Starlikeness, Convexity and Landau Type Theorem of the Real Kernel $\alpha$–Harmonic Mappings

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Abstract. In [26], Olofsson introduced a kind of second order homogeneous partial differential equation. We call the solution of this equation real kernel $\alpha$–harmonic mappings. In this paper, we study some geometric properties of this real kernel $\alpha$–harmonic mappings. We give univalence criteria and sufficient coefficient conditions for real kernel $\alpha$–harmonic mappings that are fully starlike or fully convex of order $\gamma$, $\gamma \in [0, 1)$. Furthermore, we establish a Landau type theorem for real kernel $\alpha$–harmonic mappings.

1. Introduction

Let $\mathbb{C}$ be the complex plane and $D_\rho = \{z : |z| < \rho\}$. In particular, $D$ denotes the open unit disk $D_1$. For $\alpha \in \mathbb{R}$ and $z \in D$, let

$$T_\alpha = \frac{\alpha^2}{4} (1 - |z|^2)^{\alpha-1} + \frac{\alpha}{2} (1 - |z|^2)^{\alpha-1} \left( z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}} \right) + (1 - |z|^2)^{-\alpha} \Delta$$

be the second order elliptic partial differential operator, where $\Delta$ is the usual complex Laplacian operator

$$\Delta := 4 \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad z = x + yi.$$

The corresponding partial differential equation is

$$T_\alpha(u) = 0 \quad \text{in } D$$

and its associated the Dirichlet boundary value problem are as follows

$$\begin{cases} T_\alpha(u) = 0 & \text{in } D, \\ u = u^* & \text{on } \partial D. \end{cases}$$

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Here, the boundary data \( u' \in \Sigma'(\partial \mathbb{D}) \) is a distribution on the boundary \( \partial \mathbb{D} \) of \( \mathbb{D} \), and the boundary condition in (1.2) is interpreted in the distributional sense that \( u_r \to u' \) in \( \Sigma'(\partial \mathbb{D}) \) as \( r \to 1^- \), where

\[
\quad u_r(e^{i\theta}) = u(r e^{i\theta}), \quad e^{i\theta} \in \partial \mathbb{D},
\]

for \( r \in [0, 1) \). In [26], Olofsson proved that, for parameter \( \alpha > -1 \), if a function \( u \in C^2(\mathbb{D}) \) satisfies (1.1) with \( \lim_{r \to 1^-} u_r = u' \in \Sigma'(\partial \mathbb{D}) \), then it has the form of Poisson type integral

\[
\quad u(z) = \frac{1}{2\pi} \int_0^{2\pi} K_\alpha(ze^{i\gamma})u'(e^{i\gamma})\,d\gamma, \quad \text{for } z \in \mathbb{D},
\]

where

\[
\quad K_\alpha(z) = c_\alpha \frac{(1 - |z|^2)^{\alpha + 1}}{|1 - z|^\alpha + 2},
\]

\( c_\alpha = \Gamma^2(\alpha/2 + 1)/\Gamma(1 + \alpha) \) and \( \Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,dt \) for \( s > 0 \) is the standard Gamma function. In fact, by Proposition 3.2 of [26], temperate growth of the solution is equivalent to distributional boundary value for a solution of (1.1) when \( \alpha > -1 \).

If we take \( \alpha = 2(p - 1) \), then \( u \) is polyharmonic (or \( p \)-harmonic), where \( p \in \{1, 2, \ldots\} \). For related study of polyharmonic mappings, see [1, 3, 7, 8, 11, 25, 30]). In particular, if \( \alpha = 0 \), then \( u \) is harmonic. Thus, \( u \) is a kind of generalization of classical harmonic mappings. Actually, by [27], we know that it is related to standard weighted harmonic mappings. Furthermore, since the kernel \( K_\alpha \) in (1.4) is a real-valued function, we can call \( u \) of (1.3) real kernel \( \alpha \)-harmonic mappings. For the related discussion on standard weighted harmonic mappings, see [9, 12, 16, 17, 19, 24].

The Gauss hypergeometric function is defined by the series

\[
\quad F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{x^n}{n!},
\]

for \( |x| < 1 \), and by continuation elsewhere, where \( (a)_0 = 1 \) and \( (a)_n = a(a + 1) \cdot \cdot \cdot (a + n - 1) \) for \( n = 1, 2, \ldots \) are the Pochhammer symbols. Obviously, for \( n = 0, 1, 2, \ldots \), \( (a)_n = \Gamma(a + n)/\Gamma(a) \). It is easily verified that

\[
\quad \frac{d}{dx} F(a, b; c; x) = \frac{ab}{c} F(a + 1, b + 1; c + 1; x).
\]

Furthermore, for \( \text{Re}(c - a - b) > 0 \), we have (cf.[4],Theorem 2.2.2)

\[
\quad F(a, b; c; 1) = \lim_{x \to 1} F(a, b; c; x) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}
\]

The following Lemma involves the determination of monotonicity of Gauss hypergeometric functions.

\textbf{Lemma 1.1.} [26] Let \( c > 0, a \leq c, b \leq c \) and \( ab \leq 0 \) \( (ab \geq 0) \). Then the function \( F(a, b; c; x) \) is decreasing (increasing) on \( x \in (0, 1) \).

Gauss hypergeometric function as an analytic function in the complex domain itself is widely and deeply studied [28, 29, 31–33]. Recently, the research on harmonic mapping constructed by Gauss hypergeometric function has also aroused people’s interest [5].

The following result of [26] is the homogeneous expansion of solutions of (1.1).
Theorem 1.2. [26] Let $\alpha \in \mathbb{R}$ and $u \in C^2(D)$. Then $u$ satisfies (1.1) if and only if it has a series expansion of the form
\[
u(z) = \sum_{k=0}^{\infty} c_k F(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; |z|^2) z^k, \quad z \in D,
\]
for some sequence $\{c_k\}$ of complex number satisfying
\[
\lim_{|k| \to \infty} \sup \{|c_k|^2\} \leq 1.
\]
In particular, the expansion (1.7), subject to (1.8), converges in $C^\infty(D)$, and every solution $u$ of (1.1) is $C^\infty$-smooth in $D$.

In [26], the author pointed out that if $\alpha \leq -1$, $u \in C^2(D)$ satisfies (1.1), and the boundary limit $u' = \lim_{\alpha \to -1} u$, exists in $\mathbb{D}'(\partial D)$, then $u(z) = 0$ for all $z \in D$. So, in this paper, we always assume that $\alpha > -1$.

Definition 1.3. Suppose $\alpha > -1$, $u(z)$ have the expansion of (1.7). We call $u(z)$ real kernel $\alpha$-harmonic mapping.

