SHIFTING CHAIN MAPS IN QUANDLE HOMOLOGY AND COCYCLE INVARIANTS

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ABSTRACT. Quandle homology theory has been developed and cocycles have been used to define invariants of oriented classical or surface links. We introduce a shifting chain map \( \sigma \) on each quandle chain complex that lowers the dimensions by one. By using its pull-back \( \sigma^\# \), each 2-cocycle \( \phi \) gives us the 3-cocycle \( \sigma^\# \phi \). For oriented classical links in the 3-space, we explore relation between their quandle 2-cocycle invariants associated with \( \phi \) and their shadow 3-cocycle invariants associated with \( \sigma^\# \phi \). For oriented surface links in the 4-space, we explore how powerful their quandle 3-cocycle invariants associated with \( \sigma^\# \phi \) are. Algebraic behavior of the shifting maps for low-dimensional (co)homology groups is also discussed.

1. INTRODUCTION

Quandles \cite{15, 19} are algebraic structures whose axioms encode the movements of oriented classical links in \( \mathbb{R}^3 \) through their diagrams in \( \mathbb{R}^2 \). Later they turn out to be also useful for studying oriented surface links in \( \mathbb{R}^4 \) through their diagrams in \( \mathbb{R}^3 \). Basic link invariants derived from quandles are coloring numbers, and there are enhancements of coloring numbers, called quandle cocycle invariants and shadow cocycle invariants, by using quandle homology theory. For a quandle \( X \) and an abelian group \( A \), a (quandle) \( n \)-cocycle is a map from \( X^n \) to \( A \) satisfying some conditions; see Example 2.1 when \( n = 2 \). Using a 2-cocycle \( \phi \) and a 3-cocycle \( \theta \), we have the following invariants of an oriented classical link \( L \) in \( \mathbb{R}^3 \) and an oriented surface link \( L \) in \( \mathbb{R}^4 \):

- a quandle (2-)cocycle invariant \( \Phi_{\phi}(L) \in \mathbb{Z}[A] \) of \( L \subset \mathbb{R}^3 \),
- a quandle (3-)cocycle invariant \( \Phi_{\phi}(L) \in \mathbb{Z}[A] \) of \( L \subset \mathbb{R}^4 \),
- a shadow (3-)cocycle invariant \( \Phi_{\theta}(L) \in \mathbb{Z}[A] \) of \( L \subset \mathbb{R}^3 \) for each \( x \in X \),

where \( \mathbb{Z}[A] \) is the group ring of \( A \) over \( \mathbb{Z} \). Hence it is important to find 2-cocycles and 3-cocycles for studying oriented classical or surface links.

In this paper, we introduce a shifting chain map \( \sigma \) in quandle homology theory. Using its pull-back cochain map \( \sigma^\# \), the \((n + 1)\)-cocycle can be obtained from an \( n \)-cocycle for each \( n \in \mathbb{Z} \). Although the precise definition will be given in Subsection 2.2, we now describe the explicit formula for a 2-cocycle \( \phi: X^2 \to A \). The map \( \sigma^\# \phi: X^3 \to A \) is defined by the formula

\[
\sigma^\# \phi(x, y, z) = \phi(y, z) - \phi(x, z) + \phi(x, y)
\]

for each \( x, y, z \in X \); see Example 2.4. Given a 2-cocycle \( \phi \), it is natural to compare the shadow 3-cocycle invariant \( \Phi_{\sigma^\# \phi}(L) \) with the quandle 2-cocycle invariant \( \Phi_{\phi}(L) \).
for an oriented classical link $L$ and $x \in X$, and ask how powerful the quandle 3-
cocycle invariant $\Phi_{\sigma \# \phi}(L)$ is for an oriented surface link $L$. The following are our
main results:

**Theorem 1.1.** Let $L$ be an oriented classical link in $\mathbb{R}^3$ and $\phi: X^2 \to A$ a 2-cocycle
for a quandle $X$ and an abelian group $A$. Then we have

$$\Phi_{\sigma \# \phi}(L) = \Phi_{\phi}(L) \in \mathbb{Z}[A]$$

for each $x \in X$.

**Theorem 1.2.** Let $L$ be an oriented surface link in $\mathbb{R}^4$ and $\phi: X^2 \to A$ a 2-cocycle
for a quandle $X$ and an abelian group $A$. Then $\Phi_{\sigma \# \phi}(L)$ is trivial, that is,

$$\Phi_{\sigma \# \phi}(L) = |\text{Col}_X(L)| \cdot 0_A \in \mathbb{Z}[A],$$

where $|\text{Col}_X(L)|$ is the $X$-coloring number of $L$ and $0_A$ is the identity element of $A$.

Theorem 1.1 implies that any quandle 2-cocycle invariant can be considered as
a shadow 3-cocycle invariant through the shifting chain map $\sigma$ for classical links.
It follows that if the quandle 2-cocycle invariant associated with a 2-cocycle $\phi$ is
non-trivial for some classical link then the cohomology class of the 3-cocycle $\sigma \# \phi$ is
non-trivial. On the other hand, Theorem 1.2 implies that the 3-cocycle $\sigma \# \phi$ gives
us a trivial quandle 3-cocycle invariant for surface links, even if the cohomology
class of $\sigma \# \phi$ is non-trivial.

This paper is organized as follows. After reviewing quandles and their homology
theory, we define shifting chain maps, which are our main subjects, in Section 2.
Using a generalization of quandle homology theory reviewed in Section 3 we define
the quandle cocycle invariants and the shadow cocycle invariants in Section 4.
Key ingredients, called the fundamental quandles and the fundamental classes, for
defining cocycle invariants are reviewed in Appendix A. Section 5 is devoted to
proving Theorem 1.1 and 1.2. Algebraic behavior of the shifting maps for low-
dimensional (co)homology groups is studied in Section 6.

## 2. Shifting chain maps

We review a quandle [15,19] and its homology theory [4], and introduce a shifting
chain map on a quandle chain complex that lowers the dimensions by one.

### 2.1. Quandle homology

A *quandle* is a set $X$ equipped with a binary operation $*: X \times X \to X$ satisfying the following three axioms for any $x, y, z \in X$: (Q1) $x * x = x$, (Q2) the map $s_x: X \to X$, defined by $\bullet \mapsto \bullet * x$, is bijective, and (Q3) $(x * y) * z = (x * z) * (y * z)$. Typical examples such as the dihedral quandles and the tetrahedral quandle will be reviewed in Section 6.

Let $C^R_n(X)$ be the free abelian group generated by $X^n$ for $n \geq 1$ and $C^R_n(X) = 0$
for $n \leq 0$. Define two homomorphisms $\partial_n^0, \partial_n^1: C^R_n(X) \to C^R_{n-1}(X)$ by

$$\partial_n^0(x_1, \ldots, x_n) = \sum_{i=1}^{n} (-1)^i (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

$$\partial_n^1(x_1, \ldots, x_n) = \sum_{i=1}^{n} (-1)^j (x_1 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n)$$
for $n \geq 2$ and $\partial_n^0 = \partial_n^1 = 0$ for $n \leq 1$. By a direct computation, we have

\[ \partial_{n-1}^0 \circ \partial_n^0 = 0, \quad \partial_n^1 \circ \partial_n^1 = 0 \quad \text{and} \quad \partial_{n-1}^1 \circ \partial_n^0 + \partial_n^0 \circ \partial_n^1 = 0. \]

Let $\partial_n : C_n^R(X) \to C_{n-1}^R(X)$ be a homomorphism defined by $\partial_n = \partial_n^0 - \partial_n^1$. It follows from Equation (2) that $\partial_{n-1} \circ \partial_n = 0$. Then $C_n^R(X) = (C_n^R(X), \partial_n)$ is a chain complex.

Let $C_n^D(X)$ be the subgroup of $C_n^R(X)$ generated by

\[ \{ (x_1, \ldots, x_n) \in X^n \mid x_i = x_{i+1} \text{ for some } i \} \]

for $n \geq 2$ and $C_n^D(X) = 0$ for $n \leq 1$. We have $\partial_n^0(C_n^D(X)) \subset C_{n-1}^D(X)$ and $\partial_n^1(C_n^D(X)) \subset C_{n-1}^D(X)$, hence $C_n^D(X) = (C_n^D(X), \partial_n)$ is a subcomplex of $C_n^R(X)$.

