The Bethe ansatz approach for factorizable centrally extended $\mathfrak{su}(2|2)$ $S$-matrices

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Abstract

We consider the Bethe ansatz solution of integrable models interacting through factorized $S$-matrices based on the central extension of the $\mathfrak{su}(2|2)$ symmetry. The respective $\mathfrak{su}(2|2)$ $R$-matrix is explicitly related to that of the covering Hubbard model through a spectral parameter dependent transformation. This mapping allows us to diagonalize inhomogeneous transfer matrices whose statistical weights are given in terms of $\mathfrak{su}(2|2)$ $S$-matrices by the algebraic Bethe ansatz. As a consequence of that we derive the quantization condition on the circle for the asymptotic momenta of particles scattering by the $\mathfrak{su}(2|2) \otimes \mathfrak{su}(2|2)$ $S$-matrix. The result for the quantization rule may be of relevance in the study of the energy spectrum of the $AdS_5 \times S^5$ string sigma model in the thermodynamic limit.

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1 Introduction

Nowadays there exists a considerable amount of evidences that integrable structures appear in both planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [1, 2] and free IIB superstring theory on the $AdS_5 \times S^5$ curved space [3, 4]. Integrability in the planar $\mathcal{N} = 4$ gauge theory\(^1\) is related to the fact that the spectrum of the anomalous dimensions of the conformal operators can be obtained by diagonalizing long-range integrable spin chains [6, 7]. For the $AdS_5 \times S^5$ string theory, signatures of integrability have been found on the context of semi-classical string states [8, 9, 10]. There is, however, indications that integrability may survive when quantum corrections are taken into account [11]. There exists also arguments in favour of absence of particle production at tree level for the $AdS_5 \times S^5$ gauge-fixed sigma model [12] and that the respective quantum world-sheet energies may arise from quantization conditions of the form of Bethe ansatz equations [13, 14, 15, 16].

Assuming that such gauge and string theories are indeed quantum integrable it is natural to expect that much of their properties could be inferred within the factorized $S$-matrix framework [17]. In the $\mathcal{N} = 4$ gauge theory the $S$-matrix is supposed to encode the interactions of a long-rang spin chain and it is expected to satisfy the dynamical form of the Yang-Baxter equation [18, 19]. From the $AdS_5 \times S^5$ string theory perspective the $S$-matrix should describe the scattering amplitudes between the world-sheet excitations and therefore has to obey the standard Yang-Baxter equation [20, 21]. The basic form of these two matrix operators have recently been argued to be constrained by the requirement that they are invariant under the centrally extended $\mathfrak{su}(2|2) \otimes \mathfrak{su}(2|2)$ superalgebra [18, 19, 20]. The only remaining freedom consists in an abelian phase factor that has been argued to be constrained with the help of crossing symmetry [21].

The next step would then be to determine the quantization conditions for the particle momenta excitations on a ring of size $L$. This requisite is necessary to perform non-perturbative investigations of the energy and momenta eigenstates associated to the $AdS_5 \times S^5$ gauged-fixed

\(^1\)For a review on integrable properties of general four-dimensional gauge theories see [5].
sigma model in the infinite volume limit. The solution of this problem is directly related to the diagonalization of an inhomogeneous two-dimensional vertex model of statistical mechanic whose Boltzmann weights are the matrix elements of the underlying factorized $S$-matrix. In fact, this task has recently been discussed by Beisert [19] in the context of the $\mathcal{N} = 4$ gauge theory. The respective transfer matrix eigenvalues were proposed by means of the analytical Bethe ansatz method and apparently also with the help of a mapping to the Hubbard model. We recall that the construction of eigenvectors is unfortunately beyond the scope of the analytical Bethe ansatz.

It seems therefore of interest to tackle the problem of diagonalization of transfer matrices based on centrally extended $\mathfrak{su}(2|2)$ $S$-matrices by an independent, first principle approach, such as the algebraic Bethe ansatz method [23, 24]. This framework can provide us in an unambiguous way both the transfer matrix eigenvalues and eigenvectors as well as the respective Bethe ansatz equation. To this end, the string theory point of view appears to be the ideal one, since the respective $S$-matrix satisfies canonical properties expected from $(1 + 1)$-dimensional integrable theories [20]. The exact knowledge of the eigenvectors, in a near future, could be of utility in the understanding of the Hilbert space structure of the $AdS_5 \times S^5$ string theory in the thermodynamic limit.

In this paper we are going to consider the latter approach in order to present the Bethe ansatz solution of the monodromy problem associated to the $\mathfrak{su}(2|2) \otimes \mathfrak{su}(2|2)$ string $S$-matrix. We have organized this paper as follows. In next section we describe the centrally extended $\mathfrak{su}(2|2)$ $R$-matrix in a convenient basis. We show that this operator is related to the $R$-matrix of the covering Hubbard model by means of a unitary transformation depending on the spectral parameters. In section 3, with the help of this mapping, we formulate the diagonalization of an inhomogeneous transfer matrix based on the $\mathfrak{su}(2|2)$ $R$-matrix weights by the algebraic Bethe ansatz approach. In section 4 we use this result to derive the quantization rule on a ring of size $L$ for the particle momenta within the asymptotic Bethe ansatz framework [22]. Our conclusions are summarized in section 5 and in Appendix A we have collected Shastry’s $R$-matrix.
2 The $\text{su}(2|2)$ $R$-matrix

In this section we are going to discuss a family of solutions of the Yang-Baxter equation,

$$R_{23}(\lambda_1, \lambda_2)R_{12}(\lambda_1, \lambda_3)R_{23}(\lambda_2, \lambda_3) = R_{12}(\lambda_2, \lambda_3)R_{23}(\lambda_1, \lambda_3)R_{12}(\lambda_1, \lambda_2),$$

(1)

where the $R$-matrix $R_{ab}(\lambda, \mu)$ acts on the tensor product of two $\mathbb{Z}_2$ graded spaces $V_a^{(0)} \oplus V_a^{(1)}$ and $V_b^{(0)} \oplus V_b^{(1)}$. As usual, the $\alpha$th element of the even $V^{(0)}$ and odd $V^{(1)}$ subspaces are distinguished by its $\mathbb{Z}_2$ parity $\theta_\alpha$,

$$\theta_\alpha = \begin{cases} 
0 & \text{for } \alpha \in V^{(0)} \text{ (even)} \\
1 & \text{for } \alpha \in V^{(1)} \text{ (odd)}
\end{cases}.$$

(2)

Here we shall consider the case where the graded vector space is that of a central extension of the $\text{su}(2|2)$ superalgebra [25] on its fundamental four-dimensional representation [19, 26]. In what follows, it will be convenient to represent the $R$-matrix $R_{12}(\lambda, \mu)$ in terms of its matrix elements and we shall use the following representation,

$$R_{12}(\lambda, \mu) = \sum_{a,b,c,d=1}^4 R_{c,d}^{e,a} e_{ac} \otimes e_{bd},$$

(3)

where $e_{ab}$ denotes standard Weyl matrices.

