Spectral Asymptotics of Eigen-value Problems with Non-linear Dependence on the Spectral Parameter

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Abstract

We study asymptotic distribution of eigen-values $\omega$ of a quadratic operator polynomial of the following form $(\omega^2 - L(\omega))\phi_\omega = 0$, where $L(\omega)$ is a second order differential positive elliptic operator with quadratic dependence on the spectral parameter $\omega$. We derive asymptotics of the spectral density in this problem and show how to compute coefficients of its asymptotic expansion from coefficients of the asymptotic expansion of the trace of the heat kernel of $L(\omega)$. The leading term in the spectral asymptotics is the same as for a Laplacian in a cavity. The results have a number of physical applications. We illustrate them by examples of field equations in external stationary gravitational and gauge backgrounds.

Key words: polynomial operator pencils, spectral asymptotics, quantum effects in external fields

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1 Introduction

This work is motivated by studying quantum effects in stationary gravitational or gauge classical background fields \[1\], \[2\]. If quantum fields $\phi$ are free or one is restricted by one-loop approximation the fields obey second order differential equations. In the corresponding coordinates $(x^0, x^i)$ these equations can be brought to the following general form:

$$\left[ \partial_0^2 + L(i\partial_0) \right] \phi = 0. \quad (1.1)$$

$L(i\partial_0)$ is the differential operator which includes both time derivative $\partial_0 = \partial/\partial x^0$ and space derivatives,

$$L(i\partial_0) = L_2 + (i\partial_0)L_1 + (i\partial_0)^2L_0, \quad (1.2)$$

$L_k$ are $k$th order differential operators with the space derivatives only. This structure of $L(i\partial_0)$ can be inferred by analyzing dimensionalities. In what follows we assume that $L_k$ act over a $d$-dimensional compact space $\mathcal{M}_d$. The concrete form of (1.1) will be specified in next section.

In time-independent backgrounds, field excitations $\phi_\omega(x^0, x^i) = e^{-i\omega x^0}\phi_\omega(x^i)$ with energy $\omega$ are solutions to

$$\left[ \omega^2 - L(\omega) \right] \phi_\omega = 0, \quad (1.3)$$

$$L(\omega) = L_2 + \omega L_1 + \omega^2 L_0. \quad (1.4)$$

Equation (1.3) is an eigen-value problem on the spectrum of $\omega$. If $L_1$ and $L_0$ in (1.2) are absent one has a standard problem of finding eigen-values $\Lambda = \omega^2$ of a second order differential operator $L_2$. In general, however, operators $L_1$ and $L_0$ in (1.3) are non-trivial and all three operators $L_k$ do not commute. Thus, (1.3) depends polynomially on the spectral parameter $\omega$.

In this paper we call (1.3) the non-linear spectral problem (NLSP) (this should not be confused with problems which depend non-linearly on eigen-functions). Equations like (1.3), (1.4) belong to the spectral theory of polynomial operator pencils \[3\], a field of mathematics where important pioneering results were established by Keldysh \[4\] fifty years ago. In addition to already mentioned applications, quadratic and more general operator polynomials appear in other physical problems, for instance, oscillations of a viscous fluid, Schrödinger equation with energy-dependent potential and etc.

The aim of this work is to analyze asymptotic distribution of the spectrum of (1.3) at large $\omega$. This can be done by studying asymptotics of the spectral density. If the spectrum is real the information about its properties can be also derived from the function

$$K(t) = \frac{1}{2} \sum_\omega e^{-t\omega^2}, \quad t > 0, \quad (1.5)$$

where $\omega$ are eigen-values of (1.3). The behavior of $K(t)$ at small $t$ is connected with the distribution of large $\omega$. In next sections we show that at small $t$, if $L(\omega)$ is a positive
elliptic operator,

\[ K(t) \sim \frac{1}{(4\pi t)^{d/2}} \sum_{n=0}^{\infty} \left[ a_n t^n + b_n t^{n+1/2} \right]. \] (1.6)

This expansion has the same form as asymptotics of the trace of the heat kernel of a second order differential operator. Moreover, the coefficient \( a_0 \) is proportional to the volume of \( \mathcal{M}_d \) while other \( a_n \) and \( b_n \) are local functionals of the external background fields and can be computed by using heat-kernel coefficients of the operator \( L(\omega) \) in (1.3). Thus, the behavior of the spectrum of the NLSP at large \( \omega^2 \) is very similar to behavior of the spectrum of a Laplacian in a cavity.

There are several important applications of (1.6) in quantum field theory. For instance, the spectrum of \( \omega \) defines the vacuum energy \( E_0 \) and, in case of a finite temperature \( \beta^{-1} \), the free-energy of system \( F(\beta) \). For Bose statistics

\[ E_0 = \frac{1}{2} \sum \omega, \quad F(\beta) = \beta^{-1} \sum \ln \left( 1 - e^{-\beta \omega} \right). \] (1.7)

Asymptotics (1.6) determine the form of divergence of \( E_0 \) at large \( \omega \). One can also use (1.6) to derive the behaviour of \( F(\beta) \) at large temperatures. For further discussion see [2].

There are two main approaches to the operator polynomials discussed in the literature (see [3]): the method of linearization and the method of factorization. The method of linearization reduces the spectral problem like (1.3) to a standard linear problem but in an extended Hilbert space. This method is analogous to the reduction of an \( n \)th order differential equation to a system of \( n \) first order equations. The factorization method reduces an operator polynomial to a product of pseudo-differential operators each of which depends linearly on the spectral parameter.

The method used in this paper is different. Our idea is to find the spectrum of (1.3) from the spectrum of \( L(\omega) \), by considering at first \( \omega \) as an independent parameter. This way seems to be more simple (at least for discussed class of polynomials) and it enables us to use the results of the spectral theory of elliptic operators. Another new element which is absent in the Keldysh approach is in using the inner product \( <,> \) analogous to the Klein-Gordon product known in relativistic field theory. This step is motivated by physical applications and is important for our analysis. The eigen-functions \( \phi_\omega \) with different \( \omega \) are orthogonal with respect to \( <,> \) and the set of eigen-functions can be orthonormalized.

Asymptotics (1.6) were first reported in our earlier publication [2] for particular problem related to fields in stationary but not static backgrounds. The aim of this work is to make arguments leading to (1.6) more general.

The paper is organized as follows. In next section we give more detailed definition of the class of NLSP which will be considered here. Conditions which restrict this class are dictated by physical requirements to equation (1.1). We then derive a relation between the spectral density of eigen-values of (1.3) and the spectral density of operators \( L(\omega) \).
This relation is used in section 3 in studying spectral asymptotics of the NLSP and for calculation of coefficients \(a_n\) and \(b_n\) in (1.10). One of the consequences of these results is that for certain \(n\), depending on the dimension \(d\), coefficients \(a_n, b_n\) are related to heat kernel coefficients of the operator \(\partial_0^2 + L(i\partial_0)\), see section 4. Section 5 is devoted to examples of NLSP where the spectrum has a simple form which allows one a direct check of the obtained formulas. One example is a charged field in a constant gauge potential, the other is a field in a rotating Einstein universe \(R^1 \times S^3\). Finally, section 6 is devoted to spectral problems related to the Dirac equation. Although our results are applicable to these problems additional consideration is needed because the corresponding operator \(L(\omega)\) is not self-adjoint.

2 Formulation of the problem

We consider free fields \(\phi\) in a stationary space-time of a dimension \(D = d + 1\). Equation (1.1) for \(\phi\) is reduced to problem (1.3), (1.4) where \(L(\omega)\) will be taken in the following form

\[
L(\omega) = -(\nabla_k + iA_k + i\omega a_k)(\nabla^k + iA^k + i\omega a^k) + \omega B + V. \tag{2.1}
\]

It will be assumed that \(L(\omega)\) acts on sections to some vector bundle over \(M_d\) where \(M_d\) is a compact \(d\)-dimensional Riemannian manifold with metric \(h_{ik}\). Index \(k\) in (2.1) is raised and lowered by using the metric tensor \(h_{ik}\), \(\nabla_k\) are the corresponding connections on \(M_d\) and \(A_k, a_k, B\) and \(V\) are some matrix-valued fields.

