Coordinating users of shared facilities via data-driven predictive assistants and game theory*

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Abstract

We study data-driven assistants that provide congestion forecasts to users of shared facilities (roads, cafeterias, etc.), to support coordination between them, and increase efficiency of such collective systems. Key questions are: (1) when and how much can (accurate) predictions help for coordination, and (2) which assistant algorithms reach optimal predictions?

First we lay conceptual ground for this setting where user preferences are a priori unknown and predictions influence outcomes. Addressing (1), we establish conditions under which self-fulfilling prophecies, i.e., “perfect” (probabilistic) predictions of what will happen, solve the coordination problem in the game-theoretic sense of selecting a Bayesian Nash equilibrium (BNE). Next we prove that such prophecies exist even in large-scale settings where only aggregated statistics about users are available. This entails a new (nonatomic) BNE existence result. Addressing (2), we propose two assistant algorithms that sequentially learn from users’ reactions, together with optimality/convergence guarantees. We validate one of them in a large real-world experiment.

1 INTRODUCTION

Data-driven interventions on social/economic systems are on the rise, but it remains a challenge to understand when and how they can improve such systems in terms of peoples’ actual utilities and overall resource efficiency. Here we consider central predictive coordination assistants, that, in the simplest case, work as follows: The assistant provides a congestion forecast \(A\) to users of some facility, based on past observations. The users trust \(A\) to be a good forecast, and individually optimize their facility use based on it, e.g., their arrival time slot, to coordinate and avoid crowds. Thereby they generate an observable outcome \(Y\), which \(A\) is a forecast for. In particular, forecast \(A\) influences outcome \(Y\). Versions of such assistants exist for roads, trains, swimming pools, etc. [Google, 2019, ASFA, 2019, DB, 2019], or, in our experiment, a cafeteria, see Figure 1.

**Figure 1:** Example of \(A \in \mathbb{R}^{18}\), the assistant’s (point) forecast, and outcome \(Y \in \mathbb{R}^{18}\), in our cafeteria experiment (\(A\) updated between but not within days, Section 6).

**Main goals and contributions:** We aim at (1) understanding to what extent optimally accurate assistant predictions can help coordination between users (Goal 1), and (2) designing sequential assistant algorithms that achieve optimal predictions (Goal 2). Our contributions:

- Introducing new concepts for this setting, we analyze when the assistant achieving a “perfect” (probabilistic) prediction \(A\) of \(Y\), i.e., a “self-fulfilling prophecy”, is equivalent to “solving” coordination in the sense of selecting a Bayesian Nash equilibrium (BNE) (Theorem 1).
- We establish conditions under which such a prophecy exists even in large-scale settings with only population-
level aggregated user data (Theorem^2), using the Leray-Schauder-Tychonoff fixed point theorem. This entails a new nonatomic game BNE existence result (Corollary^3).

• We propose learning assistant Algorithms 1 and 2 (controllers), for large-/small-scale settings, with optimality/convergence guarantees (Propositions 1 and 2).

• We report positive evaluation of Algorithm 1 in a large-scale real-world cafeteria experiment (Section 6).

Closest related research overview: Within game theory, dynamics/equilibria of multiple agents are studied that learn about each other by repeatedly interacting, but without central assistant [Shoham and Leyton-Brown, 2008]. Besides this, the following game-theoretic work usually assumes that agents reason fully rationally based on their own a priori given beliefs about other agents, instead of using a predictive assistant informed by past behavioral data: Congestion games [Nisan et al., 2007] formalize coordination in certain shared facilities. (Allocation) mechanisms are designed [Nisan et al., 2007] that maximize social welfare (which is defined in terms of agent’s a priori unknown preferences), in spite of agents being self-interested, by using incentives. Unlike our assistant, these mechanisms often fully control the outcome. And we consider “solving coordination” in game-theoretic (equilibrium selection) rather than in social welfare terms. Beyond game theory, certain smart cities research [Mareček et al., 2015] uses a control-theoretic approach for congested facilities, but they fix an objective that does not in general account for users’ individual, a priori unknown preferences. For further related work, see Section 7 and Section 8.

2 PRELIMINARIES AND SETTING

Notation: For a vector $b$, $b_i$ or $[b]_i$ is the $i$-th component, $b_{−i}$ means dropping $b_i$, and $(b_i, b_{−i})$ reads $b$. For a variable $Z$, range$_Z$ denotes the (implicitly given) range.

2.1 General setting and assistant-based system

Let us first introduce the general users’ decision problem. We leave it fairly abstract so that later on we can consider different forms of decision making scenarios based on it.

Setting 1 (General (one-stage) setting). There is a finite set $K = \{0, \ldots, |K| − 1\}$ of slots^4 and a set $I$, interpreted as users (here and in Section 3.2) or types of users (in Section 3.3), respectively. Each user $i \in I$:

• receives a (private) signal $W_i$,

• as (private) action $B_i$ chooses a slot in $K$, and

• experiences (private) utility $U_i$ he wants to maximize.

^1$K$ can be e.g., several facilities, or time slots in one facility.

Let $W = (W_i)_{i \in I}$, $B = (B_i)_{i \in I}$ and $U = (U_i)_{i \in I}$. Besides the private signals, there is a publicly available signal $V$, and some underlying (latent) state $X$. And there is a publicly observable outcome $Y = Y(X, B)$, for some function $Y$. We assume there is a “true” distribution $P(X, V, W)$. If not stated otherwise, we assume that all users $i$ are inference-assistable, i.e.,

$$U_i = \tilde{U}_i(W_i, B_i, h_i(Y(X, B))),$$

for (continuous) functions $\tilde{U}_i, h_i$ such that $h_i(\tilde{Y}(x, (b_i, b_{−i})))$ does not depend on $b_i$, for all $b, x \in B$ And let all users $i$ be assistant-separable, i.e., $h_i(Y) \perp W_i|V$ (for any possible mechanism that generates $B$ from $V, W$).

Setting 1 leaves open how users reason/decide. Our main object of study is a system that enriches this setting: user $i$ chooses $B_i$ that maximizes her expected utility, given $Y$ is distributed according to a central assistant’s forecast:

**Definition 1 (Assistant-based system $M$).** Based on Setting 1 or any restricted version, let the assistant-based (one-stage) system $M$ be defined by the following objects and assumptions additional to Setting 1, as depicted by the (causal) Bayes net [Pearl, 2000] in Figure 2. There is an assistant that takes public signal $V$ as input and outputs $A$, a probabilistic forecast for the public outcome $Y$.^5

^5This models the fact that actions $B$ may no be observed publicly, but just, say, some stochastic aggregation of them.

^6The intuition behind this constraint on the utility functions is that users’ decision making can be discerned into (1) an optimization performed by the users and (2) the task of predicting $Y$ which can be “outsourced” to an assistant.

^7This means, roughly, that the users do not know more about each other than is contained in the public $V$. Figure 2: (Causal) diagram of the assistant-based system $M$ (without $U$). The dashed gray arrows indicate the dynamic extension $M^\delta$ we will introduce in Section 4.
based on policy $\pi$, i.e., $A = \pi(V)$. That is, $A$ is a distribution over $Y$ (later we also consider point forecasts). User $i \in I$ takes forecast $A$ (besides her private signal $W_i$) as input, and acts assistant-best-respondingly, i.e.,

$$B_i \in \arg \max_{b_i} \mathbb{E}_{Y \sim A}(U_i(W_i, b_i, h_i(Y')))$$

(breaking ties via $K$). A joint $P_M(X, V, W, A, B, Y, U)$ is induced by all the above (measurable) equations and $P(X, V, W)$. We may write $P_M, \pi$ and $\mathbb{E}_{M, \pi}$ to make the dependence on $\pi$ explicit. $Y$ is observed, but the specific $P(X, V, W)$ and $U_i$’s are a priori unknown, and $X, W, B$ and utilities $U$ are unobserved by the assistant.

2.2 Game-theoretic tools to characterize efficiency

We want to analyze the degree of efficiency that assistant-based coordination can achieve. For this, we now define what a “solution” of the coordination problem would be (a BNE), accounting for users’ preferences. This is based on an idealized, assistant-free version of Setting 1 (a Bayesian game), where users $i$ have informed priors and unlimited inference abilities themselves, using $W_i$ and $V$ as input. Then, for any user behavior that arises in the assistant-based system, we can check if it is (or rather: corresponds to) such a solution. For background on game theory and the Bayesian game definition we use, see Section A.

Definition 2 (Benchmark (assistant-free) game $G$). Based on Setting 1 or any restriction of it, let the benchmark game $G$ be defined as the Bayesian game canonically associated to this setting: Each user $i \in I$ is a player who has: signal $(W_i, V) \in \text{range}(W_i, V)$, (measurable) utility function $\text{range}(X, W_i, B) \to \mathbb{R}$ given by Eq. 2 and action $B_i \in K$. The utility functions are common knowledge and $P(X, V, W)$ is the common prior.

As usual, a (pure) strategy profile for $G$ is a tuple $s = (s_i)_{i \in I}$ of (measurable, pure) strategies $s_i : \text{range}(W_i, V) \to K$, $i \in I$. A strategy profile $s$ is a Bayesian Nash equilibrium (BNE) of $G$ [large], if

$$s_i(w_i, v) \in \arg \max_{b_i} \mathbb{E}_{G,(b_i, s_{-i})}(U_i(w_i, v))$$

for (almost) all $i, w_i, v$; with $U_i$ as in Eq. 1 and $\mathbb{E}_{G,(b_i, s_{-i})}$ the expectation under $P_G(b_i, s_{-i})(X, \ldots, U)$ obtained by “plugging” strategy profile $(b_i, s_{-i})$ into game $G$ (here means the constant strategy).\(^5\) We call the BNE strict if the argmax is unique.

To relate $M$ to $G$, given an assistant policy $\pi$ of the assistant-based system $M$, we define the corresponding strategy profile $s_\pi$ by the composition of $\pi$ and users’ subsequent (deterministic) “best-response” action, i.e.,

$$[s_\pi]_i(w_i, v) := \mathbb{E}_{M, \pi}(B_i|w_i), \text{ for all } i, w_i, v.$$ (4)

Conversely, given a strategy profile $s$ of the benchmark game $G$, we define the corresponding assistant policy $\pi_s$, \(\pi_s(v) := P_M(Y|v)$, for all $v$. (5)

2.3 Objective functions

We consider the following two objective functions for the assistant’s policy $\pi$, where, as we will see, the former can be seen as a directly measurable “proxy” to the latter:

• (probabilistic) prediction accuracy objective (loss):

$$L_{\pi}^{\text{pred}} := \mathbb{E} \langle d(P_{M, \pi}(Y|V), \pi(Y)) \rangle, \text{ for all } \pi.$$ (6)

with $d(\cdot, \cdot)$ some arbitrary but fixed statistical distance which is 0 iff both distributions coincide;

• equilibrium selection objective\(^6\) $\pi$ is optimal iff the corresponding strategy profile $s_\pi$ (Eq. 4) is a BNE of the benchmark game $G$.

We call a policy $\pi$ that tries to optimize $L_{\pi}^{\text{pred}}$ loosely a (predictive coordinator) assistant, and a $\pi$ that achieves $L_{\pi}^{\text{pred}} = 0$ a (formally) self-fulfilling prophecy (policy).

3 THE UTILITY OF PREDICTIONS FOR COORDINATION – ANALYSIS

In this section, we pursue the following goal, for which the one-stage setting we introduced in Section 2 is sufficient (we will introduce a repeated version in Section 4).

Goal 1. Understand the conditions when, and the degree to which, assistants, that achieve $\pi \in \arg \max_{\pi^*} L_{\pi}^{\text{pred}}$, help solve the problem of coordination between users of facilities (here: in terms of equilibrium selection).

3.1 Characterization step in general setting

Theorem 1 (Self-Fulfilling Prophecy Characterization). We have, in the general setting (Setting 1 with all users being inference-assistable and assistant-separable):

• If the assistant policy $\pi$ in the assistant-based system $M$ (where all users are assistant-best-responding) is a

\(^5\)I.e., the strategy $s_i$ maps player $i$’s signal to her action

\(^6\)I.e., each player’s strategy is a best response to the others.
self-fulfilling prophecy (i.e., \( L^\text{pred} = 0 \)), then the corresponding strategy profile \( s_\pi \) is a Bayesian Nash equilibrium (BNE) of the benchmark game \( G^\text{large} \).

- Conversely, if the strategy profile \( s \) is a strict BNE of the benchmark game \( G^\text{large} \), then the corresponding assistant policy \( \pi_s \) is a self-fulfilling prophecy.

The proof is in Section D.2.1 Since in Section 2 we were very brief regarding some of the (measurability) assumptions and definitions underlying the theorem, we give a detailed elaboration of these assumptions and definitions, and their soundness, in Section C. For a justification of some of the theorem’s assumptions see Section 2. Note that Setting 1 is formulated pretty generally: the slots \( K \) can be any set of options the users have. \( K \) can be time slots in one shared facility, like a road section; or \( K \) can be several facilities that provide the same service, say citizen centers in a city; or it can be a combination, i.e., time slots in several facilities. (Our “slot” is similar to “facility”, or, to some extent, “feasible combination of facilities”, in congestion games.) The main limitation may be seen in the assumptions of inference-assistability and assistant-best-responding, saying that users can meaningfully evaluate the utility of their choices based on only \( Y \), which \( A \) is a forecast for.

### 3.2 Existence step in small-scale setting

In Theorem 1 we characterized the type of solution (a BNE of game \( G \)) that is implemented by the assistant-based system (with significantly lower requirements on users’ knowledge/inference capacities than in the game \( G \)), if the assistant reaches a self-fulfilling prophecy policy (“characterization step”). We established this result for the general setting (Setting 1). As second step towards Goal 1 it remains to understand when such a self-fulfilling prophecy exists (“existence step”). This second step we perform separately for two instructive subsets of the general setting, which are each still reasonably general. As a warm-up exercise, we start in a setting where we can easily build on game-theoretic results – because it corresponds to a classical finite Bayesian game.

#### 3.2.1 Introducing the setting

**Setting 2 (Small-scale setting).** As a restricted form of Setting 1 consider the following small-scale setting: \( I = \{1, \ldots, n\} \) is finite and we interpret its elements here as users (not types), and the individual actions of the users are directly publicly observable, i.e., \( Y = B \), and \( h_i(B) = B_{-i} \) in Eq. 1. \( X, V, W \) all have finite range.

We may write \( M^\text{small} \) and \( G^\text{small} \) to denote assistant-based system (Definition 1) and benchmark game (Definition 2), respectively, canonically associated to this particular small-scale setting.

**Corollary 1.** Setting 2 is a special case of Setting 1. In particular, it satisfies the conditions of Theorem 1 and hence the theorem’s implications hold for \( M^\text{small} \) and \( G = G^\text{small} \).

#### 3.2.2 Self-fulfilling prophecy existence

To understand the conditions under which a self-fulfilling prophecy exists, based on the second part of Theorem 1 (or rather: Corollary 1) it is enough to understand when a strict BNE of \( G \) exists. But in the current small-scale setting, \( G^\text{small} \) is the classical finite (Bayesian) game, which is well understood. For example, Harsanyi [1973] Theorems 3, 4 showed that when assuming that the players’ utilities are contaminated by a small additive noise, then there exists a strict equilibrium with probability one. Furthermore, Bilancini and Boncinelli [2016] establish conditions under which all BNE are (essentially) strict, which entails existence of such strict BNE when combined with general BNE existence results.

### 3.3 Existence step in large-scale setting

While the above small-scale setting is easy to understand, it has significant limitations: first, the users’ actions \( B \) have to be fully observable for the (loss \( L^\text{pred} \) of the assistant, which is often impossible due to data privacy regulations; and second, there has to be a fixed set of unique users, while in practice the set of users may change of course. Therefore we perform the second step towards Goal 1 also for the following large-scale setting (again a subsetting of the general setting of Section 2 different from the small-scale setting). It corresponds to nonatomic games Schmeidler [1973], and is mathematically more involved, but abstracts away from individual users and in particular only requires a cross-user aggregate of actions to be publicly observed.

