LEFSCHETZ FIBRATIONS ON COTANGENT BUNDLES AND SOME PLUMBINGS

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Abstract. We introduce an idea of constructing Lefschetz fibrations of Weinstein manifolds from Weinstein handle decompositions on them. We prove theorems which formulate the idea for the cases of cotangent bundles and some plumbings. As a corollary, we give diffeomorphic families of plumbing spaces. Those diffeomorphic families contain some plumbing spaces with names. For example, Milnor fibers of $A_{4k+3}$ and $D_{4k+3}$ singularities are diffeomorphic if their dimension is $2n$ with odd $n \geq 3$.

1. Introduction

1.1. Motivation. Lefschetz fibrations are powerful tools in Symplectic topology. For example, in [10], [11], and [1], Lefschetz fibrations are used to construct exotic pairs of Weinstein manifolds, i.e., pairs of Weinstein manifolds which are diffeomorphic, but are different from each other as Weinstein manifolds. Another example of using Lefschetz fibrations is described in [16, Sections 2 and 3]. The example is using a Lefschetz fibration on a Weinstein manifold in order to describe the symplectic mapping class group of the Weinstein manifold. In [13], McLean gives a systematic way to compute the symplectic cohomology of the total space of a Lefschetz fibration, by using the Lefschetz fibration. Also, there are much more ways of using Lefschetz fibrations in symplectic topology than what we listed above.

Since Lefschetz fibrations are useful, it would be natural to ask which symplectic manifolds admit Lefschetz fibrations. Giroux and Pardon [7] gave a wonderful answer for the question. They proved that every Stein manifold should admit a Lefschetz fibration. They also proved that every Weinstein manifold should admit a Lefschetz fibration indirectly, by using the equivalence between Stein and Weinstein structures, given in [5].

In the present paper, we introduce an idea of constructing Lefschetz fibrations of Weinstein manifolds, from Weinstein structures directly. More precisely, we construct Lefschetz fibrations from Weinstein handle decompositions.

Unfortunately, the present paper does not consider a general Weinstein manifold. We consider Weinstein manifolds of two types. The Weinstein manifolds of the first type are cotangent bundles, and those of the second type are Weinstein manifolds constructed by plumbing copies of cotangent bundles of spheres along trees, or by plumbing two cotangent bundles. For the case of general Weinstein manifolds, we are working on progress.

As an application, we give diffeomorphic families of Weinstein manifolds. Especially, some of the given diffeomorphic families contain the Milnor fibers of simple singularities.

More specific results will appear in Section 1.2.
1.2. Results. The current paper consists of two parts, except Sections 1 and 2 which are the introduction and the preliminaries.

In the first part, we discuss the case of cotangent bundles. This is because it is easy to consider the Weinstein handle decompositions of cotangent bundles.

To be more precise, let \( M \) be a smooth manifold. Then, the cotangent bundle \( T^*M \) admits a natural Liouville structure. This Liouville structure is not a Weinstein structure, since the zeros of the standard Liouville 1-form are not isolated. However, one could easily obtain a Weinstein structure of \( T^*M \) by perturbing the standard Liouville structure together with a Morse function on \( M \).

Motivated from this, we construct an algorithm producing a Weinstein handle decomposition of \( T^*M \) from a handle decomposition of \( M \). We note that a Morse function on \( M \) induces a handle decomposition of \( M \). Then, Theorem 1.1 constructs a Lefschetz fibration on \( T^*M \) using the given Weinstein handle decomposition.

**Theorem 1.1** (Technical statement is Theorem 5.1). Let \( M \) be a smooth manifold. We give an algorithm producing a Lefschetz fibration on \( T^*M \) from a handle decomposition of \( M \).

When we have two different handle decompositions of a manifold \( M \), then Theorem 1.1 gives different Lefschetz fibrations on \( T^*M \). On the other hand, it is well-known that every handle decomposition of the same manifold is connected to each other by handle moves. Thus, it would be natural to ask the relation between different Lefschetz fibrations which are obtained by applying Theorem 1.1 to different handle decompositions of \( M \). For the case of 2 dimensional manifold \( M \), we answer the question, i.e., we prove Proposition 1.2.

**Proposition 1.2** (Technical statement is Proposition 7.2). If \( M \) is a 2 dimensional smooth manifold, then every Lefschetz fibration on \( T^*M \) obtained by applying Theorem 1.1 is unique up to four moves given in Section 7.1.

After proving Proposition 1.2 we move on to the second part which considers Weinstein manifolds obtained by plumbing cotangent bundles. However, since Weinstein handle decompositions on plumbings are more complicated to be described than those of cotangent bundles, we restricts our interest to two cases. The first case is plumbings of two cotangent bundles, and the second case is plumbings of copies of the cotangent bundle of a sphere along trees.

We start the second part by proving Theorem 1.3.

**Theorem 1.3** (Technical statement is Theorem 8.2). Let \( M_1 \) and \( M_2 \) be smooth manifolds of the same dimension. There is an algorithm producing a Lefschetz fibration on the plumbing of \( T^*M_1 \) and \( T^*M_2 \) at one point from a pair of handle decompositions of \( M_1 \) and \( M_2 \).

Also, we give Theorem 1.4.

**Theorem 1.4** (Technical statement is Theorem 11.4). Let \( P \) be a Weinstein manifold obtained by plumbing \( T^*S^n \) along a tree \( T \). Then, we give an algorithm producing a Lefschetz fibration defined on \( P \).

Even though we consider some restricted examples, there are possible applications. One of the possible applications is Corollary 13.2 which gives diffeomorphic families of Weinstein manifolds. The members of a diffeomorphic family are
plumbings of cotangent bundles of spheres. Moreover, the families contain some Milnor fibers of simple singularities. For example, one could check that Minor fibers of \( A_{4k+3} \), \( D_{4k+3} \)-singularities are diffeomorphic to each other if their dimension is \( 2n \) with odd \( n \geq 3 \). For more detail, see Corollary 13.2.

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2. Preliminaries

In Section 2, we review preliminaries and partially set notation.

2.1. **Handle decomposition.** In the present subsection, we explain what notion we mean by “handle decomposition”.

**Definition 2.1.**

1. An \( n \) dimensional standard handle \( h^i \) of index \( i \) is a subspace

\[
 h^i = D^i \times D^{n-i}
\]

in \( \mathbb{R}^n \), where \( D^k \) is the disk of radius 1 in \( \mathbb{R}^k \).

2. The attaching region of \( h^i \) is \( \partial D^i \times D^{n-i} = S^{i-1} \times D^{n-i} \). Let \( \partial_R h^i \) denote the attaching region of \( h^i \).

If there is no chance of confusion, we sometimes omit its dimension and simply call it the standard \( i \)-handle.

Let \( M \) be an \( n \) dimensional manifold with boundary. If there is a map \( \phi : \partial_R h^i \rightarrow \partial M \), then one can attach the \( n \) dimensional standard handle \( h^i \) to \( M \). As the result of the attaching, one obtains another \( n \) dimensional manifold, given as follows:

\[
 M \cup h^i / \sim, x \sim \phi(x) \text{ for all } x \in \partial_R h.
\]

Based on this, the notion of handle decomposition of \( M \) mean data explaining the construction of \( M \) as a union of handles. More precise definition is following below.

**Definition 2.2.** By a handle decomposition of an \( n \) dimensional smooth manifold \( M \), we mean a finite, ordered set of \( n \) dimensional handles \( h_0, \ldots, h_m \) together with the injective maps \( \phi_i : \partial_R h_i \rightarrow \partial(\cup_{j=0}^{i-1} h_j) \) satisfying the followings

- \( h_0 \) is the unique index 0 handle,
- there is a natural number \( N \) such that for \( i \leq N \) (resp. \( i > N \)), \( h_i \) is sub-critical (resp. critical), i.e., \( \text{ind}(h_i) < n \) (resp. \( \text{ind}(h_i) = n \)),
- two different critical handles are disjoint, or equivalently, every critical handle are attached to the union of subcritical handles, and
- \( \cup_{i=0}^m h_i \) is diffeomorphic to \( M \).

The maps \( \phi_i \) are called gluing maps.

We note that the unions in the above definition are “not” disjoint unions of standard handles. The unions mean the gluing by the gluing maps \( \phi_i \).

**Remark 2.3.** We also note that Definition 2.2 is not a definition which is usually used in literature. However, we use Definition 2.2 for some technical reasons which will be appeared later.
We also define the following notation for the later use.

Definition 2.4. Let $\mathcal{H}(M)$ be the set of handle decomposition of a smooth manifold $M$.

2.2. Weinstein Handle. We review the notion of Weinstein handle and their attachment in Section 2.2. For more detail, we refer the reader to Weinstein [15].

In order to define a standard Weinstein handle, we fix a smooth function $F : \mathbb{R}^2 \to \mathbb{R}$ such that

- $F(0, 0) \neq 0$,
- whenever $F(x, y) = 0$, the partial derivatives of $F$, $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ do not have the same sign,
- $\frac{\partial F}{\partial y} \neq 0$ when $y = 0$, and
- $\frac{\partial F}{\partial x} \neq 0$ when $x = 0$.

Let fix an integer $i$, in order to define the Weinstein handle of index $i$. Let the standard symplectic Euclidean space $\big(\mathbb{R}^{2n}, \omega_{\text{std}}\big)$ be equipped with a Liouville form

\begin{equation}
\lambda_i = \sum_{j=1}^{i} - (x_j dy_j + 2y_j dx_j) + \sum_{j=1}^{n-i} \frac{1}{2} (p_j dq_j - q_j dp_j). \tag{2.1}
\end{equation}

Here $(x_1, \cdots, x_i, y_1, \cdots, y_i, p_1, \cdots, p_{n-i}, q_1, q_{n-i})$ are coordinates of $\mathbb{R}^{2n}$. Then, the Liouville vector field corresponding to $\lambda_i$ is the gradient vector field, with respect to the standard Euclidean metric, of the Morse function

$$f_i = \sum_{j=1}^{i} (y_j^2 - \frac{1}{2} x_j^2) + \sum_{j=1}^{n-i} \frac{1}{4} (p_j^2 + q_j^2).$$

Weinstein [15] defined the notion of Weinstein handle as follows.

Definition 2.5. The standard $2n$ dimensional Weinstein $i$ handle $H^i$ is a region of $\big(\mathbb{R}^{2n}, \omega_{\text{std}}, \lambda_i\big)$ satisfying

- the region is bounded by
  $$\{ f_i^{-1}(\frac{1}{2}) \} \text{ and } \{ F(\sum_{j=1}^{i} x_j^2, \sum_{j=i+1}^{n} x_j^2 + \sum_{j=1}^{n} y_j^2) = 0 \},$$
  - the region contains the origin point.

[15] Lemma 3.1] proved that the choice of a specific function $F$ does not change a standard handle up to symplectic completion.

Remark 2.6. It is easy to check that as a smooth manifold, the $2n$ dimensional standard Weinstein $i$ handle $H^i$ is diffeomorphic to a smooth $2n$ dimensional $i$ handle $h^i$. In order to avoid confusion, we will use the uppercase $H$ (resp. the lower case $h$) for a Weinstein handle (resp. smooth handle).

The following notion are necessarily to discuss the attachment of Weinstein handles.

Definition 2.7.

1. The attaching region of $H^i$ is the intersection of $\partial H^i$ and $f_i^{-1}(\frac{1}{2})$. As similar to the case of smooth handles, let $\partial_R H^i$ denote the attaching region.
(2) The attaching sphere of $H_i$ is the intersection of $\partial RH_i$ and the isotropic subspace
\[ \{ y_1 = \cdots = y_i = p_1 = \cdots = p_{n-i} = q_1 = \cdots = q_{n-i} = 0 \} \subset \mathbb{R}^{2n}. \]
Let $\partial S H_i$ denote the attaching sphere.

In order to attach a Weinstein handle $H$ to a Weinstein domain $W$, one needs a gluing map $\phi : \partial RH \to \partial W$. The difference from the smooth handle attachment is that one should consider the Weinstein structures on $H$ and $W$. Thus, the gluing map should preserve the contact structure, or more precisely, $\phi$ should be a contactomorphism between $\partial RH$ and the image of $\phi$.

**Remark 2.8.** Let $W$ be a Weinstein manifold. Let assume that there are two gluing maps $\phi_0, \phi_1 : \partial RH \to \partial W$ which are contactoisomorphic in the following sense: there is a one parameter family $f_t : W \to W$ of symplectomorphisms, such that $f_0$ is the identity and $\phi_1 = f_1 \circ \phi_0$.

If $W_i$ denotes the Weinstein manifold obtained by attaching $H$ to $W$ with $\phi_i$, it is easy to check that $W_0$ and $W_1$ have symplectomorphic symplectic completions. One can show that by using the one parameter family $W_t$ of Weinstein manifolds which are obtained by attaching $H$ to $W$ with $f_t \circ \phi_0$.

[15] showed that in order to attach a Weinstein handle $H_i$ of index $i$, it is enough to remember some local information, rather than the gluing map defined on the attaching region. More precise statement will appear at the last part of the present subsection.

The local information consist of a pair of an isotropic $(i-1)$ sphere $\Lambda$, which the attaching sphere of $H_i$ will be attached along, and a trivialization of the “conformal symplectic normal bundle of $\Lambda$”. In the rest of Section 2.2, we review the notion of conformal symplectic normal bundle.

Let $(X, \xi)$ be a $(2n-1)$ dimensional contact manifold where $\xi$ is the given contact structure. (Or one could consider a $2n$ dimensional Weinstein domain and let $X$ be the boundary of $W$..) If $\alpha$ is a contact 1 form on $X$, then, it is well-known that $(\xi_x, d\alpha)$ is a symplectic vector space.

Let $\Lambda$ be an isotropic $(i-1)$ dimensional sphere in $X$. Then, $T_x \Lambda$ is an isotropic subspace of a symplectic vector space $(\xi_x, d\alpha)$. Thus, if $T_x \Lambda^\perp$ means the symplectic dual of $T_x \Lambda$, i.e.,
\[
T_x \Lambda^\perp := \{ v \in \xi_x \mid d\alpha(v, w) = 0 \text{ for all } w \in T_x \Lambda \},
\]
then,
\[
T_x \Lambda \subset T_x \Lambda^\perp.
\]

One can easily check that the quotient
\[
T \Lambda^\perp / T \Lambda
\]
is a $(2n-2i)$ dimensional vector bundle over $\Lambda$ which carries a conformal symplectic structure naturally induced from $d\alpha$.

**Definition 2.9.** The quotient in Equation (2.2) is called the conformal symplectic normal bundle of $\Lambda$. Let $CSN(\Lambda)$ denote the conformal symplectic normal bundle of $\Lambda$. 

The result of \cite{15} is to determine a contact isotopy class of a gluing map \( \phi : \partial_R H \to X \) from a pair of \( \Lambda \) and \( CSN(\Lambda) \). Thus, one could attach a Weinstein handle from the information given by the pair \((\Lambda, CSN(\Lambda))\) uniquely up to symplectomorphic symplectic completion. Remark 2.8 explains briefly how the contact isotopy class induces the uniqueness.

Conversely, if there is a gluing map \( \phi : \partial_R H \to X \), then \( \phi \) induces an isotropic sphere \( \Lambda := \phi(\partial_S H) \) and the differential \( D\phi \) induces a trivialization of \( CSN(\Lambda) \), which the pair recovers the contact isotopy class of \( \phi \).

2.3. **Weinstein handle decomposition.** It is well-known that every Weinstein domain can be broken down into Weinstein handles, or equivalently, every Weinstein domain admits a Weinstein handle decomposition. In Section 2.3, we defined the notion of Weinstein handle decomposition, which we use in the present paper.

We recall that Definition 2.2 defines a handle decomposition of \( M \) as a collection of handles and gluing information of them. In other words, a handle decomposition of \( M \) explains how to construct \( M \) as an attachment of handles to a unique 0 handle. In the context, constructing \( M \) actually means that constructing a smooth manifold which is diffeomorphic to \( M \), i.e., Definition 2.2 is defined up to diffeomorphisms.

As similar to Definition 2.2, we define a handle decomposition of a Weinstein domain \( W \) as a collection of Weinstein handles together with gluing information. Thus, a Weinstein handle decomposition of \( W \) gives a Weinstein domain which is equivalent to \( W \). Before defining the notion of a Weinstein handle decomposition, we discuss which equivalence we consider in the current paper.

A technical difficulty of studying Weinstein domains arises from the incompleteness of Weinstein domains. In order to resolve the difficulty, one could take the symplectic completions of them. For more details, we refer the reader to \cite[Section 11]{8}. Based on this, we define the equivalence as follows.

**Definition 2.10.** We say that two Weinstein domains are equivalent to each other if their symplectic completions are exact symplectomorphic.

We note that if two finite type Weinstein manifolds are symplectomorphic, then they are exact symplectomorphic by \cite[Theorem 11.2]{5}.

**Definition 2.11.** By a Weinstein handle decomposition of a Weinstein domain \( W \), we mean a finite, ordered set of Weinstein handles \( H_0, \cdots, H_m \) together with the injective maps \( \Phi_i : \partial S H_i \to \partial(\cup_{j=0}^{i-1} H_j) \) whose images are isotropic spheres, and trivializations of \( \Phi_i(\partial S H_i) \) satisfying the followings

- \( H_0 \) is the unique index 0 handle,
- there is a natural number \( N \) such that for \( i \leq N \) (resp. \( i > N \)), \( H_i \) is subcritical (resp. critical), i.e., \( \text{ind}(H_i) < n \) (resp. \( \text{ind}(H_i) = n \)), and
- \( \cup_{i=0}^{m} H_i \) and \( W \) have symplectomorphic symplectic completions.

We note that the gluing information in Definition 2.2 are given by gluing maps, defined on the whole attaching regions of each handle. However, in Definition 2.11, the gluing information are given as maps on attaching spheres and trivializations of the conformal symplectic normal bundles.

2.4. **Lefschetz fibration.** We move on to our main interest, Lefschetz fibrations.
Definition 2.12. Let $(W, \omega = d\lambda)$ be a finite type Liouville manifold. A Lefschetz fibration on $W$ is a map $\pi : W \to \mathbb{C}$ satisfying the following properties:

• (Triviality near the horizontal boundary.) There exists a contact manifold $(B, \xi)$, an open set $U \subset W$ such that $\pi : W \setminus U \to \mathbb{C}$ is proper and a codimension zero embedding $\Phi : U \to S^1 B \times \mathbb{C}$ such that $pr_2 \circ \Phi = \pi$ and $\Phi^* \lambda = pr_1^* \lambda + pr_2^* \mu$ where $\mu = \frac{1}{2} r^2 d\theta$.

• (Lefschetz type critical points.) There are only finitely many points where $d\pi$ is not surjective, and for any such critical point $p$, there exist complex Darboux coordinates $(z_1, \ldots, z_n)$ centered at $p$ so that $\pi(z_1, \ldots, z_n) = \pi(p) + z_1^2 + \cdots + z_n^2$. Moreover, there is at most one critical point in each fiber of $\pi$.

