Controller-jammer game models of Denial of Service in
control systems operating over packet-dropping links

V. Ugrinovskii\textsuperscript{a} and C. Langbort\textsuperscript{b}

\textsuperscript{a}School of Engineering and IT, University of NSW at the Australian Defence Force Academy, Canberra, ACT, 2600, Australia.
\textsuperscript{b}Department of Aerospace Engineering, University of Illinois at Urbana-Champaign.

Abstract

The paper introduces a class of zero-sum games between the adversary and controller as a scenario for a ‘denial of service’ in a networked control system. The communication link is modeled as a set of transmission regimes controlled by a strategic jammer whose intention is to wage an attack on the plant by choosing a most damaging regime-switching strategy. We demonstrate that even in the one-step case, the introduced games admit a saddle-point equilibrium, at which the jammer’s optimal policy is to randomize in a region of the plant’s state space, thus requiring the controller to undertake a nontrivial response which is different from what one would expect in a standard stochastic control problem over a packet dropping link. The paper derives conditions for the introduced games to have such a saddle-point equilibrium. Furthermore, we show that in more general multi-stage games, these conditions provide ‘greedy’ jamming strategies for the adversary.

Key words: Adversarial zero-sum games, control over adversarial channels, security of control systems, control over packet-dropping links.

1 Introduction and Motivation

The topic of control over a communication link has been extensively studied in the past decade, with issues such as the minimum data rate for stabilization \cite{1,2} and optimal quadratic closed-loop performance \cite{3,4} being the main focus. Other issues of interest concern effects of channel-induced packet loss and/or time-varying delays on closed-loop performance.

The majority of papers concerned with control over networks regards the mechanism of information loss in the network as probabilistic but not strategic. In contrast, in the \textit{adversarial} networked control problem, the communication link is controlled by a strategic jammer who actively modifies the link to disrupt the control goal. We broadly refer to such modifications/disruptions as Denial of Service attacks; cf. \cite{14}.

\textsuperscript{*}This work was supported by the Australian Research Council under Discovery Projects funding scheme (project DP120102152) and, in parts, by US Air Force Office of Scientific Research (AFOSR) under grant number MURI FA9550-10-1-0573, and the US National Science Foundation under award #1151076.

\textit{Email addresses:} v.ugrinovskii@gmail.com (V. Ugrinovskii), langbort@illinois.edu (C. Langbort).

A natural way to describe the adversarial networked control problem is to employ a game-theoretic formulation. While the most immediate purpose of the game-theoretic analysis may be to devise the best defence against strategic attacks, it can also be used to predict possible attack strategies. Originally proposed in \cite{5}, the game-theoretic formulation of adversarial networked control has been followed upon in a number of recent papers including \cite{6,10}. A zero-sum dynamic game between a controller performing a finite horizon linear-quadratic control task and a jammer, proposed in \cite{6}, specifically accounted for the jammer’s strategic intentions and limited actuation capabilities. A startling conclusion of \cite{6} was that in order to maximally disrupt the control task, the jammer had to act in a markedly different way than a legitimate, non-malicious, packet-dropping channel. More precisely, the jammer’s saddle-point strategy was to deterministically drop packets whenever the plant state was crossing certain thresholds. Once this deterministic behavior is observed by the controller, it can establish with certainty that an attack has taken place.

The chief motivation behind the present work is to investigate whether it is possible in principle for the attacker to be able to conceal its actions by disguising as a packet dropping link. Naturally, this implies that the controller has no prior information of the attack taking
place. In [10], to demonstrate such a possibility, we introduced a model of adversarial networked control (ANC), which, while capturing the same fundamental aspects of the problem as in [6], modified the jammer’s action space so that each jammer’s decision corresponded to a choice of a binary automaton governing the transmission rather than to passing/blocking transmission directly. The corresponding one-step zero-sum game was shown to have a unique saddle point in the space of mixed jammer’s strategies. The optimal jammer’s strategy was shown to randomly choose between two automata, each having a nonzero probability to be selected. In turn, the controller’s best response to the jammer’s optimal strategy was to act as if it was operating over a packet-dropping channel whose statistical characteristics were controlled by the jammer. Since under normal circumstances the controller cannot be aware of these characteristics, and cannot implement such a best response strategy, the system performance is likely to be adversely affected when the jammer follows its optimal strategy; see Section 4.1.

In this paper, we show that such a situation is not specific to the ANC problem considered in [10], and it arises in a much more general zero-sum stochastic game setting. The only common feature between our problem formulation and that in [10] is the general mechanism of decision making adopted by the jammer. All other attributes of the problem (the plant model, the assumptions on the stage cost, etc.) are substantially more general, to the extent that unlike [10], the saddle point cannot be computed directly. Instead, for the one-step game, we obtain sufficient conditions under which optimal jammer’s strategies in the class of mixed strategies are to randomly choose between two actions. That is, to make a maximum impact on the control performance, the jammer must act randomly, in contrast to [6].

Our conditions for the one-step game are quite general, they apply to nonlinear systems and draw on standard convexity/coercivity properties of payoff functions. Under additional smoothness conditions, these conditions are also necessary and sufficient. Also, we specialize these conditions to three linear-quadratic control problems over a packet-dropping link. In two of these problems, our conditions allow for a direct characterization of a set of plant’s initial states for which optimal jammer’s randomized strategies exist. We also compute controller’s optimal responses to those strategies, which turn out to be nonlinear. The third example revisits the problem setting in [6], showing that our conditions naturally rule out randomized jammer’s behaviour in that problem.

Our analysis of the one-step game can be thought of as reflecting a more general situation where one is dealing with a one-step Hamilton-Jacobi-Bellman-Isaacs (HJBI) min-max problem associated with a multi-step ANC problem. Also, even the one-step formulation provides a rich insight into a possible scenario of attacks on controller networks. For instance, dynamic multistep jamming attacks can be planned so that at each step the jammer chooses its actions based on the proposed formulation. Greedy jamming strategies where at each time step the jammer pursues a strategy which is optimal only at this particular time are discussed in Section 5.

Compared with the conference version [11], here we have obtained a new sufficient condition for the one-step ANC game to have a nontrivial equilibrium. This condition does not require the stage cost to be a smooth function, which potentially makes this result applicable to the mentioned one-step HJBI min-max problems where in general the smoothness of the value function cannot be guaranteed in advance; see Theorem 4. Also, the paper introduces a multi-stage game model to study intelligent jamming attacks on linear control systems.

The paper is organized as follows. A general controller-jammer ANC game and its connections with intelligent jamming models are presented in Section 2. The conditions for the one-step ANC game to have a nonpure saddle point are presented in Section 3. First we present a general sufficient condition suitable for analysis of general multi-input networked control systems. We then show that in the case of single-input systems they are in fact necessary and sufficient (under an additional smoothness assumption). Next, in Section 4 we demonstrate applications of these results to three linear-quadratic static problems. In one of these problems, which is an extension of the problem in [10], the jammer is offered an additional reward for undertaking actions concealing its presence. In another problem, the jammer’s actions take into account the cost the controller must pay to mitigate the jammer’s presence should the jammer reveal itself. The third problem revisits the problem in [6]. Section 5 generalizes some of the results of Section 3 about the existence of saddle-point strategies to the case of multistage ANC games; this generalization requires the plant to be linear. The Appendix (Section 7) contains proofs of the results. Conclusions are given in Section 6.

Notation \( \mathbb{R}^n \) is the \( n \)-dimensional Euclidean space of real vectors, \( \mathbb{R}^n_+ \) is the cone in \( \mathbb{R}^n \) consisting of vectors whose all components are non-negative. The unit simplex in \( \mathbb{R}^n \) is denoted \( \mathcal{F}_N \); i.e., \( \mathcal{F}_N = \{ p \in \mathbb{R}^n : 0 \leq p_j \leq 1, \sum_{j=1}^N p_j = 1 \} \). \( |\mathcal{F}| \) is the cardinality of a finite set \( \mathcal{F} \). \( \delta_{i,j} \) is the Kronecker symbol, i.e., \( \delta_{i,j} = 1 \) if \( i = j \), otherwise \( \delta_{i,j} = 0 \). For two sets \( M, U \), \( M \cup U = \{ x \in M : x \notin U \} \). The symbol \( \Pr(A|z) \) denotes the conditional probability of an event, and \( \Pr(A|z) \) denotes the conditional probability of an event \( A \) given \( z \).

2 Adversarial Network Control Games

2.1 The general system setup

We consider a general setup, within which the evolution of a plant controlled over a communication link subject
to adversarial interference is described by a mapping \( \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \),
\[
x^{t+1} = F_t(x^t, v^t),
\]
describing the response of the plant, when it is at state \( x^t \), to an actuator signal \( v^t \) at time \( t \). The actuator signal is determined by the controller who transmits control packets over a randomly varying communication link.

The link can operate in one of several transmission regimes, and randomly switches between them. The mechanism of the regime switching is controlled by a strategic jammer, unbeknownst to the controller. For simplicity, we identify the set of transmission regimes with the set of integers, so that \( \mathcal{F} = \{1, \ldots, |\mathcal{F}|\} \). Once the link updates its transmission regime, it randomly assumes either the passing or blocking state. Thus, the transmission state of the communication link is a binary random variable \( b_t \), taking values 0 and 1 associated with blocking and passing control packets, and \( v^t = b^t u^t \).

Evolution of the system over a time interval \( \{0, \ldots, T\} \) is captured by a bivariate stochastic process \( \{ (x^t, s^t) \}_{t=0}^{T} \), with the state space \( \mathcal{X} = \mathbb{R}^n \times \mathcal{F} \). We now describe the dynamics of this bivariate process.

**The initial state.** Initially, at time \( t = 0 \) the plant is at a state \( x^0 = x_0 \), and the initial transmission regime of the communication link is \( s^0 = s \). It is assumed that at the time when the controller and the attacker make their decisions, both of them know the current state of the plant and the transmission regime of the link. In accordance with this assumption, they know \( x^0, s^0 \).

**State updates.** The system state and the transmission regime of the communication link are updated at every time \( t \), in response to controller’s and jammer’s actions, as described below.

**Controller actions.** At every time instant \( t = 0, \ldots, T-1 \), the controller observes the current system state \( x^t \) and the transmission regime \( s^t \) of the communication link. Based on this information, it generates a control input \( u^t \in \mathbb{R}^m \) and sends it via the communication link.

**Jammer actions.** At every time step \( t = 0, \ldots, T-1 \), the jammer observes the plant state \( x^t \), the transmission regime of the communication link \( s^t \), and the control signal \( u^t \). Using this information, the jammer selects a matrix from a predefined finite set of row stochastic matrices \( \mathcal{A} = \{ P^t(a) \in \mathbb{R}^{|\mathcal{F}| \times |\mathcal{F}|} : a \in \mathcal{A} = \{1, \ldots, N\} \} \). That is, its action \( a^t \) is to draw a matrix \( P^t(a^t) \in \mathcal{A} \).

**Communication link update.** After the jammer has chosen its action \( a^t = a \) and the corresponding row stochastic matrix \( P^t(a) \), the transmission regime of the link changes from \( s^t \) to \( s^{t+1} \) according to a Markov chain with \( \mathcal{F} \) as a state space and \( P^t(a^t) = P^t(a) \) as the transition probability matrix, i.e.,
\[
\Pr(s^{t+1} = i | s^t = j, a^t = a) = P_{ji}^t(a) \quad i, j \in \mathcal{F}.
\]
The initial probability distribution of this Markov chain reflects that the initial transmission regime is given to be \( s^0 = s \), i.e., \( \Pr(s^0 = i) = \delta_{i,s}, i \in \mathcal{F} \).

