ON HOMOTOPY BRAIDS

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ABSTRACT. Homotopy braid group $\tilde{B}_n$ is the subject of the paper. First, linearity of $\tilde{B}_n$ over the integers is proved. Then we prove that the group $\tilde{B}_3$ is torsion free.

1. Introduction

Homotopy braid groups are one of the interesting variations of classical braid groups.

Two geometric braids with the same endpoints are called homotopic if one can be deformed to the other by homotopies of the braid strings which fix the endpoints, so that different strings do not intersect. If two geometric braids are isotopic, they are evidently homotopic. E. Artin [3] posed the question of whether the notions of isotopy and homotopy of braids are different or the same. Deborah Goldsmith [9] gave an example of a braid which is not trivial in the isotopic sense, but is homotopic to the trivial braid. This braid is expressed in the canonical generators of the classical braid group in the following form:

$$\sigma_1^{-1}\sigma_2^{-2}\sigma_1^{-2}\sigma_2^2\sigma_1\sigma_2^{-2}\sigma_1^3\sigma_2^2\sigma_1^{-1}.$$ 

Thus, the study of homotopy braids is interesting in itself.

In this work we consider several questions concerning homotopy braids. The paper is organized as follows. In section 2 we recollect necessary facts about the reduced free groups. In section 3 we prove the linearity of $\tilde{B}_n$ over $\mathbb{Z}$. Unfortunately we don’t know a concrete representation in $GL_m(\mathbb{Z})$. We then prove that the factorization of Burau representation cannot serve for these purposes. In section 4 we prove that the group $\tilde{B}_3$ is torsion free. Several questions are proposed in the last section.

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2. Reduced free group and homotopy braid group

Let \( F_n = F(x_1, \ldots, x_n) \) be a free group on generators \( x_1, \ldots, x_n \). Consider a subgroup generated by the commutators

\[ gx_1g^{-1}x_1^{-1}, \ldots, gx_ng^{-1}x_n^{-1}, \]

where \( g \) is an arbitrary element of \( F_n \). It is a normal subgroup of \( F_n \); let us denote it by \( N_n \). The quotient group \( K_n = F_n/N_n \) is called the reduced free group. It was introduced by J. Milnor [16] and studied by Habegger-Lin [10], F. Cohen [6] and F. Cohen & Jie Wu [7].

For elements \( a, b \) of arbitrary group \( G \) we will use the following notations

\[ a^b = b^{-1}ab, \quad [a, b] = a^{-1}b^{-1}ab. \]

Let \( a_{i,j} \) be the standard (Burau) generators of the pure braid group. Recall that the homotopy braid group \( \tilde{B}_n \) is the quotient of the braid group \( B_n \) by the relations

\[ [a_{ik}, a_{ik}^g] = 1, \quad \text{where} \quad g \in \langle a_{1k}, a_{2k}, \ldots, a_{k-1,k} \rangle, \quad 1 \leq i < k \leq n. \]

The quotient of the pure braid group \( P_n \) by the same relations gives the homotopy pure braid group \( \tilde{P}_n \) and from the standard short exact sequence for \( B_n \) we have the following short exact sequence

\[ 1 \longrightarrow \tilde{P}_n \longrightarrow \tilde{B}_n \longrightarrow S_n \longrightarrow 1, \]

for \( \tilde{B}_n \), where \( S_n \) is the symmetric group. Group \( \tilde{P}_n \) has a decomposition \( \tilde{P}_n = \tilde{U}_n \times \tilde{P}_{n-1} \), where \( \tilde{U}_n \) is the quotient of the free group \( U_n = \langle a_{1n}, a_{2n}, \ldots, a_{n-1,n} \rangle \) of rank \( n - 1 \) by the relations

\[ [a_{in}, a_{ik}^g] = 1, \quad \text{where} \quad g \in U_n, \quad 1 \leq i < k \leq n, \]

(see [10]). Note, that \( \tilde{U}_n \) is isomorphic to \( K_{n-1} \). In particular, \( \tilde{U}_2 \) is isomorphic to the infinite cyclic group and \( \tilde{U}_3 \) is the quotient of \( U_3 = \langle a_{13}, a_{23} \rangle \) by the relations

\[ a_{13} \cdot a_{23}^{-1}a_{13}a_{23} = a_{23}^{-1}a_{13}a_{23} \cdot a_{13}, \]

\[ a_{23} \cdot a_{13}^{-1}a_{23}a_{13} = a_{13}^{-1}a_{23}a_{13} \cdot a_{23}. \]

