Modeling and adaptive robust wavelet control for a liquid container system under slosh and uncertainty

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Abstract
Liquid sloshing in moving or stationary containers and flexible uncertainty caused by the slosh are considered to be the most probable causing unexpected coupling effects on the dynamics of many systems such as aerospace, ground vehicles, and high speed industries arms.

The coupling of dynamic liquid slosh in a container system with the uncertainty caused by the sensors or dampers is rare documented and this coupling can be considered as a highly nonlinear system.

In this paper, an investigation is presented to demonstrate a new approach for enabling the reduction of the liquid slosh and uncertainty by implementing adaptive robust wavelet control technique.

Starting by creating the mathematical dynamic model for the nonlinear slosh coupled by uncertainty, adaptive robust control based wavelet transform is applied for calculating optimal motion that minimize residual slosh and uncertainty.

Subsequently the adaptive robust control based wavelet network approximation and the appropriate parameter algorithms for the container system with slosh and uncertainty are derived to achieve the feedback linearization, adaptive control, and $H_{\infty}$ tracking performance.

The simulation results show that the effects of slosh errors and external uncertainty can be successfully attenuated within a desired attenuation level.

Keywords
Slosh-container system, wavelet control, nonlinear $H$-infinity control, nonlinear optimal control, nonlinear systems, robust control, Riccati equation, uncertainty

Introduction
The liquid sloshing in a partially filled container has been a significant problem in many engineering applications. The liquid slosh in the container system can deliver an additional uncertainties to the sensors or dampers. These additional uncertainties can upset the performance of the system and lead to instabilities in some cases.

The uncertainties caused by the liquid slosh in the container system are the result of three error categories:

- Systematic errors which may be due to the models used to quantify the slosh process in the container system under consideration.
- Random errors which may be due to natural variability of the slosh process.
- Spurious errors which may be due to unclear or incomplete mathematical definitions of the slosh.

Thus, it is important to design a controller that can effectively suppress the coupling of slosh and uncertainty to maintain system stability and safe operations.

Numerous researches have tried to provide control solutions to the challenging problems caused by the sloshing dynamics in the container system.

In several cases simplified feed-forward control models of container system are used to design stabilizing controllers without using any feedback sensors and robust input shaping in some cases.
Additionally, there have been several approaches based on numerical analysis, modeling, and simulation of parametric liquid sloshing in the container.5–7

Due to the fact that the feed-forward control model are extremely sensitive to the external errors which occurred in the container system, thus the feed-back control has been applied to eliminate the slosh of the liquid in the container system. The feed-back control methods are used by creating a specified motion with the using of a feedback sensors, which will reduce the residual amount of the slosh of the liquid in the container system.

These methods include PI/PID control,8–11 H-infinity control,12 sliding mode control,13–15 predictive controller,16 active force control,17 genetic algorithm,18 control the robot end-effector movement by using Lyapunov-based feedback controllers19 and variable gain by using super-twisting algorithm.20

There also exist a passive control methods which tried to smooth the liquid’s sloshing without direct measurements of the sloshing state such as intermittent air-bubble injection method,21 inverse dynamic control technique by generating a computer model to estimate the modes of oscillation of the liquid in the container system,22 guidance, navigation, and control (GNC) algorithm commands thruster firings to counter the fluid slosh forces23 and a shallow-depth sloshing absorber.24

Apart from the above reviewed control methods, it is noticed that the solutions for the problem of the liquid slosh produced by many researches had been suffered from the following problems:

1. The slosh and uncertainty combination was unavailable during the design of the mathematical model.
2. When the control algorithms were designed, the modeling of the slosh had not been considered as an additional source of uncertainty.
3. Many researches are based on the use of the linearized dynamic models to control the slosh. This traditional methods are extremely produced an additional uncertainty errors in the unmodeled dynamics of the container system.

In this paper an adaptive robust wavelet control technique is suggested to perfectly approximate the nonlinear combination of the slosh and uncertainty.

The suggested method considered the adaptive wavelet networks as a rough tuning control for the approximation of the nonlinear combination of slosh and uncertainty, and the $H_\infty$ control considered to be as a fine tuning control used to filter the approximation errors caused by the slosh and external uncertainties.

