this equivalence relation.
Scholium 4.1. A transmission \( \alpha : U \rightarrow V \) factors through the ideal compression \( \pi_{E(U,S)} \) (in a unique way), iff the ghost kernel \( \mathfrak{A}_\alpha \) contains the set \( S \).

We now define a subset \( S(U) \) of \( \mathcal{T}(U) \), for which the relation \( E(U,S(U)) \) will play a central role for most of the rest of the paper.

Definition 4.2.

(a) We call an element \( x \) of \( U \) an \textbf{NC-product} (in \( U \)), if there exist elements \( y, z \) of \( U \) and \( y' \) of \( M \) with

\[ x = yz, \quad y' < ey, \quad y'z = eyz. \]

Here the label “NC” alludes to the fact that we meet a non-cancellation situation in the monoid \( M \): We have \( y' \neq ey \), but \( y'z = eyz \).

(b) We denote the set of all NC-products in \( U \) by \( D_0(U) \) and the set \( D_0(U) \cap M \) by \( D(U) \).

We finally put

\[ S(U) := D(U) \setminus M = D_0(U) \cap \mathcal{T}(U). \]

This is the set of tangible NC-products in \( U \).

Clearly \( D_0(U) \cdot U \subset D_0(U) \). Thus \( D(U) \) is an ideal of \( U \) containing \( M \). We have

\[ E(U,S(U)) = E(U,D_0(U)) = E(U,D(U)). \]

Theorem \( \text{[1.2]} \) tells us that \( U \) is a semiring iff \( S(U) = \emptyset \), i.e., \( D(U) = M \).

We compare the set \( S(U) \) with \( S(V) \) for \( V \) an ideal compression of \( U \).

Lemma 4.3. Let \( \mathfrak{A} \) be an ideal of \( U \) containing \( M \), \( \mathfrak{A} = M \cup S \) with \( S \subset \mathcal{T}(U) \). We regard \( V := U/E(U, \mathfrak{A}) \) as a subset of \( U \), as explained in Convention \( \text{[3.3]} \). Then

\[ S(V) = S(U) \setminus S. \]

Proof. It is obvious from the description of \( V \) in Convention \( \text{[3.3]} \), that \( S(V) = S(U) \cap \mathcal{T}(V) \), and we have \( \mathcal{T}(V) = \mathcal{T}(U) \setminus S \).

Lemma 4.4. If \( T \) is a tangible MFCE-relation then

\[ S(U/T) = S(U)/T. \]

Proof. Look at the description of \( U/T \) in Convention \( \text{[3.3]} \).

Theorem 4.5.

(i) The supertropical monoid \( \widehat{U} := U/E(U, S(U)) \) is a semiring.

(ii) The ideal compression

\[ \sigma_U := \pi_{E(U,S(U))} : U \rightarrow \widehat{U} \]

is universal among all fiber contractions \( \alpha : U \rightarrow V \) with \( V \) a semiring. More precisely, given such a fiber contraction \( \alpha \), we have a (unique) fiber contraction \( \beta : \widehat{U} \rightarrow V \) with \( \alpha = \beta \circ \sigma_U \). \{N.B. If \( \alpha \) is a fiber contraction over \( M \), the same holds for \( \beta \).\}

Proof. (i): By Lemma \( \text{[4.3]} \), the set \( S(\widehat{U}) \) is empty; hence \( \widehat{U} \) is a semiring.

(ii): We may assume that \( \alpha : U \rightarrow V \) is a fiber contraction over \( M \), and then that \( \alpha = \pi_T \circ \pi_{E(U,\mathfrak{A})} \) with an ideal \( \mathfrak{A} \triangleright M \) of \( U \) and \( T \) a tangible equivalence relation on \( \overline{U} := U/E(U, \mathfrak{A}) \). By Lemma \( \text{[4.3]} \) the set \( S(V) \) is empty iff \( S(\overline{U}) \) is empty, and by Lemma \( \text{[4.3]} \)
this happens iff $S(U) \subset \mathfrak{A}$. Then $\pi_{E(U,\mathfrak{A})}$, and hence $\alpha$, factors through $\pi_{E(U,S(U))} = \sigma_U$ (cf. Scholium [4.1]). Conversely, if $\alpha = \beta \circ \sigma_U$ with $\beta : \hat{U} \to V$ a fiber contraction, then we know already by Theorem [1.6.ii] that $V$ is a semiring, since $\hat{U}$ is a semiring. □

We call $\hat{U}$ the semiring associated to the supertropical monoid $U$.

**Theorem 4.6.** Assume that $U$ is a supertropical semiring and $\gamma$ is a surjective homomorphism from $M := eU$ to a (bipotent) semiring $N$. Let $V := U/F(U,\gamma)$, which may be only a supertropical monoid. Then

$$\alpha := \sigma_V \circ \pi_{F(U,\gamma)} : U \to V \to \hat{V}$$

(with $\hat{V}$ and $\sigma_V$ as defined in the preceding theorem) is the initial transmission from $U$ to a supertropical semiring covering $\gamma$ (cf. [IKR2, Definition 1.3]). In the Notation 1.7 of [IKR2] this reads

$$\sigma_V \circ \pi_{F(U,\gamma)} = \alpha_U, \gamma.$$

**Proof.** Let $\beta : U \to W$ be a transmission to a supertropical semiring $W$ covering $\gamma$ (in particular, $eW = N$). Since $\pi_{F(U,\gamma)}$ is an initial transmission in the category $\text{STROP}_m$ covering $\gamma$, we have a (unique) transmission $\eta : V \to W$ over $N$, hence fiber contraction over $N$, with $\beta = \eta \circ \pi_{F(U,\gamma)}$.

Theorem 4.5 gives us a factorization $\eta = \varphi \circ \sigma_V$ with $\varphi : \hat{V} \to W$ again a fiber contraction over $N$. Then

$$\beta = \varphi \circ \sigma_V \circ \eta = \varphi \circ \alpha$$

is the desired factorization of $\beta$ in the category $\text{STROP}$. Of course, the factor $\varphi$ is unique, since $\alpha$ surjective. □

We want to find the canonical factorization of $\alpha_U, \gamma$. More generally we look for the canonical factors of

$$\alpha := \pi_{E(V,\mathfrak{B})} \circ \pi_{F(U,\gamma)}$$

with $V := U/F(U,\gamma)$ and $\mathfrak{B}$ an ideal of $V$ containing $N = eV$. We allow $U$ to be any supertropical monoid.

We write $\mathfrak{B} = N \cup S$ with $S \subset \mathcal{T}(V)$. Similarly to Convention 3.3b (which treats a special case) we have a natural identification

$$\mathcal{T}(V) = \mathcal{T}(U) \setminus \mathfrak{a}_{U, \gamma}$$

in such a way that for every $x \in U$

$$\pi_{F(U,\gamma)}(x) = \left\{ \begin{array}{ll} x & \text{if } x \in \mathcal{T}(U) \setminus \mathfrak{a}_{U, \gamma}, \\ \gamma(ex) & \text{otherwise} \end{array} \right.$$

We then obtain the following generalization of Proposition 3.10, arguing essentially in the same way as in [3]

**Lemma 4.7.** Let $V := U/F(U,\gamma)$ and $\beta := \pi_{F(U,\gamma)}$.

(i) The ideals $\mathfrak{B}$ of $V$ containing $N = eV$ correspond uniquely with the ideals $\mathfrak{A}$ of $U$ containing $M \cup \mathfrak{a}_{U, \gamma}$ via $\mathfrak{A} = \beta^{-1}(\mathfrak{B})$, $\mathfrak{B} = \beta(\mathfrak{A})$. Writing $\mathfrak{B} = N \cup S'$, with $S' \subset \mathcal{T}(V) = \mathcal{T}(U) \setminus \mathfrak{a}_{U, \gamma}$,
we have $\mathfrak{A} = M \cup S$ with

$$S := \{x \in T(U) \mid \gamma(ex) = 0\} \cup S' \subset T(U).$$

(ii) $\pi_{E(V,\mathfrak{A})} \circ \pi_{F(U,\gamma)} = \pi_{F(\mathfrak{U},\gamma)} \circ \pi_{E(U,\mathfrak{A})}$, with $\mathfrak{U} := U/E(U,\mathfrak{A}) = (T(U) \setminus S) \cup M$. \hfill $\square$

In the case $\mathfrak{B} = D(V)$ we have $S' = S(V)$. Thus the elements of $S'$ are the products $yz \in T(V) \subset T(U)$ with $\gamma(y') < \gamma(ey)$ and $\gamma(y'z) = \gamma(eyz)$ for some $y' \in M$. Notice that this forces $\gamma(y') \neq 0$.

**Definition 4.8.** Let $U$ be any supertropical monoid. We call an element $x$ of $U$ a $\gamma$-NC-product (in $U$), if there exist elements $y' \in M$, $y \in U$, $z \in U$ with $x = yz$ and $\gamma(y') < \gamma(ey)$, $\gamma(y'z) = \gamma(eyz)$. We denote the set of these elements $x$ by $D_0(U,\gamma)$ and the set $D_0(U,\gamma) \cap T(U)$ of tangible $\gamma$-NC-products by $S(U,\gamma)$.

Notice that $D_0(U,\gamma)$ is an ideal of $U$. We further define

$$D(U,\gamma) := M \cup D_0(U,\gamma) = M \cup S(U,\gamma),$$

which is an ideal of $U$ containing $M$.

In this terminology we have $S' = S(U,\gamma)$. If $U$ is a semiring, then we read off from Theorem 4.6 and Lemma 4.7 the following fact.

**Theorem 4.9.** Let $U$ be a semiring and $\gamma : eU = M \rightarrow N$ a surjective homomorphism from $M$ to a semiring $N$. Then $\alpha_{U,\gamma}$ has the canonical factorization

$$\alpha_{U,\gamma} = \pi_{F(\mathfrak{U},\gamma)} \circ \pi_{E(U,S)}$$

with

$$S = \{x \in T(U) \mid \gamma(ex) = 0\} \cup S(U,\gamma)$$

and $\mathfrak{U} = U/E(U,S)$.

It is now easy to write down the equivalence relation $E(\alpha_{U,\gamma}) = E(U,\gamma)$ (cf. Notation 1.7 in [IKR2]). We obtain

**Corollary 4.10.** For $U$ and $\gamma$ as above, the equivalence relation $E(U,\gamma)$ reads as follows ($x_1, x_2 \in U$):

$$x_1 \sim_{E(U,\gamma)} x_2 \iff \begin{cases} \text{either} & x_1 = x_2, \\ \text{or} & x_1, x_2 \in D(U,\gamma), \gamma(ex_1) = \gamma(ex_2), \\ \text{or} & \gamma(ex_1) = \gamma(ex_2) = 0. \end{cases}$$

$\square$

If $N$ is cancellative then $S(U,\gamma) = \emptyset$, and we fall back to the description of $E(U,\gamma)$ in [IKR2] Theorem 1.11.

Our arguments leading to Theorems 4.6 and 4.9 make sense if we only assume that $U$ is a supertropical monoid. To spell this out we introduce an extension of Notation 1.7 in [IKR2].

**Definition 4.11.** Let $U$ be a supertropical monoid with ghost ideal $M := eU$, and let $\gamma : M \rightarrow N$ be a surjective semiring homomorphism.

(i) We define $U_\gamma = \hat{V}$ with $V := U/F(U,\gamma)$. Thus $U_\gamma$ is a supertropical semiring.
Example 4.14.

Remark 4.13. Assume further that \( G \) is cancellative and then have \( U_\gamma = U/E(U, \gamma) \).

The arguments leading to Theorems 4.6 and 4.9 give more generally the following

**Theorem 4.12.**

(i) Given a transmission \( \beta : U \to W \) from a supertropical monoid \( U \) to a supertropical semiring \( W \) covering \( \gamma \) (in particular \( eW = N \)), there exists a unique semiring homomorphism \( \eta : U_\gamma \to W \) over \( N \) such that \( \beta = \eta \circ \alpha_{U, \gamma} \).

(ii) \( \alpha_{U, \gamma} \) has the same canonical factorization as given in Theorem 4.9 for \( U \) a semiring, and \( E(U, \gamma) \) has the description written down in Corollary 4.10.

\[ \square \]

Given a further semiring homomorphism \( \delta : N \to L \) we may ask whether there exists a transmission \( \eta : U_\gamma \to U_{\delta \gamma} \) covering \( \delta \). In other words, is \( E(U, \delta \gamma) \supseteq E(U, \gamma) \)?

