The value of power-related options under spectrally negative Lévy processes

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Abstract We provide analytical tools for pricing power options with exotic features (capped or log payoffs, gap options ...) in the framework of exponential Lévy models driven by one-sided stable or tempered stable processes. Pricing formulas take the form of fast converging series of powers of the log-forward moneyness and of the time-to-maturity; these series are obtained via a factorized integral representation in the Mellin space evaluated by means of residues in \( \mathbb{C} \) or \( \mathbb{C}^2 \). Comparisons with numerical methods and efficiency tests are also discussed.

Keywords Lévy Process · Stable Distribution · Tempered Stable Distribution · Digital option · Power option · Gap option · Capped option

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1 Introduction

Spectrally negative Lévy processes are Lévy processes (see the classical textbook \[7\] for a complete introduction to the theory of Lévy processes, and, among many other references, \[21,13,41\] for their applications in financial modeling) whose Lévy measure is supported by the real negative axis, i.e., processes without positive jumps \[28\]: they include Brownian motion with drift, asymmetric \(\alpha\)-stable \[11,12\] or asymmetric tempered-stable \[37\] processes and their particular cases, such as negative Gamma and Inverse Gamma processes. Such one-sided processes have been shown to be effective for modeling the price of financial assets, because their heavy-tail induces a leptokurtosis in the distribution of returns (whose empirical evidence is known since \[17\]), and their skewed behavior traduces the asymmetry in the occurrence of upward
and downward jumps (see [12,32,15] for more recent discussions and justifications). Moreover, in the context of exponential market models [41], they generate a wide range of dynamics for the log returns, from almost surely continuous trajectories in the Brownian motion case [9], to highly discontinuous realizations with a potentially infinite number of downward jumps on any given time interval.

For the specific purpose of option pricing, spectrally negative Lévy processes have been introduced in [12] in the case of a totally skewed $\alpha$-stable dynamics, the strong asymmetry of the model combined with the presence of fat tails capturing volatility patterns for longer observable horizons more accurately than Gaussian models. They have subsequently been employed in the calibration of index options on some major equity indices (it is shown in [15] that positive jumps are not needed for long term options on most index markets); concerning path-dependent instruments, the impact of one-sided Lévy dynamics on Asian and Barrier options has also been investigated [36,4]. Let us also mention that spectrally negative Lévy processes have been successfully applied in other areas of Quantitative Finance, notably in default modeling and credit exposure [32], as the default of a firm is often linked to brutal losses in their assets’ value.

When it comes to practical evaluation however, things are more complicated under a Lévy dynamics than in the usual Black-Scholes framework; the literature is dominated by numerical (finite difference) schemes for Partial Integro-Differential Equations [14], by Monte Carlo simulations [37] or Fourier transforms of option prices [10]. The latter approach is particularly popular, because in most exponential Lévy models, the characteristic function of the asset’s log price is available in a closed and relatively compact form; several refinements of the method have been introduced to accelerate the evaluation of Fourier integrals, notably by means of other integral transforms (among others, Fourier-cosine transform [18] or Hilbert transform [19]) or, more recently, by application of frame duality properties [26].

In this paper, we would like to take profit of the properties of another Fourier-related transform, namely the Mellin transform [20]. First, let us mention that the Mellin transform has been previously implemented in many areas of financial modeling, from providing representations for vanilla or basket options in the Black-Scholes model [34], to quantifying the at-the-money implied volatility slope in various Lévy models [22]. In our approach, we will focus on expressing Mellin integrals as a sum of residues in $\mathbb{C}$ or $\mathbb{C}^2$, so as to obtain simple series expansions for option prices. More precisely, we will show that, in the framework of exponential Lévy models driven by spectrally negative processes, option prices have a factorized form in the Mellin space (in terms of maturity and log-forward moneyness); inverting the transform, the prices can be conveniently computed by a straightforward series of residues, allowing for a very simple and fast evaluation of the options.

The Mellin residue technique has been used to derive fast convergent series for European options prices and Greeks, in the Black-Scholes [2] and FMLS [3] models; in this article, we will show that the technique successfully applies
to a more general range of exotic power-related options (Digital, Log, Gap, European with cap . . .). This family of options offers a higher (and nonlinear) payoff than the vanilla options, and thus increases the leverage ratio of the strategies. In the Gaussian context, closed formulas for pricing and hedging standard power options are known since [24], and have been recently generalized to include some barrier features [25]; studies have also been made in the setup of local volatility models, or for more generic polynomial options (decomposed a sum of power options) in [30]. The present paper will be devoted to establishing efficient pricing formulas in the context of an asymmetric α-stable exponential Lévy model, and to show that it is possible to extend them to the more generic class of tempered stable processes.

The paper is organised as follows: in section 2 we start by recalling some basic facts about option pricing in exponential Lévy models; then, in section 3 we establish a factorized form for option prices in the case of a spectrally negative α-stable dynamics. This factorized form enables us to derive several pricing formulas for power-related instruments in section 4, under the form of fast convergent series of powers of the time-to-maturity and of the moneyness; in this section, we also test the results numerically, and provide efficiency tests. In section 5 we show that similar formulas can also be derived if the stable distribution is tempered, and study the impact of the tempering parameter in the case of a digital option. Finally, section 6 is devoted to concluding remarks and perspectives.

2 Option pricing in exponential Lévy Models

2.1 Model specification

Notations. Given a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\), recall that a process \(\{X_t\}_{t \geq 0}\) is a Lévy process [7,29] if there exists a triplet \((a, b, \nu)\) such that the characteristic exponent \(\Psi(k) := -\log \mathbb{E}[e^{ikX_1}]\) of \(X_t\) admits the representation:

\[
\Psi(k) = iak + \frac{1}{2}bk^2 + \int_{\mathbb{R}} (1 - e^{ikx} + ikx 1_{\{|x|<1\}}) \nu(dx)
\]

(1)

where \(a, b \in \mathbb{R}\) and \(\nu\) is a measure concentrated on \(\mathbb{R}\setminus\{0\}\) satisfying

\[
\int_{\mathbb{R}} \min(1, x^2) \nu(dx) < \infty
\]

(2)

is known as the Lévy-Khintchine formula; \(a\) is the drift, \(b\) is the Brownian (or diffusion) coefficient and \(\nu\) is the Lévy measure of the process.

If \(\nu(\mathbb{R}) < \infty\), one speaks of a process with finite activity or intensity; this corresponds to processes whose realizations have a finite number of jumps on every finite interval, like in jump-diffusion models such as the Merton model [33] or the Kou model [27]. If \(\nu(\mathbb{R}) = \infty\), then one speaks of a process with infinite activity or intensity, and in this case an infinite number of jumps occur
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on every finite interval; this gives birth to a very rich dynamics and such processes do not need Brownian component to generate complex behaviors. When furthermore \( \nu(\mathbb{R}^+) = 0 \) (resp. \( \nu(\mathbb{R}^-) = 0 \)), the process is said to be spectrally negative (resp. spectrally positive).