• (Transversality to the vertical boundary.) There exists $R > 0$ such that the Liouville vector field $X$ lifts the vector field $\frac{1}{2} r \partial_r$ near the region $\{|\pi| \geq R\}$.

• (Symplectic fiber.) Away from the critical points, $\omega$ is non-degenerate on the fibers of $\pi$.

We note that it would be more precise to use the term ‘Liouville Lefschetz fibration’ in Definition 2.12. However, in this paper, this is the only type which we considered here. Thus, we omit the adjective for convenience.

Definition 2.12 is classical, but [7] suggested an alternative definition.

Definition 2.13. An abstract Weinstein Lefschetz fibration is a tuple

$$W = (F : L_1, \ldots, L_m)$$

consisting of a Weinstein domain $F^{2n-2}$ (the “central fiber”) along with a finite sequence of exact parameterized Lagrangian spheres $L_1, \ldots, L_m \subset F$ (the “vanishing cycles”).

Definitions 2.12 and 2.13 are interchangeable. In the rest of Section 2.4, we explain how to obtain a Lefschetz fibration of a Weinstein manifold when an abstract Weinstein Lefschetz fibration is given briefly. For more details on the equivalence of Definitions 2.12 and 2.13, we refer the reader to [3, Section 8].

Let $W = (F : L_1, \ldots, L_m)$ be a given abstract Weinstein Lefschetz fibration. Then, one can construct a Weinstein domain as follows: first, we consider the product of $F$ and $\mathbb{D}^2$. Then, the vertical boundary $F \times \partial \mathbb{D}^2$ admits a natural contact structure. Moreover, the vanishing cycle $L_i$ can be lifted to a Legendrian $\Lambda_i$ near $2\pi i/m \in S^1$. The lifting procedure is given in Section 6.1. We note that by assuming that the disk $\mathbb{D}^2$ has a sufficiently large radius, one could assume that the projection images of $\Lambda_i$ onto the $S^1$ factor are disjoint to each other. Finally, one could attach critical Weinstein handles along $\Lambda_i$ for all $i = 1, \ldots, m$. Then, the completion of the resulting Weinstein domain admits a Lefschetz fibration satisfying that the regular fiber is $F$, and that there are exactly $m$ singular values near $2\pi i/m \in S^1$. 
Part 1. Lefschetz fibrations on cotangent bundles

The main goal of the first part is to introduce the idea of obtaining a Lefschetz fibration from a Weinstein handle decomposition. In order to introduce the idea, we consider the case of cotangent bundles in the first part.

After introducing the idea, it would be natural to ask about the relation between two Lefschetz fibrations on the same Weinstein manifold, which come from different Weinstein handle decompositions. Section 7 discusses the question.

3. Weinstein handle decompositions of cotangent bundles

Before discussing our construction of Lefschetz fibrations on cotangent bundles, we discuss an algorithm producing Weinstein handle decomposition of a cotangent bundle \( T^*M \) from a handle decomposition of a smooth manifold \( M \).

Section 3.1 introduces the notion of attaching Legendrian. We use the notion in Section 3.2, to construct a Weinstein domain \( W_D \) from a handle decomposition \( D \) of a smooth manifold \( M \) by gluing of Weinstein handles. In other words, we construct \( W_D \) together with a Weinstein handle decomposition of it. In Section 3.3, we prove that the symplectic completion of \( W_D \) is exact symplectomorphic to \( T^*M \).

3.1. Attaching Legendrian. The attaching Legendrian (resp. core Lagrangian) is defined on a standard Weinstein handle \( H_k \subset \mathbb{R}^{2n} \), where \( \mathbb{R}^{2k} \times \mathbb{R}^{2(n-k)} \) is coordinated by \[(x_1, \ldots, x_k, y_1, \ldots, y_k, p_1, \ldots, p_{n-k}, q_1, \ldots, q_{n-k}),] as we did in Equation (6.5).

Definition 3.1.

(1) The attaching Legendrian \( \partial_L H^k \) of the standard 2n dimensional Weinstein handle \( H^k \) is the intersection of \( \partial_R H^k \) and the region \( \{ y_1 = \cdots = y_k = 0 = q_1 = \cdots = q_{n-k} \} \).

(2) The core Lagrangian of the standard 2n dimensional Weinstein handle \( H^k \) is the intersection of the handle and the region \( \{ y_1 = \cdots = y_k = 0 = q_1 = \cdots = q_{n-k} \} \).

Remark 3.2. We note that the attaching Legendrian and the core Lagrangian are not intrinsic in Weinstein handles, different from the notion of attaching spheres. More precisely, one can observe that the Liouville vector field has only one zero in a Weinstein handle, and that the attaching sphere is the boundary of the stable manifold of the unique zero with respect to the Liouville vector flow. Thus, the attaching sphere of a Weinstein handle could be defined by using the Liouville structure on the Weinstein handle without using coordinates. However, in order to define the notions of attaching Legendrians and core Lagrangians, a choice of coordinate charts is necessarily. Thus, for a general Weinstein handle \( H, \partial_L H \) is defined with respect to an identification with \( H \) and the standard handle. For convenience, we use the notions of attaching Legendrians and core Lagrangians without mentioning a choice of identifications.

Lemma 3.3. Let \( X \) be a \((2n - 1)\) dimensional contact manifold. If there is a map \( \phi : \partial_L H^k \hookrightarrow X \) such that

- \( \phi \) is an embedding, and
• \( \text{Im}(\phi) \) is a Legendrian in \( X \),

then \( \phi \) induces a trivialization on \( CSN(\Lambda) \) where \( \Lambda := \phi(\partial S H^k) \).

**Proof.** Simply, this is because of [15, Proposition 4.2].

More precisely, for any Legendrian \( \Lambda \) in a contact manifold, there is a neighborhood of \( \Lambda \) which is contactomorphic to a neighborhood of \( \Lambda \) in the Jet 1 bundle of \( \Lambda \). Since \( \phi \) identifies two Legendrians \( \partial L H^k \) and it’s image, there are neighborhoods of them which are identified to each other. Then, the standard trivialization on \( \partial L H^k \) induces a trivialization on the other side of identification. It induces a trivialization of \( CSN(\Lambda) \).

\[ \blacksquare \]

**Remark 3.4.** Lemma 3.3 concludes that if there is a map \( \phi \) satisfying the setting in Lemma 3.3, then one could attach the standard handle.

Together with Lemma 3.3 and Remark 3.4, we will use the notion of attaching Legendrians to encode gluing information of Weinstein handles in the rest of the present paper. To be more precise, Lemma 3.5 is necessarily.

**Lemma 3.5.** Let \( W \) be a Weinstein domain and there is a map \( \phi : \partial L H^k \to \partial W \) satisfying the conditions in Lemma 3.3. Let \( \Lambda_i \) be an Legendrian isotopy connecting \( \Lambda_0 := \phi(\partial L H^k) \) and \( \Lambda_1 \). If \( W_i \) denotes the Weinstein domain obtained by attaching \( H^k \) along \( \Lambda_i \) for \( i = 0, 1 \), then \( W_0 \) and \( W_1 \) have symplectomorphic symplectic completions.

**Proof.** On the contact manifold \( \partial W \), the Legendrian isotopy \( \Lambda_t \) can be extended to the contact isotopy \( \psi_t \) of \( \partial W \). For the extension procedure, we refer the reader to [6, Section 2.5]. By [5, Lemma 12.5], there is a Liouville structure on \( \partial W \times [0, 1] \) such that Weinstein homotopic to the \( e^t \alpha \) where \( t \) is the coordinate for \( [0, 1] \)-factor, and such that the homology from \( \partial W \times \{0\} \) to \( \partial W \times \{1\} \) is the contact isotopy \( \psi_1 \). Since a Weinstein homotopic change does not affect on the equivalence class of the symplectic completion, it completes the proof. \[ \blacksquare \]

### 3.2. Construction of \( W_D \)

Let \( M \) be a smooth \( n \)-dimensional manifold. Let \( D \) be a handle decomposition of \( M \). We construct a Weinstein domain \( W_D \) by gluing Weinstein handles in Section 3.2. Before that, we set notation for convenience.

**Notation.** A handle decomposition \( D \) is an ordered collection of handles \( \{h_0, \ldots, h_m\} \) together with the gluing information, satisfying the conditions in Definition 2.2. The gluing information could be encoded as injective maps defined on \( \partial R h_i \) for all \( i \). We use the following notation to denote them.

\[ \phi_i : \partial R h_i \to \partial (\cup_{j<i} h_j) \]

For a given handle decomposition \( D \) of \( M \), let \( D \) denote a collection of Weinstein handles \( \{H_0, \ldots, H_m\} \) such that

\[ \text{ind}(H_i) = \text{ind}(h_i) \]

for all \( i = 0, \ldots, m \).

Since \( H_i \) could be identified with a closed subset of \( (\mathbb{R}^{2n}, \lambda_k) \) where \( k = \text{ind}(H_i) \), one can easily construct an embedding \( \iota_i : h_i \to H_i \) such that

1. \( \iota_i(h_i) \) is the core Lagrangian of \( H_i \),
2. \( \iota_i \) sends \( \partial R h_i \) to the attaching Legendrian of \( H_i \).
The core Lagrangian and the attaching Legendrian are defined in Definition 3.1.

**Attaching information.** As discussed in Section 3.1, the gluing information for Weinstein handles can be given by the maps defined on the attaching Legendrians of Weinstein handles. Then, the following map

$$\Phi_i : \partial L_{H_i}^{\alpha - 1} \to \partial R_{H_i}^{\partial \mathcal{V}_i} \to \partial (\cup_{j<i} \mathcal{D}) \cup_{j<i} \partial \mathcal{H}_j.$$

The maps $\Phi_i$ for $i = 1, \ldots, m$ contain the information explaining how to attach Weinstein handles in $D$. Let $W_D$ denote the resulting Weinstein domain by attaching Weinstein handles in $D$.

In the following section, we prove that $W_D$ and $T^* M$ have the symplectomorphic symplectic completions.

### 3.3. Weinstein handle decomposition of $T^* M$.

Let $W_D$ be the Weinstein manifold constructed in Section 3.2 when a smooth manifold $M$ admits a handle decomposition $D = \{h_0, \ldots, h_m\}$.

**Lemma 3.6.** The cotangent bundle $T^* M$ and $W_D$ have the symplectomorphic symplectic completions.

**Proof.** By definition, $W_D$ admits a Weinstein handle decomposition $D = \{H_0, \ldots, H_m\}$. Then, $W_D$ has a Lagrangian skeleton with respect to the induced Liouville structure.

If $H_i$ is a critical handle, i.e., $\text{ind}(H_i) = n$, then the intersection of the Lagrangian skeleton and a Weinstein handle $H_i$ is the stable manifold of the unique zero of the Liouville vector field on $H_\alpha$, i.e., the core Lagrangian of $H_i$.

If $H_i$ is a subcritical handle of index $(n-1)$, then the intersection of the Lagrangian skeleton and $H_i$ is

$$\cup_{t>0} \Psi_t \text{(attaching Legendrians of critical handles attached on } \partial H_i),$$

where $\Psi_t$ means the Liouville flow on $H_i$.

From the construction in Section 3.2, for all $i$ such that $\text{ind}(H_i) = n - 1$, the attaching Legendrian attached on $\partial L_{H_i}$ is exactly the boundary of the core Lagrangian of $H_i$ except the attaching Legendrian $\partial L_{H_i}$. Then, the intersection of Lagrangian skeleton and $H_i$ is the core Lagrangian of $H_i$. One can easily check this in the standard handle.

Inductively, one could show that, for every subcritical handle $H_i$, the intersection of the Lagrangian skeleton and $H_i$ is the core Lagrangian of $H_i$. Thus, the skeleton of $W_D$ is the union of all core Lagrangians. Thus, the Lagrangian skeleton admits a handle decomposition

$$D_0 := \{\tau_0(h_0), \ldots, \tau_m(h_m)\},$$

with the notation in Section 3.2.

This means that the Lagrangian skeleton is diffeomorphic to $M$. Since a Weinstein manifold and a small neighborhood of the Lagrangian skeleton have symplectomorphic symplectic completions, Weinstein Lagrangian neighborhood theorem completes the proof. \qed
4. Weinstein handle decompositions admitting Lefschetz fibrations

Let $W$ be a $2n$ dimensional Weinstein manifold equipped with a Lefschetz fibration $\pi$. Then, $\pi$ induces a decomposition of $W$ into two parts, one is a subcritical part $F \times \mathbb{D}^2$ where $F$ is the regular fiber of $\pi$, and the other is a collection of $m$ critical handles where $m$ is the number of critical values of $\pi$ by [3].

If a Weinstein handle decomposition of $W$ is given, then there is a natural decomposition of $W$ into a union of subcritical handles and a union of critical handles. Moreover, by [4] or [5, Theorem 14.16], the subcritical part, i.e., the union of all subcritical handles, can be identified with a product of a $(2n-2)$ dimensional Weinstein manifold and $\mathbb{D}^2$.

Based on the above arguments, it would be natural to ask whether a Weinstein handle decomposition of $W$ induces a Lefschetz fibration such that the number of singular values are the same to the number of critical handles in the Weinstein handle decomposition.

In Section 4, first, we give an example of a Weinstein handle decomposition which does not induce a Lefschetz fibration. Then, we discuss when a Weinstein handle decomposition induces a Lefschetz fibration.

4.1. Example: the case of $T^*S^n$. It is easy to prove that $T^*S^n$ admits a Weinstein handle decomposition consisting of one Weinstein 0-handle and one Weinstein $n$-handle. This is because $S^n$ admits a decomposition into one 0-handle and one $n$-handle. Then, Lemma 5.6 gives the desired Weinstein handle decomposition of $T^*S^n$.

Let assume that the Weinstein handle decomposition induces a Lefschetz fibration $\pi$. Then, the product of the regular fiber $F$ of $\pi$ and $\mathbb{D}^2$ should be equivalent to the unique subcritical handle, i.e., the 0 handle. This means that $F$ should be a disk of dimension $(2n-2)$.

Since the Weinstein handle decomposition has one critical handle, the Lefschetz fibration $\pi$ has one critical value. Let $L$ be the vanishing cycle corresponding to the critical value. Then, $L$ should be an exact Lagrangian submanifold of $F$. However, it is well-known that there is no exact Lagrangian in $\mathbb{D}^{2n-2}$. Thus, it is a contradiction.

Remark 4.1. From the above arguments, one can conclude that every Lefschetz fibration of $T^*S^n$ has at least 2 or more critical values. Since it is well-known that there exist a Lefschetz fibration of $T^*S^n$ having exactly 2 critical values, 2 is the minimal number of critical values of a Lefschetz fibration of $T^*S^n$.

Moreover, the same arguments work for the case of milnor fibers having $A_n$-singularities. Thus, any Lefschetz fibrations of those Milnor fibers have at least $n$ critical values and $n$ is the minimum number of critical values.

4.2. A Weinstein handle decomposition admitting a Lefschetz fibration. In Section 4.2, we introduce some conditions such that if a Weinstein handle decomposition satisfies them, then it induces a Lefschetz fibration naturally.

Let $D = \{H_0, \cdots, H_m\}$ be a Weinstein handle decomposition of a Weinstein manifold $W$. By Definition 2.11 there is a number $N$ such that $H_i$ is a subcritical (resp. critical) handle if $i \leq N$ (resp. $> N$).

As mentioned above, the union $\bigcup_{i=0}^{N} H_i$ of subcritical handles is equivalent to a product $F \times \mathbb{D}^2$ where $F$ is a $(2n-2)$ dimensional Weinstein domain. We note that the union is not a disjoint union, but a gluing of handles by the gluing maps.
If critical handles $H_i$ for $i > N$ satisfy the critical handle condition which is given below, then [3 Proposition 8.1] induces a Lefschetz fibration corresponding to $D$.

(Critical handle condition.) For all $i > N$, the image of gluing map $\Phi_i$ of $H_i$ is contained in $F \times \partial \mathbb{D}^2$. Moreover, the projections of the images of all $\Phi_i$ are disjoint intervals on $\partial \mathbb{D}^2 \simeq S^1$.

Remark 4.2. We remark that the critical handle condition is dependent on the identification with the subcritical part and the product space $F \times D^2$. Thus, “the critical handle condition with respect to the identification” is a better term, but for convenience, we omit it.

By [3], if $D$ satisfies the critical handle condition, then $D$ induces a Lefschetz fibration. However, as mentioned in Remark 4.2, there is a technical problem which is to fined an identification between the subcritical part and the product space. In order to reduce this technical difficulty, we introduce an extra condition, which we call subcritical handle condition.

Before introducing the statement of the subcritical handle condition, we discuss the main idea of the condition. From Definition 2.5 one can observe that a 2-dimensional subcritical Weinstein handle $H$ can be identified with a product Weinstein domain $\hat{H} \times D^2$ up to the equivalence defined in Definition 2.10, where $\hat{H}$ is a $(2n - 2)$ dimensional Weinstein handle of index $\text{ind}(H)$, and where $D^2$ is the Weinstein domain having the radial Liouville vector field. Thus, if the subcritical handles are glued to each other in a way “respecting the product structures”, then, the union of all subcritical handles admits a product structure naturally.

More precisely, let assume that $H_0$ and $H_1$ in $D$ can be decomposed into products $\hat{H}_0 \times D^2$ and $\hat{H}_1 \times D^2$, so that the gluing map $\Phi_1 : \partial R H_1 = \partial (\cup_{j=0}^{i-1} H_j) \rightarrow \partial H_0$ could written as a product of $\Phi_1 : \partial R H_1 \rightarrow \partial H_0$ and a symplectomorphism $f_1 : D^2 \rightarrow D^2$ preserving the radial Liouville vector field of $D^2$, i.e.,

$$\Phi_1 = \Phi_1 \times f_1.$$

Then, the attaching of $H_1$ to $H_0$ is a product of $D^2$ and the $(2n - 2)$ dimensional Weinstein domain which is the attachment of $H_1$ to $\hat{H}_0$ via $\Phi_1$, i.e.,

$$(4.3) \quad H_0 \cup H_1 = (\hat{H}_0 \cup \hat{H}_1) \times D^2.$$

We note that the unions do not mean disjoint unions, but the gluings via $\Phi_1$ and $\hat{\Phi}_1$.

Inductively, a Weinstein handle decomposition $D$ satisfies the subcritical handle condition if the following holds:

(Subcritical handle condition.) For all subcritical handle $H_i \in D$, $H_i$ and the attaching map $\Phi_i : \partial R H_i \rightarrow \partial (\cup_{j=0}^{i-1} H_j)$ are decomposed into products $\hat{H}_i \times D^2$ and $\Phi_i \times f_i$ respectively, where $\hat{H}_i$ is a $(2n - 2)$ dimensional Weinstein handle of index $\text{ind}(H_i) = \text{ind}(\hat{H}_i)$ so that

- $\Phi_i$ is a gluing map for $\hat{H}_i$, and
- $f_i$ is a symplectomorphism of $D^2$ preserving the radial Liouville vector field.