In general the answer will be negative. Assume for simplicity that \( a_{U, \gamma} = a_{U, \delta \gamma} = \{0\} \) (or even, that \( \gamma^{-1}(0) = \{0\} \), \( \delta^{-1}(0) = \{0\} \)). We have to study the commutative diagram

\[
\begin{array}{ccc}
U & \xrightarrow{\alpha_{U, \gamma}} & U_{\gamma} \\
\downarrow \gamma & & \downarrow \delta \\
M & \xrightarrow{U/F(U, \gamma)} & U/F(U, \delta \gamma) \\
\downarrow \gamma & & \downarrow \delta \\
N & \xrightarrow{U/F(U, \gamma)} & L \\
\end{array}
\]

where the unadorned arrows are the obvious natural maps. Assume further that \( L \) is cancellative. Using Convention 3.3(b) we have

\[ T(U) = T(U/F(U, \gamma)) = T(U/F(U, \delta \gamma)) = T(U_{\delta \gamma}), \]

but \( T(U_\gamma) = T(U) \setminus S(U, \gamma) \). If \( \eta \) would exist then \( \eta \circ \alpha_{U, \gamma} \) would restrict to the identity on \( T(U) \). But this cannot happen as soon as \( S(U, \gamma) \) is not empty. In particular we realize the following:

**Remark 4.13.** If \( U \) is a semiring, \( \gamma^{-1}(0) = \{0\} \), but \( S(U, \gamma) \neq \emptyset \), and if there exist a homomorphism \( \delta : N \to L \) of semirings with \( L \) cancellative and \( \delta^{-1}(0) = \{0\} \), then the initial transmission \( \alpha_{U, \gamma} \) in STROP is **not** a pushout transmission.

It is not difficult to find cases where the situation described here is met.

**Example 4.14.**

(a) We choose a totally ordered abelian group \( G \) and a convex subgroup \( H \) of \( G \) with \( H \neq \{1\} \), \( H \neq G \). The group \( G/H \) is again totally ordered in a unique way such that the map \( G \to G/H, \ g \mapsto gH \), is order preserving. Thus we have bipotent semifields \( \bar{M} := G \cup \{0\} \) and \( \bar{L} := (G/H) \cup \{0\} \) at hand. Let

\[ A := \{a \in G \mid a < H\} \quad \text{and} \quad A/H := \{aH \mid a \in A\}. \]

Then

\[ M := A \cup H \cup \{0\}, \quad L := (A/H) \cup \{1 \cdot H\} \cup \{0\}. \]
are subsemirings of \( \hat{M} \) and \( \hat{L} \), and hence are cancellative bipotent semidomains.

(b) We construct a noncancellative bipotent semiring \( N \) as follows. As an ordered set, we put

\[
N := (A/H) \cup H \cup \{0\}
\]

with \( 0 < A/H < H \), keeping the given orderings on \( A/H \) and \( H \). We decree that the multiplication on \( N \) extends the given multiplication on \( A/H \) and \( H \), and, of course, \( 0 \cdot x = x \cdot 0 = 0 \) for all \( x \in N \), while \( (aH) \cdot h := aH \) for \( a \in A \), \( h \in H \). This multiplication clearly is associative and commutative, has the unit element \( 1 \in H \), and is compatible with the ordering on \( N \). Thus \( N \) can be interpreted as a supertropical semiring.

(c) We define maps \( \gamma : M \to N \) and \( \delta : N \to L \) by putting \( \gamma(0) := 0 \) and \( \delta(0) := 0 \), \( \gamma(a) := aH \), \( \gamma(h) := h \), \( \delta(aH) := aH \), \( \delta(h) := 1 \cdot H = 1_L \) for \( a \in A \), \( h \in H \). Clearly \( \gamma \) and \( \delta \) are order preserving surjective monoid homomorphisms, hence are surjective semiring homomorphisms. We have \( \gamma^{-1}(0) = \{0\} \), \( \delta^{-1}(0) = \{0\} \).

(d) We choose a homomorphism \( \hat{\nu} : \hat{T} \to G \) from an abelian group \( \hat{T} \) onto \( G \). Then, by \cite{IKR1} Construction 3.16], we have a supertropical semifield

\[
\hat{U} := \text{STR}(\hat{T}, G, \hat{\nu})
\]

at hand with \( \mathcal{T}(U) = \hat{T} \), \( G(\hat{U}) = G \), \( ex = \hat{\nu}(x) \) for \( x \in \hat{T} \). Let

\[
T := \hat{\nu}^{-1}(A \cup H),
\]

and let \( v : T \to A \cup H \) denote the monoid homomorphism obtained from \( \hat{\nu} \) by restriction. The subsemiring

\[
U := \text{STR}(T, A \cup H, v)
\]

of \( \hat{U} \) is a supertropical domain with ghost ideal \( M \) and \( \mathcal{T}(U) = T \).

(e) We take elements \( h_1 < h_2 \) in \( H \) and \( a \in A \). Then we take elements \( x_1, x_2, y \in T \) with \( v(x_1) = h_1 \), \( v(x_2) = h_2 \), \( v(y) = a \). Now we have

\[
\gamma(ex_1) = h_1 < \gamma(ex_2) = h_2 \quad \text{and} \quad \gamma(ex_1y) = \gamma(ex_2y) = aH.
\]

Also \( x_2y \in \mathcal{T}(U) \). Thus \( x_2y \in S(U, \gamma) \). Since for every \( h_2 \in H \) there exists some \( h_1 \in H \) with \( h_1 < h_2 \), this shows that

\[
v^{-1}(H) \cdot v^{-1}(A) \subseteq S(U, \gamma).
\]

In particular, \( S(U, \gamma) \neq \emptyset \). We conclude by Remark 4.13 that the initial transmission \( \alpha_{U, \gamma} : U \to U \gamma \) in \( \text{STROP} \) is not pushout in \( \text{STROP} \) (and all the more not pushout in \( \text{STROP}_m \)).

5. \( h \)-TRANSMISSIONS

In \cite{IKR2} §6 the equivalence homomorphic relations on a supertropical semiring \( U \) have been studied in detail. These are the TE-relations on \( U \) such that the supertropical monoid \( U/E \) is a semiring and \( \pi_E : U \to U/E \) is a homomorphism of semirings. It turned out that these relations can be completely characterized in terms of \( U \) as a supertropical monoid, cf. \cite{IKR2} Proposition 6.4, where the crucial compatibility of \( E \) with addition is characterized in this way.

Having this in mind we define “\( h \)-transmissions” for supertropical monoids,
Definition 5.1. We call a map \( \alpha : U \to V \) between supertropical monoids an **h-transmission**, if \( \alpha \) is a transmission and has also the following property

\[
\forall x, y \in U : \text{ If } ex < ey \text{ and } \alpha(ex) = \alpha(ey), \text{ then } \alpha(y) \in eV.
\]

We can read off the following result from [IKR2, Proposition 6.4]:

**Proposition 5.2.** Assume that \( U \) and \( V \) are supertropical semirings. Then a map \( \alpha : U \to V \) is an h-transmission iff \( \alpha \) is a semiring homomorphism.

**Remark 5.3.** We note in passing that in Definition 5.1 the condition can be formally relaxed as follows.

\[
\forall x, y \in U : \text{ If } 0 < ex < ey \text{ and } \alpha(ex) = \alpha(ey), \text{ then } \alpha(y) \in eV.
\]

Indeed if \( \alpha \) is a transmission and \( ex = 0 \), \( \alpha(ex) = \alpha(ey) \), we conclude right away that \( 0 = \alpha(ex) = \alpha(ey) \), hence \( \alpha(y) = 0 \in eV \).

We now study h-transmissions between supertropical monoids with the primary goal to gain a more insight into the variety of homomorphisms between supertropical semirings. If nothing else is said, letters \( U, V, W \) will denote supertropical monoids.

**Example 5.4.** Every transmission \( \alpha : U \to V \), such that \( \gamma := \alpha^r \) is injective on \( (eU) \setminus \{0\} \), is an h-transmission. Indeed, now the condition (HT) is empty.

The functorial properties of transmissions stated in Proposition 1.11 have a counterpart for h-transmissions.

**Proposition 5.5.** Let \( \alpha : U \to V \) and \( \beta : V \to W \) be maps between supertropical monoids.

(i) If \( \alpha \) and \( \beta \) are h-transmissions, then \( \beta \alpha \) is an h-transmission.

(ii) If \( \alpha \) and \( \beta \alpha \) are h-transmissions and \( \alpha \) is surjective, then \( \beta \) is an h-transmission.

**Proof.** By Proposition 1.11 we may already assume that \( \alpha \) and \( \beta \) are transmissions.

(i): Assume that \( x, y \in U \) are given with \( 0 < ex < ey \) and \( \beta \alpha(ex) = \beta \alpha(ey) \). We have to verify that \( \beta \alpha(y) \in eW \).

*Case 1:* \( \alpha(ex) = \alpha(ey) \). Now \( \alpha(y) \in eV \), since \( \alpha \) is an h-transmission. This implies \( \beta \alpha(y) \in eW \).

*Case 2:* \( \alpha(ex) < \alpha(ey) \). Since \( \beta \alpha(ex) = \beta \alpha(ey) \) and \( \beta \) is an h-transmission, again \( \beta \alpha(y) \in eW \).

(ii): Let \( x, y \in U \) be given with \( 0 < \alpha(ex) < \alpha(ey) \) and \( \beta \alpha(ex) = \beta \alpha(ey) \). Then \( 0 < ex < ey \). We conclude that \( \beta \alpha(y) \in eW \). Since \( \alpha \) is surjective and \( \alpha(ex) = e\alpha(x) \), \( \alpha(ey) = e\alpha(y) \), this proves that \( \beta \) is an h-transmission. \( \square \)

**Notations 5.6.**

(a) We introduce two new categories:

i) Let \( \text{STROPH}_m \) denote the category whose objects are the supertropical monoids and morphisms are the h-transmissions. Notice that this makes sense by Proposition 5.5.ii.

ii) Let \( \text{STROPH} \) denote the category whose objects are the supertropical semirings and morphisms are the semiring homomorphisms between supertropical semirings.
(b) We further denote by $\text{Sring}$ the category of all semirings and semiring homomorphisms.

$\text{STROPH}$ is a full subcategory of $\text{Sring}$ and, due to Proposition 5.2 also of $\text{STROPH}_m$. Thus we have the following chart of categories, where "⊂" means "subcategory" and "⊂_{\text{full}}" means "full subcategory"

\[
\begin{array}{ccc}
\text{STROP} & \subset_{\text{full}} & \text{STROP}_m \\
\cup & & \cup \\
\text{Sring} & \subset_{\text{full}} & \text{STROPH} & \subset_{\text{full}} & \text{STROPH}_m
\end{array}
\]

Moreover, in slightly symbolic notation,

\[
\text{STROPH} = \text{STROP} \cap \text{Sring} = \text{STROP}_m \cap \text{Sring} = \text{STROP} \cap \text{STROPH}_m.
\]

Our main concern will be to understand relations between $\text{STROPH}_m$ and $\text{STROP}$ within the category $\text{STROP}_m$, in order to get an insight into $\text{STROPH} \subset_{\text{AG}} \text{STROP} \subset_{\text{CG}} \text{Sring} \subset_{\text{AG}} \text{STROP} \subset_{\text{CG}} \text{STROPH}_m$.

**Theorem 5.7.** Assume that $\alpha : U \to V$ is a surjective $h$-transmission and $U$ is a semiring. Then $V$ is a semiring.

**Proof.** Let $M := eU$, $N := eV$, and $\gamma := \alpha^\circ$. We check the condition (Dis) in Theorem 1.2 for the supertropical monoid $V$. Since $\alpha$ and hence $\gamma$ is surjective, this means the following.

Let $y, z \in U$ and $y' \in M$ be given with $0 < \gamma(y') < \gamma(ey)$ and $\gamma(y'z) = \gamma(eyz)$. Verify that $\alpha(yz) \in N$!

We have $y' < ey$. If $y'z = eyz$ then $yz \in M$, since $U$ is a semiring, and we conclude that $\alpha(yz) \in N$.

There remains the case that $y'z < eyz$. Since $\alpha$ is an $h$-transmission, we conclude again that $\alpha(yz) \in N$. $\square$

In the following we assume that $U$ is a supertropical monoid and $\gamma$ is a homomorphism from $M := eU$ onto a (bipotent) semiring $N$. We look for $h$-transmissions $\alpha : U \to V$ which cover $\gamma$. We introduce the set

$$
\Sigma_0(U, \gamma) := \{x \in T(U) \mid \exists x_1 \in M : x_1 < ex, \gamma(x_1) = \gamma(ex) \neq 0\}.
$$

**Proposition 5.8.** A transmission $\alpha : U \to V$ covering $\gamma$ is an $h$-transmission iff the ghost kernel $\mathfrak{A}_\alpha$ contains the set $\Sigma_0(U, \gamma)$.

