The classification of abelian groups generated by time-varying automata and by Mealy automata over the binary alphabet

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Abstract
For every natural number $n$, we classify abelian groups generated by an $n$-state time-varying automaton over the binary alphabet, as well as by an $n$-state Mealy automaton over the binary alphabet.

1. Introduction
In the theory of computation time-varying automata over a finite alphabet are finite-state transducers which constitute a natural generalization of Mealy-type automata as they allow to change both the transition function and the output function in successive steps of processing input sequences of letters into output sequences (see [10]). These automata in turn constitute a subclass in the class of time-varying automata over a changing alphabet (see [14]). In group theory all these types of transducers turned out to be a useful tool for defining and studying groups of automorphisms of certain rooted trees, which helped in the discovery of interesting geometric and algebraic properties and the dynamics of various types of groups associated with their actions on these trees. The intensive study of Mealy automata in relation to automorphism groups has continued for last three decades, and the extraordinary properties of these groups are discussed in a great deal of papers (for the comprehensive works see [6, 7, 8, 9, 11]). On the other hand, the investigation of automorphism groups as groups defined by time-varying automata is a quite new approach, which we introduced in [14] (for other results on this subject see [3, 4, 15, 16, 18, 19]).

If an automaton $A$ is invertible, then each of its states defines an automorphism of the corresponding rooted tree. The group generated by these automorphisms (i.e. by automorphisms corresponding to all the states of $A$) is called the group generated by the automaton $A$ and is usually denoted by $G(A)$.

**Definition 1.** Let $n$ and $k$ be natural numbers and let $G$ be an abstract group. We say that $G$ is generated by an $n$-state time-varying (Mealy) automaton over
a $k$-letter alphabet if there is an invertible time-varying (Mealy) automaton $A$ with an $n$-element set of states which works over a $k$-letter alphabet such that $G$ is isomorphic to the group $G(A)$ generated by this automaton.

Given two natural numbers $n$, $k$ and a class of abstract groups, the natural question arises: which groups from this class are generated by an $n$-state (time-varying or Mealy) automaton over a $k$-letter alphabet? This problem was studied so far only in the case of Mealy-type automata and it turned out to be difficult even for small values of $n$ and $k$ and for classes containing algebraically well known constructions, such as abelian groups, free groups, free products of finite groups, and many others.

It is worth noting that there is no general methods for deducing even the basic algebraic properties of the group generated by an automaton directly from the structure of this automaton. For example, for a long time it was an open problem whether a free non-abelian group is generated by a Mealy automaton (see [5] solving this problem). However, a candidate for the solution was known and studied since 80’s last century (the so-called Aleshin-Vorobets automaton – see [1, 13]). Until now, we do not know if the free non-abelian group of rank two is generated by a 2-state Mealy automaton or by a 2-state time-varying automaton over a finite alphabet. On the other hand, we provided for this group a natural construction of a 2-state time-varying automaton over an unbounded changing alphabet (see [16]).

The full classification of groups generated by Mealy automata is known only in the simplest non-trivial case, i.e. when $n = k = 2$. Apart from that, for the classes of abelian groups and finite groups the solutions for Mealy automata are known only in the case $(n, k) = (3, 2)$ (see Theorems 3,4 in the extensive work [2] devoted to the classification of all groups generated by 3-state Mealy automata over the binary alphabet).

2. The results

The goal of this paper is providing for every natural number $n$ the full classification of abelian groups generated by an $n$-state time-varying automaton over the binary alphabet as well as by an $n$-state Mealy automaton over the binary alphabet.

Let us denote the following classes of abelian groups:

- $TV\mathcal{A}(n)$ – the class of abelian groups generated by an $n$-state time-varying automaton over the binary alphabet,
- $MA(n)$ – the class of abelian groups generated by an $n$-state Mealy automaton over the binary alphabet,
- $AB_2(n)$ – the class of abelian groups of rank not greater than $n$ in which the torsion part is a 2-group,
- $FA(n)$ – the class of free abelian groups of rank not greater than $n$,
The main result of the paper is the following characterization.

**Theorem 1.** $\mathcal{TVA}(1) = \mathcal{EA}_2(1)$, and if $n > 1$, then $\mathcal{TVA}(n) = \mathcal{AB}_2(n)$.

**Theorem 2.** $\mathcal{MA}(n) = \mathcal{FA}(n-1) \cup \mathcal{EA}_2(n)$. In particular, there are exactly $2n$ abelian groups generated by an $n$-state Mealy automaton over the binary alphabet.

For the proof of Theorem 1 (Section 5), we find at first some restrictions on abelian groups generated by time-varying automata over the binary alphabet. These restrictions imply the inclusion $\mathcal{TVA}(n) \subseteq \mathcal{AB}_2(n)$ for each $n \geq 1$, as well as the equality $\mathcal{TVA}(1) = \mathcal{EA}_2(1)$. Next, for every $n \geq 1$ and every group $G \in \mathcal{AB}_2(n)$ of rank $n$, we provide an explicit construction of a time-varying automaton $A$ over the binary alphabet which generates $G$ (see Propositions 5–7). In the construction of the automaton $A$, we distinguish between three cases: where the group $G$ is cyclic, where $G$ is non-cyclic and has a non-trivial torsion part, and finally, where $G$ is a non-cyclic, torsion-free group. Note that in the last case $G$ must be a free abelian group, which follows from the property that every finitely generated, abelian group is a direct sum of cyclic groups. To derive the required property of our construction, we use the language of wreath recursions, which is popular in the study of groups generated by automata, and which arises from an embedding of the group $G(A)$ into the permutational wreath product (for more on the method involving wreath recursions in the case of Mealy automata see, for example, [7, 11], and in the case of time-varying automata – see [19]).

In the proof of Theorem 2 (Section 6), for an arbitrary group $G$ generated by a Mealy automaton over the binary alphabet, we deduce certain relations between the first level stabilizer of $G$, the set $I_G$ of involutions and the set $G^2$ of squares. We show (Proposition 8) a quite unexpected dichotomy, which holds in the case when $G$ is abelian. Namely, the relations imply in this case that $G$ is either a free abelian group or an elementary abelian 2-group. Next, we introduce and study a construction of the Mealy automata generating elementary abelian 2-groups. We also use the known construction of the Mealy automata generating free abelian groups (so-called “sausage” automata – see [7]), as well as the restriction for the rank of free abelian groups generated by Mealy automata over the binary alphabet (see Proposition 3.1 from [17]).

We hope that the above characterizations, together with the involved constructions and the methods presented in the proofs, will encourage to develop further study of the groups generated by time-varying automata. This may be useful when investigating some important, computational problems, which are open in the class of groups generated by time-varying automata, but have a simple solution in the class of groups generated by Mealy automata. Let us mention the word problem for groups (which is decidable in the latter class). It is known...
that there is a time-varying automaton $A$ over an arbitrary, unbounded changing alphabet such that the group $G(A)$ has undecidable word problem. This follows from the fact that there exist finitely generated, residually finite groups with undecidable word problem, as well as from some unconstructive method of defining an automaton realization for an arbitrary, finitely generated, residually finite group (the so-called diagonal realization – see [14, 18]). However, there is not known any explicit and naturally defined construction of an automaton $A$ such that the group $G(A)$ has undecidable word problem. Moreover, in the class of time-varying automata over a finite alphabet, even the question on the existence of such an automaton is open.

