The spectral radius of subgraphs of regular graphs

Vladimir Nikiforov
Department of Mathematical Sciences, University of Memphis, Memphis TN 38152, USA

Submitted: May 25, 2007; Accepted: Sep 30, 2007; Published: Oct 5, 2007
Mathematics Subject Classification: 05C50

Abstract
Let $\mu(G)$ and $\mu_{\min}(G)$ be the largest and smallest eigenvalues of the adjacency matrix of a graph $G$. Our main results are:

(i) Let $G$ be a regular graph of order $n$ and finite diameter $D$. If $H$ is a proper subgraph of $G$, then
$$\mu(G) - \mu(H) > \frac{1}{nD}.$$ 

(ii) If $G$ is a regular nonbipartite graph of order $n$ and finite diameter $D$, then
$$\mu(G) + \mu_{\min}(G) > \frac{1}{nD}.$$ 

Keywords: smallest eigenvalue, largest eigenvalue, diameter, connected graph, nonbipartite graph

Main results

Our notation follows [1]. Specifically, $\mu(G)$ and $\mu_{\min}(G)$ stand for the largest and smallest eigenvalues of the adjacency matrix of a graph $G$.

The aim of this note is to improve some recent results on eigenvalues of subgraphs of regular graphs. Cioabă ([2], Corollary 2.2) showed that if $G$ is a regular graph of order $n$ and $e$ is an edge of $G$ such that $G - e$ is a connected graph of diameter $D$, then
$$\mu(G) - \mu(G - e) > \frac{1}{nD}.$$ 

The approach of [3] helps improve this assertion in a natural way:

Theorem 1 Let $G$ be a regular graph of order $n$ and finite diameter $D$. If $H$ is a proper subgraph of $G$, then
$$\mu(G) - \mu(H) > \frac{1}{nD}.$$
Since \( \mu(H) \leq \mu(H') \) whenever \( H \subseteq H' \), we may assume that \( H \) is a maximal proper subgraph of \( G \), that is to say, \( V(H) = V(G) \) and \( H \) differs from \( G \) in a single edge. Thus, we can deduce Theorem 1 from the following assertion.

**Theorem 2** Let \( G \) be a regular graph of order \( n \) and finite diameter \( D \). If \( uv \) is an edge of \( G \), then

\[
\mu(G) - \mu(G - uv) > \begin{cases} 1/(nD), & \text{if } G - uv \text{ is connected;} \\ 1/(n - 3)(D - 1), & \text{otherwise.} \end{cases}
\]

Furthermore, Theorem 1 implies a result about nonbipartite graphs.

**Theorem 3** If \( G \) is a regular nonbipartite graph of order \( n \) and finite diameter \( D \), then

\[
\mu(G) + \mu_{\min}(G) > \frac{1}{nD}.
\]

Finally, we note the following more general version of the lower bound in Corollary 2.2 in [2].

**Lemma 4** Let \( G \) be a connected regular graph and \( e \) be an edge of \( G \). If \( H \) is a component of \( G - e \) with \( \mu(H) = \mu(G - e) \), then

\[
\mu(G) - \mu(H) > \frac{1}{\text{Diam}(H) |H|}.
\]

This lemma follows easily from Theorem 2.1 of [2] and its proof is omitted.

**Proofs**

**Proof of Theorem 2** Write \( \text{dist}_F(s,t) \) for the length of a shortest path joining two vertices \( s \) and \( t \) in a graph \( F \). Write \( d \) for the degree of \( G \), let \( H = G - uv \), and set \( \mu = \mu(H) \).

**Case (a): \( H \) is connected.**

Let \( x = (x_1, \ldots, x_n) \) be a unit eigenvector to \( \mu \) and let \( x_w \) be a maximal entry of \( x \); we thus have \( x_w^2 \geq 1/n \). We can assume that \( w \neq v \) and \( w \neq u \). Indeed, if \( w = v \), we see that

\[
\mu x_v = \sum_{i \in E(G)} x_i \leq (d - 1) x_v,
\]

and so \( d - \mu \geq 1 \), implying (1). We have

\[
d - \mu = d \sum_{i \in V(G)} x_i^2 - 2 \sum_{ij \in E(G)} x_i x_j = \sum_{ij \in E(G)} (x_i - x_j)^2 + x_u^2 + x_v^2.
\]
Assume first that \( \text{dist}_H (w, u) \leq D - 1 \). Select a shortest path \( u = u_1, \ldots, u_k = w \) joining \( u \) to \( w \) in \( H \). We see that

\[
d - \mu = \sum_{ij \in E(G)} (x_i - x_j)^2 + x_u^2 + x_v^2 > \sum_{i = 1}^{k-1} (x_{u_i} - x_{u_{i+1}})^2 + x_u^2 \geq \frac{1}{k-1} (x_{u_i} - x_{u_{i+1}})^2 + x_u^2 \geq \frac{1}{k-1} (x_{w} - x_{u})^2 + x_u^2 \geq \frac{1}{k} x_w^2 \geq \frac{1}{nD},
\]

completing the proof.