One important feature of equation (1) is that it is insensitive to the parities of the elements of the $\text{su}(2|2)$ superalgebra. In principle, from a given $R$-matrix satisfying equation (1) it is possible to define two types of $S$-matrices [27] \footnote{Strictly speaking it is required that $R_{cd}^{ab}(\lambda, \mu) = 0$ for any $\theta_a + \theta_b + \theta_c + \theta_d = 1, 3$. This condition is satisfied by the $\text{su}(2|2)$ $R$-matrix to be discussed in this section.}. One of them is the standard $S$-matrix,

$$S_{12}(\lambda, \mu) = P_{12} R_{12}(\lambda, \mu),$$

(4)

while the other possibility is the so-called graded $S$-matrix,

$$S_{12}(\lambda, \mu) = P_{12}^{(g)} R_{12}(\lambda, \mu),$$

(5)

where the standard and the graded permutators are given by,

$$P_{12} = \sum_{a,b=1}^4 e_{ab} \otimes e_{ba}, \quad P_{12}^{(g)} = \sum_{a,b=1}^4 (-1)^{\theta_a \theta_b} e_{ab} \otimes e_{ba}.$$
The $S$-matrices defined by Eqs.(4,5) satisfy the canonical form of the Yang-Baxter equation,

$$S_{12}(\lambda_1, \lambda_2)S_{13}(\lambda_1, \lambda_3)S_{23}(\lambda_2, \lambda_3) = S_{23}(\lambda_2, \lambda_3)S_{13}(\lambda_1, \lambda_3)S_{12}(\lambda_1, \lambda_2),$$

(7)

where in the case of the graded $S$-matrix the tensor products take into account the graduation of the subspaces.

The framework of $R$-matrices, however, will be the most convenient one for the purposes of this paper. In fact, it is the $R$-matrix that plays the decisive role in an algebraic formulation of the Bethe ansatz. It will also help us to make a clear relationship between the standard $\text{su}(2|2)$ $S$-matrix discussed by Arutynov et al [20] and the $R$-matrix of the Hubbard model [28]. This connection is expected, since the existence of such mapping has been predicted by Beisert [19] in the context of the $\mathcal{N} = 4$ gauge theory. It should be stressed, however, that in this paper we intend to search a connection between $R$-matrices satisfying the same canonical Yang-Baxter relations (1).

In order to accomplish this task one has to perform certain unitary transformations in the $\text{su}(2|2)$ $S$-matrix $S_{12}(\lambda, \mu)$ given by Arutynov et al [20]. More specifically, we shall use the property that the Yang-Baxter equation (7) is invariant by spectral dependent transformations of the following type,

$$\tilde{S}_{12}(\lambda, \mu) = G_1(\lambda)G_2(\mu)S_{12}(\lambda, \mu)G_1^{-1}(\lambda)G_2^{-1}(\mu),$$

(8)

where $G(\lambda)$ is an arbitrary invertible matrix.

It turns out that the suitable transformation we need to make the above mentioned connection can be decomposed as,

$$G(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t(\lambda) & 0 & 0 \\ 0 & 0 & t(\lambda) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

(9)

where $t(\lambda)$ plays the role of an external field.

By transforming the centrally extended $\text{su}(2|2)$ standard $S$-matrix given by Arutynov et al [20] according to Eqs.(8,9) and by taking into account Eq.(4) we find that the corresponding
\( R \)-matrix can be written as,

\[
\tilde{R}_{12}(\lambda, \mu) = a_1(\lambda, \mu)[e_{11} \otimes e_{11} + e_{14} \otimes e_{41} + e_{41} \otimes e_{14} + e_{44} \otimes e_{44}]
+ a_2(\lambda, \mu)[-e_{11} \otimes e_{44} + e_{14} \otimes e_{41} + e_{41} \otimes e_{14} - e_{44} \otimes e_{11}]
+ a_3(\lambda, \mu)[e_{22} \otimes e_{22} + e_{23} \otimes e_{32} + e_{32} \otimes e_{23} + e_{33} \otimes e_{33}]
+ a_4(\lambda, \mu)[-e_{22} \otimes e_{33} + e_{23} \otimes e_{32} + e_{32} \otimes e_{23} - e_{33} \otimes e_{22}]
+ a_5(\lambda, \mu)[e_{21} \otimes e_{12} + e_{31} \otimes e_{13} + e_{24} \otimes e_{42} + e_{34} \otimes e_{43}]
+ a_6(\lambda, \mu)[e_{12} \otimes e_{21} + e_{13} \otimes e_{31} + e_{42} \otimes e_{24} + e_{43} \otimes e_{34}]
+ a_7(\lambda, \mu)\frac{1}{t(\lambda)t(\mu)}[-e_{12} \otimes e_{43} + e_{13} \otimes e_{42} + e_{42} \otimes e_{13} - e_{43} \otimes e_{12}]
+ a_8(\lambda, \mu)\frac{t(\mu)}{t(\lambda)}[-e_{21} \otimes e_{34} + e_{31} \otimes e_{24} + e_{24} \otimes e_{31} - e_{34} \otimes e_{21}]
+ a_9(\lambda, \mu)\frac{t(\mu)}{t(\lambda)}[e_{22} \otimes e_{11} + e_{22} \otimes e_{44} + e_{33} \otimes e_{11} + e_{33} \otimes e_{44}]
+ a_{10}(\lambda, \mu)\frac{t(\lambda)}{t(\mu)}[e_{11} \otimes e_{22} + e_{11} \otimes e_{33} + e_{44} \otimes e_{22} + e_{44} \otimes e_{33}]
\] (10)

The ten distinct weights \( a_i(\lambda, \mu) \) are obtained directly from [20] and they are given by,

\[
\begin{align*}
    a_1(\lambda, \mu) &= \frac{[x^-(\mu) - x^+(\lambda)]\eta(\mu)}{[x^+(\mu) - x^-(\lambda)]\eta(\lambda)} \\
    a_2(\lambda, \mu) &= \frac{[x^-(\lambda) - x^+(\lambda)][x^-(\mu) + x^+(\lambda)][x^-(\mu) - x^+(\mu)]\eta(\mu)}{[x^-(\lambda) - x^+(\mu)][x^-(\lambda)x^-(\mu) - x^+(\lambda)x^+(\mu)]\eta(\lambda)} \\
    a_3(\lambda, \mu) &= -1 \\
    a_4(\lambda, \mu) &= \frac{[x^-(\lambda) - x^+(\mu)]}{[x^-(\lambda) - x^+(\mu)]}\eta(\mu) \\
    a_5(\lambda, \mu) &= \frac{[x^+(\mu) - x^-(\lambda)]}{[x^+(\mu) - x^-(\lambda)]}\eta(\mu) \\
    a_6(\lambda, \mu) &= \frac{[x^+(\lambda) - x^+(\mu)]}{[x^+(\lambda) - x^+(\mu)]}\frac{1}{\eta(\lambda)} \\
    a_7(\lambda, \mu) &= \frac{\sqrt{|x^+(\lambda) - x^+(\mu)]|}[x^-(\lambda) - x^-(\mu)]^{1/2}[x^+(\lambda) - x^+(\mu)]}{[x^-(\lambda) - x^+(\mu)]}\eta(\lambda) \\
    a_8(\lambda, \mu) &= \frac{\sqrt{|x^+(\lambda) - x^+(\mu)]|}[x^-(\lambda) - x^+(\mu)]^{1/2}[x^+(\lambda) - x^+(\mu)]}{[x^-(\lambda) - x^+(\mu)]}\eta(\lambda) \\
    a_9(\lambda, \mu) &= \frac{\sqrt{|x^-(\lambda) - x^+(\lambda)]|}[x^-(\mu) - x^+(\mu)]}{[x^+(\mu) - x^-(\lambda)]}
\end{align*}
\]
\[ a_{10}(\lambda, \mu) = \frac{\sqrt{[x^-(\lambda) - x^+(\lambda)][x^-(\mu) - x^+(\mu)]} \eta(\mu)}{[x^+(\mu) - x^-(\lambda)]} \eta(\lambda), \]  

