Classification and monogamy of three-qubit biseparable Bell correlations

Michael Seevinck
Institute for History and Foundations of Science, Utrecht University PO Box 80.000, 3508 TA Utrecht, the Netherlands
(Dated: August 15, 2018)

We strengthen the set of Bell-type inequalities presented by Sun & Fei [Phys. Rev. A 74, 032335 (2006)] that give a classification for biseparable correlations and entanglement in tripartite quantum systems. We will furthermore consider the restriction to local orthogonal spin observables and show that this strengthens all previously known such tripartite inequalities. The quadratic inequalities we find indicate a type of monogamy of maximal biseparable tripartite quantum correlations, although the nonmaximal ones can be shared. This is contrasted to recently found monogamy inequalities for bipartite Bell correlations in tripartite systems.

PACS numbers: 03.65.Ud, 03.67.Mn

I. INTRODUCTION

Although Bell inequalities have been originally proposed to test quantum mechanics against local realism, nowadays they also serve another purpose, namely investigating quantum entanglement. Indeed, Bell inequalities were used to give detailed characterisations of multipartite entangled states by giving bounds on the correlations that these states can give rise to \( i \).

Recently a set of Bell-type inequalities was presented by Sun & Fei \( 2 \) that gives a finer classification for entanglement in tripartite systems than was previously known. The inequalities distinguish three different types of bipartite entanglement that may exist in tripartite systems. They not only determine if one of the three parties is separable with respect to the other two, but also which one. It was shown that the three inequalities give a bound that can be thought of as tracing out a sphere in the space of expectations of the three Bell operators that were used in the inequalities. Here we strengthen this bound by showing that all states are confined within the interior of the intersection of three cylinders and the already mentioned sphere.

Furthermore, in Refs. \( 3, 4 \) it was shown that considerably stronger separability inequalities for the expectation of Bell operators can be obtained if one restricts oneselfs to local orthogonal spin observables (so-called LOO’s). We will show that the same is the case for the Bell operators considered here by strengthening all above mentioned tripartite inequalities under the restriction of using orthogonal observables.

The relevant tripartite inequalities are included in the \( N \)-particle inequalities derived in \( 4 \). It was shown that these \( N \)-particle inequalities can be violated maximally by the \( N \)-particle maximally entangled Greenberger-Horne-Zeilinger (GHZ) states\( 6 \), but, as will be shown here, they can also be maximally violated by states that contain only \( (N-1) \)-partite entanglement. Although these inequalities thus allow for further classification of multipartite entanglement (besides some other interesting properties), they can not be used to distinguish full \( N \)-particle entanglement from \( (N-1) \)-particle entanglement in \( N \)-particle states. It is shown that this is neither the case for the stronger bounds that are derived for the case of LOO’s.

In section II the analysis for unrestricted spin observables is performed and in section III for the restriction to LOO’s. Lastly, in the discussion of section IV we will interpret the presented quadratic inequalities as indicating a type of monogamy of maximal biseparable three-particle quantum correlations. Nonmaximal correlations can however be shared. This is contrasted to the recently found monogamy inequalities of Toner & Verstraete \( 5 \).

II. UNRESTRICTED OBSERVABLES

Chen et al. \( 6 \) consider \( N \)-parties that each have two alternative dichotomic measurements denoted by \( A_j \) and \( A_j' \) (outcomes \( \pm1 \)) and show that local realism (LR) requires that

\[
|\langle D^{(i)}_{LR} \rangle| = \frac{1}{2} |B_{N-1}^{(i)}(A_i + A_i') + (A_i - A_i')|_{LR} \leq 1, \quad (1)
\]

for \( i = 1, 2, \ldots, N \), where \( B_{N-1}^{(i)} \) is the Bell polynomial of the Werner-Wolf-Zukowski-Brukner (WWZB) inequalities\( 7 \) for the \( N-1 \) parties, except for party \( i \). These Bell-type inequalities have only two different local settings and are contained in the general inequalities for \( N > 2 \) parties that have more than two alternative measurement settings derived by Laskowski et. al \( 8 \). Indeed, they follow from the latter when choosing certain settings equal. Note furthermore that the WWZB inequalities are contained in the inequalities of Eq. \( 1 \) by choosing \( A_N = A_N' \).

