Derivation of the total twist from Chern-Simons theory

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Abstract

The total twist number, which represents the first non-trivial Vassiliev knot invariant, is derived from the second order expression of the Wilson loop expectation value in the Chern-Simons theory. Using the well-known fact that the analytical expression is an invariant, a non-recursive formulation of the total twist based on the evaluation of knot diagrams is constructed by an appropriate deformation of the knot line in the three-dimensional Euclidian space. The relation to the original definition of the total twist is elucidated.

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1 Introduction

The use of Chern-Simons theory as a tool for knot theory has proved to be very fruitful. Some of the background of the relationship of these two areas is recalled very briefly in order to fix the starting point for the present work.

One considers the Chern-Simons quantum field theory on the manifold $S^3$ with some gauge group, say $SU(N)$. It is characterized by a generating functional

$$Z[J] = \int DA \exp \left\{ iS_{CS}[A] + \int_{S^3} d^3x \ J_\mu(x)A_\mu^a(x) \right\}, \quad (1.1)$$

where $S_{CS}$ is the integral of the Chern-Simons differential form:

$$S_{CS}[A] = \frac{k}{4\pi} \int_{S^3} \text{tr} \left\{ A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\} \quad (1.2)$$

and $A = A_\mu^aT_a dx^\mu$ is a gauge field in some appropriately normalized representation $T$. Here $k$ must be an integer in order to maintain gauge invariance. The observables of the theory are Wilson-line operators along framed links. The concept of framing was introduced in the original work of Witten [1]. The Wilson loops are symbolically written as path-ordered exponentials of line integrals along knots:

$$W(K) = \text{tr} P \exp \left\{ \oint_K A \right\} \quad \text{for a knot } K \text{ embedded in } S^3, \quad (1.3)$$

$$W(L) = W(K_1) \ldots W(K_n) \quad \text{for a link with } n \text{ components } L = \{K_1, \ldots, K_n\}. \quad (1.4)$$

In the framework of quantum field theory the expectation values of $W(L)$ are defined as

$$\langle W(L) \rangle_f = \frac{\int DA \ W(L) \exp\{iS_{CS}[A]\}}{\int DA \ \exp\{iS_{CS}[A]\}}, \quad (1.5)$$

where $f$ symbolizes a framing prescription used in the evaluation of the Wilson loop. The expectation values are dependent on the coupling constant $k$ and on the order $N$ of the gauge group (and on the framing $f$). It has been shown that $\langle W(L) \rangle$ associated to a link $L$ fulfills the skein relation of a so-called generalized Jones polynomial. A comprehensive exposition of this subject can be found in a monograph by E. Guadagnini [2].

The Wilson loops can be expanded in a perturbation series. Then $\langle W(L) \rangle$ is a power series in $1/k$ and the Casimir factors of the gauge group $c_2(T), c_v, c_4(T)$ etc. Its coefficients are multi-dimensional path-ordered line integrals along the knot. Since $1/k$ and the Casimir factors can be considered as linearly independent the coefficient of every monomial is a link invariant. These line integrals can be calculated either numerically or by expanding the generalized Jones polynomial mentioned above, as demonstrated in a recent paper by Alvarez and Labastida [3].

We present here a direct method which does not rely on recursive procedures. Our starting point is the analytical expression for $\langle W(K) \rangle$ in the second order of $1/k$ which has been calculated by Guadagnini et al. in [4]:

$$\langle W(K) \rangle_{f, \text{2.order}} = \dim T \left( \frac{2\pi}{k} \right)^2 \left\{ \frac{1}{2} c_2^2(T)\varphi_f^2(K) + c_v c_2(T)[\rho_1(K) + \rho_2(K)] \right\}, \quad (1.6)$$

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where \( \varphi_f(K) \) is the framing number of the framed knot. The line integrals \( \rho_1(K) \) and \( \rho_2(K) \) are given by

\[
\rho_1(K) = -\frac{1}{32\pi^3} \int_K dx_1 \int_{BP} dx_2 \int_{BP} dx_3 \epsilon_{\mu_1\nu_1\sigma_1} \epsilon_{\mu_2\nu_2\sigma_2} \epsilon_{\mu_3\nu_3\sigma_3} \\
\times \int d^3z \frac{(z - x_1)^{\sigma_1} (z - x_2)^{\sigma_2} (z - x_3)^{\sigma_3}}{|z - x_1|^3 |z - x_2|^3 |z - x_3|^3},
\]

\[(1.7)\]

\[
\rho_2(K) = \frac{1}{8\pi^2} \int_K dx_1 \int_{BP} dx_2 \int_{BP} dx_3 \int_{BP} dx_4 \epsilon_{\mu_4\mu_2\sigma_2} \epsilon_{\mu_3\nu_1\sigma_1} \\
\times \frac{(x_4 - x_2)^{\sigma_2} (x_3 - x_1)^{\sigma_1}}{|x_4 - x_2|^3 |x_3 - x_1|^3},
\]

\[(1.8)\]

They are frame-independent, as shown in [4]. The knot invariant we are interested in is

\[
\rho^{II}(K) := \rho_1(K) + \rho_2(K).
\]

\[(1.9)\]

Our method to calculate \( \rho^{II} \) is based on the evaluation of knot diagrams \( K \) only, without recourse to skein relations. The program can be summed up in the following steps.

- We are interested of course in the evaluation of \( \rho^{II} \) for arbitrary knots embedded in three-dimensional space. To achieve this it is convenient to work with the knot diagrams. The formalism appropriate for our purposes is developed in sections 2 to 4.

- The sense in which knot diagrams are flattened knots is clarified in section 5.

- The integral \( \rho_2 \) is calculated for the limit of flat knots. The result is an expression for \( \rho_2 \) based on the evaluation of knot diagrams (section 6).

- A similar expression for \( \rho_1 \) is constructed using some properties of the line integral \( \rho_1(K) \) and the invariance of \( \rho^{II}(K) \). The knot theoretical formulation of \( \rho^{II}(K) \) is thereby completed (section 7).

- It is known from [4] that \( \rho^{II}(K) \) is closely related to the total twist \( \tau \) defined by Lickorish and Millet in [5]. This relation is examined for our diagrammatical version of \( \rho^{II} \) (section 8).

- The procedure for calculating the invariants \( \rho^{II} \) and \( \tau \) for a given knot using our expressions (7.14) and (8.1), which are based on the crossing numbers introduced in section 4.1, is illustrated in section 9.
We should emphasize that the important point in this paper is not just to find another expression for the total twist, but to find a method of constructing it which can be generalized to other link invariants. As has been argued in [3] the line integrals from the Wilson loop expansion are Vassiliev invariants, if normalized correctly. These have been introduced by Vassiliev in [6]. For an axiomatic approach to this topic see Birman and Lin [7]. The total twist, referred to above, is the simplest non-trivial invariant of this kind [8]. The method presented here, extended to higher orders, should allow interesting insights concerning this large class of knot invariants.

2 Definition of the knot diagrams

In this section we shall give the definitions of the knot diagrams used in this paper. Based, oriented knots are considered. Their diagrams will be described as mappings from a circle to a plane. First, define the unit interval \( I = [0, 1] \) and an equivalence relation which identifies \( 0 \sim 1 \). Then \( I/\sim \) is homeomorphic to a circle. The projection plane is \( \mathbb{R}^2 \). The shadow diagram of some knot is given by a mapping

\[
\pi : I/\sim \to \mathbb{R}^2.
\]

(2.1)

The point \( \pi(0) = \pi(1) \) is called the basepoint. The diagram itself is

\[
K = \text{image}(\pi) \subset \mathbb{R}^2.
\]

(2.2)

The set of crossings is given by

\[
C = \{ x \in \mathbb{R}^2 | \exists t_1 \neq t_2 \in I/\sim : \pi(t_1) = \pi(t_2) = x \} \subset \mathbb{R}^2.
\]

(2.3)

The over/under crossing information is given by a function

\[
\epsilon : C \to \{ -1, +1 \},
\]

(2.4)

defined as in the accompanying figure. Then \( \pi^{-1}(C) \) is a set \( \{ s_0, \ldots, s_{2n-1} \} \), with \( s_i \in I/\sim \) and \( n \) the number of crossings. More generally, we set \( \pi^{-1}(C) = \{ s_i | i \in \mathcal{I} \} \), where \( \mathcal{I} \) is some index set. The mapping \( \pi \) may be chosen such that for all \( i \in \mathcal{I} \) \( s_i \not\sim 0 \), i.e. the basepoint does not coincide with any crossing. The knot diagram is now defined by

\[
\mathcal{K} = (\pi, \epsilon, \mathcal{I}).
\]

