Borel-Ecalle resummation of a two-point function

Pierre J. Clavier\textsuperscript{1,2}

\textsuperscript{1} Institute of Mathematics, University of Potsdam, D-14476 Potsdam, Germany

\textsuperscript{2} Mathematics Laboratory, Technische Universität, 10623 Berlin, Germany

email: clavier@math.uni-potsdam.de

Abstract

We provide an overview of the tools and techniques of resurgence theory used in the Borel-Ecalle resummation method, which we then apply to the massless Wess-Zumino model. Specifically, we discuss the notion of well-behaved averages and the describe the spaces involved in their definition. These tools are then used to solve the renormalisation group equation of the Wess-Zumino model for the two point function in a space of formal series. We show that this solution is 1-Gevrey and that its Borel transform is resurgent. The Schwinger-Dyson equation of the model is then used to prove an asymptotic exponential bound for the Borel transformed two point function on the principal branch of a suitable ramified complex plane.

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1 Introduction

1.1 State of the art and goals of the paper

Recently, much progress has been made towards a better analytic understanding of Quantum Field Theories (QFTs), both in perturbative and non-perturbative approaches. To illustrate this, for the perturbative approach let us quote [1], where the authors proved that the regularised Feynman rule of a QFT has its image in a space of meromorphic families of distributions with linear poles. For non-perturbative approaches, let us mention [2], where an explicit solution to the Schwinger-Dyson equation is found, a more powerful result than the ones previously obtained in [3, 4] for similar Schwinger-Dyson equations. Of particular interest to us are the results of the articles [5], [6] and [7] which undertake a resurgent study of some Quantum Field Theories.

Resurgence theory was developed in the late 70s and early 80s almost single-handedly by Ecalle [8, 9, 10]. It was initially applied to problems from the theory of dynamical systems (Dulac’s problem, see [11]) but quickly found applications in other branches of mathematics. The theory of averages is one important aspect of resurgence theory, developed by Frédéric Menous [12, 13] in the late 90s and much more recently by Emmanuel Vieillard-Baron [14].

Notwithstanding its successes, the theory of resurgence has not reached a very large audience until fairly recently. In the early 2010, Inês Aniceto and Riccardo Schiappa started in [15], a program to apply various aspects of resurgence theory to physics. In particular they use alien calculus to compute non-perturbative contributions to physical theories.

This approach has proved to be successful and nowadays resurgence theory is becoming part of the standard toolkit available to physicists. Instead of listing the various topics and articles of the physics literature applying resurgence theory, we refer the reader to the review [16] and the introductory article [17] for a presentation of the physicists’ point of view about resurgence.

Bearing this in mind, the present paper has two main goals:

- To present a self-contained introduction to the Borel-Ecalle resummation method, useful for the reader who wishes to apply it to specific problems, e.g. those coming from physics.
- To illustrate the applicability of this method to physical problems, and in particular in QFT, in which context we explicitly work out a resurgent analysis of a Wess-Zumino model.

1.2 Summary of the paper

The aim of the paper is to prove the following result:

**Theorem 1.1.** The solution $\tilde{G}(a, L)$ of the Schwinger-Dyson equation (4) and the renormalisation group equation (5) is Borel-Ecalle resummable along the positive real axis. For any real value of the kinematic parameter $L$, the resummed function $a \mapsto G^{\text{res}}(a, L)$ is analytic in the domain

$$\left| a - \frac{1}{20L} \right| < \frac{1}{20L}.$$

Before proving the theorem, we give a brief introduction to some aspects of Ecalle’s resurgence theory, which allows to review crucial concepts of the Borel-Ecalle resummation method. We first define the Borel-Laplace resummation operator (Definition 2.6), which is generalised by the Borel-Ecalle resummation method. We then introduce the notion of resurgent functions (Definition 2.10) and state some bounds on their convolution products that will be of crucial use later on (Equation 2 and Theorem 2.13). Finally we present the notion of well-behaved averages (Definition 2.19) which allows to state the main result of Borel-Ecalle resummation method, namely Theorem 2.21.

In the next section we start by introducing the model we will focus on: the massless Wess-Zumino model. It is a massless supersymmetric model in four dimensions. Supersymmetry prevents the need for a vertex renormalisation, thus drastically simplifying the Schwinger-Dyson equations. This makes this model a true QFT model simple enough to be used as a testing ground for the Borel-Ecalle resummation method. We first introduce the equations whose solution we will study.
using the tools of resurgence theory: a truncated version of the Schwinger-Dyson equation (4) and
the renormalisation group equation (5). We then state the known results we base this study upon,
in particular Theorem 3.1.

The following section focuses on the study of the renormalisation group equation. We start by
building a solution to the renormalisation group equation in a space of formal series (Proposition
4.2). We then show that this solution is 1-Gevrey (Proposition 4.5), and thus that its Borel
transform is analytic in a disc around the origin. The main result of this section is Theorem 4.12,
which states that the studied solution is indeed resurgent. Using basic results of the theory of
resurgent function and previous results on the Wess-Zumino model, the proof of this statement is
reduced to a proof of normal convergence of a series of functions. The proof of this property relies
on Sauzin’s non-linear analysis of resurgent functions [18].

The last section is a study of the asymptotic behavior of the Borel transform of the solution
of the Schwinger-Dyson equation and the renormalisation group equation. We are concerned with
its behavior at infinity on the principal branch of a suitable cover of the complex plane. We first
explain that a naive approach give the asymptotic bound

$$|\hat{G}(\zeta, L) | \leq K \exp(c|\zeta|^g(\zeta)L),$$

with $g$ an asymptotic bounds of the Borel transform of the anomalous dimension of the Wess-
Zumino model. This bound is not satisfactory since numerical studies of [6] suggest that $g$ does
not vanishes at infinity. To obtain a better bound, and to study the asymptotic behavior of $g$,
we need to make use of the Schwinger-Dyson equation. We start by expanding it (Equation (17))
and finding bounds on the numbers that appear in it (Lemma 5.4). We then use the Schwinger-Dyson
equation and the renormalisation group equation to find improved bounds on the functions whose
series is the Borel transform of the two point function (Proposition 5.8).

In the last subsection of this paper, we prove an asymptotic bound for the Borel transform
$\hat{g}$ of the anomalous dimension of the Wess-Zumino model (Proposition 5.9). This can then be
used together to derive an asymptotic bound for the two point function of the theory: Theorem
5.10. These bounds hold on the principal branch of the ramified plane $\mathbb{C}/\Omega$. Theorem 5.10,
together with Theorem 4.12 implies that the solution of the renormalisation group equation and
Schwinger-Dyson equation is Borel-Ecalle resummable (Corollary 5.11). The proof of Theorem 1.1
is concluded by Proposition 5.12 which precis the analyticity domain of the resummed function.

1.3 Some open questions

This paper aiming at being a gentle introduction to the basic concepts of Borel-Ecalle resummation
procedure we have tried to provide examples of the key concepts arising in the initial discussion.
The rest of the paper, which is more technical, also has the objective to convince the reader that
this procedure can actually be carried out in non trivial problems of mathematical physics. We
therefore try to motivate each computation in order to guide the reader through the sometimes
cumbersome computations. We feel that the results of this paper open new and exciting directions
of research which we now briefly describe.

The proof of the Borel-Ecallesummability of a simple, but non trivial QFT is a first step
towards physically more relevant models, a long-term goal being.

**Question 1.** Are the Yang-Mills model Borel-Ecalle (accelero-)summable?

There are still quite a few technical issues to be tackled before reaching this aim. For example,
the Schwinger-Dyson equations of non supersymmetric models generally do not close. One then
imposes one further equation to study the system, whose compatibility with the gauge symmetry
is still open. Also, the study of a asymptotically free QFT would probably require the more
sophisticated accelero-summation method.

However, other less ambitious questions seem to be within short term reach. One concerns the
transseries expansion of Borel-Ecalle resummed functions. There is a very precise analytical link
between a Borel summable series and the associated Borel resummed function, known as Watson’s
theorem (which was generalised by Sokal). To the best of the author’s knowledge, the equivalent
theorem for the Borel-Ecalle resummation method is not yet available:

**Question 2.** Is there a Watson’s theorem for the Borel-Ecalle resummation method?

Such a theorem would be of importance for the physical implications of the Borel-Ecalle resum-
ination method. Indeed, for these applications only a transseries expansions of the Borel-Ecalle
resummed function were computed. These transseries are not the full Borel-Ecalle resummed functions but rather a good approximation which can then be compared to experimental results. A Watson’s theorem for Borel-Ecalle resummation which would be formulated with transseries would provide a more precise meaning to the word “good” in the previous sentence and allow to have estimates for error margins coming from the truncations of the transseries.

Another reason why such a theorem would be of importance lies in the details of the physical applications of resurgence theory to physics. The coefficients of the transseries expansion are computed using the so-called median average, which can be expressed in terms of the alien derivatives of the formal series to be resummed. The median average is one special average, a notion that will be introduced below. However, it is not a “well-behaved average”, which are the ones that should be used for the Borel-Ecalle resummation method. Nonetheless, one could expect the transseries expansion of a function to be unique. Thus Watson’s theorem for Borel-Ecalle resummation would give a better mathematical ground to physical computations.

As we shall see later, in order to perform a Borel-Ecalle resummation on a formal series, a choice of a well-behaved average is required. This choice is not unique which raise a natural and important question:

Question 3. How does the Borel-Ecalle resummed function depend on the choice of the well-behaved average?

One could conjecture that it actually does not depends on the choice made and that changing averages amounts to a reparametrisation of the solution. This conjecture is motivated by an observation of [19] that it indeed holds for a specific problem and from the fact that two averages are always related by a so-called passage automorphism. Even if the choice of the average changes the resummed function, one should expect stability of some physically relevant properties, for example the poles of the resummed function. This observation relates this question with question 2: one should not expect the transseries expansion to depend on a specific choice of an average.

One last important question lying outside the scope of the present article is

Question 4. How and in which extent can one characterise the Borel-Ecalle resummed function solving a given problem?

It was argued in [6] that the Borel-Ecalle resummation method applied to QFT could give a non-perturbative mass generation mechanism. In order to study the relevance of this mechanism, one needs to study the poles of the resummed function. This rather ambitious question is linked to the question 2 and 4. Let us finally mention that this last question has motivated the present study.

2 Elements of resurgence theory

2.1 The Borel-Laplace resummation method

Many excellent introductions of the classical theory of Borel-Laplace resummation can be found in the literature. In particular, the PhD thesis [20] offers a well-written and short presentation of this topic (in French), while and the article [21] is a very thorough introduction. Nonetheless, Borel-Laplace resummation method will shortly be presented below in order to obtain a self-contained paper.

Definition 2.1. The formal Borel transform is defined on formal series as

\[ B : (z^{-1} \mathbb{C}[[z^{-1}]],.) \longrightarrow (\mathbb{C}[[\xi]], \star) \]

\[ \tilde{f}(z) = \frac{1}{z} \sum_{n=0}^{+\infty} \frac{c_n}{z^n} \quad \longrightarrow \quad \hat{f}(\xi) = \sum_{n=0}^{+\infty} \frac{c_n}{n!} \xi^n. \]

The formal Borel transform enjoys many useful properties, easy to prove by manipulation of formal series (see for example [21], §4.3 and 5.1).