The quantum mechanical counterpart of the Bell-type inequality of Eq. \( 1 \) is obtained by introducing dichotomic observables \( A_k, A_k' \) for each party \( k \). Let us...
define analogously to Sun & Fei [2] the operator
$$D^{(i)}_N = B^{(i)}_{N-1} \otimes (A_i + A'_i)/2 \pm \mathbb{1}_{N-1} \otimes (A_i - A'_i)/2, \quad (2)$$
for \(i = 1, 2, \ldots, N\). Here \(B^{(i)}_{N-1}\) and \(\mathbb{1}_{N-1}\) are respectively the Bell operator of the WWZB inequalities and the identity operator both for the \(N - 1\) qubits not involving qubit \(i\).

The quantum mechanical counterpart of the local realism inequalities of Eq. (1) for all \(i\) is then
$$|\langle D^{(i)}_N \rangle| \leq 1, \quad (3)$$
where \(\langle D^{(i)}_N \rangle := \text{Tr}[D^{(i)}_N \rho]\) and \(\rho\) is a \(N\)-party quantum state.

Since the Bell inequality of Eq. (3) uses only two alternative dichotomic observables for each party the maximum violation of this Bell inequality is obtained for an \(N\)-particle pure qubit state and furthermore for projective observables, as proven recently by Masanes [10] and by Toner & Verstraete [7]. In the following we will thus consider qubits only and the observables will be represented by the spin operators \(A_k = a_k \cdot \sigma\) and \(A'_k = a'_k \cdot \sigma\) with \(a\) and \(a'\) unit vectors that denote the measurement settings and \(\sigma = \sum a_i \sigma_i\) where \(\sigma_i\) are the familiar Pauli spin observables for \(l = x, y, z\) on \(\mathcal{H} = \mathbb{C}^2\). In fact, it suffices [7] to consider only real and traceless observables, so we can set \(a_y = 0\) for all observables.

An interesting feature of the inequalities in Eq. (3) is that all generalised GHZ states \(|\Psi^{(N)}\rangle = \cos \alpha |0\rangle^{\otimes N} + \sin \alpha |1\rangle^{\otimes N}\) can be made to violate them for all \(\alpha\) [4, 9], which is not the case for the WWZB inequalities. Furthermore, the maximum is given by
$$\max_{A_i, A'_i} |\langle D^{(i)}_N \rangle| = 2^{(N-2)/2}, \quad (4)$$
as was proven by Chen et al. [3]. They also noted that this maximum is obtained for the maximally entangled \(N\)-particle GHZ state \(|\text{GHZ}_N\rangle\) (i.e., \(\alpha = \pi/4\)) and for all local unitary transformations of this state. However, not noted in [3] is the fact that the maximum is also obtainable by \(N\)-partite states that only have \((N - 1)\)-particle entanglement, which is the content of the following theorem.

**Theorem 1.** Not only can the maximum value of \(2^{(N-2)/2}\) for \(\langle D^{(i)}_N \rangle\) be reached by fully \(N\)-particle entangled states (proven in [3]) but also by \(N\)-partite states that only have \((N - 1)\)-particle entanglement.

**Proof:** Firstly, \(\langle B^{(i)}_{N-1} \rangle^2 \leq 2^{(N-2)/2} \mathbb{1}_{N-1} \) (as proven in [6]). Here \(X \leq Y\) means that \(Y - X\) is semipositive definite. Thus the maximum possible eigenvalue of \(B^{(i)}_{N-1}\) is \(2^{(N-2)/2}\). Consider a state \(|\Psi^{(N)}_{i}\rangle\) for which \(\langle B^{(i)}_{N-1} |\Psi^{(i)}_{N-1}\rangle = 2^{(N-2)/2}\). This must be [6] a maximally entangled \((N - 1)\)-particle state (for the \(N\) particles except for particle \(i\), such as the state \(|\text{GHZ}_{N-1}\rangle\). Next consider the state \(|\xi^{(i)}\rangle = |\Psi^{(i)}_{N-1}\rangle \otimes |0_i\rangle\), with \(|0_i\rangle\) an eigenstate of the observable \(A_i\) with eigenvalue 1. This is an \(N\)-partite state that only has \((N - 1)\)-particle entanglement. Furthermore choose \(A_i = A'_i\) in Eq. (2). We then obtain \(\langle D^{(i)}_N \rangle = \langle B^{(i)}_{N-1} \mid\Psi^{(i)}_{N-1}\rangle \langle A_i \mid 0_i\rangle = 2^{(N-2)/2}\), which was to be proven. \(\square\)

This theorem thus shows that the Bell inequalities of Eq. (3) can not distinguish between full \(N\)-partite entanglement and \((N - 1)\)-partite entanglement, and thus can not serve as full \(N\)-particle entanglement witnesses.

