An instanton-like excitation of a discrete time crystal

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Spontaneous symmetry breaking and elementary excitation are two of the pillars of condensed matter physics that are closely related to each other. The symmetry and its spontaneous breaking not only control the dynamics and spectrum of elementary excitations, but also determine their underlying structures. In this paper, we study the excitation properties of a non-equilibrium quantum matter: a discrete time crystal phase that spontaneously breaks the temporal translational symmetry. It is shown that such an intriguing symmetry breaking allows an instanton-like excitation that represents a tunneling between two “degenerate” time crystal phases. Furthermore, we also observe a dynamical transition point at which the instanton “size” diverges, a reminiscence of the critical slowing down phenomenon in nonequilibrium statistical physics. A phenomenological theory has been proposed to understand the phase dynamics of the proposed system and the experimental realization and detection have also been discussed.

Usually, the ground state energy of an equilibrium system does not have much to do with its observable behavior. What’s physically important are the properties of low-lying excited states, which are likely to be excited owing to weak external fields or relatively low temperatures. For example, the thermal and elastic properties of a solid are determined by a few number of lattice wave excitations known as phonons [1]. Non-equilibrium quantum matter fundamentally differs from its equilibrium counterparts, and has received considerable interest in various fields ranging from ultracold atoms [2] to solid state physics [3] over the past decade. However, compared to equilibrium cases, much less is known about the “excitation” of non-equilibrium quantum matter, even its definition may be questionable. For example, “exciting” such a state does not necessarily mean increasing its energy. Even if the excitation can be well-defined, its properties are far from clear. For example, one may wonder whether they can be decoupled into superpositions of “elementary” excitations similar to equilibrium systems [1]. If so, how to characterize such elementary excitations and build up their relationship with fundamental properties of non-equilibrium quantum matter, e.g. the symmetries and their spontaneous breaking.

As a prototypical example of nonequilibrium quantum matter, the time crystal (TC) phase has allowed new possibilities for the spontaneous symmetry breaking (SSB) paradigm [4], and have attracted considerable interest in its different forms [5–17]. Such a state with a temporal translational symmetry breaking(TSB), despite being proven to be forbidden in thermodynamic equilibrium states [18–19], has been experimentally realized in intrinsically non-equilibrium settings with periodic driving [20–21]. Its physical observables develop persistent oscillations whose periods are an integer multiple of the Hamiltonian period, and thus they spontaneously break the discrete temporal translational symmetry. In equilibrium systems, the SSB is closely related to the elementary excitation. In particular, it does not only affect the dynamics and spectrum of an elementary excitation, but also determines its structure. For example, for a one-dimensional (1D) system, a spontaneous discrete spatial TSB allows certain soliton-like excitation: a topological defect that usually carries fractionalized quantum number [22]. A profound question is then to understand the relationship between the SSB and elementary excitation in non-equilibrium quantum matters like the time crystal, for example, what’s kind of elementary excitation permitted by the discrete temporal TSB?

In this paper, we address this issue by studying the dynamics of a periodically driven 1D interacting bosonic model that can manifest in a sub-harmonic response for physical quantities, which is an indication of a TC phase. To “excite” this TC phase, we impose a time-dependent perturbation, which transiently breaks the original time translational symmetry, and we then monitor the response of the physical observable. It is found that a slow perturbation (ramping) may induce an instanton-like excitation sandwiched between two “degenerate” TC phases, as shown in Fig.1. By tuning the ramping velocity, one can observe a transition point, at which the “size” (lifetime) of such an excitation diverges, a reminiscence of the critical slowing down phenomenon [23–24]. A phenomenological theory has been proposed to explain our numerical results. It is worth emphasizing that the system we studied is a closed system, which makes its excitations fundamentally differ from those diffusive Goldstone modes studied in dissipative TC phases [25].

