Virtual Compton scattering in the generalized Bjorken region and positivity bounds on generalized parton distributions

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The positivity bounds on generalized parton distributions are derived from the positivity properties of the absorptive part of the amplitude of the virtual Compton scattering in the generalized Bjorken region.

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I. INTRODUCTION

Generalized parton distributions (GPDs) [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] appear in the context of the QCD factorization in various hard exclusive phenomena including deep virtual Compton scattering and hard exclusive meson production. Among several general constraints on GPDs an important role is played by the so-called positivity bounds. Various inequalities for GPDs have been derived in Refs. [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22].

As shown in Ref. [23], these inequalities can be considered as particular cases of a general positivity bound which has a relatively simple form in the impact parameter representation for GPDs [12, 21, 22]. This positivity bound is stable under the one-loop evolution of GPDs to higher normalization points [23]. The positivity bound of Ref. [24] was explicitly checked for one-loop GPDs in various perturbative models [25]. The solutions of this positivity bound are studied in Refs. [24, 26]. The positivity bounds can be used for self-consistency checks of models of GPDs [27].

The derivation of the positivity bounds on GPDs in Ref. [23] was based on the positivity of the norm in the Hilbert space of states

\[ \left\| \sum_{\sigma} \int \frac{dP^+ dP^- d\lambda}{2P^+} g_\sigma(\lambda, P) \psi(\lambda n) |P, \sigma\rangle \right\|^2 \geq 0. \]  

(1)

Here $|P, \sigma\rangle$ is the hadron state with momentum $P$ and spin $\sigma$. The quark field $\psi$ is taken at the point $\lambda n$ defined by the light-cone vector $n$, and the “good” spin components of the field $\psi$ are implied. The superposition of quark-hadron states is weighted with arbitrary functions $g_\sigma$. Expanding the square on the left-hand side (LHS) of the inequality (1), one obtains the following inequality:

\[ \sum_{\sigma_1, \sigma_2} \int \frac{dP^+_1 dP^-_1 d\lambda_1}{2P^+_1} \int \frac{dP^+_2 dP^-_2 d\lambda_2}{2P^+_2} g_{\sigma_1}(\lambda_1, P_1) \times g_{\sigma_2}(\lambda_2, P_2) |P_2, \sigma_2\rangle \bar{\psi}(\lambda_2 n)(n\gamma)\psi(\lambda_1 n) |P_1, \sigma_1\rangle \geq 0. \]  

(2)

This inequality contains the matrix elements which enter the definition of GPD. For simplicity we restrict the consideration to the case of spin-0 hadrons. In this case the GPD is defined as follows:

\[ H(x, \xi, t) = \int \frac{d\lambda}{4\pi} \exp \left\{ \frac{1}{2} i x \lambda [n(P_1 + P_2)] \right\} \times \langle P_2 | \bar{\psi} \left( -\frac{\lambda n}{2} \right) (n\gamma) \psi \left( \frac{\lambda n}{2} \right) |P_1\rangle. \]  

(3)

Here we use the standard Ji variables $x, \xi, t$ [15]:

\[ \xi = \frac{n(P_2 - P_1)}{n(P_1 + P_2)}, \quad t = (P_2 - P_1)^2. \]  

(4)

Taking into account that inequality (2) should hold for arbitrary functions $g_\sigma$, one can derive positivity bounds on the GPD $H$ [23].

The approach based on inequality (1) raises a number of questions. The bilinear quark operators have light-cone singularities which have to be renormalized. The renormalization involves subtractions which can violate the inequality. Next, the quark-hadron states in the inequality (1) do not belong to the physical sector and one can wonder whether the positivity of the norm in the Hilbert space is guaranteed. Intuitively one can expect that at large normalization points (where the renormalization is effectively equivalent to the regularization) the positivity is not destroyed by the renormalization. One can also think that the insertion of the gluon $P$ between the quark fields in the inequality (1) will also protect us from the violations of the positivity of the norm which can be met in the nonphysical sector. On the other hand, these arguments in favor of positivity are not completely impeccable and it makes sense to look for an alternative derivation of the positivity bounds on GPDs which would avoid the problems related to the light-cone singularities, renormalization and properties of the nonphysical quark-hadron states.

At this point it is useful to recall the similar problems and their solution in the context of the forward parton distributions. It is well known that in the case of the deep inelastic scattering (DIS) one should distinguish between the parton distributions and the structure functions.

The parton distributions are defined in terms of matrix elements of quark fields

\[ q(x) = \int \frac{d\lambda}{4\pi} e^{i x \lambda n P} \langle P | \bar{\psi}(\lambda n)(n\gamma)\psi(0) | P \rangle. \]  

(5)
Inequality (7) can be rewritten in the form

\[ W_{\mu\nu}(P, q) = \frac{1}{4\pi} \int d^4 z e^{iqz}(P|j_\mu(z)j_\nu(0)|P) . \] (6)

The positivity of parton distributions is usually associated with their physical meaning in the infinite momentum frame as the probability to find a quark with a given momentum fraction. This physical interpretation makes sense only at large normalization points \( \mu \) whereas at low normalization points the positivity of parton distributions can be violated. In contrast, the structure functions are directly related to DIS cross sections and the positivity of parton distributions is usually associated with their physical meaning in the infinite momentum frame as the probability to find a quark with a given momentum fraction.

