BALANCED METRICS AND CHOW STABILITY OF
PROJECTIVE BUNDLES OVER KÄHLER MANIFOLDS

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Abstract. In 1980, I. Morrison proved that slope stability of a
vector bundle of rank 2 over a compact Riemann surface implies
Chow stability of the projectivization of the bundle with respect
to certain polarizations. Using the notion of balanced metrics and
recent work of Donaldson, Wang, and Phong-Sturm, we show that
the statement holds for higher rank vector bundles over compact
algebraic manifolds of arbitrary dimension that admit constant
scalar curvature metric and have discrete automorphism group.

1. Introduction

A central notion in geometric invariant theory (GIT) is the con-
cept of stability. Stability plays a significant role in forming quotient
spaces of projective varieties for which geometric invariant theory was
invented. One can define Mumford-Takemoto slope stability for hol-
morphic vector bundles, and also there is a notion of Gieseke stability
which is more in the realm of geometric invariant theory. It is well-
known that over algebraic curves, all of these different notions coincide.
It was known from the work of Narasimhan and Seshadri that a hol-
omorphic vector bundle over a compact Riemann surface is poly-stable if
and only the bundle admits a projectively flat connection. The picture
became complete with the later work of Donaldson, Uhlenbeck and Yau
([D1], [D2], [UY]). They proved that over a compact Kähler manifold,
a holomorphic vector bundle is poly-stable if and only if it admits a
Hermitian-Einstein metric. This is known as the Hitchin-Kobayashi
 correspondence. By a conjecture of Yau, one would also expect such a
correspondence for polarized algebraic manifolds. In other words, the
existence of extremal metrics on such a manifold should be equivalent
to being stable in some GIT sense. In [Zh], Zhang introduced the con-
cept of balanced embedding and proved that the existence of balanced
embedding of a polarized algebraic variety is equivalent to stability of
Chow point of the variety. Zhang’s result has been reproven by Lu
in [L] and Phong and Sturm in [PS1]. The same correspondence was

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proven for vector bundles by Wang in [W1]. Later in [D3], Donaldson proved that the existence of constant scalar curvature Kähler metrics implies existence of balanced metrics and hence asymptotic Chow stability. The converse is not yet known.

Earlier, in [M], Morrison proved that for the projectivization of a rank two holomorphic vector bundle over a compact Riemann surface, Chow stability is equivalent to the stability of the bundle. Using ideas from the recent research discussed above, in this article we generalize one direction of Morrison’s result for higher rank vector bundles over compact algebraic manifolds of arbitrary dimension that admit constant scalar curvature metric and have discrete automorphism group.

To state the precise result, let $X$ be a compact complex manifold of dimension $m$ and $\pi : E \to X$ a holomorphic vector bundle of rank $r$ with dual bundle $E^*$. This gives a holomorphic fibre bundle $\mathbb{P}E^*$ over $X$ with fibre $\mathbb{P}^{r-1}$. One can pull back the vector bundle $E$ to $\mathbb{P}E^*$. We denote the tautological line bundle on $\mathbb{P}E^*$ by $O_{\mathbb{P}E^*}(-1)$ and its dual by $O_{\mathbb{P}E^*}(1)$. Let $L \to X$ be an ample line bundle on $X$ and $\omega \in 2\pi c_1(L)$ be a Kähler form. Since $L$ is ample, there is an integer $k_0$ so that for any $k \geq k_0$, $O_{\mathbb{P}E_k^*}(1)$ is very ample over $\mathbb{P}E^*$, where $E_k = E \otimes L^\otimes k$. Note that $\mathbb{P}E_k^* \cong \mathbb{P}E^*$ and $O_{\mathbb{P}E_k^*}(1) \cong O_{\mathbb{P}E^*}(1) \otimes \pi^*L^k$. The theorem we shall prove is the following:

**Theorem 1.1.** Suppose that $\text{Aut}(X)$ is discrete and $X$ admits a constant scalar curvature Kähler metric in the class of $2\pi c_1(L)$. If $E$ is Mumford stable, then there exists $k_0$ such that

$$(\mathbb{P}E^*, O_{\mathbb{P}E^*}(1) \otimes \pi^*L^k)$$

is Chow stable for $k \geq k_0$.

One of the earliest results in this spirit is the work of Burns and De Bartolomeis in [BD]. They construct a ruled surface which does not admit any extremal metric in certain cohomology class. In [H1], Hong proved that there are constant scalar curvature Kähler metrics on the projectivization of stable bundles over curves. In [H2] and [H3], he generalizes this result to higher dimensions with some extra assumptions. Combining Hong’s results with Donaldson’s, $(\mathbb{P}E^*, O_{\mathbb{P}E_m^*}(n))$ is Chow stable for $m, n \gg 0$ when the bundle $E$ is stable. Note that it differs from our result, since it implies the Chow stability of $(\mathbb{P}E^*, O_{\mathbb{P}E_m^*}(n))$ for $n$ big enough.

In [RT], Ross and Thomas developed the notion of slope stability for polarized algebraic manifolds. As one of the applications of their theory, they proved that if $(\mathbb{P}E^*, O_{\mathbb{P}E^*}(1) \otimes \pi^*L^k)$ is slope semi-stable for $k \gg 0$, then $E$ is a slope semistable bundle and $(X, L)$ is a slope
semistable manifold. Again note that they look at stability of $\mathbb{P}E^*$ with respect to polarizations $\mathcal{O}_{\mathbb{P}E^*}(n)$ for $n$ big enough. For the case of one dimensional base, however they showed stronger results. In this case they proved that if $(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes \pi^*L)$ is slope (semi, poly) stable for any ample line bundle $L$, then $E$ is a slope (semi, poly) stable bundle.

In order to prove Theorem 1.1 we use the concept of balanced metrics (See Definition 2.1). Combining the results of Luo, Phong, Sturm and Zhang on the relation between balanced metrics and stability, it suffices to prove the following

**Theorem 1.2.** Let $X$ be a compact complex manifold and $L \rightarrow X$ be an ample line bundle. Suppose that $X$ admits a constant scalar curvature Kähler metric in the class of $2\pi c_1(L)$ and $\text{Aut}(X)$ is discrete. Let $E \rightarrow X$ be a holomorphic vector bundle on $X$. If $E$ is Mumford stable, then $\mathcal{O}_{\mathbb{P}E^*}(1) \otimes \pi^*L^k$ admits balanced metrics for $k \gg 0$.

The balanced condition may be formulated in terms of Bergman kernels. First, we show that there exists an asymptotic expansion for the Bergman kernel of $(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes \pi^*L^k)$. Fix a positive hermitian metric $\sigma$ on $L$ such that $\text{Ric}(\sigma) = \omega$. For any hermitian metric $g$ on $\mathcal{O}_{\mathbb{P}E^*}(1)$, we define the sequence of volume forms $d\mu_{g,k}$ on $\mathbb{P}E^*$ as follows

$$d\mu_{g,k} = k^{-m}(\omega_g + k\pi^*\omega)^{m+r-1}$$

where $\omega_g = \text{Ric}(g)$.

Let $\rho_k(g, \omega)$ be the Bergman kernel of $H^0(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes \pi^*L^k)$ with respect to the $L^2$-inner product $L^2(g \otimes \sigma^\otimes k, d\mu_{g,k})$. We prove the following

**Theorem 1.3.** For any hermitian metric $h$ on $E$ and Kähler form $\omega \in 2\pi c_1(L)$, there exist smooth endomorphisms $\tilde{B}_k(h, \omega)$ such that

$$\rho_k(g, \omega)([v]) = C_k^{-1} tr(\lambda(v, h)\tilde{B}_k(h, \omega)),$$

where $g$ is the Fubini-Study metric on $\mathcal{O}_{\mathbb{P}E^*}(1)$ induced by the hermitian metric $h$. Moreover,

1. There exist smooth endomorphisms $A_i(h, \omega) \in \Gamma(X, E)$ such that the following asymptotic expansion holds as $k \rightarrow \infty$,

$$\tilde{B}_k(h, \omega) \sim k^m + A_1(h, \omega)k^{m-1} + \ldots .$$

2. In particular

$$A_1(h, \omega) = \frac{i}{2\pi} \Lambda F_{(E, h)} - \frac{i}{2\pi r} tr(\Lambda F_{(E, h)})I_E + \frac{(r + 1)}{2r} S(\omega)I_E,$$
where $\Lambda$ is the trace operator acting on $(1,1)$-forms with respect to the Kähler form $\omega$ and $F_{(E,h)}$ is the curvature of $(E,h)$ and $S(\omega)$ is the scalar curvature of $\omega$.

(3) The asymptotic expansion holds in $C^\infty$. More precisely, for any positive integers $a$ and $p$, there exists a positive constant $K_{a,p,\omega,h}$ such that

$$\left| \tilde{B}_k(h,\omega) - (k^m + \cdots + A_p(h,\omega)k^{m-p}) \right|_{C^a} \leq K_{a,p,\omega,h} k^{m-p-1}.$$

Moreover the expansion is uniform in the sense that there exists a positive integer $s$ such that if $h$ and $\omega$ run in a bounded family in $C^s$ topology and $\omega$ is bounded from below, then the constants $K_{a,p,\omega,h}$ are bounded by a constant depending only on $a$ and $p$.

Finding balanced metrics on $O_{\mathbb{P}E^*}(1) \otimes \pi^*L^k$ is basically the same as finding solutions to the equations $\rho_k(g,\omega) = \text{Constant}$. Therefore in order to prove Theorem 1.2, we need to solve the equations $\rho_k(g,\omega) = \text{Constant}$ for $k \gg 0$. Now if $\omega$ has constant scalar curvature and $h$ satisfies the Hermitian-Einstein equation $\Lambda_\omega F_{(E,h)} = \mu I_E$, then $A_1(h,\omega)$ is constant. Notice that in order to make $A_1$ constant, existence of Hermitian-Einstein metric is not enough. We need the existence of constant scalar curvature Kähler metric as well. Next, the crucial fact is that the linearization of $A_1$ at $(h,\omega)$ is surjective. This enables us to construct formal solutions as power series in $k^{-1}$ for the equation $\rho_k(g,\omega) = \text{Constant}$. Therefore, for any positive integer $q$, we can construct a sequence of metrics $g_k$ on $O_{\mathbb{P}E^*}(1) \otimes \pi^*L^k$ and bases $s_1^{(k)}, ..., s_N^{(k)}$ for $H^0(\mathbb{P}E^*, O_{\mathbb{P}E^*}(1))$ such that

$$\sum |s_i^{(k)}|^2_{g_k} = 1,$$

$$\int \langle s_i^{(k)} \bar{s_j}^{(k)} \rangle_{g_k} dvol_{g_k} = D_k I + M_k,$$

where $D_k \to C_r$ as $k \to \infty$ (See [5,1] for definition of $C_r$), and $M_k$ is a trace-free hermitian matrix such that $||M_k||_{op} = o(k^{-q-1})$ as $k \to \infty$.

The next step is to perturb these almost balanced metrics to get balanced metrics. As pointed out by Donaldson, the problem of finding balanced metric can be viewed also as a finite dimensional moment map problem solving the equation $M_k = 0$. Indeed, Donaldson shows that $M_k$ is the value of a moment map $\mu_D$ on the space of ordered bases with the obvious action of $SU(N)$. Now, the problem is to show that if for some ordered basis $s$, the value of moment map is very small, then we can find a basis at which moment map is zero. The standard technique is flowing down $s$ under the gradient flow of $||\mu_D||^2$ to reach a zero of $\mu_D$. We need a Lojasiewicz type inequality to guarantee that
the flow converges to a zero of the moment map. We do this in Section 3 by adapting Phong-Sturm proof to our situation.

Here is the outline of the paper: In Section 2, we review Donaldson’s moment map setup. We follow Phong and Sturm treatment from ([PS2]). In Section 3, we obtain a lower bound for the derivative of the moment map by adapting the argument in ([PS2]) to our setting. In Section 4, we show how to perturb almost balanced metrics to obtain balanced metrics in the general setting of Section 3. In order to do that, we use the estimate obtained in Theorem 3.2 to apply the Donaldson’s version of inverse function theorem (Proposition 2.2). In Section 5, we prove the existence of an asymptotic expansion for the Bergman kernel of $\mathcal{O}_{\mathbb{P}^E^*}(1) \otimes \pi^*L^k$ using results of Catlin and Zelditch. Section 6 is devoted to constructing almost balanced metrics on $\mathcal{O}_{\mathbb{P}^E^*}(1) \otimes \pi^*L^k$ using the asymptotic expansion obtained in Section 5.

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2. Moment Map Setup

In this section, we review Donaldson’s moment map setup. We follow the notation of [PS2].

Let $(Y, \omega_0)$ be a compact Kähler manifold of dimension $n$ and $O(1) \to Y$ be a very ample line bundle on $Y$ equipped with a Hermitian metric $g_0$ such that $\text{Ric}(g_0) = \omega_0$. Since $O(1)$ is very ample, using global sections of $O(1)$, we can embed $Y$ into $\mathbb{P}(H^0(Y, O(1))^*)$. A choice of ordered basis $\mathfrak{s} = (s_1, ..., s_N)$ of $H^0(Y, O(1))$ gives an isomorphism between $\mathbb{P}(H^0(Y, O(1))^*)$ and $\mathbb{P}^{N-1}$. Hence for any such $\mathfrak{s}$, we have an embedding $\iota_\mathfrak{s} : Y \hookrightarrow \mathbb{P}^{N-1}$ such that $\iota_\mathfrak{s}^* O_{\mathbb{P}^N}(1) = O(1)$. Using $\iota_\mathfrak{s}$, we can pull back the Fubini-Study metric and Kähler form of the projective space to $O(1)$ and $Y$ respectively.

Definition 2.1. An embedding $\iota_\mathfrak{s}$ is called balanced if

$$\int_Y \langle s_i, s_j \rangle \iota_\mathfrak{s}^* h_{FS} \frac{\omega_{FS}}{n!} = V \frac{\delta_{ij}}{N},$$

where $V = \int_Y \frac{\omega^n}{n!}$. A hermitian metric (respectively a Kähler form) is called balanced if it is the pull back $\iota_\mathfrak{s}^* h_{FS}$ (respectively $\iota_\mathfrak{s}^* \omega_{FS}$) where $\iota_\mathfrak{s}$ is a balanced embedding.
There is an action of $SL(N)$ on the space of ordered bases of $H^0(Y,\mathcal{O}(1))$. Donaldson defines a symplectic form on the space of ordered bases of $H^0(Y,\mathcal{O}(1))$ which is invariant under the action of $SU(N)$. So there exists an equivariant moment map on this space such that its zeros are exactly balanced bases.

