Boundaries and junctions in two parity
violating models in 2+1 dimensions

December 28, 2021

Mark Burgess

Oslo College of Engineering, Cort Adelersgata 30
N-0254 Oslo, Norway
and
Institute of Physics, University of Oslo
P.O. BOX 1048, Blindern, N-0316 OSLO 3, Norway

Margaret Carrington

Department of Physics and Winnipeg Institute for Theoretic al Physics
University of Winnipeg, Winnipeg, Manitoba
Canada R3B 2E9

Abstract

Recently it has been suggested that junctions between materials
with different parity violating properties would be characterized by
diffusion layers, analogous to those in the p-n junction\[9, 10\]. This
remark is amplified by a fuller investigation of two related parity vi-
olating effective Lagrangians, which possess a kind of duality. It is
shown that gauge invariance and energy conservation are sufficient
to determine the behaviour at the interface. This leads to modifi-
cations of normal parity-violating electrodynamics. The coupling of
an interface to an external system is a natural solution to the deficiencies of Maxwell-Chern-Simons theory. A heuristic model of a transistor-like device is discussed which relates to recent experiments in device technology. Radiative corrections to Chern-Simons theory induce a local magnetic moment interaction whose lagrangian is everywhere gauge invariant. The effects of this interaction are compared to Maxwell-Chern-Simons theory. The dispersion of classical waves for these models is computed and the laws of reflection and refraction are found to hold despite the lack of $P$ and $T$ invariance. The magnetic moment dispersion is gapless in contrast to the Chern-Simons dispersion except in the case of a scalar field which is covariantly constant. Both models exhibit optical activity (Faraday effect).

1 Introduction

The Chern-Simons term is widely exploited in the construction of effective theories breaking parity and time-reversal invariance. Although originally the term was introduced as a way of providing the Yang-Mills gauge field with a gauge invariant mass\[1, 2, 3\], it has since appeared more often in the condensed matter literature in discussions of systems like the fractional quantum Hall effect and anyon superconductivity\[4\]. These models postulate the Chern-Simons term as an effective action for an unknown microscopic theory, the coefficient of which takes on a constant value which is chosen or, in principle, determined from the underlying physics.

In this paper we explore more carefully the possible roles of the Chern-Simons Lagrangian by investigating systems in which its coefficient – the physical potency and sign of the parity violating effects – takes on different values in different parts of the system. An additional model which couples a gauge invariant current to the dual of the field strength is also considered and compared to the usual Chern-Simons term. This model reduces to the Chern-Simons expression in a special case.

The junction scenario described in this paper, although motivated on theoretical grounds, could have some bearing on recent experiments in which participating electrons are characterized by predominantly a single spin direction\[5, 6\]. Redlich has shown that spin polarized Dirac fermions give rise to an effective Chern-Simons theory when quantum corrections are accounted for\[7\] and thus a Chern-Simons theory describes such a two-dimensional electron
gas.

We begin by formulating the simplest junction in terms of an action principle. This consists of two regions in a (2+1)-dimensional space, which meet on the line \(x_1 = 0\). The two regions are considered to have physically disparate properties so that a boundary condition is implied for the physical fields in addition to the relevant field equations on either side of the boundary.

2 Formalism

To illustrate the variational formalism we are using, consider the simplest case of plain electromagnetism at a material junction. In Maxwell theory, the only variables in a physical medium are the dielectric properties of matter; it is sufficient to describe these properties in terms of a conserved electric current variable \(J^\mu_{TOT}(x)\) whose value is position dependent and a dipole field \(P_{\mu\nu} \equiv D_{\mu\nu} - F_{\mu\nu}\). The action is given by

\[
S = \int dV_x \left\{ -\frac{1}{4} D^{\mu\nu} F_{\mu\nu} - J^\mu_{TOT} A_{\mu} \right\}
\]

where \(F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}\) (\(F_1^2 = B\), \(F_{30} = E^i\)), \(dV_x = dtdx_1dx_2\) and the metric signature is \(\eta_{\mu\nu} = (+- -)\). Units are chosen such that \(\hbar = c = \epsilon_0 = \mu_0 = 1\) and latin indices refer to spatial dimensions only. We shall have occasion to consider both sharp and soft interfaces. To take account of the polarization of media, one must allow the possibility that currents will run along the surface of a sharp interface at \(x_1 = 0\). One can, with a certain freedom, either include this effect in \(P_{\mu\nu}\) or introduce the current explicitly as a surface current. In the following we include both, anticipating the work in later sections. Since we are modelling the interface as an abrupt change in the physical properties of our system, these currents will be proportional to a delta-function in the \(x_1\) direction. We write

\[
J^\mu_{TOT} = J^\mu + j^\mu \delta(x_1) = J^\mu + J^\mu_s
\]

\(j^1 = 0\) is expected from the geometry (this is not the case in the Chern-Simons theory) and the total current will be conserved \(\partial_{\mu} J^\mu_{TOT} = 0\). Varying the action and integrating by parts leads to

\[
\delta S = \int dV_x \left\{ \delta A^\nu \partial^\mu (F_{\mu\nu} + P_{\mu\nu}) - J^\mu \delta A_\mu \right\}
\]
\[
+ \int dV \{ \partial^{\mu} \left[ -\delta A^{\nu} (F_{\mu\nu} + P_{\mu\nu}) \right] - \delta A^{\nu} j_{\nu} \delta(x_1) \} = 0 \quad (3)
\]

Requiring that \( A_{\mu} \to 0 \) as \( |x_{\mu}| \to \infty \) leads to

\[
\delta S = \int dV \{ \delta A^{\nu} \partial^{\mu} D_{\mu\nu} - J^{\mu} \delta A_{\mu} \}
+ \int dt dx_2 \int_{-\epsilon}^{+\epsilon} dx_1 \{ \partial^{1} \left[ -\delta A^{\nu} D_{1\nu} \right] - \delta A^{\nu} j_{\nu} \delta(x_1) \} = 0. \quad (4)
\]

Moreover, if the regular field equations (in the absence of a boundary) are to apply arbitrarily close to \( x_1 = 0 \), then the boundary condition

\[
\delta S_{\text{boundary}} = \lim_{\epsilon \to 0} \int dt dx_2 \int_{-\epsilon}^{+\epsilon} dx_1 \{ \partial^{1} \left[ -\delta A^{\nu} (F_{1\nu} + P_{1\nu}) \right] - \delta A^{\nu} j_{\nu} \delta(x_1) \} = 0 \quad (5)
\]

is implied. Here it is assumed that the components of the vector potential \( A^{\mu} \) are continuous across the boundary. This assumption may be unnecessarily restrictive if one considers the case in which a contact potential characterizes the junction and will be relaxed later. This leads to the form

\[
\Delta(F_{1\mu} + P_{1\mu}) + j_{\mu} = 0 \quad \mu = 0, 2 \quad (6)
\]

where \( \Delta \) means the change in value across the boundary. The field equations outside of the boundary are given by

\[
\partial^{\mu} D_{\mu\nu} = J_{\nu} \quad (7)
\]

Clearly one can absorb the contribution due to surface currents into the more usual form of a polarization tensor. The surface polarization tensor may be defined by

\[
\partial^{\mu} P_{1\mu} = J^s_{\nu} \quad (8)
\]

which is added to \( P_{\mu\nu} \). The antisymmetric components of the total polarization are \( P_{1}{}^{2} = -M, \quad P^{10} = P^{1} \), enabling (3) to be written in the standard form:

\[
\Delta D^{1} = \Delta(E^{1} + P^{1}) = 0 \\
\Delta H = \Delta(B - M) = 0. \quad (9)
\]

where \( P^{1} \) and \( M \) are the polarization and magnetization respectively. It also follows immediately from the assumption about the continuity of the
vector potential that $\Delta E_2 = \Delta(\partial_2 A_0 - \partial_0 A_2) = 0$. The conservation of the total current gives us a relation between $j_\mu$ and the change in $J_\mu$ across the boundary ($\Delta J_\mu$).

$$\partial^\mu (J_\mu + j_\mu \delta(x_1)) = 0 \quad (10)$$

Integrating this equation with respect to $x_1$ from $-\epsilon$ to $+\epsilon$ and consider the limit $\epsilon \to 0$, one obtains

$$\Delta J_1 + \partial^\mu j_\mu = 0, \quad (j_1 = 0) \quad (11)$$

The main point to note from this exercise is that it is unnecessary to explicitly introduce fields $D$ and $H$ to account for polarization and magnetization effects specifically on the boundary: it is sufficient to include the possibility of surface currents. We shall therefore not refer to these fields again.