#### 3.3.1 Introducing the setting

**Setting 3 (Large-scale setting).** As a restricted form of Setting 1 consider the following (aggregated) large-scale setting:

- For simplicity, we assume there are only two slots, \( K = \{0, 1\} \), and that \( V, W \) are constant.\footnote{We will prove the main results, Theorem 2 for an arbitrary number \(|K|\) of slots though. The extension to stochastic \( V, W \) is less obvious due to measure-theoretic issues.} We assume \( I = [0, 1] \) with the Borel sets as \( \sigma \)-algebra \( \mathcal{I} \), and interpret \( i \in I \) as a type of user with a certain form of utility function and private signal (similar as Kim and Yannelis [1997]). Let range\(_B \) (i.e., the set of possible joint user
actions \((B_i)_{i \in I}\) be the set of \(\{0, 1\}\)-valued Lebesgue-measurable functions on \(I\).

- Let \(\text{range}_Y = [0, 1]\) and \(Y_1 := \int B_i r(i|x) \, di\), for \(r(\cdot|x) \in \text{range}_X\) a family of continuous (Lebesgue) densities on \((I, \mathcal{I})\), continuous also in \(x\). And let \(Y_0 := 1 - Y_1\). The interpretation is that \(Y_1\) is the fraction of users that choose slot \(1\), i.e., a (stochastic) aggregate of \(B\), and \(Y_0\) is the remaining amount of users, that choose slot \(0\). Since \(Y = (Y_0, Y_1)\) is fully parameterized by \(Y_1\), from now on we consider \(Y\) to be 1-dimensional and stand for \(Y_1\).

- Regarding users \(i \in I\) and utilities, let \(h_i\) (Eq. 7) be the identity, and let \((i, y) \mapsto \tilde{U}_i(k, y)\) be a polynomial in \(i, y\) for all \(k \in K\) (we dropped \(W_i, h_i\) from the general \(U_i\), Eq. 7). This means, in particular, that the utilities only depend on the amount of users at the various slots, not on their identities. For any \(k \neq l \in K\), let \(\tilde{U}_i(k, y) - \tilde{U}_i(l, y) = \sum_m \epsilon^{m}_q m_i(y)\) be such that, for at least one \(m \geq 1\), \(q_m(y)\) is nonzero and constant in \(y\).

Note that, while in practice the course the set of (simultaneous) users and thus also (simultaneous) types of users is finite, having \(I = [0, 1]\) can be seen as an approximation with nice theoretical properties to real settings with many users. We may write \(M^{\text{large}}\) and \(G^{\text{large}}\) to denote assistant-based system (Definition 1) and benchmark game (Definition 2), respectively, for this particular large-scale setting. \(G^{\text{large}}\) can be seen as an incomplete-information nonatomic game, related to [Kim and Yannelis, 1997] but different in that our state can have uncountable range, see also Section 5.1. For the sake of completeness, let us formally state a version of Theorem 1 for this setting, proved in Section D.2.

**Corollary 2.** Setting 3 is a special case of Setting 4. In particular, it satisfies the conditions of Theorem 1 and hence the theorem’s implications hold for \(M = M^{\text{large}}\) and \(G = G^{\text{large}}\).

### 3.3.2 Self-fulfilling prophecy existence

In contrast to the small-scale setting, for the large-scale setting and the corresponding benchmark game \(G^{\text{large}}\) there is less established work that helps to understand existence of a self-fulfilling prophecy policy. Intuitively, a key question in this large-scale setting is: can a forecast that only forecasts an aggregate of the users’ actions (the name “nonatomic” comes from the fact that one considers nonatomic measures on the type space \(I = [0, 1]\).)

Y of Setting 3 actually be a self-fulfilling prophecy and thus help for coordination? For instance, as observed by Mareček et al. (2016), if the population of users is completely homogeneous, they will all respond in the same way upon receiving the same input, making coordination difficult. Here is our answer for this question – the second of our two main theoretical results.

**Theorem 2** (Large-Scale Self-Fulfilling Prophecy Existence). There exists a self-fulfilling prophecy policy \(\pi\) in the assistant-based system \(M^{\text{large}}\) (in Setting 3).

This implies, based on Corollary 2.

**Corollary 3** (Large-Scale Bayesian Nash Equilibrium Existence). The benchmark game \(G^{\text{large}}\) (for Setting 3) has a Bayesian Nash equilibrium (BNE).

**Proof idea and interpretation:** The proof of Theorem 2 which is given in Section D.3 for an arbitrary number of slots \(K\), is based on the Leray-Schauder-Tychonoff fixed point theorem, harnessing the compactness of the set of Borel measures, \(\text{range}_A\), under a weak topology. The most important implication of the theorem is that \(\min_{\pi} L^{\text{pred}} = 0\). And therefore, together with the first step in the form of Theorem 1, it shows that an assistant that only forecasts an aggregate can nonetheless, when it achieves its optimum, help “solve” the coordination problem – select a BNE. The intuition behind the assumptions is that types and their utility functions have to be diverse. Corollary 3 can be seen as stand-alone, purely game-theoretic result for \(G^{\text{large}}\).

### 3.3.3 An instructive linear special case

Let us consider a simple special case of the large-scale setting (which is not central to understand the rest of the paper and can be skipped). On the one hand, this helps to get an intuition for Theorem 1 on the other hand this will justify assumptions we will make in the analysis of our algorithm in Section 5.4. Assume the utility \(U_i(k, y)\) of Setting 3 is linear in \(i, y\) for all \(k \in K\) (making the users “risk-neutral”). So \(\tilde{U}_i(1, y) - \tilde{U}_i(0, y) = i + \varphi y + \chi\), for \(\varphi, \chi \in \mathbb{R}\). Let \(r(i|x) := \frac{1}{2\delta}[x - \delta \leq i \leq x + \delta]\), \(i \in I, x \in \text{range}_X\), with \(\lfloor \cdot \rfloor\) the Iverson bracket (i.e., density of the uniform on \([x - \delta, x + \delta]\), and let \(P_X\) be the uniform on \([\delta, 1 - \delta]\). Then the value of \(Y\) as a function of \(A = a, X = x\) is, for \(H\) the Heaviside function, given by

\[
\begin{align*}
\int H \left( \int \tilde{U}_i(1, y) - \tilde{U}_i(0, y) \, da(y) \right) r(i|x) \, di \\
= \int H (i + \varphi E_{Y' \sim a}(Y') + \chi) r(i|x) \, di \\
= \frac{\varphi}{2\delta} E_{Y' \sim a}(Y') + \frac{1}{2\delta} x + \frac{\delta + \chi}{2\delta},
\end{align*}
\]
for \( x - \delta \leq -\varphi E_{Y|\sim_0}(Y') - \chi \leq x + \delta \), and 0 or 1, respectively, otherwise – a piece-wise linear function in \( E_{Y|\sim_0}(Y'), x \).

First, this shows that under the mentioned assumptions, \( Y \) (and its distribution) only depends on the mean \( E_{Y|\sim_0}(Y') \), but no other properties of \( a \). In particular, \( L^\text{pred}_\pi = 0 \) iff \( L^\text{point}_\pi = 0 \), for \( \pi \) an appropriate probabilistic extension of \( \pi' \), and
\[
L^\text{point}_\pi := E_{M,\pi'}(\| A - E(Y|V) \|_2^2) \tag{10}
\]
a point prediction version of the probabilistic prediction accuracy loss \( L^\text{pred}_\pi \). This justifies for the assistant to provide point forecasts under the above assumptions. Second, this justifies a (locally) linear model for \( Y \) in \( (E_{Y|\sim_0}(Y'), x) \) and noise. Note that Theorem 2 restricted to this simple linear case is immediate based on the intuitive fact that a generic linear function has a fixed point.

4 SETTING FOR ALGORITHM PART – CONTROL DYNAMICS

To prepare the algorithm part of the paper, let us extend the general one-stage setting (Setting 1) and the assistant-based one-stage system (Definition 1) to a general dynamic setting and an assistant-based dynamic system \( M^{\text{dyn}} \), respectively, in the following “natural” way. This directly implies also dynamic extensions of small-scale and large-scale setting (Settings 2 and 3) and the corresponding assistant-based systems (we do not introduce explicit symbols for them though).

The dynamic extensions consists of \( \mathbb{N} \) copies of the one-stage versions, called stages/repetitions. We denote variables, say \( A \), in the \( t \)-th repetition by \( A_t \), \( t \in \mathbb{N} \). Furthermore, the dynamic extensions contains the following equations that replace/extend the ones of repetition \( t \) – think of it as a form of feedback control model, a partially observable Markov decision process (POMDP) [Sutton and Barto, 1998] (from the perspective of the assistant):
\[
X^t = \bar{X}(X^{t-1}, E^t) \tag{11}
\]
\[
A^t = \pi(V^{0:t}, A^{0:t-1}, Y^{0:t-1}), \tag{12}
\]
with \( E^t \) independent stochastic error terms, (measurable) function \( \bar{X} \), and (measurable) dynamic assistant policy \( \pi \). The gray, dashed arrows of Figure 2 indicates this dynamic extension. Regarding the assistant’s objectives, let \( L_{t,\text{pred}}^\pi, L_{t,\text{point}}^\pi \) be defined similarly as \( L_{\text{pred}}^\pi, L_{\text{point}}^\pi \), but additionally conditioning on the observed past:
\[
L_{t,\text{pred}}^\pi = E_{M,\pi'} \left( d(P_{M,\pi}(Y^t|A^{0:t-1}, Y^{0:t-1}, Y^{0:t-1}), A^t) \right), \tag{13}
\]
for all \( \pi \), and similarly \( L_{t,\text{point}}^\pi \). Remember: stage \( t \) must not be confused with (time) slot \( k \) within one stage. To motivate the algorithmic part below, let us give two examples of naïve dynamic assistant policies that fail.

Example 1 (Naive assistant yields oscillation). Consider a toy scenario of two users, \( i = 1, 2 \), two slots, \( K = \{0, 1\} \), \( W, V, X \) constant, and \( (0, 1) \) and \( (1, 0) \) the (pure) Nash equilibria of the induced complete-information benchmark game. For simplicity, let \( B \) be directly observed \((Y = B)\), let \( A \) be a point forecast. As usual, assume each day \( t \) both users best-respond to \( A_t \). The assistant starts with, say, \( A_0 = (0, 0) \) and then, naively, each day takes yesterday’s outcome \( B^{t-1} \) as forecast for today, \( A^t \). It is easy to see that this will lead to an overshooting and oscillating system \( B_0 = (1, 1), B_1 = (0, 0), B_2 = (1, 1), \ldots \) (called flapping by [Mareček et al., 2013]).

Example 2 (“I.i.d.” assistant is sub-optimal). Classical forecasting applied to the sequence \( B^1, B^2, \ldots \) from Example 1 would yield the empirical distribution \( P(B = b) = \frac{1}{2}(\delta_{(0,0),b} + \delta_{(1,1),b}) \), with \( \delta \) the Dirac delta, as optimal probabilistic forecast \( A^t \) – under some stationarity assumption. But the actual best forecast would be a Dirac delta on one of the two Nash equilibria \((0, 1) \) and \((1, 0) \) (Theorem 2: we ignore mixed equilibria here).

5 PREDICTIVE ASSISTANT ALGORITHMS WITH GUARANTEES

In the first part, we analyzed conditions under which predictive assistants help coordination (in terms of the equilibrium selection objective, Section 2.3), if they manage to optimize prediction accuracy, leaving open the “how”. Therefore, as second part of the paper, we address:

Goal 2. Design algorithms for the assistant policy \( \pi \) in the dynamic assistant-based system \( M^{\text{dyn}} \) that optimize prediction accuracy \( L_{t,\text{pred}}^\pi \) (and asymptotically select an equilibrium, if possible), learning from past interactions.

We will consider dynamic versions of the two settings for which we established in Section 3 that predictions can help coordination: large-scale setting (in Section 5.1) and small-scale setting (in Section 5.2). For each setting, we propose an assistant algorithm \( \hat{\pi} \), and provide a
theoretical analysis of its dynamics/convergence. A unifying idea behind both algorithms is that they mitigate certain bad user behavior, e.g., “overshooting” due to too many users jumping to the same purportedly “good” slot, helping convergence to a Nash equilibrium (of the stage benchmark game). Recall that users’ utilities (functions) are hidden from the assistant (Definition 1), so the assistant’s inference (about the equilibrium) is mainly based on behavioral data of how users react to forecasts.

5.1 Expodamp for large-scale setting

Consider the dynamic large-scale setting\(^{16}\) (Section 4) and let \(A\) be a point forecast for \(Y\), i.e., \(\text{range}_A = \text{range}_Y\), and consider \(L_{\text{point}}^\alpha\) as loss (dynamic version of Eq. 10) as described in Section 4. Recall that in Section 3.3.3 we gave conditions that justify this point prediction approach.

We propose Expodamp as described in Algorithm 1 as the assistant’s dynamic policy \(\pi\). The intuition behind Expodamp is that this formula can dampen oscillations due to “overshooting” user behavior (Example 1) but it can also accommodate for non-stationarities in user preferences. These intuitions will be made rigorous in the proposition below.\(^{13}\)

**Assumption 1.** Let the following equations hold for the dynamic assistant-based system \(M_{\text{dyn}}\), \(t \geq 1\):

\[
X^t = X^{t-1} + E^Y_X,
\]

\[
Y^t = \beta A^t + \gamma X^t + E^Y_Y,
\]

with \(E^Y_X\), \(E^Y_Y\) noise terms that are independent of the past and each other. (This is a state-space model known from the Kalman filter [Lütkepohl, 2006].)

Recall that in Section 3.3.3 we gave conditions, in the large-scale setting, that justify the linearity in Assumption 1 (note that the \(X\) in Assumption 1 would correspond to a parameter of the distribution of \(X\) rather than \(X\) itself in Section 3.3.3 but we neglect this detail for simplicity of notation). Also note that Assumption 1 is a linear approximation which facilitates the theoretical analysis but comes at the cost of a mismatch to the actual setting: \((Y^t)_{t \in \mathbb{N}}\) in Assumption 1 can leave \([0, 1]\) in the long run, so the model should rather be seen as a local approximation. Note that, due to convexity, Expodamp will always output \(A^t \in [0, 1]\) upon \(Y^t-j \in [0, 1]\) (\(j \geq 1\)) though. Keep in mind that the fixed point (self-fulfilling prophecy) of the linear function \(a \mapsto \beta a + \gamma x\) (ignoring the noise term) is \(\gamma (1-\beta)^{-1} x\) (exists whenever \(\beta \neq 1\)). In particular, if \(\beta = (1-\gamma)\), then the fixed point (corresponding to the self-fulfilling prophecy/BNE) is \(x\). We can give the following guarantees, for which we prove a generalization\(^{14}\) in Section 4.1.

**Proposition 1 (Optimality and Convergence Rate of Expodamp).** In the dynamic large-scale setting (Section 4), let Assumption 1 hold true. Let the assistant’s policy \(\pi\) be Expodamp (Algorithm 1).

- **Stochastic case:** In Expodamp, let \(\alpha := (1-\beta)^{-1}\), for the true \(\beta\) of Eq. 15. Assume \(E^Y_Y = 0\), \(t \geq 1\). Then, at each stage \(t\), \(L_{\text{point}}^\alpha = 0\) and

\[
A^t = \arg \min_{\alpha'} \mathbb{E} \left( \| A^t - Y^t \|^2_2 \mid A^0 = a', A^{0:t-1}, Y^{0:t-1} \right).
\]

(16)

- **Deterministic case:** Assume that \(X^t = x\) is constant, that \(\beta = (1-\gamma)\) and that \(E^Y_X = E^Y_Y = 0\). Then

\[
Y^t = x + (1-\gamma)(A^0 - x)(1-\alpha \gamma)^t,
\]

for all \(t \geq 0\).