Let us now concentrate on the tri-partite case \((N = 3\) and \(i = 1, 2, 3\)). Sun & Fei [2] obtain that for fully separable three particle states it follows that \(|\langle D^{(i)}_3 \rangle| \leq 1\), which does not violate the local realistic bound of Eq. (1). General three particle states give \(|\langle D^{(i)}_3 \rangle| \leq \frac{3}{2}\), which follows from Theorem 1 this can be saturated by both fully entangled three particle states as well as for bi-separable entangled three particle states (e.g., two-partite entangled three particle states).

Sun & Fei have furthermore presented a set of Bell inequalities that distinguish three possible forms of bi-separable entanglement. They consider biseparable states that allow for the partitions \(1\) to \(23\), \(2\) to \(13\) and \(3\) to \(12\) where the set of states in these partitions is denoted as \(S_{123}, S_{213}, S_{312}\) and which we label by \(j = 1, 2, 3\) respectively. These sets contain states such as \(\rho_1 \otimes \rho_2\), \(\rho_2 \otimes \rho_3\), and \(\rho_3 \otimes \rho_1\) respectively. For states in partition \(j\) (and for \(i = 1, 2, 3\)) Sun & Fei obtained
$$|\langle D^{(j)}_3 \rangle| \leq \chi_{i,j}, \quad (5)$$
with \(\chi_{i,j} = \sqrt{2}\) for \(i = j\) and \(\chi_{1,2} = 1\) otherwise.

They furthermore proved that for all three qubit states
$$\langle D^{(1)}_3 \rangle^2 + \langle D^{(2)}_3 \rangle^2 + \langle D^{(3)}_3 \rangle^2 \leq 3, \quad \forall \rho. \quad (6)$$
Although this inequality is stronger than the set above (for details see Fig. 1 in [2]), it can be saturated by fully separable states. For example, choose the state \(|000\rangle\) and choose all observables to be projections onto this state. Then we get \(\langle D^{(3)}_3 \mid 000\rangle^2 + \langle D^{(3)}_3 \mid 000\rangle^2 + \langle D^{(3)}_3 \mid 000\rangle^2 = 3\).

Let us consider \(D^{(i)}_3\) (for \(i = 1, 2, 3\)) to be three coordinates of a space in the same spirit as Sun & Fei [2] did. They showed that the fully separable states are confined to a cube with edge length 2 and the biseparable states in partition \(j = 1, 2, 3\) are confined to cuboids with size either \(2\sqrt{2} \times 2 \times 2\), \(2 \times 2 \times 2 \times 2\), or \(2 \times 2 \times 2 \times 2\). Note that states exist that are biseparable with respect to all three partitions (and thus must lie within the cube with edge length 2), but which are not fully separable [11]. Furthermore, all three-qubit states are in the intersection of the cube with size \(2\sqrt{2}\) and of the sphere with radius \(\sqrt{3}\). Sun & Fei note that this sphere is just the external sphere of the cube with edge 2, which is consistent with the above observation that fully separable states can lie on this sphere. If we look at the \(D^{(i)}_3 - D^{(i+1)}_3\) plane we get Fig. 1. The fully separable states are in region I; region II belongs to the biseparable states of partition
$j = i + 1$; and region III belongs to states of partition $j = i$. Other biseparable states and fully entangled states are outside these regions but within the circle with radius $\sqrt{3}$. However, in the following theorem we show a quadratic inequality even stronger than Eq. \((8)\) which thus strengthens the bound in Fig. \(1\) given by the circle of radius $\sqrt{3}$ and which forces the states just mentioned into the black regions.

**Theorem 2.** For the case where each observer chooses between two settings all three qubit states obey the following inequality:

$$\langle D_3^{(i)} \rangle^2 + \langle D_3^{(i+1)} \rangle^2 \leq \frac{5}{2}, \quad \forall \rho, \quad (7)$$

for $i = 1, 2, 3$ and where $i$ and $i + 1$ are both modulo 3.

