BOUNDED GEOMETRY AND CHARACTERIZATION OF POST-SINGULARLY FINITE \((p, q)\)-EXPONENTIAL MAPS

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ABSTRACT. In this paper we define a topological class of branched covering maps of the plane called \textit{topological exponential maps of type \((p, q)\)} and denoted by \(TE_{p,q}\), where \(p \geq 0\) and \(q \geq 1\). We follow the framework given in [Ji] to study the problem of combinatorially characterizing an entire map \(Pe^Q\), where \(P\) is a polynomial of degree \(p\) and \(Q\) is a polynomial of degree \(q\) using an \textit{iteration scheme defined by Thurston} and a \textit{bounded geometry condition}. We first show that an element \(f \in TE_{p,q}\) with finite post-singular set is combinatorially equivalent to an entire map \(Pe^Q\) if and only if it has bounded geometry with compactness. Thus to complete the characterization, we only need to check that the bounded geometry actually implies compactness. We show this for some \(f \in TE_{p,1}, p \geq 1\).

Our main result in this paper is that a post-singularly finite map \(f\) in \(TE_{0,1}\) or a post-singularly finite map \(f\) in \(TE_{p,1}, p \geq 1\), with only one non-zero simple branch point \(c\) such that either \(c\) is periodic or \(c\) and \(f(c)\) are both not periodic, is combinatorially equivalent to a post-singularly finite entire map of either the form \(e^{\lambda z}\) or the form \(\alpha z e^{\lambda z}\), where \(\alpha = (\lambda/p)^pe^{-\lambda(-p/\lambda)^p}\), respectively, if and only if it has bounded geometry. This is the first result in this direction for a family of transcendental holomorphic maps with critical points.

1. INTRODUCTION

Thurston asked the question “when can we realize a given branched covering map as a holomorphic map in such a way that the post-critical sets correspond?” and answered it for post-critically finite degree \(d\) branched covers of the sphere [DH]. His theorem is that a postcritically finite degree \(d \geq 2\) branched covering of the sphere, with hyperbolic orbifold, is either combinatorially equivalent to a rational map or there is a topological obstruction, now called a “Thurston obstruction”. The proof uses an iteration scheme defined for an appropriate Teichmüller space. The rational map, when it exists, is unique up to conjugation by a Möbius transformation.

Although Thurston’s iteration scheme is well defined for transcendental maps, his theorem does not naturally extend to them because the proof uses the finiteness of both the degree and the post-critical set in a crucial way. In this paper, we study a class of entire maps: maps of the form \(Pe^Q\) where \(P\) and \(Q\) are polynomials of degrees \(p \geq 0\) and \(q \geq 0\) respectively. This class, which we denote by \(E_{p,q}\) includes the exponential and polynomials. We call the topological family analogue to these analytic maps \(TE_{p,q}\) and define it below. Thurston’s question makes sense for this family and our main theorem is an answer to this question. Convergence
of the iteration scheme depends on a compactness condition defined in section 9. We approach the question of compactness, using the idea of “bounded geometry”. This point of view was originally outlined in [Ji] where the bounded geometry condition is an intermediate step that connects various topological obstructions with the characterization of rational maps. The introduction of this intermediate step makes understanding the characterization of rational maps relatively easier and the arguments are more straightforward (see [Ji, JZ, CJ]). It also gives insight into the characterization of entire and meromorphic maps.

In this paper we apply our techniques to characterize the class of post-singularly finite entire maps with exactly one asymptotic value and finitely many critical points, the model topological space $T_{E_{p,q}}$. Our first result is

**Theorem 1.** A post-singularly finite map $f$ in $T_{E_{p,q}}$ is combinatorially equivalent to a post-singularly finite entire map of the form $Pe^{\lambda z}$ if and only if it has bounded geometry and satisfies the compactness condition.

We next prove that the compactness hypothesis holds for a special family. Our main result is

**Theorem 2 (Main Theorem).** A post-singularly finite map $f$ in $T_{E_{0,1}}$ or a post-singularly finite map $f$ in $T_{E_{p,1}}$, $p \geq 1$, with only one non-zero simple branch point $c$ such that either $c$ is periodic or $c$ and $f(c)$ are both not periodic, is combinatorially equivalent to a post-singularly finite entire map of either the form $e^{\lambda z}$ or the form $\alpha z e^{\lambda z}$, where $\alpha = (-\lambda/p)p e^{-\lambda(-p/\lambda)^p}$, respectively, if and only if it has bounded geometry.

Our techniques involve adapting the Thurston iteration scheme to our situation. We work with a fixed normalization. There are two important parts to the proof of the main theorem (Theorem 2). The first is the proof of sufficiency under the assumptions of both bounded geometry and compactness. This part shows that bounded geometry together with the compactness assumption implies the convergence of the iteration scheme to an entire function of the same type (see section 9). Its proof involves an analysis of quadratic differentials associated to the functions in the iteration scheme. The proof of the first part works for a more general post-singular finite map $f \in T_{E_{p,q}}$. The second part of the proof of the main theorem is in section 10 where we define a topological constraint. We prove in this part that the bounded geometry together with the topological constraint implies compactness.

**Remark 1.** Our main result is the first result to apply the Thurston iteration scheme to characterize a family of transcendental holomorphic maps with critical points. Post-singularly finite maps $f$ in $T_{E_{0,1}}$ were also studied by Hubbard, Schleicher, and Shishikura [HSS] who used them to characterize a family of holomorphic maps with one asymptotic value, that is, the exponential family. In their study, they used a different normalization; their functions take the form $\lambda e^z$ so that they all have period $2\pi i$. They study the limiting behavior of quadratic differentials associated
to the exponential functions with finite post-singular set. They use a Levy cycle condition (a special type of Thurston’s topological condition) to characterize when it is possible to realize a given exponential type map with finite post-singular set as an exponential map by combinatorial equivalence. Their calculations involve hyperbolic geometry.

In this paper, by contrast, we normalize the maps in the Thurston iteration scheme by assuming they all fix 0, 1, ∞. We can therefore use spherical geometry in most our calculations rather than hyperbolic geometry. We are able to consider the more general exponential maps in $TE_{p,q}$. The characterization theorem for $f \in TE_{p,1}$ in this paper, is completely new.

The paper is organized as follows. In §2, we review the covering properties of $(p,q)$-exponential maps $E = Pe^Q$. In §3, we define the family $TE_{p,q}$ of $(p,q)$-topological exponential maps $f$. In §4, we define the combinatorial equivalence between post-singularly finite $(p,q)$-topological exponential maps and prove there is a local quasiconformal $(p,q)$-topological exponential map in every combinatorial equivalence class of post-singularly finite $(p,q)$-topological exponential maps. In §5, we define the Teichmüller space $T_f$ for a post-singularly finite $(p,q)$-topological exponential map $f$ and in §6, we introduce the induced map $\sigma_f$ from the Teichmüller space $T_f$ into itself; this is the crux of the Thurston iteration scheme. In §7, we define the concept of “bounded geometry” and in §8 we prove the necessity of the bounded geometry condition. In §9, we give the proof of sufficiency assuming compactness. The proofs we give in §2-9 are for $(p,q)$-topological exponential maps $f$, $p \geq 0$ and $q \geq 0$. In §10.1, §10.2, and §10.3, we define a topological constraint for the maps in our main theorem; this involves defining markings and a winding number. We prove that the winding numbers are unchanged during iteration of the map $\sigma_f$. Furthermore, in §10.4 and §10.5, we prove that the bounded geometry together with the topological constraint implies the compactness. This completes the proof of our main result.

In §11, we make some remarks about the relations between “bounded geometry” and “canonical Thurston obstructions” and between “bounded geometry” and “Levy cycles” in the context of $TE_{p,q}$.

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2. The Space $\mathcal{E}_{p,q}$ of $(p,q)$-Exponential Maps

We use the following notation: $\mathbb{C}$ is the complex plane, $\hat{\mathbb{C}}$ is the Riemann sphere and $\mathbb{C}^*$ is the complex plane punctured at the origin.

A $(p,q)$-exponential map is an entire function of the form $E = Pe^Q$ where $P$ and $Q$ are polynomials of degrees $p \geq 0$ and $q \geq 0$ respectively such that $p + q \geq 1$. We use the notation $\mathcal{E}_{p,q}$ for the set of $(p,q)$-exponential maps.

Note that if $P(z) = a_0 + a_1 z + \ldots a_p z^p$, $Q(z) = b_0 + b_1 z + \ldots b_q z^q$, $\hat{P}(z) = e^{b_0}P(z)$ and $\hat{Q}(z) = Q(z) - b_0$ then $P(z)e^{Q(z)} = \hat{P}e^{\hat{Q}(z)}$.

To avoid this ambiguity we always assume $b_0 = 0$. If $q = 0$, then $E$ is a polynomial of degree $p$. Otherwise, $E$ is a transcendental entire function with essential singularity at infinity.

The growth rate of an entire function $f$ is defined as $\limsup_{r \to \infty} \frac{\log \log M(r)}{\log r}$ where $M(r) = \sup_{|z| = r} |f(z)|$. It is easy to see that the growth rate of $E$ is $q$.

Recall that an asymptotic tract $V$ for an entire transcendental function $g$ is a simply connected unbounded domain such that $g(V) \subset \hat{\mathbb{C}}$ is conformally a punctured disk $D \setminus \{a\}$ and the map $g : V \to g(V)$ is a universal cover. The point $a$ is the asymptotic value corresponding to the tract. For functions $E$ in $\mathcal{E}_{p,q}$ we have

**Proposition 1.** If $q \geq 1$, $E$ has $2q$ distinct asymptotic tracts that are separated by $2q$ rays. Each tract maps to a punctured neighborhood of either zero or infinity and these are the only asymptotic values.

**Proof.** From the growth rate of $E$ we see that for $|z|$ large, the behavior of the exponential dominates. Since $Q(z) = b_q z^q + \text{lower order terms}$, in a neighborhood of infinity there are $2q$ branches of $\Re Q = 0$ asymptotic to equally spaced rays. In the $2q$ sectors defined by these rays the signs of $\Re Q$ alternate. If $\gamma(t)$ is a curve such that $\lim_{t \to \infty} \gamma(t) = \infty$ and $\gamma(t)$ stays in one sector for all large $t$, then either $\lim_{t \to \infty} E(\gamma(t)) = 0$ or $\lim_{t \to \infty} E(\gamma(t)) = \infty$, as $\Re Q$ is negative or positive in the sector. It follows that there are exactly $q$ sectors that are asymptotic tracts for 0 and $q$ sectors that are asymptotic tracts for infinity. Because the complement of these tracts in a punctured neighborhood of infinity consists entirely of these rays, there can be no other asymptotic tracts. \qed

**Remark 2.** The directions dividing the asymptotic tracts are called Julia rays or Julia directions for $E$. If $\gamma(t)$ tends to infinity along a Julia ray, $E(\gamma(t))$ remains in a compact domain in the plane. It spirals infinitely often around the origin.

Two $(p,q)$-exponential maps $E_1$ and $E_2$ are conformally equivalent if they are conjugate by a conformal automorphism $M$ of the Riemann sphere $\hat{\mathbb{C}}$, that is, $E_1 = M \circ E_2 \circ M^{-1}$. The automorphism $M$ must be a Möbius transformation and it must
fix both 0 and \( \infty \) so that it must be the affine stretch map \( M(z) = az, a \neq 0 \). We are interested in conformal equivalence classes of maps, so by abuse of notation, we treat conformally equivalent \((p, q)\)-exponential maps \( E_1 \) and \( E_2 \) as the same.

The critical points of \( E = Pe^Q \) are the roots of \( P' + PQ' = 0 \). Therefore, \( E \) has \( p + q - 1 \) critical points counted with multiplicity which we denote by

\[
\Omega_E = \{c_1, \cdots, c_{p+q-1}\}.
\]

Note that if \( E(z) = 0 \) then \( P(z) = 0 \). This in turn implies that if \( c \in \Omega_E \) maps to 0, then \( c \) must also be a critical point of \( P \). Since \( P \) has only \( p - 1 \) critical points counted with multiplicity, there must be at least \( q \) points (counted with multiplicity) in \( \Omega \) which are not mapped to 0. Denote by

\[
\Omega_{E,0} = \{c_1, \cdots, c_k\}, \quad k \leq p - 1,
\]

the (possibly empty) subset of \( \Omega_E \) consisting of critical points such that \( E(c_i) = 0 \). Denote its complement in \( \Omega_E \) by

\[
\Omega_{E,1} = \Omega_E \setminus \Omega_{E,0} = \{c_{k+1}, \cdots, c_{p+q-1}\}.
\]

When \( q = 0 \), \( E \) is a polynomial. The post-singular set in this special case is the same as the post-critical set. It is defined as

\[
P_E = \bigcup_{n \geq 1} E^n(\Omega_E) \cup \{\infty\}.
\]

To avoid trivial cases here we will assume that \( \#(P_E) \geq 4 \). Conjugating by an affine map \( z \to az + b \) of the complex plane, we normalize so that 0, 1 \( \in P_E \).

When \( q = 1 \) and \( p = 0 \), \( \Omega_E = \emptyset \) and \( E_{0,1} \) consists of exponential maps \( \alpha e^{\lambda z} \), \( \alpha, \lambda \in \mathbb{C}^* \). The post-singular set in this special case is defined as

\[
P_E = \bigcup_{n \geq 0} E^n(0) \cup \{\infty\}.
\]

Conjugating by an affine stretch \( z \to \alpha z \) of the complex plane, we normalize so that \( E(0) = 1 \). Note that after this normalization the family takes the form \( e^{\lambda z} \), \( \lambda \in \mathbb{C}^* \).

When \( q \geq 2 \) and \( p = 0 \) or when \( q \geq 1 \) and \( p \geq 1 \), \( \Omega_{E,1} \) is a non-empty set. Let

\[
\mathcal{V} = E(\Omega_{E,1}) = \{v_1, \cdots, v_m\}
\]

denote the set of non-zero critical values of \( E \). The post-singular set for \( E \) in the general case is now defined as

\[
P_E = \bigcup_{n \geq 0} E^n(\mathcal{V} \cup \{0\}) \cup \{\infty\}.
\]

We normalize as follows:

If \( E \) does not fix 0, which is always true if \( q \geq 2 \) and \( p = 0 \), we conjugate by an affine stretch \( z \to az \) so that \( E(0) = 1 \).

If \( E(0) = 0 \), there is a critical point in \( \Omega_{E,1} \) with \( c_{k+1} \neq 0 \) and \( v_1 = E(c_{k+1}) \neq 0 \). In this case we normalize so that \( v_1 = 1 \). The family \( E_{1,1} \) consists of functions of the form \( \alpha ze^{\lambda z} \). After normalization they take the form

\[
-\lambda e^{\lambda z}.
\]
An important family we consider in this paper is the family in $\mathcal{E}_{p,1}$, $p \geq 1$, where each map in this family has only one non-zero simple critical point. After normalization, the functions in this family take the form

$$E(z) = \alpha z^p e^{\lambda z}, \quad \alpha = \left(-\frac{\lambda}{p}\right)^p e^{-\lambda \left(-\frac{1}{p}\right)^p}.$$ 

This is the main family we will study in this paper.

3. Topological Exponential Maps of Type $(p, q)$

We use the notation $\mathbb{R}^2$ for the Euclidean plane. We define the space $\mathcal{T}E_{p,q}$ of topological exponential maps of type $(p, q)$ with $p + q \geq 1$. These are branched coverings with a single finite asymptotic value, normalized to be at zero, modeled on the maps in the holomorphic family $\mathcal{E}_{p,q}$. For a full discussion of the covering properties for this family see Zakeri [Z], and for a more general discussion of maps with finitely many asymptotic and critical values see Nevanlinna [N].

If $q = 0$, then $\mathcal{T}E_{p,0}$ consists of all topological polynomials $P$ of degree $p$: these are degree $p$ branched coverings of the sphere such that $f^{-1}(\infty) = \{\infty\}$.

If $q = 1$ and $p = 0$, the space $\mathcal{T}E_{0,1}$ consists of universal covering maps $f : \mathbb{R}^2 \to \mathbb{R}^2 \setminus \{0\}$. These are discussed at length in [HSS] where they are called topological exponential maps.

The polynomials $P$ and $Q$ contribute differently to the covering properties of maps in $\mathcal{E}_{p,q}$. As we saw, the degree of $Q$ controls the growth and behavior at infinity. Using maps $e^Q$ as our model we first define the space $\mathcal{T}E_{0,q}$.

**Definition 1.** If $q \geq 2$ and $p = 0$, the space $\mathcal{T}E_{0,q}$ consists of topological branched covering maps $f : \mathbb{R}^2 \to \mathbb{R}^2 \setminus \{0\}$ satisfying the following conditions:

i) The set of branch points, $\Omega_f = \{c \in \mathbb{R}^2 \mid \deg_c f \geq 2\}$ consists of $q - 1$ points counted with multiplicity.

ii) Let $V = \{v_1, \cdots, v_m\} = f(\Omega_f) \subset \mathbb{R}^2 \setminus \{0\}$ be the set of distinct images of the branch points. For $i = 1, \ldots, m$, let $L_i$ be a smooth topological ray in $\mathbb{R}^2 \setminus \{0\}$ starting at $v_i$ and extending to $\infty$ such that the collection of rays $\{L_1, \cdots, L_m\}$ are pairwise disjoint. Then

1) $f^{-1}(L_i)$ consists of infinitely many rays starting at points in the preimage set $f^{-1}(v_i)$. If $x \in f^{-1}(v_i) \cap \Omega_f$, there are $d_x = \deg_x f$ rays meeting at $x$ called critical rays. If $x \in f^{-1}(v_i) \setminus \Omega_f$, there is only one ray emanating from $x$; it is called a non-critical ray. Set

$$W = \mathbb{R}^2 \setminus (\bigcup_{i=1}^m L_i \cup \{0\}).$$

2) The set of critical rays meeting at points in $\Omega_f$ divides $f^{-1}(W)$ into $q = 1 + \sum_{c \in \Omega_f} (d_c - 1)$ open unbounded connected components $W_1, \cdots, W_q$.

3) $f : W_i \to W$ is a universal covering for each $1 \leq i \leq q$. 

Note that the map restricted to each \( W_i \) is a topological model for the exponential map \( z \mapsto e^z \) and the local degree at the critical points determines the number of \( W_i \) attached at the point.

We now define the space \( T_E_{p,q} \) in full generality where we assume \( p > 0 \) and there is additional behavior modeled on the role of the new critical points of \( Pe^Q \) introduced by the non-constant polynomial \( P \).

**Definition 2.** If \( q \geq 1 \) and \( p \geq 1 \), the space \( T_E_{p,q} \) consists of topological branched covering maps \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) satisfying the following conditions:

i) \( f^{-1}(0) \) consists of \( p \) points counted with multiplicity.

ii) The set of branch points, \( \Omega_f = \{ c \in \mathbb{R}^2 \mid \deg c f \geq 2 \} \) consists of \( p + q - 1 \) points counted with multiplicity.

iii) Let \( \Omega_{f,0} = \Omega_f \cap f^{-1}(0) \) be the \( k < p \) branch points that map to 0 and \( \Omega_{f,1} = \Omega_f \setminus \Omega_{f,0} \) the \( p + q - 1 - k \) branch points that do not. Note that \( \Omega_{f,1} \) contains at least \( q \) points and \( V = \{ v_1, \ldots, v_m \} = f(\Omega_{f,1}) \) is contained in \( \mathbb{R}^2 \setminus \{0\} \). For \( i = 1, \ldots, m \), let \( L_i \) be a smooth topological ray in \( \mathbb{R}^2 \setminus \{0\} \) starting at \( v_i \) and extending to \( \infty \) such that the collection of rays \( \{ L_1, \ldots, L_m \} \) are pairwise disjoint. Then

1. \( f^{-1}(L_i) \) consists of infinitely many rays starting at points in the pre-image set \( f^{-1}(v_i) \). If \( x \in f^{-1}(v_i) \setminus \Omega_{f,1} \), there is only one ray emanating from \( x \); this is a non-critical ray. If \( x \in f^{-1}(v_i) \cap \Omega_{f,1} \), there are \( d_x = \deg_x f \) critical rays meeting at \( x \). Set

\[
W = \mathbb{R}^2 \setminus (\bigcup_{i=1}^m (L_i) \cup \{0\}).
\]

2. The collection of all critical rays meeting at points in \( \Omega_{f,1} \) divides \( f^{-1}(W) \) into \( l = p + q - k = 1 + \sum_{c \in \Omega_{f,1}} (d_c - 1) \) open unbounded connected components.

3. Set \( f^{-1}(0) = \{ a_i \}_{i=1}^{p-k} \) where the \( a_i \) are distinct. Each \( a_i \) is contained in a distinct component of \( f^{-1}(W) \); label these components \( W_{i,0} \), \( i = 1, \ldots, p - k \). Then the restriction \( f : W_{i,0} \setminus \{ a_i \} \to W \) is an unbranched covering map of degree \( d_i = \deg_{a_i} f \) where \( d_i > 1 \) if \( a_i \in \Omega_{f,0} \) and \( d_i = 1 \) otherwise.

4. Label the remaining \( q \) connected components of \( f^{-1}(W) \) by \( W_{j,1} \), \( j = 1, \ldots, q \). Then the restriction \( f : W_{j,1} \to W \) is a universal covering map.

In section 3 of [Z], Zakeri proves that the \((p,q)\)-exponential maps are topological exponential maps of type \((p,q)\). The converse is also true.

**Theorem 3.** Suppose \( f \in T_E_{p,q} \) is analytic. Then \( f = Pe^Q \) for two polynomials \( P \) and \( Q \) of degrees \( p \) and \( q \). That is, an analytic topological exponential map of type \((p,q)\) is a \((p,q)\)-exponential map.

**Proof.** If \( q = 0 \), then \( f \) is a polynomial \( P \) of degree \( p \).
If $q \geq 1$, then $f$ is an entire function with $p$ roots, counted with multiplicity. Every such function can be expressed as

$$f(z) = P(z)e^{g(z)}$$

where $P$ is a polynomial of degree $p$ and $g$ is some entire function (see [Al, Section 2.3]).

Consider

$$f'(z) = (P(z)g'(z) + P'(z))e^{g(z)}.$$ 

It is also an entire function, and by assumption it has $p+q-1$ roots so that $Pg' + P'$ is a polynomial of degree $p+q-1$. It follows that $g'$ is a polynomial of degree $q-1$ and $g = Q$ is a polynomial of degree $q$. \[\square\]

Note that if $f \in \mathcal{T}E_{p,q}$, $q \neq 0$, the origin plays a special role: it is the only point with no or finitely many pre-images. The conjugate of $f$ by $z \mapsto az$, $a \in \mathbb{C}^*$, is also in $\mathcal{T}E_{p,q}$; conjugate maps are conformally equivalent.

For $f \in \mathcal{T}E_{p,q}$, we define the post-singular set as follows:

i) When $q = 0$, $E$ is a polynomial and, as mentioned in the introduction, is treated elsewhere. We therefore always assume $q \geq 1$.

ii) When $q = 1$ and $p = 0$, the post-singular set is

$$P_f = \bigcup_{n \geq 0} f^n(0) \cup \{\infty\}.$$ 

We normalize so that $f(0) = 1 \in P_f$.

iii) When $q \geq 1$ and $p \geq 1$, the set of branch points is

$$\Omega_f = \{c \in \mathbb{R}^2 \mid \text{deg}_c f \geq 2\}$$

and the post-singular set is

$$P_f = \bigcup_{n \geq 0} f^n(\mathbb{V} \cup \{0\}) \cup \{\infty\}.$$ 

If $q > 1$ or if $q = 1$ and $f(0) \neq 0$, we normalize so that $f(0) = 1 \in P_f$. If $f(0) = 0$, then, by the assumption $q \geq 1$, there is a branch point $c_{k+1} \neq 0$ such that $v_1 = f(c_{k+1}) \neq 0$. We normalize so that $v_1 = 1$.

To avoid trivial cases we assume that $\#(P_f) \geq 4$.

It is clear that, in any case, $P_f$ is forward invariant, that is,

$$f(P_f \setminus \{\infty\}) \cup \{\infty\} \subseteq P_f$$

or equivalently,

$$f^{-1}(P_f \setminus \{\infty\}) \cup \{\infty\} \supset P_f.$$ 

Note that since we assume $q \geq 1$, $f^{-1}(P_f \setminus \{\infty\}) \setminus (P_f \setminus \{\infty\})$ contains infinitely many points.

**Definition 3.** We call $f \in \mathcal{T}E_{p,q}$ post-singularly finite if $\#(P_f) < \infty$. 
4. Combinatorial Equivalence

Definition 4. Suppose $f, g$ are two post-singularly finite maps in $\mathcal{T}E_{p,q}$. We say that they are combinatorially equivalent if they are topologically equivalent so that there are homeomorphisms $\phi$ and $\psi$ of the sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$ fixing 0 and $\infty$ such that $\phi \circ f = g \circ \psi$ on $\mathbb{R}^2$ and if they satisfy the additional condition, $\phi^{-1} \circ \psi$ is isotopic to the identity of $S^2$ rel $P_f$.

The commutative diagram for the above definition is

$$
\begin{array}{ccc}
\mathbb{R}^2 & \xrightarrow{\psi} & \mathbb{R}^2 \\
\downarrow{f} & & \downarrow{g} \\
\mathbb{R}^2 & \xrightarrow{\phi} & \mathbb{R}^2
\end{array}
$$

The isotopy condition says that $P_g = \phi(P_f)$.

Consider $\mathbb{R}^2 \cup \{\infty\}$ equipped with the standard conformal structure as the Riemann sphere. Then $f \in \mathcal{T}E_{p,q}$ is a map from $\hat{\mathbb{C}}$ into itself. We say $f \in \mathcal{T}E_{p,q}$ is locally $K$-quasiconformal for some $K > 1$ if for any $z \in \hat{\mathbb{C}} \setminus \Omega_f \cup \{0\}$, there is a neighborhood $U$ of $z$ such that $f : U \to f(U)$ is $K$-quasiconformal. Since we are working with isotopies rel a finite set, the following lemma is standard.

Lemma 1. Any post-singularly finite $f \in \mathcal{T}E_{p,q}$ is combinatorially equivalent to some locally $K$-quasiconformal map $g \in \mathcal{T}E_{p,q}$.

Proof. Recall that $\Omega_f$ is the set of branch points of $f$ in $\mathbb{C}$ and 0 is only asymptotic value in $\mathbb{C}$. Consider the space $X = \mathbb{C} \setminus \Omega_f$. For every $p \in X$, let $U_p$ be a small neighborhood about $p$ such that $\phi_p = f|_U : U \to f(U) \subset \mathbb{C}$ is injective. Then $\alpha = \{(U_p, \phi_p)\}_{p \in X}$ defines an atlas on $X$ with charts $(U_p, \phi_p)$. If $U_p \cap U_q \neq \emptyset$, then $\phi_p \circ \phi_q^{-1}(z) = z : \phi_q(U_p \cap U_q) \to \phi_p(U_p \cap U_q)$. Thus all transition maps are conformal (1 - 1 and analytic) and the atlas $\alpha$ defines a Riemann surface structure on $X$ which we again denote by $\alpha$. Denote the Riemann surface by $S = (X, \alpha)$. From the uniformization theorem, $S$ is conformally equivalent to the Riemann surface $\mathbb{C} \setminus A$ with the standard complex structure induced by $\mathbb{C}$, where $A$ consists of $n = \#(\Omega_f) + 1$ points. The homeomorphism $h : \mathbb{C} \to \mathbb{C}$ with $h(0) = 0$ and $h : S = (X, \alpha) \to \mathbb{C} \setminus A$ is conformal so that $R = f \circ h^{-1} : \mathbb{C} \to \mathbb{C}$ is holomorphic with critical points at $h(\Omega_f)$ and one asymptotic value at 0. Since the set $P_f$ is finite, following the standard procedure in quasiconformal mapping theory, there is a $K$-quasiconformal homeomorphism $k : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $h$ is isotopic to $k$ rel $P_f$. The map $g = R \circ k$ is a locally $K$-quasiconformal map in $\mathcal{T}E_{p,q}$ and combinatorially equivalent to $f$. This completes the proof of the lemma. \qed

Thus without loss of generality, in the rest of the paper, we will assume that any post-singularly finite $f \in \mathcal{T}E_{p,q}$ is locally $K$-quasiconformal for some $K \geq 1$. 
5. Teichmüller Space \( T_f \)

Recall that we denote \( \mathbb{R}^2 \cup \{ \infty \} \) equipped with the standard conformal structure by \( \hat{\mathbb{C}} \). Let \( \mathcal{M} = \{ \mu \in L^\infty(\hat{\mathbb{C}}) \mid \| \mu \|_\infty < 1 \} \) be the unit ball in the space of all measurable functions on the Riemann sphere. Each element \( \mu \in \mathcal{M} \) is called a Beltrami coefficient. For each Beltrami coefficient \( \mu \), the Beltrami equation

\[
 w(z) = \mu w(z)
\]

has a unique quasiconformal solution \( w^\mu \) which maps \( \hat{\mathbb{C}} \) to itself fixing 0, 1, \( \infty \). Moreover, \( w^\mu \) depends holomorphically on \( \mu \).

Let \( f \) be a post-singularly finite map in \( TE_{p,q} \) with post-singular set \( P_f \). The Teichmüller space \( T(\hat{\mathbb{C}}, P_f) \) is defined as follows. Given Beltrami differentials \( \mu, \nu \in \mathcal{M} \) we say that \( \mu \) and \( \nu \) are equivalent in \( \mathcal{M} \), and denote this by \( \mu \sim \nu \), if \( (w^\mu)^{-1} \circ w^\nu \) is isotopic to the identity map of \( \hat{\mathbb{C}} \) rel \( P_f \). The equivalence class of \( \mu \) under \( \sim \) is denoted by \( [\mu] \). We set

\[
 T_f = T(\hat{\mathbb{C}}, P_f) = \mathcal{M} / \sim.
\]

It is easy to see that \( T_f \) is a finite-dimensional complex manifold and is equivalent to the classical Teichmüller space \( Teich(\hat{\mathbb{C}} \setminus P_f) \) of Riemann surfaces with basepoint \( \hat{\mathbb{C}} \setminus P_f \). Therefore, the Teichmüller distance \( d_T \) and the Kobayashi distance \( d_K \) on \( T_f \) coincide.

6. Induced Holomorphic Map \( \sigma_f \)

For any post-singularly finite \( f \) in \( TE_{p,q} \), there is an induced map \( \sigma = \sigma_f \) from \( T_f \) into itself given by

\[
 \sigma([\mu]) = [f^* \mu],
\]

where

\[
 f^* \mu(z) = \frac{\mu_f(z) + \mu_f((f(z))\theta(z))}{1 + \mu_f(z)f_f(z)\theta(z)}, \quad \theta(z) = \frac{\bar{f}_z}{f_z}.
\]

It is a holomorphic map so that

**Lemma 2.** For any two points \( \tau \) and \( \bar{\tau} \) in \( T_f \),

\[
 d_T(\sigma(\tau), \sigma(\bar{\tau})) \leq d_T(\tau, \bar{\tau}).
\]

The next lemma follows directly from the definitions.

**Lemma 3.** A post-singularly finite \( f \) in \( TE_{p,q} \) is combinatorially equivalent to a \((p,q)\)-exponential map \( E = PeQ \) iff \( \sigma \) has a fixed point in \( T_f \).
7. Bounded Geometry

For any $\tau_0 \in T_f$, let $\tau_n = \sigma^n(\tau_0)$, $n \geq 1$. The iteration sequence $\tau_n = [\mu_n]$ determines a sequence of finite subsets

$$P_{f,n} = w^{\mu_n}(P_f), \quad n = 0, 1, 2, \ldots.$$  

Since all $w^{\mu_n}$ fix 0, 1, $\infty$, it follows that 0, 1, $\infty$ $\in P_{f,n}$.

**Definition 5** (Spherical Version). We say $f$ has bounded geometry if there is a constant $b > 0$ and a point $\tau_0 \in T_f$ such that

$$d_{sp}(p_n, q_n) \geq b$$  

for $p_n, q_n \in P_{f,n}$ and $n \geq 0$. Here

$$d_{sp}(z, z') = \frac{|z - z'|}{\sqrt{1 + |z|^2} \sqrt{1 + |z'|^2}}$$  

is the spherical distance on $\hat{C}$.

Note that $d_{sp}(z, \infty) = \frac{|z|}{\sqrt{1 + |z|^2}}$. Away from infinity the spherical metric and Euclidean metric are equivalent. Precisely, for any bounded $S \subset \mathbb{C}$, there is a constant $C > 0$ which depends only on $S$ such that

$$C^{-1}d_{sp}(x, y) \leq |x - y| \leq C d_{sp}(x, y) \quad \forall x, y \in S.$$

Consider the hyperbolic Riemann surface $R = \hat{C} \setminus P_f$ equipped with the standard complex structure as the basepoint $\tau_0 = [0] \in T_f$. A point $\tau$ in $T_f$ defines another complex structure $\tau$ on $R$. Denote by $R_\tau$ the hyperbolic Riemann surface $R$ equipped with the complex structure $\tau$.

A simple closed curve $\gamma \subset R$ is called non-peripheral if each component of $\hat{C} \setminus \gamma$ contains at least two points of $P_f$. Let $\gamma$ be a non-peripheral simple closed curve in $R$. For any $\tau = [\mu] \in T_f$, let $l_\tau(\gamma)$ be the hyperbolic length of the unique closed geodesic homotopic to $\gamma$ in $R_\tau$. The bounded geometry property can be stated in terms of hyperbolic geometry as follows.

**Definition 6** (Hyperbolic version). We say $f$ has bounded geometry if there is a constant $a > 0$ and a point $\tau_0 \in T_f$ such that $l_{\tau_0}(\gamma) \geq a$ for all $n \geq 0$ and all non-peripheral simple closed curves $\gamma$ in $R$.

The above definitions of bounded geometry are equivalent because of the following lemma and the fact that we have normalized so that 0, 1, $\infty$ always belong to $P_f$.

**Lemma 4.** Consider the hyperbolic Riemann surface $\hat{C} \setminus X$, where $X$ is a finite subset of $\hat{C}$ such that 0, 1, $\infty \in X$, equipped with the standard complex structure. Let $a > 0$ be a constant. If every simple closed geodesic in $\hat{C} \setminus X$ has hyperbolic length greater than $a$, then the spherical distance between any two distinct points in $X$ is bounded below by a bound $b > 0$ which depends only on $a$ and $m = \#(X)$. 
Proof. If $m = 3$ there are no non-peripheral simple closed curves so in the following argument we may assume that $m \geq 4$. Let $X = \{x_1, \cdots, x_{m-1}, x_m = \infty\}$ and let $| \cdot |$ denote the Euclidean metric on $\mathbb{C}$.

Suppose $0 = |x_1| \leq \cdots \leq |x_{m-1}|$. Let $M = |x_{m-1}|$. Then $|x_2| \leq 1$, and we have

$$ \prod_{2 \leq i \leq m-2} \frac{|x_{i+1}|}{|x_i|} = \frac{|x_{m-1}|}{|x_2|} \geq M.$$ 

Hence

$$\max_{2 \leq i \leq m-2} \left\{ \frac{|x_{i+1}|}{|x_i|} \right\} \geq \frac{1}{M^{m-3}}.$$ 

Let

$$A_i = \{ z \in \mathbb{C} \mid |x_i| < z < |x_{i+1}| \}$$

and let $\mod (A_i) = \frac{1}{2\pi} \log \frac{|x_{i+1}|}{|x_i|}$ be its modulus. Then for some integer $2 \leq i_0 \leq m - 2$ it follows that

$$\mod (A_{i_0}) \geq \frac{\log M}{2\pi(m-3)}.$$ 

Denote the extremal length of the core curve $\gamma_{i_0}$ in $A_{i_0} \subset \hat{\mathcal{C}} \setminus X$ by $\|\gamma_{i_0}\|$. By properties of extremal length,

$$\|\gamma_{i_0}\| = \frac{1}{\mod (A_{i_0})} \leq \frac{2\pi(m-3)}{\log M}.$$ 

Since extremal length is defined by taking a supremum over all metrics and the area of $\hat{\mathcal{C}} \setminus X$ is $2\pi(m-2)$ for every hyperbolic metric,

$$\|\gamma_{i_0}\| \geq \frac{L^2_{\gamma_{i_0}}(\gamma)}{2\pi(m-2)} \geq \frac{a^2}{2\pi(m-2)}.$$ 

Setting $a' = \frac{a^2}{2\pi(m-2)}$, these inequalities imply

$$M \leq M_0 = e^{\frac{2\pi(m-3)}{a'^2}}.$$ 

Thus the spherical distance between $\infty$ and any finite point in $X$ has a positive lower bound $b$ which depends only on $a$ and $m$.

Next we show that the spherical distance between any two finite points in $X$ has a positive lower bound depending only on $a$ and $m$. By the equivalence of the spherical and Euclidean metrics in a bounded set in the plane, it suffices to prove that $|x - y|$ is greater than a constant $b$ for any two finite points in $X$.

First consider the map $\alpha(z) = 1/z$ which is a hyperbolic isometry from $X$ to $\alpha(X)$. It preserves the set $\{0, 1, \infty\}$ so that $0, 1, \infty \in \alpha(X)$. For any $2 \leq i \leq m - 1$, the above argument implies that $1/|x_i| \leq M_0$ and hence $|x_i| \geq 1/M_0$. Similarly, for any $x_i \in X, 2 \leq i \leq m - 1$, consider the map $\beta(z) = z/(z - x_i)$. It maps $\{0, \infty, x_i\}$ to $\{0, 1, \infty\}$ so that $\beta(X)$ contains $\{0, 1, \infty\}$ and it is also a hyperbolic isometry. For any $2 \leq i \leq m - 1$, the above argument implies that $|x_j|/|x_j - x_i| \leq M_0$ which in turn implies that $|x_j - x_i| \geq 1/M_0^2$ proving the lemma. \[\square\]
8. The Main Result—Necessity

Our main result (Theorem 2) has two parts: necessity and sufficiency. The necessity is relatively easy and can be proved for the general case. We prove the following statement.

**Theorem 4.** A post-singularly finite topological exponential map \( f \in T_{E_{p,q}} \) is combinatorially equivalent to a unique \((p,q)\)-exponential map \( E = Pe^Q \) if and only if \( f \) has bounded geometry.

**Proof of Necessity.** If \( f \) is combinatorially equivalent to \( E = Pe^Q \), then \( \sigma \) has a unique fixed point \( \tau_0 \) so that \( \tau_n = \tau_0 \) for all \( n \). The complex structure on \( \hat{C} \setminus P_f \) defined by \( \tau_0 \) induces a hyperbolic metric on it. The shortest geodesic in this metric gives a lower bound on the lengths of all geodesics so that \( f \) satisfies the hyperbolic definition of bounded geometry. \( \square \)

9. Sufficiency under Compactness

The proof of sufficiency of our main theorem (Theorem 2) is more complicated and needs some preparatory material. There are two parts: one is a compactness argument and the other is a fixed point argument. Once one has compactness, the proof of the fixed point argument is quite standard (see [Ji]) and works for any \( f \in T_{E_{p,q}} \). This is the content of Theorem 1 which we prove in this section.

The normalized functions in \( \mathcal{E}_{p,q} \) are determined by the \( p + q + 1 \) coefficients of the polynomials \( P \) and \( Q \). This identification defines an embedding into \( \mathbb{C}^{p+q+1} \) and hence a topology on \( \mathcal{E}_{p,q} \).

Given \( f \in T_{E_{p,q}} \) and given any \( \tau_0 = [\mu_0] \in T_f \), let \( \tau_n = \sigma^n(\tau_0) = [\mu_n] \) be the sequence generated by \( \sigma \). Let \( w^{\mu_n} \) be the normalized quasiconformal map with Beltrami coefficient \( \mu_n \). Then
\[
E_n = w^{\mu_n} \circ f \circ (w^{\mu_{n+1}})^{-1} \in \mathcal{E}_{p,q}
\]
since it preserves \( \mu_0 \) and hence is holomorphic. This gives a sequence \( \{E_n\}_{n=0}^{\infty} \) of maps in \( \mathcal{E}_{p,q} \) and a sequence of subsets \( P_{f,n} = w^{\mu_n}(P_f) \). Note that \( P_{f,n} \) is not, in general, the post-singular set \( P_{E_n} \) of \( E_n \). If it were, we would have a fixed point of \( \sigma \).

**The compactness condition.** We say \( f \) satisfies the compactness condition if the sequence \( \{E_n\}_{n=1}^{\infty} \) generated in the Thurston iteration scheme is contained in a compact subset of \( \mathcal{E}_{p,q} \).

From a conceptual point of view, the compactness condition is very natural and simple. From a technical point of view, however, it is not at all obvious. We give a detailed proof showing how to get a fixed point from the condition of bounded geometry with compactness.

Suppose \( f \) is a post-singularly finite topological exponential map in \( T_{E_{p,q}} \). For any \( \tau = [\mu] \in T_f \), let \( T_{\tau}T_f \) and \( T_{\tau}^*T_f \) be the tangent space and the cotangent space of...
Let $w^\mu$ be the corresponding normalized quasiconformal map fixing 0, 1, $\infty$. Then $T^*_\tau T_f$ coincides with the space $Q_\mu$ of integrable meromorphic quadratic differentials $q = \phi(z)dz^2$. Integrability means that the norm of $q$, defined by

$$||q|| = \int_C |\phi(z)|dzd\bar{z}$$

is finite. This condition implies that the poles of $q$ must occur at points of $w^\mu(P_f)$ and that these poles are simple.

Set $\tilde{\tau} = \sigma(\tau) = [\tilde{\mu}]$ and denote by $w^\mu$ and $w^{\tilde{\mu}}$ the corresponding normalized quasiconformal maps. We have the following commutative diagram:

$$\begin{array}{c}
\hat{C} \setminus f^{-1}(P_f) \xrightarrow{w^\mu} \hat{C} \setminus \tilde{\phi}(f^{-1}(P_f)) \\
\downarrow f \quad \downarrow E_{\mu,\tilde{\mu}} \\
\hat{C} \setminus P_f \xrightarrow{w^\mu} \hat{C} \setminus w^\mu(P_f).
\end{array}$$

Note that in the diagram, by abuse of notation, we write $f^{-1}(P_f)$ for $f^{-1}(P_f) \cup \{\infty\}$. Since by definition $\tilde{\mu} = f^*\mu$, the map $E = E_{\mu,\tilde{\mu}} = w^\mu \circ f \circ (w^{\tilde{\mu}})^{-1}$ defined on $\hat{C}$ is analytic. By Theorem 3, $E_{\mu,\tilde{\mu}} = P_{\tau,\tilde{\tau}} \circ Q_{\tau,\tilde{\tau}}$ for a pair of polynomials $P = P_{\tau,\tilde{\tau}}$ and $Q = Q_{\tau,\tilde{\tau}}$ of respective degrees $p$ and $q$.

Let $\sigma_* : T_f T_f \to T_f T_f$ and $\sigma^* : T_f^* T_f \to T_f^* T_f$ be the tangent and co-tangent map of $\sigma$, respectively.

Let $\beta(t) = [\mu(t)]$ be a smooth path in $T_f$ passing though $\tau$ at $t = 0$ and let $\eta = \beta'(0)$ be the corresponding tangent vector at $\tau$. Then the pull-back $\tilde{\beta}(t) = [f^*\mu(t)]$ is a smooth path in $T_f$ passing though $\tilde{\tau}$ at $t = 0$ and $\tilde{\eta} = \sigma_* \eta = \tilde{\beta}'(t)$ is the corresponding tangent vector at $\tilde{\tau}$. We move these tangent vectors to the origin in $T_f$ obtaining the vectors $\xi, \tilde{\xi}$ using the maps

$$\eta = (w^\mu)^* \xi \quad \text{and} \quad \tilde{\eta} = (w^{\tilde{\mu}})^* \tilde{\xi}.$$ 

This gives us the following commutative diagram:

$$\begin{array}{ccc}
\tilde{\eta} & \xleftarrow{(w^{\tilde{\mu}})^*} & \tilde{\xi} \\
\uparrow f^* & & \uparrow E^* \\
\eta & \xleftarrow{(w^\mu)^*} & \xi
\end{array}$$

Now suppose $\tilde{q}$ is a co-tangent vector in $T^*_f$ and let $q = \sigma^* \tilde{q}$ be the corresponding co-tangent vector in $T^*_f$. Then $\tilde{q} = \tilde{\phi}(w)dw^2$ is an integrable quadratic differential on $\hat{C}$ that can have at worst simple poles along $w^{\tilde{\mu}}(P_f)$ and $q = \phi(z)dz^2$ is an integrable quadratic differential on $\hat{C}$ that can have at worst simple poles along $w^\mu(P_f)$. This implies that $q = \sigma_* \tilde{q}$ is also the push-forward integrable quadratic differential

$$q = E_* \tilde{q} = \phi(z)dz^2$$

of $\tilde{q}$ by $E$. To see this, recall from section 3 that $E$, and a choice of curves $L_i$ from the branch points, determine a finite set of domains $W_i$ on which $E$ is an unbranched covering to a domain homeomorphic to $\hat{C}^*$. Since $E$ restricted to each
$W_i$ is either a topological model for $e^z$ or $z^k$, we may divide each $W_i$ into a collection of fundamental domains on which $E$ is bijective. Therefore the coefficient $\phi(z)$ of $q$ is given by the formula

$$ (2) \quad \phi(z) = (L\tilde{\phi})(z) = \sum_{E(w)=z} \frac{\tilde{\phi}(w)}{(E'(w))^2} = \frac{1}{z^2} \sum_{E(w)=z} \frac{\tilde{\phi}(w)}{(P'(w) + Q'(w))^2} $$

where $L$ is the standard transfer operator and $\tilde{\phi}$ is the coefficient of $\tilde{q}$. Thus

$$ (3) \quad q = \phi(z) dz^2 = \frac{dz^2}{z^2} \sum_{E(w)=z} \frac{\tilde{\phi}(w)}{(P'(w) + Q'(w))^2} $$

It is clear that as a quadratic differential defined on $\hat{C}$, we have

$$ ||q|| \leq ||\tilde{q}||. $$

Since $q$ is integrable and 0 and $\infty$ are isolated singularities, it follows that $q$ has at worst possible simple poles at these points so that the inequality holds on all of $\hat{C}$.

By formula (2), we have

$$ \langle \tilde{q}, \tilde{\xi} \rangle = \langle q, \xi \rangle $$

which implies

$$ ||\tilde{\xi}|| \leq ||\xi|| $$

where this is the $L^\infty$ norm. This gives another proof of Lemma 2. Furthermore, we have the following stronger assertion

**Lemma 5.**

$$ ||q|| < ||\tilde{q}|| $$

and

$$ ||\tilde{\xi}|| < ||\xi||. $$

**Proof.** Suppose there is a $\tilde{q}$ such that $||q|| = ||\tilde{q}|| \neq 0$. Using the change of variable $E(w) = z$ on each fundamental domain we get

$$ \int_{\hat{C}} \left| \sum_{E(w)=z} \frac{\tilde{\phi}(w)}{(E'(w))^2} \right| d\bar{z} d\bar{w} = \int_{\hat{C}} |\phi(z)| d\bar{z} d\bar{w} = \int_{\hat{C}} |\tilde{\phi}(w)| dw d\bar{w} $$

$$ = \sum_i \int_{W_i} |\tilde{\phi}(w)| dw d\bar{w} = \int_{\hat{C}} \sum_i \left| \frac{\tilde{\phi}(w)}{(E'(w))^2} \right| d\bar{z} d\bar{w}. $$

By the triangle inequality, all the factors $\frac{\tilde{\phi}(w)}{(E'(w))^2}$ in $\sum_{E(w)=z} \frac{\tilde{\phi}(w)}{(E'(w))^2}$ have the same argument. That is, there is a real number $a_z$ for every $z$ such that for any pair of points $w, w'$ with $E(w) = E(w') = z$,

$$ \frac{\tilde{\phi}(w)}{(E'(w))^2} = a_z \frac{\tilde{\phi}(w')}{(E'(w'))^2}. $$
Now formula (2) implies $\phi(z) = \infty$ which cannot be; this contradiction proves the lemma. □

Remark 3. The real point here is that $E$ has infinite degree and any $q$ has finitely many poles. If there were a $\tilde{q}$ with $||q|| = ||\tilde{q}|| \neq 0$ and if $Z$ is the set of poles of $\tilde{q}$, then the poles of $q$ would be contained in the set $E(Z) \cup \mathcal{V}_E$, where $\mathcal{V}_E$ is the set of critical values of $E$. Thus, by formula (2),

$$E^*q = \phi(E(w))dw^2 = dq(w),$$

where $d$ is the degree of $E$. Furthermore,

$$E^{-1}(E(Z) \cup \mathcal{V}_E) \subseteq Z \cup \Omega_E.$$

Since $d$ is infinite, the last inclusion formula can not hold since the left hand side is infinite and the right hand side is finite.

An immediate corollary is

Corollary 1. For any two points $\tau$ and $\tilde{\tau}$ in $T_f$,

$$d_T(\sigma(\tau), \sigma(\tilde{\tau})) < d_T(\tau, \tilde{\tau}).$$

Furthermore,

Lemma 6. If $\sigma$ has a fixed point in $T_f$, then this fixed point must be unique. This is equivalent to saying that a post-singularly finite $f$ in $\mathcal{T}E_{p,q}$ is combinatorially equivalent to at most one $(p,q)$-exponential map $E = Pe^Q$.

We can now finish the proof of the sufficiency in Theorem 1.

Proof of Theorem 1. Suppose $f \in \mathcal{T}E_{p,q}$ has bounded geometry with compactness. Recall that the map defined by

$$(4) \quad E_n = w^{\mu_n} \circ f \circ (w^{\mu_n+1})^{-1}$$

is a $(p,q)$-exponential map.

If $q = 0$, $E_n$ is a polynomial and the theorem follows from the arguments given in [CJ] and [DH]. Note that if $P_f = \{0,1,\infty\}$, then $f$ is a universal covering map of $\mathbb{C}^*$ and is therefore combinatorially equivalent to $e^{2\pi iz}$. Thus in the following argument, we assume that $\#(P_f) \geq 4$. Then, given our normalization conventions and the bounded geometry hypothesis we see that the functions $E_n, n = 0,1,\ldots$ satisfy the following conditions:

1) $m = \#(w^{\mu_n}(P_f)) \geq 4$ is fixed.
2) $0,1,\infty \in w^{\mu_n}(P_f)$.
3) $\Omega_{E_n} \cup \{0,1,\infty\} \subseteq E_n^{-1}(w^{\mu_n}(P_f))$.
4) there is a $b > 0$ such that $d_{sp}(p_n, q_n) \geq b$ for any $p_n, q_n \in w^{\mu_n}(P_f)$. 
As a consequence of the compactness, we have that in the sequence \( \{E_n\}_{n=1}^{\infty} \), there is a subsequence \( \{E_n\}_{n=1}^{\infty} \) converging to a map \( E = Pe^Q \in \mathcal{E}_{p,q} \) where \( P \) and \( Q \) are polynomials of degrees \( p \) and \( q \) respectively.

Any integrable quadratic differential \( q_n \in T^*_\tau T_f \) has, at worst, simple poles in the finite set \( P_{n+1,f} = w^{m+1}(P_f) \). Since \( T^*_\tau T_f \) is a finite dimensional linear space, there is a quadratic differential \( q_{n,max} \in T^*_\tau T_f \) with \( \|q_{n,max}\| = 1 \) such that

\[
0 \leq a_n = \sup_{\|q_n\|=1} \|(E_n)_*q_n\| = \|(E_n)_*q_{n,max}\| < 1.
\]

Moreover, by the bounded geometry condition, the possible simple poles of \( \{q_{n,max}\}_{n=1}^{\infty} \) lie in a compact set and hence these quadratic differentials lie in a compact subset of the space of quadratic differentials on \( \hat{\mathbb{C}} \) with, at worst, simples poles at \( m = \#(P_f) \) points.