Now one can ask the question whether this picture of the relations between the structure functions and parton distributions can be generalized from the forward case to the case of GPDs? The answer to this question is rather simple: one should start the analysis from the positivity of the norm the following state

\[ \left\| \int d^4 z \sqrt{2P_1^+} f^{\mu}(z, P) j_\mu(z)|P\right\|^2 \geq 0 \] (7)

where \( j_\mu \) is the color singlet quark current

\[ j_\mu = \bar{\psi} \gamma_\mu \psi \] (8)

Inequality (8) can be rewritten in the form

\[ \int d^4 z_1 \int d^3 P_1 f^{\mu_1}(z_1, P_1) \int d^4 z_2 \int d^3 P_2 \sqrt{2P_2^+} \times f^{\mu_2^*}(z_2, P_2) \langle P_2|j_{\mu_2}(z_2)j_{\mu_1}(z_1)|P_1 \rangle \geq 0 \] (9)

Here we deal with color singlet currents \( j_\mu \) and physical hadronic states. Therefore this inequality should certainly hold for any functions \( f^\mu(z, P) \).

Note that the matrix element \( \langle P_2|j_{\mu_2}j_{\mu_1}|P_1 \rangle \) appearing in the inequality (7) contains the usual product of currents \( j_\mu \). In the momentum representation it can be expressed in terms of the discontinuities of the corresponding matrix element with the time-ordered product \( \langle P_2|T\{j_{\mu_2}j_{\mu_1}\}|P_1 \rangle \). These time-ordered matrix elements are directly related to the “scattering amplitudes” involving two virtual photons. In the so-called generalized Bjorken region [1, 23] these amplitudes can be expressed in terms GPDs.

Similarly, in the case of the hard kinematics, we can express the matrix elements \( \langle P_2|j_{\mu_2}j_{\mu_1}|P_1 \rangle \) in terms of the GPDs. Our aim is to choose functions \( f^\mu(z, P) \) so that the integral on the LHS of inequality (7) is saturated by the hard kinematics where the matrix element \( \langle P_2|j_{\mu_2}j_{\mu_1}|P_1 \rangle \) can be reduced to the GPD. This will lead us to the positivity bounds on GPDs.

The structure of this paper is as follows. In Section I we determine kinematical regions relevant for the derivation of the positivity bounds. In Section II we describe the constraints on functions \( f^\mu \) which allow us to express matrix elements \( \langle P_2|j_{\mu_2}j_{\mu_1}|P_1 \rangle \) in terms of GPDs. The corresponding inequality for GPDs is derived in Section III where we establish the equivalence of the approach based on the quark-hadron inequality (I) and of the current method relying on the positivity property (8) of the matrix element of currents \( j_\mu \). The equivalence of the two approaches is established without using the explicit form of the positivity bound on GPDs in the impact parameter representation which is briefly described in Section IV (the technical details of the derivation can be found in Appendix B).

II. POSITIVITY BOUNDS FOR THE MATRIX ELEMENT OF CURRENTS

In the momentum representation inequality (I) takes the form

\[ \int d^3 P_1 d^4 q_1 \delta^{\mu_1}(P_1, q_1) \int d^3 P_2 d^4 q_2 \delta^{\mu_2^*}(P_2, q_2) x (2\pi)^4 \delta^{(4)}(P_1 + q_1 - P_2 - q_2) A_{\mu_2\mu_1}(q_1, q_2; P_1, P_2) \geq 0 \] (10)

Here \( h^{\mu}(P, q) \) are arbitrary functions,

\[ A_{\mu_2\mu_1}(q_1, q_2; P_1, P_2) = \int d^4 z e^{iqz} \langle P_2|j_{\mu_2}(z)j_{\mu_1}(0)|P_1 \rangle \] (11)

and the momentum conservation reads:

\[ P_1 + q_1 = P_2 + q_2 \] (12)

Let us separate the connected part of the matrix element

\[ \langle P_2|j_{\mu_2}(z_2)j_{\mu_1}(z_1)|P_1\rangle^{conn} \]

\[ \equiv \langle P_2|j_{\mu_2}(z_2)j_{\mu_1}(z_1)|P_1\rangle - \langle P_2|P_1\rangle \langle 0|j_{\mu_2}(z_2)j_{\mu_1}(z_1)|0 \rangle \] (13)

and define

\[ A_{\mu_2\mu_1}^{conn}(q_1, q_2; P_1, P_2) = \int d^4 z e^{iqz} \langle P_2|j_{\mu_2}(z)j_{\mu_1}(0)|P_1\rangle^{conn} \] (14)

Note that in the hard limit only the connected part \( A_{\mu_2\mu_1}^{conn} \) reduces to the GPD. In order to avoid the contamination
of inequality (10) by the vacuum part \( \langle 0|j_{\mu_2}j_{\mu_1}|0 \rangle \) we must choose functions \( h^\mu \) so that the vacuum part does not contribute. Let \( m_0 \) be the mass of the lightest intermediate state contributing to \( \langle 0|j_{\mu_2}j_{\mu_1}|0 \rangle \). The vacuum part \( \langle 0|j_{\mu_2}j_{\mu_1}|0 \rangle \) will be suppressed if for

\[
q^2 > m_0^2, \quad q^0 > 0
\]  
(15)

we have

\[
h^\mu(P, q) = 0.
\]  
(16)

Next, the matrix element \( A_{\mu_2\mu_1}(q_1, q_2; P_1, P_2) \) does not vanish only in a certain kinematical region of variables \( P_i, q_i \). Let us assume for simplicity that the lightest intermediate state \( |n \rangle \), which can contribute to the matrix element

\[
\langle P_2|j_{\mu_2} (z_2) j_{\mu_1} (z_1)|P_1 \rangle
\]  
(17)

has the same mass as \( P_1 \) and \( P_2 \). Then the following condition is necessary if one wants to have nonvanishing \( A_{\mu_2\mu_1}(q_1, q_2; P_1, P_2) \):

\[
2P_1q_1 + q_1^2 \geq 0, \quad (P_1 + q_1)^0 \geq 0
\]  
(18)