Model and method: We consider a 1D hard-core bosonic model with an infinite long-range interaction. The Hamiltonian reads as follows:

$$H = -J \sum_i \langle \hat{b}_i \hat{b}_{i+1} \rangle + h.c - \frac{V(t)}{L} \sum_{ij} (-1)^{i-j} \hat{n}_i \hat{n}_j \quad (1)$$

where $J$ is the single-particle nearest-neighbor(NN) hopping amplitude, $\hat{n}_i = \hat{b}_i^\dagger \hat{b}_i$ is the particle number oper-
ator at site $i$. $V(t)$ is the strength of the all-to-all interaction, which is time-dependent but does not decay with distance. $L$ is the system size of the 1D lattice, and the prefactor $\frac{1}{t}$ in front of the interaction terms of Eq. (1) guarantees that the total interacting energy linearly scales with the system size. In a bipartite lattice (e.g., a 1D lattice as in our case), Eq. (1) indicates that the interaction between a pair of bosons are attractive (repulsive) if they are located in the same (different) sublattice. Such an infinitely long-range interaction has been realized in recent high-finesse cavity experiments \cite{26, 27} by coupling bosons to the cavity vacuum mode whose period is twice of that of the optical lattice. The total particle number $N$ is conserved in our system, and we focus on the case of half-filling ($N = L/2$) throughout this paper. In the equilibrium case ($V(t) = V_0$), the ground state of the 1D Ham. (1) is always a Mott-insulator with a charge-density-wave (CDW) order for arbitrary positive $V_0$. 

Owing to the all-to-all coupling of the interaction in Ham. (1), in the thermodynamical limit the mean-field method is not an approximation but an exact method. In particular, it provides an exact description of both the equilibrium properties and real-time dynamics of the system \cite{28, 29} (see also Supplementary Material (SM) \cite{31}). The mean-field method allows us to decouple the all-to-all interaction by introducing an auxiliary staggered field which is self-consistently determined during the time evolution, and the Ham. (1) can be expressed as:

$$\bar{H}(t) = -J \sum_i [b_i^\dagger b_{i+1} + h.c] + m(t)V(t) \sum_i (-1)^i \hat{n}_i \quad (2)$$

where $m(t) = \langle \Psi(t)|\frac{1}{V}|(-1)^i \hat{n}_i|\Psi(t)\rangle$ and $|\Psi(t)\rangle$ is the wavefunction of the system at time $t$. The time evolution under the Hamiltonian (2) can be solved exactly by performing the Jordan-Wigner transformation to transfer 1D hard-core bosons into spinless fermions, and the Ham. (2) transform into a non-interacting fermionic model.

Throughout this paper, we consider the periodic boundary condition (PBC), which allows us to perform the Fourier transformation, after which the fermionic Hamiltonian turns to

$$\bar{H}(t) = \sum_k \begin{bmatrix} c_k^\dagger & c_{k+\pi}^\dagger \end{bmatrix} \begin{bmatrix} \varepsilon_k & m(t) \\ m(t) & \varepsilon_{k+\pi} \end{bmatrix} \begin{bmatrix} c_k \\ c_{k+\pi} \end{bmatrix} \quad (3)$$

where the summation is over the momentum in the first Brillouin zone of Ham. (2) ($k \in [-\frac{\pi}{2}, \frac{\pi}{2}]$), and $c_k$ ($c_k^\dagger$) denotes the annihilation (creation) operator of the spinless fermion. $\varepsilon_k = -2J\cos k$, thus $\varepsilon_k = -\varepsilon_{k+\pi}$. Eq. (3) indicates that the dynamics of the system can be considered as a collective behavior of different $k$ modes, each of which is a two-level quantum system subjected to time-dependent field that was self-consistently determined as $m(t) = \frac{1}{\pi} \sum_k \langle \Psi_k(t)|c_k^\dagger c_{k+\pi}|\Psi_k(t)\rangle$.

*Discrete time crystal.* Despite the triviality of the ground state phase diagram, the system can exhibit rich dynamical behavior in the presence of a time-dependent $V(t)$. For instance, in a quantum quench protocol, one can start from a ground state of Ham. (1) with $V(t = 0) = V_i$, suddenly change it to a different value $V(t > 0) = V_f$ and let the system evolve under this new Hamiltonian. It has been shown that the long-time dynamics of this model can exhibit either persistent oscillations or thermalization depending on the choice of initial states \cite{31} (see SM \cite{31}). The dynamical behavior is even richer and more interesting when we introduce periodical driving into Ham. (1), e.g., $V(t > 0) = V_f + \delta \cos 2\pi t$ with $\delta$ the driving amplitude. We observe that \cite{33} depending on the different choices of the initial states and driving amplitude $\delta$, the long-time dynamics of $m(t)$ could exhibit a periodic oscillation with a frequency that is either identical to or independent of the driving frequency; the former can be considered as a synchronization phenomenon which has been observed in periodically-driven integrable systems \cite{34}. In addition, it can also exhibit quasi-periodic oscillations with a multi-period structure \cite{35}.