More precisely we define

$$\tilde{Z} = \{ s = (s_1,...,s_N) \mid s_1,...,s_N \text{ a basis of } H^0(Y,\mathcal{O}(1)) \} / \mathbb{C}^*$$

and $Z = \tilde{Z} / \mathbb{P}Aut(Y,\mathcal{O}(1))$. Donaldson defines a symplectic form $\Omega_D$ on $Z$. There is a natural action of $SU(N)$ on $(Z,\Omega_D)$ which preserves the symplectic form $\Omega_D$. The moment map for this action is defined by

$$\mu_D(s) = i\langle (s_\alpha, s_\beta) h_s - \frac{1}{N} \delta_{\alpha,\beta} \rangle,$$

where $h_s$ is the $L^2$- inner product with respect to the pull back of Fubini-Study metric and Fubini-Study Kähler form via the embedding $i_s$. Also we identify $su(N)^*$ with $su(N)$ using the invariant inner product on $su(N)$, where $su(N)$ is the Lie algebra of the group $SU(N)$ and $su(N)^*$ is its dual. (For construction of $\Omega_D$ and more details see [D3] and [PS2].)

Using Deligne’s pairing, Phong and Sturm construct another symplectic form on $Z$ as follows:

Let

$$\tilde{Y} = \{ (x,s) \mid x \in \mathbb{P}^{N-1}, s = (s_1,...,s_N), x \in i_s(Y) \}$$

and $Y = \tilde{Y} / \mathbb{P}Aut(Y,\mathcal{O}(1))$. One obtains a holomorphic fibration $Y \to Z$ where every fibre is isomorphic to $Y$. Let $p : Y \to \mathbb{P}^{N-1}$ be the projection on the first factor. Then define a hermitian line bundle $\mathcal{M}$ on $Z$ by

$$\mathcal{M} = \langle p^*\mathcal{O}_{\mathbb{P}^{N-1}}(1),...,p^*\mathcal{O}_{\mathbb{P}^{N-1}}(1) \rangle (\frac{Y}{Z})$$

which is the Deligne’s pairing of $(n + 1)$ copies of $p^*\mathcal{O}_{\mathbb{P}^{N-1}}(1)$. Denote the curvature of this hermitian line bundle by $\Omega_{\mathcal{M}}$. It follows from properties of Deligne’s pairing that

$$\Omega_{\mathcal{M}} = \int_{Y/Z} \omega_{FS}^{n+1}. \tag{2.1}$$

Since $SU(N)$ is semisimple, there is a unique equivariant moment map $\mu_{\mathcal{M}} : Z \to su(N)$ for the action of $SU(N)$ on $(Z,\Omega_{\mathcal{M}})$.

**Theorem 2.1.** ([PS2 Theorem 1]) $\Omega_{\mathcal{M}} = \Omega_D$ and $\mu_{\mathcal{M}} = \mu_D$. 
Let $\xi$ be an element of the Lie algebra $su(N)$. Since $SU(N)$ acts on $Z$, the infinitesimal action of $\xi$ defines a vector field $\sigma_Z(\xi)$ on $Z$. Fixing a point $z \in Z$, we have a linear map $\sigma_z : su(N) \to T_z Z$. Let $\sigma_z^*$ be its adjoint with respect to the metric on $TZ$ and the invariant metric on $su(N)$. Then we get the operator

$$Q_z = \sigma_z^* \sigma_z : su(N) \to su(N).$$

Define $\Lambda_z^{-1}$ as the smallest eigenvalue of $Q_z$. In [D3], Donaldson proves the following.

**Proposition 2.2.** ([D3, Proposition 17]) Suppose given $z_0 \in Z$ and real numbers $\lambda, \delta$ such that for all $z = e^{i\xi}z_0$ with $|\xi| \leq \delta$ and $\xi \in su(N)$, $\Lambda_z \leq \lambda$. Suppose that $\lambda |\mu(z_0)| \leq \delta$, then there exists $w = e^{in}$ with $\mu(w) = 0$, where $|\eta| \leq \lambda |\mu(z_0)|$.

3. Eigenvalue Estimates

In this section, we obtain a lower bound for the derivative of the moment map $\mu_D$. This is equivalent to an upper bound for the quantity $\Lambda_z$ introduced in the previous section. In order to do this, we adapt the argument of Phong and Sturm to our setting. The main result is Theorem 3.2.

Let $(Y, \omega_0)$ and $\mathcal{O}(1) \to Y$ be as in the previous section. Let $(L, h_\infty)$ be a Hermitian line bundle over $Y$ such that $\omega_\infty = Ric(h_\infty)$ is a semi positive $(1,1)$-form on $Y$. Define $\tilde{\omega}_0 = \omega_0 + k\omega_\infty$. For the rest of this section and next section let $m$ be the smallest integer such that $\omega^{m+1}_\infty = 0$. Also assume that $\omega^{n-m}_0 \wedge \omega^m_\infty$ is a volume form and there exist positive constant $n_1$ and $n_2$ such that

$$N_k = \dim H^0(Y, \mathcal{O}(1) \otimes L^k) = n_1 k^m + O(k^{m-1}).$$

$$V_k = \int_Y (\omega_0 + k\omega_\infty)^n = n_2 k^m + O(k^{m-1}).$$

Notice that (3.2) is implied from the fact that $\omega^{n-m}_0 \wedge \omega^m_\infty$ is a volume form and $\omega^{m+1}_\infty = 0$.

The case important for this paper is the following:

**Example 3.1.** Let $(X, \omega_\infty)$ be a compact Kähler manifold of dimension $m$ and $L$ be a very ample holomorphic line bundle on $X$ such that $\omega_\infty \in 2\pi c_1(L)$. Let $E$ be a holomorphic vector bundle on $X$ of rank $r$ such that the line bundle $\mathcal{O}_{\mathbb{P}E^*}(1) \to Y = \mathbb{P}E^*$ is an ample line bundle. We denote the pull back of $\omega_\infty$ to $\mathbb{P}E^*$ by $\omega_\infty$. Then $\omega^{m+1}_\infty = 0$ and by
Riemann-Roch formula we have
\[ \dim H^0(Y, \mathcal{O}(1) \otimes L^k) = \dim H^0(X, E \otimes L^k) = \frac{r}{m!} \int_X c_1(L)^m k^m + O(k^{m-1}). \]

The following lemma is clear.

**Lemma 3.1.** Let \( h_k \) be a sequence of hermitian metrics on \( \mathcal{O}(1) \otimes L^k \) and let \( s^{(k)} = (s_1^{(k)}, ..., s_N^{(k)}) \) be a sequence of ordered bases for \( H^0(Y, \mathcal{O}(1) \otimes L^k) \). Suppose that for any \( k \)
\[ \sum |s_i^{(k)}|^2_{h_k} = 1 \]
and
\[ \int_Y \langle s_i^{(k)}, s_j^{(k)} \rangle_{h_k} d\text{vol}_{h_k} = D^{(k)} \delta_{ij} + M^{(k)}_{ij}, \]
where \( D^{(k)} \) is a scalar and \( M^{(k)} \) is a trace-free hermitian matrix. Then
\[ D^{(k)} = \frac{V_k}{N_k} \rightarrow \frac{n_2}{n_1} \text{ as } k \rightarrow \infty, \]
where the constants \( n_1 \) and \( n_2 \) are defined by (3.1) and (3.2).

We start with the notion of \( R \)-boundedness introduced originally by Donaldson in [D3].

**Definition 3.2.** Let \( R \) be a real number with \( R > 1 \) and \( a \geq 4 \) be a fixed integer and let \( s = (s_1, ..., s_N) \) be an ordered basis for \( H^0(Y, \mathcal{O}(1) \otimes L^k) \). We say \( s \) has \( R \)-bounded geometry if the Kähler form \( \tilde{\omega} = \iota^*_s \omega_{\text{FS}} \) satisfies the following conditions
\[ \begin{align*}
&\|\tilde{\omega} - \tilde{\omega}_0\|_{C^a(\tilde{\omega}_0)} \leq R, \text{ where } \tilde{\omega}_0 = \omega_0 + k\omega_{\infty}. \\
&\tilde{\omega} \geq \frac{1}{R} \tilde{\omega}_0.
\end{align*} \]

Recall the definition of \( \Lambda_\sigma \) from the previous section. The main result of this section is the following.

**Theorem 3.2.** Assume \( Y \) does not have any nonzero holomorphic vector fields. For any \( R > 1 \), there are positive constants \( C \) and \( \epsilon \leq n_2/10n_1 \) such that, for any \( k \), if the basis \( s = (s_1, ..., s_N) \) of \( H^0(Y, \mathcal{O}(1) \otimes L^k) \) has \( R \)-bounded geometry, and if \( \|\mu_D(s)\|_{op} \leq \epsilon \), then
\[ \Lambda_\sigma \leq Ck^{2m+2}. \]

The rest of this section is devoted to the proof of Theorem 3.2. Notice that the estimate \( \Lambda_\sigma \leq Ck^{2m+2} \) is equivalent to the estimate
\[ (3.3) \quad |\sigma_\sigma(\xi)|^2 \geq ck^{-(2m+2)}||\xi||^2. \]
On the other hand (2.1) and Theorem 2.1 imply that

\[(3.4) \quad |\sigma_Z(\xi)|^2 = \int_Y \iota_{Y_\xi} \omega^{n+1}_{FS}.\]

Hence, in order to establish Theorem 3.2, we need to estimate the quantity \(\int_Y \iota_{Y_\xi} \omega^{n+1}_{FS}\) from below.

For the rest of this section, fix an ordered basis \(s^{(k)} = (s_1, \ldots, s_N)\) of \(H^0(Y, \mathcal{O}(1) \otimes L^k)\) and let \(M^{(k)} = -i\mu_D(s^{(k)})\). It gives an embedding \(\iota = \iota_{s^{(k)}}: Y \to \mathbb{P}^{N-1}\), where \(N = N_k = \dim H^0(Y, \mathcal{O}(1) \otimes L^k)\). For any \(\xi \in su(N)\), we have a vector field \(Y_\xi\) on \(\mathbb{P}^{N-1}\) generated by the infinitesimal action of \(\xi\).

Every tangent vector to \(\mathbb{P}^{N-1}\) is given by pairs \((z, v)\) modulo an equivalence relation \(\sim\). This relation is defined as follows:

\((z, v) \sim (z', v')\) if \(z' = \lambda z\) and \(v' - \lambda v = \mu z\) for some \(\lambda \in \mathbb{C}^*\) and \(\mu \in \mathbb{C}\).

For a tangent vector \([(z, v)]\), the Fubini-Study metric is given by

\[||[(z, v)]||^2 = \frac{v^* v z^* z - (z^* v)^2}{(z^* z)^2}.\]

Since the vector field \(Y_\xi\) is given by \([z, \xi z]\), we have

\[||(Y_\xi(z)||^2 = \frac{-(z^* \xi z)^2 + (z^* \xi^2 z)(\xi z)}{(z^* z)^2}.\]

We have the following exact sequence of vector bundles over \(Y\)

\[0 \to TY \to \iota^* T\mathbb{P}^{N-1} \to Q \to 0.\]

Let \(\mathcal{N} \subset \iota^* T\mathbb{P}^{N-1}\) be the orthogonal complement of \(TY\). Then as smooth vector bundles, we have

\[\iota^* T\mathbb{P}^{N-1} = TY \oplus \mathcal{N}.\]

We denote the projections onto the first and second component by \(\pi_T\) and \(\pi_N\) respectively. Define

\[\sigma_t(z) = \exp(it\xi)z,\]

\[\varphi_t(z) = \log \frac{||\sigma_t(z)||}{|z|}.\]

Direct computation shows that

\[(3.6) \quad \frac{d}{dt}\bigg|_{t=0} \varphi_t(z) = 2i \frac{z^* \xi z}{z^* z}.\]
The following is straightforward.

**Proposition 3.3.** For any $\xi \in su(N)$, we have

$$||\pi_N Y_\xi||_{L^2(Y,TY)}^2 = \int_Y t_{Y_\xi Y_\xi} \omega^{n+1}_{FS}.$$  

Therefore, the estimate in Theorem 3.2 will follow from:

$$||\xi||^2 \leq c_R k^m ||Y_\xi||^2$$  \hspace{1cm} (3.8)

$$c'_R ||\pi T Y_\xi||^2 \leq k^{m+2} ||\pi N Y_\xi||^2$$  \hspace{1cm} (3.9)

$$||Y_\xi||^2 = ||\pi T Y_\xi||^2 + ||\pi N Y_\xi||^2$$  \hspace{1cm} (3.10)

We will prove (3.8) in Proposition 3.6 and (3.9) in Proposition 3.9. Assuming these, we give the Proof of Theorem 3.2.