**Proof.** By Scholium 4.1 it is evident that $\Sigma_0(U, \gamma) \subset \mathfrak{A}_\alpha$ iff $\alpha$ obeys the condition (HT') from above. $\square$

We further introduce the set

$$
\Sigma(U, \gamma) := \Sigma_0(U, \gamma) \cup \{x \in T(U) \mid \gamma(ex) = 0\},
$$

and the supertropical monoids

$$
\overline{U} := U/E(U, \Sigma(U, \gamma)),
$$

$$
U^h_\gamma := \overline{U}/F(\overline{U}, \gamma),
$$

finally the transmission $\alpha^h_{U, \gamma} : U \to U^h_\gamma$, given by

$$
\alpha^h_{U, \gamma} := \pi_E(\overline{U}, \gamma) \circ \pi_E(U, \Sigma(U, \gamma)) \cdot (*).
$$
Notice that $\alpha_{U,\gamma}^h$ covers $\gamma$ and is the product of an ideal compression and a strict ghost contraction, so that \((*)\) gives already the canonical factorization of $\alpha_{U,\gamma}^h$. {N.B. The ideal $U \cdot \Sigma(U, \gamma)$ contains $a_{U,\gamma}$, hence $a_{U,\gamma} \subseteq M$.}

The ghost kernel of $\alpha_{U,\gamma}^h$ contains the set $\Sigma_0(U, \gamma)$, and thus we know by Proposition 5.8 that $\alpha_{U,\gamma}^h$ is an $h$-transmission.

We call $\alpha_{U,\gamma}^h$ a **pushout in the category** STROP$_m$, since the following holds:

**Theorem 5.9.** Assume that $\delta : N \to L$ is a surjective homomorphism from $N$ to a semiring $L$ and $\beta : U \to W$ is an $h$-transmission covering $\delta \gamma$ (in particular $eW = L$). Then there exists a (unique) $h$-transmission $\eta : U^h_\gamma \to W$ covering $\delta$ such that $\beta = \eta \circ \alpha_{U,\gamma}^h$.

**Proof.** Let $\alpha := \alpha_{U,\gamma}^h$ and $\lambda := \pi_{E(U,\Sigma(U,\gamma))}$. We retain the notations from above, hence have $U^h_\gamma = \overline{U}/F(U, \gamma)$ with $\overline{U} = U/E(U, \Sigma(U, \gamma))$.

Now observe that $\Sigma(U, \gamma)$ is contained in $\Sigma(U, \delta \gamma)$. Indeed, let $x \in \Sigma(U, \gamma)$. If $\gamma(ex) = 0$, then $\delta \gamma(ex) = 0$. If there exists some $x_1 \in M$ with $x_1 < ex$ and $\gamma(x_1) = \gamma(ex)$, then either $\delta \gamma(ex) = 0$, or $\delta \gamma(ex) \neq 0$, and then $x \in \Sigma_0(U, \delta \gamma)$. Thus $x \in \Sigma(U, \delta \gamma)$ in all cases.

Since $\beta$ is an $h$-transmission covering $\delta \gamma$, the ghost kernel of $\beta$ contains $\Sigma(U, \delta \gamma)$ and hence $\Sigma(U, \gamma)$. Thus we have a factorization

$$
\beta : U \xrightarrow{\lambda} \overline{U} \xrightarrow{\beta} W
$$

with $\beta$ a transmission again covering $\delta \gamma$.

We have a commuting diagram (solid arrows)

```
\begin{array}{cccccc}
U & \xrightarrow{\alpha} & U^h_\gamma & \xrightarrow{\eta} & W \\
\downarrow{\lambda} & & \downarrow{\gamma} & & \downarrow{\delta} \\
M & \xrightarrow{\gamma} & N & \xrightarrow{\delta} & L
\end{array}
```

Since $\pi_{F(U, \gamma)}$ is a pushout in the category STROP$_m$ (Theorem 1.13), we have a transmission $\eta : U^h_\gamma \to W$ covering $\delta$ such that $\eta \circ \pi_{F(U, \gamma)} = \overline{\beta}$, hence $\eta \circ \alpha = \overline{\beta} \circ \lambda = \beta$. Since both $\alpha$ and $\beta$ are $h$-transmissions, also $\eta$ is an $h$-transmission (Proposition 5.3i).

**Corollary 5.10.** (Entering the category STROPH.) Let $V := U^h_\gamma$ and $\alpha := \alpha_{U,\gamma}^h : U \to V$. Then

$$
\hat{\alpha} := \sigma_V \circ \alpha : U \to \hat{V}
$$

is the initial $h$-transmission from $U$ to a semiring covering $\gamma$, i.e., given an $h$-transmission $\beta : U \to W$ covering $\gamma$ with $W$ a semiring, there exists a (unique) fiber contraction $\zeta : \hat{V} \to W$ over $N$ with $\beta = \varphi \circ \hat{\alpha}$.

**Proof.** By Theorem 5.9, applied with $\delta = \text{id}_N$, we have fiber contraction $\eta : V \to W$ such that $\beta = \eta \circ \alpha$. By Theorem 4.5 there exists a fiber contraction $\zeta : \hat{V} \to W$ over $N$ such that $\eta = \zeta \circ \sigma_V$. Thus $\beta = \zeta \circ \sigma_V \circ \alpha = \zeta \circ \hat{\alpha}$. □
**Theorem 5.11.** Assume that \( \alpha : U \to V \) is a surjective \( h \)-transmission, and
\[
\alpha : U \xrightarrow{\lambda} \overline{U} \xrightarrow{\beta} W \xrightarrow{\mu} \overline{W} \xrightarrow{\rho} V
\]
is the canonical factorization of \( \alpha \) (cf. Theorem 1.2).

(i) The factors \( \lambda, \beta, \mu, \rho \) are again \( h \)-transmissions.

(ii) If \( U \) is a semiring, then the supertropical monoids \( U, W, \overline{W}, V \) are semirings, and the maps \( \lambda, \beta, \mu, \rho \) are semiring homomorphisms.

*Proof.* i): We know already by Example 5.4 that \( \lambda, \mu, \rho \) are \( h \)-transmissions since they cover the identities \( \text{id}_M \) and \( \text{id}_L \) respectively (and \( \rho \) is even an isomorphism). We have \( \overline{U} = U/E(U, \Sigma(U, \gamma)) \). Now observe that, if \( a' \) and \( a \) are elements of \( M \) with \( a' < a, \gamma(a') = \gamma(a) \neq 0 \), then the fiber \( \overline{U}_a = \gamma^{-1}(a) \) contains no tangible elements. \( \{ \text{Recall the definition of the set } \Sigma_0(U, \gamma) \subset \Sigma(U, \gamma) \} \) Thus every transmission \( \beta' : U \to W' \) covering \( \gamma \) trivially obeys the condition (HT') from above (Remark 5.3), hence is an \( h \)-transmission. In particular, \( \beta \) is an \( h \)-transmission.

ii): If \( U \) is a semiring, we conclude by Theorem 5.13 successively, that \( U, W, \overline{W}, V \) are semirings. Now invoke Proposition 5.2 to conclude that \( \lambda, \beta, \mu, \rho \) are semiring homomorphisms. \( \square \)

We strive for an explicit description of the initial \( h \)-transmission \( \alpha^h_{U, \gamma} \) covering \( \gamma \). Let \( \mathcal{H}(U, \gamma) \) denote the ideal of \( U \) generated by \( \Sigma(U, \gamma) \cup M \), i.e.,
\[
\mathcal{H}(U, \gamma) := (U \cdot \Sigma_0(U, \gamma)) \cup a_{U, \gamma} \cup M.
\]
We have
\[
\alpha^h_{U, \gamma} = \pi_{F(U, \gamma)} \circ \pi_{E(U, \mathcal{H})}
\]
with \( \mathcal{H} := \mathcal{H}(U, \gamma) \) and \( U = U/E(U, \mathcal{H}) \). Invoking Example 2.15 we learn that
\[
\alpha^h_{U, \gamma} = \pi_{E(U, \mathcal{H}(U, \gamma), \gamma)}.
\]
We denote the equivalence relation \( E(U, \mathcal{H}(U, \gamma), \gamma) \) more briefly by \( H(U, \gamma) \).

Our task is to describe this TE-relation explicitly. We will succeed if \( U \) is a semiring.

**Lemma 5.12.** If \( U \) is a semiring, then
\[
\mathcal{H}(U, \gamma) = \Sigma_0(U, \gamma) \cup a_{U, \gamma} \cup M.
\]

*Proof.* \( \mathcal{H}(U, \gamma) \) contains the set on the right hand side. We are done, if we verify that a product \( xy \) with \( x \in \Sigma_0(U, \gamma), y \in T(U) \), \( xy \in T(U) \backslash a_{U, \gamma} \) lies in \( \Sigma_0(U, \gamma) \).

We have \( x \in T(U) \). By definition of \( \Sigma_0(U, \gamma) \) there exists some \( x' \in M \) with \( x' < ex \) and \( \gamma(x') = \gamma(ex) \neq 0 \). Now \( x'y \leq exy \), but equality here would imply that \( xy \in M \), since \( U \) is a semiring (cf. Theorem 1.2). Thus \( x'y < exy \). Further \( \gamma(x'y) = \gamma(exy) \neq 0 \). This shows that indeed \( xy \in \Sigma_0(U, \gamma) \). \( \square \)

Starting from this lemma and the general description of the relations \( E(U, \mathfrak{A}, \gamma) \) in Theorem 1.17 it is now easy to write out the TE-relation \( H(U, \gamma) \). We obtain a theorem which runs completely in the category STROPH.

**Theorem 5.13.** Assume that \( U \) is a supertropical semiring. The initial semiring homomorphism \( \alpha^h_{U, \gamma} \) covering \( \gamma \) is the map
\[
\pi_{H(U, \gamma)} : U \to U/H(U, \gamma)
\]
corresponding to the following equivalence relation \( H(U, \gamma) \) on \( U \):

If \( x_1, x_2 \in U \), then \( x_1 \sim_{H(U, \gamma)} x_2 \iff \gamma(ex_1) = \gamma(ex_2) \) and either \( x_1 = x_2 \), or \( x_1, x_2 \in M \cup \Upsilon_0(U, \gamma) \), or \( \gamma(ex_1) = 0 \).

\[ \square \]

6. Ordered supertropical monoids

In the paper [IKR3] the present authors studied supervaluations with values in a “totally ordered supertropical semiring” [IKR3, Definition 3.1] and obtained – as we believe – natural and useful examples of such supervaluations. This motivates us now to define “ordered supertropical monoids”.

**Definition 6.1.** An ordered supertropical monoid, or \( \text{OST} \)-monoid for short, is a supertropical monoid \( U \) equipped with a total ordering \( \leq \) of the set \( U \), such that the following hold:

\begin{align*}
\text{(OST 1)}: & \quad \text{The ordering } \leq \text{ is compatible with multiplication, i.e., for } x, y, z \in U, \ x \leq y \Rightarrow xz \leq yz; \\
\text{(OST 2)}: & \quad \text{The ordering } \leq \text{ extends the natural total order of the bipotent semiring } M := eU, \ i.e., if } x, y \in M, \text{ then } x \leq y \iff x \leq_M y; \\
\text{(OST 3)}: & \quad 0 \leq 1 \leq e.
\end{align*}

**Lemma 6.2.** Let \( x \in U \).

(a) Then \( 0 \leq x \leq ex \).

(b) If \( x \in T(U) \), then \( x < ex \).

**Proof.** (a): This follows by multiplying the inequality \( 0 \leq 1 \leq e \) by \( x \).

(b): We have \( x \leq ex \) and \( x \neq ex \); hence \( x < ex \). \[ \square \]

As common, we call a subset \( C \) of a totally ordered set \( X \) convex (in \( X \)) if for all \( x, y \in C, \ z \in X \) with \( x \leq z \leq y \) also \( z \in C \). (This definition still makes sense if \( X \) is only partially ordered, but now we do not need this generality.)

**Proposition 6.3.**

(a) For every \( c \in M \setminus \{0\} \) both the fiber \( U_c := \nu_U^{-1}(c) \) and the tangible fiber \( T(U)_c := T(U) \cap U_c \) are convex in \( U \).

(b) If \( c, d \in M \) and \( c < d \), then

\[ c \in T(U)_d < d \]

(i.e., \( c < x < d \) for every \( x \in T(U)_d \)).

**Proof.** (a): Let \( x, y \in U_c, \ z \in U, \) and \( x \leq z \leq y \). We can conclude from \( c = ex \leq ez \leq ey = c \) that \( ez = c \); hence \( z \in U_c. \) Now assume that in addition \( x, y \in T(U) \). If \( z \) were ghost, hence \( ez = c \), it would follow by Lemma 6.2.b that \( y < ey = z \). Thus \( z \in T(U) \).

(b): Let \( x \in T(U)_d \). Then \( x < ex = d \), by Lemma 6.2.b. Suppose \( x \leq c \). Then it would follow that \( ex = d \leq c \), which is not true. Thus \( c < x \). \[ \square \]

**Theorem 6.4.** If \( (U, \leq) \) is an \( \text{OST} \)-monoid, then \( U \) is a semiring.
Proof. We verify condition (Dis) in Theorem 1.2. Let \( x, y \in U, x' \in M \), and assume that \( x' < ex \), but \( x'y = exy \). From \( x' < ex \) we conclude by Proposition 6.3 that \( x' \leq x \). Furthermore \( x \leq ex \) by Lemma 6.2.a. Multiplying by \( y \), we obtain
\[
x'y \leq xy \leq exy \leq x'y,
\]
and we conclude that \( xy = exy \). \( \square \)

**Theorem 6.5.** If \( (U, \leq) \) is an OST-monoid, then addition\(^3\) in the semiring \( U \) is compatible with the ordering \( \leq \), i.e., \( (x, y, z \in U) \)
\[
x \leq y \implies x + z \leq y + z.
\]

**Proof.** We conclude from \( x \leq y \) that \( ex \leq ey \). We distinguish the cases \( ex < ey \) and \( ex = ey \), and go through various subcases.