It was proved by Cohen and Wu [7] that the canonical Artin monomorphism

\[ \nu_n : B_n \hookrightarrow \text{Aut} \ F_n \]

generates a homomorphism

\[ \tilde{\nu}_n : \tilde{B}_n \rightarrow \text{Aut} \ K_n. \]

It is known that \( \nu_n \) is a monomorphism. See, for example, the thesis of Liu Minghui [17].
Since $K_n$ is a finitely generated nilpotent group of class $\leq n$ ([10, Lemma 1.3]), then from result of A.I. Mal’cev [14] follows that the word problem is decidable in $K_n$. From the fact that $\tilde{B}_n$ is a finite extension of $\tilde{P}_n$ follows that the word problem is decidable in $\tilde{B}_n$. In the next section we prove stronger result that $\tilde{B}_n$ is a linear group over $\mathbb{Z}$.

3. Linearity problem

3.1. Existence. Recall that a group $G$ is called linear if it has a faithful representation into the general linear group $GL_m(k)$ for some $m$ and a field $k$. In the works [4] and [12] it was proved that the braid group $B_n$ is linear for every $n \geq 2$. So, it is natural to ask the question of linearity of $\tilde{B}_n$.

**Theorem 3.1.** The homotopy braid group $\tilde{B}_n$ is linear for all $n \geq 2$. Moreover, for every $n \geq 2$ there is a natural $m$ such that there exists a faithful representation $\tilde{B}_n \to GL_m(\mathbb{Z})$

**Proof.** As it was mentioned above, the reduced free group $K_n$, $n \geq 2$ is polycyclic. Finitely generated nilpotent groups are polycyclic and hence they are represented by integer matrices [1, 18]. It was proved in [15] that the holomorph of every polycyclic group has a faithful representation into $GL_m(\mathbb{Z})$ for some $m$. Hence, holomorph $Hol(K_n)$ has a faithful representation into $GL_m(\mathbb{Z})$ for some $m$. But $Hol(K_n)$ contains $Aut(K_n)$ as a subgroup and $\tilde{B}_n$ is embedded into $Aut(K_n)$. □

It is interesting to find a faithful linear representation of $\tilde{B}_n$ explicitly. One can try to factor through $\tilde{B}_n$ the known representations of $B_n$, for example, Burau representation, Lawrence-Krammer-Bigelow representation or other ones.

3.2. Factorization of the Burau representation through $\tilde{B}_n$. Let

$$\rho_B : B_n \to GL(W_n)$$

be the Burau representation of $B_n$, where $W_n$ is a free $\mathbb{Z}[t^{\pm 1}]$-module of rank $n$ with the basis $w_1, w_2, \ldots, w_n$. Let $n = 3$. In this case the automorphisms $\rho_B(\sigma_i), i = 1, 2$, of module $W_3$ act by the rule

$$\sigma_1 : \begin{cases} w_1 \to (1 - t)w_1 + tw_2, \\ w_2 \to w_1, \\ w_3 \to w_3 \end{cases} \quad \sigma_2 : \begin{cases} w_1 \to w_1, \\ w_2 \to (1 - t)w_2 + tw_3, \\ w_3 \to w_2, \end{cases}$$

where we write for simplicity $\sigma_i$ instead of $\rho_B(\sigma_i)$. Let us find the action of the generators of $P_3$ on the module $W_3$. Recall, that $P_3 = U_2 \ltimes U_3$, 

where $U_2$ is the infinite cyclic group with the generator $a_{12} = \sigma_1^2$, $U_3$ is the free group of rank 2 with the free generators $a_{13} = \sigma_2\sigma_1^2\sigma_2^{-1}$, $a_{23} = \sigma_2^2$.

These elements define the following automorphisms of $W_3$:

\begin{align*}
(3.1) \quad a_{12} & : \begin{cases}
   w_1 \mapsto (1 - t + t^2)w_1 + t(1 - t)w_2, \\
   w_2 \mapsto (1 - t)w_1 + tw_2, \\
   w_3 \mapsto w_3,
\end{cases} \\
(3.2) \quad a_{13} & : \begin{cases}
   w_1 \mapsto (1 - t + t^2)w_1 + t(1 - t)w_3, \\
   w_2 \mapsto (1 - t)^2w_1 + w_2 - (1 - t)^2w_3, \\
   w_3 \mapsto (1 - t)w_1 + tw_3,
\end{cases} \\
(3.3) \quad a_{23} & : \begin{cases}
   w_1 \mapsto w_1, \\
   w_2 \mapsto (1 - t + t^2)w_2 + t(1 - t)w_3, \\
   w_3 \mapsto (1 - t)w_2 + tw_3,
\end{cases} \\
(3.4) \quad a_{23}^{-1}a_{13} & : \begin{cases}
   w_1 \mapsto w_1, \\
   w_2 \mapsto t^{-1}w_2 + (1 - t^{-1})w_3, \\
   w_3 \mapsto t^{-1}(1 - t^{-1})w_2 + (1 - t^{-1} + t^{-2})w_3.
\end{cases}
\end{align*}