**Proof of Theorem 3.2.** By (3.4), we have

$$|\sigma Z(\xi)|^2 = \int_Y t_{Y_\xi Y_\xi} \omega^{n+1}_{FS}.$$  

Applying Proposition 3.3 we get

$$|\sigma Z(\xi)|^2 = ||\pi N Y_\xi||^2.$$  

Thus, in order to prove Theorem 3.2 we need to show that

$$||\pi N Y_\xi||^2 \geq c_R k^{-(m+3)} ||\xi||^2.$$  

By (3.8), we have

$$||\xi||^2 \leq c_R k^m ||Y_\xi||^2 = c_R k^m ||\pi N Y_\xi||^2 + c_R k^m ||\pi T Y_\xi||^2.$$  

Hence (3.9) implies that

$$||\xi||^2 \leq c_R k^m ||\pi N Y_\xi||^2 + c_R c'_R k^{2m+2} ||\pi N Y_\xi||^2 \leq c''_R k^{2m+2} ||\pi N Y_\xi||^2.$$  

□

**Lemma 3.4.** There exists a positive constant $c$ independent of $k$ such that for any $f \in C^\infty(Y)$, we have

$$c \int_Y f^2 \tilde{\omega}_0^n \leq k^m \int_Y \bar{\partial} f \wedge \partial f \wedge \tilde{\omega}_0^{n-1} + k^{-m} \left( \int_Y f \tilde{\omega}_0^n \right)^2.$$  

By (3.8), we have

$$||\xi||^2 \leq c_R k^m ||Y_\xi||^2 = c_R k^m ||\pi N Y_\xi||^2 + c_R k^m ||\pi T Y_\xi||^2.$$  

Hence (3.9) implies that

$$||\xi||^2 \leq c_R k^m ||\pi N Y_\xi||^2 + c_R c'_R k^{2m+2} ||\pi N Y_\xi||^2 \leq c''_R k^{2m+2} ||\pi N Y_\xi||^2.$$  

□
Proof. In the proof of this Lemma, we put $\omega_k = \omega_0 + k\omega_\infty$ and $\alpha = \omega_1 = \omega_0 + \omega_\infty$. For $k \geq 1$, we have

$$k^{-m}\omega_k^n \leq \alpha^n \leq \omega_k^n.$$ 

Assume that the statement is false. So, there exists a subsequence $k_j \to \infty$ and a sequence of functions $f_j$ such that $\int_Y f_j^2 \omega_{k_j}^n = 1$ and

$$k^m \int_Y \nabla f_j \wedge \nabla f_j \wedge \omega_{k_j}^{n-1} + k_j^{-m} \left( \int_Y f_j \omega_{k_j}^n \right)^2 \to 0$$

as $k \to \infty$. We define $||f||^2 = \int_Y f^2 \alpha^n$. Hence

$$||f_j||^2 = \int_Y f_j^2 \alpha^n \geq k_j^{-m} \int_Y f_j^2 \omega_{k_j}^n = k_j^{-m}.$$ 

Let $g_j = f_j / ||f_j||$. We have

$$\int_Y |\nabla g_j|^2 \alpha^n = \int_Y \nabla g_j \wedge \nabla g_j \wedge \alpha^{n-1}$$

$$= ||f_j||^{-2} \int_Y \nabla f_j \wedge \nabla f_j \wedge \alpha^{n-1}$$

$$\leq k_j^m \int_Y \nabla f_j \wedge \nabla f_j \wedge \omega_{k_j}^{n-1} \to 0 \quad \text{as} \quad k \to \infty.$$ 

On the other hand $\int_Y g_j^2 \alpha^n = 1$ which implies that the sequence $g_j$ is bounded in $L^2_1(\alpha^n)$. Hence, $g_j$ has a subsequence which converges in $L^2(\alpha^n)$ and converges weakly in $L^2_1(\alpha^n)$ to a function $g \in L^2_1(\alpha^n)$. Without loss of generality, we can assume that the whole sequence converges. Since $\int_Y |\nabla g_j|^2 \alpha^n \to 0$ as $k \to \infty$, it can be easily seen that $g$ is a constant function. We have

$$k_j^{-m} \int_Y (g_j - g) \omega_{k_j}^n \leq k_j^{-m} \int_Y |g_j - g| \omega_{k_j}^n$$

$$\leq \int_Y |g_j - g| \alpha^n$$

$$\leq C \left( \int_Y |g_j - g|^2 \alpha^n \right)^{\frac{1}{2}} \to 0,$$

where $C^2 = \int_Y \alpha^n$ does not depend on $k$. Hence

$$k_j^{-m} \int_Y (g_j - g) \omega_{k_j}^n \to 0.$$ 

Since $g$ is a constant function and $\int_Y \omega_{k_j}^n = n_2 k_j^m + O(k_j^{m-1})$, we get

$$k_j^{-m} \int_Y g_j \omega_{k_j}^n \to n_2 g.$$
where \( n_2 \) is defined by (3.2). On the other hand

\[
\left( k_j^{-m} \int_Y g_j \omega^n_{k_j} \right)^2 = k_j^{-2m} ||f_j||^{-2} \left( \int_Y f_j \omega^n_{k_j} \right)^2 \\
\leq k_j^{-m} \left( \int_Y f_j \omega^n_{k_j} \right)^2 \to 0
\]

which implies \( g \equiv 0 \). It is a contradiction since \( ||g_j|| = 1 \) and \( g_j \to g \) in \( L^2(\alpha^n) \).

\[\square\]

The proof of the following lemma can be found in ([PS2, p. 704]). For the sake of completeness, we give the details.

**Lemma 3.5.** There exists a positive constant \( c_R \) independent of \( k \) such that for any Kähler form \( \tilde{\omega} \in c_1(\mathcal{O}(1) \otimes L^k) \) having \( R \)-bounded geometry and any \( f \in C^\infty(Y) \), we have

\[ c_R \int_Y f^2 \tilde{\omega}^n \leq k^m \int_Y \partial \bar{\partial} f \wedge \bar{\partial} f \wedge \tilde{\omega}^{n-1} + k^{-m} \left( \int_Y f \tilde{\omega}^n \right)^2. \]

**Proof.** Since \( \tilde{\omega} \) has \( R \)-bounded geometry, we have

\[ R^{-1} \tilde{\omega}_0 \leq \tilde{\omega} \leq 2R \tilde{\omega}_0. \]

Therefore,

\[ c(2R)^{-n} \int_Y f^2 \tilde{\omega}^n \leq c \int_Y f^2 \tilde{\omega}^n \leq k^m \int_Y \partial \bar{\partial} f \wedge \bar{\partial} f \wedge \tilde{\omega}_0^{n-1} + k^{-m} \left( \int_Y f \tilde{\omega}_0^n \right)^2. \]

On the other hand, there exists a unique function \( \phi \) such that \( \tilde{\omega} - \tilde{\omega}_0 = \partial \bar{\partial} \phi \) and \( \int_Y \phi \tilde{\omega}_0^n = 0 \). Hence,

\[ \tilde{\omega}^n - \tilde{\omega}_0^n = \partial \bar{\partial} \phi \wedge \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \tilde{\omega}_0^{n-j-1}. \]
We have,
\[
\left| \int f(\tilde{\omega}^n - \tilde{\omega}_0^n) \right| = \left| \int f \partial \bar{\partial} \phi \wedge \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \tilde{\omega}_0^{n-j-1} \right|
\]
\[
= \left| \int \partial f \wedge \partial \phi \wedge \sum_{j=0}^{n-1} \tilde{\omega}^j \wedge \tilde{\omega}_0^{n-j-1} \right|
\]
\[
\leq \sum_{j=0}^{n-1} \int |\partial f|_{\tilde{\omega}_0^n} |\partial \phi|_{\tilde{\omega}_0^n} \left( \frac{\tilde{\omega}}{\tilde{\omega}_0^n} \right)^p \tilde{\omega}_0^n
\]
\[
\leq n(2R)^n \int |\partial f|_{\tilde{\omega}_0^n} |\partial \phi|_{\tilde{\omega}_0^n}
\]
\[
\leq C_1 \left( \sqrt{\int |\partial f|^2_{\tilde{\omega}_0^n} \tilde{\omega}_0^n} \right)^{\frac{1}{2}} \left( \int |\partial \phi|^2_{\tilde{\omega}_0^n} \tilde{\omega}_0^n \right)^{\frac{1}{2}}
\]
\[
= C_1 \left( \left( \int \partial f \wedge \partial \phi \wedge \tilde{\omega}_0^{n-1} \right)^{\frac{1}{2}} \left( \int |\partial \phi|^2_{\tilde{\omega}_0^n} \tilde{\omega}_0^n \right)^{\frac{1}{2}} \right.
\]
\[
\left. \leq C_2 k^{m} \left( \int \frac{1}{2} k^n \left( \int |\partial \phi|^2_{\tilde{\omega}_0^n} \tilde{\omega}_0^n \right) \frac{1}{2} \right) \right)
\]
\[
\leq Ck^m \left( \int |\partial \phi|^2_{\tilde{\omega}_0^n} \tilde{\omega}_0^n \right)^{\frac{1}{2}}
\]

We will show that
\[
\int |\partial \phi|^2_{\tilde{\omega}_0^n} \tilde{\omega}_0^n \leq C_2 k^m.
\]

Since \( \tilde{\omega} - \tilde{\omega}_0 = \partial \bar{\partial} \phi \) and \( ||\tilde{\omega} - \tilde{\omega}_0||_{C^\alpha(\tilde{\omega}_0)} \leq R \), we have \( ||\partial \bar{\partial} \phi||_{C^\alpha(\tilde{\omega}_0)} \leq R \).

This implies that
\[
||\tilde{\omega} - \tilde{\omega}_0||_{C^\alpha(\tilde{\omega}_0)} \leq R.
\]

Applying Lemma 3.3 to \( \phi \), we get
\[
c \int_Y \phi^2 \tilde{\omega}_0^n \leq k^m \int_Y \bar{\partial} \phi \wedge \partial \phi \wedge \tilde{\omega}_0^{n-1}
\]

On the other hand
\[
\int_Y |\partial \phi|^2_{\tilde{\omega}_0^n} \tilde{\omega}_0^n = \int_Y \bar{\partial} \phi \wedge \partial \phi \wedge \tilde{\omega}_0^{n-1} = \left| \int_Y \phi \Delta_{\tilde{\omega}_0} \phi \tilde{\omega}_0^n \right|
\]
\[
\leq \left( \int_Y \phi^2 \tilde{\omega}_0^n \right)^{\frac{1}{2}} \left( \int_Y |\Delta_{\tilde{\omega}_0} \phi|^2 \tilde{\omega}_0^n \right)^{\frac{1}{2}}
\]
\[
\leq c \frac{1}{2} k^m \left( \int_Y |\partial \phi|^2_{\tilde{\omega}_0^n} \tilde{\omega}_0^n \right)^{\frac{1}{2}} \left( R^2 \int_Y \tilde{\omega}_0^n \right)^{\frac{1}{2}}
\]
\[
= Ck^m \left( \int_Y |\partial \phi|^2_{\tilde{\omega}_0^n} \tilde{\omega}_0^n \right)^{\frac{1}{2}}
\]

Therefore,
\[
\int |\partial \phi|^2_{\tilde{\omega}_0^n} \tilde{\omega}_0^n \leq C_2 k^m.
\]
So, we get
\[
\left| \int f(\bar{\omega}^n - \bar{\omega}_0^n) \right| \leq C k^m \left( \int_Y \overline{\partial f} \wedge \partial f \wedge \bar{\omega}_0^{n-1} \right)^{1/2}
\]

On the other hand
\[
\frac{1}{2} \left( \int_Y f \bar{\omega}_0^n \right)^2 \leq \left( \int_Y f \bar{\omega}^n \right)^2 + \left( \int_Y f (\bar{\omega}^n - \bar{\omega}_0^n) \right)^2
\]

Hence,
\[
\tilde{C} \int_Y f^2 \bar{\omega}^n \leq k^m \int_Y \overline{\partial f} \wedge \partial f \wedge \bar{\omega}_0^{n-1} + 2 k^{-m} \left( \int_Y f \bar{\omega}^n \right)^2 + \left( \int_Y f (\bar{\omega}^n - \bar{\omega}_0^n) \right)^2
\]
\[
\leq k^m \int_Y \overline{\partial f} \wedge \partial f \wedge \bar{\omega}_0^{n-1} + 2 k^{-m} \left( \int_Y f \bar{\omega}^n \right)^2 + C_3 k^m \int_Y \overline{\partial f} \wedge \partial f \wedge \bar{\omega}_0^{n-1}
\]
\[
\leq C_4 k^m \int_Y \overline{\partial f} \wedge \partial f \wedge \bar{\omega}_0^{n-1} + k^{-m} \left( \int_Y f \bar{\omega}^n \right)^2.
\]

\[\square\]

**Proposition 3.6.** There exists a positive constant \( c_R \) such that for any \( \xi \in su(N) \), we have
\[
||\xi||^2 \leq c_R k^m ||Y_\xi||^2,
\]
where \( ||.|| \) in the right hand side denotes the \( L^2 \)-norm with respect to the Kahler form \( \bar{\omega} \) on \( Y \) and Fubini-Study metric on the fibres.

**Proof.** By (3.5), we have
\[
|Y_\xi|^2 = \frac{-4(z^* \xi z)^2 - (z^* \xi z^2 z) (z^* z)}{(z^* z)^2}
\]
This implies that
\[
||Y_\xi||^2_{L^2(\bar{\omega})} = tr(\xi^* \xi \int z z^* \bar{\omega}^n) - \int \frac{(z^* \xi z)^2}{(z^* z)^2} \bar{\omega}^n
\]
\[
= tr(\xi^* \xi \int z z^* \bar{\omega}^n) - \int \phi^2 \bar{\omega}^n.
\]
We can write
\[
\int_{Y} z z^* \bar{\omega}^n = D^{(k)} I + M^{(k)},
\]
where \( D^{(k)} \to n_2/n_1 \) as \( k \to \infty \) and \( M^{(k)} \) is a trace free hermitian matrix with \( ||M^{(k)}||_{op} \leq \epsilon \). Therefore,
\[
||Y_\xi||^2 = ||\xi||^2 D^{(k)} + tr(\xi^* \xi M^{(k)}) - \int \phi^2 \bar{\omega}^n.
\]
Hence
\[
|tr(\xi^* \xi M^{(k)})| = |tr(\xi M^{(k)})| \leq ||\xi||^2 ||M^{(k)}||_{op} \leq \epsilon ||\xi||^2.
\]
Since $D^{(k)} \to n_2/n_1$ as $k \to \infty$, there exists a positive constant $c$ such that

$$||Y_\xi||^2 \geq c||\xi||^2 - \int \phi^2 \tilde{\omega}^n.$$ 

On the other hand

$$\left| \int \phi \tilde{\omega}^n \right| = |tr(\xi M^{(k)})| \leq \sqrt{N}||\xi||||M^{(k)}||_{op}$$

$$\leq ck \frac{m}{2} ||\xi||||M^{(k)}||_{op}.$$ 

Now applying Lemma 3.5, we get

$$C \int_Y \phi^2 \tilde{\omega}^n \leq k^m \int_Y \tilde{\partial} \phi \wedge \partial \phi \wedge \tilde{\omega}^{n-1} + k^{-m} \left( \int_Y \phi \tilde{\omega}^n \right)^2$$

$$\leq k^m \int_Y \tilde{\partial} \phi \wedge \partial \phi \wedge \tilde{\omega}^{n-1} + c_2 ||\xi||^2 ||M^{(k)}||_{op}^2.$$ 

This implies

$$(c_1 - C_2 ||M^{(k)}||_{op}^2) ||\xi||^2 \leq ||Y_\xi||^2 + k^m \int_Y \tilde{\partial} \phi \wedge \partial \phi \wedge \tilde{\omega}^{n-1}.$$ 

Since $||M^{(k)}||_{op} \leq \epsilon$ and $\epsilon$ is small enough, there exists a positive constant $c$ such that

$$c ||\xi||^2 \leq ||Y_\xi||^2 + k^m \int_Y \tilde{\partial} \phi \wedge \partial \phi \wedge \tilde{\omega}^{n-1}$$

$$= ||Y_\xi||^2 + k^m \int_Y |\tilde{\partial} \phi|^2 \tilde{\omega}^n.$$ 

We know that $\tilde{\partial} \phi|_Y = \iota_{\pi^* Y_\xi} \tilde{\omega}$ which implies

$$c ||\xi||^2 \leq ||Y_\xi||^2 + k^m ||\pi^* Y_\xi||^2.$$ 

$\square$

**Lemma 3.7.** For $k \gg 0$, we have

$$|S(\omega_0 + k\omega_\infty)| \leq C \log k,$$

where $S$ is the scalar curvature.

