Non-hermitian radial momentum operator and path integrals in polar coordinates

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Abstract

A salient feature of the Schrödinger equation is that the classical radial momentum term $p^2$ in polar coordinates is replaced by the operator $\hat{P}_r^\dagger \hat{P}_r$, where the operator $\hat{P}_r$ is not hermitian in general. This fact has important implications for the path integral and semi-classical approximations. When one defines a formal hermitian radial momentum operator $\hat{p}_r = (1/2) \left( (\hat{\vec{x}}_r)^2 \hat{\vec{p}} + \hat{\vec{p}} (\hat{\vec{x}}_r) \right)$, the relation $\hat{P}_r^\dagger \hat{P}_r = \hat{p}_r^2 + \hbar^2 (d-1)(d-3)/(4r^2)$ holds in $d$-dimensional space and this extra potential appears in the path integral formulated in polar coordinates. The extra potential, which influences the classical solutions in the semi-classical treatment such as in the analysis of solitons and collective modes, vanishes for $d = 3$ and attractive for $d = 2$ and repulsive for all other cases $d \geq 4$. This extra term induced by the non-hermitian operator is a purely quantum effect, and it is somewhat analogous to the quantum anomaly in chiral gauge theory.

1 Introduction

It is known that one needs to add an extra potential term to the naive kinetic term in $d = 2$ dimensions when one defines the path integral in polar coordinates [1]. This extra term has been later analyzed from a point of view of path integrals in curved space [2, 3]. A connection of this extra potential with the hermitian radial momentum operator has also been discussed in [4]. This extra term is important in the quantum analyses of solitons [5] and collective modes [6]. The importance of this extra term was recently re-emphasized by Jackiw [7].

In the present note, we discuss this problem from a general point of view of the treatment of non-hermitian radial momentum operators in arbitrary dimensions. Our basic observation is that the classical Hamiltonian in polar coordinates

$$ H_{cl} = \frac{1}{2m} [p_r^2 + \frac{\vec{L}^2}{r^2}] + V(r) $$

(1)
is replaced by the quantized Hamiltonian

\[ \hat{H} = \frac{1}{2m} [\hat{P}_r^\dagger \hat{P}_r + \frac{\hat{L}^2}{r^2}] + V(r) \]  

(2)

where the operator \( \hat{P}_r \) is not hermitian in general and \( \hat{L}^2 \) stands for the quadratic Casimir operator of the rotation group in \( d \)-dimensional space. It is then shown that

\[ \hat{P}_r^\dagger \hat{P}_r = \hat{p}_r^2 + \frac{\hbar^2 (d-1)(d-3)}{4r^2} \]  

(3)

when one defines the formal hermitian radial operator \( \hat{p}_r = (1/2) \left( \frac{\hat{\mathbf{x}}}{r} \hat{\mathbf{p}} + \hat{\mathbf{p}} \frac{\hat{\mathbf{x}}}{r} \right) \) in general \( d \)-dimensional space. We then present an explicit construction of the path integral in polar coordinates starting with the quantum evolution operator. It is shown that the use of the hermitian or non-hermitian radial operator does not matter in the time slicing of the quantum evolution operator. But the formal hermitian operator gives a natural definition of the “radial plane wave”, and thus it is essential to write the path integral in the conventional form.

We also briefly mention that the appearance of an extra term induced by the non-hermitian operator and a technical aspect of the analysis are somewhat analogous to the quantum anomaly in chiral gauge theory.

2 Hermitian radial momentum operator

We start with the identity in the classical level in general \( d \)-dimensional space

\[ \left( \sum_i x_i p_i \right)^2 + \sum_{i \neq j} \frac{1}{2} (x_i p_j - x_j p_i)^2 = \sum_{i,j} x_i^2 p_j^2 = (\sum_i x_i^2)(\sum_j p_j^2) \]  

(4)

Namely, we have the relation

\[ (\hat{\mathbf{p}})^2 = (\sum_i \frac{x_i}{r} p_i)^2 + \sum_{i \neq j} \frac{1}{2} \frac{1}{r^2} L_{i,j}^2 \]  

(5)

with

\[ r^2 = \sum_i x_i^2, \]

\[ L_{i,j} = x_i p_j - x_j p_i, \quad i \neq j. \]  

(6)
We thus have the classical Hamiltonian

\[ H_{cl} = \frac{1}{2m}(\vec{p})^2 + V(r) \]
\[ = \frac{1}{2m}p_r^2 + \frac{1}{2m} \sum_{i \neq j} \frac{1}{2} \frac{1}{r^2} L_{i,j}^2 + V(r) \]  

(7)

with

\[ p_r = \sum_i \frac{x_i}{r} \hat{p}_i. \]  

(8)

One may define a general form of the quantized Hamiltonian by

\[ \hat{H} = \frac{1}{2m} \hat{P}_r^\dagger \hat{P}_r + \frac{1}{2m} \sum_{i \neq j} \frac{1}{2} \frac{1}{r^2} \hat{L}_{i,j}^2 + V(r) \]  

(9)

where \( \hat{P}_r \) stands for the quantized version of \( p_r \) in (8) which contains an operator ordering ambiguity. The quantized \( \hat{P}_r \) is not hermitian in general, and we transcribe the radial kinetic term by a quantized version \( \hat{P}_r^\dagger \hat{P}_r \). The second term on the right-hand side of (9) does not contain any operator ordering problem since \( \hat{L}_{i,j} \) itself has no ordering ambiguity and

\[ [\hat{L}_{i,j}, r] = 0 \]  

(10)

as \( \hat{L}_{i,j} \) generates the rotation in the \( i - j \) plane.

By noting the relation

\[ (\hat{p})^2 = \hat{P}_r^\dagger \hat{P}_r + \sum_{i \neq j} \frac{1}{2} \frac{1}{r^2} \hat{L}_{i,j}^2 \]  

(11)

the term \( \hat{P}_r^\dagger \hat{P}_r \) needs to be positive semi-definite, which is indeed the case by its construction since \( \hat{P}_r^\dagger \hat{P}_r \geq 0 \) independently of the detailed definition of \( \hat{P}_r \). Secondly, \([ (\hat{p})^2, c] = 0 \) for any constant \( c \) and thus

\[ [\hat{P}_r^\dagger \hat{P}_r, c] = 0 \]  

(12)

should hold. This suggests that

\[ \hat{P}_r = \sum_i \frac{\hat{x}_i}{r} \hat{p}_i = \frac{\hbar}{i} \frac{\partial}{\partial r} \]  

(13)
and thus
\[ \hat{\mathbf{P}}^\dagger_r = \sum_i \hat{p}_r \hat{\mathbf{x}}_i = \hbar \left( \frac{\partial}{\partial r} + \frac{d-1}{r} \right). \] (14)

We note that
\[ [\hat{\mathbf{P}}_r, r] = [\hat{\mathbf{P}}^\dagger_r, r] = \frac{\hbar}{i} \] (15)

and the general definition of \( \hat{\mathbf{P}}_r \) satisfies the canonical commutation relation.

The quantized Hamiltonian is thus fixed to be
\[
\hat{H} = \frac{1}{2m} \left( \frac{\hbar^2}{i} \left( \frac{\partial}{\partial r} + \frac{d-1}{r} \right) \frac{\partial}{\partial r} + \frac{1}{2} \sum_{i \neq j} \frac{1}{2 r^2} \hat{L}^2_{i,j} + V(r) \right)
\]
\[
= \frac{1}{2m} \left( \frac{\hbar^2}{i} \frac{1}{r^{(d-1)}} \frac{\partial}{\partial r} \frac{1}{r^{(d-1)}} \right) + \frac{1}{2} \sum_{i \neq j} \frac{1}{2 r^2} \hat{L}^2_{i,j} + V(r),
\] (16)

the radial part of which agrees with the radial part of the Laplacian in polar coordinates in general d-dimensional space.

It is shown later that a formal hermitian radial momentum operator \( \hat{p}_r \), which defines the “radial plane wave” naturally, is essential to define the conventional form of the path integral for the radial component starting with the quantum evolution operator. One may define the formal hermitian operator by
\[ \hat{p}_r = \frac{1}{2} \sum_i \left\{ \left( \frac{\hat{\mathbf{P}}_r}{r} \right) \hat{\mathbf{p}}_i + \hat{\mathbf{p}}_i \left( \frac{\hat{\mathbf{P}}_r}{r} \right) \right\} \]
\[ = \frac{\hbar}{i} \frac{1}{r^{(d-1)}} \frac{\partial}{\partial r} r^{(d-1)} \]
\[ = \hat{p}_r^\dagger \] (17)
in d-dimensional space. This operator \( \hat{p}_r = (\hat{P}_r + \hat{P}_r^\dagger)/2 \) also satisfies the canonical commutation relation
\[ [\hat{p}_r, r] = \frac{\hbar}{i}. \] (18)

By using the relation
\[
(\hat{p}_r)^2 = -\hbar^2 \frac{1}{r^{(d-1)}} \frac{\partial^2}{\partial r^2} \frac{1}{r^{(d-1)}} \frac{1}{r^2}
\]
\[ = -\hbar^2 \frac{1}{r^{(d-1)}} \frac{\partial}{\partial r} \frac{1}{r^{(d-1)}} \frac{\partial}{\partial r} - \hbar^2 \frac{(d-1)(d-3)}{4} \frac{1}{r^2}
\]
\[ = \hat{P}_r^\dagger \hat{P}_r - \hbar^2 \frac{(d-1)(d-3)}{4} \frac{1}{r^2}, \] (19)
the quantized Hamiltonian (16) is finally written as

\[ \hat{H} = \frac{1}{2m} \hat{p}_r^2 + \frac{1}{2m} \hbar^2 (d-1)(d-3) \frac{1}{4r^2} + \frac{1}{2m} \sum_{i \neq j} \frac{1}{2r^2} \hat{L}_{i,j}^2 + V(r) \]  

(20)
in terms of the hermitian radial momentum operator \( \hat{p}_r \). We note that \( r \) in coordinate space and \( p_r \) in momentum space are not quite symmetric; \( r = |\vec{r}| \geq 0 \) but \( \infty > p_r = \sum_i (x_i/r)p_i > -\infty \) and \( |p_r| \neq |\vec{p}| \) in general.

To find the values of the quadratic Casimir operator of the rotation group \( SO(d) \)

\[ \hat{C}_2 = \sum_{i \neq j} \frac{1}{2} \hat{L}_{i,j}^2, \]  

(21)

we recall that the basis set defined by the \( l \)-th order homogeneous terms of \( x_1, ..., x_d \), namely, \( x_1 \ x_2^{l_2} ... x_d^{l_d} \) with \( \sum_k l_k = l \), span an invariant space under the action of \( \hat{L}_{i,j} \) which keeps \( r \) invariant. We thus consider \( (a_1 x_1 + ... + a_d x_d)^l = r^l Y_1 \) with complex numbers \( a_1 \sim a_d \) which satisfy the condition \(^1\) \( a_1^2 + ... + a_d^2 = 0 \). We then have

\[ -\hbar^2 \Delta (a_1 x_1 + ... + a_d x_d)^l = \left[-\hbar^2 \frac{1}{r^{(d-1)}} \frac{\partial}{\partial r} r^{(d-1)} \frac{\partial}{\partial r} + \hat{C}_2 \frac{1}{r^2}\right] r^l Y_1 \]

\[ = 0 \]  

(22)

and thus

\[ \hat{C}_2 Y_1 = \hbar^2 l (l + d - 2) Y_1. \]  

(23)

The radial part of the Hamiltonian is thus written as

\[ \hat{H}_l = \frac{1}{2m} \hat{p}_r^2 + \frac{1}{2m} \hbar^2 (d-1)(d-3) \frac{1}{4r^2} + \frac{1}{2m} \hbar^2 l (l + d - 2) \frac{1}{r^2} + V(r). \]  

(24)

The Casimir operator \( \hat{C}_2 \) is explicitly written in terms of angular variables in the polar coordinates such as

\[ x_1 = r \cos \theta_1, \]
\[ x_2 = r \sin \theta_1 \cos \theta_2, \]
\[ x_3 = r \sin \theta_1 \sin \theta_2 \cos \theta_3, \]
\[ \ldots \]
\[ x_{d-1} = r \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{(d-2)} \cos \phi, \]
\[ x_d = r \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{(d-2)} \sin \phi \]  

(25)

\(^1\)If \( a_1^2 + ... + a_d^2 = 0 \) is not satisfied, \( Y_l \) is mixed with \( Y_{l-2} \) under the action of \( \hat{C}_2 \).
with
\[ 0 \leq \theta_1 \leq \pi, \quad 0 \leq \theta_2 \leq \pi, \quad \ldots, \quad 0 \leq \theta_{(d-2)} \leq \pi, \]
\[ 0 \leq \phi \leq 2\pi. \]  

From the final expression of the quantized Hamiltonian (20) with the formal hermitian radial momentum operator, we recognize that \( d = 3 \) is exceptional in that the extra potential vanishes, and \( d = 2 \) is special in that the extra potential is attractive. For all other cases \( d \geq 4 \), the extra potential is repulsive. This feature will be important when one considers a classical solution in the path integral as a starting point of the semi-classical analysis [5, 6].

3 Path integrals

To analyze the path integral for the radial coordinate, we start with the definition of the eigenstates for the formal hermitian radial momentum \( \hat{p}_r \) in (17) by
\[ \langle r | p_r \rangle = \frac{1}{r^{(d-1)/2}} \frac{1}{\sqrt{R}} e^{ip_r r/\hbar} \]  
which satisfies
\[ \langle r | \hat{p}_r | p_r \rangle = p_r \langle r | p_r \rangle = \frac{\hbar}{i} r^{(d-1)/2} \frac{\partial}{\partial r} \langle r | p_r \rangle. \]  

The boundary condition for \( \langle r | p_r \rangle \) may be chosen to be “periodic” inside a ball with a radius \( R \), \( 0 \leq r \leq R \), in the sense that
\[ e^{ip_r 0/\hbar} = e^{ip_r R/\hbar}, \quad p_r = \frac{2\pi \hbar n}{R}, \quad n = 0, \pm 1, \pm 2, \ldots, \]
\[ \int_0^R r^{(d-1)} dr \langle r | p'_r \rangle^* \langle r | p_r \rangle = \int_0^R dr \left( \frac{1}{\sqrt{R}} e^{ip'_r r/\hbar} \right)^* \frac{1}{\sqrt{R}} e^{ip_r r/\hbar} = \delta_{p_r,p'_r} \]  
and let \( R \to \infty \) later. This boundary condition ensures the hermiticity of \( \hat{p}_r \). \(^2\) We also have
\[ \int_{-\infty}^{\infty} Rdp_r \langle r_1 | p_r \rangle \langle p_r | r_2 \rangle = \frac{1}{(r_1 r_2)^{(d-1)/2}} \delta(r_1 - r_2). \]

\(^2\)When one imposes the conditions \( r^{(d-1)/2} \psi(r) = 0 \) at \( r = 0 \) and \( r = \infty \), one can ensure the hermiticity of \( \hat{p}_r \) in the sense \( \langle \psi, \hat{p}_r \psi \rangle = \langle \hat{p}_r \psi, \psi \rangle \). But the eigenstates of \( \hat{p}_r \) do not satisfy the boundary conditions. See, for example, [8]. We thus define the complete set by (27) and (29) with a periodic boundary condition for \( r^{(d-1)/2} \psi(r) \) in the interval \( 0 \leq r \leq R \). This complete set provides the “radial plane waves” to define the path integral (33), and it gives radial path integrals as defined in [1, 4] after the integral over momentum variables. We do not assign a physical significance to eigenvalues \( p_r \), and we use the radial plane waves just to define the path integral.
The completeness relations are then written as
\[
\sum_l \int_{-\infty}^{\infty} \frac{Rdp_r}{2\pi\hbar} |p_r,l\rangle\langle p_r,l| = 1, \\
\int_0^R r^{(d-1)} dr d\Omega |r,\Omega\rangle\langle r,\Omega| = 1, \tag{31}
\]
where the symbols \(\Omega\) and \(l\) collectively stand for all the angular variables and all the quantum numbers associated with angular freedom, respectively. Note that \(r = |\vec{r}|\) but \(p_r \neq |\vec{p}|\) and in fact we have \(-\infty < p_r < \infty\).

The path integral formula is written for \(t = 2\Delta t\), for example, in the following way. We first define
\[
\hat{H}_l = \frac{1}{2m}p_r^2 + \frac{\hbar^2}{2m} \frac{(d-1)(d-3)}{4r^2} + \frac{\hbar^2 l(l+d-2)}{2m r^2} + V(r) \\
\equiv \frac{1}{2m}p_r^2 + \tilde{V}_l(r) \tag{32}
\]
and then the conventional procedure by using the completeness relations (30) and (31) gives
\[
\langle r_f, l|e^{-\frac{i}{\hbar}\hat{H}_2\Delta t}|r_i, l\rangle \\
= \langle r_f|e^{-\frac{i}{\hbar}\hat{H}_1\Delta t}|r_i\rangle \\
= \int_0^R r_1^{(d-1)} dr_1 \langle r_f|e^{-\frac{i}{\hbar}\hat{H}_1\Delta t}|r_1\rangle \langle r_1|e^{-\frac{i}{\hbar}\hat{H}_1\Delta t}|r_i\rangle \\
= \int_0^R r_1^{(d-1)} dr_1 \int_{-\infty}^{\infty} \frac{Rdp_{r_2}}{2\pi\hbar} \frac{Rdp_{r_1}}{2\pi\hbar} \langle r_f|e^{-\frac{i}{\hbar}\hat{H}_1\Delta t}|p_{r_2}\rangle \langle p_{r_2}|p_{r_1}\rangle \langle r_1|e^{-\frac{i}{\hbar}\hat{H}_1\Delta t}|p_{r_1}\rangle \langle p_{r_1}|r_i\rangle \\
= \frac{1}{\sqrt{(r_f r_i)^{d-1}}} \int_0^R dr_1 \int_{-\infty}^{\infty} \frac{dp_{r_2}}{2\pi\hbar} \frac{dp_{r_1}}{2\pi\hbar} \exp\left\{ \frac{i}{\hbar}[(r_f - r_1)p_{r_2} + (r_1 - r_i)p_{r_1} - \frac{p_{r_2}^2}{2m} - \frac{p_{r_1}^2}{2m} + \tilde{V}_l(r_f) + \tilde{V}_l(r_i)] \Delta t \right\} \\
= \frac{1}{\sqrt{(r_f r_i)^{d-1}}}(\sqrt{\frac{m}{2\pi\hbar\Delta t}})^2 \int_0^R dr_1
\]