(2.5)

The order relations \( > \) and \( < \) in \( I \) induce relations \( > \) and \( < \) in the index set \( \mathcal{I} \) according to

\[
s_i > s_j \Rightarrow i > j, \quad i, j \in \mathcal{I}.
\]

(2.6)

If not denoted otherwise \( \mathcal{I} = \{ 0, \ldots, 2n - 1 \} \) with its common order will be used. In this case the addition (modulo 2n) of elements of \( \mathcal{I} \) with integers is defined and will be used.
On $I_{/\sim}$ intervals will be denoted as follows. For $s_i \not \sim s_j$, $[s_i, s_j]$ is the closed arc from $s_i$ to $s_j$ following the orientation of $I_{/\sim}$. Hence

$$[s_i, s_j] \cup [s_j, s_i] = I_{/\sim} \quad \text{and} \quad [s_i, s_j] \cap [s_j, s_i] = \{s_i, s_j\}. \quad \text{(2.7)}$$

From this, ordered subsets of the index set are defined, and denoted as

$$[i_a, i_b] = \{i \in I \mid s_i \in [s_{i_a}, s_{i_b}]\} \text{ for } i_a \neq i_b, \quad \text{(2.8)}$$

with the property

$$[i_a, i_b] \cup [i_b, i_a] = I \quad \text{and} \quad [i_a, i_b] \cap [i_b, i_a] = \{i_a, i_b\}. \quad \text{(2.9)}$$

We define open subsets of the index set by

$$]i_a, i_b[ = [i_a, i_b] \setminus \{i_a, i_b\}. \quad \text{(2.10)}$$

The following alternative notations for the crossing function will also be used:

For all $i_1 \in I$ define $\epsilon(i_1) := \epsilon(\pi(s_{i_1}))$

For all $i_1, i_2 \in I$ define $\epsilon(i_1, i_2) := \begin{cases} \epsilon(i_1) & \text{if } \pi(s_{i_1}) = \pi(s_{i_2}) \text{ and } i_1 \neq i_2 \\ 0 & \text{otherwise} \end{cases} \quad \text{(2.11)}$

An example for a knot diagram $K = (\pi, \epsilon, I)$ defined in this way is shown in figure ??. Here, $n = 4$ (figure-eight knot) and $I = \{0, 1, \ldots, 7\}$.

**Definition of pieces of the knot diagram.** We consider a knot diagram $K = (\pi, \epsilon, I)$. A piece of the knot diagram will be defined as an open connection between two crossings or one crossing with itself. Consider two indices $i_1 \neq i_2 \in I$. Then a set such as

$$S = \pi([s_{i_1}, s_{i_2}]) \subset \mathbb{R}^2 \quad \text{(2.12)}$$

is called a piece of the knot diagram. To this piece an index subset is associated:

$$I(S) = ]i_1, i_2[ = \{i_1 + 1, i_1 + 2, \ldots, i_2 - 1\}. \quad \text{(2.13)}$$

Of course, a piece is not necessarily free of self-crossings.

Two pieces $S$ and $T$ will be called non-overlapping if $I(S) \cap I(T) = \emptyset$. In this case $S$ and $T$ can intersect each other, but the set of common points only consists of crossings.

### 3 Description of the Reidemeister moves

It is necessary for the following to formulate the Reidemeister moves in terms of $\pi$, $\epsilon$, and $I$. In this section $K = (\pi, \epsilon, I)$ will always denote the the knot diagram before the Reidemeister move, and $K' = (\pi', \epsilon', I')$ will be the diagram afterwards. The sets of
crossings are denoted by $C$ and $C'$. The Reidemeister moves can be formulated in the following way. In the diagram $L^+_1$, the indices which form the crossing are $k_1$ and $k_1 + 1$, and the situation can obviously be described by

$$\pi(s_{k_1}) = \pi(s_{k_1+1}). \quad (3.1)$$

After performing the Reidemeister-I move (see diagram $L^0_1$ in figure (??)), i.e. replacing $\pi$ by a mapping $\pi'$, the situation is described by a new index set $I'$ and a new crossing function $\epsilon' : C' \rightarrow \{-1, +1\}$, where $C' = C \setminus \{\pi(s_k)\}$.

$$I' = I \setminus \{k_1, k_1 + 1\}, \quad \epsilon' = \epsilon \mid_{C'}. \quad (3.2)$$

For the second Reidemeister move two cases have to be distinguished. First we consider the situation $L_{II-A}$ in figure ?? with the indices $k_1, k_1 + 1, k_2$, and $k_2 + 1$. It is characterized by

$$\pi(s_{k_1}) = \pi(s_{k_2}), \quad \pi(s_{k_1+1}) = \pi(s_{k_2+1}),$$

$$\epsilon(k_1, k_2) = -\epsilon(k_1 + 1, k_2 + 1). \quad (3.3)$$

In the situation $L^0_{II-A}$, i.e. after the Reidemeister-II-A move, we have

$$I' = I \setminus \{k_1, k_1 + 1, k_2, k_2 + 1\}, \quad \epsilon' = \epsilon \mid_{C'}. \quad (3.4)$$

For $L_{II-B}$ (see figure ??) with the same indices as in $L_{II-A}$ we have

$$\pi(s_{k_1}) = \pi(s_{k_2+1}), \quad \pi(s_{k_2}) = \pi(s_{k_1+1}),$$

$$\epsilon(k_1, k_2 + 1) = -\epsilon(k_1 + 1, k_2). \quad (3.5)$$

Performing the Reidemeister-II-B move we get the same $I'$ and $\epsilon'$ as for the Reidemeister-II-A move.

A Reidemeister-III situation as shown in figure (??) with indices $k_1, k_1 + 1, k_2, k_2 + 1, k_3$, and $k_3 + 1$, and a cyclic orientation is described by

$$\pi(s_{k_1}) = \pi(s_{k_3+1}), \quad \pi(s_{k_2}) = \pi(s_{k_1+1}), \quad \pi(s_{k_3}) = \pi(s_{k_2+1})$$

$$\epsilon(k_1)\epsilon(k_2) + \epsilon(k_2)\epsilon(k_3) + \epsilon(k_3)\epsilon(k_1) = -1. \quad (3.6)$$

The Reidemeister-III move does not change the index set, i.e. $I' = I$. The relation between the crossing functions $\epsilon$ and $\epsilon'$ is:

$$\epsilon'(k_1) = \epsilon(k_2), \quad \epsilon'(k_2) = \epsilon(k_3), \quad \epsilon'(k_3) = \epsilon(k_1). \quad (3.7)$$

The relation (3.6) is also valid for $\epsilon'$. 

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4 Crossing numbers

4.1 Definitions

We shall now define some functions of knot diagrams. Later they will be used to formulate the invariant $\rho^\Pi$ for diagrams. Let $\mathcal{K} = (\pi, \epsilon, \mathcal{I})$ be a knot diagram, with the set of crossings $\mathcal{C}$ and the number of crossings $n$. The behaviour of $n$ under the Reidemeister moves can be easily found by counting the crossings. It is

$$
\begin{align*}
n(\mathcal{L}_1^{+/-}) - n(\mathcal{L}_1^0) &= 1 \\
n(\mathcal{L}_{II-A/B}) - n(\mathcal{L}_{II-A/B}^0) &= 2 \\
n(\mathcal{L}_{III}^+) - n(\mathcal{L}_{III}^-) &= 0.
\end{align*}
$$

(4.1)

The writhe number, which is a regular isotopy invariant, is defined by

$$
\chi_1(\mathcal{K}) = \sum_{c \in \mathcal{C}} \epsilon(c).
$$

(4.2)

This can also be written as

$$
\chi_1(\mathcal{K}) = \sum_{j_1, j_2 \in \mathcal{I}, j_1 > j_2} \epsilon(j_1, j_2).
$$

(4.3)

For two non-overlapping pieces $S$ and $T$ of the diagram the first crossing number is defined as

$$
\chi_1(S, T) = \sum_{i_S \in \mathcal{I}(S)} \sum_{i_T \in \mathcal{I}(T)} \epsilon(i_S, i_T).
$$

(4.4)

Instead of writing $\chi_1(S, T)$ the notation using index sets $\chi_1(\mathcal{I}(S), \mathcal{I}(T)) := \chi_1(S, T)$ will also be used. If $S$ and $T$ start and end at the same point $c$ (see figure ???), $\chi_1(S, T)$ is closely related to the linking number $\lambda$ of the two link components arising from nullifying $c$. Denoting these components as $L_S$ and $L_T$ one has

$$
\frac{1}{2} \chi_1(S, T) = \lambda(L_S, L_T).
$$

(4.5)

