Compartmental model with loss of immunity: analysis and parameters estimation for Covid-19

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Abstract

The outbreak of Covid-19 led the world to an unprecedented health and economical crisis. In an attempt to respond to this emergency researchers worldwide are intensively studying the Covid-19 pandemic dynamics. In this work, a SIRSi compartmental model is proposed, which is a modification of the known classical SIR model. The proposed SIRSi model considers differences in the immunization within a population, and the possibility of unreported or asymptomatic cases. The model is adjusted to three major cities of S\~{a}o Paulo State, in Brazil, namely, S\~{a}o Paulo, Santos and Campinas, providing estimates on the duration and peaks of the outbreak.

Keywords: Covid-19, Compartmental models, Equilibrium analysis, Parameter fitting.

1. Introduction

The Wuhan Municipal Health Commission reported a cluster of 27 pneumonia cases on 31st December 2019, in the Wuhan city, Hubei Province in China. On 1st January 2020 the World Health Organization (WHO) set up the Incident Management Support Team putting the organization to an emergency level for

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dealing with the outbreak. On 5th January 2020 WHO published the first outbreak news on the new virus. On 7th January the causative agent was identified and named Severe Acute Respiratory Syndrome Coronavirus 2 (SARS-CoV-2) (WHO named the disease Covid-19). On 13th January 2020 the first case outside China was reported in Thailand. On 22nd January 2020 WHO stated that there was evidence of human-to-human transmission, and approximately seven weeks later, on 13th March 2020, WHO characterized the Covid-19 as a pandemic. 

In Brazil, the first confirmed Covid-19 case was reported on the 26th February 2020, and up to 25th June 2020 there was 1,188,631 confirmed cases with 53,830 deaths. Globally, according to there was 9,292,202 confirmed cases, and 479,133 deaths up to 25th June 2020.

Most cases are asymptomatic carriers and are spontaneously resolved, however, some developed various fatal complications, notably for patients with comorbidities, and, allied to the fastly spread of covid-19, the worldwide emergency state brings up with important, and yet unanswered, questions related to the contagion dynamical behavior and its mitigation and control strategies. As a result, strategies to contain the contagion such as social distancing, quarantine and complete lockdown of areas have been studied.

In Mexico, on 18th March 2020, the Mexican Health Secretariat reported that the pandemic was going to last 90 days, with 250,656 expected cases. On the next day, the Health Secretariat informed that approximately 9.8% (24,594) of the cases would need hospitalization, and 4.2% (10,528) of the total cases would be critical patients, needing intensive care. On the same date the number of available Health Care units at that time was 4,291 with 2,053 ventilators. The situation in Mexico led to the adoption of non-pharmaceutical interventions, such as washing hands, social distancing, cough/sneeze etiquette, and so on.

In Rio de Janeiro, Brazil, the social distancing started on 17 March 2020, and the government of the state of São Paulo, Brazil, decreed quarantine on 2

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1Reported on 2:25pm CEST.
23rd March 2020. Up to 25th June 2020, the State of São Paulo had 248,587 confirmed cases with 13,759 deaths.

One important question is related to the patient’s immunity after recovery. The difference in immunity after recovery have been reported in humans in [12], and in experiments with rhesus macaques [13]. The experiment in [12] collected plasma from 175 Covid-19 recovered patients, and SARS-CoV-2-specific neutralizing antibodies (NAbs) were detected from day 10 to 15 after the onset of the disease and remained thereafter. Nonetheless, the NAbs levels were variable in the cohort, 52 (≈30%) of the patients developed low levels of these antibodies, from which with 10 (≈6% of the recovered patients) the NAbs levels were undetectable, 25 (≈14%) developed high levels.

In [14] the data suggests that the response to SARS-CoV-2 is imbalanced regarding to controlling the virus replication versus the activation of the adaptive immune response. In some cases the immune system doesn’t work properly and lung cells remain vulnerable to infection. The virus continues replicating while the immune response system attacks infected cells, killing even healthy nearby cells, and the lung tissue becomes seriously inflamed. This seems to be the mechanism that make some patients become severely ill weeks after their initial infection. Additionally, SARS-CoV-2 probably induces immunity like other coronaviruses, however, it is so new that this mechanism isn’t fully understood [15].

The economic crisis due to the pandemics is another important issue. According to [16] the mortality rate of Covid-19 is not necessarily correlated with the economic risk to global economy since governments, companies, consumers and media reacted to the economic shock. However, a global recession seems to be inevitable, its duration and intensity will depende on the success of measures to prevent the spread of Covid-19.