In addition, a pendulum model was developed to represent the slosh in the container system in presence of mismatched uncertainties.

![A simple pendulum representation for the slosh container system.](image)

The modeling of uncertainty which is due to the liquid slosh in the container system is considered to be an additional random errors that is compensated asymptotically by the robustness of the control algorithm. Finally an adaptive robust wavelet controller is designed to achieve system stabilizing, $H_\infty$ tracking and reducing the effect of both the slosh and uncertainty.

**Modeling of liquid slosh**

A commonly used model to represent the slosh in the container system is a simple pendulum24 as shown in Figure 1.

From Figure 1 it can be shown that the nonlinear model of the pendulum is configured from the horizontally movable cart with mass $M_m$, pendulum mass $m_p$, and $h$ is the pendulum rod length with damping constant $\delta$. In addition $u(t)$ represents the externally force on the cart in the x-direction and $g$ is the gravity force.

Subsequently the system coordinate can be defined with two variables, $x(t)$ which represents the position of the cart and $\theta(t)$ which indicates the tilt angle suggested to the vertically upward direction.

In the $x$-direction, the force equilibrium gives that the external force on the system must equal the $x$-directed force that affect the pendulum mass plus the force on the cart. In equation form it can be represented as:

$$M_m \ddot{x} + m_p \ddot{x}_i = u + \delta + \omega$$  \hspace{1cm} (1)

Where $\omega$ is an additional disturbance input due to the slosh effects which represents the uncertainty.

Denoting that the combination of slosh and uncertainty can be defined as:

$$\xi = \delta + \omega$$  \hspace{1cm} (2)

The time-dependent center of gravity location of the pendulum mass is given by the coordinates $(x_i, y_i)$. Thus the location of the center of gravity of the pendulum mass is simply written as:

$$x_i = x + h \sin \theta$$

and

$$y_i = h \cos \theta$$  \hspace{1cm} (3)
Substituting in equation (1) yields

\[ M_m \ddot{x} + m_p \frac{d^2}{dt^2} (x + h \sin \theta) = u + \xi \] (4)

By noting the following definitions:

\[ \frac{d}{dt} \sin \theta = u (\cos \theta) \dot{\theta} + \delta \] and
\[ \frac{d}{dt} \cos \theta = - (\sin \theta) \dot{\theta} + (\cos \theta) \dot{\theta} \]

also

\[ \frac{d}{dt} \sin \theta = - (\sin \theta) \dot{\theta} \quad \text{and} \quad \frac{d}{dt} \cos \theta = - (\cos \theta) \dot{\theta} \]

Thus

\[ (M_m + m_p) \ddot{x} - m_p h \sin \theta \dot{\theta}^2 + m_p \dot{h} \cos \theta \theta = u + \xi \] (7)

Since the perpendicular component of the force and the pendulum length are available, the resultant balance on the pendulum can be also achieved. The torque balance on the pendulum can be written as:

\[ (f_x \cos \theta) h - (f_y \sin \theta) h = (m_p g \sin \theta) h \] (8)

Where the force components \( f_x \) and \( f_y \) can be written as:

\[ f_x = \frac{d^2}{dt^2} x_i = m_p \left[ \ddot{x} - h \sin \theta \dot{\theta}^2 + h \cos \theta \dot{\theta} \right] \]

\[ f_y = m_p \frac{d^2}{dt^2} y_i = - m_p \left[ h \sin \theta \dot{\theta}^2 + h \cos \theta \dot{\theta} \right] \] (9)

\[ M_m \ddot{x} \cos \theta + m_p \ddot{h} \cos \theta = m_p g \sin \theta + \xi \cos \theta \] (10)

The nonlinear equation (10) includes the combination of slosh and uncertainty and can be reformed and put into the standard state form as:

\[ m_p h \theta = m_p g \sin \theta + \xi \cos \theta - m_p \ddot{x} \cos \theta \] (11)

And

\[ [M_m + m_p - m_p \cos^2 \theta] \ddot{x} = u + m_p h \cos \theta \dot{\theta}^2 - m_p g \sin \theta \cos \theta + \xi \sin^2 \theta \] (12)