Excluding the region \( \mathbb{R}^3 \), we see that we must deal with functions \( h^\mu(P, q) \) vanishing if at least one of the following conditions holds:

\[
h^\mu(P, q) = 0 \begin{cases} \text{if } q^2 \geq m_0^2 \text{ and } q^0 \geq 0 \\ \text{if } 2Pq + q^2 \geq 0 \\ \text{if } (P + q)^0 \leq 0 \end{cases}
\]  
(19)

Obviously we can replace the zero components by the projections on any time- or light-like vector \( n \) (with \( n^0 > 0 \)):

\[
h^\mu(P, q) = 0 \begin{cases} \text{if } q^2 \geq m_0^2 \text{ and } nq \geq 0 \\ \text{if } 2Pq + q^2 \geq 0 \\ \text{if } n(P + q) \leq 0 \end{cases}
\]  
(20)

\section{III. HARD KINEMATICS}

We want to choose functions \( h^\mu(P, q) \) so that matrix elements \( A_{\mu_2\mu_1}(q_1, q_2; P_1, P_2) \) appear in the inequality (11) only in the hard kinematics (corresponding to the generalized Bjorken region of Refs. [1, 23]):

\[
-q_i^2 \sim (P_i q_n) \gg (P_i P_n) \sim \Lambda_{CD}^2.
\]  
(21)

This kinematics will allow us to reduce the matrix element \( A_{\mu_2\mu_1} \) to the GPDs and to derive the positivity bounds on GPDs from the inequality (10). To this aim we take

\[
q_i = p_n - k_i \quad (i = 1, 2)
\]  
(22)

where \( n \) is a light-light-cone vector

\[
n^2 = 0
\]  
(23)

with the positive time component \( n^0 > 0 \) so that

\[
(p_1), \quad (n_2) > 0.
\]  
(24)

Parameter \( n \) is assumed to be large:

\[
r \to \infty
\]  
(25)

and momenta

\[
k_i, p_i = \text{const}
\]  
(26)

are fixed in this limit. In this hard limit the constraint (24) takes the form

\[
h^\mu(P, pq - k) = 0 \begin{cases} \text{if } \rho(nk) \leq 0 \text{ and } (nk) \leq 0 \\ \text{if } pm(P - k) \leq 0 \\ \text{if } n(P - k) \leq 0 \end{cases}
\]  
(27)

If \( \rho < 0 \) then the last two lines lead to the completely vanishing function \( h^\mu = 0 \). Therefore the limit \( \rho \to \infty \) should be understood as \( \rho \to +\infty \). In this case the above constraints on \( h^\mu \) simplify as follows:

\[
h^\mu(P, pq - k) = 0 \begin{cases} \text{if } (nk) \leq 0 \\ \text{if } n(P - k) \leq 0 \\ \text{if } n(P - k) \leq 0 \end{cases}
\]  
(28)

Taking into account condition (24) we conclude that

\[
h^\mu(P, pq - k) \neq 0 \quad \text{only if} \quad 0 \leq (nk) \leq (nP).
\]  
(29)

Certainly we also assume that \( h^\mu(P, pq - k) = 0 \) if \( \rho \) is not large enough.

Let us introduce the following kinematical variables:

\[
x = -\frac{q_i^2 + q_i^2}{(P_2 + P_1)(q_1 + q_2)},
\]  
(30)

\[
\xi = \frac{(P_1 - P_2)(q_1 + q_2)}{(P_2 + P_1)(q_1 + q_2)}.
\]  
(31)

The choice of notation for these variables is motivated by the compatibility with Ji variables \( x, \xi \) [see Eqs. (3), (4)] in the hard limit.

In the hard limit (22), (23) we have

\[
x = \frac{n(k_2 + k_2)}{n(P_1 + P_2)},
\]  
(32)

\[
\xi = \frac{(P_1 - P_2)(p_1 - k_2)}{(p_1 + P_2)(p_1 + k_2)} = \frac{n(k_1 - k_2)}{n(P_1 + P_2)}.
\]  
(33)

The property (29) of functions \( h^\mu \) guarantees that in the integral (10) we deal only with the case

\[
0 \leq (nk_i) \leq (nP_i).
\]  
(34)

This means that the corresponding parameters \( x, \xi \) are constrained to the following region:

\[
|\xi| \leq x \leq 1.
\]  
(35)
IV. REDUCTION OF THE MATRIX ELEMENT OF CURRENTS TO GPD

The next step is to notice that the matrix element $A^{\text{conn}}_{\mu_2 \mu_1}(q_1, q_2; P_1, P_2)$ can be reduced to GPD (3) in the hard limit (22), (23). Indeed, in this limit we have

$$A^{\text{conn}}_{\mu_2 \mu_1}(q_1, q_2; P_1, P_2) = \int d^4 z \, e^{i q_z z} \langle P_2 | j_{\mu_2} (z) j_{\mu_1} (0) | P_1 \rangle^{\text{conn}}$$

$$\to \frac{1}{2} \int d \lambda \exp \left\{ \frac{i}{2} i x \lambda [n(P_1 + P_2)] \right\}$$

$$\times \langle P_2 | \bar{\psi} \left( - \frac{\lambda n}{2} \right) \gamma_{\mu_2} (n \gamma) \gamma_{\mu_1} \psi \left( \frac{\lambda n}{2} \right) | P_1 \rangle$$

$$- \frac{1}{2} \int d \lambda \exp \left\{ \frac{1}{2} i x \lambda [n(P_1 + P_2)] \right\}$$

$$\times \langle P_2 | \bar{\psi} \left( - \frac{\lambda n}{2} \right) \gamma_{\mu_1} (n \gamma) \gamma_{\mu_2} \psi \left( \frac{\lambda n}{2} \right) | P_1 \rangle.$$ \hspace{1cm} (36)