As a Lévy process has stationary independent increments, its characteristic function can be written down as

\[
F[X_t](k) := \mathbb{E}^P[e^{ikX_t}] = e^{-t\Psi(k)}
\]  

(3)

and its moment generating function, whenever it converges, as:

\[
M[X_t](p) := \mathbb{E}^P[e^{pX_t}] = e^{t\phi(p)} , \quad \phi(p) = -\Psi(-ip)
\]  

(4)

The function \( \phi(p) \) is the Laplace exponent or cumulant generating function of the process, and its existence depends on the asymptotic behavior of the Lévy measure; in particular, in the case of a spectrally negative process, the absence of positive fat tail ensures that \( \phi(p) \) exists in the whole complex half-plane \( \{Re(p) > 0\} \).

**Exponential processes.** Let \( T > 0 \), and let \( S_t \) denote the value of a financial asset at time \( t \in [0, T] \); we assume that it can be modeled as the realization of a stochastic process \( \{S_t\}_{t \geq 0} \) on the canonical space \( \Omega = \mathbb{R}^+ \) equipped with its natural filtration, and that, under the risk-neutral measure \( Q \), its instantaneous variations can be written down in local form as:

\[
\frac{dS_t}{S_t} = (r - q) dt + dX_t
\]  

(5)

In the stochastic differential equation (5), \( r \in \mathbb{R} \) is the continuously compounded risk-free interest rate and \( q \in \mathbb{R} \) is the dividend yield, both assumed to be deterministic, and \( \{X_t\}_{t \geq 0} \) is a Lévy process; for the simplicity of notations, we will assume that \( q = 0 \), but all the results of the paper remain valid when replacing \( r \) by \( r - q \).

The solution to (5) is the exponential Lévy process [11] defined by:

\[
S_T = S_t e^{(r+\mu)\tau + X_{\tau}}
\]  

(6)

where \( \tau := T - t \) is the horizon (or time-to-maturity), and \( \mu \) is a convexity adjustment computed in a way that the discounted stock price is a \( Q \)-martingale, which resumes to the condition:

\[
\mathbb{E}^Q \left[e^{\mu\tau + X_{\tau}}\right] = 1
\]  

(7)

or, equivalently, in terms of Laplace exponent:

\[
\mu = -\phi(1)
\]  

(8)
2.2 Option pricing

Let \( N \in \mathbb{N} \) and \( \mathcal{P} : \mathbb{R}^{1+N}_+ \to \mathbb{R} \) be a non time-dependent payoff function depending on the terminal price \( S_T \) and on some positive parameters \( K_n, \ n = 1 \ldots N \):

\[
\mathcal{P} : (S_T, K_1, \ldots, K_N) \to \mathcal{P}(S_T, K) := \mathcal{P}(S_T, K)
\]

The value at time \( t \) of an option with maturity \( T \) and payoff \( \mathcal{P}(S_T, K) \) is equal to the risk-neutral conditional expectation of the discounted payoff:

\[
C(S_t, K, \mu, \tau) = \mathbb{E}[^{Q_t}[e^{-r\tau}\mathcal{P}(S_T, K)]]
\]

In the case where the \( \text{Lévy process admits a} \ Q \)-density \( g(x, t) \), then, using (6), we can re-write (10) by integrating all possible realizations for the terminal payoff over the martingale measure:

\[
C(S_t, K, \mu, \tau) = e^{-r\tau} \int_{-\infty}^{\infty} \mathcal{P}(S_t e^{(r+\mu)\tau+x}, K) g(x, \tau) \, dx
\]

In all the following and to simplify the notations, we will forget the \( t \) dependence in the stock price \( S_t \).

3 Spectrally negative \( \alpha \)-stable process (FMLS process)

3.1 Lévy-stable process

A \textit{Lévy-stable} process \cite{39, 43} is a Lévy process whose Lévy-Khintchine triplet has the form \((\alpha, 0, \nu_{\text{stable}})\), with:

\[
\nu_{\text{stable}}(x) = \frac{\gamma_-}{|x|^{1+\alpha}} \mathbb{1}_{\{x<0\}} + \frac{\gamma_+}{x^{1+\alpha}} \mathbb{1}_{\{x>0\}}
\]

where \( \alpha \in (0, 2) \) and \( \gamma_{\pm} \in \mathbb{R} \). It is known that, introducing \( \gamma \) and \( \beta \) defined by

\[
\begin{align*}
\gamma^\alpha &:= - (\gamma_+ + \gamma_-) \Gamma(-\alpha) \cos \frac{\pi \alpha}{2} \\
\beta &:= \frac{\gamma_+ - \gamma_-}{\gamma_+ + \gamma_-}
\end{align*}
\]

then for \( \alpha \in (0, 1) \cup (1, 2] \) the characteristic exponent of the process admits the parametrization:

\[
\Psi_{\text{stable}}(k) = \gamma^\alpha |k|^\alpha \left( 1 - i\beta \tan \frac{\alpha \pi}{2} \text{sgn} k \right) + i\eta k
\]

for some constant \( \eta \in \mathbb{R} \) (see, for instance, exercise 1.4 in the textbook \cite{29}). A Lévy-stable process can therefore be represented as a 4-parameter process.
$L(\eta, \gamma^\alpha, \beta)$: $\alpha$ controls the behavior of the tails and $\beta \in [-1, 1]$ their asymmetry, $\gamma$ is a scale parameter, and $\eta$ is a location parameter. In particular, when $\alpha \in (1, 2]$ then it follows from (14) that $\eta$ equals the mean $E^Q[X_t]$.

It is interesting to note that when $\alpha = 2$ and $\eta = 0$ then the characteristic function (14) degenerates into the characteristic function of the centered normal distribution:

$$L(0, \sigma^2, \beta) = N(0, (\sigma\sqrt{2})^2) \quad \forall \beta \in [-1, 1]$$

and therefore the Black-Scholes model is a particular case of a Lévy-stable model for $\alpha = 2$.

3.2 Fully asymmetric process

It follows from the definition of the Lévy measure (12), that the moment generating function $M[X_t](p)$ of a Lévy-stable process exists if and only if $\gamma + = 0$, or equivalently $\beta = -1$ that is, in the case of a spectrally negative Lévy-stable process, because it has only one fat-tail located in the real negative axis; one also speaks of a fully asymmetric process, and the condition $\beta = -1$ is known as the maximal negative asymmetry hypothesis. In this context, choosing $\eta = 0$ (process with zero mean), we have:

$$-\Psi_{\text{stable}}(-ip) = \gamma_- \int_{-\infty}^{0} (e^{px} - 1) \frac{dx}{|x|^{1+\alpha}} = \gamma_- \Gamma(-\alpha) p^\alpha = -\frac{\gamma^\alpha}{\cos \frac{\pi \alpha}{2}} p^\alpha$$

which is valid for $p > 0$. It follows from definition (8) that the convexity adjustment reads:

$$\mu = \frac{\gamma^\alpha}{\cos \frac{\pi \alpha}{2}}$$

It is in [12] that an exponential Lévy model (5) for a process $\{X_t\}_{t \geq 0}$ being a spectrally negative Lévy-stable process $L(0, \sigma^\alpha, -1)$ was first introduced for the purpose of option pricing. The authors gave it the name of Finite Moment Log Stable (FMLS) process, in reference to the existence of the cumulant generating function in this case. Note that the process has infinite activity, the integral of the stable measure being divergent in 0.