**Case 1:** \( ex < ey \).

(a) If \( ez \leq ex \), then \( e(x + z) = ex + ez = ex \) and \( y + z = y \). Since \( ex < ey \), we conclude that \( x + z \leq e(x + z) < y + z \) (cf. Proposition 6.3.b).

(b) If \( ex < ez < ey \), then \( x + z = z, y + z = y \), and we conclude from \( ez < ey \) that \( x + z < y + z \).

(c) If \( ex = ey \), then \( x + z = z, y + z = ey \). Since \( z \leq ez \) we obtain that \( x + z \leq y + z \).

(d) If \( ey < ez \), then \( x + z = z, y + z = z \), hence \( x + z = y + z \).

**Case 2:** \( ex = ey \).

(a) If \( z < ex \), then \( x + z = x, y + z = y \).

(b) If \( z = ex \), then \( x + z = ex, y + z = ey \).

(c) If \( ex < z \), then \( x + z = y + z \).

Thus in all three cases \( x + z \leq y + z \). \( \square \)

Starting from now we denote an OST-monoid \( (U, \leq) \) by the single letter \( U \). From Theorems 6.4 and 6.5 it is obvious that the present OST-monoids are the same objects as the totally ordered supertropical semirings defined in [IKR3, Definition 3.1]. Examples of these structures can be found in [IKR3, §3, §4, §6].

**Definition 6.6** \((= [IKR3, Definition 5.1])\). Assume that \( U \) and \( V \) are OST-monoids. We call a transmission \( \alpha : U \to V \) (cf. Definition 1.4) **monotone**, if \( \alpha \) is compatible with the ordering on \( U \) and \( V \), i.e.,
\[
\forall x, y \in U : \ x \leq y \implies \alpha(x) \leq \alpha(y).
\]

**Theorem 6.7** \((\text{cf. [IKR3, Theorem 5.3]}\)). Every monotone transmission \( \alpha : U \to V \) is a semiring homomorphism.

**Proof.** We verify condition (HT) in Definition 5.1 and then will be done by Proposition 5.2.

Let \( x, y \in U \) with \( ex < ey \) and \( \alpha(ex) = \alpha(ey) \). By Proposition 6.3 we have \( ex < y < ey \). Applying \( \alpha \), we obtain
\[
\alpha(ex) \leq \alpha(y) \leq \alpha(ey) = \alpha(ex),
\]
hence \( \alpha(ex) = \alpha(ey) \). But \( \alpha(ey) = e\alpha(y) \) (cf. Definition 5.1), and we conclude that \( \alpha(y) \in eV \), as desired. \( \square \)

\(^3\)Recall the formulas for \( x + y \) in \([1] \) preceding Theorem 1.2.
Another proof, which relies more on the semiring structure of $U$ and $V$, can be found in [IKR3, §5].

**Example 6.8.** Every bipotent semiring can be regarded as an OST-monoid. (This is the case $1 = e$.) If $U$ is an OST-monoid, $M = eU$ (our present overall assumption), then $\nu_U : U \to M$ is a monotone transmission.

We indicate a way how to obtain new OST-monoids from given ones. First we quote a general fact about total orderings (cf. e.g. [IKR2, Remark 4.1]).

**Lemma/Definition 6.9.** Let $X$ be a totally ordered set and $f : X \to Y$ a map from $X$ onto a set $Y$. Then there exists a (unique) total ordering on $Y$, such that $f$ is order preserving, iff all fibers $f^{-1}(y), y \in Y$, are convex in $X$. We call this total ordering the **ordering on $Y$ induced by $f$**.

N.B. This ordering on $Y$ can be characterized as follows: For $x_1, x_2 \in X$

$$f(x_1) < f(x_2) \implies x_1 < x_2 \implies f(x_1) \leq f(x_2).$$

Alternatively, we can state:

- If $x_1 < x_2$, then $f(x_1) \leq f(x_2)$,
- If $x_1 > x_2$, then $f(x_1) \geq f(x_2)$.

**Theorem 6.10.** Assume that $U$ is an OST-monoid, $V$ is a supertropical monoid, and $\alpha : U \to V$ is a surjective transmission. Assume further that for every $p \in V$ the fiber $\alpha^{-1}(p)$ is convex in $U$. Then $V$, equipped with the total ordering induced by $\alpha$, is again an OST-monoid.

**Proof.** We verify the axioms OST 1-OST 3 in Definition 6.1 for the induced ordering $\leq_V$ on $V$.

(OST 1) : Let $x, y, z \in U$ and $\alpha(x) < \alpha(y)$. Then $x < y$, hence $xz \leq yz$, hence

$$\alpha(x)\alpha(z) = \alpha(xz) \leq_V \alpha(yz) = \alpha(y)\alpha(z).$$

(OST 2) : Let $M := eU$, $N := eV$. On $U$ and $V$ we have the given orderings $\leq_U$, $\leq_V$, and on $M$ and $N$ we have the natural orderings $\leq_M$, $\leq_N$ as bipotent semirings. The ordering $\leq_U$ restricts on $M$ to $\leq_M$. We have to verify that $\leq_V$ restricts on $N$ to $\leq_N$.

The map $\alpha : U \to V$ restricts to a semiring homomorphism $\gamma : M \to N$, which consequently is compatible with $\leq_M$ and $\leq_N$. Let $x, y \in M$. If $\alpha(x) <_V \alpha(y)$ then $x <_U y$, hence $x <_M y$, hence $\gamma(x) <_N \gamma(y)$.

Conversely, if $\gamma(x) <_N \gamma(y)$, then $x <_M y$, hence $x <_U y$, hence $\alpha(x) \leq_V \alpha(y)$. Thus

$$\alpha(x) \leq_V \alpha(y) \implies \gamma(x) \leq_N \gamma(y).$$

This proves that indeed the ordering $\leq_V$ restricts to $\leq_N$ on $N$.

(OST 3) : Applying $\alpha$ to $0 \leq 1 \leq e$ in $U$, we obtain $0 \leq 1 \leq e$ in $V$. \hfill $\square$

We are ready for the main result of this section, which roughly states that, given a monotone transmission $\alpha : U \to V$, the canonical factors of $\alpha$ may be viewed as monotone transmissions in a unique way. We will rely on three easy lemmas.
Lemma 6.11. Assume that $U, V, W$ are OST-monoids, and $\alpha : U \to V$, $\beta : V \to W$ are transmissions. Assume further that $\alpha$ and $\beta \circ \alpha$ monotone and $\alpha$ surjective. Then $\beta$ is monotone.

Proof. If $x, y \in U$ and $\alpha(x) < \alpha(y)$, then $x < y$, hence $\beta \alpha(x) \leq \beta \alpha(y)$. Thus $\alpha(x) \leq \alpha(y)$ implies $\beta(\alpha(x)) \leq \beta(\alpha(y))$. \hfill \square

Lemma 6.12. Assume that $\alpha : U \to V$ is a monotone transmission (between OST-monoids). Let $\mathcal{A}$ denote the ghost kernel of $\alpha$, $\mathcal{A} = \mathcal{A}_c$. Then for any $c \in eU$ the fiber $\mathcal{A}_c := \mathcal{A} \cap U_c$ is an upper set of the totally ordered set $U_c$.

Proof. Assume that $x \in \mathcal{A}_c$, $y \in U_c$, and $x < y$. Then $x < y \leq c$, hence $\alpha(x) \leq \alpha(y) \leq \alpha(c)$. Since $x$ lies in the ghost kernel $\mathcal{A}$ of $\alpha$ we have

$$\alpha(x) = e\alpha(x) = \alpha(ex) = \alpha(c).$$

It follows that $\alpha(y) = \alpha(c) \in eV$, hence $y \in \mathcal{A}$, hence $y \in \mathcal{A}_c$. \hfill \square

Lemma 6.13. Assume that $\alpha : U \to V$ is a monotone transmission with trivial ghost kernel. Let $\gamma = \alpha' : M \to N$ denote the ghost part of $\alpha$. Then $U_c = \{c\}$ for any $c \in M$ such that there exists some $c_1 < c$ in $M$ with $\gamma(c_1) = \gamma(c)$.

Proof. Precisely this has been verified in the proof of Theorem 6.7. \hfill \square

Theorem 6.14. Assume that $U, V$ are an OST-monoids, and $\alpha : U \to V$ is a surjective monotone transmission. Assume further that

$$\alpha : U \xrightarrow{\lambda} \overline{U} \xrightarrow{\beta} W \xrightarrow{\rho} \overline{W} \xrightarrow{\rho} V$$

is a canonical factorization (cf. 3.2) of the transmission $\alpha$. Then the monoids $\overline{U}, \overline{W}, W$ can be equipped with total orderings (in a unique way), such that they become OST-monoids and all factors $\lambda, \beta, \mu, \rho$ are monotone transmissions.

Proof. a) Let $\gamma := \alpha' : M \to N$ denote the ghost part of the transmission $\alpha$ and $\mathcal{A}$ denote the ghost kernel of $\alpha$. Without loss of generality we may assume that

$$\overline{U} = U/E(U, \mathcal{A}), \quad \lambda = \pi_{E(U, \mathcal{A})}, \quad W = \overline{U}/F(\overline{U}, \gamma),$$

$$\beta = \pi_{F(\overline{U}, \gamma)}, \quad \overline{W} = V, \quad \rho = \text{id}_V.$$

For any $c \in M$ we have $\lambda^{-1}(c) = \mathcal{A}_c$, which by Lemma 6.12 is an upper set $U_c$, hence is convex in $U_c$. Since $U_c$ is convex in $U$, it follows that $\lambda^{-1}(c)$ is convex in $U$.

Invoking Lemma 6.9 we equip the monoid $\overline{U}$ with the total ordering induced by $\lambda$, and then know by Theorem 6.10 that $\overline{U}$ has become an OST-monoid and $\lambda$ has become a monotone transmission. By Lemma 6.11 also $\mu \circ \beta : \overline{U} \to V$ is monotone.

b) Replacing $U$ by $\overline{U}$, we are allowed to assume henceforth that $\alpha : U \to V$ has trivial ghost kernel, and may focus on the canonical factorization $\alpha = \mu \circ \beta$ with $\beta = \pi_{F(\overline{U}, \gamma)}$ and $\mu : W \to V$ a tangible fiber contraction.

We use the identifications in Convention 3.3b to handle $W = U/F(\overline{U}, \gamma)$ and $\beta = \pi_{F(\overline{U}, \gamma)}$.

For any $d \in N = eW$ the tangible fiber $\mathcal{T}(W)_d$ is the union of all fibers $\mathcal{T}(U)_c$ with $c \in \gamma^{-1}(d)$. Let $L(\gamma)$ denote those $c \in M$ such that $c \neq 0$ and $c$ is the smallest element of $\gamma^{-1}(\gamma(c))$. Lemma 6.13 tells us that $\mathcal{T}(U)_c \neq \emptyset$ if $c \in M \setminus L(\gamma)$. Thus we have the following picture: If $d \in N$ then $\mathcal{T}(W)_d = \mathcal{T}(U)_c$ if there exists $c \in L(\gamma)$ with $\gamma(c) = d$, and this $c$ is then unique. Otherwise $\mathcal{T}(W)_d = \emptyset$. 

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c) Looking again at Convention 3.3.b we see that \( \beta \) has the following fibers: If \( p \in T(W)_d \), \( d = \gamma(c) \) with \( c \in L(\gamma) \), then \( \beta^{-1}(p) = \{p\} \) (using the identifications in Convention 3.3.b). If \( d \in N \), then \( \alpha^{-1}(d) = \gamma^{-1}(d) \). Recalling Proposition 6.3, we see that \( \gamma^{-1}(d) \) is convex in \( U \). Thus all fibers of \( \beta \) are convex in \( U \).

Invoking again Lemma 6.9 and Theorem 6.10, we equip \( W \) with that total ordering induced by \( \beta \), which makes \( W \) an OST-monoid and \( \beta \) a monotone transmission. By Lemma 6.11, we conclude that also \( \mu \) is a monotone transmission. \( \square \)

7. \( m \)-SUPervaluations

**Definition 7.1.** Let \( R \) be a semiring. An \( m \)-supervaluation on \( R \) is a map \( \phi : R \to U \) to a supertropical monoid which fulfills the axioms SV1-SV4 required for a supervaluation in [IKR1, Definition 4.1], there for \( U \) a supertropical semiring. To repeat,

\[
\begin{align*}
SV1: & \quad \phi(0) = 0, \\
SV2: & \quad \phi(1) = 1, \\
SV3: & \quad \forall a, b \in R : \phi(ab) = \phi(a)\phi(b), \\
SV4: & \quad \forall a, b \in R : e\phi(a + b) \leq e(\phi(a) + \phi(b)) \quad (= \max(e\phi(a), e\phi(b))).
\end{align*}
\]

We then say that \( \phi \) covers the \( m \)-valuation

\[
e\phi : R \to eU, \quad a \mapsto e\phi(a).
\]

Most notions developed for supervaluations in [IKR1, IKR2, §2] make sense for \( m \)-supervaluations in the obvious way and will be used here without further explanation, but we repeat the definition of dominance.

