Weak Poisson structures on infinite dimensional manifolds and hamiltonian actions

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Abstract We introduce a notion of a weak Poisson structure on a manifold $M$ modeled on a locally convex space. This is done by specifying a Poisson bracket on a subalgebra $\mathcal{A} \subseteq C^\infty(M)$ which has to satisfy a non-degeneracy condition (the differentials of elements of $\mathcal{A}$ separate tangent vectors) and we postulate the existence of smooth Hamiltonian vector fields. Motivated by applications to Hamiltonian actions, we focus on affine Poisson spaces which include in particular the linear and affine Poisson structures on duals of locally convex Lie algebras. As an interesting byproduct of our approach, we can associate to an invariant symmetric bilinear form $\kappa$ on a Lie algebra $\mathfrak{g}$ and a $\kappa$-skew-symmetric derivation $D$ a weak affine Poisson structure on $\mathfrak{g}$ itself. This leads naturally to a concept of a Hamiltonian $G$-action on a weak Poisson manifold with a $\mathfrak{g}$-valued momentum map and hence to a generalization of quasi-hamiltonian group actions.

1 Introduction

In geometric mechanics symplectic and Poisson manifolds form the basic underlying geometric structures on manifolds. In the finite dimensional context, this provides a perfect setting to model systems whose states depend on finitely many parameters ([MR99]). In the context of symplectic geometry, resp., Hamiltonian flows, Banach manifolds were introduced by Marsden ([Mar67]), and Weinstein obtained
a Darboux Theorem for strong symplectic Banach manifolds ([Wei69]). Schmid’s monograph [Sch87] provides a nice introduction to infinite dimensional Hamiltonian systems. For more recent results on Banach–Lie–Poisson spaces we refer to the recent work of Ratiu, Odzijewicz and Beltita ([OdR03, OdR04], [BR05], [OdR08], [Ra11]) and in particular for [Gl08] for certain classes of locally convex spaces.

In the present note we describe a possible approach to Poisson structures on infinite dimensional manifolds that works naturally for smooth manifolds modeled on locally convex spaces, such as spaces of test functions, smooth sections of bundles and distributions ([Ha82], [Ne06]). Our requirements are minimal in the sense that any other concept of an infinite dimensional Poisson manifold should at least satisfy our requirements.

In the finite dimensional case, the main focus of the theory of Poisson manifolds lies on the Poisson tensor $\Lambda$ which is a section of the vector bundle $\Lambda^2(T(M))$ and defines a skew-symmetric form on each cotangent space $T^*_m(M)$. This does not generalize naturally to infinite dimensional manifolds because continuous bilinear maps may be of infinite rank. Our main point is to define a weak Poisson structure on a smooth manifold $M$ by a Poisson bracket $\{\cdot,\cdot\}$ on a unital subalgebra $\mathcal{A} \subseteq C^\infty(M)$ satisfying the Leibniz rule and the Jacobi identity. In addition to that, we require that $\mathcal{A}$ is large in the sense that, for every $m \in M$, the differentials $dF(m)$, $F \in \mathcal{A}$, separate the points in the tangent space $T_m(M)$. We also require for each $H \in \mathcal{A}$ the existence of a smooth Hamiltonian vector field $X_H$ determined by $\{F,H\} = X_H F$ for every $F \in \mathcal{A}$. The main difference to the traditional approaches is that we do not require the Poisson bracket to be defined on all smooth functions, instead we restrict the class of admissible differentials to define Poisson brackets. It turns out that this rather algebraic approach is strong enough to capture the main formal features of momentum maps and affine Poisson structures on locally convex space as well as their relations with Lie algebras and their duals. In the affine case $M = V$, the minimal choice of $\mathcal{A}$ is the subalgebra generated by a point separating subspace $V_\epsilon$ of the topological dual space $V'$. In this context one can also enlarge the algebra $\mathcal{A}$ by adding certain exponential functions and extend the Poisson bracket appropriately; see [Wa13] for such constructions.

Although our approach largely ignores geometric difficulties we hope that it provides a natural language for dealing with Poisson structures on rather general infinite dimensional manifolds and that this leads to precise specifications of the key difficulties arising for concrete examples. A discussion of similar structures is used in the context of hydrodynamics ([Ko07]) and for free boundary problems ([L*86]).

One of our main objectives was to understand the nature of the affine Poisson structures arising implicitly on Lie algebras of smooth loops in the context of Hamiltonian actions of loop groups and quasihamiltonian actions [AMM98] (Section 4).

Although the construction of the tangent bundle $T(M)$ of a locally convex manifold $M$ and the Lie algebra $\mathfrak{V}(M)$ of smooth vector fields on $M$ follows pretty much the constructions from finite dimensional geometry (cf. [HN11, Ch. 8]), serious difficulties arise when one wants to put a smooth manifold structure on the cotangent

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1 A symplectic form $\omega$ on $M$ is called strong if, for every $p \in M$, every continuous linear functional on $T_p(M)$ is of the form $\omega_p(v, \cdot)$ for some $v \in T_p(M)$.
bundle $T'(M) := \bigcup_{p \in M} T_p(M)'$ whose elements are continuous linear functionals on the tangent spaces $T_p(M)$ of $M$. This works well for Banach manifolds when the dual spaces carry the norm topology, but if $M$ is not modeled on a Banach space, there may not be any topology for which the natural chart changes for $T'(M)$ are smooth. Accordingly, cotangent bundles can be constructed naturally if $M$ is a locally convex space or if the tangent bundle $T(M)$ is trivial, in which case $T'(M) \cong M \times V$ leads to $T'(M) \cong M \times V'$, so that any locally convex topology on $V'$ leads to a manifold structure on $T'(M)$. This works in particular for Lie groups.

Since our main concern is with the algebraic framework for Poisson structures, we do not go into analytical aspects of symplectic leaves which are already subtle for Poisson manifolds not modeled on Hilbert spaces ([BR05, BRT07, Ra11]).

The structure of this paper is as follows. In Section 2 we introduce the notion of a weak Poisson manifold and discuss various types of examples, in particular affine ones and weak symplectic manifolds. We also take a brief look at Poisson maps arising from inclusions of submanifolds and from submersions. In Section 3 we then turn to momentum maps, which we consider as Poisson morphisms into affine Poisson spaces which arise naturally as subspaces of the dual of a Lie algebra $\mathfrak{g}$. If $\mathfrak{g}$ is the Lie algebra of a Lie group, we also have a global structure coming from the corresponding coadjoint action, but unfortunately there need not be any locally convex topology on $\mathfrak{g}'$ for which the coadjoint action is smooth.

As an interesting byproduct of our approach, one can use an invariant symmetric bilinear form $\kappa$ and a $\kappa$-skew-symmetric derivation $D$ on a Lie algebra $\mathfrak{g}$ to obtain a weak affine Poisson structure on $\mathfrak{g}$ itself. This leads naturally to a concept of a Hamiltonian $G$-action on a weak Poisson manifold with a $\mathfrak{g}$-valued momentum map. For the classical case where $G$ is the loop group $L(K) = C^\infty(S^1, K)$ of a compact Lie group and the derivation is given by the derivative, we thus obtain the affine action on $\mathfrak{g} = \mathfrak{L}(\mathfrak{t})$ which corresponds to the natural action of the gauge group $\mathfrak{L}(K)$ on gauge potentials on the trivial $K$-bundle over $S^1$. At this point we obtain a natural concept of a Hamiltonian $\mathfrak{L}(K)$-space generalizing the one used in the context of quasi-hamiltonian $K$-spaces, where it is only defined for weak symplectic manifolds ([AMM98], [Me08]).

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### 2 Infinite dimensional Poisson manifolds

In this section we introduce the concept of a weak Poisson structure on a locally convex manifold. Our requirements are minimal in the sense that any other concept of an infinite dimensional Poisson manifold should at least satisfy our requirements. The concept discussed below is strong enough to capture the main algebraic features of momentum maps and the Poisson structure on the dual of a Lie algebra.
2.1 Locally convex manifolds

We first recall the basic concepts concerning infinite dimensional manifolds modeled on locally convex spaces. Throughout these notes all topological vector spaces are assumed to be Hausdorff.

Let $E$ and $F$ be locally convex spaces, $U \subseteq E$ open and $f: U \to F$ a map. Then the derivative of $f$ at $x$ in the direction $h$ is defined as

$$df(x)(h) := \left( \frac{d}{dt} \right)_{t=0} f(x + th) = \lim_{t \to 0} \frac{1}{t} (f(x + th) - f(x))$$

whenever it exists. The function $f$ is called differentiable at $x$ if $df(x)(h)$ exists for all $h \in E$. It is called continuously differentiable, if it is differentiable at all points of $U$ and

$$df: U \times E \to F, \quad (x, h) \mapsto df(x)(h)$$

is a continuous map. The map $f$ is called a $C^k$-map, $k \in \mathbb{N} \cup \{\infty\}$, if it is continuous, the iterated directional derivatives

$$d^j f(x)(h_1, \ldots, h_j) := \left( \frac{\partial}{\partial t} \right)_{t=0}^{j} f(x + th_1 \cdots th_j)$$

exist for all integers $1 \leq j \leq k$, $x \in U$ and $h_1, \ldots, h_j \in E$, and all maps $d^j f: U \times E^j \to F$ are continuous. As usual, $C^\infty$-maps are called smooth.

Once the concept of a smooth function between open subsets of locally convex spaces is established, it is clear how to define a locally convex smooth manifold. The tangent bundle $T(M)$ and the Lie algebra $\mathcal{Y}(M)$ of smooth vector fields on $M$ are now defined as in the finite dimensional case (cf. [HN11, Ch. 8]) and differential $p$-forms are defined as smooth functions on the $p$-fold Whitney sum $T(M)^p$. Although it is clear what the cotangent bundle is as a set, namely the disjunct union $T^*(M) := \bigcup_{p \in \mathbb{N}} T_p(M)^*$ of the topological dual spaces of the tangent spaces, in general it is not clear how to put a smooth manifold structure on $T^*(M)$. This is due to the fact that the dual $V^*$ of the model space $V$ need not carry a locally convex topology for which the chart changes for $T^*(M)$ are smooth. For a Banach manifold this works with the natural Banach space structure on the dual, and it also works for manifolds with a single chart and the weak-$*$-topology on the dual, but for general locally convex manifolds $M$ there seems to be no natural smooth structure on $T^*(M)$ (see [Ne06, Ha82] for more details).

2.2 Weak Poisson manifolds

Definition 2.1. Let $M$ be a smooth manifold modeled on a locally convex space. A weak Poisson structure on $M$ is a unital subalgebra $\mathcal{A} \subseteq C^\infty(M, \mathbb{R})$, i.e., it contains the constant functions and is closed under pointwise multiplication, with the following properties:

(P1) $\mathcal{A}$ is endowed with a Poisson bracket $\{\cdot, \cdot\}$, this means that it is a Lie bracket, i.e.,

$$\{F, G\} = -\{G, F\}, \quad \{F, \{G, H\}\} = \{\{F, G\}, H\} + \{G, \{F, H\}\}.$$  \hspace{1cm} (J)

and it satisfies the Leibniz rule

$$\{F, GH\} = \{F, G\}H + G\{F, H\}.$$  \hspace{1cm} (L)

(P2) For every $m \in M$ and $v \in T_m(M)$ satisfying $dF(m)v = 0$ for every $F \in \mathcal{A}$ we have $v = 0$.

(P3) For every $F \in \mathcal{A}$, there exists a smooth vector field $X_F \in \mathcal{Y}(M)$ with $X_FF = \{F, H\}$ for $F, H \in \mathcal{A}$. It is called the corresponding Hamiltonian vector field.

If (P1-3) are satisfied, then we call the triple $(M, \mathcal{A}, \{\cdot, \cdot\})$ a weak Poisson manifold.
Remark 2.2. (a) (P2) implies that the vector field $X_H$ in (P3) is uniquely determined by the relation 
\[ \{ F, H \}(m) = [X_H F](m) = d F(m) X_H (m) \] for every $F \in \mathcal{A}$.

