Chaotic scattering around black holes

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Abstract

It is shown that the scattering of a charged test particle by a system of four extreme Reissner-Nordström black holes is chaotic in some cases. The fractal structure of the scattering angle and time delay functions is another manifestation of the existence of a nonattracting chaotic set: fractal basin boundaries were previously known.

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I. INTRODUCTION

The appearance of deterministic chaos in general relativity has been analyzed mainly in two different contexts: the evolution of cosmological models [1] and the motion of test particles in prescribed backgrounds. In the last context we may mention the study of charged particles in interaction with gravitational waves [2] and in the Ernst spacetime [3] and the chaotic motion of spinless [4] and spinning particles [5] around a Schwarzschild black hole and a Schwarzschild black hole with quadrupolar and octupolar contributions of arbitrary strength [6], but the main interest has been deserved by the system of two extreme Reissner-Nordström black holes described by the Majumdar-Papapetrou (MP) static solutions [7,8].

Chandrasekhar [9] and Contopoulos [10] analyzed the timelike and null geodesics around these two black holes and described the appearance of chaotic trajectories and their geometric source has been discussed by Yurtsever [11]. Another aspect of chaos, fractal basin boundaries, have been studied by Dettmann, Frankel and Cornish [12] for the case of two and three static black holes in the MP geometry and by Drake, Dettmann, Frankel and Cornish [13] in the context of Special Relativity. These fractal boundaries are consequence of the existence of a nonattracting chaotic set whose dimension provides a coordinate independent measure of chaos [14].

The goal of this paper is to show that, for a different range of parameters, another consequence of the existence of a nonattracting chaotic set arises in multi-black-hole spacetimes: chaotic scattering [15]. We will consider a static system of four extreme Reissner-Nordström black holes and a test particle, with $e/m > 1$. For certain energy ranges, the scattering is chaotic both in the relativistic system and in the Newtonian approximation. To the best of our knowledge this is the first time that chaotic scattering angle and time delay functions are described in general relativity [16].
II. THE EQUATIONS OF MOTION

Majumdar [7] and Papapetrou [8] independently showed that the metric

\[ ds^2 = -U^{-2} dt^2 + U^2 \left( dx^2 + dy^2 + dz^2 \right) \]  

(1)

is a static solution of the Einstein-Maxwell equations corresponding to the electrostatic potential \( A = U^{-1} dt \) if the function \( U(x, y, z) \) satisfies Laplace’s equation:

\[ U_{xx} + U_{yy} + U_{zz} = 0. \]  

(2)

Hartle and Hawking [17] showed that if one chooses

\[ U = 1 + \sum_{i=1}^{N} \frac{m_i}{|x - x_i|}, \]  

(3)

the MP solution represents \( N \) extremal \((q_i = m_i)\) Reissner-Nordström black holes held at the fixed positions, \( x = x_i \), under the combined action of gravitational attraction and electrostatic repulsion. The apparent singularity of \( U \) at the points \( x_i \) corresponds to the usual coordinate singularity at an event horizon.

Let us now consider a test particle of mass \( m \) and charge \( e \) moving in the above multi-black-hole geometry. Following Dettmann, Frankel and Cornish [12] we can write its equations of motion in the form

\[ \dot{x} = U^{-1} v, \]  

(4)

\[ \dot{v} = U^{-2} \left[ \left( 1 + 2 v \cdot v - \frac{e}{m} \gamma \right) \nabla U - (v \cdot \nabla U) v \right], \]  

(5)

\[ \dot{t} = U \gamma, \]  

(6)

\[ \gamma \equiv \sqrt{1 + v \cdot v}, \]  

(7)

where a dot indicates the derivative with respect to the proper time and \((\gamma, v)\) are the components of the four-velocity in an orthonormal frame. The conserved energy per unit mass is

\[ E = U^{-1} \left( \frac{\gamma - \frac{e}{m}}{2} \right). \]  

(8)
III. CHAOTIC SCATTERING

Contopoulos [10] analyzed the case of an uncharged particle \((e/m = 0)\) moving around \(N = 2\) static black holes and discussed the role of periodic orbits and the route to chaos through a cascade of period-doubling bifurcations. Dettmann, Frankel and Cornish [12] considered the cases \(N = 2, 3\) for a particle with \(e/m \leq 1\) which was released from rest. Under these circumstances the gravitational attraction overcomes the electrostatic repulsion and the particle will fall into one of the black holes or orbit indefinitely, but will not escape to infinity. Dettmann, Frankel and Cornish showed that the boundary separating in phase-space the different asymptotic behaviors is fractal, which indicates the presence of a nonattracting chaotic set [15] whose dimension provides a coordinate independent measure of chaos [14].

