FedGiA: An Efficient Hybrid Algorithm for Federated Learning

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Abstract—Federated learning has shown its advances recently but is still facing many challenges, such as how algorithms save communication resources and reduce computational costs, and whether they converge. To address these critical issues, we propose a hybrid federated learning algorithm (FedGiA) that combines the gradient descent and the inexact alternating direction method of multipliers. The proposed algorithm is more communication- and computation-efficient than several state-of-the-art algorithms theoretically and numerically. Moreover, it also converges globally under mild conditions.

Index Terms—Federated learning, gradient descent, inexact ADMM, communication-efficiency, computational efficiency, global convergence

I. INTRODUCTION

FEDERATED learning (FL) is burgeoning into an advanced approach in machine learning presently due to the ability to deal with various issues like data privacy, data security, and data access to heterogeneous data. Typical applications include vehicular communications [1], [2], [3], [4], digital health [5], and smart manufacturing [6], just naming a few. The earliest work for FL can be traced back to [7] in 2015 and [8] in 2016. It is still undergoing development and facing many challenges [9], [10], [11].

A. Related work

Gradient descent-based learning. Lately, there is an impressive body of work on developing FL algorithms. One of the most popular approaches benefits from the stochastic gradient descent (SGD). The general framework is to run certain steps of SGD in parallel by clients and then average the resulting parameters from clients by a central server once in a while. Representatives of SGD family consist of the famous federated averaging (FedAvg [12]), its modified version [13], and local SGD (LocalSGD [14], [15]). Other state-of-the-art ones can be found in [16], [17], [18]. These algorithms execute global aggregation (or averaging) periodically and thus can reduce the communication rounds (CR), thereby saving resources (e.g., transmission power and bandwidth in wireless communication) for real-world applications.

However, to establish the convergence theory, most SGD algorithms assume that the local data is identically and independently distributed (i.i.d.), which is unrealistic for FL applications where data is usually heterogeneous. More details can be referred to the local SGD [14], K-step averaging SGD [19], and cooperative SGD [18].

A parallel line of research aims to investigate gradient descent (GD) based-FL algorithms. Since full data is used to construct the gradient, these algorithms do not impose assumptions on distributions of the involved data [20], [21], [22], [23], [24]. Nevertheless, strong conditions on the objective functions of the learning optimization problems are frequently assumed to guarantee convergence, such as the gradient Lipschitz continuity (also known as L-smoothness), convexity, or strong convexity.

ADMM-based learning. The alternating direction method of multipliers (ADMM) has been rapidly developed in theoretical and numerical aspects over the last few decades, with extensive applications into various disciplines. In particular, there is a success of implementation of ADMM in distributed learning [25], [26], [27]. Fairly recently, ADMM-based FL algorithms draw much attention due to its simple structure and easy implementation. These algorithms can be categorized into two classes: exact and inexact ADMM. The former aims at updating local parameters through solving sub-problems exactly, which hence brings more computational burdens for local clients [28], [29], [30], [31], [32].

Therefore, inexact ADMM provides a promising solution to reduce the computational complexity [33], [34], [35], [36], where clients update their parameters via solving sub-problems approximately, thereby alleviating the computational burdens and accelerating the learning speed. Again, we shall emphasize that most of these algorithms need restrictive assumptions to ensure convergence. To overcome this, an algorithm from the primal-dual optimization perspective has been cast in [36] and turns out to be a member of inexact ADMM-based FL algorithms. It is shown that the algorithm converges under weaker assumptions.

B. Our contributions

The main contribution of this paper is to develop a new FL algorithm that is capable of saving communication resources, reducing computational burdens, and converging under relatively weak assumptions.

I) The proposed algorithm, FedGiA in Algorithm 1, has a novel framework. When iteration $k$ is a multiple of a given integer $k_0$, communication occurs between each client and the central server. At the same time, all clients are split into two groups randomly. One group adopts the scheme of the inexact ADMM to update their parameters $k_0$ times, while the second group exploits the GD to update their parameters just once. In summary, FedGiA possesses three advantages as follows.
TABLE I: Comparisons of different algorithms. Assumptions: ① Gradient Lipschitz continuity; ② Bounded level set; ③ Strong convexity; ④ KL property; ⑤ Bounded gradient dissimilarity. Here, \( \beta_1, \beta_2 \), and \( \beta_3 \) are defined in Remark IV.5.

| Algs. | Ref. | Convergence rate | Communication rounds | Assumptions | Computational complexity (for \( k_0 \) steps) |
|-------|------|------------------|---------------------|-------------|---------------------------------------------|
| FedPorx | [37] | \( O\left(\frac{1}{k}\right) \) | \( O\left(\frac{1}{k}\right) \) | ① ⑤ | \( O\left((\beta_3+n)mk_0\right) \) |
| FedAvg | [38] | \( O\left(\frac{1}{\sqrt{m}}\right) \) | \( O\left(\frac{1}{\sqrt{m}}\right) \) | ① ⑤ | \( O\left((\beta_1+n)mk_0\right) \) |
| SCAFFOLD | [38] | \( O\left(\frac{1}{\sqrt{m}}\right) \) | \( O\left(\frac{1}{\sqrt{m}}\right) \) | ① | \( O\left((\beta_2+nl)mk_0\right) \) |
| FedPD | [36] | \( O\left(\frac{1}{\sqrt{m}}\right) \) | \( O\left(\frac{1}{\sqrt{m}}\right) \) | ① | \( O\left((\beta_1+nl)mk_0\right) \) |
| FedGiA | Our | \( O\left(\frac{1}{\sqrt{m}}\right) \) | \( O\left(\frac{1}{\sqrt{m}}\right) \) | ① | \( O\left((\beta_1/n)mk_0\right) \) |

Type-I convergence: \( \| \nabla f(x^k) \|^2 \leq \epsilon \)

| Algs. | Ref. | Convergence rate | Communication rounds | Assumptions | Computational complexity (for \( k_0 \) steps) |
|-------|------|------------------|---------------------|-------------|---------------------------------------------|
| LocalSGD | [14] | \( O\left(\frac{1}{k}\right) \) | \( O\left(\frac{1}{k}\right) \) | ① ⑤ | \( O\left((\beta_2+n)mk_0\right) \) |
| FedAvg | [39] | \( O\left(\frac{1}{nk}\right) \) | \( O\left(\frac{1}{nk}\right) \) | ① ⑤ | \( O\left((\beta_2+n)mk_0\right) \) |
| FedAvg | [38] | \( O\left(\frac{1}{nk}\right) \) | \( O\left(\frac{1}{nk}\right) \) | ① ⑤ | \( O\left((\beta_2+n)mk_0\right) \) |
| SCAFFOLD | [38] | \( O\left(\frac{1}{nk}\right) \) | \( O\left(\frac{1}{nk}\right) \) | ① ⑤ | \( O\left((\beta_2+n)mk_0\right) \) |
| FedGiA | Our | \( O\left(\frac{1}{nk}\right) \) | \( O\left(\frac{1}{nk}\right) \) | ① ⑤ | \( O\left((\beta_2+n)mk_0\right) \) |

Type-II convergence: \( f(x^k) - f^* \leq \epsilon \)

- It is communication-efficient since CR can be controlled by setting \( k_0 \). Our numerical experiments have shown that CR decline when \( k_0 \) increases, see Figure 2.
- It is computation-efficient, which results from two aspects. All local clients take advantage of inexact updates and the key item (i.e., the gradient) need to be computed only once for \( k_0 \) steps. The computational efficiency can be found in Table I and in our numerical comparisons in Table IV.
- It is possible to cope with situations where a portion of clients are in bad conditions. The sever could put them into the second group where less effort is required to update their parameters.

II) The assumptions to guarantee convergence are mild. FedGiA is proven to converge to a stationary point of (4) and to enjoy two types of convergence rate, as shown in Table I. For type-I convergence, FedGiA achieves the sub-linear convergence rate (i.e., \( O(k_0/k) \)) only under the assumption of the gradient Lipschitz continuity. If we further assume the boundedness of a level set and Kurdyka-Lojasiewicz (KL) property, a weaker condition than the strong convexity, then it has the type-II convergence rates better than \( O(k_0/k) \) enjoyed by the other algorithms. However, it is worth mentioning that these rates of convergence established under the KL property are in a local sense, that is, FedGiA converges with such rates when its generated sequence is quite close to the stationary point. Moreover, with the help of the strongly convexity, other algorithms only converge sub-linearly while FedGiA converges linearly, as shown in the last row in Table I. To summarize, in comparison with these algorithms in the table, FedGiA can achieve the fastest convergence under the weakest conditions, thereby consuming the fewest CR.

**C. Organization and notation**

This paper is organized as follows. In the next section, we introduce FL and the framework of ADMM. In Section III, we present the algorithmic framework of FedGiA and highlight its advantages. The global convergence and convergence rate are established in Section IV. We conduct some numerical experiments and comparisons with three leading solvers to demonstrate the performance of FedGiA in Section V. Concluding remarks are given in the last section.

We end this section with summarizing the notation that will be employed throughout this paper. We use plain, bold, and capital letters to present scalars, vectors, and matrices, respectively, e.g., \( m, r \), and \( \sigma \) are scalars, \( x, x_i \), and \( x_i^k \) are vectors, \( X \) and \( X^k \) are matrices. Let \( |t| \) represent the largest integer strictly smaller than \( t + 1 \) and \( |m| := \{1, 2, \ldots, m\} \) with ‘:=’ meaning define. In this paper, \( \mathbb{R}^n \) denotes the \( n \)-dimensional Euclidean space equipped with the inner product \( \langle \cdot, \cdot \rangle \) defined by \( \langle x, y \rangle := \sum_i x_i y_i \). Let \( \| \cdot \| \) be the Euclidean norm for vectors (i.e., \( \|x\| = \langle x, x \rangle \)) and Spectral norm for matrices, and \( \| \cdot \|_H \) be the weighted norm defined by \( \|x\|_H := (Hx, x) \). Write the identity matrix as \( I \) and a positive semidefinite matrix \( A \) as \( A \succeq 0 \). In particular, \( A \succeq B \) represents \( A - B \succeq 0 \). A function, \( f \), is said to be gradient Lipschitz continuous with a constant \( r > 0 \) if

\[
\| \nabla f(x) - \nabla f(z) \| \leq r \| x - z \|. 
\]  

for any \( x \) and \( z \), where \( \nabla f(x) \) represents the gradient of \( f \) with respect to \( x \). Finally, throughout the paper, let \( X := (x_1, x_2, \ldots, x_m) \), \( \Pi := (\pi_1, \pi_2, \ldots, \pi_m) \)
II. GD AND INEXACT ADMM-BASED FL

Given $m$ clients, each client $i \in [m]$ has its local dataset $D_i$ and loss function $f_i(x) := \frac{1}{d_i} \sum_{(a,b) \in D_i} \ell_i(a, b(x, b))$ on that $D_i$, where $\ell_i(a, b(x) : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and bounded from below, $d_i$ is the cardinality of $D_i$, and $x \in \mathbb{R}^n$ is the parameter to be learned. Below are two examples used for our numerical experiments.