(b) For $F, G, H \in \mathcal{A}$,
\[ [X_F, X_G] H = \{ \{ H, G \}, F \} - \{ \{ H, F \}, G \} = \{ H, \{ G, F \} \} = X_{\{ G, F \}} H, \]
so that
\[ [X_F, X_G] = X_{\{ G, F \}} \quad \text{for} \quad F, G \in \mathcal{A}. \tag{1} \]

We also note that the Leibniz rule leads to
\[ X_{FG} = FX_G + GX_F \quad \text{for} \quad F, G \in \mathcal{A}. \tag{2} \]

(c) If $\{ \cdot, \cdot \}: \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is a skew-symmetric bracket satisfying the Leibniz rule, then the Jacobiator
\[ J(F, G, H) := \{ F, \{ G, H \} \} + \{ G, \{ H, F \} \} + \{ H, \{ F, G \} \} - \{ \{ F, G \}, H \} \]
defines an alternating map $\mathcal{A}^3 \to \mathcal{A}$ which satisfies the Leibniz rule in every argument. It vanishes if and only if $\{ \cdot, \cdot \}$ is a Lie bracket, i.e., if (P1) is satisfied. For a subset $\mathcal{F} \subseteq \mathcal{A}$ generating $\mathcal{A}$ as a unital algebra, this observation implies that $J$ vanishes if it vanishes for $F, G, H \in \mathcal{F}$.

(d) If (P1) and (P2) are satisfied, then (2) implies that the subspace of all elements $X \in \mathcal{A}$ for which $X_H$ as in (P3) exists is a subalgebra with respect to the pointwise product. Therefore it suffices to verify (P3) for a generating subset $\mathcal{F} \subseteq \mathcal{A}$.

Remark 2.3. From (P3) it follows that the value of the Poisson bracket
\[ \{ F, G \}(p) = dF(p)X_G(p) = -dG(p)X_F(p) \]
in $p \in M$ only depends on $dF(p)$, resp., $dG(p)$. On the separating subspace
\[ T_p(M)_* := \{ dF(p) : F \in \mathcal{A} \} \subseteq T_p(M)' \]
we thus obtain a well-defined skew-symmetric bilinear map
\[ A_p : T_p(M)_* \times T_p(M)_* \to \mathbb{R}, \quad A_p(\alpha, \beta) := \{ F, G \}(p) \quad \text{for} \quad \alpha = dF(p), \beta = dG(p). \]
This suggests an extension of the Poisson bracket to the subalgebra $\mathcal{B} \subseteq C^\infty(M)$ of those functions $F$, for which $dF(p) \in T_p(M)_*$ holds for every $p \in M$, by the formula
\[ \{ F, G \}(p) := A_p(\alpha, \beta) = A_p(\alpha)(dG(p)). \]
At this point it is not clear that this results in a smooth function $\{ F, G \}$ nor that, for $G \in \mathcal{B}$, there exists a smooth vector field $X_G$ on $M$ such that $\{ F, G \} \equiv X_G F$ holds for $F \in \mathcal{B}$ (cf. Example 2.13 below for criteria). If both these conditions are satisfied and, in addition, the Poisson bracket on $\mathcal{B}$ satisfies the Jacobi identity, then we can also work with the larger algebra $\mathcal{B}$ instead of $\mathcal{A}$.

Remark 2.4. Suppose that $M$ is a Banach manifold. The notion of a Banach–Poisson manifold used in [OdR08, Ra11] differs from our concept of a weak Poisson structure on $M$ in the sense that it is required that $\mathcal{A} = C^\infty(M)$ and that every continuous linear functional on the dual space $T_p(M)'$ of the form $\alpha^\#: A_p(\alpha, \cdot) \in T_p(M)'$ can be represented by an element of $T_p(M)$.

Remark 2.5. Let $(M, \mathcal{A}, \{ \cdot, \cdot \})$ be a weak Poisson manifold. For $p \in M$, we call
\[ C_p(M) := \{ X_F(p) : F \in \mathcal{A} \} \subseteq T_p(M) \]
the characteristic subspace in $p$. Then
\[ \omega_p : C^p(M) \times C^p(M) \to \mathbb{R}, \quad \omega_p(X_F(p),X_G(p)) := \{F,G\}(p) = dF(p)X_G(p) = -dG(p)X_F(p) \]

is a well-defined skew-symmetric form. On the Lie algebra

\[ \text{ham}(M,\mathfrak{X}) := \{X_F : F \in \mathfrak{X}\} \subseteq \mathcal{Y}(M) \]

of Hamiltonian vector fields, every form \( \omega_p \) defines a 2-cocycle

\[ \bar{\omega}_p(X,Y) := \omega_p(X(p),Y(p)) \]

because

\[ \bar{\omega}_p([X_F,X_G],X_H) = \bar{\omega}_p(X_{\{G,F\}},X_H) = \{\{G,F\},H\}(p) \]

and \( \{\cdot,\cdot\} \) satisfies the Jacobi identity.

### 2.3 Examples of weak Poisson manifolds

We now turn to natural examples of weak Poisson manifolds.

**Example 2.6.** (Finite dimensional Poisson manifolds) Every finite dimensional (paracompact) Poisson manifold \( (M,\Lambda) \) carries a natural weak Poisson structure with \( \mathfrak{X} := C^\infty(M) \) and \( \{F,G\}(m) := \Lambda_m(dF(m),dG(m)) \). Then \( T_0(M)^* = \{dF(m) : F \in \mathfrak{X}\} \) implies (P2) and the existence of \( \chi_H \in \mathcal{Y}(M) \) follows from the fact that every derivation of the algebra \( C^\infty(M) \) is of the form \( F \mapsto XF \) for some smooth vector field \( X \in \mathcal{Y}(M) \) ([HN11, Thm. 8.4.18]).

**Remark 2.7.** Let \( V \) be a real vector space. We call a linear subspace \( V_\alpha \subseteq V^* \) separating if \( \alpha(v) = 0 \) for every \( \alpha \in V_\alpha \) implies \( v = 0 \). This implies that, for every finite dimensional subspace \( F \subseteq V \), the restriction map \( V_\alpha \to F^* \) is surjective, and this in turn implies that the natural map \( S(V) \to \mathbb{R}^V \) of the symmetric algebra \( S(V) \) over \( V \) to the algebra of functions on \( V \) is injective.

**Theorem 2.8.** (Affine Poisson structures) Let \( V \) be a locally convex space and \( V_\alpha \subseteq V^* \) be a separating subspace. Further, let

(a) \( \Lambda : V_\alpha \times V_\beta \to \mathbb{R} \) be a skew-symmetric bilinear map with the property that, for every \( \alpha \in V_\alpha \), there exists an element \( \alpha^2 \in V \) with \( \Lambda(\alpha,\alpha^2) = \beta(\alpha^2) \) for every \( \beta \in V_\beta \), and

(b) let \([\cdot,\cdot]_0\) be a Lie bracket on \( V_\alpha \) for which

(i) \( \Lambda \) is a 2-cocycle, i.e., \( \Lambda(\{\alpha,\beta\},\gamma) + \Lambda([\beta,\gamma],\alpha) - \Lambda([\gamma,\alpha],\beta) = 0 \) for \( \alpha,\beta,\gamma \in V_\alpha \).

(ii) The linear maps \( \text{ad}_0 \alpha : V_\alpha \to V_\beta \) and \( \text{ad}_0 \beta : V_\beta \to V_\alpha \) have continuous adjoint maps

\[ \text{ad}_0^* \alpha : V \to V \text{ defined by } \beta(\text{ad}_0^* \alpha v) = \{\alpha,\beta\}_0(v) \text{ for } \alpha,\beta \in V_\alpha \text{ and } v \in V. \]

This leads to a Lie algebra structure on the space \( \bar{V}_\alpha := \mathbb{R}1 \oplus V_\alpha \) of affine functions on \( V \) by

\[ [t+\alpha,s+\beta] := \Lambda(\alpha,\beta) + [\alpha,\beta]_0 \quad \text{for} \quad t,s \in \mathbb{R}, \alpha,\beta \in V_\alpha. \]

Let \( \mathfrak{X} \cong S(V_\alpha) \subseteq C^\infty(V) \) denote the unital subalgebra generated by \( V_\alpha \). Then \( dF(v) \in V_\alpha \) for \( F \in \mathfrak{X} \) and \( v \in V \), and

\[ \{F,G\}(v) := \langle dF(v),dG(v)\rangle, v \quad \text{for} \quad v \in V, F,G \in \mathfrak{X} \]

defines a weak Poisson structure on \( V \).

This weak Poisson structure is affine in the sense that, for \( \alpha,\beta \in V_\alpha \), the function \( \{\alpha,\beta\} \) on \( V \) is affine.
Proof. First we observe that, for every $F \in \mathfrak{A}$ and $v \in V$, the Leibniz rule implies that the differential $dF(v)$ is contained in $V$. Therefore $\{\cdot, \cdot\}$ defines a skew-symmetric bracket $\mathfrak{A} \times \mathfrak{A} \to \mathbb{R}$ satisfying the Leibniz rule. For $\alpha, \beta \in V$, the function $\{\alpha, \beta\}$ is continuous in $V \subseteq \mathfrak{A}$, and this implies that $\{\mathfrak{A}, \mathfrak{A}\} \subseteq \mathfrak{A}$. To verify the Jacobi identity, it suffices to do this on the generating subspace $V \subseteq \mathfrak{A}$ (Remark 2.2(c)). For $\alpha, \beta, \gamma \in V$, we have $\{\alpha, \{\beta, \gamma\}\} = \{\alpha, [\beta, \gamma]\}$, so that (P1) follows from the Jacobi identity in the Lie algebra $V$. Condition (P2) follows from the fact that $V \subseteq \mathfrak{A}$ separates the points of $V$. To verify (P3), we first observe that, for $\alpha \in V$ and $F \in \mathfrak{A}$, we have

$$\{F, \alpha\}(v) = ([dF(v), \alpha], v) = \Lambda(dF(v), \alpha) + [dF(v), \alpha]_0(v) = dF(v)(\alpha^2) - dF(v)(\text{ad}_0 \alpha)^*v.$$ 

Therefore the affine vector field

$$X_\alpha(v) := \alpha^2 - (\text{ad}_0 \alpha)^*v \quad (3)$$

is a smooth vector field satisfying (P3). Now (P3) follows from an easy induction and (2) (cf. Remark 2.2(d)). This completes the proof. $\square$

Specializing to the two particular cases $\{\cdot, \cdot\}_0 = 0$ and $A = 0$, we obtain constant, resp., linear Poisson structures as special cases.

**Corollary 2.9.** (Constant Poisson structures) Let $V$ be a locally convex space, $V \subseteq V'$ be a separating subspace and $\Lambda : V \times V \to \mathbb{R}$ be a skew-symmetric bilinear map with the property that, for every $\alpha \in V$, there exists an element $\alpha^2 \in V$ with $\Lambda(\beta, \alpha) = \beta \times \alpha^2$ for every $\beta \in V$. Let $\mathfrak{A} \subseteq C^\infty(V)$ denote the initial subalgebra generated by the linear functions in $V$. Then

$$\{F, G\}(v) := \Lambda(dF(v), dG(v)) \quad \text{for} \quad v \in V, F, G \in \mathfrak{A}$$

defines a weak Poisson structure on $V$.

**Example 2.10.** (Canonical Poisson structures) Let $V$ be a locally convex space and $V \subseteq V'$ be a separating subspace, endowed with a locally convex topology for which the pairing $V \times V \to \mathbb{R}$ is separately continuous. We consider the product space $W := V \times V_*$. Then $W_* := V_* \times V$ is a separating subspace of $W' \cong V' \times (V_*)'$,

$$\Lambda((\alpha, v), (\alpha', v')) := \alpha(v') - \alpha'(v)$$

is a skew-symmetric bilinear form on $W_*$, and for $(\alpha, v)^2 := (v, -\alpha) \in W$, we have

$$\Lambda((\alpha, v), (\alpha', v')) = ((\alpha, v), (v', -\alpha')) = \langle (\alpha, v), (\alpha', v') \rangle.$$ 

Therefore we obtain with Corollary 2.9 on $W$ a constant weak Poisson structure with $\mathfrak{A} \cong S(W_*)$ which is given on $W_* \times W_*$ by $\Lambda$.

