Dual formulation of the Lie algebra $S$-expansion procedure

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The expansion of a Lie algebra entails finding a new bigger algebra $\mathcal{G}$ through a series of well-defined steps from an original Lie algebra $\mathfrak{g}$. One incarnation of the method, the so-called $S$-expansion, involves the use of a finite Abelian semigroup $S$ to accomplish this task. In this paper we put forward a dual formulation of the $S$-expansion method, which is based on the dual picture of a Lie algebra given by the Maurer–Cartan forms. The dual version of the method is useful in finding a generalization to the case of a gauge free differential algebra, which, in turn, is relevant for physical applications in, e.g., supergravity. It also sheds new light on the puzzling relation between two Chern–Simons Lagrangians for gravity in 2+1 dimensions, namely, the Einstein–Hilbert Lagrangian and the one for the so-called “exotic gravity.” © 2009 American Institute of Physics. [DOI: 10.1063/1.3171923]

I. INTRODUCTION

Lie algebra expansions were introduced for the first time in Ref. 1, and the method was subsequently studied in general in Refs. 2–4. The idea is to perform a rescaling by a parameter $\lambda$ of some of the group coordinates $g^i, i=1, \ldots, \dim \mathfrak{g}$. Consequently, the Maurer–Cartan (MC) one-form $\omega^i(g, \lambda)$ of $\mathfrak{g}$ are expanded as a power series in $\lambda$. Inserting these expansions back in the original MC equations for $\mathfrak{g}$, one obtains the MC equations of a new finite-dimensional expanded Lie algebra.

An alternative expansion procedure to the method of power series expansion is the $S$-expansion method.5,6 The $S$-expansion method allows us also to obtain new Lie algebras starting from an original one by choosing an Abelian semigroup $S$ and applying the general Theorems 4.2 and 6.1 from Ref. 5, which give us, following the terminology introduced in Ref. 5, “resonant subalgebras” and what has been dubbed as “reduced algebras.”

In spite of the examples given in Ref. 5, the relation between both procedures has remained as an interesting open problem basically because (i) the $S$-expansion is defined as the action of a semigroup $S$ on the generators $T_A$ of the algebra and the power series expansion is carried out on the MC forms of the original algebra, and (ii) the $S$-expansion is defined on the algebra $\mathfrak{g}$ without referring to the group manifold, whereas the power series expansion is based on a rescaling of the group coordinates.

It is the purpose of this paper to study the $S$-expansion procedure in the context of the group manifold and then to find the dual formulation of such $S$-expansion procedure.

This article is organized as follows. In Sec. II we review the main aspects of the $S$-expansion procedure. In Sec. III we shall construct the dual formulation of the Lie algebra $S$-expansion procedure. The expansion of gauge free differential algebras (FDAs) is considered in Sec. IV. In

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Sec. V we obtain the (2+1)-dimensional Chern–Simons (CS) AdS gravity from the so-called “exotic gravity” as an example of the application of the dual $S$-expansion method. Section VI concludes the work with an outlook to further applications in gravity and supergravity.

II. S-EXPANSION OF THE LIE ALGEBRAS

In this section we shall review the main aspects of the $S$-expansion procedure introduced in Ref. 5. The $S$-expansion method is based on combining the structure constants of a Lie algebra $g$ with the inner multiplication law of a semigroup $S$ to define the Lie bracket of a new $S$-expanded algebra $\mathcal{G}=S\times g$.