That is, \(Y^t\) converges exponentially with rate \(\gamma \alpha\) towards the “optimum”/fixed point \(x\) (and thus also \(A^t\) converges to \(x\) based on Expodamp’s formula) if \(0 \leq \gamma \alpha < 2\).

When applying Algorithm 1 in practice, often one does not know the parameter \(\alpha\) a priori and has to infer it. As a first approximation, it may be learned by naively fitting Algorithm 1 to past observational data as if it were a classical (non-influential) forecasting method [Hyndman et al., 2008]. In principle however, without going into detail, \(\alpha\) rather has to be learned like a control policy, based on how the environment responds to it.

5.2 Partpred for small-scale setting

While Expodamp is the main algorithm of this paper, here we also provide a proof-of-concept algorithm for the repeated small-scale setting (Section 4). Assume \(X^t\) to be independent of \(X^{1:t-1}\), i.e., the special case where the \(X^t, t \in \mathbb{N}\) are i.i.d. The algorithm, Partpred, is sketched – for the case that \(V\) is constant – in Algorithm 2 and fully described in Section 4.2.

\(^{14}\)In particular, \(Y\) is considered 1-dimensional (since \(Y_1\) determines \(Y_0\)). The extension to more slots is straightforward.

\(^{13}\)The formula in Algorithm 1 is a case of a so-called exponential smoothing method [Hyndman et al., 2008]. However, so far (to the best of our knowledge) it has only been applied to classical forecasts that do not influence the outcome. In a sense, we generalize the established method to this new setting.

\(^{16}\)It is formulated slightly cleaner, using the do-operator.

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**Algorithm 1: Expodamp (large-scale setting)**

1. **Input:** parameter: \(\alpha\)
2. for each stage \(t \geq 1\) do
3.  **Input:** \(A^{t-1}, Y^{t-1}\)
4.  **Output:** \(A^t := A^{t-1} + \alpha(Y^{t-1} - A^{t-1})\)
The basic idea is as follows: as long as there is (significant) uncertainty about where the optimum (self-fulfilling prophecy/BNE) would be, the algorithm tries to make a prediction that is at least partially correct (i.e., makes the correct prediction at least w.r.t. the behavior of one player). The algorithm combines ideas from best-response dynamics and congestion games [Roughgarden, 2016] with random exploration whenever the best-response dynamics would cycle. Let $\bar{A}$ be the (finite) set of all distributions $P_{\bar{A},s}(B)$ that arise from (deterministic) strategy profiles $s$ of $G^{\text{small}}$. For simplicity, we assume $\bar{A}$ to be given, but in a next step this could be inferred as well. We give the following guarantee, sketched for $V$ constant, whose general version is proved in Section 5.2.

**Proposition 2** (Convergence of Algorithm 2 (Sketch)). In the setting described above, assume $G^{\text{small}}$ has a strict BNE. Let the assistant’s policies $\pi, r \in \mathbb{N}$ be given by Algorithm 2 with parameter $\bar{A}$ as defined above. Then, for any $\varepsilon > 0$, there exists $R,T$ such that for all $r > R, t > T$, it holds that $P(L_{L_{t}^{\text{pred}}} = 0) > 1 - \varepsilon$ and $P(s_{\pi, r}, s_{\pi, r} \in \text{BNE of } G^{\text{small}}) > 1 - \varepsilon$.

### 6 EXPERIMENT

Here we empirically evaluate Expodamp (Algorithm 1) for the large-scale setting and a baseline.

**Experimental setup:** We conducted our experiment in a real-world congested campus cafeteria with around 400 users per day. Here, observation $[Y^{t}]_{k}$ is (a proxy to) the number of people in the queue at time $k$ of day $t$.

---

**Table 1:** Evaluation shows that Expodamp has higher prediction accuracy.

| Method                      | $\bar{L}_{t}^{\text{point}}$ (MSE; Eq. (18)) |
|-----------------------------|---------------------------------------------|
| Expodamp (Algorithm 1)      | 69.56                                       |
| Average (baseline; Eq. 17)  | 74.25                                       |

The coordination assistant in this experiment is a web app which provides the daily forecast (i.e., the forecast is updated once per day, in the morning – more dynamic versions are future work) to the cafeteria users, to inform their decisions in terms of when to go to the cafeteria. The web app is used by between 15 and 45 users per day but may influence more (slightly deviating from our model). Besides Expodamp (with parameter $\alpha$ tuned based on a previous observational sample), we evaluate the baseline method Average defined by

$$a_{t+1}^{t} := \frac{1}{t} \sum_{s=1}^{t} y^{s}, t \geq 2$$

(i.e., treating $y^{1:t}$ as purely observational i.i.d. sample). Expodamp and Average are run as the policy that generates the forecast (which is then provided via the web app to the users of the cafeteria), each for a period of $T = 35$ days. See Figure 3 for an illustration of the experimental protocol. As metric, we use the mean squared error

$$\bar{L}_{t}^{\text{point}} := \frac{1}{T} \sum_{t=1}^{T} \|a_{t}^{t} - y^{t}\|^{2}_{2},$$

### Figure 3: Protocol of our real-world interventional experiment in a large campus cafeteria; steps along Y-axis.

The outcome is in Table 1 showing that Expodamp outperforms Average in this experiment. For illustration, we also show a sample of Expodamp’s output $A^{t}$ and actual outcome $Y^{t}$, for one day $t$, in Figure 1.
7 REMARKS AND FURTHER RELATED WORK

This section discusses additional aspects of the main results and further related work.

Why prediction accuracy / equilibrium selection as objective. Alternative to our approach in this paper, one could start from some (somewhat legitimized) social welfare [Nisan et al. 2007] as a function of users’ preferences, and design assistants that try to optimize it. This would be somewhat more in line with the economic notion of optimizing efficiency. Here, we rather follow a heuristic approach of starting with the “natural” prediction accuracy objective, because it compares well to the benchmark of equilibrium selection (Theorem 1), and for the following reasons: First, prediction accuracy can be directly measured, while social welfare seems hard to infer/identify from the incomplete information contained in the behavioral data available in our setting. Second, it is easy to interpret for users and leads to a form of “incentive compatibility” of users’ assistant-best-response (see remark below). Third, we feel that in our coordinative setting, equilibrium outcomes can be quite efficient in terms of social welfare. Generally, social welfare functions of course are hard to pick and impose in the first place. Nonetheless, equilibrium outcomes can of course be significantly inefficient, which has extensively been studied under the name of price of anarchy [Roughgarden 2005, Nisan et al. 2007]. But even in this regard, Theorem 1 can be helpful in that it makes predictive assistant-based settings amenable to such studies.

Remarks on our model assumptions: To justify our assumption of users “blindly” best-responding to the assistant’s forecast (Definition 1) observe that it can be seen as consistent with (instrumental) rationality [in the following sense: if only considering the asymptotic utility (once the assistant converged), then deviating from this behavior means deviating from a BNE, based on Theorem 1] Furthermore, all users best-responding simultaneously can sometimes be a too strong assumption, but we feel that it is a situation that can happen (more or less) at least sometimes, and therefore is worth analyzing. This being said, the assistant-best-responding assumption should be seen as a pragmatic first step that can be refined in future work. Generally, Theorem 1 shows that assistant-based systems can achieve coordination comparable to the benchmark game (additionally, it serves as a mechanism for equilibrium selection if there are several) – but at a significantly lower cost, since the inference task is centrally done by the assistant. (Obviously, it is only cheaper when inference comes at a cost – otherwise raw data $V$ could simply be provided to users directly.)

Further general related work: Let us mention that for the various versions of assistants we mentioned in Section 1 that are publicly available [Google, 2019, DB, 2019, ASFA, 2019], we could not find out what algorithms or theory they rely on. Research-wise, in mechanism design [22], a related direction has been emerging that studies how to design the information structure [Taneva, 2015, Bergemann and Morris, 2017] instead of the allocation/payment structure. Furthermore, data-driven approaches to mechanism design have gained momentum [Balcan et al. 2016, Duetting et al. 2019, Tang 2017, Kearns et al. 2014]. But these lines of research differ from ours – often additionally to what we already mentioned in Section 1 (bounded rationality of our users and limited power of our mechanism) as follows: either they assume that agents input their (true, if “incentive compatible”) preferences explicitly (instead of behavioral data), or they neglect, to some extent, agent’s actual preferences (which can be appropriate for revenue maximization of course).

8 CONCLUSIONS

In this work, we studied when and how parts of the coordination process of users of shared resources can be “outsourced” to a central data-driven predictive assistant. Our theoretical analysis showed that such assistants can help solve this multi-agent coordination problem in a game-theoretic sense, but non-trivial conditions have to be met: in terms of the information and preference structure of users, and stochasticity of their preferences in case only large-scale aggregated information is available to the assistant. Based on this analysis, we proposed two machine learning coordination assistant algorithms on behavioral data. We used linear dynamical systems models to prove their optimality/convergence, accounting for the fact that there is a feedback loop from predictions to outcomes. And we conducted a large-scale interventional experiment in a real campus cafeteria that provided empirical hints for the validity of our main algorithm.

Generally, the mentioned related work and our work in...
dicate that there is a plethora of possible computational mechanisms for collective decision making, in terms of inputs (high-level information, behavioral data, and beyond) and influences (full control over the outcome, money incentives, pure information/predictions, and beyond), many of which may still be unexplored.

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Appendix

A Background on game theory

Game theory [Osborne and Rubinstein, 1994, Shoham and Leyton-Brown, 2008] models the interaction between strategic agents, that is, settings with several such agents and where the utility of any one of them is influenced by the actions of one or several of the others. Since each agent’s utility depends on the other agents’ actions, each agent has to reason about how the others act when deciding on its own action.

The modeling in game theory is usually split into two parts: First, a game formalizes, in a sense, the decision making problem, that the agents (also called “players”) are facing. Note that, in a sense, there are two problems, but often they are treated simultaneously: the descriptive problem of predicting which strategies the players will chose when facing the game, and the prescriptive problem that each player faces – choosing the strategy that best serves her objective. Second, game theory considers solution concepts [Shoham and Leyton-Brown, 2008] that formalize how the agents will (in the descriptive interpretation) or should (in the prescriptive interpretation) approach this problem (game).

Now essentially, a game, as used by game theory, represents each agent by (1) a utility function, that models her interest/preferences/goals, and (2) a set of possible actions that she can take and has full control over. In the simplest case, called a (complete-information) normal-form game [Shoham and Leyton-Brown, 2008], this is essentially already the full model. In this case it is assumed that all utility functions are fixed and each agent knows the utility function of all other agents.

Let us give an example of such a complete-information normal-form game with two players and two actions each. We consider a simple coordination problem where the players can chose between going at 12noon or at 1pm to a cafeteria, and aim to avoid each others, say to avoid queuing. Additionally, assume that going early is favored by both. Specifically, let the game be given by the payoff matrix in Table 2 (payoff matrix is just another term for “utility matrix”). This representation has to be read as follows: player 1’s utility, in case player 1 chooses action $i$ and player 2 chooses action $j$, is given by the $i$-th entry of the tuple at column $i$, row $j$ of the matrix. For instance, if player 1 goes at 12noon and player 2 goes at 1pm, then player 1 has utility 2 and player one has utility 1.

Given such a game, each player can chose an action, and we can also consider jointly the actions of the players. We formalize such joint actions by action profiles, i.e., tuples of actions, one for each player. For any such action profile, we can ask if it is a solution to the game, according to some solution concept, as mentioned above. The most common solution concept for complete-information normal-form games is the (pure) Nash equilibrium. In the two-player, two-action case, it is defined as any action profile $(i, j)$, such that no player can improve her utility by unilaterally deviating to an action $i'$, i.e., every player chooses her optimal action (“best-responds”) given the other action in the tuple is fixed. For instance, in Table 2 the action profile (12noon, 1pm) is a (pure) Nash equilibrium.

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23“Strategic” means that they have goals/objectives/preferences/interests/utilities and take the best possible means to achieve them, accounting for the (multi-agent) context; a common alternative expressions are “(instrumentally) rational”, “self-minded” or “self-interested”.

24The name “game” likely comes from games of parlor being a special case of such games, but also based on them being a metaphor for general multi-agent situations, a metaphor that helps for the formal abstraction.

25Alternative formulations use preference relations instead of utility functions.

26This is a toy version of the setting of our cafeteria experiment, Section 6 of the main paper.
Clearly, “complete-information” is a strong assumption and therefore a generalization of these complete-information games has been proposed [Harsanyi, 1967], to model the case where the utility functions (more specifically: the precise influence of the joint action on the utilities of the agents) are a priori unknown to the agents (so in a sense: the game is a priori unknown). Intuitively, this absence of knowledge comes from agents’ preferences as well as relevant external events not being determined/known a priori. Instead, it is assumed that each agent, before choosing its action, observes a (private) signal and then uses a Bayesian prior distribution over this signal and the other relevant variables for its inference, according to Bayes rule.

Let us give a formal definition, based on [Osborne and Rubinstein, 1994] but adapted for our purposes (Remark 1).

**Definition 3.** A Bayesian game consists of a set I of players, a state of the world X, and for each player i ∈ I:

1. a signal Θi out of a set of possible signals $\text{range}_{\Theta_i}$,
2. an action $B_i$ out of a set of possible actions $\text{range}_{B_i}$,
3. a utility function $U_i : \text{range}_{(X, \Theta_i, B_i)} \rightarrow \mathbb{R}$,

and a common (“objective”) prior distribution $P(X, \Theta_i, i \in I)$.

**Remark 1.** We adapted the definition in [Osborne and Rubinstein, 1994] to our purposes in three ways: First, we dropped the assumption of finite cardinality of the sets (somewhat similar to [Kim and Yannelis, 1997]). Second, we formulate it more in the spirit of random variables (we event treat actions as random variables, in the sense that choosing a specific action corresponds to intervening on the action variable, in the sense of causal models [Pearl, 2000]), instead of just specifying their ranges. Third, our “state of the world” X is a random variable and it does not have to determine the value of the other variables. In contrast, in [Osborne and Rubinstein, 1994] the “state” is the outcome, in the probability theoretic sense (as element of the sample space, typically denoted by $\omega \in \Omega$) [Klenke, 2013], instead of a random variable. Accordingly, in their definition, the utility function does not have to depend on the signal, and the prior is already specified by a distribution over the state. But a Bayesian game according to our definition can be mapped to one according to the definition in [Osborne and Rubinstein, 1994] in the obvious way (essentially replacing our X by the state in the sense of the outcome), and vice versa.

Now, while there is no single one established solution concept (mentioned above) for a Bayesian game, the most common one is the Bayesian Nash equilibrium as we define it in Section 2 of the main paper. Note that in the case of the Bayesian game, potential solutions are given in the form of strategy profiles (as we also introduce in Section 2 of the main paper), i.e., tuples $(s_i)_{i \in I}$, where each $s_i$ is a strategy – a mapping from the set $\text{range}_{\Theta_i}$ of possible signals of player i, to the set of possible actions $\text{range}_{B_i}$ of player i. This generalizes the notion of an action profile introduced above, accounting for the fact that the player’s behavior is only fully specified once we determine her action for each possible observed signal.

### B Additional related work and comments

**Further general related work in game theory:** Within game theory, note that correlated equilibria were studied [Osborne and Rubinstein, 1994] that require a correlated (i.e., central) signal. Inference of preferences from behavioral data has been studied [Ling et al., 2018], but they do not feed the results back into the multi-agent system. Interpretations of the Nash equilibrium as self-fulfilling prophecy have been discussed, often informally, in epistemic game theory [Pacuit and Roy, 2017] [Spohn, 1982]. But they do not give a rigorous analysis of the specific conditions on information/utility/response structure for a concrete setting where the prophecy comes from an “external” agent. Influenstellar forecasts have been studied, also using fixed-point formulations but for election predictions, by [Simon, 1954]. Bayesian games with discrete actions but continuous states and signals have been studied by [Hellman and Levy, 2017].