**Corollary 2.11.** (Linear Poisson structures) Let $V$ be a locally convex space, $V \subseteq V'$ be a separating subspace and $\{\cdot, \cdot\}$ be a Lie bracket on $V$, for which the linear maps $\text{ad}_\alpha : V \to V$, have continuous adjoint maps $\text{ad}^* \alpha : V \to V$. Let $\mathfrak{A} \subseteq C^\infty(V)$ denote the initial subalgebra generated by $V_*$. Then

$$\{F, G\}(v) := \langle dF(v), dG(v) \rangle, v \quad \text{for} \quad v \in V, F, G \in \mathfrak{A}$$

defines a weak Poisson structure on $V$.

For a version of the preceding corollary for Banach spaces, we refer to [Ra11, Thm. 3.2] and [OdR03]. In this context $V$ is a Banach space and $V_* := V'$ is the dual Banach space. Typical examples of Banach–Lie–Poisson space are the duals of $C^*$-algebras and preduals of $W^*$-algebras. Here the example of the space $V = \text{Hermitian}(\mathcal{H})$ of hermitian trace class operators on a Hilbert space $\mathcal{H}$ is of particular importance in Quantum Mechanics. By the trace pairing, its dual can be identified with the Lie algebra of skew-hermitian compact operators.
Remark 2.12. In the context of Theorem 2.8 one can enlarge the algebra $\mathcal{A} \subseteq C^\infty(V)$ under the following topological assumptions. We assume that $V_\ast$ carries a locally convex topology for which

(A1) the pairing $(\cdot, \cdot): V_\ast \times V \rightarrow \mathbb{R}$ is continuous,

(A2) the Lie bracket $[\cdot, \cdot]: V_\ast \times V_\ast \rightarrow V_\ast$ is continuous,

(A3) the map $V_\ast \times V \rightarrow V_\ast (\alpha, v) \mapsto (ad_\alpha)^* v$ is continuous, and

(A4) the map $\sharp: V_\ast \rightarrow V$ is continuous.

Then

$$\mathcal{B} := \{ F \in C^\infty(V): \ dF \in C^\infty(V, V_\ast) \}$$

is a subalgebra of $C^\infty(V)$ with respect to the pointwise multiplication. For $F, G \in \mathcal{B}$, the function

$$\{F, G\}(v) := [dF(v), dG(v)](v) = \langle [dF(v), dG(v)]_0, v \rangle + \Lambda(v, dF(v), dG(v))$$

is smooth and so is the vector field

$$X_G(v) = -(ad_0 dG(v))^* v + dG(v)^2$$

on $V$ (cf. (3)) which satisfies

$$\{F, G\} = X_G F = \langle dF, X_G \rangle \quad \text{and} \quad \{\alpha, X_G(v)\} = \langle [\alpha, dG(v)], v \rangle \quad \text{for} \ \alpha \in V_\ast, v \in V.$$

For every $F \in \mathcal{B}$, we now identify $d^2 F$ with a smooth function $\tilde{d}^2 F: V \times V \rightarrow V_\ast$ which is linear in the second argument. The symmetry of the second derivative then leads to the relation

$$d^2 F_v (w, u) = \langle \tilde{d}^2 F_v (w), u \rangle = \langle \tilde{d}^2 F_v (u), w \rangle.$$

We now show that $\{F, G\} \in \mathcal{B}$. The calculation

$$d\{F, G\}(v)(h) = [d^2 F_v (h), dG(v)](v) + [dF(v), d^2 G_v (h)](v) + \langle [dF(v), dG(v)]_0, h \rangle$$

$$= \langle \tilde{d}^2 F_v (X_G(v), h) - d^2 G_v (X_F(v), h) + \langle [dF(v), dG(v)]_0, h \rangle,$$

shows that

$$d\{F, G\}(v) = \tilde{d}^2 F_v (X_G(v)) - d^2 G_v (X_G(v)) + [dF(v), dG(v)]_0$$

is a smooth $V_\ast$-valued function. Therefore the Poisson bracket extends to $\mathcal{B}$. From

$$\langle [dF(v), dG(v)]_0, X_H(v) \rangle = \langle [dF(v), dG(v)]_0, -(ad_0 dH(v))^* v \rangle + dH(v)^2$$

$$\quad = \langle [dF(v), dG(v)]_0, dH(v) \rangle + \Lambda(v, dF(v), dG(v))$$

we now derive

$$\{\{F, G\}, H\}(v) = \tilde{d}^2 F_v (X_H(v), X_G(v)) - d^2 G_v (X_H(v), X_F(v)) + \langle [dF(v), dG(v)], dH(v) \rangle.$$

Now the symmetry of the second derivative implies that the Poisson bracket on $\mathcal{B}$ satisfies the Jacobi identity, so that $(V, \mathcal{B}, \{\cdot, \cdot\})$ also is a weak Poisson structure on $V$.

If $V$ is a Banach space with $V_\ast = V'$ (in particular if dim $V < \infty$), then the preceding construction actually leads to all smooth functions $\mathcal{B} = C^\infty(V)$, so that we are in the context of Banach–Lie–Poisson spaces. However, one can do better:

Remark 2.13. (Glockner’s locally convex Poisson vector spaces) To obtain Poisson structures on $V$ for the algebra $\mathcal{A} = C^\infty(V)$ of all smooth functions, one has to impose stronger assumptions on topologies on $V'$ and $V_\ast$. In [Gl08, Def. 16.35] these are encoded in the concept of a locally convex Poisson vector space, which requires that the locally convex space $V$ has the following properties:
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(a) For the topology of uniform convergence on compact \((S = c)\), resp., bounded \((S = b)\) subsets of \(V\) (or even more general classes \(S\) of subsets) the linear injection \(\eta_V : V \to \bigcap_{\alpha \in \mathcal{S}} \eta_V(\alpha) = \alpha(v)\) is a topological embedding.
(b) The topology on every product space \(V^n\) is determined by its restriction to compact subsets \((V\) is a \(k^n\) space).
(c) The dual space \(V'_\circ\) carries an \(S\)-hypocontinuous Lie bracket \([\cdot, \cdot]\), i.e., it is separately continuous and continuous on all subsets of the form \(V'_\circ \times B, B \in S\).
(d) The Lie bracket on \(V'_\circ\) satisfies \(\eta_V(\alpha) \circ \text{ad}_\alpha \in \eta_V(V)\) for \(\alpha \in V'_\circ\).

If these conditions are satisfied, then \([\text{GI08, Thm. 16.40}]\) asserts that, for two smooth functions \(F, G \in C^\omega(V)\), their Poisson bracket
\[
\{F, G\}(v) := \langle [dF(v), dG(v)], v \rangle
\]
is smooth and that
\[
X_{\{F, G\}}(v) := -\eta_V^{-1}(\eta_V(\alpha) \circ \text{ad}(dF(v)))
\]
is a smooth vector field satisfying \(\{G, F\} = X_{\{F, G\}}\). As in the preceding remark it now follows that \(\{V, C^\omega(V), \{\cdot, \cdot\}\}\) is a weak Poisson structure on \(V\). This is the special case of Corollary 2.11, where \(V' = V'_\circ\).

Example 2.14. (a) Let \(\mathfrak{g}\) be a locally convex Lie algebra, i.e., a locally convex space with a continuous Lie bracket. We write \(\mathfrak{g}'\) for its topological dual space, endowed with the weak-*topology. Then Corollary 2.11 applies to \(V := \mathfrak{g}'\) and \(V_* := \mathfrak{g}\) because, for each \(X \in \mathfrak{g}\), the bracket map \(\text{ad}X : \mathfrak{g} \to \mathfrak{g}\) has a continuous adjoint \(\text{ad}^* : \mathfrak{g}' \to \mathfrak{g}'\). If \(\mathfrak{g}\) is finite dimensional, we thus obtain the KKS (Kirillov–Kostant–Souriau) Poisson structure on \(\mathfrak{g}' = \mathfrak{g}'\).

(b) The preceding construction can be varied by changing the topology on \(\mathfrak{g}'\) and by passing to a smaller subspace. Let \(\mathfrak{g}_s \subseteq \mathfrak{g}'\) be a separating subspace on which the adjoint maps \(\text{ad}^* X \alpha := \alpha \circ \text{ad}X\) induce for each \(X \in \mathfrak{g}\) a continuous linear map. Then Corollary 2.11 applies with \(V := \mathfrak{g}_s\) and \(V_* := \mathfrak{g}\), and we thus obtain a weak Poisson structure on \(\mathfrak{g}_s\) for which the Hamiltonian functions \(H_X(\alpha) = \alpha(X)\) satisfy
\[
\{H_X, H_Y\} = H_{[X,Y]} \quad \text{for} \quad X, Y \in \mathfrak{g}_s.
\]

(c) Suppose that \(\mathfrak{g}\) is a locally convex Lie algebra and \(K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}\) is a continuous non-degenerate symmetric bilinear form which is invariant under the adjoint representation, i.e.,
\[
K([x, y], z) + K(y, [x, z]) = 0 \quad \text{for} \quad x, y, z \in \mathfrak{g}.
\]

Then the natural map
\[
\hat{\cdot} : \mathfrak{g} \to \mathfrak{g}' = X^\omega(\mathfrak{g}) = \kappa(X, Y)
\]
is injective and \(\mathfrak{g}\)-equivariant with respect to the adjoint and coadjoint representation, respectively. We may thus apply (b) with \(\mathfrak{g}_s := \mathfrak{g}' = \{X^\omega : X \in \mathfrak{g}\} \cong \mathfrak{g}\) to obtain a linear weak Poisson structure on \(\mathfrak{g}\) with \(\mathfrak{g}' \cong S(\mathfrak{g})\). The Hamiltonian functions \(X^\omega(Y) = \kappa(X, Y)\) satisfy
\[
\{X^\omega, Y^\omega\} = [X, Y]_{\mathfrak{g}} \quad \text{for} \quad X, Y \in \mathfrak{g}.
\]

(d) Let \(\mathfrak{g}\) be a locally convex Lie algebra and \(\omega : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}\) be a continuous 2-cocycle, i.e.,
\[
\omega([X, Y], Z) + \omega([Y, Z], X) + \omega([Z, X], Y) = 0,
\]
so that \(\hat{\mathfrak{g}} = \mathbb{R} \oplus \mathfrak{g}\) is a locally convex Lie algebra with respect to the Lie bracket
\[
[(t, X), (s, Y)] := (\omega(X, Y), [X, Y]).
\]
We call it the central extension defined by \(\omega\). Identifying the element \((t, X) \in \hat{\mathfrak{g}}\) with the affine function \(\alpha \mapsto t + \alpha(X)\) on \(\mathfrak{g}'\), we obtain with Theorem 2.8 (for \(V = \mathfrak{g}'\) and \(V_* = \mathfrak{g}\)) an affine weak Poisson structure on \(\mathfrak{g}'\), for which the Hamiltonian functions \(H_X(\alpha) = \alpha(X), X \in \mathfrak{g}\), satisfy
\[ \{H_X, H_Y\} = H_{[X,Y]} + \omega(X, Y) \quad \text{for} \quad X, Y \in \mathfrak{g}. \]

The assumptions of Theorem 2.8 are satisfied with \( \Lambda = \omega \).

More generally, suppose that \( \mathfrak{g}_0 \subseteq \mathfrak{g}' \) is subspace separating the points of \( \mathfrak{g} \) and on which the adjoint maps \( \text{ad}^* X, \ X \in \mathfrak{g} \), induce continuous endomorphisms. Assume further that it contains all functionals \( i_X \omega, X \in \mathfrak{g} \). Then Theorem 2.8 yields an affine weak Poisson structure on \( \mathfrak{g}_0 \) with

\[ \{H_X, H_Y\} = H_{[X,Y]} + \omega(X, Y) \quad \text{for} \quad X, Y \in \mathfrak{g}. \]

(c) To combine (c) and (d), we assume that, in addition to \( \mathfrak{g} \) and \( \kappa \) as in (c), we are given a \( \kappa \)-skew symmetric continuous derivation \( D : \mathfrak{g} \to \mathfrak{g} \), so that \( \omega(X, Y) = \kappa(DX, Y) \) is a 2-cocycle. Then we obtain an affine weak Poisson structure \( \{\mathcal{A}, \{\cdot, \cdot\}_D\} \) on \( \mathfrak{g} \) with \( \mathcal{A} \equiv S(\mathfrak{g}) \). The Hamiltonian functions \( X^\alpha(Y) := \kappa(X, Y) \) satisfy

\[ \{X^\alpha, Y^\beta\}_D = [X, Y] + \kappa(DX, Y) \quad \text{for} \quad X, Y \in \mathfrak{g}. \]

An important concrete class of examples to which the preceding constructions apply arise from loop algebras. We shall return to this example later, when we connect with Hamiltonian actions of loop groups (cf. Definition 4.3).