Proposition 2.2. Let \( \tilde{f}(z), \tilde{g}(z) \in z^{-1} \mathbb{C}[[z^{-1}]] \) be two formal series and \( \hat{f}, \hat{g} \in \mathbb{C}[[\xi]] \) be their Borel transforms. Then the following hold

- \( B(\tilde{f} \cdot \tilde{g}) = \hat{f} \star \hat{g} \).
\( \mathcal{B}(\partial \hat{f}) = -\zeta \hat{f} \);

\( \mathcal{B}(z^{-1} \hat{f}) = \hat{f} \);  

if \( \hat{f}(z) \in \mathbb{C}[z^{-1}] \), then \( \mathcal{B}(z \hat{f}) = \frac{d \hat{f}}{d\zeta} \);

where the derivatives and the integral are formal (i.e. defined term by terms) and \( \ast \) stands for the convolution product of formal series. These properties stay true in the case where \( \hat{f}, \hat{g} \) are convergent. In this case, the first property becomes

\[
\mathcal{B}(\hat{f}, \hat{g})(\zeta) = (\hat{f} \ast \hat{g})(\zeta) = \int_0^\zeta \hat{f}(\eta) \hat{g}(\zeta - \eta) d\eta
\]

for \( \zeta \) in the intersection of the convergence domains of \( \hat{f} \) and \( \hat{g} \).

We will in fact study the case where the Borel transform is convergent. There exists a simple necessary and sufficient condition of the convergence of the Borel transform, but we need one more definition.

**Definition 2.3.** A formal series \( \hat{f}(z) = \frac{1}{\zeta} \sum_{n=0}^{+\infty} \frac{a_n}{\zeta^n} \) is 1-Gevrey if

\[
\exists A, B > 0 : |a_n| \leq AB^n n! \ \forall n \in \mathbb{N}.
\]

In this case, we write \( \hat{f}(z) \in \mathcal{C}[z^{-1}] \). An easy but important result (see for example [21], §4.2) is then:

**Theorem 2.4.** Let \( \hat{f}(z) \in \mathcal{C}[z^{-1}] \) be a formal series. Its Borel transform has a finite radius of convergence (in this case we write \( \hat{f} \in \mathcal{C}\{\zeta\} \)) if and only if \( \hat{f} \) is 1-Gevrey.

One can make other statements relating for example the Borel transform \( \hat{f} \) and the convergence of its associated formal series \( \hat{f} \), however such considerations will play no role here. The importance of the Borel transform for us lies in particular in the existence of an inverse operation: the Laplace transform.

**Definition 2.5.** Let \( \theta \in [0, 2\pi] \) and set \( \Gamma_\theta := \{R e^{i\theta}, R \in [0, +\infty]\} \). Let \( \hat{f} \in \mathcal{C}\{\zeta\} \) be a germ admitting an analytic continuation in an open subset of \( \mathbb{C} \) containing \( \Gamma_\theta \) and such that

\[
\exists c \in \mathbb{R}, \ K > 0 : |\hat{f}(\zeta)| \leq Ke^{c|\zeta|} \quad (1)
\]

for any \( \zeta \) in \( \Gamma_\theta \). Then the Laplace transform of \( \hat{f} \) in the direction \( \theta \) is defined as

\[
\mathcal{L}^\theta[\hat{f}](z) = \int_0^{+\infty} \hat{f}(\zeta) e^{-\zeta z} d\zeta.
\]

The Laplace integral of this definition is finite for \( z \) in an open subset of \( \mathbb{C} \) to be specified later on. For the time being, let us say that the composition of the Laplace and the Borel transforms is the so-called Borel-Laplace resummation method.

**Definition 2.6.** Let \( \theta \in [0, 2\pi] \), and \( \hat{f}(z) \in \mathcal{C}[z^{-1}] \) such that the Laplace transform of its Borel transform exists in the direction \( \theta \). Then \( \hat{f}(z) \) is said to be Borel summable in the direction \( \theta \).

The Borel-Laplace resummation operator in the direction \( \theta \) is defined on the functions Borel summable in the direction \( \theta \) as

\[
S_\theta = \mathcal{L}^\theta \circ \mathcal{B}.
\]

For a Borel summable formal series \( \hat{f} \), the function \( z \mapsto S_\theta[\hat{f}](z) \) is called the Borel sum of \( \hat{f} \).

Varying the direction \( \theta \) of the resummation leads to interesting concepts and phenomena such as sectorial resummation and the Stokes phenomenon, however we will not be interested in them here.

**Remark 2.7.** It is easy to see that a formal series \( \hat{f}(z) \in \mathcal{C}[z^{-1}] \) with a finite non-zero radius of convergence has a Borel transform admitting an exponential bound \( (1) \) for all \( \theta \in [0, 2\pi] \) and that its the Borel sum in any direction coincide with the usual summation of series. Thus the Borel-Laplace resummation method is an extension of the usual summation of series.
We claimed in Definition 2.6 that the Borel sum of a Borel summable function is a function. This is a consequence of the following theorem, which is itself a consequence of classical results of the theory of the Laplace transformation.

**Theorem 2.8.** Let \( \hat{f} \) be a formal series, Borel summable in the direction \( \theta \) with the exponential bound \( 1 \):

\[
\exists c \in \mathbb{R}, \ K > 0: |\hat{f}(\xi)| \leq Ke^{c|\xi|}.
\]

Then its Borel sum is analytic as a function of \( z \) in the half-plane \( \Re(ze^{i\theta}) > c \).

We have seen that one can perform the Borel-Laplace resummation method in non-singular directions of the Borel transform only. However, in many problems of interest (in particular, of interest to physicists), the Borel transform will have singularities in the direction where we wish to perform the resummation. Ecalle defines objects where the poles have a specified location (resurgent functions) and objects allowing to compute these singularities (Alien derivatives). The introduction of these concepts is the subject of the remaining part of this section.

### 2.2 Resurgent functions

In the rest of this text, we take \( \Omega \) a non-empty, discrete and closed subset of \( \mathbb{C} \). We recall that a function (or a germ) \( f \) holomorphic in a disc \( D \), around the origin is continuable along a path \( \gamma \) in \( \mathbb{C} \) starting within the disc of convergence of the function if there is a finite family \( (D_i)_{i \in \{1, \ldots, n\}} \) of convex open subset of \( \mathbb{C} \) covering \( \gamma \) such that \( f \) is analytically continuable to \( D \cup D_1 \cup \cdots \cup D_n \).

**Remark 2.9.** Being continuable along a path is much less strict than being continuab le. In particular, a function continuable along a family of paths can be seen as a function over an open subset of a cover of \( \mathbb{C} \) rather than a function of \( \mathbb{C} \).

**Definition 2.10.** A germ \( \varphi \in \hat{\mathbb{C}} \{\xi\} \) is said to be an \( \Omega \)-resurgent function if it is continuable along any rectifiable (i.e. of finite length) path in \( \mathbb{C} \setminus \Omega \). We set

\[
\hat{\mathbb{R}}_\Omega := \{\text{all } \Omega\text{-continuable germs}\} \subset \hat{\mathbb{C}} \{\xi\}.
\]

Now, the convolution product of two \( \Omega \)-resurgent function is well-defined inside the intersection of their convergence discs. A difficult question is whether or not this convolution product defines an \( \Omega \)-resurgent function. The following theorem is a cornerstone of resurgence theory, as it states when this is indeed the case and thus that resurgent functions are stable under an extension of the convolution product, and therefore to suited to the study of non-linear differential equations.

**Theorem 2.11.** (Ecalle, Sauzin [21][Theorem 21.1])

Let \( \Omega \subset \mathbb{C} \) be non-empty, discrete and closed. Then \( \hat{\mathbb{R}}_\Omega \) is stable under the convolution product if, and only if, \( \Omega \) is closed under addition.

The example (present for example in [21]) below is already enough to show that \( \Omega \) being closed under addition is a necessary condition. The hard part of the Theorem is thus to show that it is sufficient.

**Example 2.12.** Take \( \omega_1, \omega_2 \in \Omega \) and two meromorphic (and therefore resurgent) functions defined by \( \hat{f}_1(\xi) = \frac{1}{\xi - \omega_1} \), \( \hat{f}_2(\xi) = \frac{1}{\xi - \omega_2} \). Then a direct computation gives

\[
(\hat{f}_1 \ast \hat{f}_2)(\zeta) := \int_0^\zeta \hat{f}_1(\eta)\hat{f}_2(\zeta - \eta)d\eta
\]

\[
= \frac{1}{\zeta - \omega_1 - \omega_2} \left[ \int_0^\zeta \frac{d\eta}{\eta - \omega_1} + \int_0^\zeta \frac{d\eta}{\eta - \omega_2} \right]
\]

One can check that the R.H.S. has indeed a pole in \( \omega_1 + \omega_2 \). Therefore if \( \omega_1 + \omega_2 \) is not an element of \( \Omega \), \( \hat{f}_1 \ast \hat{f}_2 \) is not \( \Omega \)-resurgent.

For \( \Omega \subset \mathbb{C} \) non-empty, discrete and closed we set \( \rho(\Omega) := \min\{|\omega| : \omega \in \Omega^*\} \), with \( \Omega^* = \Omega \) if \( 0 \notin \Omega \) (this will be the case we will work with) and \( \Omega^* = \Omega \setminus \{0\} \) otherwise.
Finally, we will use here bounds on convolution products of resurgent functions. First, recall that for any open set $U \subset \mathbb{C}$ containing the origin and star-shaped with respect to the origin, the following bound holds by direct computation:

$$|(\hat{\phi}_1 \ast \cdots \ast \hat{\phi}_n)(\zeta)| \leq \frac{|\zeta|^{n-1}}{(n-1)!} \max_{[0,\zeta]} |\hat{\phi}_1| \cdots \max_{[0,\zeta]} |\hat{\phi}_n|$$

(2)

for any $\hat{\phi}_1, \cdots, \hat{\phi}_n$ holomorphic on $U$ and $\zeta$ in $U$. We used $[0, \zeta]$ to denote the straight line between $0$ and $\zeta$.

This bound will be useful to show that the two-point function has the right type of bound at infinity on the principal sheet and converges near the origin. However, it will not allow us to prove that it is resurgent. For this we will need to prove the normal convergence of a series of analytic functions along any paths avoiding $\Omega$. It will require the refined results of [18], specific to resurgence theory. In order to state this result, we need to introduce some notations, the same as in [18].

First, let $\mathcal{A}_\Omega$ be the set of homotopy classes with fixed endpoints of path $\gamma : [0, l] \to \mathbb{C} \setminus \Omega^*$ such that $\gamma(0) = 0$. Then, for $\delta, L \geq 0$ we set

$$K_{\delta, L}(\Omega) := \{ \zeta \in \mathcal{A}_\Omega | \exists \gamma \in \mathcal{A}_\Omega : \gamma(l) = \zeta, \text{ } \gamma \text{ of length } \leq L, \text{ } \text{dist}(\gamma(t), \Omega^*) \geq \delta \forall t \in [0, l] \}.$$ 

It was shown in [22] that $\mathcal{A}_\Omega$ has the structure of a Riemann surface, which is a cover of $\mathbb{C} \setminus \Omega$. Then $K_{\delta, L}(\Omega)$ can be described as the set of point of this cover which can be reached by paths of length less than $L$ and staying at a distance at least $\delta$ of $\Omega^*$. One can in particular see the set of $\Omega$-resurgent functions as the set of locally integrable maps $f : \mathcal{A}_\Omega \to \mathbb{C}$. This observation will become important to define the notion of average.