Hereafter, we assume that \( \text{dist}_H (w, u) \geq D \) and, by symmetry, \( \text{dist}_H (w, v) \geq D \).

Let \( P(u, w) \) and \( P(v, w) \) be shortest paths joining \( u \) and \( v \) to \( w \) in \( G \). If \( u \in P(v, w) \), then there exists a path of length at most \( D - 1 \), joining \( w \) to \( u \) in \( G \), and thus in \( H \), a contradiction. Hence, \( u \notin P(v, w) \) and, by symmetry, \( v \notin P(u, w) \). Therefore, the paths \( P(u, w) \) and \( P(v, w) \) belong to \( H \), and we have

\[
\text{dist}_H (w, u) = \text{dist}_H (w, v) = D.
\]

Let \( Q(u, z) \) and \( Q(v, z) \) be the longest subpaths of \( P(u, w) \) and \( P(v, w) \) having no internal vertices in common. Clearly \( Q(u, z) \) and \( Q(v, z) \) have the same length. Write \( Q(z, w) \) for the subpath of \( P(u, w) \) joining \( z \) to \( w \) and let

\[
Q(u, z) = u_1, \ldots, u_k, \quad Q(v, z) = v_1, \ldots, v_k, \quad Q(z, w) = w_1, \ldots, w_l,
\]

where

\[
u_1 = u, \quad u_k = v_k = w_1 = z, \quad w_l = w, \quad k + l - 2 = D.\]

The following argument is borrowed from [2]. Using the AM-QM inequality, we see that

\[
d - \mu \geq \sum_{i=1}^{k-1} (x_{v_i} - x_{v_{i+1}})^2 + x_v^2 + \sum_{i=1}^{k-1} (x_{u_i} - x_{u_{i+1}})^2 + x_u^2 + \sum_{i=1}^{l-1} (x_{w_i} - x_{w_{i+1}})^2 \geq \frac{2}{D - l + 2} x_z^2 + \frac{1}{l - 1} (x_w - x_z)^2 \geq \frac{2}{D + l - 1} x_w^2 \geq \frac{1}{Dn},
\]

completing the proof.

**Case (b):** \( H \) is disconnected.

Since \( G \) is connected, \( H \) is union of two connected graphs \( H_1 \) and \( H_2 \) such that \( u \in H_1 \), \( v \in H_2 \). Assume \( \mu = \mu (H_1) \), set \( |H_1| = k \) and let \( x = (x_1, \ldots, x_k) \) be a unit eigenvector to \( \mu \). Since \( d \geq 2 \), we see that \( |H_2| \geq 3 \), and so, \( k \leq n - 3 \).

Let \( x_w \) be a maximal entry of \( x \); we thus have \( x_w^2 \geq 1/k \geq 1/(n - 3) \). Like in the previous case, we see that \( w \neq u \). Since \( d \geq 2 \), there is a vertex \( z \in H_2 \) such that \( z \neq v \). Select a shortest path \( u = u_1, u_2, \ldots, u_l = w \) joining \( u \) to \( w \) in \( H_1 \). Since \( \text{dist}_G (z, w) \leq \text{diam} G = D \), we see that \( l \leq D - 1 \). As above, we have

\[
d - \mu = \sum_{ij \in E(G)} (x_i - x_j)^2 + x_a^2 + x_a^2 > \sum_{i=1}^{l-1} (x_{u_i} - x_{u_{i+1}})^2 + x_u^2 \geq \frac{1}{l-1} (x_{w} - x_{u})^2 + x_u^2 \geq \frac{1}{l} x_w^2 \geq \frac{1}{(n - 3) (D - 1)},
\]
completing the proof. \hfill \square

**Proof of Theorem 3** Let $\mathbf{x} = (x_1, \ldots, x_n)$ be an eigenvector to $\mu_{\min}(G)$ and let $U = \{u : x_u < 0\}$. Write $H$ for the bipartite subgraph of $G$ containing all edges with exactly one vertex in $U$; note that $H$ is a proper subgraph of $G$ and $\mu_{\min}(H) < \mu_{\min}(G)$. Hence,

$$\mu(G) + \mu_{\min}(G) > \mu(G) + \mu_{\min}(H) = \mu(G) - \mu(H),$$

and the assertion follows from Theorem 1. \hfill \square

**Acknowledgment** A remark of Lingsheng Shi initiated the present note and a friendly referee helped complete it.

**References**

[1] B. Bollobás, *Modern Graph Theory*, Graduate Texts in Mathematics, 184, Springer-Verlag, New York (1998), xiv+394 pp.

[2] S. M. Cioaba, The spectral radius and the maximum degree of irregular graphs, Electronic J. Combin., 14 (2007), R38.

[3] V. Nikiforov, Revisiting two classical results on graph spectra, Electronic J. Combin., 14 (2007), R14.