Do Neutron Star Gravitational Waves Carry Superfluid Imprints?

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Isolated neutron stars undergoing non-radial oscillations are expected to emit gravitational waves in the kilohertz frequency range. To date, radio astronomers have located about 1,300 pulsars, and can estimate that there are about $2 \times 10^8$ neutron stars in the galaxy. Many of these are surely old and cold enough that their interiors will contain matter in the superfluid or superconducting state. In fact, the so-called glitch phenomenon in pulsars (a sudden spin-up of the pulsar’s crust) is best described by assuming the presence of superfluid neutrons and superconducting protons in the inner crusts and cores of the pulsars. Recently there has been much progress on modelling the dynamics of superfluid neutron stars in both the Newtonian and general relativistic regimes. We will discuss some of the main results of this recent work, perhaps the most important being that superfluidity should affect the gravitational waves from neutron stars (emitted, for instance, during a glitch) by modifying both the rotational properties of the background star and the modes of oscillation of the perturbed configuration. Finally, we present an analysis of the so-called zero-frequency subspace (i.e. the space of time-independent perturbations) and determine that it is spanned by two sets of polar (or spheroidal) and two sets of axial (or toroidal) degenerate perturbations for the general relativistic system. As in the Newtonian case, the polar perturbations are the g-modes which are missing from the pulsation spectrum of a non-rotating configuration, and the axial perturbations should lead to two sets of r-modes when the degeneracy of the frequencies is broken by having the background rotate.

I. INTRODUCTION

Jacob Bekenstein is one of those rare individuals who can make significant, original contributions to diverse areas of theoretical physics. He is also a man of great integrity and, I
believe, has a humility that serves him well in advising and supporting students and young scientists. I am profoundly grateful that fate allowed me to be one of those young scientists and now lets me participate in this celebration of his career. One of the areas of theoretical physics that Jacob has worked on is relativistic fluid dynamics. This is an important component of my current area of research, which is to develop models of Newtonian and general relativistic superfluid neutron stars. My original interest in superfluids, appropriately enough, was sparked by Jacob, when he suggested that I look at superfluid analogs of effects predicted for quantum fields in curved spacetimes (the Hawking and Fulling-Davies-Unruh effects). My current interest in superfluids is to determine how the dynamics of superfluid neutron stars differ from their ordinary, or perfect, fluid counterparts and if the different dynamics can lead to observable effects in gravitational waves. In the remainder of this article, I will give an overview of what my collaborators and I have accomplished so far, including some new results (from work with Nils Andersson) on the structure of the so-called zero-frequency subspace (i.e. the space of time-independent perturbations). The main purpose is to show that superfluidity in neutron stars should affect their gravitational waves in two ways, by modifying the rotational properties of the background star and the modes of oscillation of the perturbed configuration.

While there are many mysteries about neutron stars that remain to be explained, we do have some significant observational facts to work with. For instance, Lorimer \[1\] reports that nearly 1300 pulsars (i.e. rotating neutron stars) have now been observed. By extrapolating the data on the local population, he can estimate that there are about $1.6 \times 10^5$ normal pulsars and around $4 \times 10^4$ millisecond pulsars in our galaxy. Of course, there are also neutron stars that are no longer active pulsars. To get a handle on their number Lorimer takes the observed supernova rate, which is about 1 per 60 years, and the age of the universe to find about $2 \times 10^8$ neutron stars in the galaxy. The overwhelming majority of these objects must be very cold in the sense that their (local) temperatures are much less than the (local) Fermi temperatures of the independent species of the matter. One can estimate the Fermi temperature to be about $10^{12}$ K for neutrons at supra-nuclear densities, and it is generally accepted that within the first year (and probably much sooner than that) nascent neutron stars should cool to temperatures less than $10^9$ K. This is an interesting fact, in that nuclear physics calculations of the transition temperature for neutrons and protons to become superfluid and superconducting, respectively, consistently yield a value that is
$10^9$ K in order of magnitude (for recent reviews see [2, 3]). Thus we can expect that a significant portion of the neutron stars in our galaxy will have at least two (and perhaps more) superfluids in their cores.

In addition to nuclear physics theory and experiment, the well-established glitch phenomenon in pulsars (e.g. Vela and Crab) [4, 5] is perhaps the best piece of evidence that supports the existence of superfluids in neutron stars. A glitch is a sudden spin-up of the observed rotation rate of a neutron star, and can have a relaxation time of weeks to months [6]. Baym et al [7] have noted that a relaxation mechanism based on ordinary fluid viscosity would be much too short to explain a weeks to months timescale and so they argue that this signals the presence of a neutron superfluid. Now, a mainstay idea for explaining glitches is that of superfluids and their vortex dynamics, i.e. how the vortices get pinned, unpinned, and then repinned [8, 9, 10] to nuclei in the inner crusts of the glitching pulsars. This is known as the vortex creep model and in it glitches are a transfer of momentum via vortices from one angular momentum carrying component of the star to another. The model has worked well to describe both the giant glitches in Vela and the smaller ones of the Crab. The vortex creep model can also be used to infer the internal temperature of a glitching pulsar, and for Vela it implies a temperature of $10^7$ K [10]. It is also interesting to note the work of Tsakadze and Tsakedze [11] who have experimented with rotating superfluid Helium II and find behaviour very much like glitches in pulsars.

The classic description of superconductivity in ordinary condensed matter systems is based on the so-called “BCS” mechanism (see, for instance, [12] or [13] for excellent presentations): the particles that become superconducting must be fermions, and below a certain transition temperature there must be an (usually effective) attractive interaction between them (at the Fermi surface with zero total momentum). The interaction leads to so-called Cooper-pairing where a pair of fermions act like a single boson and a collection of them can behave as a condensate. The mechanism is very robust, which is why it also forms the basis for discussion of nucleon superfluidity and superconductivity [14]; i.e. nucleons are fermions and the effective interaction between them at nuclear and supra-nuclear densities can be attractive. For instance, it is known experimentally that the lowest excited states in even-even nuclei are systematically higher than other nuclei because of pairing between nucleons which must be broken [12].

After many years of development, beginning with the work of Migdal [15], a consistent
picture has emerged (in part, from gap calculations \[2, 3\]): At long-range the nuclear force is attractive and leads to neutron “Cooper” pairing in \(^1S_0\) states in the inner crust, but because of short-range repulsion in the nuclear force and the spin-orbit interaction neutrons pair into \(^3P_2\) states in the more dense regions of the core \[16\]. In the crust protons are locked inside of neutron rich nuclei embedded in a degenerate normal fluid of electrons. In the inner crust the nuclei are also embedded in, and even penetrated by, the superfluid neutrons. In the core, however, the nuclei have dissolved and the protons remain dilute enough that they feel only the long-range attractive part of the nuclear force and pair in \(^1S_0\) states. There is no pairing between neutrons and protons anywhere in the core since their respective Fermi energies are so different. The core superfluid neutrons and superconducting protons are also embedded in a highly degenerate normal fluid of electrons. Other possibilities, such as pion or hyperon condensates, have been put forward but we will keep to the simplest scenario that considers only superfluid neutrons, superconducting protons, crust nuclei, and normal fluid electrons.

There are several ways in which the dynamics of a superfluid differ from its ordinary fluid counterpart, and each difference should have some impact on the gravitational waves that a superfluid neutron star emits. One key difference is that a pure superfluid is locally irrotational. A superfluid, however, can mimic closely ordinary fluid rotation by forming a dense array of (quantized) vortices. In the core of each vortex the superfluidity is destroyed and the particles are in an ordinary fluid state, and can carry non-zero vorticity. A second, very important difference is when there are several species of matter in a superfluid, or superconducting, state. The superfluids of all the species will interpenetrate and each superfluid will be dynamically independent having its own unit four-vector and local particle number density. Lastly, superfluids have zero viscosity, but when vortices and excitations are present, then dissipative mechanisms can exist. For instance, the scattering of excitations off of the normal fluid in the vortex cores can lead to dissipative momentum exchange between the excitations and the superfluid, the net effect being that the superfluid motion becomes dissipative. This form of dissipative mechanism is known as mutual friction.

In neutron stars there is a very efficient form of mutual friction \[14, 17, 18\] that depends on the entrainment effect \[19, 20\], which Sauls \[14\] describes as follows: even though the neutrons are superfluid and the protons are superconducting both will still feel the long-range attractive component of the nuclear force. In such a system of interacting fermions
the resulting excitations are quasiparticles. This means that the bare neutrons (or protons) are “dressed” by a polarization cloud of nucleons comprised of both neutrons and protons. Since both types of nucleon contribute to the cloud the momentum of the neutrons, say, is modified so that it is a linear combination of the neutron and proton particle number density currents. The same is true of the proton momentum. Thus when one of the nucleon fluids starts to flow it will, through entrainment, induce a momentum in the other fluid. Alpar et al [17] have shown that the electrons track very closely the superconducting protons (because of electromagnetic attraction). Around each vortex is a flow of the superfluid neutrons. Because of entrainment, a portion of the protons, and thus electrons too, will be pulled along with the superfluid neutrons. The motion of the plasma leads to magnetic fields being attached to the vortices. The mutual friction in this case is the dissipative scattering of the normal fluid electrons off of the magnetic fields attached to the vortices.

There has been much effort put forward to develop Newtonian [21, 22, 23, 24, 25] and general relativistic formalisms [26, 27, 28, 29, 30, 31, 32, 33, 34] for describing superfluid neutron stars. In the simplest, but still physically interesting, formalism one has a system that consists of two interpenetrating fluids—the superfluid neutrons in the inner crust and core and the remaining charged constituents (i.e. crust nuclei, core protons, and crust and core electrons) that will be loosely referred to as “protons”—and the entrainment effect that acts between them. In principle the model can be expanded to have more than two interpenetrating fluids (see [35] for instance). As well a given superfluid can be confined to a distinct region in the star [36]. In this way the proton fluid, say, can be made to extend out farther than the superfluid neutrons. This is a first approximation at incorporating the fact that the superfluid neutrons do not extend all the way to the surface of the star.

Our primary goal is to show that superfluidity will affect gravitational wave emission from neutron stars. We will see that a suitably advanced, but plausible, detector will have enough sensitivity at high frequency to see modes excited during a glitch. With such detections we will be able to place constraints, say, on the parameters that describe entrainment. But in addition to studying glitches, we also need to analyze further the recently discovered instability in the r-modes of neutron stars [37, 38]. The instability is driven by gravitational wave emission (the CFS mechanism [39, 40, 41]) and the waves are potentially detectable by LIGO II [42, 43, 44]. The conventional wisdom early on stated that mutual friction would act against the instability in a superfluid neutron star and thus effectively suppress the
gravitational radiation. But Lindblom and Mendell [45] have found that mutual friction is largely ineffective at suppressing the r-mode instability. However, there are many questions about the spectrum of oscillation modes allowed by a rotating superfluid neutron star and the analysis of instabilities is very likely to be much richer than the ordinary fluid case. Thus, another goal here is to lay some groundwork for a future detailed study of the CFS mechanism in superfluid neutron stars. Specifically, we will demonstrate that the zero frequency subspace is spanned by two sets of polar (or spheroidal) and two sets of axial (or toroidal) degenerate perturbations for the general relativistic system. Like the Newtonian case [46], the polar perturbations are the g-modes which are missing from the pulsation spectrum of a non-rotating configuration, and the axial perturbations should lead to two sets of r-modes when the degeneracy of the frequencies is broken by having the background rotate.