**Definition 7.2.** Assume that \( \phi : R \to U \) and \( \psi : R \to V \) are \( m \)-supervaluations. We say that \( \phi \) dominates \( \psi \) and write \( \phi \triangleright \psi \), if for all \( a, b \in R \) the following holds.

\[
\begin{align*}
D1. & \quad \phi(a) = \phi(b) \implies \psi(a) = \psi(b), \\
D2. & \quad e\phi(a) \leq e\phi(b) \implies e\psi(a) \leq e\psi(b), \\
D3. & \quad \phi(a) \in eU \implies \psi(a) \in eV.
\end{align*}
\]

If \( \phi : R \to U \) is an \( m \)-supervaluation and \( \alpha : U \to V \) is a transmission, then clearly \( \alpha \circ \phi \) is an \( m \)-supervaluation dominated by \( \phi \).

Conversely, if \( \phi : R \to U \) is an \( m \)-supervaluation which is surjective (i.e., \( U = \phi(R) \cup e\phi(R) \), cf. [IKR1, Definition 4.3]), and \( \psi : R \to V \) is an \( m \)-supervaluation dominated by \( \phi \), there exists a (unique) transmission \( \alpha : U \to V \) with \( \psi = \alpha \circ \phi \). This can be proved in exactly the same way as done in [IKR1, §5] for supervaluations. If \( \phi \) and \( \psi \) cover the same \( m \)-valuation \( v : R \to M \), then \( \alpha \) is a fiber contraction over \( M \) (hence an \( h \)-transmission).

Let now \( v : R \to M \) be a fixed \( m \)-valuation. We call any \( m \)-supervaluation \( \phi \) with \( e\phi = v \) an \( m \)-cover of \( v \). We call two \( m \)-covers \( \phi, \psi : R \to U \) and \( \psi : R \to V \) equivalent, if \( \phi \triangleright \psi \) and \( \psi \triangleright \phi \). If \( \phi \) and \( \psi \) are surjective this means that \( \psi = \alpha \circ \phi \) with \( \alpha : U \to V \) an isomorphism over \( M \).

We further denote the equivalence class of an \( m \)-cover \( \phi \) of \( v \) by \( [\phi] \), and the set of all these classes by \( \text{Cov}_m(\varphi) \). This set is partially ordered by declaring

\[
[\varphi] \triangleright [\psi] \text{ iff } \varphi \triangleright \psi.
\]
We now assume for simplicity and without loss of generality that \( v \) is surjective. Then every class \( \zeta \in \text{Cov}_m(v) \) can be represented by a surjective \( m \)-supervaluation.

**Proposition 7.3.** If \( \varphi : R \to U \) is an \( m \)-cover of \( v \), the subset

\[
C(\varphi) := \{ [\psi] \in \text{Cov}_m(v) \mid \varphi \geq \psi \}
\]

of the poset \( \text{Cov}_m(v) \) is a complete lattice. It has the top element \([\varphi]\) and the bottom element \([v]\).

**Proof.** We may assume that the \( m \)-supervaluation \( \varphi : R \to U \) is surjective. Let \( \text{MFC}(U) \) denote the set of all \( \text{MFC} \)-relations on \( U \). This set is partially ordered by inclusion,

\[
E_1 \leq E_2 \iff E_1 \subseteq E_2.
\]

{We view the equivalence relations \( E_i \) as subsets of \( U \) in the usual way.} We have a bijection

\[
\text{MFC}(U) \to C(\varphi), \quad E \mapsto [\pi_E \circ \varphi],
\]

since every fiber contraction \( \alpha \) over \( M \) is of the form \( \rho \circ \pi_E \), with \( E \in \text{MFC}(U) \) uniquely determined by \( \alpha \) and \( \rho \) an isomorphism over \( M \). Clearly the bijection reverses the partial orders on \( \text{MFC}(U) \) and \( C(\varphi) \). Now it can be proved exactly as in \([\text{IKR1}, \S 7]\) for \( U \) a supertropical semiring, that the poset \( \text{MFC}(U) \) is a complete lattice. Thus \( C(\varphi) \) is a complete lattice. \( \square \)

We construct a supertropical monoid \( U \) which will be the target of an \( m \)-cover \( \varphi : R \to U \) dominating all other \( m \)-covers.

Let \( q := v^{-1}(0) = \text{supp}(v) \). As a set we define \( U \) to be the disjoint union of \( R \setminus q \) and \( M \),

\[
U = (R \setminus q) \cup M.
\]

We introduce on \( U \) the following multiplication: For \( x, y \in U \)

\[
x \cdot_U y = y \cdot_U x = \begin{cases} x \cdot_R y & \text{if } x, y, xy \in R \setminus q, \\ 0 & \text{if } x, y \in R \setminus q, xy \in q, \\ v(x) \cdot_M y & \text{if } x \in R \setminus q, y \in M, \\ x \cdot_M y & \text{if } x, y \in M. \\ \end{cases}
\]

It is readily checked that \( U \) with this multiplication is a monoid with unit element \( 1_U = 1_R \) and absorbing idempotent \( 0_U = 0_M \). Moreover \( e := 1_M \) is an idempotent of \( U \) such that \( M = e \cdot U \) and \( M \) in its given multiplication is a submonoid of \( U \). Finally \( 0_M \) is the only element \( x \) of \( U \) with \( 0_M \cdot x = 0_M \). Thus, if we choose the given total ordering on the submonoid \( M \) of \( U \), we have established on \( U \) the structure of a supertropical monoid (cf. Definition 1.1). We denote this supertropical monoid now by \( U^0(v) \).

**Theorem 7.4.**

(i) The map \( \varphi^0_v : R \to U^0(v) \) with

\[
\varphi^0_v(a) = \begin{cases} a & \text{if } a \in R \setminus q, \\ 0_M & \text{if } a \in q, \\ \end{cases}
\]

is a surjective \( m \)-valuation covering \( v \).

(ii) \( \varphi^0_v \) dominates every other \( m \)-cover of \( v \).
Corollary 7.5. The poset Cov\(_m(v)\) is a complete lattice with top element \([\varphi_v^0]\).

Proof. By Theorem 7.4 we have Cov\(_m(v) = C(\varphi_v^0)\), and this is a complete lattice by Proposition 7.3. □

Proposition 7.6. If \(\zeta \in\) Cov\(_v\), \(\eta \in\) Cov\(_m(v)\) and \(\zeta \geq \eta\), then \(\eta \in\) Cov\(_v\).

Proof. We choose surjective m-valuations \(\varphi : R \to U, \psi : R \to V\) with \(U\) a semiring and \(\zeta = [\varphi], \eta = [\psi]\). There exists a fiber contraction \(\alpha : U \to V\) over \(M\) with \(\alpha \circ \varphi = \psi\). Since \(\varphi\) and \(\psi\) are surjective, \(U = \varphi(R) \cup e\varphi(R)\) and \(V = \psi(R) \cup e\psi(R)\). We conclude that \(\alpha(U) = \alpha\varphi(R) \cup e\alpha\varphi(R) = \psi(R) \cup e\psi(R) = V\).

Since \(U\) is a semiring, it follows by Theorem 5.7 or already by Theorem 1.6 ii, that \(V\) is a semiring, hence \(\eta \in\) Cov\(_v\). □

Recall from §4 that every supertropical monoid \(U\) gives us a supertropical semiring \(\hat{U} = U/E(U, S(U))\) together with an ideal compression \(\sigma_U := \pi_{E(U, S(U))} : U \to \hat{U}\). Here \(S(U)\) is the set of tangible NC-elements in \(U\) (cf. Definition 4.2).

Definition 7.7. For every m-supervaluation \(\varphi : R \to U\) we define a supervaluation
\[
\hat{\varphi} := \sigma_U \circ \varphi : R \to \hat{U}.
\]

Proposition 7.8. Let \(\varphi\) be an m-cover of \(v\).

(i) \(\hat{\varphi}\) is a cover of \(v\) and \(\varphi \geq \hat{\varphi}\).

(ii) If \(\psi\) is a cover of \(v\) with \(\varphi \geq \psi\), then \(\hat{\varphi} \geq \hat{\psi}\).

(iii) If \(\psi\) is an m-cover of \(v\) with \(\varphi \geq \psi\), then \(\hat{\varphi} \geq \hat{\psi}\).

Proof. i): This is obvious.

ii): We may assume that \(\varphi\) is a surjective m-supervaluation. Then we have a fiber contraction \(\alpha : U \to V\) over \(M\) with \(\psi = \alpha \circ \varphi\). By Theorem 4.5 we have a factorization \(\alpha = \beta \circ \sigma_U\) with \(\beta\) another fiber contraction over \(M\). We conclude that \(\psi = \beta \sigma_U \varphi = \beta \hat{\varphi}\).

iii): \(\varphi \geq \psi \geq \hat{\psi}\) by i), hence \(\hat{\varphi} \geq \hat{\psi}\) by ii). □

Theorem 7.9. As before assume that \(v : R \to M\) is a surjective m-valuation. Let
\[
U(v) := (U^0(v))^\wedge
\]
and
\[
\varphi_v := (\varphi_v^0)^\wedge : R \to U(v).
\]
The supervaluation $\varphi_v$ is an initial cover of $v$; i.e., given any supervaluation $\psi : R \to V$ covering $v$, there exists a (unique) semiring homomorphism $\alpha : U \to V$ over $M$ with $\psi = \alpha \circ \varphi_v$.

Proof. We may assume that $\psi$ is a surjective supervaluation covering $v$. We know by Theorem $[7.3]$ that $\varphi_v^0 \geq \psi$, and conclude by Proposition $[7.5]$ that $\varphi_v = (\varphi_v^0)^\wedge \geq \psi$. Thus, there exists a fiber contraction $\alpha : U(v) \to V$ over $M$ with $\psi = \alpha \circ \varphi$. Since $U(v)$ and $V$ are semirings, $\alpha$ is a semiring homomorphism over $M$ (cf. Proposition $[5.4]$).

**Corollary 7.10.** The poset $\text{Cov}(v)$ is a complete lattice with top element $[\varphi_v]$.

We proved in [IKR1] §7 that Cov($v$) is a complete lattice, but – except in the case that $v$ is a valuation – there we have only proved that the a top element $[\varphi_v]$ exists (loc. cit, Proposition 7.5), without giving an explicit description of $\varphi_v$. Starting from the formula $\varphi_v = (\varphi_v^0)^\wedge$, this is now possible.

Let $U := U^0(v)$. Then $\mathcal{T}(U) = R\setminus q$ and $eU = M$. We have

$$\hat{U} = U/E(U, D(U)) = U/E(U, S(U))$$

with $S(U)$ the set of tangible NC-products in $U$ (cf. Definition $[4.2]$ and $D(U) = S(U) \cup M$, which is an ideal of $U$ (cf. $[3]$). We view $\hat{U}$ as a subset of $U$, as indicated in Convention $[3.3]$a. Thus $e\hat{U} = M$ and $\mathcal{T}(\hat{U}) = \mathcal{T}(U)/S(U)$. Recalling the description of the supertropical monoid $U^0(v)$ from above, we see that $S(U)$ is the following subset $Y(v)$ of $R\setminus q$:

$$Y(v) := \{ab \mid a, b \in R, \exists d' \in R \text{ with } v(a) < v(b), v(ab) = v(ab) \neq 0\}.$$ 

Thus $\mathcal{T}(\hat{U}) = R\setminus q'$ with $q' := q \cup Y(v)$. Clearly $R \cdot Y(v) \subseteq q \cup Y(v)$, and thus $q'$ is an ideal of $R$.

Looking again at Convention $[3.3]$a we obtain a completely explicit description of $U(v)$ and $\varphi_v$ as follows.

**Scholium 7.11.** Assume that $v : R \to M$ is a surjective m-valuation with support $q = v^{-1}(0)$. Let $q' := q \cup Y(v)$ with the set $Y(v) \subseteq R\setminus q$ given above. Then $U(v)$ is the subset $(R\setminus q') \cup M$ of $U^0(v) = (R\setminus q) \cup M$, with the following new product $\odot$: If $x, y \in U(v)$, then

$$x \odot y = \begin{cases} 
xy & \text{if } x, y, xy \in R\setminus q', \\
\text{exy} & \text{otherwise},
\end{cases}$$

the products on the right hand side taken in $U^0(v)$. \{Here $e = e_{U^0(v)} = e_{U(v)}$.$\}$ The initial covering $\varphi_v : R \to U(v)$ is given by

$$\varphi_v(a) = \begin{cases} 
a & \text{if } a \in R\setminus q', \\
v(a) & \text{if } a \in q'.
\end{cases}$$

**Theorem 7.12.** As before assume that $v : R \to M$ is a surjective m-valuation.

(i) If $\varphi$ is any supervaluation covering $v$, then $\varphi(a)$ is ghost for every $a \in Y(v)$.