This expression can be obtained by calculating the discontinuities of the time-ordered matrix elements studied in Ref. [1]. Alternatively one can derive Eq. (36) by treating the quark fields as free and using Wick theorem.

Let us introduce a light-cone vector $p$ dual to $n$

$$p^2 = 0, \quad (p n) \neq 0.$$ \hspace{1cm} (37)

Using the light-cone vectors $p$ and $n$, we define the projector

$$\Pi_{\mu \nu} = g_{\mu \nu} - \frac{1}{(p n)} (p_{\mu} n_{\nu} + n_{\mu} p_{\nu}).$$ \hspace{1cm} (38)

with the properties

$$\Pi_{\mu \nu} \Pi^{\nu \rho} = \delta^\rho_{\mu},$$ \hspace{1cm} (39)

$$\Pi_{\mu \nu} n^\nu = 0, \quad \Pi_{\mu \nu} p^\nu = 0.$$ \hspace{1cm} (40)

Obviously $\Pi_{\mu \nu}$ is a projector on the transverse plane where one can choose basis $e^{(1)}, e^{(2)}$:

$$- \Pi_{\mu \nu} = \sum_{a=1,2} e_{\mu}^{(a)} e_{\nu}^{(a)}.$$ \hspace{1cm} (41)

$$e^{(1)} e^{(1)} = e^{(2)} e^{(2)} = -1,$$

$$(e^{a} n) = (e^{a} p) = 0.$$ \hspace{1cm} (42)

Taking into account that

$$- \Pi_{\mu_2 \mu_1} (n \gamma) \gamma_{\mu_1} = 2 (n \gamma),$$ \hspace{1cm} (44)

we find from Eq. (36) in the hard limit

$$\frac{1}{4 \pi} \sum_{a=1,2} \int d^4 z e^{i q_z z} \langle P_2 | [e^{a} \cdot j (z)] [e^{a} \cdot j (0)] | P_1 \rangle$$

$$\to H(x, \xi, t) - H(-x, \xi, t).$$ \hspace{1cm} (45)

The GPD $H(x, \xi, t)$ was defined in Eq. (10). The LHS of Eq. (45) obeys the positivity bound (10). Taking in inequality (10)

$$h^{\mu} (P, n m - k) \equiv e^{(a) \mu} s(P, k)$$ \hspace{1cm} (46)

and summing over $a = 1, 2$, we derive using Eq. (45)

$$\int \frac{d^3 P_1 d^4 k_1}{2 P_1^+} s(P_1, k_1) \int \frac{d^3 P_2 d^4 k_2}{2 P_2^+} s^*(P_2, k_2)$$

$$\times (2 \pi)^4 \delta^{(4)} (P_1 - k_1 - P_2 + k_2)$$

$$\times | H(x, \xi, t) - H(-x, \xi, t) | \geq 0.$$ \hspace{1cm} (47)

The variables $x, \xi$ on the right-hand side are assumed to be expressed in terms of $P_1, k_1$ according to Eqs. (52), (53). Function $s(P, k)$ is arbitrary up to the constraint (29):

$$s(P, k) \neq 0 \quad \text{only if} \quad 0 \leq (n k) \leq (n P).$$ \hspace{1cm} (48)

This means that inequality (47) covers only the region $|\xi| \leq x \leq 1$ (24). The GPD $H(x, \xi, t)$ should be taken at the normalization point $\mu$ determined (with the leading logarithm accuracy) by the hard scale (21)

$$\mu^2 \sim - q^2 \gg - t \sim \Lambda_{\text{QCD}}^2.$$ \hspace{1cm} (49)

Note that the inequality (47) contains two terms

$$H(x, \xi, t) - H(-x, \xi, t).$$ \hspace{1cm} (50)

At $x > |\xi|$ the first term $H(x, \xi, t)$ can be interpreted as the quark contribution whereas $H(-x, \xi, t)$ corresponds to the antiquarks. Actually inequality (47) is a sum of two independent positivity bounds for the quark and antiquark distributions

$$\int \frac{d^3 P_1 d^4 k_1}{2 P_1^+} s(P_1, k_1) \int \frac{d^3 P_2 d^4 k_2}{2 P_2^+} s^*(P_2, k_2)$$

$$\times (2 \pi)^4 \delta^{(4)} (P_1 - k_1 - P_2 + k_2) \left[ \pm H(\pm x, \xi, t) \right] \geq 0.$$ \hspace{1cm} (51)

The reason, why the positivity bounds for quarks and antiquarks mix in inequality (47), can be understood already at the level of the forward parton distributions: it
is well known that in the electromagnetic DIS the structure functions contain the sum of the quark and antiquark distributions with squared electric charges so that for any flavor the quark and antiquark distributions appear together with the same weight.

In order to separate the quark contribution from the antiquark part, one can consider the positivity properties of the left currents. This is done in Appendix B, where inequality (51) is derived.