Then $C_n^Q(X) = (C_n^Q(X), \partial_n)$ is defined to be the quotient complex $C_n^R(X)/C_n^D(X)$ and called the \textit{quandle chain complex} of $X$, where all the induced boundary maps are again denoted by $\partial_n$’s. The $n$th group of cycles in $C_n^Q(X)$ is denoted by $Z_n^Q(X)$, and the $n$th homology group of this complex is called the $n$th \textit{quandle homology group} and is denoted by $H_n^Q(X)$.

For an abelian group $A$, define the cochain complex

\[ C^*_Q(X; A) = \text{Hom}_\mathbb{Z}(C^*_Q(X), A) \quad \text{and} \quad \delta^* = \text{Hom}(\delta_*, \text{id}) \]

in the usual way. The $n$th group of cocycles in $C^n_Q(X; A)$ is denoted by $Z^n_Q(X; A)$, and the $n$th cohomology group of this complex is called the $n$th \textit{quandle cohomology group} and is denoted by $H^n_Q(X; A)$.

**Example 2.1.** When $C^n_R(X; A) = \text{Hom}_\mathbb{Z}(C^n_R(X), A)$ is canonically identified with the set of all maps from $X^n$ to $A$, a map $\phi : X^2 \to A$ is a quandle 2-cocycle if it satisfies

\[ \phi(x, x) = 0_A \quad \text{and} \quad \phi(x, z) - \phi(x, y) - \phi(x * y, z) + \phi(x * z, y * z) = 0_A \]

for any $x, y, z \in X$, where the first and second condition follow from

\[ \phi(C^2_Q(X)) = \{0_A\} \quad \text{and} \quad \phi(\delta_3(C^1_Q(X))) = \{0_A\}, \]

respectively.

### 2.2. Shifting Chain Map

Let $\sigma_n : C_n^Q(X) \to C_{n-1}^Q(X)$ be two homomorphisms defined by

\[ \sigma_n = (-1)^n \partial_n^0 \quad \text{and} \quad \tilde{\sigma}_n = (-1)^n \partial_n^1 \]

for a quandle $X$. It follows from Equation (2) that the set of maps $\sigma_n$ and that of maps $\tilde{\sigma}_n$ form two chain maps $\sigma, \tilde{\sigma} : C^*_Q(X) \to C^*_{Q-1}(X)$. These two chain maps are closely related as follows.

**Proposition 2.2.** The chain map $\sigma$ is chain homotopic to $\tilde{\sigma}$.

**Proof.** Let $P_n : C_n^Q(X) \to C_n^Q(X)$ be a homomorphism defined by $P_n = (-1)^n n \cdot \text{id}$. Direct computations show that

\[ P_{n-1} \circ \partial_n = (-1)^{n-1} (n-1) \cdot \partial_n \quad \text{and} \quad \partial_n \circ P_n = (-1)^n n \cdot \partial_n, \]

hence we have

\[ P_{n-1} \circ \partial_n + \partial_n \circ P_n = (1)^n \cdot \partial_n = \sigma_n - \tilde{\sigma}_n. \]

This implies that the set of maps $P_n$ is a chain homotopy between $\sigma$ and $\tilde{\sigma}$. \qed

**Definition 2.3.** The chain map $\sigma : C^*_Q(X) \to C^*_{Q-1}(X)$ is called a \textit{shifting (chain) map} on the quandle chain complex $C^*_Q(X)$. 
The shifting map $\sigma$ induces a cochain map $\sigma^\#: C^{-1}_Q(X; A) \to C^{-1}_Q(X; A)$ for an abelian group $A$, and then induces homomorphisms

$$\sigma_* : H^0_Q(X) \to H^0_Q(X) \quad \text{and} \quad \sigma^* : H^{-1}_Q(X; A) \to H^{-1}_Q(X; A),$$

in the usual way. Behavior of these homomorphisms will be discussed in Section 6.

**Example 2.4.** A quandle 2-cocycle $\phi : X^2 \to A$ is sent to the quandle 3-cocycle $\sigma^\# \phi : X^3 \to A$ such that

$$\sigma^\# \phi(x, y, z) = \phi(\sigma_3(x, y, z)) = \phi((-1)^3 \partial^0_3(x, y, z))$$

$$= \phi((y, z) - (x, z) + (x, y)) = \phi(y, z) - \phi(x, z) + \phi(x, y)$$

for each $x, y, z \in X$. This is nothing but Formula (1) shown in Section 1.

**Remark 2.5.** We mention here a few related topics. Another kind of shifting chain map lowering the dimension by one was discussed in [7]. However, as pointed out in [21, Remark 23], their map is not a chain map. Shifting chain maps raising the dimension by one and two were defined in [21] for some class of quandles including dihedral quandles of odd order. It might be interesting to investigate the composite of our shifting chain map and their shifting maps.

3. Generalized quandle homology

We review quandle homology theory with local coefficients, called generalized quandle homology theory [11, 3], which will be a bridge between quandle cocycle invariants and shadow cocycle invariants in Section 5. Although our notational conventions for generalized quandle homology theory are based on [17], they are essentially the same, as pointed out in [14, Remark 1 and 2]. We start with reviewing the associated group [12, 15, 19] of a quandle and its action.

3.1. Associated group of a quandle. The associated group $\text{As}(X)$ of a quandle $X$ is defined by

$$\text{As}(X) := \langle x \mid (x, x) \cdot y = y^{-1}xy \ (x, y \in X) \rangle.$$ 

A set equipped with a right action of $\text{As}(X)$ is called an $X$-set. The following are typical examples of $X$-sets.

**Example 3.1.** These five $X$-sets will appear in what follows.

1. The set $X$ itself is naturally an $X$-set. A right action of $\text{As}(X)$ on $X$ is given by $y \cdot x = y \ast x$ and $y \cdot x^{-1} = y^{-1} \ast x$ for $x, y \in X$, where $x$ is regarded as an element in $\text{As}(X)$.
2. The set $\text{As}(X)$ is naturally an $X$-set. A right action of $\text{As}(X)$ on $\text{As}(X)$ is given by $h \cdot g = hg$ for $h, g \in \text{As}(X)$.
3. A singleton $Y = \{y_0\}$ is an $X$-set. A right action of $\text{As}(X)$ on $Y$ is given by $y_0 \cdot g = y_0$ for $g \in \text{As}(X)$.
4. The set $\mathbb{Z}$ of all integers is an $X$-set. A right action of $\text{As}(X)$ on $\mathbb{Z}$ is given by $y \cdot x = x + 1$ and $y \cdot x^{-1} = y - 1$ for $x \in X$ and $y \in \mathbb{Z}$, where $x$ is regarded as an element in $\text{As}(X)$.
5. For two $X$-sets $Y_1$ and $Y_2$, the product $Y_1 \times Y_2$ is an $X$-set. A right action of $\text{As}(X)$ on $Y_1 \times Y_2$ is given by $(y_1, y_2) \cdot g = (y_1 \cdot g, y_2 \cdot g)$ for $y_1 \in Y_1$, $y_2 \in Y_2$ and $g \in \text{As}(X)$.
Let $Y, Z$ be $X$-sets. A map $p: Y \to Z$ is called an $X$-map if it satisfies

$$p(y \cdot g) = p(y) \cdot g$$

for any $y \in Y$ and $g \in \text{As}(X)$. The following are typical examples of $X$-maps.

**Example 3.2.** These three $X$-maps will appear in what follows.

1. Let $Y$ be an $X$-set. For each element $y \in Y$, a map $s^y: \text{As}(X) \to Y$, defined by $s^y(g) = y \cdot g$ for $g \in \text{As}(X)$, is an $X$-map.
2. Let $Y_1, Y_2$ be two $X$-sets. For the product $Y_1 \times Y_2$, the projection $p_i: Y_1 \times Y_2 \to Y_i$ onto the $i$-th factor is an $X$-map for each $i = 1, 2$.
3. Let $Y$ be an $X$-set. For a singleton $\{y_0\}$, the unique map $q: Y \to \{y_0\}$ is an $X$-map.