Example 2.15. Let \( \mathfrak{t} \) be a Lie algebra which carries a non-degenerate invariant symmetric bilinear form \( \langle \cdot, \cdot \rangle \). Then the loop algebra of \( \mathfrak{t} \) is the Lie algebra \( \mathfrak{g} := \mathcal{L}(\mathfrak{t}) := C^\infty(\mathbb{S}^1, \mathfrak{t}) \), endowed with the pointwise bracket. We identify the circle \( \mathbb{S}^1 \) with \( \mathbb{R}/\mathbb{Z} \) and, accordingly, elements of \( \mathfrak{g} \) with 1-periodic functions on \( \mathbb{R} \). Then \( \kappa(\xi, \eta) = \int_0^1 \langle \xi(t), \eta(t) \rangle \, dt \) is a non-degenerate invariant symmetric bilinear form on \( \mathfrak{g} \) and \( D\xi = \xi' \) is a skew-symmetric derivation. We thus obtain on \( \mathfrak{g} \) with Example 2.14(e) an affine weak Poisson structure with

\[ \{\xi^\alpha, \eta^\beta\} = \kappa(D\xi^\alpha, \eta^\beta) + \int_0^1 \langle \xi^\alpha(t), \eta(t) \rangle \, dt. \]

Remark 2.16. Typical predual spaces \( \mathfrak{g}_0 \subseteq \mathfrak{g}' \) arise from geometric situations as follows (cf. [KW09]):

(a) If \( \mathfrak{g} = C^\infty(M, \mathfrak{t}) \), where \( \mathfrak{t} \) is finite dimensional with a non-degenerate invariant symmetric bilinear form \( \langle \cdot, \cdot \rangle \) and \( \mu \) is a measure on \( M \) which is equivalent to Lebesgue measure in charts, then we have an invariant pairing \( \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, (\xi, \eta) \mapsto \int_M \langle \xi, \eta \rangle \, d\mu \) which leads to \( \mathfrak{g}_0 \cong \mathfrak{g} \).

(b) If \( M \) is a compact smooth manifold and \( \mathfrak{g} = \mathcal{Y}(M) \), the Fréchet–Lie algebra of smooth vector fields on \( M \), then the space \( \mathfrak{g}_0 \) of density-valued 1-forms on \( M \) has a natural Diff(M) invariant pairing given by \( \langle X, \alpha \rangle \mapsto \int_M \alpha(X) \). Locally the elements of \( \mathfrak{g}_0 \) are represented by smooth 1-forms, so that \( \mathfrak{g}_0 \) is much smaller than the dual space \( \mathfrak{g}' \) whose elements are locally represented by distributions.

In finite dimensions, symplectic manifolds provide the basic building blocks of Poisson manifolds because every Poisson manifold is naturally foliated by symplectic leaves. In the infinite dimensional context the situation becomes more complicated because a symplectic form \( \omega : V \times V \to \mathbb{R} \) on a locally convex space needs not represent every continuous linear functional on \( V \). If it does, \( \omega \) is called strong, and weak otherwise. Accordingly, a 2-form \( \omega \) on a smooth manifold \( M \) is called strong if all forms \( \omega_p, p \in M \), are strong, and weak otherwise.

Definition 2.17. A weak symplectic manifold is a pair \( (M, \omega) \) of a smooth manifold \( M \) and a closed non-degenerate 2-form \( \omega \). For a weak symplectic manifold we write

\[ \text{ham}(M, \omega) := \{X \in \mathcal{Y}(M) : (\exists H \in C^\infty(M)) i_X \omega = dH\} \]

for the Lie algebra of Hamiltonian vector fields on \( M \) and

\[ \text{sp}(M, \omega) := \{X \in \mathcal{Y}(M) : \mathcal{L}_X \omega = d(i_X \omega) = 0\} \]
for the larger Lie algebra of symplectic vector fields (cf. [NV10] for related constructions).

**Proposition 2.18. (Poisson structure on weak symplectic manifolds)** Let \((M, \omega)\) be a weak symplectic manifold. Then

\[
\mathcal{A} := \{ H \in C^\infty(M) : (\exists X_H \in \mathfrak{X}(M)) \ dH = i_{X_H} \omega \}
\]

is a unital subalgebra of \(C^\infty(M)\) and

\[
\{ F, G \} := \omega(X_F, X_G) = dF(X_G) = X_G F
\]

defines on \(\mathcal{A}\) a Poisson bracket satisfying (P1) and (P3).

If, in addition, for \(v \in T_m(M)\), the condition \(\omega(X(m), v) = 0\) for every \(X \in \text{ham}(M, \omega)\), implies \(v = 0\), then (P2) is also satisfied.\(^2\)

**Proof.** Since \(\omega\) is non-degenerate, the vector field \(X_H\) is uniquely determined by \(H\). For \(F, G \in \mathcal{A}\) we have

\[
d(FG) = F dG + G dF = i_{X_G i_{X_F}} \omega,
\]

which implies that \(\mathcal{A}\) is a unital subalgebra of \(C^\infty(M)\).

The closedness of the 1-forms \(i_{X_H} \omega\) implies that \(\mathcal{L}_{X_H} \omega = 0\). Further, \([\mathcal{L}_{X_F}, i_{X_G}] = i_{X_{[F,G]}}\] leads to

\[
i_{X_{[F,G]}} \omega = [\mathcal{L}_{X_F}, i_{X_G}] \omega = \mathcal{L}_{X_F} (i_{X_G} \omega) = \mathcal{L}_{X_F} dG = d(i_{X_F} dG) = d\{i_{X_F} dG\} = d\{G, F\}.
\]

Since \(\omega\) is non-degenerate, this implies \(\{\mathcal{A}, \mathcal{A}\} \subseteq \mathcal{A}\) with

\[
[X_F, X_G] = X_{\{G,F\}} \quad \text{for} \quad F, G \in \mathcal{A}. \quad (4)
\]

It is clear that \(\{\cdot, \cdot\}\) is bilinear and skew-symmetric, and from \(d(FG) = F dG + G dF\) we conclude that it satisfies the Leibniz rule. So it remains to check the Jacobi identity. This is an easy consequence of (4):

\[
\{ F, \{ G, H \} \} = X_{\{G,H\} F} = -\{ X_G, X_H F \} = -\{ X_G i_{X_H} F \} + X_H (X_G F) = \{ G, \{ F, H \} \} + \{ \{ F, G \}, H \}.
\]

We have thus verified (P1) and (P3). For (P2) we further need that, for every \(v \in T_m(M)\), the condition that \(\omega(X(m), v) = 0\) for every \(X \in \text{ham}(M, \omega)\) implies \(v = 0\). \(\square\)

**Example 2.19.** If \((V, \omega)\) is a symplectic vector space, then a linear functional \(\alpha : V \to \mathbb{R}\) is contained in the Poisson algebra \(\mathcal{A}\) if and only if there exists a vector \(v \in V\) with \(i_v \omega = \alpha\). Then 

\[
H_v = \alpha = i_v \omega
\]

is the Hamiltonian function of the constant vector field \(v\). Accordingly, the Poisson structure on \(V\) is determined by

\[
\{ H_v, H_w \} = dH_v(w) = \omega(v, w) \quad \text{for} \quad v, w \in V. \quad (5)
\]

Here (P2) follows from the non-degeneracy of \(\omega\).

\(^2\) This condition is satisfied for finite dimensional symplectic manifolds, for strongly symplectic smoothly paracompact Banach manifolds (cf. [KM97]) and for symplectic vector spaces.
2.4 Poisson maps

It is now clear how to define the notion of a Poisson map between two weak Poisson manifolds. Here we take a closer look at Poisson maps arising from inclusions of submanifolds and from submersions which correspond to regular Poisson reduction. In the context of Hamiltonian actions, Poisson maps to weak affine Poisson space arise as momentum maps.

Definition 2.20. Let \((M_j, \mathcal{A}_j, \{\cdot, \cdot\}_j), j = 1, 2,\) be weak Poisson manifolds. A smooth map \(\varphi : M_1 \to M_2\) is called a Poisson map, or morphism of Poisson manifolds, if \(\varphi^* \mathcal{A}_1 \subseteq \mathcal{A}_2\) and \(\varphi^* \{F, G\} = \{\varphi^* F, \varphi^* G\}\) for \(F, G \in \mathcal{A}_2\).

Proposition 2.21. (Poisson submanifolds) Let \((M, \mathcal{A}, \{\cdot, \cdot\})\) be a weak Poisson manifold and \(N \subseteq M\) be a submanifold with the property that, for every \(F \in \mathcal{A}\), the restriction of \(X_F\) to \(N\) is tangent to \(N\). Then \(\mathcal{I}_N := \{F \in \mathcal{A} : F|_N = 0\}\) is an ideal with respect to the Poisson bracket, i.e., \(\mathcal{I}_N, \mathcal{A}\) is a Poisson ideal. So let \(\mathcal{J}_N := \mathcal{A}/\mathcal{I}_N \subseteq \mathcal{C}^\infty(N)\) defines a weak Poisson structure on \(N\) such that the inclusion \(N \hookrightarrow M\) is a morphism of weak Poisson manifolds.

Proof. First we show that \(\mathcal{J}_N\) is a Poisson ideal. So let \(F \in \mathcal{J}_N\) and \(G \in \mathcal{A}\). Then, for \(n \in N\), \(\{F, G\}(n) = dF(n)X_G(n) = 0\) because \(F\) vanishes on \(N\) and \(X_G(n) \in T_n(N)\). This implies that \(\mathcal{J}_N\) inherits the structure of a Poisson algebra by

\[
\{F|_N, G|_N\} := \{F, G\}|_N,
\]

and that (P1) is satisfied.

If \(v \in \mathcal{T}_n(N), n \in N\), satisfies \(dv(n)v = 0\) for every \(F \in \mathcal{A}_N\), then the same holds for \(F \in \mathcal{A}\), so that (P2) for \(\mathcal{A}\) implies (P2) for \(\mathcal{A}_N\).

To verify (P3), we simply observe that our assumption implies that

\[
\{F|_N, G|_N\} = \{F, G\}|_N = (X_GF)|_N = (X_G|_N)F|_N.
\]

\[\square\]

Remark 2.22. (a) Let \(\mathfrak{g}\) be a locally convex Lie algebra and endow \(\mathfrak{g}'\) with the weak Poisson structure from Corollary 2.11 above. Let \(C \in \mathfrak{g}(\mathfrak{g})\) be a central element. Then the hyperplane

\[N := \{\alpha \in \mathfrak{g}' : \alpha(C) = 1\}\]

is a submanifold of \(\mathfrak{g}'\), and for every \(F \in \mathcal{A}_{\mathfrak{g}'}\) and \(\alpha \in N\), we have

\[0 = X_F(\alpha)H_C = (X_F(\alpha), C),\]

so that \(X_F \in \mathcal{T}(N)\). Therefore the assumptions of Proposition 2.21 are satisfied, so that \(\mathcal{J}_N := \mathcal{A}/\mathcal{I}_N\) yields a weak Poisson structure on the hyperplane \(N\).