Let $S=\{\lambda_{\alpha}, \alpha=1, \ldots, N\}$ be a finite Abelian semigroup with “two-selector” $K_{\alpha\beta}^\gamma$ defined by

$$
K_{\alpha\beta}^\gamma = \begin{cases} 
1 & \text{when } \lambda_{\alpha}\lambda_{\beta} = \lambda_{\gamma} \\
0 & \text{otherwise}, 
\end{cases}
$$

and let $g$ be a Lie algebra with structure constants $C_{AB}^C$.

$$
[T_A, T_B] = C_{AB}^C T_C. 
$$

Then it may be shown\(^5\) that the product $\mathcal{G} = S \times g$ corresponds to the Lie algebra given by

$$
[T_{(A,\alpha)}, T_{(B,\beta)}] = K_{\alpha\beta}^\gamma C_{AB}^C T_{(C,\gamma)}. 
$$

**Theorem 1:** The product $[\cdot, \cdot]$ defined in Eq. (3) is also a Lie product because it is linear, antisymmetric, and satisfies the Jacobi identity. This product defines a new Lie algebra characterized by the pair $(\mathcal{G}, [\cdot, \cdot])$, which is called $S$-expanded Lie algebra.

**Proof:** The proof is direct and may be found in Ref. 5. \(\square\)

III. DUAL FORMULATION OF THE S-EXPANSION PROCEDURE

Theorem 1 implies that for every Abelian semigroup $S$ and Lie algebra $g$, the product $\mathcal{G}=S \times g$ is also a Lie algebra, with a Lie bracket given by Eq. (3). This, in turn, means that it must be possible to look at this $S$-expanded Lie algebra $\mathcal{G}$ from the dual point of view of the MC forms.

**Theorem 2:** Let $S=\{\lambda_{\alpha}, \alpha=1, \ldots, N\}$ be a finite Abelian semigroup and let $\omega^A$ be the MC forms for a Lie algebra $g$. Then, the MC forms $\omega^{(A,\alpha)}$ associated with the $S$-expanded Lie algebra $\mathcal{G}=S \times g$ (cf. Theorem 1) are related to $\omega^A$ by

$$
\omega^{A,\alpha} = \lambda_{\alpha} \omega^{(A,\alpha)}. 
$$

By definition, these forms satisfy the MC equations

$$
d[\omega^{(C,\gamma)}] + \frac{1}{2} K_{\alpha\beta}^\gamma C_{AB}^C \omega^{(A,\alpha)} \omega^{(B,\beta)} = 0. 
$$

**Proof:** The simplest way to check the validity of Eq. (4) is by multiplying Eq. (5) by $\lambda_{\gamma}$ and using the defining relation for the two-selector $K_{\alpha\beta}^\gamma$, namely, $\lambda_{\alpha}\lambda_{\beta}=K_{\alpha\beta}^\gamma\lambda_{\gamma}$. We get

$$
\lambda_{\gamma} [d[\omega^{(C,\gamma)}] + \frac{1}{2} K_{\alpha\beta}^\gamma C_{AB}^C \omega^{(A,\alpha)} \omega^{(B,\beta)}] = 0, 
$$

and

$$
d[\lambda_{\gamma} \omega^{(C,\gamma)}] + \frac{1}{2} C_{AB}^C [\lambda_{\alpha} \omega^{(A,\alpha)} \lambda_{\beta} \omega^{(B,\beta)}] = 0. 
$$

The required identification is obtained by matching Eq. (6) with the MC equations for $g$. This concludes the proof. \(\square\)

It is perhaps interesting to notice that the relation shown in Eq. (4) is analogous to the method of power series expansion developed in Ref. 2.
A. 0₅-reduction in S-expanded Lie algebras

The concept of reduction in Lie algebras and, in particular, 0₅-reduction, was introduced in Ref. 5. As implied by the name, it involves the extraction of a smaller algebra from a given Lie algebra g when certain conditions are met. In spite of the superficial similarity of the concepts, a reduced algebra is not, in general, a subalgebra of g. ⁵

In this section we present the dual formulation for the 0₅-reduction in an S-expanded Lie algebra ₅, formulated in the language of the MC forms.

Let ₅=₁, ..., N∪{₅₅+₁=0₅} be an Abelian semigroup with zero. The expanded MC forms ₆ ₅ are then given by

\[ ₖ = λₖ ₆ + 0₅ ₖₖ, \]

where ₖₖ = ₆ₖₖ₅+₁. We shall show that the MC forms ₆ ₖ are themselves (without including ₖₖ) are those of a Lie algebra—the 0₅-reduced algebra ₅.

