Critical behavior of two-dimensional fully frustrated XY systems

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In celebration of Tony Guttmann’s 60th birthday

Abstract. We study the phase diagram of the two-dimensional fully frustrated XY model (FFXY) and of two related models, a lattice discretization of the Landau-Ginzburg-Wilson Hamiltonian for the critical modes of the FFXY model, and a coupled Ising-XY model. We present Monte Carlo simulations on square lattices \( L \times L \), \( L \lesssim 10^3 \). We show that the low-temperature phase of these models is controlled by the same line of Gaussian fixed points as in the standard XY model. We find that, if a model undergoes a unique transition by varying temperature, then the transition is of first order. In the opposite case we observe two very close transitions: a transition associated with the spin degrees of freedom and, as temperature increases, a transition where chiral modes become critical. If they are continuous, they belong to the Kosterlitz-Thouless and to the Ising universality class, respectively. Ising and Kosterlitz-Thouless behavior is observed only after a preasymptotic regime, which is universal to some extent. In the chiral case, the approach is nonmonotonic for most observables, and there is a wide region in which finite-size scaling is controlled by an effective exponent \( \nu_{\text{eff}} \approx 0.8 \). This explains the result \( \nu \approx 0.8 \) of many previous studies using smaller lattices.

1. Fully frustrated systems

In the last few decades there has been a considerable interest in the consequences of frustration on the critical behavior of statistical systems. The simplest example is the antiferromagnetic Ising model on a triangular lattice, whose Hamiltonian is

\[
\mathcal{H} = J \sum_{\langle ij \rangle} \sigma_i \sigma_j,
\]

where \( J \) is positive, \( \sigma_i = \pm 1 \), and the sum is extended over all lattice nearest neighbors \( \langle ij \rangle \). For \( T \to 0 \) the system tends to be antiferromagnetically ordered, i.e. spins on nearest-neighbor sites prefer to be oppositely aligned. However, this is not possible everywhere. For instance, see Fig. 1, on any lattice triangle one link must be frustrated, i.e. spins on the corresponding sites must be parallel so that the local energy assumes its maximum value. The presence of frustration has an important consequence. At
The Ising model can be generalized by considering the $N$-vector model on a triangular lattice. In this case one considers unit $N$-component vectors $\vec{s}_i$ and the Hamiltonian

$$\mathcal{H} = J \sum_{<ij>} \vec{s}_i \cdot \vec{s}_j. \tag{1.2}$$

Also this model is frustrated: There is no configuration in which all neighboring spins are antiparallel. However, at variance with the Ising case, here the entropy vanishes at zero temperature. Indeed, once rotational invariance has been broken by fixing the direction of one spin, there is a finite number of configurations that are global minima of the Hamiltonian [6]. For $N = 2$, the only case we will consider, if $\vec{s}_i = (\cos \theta_i, \sin \theta_i)$, one must have $|\theta_i - \theta_j| = 2\pi/3$ or $4\pi/3$ when $i$ and $j$ are nearest-neighbor sites. It is easy to verify that the degeneracy of the ground state is $\mathbb{Z}_2 \otimes O(2)$, where $O(2)$ is the invariance rotation group. The group $\mathbb{Z}_2$ is due to the possibility of two (chirally) different configurations. As shown in Fig. 2, the ground state is uniquely determined once one breaks rotational invariance (by setting, for instance, $\theta_A = 0$) and chooses the chirality of triangle ABC (by setting $\theta_B = 120^\circ$ or $240^\circ$). An observable that distinguishes between the two ground states is the chirality. Given a lattice triangle, see Fig. 2, we can consider [6]

$$C_n \equiv \frac{2}{3\sqrt{3}}[\sin(\theta_A - \theta_B) + \sin(\theta_B - \theta_C) + \sin(\theta_C - \theta_A)], \tag{1.3}$$

which assumes the values $\pm 1$ on any lattice triangle in a ground-state configuration. A good order parameter is obtained as follows. We assign $s_n = \pm 1$ to each lattice triangle so that $s_n = -s_m$ if triangles $n$ and $m$ share a lattice link. The order parameter, the chirality magnetization, is simply

$$M_C \equiv \sum_n s_n C_n, \tag{1.4}$$

where the sum is extended over all lattice triangles.

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1 It is interesting to note that this is not true for the spin-$S$ antiferromagnetic Ising model if $S$ is large enough; see [3, 4, 5] and references therein.
Figure 2. Two inequivalent ground states related by a chiral transformation. They are obtained as follows: one first fixes \( \theta_A = 0^\circ \), breaking rotational invariance. Then, there are two possible choices: on the left we choose \( \theta_B = 240^\circ, \theta_C = 120^\circ \); on the right we make the opposite choice. All other lattice spins are univocally defined.

Figure 3. The couplings \( j_{ij} \) in Hamiltonian (1.5):
\( j_{ij} = 1 \) on thin lines, \( j_{ij} = -\alpha \) on thick lines.