Below we will alternate between discussions based on Newtonian gravity and those using general relativity. Accuracy demands that a fully relativistic formalism be employed, however there are some questions of principle for which a Newtonian formalism can suffice. For instance, in determining the number of different modes of oscillation that a superfluid neutron star can undergo it is much more tractable to use the Newtonian equations. But, ultimately there is the need for a general relativistic formalism. Newtonian gravity does not include gravitational waves and so one needs a fully relativistic formalism to get an accurate damping time of a mode of oscillation due to gravitational wave emission. Also, there is the well-known problem that Newtonian models do not produce reliable values for the mass and radius, in that, for a given central density, the predicted mass and radius in Newtonian models may differ considerably from those of general relativity. This is a crucial point since the superfluid phase transition in a neutron star is sensitive to density, as are the parameters relevant to entrainment, and the oscillation frequencies can depend sensitively on mass and radius.

II. GENERAL RELATIVISTIC AND NEWTONIAN SUPERFLUID FORMALISMS

Based on the preceding discussion, our formalism for modelling superfluid neutron stars must allow for two interpenetrating fluids (i.e. the neutrons and “protons”) and the entrain-
ment effect. This must be the case whether we are working in the general relativistic regime or the Newtonian limit. A general relativistic superfluid formalism has been developed by Carter and Langlois [26, 27, 30, 31, 32] and their collaborators [28, 29, 33, 34]. Here an action principle will be outlined that yields the equations of motion and the stress-energy tensor for this system. Although a variational principle also exists for the Newtonian regime [25], we will briefly indicate how to get the Newtonian equations by taking the appropriate limit of the general relativistic equations (see [46] for full details). In the same spirit, we will use a “slow” velocity approximation to motivate an expansion that can be used to incorporate existing models of the entrainment effect [36].

A. The General Relativistic Formalism

In this subsection the equations of motion and stress-energy tensor for a two-component general relativistic superfluid are obtained from an action principle. Specifically a so-called “pull-back” approach (for instance, see [28, 29]) is used to construct Lagrangian displacements of the number density four-currents that form the basis of the variations of the fluid variables in the action principle (they will also be used later in the analysis of the zero-frequency subspace). It will generalize to the superfluid case some of the techniques used to analyze the CFS mechanism in ordinary fluid neutron stars [40, 41]. Finally, it can serve as a starting point for generalizing the Hamiltonian formalism developed by Comer and Langlois [29] who limited their discussion to a superfluid in the Landau state [12] (i.e. purely irrotational).

For both the rotation and mode calculations effects such as “transfusion” [33] of one component into the other (because of the weak interaction, for instance) will be ignored. The neutron and proton number density four-currents, to be denoted $n^\mu$ and $p^\mu$ respectively, are thus taken to be separately conserved, meaning

$$\nabla_\mu n^\mu = 0, \quad \nabla_\mu p^\mu = 0.$$  \hspace{1cm} (1)

Such an approximation is reasonable for the mode calculations since the time-scale of the oscillations (which is milliseconds) is much less than the weak interaction time-scale in neutron stars, as long as the amplitudes of the oscillations remain small enough [47]. In the case of slowly rotating neutron stars, it has been shown [48, 49] that when the neutrons
and protons rotate rigidly at different rates then chemical equilibrium cannot exist between them. Of course, the energy associated with the relative rotation could be dissipated through a process like transfusion. But Haensel [50] has demonstrated that such a process would take months to years in mature neutron stars, and so again it will be neglected (since ultimately we are interested in gravitational waves emitted during a glitch, say, the timescale of which would be much shorter than the transfusion timescale).

By introducing the duals to $n^\mu$ and $p^\mu$, i.e.

$$n_{\nu\lambda\tau} = \epsilon_{\nu\lambda\tau\mu} n^\mu, \quad n^\mu = \frac{1}{3!} \epsilon^{\mu\nu\lambda\tau} n_{\nu\lambda\tau},$$

(2)

and

$$p_{\nu\lambda\tau} = \epsilon_{\nu\lambda\tau\mu} p^\mu, \quad p^\mu = \frac{1}{3!} \epsilon^{\mu\nu\lambda\tau} p_{\nu\lambda\tau},$$

(3)

respectively, then the conservation rules are equivalent to having the two three-forms be closed, i.e.

$$\nabla_{[\mu} n_{\nu\lambda\tau]} = 0, \quad \nabla_{[\mu} p_{\nu\lambda\tau]} = 0.$$  

(4)

The reason for introducing the duals is that it is straightforward to construct particle number density three-forms that are automatically closed. The point is that the conservation of the particle number density currents should not—speaking from a strict field theoretic point of view—be a part of the system of equations of motion, rather they should be automatically satisfied when the “true” system of equations is imposed.

This can be made to happen by introducing two three-dimensional abstract spaces which can be labelled by coordinates $X^A$ and $Y^A$, respectively, where $A, B, C$ etc $= 1, 2, 3$. By “pulling-back” each three-form onto its respective abstract space we can construct three-forms that are automatically closed on spacetime, i.e. let

$$n_{\nu\lambda\tau} = N_{ABC}(X^D) \nabla_\nu X^A \nabla_\lambda X^B \nabla_\tau X^C, \quad p_{\nu\lambda\tau} = P_{ABC}(Y^D) \nabla_\nu Y^A \nabla_\lambda Y^B \nabla_\tau Y^C$$  

(5)

where $N_{ABC}$ and $P_{ABC}$ are completely antisymmetric in their indices. Because the abstract space indices are three-dimensional and the closure condition involves four spacetime indices, and also that the $X^A$ and $Y^A$ are scalars on spacetime (and thus two covariant differentiations commute), the pull-back construction does indeed produce a closed three-form:

$$\nabla_{[\mu} n_{\nu\lambda\tau]} = \nabla_{[\mu} \left( N_{ABC}(X^D) \nabla_\nu X^A \nabla_\lambda X^B \nabla_\tau X^C \right) \equiv 0,$$  

(6)
and similarly for the protons. In terms of the scalar fields $X^A$ and $Y^A$, we now have particle number density currents that are automatically conserved, and so another way of viewing the pull-back construction is that the fundamental fluid field variables are the spacetime functions $X^A$ and $Y^A$. The variations of the three-forms can now be derived by varying them with respect to $X^A$ and $Y^A$.

Let us introduce two Lagrangian displacements on spacetime for the neutrons and protons, to be denoted $\xi^\mu_n$ and $\xi^\mu_p$, respectively. These are related to the variations $\delta X^A$ and $\delta Y^A$ via a "push-forward" construction:

$$\delta X^A = - \left( \nabla_\mu X^A \right) \xi^\mu_n, \quad \delta Y^A = - \left( \nabla_\mu Y^A \right) \xi^\mu_p.$$  \hspace{1cm} (7)

Using the fact that

$$\nabla_\nu \delta X^A = - \nabla_\nu \left( \left[ \nabla_\mu X^A \right] \xi^\mu_n \right) = - \left( \nabla_\mu X^A \right) \nabla_\nu \xi^\mu_n - \left( \nabla_\mu \nabla_\nu X^A \right) \xi^\mu_n,$$  \hspace{1cm} (8)

and similarly for the proton variation, then we find

$$\delta n_{\nu\lambda\tau} = - \left( \xi^\sigma_n \nabla_\sigma n_{\nu\lambda\tau} + n_{\sigma\lambda\tau} \nabla_\nu \xi^\sigma_n + n_{\nu\sigma\tau} \nabla_\lambda \xi^\sigma_n + n_{\nu\lambda\sigma} \nabla_\tau \xi^\sigma_n \right) = - \mathcal{L}_{\xi_n} n_{\nu\lambda\tau},$$

where $\mathcal{L}$ is the Lie derivative. We can thus infer that

$$\delta n^\mu = n^\sigma \nabla_\sigma \xi^\mu_n - \xi^\sigma_n \nabla_\sigma n^\mu - n^\mu \left( \nabla_\sigma \xi^\sigma_n + \frac{1}{2} g^{\sigma\rho} \delta g_{\sigma\rho} \right),$$

$$\delta p^\mu = p^\sigma \nabla_\sigma \xi^\mu_p - \xi^\sigma_p \nabla_\sigma p^\mu - p^\mu \left( \nabla_\sigma \xi^\sigma_p + \frac{1}{2} g^{\sigma\rho} \delta g_{\sigma\rho} \right).$$  \hspace{1cm} (10)

By introducing the two decompositions

$$n^\mu = n u^\mu, \quad u^\mu u^\nu = -1,$$

$$p^\mu = p u^\mu, \quad v^\mu v^\nu = -1,$$  \hspace{1cm} (11)

then we can show furthermore that

$$\delta n = - \nabla_\sigma \left( n \xi^\sigma_n \right) - n \left( u^\nu u^\sigma \nabla_\sigma \xi^\nu_n + \frac{1}{2} \left[ g^{\sigma\rho} + u^\sigma u^\rho \right] \delta g_{\sigma\rho} \right),$$
\[ \delta p = -\nabla_\sigma \left( p \xi^\sigma \right) - p \left( v_\nu v^\sigma \nabla_\sigma \xi^\nu + \frac{1}{2} \left[ g^{\sigma \rho} + v^\sigma v^\rho \right] \delta g_{\sigma \rho} \right), \tag{12} \]

and

\[ \delta u^\mu = \left( \delta^\mu_\rho + u^\mu u_\rho \right) \left( u^\sigma \nabla_\sigma \xi^\rho_n - \xi^\sigma_n \nabla_\sigma u^\rho \right) + \frac{1}{2} u^\mu u^\sigma u^\rho \delta g_{\sigma \rho}, \]

\[ \delta v^\mu = \left( \delta^\mu_\rho + v^\mu v_\rho \right) \left( v^\sigma \nabla_\sigma \xi^\rho_p - \xi^\sigma_p \nabla_\sigma v^\rho \right) + \frac{1}{2} v^\mu v^\sigma v^\rho \delta g_{\sigma \rho}. \tag{13} \]

