Maximum Forces in Modified Gravity Theories

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Abstract

We investigate the existence and nature of classical maximum force constraints in some gravity theories other than Einstein’s. We calculate the maximum force bounds in several of these gravity theories. We are interested in mass-independent maximum force bounds between black holes with touching horizons and find that this mass-independent feature of general relativity in three dimensions is present in Moffat’s gravity theory, Brans Dicke theory, and the pure Lovelock theory in which only one term in the lagrangian sum for the general Lovelock lagrangian is retained but not the sum over lower-order terms. Only pure Lovelock theory has this property in higher dimensions when the spatial dimension is equal to three times the Lovelock order contributing the single lagrangian to define the Lovelock lagrangian. We determine whether the maximum force is bigger or smaller than in general relativity. The absence of mass dependence in the maximum force relations may have relevance for the formation of naked singularities.

1 Introduction

It has been proposed and demonstrated in a wide range of situations, \cite{1,2} that in general relativity (GR) there should be a maximum value to any physically attainable force, or tension, given by

\[ F_{\text{max}} = \frac{c^4}{4G} , \]  

where \( c \) is the velocity of light and \( G \) is the Newtonian gravitational constant. This motivates closely related conjecture that there is a maximum power defined by

\[ P_{\text{max}} = cF_{\text{max}} = \frac{c^5}{4G} , \] 

the so-called Dyson Luminosity \cite{3}, or some multiple of it to account for geometrical factors \( O(1) \). This limits maximum possible luminosity in gravitational waves, or indeed other forms of radiation that an isolated system may emit, \cite{4,5}. Schiller has come to the same conclusion and proposed a stronger thesis: that the existence of a
maximum force implies general relativity, just as a maximum velocity characterises special relativity. This claim is much less clear since it requires in effect a proof of cosmic censorship. It is also necessary to choose quite subtle energy conditions in order to avoid the formation of sudden singularities \[6, 7, 8\], where unbounded pressure forces (or its time-derivatives) will occur. The origin of the maximum force and luminosity bounds lies in the fact that the Planck units of these quantities do not contain Planck’s constant: they are entirely classical and exit in the presence of a cosmological constant \[2\]. In \(N\)-dimensional space, the Planck unit of force in powers of the fundamental constants \(G\), \(c\), and \(h\) is

\[
F_{pl} = \frac{G^2}{(1-N)(5+N)(N-1)h^{(3-N)/(1-N)}}.
\]

and we see the disappearance of Planck’s constant of action, \(h\), when and only when \(N = 3\). A further example, the magnetic moment to angular momentum, has also been identified \[9\]. This signals something fundamental about these non-quantum natural units that is unique to three-dimensional space \[10\]. Note that this bound on forces does not exist in Newtonian gravity \[11\], where point masses can approach arbitrarily closely and the inverse-square gravitational force can become arbitrarily large. It is the formation of an event horizon around these mass points in general relativity that gives rise to the maximum force: it corresponds to the inverse square law force between two Schwarzschild black holes of the same mass touching at their horizons. If their masses are unequal an inequality ensures bounded by the maximum force \[10\]. The force bound between two black holes in higher-dimensional extensions of GR is only independent of the masses is only mass-independent in three-dimensional spaces. We will investigate whether this feature is shared in Lovelock’s extensions of GR in higher dimensions.

In this paper we will extend these analyses of the existence of a maximum force to some other gravity theories where static spherically symmetric solutions are available to do an analogous exact calculation of the sort performed in general GR. We identify theories that give the same mass-independent maximum force as GR. We set the speed of light, \(c\), equal to unity unless we specify otherwise.

## 2 Gravity Theories Beyond Einstein

In this section we will study the existence of maximum forces and compare them with the bound derived for general relativity. In general, we will evaluate the magnitude of the Newtonian inverse-square law force between two vacuum static spherically symmetric black holes that touch horizons.