Denoting that:

\[ \xi \sin^2 \theta = \xi - \xi \cos^2 \theta \] (13)

Then:

\[ \ddot{x} = \frac{m_p g \sin \theta + \xi \cos \theta - m_p \ddot{h} \cos \theta}{m_p \cos \theta} \] (14)

Multiplying both sides by \( \cos(\theta) \) gives

\[ [m_p \cos^2 \theta - (M_m + m_p) \dot{\theta}] \ddot{\theta} = u \cos \theta - (M_m + m_p) g \sin \theta + m_p h \cos \theta \sin \theta \dot{\theta} - \left( \frac{M_m + m_p}{m_p} \right) \xi \cos \theta + \xi \cos \theta \] (15)

If \( x \) is denoted as the position of the slosh with respect to the vertical axis, and \( v \) be the velocity of the slosh, then the dynamic equations of the slosh in the container system can be represented in following form:

\[ \ddot{x} = v \]

\[ v = g \sin(x) - \frac{m_p h^2 \cos(x) - \sin(x)}{M_m + m_p} \]

\[ + \frac{\cos(x)}{M_m + m_p} \left( \frac{m_p \cos(x)}{M_m + m_p} \right) u + \xi \] (16)

Adaptive feedback linearization control for the slosh in the container system

The approximately linearized model of the slosh in the container system can be written in a simple form as:

\[ \ddot{x} = A(x) x_n + D(x) x_n \dot{u} + \xi \] (17)

With the control input \( u \in \mathbb{R}^n \) and \( A, D : \mathbb{R}^n \to \mathbb{R}^n \)

if \( x \in \mathbb{R}^n \) is considered to be the desired trajectory reference which must be uniformly bounded and continuously differentiable, then \( \varepsilon \) will represent the tracking error and can be written as:

\[ \varepsilon = x_r - x \] (18)

To achieve the tracking control goal, the appropriate control law can be derived based on the using of the adaptive feedback linearization technique and defining \( u \) as:

\[ u = -A(x) + u_{aux} + v \frac{D(x)}{\dot{D}(x)} \] (19)

Where \( u_{aux} \) represents the auxiliary control variable and will be specified later, and \( v \) can be defined as a polynomial in the following:

\[ v = x^n + \alpha_1 (x^{n-1} - x^{n-1}) + \ldots + \alpha_n (x_r - x) \] (20)

Note that the coefficients \( \alpha_1, ..., \alpha_n \) are positive constant to be assigned such that the polynomial in equation (20) will be Hurwitz.

Thus the error dynamic of the container system can produce the following formula:

\[ \varepsilon^n = \alpha_1 \varepsilon^{n-1} + \ldots + \alpha_n \varepsilon = u_{aux} + \xi \] (21)

The error dynamic \( \varepsilon^n \) can also be represented in the state space form as:

\[ \varepsilon^n = \alpha_1 \varepsilon^{n-1} + \ldots + \alpha_n \varepsilon = u_{aux} + \xi \]
\[
\dot{e} = f\dot{e} + g(u_{aux} + \xi)
\]  
(22)

Where:

\[
f = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1
\end{bmatrix}
\]

And

\[
g = [0 \ldots 0]^T
\]

Because of the high nonlinear impact of combination of the slosh and uncertainty \(\xi\), the sub-boundaries of both \(A(x)\) and \(D(x)\) will be permanently uncertain. Therefore the adaptive strategies must be adopted as a wavelet network based adaptive algorithm by using the following wavelet networks\(^2\):

\[
\tilde{A}(x, \Psi_A) = \Psi_A^TW_A(c_A^T x)
\]

\[
\tilde{D}(x, \Psi_D) = \Psi_D^TW_D(c_D^T x)
\]  
(23)

The optimal parameter \(\Psi^*\) can be defined as the best approximation of \(\Psi_A\) and \(\Psi_D\) respectively and can be represented by\(^2\):

\[
\Psi_A^* = \arg\min_{\Psi_A} \max_x |\tilde{A}(x, \Psi_A) - A(x)|; \]

\[
\Psi_D^* = \arg\min_{\Psi_D} \max_x |\tilde{D}(x, \Psi_D) - D(x)|;
\]  
(24)