(b) The preceding restriction is of particular importance if we are dealing with a central extension \(\tilde{\mathfrak{g}} = \mathbb{R} \oplus \mathfrak{g}\) of the Lie algebra \(\mathfrak{g}\) with the bracket

\[(z, X), (z', X') = (\omega(X, X'), [X, X']),\]

where \(\omega : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}\) is a continuous 2-cocycle. Then \(C := (1, 0)\) is a central element of \(\tilde{\mathfrak{g}}\) and

\[H_C^{-1}(1) = \{1\} \times \mathfrak{g}' \subseteq \tilde{\mathfrak{g}}',\]

inherits a Poisson structure from \(\mathcal{A}_{\tilde{\mathfrak{g}}'}\). Identifying the affine space \(\mathfrak{g}'\) in the canonical fashion with the affine space \(\{1\} \times \mathfrak{g}'\), we thus obtain a weak Poisson structure on \(\mathfrak{g}'\), where \(\mathcal{A} \subseteq \mathcal{C}^\infty(\mathfrak{g}')\) is generated by the continuous affine functions, i.e., \(\mathcal{A} \cong \mathcal{S}(\mathfrak{g})\) as an associative algebra, and the Poisson bracket on \(\mathcal{A}\) is determined by
Proposition 2.23. (Smooth Poisson quotients) Let $(M, \mathfrak{g}_M, \{\cdot, \cdot\})$ be a weak Poisson manifold and $q: M \rightarrow N$ be a submersion. Then a Poisson subalgebra $\mathfrak{B} \subseteq q^*\mathfrak{g}_N \cap \mathfrak{g}_M$ is the image under $q^*$ of a weak Poisson structure on $N$ for which $q$ is a Poisson map if and only if

$$
\ker T_m(q) = \{v \in T_m(M): (\forall F \in \mathfrak{B}) \partial F(m)v = 0\}.
$$

Proof. Suppose first that $q$ is a Poisson map w.r.t. the weak Poisson structure $(\mathfrak{g}_N, \{\cdot, \cdot\})$ on $N$. Then $\mathfrak{B} := q^*\mathfrak{g}_N \subseteq \mathfrak{g}_M$ is a Poisson subalgebra and property (P2) of $\mathfrak{g}_N$ implies (6).

Suppose, conversely, that $\mathfrak{B} \subseteq q^*\mathfrak{g}_N \cap \mathfrak{g}_M$ is a Poisson subalgebra satisfying (6). Let $\mathfrak{g}_N \subseteq C^\infty(N)$ be the subalgebra with $q^*\mathfrak{g}_N = \mathfrak{B}$. Since $q^*$ is injective, $\mathfrak{g}_N$ inherits a natural Poisson algebra structure from $\mathfrak{B}$. Hence $(N, \mathfrak{g}_N, \{\cdot, \cdot\})$ satisfies (P1), and (P2) follows from (6). To see that (P3) also holds, let $f \in \mathfrak{g}_N$ and $F = q^*f \in \mathfrak{B}$. Then the corresponding Hamiltonian vector field $X_F \in T^*(M)$ satisfies for every $G = q^*g \in \mathfrak{B}$ the relation

$$
\partial g(q(m))T_m(q)X_F(m) = \partial G(m)X_F(m) = \{G, F\}(m) = \{g, f\}(q(m)).
$$

For $m' \in M$ with $q(m) = q(m')$, this leads to

$$
\partial g(q(m))T_m(q)X_F(m) = \partial g(q(m))T_{m'}(q)X_F(m')
$$

for every $g$, so that (P2) implies $T_m(q)X_F(m) = T_{m'}(q)X_F(m')$. Hence $X_F$ is projectable to a vector field $Y \in T^*_N$ which is $q$-related to $X_F$. We then have for every $g \in \mathfrak{g}_N$ the relation $\{g, F\} = Y g$, so that (P3) is also satisfied.

Remark 2.24. If, in the context of Proposition 2.23, the subalgebra $\mathfrak{B}$ is Poisson commutative, then (i) implies that the vector fields $X_F, F \in \mathfrak{B}$, are tangential to the fibers of $q$, hence projectable to 0. We thus obtain the trivial Poisson structure on $N$ for which all Poisson brackets vanish.

3 Momentum maps

We now turn to momentum maps, which we consider as Poisson morphisms to affine Poisson spaces which arise naturally as subspaces of the duals of Lie algebras $\mathfrak{g}$. If $\mathfrak{g}$ is the Lie algebra of a Lie group, we also have a global structure coming from the corresponding coadjoint action, but unfortunately there need not be any locally convex topology on $\mathfrak{g}^*$ for which the coadjoint action is smooth.

3.1 Momentum maps as Poisson morphisms

Since momentum maps are Poisson maps $\Phi: M \rightarrow V$, where $V$ carries an affine weak Poisson structure (Theorem 2.8), we start with a characterization of such maps.
Proposition 3.1. Let \((V, \mathcal{A}_V)\) be an affine Poisson manifold corresponding to a Lie algebra structure on the space \(\mathcal{A}_V = \mathcal{A}_\mathfrak{g} + \mathbb{R}1\) of affine functions on \(V\), \((M, \mathcal{A}_M)\) a weak Poisson manifold and \(\Phi : M \to V\) a smooth map such that \(\varphi(\alpha) := \Phi^* \alpha = \alpha \circ \Phi \in \mathcal{A}_M\) for every \(\alpha \in V\). Then the following are equivalent

(i) \(\Phi^* : \mathcal{A}_M \to \mathcal{A}_V\) is a homomorphism of Lie algebras, i.e., \(\Phi\) is a Poisson map.
(ii) \(\varphi : V \to \mathcal{A}_M\) satisfies \(\varphi(\{\alpha, \beta\}) = \{\varphi(\alpha), \varphi(\beta)\}\) for \(\alpha, \beta \in V\).
(iii) \(\Phi : M \to V\) satisfies the equivariance condition

\[ T_m(\Phi)X_{\varphi(\alpha)}(m) = X_{\alpha}(\Phi(m)) \quad \text{for} \quad m \in M, \alpha \in V. \quad (7) \]

Proof. (i) \(\Rightarrow\) (ii): Clearly, \(\Phi^* : \mathcal{A}_V \to \mathcal{A}_M\) is a homomorphism of commutative algebras because \(\Phi^*(V) \subseteq \mathcal{A}_M\) and \(\mathcal{A}_V\) is generated by \(V\). Let \(F, G \in \mathcal{A}_V\). For \(p \in M\) we put \(\alpha := dF_{\Phi(p)}\) and \(\beta := dG_{\Phi(p)}\), which are elements of \(V\). Then

\[ d(F \circ \Phi)_p = \alpha \circ T_p(\Phi) = d(\Phi^* \alpha)_p = (d\varphi(\alpha))_p, \]

and we thus obtain

\[ \{\Phi^* F, \Phi^* G\}(p) = d\varphi(\alpha)_pX_{\Phi^* G}(p) = \{\varphi(\alpha), \Phi^* G\}(p) = \{\varphi(\alpha), \varphi(\beta)\}(p) \]

and

\[ \varphi(\{\alpha, \beta\})(p) = \{\Phi^* \alpha, \Phi^* \beta\}(p) = \{F, G\}(\Phi(p)). \]

This proves that (ii) implies (i).

(ii) \(\Leftrightarrow\) (iii): The equivariance relation (7) is an identity for elements of \(V\). Hence it is satisfied if and only if it holds as an identity of real numbers when we apply elements of the separating subspace \(V\). This means that

\[ d\varphi(\beta)_mX_{\varphi(\alpha)}(m) = \{\beta, \alpha\}(\Phi(m)) \quad \text{for} \quad m \in M, \alpha, \beta \in V. \]

Since the left hand side equals \(\{\varphi(\beta), \varphi(\alpha)\}(m)\), this relation is equivalent to (ii). \(\square\)

The classical case of the preceding proposition is the one where \(V = g^*\) is the dual of locally convex Lie algebra, endowed with the weak-* topology.

Corollary 3.2. Let \(g\) be a locally convex Lie algebra, endow \(g^*\) with the canonical linear Poisson structure \(\mathcal{A}_{g^*}\), let \((M, \mathcal{A}_M)\) be a weak Poisson manifold and \(\Phi : M \to g^*\) be a map such that all functions \(\varphi_\alpha(m) := \Phi(m)(X)\) are contained in \(\mathcal{A}_M\). Then the following are equivalent

(i) \(\Phi^* : \mathcal{A}_{g^*} \to \mathcal{A}_M\) is a homomorphism of Lie algebras, i.e., \(\Phi\) is a Poisson map.
(ii) \(\varphi : g \to \mathcal{A}_M\) satisfies \(\varphi([X, Y]) = \{\varphi(X), \varphi(Y)\}\) for \(X, Y \in g\).
(iii) \(\Phi : M \to g^*\) satisfies the equivariance condition

\[ T_m(\Phi)X_{\varphi(X)}(m) = -\Phi(m) \circ \text{ad} X \quad \text{for} \quad m \in M, X \in g. \quad (8) \]

Remark 3.3. If we endow \(g^*\) with an affine Poisson structure corresponding to a Lie algebra cocycle \(\omega\), then the condition Corollary 3.2(ii) has to be modified to

\[ \{\varphi(X), \varphi(Y)\} = \varphi([X, Y]) + \omega(X, Y) \quad \text{for} \quad X, Y \in g. \]

Definition 3.4. An infinitesimal action of the locally convex Lie algebra \(g\) on the smooth manifold \(M\) is a Lie algebra homomorphism \(\beta : g \to \mathcal{Y}(M)\) for which all maps \(\beta_\alpha : g \to T_p(M), X \mapsto \beta(X)_m\)

are continuous.

If \((M, \mathcal{A}_M, \{\cdot, \cdot\})\) is a weak Poisson manifold, then an infinitesimal action \(\beta : g \to \mathcal{Y}(M)\) of a locally convex Lie algebra on \(M\) is said to be Hamiltonian if there exists a homomorphism \(\varphi : g \to \mathcal{A}_M\) of Lie algebras satisfying \(X_{\varphi(Y)} = -\beta(Y)\) for every \(Y \in g\). Then the map
\[ \Phi : M \to g^* , \quad \Phi(m)(Y) := \phi_Y(m) \]
is called the corresponding momentum map. Note that \(\Phi(M) \subseteq g^*\) is equivalent to the requirement that, for every \(m \in M\), the linear functional \(g \to \mathbb{R}, Y \mapsto \Phi_Y(m)\) is continuous.

**Corollary 3.5.** If \(\Phi : M \to g^*\) is a momentum map for a Hamiltonian action of \(g\) on the weak Poisson manifold \((M, \mathcal{A}_M, \{\cdot, \cdot\})\), then \(\Phi\) is a Poisson map.

**Example 3.6.** For a locally convex Lie algebra \(g\), the infinitesimal coadjoint action \(B : g \to \mathcal{Y}(g^*)\) is given by the vector fields \(B(X)(\alpha) := \alpha \circ \text{ad} X = (\text{ad} X)^* \alpha\). In view of Corollary 3.2, this action is Hamiltonian with momentum map \(\Phi = \text{id}_{g^*}\).

**Remark 3.7.** (From symplectic actions to Hamiltonian actions) Let \((M, \omega)\) be a connected weak symplectic manifold and \(\mathcal{A}\) be as in Proposition 2.18. Further, let \(\beta : g \to \text{sp}(M, \omega)\) be an infinitesimal action by symplectic vector fields (cf. Definition 2.17). For \(\beta\) to be a Hamiltonian action requires a lift of this homomorphism to a Lie algebra homomorphism

\[ \varphi : g \to (\mathcal{A}, \{\cdot, \cdot\}) \]

A necessary condition for such a lift to exist is that \(\beta(g) \subseteq \text{ham}(M, \omega)\). Even if this is the case, such a lift does not always exist. To understand the obstructions, we recall the short exact sequence

\[ 0 \to \mathbb{R} \to \mathcal{A} \to \text{ham}(M, \omega) \to 0, \]

which exhibits the Lie algebra \(\mathcal{A}\) as a central extension of the Lie algebra \(\text{ham}(M, \omega)\) (cf. [NV10] for an in depth discussion of related central extensions).

Assuming that \(\beta(g) \subseteq \text{ham}(M, \omega)\), we consider the subspace

\[ \hat{g} := \{(X, F) \in g \oplus \mathcal{A} : \beta(X) = -X_F\} \]

and observe that this is a Lie subalgebra of the direct sum \(g \oplus \mathcal{A}\). Moreover, the projection \(p(X, F) := X\) is a surjective homomorphism whose kernel consists of all pairs \((0, F)\), where \(F\) is a constant function. We thus obtain the central extension

\[ \mathbb{R} \cong \mathbb{R}(0, 1) \to \hat{g} \xrightarrow{p} g. \]

The existence of a homomorphic lift \(\varphi : g \to \mathcal{A}\) is equivalent to the existence of a splitting \(\sigma : g \to \hat{g}\). Therefore the obstruction to the existence of \(\varphi\) is a central \(\mathbb{R}\)-extension of \(g\), resp., a corresponding cohomology class in \(H^2(g, \mathbb{R})\) (cf. [Net02]).