Finally, we define an object called the second self-crossing number which is not an invariant:

$$
\chi_2(\mathcal{K}) = \sum_{j_1, j_2, j_3, j_4 \in \mathcal{I}, j_1 > j_2 > j_3 > j_4} \epsilon(j_1, j_3)\epsilon(j_2, j_4).
$$

(4.6)

It can easily be shown that $\chi_2$ can also be written as

$$
\chi_2(\mathcal{K}) = \frac{1}{4} \sum_{j_1, j_3 \in \mathcal{I}} \epsilon(j_1, j_3)\chi_1(\lfloor j_1, j_3 \rfloor, \lfloor j_3, j_1 \rfloor).
$$

(4.7)

In this last formulation neither the basepoint nor the order of the elements of $\mathcal{I}$, which was fixed by the basepoint, appear. Therefore $\chi_2(\mathcal{K})$ is independent of the basepoint. Now some properties of $\chi_2(\mathcal{K})$ will be examined.
4.2 Behaviour of $\chi_2$ under change of one crossing

An important property of $\chi_2$, which will be used later, is its behaviour if the sign of one crossing is changed. Consider two knot diagrams $K = (\pi, \epsilon, I)$ and $K' = (\pi, \epsilon', I)$ which differ only in one crossing $c$, i.e.

$$
\epsilon'(d) = \begin{cases} 
-\epsilon(d) & \text{if } d = c \\
+\epsilon(d) & \text{if } d \neq c. 
\end{cases}
$$

(4.8)

The crossing is formed by two indices $i^+_c, i^-_c \in I$. We want to calculate $\chi_2(K) - \chi_2(K')$. It is convenient to use the representation (4.7). The range of summation $I$ is divided into $\{i^+_c, i^-_c\}$ and $I \setminus \{i^+_c, i^-_c\}$. Then

$$
\chi_2(K) - \chi_2(K') = \frac{1}{2} \epsilon(i^+_c, i^-_c) \chi_1(\ ]i^+_c, i^-_c[ , ]i^+_c, i^-_c[ ; \epsilon) \\
+ \frac{1}{4} \sum_{j_1, j_3 \in I \setminus \{i^+_c, i^-_c\}} \epsilon(j_1, j_3) \chi_1(\ ]j_1, j_3[ , ]j_3, j_1[ ; \epsilon) \\
- \frac{1}{2} \epsilon'(i^+_c, i^-_c) \chi_1(\ ]i^+_c, i^-_c[ , ]i^+_c, i^-_c[ ; \epsilon') \\
- \frac{1}{4} \sum_{j_1, j_3 \in I \setminus \{i^+_c, i^-_c\}} \epsilon'(j_1, j_3) \chi_1(\ ]j_1, j_3[ , ]j_3, j_1[ ; \epsilon'),
$$

(4.9)

where the additional argument of $\chi_1$ indicates which crossing function is used. The first and the third summands give

$$
\epsilon(i^+_c, i^-_c) \chi_1(\ ]i^+_c, i^-_c[ , ]i^+_c, i^-_c[ ; \epsilon).
$$

(4.10)

In the second and fourth terms the $\epsilon$-part can be factorized, since $\epsilon(j_1, j_3) = \epsilon'(j_1, j_3)$ in this subset of $I$. The remaining factor is the difference of the $\chi_1$-parts:

$$
\chi_1(\ ]j_1, j_3[ , ]j_3, j_1[ ; \epsilon) - \chi_1(\ ]j_1, j_3[ , ]j_3, j_1[ ; \epsilon') = \begin{cases} 
2\epsilon(i^+_c, i^-_c) & \text{if } i^+_c \in ]j_1, j_3[ \text{ and } i^-_c \in ]j_3, j_1[ \\
0 & \text{otherwise}.
\end{cases}
$$

(4.11)

This follows immediately from the definition of $\chi_1$ and the relation (4.8). The condition

$$
i^+_c \in ]j_1, j_3[ \text{ and } i^-_c \in ]j_3, j_1[.
$$

(4.12)

is equivalent to the condition

$$
\hat{j}_1 \in ]i^-_c, i^+_c[ \text{ and } \hat{j}_3 \in ]i^+_c, i^-_c[.
$$

(4.13)
Therefore the second and the fourth terms give
\[
\left[ \sum_{j_1 \in \{j_1^+, j_1^-\}} + \sum_{j_4 \in \{j_4^+, j_4^-\}} \right] \frac{1}{4} \epsilon(j_1, j_4) 2\epsilon(i_c^+, i_c^-) \\
= \chi_1([]i_c^+, i_c^- [ ; ]i_c^+, i_c^- [ ; ]i_c^-, i_c^+ [ ; \epsilon]).
\] (4.14)

Combining (4.11) and (4.14) we get
\[
\chi_2(K) - \chi_2(K') = 2\epsilon(i_c^+, i_c^-) \chi_1([]i_c^+, i_c^- [ ; ]i_c^-, i_c^+ [ ; \epsilon]).
\] (4.15)

### 4.3 Behaviour of $\chi_2$ under the Reidemeister moves

In the following the behaviour of $\chi_2$ under the moves RM-I, RM-II-A, RM-II-B, and RM-III will be examined. It is not necessary to calculate the behaviour for other versions of RM-III, e.g. with reversed orientations for some lines, because these moves can be composed from the moves mentioned above.

**Behaviour of $\chi_2$ under RM-I.** Consider two knot diagrams $K = (\pi, \epsilon, I)$ and $K' = (\pi', \epsilon', I')$ which are equal up to one Reidemeister-I move, so that $K$ contains a situation $L_i^{+/-}$ at the indices $k, k + 1 \in I$ and $K'$ contains instead a situation $L_0^0$, as shown in figure 9. This means that $I' = I \setminus \{k, k + 1\}$. The basepoint is assumed to be outside the Reidemeister situation, i.e. somewhere in $\pi([s_{k+1}, s_k])$, so that $k + 1 \mod 2n > k$. We now calculate $\chi_2(K) - \chi_2(K')$. There are two sums over four summation variables, each of which satisfies $j_1 > j_2 > j_3 > j_4$. Since the sole difference between $K$ and $K'$ is the crossing at $k$ and $k + 1$, all cases in which none of the summation variables is $k$ or $k + 1$ cancel out. Consider the summands with $j_1 = k$. Due to the factor $\epsilon(j_1, j_3)$ the summands can only be non-vanishing if $j_3 = k + 1$ which is not possible because the summation variables fulfil the condition $j_1 > j_3$. Consider now the summands with $j_1 = k + 1$. Due to the factor $\epsilon(j_1, j_3)$ there can be a contribution only if $j_3 = k$. But in this case the condition $j_1 > j_2 > j_3$ cannot be fulfilled and hence there are no non-vanishing summands. The same argument is valid for all other cases. Therefore, the behaviour of $\chi_2$ under the first Reidemeister move is
\[
\chi_2(K) - \chi_2(K') = 0.
\] (4.16)

**Behaviour of $\chi_2$ under RM-II-A and RM-II-B.** Consider a knot $K = (\pi, \epsilon, I)$ which contains a situation $L_{II-A}$ with the indices $k_1, k_1 + 1, k_2$, and $k_2 + 1$, where $k_2 + 1 > k_1$. Consider another knot $K' = (\pi', \epsilon', I')$ which is equal to $K$ except for containing $L_{II-A}^0$ instead of $L_{II-A}$, as shown in figure 9. We wish to calculate $\chi_2(K) - \chi_2(K')$. Among all summands it is sufficient to consider those with at least one summation variable in $\{k_1, k_1 + 1, k_2, k_2 + 1\}$. However, if in the product $\epsilon(j_1, j_3)\epsilon(j_2, j_4)$ one of the arguments, e.g. $j_1$ belongs to $\{k_1, k_1 + 1, k_2, k_2 + 1\}$, the product is non-zero if and only if $j_3$ assumes
the corresponding value so that \( j_1 \) and \( j_3 \) form a crossing. This reduces the number of summands which have to be considered.

\[
\chi_2(\mathcal{K}) - \chi_2(\mathcal{K}') = \sum_{j_2,j_4 \in \mathcal{I}} \varepsilon(k_2,k_1)\varepsilon(j_2,j_4)
+ \sum_{k_2,j_4 \in \mathcal{I}} \varepsilon(k_2+1,k_1+1)\varepsilon(j_2,j_4)
+ \sum_{j_3,k_1 \in \mathcal{I}} \varepsilon(j_1,j_3)\varepsilon(k_2,k_1)
+ \sum_{j_3,k_1 \in \mathcal{I}} \varepsilon(j_1,j_3)\varepsilon(k_2+1,k_1+1)
+ \varepsilon(k_2+1,k_1+1)\varepsilon(k_2,k_1)
\]