Then by the certainty equivalent principle, the adaptive wavelet control law can be implemented as follows\(^2\):

\[
u = \frac{-\tilde{A}(x, \Psi_A) + u_{aux} + v}{\tilde{D}(x, \Psi_D)}
\]  
(25)

by substituting in equation (17) yields:

\[
\dot{s}_n = A(x) + D(x)u - \tilde{D}(x, \Psi_D)u + \tilde{D}(x, \Psi_D)u + \xi
\]

\[
= (A(x) - \tilde{A}(x, \Psi_A)) + (D(x) - \tilde{D}(x, \Psi_D))u + u_{aux} + v + \xi
\]  
(26)

Thus, the corresponding error dynamic can be rewritten as:

\[
\dot{\tilde{e}} = f\tilde{e} + gu_{aux} + g[(A(x) - \tilde{A}(x, \Psi_A)) + (D(x) - \tilde{D}(x, \Psi_D))u] + g\xi
\]  
(27)

For achieving the best approximation, equation (27) can be rewritten in the form:

\[
\dot{\tilde{e}} = f\tilde{e} + gu_{aux} + g[(A(x, \Psi_A') - \tilde{A}(x, \Psi_A))
\]

\[
+ \tilde{D}(x, \Psi_D') - \tilde{D}(x, \Psi_D))u] + g\theta
\]  
(28)

Where \(\theta\) denoting the sum of the approximation errors due to the slosh and external uncertainty. This parameter is crucial and will be attenuated by \(u_{aux}\). However \(\theta\) can be represented by the formula:

\[
\theta = (A(x) - \tilde{A}(x, \Psi_A')) + (D(x) - \tilde{D}(x, \Psi_D'))u + \xi
\]  
(29)

The wavelet networks \(\tilde{A}(x, \Psi_A')\) and \(\tilde{D}(x, \Psi_D')\) ought to be prepared to realize \(\tilde{A}(x, \Psi_A')\) and \(\tilde{D}(x, \Psi_D')\) respectively to achieve the best tracking to the desired signal \(x_r\), so that the term in equation (28) must be equal to zero, such that:

\[
[(\tilde{A}(x, \Psi_A') - \tilde{A}(x, \Psi_A)) + (\tilde{D}(x, \Psi_D') - \tilde{D}(x, \Psi_D))u] = 0
\]  
(30)

**Design \(H_\infty\) tracking control based on the adaptive wavelet network**

According to equation (23), the tacking control law in equation (19) could have an adaptive wavelet network control law as\(^2\):

\[
u = -\Psi_A^TW_A(c_A^T x) + u_{aux} + v \overline{\Psi_D^TW_D(c_D^T x)}
\]  
(31)

With:

\[
u_{aux} = -\frac{1}{K}g^T P\tilde{e}
\]  
(32)

And the optimal approximation of:

\[
\dot{\Psi}_A = \gamma_A W_A(c_A^T x)g^T P\tilde{e}
\]  
(33)

\[
\dot{\Psi}_D = \gamma_D W_D(c_D^T x)g^T P\tilde{e}
\]  
(34)

Where \(r\) is defined in equation (20); \(K\) is a positive scalar and the positive definite matrix \(P = P^T\) is obtained from the solution of the Riccati-eq414140uation at steady-state such that:

\[
Pf + f^TP + \frac{2}{K}Pgg^TP + \frac{1}{r^2}Pgg^TP + Q = 0
\]  
(35)

where \(Q\) is a positive symmetric matrix and \(r\) represents the attenuated level.

The solution of the Riccati-equation equation (35) is achieved when the matrices \(P\) and \(Q\) are definite and symmetric.\(^3\)
Based on the adaptive scheme for $u_{aux}$, $\Psi_A$, and $\Psi_D$ and to achieve the optimal reduction of $\theta$ in equation (30), the $H_\infty$ tracking performance can be implemented by the following formula (Appendix A):

$$
\begin{align*}
\bar{e}^T \dot{Q} \bar{e} dt &\leq e^T(0) P e(0) + \frac{1}{\gamma_A} \bar{\Psi}^T_A(0) \bar{\Psi}_A(0) \\
&\quad + \frac{1}{\gamma_D} \bar{\Psi}_D^T(0) \bar{\Psi}_D(0) + r^2 \int_0^T \theta^T \theta dt
\end{align*}
$$

(36)

Where $T$ represents the time of the control process.