We would like to point out that Equation (4.3) is obtained by using the gluing maps defined on the attaching regions. However, one could use the gluing maps
Let $\Phi'_1$ and $\hat{\Phi}'_1$, which are defined on the attaching Legendrians of $H_1$ and $\hat{H}_1$. To be more precise, we use the fact that $\partial_l H_i = \partial_l \hat{H}_i \times I_i$, where $I_i$ is a diameter of $\mathbb{D}^2$ for $i = 0, 1$. Let a gluing map $\Phi'_1 : \partial_l H_1 \rightarrow \partial H_0$ satisfy

$$\Phi'_1 = \Phi'_1 \times f'_1,$$

where $\Phi'_1$ is a gluing map defined on $\partial_l H_1$ and $f'_1 : I_1 \rightarrow \mathbb{D}^2$ such that $\text{Im}(f'_1)$ is a diameter on $\mathbb{D}^2$. Then, by Lemma 3.3 and [15], $\Phi'_1$ (resp. $\hat{\Phi}'_1$) extends to a gluing map $\Phi_1$ (resp. $\hat{\Phi}_1$). Moreover, through the extension, it is easy to obtain $\Phi'_1$ and $\hat{\Phi}'_1$ satisfying Equation 4.3. We note that, as mentioned in Remark 3.2, $\partial_l H_1$ is defined with respect to an identification with the standard handle. The diameter $I_1$ depends on the choice of identification.

The given subcritical handle condition is using the gluing maps defined on the attaching regions, but one could define by using the gluing maps defined on the attaching Legendrians.

(Subcritical handle condition') For all subcritical handle $H_i \in \mathcal{D}, H_i$ and the attaching map $\Phi_i : \partial_l H_i \rightarrow \partial(\bigcup_{j=0}^{i-1} H_j)$ are decomposed into products $H_i \times \mathbb{D}^2$ and $\Phi_i \times f_i$, where $H_i$ is a $(2n - 2)$ dimensional Weinstein handle of index $\text{ind}(H_i) = \text{ind}(\hat{H}_i)$ so that

- $\hat{\Phi}_i$ is a gluing map for $\hat{H}_i$,
- $f_i$ is defined on a diameter of $\mathbb{D}^2$, so that the image of $f_i$ is a diameter of $\mathbb{D}^2$, and $f_i$ sends the center of $\mathbb{D}^2$ to the center of $\mathbb{D}^2$.

If a Weinstein handle decomposition $\mathcal{D}$ of a Weinstein domain $W$ satisfies the subcritical handle condition, then the union of all subcritical handles is equivalent to a product Weinstein domain $F \times \mathbb{D}^2$ where $F$ is obtained by gluing $\hat{H}_i$. Moreover, if $\mathcal{D}$ satisfies the critical handle condition with respect to the product structure given by the subcritical handle condition, then [3, Proposition 8.1] gives a Lefschetz fibration defined on $W$. To summarize it, one obtains the following Definition 4.3 and Lemma 4.4.

Definition 4.3. Let $W$ be a Weinstein domain. A Weinstein handle decomposition $\mathcal{D}$ of $W$ admits a Lefschetz fibration if $\mathcal{D}$ satisfies the subcritical handle condition and the critical handle condition with respect to the product structure on the subcritical part, which is induced from the subcritical handle condition.

Lemma 4.4 follows Definition 4.3.

Lemma 4.4. Let $W$ be a Weinstein domain and let $\mathcal{D}$ be a Weinstein handle decomposition of $W$, which is admitting a Lefschetz fibration. Then, there is a Lefschetz fibration $\pi_D : W \rightarrow \mathbb{C}$ such that the number of critical values of $\pi$ is the same as the number of critical handles in $\mathcal{D}$.

Remark 4.5. To be more precise, we would like to point out that, by Definition 2.11, one obtains a Weinstein domain $W'$ which is equivalent to $W$ by gluing a Weinstein handle decomposition of $W$. Thus, when a Weinstein handle decomposition of $W$ admits a Lefschetz fibration, it gives a Lefschetz fibration defined on $W'$, not $W$. Then, the equivalence between $W$ and $W'$, together with the Lefschetz fibration on $W'$, gives a Lefschetz fibration defined on the symplectic completion of $W$. This remedies the gap between Definition 4.3 and Lemma 4.4.
Similar to Definition 2.4, we define a notation for the set of Weinstein handle decomposition admitting a Lefschetz fibration of a Weinstein domain $W$, for the future use.

**Definition 4.6.** Let $\text{WHL}(W)$ be the set of Weinstein handle decomposition admitting Lefschetz fibrations of a Weinstein domain $W$.

5. The algorithm

We give the technical statement of Theorem 1.1 in Section 5.1, which will be proven in Sections 5 and 6. Section 5.2 is the proof of Theorem 5.1 except a technical part. The technical part will be discussed in Section 6.

5.1. Technical statement of Theorem 1.1

The technical statement of Theorem 1.1 which uses Definitions 2.4 and 4.6 is the following.

**Theorem 5.1.** There is an algorithm $\mathcal{A} : \mathcal{H}(M) \to \text{WHL}(M)$, so that the number of critical handles of $\mathcal{A}(D) \in \text{WHL}(M)$ is the same as the number of handles of $D \in \mathcal{H}(M)$.

By Lemma 4.4 and Theorem 5.1, one could obtain a Lefschetz fibration $\pi_D$ of $T^*M$ from a handle decomposition $D$ of $M$. Moreover, the number of singular values of $\pi_D$ is the same as the number of handles in $D$.

The algorithm $\mathcal{A}$ consists of two steps. Before stating the algorithm, we fix notations. Let $D = \{h_0, \ldots, h_m\}$ be a handle decomposition of an $n$ dimensional manifold $M$, i.e., $D \in \mathcal{H}(M)$. Let $N$ be the natural number such that $h_i$ is subcritical (resp. critical) if $i \leq N$ (resp. $i > N$). For a given $D \in \mathcal{H}(M)$, let $W_D$ denote the Weinstein handle decomposition of $T^*M$ which we constructed in Section 3 by applying Lemma 3.6 to $D$.

**Step 1.** The first step is to construct another handle decomposition $\tilde{D}$ of $M$ from $D$, as follows: For every subcritical handle $h_i$, we consider the division of $h_i$ into three handles, one of index $\text{ind}(h_i)$, denoted by $h_i^{\text{ori}}$, and a canceling pair of indices $n - 1$ and $n_i$, denoted by $h_i^{-1}$ and $h_i^0$ respectively, satisfying the followings:

(i) the attaching region $\partial_R h_i$ of the original handle $h_i \in D$ intersects the attaching region $\partial_R h_i^n$ of $h_i^n$ in the interior of $\partial_R h_i$ itself. To be more precise, we identify $\partial_R h_i$ with $S^{k-1} \times \mathbb{D}^{n-k}$ where $k := \text{ind}(h_i)$, i.e., $\partial_R h_i \cong S^{k-1} \times \mathbb{D}^{n-k}$, so that $\partial_R h_i \cap \partial_R h_i^n \cong S^{k-1} \times \mathbb{D}^{n-k}$, where $\mathbb{D}^{n-k}$ is a smaller disk with a radius $\epsilon < 1$.

(ii) $\partial h_i \setminus \partial_R h_i$ does not intersect the added critical handle.

An example of 3 dimensional 1 handle is given in Figure 1.

**Remark 5.2.** We note that if $\text{ind}(h_i) = n - 1$, then there are two $(n - 1)$ handles after dividing. Thus, in order to use the notation $h_i^{\text{ori}}$ and $h_i^{-1}$, it is necessarily to choose one of two possibilities. However, at the end, the choice does not effect on the resulting Lefschetz fibration.

After dividing all subcritical handles in $D$, one obtains another handle decomposition $\tilde{D}$ of $M$ as follows:

$$\tilde{D} := \{h_0^{\text{ori}}, h_0^{n-1}, h_1^{\text{ori}}, h_1^{n-1}, \ldots, h_N^{\text{ori}}, h_N^{n-1}, h_0^0, \ldots, h_N^0, h_{N+1}, \ldots, h_m\}.$$  

We note that $\tilde{D}$ consists of $2N$ subcritical handles and $m$ critical handles.
Step 2. The second step of the algorithm is to apply Lemma 3.6 for $\tilde{D}$. Then, one obtains a Weinstein handle decomposition $W_{\tilde{D}}$ of $T^*M$. For the future use, we use the following notation for $W_{\tilde{D}}$,

$$W_{\tilde{D}} = \{H_0^{ori}, H_1^{n-1}, H_1^{ori}, H_2^{n-1}, \ldots, H_N^{ori}, H_N^{n-1}, H_N^n, H_{N+1}, \ldots, H_m\}.$$ 

We remark that there is a one to one relation between the handles in $\tilde{D}$ and Weinstein handles in $W_{\tilde{D}}$, so that $h_{ori}^i, h_{n-1}^i, h_n^i$ correspond to $H_{ori}^i, H_{n-1}^i, H_n^i$.

Remark 5.3. Before going further, we would like to explain why we consider $\tilde{D}$ instead of $D$. The reason is that there is a possibility of obtaining a Weinstein handle decomposition $W_D$ which does not admit a Lefschetz fibration. The simplest example is given in Section 4.1.

5.2. **The proof of Theorem 5.1**. In Section 5.2, we prove Theorem 5.1 except a technical part. The technical part is to modify Legendrians.

Setting. We use the same notation as we used in the previous sections.

From the handle decomposition $\tilde{D}$, there is an increasing collection of closed subsets

$$M_0 \subset M_1 \subset \cdots \subset M_{N} \subset M_{N+1} = M,$$

by setting

$$(5.4) \quad M_i := \bigcup_{j=0}^i (h_{ori}^j \cup h_{n-1}^j), \text{ if } i \leq N, \text{ and } M_{N+1} := M.$$ 

We note that $M_i$ admits a handle decomposition induced from Equation (5.4). Then, Section 5.2 and Lemma 5.6 explain how to obtain Weinstein handle decomposition of $T^*M_i$ for $i = 0, \cdots, N + 1$, which comprise an increasing sequence

$$T^*M_0 \subset T^*M_1 \subset \cdots T^*M_N \subset T^*M_{N+1} = T^*M.$$
The base step. In order to prove Theorem 5.1 from the given Weinstein handle decomposition of $T^* M_i$, we inductively construct a Weinstein domain $W_i$ for any $i = 0, \ldots, N + 1$, satisfying

- $T^* M_i$ and $W_i$ have symplectomorphic symplectic completions, and
- $W_i$ admits a Weinstein handle decomposition admitting a Lefschetz fibration.

The base step is to construct a Lefschetz fibration for $T^* M_0$. By the above construction, $M_0$ is a $n$ dimensional disk removed a smaller disk inside, i.e., $M_0 \simeq S^{n-1} \times [0, 1]$. Thus, $T^* M_0$ is equivalent to the product of $T^* S^{n-1}$ and $\mathbb{D}^2$.

Let $W_0$ be the total space of an abstract Lefschetz fibration $\pi_0$ given as $\pi_0 := (F_0 = T^* S^{n-1}; \emptyset)$. Since $T^* M_0$ and $W_0$ both are equivalent to $T^* S^{n-1} \times \mathbb{D}^2$, $T^* M_0$ is equivalent to $W_0$.

Remark 5.4. Before going further, we remark the following: there are handles in $W_D$, which are attached to $T^* M_0$. One can observe that the attaching Legendrians of the handles are attached along $\partial M_0 \subset T^* M_0$. Since $M_0$ is homeomorphic to $S^{n-1} \times [0, 1], \partial M_0$ has two components and each component is an $(n - 1)$ dimensional sphere. One can also observe that along one component of $\partial M_0$, there is only one handle $H_0^i$ is attached along the component. Moreover, under the identification of $T^* M_0$ with the total space $W_0$ of $\pi_0$, $\partial M_0$ are identified to the zero sections of fibers $\pi_0^{-1}(\pm 1) \simeq T^* S^{n-1}$ where the base $\mathbb{D}$ is the unit disk in $\mathbb{C}$. We assume that the component of $\partial M_0$, which $H_0^i$ is attached along, is identified to the zero section of $\pi_0^{-1}(1)$ without loss of generality. Let $\Lambda_0$ denote the zero section of $\pi_0^{-1}(1)$.

Construction of $W_1$ from $W_0$. We construct $W_1$ by attaching subcritical handles $H_1^{ori}$ and $H_1^{n-1}$ to $W_0$.

The handles $H_1^{ori}$ and $H_1^{n-1}$ would be attached along the zero section of $\pi_0^{-1}(1)$, or equivalently $\Lambda_0$ by using the notation defined in Remark 5.4. However, if one attaches the handles along $\Lambda_0$, then after attachment, the Lefschetz fibration $\pi_0$ could not extended to the resulting Weinstein domain. Thus, we modify $\Lambda_0$ in a specific way. The specific way will be given in Section 6 but we explain what we would like to achieve by the modification here.

We would like to construct $W_1 \simeq T^* M_1$ admitting a Lefschetz fibration. Since $W_1$ consists of subcritical handles, we consider the subcritical handle condition in Section 5.2.

Based on the above arguments, by modifying $\Lambda_0$, we would like to obtain a Legendrian satisfying the followings: let $\theta_0$ be a positive small number. Then,

(i) the parts of the modified Legendrian, which $H_1^{ori}, H_1^{n-1}, H_1^n$ are “not” attached along, are lying on the vertical boundary of $W_0$. Moreover, those parts are projected by $\pi_0$ to $\{e^{i\theta} \mid \theta \in (-\theta_0, 0]\}$.

(ii) The parts of the modified Legendrian, which $H_1^{ori}, H_1^{n-1}$ are attached along, are lying on the horizontal boundary of $W_0$. Moreover, those parts are projected by $\pi_0$ to a diameter of the base. For the future use, let $\theta_1$ be a number such that the diameter connecting $e^{-i\theta_1}$ and $e^{i(-\theta_1+\pi)}$.

(iii) The parts of the modified Legendrian, which $H_1^n$ are attached along, are lying on the vertical boundary of $W_0$. Moreover, those parts are projected by $\pi_0$ to $\{e^{i\theta} \mid \theta \in (-\theta_0 + \pi, \pi]\}$.
Also, we use a small $\theta_0$ for attaching critical handles later. A conceptual picture for the lowest dimensional case is given in Figure 2.

Figure 2. a). An example of handle decomposition $D$ having an index 0 handle $h_0$ and an index 1 handle $h_1$. b). A handle decomposition $\tilde{D}$ constructed from $D$. c). The Lefschetz fibration $\pi_0$ together with the zero sections of the fibers $\pi_0^{-1}(\pm 1)$, which are Legendrians corresponding to the boundary of $h_2^0$ (in $\pi_0^{-1}(-1)$) and the boundary of $h_0$ (in $\pi_0^{-1}(1)$). d). The projected image of the modified Legendrian under $\pi_0$.

One always can modify $\Lambda_0$ by using the following two facts. The first fact is that, near the boundary of $W_0$, the Liouville 1 form of $W_0$ is given as the product of Liouville 1 forms of the fiber and the base. The second fact is that exact Lagrangians in the regular fiber could be lifted to Legendrians in the vertical boundary of $W_0$. For more details, see Section 6 which contains examples with detailed computations.

Together with the modified Legendrian, one can attach $H_1^{ori}$ and $H_1^{n-1}$ along the horizontal boundary of $W_0$. In the process of attaching handles, one can attach them in the way satisfying the subcritical condition. More precisely, one can attach in the following way.

(i) $H_1^{ori}$ (resp. $H_1^{n-1}$) can be identified with the product of $\tilde{H}_1^{ori}$ (resp. $\tilde{H}_1^{n-1}$) and $\mathbb{D}^2$, where $\tilde{H}_1^{ori}$ (resp. $\tilde{H}_1^{n-1}$) is a $(2n-2)$ dimensional Weinstein handle such that $\text{ind}(H_1^{ori}) = \text{ind}(\tilde{H}_1^{ori})$ (resp. $\text{ind}(H_1^{n-1}) = \text{ind}(\tilde{H}_1^{n-1})$).

(ii) Under the identification in (i), $\partial_L H_1^{ori}$ (resp. $\partial_L H_1^{n-1}$) is a product of $\partial_L \tilde{H}_1^{ori}$ (resp. $\partial_L \tilde{H}_1^{n-1}$) and a diameter of $\mathbb{D}^2$, which connects $e^{-i\theta_1}$ and $e^{i(-\theta_1+\pi)}$.

Let $W_1$ be the resulting Weinstein domain obtained by attaching Weinstein handles to $W_0$. Since $W_0$ is a product of $F_0$ and $\mathbb{D}^2$, $W_1$ is the product of $F_1$ and $\mathbb{D}^2$ where $F_1$ is obtained by attaching $\tilde{H}_1^{ori}$ and $\tilde{H}_1^{n-1}$ to $F_0$. Thus, $W_1$ admits a product...
Lefschetz fibration $\pi_1$. Moreover, since $W_0$ is equivalent to $T^*M_0$, the construction of $W_1$ concludes that $W_1$ is equivalent to $T^*M_1$ by Lemma 3.5.

Construction of $W_2$ from $W_1$. In order to construct $W_2$ from $W_1$ by attaching handles, as similar to the construction of $W_1$ from $W_0$, it is necessarily to find Legendrians on the boundary of $W_1$, which is identified to $\partial M_1$. We note that $\partial M_1$ is divided into two parts. One is the part of $\partial M_0$, which is “not” used to attach $H_1^{\text{ori}}, H_1^{n-1}$. The other is the part of boundary of core Lagrangians of the attached handles $H_1^{\text{ori}}, H_1^{n-1}$. This because the union of the core Lagrangians of

$$\{H_0, H_0^{n-1}, H_1^{\text{ori}}, H_1^{n-1}\}$$

is $M_1$.

Based on the argument, inside $W_1$, the Legendrian corresponding to $\partial M_1$ can be decomposed into two parts. The first part is the part of the modified Legendrian in $W_0$ such that the part is lying on the vertical boundary. This part is given in Figure 2, as thick blue and black curves. In order to find the second part, we consider the boundaries of the core Lagrangians of the attached handles $H_1^{\text{ori}} \simeq H_1^{\text{ori}} \times \mathbb{D}^2$ and $H_1^{n-1} \simeq H_1^{n-1} \times \mathbb{D}^2$. Based on the product structure, the boundaries of the core Lagrangians of $H_1^{\text{ori}}, H_1^{n-1}$ are the core Lagrangians of $H_1^{\text{ori}}, H_1^{n-1}$ in $H_1^{\text{ori}} \times \{e^{-i\theta_1}, e^{i(-\theta_1+\pi)}\}, H_1^{n-1} \times \{e^{-i\theta_1}, e^{i(-\theta_1+\pi)}\}$.