3 Maxwell-Chern-Simons Theory

3.1 Gauge invariance at a junction

The action formalism does not yield an explicitly gauge invariant boundary condition when the Chern-Simons term is considered. The Abelian Chern-Simons term is gauge invariant only in the absence of boundaries, being quadratic in $A_\mu$ but only linear in the derivatives. Consider the action,

$$S = \int dV_x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\mu}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - J^\mu_{TOT} A_\mu \right\} \quad (12)$$

The variation of this action, without further stipulation, leads to the field equations and associated boundary condition

$$\partial^\mu F_{\mu\nu} + \mu \epsilon_{\nu\mu\lambda} \partial^\nu A^\lambda = J_\nu \quad (13)$$

$$\Delta [-F_{1\lambda} + \frac{1}{2} \mu \epsilon_{\mu\lambda\lambda} A^\mu] - j_\lambda = 0 \quad (14)$$

Gauge invariance of the boundary condition implies a restriction on $A_\mu$. One is therefore led to consider the effect of the gauge transformation $A'_\mu = A_\mu - \partial_\mu \theta$ on the action (12) over a region $-\epsilon < x_1 < \epsilon$ as $\epsilon \to 0$.

$$\delta S = S[A'_\mu] - S[A_\mu] = \int dt dx_2 \int_{-\epsilon}^{+\epsilon} dx_1 \{ -\partial_1 [\frac{1}{2} \mu \epsilon_{\nu\lambda\lambda} \partial_\nu A^\lambda + J_1] - (\partial^\mu j_\mu) \delta(x_1) \} \theta \quad (15)$$
where \( j_1 = 0, j_0, j_2 \) are independent of \( x_1 \) and \( E^i = F^{i0} \). If the gauge choice \( \theta \) is to be arbitrary one has

\[
\Delta \left( \frac{1}{2} \mu E_2 + J_1 \right) + \partial^\mu j_\mu = 0. \tag{16}
\]

On use of (11) this simply becomes

\[
\Delta \mu E_2 = 0 \tag{17}
\]

since the continuity of the vector potential implies that \( \Delta E_2 = 0 \). This result indicates that the only valid boundary condition between Chern-Simons media with different coefficients is one in which \( E_2 = 0 \) on the boundary. This boundary condition does not permit the passage of electromagnetic waves or information. The vanishing of \( E_2 \) implies the form

\[
A_\mu = \partial_\mu \xi(x) \quad \mu \neq 1 \tag{18}
\]

for some scalar field \( \xi(x) \). This solution may be used in the action as a restriction on the allowed variations. The analogue of equation (5) is then

\[
\delta S_{\text{boundary}} = \int dt \, dx_2 \int_{-\epsilon}^{+\epsilon} dx_1 \{ \partial^1 \partial^\nu F_{1\nu} - \frac{1}{2} \partial_1 (\mu \epsilon_{\mu1\lambda} \partial^\lambda A^\nu) + \partial^\mu j_\mu \delta(x_1) \} \delta \xi = 0 \tag{19}
\]

which, on use of the current conservation equation, gives the gauge invariant boundary condition

\[
\Delta (\partial^\mu F_{\mu1} + \frac{1}{4} \mu \epsilon_{\mu1\lambda} F^{\lambda\mu} - J_1) = 0 \tag{20}
\]

This is seen to be consistent with the expression obtained from the field equations (13), integrated directly over an infinitesimal region.

### 3.2 Completing the action

The gauge non-invariance of the Chern-Simons action at non-reflecting junctions is an indication of the incompleteness of the Chern-Simons theory at the boundary. Physically, in the region of a boundary one expects short wavelength modifications to come into play which will either modify the values of physical fields or modify the action itself. To rectify the omission one
must either return to a more fundamental theory and rederive the correct
effective description, or postulate the remainder of the degrees of freedom in
such a way that gauge invariance is restored. A general requirement of the
latter is the introduction of new variables. We now wish to discuss this latter
problem in more detail. The general solution has been discussed by one of
us in [9, 10], ignoring contact potentials. Consider the action

\[ S = \int dV \left\{ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \mu(x) \epsilon_{\nu\rho\lambda} A^\nu \partial^\rho A^\lambda + f(x) J^\mu A_\mu \right\}. \]  (21)

Gauge invariance implies the restriction

\[ \frac{\delta S}{\delta \theta} = \int dV \left\{ -\frac{1}{2} (\partial^\nu \mu) \epsilon_{\nu\rho\lambda} \partial^\rho A^\lambda - (\partial_\nu f) J^\nu \right\} = 0 \]  (22)

which may be written

\[ \frac{1}{2} (\partial_t \mu) B + \frac{1}{2} (\partial_1 \mu) E^2 = (\partial_t f) J^0 + (\partial_1 f) J^1. \]  (23)

The conserved external source \( J_\mu \) can, if necessary, be used to obtain a solv-
able equation. If \( \mu(x) \) is a function of say \( x_1 \) and \( t \) then there are sufficient
terms to find a solution for \( \mu \) in terms of \( E^2 \) and \( B \) without introducing the
source. The solution by this method will in the general case result in the
introduction of higher derivative terms in the action.

The role of \( f(x) \) is to act as a mediating ‘leaky membrane’ which separates
the external source from the two dimensional system. Thus although \( J_\mu \) is
conserved in total, it may appear to be non-conserved via its contact with the
two dimensional junction. Since the sole function of this term is to balance
the gauge invariance equation, a natural boundary condition is the vanishing
of \( f(x) \) when \( \partial_\nu \mu(x) = 0 \). Taking the variation to be with respect to \( x_1 \) and
\( t \) only, this is satisfied by

\[ f(x_1, t) = \frac{1}{2} \alpha (\partial_1 \mu) + \frac{1}{2} \beta (\partial_t \mu) \]  (24)

for constants \( \alpha \) and \( \beta \), hence

\[ \left( \alpha J^1 \partial_t^2 - E^2 \partial_t + \beta J^0 \partial_t^2 - B \partial_t \right) \mu(x_1, t) + (\alpha J^0 + \beta J^1) \partial_1 \partial_t \mu(x_1, t) = 0. \]  (25)
This equation is too difficult to solve in the most general case, but some insight can be gained by noting that space and time appear symmetrically. Considering time-independent solutions, representing steady state solutions one has

\[ \alpha J^1(\partial_1^2 \mu) - E^2(\partial_1 \mu) = 0. \]  
(26)

Letting \( y = \partial_1 \mu \) and \( P(x_1) = E^2/\alpha J^1(x_1) \), one has

\[ \frac{dy}{dx_1} - P(x_1)y = 0 \]  
(27)

which can be cast as a total differential by introducing the integrating factor \( \exp(-\int_0^{x_1} P(z) dz) \),

\[ \frac{d}{dx_1} \left( ye^{-\int_0^{x_1} P(z) dz} \right) = 0 \]  
(28)

thus one has

\[ \frac{d\mu}{dx_1} = \frac{\mu_c}{L} e^{\int_0^{x_1} P(z) dz} \equiv \frac{d}{dx_1} \left( g(x_1) e^{\int_0^{x_1} P(z) dz} \right) \]  
(29)

where \( \mu_c \) and \( L \) are constants. Comparison of these equations leads to a differential equation for \( g(x_1) \),

\[ \frac{dg(x_1)}{dx_1} + g(x_1)P(x_1) = \frac{\mu_c}{L}. \]  
(30)

If, not unreasonably, \( E^2 \) and \( J^1 \) are approximately correlated, then one may write \( P(x) = p + \Delta P(x) \), for constant \( p \) and the equation for \( g(x_1) \) takes the form

\[ \frac{dg(x_1)}{dx} + pg(x_1) = \frac{\mu_c}{L} - \Delta P(x_1)g(x_1) \]  
(31)

whose general solution is given by

\[ g(x_1) = \int_0^{x_1} dz e^{p(z-x_1)} \left( \frac{\mu_c}{L} - \Delta P(z)g(z) \right) + g_0 e^{-px_1}. \]  
(32)

When \( \Delta P(x_1) = 0 \), this is simply the exponential decay law. In general, the exponential behaviour is modulated by a self-consistently defined function \( \Delta P \), but for any ‘physical’ function \( \Delta P \), the long term behaviour will always be dominated by the exponential factors. Further assumptions of symmetry
can be made in order to simplify even more\cite{10}. The key observation is that
the diffusion-like equation \((26)\) implies that the Chern-Simons parameter
must fall off exponentially in space. Clearly, from the symmetry of \((25)\),
the same argument also applies for the space-independent equation for time-
variation. The field equations in the varying region must now be solved self-
consistently so as to record the effect of the constraint introduced through
the solution of the gauge-invariance condition.

Had the boundary between the regions been curved rather than linear
then it is natural to suppose that the decay law would be modified by the
extrinsic curvature of the interface\cite{16}. Since the concentration of field is
greater around a ‘corner’ like feature, this would lead to an intensification of
\(E^1\) in this region. Although this would not seem to modify the invariance
constraint it will, though the field equations, exacerbate the current \(J^1\) in
the vicinity of the interface. This increased activity must be answered by the
sources, thus one would expect increased dissipation in such a region.

3.3 Examples

A simple example of the foregoing procedure can be computed in the absence
of sources\cite{10}. When \(\mu = \mu(x_1, t)\) and \(E_2\) and \(B\) are assumed to be constants
throughout the region of interest, the gauge invariance constraint may be
solved together with the Bianchi identity to show that both the Chern-Simons
coefficient and \(E_1\) are arbitrary functions of \(\gamma(E_2^{-1}x_1 - B^{-1}t)\), for constant \(\gamma\).
This solution is extremely general and admits both longitudinal waves and
exponential decay, but does not correspond to any obvious physical situation.
One possibility is that it is a heuristic representation of periodic impurities
in a quantum Hall system.