We will take one more step ahead and associate a notion of Lagrangian variation with each Lagrangian displacement. These are defined to be

\[ \Delta_n \equiv \delta + \mathcal{L}_{\xi_n}, \quad \Delta_p \equiv \delta + \mathcal{L}_{\xi_p}, \tag{14} \]

so that it now follows that

\[ \Delta_n n_{\mu \lambda \tau} = 0, \quad \Delta_p p_{\mu \lambda \tau} = 0, \tag{15} \]

which is entirely consistent with the pull-back construction. We also find that

\[ \Delta_n u^\mu = \frac{1}{2} u^\mu u^\sigma u^\rho \Delta_n g_{\sigma \rho}, \quad \Delta_p v^\mu = \frac{1}{2} v^\mu v^\sigma v^\rho \Delta_p g_{\sigma \rho}, \tag{16} \]

\[ \Delta_n \epsilon_{\nu \lambda \tau \sigma} = \frac{1}{2} \epsilon_{\nu \lambda \tau \sigma} g^{\mu \rho} \Delta_n g_{\mu \rho}, \quad \Delta_p \epsilon_{\nu \lambda \tau \sigma} = \frac{1}{2} \epsilon_{\nu \lambda \tau \sigma} g^{\mu \rho} \Delta_p g_{\mu \rho}, \tag{17} \]

and

\[ \Delta_n n = -\frac{n}{2} \left( g^{\sigma \rho} + u^\sigma u^\rho \right) \Delta_n g_{\sigma \rho}, \quad \Delta_p p = -\frac{p}{2} \left( g^{\sigma \rho} + v^\sigma v^\rho \right) \Delta_p g_{\sigma \rho}. \tag{18} \]

However, in contrast to the ordinary fluid case \cite{40, 41}, there are many more options to consider. For instance, we could also look at the Lagrangian variation of the neutron number density with respect to the proton flow, i.e. \( \Delta_p n \), or the Lagrangian variation of the proton number density with respect to the neutron flow, i.e. \( \Delta_n p \). It is not clear at this point how the existence of two preferred rest frames—one that is attached to the neutrons and the other that is attached to the protons—will affect an analysis of the CFS mechanism in superfluid neutron stars.

Nevertheless, with a general variation of the conserved four-currents in hand, we can now use an action principle to derive the equations of motion and the stress-energy tensor. The central quantity is the so-called “master” function \( \Lambda \), which is a function of all the different
scalars that can be formed from \( n^\mu \) and \( p^\mu \), i.e. the three scalars \( n^2 = -n_\mu n^\mu \), \( p^2 = -p_\mu p^\mu \), and \( x^2 = -p_\mu n^\mu \). In the limit where the two currents are parallel, i.e. the two fluids are comoving, then the master function is such that \(-\Lambda \) corresponds to the local thermodynamic energy density. In the action principle, the master function is the Lagrangian density for the two fluids.

An unconstrained variation of \( \Lambda(n^2,p^2,x^2) \) with respect to the independent vectors \( n^\mu \) and \( p^\mu \) and the metric \( g_{\mu\nu} \) takes the form

\[
\delta \Lambda = \mu_\mu \delta n^\mu + \chi_\mu \delta p^\mu + \frac{1}{2} (n^{\mu\nu} + p^{\mu\nu}) \delta g_{\mu\nu},
\]

where

\[
\mu_\mu = B n_\mu + A p_\mu , \quad \chi_\mu = C p_\mu + A n_\mu ,
\]

and

\[
A = -\frac{\partial \Lambda}{\partial x^2} , \quad B = -2 \frac{\partial \Lambda}{\partial n^2} , \quad C = -2 \frac{\partial \Lambda}{\partial p^2} .
\]

The momentum covectors \( \mu_\mu \) and \( \chi_\mu \) are dynamically, and thermodynamically, conjugate to \( n^\mu \) and \( p^\mu \), and their magnitudes are, respectively, the chemical potentials of the neutrons and the protons. The two momentum covectors also show the entrainment effect since it is seen explicitly that the momentum of one constituent carries along some mass current of the other constituent (for example, \( \mu_\mu \) is a linear combination of \( n^\mu \) and \( p^\mu \)). We also see that entrainment vanishes if \( \Lambda \) is independent of \( x^2 \) (because then \( A = 0 \)).

If the variations of the four-currents were left unconstrained, the equations of motion for the fluid implied from the above variation of \( \Lambda \) would require, incorrectly, that the momentum covectors should vanish in all cases. This reflects the fact that the variations of the conserved four-currents must be constrained. In terms of the constrained Lagrangian displacements, a variation of \( \Lambda \) now yields

\[
\delta \left( \sqrt{-g} \Lambda \right) = \frac{1}{2} \sqrt{-g} \left( \Psi g^{\mu\nu} + p^{\mu} \chi^{\nu} + n^{\mu} \mu^{\nu} \right) \delta g_{\mu\nu} - 2 \sqrt{-g} \left( n^{\mu} \nabla_{[\mu\mu]} \xi_n^{\nu} + p^{\mu} \nabla_{[\mu\chi]} \xi_p^{\nu} \right) + T.D.
\]

where the \( T.D. \) is a total divergence and thus does not contribute to the field equations or stress-energy tensor. At this point we can return to the view that \( n^\mu \) and \( p^\mu \) are the fundamental variables for the fluids. Thus the equations of motion consist of the two original conservation conditions of Eq. (1) plus two Euler type equations

\[
n^{\mu} \nabla_{[\mu\mu]} = 0 , \quad p^{\mu} \nabla_{[\mu\chi]} = 0 ,
\]

(23)
since the two Lagrangian displacements are independent. We see that the stress-energy tensor is

\[ T_{\mu}^{\nu} = \Psi \delta_{\nu}^{\mu} + p^{\mu} \chi_{\nu} + n^{\mu} \mu_{\nu} , \]  

(24)

where the generalized pressure \( \Psi \) is defined to be

\[ \Psi = \Lambda - n^{\mu} \mu_{\mu} - p^{\mu} \chi_{\mu} . \]  

(25)

When the complete set of field equations is satisfied then it is automatically true that \( \nabla_{\mu} T^{\mu}_{\nu} = 0 \).

In a later section we will be interested in the linearized version of the combined Einstein and superfluid equations. It will thus be convenient to write down the variations of the momentum covectors in terms of the variations of the particle number density currents. Following the scheme of Carter \[27\], and Comer et al \[52\], the variations of \( \mu_{\rho} \) and \( \chi_{\rho} \) due to a generic variation of \( n^{\mu} \), \( p^{\mu} \), and \( g_{\mu\nu} \) take the form

\[ \delta \mu_{\rho} = A_{\rho}^{\sigma} \delta p_{\sigma} + B_{\rho}^{\sigma} \delta n_{\sigma} + (\delta g A) p_{\rho} + (\delta g B) n_{\rho} , \]  

(26)

\[ \delta \chi_{\rho} = C_{\rho}^{\sigma} \delta p_{\sigma} + A_{\rho}^{\sigma} \delta n_{\sigma} + (\delta g C) p_{\rho} + (\delta g A) n_{\rho} , \]  

(27)

with

\[ A_{\mu\nu} = A g_{\mu\nu} - 2 \frac{\partial B}{\partial \nu} n_{\mu} p_{\nu} - 2 \frac{\partial A}{\partial n^{2}} n_{\mu} n_{\nu} - 2 \frac{\partial A}{\partial p^{2}} p_{\mu} p_{\nu} - \frac{\partial A}{\partial x^{2}} p_{\mu} n_{\nu} , \]  

\[ B_{\mu\nu} = B g_{\mu\nu} - 2 \frac{\partial B}{\partial \nu} n_{\mu} n_{\nu} - 4 \frac{\partial A}{\partial n^{2}} P_{\mu} (n_{\nu}) - \frac{\partial A}{\partial x^{2}} p_{\mu} p_{\nu} , \]  

\[ C_{\mu\nu} = C g_{\mu\nu} - 2 \frac{\partial C}{\partial p^{2}} p_{\mu} p_{\nu} - 4 \frac{\partial A}{\partial p^{2}} P_{\mu} (n_{\nu}) - \frac{\partial A}{\partial x^{2}} n_{\mu} n_{\nu} , \]  

(28)

and the terms \( \delta g A \), \( \delta g B \) and \( \delta g C \) are given by

\[ \delta g A = \left[ \frac{\partial A}{\partial n^{2}} n^{\mu} n^{\nu} + \frac{\partial A}{\partial p^{2}} p^{\mu} p^{\nu} + \frac{\partial A}{\partial x^{2}} n^{\mu} p^{\nu} \right] \delta g_{\mu\nu} \]  

(29)

(\( \delta g B \) and \( \delta g C \) being given by analogous formulas, with \( A \) replaced by \( B \) and \( C \) respectively).

B. The Newtonian Limit

The Newtonian superfluid equations can be obtained by writing the general relativistic field equations to order \( c^{0} \), where \( c \) is the speed of light, and then taking the limit that \( c \)
becomes infinite. To order \( c^0 \) the metric can be written as

\[
ds^2 = -c^2 \left( 1 + \frac{2\Phi}{c^2} \right) dt^2 + \delta_{ij} dx^i dx^j ,
\]

(30)

where the \( x^i \) \((i = 1, 2, 3)\) are Cartesian-like coordinates, and the gravitational potential \( \Phi \) is assumed to be small in the sense that \(-1 << \Phi/c^2 \leq 0\). To the same order the unit four-velocity components defined earlier are given by

\[
u^t = 1 - \frac{\Phi}{c^2} + \frac{v_n^2}{2c^2} , \quad \nu^i = v_n^i ,
\]

(31)

and

\[
u^t = 1 - \frac{\Phi}{c^2} + \frac{v_p^2}{2c^2} , \quad \nu^i = v_p^i ,
\]

(32)

where \( v_{n,p}^2 = \delta_{ij} v_{n,p}^i v_{n,p}^j \), and \( v_n^i \) and \( v_p^i \) are the Newtonian three-velocities of the neutron and proton fluids, respectively. The three-velocities are assumed to be small with respect to the speed of light. The entrainment variable \( x^2 \) takes the limiting form

\[
x^2 = n_n n_p \left( 1 + \frac{w^2}{2c^2} \right) ,
\]

(33)

where

\[
w^2 = \delta_{ij} \left( v_n^i - v_p^i \right) \left( v_n^j - v_p^j \right)
\]

(34)

and we have introduced the new notation \( n_n = n \) and \( n_p = p \) (to distinguish between Newtonian quantities and their general relativity counterparts). Finally, we separate the “master” function \( \Lambda \) into its mass part and a much smaller internal energy part \( E \), i.e. we write it as

\[
\Lambda = -(m_n n_n + m_p n_p) c^2 - E(n_n^2, n_p^2, x^2) ,
\]

(35)

where \( m_n \) (\( m_p \)) is the neutron (proton) mass.