In view of the above study, also the following problem is interesting and far from trivial: given $n$ and $k$, how many (pairwise non-isomorphic) groups are contained in the class $\mathcal{GT}(n,k)$ of the groups generated by an $n$-state time-varying automaton over a $k$-letter alphabet? Obviously, if $n = 1$ or $k = 1$, then the class $\mathcal{GT}(n,k)$ is finite. Further, if $n \leq n'$ and $k \leq k'$, then $\mathcal{GT}(n,k)$ is contained in $\mathcal{GT}(n',k')$ (see the last paragraph of Section 5 for the corresponding reasoning). Directly by Theorem 1 we obtain $\mathcal{AB}_2(2) \subseteq \mathcal{GT}(2,2)$, which implies that $\mathcal{GT}(2,2)$ contains infinitely many groups. But, if $k > 1$ or $n > 1$, then there are uncountable many $n$-state time-varying automata over a $k$-th letter alphabet. Thus the natural problem arises: find the smallest $n$ and $k$ such that the class $\mathcal{GT}(n,k)$ is uncountable. In particular, is the class $\mathcal{GT}(2,2)$ uncountable? Note that the class of all groups generated by an $n$-state Mealy automaton over a $k$-letter alphabet is finite (there are exactly $(n^k \cdot k!)^n$ invertible Mealy automata with the $n$-element set of states over a $k$-th letter alphabet), which implies that the class of all groups generated by Mealy automata is countable.

The paper is organized as follows. Section 3 contains definitions of an automaton over a finite alphabet and the group generated by the automaton transformations of the corresponding rooted tree. In Section 4 we describe the notion of the wreath recursion. In the last two sections, we present the proofs of our characterization.

3. Automata and groups generated by automata

Let $\mathbb{N} = \{1, 2, \ldots \}$ be the set of natural numbers, and let $X$ be a nonempty, finite set (a finite alphabet). A time-varying automaton over $X$ is a quadruple $A = (X, Q, \varphi, \psi)$, where $Q$ is a finite set (set of states), $\varphi = (\varphi_i)_{i \in \mathbb{N}}$ is a sequence of so-called transition functions $\varphi_i : Q \times X \to Q$, and $\psi = (\psi_i)_{i \in \mathbb{N}}$ is a sequence of so-called output functions $\psi_i : Q \times X \to X$. If the sequences $\psi$ and $\varphi$ are constant, then the automaton $A$ is called a Mealy automaton. In this case the sequences $\varphi$, $\psi$ are identified with, respectively, the transition function $\varphi_1$ and the output function $\psi_1$, i.e. we can write $A = (X, Q, \varphi_1, \psi_1)$. If for every $i \in \mathbb{N}$ and every $q \in Q$ the mapping $\sigma_{i,q} : x \mapsto \psi_i(q, x)$ (the so-called labeling of the state $q$ in
the $i$-th transition of $A$) is a bijection on the alphabet, then the automaton $A$ is called invertible.

The tree $X^*$ of finite words over a finite alphabet $X$ consists of finite sequences of the form $x_1 x_2 \ldots x_l$ ($l \in \mathbb{N}$), where $x_i \in X$ ($1 \leq i \leq l$), together with the empty sequence (empty word) denoted by $\epsilon$. The tree $X^*$ is an example of a rooted tree with the empty word as the root, and two words are adjacent if and only if they are of the form $w$ and $wx$ for some $w \in X^*$, $x \in X$.

Let $A = (X, Q, \varphi, \psi)$ be a time-varying automaton over $X$. We consider the transformations

$$q_i : X^* \to X^*, \quad q \in Q, \quad i \in \mathbb{N}$$

defined in the following recursive way:

- $q_i(\epsilon) = \epsilon$ for all $q \in Q, i \in \mathbb{N}$,
- $q_i(xw) = x'q_{i+1}'(w)$ for any $q \in Q$, $i \in \mathbb{N}$, $x \in X$, $w \in X^*$, where $x' = \psi_i(q, x) = \sigma_{i,q}(x)$, $q' = \varphi_i(q, x)$.

The image $q_i(w)$ of any word $w \in X^*$ can be easily found directly from the diagram of the automaton $A = (X, Q, \varphi, \psi)$. We define such a diagram as an infinite, directed, locally finite graph with the set $Q \times \mathbb{N}$ as the set of vertices. Each vertex $(q, i) \in Q \times \mathbb{N}$ is labeled by the labeling $\sigma_{i,q}$ of the state $q$ in the $i$-th transition of $A$. Two vertices $(q, i), (q', i') \in Q \times \mathbb{N}$ are connected with a directed edge which starts at $(q, i)$ and ends at $(q', i')$ if and only if $i' = i + 1$ and there is $x \in X$ such that $\varphi_i(q, x) = q'$; we label this edge by the letter $x$. In particular, for every $i \in \mathbb{N}$, $q \in Q$ and $x \in X$, there is a unique edge which is labeled by $x$ and starts at $(q, i)$. Now, if $w = x_1 \ldots x_l \in X^*$ is any word and if $\tau_1, \ldots, \tau_l$ are the labels of the consecutive vertices on the directed path which is labeled by $w$ and which starts at $(q, i)$, then we have $q_i(w) = \tau_1(x_1) \ldots \tau_l(x_l)$.

The transformation $q_i$ is called the automaton transformation corresponding to the state $q$ in its $i$-th transition. If the automaton $A$ is invertible, then each $q_i$ is an element of the automorphism group $\text{Aut}(X^*)$ of the tree $X^*$, i.e. $q_i$ is a permutation of the set $X^*$ preserving the root $\epsilon$ and the vertex adjacency. For every $i \in \mathbb{N}$ we denote by $G(A_i)$ the subgroup of $\text{Aut}(X^*)$ generated by automorphisms $q_i$ for $q \in Q$:

$$G(A_i) := \langle q_i : q \in Q \rangle.$$

The group

$$G(A) := G(A_1)$$

is called the group generated by the automaton $A$.

4. The language of wreath recursions

Let $A = (Q, X, \varphi, \psi)$ be a time-varying automaton over a finite alphabet $X$. We define the transformations

$$q_i|_w : X^* \to X^*, \quad i \in \mathbb{N}, \quad q \in Q, \quad w \in X^*$$

in the following recursive way:
• $q_i|_e = q_i$ for all $q \in Q$, $i \in \mathbb{N}$,
• $q_i|_{xw} = q'_i|_w$ for all $q \in Q$, $i \in \mathbb{N}$, $x \in X$, $w \in X^*$, where $q' = \varphi_i(q, x)$.