An example of more physical relevance is the case of a time-varying Chern-
Simons coefficient in a time-varying magnetic field. Since it is known that
a system of spin-polarized fermions gives rise to an effective field theory
involving a Chern-Simons term\cite{7} in the long wavelength limit, this situation
should correspond directly to spin relaxation, or spin pumping in a 2+1
dimensional system. Also, the construction of a model of two coupled systems
in which spin migrates from one half-plane to the other could be composed
of two systems of this kind, with a third junction layer of space- and time-
varying \(\mu\) describing the contact region. The constraint that \(\int d^2x \mu(x)\) be
conserved for all times is a natural addition, implying that what leaves one
half-plane must end up in the other. This corresponds to the conservation of
spin in the spin picture. Of course this is only one physical interpretation of
such a system: the effective field equations know nothing of any microscopic
origins and are therefore not prejudiced by the association with spin or any
other parity violating effect.

To solve the formal constraint, let \( \mu = \mu(B(t), t, \rho) \), \( \rho = J^0 \), where \( J^\mu \) is
the source in (21). The constraints on the variables in the action are Bianchi
identity

\[
\partial_t B + \epsilon_{ij} \partial_i E^j = 0
\]

and the gauge invariance identity (23)

\[
\frac{\delta S}{\delta \theta} = \int dV x \left\{ \beta \rho \ddot{\mu} - B \dot{\mu} \right\} = 0
\]

where dots represent partial time derivatives and the boundary condition
(24) implies that

\[
f(t) = \frac{1}{2} \beta \dot{\mu}.
\]

Eqn. (34) may be solved for \( \mu \) as in the preceding subsection. It is important
to note that the role of the gauge invariance identity is two-fold here. It is
both an algebraic relation between variables in the action and a condition on
the possible variations of the action. Since \( J_\mu \) is an external source, one has
that \( \delta J_\mu = 0 \), thus a variation of the action leads to

\[
\delta S = \int dV x \left\{ \delta A^\nu \partial^\mu F_{\mu \nu} + \frac{1}{2} \frac{\delta \mu}{\delta B} \beta B e^{\mu \nu \lambda} A_\mu \partial_\nu A_\lambda + \mu e^{\mu \nu \lambda} \delta A_\mu \partial_\nu A_\lambda \\
+ \frac{1}{2} (\partial_\nu \mu) e^{\mu \nu \lambda} \delta A_\mu A_\lambda + \frac{\delta f(t)}{\delta B} B (J^\mu A_\mu) + f(t) J^\mu \delta A_\mu \right\},
\]

where \( \delta B = \epsilon_{ij} \partial_i \delta A^j \). The variation of the constraint (34) yields

\[
\left( \beta \rho \frac{\delta \ddot{\mu}}{\delta B} - \dot{\mu} - B \frac{\delta \dot{\mu}}{\delta B} \right) \delta B = 0.
\]

The only apparent solution is \( \delta B = 0 \), which implies that the vector potential
may be written \( \delta A^j = \partial^j \xi \) for some scalar function \( \xi \) and \( j \neq 0 \). The zeroth
component \( A^0 \) is unrestricted and hence there is no contradiction with (33).
Substituting this into the variation of the action \(36\) gives the field equations for the system. For \(A^0\) one has,

\[
\frac{\delta S}{\delta A^0} = \int dV_x \left\{ \partial_i E^i - \mu(B,t)B + f(t)\rho \right\} = 0 \tag{38}
\]

which is identical in form to the usual result for constant \(\mu\). For \(A^j\) one has

\[
\delta S = \int dV_x \left\{ \partial^j \xi \left( \partial^\mu F_{\mu i} \right) + \mu \epsilon^{i\nu\lambda} \partial_\nu A_\lambda - \frac{1}{2} \dot{\mu} \epsilon^{ij} (\partial_i \xi) A_j + f(t) J^i (\partial_i \xi) \right\} = 0 \tag{39}
\]

Integrating by parts and using the conservation equation \(\partial^i J_i = -\dot{\rho}\) gives

\[
B + \beta \dot{\rho} = 0 \tag{40}
\]

which agrees precisely with the gauge invariance condition \(34\) up to a total time-derivative. Thus the consistency between the field equation and the gauge invariance condition is restored, by analogy with \(20\). It is interesting to observe that the field equation is independent of the Chern-Simons coefficient, so that linearity is preserved. The equation \(34\) determining the coefficient takes the form

\[
\dot{\mu} = C \exp \int \frac{B(t)}{\beta \rho} dt \tag{41}
\]

which exhibits exponential behaviour. The connection with spin relaxation can now be noted as follows. The variable \(B\) in the action is the effective electromagnetic field, not the microscopic field felt by the spins. This includes the effect of the spin degrees of freedom. Since the coefficient of the Chern-Simons term is proportional in some sense to the sign of the spin and depends on the chemical potential of the spins\[4, 9\] this is the relevant variable to consider. One is thus interested in consistent solutions of \(41\). As the effective field \(B\) tends to zero, the Chern-Simons coefficient tends to a constant value (typically zero). For changing \(B\) and \(\mu\) the external source is needed to drive the system. The simplest solution is for exponentially decaying \(\mu\) and \(B\) which corresponds to spin relaxation in the absence of an external field. If one drives the spin system with an adiabatically sinusoidal time-varying magnetic field, this is reflected by an oscillatory part for \(B\). This is coupled to the time-variation of \(\mu\) through \(38\) and leads to an exponential lag in the response, corresponding to a hysteresis effect.
3.4 Contact potential: a switching junction

A further example of physical interest arises in the case of a sharp boundary supporting a contact potential $\Delta A_0$. Taking the action (12) and boundary condition (14) one has, in components

$$\Delta(B - \frac{1}{2}\mu A_0) - j^2 = 0$$
$$\Delta(-E_1 + \frac{1}{2}\mu A_2) - j^0 = 0$$

and there is a step $\Delta \mu$ at $x_1 = 0$. This system possesses certain qualities resembling those of a transistor or switching device: two regions of differing properties separated by a thin junction to which an external (bias) current is applied. It is straightforward to show that the picture has the properties of a switching device. Taking $j^\mu$ to be an external source (coupling to the third dimension, for instance) which acts only at the junction, conventional electromagnetic boundary conditions are obtained (and gauge invariance is restored) provided

$$\frac{1}{2}\Delta \mu A_0 + \pi \Delta A_0 = -j^2$$
$$\frac{1}{2}\Delta \mu A_2 = j^0$$

where barred quantities signify the mean values of the respective parameters at the discontinuity ($\overline{A}_2 = A_2$). Let $\phi \equiv \Delta A_0$. Since the step in the Chern-Simons coefficient is a physical quantity, it should be gauge invariant, thus equating (43) and (44) one has

$$-j^0 = j^2 + \pi \phi$$

and for gauge invariance

$$\overline{A}_0 = \overline{A}_0 + \partial_0 \theta$$
$$\overline{A}_2 = \overline{A}_2 + \partial_2 \theta$$

This implies a restriction on the gauge invariance of the theory at the boundary to transformations of the form\footnote{For instance, $\theta$ might be of the form $\exp(i(kx^2 + \omega t))$ in which case it plays the role of a massless excitation.}

$$\theta = \theta(\overline{A}_2 x^2 + \overline{A}_0 t).$$

(47)
This condition is satisfied by wavelike solutions, for instance. The gauge invariant ratio \( \alpha \equiv A_0/A_2 \) must be regarded as a property of a given interface and the gauge invariance requirement becomes

\[
\phi = -\frac{j^2 + j^0 \alpha}{\mu}.
\]  

(48)

The contact potential or \( \Delta \mu \) is seen to depend on the external current \( j^2 \) as well as the density at the boundary \( j^0 \). In particular, in the absence of current, the step must collapse.

Now suppose that an external electrostatic potential \( V \) is held across this junction in reverse bias (opposing the potential \( \phi \)). Donor charges (i.e. those not taken into account by the effective theory) or quasi-particles will only be able to surmount the potential barrier only if \( V > \phi \). However the size of \( \phi \) is controlled by the external source and thus the junction can be made to switch a small electric current. It is noted, on the other hand, that this ‘transistor’ is somewhat unusual since the apparent magnetic field leads also to a Hall drift of the electrons, and thus they drift parallel to the interface as well as across it. The generic behaviour described here is clearly relevant in a device which is populated largely by electrons of the same spin, or in an ordinary device with differing magnetizations in a strong magnetic field. It is known that, in a system of fermions with predominantly one spin direction, a Chern-Simons term is induced by radiative corrections\[7\] in the long wavelength limit, the coefficient of which is proportional to the two-dimensional spin eigenvalue. The above scenario then describes a spin-switch, i.e. a semi-conductor-like switch which is controlled entirely by parity-violating spin effects. Some experimental evidence for this exists already.