Taking the whole scenario into account researchers worldwide are intensively studying and developing mathematical models of the Covid-19 outbreak. The knowledge on this pandemic dynamics is important for providing estimates on the duration and peaks of the outbreak.
The macro-modelling of infectious disease spread have been a field of research from many years since the simple deterministic model of Kermack and McKendrick [17, 18, 19, 20], that provides a dynamical model classifying individuals in a population as Susceptible - Infected - Removed (SIR) [21]. A number of classical models such as SIR [22] and Susceptible - Exposed - Infected - Removed (SEIR) [23, 24] have been proposed for epidemic modelling.

In addition, works considering time delayed and fractional order dynamical systems applied to Covid-19 outbreak have also been proposed.

In [27] a SIRD model had been adjusted to Covid-19 spreads in China, Italy and France. The results have shown that the recovery rate were similar for the situations of China, Italy and France. In [28], a mixed analytical-statistical inverse problem is used to predict the Covid-29 progression in Brazil, a SIRU model, where U stands for Unreported cases, was used for the direct problem, and a mixture of parameter estimation with Bayesian inference for the inverse problem analysis.

Compartmental models considering immigration and home isolation, or quarantine, are proposed on [10], all the situations presented infection-endemic equilibrium, the results demonstrated that home isolation, or lockdown, mitigates the chance of infection.

In this work the proposed model is a modification of the original compartmental SIR model of Kermack and McKendrick [17, 18, 19], including a sick (S\textit{ick}) population compartment representing nodes of the network that manifest the symptoms of the disease. The proposed Susceptible - Infected - Removed - Sick (SIRSi) model also considers the birth and death of individuals in the given population. In addition, a feedback from those recovered who did not gain immunity or loss their immunity after a period of time and become susceptible again is also introduced.

The proposed SIRSi model presents both a disease free and an endemic equilibrium, and the influence of the re-susceptibility feedback is investigated both analytically and numerically.

The parameters of the proposed SIRSi model are numerically fitted to the
epidemic situation for three cities of the São Paulo State, Brazil, namely, São Paulo, the capital of the State; Santos, on the coast and approximately 80 Km away from São Paulo; and Campinas, in the interior of the state and approximately 90 Km distant from the capital São Paulo, providing estimates for diagnosis and forecasting of the Covid-19 epidemics spread.

The paper is organized as follows, on section 2 the SIRSi compartmental mathematical model is presented. The equilibrium points existence and stability conditions are discussed in section 3, showing the possibility of both endemic and disease-free equilibrium. The parameter fitting of the SIRSi model for the cities of São Paulo, Santos and Campinas is shown in section 4 and the numerical experiments results in section 5. The concluding remarks can be seen in section 6.

2. A modified SIR model with birth and death cases

The proposed SIRSi model can be seen in Fig. 1. In this model, the susceptible population $S$ is infected at a rate $\alpha$ when contact infected individuals from $I$. The susceptible compartment also receives a population, at a rate $1/\gamma$, who didn’t gain complete immunity after recovering or who loss their immunity after a period of time.

The compartment $I$ represents the infectious population in incubation stage prior the onset of symptoms. Infection transmission during this period has been reported in [29, 30, 31, 32]. The infected population can be asymptomatic or symptomatic. The period between infection and onset of symptoms, $1/\beta_2$, ranges from 3 to 38 days with median of 5.2. Once the infected individual is tested positive and the case is documented, the case is moved to the Sick compartment. Those who don’t develop severe symptoms become asymptomatic.

In [33] the estimative of the infections originated from undocumented cases is as high as 86%, which include mild, limited and lack of symptoms infectious individuals. In other recent studies [34, 35], it was found that, 20% - 40% of positive tested patients were asymptomatic.
In this work, and according to [36], the asymptomatic population is considered as those of under-reporting cases. This population could be as 7 times bigger than the size of the documented cases. This group, under-reporting cases, become recovered with the period $1/\beta_1$.

Some of the individuals within this population could eventually develop symptoms. In [37], it has been reported that the average time between infection and the onset of symptoms can be 4.6 days. Once the case becomes documented it should be moved to the $S_{ick}$ compartment. $S_{ick}$ are those infected which seek for medical attention with severe symptoms. This population are those who tested positive for COVID-19. In [38] it was reported that this population could represent up to 19.9% of the total documented cases, of which 13.8% are severe cases and 6.1% require intensive care. The sick population become recovered with the period $1/\beta_3$ or removed at a rate $\sigma$ (see Fig. 1).