(4.17)

Using the relation (3.3), the first four terms cancel pairwise, and we are left with

\[
\chi_2(\mathcal{K}) - \chi_2(\mathcal{K}') = \varepsilon(k_2+1,k_1+1)\varepsilon(k_2,k_1) = -1.
\]

(4.18)

For the Reidemeister-II-B move the calculation is similar, except that the only non-vanishing term is missing. Hence in this case

\[
\chi_2(\mathcal{K}) - \chi_2(\mathcal{K}') = 0.
\]

(4.19)

**Behaviour of \( \chi_2 \) under RM-III.** The calculation of the behaviour of \( \chi_2 \) under the third Reidemeister move is longer, but in principle no more complicated than for the preceding cases. As in the previous cases we consider two knots: \( \mathcal{K} = (\pi,\epsilon,\mathcal{I}) \) with a situation \( L_{\text{III}}^+ \) consisting of the indices \( \{k_1,k_1+1,k_2,k_2+1,k_3,k_3+1\} \) and \( \mathcal{K}' = (\pi',\epsilon',\mathcal{I}') \) with a situation \( L_{\text{III}}^- \) at the same location, as shown in figure ???. The case \( k_3 > k_2 > k_1 \) is considered. For simplicity we shall now assume\(^2\) that the basepoint lies between the index \( k_3+1 \) and the next index \( k_3 + 2 \mod 2n \). In this situation there are no summands with variables \( j_i > k_3 + 1 \) because \( k_3 + 1 \) is the greatest index in \( \mathcal{I} \). Only summands with two or four summation variables in \( \{k_1,k_1+1,k_2,k_2+1,k_3,k_3+1\} \) can contribute.

\[
\chi_2(\mathcal{K}) - \chi_2(\mathcal{K}') = \sum_{j_2,j_4 \in \mathcal{I}} \varepsilon(k_2,k_1+1)\varepsilon(j_2,j_4)
+ \sum_{j_2,j_4 \in \mathcal{I}} \varepsilon(k_3,k_2+1)\varepsilon(j_2,j_4)
+ \sum_{j_2,j_4 \in \mathcal{I}} \varepsilon(k_3+1,k_1)\varepsilon(j_2,j_4)
\]

\( ^2 \)This could have been done for the Reidemeister-II move as well. We renounced this in order to illustrate which terms can appear in sums of this type.
We shall now define the limit of flat knots. As projection space \( \mathbb{R}^5 \), the limit of flat knots \( K = \text{image} \) is used in the previous definition means that only one crossing is contained in every \( U \). For all \( \pi \in C \), the set of crossings is \( C \subset \mathbb{R}^2 \). We will make some assumptions for simplicity, but without loss of generality. The first of them is formulated as

\[
\chi_2(K) - \chi_2(K') = - \epsilon'(k_2 + 1, k_1)\epsilon'(k_3 + 1, k_2) - \epsilon'(k_3 + 1, k_2)\epsilon'(k_3, k_1 + 1) - \epsilon'(k_3, k_1 + 1)\epsilon'(k_2 + 1, k_1) = +1,
\]

using the relations (3.7) between \( \epsilon \) and \( \epsilon' \) for the Reidemeister-III move it is easy to see that most of these terms cancel, and we are left with

\[
\chi_2(K) = \chi_2(K').
\]

due to the cyclic relation (3.6) for \( \epsilon' \).

5 The limit of flat knots

We shall now define the limit of flat knots. As projection space \( \mathbb{R}^2 \times \{0\} = \{(x, y, 0) \in \mathbb{R}^3\} \) will now be used. Consider some knot diagram \( K = (\pi, \epsilon, \mathcal{I}) \); the shadow diagram is \( K = \text{image}(\pi) \), the set of crossings is \( C \subset \mathbb{R}^2 \). We will make some assumptions for simplicity, but without loss of generality. The first of them is formulated as

\[
|\hat{x}(s)| = \text{const} \quad \text{for all} \quad s \in I_{\infty}.
\]

For all \( c \in C \) let \( U_c \subset \mathbb{R}^2 \) be a sufficiently small, open disk with center \( c \). Furthermore, we define

\[
U = \mathbb{R}^2 \setminus \bigcup_{c \in C} U_c.
\]

Sufficiently small in the previous definition means that only one crossing is contained in every \( U_c \) and the basepoint \( \pi(0) = \pi(1) \) lies outside every \( U_c \). The projection space is a deformation retract of \( \mathbb{R}^3 \) with respect to the homotopy

\[
H_t : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \mapsto (x, y, tz), \quad t \in [0, 1].
\]
Hence \( H_0 \) projects the whole space onto the projection space and \( H_1 \) is the identity in \( \mathbb{R}^3 \). Now we consider a knot \( K_0 \subset \mathbb{R}^3 \) parametrized by \( \pi_0 : I_{/\sim} \rightarrow \mathbb{R}^3 \) with the following properties:

(i) \( H_0 \circ \pi_0 = \pi \), therefore \( H_0(K_0) = K \).

(ii) For every \( s \in I_{/\sim} \) with \( \pi_0(s) \in U_c \times \mathbb{R} \) the third component of \( \dot{\pi}_0(s) \) vanishes and \( \ddot{\pi}_0(s) = 0 \), this means that the knot line is straight in this region.

Every cylinder \( U_c \times \mathbb{R} \) is crossed by two straight lines. They will be called \( g^+ \) and \( g^- \) so that

\[
\begin{align*}
  s_+ &> s_- \text{ for all } s_+ \in \pi_0^{-1}(g^+) \text{ and } s_- \in \pi_0^{-1}(g^-). 
\end{align*}
\]

(5.4)

We introduce another condition for convenience, namely \( H_0(g^+) \perp H_0(g^-) \). The situation is shown in figure ??.

Consider a diagram \( \mathcal{K} = (\pi, \epsilon, \mathcal{I}) \) and an analytically formulated ambient isotopy invariant \( f(K_0) \) which is based on the evaluation of an expression defined using the parametrization \( \pi_0 \) with \( K_0 = \text{image}(\pi_0) \) and \( \pi = H_0 \circ \pi_0 \). Then

\[
\begin{align*}
  f(K_0) &= \lim_{t \to 0} f(H_t(K_0)) =: f(\mathcal{K}) 
\end{align*}
\]

(5.5)

since \( H_t \) is an ambient isotopy for every \( t \in [0, 1] \). This limit will be used for calculating the line integral \( \rho_2 \).

6 Calculation of \( \rho_2(K_0) \) for the limit of flat knots

In this section we calculate \( \rho_2(K_0) \) for the limit of flat knots. For any knot diagram \( \mathcal{K} \) a knot \( K_0 \) constructed as in section ?? may be given. The integral \( \rho_2 \) for this knot is given by

\[
\rho_2(K_0) = \frac{1}{8\pi^2} \int_{K_0} dx_1^{\mu_1} \int_{\text{BP}} dx_2^{\mu_2} \int_{\text{BP}} dx_3^{\mu_3} \int_{\text{BP}} dx_4^{\mu_4} \epsilon_{\mu_1\mu_2\sigma_2} \epsilon_{\mu_3\mu_1\sigma_1} (x_4 - x_2)^{\sigma_2} (x_3 - x_1)^{\sigma_1} \frac{1}{|x_4 - x_2|^3} \frac{1}{|x_3 - x_1|^3},
\]

(6.1)

where BP denotes the basepoint. Using a parametrization

\[
x : I_{/\sim} \rightarrow \mathbb{R}^3
\]

(6.2)

it is written as

\[
\rho_2(K_0) = \frac{1}{8\pi^2} \int_0^1 ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 \int_0^{s_3} ds_4 \, \hat{x}(s_1)^{\mu_1} \hat{x}(s_2)^{\mu_2} \hat{x}(s_3)^{\mu_3} \hat{x}(s_4)^{\mu_4} \times \epsilon_{\mu_1\mu_2\sigma_2} \epsilon_{\mu_3\mu_1\sigma_1} \frac{(x(s_4) - x(s_2))^{\sigma_2} (x(s_3) - x(s_1))^{\sigma_1}}{|x(s_4) - x(s_2)|^3 |x(s_3) - x(s_1)|^3}.
\]