After the communication link randomly switches to the transmission regime \( s^{t+1} \in \mathcal{F} \), the transmission state of the link is determined randomly and, given \( s^{t+1} \), it is conditionally independent of the previous transmission regimes, the current and past states of the plant and the controller’s and jammer’s actions. Thus, the probability of the link to become passing is determined by a stochastic kernel \( q^t \) on \( \{0,1\} \) given \( \mathcal{F} \):
\[
q_{ij}^t = q^t(b^t = 1 | j) \hat{=} \Pr(b^t = 1 | s^{t+1} = j), \quad j \in \mathcal{F}.
\]
We denote \( q^t \hat{=} (q_{1}^t, \ldots, q_{|\mathcal{F}|}^t) \). It is worth noting that the pair of processes \( (s^{t+1}, b^t) \) form a controlled hidden Markov process, with \( \mathcal{F} \) and \( \{0,1\} \) as its state space and the output alphabet, respectively, \( \{\delta_{s^0,j}, j \in \mathcal{F}\} \) as its initial probability distribution, the sequence of matrices \( \{P^i(a^i)^{T-1}\}_{i=0}^{T-2} \) as the sequence of state transition probability matrices, and the the sequence of stochastic kernels \( \{q^i\}_{i=0}^{T-2} \) on \( \{0,1\} \) given \( \mathcal{F} \) as the output probabilities. This hidden Markov process is denoted \( \mathcal{M} \).

**Plant state updates.** Next, the plant state is updated according to (1), in response to control \( u^t \) and the jammer’s action \( a^t \). Specifically, the transmission state of the link between the controller and the plant is determined by the binary random variable \( b^t \), whose probability distribution given \( (x^t, s^t) \), \( u^t \) and \( a^t \) can now be expressed as
\[
Pr(b^t = 1 | s^t = j, a^t = a, x^t = x, u^t = u) = (P^t(a) q^t), j (4)
Pr(b^t = 0 | s^t = j, a^t = a) = 1 - (P^t(a) q^t). \quad (5)
\]
Then the actuator signal \( v^t \) becomes \( v^t = b^t u^t \), and according to (1), the plant’s new state becomes \( x^{t+1} \).

**Proposition 1** Given a sequence of controller’s and jammer’s actions, the joint state-link process \( \{ x^t, s^t \}_{t=0}^{T} \) is a Markov process.

The proof of this and all subsequent propositions is deferred to the Appendix.

The transition probability measure of the Markov process \( \{ x^t, s^t \}_{t=0}^{T} \) defines a sequence of stochastic kernels on \( \mathcal{F} \), given \( u \in \mathbb{R}^m \), and \( a \in \mathcal{A} \),
\[
Q_{t+1}(A \times S | x, j, u, a) = \Pr(x^{t+1} \in A, s^{t+1} \in S | x^t = x, s^t = j, u^t = u, a^t = a) = \sum_{i \in \mathcal{S}} P_{ji}^t(a) (q_i^t \chi_A(F_i(x, u)) + (1 - q_i^t) \chi_A(F_i(x, 0))) \quad (6)
\]
for $t = 0, \ldots, T - 1$; here $\Lambda, S$ are Borel subsets of $\mathbb{R}^n$, $\mathcal{F}$ respectively, and $\chi_A(\cdot)$ is the indicator function of the set $A$. Let $\Xi$ be an action space of the controller, and the jammer, respectively.

The sequence of stochastic kernels (6) on $\mathcal{X}$ given $\mathcal{X} \times \mathbb{R}^n \times \mathcal{A}$ and the Markov process $\{(x^t, s^t)\}_{t=0}^{T}$ associated with given sequences of controller’s and jammer’s actions.

A real-valued finite horizon cost

$$\mathcal{Y} \left( \{x^t\}_{t=0}^{T}, \{u^t\}_{t=0}^{T-1} \right) = \sum_{t=0}^{T-1} \sigma^t(x^t, u^t) + \sigma^T(x^T). \quad (7)$$

The stage cost $\sigma^t(x^t, u^t)$ in (7) symbolizes the performance loss incurred by the system (1) when the control $u^t$ is applied to the system while it is in state $x^t$. In the absence of the jammer, the controller would be expected to minimize this loss.

A real-valued function representing the cost of jammer’s actions $a^t$, $t = 0, \ldots, T$:

$$\Gamma \left( \{a^t\}_{t=0}^{T-1}, \{s^t\}_{t=0}^{T-1} \right) = \sum_{t=0}^{T-1} g^t(a^t, s^t). \quad (8)$$

It describes how the jammer’s resources and incentives are affected, when it takes its actions to control the communication link. The stage cost $g^t(a^t, s^t)$ in (8) symbolizes the cost incurred by the jammer when at time $t$ it selects $P^t(a^t)$ as a state transition probability matrix for the Markov process $\mathcal{M}$. Normally, the jammer would be expected to minimize this cost.

A total cost of the controller’s and jammer’s actions

$$\Sigma \left( \{x^t\}_{t=0}^{T}, \{u^t\}_{t=0}^{T-1}, \{a^t\}_{t=0}^{T-1} \right) = \mathcal{Y} \left( \{x^t\}_{t=0}^{T}, \{u^t\}_{t=0}^{T-1} \right) - \Gamma \left( \{a^t\}_{t=0}^{T-1}, \{s^t\}_{t=0}^{T-1} \right). \quad (9)$$

The minus sign in (9) indicates that degrading the control performance comes at a cost to the jammer, who must spend its resources to achieve its goals.

In what follows we consider a class of control input sequences $\{u^t\}_{t=0}^{T}$ and a class of jammer’s strategies consisting of sequences $\{p^t\}_{t=0}^{T-1}$ of probability vectors $p^t$ over $\mathcal{A}$. Associated with each such sequence of the controller’s and jammer’s strategies $\{u^t\}_{t=0}^{T-1}, \{p^t\}_{t=0}^{T-1}$ and system and link initial conditions $x^0, s^0$ is a probability distribution on the space of sequences of system and communication link states endowed with the $\sigma$-algebra of all Borel subsets $\mathcal{F}$.

$Q^u\cdot(dx_1, \ldots, dx_T, s_1, \ldots, s_T|x_0, s_0)$

$= Q^u\cdot(dx_1, s_1|x_0, s_0) \cdot \cdots \cdot Q^u\cdot(dx_T, s_T|x_{T-1}, s_{T-1})$, \quad (10)

generated by the sequence of stochastic kernels

$$Q^p_{T+1}(dx, s|x, j) = \sum_{a \in \mathcal{A}} p^T_a Q_{T+1}(dx, s|x, j, u^T, a), \quad (11)$$

where $Q_t$ is the stochastic kernel defined in (6). The expectation with respect to the probability measure $Q^u\cdot$ will be denoted $E^u\cdot$. We will omit the superscripts $u\cdot$ when this does not lead to a confusion.

**Definition 1** The zero-sum stochastic game with the state space $\mathcal{X}$, action spaces $\mathbb{R}^n$ for the controller and $\mathcal{A}$ for the jammer, the stochastic kernels $\{Q^u, t = 0, \ldots, T\}$ defined in (6) and the payoff function $\Sigma$ is called the adversarial networked control (ANC) game associated with the Markov process $\{(x^t, s^t)\}_{t=0}^{T}$ and costs (7), (8), (9).

For a given initial condition $x^0 = x$ and an initial transmission regime of the link $s^0 = s$, solving the ANC game requires one to find (if they exist) strategies for the jammer $p^* = \{(p^*)^t\}_{t=0}^{T-1}$ and for the controller $u^* = \{(u^*)^t\}_{t=0}^{T-1}$ such that

$$E^{u\cdot, p^*} \left[ \Sigma \left( \{x^t\}_{t=0}^{T}, \{(u^*)^t\}_{t=0}^{T-1}, \{a^t\}_{t=0}^{T-1} \right) \right] = \inf_{\{u^t\}_{t=0}^{T-1}} \sup_{\{p^t\}_{t=0}^{T-1}} E^{u\cdot, p^*} \left[ \Sigma \left( \{x^t\}_{t=0}^{T}, \{u^t\}_{t=0}^{T-1}, \{a^t\}_{t=0}^{T-1} \right) \right] \quad (12)$$

**2.3 Additional comments and special cases**

**2.3.1 Relation to conventional DoS attacks on communications**

In the model presented in Section 2.1, the adversary controls information transmission strategically and indirectly, unbeknownst to the controller. The motivation for introducing the indirect jamming model is to reduce the possibility of attack detection by adding intelligence to jamming strategy and attempting to force the controller to believe that performance degradation is due to poor link conditions and not due to the presence of the attack, i.e., our model is consistent with objectives of intelligent Denial-of-Service jamming [14]. We now present two intelligent jamming scenarios which our model captures.

---

1. In (10), $x_t$ is an arbitrary vector in $\mathbb{R}^n$, in contrast to the state $x^t$ of the plant at time $t$. Likewise, $s_t$ is an integer in $\mathcal{F}$, in contrast to the random state $s^t$ of the Markov process $\mathcal{M}$ with values in $\mathcal{F}$.
An attack by re-routing. In this scenario, the controller and the plant are connected by a multi-hop network (i.e., a graph), with each transmission regime corresponding to a path connecting the controller and the plant vertices in that graph. Assuming that the graph is acyclic, the set of such paths has finite (yet possibly large) cardinality, so it can be modeled by our set $\mathcal{F}$. When a jammer intercepts a message at a vertex and re-routes it on an outgoing edge of its choosing, it essentially, modifies the characteristics (packet-drop rate, delay, etc.) of the communication channel experienced by the controller. It can thus, in effect, be thought of as controlling the characteristics of a communication channel or as ‘switching’ channels. Accordingly, the jammer can be thought of as having the action set $\mathcal{A} = \mathcal{F}$, and the row stochastic matrices (2) describing changes of the transmission regimes in this example are defined as

$$P^t_{ji}(a) = \begin{cases} 1, & \text{if } i = a, \\ 0, & \text{if } i \neq a, \end{cases}$$  \hspace{1cm} (13)

i.e., the control information is transmitted through the channel selected by the jammer.

The controller may know the previous channel used for transmission (e.g., from an acknowledgement from the plant), and therefore in general its decision $u^t$ can depend on $s^t$. However, even though the controller has knowledge about the previously used channel and can use this knowledge to select a channel which it wants to use, the attacker overrules this selection and redirects control packets through a different channel. The actual mechanism through which (re)routing, and the ensuing ‘switching’, is performed in the network is not important for our purposes. All that matters is that jammer’s actions eventually affect the row stochastic matrix describing transmission, as (13) shows. In that sense, the picture in Fig. 1 is a good metaphor (it is not an exact technical description) of the jamming process.

Letting the jammer select the channel randomly, according to a probability vector $p^t = (p_1^t, \ldots, p_N^t)$ over $\mathcal{F}$ will allow it to control the probability of packet dropouts, since in this case

$$\Pr(b^t = 1|s^t = j) = (p^t)'q^t.$$  \hspace{1cm} (14)

As a result, the end-to-end statistical characteristics of the communication link depend on the jammer’s selection of the probability distribution $p^t$ but are independent of the past channel $s^t$ known to the controller. Hence, such characteristics are difficult for the controller to predict. Implications of this observation will be considered in detail in the next sections.

An attack by modifying channel characteristics. Where there is a single physical channel between the controller and the plant, the adversary can interfere with transmission by means of intercepting control packets and ‘filtering’ them through one of several binary automata available to it. Each ‘filter’-automaton determines the transmission regime in this case. Indeed, let again $\mathcal{A} = \mathcal{F}$ and let $P^t_{ji}(a)$ be defined as in (13). Then the probability for the link to become passing when the jammer selects automaton $a^t = i$ becomes the function of the jammer’s action, since according to (4) we have:

$$\Pr(b^t = 1|s^t = j, a^t = i) = q_i^t = \Pr(b^t = 1|a^t = s^{t+1} = i).$$

Again, we see that this probability does not depend on the transmission regime $s^t$ utilized at the previous time step but depends on the jammer’s action. As in the previous attack model considered, the end-to-end statistical characteristics of the communication link depend on the jammer’s actions and hence are difficult for the controller to predict. Letting the jammer select its actions according to a probability distribution $p^t = (p_1^t, \ldots, p_N^t)$ will allow it to exercise even greater control over the distribution of $b^t$, according to (14).