The above arguments mean that under the identification of $W_1$ and $T^*M_1, \partial M_1$ is Legendrians lying on the vertical boundary of $W_1$. We divide this Legendrian into three parts based on which Weinstein handles are attached along it, as follow:

(i) The first one is projected to $-1 \in \mathbb{D}^2$ by $\pi_1$. This component is the part of $\partial M_0$, which is $H_0^{n-1}$ will be attached along.

(ii) The second one is projected to an interval in $\partial \mathbb{D}^2$, where the interval is contained in $\{e^{i\theta} | \theta \in (-\theta_0 + \pi, \pi]\}$. Or roughly, one could say that the interval is a small interval containing $e^{i(-\theta_1+\pi)}$. Along this component, the critical handle $H_1^{\text{ori}}$ will be attached.

(iii) The last one is projected to an interval in $\partial \mathbb{D}^2$, where the interval is contained in $\{e^{i\theta} | \theta \in (-3\theta_0, 0]\}$. We take a negative Reeb flow of this Legendrian. It gives a Legendrian isotopic change of the Legendrian so that after the Legendrian isotopy, it is projected to the interval contained in $\{e^{i\theta} | \theta \in (-3\theta_0, 0]\}$. This is easy to achieve, since the Reeb vector field is the rotational vector along the boundary of the base. We call this Legendrian after isotopy as $\Lambda_1$.

Similar to the previous step, $H_2^{\text{ori}}$ and $H_2^{n-1}$ will be attached to $W_1$ along $\Lambda_1$ in order to construct $W_2$ such that $W_2$ is equivalent to $T^*M_2$. However, if one attaches Weinstein handles along $\Lambda_1, \pi_1$ does not extend to $W_2$. Thus, we modify $\Lambda_1$ in the same way we did for $\Lambda_0$. More precisely, we modify $\Lambda_1$ so that the part of $\Lambda_1$ which $H_2^{\text{ori}}$ and $H_2^{n-1}$ will be attached along should be lying on the horizontal boundary of $W_1$, and so that they are projected to a diameter of the base. Moreover, the part of $\Lambda_1$ which $H_2^{\text{ori}}$ will be attached along should be lying on the vertical boundary of $W_2$ so that the part is projected to an interval contained in

$$\{e^{i\theta} | \theta \in (-3\theta_0 + \pi, -2\theta_0 + \pi]\}.$$
After this modification, we attach $H_{2i}^n$ and $H_{2i+1}^n$ to $W_1$ in the way constructing a product space $F_2 \times \mathbb{D}^2 \simeq W_2$. Then, $W_2$ is equivalent to $T^* M_2$. Also, $W_2$ is equipped with a product Lefschetz fibration.

**Inductive steps for subcritical handles.** By repeating this for $i = 3, \ldots, N$, one could obtain $W_i$ satisfying

- $W_i$ is equivalent to $T^* M_i$,
- $W_i$ is a product space so that there is a project Lefschetz fibration $\pi_i$, defined on $W_i$, and
- under the identification with $W_j$ and $T^* M_j$ for $j > i$, the part of $\partial M_j$ where $h_i^n$ is attached along would be identified to a Legendrian lying on the vertical boundary of $W_j$, and the Legendrian is projected to an interval contained in $\{ e^{i\theta} \mid \theta \in (-2i - 1)\theta_0 + \pi, -(2i - 2)\theta_0 + \pi \}$.

The last statement will be crucial to attach the critical handles.

**Attaching critical handles.** By the above arguments, one obtains a product Weinstein domain $W_N$. The product space is equivalent to $T^* M_N$ which is the union of all subcritical handles in $W_D$. Thus, in order to finish the proof, we would like to attach critical handles to $W_N$.

The attachments of critical handles are studied well, for example, \cite[Proposition 8.1]{3}. Based on \cite[Proposition 8.1]{3}, it is enough to show that the attaching spheres of critical handles are lying on the vertical boundary, and also they are projected to disjoint intervals of the boundary of the base $\mathbb{D}^2$ by the product Lefschetz fibration.

We recall that under the identification of $T^* M_N$ and $W_N$, $\partial M_N$ are identified to Legendrians lying on the vertical boundary of $W_N$. For convenience, we set notation. Let $A_i$ denote the Legendrian which $H_{i}^n$, for $i \leq N$, or $H_{i}$, for $i > N$ is attached along. The product Lefschetz fibration on $W_N = F_N \times \mathbb{D}^2$ is the projection to the second component of the product. Let $\pi$ denote the projection to the first component $F_N$, or equivalently, the regular fiber.

One can easily check that $A_0 \subset \pi_N^{-1}(-1)$. Similarly, if $i \in [1, N]$, then

$$\pi_N(A_i) \supset \{ e^{i\theta} \mid \theta \in (-2i \theta_0 + \pi, -(2i - 1)\theta_0 + \pi) \}.$$ 

Thus, by choosing a sufficiently small $\theta_0$,

$$\pi_N(A_0), \ldots, \pi_N(A_N) \subset \pi_N^{-1}\{ e^{i\theta} \mid \theta \in [0, \pi] \}.$$ 

Also, one could check that $\pi_N(A_0), \ldots, \pi_N(A_N)$ are disjoint from the inductive steps.

Also, by choosing a small $\theta_0$, one could observe that the other parts of $\partial M_N$, which critical handles $H_{N+1}, \ldots, H_m$ will be attached along, i.e., $A_{N+1}, \ldots, A_m$ satisfy

$$\pi_N(A_{N+1}), \ldots, \pi_N(A_m) \subset \{ e^{i\theta} \mid \theta \in (-\pi, 0) \}.$$ 

It means that $\pi_N(A_i)$ and $\pi_N(A_j)$ are disjoint if $i \leq N < j$. Thus, it is enough to prove that $\pi_N(A_i)$ and $\pi_N(A_j)$ are disjoint for $i, j > N$.

Unfortunately, $\pi_N(A_i)$ and $\pi_N(A_j)$ are not necessarily to be disjoint, but one could modify $A_i$ and $A_j$ by Legendrian isotopies so that $\pi_N(A_i)$ and $\pi_N(A_j)$ are disjoint after the modification. To prove this, we observe that $pr(A_i)$ and $pr(A_j)$ are disjoint. If they intersect, then it means that $H_i$ and $H_j$ are attached to the same part of $\partial M_N$. It means that in the handle decomposition $D$ of $M$, $h_i$ and $h_j$ are...
intersect along their boundaries. This is contradict since two critical handles in a handle decomposition are disjoint each other by Definition 2.2.

Since \( pr(A_i) \) and \( pr(A_j) \) are disjoint, when one considers the time \( t \) Reeb flow of \( A_i \), the Reeb flow image is disjoint from \( A_j \). Thus, one could modify so that \( \pi_N(A_i) \) and \( \pi_N(A_j) \) are disjoint.

Finally, we could construct a Lefschetz fibration \( \pi_D \) by attaching critical Weinstein handles along the modified \( \pi(A_i) \). \( \square \)

6. Modification of Legendrians

Section 6 discusses the technical part which we omitted in Section 5. In the proof of Theorem 5.1, when a handle decomposition \( D = \{ h_0, \cdots, h_m \} \) of a smooth manifold \( M \) is given, we constructed a sequence of Weinstein domains \( W_0, \cdots, W_{N+1} \). The sequence is constructed in an inductive way. More precisely, \( W_{i+1} \) is obtained by attaching Weinstein handles to \( W_i \). In order to attach Weinstein handles to \( W_i \) in a proper way, we should modify the Legendrians which the Weinstein handles are attached along.

The modifications of Legendrians are missing in Section 5 and we discuss in the present section.

6.1. Notation. In Section 6.1 we set notation before modifying Legendrians.

Product structure of \( W_i \). Since we would like to modify Legendrians in \( \partial W_i \) for \( i = 0, \cdots, N \), we need to review the contact structure on \( \partial W_i \). The contact structure is the restriction of the Liouville structure, thus we start from the Liouville structure of \( W_i \).

For \( i \leq N \), \( W_i \) admits a product Lefschetz fibration \( \pi_i \). Thus, \( W_i \) is equivalent to a product space \( F_i \times D^2 \), where \( F_i \) is the regular fiber of \( \pi_i \). The equivalence is not correct technically since the product is a manifold with corners. However, we use the equivalence in the sense that the symplectic completion of \( W_i \) is symplectomorphic to the product of symplectic completions of \( F_i \) and \( D^2 \). Also from this viewpoint, \( W_i \) is equivalent to \( F_i \times D^2_R \) where \( D^2_R \) means the 2 dimensional disk of radius \( R \).

Because of the product structure, the Liouville 1 form of \( W_i \) is given by

\[
\lambda_{F_i} + \frac{1}{2}(xdx - ydy),
\]

where \( \lambda_{F_i} \) is a Liouville 1 form of \( F_i \), and where \( x, y \) are the standard coordinates of \( D^2_R \subset \mathbb{R}^2 \). For convenience, we simply use \( \lambda_i \) for \( \lambda_{F_i} \) if there is no chance of confusion. Also by rescaling, we assume that \( D^2_R \) has the radius 1, instead we replace Equation (6.5) with

\[
\lambda_i + \frac{1}{c}(xdy - ydx),
\]

where \( c \) is a positive real number.

From the product structure, there is a natural projection map

\[
pr_i : W_i \cong F_i \times D^2 \to F_i.
\]

Contact topology on \( \partial W_i \). Under the product structure, \( \partial W_i \) consists of two parts, the vertical boundary \( F_i \times \partial D^2 \) and the horizontal boundary \( \partial F_i \times D^2 \). Or more
precisely, the asymptotic boundary of the completion of $W_i$ can be divided into two parts, one is contactomorphic to $F_i \times \partial \mathbb{D}^2$, and the other is contactomorphic to $\partial F_i \times \mathbb{D}^2$. The contact forms on the vertical boundary and the horizontal boundary are given by

$$\lambda_i + \frac{1}{c} d\theta,$$

$$\alpha_{F_i} + \frac{1}{c} (xdy - ydx),$$

where $\theta \in \mathbb{R}/2\pi$ is the standard coordinate of $\partial \mathbb{D}^2$, and where $\alpha_{F_i}$ denotes the restriction of $\lambda_{F_i}$ on $\partial F_i$. We simply use $\alpha_i$ instead of $\alpha_{F_i}$ if there is no chance of confusion.

Let $L$ be an exact Lagrangian of $F_i$, i.e., there is a function $f : L \to \mathbb{R}$ such that $df = \lambda_i|_L$. Then, together with a choice of $\theta_0 \in \mathbb{R}/2\pi$, one could lift $L$ to a Legendrian $\Lambda$ in the vertical boundary, which is defined by setting as

$$\Lambda := \{(p, \cos(-cf(p) - \theta_0), \sin(-cf(p) - \theta_0)) \in F_i \times \partial \mathbb{D}^2 \mid p \in L\}.$$

We note that $\partial \mathbb{D}^2$ factors are coordinated by the standard $x,y$ coordinates of $\mathbb{D}^2$.

To prove that $\Lambda$ is a Legendrian, we observe that $TL$ is identified with $T\Lambda$ by

$$V \in TL \mapsto V + cV(f) \sin(-cf(p) - \theta_0)\partial x - cV(f) \cos(-cf(p) - \theta_0)\partial y.$$

By plunging the vector in the contact form of the vertical boundary, i.e., a form in Equation (6.8), one obtains

$$\lambda_i(V) - \frac{1}{c} cV(f) = df(V) - V(f) = 0.$$

We note that the second equality comes from $\lambda_i|_L = df$. Then, it proves that $\Lambda$ is a Legendrian.

**Definition 6.1.** The Legendrian lift of $L$ with respect to $\lambda_i$ and $\theta_0$ is $\Lambda$ in Equation (6.10).

**Lemma 6.2.** Let $L$ be an exact Lagrangian in $F_i$, let $\lambda_i$ and $\lambda'_i$ be two Liouville 1 forms on $F_i$ such that $\lambda_i - \lambda'_i$ is an exact 1 form, and let $\theta_0$ and $\theta'_0$ be arbitrary real numbers. If $\Lambda$ (resp. $\Lambda'$) is the Legendrian lift of $L$ with respect to $\lambda_i$ and $\theta_0$ (resp. $\lambda'_i$ and $\theta'_0$), then there is a contact isotopy connecting two triples $(\partial W_i, (\lambda_i + \frac{1}{2} (xdy - ydx))|_{\partial W_i}, \Lambda)$ and $(\partial W_i, (\lambda'_i + \frac{1}{2} (xdy - ydx))|_{\partial W_i}, \Lambda')$.

**Proof.** By applying the contact isotopy induced from the Reeb flow, one could assume that $\theta_0 = \theta'_0$ up to contact isotopies. Since $\lambda_i - \lambda'_i$ is an exact 1 form, there is a 1 parameter family of Liouville 1 forms. Then, by Gray’s Stability Theorem, the family of 1 forms induces the desired contact isotopy on $\partial W_i$. This completes the proof. □

By Lemma 6.2, the lifted Legendrian $\Lambda$ of an exact Lagrangian $L$ is unique up to Legendrian isotopy. Based on this, we simply call $\Lambda$ a lifted Legendrian of $L$ without mentioning $\lambda_i$ or $\theta_0$.

We end the current subsection by defining a Hamiltonian flow on $F_i$. Since $F_i$ is a Weinstein domain, there is a small tubular neighborhood of $\partial F_i$ which is symplectomorphic to $\partial F_i \times (-\epsilon, 0]$. The symplectic form on $\partial F_i \times (-\epsilon, 0]$ is $d(e^r \alpha_i)$ where $r \in (-\epsilon, 0]$. Moreover, the Liouville 1 form $\lambda_i$ agrees with $e^r \alpha_i$ on $\partial F_i \times (-\epsilon, 0]$.

Let $H : F_i \to \mathbb{R}$ be a function such that
Let $\Phi^t_i$ denote the time $t$ Hamiltonian flow associated to $H$.

**Remark 6.3.** It is easy to check that on $\partial F_i$, $\Phi^t_i$ is the time $t$ Reeb flow of $\partial F_i$ with respect to the contact 1 form $\alpha_i$.

### 6.2. An example of Theorem 5.1

We give a specific example with figures, of Theorem 5.1. Also, Remarks 6.4–6.8 discuss the general case.

The example manifold $M$ we consider is the 2 dimensional torus equipped with a handle decomposition $D$ consisting of one 0 handle, two 1 handles, and one 2 handle. The handle decomposition $D$ and the induced handle decomposition $\tilde{D}$ of $M$ are given in Figure 3, a) and b), respectively. Figure 4 describes $M_1, \cdots, M_3$

![Figure 3](image1.png)

*Figure 3. a). The square, both side (resp. the top and the bottom) are identified to each other, is the torus which is decomposed into a 0 handle (center circle), two 1 handles $h_1, h_2$ whose boundaries are red and blue lines respectively, and a 2 handle (the rest). b). It describes the induced handle decomposition of a torus, so that a 1 handle $h_i$ is divided into two 1 handles $h^1_i, h^2_i$ and a 2 handle $h^2_i$. Defined in Equation 5.4 for the given $\tilde{D}$. The base step is to construct a product space $W_0 = D^* S^1 \times D^2$ which is equivalent to $D^* M_0$, where $D^* M$ means the disk cotangent bundle of $M$.

Then, under the equivalence $D^* M_0 \simeq W_0$, the outer (resp. inner) boundary of $M_0$ is identified with the zero section of the fiber $\pi_0^{-1}(1)$ (resp. $\pi_0^{-1}(-1)$). Since the fiber is the cotangent bundle $D^* S^1$, the zero section makes sense here. By using the notation in Section 5 let $\Lambda_0$ denote the outer boundary of $M_0$ in $\pi_0^{-1}(1)$. Then, $\Lambda_0$ is a Legendrian.

In order to construct $W_1$ from $W_0$, we should modify $\Lambda_0$. We observe that $\Lambda_0$ is a lifted Legendrian of an exact Lagrangian $L_0 := pr_0(\Lambda_0)$ in the regular fiber. We note that $pr_0$ is defined in Equation 6.7. Our plan is to modify $L_0$, instead of $\Lambda_0$, via an exact Lagrangian isotopy. Then, by lifting the Lagrangian isotopy, one could obtain a Legendrian isotopy starting from $\Lambda_0$.

**Remark 6.4.** In this example, $L_0 = \Lambda_0$, so that there is no reason to distinguish them. However in a general case, i.e., for a general dimension and for a general $i$, 

\begin{itemize}
  \item $H|_{F_i \setminus \partial F_i \times (-\epsilon, 0]} \equiv 0$, and
  \item $H|_{\partial F_i \times (-\frac{\epsilon}{2}, 0]} \equiv e^t /$.
\end{itemize}
Figure 4. a). $M_0$, i.e., union of $h_0^{ori}$ and $h_1^0$ is given. Similarly, in b), c), and d), $M_1$, $M_2$ and $M_3$ are given respectively. For each $M_i$, the added handles compared to $M_{i-1}$ are labeled.

$L_i \neq \Lambda_i$. We use $L_0$ and $\Lambda_0$ and distinguish them to be compatible with the general cases.

**Push to the horizontal boundary.** We recall that, after the modification, the parts of $\Lambda_0$ which $H_1^{ori}$ and $H_1^1$ will be attached along should be on the horizontal boundary of $W_0$. Thus, the starting point of the modification is pushing the corresponding part of $L_0$ to the boundary of $F_0$. In order to do this, we specify the corresponding part of $L_0$. Since $h_1$ is a 1 handle, the attaching boundary is homeomorphic to $S^0 \times D^1$. One can observe that the attaching boundary of $h_1$ can be divided into three parts, each of them corresponds to $h_1^{ori}$, $h_1^1$ and $h_1^2$. Moreover, without loss of generality, one could identify $\partial_R h_1$ with $S^0 \times D^1_2$, where $D^k_r$ means a $k$ dimensional disk of the radius $r$, so that the part of $\partial_R h_1$ corresponding to $h_1^2$ is identified with $S^0 \times D^1_r$. We note that the identifications also preserves the orientations. Under the identification $\partial h_0 \simeq \Lambda_0 \simeq L_0$, one could embed $\partial_R h_1$ into $L_0$. For convenience, let $S^0 \times D^1_r$ denote the image of the embedding. Moreover, we choose a small neighborhood of the image, and let $S^0 \times D^3_{3r}$ denote the neighborhood. Figure 5 describes this.

**Remark 6.5.** For the general case, $S^0 \times D^1_r$ would be replaced with $S^{k-1} \times D^{n-k}_r$ where $h_i$ is a $n$ dimensional $k$ handle.
In order to modify the specified part $S^0 \times \mathbb{D}^1_{3\epsilon}$, we fix an auxiliary function $\varphi : [0, 3\epsilon] \to \mathbb{R}$ so that

- $\varphi(3\epsilon) = 0$, and
- the graph of the derivative $\varphi'$ is given in Figure 6.

