Simulating interaction of nonlinear spatial waves on a free surface of the shallow viscous liquid layer

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Abstract. This paper deals with the combined approach to describing the evolution of weakly nonlinear three-dimensional moderately long perturbations of free surface of viscous liquid. The initial system of hydrodynamic equations is reduced to the novel model system of equations. The first of them is integro-differential equation for nonlinear perturbation of the free surface, taking into account non-stationary shear stress on a weakly sloping bottom. Another equation is an auxiliary linear equation for determining the liquid horizontal velocity vector, averaged over the layer depth. This vector is present in the main equation only in the term of the second order of smallness. The proposed model is suitable for finite-amplitude waves, traveling in different directions in the horizontal plane. Some problems of interactions and collisions of such perturbations over the horizontal and weakly sloping bottom are solved numerically.

1. Introduction

Gravity waves in fluids are among the classical objects of the nonlinear phenomena mechanics. In particularly, Korteweg and de Vries derived their famous equation precisely for perturbations of the free surface of a shallow water layer (see, for example, [1]). Kadomtsev – Petviashvili equation [2] was surely the next major step in the development of evolutionary equations for perturbations of small but finite amplitude. The model assumes that the horizontal scale of the wave processes under consideration is appreciably smaller in the longitudinal rather than in the transverse coordinate.

In recent decades, much attention of researchers has been attracted to essentially three-dimensional finite-amplitude perturbations of the water surface. However, almost all simplified models (e. g., articles [3–5]) are applicable only to nonlinear waves, propagating mainly in one direction (therein lies the fundamental restriction of the above-mentioned equations resulting from the fact that the equations of motion involve convective terms). Only in these cases, the fluid velocity can be eliminated from these terms, and the problem may be reduced to one equation for the perturbation of the free surface. For this reason, finite-amplitude waves traveling simultaneously in different directions (at arbitrary angles to each other) can be described only by the systems of equations incorporating both the perturbation of the free boundary and the liquid velocity. In the systems proposed before (for example, [6–8]), even the linear terms of all equations involve terms depending on the fluid velocity.

The equation of the second order in time, assuming head-on collision of plane nonlinear waves, was suggested in the article [9]. Its author managed to avoid the liquid velocity in the first order
of smallness and as a result to use the linearized equation of continuity for its determination. This idea served as a basis for the present work.

The purpose of this research is to propose a system of equations, which would be convenient for analysis of the spatial nonlinear waves interactions. The first version of such system was suggested in the report [10], where the results of only two simple calculations for an inviscid liquid with the horizontal bottom were given.

2. Problem statement and model equations

Since the propagation rate of gravity perturbations is much lower than the speed of sound in liquids, we may use the incompressible liquid approximation. Moreover, we assume that (i) the stationary components of the liquid flow equal zero; (ii) the typical horizontal scale of the free surface perturbations $l_w$ is noticeably larger than the typical equilibrium depth of the liquid layer $h_0$ ($h_0/l_w \sim \sqrt{\varepsilon}$, where $\varepsilon$ is a small parameter); (iii) the perturbation amplitude $\eta$ is considerably small ($\eta/h_0 \sim \varepsilon$); (iv) the bottom of the pool is gently sloping ($\nabla h \sim \eta/l_w$, where $\nabla = (\partial/\partial x, \partial/\partial y)$, $x$ and $y$ are the horizontal coordinates), $h(x,y)$ is the undisturbed depth of the liquid layer); (v) the capillary effects are moderate (Bond number $Bo = \rho gh_0^2/\sigma \sim 1$, where $\rho$ is the liquid density, $g$ is the acceleration of free fall, $\sigma$ is the surface tension); (vi) the emerging boundary layers remain thin, that is, the time of expansion of boundary layers over the entire thickness of liquid is much greater than the typical time of perturbation propagation over any considered point of the pool $t_w$ (the hydrodynamical homochronicity number $Ho_v = \nu t_w/h_0^2 \sim \varepsilon^2$, where $\nu$ is the kinematic viscosity). We note that $t_w = l_w/c$, where $c$ is the propagation rate of very long linear gravitational waves. Hence, the emerging flow should be potential everywhere except for the thin boundary layer.

By analogy with the approach used in [10] for an inviscid liquid, we reduced the initial system of the continuity equation and the Stokes equations as well as the conventional kinematic and dynamic boundary conditions on the free surface and the no-slip boundary condition on the fixed bottom to the basic evolution equation of the wave type in the following form:

$$
\frac{\partial^2 \eta}{\partial t^2} - gh \nabla^2 \eta - \frac{g}{2} \nabla^2 (\eta^2) - h \nabla^2 \left(\frac{u^2}{\rho}\right) - \frac{h^2}{3} \nabla^2 \frac{\partial^2 \eta}{\partial t^2} + \frac{\sigma h}{\rho} \nabla^4 \eta - g \nabla h \cdot \nabla \eta + g \sqrt{\frac{\nu}{\pi}} \int_0^t \frac{\nabla^2 \eta}{\sqrt{t-t'}} dt' = \frac{1}{h} \sqrt{\frac{\nu}{\pi t}} \nabla \cdot u_0,
$$