In this scenario, since the transmission regimes are fully controlled by the jammer, the controller is unlikely to be able to observe past jammer’s actions. In view of this restriction, the controller’s feasible strategies are limited to state feedback strategies $u^t = u^t(x^t)$, which are a special case of more general control strategies dependent on $s^t$ mentioned earlier.

In both attack scenarios presented above, direct jamming has been forsaken in favour of intelligently controlling the probability of control packets delivery. As was mentioned, a direct control over transmission was shown in [6] to result in a deterministically predictable jamming strategy. We will show in this paper that the proposed intelligent jamming strategies lead to a distinctly different jammer behaviour while forcing the controller to respond in a manner distinct from what one would expect in the situation where packet loss is benign.
In the ANC game (12), an optimal jammer’s strategy is sought in the set of mixed strategies, i.e., probability distributions on \( \mathcal{A} \). On the other hand, the controller does not use mixed strategies. Indeed, in a number of control problems considering benign loss of communication, optimal control laws are sought in the class of linear non-randomized functions of the (estimated) state of the plant; e.g., see [3]. It is this sort of situation that we consider as a target for stealthy intelligent jamming — unless the controller knows that the system is under attack, it has no reason to deviate from its optimal policy.

2.3.3 The actuation model and control information

The proposed actuation model assumes that when the link drops control packets, the actuator signal \( v_t \) is set to 0. The ‘hold’ situation where \( v_t \) is set to the last received control signal \( u^- \) whenever \( u^+ \) is dropped, can be reduced to this case by the change of control variables.

Our control model includes feedback controls that use the full information about the state of the underlying Markov process \( \mathcal{M} \). In practice, it may not always be possible for the controller to observe this state; the second example in Section 2.3.1 alludes to such situation. Games with partial information are known to be much more difficult to solve, and the existence of a saddle point is much more difficult to establish if possible at all. Considering a larger class of controllers which includes Markov control laws allows us to circumvent this difficulty, and obtain conditions under which the ANC game has an equilibrium (as will be shown, under these conditions, the controller’s optimal response is indeed Markov). The upper value of this game then sets the lower bound on control performance which can be achieved when, for instance, the controller is agnostic about the transmission characteristics of the communication link.

3 Static ANC games

3.1 One-stage ANC games

Our aim is to determine conditions under which an ANC game admits non-pure saddle points. By non-pure saddle point we mean a saddle point of the game (12) consisting of sequences of non-pure strategy vectors \((p^*)^t\), each being a linear combination of two or more vertices of the simplex \( \mathcal{F}_{N-1} \), and the corresponding controller’s best responses \((u^*)^t\). As demonstrated in Section 2.3.1, non-pure policies allow the attacker to exercise an intelligent DoS jamming by randomizing its actions and controlling the probability of control packet transmission. To facilitate characterization of non-pure ANC saddle-point strategies, in this section we consider one-stage ANC games which can be regarded as a special case of the ANC game (12) played over a one time step horizon. To emphasize the one-stage nature of the game, we temporarily suppress the time variable \( t \), and adopt a simplified notation, where the variables current at time \( t \) are not indexed, while the variables which are updated and become current at time \( t+1 \) are marked with the superscript +. With this convention, a one-step version of the general model presented in Section 2 involves:

- A plant initial state \( x \) and an initial transmission regime of the communication link \( s \), which captures characteristics of the link between the controller and the plant before the jamming attack is undertaken.
- Controller and jammer actions \( u \) and \( a \), with \( a \) selected according to a probability distribution \( p \) over \( \mathcal{A} \).
- A transmission mode \( s^+ \) of the link, randomly drawn from \( \mathcal{F} \) according to the probability distribution \( P_s(a) \) corresponding to the \( s \)-th row of the matrix \( P(a) \) selected by the jammer; see (2).
- The probability of the link operating in the transmission regime \( s^+ \in \mathcal{F} \) to become passing, determined by equation (3). The vector of these probabilities is denoted \( q = (q_1, \ldots, q_{|\mathcal{F}|})^t \).
- The binary random variable \( b \) capturing the transmission state of the communication link at the time of plant state update. According to (4), when the jammer’s actions have a distribution \( p \), the probability of the link to become passing is \( \sum_{a=1}^N p_a(P(c,a)q)_s \).
- The updated state of the plant \( x^+ \), which is the terminal plant state in the one-stage ANC game. From (1),

\[
x^+ = F(x, bu).
\]

The payoff function associated with the controller’s and jammer’s actions and the given initial system state \( x \) and the link transmission regime \( s \),

\[
\Sigma(x^+, u, a) = \sigma^0(x, u) + \sigma^1(x^+) - g^0(a, s).
\]

The latter is a special case of the payoff (9).

In the one-step case, the probability measure (10) reduces to the conditional probability measure on \( \mathcal{F} \) given that the plant starts at \( x \) and the link is initiated at an initial transmission regime \( s \)

\[
Q^{u,p}(dx^+, s^+|x, s) = \sum_{a=1}^N p_a P_{ss^+}(a) \left[q_{s^+} \delta(x^+ - F(x, u)) + (1 - q_{s^+}) \delta(x^+ - F(x, 0))\right] dx^+.
\]

**Definition 2** The one-stage ANC game is to find (if they exist) a strategy for the jammer \( p^* \in \mathcal{F}_{N-1} \) and a feedback strategy for the controller \( u^* \) such that

\[
\mathbb{P}^{u^*, p^*} \Sigma(x^+, u^*, a) = J_1 = J_2,
\]

where \( J_1 \) and \( J_2 \) are the upper and the lower values of the
in a static zero-sum convex-concave game [9], the strat-
 Unlike standard results on the existence of a saddle point
where $h$ convex and coercive for every
Under Assumption 1, for every initial pair
According to (17), the expected cost in (18) has the form
The pair $(u^*, p^*)$ is then a saddle point of the game.
We wish to determine whether the ANC game has non-
Our result on the existence of an equilibrium of the one-
Theorem 2 asserts the existence of a non-pure saddle
Theorem 2
Our first result about the ANC game (18) now follows.
According to (17), the expected cost in (18) has the form
where $h^x,s(\cdot) is a vector function $R^m \rightarrow R^N$, whose $i$-
h_i^x,s(u) = E^{u,p}[\Sigma(x^+, u, a) | a = i]
Under Assumption 1, each function $h_i^x,s(\cdot)$ is continuous, convex and coercive for every $s \in \mathcal{S}$ and $x \in \mathbb{R}^n$.
Theorem 1 Under Assumption 1, for every initial pair $(x, s)$, the one-stage ANC game has a finite value, i.e., $-\infty < J_1 = J_2 < \infty$, and the game has a (possibly non-
our first result about the ANC game (18) now follows.
Clearly, under Assumption 2, the functions \( h_{x,s}^{a,i}(\cdot) \) are smooth on \( \mathcal{G} \). Theorem 3

Suppose Assumption 2 holds. This will be demonstrated in the next section. In the remaining part of this Section and in Section 4, we will further particularize conditions of Theorem 2 to provide a better insight into how Theorem 2 enables such an analysis.

Lemma 1 and Theorem 2 make no assumption regarding the smoothness of the function \( H \), which is useful for application to dynamic ANC games, where the game (19), (20) may arise as an Isaacs equation from application of the Dynamic Programming. The value function in such an equation may not be smooth (e.g., in [10] it was shown to be only piece-wise smooth). However, Theorem 2 can be sharpened when local differentiability holds. This will be demonstrated in the next section.

3.4 Necessary and sufficient conditions for strategic jamming

The analysis in this section requires control inputs \( u \) to be one-dimensional, and the functions \( h_{x,s}^{a,i}(\cdot) \), \( i \in \mathcal{G} \) must be piece-wise smooth. Furthermore, we assume that the set \( \mathcal{G} \) consists of only two actions, \( \mathcal{G} = \{a_1, a_2\} \).

Assumption 2 Suppose the space of control inputs is one-dimensional. Also, for every \( x \in \mathbb{R}^n \), \( a \in \mathcal{A} \), the functions \( \Sigma(F(x,\cdot),\cdot,a) \) and \( \Sigma(F(x,0),\cdot,a) \) are convex functions defined on \( \mathbb{R}^1 \), which are continuously differentiable on \( \mathbb{R}^1 \), perhaps with the exception of a finite number of points; let \( U^d(x) \) be the set of such points.

Clearly, under Assumption 2, the functions \( h_{x,s}^{a,i}(\cdot) \), \( h_{x,s}^{a,i}(\cdot) \) are convex. Also, these functions are continuously differentiable at every \( u \), except for \( u \in U^d \).

Theorem 3 Suppose Assumption 2 holds.

(i) If for every \( x,s \), there exists \( \bar{u} \notin U^d \) such that

\[
\begin{align*}
\hspace{0.5cm} h_{x,s}^{a_1}(\bar{u}) & = h_{x,s}^{a_2}(\bar{u}), \\
\hspace{0.5cm} h_{x,s}^{a_1}(\bar{u}) & \geq h_{x,s}^{a_2}(\bar{u}), \quad \forall i \in \mathcal{G}, a \notin \mathcal{G},
\end{align*}
\]

and one of the following conditions hold: either

\[
\frac{dh_{x,s}^{a_1}(\bar{u})}{du} \left( \frac{dh_{x,s}^{a_2}(\bar{u})}{du} \right) < 0,
\]

or

\[
\frac{dh_{x,s}^{a_1}(\bar{u})}{du} = \frac{dh_{x,s}^{a_2}(\bar{u})}{du} = 0,
\]

then the zero-sum game (19) admits a non-pure saddle point \((u^*,p^*)\), \( u^* \notin U^d \), with \( p^* \) supported on \( \mathcal{G} = \{a_1, a_2\} \).

(ii) Conversely, if the zero-sum game (19) admits a non-pure saddle point \((u^*,p^*)\), \( u^* \notin U^d \), with \( p^* \) supported on \( \mathcal{G} = \{a_1, a_2\} \), then there exists \( \bar{u} \notin U^d \) such that either (25), (27) or (25), (28) hold.

The conditions of Theorem 3 are illustrated in Figure 2. We note that conditions (25), (26) are a special case of condition (a) of Theorem 2. Also, it will be shown in the proof of this theorem in the Appendix that conditions (b) and (c) of Theorem 2 follow from condition (27) when \( h_{x,s}^{a_1}(\cdot) \) and \( h_{x,s}^{a_2}(\cdot) \) are smooth and strictly convex. On the other hand, condition (28) was ruled out in Theorem 2; see condition (c) in that theorem. Indeed, conditions (27) and (28) are mutually exclusive.

3.5 Rewarding certain actions leads to randomized jamming strategies

Attacker’s decisions to pass/block transmission of control information may be based on considerations other than cost to the controller. These considerations encoded in the jamming cost \( g^d(a,s) \) in (16) may either discourage the jammer from launching an attack, or conversely encourage it to undertake a denial-of-service attack. The cost of link switching may be one consideration behind the jammer’s decisions whether to launch an attack. Also, the jammer may be rewarded for being inactive to ensure it is not detected; e.g., this situation may occur when the controller monitors the communication link, and an anomaly in the link behaviour can reveal the jammer. In this section we show that when the attacker is rewarded for undertaking certain actions, a set \( \mathcal{G} \) considered in Lemma 1 and Theorem 2 arises naturally.

Most generally, the reward scenario can be captured by reserving special actions in the jammer’s action space \( \mathcal{A} \); such actions will typically attract a distinctly different cost. For simplicity let \( a_o \) be the only reserved ac-
tion. Next, suppose the remaining actions are ranked in accordance with their contribution to the game payoff.

