A PROOF OF GROTHENDIECK’S BASE CHANGE THEOREM

E. TENGAN

Abstract. We give an elementary short proof of Grothendieck’s base change theorem for the cohomology of flat coherent sheaves.

1. Introduction

The purpose of this note is to give a short alternative proof of

Theorem 1.1 (Grothendieck). Let \( f: X \rightarrow Y \) be a proper map of noetherian schemes, and \( \mathcal{F} \) be a coherent sheaf on \( X \) which is flat over \( Y \). For \( y \in Y \) let \( X_y = X \times_Y \text{Spec} \kappa(y) \) be the fiber of \( y \), and \( \mathcal{F}_y \) be the pullback of \( \mathcal{F} \) to \( X_y \).

(a) The base change map
\[
\varphi^p(y): R^pf_*\mathcal{F} \otimes_{O_Y} \kappa(y) \to H^p(X_y, \mathcal{F}_y)
\]
is surjective if and only if it is an isomorphism.

(b) Suppose that \( \varphi^p(y) \) is surjective. Then the following conditions are equivalent:

(i) \( \varphi^{p-1}(y) \) is also surjective;

(ii) \( R^pf_*\mathcal{F} \) is a free sheaf in a neighborhood of \( y \).

Furthermore, if these conditions hold for all \( y \in Y \), then the formation of \( R^pf_*\mathcal{F} \) commutes with arbitrary base change.

The traditional proofs found in [Har77] (theorem 12.11, p.290) and [Gro63] (théorème 7.7.5, p.67, proposition 7.7.10, p.71, proposition 7.8.4, p.73) rely on either the formal functions theorem or completion methods (in the spirit of the proof of the local criterion of flatness). On the other hand, Mumford [Mum08] (§5, p.46) has given streamlined proofs of all the main results in cohomology of base change except for above one. Mumford’s methods can be readily adapted to prove theorem 1.1 as well, and in a quite elementary fashion. Surprisingly, I could not find any written account thereof; that is the reason why I decided to write one.

2. Linear algebra over local rings

All proofs of results in cohomology of base change are based on the following key technical result (see [Mum08], §5, p.46):

Theorem 2.1 (The Grothendieck complex). Let \( f: X \rightarrow \text{Spec} A \) be a proper map of noetherian schemes, and \( \mathcal{F} \) be a coherent sheaf on \( X \) which is \( A \)-flat. There exists a finite complex \( (F^\bullet, d^\bullet) \) of finitely generated \( A \)-flat modules such that for any \( A \)-algebra \( B \) we have an isomorphism, functorial in \( B \),
\[
H^p(X \otimes_A B, \mathcal{F} \otimes_A B) = H^p(F^\bullet \otimes_A B)
\]

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1No, this is not any disorder suffered by those who unsuccessfully tried to learn scheme theory
By the above remark, the matrix \( (F) \) is invertible. Indeed, clearly the matrix \( (F) \) is invertible since both the \( \pi_j \) and the \( T_i \) form \( k \)-bases of \( F/mF \), hence \( \det((\pi_{ij})) \neq 0 \) for some \( i, j \). This shows that \( (a_{ij}) \) is also invertible, and therefore the \( e_i \) also form an \( A \)-basis of \( F \).

**Lemma 2.2.** Let \( (A, m, k) \) be a local ring, and \( d: F \to F' \) be a map of free \( A \)-modules of finite rank. There exist decompositions

\[
F = V \oplus W \quad F' = W' \oplus U
\]

such that \( d(V) \subseteq mF' \) and \( d \) restricts to an isomorphism \( d: W \approx W' \). In other words, there exist \( A \)-bases of \( F \) and \( F' \) with respect to which

\[
d = \begin{pmatrix}
M_{s \times r} & Id_{s \times s} \\
N_{t \times r} & 0_{t \times s}
\end{pmatrix}
\]

with all entries of \( M, N \) belonging to \( m \).

**Proof.** Consider the \( k \)-linear map \( d \otimes 1: F \otimes_A k \to F' \otimes_A k \), and choose \( e_1, \ldots, e_{r+s} \in F \) so that \( e_1, \ldots, e_r \) is a \( k \)-basis of \( F \otimes_A k \), with the first \( r \) vectors generating \( \ker(d \otimes 1) \). Hence \( d(e_{r+1}), \ldots, d(e_{r+s}) \in F' \otimes_A k = F'/mF' \) is a \( k \)-basis of \( \text{im}(d \otimes 1) \).