**Example II.1** (Least square loss). Suppose client $i$ has dataset $D_i = \{(a_{ij}, b_j^i), \ldots, (a_{ij}, b_j^i)\}$, where $a_j^i \in \mathbb{R}^n$, $b_j^i \in \mathbb{R}$. Then the least square loss is

$$f_i(x) = \frac{1}{2d_i} \sum_{j=1}^{d_i} (x_j - b_j^i)^2. \quad (2)$$

**Example II.2** ($\ell_2$ norm regularized logistic loss). Similarly, client $i$ has dataset $D_i$ but with $b_j^i \in \{0, 1\}$. The $\ell_2$ norm regularized logistic loss is given by

$$f_i(x) = \frac{1}{d_i} \sum_{j=1}^{d_i} \left( \ln(1 + e^{a_j^i x_j}) - b_j^i a_j^i \cdot x_j \right) + \frac{\mu}{2d_i} \|x\|^2, \quad (3)$$

where $\mu > 0$ is a penalty parameter.

The overall loss function can be defined as

$$f(x) := \frac{1}{m} \sum_{i=1}^m f_i(x).$$

Federated learning aims to learn a best parameter $x^*$ that attains the minimal overall loss, namely,

$$x^* := \arg\min_{x \in \mathbb{R}^n} f(x). \quad (4)$$

Since $f_i$ is bounded from below, we have

$$f^* := f(x^*) > -\infty. \quad (5)$$

By introducing auxiliary variables, $x_i = x$, problem (4) can be equivalently rewritten as

$$\min_{x_i, x} F(X) := \frac{1}{m} \sum_{i=1}^m f_i(x_i), \quad (6)$$

s.t. $x_i = x$, $i \in [m]$.

In this paper, we focus on the above problem. It is easy to see that $f(x) = F(X)$ if $X = (x, x, \ldots, x)$.

**A. ADMM**

The backgrounds of ADMM can be referred to the earliest work [40] and a nice book [41]. To apply ADMM for (6), we introduce the augmented Lagrange function defined by,

$$\mathcal{L}(x, \pi_i) := \sum_{i=1}^m L(x, x_i, \pi_i)$$

$$L(x, x_i, \pi_i) := \frac{1}{m} f_i(x_i) + \langle x_i - x, \pi_i \rangle + \frac{\sigma}{2} \|x_i - x\|^2,$$

where $\pi_i \in \mathbb{R}^n$, $i \in [m]$ are the Lagrange multipliers and $\sigma > 0$. The framework of ADMM for problem (6) is given as follows: for an initialized point $(x^0, X^0, \Pi^0)$, perform the following updates iteratively for every $k \geq 0$,

$$x^{k+1} = \arg\min_{x \in \mathbb{R}^n} \mathcal{L}(x, X^k, \Pi^k) = \frac{1}{m} \sum_{i=1}^m (x_i^k + \pi_i^k),$$

$$x_i^{k+1} = \arg\min_{x_i \in \mathbb{R}^n} L(x_i^{k+1}, x_i, \pi_i^k), \quad i \in [m],$$

$$\pi_i^{k+1} = \pi_i^k + \sigma (x_i^{k+1} - x_i^{k+1}), \quad i \in [m]. \quad (8)$$

**B. Stationary points**

To end this section, we present the optimality conditions of problems (6) and (4).

**Definition II.1.** A point $(x^*, X^*, \Pi^*)$ is a stationary point of problem (6) if it satisfies

$$\begin{cases} \frac{1}{m} \nabla f_i(x_i^*) + \pi_i^* = 0, & i \in [m], \\ x_i^* - x^* = 0, & i \in [m], \\ \sum_{i=1}^m \pi_i^* = 0. \end{cases} \quad (9)$$

A point $x^*$ is a stationary point of problem (4) if it satisfies

$$\nabla f(x^*) = 0. \quad (10)$$

Note that any locally optimal solution to problem (6) (resp. (4)) must satisfy (9) (resp. (10)). If $f_i$ is convex for every $i \in [m]$, then a point is a globally optimal solution to problem (6) (resp. (4)) if and only if it satisfies condition (9) (resp. (10)). Moreover, it is easy to see that a stationary point $(x^*, X^*, \Pi^*)$ of problem (6) indicates

$$\nabla f(x^*) = \frac{1}{m} \sum_{i=1}^m \nabla f_i(x_i^*) = -\frac{1}{m} \sum_{i=1}^m \pi_i^* = 0.$$ 

That is, $x^*$ is also a stationary point of the problem (4).

**III. ALGORITHMIC DESIGN**

The framework of ADMM in (8) encounters three drawbacks in reality. (i) It repeats the three updates at every step, leading to communication inefficiency. In FL, the framework manifests that local clients and the central server have to communicate at every step. However, frequent communications would come at a huge price, such as a long learning time and large amounts of resources. (ii) Solving the second subproblem in (8) would incur expensive computational cost as it generally does not admit a closed-form solution. (iii) In real applications, some clients may suffer from bad conditions (e.g., limited computational capacity), which leads to computational difficulties. It is necessary to leave them more time to update their parameters. Therefore, to overcome these drawbacks, we cast a new algorithm in Algorithm 1, where

$$\overline{g}_{i}^{k+1} := \frac{1}{m} \nabla f_i(x_i^{T_{k+1}}).$$

The merits of Algorithm 1 are highlighted as follows.

**A. Communication efficiency**

Algorithm 1 shows that communications only occur when $k \in \mathcal{K} = \{0, k_0, 2k_0, \ldots\}$, where $k_0$ is a predefined positive integer. Therefore, CR can be reduced if setting a big $k_0$, thereby saving the cost vastly. In fact, such an idea has been extensively used in literature [14], [15], [16], [17], [18].

**B. Fast computation**

We update $x_i^{k+1}$ by (12) instead of solving the second subproblem in (8). It can accelerate the computation for local clients significantly, as the computation is relatively cheap if
Algorithm 1: FL via GD and inexact ADMM (FedGiA)

Given an integer $k_0 > 0$ and a constant $\sigma > 0$, every client $i$ initializes $H_i \geq 0, x_i^0, \pi_i^0$ and $z_i^0 = x_i^0 - \pi_i^0/\sigma, i \in [m]$. Let $\tau_k$ be a function of $k$ as $\tau_k := \lfloor k/k_0 \rfloor$.

for $k = 0, 1, 2, 3, \ldots$ do
  if $k \in K := \{0, k_0, 2k_0, 3k_0, \ldots \}$ then
    Weights upload: (Communication occurs)
      All clients upload $\{z_i^k, \ldots, z_m^k\}$ to the server.
    Global aggregation:
      The server calculates average parameter $x^{k+1}$ by
      \[ x^{k+1} = \frac{1}{m} \sum_{i=1}^{m} x_i^k. \] (11)
    Weights broadcast: (Communication occurs)
      The server broadcasts $x^{k+1}$ to all clients.
  end
  for every $i \in C^{k+1}$ do
    Local update: Client $i$ updates its parameters by
    \[ x_i^{k+1} = x_i^{k+1} - (H_i/m + \sigma I)^{-1} (g_i^{k+1} + \pi_i^k), \] (12)
    \[ \pi_i^{k+1} = \pi_i^k + \sigma (x_i^{k+1} - x^{k+1}), \] (13)
    \[ z_i^{k+1} = x_i^{k+1} + \pi_i^{k+1}/\sigma. \] (14)
  end
  for every $i \notin C^{k+1}$ do
    Local invariance: Client $i$ keeps parameters by
    \[ x_i^{k+1} = x_i^{k+1}, \] (15)
    \[ \pi_i^{k+1} = \pi_i^k, \] (16)
    \[ z_i^{k+1} = x_i^{k+1} - \pi_i^{k+1}/\sigma. \] (17)
  end
end

$H_i$ is chosen properly (e.g., diagonal matrices). We point out that (12) is a result of
\[ x_i^{k+1} = \arg\min_{x_i} \left( \frac{1}{m} \sum_{i=1}^{m} h_i(x_i; x^{k+1}) \right) \]
\[ + \langle \nabla f_i(x_i), x_i - z_i \rangle + \frac{\sigma}{2} \|x_i - x^{k+1}\|^2 \]
\[ = x_i^{k+1} - (H_i/m + \sigma I)^{-1} (g_i^{k+1} + \pi_i^k), \] (18)
where $h_i(x_i; z)$ is an approximation of $f_i(x_i)$, namely,
\[ h_i(x_i; z) := f_i(z) + \langle \nabla f_i(z), x_i - z \rangle + \frac{\sigma}{2} \|x_i - z\|^2. \] (19)
The fast computation comes from two aspects. First, we take advantage of the inexact updates, where $(H_i/m + \sigma I)^{-1}$ needs to be computed only once for the entire learning as it is independent of $k$. In addition, one can observe that for every consecutive $k_0$ iterations, $g_i^{k+1} = \frac{1}{m} \nabla f_i(x^{k+1})$ needs to be calculated only once due to $\tau_{k+1} \equiv \tau_{k_0}$ for any $k = sk_0, sk_0 + 1, \ldots, sk_0 + k_0 - 1$. Overall, such strategies allow clients to update their parameters quickly.

C. Mixed updates

At every $k \in K$ in Algorithm 1, all clients are divided into two groups. For clients in $C^{k+1}_k$, they update their parameters $k_0$ times iteratively based on the inexact ADMM, while for clients outside $C^{k+1}_k$, they update their parameters just once based on the GD. The motivation for using such a mixed scheme is twofold.

- In practice, it is unlikely to equip all clients with strong computational capacity. So, this scheme allows clients outside $C^{k+1}_k$ (corresponding to clients with weak computational capacity) to have more time to update their parameters by (15)-(17) just once, which does not require much computational endeavour.
- In addition, the scheme would avoid scenarios where the training for some clients is insufficient. If we only let clients in $C^{k+1}_k$ update their parameters and clients outside $C^{k+1}_k$ keep their parameters unchanged, then there is a small possibility where the selected times of some clients are not enough to train a good shared parameter. For instance, a worst case is that some clients are never chosen to join in the training. Then the trained parameter may not be appropriate for them as the training does not use their data at all.

As mentioned above, for every consecutive $k_0$ steps, all (selected or non-selected) clients carry out the update at least once, namely all clients join in the training. Thanks to this, the total objective function values of the generated sequence can be guaranteed to decrease with a declining scale at every step, as shown by Lemma IV.1. This thus accelerates the convergence so as to reduce CR, see the reduced CR in Table I.

One can check that (12)-(14) imply
\[ z_i^{k+1} = x_i^{k+1} - g_i^{k+1}/\sigma - H_i (x_i^{k+1} - x^{k+1})/(m\sigma). \]
Therefore, the inexact ADMM update can be deemed as a proximal GD update if $H_i \neq 0$ and a GD update if $H_i = 0$.

We would like to point out that FedAvg [12], [39], FedProx [37], and FedADMM [42] select partial devices to join in the training in each communication round. That is, if $k \in K$, they randomly select a subset $C^{k+1}_k$ of clients. Only clients in $C^{k+1}_k$ update their parameters and the rest clients remain unchanged. However, differing from that, FedGiA let clients outside $C^{k+1}_k$ also update their parameters but only once for every consecutive $k_0$ steps.

IV. Convergence Analysis

We first present all the assumptions used to establish the convergence properties.

Assumption IV.1. Every $f_i, i \in [m]$ is gradient Lipschitz continuous with a constant $r_i > 0$.

Assumption IV.1 implies that there is always a $\Theta_i$ satisfying $r_i I \succeq \Theta_i \succeq 0$ such that
\[ f_i(x_i) \leq f_i(z_i) + \langle \nabla f_i(z_i), x_i - z_i \rangle + \frac{\sigma}{2} \|x_i - z_i\|^2. \] (20)
for any $x_i, z_i \in \mathbb{R}^n$. Apparently, many $\Theta_i$s satisfy the above condition (e.g., $\Theta_i = r_i I$). In the subsequent convergence analysis, we suppose that every client $i$ chooses $H_i = \Theta_i$. 