**Assumption 3** For any two actions \( j, k \in \mathcal{A} \), \( j \neq k \), if \( j < k \) then for all \( u \in \overline{U}(x) \)
\[
\mathbb{E}^{u,p}[\Sigma(x^+, u, a)|a = j] > \mathbb{E}^{u,p}[\Sigma(x^+, u, a)|a = k],
\]
here \( \overline{U}(x) \) is the interior of the compact set \( U(x) \) from Theorem 1.

Under Assumption 3, the jammer who seeks a higher value of the game will favour actions with lower numbers, since these actions generate larger payoff. In contrast, the controller should be forcing the jammer into utilizing actions with higher numbers. The reserved action \( a_o \) has been excluded from ranking. Doing so is instrumental to provide the jammer with a genuine choice between taking the reward and blocking/passing transmission.

We now show that using the action ranking from Assumption 3, the value and the saddle points of the game (19) can be characterized by solving a game over a reduced jammer’s action space \( \mathcal{G} \) consisting of the reserved action \( a_o \) and one of the remaining actions which delivers the highest payoff to the jammer when it seeks to block communications between the controller and the plant.

Note that when \( a_o \neq 1 \), then according to (29), the jammer’s highest payoff among ‘regular’ (i.e., not reserved) actions is associated with action \( a = 1 \). Alternatively, if \( a_o = 1 \), then according to (29), the highest payoff among the actions from \( \mathcal{A}/\{a_o\} \) is delivered when the jammer selects \( a = 2 \). In both cases, the reduced action space will contain only two actions from \( \mathcal{A} \); in the first case, \( \mathcal{G} = \{1, a_o\} \), or \( \mathcal{G} = \{1, 2\} \) in the second case. The analysis of the reduced game is simplified in both cases, therefore we only consider the case \( \mathcal{G} = \{1, a_o\} \). In Section 4, detailed examples will be given to illustrate this case.

Now that the set \( \mathcal{G} \) has been established, the existence of non-pure strategies in the ANC game with reserved actions can be derived from the results in the previous section. The first result is a corollary from Theorem 2 and is applicable when the control input is a vector.

**Corollary 1** Suppose Assumptions 1 and 3 hold. Furthermore, suppose that for given initial \( x \), there exists \( u^* = u^*(x, s) \in \overline{U} \) such that the functions \( h_{1,s}^x(\cdot) \), \( h_{a_o}^x(\cdot) \) satisfy the conditions
\[
(a) \quad h_{1,s}^x(u^*) = h_{a_o}^x(u^*);
(b) \quad \text{For all } u \neq u^*, h_{1,s}^x(u) > h_{1,s}^x(u^*) \text{ or } h_{a_o}^x(u) > h_{a_o}^x(u^*);
(c) \quad u^* \text{ is not a minimum of } h_{1,s}^x(\cdot), h_{a_o}^x(\cdot).
\]

Then, the zero-sum game (19), (20) admits a non-pure saddle point \((u^*, p^*)\) with non-pure vector \( p^* \) supported on \( \mathcal{G} = \{1, a_o\} \).

**Proof:** Under Assumption 3, \( h_{1,s}^x(u^*) > h_{k,s}^x(u^*) \forall k \notin \{1, a_o\} \). Together with condition (a), this observation verifies condition (a) of Lemma 1, with \( \mathcal{G} = \{1, a_o\} \). Conditions (b) and (c) of that lemma trivially follow from (b) and (c) in this corollary. Thus, from Lemma 1, the ANC game has a saddle point \((u^*, p^*)\) with \( p^* \) supported on \( I(p^*) \subseteq \mathcal{G} \). Since \( |\mathcal{G}| = 2 \), and \( p^* \) is not pure, then \( |I(p^*)| = 2 \), i.e, \( 0 < p^*_1 < 1 \) and \( 0 < p^*_a < 1 \). \( \square \)

The second corollary follows from Theorem 3 and applies when the control input is scalar. It eliminates condition (26) which is the gap between the necessity and sufficiency statements in that theorem.

**Corollary 2** Suppose Assumptions 2, 3 hold. The zero-sum game (19) admits a non-pure saddle point \((u^*, p^*)\), \( u^* \notin U^d \), with \( p^* \) supported on \( \mathcal{G} = \{1, a_o\} \), if and only if there exists \( \bar{u} \in \overline{U}(x) \setminus U^d \) satisfying (25) and either (27) or (28) with \( a_1 = 1 \) and \( a_2 = a_o \).

**Proof:** The corollary directly follows from Theorem 3 since condition (26) of the sufficiency part of that theorem is trivially satisfied under Assumption 3.

### 3.6 Remarks on the single-stage payoff

The foregoing analysis of the single-stage ANC game (19) has relied on the properties of conditional payoff functions \( h_{1,s}^x(\cdot) = \mathbb{E}^{u,p}[\Sigma(x^+, \cdot, a)|a = i] \) which in turn follow from coercivity, convexity and continuity of \( \Sigma(F(x, \cdot, \cdot), a) \) and \( \Sigma(F(x, 0), a) \). Of course these properties can be readily validated from the corresponding properties of \( \sigma^0 \) and \( \sigma^1 \) through (16). However, we stress that the results in this section are in fact more general in that they apply to functions \( \Sigma \) more general than (16); this observation will be useful in the next section.

### 4 Linear-quadratic controller-jammer games

In this section, we specialize the results of Section 3 to one-stage controller-jammer ANC games where the plant (1) is linear, and hence (15) becomes
\[
x^+ = Ax + bBu,
\]
and the cost of control is quadratic. Also, we give examples of the situations alluded in Section 3, where rewarding certain actions leads to randomized jamming.

Two such games will be considered. In the first game the jammer is rewarded for remaining stealthy, while in the second game its decisions are determined by the premium the controller must pay for terminating the game.
(e.g., as a cost of repair or cost of re-routing control signals). We show that in both games, there is a region in the plant state space where the jammer’s optimal policy is to randomize among its actions, and an optimal control response to this jamming policy is nonlinear. We will also revisit the game from [6] and will show that the deterministically predictable behaviour of the jammer observed in [6] can be predicted using our results.

In order to provide a clear context of the attack strategies resulting from these games we revisit the intelligent DoS by re-routing; see Section 2.3.1. That is, we assume that the jammer’s actions are to randomly select one of the available communication channels, i.e., \( \mathcal{A} = \mathcal{F} \), and \( P(a) \) is defined as in (13). In addition, we will assume that \( u \in \mathbb{R}^1 \); this will allow us to apply Theorem 3 and, under Assumption 3, Corollary 2.

Thanks to the linear-quadratic nature of the games, in all three problems the conditional expected payoff (22), given the jammer’s action \( a = j \), will have the form:

\[
h_j(u) = \gamma_j(x) + u^2 + r_jq_j(u + 2\beta(x)),
\]

where \( r_j \geq 0 \) is a constant, and \( \gamma_j(x) \geq 0 \) for all \( j \). Also, all available channels are assumed to be ordered according to their probability to become passing, that is, the probabilities (3) are assumed to be ordered as

\[
q_1 < q_2 < \ldots < q_n.
\]

Lemma 2 The set \( U(x) = \{ u : u(u + 2\beta(x)) \leq 0 \} \) verifies properties stated in Theorem 1. Also, let \( a_0 \neq 1 \) denote a reserved action, and suppose \( r_j > 0 \) and \( \gamma_j(x) = \gamma(x) \) for all \( j \neq a_0 \). Then under condition (32), Assumption 3 is also satisfied with \( U(x) \) defined above.

Proof: We fix \( x \) and assume \( \beta(x) > 0 \). The case \( \beta(x) < 0 \) can be analyzed in a similar manner, while the case \( \beta(x) = 0 \) is trivial.

The functions \( h_j(u) \) defined in (31) are continuous and strictly convex in \( u \). They are monotone decreasing on the interval \([−\infty, −2\beta(x)]\) and are monotone increasing on the interval \([0, +\infty)\). Therefore, \( h(u) = \max_j h_j(u) \) and \( \hat{h}(u) = \sum_j p_j h_j(u) \) are also monotone decreasing on the interval \([−\infty, −2\beta(x)]\) and are monotone increasing on \([0, +\infty)\). Since both functions are continuous,

\[
\begin{align*}
\min_{u \leq −2\beta(x)} \hat{h}(u) &= \hat{h}(−2\beta(x)), & \min_{u \geq 0} \hat{h}(u) &= \hat{h}(0), \\
\min_{u \leq −2\beta(x)} h(u) &= \bar{h}(−2\beta(x)), & \min_{u \geq 0} h(u) &= \bar{h}(0).
\end{align*}
\]

Therefore, since the set \( U(x) \) is closed and contains the points 0 and \( -2\beta(x) \), then

\[
\begin{align*}
\inf_u \sup_{p \in \mathcal{F}_{N−1}} \bar{h}(u) &= \inf_u \hat{h}(u) \\
&= \min \left[ \inf_{u \leq −2\beta(x)} \hat{h}(u), \inf_{u \in U(x)} \hat{h}(u), \inf_{u \geq 0} \hat{h}(u) \right] \\
&= \inf_{u \in U(x)} \hat{h}(u) = \inf_{u \in U(x)} \sup_{p \in \mathcal{F}_{N−1}} \bar{h}(u).
\end{align*}
\]

Also, for every \( p \in \mathcal{F}_{N−1} \), \( \inf_u \bar{h}(u) \) is attained at

\[
\begin{align*}
\inf_{u \in U(x)} \hat{h}(u) &= \sup_{u \in U(x)} \inf_{p \in \mathcal{F}_{N−1}} \bar{h}(u).
\end{align*}
\]

This proves that an optimal response of the minimizing player in the game (19) lies within the interval \([-2\beta(x), 0]\).

To verify the second claim of the lemma, we note that for any \( j, k \neq a_0 \), when \( j > k \) then due to (32) \( h_j(u) < h_k(u) \) for all \( u \in (-2\beta(x), 0) \) i.e., (29) is satisfied. \( \square \)

4.1 LQ control under reward for stealthiness

Consider a controller-jammer game for the plant (30) with initial conditions \( x^0 = x \) and \( s^0 = s \) and the quadratic payoff (16), with \( \sigma^0(x, u) = \|x\|^2 + u^2 \), \( \sigma^1(x^+) = \|x^+\|^2 \), \( g^0(a, s) = -\tau \delta_{a,s} \). Here, \( \tau > 0 \) is the constant payoff which the jammer receives if it does not re-route control packets through a different channel. The rationale here is to reward the jammer for not switching channels when excessive switching may reveal its presence (hence rewarding stealthiness), or may drain its resources. Thus, the reserved action is to maintain transmission through the initial channel \( s \), i.e., \( a_0 = s \). The corresponding function \( \Sigma(\cdot) \) in this case is

\[
\Sigma(x^+, u, a) = \begin{cases} 
\|u\|^2 + u^2 + \|x^+\|^2, & a \neq s, \\
\|x\|^2 + u^2 + \|x^+\|^2 + \tau, & a = s,
\end{cases}
\]

and the static LQ ANC game is to find a control strategy \( u^* \) and a jammer’s non-pure strategy \( p^* \) which form a saddle point of the game (19) for the plant (30), with the payoff (33). We now show that Corollary 2 can be used for that. Indeed, the functions \( h_a(u) = E^{u,p}[\Sigma(x^+, u, a) | a = j] \) have the form (31), with \( \gamma_a(x) = x'((I + A'A)x, j \neq s \gamma_a(x) = x'(I + A'A)x + \tau \), and \( \beta(x) = \|B\|^2 \). Thus, \( \gamma_a(x) \) is monotone increasing on \([0, +\infty)\) and \( \beta(x) = \|B\|^2 \). According to Lemma 2, with \( a_0 = s \), Assumption 3 is satisfied in this special case. Furthermore, Assumption 2 is also satisfied, due to linearity of the plant and a quadratic nature of the payoff. Then the analysis of the ANC game can be reduced to verifying whether the payoff functions for the reduced zero-sum game, namely \( \tilde{h}_a(u) = h_1(u) \) and \( \tilde{h}_a(u) = h_2(u) \) satisfy the conditions of Corollary 2.
Using (31), condition (25) reduces to the equation to be solved for \( \bar{u} \in (-2\beta(x), 0) \),

\[
\bar{u} \left( \bar{u} + \frac{2}{\|B\|^2} B'Ax \right) = \frac{\tau}{\|B\|^2(q_1 - q_s)}.
\]

(34)

which admits real solutions if \( \frac{1}{\|B\|^2} x' A' B' Ax \geq \frac{\tau}{q_s - q_1} \).