**Proof**: The proof uses the exact same steps of the proof of Eq. \((8)\) as performed by Sun & Fei (i.e., proof of Theorem 2 in \([2]\)) and can be easily performed for the left hand side of Eq. \((7)\) that contains only two terms instead of the three terms on the right hand side of Eq. \((8)\). This results in only a minor change in calculations \([14]\).

Case (3) in this proof then has the highest bound of $5/2$, whereas the other three cases give a lower bound equal to $2$. \(\square\)

Note that in contrast to Eq. \((8)\) the inequality of Eq. \((7)\) can not be saturated by separable states, since the latter have a maximum of 2 for the left hand expression in Eq. \((7)\).

If we again look at the space given by the coordinates $D_3^{(i)}$ (for $i = 1, 2, 3$), we have thus found that all states are, firstly, confined within the intersection of the three orthogonal cylinders $\langle D_3^{(i)} \rangle^2 + \langle D_3^{(i+1)} \rangle^2 \leq 5/2$ (with $i + 1$ and $i + 2$ both modulo 3) each with radius $\sqrt{5/2}$ and, secondly, they must furthermore still lie within the cube of edge length $2\sqrt{2}$, and thirdly they must also lie within the sphere with radius $\sqrt{3}$. In Fig. \(1\) we see the strengthened bound of Eq. \((7)\) as compared to the bound of Sun & Fei. However, we see from this figure that neither the intersection of the three cylinders, nor the sphere, nor the cube give tight bounds.

The black areas in Fig. \(1\) are nonempty. For the case of Eq. \((7)\) states thus exist that have both $\langle D_3^{(i)} \rangle > 1$ and $\langle D_3^{(i+1)} \rangle > 1$ (for some $i$). For example, the so-called $W$-state

$$|W\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}, \quad (8)$$

gives $\langle D_3^{(i)} \rangle = 1.022$ for all $i$ when the observables are chosen as follows: $A_i = \cos \alpha_i \sigma_x + \sin \alpha_i \sigma_z$ with $\alpha_i = -0.133$ and $A'_i = \cos \beta_i \sigma_x + \sin \beta_i \sigma_z$ with $\beta_i = 0.460$.

**III. RESTRICTION TO LOCAL ORTHOGONAL SPIN OBSERVABLES**

In Refs. \([2, 4]\) it was shown that considerably stronger separability inequalities for the expectation of the bipartite Bell operator $B_2$ can be obtained if one restricts oneself to local orthogonal observables (LOO’s). We will now show that the same is the case for the Bell operator $D_3^{(i)}$. The following theorem strengthens all previous bounds of section II for general observables.

**Theorem 3.** Suppose all local observables are orthogonal, i.e., $a_i \cdot a'_i = 0$, then the following inequalities hold:

(i) For all states: $|\langle D_3^{(i)} \rangle| \leq \sqrt{3/2} \approx 1.225$.

(ii) For fully separable states: $|\langle D_3^{(i)} \rangle| \leq \sqrt{3/4} \approx 0.866$.

(iii) For biseparable states in partition $j = 1, 2, 3$:

$$|\langle D_3^{(i)} \rangle| \leq \chi_{i,j}, \quad (9)$$

with $\chi_{i,j} = \sqrt{3/2} \approx 1.225$ for $i = j$ and $\chi_{i,j} = \sqrt{3/4} \approx 0.866$ otherwise.

(iv) Lastly, for all states:

$$\langle D_3^{(i)} \rangle^2 + \langle D_3^{(i+1)} \rangle^2 \leq 2.\quad (10)$$

**Proof**: (i) The square of $D_3^{(i)}$ is given by

$$\langle D_3^{(i)} \rangle^2 = \langle B_2^{(i)} \rangle^2 \otimes \frac{1}{2} (1 + a_i \cdot a'_i), \quad (11)$$

where $B_2^{(i)}$ is the identity operator for the 2 qubits not including qubit $i$. For orthogonal observables we get $a_i \cdot a'_i = 0$, and $\langle B_2^{(i)} \rangle^2 \leq 2I_2^{(i)}$ (as proven in \([2, 4]\)). The maximum eigenvalue of $\langle B_2^{(i)} \rangle^2$ is thus $3/2$, which implies that $|\langle D_3^{(i)} \rangle| \leq \sqrt{3/2}$.