The transformation $q_i|_w$ is called the section of the transformation $q_i$ at the word $w$. Assuming that $A$ is invertible, we can use the labelings of the states and the sections at one-letter words to describe the elements of the groups $G(A_i) (i \in \mathbb{N})$ in the algebraical language of wreath products. Namely, if $X = \{x_1, \ldots, x_k\}$, then for every $i \in \mathbb{N}$ the mapping

$$q_i \rightarrow (q_i|x_1, \ldots, q_i|x_k)\sigma_{i,q}, \quad q \in Q$$

induces an embedding

$$\phi_i : G(A_i) \rightarrow G(A_{i+1}) \wr \text{Sym}(X)$$

of the group $G(A_i)$ into the permutational wreath product of the group $G(A_{i+1})$ and the symmetric group of the alphabet, that is into the semidirect product

$$G(A_{i+1})^X \rtimes \text{Sym}(X)$$

with the natural action of $\text{Sym}(X)$ on the direct power $G(A_{i+1})^X$ by permuting the factors. Further, we identify any element $g \in G(A_i)$ with its image $\phi_i(g)$ and call the relation $g = \phi_i(g)$ the wreath recursion of $g$. In particular, for the wreath recursion of any generator $q_i \in G(A_i)$ ($q \in Q$) we have:

$$q_i = ((r_1)_{i+1}, \ldots, (r_k)_{i+1})\sigma_{i,q},$$

where $r_j = \varphi_i(q, x_j)$ for $1 \leq j \leq k$. In general, if

$$g = (h_1, \ldots, h_k)\pi$$

is the wreath recursion of an arbitrary element $g \in G(A_i)$, then the element $h_j \in G(A_{i+1})$ ($1 \leq j \leq k$), denoted further by $g|_{x_j}$, is called the section of $g$ at the letter $x_j \in X$, and the permutation $\pi$, denoted further by $\sigma_{g}$, is called the root permutation of $g$. In particular, for the wreath recursion of the inverse $g^{-1}$ we have

$$g^{-1} = ((g|_{y_1})^{-1}, \ldots, (g|_{y_k})^{-1})\pi^{-1},$$

where $y_j = \pi^{-1}(x_j)$ ($1 \leq j \leq k$), and if an element $g' \in G(A_i)$ is defined by the wreath recursion

$$g' = (g'|_{x_1}, g'|_{x_2}, \ldots, g'|_{x_k})\pi',$$

then the wreath recursion of the product $g \cdot g'$ satisfies:

$$g \cdot g' = (g|_{x_1} \cdot g'|_{z_1}, \ldots, g|_{x_k} \cdot g'|_{z_k})\pi\pi',$$

where $z_j = \pi(x_j)$ for $1 \leq j \leq k$.  

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Remark 1. If the root permutation $\sigma_g$ is trivial, then it is usually omitted in the wreath recursion, and we write $g = (g|_{x_1}, \ldots, g|_{x_k})$; also, if the sections $g|_{x_j}$, $(1 \leq j \leq k)$ are all trivial, then $g$ is identified with its root permutation, and we write $g = \sigma_g$.

We also define the sections $g|_w$ ($g \in G(A_i)$, $w \in X^*$) recursively: $g|_e = g$, $g|_{xw} = (g|_x)|_w$ for all $x \in X$, $w \in X^*$.

It is worth noting that if $A$ is a Mealy automaton, then for any $i \in \mathbb{N}$ and any $q \in Q$ the automaton transformation $q_i : X^* \to X^*$ of the state $q$ in its $i$-th transition coincides with the automaton transformation $q_1$. In particular, if $A$ is invertible, then $G(A_i) = G(A)$ for every $i \in \mathbb{N}$, and if $g \in G(A)$ and $w \in X^*$, then $g|_w \in G(A)$.

Given a nonempty word $w = x_1x_2 \ldots x_l \in X^*$ and an element $g \in G(A)$, we can use the notions of a section and a root permutation to compute the image $g(w)$ of $w$ under $g$ in the following way:

$$g(w) = g_0(x_1)g_1(x_2) \ldots g_{l-1}(x_l) = \sigma_{g_0}(x_1)\sigma_{g_1}(x_2) \ldots \sigma_{g_{l-1}}(x_l),$$

where $g_0 := g$ and $g_i := g|_{x_1x_2 \ldots x_i}$ for $1 \leq i \leq l - 1$.

5. The proof of Theorem 1

The inclusion $\mathcal{TV}A(n) \subseteq \mathcal{AB}_2(n)$ follows from the obvious observation that the rank of a group generated by an $n$-state time-varying automaton is not greater than $n$, as well as from the following result

Proposition 3. In the group $\text{Aut}(\{0,1\}^*)$ the order of any element of finite order is a power of two.

Proof. Let $g \in \text{Aut}(\{0,1\}^*)$ be an element of a finite order $d := o(g)$. For every $1 \leq i < d$ there is a word $w_i \in \{0,1\}^*$ such that $g^i(w_i) \neq w_i$. Let $m = \max\{|w_i| : 1 \leq i < d\}$ and let $\overline{g}$ be the restriction of $g$ to the subtree $T := \{0,1\}^{\leq m}$ consisting of all binary words of length not greater than $m$. Then $\overline{g} \in \text{Aut}(T)$ and $o(\overline{g}) = o(g) = d$. It is well known that the group $\text{Aut}(T)$ is isomorphic to the $m$-iterated wreath product $C_2 \wr C_2 \wr \cdots \wr C_2$ of the cyclic group of order two, and that the order of this wreath product is equal to $2^{2^m-1}$.

In particular, we have the divisibility $d|2^{2^m-1}$, which implies the claim. \hfill $\Box$

The next two propositions concern the generation of cyclic groups by time-varying automata over the binary alphabet equipped, respectively, with a single state and two states.

Proposition 4. The only groups generated by a time-varying automaton over the binary alphabet with a single state is the trivial group and the group of order two, i.e. we have $\mathcal{TV}A(1) = \mathcal{EA}_2(1)$. 

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PROOF. If $A = \{\{0,1\}, \{a\}, \varphi, \psi\}$ is a time-varying automaton with a single state $a$, then the corresponding automaton transformations $a_i: \{0,1\}^* \rightarrow \{0,1\}^*$ ($i \in \mathbb{N}$) satisfy the following wreath recursions: $a_i = (a_{i+1}, a_{i+1})\pi_i$, where $\pi_i \in \text{Sym}(\{0,1\})$. In particular, we have $a_i^2 = (a_{i+1}, a_{i+1})$ for every $i \in \mathbb{N}$, which implies: $a_i^2 = \text{id}$ for every $i \in \mathbb{N}$. Now, if $\pi_i = \text{id}$ for every $i \in \mathbb{N}$, then the group $G(A) = \langle a_1 \rangle$ is trivial, and if $\pi_i \neq \text{id}$ for some $i \in \mathbb{N}$, then $G(A)$ is the group of order two. \[
abla \]

**Proposition 5.** Let $G$ be an infinite cyclic group $C_\infty$ or a finite cyclic group $C_{2^r}$ of order $2^r$, where $r \in \mathbb{N}$. Then there is a 2-state time-varying automaton $A$ over the binary alphabet such that $G \simeq G(A)$.