Let us denote by $\tilde{\rho}_B$ the representation

$$
\tilde{\rho}_B : \tilde{B}_n \rightarrow GL(W_n)
$$

which is the factorization of $\rho_B$ throw $\tilde{B}_n$.

**Proposition 3.2.** The factorization of the representation $\tilde{\rho}_B$ on $\tilde{P}_3$ is trivial. Hence, the image $\tilde{\rho}_B(B_3)$ is isomorphic to the symmetric group $S_3$.

**Proof.** To get a representation of $\tilde{\rho}_B(B_3)$ we must have the following relations among the automorphisms $a_{i,j}$ (3.1)-(3.3) of $W_3$:

$$
[a_{13}, a_{13}^{a_{23}}] = 1, \quad [a_{23}, a_{23}^{a_{13}}] = 1,
$$

which are equivalent to the relations

$$
a_{13}a_{13}^{a_{23}} = a_{13}^{a_{23}}a_{13}, \quad a_{23}a_{23}^{a_{13}} = a_{23}^{a_{13}}a_{23}.
$$

From the definitions the automorphisms (3.1)-(3.4) we obtain

$$
a_{23}^{-1}a_{13}a_{23} : \begin{cases}
   w_1 \mapsto (1 - t + t^2)w_1 + t(1 - t)^2w_2 + t^2(1 - t)w_3, \\
   w_2 \mapsto w_2, \\
   w_3 \mapsto t^{-1}(1 - t)w_1 - t^{-1}(1 - t)^2w_2 + tw_3,
\end{cases}
$$
and

\[
\begin{align*}
\alpha_{13} & : \\
\alpha_{23} & : \\
\end{align*}
\]

In order to satisfy relation \(\alpha_{13}\alpha_{23} = \alpha_{23}\alpha_{13}\) the following system of equations should have a solution

\[
\begin{align*}
1 - 3t + 4t^2 - 4t^3 + 3t^4 - t^5 &= 0, \\
(1 - t)^2(-1 - t - t^2 + t^3) &= 0, \\
t(1 - t)^5 &= 0, \\
(1 - t)^2(-t^{-1} + 1 - t + t^2) &= 0, \\
t^{-1}(1 - t)^4(1 + t^2) &= 0, \\
(1 - t)^2(1 - t + t^2 - t^3) &= 0, \\
(1 - t)^2(-1 + t - t^2 + t^{-1}) &= 0, \\
1 - t - 4t^2 + 8t^3 - 5t^4 + t^5 &= 0.
\end{align*}
\]

This system has a solution only if \(t = 1\). In this case, automorphisms \(\alpha_{12}, \alpha_{13}, \alpha_{23}\) are equal to the identity automorphism. \(\square\)

3.3. **Linear representation of** \(K_n\). We know that \(K_2\) is the free 2-step nilpotent group of rank 2. Hence, for every non-zero integers \(a\) and \(b\) the map of \(K_2\) defined on the generators \(x_1, x_2\) by the formulas

\[
\begin{align*}
x_1 \mapsto A = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \quad x_2 \mapsto B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}
\end{align*}
\]

defines a faithful representation to the unitriangular group

\(K_2 \rightarrow UT_3(\mathbb{Z})\).

Note that in this case

\[
[x_1, x_2] \mapsto [A, B] = \begin{pmatrix} 1 & 0 & ab \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

**Question 3.3.** Is there a faithful representation of \(K_n\), \(n > 2\) into \(UT_n(\mathbb{Z})\)?
Lemma 3.4. The following basis commutators of weight 3 are trivial in $K_3$

\[ [[x_2, x_1], x_1], [[x_2, x_1], x_2], [[x_3, x_1], x_1], [[x_3, x_1], x_3], [[x_3, x_1], x_2], [[x_3, x_2], x_2], [[x_3, x_2], x_3], \]

and the following basis commutators are non-trivial

\[ [[x_2, x_1], x_3], [[x_3, x_1], x_2]. \]