Definition 1.4. A univalent and sense-preserving real kernel $\alpha$-harmonic mapping $u$, with $u(0) = 0$, is said to be fully starlike of order $\gamma$, $\gamma \in (0, 1)$, in $D$ if
\[
\frac{\partial (\arg u(re^{i\theta}))}{\partial \theta} = \Re \left( \frac{D u}{u} \right) > \gamma
\]
for all $z \neq 0$ and $r \in (0, 1)$, where $D$ is a linear operator defined by
\[
D = z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}}.
\]
In particular, when $\gamma = 0$ in (1.9), $u$ is said to be fully starlike.

Definition 1.5. A univalent and sense-preserving real kernel $\alpha$-harmonic mapping $u$ with $u(0) = 0$ is said to be fully convex of order $\gamma$, $\gamma \in [0, 1)$, in $D$ if
\[
\frac{\partial (\arg \frac{\partial u}{\partial \alpha} u(re^{i\theta}))}{\partial \theta} = \Re \left( \frac{D^2 u}{Du} \right) > \gamma
\]
for all $z \neq 0$ and $r \in (0, 1)$, where $D^2 = D(D)$ is the composition of $D$ and itself. In particular, when $\gamma = 0$ in (1.10), $u$ is said to be fully convex.

The starlikeness and convexity with order $\gamma$ of functions are widely and deeply studied in analytic functions, harmonic functions and polyharmonic functions, see [13–15, 21–23].

The classical Landau’s theorem states that if $f$ is an analytic function on the unit disk $D$ with $f(0) = f'(0) - 1 = 0$ and $|f(z)| < M$ for $z \in D$, then $f$ is univalent in the $D_0 = \{z : |z| < r_0\}$ with
\[
r_0 = \frac{1}{M + \sqrt{M^2 - 1}},
\]
and $f(D_0)$ contains a disk $|\omega| < R_0$ with $R_0 = Mr_0^2$. This result is sharp, with the extremal function $f_0(z) = Mz \frac{1-Mz}{Mz - 1}$. For other types of functions, Landau-type theorem were also studied. See [20] for harmonic mappings, [1, 6, 10, 11] for polyharmonic mappings, [2] for logharmonic mappings, [18] for log-$p$-harmonic mappings, [9] for weighted harmonic mappings.

The main purpose of this paper is to study the properties of the real kernel $\alpha$-harmonic mappings. In section 2, for the real kernel $\alpha$–harmonic mappings, we give a necessary and sufficient condition for the relationship between full starlikeness and full convexity. Furthermore, We give univalence criteria and sufficient coefficient conditions for real kernel $\alpha$–harmonic mappings that are starlike or convex of order $\gamma$, $\gamma \in [0, 1)$. In section 3, we get a Landau type theorem for real kernel $\alpha$–harmonic mappings.
2. Starlikeness and convexity

In the rest of this paper, we use the following denotations.

Let $z = re^{i\theta}$,

$$t = |z|^2 = r^2,$$  \hspace{1cm} (2.1)

$$F = F_k = F(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; t),$$  \hspace{1cm} (2.2)

and

$$F_t = F_k(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; t) = \frac{dF_k}{dt} = \frac{dF}{dt}.$$  \hspace{1cm} (2.3)

**Proposition 2.1.** The operator $D$ is a real kernel $\alpha$-harmonic mapping preserving operator.

**Proof.** Let $u$ be a real kernel $\alpha$-harmonic mapping with the series expansion of (1.7). Then by direct computation, we have

$$Du = zu_x - \bar{z}u_z = z \left[ \sum_{k=1}^{\infty} c_k (F_k z^k + F_k z^{k-1}) + \sum_{k=1}^{\infty} c_{-k} F_k z^{k+1} \right] - \bar{z} \left[ \sum_{k=1}^{\infty} c_k F_k \bar{z}^{k+1} + \sum_{k=1}^{\infty} c_{-k} (F_k \bar{z}^k + F_k \bar{z}^{k-1}) \right] = \sum_{k=1}^{\infty} k c_k F_k z^k - \sum_{k=1}^{\infty} k c_{-k} F_k \bar{z}^k.

Furthermore, for sequence $\{c_k\}_{-\infty}^{\infty}$ if (1.8) holds, then

$$\lim_{k \to \infty} \sup \frac{|kc_k|}{|k|} \leq 1$$

and

$$\lim_{k \to \infty} \sup |k c_{-k}| \leq 1.

Therefore, by Theorem 1.2, we get that $Du$ is a real kernel $\alpha$-harmonic mapping.

**Theorem 2.2.** Suppose real kernel $\alpha$-harmonic mappings

$$u(z) = \sum_{k=1}^{\infty} c_k F_k z^k + \sum_{k=1}^{\infty} c_{-k} F_k z^k$$

and

$$v(z) = \sum_{k=1}^{\infty} c_k F_k z^k - \sum_{k=1}^{\infty} c_{-k} F_k z^k$$

are univalent in $D$. Then $u(z)$ is fully starlike of order $\gamma$ if and only if $v(z)$ is fully convex of order $\gamma$.

**Proof.** Direct computation leads to

$$\frac{Du}{u} = \frac{\sum_{k=1}^{\infty} k c_k F_k z^k - \sum_{k=1}^{\infty} k c_{-k} F_k z^k}{\sum_{k=1}^{\infty} c_k F_k z^k + \sum_{k=1}^{\infty} c_{-k} F_k z^k}.$$
and
\[
\frac{D^2v}{Dv} = D\left(\sum_{k=1}^{\infty} c_k F_z^{2k} + \sum_{k=1}^{\infty} c_k F_z^{2k}\right) = \frac{\sum_{k=1}^{\infty} k c_k F_z^{2k} - \sum_{k=1}^{\infty} k c_k F_z^{2k}}{\sum_{k=1}^{\infty} c_k F_z^{2k} + \sum_{k=1}^{\infty} c_k F_z^{2k}}.
\]

It follows that
\[
\frac{Dv}{Dv} = \frac{D^2v}{Dv}.
\]

Therefore, \(\Re\left(\frac{Dv}{Dv}\right) > \gamma\) is equivalent to \(\Re\left(\frac{D^2v}{Dv}\right) > \gamma\). The proof is completed. \(\square\)

**Lemma 2.3.** [29, 34] Let \(r_n\) and \(s_n\) \((n = 0, 1, 2, \ldots)\) be real numbers, and let the power series
\[
R(x) = \sum_{n=0}^{\infty} r_n x^n \quad \text{and} \quad S(x) = \sum_{n=0}^{\infty} s_n x^n
\]
be convergent for \(|x| < r\), \((r > 0)\) with \(s_n > 0\) for all \(n\). If the non-constant sequence \((r_n/s_n)\) is increasing (decreasing) for all \(n\), then the function \(x \mapsto R(x)/S(x)\) is strictly increasing (resp. decreasing) on \((0, r)\).