The flow in and out of the system by sources at the boundary, in our notation, has the appearance of a charged current. This is because \( J_\mu \) is formally conserved. However, the variable coupling \( f(x) \) implies that \( J_\mu \) is not conserved on the boundary. There, the lack of manifest gauge invariance implies that the compensating current could appear neutral, as seen from the junction’s perspective. Thus \( J_\mu \) need not be interpreted as a current of electrons. The dissipation could relate to radiated energy generated by spin flip transitions in moving from one region to the other. If an electric current is prevented from flowing freely then it would be expected, from the preceding discussion, that there would be some resistance to the transport across the junction, since a tendency toward a reflective boundary condition
would arise. This could lead to a component of $J_1$ arising, implying through an accumulation of spins at the junction. While these would decay by diffusion eventually, there is the possibility of some hysteresis depending on the conductivity of the material at the boundary.

Finally, it is important to bear in mind that the present theoretical model exists only at the level of an effective field theory, not a microscopic one. The motion of dressed quasi-particles through an interface of changing physical properties must take place through the interaction with some third party, since the particles must be ‘undressed’ and ‘redressed’ in order to make the transition across the barrier. In the above construction, this occurs through an interaction with the source. The source itself consists formally of quasi-particles characterized by the mediating value of the physical parameter $\mu$. In the above discussion we have adopted the classical viewpoint of an effective field theory to avoid dealing with the subtleties of quantum tunneling, but in the final analysis one is interested in dispensing with the effective theory and finding the appropriate microscopic one. Tunneling processes will be relevant at this level.

### 3.5 Experimental evidence

After predicting the diffusive junction behaviour and the above switch model, we learned of two separate experiments which have a direct relevance to these findings. Experiments by Kane et al.[5] consider the interface between two quantum Hall systems with different filling fractions. This corresponds to differing Chern-Simons coefficients according to the arguments in refs. [9, 23]. The key observation is that the aligned spins break parity invariance and that the magnitude of this breaking depends on the number of spins[4]. Here one observes a diode-like behaviour, as expected from the general arguments above. It is interesting that this experiment shows evidence for a polarization of nuclear spins at the junction. This coupling to nuclear spin acts as source (sink) of the polarization in crossing the junction and constitutes the ‘external system’ in the preceding source language[4, 10].

A second set of experiments by Johnson relates more directly to the switching phenomenon[6]. Here a device is constructed in which spin-polarized charge carriers inhabit a thin gold layer with a two dimensional symmetry.

\[\text{We thank B. Halperin and T. Finstad for pointing out these references.}\]
Spins subsequently migrate across a junction into a region of different or opposite spin by passing through a finite width junction which is coupled to an external source of current. A ‘spin bottleneck’ phenomenon is observed which prevents the non-equilibrium junction from decaying when the source is active and switching is indeed observed, in accordance with the above discussion.

4 Non-minimal coupling

In this section we study a related system which avoids some of the gauge invariance problems of the Chern-Simons system.

4.1 Gauge Invariance

Consider the action

\[ S = \int dV \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \kappa(x) \epsilon^{\mu\nu\lambda} J_\mu F_{\nu\lambda} \right\} \] (49)

for some conserved current \( J^\mu \). This model has been studied in a number of recent works\[17, 18, 19, 20, 21\]. The coupling of the current to the dual of the field strength leads to a local magnetic moment type interaction for scalar particles\[18\]. In 2+1 dimensions this leads to an induced phase analogous to the anyon phase, known as the Aharonov-Casher phase\[21\].

An interesting duality exists between this term and the usual Chern-Simons type parity breaking term. If one begins with scalar electrodynamics coupled to a Chern-Simons term, the non-minimal coupling is induced by radiative corrections\[18\]. The coefficient is related to the inverse of the Chern-Simons coefficient and the electric charge and does not vanish in the long wavelength limit. Similarly, if one begins with the non-minimal coupling and computes one-loop corrections, a regular Chern-Simons term is induced. Apart from these properties, this non-minimal coupling is interesting in the present context due to its manifest gauge invariance, even in the presence of a boundary. It is natural to compare the boundary properties of the above model with the more usual Chern-Simons term.

Applying the action principle to (49), one has

\[ \partial_\mu \left[ F^{\mu\lambda} + \kappa(x) \epsilon^{\mu\nu\lambda} J_\nu \right] = 0 \] (50)
and associated boundary condition, for a junction at $x_1 = 0$,

$$\Delta(F^{1\lambda} + \kappa(x)e^{1\nu\lambda}J_\nu) = 0. \quad (51)$$

It is now possible to compare the action (49) to the Chern-Simons action coupled to a source (21). A direct comparison is only possible if the current itself includes terms involving the gauge field. This is the case for a complex scalar field, for example. Then one has

$$J_\mu = i[\Phi(D_\mu \Phi) - \Phi(D_\mu \Phi)^\dagger], \quad (52)$$

where $D_\mu = \partial_\mu - ieA_\mu$, or in the unitary parameterization $\Phi = \rho e^{i\theta}$,

$$J_\mu = 2\rho^2[\partial_\mu \theta - eA_\mu]. \quad (53)$$

The non-minimal term in (49) then takes the form

$$2\kappa(x)\rho^2 F^{*\mu}(\partial_\mu \theta - eA_\mu) \quad (54)$$

where $F^{*\mu} = \frac{1}{2}\epsilon^{\mu\nu\lambda}F_{\nu\lambda}$. Comparing this to (21) and derived quantities, it is possible to identify

$$\mu(x) = -4e\kappa(x)\rho^2 \quad (55)$$

$$f(x)J^\lambda = -2\epsilon^{\mu\nu\lambda}(\partial_\nu \kappa \rho)(\partial_\mu \theta) \quad (56)$$

which indicates that the non-minimal coupling can be regarded as a normal Chern-Simons term with a variable coefficient plus an external massless scalar field $\theta$ which lives in the region $\nabla_\mu (x) \neq 0$ (for example on boundaries). This is consistent with the discussion in section 3.2 and in refs. [9, 10].

4.2 Atomic spring model

For the purpose of determining bound state properties for comparison with ref. [11], it is interesting to consider a toy model of a two dimensional medium in which electrons are bound to their parent atoms by means of a pseudo-harmonic potential. If $s^i$ is the displacement vector of a single electron then under a Lorentz force,

$$\left(\partial_t^2 + \gamma \partial_t + \omega_0^2\right)s^i = -\frac{e}{m}(E^i + \epsilon_{ij}B_c \partial_t s^j) \quad (57)$$

16
where \( B_c \) is a constant external magnetic field, \( \omega_0^2 = k/m \) for spring constant \( k \) and \( \gamma \) is a damping factor. The current arising from this motion can be characterized by

\[
J^\mu = \left( \begin{array}{cc}
\rho & 0 \\
0 & -N e \partial_i s
\end{array} \right)
\]  

(58)

where \( \rho \) is the total charge density (which is usually zero, including the effect of background charges) and \( N \) is the number of optically active electrons. Adding a minimal gauge coupling \( J^\mu A_\mu \) to (49), and omitting dielectric polarization effects one has the field equation for constant \( \kappa \)

\[
\partial_\mu F^{\mu\lambda} + \kappa \epsilon^{\mu
u\lambda} \partial_\mu J_\nu = J_\lambda
\]  

(59)

and boundary condition given by (51). We shall use this model below in order to determine the optical properties of this system. Such properties were previously calculated for the usual Chern-Simons term in ref. \[11\].

4.3 Complex Scalar Field

A more realistic quantizable field theory can be obtained by considering a scalar field coupled minimally to a gauge field. This is closely related to the effective \( P \) and \( T \) breaking theory introduced in ref. \[22\] for superconductivity, and the to the Landau-Ginsburg theory of the Hall effect\[23\]. The scalar field represents the collective excitations of an unknown microscopic system in the long wavelength limit. In the Chern-Simons case, this model has already been shown to lead to a Faraday rotation in reflected waves\[22\]. The principal difference from the atomic spring model is that the scalar field is a superconductor-insulator. There is no in-built mechanism for dissipation at zero temperature, hence no finite conductivity. Moreover, the model does not describe bound states. In the superconductive regime, waves will not propagate inside the material, but extra-planar waves can be reflected off a two-dimensional surface in which the scalar field lives. In the insulating regime, wavlike-solutions can only exist under special circumstances which will be described below.

