SPREADING SPEED AND TRAVELING WAVES FOR A NON-LOCAL DELAYED REACTION-DIFFUSION SYSTEM WITHOUT QUASI-MONOTONICITY

ZHENGUO BAI
School of Mathematics and Statistics, Xidian University
Xi’an, Shaanxi 710126, China

TINGTING ZHAO
School of Mathematics, Northwest University
Xi’an, Shaanxi 710127, China

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Abstract. A non-local delayed reaction-diffusion model with a quiescent stage is investigated. It is shown that the spreading speed of this model without quasi-monotonicity is linearly determinate and coincides with the minimal wave speed of traveling waves.

1. Introduction. The notion of asymptotic speed of spread (spreading speed for short) was first introduced by Aronson and Weinberger [1, 2] for reaction-diffusion equations. Since then, there have been many works to reveal the link between the traveling wave solutions and the spreading speeds for various types of evolution systems (see, e.g., [20, 35] and references therein). A general theory of spreading speeds and traveling wave has been greatly developed in [14, 15, 16, 17, 25] for monotone semiflows so that it can be applied to various evolution equations admitting the comparison principle. For the results about the asymptotic speed of spread for non-monotone problems, we can refer to [5, 6, 12, 22, 23, 27, 29, 34] and their references.

To describe a population where the individuals alternate between mobile and non-mobile states, Hadeler and Lewis [9] presented and discussed briefly the following single population model with mobile and non-mobile stages

\[
\begin{align*}
    u_t &= Du_{xx} + f(u(x,t)) - \gamma_1 u(x,t) + \gamma_2 v(x,t), \\
    v_t &= \gamma_1 u(x,t) - \gamma_2 v(x,t),
\end{align*}
\]

(1)

where \(u(x,t)\) and \(v(x,t)\) are the densities of the mobile and non-mobile subpopulations at location \(x\) and time \(t\), respectively, \(D\) is the diffusion coefficient, \(\gamma_1\) and \(\gamma_2\) are the switching rates, and \(f\) is the reproduction function. All of the parameters in

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this model are positive constants. Zhang and Zhao [32] investigated the asymptotic behavior for system (1) in both unbounded and bounded spatial domains. Zhang and Li [31] further established the monotonicity and uniqueness of traveling waves for system (1).

In model (1), it is assumed that the growth rate of the population acts instantaneously. However, due to the duration of gestation, maturation and hatching period, there may be a time delay in a population model. Thus, Wu and Zhao [28] considered the following delayed reaction-diffusion system

\[
\begin{align*}
    u_t &= Du_{xx} + f(u(x,t), u(x,t-\tau)) - \gamma_1 u(x,t) + \gamma_2 v(x,t), \\
    v_t &= \gamma_1 u(x,t) - \gamma_2 v(x,t),
\end{align*}
\]

(2)

where the reproduction function is \( f(u(x,t), u(x,t-\tau)) \), where \( \tau > 0 \) is a constant. They established the existence of the minimal wave speed, monotonicity and uniqueness (up to translation) of the traveling wave fronts under the assumption that \( f(u,v) \) is monotone with respect to \( v \). However, \( f(u,v) \) may not be monotone with respect to \( v \) in practical problems. For example, if \( f \) is chosen as

\[ f(u(x,t), u(x,t-\tau)) = -du(x,t) + pu(x,t-\tau)e^{-au(x,t-\tau)} \]

for \( d,p,a > 0 \), then it is not monotone with respect to the second variable. Recently, by using the method in \([6,12,27]\), Zhao and Liu [34] obtained the spreading speed and its coincidence with the minimal wave speed of traveling wave solutions of system (2) in non-quasi-monotone case.

In ecology, since populations took time to move in space and usually were not at the same position in space at previous times, sometimes it is not sufficient to include only a discrete delay or a finite delay in a population model [13]. To address this difficulty, Britton [3, 4] considered comprehensively the two factors and introduced the so-called nonlocal delay, that is, the delay term involves a weighted spatiotemporal average over the whole infinite spatial domain and the whole of previous times. Since then, there have been many works of studying the spatial dynamics for reaction-diffusions with nonlocal delay \([13,24,27,29,30]\) (see also the references cited therein). Recently, Wu and Hsu [26] considered the general delayed non-local reaction-diffusion equation

\[
\begin{align*}
    u_t &= Du_{xx} + f \left( u(x,t), \int_{\mathbb{R}} J(x-y)S(u(y,t-\tau))dy \right), \\
    v_t &= \gamma_1 u(x,t) - \gamma_2 v(x,t),
\end{align*}
\]

(3)

which can describe the evolution of the mature population of a single species. They investigated the entire solutions of (3) and extended the arguments to two specific non-quasi-monotone delayed differential equations.

Note that the delay term in the first equation of (2) models the duration of gestation or hatching period, in which mobile subpopulations are not moving very much or not at all. Therefore, the use of a local time-delay term is probably reasonable. However, since model (2) is also applicable to many other species that have a maturation phase when the individuals may indeed move about. For such cases, nonlocal delays are indeed essential.

Based on the above consideration, we consider the following model

\[
\begin{align*}
    u_t &= Du_{xx} + f \left( u(x,t), \int_{\mathbb{R}} J(x-y)S(u(y,t-\tau))dy \right) - \gamma_1 u(x,t) + \gamma_2 v(x,t), \\
    v_t &= \gamma_1 u(x,t) - \gamma_2 v(x,t),
\end{align*}
\]

(4)
where $x \in \mathbb{R}, t \geq 0$. When $S(u) = u$ and $J(x) = \delta(x)$ with $\delta(\cdot)$ is the Dirac delta function, (4) reduces to system (2). A typical example of (4) is the delayed diffusive Nicholson’s blowflies model with a quiescent stage [8]. Throughout the paper, we always assume that $J, S$ and $f$ satisfy the following assumptions:

(C1) $J(-x) = J(x) \geq 0$ for $x \in \mathbb{R}, \int_{\mathbb{R}} J(y)dy = 1$ and there exists a $\lambda_0 > 0 (\lambda_0$ may be $+\infty$) such that

$$\int_{\mathbb{R}} e^{-\lambda y}J(y)dy < +\infty \text{ for } \lambda \in [0, \lambda_0) \text{ and } \lim_{\lambda \to \lambda_0^-} \int_{\mathbb{R}} e^{-\lambda y}J(y)dy = +\infty.$$

(C2) $S \in C^2(\mathbb{R}_+, \mathbb{R}_+)$ with $S(0) = 0$ and $S(u) > 0$ for $u > 0$, and $f \in C^2(\mathbb{R}_+^2, \mathbb{R})$, $f(0, S(0)) = f(K, S(K)) = 0$, $f(u, S(u)) > 0$ for $u \in (0, K)$, $f(u, S(u)) < 0$ for $u > K$, and $\partial_t f(0,0) < 0$.

(C3) $f$ is strictly subhomogeneous in the sense that $f(\delta u, \delta v) > \delta f(u, v)$ for any $\delta \in (0, 1), u, v > 0$, and $S$ is strictly subhomogeneous.

In this paper, we shall study the spreading speed and traveling wave solutions of system (4) without quasi-monotonicity. We first construct two auxiliary quasi-monotone systems to “trap” system (4) and then establish a comparison argument for the auxiliary and original systems. To this end, we need the following additional assumptions:

(C4) There exist $K^\pm$ and $K$ with $0 < K^- \leq K \leq K^+$ and twice piecewise continuously differentiable functions $S^\pm : [0, K^+] \to \mathbb{R}_+$ and $f^\pm : \bar{I} \to \mathbb{R}$ such that

(i) $f^\pm(0, S^\pm(0)) = 0, f^\pm(K^\pm, S^\pm(K^\pm)) = 0, (S^\pm)'(0) = S'(0)$ and $\partial_t f^\pm(0,0)$ $= \partial_t f(0,0), i = 1, 2$ and $\partial_t f^\pm \in C(\bar{I}, \mathbb{R})$;

(ii) $S^\pm(u)$ are non-decreasing on $[0, K^\pm], S^\pm(0) = 0$ and $0 < S^-(u) \leq S(u) \leq S^+(u)$ for $u \in (0, K^+]$. $f^\pm(u, v)$ are non-decreasing with respect to $v$ on $\bar{I}$ and

$$f^-(u, v) \leq f(u, v) \leq f^+(u, v), \forall (u, v) \in \bar{I};$$

(iii) $f^\pm$ is strictly subhomogeneous on $\bar{I}$ and $S^\pm$ is strictly subhomogeneous on $[0, K^+]$.

Here, and in what follows, we define $\bar{I} := [0, K^+] \times [0, S^+(K^+)]$.

**Remark 1.** (1) From (C2) and (i) and (ii) of (C4), one finds that $f^\pm(0, 0) = f(0,0) = 0$.

(2) The condition $\partial_t f^\pm \in C(\bar{I}, \mathbb{R})$ in (i) of (C4) is used to construct two auxiliary monotone systems in Section 3.

(3) The assumptions (i) and (iii) of (C4) imply that for any $\delta \in (0, 1), f^\pm(\delta K^+, S^\pm(\delta K^+)) > \delta f^\pm(K^+, S^\pm(K^+)) = 0$, which shows that $f^\pm(u, S^\pm(u)) > 0$ for $u \in (0, K^+)$, and hence there is no other positive zeros of $f^\pm$ on $[0, K^+] \times [0, S^+(K^+)]$.

(4) In view of (C3) and (C4), it follows from [36, Lemma 2.3.2] that

$$0 \leq S^-(u) \leq S(u) \leq S^+(0)u, \forall u \in [0, K^+],$$

$$f^-(u, v) \leq f(u, v) \leq f^+(u, v) \leq \partial_1 f(0,0)u + \partial_2 f(0,0) v, \forall (u, v) \in \bar{I}.$$
By means of these two inequalities, together with (C2), we have
\[ 0 < S(u) \leq S'(0)u, \quad \forall \ u \in (0, K), \]
\[ 0 < f(u, S(u)) \leq (\partial_1 f(0, 0) + \partial_2 f(0, 0)S'(0))u, \quad \forall \ u \in (0, K), \]
which imply that
\[ S'(0) > 0 \quad \text{and} \quad \partial_1 f(0, 0) + \partial_2 f(0, 0)S'(0) > 0. \]

The remainder of this paper is organized as follows. In Section 2, we present preliminaries. In Section 3, we use comparison arguments and a fluctuation method to establish the existence of spreading speed \( c^* \) of (4) with non-monotone reaction terms. In Section 4, we use Schauder’s fixed point theorem to obtain the existence of traveling waves with \( c > c^* \), and get the existence of traveling waves with \( c = c^* \) based on a limiting argument.

2. Preliminaries. In this section, we study the spreading speed and traveling wave solutions of (4) in quasi-monotone case. For this, we make the following assumption:
(C4)’ \( \partial_2 f(u, v) > 0 \) for all \( (u, v) \in [0, K] \times [0, S(K)] \) and \( S(u) \) is non-decreasing on \([0, K]\).

Note that if (C4)’ holds, then we can choose \( f^\pm = f, \ S^\pm = S \) and \( K^\pm = K \).

From the assumption (C2), we see that system (4) has two equilibria \( 0 = (0, 0) \) and \( \mathbf{K} := (K, \bar{K}) \) with \( \gamma_1 K = \gamma_2 \bar{K} \).