**Lemma 2.4.** Let \(\frac{q}{2} \in (0, 1]\). Then it holds that

1. \(\frac{f_k}{f_1} \leq 1\) for \(k = 2, 3, \ldots\) and \(t \in [0, 1)\);
2. \(\frac{|f_k|}{f_1} \leq \frac{(k - \frac{q}{2})!}{(k+1)!(2 + \frac{q}{2})} \frac{1}{2!} \frac{k}{(k+1) + \frac{q}{2}}\) for \(k = 1, 2, \ldots\) and \(t \in (0, 1)\).

**Proof.** (1) We divide it into two subcases to discuss.

If \(\frac{q}{2} = 1\), then we can get \(f_1 \equiv 1\) and \(f_k\) is decreasing for \(t \in [0, 1)\) and \(k = 2, 3, \ldots\) by Lemma 1.1. Thus, \(\frac{f_k}{f_1}\) is decreasing for \(t \in [0, 1)\).

If \(0 < \frac{q}{2} < 1\), let
\[
A_n = \frac{(-\frac{q}{2})_n(k - \frac{q}{2})_n}{(k+1)_n n!}, \quad B_n = \frac{(-\frac{q}{2})_n(1 - \frac{q}{2})_n}{(2)_n n!}.
\]

Then it follows that \(B_n > 0\) for \(n = 1, 2, \ldots\) and
\[
\frac{A_n}{B_n} = \frac{(-\frac{q}{2})_n(k - \frac{q}{2})_n}{(k+1)_n n!} \quad \frac{(2)_n n!}{(-\frac{q}{2})_n(1 - \frac{q}{2})_n} = \frac{(k - \frac{q}{2})_n(2)_n}{(k+1)_n(1 - \frac{q}{2})_n}.
\]

It can be verified that
\[
\frac{\frac{A_n+1}{B_n+1}}{\frac{A_n}{B_n}} = \frac{(k - \frac{q}{2} + n)(2 + n)}{(k+1 + n)(1 - \frac{q}{2} + n)} > 1
\]
for \(k = 2, 3, \ldots\). Thus \(\frac{A_n}{B_n}\) is strictly increasing for all \(n = 1, 2, \ldots\). By Lemma 2.3, we get that
\[
f(t) = \sum_{n=1}^{\infty} \frac{A_n t^n}{\sum_{n=1}^{\infty} B_n t^n}
\]
is strictly increasing for \(t \in (0, 1)\). Furthermore, we have
\[
f(0) = \lim_{t \to 0} f(t) = \frac{A_1}{B_1} = \frac{2(k - \frac{q}{2})}{(k+1)(1 - \frac{q}{2})} > 1
\]
for $k = 2, 3, \ldots$. It follows that

$$f(t) > 1$$

for $t \in [0, 1)$. Observe that

$$F_k = \frac{1 + \sum_{n=1}^{\infty} \left(\frac{-2k}{2k+1}\right)^n \frac{t^n}{n!}}{1 + \sum_{n=1}^{\infty} \left(\frac{-2k}{2k+1}\right)^n \frac{t^n}{n!}} = \frac{1 - f(t) \sum_{n=1}^{\infty} B_n t^n}{1 - \sum_{n=1}^{\infty} B_n t^n}. $$

Noting that $F_1$ is increasing and considering the monotonicity of $f$ and (2.5), we can get

$$\frac{d}{dt} \left(\frac{F_k}{F_1}\right) = \frac{-(\sum_{n=1}^{\infty} B_n t^n) f'(t) \sum_{n=1}^{\infty} B_n t^n (f(t) - 1)(\sum_{n=1}^{\infty} B_n t^n)' < 0. $$

Thus, $\frac{F_k}{F_1}$ is strictly decreasing for $t \in (0, 1)$.

Therefore, for $\frac{t}{2} \in (0, 1]$ we have

$$\frac{F_k}{F_1} \leq \lim_{t \to 0} F_k = 1.$$  

(2) If $0 < \frac{t}{2} < 1$, then $F_1$ is negative for $t \in (0, 1)$. By (1.5), we have

$$|F_1| = -F_1 = \frac{\frac{k}{2} - \frac{k}{2}}{k + 1} F(1 - \frac{\alpha}{2}; k + 1 - \frac{\alpha}{2}; k + 2; t).$$

Observe that both $F(1 - \frac{\gamma}{2}; k + 1 - \frac{\gamma}{2}; k + 2; t)$ and $F_1$ are positive. Furthermore, $F(1 - \frac{\gamma}{2}; k + 1 - \frac{\gamma}{2}; k + 2; t)$ is increasing with respect to $t \in [0, 1)$ as well as $F_1$ is decreasing with respect to $t \in [0, 1)$. Therefore, we have

$$|F_1| < \lim_{t \to 1} \left(\frac{\frac{k}{2} - \frac{k}{2}}{k + 1} F(1 - \frac{\gamma}{2}; k + 1 - \frac{\gamma}{2}; k + 2; t) \right)$$

$$= \frac{\frac{k}{2} - \frac{k}{2}}{k + 1} \frac{\Gamma(k + 1)\Gamma(1 + \gamma)}{\Gamma(1 + 2\gamma)} = \frac{(k - \gamma)\Gamma(k + 1)(2 + \gamma)}{2\Gamma(k + 1 + \gamma)}.$$  

(2.6)

The above first equality holds because of (1.6).

If $\frac{t}{2} = 1$, then $F_1 \equiv 1$ and $F_k = 1 - \frac{1}{2} \frac{t}{2}$. Then it is easy to see that the equality of Lemma 2.4 (2) holds.

**Theorem 2.5.** Suppose $\gamma \in [0, 1)$, $\frac{t}{2} \in (0, 1]$ and $u(z)$ be a real kernel $\alpha$-harmonic mapping that has series expansion of (1.7) with $c_1 = 1$, $c_0 = 0$ and $|c_{-1}| < \min\{\frac{1-\gamma}{1+\gamma}, \frac{\gamma}{1+\gamma}\}$. Let

$$\sum_{k=2}^{\infty} (A_k|c_k| + B_k|c_{-1}|) \leq C,$$

(2.7)

where

$$A_k = \frac{(k - \frac{\gamma}{2})\Gamma(k + 1)\Gamma(2 + \frac{\gamma}{2})}{\Gamma(k + 1 + \frac{\gamma}{2})} \frac{1}{1 - |c_{-1}|} + \frac{k - \gamma}{(1 - \gamma)(1 + \gamma)|c_{-1}|}$$