To more closely agree with the Newtonian superfluid equations derived by other means [25], a different choice for the independent variables is used which is the triplet of variables \( (n_n^2, n_p^2, w^2) \). In this case \( E = E(n_n^2, n_p^2, w^2) \) and the analog of the combined First and Second Law of Thermodynamics for the system takes the form

\[
dE = \mu_n dn_n + \mu_p dn_p + \alpha dw^2 ,
\]

(36)

where

\[
\mu_n = \frac{\partial E}{\partial n_n} , \quad \mu_p = \frac{\partial E}{\partial n_p} , \quad \alpha = \frac{\partial E}{\partial w^2} .
\]

(37)
The generalized pressure $\Psi$, to be renamed $P$, takes the limiting form

$$P = -E + \mu_n n_n + \mu_p n_p .$$  \hspace{1cm} (38)

Formally letting the speed of light become infinite in the combined Einstein and general relativistic superfluid field equations, results in the following set of 9 equations:

$$0 = \frac{\partial n_n}{\partial t} + \partial_i \left( n_n v_n^i \right) ,$$
$$0 = \frac{\partial n_p}{\partial t} + \partial_i \left( n_p v_p^i \right) ,$$ \hspace{1cm} (39)

and

$$0 = \frac{\partial}{\partial t} \left( v_n^i + \frac{2\alpha}{m_n n_n} \left[ v_p^i - v_n^i \right] \right) + v_n^j \partial_j \left( v_n^i + \frac{2\alpha}{m_n n_n} \left[ v_p^i - v_n^i \right] \right) + \delta^{ij} \partial_j \left( \Phi + \frac{\mu_n}{m_n} \right) +$$
$$\frac{2\alpha}{m_n n_n} \delta^{ij} \delta_{kl} \left( v_p^l - v_n^l \right) \partial_j v_n^k ,$$
$$0 = \frac{\partial}{\partial t} \left( v_p^i + \frac{2\alpha}{m_p n_p} \left[ v_n^i - v_p^i \right] \right) + v_p^j \partial_j \left( v_p^i + \frac{2\alpha}{m_p n_p} \left[ v_n^i - v_p^i \right] \right) + \delta^{ij} \partial_j \left( \Phi + \frac{\mu_p}{m_p} \right) +$$
$$\frac{2\alpha}{m_p n_p} \delta^{ij} \delta_{kl} \left( v_n^l - v_p^l \right) \partial_j v_p^k .$$ \hspace{1cm} (40)

The gravitational potential $\Phi$ is obtained from

$$\partial_i \partial^i \Phi = 4\pi G \left( m_n n_n + m_p n_p \right) .$$ \hspace{1cm} (41)

For this system having no entrainment means setting the coefficient $\alpha$ to zero.

These equations are equivalent to those derived independently by Pritz [25], using a Newtonian variational principle. We also note that they are formally equivalent to the two-fluid set developed by Landau [53] for superfluid He II. The two fluids in the Landau case are traditionally taken to be the normal fluid (i.e. the phonons, rotons, and other excitations) that carries the entropy and the rest of the fluid (i.e. the superfluid) that carries no entropy.

\section*{C. An analytical equation of state with entrainment}

The item that connects the microphysics to the global structure and dynamics of superfluid neutron stars is the master function, since it incorporates all of the information about
the local thermodynamic state of the matter. Ultimately, realistic models of superfluid neutron stars must be built using realistic master functions (i.e. equations of state). Only in this way can gravitational wave data be used to greatest effect to constrain the microphysics, such as parameters that are important for entrainment. Unfortunately, there is not yet a fully relativistic determination of entrainment (although Comer and Joynt are currently working on this), and the best that can be done is to adapt models that have been used in the Newtonian limit. Here we will describe an analytic expansion of the master function that facilitates the process (see [36] for all the details).

In the limit where the fluid velocities are small with respect to $c$ we see from Eq. (33) that the combination $x^2 - np$ is small. Thus, it makes sense to consider an expansion of the master function of the form [36]

$$\Lambda(n^2, p^2, x^2) = \sum_{i=0}^{\infty} \lambda_i(n^2, p^2) \left(x^2 - np\right)^i.$$ (42)

The $A$, $B$, and $C$ coefficients that appear in the definitions of the momentum covectors become

$$A = -\sum_{i=1}^{\infty} i \lambda_i(n^2, p^2) \left(x^2 - np\right)^{i-1},$$

$$B = -1 \frac{\partial \lambda_0}{n \partial n} - p \frac{\partial A}{n} - 1 \sum_{i=1}^{\infty} \frac{\partial \lambda_i}{n \partial n} \left(x^2 - np\right)^i,$$

$$C = -1 \frac{\partial \lambda_0}{p \partial p} - n \frac{\partial A}{p} - 1 \sum_{i=1}^{\infty} \frac{\partial \lambda_i}{p \partial p} \left(x^2 - np\right)^i.$$ (43)

The expansion is especially useful for both the rotation and mode calculations, since $x^2 = np$ for any zeroth-order, or background, quantity. In practice this means that only the first few $\lambda_i$ contribute, and for the mode calculations we need retain only $\lambda_0$ and $\lambda_1$. The first coefficient $\lambda_0$ is directly related to equations of state that describe a mixture of neutrons and protons that are locally at rest with respect to each other. The other coefficient $\lambda_1$ contains the information concerning the entrainment effect.

We will use a sum of two polytropes for $\lambda_0$, i.e.

$$\lambda_0(n^2, p^2) = -m_n n - \sigma_n n^{\beta_n} - m_p p - \sigma_p p^{\beta_p}.$$ (44)

In Table I are given various parameter values that have been used in specific applications [36, 48, 52]. Model I corresponds to the values used in the initial study of modes on a non-rotating background by Comer et al [52], and is meant to describe only a neutron star core
without an outer envelope of ordinary fluid. Thus, the neutron and proton number densities vanish at the same radius. Model II is used in the mode study of Andersson et al \cite{36} and is the most realistic. Its parameter values given in Table I have been chosen specifically to yield a canonical static and spherically symmetric background model having the following characteristics (see \cite{36} for complete details): (i) A mass of about $1.4M_\odot$, (ii) a total radius of about 10 km, (iii) an outer envelope of ordinary fluid of roughly one kilometer thickness, and (iv) a central proton fraction of about 10%. These are values that are considered to be representative of a typical neutron star. The background distribution of particles for this model is determined in Andersson et al \cite{36} and their graph and caption is reproduced here in Fig. 1. \textbf{Finally, model III is used by Andersson and Comer \cite{48} to study slowly rotating configurations and is meant to be more realistic than model I by extending the radius out further and having a better total mass value, but with no distinct envelope so that the neutron and proton number densities vanish at the same radius.}

Our strategy for incorporating entrainment is to adapt models that have been used in Newtonian calculations. The particular model we will use is that of Lindblom and Mendell \cite{45}. It is a parametrized approximation of the more detailed model based on Fermi liquid theory \cite{55} developed by Borumand, Joynt, and Kluzniak \cite{56}. As demonstrated in Andersson et al \cite{36} the Lindblom and Mendell model translates into the relation

$$\lambda_1 = -\frac{m_n m_p}{\rho_{np}^2 - \rho_{nn} \rho_{pp}} \rho_{np}, \quad (45)$$

where

$$m_n n = \rho_{nn} + \rho_{np}, \quad m_p p = \rho_{pp} + \rho_{np}, \quad (46)$$

and

$$\rho_{np} = -\epsilon m_n n. \quad (47)$$

Here $\epsilon$ is taken to be a constant, which is the approximation introduced by Lindblom and Mendell \cite{45}. Prix, Comer and Andersson \cite{49} argue that

$$\epsilon = \frac{m_p p}{m_n n} \left( \frac{m_p}{m_p^*} - 1 \right), \quad (48)$$

where $m_p^*$ is the proton effective mass. But, Sjöberg \cite{57} has determined that $0.3 \leq m_p^*/m_p \leq 0.8$. Assuming a proton fraction of about 10\% we can take as “physical” for a neutron star core those values that lie in the range $0.04 \leq \epsilon \leq 0.2$. 

III. SLOWLY ROTATING SUPERFLUID NEUTRON STARS

The first application to consider is to the problem of axisymmetric, stationary, and asymptotically flat configurations, i.e. the standard assumptions made for rotating neutron stars [58, 59, 60]. Although accurate codes exist for looking at rapidly rotating neutron stars in the ordinary fluid case [60, 61, 62], and can in principle be adapted to the superfluid case (Prix, Novak, and Comer, work in progress), we will limit our discussion to situations where rapid rotation accuracy is not needed. That is, we will describe models of rotating superfluid neutron stars that have been developed [48, 49, 63] using a slow-rotation approximation. In the general relativistic regime, we discuss results of Andersson and Comer [48] who have adapted the one-fluid formalism of Hartle [64] and Hartle and Thorne [65] to the superfluid case, whereas in the Newtonian limit it is the work Prix et al [49] (who have built on the Chandrasekhar-Milne approach [66, 67]) that will be reviewed. The most important aspect of the superfluid case in both the general relativistic regime and Newtonian limit is that the neutrons can rotate at a rate different from that of the protons, and this obviously has no analog in the ordinary fluid case.

At the heart of the slow-rotation approximation is the assumption that the star is rotating slowly enough that the fractional changes in pressure, energy density, and gravitational field induced by the rotation are all relatively small [64]. If $M$ and $R$ represent the mass and radius, respectively, of the non-rotating configuration, and $\Omega_n$ and $\Omega_p$ the respective constant angular speeds of the neutrons and protons, then “slow” can be defined to mean rotation rates that satisfy the inequality

$$\Omega_n^2 \text{ or } \Omega_p^2 \text{ or } \Omega_n \Omega_p \ll \left( \frac{c}{R} \right)^2 \frac{GM}{Rc^2}. \tag{49}$$

Since $GM/Rc^2 < 1$, the inequality also implies $\Omega_{n,p} R \ll c$, i.e. that the linear speed of the matter must be much less than the speed of light.

This is not as restrictive as one might guess, especially for the astrophysical scenarios we have in mind. For a solar mass neutron star of radius 10 km, the square root of the combination on the right-hand-side of Eq. (49) works out to 11 500 s$^{-1}$. The Kepler limit (i.e. the rotation rate at which mass-shedding sets in at the equator) for the same star is roughly 7700 s$^{-1}$. The fastest known pulsar rotates with a period of 1.56 ms, which translates into a rotation rate of 4000 s$^{-1}$ or nearly half the Kepler limit. This is why the slow-rotation approximation is still accurate to (say) 15 – 20% for stars rotating at the Kepler limit.
This is seen in Fig. 2, which is taken from Prix et al [49]. It shows a one-fluid rotating star’s equatorial ($R_{\text{equ}}$) and polar ($R_{\text{pole}}$) radii, determined using both the slow-rotation approximation and the very accurate LORENE code developed by the Meudon Numerical Relativity group [60, 62]. Notice that the slow-rotation approximation works quite well up to and including rotation rates equal to the fastest known pulsar.