(11)

where \( \eta(\lambda) = \sqrt{\frac{x^+(\lambda)}{x^-(\lambda)}} \). The functions \( x^\pm(\lambda) \) depend on both the string sigma model coupling constant \( g \) and the world-sheet rapidity \( \lambda \). They are constrained to satisfy the following relations [18, 19],

\[ \frac{x^+(\lambda)}{x^-(\lambda)} = e^{i\lambda}, \quad x^+(\lambda) + \frac{1}{x^+(\lambda)} - x^-(\lambda) - \frac{1}{x^-(\lambda)} = \frac{i}{g}. \]  

(12)

Before proceeding we note that the constant part of the transformation (9) only relabels the Weyl basis. This means that for \( t(\lambda) = 1 \) the operator \( \bar{S}_{12}(\lambda, \mu) = P_{12}R_{12}(\lambda, \mu) \) is just the original \( su(2|2) \) S-matrix discussed by Arutynov et al [20] written in a different basis. We emphasize that in this new basis the Grassmann parities \( \theta_\alpha \) associated to the \( R \)-matrix (10-12) is,

\[ \theta_\alpha = \begin{cases} 
1 & \text{for } \alpha = 2, 3 \\
0 & \text{for } \alpha = 1, 4 
\end{cases}. \]  

(13)

The first step to compare the \( R \)-matrix (10-12) with the graded \( R \)-matrix of the Hubbard model [28, 29] is to rewrite the restriction (12) in terms of Shastry’s original coupling constraint. Let us denote this latter \( R \) matrix by \( R^{(s)}_{12}(\lambda, \mu) \) which for sake of completeness has been summarized in Appendix A. It turns out that a convenient parameterization for the variables \( x^\pm(\lambda) \) is \(^3\)

\[ x^+(\lambda) = i \frac{a(\lambda)}{b(\lambda)} e^{2h(\lambda)}, \quad x^-(\lambda) = -i \frac{b(\lambda)}{a(\lambda)} e^{2h(\lambda)} \]  

(14)

where \( a(\lambda), b(\lambda) \) and \( h(\lambda) \) are the free-fermion weights and the coupling entering Shastry’s \( R \) matrix defined by Eq.(A.3).

By substituting Eq.(14) into the \( R \)-matrix weights (11) and by comparing them with Shastry’s weights given by Eqs.(A.2) we see that a perfect matching occurs for a special choice of function \( t(\lambda) \), namely

\[ t(\lambda) = \sqrt{\eta(\lambda)} = \left( \frac{x^+(\lambda)}{x^-(\lambda)} \right)^{1/4} \]  

(15)

\(^3\)We recall that this type of parameterization has first appeared in [28, 30] to simplify the transfer matrix eigenvalues of the covering Hubbard model.
We see that for this particular value of \( t(\lambda) \) the Boltzmann weights of the \( R \)-matrix (10-12) are brought to the most possible symmetrical form. For the specific value (15) we found the following relationship,

\[
\tilde{R}_{12}(\lambda, \mu) = \frac{R_{12}^{(s)}(\lambda, \mu)}{R^{(s)}(\lambda, \mu)}
\] (16)

The above discussion reveals us that both the centrally extended \( \text{su}(2|2) \) \( R \)-matrix and Shastry’s graded \( R \)-matrix are indeed special cases of a family of \( R \)-matrices \( \tilde{R}_{12}(\lambda, \mu) \) having a spectral dependent free-parameter. This result brings extra support to the claim by Beisert [19] that the symmetry underlying the integrability of the Hubbard model should be that of the \( \text{su}(2|2) \) superalgebra. Indeed, our result makes it precise the way one can go from the \( \text{su}(2|2) \) \( R \)-matrix to that of the Hubbard model within a common integrable structure satisfying the canonical Yang-Baxter relation (1). This comparison will be of great help to formulated the corresponding algebraic Bethe ansatz solution in next section.

3 The algebraic Bethe ansatz

In this section we shall study the problem of diagonalizing an inhomogeneous row-to-row transfer matrix by an algebraic Bethe ansatz,

\[
T(\lambda, \{p_i\}) |\Phi\rangle = \Lambda(\lambda, \{p_i\}) |\Phi\rangle
\] (17)

where \( p_1, \ldots, p_N \) are the inhomogeneities.

We will consider the situation in which the Boltzmann weights \( \tilde{S}_{12}(\lambda, \mu) \) of the transfer matrix \( T(\lambda, \{p_i\}) \) take into account the graduation of the degrees of freedom,

\[
\tilde{S}_{12}(\lambda, \mu) = P_{12}^{(g)} \tilde{R}_{12}(\lambda, \mu)
\] (18)

where the parities of the graded permutator are defined in Eq.(13).

As usual the transfer matrix \( T(\lambda, \{p_i\}) \) can be written as the supertrace of an operator denominated monodromy matrix [27],

\[
T(\lambda, \{p_i\}) = \text{Str}_A[T_A(\lambda, \{p_i\})] = \sum_{\alpha=1}^{4} (-1)^{\theta_\alpha} T_{\alpha\alpha}(\lambda, \{p_i\})
\] (19)
where $T_{\alpha\beta}(\lambda, \{p_i\})$ denotes the matrix elements on the auxiliary space $\mathcal{A} \equiv \mathbb{C}^4$ of the following ordered product of $S$-matrices,

$$T_{\mathcal{A}}(\lambda, \{p_i\}) = \tilde{S}_{AN}(\lambda, p_N) \tilde{S}_{AN-1}(\lambda, p_{N-1}) \ldots \tilde{S}_{A1}(\lambda, p_1)$$  \hspace{1cm} (20)

An essential ingredient to establish an algebraic Bethe solution is the existence of a reference state $|\omega\rangle$ such that the action of the monodromy operator (20) in this state gives as a result a triangular matrix. In our case this state is easily built by the following product of local vectors,

$$|\omega\rangle = \prod_{j=1}^{n} \otimes |\omega\rangle_j, \quad |\omega\rangle_j = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}_j,$$  \hspace{1cm} (21)

which is an exact eigenstate of $T(\lambda, \{p_i\})$.

In order to construct other eigenvectors other than $|\omega\rangle$ we need the help of the quadratic algebra satisfied by the monodromy operator, namely

$$\tilde{R}_{12}(\lambda, \mu) T(\lambda, \{p_i\}) \otimes T(\mu, \{p_i\}) = T(\mu, \{p_i\}) \otimes T(\lambda, \{p_i\}) \tilde{R}_{12}(\lambda, \mu),$$  \hspace{1cm} (22)

where the symbol $\otimes$ stands for the supertensor product [27] whose parities are defined in Eq.(13).

From this algebra one can in principle derive suitable commutation rules between the monodromy matrix elements acting on the quantum space $\prod_{j=1}^{n} \otimes \mathbb{C}^4$. The diagonal monodromy operators define the transfer matrix eigenvalue problem while the off-diagonal ones play the role of creation and annihilation fields over the pseudovacuum $|\omega\rangle$. A convenient representation of $T(\lambda, \{p_i\})$ in terms of these fields turns out to be,

$$T(\lambda, \{p_i\}) = \begin{pmatrix} B(\lambda) & \tilde{B}(\lambda) & F(\lambda) \\ \tilde{C}(\lambda) & \hat{A}(\lambda) & \tilde{B}^*(\lambda) \\ C(\lambda) & \tilde{C}^*(\lambda) & D(\lambda) \end{pmatrix}_{4 \times 4},$$  \hspace{1cm} (23)

where $\tilde{B}(\lambda)$ ($\tilde{B}^*(\lambda)$) and $\tilde{C}^*(\lambda)$ ($\tilde{C}(\lambda)$) are two-component row (column) vectors, $\hat{A}(\lambda)$ is a $2 \times 2$ matrix. The fields $\tilde{B}(\lambda)$, $\tilde{B}^*(\lambda)$ and $\tilde{C}(\lambda)$, $\tilde{C}^*(\lambda)$ are creation and annihilation operators over the state $|\omega\rangle$, respectively.
The construction of the eigenvectors in terms of the creation fields will depend much on
the form of the Boltzmann weights of the R-matrix $\bar{R}_{12}(\lambda, \mu)$. Considering that the main
structure of $\bar{R}_{12}(\lambda, \mu)$ resembles that of the Hubbard model one expects that the algebraic
Bethe ansatz solution of the eigenvalue problem (17-20) will be similar to that developed for
the Hubbard chain [30]. For arbitrary $t(\lambda)$, however, not all the weights of $\bar{R}_{12}(\lambda, \mu)$ are exactly
the same as that of Shastry’s graded R-matrix. It turns out that some of them are crucial in the
construction of the eigenvectors and this means that we need to implement few adaptations
on the results of [30] before using them in our situation. In what follows, we will describe
such modifications in terms of the general matrix elements $\bar{R}(\lambda, \mu)_{c,d}^{b,a}$ in order to make the
construction of [30] more widely applicable.