2.1. The spatially homogeneous system. We start with the global dynamics of the following spatially homogeneous system:
\[
\begin{align*}
\frac{du}{dt} &= f(u(t), S(u(t - \tau))) - \gamma_1 u(t) + \gamma_2 v(t), \\
\frac{dv}{dt} &= \gamma_1 u(t) - \gamma_2 v(t).
\end{align*}
\]
(5)

Let \( W := C([-\tau, 0], [0, K]) \times [0, \bar{K}] \). Then we have the following result.

Lemma 2.1. Let (C2)–(C3) and (C4)’ hold. For any \( \phi \in W \), system (5) has a unique solution \( w(t, \phi) \) with \( w_0 = \phi \), and \( w_t(\phi) := (u_t(\phi), v(t, \phi)) \in W \) for all \( t \geq 0 \).

Proof. For any \( \phi = (\phi_1, \phi_2) \in W \), define \( \tilde{f} : W \to \mathbb{R}^2 \) by
\[
\tilde{f}(\phi_1, \phi_2) = \left( \begin{array}{c} f(\phi_1(0), S(\phi_1(-\tau))) - \gamma_1 \phi_1(0) + \gamma_2 \phi_2 \\ \gamma_1 \phi_1(0) - \gamma_2 \phi_2 \end{array} \right).
\]
Since \( \tilde{f} \) is continuous in \( \phi \in W \), and \( \tilde{f} \) is Lipschitz in \( \phi \) on each compact subset of \( W \), it then follows that system (5) has a unique solution \( w(t, \phi) \) with \( w_0 = \phi \) on its maximal interval \([0, \sigma_\phi]\) of existence (see, e.g., [10, Theorems 2.2.1 and 2.2.3]).

For any given \( l \geq 1 \), define the set \([0, lK]_W \) by
\[
[0, lK]_W := \{ \phi \in W : 0 \leq \phi_1(\theta) \leq lK, \forall \theta \in [-\tau, 0] \text{ and } 0 \leq \phi_2 \leq l\bar{K} \}.
\]
The assumption (C2) implies that \([0, lK]_W \) is positively invariant for system (5). Since \( l \) can be chosen as large as we wish, this prove the positivity and boundedness of solutions in \( W \), and hence, \( \sigma_\phi = \infty. \) □

It is easy to see that \( \tilde{f} : W \to \mathbb{R}^2 \) is continuously differentiable and cooperative in the sense that for any \( \psi \in W \), the linear operator \( \tilde{L} := d\tilde{f}(\psi) \) satisfies \( \tilde{L}_1(\phi) \geq 0 \) for all \( \phi \in W \) with \( \phi_1(0) = 0 \) and \( \tilde{L}_2(\phi) \geq 0 \) for all \( \phi \in W \) with \( \phi_2 = 0 \). Further,
\( \dot{f} \) is subhomogeneous on \( W \), that is, \( \dot{f}(\delta \phi) \geq \delta \dot{f}(\phi) \) for any \( \delta \in (0, 1) \) and \( \phi \in W \). For any \( x = (x_1, x_2) \in \mathbb{R}^2 \), we define

\[
F(x) := \left( f(x_1, S(x_1)) - \gamma_1 x_1 + \gamma_2 x_2 \right) 
\]

By assumption (C3), we see that \( F(\delta x) > \delta F(x) \) for any \( \delta \in (0, 1) \) and \( x \in [0, K] \times [0, \bar{K}] \) with \( x \gg 0 \). Note that the Jacobian matrix \( DF(x) \) is cooperative and irreducible for any \( x \in [0, K] \times [0, \bar{K}] \). In particular, we have

\[
DF(0) = \begin{pmatrix} \kappa - \gamma_1 & \gamma_2 \\ \gamma_1 & -\gamma_2 \end{pmatrix},
\]

where \( \kappa = \partial_x f(0, 0) + \partial_x f(0, 0) S'(0) > 0 \). Let \( s(DF(0)) := \max \{ \text{Re} \lambda : \det(\lambda I - DF(0)) = 0 \} \). We then have

\[
s(DF(0)) = \frac{\kappa - \gamma_1 - \gamma_2 + \sqrt{(\kappa - \gamma_1 - \gamma_2)^2 + 4\kappa \gamma_2}}{2} > 0.
\]

It then follows from [33, Theorem 3.2(b)] that \( \mathcal{K} = (K, \bar{K}) \) is globally asymptotically stable for system (5) in \( W \setminus \{ 0 \} \).

2.2. Results for monotone delayed system. Since system (4) is cooperative and its solution maps are monotone, we can use the general theory developed in [16] to study the spreading speeds for (4). We start with some basic notations.

Let \( X := \text{BUC}(\mathbb{R}, \mathbb{R}^2) \) be the Banach space of all bounded and uniformly continuous functions from \( \mathbb{R} \) into \( \mathbb{R}^2 \) with the usual supremum norm \( \| \cdot \|_X \), and \( X^+ := \{ \phi \in X : \phi(x) \geq 0, \forall x \in \mathbb{R} \} \). The space \( \text{BUC}(\mathbb{R}, \mathbb{R}) \) is defined similarly. Clearly, any vector in \( \mathbb{R}^2 \) can be regarded as a function in \( X \).

Let \( Y \) be the space of all continuous functions from \( [-\tau, 0] \) to \( \mathbb{R}^2 \) with the usual supremum norm \( \| \cdot \|_Y \)i.e., \( Y = \mathcal{C}([-\tau, 0], \mathbb{R}^2) \), and \( Y_+ = \mathcal{C}([-\tau, 0], \mathbb{R}^2_+) \). Then \( (Y, Y_+) \) is an ordered Banach space. For \( u, v \in Y \), we write \( u \geq v \) if \( u - v \in Y_+ \), \( u > v \) if \( u - v \in Y_+ \setminus \{ 0 \} \), and \( u \gg v \) if \( u - v \in \text{Int}(Y_+) \).

Let \( \mathcal{C} \) be the set of all bounded and continuous functions from \( \mathbb{R} \) to \( Y \) equipped with the compact open topology, that is, \( u_m \rightarrow u \) in \( \mathcal{C} \) means that the sequence of \( u_m(x) \) converges to \( u(x) \) in \( Y \) as \( m \rightarrow \infty \) uniformly for \( x \) in any compact subset of \( \mathbb{R} \). For \( u, v \in \mathcal{C} \), we write \( u \geq v (u \gg v) \) provided \( u(x) \geq v(x) (u(x) > v(x)) \), \( \forall x \in \mathbb{R} \) and \( u > v \) provided \( u \gg v \) but \( u \neq v \). Clearly, any element in \( Y \) can be regarded as a constant function in \( \mathcal{C} \). Moreover, for each \( r \in Y \) with \( r \gg 0 \), we define \( Y_r := \{ u \in Y : 0 \leq u \leq r \} \) and \( \mathcal{C}_r := \{ u \in \mathcal{C} : u(x) \in Y_r, \forall x \in \mathbb{R} \} \). As usual, we identify an element \( \phi \in \mathcal{C} \) as a function from \( \mathbb{R} \times [-\tau, 0] \) into \( \mathbb{R}^2 \) by \( \phi(x, \theta) = \phi(x)(\theta) \).

Define the reflection operator \( R \) by \( R[u](x, \theta) = u(-x, \theta) \), and the translation operator \( T_y \) by \( T_y[u](x, \theta) = u(x - y, \theta) \) for each \( y \in \mathbb{R} \). Let \( Q : \mathcal{C}_r \rightarrow \mathcal{C}_r \) be a given map. The following assumptions on map \( Q \) will be referred to:

(A1) \( Q[R[u]] = R[Q[u]], T_y[Q[u]] = Q[T_y[u]], \forall u \in \mathcal{C}_r, y \in \mathbb{R} \).

(A2) \( Q : \mathcal{C}_r \rightarrow \mathcal{C}_r \) is continuous with respect to the compact open topology.

(A3) \( \{ Q[u](x, \cdot) : u \in \mathcal{C}_r, x \in \mathbb{R} \} \) is precompact in \( Y \).

(A4) \( Q \) is monotone (order preserving) in the sense that \( Q[u] \geq Q[v] \) whenever \( u \gg v \) in \( \mathcal{C}_r \).

(A5) \( Q : Y_r \rightarrow Y_r \) admits exactly two fixed points \( 0 \) and \( r \) and \( \lim_{n \rightarrow \infty} Q^n[y] = r \) for any \( y \in Y_r \) with \( y \gg 0 \).
Let \( T(t) = \text{diag}(T_1(t), T_2(t)) \) be a family of linear operators defined for \( t \geq 0 \) with \( T(0) = I \) and
\[
T_i(t)\phi(x) = \int_\mathbb{R} \Gamma_i(x - y, t)\phi(y)dy, \quad \forall \phi \in \text{BUC}(\mathbb{R}, \mathbb{R}), t > 0,
\]
where \( \Gamma_1(x, t) = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{x^2}{4Dt}}, \Gamma_2(x, t) = e^{-\gamma t}\delta(x) \), and \( \delta(\cdot) \) is the Dirac function. Define \( B = (B_1, B_2) : \mathcal{C} \rightarrow X \) by
\[
B_1(\phi_1, \phi_2)(x) := f(\phi_1(x, 0), \int \mathbb{R} J(x - y)S(\phi_1(y, -\tau))dy) + \gamma \phi_2(x, 0), \quad \forall x \in \mathbb{R},
\]
\[
B_2(\phi_1, \phi_2)(x) := \gamma_1 \phi_1(x, 0), \quad \forall x \in \mathbb{R}.
\]
It is convenient to write (4) into the following integral form
\[
w(x, t) = T(t)w(x, 0) + \int_0^t T(t - s)B(w_s)(x)ds, \quad \forall (x, t) \in \mathbb{R} \times (0, \infty),
\]
where \( w(x, t) := (u(x, t), v(x, t)) \). By using the theory of abstract functional differential equations developed in [19], we can establish the following result.

**Lemma 2.2.** Assume that (C1)–(C3) and (C4)' hold. For any \( \phi = (\phi_1, \phi_2) \in \mathcal{C}_K \), system (4) has a unique solution \( w(x, t; \phi) := (u(x, t; \phi), v(x, t; \phi)) \) which exists globally in time \( t \geq -\tau \) such that \( w_0 = \phi \) and \( w_t \in \mathcal{C}_K \) for \( t \geq 0 \), where \( w_t \) is defined by \( w_t(x; \theta; \phi) = w(x, t + \theta; \phi), \quad \forall t > 0, x \in \mathbb{R}, \theta \in [-\tau, 0] \).

Now we let \( w(x, t; \phi) \) be the solution of (4) with initial value \( \phi \in \mathcal{C}_K \). Define \( Q_t : \mathcal{C}_K \rightarrow \mathcal{C}_K \) by
\[
[Q_t(\phi)](x, \theta) = w_t(x, \theta; \phi), \quad \forall t \geq 0, x \in \mathbb{R}, \theta \in [-\tau, 0].
\]
It is easy to see that \( Q_0 = I \), and \( Q_{t+s} = Q_t \circ Q_s \) for all \( t, s \geq 0 \). By arguments similar to those in [7, Lemma 4.3], we can show that \( \{Q_t\}_{t \geq 0} \) is a monotone semiflow on \( \mathcal{C}_K \) with time-one map \( Q_1 \) satisfying (A1)–(A5). It then follows from [16, Theorems 2.11 and 2.15] that \( Q_1 \) admits a spreading speed \( c^* > 0 \). The following result shows that \( c^* \) is also the spreading speed of (4).