In what follows we assume that the spectrum of (1.3) is real and consider \(\omega\) as a real parameter. We also make two additional assumptions: i) \(L(\omega)\) is a positive elliptic operator which is self-adjoint on a Hilbert space \(L^2(M_d)\) on \(M_d\) and \(L_2 = L(0)\) has strictly positive spectrum; ii) for solutions of (1.1) there can be defined a time-independent product \(\langle \cdot, \cdot \rangle\) such that all positive frequency eigen-functions \(\phi_\omega(x^0, x^i) = e^{-i\omega x^0} \phi_\omega(x^i)\) \((\omega > 0)\) have positive norm \((\langle \phi_\omega, \phi_\omega \rangle > 0)\) while the norm of negative-frequency functions \((\omega < 0)\) is negative. According to (i) there are no zero modes \((\omega = 0)\) in the spectrum.

Positivity of \(L(\omega)\) implies that \(M_d\) is a Euclidean manifold. The second assumption is dictated by physical considerations. To quantize the theory the field \(\phi\) is divided onto two parts. The part which has a positive norm is connected with particles while the negative norm part corresponds to anti-particles. Our assumption guarantees that energies \(\omega\) of all physical particles are non-negative.

The Hilbert space \(L^2(M_d)\) is equipped with the standard inner product \((\phi, \psi) = \int \phi^* \psi \sqrt{h} d^d x\). The product \(\langle \cdot, \cdot \rangle\) is defined as follows

\[
\langle \phi, \psi \rangle = i(\phi, \dot{\psi}) - i(\dot{\phi}, \psi) - (\phi, L_1 \psi) - i(\phi, L_0 \dot{\psi}) + i(\dot{\phi}, L_0 \psi), \tag{2.2}
\]

where \(\dot{\phi} = \partial_0 \phi\) and \(L_k\) are Hermitian operators defined for (2.1) by (1.4). One can check that for any two solutions \(\phi\) and \(\psi\) of (1.1) which belong to \(L^2(M_d)\) (2.2) does not depend
on \(x^0, \partial_0(\phi, \psi) = 0\). In a covariant theory on a globally hyperbolic space-time the latter property is extended to independence on the choice of a space-like hyper-surface. For scalar fields (2.2) coincides with the Klein-Gordon product.

Consider the eigen-value problem for operator \(L(\omega)\)

\[
L(\omega)\phi^{(\omega)}_\Lambda = \Lambda(\omega)\phi^{(\omega)}_\Lambda, \quad (2.3)
\]

where \(\omega\) is a real parameter. If the spectrum \(\Lambda(\omega)\) is known the spectrum of NLSP (1.3) is determined by roots of equation

\[
\chi(\omega, \Lambda) = 0, \quad (2.4)
\]

\[
\chi(\omega, \Lambda) = \omega^2 - \Lambda(\omega). \quad (2.5)
\]

This also yields the eigen functions of the NLSP, \(\phi_\omega = C_\omega \phi^{(\omega)}_\Lambda\), where \(\omega\) is a root of (2.4) and \(C_\omega\) is a normalization coefficient. (Recall that (2.4) is assumed to have only real roots.) It is not difficult to see that for eigen-functions of (1.3) the following property takes place

\[
\langle \phi_\omega, \psi_\sigma \rangle = \delta_{\omega\sigma} \chi'(\omega)(\phi_\omega, \psi_\omega), \quad (2.6)
\]

where \(\delta_{\omega\sigma} = 0\) if \(\omega \neq \sigma\) and \(\delta_{\omega\sigma} = 1\) if \(\omega = \sigma\). \(\chi'(\omega)\) denotes the derivative \(\partial_\omega \chi(\omega, \Lambda)\) computed at \(\omega\) which are roots of (2.4). It follows from (2.6) that any two eigen-functions of (1.3) with different eigen-values \((\omega \neq \sigma)\) are automatically orthogonal with respect to \(\langle , \rangle\); functions with equal eigen-values are orthogonal if they are orthogonal with respect to the standard inner product \(\langle , \rangle\). Thus, the product \(\langle , \rangle\) can be used to construct ortho-normalized set of eigen-functions \(\phi_\omega(x^0, x^i)\) for problem (1.3).

Because the norm \(\langle \phi_\omega, \phi_\omega \rangle\) is positive the sign of the norm \(\langle \phi_\omega, \phi_\omega \rangle\), according to (2.6), coincides with the sign of \(\chi'(\omega)\). Therefore, condition (ii) is equivalent to the requirement

\[
\chi'(\omega) = \varepsilon(\omega)|\chi'(\omega)|, \quad (ii)
\]

where \(\varepsilon(\omega)\) is the sign function. Let us also note that verification of (ii) is simplified when field equations are invariant with respect to the charge conjugation. In this case the eigen-values \(\Lambda(\omega)\) are symmetric functions of \(\omega\).

Consider now function \(K(t)\) defined by (1.5). By following [2] we call it pseudo-trace, keeping in mind that \(K(t)\) may not be a trace of any operator. In particular cases, when \(L(\omega) = L\), the pseudo-trace coincides with the trace of the heat kernel of \(L\). This is the reason why normalization coefficient \(1/2\) is introduced in (1.5). According to definitions (2.3)–(2.5) \(K(t)\) can be represented in the integral form

\[
K(t) = \frac{1}{2} \int_{-\infty}^{\infty} d\omega \sum_{\Lambda(\omega)} \delta(\chi(\omega, \Lambda)) |\partial_\omega \chi(\omega, \Lambda)| e^{-t\omega^2}. \quad (2.7)
\]

By using the integral representation for the Dirac \(\delta\)-function and the fact that the signs of \(\chi'(\omega)\) and \(\omega\) coincide (2.7) can be transformed further

\[
K(t) = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega \varepsilon(\omega) \sum_{\Lambda(\omega)} \int_{-\infty}^{\infty} dx e^{ix\chi(\omega, \Lambda)} \partial_\omega \chi(\omega, \Lambda) e^{-t\omega^2}
\]
\[
\frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega \varepsilon(\omega) \int_C dz \ e^{-\omega^2(t-iz)} \left(2\omega + \frac{1}{iz}\partial_\omega\right) K_\omega(iz),
\]

where \(K_\omega(t)\) is the trace of the heat kernel of the operator \(L(\omega)\)

\[
K_\omega(t) = \text{Tr} e^{-tL(\omega)} = \sum_{\Lambda(\omega)} e^{-t\Lambda(\omega)}.
\]

The integration over \(x\) in (2.8) is shifted to the complex plane. Contour \(C\) goes from \(-i\epsilon - \infty\) to \(-i\epsilon + \infty\) where \(\epsilon\) is a small positive parameter. Equation (2.8) enables one to relate the pseudo-trace to the trace of an elliptic operator. To proceed we represent \(K_\omega(t)\) in the integral form

\[
K_\omega(t) = \int_\mu^\infty e^{-t\lambda} \varphi(\lambda, \omega) d\lambda,
\]

where \(\varphi(\lambda, \omega)\) is the spectral density, which can be written as the sum of delta-functions \(\delta(\lambda - \Lambda(\omega))\). Parameter \(\mu\) is chosen to be smaller than the lowest eigen-value \(\Lambda_0(\omega)\). In principle, the choice of \(\mu\) depends on \(\omega\). If the spectrum \(\Lambda(\omega)\) is strictly positive it is convenient to put \(\mu = 0\). Let us also introduce the counting function

\[
N(\lambda, \omega) = \int_\mu^\lambda d\sigma \varphi(\sigma, \omega),
\]

which is equal to the total number of eigen-values \(\Lambda(\omega)\) which do not exceed \(\lambda\). We use now (2.10) in (2.8) and take into account (2.11)

\[
K(t) = 1 + \frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega \varepsilon(\omega) \int_C dz \ e^{-\omega^2(t-iz)} \int_\mu^{\infty} d\lambda e^{-iz\lambda} \left(2\omega \varphi(\lambda, \omega) + \frac{1}{iz}\partial_\omega\partial_\lambda N(\lambda, \omega)\right).
\]

The last term in the brackets can be integrated by parts over \(\lambda\). Then the integral over \(z\) results in the delta-function \(\delta(\lambda - \omega^2)\) and we get the final expression

\[
K(t) = \int_0^\infty d\lambda e^{-\lambda t} \varphi(\lambda),
\]

\[
\varphi(\lambda) = \frac{1}{2} \left(\bar{\varphi}(\lambda, \sqrt{\lambda}) + \varphi(\lambda, -\sqrt{\lambda})\right),
\]

\[
\bar{\varphi}(\lambda, \omega) = \varphi(\lambda, \omega) + \frac{1}{2\omega}\partial_\omega N(\lambda, \omega).
\]

Note that (2.13) does not depend on \(\mu\). Representation (2.13) \(\dagger\) is convenient because it has the same form as integral representation (2.10) for an ordinary elliptic operator. Together with (2.14), (2.15) it will be our basic equation.