Then, one can define a function $g$ on $L_0$ as follows:

$$g : L_0 \to \mathbb{R},$$

$$g(x) = \begin{cases} -\varphi(|t|), & \text{if } x = (p, q) \in S^0 \times \mathbb{D}^1_{3\epsilon}, \\ 0, & \text{otherwise}. \end{cases}$$

Let $L'_0$ be the graph of the 1 form $dg$ in $F_0 = D^*S^1$ and let $\Lambda'_0$ be a lift of $L'_0$ such that $\Lambda'_0$ agrees with $\Lambda_0$ outside of $S^0 \times \mathbb{D}^1_{3\epsilon}$. It is easy to check that $L_0$ and $L'_0$ are Hamiltonian isotopic, and that the Hamiltonian isotopy induces a Legendrian isotopy connecting $\Lambda_0$ and $\Lambda'_0$. Figure 7, a). is $L'_0$ in $F_0$ and b). is $\pi_0(\Lambda'_0)$ on the base.

By abuse of notation, we set $L'_0$ as a map from $S^0 \times \mathbb{D}^1_{3\epsilon}$ to $F_0$, whose image is the specified part of the Lagrangian $L'_0$. Similarly, $\Lambda'_0$ is also a function defined on
In Equation (6.11), the first component is a point in $F_0$. The second and the last components are coordinated by the standard $(x, y)$-coordinates of $D^2$. We also note that in Equation (6.11), $g(L'_0(p, q))$ is simply written as $g(p,q)$ for convenience.

**Remark 6.6.** For the example case, we used the function $g$ for pushing the Legendrian to the horizontal boundary. For the general cases described in Remarks 6.4 and 6.5, the function $g$ is generalized as follows:

$$g : L_i \to \mathbb{R},$$

$$g(x) = \begin{cases} -\varphi(|t|), & \text{if } x = (p,q) \in S^k \times D_{3\epsilon}^{n-k-1}, \\ 0, & \text{otherwise}. \end{cases}$$

For the example case, $L'_1$ is the graph of $dg$. This is using the fact that $F_0$ is the disk cotangent bundle of $L_1$. However, for a general case, a fiber does not admit a cotangent bundle structure. Thus, the way of pushing the Legendrian to the horizontal boundary by using the generalized $g$ is more complicated than the example case. The way is given in Section 6.3.

**Crossing the base.** The next step of the modification is to modify $\Lambda'_0$ in order to obtain another Legendrian whose image under $\pi_0$ contains a diameter of the base disk. If one obtains such Legendrian, then one attaches $H^{or}_1$ and $H^1_1$ along the Legendrian on the horizontal boundary.
In order to do that, we will construct two 1 parameter families of maps $\gamma^1_s$ and $\gamma^2_s$ for all $s \in [0, \pi]$. Those two families are defined on $S^0 \times \partial D^{\mathbb{R}_+} \times [0, 1]$ and $S^0 \times \partial D^{\mathbb{R}_+}$, respectively. At the end, the concatenation of them will give a Legendrian isotopy connecting $\Lambda'_0$ and the desired Legendrian.

The first family $\gamma^1_s$ is defined as follows:

\begin{equation}
\gamma^1_s : S^0 \times \partial D^{\mathbb{R}_+} \times [0, 1] \to \partial(F_0 \times \mathbb{R}^2),
\end{equation}

\[
(p, q, t) \mapsto \left(\Phi^s_0\frac{-t}{c_1}\sin s(L'_0(p, q)), (1 - t) \cos(-cg(p, q)) + t \cos(-cg(p, q) + s),
(1 - t) \sin(-cg(p, p, -\pi)) + t \sin(-cg(p, p, \pi) + s)\right).
\]

We note that $L'_0(p, q)$ are defined right above of Equation (6.11).

One could check the followings:

(i) $\text{Im}(\gamma^1_s)$ is a Legendrian for any $s \in [0, \pi]$, and
(ii) $\gamma^1_s(p, q, 0) = \Lambda'_0(p, q)$.

The second is easy to check. In order to prove (i), one need to compute

\begin{equation}
(\alpha_0 + \frac{1}{c}(x dy - y dx))(\gamma^1_s(\partial t)) = 0.
\end{equation}

By definition,

\[
\gamma^1_s(\partial t) = \frac{\partial}{\partial t} \Phi^s_0\frac{-t}{c_1}\sin s(L'_0(p, q)) + (- \cos(-cg(p, q)) + \cos(-cg(p, q) + s))\partial x + (- \sin(-cg(p, q, -\pi)) + \sin(-cg(p, q, \pi) + s))\partial y.
\]

When one plug this vector into the contact 1 form on the horizontal boundary, i.e., the 1 form in Equation (6.9), one obtains

\[
a_0\left(\frac{\partial}{\partial t} \Phi^s_0\frac{-t}{c_1}\sin s(L'_0(p, q))\right) + \frac{1}{c}\sin s = -\frac{1}{c}\sin s + \frac{1}{c}\sin s = 0.
\]

The first equality holds since $\Phi^t$ is the Reeb flow on $\partial F_1$. Thus, Equation (6.13) holds.

**Remark 6.7.** For a general case described in Remark 6.4, $\gamma^1_\ast$ is generalized as follows:

\begin{equation}
\gamma^1_\ast : S^k \times \partial D^{\mathbb{R}}^{n-k-1} \times [0, 1] \to \partial(F_1 \times \mathbb{R}^2),
\end{equation}

\[
(p, q, t) \mapsto \left(\Phi^s_1\frac{-t}{c_1}\sin s(L'_1(p, q)), (1 - t) \cos(-cg(p, q)) + t \cos(-cg(p, q) + s),
(1 - t) \sin(-cg(p, p, -\pi)) + t \sin(-cg(p, p, \pi) + s)\right).
\]

One can check that the above (i)--(ii) hold for the generalized $\gamma_\ast$. One could check that (ii) holds by the same way we did for the example case. However, for the case of either $k \geq 1$ or $n - k - 1 \geq 2$, an extra work is necessarily to prove (i).

The extra work is the following: Let $V_0$ be a tangent vector on $S^k \times \partial D^{\mathbb{R}}^{n-k-1}$. Then, there exists a tangent vector $V_\ast$ on $L'_1$ such that $V : L'_1 = (V_0)$. By Equation (6.12),

\[
\gamma^1_\ast(V_0) = \left(\Phi^s_1\frac{-t}{c_1}\sin s\right)_\ast(V)
\]

\[
+ \left((1 - t) \sin(-cg(p, q))cV(g) + t \sin(-cg(p, q) + s)cV(g)\right)\partial x
\]

\[
+ \left(- (1 - t) \cos(-cg(p, q))cV(g) - t \cos(-cg(p, q) + s)cV(g)\right)\partial y
\]

\[
= \Phi^s_1\frac{-t}{c_1}\sin s(V)
\]

The last equality comes from the fact that $g$ is constant on $L'_1(S^k \times \partial D^{\mathbb{R}}^{n-k-1})$.\]
The vector \( \gamma_i^1(V_0) \) is contained in the contact structure, since
\[
(\alpha_i + \frac{1}{c}(xdy - ydx))(\gamma_i^1(V_0)) = \alpha_i((\Phi_i - \frac{t}{s}\sin s)_*)(V)
\]
\[
= ((\Phi_i - \frac{t}{s}\sin s)_*\alpha_i)(V) = \alpha_i(V) = \lambda_i(V) = V(g) = 0.
\]
The third equality holds since \( \Phi_i \) is the Reeb flow on \( \partial F_i \), and the others hold by definitions. This proves that (i) holds for the general cases.

In order to construct the second 1 parameter family \( \gamma_2^s \), we observe the following: since \( \Phi_i - \frac{t}{s}\sin s \) is a symplectomorphism, by [5, Lemma 11.2], there is a function \( h_s : F_i \rightarrow \mathbb{R} \) such that
\[
(6.15) \quad \Phi_i - \frac{t}{s}\sin s)_* (\lambda_i) = \lambda_i + dh_s.
\]
Moreover, \( h_s|_{F_i} \) is a constant function, since on \( \partial F_i \), \( \Phi_i - \frac{t}{s}\sin s \) is the Reeb flow, so that
\[
(\Phi_i - \frac{t}{s}\sin s)_* (\lambda_i) = \lambda_i.
\]
Since \( h_s \) is unique up to constant in Equation (6.15), we can choose \( h_s \) such that \( h_s|_{\partial F_i} \equiv 0 \).

We set \( \gamma_2^s \) for \( s \in [0, \pi] \) as follows:
\[
(6.16) \quad \gamma_2^s : S^0 \times D_2^1 \rightarrow \partial(F_0 \times D^2),
(\gamma_2^s(p,q)) = (\Phi_0 - \frac{t}{s}\sin s(p,q), \cos(-cg(p,q) + s + h_s(p,q)), \sin(-cg(p,t) + s + h_s(p,q))).
\]
As similar to the case of \( \gamma_1^s \), the following facts hold:
(iii) \( \text{Im}(\gamma_2^s) \) is a Legendrian, and
(iv) \( \gamma_2^s(p,q,1) = \gamma_2^s(p,q) \) for \( (p,q) \in S^0 \times \partial D_2^1 \).
Since \( \gamma_2^s \) is in the form of a lifted Legendrian, (iii) holds, and since \( h_s|_{\partial F_0} \equiv 0 \), (iv) holds.

**Remark 6.8.** As similar to Remark 6.7, \( \gamma_2^s \) is defined as follows for the general cases.
\[
(6.17) \quad \gamma_2^s : S^k \times D_2^{n-k-1} \rightarrow \partial(F_i \times D^2),
(\gamma_2^s(p,q)) = (\Phi_i - \frac{t}{s}\sin s(p,q), \cos(-cg(p,q) + s + h_s(p,q)), \sin(-cg(p,t) + s + h_s(p,q))).
\]
Also, (iii)–(iv) hold for the general \( \gamma_2^s \).

From (i) – (iv), by removing \( \text{Im} \gamma_0^0 \cup \text{Im} \gamma_0^0 \) from \( \Lambda_0^* \) and attaching \( \text{Im} \gamma_1^s \cup \text{Im} \gamma_2^s \) and by smoothing them, one could obtain an 1 parameter family of Legendrians. Let \( \Lambda_0 \) be the Legendrian obtained when \( s = \pi \). After a small reparameterization of \( \Lambda_0 \), the image of \( \Lambda_0 \) under \( \pi_0 \) is given in Figure 8.

**Attaching subcritical handles.** The next step is to attach subcritical handles \( H_i^{ori} \) and \( H_i^1 \). We attach them along the part of \( \Lambda_0 \), which are contained in the horizontal boundary. More precisely, from the starting data, i.e., the handle decomposition \( D \) of \( M \), one obtains gluing maps for \( H_i^{ori} \) and \( H_i^1 \). The gluing maps send the attaching Legendrians of \( H_i^{ori} \) and \( H_i^1 \) to some parts of \( \Lambda_0 \). By composing the gluing maps with the Legendrian isotopy connecting \( \Lambda_0 \) and \( \Lambda_0^* \), one obtains gluing maps attaching \( H_i^{ori} \) and \( H_i^1 \) along the corresponding parts of \( \Lambda_0 \).

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After attaching the subcritical handles, the resulting Weinstein domain $W_1$ admits the product Lefschetz fibration $\pi_1$. The regular fiber $F_1$ of $\pi_1$ is given in Figure 9. The construction $W_1$ induces that $W_1$ is equivalent to $T^* M_1$.

In Figure 9 one could observe that $\partial M_1$ has four components. After smoothing, the images of four components of $\partial M_1$ under $pr_1 : W_1 \to F_1$ are given in Figure 9. Also one could observe that two of the four components are used for attaching critical handles $H^0_2, H^1_1$. Moreover, it is easy to check that $H^0_2$ (resp. $H^1_1$) is attached along the component of $\partial M_1$, which is projected to near $-1$ (resp. $e^{-\theta_1}+\pi$ where $\theta_1$ is a constant depending on the choice of the small positive number $c$ and the auxiliary function $\phi$ above). The component for $H^2_0$ (resp. $H^2_1$) corresponds to the dashed (resp. blue) curve in Figure 9.
Remark 6.9. More precisely, one can define $\theta_1$ as setting $\theta_1 = c\phi(2\epsilon)$. We omit the detail, but we would like to point out that one can obtain an arbitrarily small $\theta_1$ by choosing sufficiently small $c$.

The other two components are projected down to the interval
\[
\{e^{-i\theta} \mid \theta \in [-\theta_1, 0]\} \subset \partial \mathbb{D}^2.
\]
By Legendrian isotoping, one could move them so that after moving the Legendrians are projected to
\[
\{e^{-i\theta} \mid \theta \in (-3\theta_0, -2\theta_0]\} \subset \partial \mathbb{D}^2,
\]
for some $\theta_0$ such that $\theta_0 > \theta_1$. The desired Legendrian isotopy is obtained by taking negative Reeb flows of the Legendrians.

Remark 6.10. The condition that $\theta_0 > \theta_1$ will be used when we attach critical handles $H_2^1$ and $H_2^2$.

Let $\Lambda_1$ be the Legendrian which one obtains after the Legendrian isotopy. One could repeat the procedure for $\Lambda_1$, which we did with $\Lambda_0$. Then one obtains $W_2$ equipped with the product Lefschetz fibration $\pi_2$ by attaching subcritical Weinstein handles to $W_1$ along the modified Legendrian.

**Attaching critical handles.** The constructed $W_2$ can be identified with $T^*M_2$. Then, $\partial M_2$ are identified with a union of Legendrians under the identification. The projected images of those four Legendrians, under $pr_2$ and $\pi_2$, are given in Figure 10. With Figure 10, one could attach critical handles $H_0^2$, $H_1^2$, $H_2^2$ and $H_3$ along $\partial M_2$, by [3, Proposition 8.1].

Remark 6.11. The resulting Lefschetz fibration in this subsection is the same as the Lefschetz fibration which [8] constructed. Moreover, when a smooth manifold $M$ is of dimension 2 and the starting handle decomposition $D$ comes from a self-indexing Morse function of $M$, then the Lefschetz fibration obtained by applying Theorem 5.1 is the same Lefschetz fibration which [8] constructed.

6.3. The general case. In Section 6.2, we discussed a specific example, and Remarks 6.4–6.8 discussed the general cases. The missing part for the general case is to push Legendrians to the horizontal boundary. More precise statement for the missing part is given in Remark 6.6. In the current subsection, we discuss the missing part.

The advantage of fixing an example case is that we have a concrete description for the fibers of $\pi_i$. Thus, in Section 6.2, we could use the cotangent bundle structure on the fiber in order to push the Legendrian to the boundary. For the general cases, we consider the following lemma instead of it.

**Lemma 6.12.** Let $(F, \omega = d\lambda_0)$ be a Weinstein domain and let $L$ be a compact exact Lagrangian of $F$, i.e., there exists a function $f : L \to \mathbb{R}$ such that $df = \lambda|_L$. Let $g$ be a real-valued function defined on $L$. Then, there is a Hamiltonian isotopy $\Psi^t$ and a Liouville 1 form $\tilde{\lambda}$ on $F$ such that

(i) $\tilde{\lambda} - \lambda$ is an exact 1 form on $F$, and

(ii) if $\psi = (f + g) \circ \Psi^{-1} : \Psi^1(L) \to \mathbb{R}$, then $d\psi = \tilde{\lambda}|_{\Psi^1(L)}$. 


Figure 10. a). is the fiber $F_2$ together with the images of $\partial M_2$ under $pr_2$. b). is the base $D^2$ together with the images of $\partial M_2$ under $\pi_2$. The images of the same component of $\partial M_2$ are in the same color in a). and b).

Proof. Let $G : F \to \mathbb{R}$ be a compactly supported function obtained by extending $g$. Then, there is a Hamiltonian vector field $X_G$. Since $G$ is compactly supported, $X_G$ is complete. Let $\Psi^t$ be time $t$ flow of $X_G$ and let $\lambda_t := (\Psi^t)^* \lambda$. Then, the followings
hold.

\[
\lambda_1 - \lambda_0 = \int_0^1 \frac{d}{dt} \lambda_t dt = \int_0^1 (X_{G_i} d\lambda_t + d(X_{G_i} \lambda_t)) dt = dG + d \int_0^1 (X_{G_i} \lambda_t) dt.
\]

Let \( H = \int_0^1 (X_{G_i} \lambda_t) dt \). Then, for \( V \in T\Psi^{-1}(L) \),

\[
\lambda_0(V) = \lambda_0(\Psi_*(\Psi^{-1}_*(V)) = (\Psi^1)^* \lambda_0)(\Psi_*(\Psi^{-1}_*(V))
\]

\[
= \lambda_1(\Psi^{-1}_*(V)) = \lambda_0(\Psi^{-1}_*(V) + dG(\Psi^{-1}_*(V) + dH(\Psi^{-1}_*(V)
\]

\[
= df + g(\Psi^{-1}_*(V) + dH(\Psi^{-1}_*(V)
\]

\[
= d\psi(V) + d(H \circ \Psi^{-1})(V).
\]

If one sets \( \tilde{\lambda} = \lambda - d(H \circ \Psi^{-1}) \), then the conditions (i) and (ii) hold. \( \square \)

We use the same notation which we used in Section 6.2. Let \( \Lambda_i \) be the Legendrian corresponding to \( \partial M_i \) and let \( S^{k-1} \times D^{n-k}_r \) denote a part of \( \Lambda_i \), where subcritical handles \( H^{ori}_r \) and \( H^{n-1}_r \) will be attached along. Also, \( S^{k-1} \times D^{n-k}_r \) is a neighborhood of \( S^{k-1} \times D^{n-k}_r \). See Remark 6.4 for the notation \( S^{k-1} \times D^{n-k}_r \).

Let \( g \) be the function defined on \( L_i := \text{pr}_i(\Lambda_i) \) defined in Remark 6.6 for the general cases. We apply Lemma 6.12 to the exact Lagrangian \( L_i \) and \( g \). Then, one obtains a \( \tilde{L}_i \), or equivalently, \( \Psi^1(\tilde{L}_i) \) with the notation in Lemma 6.12 and a Liouville 1-form \( \tilde{\lambda}_i \). We note that by choosing a proper \( G \), one can assume that \( \tilde{L}_i \) is obtained by pushing \( L_i \) to the boundary of \( F_i \). This is because asymptotically \( L_i \) is a part of the core, or equivalently the Lagrangian skeleton, of \( F_i \).

The Legendrian lift of \( L_i \) with respect to \( \lambda_i \) and that of \( \tilde{L}_i \) with respect to \( \tilde{\lambda}_i \) are Legendrian isotopic to each other by Lemmas 6.2 and 6.12. This completes the modification of Legendrians.