(6.3)
The total integration range is therefore

\[ \Delta_4 := \{(s_1, s_2, s_3, s_4) | 1 > s_1 > s_2 > s_3 > s_4 \geq 0 \}, \quad (6.4) \]

and it will be divided into several parts, classified by the location of the \(x(s_i)\) with respect to the crossings. For every crossing \(c\) we define intervals \(I_c^+\) and \(I_c^-\) in \(I_{I/\sim}\) with the following properties (cf. figure ??):

\[ x(I_c^+) = g_c^+ \text{ and } x(I_c^-) = g_c^- . \quad (6.5) \]

Due to the construction of the \(U_c\), \(\pi(0)\) is not contained in any of the \(I_c^{+/-}\) and all intervals \(I_c^{+/-}\) are disjoint. The following cases are considered:

1. \(x(s_1), x(s_3) \in U_c \times \mathbb{R}, \text{ with } x(s_1) \in g_c^+ \text{ and } x(s_3) \in g_c^-;\)
   \(x(s_2), x(s_4) \in U_d \times \mathbb{R}, \text{ with } x(s_2) \in g_d^+ \text{ and } x(s_4) \in g_d^-;\)
   i.e. two pairs of integration variables meet at different crossings \(c\) and \(d\) (see figure ??). This part will be denoted by \(\Delta_4^{(1)}(c, d)\) and is defined as

\[ \Delta_4^{(1)}(c, d) = \{(s_1, s_2, s_3, s_4) \in I_c^+ \times I_d^+ \times I_c^- \times I_d^- | s_1 > s_2 > s_3 > s_4 \}. \quad (6.6) \]

Note that this set can be empty for certain choices of \(c\) and \(d\).

2. \(x(s_1), x(s_3) \in U_c \times \mathbb{R}, \text{ with } x(s_1) \in g_c^+ \text{ and } x(s_3) \in g_c^-;\)
   \(x(s_2), x(s_4) \in U_c \times \mathbb{R}, \text{ with } x(s_2) \in g_c^+ \text{ and } x(s_4) \in g_c^-;\)
   i.e. all integration variables meet at the same crossing \(c\) in the way shown in figure ??). This part will be denoted by \(\Delta_4^{(2)}(c)\) and is defined as

\[ \Delta_4^{(2)}(c) = \{(s_1, s_2, s_3, s_4) \in I_c^+ \times I_c^+ \times I_c^- \times I_c^- | s_1 > s_2 > s_3 > s_4 \}. \quad (6.7) \]

3. Other cases, which do not contribute to \(\rho_2\). This part will be called \(\Delta_4^{\text{Rest}}\). These cases contain at least one pair of integration variables \((x(s_1), x(s_3))\) or \((x(s_2), x(s_4))\) which do not encounter at any crossing in the way described in the previous cases.

The total integration range can then be written as

\[ \Delta_4 = \bigcup_{c, d \in \mathcal{C}} \Delta_4^{(1)}(c, d) \cup \bigcup_{c \in \mathcal{C}} \Delta_4^{(2)}(c) \cup \Delta_4^{\text{Rest}}, \quad (6.8) \]

and the integral over \(K_0\) is written as

\[ \rho_2(K_0) =: \rho_2(K_0; \Delta_4) = \sum_{c, d \in \mathcal{C}} \rho_2(K_0; \Delta_4^{(1)}(c, d)) + \sum_{c \in \mathcal{C}} \rho_2(K_0; \Delta_4^{(2)}(c)) + \rho_2(K_0; \Delta_4^{\text{Rest}}), \quad (6.9) \]

where the second argument of \(\rho_2\) denotes the respective integration range.
Case 1: Four variables encounter at two crossings. The corresponding integration range containing two crossings $c$ and $d$ is $\Delta^{(1)}_4(c, d)$. The integral $\rho_2(K_0; \Delta^{(1)}_4(c, d))$ can be split into two factors:

$$\rho_2(K_0; \Delta^{(1)}_4(c, d)) = \frac{1}{8\pi^2} \int_{I_c^+} ds_1 \int_{I_c^-} ds_3 \quad \hat{x}(s_1)^\mu_1 \hat{x}(s_3)^\mu_3 \epsilon_{\mu_3 \mu_1 \sigma_1} \frac{(x(s_3) - x(s_1))^\sigma_1}{|x(s_3) - x(s_1)|^3}$$

$$\times \int_{I_d^+} ds_2 \int_{I_d^-} ds_4 \quad \hat{x}(s_2)^\mu_2 \hat{x}(s_4)^\mu_4 \epsilon_{\mu_4 \mu_2 \sigma_2} \frac{(x(s_4) - x(s_2))^\sigma_2}{|x(s_4) - x(s_2)|^3}. \tag{6.10}$$

We now calculate one of these factors for the limit of flat knots. The expression

$$\int_{I_c^+} ds_1 \int_{I_c^-} ds_3 \quad \hat{x}(s_1)^\mu_1 \hat{x}(s_3)^\mu_3 \epsilon_{\mu_3 \mu_1 \sigma_1} \frac{(x(s_3) - x(s_1))^\sigma_1}{|x(s_3) - x(s_1)|^3} \tag{6.11}$$

is reparametrizable. We use parameters $s'_1, s'_3 \in ]-1, +1[$ and a corresponding parametrization $x'_1, x'_3 : ]-1, +1[ \to \mathbb{R}^3$. The expression (6.11) is invariant with respect to scaling, rotating, and translating the coordinate frame. So the coordinates can be chosen in such a way that

$$x'_1(s'_1)^\mu_1 = (s'_1, 0, z_1) \quad \text{and} \quad x'_3(s'_3)^\mu_3 = (0, s'_3, z_3) \tag{6.12}$$

and hence

$$\hat{x}'_1(s'_1)^\mu_1 = (1, 0, 0) \quad \text{and} \quad \hat{x}'_3(s'_3)^\mu_3 = (0, 1, 0). \tag{6.13}$$

After defining $h = z_1 - z_3$ the crossing information of $c$ is simply

$$\epsilon(c) = \text{sgn}(h). \tag{6.14}$$

The evaluation of the vector products gives

$$\int_{-1}^{+1} ds'_1 \int_{-1}^{+1} ds'_3 \quad \frac{h}{\sqrt{s'^2_1 + s'^2_3 + h^2}}. \tag{6.15}$$

The limit of a flat knot is now equivalent to the limit $h \to 0$. In this limit the integral assumes the value

$$2\pi \epsilon(c), \tag{6.16}$$

and therefore

$$\rho_2(K_0; \Delta^{(1)}_4(c, d)) = \frac{1}{2} \epsilon(c)\epsilon(d) \tag{6.17}$$

if for the crossings $c$ and $d$ the range $\Delta^{(1)}_4(c, d)$ is not empty.
Case 2: Four variables encounter at one crossing. The integration range is $\Delta_4^{(2)}(c)$. For this range the integral cannot be factorized as in the previous case.

$$
\rho_2(K_0; \Delta_4^{(2)}(c)) = \frac{1}{8\pi^2} \int_{I_+^c \times I_+^c, s_2 > s_1} ds_1 ds_2 \int_{I_-^c \times I_-^c, s_3 > s_4} ds_3 ds_4 \frac{\dot{x}(s_1)^\mu \dot{x}(s_2)^\mu \dot{x}(s_3)^\mu \dot{x}(s_4)^\mu}{|x(s_1) - x(s_2)|^3 |x(s_3) - x(s_4)|^3} \times \epsilon_{\mu_1 \mu_2 \sigma_1} \epsilon_{\mu_3 \mu_4 \sigma_2} (\sigma_1^1 + \sigma_2^1) \times \epsilon_{\mu_1 \mu_2 \sigma_1} \epsilon_{\mu_3 \mu_4 \sigma_2} (\sigma_1^2 + \sigma_2^2)
$$

(6.18)

Again, as in case 1, the parametrizations $x_1'$ and $x_2'$ are used, with parameters in the range $[-1, +1]$. Under the reparametrization we replace

$$
x(s_1) \to x_1'(s_1') \quad \text{and} \quad x(s_2) \to x_1'(s_2')
$$

$$
x(s_3) \to x_3'(s_3') \quad \text{and} \quad x(s_4) \to x_3'(s_4')
$$

(6.19)

since $s_1$ and $s_2$ relate to the same piece $g_+^c$, and $s_3$ and $s_4$ relate to $g_-^c$. We then get

$$
\frac{1}{8\pi^2} \int_{-1}^{+1} ds_1' \int_{-1}^{+1} ds_2' \int_{-1}^{s_1'} ds_3' \int_{-1}^{s_2'} ds_4' \frac{h}{\sqrt{s_1'^2 + s_2'^2 + h^2}} \frac{h}{\sqrt{s_3'^2 + s_4'^2 + h^2}} = \frac{1}{8}.
$$

(6.20)

Case 3: At least one pair of variables does not encounter at any crossing. We assume that $x(s_1)$ and $x(s_3)$ lie in different parts of the knot, e.g. in $U \times \mathbb{R}$ and some $U_c \times \mathbb{R}$, or in cylinders associated to different crossings. Then the triple product

$$
\dot{x}(s_1)^\mu \dot{x}(s_3)^\mu \epsilon_{\mu_3 \mu_4 \sigma_1} \epsilon_{\mu_3 \mu_4 \sigma_2} (x(s_3) - x(s_1))^{\sigma_1} |x(s_3) - x(s_1)|^3
$$

(6.23)

and therefore the whole integral in this range vanishes in the limit of flat knots. If $x(s_1)$ and $x(s_3)$ lie in the same cylinder or in $U \times \mathbb{R}$, and $x(s_3)$ lies near $x(s_1)$ the expression (6.23) vanishes as well, as can be shown by expanding $x(s_3)$ in a power series near $x(s_1)$.