### 3.2 Infinite dimensional Lie groups

Before we turn to momentum maps and Hamiltonian actions, we briefly recall the basic concepts underlying the notion of an infinite dimensional Lie group. A (locally convex) Lie group \(G\) is a group equipped with a smooth manifold structure modeled on a locally convex space for which the group multiplication and the inversion are smooth maps. We write \(\mathbf{1} \in G\) for the identity element. Then each \(x \in T\mathbf{1}(G)\) corresponds to a unique left invariant vector field \(x_l\) with \(x_l(\mathbf{1}) = x\). The space of left invariant vector fields is closed under the Lie bracket of vector fields, hence inherits a Lie algebra structure. We thus obtain on \(g := T\mathbf{1}(G)\) a continuous Lie bracket which is uniquely determined by \([x, y] = [x_l, y_l]|_{\mathbf{1}}\) for \(x, y \in g\). We shall also use the functorial notation \(L(G) := (g, \{\cdot, \cdot\})\) for the Lie algebra of \(G\) and, accordingly, \(L(\varphi) = T\mathbf{1}(\varphi) : L(G_1) \to L(G_2)\) for the Lie algebra homomorphism associated to a smooth homomorphism \(\varphi : G_1 \to G_2\) of Lie groups. Then \(L\) defines a functor from the category of locally convex Lie groups to the category of locally convex
Lie algebras. If \( \mathfrak{g} \) is a Fréchet, resp., a Banach space, then \( G \) is called a Fréchet–, resp., a Banach–Lie group.

A smooth map \( \exp_{\mathfrak{g}} : \mathfrak{L}(G) \to G \) is called an exponential function if each curve \( \gamma_{t} := \exp_{\mathfrak{g}}(tx) \) is a one-parameter group with \( \gamma'_{0}(0) = x \). Not every infinite dimensional Lie group has an exponential function ([Net96, Ex. II.5.5]), but exponential functions are unique whenever they exist.

With the left and right multiplications \( \lambda_{g}(h) := p_{0}(g) := gh \) we write \( g.X = T_{1} = T_{1}(p_{0})X \) for \( g \in G \) and \( X \in \mathfrak{g} \). Then the two maps

\[
G \times \mathfrak{g} \to TG, \quad (g,X) \mapsto g.X \quad \text{and} \quad G \times \mathfrak{g} \to TG, \quad (g,X) \mapsto Xg
\]

trivialize the tangent bundle \( TG \).

### 3.3 Coadjoint actions and affine variants

To add some global aspects to the Poisson structures on the dual \( \mathfrak{g}' \) of a Lie algebra \( \mathfrak{g} \), we assume that \( \mathfrak{g} = \mathfrak{L}(G) \) for a Lie group \( G \). Then the adjoint action of \( G \) on \( \mathfrak{g} \) is defined by \( \text{Ad}(g) := \mathfrak{L}(c_{g}) \), where \( c_{g}(x) = gxg^{-1} \) is the conjugation map. The adjoint action is smooth in the sense that it defines a smooth map \( G \times \mathfrak{g} \to \mathfrak{g} \). The coadjoint action on the topological dual space \( \mathfrak{g}' \) is defined by

\[
\text{Ad}^*(g)x := x \circ \text{Ad}(g)^{-1}.
\]

The maps \( \text{Ad}^*(g) \) are continuous with respect to the weak-* topology on \( \mathfrak{g}' \) and all orbit maps for \( \text{Ad}^* \) are smooth because, for every \( X \in \mathfrak{g} \) and \( \alpha \in \mathfrak{g}' \), the map \( g \mapsto \alpha(\text{Ad}(g)^{-1}X) \) is smooth. If \( G \) is a Banach–Lie group, then the coadjoint action is smooth with respect to the norm topology on \( \mathfrak{g}' \), but in general it is not continuous, as the following example shows. \(^3\)

**Example 3.8.** Let \( V \) be a locally convex space and \( \alpha_{v} := e^{t}v \). Then the semidirect product

\[
G := V \rtimes_{\alpha} \mathbb{R}, \quad (v,t)(v',t') = (v + e^{t'}v, t + t')
\]

is a Lie group. From \( c_{(v,t)}(w,s) = ((1 - e^{s})v + e^{s}w, s) \) we derive that

\[
\text{Ad}(v,t)(w,s) = (e^{s}w - sv, s).
\]

Accordingly, we obtain

\[
\text{Ad}^*(v,t)(\alpha, u) = (e^{-t}\alpha + e^{t}\alpha_{v}).
\]

If \( \text{Ad}^* \) is continuous, restriction to \( t = 1 \) implies that the evaluation map

\[
V' \times V \to \mathbb{R}, \quad (\alpha, v) \mapsto \alpha(v)
\]

is continuous, but w.r.t. the weak-* topology on \( V' \), this happens if and only if \( V \) is finite dimensional. Therefore \( \text{Ad}^* \) is not continuous if \( \dim V = \infty \). \(^4\)

\(^3\) By definition of the weak-* topology on \( \mathfrak{g}' \), which corresponds to the subspace topology with respect to the embedding \( \mathfrak{g}' \hookrightarrow \mathbb{R}^d \), a map \( \varphi : M \to \mathfrak{g}' \) is smooth with respect to this topology if and only if all functions \( \varphi_{m} : = \varphi(m) : \mathfrak{g}' \to \mathbb{R} \) are smooth on \( M \).

\(^4\) One can ask more generally, for which locally convex spaces \( V \) and which topologies on \( V' \) the evaluation map \( V \times V' \to \mathbb{R} \) is continuous. This happens if and only if the topology on \( V \) can be defined by a norm, and then the operator norm turns \( V' \) into a Banach space for which the evaluation map is continuous.
Remark 3.9. (a) If $g'$ is endowed with the affine Poisson structure corresponding to a 2-cocycle $\omega: g \times g \to \mathbb{R}$, then the corresponding infinitesimal action $\beta: g \to \mathcal{Y}(g')$ of the Lie algebra $g$ by affine vector fields need not integrate to an action of a connected Lie group $G$ with $L(G) = g$, but if $G$ is simply connected, then it does (cf. [Ne02, Prop. 7.6]).

(b) The situation is much better for the Poisson structures on $g$ discussed in Example 2.14(e). Then the Hamiltonian vector field associated to $\omega$ is simply connected. Then we obtain a homomorphism of Lie algebras

$$X_{\Phi_p}(Z) = [Y, Z] - DY.$$

Let $G$ be a Lie group with Lie algebra $g$ and $\gamma_0: G \to g$ be a 1-cocycle for the adjoint action with $T_1(\gamma_0) = D$. Here the cocycle condition is

$$\gamma_0(gh) = \gamma_0(g) + Ad_g \gamma_0(h) \quad \text{for} \quad g, h \in G.$$

Since the adjoint action is smooth, such a cocycle exists if $G$ is simply connected. Then we obtain an affine action of $G$ on $g$ by

$$Ad^D_{\gamma} X := Ad_{\gamma} X - \gamma_0(g)$$

integrating the given infinitesimal action of $g$ determined by (10).

Definition 3.10. Let $(M, \omega)$ be a weak Poisson manifold, $G$ a connected Lie group, and $\sigma: G \times M \to M$ a smooth (left) action. We also write $g.p := g(\sigma_p) := \sigma^\sharp(g) := \sigma(g, p)$ and define the vector fields

$$X_{\sigma}(p) := T_{(g, p)}(\sigma)(X, 0) \quad \text{for} \quad X \in g.$$

Then we have a homomorphism

$$L(\sigma): g \to \mathcal{Y}(M) \quad \text{with} \quad X \mapsto -X_{\sigma}$$

which defines an infinitesimal action of $g$ on $M$.

The action $\sigma$ is called Hamiltonian if its derived action $L(\sigma)$ is Hamiltonian, i.e., if there exists a homomorphism of Lie algebras $\Phi: g \to \mathcal{A}$ with $X_{\Phi(Y)} = Y$ for $Y \in g$ such that, for every $m \in M$, the linear map $\Phi(m): g \to \mathbb{R}, Y \mapsto \Phi(Y)(m)$ is continuous. Then $\Phi: M \to g'$ is called the corresponding momentum map (cf. Definition 3.4).

Remark 3.11. For any smooth left action $\sigma: G \times M \to M$ and $p \in M$, the right invariant vector field $X_{\sigma}(g) = X_g$ on $G$ and the corresponding vector field $X_{\sigma}(p) = \sigma^\sharp(X_g)$ are $\sigma^\sharp$-related. This follows from the relation $\sigma^\sharp(hg) = h.\sigma^\sharp(g)$ for $h, g \in G$. Combining this observation with the “Related Vector Field Lemma”, one obtains a proof for $L(\sigma): g \to \mathcal{Y}(M)$ being a homomorphism of Lie algebras.

Example 3.12. Let $(V, \omega)$ be a locally convex symplectic vector space and $G = (V, +)$ the translation group of $V$. Then the translation action $\sigma(v, w) := v + w$ of $V$ on itself is symplectic and every constant vector field $v_\sigma(w) = v$ is Hamiltonian (cf. Example 2.19). The relation

$$\{v, w\} = \omega(v, w)$$

shows that there is no homomorphism $\Phi: g \to \mathcal{A}$ with $X_{\Phi(v)} = v_\sigma$ for every $v \in g$.

Remark 3.13. Of particular interest with respect to Poisson structures are Lie groups $G$ whose Lie algebras $g$ can be approximated in a natural way by finite dimensional ones. This can be done by direct or projective limits.

(a) If $G = \lim_{n} G_n$ is a Lie group whose Lie algebra $g$ is a directed union of a sequence of finite dimensional subalgebras $g_n = L(G_n), n \in \mathbb{N}$, then $g$ carries the finest locally convex topology which actually coincides with the direct limit topology (see [Gl03, Gl05] for direct limit manifolds and Lie groups). Then its topological dual $V := g'$, endowed with the topology of uniform convergence of bounded or compact subsets is a Fréchet space (isomorphic to a product $\mathbb{R}^N$) and all assumptions (a)-(d) from Example 2.13 are satisfied ([Gl08, Rem. 16.34]), so that we obtain a
linear Poisson structure on $V = g'_c = g'_c$. In this case the coadjoint orbits of $G$ are unions of finite dimensional manifolds, which can be used to obtain symplectic manifold structures on them (cf. [Gi03], [CL13]).

(b) The opposite situation is obtained for Lie groups $G = \lim G_n$ which are projective limits of finite dimensional Lie groups $G_n$ (see [HN09]). Typical examples are groups of infinite jets of diffeomorphisms. Here $g$ is a Fréchet space (isomorphic to $\mathbb{R}^\infty$) and the dual space $g^*$ is the union of the dual spaces $g^*_n$. Endowed with the topology of uniform convergence of bounded or compact subsets the space $V = g'$ satisfies all assumptions (a)-(d) from Example 2.13 ([Gi08, Rem. 16.34]). In this case all coadjoint orbits are finite dimensional because they can be identified with coadjoint orbits of some $G_n$.

In both cases we obtain weak Poisson structures on $g'$ for which $\mathcal{A} = C^\infty(g')$ is the full algebra of smooth functions for a suitable topology which is the weak-$\ast$-topology in the first case and the finest locally convex topology in the second.

Example 3.14. Let $G$ be a Lie group and $g = L(G)$. Further, let $g_* \subseteq g'$ be an $Ad^*(G)$-invariant separating subspace endowed with a locally convex topology for which the coadjoint action $Ad_*(g) := Ad^*(g)|_{g_*}$ on $g_*$ is smooth. Then $g_*$ carries a natural linear weak Poisson structure with $\mathcal{A} \cong S(g)$ and

$$\{F, H\}(\alpha) = \langle \alpha, [dF(\alpha), dH(\alpha)] \rangle$$

for $\alpha \in g_*, F, H \in \mathcal{A}$

(Example 2.14(b); see also [Ra11, Sect. 4.2] for similar requirements in the context of Banach spaces).