(ii) If $Y(v) = \emptyset$, then every surjective m-supervaluation covering $v$ is a supervaluation. In short $\text{Cov}_m(v) = \text{Cov}(v)$.

*Proof.* i): We read off from Scholium $[7.11]$ that $\varphi_v(a) = v(a) \in M$ for any $a \in Y(v)$. Since $\varphi_v$ dominates $\varphi$, also $\varphi(a) = v(a) \in M$.

ii): If $Y(v) = \emptyset$, it follows from Scholium $[7.11]$ that $\varphi_v^0 = \varphi_v$. Thus $[\varphi_v^0] \in \text{Cov}(v)$. If $\varphi$ is any m-supervaluation covering $v$, then $\varphi_v^0 \geq \varphi$, hence $[\varphi] \in \text{Cov}(v)$ by Proposition $[7.6]$. □
In particular $Y(v)$ is empty if $v$ is a valuation, since this means that the bipotent semiring $M$ is cancellative. Then we fall back on the explicit description of the initial cover $\varphi_v$ in [IKR1] which in present notation says that $\varphi_v = \varphi_v^0$ (loc. cit., Example 4.5).

In large parts of [IKR1] and whole [IKR2] §2, where we studied coverings of valuations, it was important that we have tangible supervaluations at our disposal. There remains the difficult task to develop a similar theory for coverings of m-valuations, which are not valuations. We leave this to the future. But we mention that there exist many natural and beautiful m-valuations which are not valuations, as is already clear from [HV] and [Z]. More on this can be found in a recent paper [IKR3].

8. LIFTING GHOSTS TO TANGIBLES

**Definition 8.1.** We call a supertropical monoid $U$ **unfolded**, if the set $\mathcal{T}(U)_0 := \mathcal{T}(U) \cup \{0\}$ is closed under multiplication.

If $U$ is unfolded, then $N := \mathcal{T}(U)_0$ is a monoid under multiplication with absorbing element 0. Further $M := eU$ is a totally ordered monoid with absorbing element 0, and the restriction

$$\rho := \nu|U : N \to M$$

is a monoid homomorphism with $\rho^{-1}(0) = \{0\}$. Observing also that $e_U = 1_M = \rho(1_N)$, we see that the supertropical monoid $U$ is completely determined by the triple $(N, M, \rho)$. This leads to a way to construct all unfolded supertropical monoids up to isomorphism.

**Construction 8.2.** Assume that we are given a totally ordered monoid $M$ with absorbing element $0_M \leq x$ for all $x \in M$, i.e., a bipotent semiring $M$, further an (always commutative) monoid $N$ with absorbing element $0_N$, and a multiplicative map $\rho : N \to M$ with $\rho(1_N) = \rho(1_M)$, $\rho^{-1}(0_M) = \{0_N\}$. Then we define an unfolded supertropical monoid $U$ as follows: As a set $U$ is the disjoint union of $M \setminus \{0_M\}$, $N \setminus \{0_N\}$, and a new element 0. We identify $0_M = 0_N = 0$, and then write

$$U = M \cup N, \quad \text{with } M \cap N = \{0\}.$$ 

The multiplication on $U$ is given by the rules, in obvious notation,

$$x \cdot y = \begin{cases} x \cdot_N y & \text{if } x, y \in N, \\ \rho(x) \cdot_M y & \text{if } x \in N, y \in M, \\ x \cdot_M \rho(y) & \text{if } x \in M, y \in N, \\ x \cdot_M y & \text{if } x, y \in M. \end{cases}$$

It is easy to verify that $(U, \cdot)$ is a (commutative) monoid with $1_U = 1_N$ and absorbing element 0. Let $e := 1_M$. Then $eU = M$ and $\rho(x) = ex$ for $x \in M$, further $ex = 0$ iff $x = 0$ for any $x \in U$, since $\rho^{-1}(0) = \{0\}$. Thus $(U, \cdot, e)$, together with the given ordering on $M = eU$, is a supertropical monoid. It clearly is unfolded. We denote this supertropical monoid $U$ by $\text{STR}(N, M, \rho)$.

This construction generalizes the construction of supertropical domains [IKR1] (loc. cit. Construction 3.16). There we assumed that $N \setminus \{0\}$ and $M \setminus \{0\}$ are closed under multiplication, and that the monoid $M \setminus \{0\}$ is cancellative, and we obtained all supertropical predomains up to isomorphism. Dropping here just the cancellation hypothesis would give us a class of supertropical monoids not broad enough for our work below.
The present notation \( \text{STR}(N, M, \rho) \) differs slightly from the notation \( \text{STR}(\mathcal{T}, \mathcal{G}, v) \) in [IKR1]. Construction 3.16. Regarding the ambient context this should not cause confusion.

We add a description of the transmissions between two unfolded supertropical monoids.

**Proposition 8.3.** Assume that \( U' = \text{STR}(N', M', \rho') \) and \( U = \text{STR}(N, M, \rho) \) are unfolded supertropical monoids.

(i) If \( \lambda : N' \to N \) is a monoid homomorphism with \( \lambda(0) = 0 \), and \( \mu : M' \to M \) is a semiring homomorphism, and if \( \rho' \lambda = \mu \rho \), then the well-defined map

\[
\text{STR}(\lambda, \mu) : U' = N' \cup M' \to U = N \cup M,
\]

which sends \( x' \in N' \) to \( \lambda(x') \) and \( y' \in M' \) to \( \mu(y') \), is a tangible transmission (cf. Definition 2.3).

(ii) In this way we obtain all tangible transmissions from \( U' \) to \( U \).

**Proof.** (i): A straightforward check. (ii): Obvious.

We mention in passing, that given an \( m \)-valuation \( v : R \to M \) with support \( v^{-1}(0) = q \), the supertropical semiring \( U^0(v) \) occurring in Theorem 7.4 may be viewed as an instance of Construction 8.2, as follows.

**Example 8.4.** Let \( E \) denote the equivalence relation on \( R \) with equivalence classes \([0]_E = q\) and \([x]_E = \{x\} \) for \( x \in R \setminus q \). It is multiplicative, hence gives us a monoid \( R/E \) with absorbing element \([0]_E = 0\), which we identify with the subset \((R \setminus q) \cup \{0\}\) in the obvious way. The map \( v : R \to M \) induces a monoid homomorphism \( \tilde{v} : R/E \to M \) with values \( \tilde{v}(x) = v(x) \) for \( x \in R \setminus q \), \( \tilde{v}(0) = 0 \). We have \( \tilde{v}^{-1}(0) = \{0\} \) and

\[
U^0(v) = \text{STR}(R/E, M, \tilde{v}).
\]

We now look for ways to “unfold” an arbitrary supertropical monoid \( U \). By this we roughly mean a fiber contraction \( \tau : \tilde{U} \to U \) with \( \tilde{U} \) an unfolded supertropical monoid and fibers \( \tau^{-1}(x), x \in U \) as small as possible. More precisely we decree

**Definition 8.5.** Let \( M := eU \), and let \( N \) be a submonoid of \( (U, \cdot) \) which contains the set \( \mathcal{T}(U)_0 \). An **unfolding of \( U \) along \( N \)** is a fiber contraction \( \tau : \tilde{U} \to U \) over \( M \) (in particular \( e\tilde{U} = M \)), such that

\[
\tau^{-1}(x) = \begin{cases} 
\{x, \hat{x}\} & \text{if } x \in M \cap N, \\
\{\hat{x}\} & \text{if } x \in N \setminus M, \\
\{x\} & \text{if } x \in M \setminus N,
\end{cases}
\]

with \( \hat{x} \in \mathcal{T}(U)_0 \). For any \( x \in N \) we call \( \hat{x} \) the **tangible lift** of \( x \) (with respect to \( N \)).

Notice that this forces \( \tau(\mathcal{T}(\tilde{U})_0) = N \), and that moreover for any \( x \in N \) the tangible fiber \( \hat{x} \) is the unique element of \( \mathcal{T}(\tilde{U})_0 \) with \( \tau(\hat{x}) = x \), hence \( \mathcal{T}(\tilde{U})_0 = \{\hat{x} \mid x \in N\} \).

Thus, if \( \tau : \tilde{U} \to U \) is an unfolding along \( N \), then the map \( \hat{x} \mapsto x \) from \( \mathcal{T}(\tilde{U})_0 \) to \( N \) obtained from \( \tau \) by restriction is a monoid isomorphism, and \( \tau \) itself is an ideal compression with ghost kernel \( (M \cap N)^\sim \cup M \), where \( (M \cap N)^\sim := \{\hat{x} \mid x \in M \cap N\} \).

**Theorem 8.6.**
(i) Given a pair \((U, N)\) consisting of a supertropical monoid \(U\) and a multiplicative submonoid \(N \supseteq \mathcal{T}(U)_0\), there exists an unfolding \(\tau : \tilde{U} \to U\) of \(U\) along \(N\).

(ii) If \(\tau' : \tilde{U}' \to U\) is a second unfolding of \(U\) along \(N\), then there exists a unique isomorphism of supertropical monoids \(\alpha : \tilde{U} \cong \tilde{U}'\) with \(\tau' \circ \alpha = \tau\).

Proof. i) Existence: Since \(M\) is an ideal of \(U\), the set \(M \cap N\) is a monoid ideal of \(N\). We have \(U = N \cup M\), since \(N \supseteq \mathcal{T}(U)_0\). Let \(\rho : N \to M\) denote the restriction of \(\nu_U\) to \(N\). It is a monoid homomorphism with \(\rho^{-1}(0) = \{0\}\).

Let \(\tilde{N}\) denote a copy of the monoid \(N\) with copying isomorphism \(x \mapsto \tilde{x}\) \((x \in N)\), and let \(\tilde{\rho} : \tilde{N} \to M\) denote the monoid homomorphism from \(\tilde{N}\) to \(M\) corresponding to \(\rho : N \to M\). Thus \(\tilde{\rho}(\tilde{x}) = \rho(x) = ex\) for \(x \in N\). Now define the unfolded supertropical monoid

\[
\tilde{U} := \text{STR}(\tilde{N}, M, \tilde{\rho}) = \tilde{N} \cup M.
\]

In \(\tilde{U}\) we obtain \(\tilde{0}_U = 0\) and \(\tilde{N} \cap M = \{0\}\). Further \(\mathcal{T}(\tilde{U})_0 = \tilde{N}\) and \(\tilde{e}_U = eU = M\).

We obtain a well-defined surjective map \(\tau : \tilde{U} \to U\) by putting \(\tau(\tilde{x}) := x\) for \(x \in N\), \(\tau(y) := y\) for \(y \in M\). This map \(\tau\) is multiplicative, as checked easily, sends \(0\) to \(0\), \(1\) to \(1\) \((\in \mathcal{T}(\tilde{U}))\) to \(1 \in N\), and restricts to the identity on \(M\). Thus \(\tau\) is a fiber contraction (cf. Definition 2.1). The fibers of \(\tau\) are as indicated in Definition 8.3, hence \(\tau\) is an unfolding of \(U\) along \(N\).

ii) Uniqueness: Let \(\tilde{\tau} : \tilde{U} \to U\) and \(\tilde{\tau}' : \tilde{U}' \to U\) be unfoldings of \(U\) along \(N\) with tangible lifts \(x \mapsto \tilde{x}\) and \(x \mapsto \tilde{x}'\) respectively. Without loss of generality we assume that \(\tilde{U} = \text{STR}(\tilde{N}, M, \tilde{\rho})\) and \(\tilde{U}' = \text{STR}(\tilde{N}', M, \tilde{\rho}')\) with tangible lifts \(x \mapsto \tilde{x}\) and \(x \mapsto \tilde{x}'\) \((x \in N)\). Then \(\tilde{\rho}(\tilde{x}) = \tilde{\rho}'(\tilde{x}') = ex\) for every \(x \in N\). The map \(\lambda : \tilde{N} \to \tilde{N}',\) given by \(\lambda(\tilde{x}) = \tilde{x}'\) for \(x \in \tilde{N}\), is a monoid isomorphism with \(\tilde{\rho}' \circ \lambda = \tilde{id}_M \circ \tilde{\rho}\). Thus we have a well defined transmission

\[
\alpha := \text{STR}(\lambda, \tilde{id}_M) : \tilde{U} \to \tilde{U}'
\]

at hand (cf. Proposition 8.3). \(\alpha\) is an isomorphism over \(U\), i.e., an isomorphism with \(\tau' \circ \alpha = \tau\), clearly the only one. \(\square\)

Notation 8.7. We call the map \(\tau : \tilde{U} \to U\) constructed in part i) of the proof of Theorem 8.6 “the” unfolding of \(U\) along \(N\) and write this map more precisely as

\[
\tau_{U,N} : \tilde{U}(N) \to U
\]

if necessary. But sometimes we abusively will denote any unfolding of \(U\) along \(N\) in this way, justified by part ii) of Theorem 8.6.