**Theorem 2.3.** Assume (C1)–(C3) and (C4)' hold. Then the following statements are valid:

(i) For any \( c > c^* \), if \( 0 \leq \phi \ll K \) and \( \phi(x, \cdot) = \theta \) for \( x \) outside a bounded interval, then
\[
\lim_{t \to \infty, |x|_2 \geq ct} w(x, t; \phi) = 0.
\]
(ii) For any \( c \in (0, c^*) \), if \( \phi := (\phi_1, \phi_2) \in \mathcal{C}_K \) and either \( \phi_1 \not\equiv 0 \) or \( \phi_2(\cdot, 0) \not\equiv 0 \) holds, then
\[
\lim_{t \to \infty, |x|_2 \leq ct} w(x, t; \phi) = K.
\]

**Proof.** We use the similar arguments in the proof of [30, Theorem 4.1]. Statement (i) follows from [16, Theorem 2.17(i)]. To prove statement (ii), we first make the following claim:

**Claim.** For any \( \phi = (\phi_1, \phi_2) \in \mathcal{C}_K \), if either \( \phi_1 \not\equiv 0 \) or \( \phi_2(\cdot, 0) \not\equiv 0 \), then
\[
w(x, t; \phi) \gg 0 \quad \text{for all} \quad x \in \mathbb{R}, t > \tau, \quad \text{and hence,} \quad Q_t(\phi) \gg 0 \quad \text{for} \quad t > 2\tau.
\]
If \( \phi_1 \not\equiv 0 \), then there exists a number \( \eta > 0 \) and an interval \( [a_1, a_2] \times [b_1, b_2] \subset \mathbb{R} \times [-\tau, 0] \) such that
\[
\phi_1(x, \theta) \geq \eta, \quad \forall (x, \theta) \in [a_1, a_2] \times [b_1, b_2].
\]
By Lemma 2.2, \( w(x, t; \phi) \geq 0 \) for all \( x \in \mathbb{R} \) and \( t \geq 0 \). Then the first equation of (4) satisfies

\[
  u_t = Du_{xx} + (h(x, t) - \gamma_1)u(x, t) + g(x, t) + \gamma_2v(x, t) \\
  \geq Du_{xx} - (m + \gamma_1)u(x, t) + g(x, t),
\]

where \( m = \max_{(u, v) \in \overline{[0, K]} \times [0, S(K)]} [\partial_1 f(u, v)] \) and

\[
  h(x, t) = \int_0^1 \partial_1 f \left( \theta u(x, t), \theta \int_\mathbb{R} J(x - y)S(u(y, t - \tau))dy \right) \, d\theta, \\
  g(x, t) = \int_0^1 \partial_2 f \left( \theta u(x, t), \theta \int_\mathbb{R} J(x - y)S(u(y, t - \tau))dy \right) \\
  \times \int_\mathbb{R} J(x - y)S(u(y, t - \tau))dy \, d\theta.
\]

In view of (6), we have

\[
  u(x, \tau; \phi) \geq \int_\mathbb{R} \Gamma(x - y, \tau)\phi_1(y, 0)dy + \int_0^\tau \int_\mathbb{R} \Gamma(x - y, \tau - s)g(y, s)dyds \\
  \geq \int_0^\tau \int_\mathbb{R} \Gamma(x - y, \tau - s) \int_0^1 [\partial_2 f \left( \theta u(y, s), \theta \int_\mathbb{R} J(y - \xi)S(u(\xi, s - \tau))d\xi \right) \\
  \times \int_\mathbb{R} J(y - \xi)S(u(\xi, s - \tau))d\xi] \, d\theta dyds' \\
  = \int_{-\tau}^\tau \int_\mathbb{R} \Gamma(x - y, -s') \int_0^1 [\partial_2 f \left( \theta u(y, s' + \tau), \theta \int_\mathbb{R} J(y - \xi)S(u(\xi, s'))d\xi \right) \\
  \times \int_\mathbb{R} J(y - \xi)S(u(\xi, s'))d\xi] \, d\theta dyds' \\
  \geq S(\eta) \int_{b_2}^{a_2} \int_\mathbb{R} \Gamma(x - y, -s') \int_0^1 [\partial_2 f \left( \theta u(y, s' + \tau), \theta \int_\mathbb{R} J(y - \xi)S(u(\xi, s'))d\xi \right) \\
  \times \int_\mathbb{R} J(y - \xi)S(u(\xi, s'))d\xi] \, d\theta dyds' > 0, \forall \ x \in \mathbb{R},
\]

where \( \Gamma(x, t) := \frac{1}{\sqrt{4Dt}} e^{-\frac{x^2}{4Dt}-(m+\gamma_1)t} \). Then,

\[
  u(x, t; \phi) \geq \int_\mathbb{R} \Gamma(x - y, t - \tau)u(y, \tau; \phi)dy \\
  + \int_{\tau}^{t} \int_\mathbb{R} \Gamma(x - y, t - s)g(y, s)dyds > 0, \forall \ x \in \mathbb{R}, t > \tau.
\]

By the integral form of the second equation of (4), we obtain

\[
  v(x, t; \phi) = e^{-\gamma_2t} \phi_2(x, 0) + \gamma_1 \int_0^t e^{-\gamma_2(t-s)}u(x, s; \phi)ds > 0, \forall \ x \in \mathbb{R}, t > \tau.
\]

If \( \phi_2(x, 0) \geq 0 \) with \( \phi_2(x, 0) \neq 0 \), then

\[
  v(x, t; \phi) = e^{-\gamma_2t} \phi_2(x, 0) + \gamma_1 \int_0^t e^{-\gamma_2(t-s)}u(x, s; \phi)ds \geq 0(\neq 0), \forall \ x \in \mathbb{R}, t \geq 0.
\]
Since $u_t \geq Du_{xx} + (m - \gamma_1)u(x, t) + \gamma_2 v(x, t)$, we get

$$u(x, t; \phi) \geq \gamma_2 \int_0^t \int_{\mathbb{R}} (x - y, t - s)v(y, s; \phi)dyds \geq 0 \not\equiv 0, \quad \forall x \in \mathbb{R}, t > 0.$$  

Hence, by [21, Theorem 1.4.5], we get $u(x, t; \phi) > 0, \forall x \in \mathbb{R}, t > 0$. It then follows that $v(x, t; \phi) > 0, \forall x \in \mathbb{R}, t > 0$. Therefore, $w(x, t; \phi) \not\equiv 0, \forall x \in \mathbb{R}, t > \tau$, and hence, $Q_t(\phi) \not\equiv 0$ for all $t > 2\tau$.

Since $Q_t$ is sub-homogeneous, we can choose the number $r_\sigma$ in [16, Theorem 2.17(ii)] to be the number $\bar{r}$, so that $r_\sigma = \bar{r}$ is independent of $\sigma \gg 0$. If $\phi \in C_\mathbb{K}$ and $\phi(x, \theta) \not\equiv 0$ for all $x$ on an interval $I$ of length $2\bar{r}$ and $\theta \in [-\tau, 0]$, then there exists a vector $\sigma \gg 0$ in $\mathbb{R}^2$ such that $\phi(x, \theta) \gg \sigma, \forall (x, \theta) \in I \times [-\tau, 0]$, and hence, [16, Theorem 2.17(ii)] implies that $\lim_{t \to \infty, |x| \leq c} w(x, t; \phi) = \mathbb{K}$. By the claim, we can fix $t_0 > 2\tau$ such that $u_{t_0}(\phi) \not\equiv 0$ for any given $\phi = (\phi_1, \phi_2) \in C_\mathbb{K}$ with either $\phi_1 \not\equiv 0$ or $\phi_2(., 0) \not\equiv 0$. By taking $u_{t_0}$ as a new initial data, we see that statement (ii) is satisfied.

Now, we look for the non-trivial traveling wave solutions

$$(u(x, t), v(x, t)) = (\phi_c(x + ct), \psi_c(x + ct))$$

of (4) satisfying the following condition

$$(\phi_c(-\infty), \psi_c(-\infty)) = 0, \ (\phi_c(+\infty), \psi_c(+\infty)) = \mathbb{K}. \quad (7)$$

Let $\xi = x + ct$, then $(\phi_c(\xi), \psi_c(\xi))$ satisfies

$$\begin{cases}
  c\phi_c'(\xi) = D\phi_c''(\xi) + f\left(\phi_c(\xi), \int_{\mathbb{R}} J(y)S(\phi_c(\xi - y - ct))dy\right) - \gamma_1 \phi_c(\xi) + \gamma_2 \psi_c(\xi), \\
  c\psi_c'(\xi) = \gamma_1 \phi_c(\xi) - \gamma_2 \psi_c(\xi).
\end{cases} \quad (8)$$

For $\lambda \geq 0$ and $c > 0$, define a function as follows

$$\Delta(c, \lambda) = D\lambda^2 - 2c\lambda + \partial_1 f(0, 0) + \partial_2 f(0, 0)S'(0)e^{-\lambda t} \int_{\mathbb{R}} J(y)e^{-\lambda y}dy - \gamma_1 + \frac{\gamma_1 \gamma_2}{c\lambda + \gamma_2}.$$  

Then we have the following result.

**Lemma 2.4.** Assume (C1)–(C3) and (C4) hold. There exists a positive pair of $(c_*, \lambda_*)$ such that

$$\Delta(c_*, \lambda_*) = 0, \quad \frac{\partial \Delta(c_*, \lambda_*)}{\partial \lambda} = 0.$$

Furthermore, 

(i) if $0 < c < c_*$, then $\Delta(c, \lambda) > 0$ for all $\lambda \in [0, \lambda_0]$;

(ii) if $c > c_*$, then $\Delta(c, \lambda) = 0$ has two positive real roots $\lambda_1 := \lambda_1(c)$ and $\lambda_2 := \lambda_2(c)$ with $\lambda_1(c) < \lambda_* < \lambda_2(c)$ such that

$$\Delta(c, \cdot) > 0 \text{ in } \mathbb{R} \setminus (\lambda_1(c), \lambda_2(c)), \ \Delta(c, \cdot) < 0 \text{ in } (\lambda_1(c), \lambda_2(c)).$$
Proof. Note that \( \partial_1 f(0,0) > 0 \) and \( \partial_1 f(0,0) + \partial_2 f(0,0) S'(0) > 0 \). By direct calculation, we get

\[
\Delta(c, 0) = \partial_1 f(0,0) + \partial_2 f(0,0) S'(0) > 0,
\Delta(\pm \infty, \lambda) = -\infty, \quad \forall \lambda \in [0, \lambda_0),
\lim_{\lambda \to \lambda_0^-} \Delta(c, \lambda) = +\infty, \quad \forall c > 0,
\Delta(0, \lambda) = D \lambda^2 + \partial_1 f(0,0) + \partial_2 f(0,0) S'(0) \int_{\mathbb{R}} J(y) e^{-\lambda y} dy > 0.
\]