### 3.2. Generalized quandle homology

For a quandle $X$ and an $X$-set $Y$, let $C^R_n(X)_Y$ be the free abelian group generated by $Y \times X^n$ for $n \geq 1$ and $C^R_0(X)_Y$ the free abelian group generated by $Y$. Put $C^R_n(X) = 0$ for $n \leq -1$. Define two homomorphisms $\partial^0_n, \partial^1_n: C^R_n(X)_Y \to C^R_{n-1}(X)_Y$ by

$$\partial^0_n(y; x_1, \ldots, x_n) = \sum_{i=1}^{n} (-1)^i (y; x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$$

$$\partial^1_n(y; x_1, \ldots, x_n) = \sum_{i=1}^{n} (-1)^i (y \cdot x_i; x_1, x_i, \ldots, x_{i-1} \cdot x_i, x_{i+1}, \ldots, x_n)$$

for $n \geq 1$ and $\partial^0_n = \partial^1_n = 0$ for $n \leq 0$. Let $\partial_n: C^R_n(X)_Y \to C^R_{n-1}(X)_Y$ be a homomorphism defined by $\partial_n = \partial^0_n - \partial^1_n$. We can check that $\partial_{n-1} \circ \partial_n = 0$ in the same way, hence $C^R_n(X)_Y = (C^R_n(X)_Y, \partial_n)$ is a chain complex.

Let $C^D_n(X)_Y$ be the subgroup of $C^R_n(X)_Y$ generated by

$$\{(y; x_1, \ldots, x_n) \in Y \times X^n \mid x_i = x_{i+1} \text{ for some } i\}$$

for $n \geq 2$ and $C^D_n(X)_Y = 0$ for $n \leq 1$. We have $\partial^0_n(C^D_n(X)_Y) \subset C^D_{n-1}(X)_Y$ and $\partial^1_n(C^D_n(X)_Y) \subset C^D_{n-1}(X)_Y$, hence $C^D_n(X)_Y = (C^D_n(X)_Y, \partial_n)$ is a subcomplex of $C^R_n(X)_Y$. Then $C^Q_n(X)_Y = (C^Q_n(X)_Y, \partial_n)$ is defined to be the quotient complex $C^R_n(X)_Y/C^D_n(X)_Y$ and called the *generalized quandle chain complex* of $X$, where all the induced boundary maps are again denoted by $\partial_n$’s. The $n$th group of cycles in $C^Q_n(X)_Y$ is denoted by $Z^Q_n(X)_Y$, and the $n$th homology group of this complex is called the $n$th *generalized quandle homology group* and is denoted by $H^Q_n(X)_Y$.

When $Y$ is a singleton $\{y_0\}$, a set of natural maps $C^Q_n(X)_{\{y_0\}} \to C^Q_n(X)$ defined by $(y_0; x) \mapsto (x)$ becomes a chain isomorphism.

For an abelian group $A$, define the cochain complex

$$C^Q_n(X; A)_Y = \text{Hom}_\mathbb{Z}(C^Q_n(X)_Y, A), \quad \delta^* = \text{Hom}(\partial_n, \text{id})$$

in the usual way. The $n$th group of cocycles in $C^n(X; A)_Y$ is denoted by $Z^n_Q(X; A)_Y$, and the $n$th cohomology group of this complex is called the $n$th *generalized quandle cohomology group* and is denoted by $H^n_Q(X; A)_Y$.

### 3.3. Two types of induced chain maps

We introduce two types of induced chain maps, which will be essentially used to define various cocycle invariants in the next section.
First, we define a chain map induced from a quandle homomorphism. Let $Q, X$ be two quandles. Any quandle homomorphism $f: Q \to X$ induces a group homomorphism $\text{As}(f): \text{As}(Q) \to \text{As}(X)$ defined by $\text{As}(f)(x) = f(x)$ and $\text{As}(f)(x^{-1}) = f(x)^{-1}$ for $x \in X$, where $x$ and $f(x)$ are regarded as an element in $\text{As}(Q)$ and $\text{As}(X)$ respectively. Let $f_{\#,n}: C^n_\#(Q)_{\text{As}(Q)} \to C^n_\#(X)_{\text{As}(X)}$ be a homomorphism defined by
\[
 f_{\#,n}(g; x) = \left( \text{As}(f)(g); (f \times \cdots \times f)(x) \right)
\]
for each $n \geq 1$ and $f_{\#,0}(g) = (\text{As}(f)(g))$, where $g \in \text{As}(X)$ and $x \in X^n$. Put $f_{\#,n} = 0$ for $n \leq -1$. Then the set of maps $f_{\#,n}$ becomes a chain map $f_\#: C^\#(Q)_{\text{As}(Q)} \to C^\#(X)_{\text{As}(X)}$.

Second, we define a chain map induced from an X-map for a quandle $X$. Let $Y, Z$ be two $X$-sets, and $p: Y \to Z$ an $X$-map. We define a homomorphism $p_{\#,n}: C^n_\#(X)_Y \to C^n_\#(X)_Z$ by $p_{\#,n}(g; x) = (p(g); x)$ for $n \geq 1$ and $p_{\#,0}(g) = (p(g))$. Put $p_{\#,n} = 0$ for $n \leq -1$. Then the set of maps $p_{\#,n}$ becomes a chain map $p_\#: C^\#(X)_Y \to C^\#(X)_Z$. For each element $y \in Y$, we have two $X$-maps $s^y_\#: \text{As}(X) \to Y$ and $s^{p(y)}_\#: \text{As}(X) \to Z$ as in Example 3.2 (1). The following is easy to prove.

**Lemma 3.3.** We have $s^{p(y)}_\# = s^y \circ p: \text{As}(X) \to Z$ for any $y \in Y$. We also have $s^{p(y)}_{\#'} = p_\# \circ s^y_\#: C^\#(X)_{\text{As}(X)} \to C^\#_\#(X)_Z$ for any $y \in Y$.

### 4. Cocycle invariants

We define the quandle cocycle invariant [4] and shadow cocycle invariant [8] as two types of specializations of the generalized quandle cocycle invariant [3]. We start with recalling the fundamental classes (of an oriented classical link [8] and surface link [15]) valued in the generalized homology group of the fundamental quandles (of the classical link [15] and surface link [16]). The fundamental class is considered as a universal object for the generalized quandle cocycle invariant, and naturally derived from homological interpretation [9] of the invariant.

#### 4.1. Fundamental quandle and fundamental class

To each diagram $D$ of an oriented classical link (or surface link), we can associate a quandle and a homology class as invariants of the link; the fundamental quandle $Q_D$ and the fundamental class $[D]$, where $[D]$ takes its value in the 2nd (or 3rd) generalized quandle homology group $H^Q_2(Q_D)_{\text{As}(Q_D)}$. Since we do not use the precise constructions of both invariants in the rest of the paper, we postpone their definition to Appendix A and here introduce their important properties instead. For another diagram $D'$, if the oriented classical or surface link represented by $D'$ is equivalent to that represented by $D$, then there exists a quandle isomorphism from $Q_D$ to $Q_{D'}$ such that the induced group isomorphism on the homology groups sends $[D]$ to $[D']$; see Theorem A.1 and A.2.