The structure of the eigenvectors is that of multiparticle states parameterized by variables
$\lambda_1, \ldots, \lambda_{m_1}$ that are fixed by Bethe ansatz equations. Formally, they can be written by the
following scalar product [30],

$$|\Phi\rangle = \varphi_{m_1}(\lambda_1, \ldots, \lambda_{m_1}) \cdot \bar{F} |\omega\rangle \quad (24)$$

where the components of the vector $\bar{F} \in \prod_{j=1}^{m_1} \otimes \mathbb{C}_j^2$ shall be fixed by a second Bethe ansatz. The
vector $\varphi_{m_1}(\lambda_1, \ldots, \lambda_{m_1})$ carries the dependence on the creation fields and obeys a second order
recursion relation given by,

$$\varphi_{m_1}(\lambda_1, \ldots, \lambda_{m_1}) = \bar{B}(\lambda_1) \otimes \varphi_{m_1-1}(\lambda_2, \ldots, \lambda_{m_1}) + \sum_{j=2}^{m_1} \bar{R}(\lambda_1, \lambda_j)^{2,3}_{4,1} \prod_{k=2}^{m_1} \bar{R}(\lambda_k, \lambda_j)^{1,1}_{2,1} \times \bar{R}(\lambda_k, \lambda_j)^{2,3}_{4,1} \cdot \varphi_{m_1-1}(\lambda_2, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_{m_1}) B(\lambda_j) \prod_{k=2}^{j-1} \frac{\bar{R}(\lambda_k, \lambda_j)^{2,2}_{1,1}}{\bar{R}(\lambda_k, \lambda_j)^{1,1}_{1,1}} \tilde{r}_{k+1}(\lambda_k, \lambda_j). \quad (25)$$

The auxiliary four-dimensional vector $\tilde{\xi}$ and the $4 \times 4$ R-matrix $\tilde{r}_{12}(\lambda, \mu)$ are given by,

$$\tilde{\xi} = (0 \quad 1 \quad -1 \quad 0) \quad \tilde{r}_{12}(\lambda, \mu) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \bar{a}(\lambda, \mu) & \bar{b}(\lambda, \mu) & 0 \\
0 & \bar{b}(\lambda, \mu) & \bar{a}(\lambda, \mu) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \quad (26)$$
where the expressions for the elements $\bar{a}(\lambda, \mu)$ and $\bar{b}(\lambda, \mu)$ are,

$$\bar{a}(\lambda, \mu) = \frac{R(\lambda, \mu)^{2,3} R(\lambda, \mu)^1_4 - R(\lambda, \mu)^{2,3} R(\lambda, \mu)^{4,1}}{R(\lambda, \mu)^{2,2} R(\lambda, \mu)^{4,1}}$$

$$\bar{b}(\lambda, \mu) = \frac{\tilde{R}(\lambda, \mu)^{3,2} \tilde{R}(\lambda, \mu)^1_4 - \tilde{R}(\lambda, \mu)^{3,2} \tilde{R}(\lambda, \mu)^{4,1}}{R(\lambda, \mu)^{2,2} R(\lambda, \mu)^{4,1}}. \quad (27)$$

From the structure of the $R$-matrix (10) we see that the eigenvectors dependence on the free-parameter $t(\lambda)$ is encoded only through the weight $\tilde{R}(\lambda, \mu)^{2,3}$. The functions $\bar{a}(\lambda, \mu)$ and $\bar{b}(\lambda, \mu)$ does not carry any dependence on this extra parameter since it is canceled out in the product $\tilde{R}(\lambda, \mu)^{2,3} \tilde{R}(\lambda, \mu)^{4,1}$. From now on the analysis becomes fairly parallel to that already carried out for the Hubbard model [30]. In particular, one expects that the auxiliary $R$-matrix $\tilde{r}_{12}(\lambda, \mu)$ should be that of an isotropic six-vertex model. This can be seen with the help of the following change of variables,

$$\bar{\lambda} = x^+(\lambda) + \frac{1}{x^+(\lambda)} - \frac{i}{2g} = x^-(\lambda) + \frac{1}{x^-(\lambda)} + \frac{i}{2g} \quad (28)$$

After some simplifications it is possible to rewrite $\tilde{r}_{12}(\bar{\lambda}, \bar{\mu})$ as

$$\tilde{r}_{12}(\bar{\lambda}, \bar{\mu}) = \frac{1}{\bar{\mu} - \bar{\lambda} + \frac{2}{g}} \left[ (\bar{\mu} - \bar{\lambda}) \sum_{\alpha=1}^{2} e_{\alpha\beta} \otimes e_{\beta\alpha} + \frac{i}{g} \sum_{\alpha=1}^{2} e_{\alpha\beta} \otimes e_{\alpha\beta} \right] \quad (29)$$

which is exactly the $R$-matrix of the rational six-vertex model.

As explained in [30] the state $|\Phi\rangle$ defined by the expressions (24-27) becomes an eigenvector of the transfer matrix $T(\lambda, \{p_i\})$ under the requirement that the vector $\tilde{\mathbf{F}}$ is an eigenstate of yet another inhomogeneous transfer matrix whose weights are that of the isotropic six-vertex model (29). The form of the eigenvalues $\Lambda(\lambda, \{p_i\})$ depends also on extra variables $\mu_1, \ldots, \mu_m$ that are needed in the diagonalization of such inhomogeneous six-vertex transfer matrix. Its final expression in terms of the matrix elements is,

$$\Lambda(\lambda, \{p_i\}; \{\lambda_j, \mu_i\}) = \prod_{i=1}^{N} \tilde{R}(\lambda, p_i)^{1,1}_{1,1} \prod_{j=1}^{m_1} \tilde{R}(\lambda_j, \lambda)^{1,1}_{1,1} \tilde{R}(\lambda_j, \lambda)^{2,2}_{2,1} \prod_{l=1}^{m_2} \frac{1}{b(\mu_l, \lambda)} + \prod_{j=1}^{m_1} \tilde{b}(\lambda, \lambda_j)^{m_2}_{1,1} \prod_{l=1}^{m_2} \frac{1}{b(\lambda, \mu_i)} \quad (30)$$
provided that the rapidities \( \{ \lambda_j \} \) and \( \{ \mu_j \} \) satisfy the nested Bethe ansatz equations,

\[
\prod_{i=1}^{N} \frac{\tilde{R}(\lambda_j, p_i)}{R(\lambda_j, p_i)} = \prod_{i=1}^{m_1} \frac{\tilde{R}(\lambda_j, \lambda_k)}{R(\lambda_j, \lambda_k)} \prod_{l=1}^{m_2} \frac{1}{b(\mu_l, \lambda_j)} \quad j = 1, \ldots, m_1,
\]

\[
\prod_{j=1}^{m_1} b(\mu_l, \lambda_j) = \prod_{j=1}^{m_2} \frac{\tilde{b}(\mu_l, \mu_k)}{b(\mu_k, \mu_l)} \quad l = 1, \ldots, m_2.
\]

(31)

From the above expressions we see that both the eigenvalues (30) and the Bethe ansatz equations (31) depend only on Boltzmann weights \( \tilde{R}(\lambda, \mu)_{b,a} \) that are independent of free-parameter \( t(\lambda) \). This is not surprising since we have shown that this family of models are related by the transformation (8) which clearly preserves transfer matrix eigenvalues. The eigenvectors, however, carry a non-trivial dependence on the parameter \( t(\lambda) \) explicitly exhibited in expression (25).