\dagger\ Because \(L(\omega)\) is a positive elliptic operator in a finite volume its spectrum is bounded from below.

\(\ddagger\) Formula (2.13) can be also written as \(K(t) = \int_0^\infty d\omega e^{-\omega^2} \Phi(\omega)\) where \(\Phi(\omega) = 2\omega \varphi(\omega^2)\). This definition is used in \(\ddagger\).
3 Short $t$ expansions and spectral asymptotics

We begin with short $t$ expansion of the trace $K_\omega(t)$, which is standard,

$$K_\omega(t) \sim \frac{1}{(4\pi t)^{d/2}} \sum_{n=0}^{\infty} \left[ a_n(\omega) t^n + b_n(\omega) t^{n+1/2} \right], \quad (3.1)$$

where $a_n(\omega)$ and $b_n(\omega)$ are Hadamard–Minackshisundaram–DeWitt–Seeley coefficients. It follows from (1.2), (2.1) that

$$a_n(\omega) = \sum_{m=0}^{n} a_{m,n} \omega^m, \quad b_n(\omega) = \sum_{m=0}^{n} b_{m,n} \omega^m. \quad (3.2)$$

The highest power of $\omega$ in $a_n(\omega)$ and $b_n(\omega)$ can be found by using (2.1) and analyzing dimensionalities. It is easy to see that this power is determined by the term $\omega B$ in $L(\omega)$, see (2.1).

We are interested in (3.1) because short $t$ limit is related to distribution of large eigenvalues $\Lambda$ in (2.9). For instance, the first term in series (3.1) corresponds to the leading asymptotics of the spectral function $N(\lambda, \omega)$

$$N(\lambda, \omega) \sim \frac{\lambda^{d/2} a_0}{(4\pi)^{d/2} \Gamma(d/2 + 1)} \sim r \frac{\lambda^{d/2} V_d}{(4\pi)^{d/2} \Gamma(d/2 + 1)}, \quad (3.3)$$

where $V_d$ is the volume of $\mathcal{M}_d$ and $r$ is the dimensionality of the representation of the field $\phi$. Equation (3.3) is known as the Weyl formula. It is impossible, however, to define all other sub-leading terms in $N(\lambda, \omega)$ corresponding to (3.1). The reason is that starting with certain $n$ these terms become smaller then fluctuations of $N(\lambda, \omega)$ when $\lambda$ goes from one eigen-value to the next one [3]. The way out of this difficulty is to work with smoothed functions $N(\lambda, \omega)$ and $\varphi(\lambda, \omega)$, see, e.g., [3], [4].

The smoothing can be done by different ways and one of them is to use the Riesz means [8]. Let us define a “smoothed” spectral function $\varphi_\alpha(\lambda, \omega)$ by formula

$$K_\omega(t) t^{-\alpha} = \int_\mu^\infty e^{-t\lambda} \varphi_\alpha(\lambda, \omega) d\lambda, \quad (3.4)$$

where $\alpha$ is a complex parameter, $\Re \alpha > 0$. In particular, $\varphi_1(\lambda, \omega) = N(\lambda, \omega)$ and in the limit $\alpha \to 0$ $\varphi_0(\lambda, \omega) = \varphi(\lambda, \omega)$. By using the inverse Laplace transform in (3.4) one gets

$$\varphi_\alpha(\lambda, \omega) = \frac{1}{\Gamma(\alpha)} \int_\mu^\lambda (\lambda - \sigma)^{\alpha-1} \varphi(\sigma, \omega) d\sigma. \quad (3.5)$$

In fact, $\varphi_\alpha(\lambda, \omega)$ coincides with the fractional derivative $\partial_\lambda^{-\alpha} \varphi(\lambda, \omega)$ of $\varphi(\lambda, \omega)$ of the order $-\alpha$, see [3]. Suppose that $\alpha \neq k/2$ where $k$ is an integer. Then at large $\lambda$ the spectral function $\varphi_\alpha(\lambda, \omega)$ is represented by the asymptotic series

$$\varphi_\alpha(\lambda, \omega) \sim \frac{1}{(4\pi)^{d/2}} \sum_{n=0}^{\infty} \left[ a_n(\omega) \frac{\lambda^{d/2-n+\alpha-1}}{\Gamma \left( \frac{d}{2} - n + \alpha \right)} + b_n(\omega) \frac{\lambda^{(d-1)/2-n+\alpha-1}}{\Gamma \left( \frac{d-1}{2} - n + \alpha \right)} \right]. \quad (3.6)$$
If \( \mu = 0 \) one can substitute (3.6) in (3.4) and check that this series corresponds to the short \( t \) expansion (3.1). This result is valid also if \( \mu < 0 \) at \( \lambda > |\mu| \), see Appendix.

The difficulty which appears when one wants to use (3.6) to get asymptotics of \( \varphi(\lambda, \omega) \) or \( N(\lambda, \omega) \) is that some terms in (3.6) disappear in the limit \( \alpha = 0 \) or \( \alpha = 1 \). Thus, these expansions cannot reproduce the entire series (3.1). It was recently pointed out by Dowker [10] that for \( \mu = 0 \) the problem can be formally avoided if in this limit the sub-leading terms are treated as generalized functions\(^3\). The recipe of [10] uses the fact that [11] that for \( \mu = 0 \) the problem can be formally avoided if in this limit the sub-leading terms are treated as generalized functions\(^3\).

To get asymptotics of the spectral density \( \varphi(\lambda) \) of the pseudo-trace \( K(t) \) we first define the functions

\[
\tilde{\varphi}_\alpha(\lambda, \omega) = \varphi_\alpha(\omega, \lambda) + \frac{1}{2\omega} \partial_\omega \varphi_{\alpha+1}(\lambda, \omega)
\]

for \( \alpha \) complex. Then \( \tilde{\varphi}(\lambda, \omega) \) introduced in (2.15) can be obtained from \( \tilde{\varphi}_\alpha(\lambda, \omega) \) in the limit \( \alpha \to 0 \). It follows from (3.6) that at large \( \lambda \)

\[
\frac{1}{2} \left( \tilde{\varphi}_\alpha(\lambda, \omega) + \tilde{\varphi}_\alpha(\lambda, -\omega) \right)
\]

\[
\sim \frac{1}{(4\pi)^{d/2}} \sum_{n=0}^{\infty} \left[ \tilde{a}_n(\omega) \lambda^{d/2 - n + \alpha - 1} \Gamma \left( \frac{d}{2} - n + \alpha \right) + \tilde{b}_n(\omega) \lambda^{(d-1)/2 - n + \alpha - 1} \Gamma \left( \frac{d-1}{2} - n + \alpha \right) \right],
\]

\[
\tilde{a}_n(\omega) = \frac{1}{2} \left[ a_n(\omega) + a_n(-\omega) + \frac{1}{2\omega} \partial_\omega (a_{n+1}(\omega) + a_{n+1}(-\omega)) \right],
\]

\[
\tilde{b}_n(\omega) = \frac{1}{2} \left[ b_n(\omega) + b_n(-\omega) + \frac{1}{2\omega} \partial_\omega (b_{n+1}(\omega) + b_{n+1}(-\omega)) \right],
\]

where we have taken into account that \( \partial_\omega a_0(\omega) = \partial_\omega b_0(\omega) = 0 \). According to (3.2) one can write

\[
\tilde{a}_n(\omega) = \sum_{m=0}^{\infty} \tilde{a}_{2m,n}\omega^{2m}, \quad \tilde{b}_n(\omega) = \sum_{m=0}^{\infty} \tilde{b}_{2m,n}\omega^{2m},
\]

\[
\tilde{a}_{2m,n} = a_{2m,n} + (m+1)a_{2(m+1),n+1}, \quad \tilde{b}_{2m,n} = b_{2m,n} + (m+1)b_{2(m+1),n+1},
\]

where \( a_{2m,n}, b_{2m,n} \) are assumed to be equal to zero for \( 2m > n \). Then

\[
\frac{1}{2} \left( \tilde{\varphi}_\alpha(\lambda, \sqrt{\lambda}) + \tilde{\varphi}_\alpha(\lambda, -\sqrt{\lambda}) \right)
\]

\[
\sim \frac{1}{(4\pi)^{d/2}} \sum_{n=0}^{\infty} \left[ \tilde{a}_n^{(\alpha)}(\omega) \lambda^{d/2 - n + \alpha - 1} \Gamma \left( \frac{d}{2} - n + \alpha \right) + \tilde{b}_n^{(\alpha)}(\omega) \lambda^{(d-1)/2 - n + \alpha - 1} \Gamma \left( \frac{d-1}{2} - n + \alpha \right) \right],
\]