Moreover,

\[
\frac{\partial \Delta(c, 0)}{\partial \lambda} = -c - \partial_2 f(0,0) S'(0) c \tau - c \frac{\gamma_1}{\gamma_2} < 0,
\frac{\partial \Delta(c, \lambda)}{\partial c} = -\lambda - \partial_2 f(0,0) S'(0) \lambda \tau \int_{\mathbb{R}} J(y) e^{-\lambda y} dy - \frac{\lambda \gamma_1 \gamma_2}{(c \lambda + \gamma_2)^2} < 0, \quad \forall \lambda \in (0, \lambda_0),
\frac{\partial^2 \Delta(c, \lambda)}{\partial \lambda^2} = 2D + \partial_2 f(0,0) S'(0) (c \tau)^2 e^{-\lambda c \tau} \int_{\mathbb{R}} J(y) e^{-\lambda y} dy
+ 2 \partial_2 f(0,0) S'(0) c \tau e^{-\lambda c \tau} \int_{\mathbb{R}} y J(y) e^{-\lambda y} dy
+ \partial_2 f(0,0) S'(0) e^{-\lambda c \tau} \int_{\mathbb{R}} y^2 J(y) e^{-\lambda y} dy + \frac{2 c^2 \gamma_1 \gamma_2}{(c \lambda + \gamma_2)^3}
= 2D + \partial_2 f(0,0) S'(0) e^{-\lambda c \tau} \int_{\mathbb{R}} (c \tau + y)^2 e^{-\lambda y} J(y) dy + \frac{2 c^2 \gamma_1 \gamma_2}{(c \lambda + \gamma_2)^3} > 0.
\]

Define

\[
c_* := \inf \{ c > 0 : \Delta(c, \lambda) = 0 \text{ for some } \lambda \in (0, \lambda_0) \}.
\]

Clearly, \( c_* > 0 \) and \( \Delta(c_*, \lambda(c_*)) = 0 \). Now we claim that \( \lambda(c_*) \) is uniquely determined. Otherwise, there exist \( (c_1, \lambda_1(c_*)) \) and \( (c_2, \lambda_2(c_*)) \) with \( \lambda_1(c_*) < \lambda_2(c_*) \) such that

\[
\Delta(c_1, \lambda_1(c_*)) = 0 \quad \text{and} \quad \Delta(c_2, \lambda_2(c_*)) = 0.
\]

Since \( \Delta(c, \lambda) \) is strictly convex down for each \( c > 0 \), we have

\[
\Delta(c_1, \theta \lambda_1(c_*)) + (1 - \theta) \Delta(c_2, \lambda_2(c_*)) < \theta \Delta(c_1, \lambda_1(c_*)) + (1 - \theta) \Delta(c_2, \lambda_2(c_*)) = 0, \quad \theta \in (0, 1).
\]

Choose a \( \hat{\lambda} \in (\lambda_1(c_*), \lambda_2(c_*)) \) and then \( \Delta(c_*, \hat{\lambda}) < 0 \). Since \( \Delta(0, \hat{\lambda}) > 0 \), and \( \frac{\partial \Delta(c, \lambda)}{\partial c} < 0 \) for any \( \lambda \in (0, \lambda_0) \), it follows from the intermediate value theorem that there is a unique \( \hat{c} \in (0, c_*) \) such that \( \Delta(\hat{c}, \hat{\lambda}) = 0 \), which contradicts the definition of \( c_* \). Thus, the claim holds.

Moreover, it is easy to see that \( \frac{\partial \Delta(c_*, \lambda_*)}{\partial \lambda} = 0 \), where \( \lambda_* := \lambda(c_*) \). Combining the above properties of the function \( \Delta(c, \hat{\lambda}) \), the conclusion follows.

\[\square\]

Remark 2. By using the linear operators approach (see [16, Theorem 3.10]), we can show that \( c_* \) in Theorem 2.3 is coincident with \( c_* \) in Lemma 2.4, i.e., \( c_* = c_* \).

In the remainder of the paper, we will use \( c_* \) instead of \( c_* \).

Following the idea in [18, Theorem 2.2], the existence of solutions of (8) can be reduced to the existence of a pair of super-solution and sub-solution of (8). By Lemma 2.4, we will construct a pair of super-solution and sub-solution of (8).
For the sake of notational convenience, we define
\[
(J \ast S(u))(x, t) := \int_{\mathbb{R}} J(y)S(u(x - y, t - \tau))dy, \ \forall \ u \in C(\mathbb{R}^2),
\]
\[
(J \ast \varphi)(x) := \int_{\mathbb{R}} J(y)\varphi(x - y - ct)dy, \ \forall \ \varphi \in C(\mathbb{R}).
\]

**Lemma 2.5.** Let (C1)-(C3) and (C4) hold, \( c > c^* \), \( q > 0 \) and \( \mu \in (1, \min\{2, \lambda_2/\lambda_1\}) \). Define
\[
\Phi^+(\xi) = (\phi^+(\xi), \psi^+(\xi)) := (\min\{K, e^{\lambda_1\xi} \}, \min\{\bar{K}, b(c)e^{\lambda_1\xi} \}),
\]
\[
\Phi^-_c(\xi) = (\phi^-_c(\xi), \psi^-_c(\xi)) := (\max\{0, e^{\lambda_1\xi} - qe^{\mu_1\xi} \}, \max\{0, b(c)e^{\lambda_1\xi} - qB(c,\mu)e^{\mu_1\xi} \}),
\]
where \( b(c) = \frac{\gamma_1}{c\lambda_1 + \gamma_2} \) and \( B(c,\mu) = \frac{\gamma_1}{c\mu_1 + \gamma_2} \). Then \( \Phi^+_c(\xi) \) is a super-solution of (8), and \( \Phi^-_c(\xi) \) is a sub-solution of (8) provided \( q \geq \max\{1, b(c)/B(c,\mu) \} \) is large enough.

**Proof.** For convenience, let's denote
\[
H_1(\phi, \psi)(\xi) := c\phi'(\xi) - D\phi''(\xi) - f(\phi(\xi), (J \ast S(\phi))(\xi)) + \gamma_1\phi(\xi) - \gamma_2\psi(\xi),
\]
\[
H_2(\phi, \psi)(\xi) := c\psi'(\xi) - -\gamma_1\phi(\xi) + \gamma_2\psi(\xi).
\]
Then \( \Phi(\xi) := (\phi(\xi), \psi(\xi)) \) is a super-solution (or sub-solution) of (8) if \( \Phi'(\xi^+) \leq \Phi'(\xi^-) \) (or \( \Phi'(\xi^+) \geq \Phi'(\xi^-) \)) for \( \xi \in \mathbb{R} \), and there exist \( \xi_1, \cdots, \xi_m \in \mathbb{R} \) such that
\[
H_i(\phi, \psi)(\xi) \geq 0 \text{ (or } H_i(\phi, \psi)(\xi) \leq 0 \text{) for } \xi \in \mathbb{R} \setminus \{\xi_1, \cdots, \xi_m\}, i = 1, 2.
\]
Here \( \Phi'(\xi^+) \) and \( \Phi'(\xi^-) \) are the right-hand derivative and the left-hand derivative of \( \Phi \) at \( \xi \), respectively.

First, we show that \( \Phi^+_c(\xi) \) is a super-solution of (8). Let \( \xi^+_1 := (\ln K)/\lambda_1, \xi^+_2 := (\ln \bar{K} - \ln b(c))/\lambda_1 \). Obviously, \( \Phi^+_c(\xi^+) \leq \Phi^+_c(\xi^-) \) for \( \xi \in \mathbb{R} \). Thus, it suffices to show that \( H_1(\Phi^+_c)(\xi) \geq 0 \) for \( \xi \in \mathbb{R} \setminus \{\xi^+_1, \xi^+_2\}, i = 1, 2 \). For \( \xi > \xi^+_1 \), we have
\[
H_1(\Phi^+_c)(\xi) = -f(K, (J \ast S(\phi^+_c))(\xi)) + \gamma_1 K - \gamma_2 \psi^+_c(\xi)
\geq -f(K, S(K)) + \gamma_1 K - \gamma_2 \bar{K} = 0. \tag{9}
\]
On the other hand, for \( \xi < \xi^+_1 \), using the \( S(u) \leq S'(0)u \) for \( u \in [0, K] \), we have
\[
H_1(\Phi^+_c)(\xi) \geq e^{\lambda_1\xi}[c\lambda_1 - D\lambda^2_1 + \gamma_1 - \gamma_2 b(c)] - f(\phi^+_c, (J \ast S(\phi^+_c))(\xi))
\geq e^{\lambda_1\xi}[c\lambda_1 - D\lambda^2_1 + \gamma_1 - \gamma_2 b(c)] - \partial_t f(0, 0)
\geq -\partial_2 f(0, 0)S'(0)\int_{\mathbb{R}} J(y)e^{-\lambda_1(y+\tau)}dy = 0. \tag{10}
\]
Note that in the above last equality, we have used the fact \( \Delta(c, \lambda_1) = 0 \).

Similarly, if \( \xi > \xi^+_2 \), then
\[
H_2(\Phi^+_c)(\xi) = -\gamma_1 \phi^+_c(\xi) + \gamma_2 \bar{K} \geq -\gamma_1 K + \gamma_2 \bar{K} = 0. \tag{11}
\]
On the other hand, if \( \xi < \xi^+_2 \), then
\[
H_2(\Phi^+_c)(\xi) = e^{\lambda_1\xi}[c\lambda_1 b(c) + \gamma_2 b(c)] - \gamma_1 \phi^+_c(\xi)
\geq e^{\lambda_1\xi}[c\lambda_1 b(c) + \gamma_2 b(c) - \gamma_1] = 0. \tag{12}
\]
It follows from (9)-(12) that \( \Phi^+_c(\xi) \) is a super-solution of (8).
Next, we show that $\Phi_\ast^-(\xi)$ is a sub-solution of (8) provided $q > b(c)/B(c,\mu)$ is large enough. Let
\[
\xi_1 := -\frac{\ln q}{(\mu - 1)\lambda_1} < 0, \quad \xi_2 := -\frac{\ln qB(c,\mu) - \ln b(c)}{(\mu - 1)\lambda_1} < 0.
\]

It is easy to see that $\Phi_\ast^-(\xi) \geq \Phi_\ast^-(\xi)$ for $\xi \in \mathbb{R}$. Thus, it needs to show that $H_i(\Phi_\ast^-)(\xi) \leq 0$ for all $\xi \in \mathbb{R} \setminus \{\xi_1, \xi_2\}$, $i = 1, 2$. If $\xi > \xi_1$, then
\[
H_1(\Phi_\ast^-)(\xi) = -f(0, (J * S(\phi_\ast^-)))(\xi) - \gamma_2\psi_\ast^-(\xi) \leq \gamma_2\psi_\ast^-(\xi) \leq 0.
\]

We now consider the case $\xi < \xi_1 < 0$. It is easy to see that
\[
e^{-\lambda_1 \xi} \geq \phi_\ast^-(\xi) \geq e^{\lambda_1 \xi} - q e^{\mu \lambda_1 \xi}, \quad \xi \in \mathbb{R}. \tag{13}
\]

By using the Taylor’ formula, there exist positive constants $D_1$ and $D_2$ such that
\[
f(u, v) \geq \partial_1 f(0, 0)u + \partial_2 f(0, 0)v - D_1 u^2 - D_2 v^2 \quad \text{for } u, v \in [0, K] \times [0, S(K)]. \tag{14}
\]