**Theorem 2.13.** [18, Theorem 1/]

Let $\Omega \subset \mathbb{C}$ be discrete, closed and stable under addition. Let $\delta, L > 0$ with $\delta < \rho(\Omega)$. Set

$$C := \rho(\Omega) \exp \left(3 + \frac{6L}{\delta}\right), \quad \delta' := \frac{1}{2} \rho(\Omega) \exp \left(-2 - \frac{4L}{\delta}\right), \quad L' := L + \frac{\delta}{2}. $$

Then, for any any $n \geq 1$ and $\hat{\phi}_1, \cdots, \hat{\phi}_n \in \hat{\mathcal{R}}_\Omega$

$$\max_{K_{\delta, L}(\Omega)} |\hat{\phi}_1 \ast \cdots \ast \hat{\phi}_n| \leq \frac{2 C^n}{n!} \max_{K_{\delta', L'}(\Omega)} |\hat{\phi}_1| \cdots \max_{K_{\delta', L'}(\Omega)} |\hat{\phi}_n|. $$

(3)

**Remark 2.14.** In subsequent work [22], Sauzin and Kamimoto have generalised this result to the cases where $\Omega$ is not stable under addition. One could in principle use the result of [22] to prove resurgence of the two-point functions on $\mathbb{Z}^* / 3$ rather than $\mathbb{N}^* / 3$. However this is not needed for the Ecalle-Borel resummation procedure along the positive real axis, and we will satisfy ourselves with using the above bound, which is of simpler use.

**Theorem 2.13** implies that the convolution product is bicontinuous for the natural topology induced by the family of semi-norms

$$||\hat{\phi}||_{\delta, L} := \max_{\zeta \in K_{\delta, L}(\Omega)} |\hat{\phi}(\zeta)|.$$

More precisely we have

**Corollary 2.15.** [18, Theorem 2, Remark 3.2]

$(\hat{\mathcal{R}}_\Omega, \ast)$ is a Fréchet algebra.

### 2.3 Borel-Ecalle resummation method

In practice we do not need to consider path going backward to perform a Borel-Ecalle resummation. To simplify the statements we take from now on $\Omega$ to be a subset of $\mathbb{R}^*_+$. 

**Definition 2.16.** Let $\mathbb{C} / \Omega$ be the $\Omega$-ramified plane, namely the space of homotopy classes $[\gamma]$ of rectifiable paths $\gamma : [0, 1] \to \mathbb{C} \setminus \Omega$ such that $\forall t, t' \in [0, 1], \ t < t' \Rightarrow \Re(\gamma(t)) < \Re(\gamma(t'))$.

\[\text{In order to avoid confusion between the kinematic parameter of the two-points function and the length of the path we will denote the former by the letter } \Lambda.\]
One can show that $\mathbb{C}/\Omega$ has the structure of a Riemann surface, see [22]. $\mathbb{C}/\Omega$ is a cover of $\mathbb{C}\setminus \Omega$. We call $\pi: \mathbb{C}/\Omega \to \mathbb{C}\setminus \Omega$ the canonical local biholomorphism associated to this Riemann surface. We refer the reader to [22, Section 3] for a precise definition of this geometric object. We omit these definitions as they will play only a minor role in the present work.

Let $\zeta \in \mathbb{C}\setminus \Omega$ and $\zeta' \in \mathbb{C}/\Omega$ such that $\pi(\zeta') = \zeta$. If $\Omega = \{\omega_1, \omega_2, \ldots\} \subset \mathbb{R}_+^*$ with $\omega_0 := 0 < \omega_1 < \omega_2 < \cdots$, we write $\zeta^{\epsilon_1, \ldots, \epsilon_n}$ instead of $\zeta$, with $\epsilon_i \in \{+,-\}$, $(\epsilon_1, \ldots, \epsilon_n)$ the signature of the branch of $\mathbb{C}/\Omega$ on which $\zeta^{\epsilon_1, \ldots, \epsilon_n}$ stands and $|\pi(\zeta^{\epsilon_1, \ldots, \epsilon_n})| \in [\omega_n, \omega_{n+1})$. From now on we will make the simplifying assumption that $\Omega = \omega N^*$ for some $\omega \in \mathbb{R}_+^*$.

While performing a Borel-Ecalle resummation, it will be useful to see $\Omega$-resurgent functions as locally integrable functions from $\mathbb{C}/\Omega \to \mathbb{C}$. We refer the reader to [22, Section 3] for a precise definition of this geometric object. We make the simplifying assumption that $\Omega = \mathbb{C}$ locally integrable functions from $\mathbb{C}/\Omega \to \mathbb{C}$.

We call $\phi$ a Catalan average: Let $\Omega$ be a real function.

The notion of average is too weak to be used as such. Indeed, we want the averaged function $\phi$ to

- solve the same equation as $\phi$;
- be a real function;
- admit a Laplace transform provided that $\phi$ had a reasonable behavior at infinity.

These requirements are formalized by the notion of well-behaved average.

**Example 2.18.**

- Left lateral average:
  \[ \text{mul}^{\varepsilon_1, \ldots, \varepsilon_n} = \begin{cases} 1 & \text{if } \varepsilon_1 = \cdots = \varepsilon_n = + \\ 0 & \text{otherwise}. \end{cases} \]

- Median average:
  \[ \text{mun}^{\varepsilon_1, \ldots, \varepsilon_n} = \frac{(2p)!(2q)!}{4^{p+q}(p+q)!(p!q!)}, \]

with $p$ (resp. $q$) the number of $+$ (resp. $-$) in $\{\varepsilon_1, \ldots, \varepsilon_n\}$.

- Catalan average: Let $Ca_n$ be the $n$-th Catalan number, $Qa_n(x)$ the $n$-th Catalan polynomial, $\alpha, \beta \in \mathbb{R}$, $\alpha + \beta = 1$.

Write $\underline{\varepsilon} = \varepsilon_1 \cdots \varepsilon_n = (\pm)^{n_1} (\mp)^{n_2} \cdots (\varepsilon_n)^{n_n}$, set

\[ \text{man}^{\underline{\varepsilon}}_{(\alpha, \beta)} = (\alpha \beta)^n Ca_{n_1} \cdots Ca_{n_{n-1}} Qa_n ((\alpha/\beta)^{\varepsilon_n}) . \]

The notion of average is too weak to be used as such. Indeed, we want the averaged function $\text{m} \phi$ to

- solve the same equation as $\phi$;
- be a real function;
- admit a Laplace transform provided that $\phi$ had a reasonable behavior at infinity.

**Definition 2.19.** An average $m$ is called well-behaved if

- (P1) It preserves the convolution $m(\phi \ast \psi) = (m\phi) \ast (m\psi)$.
- (P2) It preserves reality: $m^{\varepsilon_1, \ldots, \varepsilon_n} = m^{\varepsilon_1, \ldots, \varepsilon_n}$, with $\pm = \mp$.
- (P3) It preserves exponential growths: $\forall \phi \in \mathcal{H}_\Omega, \zeta \in \mathbb{C}\setminus \Omega$

\[ |\phi(\zeta^{\pm_1, \ldots, \pm_n})| \leq Ke^{c|\zeta|} \implies |(m\phi)(\zeta)| \leq Ke^{c|\zeta|} \]

\[ f: U \subset \mathbb{C} \to \mathbb{C} \text{ is real if } f(\Omega) = f(\overline{\Omega}) \text{ whenever both sides of the equation make sense. We require this condition since we want the resummed function to represent a physical quantity.} \]

---
Remark 2.20. In general the equation one is studying with the Borel-Ecalle resummation method is a differential equation. However, averages naturally preserve the differential structure: since $B(\partial_x f)(\zeta) = -\zeta \hat{f}(\zeta)$ and since $\zeta \mapsto -\zeta$ is in $\hat{U}_\Omega$, $L[mB(\partial_x f)](z) = \partial_x L[mB(f)](z)$. We used the variable $z = 1/a$ for the Borel transform for simplicity.

The following table lists the properties of the averages of Example 2.18.

```
|   | (P1) | (P2) | (P3) |
|---|------|------|------|
| mul | N    | N    | ✓    |
| mun | ✓    | ✓    | N    |
| man | ✓    | ✓    | ✓    |
```

In particular, the fact that the Catalan average is a well-behaved average is a highly non-trivial result of [13]. A finite number of other families of well-behaved averages are known. It is conjectured there are no more than the ones already known. Progresses toward a classification of well-behaved averages were recently made in [23], using methods from the theory of Rota-Baxter algebras.

Finally, the core of the Borel-Ecalle resummation method can be summed up in the following theorem:

**Theorem 2.21.** Let $(E)$ a differential equation admitting a solution $\hat{f} \in \mathcal{C}[a]_1$ such that $\hat{f} \in \mathcal{R}_\Omega$ for some $\Omega = \omega N^+ \subset \mathbb{R}^+$ and $|\phi(\zeta^n \cdots \hat{\zeta})| \leq K e^{c|\zeta|}$ for $|\zeta|$ big enough. Let $m$ be a well-behaved average. Then $f_{\text{res}} := L \circ m \circ B \circ \hat{f}$ is a solution of $(E)$ analytic in the open set $U = \{ a \in \mathbb{C} : |a - c/2| < c/2 \}$.

3 The Wess-Zumino model

We introduce the model we are going to study and state some known facts about it. Some of these results are well-known (e.g. the derivation of the Schwinger-Dyson and renormalisation group equations) while other are more recent. These can all be found in the PhD thesis [24].

3.1 Presentation of the model

The Wess-Zumino is one of the simplest possible supersymmetric model: it is massless and exactly supersymmetric. It was first introduced and studied in the papers [25, 26], seminal to supersymmetry. This model has two features that make it suitable as a first QFT to study within the framework of resurgence theory.

First, the $\beta$ and $\gamma$ functions are proportional: $\beta = 3\gamma$. This can in particular be shown using Hopf-algebraic techniques. It also presents the striking feature that it needs no vertex renormalisation, due to its (exact) supersymmetry. Therefore the Schwinger-Dyson equation for the two point function, truncated to the first loop, actually decouples from the Schwinger-Dyson equations for higher point functions. It reads

$$
(1-a)
$$

The other equation we are going to study is the renormalisation group equation. It takes the particularly simple form

$$
\partial_L G(L, a) = \gamma(a)(1 + 3a\partial_a)G(L, a)
$$

with $\gamma(a) := \partial_L G(a, L)|_{L=0}$ the anomalous dimension of the theory.

Using some known results of this model which are going to be listed in the next subsection, we will study the system composed of the renormalisation group and the Schwinger-Dyson equations. Let us emphasize that this study will be purely mathematical. Within the assumption that this study actually carries most of the information of the non perturbative regime of the Wess-Zumino model, we will then derive some physical interpretations of our work at the very end of this paper.
3.2 State of the art

Writing \( G \) as a formal series in \( L \)

\[
G(L, a) = \sum_{k=0}^{+\infty} \gamma_k(a) \frac{L^k}{k!},
\]

(with \( \gamma_0(a) = 1 \) and \( \gamma_1(a) =: \gamma(a) \)) we can easily write the RGE \( [5] \) as an induction relation on the \( \gamma_k \)'s:

\[
\gamma_{k+1}(a) = \gamma(1 + 3a \partial_a) \gamma_k.
\]

This justifies that we look for an equation over \( \gamma \) rather than an equation over \( G \). Plugging the expansion \( [6] \) into the Schwinger-Dyson equation \( [4] \) and computing the Feynman integral we obtain

\[
\gamma(a) = a \left( 1 + \sum_{n=1}^{+\infty} \frac{\gamma_n(a)}{n!} \frac{d^n}{dx^n} \right) \left( 1 + \sum_{m=1}^{+\infty} \frac{\gamma_m(a)}{m!} \frac{d^m}{dx^m} \right) H(x, y) \bigg|_{x=y=0},
\]

with \( H \) the one-loop Mellin transform:

\[
H(x, y) = \frac{\Gamma(1+x)\Gamma(1+y)\Gamma(1-x-y)}{\Gamma(1-x)\Gamma(1-y)\Gamma(2+x+y)} = \frac{1}{1+x+y} \exp \left( \sum_{k=1}^{+\infty} \frac{2k+1}{2k+1} ((x+y)^{2k+1} - x^{2k+1} - y^{2k+1}) \right).
\]

We will study the Borel transform of this equation. It maps the usual product of formal series to a convolution product and the identity function to the constant function \( \zeta \mapsto 1 \). Separating the 1 in the equation above from the rest we end up with

\[
\hat{\gamma}(\zeta) = 1 + 2 \sum_{n=1}^{+\infty} \frac{(1 \ast \hat{\gamma}_n)(\zeta)}{n!} \frac{d^n}{dx^n} H(x, y) \bigg|_{x=y=0} + \sum_{n, m=1}^{+\infty} \frac{(1 \ast \hat{\gamma}_n \ast \hat{\gamma}_m)(\zeta)}{n!m!} \frac{d^n}{dx^n} \frac{d^m}{dy^m} H(x, y) \bigg|_{x=y=0}.
\]

Similarly, taking the Borel transform of the renormalisation group equation \( [7] \) one obtains

\[
\hat{\gamma}_{n+1}(\zeta) = \hat{\gamma} \ast (4 + 3\zeta \partial_\zeta) \hat{\gamma}_n(\zeta).
\]

Now, \( \gamma(a) \) is a formal series with coefficients in \( \mathbb{C} \), without constant term:

\[
\gamma(a) = \sum_{n=1}^{+\infty} c_n a^n.
\]

A result of \([27]\) (with more orders computed in \([28]\)) is the following asymptotic behavior of the coefficients \( c_n \):

\[
c_{n+1} = -(3n + 2 + O(n^{-1})) c_n.
\]

Furthermore, one easily check that the first terms of this expansion are given by \( c_1 = 1 \) and \( c_2 = -2 \).