The slow rotation scheme is based upon an expansion in terms of the constant rotation rates of the fluids. The first rotationally induced effect that one encounters in general relativity is the linear order frame-dragging (i.e. local inertial frames close to and inside the star are rotating with respect to inertial frames at infinity). It is only at the second order that rotationally induced changes in the total mass, shape, and distribution of the matter of the star are produced. Since the frame-dragging is a purely general relativistic effect, the Newtonian slow rotation scheme yields nothing at linear order, but does have second order changes that parallel those of general relativity.

Andersson and Comer [48] apply their slow rotation scheme using the two-polytrope equation of state with the model III parameters listed in Table I. Although their formalism is general enough to allow for entrainment, they do not consider it in their numerical solutions to the field equations, since their main emphasis is to extract effects due to two rotation rates that can be independently specified. For instance, reproduced here in Fig. 3 are some of their typical results for the frame-dragging for modest values of the relative rotation $\Omega_n/\Omega_p$ between the neutrons and protons. The solutions exhibit a characteristic monotonic decrease of the frame-dragging from the center to the surface of the star. Although likely very unrealistic, they also considered an extreme case of having the neutrons and protons counterrotate, with the result shown in Fig. 3 (reproduced, again, from [48]). Clearly the frame-dragging is no longer monotonic, and even changes sign. One can understand this behavior as follows: In the inner core of the star, the angular momentum in the protons dominates so that the frame-dragging is positive. But in the outer layers of the star the fact that the neutrons contain roughly 90% of the mass means they begin to dominate so that the frame-dragging reverses. Finally, we reproduce here one other result of Andersson and Comer in Fig. 3, which is the effect of relative rotation on the Kepler limit. When the relative rotation is larger than one there is little change in the Kepler limit, which is due simply to the fact that the neutrons contain most of the mass. On the other hand, when the relative rotation is being decreased toward zero, the Kepler limit rises because the frequency
of a particle orbiting at the equator is approaching the non-rotating limit (again because the neutrons carry most of the mass).

Prix et al [49] apply their Newtonian slow rotation formalism in a manner similar to Andersson and Comer. There are important differences, however, even beyond the exclusion of general relativistic effects, and these are (i) they use an equation of state that includes entrainment and terms related to symmetry energy (i.e. terms \[ \sigma \] which tend to force the system to have as many neutrons as protons), and (ii) an exact solution to the slow rotation equations is used for the analysis. Thus, they are able to explore how entrainment and the symmetry energy affects the rotational configuration of the star. We reproduce here in Fig. 6 their result for the Kepler limit as the relative rotation is varied, for different values of entrainment (denoted \( \varepsilon \)) and symmetry energy (denoted as \( \sigma \)). The big surprise is that the symmetry energy has as much an impact as the entrainment. This fact has not been noticed before and should be explored in more depth, using a more realistic equation of state (like [68]).

IV. THE LINEARIZED NON-RADIAL OSCILLATIONS

The second application of our superfluid formalism is to the problem of non-radial oscillations. The ultimate goal is to calculate such oscillations, and the gravitational waves that result from them, for rotating neutron stars in the general relativistic regime. This is not an easy task, and the problem has not been solved fully (for rapidly rotating backgrounds) even for the “simpler” case of the ordinary perfect fluid. That is, while some recent progress has been made to calculate the frequency of oscillations [69], there are as yet no complete determinations of the damping rates of the modes because of gravitational wave emission. Even using the slow rotation approximation there are complications, due to questions about the basic nature of the modes and if they can be separated into purely polar and axial parts, or if they are of the inertial hybrid mode class (as in [70, 71]). Nevertheless, we can gain valuable insight by considering non-radial oscillations on non-rotating backgrounds. We will use the Newtonian equations to reveal the nature of the various modes of oscillation, by given the highlights of the recent analysis of Andersson and Comer [46], and we summarize the main results of general relativistic calculations [36, 52] of mode frequencies and damping rates.
A. Linearized oscillations in Newtonian theory

The investigation of the nature of the modes of oscillation in both the Newtonian and general relativistic regimes has now a decade and a half of history. Epstein’s work [47] is the beginning, since he is the first to suggest that there should be new oscillation modes because superfluidity allows the neutrons to move independently of the protons, and thereby increases the fluid degrees of freedom. Mendell [22] reaches the same conclusion and moreover argues, using an analogy with coupled pendulums, that the new modes should have the characteristic feature of a counter-motion between the neutrons and protons, i.e. in the radial direction as the neutrons are moving out, say, the protons will be moving in, which is to be contrasted with the ordinary fluid modes that have the neutrons and protons moving in more or less “lock-step”. This basic picture has been confirmed by analytical and numerical studies [23, 36, 46, 52, 72, 73, 74] and the new modes of oscillation are known as superfluid modes. As we will see below they are predominately acoustic in nature, and have a sensitive dependence on entrainment parameters. It is worthwhile to mention again that our equations are formally equal to the two-fluid set of Landau for superfluid He II. With some hindsight, one recognizes that the superfluid modes could have perhaps been inferred to exist from a thermomechanical effect [12] in which the normal fluid and superfluid are forced into counter-motion by passing an alternating current through a resistor placed in the container that holds the fluid.

The existence of the superfluid modes seems to confirm one’s intuition that a doubling of the fluid degrees of freedom should lead to a doubling of modes. A review of the spectrum of modes for the ordinary fluid should then give one an idea of what to expect in the superfluid. McDermott et al [75] have given an excellent discussion of many of the possible modes in neutron stars, and these include the polar (or spheroidal) f-, p-, and g-modes and the axial (or toroidal) r-modes. But a puzzling aspect in all of this is that Lee’s [72] numerical analysis does not reveal a new set of g-modes, and in fact does not find any g-modes of non-zero frequency. Of course, the model he considers is that of a zero-temperature neutron star, and so one would not really expect g-modes like those of the sun (which exist because of an entropy gradient) to be important in a mature neutron star. However, Reisenegger and Goldreich [76] have shown conclusively that a composition gradient, such as the proton fraction in a neutron star, will also lead to g-modes, and the model of Lee does have a
composition gradient. Fortunately, this issue has been clarified by Andersson and Comer [46] who use a local analysis of the Newtonian equations to confirm Lee’s numerical result that there are no g-modes of non-zero frequency. Their analysis of the zero-frequency subspace reveals two sets of degenerate spheroidal modes, which they take to be the missing g-modes. They also find two sets of degenerate toroidal modes, which they interpret to be r-modes. And in fact when they add in rotation they find that the degeneracy is lifted and two sets of non-zero frequency r-modes exist.

Apart from questions about the g-modes, Andersson and Comer [46] also illuminate the character of the superfluid modes. Although they do not solve the linearized equations for global mode frequencies, they are able to demonstrate that the equation that describes the radial behaviour of the superfluid modes is of the Sturm-Liouville form for large frequencies $\omega$. Thus, one can expect there to be a set of modes for which $\omega^2_n \to \infty$ as the index $n \to \infty$. The equation that describes the ordinary fluid modes is also of the Sturm-Liouville form for large frequencies and so it also has a set of modes with the same mode frequency behavior. That is both sets of modes will be interlaced in the pulsation spectrum of the neutron star.

Finally, Andersson and Comer [46] also use a local analysis of the Newtonian equations to find a (local) dispersion relation for the mode frequencies. Letting

$$c_n^2 = \frac{n_n}{m_n} \frac{\partial \mu_n}{\partial n_n}, \quad c_p^2 = \frac{n_p}{m_p} \frac{\partial \mu_p}{\partial n_p},$$

and assuming that the proton fraction (i.e. the ratio of the proton number density over the total number density) is small, then they find that one solution to the dispersion relation is

$$\omega_o^2 \approx l_i^2,$$

where

$$l_i^2 \approx \frac{l(l+1)c_n^2}{r^2}.$$  

Here $c_n$ is essentially the speed of sound in the neutron fluid, $l$ is the index of the associated spherical harmonic $Y_l^m$ of the mode, and $r$ is the radial distance from the center of the star, and so this is the classic ordinary fluid solution in terms of the Lamb frequency $L_l$ [77]. Likewise, they find another solution of the form

$$\omega_s^2 \approx \frac{m_p}{m_p^*} \frac{l(l+1)}{r^2} c_p^2,$$
where \( c_p \) is roughly the speed of sound in the proton fluid. Thus both solutions are of predominately acoustic nature, but we see that the second solution, which corresponds to the superfluid mode, has a sensitive dependence on entrainment (through the appearance of the proton effective mass). An observational determination of the superfluid mode frequency via gravitational waves, say, could be used to constrain the proton effective mass [36, 78]. This would translate into a deeper understanding of superfluidity at supra-nuclear densities since the effective proton mass is part of the input information for BCS gap calculations [79].

### B. Quasinormal modes in general relativity

We have just seen that the spectrum of mode pulsations in a superfluid neutron star is significantly different from its ordinary fluid counterpart. Moreover, we have also seen that the superfluid modes have a sensitive dependence on entrainment. We will now confirm, and build on, this basic picture in a quantitative way by looking at numerical results for the modes obtained using the general relativistic formalism. The key results to be described come from the work of Comer et al [52] and Andersson et al [36]. It is worth noting that the set of linearized equations that describe the mode oscillations of superfluid neutron stars has much in common with the ordinary fluid set, and so many of the computational and numerical techniques that have been developed for the ordinary fluid [80, 81, 82, 83, 84, 85, 86, 87, 88] can be adapted to the superfluid case. That being said, Andersson et al [36] have developed a new technique for calculating the damping rates of the modes due to gravitational wave emission.

Before we get to the ordinary fluid and superfluid modes, there is another set of modes, called w-modes, that we will discuss that exist only in a general relativistic setting. They were first discovered by Kokkotas and Schutz [89] and are due mostly to oscillations of spacetime, coupling only very weakly to the matter. For instance, Andersson et al [90] use an Inverse Cowling Approximation where all the fluid degrees of freedom are frozen out and were able to find the w-modes. Comer et al [52] have obtained w-modes in the superfluid neutron star case and find that they look very much like those of ordinary fluid neutron stars. As well they do not find a second set of w-modes because of superfluidity. Both results are due to the fact that the w-modes are primarily oscillations of spacetime.
From the previous subsection we expect that there will be no g-modes in the pulsation spectrum (cf. Sec. V where we find two sets of polar perturbations in the zero frequency subspace), but there should be interlaced in it the ordinary and superfluid modes. In general relativity one obtains quasinormal modes because each frequency will have an imaginary part due to dissipation via gravitational wave emission. Such modes correspond to those particular solutions that have no incoming gravitational waves at infinity. In Fig. 7, taken from [36], is graphed the asymptotic amplitude of the incoming wave versus the real part of the mode frequency for model I of Table I. The zeroes of the asymptotic incoming wave amplitude correspond with the deep minima of the figure. As expected, the ordinary and superfluid modes are interlaced in the spectrum and the lowest few have been identified in the figure.