In order to consider the social distancing measure effect in the number of infected and deaths toll shown in Fig. 1 the parameter $\theta$ is introduced on the mathematical model (1), where $\theta$ is subject to the constraint $0 < \theta < 1$. Consequently, given these facts, the mathematical SIRSi model is given by (1).
\[ \dot{S} = \lambda - \alpha (1 - \theta)SI - \delta S + \gamma R, \]
\[ \dot{I} = \alpha (1 - \theta)SI - (\beta_1 + \beta_2)I, \]
\[ \dot{S}_{icke} = \beta_2 I - (\beta_3 + \sigma)S_{icke} \]
\[ \dot{R} = \beta_1 I + \beta_3 S_{icke} - (\gamma + \delta)R \]

where \( \lambda \) and \( \delta \) are the birth and death rates, respectively.

It is important to notice that the number of documented cases is a key information that should emerge, somehow, from (1). The reason is the fact that the accumulated number of confirmed cases is publicly available, and will be used to fine-tuning the model. The number of daily new infections is also available, but it tends to be noisy and less representative of the dynamics.

3. Disease-free and endemic equilibrium points

Considering (1), such that \( \dot{x} = f(x) \), where \( x = (S, I, S_{icke}, R)^T \), \( x \in U \subset (\mathbb{R}_0^+)^4 \), \( f : U \to U \) is the right-hand side of (1), and parameters \( \alpha, \beta_1, \beta_2, \beta_3, \sigma, \gamma, \theta \in \mathbb{R}^+ \).

To investigate the influence of the introduction of the feedback from those recovered with no immunity which become susceptible again, and dividing the population into groups, the equilibrium points related to the model described by (1) must be determined and their stability discussed.

Assuming \( \alpha \neq 0 \), i.e., susceptible can be converted into infected, and despite being an assumption it is realistic for a spreading disease, the equilibrium points are such that \( f(x^*) = 0 \).

Using the Hartman-Grobman Theorem \[39\] the local stability of the equilibrium points can be determined by the eigenvalues of the Jacobian matrix computed on each equilibrium point. The Jacobian \( J = \frac{\partial f}{\partial x} \bigg|_{x^*} \) of (1) is given by (2).
In the following the disease-free (section 3.1) and the endemic (section 3.2) equilibrium points are determined.

3.1. Disease-free equilibrium points

The disease-free equilibrium is a state corresponding to the absence of infected individuals, i.e., \( I^* = 0 \). Applying this condition to the equilibrium of (1), the point can be determined.

Assume that there exists a disease-free equilibrium \( (I^*, x^*) \in \mathbb{U} \), such that \( f(x^*) = 0 \). This equilibrium point \( x^* = (S^*, I^*, S^*_{ick}, R^*)^T \) with \( x^* \) in the first octant of \( \mathbb{R}^4 \) is given by:

\[
P_1 = (S^*, I^*, S^*_{ick}, R^*)^T = (\lambda/\delta, 0, 0, 0)^T.
\] (3)

Considering \( P_1 \), the corresponding linear system Jacobian \( J_{P_1} = \left. \frac{\partial f}{\partial x} \right|_{x^*} \) is given by (4).

\[
J_{P_1} = \begin{bmatrix}
-\delta & -\alpha(1 - \theta)(\lambda/\delta) & 0 & \gamma \\
0 & \alpha(1 - \theta)(\lambda/\delta) - (\beta_1 + \beta_2) & 0 & 0 \\
0 & \beta_2 & -\beta_3 + \sigma & 0 \\
0 & \beta_1 & \beta_3 & -\delta + \gamma
\end{bmatrix}.
\] (4)

By the Laplace determinant development, the eigenvalues of (4) are the elements in the diagonal, that is, \( \xi_1 = -\delta \), \( \xi_2 = \alpha(1 - \theta)(\lambda/\delta) - (\beta_1 + \beta_2) \), \( \xi_3 = -\beta_3 + \sigma \) and \( \xi_4 = -\delta + \gamma \).

The eigenvalues above with the Hartman-Grobman Theorem indicates that (1) presents one disease-free equilibrium point, and considering the condition
given by the eigenvalue $\xi_2 = \alpha(1 - \theta)\frac{\lambda}{\delta} - (\beta_1 + \beta_2)$, if $\alpha(1 - \theta)\frac{\lambda}{\delta} < (\beta_1 + \beta_2)$ the eigenvector associated indicates an asymptotically stable direction. Consequently, if $\alpha(1 - \theta)\frac{\lambda}{\delta} > (\beta_1 + \beta_2)$ the equilibrium point $P_1$ is unstable, indicating a bifurcation in the parameter space.