**Further related work within smart cities research:** Besides the work already discussed in Sections 1 of the main paper and 3.3.2 of the main paper, also the following work in the area of smart cities and control is on congestion/coordination in shared facilities (often with some form of central assistant or signal): [Wirth et al., 2019] consider agents that share a constraint resource and receive a central capacity signal. For the case that the agents behave according to a (randomized) so-called additive-increase multiplicative decrease (AIDM) algorithm (alongside additional assumptions), their theoretical analysis shows convergence against the optimum under an overall objective function given by the sum of the individual agents’ utilities. [Häusler et al., 2014] also discuss the problem of flapping for the case of coordinated (balanced) routing of cars in road networks, and present a randomized approach to it.
[Schlote et al., 2014] consider users of bike sharing stations and their decision making in terms of which station to go to for renting/returning a bike. They present an approach that combines providing users with occupancy data and a random assignment based on it, for the sake of balancing. The main differences between these works and ours are that (1) we focus on game-theoretic Bayesian Nash equilibrium solutions to the coordination/congestion problem (and the conditions under which it exists, in Theorem 2 of the main paper), and (2) our results focus more on the conditions under which the assistant can solve certain inference/prediction tasks (like assistant-separability in Theorem 1 of the main paper).

Further related work for Theorem 2: Our setting relates to nonatomic games [Schmeidler, 1973] studied in game theory. However, we are only aware of two lines of work that study the incomplete-information case in the setting of a nonatomic continuum of types: [Sabourian, 1990], but they do not focus on Bayesian Nash equilibrium existence in the stage game itself. More closely related is [Kim and Yannelis, 1997]: they study existence of a Bayesian Nash equilibrium in incomplete-information nonatomic games in quite general terms. But they do not cover our case where the “state of the world” (X) has an uncountable range. Furthermore, existence of a self-fulfilling aggregate prophecy is not entailed by their results (using our equivalence in Theorem 1), due to potential non-strictness of their Bayesian Nash equilibrium. In this sense, our Corollary 3 may also be of value for the game-theoretic side. Let us also mention [Rathi, 1992], who, in one part of their proof of their Theorem 1, also reduces the Nash equilibrium existence problem to existence of a form of self-fulfilling prophecy on the aggregate level (without considering it as such). But they restrict to the complete-information case. From the smart cities research side, we already mentioned [Mareček et al., 2016, 2015] above. They essentially propose two solutions: either sending different signals to different agents (which we, in a different sense, also do in the small-scale setting, Setting 2) or the population of agents has to be heterogeneous, which relates to our assumption of random types. But their heterogeneity is rather in the behavior, not in the form of individually differing utility functions, as in our case.

Remarks on Algorithm 2 and Proposition 2: Algorithm 2 is mainly a proof-of-concept to illustrate several points: An assistant can handle simultaneous/imperfectly orchestrated user responses. And while most assistant-free dynamics of “learning in games” (mentioned in Section 1), such as best-response dynamics or fictitious play [Shoham and Leyton-Brown, 2008], only converge in special cases, an assistant can help to overcome cycling and oscillations, use exploration, and make the system always converge (with high probability; in the finite setting under consideration). Furthermore, as the extension of Proposition 2 in Section E.2 of the main paper will make more clear, the assistant can use prior knowledge of the utility functions, e.g. that they form a congestion game [Nisan et al., 2007], to speed up convergence. Generally, the area of “learning in games” is related to ours in that they also consider the case where agents do not know the preferences of others. Note that in “learning in games”, often agents first build a model/belief about the other agents behavior and then optimize their decision under it – referred to as model-based decision making [Shoham and Leyton-Brown, 2008]. Our setting can be seen as a version of such model-based decision making, where the data-driven modeling task is “outsourced” to the central assistant.

Additional remarks: Besides the price of anarchy [Roughgarden, 2005, Nisan et al., 2007], further limitations can occur when extending the setting: for instance it could happen that the assistant would figure out that making users not use the assistant (e.g., by deliberately providing poor forecasts for some time) could yield more predictable outcomes than other strategies (although, based on our results, never as good ones as Nash equilibria) – possibly yielding completely undesired assistant behavior. Note that, instead of making a statement about the reasonableness of $L_{\text{pred}}$ in isolation, which is impossible, rather here we analyzed the combination $(L_{\text{pred}}, \sigma)$, for a certain joint user behavior $\sigma = (\sigma_i)_{i \in I}$. Also note that so far we only consider classical (Bayesian) Nash equilibria, but the results may be extendable to harness the assistant to also announce correlated equilibria [Osborne and Rubinstein, 1994].

Additional related work for algorithmic aspects: Regarding Algorithm 1 of the main paper, Zhang et al. [2013] apply various machine learning methods to wait time prediction, similar as we do, but not considering influential predictions or non-stationarities. Smyrnakis and Leslie [2010] model multi-agent dynamics using latent-state models, but from the view of one of the players and in a non-aggregate setting.
C  Some additional notation and details on measurability assumptions etc. in Section 2 of the main paper

C.1  Additional notation

For the following sections, let us introduce explicit names for certain mechanisms that are part of our basic model of Section 2 of the main paper, for which we have not given explicit names there:

• We use $\sigma_i$ to denote user $i$'s behavior in the assistent-based system, i.e., $i$'s policy that generates her action $B_i$ from the input $(W_i, A)$ in the assistant-based system, obeying Eq. 2 of the main paper, i.e., the “best response” to forecast $A$ (it is uniquely defined by Eq. 2 together with the tie-breaking rule we give in Section C.2). And we let $\sigma = (\sigma_i)_{i \in I}$.

• We denote by $\bar{U}_i$ the generic utility function, the mechanism that generates user $i$’s utility $U_i$, i.e., $U_i = \bar{U}_i(X, B)$. (Recall that in the main paper we generally assume inference-assistability and thus solely use the restricted form of the mechanism in the form of the function $\hat{U}$. In particular, $\bar{U}_i(X, (B_i, B_{-i})) = \bar{U}_i(W_i, B_i, h_i(Y))$ under the assumption of inference-assistability.)

Generally, keep in mind that, for random variables $Z_1, Z_2$, $P(Z_1|z_2)$ is shorthand for the (regular) conditional distribution $P(Z_1|Z_2 = z_2)$, for $z_2$ a value of $Z_2$.

C.2  Details on measurability assumptions etc. and soundness of definitions in Section 2 of the main paper

We were very brief in Section 2 of the main paper regarding measurability assumptions etc. Here we explicate the assumptions we meant there in detail.

We will have somewhat different assumptions regarding ranges, $\sigma$-algebras, measurability etc. of variables for two different cases: (1) finitely many users and (2) infinitely many. In the proofs, we will treat both cases simultaneously where we can, and treat them separately where we have to.

C.2.1  Case of finitely many users

In case the set $I$ (the set of users) is finite, we assume that (these assumptions are, in a sense, a generalization of Setting 2 of the main paper):

• there is some underlying probability space $(\Omega, \mathcal{F}, P)$,

• regarding ranges $\text{range}_Z$ of the variables $Z \in \{X, V, W_i : i \in I\}$, we assume that they are either all finite or all continuous (meaning compact subsets of a Euclidean space),

• the ranges $\text{range}_Z$ of all the variables $Z \in \{X, V, W_i, B_i, Y, h_i(Y), U_i : i \in I\}$ are equipped with a respective $\sigma$-algebra denoted by $\sigma_{\text{range}_Z}$ (and Cartesian products of ranges are equipped with the respective product $\sigma$-algebras); in particular, $\text{range}_{B_i}, i \in I$ is $K$ and is equipped discrete $\sigma$-algebra, and $\text{range}_{U_i}, i \in I$ is $\mathbb{R}$ and is equipped with the Borel sets, and the others are either, in the case of discrete ranges, equipped with the discrete $\sigma$-algebras or, in the case of continuous ranges, with the Euclidean topology and the Borel sets,

• $X, V, W$ are random variables on $(\Omega, \mathcal{F})$,

• the variable $A$ as range $\text{range}_A$ has $\mathcal{M}_1 = \mathcal{M}_1^{\text{range}_Y}$, the set of Borel measures $^{28}$Klenke [2013] on $\text{range}_Y$, and it is equipped with $\tau_A$ the weak topology $^{28}$Klenke [2013], Remark 13.14(ii)] and $\sigma_{\text{-alg}_A}$ the Borel sigma algebra induced by the weak topology,

• the range of the public outcome variable $Y$ is finite,

\text{28}These assumptions are more general than, but still somewhat tailored to, the two main settings we consider (small-scale and large-scale). For our purposes this is enough. We do believe that Theorem 1 of the main paper holds more generally; but the measurability side of things becomes rather involved.

\text{28}In the case of finitely many users, $Y$ is assumed to be finite and then the Borel measures are just the usual simplex in the Euclidean space. However, with the formulation in terms of Borel measures we can simultaneously cover the case of infinitely many users, where $Y$ is potentially continuous.
• the function $\bar{Y}$ is measurable w.r.t. the respective (product) $\sigma$-algebras,

• the functions $y \mapsto h_i(y), i \in I,$ and $(i, w_0, y) \mapsto \tilde{U}_i(w_0, k, h_i(y))$ are continuous and bounded in all arguments with continuous range for all $k \in K,$

• the assistant policy $\pi$ is measurable w.r.t. $\sigma$-alg $V$ and $\sigma$-alg $A$,

• we assume that users break ties by preferring the slot $k \in K \subset \mathbb{N}$ with the lower number (i.e., take the natural ordering of $\mathbb{N}$ as the tie breaking preference ordering whenever two slots yield the same (expected) utility $U_i$ for them), this together with Eq. 2 uniquely determines the user behavior $\sigma$,

• for a strategy profile $s$ (and in particular a BNE $s$), we assume that $s$ is measurable w.r.t. the product $\sigma$-algebra $\sigma$-alg $W \otimes \sigma$-alg $V$ to $\sigma$-alg $A$.

C.2.2 Case of infinitely many users

In case the set $I$ is infinite (interpreted as types of users in this case), we make the following assumptions (these assumptions are, in a sense, a generalization of Setting 3 of the main paper), as modifications of those for the case of finite $I$ stated above (Section C.2.1):

• we now assume $V, W$ to be constant (corresponding to Setting 3 of the main paper), with $W_i$ also being constant in $i$,

• we let $I = [0, 1]$ (in this case interpreted as types of users), and equip with the Borel sets as $\sigma$-algebra $\mathcal{I},$

• we consider the variables $A, B$ not to be random variables but only variables, in particular, only have a range but not a $\sigma$-algebra,

• $\pi$ takes the values of $V$ as argument, but we do not assume measurability in this argument anymore (alternatively one can consider $\pi$ not to take any argument),

• $\sigma_i, i \in I$ takes the values of $W_i, A$ as arguments, but we do not consider it as a potentially measurable function anymore (alternatively one can consider $\sigma_i$ to only have argument $A$),

• for the variable $B$, we introduce an explicit range $\text{range}_B$, which is a subset of $\times_{i \in I} \text{range}_{B_i}$, and can be a proper subset (in particular, this constrains the range of $\sigma = (\sigma_i)_{i \in I}$),

• the range of the public outcome variable $Y$ can be continuous,

• we consider $\bar{Y}$ to be a function with domain $\text{range}_X \times \text{range}_B$, for which we do not require measurability in all arguments but only that, for fixed $b, x \mapsto \bar{Y}(x, b)$ is measurable,

• for a strategy profile (and in particular a BNE) $s$, (which we still consider to take the values of $V, W$ as input, but not to be measurable in them anymore) the mapping $i \mapsto s_i$ is measurable w.r.t. $\mathcal{I}$ and $2^K$.

C.3 Well-definedness in terms of measurability etc.

Above (in the case of finitely many users), we assumed measurability of all of the relevant “primitive” mappings that occur in Section 2 of the main paper. However, for $\sigma$ (defined in Section C.1 as shorthand for Eq. 2 of the main paper and tie-breaking), the user behavior in the assistant-based system (Definition 1 of the main paper), we have to prove measurability, because it is not a “primitive” mapping, but rather defined based on other mappings. This, together with measurability of $\pi$, also establishes the soundness of the definition of the corresponding strategy profile (Eq. 4 of the main paper); we will use this in the proof of Theorem 1 of the main paper.

\[29\] Again this is a formulation to cover the case of finitely and infinitely many users simultaneously.
**Lemma S1.** The following mappings are measurable w.r.t. the respective (product) σ-algebras (in the general setting and the canonically associated assistant-based system): in the case of finitely many users, for fixed \(i\),

\[
(w_i, a) \mapsto \sigma_i(w_i, a) \tag{19}
\]

and, in the case of infinitely many users, for fixed \(w, a\),

\[
i \mapsto \sigma_i(w_i, a) \tag{20}
\]

**Proof of Lemma S1.** Note that, in the case of discrete ranges, everything is measurable, so let us focus on the case of continuous ranges.

We explicate the proof for the case of two slots, i.e., \(K = \{0, 1\}\). The case of more slots works similarly.

**Measurability of the mapping in Eq. (19).**

Recall that \(\text{range}_A\) is \(\mathcal{M}_1 = \mathcal{M}_1^\text{range}_Y\), the set of Borel measures on \(\text{range}_Y\), and we equip it with \(\tau_A\) the weak topology (with the bounded continuous functions as “test functions” [Klenke, 2013, Remark 13.14(ii)] and \(\sigma\)-alg\(\_A\) the Borel sigma algebra induced by the weak topology.

Let \(H\) be the Heaviside step function. Let \(i\) be arbitrary but fixed. Let

\[
Q(w, y) := \tilde{U}_i(w, 1, h_i(y)) - \tilde{U}_i(w, 0, h_i(y)), w \in \text{range}_W, y \in \text{range}_Y
\]

(here \(w\) is a value of \(W\), not of \(W\), for ease of notation).

We have to show that (keep in mind that values \(a\) of \(A\) are elements of \(\mathcal{M}_1\), i.e., measures)

\[
(w, a) \mapsto \sigma_i(w, a) = H(\int Q_i(w, y) da(y)) \tag{21}
\]

is measurable, as a mapping from the product \(\sigma\)-algebra \(\sigma\)-alg\(\_W\) \(\otimes\) \(\sigma\)-alg\(\_A\) to the discrete \(\sigma\)-algebra on \(K\). For this it is enough show that

\[
(w, a) \mapsto \int Q(w, y) da(y) \tag{22}
\]

is continuous w.r.t. the respective (product) topologies, because continuity implies measurability and furthermore, the Heaviside function \(H\) is measurable, and so their concatenation is [Klenke, 2013].

So let \((w_m, a_m) \xrightarrow{m \to \infty} (w, a)\) w.r.t. the product topology of \(\tau_W\) and \(\tau_A\), which implies convergence \(w_m \xrightarrow{m \to \infty} w\) and \(a_m \xrightarrow{m \to \infty} a\) w.r.t. the individual topologies as well. Let \(R_m(y) := Q(w_m, y)\) and \(R(y) := Q(w, y)\) for all \(y\).

Then (using an argument similar to [Brezis, 2010, Proposition 3.13])

\[
\left| \int Q(w_m, y) da_m(y) - \int Q(w, y) da(y) \right| \leq \left| \int R_m da_m - \int R da \right| \tag{23}
\]

\[
\leq \left| \int (R_m - R) da_m \right| + \left| \int R da_m - \int R da \right| \tag{24}
\]

The second term converges to zero by definition of the weak convergence (because \(R\) is continuous and bounded, thus qualifies as a “test function”). The first term can be bounded by

\[
\int \|R_m - R\|_\infty da_m \leq \|R_m - R\|_\infty \int 1 da_m = \|R_m - R\|_\infty \xrightarrow{m \to \infty} 0, \tag{25}
\]

\[30\text{We formulate this proof more generally than we would have to: we formulate it for } A \text{ being a general Borel measure, although we only consider the case of finitely many users where } A \text{ is actually always a measure over a finite set.}
since $\hat{U}_i$ and thus $Q$ is uniformly continuous (based on continuity and range$_{W_i}$ and any other range to being compact) and thus $R_m$ converges uniformly to $R$.