The action for a complex scalar field is defined by

\[
S = \int dV_x \left\{ (D^\mu \Phi)^\dagger (D_\mu \Phi) - m^2 \Phi^\dagger \Phi - \frac{\lambda}{6} (\Phi^\dagger \Phi)^2 - \frac{1}{4} F^{\mu \nu} F_{\mu \nu} + \frac{1}{2} \kappa \epsilon^{\mu \nu \lambda} J_\mu F_{\nu \lambda} \right\}
\]  

(60)
where the current is given by \( (52) \) and \( D_\mu = \partial_\mu - ieA_\mu \). It is noteworthy that the matter coupling involves a quadratic dependence on the gauge field that cannot be written strictly in the form \( J^\mu A_\mu \). The field equation for the scalar field is given by

\[
(D^2 + m^2)\Phi + 2i\kappa F^*\mu(D_\mu \Phi) + \frac{\lambda}{3}(\Phi^\dag \Phi)\Phi + i(\partial^\mu \kappa)F^*\Phi = 0 \quad (61)
\]

with associated boundary condition at \( x_1 = 0 \),

\[
\Delta(D_1 + \frac{i}{2}\kappa\epsilon_{1\nu\lambda}F^{\nu\lambda})\Phi = 0. \quad (62)
\]

For the gauge field, one obtains

\[
\partial^\mu F^{\mu\lambda} + \kappa\epsilon^{\mu\nu\lambda}\partial_\mu J_\nu - 2e\kappa F^*\lambda\Phi^\dag \Phi = eJ^\lambda \quad (63)
\]

provided

\[
\Delta(F^{1\lambda} + \kappa\epsilon^{1\mu\lambda}J_\mu) = 0. \quad (64)
\]

5 Energy Momentum Tensor

The gauge non-invariance of the Chern-Simons Lagrangian at a boundary is accompanied by a discontinuity in the electromagnetic Poynting vector \( \Delta S_1 = \Delta(E_2 B) \). This is another indication that the theory is not complete at the boundary. To properly understand the energy flow in the present models one should supplement the usual electromagnetic Poynting flow by a contribution from the parity violating term. A consideration of the energy-momentum tensor as defined through Nöther’s theorem leads to the relevant conserved quantities. The generalized force law for the system is given by

\[
f^\mu = \int dv_x \partial_\nu \theta^{\nu\mu} \quad (65)
\]

which, in the case of the electromagnetic field, gives rise to the Lorentz force law \( f^i = -\int dv_x J^\mu F^i_\mu \), where \( dv_x \) is a spatial volume element. Any anomalies in \( \theta^{\mu\nu} \) could give rise to corrections to this force law and are therefore

\(^3\)If one defines the energy-momentum tensor by \( T^{\mu\nu} = \frac{\delta S}{\delta g_{\mu\nu}} \) for the metric \( g_{\mu\nu} \) then there is no contribution from parity violating terms, since \( \epsilon^{\mu\nu\lambda} \) transforms like a tensor density and is subsequently independent of the metric. However, this should be regarded as a failure of the variational definition rather than a reason to disregard the extra terms.
important to the discussion of boundary effects. A generic feature of the energy-momentum tensor in the present models is that it is non-symmetric and gauge-dependent. This signals a breakdown of Lorentz invariance as well as gauge invariance. That these two principles should be broken simultaneously is reminiscent of the deficiencies of the canonical energy momentum tensor\[12, 13\] and thus in what follows we shall adopt the covariant procedure of refs. \[12\] and \[13\] to obtain directly the normally symmetric ‘Belifante energy-momentum tensor’.

To construct $\theta^{\mu\nu}$, one notes that the covariant vector potential $A^\mu$ transforms like a 3-vector only up to a gauge transformation. Nöther’s theorem implies that under a Lorentz transformation

\[
x^{\mu} \rightarrow x^{\mu'} = x^{\mu} + \delta x^{\mu}
\]
\[
A^{\mu}(x) \rightarrow A^{\mu'}(x) = A^{\mu}(x) + \delta A^{\mu}(x)
\]

(66) (67)
a symmetric theory is characterized by the continuity equation $\partial_{\mu} C^{\mu} = 0$, where

\[
C^{\mu} = \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} A_{\sigma})} \delta A_{\sigma} + \mathcal{L} \delta x^{\mu}
\]

(68)
where $S = \int dV \mathcal{L}$. The defining equation for $\theta^{\mu\nu}$ is then

\[
C^{\mu} = -\theta^{\mu\nu} \delta x_{\nu}.
\]

(69)
The gauge-Lorentz invariant restriction $\delta A_{\mu} = F^{\sigma}_{\mu \delta} \delta x_{\sigma}$ implies that

\[
\theta^{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial (\partial_{\mu} A_{\sigma})} F^{\sigma}_{\nu} - \eta^{\mu\nu} \mathcal{L}.
\]

(70)
If a boundary is introduced then Lorentz invariance is explicitly violated and this expression loses its immediate interpretation as a consequence of the lack of translational invariance. In the absence of a parity violating term one has the usual electromagnetic energy momentum tensor

\[
\theta^{\mu\nu}_{EM} = F^{\sigma\mu} F^{\nu}_{\sigma} + \frac{1}{4} \eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}.
\]

(71)
The corrections $\Delta \theta^{\mu\nu}$ to this due to the parity violating terms will now be discussed.
5.1 Chern-Simons

Use of the Chern-Simons Lagrangian in (70) gives
\[ \Delta \theta^{\mu\nu} = \frac{1}{2} \mu \epsilon^{\rho\mu\sigma} A_\rho F_\sigma^{\nu} - \frac{1}{4} \mu \eta^{\mu\nu} \epsilon^{\rho\sigma\lambda} A_\rho F_{\sigma\lambda}. \] (72)

This contribution is not symmetric, but, using the Bianchi identity \[ \partial_\mu F_{\nu\lambda} = 0 \] it is seen to be gauge-invariant provided the coefficient \( \mu \) is a constant. The integral of the divergence of the full \( \theta^{\mu\nu} = \theta^{\mu\nu}_{EM} + \Delta \theta^{\mu\nu} \) is conserved in the absence of sources,
\[ \int dV_x \partial_\mu \theta^{\mu\nu} = 0 \] (73)
where the field equations (13) have been used. Adding sources produces the conventional modification
\[ \int dV_x \partial_\mu \theta^{\mu\nu} = - \int dV_x J_\sigma F_\sigma^{\nu} \] (74)
which is the standard Lorentz force law. The latter result implies that no modification of the force law is required in Maxwell-Chern-Simons theory, except at boundaries or in regions of variable \( \mu(x) \) in which case one finds that
\[ \int dV_x \partial_\mu \theta^{\mu\nu} = \int dV_x \left\{ \mu(x) F^{*\sigma}(\frac{1}{2} F_\sigma^{\nu} + \frac{1}{2} (\partial_\nu A_\sigma)) \right\}. \] (75)
Under a gauge transformation \( A_\mu \to A_\mu + \partial_\mu \chi \), this term changes by the gradient of \( -\frac{1}{2} (\partial_\mu \mu(x)) F_\sigma^{*\sigma} \chi \), which may be compared to (22) for zero source. This identification shows explicitly that the lack of energy conservation at the boundary is related to the lack of gauge invariance, as previously argued in ref. [9] and leads to a generalized force on the system.

In ref. [14] the presence of an instability in the dispersion was found. Although some evidence for this can be seen in \( \theta^{00} \) (which has no definite sign), this does not cause any problems in the present work.

5.2 Non-minimal term

For the non-minimal Lagrangian, one has
\[ \Delta \theta^{\mu\nu} = \kappa \epsilon^{\rho\mu\sigma} J_\rho F_\sigma^{\nu} - \frac{1}{2} \eta^{\mu\nu} \kappa \epsilon^{\alpha\beta\lambda} J_\alpha F_{\beta\lambda}. \] (76)
and it is assumed that \( J_\mu \) is independent of \( A_{\mu\nu} \). This correction is non-symmetric, but is explicitly gauge invariant. The integral over the divergence of the total energy-momentum tensor is conserved, provided \( \kappa \) is constant, and the addition of sources leads to the usual Lorentz modification (74). For non-constant \( \kappa(x) \), one has

\[
\int dV_x \partial_\mu \theta^{\mu\nu} = -\int dV_x \kappa \epsilon^{\mu\nu\rho\sigma} \left\{ (\partial_\mu J_\rho) F_{\nu\sigma} - \frac{1}{2} (\partial_\nu J_\rho) F_{\mu\sigma} \right\}.
\]

(77)

This term is gauge invariant and represents a modification of the standard Lorentz force wherever \( \kappa \) is a function of \( x^\mu \).

In ref. [9] it was shown that a Casimir force is necessarily present on a gauge-invariant reflective boundary due to quantum or thermal fluctuations in Maxwell-Chern-Simons theory. The force arises because there is a mass or gap-mismatch in the spectra for the two sides of the junction. Here the spectrum of the non-minimal coupling is notably gapless (massless) in the absence of an ordered phase (spontaneous symmetry breaking). See equation (106). This seems also to be confirmed by refs. [19, 20] at one loop, but here one notes that scalar and gauge loops are inextricably linked by the non-minimal term so that it is not possible to consider the gauge field in isolation. Further investigations would be required to determine the absence of a gap due to fluctuations in general.

Finally, \( \theta^{00} \) does not have a definite sign, which suggests the possibility of an instability for certain values of the sources. No instability is found in the dispersion (106, 120) however. In the case of normally impinging waves for inhomogeneous currents, this is less clear (123).

### 6 Electromagnetic waves

In an earlier paper [11] one of us has considered the properties of wavelike solutions of the Lagrangian (12). Related work has also been carried out in 3 + 1 dimensions [14]. There it was remarked that a future problem would be to consider the analogue of Fresnel’s equations at a material interface. Since the naive boundary condition for the Chern-Simons model is not gauge invariant, it is not immediately clear how to proceed. To satisfy the gauge invariant boundary condition (17), there must be total reflection from the line \( x_1 = 0 \). If on the other hand one couples to an external source as in (21),
then the gauge non-invariant parts of the boundary condition vanish and one is left with the normal electromagnetic boundary conditions. The virtue of the non-minimal model is its automatic gauge invariance at the boundary.

6.1 Maxwell-Chern-Simons atomic spring model

We begin with the unmodified Chern-Simons theory and consider the case in which the coefficient is constant.