By the above remark, the \( e_i \) form an \( A \)-basis of \( F \), and we may find an \( A \)-basis \( f_1, \ldots, f_{s+t} \) of \( F' \) with \( f_1 = d(e_{r+1}), \ldots, f_s = d(e_{r+s}) \). Now set

\[
V = Ae_1 \oplus \cdots \oplus Ae_r \\
W = Ae_{r+1} \oplus \cdots \oplus Ae_{r+s} \\
W' = Af_1 \oplus \cdots \oplus Af_s \\
U = Af_{s+1} \oplus \cdots \oplus Af_{s+t}
\]

and we are done. \( \square \)

**Lemma 2.3.** In the notation above, the following conditions are equivalent:

(i) \( \varphi^p \) is an isomorphism;

(ii) \( \varphi^p \) is surjective;

(iii) \( d^p \) can be put in matrix form

\[
d^p = \begin{pmatrix}
0 & Id \\
0 & 0
\end{pmatrix}
\]

for some choice of \( A \)-bases of \( F^p \) and \( F^{p+1} \).

(iv) \( \ker(d^p) \) and \( \text{im}(d^p) \) are direct summands of \( F^p \) and \( F^{p+1} \), respectively (in particular, they are free since \( A \) is local noetherian).
Proof. Clearly (i) ⇒ (ii) and (iii) ⇔ (iv). Next, observe that \((F^* \otimes_A k, d^* \otimes 1)\) can be written as
\[
\cdots \longrightarrow \frac{F^p}{mF^p} \longrightarrow \frac{F_p}{mF_p} \longrightarrow \frac{F^{p+1}}{mF^{p+1}} \longrightarrow \cdots
\]
and the base change map \(\varphi^p : \mathcal{H}^p(F^*) \otimes_A k \to \mathcal{H}^p(F^* \otimes_A k)\) as the natural map
\[
\varphi^p : \frac{\text{im } d^p}{\text{im } d^{p-1} + m \ker d^p} \to \frac{(d^p)^{-1}(mF^{p+1})}{\text{im } d^{p-1} + mF_p}
\]
This shows that (iii) ⇒ (i): if \(F^p = V \oplus W\) is the corresponding decomposition in (iii) with \(V = \ker d^p\), we have
\[
\frac{(d^p)^{-1}(mF^{p+1})}{\text{im } d^{p-1} + mF_p} = \frac{V \oplus mW}{\text{im } d^{p-1} + mV} = \frac{V}{\text{im } d^{p-1} + mV}
\]
since \(\text{im } d^{p-1} \subseteq V = \ker d^p\), and thus \(\varphi^p\) is an isomorphism.

Finally, to prove that (ii) ⇒ (iii), notice first that from (i) we get
\[
\varphi^p \text{ is surjective } \iff \ker d^p + mF_p = (d^p)^{-1}(mF^{p+1})
\]
Now applying the previous lemma to \(d^p\), there are decompositions \(F^p = V \oplus W\) and \(F^{p+1} = W' \oplus U\) with respect to which \(d^p\) has matrix
\[
d^p = \begin{pmatrix} M & \text{Id} \\ N & 0 \end{pmatrix}
\]
where all entries of \(M\) and \(N\) belong to \(m\). Therefore
\[
(d^p)^{-1}(mF^{p+1}) = V \oplus mW \quad \text{and} \quad \ker d^p = \{(v, -Mv) \in V \oplus W \mid Nv = 0\}
\]
and the right hand side of (***) becomes
\[
\ker N + mV = V \overset{\text{Nakayama}}{\iff} \ker N = V \iff N = 0
\]
But now \(\ker d^p = \{(v, -Mv) \in V \oplus W \mid v \in V\}\) is a free summand of \(F^p\), and a final change of basis puts \(d^p\) in the desired format: just right multiply it by the following invertible matrix, whose columns on the left form a basis of \(\ker d^p\).
\[
\begin{pmatrix} \text{Id} & 0 \\ -M & \text{Id} \end{pmatrix}
\]
\[\square\]

Lemma 2.4. Suppose that \(\varphi^p\) is surjective. Then the following are equivalent:

(i) \(\varphi^{-1}\) is surjective;

(ii) \(\mathcal{H}^p(F^*)\) is free.