![Figure 1: $D_3^{(i)} - D_3^{(i+1)}$ plane with the stronger bound given by the circle with radius $\sqrt{3/2}$ which strengthens the less strong bound with radius $\sqrt{3}$ that is given by the dashed circle.](image)
(ii) For fully separable states we have from Eq. (2) that
\[
\langle D_{3}^{(i)} \rangle = \frac{1}{2} \left( \langle B_{2}^{(i)} \rangle \langle (A_{i} + A'_{i}) \rangle + \langle (A_{i} - A'_{i}) \rangle \right).
\] (12)

Furthermore for the case of orthogonal observables \( \langle B_{2}^{(i)} \rangle \leq 1/\sqrt{2} \) [3, 4]. Thus \( \langle D_{3}^{(i)} \rangle \leq 1/2 \times (\langle (A_{i} + A'_{i}) \rangle + \langle (A_{i} - A'_{i}) \rangle) \). Since the averages are linear in the state \( \rho \) the maximum is obtained for a pure state of qubit \( i \). This state can be represented as \( 1/2 (|o + o'\sigma) \), with \( |o| = 1 \) and \( o \cdot \sigma = \sum_{k} o_{k}o'_{k} \) (\( k = x, y, z \)). Take \( C = (A_{i} + A'_{i}) \), \( D = (A_{i} - A'_{i}) \) and \( s = a_{i} + a'_{i} \), \( t = a_{i} - a'_{i} \). We get \( |s| = |t| = \sqrt{2} \). Choose now without losing generality \( \theta \)
\[
\sqrt{2} \cos \theta, 0, \sin \theta \) and \( t = \sqrt{2}(-\sin \theta, 0, \cos \theta) \). Then
\[
\langle D_{3}^{(i)} \rangle \leq \left| \langle s - o/\sqrt{2} + t \cdot o \rangle \right|/2
\]
\[
= \frac{1}{2} \left( (o_{x} - \sqrt{2}a_{x}) \sin \theta + (a_{x} + \sqrt{2a_{x}} \cos \theta) \right) .
\]

Maximizing over \( \theta \) (i.e., \( \max_{\theta} 2(\cos \theta + \sin \theta) = \sqrt{2} \)) and using \( a_{x}^{2} + a_{y}^{2} + a_{z}^{2} = 1 \) we finally get
\[
\langle D_{3}^{(i)} \rangle \leq \left| \sqrt{3/4} (a_{x}^{2} + a_{y}^{2}) \right| \leq \sqrt{3/4} .
\] (13)

(iii) For biseparable states in partition \( j = i \) we get the same as in Eq. (12), but now \( \langle D_{3}^{(i)} \rangle \) or \( \langle D_{3}^{(i)} \rangle \) using the method of (ii) we get
\[
\langle D_{3}^{(i)} \rangle \leq \left| \langle \sqrt{2} s - o + t \cdot o \rangle \right|/2 \leq \sqrt{3/2} .
\] (14)

For biseparable states in partition \( i + 1 \) and \( i + 2 \) a somewhat more elaborated proof is needed. Let us set \( i = 1 \) and \( j = 3 \) for convenience (for the other partition \( j = 2 \) we get the same result). The maximum is again obtained for pure states. Every pure state in partition \( j = 3 \) can be written as \( |\psi\rangle = |\psi\rangle_{12} \otimes |\psi\rangle_{3} \). Then
\[
\langle D_{3}^{(i)} \rangle = -\frac{1}{4} \left( \langle A_{1} + A'_{1} \rangle \langle A_{2} + A'_{2} \rangle |\psi_{12}\rangle \langle A_{3} \rangle_{3} \langle \psi_{3} \rangle_{3} \right)
\]
\[
+\frac{1}{4} \left( \langle A_{1} + A'_{1} \rangle \langle A_{2} - A_{2} \rangle |\psi_{12}\rangle \langle A'_{3} \rangle_{3} \langle \psi_{3} \rangle_{3} \right)
\]
\[
+\frac{1}{2} \left( \langle A_{1} - A'_{1} \rangle \otimes |12\rangle \langle \psi_{12} \rangle \right) .
\]