Inequality (51) can be easily reduced to inequality (2) which was used as a starting point for the derivation of the positivity bounds on GPDs in Ref. [23]. Thus we see that the current approach (based on the positivity properties of matrix elements of currents) and the method of Ref. [24] (relying on the positivity of the norm of the quark-hadron states) lead to the same bounds on GPDs.

V. POSITIVITY BOUNDS IN THE IMPACT PARAMETER REPRESENTATION

In the previous section it was explained that the quark-hadron method of Ref. [23] and the current approach lead to the same result. In Ref. [23], it was shown that the positivity bounds can be simplified by using the impact parameter representation for GPDs. In this section we present only the result. The technical details can be found in Appendix B.

In the frame, where \((P_1 + P_2)^\perp = 0\) and \(n^\perp = 0\), the transverse component of the transferred momentum \(\Delta^\perp = P_2^\perp - P_1^\perp\) is connected with the variable \(t\) by the following relation:

\[
t = -\frac{|\Delta^\perp|^2 + 4\xi^2M^2}{1 - \xi^2}.
\]  

Let us define the GPD in the impact parameter representation via the Fourier transformation in \(\Delta^\perp\):

\[
\hat{F}(x, \xi, b^\perp) = \int \frac{d^2\Delta^\perp}{(2\pi)^2} \exp\left[i(\Delta^\perp b^\perp)\right]
\times H\left(x, \xi, -\frac{|\Delta^\perp|^2 + 4\xi^2M^2}{1 - \xi^2}\right).
\]  

In Appendix B, the following inequality is derived from inequality (51):

\[
\int_1^{-1} \frac{d\xi}{|\xi|} \frac{dx}{(1-x)^3} \rho^*(\frac{1-x}{1-\xi}) p\left(\frac{1-x}{1+\xi}\right)
\times \left[\pm \tilde{F}\left(\pm x, \xi, \frac{1-x}{1-\xi^2}b^\perp\right)\right] \geq 0.
\]  

This inequality should hold for any function \(p\). It coincides with the positivity bound derived in Ref. [23].

VI. CONCLUSIONS

In this paper the alternative derivation of the positivity bounds on GPDs is considered. The advantage of this method is that it is based on quite transparent positivity properties of matrix elements of color singlet currents over physical hadronic states. This allows us to avoid a number of problems which arise in the original derivation of the positivity bounds based on the positivity properties of the nonphysical quark-hadronic states. From this point of view the derivation of the positivity bounds described in this paper is more favorable. Another advantage of the new derivation is that it makes clear certain physical restrictions on the region where the positivity bounds should hold. We see from Eq. (49) that the normalization point \(\mu\) should be large not only compared to \(\Lambda_{QCD}\) but also the condition \(\mu^2 \gg |t|\) should hold.

In terms of the impact parameter \(b^\perp\) used in the explicit formulation of the positivity bounds (54) this means that for the validity of the positivity bounds we need the condition \(\mu \gg |b^\perp|^{-1}\).

For simplicity our analysis was restricted to the case of spin-0 hadrons. The generalization to hadrons with nonzero spins is straightforward and the explicit form of the corresponding positivity bounds can be found in Ref. [23].

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APPENDIX A: POSITIVITY BOUNDS ON GPDS AND LEFT CURRENTS

In this appendix we derive inequality (51) using the positivity properties of the matrix element of the left currents

\[
j_{\mu}^L = \bar{\psi}g_{\mu}(1 - \gamma_5)\psi.
\]  

By analogy with Eq. (36) we have in the hard limit

\[
\int d^4ze^{iq\cdot z}\langle P_2|j_{\mu_2}^L(z)j_{\mu_1}^L(0)|P_1\rangle
\rightarrow \int d\lambda \exp\left\{\frac{1}{2}ix\lambda[n(P_1 + P_2)]\right\}
\times \langle P_2|\bar{\psi}\left(-\frac{\lambda n}{2}\right)\gamma_\mu_2(n\gamma)\gamma_{\mu_1}(1 - \gamma_5)\psi\left(\frac{\lambda n}{2}\right)|P_1\rangle
\times \langle P_2|\tilde{\psi}\left(-\frac{\lambda n}{2}\right)\gamma_{\mu_1}(n\gamma)\gamma_{\mu_2}(1 - \gamma_5)\psi\left(\frac{\lambda n}{2}\right)|P_1\rangle.
\]  

Let us introduce the vector describing the helicity eigenstate of the virtual photon

\[ e = \frac{1}{\sqrt{2}} \left( e^{(1)} + ie^{(2)} \right) \]  

(A3)

with the properties

\[ (en) = 0, \quad (ee) = 0, \quad (e^*e) = -1, \]  

(A4)

\[ \varepsilon_{\lambda\nu\rho}e^{\lambda\nu}e^\rho n = -in. \]  

(A5)

Using relations

\[ (e\gamma)(e\gamma)(n\gamma) = -(1 + \gamma_5)(n\gamma), \]  

(A6)

\[ (e^*\gamma)(e^*\gamma)(n\gamma) = -(1 - \gamma_5)(n\gamma), \]  

(A7)

we find from Eq. (A2) that in the hard limit

\[ \int d^4ze^{iqz} \langle P_2 | [e^{\mu_2}j_{\mu_2}(z)]^+ [e^{\mu_1}j_{\mu_1}(0)] | P_1 \rangle \]

\[ \to 2 \int d\lambda \exp \left\{ \frac{1}{2}ix\lambda [n(P_1 + P_2)] \right\} \]

\[ \times \langle P_2 | \bar{\psi} \left( -\frac{\lambda n}{2} \right) (n\gamma)\psi \left( \frac{\lambda n}{2} \right) | P_1 \rangle \]

\[ = 8\pi H(x, \xi, t). \]  

(A8)

Using this relation instead of Eq. (A3), we obtain inequality (51) with the upper sign choice in ± by analogy with the derivation of inequality (51).

