Higher Anomalies, Higher Symmetries, and Cobordisms II:
Applications to Quantum Gauge Theories

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Abstract

We discuss the topological terms, the global symmetries and their 't Hooft anomalies of pure gauge theories in various dimensions, with dynamical gauge group \(G\), the Lorentz symmetry group \(G_{\text{Lorentz}}\), and the internal global symmetry \(G_{e,[1]} \times G_{m,[d-3]}\) which consists of 1-form electric center symmetry \(G_{e,[1]}\) and \((d-3)\) form magnetic symmetry \(G_{m,[d-3]}\). The topological terms are determined by the cobordism invariants \((\Omega^d)^G'\) where \(G'\) is the group extension of \(G_{\text{Lorentz}}\) by \(G\), which also characterize the invertible TQFTs or SPTs with global symmetry \(G'\). The 't Hooft anomalies are determined by the cobordism invariants \((\Omega^{d+1})^{G''}\) where \(G''\) is the symmetry extension of \(G_{\text{Lorentz}}\) by the higher form symmetry \(G_{e,[1]} \times G_{m,[d-3]}\). Different symmetry extensions correspond to different fractionalizations of \(G_{\text{Lorentz}}\) quantum numbers on the symmetry defects of \(G_{e,[1]} \times G_{m,[d-3]}\). We compute the cobordism groups/invariants described above for \(G = U(1), SU(2)\) and \(SO(3)\) in \(d \leq 5\), thus systematically classifies all the topological terms and the 't Hooft anomalies of \(d\) dimensional quantum gauge theories with the above gauge groups.

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1 Introduction and Summary

Global symmetries and ’t Hooft anomalies are key ingredients characterizing topological phases of quantum matters [1–9]. For a generic quantum field theory $Q_d$ in $d$ spacetime dimensions with global symmetry $G$, if one modifies $Q_d$ to $Q'_d$ by $G$-invariant deformations, it has been shown [10] that the ’t Hooft anomaly of $Q_d$ and $Q'_d$ are the same. In particular, the ’t Hooft anomaly is invariant under renormalization group flow. Recently it has been conjectured [11] that given two quantum field theories $Q_d$ and $Q'_d$ in the same spacetime dimension, with the same global symmetries and ’t Hooft anomalies, one can always add degrees of freedom at short distances to interpolate between $Q_d$ and $Q'_d$. Hence the global symmetries and the ’t Hooft anomalies classify the deformation classes in the space of quantum field theories.

It is widely believed that the ’t Hooft anomaly of $Q_d$ can be cancelled by the anomaly inflow from an anomaly polynomial in $(d + 1)$ dimension. Physically, the anomaly polynomial is characterized by a $G$-Symmetry Protected Topological state ($G$-SPTs) [1]. Mathematically, the anomaly polynomial is characterized by a Cobordism invariant, or an invertible topological quantum field theory (iTQFT) [7,12–15]. The purpose of this work is to derive the cobordism invariants for various $G$ in various dimensions, hence potentially classifying the deformation classes for given symmetries.

The examples we study in this work are the pure gauge theories whose gauge groups are small rank Lie groups $G$. Following [7,16], there are different $d$-dimensional gauge theories for the same gauge group $G$, which are obtained by gauging different $d$-dimensional $G$-SPTs. Thus in this work, we systematically construct the pure gauge theories in two steps:

1. We first classify the $d$-dimensional $G$-SPTs, by computing the cobordism invariants in the same dimension. Here $G$ is a unitary global symmetry, taken to be $G = U(1), SU(2), SO(3)$ and $SU(3)$.

2. We further gauge the global symmetry $G$ to obtain a pure gauge theory with dynamical gauge group $G$. After gauging, there are emergent global symmetries. We further compute the ’t Hooft anomaly of the gauge theory by computing the cobordism invariant of the emergent symmetries in $(d + 1)$ dimensions.

In the rest of this introduction section, we explain the above two steps in detail.

1.1 $G$-SPTs

We first consider symmetry protected topological phases with the internal unitary global symmetry $G$. In this work, we take $G$ to be continuous small rank Lie groups, $G = U(1), SU(2), SO(3)$ and $SU(3)$ which are all zero form symmetries. By internal, we demand that $G$ does not act on the coordinates. These symmetries are related to the Standard models [17–22]. Beyond the internal symmetries, we assume the SPTs preserve the Lorentz symmetry $G_{\text{Lorentz}}$. Depending on whether the theory depends on the spin structure, and whether the theory has time reversal symmetry, we
consider the following choices of $G_{\text{Lorentz}}$ [16, 23].

$$
G_{\text{Lorentz}} = \begin{cases} 
\text{SO}(d), & \text{bosonic,} \\
\text{Spin}(d), & \text{fermionic,} \\
O(d), & \text{bosonic}, \quad T^2 = 1, \\
E(d), & \text{bosonic,} \quad T^2 = -1, \\
\text{Pin}^+(d), & \text{fermionic,} \quad T^2 = -1, \\
\text{Pin}^-(d), & \text{fermionic,} \quad T^2 = 1. 
\end{cases} \quad (1.1)
$$

We explain the terminology in (1.1) below.

1. If a theory depends (does not depend) on the spin structure, this theory can emerge from a UV system whose fundamental degrees of freedom are fermions (bosons), hence we say such a theory is fermionic (bosonic).

2. If $G_{\text{Lorentz}} = \text{SO}(d)$ or $\text{Spin}(d)$, such a theory does not have time reversal symmetry. However, for the remaining four choices of $G_{\text{Lorentz}}$, the theory is time reversal symmetric. Differential implementations gives rise to different $G_{\text{Lorentz}}$. For $G_{\text{Lorentz}} = O(d), E(d), \text{Pin}^+(d)$ and $\text{Pin}^-(d)$, there is a $\mathbb{Z}_2$ center. $T^2$ in (1.1) indicates the $T^2$ eigenvalue acting on the field with odd charge under $\mathbb{Z}_2$ center in the theory.

3. If $G_{\text{Lorentz}} = \text{SO}(d)$, we denote such a theory has SO structure. Similarly, we define $\text{Spin}, O, E, \text{Pin}^+, \text{Pin}^-$ structures.

Given the internal global symmetry group $G$, and the Lorentz symmetry group $G_{\text{Lorentz}}$, the total group is given by the extension

$$
1 \to G \to G' \to G_{\text{Lorentz}} \to 1 \quad (1.2)
$$

In particular, the total group $G'$ does not have to be the simply direct product of the internal and Lorentz groups $G \times G_{\text{Lorentz}}$. For example, let us consider a bosonic theory without time reversal, hence $G_{\text{Lorentz}} = \text{SO}(d)$, and let us take the internal symmetry to be SU(2). There are two choices of the total group, $G' = \text{SU}(2) \times \text{SO}(d)$, and $G' = \text{SU}(2) \times \mathbb{Z}_2 \text{Spin}(d)$. The latter means that the field carrying charge 1 under $\mathbb{Z}_2 \subset \text{SU}(2)$ also transforms under $\mathbb{Z}_2 \subset \text{Spin}(d)$, which means that it is a fermion. Notice however although there is a Spin(d) in the total group, it does not mean that the theory is fermionic. If the internal symmetry SU(2) is broken, the total group is still Spin(d)/$\mathbb{Z}_2 = \text{SO}(d)$ hence it is still bosonic.

In the main text, when $G_{\text{Lorentz}} = \text{Spin}(d), \text{Pin}^+(d)$ and $\text{Pin}^-(d)$, we will use a slightly different group extension compared with (1.2). We move the fermi parity $\mathbb{Z}_2^F$ into the $G$, and considered a modified group extension

$$
1 \to G \times \mathbb{Z}_2^F \to G' \to \frac{G_{\text{Lorentz}}}{\mathbb{Z}_2^F} \to 1 \quad (1.3)
$$

Hence in the fourth component of the sequence (1.3), we only need to consider $\text{SO}(d)$ and $O(d)$. Notice that such a modification does not change the total group $G'$.

Given the total group $G'$, the SPTs protected by $G'$ is given by the cobordism invariant

$$
(\Omega^d)^G = \text{Hom} \left( \Omega^d_G, U(1) \right) \quad (1.4)
$$
The majority part of this work is to use Adams spectral sequence to compute $\Omega^G_d$ from which $(\Omega^d)^G$ can be inferred. Physically, the $G$-SPT given by $(\Omega^d)^G$ is the precursor of the gauge theory with dynamical gauge group $G$. We can also view such $G$-SPT as the anomaly polynomial for $(d-1)$ dimensional quantum field theory with the 0-form global symmetry $G$.

The following fact is useful to infer $(\Omega^d)^G$ from $\Omega^G_d$. For all the examples in this work, $\Omega^G_d$ is a tensor product of finite order cyclic groups $\mathbb{Z}$ and infinite order cyclic group $\mathbb{Z}$. If $\mathbb{Z}_p$ is a subgroup of $\Omega^G_d$, then $\mathbb{Z}_p$ is also a subgroup of $(\Omega^d)^G$. If $\mathbb{Z}$ is a subgroup of $\Omega^G_d$, then $\mathbb{Z}$ is a subgroup of $(\Omega^{d-1})^G$ instead. In summary,

$$
\begin{align*}
\mathbb{Z}_p \subset \Omega^G_d & \implies \mathbb{Z}_p \subset (\Omega^d)^G, \\
\mathbb{Z} \subset \Omega^G_d & \implies \mathbb{Z} \subset (\Omega^{d-1})^G.
\end{align*}
$$

\section{A Mathematical Primer For $\Omega^G_d$}

We give a mathematical primer for computing the bordism group $\Omega^G_d$, see [20,24] for details. We will use the generalized Pontryagin-Thom isomorphism,

$$
\Omega^G_d = \pi_d(MTG')
$$

(1.5)

to identify the cobordism group $\Omega^G_d$ with the homotopy group of the Madsen-Tillmann spectrum $MTG'$. Hence computing $\Omega^G_d$ is equivalent to computing $\pi_d(MTG')$. To compute $\pi_d(MTG')$, we use the Adams spectral sequence

$$
\text{Ext}^{s,t}_{\mathcal{A}_p}(H^*(Y,\mathbb{Z}_p),\mathbb{Z}_p) \Rightarrow \pi_{t-s}(Y)^\wedge.
$$

(1.6)

Here $\mathcal{A}_p$ is the mod $p$ Steenrod algebra, $Y$ is any spectrum. For any finitely generated abelian group $G$, $G^\wedge_p = \lim_{n \to \infty} G/p^nG$ is the $p$-completion of $G$. In particular, $\mathcal{A}_2$ is generated by Steenrod squares $Sq^i$. To apply to our problem, we demand $Y = MTG'$ in Adams spectral sequence, and we focus on $p = 2$. For example, if $MTG' = MSpin \wedge X$ where $X$ is any spectrum, by Corollary 5.1.2 of [25], we have

$$
\text{Ext}^{s,t}_{\mathcal{A}_2}(H^*(MSpin \wedge X,\mathbb{Z}_2),\mathbb{Z}_2) = \text{Ext}^{s,t}_{\mathcal{A}_2(1)}(H^*(X,\mathbb{Z}_2),\mathbb{Z}_2)
$$

(1.7)

for $t - s < 8$. Here $\mathcal{A}_2(1)$ is the subalgebra of $\mathcal{A}_2$ generated by $Sq^1$ and $Sq^2$. Hence for the dimension $d = t - s < 8$, we have

$$
\text{Ext}^{s,t}_{\mathcal{A}_2(1)}(H^*(X,\mathbb{Z}_2),\mathbb{Z}_2) \Rightarrow (\Omega^G_d)^\wedge_{t-s}
$$

(1.8)

The $H^*(X,\mathbb{Z}_2)$ is an $\mathcal{A}_2(1)$-module whose internal degree $t$ is given by the $\ast$.

Our computation of $E_2$ pages of $\mathcal{A}_2(1)$-modules is based on Lemma 11 of [24]. More precisely, we find a short exact sequence of $\mathcal{A}_2(1)$-modules $0 \to L_1 \to L_2 \to L_3 \to 0$, then apply Lemma 11 of [24] to compute $\text{Ext}^{s,t}_{\mathcal{A}_2(1)}(L_2,\mathbb{Z}_2)$ by the data of $\text{Ext}^{s,t}_{\mathcal{A}_2(1)}(L_1,\mathbb{Z}_2)$ and $\text{Ext}^{s,t}_{\mathcal{A}_2(1)}(L_3,\mathbb{Z}_2)$. Our strategy is choosing $L_1$ to be the direct sum of suspensions of $\mathbb{Z}_2$ on which $Sq^1$ and $Sq^2$ act trivially, then we take $L_3$ to be the quotient of $L_2$ by $L_1$. We can use this procedure again and again until $\text{Ext}^{s,t}_{\mathcal{A}_2(1)}(L_3,\mathbb{Z}_2)$ is determined.
1.3 Gauging $G$, Emergent Symmetries and Anomalies

We further gauge the global symmetry $G$ in the $G$-SPT computed from (1.4). Suppose the $G$-background field is $A$. Denote the partition function of the SPT $Z_{\text{SPT}}[A]$. Because $G$-SPT is also an iTQFT, $Z_{\text{SPT}}[A]$ is a $U(1)$ phase. Gauging $G$ amounts to summing over $A$ in the path integral. The partition function after gauging $G$ is [7,9,17,19,26–29]

$$Z = \int [DA] \ Z_{\text{SPT}}[A] \ Z_{\text{dyn}}[A]$$  \hspace{1cm} (1.10)

where $Z_{\text{dyn}}[A]$ is the partition function of a non-topological theory, which has nontrivial coupling constants and can flow under the renormalization group. Since in this work we focus on pure gauge theories, we will exclusively take $Z_{\text{dyn}}[A]$ to be the Yang-Mills action $Z_{\text{YM}}[A]$, where

$$Z_{\text{YM}}[A] = \exp \left( -\frac{i}{4g^2} \int \text{Tr} (F \wedge \ast F) \right)$$  \hspace{1cm} (1.11)

After gauging the 0-form symmetry $G$, the dynamical gauge theory exhibits emergent global symmetries. If $G$ has nontrivial center, there is an emergent 1-form global symmetry $G_{e,[1]}$. If either $u_2(V_G) \in H^2(G, U(1))$ or $u_2(V_G) \in H^2(G, \mathbb{Z}_p)$ for some integer $p$ is nontrivial, then there is an emergent $(d-3)$-form global symmetry, i.e. $G_{m,[d-3]}$. [30–32] 1 For the gauge groups we discuss in this work, we enumerate the emergent global symmetries in the table below:

| $G$     | $G_{e,[1]}$ | $G_{m,[d-3]}$ |
|---------|------------|---------------|
| U(1)    | U(1)$_{[1]}$ | U(1)$_{[d-3]}$ |
| SU(2)   | $\mathbb{Z}_2[1]$ | 0 |
| SO(3)   | 0          | $\mathbb{Z}_2[1]$ |
| SU(3)   | $\mathbb{Z}_3[1]$ | 0 |

Since the new global symmetries after gauging $G$ contains the $G_{\text{Lorentz}},\ G_{e,[1]}$ and $G_{m,[d-3]}$, the total global symmetry is determined by the exact sequence,

$$1 \rightarrow G_{e,[1]} \times G_{m,[d-3]} \rightarrow G' \rightarrow G_{\text{Lorentz}} \rightarrow 1$$  \hspace{1cm} (1.12)

Physically [16,33,34], different extensions (1.12) correspond to different Lorentz symmetry fractionalizations on the symmetry defects of $G_{e,[1]} \times G_{m,[d-3]}$.

The 't Hooft anomaly of the global symmetry $G'$ in (1.12) can be derived from the cobordism group

$$ (\Omega^d)^{G'} = \text{Hom} \left( \Omega^d_G, U(1) \right)$$  \hspace{1cm} (1.13)

which should be distinguished from (1.4) because in (1.13) the $G'$ is a higher group. We emphasize that different choices of $Z_{\text{SPT}}[A]$ in the path integral (1.10) can dramatically affect the dynamics of gauge theories after gauging $G$. For example, for $d = 4$, we consider $G = SU(2)$ and $G_{\text{Lorentz}} = O(4)$. The $G$-SPT includes a 4d theta term with $\theta = 0$ and $\pi$. It has been extensively discussed [5,16,34] that after $SU(2)$ is gauged, there is a mixed anomaly between the emergent 1-form symmetry $\mathbb{Z}_2[1]$ and $O(4)$ for $\theta = \pi$ while no anomaly for $\theta = 0$. Hence as a consequence, the dynamics for $\theta = 0$ and $\pi$ are dramatically different.

1This is to be contrasted with the discrete symmetry. If $G$ is a discrete 0-form symmetry, gauging $G$ in $d$ dimension will induce a $(d-1)$-form global symmetry. In our case, $G$ is a continuous 0-form symmetry. Gauging it will induce a $(d-3)$-form global symmetry.
1.4 Outline

The rest of this work computes the bordism groups/invariants for various internal symmetries \( G \) and the Lorentz symmetries \( G_{\text{Lorentz}} \), as well as their extensions. We further gauge \( G \) to obtain \( G \) dynamical gauge theories and compute the bordism groups/invariants for the extension of the Lorentz symmetry \( G_{\text{Lorentz}} \) by the emergent higher symmetries \( G_1 \times G_m[d-3] \). In Sec. 2, we discuss \( G = SU(2) \), and \( G_{\text{Lorentz}} = SO(d), O(d) \) and \( E(d) \). In Sec. 3, we still discuss \( G = SU(2) \), but \( G_{\text{Lorentz}} = \text{Spin}(d), \text{Pin}^+(d) \) and \( \text{Pin}^-(d) \). In Sec. 4, we discuss \( G = SO(3) \), and \( G_{\text{Lorentz}} = SO(d), O(d) \) and \( E(d) \). In Sec. 5, we still discuss \( G = SO(3) \), but \( G_{\text{Lorentz}} = \text{Spin}(d), \text{Pin}^+(d) \) and \( \text{Pin}^-(d) \). In Sec. 6, we discuss \( G = U(1) \), and \( G_{\text{Lorentz}} = SO(d), O(d) \) and \( E(d) \). In Sec. 7, we still discuss \( G = U(1) \), but \( G_{\text{Lorentz}} = \text{Spin}(d), \text{Pin}^+(d) \) and \( \text{Pin}^-(d) \).

2 SU(2) extension

2.1 \( 1 \rightarrow SU(2) \rightarrow G' \rightarrow SO(d) \rightarrow 1 \)

We consider the group extension problem

\[
1 \rightarrow SU(2) \rightarrow G' \rightarrow SO(d) \rightarrow 1.
\] (2.1)

There are two solutions, \( G' = SO(d) \times SU(2) \) or \( \text{Spin}(d) \times \mathbb{Z}_2 \cdot SU(2) \).