One important result we will build upon is

**Theorem 3.1.** \([27]\) \( \hat{\gamma} \) is \( \mathbb{Z}^*/3\)-resurgent.

Since we want to resum the two-point function in the direction \( \theta = 0 \), we will focus on this direction. Therefore we will set \( \Omega = \mathbb{N}^*/3 \) and move on to prove that the two-point function is \( \Omega \)-resurgent.

4 Resurgent analysis of the RGE

4.1 Solution of the renormalisation group equation

We want to study the two-points function \( G(L, a) \) as a formal series in \( a \). We first show that \( G(L, a) \) is indeed such a formal series thanks to the following lemma.
Lemma 4.1. For any \( L \in \mathbb{C} \); the formula (13) defines a formal series in \( a \) with coefficients depending on \( L \).

Proof. Since \( \gamma_0(a) = 1 \) by definition and \( \gamma_1(a) = \gamma(a) \) lies in \( a\mathbb{C}[[a]] \) as a result of \( [29] \), we obtain from (7) with a trivial induction that, for any \( k \in \mathbb{N} \), \( \gamma_k(a) \in a^k\mathbb{C}[[a]] \). Then, for \( n \geq 1 \), contributions to \( a^n \) in \( G(L, a) \) can only come from \( \gamma_1(a), \ldots, \gamma_n(a) \) and their sum is therefore finite.

The fact that we had to make this small manipulation indicates that the expansion (13) is not suited to the study of \( G(L, a) \) as a formal series in \( a \). We will therefore use the following alternative expansion of the two-points function.

\[
G(L, a) = \sum_{n=0}^{+\infty} g_n(L)a^n \in A[[a]] 
\]

with \( A \) some suitable algebra of smooth functions or formal series.

Proposition 4.2. The renormalisation group equation (5) admits a solution of the form (13), with \( A = \mathbb{C}[L] \), explicitly given by \( g_0(L) = 1 \) and

\[
g_n(L) = \sum_{q=1}^{n} \left( \sum_{i_q \geq 0 \atop i_q + \cdots + i_1 = n} c_{i_1} \cdots c_{i_q} K_{i_1 \cdots i_q} \right) \frac{L^q}{q} \tag{14}
\]

with the \( c_n \) the coefficients of \( \gamma(a) \) and \( K_{i_1 \cdots i_q} \) real numbers inductively defined for any \( n \in \mathbb{N} \) and \( q \in \{2, \ldots, n+1\} \) by \( K_n = 1 \) and

\[
K_{i_1 \cdots i_q} = (1 + 3(n+1-i_q))K_{i_1 \cdots i_q-1}
\]

with \( i_1 + \cdots + i_q = n+1 \).

Proof. First, observe that the SDE (4) taken at \( a = 0 \) gives \( G(L, 0) = g_0(L) = 1 \). Furthermore, the RGE (5) implies the following family of differential equations (with \( n \geq 1 \)) when one replaces \( G(L, a) \) by its representation (13)

\[
g'_n(L) = \sum_{p=1}^{n} c_p(1 + 3(n-p))g_{n-p}(L).
\]

Notice that at this stage the derivative can be the derivative of function or the derivative of formal series.

We now prove that these equations are solved as claimed by (13) by induction. For the case \( n = 1 \), the equation reduces to \( g'_1(L) = 1 \) since \( c_1 = 1 = g_0(L) \). This is solved to \( g_1(L) = L \) since by the expansion (13), \( G(L, a) \) has only \( 1 = g_0(L) \) as a term independent of \( L \). We thus find \( K_1 = 1 \) as claimed.

It will be important for the induction step to have performed the case \( n = 2 \). Observing that \( g_1(L) = c_1L \) since \( c_1 = 1 \) we find for \( g_2 \) the equation \( g'_2(L) = c_1(1 + 3(2-1))c_1L + c_2 \). This integrates to

\[
g_2(L) = (c_1)^2(1 + 3(2 - 1))\frac{L^2}{2} + c_2L
\]

without constant term for the same reason than the case \( n = 1 \) treated above. We then find \( K_2 = 1 \) and \( K_{11} = (1 + 3(2 - 1))\frac{K_2}{2} \) as claimed.

Let us now assume that the statement of the proposition holds for \( n \geq 2 \). Writing aside the term \( p = n + 1 \), integrating and switching the sum over \( q \) by one we find

\[
g_{n+1}(L) = c_{n+1}L + \sum_{p=1}^{n} c_p(1 + 3(n + 1 - p))\sum_{q=2}^{n+1-(p-1)} \frac{L^q}{q} \sum_{i_q \geq 0 \atop i_q + \cdots + i_1 = n+1-p} c_{i_1} \cdots c_{i_{q-1}} K_{i_1 \cdots i_{q-1}}.
\]

As before, we do not have a constant term thanks to the expansion (13).
Noticing that $\sum_{p=1}^{n} \sum_{q=2}^{n+1-(p-1)} = \sum_{q=2}^{n+1} \sum_{p=1}^{n+1-(q-1)}$ we can rewrite $g_{n+1}(L)$ as

$$c_{n+1}L + \frac{L^q}{q} \sum_{p=1}^{n+1} \sum_{q=2}^{n+1-(p-1)} (\cdots).$$

Now we can relabel the sum over $p$ as a sum over $i_q$. Thus the sums over $p$ and $i_1, \cdots, i_{q-1}$ can be merged. We obtain

$$g_{n+1}(L) = c_{n+1}L + \frac{L^q}{q} \left( \sum_{i_1, \cdots, i_q > 0} c_{i_1} \cdots c_{i_q} (1 + 3(n + 1 - i_q))K_{i_1, \cdots, i_{q-1}} \right).$$

We therefore have the right form for $g_{n+1}(L)$, $K_{n+1} = 1$ and the induction relation over the $K_{i_1, \cdots, i_q}$ claimed in the Proposition.

### 4.2 The two-point function is 1-Gevrey

To prove that the formal series (13) is indeed 1-Gevrey, we first need a reformulation of the formula (12).

**Lemma 4.3.** For any $n \in \mathbb{N}^*$, the following bounds hold

$$(3\delta)^{n-1}(n-1)! \leq |c_n| \leq (3K)^n n!$$

for some $K > 1$ and $\delta \in [0, 1]$.

**Proof.** The proof is by induction. The case $n = 1$ holds since $c_1 = 1$. Assuming both inequalities hold for $n \in \mathbb{N}^*$, we first have

$$|c_{n+1}| = |3n + 2 + \mathcal{O}(n^{-1})||c_n| \leq 3K(n + 1)|c_n|$$

provided $K$ has been chosen large enough. The upper bound of $|c_{n+1}|$ then follows from the upper bound of $|c_n|$. For the lower bound, one writes

$$|c_{n+1}| = |3n + 2 + \mathcal{O}(n^{-1})||c_n| \geq 3n\delta|c_n|$$

(provided $\delta$ has been chosen small enough) and the lower bound of $|c_{n+1}|$ then follows from the lower bound of $|c_n|$.

One can without too much trouble show that

$$\frac{1}{q} K_{i_1, \cdots, i_q} \leq \frac{1}{n} K_{1, \cdots, 1} = (3n - 2)!!.$$

with $n = i_1 + \cdots + i_q$ and $(3n - 2)!! = \prod_{i=1}^{3n-2} (3n - 2 - i)$. However this bound is too crude: we need a bound that is not uniform in $q$. Indeed, one obtain from the Lemma 4.3 that the term $c_{i_1} \cdots c_{i_q}$ in the solution (14) is dominated by the case $q = 1$ while the term $K_{i_1, \cdots, i_q}$ is dominated by the term $q = n$. It is the fact that these two bounds cannot be reached together that will allow to prove that the solution (14) is 1-Gevrey.

Recall that for $n \in \mathbb{N}^*$, a **composition** of $n$ is a finite sequence $(i_1, \cdots, i_q)$ of strictly positive integers such that $i = 1 + \cdots + i_q = n$. For any composition $(i_1, \cdots, i_q)$ of $n \in \mathbb{N}^*$ recall that the **multinomial number** $\binom{n}{i_1, \cdots, i_q}$ is defined by

$$\binom{n}{i_1, \cdots, i_q} := \frac{n!}{i_1! \cdots i_q!}.$$

These numbers famously appear in the multinomial theorem and have many important combinatorics properties.

**Lemma 4.4.** For any $n \in \mathbb{N}^*$ and composition $(i_1, \cdots, i_q)$ of $n$, we have

$$\frac{1}{q} K_{i_1, \cdots, i_q} \leq \frac{3^n}{n} \binom{n}{i_1, \cdots, i_q}.$$
Proof. First, observe that, for any \( n \in \mathbb{N}^* \), the case \( q = 1 \) trivially holds since \( K_n = 1 = \binom{n}{1} \). We now prove that the result holds for every \( n \) and every \( q \) by induction over \( n \).

For \( n = 1 \), the inequality trivially holds (it is the equality case). Assume it holds for all \( p \in \{1, \ldots, n\} \) for some \( n \in \mathbb{N}^* \) and let \((i_1, \ldots, i_q)\) be a composition of \( n + 1 \). We have already seen if \( q = 1 \) the result holds. If \( q \geq 2 \) we then have

\[
\frac{1}{q} K_{i_1, \ldots, i_q} \leq (1 + 3(n + 1 - i_q)) \frac{K_{i_1 \ldots i_{q-1}}}{q - 1} \leq (1 + 3(n + 1 - i_q)) \frac{3^{n+1-i_q}}{n + 1 - i_q} \binom{n + 1 - i_q}{i_1, \ldots, i_{q-1}}
\]

by the induction hypothesis, which we can use since \( q \geq 2 \) and thus \( i_q \in \{1, \ldots, n\} \).

From the definition of the multinomial numbers, we have

\[
\binom{n + 1 - i_q}{i_1, \ldots, i_{q-1}} = \binom{n + 1}{i_q} \binom{n + 1}{i_1, \ldots, i_{q-1}}.
\]

The result on rank \( n + 1 \) then follows from the observation that

\[
\left(3 + \frac{1}{n + 1 - i_q}\right) \binom{n + 1}{i_q}^{-1} \leq 3^q
\]

for every \( n \in \mathbb{N}^* \) and \( i_q \in \{1, \ldots, n\} \).