We recall that the ordinary fluid modes should be characterized by the neutrons and protons flowing in “lock-step” whereas the superfluid modes should have the particles in counter-motion. As well, the superfluid modes should have a sensitive dependence on entrainment. Both are confirmed by the next two figures (Figs. 8 and 9 both taken from [36]), which also reveal a phenomenon known as avoided crossings. Fig. 8 graphs the real part of the first few ordinary and superfluid mode frequencies as a function of the entrainment parameter $\epsilon$ (for the “physical” range discussed earlier in Sec. II). The solid lines in the figure are for the ordinary fluid modes, and we see that the first few are essentially flat as the entrainment parameter is varied, but the superfluid modes (the dashed lines) are clearly dependent on the entrainment parameter. Fig. 9 is a graph of the Lagrangian variations in the neutron and proton number densities. The two plots on the far left of the figure are for the modes whose frequencies are labelled by $a_0$ (for the superfluid mode) and $b_0$ (for the ordinary fluid mode) in Fig. 8. The $a_0$ graph in Fig. 9 clearly indicates a counter-motion of the neutrons with respect to the protons, whereas the $b_0$ graph shows the opposite behavior.

Another obvious feature of both figures is the avoided crossings phenomenon. Near the top of Fig. 8 we see that there are points in the $(\epsilon, \text{Re} \omega M)$ plane where the solid and dashed lines approach each other, but just before crossing they diverge away from each other. The most interesting aspect of an avoided crossing is how the mode functions behave before, during, and after the avoided crossing. This is shown in Fig. 9, where the middle and right-hand-side graphs are for the modes of Fig. 8 labelled by $\{a_{0.1}, b_{0.1}\}$ and $\{a_{0.2}, b_{0.2}\}$, respectively. In the middle graphs we see that the modes no longer have a clear distinction as
to whether or not the neutrons and protons are flowing together or in counter-motion. But in the $\{a_{0.2}, b_{0.2}\}$ graphs of Fig. 8 we see that now it is the superfluid modes that have the particles flowing together and the ordinary fluid modes show the counter-motion. Although we will not go into details here, Andersson and Comer [46] and Andersson et al [43] have suggested that this may explain one of the interesting results of Lindblom and Mendell [42] on the effect of mutual friction damping on the r-modes in superfluid neutron stars, and that is that mutual friction damping is negligible for the r-modes except for a small subset of the values of the entrainment parameter $\epsilon$. Mutual friction should be most effective when the neutrons and protons are in counter-motion, as in the superfluid modes, and what may be happening is that as the entrainment parameter is varied it is possible that they find modes beyond an avoided crossing where the ordinary fluid modes take on the characteristic of counter-motion.

C. Detectable Gravitational Wave Signals?

While the study of modes in superfluid neutron stars is a fascinating and complex mathematical and theoretical physics problem, the real hope is that one can use the results to make scientific progress. This is why it is important to understand superfluid neutron star dynamics for realistic astrophysical scenarios, because one wants to know if there are detectable gravitational waves that carry imprints of superfluidity. In other words, one wants to develop a gravitational wave asteroseismology as a probe of neutron star interiors, in much the same way that helioseismology is already a probe of the sun and asteroseismology is a probe of distant stars [71]. This exciting possibility is being discussed [78, 72, 73, 74, 75] in the literature and already some quantitative statements are in hand. Unfortunately, estimates for LIGO II suggest that one needs modes of unrealistic amplitudes to ensure detection. The best possibility is for detection of modes following neutron star formation from gravitational collapse, but even this has to be qualified [72, 73] because of low event rates and uncertainties in the energy that will get deposited in the oscillations. However, there is no reason why one should only be pessimistic, since clearly gravitational wave detection will improve as more experience is gained, and new technology will also lead to improved sensitivities. For instance, there is already the so-called EURO [76] detector being discussed, which is a configuration of several narrow-banded (cryogenic) detectors operating as a “xylophone”
that should allow high sensitivity at high frequencies.

Andersson and Comer \[78\] and Andersson et al \[36\] have taken the estimated spectral noise density for such a configuration and have used it to determine the signal-to-noise for a detection of oscillation modes from superfluid neutron stars that have been excited during a glitch. These estimates have been for two of the best studied glitching pulsars, which are the Crab and Vela pulsars. One assumes that a typical gravitational wave signal from a mode takes the form of a damped sinusoidal, where the damping time is that of the mode itself \[36, 78\]. The amplitude of the signal can be expressed in terms of the total energy radiated through the mode. Andersson and Comer and Andersson et al assume that this total energy is comparable to the amount of energy released during a glitch, which for the Crab and Vela pulsars can be of order $10^{-12} - 10^{-13} M_\odot c^2 \[97, 98\]$. Unfortunately, an error in the signal-to-noise calculation of Andersson and Comer, to be corrected in Andersson et al, has incorrectly estimated the predicted signal-to-noise for the EURO configuration. Fortunately, the revised data still indicate sufficient sensitivity to expect detection for Crab- and Vela-like glitches.

V. THE GENERAL RELATIVISTIC ZERO-FREQUENCY SUBSPACE

Andersson and Comer \[46\] have determined that the zero-frequency subspace in Newtonian theory is spanned by two sets of polar and two sets of axial degenerate perturbations. These solutions are time-independent convective currents. They were stated to be the two missing sets of g-modes and the two sets of r-modes that become non-degenerate when rotation is added. We will extend this analysis to the general relativistic case and show that there are two sets of polar perturbations, thus supporting our earlier claim that there will be no non-zero frequency g-modes in the pulsation spectrum of a general relativistic superfluid neutron star. We will also find two sets of axial perturbations, which could presumably lead to two sets of r-modes (or more general hybrid modes) when the background rotates.
A. The background spacetime and fluid configuration

The background is treated exactly as in Comer et al. \cite{52}, i.e. it is spherically symmetric and static, and thus the metric can be written in the Schwarzschild form
\[ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right).\] (54)
The two conserved currents \(n^\nu\) and \(p^\nu\) are parallel with the timelike Killing vector \(t^\nu = (1, 0, 0, 0)\), and are thus of the form
\[n^\nu = n(r)U^\nu, \quad p^\nu = p(r)U^\nu,\] (55)
where \(U^\nu = t^\nu/|t|\). Likewise, the chemical potential covectors \(\mu_\nu\) and \(\chi_\nu\) become
\[\chi_\nu = \chi(r)U_\nu, \quad \mu_\nu = \mu(r)U_\nu,\] (56)
where \(\mu = Bn + Ap\) and \(\chi = Cp + An\). Explicit solutions for the background configurations can be constructed following the procedure of Comer et al \cite{52}.

B. The linearized fluid and metric variables

Making no assumptions yet on the metric and matter variations, we will first insert the background metric and matter variables into Eqs. (12), (13) and (27). Without too much effort, one finds
\[\delta u^0 = \frac{1}{2}e^{-3\nu/2}\delta g_{00}, \quad \delta u^i = e^{-\nu/2}\frac{\partial}{\partial t}\xi^i_n,\]
\[\delta v^0 = \frac{1}{2}e^{-3\nu/2}\delta g_{00}, \quad \delta v^i = e^{-\nu/2}\frac{\partial}{\partial t}\xi^i_p,\] (57)
for the fluid velocity perturbations,
\[\delta n = -\frac{n}{2} \left(e^{-\lambda}g_{rr} + \frac{1}{r^2} \left[\delta g_{\theta\theta} + \frac{1}{\sin^2\theta}\delta g_{\phi\phi}\right]\right) - \frac{1}{r^2 e^{\lambda/2}} \frac{\partial}{\partial r} \left(n r^2 e^{\lambda/2}\xi^r_n\right) - n \left(\frac{\partial}{\partial \theta}\xi^\theta_n + \frac{\partial}{\partial \phi}\xi^\phi_n + \cot\theta\xi^\theta_n\right),\]
\[\delta p = -p \left(e^{-\lambda}g_{rr} + \frac{1}{r^2} \left[\delta g_{\theta\theta} + \frac{1}{\sin^2\theta}\delta g_{\phi\phi}\right]\right) - \frac{1}{r^2 e^{\lambda/2}} \frac{\partial}{\partial r} \left(p n r^2 e^{\lambda/2}\xi^r_p\right) -\]
\[
p \left( \frac{\partial}{\partial \theta} \xi^\theta_p + \frac{\partial}{\partial \phi} \xi^\phi_p + \cot \theta \xi^\theta_p \right),
\]
for the density variations and
\[
\delta \mu_0 = \frac{1}{2} \mu e^{-\nu/2} \delta g_{00} - e^{\nu/2} \left( B_0^0 \delta n + A_0^0 \delta p \right),
\]
\[
\delta \mu_i = \mu e^{-\nu/2} \delta g_{0i} + e^{-\nu/2} g_{ij} \left( B_n \frac{\partial}{\partial t} \xi^j_n + A_p \frac{\partial}{\partial t} \xi^j_p \right),
\]
\[
\delta \chi_0 = \frac{1}{2} \chi e^{-\nu/2} \delta g_{00} - e^{\nu/2} \left( C_0^0 \delta p + A_0^0 \delta n \right),
\]
\[
\delta \chi_i = \chi e^{-\nu/2} \delta g_{0i} + e^{-\nu/2} g_{ij} \left( C_p \frac{\partial}{\partial t} \xi^j_p + A_n \frac{\partial}{\partial t} \xi^j_n \right),
\]
for the momentum covector variations, where \( A_0^0, B_0^0, \) and \( C_0^0 \) can be calculated from Eq. (28).

One can also show that
\[
\delta \Lambda = -\mu \delta n - \chi \delta p , \quad \delta \Psi = \left( B_0^0 n + A_0^0 p \right) \delta n + \left( C_0^0 p + A_0^0 n \right) \delta p .
\]

Since the four velocity perturbations must be time-independent we see that the Lagrangian displacements must take the form
\[
\xi^i_n = e^{\nu/2} t \delta u^i + \zeta^i_n , \quad \xi^i_p = e^{\nu/2} t \delta v^i + \zeta^i_p ,
\]
where \( \zeta^i_n \) and \( \zeta^i_p \) are “integration constants” (i.e. they are independent of time but depend on the spatial coordinates). We will see below that their only role is that once they have been determined they specify \( \delta n \) and \( \delta p \) via Eq. (58).