For practical purposes it is relevant to present the expressions for the eigenvalues and Bethe ansatz equations in terms of the kinematical string variables \( x^\pm(\lambda) \). Since they do not depend on \( t(\lambda) \) the simplifications that we need to perform on Eqs.(30,31) follow closely that carried out for the Hubbard model [30]. Omitting here such technicalities we find the eigenvalues are given by,

\[
\Lambda(\lambda, \{ p_i \}; \{ \lambda_j, \mu_l \}) = \prod_{i=1}^{N} \left[ \frac{x^-(p_i) - x^+(\lambda)}{x^+(p_i) - x^-(\lambda)} \right] \frac{\eta(p_i)}{\eta(\lambda)} \prod_{j=1}^{m_1} \eta(\lambda) \frac{x^-(\lambda) - x^+(\lambda)}{x^+(\lambda) - x^+(\lambda)}
\]

\[
- \prod_{i=1}^{N} \left[ \frac{x^+(\lambda) - x^+(p_i)}{x^-(\lambda) - x^+(p_i)} \right] \frac{1}{\eta(\lambda)} \left\{ \prod_{j=1}^{m_1} \eta(\lambda) \left[ \frac{x^-(\lambda) - x^+(\lambda)}{x^+(\lambda) - x^+(\lambda)} \right] \prod_{l=1}^{m_2} \frac{1}{x^+(\lambda)} \right\}
\]

\[
+ \prod_{j=1}^{m_1} \eta(\lambda) \left[ \frac{x^+(\lambda) - x^+(\lambda)}{x^+(\lambda) - x^+(\lambda)} \right] \prod_{l=1}^{m_2} \frac{1}{x^+(\lambda)} \right\} \prod_{j=1}^{m_1} \eta(\lambda) \left[ \frac{x^+(\lambda) - x^+(\lambda)}{x^+(\lambda) - x^+(\lambda)} \right]
\]

(32)

and the corresponding Bethe ansatz equations become,

\[
\prod_{i=1}^{N} \left[ \frac{x^+(\lambda_j) - x^-(p_i)}{x^+(\lambda_j) - x^+(p_i)} \right] \eta(p_i) = \prod_{l=1}^{m_2} \frac{x^+(\lambda_j)}{x^+(\lambda_j) + x^+(\lambda)} - \mu_l + \frac{i}{2g}
\]

\[
\prod_{j=1}^{m_1} \tilde{\mu}_l - x^+(\lambda_j) - \frac{1}{x^+(\lambda_j)} + \frac{i}{2g}
\]

\[
\prod_{k \neq l} \tilde{\mu}_l - x^+(\lambda_j) - \frac{1}{x^+(\lambda_j)} - \frac{i}{2g}
\]

(33)
We conclude this section with the following comment. Our result for the eigenvalues (32) is consistent with that proposed by Beisert [19] provided that one takes into account the following change of variables
\[ x^\pm(\lambda) \rightarrow x^\mp(\lambda) \] and 
\[ g \rightarrow -g. \]
Though the Bethe ansatz equations (33) are invariant by such transformation the ratio \( \frac{x^+(\lambda)}{x^-(\lambda)} \) defining the pseudo-momenta is now inverted. While this ambiguity can appear in analytical Bethe ansatz analysis it certainly does not occur in our algebraic Bethe ansatz framework.

4 The asymptotic Bethe ansatz

The purpose of this section is to derive the quantization rule for the momenta of the particles interacting through a \( \mathfrak{su}(2|2) \otimes \mathfrak{su}(2|2) \) factorizable \( S \)-matrix by using the asymptotic Bethe ansatz [22]. This form of scattering has been argued to be the one relevant for the world-sheet excitations of the \( \text{AdS}_5 \times S^5 \) sigma model [18, 19, 20]. Formally, the respective scattering matrix can be written as,

\[ S^*_{12}(p_1,p_2) = \hat{S}_{12}(p_1,p_2) \otimes \hat{S}_{12}(p_1,p_2), \]  

where \( p_1, \ldots, p_N \) denote the momenta of an interacting \( N \)-particle system.

The basic structure of the building block \( S \)-matrix \( \hat{S}_{12}(p_1,p_2) \) is constrained by the invariance of the solution of the Yang-Baxter equation under the centrally extended \( \mathfrak{su}(2|2) \) symmetry. In principle, one can choose either the standard or the graded \( \mathfrak{su}(2|2) \) \( S \)-matrix discussed in section 2. Our context here, however, is that the \( S \)-matrix \( S^*_{12}(p_1,p_2) \) plays the role of an operator describing the monodromy of the \( N \)-particle wave function. When periodic boundary conditions are considered one has to take into account the different particle statistics under cyclic permutations. This compatibility condition leads us to single out the graded \( \mathfrak{su}(2|2) \) \( S \)-matrix,

\[ \hat{S}_{12}(p_1,p_2) = S_0(p_1,p_2) \left[ P^{(g)}_{12} \bar{R}_{12}(p_1,p_2) \right]. \]  

where \( S_0(p_1,p_2) \) is a scalar factor that can not be determined on basis of the \( \mathfrak{su}(2|2) \) invariance.

The possible functional form of \( S_0(p_1,p_2) \) has been argued to be restricted by unitarity and
an extention of the crossing property to $S$-matrices depending on both momenta $p_1$ and $p_2$ [21]. This factor was first proposed on the context of the $AdS^5 \times S^5$ string spectrum [14] and since then has been further investigated by several authors [31, 32, 33]. The general structure of this factor is believed to be given by the expression,

$$[S_0(p_1, p_2)]^2 = \frac{x^+(p_2) - x^-(p_1)}{x^-(p_2) - x^+(p_1)} \frac{1}{x^+(p_1)x^-(p_2)}[\sigma(p_1, p_2)]^2$$

where $\sigma(p_1, p_2)$ is the so-called string dressing term.

This dressing factor can in general be expressed in terms of a standard phase-shift $\sigma(p_1, p_2) = \exp[i\theta(p_1, p_2)]$ where the phase $\theta(p_1, p_2)$ is an anti-symmetric function of the two momenta $p_1$ and $p_2$. For recent closed formula representations of the gauge independent part of this function see for instance [34]. Here we recall that the property $\theta(p_1, p_2) = -\theta(p_2, p_1)$ together with the expression for the $R$-matrix given in section 2 implies that the $S$-matrix $S_{12}^*(p_1, p_2)$ obeys the unitarity condition, namely

$$S_{12}^*(p_1, p_2)S_{21}^*(p_2, p_1) = Id$$

where $Id$ is the identity matrix.