### 2.1 Moffat’s theory

Moffatt’s extension of GR \[12, 13\] has a black hole solution with static spherically symmetric metric:

\[
ds^2 = h(r)dt^2 - \frac{dr^2}{h(r)} - r^2d\Omega^2,
\]
where

$$h(r) = 1 - \frac{2(1 + \alpha)G_NM}{r} + \frac{\alpha(1 + \alpha)M^2G_N^2}{r^2},$$ \hspace{1cm} (5)

where

$$G = G_N(1 + \alpha),$$ \hspace{1cm} (6)

with $G$ the effective gravitation constant, and $G_N$ the Newtonian constant (the speed of light is unity). This solution approaches the Schwarzschild metric of GR. Its event horizons are at

$$R_0^\pm = G_NM \left[ 1 + \alpha \pm \sqrt{1 + \alpha} \right],$$ \hspace{1cm} (7)

and we take the positive sign to obtain the GR result as $\alpha \to 0$.

The inverse-square force between two equal-mass black holes (M) touching at this radius for each is

$$F_{moff} = \frac{(1 + \alpha)G_NM^2}{(R_g^2)^2} = \frac{(1 + \alpha)}{G_N(1 + \alpha + \sqrt{1 + \alpha})^2},$$ \hspace{1cm} (8)

On restoring the speed of light this reduces to the GR result when $\alpha = 0$:

$$F_{GR} = \frac{\alpha^4}{4G_N}.$$ \hspace{1cm} (9)

However, when $\alpha > 0$, it can be less than the GR bound. For example, at large $\alpha$,

$$F_{moff} \to \frac{\alpha^4}{\alpha G_N} < F_{GR}.$$ \hspace{1cm} (10)

For small $\alpha > 0$, we have

$$F_{moff} \to F_{GR} \left( 1 - \frac{\alpha}{2} + O(\alpha^2) \right) < F_{GR}.$$ \hspace{1cm} (11)

In general, the maximum force in eq. (9) arises when $\alpha = 0$ and $F_{moff} < F_{GR}$ for all $\alpha > 0$.

### 2.2 Higher-order, power-law gravity

If we consider a power-law generalisation of the Einstein-Hilbert lagrangian to

$$L_g = R^{1+\delta},$$ \hspace{1cm} (12)

then the static spherically symmetric vacuum solution is \[^{[?]}\]

\[^{[?]}\]The bound is stronger for unequal masses: to see this, use the inequality $(\sqrt{M_1} - \sqrt{M_2})^2 = M_1 + M_2 - 2\sqrt{M_1M_2} \geq 0$. Hence, $M_1M_2 \leq \frac{1}{4}(M_1 + M_2)^2$ in the formula for $F_N(M_1, M_2)$ and we find for the maximum force $F_N(M_1, M_2) \leq F_N(M_1, M_1)$ always.\]
\[ ds^2 = A(r) dt^2 - \frac{dr^2}{B(r)} - r^2 d\Omega^2, \quad (13) \]

with
\begin{align*}
A(r) &= r^{\frac{2(1+2\delta)}{1-\delta}} + \frac{C}{r^{(1-4\delta)/(1-\delta)}}, \\
B(r) &= \frac{(1 - \delta)^2}{(1 - 2\delta + 4\delta^2)(1 - 2\delta - 2\delta^2)} \left[ 1 + \frac{C}{r^{\frac{1-2\delta+4\delta^2}{1-2\delta}}} \right],
\end{align*}

and \( C \) is constant. This reduces to the Schwarzschild solution of GR when \( \delta \to 0 \) so \( C \equiv 2GM \). Hence the event horizon is at
\[ R_g = (2GM)^{\frac{1+\delta}{2(1+2\delta)}}. \quad (16) \]

The Newtonian force between two black holes of mass \( M \) is then
\[ F_{\text{Clif}} = \frac{GM^2}{(2GM)^\frac{2(1+\delta)}{2(1-\delta)-6\delta^2}}, \quad (17) \]

and we see that \( F_{\text{Clif}} \to F_{\text{GR}} \) as \( \delta \to 0 \) where
\[ F_{\text{Clif}} \to \frac{GM^2}{(2GM)^{2(1+\delta)-6\delta^2}} = F_{\text{GR}} \frac{1}{(2GM)^{2\delta}} \text{ as } \delta \to 0. \quad (18) \]

When \( \delta > 0 \) we see a new effect: the force bound is no longer independent of the black hole masses and so does not only depend on the universal constants, \( c \) and \( G \), as in GR. From eq. (18) we also see that the maximum force is much larger than in GR as \( \delta \to 0 \). When \( \delta \to \infty \), we have
\[ F_{\text{Clif}} \to F_{\text{Clif}} = GM^2 (2GM)^{\frac{2}{2\delta}} = F_{\text{GR}}(2GM)^{2+1/2\delta} \to F_{\text{GR}}(2GM)^2. \quad (19) \]

Again, it exceeds the GR maximum force. Only when \( \delta = 0 \) is the maximum force mass-independent. Similar results are expected with more general polynomial lagrangians of the form \( L = f(R) \) since any vacuum solution of GR (like Schwarzschild) will also be a solution of the \( f(R) \) gravity theory if \( f(0) \neq 0 \). Hence the basic calculation of inter-black hole forces remains possible.