For $X, Y \in g$, we have $\{H_X, H_Y\} = H_{[X,Y]}$, and the corresponding Hamiltonian vector fields are $X_H(\alpha) = -\alpha \circ ad Y$. Therefore the coadjoint action $Ad_*$ on $g_*$ is Hamiltonian and its momentum map is the inclusion $g_* \hookrightarrow g'$.

For the coadjoint action $Ad_*$ on $g_*$, the “tangent space” to the orbit of $\alpha \in g_*$ is the space $\{X_{Ad_*(\alpha)}(\alpha) : X \in g\} = \alpha \circ ad(g)$. This is also the characteristic subspace of the Poisson structure (cf. Remark 2.5) and the corresponding skew-symmetric form is given by

$$\omega_{\alpha}(X_f(\alpha), X_H(\alpha)) = \{F, H\}(\alpha) = dF(\alpha)X_H(\alpha),$$

resp.,

$$\omega_{\alpha\circ ad X, \alpha \circ ad Y} = \{H_X, H_Y\}(\alpha) = H_{[X,Y]}(\alpha) = \alpha([X,Y]).$$

Fix $\alpha \in g_*$. Then we obtain on $G$ a 2-form by

$$\Omega_{\alpha}(X, g, Y, g) := \omega_{g, \alpha}(X_{Ad_*(g, \alpha)}, Y_{Ad_*(g, \alpha)}) = \omega_{\alpha}(g \circ ad X, g \circ ad Y)$$

$$= \alpha([X,Y]).$$

This means that $\Omega$ is a left-invariant 2-form on $G$. Since $(\Omega_{\alpha, 1})(X, Y) = \alpha([X,Y])$ is a 2-cocycle, $\Omega$ is closed, the radical of $\Omega_1$ coincides with the Lie algebra of the stabilizer subgroup $G_{\alpha}$.

If $\mathcal{O}_G := Ad_*(G)\alpha$ carries a manifold structure for which the orbit map $G \to \mathcal{O}_G$ is a submersion, we thus obtain on $\mathcal{O}_\alpha$ the structure of a weak symplectic manifold. However, if the Lie algebra $g$ is not a Hilbert space, then it is not clear how to obtain a manifold structure on $\mathcal{O}_\alpha$, resp., the homogeneous space $G/G_{\alpha}$. In any case, we may consider the pair $(G, \Omega_{\alpha})$ as a non-reduced variant of the symplectic structure on the coadjoint orbit.

3.4 Cotangent bundles of Lie groups and their reduction

Let $G$ be a Lie group, $g = L(G)$ and $g_* \subseteq g'$ be as in Example 3.14, so that the coadjoint action $Ad_*$ on $g_*$ is smooth. Then the “cotangent bundle”
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\[ T_c(G) := \bigcup_{g \in G} \{ \alpha \in T^*_g(G) : \alpha \circ T_1(\rho_g) \in \mathfrak{g}_* \} \]

carries a natural Lie group structure for which it is isomorphic to the semidirect product \( \mathfrak{g}_* \rtimes \text{Ad}_G \). Here we identify \((\alpha, g)\) with the element \( \alpha \circ T_1(\rho_g)^{-1} \in T^*_g(G) \)', which leads to an injection \( T_c(G) \hookrightarrow T^*(G) \).

The lift of the left, resp., right multiplications to \( T_c(G) \) is given by

\[ \sigma'_g(\alpha, h) = (\alpha \circ \text{Ad}^{-1}_g, gh) \quad \text{and} \quad \sigma'_g(\alpha, h) = (\alpha, hg). \quad (11) \]

The corresponding infinitesimal action is given by the vector fields

\[ X_{\alpha'}(\alpha, h) = (-\alpha \circ \text{ad}(X, h)) \quad \text{and} \quad X_{\alpha'}(\alpha, h) = (0, hX). \]

The smooth 1-form defined by

\[ \Theta(\alpha, g)(\beta, X, g) := \alpha(X) \]

is an analog of the Liouville 1-form. It follows from (11) that it is invariant under both actions \( \sigma' \) and \( \sigma' \). Note that

\[ \Theta(X_{\alpha'})(\alpha, h) = \alpha(X) \quad \text{and} \quad \Theta(X_{\alpha'})(\alpha, h) = \alpha(\text{Ad}_h X). \]

Now

\[ \Omega := -d\Theta \]

is closed smooth 2-form on \( T_c(G) \). To see that it is non-degenerate, we observe that its invariance under left and right translations and the Cartan formulas imply

\[ (ix_{\alpha'}(\beta, Y, g) = d(ix_{\alpha'}(\Theta)(\beta, Y, g)) = \beta(X) \quad (12) \]

and, for the constant vertical vector field \( Z(\alpha, g) = \gamma \in \mathfrak{g}_* \), the relation \( \Theta(Z) = 0 \) leads to

\[ (i_2 \Omega)(\alpha, g)(\beta, Y, g) = -(\mathcal{L}_Z(\Theta)(\alpha, g)(\beta, Y, g) = \gamma(Y). \quad (13) \]

We conclude that \( (T_c(G), \Omega) \) is a weak symplectic manifold.

We thus obtain by Proposition 2.18 on \( T_c(G) \) a weak Poisson structure on the subalgebra

\[ \mathcal{A}' := \{ H \in C^\infty(T_c(G)) : (\exists X_H \in \mathcal{Y}(T_c(G)) \ dH = i_{X_H} \Omega \} \subseteq C^\infty(T_c(G)). \]

Let \( C^\infty_c(G) \subseteq C^\infty(G) \) denote the subalgebra of smooth functions \( H \) whose differential \( dH \) defines a smooth section \( G \rightarrow T_c(G) \), resp., a smooth function

\[ \delta H : G \rightarrow \mathfrak{g}_* \quad (\delta H)_g(X) := (dH)_g(X, g). \]

Then (13) shows that, for \( H \in C^\infty_c(G) \), the vertical vector field on \( T_c(G) \) defined by \( X_H(\alpha, g) := (\delta H(g), 0) \) satisfies

\[ (i_{X_H} \Omega)(\alpha, g)(\beta, Y, g) = (\delta H)_g(Y) = (dH)_g(Y, g). \]

For the corresponding function \( \tilde{H} \) on \( T_c(G) \), we therefore have \( d\tilde{H} = i_{X_H} \Omega \), so that \( H \in \mathcal{A}' \). On the other hand, we have seen above that, for \( X \in \mathfrak{g} \), the function \( H_X(\alpha, g) = \alpha(X) \) on \( T_c(G) \) satisfies \( dH_X = i_{X^T} \Omega \). This shows that \( \mathcal{A}' \) contains the subalgebra \( C^\infty_c(G) \) and the algebra \( S(\mathfrak{g}) \) of polynomial functions on the first factor \( \mathfrak{g}_* \), generated by the functions \( H_X, X \in \mathfrak{g} \). We therefore have \( S(\mathfrak{g}) \otimes C^\infty_c(G) \subseteq \mathcal{A}' \).

The Poisson bracket vanishes on \( C^\infty_c(G) \), and, for \( X \in \mathfrak{g} \) and \( F \in C^\infty_c(G) \), we have

\[ \{ \tilde{F}, H_X \}(\alpha, g) = df_{\tilde{F}}(X, g) = (X_{\tilde{F}})(g) = \tilde{X}_F(\alpha, g). \]

We also note that, for \( X, Y \in \mathfrak{g} \), we have by (12)
\[ \{ H_\kappa, H_\gamma \}(\alpha, g) = \Omega(X_{\kappa'}, Y_{\alpha'}) (\alpha, g) = (i_{X_{\kappa'}} \Omega)_{(\alpha, g)} (-\alpha \circ \text{ad} Y, Y, g) = -\alpha \circ \text{ad} Y)(X) = \alpha([X, Y]) = H_{[X, Y]}(\alpha, g). \]

This implies that \( \mathcal{S}(g) \) and \( \mathfrak{R} := \mathcal{S}(g) \otimes C^\omega_c(G) \) are Poisson subalgebras of \( \mathfrak{R} \). In particular, \( \mathfrak{R} \) defines a weak Poisson structure on \( T_\kappa(G) \).

Consider the submersion \( q : T_\kappa(G) \rightarrow \mathfrak{g}_s, (\alpha, g) \mapsto \alpha \). Then \( \mathfrak{R} \cap q' C^\omega(\mathfrak{g}_s) \cong S(\mathfrak{g}) \), and since \( H_\kappa(\alpha, g) = (q(\alpha, g), X) \), condition (6) in Proposition 2.23 is satisfied. Therefore \( q \) is a Poisson map if we endow \( \mathfrak{g}_s \) with the Poisson structure determined on \( \mathfrak{R}_s = S(\mathfrak{g}) \) by \( \{ H_\kappa, H_\gamma \} = H_{[X, Y]} \) for \( X, Y \in \mathfrak{g} \).

The fibers of \( q \) are the orbits of the right translation action \( \sigma' \), which is a Hamiltonian action of \( G \) on \( T_\kappa(G) \) and \( \mathfrak{R}\sigma'(G) \cong S(\mathfrak{g}) \) is the subalgebra of invariant functions in \( \mathfrak{R} \). On the other hand, \( q \) is a momentum map for the left action \( \sigma' \) of \( G \) on \( T_\kappa(G) \). Therefore the passage to the orbit space \( G / \sigma'(G) \) is an example of Poisson reduction from the Hamiltonian action \( \sigma' \) to the coadjoint action \( \text{Ad}_\kappa \) on \( \mathfrak{g}_s \). (cf. [MR99, Thm. 13.1.1] for the finite dimensional case).

Remark 3.15. (Magnetic cotangent bundles) A natural variation of this construction is obtained by using a continuous 2-cocycle \( b : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R} \) to get a closed right invariant 2-form \( B \in \Omega^2(G) \). If \( \pi : T_\kappa G \rightarrow G \) is the bundle projection, then

\[ \Omega_b := \Omega + \pi^* B \]

is a closed right invariant 2-form on \( T_\kappa(G) \). Since its values in vertical directions are the same as for \( \Omega \), the form \( \Omega_b \) is also non-degenerate. We thus obtain an infinite dimensional version of a magnetic cotangent bundle (cf. [MR99, §6.6], [M*07, §7.2]).

The Poisson bracket on \( C^\omega_c(G) \) still vanishes, and, for \( X \in \mathfrak{g} \) and \( F \in C^\omega_c(G) \), we still have \( \{ F, H_\kappa \} = X F \). But for \( X, Y \in \mathfrak{g} \) we obtain

\[ \{ H_\kappa, H_\gamma \} = \Omega(X_{\kappa'}, Y_{\gamma'}) + b(X, Y) = H_{[X, Y]} + b(X, Y). \]

Therefore the quotient Poisson structure on \( \mathfrak{g}_s \cong T_\kappa(G) / \sigma'(G) \) is the affine Poisson structure from Example 2.14(b) (see [GBR08] for applications of these techniques).

4 Lie algebra-valued momentum maps

We have already seen in Example 2.14(e) how to obtain from an invariant symmetric bilinear form \( \kappa \) and a \( \kappa \)-skew-symmetric derivation \( D \) a weak affine Poisson structures on a Lie algebra \( \mathfrak{g} \). This leads naturally to a concept of a Hamiltonian \( G \)-action with a \( \mathfrak{g} \)-valued momentum map. For the classical case where \( G \) is the loop group \( \mathcal{L}(K) = C^\omega([S^1, K]) \) of a compact Lie group and the derivation is given by the derivative, we thus obtain the affine action on \( \mathfrak{g} = \mathcal{L}(t) \) which corresponds to the action of \( \mathcal{L}(K) \) on gauge potentials on the trivial \( K \)-bundle over \( S^1 \).