We now proceed under the assumption of the existence of an abelian phase $\theta(p_1, p_2)$ such that the $S$-matrix (34) indeed encodes the interactions of the many-body problem associated to the $AdS^5 \times S^5$ string sigma model. Under this hypothesis it is possible to find the respective particles momenta quantization by means of the asymptotic Bethe ansatz framework [22]. In this method one assumes that there exists particle coordinate regions $|x_i - x_j| \geq R_c$ where the particles do not interact and that possible off-mass-shell effects can be neglected. In such asymptotic regions the wave function is simply the linear combination plane waves with asymptotic momenta $p_1, \ldots, p_N$. More precisely, the state vector, in an asymptotic region where the particles coordinates are ordered as $0 \leq x_{Q_1} < x_{Q_2} < \ldots < x_{Q_N} \leq L$, has the form of a generalized Bethe ansatz [35, 36],

$$|\Psi\rangle = \int dx_1 \ldots dx_N \sum_P A_{\sigma_1, \ldots, \sigma_N}(P|Q)e^{i\sum_{j=1}^N p_{Q_j} x_{Q_j}} \prod_{k=1}^N \psi_{\sigma_k}^\dagger(x_k) |0\rangle,$$
where $\psi^{\dagger}_{\sigma k}(x_k)$ creates a particle with internal quantum number $\sigma_k$ on the vacuum $|0\rangle$. The sum runs over all $N!$ permutations $P$ of numbers $\{1, \cdots, N\}$ that index the particles.

The interaction between the particles permit them to cross the various regions and the corresponding amplitudes $A_{\sigma_1, \cdots, \sigma_N}(P|Q)$ are related to one another by the $S$-matrix (34). For instance, the amplitudes of two distinct regions $(P|Q)$ and $(\bar{P}|\bar{Q})$ differing by the permutation of neighboring $ith$ and $jth$ particles are connected by

$$A_{\sigma_1, \cdots, \sigma_i, \sigma_j, \cdots, \sigma_N}(\bar{P}|\bar{Q}) = S^*_{ijk}(p_i, p_j) A_{\sigma_1, \cdots, \bar{\sigma}_i, \bar{\sigma}_j, \cdots, \sigma_N}(P|Q),$$

where $S^*_{ijk}(p_i, p_j)$ are the matrix elements of the $S$-matrix (34).

The total energy $E$ and momenta $P$ of the Bethe wave function (38) are given by the free particle expressions, namely

$$P = \sum_{k=1}^{N} p_k, \quad E = \sum_{k=1}^{N} \varepsilon(p_k),$$

where $\varepsilon(p_i)$ is the one-particle dispersion relation.

From Eqs.(38,39) one clearly sees that the role of the scattering theory is to provide the conditions to match the wave function in adjacent free distinct regions. Therefore, the information that the internal wave function degrees of freedom satisfy both commuting and anticommuting rules of permutation should then be encoded in the $S$-matrix (34). This feature is guaranteed when one takes as $\hat{S}_{12}(p_1, p_2)$ the graded $\mathfrak{su}(2|2)$ $S$-matrix equation (35). The next step is to quantize the asymptotic momenta $p_k$ by imposing periodic boundary conditions to the wave function (38,39) on a ring of size $L$. This is accomplished by the successive use of Eq.(39) in order to relate different regions in the configuration space. The graded approach assures us strictly periodic boundary conditions in all sectors for both bosonic and fermionic variables. It turns out that the one-particle momenta $p_k$ are required to satisfy the following condition,

$$e^{-ip_kL} = \frac{\Lambda^*(\lambda = p_k, \{p_i\})}{S^*_{1,1}(p_k, p_k)}, \quad k = 1, \ldots, N$$

where $\Lambda^*(\lambda, \{p_i\})$ are the eigenvalues of the transfer matrix operator $T^*(\lambda, \{p_i\})$ given by

$$T^{(*)}(\lambda, \{p_i\}) = \text{Str}_A[S^{(s)}_{AN}(\lambda, p_N)S^{(s)}_{AN-1}(\lambda, p_{N-1}) \cdots S^{(s)}_{A1}(\lambda, p_1)]$$

14
There is no need of extra effort to compute the eigenvalues $\Lambda^*(\lambda, \{p_i\})$ because of the tensor product character of the $S$-matrix $S_{12}^{(a)}(p_1, p_2)$. They are simply given in terms of the product of the eigenvalues of the transfer matrix diagonalized in section 2. More specifically we have,

$$
\Lambda^*(\lambda, \{p_i\}) = \prod_{i=1}^{N} [S_0(\lambda, p_i)]^2 \Lambda(\lambda, \{p_i\}; \{\lambda_j^{(1)}, \mu_i^{(1)}\})\Lambda(\lambda, \{p_i\}; \{\lambda_j^{(2)}, \mu_i^{(2)}\})
$$

where $\{\lambda_j^{(\alpha)}\}$ and $\{\mu_j^{(\alpha)}\}$ $\alpha = 1, 2$ denote the Bethe ansatz roots used to diagonalize the transfer matrix based on the two $su(2|2)$ $S$-matrices of the tensor product (34).

Taking into account the explicit expression given in Eq. (32) as well as the structure of the abelian factor (36) one finds that the nested Bethe ansatz equations for one-particle momenta $p_k$ are,

$$
e^{ip_k \left(-L+N-\frac{m^{(1)}}{2}\right)} = e^{i\varepsilon_p N} \prod_{i=1}^{N} \left[ \frac{x^+(p_i) - x^-(p_k)}{x^+(p_i) - x^-(p_k)} \right] \left[ 1 - \frac{1}{x^+(p_k) x^-(p_i)} \right] \frac{1}{x^+(p_k) x^-(p_i)} \left[ \sigma(p_k, p_i) \right]^2 \\
\times \prod_{\alpha=1}^{2} \prod_{j=1}^{m_\alpha^{(1)}} \left[ \frac{x^+(\lambda_j^{(\alpha)}) - x^-(p_k)}{x^+(\lambda_j^{(\alpha)}) - x^+(p_k)} \right] \quad k = 1, \ldots, N,
$$

$$
e^{i\varepsilon_p} \prod_{i=1}^{N} \left[ \frac{x^+(\lambda_i^{(\alpha)}) - x^-(p_i)}{x^+(\lambda_i^{(\alpha)}) - x^+(p_i)} \right] = \prod_{i=1}^{m_\alpha^{(2)}} \frac{x^+(\lambda_j^{(\alpha)})}{x^+(\lambda_j^{(\alpha)})} + \frac{1}{x^+(\lambda_j^{(\alpha)})} \frac{1}{x^+(\lambda_j^{(\alpha)})} - \hat{\mu}_l^{(\alpha)} + \frac{i}{2g} \\
\times \prod_{j=1}^{m_\alpha^{(1)}} \left[ \frac{x^+(\lambda_j^{(\alpha)}) - x^+(\lambda_j^{(\alpha)})}{x^+(\lambda_j^{(\alpha)}) - x^+(\lambda_j^{(\alpha)})} \right] \quad j = 1, \ldots, m_\alpha^{(1)}; \alpha = 1, 2,
$$

$$
\prod_{j=1}^{m_\alpha^{(1)}} \frac{\tilde{\mu}_l^{(\alpha)} - x^+(\lambda_j^{(\alpha)}) - \frac{i}{2g}}{\hat{\mu}_l^{(\alpha)} - x^+(\lambda_j^{(\alpha)}) - \frac{i}{2g}} = \prod_{k=1}^{m_\alpha^{(2)}} \frac{\tilde{\mu}_l^{(\alpha)} - \tilde{\mu}_k^{(\alpha)} + \frac{i}{2g}}{\hat{\mu}_l^{(\alpha)} - \hat{\mu}_k^{(\alpha)} - \frac{i}{2g}} \\
l = 1, \ldots, m_\alpha^{(2)}; \alpha = 1, 2.
$$

(46)