### 2.3 Brans-Dicke Theory

There are four varieties of static spherically symmetric vacuum metrics in Brans-Dicke (BD) gravity \[16\,17\,18\] and only one is physically realistic with positive Brans-Dicke parameter, \( \omega \), and describes a black hole or naked singularity:
\[ ds^2 = -dt^2(1 - \frac{B}{r})^{n+1} - dr^2(1 - \frac{B}{r})^{n-1} - r^2(1 - \frac{B}{r})^n d\Omega^2, \quad (20) \]
and the B-D scalar field, $\phi$, evolves as

$$\phi(r) = \phi_0 (1 - \frac{B}{r})^{-(m+n)/2}. \quad (21)$$

The BD coupling constant is related to $m$ and $n$ by

$$\omega = -2 \left[ \frac{m^2 + n^2 + mn + m - n}{(m+n)^2} \right]. \quad (22)$$

A black hole solution is allowed for $n < 1$. The radial transformation $r \to \rho$ to isotropic coordinates defined by

$$r = \rho \left( 1 + \frac{B}{r} \right)^2, \quad (23)$$

reveals the more familiar form of the solution. Putting

$$m = \frac{1}{\lambda} - 1, \quad (24)$$
$$n = 1 - \frac{C + 1}{\lambda}, \quad (25)$$

the scalar curvature invariant, $\mathcal{R}$, is

$$\mathcal{R} = \frac{4\omega B^2 C^2}{\lambda^2 \rho^4 (1 + \frac{B}{\rho})^8} \left( \frac{1 - B/\rho}{1 + B/\rho} \right)^{\frac{2(2\lambda - C - 1)}{\lambda}}. \quad (26)$$

When $n < -1$, i.e. for $2\lambda - C - 1 > 0$, $\mathcal{R} \to \infty$ as $\rho \to B$ and $\rho \to 0$ and there is a naked singularity. But, when $(C + 2 - \lambda)/\lambda > 0$ the curvature invariants are non-singular and $\rho = B$ is an event horizon. However, this results in a violation of the weak energy condition because it requires negative $\omega$, with

$$-2 < \omega < -2(1 + \frac{1}{\sqrt{3}}). \quad (27)$$

In the naked singularity case the forces between point particles can become arbitrarily large on approach to the singularity. In the black hole case the only allowed black holes are the Schwarzschild black holes of GR \[19\] and so the maximum force between them will still be $F_{GR}$.

### 3 Dimensional features

We have found earlier that there is a significant effect of spatial dimension on the maximum force bounds. Only in three space dimensions is the force bound independent of the masses of the gravitating objects. Moreover, in $N$ space dimensions the fundamental classical 'Planck' quantity depending in $G$, and $c$ (but not $\hbar$) has dimensions of mass $\times$ (acceleration)$^N$, and is only a force when $N = 3$, see ref. [10].
The maximum force between two touching $N$-dimensional black holes is

\[
F_N = \frac{(N - 2)8\pi GM^2}{(N - 1)\Omega_{N-1}(2GM/c^2)^{(N-3)/2}} \propto G^{N-1} M^{N-3} \frac{(N-2)^{(N-1)}}{N-2}. \tag{29}
\]

An interesting feature is the appearance of the mass of the attracting black holes in higher-dimensional general relativity when (and only when) $N = 3$. This may have some significance for the easier appearance of naked singularities in $N > 3$ dimensions. However, we are interested to see if this feature persists in interesting generalizations of GR that retain second-order field equations, like versions of Lovelock’s theory [20]. In the next section we will determine the existence of a maximum force in a preferred version of Lovelock’s theory to see if it is mass-independent when $N > 3$.