PROOF. Let us define the subsets $N_1, N_2 \subseteq \mathbb{N}$ as follows: $N_1 = \{1,2,\ldots,r\}$, $N_2 = \mathbb{N}$. For each $k \in \{1,2\}$ let

$$A[k] = \{0,1\}, \{a_1, a_2\}, \varphi, \psi$$

be the 2-state time-varying automaton over the binary alphabet in which the sequences $\varphi$, $\psi$ of transition and output functions are defined as follows:

$$\varphi_i(a_j, x) = \begin{cases} a_1, & \text{if } j = 2, i \in N_k, x = 1, \\ a_j, & \text{otherwise}, \end{cases} \quad \psi_i(a_j, x) = \begin{cases} \tau(x), & \text{if } j = 2, i \in N_k, \\ x, & \text{otherwise} \end{cases}$$

for all $i \in \mathbb{N}$, $j \in \{1,2\}$, $x \in \{0,1\}$, where $\tau \in \text{Sym}(\{0,1\})$ is a transposition.

Let $a_{2,i} := (a_j)_i \in G(A[k]_i)$ ($j \in \{1,2\}, i \in \mathbb{N}$) be the automaton transformation of the state $a_j$ in its $i$-th transition.

In the case $k = 1$ we obtain by the above formulae the following wreath recursions:

$$a_{1,i} = (a_{1,i+1}, a_{1,i+1}), \quad a_{2,i} = \begin{cases} (a_{1,i+1}, a_{2,i+1})\tau, & \text{if } 1 \leq i \leq r, \\ (a_{2,i+1}, a_{2,i+1}), & \text{if } i > r. \end{cases}$$

Hence $a_{1,i} = \text{id}$ for $i \in \mathbb{N}$, $a_{2,i} = (\text{id}, a_{2,i+1})\tau$ for $1 \leq i \leq r$, and $a_{2,i} = \text{id}$ for $i > r$. In particular, the order $o(a_{2,r})$ of the generator $a_{2,r} \in G(A[1]_r)$ is equal to $2$, and for every $i \in \{1,\ldots,r\}$ we obtain by easy induction on $i$ the equality $o(a_{2,r+1-i}) = 2^i$. Hence $o(a_{2,1}) = 2^r$ and $G(A[1]) = \langle a_{1,1}, a_{2,1} \rangle = \langle a_{2,1} \rangle \simeq C_{2^r}$.

In the case $k = 2$ we have: $a_{1,i} = \text{id}$ for $i \in \mathbb{N}$ and $a_{2,i} = (\text{id}, a_{2,i+1})\tau$ for $i \in \mathbb{N}$. From the last wreath recursion we obtain for any $i, s \in \mathbb{N}$ the following wreath recursion for the $s$-th power of $a_{2,i}$:

$$a_{2,i}^s = (a_{2,i+1}^{s/2}, a_{2,i+1}^{s/2})\tau^s.$$ 

In particular, if $a_{2,i}^s = \text{id}$ for some $i, s \in \mathbb{N}$, then $\tau^s = \text{id}$. Hence the number $s$ is even and $a_{2,i}^s = (a_{2,i+1}^{s/2}, a_{2,i+1}^{s/2}) = \text{id}$. For every $j \in \mathbb{N}$ we obtain by easy induction on $j$ the divisibility $2^j \mid s$. In particular, the generator $a_{2,1} \in G(A[2])$ is of infinite order and $G(A[2]) = \langle a_{1,1}, a_{2,1} \rangle = \langle a_{2,1} \rangle \simeq C_\infty$. \[
\Box \]
Further, let \( n \geq 2 \) and let \( G \) be an arbitrary abelian group from the class \( \text{AB}_2(n) \) such that the rank of this group is equal to \( n \). We provide the construction of an \( n \)-state time-varying automaton \( A \) over the binary alphabet such that \( G \) is isomorphic to the group \( G(A) \) generated by this automaton. As we see in the following two propositions, our construction depends on whether \( G \) contains a non-trivial torsion part.

**Proposition 6.** Let \( n \in \mathbb{N} \) and let \( G \) be an abelian group of rank \( n \) such that the torsion part of \( G \) is a non-trivial \( 2 \)-group. If \( n \geq 2 \), then there is an \( n \)-state time-varying automaton \( A \) over the binary alphabet such that \( G \) is isomorphic to \( G(A) \).

**Proof.** By the fundamental theorem of finitely generated abelian groups there are integers \( 1 \leq d \leq n \) and \( 0 \leq d' \leq n \) such that \( d + d' = n \) and the group \( G \) is isomorphic to the direct sum

\[
C_{2r_1} \oplus \ldots \oplus C_{2r_d} \oplus C_{d'}^d
\]

for some \( r_j \in \mathbb{N} \) \((1 \leq j \leq d)\), where \( C_{d'}^d \) denotes the free abelian group of rank \( d' \) in the case \( d' > 0 \) or the trivial group in the case \( d' = 0 \). Let us denote:

\[
M = \begin{cases} 
\mathbb{N}, & \text{if } d' > 0, \\
\{1, 2, \ldots, R\}, & \text{if } d' = 0,
\end{cases}
\]

where \( R = r_1 + \ldots + r_d \), and let \( N_1 \cup \ldots \cup N_n \) be an arbitrary partition of the set \( M \) in which: \( N_1 = \{1, \ldots, r_1\} \), \( |N_j| = r_j \) for \( 1 < j \leq d \) and \( |N_j| = \infty \) for \( d < j \leq n \).

Let us assume that \( n \geq 2 \) and let \( A \) be an \( n \)-state time-varying automaton over the binary alphabet in which the set \( Q \) of states consists of the symbols \( a_1, \ldots, a_n \), and the sequences \( \varphi = (\varphi_i)_{i \in \mathbb{N}}, \psi = (\psi_i)_{i \in \mathbb{N}} \) of transition and output functions are defined as follows:

\[
\varphi_i(a_j, x) = \begin{cases} 
a_1, & \text{if } j \neq 1, i \in N_j, \ x = 1, \\
a_2, & \text{if } j = 1, i \in N_1 \setminus \{r_1\}, \ x = 1, \\
a_2, & \text{if } j = 1, i = r_1, \\
a_j, & \text{otherwise},
\end{cases}
\]

\[
\psi_i(a_j, x) = \begin{cases} 
\tau(x), & \text{if } i \in N_j, \\
x, & \text{otherwise}
\end{cases}
\]

for all \( i \in \mathbb{N}, j \in \{1, 2, \ldots, n\}, x \in \{0, 1\} \), where \( \tau \in \text{Sym}(\{0, 1\}) \) is a transposition. In the case \( n = 4, d = 3, r_1 = 3, r_2 = 1, r_3 = 2, N_1 = \{1, 2, 3\}, N_2 = \{4\}, N_3 = \{5, 6\}, N_4 = \mathbb{N} \setminus \{1, 2, 3, 4, 5, 6\} \), the diagram of the automaton \( A \) is depicted in Figure 4 (for clarity, we replaced a large number of edges connecting two given vertices and having the same direction with a one multi-edge and, since every multi-edge is labeled by the both letters of the alphabet, we omitted its labeling).
Let $a_{j,i} := (a_j)_i$ ($1 \leq j \leq n$, $i \in \mathbb{N}$) be the automaton transformation of the state $a_j \in Q$ in its $i$-th transition. Directly by the above formulae, we have the following wreath recursions for the automorphisms $a_{j,i}$ ($i \in \mathbb{N}$):

$$a_{j,i} = \begin{cases} (a_{j,i+1}, a_{j,i+1}) \tau, & \text{if } i \in N_1 \setminus \{r_1\}, \\ (a_{j,i+1}, a_{j,i+1}), & \text{if } i \notin N_1. \end{cases}$$ (3)