\(^{3}\text{Another possible way to avoid this problem is to use a dimensional regularization [1].}\)
Coefficients $a_n^{(a)}$, $b_n^{(a)}$ can be found with the help of (3.12), (3.13). After some algebra one can represent them in the form

$$a_n^{(a)} = \sum_{m=n}^{2n} (-1)^{n-m} \frac{\Gamma \left(-\frac{d}{2} + m - \alpha\right)}{\Gamma \left(-\frac{d}{2} - n - \alpha\right)} a_{2(m-n),m},$$

\hspace{1cm} (3.15)

$$b_n^{(a)} = \sum_{m=n}^{2n} (-1)^{n-m} \frac{\Gamma \left(-\frac{d+1}{2} + m - \alpha\right)}{\Gamma \left(-\frac{d+1}{2} - n - \alpha\right)} b_{2(m-n),m}.$$ \hspace{1cm} (3.16)

Finally, asymptotics of the spectral density (2.14) is obtained from (3.14) in the limit $\alpha \to 0$ by treating its coefficients as generalized functions, see (3.8),

$$\varphi(\lambda) = \frac{1}{2} \lim_{\alpha \to 0} \left( \tilde{\varphi}_\alpha(\lambda, \sqrt{\lambda}) + \tilde{\varphi}_\alpha(\lambda, -\sqrt{\lambda}) \right)$$

$$\sim \frac{1}{(4\pi)^{d/2}} \sum_{n=0}^{\infty} \left[ a_n \frac{\lambda^{n-d/2} + b_n \frac{\lambda^{n-d+1/2}}{\Gamma \left(\frac{d+1}{2} - n + \alpha\right)} \right] \delta(\lambda),$$

\hspace{1cm} (3.17)

where $a_n = a_n^{(0)}$, $b_n = b_n^{(0)}$ and the symbol $\partial_\lambda^{\gamma}$ for $\gamma \neq n$ denotes the fractional derivative. Note that $L(\omega)$ may have negative eigen-values at some $\omega$. In this case there is the restriction $\lambda > |\mu| > 0$ in (3.6) where $\mu = \mu(\omega)$ is determined by the lowest eigen-value $\Lambda_0(\omega)$ of $L(\omega)$, $\mu(\omega) < \Lambda_0(\omega)$. In (3.14) this restriction is absent because the lowest eigen-value and $\mu(\omega)$ depend on parameter $\lambda$. When $\lambda$ decreases and goes to zero so does $|\mu|$ (recall that spectrum of $L_2 = L(0)$ is strictly positive). Therefore, using formula (3.7) in (3.14) is legitimate.

Equation (3.17) can be used to get the normal asymptotic series for a smoothed spectral density at large $\lambda$. For the Riesz means with Re $\alpha > 0$, $\alpha \neq 0$,

$$\varphi_\alpha(\lambda) = \frac{1}{\Gamma(\alpha)} \int_0^\lambda (\lambda - \sigma)^{\alpha-1} \varphi(\sigma) d\sigma,$$ 

\hspace{1cm} (3.18)

$$\varphi_\alpha(\lambda) \sim \frac{1}{(4\pi)^{d/2}} \sum_{n=0}^{\infty} \left[ a_n \frac{\lambda^{d/2-n+\alpha-1}}{\Gamma \left(\frac{d}{2} - n + \alpha\right)} + b_n \frac{\lambda^{(d-1)/2-n+\alpha-1}}{\Gamma \left(\frac{d+1}{2} - n + \alpha\right)} \right],$$ 

\hspace{1cm} (3.19)

$$a_n = \sum_{m=n}^{2n} (-1)^{n-m} \frac{\Gamma \left(-\frac{d}{2} + m\right)}{\Gamma \left(-\frac{d}{2} + n\right)} a_{2(m-n),m},$$ 

\hspace{1cm} (3.20)

$$b_n = \sum_{m=n}^{2n} (-1)^{n-m} \frac{\Gamma \left(-\frac{d+1}{2} + m\right)}{\Gamma \left(-\frac{d+1}{2} + n\right)} b_{2(m-n),m}.$$ 

\hspace{1cm} (3.21)

The equivalent and regularization independent way to represent this result is the short $t$ expansion of the pseudo-trace $K(t)$ which is the direct consequence of (2.13) and (3.17)

$$K(t) \sim \frac{1}{(4\pi t)^{d/2}} \sum_{n=0}^{\infty} \left[ a_n t^n + b_n t^{n+1/2} \right].$$ 

\hspace{1cm} (3.22)
Brief comments about (3.19) and (3.22) are in order. 1) It is important fact that $a_n$ and $b_n$ are finite combinations of some pieces, $a_{2m,n}$, $b_{2m,n}$, of the heat kernel coefficients of the associated operator polynomial $L(\omega)$, see (2.1), (3.1), (3.2). Thus, studying the spectral asymptotics of the considered class of non-linear spectral problems is reduced to standard computations. 2) One can use (3.19) to get the leading asymptotics of the counting function $N(\lambda) = \varphi_1(\lambda)$ for the given class of NLSP

$$N(\lambda) \sim r \frac{\lambda^{d/2}}{(4\pi)^{d/2} \Gamma(d/2 + 1)}.$$  

This equation follows from (3.20) and the fact that $a_{0,0} = a_0 = r V_d$. It means that behavior of $N(\lambda)$ is described by the Weyl formula. The difference between NLSP spectrum and spectrum of a Laplacian appears only in the sub-leading corrections to (3.23). 3) The form of asymptotics (3.19), (3.22) strongly depends on the form of operator polynomial. The heat kernel coefficients for $L(\omega)$ defined by (2.1) are polynomials (3.2) of the certain order which guarantees that $a_n$, $b_n$ are finite series given by (3.20), (3.21). These formulas may not hold for other operator polynomials. The leading term in spectral asymptotics for some class of operator polynomials is discussed also in [3].

4 Dimensional reduction

Spectral problem (1.3) is related to wave equation (1.1) in a stationary $d + 1$ dimensional Lorentzian space-time $\mathcal{M}_{d+1}$. One can introduce a differential operator $P$ acting over $\mathcal{M}_{d+1}$

$$P = \partial_0^2 + L(i\partial_0),$$  

where $L(i\partial_0)$ is defined in (1.2). The heat kernel of $P$ and its asymptotics [12]

$$K_{d+1}(s) = \text{Tr} e^{-sP} \sim \frac{1}{i(4\pi s)^{(d+1)/2}} \sum_{n=0}^{\infty} \left[ A_n s^n + B_n s^{n+1/2} \right]$$

at small $|s|$ play an important role in quantum field theory on $\mathcal{M}_{d+1}$. (The parameter $s$ in (4.2) is imaginary because $P$ is a hyperbolic operator [12] and, strictly speaking, $K_{d+1}(s)$ should be solution to a Schrödinger problem.) There is a relation between asymptotics (4.2) and asymptotics of the related NLSP. To see this let us first regularize integrals over time $x^0$ by requiring that $-T/2 < x^0 < T/2$, where $T$ is a large parameter. Then we can write

$$K_{d+1}(s) = \frac{T}{2\pi} \int_{-\infty}^{\infty} d\omega \sum_{\Lambda(\omega)} e^{-s(-\omega^2 + \Lambda(\omega))} = \frac{T}{2\pi} \int_0^{\infty} d\omega \ e^{s\omega^2} \left[ K_\omega(s) + K_{-\omega}(s) \right].$$