We are now ready to prove the main result of this subsection, namely that the two-point function is 1-Gevrey

**Proposition 4.5.** The two-point function \( G(L, a) \) is a 1-Gevrey as a formal series in \( a \): for any \( L \in \mathbb{R} \)

\[
|g_n(L)| \leq \frac{3}{2}(18K^2\bar{L})^n n!
\]

with \( \bar{L} := \max\{L, 1\} \) and \( K \) the constant appearing in the upper bound of \( |c_n| \) in Lemma 4.3.

**Remark 4.6.** In practice, we are interested in the non perturbative regime which in the WZ model appears for \( p^2 = \mu^2 \exp(L) \to \infty \). In this regime, we see that the locus of the first singularity of the two-point function could depend on \( L \) and in particular go to zero as \( L \to \infty \). We will see later that this is not the case. However the first singularities of \( G(\zeta, L) \) can move in an intermediate regime. This indicates that the singularities of the Borel transform contains non perturbative information of the theory (which is not a new observation: see for example [30]). Therefore resurgence theory has to be an important tool to unravel non perturbative aspects of QFTs.

**Proof.** Using Lemma 4.3 we have

\[
\left| \frac{c_n}{c_{i_1, \ldots, i_q}} \right| \geq \frac{(3\delta)^{n-1}(n-1)!}{(3K^3)^{i_1!} \cdots (3K^3)^{i_q!}} = \frac{3^{n-1}}{3n} \binom{n}{i_1, \ldots, i_q} = \frac{1}{3n} K^n \binom{n}{i_1, \ldots, i_q}.
\]

Using this as an upper bound for \( |c_{i_1, \ldots, i_q}| \) together with the bound for \( \frac{1}{q} K_{i_1, \ldots, i_q} \) of Lemma 4.3 we obtain

\[
|g_n(L)| \leq 3 \sum_{q=1}^{n} \binom{n}{i_1, \ldots, i_q, i_{q+1}=n} (3K)^n |c_n| \left( \sum_{i_1+\ldots+i_q=n} L^q \right) = 3(3K)^n |c_n| \sum_{q=1}^{n} \binom{n-1}{q-1} L^q
\]

where we have used the simple combinatorial result that there are \( \binom{n-1}{q-1} \) compositions of \( n \) with length \( q \). Using that \( L^q \leq \bar{L}^n \) for any \( q \in \{1, \ldots, n\} \) and once more the upper bound for \( |c_n| \) of Lemma 4.3 we find the result of the Theorem since \( \sum_{q=1}^{n} \binom{n-1}{q-1} = 2^{n-1} \).

**Remark 4.7.** One can use the bound (15) more directly to find a more precise bound:

\[
|g_n(L)| \leq 3(9K^2)^n L(L + 1)^{n-1} n!
\]

which holds for all \( L \). This bound indicates that the first singularities of the Borel transform is rejected to infinity in the perturbative limit \( L \to 0 \) (but not that \( G(L, a) \) is analytic in this limit), and therefore that the non perturbative effects encoded in the singularities of the Borel transform vanish as expected in the perturbative limit \( L \to 0 \).

\(^3\)at least the first one, but since a singularities in \( \omega \in \mathbb{C}^* \) generally produces new singularities in \( \omega \mathbb{N}^* \) (as in Example 2.12), we expect that all singularities will depend on \( L \), at least in some non perturbative regime.
4.3 The two-point function is resurgent

We start with an easy Lemma:

Lemma 4.8. The function $\tilde{\gamma}_n$ is $\Omega$-resurgent for all $n$ in $\mathbb{N}^*$.

Proof. This result is a direct consequence of the fact that the space of $\Omega$-resurgent functions is stable under convolution, derivation and multiplication by an analytic function together with the fact that $\tilde{\gamma}$ is resurgent (Theorem 3.1). This Lemma is then an easily shown by induction using the renormalisation group equation (11). \hfill \Box

The space of resurgent functions is stable by sums, but the above Lemma is not enough to prove that $\sum_{n \geq 1} \tilde{\gamma}_n(\zeta) \frac{1}{n} =: \hat{G}(\zeta, \Lambda)$ is $\Omega$-resurgent. In order to tame the combinatorics of the objects appearing in the proof, let us introduce some intermediate objects.

Definition 4.9. For any $n \in \mathbb{N}^*$ define the set $W_n$ as the subset of words written in the alphabet $\{*, \}$ such that

$$W_1 := \{\emptyset\}, \quad W_{n+1} := \{(* \sqcup w) \mid w \in W_n\} \cup \{(* \sqcup w) \mid w \in W_n\}$$

with $\sqcup$ the concatenation product of words. We further set $W := \bigcup_{n \in \mathbb{N}^*} W_n$.

Lemma 4.10. For any $n \in \mathbb{N}^*$ we have $|W_n| = 2^{n-1}$.

Proof. For any $n \in \mathbb{N}^*$ write $W_{n+1} = A_n \sqcup B_n$ with $A_n := \{(* \sqcup w) \mid w \in W_n\}$ and $B_n := \{(* \sqcup w) \mid w \in W_n\}$. Let us check that $A_n \cap B_n = \emptyset$. Let $W_{n+1} \ni w \in A_n \cap B_n$. Then it exists $w_1 \in A_n$ and $w_2 \in B_n$ such that

$$w = (* \sqcup w_1) = (* \sqcup w_2).$$

This implies that $w_1 \neq \emptyset$ and since every nonempty word in $W$ starts with $*$ we can write $w_1 = (* \sqcup w_3)$ for some word $w_3$ not necessarily in $W$. We then have $w = (* * \sqcup w_3) = (* \sqcup w_2)$ which a contradiction. Then $A_n \cap B_n = \emptyset$ and $|W_{n+1}| = 2|W_n|$. The result then follows from $|W_1| = 1 = 2^0$. \hfill \Box

Finally, let us prove a simple but useful lemma about analytic continuation of series.

Lemma 4.11. Let $U \subset V$ be two open subsets of $\mathbb{C}$. Let $f_n : U \mapsto \mathbb{C}$ be a sequence of holomorphic functions such that:

1. $f := \sum_{n = 0}^{\infty} f_n$ is holomorphic in $U$;
2. $f_n$ admits an analytic continuation $\tilde{f}_n$ to $V$;
3. $\tilde{f}_n$ is bounded on $V$ by an analytic function $F_n$: $|\tilde{f}_n| \leq F_n$;
4. The series $F = \sum_{n = 0}^{\infty} F_n$ converges in $V$.

Then $f$ admits an analytic continuation $\tilde{f}$ to $V$ and $|\tilde{f}| \leq F$.

Proof. For any $z \in V$, let us set

$$S_N(z) := \sum_{n = 0}^{N} |\tilde{f}_n(z)| \leq \sum_{n = 0}^{N} F_n(z) \longrightarrow F(z)$$

as $N \rightarrow \infty$. Then $S_N(z)$ is increasing and bounded and therefore convergent. Therefore the series $f(z) := \sum_{n = 0}^{\infty} f_n(z)$ is absolutely convergent and thus convergent. This series by definition is an analytic continuation of $f$ to $V$ and is bounded by $F$. \hfill \Box

We are now ready to prove the main result of this section.

Theorem 4.12. For any $\Lambda \in \mathbb{R}$, the map $\zeta \mapsto \hat{G}(\zeta, \Lambda)$ is $\Omega$-resurgent.
Proof. Let \( \delta, L > 0 \) with \( \delta < \rho(\Omega)/2 \). Let \( \gamma \) be a path in \( \mathcal{K}_{\delta,L}(\Omega) \). According to Lemma 4.11 we only need to prove that the series

\[
\sum_{n \geq 1} \frac{(\text{cont}_\gamma \hat{\gamma}_n)(\zeta) A^n}{n!}
\]

converges normally. Indeed, in this case, it will be equal to

\[
(\text{cont}_\gamma \hat{\gamma})(\zeta, \Lambda) := \left( \text{cont}_\Lambda \sum_{n=1}^{\infty} \hat{\gamma}_n \right)(\zeta).
\]

For any \( N \in \mathbb{N}^* \), we will deduce from a bound on \( \hat{\gamma} \) a bound on \( \hat{\gamma}_{N+1} \) in the domain \( \mathcal{K}_{\delta,L}(\Omega) \) which contain the path \( \gamma \). So, fix \( N \in \mathbb{N}^* \) and for \( n \in \{1, \cdots, N + 1\} \), set

\[
\delta_n := \frac{\delta}{2} + (n - 1) \frac{\delta}{2N}, \quad L_n := L + \frac{\delta}{2} - (n - 1) \frac{\delta}{2N}.
\]

We did not write the dependence on \( N \) of \( \delta_n \) and \( L_n \) to lighten the notations. Notice however that \( \delta_1 = \delta/2 \) and \( L_1 = L + \delta/2 \) for any \( N \in \mathbb{N}^* \).

We now define a map

\[
f : W \rightarrow \hat{\mathcal{R}}_\Omega
\]

\[
w \mapsto f_w
\]

recursively by

\[
f_\emptyset(\zeta) := |\hat{\gamma}(\zeta)| + S, \quad f_{(\ast),\text{lw}}(\zeta) := 4(f_\emptyset \ast f_w)(\zeta), \quad f_{(\ast),\text{lw}}(\zeta) := \frac{6NK}{\delta} (f_\emptyset \ast f_w)(\zeta)
\]

where we have set

\[
S := \max_{\zeta \in K_{\delta_n,L_n}(\Omega)} |\hat{\gamma}(\zeta)| \quad \text{and} \quad K := \max_{\zeta \in K_{\delta_n,L_n}(\Omega)} |\zeta|.
\]

The map \( f \) is well-defined due to the proof above that the sets \( A_n \) and \( B_n \) do not intersect. Furthermore its image is a subset of the \( \Omega \)-resurgent functions since they are stable by convolution and by multiplication by analytic functions.

The analytical part of this proof is now essentially contained is the next Lemma.

**Lemma 4.13.** For any \( N \in \mathbb{N}^* \) and \( n \in \{1, \cdots, N + 1\} \) we have

\[
|\hat{\gamma}_n(\zeta)| \leq \sum_{w \in W_n} f_w(\eta)
\]

for any \( \zeta, \eta \in K_{\delta_n,L_n}(\Omega) \).

**Proof.** We prove this result by induction on \( n \). For \( n = 1 \) we have \( f_\emptyset(\zeta) \geq S = \max_{\zeta \in K_{\delta_1,L_1}(\Omega)} |\hat{\gamma}(\zeta)| \) and therefore the lemma holds. Assume it holds for \( n \in \{1, \cdots, N\} \). We then have, for any \( \zeta \in K_{\delta_{n+1},L_{n+1}}(\Omega) \)

\[
|\hat{\gamma}_{n+1}(\zeta)| \leq 4|\hat{\gamma}| * |\hat{\gamma}_n|(|\zeta| + 3|\hat{\gamma}| * |\zeta\partial\hat{\gamma}_n|(|\zeta|).
\]

Then using the induction hypothesis and the continuity of the convolution product we have

\[
4|\hat{\gamma}| * |\hat{\gamma}_n| \leq \sum_{w \in W_n} 4(f_\emptyset \ast f_w)(\eta) = \sum_{w \in W_n} f_{(\ast),\text{lw}}(\eta)
\]

for any \( \eta \in K_{\delta_n,L_n}(\Omega) \subset K_{\delta_{n+1},L_{n+1}}(\Omega) \).