Based on equation (36) and with appropriate positive definite symmetric matrices $Q = Q^T$, $P = P^T$, positive weighting factors $\gamma_A$ and $\gamma_D$, it is clear that the impact of the approximation errors summation $\theta$ on the error tracking $\bar{e}$ can be reduced by the factor $r$. Likewise the initial errors in equation (36) $[e(0), \bar{\Psi}_A(0)$ and $\bar{\Psi}_D(0)]$ are viewed to be an additional uncertainties which can influence with the tracking error $\bar{e}$.

In order to achieve solution of Riccati-equation, it tends to be demonstrated that the following accompanying condition must be likewise fulfilled:

$$
K \leq 2r^2
$$

(37)

The $H_\infty$ tracking performance with specific reduction level of $r$ for the slosh container system described in equation (17) based on the use of the adaptive wavelet network control in equations (31) – (36) can generally be accomplished only if the accompanying condition described in equation (37) is satisfied.

In addition, the control variable $u_{aux}$ described in equation (32) must increase to achieve the attenuation of $\theta$ for a desired level of $r$.

Results and discussion

In this study, a numerical example is proposed to demonstrate the feasibility and effectiveness of the developed method used by considering the slosh container system in equation (16). The combination of slosh and uncertainty $\xi$ used in the simulation was defined by the following expression:

$$
\xi(x, v, t) = \frac{1 - 2 \cos(2t)x^2v^2}{2 + 0.5 \sin(t) + 1.5x^2 + 1.5v^2}
$$

(38)

Based on equation (16) it is assumed that the position state $x$ and the velocity state $v$ are measurable with reference states $x_r$ and $v_r$, respectively.

The reference states for both position and velocity used in this simulation is assumed to be in the following form:

$$
ref_{state}(t) = (\pi/6) \sin(t)
$$

Indeed, the following data used in the simulation:

- The mass of the cart $M_m = 1$ kg, the mass of the pendulum $m_p = 0.2$ kg, the length of the pendulum $h = 0.4$ m and the gravity acceleration $g = 9.81$ m/sec².

To construct the matrix $f$ in equation (22), $\alpha$ is randomly chosen between $\alpha_{min} = 0.1$ and $\alpha_{max} = 1.3$.

Furthermore, in this simulation 100 hidden layer nodes are formed by the wavelet function.

According to equation (23) the initial values of the Gaussian function centers $c_i$ are uniformly randomly assigned inside a hyper-cubic $[-2\pi, 2\pi]^d$ with $c_i(0) = 0.5\pi$.

Additionally, $W_i$ is randomly assigned with $W_{max} = 1$ and $W_{min} = 3$ and $W_i(0)$ is assumed to be zero for all $i = 1, ..., 100$.

Finally the positive weighting factors $\gamma_A$ and $\gamma_D$ are assumed to be equal to five for all $i = 1, ..., 100$.

**Case 1**: by using the adaptive feedback linearization control for the slosh container system, the position, velocity, and input control will take the shape shown in Figure 2.

It is obvious from Figure 2 that the position state couldn’t track the reference state impeccably due to the nonlinear impact of the combination of the slosh and uncertainty. Moreover the impact of the control input stayed consistent and with no change regardless of non-tracking. This implies that the control input from the moment $t = 1.8$ s was unable to lessen the effect of the errors caused by the combination of both slosh and uncertainty. Thus the tracking error for the position state can be shown in Figure 3.