Most interesting dynamics can be observed in the intermediate driving regime (see Fig 2b), where $m(t)$ exhibits a persistent oscillation whose period is twice that of the external driving period: a signature of “discrete time crystal” \cite{33} that the discrete time translational symmetry in Ham. (1) $H(t) = H(t + T)$ has been spontaneously broken ($m(t) = m(t + 2T) \neq m(t + T)$) with $T = 1$ is the period of driving). This phenomena is rooted in the non-linearity of the self-consistent mean-field equation of motion. However, unlike the period doubling phenomena in the non-linear classical systems (e.g. a driven-dissipative pendulum \cite{34}) or the "dissipative time crystals" in open quantum many-body systems \cite{36}, our system is a closed system, therefore there is no entropy generation. Ham. (1) is free from disorder, hence it is the integrability \cite{33} rather than the many-body localization \cite{30} that prevents our model from being heated to an infinite temperature state.

![](image) FIG. 1: (Color online). (a) Schematic diagram of a period doubling dynamics in the presence of periodical driving and two degenerate TC phases; (b) Schematic diagram of the phase ramping protocol in our model and an instanton-like excitation induced by it.
The phase on top of the periodic driving: $V(t > 0) = V_f + \delta \cos 2\pi t$ with a period $T_0 = 1$. (b) Time crystal dynamics with a period 2 in the absence of ramping. (c) Long-time dynamics of $m(t)$ with slow and fast ramping, which correspond to two “degenerate” states. To realize such an object, we consider the excitation as a domain wall sandwiched between two “degenerate” TC phases. (d) Long-time dynamics of $m(t)$ with an intermediate ramping rate close to the dynamical transition point. The parameters are chosen as $V_i = 10J$, $V_f = 3J$, $\delta = 0.8J$ and $L = 5000$.

**Instanton-like excitations**: Despite the richness of the dynamics behavior, here we will study neither the global non-equilibrium phase diagram of our model, nor the mathematical origin behind this DTC phase. Instead, we will use this exactly solvable model as a starting point to study the “excitations” of the DTC phase. Owing to the spontaneous breaking of the discrete time TTS, the TC phase are supposed to be two-fold “degenerate”, each of which has a period 2 and can be connected to the other one by shifting a half-period of the DTC along the temporal direction. By making an analogy with the 1D CDW system, we can consider the excitation as a domain wall in the form of a topological soliton that separates these two “degenerate” states. To realize such an object, one need to “excite” the system by introducing a temporal perturbation that transiently breaks the original time translational symmetry.

Here, we introduce an additional linear ramping of the phase on top of the periodic driving: $V(t) = V_i + \delta \cos 2\pi [t + \theta(t)]$, where $\theta(t) = \frac{2\pi t_1}{T_0}$ for $t \in [t_1, t_1 + T_0]$, and $\theta(t) = 0$ otherwise. $t_1$ is the initial time of the ramping, and $T_0$ is its duration, after which the external driving accumulates an additional $2\pi$ phase compared with the case without ramping. Therefore, the Hamiltonian is identical the one before the ramping. We assume that prior to the ramping ($t < t_1$), the system is in one of the degenerate DTC phase, which is “excited” by the additional time-dependent perturbation once the ramping is switched on. After the ramping ($t > t_1 + T_0$), it still takes some time for the system to relax before it enters into another “stable” dynamical regime.