**Measurability of the mapping in Eq. 20**

Let $a, w$ be arbitrary but fixed. First, observe that for all $k \in K$,

$$i \mapsto f_k(i) := \mathbb{E}_{Y' \sim a}\left(\hat{U}_i(w_i, k, h_i(Y'))\right)$$

(27)

is measurable since we assumed

$$(i, y) \mapsto \hat{U}_i(w_i, k, h_i(y))$$

(28)

to be measurable (w.r.t. the product $\sigma$-algebra) and then we can apply standard arguments involved in Fubini’s theorem (more specifically: [Klenke 2013, Theorem 14.16, Eq. 14.6]). Now, observe that for all $i$

$$\sigma_i(a, w) = 0$$

(29)

iff

$$0 \in \arg\max_{b_i} \mathbb{E}_{Y' \sim a}\left(\hat{U}_i(w_i, b_i, h_i(Y'))\right)$$

(30)

iff

$$\mathbb{E}_{Y' \sim a}\left(\hat{U}_i(w_i, 0, h_i(Y'))\right) - \max\left(\mathbb{E}_{Y' \sim \pi_{\mathbb{Y}}}\left(\hat{U}_i(w_i, 0, h_i(Y'))\right), \mathbb{E}_{Y' \sim \pi_{\mathbb{Y}}}\left(\hat{U}_i(w_i, 1, h_i(Y'))\right)\right) = 0$$

(31)

But the l.h.s. of the latter equation is a composition ($f_0(i) = \max(f_0(i), f_1(i))$) of measurable functions (recall that we showed $i \mapsto f_k(i) = \mathbb{E}_{Y' \sim a}\left(\hat{U}_i(w_i, k, h_i(Y'))\right)$ to be measurable) that is measurable again [Klenke 2013].

Also keep in mind the following statement, which guarantees measurability of assistant policies induced by strategy profiles.

**Lemma S2.** In the case of finitely many users, given a strategy profile $s$, the corresponding assistant policy $\pi_s$ (Eq. 5) is measurable from $\sigma$-alg$_{Y'}$ to $\sigma$-alg$_A$.

**Proof.** We have to show that $v \mapsto P_{G,s}(Y|v)$ is measurable, as a mapping from range$_{Y'}$ equipped with $\sigma$-alg$_{Y'}$, to range$_A$ (generally: the set of Borel measures on $Y$), equipped with $\sigma$-alg$_A$ (generally: the Borel sets induced by the weak topology on range$_A$).

Let $f(x, v, v) := \bar{Y}(x, s(v, w))$.

Since we assumed range$_{Y'}$ to be finite in the case of finitely many users, range$_A$ (generally the Borel measures on range$_{Y'}$) is simply a subspace of the Euclidean space, and $\sigma$-alg$_A$ are simply the Borel sets on it. So $P(Y|v)$ can be seen as a finite-dimensional vector in the Euclidean space with components $P(Y = y^l|v) = P((X, W, V) \in f^{-1}(\{y^l\})|v)$, $l = 1, \ldots, m$, assuming range$_{Y'} = \{y^1, \ldots, y^m\}$.

So to show that $v \mapsto P_{G,s}(Y|v)$ is measurable, it is enough to show that each of its components $P((X, W, V) \in S|v)$ is measurable in $v$. But this holds true since given any measurable set $S$, we have that $P((X, W, V) \in S|v)$ is measurable in $v$ (by the definition of conditional expectations/distributions [Klenke 2013]).

Also keep in mind the following observation.

**Remark 2.** Recall how we defined the corresponding assistant policy $\pi_s(v)$ in Eq. 5 as $P_{G,s}(Y|V = v)$. Note that, since we assumed range$_{Y'}$ to be equipped with the Borel sets as $\sigma$-algebra, the (regular) conditional distribution $P_{G,s}(Y|V = v)$ is a Borel measure on $Y$. This implies that the output of $\pi_s(v)$ is guaranteed to be contained in the range range$_A$ we assumed for it – the Borel measures (Section C.2).
D  Proofs for Section 3 of the main paper

D.1  Theorem 1 of the main paper

Before proving it, let us restate the result.

**Theorem 1 of the paper.** We have, in the general setting (Setting 1 of the main paper, with all users being inference-assistable and assistant-separable):

- If the assistant policy \( \pi \) in the assistant-based system \( M \) (where all users are assistant-best-responding) is a self-fulfilling prophecy (i.e., \( L^\text{pred}_\pi = 0 \)), then the corresponding strategy profile \( s_\pi \) is a Bayesian Nash equilibrium (BNE) of the benchmark game \( G^\text{small} \).
- Conversely, if the strategy profile \( s \) is a strict BNE of the benchmark game \( G^\text{small} \), then the corresponding assistant policy \( \pi_s \) is a self-fulfilling prophecy.

**Proof of Theorem 1 of the main paper.** Let \( Z_i := h_i(Y) \), \( i \in I \).

**Claim 1.** If \( L^\text{pred}_\pi = 0 \), then \( s_\pi \) is a BNE of \( G \).

**Proof of Claim 1.** First note that \( s_\pi \) can be written slightly more compactly than in Eq. 4, using \( \sigma \) as defined in Section C.1 in the following way, for all \( w_i, v \):

\[
[s_\pi]_i(w_i, v) := \sigma_i(w_i, \pi(v)).
\]  

(32)

**Main derivation:**

We state the following sequence of equalities for the case of finite \( I \) with stochastic \( V, W \); the case of infinite \( I \), where \( V, W \) are constant, is analogous but even simpler (essentially one has to drop all the occurring \( V, W, v, w \)). We have, for all \( i, w_i \) and almost all \( v \),

\[
[s_\pi]_i(v, w_i) = \sigma_i(w_i, \pi(v))
\]

(33)

\[
\in \arg \max_{b'_i} \mathbb{E}_{Y \sim \pi(v)} \left( \tilde{U}_i(w_i, b'_i, h_i(Y')) \right)
\]

(34)

\[
= \arg \max_{b'_i} \mathbb{E}_{Y \sim P_M, \pi(Y | v)} \left( \tilde{U}_i(w_i, b'_i, h_i(Y')) \right)
\]

(35)

\[
= \arg \max_{b'_i} \mathbb{E}_{Z'_i \sim P_G, \pi(Z_i | v, w_i)} \left( \tilde{U}_i(w_i, b'_i, Z'_i) \right)
\]

(36)

\[
= \arg \max_{b'_i} \mathbb{E}_{Z'_i \sim P_G, \pi(Z_i | v, w_i)} \left( \tilde{U}_i(w_i, b'_i, Z'_i) \right)
\]

(37)

\[
= \arg \max_{b'_i} \mathbb{E}_{Z'_i \sim P_G, \pi(Z_i | v, w_i)} \left( \tilde{U}_i(w_i, b'_i, Z'_i) \right)
\]

(38)

\[
= \arg \max_{b'_i} \mathbb{E}_{Z'_i \sim P_G, \pi(Z_i | v, w_i)} \left( \tilde{U}_i(w_i, b'_i, Z'_i) \right)
\]

(39)

\[
= \arg \max_{b'_i} \mathbb{E}_{Z'_i \sim P_G, \pi(Z_i | v, w_i)} \left( \tilde{U}_i(w_i, b'_i, Z'_i) \right)
\]

(40)

\[
= \arg \max_{b'_i} \mathbb{E}_{Z'_i \sim P_G, \pi(Z_i | v, w_i)} \left( \tilde{U}_i(w_i, b'_i, Z'_i) \right)
\]

(41)

where:

- Eqs. 34, 35 hold by definition of \( s_\pi \) and the definition of the user behavior \( \sigma_i \) (Section C.1, Eq. 2) of the main paper – assumption “assistant-best-responding”), respectively.

---

31 Note that even when not assuming assistant-separability, a BNE may be achieved. However, this would be a BNE w.r.t. a different game, where players would not use the full information available to them – \( V, W \).

32 There we used the notation based on the conditional expectation not because we refer to some average \( B_i \), but only to rigorously refer to the value of \( B_i \) conditioned on \( W_i, V \), which is actually fully determined by these variables.
• To understand Eq. 36 let us look at what our assumption $L^\text{pred}_\pi = 0$ implies. Based on its very definition, it implies
\[ \pi(v) = P_{M,\pi}(Y|v) \] (42)
for almost all $v$.

• Eq. 39 follows from our assumption (“assistant-separability”) that $Z_i \perp W_i|V$ for any $\pi, \sigma$.

• To understand Eq. 40 observe that the only thing that can be different between $M_\pi$ (when ignoring $A$) and $G$ (with a “plugged in” strategy profile) is the mechanism that generates $B$ from $V, W$. But, by our definition of $s_\pi$, this mechanism is in fact the same in $M_\pi$ and $G$ with “plugged in” $s_\pi$. Therefore, all (random) variables, in particular $Z_i$, coincide between $M_\pi$ and $G$ with $s_\pi$.

• To understand Eq. 41 note that $Z_i$ is defined without $B_i$ needing to be defined. Therefore, it is already defined in $G$ with “incomplete strategy profile” $[s_\pi]_{-i}$ alone.

• Generally, note that terms like “$P_{M,\pi}(Y|v)$” – a regular conditional distribution – though we would not necessarily always need them, are well-defined and exist in our setting (of discrete or Euclidean ranges) [Klenke, 2013, Theorem 8.37].

But Eqs. 33 through 41 mean that for almost no $v, w_i$, player $i \in I$ could improve his utility by deviating from $[s_\pi]_{i}(v, w_i)$.

**Measurability discussion:**

Generally, note that $\bar{U}_i, i \in I$ is measurable also when we fix some of its arguments, because we assumed it to be measurable w.r.t. the respective product $\sigma$-algebra [Klenke, 2013, Lemma 14.13 and Theorem 14.16].

Still for the case of $I$ finite, note that $s_\pi$ is measurable w.r.t. the product $\sigma$-algebra $\sigma\text{-alg}_W \otimes \sigma\text{-alg}_V$ to $\sigma\text{-alg}_B$ (which is necessary for it to be a strategy profile), for the following reasons: We assumed $\pi$ to be measurable w.r.t. $\sigma\text{-alg}_V$ to $\sigma\text{-alg}_A$ and $\sigma = (\sigma_i)_{i \in I}$ is measurable w.r.t. $\sigma\text{-alg}_W \otimes \sigma\text{-alg}_A$ to $\sigma\text{-alg}_B$ due to the first part of Lemma ST13. But $s_\pi$ is just the composition $\sigma(\cdot, \pi(\cdot))$.

It remains to be shown that $s_\pi$ is a strategy profile, in terms of measurability (in the sense of Section C.2), also for the case of $I$ infinite and $V, W$ constant. Specifically, we have to show that for any values $v, w$ that $V, W$ are fixed to, that
\[ i \mapsto [s_\pi]_i(w_i, v) = \sigma_i(w_i, \pi(v)) \] (43)
is measurable in $i$ (w.r.t. codomain $K = \{0, 1\}$ equipped with the power set as $\sigma$-algebra). But this directly follows from the second part of Lemma ST13 (plugging in $\pi(v)$ for $a$).

Everything together implies that $s_\pi$ is a BNE of $G$.

\[ \square \]

**Claim 2.** Conversely, if $s$ is a strict BNE of $G$, then $L^\text{pred}_\pi = 0$.

**Proof of Claim**

Let $s = (s_i)_{i \in I}$ be a strict BNE of $G$. That is, for all $i, w_i, v,
\[ s_i(w_i, v) \in \arg\max_{b'_i} \mathbb{E} \left( \bar{U}_i(X_i, (b_i, (s_j(W_j, V))_{j \in I \setminus \{i\}} | w_i, v) \right), \] (44)

with the argmax being unique.

Similar as above, we state the following derivation for the case of finite $I$ with stochastic $V, W$; the case of infinite $I$, where $V, W$ are constant, is analogous but even simpler (essentially one has to drop all the occurring $V, W, v, w$).

**First**, for the case of the assistant’s policy being $\pi_s$, we have for all $i, v, w_i$,
\[ \sigma_i(w_i, \pi_s(v)) \] (45)
\[ \in \arg\max_{b'_i} \mathbb{E}_{Y \sim \pi_s(v)} \left( \bar{U}_i(w_i, b'_i, h_i(Y')) \right) \] (46)

\[ \text{Since } \sigma\text{-alg}_B \text{ is the product } \sigma\text{-algebra, } \sigma \text{ measurable is equivalent to } \sigma_i \text{ measurable for all } i \text{ [Klenke, 2013, Corollary 1.82].} \]
\[= \arg \max \mathbb{E}_{Y' \sim P_{G,s}(Y'|v)} \left( \tilde{U}_i(w_i, b'_i, h_i(Y')) \right) \]  
\[
= \arg \max \mathbb{E}_{Z'_i \sim P_{G,s}(Z_i|v)} \left( \tilde{U}_i(w_i, b'_i, Z'_i) \right) \]  
\[
= \arg \max \mathbb{E}_{Z'_i \sim P_{G,s}(Z_i|v,w_i)} \left( \tilde{U}_i(w_i, b'_i, Z'_i) \right) \]  
\[
= \arg \max \frac{Z_i}{v,w_i} \left( \tilde{U}_i(W_i, b'_i, Z_i) \right) \]  
\[
= \arg \max \mathbb{E}_{h,s_{-i}} \left( \tilde{U}_i(W_i, b'_i, Z_i) | w_i, v \right) \]  
\[
= \arg \max \mathbb{E}_{h,s_{-i}} \left( \tilde{U}_i(W_i, b'_i, \tilde{Y}(X, b'_i, (s_j(W_j, V))_{j \in I \setminus \{i\}})) | w_i, v \right) \]  
\[
\ni \in s_i(w_i, v), \]  

where:

- Eq. 56 is our assumption that users are assistant-best-responding.
- Eq. 50 follows from our assumption (“assistant-separability”) that \( Z_i \perp W_i | V \) for any \( \pi, \sigma \) and thus also in \( G \) with any \( s \).
- Eq. 53 follows from how we defined \( Z_i \).
- Eq. 54 is due to \( s \) being a BNE (and our assumption that users are inference-assistable (Eq. 1 of the main paper)).

Since we assumed the above argmax to be unique, we get that for all \( i, w_i, v \),

\[
\sigma_i(w_i, \pi_s(v)) = s_i(w_i, v). \tag{55} \]

**Second**, we have, for all \( i, v \),

\[
P_{M,\pi,s}(Y|v) \tag{56} \]
\[
= P \left( \tilde{Y}(X, (\sigma_i(W_i, \pi_s(V)))_{i \in I}) | v \right) \tag{57} \]
\[
= P \left( \tilde{Y}(X, (s_i(W_i, V))_{i \in I}) | v \right) \tag{58} \]
\[
= P_{G,s}(Y|v) \tag{59} \]
\[
= \pi_s(v), \tag{60} \]

where:

- Eq. 58 follows from above’s derivation ending with Eq. 55
- Eq. 60 is just the definition of \( \pi_s \).

This implies \( L_{\pi_s}^{\text{pred}} = 0 \), which is what had to be shown.

**Measurability discussion:**

For the case of finitely many users, see Lemma 2. In the case of infinitely many, nothing has to be shown (regarding the correctness of the codomain of \( \pi_s \), see Remark 2).

**D.2 Corollary 2 of the main paper**

Before proving it, let us restate the result form the main paper:
Corollary 2 of the paper. Setting 3 of the main paper is a special case of Setting 1 of the main paper. In particular, it satisfies the conditions of Theorem 1 of the main paper and hence the theorem’s implications hold for $M = M^{\text{large}}$ and $G = G^{\text{large}}$.