Replacing \( e \to e^* \) in the LHS of Eq. (A8) we find

\[ \int d^4ze^{iqz} \langle P_2 | [e^{\mu_2}j^L_{\mu_2}(z)]^+ [e^{\mu_1}j^L_{\mu_1}(0)] | P_1 \rangle \]

\[ \to -2 \int d\lambda \exp \left\{ \frac{1}{2}ix\lambda [n(P_1 + P_2)] \right\} \]

\[ \times \langle P_2 | \bar{\psi} \left( -\frac{\lambda n}{2} \right) (n\gamma)\psi \left( \frac{\lambda n}{2} \right) | P_1 \rangle \]

\[ = -8\pi H(-x, \xi, t). \]  

(A9)

This result allows us to derive inequality (51) with the minus sign choice.

\[ \text{APPENDIX B: DERIVATION OF POSITIVITY} \]

\[ \text{BOUNDS ON GPDS IN THE IMPACT} \]

\[ \text{PARAMETER REPRESENTATION} \]

In this appendix we derive the positivity bound in the impact parameter representation (6) from the inequality (51).

Let us choose the light-cone coordinates so that for any vector \( X^\mu \)

\[ X^+ = \text{const} (nX). \]  

(B1)

Then according to Eqs. (52), (53)

\[ x = \frac{k_1^+ + k_2^+}{P_1^+ + P_2^+}, \quad \xi = \frac{P_1^+ - P_2^+}{P_1^+ + P_2^+}. \]  

(B2)

We can rewrite inequality (57) in the following form (in the case of the upper sign ±)

\[ \int d^3P d^3k_1 \int d^3P d^3k_2 s(P_1, k_1) \int d^3P d^3k_2 s^*(P_2, k_2). \]

\[ \times (2\pi)^4\delta^4(P_1 - k_1 - P_2 + k_2) \]

\[ \times H \left[ \frac{k_1^+ + k_2^+ - P_1^+ - P_2^+}{P_1^+ + P_2^+}, (P_2 - P_1)^2 \right] \geq 0. \]  

(B3)

Using the Fourier representation for the delta function

\[ 2\pi\delta(P_1^+ - k_1^+ - P_2^+ + k_2^+) = \int dy \]

\[ \times \exp \left[ iy \left( \frac{|P_1^+|^2 + M^2}{2P_1^+} - k_1^+ - \frac{|P_2^+|^2 + M^2}{2P_2^+} + k_2^+ \right) \right] \]  

we can reduce inequality (B3) to the form

\[ \int dy \int d^3P d^3k_1 \frac{s_2(P_1, k_1^+, k_1^-, y)}{2P_1^+} \int d^3P d^3k_2 \frac{s_2^*(P_2, k_2^+, k_2^-, y)}{2P_2^+} \]

\[ \times s_2^*(P_2, k_2^+, k_2^-, y)(2\pi)^3\delta(P_1^+ - k_1^+ - P_2^+ + k_2^+) \]

\[ \times \delta^2(P_1^+ - k_1^+ - P_2^+ + k_2^+) \]

\[ \times H \left[ \frac{k_1^+ + k_2^+ - P_1^+ - P_2^+}{P_1^+ + P_2^+}, (P_2 - P_1)^2 \right] \geq 0, \]  

(B5)

where

\[ s_2(P, k^+, k^-, y) = \int \frac{dk^-}{2\pi} s(P, k) \]

\[ \times \exp \left[ iy \left( \frac{|P|^2 + M^2}{2P^+} - k^- \right) \right] \]  

(B6)
In inequality \((\text{B5})\) the dependence of functions \(s_2(P, k^+, k^-, y)\) on \(y\) is arbitrary. Therefore inequality \((\text{B5})\) should hold before the integration over \(y\) for any fixed \(y\). Thus we conclude that for any function \(s_3(P, k^+, k^-)\)

\[
\int \frac{d^3 P_1 d^3 k_1}{2P_1^+} s_3(P_1, k_1^+, k_1^-) \int \frac{d^3 P_2 d^3 k_2}{2P_2^+} s_3^*(P_2, k_2^+, k_2^-)
\]

\((2\pi)^3 \delta(P_1^+ - k_1^+ - P_2^+ + k_2^+ + k^-) \delta(\mathbf{P}_1 - k_1 - P_2^+ + k_2^-) \times H \left[ k_1^+ + k_2^+ \frac{P_1^+ - P_2^+}{P_1^+ + P_2^+}, (P_2 - P_1)^2 \right] \geq 0.\) \((\text{B7})\)

Similarly we define the function

\[
s_4(P, k^+, y^+) = \int \frac{d^2 k^+}{(2\pi)^2} s_3(P, k^+, k^-) \times \exp \left[ i y^+ (P^+ - k_1^-) \right].\) \((\text{B8})\)

Then we find from the inequality \((\text{B3})\)

\[
\int d^2 y^+ \int \frac{d^3 P_1 d^3 k_1}{2P_1^+} s_4(P_1, k_1^+, y^+) \int \frac{d^3 P_2 d^3 k_2}{2P_2^+} s_4^*(P_2, k_2^+, y^+)
\]