This implies: $a_{1,i} = id$ for $i > r_1$. Similarly, for every $2 \leq j \leq n$ the wreath recursions for $a_{j,i}$ ($i \in \mathbb{N}$) satisfy: $a_{j,i} = (a_{j,i+1}, a_{1,i+1}) \tau$ for every $i \in N_j$ and $a_{j,i} = (a_{j,i+1}, a_{j,i+1})$ for $i \notin N_j$. If $j \neq 1$ and $i \in N_j$, then $i \notin N_1$ and hence $i > r_1$. Consequently, we have:

$$a_{j,i} = \begin{cases} (a_{j,i+1}, id) \tau, & \text{if } j \neq 1 \text{ and } i \in N_j, \\ (a_{j,i+1}, a_{j,i+1}), & \text{if } j \neq 1 \text{ and } i \notin N_j. \end{cases}$$ (4)

By (4), we obtain for all $i, s \in \mathbb{N}$ and every $j \neq 1$:

$$a_{j,i}^s = \begin{cases} (a_{j,i+1}^{[s/2]}, a_{j,i+1}^{[s/2]}) \tau^s, & \text{if } i \in N_j, \\ (a_{j,i+1}^{s}, a_{j,i+1}^{s}), & \text{if } i \notin N_j. \end{cases}$$

The above formula implies that the generator $a_{j,1} \in G(A)$ ($j \neq 1$) is of infinite order if and only if $|N_j| = \infty$, and if $|N_j| < \infty$ (which implies $|N_j| = r_j$ by our assumption), then $o(a_{j,1}) = 2^{r_j}$.

Since $a_{2,i} = (a_{2,i+1}, a_{2,i+1})$ for every $i \in N_1$, the wreath recursion of the element $b_{1,i} := a_{1,i}a_{2,i}^{-1}$ ($i \in N_1$) satisfies:

$$b_{1,i} = \begin{cases} (b_{1,i+1}, id) \tau, & \text{if } i \in N_1 \setminus \{r_1\}, \\ (id, id) \tau, & \text{if } i = r_1. \end{cases}$$ (5)

In particular, we obtain $o(b_{1,1}) = 2^{r_1}$. 

Figure 1: The automaton from Proposition 6 for $n = 4$, $d = 3$, $r_1 = 3$, $r_2 = 1$, $r_3 = 2$. 

$$\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\tau & \tau & id & id & id & id & id & id & \ldots \\
1 & 1 & \tau & 0 & id & id & id & id & \ldots \\
0 & id & id & id & id & id & id & id & \ldots \\
0 & id & id & id & id & id & id & id & \ldots \\
0 & id & id & id & id & id & id & id & \ldots \\
0 & id & id & id & id & id & id & id & \ldots \\
0 & id & id & id & id & id & id & id & \ldots \\
\end{array}$$
By \([3]\) and \([4]\), we also obtain the following condition: if \(a_{j,i} \neq (a_{j,i+1}, a_{j,i+1})\) for some \(i \in \mathbb{N}\) and \(1 \leq j \leq n\), then \(a_{j,i} = (a_{l,i+1}, a_{l,i+1})\) for some \(l, l' \in \{1, 2, j\}, \pi \in \{id, \tau\}\), and then for every \(j' \neq j\) the element \(a_{j',i}\) has the following wreath recursion: \(a_{j',i} = (a_{j',i+1}, a_{j',i+1})\). This implies that for any \(i \in \mathbb{N}\), \(1 \leq j, j' \leq n, x \in \{0, 1\}\) there are \(\pi \in \{id, \tau\}\) and \(l, l' \in \{1, 2, j, j'\}\) such that

\[
 a_{j,i}a_{j',i} x = a_{j,i+1}a_{j',i} x, \quad a_{j,i}a_{j',i} x = a_{j',i+1}a_{j,i+1} x,
\]

\[
 \sigma_{a_{j,i}, a_{j',i}} = \sigma_{a_{j',i}, a_{j,i}} = \pi.
\]

Consequently, the elements \(a_{j,i}\) and \(a_{j',i}\) commute for all \(i \in \mathbb{N}\) and \(1 \leq j, j' \leq n\). Thus the group \(G(A) = \langle a_{1,1}, \ldots, a_{n,1} \rangle = \langle b_{1,1}, a_{2,1}, \ldots, a_{n,1} \rangle\) is abelian.

Let \(k_j\) \((1 \leq j \leq n)\) be arbitrary integers for which the product \(g = b_{1,1}^{k_1} \cdot a_{2,1}^{k_2} \cdots a_{n,1}^{k_n}\) represents the trivial element in \(G(A)\). By \([3]\)–\([4]\), we see that the root permutation \(\sigma_g\) is equal to \(\sigma^{k_1}\). Hence \(2 \mid k_1\) and consequently \(b_{1,1}^{k_1} = (b_{1,1}^{k_1/2}, b_{1,1}^{k_1/2})\). By \([3]\), we have: \(a_{j,j+1}^{k_j} = (a_{j,j+2}^{k_j}, a_{j,j+2}^{k_j})\) for \(2 \leq j \leq n\). Thus \(g_0 = g_{1,1} = b_{1,1}^{k_1/2} \cdot a_{2,1}^{k_2} \cdots a_{n,1}^{k_n}\). Since \(g_{w} = id\) for every \(w \in \{0, 1\}^*\), a trivial induction on \(i\) gives for every \(i \in N_1\) and every \(w \in \{0, 1\}^{i-1}\) the divisibility \(2^{i-1} \mid k_1\) and the equality \(g_{w} = b_{1,1}^{k_1/2^{i-1}} \cdot a_{2,1}^{k_2} \cdots a_{n,1}^{k_n}\). In particular \(g_{w} = b_{1,1}^{k_1/2^{i-1}} \cdot a_{2,1}^{k_2} \cdots a_{n,1}^{k_n}\) for every \(w \in \{0, 1\}^{i-1}\). Again by \([4]\)–\([5]\), we see that the root permutation of the last product is equal to \(\sigma^{k_1/2^{i-1}}\), which implies \(2^{i} \mid k_1\). Since \(b_{1,1}^{k_1} = b_{1,1}^{k_1+i} = id\), we obtain: \(g_{w} = a_{2,1}^{k_2} \cdots a_{n,1}^{k_n}\) for every \(w \in \{0, 1\}^{i+1}\) and hence:

\[
 a_{2,1}^{k_2} \cdots a_{n,1}^{k_n} = id.
\]