Also, condition (27) reduces to the condition

\[
- \frac{\|B\|^2 q_s}{1 + \|B\|^2 q_1} < \bar{u} < - \frac{\|B\|^2 q_1}{1 + \|B\|^2 q_1}.
\]

(35)

The analysis of conditions (34), (35) shows that only one of the solutions of equation (34), namely

\[
\tilde{u} \triangleq u^* = -\frac{1}{\|B\|^2} B'Ax \left( 1 - \sqrt{\tau \|B\|^2 - (q_1 - q_s)x' A' B' Ax} \right)
\]

(36)

satisfies (35) provided

\[
R_1 < x' A' B' Ax < R_s,
\]

(37)

\[
R_1 \triangleq \frac{(1 + \|B\|^2 q_1)^2}{(1 + \|B\|^2 q_1)^2 - 1 - \tau \|B\|^2 q_1},
\]

\[
R_s \triangleq \frac{(1 + \|B\|^2 q_2)^2}{(1 + \|B\|^2 q_2)^2 - 1 - \tau \|B\|^2 q_2}.
\]

Condition (37) describes the region in the state space in which the jammer’s optimal policy is to choose randomly between the initial channel \( s \) and the most blocking channel 1. In the case where the plant (30) is scalar and \( B = 1 \), we recover exactly the condition obtained in [10] by direct computation. That is, Corollary 3 confirms the existence of the jammer’s non-pure optimal strategy for this region. We refer the reader to [10] for the exact value of the optimal vector \( p^* \); the calculation for the multidimensional plant (30) follows same lines, and is omitted for the sake of brevity. We also point out that the optimal controller’s policy (36) is nonlinear.

Let us now compare quadratic performance of the optimal response control \( u^* \) with performance of the optimal guaranteed cost control law designed to control the plant (30) over a bona fide packet dropping link with unknown probability distribution of packet dropouts. Since the controller is unaware of the attack and is also unaware of the precise statistical characteristics of the channel, such a control strategy is a natural choice. Let \( d \) denote the unknown probability distribution of the transmission state of a packet dropping link perceived by the controller. The set of all feasible probability distributions \( d \) is denoted \( D \). The optimal guaranteed performance control \( u^{\text{LQR}} \) is characterized by the condition

\[
\sup_{d \in D} \mathbb{E}^d[\|x\|^2 + \|u^{\text{LQR}}\|^2 + \|x^+\|^2] = \inf_{u \in D} \sup_{d \in D} \mathbb{E}^d[\|x\|^2 + \|u\|^2 + \|x^+\|^2];
\]

(38)

here \( \mathbb{E}^d \) denotes the expectation with respect to the probability vector \( d \). In particular, when the link is optimally controlled by the jammer, we have \( d^* = [(p^*)' q, (1 - (p^*)' q)]^T \). Hence when \( d^* \in D \), then

\[
\mathbb{E}^d |\mathbb{E}^{u^{\text{LQR}}, p^*}| [(\|x\|^2 + \|u^{\text{LQR}}\|^2 + \|x^+\|^2] \leq \sup_{d \in D} \mathbb{E}^d [\|x\|^2 + \|u^{\text{LQR}}\|^2 + \|x^+\|^2]
\]

\[
= \inf_{u \in D} \sup_{d \in D} \mathbb{E}^d [\|x\|^2 + \|u\|^2 + \|x^+\|^2].
\]

(39)

On the other hand, since \( (u^*, p^*) \) is the saddle point of the one-stage min-max ANG game which we have shown to admit the upper value, we conclude that for all \( u \),

\[
\mathbb{E}^d [\|x\|^2 + \|u^\| \leq \|u^*\|^2 + \|x^+\|^2 + \delta_{a,s} \tau] = \mathbb{E}^d [\|x\|^2 + \|u^*\|^2 + \|x^+\|^2 + \delta_{a,s} \tau].
\]

(40)

Drop the term \( \mathbb{E}^d [\|x\|^2 + \|u^*\|^2 + \|x^+\|^2] \) on both sides of (40), it then follows from (39), (40)

\[
\mathbb{E}^d [\|x\|^2 + \|u^*\|^2 + \|x^+\|^2] \leq \inf_{u \in D} \sup_{d \in D} \mathbb{E}^d [\|x\|^2 + \|u\|^2 + \|x^+\|^2].
\]

(41)

This argument shows that when the jammer exercises its optimal randomized strategy, unbeknownst to the controller, and forces the latter to perceive the communication link as a benign uncertain packet dropping channel, the guaranteed cost control law designed under this assumption will have an inferior performance, compared with \( u^* \), provided \( d^* \in D \). We interpret this situation as a signature of a successful DoS attack by the jammer.

From the above analysis, it appears that a possible line of defence against the DoS attack analyzed in this section could be controlling the system so as to avoid the region defined by (37). In this case, the jammer will be forced to deterministically route control packets over the genuine, albeit most damaging, packet dropping channel. An impact of this defense on the system performance is an open problem which will be studied in future work.

4.2 Linear quadratic game with cost on loss of control

We again consider the plant (30), but with the payoff

\[
\|x\|^2 + v^2 + \|x^+\|^2 = \|x\|^2 + t^2 v^2 + \|x^+\|^2.
\]

This is a one-step version of the payoff considered in [6]. The payoff in this game directly depends on whether the control input is blocked. In this case, the functions \( h_j \) have a form slightly different from those in Section 4.1:

\[
h_j(u) = x' (I + A' A) x + q_j (1 + \|B\|^2) u \left( u + \frac{2}{1 + \|B\|^2} B' A x \right).
\]
According to Lemma 2, Assumption 3 is still satisfied (with the reserved action \(a_0 = s\)), and we can attempt to apply Corollary 2. For every \(x \neq 0\), two points solve condition (25), \(\bar{u} = 0\) and \(\bar{u} = -\frac{1}{\|B\|^2} B'Ax\), but none of them satisfy conditions (27) or (28). Hence, the jammer’s optimal strategy is to switch to the most blocking channel 1, instead of randomizing between \(s\) and 1. This finding is consistent with the result obtained in [6].

### 4.3 Linear-quadratic game with termination payoff

In this example, suppose the controller chooses a channel \(s\) to transmit information and receives messages from the plant as to which channel was used for transmission. Once the controller detects change, it terminates the game and pays a termination fee \(T\), otherwise its terminal cost is a regular state-dependent cost \(\sigma(x)\). The jammer has a choice between disrupting that channel or refusing to transmit information and receiving messages from the plant as to which channel was used for transmission, and thus revealing itself, and holding off the attack to remain undetected. The game payoff is then

\[
\Sigma(x^+, u, a) = \|x\|^2 + u^2 + (1 - \delta_{a,s}) T + \delta_{a,s} \|x^+\|^2.
\]

Owing to the term \(\delta_{a,s} \|x^+\|^2\), this payoff is not of the form (16), however its conditional expectation given \(a = j\) has the form (31), with \(\gamma_j(x) = \|x\|^2 + T, \tau_j = 0, j \neq s, \gamma_s(x) = x'(I + A'A)x, \tau_s = \|B\|^2, \text{ and } \beta(x) = \|B\|^2 B'A\); it satisfies all the convexity and coercivity conditions required in Section 3, hence the results developed in Section 3 can be applied in this problem; see the remarks in Section 3.6. Without loss of generality, we only consider the case where \(B'Ax \leq 0\). The analysis of the case \(B'Ax \geq 0\) follows the same lines.

In the game considered, all the channels except \(s\) have equal value for the jammer. Essentially, the jammer has to choose between two actions: allow the controller to use its chosen channel \(s\) or reveal itself by selecting some other channel, e.g., channel 1. For that reason, we select \(G = \{1, s\}\) and proceed using Theorem 3. Naturally, we assume \(s \neq 1\), since the case \(s = 1\) is trivial.

According to our selection of the set \(G\), the functions \(h_{\alpha_1}^{x,s} \text{ and } h_{\alpha_2}^{x,s}\) in Theorem 3 are \(h_1\) and \(h_s\) from (31), respectively. It can be shown that if \(B'Ax \leq 0\), \(T \leq x'A(I - \frac{1}{\|B\|^2} BB')Ax\), then

\[
\bar{u} = u^* = -\frac{1}{\|B\|^2} B'Ax
\]

\[
+ \sqrt{\frac{\|B\|^2 T - x'A'(\|B\|^2 T - (q_s)BB')Ax}{\|B\|^4 q_s}}.
\]

(42)

validates conditions (25), (27) of Theorem 3 when

\[
\bar{u} < 0 \text{ and } (1 + \|B\|^2 q_s)\bar{u} + (q_s)x'AB > 0.
\]

Substituting (42) into (43) yields

\[
x'A'Ax > T, \quad T > x'A'(I - \frac{2 + \|B\|^2 q_s}{\|B\|^2 q_s})BB'Ax.
\]

Note that \(I - \frac{2 + \|B\|^2 q_s}{\|B\|^2 q_s})BB' > I - \frac{\|B\|^2 q_s}{\|B\|^2 q_s})BB' > 0\). Therefore the region in the plant’s state space where the game has a non-pure saddle point is the intersection of the interior of the ellipsoid defined by condition (44) and the set \(\{x : \|Ax\|^2 > T\}\). Such an intersection is clearly not an empty set, since \(I - \frac{2 + \|B\|^2 q_s}{\|B\|^2 q_s})BB' < I\).

Once again, as in Section 4.1, our analysis discovers a region in the state space where the jammer’s optimal strategy is to act randomly. In fact, in this problem the choice is between allowing transmission through the channel \(s\) selected by the controller and switching to some other channel — no matter which channel is selected to replace \(s\), the value of the game is not affected by this selection. The jammer’s optimal policy vector \(p_s^*\) can be computed directly in this problem to be an arbitrary vector in \(\mathcal{F}_N\) with the \(s\)-th component being equal to

\[
p_s^* = -\frac{1}{\|B\|^2 q_s} \times \left(1 - \sqrt{\frac{(q_s)x'A'Ax}{\|B\|^2 T - x'A'(\|B\|^2 T - (q_s)BB')Ax}}\right).
\]

Again, as in Section 4.1, the controller’s best response is nonlinear in \(x\).