Assumption IV.2. The following level set is bounded,
\[ S := \{x \in \mathbb{R}^n : f(x) \leq L(x^0, X^0, \Pi^0) \}. \]  
(21)

We point out that the boundedness of the level set is frequently used in establishing the convergence properties of optimization algorithms. The main purpose is to bound the generated sequence. There are many functions satisfying this condition, such as the coercive functions. It is noted that the boundedness of the level set and the convexity does not imply each other. For example, \(|t^2-1|\) is non-convex on \(\mathbb{R}\) but has the bounded level set while \(e^t\) is convex on \(\mathbb{R}\) but does not have a bounded level set. However, if a function is strongly convex, then all its level sets are bounded.

Assumption IV.3. Every \(f_i, i \in [m]\) is a KL function, defined by Definition A.2.

KL functions are general enough [43] and include most commonly used functions, such as the real polynomial functions, logistic loss function, norm functions, indicator functions, semi-algebraic sets (e.g., cone of positive semi-definite matrices, Stiefel manifolds, and constant rank matrices). Moreover, most convex functions (e.g., functions in (2) and (3) and strongly convex functions) in finite dimensional applications satisfy the KL property as well. Hence, there are various convex and non-convex functions belong to KL functions.

A. Global convergence

For notational convenience, hereafter we let \(a^k \rightarrow a\) a stand for \(\lim_{k \rightarrow \infty} a^k = a\) and denote
\[
Z^k := (x^a, X^k, \Pi^k), \quad Z^\ast := (x^a, X^\ast, \Pi^\ast),
\]
\[
Z^\infty := (x^\infty, X^\infty, \Pi^\infty), \quad r := \max_{i \in [m]} r_i,
\]
\[
\Delta x_i^{k+1} := x_i^{k+1} - x_i^k, \quad \Delta x_i^{k+1} := x_i^{k+1} - x^{\infty}.
\]

With the help of Assumption IV.1, our first result shows the decreasing property of the objective function values of the generated sequence.

Lemma IV.1. Let \(\{Z^k\}\) be the sequence generated by Algorithm 1 with \(H_i = \Theta_i, i \in [m]\) and \(\sigma \geq 6r/m\). If Assumption IV.1 holds, then the following statements are true.

i) For any \(k \geq 0\),
\[
L(Z^{k+1}) - L(Z^k) \leq -\frac{\sigma 1}{2} \omega_{k+1},
\]
where
\[
\omega_{k+1} := \sum_{i=1}^m (\|\Delta x_i^{k+1}\|^2 + \|\Delta x_i^{k+1}\|^2).
\]

ii) For any \(s \in K\),
\[
\|\nabla L(Z^{s+1})\| \leq (2\sigma + 1)\sqrt{m}\infty_{s+1}.\]

The above lemma allows us to prove whole sequences \(\{L(Z^{k})\}, \{F(X^{k})\}\), and \(\{f(x^{\infty})\}\) converge.

Theorem IV.1. Let \(\{Z^k\}\) be the sequence generated by Algorithm 1 with \(H_i = \Theta_i, i \in [m]\) and \(\sigma \geq 6r/m\). The following results hold under Assumption IV.1.

i) Three sequences \(\{L(Z^k)\}, \{F(X^k)\}, \) and \(\{f(x^{\infty})\}\) converge to the same value, namely,
\[
\lim_{k \rightarrow \infty} L(Z^k) = \lim_{k \rightarrow \infty} F(X^k) = \lim_{k \rightarrow \infty} f(x^{\infty}).
\]

ii) \(\nabla F(X^k)\) and \(\nabla f(x^{\infty})\) eventually vanish, namely,
\[
\lim_{k \rightarrow \infty} \nabla F(X^k) = \lim_{k \rightarrow \infty} \nabla f(x^{\infty}) = 0.
\]

Besides the convergence of the objective function values in the above theorem, we would like to see the convergence performance of sequence \(\{Z^k\}\) itself in the following theorem.

Theorem IV.2. Let \(\{Z^k\}\) be the sequence generated by Algorithm 1 with \(H_i = \Theta_i, i \in [m]\) and \(\sigma \geq 6r/m\). The following results hold under Assumptions IV.1 and IV.2.

i) Then sequence \(\{Z^k\}\) is bounded, and any its accumulating point, \(Z^\infty\), is a stationary point of (6), where \(x^\infty\) is a stationary point of (4).

ii) If further assume that either \(x^\infty\) is isolated or Assumption IV.3 holds, then whole sequence \(\{Z^k\}\) converges to \(Z^\infty\).

Remark IV.1. We give some comments on the conditions in Theorem IV.2.

- According to the proof, apart from the choice of \(H_i = \Theta_i, i \in [m]\), any matrix \(H_i\) satisfying \(r_i I \succeq H_i \succeq 0\) suffices to ensure the convergence property. This means \(H_i\) can be chosen flexibly. However, for the sake of reducing the computational complexity, equation (12) suggests that we should set \(H_i\) to be a matrix enabling the easy calculation of \((H_i/m + \sigma I)^{-1}\). Typical choices include the gram matrix or diagonal matrix, see Table III.

- Condition \(\sigma \geq 6r/m\) is essential. First, since the problem may be non-convex, a properly large \(\sigma\) can guarantee the strong convexity of (18) so as to ensure a unique solution for the case of \(H_i = 0\). In addition, the big value can prevent over-updating for \(x_i^{k+1}\), namely, avoiding it stepping too further from global parameter \(x^{\infty}\).

- It is noted that if \(f\) is locally strongly convex at \(x^\infty\), then \(x^\infty\) is unique and hence is isolated. However, being isolated is a weaker assumption than locally strong convexity. In addition, there are extensive functions satisfying Assumption IV.3 since KL functions are general enough.

It is worth mentioning that the establishment of Theorem IV.2 does not require the convexity of \(f_i\) or \(f\), because of this, the sequence is guaranteed to converge to the stationary point of problems (6) and (4). In this regard, if we further assume the convexity of \(f\), then the sequence is capable of converging to the optimal solution to problems (6) and (4), which is stated by the following corollary.

Corollary IV.1. Let \(\{Z^k\}\) be the sequence generated by Algorithm 1 with \(H_i = \Theta_i, i \in [m]\) and \(\sigma \geq 6r/m\). The following results hold under Assumptions IV.1 and IV.2, and the convexity of \(f\).
i) Three sequences \( \{L(Z^k)\}, \{F(X^k)\}, \{f(x^k)\} \) converge to the optimal function value of (4), namely
\[
\lim_{k \to \infty} L(Z^k) = \lim_{k \to \infty} F(X^k) = \lim_{k \to \infty} f(x^k) = f^*. \tag{28}
\]

ii) Any accumulating point \( Z^\infty \) of sequence \( \{Z^k\} \) is an optimal solution to (6) and \( x^\infty \) is an optimal solution to (4).

iii) If further assume \( f \) is strongly convex. Then whole sequence \( \{Z^k\} \) converges to unique optimal solution \( Z^* \) to (6) and \( x^* \) is the unique optimal solution to (4).

**Remark IV.2.** Regarding the assumption in Corollary IV.1, \( f \) being strongly convex does not require the strong convexity for every \( f_i, i \in [m] \). If one \( f_i \) is strongly convex and the remaining is convex, then \( f = \sum_{i=1}^m w_i f_i \) is strongly convex. Moreover, the strongly convex suffices to the boundedness of level set \( S \). Therefore, under the strongly convex, the assumption on the boundedness of \( S \) can be exempted.

### B. Convergence rate

In this part, we investigate the convergence rate of Algorithm 1. The following result states that the minimal value among \( ||\nabla f(x^r)||^2, j \in [k] \) vanishes with rate \( O(rk_0/k) \).

**Theorem IV.3.** Let \( \{(x^r, X^k, \Pi^k)\} \) be the sequence generated by Algorithm 1 with \( H_t = \Theta_t, i \in [m] \) and \( \sigma \geq 6r/m \). If Assumption IV.1 holds, then it follows
\[
\min_{j \in [k]} ||\nabla f(x^r)||^2 \leq \frac{100m\sigma k_0 (L(Z^0) - f^*)}{k}.
\]

The establishment of such a convergence rate only requires the assumption of gradient Lipschitz continuity, namely, Assumption IV.1. Moreover, since \( \sigma = tr/m \), we have
\[
\min_{j \in [k]} ||\nabla f(x^r)||^2 = O(rk_0/k).
\]

This is what we expected. The larger \( k_0 \) is, the more iterations are required to converge.

**Remark IV.3.** Theorem IV.3 hints that Algorithm 1 can be terminated if
\[
||\nabla f(x^r)||^2 \leq \epsilon, \tag{29}
\]
where \( \epsilon \) is a given tolerance. Therefore, after
\[
k = \left[ \frac{\rho k_0 (L(Z^0) - f^*)}{\epsilon} \right] = O\left(\frac{rk_0}{\epsilon}\right) \tag{30}
\]
iterations, Algorithm 1 meets (29) and the total CR are
\[
CR := \left[ \frac{2k}{k_0} \right] = \left[ \frac{2\rho (L(Z^0) - f^*)}{\epsilon} \right] = O\left(\frac{1}{\epsilon}\right). \tag{31}
\]

If we further assume the KL property, then we can achieve a better convergence rate as follows.

**Theorem IV.4.** Let \( \{Z^k\} \) be the sequence generated by Algorithm 1 with \( H_i = \Theta_i, i \in [m] \) and \( \sigma \geq 6r/m \), and \( Z^\infty \) be its limit. Suppose that Assumptions IV.1 and IV.2 hold, and Assumption IV.3 hold with a desingularizing function (see Definition A.1) \( \varphi(z) = \frac{c^2}{2} z^{1-\theta} \), where \( c > 0, \theta \in [0, 1] \), then the following rates of convergence hold.

i) If \( \theta = 0 \), then there is a \( k_1 \in K \) such that
\[
f(x^\infty) := f(x^\infty), \quad \text{for all } k \in K, \tag{32}
\]
ii) If \( \theta \in (0, 1/2] \), then there is a \( k_2 \in K \) and \( c_2 > 0 \) satisfying
\[
f(x^\infty) - f(x^r) \leq c_2 \left( \frac{\rho}{\epsilon} \right)^{1-k_2/k_0}, \quad \text{for all } k \in K, \tag{33}
\]
iii) If \( \theta \in (1/2, 1) \), then there is a \( k_3 \in K \) and \( c_3 > 0 \) satisfying
\[
f(x^\infty) - f(x^r) \leq \left( \frac{c_3 k_0}{k} \right) \frac{1}{\epsilon}, \quad \text{for all } k \in K. \tag{34}
\]

Note that if \( \theta = 0 \), the convergence result means that the algorithm can terminate within finitely many steps. In such a case we write
\[
f(x^\infty) - f(x^r) = O(0). \tag{35}
\]

If \( \theta \in (0, 1/2] \), then the convergence rate is linear. If \( \theta \in (1/2, 1) \), then \( 1/\sqrt{\epsilon-1} \in (1, \infty) \) and thus the convergence rate is at least sub-linear. It is worth mentioning that all semi-algebraic functions (see [43, Definition 5]) satisfy KL property with \( \varphi(z) = \frac{c^2}{2} z^{1-\theta} \). Typical semi-algebraic functions include \( ||x||_p := \sum(|x_i|^p)^{1/p}, p \geq 0 \), real polynomial functions, and those functions in [43, Example 2-4].