**Proof.** We have

$$\omega_0 + k\omega_\infty = m \sum_{j=0} \binom{n}{k} k^j \omega_0^{m-j} \wedge \omega_\infty^j = (1 + \sum_{j=1}^m k^j f_j) \omega_0^m,$$
will be some smooth nonnegative functions $f_j$ on $Y$. The function $f_m$ is positive, since $\omega_0^n \wedge \omega_\infty^m$ is a volume form. Therefore there exists a positive constant $l$ such that $f_m \geq l > 0$. We define

$$F = \sum_{j=1}^m k^{j-m} f_j.$$ 

We have

$$\nabla^2 \log(1 + k^m F) = \nabla \left( \frac{k^m \nabla F}{1 + k^m F} \right) = \frac{k^m \nabla^2 F}{1 + k^m F} - \frac{k^{2m} (\nabla F)^2}{(1 + k^m F)^2}.$$ 

Hence there exists a positive constant $C$ such that

$$\log(1 + k^m F) \leq mC \log k + C,$$

since $||F||_{C^2}$ is bounded independent of $k$ and $F \geq f_m \geq l > 0$. This implies that

$$\left| \frac{\partial \overline{\partial} \log \det(\omega_0 + k\omega_\infty)}{\partial \overline{\partial} \log \det(\omega_0 + k\omega_\infty)} \right|_{C^0} \leq \left| \log \det(\omega_0 + k\omega_\infty) \right|_{C^2} = \left| \log(\omega_0 + k\omega_\infty)^n \right|_{C^2} \leq \left| \log \omega_0^n \right|_{C^2} + \left| \log(1 + k^m F) \right|_{C^2} \leq C_1 + C_2 m \log k.$$

Fix a point $p \in Y$ and a holomorphic local coordinate $z_1, ..., z_n$ around $p$ such that

$$\omega_0(p) = i \sum d z_i \wedge d \overline{z_i},$$

$$\omega_\infty(p) = i \sum \lambda_i dz_i \wedge d \overline{z_i},$$

where $\lambda_i$’s are some nonnegative real numbers. Therefore, we have

$$|S(\omega_0 + k\omega_\infty)(p)| = \left| \sum (1 + k\lambda_i)^{-1} \partial_i \partial^{\overline{i}} \log \det(\omega_0 + k\omega_\infty) \right| \leq \sum (1 + k\lambda_i)^{-1} (C_1 + C_2 m \log k) \leq C_3 \log k,$$

for $k \gg 0$.

**Proposition 3.8.** For any holomorphic vector field $V$ on $\mathbb{P}^{N-1}$, we have

$$|\pi_N V|^2 \geq C_R k^{-1} |\overline{\partial}(\pi_N V)|^2.$$ 

**Proof.** The following is from ([PS2, pp. 705-708]). For the sake of completeness, we give the details of the proof. Fix $x \in Y$. Let $e_1, ..., e_n, f_1, ..., f_m$ be a local holomorphic frame for $\iota^* T\mathbb{P}^{N-1}$ around $x$ such that

1. $e_1(x), ..., e_n(x), f_1(x), ..., f_m(x)$ form an orthonormal basis.
(2) \( e_1, \ldots, e_n \) is a local holomorphic basis for \( TY \).

Then there exist holomorphic functions \( a_j \) and \( b_j \)'s such that

\[
V = \sum a_j e_j + \sum b_j f_j.
\]

Notice that \( \pi_N f_j - f_j \) is tangent to \( Y \) since \( \pi_N'(\pi_N f_j - f_j) = 0 \). Therefore, we can write

\[
\pi_N f_j - f_j = \sum \phi_{ij} e_j,
\]

where \( \phi_{ij} \)'s are smooth functions. Since \( e_1(x), \ldots, e_n(x), f_1(x), \ldots, f_m(x) \) form an orthonormal basis, we have \( \phi_{ij}(x) = 0 \). Then

\[
\pi_N V = \sum_{j=1}^m b_j (f_j - \sum_i \phi_{ij} e_i).
\]

It implies that

\[
\bar{\partial}(\pi_N V) = \sum_{j=1}^m b_j (-\sum_i (\bar{\partial} \phi_{ij}) e_i).
\]

So in order to establish 3.8, we need to prove that

\[
\sum_{i=1}^n \sum_{j=1}^m |b_j \bar{\partial} \phi_{ij}|^2 \leq C_R^{-1} k \sum_{j=1}^m |b_j|^2.
\]

Using the Cauchy-Schwartz inequality, it suffices to prove

\[
\sum_{i=1}^n \sum_{j=1}^m |\bar{\partial} \phi_{ij}|^2 \leq C_2 k;
\]

where \( C_2 = C_2(R) \) is independent of \( k \) (depends on \( R \)). Now the matrix \( A^* = (\bar{\partial} \phi_{ij}) \) is the dual of the second fundamental form \( A \) of \( TY \) in \( \iota^* T\mathbb{P}^{N-1} \). Let \( F_{\iota^* T\mathbb{P}^{N-1}} \) be the curvature tensor of the bundle \( \iota^* T\mathbb{P}^{N-1} \) with respect to the Fubini-Study metric. \( F_{\iota^* T\mathbb{P}^{N-1}} \) is a 2-form on \( Y \) with values in \( \text{End}(\iota^* T\mathbb{P}^{N-1}) \). Thus \( F_{\iota^* T\mathbb{P}^{N-1}}|_{TY} \) is a two form on \( Y \) with values in \( \text{Hom}(TY, \iota^* T\mathbb{P}^{N-1}) \). So, \( \pi_T \circ (F_{\iota^* T\mathbb{P}^{N-1}}|_{TY}) \) is a two form on \( Y \) with values in \( \text{End}(TY) \). Also let \( F_{TY} \) be the curvature tensor of the bundle \( TY \) with respect to the pulled back Fubini-Study metric \( \tilde{\omega} = \iota^* \omega_{FS} \). Now by computations in [PS2, 5.28], we have

\[
\sum_{i=1}^n \sum_{j=1}^m |\bar{\partial} \phi_{ij}|^2 = \Lambda_{\tilde{\omega}} Tr [\pi_T \circ (F_{\iota^* T\mathbb{P}^{N-1}}|_{TY}) - F_{TY}],
\]

where \( \Lambda_{\tilde{\omega}} \) is the contraction with the Kähler form \( \tilde{\omega} \). The formula [PS2, 5.33] gives

\[
\Lambda_{\tilde{\omega}} Tr [\pi_T \circ (F_{\iota^* T\mathbb{P}^{N-1}}|_{TY})] = n + 1.
\]
On the other hand, $\Lambda_e Tr(F_{TY})$ is the scalar curvature of the metric $\tilde{\omega}$ on $Y$. Since $\tilde{\omega}$ has $R$-bounded geometry, we have

$$|S(\tilde{\omega}) - S(\tilde{\omega}_0)| \leq R.$$ 

Lemma 3.7 implies that $|S(\tilde{\omega}_0)| \leq C \log k \leq Ck$.

The only thing we need in addition is the following

**Proposition 3.9.** Assume that there are no nonzero holomorphic vector fields on $Y$. Then there exists a constant $c'_R$ such that for any $\xi \in su(N)$, we have

$$c'_R \|\pi_T Y_\xi\|^2 \leq k^{m+2} \|\pi_N Y_\xi\|^2.$$ 

**Proof.** We define $\alpha = \omega_0 + \omega_\infty$. Since there are no holomorphic vector fields on $Y$, for any smooth smooth vector field $W$ on $Y$, we have

$$c \|W\|_{L^2(\alpha)}^2 \leq \|\partial W\|_{L^2(\alpha)}^2.$$ 

The trivial inequalities $k\alpha \geq \tilde{\omega}_0$ and $k^{-m} \tilde{\omega}_0^n \leq \alpha^n \leq \tilde{\omega}_0^n$ imply that

$$c \|W\|_{L^2(\tilde{\omega}_0)}^2 = c \int |W|^2 \tilde{\omega}_0^n \leq c k^{m+1} \int |W|^2 \alpha^n$$

$$\leq k^{m+1} \int |\partial W|^2 \alpha^n$$

$$\leq k^{m+1} \int |\partial W|^2 \tilde{\omega}_0^n$$

$$= k^{m+1} \|\partial W\|_{L^2(\tilde{\omega}_0)}^2.$$ 

Hence, there exists a positive constant $c$ depends on $R$ and independent of $k$, such that for any $\tilde{\omega}_0$ having $R$-bounded geometry, we have

$$c \|W\|_{L^2(\tilde{\omega})}^2 \leq k^{m+1} \|\partial W\|_{L^2(\tilde{\omega})}^2.$$ 

Now, putting $W = \pi_T Y_\xi$, we get

$$c \|\pi_T Y_\xi\|_{L^2(\tilde{\omega})}^2 \leq k^2 \|\partial (\pi_T Y_\xi)\|_{L^2(\tilde{\omega})}^2.$$ 

On the other hand

$$\|\pi_N V\|^2 \geq C_R k^{-1} \|\partial (\pi_N V)\|^2,$$

which implies the desired inequality.
4. Perturbing To A Balanced Metric

We continue with the notation of the previous section. The goal of this section is to prove Theorem 4.6 which gives a condition for when an almost balanced metric can be perturbed to a balanced one. In order to do this, first we need to establish Theorem 4.5. We need the following estimate.

**Proposition 4.1.** There exist positive real numbers $K_j$ depends only on $h_0$, $g_\infty$ and $j$ such that for any $s \in H^0(Y, \mathcal{O}(1) \otimes L^k)$, we have

$$|\nabla^j s|^2_{c^0(\omega_0)} \leq K_j k^{n+j} \int_Y |s|^2 \frac{\omega_0^n}{n!}.$$ 

In order to prove Proposition 4.1 we start with some complex analysis.

Let $\varphi$ be a strictly plurisubharmonic function and $\psi$ be a plurisubharmonic function on $B = B(2) \subset \mathbb{C}^n$ such that $\varphi(0) = \psi(0) = 0$. We can find a coordinate on $B(2)$ such that

$$\varphi(z) = |z|^2 + O(|z|^2) \quad \text{and} \quad \psi(z) = \sum \lambda_i |z_i|^2 + O(|z|^2),$$

where $\lambda_i \geq 0$. For any function $u : B \to \mathbb{C}$, we define $u^{(k)}(z) = u(\frac{z}{\sqrt{k}})$.

**Theorem 4.2.** (Cauchy Estimate cf. [Ho, Theorem 2.2.3]) There exist positive real numbers $C_j$ such that for any holomorphic function $u : B \to \mathbb{C}$, we have

$$|\nabla^j u|^2(0) \leq C_j \int_{|z| \leq 1} |u(z)|^2 dz \wedge d\bar{z}.$$ 

**Theorem 4.3.** There exist positive real numbers $c_j$ depends only on $j, \varphi, \psi$ and $d\mu$ such that for any holomorphic function $u : B \to \mathbb{C}$, we have

$$|\nabla^j u|^2(0) \leq c_j k^{n+j} \int_{B(1)} |u|^2 e^{-\varphi - k\psi} d\mu,$$

where $d\mu$ is a fixed volume form on $B$.

**Proof.** Applying Cauchy estimate to $u^{(k)}$, we get

$$k^{-j} |\nabla^j u|^2(0) \leq C_j \int_{|z| \leq 1} |u^{(k)}(z)|^2 dz \wedge d\bar{z} \leq C \int_{|z| \leq 1} |u^{(k)}(z)|^2 e^{-\sum (\lambda_i + 1)} |z_i|^2 dz \wedge d\bar{z},$$

since $e^{-\sum (\lambda_i + 1)} |z_i|^2$ is bounded from below by a positive constant on the unit ball. Using the change of variable $w = \frac{z}{\sqrt{k}}$ we get
\[
-k^j |\nabla^j u|^2(0) \leq C k^n \int_{|w| \leq k^{-1/2}} |u(w)|^2 e^{-k \sum (\lambda_i + 1)|w_i|^2} dw \wedge d\overline{w}
\]
\[
\leq C k^n \int_{|w| \leq k^{-1/2}} |u(w)|^2 e^{-\sum (k \lambda_i + 1)|w_i|^2} dw \wedge d\overline{w}.
\]

On the other hand, we have
\[
\varphi(z) + k \psi(z) = k \sum (\lambda_i + 1)|z_i|^2 + \mu(z) + k \sigma(z),
\]
where \(\lim_{z \to 0} \frac{\mu(z)}{|z|^2} = \lim_{z \to 0} \frac{\sigma(z)}{|z|^2} = 0\).

Let \(|w| \leq k^{-1/2}\), we have
\[
|k \sigma(w) + \mu(w)| \leq c (|w|^2 + |w|^2) \leq 2c
\]
for some constant \(c\) depending only on \(\psi\) and \(\varphi\). Hence
\[
k^{-j} |\nabla^j u|^2(0) \leq C k^n \int_{|w| \leq k^{-1/2}} |u(w)|^2 e^{-k \sum (k \lambda_i + 1)|w_i|^2 + |w|^2 - 2c} dw \wedge d\overline{w}
\]
\[
= C e^{-2c} k^n \int_{|w| \leq k^{-1/2}} |u(w)|^2 e^{-\sum (k \lambda_i + 1)|w_i|^2 - 2c} dw \wedge d\overline{w}
\]
\[
\leq C' k^n \int_{|w| \leq k^{-1/2}} |u(w)|^2 e^{-\sum (k \lambda_i + 1)|w_i|^2 - (\mu(w) + k \sigma(w))} dw \wedge d\overline{w}
\]
\[
= C' k^n \int_{|w| \leq k^{-1/2}} |u(w)|^2 e^{-(\varphi(w) + k \psi(w))} dw \wedge d\overline{w}
\]
\[
\leq C' k^n \int_{B(1)} |u|^2 e^{-\varphi - k \psi} dz \wedge d\overline{z}.
\]

Hence,
\[
|\nabla^j u|^2(0) \leq c_j k^{n+j} \int_{B(1)} |u|^2 e^{-\varphi - k \psi} d\mu.
\]

**Proof of Proposition 4.1.** Fix a point \(p\) in \(Y\) and a geodesic ball \(B \subset Y\) centered at \(p\). Let \(e_L\) be a holomorphic frame for \(L\) on \(B\) and \(e\) be a holomorphic frame for \(\mathcal{O}(1)\) such that \(||e_L||(p) = ||e||(p) = 1\). Any \(s \in H^0(Y, \mathcal{O}(1) \otimes L^k)\) can be written as \(s = ue \otimes e^\otimes_L\) for some holomorphic function \(u : B \to \mathbb{C}\). We have
\[
\nabla^j s = \sum \binom{j}{i} \nabla^i u \otimes \nabla^{j-i}(e \otimes e^\otimes_L).
\]
Therefore,
\[ |\nabla^j s|^2(p) \leq C(\sum |\nabla^i u|^2(p)||\nabla^{j-i}(e \otimes e^\otimes e_L)|^2(p)). \]
On the other hand we have
\[ ||\nabla^\alpha (e \otimes e^\otimes e_L)||^2(p) \leq \alpha \sum \left( ||\nabla^i e||^2(p) + k^{\alpha-i} ||\nabla^{\alpha-i} e_L||^2(p) \right) \leq C_k \alpha^2. \]
Hence
\[ |\nabla^j s|^2(p) \leq C'(\sum |\nabla^i u|^2(p)k^{j-i}). \]
Applying Theorem 4.3 concludes the proof.