It can be seen that ₖₖ ₖₖ ₖₖ are the structure constants for the 0₅-reduced S-expanded algebra ₅, which are generated by ₖₖ ₖₖ ₖₖ.

Theorem 3 below gives the equivalent statement in terms of MC forms.

**Theorem 3:** Let ₅=₁, ..., N∪{₅₅+₁=0₅} be an Abelian semigroup with zero and let \{₆ ₖ, ₖ=₁, ..., N\} \cup \{₆ ₖ₅ₕ+₁=₅ \} be the MC forms for the S-expanded algebra ₅=₅ × ₅ by the semigroup ₅. Then, \{₆ ₖ, ₖ=₁, ..., N\} are the MC forms for the 0₅-reduced S-expanded algebra ₅.

**Proof:** The MC forms for the S-expanded algebra ₅ satisfy the MC equations [cf. Eq. (5)],

\[ dₖ ₖ ₖ + \frac{1}{ₖ}ₖ ₖ ₖ ₖ ₖ ₖ ₖ ₖ + ₖ₂ ₖ ₖ ₖ ₖ ₖ ₖ ₖ ₖ = 0. \]

The \(γ=k\) component reads as

\[ dₖ ₖ ₖ + \frac{1}{ₖ}ₖ ₖ ₖ ₖ ₖ ₖ ₖ ₖ ₖ = 0. \]

Summing over ₖ and ₖ and noting that ₖ ₖ ₖ ₖ ₖ ₖ ₖ ₖ ₖ = ₀, we get

\[ dₖ ₖ ₖ + \frac{1}{ₖ}ₖ ₖ ₖ ₖ ₖ ₖ ₖ ₖ ₖ = 0, \]

which shows that \{₆ ₖ, ₖ=₁, ..., N\} are the MC forms for a Lie algebra whose structure constants are ₖ ₖ ₖ, as we set out to prove.

IV. GENERALIZATION TO THE CASE OF A GAUGE FDA

A FDA is a set of differential forms closed under the action of the exterior product ∧ (the wedge symbol ∧ is omitted throughout this work, although wedge product is always assumed between forms) and the exterior derivative d₈. ₇,₈

Every Lie algebra ₅ leads to a “gauge” FDA through its MC equations. To obtain a gauge FDA, we replace the MC forms ₅ by the gauge field one-form ₅ and introduce their corresponding curvatures ₅ by

\[ ₅ = d₅ + \frac{1}{₅}₅ ₅ ₅ ₅ ₅. \]

This equation expressing the exterior derivative of the gauge connection ₅ in terms of the curvature ₅ must be supplemented with the Bianchi identity, which expresses the exterior derivative of ₅ in terms of ₅ and ₅ itself,

\[ d₅ + ₅ ₅ ₅ ₅ ₅ = 0. \]

Equations (13) and (14) comprise the gauge FDA on which we shall perform the S-expansion.
Theorem 4: Let $S=\{\lambda_\alpha, \alpha=1,\ldots,N\}$ be a finite Abelian semigroup and let $\mathfrak{g}$ be a Lie algebra. Then, the “expanded” connection $A^{(A,\alpha)}$ and curvature $F^{(A,\alpha)}$ defined by [see Eq. (4)]

$$A^A = \lambda_\alpha A^{(A,\alpha)},$$

$$F^A = \lambda_\alpha F^{(A,\alpha)},$$

form a gauge FDA for the $S$-expanded Lie algebra $\mathfrak{G}=S \times \mathfrak{g}$, with the defining equations

$$dA^{(A,\alpha)} + \frac{1}{2} K^A_{\beta \gamma} C^{(A,\alpha)}_{BC} A^{(B,C,\gamma)} = F^{(A,\alpha)},$$

$$dF^{(A,\alpha)} + K^A_{\beta \gamma} C^{(A,\alpha)}_{BC} A^{(B,C,\gamma)} = 0.$$