3.3 Self-similarity and option pricing

We now derive a Mellin-Barnes representation for the density of the FMLS process, and for the corresponding option price that we will denote by $C_\alpha$. 

Lemma 1 Let $\sigma > 0$, $\alpha \in (1, 2]$ and $X_1 \sim L(0, \sigma^\alpha, -1)$. Then the density $g_\alpha(x, t)$ of the process $\{X_t\}_{t \geq 0}$ admits the following Mellin-Barnes representation:

$$g_\alpha(x, t) = \frac{1}{\alpha x} \int_{c_1 - i\infty}^{c_1 + i\infty} \frac{\Gamma(1 - s_1)}{\Gamma(1 - \frac{1}{\alpha})} \left( \frac{x}{(-\mu t)^{\frac{1}{\alpha}}} \right)^{s_1} \frac{ds_1}{2i\pi}$$  \((18)\)

where $c_1 < 1$ and $\mu = -\frac{\sigma^\alpha}{\cos \frac{\pi \alpha}{2}}$.

Proof Using eqs. (16) and (17) and the Laplace inversion formula, we have:

$$g_\alpha(x, t) = \frac{1}{2\pi i} \int_{c_p - i\infty}^{c_p + i\infty} e^{-px} e^{-\mu t p^\alpha} dp$$  \((19)\)

where $c_p > 0$. Taking the Mellin transform and making the change of variables $p^\alpha \rightarrow p$, we have:

$$g_\alpha^*(s_1, t) := \int_0^\infty g(x, t)x^{s_1-1}dx = \frac{1}{\alpha} \Gamma(s) \frac{1}{2\pi i} \int_{c_p - i\infty}^{c_p + i\infty} e^{-\mu t p^\alpha} p^{\frac{1}{\alpha}} dp$$  \((20)\)

for any $s > 0$. The remaining $p$-integral is equal to $\frac{1}{\Gamma(1 + \frac{1}{\alpha})} (-\mu t)^{1-s_1}$ on the condition that $s > 1 - \alpha$ (see for instance [6] or any monograph on Laplace transform); observe that, as $\alpha \in (1, 2]$, the two conditions on $s$ resume to $s > 0$. Finally, the integral (18) is obtained by applying the Mellin inversion formula (20) and by changing the variable $s \rightarrow 1 - s$.

(18) shows that the density is a function of the ratio $x(-\mu t)^{\frac{1}{\alpha}}$, which is actually a consequence of the self-similarity property (13) of stable processes (a scaling of time is equivalent to an appropriate scaling of space). This property allows for a nice factorization of the option price in the Mellin space; indeed, let us denote

$$G_\alpha^*(s_1) := \frac{1}{\alpha} \frac{\Gamma(1 - s_1)}{\Gamma(1 - \frac{1}{\alpha})}$$  \((21)\)

and

$$K^*(s_1) := \int_{-\infty}^{+\infty} \mathcal{P} \left( Se^{(r+\mu)x} \right)^{s_1-1} dx$$  \((22)\)

and let us assume that the integral (22) converges for $s_1 \in (c_-, c_+)$ for some real numbers $c_- < c_+$. Then, as a direct consequence of the pricing formula (11) and of lemma (1) we have:
Proposition 1 (Factorization in the Mellin space) Let $c_1 \in (\tilde{c}_-, \tilde{c}_+)$ where $(\tilde{c}_-, \tilde{c}_+) := (c_-, c_+) \cap (-\infty, -1)$ is assumed to be nonempty. Then, under the hypothesis of lemma \[, the value at time $t$ of an option with maturity $T$ and payoff $P(S_T, K)$ is equal to:

$$C_\alpha(S, K, r, \mu, \tau) = e^{-r\tau} \int_{c_1-i\infty}^{c_1+i\infty} K^*(s_1)G_\alpha^*(s_1) (-\mu\tau)^{-\frac{s_1}{2\pi}} \frac{ds_1}{2i\pi} \quad (23)$$

The factorized form (23) turns out to be a very practical tool for option pricing. Indeed, as an integral along a vertical line in the complex plane, it can be conveniently expressed as a sum of residues associated to the singularities of the integrand. As Gamma function are involved, we can control the behavior of the integrand when the contour goes to infinity by using the Stirling asymptotic formula for the Gamma function \[: if $a_k, b_k, c_j, d_j$ are real numbers, if $\delta := \sum_k a_k - \sum_j c_j$ and if $\delta < 0$ then

$$\left| \frac{\Pi_k \Gamma(a_k s + b_k)}{\Pi_j \Gamma(c_j s + d_j)} \right| \to 0 \quad \text{as} \quad s \to \infty \quad (24)$$

when $\arg s \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and the same holds for $\arg s \in (\frac{\pi}{2}, \frac{3\pi}{2})$ if $\delta > 0$. Therefore, by right or left closing the contour of integration in (23), the option price will take the form of a series:

$$e^{-r\tau} \times \sum \left[ \text{residues of } K^*(s_1)G_\alpha^*(s_1) \times \text{powers of } (-\mu\tau)^{\frac{s_1}{2\pi}} \right] \quad (25)$$

The only technical difficulty will in fact lie in the evaluation of $K^*(s_1)$: depending on the payoff’s complexity, it can be either computed directly, or via the introduction of a second Mellin complex variable $s_2$.

4 Power payoffs in a spectrally negative $\alpha$-stable environment

In all this section, $\alpha \in (1, 2]$, $\sigma > 0$, $X_t \sim L(0, \sigma^\alpha, -1)$ and $u > 0$; the log-forward moneyness is defined to be:

$$k_u := \log \frac{S}{K_u} + r\tau \quad (26)$$

and we will use the standard notation $X^+ := X \mathbb{1}_{\{X > 0\}}$.