Since $S''(0)$ exists, there is $D_3 > 0$ such that
\[
S(u) \geq S'(0)u - D_3 u^2, \quad u \in [0, K]. \tag{15}
\]

In view of (13)-(15), we have, for $\xi < \xi_1 < 0$,
\[
H_1(\Phi_\ast^-)(\xi)
= \left[e^{\lambda_1 \xi}(c\lambda_1 - D\lambda_1^2 + \gamma_1 - \gamma_2 b(c)) - q e^{\mu \lambda_1 \xi}(c\mu_1 - D(\mu_1)^2 + \gamma_1 - \gamma_2 B(c, \mu)) - f(\phi_\ast^-)(\xi), (J * S(\phi_\ast^-))(\xi)\right]
\leq e^{\lambda_1 \xi} \left[c\lambda_1 - D\lambda_1^2 + \gamma_1 - \partial_1 f(0, 0)
- \partial_2 f(0, 0)S'(0) \int_{\mathbb{R}} J(y) e^{-\lambda_1(y+\epsilon)} dy - \gamma_2 b(c)\right]
- q e^{\mu \lambda_1 \xi} \left[c(\mu_1) - D(\mu_1)^2 + \gamma_1 - \partial_1 f(0, 0)
- \partial_2 f(0, 0)S'(0) \int_{\mathbb{R}} J(y) e^{-\mu \lambda_1(y+\epsilon)} dy - \gamma_2 B(c, \mu)\right] + M e^{2\lambda_1 \xi}
= - q e^{\mu \lambda_1 \xi} \left[c(\mu_1) - D(\mu_1)^2 + \gamma_1
- \partial_1 f(0, 0) - \partial_2 f(0, 0)S'(0) \int_{\mathbb{R}} J(y) e^{-\lambda_1(y+\epsilon)} dy - \gamma_2 B(c, \mu)\right] + M e^{2\lambda_1 \xi}
\leq e^{\mu \lambda_1 \xi} \{ - q[c(\mu_1) - D(\mu_1)^2 + \gamma_1
- \partial_1 f(0, 0) - \partial_2 f(0, 0)S'(0) \int_{\mathbb{R}} J(y) e^{-\mu \lambda_1(y+\epsilon)} dy - \gamma_2 B(c, \mu)] + M\},
\]

where
\[
M = \frac{1}{2} D_1 + \frac{1}{2} D_2 S'(0)^2 \left(\int_{\mathbb{R}} J(y) e^{-\lambda_1(y+\epsilon)} dy\right)^2 + \frac{1}{2} D_3 D_2 f(0, 0) \int_{\mathbb{R}} J(y) e^{-2\lambda_1(y+\epsilon)} dy > 0.
\]

According the definition of $\Delta(c, \lambda)$, if $q > b(c)/B(c, \mu)$ is large enough, then $H_1(\Phi_\ast^-)(\xi) \leq 0$ for all $\xi \in \mathbb{R} \setminus \{\xi_1\}$.

Similarly, if $\xi > \xi_2$, then
\[
H_2(\Phi_\ast^-)(\xi) = -\gamma_1 \phi_\ast^-(\xi) \leq 0.
\]
On the other hand, if \( \xi < \xi_2^- < 0 \), then
\[
H_2(\Phi^-_c)(\xi) = c[b(c)]\lambda_1 e^{\lambda_1 \xi} - qB(c, \mu)\mu\lambda_1 e^{\mu \lambda_1 \xi} - \gamma_1 \phi^-_c(\xi) + \gamma_2 [b(c)] e^{\lambda_1 \xi} - qB(c, \mu)\mu\lambda_1 e^{\mu \lambda_1 \xi} \leq c[b(c)](\lambda_1 + \gamma_2 b(c)) - qe^{\mu \lambda_1 \xi} [cB(c, \mu)\mu\lambda_1 - \gamma_1 + \gamma_2 B(c, u)] = 0.
\]
Therefore, \( \Phi^-_c(\xi) \) is a sub-solution of (8). \( \square \)

With the aid of the super-solution and sub-solution of (8), we have the following existence and non-existence of traveling wave fronts of (4).

**Theorem 2.6.** Assume \((C1)-(C3)\) and \((C4)'\) hold. Then the following result holds.

(i) For each \( c \geq c^* \), system (4) has a traveling wave front \( \Phi_c(\xi) := (\phi_c(\xi), \psi_c(\xi)) \), which satisfies (7) and \( \Phi_c(\xi) \geq 0 \) for all \( \xi \in \mathbb{R} \). Moreover, if \( c > c^* \), then
\[
\lim_{\xi \to -\infty} \Phi_c(\xi) e^{-\lambda_1 \xi} = (1, b(c)) \quad \text{and} \quad \Phi(\xi) \leq e^{\lambda_1 \xi}(1, b(c)) \quad \text{for all} \quad \xi \in \mathbb{R}.
\]
(ii) For \( 0 < c < c^* \), system (4) has no traveling wave solutions connecting \( 0 \) and \( K \).

**Proof.** By Lemma 2.5 and [18, Theorem 2.2] (or [11, Theorem 1.1]), we can prove the existence of monotone traveling wave solutions with speed \( c > c^* \). From the constructions of the super-solution and sub-solution, it is easy to verify that (16) holds. Moreover, a limiting argument similar to that of [11, Theorem 1.1] gives the existence of the traveling wave with the wave speed \( c^* \).

The non-existence of traveling wave solutions is a direct consequence of Theorem 2.3. This completes the proof. \( \square \)

Theorems 2.3 and 2.6 show that \( c^* \) is not only the spreading speed but also the minimal wave speed when (4) is quasi-monotone.

3. Spreading speed. In this section, we establish the spreading speed, the upward convergence and the non-existence of traveling wave solutions for (4) in non-quasi-monotone case. For convenience, we denote \( K^\pm = (K^-, K^+) \) and \([0, K^\pm] = [0, K^-] \times [0, K^+] \), where \( K^\pm = K^\pm \gamma_1 / \gamma_2 \).

According to (C4), we construct two auxiliary monotone delayed systems:
\[
\begin{cases}
    u_t = Du_{xx} + f^+(u(x, t), (J * S^+(u))(x, t)) - \gamma_1 u(x, t) + \gamma_2 v(x, t), \\
    v_t = \gamma_1 u(x, t) - \gamma_2 v(x, t),
\end{cases}
\]
and
\[
\begin{cases}
    u_t = Du_{xx} + f^-(u(x, t), (J * S^-(u))(x, t)) - \gamma_1 u(x, t) + \gamma_2 v(x, t), \\
    v_t = \gamma_1 u(x, t) - \gamma_2 v(x, t),
\end{cases}
\]
Recalling \( \partial_t f^\pm \in C(I, \mathbb{R}) \), we let
\[
L := \max \left\{ \max_{(u, v) \in I} |\partial_t f^+(u, v)|, \max_{(u, v) \in I} |\partial_t f^-(u, v)| \right\}.
\]
For any \( \phi = (\phi_1, \phi_2) \in C_{K^+} \), we define \( Q(\phi) = (Q_1(\phi), Q_2(\phi)) \) by
\[
Q_1(\phi)(x) := f \left( \phi_1(x, 0), \int_{\mathbb{R}} J(x-y)S(\phi_1(y, -\tau))dy \right) + L\phi_1(x, 0) + \gamma_2 \phi_2(x, 0), \\
Q_2(\phi)(x) := \gamma_1 \phi_1(x, 0).
\]
Similarly, we define $Q^\pm(\phi) = (Q_1^\pm(\phi), Q_2^\pm(\phi))$ by

$$
Q_1^\pm(\phi)(x) := f^\pm \left( \phi_1(x, 0), \int_{\mathbb{R}} J(x - y) S^\pm(\phi_1(y, -\tau))dy \right) + L\phi_1(x, 0) + \gamma_2 \phi_2(x, 0),
$$

$$
Q_2^\pm(\phi)(x) := \gamma_1 \phi_1(x, 0).
$$

It is clear that $Q^\pm(\cdot)$ is monotone in $C_{K^+}$ and

$$
Q^-(\phi) \leq Q(\phi) \leq Q^+(\phi), \quad \forall \phi \in C_{K^+}.
$$

We start with the well-posedness for the initial-value problems of (4).

**Lemma 3.1.** For any $\varphi \in C_{K^+}$, system (4) has a unique mild solution $w(x, t; \varphi)$ with $w(x, s; \varphi) = \varphi(x, s)$ for $(x, s) \in \mathbb{R} \times [-\tau, 0]$ and $0 \leq w(x, t; \varphi) \leq K^+$ for $(x, t) \in \mathbb{R} \times \mathbb{R}_+$.

**Proof.** Let $\bar{I} = [0, K^+] \times [0, S^+(K^+)]$ and

$$
\tilde{L} := \max \left\{ \max_{(u, v) \in \bar{I}} |\partial_1 f(u, v)|, \max_{(u, v) \in \bar{I}} |\partial_2 f(u, v)|, \max_{u \in [0, K^+]} |S'(u)| \right\}.
$$

Consider the initial value problem

$$
\begin{align*}
\left\{ 
\begin{array}{ll}
\dot{w}(x, t) &= \tilde{T}(t)w(\cdot, 0)(x) + \int_0^t \tilde{T}(t - s)Q(w_s)(x)ds, & \forall x \in \mathbb{R}, t > 0, \\
w(x, s) &= \varphi(x, s), & \forall x \in \mathbb{R}, s \in [-\tau, 0],
\end{array}
\right.
\end{align*}
$$

where $\tilde{T}(t) = \text{diag}($$\tilde{T}_1(t), \tilde{T}_2(t))$ with $\tilde{T}(0) = I$ and

$$
\tilde{T}_1(t)\phi(x) := e^{-(L+\gamma_1)t} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} \phi(y)dy,
$$

$$
\tilde{T}_2(t)\phi(x) := e^{-\gamma_2 t}\phi(x),
$$

for any $\phi \in \text{BUC}(\mathbb{R}, \mathbb{R})$ and $t > 0$. Let

$$
\mathcal{D} := \{ w \in C(\mathbb{R} \times \mathbb{R}_+, [0, K^+]) : w(x, t) = \varphi(x, t) \text{ for } (x, t) \in \mathbb{R} \times [-\tau, 0] \}.
$$

Define an operator $P$ on $\mathcal{D}$ by

$$
P(w)(x, t) = \left\{ \begin{array}{ll}
\tilde{T}(t)w(\cdot, 0)(x) + \int_0^t \tilde{T}(t - s)Q(w_s)(x)ds, & x \in \mathbb{R}, t > 0, \\
\varphi(x, t), & x \in \mathbb{R}, t \in [-\tau, 0].
\end{array} \right.
$$

Note that $\tilde{T}(t)(\cdot)$ is a positive operator and monotone on $\mathcal{D}, \forall t \geq 0$. Then,

$$
0 \leq \tilde{T}_1(t)\varphi_1(\cdot, 0)(x) + \int_0^t \tilde{T}_1(t - s)Q_1(w_s)(x)ds
\leq \tilde{T}_1(t)K^+ + \int_0^t \tilde{T}_1(t - s)Q_1^+(K^+)(x)ds
= K^+,
$$

and

$$
0 \leq \tilde{T}_2(t)\varphi_2(\cdot, 0)(x) + \int_0^t \tilde{T}_2(t - s)Q_2(w_s)(x)ds
\leq \tilde{T}_2(t)K^+ + \int_0^t \tilde{T}_2(t - s)Q_2^+(K^+)(x)ds
= K^+.
$$
Lemma 3.2. (Comparison principle) For any $\phi^+ \in C_{K^+}$ and $\phi^- \in C_{K^-}$ with $\phi^-(x,s) \leq \phi(x,s) \leq \phi^+(x,s), \forall (x,s) \in \mathbb{R} \times [-\tau,0]$, we have