7. The effects of handle moves

Theorem 5.1 gives infinitely many Lefschetz fibrations of cotangent bundles \( T^* M \). In Section 7, for the case of \( \dim M = 2 \), we discuss how those Lefschetz fibrations of \( T^* M \) are related to each other. As the result, we show that all Lefschetz fibrations of \( T^* M \) constructed by Theorem 5.1 are connected by four moves which are introduced in Section 7.1. A technical statement and the proof of it are in Sections 7.2, 7.4.

7.1. Four moves. In Section 7, we use the notion of abstract Lefschetz fibration defined in Definition 2.13. This viewpoint is based on [7].

Let \( (F; L_1, \cdots, L_m) \) be an abstract Lefschetz fibration. Then, it is well-known that the total space of \( (F; L_1, \cdots, L_m) \) is equivalent to the total space of another abstract Lefschetz fibration obtained by applying one of the following four operations:

- Deformation means a simultaneous Weinstein deformation of \( F \) and exact Lagrangian isotopy of \((L_1, \cdots, L_m)\).
• **Cyclic permutation** is to replace the ordered collection \((L_1, \cdots, L_m)\) with \((L_2, \cdots, L_m, L_1)\). In other words,

\[(F; L_1, \cdots, L_m) \simeq (F; L_2, \cdots, L_m, L_1).\]

The equivalence means that their total spaces are equivalent.

• **Hurwitz moves.** Let \(\tau_i\) denote the symplectic Dehn twist around \(L_i\). **Hurwitz move** is to replace \((L_1, \cdots, L_m)\) with either \((L_2, \tau_2(L_1), L_3, \cdots, L_n)\) or \((\tau_1^{-1}(L_2), L_1, L_3, \cdots, L_m)\), i.e.,

\[(F; L_1, \cdots, L_m) \simeq (F; L_2, \tau_2(L_1), \cdots, L_m) \simeq (F; \tau_1^{-1}(L_2), L_1, \cdots, L_m).\]

• **Stabilization.** Let \(\dim F = 2n - 2\), or equivalently, the total space is of dimension \(2n\). For a parameterized Lagrangian disk \(D^{n-1} \hookrightarrow F\) with Legendrian boundary \(S^{n-2} = \partial D^{n-1} \hookrightarrow \partial F\) such that \(0 = [\lambda] \in H^1(D^{n-1}, \partial D^{n-1})\) where \(\lambda\) is the Liouville 1-form, replace \(F\) with \(\tilde{F}\), obtained by attaching a \((2n - 2)\) dimensional Weinstein \((n-1)\) handle to \(F\) along \(\partial D^{n-1}\), and replace \((L_1, \cdots, L_m)\) with \((\tilde{L}, L_1, \cdots, L_m)\), where \(\tilde{L} \subset \tilde{F}\) is obtained by gluing together \(D^{n-1}\) and the core of the handle. In other words,

\[(F; L_1, \cdots, L_m) \simeq (\tilde{F}; \tilde{L}, L_1, \cdots, L_m).\]

We note that in the stabilization, the position of \(\tilde{L}\) is not necessarily to be middle of \(L_1\) and \(L_m\) in the cyclic order. By doing a proper number of cyclic permutations before applying the stabilization, the same \(\tilde{L}\) could be located between \(L_i\) and \(L_{i+1}\) for any \(i \in \mathbb{Z}/m\).

**Remark 7.1.** As cited in [7], it is natural to ask whether any two Lefschetz fibrations of a fixed Weinstein manifold can be connected by a finite sequence of the above four moves. In the current paper, we do not claim that the four moves are enough to connect every Lefschetz fibrations of \(T^* M\), but we claim that they are enough to connect all Lefschetz fibrations obtained by applying Theorem 5.1 when \(\dim M = 2\).

7.2. **Equivalence of Lefschetz fibrations.** In Sections 7.2, 7.4 we prove the following Proposition.

**Proposition 7.2.** If \(M\) is a 2 dimensional manifold, then all Lefschetz fibration of \(T^* M\) obtained by applying Theorem 5.1 are connected to each other by a finite sequence of the four moves in Section 7.1

**Proof.** It is well-known that any two handle decomposition \(D_1\) and \(D_2\) of the same manifold are connected by a finite sequence of three operations, a **change of order of handles**, a **cancellation of a canceling pair** and a **handle sliding**. Because \(\dim M = 2\), and because every handle decomposition has only one \(0\) handle by Definition 2.2, we have only four cases for the above three handle operations.

The first case is to change orders of handles. The second case is to cancel a canceling pair consisting of a 1 handle and a 2 handle. The third (resp. the last) case is to slide a 1 handle along another 1 handle without twisting (resp. with twisting). The last three cases are described in Figure 11.

In order to discuss the first case, let \(D_1 := \{h_0, \cdots, h_m\}\) be a handle decomposition of \(M\). If \(D_2\) is another handle decomposition of \(M\) obtained by switching the order of \(h_i\) and \(h_j\), then from the second condition of Definition 2.2 one could observe that \(h_i\) and \(h_j\) both are either 1 handles or 2 handles. If \(h_i\) and \(h_j\) are
2 handles, then the construction of $\pi_1$ and $\pi_2$, where $\pi_i$ is the Lefschetz fibration obtained by applying Theorem 5.1 to $D_i$, guarantees that $\pi_1$ and $\pi_2$ are the same abstract Lefschetz fibration.

Let assume that $h_i$ and $h_j$ are 1 handles. If $i < j$, then one can observe the following facts.

(i) For all $k = i+1, \cdots, j$, $h_k$ is not attached to $h_i$. Or equivalently, $h_k \cap h_i = \emptyset$.
(ii) Similar to (i), $h_j$ is attached to $\cup_{k=i+1}^{j-1} h_k$.

From the construction of $\pi_1$ and (i), one could observe that the vanishing cycle corresponding to $h_i^2$ does not intersect with vanishing cycles corresponding to $h_k^2$ for $k = i+1, \cdots, j$. Similarly, from (ii), the vanishing cycles corresponding to $h_j^2$ does not intersect with vanishing cycles corresponding to $h_k^2$ for $k = i, \cdots, j-1$. Thus, switching $h_i$ and $h_j$ does not effect on the resulting abstract Lefschetz fibration, i.e., $\pi_1$ and $\pi_2$ are the same.

Based on the above arguments, it is enough to prove that if two handle decomposition $D_1$ and $D_2$ of $M$ are connected by moves described in Figure 11, then $\pi_1$
and $\pi_2$ are connected by the four moves in Section 7.1. Thus, the following Lemmas 7.3–7.4 prove the Proposition 7.2.

\textbf{Lemma 7.3.} If a handle decomposition $D_2$ is obtained from $D_1$ by a cancellation of a canceling pair, then $\pi_1$ and $\pi_2$ are connected to each other by four moves.

\textbf{Lemma 7.4.} If a handle decomposition $D_2$ is obtained from $D_1$ by sliding an 1 handle along another 1 handle (with or without twisting), then $\pi_1$ and $\pi_2$ are connected to each other by four moves.

\textbf{Remark 7.5.} Before proving above Lemmas, we remark two facts which we will use to prove them.

(i) According to the algorithm given by Theorem 5.1, the regular fiber $F$ is obtained by attaching 1 handles to the disk cotangent bundle $D^*S^1$. Moreover, the zero section of $D^*S^1$ corresponds to the boundary of the unique 0 handle in $D$, and the 1 handles attached to $D^*S^1$ correspond to the 1 handles in $D$. By using this fact, one could obtain a local figure of the regular fiber $F_1$ (resp. $F_2$) of $\pi_1$ (resp. $\pi_2$). We will prove Lemmas 7.3 and 7.4 by using the local figures.

(ii) Since $F_i$ is obtained by attaching 1 handles to $D^*S^1$, $F_i$ contains the zero section of $D^*S^1$. Moreover, near the zero section, one could assume that the Liouville structure is the same as to the Liouville structure of $D^*S^1$.

7.3. **Proof of Lemma 7.3** The strategy for proving Lemma 7.3 is the following: we start the proof by drawing a local figure of $\pi_1$. We point out that $\pi_1$ is an abstract Lefschetz fibration, thus, a local figure of $\pi_1$ means a local figure of the fiber $F_1$ together with vanishing cycles. Then, we operate a sequence of four moves, and it induces a sequence of Lefschetz fibrations. At the end, we stop when we have a local figure corresponding to $\pi_2$. We note that $\pi_i$ is obtained by applying Theorem 5.1 for $D_i$, and $D_1$ (resp. $D_2$) is modeled in Figure 11, a). left (resp. right).

Figure 12, a). is the local picture for $\pi_1$. In the local picture, there are four vanishing cycles which correspond to handles in Figure 11, a). left. The correspondence are given as follows:

- The black curve corresponds to the 0 handle $h^0$.
- The red curve corresponds to the 1 handle $h^1$.
- The green curve corresponds to the 2 handle $h^2$.
- The blue curve corresponds to the 2 handle which is adjacent to $h^1$, and which is not $h^2$.

One can also observe that in the cyclic order, the black comes the first since it corresponds to the 0 handle, the green and blue are the next since they comes from 2 handles, and the red is the last since it comes from the 1 handle. We note that the order between blue and green vanishing cycles are not important because they do not intersect each other.

Figure 12, b). is obtained from a). by doing Hurwitz move which applies an inverse Dehn twist around the green to the red. We note that the Liouville structure near the black is same as the standard Liouville structure of the cotangent bundle of the black curve, as explained in Remark 7.5 (ii). This fact gives a specific orientation, and a Dehn twist and the inverse of it can be distinguished with the orientation.
By a sequence of four moves, one can convert a) to e).
For each of a) – e), the lefts are local pictures of fibers together with vanishing cycles (colored curves) and the right circles indicate the cyclic order of vanishing cycles.
Figure 12 b). is also obtained by stabilizing c). along the green dashed curve in c). In order to justifying the stabilization operation, we should check that the integration of the Liouville form on the whole green dashed line is zero. This corresponds to the condition $0 = [\lambda] \in H^1(D, \partial D)$ in the definition of the stabilization. One can easily check this since along the green dashed curve, one can assume that the Liouville 1 form is the standard Liouville form on the cotangent bundle of the black.

Figure 12 d). is obtained by Hurwitz move for the red and blue curves. This is similar to the step between a) and b). Also, Figure 12 d). can be obtained from e). by operating a stabilization along the red dashed curve. In order to justify the stabilization procedure, we need the same computation which we did for the step between b) and c).

Since the local picture corresponding to the handle decomposition $D_2$, where $D_2$ is obtained by canceling handles from $D_1$, is Figure 12 d)., this completes the proof of Lemma 7.3.

7.4. Proof of Lemma 7.4. We prove Lemma 7.4 only for the first case, i.e., a 1 handle sliding along another 1 handle without twisting, because of the lengthy of the paper. The second case could be proven easily by a similar way.

We prove the first case as similar to the proof of Lemma 7.3. More precisely, we start from a local picture of $\pi_2$, the regular fiber corresponding to the handle decomposition $D_2$, where $D_2$ is described in Figure 11 b). right. Figure 13 a). is the same picture as Figure 11 b), except that it is decorated by colored curves. The colored curves can explain where the vanishing cycles in the local picture for $\pi_2$, which is given in Figure 13 b), come from. Then, Figures 13 and 14 give the following ‘step–by–step’ proof. We omit some details since the omitted details appeared in the proof of Lemma 7.3.

b). $\Rightarrow$ c). We take a stabilization with the dashed orange Lagrangian in b).
c). $\Rightarrow$ d). We take a deformation.
d). $\Rightarrow$ e). By operating a Hurwitz move changing the order of the orange and the green, one considers $\tau_0$ (green), where $\tau_0$ is a Dehn twist along the orange.
e). $\Rightarrow$ f). We operate another Hurwitz move, exchanging the blue and the orange. For the vanishing cycle, we consider $\tau_0$ (blue).
f). $\Leftarrow$ g). We take a stabilization with the dashed orange Lagrangian in g).
g). $\Rightarrow$ h). We take a deformation.
h). $\Leftarrow$ i). We operate a stabilization with the dashed orange Lagrangian in i).
i). $\Rightarrow$ j). We take a deformation.
j). $\Rightarrow$ k). We take two Hurwitz moves, so that the orange goes front of the blue and the purple. For the vanishing cycle, we consider $\tau_{-1}^{-1}$ (blue), $\tau_{-1}^{-1}$ (purple).
k). $\Leftarrow$ l). We take a stabilization with the dashed orange Lagrangian in l).

At the end, we can easily check that Figure 14 l). is the local picture for the fiber $\pi_1$ corresponding to the left of Figure 13 a). This completes the proof.

We discussed for the case of $\text{dim } M = 2$. We end the present section by mentioning why we did not consider the general dimensional case. For the case of general dimension, we expect that the generalization of Proposition 7.2 holds. However, the proof of Proposition 7.2 is based on the “case by case” method. For a general dimensional case, the method will not work generally since there is infinitely many cases.
Figure 13. a). It is the same as Figure [III b). For b). – f). the lefts are local pictures of fibers together with vanishing cycles and the right circles indicate the cyclic order of vanishing cycles. We note that the vanishing cycle corresponding to $H_2^0$ is denoted by a black dot in the right circle, but it is omitted in the fiber pictures.
Figure 14. For each of g) – l), the lefts are local pictures of fibers together with vanishing cycles (colored curves) and the right circles indicate the cyclic order of vanishing cycles.
Part 2. Lefschetz fibrations on some plumbings

In Part 2, we construct Lefschetz fibrations on some plumbings by using the idea introduced in the first part, i.e., the idea of getting a Lefschetz fibration from a Weinstein handle decomposition.

The plumbings which we consider are plumbings of two cotangent bundles in Section 8 and plumbings of $T^*S^n$, whose plumbing patterns are trees in Sections 9–12. We note that the same idea will work for the case of general plumbings. However, we only consider the restricted plumbing spaces, because of lengthy of the paper.

Section 13 introduces possible applications. Especially, Corollary 13.2 gives diffeomorphic families of plumbing spaces. The given diffeomorphic families contain some plumbing spaces with names. For example, $A_{2k+1}$ and $D_{2k+1}$ plumbings are diffeomorphic to each other.

8. Plumbing space of two cotangent bundles

8.1. Lagrangian skeletons. In the present subsection, we discuss a Lagrangian skeleton of the total space of an abstract Lefschetz fibration. For convenience, let $W$ denote the total space of a given abstract Lefschetz fibration

$$\pi = (F; V_m, \cdots, V_1, V_0).$$

We recall the construction of $W$. The total space $W$ is obtained by attaching critical Weinstein handles along $V_i$ to $F \times \mathbb{D}^2$. After attaching critical Weinstein handles, the centers of the critical handles are the singular points of the Lefschetz fibration on $W$.

Let $H$ be a critical Weinstein handle of $W$. Then, the stable manifold of the center of $H$ could be decomposed into two parts, one part is contained in $H$, and the other is contained in the subcritical part $F \times \mathbb{D}^2$. We note that in the present paper, the stable manifold means the stable manifold with respect to the negative Liouville flow.

One can also observe that the first part is given by a disk centered at the singular point, and whose boundary is the Legendrian in $\partial (F \times \mathbb{D}^2)$, which $H$ is attached along. The second part is easily obtained, because $F \times \mathbb{D}^2$ admits a product Weinstein structure. We note that $\mathbb{D}^2$ is equipped with the standard radial Liouville vector field. Figure 15 describes an example of the image of the union of stable manifolds under the Lefschetz fibration.

Since the Lagrangian skeleton is given by the union of stable manifolds of all the center of Weinstein handles, from the above arguments, one can easily check that the image of Lagrangian skeleton can be given as similar to Figure 15. Also, one can observe that the stable manifolds of the centers of subcritical handles become the boundary of those of critical handles. Thus, it is possible to say that for the case of $W$, the Lagrangian skeleton is the union of stable manifolds of the centers of critical handles.

Remark 8.1. In Figure 15 every singular value is connected to the center. This is because the center is the unique zero of the Liouville 1 form of $\mathbb{D}^2$. We would like to note that by isotoping Liouville 1 forms on $\mathbb{D}^2$, one could move the unique zero to any point. Based on this, we use the term “base point”, rather than the center.
Figure 15. The outer circle means the base of a Lefschetz fibration having 5 singular values. The star marks are singular values, then center marker is the center of the base. The interior part inside the dotted circle corresponds the subcritical parts $F \times D^2$ and the red parts are images of Legendrians in $F \times \partial D^2$, which critical handles are attached along. The shaded parts are images of the union of stable manifolds of the centers of critical handles, or equivalently the singular points, under the Lefschetz fibration.

8.2. Lefschetz fibrations on plumbings. We prove Theorem 8.2.

Theorem 8.2. Let $M_1$ and $M_2$ be smooth manifolds of the same dimension. Let $P$ be the plumbing of two cotangent bundles $T^* M_1 \# T^* M_2$ at one point. Then, there is an algorithm producing a Lefschetz fibration on $P$ from a pair of handle decomposition $D_1$ and $D_2$ of $M_1$ and $M_2$ respectively, such that the center of the unique zero handle of $D_i$ is the plumbing point in $M_i$.

Proof. In order to prove, we give an abstract Lefschetz fibration, then we show that the total space of the abstract Lefschetz fibration is equivalent to $P$.

Abstract Lefschetz fibration. An abstract Lefschetz fibration consists of a regular fiber and an ordered collection of exact Lagrangians in the fiber. We start the proof by constructing a regular fiber. For $i = 1, 2$, let $D_i$ be a given handle decomposition of $M_i$. By using the notation used in Section 8.1 let

\[(F_1; X_{m_1}, \cdots, X_0)\text{ and } (F_2; Y_{m_2}, \cdots, Y_0)\]

(8.18)

denote the abstract Lefschetz fibrations which are obtained by applying Theorem 5.1 to $D_1$ and $D_2$. 
From the proof of Theorem 5.1, we constructed $F_i$ by attaching Weinstein handles to $D^*S^{n-1}$, where $\dim M_i = n$. Moreover, $X_0$ and $Y_0$ are the zero sections of $D^*S^{n-1}$.

Let $S_+$ (resp. $S_-$) be the upper hemisphere (resp. lower hemisphere) of $S^{n-1}$. Without loss of generality, one could assume that $F_1$ (resp. $F_2$) is obtained by attaching Weinstein handles only on $D^*S_+$ (resp. $D^*S_-$) part, up to the equivalence defined in Definition 2.10. The number of Weinstein handles we attach for constructing $F_1$ (resp. $F_2$) is the same as twice of the number of subcritical handles in $D_1$ (resp. $D_2$). Let the number be $2N_1$ (resp. $2N_2$).