2.1.1 \( SO \times SU(2) \)

We have \( MT(SO \times SU(2)) = MSO \wedge (BSU(2))_+ \). Here \( X_+ \) is the disjoint union of the topological space \( X \) and a point.

By K"unneth formula,

\[
H^*(MSO \wedge (BSU(2))_+, \mathbb{Z}_2) = H^*(MSO, \mathbb{Z}_2) \otimes H^*(BSU(2), \mathbb{Z}_2).
\] (2.2)

We have used the reduced version, note that the reduced cohomology of \( X_+ \) is exactly the ordinary cohomology of \( X \).

Since there is no odd torsion, we have the Adams spectral sequence

\[
\text{Ext}^{s,t}_{A_2}(H^*(MSO, \mathbb{Z}_2) \otimes H^*(BSU(2), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega^*_{t-s} \wedge SU(2).
\] (2.3)

The localization of \( MSO \) at the prime 2 is \( MSO_2 = H\mathbb{Z}_2 \vee \Sigma^4 H\mathbb{Z}_2 \vee \Sigma^5 H\mathbb{Z}_2 \vee \cdots \), here \( H \Sigma \) is the Eilenberg-MacLane spectrum of the group \( G \), \( \Sigma \) is the suspension, \( \vee \) is the wedge sum.

So the mod 2 cohomology is

\[
H^*(MSO, \mathbb{Z}_2) = A_2/A_2 \text{Sq}^1 \oplus \Sigma^4 A_2/A_2 \text{Sq}^1 \oplus \Sigma^5 A_2 \oplus \cdots.
\] (2.4)
The projective $A_2$-resolution of $A_2/A_2 Sq^1$ (denoted by $P_\bullet$) is
\[ \cdots \to \Sigma^3 A_2 \to \Sigma^2 A_2 \to \Sigma A_2 \to A_2 \to A_2/A_2 Sq^1 \] (2.5)
where the differentials $d_1$ are induced from $Sq^1$.

We have
\[ H^*(BSU(2), \mathbb{Z}_2) = \mathbb{Z}_2[c_2]. \] (2.6)
Here $c_2$ is the second Chern class of the $SU(2)$ bundle.

Since $P_\bullet$ is actually a free $A_2$-resolution of $A_2/A_2 Sq^1$, so $P_\bullet \otimes H^*(BSU(2), \mathbb{Z}_2)$ is also a free $A_2$-resolution of $A_2/A_2 Sq^1 \otimes H^*(BSU(2), \mathbb{Z}_2)$.

The $E_2$ page of the Adams spectral sequence is shown in Figure 1.

\begin{table}[h]
\centering
\begin{tabular}{ccc}
\hline
$d$ & $\Omega^*_{d SO \times SU(2)}$ & generators \\
\hline
0 & $\mathbb{Z}$ & \\
1 & 0 & \\
2 & 0 & \\
3 & 0 & \\
4 & $\mathbb{Z}^2$ & $\sigma, c_2$ \\
5 & $\mathbb{Z}_2$ & $w_2 w_3$ \\
\hline
\end{tabular}
\caption{Bordism group. Here $\sigma$ is the signature of the 4-manifold, $c_2$ is the Chern class of the $SU(2)$ bundle, $w_i$ is the Stiefel-Whitney class of the tangent bundle.}
\end{table
2.1.2 Spin $\times \mathbb{Z}_2$ SU(2)

Let $G' = \frac{\text{Spin} \times \text{SU}(2)}{\mathbb{Z}_2}$, then by [12], we have $MTG' = M\text{Spin} \wedge \Sigma^{-3}M\text{SO}(3)$.

For $t - s < 8$, since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}^{s,t}_{A_2(1)}(H^{*+3}(\text{SO}(3), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\mathbb{Z}_2}. \quad (2.7)$$

The $A_2(1)$-module structure of $H^{*+3}(\text{SO}(3), \mathbb{Z}_2)$ below degree 5 and the $E_2$ page are shown in Figure 2, 3.

![Figure 2: The $A_2(1)$-module structure of $H^{*+3}(\text{SO}(3), \mathbb{Z}_2)$ below degree 5.](image)

![Figure 3: $\Omega_{t-s}^{\mathbb{Z}_2}$](image)
Table 2: Bordism group. Here $w_i$ is the Stiefel-Whitney class of the tangent bundle, $\sigma$ is the signature of the 4-manifold, Arf is the Arf invariant, $? \equiv P_2(x_2)$ is an undetermined cobordism invariant which also appears in [20].

\begin{table}[h]
\centering
\begin{tabular}{c|c|c}
\hline
$d$ & $\Omega_d^{\text{Spin} \times \mathbb{Z}_2 \text{SU}(2)}$ & Generators \\
\hline
0 & $\mathbb{Z}$ & \\
1 & 0 & \\
2 & 0 & \\
3 & 0 & \\
4 & $\mathbb{Z}^2$ & $\sigma, ?$ \\
5 & $\mathbb{Z}_2^2$ & $w_2w_3, w_3\text{Arf}(?)$ \\
\hline
\end{tabular}
\caption{Bordism group.}
\end{table}

2.2 $1 \to \text{SU}(2) \to G' \to \text{SO}(d) \to 1$, \textbf{Gauge SU(2) in G', end with SO $\times \mathbb{Z}_2[1]$}

We consider the group extension $1 \to \text{SU}(2) \to G' \to \text{SO}(d) \to 1$, and further promote SU(2) internal global symmetry to a gauge symmetry. The resulting theory has a emergent $\mathbb{Z}_2[1]$ electric 1-form global symmetry. Together with the Lorentz symmetry SO($d$), the total global symmetry is SO $\times \mathbb{Z}_2[1]$. Thus we compute the bordism group $\Omega_d^{\text{SO} \times \mathbb{Z}_2[1]} \equiv \Omega_d^{\text{SO}(B^2\mathbb{Z}_2)}$.

2.2.1 $\text{SO} \times \mathbb{Z}_2[1]$

This case is considered in [24]. We just summarize the bordism groups in Table 3.

\begin{table}[h]
\centering
\begin{tabular}{c|c|c}
\hline
$d$ & $\Omega_d^{\text{SO} \times B\mathbb{Z}_2}$ & Generators \\
\hline
0 & $\mathbb{Z}$ & \\
1 & 0 & \\
2 & $\mathbb{Z}_2$ & $x_2$ \\
3 & 0 & \\
4 & $\mathbb{Z} \times \mathbb{Z}_4$ & $\sigma, P_2(x_2)$ \\
5 & $\mathbb{Z}_2^2$ & $w_2w_3, x_2x_3 = x_5$ \\
\hline
\end{tabular}
\caption{Bordism group. Here $x_2$ is the generator of $H^2(B^2\mathbb{Z}_2, \mathbb{Z}_2)$, $x_3 = \text{Sq}^1 x_2$, $x_5 = \text{Sq}^2 x_3$, $P_2(x_2)$ is the Pontryagin square of $x_2$, $\sigma$ is the signature of the 4-manifold, $w_i$ is the Stiefel-Whitney class of the tangent bundle.}
\end{table}

2.3 $1 \to \text{SU}(2) \to G' \to \text{O}(d) \to 1$

We compute the bordism groups $\Omega_d^{G'}$ where $G'$ is a solution of all possible extensions of

$$1 \to \text{SU}(2) \to G' \to \text{O}(d) \to 1.$$  \hspace{1cm} (2.8)
There are four distinct extensions given by
\[ G' = \begin{cases} 
    \text{O}(d) \times \text{SU}(2), \\
    E(d) \times _Z \text{SU}(2), \\
    \text{Pin}^+ (d) \times _Z^F \text{SU}(2), \\
    \text{Pin}^- (d) \times _Z^F \text{SU}(2). 
\end{cases} \]

The bordism groups and their bordism invariants (i.e., topological invariants as iTQFT and SPTs) can be determined by
\[ \Omega_{G'} \]
(2.9)

Once we gauge SU(2), we turn all bosonic/fermionic SPTs to bosonic gauge theories. The total symmetry for the SU(2) gauge theory is
\[ Z_{2,[1]} \times \text{O}(d) \]  
(2.10)

We can also compute the following bordism invariant to classify all possible anomalies.
\[ \Omega_{Z_{2,[1]} \times O}^\text{O} \equiv \Omega_{d}^\text{O} (B^2 Z_2). \]
(2.11)

### 2.3.1 O × SU(2)

We have \( MT(O \times SU(2)) = MO \land (BSU(2))_+ \).

By Künneth formula,
\[ H^*(MO \land (BSU(2))_+, Z_2) = H^*(MO, Z_2) \otimes H^*(BSU(2), Z_2). \]
(2.12)

Since there is no odd torsion, we have the Adams spectral sequence
\[ \text{Ext}_{A_2}^s(H^*(MO, Z_2) \otimes H^*(BSU(2), Z_2), Z_2) \Rightarrow \Omega_{s}^{O \times SU(2)}. \]
(2.13)

The Thom spectrum \( MO \) is the wedge sum of suspensions of the mod 2 Eilenberg-MacLane spectrum \( HZ_2 \), \( H^*(MO, Z_2) \) is the direct sum of suspensions of the mod 2 Steenrod algebra \( A_2 \), actually Thom proved that
\[ \pi_{*}(MO) = \Omega_{*}^{O} = Z_2[x_2, x_4, x_5, x_6, x_8, \ldots] \]
(2.14)

where the generators are in each degree other than \( 2^n - 1 \).

So \( MO = HZ_2 \lor \Sigma^2 HZ_2 \lor 2 \Sigma^4 HZ_2 \lor \Sigma^5 HZ_2 \lor \cdots \) and \( H^*(MO, Z_2) = A_2 \oplus \Sigma^2 A_2 \oplus 2 \Sigma^4 A_2 \oplus \Sigma^5 A_2 \oplus \cdots \).

Hence
\[
H^*(MO, Z_2) \otimes H^*(BSU(2), Z_2) \\
= (A_2 \oplus \Sigma^2 A_2 \oplus 2 \Sigma^4 A_2 \oplus \Sigma^5 A_2 \oplus \cdots) \otimes Z_2[c_2] \\
= A_2 \oplus \Sigma^2 A_2 \oplus 3 \Sigma^4 A_2 \oplus \Sigma^5 A_2 \oplus \cdots. 
\]
(2.15)
The table below lists the manifold generators and cobordism invariants for various dimensions:

| $i$ | $\Omega^i_d$ | manifold generators | cobordism invariants |
|-----|---------------|---------------------|---------------------|
| 0   | $\mathbb{Z}_2$ |                     |                     |
| 1   | 0             |                     |                     |
| 2   | $\mathbb{Z}_2$ | $\mathbb{RP}^2$    | $w_1^2$            |
| 3   | 0             |                     |                     |
| 4   | $\mathbb{Z}_2^2$ | $\mathbb{RP}^4$, $\mathbb{RP}^2 \times \mathbb{RP}^2$ | $w_1^2$, $w_2^2$ |
| 5   | $\mathbb{Z}_2$ | Wu manifold or Dold manifold | $w_2w_3$ |

Since

$$\text{Ext}_{\Lambda_2}^s(t, \Sigma^r \Lambda_2, \mathbb{Z}_2) = \begin{cases} \text{Hom}_{\Lambda_2}^t(\Sigma^r \Lambda_2, \mathbb{Z}_2) = \mathbb{Z}_2 & \text{if } t = r, s = 0 \\ 0 & \text{else} \end{cases},$$

we have the following theorem:

| Bordism group |
|---------------|
| $d$ | $\Omega^i_d \times \text{SU}(2)$ | generators |
|-----|-------------------------------|------------|
| 0   | $\mathbb{Z}_2$                |            |
| 1   | 0                             |            |
| 2   | $\mathbb{Z}_2$                | $w_1^2$   |
| 3   | 0                             |            |
| 4   | $\mathbb{Z}_2^3$              | $w_1^4$, $w_2^2$, $c_2 \mod 2$ |
| 5   | $\mathbb{Z}_2$                | $w_2w_3$  |

Table 4: Bordism group. Here $w_i$ is the Stiefel-Whitney class of the tangent bundle, $c_2$ is the Chern class of the SU(2) bundle.

### 2.3.2 $E \times \mathbb{Z}_2 \text{SU}(2)$

Partial results on 4d and 5d are given in [16].

Recall that $E$ is defined to be the subgroup of $O \times \mathbb{Z}_4$ consisting of the pairs $(A, j)$ such that $\det A = j^2$, there is a fibration $BE \to BO \xrightarrow{w_2} B\mathbb{Z}_2$.

We can also think of the space $BE$ as the fiber of $w_1 + x : BO \times B\mathbb{Z}_4 \to B\mathbb{Z}_2$, where $x$ is the generator of $H^1(B\mathbb{Z}_4, \mathbb{Z}_2)$.

Note that $\text{SU}(2) \times \mathbb{Z}_2 \mathbb{Z}_4 = \text{Pin}^+ (3)$, we can think of the space $B(E \times \mathbb{Z}_2 \text{SU}(2))$ as the fiber of $w_1 + w_1' : BO \times \text{BPIn}^+ (3) \to B\mathbb{Z}_2$, where $w_1'$ is the generator of $H^1(\text{BPIn}^+ (3), \mathbb{Z}_2)$. Take $W$ to be the rank 3 vector bundle on $\text{BPIn}^+ (3)$ determined by $\text{BPIn}^+ (3) \to BO(3)$.

Define a map $f : BO \times \text{BPIn}^+ (3) \to BO \times \text{BPIn}^+ (3)$ by $(V, V') \to (V + W - 3, V')$, with inverse $(V, V') \to (V - W + 3, V')$. Observe $f^*(w_1) = w_1 + w_1'$, so that $BE$ is homotopy equivalent to $BO \times \text{BPIn}^+ (3)$. The canonical bundle $BE \to BO$ corresponds to $V - W + 3$ on $BO \times \text{BPIn}^+ (3)$. So $MT(E \times \mathbb{Z}_2 \text{SU}(2)) = MTSO \wedge \text{Thom(BPIn}^+ (3), 3 - W) = MSO \wedge \Sigma^{-3} \text{MIn}^+ (3)$.

Note that $\text{MIn}^+ (3) = M\text{PIn}^- (3) = MT(\text{Spin}(3) \times \mathbb{Z}_2) = M\text{Spin}(3) \wedge MT\mathbb{Z}_2$.  

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So $MT(E \times \mathbb{Z}_2 SU(2)) = MSO \wedge \Sigma^{-4}MSU(2) \wedge \Sigma^1MTO(1) \simeq MO \wedge \Sigma^{-4}MSU(2)$.

By Künneth formula,

$$H^*(MO \wedge \Sigma^{-4}MSU(2), \mathbb{Z}_2) = H^*(MO, \mathbb{Z}_2) \otimes H^{*+4}(MSU(2), \mathbb{Z}_2). \quad (2.17)$$

Since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}^{s,t}_{\mathcal{A}_2}(H^*(MO, \mathbb{Z}_2) \otimes H^{*+4}(MSU(2), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{E \times \mathbb{Z}_2 SU(2)}. \quad (2.18)$$

We have

$$H^{*+4}(MSU(2), \mathbb{Z}_2) = \mathbb{Z}_2[c_2]U \quad (2.19)$$

where $c_2$ is the Chern class of the SU(2) bundle and $U$ is the Thom class.

So

$$H^*(MO, \mathbb{Z}_2) \otimes H^{*+4}(MSU(2), \mathbb{Z}_2) = \mathcal{A}_2 \oplus \Sigma^2 \mathcal{A}_2 \oplus 2 \Sigma^4 \mathcal{A}_2 \oplus \Sigma^5 \mathcal{A}_2 \oplus \cdots \otimes \mathbb{Z}_2[c_2]U \quad (2.20)$$

| $d$ | Bordism group | $\Omega_{t-s}^{E \times \mathbb{Z}_2 SU(2)}$ | generators |
|-----|---------------|-------------------------------|------------|
| 0   | $\mathbb{Z}_2$ |                               |            |
| 1   | 0             |                               |            |
| 2   | $\mathbb{Z}_2$ | $w_1^2$                      |            |
| 3   | 0             |                               |            |
| 4   | $\mathbb{Z}_2^3$ | $w_1^4, w_2^2, c_2 \mod 2$ |            |
| 5   | $\mathbb{Z}_2$ | $w_2 w_3$                    |            |

Table 5: Bordism group. Here $w_i$ is the Stiefel-Whitney class of the tangent bundle. Since $w_1^2(TM) = w_2^2(SO(3))$, the third integral Stiefel-Whitney class of SO(3) bundle vanishes, so the SO(3) bundle lifts to a Spin$^c(3) = U(2)$ bundle, $c_2$ is the Chern class of the U(2) bundle.

**2.3.3 Pin$^+ \times \mathbb{Z}_2 SU(2)$**

Let $G' = \frac{\text{Pin}^+ \times SU(2)}{\mathbb{Z}_2^2}$, then by [7, 12], we have $MTG' = MSpin \wedge \Sigma^{-3}MO(3)$.

Here $w_2(TM) = w_2(V_{SO(3)})$, $w_3(TM) + w_1(TM) w_2(TM) = w_3(V_{SO(3)})$ and $w_1(TM)$ is nontrivial, $w_1^2(V_{SO(3)}) = 0$.

For $t - s < 8$, since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}^{s,t}_{\mathcal{A}_2(1)}(H^{*+3}(MO(3), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\text{Pin}^+ \times SU(2)} \quad (2.21)$$
The $A_2(1)$-module structure of $H^{*+3}(MO(3), \mathbb{Z}_2)$ below degree 5 and the $E_2$ page are shown in Figure 4, 5.

Figure 4: The $A_2(1)$-module structure of $H^{*+3}(MO(3), \mathbb{Z}_2)$ below degree 5.

Figure 5: $\Omega_{\frac{\text{Pin}^+ \times SU(2)}{\mathbb{Z}_2}}$
| Bordism group |
|---------------|
| $d$ | $\Omega_d^{Pin \times \mathbb{Z}_2 SU(2)}$ | generators |
| 0  | $\mathbb{Z}_2$ |          |
| 1  | 0               |          |
| 2  | $\mathbb{Z}_2$ | $w_1^2$ |
| 3  | 0               |          |
| 4  | $\mathbb{Z}_2 \times \mathbb{Z}_4$ | $w_2^2, \eta_{SU(2)}$ |
| 5  | $\mathbb{Z}_2$ | $w_2 w_3$ |

Table 6: Bordism group. Here $w_i$ is the Stiefel-Whitney class of the tangent bundle, $\eta_{SU(2)}$ is defined in [7].