(1)

where $u$ is the vector of the horizontal component of liquid velocity, averaged over the layer depth, the subscript 0 marks the initial value. The equation (1) looks unusual (it contains initial data). It is explained by the fact that in the initial moment the profile of liquid velocity is not yet related to previous dynamics and has to be set explicitly. It is seen that in the course of time, the right hand side of the equation asymptotically tends to null, and its contribution in the perturbation evolution is negligible [11, 12]. The specific feature of Eq. (1) is that the average values of horizontal components of the liquid velocity are included only in the second-order term. Thereby we can use a simple approximate linear equation to determine this velocity $u = \nabla \varphi$:

$$
\nabla^2 \varphi = -\frac{1}{k} \frac{\partial \eta}{\partial t},
$$

(2)

where $\varphi$ is the potential of the liquid horizontal velocity in the irrotational case. Equation (2) is actually the law of conservation of mass in the first approximation.

We emphasize that the derived model (Eqs. (1) and (2)) is applicable to waves traveling at arbitrary angles to each other. It is essentially easier for analysis than the known systems of differential equations (see, e.g., [12]) in which all equations involve liquid velocity in both linear and nonlinear terms.
3. Numerical modelling

The work considered solutions to the system of Eqs. (1) and (2) that are periodical on spatial coordinates. The pool depth $h(x,y)$ was supposed to have respective symmetry. Two functions $\eta_0(x,y)$ and $\partial \eta / \partial t \big|_0 (x,y)$ should be set as initial condition, since according to Eq. (2) the velocity vector $\mathbf{u}_0$ is determined by the time derivative of free surface perturbation. Numerical solution of the obtained evolution problem was found with the use of Fourier method. The unknown functions $\eta$ and $\varphi$, as well as the pool depth $h$ were presented in series:

$$
[\eta, \varphi, \mathbf{u}, h](x,y,t) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left[ \hat{\eta}_{n,m}(t), \hat{\varphi}_{n,m}(t), \hat{\mathbf{u}}_{n,m}(t), \hat{h}_{n,m} \right] e^{i\alpha nx + i\beta my} \tag{3}
$$

where $\alpha = 2\pi/L_x$, $\beta = 2\pi/L_y$, $L_x, L_y$ are the dimensions of the computing area in certain directions. Complex amplitudes of spatial harmonics are restricted by the condition of functions to be real: $\hat{\eta}_{-n,-m} = \hat{\eta}_{n,m}^*, \hat{\varphi}_{-n,-m} = \hat{\varphi}_{n,m}^*, \hat{\mathbf{u}}_{-n,-m} = \hat{\mathbf{u}}_{n,m}^*, \hat{h}_{-n,-m} = \hat{h}_{n,m}^*$, here the superscript asterisk indicates operation of complex conjugation.

Substituting expressions (3) into equation (1) and equating coefficients at each exponent lead to an infinite system of ordinary differential equations for the evolution of amplitudes $\hat{\eta}_{n,m}(t)$. Confining ourselves to some rather large number of the first spatial harmonics, we obtain its finite-dimension analogue. Equation (2) is an algebraic expression for the amplitudes of potential $\hat{\varphi}_{n,m}(t)$ at a fixed time moment. Fourier coefficients of liquid velocity vector components are determined from the condition: $\hat{\mathbf{u}}_{n,m}(t) = (i\alpha k \hat{\varphi}_{n,m}(t), i\beta l \hat{\varphi}_{n,m}(t))$. An integral in equation (1) corresponding to viscous effects was calculated by trapezoidal method. The resulting system of ODEs was solved by the Runge–Kutta method of the fourth order of accuracy.

We will first consider the case of evolution of initial peakless periodic cross-type perturbations above the horizontal bottom. Fig. 1 shows one fourth of the spatial period for transparency:

![Figure 1](image)

**Figure 1.** Shapes of free surface ($\eta^* = \eta/h$) at $t = 0$ (a) and $t^* = t (g/h)^{1/2} = 100$ (b–d) when two initial plane waves run along the $x$- and $y$-axis ($x^* = x/h, y^* = y/h$) above the horizontal bottom: calculations are performed with KP equation (b), 2DB equation (c) and our model (d).
full calculated area is obtained by mirror reflection, first relative to the plane \( x^* = 100 \), and then relative to \( y^* = 100 \). This perturbation is the superposition of four plane waves, propagating in line with growing \( x \)-axis or \( y \)-axis. Namely, at initial moment of time there is \( \eta = \eta_x \) if \( \eta_x > \eta_y \) and \( \eta = \eta_y \) if \( \eta_y > \eta_x \), where \( \eta_j = \eta_{ja}/\cosh^2([j-U_jt]/4l_j), \) \( U_j = \sqrt{gh(1+\eta_{ja}/3h)}/(1-2\eta_{ja}/3h), \) \( l_j = 2h\sqrt{1/3+h/\eta_{ja}}, \) \( j = x, y \) (\( U_j \) and \( l_j \) are the velocity and the length of plane soliton solutions). In this case, with time we observe a peak that appears at the cross of two wave fronts. In particular, at \( t = 100\sqrt{h/g} \) we have \( \eta_{max} = 0.312h \) if \( \eta_{xa} = 0.15h \) and \( \eta_{ya} = 0.1h \), that is \( \eta_{max} = 1.25(\eta_{xa} + \eta_{ya}) \) \( \eta_{max} = 0.6(\eta_{xa} + \eta_{ya}) \) for \( t = 0 \). Calculating this problem using