**Case 2**: two mother wavelets named PolyWOG1 and Shannon (Appendix B) are considered for developing the adaptive wavelet control. Since $A(x)$ and $D(x)$ are unknown as stated in the previous sections, thus in this simulation the function $A(x)$ had been approximated based on Euler-langrange method and $D(x)$ approximated by the activation wavelet functions illustrated in Appendix B.

i. by using the Adaptive wavelet Control method for the slosh container system with approximation wavelet function PolyWOG1 and $(r = 0.04)$, the position, velocity, and input control will take the shape shown in Figure 4.

ii. by using the Adaptive wavelet Control for the slosh container system with approximation wavelet function POLYWOG1 and $(r = 0.06)$, the position, velocity and input control will take the shape shown in Figure 5.

According to Figures 4 and 5 it is obvious that the position state $x$ is perfectly tracked the reference state $x_r$ at attenuation level of $r = 0.06$ regardless of the nonlinear effect of the combination of slosh and uncertainty. Consequently the adaptive wavelet tracking error is shown in Figure 6 for various value of attenuation level $r$.

iii. by using the Adaptive wavelet Control method for the slosh container system with
approximation wavelet function Shannon and \( r = 0.06 \), the position, velocity, and input control will take the shape shown in Figure 7.

Table 1 shows the comparison between the feedback linearization technique and the adaptive wavelet technique with two families of approximation wavelet transforms POLYWOG1 and Shannon.

**Conclusion**

In this study it is shown that the simplest possible approach of the linearized pendulum equation is not sufficient to describe the liquid slosh data in the container system.

The container system in this approach presented as a nonlinear uncertain system which has a combination of the slosh and uncertainties in both system and input matrices.

An adaptive wavelet robust method has been proposed to enhance the damping of the slosh dynamics in the container system. A comparison was made between the traditional feedback linearization technique and the suggested technique. Three families of approximation wavelet transforms has been used to show the effectiveness of the method used.

**Figure 2.** Adaptive feedback linearization control test. (a) Position state tracking. (b) Velocity state tracking. (c) The control input.

**Figure 3.** Tracking error for position state using adaptive feedback linearization control technique.
Figure 4. Test ($r = 0.04$) using POLYWOG1 wavelet. (a) Position state tracking, (b) Velocity state tracking, (c) The control input.

Figure 5. Test ($r = 0.06$) using POLYWOG1 wavelet. (a) Position state tracking, (b) Velocity state tracking (c) The control input.
Figure 6. Adaptive tracking error (using POLYWOG1). (a) $r = 0.04$, (b) $r = 0.06$, (c) $r = 0.08$.

Figure 7. Test ($r = 0.06$) using Shannon wavelet. (a) Position state tracking, (b) Velocity state tracking, (c) The control input.
Finally, the presented methodical approach thus allows for the provision of model for slosh motion control and optimization in the container systems.

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Table 1. Controller parameters.

| Feedback linearization technique | Attenuation level $r$ | Tracking time (s) | Control force $u_a$ (Newton) | tracking-error (Rad) |
|----------------------------------|-----------------------|-------------------|-------------------------------|----------------------|
| Adaptive Wavelet Technique       | 0.04                  | 3                 | 170                           | 0.02                 |
| using (POLYWOG1)                 | 0.06                  | 3                 | 100                           | 0.0                  |
| Adaptive Wavelet Technique       | 0.08                  | 3                 | 95                            | 0.0                  |
| using (Shannon)                  | 0.06                  | 3                 | 10                            | 0.0                  |

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**Appendix A**

**Lyapunov stability Proof**

A Lyapunov function is selected to be in the following form:

\[
L = \frac{1}{2} \dot{e}^T \dot{p} + \frac{1}{2\gamma_A} \dot{\Psi}^T \dot{\Psi_A} + \frac{1}{2\gamma_D} \dot{\Psi}^T_D \dot{\Psi_D} \tag{A-1}
\]

By taking the time derivative of equation (A-1) along the trajectory of the error dynamic in equation (30) gives that:

\[
\dot{L} = \frac{1}{2} \dot{e}^T \dot{p} + \frac{1}{2} \dot{e}^T \dot{p} + \frac{1}{2} \frac{1}{\gamma_A} \dot{\Psi}^T \dot{\Psi_A} + \frac{1}{2} \frac{1}{\gamma_A} \dot{\Psi}^T_A \dot{\Psi_A} + \frac{1}{2} \frac{1}{\gamma_D} \dot{\Psi}^T_D \dot{\Psi_D} + \frac{1}{2} \frac{1}{\gamma_D} \dot{\Psi}^T_D \dot{\Psi_D} + \frac{1}{2} \frac{1}{\gamma_D} \dot{\Psi}^T_D \dot{\Psi_D} \tag{A-2}
\]