In the following, we will study both the long-time and transient dynamics of the system and demonstrate their dependence on the ramping rates $\frac{2\pi}{T_0}$. To this end, we fix the initial states ($V_i = 10J$) and all other parameters ($V_f = 3J$, $\delta = 0.8J$) except the duration of ramping $T_0$. In the limit of $T_0 = 0$, the long-time dynamics is identical to that without ramping, because the periodic driving is abruptly changed by a phase of $2\pi$ at $t = t_1$, owing to which it remains unaltered. For a rapid ramping, the system exhibits a similar long-time dynamics, but with a weaker oscillation amplitude, as shown in Fig 2 (c). In the opposite limit of slow ramping, the peak positions of $m(t)$ are pinned to those of $V(t)$. Thus, after the ramping, the phase of $V(t)$ is pushed forward by $2\pi$, and so is $m(t)$. However, due to the period doubling feature of the DTC, $m(t)$ is shifted by a half-period, and thus falls into the other degenerate state that differs from the one before the ramping, although $V(t)$ remains unchanged. Thus, by slowly ramping the driving phase, one can create a domain wall sandwiched between two “degenerate” time crystal phases, whose properties will be studied later.

On the contrary, for a fast ramping, the system cannot follow $V(t)$ “adiabatically”, thus finally relaxes to a TC phase similar to the original one. These two distinct dynamical behaviors of slow and fast ramping indicate a transition between them. From Fig 2 (d), we can find that this transition occurs suddenly at $T_0 \approx 2.41$ and there is no crossover regime between them.
For an ideal TC phase, the peak positions of \( m(t) \) are pinned to those of \( V(t) \), whereas in realistic situations, there is some relative displacement, which could be used to study the properties of excitations. For each peak of \( m(t) \), we define its relative displacement as \( S_n(t) = |P_n(t) - 2jn| \), where \( P_n(t) \) is the position of the \( n \)-th peak of \( m(t) \), \( j_n \) is an integer number indicating the peak position of \( V(t) \) that minimizes \( S_n(t) \). From Fig. 3(a) and (b), we can find that in the TC phase the peak positions of \( m(t) \) are close to those of \( V(t) \), thus one can use \( S_n(t) \) to distinguish the two “degenerate” TC phases. For solution A (B), \( S_n(t) \) is close to 0 (1), whereas \( 0 < S_n(t) < 1 \) within the domain wall. \( S_n(t) \) for various \( T_0 \) is plotted in Fig. 4 from which we can determine that the domain wall between two different TC phases can only be observed for \( T_0 > 2.41 \). It is also interesting to notice that the closer the system approaches the transition point, the longer it takes for the system to relax. It appears that the “size” of the domain wall (or relaxation time) diverges at the phase transition point (see the inset in Fig. 3(a)), which reminds us of the critical slowing down phenomena in dynamical critical systems.

Phenomenological theory of phase dynamics: The instanton-like excitations observed previously could be understood as a macroscopic tunneling process between two “degenerate” states induced by a time-dependent perturbation. Rather than the amplitude of \( m(t) \), we focus on its phase degree of freedom \( \theta(t) \). We speculate that its dynamics can be described using a phenomenological equation of motion (EOM) as:

\[
\frac{d^2\theta(t)}{dt^2} + \gamma \frac{d\theta(t)}{dt} + \frac{\partial U(\theta, t)}{\partial \theta} = 0 \tag{4}
\]

where \( \gamma \) is a phenomenological coefficient that characterizes the dissipation, which tends to stabilize the phase dynamics. Although the system we studied is a closed system without dissipation, the phase variables could exchange energy with other degrees of freedom. Therefore its dynamics can be dissipative. \( U(\theta, t) \) can be considered as an effective double-well potential imposed through the periodic driving. To imitate the phase ramping, we choose \( U(\theta, t) = V_0 \cos[2\pi(\theta + \frac{t}{\tau})] \) for \( 0 < t < T_0 \) and \( U(\theta, t) = V_0 \cos[2\pi\theta] \) otherwise. We assume that the potential is defined in the regime \( \theta \in [-1, 1] \) with PBC \( U(\theta = 1, t) = U(\theta = -1, t) \), within which \( U(\theta, t) \) is a double well potential with two minima, each of which corresponds to one degenerate TC phase that can be connected to the other by shifting a \( 2\pi \) phase.