Proof: We must show that the expanded connection and curvature defined in Eqs. (15) and (16) satisfy equations analogous to (13) and (14) with the structure constants of $\mathfrak{g}$, $C^{(A,\alpha)}_{BC}$, replaced by the structure constants of $\mathfrak{G}$, $K^A_{\beta \gamma} C^{(A,\alpha)}_{BC}$ [cf. Eqs. (17) and (18)]. Let us start with Eq. (13): substituting Eqs. (15) and (16) into Eq. (13), we get

$$\lambda_\alpha^A A^{(A,\alpha)} = d[\lambda_\alpha^A A^{(A,\alpha)}] + \frac{1}{2} C^{(A,\alpha)}_{BC} [\lambda_\alpha^A A^{(B,\beta)}] [\lambda_\alpha^A A^{(C,\gamma)}] = \lambda_\alpha^A [dA^{(A,\alpha)} + \frac{1}{2} K^A_{\beta \gamma} C^{(A,\alpha)}_{BC} A^{(B,C,\gamma)}].$$

Equating coefficients on both sides, we readily recover Eq. (13). Replacing now Eqs. (15) and (16) into Eq. (14), we find

$$d[\lambda_\alpha^A F^{(A,\alpha)}] + C^{(A,\alpha)}_{BC} [\lambda_\alpha^A A^{(B,\beta)}] [\lambda_\alpha^A F^{(C,\gamma)}] = 0,$$

$$\lambda_\alpha^A [dF^{(A,\alpha)} + K^A_{\beta \gamma} C^{(A,\alpha)}_{BC} A^{(B,C,\gamma)}] = 0,$$

which finish the proof.

A. $0_\alpha$-reduction in FDAs

Let $S=\{\lambda_i, i=1,\ldots,N\} \cup \{\lambda_{N+1}=0\}$ be an Abelian semigroup with zero. The expanded connection and curvature $A^{(A,\alpha)}$, $F^{(A,\alpha)}$ are then given by

$$A^A = \lambda A^{(A,i)} + 0_{\alpha} \tilde{A}^A,$$

$$F^A = \lambda F^{(A,i)} + 0_{\alpha} \tilde{F}^A,$$

where $\tilde{A}^A = A^{(A,N+1)}$, $\tilde{F}^A = F^{(A,N+1)}$. We shall show that the expanded forms $A^{(A,i)}$, $F^{(A,i)}$ by themselves (without including either $\tilde{A}^A$ nor $\tilde{F}^A$) are those of a gauge FDA—a $0_\alpha$-reduced gauge FDA.  

Theorem 5: Let $S=\{\lambda_i, i=1,\ldots,N\} \cup \{\lambda_{N+1}=0\}$ be an Abelian semigroup with zero and let $\{A^{(A,i)}, i=1,\ldots,N\} \cup \{A^{(A,N+1)}=\tilde{A}^A\}$, $\{F^{(A,i)}, i=1,\ldots,N\} \cup \{F^{(A,N+1)}=\tilde{F}^A\}$ be the connection and curvature for the $S$-expanded gauge FDA obtained by expanding $(A^A, F^A)$ by the semigroup $S$. Then, $\{A^{(A,i)}, i=1,\ldots,N\}$, $\{F^{(A,i)}, i=1,\ldots,N\}$ are the connection and curvature for a new, $0_\alpha$-reduced gauge FDA, with the defining equations

$$dA^{(A,i)} + \frac{1}{2} K^A_{\beta \gamma} C^{(A,i)}_{BC} A^{(B,C,\gamma)} = F^{(A,i)},$$

$$dF^{(A,i)} + K^A_{\beta \gamma} C^{(A,i)}_{BC} A^{(B,C,\gamma)} = 0.$$
The interested reader will find that the details of the calculation are easily filled in. This concludes the proof.

V. APPLICATION TO CHERN–SIMONS THEORY OF GRAVITY

Our main interest in developing methods for expanding Lie and gauge FDAs lies on their expected applications in gravity and supergravity.