Although the Einstein equations must be analyzed using a decomposition in terms of spherical harmonics for the perturbations, we can actually completely solve the linearized Euler equations, which take the form
\[
\partial_t \delta \mu_i = \partial_i \delta \mu_t , \quad \partial_t \delta \chi_i = \partial_i \delta \chi_t .
\]
One can easily verify that the left-hand-sides of each equation is zero (by taking a time derivative of \( \delta \mu_i \) and \( \delta \chi_i \) using Eq. (59)) and it thus follows that
\[
\delta \mu_t = 0 , \quad \delta \chi_t = 0 .
\]
From Eq. (59) we see that these last two equations can be used to find \( \delta n \) and \( \delta p \) in terms of \( \delta g_{00} \). Given that the conservation equations are satisfied automatically by the Lagrangian displacements, then all of the fluid equations have been solved.
We have seen earlier that the question of chemical equilibrium is an important aspect of the types of perturbations that can be induced on a superfluid neutron star. Langlois et al \[33\] have argued that the general condition for chemical equilibrium to exist between the two fluids is that

\[ \beta \equiv v^\nu (\mu_\nu - \chi_\nu) = 0 . \]  

(64)

For perturbations that do not maintain chemical equilibrium then it is the case that \( \delta \beta \neq 0 \). Using the variations above we find

\[ \delta \beta = (A_0^0 - B_0^0) \delta n - (A_0^0 - C_0^0) \delta p . \]  

(65)

We note here that axial perturbations are such that \( \delta n = \delta p = 0 \) and thus it follows that \( \delta \beta = 0 \), that is axial perturbations on spherically symmetric and static backgrounds must necessarily maintain chemical equilibrium between the two fluids if the background fluids were in chemical equilibrium.

C. The Zero-Frequency Subspace

We have exhausted the information that can be extracted from just using the background configuration in the formulas for the variations. To finish mapping out the zero-frequency subspace we must examine the Einstein equations. In doing this it will be very convenient to consider two types of perturbations and those are the polar (or spheroidal) and the axial (or toroidal) perturbations. For the metric perturbations we will use the Regge-Wheeler gauge \[80\], in which the polar components of the metric perturbations can be written as

\[
\delta^P g_{\mu\nu} = \begin{bmatrix}
    e^{\nu(r)} H_0(r) & H_1(r) & 0 & 0 \\
    H_1(r) & e^{\lambda(r)} H_2(r) & 0 & 0 \\
    0 & 0 & r^2 K(r) & 0 \\
    0 & 0 & 0 & r^2 \sin^2 \theta K(r)
\end{bmatrix} Y_l^m(\theta, \phi) ,
\]  

(66)

and the axial components as

\[
\delta^A g_{\mu\nu} = \begin{bmatrix}
    0 & 0 & h_0(r) \left( \frac{\sin \theta}{\sin \phi} \right) \frac{\partial}{\partial \phi} & h_0(r) \sin \theta \frac{\partial}{\partial \phi} \\
    0 & 0 & h_1(r) \left( \frac{\sin \theta}{\sin \phi} \right) \frac{\partial}{\partial \phi} & h_1(r) \sin \theta \frac{\partial}{\partial \phi} \\
    h_0(r) \left( \frac{1}{\sin \phi} \right) \frac{\partial}{\partial \phi} & h_1(r) \left( \frac{1}{\sin \phi} \right) \frac{\partial}{\partial \phi} & 0 & 0 \\
    h_0(r) \sin \theta \frac{\partial}{\partial \theta} & h_1(r) \sin \theta \frac{\partial}{\partial \theta} & 0 & 0
\end{bmatrix} Y_l^m(\theta, \phi) ,
\]  

(67)
where the \( Y_l^m \) are the spherical harmonics. We write the polar unit four-velocity perturbations as

\[
\delta^P u^\mu = e^{-\nu/2} \begin{bmatrix} \frac{1}{r} H_0 \\ \frac{1}{r} W_n(r) \\ \frac{1}{r^2} V_n(r) \frac{\partial}{\partial \theta} \\ \frac{1}{r^2 \sin \theta} V_n(r) \frac{\partial}{\partial \phi} \end{bmatrix} Y_l^m(\theta, \phi), \quad \delta^P v^\mu = e^{-\nu/2} \begin{bmatrix} \frac{1}{r} H_0 \\ \frac{1}{r} W_p(r) \\ \frac{1}{r^2} V_p(r) \frac{\partial}{\partial \theta} \\ \frac{1}{r^2 \sin \theta} V_p(r) \frac{\partial}{\partial \phi} \end{bmatrix} Y_l^m(\theta, \phi),
\]

(68)

and the axial perturbations as

\[
\delta^A u^\mu = \frac{e^{-\nu/2}}{r^2 \sin \theta} \begin{bmatrix} 0 \\ 0 \\ -U_n(r) \frac{\partial}{\partial \phi} \\ U_n(r) \frac{\partial}{\partial \phi} \end{bmatrix} Y_l^m(\theta, \phi), \quad \delta^A v^\mu = \frac{e^{-\nu/2}}{r^2 \sin \theta} \begin{bmatrix} 0 \\ 0 \\ -U_p(r) \frac{\partial}{\partial \phi} \\ U_p(r) \frac{\partial}{\partial \phi} \end{bmatrix} Y_l^m(\theta, \phi).
\]

(69)

Although we have already seen that the particle number density perturbations can be solved in terms of \( \delta g_{00} \), it is worthwhile to comment just a little more on their form. The easy case is that of axial perturbations since for them \( \delta^A g_{00} = 0 \) and so \( \delta^A n \) and \( \delta^A p \) must both vanish for a generic master function. The polar perturbations in the particle number densities are a bit more complicated to determine, but in the end take a simple form. Since \( \delta^P n \) and \( \delta^P p \) must both be time independent, a setting of time derivatives of Eq. (58) to zero yields constraints on the velocity perturbation functions, i.e.

\[
l(l + 1)V_n = \frac{e^{-\lambda/2}}{n} (r n e^{\lambda/2} W_n)' \quad , \quad l(l + 1)V_p = \frac{e^{-\lambda/2}}{p} (r p e^{\lambda/2} W_p)').
\]

(70)

But recall that the Lagrangian displacements still have the “integration constant” terms. However, using the same type of decomposition for \( \zeta^i_{n,p} \) as used for \( \delta^P u^i \) and \( \delta^P v^i \) above, and in place of the \( W_{n,p}(r) \) and \( V_{n,p}(r) \) coefficients we substitute some new coefficients \( A_{n,p}(r) \) and \( B_{n,p}(r) \), say, then it follows that

\[
\delta^P n = \delta n(r) Y_l^m \quad , \quad \delta^P p = \delta p(r) Y_l^m,
\]

(71)

where the \( \delta n(r) \) and \( \delta p(r) \) are linear combinations of \( A_{n,p}(r) \) and \( B_{n,p}(r) \) and their derivatives. These new coefficients \( A_{n,p}(r) \) and \( B_{n,p}(r) \) appear nowhere else but in \( \delta^P n \) and \( \delta^P p \) and thus this is why we stated earlier that the only role of the “integration constants” is to determine the particle number density perturbations. Of course at this point we can forgo using \( A_{n,p}(r) \) and \( B_{n,p}(r) \) and just solve for the \( \delta n(r) \) and \( \delta p(r) \) instead.
After some algebra it can be shown that the perturbations for \( l \geq 2 \) result in three distinct groups of the linearized Einstein and superfluid field equations, and these are

(i) \( l \geq 1 \) Group I:

\[
0 = e^{-\lambda} r^2 K'' + e^{-\lambda} \left( 3 - \frac{r \lambda'}{2} \right) r K' - \left( \frac{l(l+1)}{2} - 1 \right) K - e^{-\lambda} r H_0' - \left( \frac{l(l+1)}{2} - 1 - 8\pi r^2 \psi \right) H_0
\]

\[
+ 4\pi r^2 \left( \left[ 3\chi - nA_0^0 - pC_0^0 \right] \delta p + \left[ 3\mu - pA_0^0 - nB_0^0 \right] \delta n \right) ,
\]

\[
0 = e^{-\lambda} \left( 1 + \frac{r \nu'}{2} \right) r K' - \left( \frac{l(l+1)}{2} - 1 \right) K - e^{-\lambda} r H_0' + \left( \frac{l(l+1)}{2} - 1 - 8\pi r^2 \psi \right) H_0
\]

\[
+ 4\pi r^2 \left( \left[ \chi - 3nA_0^0 - 3pC_0^0 \right] \delta p + \left[ \mu - 3pA_0^0 - 3nB_0^0 \right] \delta n \right) ,
\]

\[
0 = e^{-\lambda} r^2 K'' + e^{-\lambda} \left( \frac{r(\nu' - \lambda')}{2} + 2 \right) r K' - 16\pi r^2 \left( \left[ nA_0^0 + pC_0^0 \right] \delta p + \left[ pA_0^0 + nB_0^0 \right] \delta n \right)
\]

\[
- e^{-\lambda} r^2 H_0'' - e^{-\lambda} \left( \frac{r(3\nu' - \lambda')}{2} + 2 \right) r H_0' - 16\pi r^2 \psi H_0 ,
\]

\[ H_2 = H_0 , \]

\[ K' = e^{-\nu} (e^\nu H_0)' , \]

\[ \frac{\mu}{2} H_0 = A_0^0 \delta p + B_0^0 \delta n , \]

\[ \frac{\chi}{2} H_0 = A_0^0 \delta n + C_0^0 \delta p . \]  \hspace{1cm} (72)

(ii) \( l \geq 2 \) Group II:

\[
0 = H_1 + \frac{16\pi r e^\lambda}{l(l+1)} \left( \mu n W_n + \chi p W_p \right) ,
\]

\[
0 = l(l+1)V_n - \frac{e^{-\lambda/2}}{n} \left( r n e^{\lambda/2} W_n \right)' ,
\]

\[
0 = l(l+1)V_p - \frac{e^{-\lambda/2}}{p} \left( r p e^{\lambda/2} W_p \right)' ,
\]
\[ 0 = e^{-(\nu-\lambda)/2} \left( e^{(\nu-\lambda)/2} H_1 \right)' + 16\pi e^\lambda (\mu n V_n + \chi p V_p) . \] (73)

(iii) \( l \geq 2 \) Group III:

\[ 0 = h_0'' - \frac{\nu' + \lambda'}{2} h_0' + \left( \frac{2 - l^2 - l}{r^2} e^\lambda - \frac{\nu' + \lambda'}{r} - \frac{2}{r^2} \right) h_0 - 16\pi e^\lambda (\chi p U_p + \mu n U_n) , \]

\[ 0 = (l - 1)(l + 2) h_1 , \]

\[ 0 = e^{-(\nu-\lambda)/2} \left( e^{(\nu-\lambda)/2} h_1 \right)' . \] (74)

A quick counting of the number of independent functions, and the number of equations, shows that Group I appears to have more equations than unknowns. However, such a result is not unexpected because of the Bianchi identities. For the present discussion, the important point about the Group I equations is that their solutions represent a subset that map static and spherically symmetric stars, with no mass currents, into other (nearby) static and spherically symmetric stars, also having no mass currents. More interesting counting comes from the Group II and III equations. For Group II one can show that the last equation in the group is a consequence of the other three. Thus, we can specify arbitrarily \( W_n \) and \( W_p \), for instance, and then the other variables \( (H_1, V_n, \text{ and } V_p) \) can be determined from the field equations. Likewise, for Group III we can specify freely \( U_n \) and \( U_p \), and then the remaining variable \( h_0 \) is determined from its field equation (because it is clear that \( h_1 = 0 \)).