Let $X$ be a finite quandle. For a diagram $D$ of an oriented classical link $L$ or an oriented surface link $\mathcal{L}$, let $\text{Col}_X(D)$ be the set of all quandle homomorphisms from $Q_D$ to $X$. It follows from the aforementioned properties of $Q_D$ that its cardinality $|\text{Col}_X(D)|$ does not depend on the choice of the diagram $D$ for the link. Hence, when $D$ represents $L$ (or $\mathcal{L}$), it is also denoted by $|\text{Col}_X(L)|$ (or $|\text{Col}_X(\mathcal{L})|$) and called the $X$-coloring number of $L$ (or $\mathcal{L}$).
4.2. **Generalized quandle cocycle invariant.** Let $X$ be a finite quandle, $A$ an abelian group, $Y$ an $X$-set and $y$ an element in $Y$. Let $D$ be a diagram of an oriented classical link $L$ or an oriented surface link $L$. When $D$ represents the classical link (or surface link), for a given 2-cocycle (or 3-cocycle) $\lambda \in Z^*_Q(X; A)_Y$, we define a *generalized quandle cocycle invariant* $\Psi^y_\lambda (D)$ by

$$\Psi^y_\lambda (D) = \sum_{c: Q_D \to X} \langle s^y_\psi \circ c_*[D], [\lambda] \rangle \in Z[A],$$

where $c_*: H^*_Q(Q_D)_\Lambda(Q_D) \to H^*_Q(X)_\Lambda(X)$ is the induced homomorphism from a quandle homomorphism $c: Q_D \to X$ as in the first half of Subsection 4.3 the map $s^y_\psi: H^*_Q(X)_\Lambda(X) \to H^*_Q(X)_Y$ is the induced homomorphism from an $X$-map $s^y: \Lambda(X) \to Y$ as in the second half of Subsection 4.3 the element $[\lambda]$ is a cohomology class of $\lambda$, and

$$\langle , \rangle: H^*_Q(X)_Y \otimes H^*_Q(X; A)_Y \to A$$

is a Kronecker product. We note that the above summation is finite, since the cardinality of $X$ is finite. By definition, the invariant $\Psi^y_\lambda (D)$ depends only on the cohomology class $[\lambda]$ of $\lambda$. Thus, when $\lambda$ is a coboundary, the invariant $\Psi^y_\lambda (D)$ is trivial, that is,

$$\Psi^y_\lambda (D) = |\mathrm{Col}_X(D)| \cdot 0_A \in Z[A],$$

where $0_A$ is the identity element of $A$. It follows from Theorem A.1 and A.2 that the invariant $\Psi^y_\lambda (D)$ does not depend on the choice of the diagram $D$ for the link. Hence, when $D$ represents $L$ (or $L$), it is also denoted by $\Psi^y_\lambda (L)$ (or $\Psi^y_\lambda (L)$).

4.3. **Quandle cocycle invariant.** Let $X$ be a finite quandle and $A$ an abelian group. Let $L$ be an oriented classical link and $L$ an oriented surface link. For given 2-cocycle $\phi \in Z^*_Q(X; A)$ and 3-cocycle $\theta \in Z^*_Q(X; A)$, we define *quandle cocycle invariants* $\Phi_\phi(L)$ and $\Phi_\theta(L)$ by

$$\Phi_\phi(L) = \Psi^y_\phi (L) \in Z[A] \quad \text{and} \quad \Phi_\theta(L) = \Psi^y_\theta (L) \in Z[A],$$

where $\phi$ and $\theta$ are regarded as the elements in $Z^*_Q(X; A)_{\{y_0\}}$ for a singleton $\{y_0\}$ by naturally identifying $C^*_Q(X; A)_{\{y_0\}}$ with $C^*_Q(X; A)$; see Subsection 5.2.

4.4. **Shadow cocycle invariant.** For a quandle $X$, we define a homomorphism $\iota_n: C^*_Q(X)_X \to C^*_Q(X)_X$ by $\iota_n(x; x) = (-1)^n(x, x)$ for $n \geq 1$ and $\iota_0(x, 0) = (x, 0)$. Put $\iota_n = 0$ for $n \leq -1$. It follows from Equation (2) in Subsection 2.1 that the set of maps $\iota_n$ forms a chain map $\iota: C^*_Q(X)_X \to C^*_Q(X)_X$. Let $A$ be an abelian group and fix an element $x$ in $X$. Let $L$ be an oriented classical link and $L$ an oriented surface link. When $X$ is finite, for given 3-cocycle $\theta \in Z^*_Q(X; A)$ and 4-cocycle $\psi \in Z^*_Q(X; A)$, we define *shadow cocycle invariants* $\Phi_\psi(L)$ and $\Phi_\psi(L)$ by

$$\Phi_\psi(L) = \Psi^y_{\iota^\#(L)} \in Z[A] \quad \text{and} \quad \Phi_\psi(L) = \Psi^y_{\iota^\#(L)} \in Z[A],$$

where $\iota^\#: Z^*_Q(X; A) \to Z^*_Q(X; A)_X$ is the pull-back induced by the chain map $\iota$.

5. **Proof**

We prove our main theorems (Theorem 1.1 and 1.2), using a duality (Proposition 5.1) for generalized quandle cocycle invariants.
5.1. Duality for invariants. For a finite quandle \( X \), let \( Y, Z \) be two \( X \)-sets and \( p: Y \to Z \) an \( X \)-map. Let \( D \) be a diagram of an oriented classical or surface link. When \( D \) represents the classical link (or surface link), let \( \lambda \) be a 2-cocycle (or 3-cocycle) in \( \mathbb{Z}^*_q(X; A)_Z \).

**Proposition 5.1.** We have
\[
\Psi^{(y)}_\lambda(D) = \Psi^y_{\lambda p}(D) \in \mathbb{Z}[A]
\]
for any \( y \in Y \), where \( p^\#: Z^*_q(X; A)_Z \to Z^*_q(X; A)_Y \) is the pull-back induced by \( p \).

**Proof.** By a direct calculation, we have
\[
\Psi^{(y)}_\lambda(D) = \sum_e \langle \delta^p(y) \circ c_e[D], [\lambda] \rangle = \sum_e \langle (p_e \circ s^y_e) \circ c_e[D], [\lambda] \rangle = \sum_e \langle s^y_e \circ c_e[D], p^e[\lambda] \rangle = \Psi^y_{\lambda p}(D),
\]
where the fourth equality follows from the usual duality of the Kronecker product and the second equality follows from Lemma 3.3.

5.2. **Proof of Theorem 1.1.** Let \( X \) be a quandle. For the two \( X \)-sets, \( Z \) and \( X \), we have an \( X \)-set \( Z \times X \); see Example 3.1. Two maps, the projection \( p: Z \times X \to X \) and the unique map \( q: X \to \{y_0\} \), are \( X \)-maps; see Example 3.2. Even if the composite
\[
\sigma \circ \iota: C^*_q(X)_X \to C^*_q(X)
\]
of the shifting chain map \( \sigma \) and the chain map \( \iota \) is not chain homotopic to the chain map
\[
q^\#: C^*_q(X)_X \to C^*_q(X)_{\{y_0\}} (= C^*_q(X)),
\]
we can show the following by composing the chain map
\[
p^\#: C^*_q(X)_Z \times X \to C^*_q(X)_X.
\]

**Proposition 5.2.** The chain map \( (\sigma \circ \iota) \circ p^\# \) is chain homotopic to \( q^\# \circ p^\# \).

**Proof.** Let \( P_n: C^*_q(X)_Z \times X \to C^*_q(X)_{n+1}(X) \) be a homomorphism defined by
\[
P_n(a, x_0; x) = a \cdot (x_0, x)
\]
for each generator \( (a, x_0; x) \in C^*_q(X)_Z \times X \). Direct computations show that
\[
P_{n-1} \circ \partial_n(a, x_0; x) = -a \cdot \partial_{n+1}(x_0, x) + \partial_n^1(x_0, x) + (x),
\]
\[
\partial_n + P_n(a, x_0; x) = a \cdot \partial_{n+1}(x_0, x),
\]
\[
q^\# \circ p^\#(a, x_0; x) = q^\#(x_0; x) = (x),
\]
and
\[
(\tilde{\sigma} \circ \iota)_n \circ p^\#(a, x_0; x) = \tilde{\sigma}_{n+1} \circ \iota_n(x_0; x) = -\partial_{n+1}(x_0, x).
\]

Hence we have
\[
P_{n-1} \circ \partial_n + \partial_{n+1} \circ P_n = q^\# \circ p^\# - (\tilde{\sigma} \circ \iota)_n \circ p^\#,\]
and this implies that the set of maps \( P_n \) is a chain homotopy between \((\sigma \circ \iota) \circ p^\#\) and \( q^\# \circ p^\# \).
Proof of Theorem 5.3. By a direct calculation, for any \( m \in \mathbb{Z} \), we have
\[
\Phi_{\sigma \# \phi}^x(L) = \Psi_{\sigma \# \phi}[\sigma(x)](L) = \Psi_{\sigma \# \phi}[\sigma(x)](L) = \Psi_{\phi}^m(L) = \Psi_{\phi}^m(L) = \Phi_{\phi}(L),
\]
where the third and seventh equalities follow from the duality (Proposition 5.1) and the fourth equality follows from Proposition 5.2.