Proof of Corollary 2 of the main paper. Throughout this proof, let $v, w$ be arbitrary but fixed.

Part 1: show that general model assumptions of Section 2 of the main paper are satisfied

Regarding correctness of the range of $\sigma(a, \pi(v))$ (i.e., showing that it ranges within $\text{range}_{\mathcal{B}} = \{b \in \mathbb{N}^I : k \in K, \{j \in I : b_j = k\} \text{ is measurable}\}$):

This follows from the fact that Setting 3 of the main paper satisfies the requirements of Setting 1 of the main paper w.r.t. the continuity of $U_i$ (that we stated in detail in Section C.2), which was all we needed in Lemma S1. Because the (second part of the) lemma implies that for all $w, v$, $i \mapsto \sigma_i(w, \pi(v))$ is measurable, which is what had to be shown.

Regarding the measurability of all mechanisms:

Regarding product measurability of $x \mapsto Y(x, b)$

We have to show that $x \mapsto \int b_i r(i|x) di$ is measurable for fixed $b$.

We show the more general statement that $\text{range}_X \times \text{range}_{\mathcal{B}} \rightarrow \text{range}_{\mathcal{Y}} = \mathbb{R} ; (x, b) \mapsto f(x, b) := \int b_i r(i|x) di$ is $\sigma$-$\mathcal{X} \otimes \sigma$-$\mathcal{B}$-Borel measurable, where, just for the sake of this proof, we assume $\text{range}_{\mathcal{B}}$ to be equipped with a $\sigma$-$\mathcal{B}$ as will be detailed below. (This implies what needs to be shown because in our setting measurability/continuity in both arguments implies the same for the individual arguments when fixing the respective other [Klenke, 2013, Lemma 14.13].)

Here, let $L^2$ denote the Lebesgue space of square integrable functions over $[0, 1]$ w.r.t. the Lebesgue measure (usually denoted $L^2([0, 1])$).

Let $\tau_X$ denote the topology of $\text{range}_X$. Let $\tau_{L^2}$ denote the topology of $L^2$. Let $\mathcal{B}_x$ denote the Borel $\sigma$-$\mathcal{X}$ induced by a topology $\tau$.

Recapture our assumptions:

- $\sigma$-$\mathcal{X} = \mathcal{B}_x$,
- $\sigma$-$\mathcal{B} = \mathcal{B}_{\tau_{L^2}}$,
- $x \mapsto r(|x) = r_x$ is continuous from $\tau_X$ to $\tau_{L^2}$.

Let $\tau_1$ be the product topology of $\text{range}_X, L^2$ and $\tau_2$ be the product topology of $L^2, L^2$.

It follows from our assumptions that the mapping $f : (x, b) \mapsto (r_x, b)$ (62) is continuous w.r.t. source topology $\tau_1$ and target topology $\tau_2$.

But the $L^2$ inner product $\langle \cdot, \cdot \rangle$ is continuous w.r.t. source topology $\tau_2$ and target topology $\mathbb{R}$. Therefore, the concatenation $\langle \cdot, \cdot \rangle \circ f$ is continuous from $\tau_1$ to $\mathbb{R}$. Hence it is measurable w.r.t. the Borel $\sigma$-$\mathcal{B}_x$ to the Borel sets on $\mathbb{R}$.

We assumed $(\text{range}_X, \tau_X)$ to be Polish. And $L^2$, the space of $\mathcal{B}_x$ is Polish. Therefore (based on Klenke, 2013 [Theorem 14.8]), $\mathcal{B}_{\tau_2}$, the Borel $\sigma$-$\mathcal{B}$ induced by the product topology on $\text{range}_X, L^2$, coincides with the product $\sigma$-$\mathcal{X} \otimes \sigma$-$\mathcal{B}$.

Regarding correctness of $\text{range}_A$ (i.e., that it contains the distribution over $Y$ that is entailed by it): We have to show that $P(Y|A = a) \in \text{range}_A$ for any $a \in \text{range}_A$. To see this, note that we above showed that $x \mapsto Y(x, b)$ is measurable with $\sigma$-$\mathcal{Y}$ being the Borel $\sigma$-$\mathcal{B}$ on $\text{range}_Y$. Therefore the pushforward measure $P(Y|A = a) = P(g(X)|A = a)$, for $g(x) = Y(x, b)$, is a Borel measure, i.e., element of $\text{range}_A$. 
Part 2: show that the remaining conditions of the underlying theorem are satisfied

Regarding assumption inference-assistability: Let $h_i$ simply be the identity. Then, based on Setting 3 of the main paper, we have

$$\tilde{U}_i(f_{W_i}(x), b_i, \tilde{Y}(x, b)) = \tilde{U}_i(f_{W_i}(x), b_i, h_i(\tilde{Y}(x, b)))$$

and $h_i(\tilde{Y}(x, b))$ does not depend on $b_i$ since

$$Y = R_X(\{j \in I : B_j = k\}) = R_X(\{j \in I : B_j = k, j \neq i\})$$

based on the fact that $R_x$ – which is the measure induced by the density $r_x$ – has a density w.r.t. the Lebesgue measure.

Regarding assumption of assistant-separability:

Since we assumed that $W_i$ is constant for all $i \in I$, it follows trivially that $Z_i \perp W_i|V$.

\[ \square \]

D.3 Theorem 2 of the main paper

The following statement generalizes Theorem 2 of the main paper in that here we allow an arbitrary finite number $|K|$ of (time) slots, not just two (i.e., arbitrary finite $K$, not just $K = \{0, 1\}$).

**Theorem 3.** There exists a self-fulfilling prophecy policy $\pi$ in the assistant-based system $M_{\text{large}}$, in Setting 3 of the main paper but with an arbitrary finite number $|K|$ of slots.

Here is the proof of this generalized version:

**Proof for Theorem 2 of the main paper.** Keep in mind that in the current large-scale setting, $i \in I$ are interpreted as types of users (with the same utility function), not users themselves. Also keep in mind that the set of slots (i.e., actions available to the users) is $K = \{1, \ldots, |K|\}$. Since $W$ is constant, here, instead of $\tilde{U}_i(w, b, k)$, we will write $\tilde{U}_i(y, k)$ for utility of type $i$ when choosing slot $k \in K$ given amounts of types $y \in \mathbb{R}^{|K|}$ at slots 1 to $|K|$.

In what follows, we will consider the space $\mathcal{M}_1$ of Borel probability measures on the standard $|K| - 1$ simplex

$$S_{|K|} = \left\{ z \in \mathbb{R}^{|K|} \mid z \geq 0, \sum_{k=1}^{|K|} z_k = 1 \right\}. $$

Furthermore, let, for any $k, l$, $H^k_l$ be the common Heaviside step function (i.e., taking value 0 upon input below 0, and value 1 upon input above 0), defining it in a special way for the point 0 (to implement a tie breaking rule that favors lower slots $k \in K$):

$$H^k_l(0) := \begin{cases} 1, & \text{if } k \leq l, \\ 0, & \text{if } k > l. \end{cases}$$

First we note that for any measure $\mu \in \mathcal{M}_1$ the expected proportion of users choosing slot $k$ conditioned on $X = x$ with assistant prediction $A = \mu$ is

$$F^k_\mu(x) := \int \left[ \prod_{l \neq k} H^k_l \left( \int Q^k_l(i, y) d\mu(y) \right) \right] r(i|x) di,$$

with $Q^k_l(i, y) = \tilde{U}_i(y, k) - \tilde{U}_i(y, l)$. Given our measurability assumptions,

$$F_\mu := (F^1_\mu, \ldots, F^{|K|}_\mu)$$

is well defined and measurable, such that the pushforward measure of $X$ by it is also a Borel probability measure.
Let $\mu, \mu_1, \mu_2, \ldots \in \mathcal{M}_1$, we say that $(\mu_n)_{n \in \mathbb{N}}$ converges weakly to $\mu$ if for any $f$ continuous on $S_{|K|}$
\[
\int f d\mu_n \xrightarrow{n \to +\infty} \int f(y) d\mu.
\]

Weak convergence induces the weak topology $\tau$ on $\mathcal{M}_1$, and $\mathcal{M}_1$ is compact for this topology (see for example \cite{Klenk2013}, Section 13.2). As a consequence $\mathcal{M}_1$ is a non-empty compact convex set of the locally convex topological vector space of bounded signed measures on $S_{|K|}$.

In order to prove the existence of a fixed point in $\mathcal{M}_1$, according to Leray-Schauder-Tychonoff fixed point theorem \cite[1972, p151]{Reed1972}, what remains is to prove that the mapping
\[
J : \mathcal{M}_1 \to P(F_\mu(X))
\]
is continuous for the above defined weak topology.

Consider $\mu_n \to \mu$ (for this weak topology). We have to show that for any $f$ continuous on $S_{|K|}$
\[
\int f dJ(\mu_n) \to \int f dJ(\mu).
\]

We rewrite the left-hand side (using basic change of variable in the Lebesgue integral), assuming $P_X$ has a bounded density $p_X$ with respect to Lebesgue measure:
\[
\int f dJ(\mu_n) = \mathbb{E}[f(F_{\mu_n}(X))] = \int f(F_{\mu_n}(x))p_X(x) dx.
\]

Since $f$ is also uniformly continuous on this (compact) simplex, proving uniform convergence on $S_{|K|}$ of $F_{\mu_n}$ to $F_\mu$ will be enough to conclude. Let us fix an $\epsilon$, we want to bound $\|F_{\mu_n} - F_\mu\|$ by $\epsilon$ uniformly over $S_{|K|}$. Since $S_{|K|}$ is included in a finite dimensional Euclidean space, a uniform bound on each component will be enough to conclude.

We first note that for any component $k$
\[
|F^k_{\mu_n}(x) - F^k_\mu(x)| \leq \int \prod_l H^k_l \left( \int Q^k_l(i, y) d\mu_n(y) \right) - \prod_l H^k_l \left( \int Q^k_l(i, y) d\mu(y) \right) r(i|x) di \quad (64)
\]

We notice we can rewrite the difference of Heaviside products as
\[
\sum_m \prod_{l \leq m} H^k_l \left( \int Q^k_l(i, y) d\mu(y) \right) \cdot \left( H^k_m \left( \int Q^k_m(i, y) d\mu_n(y) \right) - H^k_m \left( \int Q^k_m(i, y) d\mu(y) \right) \right) \\
\cdot \prod_{l > m} H^k_l \left( \int Q^k_l(i, y) d\mu_n(y) \right) \quad (65)
\]
such that we can bound the absolute difference of Eq. \eqref{64} using (based on the terms of the product that are not differences being at most 1 anyway)
\[
|F^k_{\mu_n}(x) - F^k_\mu(x)| \leq \sum_m \int H^k_m \left( \int Q^k_m(i, y) d\mu_n(y) \right) - H^k_m \left( \int Q^k_m(i, y) d\mu(y) \right) r(i|x) di. \quad (66)
\]

We will thus focus first on bounding an arbitrary term
\[
\int H^k_m \left( \int Q^k_m(i, y) d\mu_n(y) \right) - H^k_m \left( \int Q^k_m(i, y) d\mu(y) \right) r(i|x) di, \quad (67)
\]
dropping the indices $m$ and $k$ to ease notations in the following paragraph.
Our assumptions (Setting 3 of the main paper) imply any $Q_m$ is a polynomial in $(i, y)$, that can be written $\sum_{m=0}^{d} i^m q_m(y)$. Then integrating the quantity inside each $H$ yields polynomials in $i$, $p(i)$ and $p_n(i)$, of maximum order $d$, whose coefficients are a linear combination of the moments of $\mu$ and $\mu_n$ respectively, up to some order $d'$. Convergence of $\mu$ to $\mu$ thus guarantees convergence of the coefficients of $p_n$ to those of $p$, and uniform convergence of the $p_n$ to $p$ on the unit interval.

The discontinuity of $H$ does not allow us to further use uniform continuity to bound the term of Eq. (67), but we notice that the absolute difference between the two $H$ terms is either zero or one, the later occurring only when the signs of $p_n$ and $p$ differ. Using the assumption that there exists at least one $m$ such that $q_m(y)$ is constant and non-zero implies $p$ is a non-zero polynomial. There is then only two possible cases to consider:

- If $p$ has no root on the unit interval (e.g. $p$ is constant), then uniform convergence guaranties we can choose $N$ large enough such that $p$ and $p_n$ have the same sign on the unit interval, implying the difference in Heaviside function is zero on the whole interval and the corresponding term of Eq. (67) can be ignored.

- Alternatively, $p$ has a finite number of roots, and for $n$ large enough the difference inside the integral in Eq. (67) can be non-zero only on a finite number of intervals surrounding these roots, where the sign of the $p$ and $p_n$ may differ. We will thus focus on this case and show the length of these intervals can be bounded.

Let $(i_l)$ be all the (finite) collection of roots of $p$ on the unit interval, then there exists a $\eta_0$ thus that $p$ is strictly monotonous in all right and left $\eta_0$-neighborhoods of each $i_l$ (one-sided neighborhoods are needed for roots with even multiplicity), and thus admits a family of one-sided monotonous continuous local inverse functions $(\kappa_i^+, \kappa_i^-)$, up to a change in sign, such that for each $l$,

$$
\kappa_i^+(|p(i)|) = i - i_l, \quad i - i_l \in [0, \eta_0]
$$

and

$$
\kappa_i^-(|p(i)|) = i - i_l, \quad i - i_l \in [-\eta_0, 0]
$$

(note continuity of the inverse is guaranteed by continuity and strict monotonocity, while the implicit function theorem does not directly apply at multiple roots due to vanishing of the derivative). Let $\epsilon_0'$ be the maximum radius such that the interval $[0, \epsilon_0']$ is included in the intersection of the domains of all $\kappa_i^+$ and $\kappa_i^-$. We additionally choose $\epsilon_0 \leq \epsilon_0'$ such that $|p(i)| > \epsilon_0$ for any $i$ outside the union of intervals $[i_l + \kappa_i^- (\epsilon_0), i_l + \kappa_i^+ (\epsilon_0)]$ associated to each root $l$ (this can be done by picking the minimum between $\epsilon_0'$ and the lower bound of $|p|$ outside of the neighborhoods of each root). Let us choose $N_0$ such that for $n > N_0$, $|p - p_n| < \epsilon_0$ uniformly on the unit interval. Then the Lebesgue measure $(\lambda(I_{\epsilon_0}^n))$ of $I_{\epsilon_0}^n$, the union of all intervals such that

$$
|H_m^k \int Q_m^k(i, y) d\mu_n(y) - H_m^k \int Q_m^k(i, y) d\mu(y)| = |H_m^k (p_n(i)) - H_m^k (p(i))| > 0 (= 1),
$$

is inferior to $\sum_l |\kappa_i^+(\epsilon_0) - \kappa_i^- (-\epsilon_0)|$. This is because outside of $I_{\epsilon_0}^n$, $p$ is at least as far away from 0 as $\epsilon_0$, so $p_n$ has he same sign as $p$. By (uniform) continuity of all $\kappa_i^+$ and $\kappa_i^-$, for $\eta_1$ arbitrary small, we can choose $\epsilon_1 < \epsilon_0$ and $N_1 > N_0$ such that for $n > N_1$, $\lambda(I_{\epsilon_0}^n) < \eta_1$, such that, since $r(|x|) is continuous (and thus bounded) on $[0, 1]^2$, we get

$$
\int |H_m^k \int Q_m^k(i, y) d\mu_n(y) - H_m^k \int Q_m^k(i, y) d\mu(y)| r(|x|) dx \leq \eta_1 \max(r).
$$

As this procedure can be done for all $m \neq k$, we can bound the $k$-th component of $F_m$ using

$$
|F_m^k (x) - F_m^k (x)| \leq (|K| - 1) \eta_1 \max(r),
$$

for $\eta_1$ arbitrary small. We thus get a uniform bound for $|F_m^k (x) - F_m^k (x)|$, which is enough to ensure that $J$ is continuous for the weak topology.