2.3.4 $Pin^- \times \mathbb{Z}_2 SU(2)$

Let $G' = \frac{Pin^- \times SU(2)}{\mathbb{Z}_2}$, then by [7, 12], we have $MTG' = MSpin \wedge \Sigma^3 MTO(3)$.

Here $w_2(TM) + w_1(TM)^2 = w_2'(V_{SO(3)})$, $w_3(TM) + w_2(TM)w_1(TM) = w_3'(V_{SO(3)})$ and $w_1(TM)$ is nontrivial, $w_1'(V_{SO(3)}) = 0$.

For $t - s < 8$, since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}_A^{s,t}(H^{*-3}(MTO(3), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{Pin^- \times SU(2)} \mathbb{Z}_2'. \tag{2.22}$$

The $A_2(1)$-module structure of $H^{*-3}(MTO(3), \mathbb{Z}_2)$ below degree 5 and the $E_2$ page are shown in Figure 6, 7.

![Figure 6: The $A_2(1)$-module structure of $H^{*-3}(MTO(3), \mathbb{Z}_2)$ below degree 5.](image-url)
We further promote $SU(2)$ to a dynamical gauge group. The total global symmetry after gauging is $Z_{2,[1]} \times O(d)$, hence we compute the bordism invariant $\Omega_d^{Z_{2,[1]} \times O} \equiv \Omega_d^{O(B^2Z_2)}$.

### 2.4.1 $O \times Z_{2,[1]}$

This case is considered in [24]. We enumerate the bordism groups and invariants in Table 8.
### Bordism group

| $d$ | $\Omega_d^{O \times BZ_2}$ | generators |
|-----|----------------|-------------|
| 0   | $\mathbb{Z}_2$ |             |
| 1   | 0              |             |
| 2   | $\mathbb{Z}_2^2$ | $x_2, w_1^2$ |
| 3   | $\mathbb{Z}_2$ | $x_3 = w_1 x_2$ |
| 4   | $\mathbb{Z}_2^4$ | $w_1^4, w_2^2, x_2^2, w_1^2 x_2$ |
| 5   | $\mathbb{Z}_2^4$ | $w_2, x_2 x_3, x_5, w_1^2 x_3$ |

Table 8: Bordism group. Here $x_2$ is the generator of $H^2(B^2\mathbb{Z}_2, \mathbb{Z}_2)$, $x_3 = \text{Sq}^1 x_2$, $x_5 = \text{Sq}^2 x_3$, $w_i$ is the Stiefel-Whitney class of the tangent bundle.

### 3 $SU(2) \times \mathbb{Z}_2^F$ extension

#### 3.1 $1 \to SU(2) \times \mathbb{Z}_2^F \to G' \to SO(d) \to 1$

We consider the symmetry extension problem

$$1 \to SU(2) \times \mathbb{Z}_2^F \to G' \to SO(d) \to 1$$

The solutions are $G' = SO(d) \times SU(2) \times \mathbb{Z}_2$ or $Spin(d) \times SU(2)$ or $Spin(d) \times \mathbb{Z}_2 SU(2) \times \mathbb{Z}_2$.

##### 3.1.1 $SO \times SU(2) \times \mathbb{Z}_2$

We have $MT(SO \times SU(2) \times \mathbb{Z}_2) = MSO \wedge (B(SU(2) \times \mathbb{Z}_2))_+.$

By Künneth formula,

$$H^*(MSO \wedge (B(SU(2) \times \mathbb{Z}_2))_+, \mathbb{Z}_2) = H^*(MSO, \mathbb{Z}_2) \otimes H^*(BSU(2), \mathbb{Z}_2) \otimes H^*(BZ_2, \mathbb{Z}_2).$$

Since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}_{A_{\mathbb{Z}_2}}^s(H^*(MSO, \mathbb{Z}_2) \otimes H^*(BSU(2), \mathbb{Z}_2) \otimes H^*(BZ_2, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{l-s}^{SO \times SU(2) \times \mathbb{Z}_2}.$$  \(3.3\)

We have

$$H^*(BZ_2, \mathbb{Z}_2) = \mathbb{Z}_2[a].$$  \(3.4\)

Here $a$ is the generator of $H^1(BZ_2, \mathbb{Z}_2)$.

The $E_2$ page of the Adams spectral sequence is shown in Figure 8.
Figure 8: $\Omega_{\leq 0}^{\text{SO} \times \text{SU}(2) \times \mathbb{Z}_2}$

| $d$ | $\Omega_d^{\text{SO} \times \text{SU}(2) \times \mathbb{Z}_2}$ | generators |
|-----|---------------------------------|-------------|
| 0   | $\mathbb{Z}$                       |             |
| 1   | $\mathbb{Z}_2$                     | $a$         |
| 2   | 0                                |             |
| 3   | $\mathbb{Z}_2$                     | $a^3$       |
| 4   | $\mathbb{Z}^2$                    | $\sigma, c_2$|
| 5   | $\mathbb{Z}_2^2$                  | $w_2 w_3, a^5, ac_2, aw_2^2$ |

Table 9: Bordism group. Here $a$ is the generator of $H^1(B\mathbb{Z}_2, \mathbb{Z}_2)$, $\sigma$ is the signature of the 4-manifold, $w_i$ is the Stiefel-Whitney class of the tangent bundle, $c_2$ is the Chern class of the SU(2) bundle.

3.1.2 Spin $\times$ SU(2)

We have $MT(\text{Spin} \times \text{SU}(2)) = M\text{Spin} \wedge (\text{BSU}(2))_+$. For $t-s < 8$, since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}^{s,t}_{A_2(1)}(H^*(\text{BSU}(2), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\text{Spin}}(\text{BSU}(2)).$$

(3.5)

The $A_2(1)$-module structure of $H^*(\text{BSU}(2), \mathbb{Z}_2)$ below degree 5 and the $E_2$ page are shown in Figure 9, 10.
Figure 9: The $A_2(1)$-module structure of $H^*(BSU(2), \mathbb{Z}_2)$ below degree 5.

![Graph showing the $A_2(1)$-module structure.]

Figure 10: $\Omega_{\text{Spin} \times SU(2)}$

| $d$ | $\Omega_d^{\text{Spin} \times SU(2)}$ | generators |
|-----|-------------------------------------|-------------|
| 0   | $\mathbb{Z}$                        |             |
| 1   | $\mathbb{Z}_2$                      | $\tilde{\eta}$ |
| 2   | $\mathbb{Z}_2$                      | Arf         |
| 3   | 0                                   |             |
| 4   | $\mathbb{Z}^2$                      | $\frac{\sigma}{16}, c_2$ |
| 5   | $\mathbb{Z}_2$                      | $(c_2 \text{ mod } 2)\tilde{\eta}$ |

Table 10: Bordism group. Here $\tilde{\eta}$ is the mod 2 index of 1d Dirac operator, Arf is the Arf invariant, $\sigma$ is the signature of the 4-manifold, $c_2$ is the Chern class of the SU(2) bundle.

3.1.3 Spin $\times \mathbb{Z}_2$ SU(2) $\times \mathbb{Z}_2$

We have $MT(\text{Spin } \times \mathbb{Z}_2 \text{ SU(2) } \times \mathbb{Z}_2) = M\text{Spin} \wedge \Sigma^{-3} MSO(3) \wedge (B\mathbb{Z}_2)_+$. 

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By Künneth formula,

$$H^*(\Sigma^{-3} SO(3) \wedge (BZ_2)_+, Z_2) = H^{*+3}(SO(3), Z_2) \otimes H^*(BZ_2, Z_2).$$

(3.6)

For $t - s < 8$, since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}^{s,t}_{A_2(1)}(H^{*+3}(SO(3), Z_2) \otimes H^*(BZ_2, Z_2), Z_2) \Rightarrow \Omega^\text{Spin\times Z}_t^{\text{SU(2)\times Z}_2}.$$  

(3.7)

The $A_2(1)$-module structure of $H^{*+3}(SO(3), Z_2) \otimes H^*(BZ_2, Z_2)$ below degree 5 and the $E_2$ page are shown in Figure 11, 12.

Figure 11: The $A_2(1)$-module structure of $H^{*+3}(SO(3), Z_2) \otimes H^*(BZ_2, Z_2)$ below degree 5.
3.2 \( 1 \rightarrow \text{SU}(2) \times \mathbb{Z}_2^F \rightarrow G' \rightarrow \text{SO}(d) \rightarrow 1 \), \textbf{Gauge SU}(2) in \( G' \), \textbf{end with} \( \text{SO} \times \mathbb{Z}_2 \times \mathbb{Z}_2, [1] \) \textbf{or} \( \text{Spin} \times \mathbb{Z}_2, [1] \)

After gauging, the total global symmetry becomes \( \mathbb{Z}_2, [1] \times \text{Spin}(d), \mathbb{Z}_2, [1] \times \text{SO}(d) \times \mathbb{Z}_2 \). Hence we need to compute the bordism invariants \( \Omega^\text{Spin}_d(B^2\mathbb{Z}_2) \) and \( \Omega^\text{SO}_d(B\mathbb{Z}_2 \times B^2\mathbb{Z}_2) \).

3.2.1 \( \text{SO} \times \mathbb{Z}_2 \times \mathbb{Z}_2, [1] \)

This case is considered in [24]. We just summarize the bordism groups/invariants in Table 12.

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### Table 11: Bordism group

| \( d \) | \( \Omega^\text{Spin}_d \times \mathbb{Z}_2 \times \text{SU}(2) \times \mathbb{Z}_2 \) | generators |
|---|---|---|
| 0 | \( \mathbb{Z} \) | \( a \) |
| 1 | \( \mathbb{Z}_2 \) | \( a \) |
| 2 | 0 | \( \sigma, \text{?} \) |
| 3 | \( \mathbb{Z}_2 \) | \( a^3 \) |
| 4 | \( \mathbb{Z}_2^2 \) | \( \sigma, \text{?} \) |
| 5 | \( \mathbb{Z}_2^5 \) | \( w_2 w_3, w_3 \text{Arf(?)}), a^5, aw_2^2, w_3 a \tilde{\eta} \) |

Table 11: Bordism group. Here \( a \) is the generator of \( H^1(B\mathbb{Z}_2, \mathbb{Z}_2) \), \( \sigma \) is the signature of the 4-manifold, \( w_i \) is the Stiefel-Whitney class of the tangent bundle, \( \tilde{\eta} \) is the mod 2 index of 1d Dirac operator, \( \text{Arf} \) is the Arf invariant, \( \text{?} \) is an undetermined cobordism invariant which also appears in [20].
3.2.2 \( \text{Spin} \times \mathbb{Z}_2, [1] \)

This case is considered in [24]. We just summarize the bordism groups/invariants in Table 13.

| Bordism group | \( d \) | \( \Omega_d^{\text{Spin} \times \mathbb{Z}_2} \) | generators |
|---------------|-------|---------------------------------|-------------|
| 0             | \( \mathbb{Z} \) | \( a \) | |
| 1             | \( \mathbb{Z}_2 \) | \( x_2 \) | |
| 2             | \( \mathbb{Z}_2 \) | \( ax_2, a^3 \) | |
| 3             | \( \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_4 \) | \( \sigma, ax_3 = a^2x_2, \mathcal{P}_2(x_2) \) | |
| 4             | \( \mathbb{Z}_2 \times \mathbb{Z}_4 \) | \( w_2w_3, x_2x_3 = x_5, a^5, a^3x_2, ax_2^2, aw_2^2 \) | |

Table 13: Bordism group. Here \( \tilde{\eta} \) is the mod 2 index of 1d Dirac operator, Arf is the Arf invariant, \( x_2 \) is the generator of \( H^2(\mathbb{BZ}_2, \mathbb{Z}_2) \), \( \mathcal{P}_2(x_2) \) is the Pontryagin square of \( x_2 \), \( \sigma \) is the signature of the 4-manifold.

3.3 \( 1 \rightarrow \text{SU}(2) \times \mathbb{Z}_2^F \rightarrow G' \rightarrow \text{O}(d) \rightarrow 1 \)

We consider the symmetry extension

\[
1 \rightarrow \text{SU}(2) \times \mathbb{Z}_2^F \rightarrow G' \rightarrow \text{O}(d) \rightarrow 1
\]

whose solutions are \( G' = \text{O}(d) \times \text{SU}(2) \times \mathbb{Z}_2 \) or \( E(d) \times \text{SU}(2) \) or \( \text{Pin}^+(d) \times \text{SU}(2) \) or \( \text{Pin}^-(d) \times \text{SU}(2) \) or \( E(d) \times \mathbb{Z}_2 \times \text{SU}(2) \times \mathbb{Z}_2 \) or \( \text{Pin}^+(d) \times \mathbb{Z}_2 \times \text{SU}(2) \times \mathbb{Z}_2 \) or \( \text{Pin}^-(d) \times \mathbb{Z}_2 \times \text{SU}(2) \times \mathbb{Z}_2 \).

3.3.1 \( \text{O} \times \text{SU}(2) \times \mathbb{Z}_2 \)

We have \( MT(\text{O} \times \text{SU}(2) \times \mathbb{Z}_2) = \text{MO} \wedge (\text{B(SU}(2) \times \mathbb{Z}_2))_+ \).

By K"unneth formula,

\[
H^*(\text{MO} \wedge (\text{B(SU}(2) \times \mathbb{Z}_2))_+, \mathbb{Z}_2) = H^*(\text{MO}, \mathbb{Z}_2) \otimes H^*(\text{BSU}(2), \mathbb{Z}_2) \otimes H^*(\text{BZ}_2, \mathbb{Z}_2).
\]
Since there is no odd torsion, we have the Adams spectral sequence

\[ \text{Ext}^{s,t}_{A_2}(H^*(MO, \mathbb{Z}_2) \otimes H^*(BSU(2), \mathbb{Z}_2) \otimes H^*(BZ_2, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{O \times SU(2) \times \mathbb{Z}_2}. \] \hfill (3.10)

| \( d \) | \( \Omega_d^{O \times SU(2) \times \mathbb{Z}_2} \) | generators |
|---|---|---|
| 0 | \( \mathbb{Z}_2 \) | \( a \) |
| 1 | \( \mathbb{Z}_2 \) | \( a^2, w_1^2 \) |
| 2 | \( \mathbb{Z}_2^2 \) | \( a^3, aw_1^2 \) |
| 3 | \( \mathbb{Z}_2^2 \) | \( w_1^4, w_2^2, a^4, a^2w_1^6, c_2 \text{ mod } 2 \) |
| 4 | \( \mathbb{Z}_2^5 \) | \( w_2w_3, a^5, a^3w_1^2, aw_1^4, aw_2^2, ac_2 \) |

Table 14: Bordism group. Here \( a \) is the generator of \( H^1(BZ_2, \mathbb{Z}_2) \), \( w_i \) is the Stiefel-Whitney class of the tangent bundle, \( c_2 \) is the Chern class of the SU(2) bundle.

### 3.3.2 \( E \times SU(2) \)

Recall that \( E \) is defined to be the subgroup of \( O \times \mathbb{Z}_4 \) consisting of the pairs \((A, j)\) such that \( \det A = j^2 \), there is a fibration \( BE \to BO \cong B^2Z_2 \).

We can also think of the space \( BE \) as the fiber of \( w_1 + x : BO \times BZ_4 \to BZ_2 \), where \( x \) is the generator of \( H^1(BZ_4, \mathbb{Z}_2) \). Take \( W \) to be the line bundle on \( BZ_4 \) determined by \( BZ_4 \to BZ_2 \). Define a map \( f : BO \times BZ_4 \to BO \times BZ_4 \) by \( (V, V') \to (V + W - 1, V') \), with inverse \( (V, V') \to (V - W + 1, V') \). Observe \( f^*(w_1) = w_1 + x \), so that \( BE \) is homotopy equivalent to \( BSO \times BZ_4 \). The canonical bundle \( BE \to BO \) corresponds to \( V - W + 1 \) on \( BSO \times BZ_4 \). So \( MTE = MTSO \wedge \text{Thom}(BZ_4, 1 - W) = MSO \wedge S^{-1}(S^W)_{hZ_4} = MSO \wedge \Sigma^{-1}MZ_4 \).

By Künneth formula, we have

\[ H^*(MSO \wedge \Sigma^{-1}MZ_4, \mathbb{Z}_2) = H^*(MSO, \mathbb{Z}_2) \otimes H^{*+1}(MZ_4, \mathbb{Z}_2). \] \hfill (3.11)

By the generalized Pontryagin-Thom isomorphism,

\[ \pi_d(MTE) = \Omega^E_d. \] \hfill (3.12)

Since there is no odd torsion, the Adams spectral sequence shows:

\[ \text{Ext}^{s,t}_{A_2}(H^*(MTE, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^E. \] \hfill (3.13)

The localization of the Thom spectrum \( MSO \) at the prime 2 is \( MSO(2) = HZ(2) \vee \Sigma^4HZ(2) \vee \Sigma^5HZ_2 \vee \cdots \).

We have

\[ H^*(BZ_4, \mathbb{Z}_2) = \mathbb{Z}_2[y] \otimes A_{\mathbb{Z}_2}(x) \] \hfill (3.14)
where $\Lambda_{\mathbb{Z}_2}$ is the exterior algebra, $x$ is the generator of $H^1(B\mathbb{Z}_4, \mathbb{Z}_2)$, $y$ is the generator of $H^2(B\mathbb{Z}_4, \mathbb{Z}_2)$, $x = x' \mod 2$ where $x'$ is the generator of $H^1(B\mathbb{Z}_4, \mathbb{Z}_4)$ and $\beta_{(2,4)}x' = y$. Here $\beta_{(2,4)}$ is the Bockstein homomorphism associated to the extension $\mathbb{Z}_2 \to \mathbb{Z}_8 \to \mathbb{Z}_4$.

On the other hand, by Thom isomorphism, $H^{*+1}(M\mathbb{Z}_4, \mathbb{Z}_2)$ is the shift of $H^*(B\mathbb{Z}_4, \mathbb{Z}_2)$.

By [35], there is a differential $d_2$ in the Adams spectral sequence corresponding to the Bockstein homomorphism $\beta_{(2,4)}$.

The $E_2$ page of the Adams spectral sequence is shown in Figure 13.

![Figure 13: $\Omega^E$](image)

| $d$ | $\Omega^E_{d,i}$ | generators |
|-----|-----------------|------------|
| 0   | $\mathbb{Z}_4$  |            |
| 1   | 0               |            |
| 2   | $\mathbb{Z}_4$  | $y'$       |
| 3   | 0               |            |
| 4   | $\mathbb{Z}_4^2$| $y'^2, \sigma \mod 4$ |
| 5   | $\mathbb{Z}_2$  | $w_2w_3$  |

Table 15: Bordism group. Here $y'$ is the generator of $H^2(B\mathbb{Z}_4, \mathbb{Z}_4)$, $\sigma$ is the signature of the 4-manifold, $w_i$ is the Stiefel-Whitney class of the tangent bundle.