By the same argument, we can show the one-dimensional higher version of Theorem 5.3.

Proof of Theorem 1.2. Let \( X \) be a quandle. For the \( X \)-set \( \mathbb{Z} \), the unique map \( q: \mathbb{Z} \to \{y_0\} \) is an \( X \)-map. Even if the shifting chain map
\[
\sigma: C^Q_*(X) \to C^Q_{*-1}(X)
\]

itself is not null-homotopic, we can show the following by composing the chain map
\[
q\#: C^Q_*(X) \to C^Q_*(X)_{y_0} (= C^Q_*(X)).
\]

Proposition 5.4. The chain map \( \sigma \circ q\# \) is null-homotopic.

Proof. Let \( P_n: C^Q_n(X)_{y_0} \to C^Q_n(X) \) be a homomorphism defined by
\[
P_n(a; x) = (-1)^n a \cdot (x).
\]

for each generator \((a; x) \in C^Q_n(X)_{y_0}\). Direct computations show that
\[
P_{n-1} \circ \partial_n(a; x) = (-1)^{n-1} a \cdot \partial_n(x) + (-1)^n \partial^1_n(x),
\]

\[
\partial_n \circ P_n(a; x) = (-1)^n a \cdot \partial_n(x),
\]

and
\[
\partial_n \circ q\#(a; x) = \partial_n(x) = (-1)^n \partial^1_n(x).
\]

Hence we have
\[
P_{n-1} \circ \partial_n + \partial_n \circ P_n = \partial_n \circ q\#,\]

and this implies that the set of maps \( P_n \) is a chain homotopy between \( \sigma \circ q\# \) and the 0-map.

Proof of Theorem 1.3. By a direct calculation, for any \( m \in \mathbb{Z} \), we have
\[
\Phi_{\sigma \# \phi}(L) = \Psi^m_{\sigma \# \phi}(L) = \Psi_{\sigma \# \phi}(L) = \Psi_{\phi}^m(L) = \Psi_{\phi}^m(L) = \Phi_{\phi}(L),
\]

where \( \theta_{\text{triv}} \) is the trivial 3-cocycle in \( Z^3_0(X; A) \), the third equality follows from the duality (Proposition 5.1) and the fourth equality follows from Proposition 5.4.

By the same argument, we can show the one-dimensional lower version of Theorem 1.2.
Theorem 5.5. Let $L$ be an oriented classical link in $\mathbb{R}^3$ and $\kappa: X \rightarrow A$ a 1-cocycle for a quandle $X$ and an abelian group $A$. Then $\Phi_{\sigma \ast \kappa}(L)$ is trivial, that is,
$$\Phi_{\sigma \ast \kappa}(L) = |\text{Col}_X(L)| \cdot 0_A \in \mathbb{Z}[A],$$
where $|\text{Col}_X(L)|$ is the $X$-coloring number of $L$ and $0_A$ is the identity element of $A$.

6. Behavior of the shifting maps for (co)homology groups

We study behavior of the shifting maps for low-dimensional (co)homology groups. First, we discuss behavior between 1st and 2nd (co)homology groups for connected quandles and the trivial quandle of order 2. Second, we discuss behavior between low-dimensional (co)homology groups greater than one dimension for specific connected quandles; the dihedral quandles of odd prime orders and the tetrahedral quandle (of order 4).

6.1. Behavior between 1st and 2nd (co)homology groups. Given a quandle $X$, it follows from (Q2) and (Q3) of the quandle axioms that the map $s_x: X \rightarrow X$ is an automorphism, called an inner automorphism, of $X$ for each $x \in X$. The quandle $X$ is said to be connected if the group generated by all inner automorphisms acts transitively on $X$. Then, for any $x \in X$, the homology class $[(x)]$ of $[(x)] \in H_1^Q(X) = C_1^Q(X)$ generates $H_1^Q(X) \cong \mathbb{Z}$ and hence we have $H_1^Q(X; A) \cong \text{Hom}(H_1^Q(X), A) \cong A$ for any abelian group $A$; see [9]. Behavior of the shifting maps between 1st and 2nd (co)homology groups for connected quandles is the following.

Theorem 6.1. The shifting maps
$$\sigma_*: H_2^Q(X) \rightarrow H_1^Q(X) \quad \text{and} \quad \sigma^*: H_1^Q(X; A) \rightarrow H_2^Q(X; A)$$
are the 0-maps for any connected quandle $X$ and abelian group $A$.

Proof. For any generator $(x, y)$ in $C_2^Q(X)$, we have
$$\sigma_2(x, y) = -(y) + (x) \quad \text{and} \quad \partial_2(x, y) = +(x) - (x * y)$$
in $C_1^Q(X)$ at the chain level. Since the connectedness of $X$ implies $[(x)] = [(y)] \in H_1^Q(X)$, we have $[\sigma_2(x, y)] = 0 \in H_1^Q(X)$, and hence $\sigma_*$ is the 0-map. Since the connectedness of $X$ also implies that any 1-cocycle $\kappa: X \rightarrow A$ is a constant map, we have $\sigma^* \kappa(x, y) = \kappa(\sigma_2(x, y)) = -\kappa(y) + \kappa(x) = 0$, and hence $\sigma^*$ is the 0-map.

Behavior for disconnected quandles is totally different. The trivial quandle $T_2$ of order 2 is defined to be a set $\{0, 1\}$ with the binary operation $x * y = x$ for each $x, y \in \{0, 1\}$. This is the most simplest disconnected quandle, and the set of $[\{0\}]$ and $[(1)]$ generates $H_1^Q(T_2) \cong \mathbb{Z}^2$ and that the set of $[(0, 1)]$ and $[(1, 0)]$ generates $H_2^Q(T_2) \cong \mathbb{Z}^2$; see [9]. Then we have
$$H_1^Q(T_2; A) \cong \text{Hom}(H_1^Q(T_2); A) \cong A^2 \quad \text{and} \quad H_2^Q(T_2; A) \cong \text{Hom}(H_2^Q(T_2), A) \cong A^2$$
for any abelian group $A$. Since $\sigma_2(0, 1) = -(1) + (0)$ and $\sigma_2(1, 0) = -(0) + (1)$ at the chain level, we have the following.

Proposition 6.2. The shifting map $\sigma_*: H_2^Q(T_2) \rightarrow H_1^Q(T_2)$ is a homorphism $\mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ such that $(1, 0) \mapsto (1, -1)$ and $(0, 1) \mapsto (-1, 1)$. Dually, for any abelian group $A$, the shifting map $\sigma^*: H_1^Q(X; A) \rightarrow H_2^Q(X; A)$ is a homorphism $A^2 \rightarrow A^2$ such that $(a, 0) \mapsto (a, -a)$ and $(0, a) \mapsto (-a, a)$ for each $a \in A$. 

6.2. Dihedral quandle of odd prime order. The dihedral quandle $R_n$ of order $n$ is defined to be $\mathbb{Z}_n$ with the binary operation $x * y = 2x - y$ for each $x, y \in R_n$, where $\mathbb{Z}_n$ denotes the cyclic group of order $n$. The quandle $R_n$ is connected if and only if the number $n$ is odd. Let $p$ be an odd prime throughout in this subsection. It is known that

$$H^2_Q(R_p) \cong 0, \quad H^3_Q(R_p) \cong \mathbb{Z}_p, \quad H^4_Q(R_p) \cong \mathbb{Z}_p$$

and

$$H^2_Q(R_p; \mathbb{Z}_p) \cong 0, \quad H^3_Q(R_p; \mathbb{Z}_p) \cong \mathbb{Z}_p, \quad H^4_Q(R_p; \mathbb{Z}_p) \cong \mathbb{Z}_p^2$$

for any odd prime $p$; see [20] for the 2nd (co)homology group and 3rd cohomology group, [21] for the 3rd homology group, and [10, 23] for the 4th (co)homology group. The first non-trivial shifting maps for (co)homology groups of $R_p$ might be

$$\sigma : H^2_Q(R_p) \to H^3_Q(R_p) \quad \text{and} \quad \sigma^* : H^3_Q(R_p; \mathbb{Z}_p) \to H^4_Q(R_p; \mathbb{Z}_p),$$

which we are going to study.