4.1 One complex variable payoffs

Digital power options (cash-or-nothing). The call’s payoff is:

$$P^{(C/N)}(S, K) := \mathbb{1}_{\{S^u - K > 0\}} \quad (27)$$
**Proposition 2** The value at time $t$ of a digital power cash-or-nothing call option is:

$$C_{\alpha}^{(C/N)}(S, K, r, \mu, \tau) = \frac{e^{-rt}}{\alpha} \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(1 - \frac{n}{\alpha})} (k_u + \mu \tau)^n (-\mu \tau)^{-\frac{n}{\alpha}}$$  \hspace{1cm} (28)

**Proof** As we can write:

$$P^{(C/N)}(S e^{(r+\mu)\tau + x}, K) = \mathbb{1}_{\{e^{u(k_u + \mu \tau + x)} > 1\}}$$

and

$$= \mathbb{1}_{\{x > -k_u - \mu \tau\}}$$  \hspace{1cm} (29)

then, with the notation (22), the $K^*(s_1)$ function reads:

$$K^*(s_1) = \frac{(-k_u - \mu \tau)^{s_1}}{s_1}$$  \hspace{1cm} (30)

for $s < -1$. Using proposition 1 and the functional relation $\Gamma(-s_1) = -s_1 \Gamma(1-s_1)$, the option price is:

$$C_{\alpha}^{(C/N)}(S, K, r, \mu, \tau) = \frac{e^{-rt}}{\alpha} \int_{c_1 - i\infty}^{c_1 + i\infty} \frac{\Gamma(-s_1)}{\Gamma(1 - \frac{s_1}{\alpha})} (-k_u - \mu \tau)^{s_1} (-\mu \tau)^{-\frac{s_1}{\alpha}} \frac{ds_1}{2\pi i}$$  \hspace{1cm} (31)

which converges for $s_1 < 0$. We can note that:

$$\delta = \frac{1}{\alpha} - 1$$  \hspace{1cm} (32)

is negative because $\alpha > 1$, thus, it follows from the Stirling formula (24) that the analytic continuation of the integrand vanishes at infinity in the right half plane. Therefore, the integral (31) equals the sum of residues at the poles located in this half plane; these poles are induced by the $\Gamma(-s_1)$ term at every positive integer $n$, and the associated residues are:

$$\frac{(-1)^n}{n!} \frac{1}{\Gamma(1 - \frac{n}{\alpha})} (-k_u - \mu \tau)^n (-\mu \tau)^{-\frac{n}{\alpha}}$$  \hspace{1cm} (33)

Simplifying and summing all residues yields (28).

**Log power options.** These options were introduced in [42] in the case $u = 1$, and are basically options on the rate of return of the underlying asset. The call’s payoff is:

$$P^{(Log)}(S, K) := \left[\log \left(\frac{S}{K}\right)^{u}\right]$$  \hspace{1cm} (34)

**Proposition 3** The value at time $t$ of a Log power call option is:

$$C_{\alpha}^{(Log)}(S, K, r, \mu, \tau) = \frac{ue^{-rt}}{\alpha} \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(1 + \frac{1-n}{\alpha})} (k_u + \mu \tau)^n (-\mu \tau)^{\frac{1-n}{\alpha}}$$  \hspace{1cm} (35)
Proof As we can write:

\[ P(\text{Log} (Se^{(r+\mu)\tau + x}, K) = u [k_u + \mu \tau + x]^+ \]  

(36)

then the \( K^*(s_1) \) function reads:

\[ K^*(s_1) = u \frac{(-k_u - \mu \tau)^{1+s_1}}{s_1(1+s_1)} \]  

(37)

for \( s < -1 \). Using proposition [1] and the functional relation \( \Gamma(-s_1) = -s\Gamma(1-s_1) \), the option price is:

\[ C(\text{Log}) \alpha(S, K, r, \mu, \tau) = e^{-r\tau} \alpha \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(1 - \frac{n}{\alpha})} ((k_u^- + \mu \tau)^n - (k_u^+ + \mu \tau)^n) (-\mu \tau)^{-\frac{n}{\alpha}} \]  

(38)

which converges for \( s_1 < -1 \). Again, \( \delta < 0 \), and the analytic continuation of the integrand in the right half-plane has:

- a simple pole in \( s_1 = -1 \) with residue

\[ \frac{(-\mu \tau)^{\frac{1}{\alpha}}}{\Gamma(1 + \frac{1}{\alpha})} \]  

(39)

- a series of poles at every positive integer \( s_1 = n \) with residues:

\[ -\frac{(-1)^n}{(n+1)! \Gamma(1 - \frac{n}{\alpha})} (-k_u - \mu \tau)^{1+n} (-\mu \tau)^{-\frac{n}{\alpha}} \]  

(40)

Summing the residues (39) and (40) for all \( n \) and re-ordering yields (35).

Capped power options (cash-or-nothing). For \( K_- < K_+ \), the call’s payoff is:

\[ P(\text{C/N,cap}) (S, K_+, K_-) := \mathbb{I}(K_- < S < K_+) \]  

(41)

Let us define \( k_u^\pm := \log \frac{S}{K_\pm} + r\tau \). We have:

Proposition 4 The value at time \( t \) of a capped cash-or-nothing call option is:

\[ C(\text{C/N,cap}) (S, K_+, K_-, r, \mu, \tau) = e^{-r\tau} \alpha \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(1 - \frac{n}{\alpha})} ((k_u^- + \mu \tau)^n - (k_u^+ + \mu \tau)^n) (-\mu \tau)^{-\frac{n}{\alpha}} \]  

(42)
Proof We can write:
\[
P^{(C/N, cap)}(S e^{(r+\mu)\tau + x}, K_+, K_-) = 1_{\{ -k_u^- - \mu \tau < x < -k_u^+ - \mu \tau \}}
\] (43)
and therefore the \(K^*(s_1)\) function reads:
\[
K^*(s_1) = \frac{(-k_u^+ - \mu \tau)^{s_1} - (-k_u^- - \mu \tau)^{s_1}}{s_1}
\] (44)

From proposition 1, the option price is:
\[
C^{(C/N, cap)}(S, K_+, K_-) = e^{-r\tau} \sum_{n=0}^{\infty} \frac{1}{n!} \Gamma(1 + \frac{m-n}{\alpha}) \frac{(k_u^- - \mu \tau)^n}{n!} (-\mu \tau)^{\frac{m-n}{\alpha}}
\] (45)

Like in proposition 2, summing all the residues associated to the poles of the \(\Gamma(-s_1)\) function yields the series (42).