and

$$B_k = \frac{(k - \frac{\gamma}{2})\Gamma(k + 1)\Gamma(2 + \frac{\gamma}{2})}{\Gamma(k + 1 + \frac{\gamma}{2})} \frac{1}{1 - |c_{-1}|} + \frac{k - \gamma}{(1 - \gamma)(1 + \gamma)|c_{-1}|}.$$  

and

$$C = 1 - \frac{\alpha}{2} \frac{|c_{-1}|}{1 - |c_{-1}|}.$$  

Then $u(z)$ is fully starlike of order $\gamma$ in $D$. Furthermore, the coefficient bound given by (2.7) is sharp.
Proof. Before proving this theorem, we first point out that the constraint condition \(|c_{-1}| < \min\left\{\frac{1}{k\gamma}, \frac{1}{2}\right\}\) is to ensure that the denominators in the expression of the above \(A_k\) and \(B_i\) are positive and \(C\) itself is positive. Observe that inequality (2.7) is equivalent to

\[
\sum_{k=1}^{\infty} \frac{(k - \frac{\gamma}{2})\Gamma(k + 1)\Gamma(2 + \frac{\gamma}{2})}{\Gamma(k + 1 + \frac{\gamma}{2})} |c_k| + |c_{-k}| \leq 1.
\]

So, in the following proof process, we often replace (2.7) with (2.8).

First we prove \(u(z)\) is sense-preserving in \(D\). By (1.7), direct computation leads to

\[
u_z = \sum_{k=1}^{\infty} c_k(F_1 z^k + kF z^{k-1}) + \sum_{k=1}^{\infty} c_{-k} F_1 z^{k+1},
\]

and

\[
u_z = \sum_{k=1}^{\infty} c_k F_1 z^{k+1} + \sum_{k=1}^{\infty} c_{-k} (F_1 z^k + kF z^{k-1}).
\]

It follows that

\[
|\nu_z| - |\nu_2| = \left| \sum_{k=1}^{\infty} c_k(F_1 z^k + kF z^{k-1}) + \sum_{k=1}^{\infty} c_{-k}(F_1 z^{k+1} + kF z^k) \right| - \left| \sum_{k=1}^{\infty} c_k F_1 z^{k+1} + \sum_{k=1}^{\infty} c_{-k}(F_1 z^k + kF z^{k-1}) \right|
\]

\[
\geq F_1 - \sum_{k=1}^{\infty} |c_k||F_1||z|^{k+1} - \sum_{k=2}^{\infty} |c_k||F_1||z|^{k-1} - \sum_{k=1}^{\infty} |c_{-k}||F_1||z|^{k+1} - \sum_{k=1}^{\infty} |c_{-k}||F_1||z|^k
\]

\[
- \sum_{k=1}^{\infty} |c_{-k}||F_1||z|^k + kF|z|^{k-1}
\]

\[
= (1 - |c_{-1}|)F_1 - 2\sum_{k=1}^{\infty} (|c_k| + |c_{-k}|)||F_1||z|^k - \sum_{k=2}^{\infty} k(|c_k| + |c_{-k}|)F|z|^{k-1}
\]

\[
> (1 - |c_{-1}|)F_1 - 2\sum_{k=1}^{\infty} (|c_k| + |c_{-k}|)||F_1||z|^k - \sum_{k=2}^{\infty} k(|c_k| + |c_{-k}|)F.
\]

If

\[
2\sum_{k=1}^{\infty} \frac{|c_k| + |c_{-k}|}{1 - |c_{-1}|} \frac{F_1}{F} + \sum_{k=2}^{\infty} \frac{k|c_k| + |c_{-k}|}{1 - |c_{-1}|} \frac{F}{F_1} \leq 1,
\]

then inequality (2.9) implies \( |\nu_z| > |\nu_2| \), that is to say \( u \) is sense-preserving. By Lemma 2.4, it is enough for us to prove

\[
\sum_{k=1}^{\infty} \frac{(k - \frac{\gamma}{2})\Gamma(k + 1)\Gamma(2 + \frac{\gamma}{2})}{\Gamma(k + 1 + \frac{\gamma}{2})} |c_k| + |c_{-k}| \leq 1.
\]

It can be directly verified that

\[
\frac{(k - \gamma)|c_k| + (k + \gamma)|c_{-k}|}{(1 - \gamma) - (1 + \gamma)|c_{-1}|} \geq k\frac{|c_k| + |c_{-k}|}{1 - |c_{-1}|}
\]

for \( \gamma \in [0, 1], \ |c_{-1}| < \frac{1 - \gamma}{1 + \gamma} \) and \( k = 2, 3, \ldots \). Thus if inequality (2.8) holds, then (2.11) follows from (2.12).
To show that \( u(z) \) is univalent in \( \mathbb{D} \) we need to show that \( u(z_1) \neq u(z_2) \) when \( z_1 \neq z_2 \). Suppose \( z_1, z_2 \in \mathbb{D} \) so that \( z_1 \neq z_2 \). Since \( \mathbb{D} \) is simply connected and convex, we have \( z(s) = (1 - s)z_1 + sz_2 \in \mathbb{D} \), where \( s \in [0, 1] \). Then we can write

\[
 u(z_2) - u(z_1) = \int_0^1 [(z_2 - z_1)u_z(z(s)) + (z_2 - z_1)u_z(z(s))] ds.
\]

Dividing the above equation by \( z_2 - z_1 \) and taking the real parts we obtain

\[
 \Re \left\{ \frac{u(z_2) - u(z_1)}{z_2 - z_1} \right\} = \int_0^1 \Re \left\{ \frac{u_z(z(s)) + \frac{z_2 - z_1}{z_2 - z_1} u_z(z(s))}{ds} \right\} ds.
\]

(2.13)

On the other hand

\[
 \Re u_z(z) - |u_z(z)|
\]

\[
 > F_1 - \sum_{k=1}^{\infty} |c_k||F_k| - \sum_{k=2}^{\infty} |c_k||F_k - \sum_{k=1}^{\infty} |c_{-k}||F_k| - \sum_{k=1}^{\infty} |c_k||F_k| + kF
\]

\[
 = (1 - |c_{-1}|)F_1 - 2 \sum_{k=1}^{\infty} (|c_k| + |c_{-k}|)|F_k| - \sum_{k=1}^{\infty} k(|c_k| + |c_{-k}|)F
\]

\[
 \geq (1 - |c_{-1}|)F_1 - 2 \sum_{k=1}^{\infty} (|c_k| + |c_{-k}|)|F_k| - \sum_{k=1}^{\infty} \frac{(k - \gamma)|c_k| + (k + \gamma)|c_{-k}|}{(1 - \gamma)|c_1| - (1 + \gamma)|c_{-1}|} (1 - |c_{-1}|)F
\]

\[
 \geq 0
\]

by inequality (2.12), Lemma 2.4 and inequality (2.8) in order. This in conjunction with the inequality (2.13) leads to the univalence of \( u \).