This implies the existence of a fixed point of $J$ in $M_1$ according to the Leray-Schauder-Tychonoff fixed point theorem [Reed and Simon, 1972, p151].


E Proofs and extensions for Section 5 of the main paper

E.1 Extended version of the proposition and proof for Section 5.1 of the main paper

Let us state and proof a proposition that is a slight generalization of Proposition 1 of the main paper.

Proposition S1 (Optimality and Convergence Rate of Expodamp – Generalized Version of Proposition 1 of the main paper). In the dynamic large-scale setting (Section 4 of the main paper), let Assumption 1 of the main paper hold true.

- Stochastic case: Let the assistant’s policy \( \pi \) be defined by

\[
A^0_t := \gamma (1 - \beta)^{-1} \mathbb{E}(X^0),
\]
\[
A_{\pi}^{t+1} := A_{\pi}^t + \gamma (1 - \beta)^{-1} Q_t (Y^t - A_{\pi}^t),
\]

for all \( t \geq 0 \) with \( Q_t \) some function of the covariance structure as detailed in Eq. (77) of the proof of this proposition.

In particular, \( Q_t \) is such that, if there is no observation noise in the latent-state model, i.e., \( \text{var}(E^t_x) = 0 \), then \( \pi \) coincides with Expodamp (Algorithm 1 of the main paper) when setting \( \alpha := (1 - \beta)^{-1} \) for the true \( \beta \) of Eq. (73) of the main paper.

Assume \( \mathbb{E}(E^t_Y) = 0, t \geq 1 \). Then \( \pi \) at each stage \( t \), \( L^{t, \text{point}} = 0 \) and

\[
A^t = \arg \min_{\alpha} \mathbb{E}((A^t - Y^t)^2) \text{do}(A^t = \alpha^t), A^{0:t-1}, Y^{0:t-1}).
\]

- Deterministic case: Let the assistant’s policy \( \pi \) be Expodamp (Algorithm 1 of the main paper). Assume that \( X^t = x \) is constant for \( t \geq 0 \), that \( \beta = (1 - \gamma) \) and that \( E^t_x = E^t_y = 0 \). Then

\[
Y^t = x + (1 - \gamma) (A^0 - x)(1 - \alpha) t, \text{ for all } t \geq 0.
\]

That is, \( Y^t \) converges exponentially with rate \( \gamma \alpha \) towards the “optimum”/fixed point \( x \) (and thus also \( A^t \) converges to \( x \) based on Expodamp’s formula) iff \( 0 < \gamma \alpha < 2 \).

Proof of Proposition S1

First part of the proposition – stochastic case:

Prerequisites:

Consider the complete dynamical system, consisting of Assumption 1 of the main paper, the state-space model (without assistant’s behavior), together with Eq. (12) of the main paper, the assistant’s behavior under policy \( \pi \). For this model let, for \( Z \in \{ Y, X \} \) and \( t \geq t' \),

\[
Z^{t \downarrow t'} := \mathbb{E}_\pi(Z^t | Y^{0:t'}, A^{0:t'})
\]

\[
\Sigma^{t \downarrow t'}_Z := \text{var}_\pi(Z^t | Y^{0:t'}, A^{0:t'}). \tag{73}
\]

To be as explicit as possible, note that, for \( Z \in \{ Y, X \} \) and \( t \geq t' \) (due to the causal structure)

\[
\mathbb{E}_\pi(Z^t | Y^{0:t'}, A^{0:t'} = a^{0:t'}) \left( = \mathbb{E}_\pi(Z^t | Y^{0:t'}, \text{do}(A^{0:t'} = a^{0:t'})) \right) = \mathbb{E}(Z^t | Y^{0:t'}, a^{0:t'}),
\]

where the latter expectation is taken in the POMDP model of Assumption 1 of the main paper when setting \( A^t \) to constants \( a^t \), for \( t \geq 0 \), and not plugging in any assistant policy. The analogous holds for \( \text{var}_\pi(Z^t | Y^{0:t'}, A^{0:t'}) \).

So we can use the classical Kalman filter recursive equations [Lütkepohl, 2006, Section 18.3.1], which hold for the POMDP model, and thus, based on Eq. (73) also for \( Z^{t \downarrow t'}, \Sigma^{t \downarrow t'}_Z \), for \( Z \in \{ Y, X \} \) and \( t \geq t' \) defined in the complete

\[\text{proof}\]

One can also make the more general statements about \( \pi \) minimizing the cumulative (over time) loss. To see that the “local” statement (for individual \( t \)) implies more global statements observe two things: First, \( A^t \) influences only \( Y^t \) but no future \( Y^{t'}, t' > t \), and so term-wise optimization coincides with cumulative optimization. Second, \( \pi \) as defined above does the optimal thing at stage \( t \) regardless of how \( A^t, t < t \) was picked, in case we feed what \( \pi \) would have outputted at stage \( t - 1 \), instead of the actual \( A^{t-1} \), into \( \pi \) at stage \( t \).
Now observe that the statement we need to show, Eq. 71, is equivalent to

\[ \hat{X}^{t+1|t} = \hat{X}^{t|t} \]

(74)

\[ = \hat{X}^{t|t-1} + Q_t(Y^t - \hat{Y}^{t|t-1}) \]

(75)

\[ = \hat{X}^{t|t-1} + Q_t(Y^t - \gamma \hat{X}^{t|t-1} - \beta A^t), \]

(76)

for

\[ Q_t := \gamma \Sigma^{(t-1)} (\Sigma^t)^{-1} \]

(77)

\[ = \gamma \Sigma^{(t-1)} (\gamma^2 \Sigma^t + \text{var}(E^0_Y))^{-1}. \]

(78)

Note that \( Q_t \) does not depend on the \( A^t, t \in \mathbb{N} \), when considering \( A^t, t \in \mathbb{N} \) as a parameter.

**Showing Eq. 71**

Assume the conditions of the proposition, i.e., that the assistant’s policy \( \pi \) is defined by Eq. 69 and 70 (for convenience we may drop the subscript of \( A^t, \pi \) in what follows), with \( Q_t \) from Eq. 77 and Assumption 1 of the main paper.

Let us show via induction that

\[ A^t = \gamma (1-\beta)^{-1} \hat{X}^{t|t-1}, \text{ for all } t \geq 0, \]

(79)

where we let \( X^{0|1} := \mathbb{E}(X^0) \).

**Base case:** For \( t = 0 \), the statement holds by definition.

**Induction step:** Assume the statement holds for \( t \). Then we have

\[ A^{t+1} = A^t + \gamma (1-\beta)^{-1} Q_t \{ Y^t - A^t \} \]

(80)

\[ = \gamma (1-\beta)^{-1} (\gamma^{-1}(1-\beta)A^t + Q_t(Y^t - A^t)) \]

(81)

\[ = \gamma (1-\beta)^{-1} (\gamma^{-1}(1-\beta)A^t + Q_t(Y^t - \gamma^{-1}(1-\beta)A^t - \beta A^t)) \]

(82)

\[ = \gamma (1-\beta)^{-1} (\hat{X}^{t|t-1} + Q_t(Y^t - \gamma \hat{X}^{t|t-1} - \beta A^t)) \]

(83)

\[ = \gamma (1-\beta)^{-1} \hat{X}^{t+1|t}, \]

(84)

where Eq. 83 is due to the inductive assumption, and Eq. 84 is based on Eq. 76. This completes the induction for Eq. 79.

Now observe that the statement we need to show, Eq. 71, is equivalent to

\[ 0 = \frac{d}{da} \mathbb{E}((A^t - Y^t)^2|do(A^t = a), A^{0:t-1}, Y^{0:t-1}) \]

(85)

\[ = \frac{d}{da} \mathbb{E}((A^t - \beta A^t - \gamma X^t - E^t_Y)^2|do(A^t = a), A^{0:t-1}, Y^{0:t-1}) \]

(86)

\[ = \frac{d}{da} \mathbb{E}(((1-\beta)A^t - \gamma X^t - E^t_Y)^2|do(A^t = a), A^{0:t-1}, Y^{0:t-1}) \]

(87)

\[ = \frac{d}{da} \mathbb{E}(((1-\beta)A^t - \gamma X^t - E^t_Y)^2|A^{0:t-1}, Y^{0:t-1}) \]

(88)

\[ = \mathbb{E}(\frac{d}{da}((1-\beta)A^t - \gamma X^t - E^t_Y)^2|A^{0:t-1}, Y^{0:t-1}) \]

(89)

\[ = \mathbb{E}(2((1-\beta)A^t - \gamma X^t - E^t_Y)(1-\beta)|A^{0:t-1}, Y^{0:t-1}) \]

(90)

\[ = 2(1-\beta)\mathbb{E}((1-\beta)A^t - \gamma X^t - E^t_Y|A^{0:t-1}, Y^{0:t-1}), \]

(91)

which in turn is equivalent to

\[ a = (1-\beta)^{-1} (\mathbb{E}(\gamma X^t|A^{0:t-1}, Y^{0:t-1}) + \mathbb{E}(E^t_Y)) \]

(92)
which in turn is equivalent to (based on our assumption $\mathbb{E}(E^t_Y) = 0, t \geq 1$)

$$a = \gamma(1 - \beta)^{-1}\mathbb{E}(X^{t+1} | A^{0:t-1}, Y^{0:t-1}),$$  \hspace{1cm} (93)

which in turn is equivalent to (simply plugging in the definition in Eq. 72)

$$a = \gamma(1 - \beta)^{-1}X^{t-1}. \hspace{1cm} (94)$$

Since we know, based on Eq. 79, that $A^t$ under $\pi$ satisfies Eq. 94 when plugging it in for $a$, based on the chain of equivalences above, we also know that it satisfies Eq. 71, which is what needed to be shown.

The statement that Expodamp (Algorithm 1 of the main paper) is a special case of the assistant policy defined in Eq. 69 and 70 can easily be seen as follows: If there is no observation noise in the latent-state model, i.e., $\text{var}(E^t_Y) = 0$, then Eq. 78 implies that $Q^t = \gamma^{-1}$. Hence, when setting $\alpha = (1 - \beta)^{-1}$, we have

$$A^{t+1} = A^t + (1 - \beta)^{-1}(Y^t - A^t) = A^t + \alpha(Y^t - A^t),$$  \hspace{1cm} (95)

i.e., we get Expodamp (Algorithm 1 of the main paper) as special case.

Showing that $L^{\pi,\text{point}}_t = 0$:  

To also show the first statement, $L^{\pi,\text{point}}_t = 0$, observe that this means

$$0 = \mathbb{E} \left( (A^t - \mathbb{E}(Y^t | A^t, A^{0:t-1}, Y^{0:t-1}))^2 \right)$$  \hspace{1cm} (96)

which is equivalent to

$$A^t = \mathbb{E}(Y^t | A^t, A^{0:t-1}, Y^{0:t-1})$$  \hspace{1cm} (97)

almost everywhere. This in turn is equivalent to

$$a = \mathbb{E}(Y^t | A^t = a, A^{0:t-1}, Y^{0:t-1})$$

for all $a$, which is equivalent to

$$0 = a - \mathbb{E}(Y^t | A^t = a, A^{0:t-1}, Y^{0:t-1})
= \mathbb{E}(a - Y^t | A^t = a, A^{0:t-1}, Y^{0:t-1})
= \mathbb{E}(a - \beta a - \gamma X^t - E^t_Y | A^{0:t-1}, Y^{0:t-1})
= \mathbb{E}((1 - \beta)a - \gamma X^t - E^t_Y | A^{0:t-1}, Y^{0:t-1}).$$

This is equivalent to Eq. 91, which was equivalent to Eq. 94, which, as stated above, is satisfied by $\pi$.

Second part of the proposition – deterministic case:  

In this proof (and only here) let us, for simplicity, use the following notation:

- $Y^t, t \in \mathbb{N}$ denotes a sample path (instead of a random process),
- and $Y$ denotes $(Y^t)_{t \in \mathbb{N}}$.

Let $\tilde{Y}$ denote the one-sided Z-transform of $Y$ [Proakis and Manolakis, 1996], defined as the Laurent series (considered formally without considerations on the domain of convergence)

$$\tilde{Y}(z) = \sum_{k=0}^{+\infty} Y^k z^{-k}, z \in \mathbb{C},$$

and similarly for $X, A$. The assumed dynamics equation expressed in the Z-domain leads to

$$\tilde{Y}(z) = (1 - \gamma)\tilde{A}(z) + \gamma \tilde{X}(z).$$
The equation that defines Expodamp implies (using the time-shifting formula [Proakis and Manolakis 1996, p. 208])
\[ z \left( \tilde{A}(z) - A^0 \right) = (1 - \alpha)\tilde{A} + \alpha\tilde{Y}. \]

Combining the above equations results in the following expression for \( Y \) in the Z-domain:
\[ \tilde{Y}(z) = \frac{(\gamma - 1)x}{1 - z^{-1}(1-\alpha\gamma)} + \frac{x}{1 - z^{-1}} + \frac{(1-\gamma)A^0}{1 - z^{-1}(1-\alpha\gamma)}. \]

By classical inversion formulas of the Z-transform, we finally get
\[ Y^t = x + (1 - \gamma) \left[ A^0 - x \right] (1 - \alpha\gamma)^t, \quad t \geq 0. \]

which shows the exponential convergence for any \((A^0, x)\) under the condition \(0 < \gamma \alpha < 2\).

E.2 Full algorithms, proposition and proof for Section 5.2 of the main paper

In this section, let us give the “equilibrium selection objective” (Section 2 of the main paper) a formal loss function:
\[ L_{\pi_r}^{t,Nash} := \begin{cases} 0, & \text{if } s_\pi \text{ is a BNE of } G \text{ (w.r.t. the variables of the } t\text{-th stage)} \\ 1, & \text{else.} \end{cases} \quad (98) \]

Furthermore, let NE stand for (complete-information) Nash equilibrium.

E.2.1 Detailed algorithms

Consider Algorithm S3 together with Algorithms S4 and S5 respectively, as subroutines. It is a rigorous version of Algorithm 2 of the main paper that is also more general in that it allows \( V \) to vary.

E.3 Generalized proposition and proof

Let us state a proposition that generalizes Proposition 2 of the main paper.