For the rest of this section, we fix a positive integer \( q \). We continue with the notation \( (Y, \omega_\infty, \omega_0, \tilde{\omega}_0) \) of section 3. In the rest of this section, we fix the reference metric \( \omega_0 \) on \( Y \) and recall the Definition 3.2.

**Definition 4.1.** The sequence of hermitian metrics \( h_k \) on \( \mathcal{O}(1) \otimes L^k \) and ordered bases \( s_k^{(k)} = (s_1^{(k)}, ..., s_N^{(k)}) \) for \( H^0(Y, \mathcal{O}(1) \otimes L^k) \) is called **almost balanced of order** \( q \) if for any \( k \)
\[ \sum |s_i^{(k)}|^2_{h_k} = 1 \]
and
\[ \int_Y \langle s_i^{(k)}, s_j^{(k)} \rangle_{h_k} dvol_{h_k} = D^{(k)} \delta_{ij} + M^{(k)}_{ij}, \]
where \( D^{(k)} \) is a scalar so that \( D^{(k)} \rightarrow n_2/n_1 \) as \( k \rightarrow \infty \) (See (3.1) and (3.2)), and \( M^{(k)} \) is a trace-free hermitian matrix such that \( ||M^{(k)}||_{op} = O(k^{-q-1}) \).

We state the following lemma without proof. The proof is a straightforward calculation.

**Lemma 4.4.** Let the sequence of hermitian metrics \( h_k \) on \( \mathcal{O}(1) \otimes L^k \) and ordered bases \( s_k^{(k)} = (s_1^{(k)}, ..., s_N^{(k)}) \) for \( H^0(Y, \mathcal{O}(1) \otimes L^k) \) be almost balanced of order \( q \). Suppose
\[ ||\tilde{\omega}_k - \tilde{\omega}_0||_{C^a(\tilde{\omega}_0)} = O(k^{-1}), \]
where \( \tilde{\omega}_k = \text{Ric}(h_k) \). Then for any \( \epsilon > 0 \) there exists a positive integer \( k_0 \) such that
\[ \tilde{\omega}_k \geq (1 - \epsilon)\tilde{\omega}_0 \text{ for } k \geq k_0. \]
Assume that there exist a sequence of almost balanced metrics \( h_k \) of order \( q \) and bases \( s_k^{(k)} = (s_1^{(k)}, ..., s_N^{(k)}) \) for \( H^0(Y, \mathcal{O}(1) \otimes L^k) \) which satisfies (4.1). As before \( \tilde{\omega}_k = \text{Ric}(h_k) \). Then Lemma 4.4 implies that for \( k \gg 0, \tilde{\omega}_k \) has \( R \)-bounded geometry.
Fix $k$ and let $B \in isu(N_k)$. Without loss of generality, we can assume that $B$ is the diagonal matrix $\text{diag}(\lambda_i)$, where $\lambda_i \in \mathbb{R}$ and $\sum \lambda_i = 0$. There exists a unique hermitian metric $h_B$ on $OPE(1) \otimes L^k$ such that
\[
\sum e^{2\lambda_i} |s^{(k)}_i|^2_{h_B} = 1.
\]
Let $\tilde{\omega}_B = \text{Ric}(h_B)$. In the next theorem, we will prove that there exist a constant $c$ and open balls $U_k \subset isu(N_k)$ around the origin of radius $ck^{-(n+a+2)}$ so that if $B \in U_k$, then $h_B$ is $R$-bounded. More precisely,

**Theorem 4.5.** Suppose that (4.1) holds.

- There exist $c > 0$ and $k_0 > 0$ such that if $k \geq k_0$ and $B \in isu(N_k)$ satisfies
  \[
  \| B \|_{op} \leq ck^{-(n+a+2)} R,
  \]
  then the metric $\tilde{\omega}_B$ is $R$-bounded.
- There exists $c > 0$ such that if $B \in isu(N_k)$ satisfies
  \[
  \| B \|_{op} \leq k^{-(n+a+3)},
  \]
  then
  \[
  \| M_B \|_{op} \leq ck^{-1},
  \]
  where the matrix $M_B = (M_{ij}^B)$ is defined by
  \[
  M_{ij}^B = e^{\lambda_i + \lambda_j} \int Y (s^{(k)}_i, s^{(k)}_j) \tilde{\omega}_B^n = \frac{V_k}{N_k} \delta_{ij}.
  \]

**Proof.** Let $h_B = e^{\varphi_B} h_k$. So, we have
\[
1 = \sum e^{2\lambda_i} |s^{(k)}_i|^2_{h_B} = e^{\varphi_B} \sum e^{2\lambda_i} |s^{(k)}_i|^2_{h_k}.
\]
Hence
\[
\varphi_B = - \log \sum e^{2\lambda_i} |s^{(k)}_i|^2_{h_k} = - \log \left( 1 + \sum (e^{2\lambda_i} - 1) |s^{(k)}_i|^2_{h_k} \right).
\]
If $\| B \|_{op}$ is small enough, there exists $C > 0$ so that
\[
\| \varphi_B \|_{C^{a+2}(\tilde{\omega}_0)} \leq C \| B \|_{op} \sum_{i=1}^{N_k} |\nabla^{a+2} s^{(k)}_i|^2_{C^0(\tilde{\omega}_0)}
\]
and therefore Proposition 4.1 implies that

\[ \|\varphi_{B}\|_{C^{n+a+2}(\tilde{\omega}_0)} \leq C\|B\|_{\text{op}} k^{n+a+2} \sum_{i=1}^{N_k} \int_Y |s_i^{(k)}|_{h_k}^2 \frac{\omega_0^n}{n!} \]

\[ = C\|B\|_{\text{op}} k^{n+a+2} \int_Y \sum_{i=1}^{N_k} |s_i^{(k)}|_{h_k}^2 \frac{\omega_0^n}{n!} \]

\[ = C\|B\|_{\text{op}} k^{n+a+2} \int_Y \frac{\omega_0^n}{n!} = c_1 \|B\|_{\text{op}} k^{n+a+2} \]

for some positive constant \(c_1\). Now if \(\|B\|_{\text{op}} \leq c_1^{-1} R - \frac{1}{2R} k^{-(n+a+2)}\), then

\[ (4.2) \quad \|\varphi_{B}\|_{C^{n+2}(\tilde{\omega}_0)} \leq \frac{R - 1}{2R}. \]

Therefore,

\[ \|i\partial_\phi \varphi_{B}\|_{C^n(\tilde{\omega}_0)} \leq \frac{R - 1}{2R}, \]

which implies that

\[ (4.3) \quad i\partial_\phi \varphi_{B} \geq -\frac{R - 1}{2R} \tilde{\omega}_0. \]

In order to show that \(\tilde{\omega}_B\) is \(R\)-bounded, we need to prove the following:

\[ (4.4) \quad \|\tilde{\omega} - \tilde{\omega}_0\|_{C^n(\tilde{\omega}_0)} \leq R, \]

\[ (4.5) \quad \tilde{\omega}_B \geq \frac{1}{R} \tilde{\omega}_0. \]

To prove (4.4), (4.1) and (4.2) imply that for \(k \gg 0\)

\[ \|\tilde{\omega}_B - \tilde{\omega}_0\|_{C^n(\tilde{\omega}_0)} \leq \|\tilde{\omega}_B - \tilde{\omega}_k\|_{C^n(\tilde{\omega}_0)} + \|\tilde{\omega}_k - \tilde{\omega}_0\|_{C^n(\tilde{\omega}_0)} \]

\[ \leq \|\varphi_{B}\|_{C^{n+2}(\tilde{\omega}_0)} + k^{-1} \leq \frac{R - 1}{2R} + k^{-1} \]

\[ \leq R. \]

To prove (4.5), applying Lemma 4.4 with \(\epsilon = \frac{R-1}{2R}\) gives

\[ \tilde{\omega}_k \geq \frac{R + 1}{2R} \tilde{\omega}_0, \]

and therefore (4.3) implies

\[ \tilde{\omega}_B - \frac{1}{R} \tilde{\omega}_0 = \tilde{\omega}_k + i\partial_\phi \varphi_{B} - \frac{1}{R} \tilde{\omega}_0 \geq \tilde{\omega}_k - \frac{R + 1}{2R} \tilde{\omega}_0 \geq 0, \]

for \(k \gg 0\).
In order to prove the second part, by a unitary change of basis, we may assume without loss of generality that the matrix $M^B$ is diagonal. By definition

$$M^B_{ij} = e^{\lambda_i + \lambda_j} \int_Y F(s_i, s_j) \frac{\tilde{\omega}_B^n}{n!} - \frac{V_k}{N_k} \delta_{ij},$$

where

$$F = e^{-\varphi_B} \frac{\tilde{\omega}^n}{\tilde{\omega}_k^n}.$$

We have

$$M^B_{ii} = e^{2\lambda_i} \int_Y F|s_i|^2 \frac{\tilde{\omega}^n_k}{n!} - \frac{V_k}{N_k} \delta_{ij} = e^{2\lambda_i} \int_Y F|s_i|^2 \frac{\tilde{\omega}^n_k}{n!} - \int_Y |s_i|^2 \frac{\tilde{\omega}^n_k}{n!} + (M^{(k)})_{ii}$$

$$= \int_Y (e^{2\lambda_i} F - 1)|s_i|^2 \frac{\tilde{\omega}^n_k}{n!} + (M^{(k)})_{ii}.$$

Therefore,

$$|M^B_{ii}| \leq ||e^{2\lambda_i} F - 1||_\infty (\int_Y |s_i|^2 \frac{\tilde{\omega}^n_k}{n!} + (M^{(k)})_{ii}) \leq C(||e^{2\lambda_i} F - 1||_\infty + k^{-q-1}).$$

Define $f = \frac{\tilde{\omega}^n_B}{\tilde{\omega}_k^n}$. If $||B||_\text{op} \leq k^{-(n+a+3)}$, then

$$|f - 1| = \left| \frac{\tilde{\omega}^n_B - \tilde{\omega}^n_k}{\tilde{\omega}^n_k} \right| = O(k^{-1})$$

and

$$|(e^{2\lambda_i - \varphi_B} - 1)| = O(k^{-1}).$$

Therefore,

$$||e^{2\lambda_i} F - 1|| = ||e^{2\lambda_i - \varphi_B} \frac{\tilde{\omega}^n_B}{\tilde{\omega}^n_k} - 1|| = ||e^{2\lambda_i - \varphi_B} f - 1|| \leq ||(e^{2\lambda_i - \varphi_B} - 1)(f - 1)|| + ||(f - 1)||$$

$$+ ||(e^{2\lambda_i - \varphi_B} - 1)|| = O(k^{-1}),$$

which implies that

$$||M^B||_\text{op} = O(k^{-1}).$$

\qed
Remark Proposition 2.2 to get balanced metrics for $k$ notation of Proposition 2.2, we have
and $k \epsilon$ for $| | (4.6)$
Therefore, balanced and $| |$
Theorem 4.6. Suppose that the sequence of metrics $h_k$ on $O(1) \otimes L^k$ and bases $s^k = (s^k_1, ..., s^k_k)$ for $H^0(Y, O(1) \otimes L^k)$ is almost balanced of order $q$. Suppose that \[ 4.1 \] holds for
\[ \tilde{\omega}_k = Ric(h_k) \quad \text{and} \quad \omega_k = Ric(h_k) - k \omega_\infty. \]
If $q > \frac{5m}{2} + n + a + 5$, then $(Y, O(1) \otimes L^k)$ admits balanced metric for $k \gg 0$.

Proof. Let $R > 1$ and $k$ be a fixed large positive integer. Let $\sigma \in isu(N)$, where $N = N_k = \dim H^0(Y, O(1) \otimes L^k)$. If $|||\sigma|||_{op} \leq \frac{2}{7} k^{-(n+a+3)} R$, then Theorem 4.5 implies that $e^\sigma s$ has $R$-bounded geometry and $|||M^\sigma|||_{op} \leq \epsilon$ for $k \gg 0$, where $\epsilon$ is the constant in the statement of Theorem 3.2. Thus, Theorem 3.2 implies that $\Lambda(e^\sigma s^{(k)}) \leq C k^{2m+2} = \lambda$. With the notation of Proposition 2.2, we have $\mu(z_0) = M^{(k)}$. Therefore
\[ |\mu(z_0)| = |M^{(k)}| \leq \sqrt{N_k} |||M^{(k)}|||_{op} \leq C' k^{\frac{m}{2} - q}. \]
Letting $\delta = \frac{\epsilon}{2} k^{-(n+a+3)} R$, we have $\lambda |\mu(z_0)| < \delta$ if $q > \frac{5m}{2} + n + a + 5$ and $k \gg 0$. Therefore if $q > \frac{5m}{2} + n + a + 5$ and $k \gg 0$, we can apply Proposition 2.2 to get balanced metrics for $k \gg 0$.

Remark 4.7. By Proposition 2.2 there exists $\sigma_0$ such that $e^{\sigma_0} s$ is balanced and $|\sigma_0| \leq (C k^{2m+2})(C' k^{\frac{m}{2} - q}) = C'' k^{\frac{5m}{2}+2-q}$. Since
\[ |||\varphi_{\sigma_0}|||_{C^{a+2}} \leq C_1 k^{n+a+2} |||\sigma_0|||_{op} \leq C_1 k^{n+a+2} |\sigma_0|, \]
then
\[ |||\varphi_{\sigma_0}|||_{C^{a+2}} \leq C k^{\frac{5m}{2} + n+a+4-q}. \]
Therefore,
\[ (4.6) \quad |||\omega_k^{bal} - \tilde{\omega}_k|||_{C^a(\omega_0)} = O(k^{-1}). \]

5. Asymptotic Expansion

The goal of this section is to prove Theorem 1.3. Theorem 1.3 gives an asymptotic expansion for the Bergman kernel of $(\mathbb{P} E^*, O_{\mathbb{P} E^*}(1) \otimes \pi^* L^k)$. We obtain such an expansion by using the Bergman kernel asymptotic expansion proved in ([C], [Z]). Also we compute the first nontrivial coefficient of the expansion. In the next section, we use this to construct sequence of almost balanced metrics. We start with some linear algebra.