5 Existence of saddle-point strategies for multistage finite horizon ANC games

In this section we extend some of our previous results about the existence of saddle-point strategies to the case of the multistage ANC game (12). The game was posed in Section 2, but our generalization will be concerned with a special case where the function \(F_t\) is a linear function, \(F_t(x, v) = A_tx + B_tv\), where \(A_t, B_t\) are matrices of matching dimensions. That is, we restrict attention to linear systems of the form

\[
x^{t+1} = A_tx^t + b'B_tu^t.
\]

(46)

**Assumption 4**

(i) For every \(t = 0, \ldots, T - 1\), there exist scalars \(e_t, d_t\) and functions \(\alpha_t, \beta_t : \mathbb{R}_+ \rightarrow \mathbb{R}\), with \(\lim_{y \rightarrow +\infty} \alpha_t(y) = \lim_{y \rightarrow +\infty} \beta_t(y) = +\infty\) and \(\alpha_t(y) \geq e_t, \beta_t(y) \geq d_t\) such that

\[
\sigma^t(u, x) \geq \alpha_t(||u||) + \beta_t(||x||) \quad \forall x \in \mathbb{R}^n, u \in \mathbb{R}^m.
\]

(ii) For all \(t = 0, \ldots, T - 1\), the function \(\sigma^t : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}_+\) is convex in both its arguments.
Theorem 4 Suppose Assumptions 4 and 5 hold. Then the value functions \( \{V_t\}_{t=0}^T \) of the ANC game (12) defined recursively by

\[
V_T(x, s) = \sigma^T (x) - g^T(s),
\]

\[
V_t(x, s) = \inf_{u \in \mathbb{R}^n} \sup_{p \in \mathcal{A}^-_{t-1}} p' \mathcal{Y}_t(x, s, u) \quad (47)
\]

\[
= \sup_{p \in \mathcal{A}^-_{t-1}} \inf_{u \in \mathbb{R}^n} p' \mathcal{Y}_t(x, s, u). \quad (48)
\]

for all \( t = 0, \ldots, T-1 \) and \( x \in \mathbb{R}^n, s \in \mathcal{F} \), where \( \mathcal{Y}_t \) is a function \( \mathcal{F} \times \mathbb{R}^m \rightarrow \mathbb{R}^N \) with components

\[
[\mathcal{Y}_t(x, s, u)]_a \triangleq \sigma^t(x, u) - g^t(a, s)
\]

\[+
\mathbb{E}_t \left[ V_{t+1}(x^{t+1}, s^{t+1}) \right| x^t = x, s^t = s, u^t = u, a^t = a \right] \right], \quad (49)
\]

are all well defined, convex and continuous in \( x \). Furthermore, the strategy pair \((u^*, p^*)\) defined by

\[
(u^*)_a = \arg \inf_{u \in \mathbb{R}^n} \sup_{p \in \mathcal{A}^-_{t-1}} p' \mathcal{Y}_t(x, s, u), \quad (50)
\]

\[
(p^*)_a = \arg \sup_{p \in \mathcal{A}^-_{t-1}} \inf_{u \in \mathbb{R}^n} p' \mathcal{Y}_t(x, s, u), \quad (51)
\]

is a saddle-point strategy of the ANC game (12).

Theorem 4 introduces the HJBI equation for the ANC game (12) associated with the linear plant (46), see (47), (48). Such an equation reduces finding jammer’s optimal strategies and the corresponding controller’s best responses for the multi-step ANC problem to a one-step game-type problem of the type considered in the previous sections. Unfortunately, even for linear plants it appears to be difficult to obtain an analytical answer to the question whether the jammer’s corresponding optimal strategy is randomized or not. The main difficulty appears to be obtaining a closed form expression for \( V_t(x, s) \) — of course, the tools developed in the previous sections can be used to characterize a possibility of nonpure equilibria numerically. An alternative is to consider greedy jamming strategies, where at each time step the jammer pursues a strategy which is (sub)optimal only at this particular time. For example, one possibility is to use a lower bound for \( V_{t+1} \) in the HJBI equation (49),

\[
\inf_{u \in \mathbb{R}^n} \sup_{p \in \mathcal{A}^-_{t-1}} p' \mathcal{Y}_t(x, s, u),
\]

\[
[\mathcal{Y}_t(x, s, u)]_a \triangleq \sigma^t(x, u) - g^t(a, s)
\]

\[+
\mathbb{E}_t \left[ V_{t+1} \left( \sum_{i=1}^{n} A_i x + b^T a \right) \right| s^t = s, a^t = a \right] .
\]

While not optimal, such greedy strategies may allow the jammer launch a randomized DoS attack which will likely be as difficult to detect as an optimal one. The effect of such an attack on the overall system performance and defense strategies against it are an open question.

6 Discussion and conclusions

In this paper we have analyzed a class of control problems over adversarial communication links, in which the jammer strategically disrupts communications between the controller and the plant. Initially, we have posed the problem as a static game, and have given necessary and sufficient conditions for such a game to have a nonpure saddle point. This allows a characterization of a set of plant’s initial states for which a DoS attack can be mounted that requires a nontrivial controller’s response.

For instance, in two linear quadratic problems analyzed in Section 4 the optimal control law is nonlinear. This gives the jammer an advantage over any linear control policy in those problems. The jammer achieves this by randomizing its choice of a packet-dropping transmission regime rather than direct jamming. In those problems, the part of the state space where the jammer randomizes is determined by the jammer’s cost of switching (reward for not switching), cost of termination, and transition probabilities of the current and the most blocking regimes. If these parameters can be predicted/estimated by the controller, it has a chance of mitigating the attack by either eliminating those regions, or steering the plant so that it avoids visiting those regions.

Also, a multi-stage finite-horizon game has been considered for linear plants. We have shown that equilibria for that game can be found by solving similar one-stage games, although in general it is difficult to obtain closed form solutions for these games, and one may need to resort to solving them numerically. As an alternative, a greedy suboptimal analysis has been proposed.

Future work will be directed to further understanding conditions for DoS attacks, with the aim to obtain a deeper insight into dynamic/multi-step ANC problems. Another interesting question is whether associating a distinct payoff with one of the channels is necessary for the jammer to resort to randomization. The system closed-loop stability under the proposed randomized jamming attack is also an interesting problem, namely
the question whether it is possible for the jammer to degrade control performance and avoid being caught due to causing an instability. Analysis of this problem requires a different, infinite horizon problem formulation.

7 Appendix

7.1 Proof of Proposition 1

Since the process \( s^t \) is a Markov chain, for each \( t = 0, \ldots, T-1 \) and an arbitrary measurable set \( \Lambda \times S \subseteq \mathcal{S} \),

\[
\Pr(x^{t+1} \in \Lambda, s^{t+1} \in S | \{(x^0, s^0), u^0, a^0\}_{\theta=0}^t) = \sum_{i \in S} P_{s^t,i}(a^t) \Pr(x^{t+1} \in \Lambda | s^{t+1} = i, \{(x^0, s^0), u^0, a^0\}_{\theta=0}^t).
\]

Furthermore, from (3) and (1), since given \( s^{t+1}, b_t \) is conditionally independent of \( \{(x^t, s^t), u^t, a^t\}_{\theta=0}^t \), then

\[
\Pr(x^{t+1} \in \Lambda, s^{t+1} \in S | \{(x^0, s^0), u^0, a^0\}_{\theta=0}^t) = \sum_{i \in S} \Pr(x^{t+1} \in \Lambda | s^{t+1} = i, x^t, u^t) P_{s^t,i}(a^t) = \Pr(x^{t+1} \in \Lambda, s^{t+1} \in S | x^t, s^t, u^t, a^t),
\]

proving that \( \{x^t, s^t\}_{t=0}^T \) is a Markov process.

7.2 Proof of Theorem 1

First, we observe that since the function \( h^x_s(\cdot) \) is continuous and coercive, then according to [12, Theorem 1.4.1],

\[
\inf_a h^x_s(a) > -\infty. \quad \text{Let } \alpha_x \text{ be a constant such that } \alpha_x \geq \max_a \inf_a h^x_s(a).\quad \text{Owing to the coercivity property of } h^x_s(\cdot) \text{ there exists } M^x_s(x) \text{ such that if } \|u\| > M^x_s(x) \text{ then } h^x_s(u) > \frac{1}{\alpha_x} \|u\| > \alpha_x. \quad \text{Let } M(x) = \max_a M^x_s(x), \text{ and consider the ball } U(x) = \{u : \|u\| \leq M(x)\}. \text{ Then the following facts hold}
\]

\[(i) \text{ For all } s, U^s(x) = \left\{ u : \max_a \inf_a h^s_u(a) \leq \alpha_x \right\} \subseteq U(x). \text{ Indeed, if } u \in U^s(x), \text{ then } h^s_u(u) \leq \alpha_x \text{ therefore } \|u\| \leq M^s_s(x) \leq M(x).
\]

\[(ii) \text{ Also, for all } p \in \mathcal{S} \text{ and } x \in \mathbb{R}^n, s \in \mathcal{S}, \]

\[
\{u : p'h^{x,s}(u) \leq \alpha_x \} \subseteq U(x) \quad \text{forall } p \in \mathcal{S} \Leftrightarrow \forall x \in \mathcal{S} \text{, (53)}
\]

Indeed, if \( \|u\| > M(x) \), then \( h^x_s(u) > \alpha_x \) for all \( a \in \\mathcal{A} \) and we must have \( p'h^{x,s}(u) > \alpha_x. \) Therefore \( p'h^{x,s}(u) \leq \alpha_x \) implies that \( \|u\| \leq M(x). \)

Using (i) and the fact that \( p'h^{x,s}(u) \) is linear in \( p \) and therefore \( \sup_{p \in \mathcal{S}} p'h^{x,s}(u) = \max_{a \in \mathcal{A}} \inf_a h^{x,s}(u) \), we obtain

\[
J_1 = \inf_{u \in U(x)} \max_{a \in \mathcal{A}} \inf_a h^{x,s}(u) = \inf_{u \in U(x)} \max_{a \in \mathcal{A}} \sup_{p \in \mathcal{S}} p'h^{x,s}(u), \quad \text{(54)}
\]

On the other hand, by definition, for \( u \in \mathbb{R}^m \setminus U^s(x) \),

\[
\max_{a \in \mathcal{A}} h^{x,s}(u) > \alpha_x, \text{ therefore } J_1 = \inf_{u \in U(x)} \max_{a \in \mathcal{A}} \sup_{p \in \mathcal{S}} p'h^{x,s}(u).
\]

Together with (54) this yields (23).

In a similar manner, from (53) it follows that \( \forall p \in \mathcal{S} \),

\[
\inf_{u \in U(x)} p'h^{x,s}(u) = \inf_{u \in U(x)} p'h^{x,s}(u). \quad \text{The identity } J_2 = \sup_{p \in \mathcal{S}} \inf_{u \in U(x)} p'h^{x,s}(u) \text{ immediately follows from that identity, i.e., the rightmost identity (24) holds.}
\]

We have established that both the upper value (19) and the lower value (20) can be computed by performing minimization over the compact set \( U(x) \) which is also convex. Since \( \mathcal{S} \) is also compact and convex, and the payoff function \( p'h^{x,s}(u) \) is continuous and convex in \( u \) for each \( p \in U(x) \), the zero-sum game on the right-hand side of (23) has a saddle point in \( U(x) \times \mathcal{S} \), but this saddle point may not be unique [9, Theorem 4, p.168].

That is, the leftmost identity (24) holds and the value of the zero-sum game over \( U(x) \times \mathcal{S} \) is finite. Then, using (23) and (24) we conclude that both \( J_1 \) and \( J_2 \) are finite and equal.

7.3 Proof of Lemma 1

The proof is a variation of well-known arguments regarding the connection between convex duality and the existence of a saddle-point for the Lagrangian, as well as dual characterization of minimal elements of convex sets (see, e.g., [8]). Because of our need to incorporate unboundedness of vector \( u^* \) and non-purity of vector \( p \), however, we find it clearer to provide a full derivation here than to directly resort to these arguments.

Consider the set

\[
M := \{ z \in \mathbb{R}^{|\mathcal{S}|} | \exists u \text{ such that } H_{\mathcal{S}}(u) \leq z \}; \quad \text{(55)}
\]

here the notation \( H_{\mathcal{S}}(u) \) refers to the vector comprised of the components of \( H(u) \) whose indexes belong to \( \mathcal{S} \).

Also, the inequality in (55) is understood component-wise. The interior and the boundary of \( M \) are denoted \( \overset{o}{M} \) and \( \partial M = (M \setminus \overset{o}{M}) \).