In particular, if every \( f_i \) is strongly convex, then Assumptions IV.2 and IV.3 hold with \( \varphi(z) = 2\sqrt{\sigma}z \) (see [43, Example 6]), namely \( \theta = 1/2 \) in Theorem IV.4, thereby leading to the linear convergence rate.

**Remark IV.4.** Based on Theorem IV.4, after
\[
k = \begin{cases} 
O(k_0), & \theta = 0, \\
O\left( k_0 \log \left( 2 + \frac{1}{\epsilon} \right) \right), & \theta \in (0, 1/2], \\
O\left( \frac{k_0}{\sqrt{\epsilon-1}} \right), & \theta \in (1/2, 1), 
\end{cases} \tag{32}
\]
iterations, Algorithm 1 meets
\[
f(x^\infty) - f(x^r) \leq \epsilon. \tag{33}
\]

Hence, to achieve such an accuracy, the total CR are
\[
CR := \left[ \frac{2k}{k_0} \right] = \begin{cases} 
O(1), & \theta = 0, \\
O\left( \log \left( 2 + \frac{1}{\epsilon} \right) \right), & \theta \in (0, 1/2], \\
O\left( \frac{1}{\sqrt{\epsilon-1}} \right), & \theta \in (1/2, 1). 
\end{cases} \tag{34}
\]

**Remark IV.5.** We summarize several state-of-the-art algorithms and compare their performance in Table I. Here, for \( i \in [m] \), let \( C_i^1, C_i^2 \), and \( C_i^3 \) be computational complexity of computing the gradient of \( f_i \), the stochastic gradient of \( f_i \), and an optimization problem associated with \( f_i \), respectively. Then \( \beta_i := \max_{i \in [m]} C_i, t = 1, 2, 3 \). For type-I convergence, although FedGI and FedPD have the best convergence rate \( O(k_0/k) \) under the weakest assumption. For the type-II convergence since the strong convexity implies the bounded level set and KL property, FedGI converges with a rate better than sub-linear rate \( O(k_0/k) \) for all other algorithms.
under the weakest assumptions. Moreover, it converges linearly if the strong convexity holds.

In addition to the best convergence results, FedGiA also has a low computational complexity. In Table I, we present the computational complexity for all clients to update their parameters $k_0$ times, that is, the computational complexity of local computation for consecutive $k_0$ steps between two communications. Note that the complexity for FedGiA presented in the table is under the choice of $H_i$ being chosen as a diagonal matrix. From (12) and (16), FedGiA only needs to calculate the gradient once for $k_0$ steps so it has relatively low computational complexity. By contrast, all other algorithms need to calculate the gradient $k_0$ times.

V. NUMERICAL EXPERIMENTS

In this section, we conduct some numerical experiments to demonstrate the performance of FedGiA in Algorithm 1. All numerical experiments are implemented through MATLAB (R2020b) on a laptop with 32GB memory and 2.3Ghz CPU. The source codes are available at https://github.com/ShenglongZhou/FedGiA.

A. Testing example

We use Example II.1 with synthetic data and Example II.2 with real data to conduct the numerical experiments.

Example V.1 (Linear regression with non-i.i.d. data). For this problem, local clients have their objective functions as (2). We randomly generate $d$ samples $(a, b)$ from three distributions: the standard normal distribution, the Student’s $t$ distribution with degree 5, and the uniform distribution in $[-5, 5]$. Then we shuffle all samples and divide them into $m$ parts $(A_i, b_i)$ for $m$ clients, where $A_i=(a_{i1}, \ldots, a_{id})^\top$ and $b_i=(b_{i1}, \ldots, b_{id})^\top$. Therefore, $d_1+d_2+\cdots+d_m$. The data size of each part, $d_i$, is randomly chosen from $[50, 150]$. For simplicity, we fix $n=100$ but choose $m \in [64, 96, 128, 196, 256]$. In this regard, each client has non-i.i.d. data $(A_i, b_i)$.

Example V.2 (Logistic regression). For this problem, local clients have their objective functions as (3), where $\mu=0.001$ is fixed in the numerical experiments. We use two real datasets described in Table II to generate $(a, b)$. We randomly split $d$ samples into $m$ groups corresponding to $m$ clients.

| Data | Datasets       | Source         | $n$ | $d$ |
|------|----------------|----------------|-----|-----|
| qot  | Quor oral toxicity | uci            | 1024 | 8992|
| act  | Santander customer transaction | kaggle | 200  | 200000|

Example V.3 (Non-convex regularized logistic regression). For this example, we aim at solving a non-convex problem [44], [36], where client $i \in [m]$ has the objective function as

$$f_i(x) = \frac{1}{d_i} \sum_{j=1}^{d_i} \left( \ln(1+e^{a_j^\top x}) - b_j^\top a_j^\top x \right) + \frac{\mu}{d_i} \sum_{j=1}^{d_i} \frac{x_j^2}{1+x_j^2}.$$

We fix $\mu=0.01$ in the sequel. Samples $(a, b)$ are generated the same as Example V.2.

B. Implementations

As mentioned in Remark IV.3, we terminate FedGiA if $k \geq 10^4$ or solution $x^k$ satisfies

$$\text{Error} := \| \nabla f(x^k) \|^2 \leq \text{tol},$$

and initialize $x^0 = \pi_i^0 = 0, i \in [m]$, where tol $= 10^{-7}$ for Example V.1 and tol $= (5/d) \times 10^{-6}$ for Examples V.2 and V.3. For every $k \in K$, we randomly select $\alpha m$ clients to form $C^k+1$, namely, $|C^k+1| = \alpha m$ and $\alpha \in (0, 1]$. Here, $\alpha = 1$ means all clients are chosen. Theorem IV.1 suggests that $\sigma$ should be chosen to satisfy $\sigma = tr/m$, where $t$ is given in Table III. Finally, $H_i$ is chosen as Table III, where FedGiA$_d$ and FedGiA$_0$ represent FedGiA under $H_i$ opted as a Gram and Diagonal matrix, respectively.

| Example V.1 | $H_i$ | FedGiA$_d$ | FedGiA$_0$ |
|-------------|-------|------------|------------|
| $t = 0.15$  | $H_i$ | $\frac{d_i}{4d_i} + \frac{\mu}{4d_i} I$ |
| $t = \max \left \{ \frac{0.025, 4h(d_i)}{n} \right \}$ | $H_i$ | $\frac{d_i}{4d_i} I$ |
| $t = \max \left \{ \frac{0.025, 4h(d_i)}{n} \right \}$ | $H_i$ | $\frac{d_i}{4d_i} + \frac{\mu}{4d_i} I$ |

C. Numerical performance

In this part, we conduct some simulation to demonstrate the performance of FedGiA including global convergence, convergence rate, and effect of $k_0$ and $C^k+1$. To measure the performance, we report the following factors: $f(x^k)$, error $\| \nabla f(x^k) \|^2$, CR, and computational time (in second). We only report results of FedGiA solving Example V.1 and omit ones for Examples V.2 and V.3 as the observations are similar.

1) Global convergence with rate $O(k_0/k)$: We fix $m = 128, \alpha = 0.5$, and $k_0 \in \{1, 5, 10, 15, 20\}$ and present the results in Fig. 1. From the left sub-figure, as expected, all lines eventually tend to the same objective function value, well testifying Theorem IV.1. It is clear that the bigger $k_0 > 1$ (i.e., the more steps between two global aggregations), the more iterations required to reach the optimal function value. From the right sub-figure, the trends show that all errors vanish gradually as the rising of the number of iterations. Apparently, the bigger $k_0$, the more iterations used to converge, which well justifies Theorem IV.3 that convergence rate relies on $k_0$.

2) Effect of $k_0$: Next, we would like to see how the choices of $k_0$ impact the performance of FedGiA. To proceed with that, for each dimension $(m, d_1, \ldots, d_m)$ of the dataset, we generate 20 instances of Example V.1. Each instance is solved by FedGiA with fixing $\alpha = 0.5$ and $k_0 \in [20]$. The average results are reported in Fig. 2. It can be clearly seen that CR decline first and then stabilizes at a certain level with the rising of $k_0$. To this end, it is efficient to save the communication cost if we set a proper $k_0$. However, it is unnecessary to set a very big value as the larger $k_0$ the longer computational time, as shown in the right sub-figure. In general, FedGiA$_d$ used more CR but always ran faster than FedGiA$_0$. 

TABLE III: Choices of $t$ and $H_i$, where $B_i := A_i^\top A_i$. 

| Example V.1 | $H_i$ | FedGiA$_d$ | FedGiA$_0$ |
|-------------|-------|------------|------------|
| $t = 0.15$  | $H_i$ | $\frac{d_i}{4d_i} + \frac{\mu}{4d_i} I$ |
| $t = \max \left \{ \frac{0.025, 4h(d_i)}{n} \right \}$ | $H_i$ | $\frac{d_i}{4d_i} I$ |
| $t = \max \left \{ \frac{0.025, 4h(d_i)}{n} \right \}$ | $H_i$ | $\frac{d_i}{4d_i} + \frac{\mu}{4d_i} I$ |
FedGiA distributes the computational load required to solve the sub-problem among the clients. Since all clients participate in the training process, the clients are chosen to calculate (12), leading to more expensive computations. However, when $H_i$ is chosen as a diagonal matrix, computing (12) is relatively cheap, which explains that $\alpha$ has little influence on the computational time for FedGiA. In the sequel, we fix $\alpha = 0.5$ for simplicity.

3) Effect of $C_{7k+1}$: Finally, we aim to see how choices of $C_{7k+1}$ impact the performance of FedGiA by altering $\alpha \in (0.1, 1]$. The average results are presented in Fig. 3. We observe that $\alpha$ would not have a big influence on CR when $k_0 > 5$. As expected, $\alpha$ impacts FedGiA greatly in terms of computational time. From the algorithmic framework, the larger $\alpha$ the more clients are selected to calculate (12), leading to more expensive computations. However, when $H_i$ is chosen as a diagonal matrix, computing (12) is relatively cheap, which explains that $\alpha$ has little influence on the computational time for FedGiA. In the sequel, we fix $\alpha = 0.5$ for simplicity.

D. Numerical comparison

In this part, we will compare our proposed method with FedAvg [12], FedPD [36], and FedProx [37]. For fair comparison, we initialize all algorithms with same starting point $x_0^i = 0, i \in [m]$ and terminate them if condition (35) is satisfied or CR are over 1000. Since all clients participate in the training for consecutive $k_0$ steps in FedGiA, we apply the full device participation into the other algorithms, namely, all clients are chosen to update their parameters at every step. In addition to these settings, we also set up them as follows.

- For FedAvg, we use its non-stochastic version, that is, we use full data to calculate the gradient for every client. Learning rate (i.e., the step size) is set as $\gamma_k(a) := a/log_2(k+2)$ with $a = 0.01$ for Example V.1 and $a = 0.5d/m$ for Examples V.2 and V.3.

- For FedProx, each client needs to solve a regularized optimization sub-problem at every step. The regularized penalty constant is given as $\mu = 10^{-2}$. We also employ the GD method to solve the sub-problem. The maximal number of iterations is set as $5$, and the learning rate is set as $\gamma_k(a)$ with $a = 0.001$ for Example V.1 and $a = 0.5d/m$ for Examples V.2 and V.3.