(The \(\delta\)-functional source at \(r = 0\) [8] is balanced by a sink at \(r = R\) in the complete set (29).) Note also that the plane waves in cartesian coordinates, which are essential to define the path integral in cartesian coordinates, do not necessarily satisfy the boundary condition of the relevant Schrödinger wave function. We expect that our formulation is valid at least for the semi-classical approximation, which is the main physical interest [5, 6] of the polar coordinate path integral.
\[
\times \exp \left\{ \frac{i}{\hbar} \frac{m}{2\Delta t} \left[ (r_f - r_1)^2 + (r_1 - r_i)^2 - (\tilde{V}_l(r_f) + \tilde{V}_l(r_1))\Delta t \right] \right\}
\]
\[
= \frac{1}{\sqrt{(r_f r_i)^{(d-1)}}} (r_f | e^{-\frac{i}{\hbar} \hat{H}_l 2\Delta t} | r_i)
\] (33)

with
\[
\hat{H}_l = \frac{1}{2m} \frac{\hat{p}^2}{\hbar^2} + \tilde{V}_l(r) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial r^2} + \tilde{V}_l(r)
\] (34)

which stands for the Hamiltonian in one-dimensional space defined by the interval \(0 \leq r \leq R\) with an effective potential \(\tilde{V}_l(r)\). In (33) we used the standard procedure
\[
\langle r_1 | e^{-\frac{i}{\hbar} \hat{H}_l \Delta t} | r_i \rangle = \frac{1}{\sqrt{(r_f r_i)^{(d-1)}}} \langle r_f | e^{-\frac{i}{\hbar} \hat{H}_l \Delta t} | r_i \rangle
\]
(36)

for an infinitesimal \(\Delta t\).

The expression for a general time interval is obtained by applying the composition law of the evolution operator to the expression (33), and one has
\[
\langle r_f | e^{-\frac{i}{\hbar} \hat{H}_l t} | r_i \rangle = \langle r_f | e^{-\frac{i}{\hbar} \hat{H}_l \Delta t} | r_i \rangle
\]
(36)

which relates the radial evolution operator in d-dimensional space on the left-hand side to the evolution operator in one-dimensional space on the right-hand side. Both expressions contain the effective potential with an extra term,
\[
\tilde{V}_l(r) = \frac{\hbar^2}{2m} \frac{(d-1)(d-3)}{4r^2} + \frac{\hbar^2 l (l + d - 2)}{2m r^2} + V(r).
\] (37)

The path integral representation of the evolution operator for the radial freedom is thus formally written as
\[
\langle r_f, l | e^{-\frac{i}{\hbar} \hat{H}_l t} | r_i, l \rangle = \langle r_f | e^{-\frac{i}{\hbar} \hat{H}_l t} | r_i \rangle
\]
\[
\int \mathcal{D}p_r \mathcal{D}r \exp \left\{ \frac{i}{\hbar} \int_0^t dt [p_r \dot{r} - H_t] \right\} = \int \mathcal{D}r \exp \left\{ \frac{i}{\hbar} \int_0^t dt \left[ \frac{m}{2} \dot{r}^2 - \frac{\hbar^2}{2m} \frac{(d-1)(d-3)}{4r^2} - \frac{\hbar^2 l(l + d - 2)}{2mr^2} - V(r) \right] \right\} \]

where the classical Hamiltonian \( H_t \) is defined by

\[
H_t = \frac{1}{2m} p_r^2 + \frac{1}{2m} \frac{\hbar^2 (d-1)(d-3)}{4r^2} + \frac{\hbar^2 l(l + d - 2)}{2mr^2} + V(r)
\]

and

\[
\mathcal{D}r \propto \prod_t dr(t)
\]

without the naively expected weight factor \( \prod_t r(t)^{(d-1)} \). The classical solution in the semi-classical analysis is defined by the Lagrangian

\[
L_t = \frac{m}{2} \dot{r}^2 - \frac{\hbar^2 (d-1)(d-3)}{2mr^2} \frac{1}{4r^2} - \frac{\hbar^2 l(l + d - 2)}{2mr^2} - V(r)
\]

and thus the classical solution is generally influenced by the extra term.

### 4 Non-hermitian operator in path integrals

To deal directly with the non-hermitian operator, we define the complete set of functions for the hermitian operator \( \hat{P}_r^\dagger \hat{P}_r \) defined by

\[
\hat{P}_r^\dagger \hat{P}_r \langle r | P \rangle = P^2 \langle r | P \rangle.
\]

We also use the notation \( \varphi_P(r) = \langle r | P \rangle \) which satisfies

\[
\int_0^\infty r^{(d-1)} dr \varphi_P(r) \varphi_P(r) = \frac{1}{P} \delta(P - P'),
\]

\[
\int_0^\infty P dP \varphi_P^\dagger(r_1) \varphi_P(r_2) = \frac{1}{(r_1 r_2)^{(d-1)/2}} \delta(r_1 - r_2).
\]

We note that the radial coordinate \( r \) and the radial momentum \( P \) are more symmetric in the present definition, and the explicit form of \( \varphi_P(r) \) in \( d = 2 \) is given by the Bessel function \( J_0(Pr) \) which satisfies

\[
\int_0^\infty P dP J_0(Pr_1) J_0(Pr_2) = \frac{1}{\sqrt{r_1 r_2}} \delta(r_1 - r_2).
\]
Then the evolution operator is written as

\[ \langle r_1, l | e^{-\frac{i}{\hbar}\hat{H}_t \Delta t} | r_i, l \rangle = \langle r_1 | e^{-\frac{i}{\hbar}\hat{H}_t \Delta t} | r_i \rangle \]
\[ \simeq \int_0^\infty PdP \langle r_1 | 1 - \frac{i}{\hbar}\hat{H}_t \Delta t | P \rangle \langle P | r_i \rangle \]
\[ \simeq \int_0^\infty PdP \varphi_P(r_1) \varphi_P^\dagger(r_i) \]
\[ \times \exp \left\{ -\frac{i}{\hbar} \left[ \frac{1}{2m} P^2 \frac{\hbar^2}{2m} \frac{l(l+d-2)}{r_i^2} + V(r_1) \right] \Delta t \right\} \quad (45) \]

for an infinitesimal \( \Delta t \). One can construct the evolution operator for a finite time interval from (45) by using the composition law of the evolution operator, and the expression on the right-hand side may be used for a numerical evaluation. But the expression on the right-hand side in (45) does not have an expression of the conventional path integral. Since the expressions in (35) and (45) differ only in the choice of the complete states in the time slicing of the evolution operator, these two expressions are equivalent to each other if one accepts the general operation

\[ \langle r_1 | e^{-\frac{i}{\hbar}\hat{H}_t \Delta t} | r_i \rangle \simeq \sum_n \langle r_1 | 1 - \frac{i}{\hbar}\hat{H}_t \Delta t | n \rangle \langle n | r_i \rangle \]
\[ = \sum_n \langle r_1 | 1 - \frac{i}{\hbar}\hat{H}_t(n) \Delta t | n \rangle \langle n | r_i \rangle \]
\[ \simeq \sum_n \langle r_1 | n \rangle \langle n | r_i \rangle e^{-\frac{i}{\hbar}\hat{H}_t(n) \Delta t} \quad (46) \]

for an infinitesimal \( \Delta t \). But the analysis of Edwards and Gulyaev [1] suggests that the validity of this kind of operation is not always obvious.

It is thus instructive to derive the expression (35) from the last expression of (45) directly. By this way, we can also explain why no more extra potential is induced in (33). For this purpose, we examine the object defined for Euclidean time

\[ \int_0^\infty PdP \varphi_P(r_1) \varphi_P^\dagger(r_i) \exp \left\{ -\frac{1}{\hbar} \frac{1}{2m} P^2 \Delta \tau \right\} \]
\[ = \int_0^\infty PdP \exp \left\{ -\frac{1}{\hbar} \frac{1}{2m} P^2 \Delta \tau \right\} \varphi_P(r_1) \varphi_P^\dagger(r_i) \]
\[ = \exp \left\{ -\frac{1}{\hbar} \frac{1}{2m} \hat{P}_r \hat{P}_r \Delta \tau \right\} \int_0^\infty PdP \varphi_P(r_1) \varphi_P^\dagger(r_i) \]
\[ = \exp \left\{ -\frac{1}{\hbar} \frac{1}{2m} \hat{P}_r \hat{P}_r \Delta \tau \right\} \frac{1}{(r_1 r_i)^{(d-1)/2}} \delta(r_1 - r_i) \]

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\begin{align*}
&= \exp\left\{-\frac{1}{\hbar} \frac{1}{2m} \hat{\mathbf{p}}^\dagger \hat{\mathbf{p}} \Delta \tau \right\} \int_{-\infty}^{\infty} \frac{R d p_r}{2\pi \hbar} \langle r_1 | p_r \rangle \langle p_r | r_i \rangle \\
&= \exp\left\{-\frac{1}{\hbar} \frac{1}{2m} [\hat{p}_r^2 + \frac{\hbar^2 (d-1)(d-3)}{4 r_1^2}] \Delta \tau \right\} \int_{-\infty}^{\infty} \frac{R d p_r}{2\pi \hbar} \langle r_1 | p_r \rangle \langle p_r | r_i \rangle \\
&= \int_{-\infty}^{\infty} \frac{R d p_r}{2\pi \hbar} \langle p_r | r_i \rangle \exp\left\{-\frac{1}{\hbar} \frac{1}{2m} [\hat{p}_r^2 + \frac{\hbar^2 (d-1)(d-3)}{4 r_1^2}] \Delta \tau \right\} \langle r_1 | p_r \rangle \\
&= \frac{1}{(r_1 r_i)^{(d-1)/2}} \int_{-\infty}^{\infty} \frac{d p_r}{2\pi \hbar} \\
&\times e^{-i p_r r_i / \hbar} \exp\left\{-\frac{1}{\hbar} \frac{1}{2m} [\hat{p}_r^2 + \frac{\hbar^2 (d-1)(d-3)}{4 r_1^2}] \Delta \tau \right\} e^{i p_r r_1 / \hbar}
\end{align*}

where we used (27) and the radial momentum operators act on the variable \( r_1 \). A new momentum operator

\[
\hat{p} = \frac{\hbar}{i} \frac{\partial}{\partial r_1}
\]

was introduced in the last line of (47).

We thus examine

\[
\int_{-\infty}^{\infty} d p_r e^{-i p_r r_i / \hbar} \exp\left\{-\frac{1}{\hbar} \frac{1}{2m} \hat{p}_r^2 \Delta - \mathcal{V}(r_1) \Delta \right\} e^{i p_r r_1 / \hbar}
\]

where we defined the simplifying notations

\[
\mathcal{V}(r_1) = \frac{\hbar^2 (d-1)(d-3)}{8 \hbar m r_1^2},
\]

\[
\Delta = \Delta \tau.
\]

The expression (49) is written as

\[
\int_{-\infty}^{\infty} d p_r e^{i p_r (r_1 - r_i) / \hbar} \exp\left\{-\frac{1}{\hbar} \frac{1}{2m} (p_r + \hat{p})^2 \Delta - \mathcal{V}(r_1) \Delta \right\}
\]

\[
= \int_{-\infty}^{\infty} \frac{d p_r}{\sqrt{\Delta}} e^{i p_r (r_1 - r_i) / \hbar \Delta \hbar} \exp\left\{-\frac{1}{\hbar} \sqrt{\Delta} p_r \hat{p} - \frac{1}{2 \hbar m} p_r^2 \Delta - \mathcal{V}(r_1) \Delta \right\}
\]

\[
= \int_{-\infty}^{\infty} \frac{d p_r}{\sqrt{\Delta}} e^{i p_r (r_1 - r_i) / \hbar \Delta \hbar} \exp\left\{-\frac{1}{\hbar m} \sqrt{\Delta} p_r \hat{p} - \mathcal{V}(r_1) \Delta \right\}
\]

by moving the plane wave \( e^{i p_r r_1 / \hbar} \) through the operator and then re-scaling the integration variable as \( p_r \rightarrow p_r / \sqrt{\Delta} \). We also used the fact that only the terms linear in \( \Delta \) are important at the end.
We now expand the last exponential factor as
\[
\exp\left\{-\frac{1}{\hbar m} \sqrt{\Delta p_r \hat{p} - \mathcal{V}(r_1) \Delta} \right\} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left(\frac{1}{\hbar m} \sqrt{\Delta p_r} \right)^n \hat{p}^{n-1} \mathcal{V}(r_1) \Delta
\]  
which is replaced in the integrand of (51) after a shift of the integration variable \( p_r \rightarrow p_r + i(r_1 - r_i) \frac{m}{\sqrt{\Delta}} \) by
\[
\int_{-\infty}^{\infty} \frac{dp_r}{\sqrt{\Delta}} e^{-\frac{p_r^2}{2\hbar m}} e^{-\frac{m(r_1-r_i)^2}{2\hbar \Delta}} 
\times [1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left(\frac{1}{\hbar m} \sqrt{\Delta p_r} + \frac{i}{\hbar} (r_1 - r_i)\right)^n \hat{p}^{n-1} \mathcal{V}(r_1) \Delta]
\]  
\[
= \sqrt{\frac{2\pi \hbar m}{\Delta}} e^{-\frac{m(r_1-r_i)^2}{2\hbar \Delta}} [1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} (r_1 - r_i)^n \nabla^{n-1} \mathcal{V}(r_1) \Delta]
\]  
\[
= \sqrt{\frac{2\pi \hbar m}{\Delta}} e^{-\frac{m(r_1-r_i)^2}{2\hbar \Delta}} [1 - \sum_{n=1}^{\infty} (r_1 - r_i)^n \frac{\hbar^2 (d-1)(d-3)}{8\hbar mr_1^{n+1}} \Delta]
\]  
This last expression shows that \( |r_1 - r_i| \sim \sqrt{\Delta} \) and thus only the term with \( n = 1 \) is important. We can thus replace (49) by
\[
\sqrt{\frac{2\pi \hbar m}{\Delta}} e^{-\frac{m(r_1-r_i)^2}{2\hbar \Delta}} \mathcal{V}(r_1) \Delta = \sqrt{\frac{2\pi \hbar m}{\Delta}} e^{-\frac{m(r_1-r_i)^2}{2\hbar \Delta}} - \frac{\hbar^2 (d-1)(d-3)}{8\hbar mr_1^{n+1}} \Delta
\]  
which establishes the equivalence of (45) with (35) and (33) after transforming back to the Minkowski metric, \( \Delta = \Delta \tau \rightarrow i\Delta t \).

This analysis shows that the use of the non-hermitian or hermitian radial momentum operator does not matter when defining the time slicing of the quantum evolution operator, but the formal hermitian radial operator is essential to write the path integral in the conventional form.

5 Conventional path integrals in \( d = 2 \)

We briefly discuss the conventional path integral for a free particle in cartesian coordinates in \( d = 2 \)
\[
\langle \vec{x}_f | e^{-\frac{i}{\hbar} \hat{H} \Delta t} | \vec{x}_i \rangle
\]  
\[
= \int \frac{d^2p_2}{(2\pi \hbar)^2} \frac{d^2x_1 d^2p_1}{(2\pi \hbar)^2} \times \exp\left\{ \frac{i}{\hbar} \left[ (\vec{x}_f - \vec{x}_1) \vec{p}_2 + (\vec{x}_1 - \vec{x}_i) \vec{p}_1 - \frac{1}{2m} \left( (\vec{p}_2)^2 + (\vec{p}_1)^2 \right) \Delta t \right] \right\}.
\](55)
The path integral for a general time interval is constructed from this expression by applying the composition law of the evolution operator, and it is known that this expression gives rise to an accurate evolution operator.

We thus examine the basic building block of the path integral

\[ \langle \vec{x}_1 | e^{-\frac{i\hat{H}\Delta t}{\hbar}} | \vec{x}_i \rangle = \int \frac{d^2p_1}{(2\pi\hbar)^2} \exp \left\{ \frac{i}{\hbar} (\vec{x}_1 - \vec{x}_i) \vec{p}_1 - \frac{1}{2m} (\vec{p}_1)^2 \Delta t \right\} \]  

(56)

by writing the integration variables in polar coordinates. We emphasize that this change of variables is for the ordinary integral and thus it should work without any complication. One may write (56) as

\[ \int \frac{1}{|\vec{x}_1 - \vec{x}_i|} \frac{dp_r dL}{(2\pi\hbar)^2} \exp \left\{ \frac{i}{\hbar} ||\vec{x}_1 - \vec{x}_i|| p_r - \frac{1}{2m} (p_r^2 + \frac{L^2}{|\vec{x}_1 - \vec{x}_i|^2}) \Delta t \right\} \]  

(57)

by defining the variables

\[ p_r = \frac{(\vec{x}_1 - \vec{x}_i)}{|\vec{x}_1 - \vec{x}_i|} \vec{p}_1, \]
\[ L = \{ (\vec{x}_1 - \vec{x}_i) \times \vec{p}_1 \}_3, \]  

(58)

and one obtains the equality

\[ (\sqrt{\frac{m}{2i\pi\hbar\Delta t}})^2 \exp \left\{ \frac{i}{\hbar} \frac{m}{2\Delta t} |\vec{x}_1 - \vec{x}_i|^2 \right\} = (\sqrt{\frac{m}{2i\pi\hbar\Delta t}})^2 \exp \left\{ \frac{i}{\hbar} \frac{m}{2\Delta t} [(r_1 - r_i)^2 + 2r_1 r_i (1 - \cos \Delta \phi)] \right\} \]  

(59)

where \( \Delta \phi = \phi_1 - \phi_i \). This result is the same as the direct evaluation of (56), and it is known that the fourth order term \( \Delta \phi^4 \) in

\[ 2r_1 r_i (1 - \cos \Delta \phi) = r_1 r_i [ (\Delta \phi)^2 - \frac{1}{12} (\Delta \phi)^4 ] \]  

(60)

is effectively replaced by the extra potential term \( \hbar^2 / (8mr_1^2) \) when one integrates over \( \Delta \phi \) as was shown by Edwards and Gulyaev [1].

Alternatively, one may define

\[ \vec{x}_1 \vec{p}_1 = r_1 p_r, \]
\[ \vec{x}_i \vec{p}_1 = |\vec{p}| r_i \cos (\phi + \Delta \phi) \]
\[ = |\vec{p}| r_i \cos \phi \cos \Delta \phi - |\vec{p}| r_i \sin \phi \sin \Delta \phi, \]  

(61)
and thus
\[ \vec{x}_i \vec{p}_1 = r_i p_r \cos \Delta \phi - \frac{L}{r_1} r_i \sin \Delta \phi \] (62)

with \( L = |\vec{p}| r_1 \sin \phi \), and
\[ (\vec{p})^2 = p_r^2 + \frac{L^2}{r_1^2}. \] (63)

Note that \( \phi = \phi_p - \phi_1 \) and \( \phi_p - \phi_i = \phi + \phi_1 - \phi_i = \phi + \Delta \phi \). Then
\[
\int \frac{d^2 p_1}{(2\pi \hbar)^2} \exp \left\{ \frac{i}{\hbar} \left[ (\vec{x}_1 - \vec{x}_i) \vec{p}_1 - \frac{1}{2m} (\vec{p}_1)^2 \Delta t \right] \right\}
= \int \frac{dL}{r_1 (2\pi \hbar)^2} dp_r \exp \left\{ \frac{i}{\hbar} \left[ (r_1 - r_i \cos \Delta \phi) p_r + \frac{L}{r_1} r_i \sin \Delta \phi \right. \right. \\
- \left. \left. \frac{1}{2m} \left( p_r^2 + \frac{L^2}{r_1^2} \right) \Delta t \right] \right\}
\approx \int \frac{dL}{\sqrt{r_1 r_i (2\pi \hbar)^2}} dp_r \exp \left\{ \frac{i}{\hbar} \left[ (r_1 - r_i) p_r + L \Delta \phi \right. \right. \\
- \left. \left. \frac{mr_1 r_i}{24 \Delta t} (\Delta \phi)^4 \right] - \frac{1}{2m} \left( p_r^2 + \frac{L^2}{r_1^2} \right) \Delta t \right\} \] (64)

where the last line agrees with the second line when one integrates over \( dL \) and \( dp_r \), and the term with \((\Delta \phi)^4\) is effectively replaced by the extra potential term \( \hbar^2/(8mr_1^2) \) when one integrates over \( \Delta \phi \) in the final path integral formula following the analysis of [1]. One can also write the integral over the angular momentum in (64) as
\[
\frac{1}{\sqrt{\Delta t}} \int_{-\infty}^{\infty} d\tilde{L} \exp \left\{ \frac{i}{\hbar} \left[ \tilde{L} \Delta \phi / \sqrt{\Delta t} - \frac{1}{2m} \frac{\tilde{L}^2}{r_1^2} \right] \right\} \] (65)

by defining \( \tilde{L} = \sqrt{\Delta t} L \).

The expression (64) may be compared with our result for \( d = 2 \), for example, in (35)

\[
\langle r_1, \phi_1 | e^{-\frac{i}{\hbar} \frac{1}{2m} (\vec{p})^2 \Delta t} | r_i, \phi_i \rangle \\
= \sum_M \int \frac{R dp_r}{2\pi \hbar} \langle r_1, \phi_1 | e^{-\frac{i}{\hbar} \frac{1}{2m} (\vec{p})^2 \Delta t} | p_r, M \rangle \langle p_r, M | r_i, \phi_i \rangle \\
= \sum_M \frac{1}{(2\pi \hbar)^2} \frac{1}{\sqrt{r_1 r_i}} dp_r \exp \left\{ \frac{i}{\hbar} \left[ (r_1 - r_i) p_r + \hbar M (\phi_1 - \phi_i) \right. \right. \\
- \left. \left. \frac{1}{2m} \left( p_r^2 - \frac{\hbar^2}{4r_1^2} + \frac{\hbar^2 M^2}{r_1^2} \right) \Delta t \right] \right\} \] (66)
by noting
\[
\langle \phi | M \rangle = \frac{1}{\sqrt{2\pi}} e^{iM\phi}, \quad \int_0^{2\pi} d\phi \langle M' | \phi \rangle \langle \phi | M \rangle = \delta_{M,M'},
\]
\[
\sum_{M=-\infty}^{\infty} \langle \phi | M \rangle \langle M | \phi' \rangle = \delta(\phi - \phi').
\]