\((2\pi)^3 \delta(P_1^+ - k_1^+ - P_2^+ + k_2^+ + k^-) \delta(\mathbf{P}_1 - k_1 - P_2^+ + k_2^-) \times H \left[ k_1^+ + k_2^+ \frac{P_1^+ - P_2^+}{P_1^+ + P_2^+}, (P_2 - P_1)^2 \right] \geq 0.\) \((\text{B9})\)

Again, due to the arbitrary dependence of the function \(s_4(P_1, k_1^+, y^+)\) on \(y^+\), this inequality should hold before the integration over \(y^+\) for any value of \(y^+\), i.e. for any function \(s_5(P^, k^-)\) we must have

\[
\int \frac{d^3 P_1 d^3 k_1}{2P_1^+} s_5(P_1, k_1^+) \int \frac{d^3 P_2 d^3 k_2}{2P_2^+} s_5^*(P_2, k_2^+)
\]

\((2\pi)^3 \delta(P_1^+ - k_1^+ - P_2^+ + k_2^+ + k^-) \times H \left[ k_1^+ + k_2^+ \frac{P_1^+ - P_2^+}{P_1^+ + P_2^+}, (P_2 - P_1)^2 \right] \geq 0.\) \((\text{B10})\)

The next step is to notice that

\[
|P_1^+ P_2^+ - P_2^+ P_1^+|^2 = -P_1^+ P_2^+ (P_1 - P_2)^2 - M^2 (P_1^+ - P_2^+)^2.\) \((\text{B11})\)

Taking into account the expression \((\text{B2})\) for \(\xi\), we find from Eq. \((\text{B11})\)

\[
\frac{2(P_1^+ P_2^+ - P_2^+ P_1^+)}{P_1^+ + P_2^+} = (1 - \xi^2) t - 4\xi^2 M^2.\) \((\text{B12})\)

Let us introduce notation

\[
\hat{\Delta}^\perp = \frac{2(P_1^+ P_2^+ - P_2^+ P_1^+)}{P_1^+ + P_2^+}.
\] \((\text{B13})\)

Then the variable \(t\) \((\text{B4})\) is equal to

\[
t = -|\hat{\Delta}^\perp|^2 + 4\xi^2 M^2 \frac{1}{1 - \xi^2}.\) \((\text{B14})\)

Note that in the frame where \((P_1 + P_2)^\perp = 0\), we have \(\hat{\Delta}^\perp = (P_2 - P_1)^\perp \equiv \Delta^\perp\). However, here we deal with inequalities containing the integration over \(P_1, P_2\) so that the constraint \((P_1 + P_2)^\perp = 0\) cannot be imposed. In order to avoid confusion with \(\hat{\Delta}^\perp = (P_2 - P_1)^\perp\), we use notation \(\hat{\Delta}\) for the variable \((\text{B13})\).

Now we introduce the function

\[
F(x, \xi, \hat{\Delta}) = H \left( x, \xi, \frac{|\hat{\Delta}^\perp|^2 + 4\xi^2 M^2}{1 - \xi^2} \right).\) \((\text{B15})\)

In terms of this function inequality \((\text{B10})\) takes the form

\[
\int \frac{d^3 P_1 d^3 k_1}{2P_1^+} s_5(P_1, k_1^+) \int \frac{d^3 P_2 d^3 k_2}{2P_2^+} s_5^*(P_2, k_2^+)
\]

\((2\pi)^3 \delta(P_1^+ - k_1^+ - P_2^+ + k_2^+ + k^-) \times 2\pi \delta(P_1^+ - k_1^+ - P_2^+ + k_2^-) \times F \left[ k_1^+ + k_2^+ \frac{P_1^+ - P_2^+}{P_1^+ + P_2^+}, 2(P_1^+ P_2^+ - P_2^+ P_1^+) \frac{P_1^+ + P_2^+}{P_1^+ + P_2^+} \right] \geq 0.\) \((\text{B16})\)

Taking functions \(s_5\) in the factorized form

\[
s_5(k^+, P^+, P^-) = s_6(k^+, P^+) s_7 \left( \frac{P^+}{P^+} \right),\) \((\text{B17})\)

we find

\[
\int d^2 P_1^+ \int d^2 P_2^+ s_7 \left( \frac{P_1^+}{P_1^+} \right) s_7^* \left( \frac{P_2^+}{P_2^+} \right)
\]

\((2\pi)^3 \delta(P_1^+ - k_1^+ - P_2^+ + k_2^+ + k^-) \times F \left[ k_1^+ + k_2^+ \frac{P_1^+ - P_2^+}{P_1^+ + P_2^+}, 2(P_1^+ P_2^+ - P_2^+ P_1^+) \frac{P_1^+ + P_2^+}{P_1^+ + P_2^+} \right] \geq 0.\) \((\text{B18})\)

where

\[
\tilde{s}_7(b^+) = \int d^2 P^+ \exp \left[ -i (b^+ P^+) \right] s_7(P^+).\) \((\text{B19})\)
and

\[ \tilde{F}(x, \xi, b^\perp) = \int \frac{d^2 \Delta^\perp}{(2\pi)^2} \exp \left[ i(\Delta^\perp b^\perp) \right] \times H \left( x, \xi, -\frac{\Delta^\perp + 4\xi^2 M^2}{1 - \xi^2} \right) \]

\[ = \int \frac{d^2 \Delta^\perp}{(2\pi)^2} \exp \left[ i(\Delta^\perp b^\perp) \right] F(x, \xi, \Delta^\perp). \]  