Now, by definition, for any \( \zeta \in K_{\delta_{n+1},L_{n+1}}(\Omega) \), the disc of center \( \zeta \) and radius \( \frac{\delta}{2N} \) lies in \( K_{\delta_n,L_n}(\Omega) \). Therefore, using the definition of \( K \) and Cauchy inequality on the disc of center \( \zeta \) and radius \( \frac{\delta}{2N} \) we find

\[
|\zeta\partial\hat{\gamma}_n(\zeta)| \leq \frac{2NK}{\delta} \max_{\zeta \in D(\zeta, \delta/2N)} |\hat{\gamma}_n(\zeta)| \leq \sum_{w \in W_n} \frac{2NK}{\delta} f_w(\eta)
\]

for any \( \eta \in K_{\delta_n,L_n}(\Omega) \subset K_{\delta_{n+1},L_{n+1}}(\Omega) \). Thus

\[
3|\hat{\gamma}| * |\zeta\partial\hat{\gamma}_n|(|\zeta| \leq \sum_{w \in W_n} \frac{6NK}{\delta} (f_\emptyset \ast f_w)(\eta) = f_{(\ast),\text{lw}}(\eta).
\]
for any \( \eta \in K_{\delta_n, L_n}(\Omega) \subset K_{\delta_{n+1}, L_{n+1}}(\Omega) \). Combining this bound with the one for \( 4|\hat{\gamma}| \cdot |\hat{\gamma}_n|() \) we obtain

\[
|\hat{\gamma}_{n+1}(\zeta)| \leq \sum_{w \in W_n} (f(\zeta *w(\eta)) + f(\zeta_\ast w(\eta))) = \sum_{w \in W_{n+1}} f_w(\eta)
\]

for any \( \eta \in K_{\delta_{n+1}, L_{n+1}}(\Omega) \).

We now need to bound \( f_w \). Write \( |w| \) the number of times the letter \( . \) is present in the word \( w \in W \). Then for any \( n \in \{1, \cdots, N+1\} \) and \( w \in W_n \) we have

\[
f_w(\zeta) = \left( \frac{6NK}{\delta} \right)^{|w|} \frac{4^n}{(N+1)!} f^*_n(\zeta).
\]

We can now use Sauzin’s bound (3) for \( n = N + 1 \):

\[
\max_{\zeta \in K_{K_{\delta, L}}(\Omega)} f_w(\zeta) \leq \left( \frac{6NK}{\delta} \right)^{|w|} \frac{4^{N+1} C^{N+1}}{(N+1)!} \left( \max_{\zeta \in K_{K_{\delta, L+1/2}}(\Omega)} f_0(\zeta) \right)^{N+1}
\]

where we have used that \( |w| \in \{0, 1, \cdots, N\} \). Now, using that \( \delta/2 = \delta_1 \) and \( L + \delta/2 = L_1 \) we find \( \max_{\zeta \in K_{K_{\delta, L+1/2}}(\Omega)} f_0(\zeta) = 2S \). Using Lemmas 1.13 and 1.10 we obtain

\[
\max_{\zeta \in K_{K_{\delta, L}}(\Omega)} |\hat{\gamma}_{N+1}(\zeta)| \leq \frac{\delta}{12K} \left( \frac{96}{\delta} SKC \right)^{N+1} \frac{N^N}{(N+1)!}.
\]

Using Stirling’s formula we then have the following bound, for \( N \) big

\[
\max_{\zeta \in K_{K_{\delta, L}}(\Omega)} |\hat{\gamma}_{N+1}(\zeta)| \leq \frac{\delta}{12Ke} \left( \frac{96}{\delta} SKC e \right)^{N+1} \frac{1}{\sqrt{2\pi N}} \left( 1 + O \left( \frac{1}{\sqrt{N}} \right) \right).
\]

This implies the normal convergence of the series \( \sum_{n \geq 1} (\text{cont}_\zeta \hat{\gamma}_{n})(\zeta) \frac{S^n}{n^r} =: (\text{cont}_\zeta \hat{G})(\zeta, \Lambda) \) and concludes the proof.

5  Asymptotic bound of the two-point function

We now prove that \( \hat{G}(\zeta, L) \) admits an exponential bound in the principal branch \( U_\Omega \) of \( \mathbb{C}/\Omega \).

5.1  Statement of the problem

The following lemma implies that one actually need to study the Schwinger-Dyson equation in order to find the right type of bound on the two-point function.

Lemma 5.1. Let \( g : U_\Omega \to \mathbb{R}_+ \) be an increasing analytic function such that, for any \( \zeta \in U_\Omega \)

\[
\max \left\{ \max_{\eta \in [0, \zeta]} |\hat{\gamma}(\eta)|, \max_{\eta \in [0, \zeta]} |\hat{\gamma}'(\eta)| \right\} \leq g(\zeta).
\]

Then for any \( n \in \mathbb{N}^* \) we have

\[
\max \left\{ \max_{\eta \in [0, \zeta]} |\hat{\gamma}_n(\eta)|, \max_{\eta \in [0, \zeta]} |\hat{\gamma}'_n(\eta)| \right\} \leq [(4 + 3|\zeta|)(1 + g(\zeta)|\zeta|)]^{n-1} g(\zeta).
\]

Remark 5.2. The function \( g \) exists since \( \hat{\gamma} \) and \( \hat{\gamma}' \) are analytic (but not bounded) on \( U_\Omega \).

Proof. We prove this Lemma by induction. The case \( n = 1 \) holds by definition of \( g \). Assuming the Lemma holds for some \( n \in \mathbb{N}^* \); we use the bound (2) (which we can use on \( U_\Omega \) since it is star-shaped with respect to the origin) on the renormalisation group equation (11) to obtain, for any \( \zeta \in U_\Omega \)

\[
|\hat{\gamma}_{n+1}(\zeta)| \leq g(\zeta)|\zeta|(4 \max_{\eta \in [0, \zeta]} |\hat{\gamma}_n(\eta)| + 3|\zeta| \max_{\eta \in [0, \zeta]} |\hat{\gamma}'_n(\eta)|)
\]

\[
\leq (4 + 3|\zeta|)g(\zeta)|\zeta| \max \left\{ \max_{\eta \in [0, \zeta]} |\hat{\gamma}_n(\eta)|, \max_{\eta \in [0, \zeta]} |\hat{\gamma}'_n(\eta)| \right\}
\]

\[
\leq (4 + 3|\zeta|)(1 + g(\zeta)|\zeta|) \max \left\{ \max_{\eta \in [0, \zeta]} |\hat{\gamma}(\eta)|, \max_{\eta \in [0, \zeta]} |\hat{\gamma}'(\eta)| \right\}.
\]
For any $\eta \in [0, \zeta]$ we further have
\[
|\hat{\gamma}_{n+1}(\eta)| \leq (4 + 3|\eta|)(1 + g(\eta)|\eta|) \max \left\{ \max_{\sigma \in [0, |\eta|]} |\hat{\gamma}_n(\sigma)|, \max_{\eta \in [0, |\eta|]} |\hat{\gamma}_{n}(\sigma)| \right\}
\]
\[
\leq (4 + 3|\zeta|)(1 + g(|\zeta|)|\zeta|) \max \left\{ \max_{\eta \in [0, |\zeta|]} |\hat{\gamma}(\eta)|, \max_{\eta \in [0, |\zeta|]} |\hat{\gamma}_{n}(\eta)| \right\}
\]
since we have assumed $g$ to be increasing. Therefore $\max_{\eta \in [0, |\zeta|]} |\hat{\gamma}_{n+1}(\zeta)|$ admits the bound of the Lemma.

To obtain a bound on $|\hat{\gamma}_{n+1}'(\zeta)|$ we use Leibniz’s formula
\[
\frac{d}{dt} \int_{a(t)}^{b(t)} f(t, x)dx = b'(t)f(t, b(t)) - a'(t)f(t, a(t)) + \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(t, x)dx;
\]
which holds provided $a$, $b$ and $f$ are $C^1$.

In our case this formula gives
\[
\partial_\zeta (f \ast g)(\zeta) = f(0)g(\zeta) + \int_0^\zeta f'(\zeta - \eta)g(\eta)d\eta = f(\zeta)g(0) + \int_0^\zeta f(\zeta - \eta)g'(\eta)d\eta,
\]
(one gets the second equality through an integration by part). Using $\hat{\gamma}(0) = 1$ and again the bound \(2\) one the renormalisation group equation \(11\) after hitting it with a derivative, one gets, for any $\zeta \in U_2$
\[
|\hat{\gamma}_{n+1}'(\zeta)| \leq (4 + 3|\zeta|)(1 + g(|\zeta|)|\zeta|) \max \left\{ \max_{\eta \in [0, |\zeta|]} |\hat{\gamma}_n(\eta)|, \max_{\eta \in [0, |\zeta|]} |\hat{\gamma}_{n}'(\eta)| \right\}
\]
The same bound holds for any $\eta \in [0, \zeta]$ from the same argument than the one used for $\hat{\gamma}_n$.

From these bounds, the Lemma holds by induction.

Suming these $\hat{\gamma}_n$ we end up with the following bound for the two-points function (at infinity):
\[
|\hat{G}(\zeta, L)| \leq K \exp(c|\zeta|^2g(\zeta)L),
\]
for some bound $g(\zeta)$ of $\hat{\gamma}$ and $\hat{\gamma}'$ at infinity. This is too weak a bound to apply Borel-Ecalle resummation method. The square of $|\zeta|$ comes from the $\zeta$ in the renormalisation group equation \(11\) and the $\zeta^{-1}$ in the Equation \(2\), which we used with $n = 2$. In order to apply Borel-Ecalle resummation without accelero-summation, we have two challenges to tackle:

- relate the bounds for $\hat{\gamma}_n$ and for $\hat{\gamma}'_n$ in order to get ride of one of the power of $\zeta$;
- find a specific bound on the asymptotic behavior of $\hat{\gamma}$.

The second issue will be solved using the Schwinger-Dyson equation, but the solution of the first one will actually use inputs from the Schwinger-Dyson equation as well.

5.2 Rewriting the Schwinger-Dyson equation

Expanding the sum in the Schwinger-Dyson equation in the Borel plane, and using $B(af(x)) = 1 \ast \hat{f}$ we find
\[
\hat{\gamma}(\zeta) = 1 + 2 \sum_{n=1}^{+\infty} X_{0n}(1 \ast \hat{\gamma}_n)(\zeta) + \sum_{n,m=1}^{+\infty} X_{nm}(1 \ast \hat{\gamma}_n \ast \hat{\gamma}_m)(\zeta).
\]
with
\[
X_{nm} := \frac{1}{n!m!} \frac{d^n}{dx^n} \frac{d^m}{dy^m} H(x, y)|_{x=y=0}.
\]
Using the representation \(10\) of the Mellin transform $H$, we find $X_{0n} = X_{n0} = (-1)^n$. Indeed the series $\sum_{k=1}^{+\infty} \frac{\zeta(2k+1)}{2k+1} ((x+y)^{2k+1} - x^{2k+1} - y^{2k+1})$ contains no terms of the form $x^Ny^0$ nor $x^0y^N$. Thus
\[
\left. \partial_x^n \exp \left( 2 \sum_{k=1}^{+\infty} \frac{\zeta(2k+1)}{2k+1} ((x+y)^{2k+1} - x^{2k+1} - y^{2k+1}) \right) \right|_{x=y=0} = 0;
\]
and the same holds for the derivatives with respect to $y$. We thus find the Schwinger-Dyson equation in the Borel plane:
\[
\hat{\gamma}(\zeta) = 1 + 2 \sum_{n=1}^{+\infty} (-1)^n(1 \ast \hat{\gamma}_n)(\zeta) + \sum_{n,m=1}^{+\infty} X_{nm}(1 \ast \hat{\gamma}_n \ast \hat{\gamma}_m)(\zeta).
\]
Remark 5.3. It is crucial to the rest of this proof to realise that, while Equation (17) holds for any $\zeta \in \mathbb{C}/\Omega$, the series on the R.H.S. only converge in a small open subset of $\mathbb{C}/\Omega$ which is mapped to a neighborhood of the origin in $\mathbb{C}$. Indeed, deriving (17) we obtain

$$
\hat{\gamma}'(\zeta) = 2 \sum_{n=1}^{+\infty} (-1)^n \hat{\gamma}_n(\zeta) + \sum_{n,m=1}^{+\infty} X_{nm}(\hat{\gamma}_n \ast \hat{\gamma}_m)(\zeta).
$$

The renormalisation group equation (11) together with the result of [6] that $\hat{\gamma}(\zeta) \sim A \ln(1/3 - \zeta)$ when $\zeta$ goes to $1/3$ implies that $\hat{\gamma}_n$ has the same behavior when $\zeta$ goes to $1/3$. Thus $\sum_{n=1}^{+\infty} (-1)^n \hat{\gamma}_n(\zeta)$ trivially diverges in an open set close to 1/3.