Using $n$, the self-crossing number $\chi_2$ from section 4.1 and the results (6.17) and (6.22) $\rho_2(K)$ assumes the following form

$$
\rho_2(K) := \lim_{t \to 0} \rho_2(H_t(K_0)) = \frac{1}{2} \chi_2(K) + \frac{1}{8} n.
$$

(6.24)
7 Construction of $\rho_1(K)$ and $\rho^{II}(K)$

The behaviour of $\rho_2(K)$ under the Reidemeister moves follows from the results of section 4.3:

$$
\rho_2(L_1^{+/-}) - \rho_2(L_1^0) = \frac{1}{8}, \quad \rho_2(L_{II-A}) - \rho_2(L_{II-A}^0) = -\frac{1}{4}
$$

$$
\rho_2(L_{II-B}) - \rho_2(L_{II-B}^0) = \frac{1}{4}, \quad \rho_2(L_{III}^+) - \rho_2(L_{III}^-) = \frac{1}{2}.
$$

(7.1)

Furthermore, we know that $\rho_2(U_0) = 0$, where $U_0$ is the unknot diagram without crossing. We want to construct an ambient isotopy invariant

$$
\rho^{II}(K) = \rho_1(K) + \rho_2(K),
$$

(7.2)

i.e. an object based on the evaluation of $K$, invariant under the Reidemeister moves. Therefore, we have to postulate a behaviour of $\rho_1$ opposite to that of $\rho_2$:

$$
\rho_1(L_1^{+/-}) - \rho_1(L_1^0) = -\frac{1}{8}, \quad \rho_1(L_{II-A}) - \rho_1(L_{II-A}^0) = +\frac{1}{4}
$$

$$
\rho_1(L_{II-B}) - \rho_1(L_{II-B}^0) = -\frac{1}{4}, \quad \rho_1(L_{III}^+) - \rho_1(L_{III}^-) = -\frac{1}{2}.
$$

(7.3)

From an analytical calculation in [4] we know that

$$
\rho_1(U_0) = -\frac{1}{12},
$$

(7.4)

if $U_0$ represents an unknot that lies within a plane. In the appendix it will be shown that in the limit of flat knots $\rho_1$ is independent of the values of the crossing function $\epsilon$. This means that for any two diagrams $K$ and $K'$ which differ only in the crossing functions one has

$$
\rho_1(K) = \rho_1(K').
$$

(7.5)

Therefore, instead of calculating $\rho_1$ for $K$ we can use the standard ascending diagram $\alpha(K)$ with respect to the basepoint:

$$
\rho_1(K) = \rho_1(\alpha(K)).
$$

(7.6)

For any diagram $K$ the standard ascending diagram $\alpha(K)$, as defined in [5], is obtained by passing through the knot, starting from the basepoint, and switching each crossing encountered for the first time to an undercrossing. A standard ascending diagram is a diagram representation of the unknot.

It should be possible to show directly that the analytical expression for $\rho_1$ fulfils the above conditions in the limit of flat knots. Nevertheless, the properties (7.3), (7.4), and (7.6) suffice to construct $\rho_1$ in a well-defined way and to show that it is unique. For proving the uniqueness we assume that there is another expression $\tilde{\rho}_1$ with the same properties. Let
\( K \) be a knot diagram. Then \( \alpha(K) \) can be obtained from \( U_0 \) by applying a finite sequence of Reidemeister moves \( R_1, \ldots, R_m \):

\[
\alpha(K) = R_m \cdots R_1 U_0. \tag{7.7}
\]

Since \( \tilde{\rho}_1(U_0) = \rho_1(U_0) \) and the change of both under Reidemeister moves is the same, also the result is the same. Hence

\[
\tilde{\rho}_1(K) = \tilde{\rho}_1(\alpha(K)) = \rho_1(\alpha(K)) = \rho_1(K). \tag{7.8}
\]

We now demonstrate that \( \rho_1 \) is well-defined. Again, let \( K \) be a knot which can be obtained from \( U_0 \) in two different ways, i.e. by applying a sequence \( R_1, \ldots, R_m \) or another sequence \( R_1', \ldots, R_m' \). In principle one has to show that both sequences for \( \rho_1 \) lead to the same result. One can, however, simply define

\[
\rho_1(K) = -\frac{1}{12} - \rho_2(\alpha(K)) \tag{7.9}
\]

and check that the defining conditions are fulfilled. For condition (7.4) this is clear. For condition (7.6) it is clear as well, since

\[
\alpha(\alpha(K)) = \alpha(K). \tag{7.10}
\]

For the properties (7.3) some remarks have to be made. Consider a knot \( K \) which contains a Reidemeister situation \( L^b \) and another one \( K' \) differing from \( K \) by containing a Reidemeister situation \( L_{II-A} \) of figure ??, the move cannot transform \( \alpha(K) \) into \( \alpha(K') \). Therefore we will place the basepoint in such a way that the move can be applied, i.e. outside the Reidemeister situation and prove the independence of \( \rho_1 \) of the choice of the basepoint afterwards. If the move can be applied, it is clear by construction that \( \rho_1 \) behaves correctly.

**Proposition.** \( \rho_2(\alpha(K)) \) is independent of the choice of the basepoint.

**Proof.** It is sufficient to show that the basepoint can be shifted by one crossing without changing \( \rho_2 \). Since the number of crossings \( n \) is invariant under this operation, it is sufficient to show the invariance of \( \chi_2 \).

Consider a diagram \( K = (\pi, \epsilon, I) \). The basepoint \( b \) is a point in the piece between two indices \( i_b - 1 \) and \( i_b \). The basepoint shifted by one crossing lies in the piece between \( i_b \) and \( i_b + 1 \). Let \( \alpha(K) = (\pi, \epsilon_\alpha, I) \) and \( \alpha'(K) = (\pi, \epsilon_{\alpha'}, I) \) denote the standard ascending diagrams of \( K \) with respect to the basepoints \( b \) and \( b' \). From the definition of the standard ascending diagrams it is clear that the difference between \( \alpha(K) \) and \( \alpha'(K) \) lies in the crossing functions at \( i_b \):

\[
\epsilon_\alpha(i_b) = -\epsilon_{\alpha'}(i_b). \tag{7.11}
\]
The index which forms a crossing together with \(i_b\) will be denoted by \(\bar{i}_b\in I\). We have to show that
\[
\chi_2(\alpha(\mathcal{K})) - \chi_2(\alpha'(\mathcal{K})) = 0.
\] (7.12)

In section 4.2 it has been shown that for two diagrams which differ only in one crossing (in this case \(\alpha(\mathcal{K})\) and \(\alpha'(\mathcal{K})\) differ in \(\pi(s_{ib})\)), the difference for \(\chi_2\) is
\[
\chi_2(\alpha(\mathcal{K})) - \chi_2(\alpha'(\mathcal{K})) = 2\epsilon_\alpha(i_b, \bar{i}_b)\chi_1(\bar{i}_b[i_b, i_b]; \epsilon_\alpha).
\] (7.13)

The second factor is related to the linking number between the two link components corresponding to the pieces \(L^1 = \pi([s_{ib}, s_{ib}])\) and \(L^0 = \pi([s_{ib}, s_{ib}])\) according to equation (4.5). Since \(\alpha(\mathcal{K})\) is an ascending diagram, the component \(L^1\) lies completely underneath \(L^0\), which means that they can be separated in space. Hence their linking number is zero. This completes the proof.

Finally, the invariant from the second order term of the Chern-Simons theory assumes the form
\[
\rho^{II}(\mathcal{K}) = \rho_1(\mathcal{K}) + \rho_2(\mathcal{K}) = -\frac{1}{12} + \frac{1}{2}\{\chi_2(\mathcal{K}) - \chi_2(\alpha(\mathcal{K}))\}.
\] (7.14)

This formula is the main result of the present paper.