We have considered the motion of a charged test particle in the plane \((x, y)\) around a system of four equal Reissner-Nordström black holes which are at rest at points \((\pm 1, \pm 1, 0)\) and have \(q_i = m_i = 1/3\). The energy (8) has not exactly the same structure as in Newtonian mechanics, but we can still get the same kind of qualitative information given by the turning points of the potential if we study the curves of zero velocity [10], i.e., curves in the \((x, y)\) plane which satisfy Eq. (8) for \(v = 0\). Orbits never cross the curves of zero velocity.

If \(e/m > 1\) the electrostatic repulsion acting on a stationary test particle is stronger than the gravitational attraction and for \((1 + 2\sqrt{2}/3)^{-1}(1 - e/m) < E < 0\) there exist a curve of zero velocity around each black hole. In Fig. 1 we have the right hand side of (8) for \(v = 0\) and \(e/m = 2\). The curves of zero velocity are the contours of constant height of the displayed surface. In these circumstances, we will have scattering: orbits coming from infinity will escape to infinity, they will not fall into one of the black holes. But there exist another set (of measure zero but non-null dimension) of orbits which are not trapped by a single black hole: the unstable periodic orbits in which the particle bounces forever between different repulsion centers or, more precisely, between different curves of zero velocity. This is the kind of scenario in which chaotic scattering is likely to appear [15]. Although this system
does not fit exactly in one of the classes in which chaotic scattering is usually analyzed, i.e.,
gradient systems and hard disks or spheres, we expect to have a similar qualitative behavior
because we still have curves of zero velocity (which are the curves in which the energy equals
the potential energy in the case of gradient systems and the perimeter of hard disks).

In the following, we consider a test particle with \( e/m = 2 \) which is sent from the point
\((-4, b, 0)\), for different values of the impact parameter \( b \), and with initial velocity \((v, 0, 0)\).
The \( v \) value is selected in order to have always an energy per unit mass \( E = -0.45 \). As
can be seen in Fig. 2, for this value of the energy per unit mass there exist a curve of zero
velocity around each black hole preventing orbits coming from infinity to be trapped by the
black holes.

The system (4)-(5) is numerically solved by means of an ordinary-differential-equation
solver [18]. The quality of the numerical results is tested by using different integration
schemes, ranging from the very stable embedded Runge-Kutta code of eight order due to
Dormand and Prince to very fast extrapolation routines. All codes have adaptive step size
control and we check that smaller tolerances do not change the results. Furthermore, the
constancy of the energy (8) is monitored to test the integration accuracy.

In Fig. 2 we see four orbits which correspond to very close initial conditions, but look
very different after bouncing several times on different curves of zero velocity. This sensitive
dependence on initial conditions is the hallmark of chaos. To explore this further we have
chosen a large number of points \((\sim 2 \times 10^5)\) in the interval \(0.34 \leq b \leq 0.42\) and integrated
the system for each initial condition until the particle escapes the system. (In practice we
consider that the particle has abandoned the system when \(|x| > 10\), because it cannot return
if \(E < 0\).) Then we compute the scattering angle \(\theta\) between the velocity \(v\) and the \(x\) axis. In
Fig. 3 we have plot 2000 points of the scattering function \(\theta(b)\). In some ranges the function
is continuous and the points are located on a smooth curve, but in other intervals the points
are wildly scattered because \(\theta(b)\) is discontinuous at the points of a Cantor set, as can be
seen in the blowups of Figs. 4 and 5 where smaller and smaller subintervals are analyzed.
The (approximate) scale invariance typical of fractals is apparent. This kind of phenomenon
is called “chaotic scattering” [15].

IV. TIME DELAYS AND THE UNCERTAINTY DIMENSION

Exactly as in ordinary Hamiltonian systems [15], the singularity points of the scattering function correspond to unstable periodic orbits in which the particles bounces forever between the curves of zero velocities of different black holes. This can be seen in Fig. 6 where the time the particle spends in the scattering region is plotted against the impact parameter $b$. This time goes to infinity precisely at the singularity points of $\theta(b)$. It is easy to understand that if two orbits are initially very close but then come near one of the unstable periodic orbits, they will spend a very long time inside the scattering region and the accumulated defocusing effect of the successive bounces will yield rather different exit angles.