6.1.1 Planar dispersion relation

The dispersion relation is obtained on substituting the trial solution

\begin{align*}
E^i &= E_0^i e^{i(k_i x_i - \omega t)} \\
B &= B_0 e^{i(k_i x_i - \omega t)} + B_c \\
J^i &= J_0^i e^{i(k_i x_i - \omega t)} \\
\rho &= \rho_0 e^{i(k_i x_i - \omega t)} + \rho_c
\end{align*}

into the components of the field equations

\begin{align*}
B_c &= -\frac{1}{\mu} \rho_c \\
ike_0 &= \mu B_0 + \rho_0 \\
\omega E_0^i - \mu E_0^0 &= J_0^0 \\
-\ike_0 + \omega E_0^0 + \mu E_0^0 &= J_0^1
\end{align*}

where the parallel and perpendicular projections are defined through the relations

\begin{align*}
k^i E_0^i &= k E_0^0 \\
\epsilon_{ij} k^i E_0^j &= k E_0^\perp
\end{align*}

and similar ones for the current. If one neglects the oscillatory part of the magnetic field from the equations of motion (which implies that the magnetically induced current-response is small) then these equations, together with (57) and (58), can be manipulated so as to eliminate all variables except the
electric field, which then satisfies the equation

\[
\begin{pmatrix}
i\omega mW + iNe^2\omega - i\mu eB_c & -\mu mW - eB_c k^2 + \omega^2 eB_c \\
mW - \omega^2 eB_c & (\frac{-i k^2}{\omega} + i\omega)mW + iNe^2\omega - i\mu eB_c
\end{pmatrix}
\begin{pmatrix}
E_0 \parallel \\
E_0 \perp
\end{pmatrix} = 0
\]

(87)

where \( W = (-\omega^2 - i\gamma\omega + \omega_0^2) \). Demanding the vanishing of the determinant of this matrix leads immediately to the dispersion relation

\[
k^2 = \omega^2 \left\{ 1 - \frac{\mu^2}{\omega^2} + \frac{\omega_p^2}{W^2} \left( 1 + \frac{\mu^2}{\omega^2} \right) + \frac{\omega^2}{W^2} - 2\mu\omega_L \right\}
\]

(88)

where \( \omega_p^2 = Ne^2/m \) and \( \omega_L = eB_c/m \). This result has been given in ref. \[11\]. The refractive index is given by \( n = k/\omega \). The real and imaginary parts of the refractive index are plotted against \( \omega \) for various values of \( \mu \) in Figs. 1a and 1b. At a junction between two regions, this dispersion relation applies provided the interface is sharp. Sources are then needed to balance the requirements of gauge invariance, but their introduction leads only to normal electromagnetic boundary conditions, thus there is no modification to the laws of reflection or refraction and Fresnel’s relations are given by the usual formulas (see below). Thus when two dimensional waves strike the interface, currents are set up along the interface. These currents must either disperse into the two dimensional system or out into the external system. If no sources are introduced, the dispersion is only given by \( (88) \) at \( t = 0 \). The subsequent decay of the interface then modifies the dispersion in a non-linear way until a situation of equilibrium is reached.

### 6.1.2 Extra-planar dispersion relation

To show that the Chern-Simons term gives rise to a modified Faraday effect\[4\], one embeds the planar model into a three dimensional space and directs plane polarized waves so that they impinge normally to the plane. Since the matter fields describe an ostensibly two dimensional system all currents are restricted to the plane. The embedded action is given by

\[
S = \int dV \left\{ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - J^\mu A_\mu + \frac{1}{4} \bar{\mu} \epsilon_{\mu\nu\lambda\delta} A^{\mu} F^{\nu\lambda} \right\}
\]

(89)

\[4\]This is also referred to as optical activity or circular birefringence.
where caret indices run over the third spatial dimension in addition to their usual values and \( \tilde{\mu} \) has dimension one. The components of the field equation are then given by

\[
\begin{align*}
\partial_i \hat{E}^i - \tilde{\mu} B^3 &= \rho \\
-\partial_i \hat{E}^i + \epsilon_{ijk} \partial_j B^k - \tilde{\mu} \epsilon_{ijk} \hat{E}^j &= J^i \\
\partial_i B^k + \epsilon_{ijk} \partial_j \hat{E}^j &= 0 \\
\partial_i \hat{B}^i &= 0.
\end{align*}
\] (90)

Next consider a solution of the form

\[
\hat{E}^i = \hat{E}_0^i e^{i(kz - \omega t)}
\] (91)

with corresponding expressions for the magnetic field and current. To obtain the dispersion relation between \( k \) and \( \omega \) it is sufficient to consider the planar components of the electric field. It is convenient to define complex variables \( E = E^1 + \tilde{j} E^2 \) where \( j^2 = -1 \), but \( ij \neq ji \neq -1 \)\(^5\) and corresponding variables for the displacement vector \( s \) \([15]\). With these variables the equation of motion for the electrons \((57)\) becomes (neglecting damping terms)

\[
(-\omega^2 + \omega_0^2) s = -\frac{e}{m} (E - \tilde{j} B_c \partial_t s).
\] (92)

Combining the second and third equations in (90) gives

\[
N_e \omega^2 s = (\omega^2 - k^2) E - ij \omega \mu E.
\] (93)

Finally, eliminating one of the variables from the last two equations gives the dispersion relation

\[
k^2 = \omega^2 \left\{ 1 + \omega_p^2 \frac{-\omega^2 + \omega_0^2}{(\omega^2 + \omega_0^2)^2 - \omega_L^2 \omega^2} - ij \left( \frac{\tilde{\mu}}{\omega} \omega_p^2 \frac{\omega \omega_L}{(\omega^2 + \omega_0^2)^2 - \omega_p^2 \omega^2} \right) \right\}.
\] (94)

The real and imaginary parts of the refractive index \( n = k/\omega \) are plotted against \( \omega \) for various values of \( \tilde{\mu} \) in Figs. 2a and 2b. The combination \( ij \)

\(^5\)Note that if these imaginary units are treated as anticommuting numbers they reproduce the \( \text{su}(2) \) algebra of rotations. Here, as commuting numbers they represent a realization of the group \( U(1) \times U(1) \).
guarantees energy conservation and implies that the wavenumber has the form
\[ k = k_r - i j k_{ij} \tag{95} \]
which shows that the transmitted wave suffers a rotation of its plane of polarization
\[ E = E_0 e^{i(k_r z - \omega t)} (\cos k_{ij} z + j \sin k_{ij} z). \tag{96} \]

It is normal to define the measurable rotation angle in terms of ‘Verdet’s constant’ V by
\[ \chi_{\text{rot}} = V B_c L \tag{97} \]
where L is the distance travelled through a stack of planar systems, but since the Chern-Simons term results in such a rotation even in the absence of \( B_c \), it is sufficient to consider the angle rotated per unit length \( k_{ij} \).

## 6.2 Non-minimal atomic spring model

### 6.2.1 Planar dispersion relation

Wavelike solutions to the non-minimal action (49) are obtained by the same method as in ref. [11]. It is curious to note that the presence of the curl of the current makes the field equations sensitive to anisotropy and inhomogeneities in the matter field. The components of (59) give,
\[ \partial_i E^i - \kappa \epsilon_{ij} \partial_i J^j = \rho \tag{98} \]
\[ -\partial_t E^i + \epsilon_{ij} \partial_j B + \kappa \epsilon_{ij} (\partial_j \rho + \partial_t J^j) = J^i \tag{99} \]

Using the wave ansatz (78)-(81), and considering only the oscillating parts leads to
\[ i k E_{0\|} - i k k \cdot J_{0\perp} = \rho_0 \tag{100} \]
\[ i \omega E_{0\|} - i \omega k \cdot J_{0\perp} = J_{0\|} \tag{101} \]
\[ i \omega E_{0\perp} - i k B_0 - i k k \rho_0 + i k \omega J_{0\|} = J_{0\perp} \tag{102} \]
\[ i k E_{0\perp} = i \omega B_0 \tag{103} \]

where the last equation follows from the (33). The continuity equation for the current (which is not an independent equation, but follows from (100) and (101) above) may be written
\[ k J_{0\|} = \omega \rho_0. \tag{104} \]
Eliminating the electric field in favour of the current, one obtains after some effort the matrix equation
\[
\begin{pmatrix}
    mW + Ne^2 & iNe^2\omega\kappa - i\omega eB_c \\
    i\omega eB_c - iNe^2\omega\kappa & mW + Ne^2\frac{\omega^2}{\omega^2 - \kappa^2}
\end{pmatrix}
\begin{pmatrix}
    J_{0\parallel} \\
    J_{0\perp}
\end{pmatrix} = 0
\] (105)

The vanishing of the determinant of this matrix yields the dispersion relation
\[
k^2 = \omega^2 \left[ 1 + \frac{\omega_p^2 W + \omega_{\rho}^4}{W^2 + \omega_p^2 W - \omega^2(\omega_L - \kappa\omega_{\rho}^2)^2} \right].
\] (106)

The real and imaginary parts of the refractive index \( n = k/\omega \) are plotted against \( \omega \) for various values of \( \kappa \) in Figs. 3a and 3b.