Example 8.8. We consider the very special case that \(U = e_\infty U = M\). Then \(N\) can be any submonoid of \(M\) containing \(0\). Now \(\tilde{M}(N) = \tilde{N} \cup M\) with \(\tilde{N} \cap M = \{0\}\), and

\[
\tilde{M}(N) \cong \text{STR}(N, M, i)
\]

with \(i : N \to M\) the inclusion mapping. For every \(x \in N\) there exists a unique tangible element \(\tilde{x}\) of \(\tilde{M}(N)\) with \(e\tilde{x} = x\), while for \(x \in M\setminus N\) there exists no such element.

Theorem 8.9. Assume that \(\alpha : \tilde{U}' \to U\) is a transmission between supertropical monoids, and that \(N' \supseteq \mathcal{T}(U'_0), N \supseteq \mathcal{T}(U)_0\) are submonoids of \(U'\) and \(U\) with \(\alpha(N') \subseteq N\). Then there exists a unique tangible transmission

\[
\tilde{\alpha} := \tilde{\alpha}_{N', N} : \tilde{U}'(N') \to \tilde{U}(N),
\]
called the **tangible unfolding of** \( \alpha \) **along** \( N' \) **and** \( N \), such that the diagram

\[
\begin{array}{ccc}
\tilde{U}'(N') & \xrightarrow{\tilde{\alpha}} & \tilde{U}(N) \\
\downarrow{\tau_{U',N'}} & & \downarrow{\tau_{U,N}} \\
U' & \xrightarrow{\alpha} & U
\end{array}
\]

commutes.

**Proof.** Let \( M' := eU' \), \( M := eU \), and let \( \rho' : N' \to M, \; \rho : N \to M \) denote the monoid homomorphism obtained from \( \nu_{U'} \) and \( \nu_U \) by restriction to \( N' \) and \( N \). Then

\[
\tilde{U}'(N') = \text{STR}(N', M', \rho'), \quad \tilde{U}(N) = \text{STR}(N, M, \rho).
\]

The map \( \alpha \) restricts to monoid homomorphisms \( \lambda : N' \to N \) and \( \gamma : M' \to M \) with \( \lambda(0) = 0 \), \( \gamma(0) = 0 \), and \( \gamma \) order preserving. Now \( \gamma \circ \nu_{U'} = \nu_U \circ \alpha \), hence \( \gamma \rho' = \rho \lambda \). Thus we have the tangible transmission

\[
\tilde{\alpha} := \text{STR}(\lambda, \gamma) : \tilde{U}'(N') \to \tilde{U}(N)
\]
at hand. Clearly \( \tau_{U,N} \circ \tilde{\alpha} = \alpha \circ \tau_{U',N'} \). Since any tangible transmission from \( \tilde{U}'(N') \) to \( \tilde{U}(N) \) maps \( \tilde{N}' \) to \( \tilde{N} \) and \( M' \) to \( M \), it is evident that \( \tilde{\alpha} \) is the only such map. \( \square \)

**Corollary 8.10.** Assume that \( \alpha : U' \to U \) is a transmission between supertropical monoids which is tangibly surjective, i.e., \( T(U) \subset \alpha(T(U')) \). Assume further that \( U' \) is unfolded. Let \( N := \alpha(T(U'_0)) \), which is a submonoid of \( U \) containing \( T(U)_0 \).

(i) There exists a unique tangible transmission

\[
\tilde{\alpha} : U' \to \tilde{U}(N),
\]
called the **tangible lift** of \( \alpha \), such that \( \tau_{U,N} \circ \tilde{\alpha} = \alpha \).

(ii) If \( x' \in U' \), then

\[
\tilde{\alpha}(x') = \begin{cases} 
\tilde{\alpha}(x') & \text{if } x' \in T(U'_0), \\
\alpha(x') & \text{if } x' \in eU'.
\end{cases}
\]

**Proof.** (i): applying Theorem 8.9 with \( N' := T(U'_0) \), and observe that \( \tilde{U}'(N') = U' \), since \( U' \) is unfolded.

(ii): Now obvious, since \( \tau_{U,N}(\tilde{\alpha}(x')) = \alpha(x') \) and \( \tilde{\alpha}(x') \in T(\tilde{U}) \) if \( x' \in T(U'_0) \). \( \square \)

We are ready to construct “tangible lifts” of \( m \)-supervaluations.

**Theorem 8.11.** Assume that \( \varphi : R \to U \) is an \( m \)-supervaluation which is tangibly surjective, i.e., \( T(U) \subset \varphi(R) \) {e.g. \( \varphi \) is surjective; \( U = \varphi(R) \diamond e\varphi(R) \)}. Let \( N := \varphi(R) \), which is a submonoid of \( U \) containing \( T(U) \).

(i) The map

\[
\tilde{\varphi} : R \to \tilde{U}(N), \quad a \mapsto \tilde{\varphi(a)},
\]

with \( \tilde{\varphi(a)} \) denoting the tangible lift of \( \varphi(a) \) w.r.t. \( N \), is a tangible \( m \)-supervaluation of \( \varphi \), called the **tangible lift** of \( \varphi \).

(ii) If \( \varphi' : R \to U' \) is a tangible \( m \)-supervaluation dominating \( \varphi \), then \( \varphi' \) dominates \( \tilde{\varphi} \).
Proof. (i): \( \tilde{\varphi} \) is multiplicative, \( \tilde{\varphi}(0) = 0 \), \( \tilde{\varphi}(1) = 1 \), and \( e \tilde{\varphi} = e \varphi \) is an m-valuation. Thus \( \tilde{\varphi} \) is an m-supervaluation. By construction \( \tilde{\varphi} \) is tangible.

(ii): We may assume that the m-supervaluation \( \varphi' : R \to U' \) is surjective, and hence \( \varphi'(R) \supseteq \mathcal{T}'(U)_0 \). Since \( \varphi' \) is tangible, this forces \( \varphi'(R) = \mathcal{T}'(U)_0 \). Thus \( \mathcal{T}'(U)_0 \) is a submonoid of \( U' \), i.e., \( U' \) is unfolded. Since \( \varphi' \) dominates \( \varphi \), there exists a transmission \( \alpha : U' \to U \) with \( \varphi = \alpha \circ \varphi' \). We have

\[
\alpha(\mathcal{T}'(U)_0) = \alpha(\varphi'(R)) = \varphi(R) = N.
\]

Thus we have the tangible lift

\[
\tilde{\alpha} : U' \to \tilde{U}(N)
\]

of \( \alpha \) at hand. For any \( a \in R \),

\[
\tilde{\alpha}(\varphi'(a)) = \lfloor \alpha(\varphi'(a)) \rfloor = \overline{\varphi(a)} = \tilde{\varphi}(a).
\]

Thus \( \tilde{\varphi} = \tilde{\alpha} \circ \varphi' \), which proves that \( \varphi' \) dominates \( \tilde{\varphi} \).

Addendum 8.12. As the proof has shown, if the m-valuation \( \varphi' \) is surjective, then \( U' \) is unfolded, and the transmission

\[
\alpha_{\varphi, \tilde{\varphi}} : U' \to \tilde{U}(N)
\]

(cf. [IKR1, Definition 5.3]) is the tangible lift of \( \alpha_{\varphi, \varphi'} : U' \to U \).

Corollary 8.13. If \( \varphi, \psi \) are m-supervaluations covering \( v \) and \( \varphi \leq \psi \), then \( \varphi \leq \tilde{\psi} \).

Proof. We have \( \varphi \leq \psi \leq \tilde{\psi} \). It follows by Theorem 8.11.ii that \( \tilde{\varphi} \leq \tilde{\psi} \).

9. The partial tangible lifts of an m-supervaluation

In all the following \( v : R \to M \) is a fixed m-valuation and \( \varphi : R \to U \) is a tangible surjective m-supervaluation covering \( v \). (Most often \( v \) and \( \varphi \) will both be surjective.) In \([8]\) we introduced the tangible lift \( \tilde{\varphi} : R \to \tilde{U} \) (cf. Theorem 7.10). We now strive for an explicit description of the m-supervaluations \( \psi \) covering \( v \) with \( \varphi \leq \psi \leq \tilde{\varphi} \).

We warm up with two general observations.

Definition 9.1. If \( \psi : R \to V \) is an m-supervaluation covering \( v : R \to M \), we call

\[
G(\psi) := \psi(R) \cap M = \{ \psi(a) \mid a \in R, \, \psi(a) = v(a) \}
\]

the ghost value set of \( \psi \). {Notice that \( eV = M \).}

Lemma 9.2. Let \( \psi_1, \psi_2 \) be m-supervaluations covering \( v \). If \( \psi_1 \geq \psi_2 \), then \( G(\psi_1) \subseteq G(\psi_2) \).

If \( \psi_1 \sim \psi_2 \), then \( G(\psi_1) = G(\psi_2) \).

Proof. Let \( a \in R \). If \( \psi_1 \geq \psi_2 \), then \( \psi_1(a) \in M \) implies that \( \psi_2(a) \in M \), due to condition D3 in the definition of dominance (cf. Definition 7.2). Thus, for \( \psi_1 \sim \psi_2 \) we have \( \psi_1(a) \in M \) if and only if \( \psi_2(a) \in M \).

Lemma 9.3. Assume that the m-valuation \( v : R \to M \) is surjective. Then the ghost value set \( G(\psi) \) of any m-supervaluation \( \psi \) covering \( v \) is an ideal of the semiring \( M \).

Proof. If \( x \in G(\psi) \) and \( y \in M \), there exist \( a, b \in R \) with \( \psi(a) = x, \, e\psi(b) = y \). It follows that

\[
xy = e\psi(a)\psi(b) = \psi(a)\psi(b) = \psi(ab).
\]

Thus \( xy \in \psi(R) \cap M = G(\psi) \). This proves that \( G(\psi) \cdot M \subseteq G(\psi) \). Since \( M \) is bipotent, \( G(\psi) \) is also closed under addition.
Theorem 9.4. Assume that \( \varphi : R \rightarrow U \) is an m-supervaluation covering \( v : R \rightarrow M \), and that \( \psi_1, \psi_2 \) are m-supervaluations covering \( v \) with

\[
\varphi \leq \psi_1 \leq \bar{\varphi}, \quad \varphi \leq \psi_2 \leq \bar{\varphi}.
\]

(i) \( \psi_1 \geq \psi_2 \iff G(\psi_1) \subseteq G(\psi_2) \).

(ii) \( \psi_1 \sim \psi_2 \iff G(\psi_1) = G(\psi_2) \).

Proof. We assume without loss of generality that \( \varphi \) is surjective. Then also the m-supervaluations \( \psi_1, \psi_2, \bar{\varphi} \) are surjective. By Corollary 8.13 the tangible lifts \( \tilde{\psi}_1 \) and \( \tilde{\psi}_2 \) are both equivalent to \( \bar{\varphi} \).

Again without loss of generality we moreover assume that \( \varphi = \bar{\varphi}/E := \pi_E \circ \bar{\varphi} \) with \( E \) an MFCE-relation on \( \tilde{U} \), and also \( \psi_i = \bar{\varphi}/E_i \) with an MFCE-relation \( E_i \) (\( i = 1, 2 \)). Let us describe these relations \( E, E_1, E_2 \) explicitly. We have \( \tilde{U} = \tilde{N} \cup M \), with

\[
\tilde{N} := T(\tilde{U})_0 = \bar{\varphi}(R), \quad \tilde{N} \cap M = \{0\},
\]

further \( U = N \cup M \) with

\[
N := \varphi(R), \quad N \cap M = G(\varphi),
\]

and we have a copying isomorphism

\[
s : N \rightarrow \tilde{N}
\]

of monoids (new notation!), which sends each \( x \in N \) to its tangible lift \( \tilde{x} \), as explained in [8] (Definition 8.5, Proof of Theorem 8.6.i). Notice that \( es(x) = x \) for \( x \in N \cap M = G(\varphi) \).

The relation \( E \) has the 2-point equivalence classes \( \{x, s(x)\} \) with \( x \) running through \( G(\varphi) \), while all other \( E \)-equivalence classes are one-point sets. Analogously, \( E_i \) has the 2-point set equivalence classes \( \{x, s(x)\} \) with \( x \) running through \( G(\psi_i) \subseteq G(\varphi) \), while again all other \( E \)-equivalence classes are one-point sets. Thus it is obvious that \( E_1 \subseteq E_2 \) iff \( G(\psi_1) \subseteq G(\psi_2) \). But \( E_1 \subseteq E_2 \) means that \( \psi_1 \preceq \psi_2 \). This gives claim (i), and claim (ii) follows. \( \square \)

Definition 9.5. We call the monoid isomorphism

\[
s : \varphi(R) \rightarrow T(\tilde{U})_0 = \bar{\varphi}(R),
\]

i.e., the copying isomorphism \( s : N \rightarrow \tilde{N} \) occurring in the proof of Theorem 9.4, the tangible lifting map for \( \varphi \).

Notice that for \( x \in \varphi(R) \), \( y \in T(\tilde{U})_0 \) we have \( s(x)y = s(xy) \).