4.2 Two complex variables payoffs

Digital power options (asset-or-nothing). The call's payoff is:
\[
P^{(A/N)}(A/N) = \begin{cases} S & \text{if } S > K \end{cases}
\] (46)

Proposition 5 The value at time \(t\) of a digital power asset-or-nothing call option is:
\[
C^{(A/N)}(S, K, r, \mu, \tau) = K e^{-r\tau} \sum_{n=0}^{\infty} \frac{1}{n!} \Gamma\left(1 + \frac{m-n}{\alpha}\right) u^n (k_u + \mu \tau)^n (-\mu \tau)^{\frac{m-n}{\alpha}}
\] (47)

Proof We can write:
\[
P^{(A/N)}(S e^{(r+\mu)\tau + x}, K) = K e^{u(k_u + \mu \tau + x)} 1_{\{x > -k_u - \mu \tau\}}
\] (48)

Introducing a Mellin-Barnes representation for the exponential term:
\[
e^{u(k_u + \mu \tau + x)} = \int_{c_2-i\infty}^{c_2+i\infty} (1) (-1)^{-s_2} u^{-s_2} \Gamma(s_2) (k_u + \mu \tau + x)^{-s_2} \frac{ds_2}{2\pi i}
\] (49)

for \(c_2 > 0\) and integrating over the \(x\) variable, the \(K^*(s_1)\) function reads:
\[
K^*(s_1) = \frac{(-k_u^+ - \mu \tau)^{s_1} - (-k_u^- - \mu \tau)^{s_1}}{s_1}
\] (50)
and converges for \((s_1, s_2)\) in the triangle \(\{\text{Re}(s_2) \in (0, 1), \text{Re}(s_1) < \text{Re}(s_2)\}\).

From proposition \(\square\) the option price is:

\[
C_{\alpha}^{(A/N)}(S, K, r, \mu, \tau) = \frac{K e^{-\tau}}{\alpha} \int_{c_1-i\infty}^{c_1+i\infty} \int_{c_2-i\infty}^{c_2+i\infty} (-1)^{-s_2} \frac{\Gamma(s_2) \Gamma(1 - s_2) \Gamma(-s_1 + s_2) \Gamma(1 - \frac{s_1}{2})}{\Gamma(1 - \frac{s_1}{2})} u^{-s_2} (-k_u - \mu \tau)^{s_1} (-\mu \tau)^{-\frac{s_1}{2}} \frac{ds_1 ds_2}{(2\pi)^2} \]

(51)

Poles of the integrand occur when \(\Gamma(s_2)\) and \(\Gamma(-s_1 + s_2)\) are singular; performing the change of variables \(-s_1 + s_2 \to U, s_2 \to V\) allows to compute the associated residues, which read:

\[
(-1)^m \frac{(1 - 1)^m}{n!} \frac{\Gamma(1 + m)}{\Gamma(1 + \frac{m - n}{\alpha})} u^m (-k_u - \mu \tau)^n (-\mu \tau)^{\frac{m - n}{\alpha}} \]

(52)

Simplifying and summing the residues yields the series \(\square\). The fact that one can close the \(C^2\) contour in (51) is a consequence of the multidimensional generalization of the Stirling estimate \(\square\) (see \(\square\) or the appendix of \(\square\) for details).

**Gap power options.** A gap option \(\square\), also called gap risk swap, offers a nonzero payoff on the condition that a trigger price is attained at \(t = T\). More precisely, the call’s payoff is:

\[
\mathcal{P}^{(\text{Gap})}(S, K_1, K_2) := (S_u - K_1) \mathbb{1}_{\{S_u - K_2 > 0\}} \]

(53)

where \(K_1\) is the trigger price and \(K_2\) the strike price; if the trigger is lower than the strike then a negative payoff is possible (which would not be the case with a classical knock-in barrier). From the definition of the payoff \(\square\), the value of the gap call option is equal to:

\[
C_{\alpha}^{(\text{Gap})}(S, K_1, K_2, r, \mu, \tau) = C_{\alpha}^{(A/N)}(S, K_2, r, \mu, \tau) - K_1 C_{\alpha}^{(C/N)}(S, K_2, r, \mu, \tau) \]

(54)

**European power options.** The classical European power option is a gap power option with equal strike and trigger prices \((K_1 = K_2 = K)\); observing that \(\square\) is actually a particular case of \(\square\) for \(m = 0\), it follows immediately from \(\square\) that the value of the European power call is:

\[
C_{\alpha}^{(E)}(S, K, r, \mu, \tau) = \frac{K e^{-r \tau}}{\alpha} \sum_{m=0}^{\infty} \frac{1}{m!} \Gamma(1 + \frac{m - n}{\alpha}) u^m (k_u + \mu \tau)^n (-\mu \tau)^{\frac{m - n}{\alpha}} \]

(55)
When the asset is at-the-money (ATM) forward, that is when $S = K^\frac{1}{\alpha}e^{-rt}$, or, equivalently, $k_u = 0$, then (55) becomes:

$$C_{\alpha}^{E,\text{ATM}}(S, K, r, \mu, \tau) = \frac{Ke^{-rt}}{1} \left[ u(-\mu \tau + \mu^2 \frac{\sqrt{\tau}}{\sqrt{2\pi}} - u(-\mu \tau + u^2 \frac{(-\mu \tau)}{2}) + O\left(u^2(-\mu \tau)^{1+\frac{1}{\alpha}}\right) \right]$$ \hspace{1cm} (56)

In particular, if we choose $\alpha = 2$ and the normalization $\gamma = \frac{\sigma}{\sqrt{2}}$ in the definition of the convexity adjustment (17), then (55) reads:

$$C_{2}^{E,\text{ATM}}(S, K, \sigma, \tau) = \frac{Ke^{-rt}}{2} \left[ 2u \frac{\sigma \sqrt{\tau}}{\sqrt{2\pi}} - u(1 - u) \frac{\sigma^2}{2} \tau + O\left(u^2(\sigma \sqrt{\tau})^{3}\right) \right]$$

$$= \frac{1}{\sqrt{2\pi}} S\sigma \sqrt{\tau} + O\left(\sigma \sqrt{\tau}\right)^{3}$$ \hspace{1cm} (57)

which is the well-known approximation for the ATM Black-Scholes call.

**Capped power options (asset-or-nothing, European).** For $K_- < K_+$, the payoff of a capped power asset-or-nothing call is:

$$P^{(A/N,\text{cap})}(S, K_+, K_-) := S^u \mathbb{1}_{K_- < S^u < K_+}$$ \hspace{1cm} (58)

The presence of a cap allows the seller to protect themselves against the eventuality of enormous payoffs; using the identity (43) for the indicator function, and proceeding in a similar way than for proving proposition 5, we obtain:

**Proposition 6** The value at time $t$ of a capped power asset-or-nothing call option is:

$$C_{\alpha}^{(A/N,\text{cap})}(S, K_+, K_-, r, \mu, \tau) = e^{-rt} S^u e^{u(r+\mu)\tau} \times$$

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-u)^m ((k_u^+ + \mu \tau)^{1+n+m} - (k_u^- + \mu \tau)^{1+n+m})}{(1 + n + m)!n!m! \Gamma \left(1 - \frac{1+\alpha}{\alpha}\right)} (-\mu \tau)^{-\frac{1+\alpha}{\alpha}}$$ \hspace{1cm} (59)

The value of the capped European power option is easily deduced from the values of the capped cash-or-nothing (42) and asset-or-nothing (59) options:

$$C_{\alpha}^{E/N,\text{cap}}(S, K_+, K_-, r, \mu, \tau) = C_{\alpha}^{(A/N,\text{cap})}(S, K_+, K_-, r, \mu, \tau) - K_- C_{\alpha}^{C/N,\text{cap}}(S, K_+, K_-, r, \mu, \tau)$$ \hspace{1cm} (60)

When $K_+ \to \infty$, the value of the capped option (60) coincides with the classical uncapped option (55); this situation is displayed in figure 1. We can observe that the convergence to the uncapped price is quicker when $\alpha$ decreases, which is no surprise given the overall $\frac{1}{\alpha}$ factor.
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**Fig. 1** Convergence of capped European call to the uncapped price when the cap $K_+$ goes to infinity, for different tail index parameters $\alpha$; when $\alpha$ decreases, the European (uncapped) price grows higher, given the presence of a left fat tail as soon as $\alpha < 2$. Parameters: strike $K = 4000$ and horizon $\tau = 2$ years; market parameters are set to $S = 4200$, $r = 1\%$ and $\sigma = 1\%$.