By using equation (28) yields:

\[
\dot{L} = \frac{1}{2} \dot{e}^T \dot{p} + \frac{1}{2} \dot{e}^T \dot{p} + \frac{1}{2} \frac{1}{\gamma_A} \dot{\Psi}^T \dot{\Psi_A} + \frac{1}{2} \frac{1}{\gamma_A} \dot{\Psi}^T_A \dot{\Psi_A} + \frac{1}{2} \frac{1}{\gamma_D} \dot{\Psi}^T_D \dot{\Psi_D} + \frac{1}{2} \frac{1}{\gamma_D} \dot{\Psi}^T_D \dot{\Psi_D} \tag{A-3}
\]

By substituting equations (23) and (32) in equation (A-3) and denoting that:

\[
\dot{\Psi}_A = \dot{\Psi}_A, \quad \dot{\Psi}_D = \dot{\Psi}_D \tag{A-4}
\]

yields:

\[
\dot{L} = \frac{1}{2} \dot{e}^T \dot{p} + \frac{1}{2} \dot{e}^T \dot{p} - \frac{1}{\gamma_A} \Psi_A \dot{\Psi}_A + \frac{1}{2} \dot{\Psi}_D \dot{\Psi}_D + \frac{1}{2} \dot{\Psi}_D \dot{\Psi}_D + \frac{1}{2} \dot{\Psi}_D \dot{\Psi}_D \tag{A-5}
\]

Then by using the adaptation laws in Eqs. 31 -35 yields:
\[ L = -\frac{1}{2} \left( \frac{1}{r} g^T \bar{p} - r \bar{\theta} \right)^T \left( \frac{1}{r} g^T \bar{p} - r \bar{\theta} \right) \]
\[ + \frac{1}{2} r^2 \bar{\theta}^T \bar{\theta} - \frac{1}{2} \bar{e}^T Q \bar{e} \leq \]
\[ \leq -\frac{1}{2} \bar{e}^T Q \bar{e} + \frac{1}{2} r^2 \bar{\theta}^T \bar{\theta} \]  
(A-7)

Integrating equation (A-7) from 0 to T:

\[ L(T) - L(0) \leq -\frac{1}{2} \int_0^T \bar{e}^T Q \bar{e} dt + \frac{1}{2} r^2 \int_0^T \bar{\theta}^T \bar{\theta} dt \]  
(A-8)

Since \( L(T) \geq 0 \), thus the inequality in equation (A-8) can imply that:

\[ \int_0^T \bar{e}^T Q \bar{e} dt \leq \bar{e}^T(0) \bar{p}(0) + \frac{1}{r_A} \bar{\psi}_A^T(0) \bar{\psi}_A(0) \]
\[ + \frac{1}{r_D} \bar{\psi}_D^T(0) \bar{\psi}_D(0) + \]
\[ + r^2 \int_0^T \bar{\theta}^T \bar{\theta} dt \]  
(A-9)

Where equation (A-9) represents the \( H_\infty \) tracking performance of equation (36).

### Appendix B

In general the mother wavelet function can be described by the general form as:

\[ \Psi_{a,b} = \frac{1}{\sqrt{a}} \Psi \left( \frac{t-b}{a} \right) \]  
(B-1)

Where \( a \) and \( b \) are real-values which called the scale (dilation) and shift (translation) parameters respectively. These values will generate the function set \( \Psi_{a,b} (t) \) which is called a wavelet family. In this paper two wavelets family have been adopted.

1. Approximation wavelet function PolyWOG1 which can be defined as:

\[ \Psi(x) = xe^{-\frac{x^2}{2}} \]

2. Approximation wavelet function Shannon which can be described as:

\[ \Psi(x) = \frac{\sin 2\pi x - \sin \pi x}{\pi x} \]

The wavelets network architecture used in this paper is shown below: