UNIQUENESS OF MEROMORPHIC FUNCTIONS WITH THEIR HOMOGENEOUS AND LINEAR DIFFERENTIAL POLYNOMIALS SHARING A SMALL FUNCTION

Indrajit Lahiri and Bipul Pal

Abstract. In this paper we study the uniqueness question of meromorphic functions whose certain differential polynomials share a small function.

1. Introduction, definitions and results

Let \( f \) be a meromorphic function in the open complex plane \( \mathbb{C} \). We use the standard notations of Nevanlinna’s value distribution theory such as \( m(r, f) \), \( N(r, f) \), \( \overline{N}(r, f) \), \( T(r, f) \) etc. as available in [2]. We denote by \( S(r, f) \) any quantity satisfying \( S(r, f) = o(T(r, f)) \) as \( r \to \infty \) possibly outside a set of finite linear measure.

A meromorphic function \( a = a(z) \) is called a small function of \( f \) if \( T(r, a) = S(r, f) \). We denote by \( S(f) \) the collection of all small functions of \( f \). Clearly \( \mathbb{C} \subset S(f) \).

Let \( f \) and \( g \) be two meromorphic functions in \( \mathbb{C} \) and \( a \in S(f) \cap S(g) \). We say that \( f \) and \( g \) share the function \( a = a(z) \) CM (counting multiplicities) or IM (ignoring multiplicities) if \( f - a \) and \( g - a \) have the same set of zeros counting multiplicities or ignoring multiplicities respectively.

For \( a \in \mathbb{C} \cap \{\infty\} \) the quantities

\[
\delta(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} \quad \text{and} \quad \Theta(a; f) = 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, a; f)}{T(r, f)}
\]

are respectively called the deficiency and ramification index of \( a \) for the function \( f \), where \( N(r, a; f) = N(r, \frac{1}{f-a}) \), \( \overline{N}(r, a; f) = \overline{N}(r, \frac{1}{f-a}) \), \( N(r, \infty; f) = N(r, f) \) and \( \overline{N}(r, \infty; f) = \overline{N}(r, f) \).

Also \( \rho(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \) and \( \tau(f) = \limsup_{r \to \infty} \frac{T(r, f)}{r^{\rho(f)}} \) (0 < \( \rho(f) < \infty \)) are respectively called the order and type of \( f \). A meromorphic function \( f \)
is said to be of minimal type if $\tau(f) = 0$, which can be found, for example, in [2, pp. 16–17].

In 1976 Yang [10] asked to investigate the relationship between two nonconstant entire functions $f$ and $g$ if $f$ and $g$ share the value 0 CM and $f^{(1)}$ and $g^{(1)}$ share the value 1 CM. Many authors, including Shibazaki [9], Yi [13, 14], Yang and Yi [11], Hua [4], Mues and Reinders [8], Lahiri [5, 6], studied the question. Further, Yi [16], Chen, Wang and Zhang [1], Li and Li [7] and others also worked on this question and its extensions.

In 1990 Yi [13] proved the following result.

**Theorem A** ([13]). Let $f$ and $g$ be two nonconstant entire functions such that $f$, $g$ share the value 0 CM and $f^{(n)}$, $g^{(n)}$ share the value 1 CM. If $\delta(0; f) > \frac{1}{2}$, then either $f \equiv g$ or $f^{(n)} \cdot g^{(n)} \equiv 1$.

Shibazaki [9] did not consider the sharing of zeros and proved the following theorem.

**Theorem B** ([9]). Let $f$ and $g$ be two nonconstant entire functions of finite order such that $f^{(1)}$, $g^{(1)}$ share the value 1 CM. If $\delta(0; f) > 0$ and 0 is a Picard exceptional value of $g$, then either $f \equiv g$ or $f^{(1)} \cdot g^{(1)} \equiv 1$.

Yi and Yang [17], Hua [4] and many others improved Theorem B in different manners. Yi and Yang [17] proved the following result.

**Theorem C** ([17]). Let $f$ and $g$ be two nonconstant meromorphic functions such that $f^{(n)}$ and $g^{(n)}$ share the value 1 CM. If $\Theta(\infty; f) = \Theta(\infty; g) = 1$ and $\delta(0; f) + \delta(0; g) > 1$, then either $f \equiv g$ or $f^{(n)} \cdot g^{(n)} \equiv 1$.

Also Yi [16] proved the following improvement of Theorem B.

**Theorem D** ([16]). Let $f$ and $g$ be two nonconstant meromorphic functions such that $f^{(n)}$ and $g^{(n)}$ share the value 1 CM. If $\delta(0; f) + \delta(0; g) + (n + 2)\Theta(\infty; f) > n + 3$,

then either $f \equiv g$ or $f^{(n)} \cdot g^{(n)} \equiv 1$.

In [16] Yi proved some others results which improve previous ones.

**Theorem E** ([16]). Let $f$ and $g$ be two nonconstant meromorphic functions such that $f^{(n)}$ and $g^{(n)}$ share the value 1 CM. If $2\delta(0; f) + (n + 4)\Theta(\infty; f) > n + 5$ and $2\delta(0; g) + (n + 4)\Theta(\infty; g) > n + 5$,

then either $f \equiv g$ or $f^{(n)} \cdot g^{(n)} \equiv 1$.

**Theorem F** ([16]). Let $f$ and $g$ be two nonconstant meromorphic functions such that $f^{(n)}$ and $g^{(n)}$ share the value 1 IM. If $5\delta(0; f) + (4n + 7)\Theta(\infty; f) > 4n + 11$ and
5\delta(0; g) + (4n + 7)\Theta(\infty; g) > 4n + 11,

then either \( f \equiv g \) or \( f^{(n)} \cdot g^{(n)} \equiv 1 \).

In 1990 Yi [14] considered the uniqueness of entire functions when they share the value 0 CM and that their derivatives share the value 1 CM. The following result of H. X. Yi [14] is an answer to the question of C. C. Yang under a general setting.

**Theorem G** ([14]). Let \( f \) and \( g \) be two nonconstant entire functions and let \( k \) be a positive integer. If \( f \) and \( g \) share the value 0 CM, \( f^{(k)} \) and \( g^{(k)} \) share the value 1 CM and \( \delta(0; f) > \frac{1}{2} \), then either \( f \equiv g \) or \( f^{(k)} \cdot g^{(k)} \equiv 1 \).

Recently Li and Li [7] considered the problem of replacing the derivatives by linear differential polynomials generated by entire functions.

Let \( h \) be a nonconstant meromorphic function. An expression of the form

\[(1.1) \quad P(h) = h^{(k)} + a_{k-1}h^{(k-1)} + \cdots + a_{1}h^{(1)} + a_{0}h,\]

where \( a_{0}, a_{1}, \ldots, a_{k-1} \) are complex constants and \( k \) is a positive integer, is called a linear differential polynomial generated by \( h \).

Considering following example Li and Li [7] exhibited that it is not possible to replace \( f^{(k)} \) and \( g^{(k)} \) in Theorem G respectively by \( P(f) \) and \( P(g) \).

**Example 1.1** ([7]). Let \( f = \frac{i}{2}e^{-2z} \) and \( g = e^{-2z} \). If \( P(h) = h^{(2)} + 2h^{(1)} \), then \( f, g \) share the value 0 CM, \( P(f), P(g) \) share the value 1 CM and \( \delta(0; f) = 1 \) but \( f \not\equiv g \) and \( P(f) \cdot P(g) \not\equiv 1 \).

We recall the following results from Li and Li [7].

**Theorem H** ([7]). Let \( f \) and \( g \) be two nonconstant entire functions. Suppose that \( f \) and \( g \) share the value 0 CM, \( P(f) \) and \( P(g) \) share the value 1 CM and \( \delta(0; f) > \frac{1}{2} \). If \( \rho(f) \neq 1 \), then \( f \equiv g \) unless \( P(f) \cdot P(g) \equiv 1 \).

**Theorem I** ([7]). Let \( f \) and \( g \) be two nonconstant entire functions. Suppose that \( f \) and \( g \) share the value 0 CM, \( P(f) \) and \( P(g) \) share the value 1 IM and \( \delta(0; f) > \frac{1}{2} \). If \( \rho(f) \neq 1 \), then \( f \equiv g \) unless \( P(f) \cdot P(g) \equiv 1 \).

We can easily note that in Example 1.1, \( P(f) \equiv 0 \) and \( P(g) \equiv 0 \). On the other hand, in the following example we see that if \( P(f) \) and \( P(g) \) are nonconstant, then for an entire function of order 1 the conclusion of Theorem H may hold.

**Example 1.2.** Let \( f = e^{z} \) and \( g = e^{-z} \) and \( P(h) = h^{(3)} - h^{(2)} - h^{(1)} \). Then \( f \) and \( g \) share the value 0 CM, \( P(f) = -e^{z} \) and \( P(g) = -e^{-z} \) share the value 1 CM and \( \delta(0; f) = 1 \). Also \( P(f) \cdot P(g) \equiv 1 \).

In the present paper we extend the results of Li and Li [7] by including the class of entire functions of order 1. We also extend some previous results to homogeneous differential polynomials.
Let $h$ be a nonconstant meromorphic function. An expression of the form

\begin{equation}
P(h) = \sum_{k=1}^{n} a_k \prod_{j=0}^{p}(h^{(j)})^{l_{kj}},
\end{equation}

where $a_k \in S(h)$ for $k = 1, 2, \ldots, n$ and $l_{kj}$ are nonnegative integers for $k = 1, 2, \ldots, n$; $j = 0, 1, 2, \ldots, p$ and $d = \sum_{j=0}^{p} l_{kj}$ for $k = 1, 2, \ldots, n$, is called a homogeneous differential polynomial of degree $d$ generated by $h$. Also we denote by $Q$ the quantity $Q = \max_{1 \leq k \leq n} \sum_{j=0}^{p} j l_{kj}$.

Let $f$ and $g$ be two nonconstant meromorphic functions. When we consider $P(f)$ and $P(g)$, as defined by (1.2), and generated by $f$ and $g$ respectively, then we understand that the coefficients $a_k$ ($k = 1, 2, \ldots, n$) belong to $S(f) \cap S(g)$.

We now state the results of the paper.

**Theorem 1.1.** Let $f$ and $g$ be two nonconstant meromorphic functions and $a = a(z) \in S(f) \cap S(g)$ and $a \not\equiv 0, \infty$. Suppose that $P(f)$ and $P(g)$, as defined by (1.2), are nonconstant. If $P(f)$ and $P(g)$ share $a = a(z)$ IM and (1.3)

\[
\min \left\{ 5 \delta(0; f) + \frac{4Q + 7}{d} \Theta(\infty; f), \quad 5 \delta(0; g) + \frac{4Q + 7}{d} \Theta(\infty; g) \right\} > \frac{4Q + 4d + 7}{d},
\]

then either $P(f) \equiv P(g)$ or $P(f) \cdot P(g) \equiv a^2$.

**Remark 1.** If $P(f)$ and $P(g)$ share $a = a(z)$ CM, then the condition (1.3) of Theorem 1.1 can be replaced by the following

\[
\min \left\{ 2 \delta(0; f) + \frac{Q + 4}{d} \Theta(\infty; f), \quad 2 \delta(0; g) + \frac{Q + 4}{d} \Theta(\infty; g) \right\} > \frac{Q + d + 4}{d}.
\]

**Theorem 1.2.** Let $f$ and $g$ be two nonconstant meromorphic functions and $a = a(z) \not\equiv 0, \infty \in S(f) \cap S(g)$. Suppose that $P(f)$ and $P(g)$, as defined by (1.2), are nonconstant. If $f$ and $g$ share the values $0$ CM and $\infty$ IM and $P(f)$, $P(g)$ share $a = a(z)$ IM and

\[
5 \delta(0; f) + \frac{4Q + 7}{d} \Theta(\infty; f) > \frac{4Q + 4d + 7}{d},
\]

then either $P(f) \equiv P(g)$ or $P(f) \cdot P(g) \equiv a^2$.

**Theorem 1.3.** Let $f$ and $g$ be two nonconstant entire functions and $a = a(z) \not\equiv 0, \infty \in S(f) \cap S(g)$. Suppose that $P(f)$ and $P(g)$, as defined by (1.2), are nonconstant. If $f$ and $g$ share the value $0$ CM and $P(f)$, $P(g)$ share $a = a(z)$ CM and $\delta(0; f) > \frac{1}{2}$, then either $P(f) \equiv P(g)$ or $P(f) \cdot P(g) \equiv a^2$.

**Remark 2.** If $P(f)$ and $P(g)$ share $a = a(z)$ IM, then the condition $\delta(0; f) > \frac{1}{2}$ of Theorem 1.3 has to be replaced by $\delta(0; f) > \frac{1}{5}$.

As the consequences of the main results we obtain the following corollaries.
Corollary 1.1. Let \( f \) and \( g \) be two nonconstant meromorphic functions. Suppose that \( \alpha(f^{(k)})^n \) and \( \alpha(g^{(k)})^n \) are nonconstant and share the value 1 IM, where \( \alpha(\not=0) \) is a constant and \( k, n \) are positive integers. If
\[
\min \left\{ 5\delta(0; f) + \frac{4kn+7}{n}\Theta(\infty; f), 5\delta(0; g) + \frac{4kn+7}{n}\Theta(\infty; g) \right\} > \frac{4kn+4n+7}{n},
\]
then either \( \alpha^2(f^{(k)}g^{(k)})^n \equiv 1 \) or \( f \equiv \omega g \), where \( \omega^n = 1 \).

If, in addition, \( f(z_0) = g(z_0) \neq 0 \) for some \( z_0 \in \mathbb{C} \), then \( \omega = 1 \).

Corollary 1.2. Let \( f \) and \( g \) be two nonconstant entire functions such that \( P(f) \) and \( P(g) \), as defined by (1.1), are nonconstant. Suppose that \( f \) and \( g \) share the value 0 CM and \( P(f) \), \( P(g) \) share the value 1 CM. If \( \delta(0; f) > \frac{1}{2} \), then either \( f \equiv g \) or \( P(f) \cdot P(g) \equiv 1 \) under any one of the following hypotheses:

(i) \( \rho(f) \neq 1 \),
(ii) \( \rho(f) = 1 \) and

(a) \( f \) has at most a finite number of zeros, or
(b) \( f \) has infinitely many zeros and \( f \) is of minimal type.

We now recall some well known notations of the value distribution theory. Let \( F \) and \( G \) be two nonconstant meromorphic functions, which share the value 1 IM. We denote by \( \overline{N}_L(r, 1; F) \) the reduced counting function of those zeros of \( F - 1 \) in \( \{ z : |z| < r \} \), which have larger multiplicities than those of the corresponding zeros of \( G - 1 \). Also we denote by \( N_L^1(r, 1; F) \) the reduced counting function of common simple zeros of \( F - 1 \) and \( G - 1 \) in \( \{ z : |z| < 1 \} \), and denote by \( \overline{N}_E^1(r, 1; F) \) the counting function of those common multiple zeros of \( F - 1 \) and \( G - 1 \) in \( \{ z : |z| < 1 \} \), where each such common multiple zero of \( F - 1 \) and \( G - 1 \) has the same multiplicity related to \( F - 1 \) and \( G - 1 \). Likewise we define \( \overline{N}_L(r, 1; G) \), \( N_L^1(r, 1; G) \) and \( N^1_E(r, 1; G) \).

Also we denote by \( N_{12}(r, 0; F) \) the counting function of simple zeros of \( F \) and by \( \overline{N}_{12}(r, 0; F) \) the reduced counting function of multiple zeros of \( F \) in \( \{ z : |z| < r \} \).

2. Lemmas

In this section we present some necessary lemmas.

Lemma 2.1. Let \( f \) be a nonconstant meromorphic function and \( P(f) \) be defined by (1.2). Then
\[
T(r, P) \leq dT(r, f) + Q\overline{N}(r, \infty; f) + S(r, f)
\]
and
\[
N(r, 0; P) \leq T(r, P) - dT(r, f) + dN(r, 0; f) + S(r, f)
\leq Q\overline{N}(r, \infty; f) + dN(r, 0; f) + S(r, f).
\]
Proof. Since  
\[ N(r, P) \leq dN(r, f) + QN(r, \infty; f) + S(r, f) \]  and  
\[ m(r, f) \leq m(r, \frac{P}{f^d}) + m(r, f^d) = dm(r, f) + S(r, f), \]  we get  
\[ (2.1) \quad T(r, P) \leq dT(r, f) + QN(r, \infty; f) + S(r, f). \]

Now  
\[ m(r, 0; f^d) \leq m(r, 0; P) + m(r, \frac{P}{f^d}) = m(r, 0; P) + S(r, f) \]  and so  
\[ T(r, f^d) - N(r, 0; f^d) \leq T(r, P) - N(r, 0; P) + S(r, f) \]  i.e.,  
\[ (2.2) \quad N(r, 0; P) \leq T(r, P) - dT(r, f) + dN(r, 0; f) + S(r, f). \]

The lemma follows from (2.1) and (2.2). □

Lemma 2.2 ([16]). Let \( F \) and \( G \) be two nonconstant meromorphic functions such that \( F \) and \( G \) share 1 IM. Then  
\[ T(r, F) \leq N(r, 0; F) + N(r, \infty; F) + N(r, 0; G) + N(r, \infty; G) + N_F(1; F) \]  
\[ + N_L(1; F) - N_0(r, \infty; F^{(1)}) - N_0(r, 0; G^{(1)}) + S(r, F) + S(r, G), \]  
where \( N_0(r, 0; F^{(1)}) \) denotes the counting function corresponding to the zeros of \( F^{(1)} \) that are not zeros of \( F \) and \( F - 1 \), \( N_0(r, 0; G^{(1)}) \) denotes the counting function corresponding to the zeros of \( G^{(1)} \) that are not zeros of \( G \) and \( G - 1 \).

Lemma 2.3 ([2, p. 47]). Let \( f \) be a nonconstant meromorphic function and \( a_1, a_2, a_3 \) be three distinct members of \( S(f) \). Then  
\[ T(r, f) \leq N(r, 0; f - a_1) + N(r, 0; f - a_2) + N(r, 0; f - a_3) + S(r, f). \]

Lemma 2.4 ([3]). Let \( f \) be a transcendental meromorphic function and \( P(f) \), defined by (1.2), be nonconstant and \( d \geq 1 \). Then  
\[ dT(r, f) \leq N(r, \infty; f) + N(r, 1; P(f)) + dN(r, 0; f) - N_0(r, 0; (P(f))^{(1)}) + S(r, f), \]  
where \( N_0(r, 0; (P(f))^{(1)}) \) denotes the counting function corresponding to the zeros of \( (P(f))^{(1)} \) which are not the zeros of \( P(f) \) and \( P(f) - 1 \).

Remark 3. In fact Lemma 2.4 is a special case of Lemma 1 [3].

Lemma 2.5 ([12, p. 92]). Suppose that \( f_1, f_2, \ldots, f_n \) (\( n \geq 3 \)) are meromorphic functions which are not constants except for \( f_n \). Furthermore, let \( \sum_{j=1}^n f_j \equiv 1 \).

If \( f_n \neq 0 \) and  
\[ \sum_{j=1}^n N(r, 0; f_j) + (n - 1) \sum_{j=1}^n N(r, \infty; f_j) < \{ \lambda + o(1) \} T(r, f_k), \]
where $r \in I$, a set of infinite linear measure, $k = 1, 2, \ldots, n - 1$ and $0 < \lambda < 1$, then $f_n \equiv 1$.

### 3. Proof of theorems and corollaries

**Proof of Theorem 1.1.** Let $F = \frac{P(f)}{a}$ and $G = \frac{P(g)}{a}$. Then $F$ and $G$ share 1 IM and so by Lemma 2.2 we get

$$T(r, F) \leq \overline{N}(r, 0; F) + \overline{N}(r, \infty; F) + \overline{N}(r, 0; G) + \overline{N}(r, \infty; G) + N_E^1(r, 1; F)$$

(3.1)

$$+ \overline{N}_L(r, 1; F) - N_0(r, 0; F^{(1)}) - N_0(r, 0; G^{(1)}) + S(r, F) + S(r, G).$$

Let

$$H = \left( \frac{F^{(2)}}{F^{(1)}} - \frac{2F^{(1)}}{F - 1} \right) - \left( \frac{G^{(2)}}{G^{(1)}} - \frac{2G^{(1)}}{G - 1} \right).$$

We suppose that $H \neq 0$. Then by a simple calculation we see that

$$N_E^1(r, 1; F) \leq N(r, 0; H)$$

(3.2)

$$\leq T(r, H)$$

$$\leq N(r, \infty; H) + S(r, F) + S(r, G)$$

and

$$N(r, \infty; H) \leq \overline{N}_2(r, 0; F) + \overline{N}(r, \infty; F) + \overline{N}_2(r, 0; G) + \overline{N}(r, \infty; G)$$

$$+ \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + N_0(r, 0; F^{(1)}) + N_0(r, 0; G^{(1)}).$$

(3.3)

Noting that $\overline{N}(r, 0; F) + \overline{N}_2(r, 0; F) \leq N(r, 0; F)$ and combining (3.1), (3.2) and (3.3) we get

$$T(r, F) \leq N(r, 0; F) + 2\overline{N}(r, \infty; F) + N(r, 0; G) + 2\overline{N}(r, \infty; G)$$

$$+ 2\overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + S(r, F) + S(r, G).$$

(3.4)

Now by Lemma 2.1 and (3.4) we get

$$N(r, 0; F) \leq T(r, F) - dT(r, F) + dN(r, 0; f) + S(r, f)$$

$$\leq N(r, 0; F) + 2\overline{N}(r, \infty; F) + Q\overline{N}(r, \infty; g) + dN(r, 0; g)$$

$$+ 2\overline{N}(r, \infty; g) + 2\overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) - dT(r, f)$$

$$+ dN(r, 0; f) + S(r, f) + S(r, g)$$

and so

$$dT(r, f) \leq dN(r, 0; f) + 2\overline{N}(r, \infty; f) + dN(r, 0; g) + (Q - 2)\overline{N}(r, \infty; g)$$

$$+ 2\overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + S(r, f) + S(r, g).$$

(3.5)

Again using Lemma 2.1 we obtain

$$\overline{N}_L(r, 1; F) \leq N(r, 1; F) - \overline{N}(r, 1; F)$$

$$\leq N(r, 0; F^{(1)})$$

$$\leq N(r, 0; F) + \overline{N}(r, \infty; F) + S(r, F)$$
\[
\leq dN(r, 0; f) + (Q + 1)\overline{N}(r, \infty; f) + S(r, f).
\]

Similarly
\[
\overline{N}_L(r, 1; G) \leq dN(r, 0; g) + (Q + 1)\overline{N}(r, \infty; g) + S(r, g).
\]

Combining (3.5), (3.6) and (3.7) we obtain
\[
T(r, f) \leq 3N(r, 0; f) + \frac{2Q + 4}{d}N(r, \infty; f) + 2N(r, 0; g)
\]
\[
+ \frac{2Q + 3}{d}\overline{N}(r, \infty; g) + S(r, f) + S(r, g).
\]

Likewise we have
\[
T(r, g) \leq 3N(r, 0; g) + \frac{2Q + 4}{d}N(r, \infty; g) + 2N(r, 0; f)
\]
\[
+ \frac{2Q + 3}{d}\overline{N}(r, \infty; f) + S(r, f) + S(r, g).
\]

Adding (3.8) and (3.9) we obtain
\[
T(r, f) + T(r, g) \leq 5N(r, 0; f) + \frac{4Q + 7}{d}N(r, \infty; f) + 5N(r, 0; g)
\]
\[
+ \frac{4Q + 7}{d}\overline{N}(r, \infty; g) + S(r, f) + S(r, g),
\]

which implies a contradiction to the hypothesis. Therefore \(H \equiv 0\) and so on integration we get
\[
\frac{1}{G - 1} = \frac{A}{F - 1} + B,
\]

where \(A(\neq 0)\) and \(B\) are constants. This gives
\[
G = \frac{(B + 1)F + (A - B - 1)}{BF + A - B}
\]
\[
(3.10)
\]
and
\[
F = \frac{(B - A)G + (A - B - 1)}{BG - (B + 1)}.
\]
\[
(3.11)
\]

We now consider the following three cases.

**Case 1:** Let \(B \neq 0, -1\). From (3.11) we have \(\overline{N}(r, \frac{B + 1}{B}; G) = \overline{N}(r, \infty; F)\).

Now by the second fundamental theorem and Lemma 2.2 we get
\[
T(r, G) \leq N(r, 0; G) + \frac{B + 1}{B}G + \overline{N}(r, \infty; G) + S(r, G)
\]
\[
\leq T(r, G) - dT(r, g) + dN(r, 0; g) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + S(r, g)
\]
i.e.,
\[
dT(r, g) \leq dN(r, 0; g) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + S(r, g).
\]
\[
(3.12)
\]
If \( A - B - 1 \neq 0 \), from (3.10) we have \( N(r, \frac{B+1-A}{B+1}; F) = N(r, 0; G) \). Hence by the second fundamental theorem and Lemma 2.2 we get
\[
T(r, F) \leq N(r, 0; F) + \overline{N}(r, \frac{B+1-A}{B+1}; F) + \overline{N}(r, \infty; F) + S(r, F)
\]
\[
\leq T(r, F) - dT(r, f) + dN(r, 0; f) + N(r, 0; G) + \overline{N}(r, \infty; f) + S(r, f)
\]
i.e.,
\[
dT(r, f) \leq dN(r, 0; f) + dN(r, 0; g) + \overline{N}(r, \infty; f)
\]
(3.13)
\[
+ Q\overline{N}(r, \infty; g) + S(r, f) + S(r, g).
\]
Combining (3.12) and (3.13) we obtain
\[
T(r, f) + T(r, g) \leq N(r, 0; f) + \frac{2}{d} \overline{N}(r, \infty; f) + 2N(r, 0; g)
\]
\[
\frac{Q+1}{d} \overline{N}(r, \infty; g) + S(r, f) + S(r, g),
\]
a contradiction.
Hence \( A - B - 1 = 0 \) and from (3.10) we get
\[
G = \frac{(B + 1)F}{BF + 1}.
\]
Therefore \( \overline{N}(r, 0; F + \frac{1}{B}) = \overline{N}(r, \infty; G) \). Again by the second fundamental theorem and Lemma 2.2 we obtain
\[
T(r, F) \leq N(r, 0; F) + \overline{N}(r, 0; F + \frac{1}{B}) + \overline{N}(r, \infty; F) + S(r, F)
\]
\[
\leq T(r, F) - dT(r, f) + dN(r, 0; f) + \overline{N}(r, \infty; g) + \overline{N}(r, \infty; f) + S(r, f)
\]
i.e.,
(3.14) \[
dT(r, f) \leq dN(r, 0; f) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + S(r, f).
\]
Combining (3.12) and (3.14) we have
\[
T(r, f) + T(r, g) \leq N(r, 0; f) + N(r, 0; g) + \frac{2}{d} \overline{N}(r, \infty; f)
\]
\[
+ \frac{2}{d} \overline{N}(r, \infty; g) + S(r, f) + S(r, g),
\]
a contradiction.
**Case 2:** We suppose that \( B = 0 \). From (3.10) and (3.11) we have
\[
G = \frac{F + A - 1}{A} \quad \text{and} \quad F = AG + 1 - A.
\]
If \( A - 1 \neq 0 \), then it follows that
\[
N(r, 1 - A; F) = N(r, 0; G) \quad \text{and} \quad N(r, \frac{A - 1}{A}; G) = N(r, 0; F).
\]
Using the similar argument of Case 1 we arrive at a contradiction. Therefore \( A - 1 = 0 \) and so \( P(f) \equiv P(g) \).
Case 3: We suppose that \( B = -1 \). From (3.10) and (3.11) we get
\[
G = \frac{A}{A+1} \quad \text{and} \quad F = \frac{(A+1)G-A}{G}.
\]
If \( A+1 \neq 0 \), we obtain
\[
\mathcal{N}(r, A+1; F) = \mathcal{N}(r, \infty; G) \quad \text{and} \quad N(r, \frac{A}{A+1}; G) = N(r, 0; F).
\]
Using the similar argument of Case 1 we arrive at a contradiction. Therefore \( A+1 = 0 \) and so \( P(f)P(g) \equiv a^2 \). This proves the theorem. \( \square \)

Proof of Theorem 1.2. Let \( F = \frac{P(f)}{a} \) and \( G = \frac{P(g)}{a} \). Then \( F \) and \( G \) share 1 IM and so by Lemma 2.2 and Lemma 2.5 we get
\[
dT(r, f) \leq \mathcal{N}(r, \infty; f) + \mathcal{N}(r, 1; F) + dN(r, 0; f) + S(r, f)
\]
\[
= \mathcal{N}(r, \infty; g) + \mathcal{N}(r, 1; G) + dN(r, 0; g) + S(r, f)
\]
(3.15)
\[
\leq (1 + 2d + Q)T(r, g) + S(r, f) + S(r, g).
\]
Similarly
(3.16)
\[
dT(r, g) \leq (1 + 2d + Q)T(r, f) + S(r, f) + S(r, g).
\]
From (3.15) and (3.16) we get \( S(r, f) = S(r, g) \). The rest of the proof is similar to that of Theorem 1.1. This proves the theorem. \( \square \)

Proof of Corollary 1.1. By Theorem 1.1 we get either \( a^2(f^{(k)}g^{(k)})^n \equiv 1 \) or \( (f^{(k)})^n \equiv (g^{(k)})^n \). We suppose that \( (f^{(k)})^n \equiv (g^{(k)})^n \). Then \( f^{(k)} = \omega g^{(k)} \), where \( \omega \) is a constant satisfying \( \omega^n = 1 \). Integrating \( k \) times we obtain \( f = \omega g + p \), where \( p \) is a polynomial of degree at most \( k - 1 \). From the hypothesis it is clear that \( f \) and \( g \) are transcendental meromorphic functions. If \( p \neq 0 \), by Lemma 2.3 we get
\[
T(r, f) \leq N(r, 0; f) + N(r, 0; f - p) + \mathcal{N}(r, \infty; f) + S(r, f)
\]
(3.17)
\[
= N(r, 0; f) + N(r, 0; g) + \mathcal{N}(r, \infty; f) + S(r, f)
\]
and
\[
T(r, g) \leq N(r, 0; g) + N(r, 0; g + \frac{p}{\omega}) + \mathcal{N}(r, \infty; g) + S(r, g)
\]
(3.18)
\[
= N(r, 0; g) + N(r, 0; f) + \mathcal{N}(r, \infty; g) + S(r, g).
\]
Combining (3.17) and (3.18) we obtain
\[
T(r, f) + T(r, g) \leq 2N(r, 0; f) + 2N(r, 0; g) + \mathcal{N}(r, \infty; f)
\]
\[
+ \mathcal{N}(r, \infty; g) + S(r, f) + S(r, g),
\]
which contradicts the hypothesis. Therefore \( p \equiv 0 \) and so \( f \equiv \omega g \).

If, further, \( f(z_0) = g(z_0) \neq 0 \) for some \( z_0 \in \mathbb{C} \), then clearly \( \omega = 1 \) and so \( f \equiv g \). This proves the corollary. \( \square \)
Proof of Corollary 1.2. By Theorem 1.3 we get either $P(f) \equiv P(g)$ or $P(f) \cdot P(g) \equiv 1$. Let $P(f) \equiv P(g)$ so that $P(g - f) \equiv 0$. Then

\[(3.19) \quad g - f = \sum_{j=1}^{m} p_j(z)e^{\alpha_j z},\]

where $m(\leq k)$ is a positive integer, $\alpha_j$'s are distinct complex constants and $p_j(z)$'s are nonzero polynomials.

Since $f$ and $g$ share 0 CM, we can put $g = f \cdot e^h$, where $h$ is an entire function.

Let $e^h \neq 1$, otherwise we are done. So from (3.19) we get

\[f = \frac{\sum_{j=1}^{m} p_j(z)e^{\alpha_j z}}{e^h - 1}.\]

Since $f$ is entire, we see that $N(r,0;e^h - 1) \leq N(r,0;\sum_{j=1}^{m} p_j(z)e^{\alpha_j z})$ and by the second fundamental theorem we get

\[T(r, e^h) \leq N(r,\infty; e^h) + N(r,0; e^h - 1) + S(r, e^h)\]

\[\leq N\left(r,0; \sum_{j=1}^{m} p_j(z)e^{\alpha_j z}\right) + S(r, e^h)\]

\[\leq T\left(r, \sum_{j=1}^{m} p_j(z)e^{\alpha_j z}\right) + S(r, e^h)\]

\[\leq \sum_{j=1}^{m} \{T(r,p_j(z)) + T(r, e^{\alpha_j z})\} + S(r, e^h)\]

(3.20)

\[= O(\log r) + O(r) + S(r, e^h).\]

If $h$ is transcendental or a polynomial of degree at least 2, then from (3.20) we see that $T(r, e^h) = S(r, e^h)$, contradiction. Hence $h$ is a polynomial of degree at most 1.

First we assume that $h$ is a constant. Then $P(f) \equiv P(g) \equiv e^h P(f)$ and so $e^h \equiv 1$, which contradicts our assumption.

Next we assume that $h(z) = az + b$, where $a(\neq 0)$ and $b$ are constants. Then

\[f = \frac{\sum_{j=1}^{m} p_j(z)e^{\alpha_j z}}{e^{az+b} - 1} \quad \text{and so} \quad \rho(f) \leq 1.\]

We now consider the following cases.

**Case 1:** Let $\rho(f) < 1$.

Then by Milloux basic result [2, Theorem 3.2, p. 57] we get

\[T(r,f) \leq N(r,0;f) + N(1;P(f)) + S(r,f)\]

\[= N(r,0;g) + N(1;P(g)) + S(r,f)\]

\[\leq T(r,g) + T(r,P(g)) + S(r,f)\]

\[\leq T(r,g) + T(r,P(g)) + S(r,f)\]

\[\leq T(r,g) + T(r,P(g)) + S(r,f)\]

\[\leq T(r,g) + T(r,P(g)) + S(r,f)\]

(3.21)
\[ T(r, g) + m(r, P(g)) + S(r, f) \]
\[ \leq T(r, g) + m(r, g) + m\left(\frac{P(g)}{g}\right) + S(r, f) \]
\[ = 2T(r, g) + S(r, g) + S(r, f). \]  
(3.21)

Similarly
\[ T(r, g) \leq 2T(r, f) + S(r, f) + S(r, g). \]  
(3.22)

Since \( f \) and so \( g \) is of finite order, from (3.21) and (3.22) we see that \( \rho(f) = \rho(g) \). Therefore
\[ \rho(e^{az+b}) = \rho\left(\frac{g}{f}\right) \leq \max\{\rho(f), \rho(g)\} < 1, \]
which is impossible as \( a \neq 0 \).

**Case 2:** Let \( \rho(f) = 1 \).

We now consider the following subcases.

**Subcase 2.1:** Let \( f \) have at most a finite number of zeros.

We put \( f(z) = q(z)e^{cz+d} \), where \( q(z) \) is a polynomial. Then
\[ g(z) = q(z)e^{(a+c)z+(b+d)} \]
and so \( P(f) \equiv P(g) \) implies
\[ q_1(z)e^{cz+d} = q_2(z)e^{(a+c)z+(b+d)}, \]
where \( q_1, q_2 \) are polynomials. This implies \( q_2(z)e^{az+b} = q_1(z) \), which is impossible as \( a \neq 0 \).

**Subcase 2.2:** Let \( f \) have infinitely many zeros and \( f \) be of minimal type.

We put
\[ H_j(z) = -\frac{p_j(z)e^{\alpha_j z}}{f} \quad \text{for } 1 \leq j \leq m, \quad \text{and } H_{m+1}(z) = e^{az+b}. \]
Then \( f = \frac{\sum_{j=1}^{m} p_j(z)e^{\alpha_j z}}{e^{az+b} - 1} \) implies
\[ \sum_{j=1}^{m+1} H_j(z) \equiv 1. \]  
(3.23)