Choose $\mu > \max\{2\bar{L} + \gamma_2 - \gamma_1, \gamma_1 - \gamma_2\}$ such that $\max\left\{ \frac{2\bar{L} + \gamma_2}{\bar{L} + \gamma_1 + \mu}, \frac{\gamma_1}{\gamma_2 + \mu} \right\} < 1$. It then follows that $P$ is a contraction map on $D$. By the contracting mapping theorem, $P$ has a unique fixed point $w$ in $D$, which is a unique mild solution of (4) satisfying $w(x,t) = \varphi(x,t)$ for $(x,t) \in \mathbb{R} \times [-\tau,0]$. \hfill \Box
let \(w^-(x, t; \phi^-), w(x, t; \phi)\) and \(w^+(x, t; \phi^+)\) be the solution of systems (18), (4) and (17) through \(\phi^-, \phi\) and \(\phi^+\), respectively. Then

\[
w^-(x, t; \phi^-) \leq w(x, t; \phi) \leq w(x, t; \phi^+), \quad \forall \ (x, t) \in \mathbb{R} \times \mathbb{R}_+.
\]

**Proof.** The proof is similar to that of [34, Lemma 3.1] and thus omitted. \(\square\)

Now, we state the result on the spreading speed for (4) without quasi-monotonicity. In particular, to obtain the upward convergence of spreading speed, we further make the following assumption.

(H) \(f(u, v) = -\alpha u + g(v)\) for \((u, v) \in [0, K] \times [0, S(K)]\), where \(\alpha > 0\) is a constant and \(g(\cdot)\) is a given function, \(g(S(u))/u\) is strictly decreasing for \(u \in [K^-, K^+]\), and \(b(u) := \frac{1}{s}g(S(u))\) satisfies:

(P) \(\forall u_1, u_2 \in [K^-, K^+]\) satisfying \(u_2 \leq K \leq u_1, u_2 \geq b(u_1)\) and \(u_1 \leq b(u_2)\), there holds \(u_1 = u_2\).

**Theorem 3.3.** Assume that (C1)–(C4) hold. Let \(w(x, t; \phi) = (u(x, t; \phi), v(x, t; \phi))\) be the unique global solution of (4) through the initial function \(\phi \in C_{K^+}\). Then the following statements are valid:

(i) For any \(c > c^*\), if \(0 \leq \phi \leq K^+\) and \(\phi(x, \cdot) = 0\) for \(x\) outside a bounded interval, then

\[
\lim_{t \to \infty, |x| \geq ct} w(x, t; \phi) = 0.
\]

(ii) For any \(c \in (0, c^*)\), if \(\phi := (\phi_1, \phi_2) \in C_{K^+}\) and either \(\phi_1 \not\equiv 0\) or \(\phi_2(\cdot, 0) \not\equiv 0\) holds, then

\[
K^- \leq \liminf_{t \to \infty, |x| \leq ct} w(x, t; \phi) \leq \limsup_{t \to \infty, |x| \leq ct} w(x, t; \phi) \leq K^+.
\]

Moreover, if (H) holds, then

\[
\lim_{t \to \infty, |x| \leq ct} w(x, t; \phi) = K.
\]

**Proof.** The proof is similar to those of [6, Theorem 3.3], [12, Theorem 2.2] and [34, Theorem 3.3]. For any \(\phi \in C_{K^+}\), define \(\hat{\phi} \in C_{K^-}\) by \(\hat{\phi}(x, t) = \min\{\phi(x, t), K^-\}\).

From Lemma 3.2, we have

\[
w^-(x, t; \hat{\phi}) \leq w(x, t; \phi) \leq w^+(x, t; \phi), \quad \forall \ x \in \mathbb{R}, t \geq 0. \tag{20}
\]

By Theorem 2.3, \(c^*\) is the spreading speed of solutions for (17) and (18), which together with (20) implies that \(c^*\) satisfies the statement (i) and the first part of (ii).

Next, we prove the upward convergence of the spreading speeds by the fluctuation method (see [6, 12]). First, we simplify the notation \(w(x, t; \phi) = (u(x, t; \phi), v(x, t; \phi))\) by \(w(x, t) = (u(x, t), v(x, t))\). Define a function

\[
G(u, v) := \begin{cases} 
\min_{z \in [u, v]} S(z), & \text{if } u \leq v, \\
\max_{z \in [v, u]} S(z), & \text{if } v \leq u.
\end{cases}
\]
Clearly, $G(u, v)$ is non-decreasing in $u$ and non-increasing in $v$, and $G(u, u) = S(u)$. The integral form of (4) can be written as

$$u(x, t) = \int_{\mathbb{R}} e^{-(x+y)^{2}} \Gamma_{0}(x-y, t) u(y, 0) dy$$

$$+ \int_{0}^{t} \int_{\mathbb{R}} e^{-(x+y)(t-s)} \Gamma_{0}(y, t-s) \left[ \gamma_{2} v(x-y, s) + g((J * S(u))(x-y, s)) \right] dy ds$$

$$= \int_{\mathbb{R}} e^{-(x+y)^{2}} \Gamma_{0}(x-y, t) u(y, 0) dy$$

$$+ \int_{-t}^{0} \int_{\mathbb{R}} e^{(x+y)^{2}} \Gamma_{0}(y, -s) \left[ \gamma_{2} v(x-y, t+s) + g((J * G(u, u))(x-y, t+s)) \right] dy ds,$$

where $\Gamma_{0}(x, t) := \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^{2}}{4t}}$, and

$$v(x, t) = e^{-\gamma_{2} t} v(x, 0) + \gamma_{1} \int_{0}^{t} e^{-\gamma_{2} (t-s)} u(x, s) ds$$

$$= e^{-\gamma_{2} t} v(x, 0) + \gamma_{1} \int_{-t}^{0} e^{\gamma_{2} s} u(x, t+s) ds.$$ (21)

For any $\beta \in (0, c^{*})$, define

$$U_{*}(\beta) := \liminf_{t \to \infty, |x| \leq \beta t} u(x, t) \quad \text{and} \quad U^{*}(\beta) := \limsup_{t \to \infty, |x| \leq \beta t} u(x, t),$$

$$V_{*}(\beta) := \liminf_{t \to \infty, |x| \leq \beta t} v(x, t) \quad \text{and} \quad V^{*}(\beta) := \limsup_{t \to \infty, |x| \leq \beta t} v(x, t).$$

Let $c \in (0, c^{*})$ be given, and fix a number $\gamma \in (c, c^{*})$. Define

$$U_{*}(c, \gamma) := \inf_{c<\beta<\gamma} U_{*}(\beta) \quad \text{and} \quad U^{*}(c, \gamma) := \sup_{c<\beta<\gamma} U^{*}(\beta),$$

$$V_{*}(c, \gamma) := \inf_{c<\beta<\gamma} V_{*}(\beta) \quad \text{and} \quad V^{*}(c, \gamma) := \sup_{c<\beta<\gamma} V^{*}(\beta).$$

For any $\beta \in (c, \gamma)$, we can choose two sequences $\{t_{j}\}$ in $(0, \infty)$ and $\{x_{j}\}$ in $\mathbb{R}$ such that $|x_{j}| \leq \beta t_{j}$, $t_{j} \to \infty$ as $j \to \infty$ and $\lim u(x_{j}, t_{j}) = U_{*}(\beta)$. For any given $y \in \mathbb{R}$ and $s \in \mathbb{R}^{+}$, we have $\liminf u(x_{j} - y, t_{j} - s) \geq V_{*}(\gamma)$ and

$$U_{*}(\gamma) \leq \liminf_{j \to \infty} u(x_{j} - y, t_{j} - s - \tau) \leq \limsup_{j \to \infty} u(x_{j} - y, t_{j} + s - \tau) \leq U^{*}(\gamma).$$

By Fatou’s Lemma, it follows from (21) that

$$U_{*}(\beta) \geq \liminf_{j \to \infty} \int_{-t_{j}}^{0} \int_{\mathbb{R}} e^{(x+y)^{2}} \Gamma_{0}(y, -s) \left[ \gamma_{2} v(x_{j} - y, t_{j} + s) + g((J * G(u, u))(x_{j} - y, t_{j} + s)) \right] dy ds$$

$$\geq \int_{-\infty}^{0} \int_{\mathbb{R}} e^{(x+y)^{2}} \Gamma_{0}(y, -s) \liminf_{j \to \infty} \left[ \gamma_{2} v(x_{j} - y, t_{j} + s) + g((J * G(u, u))(x_{j} - y, t_{j} + s)) \right] dy ds$$

$$\geq \left[ \gamma_{2} V_{*}(\gamma) + g(G(U_{*}(\gamma), U^{*}(\gamma))) \right] / (\alpha + \gamma_{1}),$$
which implies

\[(\alpha + \gamma_1)U_*(\beta) \geq \gamma_2 V_*(\gamma) + g\left(G(U_*(\gamma), U^*(\gamma))\right).\]

Similarly, from (21) and (22), we get

\[(\alpha + \gamma_1)U^*(\beta) \leq \gamma_2 V^*(\gamma) + g\left(G(U^*(\gamma), U_*(\gamma))\right),
\]

\[\gamma_2 V_*(\beta) \geq \gamma_1 U_*(\gamma) \quad \text{and} \quad \gamma_2 V^*(\beta) \leq \gamma_1 U^*(\gamma).\]

It then follows that

\[\begin{align*}
(\alpha + \gamma_1)U_*(c, \gamma) & \geq \gamma_2 V_*(c, \gamma) + g\left(G(U_*(c, \gamma), U^*(c, \gamma))\right), \\
\gamma_2 V_*(c, \gamma) & \geq \gamma_1 U_*(c, \gamma), \\
(\alpha + \gamma_1)U^*(c, \gamma) & \leq \gamma_2 V^*(c, \gamma) + g\left(G(U^*(c, \gamma), U_*(c, \gamma))\right), \\
\gamma_2 V^*(c, \gamma) & \leq \gamma_1 U^*(c, \gamma).
\end{align*}\]  

(23)

According to the definition of \(G\), there exists \(u_1, u_2 \in [U_*(c, \gamma), U^*(c, \gamma)] \subset [K^-, K^+]\) such that

\[\begin{align*}
G(U_*(c, \gamma), U^*(c, \gamma)) &= S(u_1), \\
G(U^*(c, \gamma), U_*(c, \gamma)) &= S(u_2).
\end{align*}\]

From (23), one easily sees that

\[U_*(c, \gamma) \geq \frac{1}{\alpha} g(S(u_1)) \quad \text{and} \quad U^*(c, \gamma) \leq \frac{1}{\alpha} g(S(u_2)).\]

In view of assumption (H), we have

\[b(u_1) = \frac{1}{\alpha} g(S(u_1)) \leq U_*(c, \gamma) \leq u_1, \quad u_2 \leq U^*(c, \gamma) \leq \frac{1}{\alpha} g(S(u_2)) = b(u_2),\]

and hence

\[\frac{g(S(u_1))}{\alpha u_1} \leq 1 = \frac{g(S(K))}{\alpha K} \leq \frac{g(S(u_2))}{\alpha u_2}.\]

Since \(g(S(u))/u\) is strictly decreasing for \(u \in [K^-, K^+]\), it follows that \(u_2 \leq K \leq u_1\).