Proof. Let us prove, for example, that $[[x_2, x_1], x_2] = 1$. Indeed, using commutator identities we have

\[
[[x_2, x_1], x_2] = [x_2^{-1}x_1^{-1}x_2x_1, x_2] = [x_2^{-1}, x_2][x_2^{x_1}, x_2]^{x_1} = 1.
\]

To prove that the commutator $[[x_2, x_1], x_3]$ is non-trivial, we define an embedding of $K_3$ into $\tilde{P}_4$ (see [10]) by the rules

\[
x_1 \rightarrow a_{14}, \ x_2 \rightarrow a_{24}, \ x_3 \rightarrow a_{34},
\]

and using the fact that the homomorphism

\[ \tilde{\nu}_n: \tilde{B}_n \rightarrow \text{Aut } K_n, \]

is a monomorphism [17], we find the automorphism $\tilde{\nu}_4([[a_{24}, a_{14}], a_{34}])$ and check that it non-identity. The proofs of the other statements are similar. \qed

4. TORSION IN $\tilde{B}_n$

V.Ya. Lin formulated the following question in Kourovka Notebook [11].

Question 4.1. (V.Lin, Question 14.102 c)) Is there a non-trivial epimorphisms of $B_n$ onto a non-abelian group without torsion?

The answer to this question can be found in [13].

We conjecture that the group $\tilde{B}_n$, $n \geq 3$, does not have torsion and since there exists the epimorphism $B_n \rightarrow \tilde{B}_n$, then $\tilde{B}_n$ is a good candidate to the other solution of Lin’s problem. In this section we prove that $\tilde{B}_3$ does not have torsion.

Let $\tilde{P}_3, \tilde{U}_2, \tilde{U}_3$ be the images of $P_3, U_2, U_3$ by the canonical epimorphism $B_3 \rightarrow \tilde{B}_3$. Denote by $b_{ij}$, $1 \leq i < j \leq 3$ the images of $a_{ij}$, $1 \leq i < j \leq 3$ by this epimorphism. Then $\tilde{U}_2 = \langle b_{12} \rangle$ is the infinite cyclic group and

\[
\tilde{U}_3 = \langle b_{13}, b_{23} \mid [b_{13}, b_{13}^{b_{23}}] = [b_{23}, b_{23}^{b_{13}}] = 1 \rangle = \langle b_{13}, b_{23} \mid [b_{13}, b_{13}[b_{13}, b_{23}]] = [b_{23}, b_{23}[b_{23}, b_{13}]] = 1 \rangle.
\]
Using commutator identities or direct calculations we see that the last two relations are equivalent to the following relation

$$[[b_{23}, b_{13}], b_{23}] = [[b_{23}, b_{13}], b_{13}] = 1.$$  

Hence, $\tilde{U}_3$ is a free 2-step nilpotent group of rank 2 and so, every element $g \in \tilde{U}_3$ has a unique presentation of the form

$$g = b_{13}^{\alpha} b_{23}^{\beta} [b_{23}, b_{13}]^\gamma$$

for some integers $\alpha, \beta, \gamma$. The same way as in the case of classical braid group, $\tilde{U}_3$ is a normal subgroup of $\tilde{P}_3$ and the action of $\tilde{U}_2$ is defined in the following lemma.

**Lemma 4.2.** The action of $\tilde{U}_2$ on $\tilde{U}_3$ is given by the formulas

$$b_{13}^{k_{12}} = b_{13}[b_{23}, b_{13}]^k, \quad b_{23}^{k_{12}} = b_{23}[b_{23}, b_{13}]^{-k}, \quad [b_{23}, b_{13}]^{k_{12}} = [b_{23}, b_{13}], \quad k \in \mathbb{Z}. \quad \Box$$

The action of the generators $\sigma_1$ and $\sigma_2$ of $\tilde{B}_3$ on $\tilde{P}_3$ is given in the next lemma.

**Lemma 4.3.** The following conjugation formulas hold in $\tilde{B}_3$

$$b_{12}^{\sigma_{12}^+} = b_{12}, \quad b_{13}^{\sigma_{13}^+} = b_{23}[b_{23}, b_{13}]^{-1}, \quad b_{23}^{\sigma_{13}^-} = b_{13}, \quad b_{13}^{\sigma_{23}^-} = b_{23}, \quad [b_{23}, b_{13}]^{\sigma_{13}^-} = [b_{23}, b_{13}]^{-1},$$

$$b_{12}^{\sigma_{12}^+} = b_{12}, \quad b_{13}^{\sigma_{13}^+} = b_{23}[b_{23}, b_{13}]^{-1}, \quad b_{23}^{\sigma_{13}^-} = b_{13}, \quad b_{13}^{\sigma_{23}^-} = b_{23}, \quad [b_{23}, b_{13}]^{\sigma_{13}^-} = [b_{23}, b_{13}]^{-1}. \quad \Box$$