Let one of \( \alpha_j \)'s, say \( \alpha_1 \) be zero. Then \( H_1 \neq 0 \) and we rewrite (3.23) as
\[ \sum_{j=2}^{m+1} H_j(z) + H_1(z) \equiv 1. \]
Now
\[ \sum_{j=1}^{m+1} N(r, 0; H_j) + m \sum_{j=1}^{m+1} N(r, \infty; H_j) = \sum_{j=1}^{m+1} N(r, 0; p_j) + m^{2}\overline{N}(r, 0; f) \]
\[ = O(\log r) + m^{2}\overline{N}(r, 0; f). \]  
(3.24)
Since $e^{\alpha z} = -\frac{H_j(z)}{p_j(z)} f$, we get
$$T(r, e^{\alpha z}) \leq T(r, H_j) + T(r, f) + O(\log r).$$
This implies
$$\frac{|\alpha|}{\pi} \leq \frac{T(r, H_j)}{r} + \frac{T(r, f)}{r} + o(1)$$
and so
$$\liminf_{r \to \infty} \frac{T(r, H_j)}{r} + \limsup_{r \to \infty} \frac{T(r, f)}{r} \geq \frac{|\alpha|}{\pi}.$$ Since $f$ is of minimal type, we get
$$\liminf_{r \to \infty} \frac{T(r, H_j)}{r} \geq K \text{ for } j = 2, 3, \ldots, m,$$
where $K = \min_{2 \leq j \leq m} \frac{|\alpha_j|}{\pi} > 0$.

Hence for $j = 1, 2, \ldots, m$ we get
$$\limsup_{r \to \infty} \frac{N(r, 0; f)}{T(r, H_j)} \leq \limsup_{r \to \infty} \frac{T(r, f)}{r} \cdot \limsup_{r \to \infty} \frac{r}{T(r, H_j)} = 0.$$ Also
$$\limsup_{r \to \infty} \frac{N(r, 0; f)}{T(r, H_{m+1})} \leq \frac{\pi}{|a|} \limsup_{r \to \infty} \frac{T(r, f)}{r} = 0.$$ So from (3.24) we see that
$$\sum_{j=1}^{m+1} N(r, 0; H_j) + m \sum_{j=1}^{m+1} N(r, \infty; H_j) < \{\lambda + o(1)\} T(r, H_k)$$
for $k = 2, 3, \ldots, m+1$, where $\lambda (0 < \lambda < 1)$ is a suitable constant.

Therefore by Lemma 2.5 we get $H_1(z) \equiv 1$, which is impossible as $\rho(f) = 1$.
So, $\alpha_j \neq 0$ for $j = 1, 2, \ldots, m$. Now adopting the same technique as above we get $H_{m+1}(z) \equiv 1$, which contradicts our assumption that $e^h \neq 1$. This proves the corollary. \hfill $\square$

**Remark 4.** It is an interesting open problem to examine the validity of corollary 1.2 for entire functions $f$ and $g$ where $f$ is of unit order with nonminimal type and $f$ has infinitely many zeros.

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**References**

[1] A. Chen, X. Wang, and G. Zhang, *Unicity of meromorphic function sharing one small function with its derivative*, Int. J. Math. Math. Sci. 2010 (2010), Article Id 507454, 11 pages.

[2] W. K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford, 1964.

[3] J. D. Hinchliffe, *On a result of Chuang related to Hayman’s alternative*, Comput. Methods Funct. Theory 2 (2002), no. 1, 293–297.
[4] X. H. Hua, A unicity theorem for entire functions, Bull. London Math. Soc. 22 (1990), no. 5, 457–462.
[5] I. Lahiri, Uniqueness of meromorphic functions as governed by their differential polynomials, Yokohama Math. J. 44 (1997), no. 2, 147–156.
[6] I. Lahiri, Differential polynomials and uniqueness of meromorphic functions, Yokohama Math. J. 45 (1998), no. 1, 31–38.
[7] J. T. Li and P. Li, Uniqueness of entire functions concerning differential polynomials, Commun. Korean Math. Soc. 30 (2015), no. 2, 93–101.
[8] E. Mues and M. Reinders, On a question of C. C. Yang, Complex Var. Theory Appl. 34 (1997), no. 1-2, 171–179.
[9] K. Shibazaki, Unicity theorems for entire functions of finite order, Mem. Nat, Defence Acad. (Japan) 21 (1981), no. 3, 67–71.
[10] C. C. Yang, On two entire functions which together with their first derivatives have the same zeros, J. Math. Anal. Appl. 56 (1976), no. 1, 1–6.
[11] C. C. Yang and H. X. Yi, A unicity theorem for meromorphic functions with deficient value, Acta Math. Sinica 37 (1994), no. 1, 62–72.
[12] C. C. Yang and H. X. Yi, Uniqueness Theory of Meromorphic Functions, Science Press, Beijing and Kluwer Academic Publishers, New York, 2003.
[13] H. X. Yi, Uniqueness of meromorphic functions and a question of C. C. Yang, Complex Var. Theory Appl. 14 (1990), no. 1-4, 169–176.
[14] H. X. Yi, A question of C. C. Yang on the uniqueness of entire functions, Kodai Math. J. 13 (1990), no. 1, 39–46.
[15] H. X. Yi, Unicity theorems for entire or meromorphic functions, Acta Math. Sin. (N.S.) 10 (1994), no. 2, 121–131.
[16] H. X. Yi, Uniqueness theorems for meromorphic functions whose nth derivatives share the same 1-points, Complex Var. Theory Appl. 34 (1997), no. 4, 421–436.
[17] H. X. Yi and C. C. Yang, A unicity theorem for meromorphic functions whose nth derivatives share the same 1-points, J. Anal. Math. 62 (1994), 261–270.

Indrajit Lahiri  
Department of Mathematics  
University of Kalyani  
West Bengal 741235, INDIA  
E-mail address: ilahiri@hotmail.com

Bipul Pal  
Department of Mathematics  
University of Kalyani  
West Bengal 741235, INDIA  
E-mail address: palbipul86@gmail.com