### 3.2. Endemic equilibrium points

The endemic equilibrium points are characterized by the existence of infected people in the population, that is, $(I^* \neq 0)$.

Therefore, assuming the existence of an endemic equilibrium point, with $x^* \in U$, such that $f(x^*) = 0$, the equilibrium point $P_2 = x^* = (S^*, I^*, R^*, S^*_\text{ick})^T$ in the first octant of $\mathbb{R}^4$ is given by (6).

$$
\begin{align*}
S^* &= \frac{\beta_1 + \beta_2}{\alpha(1 - \theta)} \\
I^* &= \frac{1}{\phi}(\delta + \gamma)(\beta_3 + \sigma)[\alpha(1 - \theta)\lambda - (\beta_1 + \beta_2)\delta] \\
S^*_\text{ick} &= \frac{1}{\phi}\beta_2(\delta + \gamma)[\lambda\alpha(1 - \theta) - (\beta_1 + \beta_2)\delta] \\
R^* &= \frac{1}{\phi}(\beta_1\beta_3 + \beta_2\beta_3 + \beta_1\sigma)[\alpha(1 - \theta)\lambda - (\beta_1 + \beta_2)\delta],
\end{align*}
$$

where $\phi = \alpha(1 - \theta)(\beta_1\beta_3\delta + \beta_2\beta_3\delta + \beta_1\delta\sigma + \beta_2\delta\sigma + \beta_2\gamma\sigma)$.

Accordingly, the existence condition of a positive endemic equilibrium $P_2 = x^* = (S^*, I^*, R^*, S^*_\text{ick})^T \in \mathbb{R}^+$ is given by (6).

$$
\alpha(1 - \theta)\frac{\lambda}{\delta} > \beta_1 + \beta_2. \tag{6}
$$

Note that condition (6) reflects the fact that, in order to the endemic equilibrium exists, the rate $\alpha$ at which, the people flux rate, represented by $\lambda/\delta$, become infected, has to be greater than the rate at which infected population leave the compartment $I$, either overcoming the disease or becoming symptomatic.

It is important to highlight that $\lambda/\delta$ could also represent the total people flux commuting from a different city in a multi-population model.

The linearization $A = J_{P_2} = \left. \frac{\partial f}{\partial x} \right|_{x^*}$ at the endemic equilibrium is:
\[
A = \begin{bmatrix}
-(\delta + I^* \alpha(1 - \theta)) & -(\beta_1 + \beta_2) & 0 & \gamma \\
I^* \alpha(1 - \theta) & 0 & 0 & 0 \\
0 & \beta_2 & -(\beta_3 + \sigma) & 0 \\
0 & \beta_1 & \beta_3 & -(\delta + \gamma)
\end{bmatrix}.
\] (7)

The characteristic polynomial \(\det(A - I_d \xi) = 0\) is

\[
\xi^4 + a_1 \xi^3 + a_2 \xi^2 + a_3 \xi + a_4 = 0,
\] (8)

with

\[
a_1 = \beta_3 + 2\delta + \gamma + \sigma + I^* \alpha(1 - \theta);
\]

\[
a_2 = I^* \alpha(1 - \theta) (\beta_1 + \beta_2 + \beta_3 + \delta + \gamma + \sigma) + 2\delta (\beta_3 + \sigma) + \gamma (\beta_3 + \delta + \sigma) + \delta^2;
\]

\[
a_3 = I^* \alpha(1 - \theta) [\beta_1 \beta_3 + \beta_2 \beta_3 + (\beta_1 + \beta_2 + \beta_3) \delta + (\beta_2 + \beta_3) \gamma + \\
(\beta_1 + \beta_2 + \delta + \gamma) \sigma] + \beta_3 \delta^2 + \delta^2 \sigma + \beta_3 \delta \gamma + \delta \gamma \sigma;
\]

\[
a_4 = I^* \alpha(1 - \theta) (\beta_1 \beta_3 \delta + \beta_2 \beta_3 \delta + \beta_1 \delta \sigma + \beta_2 \delta \sigma + \beta_2 \gamma \sigma).
\] (9)

Any further effort to analytically analyze \(\xi\) eigenvalues, becomes quite difficult due to the coefficients complexity. A possible alternative approach is to go for numerical calculations.