- For FedPD, we adopt the version with oracle choice I and option I, where the maximal number of iterations for the GD method is set as $5$. This algorithm involves two parameters $\eta$ and $\eta_l$. The former has a similar role of $1/\sigma$ in (7) and the latter is the learning rate for solving a sub-problem. For Example V.1, we set $\eta = 1$ and $\eta_l = \gamma_k(a)$. For Examples V.2 and V.3, we set $\eta = \max\{400, d/50\}$ and $\eta_l = \gamma_k(0.5d/m)$. Moreover, instead of conducting the global aggregation with a probability, we use the same scheme as the other algorithms, namely, aggregating all parameters when $k \in K$.

| Algs.     | $k_0 = 1$         | $k_0 = 5$         | $k_0 = 10$        |
|-----------|--------------------|--------------------|--------------------|
|           | Obj. | CR | Time | Obj. | CR | Time | Obj. | CR | Time |
| FedAvg    | 1.744 | 515.5 | 1.18 | 1.744 | 112.0 | 0.70 | 1.744 | 63.7 | 0.64 |
| FedProx   | 1.744 | 315.7 | 1.64 | 1.744 | 74.3 | 1.41 | 1.744 | 49.9 | 1.83 |
| FedPD     | 1.744 | 21.9 | 0.15 | 1.744 | 15.1 | 0.29 | 1.744 | 11.2 | 0.42 |
| FedGiA    | 1.744 | 4.1 | 0.11 | 1.744 | 3.0 | 0.11 | 1.744 | 3.0 | 0.11 |
| FedAvg    | 1.744 | 4.5 | 0.13 | 1.744 | 3.0 | 0.16 | 1.744 | 3.0 | 0.19 |
| FedProx   | 1.744 | 20.5 | 0.68 | 1.744 | 10.8 | 1.31 | 1.744 | 8.7 | 1.93 |
| FedPD     | 1.744 | 2.0 | 0.39 | 1.744 | 5.0 | 0.60 | 1.744 | 4.9 | 0.79 |
| FedGiA    | 1.744 | 5.2 | 0.39 | 1.744 | 5.0 | 0.60 | 1.744 | 4.9 | 0.79 |

**Fig. 1:** Objective function values and errors vs. iterations. FedGiA (solid lines) and FedGiA-D (dashed lines) solve Example V.1 with $m = 128$ and $\alpha = 0.5$.

**Fig. 2:** Effect of $k_0$ for FedGiA (solid lines) and FedGiA-D (dashed lines) solving Example V.1 with $\alpha = 0.5$.

**Fig. 3:** Effect of $\alpha$. FedGiA (solid lines) and FedGiA-D (dashed lines) solve Example V.1 with $m = 128$. **TABLE IV:** Comparison for four algorithms.
Fig. 4: $f(x^{x_k})$ v.s. CR for Example V.1.

Fig. 5: $f(x^{x_k})$ v.s. CR for Example V.2 with qot.

Fig. 6: $f(x^{x_k})$ v.s. CR for Example V.3 with qot.
1) **Solving Example V.1:** For simplicity, we fix \( m = 128 \) and \( n = 100 \). From Fig. 4, the objective function values for all methods eventually tend to the same one. Basically, the larger \( k_0 \) the fewer CR used to converge. Moreover, FedGiA and FedGiAG outperform the others as they use the fewest CR. We then run 20 independent trials and report the average results in Table IV. Clearly, there is no difference on the objective function values among these algorithms. However, in terms of using CR, FedGiA performs the best, followed by FedGiAG and FedPD. For the computational speed, FedGiA always runs the fastest.

2) **Solving Examples V.2 and V.3:** Again, we fix \( m = 128 \) for simplicity. From Fig. 5 and Fig. 6, when \( k_0 = 5 \) and 10, although FedProx and FedPD can decline very quickly at the beginning, they still need higher CR than FedGiA. The objective function values of FedAvg always decrease the slowest. Then we report the average results over 20 independent trials in Table IV, from which we can conclude several conclusions. First, FedGiA obtains the smallest objective function values and consumes the lowest CR almost for all scenarios, followed by FedPD. Evidently, FedAvg comes the last. In addition, both FedGiA and FedGiAG run the fastest. By contrast, FedAvg takes the longest time since it uses the highest CR.

VI. CONCLUSION

We developed a new FL algorithm and managed to address three critical issues in FL, including saving communication resources, reducing computational complexity, and establishing convergence property under mild assumptions. These advantages hint that the proposed algorithm might be practical to deal with many real applications, such as mobile edge computing [45], [46], [47], over-the-air computation [48], [49], vehicular communications [1], unmanned aerial vehicle online path control [50], immersive virtual reality video streaming [51], [52], and so forth. Moreover, we feel that the algorithmic schemes and techniques used to build the convergence theory could be also valid for tackling decentralized FL [27], [53]. We leave these for future research.

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For notational simplicity, hereafter, we denote

\[ \Delta x_{i}^{k+1} := x_{i}^{k+1} - x_{i}^{k}, \]
\[ \Delta \pi_{i}^{k+1} := \pi_{i}^{k+1} - \pi_{i}^{k}, \]
\[ \Delta x_{T_{k+1}} := x_{T_{k+1}}^{k+1} - x_{T_{k}}^{k}, \]
\[ \Delta x_{s_{k+1}} := x_{s_{k+1}}^{k+1} - x_{s_{k}}^{k}, \]
\[ g_{k} := 1/\alpha \nabla f_{i}(x_{i}^{k}), \]
\[ g_{k}^{k+1} := 1/\alpha \nabla f_{i}(x_{i}^{k+1}). \]

For any vectors \( a, b, a_{i}, \) matrix \( H \geq 0, \) and \( t > 0, \) we have

\[ -\|b\|^{2} = 2(a, b) + \|a\|^{2} - \|a + b\|^{2}, \]
\[ \|a + b\|^{2} \leq (1 + t)\|a\|^{2} + (1 + 1/t)\|b\|^{2}, \]
\[ \| \sum_{i=1}^{m} a_{i} \|^{2} \leq m \sum_{i=1}^{m} \|a_{i}\|^{2}, \]
\[ 2\langle Ha, b \rangle \leq t\|a\|_{H}^{2} + (1/t)\|b\|_{H}^{2}. \]
\( a) \) \( \forall k \in K, \)
\[ \sum_{i=1}^{m}(\frac{e_i^k}{\sigma} + x_i^k - x^{k+1}) = 0. \] (38)

\( b) \) \( \forall k \geq 0, \forall i \in [m], \)
\[ g_i^{k+1} + \pi_i^{k+1} + \frac{1}{m} H_i \Delta x_i^{k+1} = 0. \] (39)

\( c) \) \( \forall k \geq 0, \forall i \in [m], \)
\[ \|\Delta x_i^{k+1}\| \leq \frac{\sigma^2}{m^2} \|\Delta x_i^{k+1}\|^2 + \frac{3\sigma^2}{m^2} \|\Delta x_i^k\|^2. \] (40)

**Proof.** a) For any \( i \not\in C^{k+1} \), we have from (14) that
\[ g_i^{k+1} = \pi_i^{k+1} + x_i^{k+1}. \] (41)

For any \( i \not\in C^{k+1} \), it follows from (15)-(17) that the above relation is still valid. Hence, we have (41) for any \( i \in [m] \) and for any \( k \geq 0 \). As a result, for any \( k \in K, \)
\[ \sum_{i=1}^{m}(\frac{e_i^k}{\sigma} + x_i^k - x^{k+1}) = 0. \] (38)

b) For \( i \in C^{k+1} \), solution \( x_i^{k+1} \) in (12) satisfies (18), thereby contributing to,
\[ 0 = g_i^{k+1} + \pi_i^{k+1} + \frac{1}{m} H_i \Delta x_i^{k+1} \]
\[ \leq \|\Delta x_i^{k+1}\| \leq \frac{\sigma^2}{m^2} \|\Delta x_i^{k+1}\|^2 + \frac{3\sigma^2}{m^2} \|\Delta x_i^{k+1}\|^2. \] (40)

For any \( i \not\in C^{k+1} \), the second equation in (42) is still valid due to \( \pi_i^{k+1} = -g_i^{k+1} \) and \( \Delta x_i^{k+1} = 0 \) from (15)-(17). Hence, it is true for any \( i \in [m] \) and any \( k \in K. \)

c) It follows from (39) and \( \tau_i I \geq H_i = \Theta_i \geq 0 \) that
\[ \|\Delta x_i^{k+1}\| \]
\[ \leq \|\Delta x_i^{k+1} - g_i^{k+1}\| + \|\Delta x_i^{k+1}\| + \frac{\sigma^2}{m^2} \|\Delta x_i^{k+1}\| \]
\[ \leq \frac{\sigma^2}{m^2} \|\Delta x_i^{k+1}\|^2 + \frac{\sigma^2}{m^2} \|\Delta x_i^{k+1}\|^2. \] (45)

Estimating \( e_i^k \). We denote
\[ p_i^k := L(x^{k+1}, x_i^{k+1}, \pi_i^k) - L(x^{k+1}, x_i^k, \pi_i^k) \]
\[ \leq \frac{\sigma^2}{m^2} \|\Delta x_i^{k+1}\|^2 + \frac{3\sigma^2}{m^2} \|\Delta x_i^{k+1}\|^2. \] (40)

We will consider two cases: \( i \not\in C^{k+1} \) and \( i \in C^{k+1} \). For \( i \not\in C^{k+1} \), if \( k \in K \), then \( x_i^{k+1} \equiv x^{k+1} \) (namely \( \Delta x_i^{k+1} = 0 \)) suffices to
\[ p_i^k \leq \frac{\sigma^2}{m^2} \|\Delta x_i^{k+1}\|^2. \] (40)

Estimating \( e_i^k \). If \( k \not\in K \), then \( x_i^{k+1} = x^{k+1} \), yielding
\[ e_i^k = 0 = -\frac{\sigma^2}{m^2} \|\Delta x_i^{k+1}\|^2. \] (46)

For \( k \in K \), multiplying both sides of the first equation in (38) by \( \Delta x^{k+1} \) yields
\[ \sum_{i=1}^{m} \langle \Delta x_i^{k+1}, \pi_i^k \rangle = \sum_{i=1}^{m} \langle \Delta x_i^{k+1}, \sigma(x_i^{k+1} - x_i^k) \rangle. \] (48)

The fact allows us to derive that
\[ e_i^k = \sum_{i=1}^{m} (L(x_i^{k+1}, x_i^k, \pi_i^k) - L(x_i^{k}, x_i^k, \pi_i^k)) \]
\[ \leq \sum_{i=1}^{m} (\langle \Delta x_i^{k+1}, \pi_i^k \rangle + \frac{\sigma^2}{m^2} \|\Delta x_i^{k+1}\|^2 + \frac{3\sigma^2}{m^2} \|\Delta x_i^{k+1}\|^2) \]
\[ = -\frac{\sigma^2}{m^2} \sum_{i=1}^{m} \|\Delta x_i^{k+1}\|^2 = -\frac{\sigma^2}{m^2} \|\Delta x_i^{k+1}\|^2. \] (49)