We construct the regular fiber $F$ by attaching Weinstein handles to $D^*S^{n-1}$ as follows: We attach $2(N_1 + N_2)$ Weinstein handles, $2N_1$ Weinstein handles are attached to $D^*S_+$ in the same way as we constructed $F_1$, and the other $2N_2$ Weinstein handles are attached to $D^*S_-$ in the same way we constructed $F_2$. Then, one could understand $F_1$ and $F_2$ as subsets of $F$ so that

\begin{align}
F_1 \cup F_2 &= F, \\
F_1 \cap F_2 &= D^*S^{n-1}.
\end{align}

Figure 16 is an example. The example case is the plumbing of two $T^*T^2$ where $T^2$ is the 2 dimensional torus. The handle decompositions $D_1$ and $D_2$ are the same as the handle decomposition described in Figure 3 a). Then, the fiber $F$ is given by attaching eight 1 handles to $D^*S^3$.

From Equation (8.19), one could check that an exact Lagrangian in $F_i$ is an exact Lagrangian in $F$. Then, we can set the following abstract Lefschetz fibration $\pi$.

\begin{align}
\pi := (F; X_{m_1}, \cdots, X_1, Y_{m_2}, \cdots, Y_1, X_0 = Y_0).
\end{align}

We note that the vanishing cycle $X_0 = Y_0$ is the zero section of $D^*S^{n-1} \subset F$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure16.png}
\caption{The fiber after attaching eight 1 handles is given. In the picture, the top and bottom line segments are identified. The labels mean that the segments having the same labels should be identified to each other, and the arrow indicates the way of identification. The red and blue curves are Lagrangians in the fiber, which are obtained by modifying the zero sections of $D^*S^3$. According to the proof of Theorem 5.1, the modified Lagrangians explain how to attach 1 handles.}
\end{figure}
Equivalence to the plumbing space. Let \( W \) be the total space of the abstract Lefschetz fibration in Equation (8.21), and let \( \pi \) denote the Lefschetz fibration on \( W \). The next step is to show that \( W \) is equivalent to the plumbing space \( P \). In order to show that, we define subsets \( W_1 \) and \( W_2 \) of \( W \) such that \( W_i \cong T^* M_i \).

Let \( W_i \) be the subset of \( W \) such that the restriction of \( \pi \) on \( W_i \) is a Lefschetz fibration such that

- the regular fiber of \( \pi|_{W_i} \) is \( F_i \subset F \), and
- the target of the restriction \( \pi|_{W_i} \) (resp. \( \pi|_{W_2} \)) is the interior of the red (resp. blue) circle given in Figure 17.

![Figure 17](chart.png)

**Figure 17.** The star marks are singular values. The vanishing cycles corresponding to singular values \( x_i \) and \( y_j \) are \( X_i \) and \( Y_j \) respectively. The red and blue circles are boundaries of the targets of \( \pi|_{W_1} \) and \( \pi|_{W_2} \).

We show that \( W_1 \cup W_2 \) is equivalent to \( W \) up to symplectic completion. This is because the Lagrangian skeleton of \( W \) is contained in \( W_1 \cup W_2 \). Section 8.1 proves this. Then, \( W_1 \cup W_2 \) is equivalent to \( W \) up to symplectic completion. Thus, it is enough to show that \( W_1 \cup W_2 \) is equivalent to the plumbing space \( P \).

It is easy to check that \( W_i \) is equivalent to \( T^* M_i \). This is because \( W_i \) admits a Lefschetz fibration \( \pi|_{W_i} \), thus \( W_1 \) and \( W_2 \) are equivalent to the total spaces of abstract Lefschetz fibrations

\[
(F_1; X_{m_1}, \ldots, X_1, X_0) \text{ and } (F_2; Y_{m_2}, \ldots, Y_1, Y_0),
\]

up to symplectic completion. Since the above Lefschetz fibrations are obtained by applying Theorem 5.1 to the handle decomposition \( D_i \) of \( M_i \), they are Lefschetz fibrations on \( T^* M_i \).

The intersection of \( W_1 \) and \( W_2 \) is also a total space of an abstract Lefschetz fibration such that
• the base is the intersection of the interiors of blue and red circles in Figure [17],
• the fiber is $F_1 \cap F_2 = D^*S^{n-1}$, and
• there is one singular value $x_0 = y_0$ whose vanishing cycle is $X_0 = Y_0$.

Thus, it is easy to observe that the intersection $W_1 \cap W_2$ is equivalent to $D^*\mathbb{D}^n$.

In order to complete the proof, we reconstruct $W_i$ in the following way. First, by modifying the Liouville structure on $W_1 \cap W_2$, we can see the intersection as an index 0 Weinstein handle. Then, $W_i$ is obtained by attaching the subcritical handles and critical handles to $W_1 \cap W_2$.

The reconstruction of $W_i$ also can be understood as a modification of Weinstein structure on $W_1 \cup W_2$. The images of Lagrangian skeletons of $W_1$ and $W_2$ after the modification, under $\pi$ are given in Figure [18].

Figure 18.

**Remark 8.3.** We note that after the modification of Weinstein structure, the restrictions $\pi|_{W_i}$ do not need to be Lefschetz fibrations defined in Definition 2.12. This is because Definition 2.12 cares the Weinstein structure.

The above arguments prove that $W_1 \cup W_2$ is the plumbing of $W_1$ and $W_2$ since they are sharing a zero handle, and since their Lagrangian skeletons inside the shared zero handle two disks intersecting transversely. This completes the proof.

□

**Remark 8.4.** We note that a similar argument to Theorem 8.2 is given in [2].
9. Sketch of the proof of Theorem 1.4

In Section 8, we described a way of giving a Lefschetz fibration defined on a plumbing of two cotangent bundles. The main idea is to observe the handle decomposition and the Lagrangian skeleton of a plumbing, which are corresponding to a Lefschetz fibration.

The same idea will work for the case of general plumbing spaces. However, we only considered Weinstein manifolds obtained by plumbing only two cotangent bundles at one point. The reason is that these Weinstein manifolds admit sufficiently simple Weinstein handle decompositions, so that we can formulate in the form of an algorithm.

In the rest of the current paper, we focus on the case of more generalized plumbing spaces. The more generalized cases are plumbings of multiple $T^*S^n$, where the plumbing patterns are trees. In other words, we will prove Theorem 1.4. Since we need preparations before proving, we give a sketch of the proof in the present section. A detailed proof will be given in Section 12.

Sketch of Theorem 1.4. Let $T$ be a tree, and let $P$ denote the plumbing of $T^*S^n$ whose plumbing pattern is $T$. We will give an order on the set of vertices of $T$ in Section 10.

Let $\{v_1, \ldots, v_m\}$ be the set of vertices of $T$. The subscripts are the order of each vertices. Then, we consider the subtree $T^{(k)}$ of $T$ such that the vertices of $T^{(k)}$ is $\{v_1, \ldots, v_k\}$.

Let $P_k$ denote the plumbing space of $k$ copies of $T^*S^n$, along the plumbing pattern $T^{(k)}$. Then, by definition, $P_{k+1}$ is obtained by plumbing $P_k$ and $T^*S^n$. Since it is a plumbing space of two symplectic manifolds, one can use the same argument with the proof of Theorem 8.2. In other words, when we have Lefschetz fibrations on $P_k$, the argument gives us a Lefschetz fibration on $P_{k+1}$. Then, the induction on $k$ will completes the proof. □

10. An order on a tree

In the present section, we explain our way of giving an order to the set of vertices of a tree.

Let $T$ be a tree, i.e., a graph without a cycle. For convenience, we define the following notation.

Definition 10.1. For a tree $T$, let $V(T)$ and $E(T)$ denote the set of vertices and the set of edges of $T$, respectively.

We consider an embedding of a tree into $\mathbb{R}^2$. Since every tree is planer, Definition 10.2 is well-defined.

Definition 10.2. An embedded tree is a pair

$$(T, f : T \to \mathbb{R}^2),$$

such that $f$ is an embedding of a tree $T$.

For simplicity, by a tree $T$, we mean an embedded tree without mentioning a specific embedding $f : T \to \mathbb{R}^2$, or we mean the embedded image $f(T)$.

When one has an embedded tree $T$, by using the orientation on $\mathbb{R}^2$, one can give a cyclic order on the set

$E_v(T) := \{e \in E(T) \mid v \text{ is an end point of } e\},$
for any $v \in V(T)$.

Similarly, one can define a cyclic order on the set of all boundary vertices of $T$, where a vertex $v \in V(T)$ is a boundary vertex if there exists only one edge $e \in E(T)$ such that $v$ is an end point of $e$. This is because, there is a closed subset $D \subset \mathbb{R}^2$ such that

- $D$ is homeomorphic to a topological disk $\mathbb{D}^2$,
- $T \subset D$, and
- $\partial D \cap T$ is the set of all boundary vertices.

Then, the naturally induced orientation on $\partial D$ induces the cyclic order on the set of boundary vertices. Figure 19 gives an example of the cyclic orders.

![Figure 19](image)

**Figure 19.** An embedded tree $T$ with three boundary vertices \{a, b, c\}, a non-boundary vertex $v$, and three edges $\{e_1, e_2, e_3\}$ is given in the figure. We note that $E_v(T) = \{e_1, e_2, e_3\}$ (resp. the set of boundary vertices \{a, b, c\}) is written in the induced cyclic order.

**Remark 10.3.** The cyclic orders are given by the orientation of $\mathbb{R}^2$. Thus, for two embedded trees $(T, f_1)$ and $(T, f_2)$ of the same tree $T$, if there is an orientation preserving diffeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h \circ f_1 = f_2$, then the cyclic orders induced from two embedded trees are the same.

We also define the notion of **rooted tree**.

**Definition 10.4.** Let $T$ be a tree.

1. A root of $T$ is a pair $(v \in V(T), e \in E(T))$ such that $v$ is an end point of $e$.
2. A rooted tree $T$ is a tree equipped with a specific choice of a root.

Let $T$ be a rooted tree with a root $(v_0, e_0)$. Then, it is easy to check the following facts.

- For any $e \in E(T)$, one can equip a natural orientation on $e$ so that $e$ is “going away” from the root vertex $v_0$. According to the natural orientation on $e \in E(T)$, one could see edges as arrows. We use terms heads/tails of edges from this view point.
- For any $v \in V(T)$, there is a unique sequence of edges $e_1, \cdots, e_k$ such that the tail of $e_1$ is the root $v_0$, the head of $e_i$ is the same as the tail of $e_{i+1}$, and the head of $e_k$ is $v$.

From the above facts, we define a **distance function**.
Definition 10.5. For a rooted tree $T$ with a root $(v_0, e_0)$, there is a well defined function

$$\text{dist} : V(T) \to \mathbb{Z}_{\geq 0}, v \mapsto k,$$

where $v$ is connected to $v_0$ by $k$ edges as described above.

For a rooted embedded tree $T$ with a root $(v_0, e_0)$, we can turn the cyclic orders defined above to orders. Because, in order to turn a cyclic order to an order, it is enough to fix the first/last element, the followings define the orders on $E_v(T)$ for all $v \in V(T)$ and on the set of boundary vertices.

- If $v$ is the root vertex $v_0$, then the first edge of $E_{v_0}(T)$ is the root edge $e_0$.
- If $v$ is not a root vertex $v_0$, then the last edge of $E_v(T)$ is the unique edge $e \in E_v(T)$ such that the head of $e$ is $v$.
- The first element of the set of the boundary vertices satisfies that there is a finite sequence of edges $\{e_0, e_1, \cdots, e_k\}$ such that $e_0$ is the root edge, $e_{i+1}$ is the first edge of $E_v(T)$ where $v$ is the head of $e_i$.

We note that the root vertex $v_0$ can be a boundary vertex. For that case, the rooted vertex $v_0$ is the last element of the set of boundary vertices.

For an arbitrary vertex $v \in V(T)$, there exists a unique finite sequence of edges $\{f_1, f_2, \cdots, f_k\} \subset E(T)$ such that

- $f_1$ is the first edge of $E_v(T)$,
- $f_{i+1}$ is the first edge of $E_{v_i}(T)$ where $v_i$ is the head of $f_i$, and
- the head of the last edge $f_k$ is a boundary vertex.

Let assume that the head of $f_k$ be the $j^{th}$ element of the set of boundary vertices. Then, we define the height of $v$ as follows.

Definition 10.6. The height of a vertex $v \in V(T)$ is $j \in \mathbb{N}$ which is obtained by the above argument. This defines a function

$$\text{height} : V(T) \to \mathbb{N}.$$

We note that since the sequence $\{f_1, \cdots, f_k\}$ in the above argument is unique, Definition 10.6 is well defined.

For a rooted embedded tree $T$, we set an order on $V(T)$ as follows.

Definition 10.7. Let $v, w \in V(T)$. Then, $v < w$,

- if $\text{height}(v) < \text{height}(w)$, or
- if $\text{height}(v) = \text{height}(w)$ and $\text{dist}(v) < \text{dist}(w)$.

For convenience, we label elements of $V(T) = \{v_0, v_1, \cdots, v_m\}$ so that $v_i < v_j$ if $i < j$.

The followings are facts one can easily check.

- For any vertex $v \in V(T)$, $v_0 \leq v$.
- $v_k$ is connected to $\{v_0, \cdots, v_{k-1}\}$ by an edge whose head is $v_k$.

From the second fact, one can define an order on $E(T)$ as follows.

Definition 10.8. Let $e, f \in E(T)$. Then, $e < f$ if the head of $e$ is less than the head of $f$. We label $E(T) = \{e_0, \cdots, e_{m-1}\}$, so that the head of $e_i$ is $v_{i+1}$. We note that $m = |V(T)|$. 


**Remark 10.9.** As similar to Remark 10.3, for a rooted embedded tree $T$ with an embedding $f_1 : T \to \mathbb{R}^2$ such that

- if two embedded trees $(T, f_1)$ and $(T, f_2)$ have the same root, then they induce the same orders on $V(T)$ and $E(T)$, and
- for any $v \in V(T)$, $f_2(v) = (\text{dist}(v), \text{height}(v))$.

Since the orders on $V(T)$ and $E(T)$ are important in the rest of the paper, not a specific embedding, we assume that every embedded tree satisfies the property described in the second item. Figure 20 gives two examples of embedded trees whose base trees are the same tree.

![Figure 20](image.png)

**Figure 20.** Two different embedded trees are described in Figure 20. As trees, not embedded trees, they are the same tree which is the Dynkin diagram of $D_4$ type.

We end the present Section by defining the followings for the later use.

**Definition 10.10.** Let $T$ be a rooted embedded tree with

$$V(T) = \{v_0, \cdots, v_m\}, E(T) = \{e_0, \cdots, e_{m-1}\}.$$

(1) Let $T^{(k)}$ be the subtree of $T$ consisting of

$$V(T^{(k)}) = \{v_0, \cdots, v_k\}, E(T^{(k)}) = \{e_0, \cdots, e_{k-1}\},$$

for $1 \leq k \leq m$.

(2) Let $T$ be the tree obtained from $T$ by shrinking the first edges of $E_v(T)$, for all vertices $v \in V(T)$.

**Remark 10.11.**

(1) It is easy to check that

$$T^{(1)} \subset \cdots \subset T^{(m)} = T.$$

We will use the sequence of trees to prove Theorem 1.4

(2) For any rooted embedded tree $T$, there is a natural quotient map $q : V(T) \to V(\overline{T})$.

An example of $\overline{T}$ is given in Figure 21.
11. The algorithm for the plumbings along trees

As defined in Definition 2.13, an abstract Lefschetz fibration consists of two things, a Weinstein manifold, i.e., a fiber, and an ordered collection of exact Lagrangian spheres in the fiber, i.e., the vanishing cycles. In the present section, we give an algorithm producing an abstract Lefschetz fibration from a rooted, embedded tree. In the first step of the algorithm, we describe the fiber. The second step is to choose a vanishing cycle for each vertex of the input tree. We note that we will use the order on the set of vertices in order to give an order on vanishing cycles. The last step is to choose a vanishing cycle for some edges, but not all edges of the tree. In each step, we also provide an example, where the rooted embedded tree of the example is given in Figure 22.

Figure 21. The left is a rooted embedded tree \( T \) and the right is the corresponding \( \overline{T} \).

Figure 22. An example tree \( T \) is the right, and the corresponding \( \overline{T} \) is the left.
Step 1. The fiber: Let $T$ be a given rooted, embedded tree with $V(T) = \{v_0, v_1, \cdots, v_m\}$, $E(T) = \{e_0, e_1, \cdots, e_{m-1}\}$. The labeling on $V(T)$ and $E(T)$ respects the orders on the sets. First, we set a notation.

**Definition 11.1.** Let $P_n(T)$ denote the plumbing space of copies of $T^*S^n$, whose plumbing pattern is $T$.

For a given $T$, and for $n \geq 2$, let $P_{n-1}(T)$ be the fiber of an abstract Lefschetz fibration.

When we apply the first step to the example given in Figure 27, the fiber should be $P_{n-1}(D_4)$ where $D_4$ means the Dynkin diagram of $D_4$ type.

Step 2. Vanishing cycles corresponding to vertices: For each vertex $v_i \in V(T)$, we add one vanishing cycle.

In order to describe the vanishing cycles in the second and third steps, we observe that for each vertex $v \in V(T)$, one can choose an exact Lagrangian sphere $L_v \subset P_{n-1}(T)$ in the following way. Because $P_{n-1}(T)$ can be seen as a union of $T^*S^{n-1}$, each vertex corresponds to a cotangent bundle $T^*S^{n-1}$. Then, the zero section of the cotangent bundle corresponding to a vertex $v$ would be the exact Lagrangian sphere.

**Definition 11.2.** For a tree $T$ and $v \in V(T)$, let $L_v$ denote the Lagrangian sphere in $P_n(T)$ corresponding to the given vertex $v$.

For $v_i \in V(T)$, we choose the exact Lagrangian $L_{q(v_i)} \subset P_{n-1}(T)$. We note that the quotient map $q$ is defined in Remark 11.1. Then, we have a cyclically ordered collection of exact Lagrangian spheres

$$\{L_{q(v_0)}, L_{q(v_1)}, \cdots, L_{q(v_m)}\}.$$

For the example case in Figure 27, we have

$$P_{n-1}(D_4): L_{q(v_0)}, L_{q(v_1)}, L_{q(v_2)}, L_{q(v_3)}, L_{q(v_4)}, L_{q(v_5)}.$$

**Remark 11.3.** We note that $L_{q(v_0)} = L_{q(v_2)} = L_{q(v_3)}$. This gives some matching cycles. After finishing the algorithm, the Lagrangian spheres corresponding to the matching cycles would be the zero sections of $T^*S^n$ corresponding to the vertices $v_1$ and $v_2$ in $P_n(T)$.