We assume that initially the system is in one of the degenerate TC phases (e.g. \( \theta|_{t=0} = 0.5 \) and \( \theta|_{t=0} = 0 \)), then we turn on the ramping, which corresponds to a global shift of the external potential with a constant velocity \( \frac{\gamma}{\tau} \). Once the ramping is finished, the external potential returns to its initial value \( U(\theta, t = 0) = U(\theta, t = 1) \), and we will study the dynamics of \( \theta(t) \) based on the phenomenological EOM. Fig. 4 shows that for a quick ramping, \( \theta \) has no time to follow the movement of the potential, and thus it will maintain in its original minimum. However, a sufficiently slow shift of the potential will drive \( \theta \) from 0.5 to \(-0.5\). For an intermediate ramping rate, \( \theta \) keeps oscillating between the two minima for a long time before it relaxes to one of them. The relaxation time (or the “size” of the domain wall) depends on the ramping rate and the dissipation strength \( \gamma \), which is determined by the parameters in microscopic model. The divergence of the relaxation time at the transition point observed in our microscopic model (Fig. 3) indicates that \( \gamma \rightarrow 0 \) for \( T_0 = T_0^c \), which is determined by the phase variable and other degrees of freedom.

Experimental realization and detection: The proposed model can be realized in an experimental setup similar to the one in Ref. 20. In terms of detections, the CDW order parameter can be directly measured using the superlattice band-mapping technique 36, or indirectly through the heterodyne detection 26, 27. For an optical lattice with \( J \approx 400\text{Hz} \), one can estimate that the typical oscillation period is in the order of a magnitude of 10ms, which is a measurable time scale in the current experimental setup. The major obstacle for the experimental observation is the dephasing mechanisms caused by various experimental imperfections including the dissipation (particle loss) or decoherence effect, thermal fluctuation and the extra heating effect generated by our driving protocols, all of which will lead to the damping of oscillations. Thus, the major experimental challenge is to control the dephasing rate and make it considerably smaller than the typical energy scale of the Hamiltonian, which enables us to observe and analyze the dynamics of persistent oscillations before they damp.

Conclusion and outlook: In conclusion, we propose a protocol to realize an excitation in a discrete time crystal. We have observed a dynamical transition between the cases with fast and slow ramping, where the “size” of instanton-like excitations diverges. Future developments will include the analysis of the critical behavior of
the dynamical transition, and use them to classify different types of TC phases and the transitions between them. In addition, although it is conjectured that the divergence of the relaxation time is related to the dissipationless dynamics of phase variables, the microscopic mechanism attributing to this remains unclear and will be studied in the future. Furthermore, it is shown that the phase shifts induced by the ramping are always multiples of 2π, which reminds us of the quantization of particle transport in Thouless pumping. Therefore, one may wonder whether there is a topological origin behind it. After all, in equilibrium physics, DW in a 1D system is usually a topological object carrying fractionized quantum numbers. Finally, it is interesting to generalize our model/method to the continuous time crystal (e.g. our model with δ = 0 and Vf = 0.75J), where one may expect a “phonon”-like rather than an instanton-like excitation owing to the spontaneous breaking of continuous time translational symmetry. If it exists, such an excitation should differ from the diffusive Goldstone modes studied in the dissipative time crystal, because our system is isolated from the environment.

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Appendix A: Validity of the time-dependent self-consistent mean-field method

It is well known that for a model with infinitely long-range interaction, the mean-field method provides an exact treatment for the equilibrium properties in the thermodynamic limit. In the following, we will prove that this is also the case for the dynamical problems.

We denote \( \mathcal{F} \) is the Fock basis of hard-core bosons (e.g. \( |0110\cdots11\rangle \)), and divide the evolution period \([0, t]\) into M slices with \( \Delta t = t/M \). In the limit of \( M \rightarrow \infty \), the path integral expression of the propagators can be written as:

\[
\mathcal{K} = \sum_{\{\mathcal{F}^j\}, \ldots, \mathcal{F}^{M-1}\}} \langle \mathcal{F}^0 | e^{-i\Delta tH(t_1)} | \mathcal{F}^1 \rangle \langle \mathcal{F}^1 | \cdots e^{-i\Delta tH(t_M)} | \mathcal{F}^M \rangle
\]  