Let us fix \(i \in \mathbb{N} \setminus \{1, \ldots, r_1\}\). Let \(l_j\) \((2 \leq j \leq n)\) be any integers for which the product

\[
 h = a_{2,i}^{l_2} \cdots a_{n,i}^{l_n}
\]

represents the trivial element in the group \(G(A_i)\). Since the sets \(N_j\) \((2 \leq j \leq n)\) are pairwise disjoint, we have two possibilities: (i) there is a unique \(2 \leq j_0 \leq n\) such that \(i \in N_{j_0}\), (ii) for every \(2 \leq j \leq n\) we have \(i \notin N_j\). In the case (i), by the wreath recursion \([1]\), we see that the root permutation \(\sigma_h\) is equal to \(\sigma^{l_{j_0}}\) for \(j = j_0\). Since \(h = id\), we obtain: \(2 \mid l_{j_0}\), and consequently, the sections \(h_{0}, h_{1}\) coincide and they both arise from \([7]\) by replacing the factor \(a_{j_0,i}^{l_{j_0}}\) with \(a_{j_0,i+1}^{l_{j_0}/2}\) and the factors \(a_{j,i}^{l_{j,i}}\) for \(j \neq j_0\) with \(a_{j,i+1}^{l_{j,i+1}}\), i.e. we have:

\[
 h_{0} = h_{1} = a_{2,i+1}^{l_{j_0}+1} \cdots a_{j_0,i+1}^{l_{j_0}} a_{j_0,i+1}^{l_{j_0}/2} a_{j_0,i+1}^{l_{j_0}+1} \cdots a_{n,i+1}^{l_{j_0}}.
\]

In the case (ii) the sections \(h_{0}, h_{1}\) both arise from \(h\) by replacing each \(a_{j,i}^{l_{j,i}}\) \((2 \leq j \leq n)\) with \(a_{j,i+1}^{l_{j,i+1}}\), i.e. we have: \(h_{0} = h_{1} = a_{2,i+1}^{l_{j,i+1}} \cdots a_{n,i+1}^{l_{j,i+1}}\). Since \(h_{w} = id\) for every \(w \in \{0, 1\}^*\), we obtain by the trivial induction on \(t \in \mathbb{N}\) that for every \(t \in \mathbb{N}\) and every \(2 \leq j \leq n\) the number \(l_j\) is divisible by \(2^{t_{j,i}}\), where

\[
 t_{j,i} = |N_j \cap \{i, i+1, \ldots, i+t-1\}|.
\]
Let $\mu_j = |N_j \cap \{ i' \in \mathbb{N} : i' \geq i \}|$. By our assumption, we have $N_j \cap \{1, 2, \ldots, r_i\} = \emptyset$ and $|N_j| = r_j$ for every $2 \leq j \leq n$. In particular, if $i = r_1 + 1$, then we obtain in that case: $\mu_j = |N_j| = r_j$, and hence $o(a_{j,1}) | k_j$ for every $2 \leq j \leq n$ such that $|N_j| < \infty$. Consequently, we obtain

$$G(A) \simeq \langle b_{1,1} \rangle \oplus \langle a_{2,1} \rangle \oplus \ldots \oplus \langle a_{n,1} \rangle \simeq G.$$ 

Now, it remains to consider the case of abelian groups of rank $n$ ($n \geq 2$) without torsion elements. Note that every such a group must be the free abelian group of rank $n$, which follows from the property that every finitely generated abelian group is a direct sum of cyclic groups.

**Proposition 7.** If $n \geq 2$, then there is an $n$-state time-varying automaton $A$ over the binary alphabet which generates a group isomorphic to the free abelian group of rank $n$.

**Proof.** Let $n \in \mathbb{N} \setminus \{1\}$ and let $N_2 \cup \ldots \cup N_n$ be an arbitrary partition of the set $\mathbb{N} \setminus \{1\}$ in which $|N_j| = \infty$ for every $2 \leq j \leq n$.

Let $A$ be an $n$-state time-varying automaton over the binary alphabet in which the set $Q$ of states consists of the symbols $a_1, \ldots, a_n$, and the sequences $\phi = (\phi_i)_{i \in \mathbb{N}}, \psi = (\psi_i)_{i \in \mathbb{N}}$ of transition and output functions are defined as follows:

$$\phi_i(a_j, x) = \begin{cases} a_1, & \text{if } j \neq 1, \; i \in N_j, \; x = 1, \\ a_2, & \text{if } j = 1, \; i = 1, \; x = 1, \\ a_j, & \text{otherwise}, \end{cases}$$

$$\psi_i(a_j, x) = \begin{cases} \tau(x), & \text{if } j \neq 1, \; i \in N_j, \\ x, & \text{otherwise} \end{cases}$$

for all $i \in \mathbb{N}$, $j \in \{1, 2, \ldots, n\}$, $x \in \{0, 1\}$.

As before, we denote by $a_{j,i} := (a_j)_i$ ($1 \leq j \leq n$, $i \in \mathbb{N}$) the automaton transformation of the state $a_j \in Q$ in its $i$-th transition. By the above formulae, we have the following wreath recursions for the automorphisms $a_{1,i}$ ($i \in \mathbb{N}$):

$$a_{1,i} = \begin{cases} (a_{1,i+1}, a_{2,i+1}), & \text{if } i = 1, \\ (a_{1,i+1}, a_{1,i+1}), & \text{if } i \neq 1. \end{cases}$$

This implies that $a_{1,i} = id$ for $i \neq 1$, and consequently we obtain:

$$a_{1,i} = \begin{cases} (id, a_{2,i+1}), & \text{if } i = 1, \\ (id, id), & \text{if } i \neq 1. \end{cases} \quad (8)$$

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The wreath recursions for \( a_{j,t} \) \((i \in \mathbb{N}, j \neq 1)\) satisfy (4), and similarly as in the previous proof, we obtain that the group \( G(A) \) is abelian. Let \( k_j \) \((1 \leq j \leq n)\) be arbitrary integers for which the product \( g = a_{1,1}^{k_1} \cdots a_{n,1}^{k_n} \) represents the trivial element in \( G(A) \). By (4) and by (3), we obtain:

\[
\begin{align*}
g|_\emptyset &= a_{2,2}^{k_2} \cdots a_{n,2}^{k_n}, \\
g|_1 &= a_{2,2}^{k_1+k_2} \cdots a_{n,2}^{k_n}.
\end{align*}
\]

Since \( g|_\emptyset = id \), we can apply to the product (9) the same reasoning as in the previous proof and obtain that for every \( t \in \mathbb{N} \setminus \{1\} \) and every \( 2 \leq j \leq n \) the number \( k_j \) is divisible by \( 2^{\mu_{j,t}} \), where \( \mu_{j,t} = |N_j \cap \{2,\ldots,t\}| \). Similarly, since \( g|_1 = id \), we use the product (10) and obtain that the number \( k_1 + k_2 \) is divisible by \( 2^{\mu_{j,t}} \) for every \( t \in \mathbb{N} \setminus \{1\} \). Since each \( N_j \) is an infinite set, we obtain: \( k_j = 0 \) for every \( 1 \leq j \leq n \). Thus \( G(A) \) is isomorphic to the free abelian group of rank \( n \), which finishes the proof of the proposition. \( \square \)