4.3 Numerical tests

In this subsection, we benchmark the pricing formulas established in the previous sections by comparing them with the formulas available in the cases $\alpha = 2$, $u = 1$ (i.e., in the Black-Scholes setup); we also provide comparisons with numerical evaluation of Fourier integrals when $\alpha \neq 2$. Except otherwise stated, we choose $r = 1\%$, $\sigma = 20\%$, $K = 4000$, $\tau = 2$ years and we make the normalization $\gamma = \frac{\sigma \sqrt{\tau}}{2}$ in the convexity adjustment (17), so as to recover the Black-Scholes adjustment $-\frac{\sigma^2 \tau}{2}$ when $\alpha = 2$.

**Log options** When $\alpha = 2$ and $u = 1$, a closed pricing formula exists for the Log option [23]:

$$C_{2}^{(\text{Log})}(S, K, r, \sigma, \tau) = e^{-r\tau} \left[ n(d_2)\sigma \sqrt{\tau} + d_2 N(d_2) \right], \quad d_2 := \frac{k - \frac{\sigma^2 \tau}{2}}{\sigma \sqrt{\tau}}$$

(61)

where $k := k_1$, $n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is the Gaussian density and $N(x)$ is the Normal cumulative distribution function. In table 1 we compare this formula to various truncations of the series (35) for $\alpha = 2$ and $u = 1$, in several market situations (out-of-the-money, at-the-money and in-the-money).

**Power options ($\alpha = 2$)** For $u > 0$, recall the formula by Heynen and Kat [24] for European power options in the Black-Scholes setup:

$$C_{2}^{(E)}(S, K, r, \sigma, \tau) = S^u e^{(u-1)(r+u\frac{\sigma^2}{2})\tau} N(d_1) - K e^{-r\tau} N(d_2)$$

(62)

where

$$d_1 := \frac{k_u + (u - \frac{1}{2})\sigma^2 \tau}{\sigma \sqrt{\tau}}, \quad d_2 := d_1 - u\sigma \sqrt{\tau}$$

(63)
Table 1 Comparisons between the series (35) truncated at \(n = n_{\text{max}}\) and the closed formula (61) in the case \(\alpha = 2, u = 1\). We observe that very few terms are needed to obtain an excellent degree of precision, even in deeply out or in the money situations.

| \(n_{\text{max}} = 3\) | \(n_{\text{max}} = 5\) | \(n_{\text{max}} = 10\) | Formula (61) |
|-------------------------|-------------------------|-------------------------|--------------|
| \(S = 5000\)            | 0.238961                | 0.237465                | 0.237525     |
| \(S = 4200\)            | 0.125287                | 0.125286                | 0.125286     |
| ATM                     | 0.092106                | 0.092104                | 0.092104     |
| \(S = 3800\)            | 0.079177                | 0.079158                | 0.079158     |
| \(S = 3000\)            | 0.025250                | 0.018797                | 0.019488     |

In table 2, values obtained with formula (62) are compared to various truncations of the series (55), for various powers \(u > 0\) and in the ATM situation. The convergence is very fast; of course if one is far from the money, the convergence becomes slightly slower because the moneyness \(k_u\) grows when \(u \neq 1\) (for instance, if \(S = 4500, k_1 = 0.14\) but \(k_{1.5} = 2.90\) and \(k_3 = 5.67\)), and therefore the powers \((k_\mu + \tau)^n\) in the numerator are less quickly neutralized by the factorial/Gamma terms of the denominator.

Table 2 Comparisons between the series (55) truncated at \(n_{\text{max}} = m_{\text{max}} := \text{max}\), and the values obtained via the formula (62) for some positive powers.

| \(u = 1\)  | 439.65   | 440.93   | 440.94   | 440.94   |
|------------|----------|----------|----------|----------|
| \(u = 1.5\)| 723.00   | 729.86   | 730.06   | 730.06   |
| \(u = 2\)  | 1057.71  | 1080.49  | 1081.64  | 1081.64  |
| \(u = 3\)  | 1908.17  | 2034.41  | 2049.37  | 2049.39  |

**European options (\(\alpha \neq 2\))** As a consequence of the Gil-Pelaez inversion formula for the characteristic functions, the price of an European call can be decomposed into a sum of Arrow-Debreu securities of the form (see details e.g. in [5]):

\[
C^{(E)}(S, K, r, \mu, \tau) = S \Pi_1 - K e^{-r\tau} \Pi_2
\]  

(64)

The price of each security can be expressed in terms of the stable characteristic function:

\[
\Pi_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \frac{e^{ik \log K e^{-t\Psi_{\text{stable}}(k-i)}}}{ik} \right] dk
\]  

(65)

and

\[
\Pi_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \frac{e^{ik \log K e^{-t\Psi_{\text{stable}}(k)}}}{ik} \right] dk
\]  

(66)

Given the simple form of the stable characteristic exponent [14], the integrals in (65) and (66) can be carried out rather easily in R or any other language.
In a wide interval of prices around the money ($S \in (3000, 6000)$), it is sufficient to consider only the terms up to $n_{\text{max}} = m_{\text{max}} = 5$ to obtain an excellent level of precision.