\( M \) is clearly convex since \( H \) is. In addition, assumption (b) implies that \( H_{\mathcal{S}}(u^*) \in \partial M \). Indeed, \( H_{\mathcal{S}}(u^*) \) clearly belongs to \( M \). If, in addition, we let \( 1 \) denote the vector of all ones, then, for any \( r > 0 \) the point \( z_r := H_{\mathcal{S}}(u^*) - \frac{1}{r} \cdot 1 \) belongs to the ball \( \mathcal{B}(H_{\mathcal{S}}(u^*), r) \) of center \( H_{\mathcal{S}}(u^*) \) and radius \( r \) in \( \mathbb{R}^{|\mathcal{S}|} \) but does not belong to \( M \) since

- \( H_{\mathcal{S}}(u^*) > z_r \), and

- according to (b), for any \( u \neq u^* \), there exists \( i \) such that \( H_i(u) > H_i(u^*) > (z_r)_i \), i.e., \( H_{\mathcal{S}}(u) \notin z_r \forall u \neq u^* \).
We have thus shown that, for any \( r > 0 \), \( \mathcal{R}(H_{q}(u^*)) \) is not a subset of \( M \), which implies that \( H_{q}(u^*) \) is not in the interior of \( M \). Now, because \( H_{q}(u^*) \in \partial M \), we can use the supporting hyperplane theorem (see, e.g., p. 51 of [8]), to claim that there exists \( \bar{p} \in \mathbb{R}^{|\mathcal{G}|} \), \( \bar{p} \neq 0 \), such that
\[
\bar{p}^t H_{q}(u^*) \leq \bar{p}^t z \quad \text{for all } z \in M.
\] (56)

Now, we claim that \( \bar{p} \in \mathbb{R}^{|\mathcal{G}|} \). Indeed, note that if \( z^0 \in M \), the ray \( R_z := z^0 + \mathbb{R}_+ \mathbf{e}_1 \subset M \), for any basis vector \( \mathbf{e}_1 \). Hence, because of (56), the function \( z \mapsto \bar{p}^t z \) is lower-bounded on \( R_z \), which implies that \( \bar{p}_i \geq 0 \). Also note that \( \bar{p} \) is non-pure, for otherwise, from (56), there would exist \( i \in \mathcal{G} \) such that \( H_i(u^*) = \bar{p}^t H_{q}(u^*) \leq \bar{p}^t H_{q}(u) = H_i(u) \) for all \( u \), (because \( H_{q}(u) \in M \) for all \( u \)), which contradicts (c).

Now, if we define \( p^* \in \mathbb{R}^N \) by \( p_i^* = \frac{\bar{p}_i}{\bar{p}_1 + \ldots + \bar{p}_{|\mathcal{G}|}} \) for all \( i \in \mathcal{G} \) and \( p_i^* = 0 \) for all \( i \in \mathcal{G} \), we find that \( p^* \in \mathcal{N}_N \), and
\[
(p^*)^t H(u^*) = \frac{1}{\bar{p}_1 + \ldots + \bar{p}_{|\mathcal{G}|}} \bar{p}^t H_{q}(u^*) \leq \bar{p}^t H_{q}(u) = (p^*)^t H(u)
\]
for all \( u \), which means that \( u^* \) is a best response to \( p^* \), i.e., \( u^* \in \arg\inf_{u \in \mathbb{R}^N} (p^*)^t H(u) \). Now, note that (a) implies that \( p^* \) is a best response to \( u^* \), i.e., \( p^* \in \arg\sup_{p \in \mathcal{N}_N} p^t H(u^*) \), since
\[
(p^*)^t H(u^*) = \sum_{i \in \mathcal{G}} p_i^* H_i(u^*) = \sum_{i \in \mathcal{G}} p_i^* \max_{1 \leq j \leq N} H_j(u^*) = \max_{1 \leq j \leq N} H_j(u^*) \cdot p^t H(u^*) \quad \text{for all } p \in \mathcal{N}_N.
\]

In fact, the same proof would show that any vector with support included in \( \mathcal{G} \) is a best response to \( u^* \). Now,
\[
\inf_{u \in \mathbb{R}^N} \sup_{p \in \mathcal{N}_N} p^t H(u) \leq \sup_{p \in \mathcal{N}_N} (p^*)^t H(u^*) = (p^*)^t H(u^*) = \frac{1}{p_1 + \ldots + p_{|\mathcal{G}|}} \bar{p}^t H_{q}(u^*)
\]
while it is always true that \( \sup_p \inf_u p^t H(u) \leq \inf_u \sup_p p^t H(u) \). This concludes the proof. \( \square \)

7.4 Proof of Theorem 3

Sufficiency. First we show that (25), (26) and (27) imply the existence of a non-pure saddle point for (19) supported on \( \mathcal{G} \). Without loss of generality suppose
\[
\frac{dh_{a_1}^{x,s}(u)}{du} > 0, \quad \frac{dh_{a_2}^{x,s}(u)}{du} < 0.
\] (57)
Since \( h_{a_1}^{x,s}(\cdot) \) is convex, it follows from (57) that \( \frac{dh_{a_1}^{x,s}(u)}{du} \) is non-decreasing in the region \( u \geq \bar{u} \). This is true for all points \( u \geq \bar{u} \) including the points of the set \( U^d \) where we have the inequality between the right-hand side and left-hand side derivatives, \( \frac{dh_{a_1}^{x,s}(u)}{du} \leq \frac{dh_{a_2}^{x,s}(u)}{du} \). Hence \( \frac{dh_{a_1}^{x,s}(u)}{du} > 0 \) for all \( u \geq \bar{u} \) and thus \( h_{a_1}^{x,s}(u) > h_{a_2}^{x,s}(u) \) for all \( u \geq \bar{u} \). In the same manner we can show that \( h_{a_2}^{x,s}(u) < h_{a_1}^{x,s}(u) \) for all \( u < \bar{u} \). Also, \( \bar{u} \) is not a minimum of \( h_{a_1}^{x,s}(\cdot) \) and \( h_{a_2}^{x,s}(\cdot) \) since \( \frac{dh_{a_1}^{x,s}(u)}{du} \neq 0, \frac{dh_{a_2}^{x,s}(u)}{du} \neq 0 \). The sufficiency of conditions (25), (26) and (27) now follows from Theorem 2.

We now consider the second alternative case where (25), (26) and (28) hold. Since \( h_{a_2}^{x,s}(u) \) and \( h_{a_2}^{x,s}(u) \) are convex, (28) implies that \( u \) is a global minimum of both \( h_{a_1}^{x,s}(u) \) and \( h_{a_2}^{x,s}(u) \). Therefore for an arbitrary \( p \in \mathcal{N}_N \) with \( p_0 = 0 \) for \( a \neq \mathcal{G} \), \( u \) is a global minimum of \( \mathbb{E}^{u-p}[x^+, u, a] = p^t H(u) \), where \( H(u) = h_{a_2}^{x,s}(u) \) is the vector function defined in Theorem 2. That is, \( \inf_u p^t H(u) = p^t H(u) = h_{a_2}^{x,s}(u) \). It then follows that for an arbitrary \( \bar{p}, \bar{p} \in \mathcal{N}_N \) supported on \( \mathcal{G} \) and \( u \in \mathbb{R}^1 \),
\[
\bar{p}^t H(u) \leq p^t H(u) \leq \bar{p}^t H(u).
\] (58)
The leftmost inequality holds since both expressions are equal to \( h_{a_1}^{x,s}(\bar{u}) = h_{a_2}^{x,s}(\bar{u}) \).

Next, consider an arbitrary vector \( p \in \mathcal{N}_N \). From (26) we have
\[
p^t H(u) = \sum_{j \neq a_2} p_j E^{\bar{u}-p}[x^+, \bar{u}, a] | a = j | + p_{a_2} h_{a_2}^{x,s}(\bar{u}) \leq (\sum_{j \neq a_2} p_j E^{\bar{u}-p}[x^+, \bar{u}, a] | a = a_1 | + p_{a_2} h_{a_2}^{x,s}(\bar{u}) = \bar{p}^t H(\bar{u}),
\] (59)
where \( \bar{p} \) is defined as \( \bar{p}_a = \sum_{j \neq a} p_j, \bar{p}_{a_2} = p_{a_2} \) and \( \bar{p}_a = 0 \) for \( a \neq \mathcal{G} \), and is supported on \( \mathcal{G} \). Then from (58), it follows that for all \( u \in \mathbb{R}^1 \), and \( p \in \mathcal{N}_N \), \( p^t H(u) \leq (\bar{p})^t H(\bar{u}) \). Hence for an arbitrary \( \bar{p} \in \mathcal{N}_N \) supported on \( \mathcal{G} \), \((\bar{u}, \bar{p}) \) is a saddle point of the game. Clearly, a non-pure \( \bar{p} \) exists for this purpose. Necessity. Let \( (u^*, p^*) \notin U^d \) be a non-pure saddle point of the game (19) supported on \( \mathcal{G} \). Then the inequality is true.

\[
\inf_u \max_{p \in \mathcal{N}_N} p^t H(u) \leq \max_{p \in \mathcal{N}_N} p^t H(u) = (p^*)^t H(u),
\] (60)

The last inequality implies
\[
\inf_u \max_{p \in \mathcal{N}_N} p^t H(u) \leq \max_{p \in \mathcal{N}_N} p^t H(u) = (p^*)^t H(u) = \inf_u \sup_{p \in \mathcal{N}_N} p^t H(u).
\]
However, the game has the value, therefore all the inequalities above are in fact exact identities. That is,
\[
\inf_{u} \max_{p \in \mathcal{N}} p'H(u) = \max_{p \in \mathcal{N}} p'H(u^*).
\]

In other words, \(u^* \in \arg \inf_{u} \max_{p \in \mathcal{N}} [h_{x,s}(u), h_{x,s}(u)]\)
and taking into account linearity of the payoff function in \(p\), we further have \(u^* \in \arg \inf_{u} \max [h_{x,s}(u), h_{x,s}(u)]\).
We now show that \(h_{x,s}(u^*) = h_{x,s}(u^*)\). Suppose this is not true, and \(h_{x,s}(u^*) > h_{x,s}(u^*)\). Then, since \(p_{a_1}^* > 0, p_{a_2}^* > 0\), and \(p_{a_1}^* + p_{a_2}^* = 1\) by assumption,
\[
h_{x,s}(u^*) = (p_{a_1}^* + p_{a_2}^*)h_{x,s}(u^*) > (p^*)'H(u^*).
\]
The latter condition is in contradiction with the leftmost inequality in (60). The converse inequality \(h_{x,s}(u^*) < h_{x,s}(u^*)\) is also not possible, by the same argument. This proves that \(h_{x,s}(u^*) = h_{x,s}(u^*)\), i.e., \(u^*\) satisfies the condition (25) of the theorem.

It remains to prove that \(u^*\) satisfies either (27) or (28).
We prove this by ruling out all other possibilities.

**Case 1:** \((dh_{x,s}^*(u^*))' / (dh_{x,s}^*(u^*))' > 0\).

First suppose \(dh_{x,s}^*(u^*) < 0, dh_{x,s}^*(u^*) < 0\). Then, in a sufficiently small neighbourhood of \(u^*\) one can find a point \(u < u^*\) such that
\[
h_{x,s}^*(u) < h_{x,s}^*(u^*) \quad \text{and} \quad h_{x,s}(u) < h_{x,s}(u^*)\).
\]
That is, we found a point \(u < u^*\) such that \((p^*)'H(u) < (p^*)'H(u^*)\). This conclusion is in contradiction with the rightmost inequality (60). The hypothesis that \(dh_{x,s}^*(u^*) > 0, dh_{x,s}^*(u^*) > 0\) will lead to a similar contradiction. These contradictions rule out Case 1.

**Case 2:** Either \(dh_{x,s}^*(u^*) = 0\) and \(dh_{x,s}^*(u^*)' \neq 0\), or \(dh_{x,s}^*(u^*)' = 0\) and \(dh_{x,s}^*(u^*)'' = 0\).