It is noteworthy that, on making the definitions
\[
D^i = E^i - \kappa \epsilon_{ij} J^j, \quad H = B + \kappa \rho
\] (107)

the wave equation for the transverse magnetic waves takes the usual form
\[
(\partial_t^2 - \nabla^2)H = \epsilon_{ij} \partial_i J^j,
\] (108)

and the Bianchi identity is unchanged, implying that the normal laws of reflection and refraction apply:
\[
\frac{\sin \theta_t}{\sin \theta_i} = \frac{n_1}{n_2}, \quad \theta_i = \theta_r
\] (109)

where the angles \( \theta \) refer to the incident, transmitted and reflected angles to the normal. It is quite possible that a non-linear scalar field theory (for instance, with a \( \lambda \phi^4 \) term or higher power) would violate this law, if wave solutions can even by found.

From the linearity, it is possible to write
\[
(E_{0i} - E_{0r}) \cos \theta_i = E_{0t} \cos \theta_t
\] (111)

Thus, defining the wave impedance \( Z = E/H = \mu_r/n^2 \), where \( \mu_r H = B \), and noting that the boundary condition may be written \( \Delta H = 0 \), one has the familiar result that
\[
\frac{E_{0i}}{Z_1} + \frac{E_{0r}}{Z_1} = \frac{E_{0t}}{Z_2}.
\] (112)
Combining (111) and (112) gives Fresnels standard relations for the transmission and reflection coefficients

\[
\frac{E_{0r}}{E_{0i}} = \frac{Z_1 \cos \theta_i - Z_2 \cos \theta_t}{Z_2 \cos \theta_t + Z_1 \cos \theta_i} \quad (113)
\]

\[
\frac{E_{0t}}{E_{0i}} = \frac{2Z_2 \cos \theta_t}{Z_2 \cos \theta_t + Z_1 \cos \theta_i}. \quad (114)
\]

### 6.2.2 Extra-planar dispersion relation

To compute the Faraday effect, the planar theory is embedded in a three dimensional space, with waves impinging normally, as before. The action is therefore

\[
S = \int dV_x \left\{ -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} - J^\mu A_\mu + \frac{1}{2} \ddot{k} \epsilon_{\mu \nu \lambda \delta} J^\mu F^{\nu \delta} \right\} \quad (115)
\]

where caret indices include the third dimension. The currents are assumed to lie purely in the plane. The components of the field equation and Bianchi identity are now given by

\[
\partial_t E^i - \ddot{k} \epsilon_{ij} \partial_j J^j = \rho
\]

\[
-\partial_t E^i + \epsilon_{ijk} \partial_j B^k + \ddot{k} \epsilon_{ij} \left( \partial_t \rho + \partial_t J^j \right) = J^i
\]

\[
\partial_t B^k + \epsilon_{ijk} \partial_i E^j = 0
\]

\[
\partial_t B^i = 0. \quad (116)
\]

The oscillating magnetic field can be eliminated to yield a wave equation for the electric field

\[
E^i = E_{0i} e^{i(kz - \omega t)}. \quad (117)
\]

given by

\[
(-\partial_t^2 + \nabla^2) E^i + \ddot{k} \epsilon_{ij} \left( \partial_t \partial_j \rho + \partial_t^2 J^j \right) = \partial_t J^i. \quad (118)
\]

It is pertinent to note that the presence of spatial derivatives of the current in the above expression implies that the system is sensitive to inhomogeneities and anisotropy. A proper description of anisotropy cannot be obtained from the foregoing equations, since the spring constant \( k = \omega_0^2 m \) would need to be different in the \( x_1 \) and \( x_2 \) directions to make the assumption self-consistent.
6.2.3 Homogeneous isotropic medium

In a homogeneous, isotropic medium, the spatial derivatives of the current vanish identically. Using the complex coordinate representation introduced earlier one has

\[(\omega^2 - k^2)E + ij\tilde{\kappa}\omega^3Nes = \omega^2Nes.\]  (119)

Using the equation of motion for \(s\) \([92]\), it is possible to eliminate \(E\) giving immediately the dispersion relation

\[k^2 = \omega^2 \left[ 1 + \frac{\omega_p^2(1 - ij\tilde{\kappa})}{(-\omega^2 + i j \omega L + \omega_0^2)} \right].\]  (120)

or

\[k^2 = \omega^2 \left\{ 1 + \frac{\omega_p^2[\omega_0^2 - \omega^2 + \tilde{\kappa}\omega L \omega^2 - ij(\tilde{\kappa}\omega(\omega_0^2 - \omega^2) + \omega_L \omega)]}{(\omega_0^2 - \omega^2)^2 - \omega^2 \omega_L^2} \right\} \]  (121)

The real and imaginary parts of the refractive index \(n = k/\omega\) are plotted against \(\omega\) for various values of \(\tilde{\kappa}\) in Figs. 4a and 4b.

\(k_{ij}\) is the rotation per unit length through a layered system, where \(k = k_r - ijk_{ij}\).

6.2.4 Homogeneous, anisotropic medium

If the wavelength of waves is small compared to the inhomogeneities of the material medium, the dispersion becomes sensitized to the structure. Consider the case of a uni-axial crystal which permits inhomogeneities of the current in a preferred direction \(x_1\). Using the trial solution \(E = E_0 \exp(i(k_z z + k_1 x_1 - \omega t))\) one easily obtains

\[(\omega^2 - k_z^2 - k_1^2)E^1 - iNe\kappa\omega^3s^2 = Ne\omega^2s^1\]
\[(\omega^2 - k_z^2 - k_1^2)E^2 + i\omega\tilde{\kappa}Ne(\omega^2 - k_1^2)s^1 = Ne\omega^2s^2.\]  (122)

The complex method is not appropriate here, owing to the lack of symmetry. Nevertheless, it is possible to eliminate the electric field component-wise and solve the determinental equation for the matrix coefficient of \(s^1\) and \(s^2\), giving

\[k_z^2 = \omega^2 \left\{ 1 - \frac{k_1^2}{\omega^2} + \frac{b \mp \sqrt{b^2 - 4ac}}{2a\omega^2} \right\} \]  (123)
where \( a = \omega_0^2 - \omega^2 - \omega^2 \omega_L \tilde{\kappa} \), \( b = -2\omega_0^2\omega^2(\omega_0^2 - \omega^2) + \tilde{\kappa}\omega_L\left(\frac{k_1^2}{\omega} - \omega\right) \) and \( c = \omega_0^4\omega^4(1 + \tilde{\kappa}^2(k_1^2 - \omega^2)) \). The dispersion continues to exhibit birefringence, but now with a marked asymmetry. The \( k_1 \) terms either reduce or enhance the electron mobility, depending on the sign of the magnetic field \( \omega_L \). In particular, it is seen that \( k_1 \) acts as an effective mass gap for the dispersion.

### 6.2.5 Ohmic conductors

The conducting limit of the previous results could in principle be obtained from the \( \omega_0^2 \to 0 \) limit of the atomic spring model. It is useful to reexpress the result in terms of the more familiar conductivity. Let the anisotropic conductivity tensor be defined by its projected components

\[
J_\perp = \sigma_\perp E_\perp \\
J_\parallel = \sigma_\parallel E_\parallel,
\]

then from (100)-(103) one has

\[
\left( \begin{array}{cc}
1 + \frac{i}{\omega} \sigma_\parallel & -\kappa \sigma_\perp \\
\kappa \sigma_\parallel & 1 + \frac{i\omega \sigma_\perp}{\omega^2 - k_1^2}
\end{array} \right) \left( \begin{array}{c}
E_\parallel \\
E_\perp
\end{array} \right) = 0
\]

(125)

giving the dispersion relation

\[
k^2 = \omega^2 \left[ 1 + \frac{-\kappa^2 \sigma_\parallel^2 \sigma_\perp^2 + i(\sigma_\parallel \omega + \sigma_\parallel \sigma_\perp / \omega + \kappa^2 \omega \sigma_\parallel \sigma_\perp)}{\omega^2 (1 + \kappa^2 \sigma_\parallel \sigma_\perp)^2 + \sigma_\parallel^2} \right]
\]

(126)

which reduces to standard results on setting \( \kappa = 0, \sigma_\parallel = 0 \). What is interesting here is that the longitudinal current plays a role in the dispersion. In the vicinity of a boundary, like the edge of a finite sample, the simple split into \( \sigma_\parallel \) and \( \sigma_\perp \) must break down. Close to the edge, the longitudinal conductivity must tend toward zero and be replaced by an enhanced transverse conductivity. This corresponds to a modulation of the charge at the edge of the sample, which then spreads out to form surface density waves.

The Faraday effect is straightforwardly obtained from (119) on substituting \( J = \sigma E \); we ignore the role of anisotropy and inhomogeneities here. Then, straightforwardly

\[
k^2 = \omega^2 \left[ 1 + j \kappa \sigma + \frac{\sigma_\parallel}{\omega} \right]
\]

(127)
which indicates that the rotation of the polarization plane is accompanied by damping and reflection. It is interesting to note that the combination $j\kappa$ implies that the reflective properties are unaffected by $\kappa = 0$ whereas the polarization effect is entirely due to $\kappa \neq 0$.