In Table 3, we compare the values obtained with this method with several truncations of the series (55), for a tail-index $\alpha = 1.7$. The convergence is very fast, in particular for ITM long term options.

|                     | $\max = 3$ | $\max = 10$ | $\max = 20$ | $\max = 30$ | Gil-Pelaez (64) |
|---------------------|------------|-------------|-------------|-------------|---------------|
| **Long term options** ($\tau = 2$) |            |             |             |             |               |
| $S = 5000$          | 1302.92    | 1309.86     | 1309.86     | 1309.86     | 1309.86       |
| $S = 4200$          | 679.32     | 681.56      | 681.56      | 681.56      | 681.56        |
| ATM                 | 496.87     | 498.07      | 498.07      | 498.07      | 498.07        |
| $S = 3800$          | 425.76     | 426.44      | 426.44      | 426.44      | 426.44        |
| $S = 3000$          | 128.50     | 92.46       | 96.50       | 96.50       | 96.50         |
| **Short term options** ($\tau = 0.5$) |            |             |             |             |               |
| $S = 5000$          | 1089.70    | 1075.64     | 1075.63     | 1075.63     | 1075.63       |
| $S = 4200$          | 383.17     | 383.30      | 383.30      | 383.30      | 383.30        |
| ATM                 | 230.47     | 203.49      | 203.49      | 203.49      | 203.49        |
| $S = 3800$          | 143.53     | 143.09      | 143.09      | 143.09      | 143.09        |
| $S = 3000$          | 211.44     | -27.24      | 1.04        | 1.39        | 1.39          |

Like before, the convergence is very fast, and goes even faster in the ITM region; this is because the log-forward moneyness (26) is positive in this zone, and therefore, as $\mu < 0$, $(k_{\mu} + \mu \tau)$ is closer to 0 than in the OTM zone, which accelerates the convergence of the series (55). This situation is displayed in Figure 2.
5 Extension to one-sided tempered stable processes

Tempered stable Lévy processes, which are known in Physics as truncated Lévy flights, combine α-stable and Gaussian trends, and are an alternative solution to achieve finite moments (see details and further references in [38]). Their Lévy-Khintchine triplet has the form $(a, 0, \nu_{TS})$ where

\[
\nu_{TS}(x) = \gamma_+ e^{-\lambda_+ |x|} \mathbb{1}_{\{x < 0\}} + \frac{\gamma_- e^{-\lambda_- x}}{x^{1+\alpha}} \mathbb{1}_{\{x > 0\}}
\]  

(67)

for $\gamma_\pm, \lambda_\pm \geq 0$ and $0 < \alpha_\pm < 2$. When $\gamma_- = \gamma_+$ and $\alpha_- = \alpha_+$, we recover the CGMY process [11] (sometimes named classical tempered stable process) and when furthermore $\alpha_- = \alpha_+ = 0$, the Variance Gamma process [31]. In the case where $\lambda_\pm = 0$, there is no more tempering and the process is simply a Lévy-stable process like in section 3. When $\alpha_\pm \in (0, 1) \cup (1, 2)$, the Laplace exponent of the tempered stable process can be easily computed: for $p \in (-\lambda_-, \lambda_-)$ one has

\[
\phi(p) = \eta p + \gamma_- \Gamma(-\alpha_-)(-\lambda_-^\alpha + (\lambda_- + p)^\alpha_-) + \gamma_+ \Gamma(-\alpha_+)(-\lambda_+^\alpha + (\lambda_+ - p)^\alpha_+)
\]

(68)

where $\eta$ is a constant depending on the drift $a$ and the choice of truncation function for the characteristic function of the process; without loss of generality we choose it to be equal to 0.

5.1 Tempered stable densities

Let us denote by $\nu^-_{TS}(x)$ (resp. $\nu^+_{TS}(x)$) the negative (resp. positive) part of the Lévy measure (67), and by $TS^\pm(\gamma_\pm, \lambda_\pm, \alpha_\pm)$ the associated one-sided tempered stable processes.

**Lemma 2** Let $\alpha_\pm \in (1, 2)$ and $\mu_\pm := -\gamma_\pm \Gamma(-\alpha_\pm)$.

(i) If $X_t \sim TS^-(\gamma_-, \lambda_-, \alpha_-)$, then its density $g^-(x, t)$ admits the Mellin-Barnes representation:

\[
g^-(x, t) = \frac{e^{\lambda_- \mu_- t + \lambda_- x}}{\alpha_- x^{1+\alpha}} \int_{c_1 - i\infty}^{c_1 + i\infty} \frac{\Gamma(1 - s_1)}{\Gamma(1 - \frac{1}{\alpha_-})} \left( \frac{x}{(-\mu_- t)^{1/\alpha_-}} \right)^{s_1} \frac{ds_1}{2i\pi}
\]

(69)

for any $c_1 < 1$;

(ii) If $X_t \sim TS^+(\gamma_+, \lambda_+, \alpha_+)$, then its density $g^+(x, t)$ admits the Mellin-Barnes representation:

\[
g^+(x, t) = -\frac{e^{\lambda_+ \mu_+ t - \lambda_+ x}}{\alpha_+ x^{1+\alpha}} \int_{c_2 - i\infty}^{c_2 + i\infty} \frac{\Gamma(1 - s_2)}{\Gamma(1 - \frac{1}{\alpha_+})} \left( \frac{-x}{(-\mu_+ t)^{1/\alpha_+}} \right)^{s_2} \frac{ds_2}{2i\pi}
\]

(70)

for any $c_2 < 1$. 
Proof It follows from (68) and from the Laplace inversion formula that:

\[
g^-(x, t) = e^{\lambda_+ \alpha_+ - \mu_+ t} e^{-\mu_- t (\lambda_+ + p)^+} \int_{c_p + i \infty}^{c_p - i \infty} e^{-p x} e^{-\mu_- t (\lambda_+ + p)^+} \frac{dp}{2i \pi} \tag{71}
\]

for \( c_p > 0 \). From the frequency shifting property of the Laplace transform, we can write:

\[
g^-(x, t) = e^{\lambda_+ \alpha_+ - \mu_+ t} e^{\lambda_- x} g_{\alpha_-}(x, t) \tag{72}
\]

where \( g_{\alpha_-}(x, t) \) is the stable density (18), and (i) is proved. A similar approach can be used to prove (ii).

5.2 Option pricing for negative tempered stable processes

Let \( \alpha_- \in (1, 2) \) and \( X_t \sim TS^-((\gamma_-, \lambda_-, \alpha_-)); \) from definition (53) and the Laplace exponent (68), the convexity adjustment reads:

\[
\mu = (\lambda_- + 1)\alpha_- - \lambda_-^\alpha \mu_- \tag{73}
\]

where \( \mu_- = -\gamma_-\Gamma(-\alpha_-) \) corresponds to the FMLS convexity adjustment (16), and as expected \( \mu \to \mu_- \) when \( \lambda_- \to 0 \). From the pricing formula (11) and using the notation (21) and (22), the value at time \( t \) of an option with maturity \( T \) and payoff \( \mathcal{P}(S_T, K) \) is equal to:

\[
C_{\alpha_- \lambda_-}(S, K, r, \mu_-, \tau) = e^{-(r-\lambda_- \mu_-)\tau} \int_{c_1 - i \infty}^{c_1 + i \infty} K_{\lambda_-}^\ast(s_1) G_{\alpha_-}^\ast(s_1) \left(-\mu_- \tau\right) \frac{ds_1}{2i \pi} \tag{74}
\]

where we have defined

\[
K_{\lambda_-}^\ast(s_1) := \int_{-\infty}^{+\infty} e^{\lambda_- x} \mathcal{P}(Se^{(r+\mu)\tau + x}, K) x^{s_1 - 1} \, dx \tag{75}
\]