However, some insight for the model with feedback \(\gamma\) can be obtained, in terms of bifurcations and stability, if we analyze the eigenvalues when \(\gamma = 0\) and \(\gamma \neq 0\).

3.2.1. Eigenvalues for \(\gamma = 0\)

Note that in this case, the endemic equilibrium is still possible. Computing the eigenvalues results,
\[ \xi_1 = -\delta, \]
\[ \xi_2 = -(\beta_3 + \sigma), \]
\[ \xi_3 = \frac{1}{2(\beta_1 + \beta_2)}(-\alpha(1-\theta)\lambda + \sqrt{\Delta}), \]
\[ \xi_4 = \frac{1}{2(\beta_1 + \beta_2)}(-\alpha(1-\theta)\lambda - \sqrt{\Delta}), \]

such that \( \Delta = 4\delta(\beta_1 + \beta_2)^3 + (\alpha(1-\theta))^2\lambda^2 - 4\alpha(1-\theta)(\beta_1 + \beta_2)^2. \)

The eigenvalues \( \xi_3 \) and \( \xi_4 \) can be either complex conjugate stable, or both real. The eigenvalue \( \xi_3 \) needs to be further studied due to the possibility of bifurcation.

Analysing the eigenvalue \( \xi_3 \), if \( \alpha(1-\theta)\lambda > \sqrt{\Delta} \), \( P_2 \) is stable, and consequently equation (6) holds true.

On the other hand, if \( \alpha(1-\theta)\lambda < \sqrt{\Delta} \), the endemic equilibrium point is unstable, and the existence condition (6) is not satisfied, consequently, the endemic equilibrium point \( P_2 \), if existing, is stable.

3.2.2. Eigenvalues for \( \gamma \to \infty \)

Another insight can be obtained by looking with the case \( \gamma \to \infty \). In this case, the endemic equilibrium becomes:

\[ S^* = \frac{\beta_1 + \beta_2}{\alpha(1-\theta)}, \]
\[ I^* \to \frac{(\beta_4 + \sigma)}{\alpha(1-\theta)^2\beta_2\sigma}[\alpha(1-\theta)\lambda - (\beta_1 + \beta_2)\delta], \]
\[ S_{\text{ck}}^* \to \frac{\beta_1 + \beta_2}{\alpha(1-\theta)}, \]
\[ R^* \to 0, \]

which is subject to the same existence condition given in equation (6).

Under the assumption \( \gamma \to \infty \), the characteristic polynomials coefficients in equation (9) can be approximated by (12):
\[ a_1 \approx \gamma, \]
\[ a_2 \approx \gamma(I^* \alpha(1-\theta) + \beta_3 + \delta + \sigma) = \gamma b_2, \]  \hspace{1cm} (12)
\[ a_3 \approx \gamma(I^* \alpha(1-\theta)(\beta_2 + \beta_3 + \sigma) + \delta(\beta_3 + \sigma)) = \gamma b_3, \]
\[ a_4 = \gamma I^* \alpha(1-\theta)\beta_2 \sigma = \gamma b_4, \]

the characteristic polynomial (8) can be rewritten as in (13):

\[ \xi^4 + \gamma \xi^3 + \gamma b_2 \xi^2 + \gamma b_3 \xi + \gamma b_4 = 0, \]  \hspace{1cm} (13)
\[ \xi^4 + \gamma (\xi^3 + b_2 \xi^2 + b_3 \xi + b_4) = 0, \]
assuming that at least one root \(|\xi|\) goes to infinity as \(\gamma \to \infty\), we rewrite the polynomial as

\[ \xi^4 + \gamma \xi^3 = 0, \]
\[ \xi^3 (\xi + \gamma) = 0, \]  \hspace{1cm} (14)
then,

\[ \xi_1 = -\gamma, \text{ and } \gamma \to \infty, \]

so, one eigenvalue seem to be going to \(-\infty\) as \(\gamma \to \infty\). To find an approximation to the other three roots, the characteristic polynomial is rewritten in equation (13). It can be also assumed that the other three roots are finite,

\[ \xi^4 + \gamma \xi^3 + \gamma b_2 \xi^2 + \gamma b_3 \xi + \gamma b_4 = 0, \]
\[ \frac{1}{\gamma} \xi^4 + \xi^3 + b_2 \xi^2 + b_3 \xi + b_4 = 0, \]  \hspace{1cm} (15)
\[ \approx \xi^3 + b_2 \xi^2 + b_3 \xi + b_4 = 0. \]

Looking for insight numerical experiments are performed.
4. Parameters fitting by the least-squares method

In this section the proposed SIRSi model (1) (see Fig. 1) is numerically fitted to three major cities in the state of São Paulo - Brazil, as shown in table 1 according to SEADE [41].

| City      | Total population in 2020 |
|-----------|-------------------------|
| São Paulo | 11.869.660              |
| Campinas  | 1.175.501               |
| Santos    | 428.703                 |

Table 1: Total population collected from SEADE.

