A NEW PROOF TO THE ENERGY CONSERVATION FOR THE NAVIER-STOKES EQUATIONS

CHENG YU

ABSTRACT. In this paper we give a new proof to the energy conservation for the weak solutions of the incompressible Navier-Stokes equations. This result was first proved by Shinbrot. The new proof relies on a lemma introduced by Lions.

1. INTRODUCTION

We are interested in studying the energy conservation for the weak solutions of Navier-Stokes equations

\[ u_t + u \cdot \nabla u + \nabla P - \mu \Delta u = 0, \]
\[ \text{div} u = 0, \]

with the initial data

\[ u(0, x) = u_0 \]

for \((t, x) \in \mathbb{R}^+ \times \Omega\), where \(\Omega = \mathbb{T}^d\) is a periodic domain in \(\mathbb{R}^d\).

The existence of weak solution was proved by Leray [4] and Hopf [2]. The notion of weak solution has been introduced in [4]. As usual, a weak solution \(u\) satisfies the energy inequality

\[ \int_{\Omega} |u(t, x)|^2 \, dx + 2\mu \int_0^T \int_{\Omega} |\nabla u|^2 \, dx \, dt \leq \int_{\Omega} |u_0|^2 \, dx, \]

for any \(t \in (0, T)\). It is a natural question to ask when a weak solution satisfies the stronger version of (3), that is,

\[ \int_{\Omega} |u(t, x)|^2 \, dx + 2\mu \int_0^T \int_{\Omega} |\nabla u|^2 \, dx \, dt = \int_{\Omega} |u_0|^2 \, dx. \]

As we all known, any classical solution of the Navier-Stokes equations satisfies the energy equality (4). However, the existence of global classical solution remains open. Thus, an interesting question is how badly behaved \(u\) can keep the energy conservation. In his pioneering work [6], Serrin has proved \(u\) satisfies (4) if \(u \in L^p(0, T; L^q(\Omega))\), where

\[ \frac{2}{p} + \frac{d}{q} \leq 1, \]
where \(d\) is the dimension of space. In [7], Shinbrot has shown the same conclusion if \(u \in L^p(0,T;L^q(\Omega))\), where
\[
\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}, \quad q \geq 4.
\]
(6)
Note that, it is hard to say which condition is weaker between (5) and (6). However, an interesting point about the condition (6) is that they do not depend on the dimension \(d\).

We have to mention that a similar result to the Euler equation s, which was proved by E-Constantin-Titi [1]. It was the answer to the first part of Onsager’s conjecture [3].

The goal of this paper is to give a new proof to Shinbrot’s remarkable result in [7]. The following is the main result of this paper:

**Theorem 1.1.** Let \(u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))\) be a weak solution of the incompressible Navier-Stokes equations, that is,
\[
-\int_0^T \int_\Omega u_\varphi_t \, dx \, dt - \int_\Omega u_0 \varphi(0,x) \, dx - \int_0^T \int_\Omega \nabla \varphi u \otimes u \, dx \, dt \\
+ \mu \int_0^T \int_\Omega \nabla u \nabla \varphi \, dx \, dt = 0
\]
for any smooth test function \(\varphi \in C^\infty(\mathbb{R}^d)\) with compact support, and \(\text{div} \varphi = 0\). In addition, if \(u \in L^r(0,T;L^s(\Omega))\) for any \(\frac{1}{r} + \frac{1}{s} \leq \frac{1}{2}, \quad s \geq 4\), then
\[
\int_\Omega |u(t,x)|^2 \, dx + 2\mu \int_0^T \int_\Omega |\nabla u|^2 \, dx \, dt = \int_\Omega |u_0|^2 \, dx
\]
for any \(t \in [0,T]\).

**Remark 1.1.** The proof of Theorem 1.1 was motivated by the work of Vasseur-Yu [8], where they have shown the first existence result of weak solutions to the degenerate compressible Navier-Stokes equations in dimension 3. The same conclusion for the compressible version is established in [9].

**Remark 1.2.** The global existence of weak solution to 2d Euler equations was proved in [5], in particular, see Theorem 4.1 of book [5]. For a weak solution to Euler equation in this sense, adopting the same argument, we can conclude the energy conserve for any weak solution \(u \in C([0,\infty);W^{1,r}(\Omega))\), where \(r \geq \frac{4}{2}\). It was mentioned on page 132 in [5].

2. **Proof**

The goal of this section is to prove our main result. To this end, we need to introduce a crucial lemma. The key lemma is as follows which was proved by Lions in [5].

**Lemma 2.1.** Let \(f \in W^{1,p}(\mathbb{R}^d), \ g \in L^q(\mathbb{R}^d)\) with \(1 \leq p, q \leq \infty\), and \(\frac{1}{p} + \frac{1}{q} \leq 1\). Then, we have
\[
||\text{div}(fg) \ast \eta_\varepsilon - \text{div}(f(g \ast \eta_\varepsilon))||_{L^r(\mathbb{R}^d)} \leq C ||f||_{W^{1,p}(\mathbb{R}^d)} ||g||_{L^q(\mathbb{R}^d)}
\]
for some \(C \geq 0\) independent of \(\varepsilon, f\) and \(g\). \(r\) is determined by \(\frac{1}{r} = \frac{1}{p} + \frac{1}{q}\). In addition,
\[
\text{div}(fg) \ast \eta_\varepsilon - \text{div}(f(g \ast \eta_\varepsilon)) \to 0 \quad \text{in} \quad L^r(\mathbb{R}^d)
\]
as \(\varepsilon \to 0\) if \(r < \infty\). Here \(\varepsilon > 0\) is a small enough number, \(\eta \in C_0^\infty(\Omega)\) be a standard mollifier supported in \(B(0,1)\).
The weak solution \( u \) is uniformly bounded in \( L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \). Thus, it is possible to make use of Lemma 2.1 to handle convective term \( \text{div}(u \otimes u) \). With Lemma 2.1 in hand, we are ready to prove our main result.