Step 3. Vanishing cycles corresponding to edges: For each edge $e \in E(T)$ such that

(i) $e$ is the root edge $e_0$, or

(ii) $e$ is not the first edge of $E_v(T)$ where $v$ is the tail of $e$,

we add a vanishing cycle. For an edge $e$ satisfying the above condition (i) or (ii), the head of $e$ determines the vanishing cycle, i.e., an exact Lagrangian in the fiber $P_{n-1}(T)$. The tail of $e$ determines the cyclic order of the corresponding vanishing cycle in the collection.

To be more precise, let $h \in V(T)$ denote the head of an edge $e$ satisfying either (i) or (ii). Then, the vanishing cycle corresponding to $e$ is $L_{q(h)} \in P_{n-1}(T)$. To describe the order of the vanishing cycle $L_{q(h)}$, let the tail of $e$ be $v_i$, i.e., $(i + 1)^{th}$
vertex in $V(T)$. Then, the vanishing cycle will be located between $L_{q(v_i)}$ and $L_{q(v_i+1)}$.

Let $\{e_{i_1}, \ldots, e_{i_k}\}$ be the set of edges such that their tails are $v_i \neq v_0$, and such that they are not the first edge in $E_{v_i}(T)$. Then, every vanishing cycle corresponding to one of the edges is located between $L_{q(v_i)}$ and $L_{q(v_i+1)}$. Thus, to complete the construction of an abstract Lefschetz fibration, we need to give an order on the vanishing cycles corresponding to $\{e_{i_1}, \ldots, e_{i_k}\}$. To describe the order, let assume that

$$i_1 < i_2 < \cdots < i_k.$$ 

Then, we put the vanishing cycles in the reverse order, i.e., the first vanishing cycle corresponds to $e_{i_k}$, the second corresponds to $e_{i_{k-1}}$, etc.

Lastly, we should consider $E_{v_0}(T) = \{e_0, e_{i_1}, \cdots, e_{i_k}\}$. We should add a vanishing cycle for each edge in $E_{v_0}(T)$. We consider

$$E_{v_0}(T) \setminus \{e_0\} = \{e_{i_1}, \cdots, e_{i_k}\}.$$ 

For the set, we repeat what we did before, i.e., the vanishing cycles corresponding to the above edges are located in the front of $L_{q(v_0)}$, in the reverse order. After that, we add the vanishing cycle $L_{q(v_1)}$ which corresponds to $e_0$ at the first position of the abstract Lefschetz fibration. This is the last step of the algorithm.

For our example, one can observe that we need to add vanishing cycles for edges $e_0, e_2, e_3, e_4$. The corresponding vanishing cycles are $L_{q(v_0)}, L_{q(v_3)}, L_{q(v_4)}, L_{q(v_5)}$ in the order, since the head of $e_i$ is $v_{i+1}$.

The positions of $L_{q(v_0)}$ and $L_{q(v_4)}$, which correspond to $e_2$ and $e_3$, are located between $L_{q(v_3)}$ and $L_{q(v_1)}$ since the tails of $e_2$ and $e_3$ are the same vertex $v_1$. Since $e_2 < e_3$, $L_{q(v_4)}$ comes earlier than $L_{q(v_3)}$.

The position of $L_{q(v_0)}$ is the front of $L_{q(v_0)}$ in Equation (11.22) since the tail of edge $e_4$ is $v_0$. After that, we add $L_{q(v_1)}$ corresponding to the root edge, at the first of the collection of vanishing cycles. At the end, we construct the following abstract Lefschetz fibration:

$$(P_{n-1}(D_4); L_{q(v_1)}, L_{q(v_3)}, L_{q(v_0)}, L_{q(v_4)}, L_{q(v_5)}, L_{q(v_2)}; L_{q(v_3)}, L_{q(v_0)}, L_{q(v_4)}, L_{q(v_5)}).$$

\textbf{Matching cycles:} For a given rooted, embedded tree $T$, let $LF(T)$ be the abstract Lefschetz fibration constructed by the above algorithm. Without loss of generality, one can assume that all the singular values of the Lefschetz fibration are contained in the unit circle in $\mathbb{R}^2 \cong \mathbb{C}$. Then, for any vertex $v \in V(T)$, one can relate a matching cycle in the following way:

- If $v$ is the root vertex, i.e., $v = v_0$, then on the base, we consider the straight line segment connecting two singular values corresponding to the vertex $v = v_0$ and the root edge $e_0$.
- For $i \geq 1$, if $v_i$ is connected to the vertex $v_{i-1}$, then we consider the straight line segment connecting two singular values corresponding to the vertex $v_i$ and $v_{i-1}$.
- For $i \geq 1$, if $v_i$ is ‘not’ connected to the vertex $v_{i-1}$, then there is a singular value corresponding to the edge $e_{i-1}$. We note that the head of $e_{i-1}$ is $v_{i-1}$. Then, we consider the line segment connecting two singular values corresponding to the vertex $v_{i-1}$ and the edge $e_{i-1}$.
By definition of the algorithm, one can easily check that the line segments given above are matching cycles, i.e., the two ends of the line segments are singular values having the same vanishing cycles.

We end Section 11 by giving a technical statement of Theorem 11.4.

**Theorem 11.4.** Let $T$ be an abstract tree. For any embedding of $T$ into $\mathbb{R}^2$, and for any root of $T$, the algorithm given in Section 11 produces a semi-abstract Lefschetz fibration whose total space is equivalent to $P_n(T)$ up to symplectic completion. Moreover, the Lagrangian sphere $L_v \in P_n(T)$ for any $v \in V(T)$ is Hamiltonian equivalent to the Lagrangian sphere given by the matching cycle related to $v$ by the above relation.

**Remark 11.5.** We point out that the plumbing space $P_n(T)$ depends only on the abstract tree $T$. Since the input of the algorithm is a rooted, embedded tree, we can observe that by choosing different rooted, embedded trees of the same tree $T$, one can produce different Lefschetz fibrations on the same Weinstein manifold $P_n(T)$.

12. **Proof of Theorem 11.4**

Let $T$ be a given rooted, embedded tree with

$$V(T) = \{v_0, v_1, \cdots, v_m\}, \quad E(T) = \{e_0, e_1, \cdots, e_{m-1}\}.$$ 

The labeling on $V(T)$ and $E(T)$ respects the orders on the sets.

As mentioned in Section 9, we prove Theorem 11.4 by induction on $T^{(1)} \subset \cdots \subset T^{(m)}$, which are defined in Definition 11.14.

For the base step of the induction, we operate the algorithm for $T^{(1)}$. By definition, $T^{(1)}$ is a tree with two vertices $v_0$ and $v_1$ connected by an edge $e_0$, i.e., the Dynkin diagram of $A_2$ type.

When one operates the algorithm for $T^{(1)}$, one obtains a Lefschetz fibration such that

- the fiber is $T^* S^{n-1}$,
- there are two vanishing cycles, and
- both vanishing cycles are the zero sections of $T^* S^{n-1}$.

Since it is well-known that the total space of the described abstract Lefschetz fibration is the $A_2$ plumbing, i.e., $P_n(T^{(1)})$, this completes the base step. Also, one can easily observe that the matching cycle condition holds.

For the inductive step, let assume that the algorithm in Theorem 11.4 gives an abstract Lefschetz fibration whose total space is $P_n(T^{(k)})$, with matching cycles. For convenience, let $LF(T^{(k)})$ and $LF(T^{(k+1)})$ denote the abstracts Lefschetz fibrations being produced by the algorithm for $P_n(T^{(k)})$ and $P_n(T^{(k+1)})$.

By definition, $T^{(k+1)}$ is obtained by adding a vertex $v_{k+1}$ and an edge $e_k$ to $T^{(k)}$. Then, the difference between $LF(T^{(k)})$ and $LF(T^{(k+1)})$ occurs by $v_{k+1}$ and $e_k$. In the rest of the proof, we consider two cases. The first (resp. second) case is that $e_k$ is (resp. is not) the first edge among the edges whose tails are the tail of $e_k$.

**The first case:** For the first case, it is easy to check that the added vertex $v_{k+1}$ is connected to $v_k$. This induces that $LF(T^{(k)})$ and $LF(T^{k+1})$ have the same fiber...
\( P_{n-1}(T^{(k)}) = P_{n-1}(T^{(k+1)}) \), and \( LF(T^{(k+1)}) \) is obtained by adding one vanishing cycle

\[ L_q(v_k) = q(v_{k+1}) \]

at the end of the ordered collection of vanishing cycles.

Inside \( LF(T^{(k+1)}) \), we consider two sub-fibrations defined as follows. The first sub-fibration has the same fiber \( P_{n-1}(T^{(k+1)}) \) as \( LF(T^{(k+1)}) \). The image of the first sub-fibration is the interior of the red circle given in Figure 23. The fiber of the second sub-fibration is a small neighborhood of \( L_q(v_k) = q(v_{k+1}) \subset P_{n-1}(T^{(k+1)}) \). The image of the second sub-fibration is the interior of the blue circle given in Figure 23. We note that the image of the second sub-fibration contains two singular values. Since the vanishing cycles corresponding to these two singular values are contained in the fiber of the second sub-fibration, the definition of sub-fibrations makes sense.

We observe that the total space, or equivalently the total space of the first subfibration, corresponds to a Weinstein domain obtained by attaching critical handles corresponding to all singular values except the last one, to the subcritical parts of \( LF(T^{(k+1)}) \). Let \( W_1 \) denote the Weinstein domain. Then, \( W_1 \) is equivalent to the total space of \( LF(T^{(k)}) \), since the fibration on \( W_1 \) has the exactly same fiber and singular values with \( LF(T^{(k)}) \). Thus, \( W_1 \) is equivalent to \( P_n(T^{(k)}) \) up to symplectic completions. Similarly, it is easy to check that if \( W_2 \) denotes the total space of the second sub-fibration, then \( W_2 \) is equivalent to \( T^*S^n \).

It is also easy to check that the union of \( W_1 \) and \( W_2 \) is a Weinstein domain whose symplectic completion is the total space of \( LF(T^{(k+1)}) \). This can be proven by considering the Weinstein handle decomposition of the total space of \( LF(T^{(k+1)}) \). Then, one can observe that every handle is contained in \( W_1 \cup W_2 \).

In order to complete the first case, it is enough to show that \( W_1 \cup W_2 \) is equivalent to \( P_n(T^{(k+1)}) \). This can be proven by observing that \( W_1 \cap W_2 \) also admits a sub-fibration structure. The fiber of the sub-fibration is the same as the fiber of \( W_2 \), i.e., \( T^*S^{n-1} \) up to symplectic completion, and there is one singular value whose vanishing cycles is the zero section. Thus, the intersection \( W_1 \cap W_2 \) is \( \mathbb{D}^{2n} \), since \( \mathbb{D}^{2} \) admits the same fibration.
Finally, by using the matching cycle condition in the induction hypothesis, i.e.,
the assumption that Theorem 11.4 holds for \( T(k) \), one can easily check that \( W_1 \cup W_2 \) is the plumbing of \( W_1 = P_n(T(k)) \) and \( W_2 = T^*S^n \). The plumbing point is located at the singular point corresponding to the vertex \( v_k \). This concludes that \( W_1 \cup W_2 \) is \( P_n(T(k+1)) \).

The second case: For the second case, \( v_{k+1} \) is connected to \( v_j \) such that \( 0 \leq j < k \) by \( e_k \). When one compares the fibers of \( LF(T(k)) \) and \( LF(T(k+1)) \), one can observe that their fibers are different. More precisely, the fiber of \( LF(T(k+1)) \), i.e., \( P_{n-1}(T(k+1)) \), is obtained by plumbing \( T^*S^{n-1} \) to the fiber of \( LF(T(k)) \), i.e., \( P_{n-1}(T(k)) \). In other words, we can say that \( P_{n-1}(T(k+1)) \) is a plumbing of \( P_{n-1}(T(k)) \) and \( T^*L_q(v_{k+1}) \).

When one compares the vanishing cycles of \( LF(T(k)) \) and \( LF(T(k+1)) \), one can observe that the collections of vanishing cycles for \( LF(T(k+1)) \) is obtained by adding two more vanishing cycles to that for \( LF(T(k)) \). Both of the added vanishing cycles are \( L_q(v_{k+1}) \). One of them, which corresponds to the edge \( e_k \), is located at the right after the vanishing cycle \( L_q(v_{j-1}) \) corresponding to \( v_{j-1} \). The other added vanishing cycle corresponding to \( v_{k+1} \) is located at the last of the collection.

As we did before, we consider two sub-fibrations. The first (resp. second) sub-fibration has \( P_{n-1}(T(k+1)) \) (resp. a small neighborhood of \( L_q(v_{k+1}) \)) as the fiber. The image of the first (resp. second) sub-fibration is given as the interior of red (resp. blue) circle in Figure 24.

Let \( W_1 \) and \( W_2 \) denote the total spaces of the sub-fibrations. Similar to the above case, \( W_1 \) is equivalent to \( P_n(T(k)) \). This is because the sub-fibration of \( W_1 \) can be obtained by taking a stabilization to \( LF(T(k)) \). We note that the stabilization defined in Section 7.1 makes one extra singular value, or vanishing cycle. The added vanishing cycle is \( L_q(v_{k+1}) \), located right after \( L_q(v_{j-1}) \). Thus, \( W_1 \) is equivalent to \( P_n(T(k)) \).
It is easy to check that $W_2$ is equivalent to $T^* S^n$. Then, the same argument which we used for the first case completes the proof.

13. Possible applications

Let $T$ be a tree. Then, Theorem 11.4 gives a Lefschetz fibration on $P_\alpha(T)$. By using the Lefschetz fibration, one could study some properties of $P_\alpha(T)$.

The first possible application is to find a diffeomorphic family of plumbing spaces. In order to explain this with examples, we define Definition 13.1.

**Definition 13.1.** For any $m \in \mathbb{N}$ and any $1 \leq j \leq m$, let $T_m^j$ denote the tree which is given in Figure 25.

![Figure 25. Tree $T_m^j$.](image)

Corollary 13.2 can be easily obtained from Theorem 11.4 and arguments in [10], [11], [12].

**Corollary 13.2.** For odd $n \geq 5$ (resp. $n = 3$), $m \in \mathbb{N}$, $P_n(T_m^j)$ and $P_n(T_m^{j+4})$ (resp. $P_n(T_m^{j+2})$) are diffeomorphic.

**Proof.** We prove Corollary 13.2 for the case of odd $n \geq 5$. We apply Theorem 11.4, then it gives us a Lefschetz fibration defined on $P_n(T_m^j)$. For any $m$ and $j$, the resulting Lefschetz fibrations have the same fiber $P_{n-1}(A_2)$, where $A_2$ means the Dynkin diagram of $A_2$ type. Let $\alpha$ (resp. $\beta$) denote the Lagrangian spheres $L_\alpha(v_1) = \cdots = L_\alpha(v_m)$ (resp. $L_\beta(v_{m+1})$). Then, the Lefschetz fibrations for $P_n(T_m^j)$ and $P_n(T_m^{j+4})$ are

\[(P_{n-1}(A_2); \alpha, \alpha, \cdots, \alpha = L_\alpha(v_{j-1}), \beta, \alpha = L_\alpha(v_j), \cdots, \alpha = L_\alpha(v_m), \beta = L_\alpha(v_{m+1})),\]

\[(P_{n-1}(A_2); \alpha, \alpha, \cdots, \alpha = L_\beta(v_{j+3}), \beta, \alpha = L_\beta(v_{j+4}), \cdots, \alpha = L_\beta(v_m), \beta = L_\beta(v_{m+1})).\]

We note that the middle $\beta$ in the Lefschetz fibration for $P_n(T_m^j)$ (resp. $P_n(T_m^{j+4})$) is located at $(j+1)^{th}$ (resp. $(j+5)^{th}$) position in the collection of vanishing cycles. By taking the Hurwitz move four times, one can move the middle $\beta$ in Equation (13.23) to the right. Then, it becomes $(\tau_\alpha)^4(\beta)$, and it is located at the $(j+5)^{th}$ position, where $\tau_\alpha$ denotes a Dehn twist along $\alpha$ on $P_{n-1}(A_2)$. In other words, we have the following Lefschetz fibration for $P_n(T_m^j)$.

\[(P_{n-1}(A_2); \alpha, \alpha, \cdots, \alpha = L_\alpha(v_{j+3}), (\tau_\alpha)^4(\beta), \alpha = L_\alpha(v_{j+4}), \cdots, \alpha = L_\alpha(v_m), \beta = L_\alpha(v_{m+1})).\]

When we compare Equations (13.25) and (13.24), one can conclude that if $\beta$ and $(\tau_\alpha)^4(\beta)$ induce smooth isotopic Legendrian spheres together with canonical formal Legendrian structures, then attaching critical handles along two Legendrians...
give diffeomorphic resulting Weinstein manifolds. In other words, Corollary 13.2 holds.

Since [11] and [12] prove the above sentence, it completes the proof of the case of odd $n \geq 5$.

For the case of $n = 3$, it is simpler since the formal Legendrian structures on Legendrian sphere are unique in $\mathbb{R}^5$ as stated in [14, Proposition A.4]. □

Remark 13.3.

(1) We note that we considered some restricted cases in Corollary 13.2, but by using the same method, one can construct more diffeomorphic families of plumbing spaces.

(2) Corollary 13.2 gives diffeomorphic families of plumbings. For example, if $m = 4k + 2$ and if $n \geq 5$ is odd, then Corollary 13.2 gives the following diffeomorphic families of Weinstein manifolds

$$\{ P_n(T_{4k+2})^1, P_n(T_{4k+2})^5, \cdots, P_n(T_{4k+2})^{4k+1}\}.$$ Since $P_n(T_{4k+2})^1$ (resp. $P_n(T_{4k+2})^{4k+1}$) is the plumbing space whose plumbing pattern is the Dynkin diagram of $A_{4k+3}$-type (resp. $D_{4k+3}$-type), those Milnor fibers are diffeomorphic to each other. Similarly, the Milnor fibers of $A_k$ and $E_k$-types are diffeomorphic to each other. If one considers the case of dimension 6, then one can obtain more and bigger families.

It would be natural to ask whether those diffeomorphic families are exotic families or not as Weinstein manifolds. We are working on answering the question.

We end the present paper by mentioning another possible application. That possible application is to study some symplectic automorphisms on $P_n(T)$. To be more precise, we note that since $P_n(T)$ is obtained by plumbing multiple copies of $T^*S^n$, $P_n(T)$ has at least $|V(T)|$ many Lagrangian spheres. Thus, there exist generalized Dehn twists along them.

On the base of the Lefschetz fibration we obtained from Theorem 11.4, one has matching cycles corresponding to the Lagrangian spheres. Then, it is well-known that the braid action on the base, which is switching two singular values that is located at the end of a matching cycle, corresponds to a Dehn twist along the Lagrangian sphere corresponding to the matching cycle. From the well-known fact, one can study the Dehn twists along Lagrangian spheres by using the Lefschetz fibration. Especially, we expect that this recovers the results of the author’s thesis [9] which constructs a higher dimensional stable/unstable Lagrangian laminations.

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