The data available for the birth and death rates from the public repository was out of date, with the last report been published in 2018, therefore, for 2019 and 2020 these rates were computed using interpolation, results are shown in Fig. 2.

![Birth rate x 1000 inhabitants vs Death rate x 1000 inhabitants](image)

Figure 2: Birth and death rates per 1000 inhabitants for São Paulo, Santos and Campinas. Public data is shown in solid lines and interpolated values are shown in dashed lines.

The social distancing measure was also taken into account in the model, represented by $\theta$, for this parameter, the mean of the social distancing data was considered in each case. The time series along with the mean are shown in figure
For the parameters fitting, the feedback parameter $\gamma$ was kept constant at values $\gamma \in \{0, 0.01, 0.02, 0.03, 0.04\}$, and the trust-region reflective least-squares algorithm \cite{43, 44, 45} was used for each one of the three cities into consideration. All parameters and initial conditions computed are normalized with respect to the total population in each case. The results are shown in tables 2 to 7.

In order to assess the influence of the parameter $\gamma$ in the endemic equilibrium, the eigenvalues were plotted with the set of fitted parameters for $\gamma$ from 0 to 2. If figures 4 are shown the eigenvalues for the endemic equilibrium for each city computed for each one of the fitted sets. At $\gamma = 0$ eigenvalues are stable for...
São Paulo e Santos, as $\gamma$ increases, eigenvalues move towards the left-hand side of the complex plane, whereas for Campinas eigenvalues are unstable for $\gamma = 0$. 
|            | Fit 1     | Fit 2     | Fit 3     | Fit 4     | Fit 5     |
|------------|-----------|-----------|-----------|-----------|-----------|
| $S_0$      | 1.006e+00 | 9.927e-01 | 1.004e+00 | 1.017e+00 | 9.685e-01 |
| $I_0$      | 1.141e-05 | 1.141e-05 | 1.168e-05 | 1.852e-05 | 1.402e-05 |
| $S_{sick}$ | 0.000e+00 | 0.000e+00 | 0.000e+00 | 0.000e+00 | 0.000e+00 |
| $R_0$      | 0.000e+00 | 0.000e+00 | 0.000e+00 | 0.000e+00 | 0.000e+00 |

Table 5: Fitted initial conditions for Santos.

|            | Fit 1     | Fit 2     | Fit 3     | Fit 4     | Fit 5     |
|------------|-----------|-----------|-----------|-----------|-----------|
| $\alpha$   | 7.473e-01 | 7.472e-01 | 7.466e-01 | 7.464e-01 | 7.384e-01 |
| $\beta_1$  | 1.350e-01 | 1.352e-01 | 1.354e-01 | 1.355e-01 | 1.368e-01 |
| $\beta_2$  | 1.930e-01 | 1.934e-01 | 1.933e-01 | 1.934e-01 | 1.923e-01 |
| $\beta_3$  | 5.631e-02 | 5.582e-02 | 5.225e-02 | 5.281e-02 | 5.965e-02 |
| $\sigma$   | 1.995e-01 | 1.954e-01 | 1.899e-01 | 1.906e-01 | 1.996e-01 |
| $\lambda$  | 3.353e-05 | 3.353e-05 | 3.353e-05 | 3.353e-05 | 3.353e-05 |
| $\delta$   | 4.509e-05 | 4.509e-05 | 4.509e-05 | 4.509e-05 | 4.509e-05 |
| $\theta$   | 4.842e-01 | 4.842e-01 | 4.842e-01 | 4.842e-01 | 4.842e-01 |
| $\gamma$   | 0.000e+00 | 1.000e-02 | 2.000e-02 | 3.000e-02 | 4.000e-02 |

Table 6: Fitted parameter for Campinas.

|            | Fit 1     | Fit 2     | Fit 3     | Fit 4     | Fit 5     |
|------------|-----------|-----------|-----------|-----------|-----------|
| $S_0$      | 1.008e+00 | 1.008e+00 | 1.007e+00 | 1.007e+00 | 1.005e+00 |
| $I_0$      | 8.749e-06 | 9.148e-06 | 9.364e-06 | 9.614e-06 | 1.776e-05 |
| $S_{sick}$ | 0.000e+00 | 0.000e+00 | 0.000e+00 | 0.000e+00 | 0.000e+00 |
| $R_0$      | 0.000e+00 | 0.000e+00 | 0.000e+00 | 0.000e+00 | 0.000e+00 |

Table 7: Fitted initial conditions for Campinas.