Now we show that the inequality (1.9) holds. Direct computation yields

\[
 |Du + (1 - \gamma)u| - |Du - (1 + \gamma)u|
\]

\[
 = \sum_{k=1}^{\infty} c_k kFz^k - \sum_{k=1}^{\infty} c_{-k} kFz^k + (1 - \gamma) \left( \sum_{k=1}^{\infty} c_k kFz^k + \sum_{k=1}^{\infty} c_{-k} kFz^k \right)
\]

\[
 - \sum_{k=1}^{\infty} c_k kFz^k - \sum_{k=1}^{\infty} c_{-k} kFz^k - (1 + \gamma) \left( \sum_{k=1}^{\infty} c_k kFz^k + \sum_{k=1}^{\infty} c_{-k} kFz^k \right)
\]

\[
 = \sum_{k=1}^{\infty} (k + 1 - \gamma)c_k Fz^k + \sum_{k=1}^{\infty} (1 - \gamma - k)c_{-k} Fz^k - \sum_{k=1}^{\infty} (k - 1 - \gamma)c_{-k} Fz^k - \sum_{k=1}^{\infty} (k + 1 + \gamma)c_{-k} Fz^k
\]

\[
 \geq (2 - \gamma)F_1 r - \sum_{k=2}^{\infty} (k + 1 - \gamma)c_k Fz^k - \gamma |c_{-1}| F_1 r - \sum_{k=2}^{\infty} (k - 1 + \gamma)|c_{-k}| Fz^k - \gamma F_1 r - \sum_{k=2}^{\infty} (k - 1 - \gamma)c_k Fz^k - (2 + \gamma)|c_{-1}| F_1 r - \sum_{k=2}^{\infty} (k + 1 + \gamma)|c_{-k}| Fz^k
\]

\[
 = 2[(1 - \gamma) - (1 + \gamma)]F_1 r - 2 \sum_{k=2}^{\infty} (k - \gamma)|c_k| + (k + \gamma)|c_{-k}| Fz^k > 0
\]
for $r \in (0, 1)$ by Lemma 2.4 and inequality (2.8). Furthermore, we observe that for $u \neq 0$, it holds that

$$\frac{|D u + (1 - \gamma) u| - |D u - (1 + \gamma) u|}{u} > 0.$$ 

\(\iff\) $\frac{|D u - \gamma - 1|}{u} < \frac{|D u - \gamma + 1|}{u}$

\(\iff\) $\Re \left( \frac{D u - \gamma}{u} \right) > 0$

\(\iff\) $\Re \left( \frac{D u}{u} \right) > \gamma$.

That is to say that if (2.8) holds then (1.9) holds.

The real kernel $\alpha$-harmonic mapping

$$u(z) = F_1 z + \sum_{k=2}^{\infty} \frac{1}{A_k} x_k F_k z^k + c_{-1} F_1 \bar{z} + \sum_{k=2}^{\infty} \frac{1}{B_k} y_k F_k \bar{z}^k,$$

(2.14)

where

$$\sum_{k=2}^{\infty} (|x_k| + |y_k|) = C,$$

(2.15)

show that coefficient bound given by (2.8) is sharp. That is to say, the mapping represented by (2.14) is the corresponding extremal function of Theorem 2.5.

Now we have a look about a special case of Theorem 2.5.

**Example 2.6.** If $\frac{\alpha}{2} = 1$, then $F(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; t) = 1 - \frac{k - 1}{k + 1} |z|^2$ and the corresponding extremal function (2.14) deduce to

$$u(z) = z + \sum_{k=2}^{\infty} \frac{1}{A_k} x_k (1 - \frac{k - 1}{k + 1} |z|^2) z^k + c_{-1} z + \sum_{k=2}^{\infty} \frac{1}{B_k} y_k (1 - \frac{k - 1}{k + 1} |z|^2) \bar{z}^k,$$

where

$$A_k = \frac{2(k - 1)}{k + 1} \frac{1}{1 - |c_{-1}|} + \frac{k - \gamma}{(1 - \gamma) - (1 + \gamma)|c_{-1}|},$$

$$B_k = \frac{2(k - 1)}{k + 1} \frac{1}{1 - |c_{-1}|} + \frac{k + \gamma}{(1 - \gamma) - (1 + \gamma)|c_{-1}|},$$

and

$$\sum_{k=2}^{\infty} (|x_k| + |y_k|) = 1.$$

Actually, the above $u(z)$ is biharmonic and can be rewritten as

$$u(z) = |z|^2 H(z) + G(z),$$

where

$$H(z) = - \sum_{k=2}^{\infty} \frac{1}{A_k} x_k \frac{k - 1}{k + 1} z^k - \sum_{k=2}^{\infty} \frac{1}{B_k} y_k \frac{k - 1}{k + 1} \bar{z}^k,$$

and

$$G(z) = z + \sum_{k=2}^{\infty} \frac{1}{A_k} x_k z^k + c_{-1} z + \sum_{k=2}^{\infty} \frac{1}{B_k} y_k \bar{z}^k.$$
Theorem 2.7. Suppose \( \gamma \in [0, 1) \), \( \frac{a}{2} \in (0, 1] \) and \( u(z) \) be a real kernel \( \alpha \)-harmonic mapping that has series expansion of (1.7) with \( c_1 = 1 \), \( c_0 = 0 \) and \( |c_{-1}| < \min\{\frac{1 - \gamma}{1 + \gamma}, \frac{a}{4 - a}\} \). Let

\[
\sum_{k=2}^{\infty} (M_k|c_k| + N_k|c_{-k}|) \leq C,
\]

(2.16)

where

\[
M_k = \frac{(k - \frac{a}{2})\Gamma(k + 1)\Gamma(2 + \frac{a}{2})}{\Gamma(k + 1 + \frac{a}{2})} \frac{1}{1 - |c_{-1}|} + \frac{k(1 - \gamma)}{(1 - \gamma) - (1 + \gamma)|c_{-1}|},
\]

\[
N_k = \frac{(k - \frac{a}{2})\Gamma(k + 1)\Gamma(2 + \frac{a}{2})}{\Gamma(k + 1 + \frac{a}{2})} \frac{1}{1 - |c_{-1}|} + \frac{k(1 + \gamma)}{(1 - \gamma) - (1 + \gamma)|c_{-1}|},
\]

and

\[
C = 1 - (1 - \frac{a}{2}) \frac{1 + |c_{-1}|}{1 - |c_{-1}|},
\]

Then \( u(z) \) is fully convex of order \( \gamma \) in \( D \). Furthermore, the coefficient bound given by (2.16) is sharp.