Since there is no odd torsion, the Adams spectral sequence shows:

$$\text{Ext}^{s,t}_{A_2}(H^*(MT(E \times SU(2)), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega^E_{t-s} \times SU(2).$$

(3.15)

We have $MT(E \times SU(2)) = MTE \wedge (BSU(2))_+$. 

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The $E_2$ page of the Adams spectral sequence is shown in Figure 14.

![Figure 14: $\Omega^{E \times SU(2)}_*$](image)

| $d$ | $\Omega^E_{d \times Su(2)}$ | generators |
|-----|-----------------------------|-------------|
| 0   | $\mathbb{Z}_4$             |             |
| 1   | 0                           |             |
| 2   | $\mathbb{Z}_4$             | $y'$        |
| 3   | 0                           |             |
| 4   | $\mathbb{Z}_4^3$           | $y'^2, \sigma \mod 4, c_2 \mod 4$ |
| 5   | $\mathbb{Z}_2$             | $w_2w_3$    |

Table 16: Bordism group. Here $y'$ is the generator of $H^2(B\mathbb{Z}_4, \mathbb{Z}_4)$, $\sigma$ is the signature of the 4-manifold, $w_i$ is the Stiefel-Whitney class of the tangent bundle.

### 3.3.3 $Pin^+ \times SU(2)$

We have $MT(Pin^+ \times SU(2)) = MTPin^+ \wedge (SU(2))_+$. 

$$MTPin^+ = MSpin \wedge \Sigma^1 MTO(1).$$

By Künneth formula,

$$H^*(\Sigma^1 MTO(1) \wedge (BSU(2))_+, \mathbb{Z}_2) = H^{-1}(MTO(1), \mathbb{Z}_2) \otimes H^*(BSU(2), \mathbb{Z}_2).$$  \hspace{1cm} (3.16)

For $t - s < 8$, since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}^{s,t}_{A_2(1)}(H^{-1}(MTO(1), \mathbb{Z}_2) \otimes H^*(BSU(2), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega^*_{t-s}.$$  \hspace{1cm} (3.17)
The $A_2(1)$-module structure of $H^{*-1}(MTO(1), \mathbb{Z}_2) \otimes H^*(BSU(2), \mathbb{Z}_2)$ below degree 5 and the $E_2$ page are shown in Figure 15, 16.

![Diagram](image)

Figure 15: The $A_2(1)$-module structure of $H^{*-1}(MTO(1), \mathbb{Z}_2) \otimes H^*(BSU(2), \mathbb{Z}_2)$ below degree 5.

![Diagram](image)

Figure 16: $\Omega^\text{Pin}^+ \times SU(2)$
| $d$ | $\Omega^d_{\text{Pin}^- \times \text{SU}(2)}$ | generators |
|-----|---------------------------------|------------|
| 0   | $\mathbb{Z}_2$                   |            |
| 1   | 0                               |            |
| 2   | $\mathbb{Z}_2$                   | $w_1 \tilde{\eta}$ |
| 3   | $\mathbb{Z}_2$                   | $w_1 \text{Arf}$ |
| 4   | $\mathbb{Z}_2 \times \mathbb{Z}_{16}$ | $c_2 \mod 2, \eta$ |
| 5   | 0                               |            |

Table 17: Bordism group. Here $w_i$ is the Stiefel-Whitney class of the tangent bundle, $\tilde{\eta}$ is the mod 2 index of 1d Dirac operator, Arf is the Arf invariant, $\eta$ is the 4d eta invariant, $c_2$ is the Chern class of SU(2) bundle.

### 3.3.4 Pin$^- \times \text{SU}(2)$

We have $MT(\text{Pin}^- \times \text{SU}(2)) = MTP\text{Pin}^- \wedge (\text{BSU}(2))_+$. 

$$MTP\text{Pin}^- = M\text{Spin} \wedge \Sigma^{-1} \text{MO}(1).$$

By K"unneth formula,

$$H^*(\Sigma^{-1} \text{MO}(1) \wedge (\text{BSU}(2))_+, \mathbb{Z}_2) = H^{*+1}(\text{MO}(1), \mathbb{Z}_2) \otimes H^*(\text{BSU}(2), \mathbb{Z}_2).$$

(3.18)

For $t - s < 8$, since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}^{s,t}_{A_2(1)}(H^{*+1}(\text{MO}(1), \mathbb{Z}_2) \otimes H^*(\text{BSU}(2), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\text{Pin}^- \times \text{SU}(2)}.$$  

(3.19)

The $A_2(1)$-module structure of $H^{*+1}(\text{MO}(1), \mathbb{Z}_2) \otimes H^*(\text{BSU}(2), \mathbb{Z}_2)$ below degree 5 and the $E_2$ page are shown in Figure 17, 18.

Figure 17: The $A_2(1)$-module structure of $H^{*+1}(\text{MO}(1), \mathbb{Z}_2) \otimes H^*(\text{BSU}(2), \mathbb{Z}_2)$ below degree 5.
Figure 18: $\Omega^{\text{Pin}^\times \text{SU}(2)}_s$

| $d$ | $\Omega^{\text{Pin}^\times \text{SU}(2)}_d$ | generators |
|-----|---------------------------------|------------|
| 0   | $\mathbb{Z}_2$                   |            |
| 1   | $\mathbb{Z}_2$                   | $\tilde{\eta}$ |
| 2   | $\mathbb{Z}_8$                   | ABK        |
| 3   | 0                               |            |
| 4   | $\mathbb{Z}_2$                   | $c_2 \mod 2$ |
| 5   | $\mathbb{Z}_2$                   | $(c_2 \mod 2)\tilde{\eta}$ |

Table 18: Bordism group. Here $\tilde{\eta}$ is the mod 2 index of 1d Dirac operator, ABK is the Arf-Brown-Kervaire invariant, $c_2$ is the Chern class of the SU(2) bundle.

3.3.5 $E \times Z_2 \times SU(2) \times Z_2$

We have $MT(E \times Z_2 \times SU(2) \times Z_2) = MT(E \times Z_2 \times SU(2)) \wedge (BZ_2)_+ \simeq MO \wedge \Sigma^{-4} MSU(2) \wedge (BZ_2)_+$.

By Künneth formula,

$$H^*(MO \wedge \Sigma^{-4} MSU(2) \wedge (BZ_2)_+, Z_2) = H^*(MO, Z_2) \otimes H^{*+4}(MSU(2), Z_2) \otimes H^*(BZ_2, Z_2).$$

(3.20)

Since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}^{s,t}_{\mathbb{Z}_2}(H^*(MO, Z_2) \otimes H^{*+4}(MSU(2), Z_2) \otimes H^*(BZ_2, Z_2), Z_2) \Rightarrow \Omega^{E \times Z_2 \times SU(2) \times Z_2}_{t-s}.$$  (3.21)
Table 19: Bordism group. Here $a$ is the generator of $H^1(B\mathbb{Z}_2, \mathbb{Z}_2)$, $w_i$ is the Stiefel-Whitney class of the tangent bundle. Since $w_2^2(TM) = w_2^2(SO(3))$, the third integral Stiefel-Whitney class of SO(3) bundle vanishes, so the SO(3) bundle lifts to a Spin$^c(3) = U(2)$ bundle, $c_2$ is the Chern class of the $U(2)$ bundle.

### 3.3.6 $Pin^+ \times \mathbb{Z}_2 SU(2) \times \mathbb{Z}_2$

We have $MT(Pin^+ \times \mathbb{Z}_2 SU(2) \times \mathbb{Z}_2) = MSpin \wedge \Sigma^{-3}MO(3) \wedge (B\mathbb{Z}_2)_+$. By Künneth formula,

$$H^*(\Sigma^{-3}MO(3) \wedge (B\mathbb{Z}_2)_+, \mathbb{Z}_2) = H^{*+3}(MO(3), \mathbb{Z}_2) \otimes H^*(B\mathbb{Z}_2, \mathbb{Z}_2).$$  \hfill (3.22)

For $t - s < 8$, since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}^{s,t}_{A_2(1)}(H^{*+3}(MO(3), \mathbb{Z}_2) \otimes H^*(B\mathbb{Z}_2, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{Pin^+ \times \mathbb{Z}_2 SU(2) \times \mathbb{Z}_2}. \hfill (3.23)$$

The $A_2(1)$-module structure of $H^{*+3}(MO(3), \mathbb{Z}_2) \otimes H^*(B\mathbb{Z}_2, \mathbb{Z}_2)$ below degree 5 and the $E_2$ page are shown in Figure 19, 20.
Figure 19: The $\mathcal{A}_2(1)$-module structure of $H^{*-3}(MO(3), \mathbb{Z}_2) \otimes H^*(B\mathbb{Z}_2, \mathbb{Z}_2)$ below degree 5.
$\Omega_{s-t}$

Figure 20: $\Omega_{s-t}$

| $d$ | $\Omega^\text{Pin}^+ \times \mathbb{Z} \times SU(2) \times \mathbb{Z}$ | generators |
|-----|-------------------------------------------------|------------|
| 0   | $\mathbb{Z}_2$                                  |            |
| 1   | $\mathbb{Z}_2$                                  | $a$        |
| 2   | $\mathbb{Z}_2^2$                                | $w^2_1, a^2$|
| 3   | $\mathbb{Z}_2^2$                                | $a^3, w^2_1a$|
| 4   | $\mathbb{Z}_2^3 \times \mathbb{Z}_4$           | $w^2_2, a^4, w^2_1a^3, \eta_{SU(2)}$|
| 5   | $\mathbb{Z}_2^5$                                | $w_2w_3, a^5, w_1w_3a, w_1^2a^3, w_2^2a$|

Table 20: Bordism group. Here $a$ is the generator of $H^1(B\mathbb{Z}_2, \mathbb{Z}_2)$, $w_i$ is the Stiefel-Whitney class of the tangent bundle, $\eta_{SU(2)}$ is defined in [7].

### 3.3.7 Pin$^-$ $\times \mathbb{Z}_2$ SU(2) $\times \mathbb{Z}_2$

We have $MT(\text{Pin}^-$ $\times \mathbb{Z}_2$ SU(2) $\times \mathbb{Z}_2) = M\text{Spin} \wedge \Sigma^3 \text{MTO}(3) \wedge (B\mathbb{Z}_2)$.

By K"unneth formula,

$$H^*(\Sigma^3 \text{MTO}(3) \wedge (B\mathbb{Z}_2), \mathbb{Z}_2) = H^{*-3}((\text{MTO}(3), \mathbb{Z}_2) \otimes H^*(B\mathbb{Z}_2, \mathbb{Z}_2). \quad (3.24)$$

For $t - s < 8$, since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}^{s,t}_{\mathcal{A}_2(1)}(H^{*-3}(\text{MTO}(3), \mathbb{Z}_2) \otimes H^*(B\mathbb{Z}_2, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^\text{Pin}^-$ $\times \mathbb{Z}_2$ SU(2) $\times \mathbb{Z}_2$. \quad (3.25)$$

The $\mathcal{A}_2(1)$-module structure of $H^{*-3}(\text{MTO}(3), \mathbb{Z}_2) \otimes H^*(B\mathbb{Z}_2, \mathbb{Z}_2)$ below degree 5 and the $E_2$ page are shown in Figure 21, 22.
Figure 21: The $A_2(1)$-module structure of $H^{*-3}(MTO(3), Z_2) \otimes H^*(BZ_2, Z_2)$ below degree 5.
Table 21: Bordism group. Here $a$ is the generator of $H^1(B\mathbb{Z}_2, \mathbb{Z}_2)$, $w_i$ is the Stiefel-Whitney class of the tangent bundle, $\tilde{\eta}$ is the mod 2 index of 1d Dirac operator, Arf is the Arf invariant.

| $d$ | $\Omega_d^{\text{Pin}\times \mathbb{Z}_2\times SU(2)\times \mathbb{Z}_2}$ | generators |
|-----|-------------------------------------------------|------------|
| 0   | $\mathbb{Z}_2$                                  | $a$        |
| 1   | $\mathbb{Z}_2$                                  | $w_2, a^2$ |
| 2   | $\mathbb{Z}_2^2$                                | $a^3, w_2a$|
| 3   | $\mathbb{Z}_2^2$                                | $w_2^2, w_4, w_3\tilde{\eta}(?), a^4, w_3a$ |
| 4   | $\mathbb{Z}_2^2$                                | $w_2w_3, w_3\operatorname{Arf}(?), w_3a\tilde{\eta}, a^5, w_2a^3, aw_2^2, aw_1^4$ |

3.4 $1 \to SU(2) \times \mathbb{Z}_2^F \to G' \to O(d) \to 1$, **Gauge SU(2) in $G'$, end with $O \times \mathbb{Z}_2 \times \mathbb{Z}_{2,[1]}$, $E \times \mathbb{Z}_{2,[1]}$, or $\text{Pin}^\pm \times \mathbb{Z}_{2,[1]}$**

After gauging SU(2), the global symmetry is $O \times \mathbb{Z}_2 \times \mathbb{Z}_{2,[1]}$ or $E \times \mathbb{Z}_{2,[1]}$ or $\text{Pin}^\pm \times \mathbb{Z}_{2,[1]}$.

3.4.1 $O \times \mathbb{Z}_2 \times \mathbb{Z}_{2,[1]}$

This case is considered in [24]. We just summarize the bordism groups/invariants in Table 22.
| $d$ | $\Omega_d^{O \times BZ_2}$ | generators |
|-----|----------------|------------|
| 0   | $\mathbb{Z}_2$  |            |
| 1   | $\mathbb{Z}_2$  | $a$        |
| 2   | $\mathbb{Z}_2^3$| $x_2, a^2, w_1^2$ |
| 3   | $\mathbb{Z}_2^4$| $x_3 = w_1x_2, ax_2, aw_1^2, a^3$ |
| 4   | $\mathbb{Z}_2^5$| $w_4^2, w_2^2, a^2x_2, ax_3, x_2^2, w_1^2a^2, w_1^2x_2$ |
| 5   | $\mathbb{Z}_2^7$| $a^5, a^2x_3, a^3x_2, a^3w_1^2, x_2w_1^2, aw_2^2, ax_2x_3, w_1^2x_3, x_5, w_2w_3$ |

Table 22: Bordism group. Here $a$ is the generator of $H^1(B\mathbb{Z}_2, \mathbb{Z}_2)$, $x_2$ is the generator of $H^2(B^2\mathbb{Z}_2, \mathbb{Z}_2)$, $x_3 = \text{Sq}^1x_2$, $x_5 = \text{Sq}^2x_3$, $w_i$ is the Stiefel-Whitney class of the tangent bundle.

### 3.4.2 $E \times \mathbb{Z}_2[1]$

We have $MT(E \times B\mathbb{Z}_2) = MTE \wedge (B^2\mathbb{Z}_2)_+ = MSO \wedge \Sigma^{-1}MZ_4 \wedge (B^2\mathbb{Z}_2)_+$. 

By Künneth formula,

$$H^*(MSO \wedge \Sigma^{-1}MZ_4 \wedge (B^2\mathbb{Z}_2)_+, \mathbb{Z}_2) = H^*(MSO, \mathbb{Z}_2) \otimes H^{*+1}(MZ_4, \mathbb{Z}_2) \otimes H^*(B^2\mathbb{Z}_2, \mathbb{Z}_2).$$  \hspace{1cm} (3.26)

Since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}_{\text{A}_2}^{s,t}(H^*(MSO, \mathbb{Z}_2) \otimes H^{*+1}(MZ_4, \mathbb{Z}_2) \otimes H^*(B^2\mathbb{Z}_2, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{E \times B\mathbb{Z}_2}. \hspace{1cm} (3.27)$$

We have

$$H^*(B^2\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2[x_2, x_3, x_5, \ldots]$$  \hspace{1cm} (3.28)

where $x_2$ is the generator of $H^2(B^2\mathbb{Z}_2, \mathbb{Z}_2)$, $x_3 = \text{Sq}^1x_2$, $x_5 = \text{Sq}^2x_3$, \ldots.

We also have

$$H^{*+1}(MZ_4, \mathbb{Z}_2) = (\mathbb{Z}_2[y] \otimes \Lambda_{\mathbb{Z}_2}(x))U$$  \hspace{1cm} (3.29)

where $x$ is the generator of $H^1(BZ_4, \mathbb{Z}_2)$, $y$ is the generator of $H^2(BZ_4, \mathbb{Z}_2)$, $\Lambda_{\mathbb{Z}_2}$ is the exterior algebra, $U$ is the Thom class of the line bundle determined by $BZ_4 \to BZ_2 = BO(1)$.

We list the elements of $H^{*+1}(MZ_4, \mathbb{Z}_2) \otimes H^*(B^2\mathbb{Z}_2, \mathbb{Z}_2)$ below degree 5 as follows:

\begin{align*}
0 & \quad U \\
1 & \quad xU \\
2 & \quad yU, x_2U \\
3 & \quad xyU, xx_2U, x_3U \\
4 & \quad y^2U, yx_2U, x_3U, x_2^2U \\
5 & \quad xy^2U, yxyU, yx_3U, xx_2^2U, x_2x_3U, x_5U.
\end{align*} \hspace{1cm} (3.30)

They satisfy $\beta_{(2,4)}U = xU$, $\beta_{(2,4)}yU = xyU$, $\beta_{(2,4)}y^2U = xy^2U$, $\text{Sq}^1(x_2U) = x_3U$, $\beta_{(2,4)}(\mathcal{P}_2(x_2)U) = (x_2x_3 + x_5 + x_2^2U)$, $\text{Sq}^1(x_5U) = \text{Sq}^1(x_2x_3U) = x_2^2U$, $\text{Sq}^1(x_2U) = xx_3U$, $\text{Sq}^1(yx_2U) = yx_3U$, $\text{Sq}^1(xy_2U) = yxy_3U$, and $\beta_{(2,4)}(\mathcal{P}_2(x_2)ux) = x(x_2x_3 + x_5)U$. 

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The differentials $d_1$ are induced by $\text{Sq}^1$. Moreover, by [35], there are differentials $d_2$ corresponding to the Bockstein homomorphism $\beta_{(2,4)}$.

The $E_2$ page is shown in Figure 23.