Using the technique in (ii) above it is found that the maximum over \( |\psi_{3}\rangle \) gives
\[
\langle D_{3}^{(i)} \rangle \leq \frac{\sqrt{3}}{4} \left( \langle A_{1} + A'_{1} \rangle^{2} \right)_{12}^{1/2}
\]
\[
+ \left\langle (A_{1} + A'_{1}) \langle A_{2} + A'_{2} \rangle \right\rangle^{1/2}
\]
\[
+ \frac{1}{2} \left( \langle A_{1} - A'_{1} \rangle \otimes |12\rangle \langle \psi_{12} \rangle \right) .
\] (16)

Without losing generality we choose \( A_{1}, A'_{1} \) in the \( x - z \) plane \( \langle A_{1} + A'_{1} \rangle^{2} \) for the case of orthogonal observables. Since the observables \( A \) and \( A' \) must be orthogonal (i.e., \( a_{x} = 1/\sqrt{2} \) and \( a_{z} = -a_{z}' = 1/\sqrt{2} \), we finally get:
\[
\langle D_{3}^{(i)} \rangle \leq \frac{\sqrt{3}}{4} \left( \langle A_{1} + A'_{1} \rangle^{2} \right)_{12}^{1/2} + \langle a_{x} - a_{x}' \rangle^{2} \right) \langle 12 \rangle \langle a_{x} + a'_{x} \rangle^{1/2} .
\]

Since the observables \( A \) and \( A' \) must be orthogonal (i.e., \( a_{x} = 1/\sqrt{2} \) and \( a_{z} = -a_{z}' = 1/\sqrt{2} \), we finally get:
\[
\langle D_{3}^{(i)} \rangle \leq \frac{\sqrt{3}}{2} \langle 12 \rangle \langle a_{x} + a'_{x} \rangle^{1/2} \left\langle (a_{x} + a'_{x})^{2} \right\rangle \langle 12 \rangle \langle a_{x} + a'_{x} \rangle^{1/2} .
\]

(iv) We use the exact same steps of the proof of Sun & Fei of Eq. (9) (i.e., proof of Theorem 2 in [2]) but since the observables are orthogonal only case (4) of that proof needs to be evaluated. This can be easily performed for the left hand side of Eq. (10) that contains only two terms instead of the three terms on the right hand side of Eq. (4), thereby resulting in only a minor modification of the calculations [12] giving the result
\[
\langle D_{3}^{(i)} \rangle^{2} + \langle D_{3}^{(i+1)} \rangle^{2} \leq 2 .
\]

These results for orthogonal observables can again be interpreted in terms of the space given by the coordinates \( D_{3}^{(i)} \) (for \( i = 1, 2, 3 \). The same structure as in Fig. 1 then arises but with the different numerical bounds of Theorem 2. The fully separable states are confined to a cube with edge length \( \sqrt{3} \) and the biseparable states in partition \( j = 1, 2, 3 \) are confined to cuboids with size either \( \sqrt{3} \times \sqrt{3} \times \sqrt{3} \), \( \sqrt{3} \times \sqrt{3} \times \sqrt{3} \), or \( \sqrt{3} \times \sqrt{3} \times \sqrt{3} \). Furthermore, all three-qubit states are in the intersection of firstly the cube with edge length \( \sqrt{6} \), secondly of the three orthogonal cylinders with radius \( \sqrt{2} \), and thirdly of the sphere with radius \( \sqrt{3} \).

The corresponding \( D_{3}^{(i)} - D_{3}^{(i+1)} \) plane is drawn in Fig. 2. Compared to the case where no restriction was made to orthogonal observables (cf. Fig. 1) we see that we can still distinguish the different kinds of biseparable states, but they can still not be distinguished from fully three-particle entangled states since both types of states still have the same maximum for \( \langle D_{3}^{(i)} \rangle \). Furthermore, the ratio of the different maxima of \( \langle D_{3}^{(i)} \rangle \) for fully separable and bi-separable states is still the same, i.e., the ratio is \( \sqrt{3}/1 = (\sqrt{3}/2)/(\sqrt{3}/4) = \sqrt{2} \).

The black areas in Fig. 2 are again non empty since states exist that have both \( \langle D_{3}^{(i)} \rangle > \sqrt{3/4} \) and \( \langle D_{3}^{(i+1)} \rangle > \sqrt{3/4} \) for the case of orthogonal observables. For example, the W-state of Eq. (8) gives \( \langle D_{3}^{(i)} \rangle = 0.906 \) for all \( i \), for the local angles \( \alpha_{i} = 0.54 = \beta_{i} - \pi/2 \) in the \( x - z \) plane.