At small $|s|$ we can use (3.1), (3.2) to get

$$K_{d+1}(s) \sim \frac{T}{\sqrt{\pi i(4\pi s)^{(d+1)/2}}} \sum_{n=0}^{\infty} \sum_{m=0}^{[n/2]} (-1)^m \Gamma \left( m + \frac{1}{2} \right) \left( a_{2m,n} s^{n-m} + b_{2m,n} s^{n-m+1/2} \right).$$
By comparing (4.4) with (4.2) one finds

\[ A_n = \frac{T}{\sqrt{\pi}} \sum_{m=n}^{2n} (-1)^{n-m} \Gamma \left( m - n + \frac{1}{2} \right) a_{2(m-n),m}, \quad (4.5) \]

\[ B_n = \frac{T}{\sqrt{\pi}} \sum_{m=n}^{2n} (-1)^{n-m} \Gamma \left( m - n + \frac{1}{2} \right) b_{2(m-n),m}. \quad (4.6) \]

These formulas yield heat kernel coefficients in \( d + 1 \) dimensions in terms of some pieces of coefficients in \( d \) dimensions. Relation to coefficients of the pseudo-trace expansion follows if (4.5), (4.6) are compared with (3.20), (3.21), respectively. For theories in space-times with even dimensions, \( D = d + 1 = 2k \),

\[ A_{D/2} = Ta_{D/2} = \int dt \, a_{D/2}, \quad (4.7) \]

for odd dimensions, \( D = d + 1 = 2k + 1 \),

\[ B_{(D-1)/2} = Tb_{(D-1)/2} = \int dt \, b_{(D-1)/2}. \quad (4.8) \]

Relation (4.7) is interesting because coefficient \( A_{D/2} \) in even dimensions determines anomalous scaling of the theory. It can be shown that in conformally invariant models it coincides with the integral of the conformal anomaly (trace of renormalized stress energy tensor). An analog of formulas (4.5)–(4.8) can be also obtained in the Euclidean theory \([13]\) where \( P \) is an elliptic operator and its heat kernel is well-defined.

5 Examples

5.1 Constant gauge potential

We now consider concrete examples which illustrate results (3.20)–(3.22). These examples can be obtained by transforming standard eigen-value problems to non-linear ones and the spectrum of (1.3) can be found explicitly. Consider the problem

\[ \left[ \omega^2 - L_2 \right] \phi_\omega = 0, \quad (5.1) \]

where \( L_2 \) is a second order positive elliptic operator on a compact space. We suppose that the lowest eigen-value \( \Lambda_0 \) of \( L_2 \) is positive, \( \Lambda_0 > 0 \). Now, if \( \omega \) is replaced to \( \omega - \varrho \), where \( \varrho \) is a real parameter, the eigen-value problem becomes non-linear

\[ \left[ \omega^2 - L(\omega) \right] \phi'_\omega = 0, \quad (5.2) \]

\[ L(\omega) = L_2 - \varrho^2 + 2\varrho \omega, \quad (5.3) \]

where \( \phi'_\omega = \phi_{\omega+\varrho} \). From the point of view of the theory in space and time the shift of frequencies corresponds to a gauge-like transformation \( \phi'_\omega(x^0, x) = e^{-i\varrho x^0} \phi_\omega(x^0, x) \). Equations for \( \phi'_\omega(x^0, x) \) look as equations in external constant gauge potential \( A_\nu dx^\nu = \)
where \( \bar{\lambda} \). Another application of these results is a finite-temperature theory where \( \rho \) plays the role of a chemical potential. In what follows, to satisfy condition (ii) of section 2, we assume that \( \rho^2 < \Lambda_0 \).

The pseudo-trace corresponding to (3.2) is

\[
K(t) = \frac{1}{2} \sum_{\omega} e^{-t\omega^2} = \frac{1}{2} \sum_{\lambda} \left( e^{-t(\sqrt{\lambda} - \rho)^2} + e^{-t(\sqrt{\lambda} + \rho)^2} \right),
\]

(5.4)

where \( \lambda \) are eigen-values of \( L_2 \). One can consider (5.4) as a result of a non-linear transformation of the spectrum, \( \lambda \to (\sqrt{\lambda} \pm \rho)^2 \). The spectral density \( \varphi(\lambda) \) is defined by (2.13) and can be written as

\[
\varphi(\lambda) = \frac{1}{2\sqrt{\lambda}} \left[ \sqrt{\lambda} + \rho \right] \varphi \left( (\sqrt{\lambda} + \rho)^2 \right) + \sqrt{\lambda} - \rho \right] \varphi \left( (\sqrt{\lambda} - \rho)^2 \right),
\]

(5.5)

where \( \varphi(\lambda) \) is the spectral density of \( L_2 \). The asymptotic behavior of the smoothed density \( \bar{\varphi}_\alpha(\lambda) = \partial_{\lambda}^\alpha \varphi(\lambda) \), \( \Re \alpha > 0, \alpha \neq k/2 \), is

\[
\bar{\varphi}_\alpha(\lambda) \sim \frac{1}{(4\pi)^{d/2}} \sum_{n=0}^{\infty} \left[ \bar{a}_n \frac{\lambda^{d/2-n+\alpha-1}}{\Gamma \left( \frac{d}{2} - n + \alpha \right)} + \bar{b}_n \frac{\lambda^{(d-1)/2-n+\alpha-1}}{\Gamma \left( \frac{d-1}{2} - n + \alpha \right)} \right],
\]

(5.6)

where \( \bar{a}_n \) are heat kernel coefficients of \( L_2 \)

\[
\text{Tr} \ e^{-tL_2} \sim \frac{1}{(4\pi t)^{d/2}} \sum_{n=0}^{\infty} \left[ \bar{a}_n t^n + \bar{b}_n t^{n+1/2} \right].
\]

(5.7)

By taking into account that

\[
(x - y)^z = \sum_{p=0}^{\infty} \frac{\Gamma(-z+p)}{p!\Gamma(-z)} x^{z-p} y^p,
\]

(5.8)

\(|y| < x\), one finds from (5.6) at large \( \lambda \)

\[
\frac{1}{2\sqrt{\lambda}} \left[ \sqrt{\lambda} + \rho \right] \varphi \left( (\sqrt{\lambda} + \rho)^2 \right) + \sqrt{\lambda} - \rho \right] \varphi \left( (\sqrt{\lambda} - \rho)^2 \right)
\]

\sim \frac{1}{(4\pi)^{d/2}} \sum_{n=0}^{\infty} \left[ a_{n-\alpha} \frac{\lambda^{d/2-n+\alpha-1}}{\Gamma \left( \frac{d}{2} - n + \alpha \right)} + b_{n-\alpha} \frac{\lambda^{(d-1)/2-n+\alpha-1}}{\Gamma \left( \frac{d-1}{2} - n + \alpha \right)} \right],
\]

(5.9)

\[
a_{n-\alpha} = \sum_{k=0}^{n} \rho^{2k} c_{k,n-\alpha}(d) \bar{a}_{n-k}, \quad b_{n-\alpha} = \sum_{k=0}^{n} \rho^{2k} c_{k,n-\alpha}(d-1) \bar{b}_{n-k},
\]

(5.10)

\[
c_{k,n-\alpha}(d) = \frac{2^{2k}}{(2k)!} \frac{\Gamma \left( \frac{d+1}{2} + \alpha - n + k \right)}{\Gamma \left( \frac{d+1}{2} + \alpha - n \right)}.
\]

(5.11)

In the limit \( \alpha \to 0 \) one obtains from (5.9)–(5.11) asymptotics (3.17) for \( \varphi(\lambda) \), where \( a_n = \lim_{\alpha \to 0} a_{n-\alpha} \) and \( b_n = \lim_{\alpha \to 0} b_{n-\alpha} \). By using this one comes to short \( t \) expansion (1.6) for pseudo-trace (5.4).
It is possible to check that obtained formulas (3.20), (3.21) correctly reproduce \( a_n \) and \( b_n \). To this aim one needs to know coefficients \( a_{2m,n} \) in (3.20) and \( b_{2m,n} \) in (3.21). They are defined by heat kernel coefficients of the operator \( L(\omega) \), Eq. (5.3), and can be easily found. By using (5.3) one gets after some algebra

\[
a_{2m,n} = \frac{2^{2m}}{(2m)!} \sum_{p=2m}^{n} \frac{g^{2p-2m}}{(p-2m)!} \bar{a}_{n-p}, \quad b_{2m,n} = \frac{2^{2m}}{(2m)!} \sum_{p=2m}^{n} \frac{g^{2p-2m}}{(p-2m)!} \bar{b}_{n-p}
\]

(5.12)