Proof. Before proving this theorem, we first point out that the constraint condition \( |c_{-1}| < \min\{\frac{1 - \gamma}{1 + \gamma}, \frac{a}{4 - a}\} \) is to ensure that the denominators in the expression of \( M_k \) and \( N_k \) are positive and the above \( C \) is positive.

Observe that inequality (2.16) is equivalent to

\[
\sum_{k=1}^{\infty} \frac{(k - \frac{a}{2})\Gamma(k + 1)\Gamma(2 + \frac{a}{2})}{\Gamma(k + 1 + \frac{a}{2})} \frac{|c_k| + |c_{-k}|}{1 - |c_{-1}|} + \sum_{k=2}^{\infty} \frac{k(1 - \gamma)|c_k| + (k + \gamma)|c_{-k}|}{(1 - \gamma) - (1 + \gamma)|c_{-1}|} \leq 1.
\]

(2.17)

Because inequality (2.17) implies inequality (2.8), by Theorem 2.5, we can know that \( u(z) \) is sense-preserving and univalent.

Now we just need to prove (1.10) holds. Firstly, we observe that for \( Du \neq 0 \). Actually, if \( Du = 0 \), then \( zu_{\zeta} = zv_{\zeta} \). It follows that \( |u_{\zeta}| = |v_{\zeta}| \). This is in contradiction with the fact that \( u \) is sense-preserving. Thus \( Du \neq 0 \). We have

\[
\Re \frac{D^2u}{Du} > \gamma
\]

\[
\Leftrightarrow \Re \left( \frac{D^2u}{Du} - \gamma \right) > 0
\]

\[
\left| \frac{D^2u}{Du} - \gamma - 1 \right| < \left| \frac{D^2u}{Du} - \gamma + 1 \right|
\]

\[
\Rightarrow |Du + (1 - \gamma)D\zeta| - |Du - (1 + \gamma)D\zeta| > 0.
\]
Secondly, direct computation leads to
\[
\left| D^2 u + (1 - \gamma) Du \right| - \left| D^2 u - (1 + \gamma) Du \right|
= \left| \sum_{k=1}^{\infty} c_k k^2 F z^k + \sum_{k=1}^{\infty} c_{-k} k^2 F z^k + (1 - \gamma) \left( \sum_{k=1}^{\infty} k c_k F z^k - \sum_{k=1}^{\infty} k c_{-k} F z^k \right) \right|
- \left| \sum_{k=1}^{\infty} c_k k^2 F z^k + \sum_{k=1}^{\infty} c_{-k} k^2 F z^k - (1 + \gamma) \left( \sum_{k=1}^{\infty} k c_k F z^k - \sum_{k=1}^{\infty} k c_{-k} F z^k \right) \right|
= \left| \sum_{k=1}^{\infty} (k^2 + k(1 - \gamma)) c_k F z^k + \sum_{k=1}^{\infty} (k^2 - k(1 - \gamma)) c_{-k} F z^k \right|
- \left| \sum_{k=1}^{\infty} (k^2 - k(1 + \gamma)) c_k F z^k + \sum_{k=1}^{\infty} (k^2 + k(1 + \gamma)) c_{-k} F z^k \right|
\geq (2 - \gamma) F_1 r - \sum_{k=1}^{\infty} (k^2 + k(1 - \gamma)) c_k |F z^k| - \gamma |c_{-1}| F_1 r - \sum_{k=1}^{\infty} (k^2 - k(1 - \gamma)) c_{-k} |F z^k|
- \gamma F_1 r - \sum_{k=1}^{\infty} (k^2 - k(1 + \gamma)) c_k |F z^k| - 2 + 2 \gamma |c_{-1}| F_1 r - \sum_{k=1}^{\infty} (k^2 + k(1 + \gamma)) c_{-k} |F z^k|
= 2[(1 - \gamma) - (1 + \gamma) |c_{-1}| F_1 r - 2 \sum_{k=1}^{\infty} k[(k - \gamma) |c_1| + (k + \gamma) |c_{-k}|] |F z^k| > 0
\]
for \( r \in (0, 1) \) by Lemma 2.4 and inequality (2.17). Thus (1.10) holds.

The real kernel \( \alpha \)-harmonic mapping
\[
u(z) = F_1 z + \sum_{k=2}^{\infty} \frac{1}{M_k} \sum_{k=2}^{\infty} x_k F k z^k + c_{-1} F_1 z + \sum_{k=2}^{\infty} \frac{1}{N_k} y_k F k z^k,
\tag{2.18}
\]
where
\[
\sum_{k=2}^{\infty} (|x_k| + |y_k|) = C,
\tag{2.19}
\]
show that coefficient bound given by (2.16) is sharp. That is to say, the function represented by (2.18) is the corresponding extremal function of Theorem 2.7. The proof is completed. □

**Example 2.8.** If \( \alpha = 1 \), by (2.18), we have that the corresponding extremal function of Theorem 2.7 deduces to
\[
u(z) = z + \sum_{k=2}^{\infty} \frac{1}{M_k} x_k (1 - \frac{k - 1}{k + 1} |z|^2) z^k + c_{-1} z + \sum_{k=2}^{\infty} \frac{1}{N_k} y_k (1 - \frac{k - 1}{k + 1} |z|^2) z^k,
\]
where
\[
M_k = \frac{2(k - 1)}{k + 1} \frac{1}{1 - |c_{-1}|} + \frac{k(k - \gamma)}{(1 - \gamma) - (1 + \gamma)|c_{-1}|},
\]
\[
N_k = \frac{2(k - 1)}{k + 1} \frac{1}{1 - |c_{-1}|} + \frac{k(k + \gamma)}{(1 - \gamma) - (1 + \gamma)|c_{-1}|},
\]
and
\[
\sum_{k=2}^{\infty} (|x_k| + |y_k|) = 1.
\]
Actually, the above \( u(z) \) is biharmonic and can be rewritten as
\[
u(z) = |z|^2 A(z) + B(z),
\]
where
\[
A(z) = -\sum_{k=2}^{\infty} \frac{1}{M_k} x_k \frac{k-1}{k+1} z^k - \sum_{k=2}^{\infty} \frac{1}{N_k} y_k \frac{k-1}{k+1} z^k,
\]
and
\[
B(z) = z + \sum_{k=2}^{\infty} \frac{1}{M_k} x_k z^k + c_{-1} z + \sum_{k=2}^{\infty} \frac{1}{N_k} y_k z^k.
\]

3. The Landau type theorem

In [9], Chen and Vuorinen obtained the Landau type theorem for real kernel \( a \)-harmoinc mappings when \( a \in (-1, 0) \). In this section, we explore the Landau type theorem for real kernel \( a \)-harmoinc mappings for \( a \in (0, 2) \). We need the following Lemma 3.1 at first.