Let $z, w$ be cycles in $Z^2_Q(R_p)$ defined by

$$z = \sum_{i=1}^{p-2} \left( (0, i, i + 1) - (i, i + 1, 0) + (0, i + 1, i) - (i + 1, i, 0) \right) \in Z^2_Q(R_p),$$

$$w = -\sum_{i=1}^{p-2} \left( (0, i, i + 1, 0) + (0, i + 1, i, 0) \right) \in Z^2_Q(R_p),$$

where these two cycles are obtained from a diagram of the 2-twist spun $(2, p)$-torus knot with a (shadow) $R_p$-coloring. Let $\chi : R^2_p \to \mathbb{Z}_p$ be a 2-cochain in $C^2_Q(R_p; \mathbb{Z}_p)$ defined by

$$\chi(x, y) := \frac{2(p - x)p + x^p - 2y^p}{p} \left( \equiv \sum_{i=1}^{p-1} (-x)^i (2y)^{p-i} \pmod{p} \right).$$

Using the 2-cochain $\chi$, we define a 3-cocycle $\theta \in Z^3_Q(R_p; \mathbb{Z}_p)$, known as Mochizuki’s 3-cocycle [20], and 4-cocycles $\psi_0, \psi_1 \in Z^4_Q(R_p; \mathbb{Z}_p)$ given in [23, Example 5.9] by

$$\theta(x, y, z) = (x - y) \cdot \chi(y, z),$$

$$\psi_0(x, y, z, w) = -\theta(x - w, y - w, z - w),$$

$$\psi_1(x, y, z, w) = \chi(z - x, y - x) \cdot \chi(z, w),$$

where this explicit expression for $\theta$ was that simplified in [2]. Then it is proved in [20] that the cohomology class $[\theta]$ generates $H^3_Q(R_p; \mathbb{Z}_p) \cong \mathbb{Z}_p$, and in [23] that the set of the cohomology classes $[\psi_0]$ and $[\psi_1]$ generates $H^4_Q(R_p; \mathbb{Z}_p) \cong \mathbb{Z}_p^2$. By a direct computation, we have $\theta(z) = -2, \psi_0(w) = -\theta(-z) = -2,$ and hence the following.

**Proposition 6.3.** The homology class $[z]$ is a generator of $H^3_Q(R_p) \cong \mathbb{Z}_p$ and $[w]$ is that of $H^4_Q(R_p) \cong \mathbb{Z}_p$.

**Remark 6.4.** Let $z_0 = \sum_{i=1}^{p-1} (0, i, i + 1)$ be a 3-cycle in $C^3_Q(R_p)$, where this cycle is obtained from a diagram of the $(2, p)$-torus knot with a shadow $R_p$-coloring. It is known that $[z_0]$ is also a generator of $H^3_Q(R_p) \cong \mathbb{Z}_p$ and $[z] = 2[z_0]$, since $\theta(z_0) = -1$.

---

1There is a minor typo: the term “$(2z - w)$” should be replaced by “$(2w - z)$” in his formula of $\psi_1$. We note that the symbols $\psi_{4,0}$ and $\psi_{4,1}$ are used in [20] instead of $\psi_0$ and $\psi_1$. 
Theorem 6.5. The map $\sigma_4 : H_1^Q(R_p) \to H_3^Q(R_p)$ is an isomorphism $\mathbb{Z}_p \to \mathbb{Z}_p$ such that $1 \mapsto -1$.

Proof. By a direct computation, we have $\sigma(w) = -z$ at the chain level. □

Proposition 6.6. The map $\sigma^*: H_3^Q(R_p; \mathbb{Z}_p) \to H_3^Q(R_p; \mathbb{Z}_p)$ is an injective homomorphism $\mathbb{Z}_p \to \mathbb{Z}_2^2$ such that $1 \mapsto (-1, n)$ for some $n \in \mathbb{Z}_p$.

Proof. By a direct computation, we have $\sigma^*(\theta)(w) = \theta(\sigma(w)) = \theta(-z) = 2$. Using $\chi(x, 0) = 0 \in \mathbb{Z}_p$ for any $x \in R_p$, we have $\psi_1(w) = 0$. Combined these with $\psi_0(w) = -2$, we have $\sigma^*(\theta) = -[\psi_0] + n[\psi_1]$ for some $n \in \mathbb{Z}_p$. □

Problem 6.7. Determine $\sigma^*: H_3^Q(R_p; \mathbb{Z}_p) \to H_3^Q(R_p; \mathbb{Z}_p)$ explicitly.

6.3. Tetrahedral quandle. The tetrahedral quandle $S_4$ is defined to be the set of vertices of a regular tetrahedron, denoted by $\{0, 1, 2, 3\}$, with the binary operation

\[
\begin{align*}
0 * 0 &= 1 * 2 = 2 * 3 = 3 * 1 = 0, \\
0 * 1 &= 1 * 3 = 2 * 0 = 3 * 2 = 1, \\
0 * 2 &= 1 * 0 = 2 * 1 = 3 * 3 = 3.
\end{align*}
\]

The inner automorphism $s_x : S_4 \to S_4$ for $x \in S_4$ can be regarded as a counterclockwise rotation of the tetrahedron by the angle $2\pi/3$ looking at the bottom face from the vertex $x$. This quandle $S_4$ is known to be connected. It is known in [3, 11] that

\[H_2^Q(S_4) \cong \mathbb{Z}_2, \ H_3^Q(S_4) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4\]

and

\[H_2^Q(S_4; \mathbb{Z}_4) \cong \mathbb{Z}_2, \ H_3^Q(S_4; \mathbb{Z}_4) \cong \mathbb{Z}_2^2 \oplus \mathbb{Z}_4(= (\mathbb{Z}_2 \oplus \mathbb{Z}_4) \oplus \mathbb{Z}_2),\]

where $H_3^Q(S_4; \mathbb{Z}_4) \cong \text{Hom}(H_2^Q(S_4), \mathbb{Z}_4) \oplus \text{Ext}(H_2^Q(S_4), \mathbb{Z}_4) \cong (\mathbb{Z}_2 \oplus \mathbb{Z}_4) \oplus \mathbb{Z}_2$. The first non-trivial shifting maps for (co)homology groups of $S_4$ might be

\[\sigma_4 : H_3^Q(S_4) \to H_2^Q(S_4) \quad \text{and} \quad \sigma^*: H_2^Q(S_4; \mathbb{Z}_4) \to H_3^Q(S_4; \mathbb{Z}_4),\]

which we are going to study.

Let $z_1, z_2, w_1, w_2$ be cycles in $Z_3^Q(S_4)$ defined by

\[
\begin{align*}
z_1 &= +(0, 3) + (3, 1) + (1, 0) \quad \in Z_2^Q(S_4), \\
z_2 &= +(0, 3) - (0, 1) + (3, 0) - (3, 2) \quad \in Z_2^Q(S_4), \\
w_1 &= +(0, 3, 1) + (0, 1, 0) \quad \in Z_3^Q(S_4), \\
w_2 &= -(1, 0, 1) + (0, 3, 0) - (0, 3, 2) \quad \in Z_3^Q(S_4),
\end{align*}
\]

where cycles $z_1, w_1$ and $z_2, w_2$ are obtained from diagrams of the trefoil knot and the figure-eight knot with (shadow) $S_4$-colorings, respectively. Given an element $x \in S_4^n$, let $\chi_x : S_4^n \to \mathbb{Z}_4$ be an $n$-cochain in $C_4^n(S_4; \mathbb{Z}_4)$ defined by

\[
\chi_x(y) = \begin{cases} 1 & (y = x) \\ 0 & (y \neq x) \end{cases}
\]
for each $y \in S_4^3$. Using the above $n$-cochains, we define a 2-cocycle $\phi \in Z^2_Q(S_4; \mathbb{Z}_4)$ and 3-cocycles $\eta_1, \eta_2, \eta_{11} \in Z^3_Q(S_4; \mathbb{Z}_4)$ by