![Figure 23: $\Omega^{E \times B\mathbb{Z}_2}$](image)

| $d$ | $\Omega^{E \times B\mathbb{Z}_2}$ | generators |
|-----|----------------------------------|------------|
| 0   | $\mathbb{Z}_4$                   |            |
| 1   | $0$                              |            |
| 2   | $\mathbb{Z}_2 \times \mathbb{Z}_4$ | $x_2, y'$  |
| 3   | $\mathbb{Z}_2$                   | $x x_2$    |
| 4   | $\mathbb{Z}_2^3 \times \mathbb{Z}_2$ | $\sigma \mod 4, y^2, P_2(x_2), y x_2$ |
| 5   | $\mathbb{Z}_2^3 \times \mathbb{Z}_4$ | $w_2 w_3, x y x_2, x_2 x_3 (\text{or } x_5), x' P_2(x_2)$ |

Table 23: Bordism group. Here $x'$ is the generator of $H^1(B\mathbb{Z}_4, \mathbb{Z}_4)$, $y'$ is the generator of $H^2(B\mathbb{Z}_4, \mathbb{Z}_4)$, $x = x' \mod 2$, $y = y' \mod 2$, $x_2$ is the generator of $H^2(B^2 \mathbb{Z}_2, \mathbb{Z}_2)$, $x_3 = \text{Sq}^1 x_2$, $x_5 = \text{Sq}^2 x_3$, $P_2(x_2)$ is the Pontryagin square of $x_2$, $\sigma$ is the signature of the 4-manifold, $w_i$ is the Stiefel-Whitney class of the tangent bundle.

### 3.4.3 $\text{Pin}^+ \times \mathbb{Z}_{2,[1]}$

This case is considered in [24]. We just summarize the bordism groups/invariants in Table 24.
Table 24: Bordism group. Here $\tilde{\eta}$ is the mod 2 index of 1d Dirac operator, $\text{Arf}$ is the Arf invariant, $x_2$ is the generator of $H^2(B^2\mathbb{Z}_2, \mathbb{Z}_2)$, $x_3 = Sq^1x_2$, $x_5 = Sq^2x_3$, $q_s(x_2)$ is explained in [24], $\eta$ is the 4d eta invariant, $w_i$ is the Stiefel-Whitney class of the tangent bundle.

### 3.4.4 $\text{Pin}^- \times \mathbb{Z}_2$ [1]

This case is considered in [24]. We just summarize the bordism groups/invariants in Table 25.

| Bordism group | | generators |
|--------------|----------------|
| $d$ | $\Omega^\text{Pin}^+ \times B\mathbb{Z}_2$ | |
| 0 | $\mathbb{Z}_2$ | |
| 1 | 0 | |
| 2 | $\mathbb{Z}_2^2$ | $x_2, w_1\tilde{\eta}$ |
| 3 | $\mathbb{Z}_2^3$ | $x_3 = w_1x_2, w_1\text{Arf}$ |
| 4 | $\mathbb{Z}_4 \times \mathbb{Z}_{16}$ | $q_s(x_2), \eta$ |
| 5 | $\mathbb{Z}_2^3$ | $x_2x_3, w_1^2x_3 = x_5$ |

Table 25: Bordism group. Here $\tilde{\eta}$ is the mod 2 index of 1d Dirac operator, ABK is the Arf-Brown-Kervaire invariant, $x_2$ is the generator of $H^2(B^2\mathbb{Z}_2, \mathbb{Z}_2)$, $x_3 = Sq^1x_2$, $w_i$ is the Stiefel-Whitney class of the tangent bundle.

### 4 SO(3) extension: $\text{SO}(3) \not\supset \mathbb{Z}_2^F$

**4.1** $1 \to \text{SO}(3) \to G' \to \text{SO}(d) \to 1$

We consider the symmetry extension problem $1 \to \text{SO}(3) \to G' \to \text{SO}(d) \to 1$. The solution is $G' = \text{SO}(d) \times \text{SO}(3)$.

**4.1.1 SO \times SO(3)**

This case is considered in [24]. We just summarize the bordism groups/invariants in Table 26.
Table 26: Bordism group. Here $w_i$ is the Stiefel-Whitney class of the tangent bundle, $w'_i$ is the Stiefel-Whitney class of the SO(3) bundle, $\sigma$ is the signature of the 4-manifold, $p'_1$ is the Pontryagin class of the SO(3) bundle.

4.1.2 **Gauge SO(3) end up with** $SO \times \mathbb{Z}^m_{2,[1]}$ **in** $d = 4$

In this subsection, we restrict to $d = 4$ where the emergent magnetic symmetry is a 1-form symmetry. The result is the same as that in Sec. 2.2.1.

4.2 $1 \to SO(3) \to G' \to O(d) \to 1$

We consider the symmetry extension problem $1 \to SO(3) \to G' \to O(d) \to 1$. The solution is $G' = O(d) \times SO(3)$.

4.2.1 **$O \times SO(3)$**

This case is considered in [24]. We just summarize the bordism groups/invariants in Table 27.

Table 27: Bordism group. Here $w_i$ is the Stiefel-Whitney class of the tangent bundle, $w'_i$ is the Stiefel-Whitney class of the SO(3) bundle.

4.2.2 **Gauge SO(3) end up with** $O \times \mathbb{Z}^m_{2,[1]}$ **in** $d = 4$

In this subsection, we restrict to $d = 4$ where the emergent magnetic symmetry is a 1-form symmetry. The result is the same as that in Sec. 2.4.1.
5 SO(3) $\times \mathbb{Z}_2^F$ extension: SO(3) $\times \mathbb{Z}_2^F$

5.1 $1 \to \text{SO}(3) \times \mathbb{Z}_2^F \to G' \to \text{SO}(d) \to 1$

We consider the symmetry extension problem $1 \to \text{SO}(3) \times \mathbb{Z}_2^F \to G' \to \text{SO}(d) \to 1$. The solutions are $G' = \text{SO}(d) \times \text{SO}(3) \times \mathbb{Z}_2$ or Spin$(d) \times \text{SO}(3)$.

5.1.1 SO $\times$ SO(3) $\times Z_2 = \text{SO} \times \text{O}(3)$

This case is considered in [24]. We just summarize the bordism groups/invariants in Table 28.

| $d$ | $\Omega_d^{\text{SO} \times \text{O}(3)}$ | generators |
|-----|--------------------------------|------------|
| 0   | $\mathbb{Z}$                   |            |
| 1   | $\mathbb{Z}_2$                 | $w'_1$     |
| 2   | $\mathbb{Z}_2$                 | $w'_2$     |
| 3   | $\mathbb{Z}_2^2$               | $w'_1^3, w'_1 w'_2 = w'_3$ |
| 4   | $\mathbb{Z}_2^2 \times \mathbb{Z}_2$ | $\sigma, p'_1, w'_1^2 w'_3, \sigma, p'_1, w'_1^2 w'_3 = w'_1^3 w'_2, w'_1^5$ |
| 5   | $\mathbb{Z}_2^3$               | $w_2 w_3, w_2^2 w'_1, w_3^2 w'_1, w_1^2 w_2^2, w_1^2 w_2^2, w_1^2 w_3^2 = w_1^3 w_2, w_1^5$ |

Table 28: Bordism group. Here $w_i$ is the Stiefel-Whitney class of the tangent bundle, $w'_i$ is the Stiefel-Whitney class of the O(3) bundle, $\sigma$ is the signature of the 4-manifold, $p'_1$ is the Pontryagin class of the O(3) bundle.

5.1.2 Gauge SO(3) end up with SO $\times Z_2 \times Z_2^{m}$ in $d = 4$

In this subsection, we restrict to $d = 4$ where the emergent magnetic symmetry is a 1-form symmetry. The result is the same as that in Sec. 3.2.1.

5.1.3 Spin $\times$ SO(3)

This case is considered in [24]. We just summarize the bordism groups/invariants in Table 29.
Table 29: Bordism group. Here $\tilde{\eta}$ is the mod 2 index of 1d Dirac operator, Arf is the Arf invariant, $\sigma$ is the signature of the 4-manifold, $w'_i$ is the Stiefel-Whitney class of the SO(3) bundle, $p'_i$ is the Pontryagin class of the SO(3) bundle.

| $d$ | $\Omega^\text{Spin} \times \text{SO}(3)$ | generators |
|-----|----------------------------------|------------|
| 0   | $\mathbb{Z}$                     | $\tilde{\eta}$ |
| 1   | $\mathbb{Z}_2$                  | Arf, $w'_2$ |
| 2   | $\mathbb{Z}_2^2$                | $\sigma, p'_1$ |
| 4   | $\mathbb{Z}_2^4$                | $w_{1,2,3,4}$ |
| 5   | 0                               | $w_{1,2,3,4}$ |

5.1.4 Gauge SO(3) end up with $\text{Spin} \times \mathbb{Z}^m_{\mathbb{Z}_2[1]}$ in $d = 4$

In this subsection, we restrict to $d = 4$ where the emergent magnetic symmetry is a 1-form symmetry. The result is the same as that in Sec. 3.2.2.

5.2 $1 \rightarrow \text{SO}(3) \times \mathbb{Z}_2^F \rightarrow G' \rightarrow \text{O}(d) \rightarrow 1$

We consider the symmetry extension problem $1 \rightarrow \text{SO}(3) \times \mathbb{Z}_2^F \rightarrow G' \rightarrow \text{O}(d) \rightarrow 1$. The solutions are $G' = \text{O}(d) \times \text{SO}(3) \times \mathbb{Z}_2$ or $\text{E}(d) \times \text{SO}(3)$ or $\text{Pin}^+(d) \times \text{SO}(3)$ or $\text{Pin}^-(d) \times \text{SO}(3)$.

5.2.1 $\text{O} \times \text{SO}(3) \times \mathbb{Z}_2 = \text{O} \times \text{O}(3)$

This case is considered in [24]. We just summarize the bordism groups/invariants in Table 30.

| $d$ | $\Omega^\text{O} \times \text{O}(3)$ | generators |
|-----|----------------------------------|------------|
| 0   | $\mathbb{Z}_2$                  | $w'_1$     |
| 1   | $\mathbb{Z}_2$                  | $w_{1,2,3,4}$ |
| 2   | $\mathbb{Z}_2^2$                | $w_{1,2,3,4}$ |
| 3   | $\mathbb{Z}_2^3$                | $w_{1,2,3,4}$ |
| 4   | $\mathbb{Z}_2^4$                | $w_{1,2,3,4}$ |
| 5   | $\mathbb{Z}_2^5$                | $w_{1,2,3,4}$ |

Table 30: Bordism group. Here $w_i$ is the Stiefel-Whitney class of the tangent bundle, $w'_i$ is the Stiefel-Whitney class of the O(3) bundle.
5.2.2 Gauge SO(3) end up with $O \times \mathbb{Z}_2 \times \mathbb{Z}_2^m$ in $d = 4$

In this subsection, we restrict to $d = 4$ where the emergent magnetic symmetry is a 1-form symmetry. The result is the same as that in Sec. 3.4.1.

5.2.3 $E \times SO(3)$

We have $MT(E \times SO(3)) = MTE \wedge (BSO(3))_+ = MSO \wedge \Sigma^{-1}MZ_4 \wedge (BSO(3))_+.$

By Künneth formula,
\[
H^*(MSO \wedge \Sigma^{-1}MZ_4 \wedge (BSO(3))_+, \mathbb{Z}_2) = H^*(MSO, \mathbb{Z}_2) \otimes H^{*+1}(MZ_4, \mathbb{Z}_2) \otimes H^*(BSO(3), \mathbb{Z}_2). \tag{5.1}
\]

Since there is no odd torsion, the Adams spectral sequence shows:
\[
\operatorname{Ext}^s_{\mathcal{A}_2}(H^*(MSO, \mathbb{Z}_2) \otimes H^{*+1}(MZ_4, \mathbb{Z}_2) \otimes H^*(BSO(3), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega^{E_{SSO(3)}}_{E-s}. \tag{5.2}
\]

We have
\[
H^*(BSO(3), \mathbb{Z}_2) = \mathbb{Z}_2[w_2', w_3'] \tag{5.3}
\]
where $w_i'$ is the Stiefel-Whitney class of the SO(3) bundle.

We also have
\[
H^{*+1}(MZ_4, \mathbb{Z}_2) = (\mathbb{Z}_2[y] \otimes \Lambda_{\mathbb{Z}_2}(x))U \tag{5.4}
\]
where $x$ is the generator of $H^1(B\mathbb{Z}_4, \mathbb{Z}_2)$, $y$ is the generator of $H^2(B\mathbb{Z}_4, \mathbb{Z}_2)$, $\Lambda_{\mathbb{Z}_2}$ is the exterior algebra, $U$ is the Thom class of the line bundle determined by $B\mathbb{Z}_4 \to B\mathbb{Z}_2 = BO(1)$.

We list the elements of $H^{*+1}(MZ_4, \mathbb{Z}_2) \otimes H^*(BSO(3), \mathbb{Z}_2)$ below degree 5 as follows:
\[
\begin{align*}
0 & \quad U \\
1 & \quad xU \\
2 & \quad yU, x_2U \\
3 & \quad xyU, xw_2'U, w_3'U \\
4 & \quad y^2U, yw_2'U, xw_3'U, w_2'^2U \\
5 & \quad xy^2U, xyw_2'U, yw_3'U, xw_2'^2U, w_2'^2w_3'U.
\end{align*} \tag{5.5}
\]
They satisfy $\beta_{(2,4)}U = xU$, $\beta_{(2,4)}yU = xyU$, $\beta_{(2,4)}y^2U = xy^2U$, $\operatorname{Sq}^1(w_2'U) = w_3'U$, $\operatorname{Sq}^1(w_2'w_3'U) = w_2'^2U$, $\operatorname{Sq}^1(xw_2'U) = xw_3'U$, $\operatorname{Sq}^1(yw_2'U) = yw_3'U$, $\operatorname{Sq}^1(xyw_2'U) = xyw_3'U$, and $\beta_{(2,4)}(w_2'^2U) = xw_2'^2U$.

The differentials $d_1$ are induced by $\operatorname{Sq}^1$. Moreover, by [35], there are differentials $d_2$ corresponding to the Bockstein homomorphism $\beta_{(2,4)}$.

The $E_2$ page of the Adams spectral sequence is shown in Figure 24.
Figure 24: $\Omega_{s}^{E \times SO(3)}$

| $d$ | $\Omega_{d}^{E \times SO(3)}$ | generators |
|-----|-----------------------------|-------------|
| 0   | $\mathbb{Z}_{4}$           |             |
| 1   | 0                           |             |
| 2   | $\mathbb{Z}_{2} \times \mathbb{Z}_{4}$ | $w_2', y'$ |
| 3   | $\mathbb{Z}_{2}$           | $xw_2'$     |
| 4   | $\mathbb{Z}_{4}^{3} \times \mathbb{Z}_{2}$ | $\sigma \mod 4, y'^2, p_1'^{2}$, $\mod 4, yw'_2$ |
| 5   | $\mathbb{Z}_{2}^{3}$       | $w_2w_3, w_2'w_3, xyw_2'$ |

Table 31: Bordism group. Here $x$ is the generator of $H^1(B\mathbb{Z}_{4}, \mathbb{Z}_2)$, $y$ is the generator of $H^2(B\mathbb{Z}_{4}, \mathbb{Z}_2)$, $y'$ is the generator of $H^2(B\mathbb{Z}_{4}, \mathbb{Z}_4)$ with $y' \mod 2 = y$, $w_i$ is the Stiefel-Whitney class of the tangent bundle, $w_i'$ is the Stiefel-Whitney class of the $SO(3)$ bundle, $\sigma$ is the signature of the 4-manifold, $p_1'$ is the Pontryagin class of the $SO(3)$ bundle.

5.2.4 Gauge $SO(3)$ end up with $E \times \mathbb{Z}_{2,1}^m$ in $d = 4$

In this subsection, we restrict to $d = 4$ where the emergent magnetic symmetry is a 1-form symmetry. The result is the same as that in Sec. 3.4.2.

5.2.5 $Pin^+ \times SO(3)$

This case is considered in [24]. We just summarize the bordism groups/invariants in Table 32.
### 5.2.6 **Gauge $SO(3)$ end up with $\text{Pin}^+ \times Z^m_{2,1}$ in $d = 4$**

In this subsection, we restrict to $d = 4$ where the emergent magnetic symmetry is a 1-form symmetry. The result is the same as that in Sec. 3.4.3.

### 5.2.7 **Pin$^-$ × $SO(3)$**

This case is considered in [24]. We just summarize the bordism groups/invariants in Table 33.

| $d$ | $\Omega^\text{Pin} \times SO(3)$ | generators |
|-----|---------------------------------|------------|
| 0   | $\mathbb{Z}_2$                  |            |
| 1   | $\mathbb{Z}_2$                  | $\tilde{\eta}$ |
| 2   | $\mathbb{Z}_2 \times \mathbb{Z}_8$ | $w'_2$, $w'_1$, $w'$ |
| 3   | $\mathbb{Z}_2$                  | $w_1w'_2 = w'_3$, $w_1\text{Arf}$ |
| 4   | $\mathbb{Z}_2 \times \mathbb{Z}_{16}$ | $q_4(w'_2)$, $\eta$ |
| 5   | $\mathbb{Z}_2$                  | $w_2w_3, w'_1w' = w'_2w'_3$ |

Table 33: Bordism group. Here $\tilde{\eta}$ is the mod 2 index of 1d Dirac operator, ABK is the Arf-Brown-Kervaire invariant, $w_i$ is the Stiefel-Whitney class of the tangent bundle, $w'_i$ is the Stiefel-Whitney class of the $SO(3)$ bundle.

### 5.2.8 **Gauge $SO(3)$ end up with $\text{Pin}^- \times Z^m_{2,1}$ in $d = 4$**

In this subsection, we restrict to $d = 4$ where the emergent magnetic symmetry is a 1-form symmetry. The result is the same as that in Sec. 3.4.4.

Table 32: Bordism group. Here $w_i$ is the Stiefel-Whitney class of the tangent bundle, $\tilde{\eta}$ is the mod 2 index of 1d Dirac operator, $\text{Arf}$ is the Arf invariant, $\eta$ is the 4d eta invariant, $w'_i$ is the Stiefel-Whitney class of the $SO(3)$ bundle, $q_4(w'_2)$ is explained in [24].
6 \textbf{U(1) extension:} $U(1) \supset \mathbb{Z}_2^F$

6.1 $1 \to U(1) \to G' \to SO(d) \to 1$

We consider the symmetry extension problem. $1 \to U(1) \to G' \to SO(d) \to 1$. The solutions are $G' = SO(d) \times U(1)$ or $Spin(d) \times \mathbb{Z}_2 U(1) = Spin^c(d)$.