We define a new function \( \Phi = u \), where \( f(t,x) = f \ast \eta_\varepsilon(x) \), \( \varepsilon > 0 \) is a small enough number, \( \eta \in C_0^\infty(\Omega) \) be a standard mollifier supported in \( B(0,1) \). Note that, we have

\[
\text{div} \Phi = 0 \tag{8}
\]

Using \( \Phi \) to test Navier-Stokes equations (1), one obtains

\[
\int_{\Omega} \Phi (u_t + \text{div}(u \otimes u) + \nabla P - \mu \Delta u) \, dx = 0,
\]

which in turn gives us

\[
\int_{\Omega} \Phi (u_t + \text{div}(u \otimes u) + \nabla P - \mu \Delta u) \, dx = 0.
\]

This yields

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u|^2 \, dx + \mu \int_{\Omega} |\nabla u|^2 \, dx = \int_{\Omega} \text{div}(u \otimes u) u \, dx,
\]

and hence

\[
\int_{\Omega} |u|^2 \, dx - \int_{\Omega} |u_0|^2 \, dx + 2\mu \int_0^T \int_{\Omega} |\nabla u|^2 \, dx \, dt = 2 \int_0^T \int_{\Omega} \text{div}(u \otimes u) u \, dx \, dt. \tag{9}
\]

Next we rewrite

\[
\text{div}(u \otimes u) = \left( \text{div}(u \otimes u) - \text{div}(u \otimes \overline{u}) \right) + \left[ \text{div}(u \otimes u) - \text{div}(u \otimes \overline{u}) \right] + \text{div}(u \otimes \overline{u}) \tag{10}
\]

Thus, the right-hand side of (9) is given by

\[
\int_0^T \int_{\Omega} (R_1 + R_2 + R_3) u \, dx \, dt.
\]

By means of (8), we have

\[
\int_0^T \int_{\Omega} R_3 u \, dx \, dt = 0. \tag{11}
\]
Now we first assume that \( u \in L^p(0, T; L^q(\Omega)) \), where \( p, q \geq 4 \). This restriction will be improved at the very end. We can control the term related to \( R_2 \) in the following way

\[
\left| \int_0^T \int_\Omega R_2 u \, dx \, dt \right| = \left| \int_0^T \int_\Omega (u \otimes u - \overline{u} \otimes \overline{u}) \nabla u \, dx \, dt \right|
\leq \int_0^T \int_\Omega |u - \overline{u}| |\nabla u| \, dx \, dt
\leq C \|u - \overline{u}\|_{L^p(0, T; L^q(\Omega))} \|u\|_{L^p(0, T; L^q(\Omega))} \|\nabla u\|_{L^2(0, T; L^2(\Omega))} \rightarrow 0
\]

as \( \varepsilon \) goes to zero, where \( p, q \geq 4 \).

Meanwhile, thanks to Lemma 2.1, we find
\[
\|R_1\|_{L^p(0, T; L^q(\Omega))} \leq C \|\overline{u}\|_{L^p(0, T; L^q(\Omega))} \|\nabla u\|_{L^2(0, T; L^2(\Omega))},
\]
and it converges to zero in \( L^{\frac{2p}{2+p}}(0, T; L^{\frac{2q}{2+q}}(\Omega)) \) as \( \varepsilon \) tends to zero. Thus, the convergence of \( R_1 \) gives us, as \( \varepsilon \) goes to zero,
\[
\left| \int_0^T \int_\Omega R_1 u \, dx \, dt \right| = \left| \int_0^T \int_\Omega \left( \text{div}(u \otimes u) - \text{div}(u \otimes \overline{u}) \right) \overline{u} \, dx \, dt \right|
\leq \|R_1\|_{L^{\frac{2p}{2+p}}(0, T; L^{\frac{2q}{2+q}}(\Omega))} \|\overline{u}\|_{L^p(0, T; L^q(\Omega))} \rightarrow 0,
\]
for any \( p, q \geq 4 \).

Letting \( \varepsilon \) goes to zero in (9), using (11), (12) and (14), what we have proved is that in the limit,
\[
\int_\Omega |u|^2 \, dx + 2\mu \int_0^T \int_\Omega |\nabla u|^2 \, dx \, dt = \int_\Omega |u_0|^2 \, dx,
\]
for any weak solutions with additional condition \( u \in L^p(0, T; L^q(\Omega)) \) with \( p \geq 4, q \geq 4 \).

The final step is to improve the restriction \( p, q \geq 4 \). Note that, \( u \in L^\infty(0, T; L^2(\Omega)) \) and \( u \in L^r(0, T; L^s(\Omega)) \), thus
\[
\|u\|_{L^p(0, T; L^q(\Omega))} \leq C \|u\|_{L^\infty(0, T; L^2(\Omega))} \|u\|_{L^r(0, T; L^s(\Omega))}^{1-\theta},
\]
for any \( \theta \in (0, 1) \) such that
\[
\frac{1}{p} = 1 - \theta \frac{r}{s},
\]
\[
\frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{s}.
\]

This yields
\[
\left( \frac{1}{r} + \frac{1}{s} \right) (1-\theta) = \frac{1}{p} + \frac{1}{q} - \frac{\theta}{2} \leq \frac{1}{2} (1-\theta),
\]
and hence

\[ \frac{1}{r} + \frac{1}{s} \leq \frac{1}{2} \]

with \( s \geq 4 \).

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Department of Mathematics, The University of Texas, Austin, Texas 78712.

E-mail address: yucheng@math.utexas.edu