Therefore, the series of the R.H.S. of (17) should be read as the analytic continuation of these series when one is away from their convergent domain. This will be important since we will use bounds on $\hat{\gamma}_n$ of the form of the bounds of Lemma 5.1 which holds for any $\zeta \in U_\Omega$. Provided the series of these bounds will admit an analytic extension to the whole of $U_\Omega$, it will provide a bound for $\hat{\gamma}$ as needed.

Now, the other numbers $X_{nm}$ could be computed using the same type of argument we used to find $X_{n0}$, or directly using the Faà-di-Bruno formula. However, the result of this computation is not particularly enlightening. It will be enough for us to find a bound for $|X_{nm}|$.

Lemma 5.4. For any any $r \in ]0, 1/2]$ it exists a finite constant $K_r > 0$ such that, for any $n, m \in \mathbb{N}^*$ we have

$$
|X_{nm}| \leq K_r \frac{r^{n+m}}{r^n + m}.
$$

Proof. We use the multivariate Cauchy inequality (see for example [31 Theorem 2.2.7]); namely that if a function $f : \mathbb{C}^n \to \mathbb{C}$ is analytic and bounded by $M$ in the polydisc $\{ z : |z_i| \leq r, i = 1, \ldots, n \}$, then $|\partial^\alpha f(0)| \leq M \frac{\alpha!}{\alpha^\alpha}$ for any multi-index $\alpha \in \mathbb{N}^n$ and with obvious notations for factorial and powers. According to (1), the Mellin transform $H$ is analytic in the polydisc $\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq r \land |z_2| \leq r \}$ for any $r \in ]0, 1/2]$. For any such $r$, set $K_r := \sup_{|z_1| \leq r, |z_2| \leq r} |H(z_1, z_2)|$. The bound (18) follows then directly from the multivariate Cauchy inequality.

5.3 Intermediate bounds

We start with a common bound of $\hat{\gamma}$ and $\hat{\zeta} \partial_r \hat{\gamma}$ to find bounds on $\hat{\gamma}_n$ and $\hat{\gamma}_n'$ for any $n \in \mathbb{N}^*$.

Lemma 5.5. Let $g : U_\Omega \setminus \{0\} \to \mathbb{R}$ be a holomorphic function increasing with $|\zeta|$ such that, for any $\zeta \in U_\Omega \setminus \{0\}$,

$$
\max_{\eta \in [0, \zeta]} |\hat{\gamma}(\eta)| \leq g(\zeta) \quad \text{and} \quad \max_{\eta \in [0, \zeta]} |\hat{\gamma}'(\eta)| \leq \frac{g(\zeta)}{|\zeta|}.
$$

Let $(g_n)_{n \in \mathbb{N}^*}$ and $(h_n)_{n \in \mathbb{N}^*}$ be two sequences of functions from $U_\Omega \setminus \{0\}$ to $\mathbb{R}$ inductively defined for any $\zeta \in U_\Omega \setminus \{0\}$ by $g_1 := g(\zeta)$, $h_1 := g(\zeta)/|\zeta|$ and

$$
g_{n+1}(\zeta) := g(\zeta)|\zeta| [4g_n(\zeta) + 3|\zeta|h_n(\zeta)],
$$

$$
h_{n+1}(\zeta) := \frac{g_{n+1}(\zeta)}{|\zeta|} + 4g_n(\zeta) + 3|\zeta|h_n(\zeta).
$$

Then, for any $n \in \mathbb{N}^*$ and $\zeta \in U_\Omega \setminus \{0\}$

$$
\max_{\eta \in [0, \zeta]} |\hat{\gamma}_n(\eta)| \leq g_n(\zeta), \quad \max_{\eta \in [0, \zeta]} |\hat{\gamma}_n'(\eta)| \leq h_n(\zeta).
$$

Remark 5.6. Such a function $g$ exists since $\hat{\gamma}$ and $\hat{\zeta} \partial_r \hat{\gamma}$ are analytic on $U_\Omega$. We will later on build a bound with this property but it will defined by another $g$.

Proof. We prove this by induction: the case $n = 1$ holds by definition of $g$.

Assuming the result holds for $n \in \mathbb{N}^*$, using the renormalisation group equation (11), the bound (2) and the induction hypothesis we obtain

$$
|\hat{\gamma}_{n+1}(\zeta)| \leq g(\zeta)|\zeta| [4g_n(\zeta) + 3|\zeta|h_n(\zeta)] =: g_{n+1}(\zeta).
$$

Taking once again the derivative of the renormalisation group equation (11) we obtain, using Leibniz’s formula (10)

$$
\hat{\gamma}_{n+1}'(\zeta) = 4[\hat{\gamma}_n(\zeta) + (\hat{\gamma}' \ast \hat{\gamma}_n)(\zeta)] + 3[\hat{\gamma}_n'(\zeta) + (\hat{\gamma}' \ast (\hat{\gamma}'_n))(\zeta)].
$$
Using the bound (2) and the induction hypothesis on this equation gives the result for $\zeta$. The case of $\eta \in [0, \zeta]$ holds from the same argument than the one of Lemma 5.1 which still holds since we assume $g$ to be increasing.

We can now express together the bounds of $\tilde{\gamma}_n$ and $\tilde{\gamma}'_n$.

**Lemma 5.7.** For any $\zeta \in U_\Omega \setminus \{0\}$, set

$$\alpha(\zeta) := \frac{g(\zeta)}{g(\zeta) + 1}$$

with $g$ a bound of $\tilde{\gamma}$ and $\tilde{\zeta}'$ as in Lemma 5.5. Then, for any $n \in \mathbb{N}^*$ and any $\zeta \in U_\Omega \setminus \{0\}$

$$h_n(\zeta) \leq \frac{1}{\alpha(\zeta)} g_n(\zeta) |\zeta|.$$

**Proof.** For $n = 1$, the inequality to show is the case $n = 1$ of Lemma 5.5 since $1/\alpha(\zeta) > 1$.

For $n = 2$, direct computations give

$$\frac{1}{\alpha(\zeta)} g_2(\zeta) |\zeta| = 7g(\zeta)(g(\zeta) + 1) \geq h_2(\zeta) = 14g(\zeta)$$

since $g(\zeta) \geq \max_{\eta \in [0, \zeta]} |\tilde{\gamma}(\eta)|$.

For any $n \geq 2$ we have

$$\frac{1}{\alpha(\zeta)} g_{n+1}(\zeta) |\zeta| = (g(\zeta) + 1)(4g_n(\zeta) + 3|\zeta|h_n(\zeta)) = h_{n+1}(\zeta).$$

Therefore the result also hold for any $n \geq 2$.

We can now prove the main result of this subsection.

**Proposition 5.8.** Let $g : U_\Omega \to \mathbb{R}$ be a bound of $\tilde{\gamma}$ and $\zeta'$ as in Lemma 5.5. Then, for any $n \in \mathbb{N}^*$ and $\zeta \in U_\Omega \setminus \{0\}$

$$\max_{\eta \in [0, \zeta]} |\tilde{\gamma}_n(\eta)| \leq |(7g(\zeta) + 3)|^{n-1} g(\zeta).$$

**Proof.** By Lemma 5.5 it is sufficient to prove $g_n(\zeta) \leq |(7g(\zeta) + 3)|^{n-1} g(\zeta)$ for any $n \in \mathbb{N}^*$. We prove this by induction: the case $n = 1$ trivially holds. Assuming the result holds for $n \in \mathbb{N}^*$, we have according to Lemma 5.7

$$g_{n+1}(\zeta) \leq g(\zeta)|\zeta| \left(4 + \frac{3}{\alpha(\zeta)}\right) g_n(\zeta) = |\zeta| (7g(\zeta) + 3) g_n(\zeta)$$

by definition of $\alpha(\zeta)$.

### 5.4 Borel-Écalle resummation of the two-points function

The one quantity that we have not bounded yet and that could still give $\tilde{G}$ a surexponential behavior at infinity on the principal branch $U_\Omega$ of $\mathbb{C}/\Omega$ is the bound $g$ of $\tilde{\gamma}$. This is taken care of in the next Proposition.

**Proposition 5.9.** On the principal branch $U_\Omega$ of $\mathbb{C}/\Omega$, $|\tilde{\gamma}(\zeta)|$ is bounded in a neighborhood of infinity by 1, and $|\tilde{\gamma}'(\zeta)|$ by $1/|\zeta|$.

**Proof.** As before let $g : U_\Omega \to \mathbb{R}$ be a bound of $\tilde{\gamma}$ and $\zeta'$ as in Lemma 5.5. Using the bound (2) on the Schwinger-Dyson equation (17) with the bounds of Proposition 5.8 for $\gamma_n$ and the bounds of Lemma 5.4 for the coefficients $X_{nm}$ we find that $|\tilde{\gamma}|$ is bounded on $U_\Omega \setminus \{0\}$ by two geometric series. In the spirit of Remark 5.3 one can more properly say that $|\tilde{\gamma}|$ is bounded in $U_\Omega \setminus \{0\}$ by the analytic continuation of (product of) geometric series. To be more precise, one has

$$|\tilde{\gamma}(\zeta)| \leq 1 + 2|\zeta| \sum_{n=1}^{\infty} \max_{\eta \in [0, \zeta]} |\gamma_n(\eta)| + \frac{K_r}{2} \max_{\eta \in [0, \zeta]} |\zeta|^2 \sum_{n,m=1}^{\infty} \frac{1}{r^n m^m} \max_{\eta \in [0, \zeta]} |\gamma_n(\zeta)| \max_{\eta \in [0, \zeta]} |\gamma_m(\zeta)|$$

$$\leq 1 + \frac{2|\zeta| g(\zeta)}{1 - (7g(\zeta) + 3)|\zeta|} + K_r \left(\frac{|\zeta| g(\zeta)}{r - (7g(\zeta) + 3)|\zeta|}\right)^2 =: G(\zeta, g(\zeta))$$

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for any \( r \in [0, 1/2] \), \( \zeta \in \mathcal{U}_2 \) and with \( K_r := \sup_{|z_1| \leq r, z_2 \leq r} |H(z_1, z_2)| \). Notice that we removed the 1/2 in the last bound in order for \( G \) to have the following property: for any \( \zeta \in \mathcal{U}_2 \setminus U \)

\[
|\hat{\varepsilon}'(\zeta)| \leq \frac{G(\zeta, g(\zeta))}{|\zeta|}. \tag{19}
\]