Though the fractal nature of the scattering function is apparent from the approximate autosimilarity shown in Figs. 3, 4 and 5, we can make this fact more quantitative by using a variation of the “uncertainty exponent technique” [15], which is also used in determining the dimension of fractal basin boundaries. We select a uncertainty value $\epsilon$ and sample the interval under study by choosing points in the form $b_{i+1} = b_i + \epsilon$. If for the three values $b_{i-1}$, $b_i$ and $b_{i+1}$ the particle scatters upward or the three values correspond to a downward scattering, we say that the $b_i$ value is $\epsilon$-certain. The remaining $b_i$ points are the $\epsilon$-uncertain values and correspond to cases in which an error of magnitude $\epsilon$ in the determination of the initial conditions will prevent us from predicting if the particle will finally escape the scattering region upward or downward. A measure of the uncertainty is thus given by the fraction of $\epsilon$-uncertain values which we will call $f(\epsilon)$.

It has been found in many different contexts that this magnitude scale with respect to $\epsilon$ according to a power law:

$$f(\epsilon) \sim \epsilon^{\alpha},$$

where $\alpha$ is the “uncertainty exponent.” When the set of uncertain points (in the limit
\( e \to 0 \) is not fractal it is easy to see that \( \alpha = 1 - D \), where \( D \) is the ordinary dimension of the set. This relation is extended to fractal cases by defining the “uncertainty dimension” as \( D \equiv 1 - \alpha \) and it has been conjectured that this definition will coincide with the “box counting” dimension \( [k] \). The meaning of the uncertainty dimension is that if we divide by 10 the error in the determination of the impact parameter, we only reduce by an amount of \( 10^{1-D} \) the error in the prediction of the late-time behavior. High values of \( D \) (0 <\( D < 1 \) in our case) make very difficult improving our predictions. This is the obstacle to predictability in chaotic scattering.

In Fig. 7, some values of the fraction of \( \epsilon \)-uncertain values \( f(\epsilon) \) in the interval \( 0.34 \leq b \leq 0.42 \) are plotted on a log-log scale. One sees that the points are fitted very well by a straight line of slope \( \alpha = 0.49 \). We conclude that the power law \( (k) \) is satisfied and the uncertainty dimension of the singularity set of the system under study is

\[
D = 1 - \alpha = 0.51. \tag{10}
\]

This dimension provides a coordinate independent measure of chaos \( [l] \).

\section*{V. FINAL COMMENTS}

We have shown that a charged test particle may experience the phenomenon of chaotic scattering around a static system formed by four extreme Reissner-Nordström black holes: in some cases the scattering angle (and time delay) function is singular at the points of a Cantor set. As mentioned above, another manifestation of the existence of a nonattracting chaotic set, fractal basin boundaries, was previously known for MP solutions, but for other values of the parameters. In some sense this kind of chaotic behavior is less surprising, because it does not appear for \( N = 1, 2 \) (which are the only cases with an integrable Newtonian limit \( [l] \)). For \( N > 2 \), we have repeated the analysis made above for the Newtonian limit, and it is not difficult to find parameter ranges for which the scattering becomes chaotic. Let us also mention that the same happens in the relativistic case. For instance, we have found
that chaotic scattering also happens around three black holes, though we have chosen to present here the more symmetric case $N = 4$.

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FIGURES

FIG. 1. Energy per unit mass \( \xi \) for \( v = 0 \) and \( e/m = 2 \).

FIG. 2. Four black holes at points \((\pm 1, \pm 1, 0)\) have curves of zero velocity around them for \( E = -0.45 \) and \( e/m = 2 \). Orbits corresponding to (1) \( b = 0.34975 \), (2) \( b = 0.34971875 \), (3) \( b = 0.3496875 \) and (4) \( b = 0.34965625 \) are also depicted.

FIG. 3. The scattering function \( \theta(b) \) for \( 0.34 \leq b \leq 0.42 \).

FIG. 4. A blowup of Fig. (3) showing the scattering function \( \theta(b) \) for \( 0.36 \leq b \leq 0.44 \).

FIG. 5. A further blowup of Figs. (3) and (4).

FIG. 6. The time-delay function indicating the time the particle spends in the scattering region for each value of the impact parameter. It becomes infinite at the singularity points of Fig. (3).

FIG. 7. A log-log plot of the fraction of uncertain values \( f(\epsilon) \) for different resolutions \( \epsilon \). The straight line fit has a slope \( \alpha = 0.49 \).