Overall, for both scenarios, we obtained
\[ e_i^k = -\frac{\sigma^2}{m^2} \|\Delta x_i^{k+1}\|^2. \] (49)
where the last inequality is from $r, I \geq H_i = \Theta_i \geq 0$ and the gradient Lipschitz continuity of $f_i$. Moreover, it follows from (36) that

$$\frac{\sigma}{2} \| \Delta x_i^{k+1} \|^2 - \frac{\sigma}{2} \| x_i^k - x_i^k \|^2 = \langle \Delta x_i^{k+1}, \sigma \Delta x_i^{k+1} \rangle - \frac{\sigma}{2} \| \Delta x_i^{k+1} \|^2.$$  

Using the above two facts derives

$$p_i^k = \frac{1}{m} f_i(x_i^{k+1}) - \frac{1}{m} f_i(x_i^k) + \langle \Delta x_i^{k+1}, \sigma \Delta x_i^{k+1} \rangle - \frac{\sigma}{2} \| \Delta x_i^{k+1} \|^2 \leq \frac{1}{m} \| \Delta x_i^{k+1} \|^2 + \langle \Delta x_i^{k+1}, \sigma \Delta x_i^{k+1} \rangle + \left( \frac{\sigma}{2m} - \frac{\sigma}{2} \right) \| \Delta x_i^{k+1} \|^2 \leq \frac{2}{m} \| \Delta x_i^{k+1} \|^2.$$

Using the second condition in (54) results in

$$\| \nabla x_i \mathcal{L}(Z^{k+1}) \| \leq \frac{\| \Delta x_i^{k+1} \|^2}{\| \nabla \pi_i (Z^{k+1}) \|^2} = \frac{\| \nabla \pi_i (Z^{k+1}) \|^2}{\| \nabla \pi_i (Z^{k+1}) \|^2} \leq \frac{\| \Delta x_i^{k+1} \|^2}{\| \nabla \pi_i (Z^{k+1}) \|^2}.$$

Combining facts (55)-(57) allows us to obtain

$$\| \nabla \pi_i (Z^{k+1}) \|^2 = \| \nabla \pi_i (Z^{k+1}) \|^2 \leq \| \Delta x_i^{k+1} \|^2 \leq \frac{\| \Delta x_i^{k+1} \|^2}{\| \nabla \pi_i (Z^{k+1}) \|^2}.$$

Using the third condition in (54) results in

$$\| \Delta x_i^{k+1} \|^2 \leq \frac{\| \Delta x_i^{k+1} \|^2}{\| \nabla \pi_i (Z^{k+1}) \|^2} \leq \frac{\| \Delta x_i^{k+1} \|^2}{\| \nabla \pi_i (Z^{k+1}) \|^2} = \frac{\| \Delta x_i^{k+1} \|^2}{\| \nabla \pi_i (Z^{k+1}) \|^2}.$$

Combining facts (55)-(57) allows us to obtain

$$\| \nabla \pi_i (Z^{k+1}) \|^2 = \| \nabla \pi_i (Z^{k+1}) \|^2 \leq \| \Delta x_i^{k+1} \|^2 \leq \frac{\| \Delta x_i^{k+1} \|^2}{\| \nabla \pi_i (Z^{k+1}) \|^2}.$$

Combining facts (55)-(57) allows us to obtain

$$\| \nabla \pi_i (Z^{k+1}) \|^2 = \| \nabla \pi_i (Z^{k+1}) \|^2 \leq \| \Delta x_i^{k+1} \|^2 \leq \frac{\| \Delta x_i^{k+1} \|^2}{\| \nabla \pi_i (Z^{k+1}) \|^2}.$$

Combining facts (55)-(57) allows us to obtain

$$\| \nabla \pi_i (Z^{k+1}) \|^2 = \| \nabla \pi_i (Z^{k+1}) \|^2 \leq \| \Delta x_i^{k+1} \|^2 \leq \frac{\| \Delta x_i^{k+1} \|^2}{\| \nabla \pi_i (Z^{k+1}) \|^2}.$$

Combining facts (55)-(57) allows us to obtain

$$\| \nabla \pi_i (Z^{k+1}) \|^2 = \| \nabla \pi_i (Z^{k+1}) \|^2 \leq \| \Delta x_i^{k+1} \|^2 \leq \frac{\| \Delta x_i^{k+1} \|^2}{\| \nabla \pi_i (Z^{k+1}) \|^2}.$$

Combining facts (55)-(57) allows us to obtain

$$\| \nabla \pi_i (Z^{k+1}) \|^2 = \| \nabla \pi_i (Z^{k+1}) \|^2 \leq \| \Delta x_i^{k+1} \|^2 \leq \frac{\| \Delta x_i^{k+1} \|^2}{\| \nabla \pi_i (Z^{k+1}) \|^2}.$$

Combining facts (55)-(57) allows us to obtain

$$\| \nabla \pi_i (Z^{k+1}) \|^2 = \| \nabla \pi_i (Z^{k+1}) \|^2 \leq \| \Delta x_i^{k+1} \|^2 \leq \frac{\| \Delta x_i^{k+1} \|^2}{\| \nabla \pi_i (Z^{k+1}) \|^2}.$$

Combining facts (55)-(57) allows us to obtain

$$\| \nabla \pi_i (Z^{k+1}) \|^2 = \| \nabla \pi_i (Z^{k+1}) \|^2 \leq \| \Delta x_i^{k+1} \|^2 \leq \frac{\| \Delta x_i^{k+1} \|^2}{\| \nabla \pi_i (Z^{k+1}) \|^2}.$$

Combining facts (55)-(57) allows us to obtain

$$\| \nabla \pi_i (Z^{k+1}) \|^2 = \| \nabla \pi_i (Z^{k+1}) \|^2 \leq \| \Delta x_i^{k+1} \|^2 \leq \frac{\| \Delta x_i^{k+1} \|^2}{\| \nabla \pi_i (Z^{k+1}) \|^2}.$$

Combining facts (55)-(57) allows us to obtain

$$\| \nabla \pi_i (Z^{k+1}) \|^2 = \| \nabla \pi_i (Z^{k+1}) \|^2 \leq \| \Delta x_i^{k+1} \|^2 \leq \frac{\| \Delta x_i^{k+1} \|^2}{\| \nabla \pi_i (Z^{k+1}) \|^2}.$$

Combining facts (55)-(57) allows us to obtain

$$\| \nabla \pi_i (Z^{k+1}) \|^2 = \| \nabla \pi_i (Z^{k+1}) \|^2 \leq \| \Delta x_i^{k+1} \|^2 \leq \frac{\| \Delta x_i^{k+1} \|^2}{\| \nabla \pi_i (Z^{k+1}) \|^2}.$$

Combining facts (55)-(57) allows us to obtain

$$\| \nabla \pi_i (Z^{k+1}) \|^2 = \| \nabla \pi_i (Z^{k+1}) \|^2 \leq \| \Delta x_i^{k+1} \|^2 \leq \frac{\| \Delta x_i^{k+1} \|^2}{\| \nabla \pi_i (Z^{k+1}) \|^2}.$$

Combining facts (55)-(57) allows us to obtain

$$\| \nabla \pi_i (Z^{k+1}) \|^2 = \| \nabla \pi_i (Z^{k+1}) \|^2 \leq \| \Delta x_i^{k+1} \|^2 \leq \frac{\| \Delta x_i^{k+1} \|^2}{\| \nabla \pi_i (Z^{k+1}) \|^2}.$$
Using the above condition, we obtain
\[
\mathcal{L}(Z^{k+1}) = \sum_{i=1}^{m} p_i f_i(x^{k+1}) = f(x^{k+1}) \geq f^*(x^{k+1}) = f^* - \infty,
\]
(60)

From (23), we conclude that
\[
\sum_{k=0}^{\infty} \sigma_{k+1} \leq \sum_{k=0}^{\infty} (\mathcal{L}(Z^k) - \mathcal{L}(X^{k+1})) = \mathcal{L}(Z^0) - \lim_{k \to \infty} \mathcal{L}(Z^{k+1}) < +\infty.
\]
The above condition means \(\Delta x^{k+1} \to 0\) and \(\Delta x^{k+1} \to 0\), yielding \(|\Delta \pi^{k+1}| \to 0\) by (40) for any \(i \in [m]\). Finally, we note that \(\Delta x^{k+1} = \Delta \pi^{k+1} / \sigma \to 0\) from (13) if \(i \in C^{k+1}\) and \(\Delta x^{k+1} = 0\) from (15) if \(i \notin C^{k+1}\). Overall, \(\Delta x^{k+1} \to 0\), which completes the whole proof is finished.

C. Proof of Theorem IV.1

Proof. i) It follows from Lemma B.2 that \(\{\mathcal{L}(Z^k)\}\) is non-increasing and bounded from below. Therefore, whole sequence \(\{\mathcal{L}(Z^k)\}\) converges. For \(i \notin C^{k+1}\), we have \(x_i^{k+1} = 0\) from (15), thereby leading to
\[
L(x^{k+1}, x_i^{k+1}, \pi_i^{k+1}) = \frac{1}{m} f_i(x_i^{k+1}).
\]
For \(i \in C^{k+1}\), it follows
\[
L(x^{k+1}, x_i^{k+1}, \pi_i^{k+1}) - \frac{1}{m} f_i(x_i^{k+1}) \leq \frac{1}{m} \left( \frac{\Delta \pi_i^{k+1}}{\sigma} + \frac{\| \Delta x_i^{k+1} \|^2}{\sigma} \right) \leq \frac{1}{m} \left( \frac{\Delta \pi_i^{k+1}}{\sigma} + \frac{\| \Delta x_i^{k+1} \|^2}{\sigma} \right) \leq \frac{1}{m} \left( \frac{\Delta \pi_i^{k+1}}{\sigma} + \frac{\| \Delta x_i^{k+1} \|^2}{\sigma} \right).
\]
Using the above two conditions, we can conclude that
\[
\mathcal{L}(Z^{k+1}) - F(X^{k+1}) = \sum_{i=1}^{m} L(x^{k+1}, x_i^{k+1}, \pi_i^{k+1}) - \frac{1}{m} f_i(x_i^{k+1}) \leq \sum_{i \in C^{k+1}} \frac{1}{m} \left( \frac{\Delta \pi_i^{k+1}}{\sigma} + \frac{\| \Delta x_i^{k+1} \|^2}{\sigma} \right) \leq \sum_{i=1}^{m} \frac{1}{2m} \| \pi_i^{k+1} \|^2 + \frac{1}{4m} \| \Delta x_i^{k+1} \|^2.
\]
In addition, same reasoning to show (59) enables to derive
\[
\frac{1}{m} f_i(x_i^{k+1}) - \frac{1}{m} f_i(x^{k+1}) \leq \frac{\| \pi_i^{k+1} \|^2}{\sigma} + \| \Delta x_i^{k+1} \|^2,
\]
where (59) yields that \(q_i^k = \frac{\| \pi_i^{k+1} \|^2}{\sigma} + \| \Delta x_i^{k+1} \|^2\), which by (59) yields that \(q_i^k = \frac{\| \pi_i^{k+1} \|^2}{\sigma} + \| \Delta x_i^{k+1} \|^2\), where
\[
q_i^k := \frac{1}{m} f_i(x_i^{k+1}) - \frac{1}{m} f_i(x^{k+1}) - \frac{\| \pi_i^{k+1} \|^2}{\sigma} + \| \Delta x_i^{k+1} \|^2.
\]
Therefore, the above fact brings out
\[
\mathcal{L}(Z^{k+1}) - f(x^{k+1}) = \sum_{i=1}^{m} \left( q_i^k + \frac{\| \pi_i^{k+1} \|^2}{\sigma} + \| \Delta x_i^{k+1} \|^2 \right) \leq \sum_{i=1}^{m} \frac{3\| \pi_i^{k+1} \|^2}{\sigma} + \| \Delta x_i^{k+1} \|^2 \to 0.
\]

ii) By (38), we derive that, for any \(k \in K\),
\[
0 = \sum_{k=1}^{m} (\pi_i^k + \sigma(x_i^k - x_i^{k+1})) = \sum_{i \in C^{k+1}} (\pi_i^k + \sigma \Delta x_i^{k+1} - \Delta x_i^{k+1}) + \sum_{i \notin C^{k+1}} (\pi_i^k - \sigma \Delta x_i^{k+1} - \Delta x_i^{k+1}) \leq \sum_{i \in C^{k+1}} (\pi_i^k + \sigma \Delta x_i^{k+1} + \Delta x_i^{k+1}) + \sum_{i \notin C^{k+1}} (\pi_i^k - \sigma \Delta x_i^{k+1} + \Delta x_i^{k+1})
\]
\[
= \sum_{i=1}^{m} (\pi_i^{k+1} - \sigma \Delta x_i^{k+1}) - \sum_{i \notin C^{k+1}} \Delta x_i^{k+1} + \Delta \pi_i^{k+1},
\]
(61)
which together with (58) implies \(\lim_{k \to \infty} \sum_{i=1}^{m} \pi_i^{k+1} = 0\). Let \(s := (\tau_{k+1} - 1)k_0 \in K\). Then
\[
\lim_{s \to \infty} \sum_{i=1}^{m} \pi_i^{s+1} = 0.
\]