Interesting enough, part of the Bethe ansatz equations depend on the total momenta of a given sector as well as of the number of the Bethe rapidities $N$ and $m_1^{(\alpha)}$. The latter feature suggests that $L' = L - N + \frac{m^{(1)}}{2} + \frac{m^{(2)}}{2}$ play the role of an effective scale from which densities should probably be measured. It turns out that this scale hides the angular momenta on $S^5$ of the $AdS_5 \times S^5$ theory in the light-cone gauge [32]. This fact can be viewed by bringing the
compact Bethe equations (44-46) close to the form of those originally proposed by Beisert and Staudacher [13]. We start the analysis by Eq. (44) that carries the explicit dependence on the size of the system. This equation is translated to that written by Beisert and Staudacher [13] by using the following roots identification,

\[ x^\pm(p_k) = \frac{x^\pm_{4,k}}{g}, \quad k = 1, \ldots, K_4 \]  

\[ x^+(\lambda_j^{(1)}) = \frac{g}{x_{1,j}}, \quad j = 1, \ldots, K_1, \quad x^+(\lambda_{K_1+j}^{(1)}) = \frac{x_{3,j}}{g}, \quad j = 1, \ldots, K_3 \]  

\[ x^+(\lambda_j^{(2)}) = \frac{x_{5,j}}{g}, \quad j = 1, \ldots, K_5, \quad x^+(\lambda_{K_5+j}^{(2)}) = \frac{g}{x_{7,j}}, \quad j = 1, \ldots, K_7 \]  

where \( N \equiv K_4, \) \( m_1^{(1)} \equiv K_1 + K_3 \) and \( m_1^{(2)} \equiv K_5 + K_7. \) We recall that the rapidities \( x^\pm_{4,k} \) and \( x_{i,j} \) for \( i = 1, 3, 5, 7 \) denote the Bethe roots used by Beisert and Staudacher [13].

By taking into account the above relabeling of Bethe roots it follows that Eq.(44) is equivalent to,

\[ e^{-ip_k(L-K_4+\frac{K_3-K_1}{2}+\frac{K_5-K_7}{2})} = e^{iP} \prod_{i=1}^{K_4} \left[ \frac{x_{4,k}^+-x_{4,i}^-}{x_{4,k}^+-x_{4,i}^+} \right] \left[ 1 - \frac{g^2}{x_{4,k}^+x_{4,i}^-} \right] \left[ \sigma(p_k,p_i) \right]^2 \]

\[ \times \prod_{j=1}^{K_3} \frac{x_{3,j}^-}{x_{3,j}^+} \prod_{j=1}^{K_1} \frac{1}{1-\frac{g^2}{x_{4,k}^-x_{4,j}^+}} \]

\[ \times \prod_{j=1}^{K_5} \frac{x_{5,j}^-}{x_{5,j}^+} \prod_{j=1}^{K_7} \frac{1}{1-\frac{g^2}{x_{4,k}^-x_{7,j}^+}}, \quad k = 1, \ldots, K_4 \]  

Before proceeding we note that the right-hand side of Eq.(50) reveals us the presence of the angular momenta charge \( J = \tilde{L} - K_4 + \frac{(K_3-K_1)}{2} + \frac{(K_5-K_7)}{2} \) [13]. Recall that in the notation of [13] the length \( \tilde{L} \) refers to effective scale of the system conjugated to the momenta variables. Considering the momenta ambiguity mentioned in section 3 one has to set here \( -J = L. \) This fact is in agreement with the expected meaning of \( J \) as the world sheet thermodynamic scale of the gauge-fixed \( AdS_5 \times S^5 \) model [32]. In order to complete the mapping of the nested Bethe equations (45,46) one has to make the following extra identifications,

\[ \tilde{u}_j^{(1)} = \frac{u_{2,j}}{g}, \quad j = 1, \ldots, K_2 \equiv m_2^{(1)} \]

\[ \tilde{u}_j^{(2)} = \frac{u_{6,j}}{g}, \quad j = 1, \ldots, K_6 \equiv m_2^{(2)} \]  

(51)
as well as the definition $u_{i,j} = x_{i,j} + \frac{g^2}{x_{i,j}}$ for $i = 1, 3, 5, 7$.

By using all these identifications, Eqs.(45) can be transformed to the following four type of Bethe equations,

$$e^{-iP/2} \prod_{i=1}^{K_4} \frac{1 - \frac{g^2}{x_{1,i}^2}}{1 - \frac{g^2}{x_{1,i}^2}} = \prod_{l=1}^{K_2} \frac{u_{1,j} - u_{2,l} + \frac{i}{2}}{u_{1,j} - u_{2,l} - \frac{i}{2}}, \quad j = 1, \ldots, K_1$$

$$e^{iP/2} \prod_{i=1}^{K_4} \frac{x_{4,i}^- - x_{3,j}}{x_{4,i}^- - x_{3,j}} = \prod_{l=1}^{K_2} \frac{u_{3,j} - u_{2,l} + \frac{i}{2}}{u_{3,j} - u_{2,l} - \frac{i}{2}}, \quad j = 1, \ldots, K_3$$

$$e^{iP/2} \prod_{i=1}^{K_4} \frac{x_{5,i}^- - x_{5,j}}{x_{5,i}^- - x_{5,j}} = \prod_{l=1}^{K_6} \frac{u_{5,j} - u_{6,l} + \frac{i}{2}}{u_{5,j} - u_{6,l} - \frac{i}{2}}, \quad j = 1, \ldots, K_5$$

$$e^{-iP/2} \prod_{i=1}^{K_4} \frac{1 - \frac{g^2}{x_{7,i}^2}}{1 - \frac{g^2}{x_{7,i}^2}} = \prod_{l=1}^{K_6} \frac{u_{7,j} - u_{6,l} + \frac{i}{2}}{u_{7,j} - u_{6,l} - \frac{i}{2}}, \quad j = 1, \ldots, K_7$$

(52)

while Eqs.(46) become,

$$\prod_{j=1}^{K_3} \frac{u_{2,l} - u_{1,j} + \frac{i}{2}}{u_{2,l} - u_{1,j} - \frac{i}{2}} = \prod_{j=1}^{K_3} \frac{u_{2,l} - u_{3,j} + \frac{i}{2}}{u_{2,l} - u_{3,j} - \frac{i}{2}} = \prod_{l=1}^{K_2} \frac{u_{2,l} - u_{2,k} + i}{u_{2,l} - u_{2,k} - i}, \quad l = 1, \ldots, K_2$$

$$\prod_{j=1}^{K_7} \frac{u_{6,l} - u_{5,j} + \frac{i}{2}}{u_{6,l} - u_{5,j} - \frac{i}{2}} = \prod_{j=1}^{K_7} \frac{u_{6,l} - u_{7,j} + \frac{i}{2}}{u_{6,l} - u_{7,j} - \frac{i}{2}} = \prod_{l=1}^{K_6} \frac{u_{6,l} - u_{6,k} + i}{u_{6,l} - u_{6,k} - i}, \quad l = 1, \ldots, K_6$$

(53)

Direct comparison between Eqs.(50,52,53) for $P = 0$ with those presented by Beisert and Staudacher in table 5 of reference [13] shows us that they are indeed equivalent in the case of the grading $\eta_1 = \eta_2 = +1$ choice \(^4\). Here we emphasize that on the context of the $AdS_5 \times S^5$ string, however, is the charge $J$ that plays the role of the thermodynamic scale. As a consequence of that the thermodynamic limit $J \to \infty$ has to be taken without making reference to any particular sector of the theory. This is of special importance in nested Bethe ansatz systems since all the Bethe roots levels can in principle contribute to the physical properties in the infinite volume limit. Recall that this feature has recently been pointed out to be relevant to unveil the possible origin of the dressing phase of the $\mathcal{N} = 4$ Yang-Mills theory [37].