(A1)

where \( |\mathcal{F}^j\rangle \) is a Fock basis at the j-th time slice \( (j = 0, \cdots, M) \), with \( \sum_{\{\mathcal{F}^j\}} |\mathcal{F}^j\rangle \langle \mathcal{F}^j| \) a unit matrix. We further divide our Hamiltonian as \( H = \hat{T} + \hat{V} \) where \( \hat{T} = -J \sum_i (\hat{b}_i \hat{b}_{i+1} + h.c.) \), and the interacting energy can be rewritten as:

\[
\hat{V} = LV(t)\frac{1}{T} \sum_i (-1)^i \hat{a}_i^2.
\]  

(A2)

By performing Suzuki-Trotter decomposition, one can obtain that

\[
e^{-i\Delta tH} = e^{-i\Delta t\hat{T}} e^{-i\Delta t\hat{V}} + \mathcal{O}(\Delta t^2)
\]  

(A3)

Therefore, in the limit of \( \Delta t \rightarrow 0 \), the propagator in Eq. (A1) can be expressed as:

\[
\mathcal{K} = \sum_{\{\mathcal{F}^1\}, \ldots, \mathcal{F}^{M-1}\}} \prod_{j=1}^{M-1} T_{j,j+1} e^{-i\Delta tV_{\{\mathcal{F}^j\}}(t_j)}
\]  

(A4)

where \( T_{j,j+1} = \langle \mathcal{F}^j | e^{-i\Delta t\hat{V}} | \mathcal{F}^{j+1} \rangle \), and \( V_{\{\mathcal{F}^j\}}(t_j) = \langle \mathcal{F}^j | e^{-i\Delta t\hat{V}} | \mathcal{F}^j \rangle = e^{-i\Delta tLV(t_j)} S\{\mathcal{F}^j\}^2 \), where \( S\{\mathcal{F}^j\} = \sum \{(-1)^i n_i | n_i \rangle \langle \mathcal{F} | \} \).

The quadratic part in Eq. (A1) can now be decoupled using the Hubbard-Stratonovic transformation by introducing auxiliary fields \( \{m^j\} \) with \( j = 1, M - 1 \) as:

\[
\mathcal{K} = \prod_{j=1}^{M-1} \sum_{\{\mathcal{F}^j\}} \int \frac{dm^j}{N} T_{j,j+1} e^{-i\Delta tLV(t_j)} (m^j)^2 - 2m^j S\{\mathcal{F}^j\})
\]  

(A5)

where \( N \) is a normalization factor to keep the identity of the Gaussian integral. In the thermodynamic limit \( L \rightarrow \infty \), Eq. (A5) indicates that the saddle point approximation \( \frac{d^{\mathcal{F}^m}}{dt^m} = 0 \) with \( \mathcal{L}[m^j] = V(t_j)(m^j)^2 - 2m^j S\{\mathcal{F}^j\}) \), becomes exact, which means that the auxiliary fields \( \{m^j\} \) are given by the CDW order parameter of the state as \( m^j = S\{\mathcal{F}^j\} \). As a consequence, For the time evolution of a state \( \psi \) in each infinitesimal time step \( (\Delta t \rightarrow 0) \), the evolution operator \( e^{iH(t)\Delta t} \) are equal to the one after mean-field approximation \( e^{iH(t,m(t)\Delta t)} \) with a time-dependent CDW order parameters \( m(t) \), which is determined self-consistently during the time evolution by the saddle point method.

FIG. 5: (Color online). Quench dynamics of \( M(t) \) calculated by the self-consistent mean-field method with different \( V_f \) and the parameters \( V_f = 3J, L = 5000, \Delta t = 10^{-5}, \delta = 0; \)
Appendix B: Quantum quench dynamics without driving

In this section, we study quantum quench dynamics of the model ($\delta = 0$) by choosing an initial state as the ground state of $\hat{H}$ with $V(t) = V_i$, and suddenly change the interaction strength to a different value $V_f$ and let the system evolve under this new Hamiltonian. The long-time dynamics of this model has been analyzed in Ref.[1], here we will outline the mean-field results of Ref.[1]. We plot the time evolution of the CDW order parameter

\[ m(t) = \frac{1}{T} \sum_i (-1)^i \langle n_i \rangle \]

starting from two different initial states in Fig 8 from which we can find two distinct dynamical behaviors even though the evolution Hamiltonian are the same ($V_f = 3J$). For the quench dynamics from the initial state with $V_i = 10J$, $m(t)$ will converge to a constant after sufficiently long time, while for the other case with $V_i = 0.75J$, it persistently oscillates with a period that spontaneously emerges during the quench dynamics.