The property (P) implies that \(u_1 = u_2 = K\), and hence,

\[U_*(c, \gamma) = U^*(c, \gamma) = K.\]  

(24)

Moreover, we deduce from (23) and (24) that

\[\gamma_1 K = \gamma_1 U_*(c, \gamma) \leq \gamma_2 V_*(c, \gamma) \leq \gamma_2 V^*(c, \gamma) \leq \gamma_1 U^*(c, \gamma) = \gamma_1 K,\]

and hence \(V_*(c, \gamma) = V^*(c, \gamma) = \frac{\gamma_1 K}{\gamma_2} = K\). Consequently,

\[K = U_*(c, \gamma) \leq U^*(c, \gamma) \leq U^*(c, \gamma) = K,\]

\[K = V_*(c, \gamma) \leq V^*(c, \gamma) \leq V^*(c, \gamma) = K,\]

which imply that \(\lim_{t \to \infty, |x| \leq ct} w(x, t; \phi) = K\) for any \(c \in (0, c^*).\)

\[\square\]

**Theorem 3.4.** Assume that (C1)–(C4) hold. For any \(0 < c < c^*\), (4) has no traveling wave solution \(\Phi_c(\xi)\) with \(\lim_{\xi \to \infty} \Phi_c(\xi) \gg 0\) and \(\Phi_c(-\infty) = 0\).

**Proof.** Assume, by contradiction, that for some \(c_1 \in (0, c^*)\), (4) has a traveling wave solution \(w(x, t) = \Phi_{c_1}(x + c_1 t)\) such that \(\lim_{\xi \to \infty} \Phi_{c_1}(\xi) \gg 0\) and \(\Phi_{c_1}(-\infty) = 0\). Let

\[\phi(x, \theta) := \Phi_{c_1}(x + c_1 \theta), \quad \forall \theta \in [-\tau, 0].\]

It is easy to see that \(\phi_1 \neq 0\) and \(\phi_2(\cdot, 0) \neq 0\). By Theorem 3.3(ii), there holds

\[\lim_{t \to \infty, |x| \leq ct} w(x, t; \phi) \geq K^- \gg 0, \quad \forall \ c \in (0, c^*).\]
Let \( \bar{c} \in (c_1, c^*) \) and \( x = -\bar{c}t \). Then
\[
0 = \lim_{t \to \infty} \Phi_{c_0}(-(\bar{c} - c_1)t) = \lim_{t \to \infty} w(-\bar{c}t, t) \geq \liminf_{t \to \infty, x \leq \bar{c}t} w(x, t) \geq 0,
\]
which is a contradiction. \( \square \)

4. Traveling waves. In this section, we establish the existence of traveling waves of (4) without monotonicity by using the Schauder’s fixed point theorem. Further, we employ the properties of the monotone traveling wave solutions of the lower auxiliary system (18) to obtain the asymptotic behavior of the wave profile.

**Theorem 4.1.** Assume that (C1)–(C4) hold. For each \( c > c^* \), let \( \lambda_1(c) \) be defined as in Lemma 2.4 and \( b(c) = \frac{\lambda_1}{c_3((c) + 2)} \). Then (4) admits a traveling wave solution \( \Phi_c(\xi) \) such that and
\[
\Phi_c(-\infty) = 0, \quad 0 \ll \Phi_c(\xi) \leq K^+, \quad \forall \, \xi \in \mathbb{R},
\]
\[
K^- \leq \liminf_{\xi \to +\infty} \Phi_c(\xi) \leq \limsup_{\xi \to +\infty} \Phi_c(\xi) \leq K^+ \quad \text{and} \quad \lim_{\xi \to -\infty} \Phi_c(\xi)e^{-\lambda_1(c)\xi} = (1, b(c)).
\]
Moreover, if (H) holds, then \( \lim_{\xi \to +\infty} \Phi_c(\xi) = K \).

**Proof.** Recalling the definition of \( L \) in (19), we let
\[
\bar{L} := \max\{L, \max_{(u,v) \in I} |\partial_1 f(u,v)|, \max_{(u,v) \in I} |\partial_2 f(u,v)|\},
\]
and
\[
r_1 = c - \frac{\sqrt{c^2 + 4D(L + \gamma_1)}}{2D} \quad \text{and} \quad r_2 = \frac{c + \sqrt{c^2 + 4D(L + \gamma_1)}}{2D}.
\]
Clearly, \( r_1 < 0 < r_2 \) and \( Dr_i^2 - cr_i - (\bar{L} + \gamma_1) = 0, i = 1, 2 \). Define an operator \( T = (T_1, T_2) : C(\mathbb{R}, [0, K^+]) \to C(\mathbb{R}, \mathbb{R}^2) \) by
\[
T_1(\Psi)(\xi) = \frac{1}{D(r_2 - r_1)} \left[ \int_{-\infty}^{\xi} e^{r_2(z-s)} F_1(\Psi)(s)ds + \int_{\xi}^{+\infty} e^{r_2(\xi-s)} F_1(\Psi)(s)ds \right],
\]
\[
T_2(\Psi)(\xi) = \frac{1}{c} \int_{-\infty}^{\xi} e^{-\bar{L}z} F_2(\Psi)(s)ds,
\]
where \( \Psi(\xi) = (\phi(\xi), \psi(\xi)) \) and
\[
F_1(\Psi)(\xi) := \bar{L}\phi(\xi) + f(\phi(\xi), (J \ast S)(\phi))(\xi) + \gamma_2\psi(\xi),
\]
\[
F_2(\Psi)(\xi) := \gamma_1\phi(\xi).
\]
It is easy to verify that a fixed point of \( T \) is a solution of (8) (also see [22, Lemma 4.1]). Similarly, we define \( T^\pm = (T_1^\pm, T_2^\pm) \) as in (27) with \( F = (F_1, F_2) \) replaced by \( F^\pm = (F_1^\pm, F_2^\pm) \), where
\[
F_1^\pm(\Psi)(\xi) := \bar{L}\phi(\xi) + f^\pm(\phi(\xi), (J \ast S^\pm)(\phi))(\xi) + \gamma_2\psi(\xi),
\]
\[
F_2^\pm(\Psi)(\xi) := \gamma_1\phi(\xi).
\]
By (ii) of (C4), it follows that \( T^\pm(\Psi) \) is monotone, and for any \( \Psi \in C(\mathbb{R}, [0, K^+]) \),
\[
T^-(\Psi) \leq T(\Psi) \leq T^+(\Psi).
\]
In view of $\Delta(c, \lambda_1(c)) = 0$, we have
\[
\frac{\partial f_1(0, 0)}{\partial t} + \frac{\partial f_2(0, 0)}{\partial t} + S'(u) e^{-\lambda_1(c)c} t \int_{\mathbb{R}} J(y) e^{-\lambda_1(c)y} dy + \frac{\gamma_1 \gamma_2}{c \lambda_1(c) + \gamma_2} = -D \lambda^2_1(c) + c \lambda_1(c) + \gamma_1 > 0,
\]
and hence $\lambda_1(c) < r_2$. Define
\[
\tilde{\Phi}^+(\xi) : = (\min\{K_+^+, e^{\lambda_1(c)\xi}\}, \min\{K_+^-, b(c)e^{\lambda_1(c)\xi}\}).
\]
Noting that $f^+(u, v) \leq \frac{\partial f_1(0, 0)}{\partial t} + \frac{\partial f_2(0, 0)}{\partial t} + v$ for $(u, v) \in \tilde{I}$, we deduce that $T^+ (\tilde{\Phi}^+) (\xi) \leq \tilde{\Phi}^+(\xi)$ for $\xi \in \mathbb{R}$. Then for a given $\lambda \in (0, \min\{\lambda_1(c), r_2\})$, let
\[
X_\lambda := \{\Psi \in C(\mathbb{R}, \mathbb{R}^2) : \sup_{\xi \in \mathbb{R}} \|\Psi(\xi)\| e^{-\lambda \xi} < +\infty\}
\]
with the norm $\|\Psi\|_\lambda = \sup_{\xi \in \mathbb{R}} \|\Psi(\xi)\| e^{-\lambda \xi}$. Then $(X_\lambda, \|\cdot\|_\lambda)$ is a Banach space.

Since $\partial f^+ (0, 0) = \partial f(0, 0), i = 1, 2$, $\Delta(c, \lambda) = 0$ is also the characteristic equation of (17) and (18) with respect to the trivial equilibrium 0. It thus follows from Theorem 2.6 that for any $c > c^*$, (18) admits a non-decreasing traveling wave solution $\Phi^- (\xi) = (\phi^- (\xi), \psi^- (\xi)), \xi = x + ct$ such that
\[
\Phi^- (\xi) \to 0, \quad \Phi^- (-\infty) = 0, \quad \Phi^- (+\infty) = K^-.
\]
\[
\liminf_{\xi \to -\infty} \Phi^- (\xi) e^{-\lambda_1(c)\xi} = (1, b(c)) \quad \text{and} \quad \Phi^- (\xi) \leq e^{\lambda_1(c)(1, b(c)), \forall \xi \in \mathbb{R}}.
\]
It is easy to see that $\Phi^- \subset X_\lambda$, and hence $\Omega := \{\Psi \in X_\lambda : \Phi^- \leq \Psi \leq \tilde{\Phi}^+\}$ is a nonempty, convex and closed subset of $X_\lambda$. For any $\Psi \in \Omega$, we have
\[
\Phi^- = T^- (\Phi^-) \leq T^- (\Psi) \leq T^+ (\Psi) \leq T^+ (\tilde{\Phi}^+) \leq \tilde{\Phi}^+.
\]
and hence $T(\Omega) \subset \Omega$.