Moreover, for any \(k\), it is easy to show that
\[
s + 1 = (\tau_{k+1} - 1)k_0 + 1 \leq k + 1 \leq \tau_{k+1}k_0,
\]
\[
\tau_{s+1} = [(s + 1)/k_0] = [\tau_{k+1} - 1 - 1/k_0] = \tau_{k+1}.
\]
Based on this, we now estimate \(\pi_i^{k+1} - \pi_i^{s+1}\) for any \(k\). For \(i \in C^{k+1}\), we can show that and hence
\[
\| \pi_i^{k+1} - \pi_i^{s+1} \| \leq \frac{1}{m} \| \pi_i^{k+1} \| + \frac{1}{m} H_i (\Delta x_i^{k+1} - \Delta x_i^{s+1}) \| \leq \frac{1}{m} \| \pi_i^{k+1} \| + \frac{1}{m} \| \Delta x_i^{k+1} \| + \frac{1}{m} \| \Delta x_i^{s+1} \|.
\]
(63)

For any \(i \notin C^{k+1}\), \(\pi_i^{k+1} = -\pi_i^{s+1}\) by (16). So, the above condition is still valid. Overall, we show that
\[
\lim_{k \to \infty} \sum_{i=1}^{m} \pi_i^{k+1} = 0.
\]
(65)
This together with (39) and (58) immediately gives us
\[
\lim_{k \to \infty} \nabla f(x^{k+1}) = \lim_{k \to \infty} \sum_{i=1}^{m} \pi_i^{k+1} = 0.
\]
(66)
Finally, the above condition together with \(\Delta x_i^{k+1} = (x_i^{k+1} - x_i^{k+1}) \to 0\) and the gradient Lipschitz continuity yields that
\[
\lim_{k \to \infty} \sum_{i=1}^{m} w_i \nabla f_i(x^{k+1}) = 0,
\]
completing the whole proof.

D. Proof of Theorem IV.2

Proof. i) It follows from Lemma B.2 i) and (60) that
\[
\mathcal{L}(Z^0) \geq \mathcal{L}(Z^{k+1}) \geq \sum_{i=1}^{m} w_i f_i(x^{k+1}) = f(x^{k+1}),
\]
which implies \(x^{k+1} \in S\) and hence \(\{x^{k+1}\}\) is bounded due to the boundedness of \(S\). This calls forth the boundedness of \(\{x^{k+1}\}\) as \(\Delta x^{k+1} \to 0\) from (58). Then the boundedness of sequence \(\{\pi_i^{k+1}\}\) can be ensured because of
\[
\| \pi_i^{k+1} \| \leq \frac{1}{m} \| \pi_i^{k+1} \| + \frac{1}{m} H_i \| \Delta x^{k+1} \| \leq \frac{1}{m} \| \pi_i^{k+1} \| + \frac{1}{m} H_i \| \Delta x^{k+1} \| + \frac{1}{m} \| \Delta x^{k+1} \| < +\infty,
\]
where \(\cdot < \cdot\) is from the boundedness of \(\{x^{k+1}\}\). Overall, \(\{Z^{k+1}\}\) is bounded. Let \(Z^\infty\) be any accumulating point of the sequence, it follows from (39) and \(\Delta x^{k+1} \to 0\) that
\[
0 = \pi_i^{k+1} + \pi_i^{k+1} + \frac{1}{m} H_i \Delta x^{k+1} \to \frac{1}{m} \nabla f_i(x^\infty) + \pi_i^\infty,
\]
(67)
Moreover, (65) and (58) suffice to \( \sum_{t=1}^{m} \pi_{i}^{\infty} = 0 \) and \( x_{i}^{\infty} = x_{\infty} = 0 \). By recalling (9), \( Z_{\infty} \) is a stationary point of (6) and \( x_{\infty} \) is a stationary point of (4).

ii) We first prove the result if \( x_{\infty} \) is isolated. Since \( \Delta x_{i-1}^{t+k} \to 0 \) and \( x_{\infty} \) being isolated, whole sequence \( \{\Delta x_{i-1}^{t+k}\} \) converges to \( x_{\infty} \) by [55, Lemma 4.10]. This together with \( \Delta x_{i}^{t+k} \to 0 \) and (39) implies that \( \{X_{i}^{k}\} \) and \( \{\Pi_{i}^{k}\} \) converge to \( X_{\infty} \) and \( \Pi_{\infty} \).

Now we show the result if every \( f_{i} \) has KL property. For any \( t \in \mathbb{N} := \{0, 1, 2, \ldots\} \), denote

\[
F_{t} := \mathcal{L}(Z_{t}^{k+1}), \quad \nabla F_{t} := \nabla \mathcal{L}(Z_{t}^{k+1}).
\]

Let \( \Omega \) be the set of all accumulating points of sequence \( \{Z_{t}^{k+1} : t \in \mathbb{N}\} \). Then \( \mathcal{L}(Z), \forall Z \in \Omega \) have the same value since whole sequence \( \{\mathcal{L}(Z_{t}^{k+1})\} \) converges. Denote \( \mathcal{F}^{\infty} := \mathcal{L}(Z), \forall Z \in \Omega \).

By the non-increasing property of \( \{\mathcal{L}(Z^{k})\} \), we have

\[
F_{t+1} = \mathcal{L}(Z_{t}^{(t+1)k+1}) \leq \mathcal{L}(Z_{t}^{(t+1)k+1}) - \frac{\sigma}{24} \mathcal{L}(Z_{t}^{(t+1)k+1}) \leq \mathcal{L}(Z_{t}^{(t+1)k+1}) - \frac{\sigma}{24} \mathcal{L}(Z_{t}^{(t+1)k+1}) \leq F_{t} - \frac{\sigma}{24} \mathcal{L}(Z_{t}^{(t+1)k+1})
\]

and

\[
\|\nabla F_{t}\| = \|\nabla \mathcal{L}(Z_{t}^{k+1})\| \leq (2\sigma + 1) \sqrt{m/\mathcal{L}(Z_{t}^{k+1})}.
\]

Now for any \( \eta > 0 \) and \( \delta > 0 \), there exists \( t_{0} \in \mathbb{N} \) such that

\[
Z_{t}^{k+1} \in \{Z : \text{dist}(Z, \Omega) \leq \delta\} \cap \{Z : \mathcal{L}(Z) \leq \mathcal{L}_{\infty} + \eta\}, \quad t \geq t_{0},
\]

due to \( \lim_{t \to \infty} Z_{t}^{k+1} \in \Omega \) and \( F_{t} \to F_{\infty} \). Since every \( f_{i} \) has KL property, so has \( \mathcal{L} \). Then from the KL property, there exists a desingularizing function \( \varphi \) such that

\[
\varphi'(F_{t} - F_{\infty}) \|

\cdot \nabla F_{t} \| \geq 1,
\]

namely,

\[
\varphi'(F_{t} - F_{\infty}) \geq \frac{1}{\|\nabla F_{t}\|} \geq \frac{1}{(2\sigma + 1) \sqrt{m/\mathcal{L}(Z_{t}^{k+1})}}.
\]

The above condition and \( \varphi \) being concave bring out

\[
\varphi(F_{t+1} - F_{\infty}) - \varphi(F_{t} - F_{\infty}) \leq \varphi(F_{t} - F_{\infty})(F_{t+1} - F_{t}) \leq \frac{a}{\delta \sqrt{m/\mathcal{L}(Z_{t}^{k+1})}},
\]

where \( a := 24(2\sigma + 1) \sqrt{m/\sigma} \), which suffices to

\[
\sqrt{\mathcal{L}(t+1)k+1} \leq \sqrt{\mathcal{L}(t)k+1} + \frac{a}{\delta \sqrt{m/\mathcal{L}(Z_{t}^{k+1})}}.
\]

Summing the both sides of the above condition yields

\[
\sum_{t \geq 0} \sqrt{\mathcal{L}(t+1)k+1} \leq \sqrt{\mathcal{L}} + a \varphi(F_{0} - F_{\infty}) < \infty.
\]

We note from (24) that

\[
\varphi_{k+1} = \sum_{t=1}^{m} \varphi_{t+1} = \sum_{t=1}^{m} \varphi_{t+1} \|\Delta x_{t+1}^{k} + \|\Delta x_{t+1}^{k} - 2\|\Delta x_{t+1}^{k} \|\Delta x_{t+1}^{k} \|2 \|\Delta x_{t+1}^{k} \|2
\]

where \( \varphi_{k+1} \) is a Cauchy sequence and hence is convergent, resulting in the convergence of sequences \( \{Z_{t}^{k+1} : t \in \mathbb{N}\} \) and \( \{X_{t}^{k+1} : t \in \mathbb{N}\} \). These by (39) and \( \Delta x_{i}^{t+k} \to 0 \) indicate \( \{\Pi_{t}^{k+1}\} \) also converges. Overall, \( \{Z_{t}^{k+1}\} \) converges. Recalling (58) and \( k_{0} \) is a finite number, we can conclude \( \{Z_{k}^{k+1}\} \) has the same convergence behaviour of \( \{Z_{t}^{k+1}\} \).

\[\square\]

E. Proof of Corollary IV.1

Proof. i) The convexity of \( f \) and the optimality of \( x_{\infty} \) lead to

\[
f(x_{t}^{k}) \geq f(x_{\infty}) \geq f(x_{t}^{k}) + \langle \nabla f(x_{t}^{k}), x_{\infty} - x_{t}^{k} \rangle.
\]

Theorem IV.1 ii) states that

\[
\lim_{k \to \infty} \nabla F(x_{t}^{k}) = \lim_{k \to \infty} \nabla f(x_{t}^{k}) = 0.
\]

Using this and the boundedness of \( \{x_{t}^{k}\} \) from Theorem IV.2, we take the limit of both sides of (74) to derive that \( f(x_{t}^{k}) \to f(x_{\infty}) \), which recalling Theorem IV.1 i) yields (28).

ii) The conclusion follows from Theorem IV.2 iii) and the fact that the stationary points are equivalent to optimal solutions if \( f \) is convex.

iii) The strong convexity of \( f \) means that there is a positive constance \( \nu \) such that

\[
f(x_{t}^{k}) - f(x_{\infty}) \geq \langle \nabla f(x_{\infty}), x_{t}^{k} - x_{\infty} \rangle + \frac{\nu}{2} \|x_{t}^{k} - x_{\infty}\|^{2}
\]

where the equality is due to (9). Taking limit of both sides of the above inequality immediately shows \( x_{t}^{k} \to x_{\infty} \) since \( f(x_{t}^{k}) \to f(x_{\infty}) \). This together with (58) yields \( x_{k}^{k} \to x_{\infty} \).