Let $V$ be a hermitian vector space of dimension $r$. The projective space $\mathbb{P}V^*$ can be identified with the space of hyperplanes in $V$ via
\[ f \in V \rightarrow ker(f) = V_f \subseteq V. \]
If \( f \neq 0 \) then \( V_f \) will be a hyperplane. There is a natural isomorphism between \( V \) and \( H^0(\mathbb{P}V^*, \mathcal{O}_{\mathbb{P}V^*}(1)) \) which sends \( v \in V \) to \( \hat{v} \in H^0(\mathbb{P}V^*, \mathcal{O}_{\mathbb{P}V^*}(1)) \) such that for any \( f \in V^* \), \( \hat{v}(f) = f(v) \). Now we can see that the inner product on \( V \) induces an inner product on \( V^* \) and then a metric on \( \mathcal{O}_{\mathbb{P}V^*}(1) \). For \( v, w \in V \) and \( f \in V^* \) we define

\[
<v, w>_g = \frac{f(v) f(w)}{|f|^2}.
\]

**Definition 5.1.** For any inner product \( h \) on \( V \), we denote the induced metric on \( \mathcal{O}_{\mathbb{P}V^*}(1) \) by \( \hat{h} \).

The following is a straightforward computation.

**Proposition 5.1.** For any \( v, w \in V \) we have

\[
<v, w>_h = C_r^{-1} \int_{\mathbb{P}V^*} <\hat{v}, \hat{w}>_{\hat{h}} \frac{\omega_{FS}^{r-1}}{(r-1)!}
\]

where \( C_r \) is a constant defined by

\[
C_r = \int_{C^{r-1}} \frac{d\xi \wedge d\xi}{(1 + \sum_{j=1}^{r-1} |\xi_j|^2)^{r+1}}.
\]

Here \( d\xi \wedge d\xi = (\sqrt{-1} d\xi_1 \wedge d\xi_1) \wedge \cdots \wedge (\sqrt{-1} d\xi_{r-1} \wedge d\xi_{r-1}) \).

**Definition 5.2.** For any \( v \in V \), we define an endomorphism of \( V \) by

\[
\lambda(v, h) = \frac{1}{|v|^2_h} v \otimes v^{*h},
\]

where \( v^{*h}(.) = h(., v) \).

Let \( (X, \omega) \) be a Kähler manifold of dimension \( m \) and \( E \) be a holomorphic vector bundle on \( X \) of rank \( r \). Let \( L \) be an ample line bundle on \( X \) endowed with a Hermitian metric \( \sigma \) such that \( \text{Ric}(\sigma) = \omega \). For any hermitian metric \( h \) on \( E \), we define the volume form

\[
d\mu_g = \frac{\omega_g^{r-1}}{(r-1)!} \wedge \frac{\pi^* \omega^m}{m!},
\]

where \( g = \hat{h}, \omega_g = \text{Ric}(g) = \text{Ric}(\hat{h}) \) and \( \pi : \mathbb{P}E^* \to X \) is the projection map. The goal is to find an asymptotic expansion for the Bergman kernel of \( \mathcal{O}_{\mathbb{P}E^*}(1) \otimes L^k \to \mathbb{P}E^* \) with respect to the \( L^2 \)-metric defined on \( H^0(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes \pi^* L^k) \). We define the \( L^2 \)- metric using the fibre metric \( g \otimes \sigma^\otimes k \) and the volume form \( d\mu_{g,k} \) defined as follows

\[
d\mu_{g,k} = k^{-m} (\omega_g + k\omega)^{m+r-1} = \sum_{j=0}^{m} k^{j-m} \frac{\omega_g^{m+r-1-j}}{(m+r-j)!} \wedge \frac{\omega^j}{j!}.
\]
In order to do that, we reduce the problem to the problem of Bergman kernel asymptotics on $E \otimes L^k \to X$. The first step is to use the volume form $d\mu_g$ which is a product volume form instead of the more complicated one $d\mu_{g,k}$. So, we replace the volume form $d\mu_{g,k}$ with $d\mu_g$ and the fibre metric $g \otimes \sigma^k$ with $g(k) \otimes \sigma^k$, where the metrics $g(k)$ are defined on $\mathcal{O}_{\mathbb{P}^*}(1)$ by

$$g(k) = k^{-m}(\sum_{j=0}^{m} k^j f_j)g = (f_m + k^{-1}f_{m-1} + \ldots + k^{-m}f_0)g,$$

and

$$\frac{\omega_g^{m+r-1-j}}{(m+r-j)!} \wedge \frac{\omega^j}{j!} = f_j d\mu_g.$$  

Clearly the $L^2$-inner products $L^2(g \otimes \sigma^k, d\mu_{g,k})$ and $L^2(g(k) \otimes \sigma^k, d\mu_g)$ on $H^0(\mathbb{P}^*, \mathcal{O}_{\mathbb{P}^*}(1) \otimes \pi^* L^k)$ are the same. The second step is going from $\mathcal{O}_{\mathbb{P}^*}(1) \to \mathbb{P}^*$ to $E \to X$. In order to do this we somehow push forward the metric $g(k)$ to get a metric $\tilde{g}(k)$ on $E$ (See Definition 5.5). Then we can apply the result on the asymptotics of the Bergman kernel on $E$. The last step is to use this to get the result.

**Definition 5.3.** Let $\hat{s}^1, \ldots, \hat{s}_N^k$ be an orthonormal basis for $H^0(\mathbb{P}^*, \mathcal{O}_{\mathbb{P}^*}(1) \otimes \pi^* L^k)$ w.r.t. $L^2(g \otimes \sigma^k, d\mu_{g,k})$. We define

$$\rho_k(g, \omega) = \sum_{i=1}^{N} |\hat{s}_i^k|^2_{g \otimes \sigma^k}.$$  

**Definition 5.4.** For any $(j,j)$-form $\alpha$ on $X$, we define the contraction $\Lambda^j_{\omega} \alpha$ of $\alpha$ with respect to the Kähler form $\omega$ by

$$\frac{m!}{(m-j)!} \alpha \wedge \omega^{m-j} = (\Lambda^j_{\omega} \alpha) \omega^m.$$  

In this section we fix the Kähler form $\omega$ on $X$ and therefore simply denote $\Lambda^j_{\omega} \alpha$ by $\Lambda^j \alpha$.

**Lemma 5.2.** Let $\nu_0$ be a fixed Kähler form on $X$. For any positive integer $p$ there exists a constant $C$ such that for any $(j,j)$-form $\gamma$, we have

$$\|\nabla^p (\Lambda^j \gamma)\| \leq \frac{C}{\inf_{x \in X} |\omega(x)|^m |\nu_0(x)|} (\|\gamma\|_{C^p(\nu_0)} + \|\Lambda^j \gamma\|_{C^{p-1}(\nu_0)}) \left(\sum_{i=1}^{m} \|\omega\|_{C^i(\nu_0)}^i\right).$$
Proof. Let $\gamma$ be a $(j, j)$-form. By definition, we have

$$(\Lambda^j \gamma) \omega_m = \frac{m!}{(m-j)!} \gamma \wedge \omega^{m-j}.$$ 

Therefore for any positive integer $p$, we have

$$\nabla^p((\Lambda^j \gamma) \omega_m) = \frac{m!}{(m-j)!} \nabla^p(\gamma \wedge \omega^{m-j}).$$

Applying Leibnitz rule, we get

$$\sum_{i=0}^{p} \binom{p}{i} \nabla^i (\Lambda^j \gamma) \nabla^{p-i} \omega_m = \frac{m!}{(m-j)!} \sum_{i=0}^{p} \binom{p}{i} \nabla^i \gamma \wedge \nabla^{p-i} \omega_m.$$ 

Thus there exists a positive constant $C'$ so that

$$||\nabla^p(\Lambda^j \gamma)\omega_m||_{C^0(\nu_0)} \leq C'(||\omega_m||_{C^p(\nu_0)} ||\Lambda^j \gamma||_{C^{p-1}(\nu_0)} + ||\gamma||_{C^p(\nu_0)} ||\omega^{m-j}||_{C^p(\nu_0)}).$$

On the other hand there exists constant $c_{p,j}$ such that for any any $0 \leq j \leq m-1$,

$$||\omega^{m-j}||_{C^p(\nu_0)} \leq c_{p,j}||\omega||_{C^p(\nu_0)}^{m-j} \leq c_{p,j}(\sum_{i=1}^{m} ||\omega||_{C^p(\nu_0)}^i).$$

Hence there exists a constant $C$ such that

$$||\nabla^p(\Lambda^j \gamma)|| \leq \frac{C}{\inf_{x \in X} ||\omega||_{C^p(\nu_0)}} (||\gamma||_{C^p(\nu_0)} ||\Lambda^j \gamma||_{C^{p-1}(\nu_0)} (\sum_{i=1}^{m} ||\omega||_{C^p(\nu_0)}^i)).$$

Definition 5.5. For any hermitian form $g$ on $\mathcal{O}_{PE^*}(1)$, we define a hermitian form $\widetilde{g}$ on $E$ as follow

$$(5.6) \quad \widetilde{g}(s, t) = C_r^{-1} \int_{PE^*_r} g(\hat{s}, \hat{t}) \frac{\omega_r^{-1}}{(r-1)!},$$

for $s, t \in E_x$. (See (5.1) for definition of $C_r$.)

Notice that if $g = \hat{h}$ for some hermitian metric $h$ on $E$, Proposition 5.1 implies that $\widetilde{g} = h.$ Define hermitian metrics $\widetilde{g}_j$’s on $E$ by

$$(5.7) \quad \widetilde{g}_j(s, t) = C_r^{-1} \int_{PE^*_r} f_j g(\hat{s}, \hat{t}) \frac{\omega_r^{-1}}{(r-1)!},$$

for $s, t \in E_x$. Also we define $\Psi_j \in End(E)$ by

$$(5.8) \quad \widetilde{g}_j = \Psi_j h.$$
Proposition 5.3. Let $\nu_0$ be a fixed Kähler form on $X$ as in Lemma 5.2. For any positive numbers $l$ and $l'$ and any positive integer $p$, there exists a positive number $C_{l,l',p}$ such that if

$$||\omega||_{C^p(\nu_0)}, ||h||_{C^{p+2}(\nu_0)} \leq l$$

and

$$\inf_{x \in X} |\omega(x)|^m_{\nu_0(x)} \geq l',$$

then

$$||\Psi_i||_{C^p(\nu_0)} \leq C_{l,l',p}$$

for any $1 \leq i \leq m$.

Proof. Fix a point $p \in X$. Let $e_1, \ldots, e_r$ be a local holomorphic frame for $E$ around $p$ such that

$$\langle e_i, e_j \rangle_h(p) = \delta_{ij}, \quad d\langle e_i, e_j \rangle_h(p) = 0$$

and

$$\frac{i}{2\pi} F_h(p) = \begin{pmatrix} \omega_1 & 0 & \cdots & 0 \\ 0 & \omega_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_r \end{pmatrix}.$$ 

Let $\lambda_1, ..., \lambda_r$ be the homogeneous coordinates on the fibre. At the fixed point $p$, we have

$$\omega_g = \omega_{FS,g} + \frac{\sum \omega_i |\lambda_i|^2}{\sum |\lambda_i|^2}.$$ 

Therefore,

$$\omega_g^{r+j-1} \wedge \omega^{m-j} = \binom{r+j-1}{r-1} \omega_{FS,g}^{r-1} \wedge \left( \frac{\sum \omega_i |\lambda_i|^2}{\sum |\lambda_i|^2} \right)^j \wedge \omega^{m-j}.$$ 

Definition of $f_{m-j}$ gives

$$f_{m-j} \omega_g^{r-1} \wedge \omega^m = \binom{m}{j} \omega_g^{r-1} \wedge \left( \frac{\sum \omega_i |\lambda_i|^2}{\sum |\lambda_i|^2} \right)^j \wedge \omega^{m-j}.$$ 

Hence

$$f_{m-j} \omega_{FS,g}^{r-1} \wedge \omega^m = \binom{m}{j} \omega_{FS,g}^{r-1} \wedge \left( \frac{\sum \omega_i |\lambda_i|^2}{\sum |\lambda_i|^2} \right)^j \wedge \omega^{m-j}.$$ 

Therefore,

$$\omega_{FS,g}^{r-1} \wedge \left( f_{m-j} \omega^m - \binom{m}{j} \left( \frac{\sum \omega_i |\lambda_i|^2}{\sum |\lambda_i|^2} \right)^j \wedge \omega^{m-j} \right) = 0,$$

which implies
\[ f_{m-j} \omega^m = \binom{m}{j} \left( \frac{\sum \omega_i |\lambda_i|^2}{\sum |\lambda_i|^2} \right)^j \cdot \omega^{m-j} \]

Simple calculation gives

\[
\int_{C^{r-1}} |\lambda_\alpha|^2 |\lambda_1|^{2j_1} \cdots |\lambda_{r-1}|^{2j_{r-1}} |d\lambda \wedge d\bar{\lambda}| = \frac{C_r r! j_1! \cdots j_r! (j_\alpha + 1)}{(r+j)!},
\]

when \(j_1 + \cdots + j_r = j\) and \(1 \leq \alpha \leq r\). Hence

\[
(5.9) \quad \bar{g}_{m-j}(e_\alpha, e_\alpha) = C_r^{-1} \pi_* \left( f_{m-j} g(e_\alpha, e_\alpha) \right) \left( \frac{\omega_g^{r-1}}{(r-1)!} \right)
\]

\[
= \frac{r!}{(r+j)!} \Lambda^j \left( \sum_{j_1+\cdots+j_r=j} (j_\alpha + 1) \omega_1^{j_1} \wedge \cdots \wedge \omega_r^{j_r} \right).
\]

From theory of symmetric functions, one can see that there exist polynomials \(P_i(x_1, \ldots, x_j)\) of degree \(i\) such that

\[
\Psi_{m-j} = \Lambda^j \left( F_h^j + P_1(c_1(h), \ldots, c_j(h)) F_h^{j-1} + \cdots + P_j(c_1(h), \ldots, c_j(h)) \right),
\]

where \(c_i(h)\) is the \(i\) th chern form of \(h\). Since \(||h||_{C^{p+2}(\nu_0)} \leq l\), there exists a positive constant \(c'\) such that

\[
||F_h^j + \cdots + P_j(c_1(h), \ldots, c_j(h))||_{C^p(\nu_0)} \leq c'(1+l)^j.
\]

Therefore Lemma 5.2 implies that

\[
||\nabla^p \Psi_{m-j}|| \leq \frac{C}{l'} (c'(1+l)^j + ||\Psi_{m-j}||_{C^{p-1}(\nu_0)}) (1+l)^m,
\]

since

\[
\inf_{x \in X} |\omega(x)^m|_{\nu_0(x)} \geq l'
\]

and

\[
\sum_{i=1}^m ||\omega||^i_{C^p(\nu_0)} \leq \sum_{i=1}^m l^i \leq (1+l)^m.
\]

On the other hand

\[
||\Psi_{m-i}||_{C^p(\nu_0)} = ||\nabla^p \Psi_{m-j}|| + ||\Psi_{m-j}||_{C^{p-1}(\nu_0)}
\]

\[
\leq \frac{C}{l'} (c'(1+l)^j + ||\Psi_{m-j}||_{C^{p-1}(\nu_0)}) (1+l)^m + ||\Psi_{m-i}||_{C^{p-1}(\nu_0)}.
\]

Now we can conclude the proof by induction on \(p\). \(\square\)
Lemma 5.4. We have the following

(1) \( \Psi_m = I_E \).

(2) \( \Psi_{m-1} = \frac{i}{2\pi(r+1)} (Tr(\Lambda F h) I_E + \Lambda F h) \).

Proof. The first part is an immediate consequence of Proposition 5.1 and the definition of \( \Psi_m \). For the second part, we use the notation used in the proof of Proposition 5.3. It is easy to see that for \( \alpha \neq \beta \), we get \( \tilde{g}_{m-1}(e_\alpha, e_\beta) = 0 \). On the other hand by plugging \( j = 1 \) in (5.9), we get

\[ \tilde{g}_{m-1}(e_\alpha, e_\alpha) = \frac{1}{(r+1)} (Tr(\Lambda F h) + \Lambda \omega_\alpha). \]

\( \square \)

The following lemmas are straightforward.

Lemma 5.5. \( g \otimes \sigma^k = \tilde{g} \otimes \sigma^k \).

Lemma 5.6. Let \( s_1, ..., s_N \) be a basis for \( H^0(X, E) \). Then

\[ \sum |\tilde{s}_i([v^*])|^2_h = Tr(B \lambda(v^*, h)), \]

where \( B = \sum s_i \otimes s_i^h \).

Proof of Theorem 1.3. We define the metric \( h(k) \) on \( E \) by

\[ h(k) = \sum_{j=0}^m k^{j-m} \tilde{g}_j = (\sum_{j=0}^m k^{j-m} \Psi_j) h. \]

Let \( B_k(h(k), \omega) \) be the Bergman kernel of \( E \otimes L^k \) with respect to the \( L^2 \)-metric defined by the hermitian metric \( h(k) \otimes \sigma^k \) on \( E \otimes L^k \) and the volume form \( \frac{\omega^m}{m!} \) on \( X \). Therefore, if \( s_1, ..., s_N \) is an orthonormal basis for \( H^0(X, E \otimes L^k) \) with respect to the \( L^2(H(k) \otimes \sigma^k, \frac{\omega^m}{m!}) \), then

\[ B_k(h(k), \omega) = \sum s_i \otimes s_i^{h(k) \otimes \sigma^k}, \]

We define \( \tilde{B}_k(h, \omega) \) as follow

\[ \tilde{B}_k(h, \omega) = \sum s_i \otimes s_i^{h(k) \otimes \sigma^k}. \]

Let \( \tilde{s}_1, ..., \tilde{s}_N \) be the corresponding basis for \( H^0(\mathcal{P}E^*, \mathcal{O}_{\mathcal{P}E^*}(1) \otimes L^k) \). Hence,

\[ \int_{\mathcal{P}E^*} \langle \tilde{s}_i, \tilde{s}_j \rangle g \otimes \sigma^k d\mu_{g,k} = \int_{\mathcal{P}E^*} \langle \tilde{s}_i, \tilde{s}_j \rangle g \otimes \sigma^k (\sum_{j=0}^m k^j f_j) d\mu_g. \]
\[
\int_{\mathbb{P}E^*} \langle \hat{s}_i, \hat{s}_j \rangle_{g(k) \otimes \sigma^k} d\mu_g = C_r \int_X \langle s_i, s_j \rangle_{h(k) \otimes \sigma^k} \frac{\omega^m}{m!} = C_r \delta_{ij}.
\]

Therefore \( \frac{1}{\sqrt{C_r}} \hat{s}_1, \ldots, \frac{1}{\sqrt{C_r}} \hat{s}_N \) is an orthonormal basis for \( H^0(\mathbb{P}E^*, \mathcal{O}_{\mathbb{P}E^*}(1) \otimes L^k) \) with respect to \( L^2(g \otimes \sigma^k, d\mu_{k,g}). \) Hence Lemma 5.6 implies

\[ C_r \rho_k(g) = \text{Tr} \left( \lambda(v^*, h) \tilde{B}_k(h, \omega) \right). \]

Now, in order to conclude the proof, it suffices to show that there exist smooth endomorphisms \( A_i \in \Gamma(X, E) \) such that

\[ \tilde{B}_k(h, \omega) \sim k^m + A_1 k^{m-1} + \ldots. \]

Let \( B_k(h, \omega) \) be the Bergman kernel of \( E \otimes L^k \) with respect to the \( L^2(h \otimes \sigma^k). \) A fundamental result on the asymptotics of the Bergman kernel ([C], [Z]) states that there exists an asymptotic expansion

\[ B_k(h, \omega) \sim k^m + B_1(h) k^{m-1} + \ldots, \]

where

\[ B_1(h) = \frac{i}{2\pi} \Lambda F(E,h) + \frac{1}{2} S(\omega) I_E. \]

(See also [BBS], [W2].) Moreover this expansion holds uniformly for any \( h \) in a bounded family. Therefore, we can Taylor expand the coefficients \( B_i(h) \)'s. We conclude that for endomorphisms \( \Phi_1, \ldots, \Phi_M, \)

\[ B_k(h(I + \sum_{i=0}^M k^{-i} \Phi_i), \omega) \sim k^m + B_1(h) k^{m-1} + \ldots. \]

Note that \( B_1(h) \) in the above expansion does not depend on \( \Phi_i \)'s and is given as before by

\[ B_1(h) = \frac{i}{2\pi} \Lambda F(E,h) + \frac{1}{2} S(\omega) I_E. \]

On the other hand

\[ B_k(h(k), \omega) = \sum s_i \otimes s_i^{*_{h(\otimes \sigma^k)}} = (\sum s_i \otimes s_i^{*_{h(\otimes \sigma^k)}})(\sum_{j=0}^m k^{j-m} \Psi_j) \]

\[ = \tilde{B}_k(h, \omega)(\sum_{j=0}^m k^{j-m} \Psi_j). \]

Therefore,

\[ \tilde{B}_k(h, \omega) = B_k(h(k), \omega)(\sum_{j=0}^m k^{j-m} \Psi_j)^{-1} \sim k^m + (B_1(h) - \Psi_{m-1}) k^{m-1} + \ldots \]

Notice that Proposition 5.3 implies that if \( h \) and \( \omega \) vary in a bounded family and \( \omega \) is bounded from below, then \( \Psi_1, \ldots, \Psi_m \) vary in a bounded
family. Therefore the asymptotic expansion that we obtained for \( \tilde{B}_k(h, \omega) \) is uniform as long as \( h \) and \( \omega \) vary in a bounded family and \( \omega \) is bounded from below.

\[ \square \]

**Proposition 5.7.** Suppose that \( \omega_\infty \in 2\pi c_1(L) \) be a Kähler form with constant scalar curvature and \( h_{HE} \) be a Hermitian-Einstein metric on \( E \), i.e.

\[ \Lambda_{\omega_\infty} F_{(E, h_{HE})} = \mu I_E, \]

where \( \mu \) is the \( \omega_\infty \)-slope of the bundle \( E \). We have

\[
A_{1,1} := \left. \frac{d}{dt} \right|_{t=0} A_1(h_{HE}(I + t\phi), \omega_\infty + it\overline{\partial}\partial \eta) = \frac{r+1}{2r} D^* D \eta I_E + \frac{i}{2\pi} \left( (\Lambda_{\omega_\infty} \overline{\partial}\partial \Phi + \Lambda_{\omega_\infty}^2 (F_{h_{HE}} \wedge (i\overline{\partial}\partial \eta))) \right) - \frac{1}{r} \text{tr}(\Lambda_{\omega_\infty} \overline{\partial}\partial \Phi) + \Lambda_{\omega_\infty}^2 (F_{h_{HE}} \wedge (i\overline{\partial}\partial \eta)),
\]

where \( D^* D \) is Lichnerowicz operator (cf. [D3, Page 515]).

**Proof.** Define

\[ f(t) = \Lambda_{\omega_\infty + it\overline{\partial}\partial \eta} F_{(h_{HE}(I+t\phi))} \]

Therefore, we have

\[ mf_{(h_{HE}(t+\phi))} (\omega_\infty + it\overline{\partial}\partial \eta)^{m-1} = f(t)(\omega_\infty + it\overline{\partial}\partial \eta)^m. \]

Differentiating with respect to \( t \) at \( t = 0 \), we obtain

\[ m\overline{\partial}\partial \Phi \wedge \omega_\infty^{m-1} + m(m-1) F_{h_{HE}} \wedge (i\overline{\partial}\partial \eta) \wedge \omega_\infty^{m-2} = f'(0) \omega_\infty^m + mf(0)(i\overline{\partial}\partial \eta) \wedge \omega_\infty^{m-1}. \]

Since \( f(0) = \mu I_E \), we get

\[ f'(0) = \Lambda_{\omega_\infty} \overline{\partial}\partial \Phi + \Lambda_{\omega_\infty}^2 (F_{h_{HE}} \wedge (i\overline{\partial}\partial \eta)) - \mu \Lambda_{\omega_\infty} (i\overline{\partial}\partial \eta) I_E. \]

On the other hand (cf. [D3, pp. 515, 516].)

\[ \frac{d}{dt} \left. \right|_{t=0} S(\omega_\infty + it\overline{\partial}\partial \eta) = D^* D \eta. \]

\[ \square \]

**Corollary 5.8.** Suppose that \( \text{Aut}(X, L)/\mathbb{C}^\ast \) is discrete and \( E \) is stable. Then the map \( A_{1,1} : \Gamma_0(\text{End}(E)) \rightarrow \Gamma_0(\text{End}(E)) \) is an isomorphism, where \( \Gamma_0(\text{End}(E)) \) is the space of smooth endomorphisms \( \Phi \in E \) such that \( \int_X \text{tr}(\Phi) \omega_\infty^m = 0. \)
Proof. First, notice that $\Gamma_0(End(E)) = \Gamma_{00}(End(E)) \oplus C_0^\infty(X)$, where $\Gamma_{00}(End(E))$ is the space of trace-free endomorphisms of $E$ and $C_0^\infty(X)$ is the space of smooth functions $\eta$ on $X$ such that $\int_X \eta \omega = 0$. Assume that $A_{1,1}(\Phi, \eta) = 0$, where $\Phi \in \Gamma_{00}(End(E))$ and $\eta \in C_0^\infty(X)$. Hence

$$r + \frac{1}{2r} D^* D \eta = 0,$$

and

$$\frac{i}{2\pi} \left( (\Lambda_{\omega} \overline{\partial} \Phi + \Lambda_{\omega}^2 (F_{hHE} \wedge (i \overline{\partial} \eta))) - \frac{1}{r} tr(\Lambda_{\omega} \overline{\partial} \Phi) + \Lambda_{\omega}^2 (F_{hHE} \wedge (i \overline{\partial} \eta)) \right) = 0$$

Since $Aut(X, L)/C^*$ is discrete, the first equation implies that $\eta$ is constant and therefore $\eta = 0$. So, the second equation reduces to the following

$$\Lambda_{\omega} \overline{\partial} \Phi - \frac{1}{r} tr(\Lambda_{\omega} \overline{\partial} \Phi) = 0$$

It implies that

$$\Lambda_{\omega} \overline{\partial} \Phi = 0,$$

since $\Phi$ is traceless. Hence simplicity of $E$ implies that $\Phi = 0$ (cf. [K]).

In order to prove surjectivity let $\Psi \in \Gamma_0(End(E))$. We know that the map

$$\eta \in C_0^\infty \to D^* D \eta \in C_0^\infty$$

is surjective since $Aut(X, L)/C^*$ is discrete (cf. [D3] pp. 515, 516)). Hence we can find $\eta_0$ such that $D^* D \eta_0 = tr(\Psi)$. On the other hand

$$\frac{i}{2\pi} \left( \Lambda_{\omega}^2 (F_{h} \wedge (i \overline{\partial} \eta_0)) - \frac{1}{r} tr(\Lambda_{\omega}^2 (F_{h} \wedge (i \overline{\partial} \eta_0))) + \Psi - \frac{1}{r} tr(\Psi) \right) \in \Gamma_0(End(E)).$$

The map

$$\Phi \in \Gamma_0(End(E)) \to \frac{i}{2\pi} \Lambda_{\omega} \overline{\partial} \Phi \in \Gamma_0(End(E))$$

is surjective since $E$ is simple. Hence, we can find $\phi_0$ such that $A_{1,1}(\phi_0, \eta_0) = \Psi$.

\[\square\]

6. Constructing Almost Balanced Metrics

Let $h_\infty$ be a hermitian metric on $L$ such that $\omega_\infty = Ric(h_\infty)$ be a Kähler form with constant scalar curvature and $h_{HE}$ be the corresponding Hermitian-Einstein metric on $E$, i.e.

$$\Lambda_{\omega} F_{(E, h_{HE})} = \mu I_E,$$

where $\mu$ is the slope of the bundle $E$. Let $\omega_0 = Ric(h_{HE})$. After tensoring by high power of $L$, we can assume without loss of generality that $\omega_0$ is a Kähler form on $\mathbb{P}E^*$. We fix an integer $a \geq 4$. In order
to prove the following, we use ideas introduced by Donaldson in ([D3, Theorem 26])

**Theorem 6.1.** Suppose \( \text{Aut}(X,L) \) is discrete. There exist smooth functions \( \eta_1, \eta_2, \ldots \) on \( X \) and smooth endomorphisms \( \Phi_1, \Phi_2, \ldots \) of \( E \) such that for any positive integer \( q \) if

\[
\nu_{k,q} = \omega_\infty + i \overline{\partial} \partial \left( \sum_{j=1}^{q} k^{-j} \eta_j \right)
\]

and

\[
h_{k,q} = h_{HE}(I_E + \sum_{j=1}^{q} k^{-j} \Phi_j),
\]

then

\[
(6.1) \quad \tilde{B}_k(h_{k,q}, \nu_{k,q}) = \frac{C_r N_k}{k-m V_k} (I_E + \delta_q),
\]

where \( ||\delta_q||_{C^{a+2}} = O(k^{-q-1}) \) and \( V_k = \text{Vol}(\mathbb{P} E^*, \mathcal{O}_{\mathbb{P} E^*}(1) \otimes L^k) \) is a topological invariant.