Suppose \(dh_{x,s}^*(u^*)' > 0\) and \(dh_{x,s}^*(u^*)' = 0\). Since \(h_{x,s}^*(u^*)\), \(h_{x,s}^*(u^*)\) are continuously differentiable at \(u^*\), then for any sufficiently small \(\epsilon > 0\), there exists \(\delta > 0\) such that for any \(u \in (u^* - \delta, u^* + \delta)\), \(dh_{x,s}^*(u) > \beta - \epsilon\), \(dh_{x,s}(u) > \epsilon\). Let us choose \(\epsilon\) so that \(\epsilon < p_{a_1}^* \beta < \beta\) and consider the Taylor expansions of \(h_{x,s}(u)\), \(h_{x,s}(u)\), \(u^* - \delta < u < u^*\), with the remainders in the Cauchy form
\[
h_{x,s}^*(u) = h_{x,s}^*(u^*) + \frac{dh_{x,s}^*(u^*)}{du}(u - u^*),
\]
\[
h_{x,s}^*(u) = h_{x,s}^*(u^*) + \frac{dh_{x,s}^*(u^*)}{du}(u - u^*),
\]
where \(\xi_1, \xi_2 \in (u, u^*)\). Since \(h_{x,s}^*(u)\) is convex and \(dh_{x,s}^*(u^*)' = 0\), then \(-\epsilon < \frac{dh_{x,s}^*(u^*)}{du} \leq 0\) for \(u^* - \delta < u < u^*\). This leads us to conclude that
\[
p_{a_1}^* \frac{dh_{x,s}^*(\xi_1)}{du} + p_{a_2}^* \frac{dh_{x,s}^*(\xi_2)}{du} > p_{a_1}^* (\beta - \epsilon) - p_{a_2}^* \epsilon > 0.
\]

Then we have
\[
(p^*)'H(u) = (p^*)'H(u^*) + \left( p_{a_1}^* \frac{dh_{x,s}^*(\xi_1)}{du} + p_{a_2}^* \frac{dh_{x,s}^*(\xi_2)}{du} \right) (u - u^*) < (p^*)'H(u^*).
\]

Again, we arrive at a contradiction with the assumption that \((u^*, p^*)\) is a saddle point and must satisfy (60).

Other similar possibilities in this case will lead to a contradiction as well. This leaves two possibilities: either \(u^*\) satisfies (27), or it satisfies (28).

### 7.5 Proof of Theorem 4

We need to show that each function \(V_t\) is well-defined, i.e., that there is indeed equality between (47) and (48) for any \(x, s\); this amounts to the zero-sum game solved at time \(t\) having a value. We will proceed by backwards induction and show that the following predicate holds for all \(t = 0, \ldots, T\):

**\(P_t\):** \(V_t(x, s)\) is a well-defined continuous convex function for all \(s \in \mathcal{S}\). In addition, there exist a function \(\nu_t\) and scalar \(\xi_t\) such that
- \(\nu_t(y) \geq \nu_t \) for all \(y\),
- \(\lim_{y \to +\infty} \nu_t(y) = +\infty\) and,
- \(V_t(x, s) \geq \nu_t(||x||)\) for all \(x \in \mathbb{R}^n, s \in \mathcal{S}\).

Clearly predicate \(P_T\) holds since \(\sigma^T\) is convex and continuous and according to Assumption 4,
\[
V_T(x, s) = \sigma^T(x) - g^T(s) \geq \beta_T(||x||) - \max_{s \in \mathcal{S}} g^T(s) \triangleq \nu_T(||x||) \geq v_T
\]
where \(v_T \triangleq d_T - \max_{s \in \mathcal{S}} g^T(s)\).

Let us now assume that predicate \(P_{t+1}\) holds for \(t \leq T - 1\) and show that \(P_t\) holds. Note that
\[
\begin{align*}
|\nu_t(x, s)| & = \sigma^t(x, u) - g^t(a, s) \\
& + \sum_{i=1}^{n} P_{a_i}^t \left[(1 - q_i^t) V_{t+1}(A_i x + B_i u, s) + q_i^t V_{t+1}(A_i x + B_i u, s)\right],
\end{align*}
\]

(63)
Using Assumption 4 we then find that for all $a \in \mathcal{A}$,
\[
\left[\mathcal{V}_t(x, s, u)\right]_a \geq \alpha_t(\|x\|) + \beta_t(\|u\|) - \max_{s \in \mathcal{S}} g_t(a, s) + \sum_{i=1}^{\mathcal{F}} P_{t,i}(a) \left[ (1 - q_t^i) v_{t+1}(\|A_t x\|) + q_t^i v_{t+1}(\|A_t x + B_t u\|) \right] \\
\geq \alpha_t(\|x\|) + \beta_t(\|u\|) - \max_{s \in \mathcal{S}} g_t(a, s) + v_{t+1}.
\]  \hspace{1cm} (64)

That is, for all fixed $(x, s) \in \mathcal{X}$, $a \in \mathcal{A}$, the function $\left[\mathcal{V}_t(x, s, \cdot)\right]_a$ is coercive on $\mathbb{R}^m$. Also, since according to predicate $P_{t+1}$, $V_{t+1}(\cdot, s)$ is a continuous function, then by Assumption 5 the function $\left[\mathcal{V}_t(x, s, \cdot)\right]_a$ is continuous on $\mathbb{R}^m$ for all $x \in \mathbb{R}^n$, $a \in \mathcal{A}$, and $s \in \mathcal{S}$. Finally, we note that since $V_{t+1}(\cdot, s)$ is a convex function by assumption, and $A_t x + B_t u$ is linear with respect to $u$, then $V_{t+1}(A_t x + B_t u, s)$ is convex in $u$. Also, $\sigma^t(x, u)$ is convex in $u$ by Assumption 4. Hence we conclude that $\left[\mathcal{V}_t(x, s, \cdot)\right]_a$ is convex.

We have verified all the conditions of Lemma 1, which can now be applied to ascertain that the inf sup and sup inf expressions in (47) and (48) are equal and finite. Thus, the function $V_t(x, s, u)$ is well-defined, and there exists a saddle point pair of strategies $(u^*, y^*)$ of $(x, s)$, $(p^*)$ of $(x, s)$ defined by the static zero-sum game (47). Furthermore, it follows from (64) that $V_t(x, s)$ satisfies properties (a)-(c) stated in predicate $P_t$ with the function $\nu_t(\cdot)$ and constant $v_t$ defined as
\[
\nu_t(y) = \alpha_t(y) + d_t - \max_{a \in \mathcal{A}} \max_{s \in \mathcal{S}} g_t(a, s) + v_{t+1};
\]
\[
v_t = e_t + d_t - \max_{a \in \mathcal{A}} \max_{s \in \mathcal{S}} g_t(a, s) + v_{t+1}.
\]

It remains to prove that $V_t(\cdot, s)$ is convex and continuous. For continuity, we note that composition of $V_{t+1}(\cdot, s)$ and $A_t x + B_t u$ is continuous uniformly in $u$, since the latter function has this property and the former function is continuous. Then $P_{t+1}^a(a) q_t^i V_{t+1}(A_t x + B_t u, s)$ is also continuous uniformly in $u$. Therefore, for every $\tilde{x} \in \mathbb{R}^n$ and $\epsilon > 0$ one can find $\delta^{a,s}(\tilde{x}, \epsilon) > 0$ which does not depend on $u$ and such that $\|x - \tilde{x}\| < \delta^{a,s}(\tilde{x}, \epsilon)$ implies $\left|\left(\mathcal{V}_t(x, s, u)\right)_a - \left(\mathcal{V}_t(\tilde{x}, s, u)\right)_a\right| < \epsilon$.

Selecting $\delta(\tilde{x}, \epsilon) = \min_{a, s} \delta^{a,s}(\tilde{x}, \epsilon)$ we obtain
\[
p^t \mathcal{V}_t(\tilde{x}, s, u) + \frac{\epsilon}{2} < p^t \mathcal{V}_t(x, s, u) + p^t \mathcal{V}_t(\tilde{x}, s, u) + \frac{\epsilon}{2}
\]
\[
\forall x \in \{x : \|x - \tilde{x}\| \leq \delta(\tilde{x}, \epsilon)\}, \quad p \in \mathcal{N}_{-1}.
\]

From here we readily conclude that
\[
\inf_{u} \sup_{p \in \mathcal{N}_{-1}} p^t \mathcal{V}_t(\tilde{x}, s, u) - \epsilon \\
\leq \inf_{u} \sup_{p \in \mathcal{N}_{-1}} p^t \mathcal{V}_t(x, s, u) < \inf_{u} \sup_{p \in \mathcal{N}_{-1}} p^t \mathcal{V}_t(\tilde{x}, s, u) + \epsilon,
\]
for all $x \in \{x : \|x - \tilde{x}\| \leq \delta(\tilde{x}, \epsilon)\}$, proving that $V_t(\cdot, s)$ is continuous at an arbitrarily chosen $\tilde{x}$.

For convexity of $V_t(\cdot, s)$, we note that each function $(x, u) \rightarrow (\mathcal{V}_t(x, s, u))_a$ is convex since $\sigma^t$ is convex in $(x, u)$ and $V_{t+1}(\cdot, s)$ is convex by the induction hypothesis. As a result, the function $\sup_{p \in \mathcal{N}_{-1}} p^t \mathcal{V}_t(\tilde{x}, s, u)$ is convex in $(x, u)$; see [8, Section 3.2.3]. In turn, using identity (23) of Lemma 1, we conclude that
\[
V_t(x, s) = \inf_{u \in \mathcal{A}} \sup_{p \in \mathcal{N}_{-1}} p^t \mathcal{V}_t(\tilde{x}, s, u) \\
= \inf_{u \in \mathcal{A}} \sup_{p \in \mathcal{N}_{-1}} p^t \mathcal{V}_t(\tilde{x}, s, u)
\]
is convex since $U(x)$ is a convex set.

References

[1] S. Tatikonda and S. Mitter, Control under Communication Constraints, IEEE Trans. Automat. Contr, 49(7):1056-1066, 2004.
[2] G. Nair, F. Fagnani, S. Zampieri, and R. Evans, Feedback control under data rate constraints: an overview, Proc. of the IEEE, 95(1):108-137, 2007.
[3] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S. Sastry, Foundations of control and estimation over lossy networks, Proc. of the IEEE, 95(1):163-187, 2007.
[4] O. Imer, S. Yüksel, and T. Başar, Optimal control of LTI systems over unreliable communication links, Automatica, 42(9):1429-1439, 2006.
[5] S. Amin, A. Cárdenas, and S. Sastry, Safe and secure networked control systems under denial-of-service attacks, Hybrid Systems: Computation and Control, pp. 31-45, 2009.
[6] A. Gupta, C. Langbort, and T. Başar, Optimal control in the presence of an intelligent jammer with limited actions, in Proc. 49th IEEE CDC, pp. 1096-1101, 2010.
[7] I. Csizsár and P. Narayan, Arbitrarily varying channels with constrained inputs and states, IEEE Trans. Inf. Theory, 34(1):27-34, 1988.
[8] S. Boyd and L. Vandenberghe. Convex Optimization. Cambridge University Press, 2003.
[9] T. Başar and G. Olader, Dynamic Noncooperative Game Theory. Academic Press, 1982.
[10] C. Langbort and V. Ugrinovskii, One-shot control over an AVC-like adversarial channel, in Proc. of 2012 ACC, Montreal, 2012.
[11] V. Ugrinovskii and C. Langbort, Control over adversarial packet-dropping communication networks revisited, in Proc. of 2014 ACC, Portland, 2014.
[12] A. L. Peressini, F. E. Sullivan, and J. J. Uhl, The mathematics of nonlinear programming, Springer-Verlag, 1988.
[13] W. Rudin, Principles of Mathematical Analysis. McGraw-Hill, 1964.
[14] K. Pelechrinis, M. Bifotou, and S. V. Krishnamurthy, Denial of service attacks in wireless networks: The case of jammers, IEEE Communications Surveys & Tutorials, 13:245-257, 2011.