IV. DISCUSSION

Let us take another look at the quadratic inequalities
\[
\langle D_{3}^{(i)} \rangle^{2} + \langle D_{3}^{(i+1)} \rangle^{2} \leq 5/2 \text{ of Eq. (7) for general observables and } \langle D_{3}^{(i)} \rangle^{2} + \langle D_{3}^{(i+1)} \rangle^{2} \leq 2 \text{ of Eq. (10) for orthogonal observables. These can be interpreted as monogamy inequalities for maximal biseparable three-particle quantum correlations (i.e., biseparable correlations that violate the inequalities maximally), since the inequalities show that a state that has maximal bi-separable Bell correlations for a certain partition cannot have it maximally}.


for another partition. Indeed, when partition $i$ gives $|\langle D_3^{(i)} \rangle| = \sqrt{2}$ it must be the case according to Eq. (7) that for the other two partitions both $|\langle D_3^{(i+1)} \rangle| \leq \sqrt{1/2}$ and $|\langle D_3^{(i+2)} \rangle| \leq \sqrt{1/2}$ must hold. The latter two must thus be non-maximal as soon as the first type of biseparable correlation is maximal. And for the second inequality of Eq. (10) using orthogonal observables we get that when $|\langle D_3^{(i)} \rangle| = \sqrt{3/2}$ (this is maximal) it must be the case that both $|\langle D_3^{(i+1)} \rangle| \leq \sqrt{1/2}$ and $|\langle D_3^{(i+2)} \rangle| \leq \sqrt{1/2}$, which is non-maximal.

From this we see that the first (i.e., Eq. (7) for general observables) is a stronger monogamy relationship than the second (i.e., Eq. (10) for orthogonal observables) since the trade-off between how much the maximal value for $|\langle D_3^{(i)} \rangle|$ for one partition $i$ restricts the value of $|\langle D_3^{(i+1)} \rangle|$, $|\langle D_3^{(i+2)} \rangle|$ for the other two partitions below the maximal value is larger in the first case than in the second case.

Let us see how this compares to the monogamy inequality $\langle B_2^{(i)} \rangle^2 + \langle B_2^{(i+1)} \rangle^2 \leq 2$ which was recently obtained by Toner and Verstraete [7]. Note that $|\langle B_2^{(i)} \rangle| \leq 1$ is the ordinary Bell-CHSH inequality for the two qubits other than qubit $i$. We see that this monogamy inequality is even more strong than the ones presented here, since when $|\langle B_2^{(i)} \rangle|$ obtains its maximal value of $\sqrt{2}$ it must be that $|\langle B_2^{(i+1)} \rangle| = |\langle B_2^{(i+2)} \rangle| = 0$.

Furthermore, the monogamy relationship of Toner & Verstraete shows that the so-called nonlocality that is indicated by correlations that violate the Bell-CHSH inequality [10] cannot be shared (cf. [12]): as soon as for some $i$ one has $|\langle B_2^{(i)} \rangle| > 1$, it must be that both $|\langle B_2^{(i+1)} \rangle| < 1$ and $|\langle B_2^{(i+2)} \rangle| < 1$. However, in Ref. [12] it was nevertheless shown that a bipartite Bell-type inequality exists where it is the case that the nonlocality that this inequality allows for can be shared. Since $|\langle D_3^{(i)} \rangle| \leq 1$ are Bell-type inequalities (i.e., local realism has to obey them, see Eq. (11) whose violation can be seen to indicate some nonlocality, the inequalities considered here could possibly also allow for some nonlocality sharing.

Indeed, this is the case since it was shown that the black areas in Fig. 1 are nonempty. The Bell-type inequalities given here thus allow for sharing of the nonlocality of biseparable three-particle quantum correlations that is indicated by a violation of these inequalities.

In conclusion, we have presented stronger bounds for bi-separable correlations in three-particle systems than were given in [2] and extended this analysis to the case of the restriction to orthogonal observables which gave even stronger bounds. The quadratic inequalities for biseparable correlations gave a monogamy relationship for correlations that violate the inequalities maximally (i.e., these cannot be shared), but they nevertheless did allow for sharing of the non-maximally violating correlations.

We hope that future research will reveal more of the structure of the different kinds of partial separability in multipartite states and of the monogamy of multi-partite Bell correlations. It could therefore be fruitful to generalize this work from three to a larger number of parties.