**Figure 2.** Shapes of free surface \((\eta^* = \eta/h_\infty)\) at 5 points in time \((t^* = t(g/h_\infty)^{1/2})\) when 4 initial three-dimensional solitary bell-type perturbations are at rest above the horizontal bottom \((x^* = x/h_\infty, \ y^* = y/h_\infty)\).

**Figure 3.** Shapes of free surface at 4 points in time when 4 initial three-dimensional solitary bell-type perturbations are at rest; and shape of the bottom \((h^* = -h/h_\infty)\).
the two-dimensional Boussinesq (2DB) equation, we obtained that $\eta_{\text{max}} = 1.15(\eta_{xa} + \eta_{ya})$.
And the use of Kadomtsev–Petviashvili (KP) equation (the cross-type perturbation running approximately along the bisector of the angle $x0y$) has resulted in $\eta_{\text{max}} = 1.07(\eta_{xa} + \eta_{ya})$.
From the mathematical standpoint, these equations allow studying interactions of perturbations, propagating at any angle. However, 2DB and KP equations were derived with the assumption that nonlinear perturbations travel mainly in one direction. Besides, KP model supposes that in such wave processes a characteristic horizontal scale along the longitudinal coordinate is less than along the transversal one. Therefore for this problem, the KP model correctly describes

Figure 4. Shapes of free surface ($\eta^* = \eta/h_{\infty}$) at 5 points in time ($t^* = t(g/h_{\infty})^{1/2}$) when 4 initial three-dimensional solitary bell-type perturbations are characterized by centripetal motion above the horizontal bottom ($x^* = x/h_{\infty}, y^* = y/h_{\infty}$).

Figure 5. Shapes of free surface at 4 points in time when 4 initial three-dimensional solitary bell-type perturbations are characterized by centripetal motion; and shape of the bottom ($h^* = -h/h_{\infty}$).
neither nonlinearity nor dispersion.

Below are the results of calculations of the evolution of four bell-type perturbations 
\[ \eta_k(x, y) = 0.1 h_\infty \exp \left( - \frac{(x - x_k)^2 + (y - y_k)^2}{(20 h_\infty)^2} \right) \] that were initially at rest. Here and below \( h_\infty \) is the liquid depth at \( x^* = 100 \) and \( y^* = 100 \). Figure 2 demonstrates the interaction of perturbations above the horizontal bottom, and Fig. 3 shows such interaction above the underwater mountain \( h(x, y) = h_\infty \left[ 1 - 0.4 \exp \left( - \frac{x^2 + y^2}{(50 h_\infty)^2} \right) \right] \). Maximum dimensionless deviations of the free surface in the center of the pool are equal to 0.063 in the first case, and 0.082 in the second case. They take place for \( t^* = \sqrt{g/h_\infty} = 74 \) in the first case, and for \( t^* = 83 \) in the second case. This, of course, is due to the fact that the waves run slower, but their nonlinearity is higher above a submarine mountain than in the pool with constant depth \( h_\infty \).

One can see that if these perturbations initially move to the center of the pool their interaction is more intensive, and the diverging waves are noticeably weaker (see Figs. 4 and 5) than in previous cases. At time moments of maximal interaction of such perturbations \( (t^* = 71 \) for Fig. 4 and \( t^* = 80 \) for Fig. 5) the errors of calculations, based on 2DB equation, equal approximately 6 % for Fig. 4 and 12 % for Fig. 5. Emphasize that in the case of uneven bottom, a nonlinearity of perturbations is determined by the local ratio \( \eta(x, y)/h(x, y) \). In particular, the local nonlinearity of the perturbation equals approximately 0.3 in the center of the pool at \( t^* = 80 \) for the case in Fig. 5.

4. Conclusion
The new combined approach to the description of three-dimensional perturbation evolution on a free surface of a viscous liquid is proposed. The model consists of one basic nonlinear equation and one linear auxiliary equation for determining the horizontal liquid velocity vector. This vector is present in the main equation only in the term of the second order of smallness. The method is suitable for moderately long waves of small but finite amplitude, simultaneously running at any angle. This approach is, in essence, easier than the known systems of equations in which all equations contain both linear and nonlinear terms.

Validity of this approach for a number of planar problems of nonlinear disturbance transformation is shown by means of numerical experiments. It is important to note, that at uneven bottom (Figs. 3 and 5) the topography stronger affects the change of coefficients of the main linear and dispersive terms, compared to the term, containing the gradient of liquid depth, which is inessential.

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6. References
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