To prove this, we take the derivative of the Schwinger-Dyson equation \( \hat{\zeta}' \):

\[
\hat{\zeta}'(\zeta) = 2 \sum_{n=1}^{+\infty} (-1)^n \hat{\gamma}(\zeta) + \sum_{n,m=1}^{+\infty} X_{nm}(\hat{\gamma} \ast \hat{\gamma})(\zeta).
\]

Therefore

\[
|\hat{\zeta}'(\zeta)| \leq 2 \sum_{n=1}^{+\infty} |\hat{\gamma}(\zeta)| + \sum_{n,m=1}^{+\infty} |X_{nm}(\hat{\gamma} \ast \hat{\gamma})(\zeta)| \\
\leq 2 \sum_{n=1}^{+\infty} \max_{\eta \in [0, \zeta]} |\hat{\gamma}(\eta)| + K_r \max_{\eta \in [0, \zeta]} |\hat{\gamma} (\zeta)| \sum_{n,m=1}^{+\infty} \frac{1}{\rho^{n+m}} \max_{\eta \in [0, \zeta]} |\hat{\gamma}(\eta)| \max_{\eta \in [0, \zeta]} |\hat{\gamma}(\eta)| \\
\leq \frac{G(\zeta, g(\zeta))}{|\zeta|}.
\]

It is a cumbersome but simple exercise to study the variations of \( G \). However it is enough for the task at hand to check that \( G \) is bounded at infinity by 1. For \( \zeta \) in the principal branch \( \mathcal{U}_2 \) of \( \mathbb{C}/\Omega \), we have

\[
G(\zeta, X) \sim 1 - \frac{2X}{7X + 3} + K_r \left( \frac{X}{7X + 3} \right)^2 =: f(X)
\]

for \( |\zeta| \to \infty \). We can still choose \( r \in [0, 1/2] \). Since \( H(0, 0) = 1 \) and since \( H \) is holomorphic in a neighborhood of \((0, 0)\), we can take \( r \) small enough such that \( K_r \) goes arbitrarily close to 1 = \( H(0, 0) \). It then is a simple exercise of real analysis to show that, provided \( K_r < 7, f \) is continuous and decreases over \( \mathbb{R}^+ \).

Therefore

\[
|\hat{\gamma}(\zeta)| \lesssim f(0) = 1
\]

in a neighborhood of infinity. The bound for \( \hat{\gamma}' \) in the same neighborhood of infinity comes from the inequality \( \tag{19} \)

**Theorem 5.10.** It exists positive constants \( K, M > 0 \) such that, for any \( L \in \mathbb{R} \), the Borel transform of the solution of the Schwinger-Dyson equation \( \tag{17} \) and the renormalisation group equation \( \tag{15} \) admits the following bound in \( \mathcal{U}_2 \) around the infinity

\[
|\hat{G}(\zeta, L)| \leq K \frac{\exp(M|\zeta|L)}{|\zeta|}.
\]

**Proof.** From Proposition \( \ref{5.9} \) we can find a bound of \( g \) of \( \hat{\gamma} \) and \( \hat{\gamma}' \) which is increasing and bounded at infinity. Using such a bound in Proposition \( \ref{5.8} \) we obtain

\[
|\hat{G}(\zeta)| \leq \sum_{n=1}^{\infty} |(7g(\zeta) + 3)| \zeta|^{n-1} g(\zeta) \frac{L^n}{n!} \\
= \frac{g(\zeta)}{(7g(\zeta) + 3)\zeta} \left( \exp((7g(\zeta) + 3)|\zeta|L) - 1 \right) \\
\leq \frac{A}{|\zeta|} \exp(M|\zeta|L)
\]

for some \( K > 0 \), and where we have set \( M := 7 \sup_{\zeta \in \mathcal{U}_2} g(\zeta) + 3 \).

This result, together with Theorem \( \ref{4.12} \) directly implies

**Corollary 5.11.** The solution of the renormalisation group equation \( \tag{15} \) and the Schwinger-Dyson equation \( \tag{17} \) is Borel-Ecalle resummable.

The main Theorem \( \ref{1.1} \) is obtained with one more result.
Proposition 5.12. The Borel-Ecalle resummed function $G^{\text{res}}(a, L)$ is analytic is the open subset of $\mathbb{C}$ defined by

$$|a - \frac{1}{20L}| < \frac{1}{20L}$$

for any $L$ in $\mathbb{R}^*_+$.

Proof. The analyticity domain of the resummed function only depends on the asymptotic of the Borel transform. We can therefore subtract to $\hat{\gamma}$ a function $\psi$ with a compact support without changing the analyticity domain. Doing this, one can assume that the bound $g$ of Proposition 5.8 is bound at infinity by the function $G$. In this case we have

$$\sup_{\zeta \in U_0} g(\zeta) \leq \sup_{X \in \mathbb{R}_+} f(X) = f(0) = 1$$

and therefore $M \leq 10$. This implies the result by Theorem 2.21.

Let us finish this article by pointing out that we have shown the analyticity of a solution of the Schwinger-Dyson equation in an open disc tangent to the origin. If one accepts that this solution encodes some of the non perturbative behavior of the theory, one should be looking for simple poles on the positive real axis of this solution. Such a pole can then by interpreted as a mass, not present in the perturbative regime of the theory.

This non perturbative mass generation mechanism was proposed in [7], where it was also shown that a transseries solution of the same Schwinger-Dyson equation indeed had a simple poles on the positive real axis.

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References

[1] Bin Zhang Viet Dang. Renormalization of feynman amplitudes on manifolds by spectral zeta regularization and blow-ups. 12 2017. arXiv:arXiv:1712.03490

[2] Romain Pascalie. A Solvable Tensor Field Theory. 2019. arXiv:1903.02907

[3] D. J. Broadhurst and D. Kreimer. Exact solutions of Dyson–Schwinger equations for iterated one-loop integrals and propagator-coupling duality. Nucl. Phys., B 600:403–422, 2001. arXiv:hep-th/0012146

[4] Pierre J. Clavier. Analytic results for Schwinger–Dyson equations with a mass term. 2015. arXiv:1405.3383, doi:10.1007/s11005-015-0762-1

[5] Juan Carlos Vasquez Jahmall Bersini, Alessio Maiezza. Resurgence of the renormalization group equation. arXiv:1910.14507

[6] Marc P. Bellon and Pierre J. Clavier. A Schwinger–Dyson Equation in the Borel plane: singularities of the solution. Lett. Math. Phys., 105, 2015. arXiv:1411.7190, doi:10.1007/s11005-015-0761-2

[7] Marc P. Bellon and Pierre J. Clavier. Alien calculus and a Schwinger–Dyson equation: two-point function with a nonperturbative mass scale. 2016. arXiv:arXiv:1612.07813[hep-th]

[8] Jean Ecalle. Les fonctions résurgentes, Vol.1. Pub. Math. Orsay, 1981.

[9] Jean Ecalle. Les fonctions résurgentes, Vol.2. Pub. Math. Orsay, 1981.

[10] Jean Ecalle. Les fonctions résurgentes, Vol.3. Pub. Math. Orsay, 1981.

[11] Jean Ecalle. Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac. Hermann, 1992.
[12] Frédéric Menous. *Les bonnes moyennes uniformisantes et leurs applications à la resommation réelle*. PhD thesis, 1996. Thèse de doctorat dirigée par Ecalle, Jean Sciences et techniques communes Paris 11 1996. URL: [http://www.theses.fr/1996PA112392](http://www.theses.fr/1996PA112392).

[13] Frédéric Menous. Les bonnes moyennes uniformisantes et une application à la resommation réelle. *Annales de la Faculté des sciences de Toulouse : Mathématiques*, 6e série, 8(4):579–628, 1999. URL: [http://www.numdam.org/item/AFST_1999_6_8_4_579_0](http://www.numdam.org/item/AFST_1999_6_8_4_579_0).

[14] Emmanuel Vieillard-Baron. *From resurgent functions to real resummation through combinatorial Hopf algebras*. PhD thesis, 2014. Thèse de doctorat dirigée par Rolin, Jean-Philippe Mathématiques Dijon 2014. URL: [http://www.theses.fr/2014DIJOS005](http://www.theses.fr/2014DIJOS005).

[15] Inès Aniceto and Ricardo Schiappa. Nonperturbative ambiguities and the reality of resurgent transseries. *Communications in Mathematical Physics*, 335:183–245, 2013. [arXiv:1308.1115](http://arxiv.org/abs/1308.1115).

[16] Ricardo Schiappa Inès Aniceto, Gökçe Başar. A primer on resurgent transseries and their asymptotics. *Physics Reports*, 809, 02 2018. doi:10.1016/j.physrep.2019.02.003.

[17] Daniele Dorigoni. An introduction to resurgence, trans-series and alien calculus. *Annals of Physics*, 11 2014. doi:10.1016/j.aop.2019.167914.

[18] David Sauzin. Nonlinear analysis with resurgent functions. 2012. [arXiv:1212.4477v4](http://arxiv.org/abs/1212.4477v4).

[19] Frédéric Menous. The well-behaved catalan and brownian averages and their applications to real resummation. *Proceedings of the Symposium on Planar Vector Fields (Lleida, 1996)*. Publ. Mat., 41:209—222, 1997.

[20] Olivier Bouillot. *Invariants Analytiques des Difféomorphismes et MultiZêtas*. PhD thesis, Université Paris-Sud 11, 2011. URL: [http://tel.archives-ouvertes.fr/tel-00647909](http://tel.archives-ouvertes.fr/tel-00647909).

[21] David Sauzin. Introduction to 1-summability and resurgence. 2014. [arXiv:1405.0356v1](http://arxiv.org/abs/1405.0356v1).

[22] David Sauzin Shingo Kamimoto. Iterated convolutions and endless riemann surfaces. 2016. [arXiv:1610.05435v2](http://arxiv.org/abs/1610.05435v2).

[23] Emmanuel Viellard-Baron. Ecalle’s averages, rota-baxter algebras and the construction of moulds. 2019. [arXiv:arXiv:1904.02417v1](http://arxiv.org/abs/1904.02417v1).

[24] Pierre J. Clavier. *Analytic and Geometrical approaches of non-perturbative quantum field theories*. PhD thesis, 2015.

[25] Julius Wess and Bruno Zumino. Supergauge transformations in four dimensions. *Nucl. Phys. B*, 70:39–50, 1974.

[26] Bruno Zumino Julius Wess. A lagrangian model invariant under supergauge transformations. *Phys. Lett.*, 49B:3–5, 1974.

[27] Marc P. Bellon. An efficient method for the solution of Schwinger–Dyson equations for propagators. *Lett. Math. Phys.*, 94:77–86, 2010. [arXiv:1005.0196](http://arxiv.org/abs/1005.0196), doi:10.1007/s11005-010-0416-3.

[28] Marc P. Bellon and Pierre J. Clavier. Higher order corrections to the asymptotic perturbative solution of a Schwinger–Dyson equation. *Lett. Math. Phys.*, 104:1–22, 2014. [arXiv:1311.1160v2](http://arxiv.org/abs/1311.1160v2), doi:10.1007/s11005-014-0686-1.

[29] Marc Bellon, Gustavo Lozano, and Fidel Schaposnik. Higher loop renormalization of a supersymmetric field theory. *Physics Letters B*, 650:293–297, 2007. URL: [10.1016/j.physletb.2007.05.024](http://dx.doi.org/10.1016/j.physletb.2007.05.024).

[30] G. ’t Hooft. *Can We Make Sense Out of “Quantum Chromodynamics”?*, pages 943–982. Springer US, Boston, MA, 1979. URL: [http://dx.doi.org/10.1007/978-1-4684-0991-8_17](http://dx.doi.org/10.1007/978-1-4684-0991-8_17).

[31] Lar Hörmander. *An introduction to complex analysis in several complex variables*. Elsevier, 1966.