Combination of (3.20), (3.21) with (5.12) yields

\[
a_n = \sum_{k=0}^{n} g^{2k} c_{k,n}(d) \bar{a}_{n-k}, \quad b_n = \sum_{k=0}^{n} g^{2k} c_{k,n}(d-1) \bar{b}_{n-k},
\]

(5.13)

\[
c_{k,n}(d) = \sum_{l=0}^{k} (-1)^l \frac{2^{2l}}{(2l)!} \frac{\Gamma(l+n-d/2)}{\Gamma(n-d/2)} \frac{1}{(k-l)!}.
\]

(5.14)

The following property of \( \Gamma \)-functions \[14\]

\[
\sum_{n=-\infty}^{\infty} \Gamma(a+n)\Gamma(b+n) \Gamma(c+n)\Gamma(d+n) = \frac{\pi^2}{\sin(\pi a) \sin(\pi b)} \frac{\Gamma(c+d-a-b-1)}{\Gamma(c-a)\Gamma(d-a)\Gamma(c-b)\Gamma(d-b)} \quad \text{(5.15)}
\]

(Re \( a + b - c - d \) < −1 and \( a, b \) are not integers) can be used to show that \( c_{k,n}(d) \) in (5.14) coincide with \( c_{k,n-\alpha}(d) \) in (5.11) at \( \alpha = 0 \). This proves validity of (3.20), (3.21).

### 5.2 Spin 0 fields in rotating Einstein universe

Consider a field theory in the Einstein universe \( \mathbb{R}^1 \times S^3 \), where \( S^3 \) is a hyper-sphere of the radius \( \rho \). Because the Einstein universe is spatially compact it allows globally defined frames of reference which rigidly rotate with coordinate angular velocities \( \Omega \), provided \( \Omega < 1/\rho \). In such frames, spectrum of frequencies \( \omega \) of single-particle field excitations \( e^{-i\omega x^0} \phi_\omega(x) \) is determined by a NLSP. As a simplest example, consider conformally coupled scalar fields

\[
\left(-\nabla^2 + \frac{1}{6} R\right) \phi = 0,
\]

(5.16)

where \( R \) is the scalar curvature of the universe, \( R = 6/\rho^2 \). In what follows we put \( \rho = 1 \) and write the metric in \( \mathbb{R}^1 \times S^3 \) as

\[
ds^2 = -(dx^0)^2 + \sin^2 \theta \ d\varphi^2 + \cos^2 \theta \ d\psi^2 + d\theta^2,
\]

(5.17)

where \( 0 \leq \theta \leq \pi/2, 0 \leq \varphi, \psi \leq 2\pi \). The metric in a frame which rotates with angular velocity \( \Omega \) can be obtained from (5.17) by change \( \varphi \) to \( \varphi + \Omega x^0 \)

\[
ds^2 = -B(dx^0 + a_\varphi d\varphi)^2 + \frac{\sin^2 \theta}{B} \ d\varphi^2 + \cos^2 \theta \ d\psi^2 + d\theta^2,
\]

(5.18)

\[
B = 1 - \Omega^2 \sin^2 \theta, \quad a_\varphi = \Omega \sin^2 \theta \ B^{-1}.
\]

(5.19)
To bring the corresponding NLSP to required form (1.3), (2.1) we use conformal covariance and first consider (5.16) on a space with metric related to (5.18) by conformal transformation

\[ ds^2 = -(dx^0 + a_\varphi d\varphi)^2 + dl^2, \]  

\[ dl^2 = \frac{1}{B} \left[ \sin^2 \theta \, d\varphi^2 + \cos^2 \theta \, d\psi^2 + d\theta^2 \right] = h_{jk} dx^j dx^k. \]  

(5.20)

(5.21)

Element \( dl^2 \) defines the metric on a compact three–dimensional manifold \( \mathcal{M}_3 \) without boundaries. The wave-equation on such a space-time results to NLSP (1.3) with operator

\[ L(\omega) = -\left( \nabla^k + i\omega a^k \right) \left( \nabla_k + i\omega a_k \right) + \frac{1}{6} \bar{R} + \frac{1}{24} F_{jk} F_{jk}. \]  

(5.22)

Here \( F_{jk} = a_{k,j} - a_{j,k} \) and \( a_j dx^j = a_\varphi d\varphi \), \( \nabla_k \) are the covariant derivatives on \( \mathcal{M}_3 \), \( \bar{R} \) is the scalar curvature of \( \mathcal{M}_3 \). Operator (5.22) has form (2.1).

The spectrum of the given NLSP can be easily found. If \( \omega_n \) are energies of quanta in the non-rotating frame (5.17) the energies in the rotating frame (5.18) are \( \omega_{nm} = \omega_n + m\Omega \) where \( m \) is the projection of the angular momentum on the rotation axis. For model (5.16) the spectrum is \( \omega_n = n + 1 \), where \( n = 0, 1, \ldots \), which follows from the spectrum of the Laplacian on \( S^3 \). The number \( m \) takes values \( -n \leq m \leq n \) and \( \omega_{nm} \) have degeneracy \( d_{nm} = n - |m| + 1 \) for given \( m \) and \( n \), see, e.g. [15]. The total spectrum of the NLSP also includes negative energies \( \omega_{nm} = -\omega_n + m\Omega \). The positive (negative) energy states have positive (negative) norm defined with respect to product (2.2). (The latter property is easy to understand if we note that signs of \( \omega_{nm} \) and \( \omega_n \) are the same and (2.2) coincides with the Klein-Gordon product.) Thus, requirement (ii) of section 2 is satisfied.

Because positive and negative parts of the spectrum are symmetric the pseudo-trace, Eq. (1.3), is

\[ K(t) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (n - |m| + 1) e^{-(n+1+m\Omega)^2 t}, \]  

(5.23)

Its short \( t \) expansion should have form (3.22) where \( d = 3 \) and \( b_n = 0 \), because \( \mathcal{M}_3 \) has no boundaries. The first coefficients \( a_n \) for (5.23) can be found explicitly. For instance,

\[ a_0 = \frac{2\pi^2}{1 - \Omega^2}, \quad a_1 = -\frac{2\pi^2}{3} \frac{\Omega^2}{1 - \Omega^2}. \]  

(5.24)

The easiest way to get (5.24) is to use the generalized \( \zeta \)-function \( \zeta(\nu) = \sum_{nm} d_{nm} \omega_{nm}^{-2\nu} \).

Then \( a_0 \) is the limit \( 4\pi^2(\nu - 3/2)\zeta(\nu) \) at \( \nu = 3/2 \) and \( a_1 \) is \( 8\pi^2(\nu - 1/2)\zeta(\nu) \) at \( \nu = 1/2 \).

Expressions (5.24) are in agreement with formula (3.20). Indeed, one can check that \( a_0 \) is the volume of \( \mathcal{M}_3 \), see (5.21). For operator (5.22) definitions (3.2) yield

\[ a_{0,1} = -\frac{1}{24} \int_{\mathcal{M}_3} h^{1/2} d^3 x F^{jk} F_{jk}, \quad a_{2,2} = -\frac{1}{12} \int_{\mathcal{M}_3} h^{1/2} d^3 x F^{jk} F_{jk}. \]  

(5.25)

\(^{4}\text{NLSP which appear on stationary space-times are discussed in [1],[2] where further details can be found.}\)
Coefficient $a_1$, as defined by \((3.20)\), is

$$a_1 = a_{0,1} + \frac{1}{2}a_{2,2} = \frac{1}{12} \int_{\mathcal{M}_3} h^{1/2} d^3 x \, F^{jk} F_{jk}$$  \hspace{1cm} (5.26)$$

and it coincides exactly with \((5.24)\).