\(^4\)We also recall that here $g = \sqrt{\lambda}/(4\pi)$ where $\lambda$ is the ‘t Hooft coupling.
5 Conclusion

In this work we have studied Bethe ansatz properties of integrable models associated to centrally extended $\mathfrak{su}(2|2)$ superalgebras. The $\mathfrak{su}(2|2)$ $R$-matrix has been shown to be in the same family of Shastry’s $R$-matrix [28, 29] with the help of spectral parameter transformation. This connection made possible the solution of the eigenvalue problem associated to the $\mathfrak{su}(2|2)$ transfer matrix within the algebraic Bethe ansatz method.

The motivation to study this type of system arose from its conjectured relationship with scattering properties of the world-sheet excitations of the $AdS_5 \times S^5$ string sigma model. In this context we have been able to derive the Bethe ansatz equations for the respective particle momenta on the circle. They presented an unusual dependence on the total momenta and the number of certain rapidities that parameterize the Hilbert space. We have argued that the latter feature encodes the information that the effective $AdS^5 \times S^5$ thermodynamic scale is governed by the angular momenta $J$ charge. This identification allows, in principle, to take the thermodynamic limit considering the nested Bethe ansatz equations altogether. It remains to be investigate the configurations of the Bethe roots that dominate the $J \to \infty$ limit which is crucial for the understanding of the elementary excitations of the spectrum such as the existence of possible bound states [38, 39].

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Appendix A: Shastry’s $R$-matrix

In this appendix we present the explicit expression of Shastry’s graded $R$ matrix [28, 29].

Following the notation of [30] this $R$-matrix is given by,

$$
\tilde{R}^{(s)}_{12}(\lambda, \mu) = \alpha_2(\lambda, \mu)[e_{11} \otimes e_{11} + e_{44} \otimes e_{44}] + \alpha_4(\lambda, \mu)[e_{11} \otimes e_{44} + e_{44} \otimes e_{11}]
$$

$$
\quad + \alpha_1(\lambda, \mu)[e_{22} \otimes e_{22} + e_{33} \otimes e_{33}] + \alpha_3(\lambda, \mu)[e_{22} \otimes e_{33} + e_{33} \otimes e_{22}]
$$

$$
\quad + \alpha_7(\lambda, \mu)[e_{14} \otimes e_{41} + e_{41} \otimes e_{14}] - \alpha_6(\lambda, \mu)[e_{23} \otimes e_{32} + e_{32} \otimes e_{23}]
$$

$$
\quad - i\alpha_8(\lambda, \mu)[e_{21} \otimes e_{12} + e_{24} \otimes e_{42} + e_{31} \otimes e_{13} + e_{34} \otimes e_{43}]
$$

$$
\quad - i\alpha_9(\lambda, \mu)[e_{12} \otimes e_{21} + e_{13} \otimes e_{31} + e_{42} \otimes e_{24} + e_{43} \otimes e_{34}]
$$

$$
\quad - i\alpha_{10}(\lambda, \mu)[e_{21} \otimes e_{43} - e_{24} \otimes e_{12} - e_{31} \otimes e_{42} - e_{34} \otimes e_{13}]
$$

$$
\quad + i\alpha_{10}(\lambda, \mu)[e_{21} \otimes e_{34} - e_{24} \otimes e_{31} - e_{24} \otimes e_{13} + e_{24} \otimes e_{34}]
$$

$$
\quad + \alpha_5(\lambda, \mu)[e_{11} \otimes e_{22} + e_{11} \otimes e_{33} + e_{44} \otimes e_{22} + e_{44} \otimes e_{33}]
$$

$$
\quad + \alpha_5(\lambda, \mu)[e_{22} \otimes e_{11} + e_{22} \otimes e_{44} + e_{33} \otimes e_{11} + e_{33} \otimes e_{44}]
$$

(A.1)

The ten non-null Boltzmann weights are,

$$
\alpha_1(\lambda, \mu) = \left\{ e^{[h(\mu)-h(\lambda)]}a(\lambda)a(\mu) + e^{-[h(\mu)-h(\lambda)]}b(\lambda)b(\mu) \right\} \alpha_5(\lambda, \mu),
$$

$$
\alpha_2(\lambda, \mu) = \left\{ e^{-[h(\mu)-h(\lambda)]}a(\lambda)a(\mu) + e^{[h(\mu)-h(\lambda)]}b(\lambda)b(\mu) \right\} \alpha_5(\lambda, \mu),
$$

$$
\alpha_3(\lambda, \mu) = \frac{e^{[h(\mu)+h(\lambda)]}a(\lambda)b(\mu) + e^{-[h(\mu)+h(\lambda)]}b(\lambda)a(\mu)}{a(\lambda)b(\lambda) + a(\mu)b(\mu)} \left\{ \frac{\cosh[h(\mu) - h(\lambda)]}{\cosh[h(\mu) + h(\lambda)]} \right\} \alpha_5(\lambda, \mu),
$$

$$
\alpha_4(\lambda, \mu) = \frac{e^{-[h(\mu)+h(\lambda)]}a(\lambda)b(\mu) + e^{[h(\mu)+h(\lambda)]}b(\lambda)a(\mu)}{a(\lambda)b(\lambda) + a(\mu)b(\mu)} \left\{ \frac{\cosh[h(\mu) - h(\lambda)]}{\cosh[h(\mu) + h(\lambda)]} \right\} \alpha_5(\lambda, \mu),
$$

$$
\alpha_6(\lambda, \mu) = \frac{e^{[h(\mu)+h(\lambda)]}a(\lambda)b(\mu) - e^{-[h(\mu)+h(\lambda)]}b(\lambda)a(\mu)}{a(\lambda)b(\lambda) + a(\mu)b(\mu)} \left\{ \frac{b^2(\mu) - b^2(\lambda)}{\cosh[h(\mu) - h(\lambda)]} \right\} \alpha_5(\lambda, \mu),
$$

$$
\alpha_7(\lambda, \mu) = \frac{-e^{-[h(\mu)+h(\lambda)]}a(\lambda)b(\mu) + e^{[h(\mu)+h(\lambda)]}b(\lambda)a(\mu)}{a(\lambda)b(\lambda) + a(\mu)b(\mu)} \left\{ \frac{b^2(\mu) - b^2(\lambda)}{\cosh[h(\mu) - h(\lambda)]} \right\} \alpha_5(\lambda, \mu),
$$

$$
\alpha_8(\lambda, \mu) = \left\{ e^{[h(\mu)-h(\lambda)]}a(\lambda)b(\mu) - e^{-[h(\mu)-h(\lambda)]}b(\lambda)a(\mu) \right\} \alpha_5(\lambda, \mu),
$$

$$
\alpha_9(\lambda, \mu) = \left\{ -e^{-[h(\mu)-h(\lambda)]}a(\lambda)b(\mu) + e^{[h(\mu)-h(\lambda)]}b(\lambda)a(\mu) \right\} \alpha_5(\lambda, \mu),
$$

$$
\alpha_{10}(\lambda, \mu) = \frac{b^2(\mu) - b^2(\lambda)}{a(\lambda)b(\lambda) + a(\mu)b(\mu)} \left\{ \frac{\cosh[h(\mu) - h(\lambda)]}{\cosh[h(\mu) + h(\lambda)]} \right\} \alpha_5(\lambda, \mu),
$$

(A.2)
where the weight $\alpha_5(\lambda, \mu)$ is an overall normalization. The functions $a(\lambda), b(\lambda)$ and the coupling $h(\lambda)$ satisfy the constraints,

$$a^2(\lambda) + b^2(\lambda) = 1, \quad \sinh[2h(\lambda)] = \frac{a(\lambda)b(\lambda)}{2g} \quad (A.3)$$

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