$$
\phi = 2(\chi(0,1) + \chi(0,2) + \chi(1,0) + \chi(1,2) + \chi(2,0) + \chi(2,1)),
$$

$$
\eta_1 = 2(\chi(0,1,0) + \chi(0,2,1) + \chi(0,2,3) + \chi(0,3,0) + \chi(0,3,1) + \chi(0,3,2) + \chi(1,0,1) + \chi(1,0,3) + \chi(1,1,0) + \chi(1,1,3) + \chi(2,0,3) + \chi(2,1,0) + \chi(2,1,3) + \chi(2,3,2)),
$$

$$
\eta_2 = +\chi(0,1,2) - \chi(0,1,3) - \chi(0,2,1) + \chi(0,3,0) + \chi(0,3,1) - \chi(0,3,2) + 2\chi(1,0,1) + \chi(1,0,3) - \chi(1,2,0) + \chi(1,3,2) + \chi(2,0,1) + \chi(2,0,2) + \chi(2,0,3) + \chi(2,1,3) + \chi(2,3,0) + \chi(3,0,3) + \chi(3,1,3),
$$

$$
\eta_{11} = -\chi(0,1,0) - \chi(0,1,3) + \chi(0,3,1) + \chi(0,3,2) - \chi(1,0,1) - \chi(1,0,2) - \chi(1,0,3) + \chi(1,2,0) - \chi(1,2,1) + \chi(1,3,0) + \chi(1,3,1) + \chi(1,3,2) + \chi(2,0,3) - \chi(2,1,0) - \chi(3,0,2) + \chi(3,2,3),
$$

where the 2-cocycle and 3-cocycles were given in the proofs of Lemma 6.8 and 6.12, respectively, in the 2nd version of pre-publication paper of the arXiv. The explicit expressions for these cocycles are described in [4, Example 8.11] and [5, p.64]. Then it is known that the cohomology class $[\phi]$ generates $H^3_Q(S_4; \mathbb{Z}_4) \cong \mathbb{Z}_2$, and that the (ordered) set of the cohomology classes $[\eta_1], [\eta_2]$ and $[\eta_{11}]$ generates $H^3_Q(S_4; \mathbb{Z}_4) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$ in this order; see [4, 5].

**Proposition 6.8.** We have $[z_1] = [z_2]$ and this generates $H^2_Q(S_4) \cong \mathbb{Z}_2$.

**Proof.** Since $\phi(z_1) = \phi(z_2) = 2 \in \mathbb{Z}_4$, $[z_1]$ and $[z_2]$ are non-zero in $H^2_Q(S_4) \cong \mathbb{Z}_2$. □

**Proposition 6.9.** A set of homology classes $[w_1]$ and $[w_2]$ generates $H^2_Q(S_4) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$. Moreover, $[w_1]$ and $[w_2]$ have order 4 and 2, respectively.

**Proof.** It suffices to prove that $[w_1]$ has order 4, $[w_2]$ has order 2, and $[w_2] \neq 2[w_1]$. It follows from $\eta_2(w_1) = 1 \in \mathbb{Z}_4$ that $[w_1]$ has order 4 and $\eta_2$ sends any element of order 4 to a non-zero element in $\mathbb{Z}_4$. Since $\eta_2(w_2) = 0 \in \mathbb{Z}_4$, $[w_2]$ does not have order 4 and $[w_2] \neq 2[w_1]$. It follows from $\eta_1(w_2) = 2 \in \mathbb{Z}_4$ that $[w_2]$ has order 2. □

**Remark 6.10.** Computations similar to ours in the above proof have been appeared in the proof of [22, Proposition 4.5], where he attempted to compute $\pi_2(BS_4)$ of the quandle space $BS_4$ of $S_4$. We note that $\pi_2(BS_4)$ has been completely computed in his subsequent paper [24].

**Theorem 6.11.** The map $\sigma_* : H^3_Q(S_4) \rightarrow H^2_Q(S_4)$ is a surjective homomorphism $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \rightarrow \mathbb{Z}_2$ such that $1 \mapsto 1$ and $(0, 1) \mapsto 1$.

**Proof.** By a direct computation, we have $\sigma(w_1) = z_1$ at the chain level. It follows from $\phi(\sigma(w_2)) = 2 \in \mathbb{Z}_4$ that $[\sigma(w_2)]$ is non-zero in $H^2_Q(S_4) \cong \mathbb{Z}_2$. □

**Remark 6.12.** Although $\phi(w_2) \neq z_2$ at the chain level, we have $\phi([w_2]) = [z_2]$.

**Proposition 6.13.** The map $\sigma^* : H^3_Q(S_4; \mathbb{Z}_4) \rightarrow H^3_Q(S_4; \mathbb{Z}_4)$ is an injective homomorphism $\mathbb{Z}_2 \rightarrow (\mathbb{Z}_2 \oplus \mathbb{Z}_4) \oplus \mathbb{Z}_2$ such that $1 \mapsto (1, 2, n)$ for some $n \in \mathbb{Z}_2$.

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[2] Not only the proofs, but also the lemmas themselves have been deleted in the published version. We note that our $\phi : S_4^2 \rightarrow \mathbb{Z}_4$ is twice their $\phi_2 : S_4^3 \rightarrow \mathbb{Z}_2$, and that our $\eta_1 : S_4^1 \rightarrow \mathbb{Z}_4$ is twice their $\eta_1 : S_4^3 \rightarrow \mathbb{Z}_2$. 
Proof. By direct computations, we have \( \sigma^\# (\phi(w_1)) = \phi(\sigma(w_1)) = \phi(z_1) = 2 \) and \( \sigma^\# (\phi(w_2)) = \phi(\sigma(w_2)) = 2 \) in \( \mathbb{Z}_4 \). We also have \( \eta_1(w_1) = 0, \eta_1(w_2) = 2, \eta_2(w_1) = 1, \eta_2(w_2) = 0 \) and \( \eta_1(w_1) = \eta_1(w_2) = 0 \) in \( \mathbb{Z}_4 \). With these computations in hand, we have \( \sigma^*([\phi]) = [\eta_1] + 2[\eta_2] + n[\eta_1] \) for some \( n \in \mathbb{Z}_2 \). \qed

**Problem 6.14.** Determine \( \sigma^*: H^2_Q(S_4; \mathbb{Z}_4) \to H^3_Q(S_4; \mathbb{Z}_4) \) explicitly.

**Appendix A. Fundamental quandle and fundamental class**

The fundamental classes were explicitly written down for classical links \([11]\) and surface links \([26]\). However original quandle homology theory was used in \([11]\), and the notational convention used in \([26]\) was based on the original one \([3]\) rather than ours \([17]\). Therefore we review the fundamental classes and also fundamental quandles for completeness.

Throughout this appendix, we rely on readers’ familiarity with basic terminology of diagrams for classical links and surface links. Let \( D \) be a diagram of an oriented classical link (or surface link), and \( A_D = \{a_1, \ldots, a_n\} \) the set of all arcs (or sheets) of \( D \). We assume that each arc (or sheet) of \( D \) is given a normal vector to indicate the orientation of the link represented by \( D \). The fundamental quandle \( Q_D \) of \( D \) is a quandle generated by \( A_D \) with the following defining relations. At each crossing (or double point curve), let \( a_j \) be the over arc (or sheet), and let \( a_i \) and \( a_k \) be the under arcs (or sheets) such that the normal vector of \( a_j \) points from \( a_i \) to \( a_k \). Then the defining relation is given by \( a_i \ast a_j = a_k \) at the crossing (or double point curve). Hereafter, we basically regard an element of \( A_D \) as the element of the fundamental quandle \( Q_D \).

Let \( R_D \) be the set of all (complementary) regions of \( D \) in \( \mathbb{R}^2 \) (or \( \mathbb{R}^3 \)), and \( r_\infty \in R_D \) the unbounded region of \( D \). Then it is easy to see that there exists a unique map, temporarily denoted by \( c^R \) in this paragraph, from \( R_D \) to \( As(Q_D) \) such that

\[
c^R(r_\infty) = e \quad \text{and} \quad c^R(r_1) \cdot a = c^R(r_2)
\]

for each arc (or sheet) \( a \in A_D \), where \( r_1 \) and \( r_2 \) are the regions such that the normal vector of \( a \) points from \( r_1 \) to \( r_2 \). Hereafter, we basically regard an element of \( R_D \) as the element of the associated group \( As(Q_D) \) through the map \( c^R \).