8 Relation between \(\rho^{II}\) and the total twist

We define
\[
\tau(\mathcal{K}) = \frac{1}{2}\left(\rho^{II}(\mathcal{K}) + \frac{1}{12}\right) = \frac{1}{4}\{\chi_2(\mathcal{K}) - \chi_2(\alpha(\mathcal{K}))\}.
\] (8.1)

The aim of this section is to show that this is equivalent to the total twist which was defined by Lickorish and Millet in [9]. Consider some knot diagram \(\mathcal{K} = (\pi, \epsilon, \mathcal{I})\) and its standard ascending diagram \(\alpha(\mathcal{K}) = (\pi, \epsilon_\alpha, \mathcal{I})\), both of course with the same shadow diagram and the same set of crossings \(\mathcal{C}\). The difference between the two diagrams may consist in \(m\) crossings \(c_1, \ldots, c_m \in \mathcal{C}\) such that \(\epsilon(c_i) = -\epsilon_\alpha(c_i)\) for \(i = 1 \ldots m\). We shall here use a notation similar to that used in [9]. Let \(\sigma_i, i = 1 \ldots m\) be the operation applied to a diagram which switches the crossing \(c_i\). Now define \(\mathcal{K}_0 = \alpha(\mathcal{K})\) and \(\mathcal{K}_j = (\pi, \epsilon_j, \mathcal{I})\) with
\[
\mathcal{K}_j = \sigma_j \sigma_{j-1} \ldots \sigma_1 \mathcal{K}_0
\] (8.2)
so that \(\mathcal{K} = \mathcal{K}_m\). Then \(\tau\) can be written as
\[
\tau(\mathcal{K}) = \frac{1}{4}\{\chi_2(\mathcal{K}_m) - \chi_2(\mathcal{K}_0)\}.
\] (8.3)

For every \(\mathcal{K}_j\) and every \(c \in \mathcal{C}\) formed by the indices \(i^+_c\) and \(i^-_c\) we define the non-overlapping pieces \(S^+_j(c) = \pi([s_{i^+_c}, s_{i^+_c}])\) and \(S^-_j(c) = \pi([s_{i^-_c}, s_{i^-_c}])\) and the link components \(L^+_j(c)\)
and $L_j^-(c)$, which arise from nullifying the crossing $c$. The two slightly different notations are used to draw the connection between our formulation and the one in [5]. In section 4.2 we have already calculated the change of $\chi_2$ under the change of one crossing, or in this case the change under the operation $\sigma_j$. The result can be written as

$$\chi_2(K_j) - \chi_2(K_{j-1}) = 2\epsilon_j(c_j)\chi_1(S_j^+(c_j), S_j^-(c_j)). \quad (8.4)$$

Using relation (8.4) we write $\tau(K)$ as

$$\tau(K) = \frac{1}{4} \left\{ \chi_2(K_m) - \chi_2(K_{m-1}) + \chi_2(K_{m-1}) - \chi_2(K_{m-2}) + \ldots + \chi_2(K_1) - \chi_2(K_0) \right\}$$
$$= \frac{1}{2} \sum_{j=1}^{m} \epsilon_j(c_j)\chi_1(S_j^+(c_j), S_j^-(c_j)). \quad (8.5)$$

The link components $L_j^+(c_j)$ and $L_j^-(c_j)$, which correspond to the pieces $S_j^+(c_j)$ and $S_j^-(c_j)$ in the previous equation, coincide precisely with the definitions of $L^1_j$ and $L^0_j$ used in [5]. Therefore, using $\epsilon_j(c_j) = \epsilon(c_j)$, the definitions of $L^1_j$ and $L^0_j$, and the relation (4.5) between $\lambda$ and $\chi_1$ one obtains

$$\tau(K) = \sum_{j=1}^{m} \epsilon(c_j)\lambda(L_j^1, L_j^0), \quad (8.6)$$

which is the same expression as the one given in [5].

9 Example: Calculation of $\rho^{II}$

Finally, we shall demonstrate the calculation of $\rho^{II}$ for a specific knot. We choose the knot $5_2$ in the notation of Rolfsen in [9], shown below in figure ???. The calculation consists of listing all possible non-vanishing contributions in the fourfold sums $\chi_2$ with crossing function $\epsilon$ and its standard ascending version with crossing function $\epsilon_\alpha$. In table 1 below all significant combinations of indices for two variables $j_1$, $j_3$ or $j_2$, $j_4$ fulfilling $j_1 > j_3$ and $j_2 > j_4$ are listed, together with the corresponding values of $\epsilon$ and $\epsilon_\alpha$. From this table all non-vanishing contributions for $j_1 > j_2 > j_3 > j_4$ are constructed (see table 2). The results are

$$\chi_2(5_2) = +7 \quad \text{and} \quad \chi_2(\alpha(5_2)) = -1, \quad (9.1)$$

and therefore

$$\rho^{II}(5_2) = \frac{47}{12} \quad \text{and} \quad \tau(5_2) = 2. \quad (9.2)$$
Table 1: Encounters of two variables in the knot $5_2$.

| $j_1$ | $j_2$ | $j_3$ | $j_4$ | $\epsilon(j_1,j_3)$ | $\epsilon(j_2,j_4)$ | $\epsilon_\alpha(j_1,j_3)$ | $\epsilon_\alpha(j_2,j_4)$ |
|-------|-------|-------|-------|----------------------|----------------------|---------------------------|---------------------------|
| 5     | 0     | 5     | 0     | +                    | -                    | +                         | -                         |
| 6     | 1     | 4     | 0     | +                    | +                    | -                         | -                         |
| 7     | 4     | 3     | 0     | +                    | -                    | +                         | -                         |
| 8     | 3     | 6     | 1     | +                    | +                    | -                         | -                         |
| 9     | 2     | 5     | 0     | +                    | +                    | -                         | -                         |

Table 2: All non-vanishing contributions to $\chi_2(K)$ and $\chi_2(\alpha(K))$.

| $j_1$ | $j_2$ | $j_3$ | $j_4$ | $\epsilon(j_1,j_3)\epsilon(j_2,j_4)$ | $\epsilon_\alpha(j_1,j_3)\epsilon_\alpha(j_2,j_4)$ |
|-------|-------|-------|-------|--------------------------------------|-----------------------------------------------|
| 6     | 5     | 1     | 0     | +                                    | -                                             |
| 7     | 5     | 4     | 0     | +                                    | +                                             |
| 7     | 6     | 4     | 1     | +                                    | -                                             |
| 8     | 5     | 3     | 0     | +                                    | -                                             |
| 8     | 6     | 3     | 1     | +                                    | +                                             |
| 9     | 5     | 2     | 0     | +                                    | +                                             |
| 9     | 6     | 2     | 1     | +                                    | -                                             |

Table 2: All non-vanishing contributions to $\chi_2(K)$ and $\chi_2(\alpha(K))$.

10 Outlook

As already emphasized, the method of flattening the knot in order to calculate the complicated line integrals from the perturbative expansion of the Wilson loops can be applied to higher orders as well. We have already checked this by the use of a computer program in C++ to be run on a PC which calculates $\rho^{\Pi}(K)$ and the third order invariant $\rho^{\III}(K)$ for arbitrary knots. Whoever is interested in this program may order it via e-mail. We shall soon publish a discussion of the third order calculation which employs the formalism developed in this paper.

One may hope that a systematic examination of the higher orders will yield a convenient description of the Vassiliev invariants, which are conjectured to classify the knots uniquely [6]. The computation time for the invariants $\rho^{\Pi}$ and $\rho^{\III}$ using the procedure presented here grows as $O(n^2)$ and $O(n^3)$ respectively. This is to be compared to computations involving polynomial invariants, whose complexity grows exponentially with crossing number [6].