In a closer view of the eigenvalues around the origin are shown.

5. Numerical experiments

In this section the numerical experiments are conducted using the MATLAB-Simulink for two different conditions. Firstly, the SIRSi model is fitted with the real data for the $S_{sick}$ population, and for different values of the parameter
$\gamma$. In the sequence, the simulation for the infected population $I$, that can be inferred from the proposed model, is carried out.

The numerical experiments, as in section 4, were conducted for three major cities in the state of São Paulo, namely, São Paulo, Campinas, and Santos.

The initial condition is $x_0 = (S_0, I_0, S_{\text{ic}0}, R_0)^T$, where $S_0$ and $I_0$ are the normalized susceptible and infected populations, which are considered free parameters in the sense that they can be modified by the fitting algorithm.

5.1. Simulation results for São Paulo

In Fig. 6 it can be seen that the SIRSi model is adjusted for the confirmed cases of infected people data.

Considering that the acquired immunity is permanent, i.e., $\gamma = 0$, and that the isolation rate is constant, the peak of the infection occurs soon after July.
2020, and until the end of the the same year, the disease will not persist, since the number of confirmed cases will go down to zero.

On the other hand, assuming that immunity is not permanent and adopting a reinfection rate $\gamma = 0.01$, meaning that every 100 days a recovered person becomes susceptible again, the model predicts a decrease in the confirmed cases and a new wave of infection in the first half of 2022.

Decreasing the time interval to 50 days in which a recovered person becomes susceptible ($\gamma = 0.02$), the model indicates a second wave of infection in the first half of 2021. For the situation in which a recovered system is susceptible to each 25 days ($\gamma = 0.04$), the model simulation shows that by the end of this year the number of confirmed infected will reduce by almost two thirds and that the number of infected will continue decreasing over time, but the number of confirmed cases will remain higher than the other simulated curves.
In Fig. 6, the infected compartment $I$ inferred from the SIRSi model is presented, showing that the peak of infection is close to July 2020.

Increasing $\gamma$ will reduce the time for a recovered person to become susceptible again, causing the peaks in Fig. 7 to increase, when compared to the curves for lower values of $\gamma$. This behavior, however, cannot be observed in Fig. 6, indicating that the increase the re-susceptibility feedback gain $\gamma$ possibly contributes to the increase of asymptomatic or unreported infected cases.

In addition, it appears that if those recovered acquire permanent immunity $\gamma = 0$, the number of infected people tends to zero by the end of 2020. For $\gamma = 0.01$, it appears that there is a small increase in January 2022. For $\gamma = 0.02$ a new wave of confirmed cases can be seen in Fig. 6, and accordingly, the increase in the infected population is also observed in Fig. 7.
For São Paulo, the numerical experiments show that considering any reinfection rate, there will be confirmed infected cases and unreported infected cases until 2023, indicating the need for a control strategy, being necessary and the study of preventive inoculation.

5.2. Simulation results for Santos

Fig. 8 shows the SIRSi model adjusted to the confirmed cases of infected people data for Santos. Assuming that the immunity acquired is permanent, $\gamma = 0$, and that the isolation rate is constant, the peak of the confirmed cases in Santos will occur very close to July 2020, and similarly to São Paulo (see Fig. 6), until the end of the same year, the disease will not persist with the number of confirmed cases going down to zero.

Adopting a nonzero reinfection rate, such that one person every 100 days
becomes susceptible again ($\gamma = 0.01$), a second wave of infection is seen in the coastal city around July 2021, one year before the second wave predicted for São Paulo with the same value for the re-susceptibility feedback gain $\gamma$.

Considering $\gamma = 0.02$, for which an infected person becomes susceptible again in a time interval of 50 days, the second wave of confirmed cases occurs at the beginning of the first half of 2021 and the number of confirmed infected is reduced to one third of the peak value.

These situations should be analyzed with caution and it is suggested to study the influence of the flow of people between these cities, since in the city of Santos the second waves of infections occur before the city of São Paulo.