**Lemma 3.1.** For \( r \in [0, 1) \), let
\[
\varphi(r) = \frac{\beta}{M(2 + \alpha)} - \frac{8M}{\pi} \left[ \frac{a}{(1-r)^4(1+r)^2} - a + \frac{(2a-1)^2 + ar}{(1-r)^2(1+r)^2} + \frac{2a-1}{2} \left( 1 - r^2 \right) \right],
\]
where \( a \in (0, 2) \), \( \beta > 0 \) and \( M > 0 \) are constants, \( a = \frac{\Gamma(1+\frac{a}{2})}{\Gamma(1+\alpha)} \). Then \( \varphi \) is strictly decreasing and there is an unique \( \rho_0 \in (0, 1) \) such that \( \varphi(\rho_0) = 0 \).

**Proof.** We observe that \( \frac{\Gamma(1)}{\Gamma(2)} = \frac{1}{2} \), \( \lim_{\alpha \to 0^+} \frac{\Gamma(1+\frac{a}{2})}{\Gamma(1+\alpha)} = 1 \) and
\[
d \log \left( \frac{\Gamma(1+\frac{a}{2})}{\Gamma(1+\alpha)} \right) = \frac{1}{2} \psi(1 + \frac{\alpha}{2}) - \psi(1 + \alpha) < 0
\]
for \( \alpha \in (0, 2) \). Then we have \( \frac{1}{2} < \alpha < 1 \) for \( a \in (0, 2) \). Here, \( \psi \) is the digamma function. It is defined by \( \psi(x) = \Gamma'(x)/\Gamma(x) \) and it is well-known (cf.[4]) that \( \psi(x) \) is strictly increasing on \((0, +\infty)\). Let
\[
h_1(r) = \frac{a}{(1-r)^4(1+r)^2} - a,
\]
\[
h_2(r) = \frac{(2a-1)^2 + ar}{(1-r)^2(1+r)^2},
\]
and
\[
h_3(r) = \frac{2a-1}{2} \frac{r^2}{1 - r^2}.
\]

Then
\[
\varphi(r) = \frac{\beta}{M(2 + \alpha)} - \frac{8M}{\pi} (h_1(r) + h_2(r) + h_3(r)).
\]

It is easy to verify that \( h_1'(r) > 0 \), \( h_2'(r) > 0 \) and \( h_3'(r) > 0 \) for any \( r \in (0, 1) \) and \( a \in (0, 2) \). It follows that \( \varphi'(r) < 0 \) for \( r \in (0, 1) \). Furthermore, we can observe that \( \lim_{r \to 0^+} \varphi(r) = \frac{\beta}{M(2 + \alpha)} > 0 \) and \( \lim_{r \to 1^-} \varphi(r) = -\infty \). Therefore, the proof is completed. \( \square \)
Therefore, inequalities (3.3) and (3.4) imply that
\[
\beta \frac{M(2 + \alpha)}{\pi} - \frac{8M}{\pi} \frac{a}{(1 - \rho_0)^4(1 + \rho_0)^2} - a + \frac{(2a - 1)\rho_0^2}{1 - \rho_0^2} = 0.
\]
(3.1)

Moreover, \(u(D_{\rho_0})\) contains an univalent disk \(D_{R_0}\) with
\[
R_0 \geq \rho_0 \left[ \beta \frac{M(2 + \alpha)}{\pi} - \frac{8M}{\pi} \left( \frac{a}{(1 - \rho_0)^4(1 + \rho_0)^2} - a + \frac{(2a - 1)\rho_0^2}{1 - \rho_0^2} \right) \right].
\]

Proof. We still adopt the notations (2.1)-(2.3). For \(\alpha \in (0, 2)\) and \(k \in \mathbb{Z}^+\), we observe that
\[
F_k = F_k(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; t) = \frac{(-\frac{\alpha}{2})_n (\frac{\alpha}{2})_n}{k + 1} \sum_{n=0}^{\infty} \frac{(1 - \frac{\alpha}{2})_n (k + 1 - \frac{\alpha}{2})_n}{(k + 2)_n} \frac{t^n}{n!} < 0.
\]
That is to say that \(F(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; t)\) is decreasing on \(t \in [0, 1]\). So,
\[
F(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; t) < F(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; 1) = 1.
\]
(3.2)

It follows from equations (1.6) and (3.2) that
\[
\frac{\Gamma(k + 1)\Gamma(1 + \alpha)}{\Gamma(k + 1 + \frac{\alpha}{2})\Gamma(1 + \frac{\alpha}{2})} < 1 + \sum_{n=1}^{\infty} \frac{(-\frac{\alpha}{2})_n (k + \frac{\alpha}{2})_n}{(k + 1)_n} \frac{t^n}{n!} - \frac{\Gamma(k + 1)\Gamma(1 + \alpha)}{\Gamma(k + 1 + \frac{\alpha}{2})\Gamma(1 + \frac{\alpha}{2})}.
\]

Thus,
\[
-\sum_{n=1}^{\infty} \frac{(-\frac{\alpha}{2})_n (k - \frac{\alpha}{2})_n}{(k + 1)_n} \frac{1}{n!} \leq 1 - \frac{\Gamma(k + 1)\Gamma(1 + \alpha)}{\Gamma(k + 1 + \frac{\alpha}{2})\Gamma(1 + \frac{\alpha}{2})}.
\]

Notice that the left side of the above inequality is an infinite sum of terms, where each term is a positive for \(\alpha \in (0, 2)\). Therefore, for \(\alpha \in (0, 2)\) and \(k \in \mathbb{Z}^+\), we have
\[
\left|\frac{(-\frac{\alpha}{2})_n (k - \frac{\alpha}{2})_n}{(k + 1)_n} \frac{1}{n!}\right| \leq 1 - \frac{\Gamma(k + 1)\Gamma(1 + \alpha)}{\Gamma(k + 1 + \frac{\alpha}{2})\Gamma(1 + \frac{\alpha}{2})}.
\]
(3.3)

Corollary 1.1 of [9] shows that for \(k \in \mathbb{Z}^+\), it holds that
\[
(|c_1| + |c_{-1}|) \leq \frac{4M\Gamma(k + 1 + \frac{\alpha}{2})\Gamma(1 + \frac{\alpha}{2})}{\pi \Gamma(k + 1)\Gamma(1 + \alpha)}.
\]
(3.4)