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Appendix

Independence of $\rho_1$ with respect to the values of $\epsilon$

It will now be demonstrated that $\rho_1(K)$ is not dependent on the values of the crossing information $\epsilon$, i.e. that $\rho_1$ can be interpreted as a functional of the shadow diagram. The analytical form of $\rho_1$ for a knot $K_0$ is given by [4]

$$
\rho_1(K_0) = -\frac{1}{32\pi^3} \int_{K_0} dx_1^\mu_1 \int_{BP} dx_2^\mu_2 \int_{BP} dx_3^\mu_3 \epsilon_1 \epsilon_2 \epsilon_3 \epsilon_{\mu_1 \nu_1 \sigma_1} \epsilon_{\mu_2 \nu_2 \sigma_2} \epsilon_{\mu_3 \nu_3 \sigma_3} I^{\sigma_1 \sigma_2 \sigma_3}(x_1, x_2, x_3)
$$

(A.1)

with

$$
I^{\sigma_1 \sigma_2 \sigma_3}(x_1, x_2, x_3) = \int d^3 z \, \frac{(z - x_1)^{\sigma_1}}{|z - x_1|^{3} |z - x_2|^{3} |z - x_3|^{3}} \quad (A.2)
$$

It was also demonstrated in [4] that this integral can be solved and assumes the form

$$
I^{\sigma_1 \sigma_2 \sigma_3}(x_1, x_2, x_3) = -\frac{\partial}{\partial x_2^{\sigma_2}} \frac{\partial}{\partial x_3^{\sigma_3}} I^{\sigma_1}(x_2 - x_1, x_1 - x_3)
$$

(A.3)

with

$$
I^{\sigma_1}(c, b) = 2\pi \frac{|c + |b| - |c + b|}{|c||b - c \cdot b|} \left\{ \frac{c^{\sigma_1}}{|c|} - \frac{b^{\sigma_1}}{|b|} \right\}
$$

(A.4)

where a slightly different notation has been used in comparison to [4] in order to formulate the expression in a symmetric way. If we define

$$
a = x_3 - x_2, \quad b = x_1 - x_3, \quad c = x_2 - x_1
$$

(A.5)

so that $a = -b - c$, the expression $I^{\sigma_1}$ can be written as

$$
I^{\sigma_1}(b, c) = 4\pi \frac{1}{|a| + |b| + |c|} \left\{ \frac{c^{\sigma_1}}{|c|} - \frac{b^{\sigma_1}}{|b|} \right\}
$$

$$
= \left\{ \frac{\partial}{\partial c^{\sigma_1}} - \frac{\partial}{\partial b^{\sigma_1}} \right\} 4\pi \ln(|a| + |b| + |c|).
$$

(A.6)

$a, b, c$ will be considered as functions of $x_1, x_2, x_3$ in the following. The partial derivatives in (A.3) can be reformulated as partial derivatives with respect to $a, b, c$. Then $I^{\sigma_1 \sigma_2 \sigma_3}$ is formulated in a symmetric way:

$$
I^{\sigma_1 \sigma_2 \sigma_3}(x_1, x_2, x_3) = \left\{ \frac{\partial}{\partial b^{\sigma_1}} - \frac{\partial}{\partial c^{\sigma_1}} \right\} \left\{ \frac{\partial}{\partial c^{\sigma_2}} - \frac{\partial}{\partial a^{\sigma_2}} \right\} \left\{ \frac{\partial}{\partial a^{\sigma_3}} - \frac{\partial}{\partial b^{\sigma_3}} \right\} 4\pi \ln(|a| + |b| + |c|).
$$

(A.7)

This can be calculated straightforwardly. We use the abbreviations

$$
N = |a| + |b| + |c|, \quad m^{\sigma}(u, v) = \frac{u^{\sigma}}{|u|} - \frac{v^{\sigma}}{|v|}, \quad m^{\sigma \tau}(u) = \frac{1}{|u|} \delta^{\sigma \tau} - \frac{u^{\sigma} u^{\tau}}{|u|^3}
$$

(A.8)
and obtain
\[
I^{\sigma_1\sigma_2\sigma_3}(x_1, x_2, x_3) = 4\pi \left( \frac{2}{N^3} m^{\sigma_1}(b, c)m^{\sigma_2}(c, a)m^{\sigma_3}(a, b) \\
+ \frac{1}{N^2} (m^{\sigma_1\sigma_3}(b)m^{\sigma_2}(c, a) + m^{\sigma_3\sigma_2}(a)m^{\sigma_1}(b, c) + m^{\sigma_2\sigma_1}(c)m^{\sigma_3}(a, b)) \right).
\]

(A.9)

Using a parametrization \( x(s) \) and the abbreviations \( x_i := x(s_i), \dot{x}_i := \dot{x}(s_i), \hat{u} \) for the unit vector of any vector \( u \), the line integral \( \rho_1 \) assumes the form
\[
\rho_1(K_0) = \frac{1}{8\pi^2} \int_0^1 ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 \frac{2}{N^3} \left[ \dot{x}_1 \times m(b, c) \right] \cdot \left[ \dot{x}_2 \times m(c, a) \right] \times \left[ \dot{x}_3 \times m(a, b) \right]
+ \frac{1}{N^2} \left\{ \dot{x}_1 \times m(b, c) \cdot \frac{1}{|a|} (\dot{x}_2 \times \dot{x}_3 - \hat{a}(\dot{x}_2 \times \dot{x}_3 \cdot \hat{a}))
+ \dot{x}_2 \times m(c, a) \cdot \frac{1}{|b|} (\dot{x}_3 \times \dot{x}_1 - \hat{b}(\dot{x}_3 \times \dot{x}_1 \cdot \hat{b}))
+ \dot{x}_3 \times m(a, b) \cdot \frac{1}{|c|} (\dot{x}_1 \times \dot{x}_2 - \hat{c}(\dot{x}_1 \times \dot{x}_2 \cdot \hat{c})) \right\}.
\]

(A.10)

Some properties of the integrand will now be examined. We shall refer to it as \( h(x_1, x_2, x_3) = h(x(s_1), x(s_2), x(s_3)) \) so that
\[
\rho_1(K_0; \Delta_3) := \rho_1(K_0) = \int_{\Delta_3} ds_1 \, ds_2 \, ds_3 \, h(x(s_1), x(s_2), x(s_3)),
\]

(A.11)

where
\[
\Delta_3 = \{(s_1, s_2, s_3) \mid 1 > s_1 > s_2 > s_3 \geq 0\}.
\]

(A.12)

The property of \( h(x_1, x_2, x_3) \) which is important here is its invariance under parity transformations. As a consequence
\[
(\lim_{t \to 0^+} - \lim_{t \to 0^-}) h(H_t \circ x(s_1), H_t \circ x(s_2), H_t \circ x(s_3)) = 0,
\]

(A.13)

where the function \( H_t \) from section 4 has been used, now however with \( t \in \mathbb{R} \). The integration range can be split into three subsets.

1. \( x(s_1), x(s_2), x(s_3) \in U_c \times \mathbb{R} \) and at least one of the \( x(s_i) \) is in \( g_c^+ \) and another in \( g_c^- \). This part will be denoted as \( \Delta_3^{(1)}(c) \).

2. \( x(s_2), x(s_3) \in U_c \times \mathbb{R}, x(s_1) \not\in U_c \times \mathbb{R}, \) with \( x(s_2) \in g_c^+ \) and \( x(s_3) \in g_c^- \), and further permutations. These parts together will be denoted as \( \Delta_3^{(2)}(c) \).

3. Other cases, where no two variables cross each other. These situations are contained in \( \Delta_3^{\text{Rest}} \).
Consider now case 1. We take a knot diagram $\mathcal{K} = (\pi, \epsilon, I)$ as flat knot limit of a knot $K$ and another diagram $\mathcal{K}' = (\pi, \epsilon', I)$ as limit of $K'$ with the sole difference that $\epsilon(c) = -\epsilon'(c)$ at a crossing $c \in \mathcal{C}$. Consider now the cylinder $U_c \times \mathbb{R}$. Within this cylinder the flat knot limits of the knots $K$ and $K'$ differ only by a parity transformation.

Hence the function $h$ with arguments in $U_c \times \mathbb{R}$ is the same for $K$ and $K'$ and therefore independent of $\epsilon(c)$ and $\epsilon'(c)$ in the range $\Delta_3^{(1)}(c)$.

For case 2 with $x(s_2)$ and $x(s_3)$ encountering each other at $c$ we consider equation (A.13). Since $h$ is a continuous function in all three arguments the expression can be replaced by

$$
(\lim_{t \to 0^+} - \lim_{t \to 0^-}) h(H_0 \circ x(s_1), H_t \circ x(s_2), H_t \circ x(s_3)) = 0.
$$

The two limits correspond to flattening $K$ to $\mathcal{K}$ and $K'$ to $\mathcal{K}'$. Since the last expression vanishes, the function $h$ is independent of $\epsilon(c)$ in the range $\Delta_3^{(2)}(c)$.

We now consider case 3. In the range $\Delta_3^{\text{Rest}}$ the function $h(H_t \circ x(s_1), H_t \circ x(s_2), H_t \circ x(s_3))$ is well defined for $t = 0$ because no two variables coincide at any crossing. The independence of $h$ on the values of the crossing function is clear, because at every variable both limits $t \to 0$ lead to the same result, and because of the continuity of $h$ one can set $t = 0$.

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