(B20)

Now we find from the inequality (B16)

\[ \int d^2 b^\perp |\tilde{s}_7(b^\perp)|^2 \int dP_1^+ dk_1^+ \int dP_2^+ dk_2^+ \frac{(P_1^+ + P_2^+)^2}{P_1^+ P_2^+} \]

\[ \times s_6(k_1^+, P_1^+) s_6^*(k_2^+, P_2^+) 2\pi \delta(P_1^+ - k_1^+ - P_2^+ + k_2^+) \]

\[ \times \tilde{F} \left[ \frac{k_1^+ + k_2^+}{P_1^+ + P_2^+}, \frac{P_1^+ - P_2^+}{P_1^+ + P_2^+}, \frac{P_1^+ + P_2^+}{2P_1^+ P_2^+} b^\perp \right] \geq 0. \]  

(B21)

Here function \( \tilde{s}_7(b^\perp) \) is arbitrary. Therefore this inequality should hold before the integration over \( b^\perp \)

\[ \int_0^\infty dP_1^+ \int_0^\infty dk_1^+ \int_0^\infty dP_2^+ \int_0^\infty dk_2^+ s_6(k_1^+, P_1^+) \]

\[ \times s_6^*(k_2^+, P_2^+) \tilde{F} \left[ \frac{k_1^+ + k_2^+}{P_1^+ + P_2^+}, \frac{P_1^+ - P_2^+}{P_1^+ + P_2^+}, \frac{P_1^+ + P_2^+}{2P_1^+ P_2^+} b^\perp \right] \]

\[ \times \delta \left[ (P_2^+ - k_2^+) - (P_1^+ - k_1^+) \right] \frac{(P_1^+ + P_2^+)^2}{P_1^+ P_2^+} \geq 0. \]  

(B22)

The integration limits are taken from Eq. (B18).

The inequality (B22) should hold for any value of \( b^\perp \) and for any function \( s_6(k^+, P^+) \). The next step is to take

\[ s_6(k^+, P^+) = s_8(P^+ - k^+) s_9(P^+) . \]  

(B23)

In the limit

\[ |s_8(u)|^2 \to \delta(u - v), \quad v > 0 \]  

(B24)

we find from inequality (B22)

\[ \int_0^\infty dP_1^+ \int_0^\infty dk_1^+ \int_0^\infty dP_2^+ \int_0^\infty dk_2^+ s_8(P_1^+) s_9(P_2^+) \]

\[ \times \tilde{F} \left[ \frac{k_1^+ + k_2^+}{P_1^+ + P_2^+}, \frac{P_1^+ - P_2^+}{P_1^+ + P_2^+}, \frac{P_1^+ + P_2^+}{2P_1^+ P_2^+} b^\perp \right] \]

\[ \times \delta \left[ (P_1^+ - k_1^+) - v \right] \delta \left[ (P_2^+ - k_2^+) - v \right] \frac{(P_1^+ + P_2^+)^2}{P_1^+ P_2^+} \]

\[ = \int_v^\infty dP_1^+ \int_v^\infty dP_2^+ s_8(P_1^+) s_9(P_2^+) \frac{(P_1^+ + P_2^+)^2}{P_1^+ P_2^+} \]

\[ \times \tilde{F} \left[ \frac{P_1^+ + P_2^+ - 2v}{P_1^+ + P_2^+}, \frac{P_1^+ - P_2^+}{P_1^+ + P_2^+}, \frac{P_1^+ + P_2^+}{2P_1^+ P_2^+} b^\perp \right] \geq 0. \]  

(B25)

We change from \( P_1^+, P_2^+ \) to the new integration variables

\[ x = \frac{P_1^+ + P_2^+ - 2v}{P_1^+ + P_2^+}, \quad \xi = \frac{P_1^+ - P_2^+}{P_1^+ + P_2^+} \]  

(B26)

\[ P_1^+ = v \frac{1 + \xi}{1 - x}, \quad P_2^+ = v \frac{1 - \xi}{1 - x} \]  

(B27)

with the Jacobian

\[ \frac{D(P_1^+, P_2^+)}{D(x, \xi)} = \frac{2v^2}{(1 - x)^3} . \]  

(B28)

After this change of variables we have

\[ \frac{(P_1^+ + P_2^+)^2}{4P_1^+ P_2^+} = \frac{1}{1 - \xi^2} . \]  

(B29)

Then inequality (B27) takes the form

\[ \int_{-1}^1 dx \int_{-1}^1 \frac{1}{(1 - x)^3(1 - \xi^2)} s_9 \left( v \frac{1 - \xi}{1 - x} \right) \]

\[ \times \delta \left[ (P_2^+ - k_2^+) - (P_1^+ - k_1^+) \right] \frac{(P_1^+ + P_2^+)^2}{P_1^+ P_2^+} \geq 0 . \]  

(B30)

Since function \( s_9 \) and parameter \( b^\perp \) are arbitrary, we can set \( v = 1 \) without losing generality. With \( s_9(z) = z p(z^{-1}) \) we obtain

\[ \int_{-1}^1 dx \int_{-1}^1 \frac{1}{(1 - x)^5} p^* \left( \frac{1 - x}{1 - \xi} \right) p \left( \frac{1 - x}{1 + \xi} \right) \]

\[ \times \tilde{F} \left( x, \xi, \frac{1 - x}{1 - \xi^2} b^\perp \right) \geq 0 . \]  

(B31)

Thus the inequality (B1) with the upper \( \pm \) sign is established. The case of the other sign can be considered in a similar way.
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