The summation over $M$ in (66) is written as
\[
\frac{1}{\sqrt{\Delta t}} \sum_{M} \hbar \sqrt{\Delta t} \exp\left\{ \frac{i}{\hbar} [\hbar \sqrt{\Delta t} M (\phi_1 - \phi_i) / \sqrt{\Delta t} - \frac{1}{2m} (\hbar \sqrt{\Delta t})^2 M^2 / r_1^2] \right\}
\]
\[
\rightarrow \frac{1}{\sqrt{\Delta t}} \int_{-\infty}^{\infty} dL \exp\left\{ \frac{i}{\hbar} [\tilde{L}(\phi_1 - \phi_i) / \sqrt{\Delta t} - \frac{1}{2m} \tilde{L}^2 / r_1^2] \right\}
\]
by defining $\tilde{L} = \hbar \sqrt{\Delta t} M$ for $\Delta t \to 0$, which agrees with (65).

From the comparison of (55) and (66) one concludes the following: One can accurately translate the evolution operator into the path integral in cartesian coordinates as in (55), but one cannot distinguish $\hat{p}_r^2$ and $\hat{P}_r^\dagger \hat{P}_r$ in the classical expression $p_r^2$ and thus one cannot produce the extra potential from the \textit{classical} expression $(\vec{p})^2$. But the path integral (55) contains all the information and, in fact, the extra potential is reproduced by the integral over the angular variable $\phi$, as was shown by Edwards and Gulyaev [1]. On the other hand, the evolution operator is accurately translated into the conventional form of the path integral in polar coordinates when one defines the formal hermitian radial momentum operator as in (66). In this latter case, the extra potential is generated by the difference of $\hat{p}_r^2$ and $\hat{P}_r^\dagger \hat{P}_r$, and no more extra terms are generated from the angular variable as is shown in (68) and also in Section 4. See also the analysis by Arthurs [4].

## 6 Discussion and conclusion

We analyzed the problem of the extra potential, which was originally discovered when writing the path integral in polar coordinates in $d = 2$ [1], in a more general setting. We emphasized that the extra potential term appearing in the path integral in polar coordinates in general d-dimensional space is purely a quantum effect associated with the non-hermitian radial momentum operator in the Schrödinger problem, though the hermitian or non-hermitian radial momentum does not matter when defining the time slicing of the quantum evolution operator. We think that this phenomenon in the elementary Schrödinger problem is important for the pedagogical purpose also.
In the following, we briefly mention an interesting analogy of this extra term with the quantum anomaly in chiral gauge theory. In the original analysis of the chiral anomaly, the vector-like gauge theory was analyzed and thus no explicit connection with the non-hermitian operator appeared [9, 10]. In the path integral formulation of chiral anomalies [11], in particular, in the analysis of chiral gauge theory, the non-hermitian Euclidean operator provides an interesting intuitive picture of the origin of quantum symmetry breaking. The general chiral gauge theory is defined by the Dirac action [12]

\[
\int d^4x \mathcal{L} = \int d^4x \bar{\psi}(x) i \gamma^\mu (\partial_\mu - i V_\mu(x) - i A_\mu(x) \gamma_5) \psi(x) \tag{69}
\]

in the background of the vector-like gauge field \(V_\mu\) and the axial-vector gauge field \(A_\mu\); these fields may in general be non-Abelian Yang-Mills fields. In the Euclidean formulation of this problem, which defines the path integral more precisely, one deals with the basic operator [13]

\[
\mathcal{D} = \gamma^\mu (\partial_\mu - i V_\mu(x) - i A_\mu(x) \gamma_5) \tag{70}
\]

which is non-hermitian in the Euclidean sense

\[
\mathcal{D}^\dagger = \gamma^\mu (\partial_\mu - i V_\mu(x) + i A_\mu(x) \gamma_5) \neq \mathcal{D}. \tag{71}
\]

One can maintain hermiticity if one replaces \(A_\mu(x) \rightarrow i A_\mu(x)\), but then the axial gauge symmetry is spoiled.

One of the ways to analyze the anomaly in gauge symmetry is to analyze the response of the Euclidean path integral

\[
Z(V_\mu, A_\mu) = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\{\int d^4x \mathcal{L}\} \tag{72}
\]

under the gauge transformation of \(V_\mu\) and \(A_\mu\). Since the regularization is defined by the Euclidean operator \(\mathcal{D}\), the quantum breaking of the axial gauge symmetry may be intuitively attributed to the non-hermitian property of the basic operator though this argument by itself does not explain the appearance of quantum anomaly.

Another way to analyze the chiral gauge anomaly is to examine the covariant form of the anomaly on the basis of the hermitian operator

\[
\mathcal{D}^\dagger \mathcal{D} \varphi_n(x) = \lambda_n^2 \varphi_n(x), \quad \mathcal{D}^\dagger \phi_n(x) = \lambda_n^2 \phi_n(x), \tag{73}
\]

and the evaluation of the anomaly is reduced to the evaluation of [13]

\[
Tr(\exp\{-\mathcal{D}^\dagger \mathcal{D} \Delta r\}) = \sum_n \varphi_n^\dagger(x) e^{-\lambda_n^2 \Delta r} \varphi_n(x) \tag{74}
\]
which is analogous to (47). The extraction of the chiral anomaly is achieved by using a plane wave in the evaluation of this trace or index. In this sense, the use of the eigenvectors for the hermitian radial momentum operator (27) in the problem analyzed in the present paper, (47) and (53), corresponds to the use of the plane wave in the analysis of the chiral anomaly. We also note that the actual evaluation of chiral anomaly in field theory often uses the first quantized formulas in an essential way [14].

Although there are fundamental differences in the extra term we analyzed in the present paper and the chiral gauge anomaly in field theory, we find it interesting that the non-hermitian operator plays an important role in defining quantum theory and a technical aspect of the analyses is similar in these two quite different phenomena.

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