![Fig. 6](image)

**FIG. 6**: (Color online). Dependence of periodically driven dynamics of $M(t)$ on driving amplitudes [(a) $\delta = 0.3J$; (b) $\delta = 0.8J$; (c) $\delta = 8J$] and initial states [(d) $V_i = 0.75J$, $\delta = 0.8J$]. Other parameters are chosen as $V_i = 10J$ for (a)-(c), $L = 5000$ and $\Delta t = 10^{-5}J^{-1}$ for (a)-(d).

![Fig. 7](image)

**FIG. 7**: (Color online). (a) Quench dynamics of $M(t)$ with different system size $L$ and the parameters $\delta = 0$, $\Delta t = 10^{-5}$; (b) Periodically driven dynamics with different $\Delta t$ and $\delta = 0.8J$, $L = 5000$; $V_i = 10J$ and $V_f = 3J$ for (a) and (b).

Appendix C: Periodically driven dynamics

The periodically driven dynamical behavior in our model is very rich, while in the main text we only focus on the time crystal phase. Here we list several typical long-time behaviors, even though it is difficult to enumerate all the possibilities.

In Fig. (7), we plot the long-time dynamics of $m(t)$ with different driving amplitude $\delta$ and initial states $V_i$. For Fig. (7)(a)-(c), we fixed the initial state as the ground state of the Hamiltonian with $V_i = 10J$, while for Fig. (7)(d), we change it to $V_i = 0.75J$. As shown in Fig. (7)(a), for a weak periodical driving, the long-time dynamics becomes a persistent quasi-periodic oscillation with a multi-period structure, which can be reflected in its Fourier spectrum

\[ f(\omega) = \frac{1}{T} \int dt e^{i\omega t} m(t). \]

As shown in the inset of Fig. (7)(a), the Fourier spectrum $f(\omega)$ exhibits sharp peaks at different characteristic frequency, while the dominant one locates at the place exactly the same with the external driving frequency ($\omega = 1$), which indicates $m(t)$ is synchronous with the external periodic driving. For an intermediate driving amplitude, for instance $\delta = 0.8J$ as shown in Fig. (7)(b), the peak position of $f(\omega)$ has been shifted from 1 to 0.5, which indicates a period-doubling time crystal phase as we studied in the main text. When we further increase $\delta$ to a strongly driven regime (e.g. $\delta = 8J$ as shown in Fig. (7)(c)), the frequency spectrum exhibits a broad distribution, a signature of a chaotic dynamics without a dominant time scale. If we start
from an different initial state, for instance, the ground state of \( V_i = 0.75J \) as shown in Fig.7 (d), we can find \( m(t) \) exhibits an persistent oscillation with a frequency \( \omega = 0.46 \), which has nothing to do with the external driving frequency.

**Appendix D: Finite size effect and Convergence of the numerical results**

In our numerical simulations, we choose the lattice size as \( L = 5000 \). Here we will show that within our simulation period (\( 0 < t < 2000J^{-1} \)), such a system size is sufficiently large that the finite-size effect can be neglected. To study the finite-size effect, we calculate \( m(t) \) for smaller systems. Take the quantum quench dynamics for an instance (\( V_i = 10J, V_j = 3J, \delta = 0 \)), where \( m(t) \) is supposed to decay to a constant value after sufficiently long time. However, as shown in Fig.7 (a), for a small system, a revival of oscillation will occur at the time \( t_r \sim \frac{J}{\sqrt{2}J} \), which is the consequence of the finite size effect. Therefore, as long as our simulation time (\( 2000J^{-1} \)) is smaller than the revival time \( 5000/(\sqrt{2}J) \), the finite size effect in our simulation can be neglected.

Another parameter in our simulation is the time step \( \Delta t \), which is chosen to be \( \Delta t = 10^{-5}J^{-1} \) in our simulation. In Fig.7 (b), we check the results with different \( \Delta t \), which indicates that our results are sufficiently converged for \( \Delta t = 10^{-5}J^{-1} \).

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