A. Three-dimensional gravity revisited

As already noted by Witten in his classic 1988 paper, the Einstein field equations for general relativity in three-dimensional spacetime can be derived from the “exotic” Lagrangian,

$$L_{ex} = \omega^a d\omega^b + \frac{2}{3} \omega^a \omega^b \omega^c,$$

which is a function of the spin connection $\omega^{ab}$ only [the vielbein $e^a$ is conspicuously absent from (25)].

The Lagrangian (25) can be written as a CS form for the $(2+1)$-dimensional Lorentz algebra $\mathfrak{L}$,

$$[J_{ab}, J_{cd}] = \eta_{cb} J_{ad} - \eta_{da} J_{cb} - \eta_{ab} J_{cd}.$$  

The $\mathfrak{L}$-valued one-form gauge connection $A$ and two-form curvature $F$ read as

$$A = \frac{1}{2} \omega^{ab} J_{ab},$$

$$F = \frac{1}{2} R^{ab} J_{ab},$$

where the Lorentz curvature $R^{ab}$ is defined as usual, $R^{ab} = d\omega^{ab} + \omega^c \omega^{cb}$. A straightforward calculation shows that

$$L_{ex} = 2 \langle AdA + \frac{2}{3} A^3 \rangle,$$

where $\langle \cdots \rangle$ stands for the following rank two, symmetric invariant tensor:

$$\langle J_{ab}, J_{cd} \rangle = \eta_{ca} \eta_{bd} - \eta_{bc} \eta_{ad}.$$  

By contrast, the usual Einstein–Hilbert (EH) Lagrangian (with a negative cosmological constant $\Lambda = -3/\ell^2$, where $\ell$ is a length),

$$L_{EH} = \frac{1}{\ell} e^{abc} \left( R^{ab} + \frac{1}{3} \ell^2 e^{a} e^{b} \right) e^c,$$

requires that we consider the full AdS algebra $so(2,2)$, with the one-form gauge connection

$$A = \frac{1}{\ell} e^a P_a + \frac{1}{2} \omega^{ab} J_{ab}.$$  

In this section we cast the relationship between (25) and the usual EH Lagrangian (with a cosmological constant) in terms of an $S$-expansion of the relevant FDA.

In order to ease the comparison, it proves useful to define (here $\varepsilon$ is the Levi-Civita symbol, with $\varepsilon^{012} = \varepsilon_{012} = +1$)

$$J^a = \frac{1}{2} \varepsilon^{abc} J_{bc},$$

$$\omega_a = \frac{1}{2} \varepsilon_{abc} \omega^{bc},$$

so that $A$ and $F$ now take the forms
\[ A = \omega_a F^a, \quad (35) \]

\[ F = F_a F^a, \quad (36) \]

with

\[ F_a = \frac{1}{2} \epsilon_{abc} R^{bc} = d\omega_a - \frac{1}{2} \eta_{abc} e^{abcd} \omega_b \omega_d. \quad (37) \]

From this, \( F_a \) we can construct the following invariant polynomial:

\[ P = F^a F_a = dL_{CS}, \quad (38) \]

which shows that \( (25) \) is a CS form, quasi-invariant under Lorentz transformations.

**B. The \( Z_2 \)-expansion**

Let us consider the (semi)group \( Z_2 = \{ \lambda_0, \lambda_1 \} \), provided with the following products:

\[ \lambda_0 \lambda_0 = \lambda_0, \quad (39) \]

\[ \lambda_0 \lambda_1 = \lambda_1, \quad (40) \]

\[ \lambda_1 \lambda_0 = \lambda_1, \quad (41) \]

\[ \lambda_1 \lambda_1 = \lambda_0. \quad (42) \]

Following the ideas from Sec. IV, we define the \( S \)-expanded spin connection and curvature by

\[ \omega_a = \lambda_0 \omega_a^{(0)} + \lambda_1 \omega_a^{(1)}, \quad (43) \]

\[ F_a = \lambda_0 F_a^{(0)} + \lambda_1 F_a^{(1)}. \quad (44) \]