**Proof.** The error term in the asymptotic expansion is uniformly bounded in \( C^{a+2} \) for all \( h \) and \( \omega \) in a bounded family. Therefore there exists a positive integer \( s \) depends only on \( p \) and \( q \) such that

\[
(6.2) \quad A_p(h(1 + \Phi), \omega + i \overline{\partial} \partial \eta) = A_p(h, \omega) + \sum_{j=1}^{q} A_{p,j}(\Phi, \eta)
\]

\[
+ O(||(\Phi, \eta)||_{C^a}^{q+1}),
\]

where \( A_{p,j} \) are homogeneous polynomials of degree \( j \), depending on \( h \) and \( \omega \), in \( \Phi \) and \( \eta \) and its covariant derivatives. Let \( \Phi_1, \ldots, \Phi_q \) be smooth endomorphisms of \( E \) and \( \eta_1, \ldots, \eta_q \) be smooth functions on \( X \). We have

\[
(6.3) \quad A_p(h(1 + \sum_{j=1}^{q} k^{-j} \Phi_j), \omega + i \overline{\partial} \partial (\sum_{j=1}^{q} k^{-j} \eta_j))
\]

\[
= A_p(h, \omega) + \sum_{j=1}^{q} b_{p,j} k^{-j} + O(k^{-q-1}),
\]

where \( b_{p,j} \)'s are multi linear expression on \( \Phi_i \)'s and \( \eta_i \)'s.
Hence

\[
\tilde{B}_k (h (1 + \sum_{j=1}^q k^{-j} \Phi_j), \omega + i \bar{\partial} (\sum_{j=1}^q k^{-j} \eta_j))
\]

\[
= k^m + A_1(h, \omega) k^{m-1} + \ldots
\]

\[
+ (A_q(h, \omega) + b_{q-1,1} + \ldots + b_{1,q-1}) k^{m-q} + O(k^{m-q-1}).
\]

We need to choose \( \Phi_j \) and \( \eta_j \) such that coefficients of \( k^m, \ldots k^{m-q} \) in the right hand side of (6.4) are constant. Donaldson's key observation is that \( \eta_p \) and \( \phi_p \) only appear in the coefficient of \( k^{m-p} \) in the form of \( A_{p}(\phi_p, \eta_p) \). Hence, we can do this inductively. Assume that we choose \( \eta_1, \eta_2, \ldots, \eta_{p-1} \) and \( \Phi_1, \Phi_2, \ldots, \Phi_{p-1} \) so that the coefficients of \( k^m, \ldots k^{m-p+1} \) are constant. Now we need to choose \( \eta_p \) and \( \Phi_p \) such that the coefficient of \( k^{m-p} \) is constant. This means that we need to solve the equation

\[
A_{1,1}(\Phi_p, \eta_p) - c_p I_E = P_{p-1},
\]

for \( \Phi_p, \eta_p \) and the constant \( c_p \). In this equation \( P_{p-1} \) is determined by \( \Phi_1, \ldots, \Phi_{p-1} \) and \( \eta_1, \ldots, \eta_{p-1} \). Corollary 5.8 implies that we can always solve the equation (6.5).

\[\Box\]

Corollary 6.2. For any positive integer \( q \), there exist hermitian metrics \( g_{k,q} \) on \( O_{\bar{E}}(1) \) and Kähler forms \( \nu_{k,q} \) on \( X \) in the class of \( 2\pi c_1(L) \) so that

\[
\rho_k (g_{k,q}, \nu_{k,q}) = \frac{N_k}{k^{-m} V_k} (1 + \epsilon_{k,q}),
\]

where \( ||\epsilon_{k,q}||_{C^{a+2}} = O(k^{-q-1}) \). Moreover,

\[
||\omega_{g_{k,q}} + k \nu_{k,q} - (\omega_0 + k \omega_\infty)||_{C^{a}(\omega_0 + k \omega_\infty)} = O(k^{-1}).
\]

**Proof.** Let \( g_{k,q} = \tilde{h}_{k,q} \). By Theorem 6.1, we have

\[
\rho_k (g_{k,q}, \nu_{k,q}) = \frac{N_k}{k^{-m} V_k} Tr (\lambda (v^*, h_{k,q}) (I_E + \delta_q))
\]

\[
= \frac{N_k}{k^{-m} V_k} (1 + Tr (\lambda (v^*, h_{k,q}) \delta_q))).
\]

The first part of corollary is proved, since \( h_{k,q} \) is bounded and \( ||\delta_{k,q}||_{C^{a+2}} = O(k^{-q-1}) \). Define \( \widetilde{\omega}_0 = \omega_0 + k \omega_\infty \). For the second part, we have

\[
||\omega_{g_{k,q}} + k \nu_{k,q} - (\omega_0 + k \omega_\infty)||_{C^{a}(\omega_0 + k \omega_\infty)} \leq ||\omega_{g_{k,q}} - \omega_0||_{C^{a}(\omega_0)} + k ||\nu_{k,q} - \omega_\infty||_{C^{a}(\omega_0)}
\]

\[
\leq ||\omega_{g_{k,q}} - \omega_0||_{C^{a}(\omega_0)} + k ||\nu_{k,q} - \omega_\infty||_{C^{a}(\omega_\infty)}
\]

\[
= ||\omega_{g_{k,q}} - \omega_0||_{C^{a}(\omega_0)} + ||\nu_{k,q} - \omega_\infty||_{C^{a}(\omega_\infty)}
\]

\[
= O(k^{-1}).
\]
Notice that by definition, we have
\[ \|\omega_{k,q} - \omega_0\|_{C^\alpha(\omega_0)} = O(k^{-1}), \]
\[ \|\nu_{k,q} - \omega_\infty\|_{C^\alpha(\omega_\infty)} = O(k^{-1}). \]

7. Proof of the main theorem

In this section, we prove Theorem 1.2. In order to do that, we want to apply Theorem 4.6. Hence, we need to construct a sequence of almost balanced metrics on \( P_{\mathbb{P}^*} \), \( O_{\mathbb{P}^*}(1) \otimes L^\otimes k \). Also, we need to show that \( \mathbb{P}E^* \) has no nontrivial holomorphic vector fields.

**Proposition 7.1.** Let \( E \) be a holomorphic vector bundle over a compact Kähler manifold \( X \). Suppose that \( X \) has no nonzero holomorphic vector fields. If \( E \) is stable, then \( \mathbb{P}E^* \) has no nontrivial holomorphic vector fields.

**Proof.** Let \( TF \) be the sheaf of tangent vectors to the fibre of \( \pi \). We have the following exact sequence over \( \mathbb{P}E \):
\[ 0 \to TF \to T\mathbb{P}E^* \to \pi^*TX \to 0. \]
This gives the long exact sequence
\[ 0 \to H^0(\mathbb{P}E^*, TF) \to H^0(\mathbb{P}E^*, T\mathbb{P}E^*) \to H^0(\mathbb{P}E^*, \pi^*TX) \to \ldots \]
Since \( H^0(\mathbb{P}E^*, \pi^*TX) = 0 \), we have
\[ H^0(\mathbb{P}E^*, TF) \simeq H^0(\mathbb{P}E^*, T\mathbb{P}E^*) \]
On the other hand, \( \pi_*TF \) may be identified with the sheaf of trace free endomorphisms of \( E \). Therefore by simplicity of \( E \) (cf. [K])
\[ H^0(\mathbb{P}E^*, TF) \simeq H^0(X, \pi_*TF) = 0. \]

**Proof of Theorem 1.2.** Since Chow stability is equivalent to the existence of balanced metric, it suffices to show that \((\mathbb{P}E^*, O_{\mathbb{P}E^*}(1) \otimes \pi^*L^k)\) admits balanced metric for \( k \gg 0 \). Fix a positive integer \( q \). From now on we drop all indexes \( q \) for simplicity. Let \( \sigma_k = \sigma_{k,q} \) be a metric on \( L \) such that \( Ric(\sigma_k) = \nu_k \), where \( \nu_k = \nu_{k,q} \) is the one in the statement of Theorem 2.1. Let \( t_1, \ldots, t_N \) be an orthonormal basis for \( H^0(\mathbb{P}E^*, O_{\mathbb{P}E^*}(1) \otimes L^k) \) w.r.t. \( L^2(g_k \otimes \sigma_{k,q}^k, \frac{(\omega_{g_k} + ku_\pi)^{m+r-1}}{(m+r-1)!}) \). Thus, Corollary 6.2 implies
\[ \sum |t_i|^2_{g_k \otimes \sigma_{k,q}^k} = \frac{N_k}{V_k}(1 + \epsilon_k). \]
Define \( g'_k = \frac{V_k}{N_k}(1 + \epsilon_k)^{-1} g_k \). We have
\[
\sum |t_i|^2 g_{k_i \otimes g_{k_j}} = 1.
\]
This implies that the metric \( g'_k \) is the Fubini-Study metric on \( \mathbb{O}_{\mathbb{P}E^*}(1) \otimes L^k \) induced by the embedding \( t_k^* : \mathbb{P}E^* \to \mathbb{P}^{N-1} \), where \( t = (t_1, \ldots, t_N) \).

We prove that this sequence of embedding is almost balanced of order \( q \), i.e.
\[
\int_{\mathbb{P}E^*} \langle t_i, t_j \rangle g_{k_i \otimes g_{k_j}} \frac{(\omega_{g_k} + k\nu_k)^{m+r-1}}{(m + r - 1)!} = D^{(k)} \delta_{ij} + M_{ij},
\]
where \( M^{(k)} = [M_{ij}] \) is a trace free hermitian matrix, \( D^{(k)} \to C_r \) as \( k \to \infty \) and \( ||M^{(k)}||_\text{op} = O(k^{-q-1}) \).

\[
M_{ij}^{(k)} = \frac{V_k}{N_k} \int_{\mathbb{P}E^*} \langle t_i, t_j \rangle g_{k_i \otimes g_{k_j}} \frac{(\omega_{g_k} + k\nu_k)^{m+r-1}}{(m + r - 1)!} - \frac{V_k}{N_k} \int_{\mathbb{P}E^*} \langle t_i, t_j \rangle g_{k_i \otimes g_{k_j}} \frac{(\omega_{g_k} + k\nu_k)^{m+r-1}}{(m + r - 1)!} \frac{(\omega_{g_k} + k\nu_k)^{m+r-1}}{(m + r - 1)!},
\]
where
\[
(\omega_{g_k} + k\nu_k)^{m+r-1} = f_k(\omega_{g_k} + k\nu_k)^{m+r-1}.
\]

By a unitary change of basis, we may assume without loss of generality that the matrix \( M^{(k)} \) is diagonal. Thus
\[
||M^{(k)}||_\text{op} \leq \frac{V_k}{N_k} ||f_k(1 + \epsilon_k)^{-1} - 1||_{L^\infty}.
\]

On the other hand,
\[
||\omega_{g_k'} - \omega_{g_k}||_{C^q(\omega_0)} = ||\overline{\partial} \log(1 + \epsilon_k)||_{C^q(\omega_0)} \leq ||\log(1 + \epsilon_k)||_{C^q(\omega_0)} \leq - \log(1 - C||\epsilon_k||_{C^q(\omega_0)}) = O(k^{-q-1}).
\]

Therefore,
\[
||f_k - 1|| = \left| \frac{\omega_{g_k'}^{m+r-1} - \omega_{g_k}^{m+r-1}}{\omega_{g_k}^{m+r-1}} \right| = \left| \frac{\omega_{g_k}^{m+r-1} - \omega_{g_k}^{m+r-1}}{\omega_{g_k}^{m+r-1}} \right| \leq Ck^{-q-1} \left| \frac{\omega_{g_k}^{m+r-1}}{\omega_{g_k}^{m+r-1}} \right|.
\]
This implies that  
\[ \|f_k - 1\|_{\infty} \leq C k^{-q-1}, \]
since  
\[ \frac{\omega_0^{m+r-1}}{\omega_{\tilde{\omega}}^{m+r-1}} \]
is bounded. Hence  
\[ \|f_k(1 + \epsilon_k)^{-1} - 1\| \leq C' k^{-q-1}. \]

Therefore  
\[ \|M^{(k)}\|_{op} = O(k^{-q-1}). \]

Proposition 7.1 implies that  \( \mathbb{P}E^* \) has no nontrivial holomorphic vector fields. Therefore, applying Theorem 4.6 and (6.6) conclude the proof.

\[ \square \]

**Remark 7.2.** Since  \( \mathbb{P}E^* \) has no nontrivial holomorphic vector fields, the sequence  \( \omega_k^{\text{bal}} \) of balanced metrics in the class of  \( O_{\mathbb{P}E^*}(1) \otimes L^{\otimes k} \) is unique. Define the sequence of metrics  \( \Omega_k = \omega_k^{\text{bal}} - k \omega_\infty \) in the class of  \( O_{\mathbb{P}E^*}(1) \). A natural question is whether  \( \Omega_k \) converges to  \( \omega_0 \) as  \( k \to \infty \). If  \( \dim \mathbb{C}X = 1 \), then it is easy to see that  \( \Omega_k \) converges to  \( \omega_0 \) in  \( C^\infty \)-norm as  \( k \to \infty \). In general, we have  
\[ \|\Omega_k - \omega_0\|_{C^p(\omega_0)} = \|\omega_k^{\text{bal}} - \tilde{\omega}_0\|_{C^p(\omega_0)} \]
\[ \leq \|\omega_k^{\text{bal}} - \tilde{\omega}_k\|_{C^p(\omega_0)} + \|\tilde{\omega}_k - \tilde{\omega}_0\|_{C^p(\omega_0)} \]

The first term has order of  \( O(k^{-1}) \) by (4.6) and the second term is of order  \( O(1) \). Therefore, in higher dimension one gets  
\[ \|\Omega_k - \omega_0\|_{C^p(\omega_0)} = O(1), \]
for any positive integer  \( p \). It is not clear that whether one can find a sharper estimate with these methods.

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