We assume henceforth that the m-valuation \( v : R \rightarrow M \) is surjective, and that \( \varphi : R \rightarrow U \) is a surjective m-supervaluation with \( ev = v \). The question arises whether every ideal \( a \) of \( M \) with \( a \subseteq G(\varphi) \) occurs as the ghost value set \( G(\psi) \) of some m-supervaluation \( \psi \) covering \( v \) with \( \varphi \leq \psi \leq \bar{\varphi} \). This is indeed true.

Construction 9.6. We employ the tangible lifting map \( s : \varphi(R) \rightarrow \bar{\varphi}(R) = T(\tilde{U})_0 \) defined above. Assume that \( a \) is an ideal of \( M \) contained in \( G(\varphi) \). We have

\[
s(a) \cdot \tilde{U} \subseteq s(a) \cup M,
\]

\(^4\)As in [IKRI] we view an equivalence relation on a set \( X \) as a subset of \( X \times X \) in the usual way.
since \( s(x)y = s(xy) \in s(a) \) for \( x \in a \) and \( y \in T(U) \). We conclude that \( s(a) \cup M \) is an ideal of \( U \). Let

\[
E_a := E(\widehat{U}, s(a)) = E(\widehat{U}, s(a) \cup M)
\]

and \( \widetilde{E}_a := \widehat{U}/E_a \). We regard \( \widetilde{E}_a \) as a subset of \( \widehat{U} \), as indicated in Convention 3.3. The map \( \pi_{E_a} : \widehat{U} \to \widetilde{E}_a \) is the ideal compression with ghost kernel \( s(a) \cup M \), and

\[
\tilde{\varphi}_a := \tilde{\varphi}/E_a = \pi_{E_a} \circ \tilde{\varphi} : R \to \widetilde{E}_a
\]

is an \( m \)-supervaluation. For any \( a \in R \)

\[
\tilde{\varphi}_a(a) = \begin{cases} 
\varphi(a) = v(a) & \text{if } \varphi(a) \in a, \\
\tilde{\varphi}(a) & \text{else}.
\end{cases}
\]

Clearly \( \varphi \leq \tilde{\varphi}_a \leq \tilde{\varphi} \) and \( G(\tilde{\varphi}_a) = a \). We call \( \tilde{\varphi}_a \) the tangible lift of \( \varphi \) outside \( a \), and we call any such map \( \tilde{\varphi}_a \) a partial tangible lift of \( \varphi \).

Let \( [\varphi, \tilde{\varphi}] \) denote the “interval” of the poset \( \text{Cov}_m(v) \) containing all classes \( [\psi] \) with \( \varphi \leq \psi \leq \tilde{\varphi} \), and let \([0, G(\varphi)]\) be the set of ideals \( a \) of \( M \) with \( a \subset G(\varphi) \), ordered by inclusion. By the Lemmas 9.2 and 9.3 we have a well defined order preserving map

\[
[\varphi, \tilde{\varphi}] \to [0, G(\varphi)],
\]

which sends each class \( [\psi] \in [\varphi, \tilde{\varphi}] \) to the ideal \( G(\psi) \). By Theorem 9.4 this map is injective, and by Construction 9.6 we know that it is also surjective. Thus we have proved

**Theorem 9.7.** The map

\[
[\varphi, \tilde{\varphi}] \to [0, G(\varphi)], \quad [\psi] \mapsto G(\psi),
\]

is a well defined order preserving bijection. The inverse of this map sends an ideal \( a \subset G(\varphi) \) to the class \( [\tilde{\varphi}_a] \) of the tangible lift of \( \varphi \) outside \( a \).

The poset \( \text{Cov}_m(v) \) is a complete lattice (cf. Corollary 7.5). The poset \( I(M) \) consisting of the ideals of \( M \) and ordered by inclusion, is a complete lattice as well. Indeed, the infimum of a family \( (a_i \mid i \in I) \) in \( I(M) \) is the ideal \( \bigcap_i a_i \), while the supremum is the ideal \( \bigcup_i a_i \). {Recall once more that every subset of \( M \) is closed under addition.} The intervals \([\varphi, \tilde{\varphi}]\) and \([0, G(\varphi)]\) are again complete lattices, and thus the map \([\varphi, \tilde{\varphi}] \to [0, G(\varphi)]\) in Theorem 9.7 is an anti-isomorphism of complete lattices. This implies the following

**Corollary 9.8.** Assume that \( (\psi_i \mid i \in I) \) is a family of supervaluations covering \( v \) with \( \varphi \leq \psi_i \leq \tilde{\varphi} \) for each \( i \in I \). Let \( \bigvee_i \psi_i \) and \( \bigwedge_i \psi_i \) denote respectively representatives of the classes \( \bigvee_i [\psi_i] \) and \( \bigwedge_i [\psi_i] \) (as described in [IKR1] §7). Then

\[
G\left( \bigvee_i \psi_i \right) = \bigcap_i G(\psi_i), \quad G\left( \bigwedge_i \psi_i \right) = \bigcup_i G(\psi_i).
\]

We switch to the case where \( \varphi : R \to U \) is a supervaluation, i.e., the supertropical monoid \( U \) is a semiring. We want to characterize the partial tangible lifts \( \psi \) of \( \varphi \) which are again supervaluations; in other terms, we want to determine the subset \( [\varphi, \tilde{\varphi}] \cap \text{Cov}(v) \) of the interval \([\varphi, \tilde{\varphi}]\) of \( \text{Cov}_m(v) \).

The set \( Y(v) \) introduced near the end of §7 will play a decisive role. It consists of the products \( ab \in R \) of elements \( a, b \in R \) for which there exists some \( a' \in R \) with

\[
v(a') < v(a), \quad v(a'b) = v(ab) \neq 0.
\]
Proposition 9.10. If we have seen in \( \gamma : eR \to M \) a semiring homomorphism to a bipotent semiring \( M \). Then

\[ v := \gamma \circ \nu_R : R \to M \]

is a strict m-valuation. The \( v \)-NC-products are the products \( yz \) with \( y, z \in U \) such that there exists some \( y' \in R \) with

\[ \gamma(ey') < \gamma(ey), \quad \gamma(ey'z) = \gamma(eyz). \]

Thus \( Y(v) \) is the ideal \( D_4(R, \gamma) \) of the supertropical semiring \( R \) introduced in Definition 4.8.

Proposition 9.10. If \( \varphi \) is a supervaluation then \( \varphi(q') \) is contained in the ghost value set \( G(\varphi) \).

Proof. We have seen in \( \varphi(Y(v)) \subset M \). Since \( \varphi(q) = \{0\} \), this implies that \( \varphi(q') \subset M \cap \varphi(R) = G(\varphi) \). \( \square \)

Remark 9.11. Here is a more direct argument that \( \varphi(Y(v)) \subset M \), than given in the proof of Theorem 7.12.i. If \( x \in Y(v) \), then we have \( a', a, b \in R \) with \( x = ab, v(a') < v(a), v(a'b) = v(ab) \neq 0 \). Clearly \( \varphi(x) = \varphi(a) \varphi(b) \) is an NC-product in the supertropical semiring \( U \) (recall Definition 4.2), and thus \( \varphi(x) \) is ghost, as observed already in Theorem 7.2.

Lemma 9.12. Assume that \( \varphi : R \to U \) is a surjective tangible m-supervaluation covering \( v \). Then \( \varphi(R \setminus q) = \mathcal{T}(U), v(R) = M \), and \( \varphi(Y(v)) = S(U) \).

Proof. a) We have \( U = \varphi(R) \cup v(R), \varphi(R \setminus q) \subset \mathcal{T}(U) \), and \( v(R) \subset M \). Since \( U = \mathcal{T}(U) \cup M \), this forces \( \varphi(R \setminus q) = \mathcal{T}(U) \) and \( v(R) = M \).

b) Let \( c \in Y(v) \). There exist \( a, b, a' \in R \) with \( c = ab, v(a') < v(a), v(a'b) = v(ab) \neq 0 \). It follows that \( \varphi(c) = xy \neq 0 \) with \( x := \varphi(a), y := \varphi(b), v(a') < ex, v(d'y) = eexy \). Thus \( \varphi(c) \) is an NC-product in \( U \). Moreover \( \varphi(c) \) is tangible, hence \( \varphi(c) \in \mathcal{T}(U) \). Thus \( \varphi(Y(v)) \subset \mathcal{T}(U) \).

c) Let \( x \in S(U) \) be given. Then \( x = yz \in \mathcal{T}(U) \) with \( y, z \in U \) and \( y' < ey, y'z = eyz \neq 0 \) for some \( y' \in M \). Clearly \( y, z \in \mathcal{T}(U) \). We choose \( a, b, a' \in R \) with \( \varphi(a) = y, \varphi(b) = z, v(a') = y' \). Then \( ey = v(a), ez = v(b) \), and it follows that \( v(a') < v(a), v(a'b) = v(ab) \neq 0 \). Thus \( ab \in Y(v) \) and \( x = \varphi(ab) \). This proves that \( S(U) \subset \varphi(Y(v)) \). \( \square \)

Theorem 9.13. Assume that \( \varphi : R \to U \) is a supervaluation, i.e., \( U \) is a semiring. Let \( \tilde{\varphi} \) denote the tangible lift of \( \varphi \) outside the ideal \( v(q') = \{0\} \cup v(Y(v)) \) of \( M \),

\[ \tilde{\varphi} := (\tilde{\varphi})_{v(q')} : R \to \tilde{U} := \tilde{U}/E_{v(q')} \]

(cf. Construction 9.6).

(i) \( \tilde{\varphi} \) is again a supervaluation. More precisely, \( \tilde{\varphi} \) coincides with the supervaluation \( (\tilde{\varphi})^\wedge \) associated to the tangible lift \( \tilde{\varphi} : R \to \tilde{U} \) of \( \varphi \) (cf. Definition 7.7).

---

5 Recall that \( S(U) \) denotes the set of tangible NC-products in \( U \) (Definition 4.2).
(ii) If $\psi$ is an $m$-supervaluation covering $v$ with $\varphi \leq \psi \leq \overline{\varphi}$, then $\psi$ is a supervaluation iff $\psi \leq \overline{\varphi}$. Thus

$$[\varphi, \overline{\varphi}] \cap \text{Cov}(v) = [\varphi, \overline{\varphi}]$$

Proof. (i): $(\overline{\varphi})^\wedge$ is the map $\overline{\varphi}/E(\overline{U}, S(\overline{U})) = \pi_{E(\overline{U}, S(\overline{U}))} \circ \overline{\varphi}$ from $R$ to $\overline{U} := \overline{U}/E(\overline{U}, S(\overline{U}))$. By Lemma 9.12 applied to $\varphi$, we have

$$S(\overline{U}) \cup \{0\} = \overline{\varphi}(q') = s\varphi(q')$$

with $s : \varphi(R) \to T(\overline{U})_0$ denoting the tangible lifting map for $\varphi$. Moreover $\varphi(q') = v(q')$ by Proposition 9.10. Thus $\overline{U} = \overline{U}/E_{v(q')}^\wedge$ and $(\overline{\varphi})^\wedge = \overline{\varphi}/E_{v(q')}^\wedge = \overline{\varphi}$.

(ii): If $\psi$ is a supervaluation, then we know by Proposition 9.10 that $G(\psi) \supseteq v(q') = G(\overline{\varphi})$, and hence by Theorem 9.4 that $\psi \leq \overline{\varphi}$. Conversely, if $\psi \leq \overline{\varphi}$, then $\psi$ is a supervaluation since $\overline{\varphi}$ is a supervaluation (cf. Proposition 9.14).

Definition 9.14.

(i) Given a supervaluation $\varphi := R \to U$ covering $v$ we call

$$\widetilde{\varphi} := (\overline{\varphi})^\wedge : R \to \overline{U} = (\overline{U})^\wedge$$

the almost tangible lift of $\varphi$ (to a supervaluation) and we call $[\widetilde{\varphi}] \in \text{Cov}(v)$ the almost tangible lift (in $\text{Cov}(v)$) of the class $[\varphi] \in \text{Cov}(v)$.

(ii) If $\widetilde{\varphi} = \varphi$, we say that $\varphi$ itself is almost tangible.

Remarks 9.15.

(a) Clearly $\varphi$ is almost tangible iff $G(\varphi) = v(q')$. A subtle point here is that then there may nevertheless exist elements $a \in R\setminus q'$ with $\varphi(a)$ ghost.

(b) If $\varphi$ is any supervaluation, then $\widetilde{\varphi}$ is almost tangible.

(c) If $v$ happens to be a valuation, i.e., $M$ is cancellative, then $\widetilde{\varphi} = \overline{\varphi}$.

Proposition 9.16. If $\psi$ is any almost tangible supervaluation dominating the supervaluation $\varphi$ (but not necessarily covering $v$), then $\psi$ dominates $\overline{\varphi}$.

Proof. $\psi \succeq \overline{\varphi}$, and hence $\psi = (\overline{\psi})^\wedge \succeq (\overline{\varphi})^\wedge = \overline{\varphi}$. □

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