**Proposition 3** (Convergence of Algorithm S3; sketch). In the dynamic small-scale setting (Section 4), assume \( X^t \) to be independent of \( X^{1:t-1} \), and that in \( G_{\text{small}} \) exists a strict BNE. Then the following holds true:

- **General stochastic case:** Let, for all \( v \),
  \[ \tilde{A}_v := \{ P_{G,s}(B|v) : s \text{ is a (deterministic) strategy profile of the benchmark game } G_{\text{small}} \}. \quad (99) \]
  (Note that \( \tilde{A}_v \) is finite since the range of all variables \( W, B \) is finite.) Let \( \tilde{A} := (\tilde{A}_v)_{v \in \text{range}_V} \). Let the assistant’s policy \( \pi_r \) be Partpred(\( \tilde{A}, r, \text{UpdateFunctionGeneral} \)) as defined in Algorithm S3 with UpdateFunctionGeneral as defined in Algorithm S5 and \( \tilde{A} \) as defined above.
  Then, for any \( \varepsilon > 0 \), there exists \( R, T \) such that for all \( r > R, t > T \), it holds that \( P(L^{t,\text{pred}}_{\pi_r} = 0) > 1 - \varepsilon \) and \( P(L^{t,Nash}_{\pi_r} = 0) > 1 - \varepsilon \).
- **Directed convergence in in complete-information congestion game case:** (Note that a version of this part of the proposition can be formulated where not best, but just improving responses are assumed for the customers, which can even speed up convergence in certain cases.) Let \( W \) be fully determined by \( V \) and for each value of \( V \), let the (complete information) game \( G_{\text{small}} \) be a congestion game [Roughgarden 2016] where all Nash equilibria are strict. For simplicity, in this deterministic setting, assume \( A \in \text{range}_B \) (i.e., an action profile instead of (Dirac) distributions over action profiles). Let the assistant’s policy \( \pi_r \) be given by Partpred(\( \tilde{A}, r, \text{UpdateFunctionCongestion} \)) (Algorithm S3) with UpdateFunctionCongestion as defined in Algorithm S4 and \( \tilde{A} = \text{range}_B \) the set of all action profiles. Then \( L^{t,\text{pred}}_{\pi_r}, L^{t,Nash}_{\pi_r} \to 0 \) for \( t \to \infty \) without ever invoking line 14, i.e., without needing “undirected” search.
Algorithm S3: \textit{Partpred}

\textbf{Input:} parameters: $A = (A_v)_{v \in \text{range}_V}$, $r$, \textit{UpdateFunction}

1. For each $v \in \text{range}_V$, initialize $a_v \in A_v$ randomly

2. for each stage $t \geq 0$ do

\hspace{1cm} Input: $v := V^t$

\hspace{1cm} if $a_v$ has been announced less than $r$ times or has converged under $V = v$ then

\hspace{1cm} \hspace{1cm} Output: $A^t := a_v$

\hspace{1cm} else

\hspace{1cm} \hspace{1cm} Let $\hat{P}^r_{v,a_v}$ be the empirical distribution of $B$ in the $r$ times it was sampled under $V = v$, $A = a_v$

\hspace{1cm} \hspace{1cm} Let $a''_v := \arg\min_{a''_v \in A_v} ||a''_v - \hat{P}^r_{v,a_v}||$

\hspace{1cm} \hspace{1cm} Let $a''''_v := \text{UpdateFunction}(a_v, a'_v, v)$

\hspace{1cm} \hspace{1cm} if $a''''_v = a_v$ then

\hspace{1cm} \hspace{1cm} \hspace{1cm} Remember that for $V = v$, convergence happened

\hspace{1cm} \hspace{1cm} else if $a''''_v$ and all other $a_v \in A_v$ have been tried $r$ times then

\hspace{1cm} \hspace{1cm} \hspace{1cm} Set $a''''_v := \arg\min_{a''''_v \in A_v} ||a''''_v - \hat{P}^r_{v,a_v}||$

\hspace{1cm} \hspace{1cm} \hspace{1cm} Remember that for $V = v$, convergence happened

\hspace{1cm} \hspace{1cm} else if $a''''_v$ has been tried $r$ times then

\hspace{1cm} \hspace{1cm} \hspace{1cm} Pick unused $a''''_v \in A_v$ at random

\hspace{1cm} \hspace{1cm} \hspace{1cm} Set $a_v = a''''_v$

\hspace{1cm} \hspace{1cm} Output: $A^t := a_v$

\hspace{1cm} \hspace{1cm} else

\hspace{1cm} \hspace{1cm} \hspace{1cm} Pick unused $a''''_v \in A_v$ at random

\hspace{1cm} \hspace{1cm} \hspace{1cm} Set $a_v = a''''_v$

\hspace{1cm} \hspace{1cm} Output: $A^t := a_v$

Algorithm S4: \textit{UpdateFunctionCongestion}

// For simplicity, consider $a_v, a'_v$ as action profiles in range$_B$ instead of (Dirac) distributions over action profiles

\textbf{Input:} $a_v, a'_v, v$

1. Let $a'''_v := a_v$, and $J := \emptyset$

2. while There is $i \in I$ s.t. $[a_v]_i = [a'_v]_i$ and $[a'_v]_j \neq [a'_v]_j$ for all $j \in J$ do

3. \hspace{1cm} $J := J \cup \{i\}$

4. \hspace{1cm} $[a''''_v]_i := [a'_v]_i$

Output: $a''''_v$

Algorithm S5: \textit{UpdateFunctionGeneral}

// For simplicity, in what follows, let $[a_v]_i$ denote the marginal distribution of $B_i$ under $a_v$

\textbf{Input:} $a_v, a'_v, v$

1. Let $a''''_v := a_v$

2. if There exists an $i \in I$ s.t. $[a_v]_i = [a'_v]_i$, then

3. \hspace{1cm} pick one such $i$ at random, if there are several

4. \hspace{1cm} $[a''''_v]_i := [a'_v]_i$

Output: $a''''_v$
Proof for Proposition 3. First part of the proposition: General stochastic case:

Prerequisites.

Let \( P_{v,a} := P_M(B|V = v, A = a) \). Keep in mind that, as usual, by a fixed point/self-fulfilling prophecy under \( V = v \) we mean \( a_v \) with \( P_{v,a_v} = a_v \). By assumption, there exists a strict BNE in \( G^{\text{small}} \). Then Corollary 1 of the main paper implies that there is \( \pi \) with \( 0 = L_\pi^{\text{pred}} = \mathbb{E}(d(P_M(B|V = \pi(V)), \pi(V))) \). Hence, for each \( V = v \) there exists a fixed point.

Now let \( v \) be arbitrary but fixed. Keep in mind that by a (same-covariate, same-prediction) group (of stages) we mean the subsequence of \( R \) stages \((t^v_j)_{j} \) where \( V^t = v \) and \( A^t = a_v \) for some \( a_v \in \hat{A}_v \). Furthermore, let us say the algorithm converges at that and that group of stages with covariate \( v \), if after that group of stages it will always output the same \( a_v \). Let \( J := |\hat{A}_v| \). Let \( d := \min_{a_v, a'_v \in \hat{A}_v} \| P_{v,a_v} - P_{v,a'_v} \| \).

Observe that the algorithm certainly converges in finite time – at the latest after sampling has happened \( r \) times (corresponding to one group) under all \( a_v \in \hat{A}_v \), i.e., after \( J \cdot r \) stages. So we have to show that with growing \( R \) the probability that the reason for convergence is not that it found an actual fixed point (self-fulfilling prophecy) goes to zero. Observe that in order for it to not converge due to finding an actual fixed point either of the following two events has to happen:

- the algorithm converges at some action that is not a fixed point by wrongly taking it for a fixed point;
- it converges after the \( J \) groups of stages by the criterion to force convergence after \( J \) (lines 10 to 12), and has missed the actual fixed point (or one of the actual fixed points).

So it suffices to show for these events individually, that with growing \( R \) the probability that they happen goes to zero.

**Bound the probability that the algorithm converges at some action that is not a fixed point by wrongly taking it for a fixed point.**

Observe that during the phase where the algorithm has not converged yet, each \( a_v \in \hat{A}_v \) is chosen as action \( A \) during at most one group of stages and let us denote the corresponding empirical distribution of \( B \) by \( \hat{P}_{v,a} \).

The phase where the algorithm has not converged yet consists of at most \( J \) groups of stages, and at most \( J - 1 \) groups of stages where \( A \) is an \( a_v \) that is not a fixed point. Given any \( \varepsilon \), we have to show that there is \( R \), such that for any \( r > R \), we can bound the probability that the algorithm converges due to “wrongly taking \( a_v \) as a fixed point” at the end any of these groups of stages by \( \varepsilon \). We do so by bounding the probability that this happens at any individual group of the at most \( J - 1 \) groups where \( a_v \) is not a fixed point, and then sum them up and apply the union bound.

Let \( R \in \mathbb{N} \) be such that, for all \( r > R \) and for all \( a_v \in \hat{A}_v \), that are used during these most \( J - 1 \) groups where \( a_v \) is not a fixed point: 
\[
P(\|P_{v,a_v} - \hat{P}_{v,a_v}\| > \frac{d}{2}) < \frac{\varepsilon}{J - 1},
\]  
with \( \hat{P}_{v,a} \) for the respective used \( a_v \) as defined in Algorithm S3 (Such \( R \) exists based on the weak law of large numbers [Klenke 2013] and the fact that \( \hat{A}_v \) is finite.)

So let us fix one of these groups of stages where \( A \) is an \( a_v \) that is not a fixed point. In particular, \( a_v \neq \hat{P}_{v,a_v} \). (Keep in mind that nonetheless, \( P_{v,a_v} \in \hat{A}_v \).) For all \( r > R \), the probability that the algorithm converges at the end of this group of stages coincides with (or rather: is bounded by) the probability that \( \|a_v - \hat{P}_{v,a_v}\| \leq \|P_{v,a_v} - \hat{P}_{v,a_v}\| \). But
\[
P\left(\|a_v - \hat{P}_{v,a_v}\| \leq \|P_{v,a_v} - \hat{P}_{v,a_v}\|\right) 
\leq P(\|P_{v,a_v} - \hat{P}_{v,a_v}\| > \frac{d}{2}) < \frac{\varepsilon}{J - 1}.
\]

To see why the inequality holds true, observe that the event \( \|P_{v,a_v} - \hat{P}_{v,a_v}\| \leq \frac{d}{2} \) implies the event \( \|a_v - \hat{P}_{v,a_v}\| \geq \frac{d}{2} \leq \|P_{v,a_v} - \hat{P}_{v,a_v}\| \). (To see the first inequality, assume otherwise. Then \( \|P_{v,a_v} - a_v\| \leq \|P_{v,a_v} - \hat{P}_{v,a_v}\| + \|a_v - \hat{P}_{v,a_v}\| < \frac{d}{2} \), which contradicts what we assumed.)

So the probability that the algorithm converges at the end of any of these groups of stages (where \( A \) is an \( a_v \) that is not a fixed point) is bounded by \( (J - 1)\frac{\varepsilon}{J - 1} = \varepsilon \). This is what had to be shown.

**Bound the probability of convergence of the algorithm after the \( J \) groups of stages by the criterion to force convergence after \( J \), and having missed the actual fixed point (or one of the actual fixed points).**
What we have to do here is bound the probability that a fixed point is not taken as a fixed point. Let us be more specific. Given any trajectory of the algorithm with some ordering of the groups of stages, let $a_v \in \bar{A}_v$ be $a$ (the first one, if there are several) fixed point, i.e., $a_v = P_{v,a_v}$, which is taken as $A$ at some point during the trajectory. Given any $\varepsilon$, we have to show that there is $R$, such that for any $r > R$, we can bound the probability that $a_v$ is “not recognized as a fixed point” by $\varepsilon$.

Let $R \in \mathbb{N}$ be such that $P(\|P_{v,a_v} - \bar{P}^r_{v,a_v}\| > \frac{d}{2}) < \varepsilon$ for all $r > R$ and for all $a_v \in \bar{A}_v$. (Such $R$ exists based on the weak law of large numbers \cite{Klenke2013} and the fact that $\bar{A}_v$ is finite.) Then for all $r > R$, the probability that it is not recognized as a fixed point is

$$P \left( \|a_v' - \bar{P}^r_{v,a_v}\| < \|a_v - \bar{P}^r_{v,a_v}\| \text{ for some } a_v' \right)$$

$$= P \left( \|a_v' - \bar{P}^r_{v,a_v}\| < \|P_{v,a_v} - \bar{P}^r_{v,a_v}\| \text{ for some } a_v' \right)$$

$$\leq P(\|P_{v,a_v} - \bar{P}^r_{v,a_v}\| > \frac{d}{2}) < \varepsilon.$$  

(Since $d$ is the minimum distance between $a_v'$ and $P_{v,a_v}$ – the analogous argument as before.)

**Finally.**

Now simply take $R, T$ large enough such that:

- With high probability, each $V = v$ (with positive probability) has been observed at least $JR$ times.
- Within the event that each $V = v$ (with positive probability) has been observed at least $JR$ times: for $r > R$, under algorithm $\pi_r$, the probability that converges against a fixed point occurred under all $V = v$ (which is a product of $|\text{range}_V|$ probabilities that each go to 1 with growing $r$, based on the above) is high enough.

**Second part of the proposition: Directed convergence in complete-information congestion game case:**

We write down the proof for the case of a fixed $V$. The general case works analogously.

Let $\Phi$ denote the potential function (the bigger the utilities, the bigger the potential function) \cite{Roughgarden2016} of the congestion game (and thus potential game) $G_{\text{small}}$.

Let stage $t$, announcement $A^t = a$ and outcome $B^t = b$ be arbitrary but fixed. In what follows, we say a set $E \subset I$ of players is collision-free if (1) $b_i \neq b_j$ for any $i, j \in E$ (no two players in $E$ move to the same “target” slot), and (2) $a_i \neq a_j$ for any $i, j \in E$ (no two players in $E$ move from the same “source” slot). Let us denote

$$a_E := (b_i : b_i = b_i \text{ if } i \in E, \text{ else } b_i = a_i, i \in I) \in \mathcal{B},$$

i.e., applying all moves of players in $E$ to $a$.

**Claim:** If $E \subset F \subset I$ are collision-free, then $\Phi(a^E) \geq \Phi(a^E)$. So, roughly speaking, setting $A^{t+1} := a^E$ for any collision-free $E$, such that no superset $F \supseteq E$ is collision-free, is a reasonable policy for the assistant.

To see why this holds, let $b^1 = a, b^2, \ldots, b^k = a^E$ be a path from $a$ to $a^E$, meaning that at each step $j$ from $b^j$ to $b^{j+1}$, only one player $i_j \in E$ applies her move $[b - a]_{i_j}$ to $b^j$.

For the potential function $\Phi$ \cite{Roughgarden2016} we have

$$\Phi(b^k) - \Phi(b^1) = \sum_j \Phi(b^{j+1}) - \Phi(b^j)$$

$$= \sum_j u_{i_j}(b^{j+1}) - u_{i_j}(b^j).$$

Hence, it suffices to show that $u_{i_j}(b^{j+1}) - u_{i_j}(b^j) \geq 0$ for all $j$, because then we cannot do better than $a^E$ by taking $a^F$ for any subset $F \subset E$. To prove this, we establish that for all $j$,

$$u_{i_j}(b^{j+1}) - u_{i_j}(b^j) \geq u_{i_j}(b^{j+1}_{c_i} - b^j_{c_i}) - u_{i_j}(b^j) \geq 0.$$
The second inequality directly follows from our assumption that player \( i_j \) makes an improvement move. To prove the first inequality, we show that for all \( j \),

\[
\begin{align*}
      u_{i_j}(b^{j+1}) &\geq u_{i_j}(b_{i_j}^{j+1}, b_{-i_j}), \\
      u_{i_j}(b^j) &\leq u_{i_j}(b^j).
\end{align*}
\] (104) (105)

Keep in mind that in the congestion game, the utility only depends on the number of other players at the same slot.

For each \( j \), based on the assumption that no two players move to the same slot, either the number of other players \( l \neq i_j \) at slot \( b_{i_j}^{j+1} \) in action profile \( b^{j+1} \) is the same or it drops compared to \( (b_{i_j}^{j+1}, b_1^{i_j}) \), which implies Inequality 104.

Furthermore, for each \( j \), based on the assumption that no two players move from the same slot, the number of other players \( l \neq i_j \) at slot \( b_{i_j}^j \) in action profile \( b^j \) is the same or it increases compared to \( c^1 \), which implies Inequality 105.

This is also the reason why we cannot allow two players to move from the same slot: because it could happen, that the change of circumstances due to the second one moving renders the move of the first one a worsening move.

**Claim:** \( L_{\pi_{\text{pred}}}^t, L_{\pi_{\text{Nash}}}^t \to 0 \) for \( t \to \infty \) without ever invoking line 14, i.e., without needing “undirected” search.

This is the analogous argument of convergence of classical best-response dynamics in congestion games against a NE [Roughgarden, 2016]: also in our case the \( \Phi \) is guaranteed to strictly increase (with some constant lower bound on each decrease since the game is finite) until it reaches a “local” minimum, since we always let at least one customer improve. Therefore we will never reach the same action profile again, i.e., never invoke lines 14 to 15.

And due to the assumed strictness of the NE, we will stay at a NE once it was announced. Then apply Corollary 1 of the main paper.

**Comment.** Note that we assume \( \tilde{A} \) to be given. This is to be more modular and better express the algorithm which is on the proof-of-concept level. In principle \( \tilde{A} \) can be inferred from data as well.
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