4 Lovelock gravity

In a $D$-dimensional spacetime, gravity can be described by an action functional involving arbitrary scalar functions of the metric and curvature, but not derivatives of curvature. In general, variation of such an arbitrary Lagrangian would lead to an equation having fourth-order derivatives of the metric. For them to be of second order, the gravitational lagrangian, $L$, is constrained to be of the following Lovelock form, [20]:

\[
L = \sum_n c_n L_n = c_n \frac{1}{2^{n}} \delta^{a_1 b_1 a_2 b_2 \ldots a_n b_n} \sigma_{a_1 d_1} \sigma_{a_2 d_2} \ldots \sigma_{a_n d_n} R^{c_1 d_1} R^{c_2 d_2} \ldots R^{c_n d_n}, \tag{30}
\]

where $\delta^{a_1 b_1 a_2 b_2 \ldots a_n b_n}$ is the completely antisymmetric determinant tensor. The case $n = 1$ is the familiar Einstein-Hilbert lagrangian, while $n = 2$ is the Gauss-Bonnet lagrangian, which is quadratic in curvature, and reads

\[
L_2 \equiv L_{GB} = (1/2)(R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2). \tag{31}
\]

Lovelock’s lagrangian is a sum over $n$, where each term is a homogeneous polynomial in curvature and has an associated dimensionful coupling constant, $c_n$. Moreover, the complete antisymmetry of the $\delta$ tensor demands $D \geq 2n$, or it would vanish identically. Even for $D = 2n$ the lagrangian reduces to a total derivative. Lovelock’s lagrangian, $L_n$, is therefore non-trivial only in dimension $D \geq 2n + 1$.

Lovelock theory is the most natural and quintessential higher-dimensional generalization of GR with the remarkable property that the field equations continue to remain second order in the metric tensor despite the action being a homogeneous polynomial in the Riemann tensor. GR is the linear order Lovelock theory ($n = 1$), whilst the Gauss-Bonnet term ($n = 2$) is quadratic, and then to any order, $n$, of the polynomial action. Each order comes with a new arbitrary dimensionful coupling constant, $c_n$.

A particular minimal case of interest is that of the pure Lovelock which has only the $n$th-order term in the lagrangian without a sum over lower orders in the action.
and the equations of motion. It distinguishes itself by the property that, as for GR in space dimension \( N = 2 \), the Lovelock analogue of the Riemann tensor \([21]\) is entirely given in terms of the corresponding Ricci tensor in all critical even space dimensions, \( N = 2n \). This property is in general termed 'kinematic'; for example, GR is kinematic in \( N = 2 \times 1 = 2 \); Gauss-Bonnet would be kinematic in \( N = 2 \times 2 = 4 \); that is, the pure Gauss-Bonnet Riemann tensor is entirely given in terms of the corresponding Ricci, and this is so for all \( N = 2n \).\([21],[22]\).

Pure Lovelock gravity is kinematic in all critical odd \( D = 2n + 1 \) dimensions because the \( n^{th} \) order Riemann tensor is entirely given in terms of the corresponding Ricci tensor, hence it has no non-trivial vacuum solution. Therefore, non-trivial vacuum solutions only exist in dimensions \( D \geq 2n + 2 \). Finally, variation of the lagrangian with respect to the metric, for pure Lovelock theories, leads to the following second-order equation,

\[
-\frac{1}{2^{n+1}}e^{a_1b_1a_2b_2...a_nb_n}R^{c_1d_1}R^{c_2d_2}...R^{c_nd_n} = 8\pi G T_{ab}. \tag{32}
\]

Since no derivatives of curvature appear, this equation is of second order in derivatives of the metric tensor. Although not directly evident, the second derivatives also appear linearly and the equations are therefore quasi-linear, thereby ensuring unique evolution.

Another property that singles out pure Lovelock is the existence of bound closed orbits around a static object \([23]\). Note, that in GR, bound orbits exist around a static object in Euclidean space only in three space dimensions. In view of these remarkable features, it has been argued that pure Lovelock is an attractive gravitational equation in higher dimensions \([24]\).