For $\gamma = 0.04$, after the peak of the confirmed cases, a second wave can be observed in the numerical results before the end of 2020, delaying the reduction of the confirmed cases.
For Santos, the numerical experiments show that for $\gamma = 0$ and for $\gamma = 0.03$, the numbers of confirmed cases tend to zero in the beginning of 2023.

The infected compartment $I$ inferred from the SIRSi model is presented in Fig. 9, showing that the peak of infection is close to July 2020.

The increase in re-susceptibility feedback gain $\gamma$, will reduction the time for an infected person to be susceptible again, causing higher peaks when compared with the curves for lower values of $\gamma$. This behavior does not occur in Fig. 8 indicating that the increase in feedback possibly contributes to the increase in asymptomatic or unreported infected cases. This situation is similar to what is observed for São Paulo.

In addition, for $\gamma = 0$, the number of the infected people $I$ tends to zero before the end of the 2020 (see the curve for $\gamma = 0$ in Fig. 9).

For $\gamma = 0.01$, a new wave of infection in 2021 can be seen, and for $\gamma = 0.02$
the peak of the second wave of infection is near January 2021 (see Fig. 9, \( \gamma = 0.01 \) and \( \gamma = 0.02 \)).

Unlike São Paulo, the highest peak of infection among the unreported occurs when \( \gamma = 0.03 \) and this behavior suggests a more detailed study of the dynamics, because together with the situation in which the infected person acquires permanent immunity, these rates suggest that the saving of confirmed cases (see Fig. 8) and asymptomatic infected individuals tends to zero more quickly.

In the situation in which a recovered person is liable to a new susceptibility in 25 days, it is observed that the infection persists in the population for a longer time, as shown by the curve with \( \gamma = 0.04 \) in Fig. 9 and justifies the policy strategies public policies, including vaccination.

5.3. Simulation results for Campinas

In Fig. 10 the SIRSi model adjusted to the confirmed cases of infected people data in Campinas is shown.

For permanent acquired immunity, \( \gamma = 0 \), and constant isolation rate, the peak of confirmed infection cases occur in the beginning of the second half of 2020.

Considering the re-susceptibility feedback gain \( \gamma > 0 \), in Fig. 10, it seems that with the increasing of \( \gamma \) the time necessary for the number of confirmed infected cases go down to zero is slightly bigger, unlike the other two cities studied. In addition, Campinas does not present a second wave of infection, even with the variation \( \gamma \).

The general behavior of Campinas, concerning the sensitivity analysis for \( \gamma \), present results that differ from Santos and São Paulo. It can be noticed in Fig. 11 that the observed data are far from the peak of infection predicted by the SIRSi model. At this point, more data is needed for any further qualitative analysis.

Observing the eigenvalues for the city of Campinas (See Figs. 4 and 5) it can be noticed that they are all real, indicating that there is no oscillatory behavior.
in the dynamics of the model. Depending on the new data this situation might change.

The infected compartment of Campinas presents the peak of infection close to the beginning of the second half of 2020, Fig.11.

The increase in the reinfection parameter, causes the peak to increase and this occurs in the figure Fig.10 indicating that the increase in feedback possibly contributes to the increase in asymptomatic or unreported infected.

6. Conclusions

The proposed SIRSi model was fitted to publicly available data of the Covid-19 outbreak, providing estimates on the duration and peaks of the outbreak. In addition, the model allows to infer information related to unreported and asymptomatic cases.
The proposed model with feedback adjusted to the confirmed infection data, suggests the possibility of the recovered ones having temporary immunity $\gamma > 0$ or even permanent $\gamma = 0$.

Considering the situation in which immunity is temporary, there is a second wave of infection which, depending on the time interval for a recovered person to be susceptible again, indicates a second wave with a greater or lesser number of reinfected persons.

If the time interval is shorter (larger $\gamma$), the second wave of infection will have a greater number of infected people when compared to a shorter time interval of the feedback.

The qualitative behavior for São Paulo and Santos are similar in terms the sensitivity analysis of the re-susceptibility feedback gain $\gamma$. The bigger the $\gamma$ the shorter the time for a recovered person to become susceptible again to infection, increasing unreported or asymptomatic cases.
For the city of Campinas, it is suggested to collect more data, because as the data of the confirmed infected presents a certain distance from the peak of the infection, the dynamics of the model may undergo some significant change, given the sensitivity of the model to disturbances.

7. Availability of data and materials

Data are publicly available with [41, 42].

8. Declaration of competing interest

There is no conflict of interest between the authors.

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