### 6.3 Non-minimal Complex scalar field

Wavelike solutions for the electromagnetic field are not a general feature of the non-minimally coupled scalar field theory. Consider the case in which the collective field mode is covariantly constant i.e. $D_\mu \Phi = 0$. Then the scalar field equation implies that

$$\Phi^2 = -\frac{3}{\lambda} (i\partial^\mu \kappa F^*_{\mu} + m^2).$$

(128)

If the field strength is oscillating, this makes most sense when $\partial_\mu \kappa = 0$. In a superconducting phase there is clearly no wave propagation in 2 dimensions. There is a regime however in which propagating solutions can be obtained.

For covariantly constant $\Phi$ the current vanishes but $\Phi^\dagger \Phi$ is an invariant constant. The field equations for the electromagnetic field are then

$$\partial_\mu F^{\mu\lambda} - 2\epsilon \kappa F^{*\lambda} \Phi^\dagger \Phi = 0.$$  

(129)

Taking the spatial components of this equation together with the Bianchi identity (33) leads to a determinental equation for the dispersion relation

$$k^2 = \omega^2 - 4\epsilon^2 \kappa^2 (\Phi^\dagger \Phi)^2$$

(130)

which is notably similar to that for the superconductor model in ref. [22], indeed it is regular Chern-Simons dispersion for $\mu = 2\epsilon \kappa \Phi^\dagger \Phi$.

### 7 Discussion

We have examined some of the consequences of parity violation in the vicinity of junctions and boundaries. Using the principles of gauge invariance and energy conservation, we derive the acceptable behaviour of two models for an effective $P$ and $T$ breaking theory. In the case of regular the Chern-Simons term, it is found that dissipation must be a feature close to a boundary. This
dissipation can either be a destructive dissipation – that is, one which tends to erode the boundary itself, or a stable transfer of current to an external system by mediating sources. It is interesting to note that, in the quantum Hall system, edge currents can be measured and that dissipation is observed at the boundaries of the sample, concentrated at the corners, which is at least in qualitative agreement with the picture conveyed here. The non-minimal coupling has by nature edge currents, but these are related unambiguously to the magnetic moment coupling and result in no dissipation. On the other hand, the presence of a ‘curl’ of the current implies a mixing of charge at the boundary with the bulk charge.

The diffusion-like behaviour of a regular Chern-Simons interface, together with the need for an external source of current suggests a transistor like behaviour. While this behaviour is in itself amusing, since it is derived from a consistency argument in an effective theory, it appears to have relevance to experimental devices in which the charge carriers in different isolated regions have predominantly the same spin [5, 6]. An electron passing from one region to another must then flip spin, requiring either an input or a drain of energy.

We have considered the effect of penetration of material samples by electromagnetic radiation. Waves inside $P$ and $T$ breaking media are no longer transverse, but no essential modification of the usual laws of reflection or refraction is noted. Dispersion is qualitatively different for the two models considered. In the regular Chern-Simons theory, the Chern-Simons parameter appears mainly as a mass or gap term in the dispersion relation. The relation for the non-minimal model does not appear to possess such a gap. Both models exhibit the required optical activity, or circular birefringence.

Finally we note that similar studies of boundary effects in parity violating models have been made in refs [24, 25] and [26]. In the former case it is shown that a connection exists between the chiral anomaly in $2n + 2$ dimensions and the Chern-Simons gauge anomaly in $2n + 1$ dimensions, at least to first order in a derivative expansion; when chiral models are calculated non-pertubatively, one also finds that the Chern-Simons form is modified by higher order terms and becomes the $\eta$-invariant. In the latter case, edge currents are argued by appealing to linear response theory and gauge invariance. In both of these cases, currents are responsible for balancing the gauge invariance constraints. In this paper further solutions are found which do not require the inclusion of additional currents and a new physical interpretation is given to the gauge invariance problem.
Acknowledgement

M.B. would like to thank P. Kelly, C. Korthals-Altes, T. Finstad, B. Halperin, R. Jackiw and L. Pryadko for pointing out a number of relevant reference.

References

[1] J. F. Schonfeld, Nucl. Phys. B185 157 (1981)
[2] S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett 48 975 (1982)
[3] S. Deser, R. Jackiw and S. Templeton, Ann. Phys. 140 372 (1982)
[4] M. Burgess, Phys. Rev. D44 2552 (1991)
[5] B.E. Kane et al., Phys. Rev. B46 7264 (1992)
[6] M. Johnson, Appl. Phys. Lett. 63 1435 (1993); M. Johnson. Phys. Rev. Lett. 70 2142 (1993)
[7] A. N. Redlich, Phys. Rev. D29, 2366 (1984)
[8] E. Fradkin, Field Theories of Condensed Matter Systems, Addison Wesley (1991).
[9] M. Burgess, in Proceedings of the Third Workshop on Thermal Fields and their Applications, Banff 1993, World Scientific 1994.
[10] M. Burgess, Phys. Rev. Lett. 72, 2823 (1994)
[11] M. Burgess, J.M. Leinaas and O.M. Løvvik, Phys. Rev. B48 12912 (1993)
[12] R. Jackiw, Phys. Rev. Lett. 41, 1635 (1978)
[13] E. Eriksen and J.M. Leinaas, Phys. Scripta 22, 199 (1980)
[14] S.M. Caroll, G.B. Field and R. Jackiw, Phys. Rev. D41 1231 (1990)
[15] M. Burgess, unpublished (1985).
[16] M. Burgess and B. Jensen, Phys. Rev. A48 1861 (1993)
[17] J. Stern, Phys. Lett. B265 119 (1991)
[18] I. Kogan, Phys. Lett. B262 83 (1991)
[19] M.E. Carrington and G. Kunstatter, Phys. Lett. B321 223 (1994)
[20] M.E. Carrington and G. Kunstatter, Phys. Rev. D To appear. (1994)
[21] M.E. Carrington and G. Kunstatter, Phys. Rev. Lett. To appear. (1994)
[22] X.G. Wen and A. Zee, Phys. Rev. Lett. 62 2873 (1989)
[23] S.C. Zhang, Int. J. Mod. Phys. B6 25 (1992)
[24] C.G. Callan and J.A. Harvey, Nucl. Phys. B250 427 (1985)
[25] A. N. Redlich and L.C.R. Wijewardhana, Phys. Rev. Lett 54 (1985) 970; K. Tsokos, Phys. Rev. Lett. B187 (1986) 187; A. Niemi, Phys. Rev. Lett. 57 (1986) 1102; A. Rutherford, Phys. Lett. B182 (1986) 187
[26] X.G. Wen, Int. J. Mod. Phys. B6, 1711 (1992)

Figure Captions

1a. The real part of the refractive index $n = k/\omega$ as determined from Eqn. (88) plotted against $\omega$ for various values of $\mu$. We use $\omega_0 = 1$, $\omega_p = .1\omega_0$, $\omega_L = .01\omega_0$ and $\gamma = .01\omega_0$.

1b. The imaginary part of the refractive index $n = k/\omega$ as determined from Eqn. (88) plotted against $\omega$ for various values of $\mu$. We use $\omega_0 = 1$, $\omega_p = .1\omega_0$, $\omega_L = .01\omega_0$ and $\gamma = .01\omega_0$. 
2a. The real part of the refractive index $n = k/\omega$ as determined from Eqn. (94) plotted against $\omega$ for various values of $\tilde{\mu}$. We use $\omega_0 = 1$, $\omega_p = .1\omega_0$, $\omega_L = .01\omega_0$ and $\gamma = 0$.

2b. The imaginary part of the refractive index $n = k/\omega$ as determined from Eqn. (94) plotted against $\omega$ for various values of $\tilde{\mu}$. We use $\omega_0 = 1$, $\omega_p = .1\omega_0$, $\omega_L = .01\omega_0$, $\omega_0 = 1$, $\omega_p = .1\omega_0$, $\omega_L = .01\omega_0$ and $\gamma = 0$.

3a. The real part of the refractive index $n = k/\omega$ as determined from Eqn. (106) plotted against $\omega$ for various values of $\kappa$. We use $\omega_0 = 1$, $\omega_p = .1\omega_0$, $\omega_L = .01\omega_0$ and $\gamma = .01\omega_0$.

3b. The imaginary part of the refractive index $n = k/\omega$ as determined from Eqn. (106) plotted against $\omega$ for various values of $\kappa$. We use $\omega_0 = 1$, $\omega_p = .1\omega_0$, $\omega_L = .01\omega_0$ and $\gamma = .01\omega_0$.

4a. The real part of the refractive index $n = k/\omega$ as determined from Eqn. (121) plotted against $\omega$ for various values of $\tilde{\kappa}$. We use $\omega_0 = 1$, $\omega_p = .1\omega_0$, $\omega_L = .01\omega_0$ and $\gamma = 0$.

4b. The imaginary part of the refractive index $n = k/\omega$ as determined from Eqn. (121) plotted against $\omega$ for various values of $\tilde{\kappa}$. We use $\omega_0 = 1$, $\omega_p = .1\omega_0$, $\omega_L = .01\omega_0$ and $\gamma = .0$. 

34