As with the Schwarzschild solution for GR, there exists an exact solution for a pure Lovelock black hole \([25]\), and it is given by Eq.(4), with (restoring the speed of light, \( c \), explicitly),

\[
h(r) = 1 - 2\Phi_n(r)/c^2 \tag{33}
\]

where the Newtonian potential term is

\[
\Phi_n(r) = \frac{G_n M}{r^\alpha}, \quad \alpha = \frac{(D - 2n - 1)}{n} \tag{34}
\]

Here, \( G_n \) is the gravitational constant for the \( n^{th} \) Lovelock order, \( D = 2n + 1 \) is the spacetime dimension (\( D \neq 2n + 1 \) for a non-trivial vacuum solution).

The analogue of the Newtonian inverse-square law now reads, for two equal mass black holes of mass \( M \):

\[
F_n = \alpha G_n M^2/r^{\alpha+1}. \tag{35}
\]

Then, \( \Phi_n = c^2/2 \) gives the black hole radius:

\[
R_g = (2G_n M/c^2)^{1/\alpha}. \tag{36}
\]

Substituting this in the above force equation, we obtain,
\[ F_n = \frac{\alpha GM^2}{(2GM/c^2)(1+\alpha)/\alpha}. \] (37)

Clearly, for \( \alpha = 1 \), which implies \( D = 3n + 1 \) or \( N = 3n \), the maximum force takes the same value, \( c^4/4G \), as for GR in \( N = 3 \) dimensions. This is because the pure Lovelock black hole potential goes as \( 1/r \) when \( D = 3n + 1 \) or \( N = 3n \). Note that this is an exact result, not an asymptotic one at large \( r \), and is the same as the Newtonian potential for \( N = 3 \) [26]. In particular, this also means that the maximum force value is mass independent in Lovelock gravity when \( N = 3n \). This feature only occurs in GR when \( N = 3 \) and may have important indications for the problem of naked singularity formation [10] because if the maximum force increases with \( M \) then it is possible for arbitrarily large forces to arise when sufficiently large black holes interact. In \( N = 3 \) then in GR, and in the pure Lovelock in general for \( N = 3n \) case we have analysed, this is not possible. This is one further remarkable property of pure Lovelock black holes and reveals the effectiveness of the maximum force relation in analysing variants and generalisations of GR. The recovery of the mass-independent maximum force bound relies on the appearance of a \( 1/r \) gravitational potential in the cases we have studied an is reminiscent of the conditions needed for the Newton-Ivory spherical property and for closed bound orbits in Newtonian gravity [11].

5 Conclusions

We have searched for the existence of a maximum force in several gravity theories that generalise GR in different ways. A maximum force \( c^4/4G \) appears to exist generally in GR, unlike in Newtonian gravity where forces between points can become arbitrarily large as they approach. The GR maximum force is independent of the attracting masses only on three space dimensions. Our investigations find the following:

a. In power-law lagrangians, the mass independence of the maximum force is achieved only for the case of the linear lagrangian \((\delta = 0)\), GR, and in all other cases, \( \delta > 0 \), the maximum force is larger than in GR.

b. In Brans-Dicke theory, black holes are the same as in GR and the maximum force is the same as in GR with mass independence in three space dimensions only.

c. In Moffat’s gravity theory there is a mass independent maximum force in three space dimensions. The maximum force is smaller than in GR and this only happens in this case.

d. In pure Lovelock gravity the maximum force is the same as in GR when the space dimension is \( 3n \), where \( n \) is the order of the one lagrangian contributing to the Lovelock lagrangian. Therefore, the mass independent maximum force bound exists in three dimensions for GR and Moffat’s generalisation of it, but in higher dimensions it only exists for pure Lovelock with \( N = 3n \).

e. Modified gravity theories generally introduced other arbitrary dimensionful parameters into the lagrangian if there are additions to the Einstein-Hilbert action. These can create new Planck-like quantities (see ref. [2] for the impact of introducing a cosmological constant). We have avoided this in our power-law lagrangian example and in the pure Lovelock lagrangian case, which has a single coupling constant.
The recovery of the mass-independent maximum force bound relies on the appearance of a $1/r$ gravitational potential in the cases we have studied and is reminiscent of the conditions needed for the Newton-Ivory spherical property and closed bound orbits in Newtonian gravity [11]. We have not included a cosmological constant in our study, as was done in ref. [2], where a very similar mass-independent maximum force bound, $c^4/9G$, was found in GR. Similar modifications of the bound are to be expected in theories of gravity where new dimensionful parameters are introduced into the gravity theory. This is a direction for further investigation in other gravity theories.

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