Let

\[
a_{\tau_0} = \sup_{n \geq 0} a_n.
\]

Let \( \{n_i\} \) be a sequence of integers such that the subsequence \( a_{n_i} \to a_{\tau_0} \) as \( i \to \infty \). By compactness, \( \{E_{n_i}\}_{i=0}^{\infty} \) has a convergent subsequence, (for which we use the same notation) that converges to a holomorphic map \( E \in \mathcal{E}_{p,q} \). Taking a further subsequence if necessary, we obtain a convergent sequence of sets \( P_{n_i,\tau_0} = w^{m_0}(P_f) \) with limit set \( X \). By bounded geometry, \( \#(X) = \#(P_f) \) and \( d_{sp}(x,y) \geq b \) for any \( x,y \in X \). Thus we can find a subsequence \( \{q_{n_i,max}\} \) converging to an integrable quadratic differential \( q \) of norm 1 whose only poles lie in \( X \) and are simple. Now by lemma 5

\[
a_{\tau_0} = \|E_*q\| < 1.
\]

Thus we have proved that there is an \( 0 < a_{\tau_0} < 1 \), depending only on \( b \) and \( f \), such that

\[
\|\sigma_*\| \leq \|\sigma^\ast\| \leq a_{\tau_0}.
\]

Let \( l_0 \) be a curve connecting \( \tau_0 \) and \( \tau_1 \) in \( T_f \) and set \( l_n = \sigma^n_f(l_0) \) for \( n \geq 1 \). Then \( \mathcal{L} = \cup_{n=0}^{\infty} l_n \) is a curve in \( T_f \) connecting all the points \( \{\tau_n\}_{n=0}^{\infty} \). For each point \( \tau_0 \in l_0 \), we have \( a_{\tau_0} < 1 \). Taking the maximum gives a uniform \( a < 1 \) for all points in \( l_0 \). Since \( \sigma \) is holomorphic, \( a \) is an upper bound for all points in \( l \). Therefore,

\[
d_T(\tau_{n+1},\tau_n) \leq a d_T(\tau_n,\tau_{n-1})
\]

for all \( n \geq 1 \). Hence, \( \{\tau_n\}_{n=0}^{\infty} \) is a convergent sequence with a unique limit point \( \tau_\infty \) in \( T_f \) and \( \tau_\infty \) is a fixed point of \( \sigma \). This combining with Lemma 6 completes the proof of the sufficiency of Theorem 1. \( \square \)

From our proofs of Theorem 4 and Theorem 1 the final step in the proof of the main theorem is to prove the compactness condition holds from bounded geometry. This is in contrast to the case of rational maps (see [1]) where the bounded geometry condition guarantees the compactness condition holds. In the case of \((p,q)\)-exponential maps, the bounded geometry condition must be combined with some
to guarantee the compactness. The topological constraints, together with the bounded geometry condition control the sizes of fundamental domains so that they are neither too small nor too big. Thus, before we prove the compactness condition holds, we will describe a topological constraint for the two types of map $f$ in the Main Theorem (Theorem 2).

10. Proof of the Main Theorem (Theorem 2).

In section 3 we defined two different normalizations for functions in $T_{p,q}$ that depend on whether or not 0 is a fixed point of the map. The topological constraints for post-singularly finite maps also follow this dichotomy.

10.1. A topological constraint for $f \in T_{0,1}$ satisfying the hypotheses of Theorem 2. Any such $f$ has no branch points so $P_f = \cup_{k \geq 0} f^k(0) \cup \{\infty\}$ which is finite. Since 0 is omitted, the orbit of 0 is pre-periodic. Let $c_k = f^k(0)$ for $k \geq 0$. By the pre-periodicity, there are integers $k_1 \geq 0$ and $l \geq 1$ such that $f^l(c_{k_1+1}) = c_{k_1+1}$. This says that
\[
\{c_{k_1+1}, \ldots, c_{k_1+l}\}
\]
is a periodic orbit of period $l$. Let $k_2 = k_1 + l$. Let $\gamma$ be a continuous curve connecting $c_{k_1}$ and $c_{k_2}$ in $\mathbb{R}^2$ disjoint from $P_f$, except for its endpoints. Because
\[
f(c_{k_1}) = f(c_{k_2}) = c_{k_1+1},
\]
the image curve $\delta = f(\gamma)$ is a closed curve.

10.2. A topological constraint for $f \in T_{p,1}$ satisfying the hypotheses of Theorem 2. Any such $f$ has exactly one non-zero simple branch point which we denote by $c$; 0 is the only other branch point and it has multiplicity $p-1$. Then $f(0) = 0$ and by our normalization, $f(c) = 1$. In this case
\[
P_f = \cup_{k \geq 1} f^k(c) \cup \{0, \infty\}.
\]
Again by the hypothesis of Theorem 2 $P_f$ is finite. Set $c_k = f^k(c)$ for $k \geq 0$.

Suppose $c$ is not periodic. As above, there are integers $k_1 \geq 0$ and $l \geq 1$ such that $f^l(c_{k_1+1}) = c_{k_1+1}$. Again,
\[
\{c_{k_1+1}, \ldots, c_{k_1+l}\}
\]
is a periodic orbit of period $l$. Let $k_2 = k_1 + l$.

As above, let $\gamma$ be a continuous curve connecting $c_{k_1}$ and $c_{k_2}$ in $\mathbb{R}^2$ disjoint from $P_f$, except for its endpoints. Since
\[
f(c_{k_1}) = f(c_{k_2}) = c_{k_1+1},
\]
the image curve $\delta = f(\gamma)$ is a closed curve.
10.3. **Winding numbers.** In each of the above cases, the *winding number* $\eta$ of the closed curve $\delta = f(\gamma)$ about 0 essentially counts the number of fundamental domains between $c_{k_1}$ and $c_{k_2}$ and defines the “distance” between the fundamental domains. The following lemma is a crucial to proving that the compactness condition holds for each type of function in Theorem 2.

**Lemma 7.** The winding number $\eta$ is does not change under the Thurston iteration procedure.

**Proof.** Given $\tau_0 = [\mu_0] \in T_f$, let $\tau_n = \sigma^n(\tau_0) = [\mu_n]$ be the sequence generated by $\sigma$. Let $w^{\mu_n}$ be the normalized quasiconformal map with Beltrami coefficient $\mu_n$. Then in either of the above situations,

$$E_n = w^{\mu_n} \circ f \circ (w^{\mu_{n+1}})^{-1} \in E_{p,1}$$

since it preserves $\mu_0$ and is holomorphic. See the following diagram.

Let $c_{k,n} = w^{\mu_n}(c_k)$. The continuous curve

$$\gamma_{n+1} = w^{\mu_{n+1}}(\gamma)$$

goes from $c_{k_1,n+1}$ to $c_{k_2,n+1}$. The image curve is

$$\delta_n = E_n(\gamma_{n+1}) = w^{\mu_n}(f((w^{\mu_{n+1}})^{-1}(\gamma_{n+1}))) = w^{\mu_n}(f(\gamma)) = w^{\mu_n}(\delta).$$

Note that $w^{\mu_n}$ fixes 0, 1, $\infty$. Thus $\delta_n$ is a closed curve through the point $c_{k_1+1,n} = w^{\mu_n}(c_{k_1+1})$ and it has winding number $\eta$ around 0. □

The argument that this invariance plus bounded geometry implies the compactness is different in each of these two cases. We present these arguments in the two subsections below. Let set

$$P_n = P_{f,n} = w^{\mu_n}(P_f), \quad n = 0, 1, 2, \ldots$$

10.4. **The compactness condition for $f \in T\mathcal{E}_{0,1}$.** In this case, all the functions in the Thurston iteration have the form $E_n(z) = e^{\lambda_n^nz}$. From our normalization, we have that

$$0, 1 = E_n(0), E_n(1) = e^{\lambda_n} \in P_{n+1}.$$  

Note that in this case $E_n(1) \neq 1$. When $f$ has bounded geometry, the spherical distance between 1 and $E_n(1)$ is bounded away from zero. That is, there is a constant $\kappa > 0$ such that

$$\kappa \leq |\lambda_n|, \quad \forall n \geq 0.$$
Now we prove that the sequence \( \{ |\lambda_n| \} \) is also bounded above. We can compute
\[
\eta = \frac{1}{2\pi i} \int_{\gamma_n} \frac{1}{w} dw = \frac{1}{2\pi i} \int_{\gamma_{n+1}} \frac{E_n'(z)}{E_n(z)} \, dz = \frac{1}{2\pi i} \int_{\gamma_{n+1}} \lambda_n \, dz.
\]
The integral therefore depends only on the endpoints and we have
\[
\eta = \frac{1}{2\pi i} \int_{\gamma_{n+1}} \lambda_n \, dz = \frac{\lambda_n}{2\pi i} (c_{k_2, n+1} - c_{k_1, n+1}).
\]
Since 0 is omitted, it can not be periodic. Therefore, both \( c_{k_2, n+1} \neq c_{k_1, n+1} \in P_{n+1} \), so by bounded geometry there is a positive constant which we again denote by \( \kappa \) such that
\[
|c_{k_2, n+1} - c_{k_1, n+1}| \geq \kappa.
\]
This gives us the estimate
\[
|\lambda_n| \leq \frac{2\pi \eta}{|c_{k_2, n+1} - c_{k_1, n+1}|} \leq \frac{2\pi \eta}{\kappa},
\]
which proves that \( \{ E_n(z) \}_{n=0}^{\infty} \) is contained in a compact family in \( E_{0,1} \). This combined with Theorem 4 and Theorem 1 completes the proof of Theorem 2 for \( f \in T E_{p,1} \).