6 Spectral problems related to Dirac equation

In this section we consider spectral problems which follow from the Dirac equation in a four-dimensional space-time

$$[\gamma^\mu (\nabla_\mu - i e A_\mu) + M] \psi = 0. \hspace{1cm} (6.1)$$

Here $\nabla_\mu$ are spinor connections, $A_\mu$ is a gauge potential and $M$ is a constant or a function, $M > 0$. Suppose that gravitational and gauge background fields are stationary, i.e. there is a coordinate system where $A_\mu$ and components of the metric do not depend on time $x^0$. Without loss of generality we take the space-time metric in the form

$$ds^2 = -(dx^0 + a_j dx^j)^2 + dl^2,$$  \hspace{1cm} (6.2)

$$dl^2 = h_{jk} dx^j dx^k, \hspace{0.5cm} j, k = 1, 2, 3, \hspace{1cm} (6.3)$$

where $a_j$ and $h_{jk}$ do not depend on $x^0$. One can always make a conformal transformation in \((6.1)\) to bring a stationary metric to this form. We assume that \((6.2)\) is the metric on a Riemannian manifold $\mathcal{M}_3$ and $\mathcal{M}_3$ is compact. Let us use the basis of one-forms associated with metric \((6.2)\),

$$U_0^\mu dx^\mu = -(dx^0 - a_i dx^i), \hspace{0.5cm} U_0^\mu U_0^\nu = h_{\mu\nu}, \hspace{0.5cm} a = 1, 2, 3.$$ Then for single-particle modes $\psi_\omega(x^0, x) = e^{-i \omega x^0} \psi_\omega(x)$ the Dirac equation is reduced to a Schrödinger equation

$$\omega - H(\omega) \psi_\omega = 0,$$  \hspace{1cm} (6.4)

$$H(\omega) = \gamma_0 (\gamma^k (D_k(\omega) - i e A_k) + M) + \frac{i}{8} F - \omega A_0.$$  \hspace{1cm} (6.5)

Here $D_k(\omega) = \nabla_k + i \omega a_k$, $\nabla_k$ are spinor connections on $\mathcal{M}_3$. We define $\gamma$-matrices

$$\{\gamma_k, \gamma_j\} = 2 h_{kj}$$

and the matrix $\gamma_0$ which anti-commutes with $\gamma_k$, $\gamma_0^2 = 1$. Also, $F = \gamma^k \gamma^j F_{jk}$ and $F_{jk} = a_{k,j} - a_{j,k}$.

Operator $H(\omega)$ is a Hermitian operator in a Hilbert space $L^2(\mathcal{M}_3)$. If $\lambda(\omega)$ are eigenvalues of $H(\omega)$ the eigen-value problem \((6.4)\) for $\omega$ is reduced to equation

$$\omega - \lambda(\omega) = 0.$$  \hspace{1cm} (6.6)

By using matrix $\gamma_5$, such that $(\gamma^5)^2 = 1$ and it anti-commutes with other $\gamma$'s, let us define a quadratic operator polynomial with the same spectrum as the spectrum of the Dirac problem \((6.4)\)

$$-(\gamma_5 \gamma_0 (\omega - H(\omega)))^2 \phi_\omega = (\omega^2 - L(\omega)) \phi_\omega = 0,$$  \hspace{1cm} (6.7)
\[
L(\omega) = -(D_k(\omega) - ieA_k - ib_k)(D^k(\omega) - ieA^k - ib^k) - \left( 2eA_0 + \frac{i}{4} F \right) \omega + V \quad (6.8)
\]

\[
b_k = \frac{1}{4} \xi_{0j} F_{kj} \gamma^i, \quad (6.9)
\]

\[
V = M^2 - \gamma^k M_k + \frac{1}{4} \bar{R} + \frac{1}{16} F^{jk} F_{jk} - \left( eA_0 + \frac{i}{8} F \right)^2 + \frac{e}{2} (F - 2\xi_{0i} \gamma^i \nabla_k A_0). \quad (6.10)
\]

Here \(\bar{R}\) is the curvature of \(M_3\), \(F = 2\nabla_j A_k \gamma^j \gamma^k\). An alternative way to come to \((6.7)\) is to start with equation \((\gamma^\mu (\nabla_\mu - ieA_\mu) - M)(\gamma^\nu (\nabla_\nu - ieA_\nu) + M)\phi = 0\).

To define a quadratic operator polynomial for the Dirac problem \((6.4)\) one could also consider equation \((\omega - H(\omega))^2 \phi = 0\). The difficulty is that it results in NLSP with the operator \(L(\omega) = -H^2(\omega) + 2\omega H(\omega)\) which is not positive-definite. Another suggestion could be equation \((\omega - H(\omega))(\omega + H(\omega))\phi = (\omega^2 - H^2(\omega))\phi = 0\). However, it includes eigen-values \(\omega\) which are not eigen-values of the original problem \((6.4)\).

Operator \(L(\omega)\) defined in \((6.8)\) has form \((2.1)\) and we assume that it is an elliptic operator with positive-definite leading symbol (the case which is relevant for physical applications). What differs \(L(\omega)\) from the class of operators considered in section 2 is that it is not Hermitian operator. The problem is that \(b_k\) in \((6.8)\) and the last term in \(V\) in \((6.10)\) are anti-Hermitian matrices. Instead, \(L(\omega)\) is Hermitian with respect to the following inner product:

\[
\langle \langle \phi, \psi \rangle \rangle = \int_{M^3} h^{1/2} dx \phi^* \bar{\gamma}_0 \psi, \quad (6.11)
\]

which is not positive-definite due to \(\bar{\gamma}_0\). Yet, because \(\bar{\gamma}_0\) is Hermitian, it is possible to show by using \((6.11)\) that the spectrum \(\Lambda(\omega)\) of \(L(\omega)\) is real and by using positivity of the leading symbol of \(L(\omega)\) it is possible to show that the spectrum is bounded from below.

Let us define for spinors \(\phi(x^0, x^i), \psi(x^0, x^i)\) the following product

\[
\langle \langle \phi, \psi \rangle \rangle = i(\langle \phi, \psi \rangle) - i(\langle \phi, \psi \rangle) - (\langle \phi, L_1 \psi \rangle) - i(\langle \phi, L_0 \psi \rangle) + i(\langle \phi, L_0 \psi \rangle), \quad (6.12)
\]

It has the same form as product \((2.2)\). The difference is that \((\ , \ )\) is replaced to \(\langle \langle \ , \ \rangle \rangle\).

It is easy to show that \(\langle \langle \ , \ \rangle \rangle\) is time-independent on solutions to \((1.1)\). Also one can see that for eigen-functions of \((6.7)\)

\[
\langle \langle \phi_\omega, \psi_\omega \rangle \rangle = \delta_{\omega \sigma} \chi'(\omega) \langle \langle \phi_\omega, \psi_\omega \rangle \rangle, \quad (6.13)
\]

where \(\chi'(\omega)\) is the derivative \(\partial_\omega \chi(\omega, \Lambda)\) at roots of \((2.4)\) and \(\chi(\omega, \Lambda)\) is defined by \((2.3)\).

By assuming that the norms do not vanish condition (ii) of section 2 can be generalized in the following way: the signs of the norms \(\langle \langle \phi_\omega, \phi_\omega \rangle \rangle\) and \(\langle \langle \phi_\omega, \psi_\omega \rangle \rangle\) coincide for \(\omega > 0\) and they are opposite for \(\omega < 0\). According to \((6.13)\) this guarantees that \(\chi'(\omega) = \epsilon(\omega)|\chi'(\omega)|\), the same property we had in section 2.

If the gauge field is absent, \(A_\mu = 0\), the Dirac equation is invariant under the charge conjugation \(\psi \rightarrow \psi^c, \text{where } \psi^c = C \psi^T, \bar{\psi} = \psi^* \bar{\gamma}_0,\text{ and } C \bar{\gamma}_\mu C^{-1} = -\gamma_\mu.\) It is easy to see that in this case the spectrum of \(\omega\) is symmetric, i.e., if \(\psi_\omega\) is an eigen-function of
ψ_ω is an eigen function of the same problem for the spectral parameter −ω. In this case eigen-values of \( L(\omega) \) are symmetric functions of \( \omega \), which simplifies verification of condition (ii).

Let us emphasize that \( \langle \langle \, , \, \rangle \rangle \) does not coincide with the physical relativistic inner product \( \langle \, , \rangle \) for Dirac equation (6.1)

\[
\langle \phi, \psi \rangle = (\phi, \psi) - (\phi, H_0 \psi), \tag{6.14}
\]

where \( H_0 \) is defined by the Hamiltonian \( H(\omega) = H_1 + \omega H_0 \). If \( \phi \) and \( \psi \) are solutions to (6.1) then (6.14) does not depend on \( x^0 \). If \( \phi_\omega \) and \( \psi_\sigma \) are solutions to the Dirac problem (6.4)

\[
\langle \phi_\omega, \psi_\sigma \rangle = \delta_{\omega \sigma} \chi'(\omega) (\phi_\omega, \psi_\omega), \tag{6.15}
\]

where \( \chi' \) is the derivative \( \partial_\omega (\omega - \lambda(\omega)) \) taken at the root of equation (6.6). Because each solution \( \psi_\omega \) to (6.4) also is a solution to (6.7) there is a relation between two norms. One can show that

\[
\langle \langle \psi_\omega, \psi_\omega \rangle \rangle = \omega^{-1} \langle M \psi_\omega, \psi_\omega \rangle. \tag{6.16}
\]

Thus, if \( M = 0 \) there are solutions of (6.7) with zero norm determined with respect to (6.12). If \( M \) is a positive constant and \( \langle \psi_\omega, \psi_\omega \rangle \) is positive it follows from (6.16) that \( \langle \langle \psi_\omega, \psi_\omega \rangle \rangle \) is positive for \( \omega > 0 \) and negative for \( \omega < 0 \).