6.1.1 $SO \times U(1)$

This case is considered in [24]. We just summarize the bordism groups/invariants in Table 34.

| $d$ | $\Omega^SO\times U(1)$ generators |
|-----|---------------------------------|
| 0   | $\mathbb{Z}$                   |
| 1   | 0                               |
| 2   | $\mathbb{Z}$ $c_1$             |
| 3   | 0                               |
| 4   | $\mathbb{Z}_2^2$ $\sigma, c_1^2$ |
| 5   | $\mathbb{Z}_2$ $w_2w_3$        |

Table 34: Bordism group. Here $w_i$ is the Stiefel-Whitney class of the tangent bundle, $\sigma$ is the signature of the 4-manifold, $c_1$ is the Chern class of the $U(1)$ bundle.

6.1.2 $Spin^c$

By [25], we have $MTSpin^c = MSpin \wedge \Sigma^{-2}MU(1)$.

For $t - s < 8$, since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}^{s,t}_{A_2(1)}(H^{s+2}(MU(1), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega^{Spin^c}_{t-s}. \quad (6.1)$$

The $A_2(1)$-module structure of $H^{s+2}(MU(1), \mathbb{Z}_2)$ below degree 5 and the $E_2$ page are shown in Figure 25, 26.

![Figure 25: The $A_2(1)$-module structure of $H^{s+2}(MU(1), \mathbb{Z}_2)$ below degree 5.](image_url)
Table 35: Bordism group. Here $c_1$ is the Chern class of the U(1) bundle, $c_1$ is divided by 2 since $c_1$ mod 2 = $w_2(TM)$ while $w_2(TM) = 0$ on Spin$^c$ 2-manifolds. $\sigma$ is the signature of the 4-manifold, $F$ is a characteristic surface of the Spin$^c$ 4-manifold $M$. By Rokhlin’s theorem, $\sigma - F \cdot F$ is a multiple of 8 and $\frac{1}{8}(\sigma - F \cdot F) = \text{Arf}(M, F) \mod 2$. See [36]'s Lecture 10 for more details.

6.1.3 Gauge U(1) end up with SO $\times$ U(1)$^e$ $\times$ U(1)$^m$ in $d = 4$

In this subsection, we restrict to $d = 4$ where the emergent magnetic symmetry is a 1-form symmetry.

By [37], the mod 2 cohomology of $B^2U(1) = K(\mathbb{Z}, 3)$ is

$$H^*(B^2U(1), \mathbb{Z}_2) = \mathbb{Z}_2[\tau_3, \text{Sq}^2\tau_3, \text{Sq}^4\text{Sq}^2\tau_3, \ldots]$$

(6.2)

where the generators are of the form $\text{Sq}^{2^n}\text{Sq}^{2^n-1}\cdots\text{Sq}^2\tau_3$ and $\tau_3$ is the generator of $H^3(B^2U(1), \mathbb{Z}_2)$ with $\text{Sq}^1\tau_3 = 0$. Denote $\tau_5 = \text{Sq}^2\tau_3$, $\tau_9 = \text{Sq}^4\text{Sq}^2\tau_3$, $\ldots$

We have $MT(SO \times BU(1)^e \times BU(1)^m) = MSO \wedge (B^2U(1)^e \times B^2U(1)^m)_+$. 

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By K"unneth formula,
\[ H^* (MSO \wedge (B^2U(1)^e \times B^2U(1)^m)_+, Z_2) = H^* (MSO, Z_2) \otimes H^* (B^2U(1)^e, Z_2) \otimes H^* (B^2U(1)^m, Z_2) \] (6.3)

Since there is no odd torsion, we have the Adams spectral sequence
\[ \text{Ext}_{A_2}^{s,t} (H^* (MSO, Z_2) \otimes H^* (B^2U(1)^e, Z_2) \otimes H^* (B^2U(1)^m, Z_2), Z_2) \Rightarrow \Omega^{SO \times BU(1)^e \times BU(1)^m}_{t-s}. \] (6.4)

We have
\[ H^* (B^2U(1)^e, Z_2) \otimes H^* (B^2U(1)^m, Z_2) = Z_2 [\tilde{\tau}_3^e, \tilde{\tau}_5^m, \ldots]. \] (6.5)

Note that \( Sq^1 \tau_5 = Sq^1 Sq^2 \tau_3 = Sq^3 \tau_3 = \tau_3^2. \)

The \( E_2 \) page is shown in Figure 27.

![Figure 27: \( \Omega^{SO \times BU(1)^e \times BU(1)^m} \)]

| \( d \) | Bordism group \( \Omega_d^{SO \times BU(1)^e \times BU(1)^m} \) | generators |
|-------|----------------------------------|-------------|
| 0     | \( Z \)                          |             |
| 1     | 0                                |             |
| 2     | 0                                |             |
| 3     | \( Z^2 \)                        | \( \tilde{\tau}_3^e, \tilde{\tau}_5^m \) |
| 4     | \( Z \)                          | \( \sigma \) |
| 5     | \( Z^3_2 \)                      | \( w_2w_3, \tilde{\tau}_5^e, \tau_5^m \) |

Table 36: Bordism group. Here \( \tilde{\tau}_3 \) is the generator of \( H^3(B^2U(1), Z) \) with \( \tilde{\tau}_3 \mod 2 = \tau_3 \). \( \tau_5 = Sq^2 \tau_3 \). \( \sigma \) is the signature of the 4-manifold, \( w_i \) is the Stiefel-Whitney class of the tangent bundle.
6.2 1 → U(1) → G' → O(d) → 1

We consider the symmetry extension problem 1 → U(1) → G' → O(d) → 1. We find the distinct group extensions are given by $G' = O(d) \times U(1)$ or $O(d) \ltimes U(1)$ or $E(d) \ltimes_{Z_2} U(1)$ or $Pin^+(d) \times_{Z_2} U(1)$ or $Pin^-(d) \times_{Z_2} U(1) = Pin^{-}(d) \times_{Z_2} U(1) = Pin^+(d) \times_{Z_2} U(1)$ or $Pin^c(d) = Pin^-(d) \times_{Z_2} U(1)$ or $Pin^c(d) = Pin^-(d) \times_{Z_2} U(1)$. Below, we list the bordism groups/invariants $\Omega^G_d$ for all the above $G'$ except $O(d) \times U(1)$ and $E(d) \ltimes_{Z_2} U(1)$ which are left for future work.

6.2.1 O × U(1)

We have $MT(O \times U(1)) = MO \wedge (BU(1))_+$. By Künneth formula,

$$H^*(MO \wedge (BU(1))_+, Z_2) = H^*(MO, Z_2) \otimes H^*(BU(1), Z_2). \quad (6.6)$$

Since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}^{s,t}_{A_2}(H^*(MO, Z_2) \otimes H^*(BU(1), Z_2), Z_2) \Rightarrow \Omega^O \times U(1). \quad (6.7)$$

| d | $\Omega^O \times U(1)$ | generators |
|---|---|---|
| 0 | $\mathbb{Z}_2$ | |
| 1 | 0 | |
| 2 | $\mathbb{Z}_2^2$ | $c_1 \mod 2, w_1^2$ |
| 3 | 0 | |
| 4 | $\mathbb{Z}_2^2$ | $c_1^2 \mod 2, w_1^4, w_2^2$ |
| 5 | $\mathbb{Z}_2$ | $w_2 w_3$ |

Table 37: Bordism group. Here $w_i$ is the Stiefel-Whitney class of the tangent bundle, $c_1$ is the Chern class of the U(1) bundle.

6.2.2 Pin$^c$

$Pin^c = Pin^+ \times_{Z_2} U(1)$ or $Pin^- \times_{Z_2} U(1)$, here we regard Pin$^c$ as $Pin^- \times_{Z_2} U(1)$, so $c_1 = w_2 + w_1^2 \mod 2$. By [25], we have $MTPin^c = MSpin \wedge \Sigma^{-1} MO(1) \wedge \Sigma^{-2} MU(1)$.

By Künneth formula,

$$H^*(\Sigma^{-1} MO(1) \wedge \Sigma^{-2} MU(1), Z_2) = H^{*-1}(MO(1), Z_2) \otimes H^{*+2}(MU(1), Z_2). \quad (6.8)$$

For $t - s < 8$, since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}^{s,t}_{A_2}(H^{*-1}(MO(1), Z_2) \otimes H^{*+2}(MU(1), Z_2), Z_2) \Rightarrow \Omega^{Pin^c}. \quad (6.9)$$
The $\mathcal{A}_2(1)$-module structure of $H^{*+1}(MO(1), \mathbb{Z}_2) \otimes H^{*+2}(MU(1), \mathbb{Z}_2)$ below degree 5 and the $E_2$ page are shown in Figure 28, 29.

Figure 28: The $\mathcal{A}_2(1)$-module structure of $H^{*+1}(MO(1), \mathbb{Z}_2) \otimes H^{*+2}(MU(1), \mathbb{Z}_2)$ below degree 5.

Figure 29: $\Omega^{\text{Pin}^c}$
### Bordism group

| $d$ | $\Omega^\text{Pin}_{d}$ | generators |
|-----|------------------------|-------------|
| 0   | $\mathbb{Z}_2$         |             |
| 1   | 0                      |             |
| 2   | $\mathbb{Z}_4$         | $\text{ABK mod 4}$ |
| 3   | 0                      |             |
| 4   | $\mathbb{Z}_2 \times \mathbb{Z}_8$ | $c_1^2 \mod 2, (c_1 \mod 2)\text{ABK}$ |
| 5   | 0                      |             |

Table 38: Bordism group. Here $c_1$ is the Chern class of the U(1) bundle, ABK is the Arf-Brown-Kervaire invariant.

#### 6.2.3 Pin$^{\tilde{c}+}$

By [12], we have $MTPin^{\tilde{c}+} = MS\text{pin} \wedge \Sigma^2 MTO(2)$.

For $t - s < 8$, since there is no odd torsion, we have the Adams spectral sequence

\[
\text{Ext}^{s,t}_{A_2(1)}(H^{*-2}(MTO(2), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega^\text{Pin}^{\tilde{c}+}_{t-s}
\]  

(6.10)

By Thom isomorphism, we have

\[
H^{*-2}(MTO(2), \mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2]U
\]  

(6.11)

where $U$ is the Thom class with $\text{Sq}^1 U = w_1 U$, $\text{Sq}^2 U = (w_2 + w_1^2) U$.

The $A_2(1)$-module structure of $H^{*-2}(MTO(2), \mathbb{Z}_2)$ below degree 5 and the $E_2$ page are shown in Figure 30, 31.

![Figure 30](image-url)

Figure 30: The $A_2(1)$-module structure of $H^{*-2}(MTO(2), \mathbb{Z}_2)$ below degree 5.
Table 39: Bordism group. Here \( w_i \) is the Stiefel-Whitney class of the tangent bundle, \( \tilde{\eta} \) is the mod 2 index of 1d Dirac operator, \( \text{Arf} \) is the Arf invariant, \( c_1 \) is the Chern class of the U(1) bundle.

6.2.4 Pin\( \tilde{\omega}^{-} \)

By \cite{12}, we have \( MTPin\tilde{\omega}^{-} = MSpin \wedge \Sigma^{-2}MO(2) \).

For \( t - s < 8 \), since there is no odd torsion, we have the Adams spectral sequence

\[
\text{Ext}^{s,t}_{A_2(1)}(H^{s+2}(MO(2), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega^\text{Pin}\tilde{\omega}^{-}_{t-s}
\]

By Thom isomorphism, we have

\[
H^{s+2}(MO(2), \mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2]U
\]

where \( U \) is the Thom class with \( \text{Sq}^1 U = w_1 U \), \( \text{Sq}^2 U = w_2 U \).

The \( A_2(1) \)-module structure of \( H^{s+2}(MO(2), \mathbb{Z}_2) \) below degree 5 and the \( E_2 \) page are shown in Figure 32, 33.
Figure 32: The $A_2(1)$-module structure of $H^{*+2}(MO(2), \mathbb{Z}_2)$ below degree 5.

Figure 33: $\Omega^\text{Pin}^{s-}$

| $d$ | $\Omega_d^{\text{Pin}^{s-}}$ | generators |
|-----|-----------------------------|-------------|
| 0   | $\mathbb{Z}_2$              |             |
| 1   | 0                           |             |
| 2   | $\mathbb{Z} \times \mathbb{Z}_2$ | $w_1^2$    |
| 3   | 0                           |             |
| 4   | $\mathbb{Z}_2$              | $w_2^2$     |
| 5   | 0                           |             |

Table 40: Bordism group. Here $w_i$ is the Stiefel-Whitney class of the tangent bundle, $c_1$ is the Chern class of the $U(1)$ bundle.
6.2.5 Gauge $U(1)$ end up with $O \times U(1)^{[1]} \times U(1)^{m}$ in $d = 4$

In this subsection, we restrict to $d = 4$ where the emergent magnetic symmetry is a 1-form symmetry.

We have $MT(O \times BU(1)^e \times BU(1)^m) = MO \wedge (B^2U(1)^e \times B^2U(1)^m)_+$. 

By Künneth formula,

$$H^*(MO \wedge (B^2U(1)^e \times B^2U(1)^m)_+, Z_2) = H^*(MO, Z_2) \otimes H^*(B^2U(1)^e, Z_2) \otimes H^*(B^2U(1)^m, Z_2).$$

(6.14)

Since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}_{A_2}(H^*(MO, Z_2) \otimes H^*(B^2U(1)^e, Z_2) \otimes H^*(B^2U(1)^m, Z_2), Z_2) \Rightarrow \Omega O^{\times BU(1)^e \times BU(1)^m}.$$

(6.15)

We have

$$H^*(MO, Z_2) \otimes H^*(B^2U(1)^e, Z_2) \otimes H^*(B^2U(1)^m, Z_2) = A_2 \oplus \Sigma^2 A_2 \oplus 2\Sigma^3 A_2 \oplus 2\Sigma^4 A_2 \oplus 5\Sigma^5 A_2 \oplus \cdots$$

(6.16)

| $d$ | $\Omega_{d}^{O^{\times BU(1)^e \times BU(1)^m}}$ | generators |
|-----|----------------------------------------|------------|
| 0   | $Z_2$                                  |            |
| 1   | $0$                                    |            |
| 2   | $Z_2$                                  | $w_1^2$    |
| 3   | $Z_2^2$                                | $\tau_3^e, \tau_3^m$ |
| 4   | $Z_2^4$                                | $w_1^2, w_2^2$ |
| 5   | $Z_2^5$                                | $w_2 w_3, w_1^2 \tau_3^e, w_1^2 \tau_3^m, \tau_5^e, \tau_5^m$ |

Table 41: Bordism group. Here $w_i$ is the Stiefel-Whitney class of the tangent bundle, $\tau_3$ is the generator of $H^3(B^2U(1), Z_2)$, $\tau_5 = Sq^2 \tau_3$.

7 $U(1) \times Z_2^F$ extension: $U(1) \times Z_2^F$

7.1 $1 \to U(1) \times Z_2^F \to G' \to SO(d) \to 1$

We consider the symmetry extension problem $1 \to U(1) \times Z_2^F \to G' \to SO(d) \to 1$. The solutions are $G' = SO(d) \times U(1) \times Z_2$ or $\text{Spin}(d) \times U(1)$ or $\text{Spin}^c(d) \times Z_2$.

7.1.1 $SO \times U(1) \times Z_2$

We have $MT(SO \times U(1) \times Z_2) = MSO \wedge (B(U(1) \times Z_2))_+$. 

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By Künneth formula,

\[ H^*(MSO \wedge (B(U(1) \times \mathbb{Z}_2))_+, \mathbb{Z}_2) = H^*(MSO, \mathbb{Z}_2) \otimes H^*(BU(1), \mathbb{Z}_2) \otimes H^*(B\mathbb{Z}_2, \mathbb{Z}_2). \]  

(7.1)

We have

\[ H^*(BU(1), \mathbb{Z}_2) = \mathbb{Z}_2[c_1]. \]  

(7.2)

Since there is no odd torsion, we have the Adams spectral sequence

\[ \text{Ext}_{A_2}^{s,t}(H^*(MSO, \mathbb{Z}_2) \otimes H^*(BU(1), \mathbb{Z}_2) \otimes H^*(B\mathbb{Z}_2, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega^\text{SO\times U(1)\times Z}_2_{t-s}. \]  

(7.3)

The \(E_2\) page of the Adams spectral sequence is shown in Figure 34.

![Figure 34](image)

**Table 42**: Bordism group. Here \(a\) is the generator of \(H^1(B\mathbb{Z}_2, \mathbb{Z}_2)\), \(c_1\) is the Chern class of the \(U(1)\) bundle, \(w_i\) is the Stiefel-Whitney class of the tangent bundle, \(\sigma\) is the signature of the 4-manifold.
7.1.2 Spin × U(1)

This case is considered in [24]. We just summarize the bordism groups/invariants in Table 43.

| d | Ω_d^{Spin×U(1)} generators |
|---|-----------------------------|
| 0 | Z                           |
| 1 | Z_2                         |
| 2 | Z × Z_2                     |
| 3 | 0                            |
| 4 | Z_2^2                       |
| 5 | 0                            |

Table 43: Bordism group. Here ˜η is the mod 2 index of 1d Dirac operator, Arf is the Arf invariant, σ is the signature of the 4-manifold, c_1 is the Chern class of the U(1) bundle.

7.1.3 Spin^c × Z_2

We have MT(Spin^c × Z_2) = MSpin ∧ Σ^{-2}MU(1) ∧ (BZ_2)_+.

By Künneth formula,

$$H^*(Σ^{-2}MU(1) ∧ (BZ_2)_+, Z_2) = H^{*+2}(MU(1), Z_2) ⊗ H^*(BZ_2, Z_2).$$ (7.4)

For t − s < 8, since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}^{s,t}_{A_2(1)}(H^{*+2}(MU(1), Z_2) ⊗ H^*(BZ_2, Z_2), Z_2) \Rightarrow Ω_{t-s}^{Spin^c × Z_2}. \quad (7.5)$$

The A_2(1)-module structure of H^{*+2}(MU(1), Z_2) ⊗ H^*(BZ_2, Z_2) below degree 5 and the E_2 page are shown in Figure 35, 36.
Figure 35: The $A_2(1)$-module structure of $H^{*+2}(MU(1), \mathbb{Z}_2) \otimes H^*(B\mathbb{Z}_2, \mathbb{Z}_2)$ below degree 5.