Furthermore, if these conditions hold, then \(\mathcal{H}^p(F^*) \otimes_A B = \mathcal{H}^p(F^* \otimes_A B)\) for any \(A\)-algebra \(B\).

Proof. Since \(\varphi^p\) is surjective, by the previous lemma \(\ker d^p\) is free, hence replacing \(F^p\) by \(\ker d^p\) we may assume that \(d^p = 0\), and we have exact sequences
\[
\begin{align*}
0 & \longrightarrow \text{im } d^{p-1} \longrightarrow F^p \longrightarrow \mathcal{H}^p(F^*) \longrightarrow 0 \\
0 & \longrightarrow \ker d^{p-1} \longrightarrow F^{p-1} \longrightarrow \text{im } d^{p-1} \longrightarrow 0
\end{align*}
\]
Now if \(\varphi^{-1}\) is surjective, then by the previous lemma \(\mathcal{H}^p(F^*) = F^p / \text{im } d^{p-1}\) is free. Conversely, suppose \(\mathcal{H}^p(F^*)\) is free. Then the first sequence splits, showing that \(\text{im } d^{p-1}\) is free, and thus the second sequence splits as well. Therefore both
ker \(d^{p-1}\) and im \(d^{p-1}\) are direct summands of \(F^{p-1}\) and \(F^p\) respectively, and by the previous lemma \(\varphi^{p-1}\) is surjective.

Finally, \(H^p(F^\bullet) \otimes_A B = H^p(F^\bullet \otimes_A B) \iff (\text{coker } d^{p-1}) \otimes_A B = \text{coker}(d^{p-1} \otimes 1)\) also directly follows from the matrix form of \(d^{p-1}\) in (iii) of the previous lemma. □

3. A USEFUL COROLLARY

For completion, we include one of the main applications of theorem 1.1, namely the following corollary, “which is extremely useful, but which is unfortunately buried there in EGA III” ([MFK94], p.19).

Corollary 3.1. Let \(f: X \to Y\) be a proper map of noetherian schemes, and \(F\) be a coherent sheaf on \(X\) which is flat over \(Y\). If \(H^1(X_y, F_y) = 0\) for all \(y \in Y\), then

(a) \(R^1 f_* F = 0\)

(b) \(f_* F\) is a locally free \(O_Y\)-module, whose formation commutes with arbitrary base change.

Proof. By theorem 1.1(a) applied with \(p = 1\), \(R^1 f_* F \otimes_{O_Y} \kappa(y) = 0\) for all \(y \in Y\), and since \(R^1 f_* F\) is coherent (Serre’s theorem, see [Gro61], théorème 3.2.1, p.116), \(R^1 f_* F = 0\) by Nakayama’s lemma. Now by theorem 1.1(b) applied with \(p = 1\), we have that \(\varphi^0(y)\) is surjective for all \(y \in Y\). Finally, applying theorem 1.1(b) again with \(p = 0\) finishes the proof: \(\varphi^{-1}(y)\) is an isomorphism since both \(R^p f_* F \otimes_{O_Y} \kappa(y)\) and \(H^p(X_y, F_y)\) vanish for \(p = -1\) (alternatively, it is easy to check that \(H^0(F^\bullet)\) is free directly: since \(R^1 f_* F = 0\), the Grothendieck complex is exact at \(p = 1\), hence free; now the exact sequence \(0 \to \ker d^0 \to F^0 \to \im d^0 \to 0\) splits, showing that \(\ker d^0 = H^0(F^\bullet)\) is free). □

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REFERENCES

[A Grothendieck. Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I. Inst. Hautes Études Sci. Publ. Math., (11):167, 1961.

[B] A. Grothendieck. Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. II. Inst. Hautes Études Sci. Publ. Math., (17):91, 1963.

[H] Robin Hartshorne. Algebraic geometry. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[M] Hideyuki Matsumura. Commutative ring theory, volume 8 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[MFK] D. Mumford, J. Fogarty, and F. Kirwan. Geometric invariant theory, volume 34 of Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], Springer-Verlag, Berlin, third edition, 1994.

[M] David Mumford. Abelian varieties, volume 5 of Tata Institute of Fundamental Research Studies in Mathematics. Published for the Tata Institute of Fundamental Research, Bombay, 2008. With appendices by C. P. Ramanujam and Yuri Manin, Corrected reprint of the second (1974) edition.

Department of Mathematics, ICMC, University of São Paulo, 13566-590, Brazil.
E-mail address: etengan@icmc.usp.br