Let \( \tau \) be a crossing (or triple point) of \( D \). The sign \( \varepsilon(\tau) \in \{+1, -1\} \) is determined by whether \( \tau \) is positive (+) or negative (−). Among four (or eight) regions around \( \tau \), there is a unique region such that normal vectors of all of the arcs (or sheets) facing the regions point from the region. We call the region the specified region of \( \tau \).

**A.1. Fundamental class of a classical link.** Let \( D \) be a diagram of an oriented classical link \( L \). For a crossing \( \tau \) of \( D \), the weight \( B(\tau) \) is defined by

\[
\varepsilon(\tau)(r; x, y) \in C_2^Q(Q_D)_{As(Q_D)},
\]

where \( r \) is the specified region of \( \tau \), \( x \) is the under arc facing \( r \), and \( y \) is the over arc at \( \tau \). Let \( |D| \in C_2^Q(Q_D)_{As(Q_D)} \) be the sum of the elements \( B(\tau) \) of all crossings of \( D \). We can check that \( |D| \in Z_2^Q(Q_D)_{As(Q_D)} \) and define the fundamental class \( [D] \in H_2^Q(Q_D)_{As(Q_D)} \) by its homology class. Then this class is independent of the choice of the diagram of \( L \) in the following sense.
Theorem A.1. For any other diagram \( D' \) of the oriented classical link \( L \), there exists a quandle isomorphism \( \alpha: Q_D \to Q_{D'} \) such that \( \alpha([D]) = [D'] \), where \( \alpha: H_2^Q(Q_D)_{Asl(Q_D)} \to H_2^Q(Q_{D'})_{Asl(Q_{D'})} \) is the induced isomorphism.

A.2. Fundamental class of a surface link. Let \( D \) be a diagram of an oriented surface link \( \mathcal{L} \). For a triple point \( \tau \) of \( D \), the weight \( B(\tau) \) is defined by

\[
\varepsilon(\tau)(r; x, y, z) \in C_3^Q(Q_D)_{Asl(Q_D)},
\]

where \( r \) is the specified region of \( \tau \), \( x \) is the bottom sheet facing \( r \), \( y \) is the middle sheet facing \( r \) and \( z \) is the top sheet at \( \tau \). Let \( |D| \in C_3^Q(Q_D)_{Asl(Q_D)} \) be the sum of the elements \( B(\tau) \) of all triple points of \( D \). We can check that \( |D| \in Z_3^Q(Q_D)_{Asl(Q_D)} \) and define the fundamental class \( [D] \in H_3^Q(Q_D)_{Asl(Q_D)} \) by its homology class. Then this class is independent of the choice of the diagram of \( \mathcal{L} \) in the following sense.

Theorem A.2. For any other diagram \( D' \) of the oriented surface link \( \mathcal{L} \), there exists a quandle isomorphism \( \alpha: Q_D \to Q_{D'} \) such that \( \alpha([D]) = [D'] \), where \( \alpha: H_3^Q(Q_D)_{Asl(Q_D)} \to H_3^Q(Q_{D'})_{Asl(Q_{D'})} \) is the induced isomorphism.

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References

1. Nicolás Andruskiewitsch and Matías Graña, From racks to pointed Hopf algebras, Adv. Math. 178 (2003), no. 2, 177–243. MR 1994219
2. Soichiro Asami and Shin Satoh, An infinite family of non-invertible surfaces in 4-space, Bull. London Math. Soc. 37 (2005), no. 2, 285–296. MR 2119028
3. J. Scott Carter, Mohamed Elhamdadi, Matías Graña, and Masahico Saito, Cocycle knot invariants from rack modules and generalized quandle homology, Osaka J. Math. 42 (2005), no. 3, 499–541. MR 2166720
4. J. Scott Carter, Daniel Jelsovsky, Seiichi Kamada, Laurel Langford, and Masahico Saito, Quandle cohomology and state-sum invariants of knotted curves and surfaces, Trans. Amer. Math. Soc. 355 (2003), no. 10, 3947–3989. MR 1990571
5. J. Scott Carter, Daniel Jelsovsky, Seiichi Kamada, and Masahico Saito, Computations of quandle cocycle invariants of knotted curves and surfaces, Adv. Math. 157 (2001), no. 1, 36–94. MR 1808844
6. , Quandle homology groups, their Betti numbers, and virtual knots, J. Pure Appl. Algebra 157 (2001), no. 2-3, 135–155. MR 1812049
7. , Shifting homomorphisms in quandle cohomology and skeins of cocycle knot invariants, J. Knot Theory Ramifications 10 (2001), no. 4, 579–596. MR 1831677
8. J. Scott Carter, Seiichi Kamada, and Masahico Saito, Geometric interpretations of quandle homology, J. Knot Theory Ramifications 10 (2001), no. 3, 345–386. MR 1825963
9. , Diagrammatic computations for quandles and cocycle knot invariants, Diagrammatic morphisms and applications (San Francisco, CA, 2000), Contemp. Math., vol. 318, Amer. Math. Soc., Providence, RI, 2003, pp. 51–74. MR 1973510
10. Frans Claesens, The algebra of rack and quandle cohomology, J. Knot Theory Ramifications 20 (2011), no. 11, 1487–1505. MR 2854230
11. Michael Eisermann, *Homological characterization of the unknot*, J. Pure Appl. Algebra 177 (2003), no. 2, 131–157. MR 1954330
12. Roger Fenn and Colin Rourke, *Racks and links in codimension two*, J. Knot Theory Ramifications 1 (1992), no. 4, 343–406. MR 1194995
13. Roger Fenn, Colin Rourke, and Brian Sanderson, *Trunks and classifying spaces*, Appl. Categ. Structures 3 (1995), no. 4, 321–356. MR 1364012
14. Ayumu Inoue and Yuichi Kabaya, *Quandle homology and complex volume*, Geom. Dedicata 171 (2014), 265–292. MR 3226796
15. David Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra 23 (1982), no. 1, 37–65. MR 638121
16. Seiichi Kamada, *Wirtinger presentations for higher-dimensional manifold knots obtained from diagrams*, Fund. Math. 168 (2001), no. 2, 105–112. MR 1852735
17. Surface-knots in 4-space, Springer Monographs in Mathematics, Springer, Singapore, 2017, An introduction. MR 3588325
18. R. A. Litherland and Sam Nelson, *The Betti numbers of some finite racks*, J. Pure Appl. Algebra 178 (2003), no. 2, 187–202. MR 1952425
19. S. V. Matveev, *Distributive groupoids in knot theory*, Mat. Sb. (N.S.) 119(161) (1982), no. 1, 78–88, 160. MR 672410
20. Takuro Mochizuki, *Some calculations of cohomology groups of finite Alexander quandles*, J. Pure Appl. Algebra 179 (2003), no. 3, 287–330. MR 1960136
21. M. Niebrzydowski and J. H. Przytycki, *Homology of dihedral quandles*, J. Pure Appl. Algebra 213 (2009), no. 5, 742–755. MR 2494367
22. Takefumi Nosaka, *On homology groups of quandle spaces and the quandle homotopy invariant of links*, Topology Appl. 158 (2011), no. 8, 996–1011. MR 278669
23. On quandle homology groups of Alexander quandles of prime order, Trans. Amer. Math. Soc. 365 (2013), no. 7, 3413–3436. MR 3042590
24. Homotopical interpretation of link invariants from finite quandles, Topology Appl. 193 (2015), 1–30. MR 335078
25. Józef H. Przytycki and Witold Rosicki, *Cocycle invariants of codimension 2 embeddings of manifolds*, Knots in Poland III. Part III, Banach Center Publ., vol. 103, Polish Acad. Sci. Inst. Math., Warsaw, 2014, pp. 251–289. MR 3363816
26. Kokoro Tanaka, *On surface-links represented by diagrams with two or three triple points*, J. Knot Theory Ramifications 14 (2005), no. 8, 963–978. MR 2196624

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