4.1 Hamiltonian actions for affine Poisson structures on Lie algebras

Let \( G \) be a Lie group with Lie algebra \( \mathfrak{g} \), \( \kappa : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R} \) be a continuous \( \text{Ad}(G) \)-invariant non-degenerate symmetric bilinear form and \( D : \mathfrak{g} \rightarrow \mathbb{R} \) be a continuous derivation for which we have a smooth \( \text{Ad} \)-cocycle \( \gamma_D : G \rightarrow \mathfrak{g} \) with \( \gamma_D'(1) = D \). In Remark 3.9(b) we have seen that this leads to a smooth affine action of \( G \) on \( \mathfrak{g} \) by
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$$\text{Ad}_g^\varphi \xi := \text{Ad}_g \xi - \gamma_0(g) \quad \text{for} \quad g \in G, \xi \in \mathfrak{g}.$$ 

We recall from Example 2.14(e) that $\mathfrak{g}$ carries a weak Poisson structure $\{\cdot, \cdot\} = \{\cdot, \cdot\}_{\kappa, D}$ with $\varphi := S(\mathfrak{g})$, generated by the functions $\xi^\varphi := \kappa(\xi, \cdot)$. It is determined by

$$\{ \xi^\varphi, \eta^\varphi \} = [\xi, \eta]^\varphi + \kappa(D\xi, \eta) \quad \text{for} \quad \xi, \eta \in \mathfrak{g}.$$ 

For any $F \in \mathcal{A}$ and $\xi \in \mathfrak{g}$, the linear functional $dF(\xi) \in \mathfrak{g}'$ is represented by $\kappa$, hence can be identified with an element $\nabla F(\xi) \in \mathfrak{g}$, the $\kappa$-gradient of $F$ in $\xi$. In these terms, the Poisson structure on $\mathfrak{g}$ is given by

$$\{ F, H \}(\xi) := \kappa(\xi, [\nabla F(\xi), \nabla H(\xi)]) + \kappa(D\nabla F(\xi), \nabla H(\xi)) \quad \text{for} \quad F, H \in \mathcal{A}, \xi \in \mathfrak{g}.$$ 

The corresponding Hamiltonian vector fields are determined by

$$\{ X_H, F \}(\xi) = \{ F, H \}(\xi) = \kappa([\nabla F(\xi), \nabla H(\xi)], \xi) + \kappa(D\nabla F(\xi), \nabla H(\xi))$$

$$= dF(\xi)([\nabla H(\xi), \xi] - D\nabla H(\xi)),$$

which leads to

$$X_H(\xi) = [\nabla H(\xi), \xi] - D\nabla H(\xi).$$

For $H = \eta^\varphi$, $\eta \in \mathfrak{g}$, this specializes to

$$X_{\eta^\varphi} = \text{ad} \eta - D\eta = \eta \text{Ad}^\varphi \quad \text{for} \quad \eta \in \mathfrak{g}. \quad (14)$$

### 4.2 Loop groups and the affine action on gauge potentials

An important example arises for $S^1 = \mathbb{R}/\mathbb{Z}$ and the loop group $G = \mathcal{L}(K) := \mathcal{C}^\infty(S^1, K)$ where $K$ is a Lie group for which $\mathfrak{k}$ carries a non-degenerate $\text{Ad}(K)$-invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$. We then put $D\xi = \xi'$ and $\kappa(\xi, \eta) = \int_0^1 \langle \xi(t), \eta(t) \rangle \, dt$ as in Example 2.15. Then $\gamma_0(g) = \delta'(g) := g'g^{-1}$ is the right logarithmic derivative, so that

$$\text{Ad}_{g}^\varphi \xi = \text{Ad}_g \xi - g'g^{-1} =: \xi'$$ \quad (15)

corresponds to the natural affine action on the space $\Omega^1(S^1, \mathfrak{k}) \cong \mathcal{C}^\infty(S^1, \mathfrak{k})$ of gauge potentials of the trivial $K$-bundle $S^1 \times K$ over $S^1$.

For $\xi \in \mathcal{L}(\mathfrak{k})$, let $\gamma_{\xi}: \mathbb{R} \to K$ denote the unique solution of the initial value problem

$$\gamma(0) = 1 \quad \text{and} \quad \delta'(\gamma) := \gamma^{-1}\gamma' = \xi. \quad (16)$$

For each $s \in \mathbb{R}$ we write

$$\text{Hol}_{s} : \mathcal{L}(\mathfrak{k}) \to K, \quad \xi \to \gamma_{\xi}(s),$$

for the corresponding holonomy map. It satisfies the equivariance relation

$$\text{Hom}_{s}(\xi g) = g(0) \text{Hol}_{s}(\xi)g(s)^{-1} \quad \text{for} \quad g \in \mathcal{L}(K). \quad (17)$$

In particular $\text{Hol} := \text{Hol}_1$ is equivariant with respect to the conjugation action of $K$ on itself. This formula also implies that $\gamma_{\xi g} = g(0)\gamma_{\xi}g^{-1}$, so that the affine $\mathcal{L}(K)$-action on $\mathfrak{g}$ corresponds on the level of curves to the multiplication with the pointwise inverse on the right.
Proposition 4.1. For any Lie group $K$ for which (16) is solvable, the action (15) of the subgroup $\Omega(K) := \{ g \in L(K) : q(0) = 1 \}$ on $g$ is free and its orbits coincide with the fibers of Hol, so that Hol induces a bijection

$$\text{Hol} : L(t)/\Omega(K) \to K, \quad [\xi] \mapsto \text{Hol}_1(\xi).$$

Proof. The relation $\xi^g = \xi$ implies $\gamma_1^g = \gamma_1$, so that $g(0)\gamma_1^g = \gamma_1g$. For $g(0) = 1$ this implies that $g = 1$ is constant. Therefore the action of the subgroup $\Omega(K)$ on $g$ is free and Hol is constant on the $\Omega(K)$-orbits.

Suppose, conversely, that $\text{Hol}(\xi) = \text{Hol}(\eta)$, i.e., $g_1 := \gamma_1(1) = \gamma_1(1)$. Since $\xi$ and $\eta$ are periodic, $\gamma_1(t + 1) = g_1\gamma_1(t)$ and $\gamma_1(t + 1) = g_1\gamma_1(t)$ holds for all $t \in \mathbb{R}$. Therefore $g(t) := \gamma_1(t)^{-1}\gamma_1(t)$ is a smooth periodic curve defining an element of $\Omega(K)$ with $\gamma_1 = \gamma_1g^{-1}$. This in turn leads to the relation

$$\eta = \delta^1(\gamma_1) = \delta^1(g^{-1}) + \text{Ad}_g \delta^1(\gamma_1) = \text{Ad}_g \xi - \delta^1(g) = \xi^g.$$

\hfill \Box

Remark 4.2. (An attempt on Poisson reduction from $L(t)$ to $K$) For every connected Lie group $K$, the map $\text{Hol} : L(t) \to K$ is surjective and it is easy to see that it is a submersion. In view of Proposition 2.23, it makes sense to ask for a Poisson subalgebra $\mathcal{A} \subset \mathcal{A} \cong \mathcal{S}(L(t))$ that induces on $K$ a Poisson structure for which $q$ is a Poisson map. A natural candidate for $\mathcal{A}$ is the invariant subalgebra

$$\mathcal{B} := \mathcal{A}^\Omega(K)$$

consisting of $\Omega$-invariant functions in $\mathcal{A}$, i.e., functions that are constant on the fibers of Hol.

If $K$ is compact, then the exponential function $\exp : \mathfrak{f} \to K$ is surjective, and since $\text{Hol}|_\mathfrak{f} = \exp$, it follows that $\text{Hol}(\mathfrak{f}) = K$, which in turn means that every $\Omega(K)$-orbit meets the subspace $\mathfrak{f} \subseteq L(t)$ of constant functions. We conclude that the restriction map $\mathcal{K} : \mathcal{B} \to \mathcal{S}(\mathfrak{f})$ is injective and that its image consists of polynomial functions on $L(t)$ that are constant on the fibers of the exponential function. Let $\mathcal{T} \subseteq \mathcal{K}$ be a maximal torus and $\mathcal{L}(\mathcal{T})$ be its Lie algebra. Then every $F \in \mathcal{B}$ restricts to a polynomial $F|_\mathfrak{t}$ which is constant on the cosets of the lattice $\ker(\exp|_\mathfrak{t})$, hence constant. Since every element $X \in \mathfrak{t}$ is contained in the Lie algebra of a maximal torus, it follows that $F$ is constant on $\mathfrak{t}$, and therefore $F$ is constant on $L(t)$. We conclude that $\mathcal{B} = \mathbb{R}1$ contains only constant functions.

This shows that the algebra $\mathcal{A} \cong \mathcal{S}(\mathfrak{g})$ of polynomial functions is too small to lead to a sufficiently large algebra of $\Omega(K)$-invariant functions. It is an interesting question whether there exists a suitable larger Poisson algebra $\mathcal{A} \supset \mathcal{A}$ for which $\mathcal{A}^\Omega(K)$ satisfies the assumptions of Proposition 2.23.

Definition 4.3. A Hamiltonian $L(K)$-space is a smooth weak Poisson manifold $(M, \mathcal{A}, \{ \cdot, \cdot \})$, endowed with a smooth action $\sigma : L(K) \times M \to M$ which has a smooth momentum map

$$\Phi : M \to L(t)$$

which is a Poisson map with respect to $(\mathcal{A}, \{ \cdot, \cdot \})$.

5 This is the case for so-called regular Lie groups (cf. [Ne06]). Banach–Lie groups and in particular finite dimensional Lie groups are regular.

6 This concept depends on the choice of the invariant symmetric bilinear form $\{ \cdot, \cdot \}$ on the Lie algebra $\mathfrak{f}$. Changing this form leads to a different Poisson structure on $L(t)$.

7 In [AMM98] one finds this concept for the special case where $(M, \omega)$ is a weak symplectic manifold. In this case one requires the action $\sigma$ to be symplectic and the existence of a smooth $L(K)$-equivariant map $\Phi : M \to L(t)$ such that the functions

$$\varphi(\xi): (m) := \kappa(\Phi(m), \xi) \quad \text{satisfy} \quad i_{\Phi(m)} \omega = d(\varphi(\xi)).$$

These conditions are easily verified to be equivalent to ours (cf. Proposition 3.1).
Remark 4.4. Since the subgroup $\Omega(K) \subseteq \mathcal{L}(K)$ acts freely on $\mathfrak{g}$ and $\Phi$ is equivariant, it also acts freely on $M$, so that we can consider the holonomy space

$$\text{Hol}(M) := M/\Omega(K),$$

and obtain a commutative diagram

$$
\begin{array}{ccc}
M & \xrightarrow{\Phi} & \mathcal{L}(\mathfrak{k}) \\
\text{Hol}(M) & \xrightarrow{\overline{\Phi}} & K
\end{array}
$$

The geometric structure contained in the bottom row consists in an action of the Lie group $K \cong \mathcal{L}(K)/\Omega(K)$ on the orbit space $\text{Hol}(M)$ and an equivariant map $\overline{\Phi}: \text{Hol}(M) \to K$. If $(M, \omega)$ is weak symplectic, this is enriched by the data contained in natural differential forms on $\text{Hol}(M)$ and $K$, which leads to the concept of a quasihamiltonian $K$-space for which $\overline{\Phi}: M \to K$ plays the role of a group-valued momentum map. If $K$ is a compact Lie group and $\mathcal{L}(K)$ denotes a suitable Banach–Lie group of differentiable loops, such as $H^1$-loops, then the Equivalence Theorem in [AMM98, Thm. 8.3] asserts that quasihamiltonian actions of $K$ are in one-to-one correspondence with Hamiltonian $\mathcal{L}(K)$-actions on Banach manifolds $M$ for which the momentum map $\Phi: M \to \mathcal{L}(\mathfrak{k})$ is proper.

Since our setup for Hamiltonian $\mathcal{L}(K)$-action uses only the invariant bilinear form on $\mathfrak{k}$, it is also valid for non-compact Lie groups $K$ and even for infinite-dimensional ones, provided $\mathfrak{k}$ carries an invariant non-degenerate symmetric bilinear form.

In particular, the construction of a Lie group-valued momentum map $\mu = \exp \circ \Phi$ from a Lie algebra-valued momentum map $\Phi: M \to \mathfrak{g}$ with respect to a Poisson structure $\{\cdot, \cdot\}_\kappa, D$ on $\mathfrak{g}$ (cf. [AMM98, Prop. 3.4]) works quite generally for any pair $(\kappa, D)$ as in Subsection 4.1.

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