Let us denote by $\Lambda_3 = \{e, \sigma_1, \sigma_2, \sigma_2 \sigma_1, \sigma_1 \sigma_2, \sigma_1 \sigma_2 \sigma_1\}$ the set of representatives of $\tilde{P}_3$ in $\tilde{B}_3$. Then every element in $\tilde{B}_3$ can be written in the form

$$b_{12}^{\alpha} b_{13}^{\beta} b_{23}^{\gamma} z^\delta \lambda, \text{ where } \alpha, \beta, \gamma, \delta \in \mathbb{Z}, \quad z = [b_{23}, b_{13}], \quad \lambda \in \Lambda_3.$$

**Theorem 4.4.** The group $\tilde{B}_3$ is torsion-free.

**Proof.** The group $\tilde{P}_3$ does not have torsion. Hence, if $\tilde{B}_3$ has elements of finite order, then they have the form

$$b_{12}^{\alpha} b_{13}^{\beta} b_{23}^{\gamma} z^\delta \lambda, \quad \lambda \in \Lambda_3 \setminus \{e\}.$$  

Every element which is conjugate with an element of finite order has a finite order. Taking into account the following formulas

$$\sigma_1^{-1} \cdot \sigma_1 = b_{12}^{-1} \sigma_1 \sigma_2 \sigma_1, \quad \sigma_2 \sigma_1 \cdot \sigma_2 \cdot \sigma_1^{-1} = \sigma_1, \quad \sigma_1^{-1} \cdot \sigma_1 \sigma_2 \cdot \sigma_1 = \sigma_2 \sigma_1,$$

it is sufficient to consider only two cases: $\lambda = \sigma_2$ and $\lambda = \sigma_1 \sigma_2$.

Let $\lambda = \sigma_2$, take $g = b_{12}^{\alpha} b_{13}^{\beta} b_{23}^{\gamma} z^\delta \sigma_2$. Then we have

$$g^2 = b_{12}^{\alpha+\beta} b_{13}^{\alpha+\beta} b_{23}^{2\gamma+1} z^{\alpha+\beta(\beta-\gamma+\alpha-1)}.$$
If \( g^2 = 1 \), then \( \alpha + \beta = 0 \) and we have
\[
g^2 = b_2^{2\gamma+1}z^{2\alpha\gamma+\alpha}.
\]
Since \( 2\gamma + 1 \) cannot be zero for integer \( \gamma \), the elements of this form cannot be of finite order.

Let \( \lambda = \sigma_1\sigma_2 \). Then we have
\[
(\sigma_1\sigma_2)^2 = b_{12}\sigma_2\sigma_1, \quad (\sigma_1\sigma_2)^3 = b_{12}b_{13}b_{23}.
\]
We calculate
\[
g^3 = (b_{12}^\alpha b_{13}^\beta b_{23}^\gamma z^\delta \sigma_1\sigma_2)^3 = b_{12}^{\alpha+\beta+\gamma+1}b_{13}^{\alpha+\beta+\gamma+1}b_{23}^{\alpha+\beta+\gamma+1}z^{\alpha(2\gamma-\beta)+\beta^2+\gamma^2-\beta\gamma+3\delta+3\beta}.
\]
If \( g^3 = 1 \), then the following system of linear equations has a solution over \( \mathbb{Z} \)
\[
\begin{align*}
\alpha + \beta + \gamma + 1 &= 0, \\
\alpha(\alpha + 2\gamma - \beta) + \beta^2 + \gamma^2 - \beta\gamma + 3\delta + 3\beta &= 0.
\end{align*}
\]
From the first equation one gets: \( \alpha = -1 - \beta - \gamma \). Inserting this equality into the second equation, we have
\[
3(\beta^2 + 2\beta + \delta) + 1 = 0.
\]
However, this equation does not have integer solutions. \( \square \)

5. Open problems

The homotopy virtual braid group \( \widetilde{VB}_n \) was defined in [8].

**Question 5.1.** 1) Is it possible to construct a normal form for words, representing elements of \( \widetilde{VB}_n \)?

2) Is there a faithful representation \( \widetilde{VB}_n \rightarrow \text{Aut}(G_n) \) for some group \( G_n \)?

3) Is it possible to define Milnor invariants for homotopy virtual links?

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