Finally, we need to observe that if \( G \) is a group generated by an \( n \)-state \((n \in \mathbb{N})\) time-varying (Mealy) automaton \( A \) over a finite alphabet \( X \), then for every \( n' > n \) the group \( G \) is also generated by an \( n' \)-state time-varying (Mealy) automaton over \( X \). Indeed, the required \( n' \)-state automaton is obtained by adding certain new symbols \( q_1,\ldots,q_{n'-n} \) to the set of states of \( A \) together with the following formulae for transition and output functions: \( \varphi_i(q_j,x) = q_j \), \( \psi_i(q_j,x) = x \) for all \( i \in \mathbb{N}, j \in \{1,\ldots,n'-n\} \), \( x \in X \). Then the automaton transformations \( (q_j)_i \in Aut(X^*) \) \((i \in \mathbb{N}, 1 \leq j \leq n' - n)\) are all trivial, and the transformations corresponding to the “old” states remain unchanged. Hence, by Propositions 5, 7 we obtain the inclusion \( TVA(n) \supseteq AB_2(n) \) for every \( n > 1 \), which finishes the proof of Theorem 2.

6. The proof of Theorem 2

Let \( G = G(A) \) be the group generated by an arbitrary Mealy automaton \( A \) over the binary alphabet and let us denote the following subsets of \( G \):

- \( G_1 = \{ g \in G : \sigma_0 \neq id \} = \{ g \in G : \forall x \in \{0,1\} \ g(x) = x \} \) – the first level stabilizer,
- \( I_G = \{ g \in G : g^2 = id \} \) – the set of involutions,
- \( G^2 = \{ g^2 : g \in G \} \) – the set of squares in \( G \).

**Lemma 1.** If \( G^2 = G_1^2 \), then \( G \) is an elementary abelian 2-group.

**Proof.** Suppose that \( G^2 = G_1^2 \). We have to show that \( I_G = G \). So, let \( g \in G \) be arbitrary. It can be seen by the formula (4) that the equality \( g^2 = id \) follows from the following condition: \( g^2|_w \in G_1 \) for every \( w \in \{0,1\}^* \). But, since \( G_1^2 \subseteq G_1 \), it is enough to show that \( g^2|_w \in G_2 \) for every \( w \in \{0,1\}^* \). We use induction on the length of a word \( w \). The claim is obvious in the case \( |w| = 0 \),
Lemma 2. If \( I_G \subseteq G_1 \), then \( I_G = \{ id \} \).

Proof. Suppose that \( I_G \subseteq G_1 \) and let \( g \in I_G \) be arbitrary. We must show that \( g = id \). As before, we see by the formula (\( \square \)) that we need to show the condition: \( g|_w \in G_1 \) for every \( w \in \{0,1\}^* \). Since \( I_G \subseteq G_1 \), it is enough to show that \( g|_w \in I_G \) for every \( w \in \{0,1\}^* \). We use induction on the length of the word \( w \). The claim is obvious in the case \( |w| = 0 \). Suppose that \( g|_w \in I_G \) for some \( w \in \{0,1\}^* \). Since \( I_G \subseteq G_1 \), we have: \( g|_w = (g|_w0, g|_w1) \). Hence \( id = (g|_w)^2 = ((g|_w0)^2, (g|_w1)^2) \). Thus \( (g|_w)^2 = id \) for \( x \in \{0,1\} \), i.e. \( g|_w \in I_G \) for \( x \in \{0,1\} \). An inductive argument finishes the proof.

Proposition 8. If the group \( G \) is abelian, then \( G \) is torsion free or it is an elementary abelian 2-group.

Proof. Suppose that \( G \) is non-trivial, abelian and not torsion free. Then there is \( g \in G \) with \( 1 < o(g) < \infty \). By Proposition 3, there is \( r \in \mathbb{N} \) such that \( o(g) = 2^r \). Then \( g^{2^r-1} \) is a non-trivial element from \( I_G \), and consequently, by Lemma 1, the set \( I_G \setminus G_1 \) is not empty. For every \( g \in G \) we have: if \( g \notin G_1 \), then \( gh \in G_1 \) and \( g^2 = g^2h^2 = (gh)^2 \). Consequently, we obtain \( G^2 = G^2_1 \) and, by Lemma 1, the group \( G \) is an elementary abelian 2-group, which finishes the proof of Proposition 8.

Remark 2. The statement of Proposition 8 can also be derived by using the concept of a \( 1/2 \)-endomorphism of a group, which defines its state-closed representation on the tree \( \{0,1\}^* \) – see [12] and Proposition 3.4 therein.

Now, the inclusion \( MA(n) \subseteq FA(n - 1) \cup EA_2(n) \) \( (n \geq 2) \) follows directly by Proposition 8 and by the following proposition (see Proposition 3.1 in [17]):

Proposition 9 ([17]). For every \( n \in \mathbb{N} \) there is no \( n \)-state Mealy automaton over the binary alphabet which generates the free abelian group of rank \( n \).

Since the equality \( MA(1) = EA_2(1) \) is trivial, we see by the observation at the end of the previous section that to show for every \( n \geq 2 \) the converse inclusion (i.e. \( MA(n) \supseteq FA(n - 1) \cup EA_2(n) \)), we only need to show that the free abelian group of rank \( n - 1 \) as well as the elementary abelian 2-group of rank \( n \) is generated by an \( n \)-state Mealy automaton over the binary alphabet. The case of a free abelian group follows directly by the following well known construction of so-called “sausage” automata (see [7] for example):

Proposition 10 ([7]). Let \( A = (\{0,1\}, Q, \varphi, \psi) \) be an \( n \)-state \( (n \geq 2) \) Mealy automaton in which \( Q = \{ a_1, \ldots , a_n \} \) and the transition and output functions as then we have \( g^2|_w = g^2|_w = g^2 \in G^2 = G^2_1 \). Suppose that \( g^2|_w \in G^1 \) for some \( w \in \{0,1\}^* \). Then \( g^2|_w = h^2 \) for some \( h \in G_1 \). Since \( h = (h_0, h_1) \), we obtain: \( g^2|_w = ((h_0)^2, (h_1)^2) \), and consequently \( g^2|_w x = (h|_x)^2 \in G^2 = G^2_1 \) for \( x \in \{0,1\} \). An inductive argument finishes the proof of Lemma 1. \( \square \)
are defined as follows:

\[
\varphi(a_j, x) = \begin{cases} 
  a_{j-1}, & \text{if } 2 < j \leq n, \\
  a_n, & \text{if } j = 2, x = 0, \\
  a_1, & \text{if } j = 2, x = 1, \\
  a_1, & \text{if } j = 1,
\end{cases}
\]

\[
\psi(a_j, x) = \begin{cases} 
  \tau(x), & \text{if } j = 2, \\
  x, & \text{otherwise}
\end{cases}
\]

for all \( j \in \{1, 2, \ldots, n\} \) and \( x \in \{0, 1\} \). Then the group \( G(A) \) is isomorphic to the free abelian group of rank \( n - 1 \).