Finally, \( \pi_{i}^{k} \to \pi_{i}^{\infty} \) because of

\[
\|\pi_{i}^{k} - \pi_{i}^{\infty}\| \leq \frac{\nu}{\sqrt{m}} \|x_{t}^{k} - x_{\infty}\| + \|\Delta x_{t}^{k}\| \to 0,
\]

displaying the desired result.

\[\square\]

F. Proof of Theorem IV.3

Proof. For any \( j \geq 1 \) and (45), there is

\[
\|\Delta x_{i}^{j+1} - \Delta x_{i}^{j+1/k}\| \leq \frac{a}{\delta} \|\Delta x_{i}^{j+1/k} - \Delta x_{i}^{j+1/k}\|.
\]

We note that \( \Delta x_{i}^{j+1/k} = \Delta x_{i}^{j+1/k} \|\Delta x_{i}^{j+1/k} - \Delta x_{i}^{j+1/k}\| \) if \( i \in C_{j+1/k} \) and \( \Delta x_{i}^{j+1} = 0 \) if \( i \notin C_{j+1/k} \). Therefore,

\[
\|\Delta x_{i}^{j+1}\| \leq \|\Delta x_{i}^{j+1/k}\|, \quad \forall i \in [m].
\]

Now we focus on \( s \in C \). This by (61) results in

\[
\sum_{i=1}^{m} \pi_{i}^{j+1} = \sum_{i=1}^{m} \sigma \Delta x_{i}^{j+1} + \sum_{i \notin C_{j+1/k}} \Delta x_{i}^{j+1},
\]

which leads to

\[
\|\sum_{i=1}^{m} \pi_{i}^{j+1}\| \leq m \sum_{i=1}^{m} \|\sigma \Delta x_{i}^{j+1}\| + \|\Delta x_{i}^{j+1}\|.\]
Using this condition generates

\[
\|\nabla f(x^{t+1})\|^2 = \|\sum_{i=1}^{m} \mathbf{g}_i^{t+1}\|^2 \\
= \left\| \sum_{i=1}^{m} \left( \pi_i^{t+1} + \frac{1}{\rho_i H_i} \triangle \mathbf{x}_i^{t+1}\right) \right\|^2 \\
\leq 2\left\| \sum_{i=1}^{m} \pi_i^{t+1}\right\|^2 + 2m \sum_{i=1}^{m} \frac{r_i^2}{\rho_i} \|\triangle \mathbf{x}_i^{t+1}\|^2 \\
\leq 2\left\| \sum_{i=1}^{m} \pi_i^{t+1}\right\|^2 + \frac{m^2}{\rho} \sum_{i=1}^{m} \|\triangle \pi_i^{t+1}\|^2 \tag{76}
\]

(44),(77)

\[
\leq 5m^2 \sum_{i=1}^{m} \|\triangle x_i^{t+1}\|^2 + \|\triangle \mathbf{x}_i^{t+1}\|^2 + \|\triangle \mathbf{x}_i^{t+1}\|^2 \\
\leq \frac{100m\sigma}{\rho} (\mathcal{L}(\mathbf{z}^t) - \mathcal{L}(\mathbf{z}^{t+1})). \tag{79}
\]

Since sequence \(\{\mathcal{L}(\mathbf{z}^k)\}\) is non-increasing from Lemma B.2, it has \(\mathcal{L}(\mathbf{z}^{t+1}) \geq \mathcal{L}(\mathbf{z}^{t+1}) \geq f^*\) by Lemma B.2 for any \(t \geq 0\), thereby resulting in

\[
\sum_{k=0}^{t-1} (\mathcal{L}(\mathbf{z}^{k}) - \mathcal{L}(\mathbf{z}^{k+1})) = \mathcal{L}(\mathbf{z}^t) - \mathcal{L}(\mathbf{z}^{t+1}) = \mathcal{L}(\mathbf{z}^{t+1}) - \mathcal{L}(\mathbf{z}^{t+1}) = \mathcal{L}(\mathbf{z}^t) - f^*.
\]

We note that for \(j = 0, 1, 2, \ldots, \tau_{k+1}k_0 - 1\),

\[
\mathbf{t}_j = \begin{cases} 
1, & \text{if } j = 0, 1, \ldots, k_0 - 1, \\
2, & \text{if } j = k_0, k_0 + 1, \ldots, 2k_0 - 1, \\
\vdots & \\
\tau_{k+1}, & \text{if } j = (\tau_{k+1} - 1)k_0, \ldots, k_0, \tau_{k+1}k_0 - 1.
\end{cases}
\]

Using the above three facts and \(k < \tau_{k+1}k_0 - 1\), we derive

\[
\text{min}_{j \in [k]} \|\nabla f(x^{t_j})\|^2 \\
\leq \frac{1}{k} \sum_{j=0}^{k-1} \|\nabla f(x^{t_j})\|^2 \\
\leq \frac{1}{k} \sum_{j=0}^{k-1} \|\nabla f(x^{t_{j+1}})\|^2 \\
= \frac{k_0}{k} \sum_{j=0}^{k-1} \|\nabla f(x^{t_{j+1}})\|^2 \\
\leq \frac{100m\sigma k_0}{\rho} (\mathcal{L}(\mathbf{z}^t) - \mathcal{L}(\mathbf{z}^{t+1})) \tag{79}
\]

completing the proof. \(\square\)

G. Proof of Theorem IV.4

Proof. Again, for any \(t \in \mathbb{N} := \{0, 1, 2, \ldots\}\), denote

\[
\mathcal{F}^t := \mathcal{L}(\mathbf{z}^{t+k_0+1}), \quad \mathcal{F} := \mathcal{F}^t - \mathcal{F}^\infty.
\]

Throughout the proof, assume \(\mathcal{F}^t \neq \mathcal{F}^\infty\). Recalling \(\varphi(z) = \frac{\sqrt{t}}{1-\varphi} z^{1-\theta}\), we obtain

\[
1 \leq (\varphi'(\mathcal{F}^t)) (2\sigma + 1)^2 \frac{m^2 \sigma k_0 + 1}{\rho} \\
\leq c(\mathcal{F}^t)^{-2\theta} (2\sigma + 1)^2 m \sigma k_0 + 1 \\
\leq c(\mathcal{F}^t)^{-2\theta} \frac{24m(2\sigma + 1)^2}{\rho} (\mathcal{F}^t - \mathcal{F}^t). \tag{68}
\]

By letting \(\rho := \frac{24m(2\sigma + 1)^2}{\sigma}\), we have

\[
\rho(\mathcal{F}^t - \mathcal{F}^t) \geq (\mathcal{F}^t)^{2\theta}. \tag{80}
\]

Now we prove the results by three cases:

- If \(\theta = 0\), then \(\rho(\mathcal{F}^t - \mathcal{F}^t) \geq 1\). However, \(\mathcal{F}^t \rightarrow 0\), which leads to a contradiction. Therefore, we have that \(\mathcal{F}^t = \mathcal{F}^\infty\) when \(t\) is over a threshold \(t_1 \geq t_0 > 0\).

- If \(\theta \in (0, 1/2]\), then there is a \(t_2 \geq t_0 > 0\) such that \(\mathcal{F}^t \in [0, 1]\) for any \(t \geq t_2\) as \(\mathcal{F}^t \rightarrow 0\), which by (80) yields

\[
\rho(\mathcal{F}^t - \mathcal{F}^t) \geq (\mathcal{F}^t)^{2\theta} \geq \mathcal{F}^t.
\]

This allows us to derive that

\[
\mathcal{F}^t \leq \frac{\mathcal{F}^t}{\mathcal{F}^t + 1} \cdot \mathcal{F}^t \leq \cdots \leq (\mathcal{F}^t)^{2\theta - 1} \geq \mathcal{F}^t.
\]

- If \(\theta \in (1/2, 1]\), then \(\varphi(z) := \frac{1}{(\rho + 1)^2} z^{1-2\theta}\) is an increasing function. If \((\mathcal{F}^t)^{2\theta - 1} \geq (\mathcal{F}^t)^{2\theta - 2}/2\), then

\[
\rho(\mathcal{F}^t - \mathcal{F}^t) = \frac{\mathcal{F}^t}{\mathcal{F}^t + 1} \cdot \mathcal{F}^t = \frac{\mathcal{F}^t}{\mathcal{F}^t + 1} \cdot \mathcal{F}^t \geq \mathcal{F}^t \geq \mathcal{F}^t.
\]

If \((\mathcal{F}^t)^{2\theta - 1} \leq (\mathcal{F}^t)^{2\theta - 2}/2\), then

\[
\rho(\mathcal{F}^t - \mathcal{F}^t) = \frac{\mathcal{F}^t}{\mathcal{F}^t + 1} \cdot \mathcal{F}^t \geq \mathcal{F}^t \geq \mathcal{F}^t.
\]

which by the non-increasing property of \(\{\mathcal{F}^t\}\) suffices to

\[
(1 - 2\theta)(\rho(\mathcal{F}^t - \mathcal{F}^t)) = \rho(\mathcal{F}^t)^{1-2\theta} - \mathcal{F}^t)^{1-2\theta} \leq \rho(\mathcal{F}^t)^{1-2\theta} - \mathcal{F}^t)^{1-2\theta} \leq \rho(\mathcal{F}^t)^{1-2\theta} - \mathcal{F}^t)^{1-2\theta} = : c.
\]

Let \(a := \min\left\{ \frac{1}{2}, \frac{c}{1-2\theta} \right\}\). Both cases lead to

\[
\rho(\mathcal{F}^t - \mathcal{F}^t) \geq \mathcal{F}^t \geq a = \mathcal{F}^t.
\]

We note from (70), the KL property holds when \(t \geq t_0\). Therefore, the above inequality holds for \(t \geq t_3 = t_0\). Summing the both sides of the above inequality for \(j = t_3, t_3 + 1, \ldots, t\) derives that

\[
\rho(\mathcal{F}^t - \mathcal{F}^t) = \sum_{j=t_3}^{t} a = (t - t_3)a.
\]

This condition indicates

\[
\frac{\rho}{1-2\theta}(\mathcal{F}^t)^{1-2\theta} = \rho(\mathcal{F}^t) \leq (t - t_3)a + \frac{\rho}{1-2\theta}(\mathcal{F}^t)^{1-2\theta} \leq (t - t_3)a.
\]

Henceforth, we obtain

\[
\mathcal{F}^t \leq \frac{(2\theta - 1)a}{\rho} (t - t_3) < 2\theta - 1 \approx \rho.
\]

Finally, the above three cases, \(f(x^t) \leq \mathcal{L}(\mathbf{z}^t)\) for any \(k \geq 0\) from lemma B.2 ii), and (26) can derive the conclusion. \(\square\)