Figure 36: $\Omega^{\text{Spin}^c \times \mathbb{Z}_2}$
| $d$ | $\Omega^\text{Spin} \times \mathbb{Z}_2$ generators |
|-----|----------------------------------|
| 0   | $\mathbb{Z}$                        |
| 1   | $\mathbb{Z}_2$  $a$                |
| 2   | $\mathbb{Z}$  $\frac{c_1}{2}$      |
| 3   | $\mathbb{Z}_2$  $a(\text{ABK mod } 4)$ |
| 4   | $\mathbb{Z}^2$  $\frac{\sigma - FF}{8}, c_1^2$ |
| 5   | $\mathbb{Z}_2 \times \mathbb{Z}_8$  $ac_1^2, c_1 \text{ABK}$ |

Table 44: Bordism group. Here $a$ is the generator of $H^1(B\mathbb{Z}_2, \mathbb{Z}_2)$, ABK is the Arf-Brown-Kervaire invariant. $c_1$ is the Chern class of the $U(1)$ bundle, $c_1$ is divided by 2 since $c_1 \mod 2 = w_2(TM)$ while $w_2(TM) = 0$ on Spin$^c$ 2-manifolds. $\sigma$ is the signature of the 4-manifold, $F$ is a characteristic surface of the Spin$^c$ 4-manifold $M$. By Rokhlin’s theorem, $\sigma - F \cdot F$ is a multiple of 8 and $\frac{1}{8}(\sigma - F \cdot F) = \text{Arf}(M, F) \mod 2$. See [36]’s Lecture 10 for more details.

7.1.4 Gauge $U(1)$ end up with $SO \times \mathbb{Z}_2 \times U(1)^{e, m}$ in $d = 4$

In this subsection, we restrict to $d = 4$ where the emergent magnetic symmetry is a 1-form symmetry.

We have $MT(SO \times \mathbb{Z}_2 \times BU(1)^e \times BU(1)^m) = MSO \wedge (B\mathbb{Z}_2 \times B^2U(1)^e \times B^2U(1)^m)_+$. By Künneth formula,

$$
H^*(MSO \wedge (B\mathbb{Z}_2 \times B^2U(1)^e \times B^2U(1)^m)_+, \mathbb{Z}_2) = H^*(MSO, \mathbb{Z}_2) \otimes H^*(B\mathbb{Z}_2, \mathbb{Z}_2) \otimes H^*(B^2U(1)^e, \mathbb{Z}_2) \otimes H^*(B^2U(1)^m, \mathbb{Z}_2).$$

Since there is no odd torsion, we have the Adams spectral sequence

$$
\text{Ext}^{s,t}_{d_2}^d(H^*(MSO, \mathbb{Z}_2) \otimes H^*(B\mathbb{Z}_2, \mathbb{Z}_2) \otimes H^*(B^2U(1)^e, \mathbb{Z}_2) \otimes H^*(B^2U(1)^m, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega^\text{Spin} \times \mathbb{Z}_2 \times BU(1)^e \times BU(1)^m.
$$

We have

$$
H^*(B\mathbb{Z}_2, \mathbb{Z}_2) \otimes H^*(B^2U(1)^e, \mathbb{Z}_2) \otimes H^*(B^2U(1)^m, \mathbb{Z}_2) = \mathbb{Z}_2[a, \tau_3^e, \tau_5^e, \tau_3^m, \tau_5^m, \ldots].
$$

Note that $\text{Sq}^1 \tau_5 = \text{Sq}^1 \text{Sq}^2 \tau_3 = \text{Sq}^3 \tau_3 = \tau_3^2$.

The $E_2$ page is shown in Figure 37.
Figure 37: $\Omega^5_{SO \times \mathbb{Z}_2 \times \text{BU}(1)^e \times \text{BU}(1)^m}$

| $d$ | $\Omega^5_{SO \times \mathbb{Z}_2 \times \text{BU}(1)^e \times \text{BU}(1)^m}$ | generators |
|-----|-------------------------------------------------|------------|
| 0   | $\mathbb{Z}$              | $a$        |
| 1   | $\mathbb{Z}_2$           |            |
| 2   | 0                              |            |
| 3   | $\mathbb{Z}_2^2 \times \mathbb{Z}_2$ | $\tilde{\tau}_3^e, \tilde{\tau}_3^m, a^3$ |
| 4   | $\mathbb{Z} \times \mathbb{Z}_2^2$            | $\sigma, a\tau_5^e, a\tau_5^m$ |
| 5   | $\mathbb{Z}_2^5$           | $w_2w_3, \tau_5^e, \tau_5^m, a^5, aw_2^2$ |

Table 45: Bordism group. Here $\tilde{\tau}_3$ is the generator of $H^3(B^2U(1), \mathbb{Z})$ with $\tilde{\tau}_3 \mod 2 = \tau_3$. $\tau_5 = \text{Sq}^2\tau_3$. $a$ is the generator of $H^1(B\mathbb{Z}_2, \mathbb{Z}_2)$, $w_i$ is the Stiefel-Whitney class of the tangent bundle, $\sigma$ is the signature of the 4-manifold.

7.1.5 **Gauge U(1) end up with** $\text{Spin} \times U(1)^e_{[1]} \times U(1)^m_{[1]}$ **in** $d = 4$

In this subsection, we restrict to $d = 4$ where the emergent magnetic symmetry is a 1-form symmetry.

We have $MT(\text{Spin} \times \text{BU}(1)^e \times \text{BU}(1)^m) = M\text{Spin} \wedge (B^2U(1)^e \times B^2U(1)^m)_+$. 

For $t - s < 8$, since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}^{s,t}_{A_2(1)}(H^*(B^2U(1)^e \times B^2U(1)^m, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega^5_{\text{Spin} \times \text{BU}(1)^e \times \text{BU}(1)^m}_{t-s}. \quad (7.9)$$

The $A_2(1)$-module structure of $H^*(B^2U(1)^e \times B^2U(1)^m, \mathbb{Z}_2)$ below degree 5 and the $E_2$ page are shown in Figure 38, 39.
Figure 38: The $A_2(1)$-module structure of $H^*(B^2U(1)^e \times B^2U(1)^m, \mathbb{Z}_2)$ below degree 5.

![Diagram](image)

Figure 39: $\Omega_{s}^{\text{Spin} \times BU(1)^e \times BU(1)^m}$

| $d$ | $\Omega_{d}^{\text{Spin} \times BU(1)^e \times BU(1)^m}$ | generators |
|-----|--------------------------------------------------|-------------|
| 0   | $\mathbb{Z}$                                    |             |
| 1   | $\mathbb{Z}_2$                                  | $\tilde{\eta}$ |
| 2   | $\mathbb{Z}_2$                                  | Arf         |
| 3   | $\mathbb{Z}_2^2$                                | $\tilde{\tau}_3^e, \tilde{\tau}_3^m$ |
| 4   | $\mathbb{Z}$                                    | $\sigma$    |
| 5   | $0$                                              |             |

Table 46: Bordism group. Here $\tilde{\eta}$ is the mod 2 index of 1d Dirac operator, Arf is the Arf invariant, $\sigma$ is the signature of the 4-manifold, $\tilde{\tau}_3$ is the generator of $H^3(B^2U(1), \mathbb{Z})$ with $\tilde{\tau}_3 \mod 2 = \tau_3$.  

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7.2 \ 1 \to U(1) \times \mathbb{Z}_2^F \to G' \to O(d) \to 1

We consider the symmetry extension problem \(1 \to U(1) \times \mathbb{Z}_2^F \to G' \to O(d) \to 1\). We find the distinct group extensions are given by \(G' = O(d) \times U(1) \times \mathbb{Z}_2\) or \(O(d) \times U(1) \times \mathbb{Z}_2\) or \(E(d) \times Z_2 \times U(1) \times \mathbb{Z}_2\) or \(E(d) \times U(1) \times \text{Pin}^+(d) \times U(1) \times \mathbb{Z}_2\) or \(E(d) \times U(1) \times \text{Pin}^-(d) \times U(1) \times \mathbb{Z}_2\) or \(E(d) \times \text{Pin}^c(d) \times \mathbb{Z}_2\) or \(E(d) \times \text{Pin}^c-(d) \times \mathbb{Z}_2\).

Below, we list the bordism groups/invariants \(\Omega^G_d\) for all the above \(G'\) except \(O(d) \times U(1) \times \mathbb{Z}_2\) and \(E(d) \times Z_2 \times U(1) \times \mathbb{Z}_2\) which are left for future work.

7.2.1 \(O \times U(1) \times \mathbb{Z}_2\)

We have \(MT(O \times U(1) \times \mathbb{Z}_2) = MO \wedge (B(U(1) \times \mathbb{Z}_2))_+\).

By Künneth formula,

\[
\begin{align*}
H^*(MO \wedge (B(U(1) \times \mathbb{Z}_2))_+, \mathbb{Z}_2) &= H^*(MO, \mathbb{Z}_2) \otimes H^*(BU(1), \mathbb{Z}_2) \otimes H^*(BZ_2, \mathbb{Z}_2). 
\end{align*}
\tag{7.10}
\]

Since there is no odd torsion, we have the Adams spectral sequence

\[
\begin{align*}
\text{Ext}^{s,t}_{A_2}(H^*(MO, \mathbb{Z}_2) \otimes H^*(BU(1), \mathbb{Z}_2) \otimes H^*(BZ_2, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega^{O \times U(1) \times \mathbb{Z}_2}_d. 
\end{align*}
\tag{7.11}
\]

| \(d\) | \(\Omega^G_d\) | Generators |
|------|----------------|------------|
| 0    | \(Z_2\)        |            |
| 1    | \(Z_2\)        |            |
| 2    | \(Z_2^3\)      | \(w_1^2, a^2, c_1 \mod 2\) |
| 3    | \(Z_2^3\)      | \(a^3, aw_1^2, ac_1\) |
| 4    | \(Z_2^6\)      | \(w_1^4, w_2^2, a^4, a^2 w_1^2, a^2 c_1, c_1^2 \mod 2\) |
| 5    | \(Z_2^7\)      | \(w_2 w_3, a^5, a^3 w_1^2, aw_1^4, aw_2^3, a^3 c_1, ac_1^2\) |

Table 47: Bordism group. Here \(a\) is the generator of \(H^1(BZ_2, \mathbb{Z}_2)\), \(c_1\) is the Chern class of the \(U(1)\) bundle, \(w_i\) is the Stiefel-Whitney class of the tangent bundle.

7.2.2 Gauge \(U(1)\) end up with \(O \times \mathbb{Z}_2 \times \text{U}(1)^e_{[1]} \times \text{U}(1)^m_{[1]}\) in \(d = 4\)

In this subsection, we restrict to \(d = 4\) where the emergent magnetic symmetry is a 1-form symmetry.

We have \(MT(O \times \mathbb{Z}_2 \times BU(1)^e \times BU(1)^m) = MO \wedge (BZ_2 \times B^2U(1)^e \times B^2U(1)^m)_+\).

By Künneth formula,

\[
\begin{align*}
H^*(MO \wedge (BZ_2 \times B^2U(1)^e \times B^2U(1)^m)_+, \mathbb{Z}_2) &= H^*(MO, \mathbb{Z}_2) \otimes H^*(BZ_2, \mathbb{Z}_2) \otimes H^*(B^2U(1)^e, \mathbb{Z}_2) \otimes H^*(B^2U(1)^m, \mathbb{Z}_2). 
\end{align*}
\tag{7.12}
\]
Since there is no odd torsion, we have the Adams spectral sequence
\[
\text{Ext}^s_t(H^*(MO, Z_2) \otimes H^*(BZ_2, Z_2) \otimes H^*(B^2U(1)^m, Z_2), Z_2) \Rightarrow \Omega_{t-s}^{O \times Z_2 \times BU(1)^m \times BU(1)^m}.
\] (7.13)

We have
\[
H^*(MO, Z_2) \otimes H^*(BZ_2, Z_2) \otimes H^*(B^2U(1)^m, Z_2) \otimes H^*(B^2U(1)^m, Z_2) = (A_2 \oplus \Sigma^2 A_2 \oplus 2\Sigma^3 A_2 \oplus \Sigma^5 A_2 + \cdots) \otimes Z_2[a, \tau_3^m, \tau_5^m, \tau_5^m, \ldots]
\]
\[
= A_2 \oplus \Sigma A_2 \oplus 2\Sigma^2 A_2 \oplus 4\Sigma^3 A_2 \oplus 6\Sigma^4 A_2 \oplus 11\Sigma^5 A_2 + \cdots
\] (7.14)

| Bordism group |
|---------------|
| \(d\) | \(\Omega_d^{O \times Z_2 \times BU(1)^m \times BU(1)^m}\) | generators |
|-----|-----------------|-------------|
| 0   | \(Z_2\)         | \(a\)       |
| 1   | \(Z_2\)         | \(a^2, w_1^2\) |
| 2   | \(Z_2^2\)       | \(a^3, aw_1^2, \tau_3^m\) |
| 3   | \(Z_2^4\)       | \(w_1^4, w_2^4, a^2 w_1^2, a \tau_3^m, \tau_5^m\) |
| 4   | \(Z_2^6\)       | \(w_2^4, a^2 w_2^2, a^2 \tau_3^m, a^2 \tau_5^m, w_1^2 \tau_3^m, w_1^2 \tau_5^m, \tau_5^m\) |
| 5   | \(Z_2^8\)       | \(w_3^2, a^5, a^3 w_1^2, aw_1^4, aw_2^2, a^2 \tau_3^m, a^2 \tau_5^m, w_1^2 \tau_3^m, w_1^2 \tau_5^m, \tau_5^m\) |

Table 48: Bordism group. Here \(a\) is the generator of \(H^1(BZ_2, Z_2)\), \(w_i\) is the Stiefel-Whitney class of the tangent bundle, \(\tau_3\) is the generator of \(H^3(B^2U(1), Z_2)\), \(\tau_5 = \text{Sq}^2 \tau_3\).

### 7.2.3 \(E \times U(1)\)

We have \(MT(E \times U(1)) = MTE \wedge (BU(1))_+ = MSO \wedge \Sigma^{-1} MZ_4 \wedge (BU(1))_+\).

By Künneth formula,
\[
H^*(MSO \wedge \Sigma^{-1} MZ_4 \wedge (BU(1))_+, Z_2) = H^*(MSO, Z_2) \otimes H^{*+1}(MZ_4, Z_2) \otimes H^*(BU(1), Z_2).
\] (7.15)

Since there is no odd torsion, the Adams spectral sequence shows:
\[
\text{Ext}^s_t(H^*(MSO, Z_2) \otimes H^{*+1}(MZ_4, Z_2) \otimes H^*(BU(1), Z_2), Z_2) \Rightarrow \Omega_{t-s}^{E \times U(1)}.
\] (7.16)

We have
\[
H^*(BU(1), Z_2) = Z_2[c_1]
\] (7.17)
where \(c_1\) is the first Chern class of the U(1) bundle.

We also have
\[
H^{*+1}(MZ_4, Z_2) = (Z_2[y] \otimes \Lambda_{Z_2}(x)) U
\] (7.18)
where $x$ is the generator of $H^1(BZ_4,\mathbb{Z}_2)$, $y$ is the generator of $H^2(BZ_4,\mathbb{Z}_2)$, $\Lambda_{\mathbb{Z}_2}$ is the exterior algebra, $U$ is the Thom class of the line bundle determined by $BZ_4 \to B\mathbb{Z}_2 = BO(1)$.

We list the elements of $H^{*+1}(MZ_4,\mathbb{Z}_2) \otimes H^*(BU(1),\mathbb{Z}_2)$ below degree 5 as follows:

\[
\begin{align*}
0 & : U \\
1 & : xU \\
2 & : yU, c_1 U \\
3 & : xyU, xc_1 U \\
4 & : y^2U, c_1^2U, yc_1 U \\
5 & : xy^2U, xc_1^2U, xyc_1 U.
\end{align*}
\]

They satisfy $\beta(2,4)U = xU$, $\beta(2,4)yU = xyU$, $\beta(2,4)y^2U = xy^2U$, $\beta(2,4)c_1 U = xc_1 U$, $\beta(2,4)c_1^2 U = xc_1^2 U$ and $\beta(2,4)yc_1 U = xyc_1 U$.

By [35], there are differentials $d_2$ corresponding to the Bockstein homomorphism $\beta(2,4)$.

The $E_2$ page of the Adams spectral sequence is shown in Figure 40.

![Figure 40: $\Omega^E_{\times U(1)}$](image)
Table 49: Bordism group. Here $y'$ is the generator of $H^2(BZ_4, Z_4)$, $\sigma$ is the signature of the 4-manifold, $c_1$ is the first Chern class of the $U(1)$ bundle, $w_i$ is the Stiefel-Whitney class of the tangent bundle.

### 7.2.4 Gauge $U(1)$ end up with $E \times U(1)^e \times U(1)^m$ in $d = 4$

In this subsection, we restrict to $d = 4$ where the emergent magnetic symmetry is a 1-form symmetry.

We have $MT(E \times BU(1)^e \times BU(1)^m) = MT \wedge (B^2U(1)^e \times B^2U(1)^m)_+ = MSO \wedge \Sigma^{-1}MZ_4 \wedge (B^2U(1)^e \times B^2U(1)^m)_+$.

By Künneth formula,

$$H^*(MSO \wedge \Sigma^{-1}MZ_4 \wedge (B^2U(1)^e \times B^2U(1)^m)_+, Z_2) = H^*(MSO, Z_2) \otimes H^{*+1}(MZ_4, Z_2) \otimes H^*(B^2U(1)^e, Z_2) \times H^*(B^2U(1)^m, Z_2). \tag{7.20}$$

Since there is no odd torsion, the Adams spectral sequence shows:

$$\text{Ext}_{A_0}^{s,t}(H^*(MSO, Z_2) \otimes H^{*+1}(MZ_4, Z_2) \otimes H^*(B^2U(1)^e, Z_2) \times H^*(B^2U(1)^m, Z_2), Z_2) \Rightarrow \Omega_{t-s}^{E \times BU(1)^e \times BU(1)^m}. \tag{7.21}$$

We have

$$H^*(B^2U(1), Z_2) = Z_2[\tau_3, \tau_5, \ldots] \tag{7.22}$$

where $\tau_3$ is the generator of $H^3(B^2U(1), Z_2)$, $\tau_5 = Sq^2 \tau_3$, $\ldots$.

We also have

$$H^{*+1}(MZ_4, Z_2) = (Z_2[y] \otimes \Lambda Z_2(x))U \tag{7.23}$$

where $x$ is the generator of $H^1(BZ_4, Z_2)$, $y$ is the generator of $H^2(BZ_4, Z_2)$, $\Lambda Z_2$ is the exterior algebra, $U$ is the Thom class of the line bundle determined by $BZ_4 \to BZ_2 = BO(1)$.