For various reasons, the cases of \( l = 0, 1 \) must be distinguished from that of \( l \geq 2 \). One reason is that there is more gauge freedom, which allows us to set \( K(r) = h_1(r) = 0 \) for \( l = 1 \) and in addition \( H_1(r) = 0 \) for \( l = 0 \) (which also has no axial perturbations). Without listing all the formulas, we find that the counting of the number of equations and unknown functions is similar to \( l \geq 2 \). In particular, the \( l = 1 \) analysis of the Groups II and III equations reveals that each have two arbitrary functions that must be specified before a solution can be obtained. For \( l = 0 \), the polar currents must also vanish (otherwise they would diverge at the center of the star). Hence, there are only \( l = 0 \) solutions that map static and spherically symmetric stars into other static and spherically symmetric stars.
D. Decomposition of the zero-frequency subspace

Regardless of the form of the equation of state for the background, or for the perturbations, we can make the following conclusions: Any solution

\[ \{H_0, H_1, H_2, K, h_0, W_n, W_p, V_n, V_p, U_n, U_p, \delta n, \delta p\} \]  

(75)
to the equations governing the time-independent perturbations of a static, spherical superfluid neutron star is a superposition of (i) a solution

\[ \{H_0, 0, H_2, K, 0, 0, 0, 0, 0, 0, \delta n, \delta p\} \]  

(76)
and (ii) a solution

\[ \{0, H_1, 0, 0, h_0, W_n, W_p, V_n, V_p, U_n, U_p, 0, 0\} \] .  

(77)
The solutions in (i) are those that satisfy the Group I equations that map one static and spherically symmetric star to another (nearby) static and spherically symmetric star. The solutions in (ii) are those that satisfy the Group II and III equations. It is not difficult to see that the solutions (ii) contain two sub-classes, the purely polar solutions that satisfy the Group II equations, i.e.

\[ \{0, H_1, 0, 0, 0, W_n, W_p, V_n, V_p, 0, 0, 0, 0\} \]  

(78)
and the purely axial solutions that satisfy the Group III equations, i.e.

\[ \{0, 0, 0, 0, h_0, 0, 0, 0, 0, U_n, U_p, 0, 0\} \] .  

(79)
The first subclass is made of the g-modes because they are (1) purely polar and (2) the particle number densities (and likewise the energy density and pressure) vanish. The second subclass is made of the r-modes because they are (1) purely axial and (2) the particle number densities (and likewise the energy density and pressure) vanish. The final conclusion is that the zero-frequency subspace of superfluid neutron stars is spanned by two sets of g-modes and two sets of r-modes. This is qualitatively the same conclusion as found for the Newtonian equations [48], and we assert that there will be no non-zero frequency g-modes in the pulsation spectrum of non-rotating superfluid neutron stars. As for the axial perturbations, presumably rotation will break the degeneracy and we will find two sets of r-modes (or more general hybrid modes [70, 71]), as in the Newtonian superfluid case, but this must be verified.
VI. CONCLUDING REMARKS

We have reviewed recent work to model the rotation and oscillation dynamics of Newtonian and general relativistic superfluid neutron stars. We have seen that superfluidity affects both the background and the perturbation spectrum of neutron stars and both should therefore cause an imprint of superfluidity to be placed in the star’s gravitational waves. In particular our local analysis of the Newtonian mode equations indicates that the superfluid modes should have a sensitive dependence on entrainment parameters, something that is supported by the quasinormal mode calculations. Given a suitably advanced detector, like the EURO configuration, we have seen that gravitational waves emitted during a Vela glitch, say, should be detectable. The key conclusion is that direct detection of gravitational waves from glitching pulsars can be used to greatly improve our understanding of the local state of matter in superfluid neutron stars. This may become more important in the next few years because of indications of free precession in neutron stars [99], which if true means that the vortex creep model will have to be reconsidered.

We have also put in place some groundwork for a future analysis of the CFS mechanism in superfluid neutron stars. We have done this by using the “pull-back” formalism to motivate fluid variations in terms of constrained Lagrangian displacements. We have also used them as perturbations to help map out the zero frequency subspace, and found that it is spanned by two sets of polar and two sets of axial perturbations. It remains to be seen if adding rotation will lift the degeneracy and yield two sets of polar and two sets of axial oscillation modes, or if the more general inertial hybrid modes result.

Finally, I would like to elaborate a little more on why I continue to treasure my two years in Israel with Jacob Bekenstein. After I had completed working on the superfluid analogs of quantum field theory in curved spacetime effects I tried in vain to publish the work. True to his character, Jacob had some very kind words of advise, which were to never worry that effort is lost when a project does not play out exactly as expected, because his own experience was that one, or many, pieces of it would eventually be of direct importance for something else. For a young scientist, those were words of comfort and hope, and remain so for one that now has a little more experience.
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Figure Captions

Figure 1. The radial profiles of the neutron and proton background particle number densities, \( n \) and \( p \), respectively, for model II. The model has been constructed such that it accords well with a \( 1.4M_\odot \) neutron star determined using the modern equation of state calculated by Akmal, Pandharipande and Ravenhall [54]. For reference, we show as horizontal lines the number densities at which Akmal et al suggest that i) neutron drip occurs, ii) there is an equal number of nuclei and neutron gas, and iii) the crust/core interface is located. It should be noted that the latter should not coincide with our core/envelope interface since one would expect there to be a region where crust nuclei are penetrated by a neutron superfluid.

Figure 2. Comparison of one-fluid slow-rotation configurations with numerical results obtained using the LORENE code. \( R_{\text{equ}} \) and \( R_{\text{pole}} \) are the star’s equatorial and polar radius respectively. The stellar model is a \( N = 1 \) polytrope with mass \( M = 1.4M_\odot \) and (static) radius \( R = 10 \) km.

Figure 3. Typical results for the frame-dragging \( \omega \) for superfluid stars in which the neutrons and the protons rotate at slightly different rates.

Figure 4. The frame dragging in an extreme (rather unphysical) situation where the neutrons and the protons counterrotate in such a way that the net frame dragging is “backwards” at the surface of the star but “forwards” in the central parts.

Figure 5. The Kepler limit \( \Omega_K \) is shown as a function of the relative rotation rate \( \Omega_n/\Omega_p \) for our model star. The filled squares show the maximum allowed rotation rate for the protons \( (\Omega_p) \), while the open squares show the corresponding neutron spin rate \( (\Omega_n) \). The Kepler frequency simply corresponds to the largest of the two.

Figure 6. Plots of the neutron (n) and proton (p) Kepler limits as functions of the relative rotation \( \Omega_n/\Omega_p \), for \( \sigma = -0.5, 0, 0.5 \) and \( \varepsilon = 0, 0.4, 0.7 \).

Figure 7. This figure shows the asymptotic amplitude \( A_{\text{in}} \) as a function of the (real) frequency \( \omega M \) for our model I. The slowly-damped QNMs of the star show up as zeros of \( A_{\text{in}} \), i.e. deep minima in the figure. The first few “ordinary” and “superfluid” modes are identified in the figure.
Figure 8. This figure shows how the frequencies of the fluid pulsation modes for our model II vary with the entrainment parameter $\epsilon$. The modes shown as solid lines are such that the two fluids are essentially comoving in the $\epsilon \to 0$ limit, while the modes shown as dashed lines are countermoving. As is apparent from the data, the higher order modes exhibit avoided crossings as $\epsilon$ varies. Recall that the range often taken as “physically relevant” is $0.04 \leq \epsilon \leq 0.2$. We indicate by $a_\epsilon$ and $b_\epsilon$ the particular modes for which the eigenfunctions are shown in Fig. 9.

Figure 9. An illustration of the fact that the modes exchange properties during an avoided crossing. We consider two modes, labelled by $a_\epsilon$ and $b_\epsilon$ (cf. Figure 8). The mode eigenfunctions are represented by the two Lagrangian number density variations, $\Delta n$ and $\Delta p$ (solid and dashed lines, respectively). For mode $a$ the two fluids are essentially countermoving in the $\epsilon \to 0$ limit (it is a superfluid mode), while the two fluids comove for mode $b$ (it is similar to a standard p-mode). After the avoided crossing (which takes place roughly at $\epsilon = 0.1$) the two modes have exchanged properties.
Table Captions

Table 1. Parameters describing our stellar models I, II and III. Model I is identical to model 2 of [52], and has only a core with no envelope. Model II has an envelope of roughly 1 km and could be seen as a slightly more realistic neutron star model. Model III has no distinct envelope but does have a more realistic mass and radius than Model I.
Figures
FIG. 1:

Number density \( n(r) \) and pressure \( p(r) \) as functions of \( r/R \). The crust-core interface and equal volume of nuclei and neutron gas are indicated. The neutron drip is shown as well.
**FIG. 2:**

Rotation frequency [Hz]

| 0  | 2  | 4  | 6  | 8  | 10 | 12 | 14 |
|----|----|----|----|----|----|----|----|
| 0  | 0  | 2  | 4  | 6  | 8  | 10 | 12 |

Radius [km]

- $R_{\text{equ}}$ LORENE
- $R_{\text{pole}}$ LORENE
- $R_{\text{equ}}$ Slow Rotation
- $R_{\text{pole}}$ Slow Rotation

Kepler limit

fastest Pulsar (1.6 ms)
FIG. 3:

\[
\frac{\omega}{\Omega_p} \quad \text{vs} \quad \frac{r}{R}
\]

- \(\frac{\Omega_n}{\Omega_p} = 1.25\) (dotted line)
- \(\frac{\Omega_n}{\Omega_p} = 1\) (solid line)
- \(\frac{\Omega_n}{\Omega_p} = 0.75\) (dashed line)

The figure shows the variation of \(\frac{\omega}{\Omega_p}\) with \(\frac{r}{R}\) for different ratios of \(\Omega_n \) to \(\Omega_p\).
FIG. 4:

\[
\frac{\Omega_n}{\Omega_p} = -0.08
\]
FIG. 5:

\[ \frac{\Omega_k}{2\pi} (\text{Hz}) \]

\[ \frac{\Omega_n}{\Omega_p} \]
FIG. 6:
FIG. 7:
FIG. 8:
Tables
# TABLE I:

|                  | model I | model II | model III |
|------------------|---------|----------|-----------|
| $\sigma_n/m_n$   | 0.2     | 0.22     | 0.2       |
| $\sigma_p/m_n$   | 0.5     | 1.95     | 2         |
| $\beta_n$        | 2.5     | 2.01     | 2.3       |
| $\beta_p$        | 2.0     | 2.38     | 1.95      |
| $n_c$ (fm$^{-3}$) | 1.3     | 1.21     | 0.93      |
| $p_c$ (fm$^{-3}$) | 0.741   | 0.22     | 0.095     |
| $M/M_\odot$      | 1.355   | 1.37     | 1.409     |
| $R$ (km)         | 7.92    | 10.19    | 10.076    |
| $R_c$ (km)       | —       | 8.90     | —         |