Now, we prove that $T$ is compact on $\Omega$. We first show that $T$ is continuous on $\Omega$. For any $\Psi_1 = (\phi_1, \psi_1), \Psi_2 = (\phi_2, \psi_2) \in \Omega$, there holds
\[
|F_1(\Psi_1)(\xi) - F_1(\Psi_2)(\xi)| e^{-\lambda_1(c)} \leq \bar{L} \phi_1(\xi) - \phi_2(\xi) e^{-\lambda_1(c)} + \bar{\gamma}_2 \psi_1(\xi) - \psi_2(\xi) e^{-\lambda_1(c)}
\]
\[
+ |\partial_1 f(\eta_1(\xi), (J * S(\phi_1))(\xi))| |\phi_1(\xi) - \phi_2(\xi)| e^{-\lambda_1(c)}
\]
\[
+ |\partial_2 f(\phi_2(\xi), \eta_2(\xi))| \int_{\mathbb{R}} J(y) e^{-\lambda_1(c)(y - c\tau)} dy e^{-\lambda_1(c)}
\]
\[
\leq L_1 \|\Psi_1 - \Psi_2\|_\lambda
\]
where $L_1 := 2\bar{L} + \bar{L} \max_{u \in [0, K^+]} |S'(u)| \int_{\mathbb{R}} J(y) e^{-\lambda_1(c)(y + c\tau)} dy + \gamma_2,
\]
\[
\eta_1(\xi) := \theta_1 \phi_1(\xi) + (1 - \theta_1) \phi_2(\xi), \quad \eta_2(\xi) := \theta_2 (J * S(\phi_1))(\xi) + (1 - \theta_2) (J * S(\phi_2))(\xi),
\]
and
\[
\eta_3(\xi - \gamma \xi) := \theta_3 \phi_1(\xi - \gamma \xi) + (1 - \theta_3) \phi_2(\xi - \gamma \xi)
\]
with $\theta_i \in (0, 1), i = 1, 2, 3$. Then we have
\[
|T_1(\Psi_1)(\xi) - T_1(\Psi_2)(\xi)| e^{-\lambda_1(c)} \leq \frac{L_1 + \gamma_2}{D(\lambda - \tau_1)(\tau_2 - \lambda)} \|\Psi_1 - \Psi_2\|_\lambda.
\]
Similarly, we obtain
\[
|T_2(\Psi_1)(\xi) - T_2(\Psi_2)(\xi)| e^{-\lambda_1(c)} \leq \frac{L_1}{c \lambda_1} \|\Psi_1 - \Psi_2\|_\lambda.
\]
The inequalities (28) and (29) lead to
\[ \|T(\Psi_1) - T(\Psi_2)\| \leq \frac{L_1 + \gamma_1 + \gamma_2}{\min\{D(\lambda - r_1)(r_2 - \lambda), c\lambda + \gamma_2\}} \|\Psi_1 - \Psi_2\|, \]
which implies that \( T \) is continuous on \( \Omega \).

Next, we show that \( T : \Omega \to \Omega \) is compact with respect to the norm \( \|\cdot\| \) in \( X_\lambda \).

For any \( \Psi \in \Omega \) and \( \xi \in \mathbb{R} \), direct calculation yields
\[
\begin{align*}
|T_1(\Psi)(\xi)| & \leq \frac{L K_1 + \gamma_2 \hat{K}_1 \max_{(u,v) \in I} |f(u,v)|}{L + \gamma_1}, \\
|T_2(\Psi)(\xi)| & \leq \frac{\gamma_1 K_1}{c}, \\
|T_1(\Psi)'(\xi)| & \leq \frac{2(L K_1 + \gamma_2 \hat{K}_1 \max_{(u,v) \in I} |f(u,v)|)}{D(r_2 - r_1)}, \\
|T_2(\Psi)'(\xi)| & \leq \frac{\gamma_1 K_1 (\gamma_2 + c)}{c^2}.
\end{align*}
\]
Therefore, the family of functions \( \{T(\Psi)(\xi) : \Psi \in \Omega\} \) is uniformly bounded and equicontinuous in \( \xi \in \mathbb{R} \). By means of the method in [12, Theorem 3.1], we can prove that \( T(\Omega) \) is compact in \( X_\lambda \). As a consequence, Schauder’s fixed point theorem implies that \( T \) has a fixed point \( \Phi_c \in \Omega \), which is a traveling wave solution of (4) for \( c > c^* \). Since
\[
0 \leq \Phi^-_c(\xi) \leq \Phi_c(\xi) \leq \Phi^+_c(\xi), \quad \forall \xi \in \mathbb{R},
\]
it follows that (25) and (26) hold.

Finally, when (H) holds, by using similar arguments as in the proof of [29, Theorem 2.4], one obtains that
\[
\lim_{\xi \to +\infty} \Phi_c(\xi) = K. \tag{30}
\]

**Theorem 4.2.** Assume that (C1)–(C4) hold and \( c = c^* \). For any vector \( \sigma \gg 0 \) with \( \|\sigma\| \ll 1 \), (4) admits a non-constant traveling wave solution \( \Phi_*(\xi) \) such that
\[
\Phi_*(\xi) \leq \sigma, \quad \forall \xi \leq 0 \quad \text{and} \quad 0 \leq \Phi_*(\xi) \leq K^+, \quad \forall \xi \in \mathbb{R}, \tag{31}
\]
Moreover, \( \Phi_*(-\infty) = 0 \) and if (H) holds, then \( \lim_{\xi \to +\infty} \Phi_c(\xi) = K. \)

**Proof.** We use similar arguments as [6, Theorem 4.2] and [27, Theorem 4.13]. Choose a sequence \( \{c_j\} \subset (c^*, +\infty) \) such that \( \lim_{j \to \infty} c_j = c^* \). By Theorem 4.1, there exists a traveling wave \( (\Phi_j, c_j) \) of (4) for each \( j \) such that
\[
K^- \leq \liminf_{\xi \to +\infty} \Phi_j(\xi) \leq \limsup_{\xi \to +\infty} \Phi_j(\xi) \leq K^+.
\]
Given any \( \sigma \gg 0 \) with \( \|\sigma\| \ll 1 \). Since \( \Phi_j(\xi + h), h \in \mathbb{R} \), is also a solution satisfying \( \Phi_j(\xi + h) = 0 \), we can assume that \( \Phi_j(\xi) \leq \sigma \) for \( \xi \leq 0 \). Similar to the proof of Theorem 4.1, we can prove that the family of functions \( \{\Phi_j(\xi)\}_{j=1}^\infty \) is uniformly bounded and equi-continuous on \( \mathbb{R} \). Thus, there exists a subsequence of \( \{c_j\} \), still denoted by \( \{c_j\} \), such that \( \Phi_j(\xi) \) converges uniformly on every bounded interval, and hence pointwise on \( \mathbb{R} \) to a function \( \Phi_*(\xi) := (\phi_*(\xi), \psi_*(\xi)) \). Note that \( \Phi_j(\xi) = T(\Phi_j)(\xi), \xi \in \mathbb{R} \). Letting \( j \to \infty \) in the above equation and using the dominated convergence theorem, we get \( \Phi_*(\xi) = T(\Phi_*)(\xi) \) for \( \xi \in \mathbb{R} \), and hence, (30) and (31) hold.

Next, we show that \( \Phi_*(-\infty) = 0 \). We first prove that
\[
\int_{-\infty}^0 \phi_*(\xi)d\xi < +\infty \quad \text{and} \quad \int_{-\infty}^0 \psi_*(\xi)d\xi < +\infty. \tag{32}
\]
It is easy to see that for any \( \varepsilon > 0 \) satisfying 
\[
\partial_1 f(0, 0) + \partial_2 f(0, 0)S'(0) - (\partial_2 f(0, 0) + S'(0) + 1)\varepsilon + \varepsilon^2 > 0 \quad \text{and} \quad \partial_2 f(0, 0) - \varepsilon > 0,
\]
there exists \( \delta_i(\varepsilon) = \delta_i > 0, i = 1, 2, 3, \) such that \( f(x, y) \geq (\partial_1 f(0, 0) - \varepsilon)x + (\partial_2 f(0, 0) - \varepsilon)y, \forall (x, y) \in [0, \delta_1] \times [0, \delta_2] \) and \( S(x) \geq (S'(0) - \varepsilon)x, \forall x \in [0, \delta_3] \). Let \( \varepsilon \) be defined as above. Denote 
\[
\rho_1 = \frac{\partial_1 f(0, 0) - \partial_2 f(0, 0)S'(0) + (\partial_2 f(0, 0) + S'(0) - 1)\varepsilon - \varepsilon^2}{2},
\]
\[
\rho_2 = \frac{\partial_1 f(0, 0) + \partial_2 f(0, 0)S'(0) - (\partial_2 f(0, 0) + S'(0) + 1)\varepsilon + \varepsilon^2}{2} > 0.
\]
Note that \( \Phi_\ast(\xi) \leq \sigma \) for any \( \xi \leq 0 \). Choose \( \sigma = (\sigma_1, \sigma_2) \gg 0 \) with \( \sigma_1 < \min\{\delta_1, \delta_3, \delta_2/S'(0)\} \). It then follows from (8) that 
\[
c(\phi_\ast)(0) - \phi_\ast(y) + c[\psi_\ast(0) - \psi_\ast(y)] - D[\phi_\ast'(0) - \phi_\ast'(y)] = \rho_1 \int_y^0 [(J * \phi_\ast)(\xi) - \phi_\ast(\xi)]d\xi \geq \rho_2 \int_y^0 [\phi_\ast(\xi) + (J * \phi_\ast)(\xi)]d\xi. \tag{34}
\]
Integrating both sides of (33) from \( y \) to 0 with \( y < 0 \), we have 
\[
D[\phi_\ast'(0) - \phi_\ast'(y)] = \frac{1}{D(r_2 - r_1)} \left[ r_1 \int_{-\infty}^{\xi} e^{\tau_1(\xi-s)}F_1(\Phi_\ast)(s)ds + r_2 \int_{\xi}^{+\infty} e^{\tau_2(\xi-s)}F_1(\Phi_\ast)(s)ds \right],
\]
\[
\psi_\ast'(\xi) = -\frac{\gamma_2}{c^2} \int_{-\infty}^{\xi} e^{-\frac{\gamma_2}{c^2}(\xi-s)}F_2(\Phi_\ast)(s)ds + \frac{1}{c} F_2(\Phi_\ast)(\xi).
\]
Since \( F(\Phi_\ast) \) is bounded, the above equalities imply that \( \Phi_\ast'(\xi) \) is uniformly bounded in \( \mathbb{R} \). By Fubini’s Theorem, we have 
\[
\left| \int_y^0 [(J * \phi_\ast)(\xi) - \phi_\ast(\xi)]d\xi \right| = \left| \int_y^0 \int_{\mathbb{R}} (x + ctr)J(x) \int_0^1 \phi_\ast'(\xi - \theta(x + ctr))d\theta dx d\xi \right| = \left| \int_{\mathbb{R}} (x + ctr)J(x) \int_0^1 \phi_\ast(-\theta(x + ctr)) - \phi_\ast(y - \theta(x + ctr))d\theta dx \right|.
\]
Note that (C1) implies that \( \int_{\mathbb{R}} |x|J(x)dx < +\infty \). Hence, we have 
\[
\left| \int_y^0 [(J * \phi_\ast)(\xi) - \phi_\ast(\xi)]d\xi \right| \leq 2K^+ \left[ ct + \int_{\mathbb{R}} |x|J(x)dx \right] < +\infty.
\]
This shows that \( \int_y^0 [(J * \phi_\ast)(\xi) - \phi_\ast(\xi)]d\xi \) is bounded on \( (-\infty, 0] \). Therefore, from (34), we see that \( \int_0^y \phi_\ast(s)ds \) is bounded on \( (-\infty, 0] \). Moreover, from the second
equation of (8), we have
\[ \gamma_2 \int_0^y \psi_s(s) ds = \gamma_1 \int_0^y \phi_s(s) ds - c[\psi_s(0) - \psi_s(y)], \]
which implies that \( \int_0^y \psi_s(s) ds \) is also bounded on \((-\infty, 0]\). Hence, (32) follows.

By the uniform boundedness of \( \Phi' \), we can show that \( \Phi_s(-\infty) = 0 \). The proof of upward convergence \( \Phi_s(\infty) = K \) is essentially similar to that of Theorem 4.1 and thus omitted.

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E-mail address: zhenguobai_q2163.com
E-mail address: zhaotingting1116@163.com