Inserting (43) and (44) in the definition of curvature (37),

\[ \lambda_0 F_a^{(0)} + \lambda_1 F_a^{(1)} = d(\lambda_0 \omega_a^{(0)} + \lambda_1 \omega_a^{(1)}) - \frac{1}{2} \eta_{abc} e^{bcd} (\lambda_0 \omega_b^{(0)} + \lambda_1 \omega_b^{(1)}) (\lambda_0 \omega_c^{(0)} + \lambda_1 \omega_c^{(1)}) (\lambda_0 \omega_d^{(0)} + \lambda_1 \omega_d^{(1)}), \quad (45) \]

and using the multiplication law (39)–(42), we easily read off the expressions

\[ F_a^{(0)} = d\omega_a^{(0)} - \frac{1}{2} \eta_{abc} e^{bcd} (\omega_c^{(0)} \omega_d^{(0)} + \omega_c^{(1)} \omega_d^{(1)}), \quad (46) \]

\[ F_a^{(1)} = d\omega_a^{(1)} - \eta_{abc} e^{bcd} \omega_c^{(0)} \omega_d^{(1)}. \quad (47) \]

Now we introduce the notations

\[ \omega_a^{(0)} = \omega_a, \quad (48) \]

\[ \omega_a^{(1)} = \frac{1}{\ell} e_a, \quad (49) \]

and find that we can write
\[ F_{a}^{(0)} = \frac{1}{2} e_{abc} \left( R^{bc} + \frac{1}{\ell^2} e^b e^c \right), \]

\[ F_{a}^{(1)} = \frac{1}{\ell} T_a, \]

provided we identify \( e^a \) with the vielbein, \( R^{ab} \) with the Lorentz curvature defined above and \( T^a \) with the torsion, \( T^a = de^a + a_j^b e^j \). These last equations correspond to the curvatures of the Lie algebra \( so(2,2) \). This is a consequence of the fact that the \( so(2,2) \) algebra can be regarded as a \( Z_2 \)-expansion of \( so(2,1) \).

**C. Expansion of the action**

One of the advantages of the S-expansion method is that it provides us with an invariant tensor for the S-expanded algebra \( S \times g \) in terms of an invariant tensor for \( g \).

As shown in Ref. 5, a rank 2 symmetric invariant tensor for an \( S \)-expanded algebra takes the form

\[ \langle T_{(A,a)} T_{(B,b)} \rangle_a = \sigma_7 K^7_{\text{abc}} (T_A T_B)_g, \]

where \( \sigma_7 \) are arbitrary constants.

Using this result in the \( Z_2 \)-expansion of the Lorentz algebra \( \mathcal{L} \), we get

\[ \langle J_{\text{ab}} J_{\text{cd}} \rangle = \sigma_0 \langle J_{\text{ab}} J_{\text{cd}} \rangle_\mathcal{L}, \]

\[ \langle J_{\text{ab}} P_c \rangle = \sigma_1 \langle J_{\text{ab}} J_c \rangle_\mathcal{L}, \]

\[ \langle P_a P_b \rangle = \sigma_0 \langle J_a J_b \rangle_\mathcal{L}, \]

where \( J_a = -(1/2) \varepsilon_{\text{abc}} J^{\text{bc}} \). Inserting (30) explicitly, we find

\[ \langle J_{\text{ab}} J_{\text{cd}} \rangle = \sigma_0 (\eta_{\text{ad}} \eta_{\text{bc}} - \eta_{\text{ac}} \eta_{\text{bd}}), \]

\[ \langle J_{\text{ab}} P_c \rangle = \sigma_1 \varepsilon_{\text{abc}}, \]

\[ \langle P_a P_b \rangle = \sigma_0 \eta_{\text{ab}}. \]