and where \( \mu \) is given by (73). The \( K_{\lambda_-}^\ast(s_1) \) function can be expressed in terms of the \( K^\ast(s_1) \) function (22) by introducing a Mellin-Barnes representation for the exponential term:

\[
K_{\lambda_-}^\ast(s_1) = \int_{c_3 - i \infty}^{c_3 + i \infty} (-1)^{s_3} \lambda_-^{-s_3} \Gamma(s_3) K^\ast(s_1 - s_3) \frac{ds_3}{2i \pi} \tag{76}
\]

for \( c_3 > 0 \), and therefore, replacing in (74), we obtain:
Proposition 7 (Factorization) If \( X_t \sim TS^-(\gamma_-, \lambda_-, \alpha_-) \) and if \( \alpha_- \in (1, 2) \), then the value at time \( t \) of an option with maturity \( T \) and payoff \( \mathcal{P}(S_T, K) \) is equal to

\[
C_{\alpha_-, \lambda_-}(S, K, \gamma_-, \mu_-, \tau) = e^{-(r-\lambda_-^\alpha_- \mu_-)\tau} \times \\
\int_{c_1-i\infty}^{c_1+i\infty} \int_{c_3-i\infty}^{c_3+i\infty} (-1)^{-s_3} \lambda_-^{-s_3} \Gamma(s_3) K^*(s_1 - s_3) G_{\alpha_-}^*(s_1) (-\mu_- \tau)^{-\frac{\alpha_-}{\alpha_-}} \frac{ds_1 ds_3}{(2i\pi)^2} 
\]

(77)

Example: digital power option (cash-or-nothing) In that case, we know from (30) that:

\[
K^*(s_1 - s_3) = -\frac{\left(-k_u - \rho_- \mu_- \tau\right)^{s_1 - s_3}}{s_1 - s_3} 
\]

(78)

where \( \rho_- = (\lambda_- + 1)^{\alpha_-} - \lambda_-^\alpha_- \), and therefore it follows from (77) that the digital cash-or-nothing call reads:

\[
C^{(G/N)}_{\alpha_-, \lambda_-}(S, K, \gamma_-, \mu_-, \tau) = \frac{1}{\alpha_-} e^{-(r-\lambda_-^\alpha_- \mu_-)\tau} \int_{c_1-i\infty}^{c_1+i\infty} \int_{c_3-i\infty}^{c_3+i\infty} (-1)^{-s_3} \times \\
\frac{\Gamma(1 - s_1) \Gamma(s_3)}{(s_1 - s_3) \Gamma(1 - \frac{\alpha_-}{\alpha_-})} \lambda_-^{-s_3} \left(-k_u - \rho_- \mu_- \tau\right)^{s_1 - s_3} (-\mu_- \tau)^{-\frac{\alpha_-}{\alpha_-}} \frac{ds_1 ds_3}{(2i\pi)^2} 
\]

(79)

The double integral (79) has a simple pole in \((0, 0)\) with residue 1, and a series of simple poles in \((1 + n, m), n, m \in \mathbb{N}\) induced by the singularities of the \(\Gamma(1 - s_1)\) and \(\Gamma(s_3)\) functions. Summing all these residues yields:

\[
C^{(G/N)}_{\alpha_-, \lambda_-}(S, K, \gamma_-, \mu_-, \tau) = e^{-(r-\lambda_-^\alpha_- \mu_-)\tau} \left[ 1 + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-\lambda_-)^m}{(1 + n + m)! m! \Gamma(1 - \frac{1+\alpha_-}{\alpha_-})} (k_u + \rho_- \mu_- \tau)^{1+n+m} (-\mu_- \tau)^{-\frac{1+n}{\alpha_-}} \right] 
\]

(80)

Note that when \( \lambda_- = 0 \), only the terms for \( m = 0 \) survive and (80) degenerates into the \(\alpha\)-stable price (28), as expected. In the ATM forward case \((k_u = 0)\), the first few terms of the series (80) read:

\[
e^{-(r-\lambda_-^\alpha_- \mu_-)\tau} \alpha_- \left[ 1 - \frac{\rho_-}{\Gamma(1 - \frac{1}{\alpha_-})} (-\mu_- \tau)^{1-\frac{1}{\alpha_-}} + O\left((-\mu_- \tau)^{2-\frac{1}{\alpha_-}}\right) \right] 
\]

(81)

and can be Taylor-expanded for small \(\lambda_-\):

\[
e^{-\frac{\tau}{\alpha_-}} \alpha_- \left[ 1 - \frac{(-\mu_- \tau)^{1-\frac{1}{\alpha_-}}}{\Gamma(1 - \frac{1}{\alpha_-})} - \frac{(-\mu_- \tau)^{1-\frac{1}{\alpha_-}}}{\Gamma(1 - \frac{1}{\alpha_-})} \lambda_- + O\left(\lambda_-^{\alpha_-}\right) \right] 
\]

(82)
In the linear approximation (82), the intercept is the stable price, while the slope is governed by the negative left tail parameter $-\alpha$; the tempered stable price is therefore lower than the stable price (which is due to the tempering of the heavy tail), and the difference increases when $\alpha$ grows. This situation is displayed on fig. 3 for $\alpha = 1.7$, $K = 4000$, $r = 1\%$, $\sigma = 20\%$, $\tau = 2Y$, the series (28) and (80) being truncated to $n_{\text{max}} = m_{\text{max}} = 10$.

6 Conclusions and future work

In this article, we have derived generic representations in the Mellin space for path-independent options with arbitrary payoff, in the setup of exponential Lévy models driven by spectrally negative stable or tempered stable processes. These representations have allowed us to obtain simple series expansions for the price of options with an exotic power-related payoff (Power Digital, Log or Gap Power options, Capped Power European options), by means of residue summation in $\mathbb{C}$ or $\mathbb{C}^2$. These series contain only simple terms and converge very fast, in particular when calls are in-the-money and for longer maturities; they can be very easily used for practical evaluation without requiring any help from numerical schemes.

Future work will include the investigation of path-dependent options, like Barrier or Lookback options; spectrally negative $\alpha$-stable processes are particularly interesting in this context, because the law of the supremum on a period of time is known to be

$$
\mathbb{P}\left[ \sup_{t \in [0,T]} X_t \geq x \right] = \alpha \mathbb{P}[X_T \geq x] (83)
$$

which generalizes the reflection principle for the Wiener process ($\alpha = 2$). Regarding path-independent instrument, we would like to apply the Mellin residue technique to two-sided Lévy processes, with a particular focus on the Variance Gamma and the Normal Inverse Gamma (NIG) processes; indeed,
the technique appears very well adapted to these models too, because, like in
the spectrally negative case, their density functions can be expressed under
the form of Mellin integrals.

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