The case of an elementary abelian 2-group follows by the following

**Proposition 11.** Let \( A = (\{0, 1\}, Q, \varphi, \psi) \) be an \( n \)-state \((n \in \mathbb{N})\) Mealy automaton in which \( Q = \{a_1, \ldots, a_n\} \) and the transition and output functions are defined as follows:

\[
\varphi(a_j, x) = \begin{cases} 
  a_{j-1}, & \text{if } 2 \leq j \leq n, \\
  a_n, & \text{if } j = 1,
\end{cases}
\]

\[
\psi(a_j, x) = \begin{cases} 
  \tau(x), & \text{if } j = 1, \\
  x, & \text{otherwise}
\end{cases}
\]

for all \( j \in \{1, 2, \ldots, n\} \) and \( x \in \{0, 1\} \). Then \( G(A) \) is isomorphic to the elementary abelian 2-group of rank \( n \).

**Proof.** To simplify notation we will use the notation \( a_j \) \((1 \leq j \leq n)\) for the automaton transformation \((a_j)_i \in G(A)\). By the formulae for transition and output functions, we have the following wreath recursions for the generators \( a_j \):

\[
a_j = \begin{cases} 
  (a_{j-1}, a_{j-1}), & \text{if } 2 \leq j \leq n, \\
  (a_n, a_n), & \text{if } j = 1.
\end{cases}
\]  

(11)

In particular, for any \( 1 \leq j, j' \leq n \) and any \( x \in \{0, 1\} \) there are \( 1 \leq i, i' \leq n \) and \( \pi \in \{id, \tau\} \) such that \((a_j a_{j'})_x = a_i a_{i'} \), \((a_j a_{j'})_x = a_i a_{i'} \) and \( \sigma_{a_j a_{j'}} = \sigma_{a_i a_{i'}} = \pi \).

Thus for any \( 1 \leq j, j' \leq n \) the elements \( a_j, a_{j'} \) commute, and hence the group \( G(A) \) is abelian. For every \( 1 \leq j \leq n \) we have: \( a_j(0^n) = 0^{i-1}10^n-j, \) and hence \( a_j \neq id \). On the other hand, by \([\square]\), we see that for every \( 1 \leq j \leq n \) there is \( 1 \leq i \leq n \) such that \( a_i^2 = (a_i^2, a_i^2) \).

Thus \( a_i^2 = id \) for every \( 1 \leq j \leq n \). Finally, suppose that for some integers \( k_j \) \((1 \leq j \leq n)\) the product \( a = a_{k_1} \cdots a_{k_n} \) represents the trivial element in the group \( G(A) \). Then the root permutation \( \sigma_\pi \) is equal to \( \tau^{k_1} \), and hence \( 2 \mid k_1 \). We also see by wreath recursions \([\square]\) that for every \( t \in \{0, \ldots, n-1\} \) and every word \( w \in \{0, 1\}^t \) the section \( g|_{tw} \) is equal to

\[
a_{k_{t+1}} \cdots a_{k_{t+2}} \cdots a_{k_{n-t}} \cdots a_{k_{n-t+1}} \cdots a_{k_1}.
\]

Thus \( id = \sigma_{g|_{tw}} = \tau^{k_{t+1}} \) and consequently \( 2 \mid k_{t+1} \), which implies the isomorphism \( G(A) \cong \langle a_1 \rangle \oplus \cdots \oplus \langle a_n \rangle \cong C_2^n \). \( \square \)

**References**

[1] V. Aleshin. A free group of finite automata. *Mosc. Univ. Math. Bull.* 38(1983), no.4, 10–13.
I. Bondarenko, R. Grigorchuk, R. Kravchenko, Y. Muntyan, V. Nekrashevych, D. Savchuk, and Z. Sunik. Groups generated by 3-state automata over a 2-letter alphabet. II. *Journal of Mathematical Sciences*, Vol. 156, No. 1, 2009.

A. Erschler. Piecewise automatic groups. *Duke Math. J.*, 134:3 (2006), 591–613.

A. Erschler. Automatically presented groups. *Groups Geom. Dyn.*, (2007), 47–59.

Y. Glasner, S. Mozes. Automata and square complexes. *Geom. Dedicata* 111(2005), 43–64.

R. Grigorchuk. Solved and Unsolved Problems Around One Group. In *Infinite Groups: Geometric, Combinatorial and Dynamical Aspects. Progress in Mathematics Series*, Vol. 248, 2005.

R. Grigorchuk, V. Nekrashevych, V. Sushchanskyy. Automata, Dynamical Systems and Groups. *Proceedings of Steklov Institute of Mathematics*, 231:128–203, 2000.

R. Grigorchuk. Some topics in the dynamics of group actions on rooted trees. *Proceedings of the Steklov Institute of Mathematics*. July 2011, Volume 273, Issue 1, pp 64–175.

V.B. Kudryavtsev, S.V. Aleshin, and A.S. Podkolzin. Introduction to the Theory of Automata (Nauka, Moscow, 1985) [in Russian].

B. Mikolajczak. Algebraic and structural automata theory, translated from the *Polish Annals of Discrete Mathematics*, Vol. 44 (North-Holland Publishing Co., Amsterdam, 1991).

V. Nekrashevych. Self-similar Groups. *Am. Math. Soc., Providence, RI*, 2005, Math. Surv. Monogr. 117.

V. Nekrashevych, S. Sidki. Automorphisms of the binary tree: state-closed subgroups and dynamics of 1/2-endomorphisms, In T.W. Miller, editor, *Groups: Topological, Combinatorial and Arithmetic Aspects*, volume 311 of LMS Lecture Notes Series, pages 375–404, 2004.

M. Vorobets, Ya. Vorobets. On a free group of transformations defined by an automaton, *Geom. Dedicata*, 124 (2007), 237–249.

A. Woryna. On permutation groups generated by time-varying Mealy automata. *Publ. Math. Debrecen*, 67(1-2):115–130, 2005.

A. Woryna. On generation of wreath products of cyclic groups by two state time varying Mealy automata. *Int. J. Algebra Comput.*, 16(2):397–415, 2006.
[16] A. Woryna. The concept of duality for automata over a changing alphabet and generation a free group by such automata. *Theoret. Comput. Sci.*, 412(45):6420–6431, 2011.

[17] A. Woryna. Automaton ranks of some self-similar groups. *Lecture Notes in Computer Science*, 2012, Volume 7183/2012, 514–525.

[18] A. Woryna. The concept of self-similar automata over a changing alphabet and lamplighter groups generated by such automata. *Theoret. Comput. Sci.*, 482:96–110, 2013.

[19] A. Woryna. On some universal construction of minimal topological generating sets for inverse limits of iterated wreath products of non-Abelian finite simple groups. *J Algebr Comb*, Vol. 42, Issue 2, pp 365-390, 2015.