If we now use the invariant tensors (56)–(58) in the general expression for a CS Lagrangian, we find

\[ L_{CS} = -\frac{1}{2} \sigma_0 \left( \omega_a \omega_b \omega_a - 2 \varepsilon_{\text{abc}} e_a T^a + \frac{1}{\ell^2} \varepsilon_{\text{abc}} e^a \right) + \frac{\sigma_1}{\ell} \varepsilon_{\text{abc}} \left( R^{ab} + \frac{1}{3} \varepsilon_{\text{abc}} e^a \right) e^c + d \left( \frac{1}{8} \omega_a \omega_a \right). \]

There are two independent terms here: the one proportional to \( \sigma_0 \) provides the “exotic” Lagrangian, while the one proportional to \( \sigma_1 \) contains the EH Lagrangian with a cosmological constant. Note also the presence of the extra torsional term, which ensures the AdS invariance of the enlarged exotic Lagrangian.

**VI. COMMENTS AND POSSIBLE DEVELOPMENTS**

In this work we have found a relation between two procedures for the expansion of Lie algebras, namely, the so-called “power series expansion method” of Ref. 2 and the “S-expansion procedure” of Ref. 5. Actually a dual formulation of the S-expansion method was found based on the MC forms. It is also shown that the generalization of this procedure permits the construction
of $S$-expanded gauge FDAs. Finally, as an example of the application of the dual $S$-expansion method, we obtain the $(2+1)$-dimensional CS AdS gravity from the so-called exotic gravity.

A comparison between the results of Sec. II from Ref. 10 and those obtained in Sec. V in the present work is now in order.

In Ref. 10 it was found that three-dimensional GR, without a cosmological constant, is equivalent to a gauge theory for the ISO$(2,1)$ group with a Lagrangian consisting solely of the CS three-form, which is constructed using the symmetric invariant tensor given in Eq. (2.8) of Ref. 10. The construction was generalized to obtain, with the same invariant tensor, a CS three-form that leads precisely to the EH Lagrangian with a cosmological constant which is given in Eq. (2.22) of Ref. 10. In Sec. 2.3 of this reference it was pointed out that, in addition to the invariant tensor (2.8), there is a second invariant tensor on the Lie algebra [cf. Eq. (2.26) of Ref. 10]. Using this invariant tensor, a new, “exotic” CS Lagrangian with a cosmological constant was constructed [cf. Eq. (2.27) from Ref. 10], and added, with an arbitrary coefficient, to the original EH Lagrangian (2.22).

In our work we have shown that an application of the dual $S$-expansion procedure to the “exotic” Lagrangian (25) (without a cosmological constant term—actually without a vielbein field!) produces the CS Lagrangian (59), which is equivalent to the general Lagrangian of Ref. 10. More explicitly we can say that, by starting from the Lorentz algebra in 2+1 dimensions and applying an $S$-expansion with $S=Z_2$, we find (i) the AdS algebra in 2+1 dimensions and (ii) an AdS-invariant, symmetric invariant tensor built from one for the Lorentz algebra. The presence of the arbitrary constants $\sigma_0$ and $\sigma_1$ signals the existence of actually two independent invariant tensors for the AdS algebra, one built from $\eta_{ab}$ and the other from $\epsilon_{abc}$. The comparison of the Lagrangian (59) with the Lagrangian given in Eq. (4.5) of Ref. 10 shows that the $\sigma_0$ and $\sigma_1$ constants, which are the arbitrary constants that appear in the invariant tensor of the $S$-expanded algebra, could have a physical interpretation in the context of three-dimensional quantum gravity. A direct comparison between Eq. (59) and Eq. (4.5) of Ref. 10 permits the identification of $\sigma_0$ with $-2/\hbar$ and $\sigma_1$ with $i\ell k/8\pi$, where $k$ is an integer. Paraphrasing Witten (see Ref. 10, p. 77), we can say that the understanding of quantum gravity on a general three-manifold would mean understanding how to compute the partition function $Z(M)$ as a function of the variables $\sigma_0$ and $\sigma_1$ that appear in the Lagrangian.

Several aspects deserve consideration and many possible developments can be anticipated. A still unsolved problem is to find a relation between five-dimensional CS gravity and general relativity (work in progress).

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