Consider now the pseudo-trace for the Dirac problem which we define as earlier, as

\[
K(t) = \frac{1}{2} \sum_\omega e^{-t\omega^2}, \tag{6.17}
\]

where \( t > 0 \) and \( \omega \) are eigen-values of quadratic polynomial (6.7). If operator \( L(\omega) \), Eq. (6.8) obeys the properties formulated above in this section, \( K(t) \) is related to the spectral density of \( L(\omega) \) by formulas (2.13)–(2.15). These formulas can be used to get for (6.17) asymptotic expansion (1.6) where coefficients are determined by coefficients of the asymptotic expansion of the trace of the operator \( e^{-tL(\omega)} \) by formulas (3.20), (3.21).

In other words, the results of this paper concerning asymptotic expansion of \( K(t) \) should be valid for spin 1/2 operators. Note that operator \( e^{-tL(\omega)} \) exists if \( L(\omega) \) has a positive-definite leading symbol regardless of the fact that \( L(\omega) \) may be a not self-adjoint operator [16], as it happens in the considered case.

To illustrate these results let us return to the problem of quantum fields in the rotating Einstein universe, see section 5.2. The spectrum of \( \omega \) in (6.4) can be connected with the spectrum of the Dirac operator on \( S^3 \). For simplicity we consider massless neutral fields \( (M = 0, A_\mu = 0) \). Then for Weyl spinors one gets (see [15]) \( \omega_{nm} = \pm \omega_n + m\Omega \), where \( \omega_n = 3/2 + n, n = 0, 1, 2, ..., \) and \(-(n + 1/2) \leq m \leq n + 1/2 \). The degeneracy of \( \omega_{nm} \) is \( d_{nm} = n - |m| + 3/2 \). The pseudo-trace for this spectrum is

\[
K(t) = 2 \sum_{n=0}^{\infty} \sum_{m=-n-1/2}^{n+1/2} (n - |m| + 3/2) e^{-(n+3/2+m\Omega)^2 t}. \tag{6.18}
\]
The short $t$ expansion for (6.18) is given by (3.22) with $d = 3$, $b_n = 0$. The first coefficients can be found explicitly

$$a_0 = \frac{4\pi^2}{1 - \Omega^2}, \quad a_1 = \frac{2\pi^2}{3} \frac{\Omega^2 - 3}{1 - \Omega^2}. \quad (6.19)$$

To get the same result by our method one has to start with $M > 0$ and then go to the limit $M = 0$. This yields $a_0 = 2\text{vol}[\mathcal{M}_3]$ (factor 2 is related to dimensionality of the spinor representation) and agrees with $a_0$ in (6.19). Coefficient $a_1$ is determined by (3.20), (6.8).

For Weyl spinors ($M = 0$, $A_\mu = 0$)

$$a_{0,1} = \int_{\mathcal{M}_3} h^{1/2} d^3x \text{Tr} \left[ \frac{1}{6} \tilde{R} - V \right] = - \int_{\mathcal{M}_3} h^{1/2} d^3x \left[ \frac{1}{6} \tilde{R} + \frac{1}{16} F^j k F_{jk} \right], \quad (6.20)$$

$$a_{2,2} = \int_{\mathcal{M}_3} h^{1/2} d^3x \text{Tr} \left[ - \frac{1}{32} F^2 - \frac{1}{12} F^j k F_{jk} \right] = - \frac{1}{24} \int_{\mathcal{M}_3} h^{1/2} d^3x \ F^j k F_{jk} \quad (6.21)$$

$$a_1 = a_{0,1} + \frac{1}{2} a_{2,2} = - \int_{\mathcal{M}_3} h^{1/2} d^3x \left[ \frac{1}{6} \tilde{R} + \frac{1}{12} F^j k F_{jk} \right], \quad (6.22)$$

where $\tilde{R}$ is the curvature of $\mathcal{M}_3$, see (5.21). By using definitions (5.19), (5.21) one can check that (6.22) coincides with $a_1$ in (6.19).

Spectral asymptotics for spin 1/2 fields were studied in [1]. In this work there is a mistake in definition of $H(\omega)$. Equations (6.5) and (6.22) correct corresponding formulas of [1].

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A Spectral asymptotics

The aim of this Appendix is to demonstrate that the asymptotics (3.6) hold when the corresponding operator has a number of negative eigen-values. In this case we use definition (3.4)

\[ K_\omega(t) t^{-\alpha} = \int_{t}^{\infty} e^{-t\lambda} \varphi_\alpha(\lambda, \omega) d\lambda, \]  

where parameter \( \mu \) is smaller than the lowest eigen-value and, hence, \( \mu < 0 \). As earlier, we suppose that \( \alpha \) is a complex parameter, \( \text{Re} \alpha > 0 \). Equation (A.1) is equivalent to the following equation

\[ \bar{K}_\omega(t) t^{-\alpha} = \int_{0}^{\infty} e^{-t\lambda} \bar{\varphi}_\alpha(\lambda, \omega) d\lambda, \]  

\[ \bar{K}_\omega(t) = e^{t\mu} K_\omega(t), \]  

\[ \bar{\varphi}_\alpha(\lambda, \omega) = \varphi_\alpha(\lambda + \mu, \omega). \]  

If \( K_\omega(t) \) has asymptotics (3.3) then

\[ \bar{K}_\omega(t) \sim \frac{1}{(4\pi t)^{d/2}} \sum_{n=0}^{\infty} \left[ a_n(\omega) t^n + b_n(\omega) t^{n+1/2} \right], \]  

and, according to (A.3), the coefficients in the both expansions are related as

\[ a_n(\omega) = \sum_{p=0}^{n} (-1)^p \frac{\mu^p}{p!} \bar{a}_n(\omega), \quad b_n(\omega) = \sum_{p=0}^{n} (-1)^p \frac{\mu^p}{p!} \bar{b}_n(\omega). \]  

(A.5) defines the asymptotic expansion of \( \bar{\varphi}_\alpha(\lambda, \omega) \) at large \( \lambda \)

\[ \bar{\varphi}_\alpha(\lambda, \omega) \sim \frac{1}{(4\pi)^{d/2}} \sum_{n=0}^{\infty} \left[ \bar{a}_n(\omega) \frac{\lambda^{d/2-n+\alpha-1}}{\Gamma \left( \frac{d}{2} - n + \alpha \right)} + \bar{b}_n(\omega) \frac{\lambda^{(d-1)/2-n+\alpha-1}}{\Gamma \left( \frac{d-1}{2} - n + \alpha \right)} \right]. \]  

(A.7)

By using (A.4) and (5.8) we get from (A.7) expansion which is valid for \( \lambda > |\mu| \)

\[ \varphi_\alpha(\lambda, \omega) \sim \frac{1}{(4\pi)^{d/2}} \sum_{n=0}^{\infty} \left[ \bar{a}_n(\omega) \frac{\lambda^{d/2-n+\alpha-1}}{\Gamma \left( \frac{d}{2} - n + \alpha \right)} + \bar{b}_n(\omega) \frac{\lambda^{(d-1)/2-n+\alpha-1}}{\Gamma \left( \frac{d-1}{2} - n + \alpha \right)} \right], \]  

\[ a_n'(\omega) = \sum_{p=0}^{n} (-1)^p \frac{\mu^p}{p!} \bar{a}_n(\omega), \quad b_n'(\omega) = \sum_{p=0}^{n} (-1)^p \frac{\mu^p}{p!} \bar{b}_n(\omega). \]  

(A.9)

After comparing (A.9) with (A.6) one can replace \( a_n'(\omega), b_n'(\omega) \) in (A.8) to \( a_n(\omega), b_n(\omega) \), respectively.
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