We list the elements of $H^{*+1}(MZ_4, Z_2) \otimes H^*(B^2U(1)^e, Z_2) \times H^*(B^2U(1)^m, Z_2)$ below degree 5 as...
follows:

\[
\begin{array}{c}
0 & U \\
1 & xU \\
2 & yU \\
3 & \tau_3^m U, \tau_3^e U \\
4 & y^2U, x\tau_3^e U, x\tau_3^m U \\
5 & xy^2U, \tau_5^m U, \tau_5^e U, y\tau_3^e U, y\tau_3^m U.
\end{array}
\]

They satisfy \(\beta_{(2,4)} U = xU, \beta_{(2,4)} yU = xyU, \beta_{(2,4)} y^2U = xy^2U, \beta_{(2,4)} \tau_3^e U = x\tau_3^e U, \beta_{(2,4)} \tau_3^m U = x\tau_3^m U, \beta_{(2,4)} y\tau_3^e U = xy\tau_3^e U, \beta_{(2,4)} y\tau_3^m U = xy\tau_3^m U, \) \(\text{Sq}^1(\tau_3^e U) = (\tau_3^e)^2 U\) and \(\text{Sq}^1(\tau_3^m U) = (\tau_3^m)^2 U\).

The differentials \(d_1\) are induced by \(\text{Sq}^1\). Moreover, by [35], there are differentials \(d_2\) corresponding to the Bockstein homomorphism \(\beta_{(2,4)}\).

The \(E_2\) page of the Adams spectral sequence is shown in Figure 41.

---

**Figure 41:** \(\Omega^E_{\times BU(1)^s \times BU(1)^m}\)

---

| \(d\) | \(\Omega^E_{\times BU(1)^s \times BU(1)^m}\) generators |
|------|--------------------------------------------------|
| 0    | \(\mathbb{Z}_4\)                               |
| 1    | \(0\)                                          |
| 2    | \(\mathbb{Z}_4\)                               |
| 3    | \(\mathbb{Z}_4^2\) \(\tilde{\tau}_3^e \mod 4, \tilde{\tau}_3^m \mod 4\) |
| 4    | \(\mathbb{Z}_4^2\) \(\sigma \mod 4, y'^2\)   |
| 5    | \(\mathbb{Z}_4^3 \times \mathbb{Z}_4^2\) \(w_2 w_3, \tau_5^e, \tau_5^m, y' \tilde{\tau}_3^e, y' \tilde{\tau}_3^m\) |

**Table 50:** Bordism group. Here \(y'\) is the generator of \(H^2(B\mathbb{Z}_4, \mathbb{Z}_4)\), \(\sigma\) is the signature of the 4-manifold, \(w_i\) is the Stiefel-Whitney class of the tangent bundle, \(\tilde{\tau}_3\) is the generator of \(H^3(B^2U(1), \mathbb{Z})\), \(\tilde{\tau}_3 \mod 2 = \tau_3\) and \(\tau_5 = \text{Sq}^2 \tau_3\).
7.2.5 $\text{Pin}^+ \times \text{U}(1)$

We have $MT(\text{Pin}^+ \times \text{U}(1)) = M\text{Spin} \wedge \Sigma^1 M\text{TO}(1) \wedge (\text{BU}(1))_+$.

By Künneth formula,

$$H^*(\Sigma^1 M\text{TO}(1) \wedge (\text{BU}(1))_+, \mathbb{Z}_2) = H^{*-1}(M\text{TO}(1), \mathbb{Z}_2) \otimes H^*(\text{BU}(1), \mathbb{Z}_2).$$

(7.25)

For $t - s < 8$, since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}_{\mathcal{A}_2(1)}^{s,t}(H^{*-1}(M\text{TO}(1), \mathbb{Z}_2) \otimes H^*(\text{BU}(1), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\text{Pin}^+ \times \text{U}(1)}.$$  

(7.26)

The $\mathcal{A}_2(1)$-module structure of $H^{*-1}(M\text{TO}(1), \mathbb{Z}_2) \otimes H^*(\text{BU}(1), \mathbb{Z}_2)$ below degree 5 and the $E_2$ page are shown in Figure 42, 43.

![Diagram of A2-module structure](image)

Figure 42: The $\mathcal{A}_2(1)$-module structure of $H^{*-1}(M\text{TO}(1), \mathbb{Z}_2) \otimes H^*(\text{BU}(1), \mathbb{Z}_2)$ below degree 5.
Table 51: Bordism group. Here $\tilde{\eta}$ is the mod 2 index of 1d Dirac operator, Arf is the Arf invariant, $w_i$ is the Stiefel-Whitney class of the tangent bundle, $c_1$ is the Chern class of the $U(1)$ bundle, ABK is the Arf-Brown-Kervaire invariant, $\eta$ is the 4d eta invariant.

### 7.2.6 Gauge $U(1)$ end up with $\text{Pin}^+ \times U(1)^e_1 \times U(1)^m_1$ in $d = 4$

In this subsection, we restrict to $d = 4$ where the emergent magnetic symmetry is a 1-form symmetry.

We have $MT(\text{Pin}^+ \times BU(1)^e \times BU(1)^m) = M\text{Spin} \wedge \Sigma^1 MTO(1) \wedge (B^2U(1)^e \times B^2U(1)^m)_+$. 

For $t - s < 8$, since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}_{A_2(1)}^s(H^{* - 1}(MTO(1), \mathbb{Z}_2) \otimes H^*(B^2U(1)^e \times B^2U(1)^m, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_d^{\text{Pin}^+ \times BU(1)^e \times BU(1)^m}_{t-s}. \quad (7.27)$$

The $A_2(1)$-module structure of $H^{* - 1}(MTO(1), \mathbb{Z}_2) \otimes H^*(B^2U(1)^e \times B^2U(1)^m, \mathbb{Z}_2)$ below degree 5 and the $E_2$ page are shown in Figure 44, 45.
Figure 44: The $A_2(1)$-module structure of $H^{*-1}(MTO(1), Z_2) \otimes H^*(B^2U(1)^e \times B^2U(1)^m, Z_2)$ below degree 5.

Figure 45: $\Omega^*_{\text{Pin}^+ \times B(1)^e \times B(1)^m}$
| $d$ | $\Omega_{d}^{\text{Pin} \times \mathbf{B}U(1)^{r} \times \mathbf{B}U(1)^{m}}$ | generators |
|-----|-------------------------------------------------|-------------|
| 0   | $\mathbb{Z}_2$                                  |             |
| 1   | 0                                               |             |
| 2   | $\mathbb{Z}_2$                                  | $w_1 \tilde{\eta}$ |
| 3   | $\mathbb{Z}_2^2$                                | $w_1 \text{Arf}, \tau_3^e, \tau_3^m$ |
| 4   | $\mathbb{Z}_{16}$                               | $\eta$      |
| 5   | $\mathbb{Z}_2^2$                                | $w_1^2 \tau_3^e, w_1^2 \tau_3^m$ |

Table 52: Bordism group. Here $\tilde{\eta}$ is the mod 2 index of 1d Dirac operator, Arf is the Arf invariant, $\eta$ is the 4d eta invariant. $\tau_3$ is the generator of $H^3(B^2U(1), \mathbb{Z}_2)$. Note that $\tau_5 = Sq^2 \tau_3 = w_1^2 \tau_3$ on Pin$^+$ 5-manifolds.

### 7.2.7  Pin$^-$ × U(1)

We have $MT(\text{Pin}^- \times U(1)) = M\text{Spin} \wedge \Sigma^{-1} \text{MO}(1) \wedge (\mathbf{B}U(1))_+$. 

By Künneth formula,

$$H^*(\Sigma^{-1} \text{MO}(1) \wedge (\mathbf{B}U(1))_+, \mathbb{Z}_2) = H^{*+1}(\text{MO}(1), \mathbb{Z}_2) \otimes H^*(\mathbf{B}U(1), \mathbb{Z}_2).$$

(7.28)

For $t - s < 8$, since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}^{s,t}_{\mathcal{A}_2(1)}(H^{*+1}(\text{MO}(1), \mathbb{Z}_2) \otimes H^*(\mathbf{B}U(1), \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\text{Pin}^- \times U(1)}.$$ 

(7.29)

The $\mathcal{A}_2(1)$-module structure of $H^{*+1}(\text{MO}(1), \mathbb{Z}_2) \otimes H^*(\mathbf{B}U(1), \mathbb{Z}_2)$ below degree 5 and the $E_2$ page are shown in Figure 46, 47.

---

Figure 46: The $\mathcal{A}_2(1)$-module structure of $H^{*+1}(\text{MO}(1), \mathbb{Z}_2) \otimes H^*(\mathbf{B}U(1), \mathbb{Z}_2)$ below degree 5.
Figure 47: $\Omega^\text{Pin} \times U(1)$

| $d$ | $\Omega_d^\text{Pin} \times U(1)$ | generators |
|-----|----------------------------------|-------------|
| 0   | $\mathbb{Z}_2$                  |             |
| 1   | $\mathbb{Z}_2$                  | $\tilde{\eta}$ |
| 2   | $\mathbb{Z}_2 \times \mathbb{Z}_8$ | $c_1 \mod 2$, ABK |
| 3   | 0                                |             |
| 4   | $\mathbb{Z}_4$                  | $(c_1 \mod 2)(\text{ABK mod 4})$ |
| 5   | 0                                |             |

Table 53: Bordism group. Here $\tilde{\eta}$ is the mod 2 index of 1d Dirac operator, ABK is the Arf-Brown-Kervaire invariant, $c_1$ is the Chern class of the $U(1)$ bundle.

7.2.8 Gauge $U(1)$ end up with $\text{Pin}^- \times U(1)_{[1]}^e \times U(1)_{[1]}^m$ in $d = 4$

In this subsection, we restrict to $d = 4$ where the emergent magnetic symmetry is a 1-form symmetry.

We have $MT(\text{Pin}^- \times BU(1)^e \times BU(1)^m) = MSpin \wedge \Sigma^{-1} MO(1) \wedge (B^2U(1)^e \times B^2U(1)^m)_{+}$.

For $t - s < 8$, since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}^{s,t}_{\mathcal{A}_2}(H^{s+1}(MO(1), \mathbb{Z}_2) \otimes H^*(B^2U(1)^e \times B^2U(1)^m, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\text{Pin}^- \times BU(1)^e \times BU(1)^m}. \quad (7.30)$$

The $\mathcal{A}_2(1)$-module structure of $H^{s+1}(MO(1), \mathbb{Z}_2) \otimes H^*(B^2U(1)^e \times B^2U(1)^m, \mathbb{Z}_2)$ below degree 5 and the $E_2$ page are shown in Figure 48, 49.
Figure 48: The $\mathcal{A}_2(1)$-module structure of $H^{*+1}(MO(1), \mathbb{Z}_2) \otimes H^*(B^2U(1)^e \times B^2U(1)^m, \mathbb{Z}_2)$ below degree 5.

Figure 49: $\Omega^\text{Pin}^{-} \times BU(1)^e \times BU(1)^m$
Table 54: Bordism group. Here $\tilde{\eta}$ is the mod 2 index of 1d Dirac operator, ABK is the Arf-Brown-Kervaire invariant. $\tau_3$ is the generator of $H^3(B^2U(1), \mathbb{Z}_2)$.

7.2.9 $\text{Pin}^c \times \mathbb{Z}_2$

We have $MT(\text{Pin}^c \times \mathbb{Z}_2) = M\text{Spin} \wedge \Sigma^{-1}M\text{O}(1) \wedge \Sigma^{-2}M\text{U}(1) \wedge (B\mathbb{Z}_2)_+$. By Künneth formula,

$$H^*(\Sigma^{-1}M\text{O}(1) \wedge \Sigma^{-2}M\text{U}(1) \wedge (B\mathbb{Z}_2)_+, \mathbb{Z}_2) = H^{*+1}(M\text{O}(1), \mathbb{Z}_2) \otimes H^{*+2}(M\text{U}(1), \mathbb{Z}_2) \otimes H^*(B\mathbb{Z}_2, \mathbb{Z}_2).$$

(7.31)

For $t - s < 8$, since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}^{s,t}_{A_2(1)}(H^{*+1}(M\text{O}(1), \mathbb{Z}_2) \otimes H^{*+2}(M\text{U}(1), \mathbb{Z}_2) \otimes H^*(B\mathbb{Z}_2, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega_{t-s}^{\text{Pin}^c \times \mathbb{Z}_2}.$$  

(7.32)

The $A_2(1)$-module structure of $H^{*+1}(M\text{O}(1), \mathbb{Z}_2) \otimes H^{*+2}(M\text{U}(1), \mathbb{Z}_2) \otimes H^*(B\mathbb{Z}_2, \mathbb{Z}_2)$ below degree 5 and the $E_2$ page are shown in Figure 50, 51.
Figure 50: The $A_2(1)$-module structure of $H^{•+1}(MO(1), \mathbb{Z}_2) \otimes H^{•+2}(MU(1), \mathbb{Z}_2) \otimes H^{•}(B\mathbb{Z}_2, \mathbb{Z}_2)$ below degree 5.
Figure 51: $\Omega^{\text{Pin}^c \times \mathbb{Z}_2}_{s}$

| $d$ | $\Omega^d_{\text{Pin}^c \times \mathbb{Z}_2}$ | generators |
|-----|---------------------------------|------------|
| 0   | $Z_2$                           | $a$        |
| 1   | $Z_2$                           |            |
| 2   | $Z_2 \times Z_4$                | $a^2$, ABK mod 4 |
| 3   | $Z_2^2$                         | $w_2a$, $a^3$ |
| 4   | $Z_2 \times Z_4 \times Z_8$    | $c_1^2 \mod 2$, $a^2(\text{ABK} \mod 4)$, $(c_1 \mod 2)\text{ABK}$ |
| 5   | $Z_2^4$                         | $a^5$, $aw_1^4$, $w_2a^3$, $ac_1^2$ |

Table 55: Bordism group. Here $a$ is the generator of $H^1(B\mathbb{Z}_2, \mathbb{Z}_2)$, ABK is the Arf-Brown-Kervaire invariant, $w_i$ is the Stiefel-Whitney class of the tangent bundle, $c_1$ is the Chern class of the $U(1)$ bundle.

7.2.10 Pin$^\mathbb{Z}_2$

By [12], we have $MTPi^{\mathbb{Z}_2}$ = $MSpin \wedge \Sigma^2 MTO(2)$. Hence $MT(Pin^{\mathbb{Z}_2} \times \mathbb{Z}_2) = MTPi^{\mathbb{Z}_2} \wedge (B\mathbb{Z}_2)_+ = MSpin \wedge \Sigma^2 MTO(2) \wedge (B\mathbb{Z}_2)_+$.

For $t - s < 8$, since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}_{A_2(1)}^{s,t}(H^{*-2}(MTO(2), \mathbb{Z}_2) \otimes H^*(B\mathbb{Z}_2, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega^{\text{Pin}^c \times \mathbb{Z}_2}_{t-s}$$

(7.33)

By Thom isomorphism, we have

$$H^{*-2}(MTO(2), \mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2]U$$

(7.34)

where $U$ is the Thom class with $Sq^1U = w_1U$, $Sq^2U = (w_2 + w_1^2)U$. 

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We also have

\[ H^*(BZ_2, Z_2) = Z_2[a] \]  

(7.35)

where \( a \) is the generator of \( H^1(BZ_2, Z_2) \).

The \( A_2(1) \)-module structure of \( H^{*-2}(MTO(2), Z_2) \otimes H^*(BZ_2, Z_2) \) below degree 5 and the \( E_2 \) page are shown in Figure 52, 53.

Figure 52: The \( A_2(1) \)-module structure of \( H^{*-2}(MTO(2), Z_2) \otimes H^*(BZ_2, Z_2) \) below degree 5.
By [12], we have $MT\text{Pin}^{-} = M\text{Spin} \wedge \Sigma^{-2}M\text{O}(2)$. Hence $MT(\text{Pin}^{-} \times \mathbb{Z}_2) = MT\text{Pin}^{-} \wedge (B\mathbb{Z}_2)_+ = M\text{Spin} \wedge \Sigma^{-2}M\text{O}(2) \wedge (B\mathbb{Z}_2)_+$.

For $t - s < 8$, since there is no odd torsion, we have the Adams spectral sequence

$$\text{Ext}^{s,t}_{\mathbb{A}_2(1)}(H^{s+2}(MO(2), \mathbb{Z}_2) \otimes H^*(B\mathbb{Z}_2, \mathbb{Z}_2), \mathbb{Z}_2) \Rightarrow \Omega^{\text{Pin}^{-} \times \mathbb{Z}_2}_{t-s}$$

(7.36)

By Thom isomorphism, we have

$$H^{s+2}(MO(2), \mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2]U$$

(7.37)

where $U$ is the Thom class with $S^1U = w_1U$, $S^2U = w_2U$. 

7.2.11 $\text{Pin}^{-} \times \mathbb{Z}_2$

Table 56: Bordism group. Here $w_i$ is the Stiefel-Whitney class of the tangent bundle, $\tilde{\eta}$ is the mod 2 index of 1d Dirac operator, $\text{Arf}$ is the Arf invariant, $c_1$ is the Chern class of the $U(1)$ bundle. $a$ is the generator of $H^1(B\mathbb{Z}_2, \mathbb{Z}_2)$, $\text{ABK}$ is the Arf-Brown-Kervaire invariant.
We also have

\[ H^*(BZ_2, Z_2) = Z_2[a] \] (7.38)

where \( a \) is the generator of \( H^1(BZ_2, Z_2) \).

The \( A_2(1) \)-module structure of \( H^{*+2}(MO(2), Z_2) \otimes H^*(BZ_2, Z_2) \) below degree 5 and the \( E_2 \) page are shown in Figure 54, 55.

![Diagram](image)

Figure 54: The \( A_2(1) \)-module structure of \( H^{*+2}(MO(2), Z_2) \otimes H^*(BZ_2, Z_2) \) below degree 5.
\begin{center}

\begin{tabular}{c|c|c}
\hline
\text{Bordism group} & \text{$\Omega^\text{Pin} \times \mathbb{Z}_2$} & \text{generators} \\
\hline
0 & $\mathbb{Z}_2$ & \\
1 & $\mathbb{Z}_2$ & $a$ \\
2 & $\mathbb{Z} \times \mathbb{Z}_2$ & $c_1, w_1^2, a^2$ \\
3 & $\mathbb{Z}_2^2$ & $a^3, w_1^2a$ \\
4 & $\mathbb{Z}_2^2$ & $w_2^2, w_2^3a^2$ \\
5 & $\mathbb{Z}_2^2 \times \mathbb{Z}_4$ & $w_1^2a^3, w_2^3a, ?$ \\
\hline
\end{tabular}
\end{center}

Table 57: Bordism group. Here $w_i$ is the Stiefel-Whitney class of the tangent bundle, $c_1$ is the Chern class of the U(1) bundle. $a$ is the generator of $H^1(B\mathbb{Z}_2, \mathbb{Z}_2)$. $?$ is undetermined.

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