Therefore, inequalities (3.3) and (3.4) imply that
\[
(|c_1| + |c_{-1}|) \left|\frac{(-\frac{\alpha}{2})_n (k - \frac{\alpha}{2})_n}{(k + 1)_n} \frac{1}{n!}\right| \leq \frac{4M\Gamma(k + 1 + \frac{\alpha}{2})\Gamma(1 + \frac{\alpha}{2})}{\pi \Gamma(k + 1)\Gamma(1 + \alpha)} - 1 < \frac{4M\Gamma(k + 1 + \frac{\alpha}{2})\Gamma(1 + \frac{\alpha}{2})}{\pi (k + 1)\Gamma(1 + \alpha) - 1}
\]
(3.5)

for \(\alpha \in (0, 2), k \in \mathbb{Z}^+, \) and \(n \in \mathbb{Z}^+\).
Applying (3.6), (3.7), (3.2), (3.4) and (3.5) in turn, we obtain

\[ D - \frac{\alpha}{2} \leq u(z) - u(0) \leq D + \frac{\alpha}{2} - 2|z|^2 - 1, \tag{3.8} \]

Applying (3.6), (3.7), (3.2), (3.4) and (3.5) in turn, we obtain

\[ |u(z) - u(0)| + |u_2(z) - u_2(0)| \leq \sum_{k=2}^{\infty} k(|c_k| + |c_{-k}|)k^{-1} + 2 \sum_{k=1}^{\infty} (|c_k| + |c_{-k}|)k^{-1} \left| F\left(\frac{\alpha}{2}, 1 - \frac{\alpha}{2}; 2; |z|^2\right) - 1 \right| \]

\[ \leq \sum_{k=2}^{\infty} k k^{-1} + 2 \sum_{k=1}^{\infty} (1 + \frac{\alpha}{2})k^{-1} \left| \Gamma(1 + \frac{\alpha}{2}) \right| k^{-1} + 2 \sum_{k=1}^{\infty} \left[ \frac{4M}{\pi} \left( \Gamma(1 + \frac{\alpha}{2}) \Gamma(1 + \alpha) - 1 \right) \sum_{n=1}^{\infty} n^{2n-1} \right] k^{-1} + 2 \frac{4M}{\pi} \left( \Gamma(1 + \frac{\alpha}{2}) \Gamma(1 + \alpha) - 1 \right) \sum_{n=1}^{\infty} n^{2n-1} \]

\[ \leq 8M \left[ \frac{2.3 (3.3 + r^2) + 2(2.3 - 2)}{(1 - r^2)} \right] \frac{(2a - 1) - 2r^2}{2 r^2} \]

\[ = 8M \left[ \frac{a}{(1 - r^2)} - a + \frac{(2a - 1) - 2r^2}{2 r^2} \right] \]

where \( a = \frac{\Gamma(1 + \frac{\alpha}{2})}{\Gamma(1 + \alpha)} \).

The inequality (3.6) of [9] shows that

\[ l(D_u(0)) := ||u(0)|| \geq \frac{\beta}{M(2 + a)}. \tag{3.9} \]

In order to prove the univalence of \( u \) in \( D_{p_0} \), we choose two distinct points \( z_1, z_2 \in D_{p_0} \) and let \( [z_1, z_2] \) denote the segment from \( z_1 \) to \( z_2 \) with the endpoints \( z_1 \) and \( z_2 \), where \( p_0 \) satisfies equation (3.1). By inequalities (3.8), (3.9) and Lemma 3.1, we have

\[ |u(z_2) - u(z_1)| = \left| \int_{[z_1, z_2]} u_2(z)dz + u_2(z)dz \right| \]

\[ \geq \int_{[z_1, z_2]} u_2(z)dz + u_2(z)dz - \int_{[z_1, z_2]} (u_2(z) - u_2(0))dz + (u_2(z) - u_2(0))dz \]

\[ \geq l(D_u(0))[z_2 - z_1] - \int_{[z_1, z_2]} |u(z_2) - u_2(z_2) + u_2(z_2) - u_2(0)|dz \]

\[ > |z_2 - z_1| \left[ \frac{\beta}{M(2 + a)} - 8M \frac{a}{(1 - p_0)^2(1 + p_0)^2} + \frac{(2a - 1)^2 + p_0^2}{(1 - p_0)^2(1 + p_0)^2} + \frac{2a - 1}{2} \frac{p_0^2}{1 - p_0^2} \right] \]

\[ = 0. \]

Thus, \( u(z_1) \neq u(z_2) \). The univalence of \( u \) follows from the arbitrariness of \( z_1 \) and \( z_2 \). This implies that \( u \) is univalent in \( D_{p_0} \).
Now, for any $\zeta = \rho_0 e^{i\theta} \in \partial D_{\rho_0}$, we obtain that

$$|u(\zeta) - u(0)| = \left| \int_{0,\zeta} u_z(z) dz + u_{\bar{z}}(z) d\bar{z} \right|$$

$$\geq \left| \int_{0,\zeta} u_z(0) dz + u_{\bar{z}}(0) d\bar{z} \right| - \left| \int_{0,\zeta} (u_z(z) - u_z(0)) dz + (u_{\bar{z}}(z) - u_{\bar{z}}(0)) d\bar{z} \right|$$

$$\geq l(D_u(0)) \rho_0 - \int_{0,\zeta} |(u_z(z) - u_z(0)) + |u_{\bar{z}}(z) - u_{\bar{z}}(0))| dz|$$

$$\geq \frac{\beta \rho_0}{M(2 + a)} - \frac{8M}{\pi} \int_0^{\rho_0} \left( \frac{a}{(1 - r)^4(1 + r)^2} - a + \frac{(2a - 1)r^2 + ar^2}{2(1 - r)^2(1 + r)^2} + \frac{2a - 1}{1 - r^2} \right) dr$$

$$\geq \frac{\beta \rho_0}{M(2 + a)} - \frac{8M}{\pi} \int_0^{\rho_0} \left( \frac{a}{(1 - r^2)^2(1 + r^2)^2} - a + \frac{(2a - 1)r_0 + a \rho_0 + 2(2a - 1)r_0^2 + 6(1 - r_0^2)}{2(1 - r_0^2)^2(1 + r_0^2)^2} \right)$$

$$= \rho_0 \left[ \frac{\beta}{M(2 + a)} - \frac{8M}{\pi} \left( \frac{a}{(1 - r^2)^2(1 + r^2)^2} - a + \frac{(2a - 1)r_0 + a \rho_0 + 2(2a - 1)r_0^2 + 6(1 - r_0^2)}{2(1 - r_0^2)^2(1 + r_0^2)^2} \right) \right].$$

Hence $U(D_{\rho_0})$ contains an univalent disk $D_{R_0}$ with

$$R_0 \geq \rho_0 \left[ \frac{\beta}{M(2 + a)} - \frac{8M}{\pi} \left( \frac{a}{(1 - r^2)^2(1 + r^2)^2} - a + \frac{(2a - 1)r_0 + a \rho_0 + 2(2a - 1)r_0^2 + 6(1 - r_0^2)}{2(1 - r_0^2)^2(1 + r_0^2)^2} \right) \right].$$

The proof is complete. $\square$

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