Acknowledgements.— The author thanks Jos Uffink for fruitful discussions.

[1] N. Gisin, H. Bechmann-Pesquimucci, Phys. Lett. A 246, 1 (1998); J. Uffink, Phys. Rev. Lett. 88, 230406 (2002); M. Seevinck and G. Svetlichny, Phys. Rev. Lett. 89, 060401 (2002); S. Yu, Z.-B. Chen, J.-W. Pan and Y.-D. Zhang, Phys. Rev. Lett. 90, 080401 (2003).
[2] B-Z. Sun and S-M. Fei, Phys. Rev. A 74, 032335 (2006).
[3] S.M. Roy, Phys. Rev. Lett. 94, 010402 (2005).
[4] J. Uffink and M. Seevinck, quant-ph/0604145 (2006).
[5] O. Gühne, M. Mecher, G. Tóth and P. Adam, Phys. Rev. A 74, 010301(R) (2006); S. Yu and N.L. Liu, Phys. Rev. Lett. 95, 150504 (2005).
[6] K. Chen, S. Albeverio and S-M. Fei, Phys. Rev. A 74, 050101(R) (2006).
[7] B.F. Toner and F. Verstraete, quant-ph/0611001 (2006).
[8] R.F. Werner, M.M. Wolf, Phys. Rev. A 64, 032112 (2002). M. Žukowski and Č. Brukner, Phys. Rev. Lett. 88, 210401 (2002).
[9] W. Laskowski, T. Paterek, M. Žukowski and Č. Brukner, Phys. Rev. Lett. 93, 200401 (2004).
[10] Ll. Masanes, Phys. Rev. Lett. 97, 050503 (2006).
Masanes, quant-ph/0512100 (2005).

[11] C.H. Bennett, D.P. DiVincenzo, T. Mor, P.W. Shor, J.A. Smolin, B.M. Terhal, Phys. Rev. Lett. 82, 5385 (1999).

[12] V. Scarani and N. Gisin, Phys. Rev. Lett. 87, 117901 (2001).

[13] D. Collins and N. Gisin, J. Phys. A: Math. Gen. 37, 1775 (2004).

[14] In further detail, steps (1) to (4) of the proof in Sun & Fei [2] become (using the terminology of their proof): (1): \( \omega = 2(s_1 \otimes s_2 \otimes s_3 \cdot Q)^2 = 2(\langle C_1C_2C_3 | \Psi \rangle)^2 \leq 2 \), (2): \( \omega = 2(s_1 \otimes s_2 \otimes s_3 \cdot Q + s_1 \otimes s_2 \otimes t_3 \cdot Q)^2 = 2(\langle C_1C_2(C_3 + D_3) | \Psi \rangle)^2 \leq 2 \), (3): \( \omega = (5/4)(\cos(\theta_1 + \theta_2 + \theta_3) - \sin(\theta_1 + \theta_2 + \theta_3))^2 \leq 5/2 \), (4): \( \omega = (\cos(\theta_1 + \theta_2 + \theta_3) - \sin(\theta_1 + \theta_2 + \theta_3))^2 \leq 2 \). Here \( \omega = \langle D^{(i)}_3 \rangle^2 + \langle D^{(i+1)}_3 \rangle^2 \) (i.e., the l.h.s. of Eq. (7)), where we have chosen \( i = 1 \). Note that by symmetry the proof goes analogous for \( i = 2, 3 \). It follows that step (3) has the highest bound of 5/2.

[15] In further detail, the proof in Sun & Fei [2] for the case of orthogonal observables amounts to (using the terminology of their proof) \( |s_i| = |t_i| = \sqrt{2}/2 \). Thus only step (4) needs to be evaluated and this gives \( \omega = (\cos(\theta_1 + \theta_2 + \theta_3) - \sin(\theta_1 + \theta_2 + \theta_3))^2 \leq 2 \). As in the proof of Theorem 2 we have \( \omega = \langle D^{(i)}_3 \rangle^2 + \langle D^{(i+1)}_3 \rangle^2 \) (i.e., the l.h.s. of Eq. (10)), where again we have chosen \( i = 1 \), but by symmetry the proof goes analogous for \( i = 2, 3 \).

[16] A nonlocal correlation is a correlation that can not be reproduced by shared randomness or any other local variables. It is detected by means of a violation of a Bell-type inequality.