10.5. The compactness condition for \( f \in T E_{p,1} \) satisfying the hypotheses in Theorem 2

For such a map, \( f(0) = 0 \) and 0 is a branch point of multiplicity \( p - 1 \). And \( f \) has exactly one non-zero branch point \( c \) with \( f(c) = 1 \). All the functions in the Thurston iteration have forms
\[
E_n(z) = \alpha_n z^p e^{\lambda_n z}, \quad \alpha_n = e^{p \left( -\frac{\lambda_n}{p} \right)^p}.
\]
Note that \( E_n(0) = 0 \) and 0 is a critical point of multiplicity \( p - 1 \). It is also the asymptotic value and hence it has no other pre-images. Moreover, \( E_n(z) \) has exactly one non-zero simple critical point
\[
c_n = -\frac{p}{\lambda_n} = w^{\mu_n}(c).
\]
and \( \alpha_n \) is defined by the normalization condition \( E_n(c_n) = 1 \).

If \( c \) is periodic, then \( c \in P_f \). This implies that \( c_n \neq 0, \infty \in P_n \) and thus its spherical distance from either 0 or \( \infty \) is bounded below. That is, there are two constants \( 0 < \kappa < K < \infty \) such that
\[
\kappa \leq |\lambda_n| \leq K, \quad \forall n > 0.
\]
This implies that the sequence \( \{ E_n \}_{n=1}^{\infty} \) is contained in a compact subset.

Now suppose \( c \) is not periodic. By the the hypotheses in Theorem 2 \( f(c) = 1 \) is also not periodic. This implies that \( k_1 \geq 1 \).

Recalling the notation \( P_n = w^{\mu_n}(P_f) \), we have
\[
0, \quad 1 = E_n(c_n), \quad E_n(1) = e^{p \left( -\frac{\lambda_n}{p} \right)^p} e^{\lambda_n} \in P_n.
\]
Let $c_{k,n} = w^{\mu_n}(c_k)$. Then $c_{k,n} \in P_n$ for all $k \geq 1$. Let $\delta_n = w^{\mu_n}(\delta)$ and $\gamma_n = w^{\mu_n}(\gamma)$.

When $f$ has bounded geometry, since $E_n(1) \neq 0$, its spherical distance from 0 is bounded below. This implies that the sequence $\{|\lambda_n|\}$ is bounded below; that is, there is a constant $\kappa > 0$ such that

$$\kappa \leq |\lambda_n|, \ \forall n > 0.$$

By our hypothesis, $c_{k_1,n+1} \neq c_{k_2,n+1}$ both belong to $P_{n+1}$ and bounded geometry implies there are two constants, which we still denote by $0 < \kappa < K < \infty$, such that

$$\kappa \leq |c_{k_2,n+1}|, \ |c_{k_1,n+1}|, \ |c_{k_2,n+1} - c_{k_1,n+1}| \leq K, \ \forall n \geq 1.$$

Now we prove that the sequence $\{|\lambda_n|\}$ is also bounded above. Recall that when we chose $\gamma$, we assumed it did not go through 0 and thus by the normalization, none of the $\gamma_{n+1}$ go through 0 either. Therefore, for each $n$ we can find a simply connected domain $D_{n+1} \subset \gamma_{n+1}$ that does not contain 0. As in the previous section we compute

$$\eta = \frac{1}{2\pi i} \oint_{\delta_n} \frac{1}{w} dw = \frac{1}{2\pi i} \int_{\gamma_{n+1}} \frac{E'_n(z)}{E_n(z)} dz = \frac{1}{2\pi i} \int_{\gamma_{n+1}} \left( \frac{p}{z} + \lambda_n \right) dz$$

so that as above

$$2\pi i \eta = \int_{\gamma_{n+1}} \frac{p}{z} dz + \int_{\gamma_{n+1}} \lambda_n dz = \int_{\gamma_{n+1}} \frac{p}{z} dz + \lambda_n (c_{k_2,n+1} - c_{k_1,n+1}).$$

Rewriting we have

$$\lambda_n (c_{k_2,n+1} - c_{k_1,n+1}) = 2\pi i \eta - \int_{\gamma_{n+1}} \frac{p}{z} dz.$$

This implies that

$$\kappa |\lambda_n| \leq |\lambda_n (c_{k_2,n+1} - c_{k_1,n+1})| \leq 2\pi \eta + \left| \int_{\gamma_{n+1}} \frac{p}{z} dz \right|$$

so that if we can bound the integral on the right we will be done.

Notice that $\log z$ can be defined as an analytic function on the simply connected domain $D_{n+1}$ containing $\gamma_{n+1}$ that we chose above. We take $\log z = \log |z| + 2\pi i \arg(z)$ as the principal branch, with $0 \leq \arg(z) < 2\pi$. We then estimate

$$\left| \int_{\gamma_{n+1}} \frac{p}{z} dz \right| = |\log c_{k_2,n+1} - \log c_{k_1,n+1}|$$

$$\leq |\log |c_{k_2,n+1}| - \log |c_{k_1,n+1}|| + |\arg(c_{k_2,n+1}) - \arg(c_{k_1,n+1})|$$

$$\leq 2\pi \eta + (\log K - \log \kappa) + 4\pi.$$
which proves that \( \{ E_n(z) \}_{n=0}^{\infty} \) is contained in a compact subset in \( \mathcal{E}_{p,1} \). This combined with Theorem 4 and Theorem 1 completes the proof of Theorem 2 for \( f \in \mathcal{T}_{E_{p,1}} \).

11. Some Remarks

One can formally define a Thurston obstruction for a post-singularly finite \((p,q)\)-topological exponential map \( f \) with \( p \geq 0 \) and \( q \geq 1 \). Because such an \( f \) is a branched covering of infinite degree, however, many arguments in the proof of the Thurston theorem that use the finiteness of the covering in an essential way do not apply (see [DH, JZ, Ji]). Thus, how to define an analog of the Thurston obstruction to characterize a \((p,q)\)-topological exponential map \( f \) is not clear to us. We can, however, define an analog of the canonical Thurston obstruction for a \((p,q)\)-topological exponential map \( f \) which depends on the hyperbolic lengths of curves (see to [Pi, CJ, Ji]).

Let \( \sigma \) be the induced map on the Teichmüller space \( T_f \). For any \( \tau_0 \in T_f \), and for \( n \geq 1 \), let \( \tau_n = \sigma^n(\tau_0) \). Let \( \gamma \) denote a simple closed non-peripheral curve in \( \mathbb{C} \setminus \mathcal{P}_f \). Define

\[
\Gamma_c = \{ \gamma \mid \forall \tau_0 \in T_f, \ l_{\tau_n}(\gamma) \to 0 \quad \text{as} \quad n \to \infty \}.
\]

We have

**Corollary 2.** If \( \Gamma_c \neq \emptyset \), then \( f \) has no bounded geometry and therefore, \( f \) is not combinatorially equivalent to a \((p,q)\)-exponential map.

The converse should also be true but we have no proof at this time. The difficulty is that in the characterization of post-critically finite rational maps, many arguments in the proof of the converse statement use the finiteness of the covering in an essential way (see [Pi, CJ]).

A Levy cycle is a special Thurston obstruction for rational maps. It can be defined for a \((p,q)\)-topological exponential map \( f \) as follows. A set

\[
\Gamma = \{ \gamma_1, \ldots, \gamma_n \}
\]

of simple closed non-peripheral curves in \( \mathbb{C} \setminus \mathcal{P}_f \) is called a Levy cycle if for any \( \gamma_i \in \Gamma \), there is a simple closed non-peripheral curve component \( \gamma' \) of \( f^{-1}(\gamma_i) \) such that \( \gamma' \) is homotopic to \( \gamma_{i-1} \) (we identify \( \gamma_0 \) with \( \gamma_n \) rel \( \mathcal{P}_f \) and \( f : \gamma' \to \gamma_i \) is a homeomorphism. Following a result in [HSS], we have

**Corollary 3 (HSS).** Suppose \( f \) is a \((0,1)\)-topological exponential map with finite post-singular set. Then \( f \) has no Levy cycle if and only if \( f \) has bounded geometry.

We believe a similar result holds for all post-singularly finite maps in \( \mathcal{T}_{E_{p,q}} \) but we do not have a proof at this time.
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