It is possible to define a frustrated model also on the square lattice. The relevant model is not the antiferromagnetic Ising or \( N \)-vector model, since no frustration occurs on the square lattice or, in general, on any bipartite lattice. To obtain a frustrated model we consider a Hamiltonian of the form [6]

\[
H_{FFXY} = -J \sum_{\langle ij \rangle} j_{ij} \vec{s}_i \cdot \vec{s}_j, \tag{1.5}
\]

where the two-component spins \( \vec{s}_i \) satisfy \( \vec{s}_i \cdot \vec{s}_i = 1 \). \( j_{ij} = 1 \) along all horizontal lines, while along vertical lines ferromagnetic \( j_{ij} = 1 \) and antiferromagnetic \( j_{ij} = -\alpha (\alpha > 0) \) couplings alternate, see Fig. 3. This model is frustrated for any positive \( \alpha \). Maximal frustration is obtained by taking \( \alpha = 1 \); for this reason this particular model, the only one we shall consider in the following, is called fully frustrated XY (FFXY) model. The square-lattice FFXY model admits two chirally different ground states, see Fig. 4, and thus it has the same ground-state degeneracy of the antiferromagnetic model on the triangular lattice. These two models are particular examples of a general class of systems that all have a \( \mathbb{Z}_2 \otimes O(2) \) ground-state degeneracy. We will collectively call them FFXY systems.

Even though the symmetry of the FFXY systems (1.2) and (1.5) is the same as that of the ferromagnetic XY model, we expect the critical behavior to be different. Indeed, the universality class is not only determined by the symmetry of the order parameter but also by the symmetry breaking pattern that is different in the two cases.
Figure 4. Ground states of the square-lattice FFXY model: in this case nearest-neighbor spins must satisfy \( \theta_i - \theta_j = 45^\circ \) or \( 315^\circ \) if they are connected by a ferromagnetic link, and \( \theta_i - \theta_j = 135^\circ \) or \( 225^\circ \) if they are connected by an antiferromagnetic link. Once we fix \( \theta_P = 0^\circ \), there are two (chirally) inequivalent possibilities: (left) \( \theta_A = 45^\circ \) or (right) \( \theta_B = 45^\circ \). All other spins are fixed. Note that the ground-state configurations are invariant under translations of two lattice spacings.

In order to determine the critical behavior we can perform a direct numerical study of the model. There is however another possibility, which is the basis of the field-theoretical approach to critical phenomena. In this case one first identifies the critical modes of the microscopic Hamiltonian and then writes down an effective coarse-grained (continuum) Hamiltonian for them. The model one obtains is no longer frustrated; still, it is expected to have the same critical behavior as the original one. In order to derive the effective theory, let us consider the antiferromagnetic model on a triangular lattice. As is evident from Fig. 2, in the ground state spins rotate by \( 120^\circ \) when moving in the \( x \) direction from one site to its neighbor. Thus, critical modes are associated with fluctuations close to the complex Fourier component \( \vec{s}(Q) \) with \( Q = [2\pi/(3a), 0] \), where \( a \) is the lattice spacing. These fluctuations are parametrized by a complex vector \( \vec{\Phi} = \vec{\phi}_1 + i\vec{\phi}_2 \). Note that the appearance of two real two-component fields is at variance with the ferromagnetic case. Indeed, in that case, the relevant modes are associated with the zero-momentum component \( \vec{s}(q = 0) \). As a consequence of the reality condition \( \vec{s}(q) = \vec{s}^*(-q) \), fluctuations are real and are parametrized by a single real two-component field. A standard calculation \([7, 8, 9]\) gives the effective Hamiltonian for the fields \( \vec{\phi}_a \):

\[
\mathcal{H}_{LGW} = \int d^d x \left\{ \frac{1}{2} \sum_{a=1,2} \left[ (\partial_\mu \vec{\phi}_a)^2 + r \vec{\phi}_a^2 \right] + \frac{1}{4!} v_0 \left( \sum_{a=1,2} \phi_a^2 \right)^2 + \frac{1}{4} v_0 \phi_1^2 \phi_2^2 \right\}.
\] (1.6)

A similar argument applies to the square-lattice FFXY model and gives the same effective Hamiltonian (1.6).

Hamiltonian (1.6) has a larger symmetry than the original one. Indeed, the symmetry group is \([O(2) \oplus O(2)] \otimes Z_2\): the \( O(2) \) groups are related to independent rotations of the two fields, while the \( Z_2 \) group corresponds to the field-interchange symmetry. Nonetheless—and this is the only property that matters—Hamiltonian (1.6) for \( v_0 > 0 \) and FFXY models have the same symmetry-breaking pattern. Indeed, for \( v_0 > 0 \), the ground state corresponds to \( \phi_1^2 = 0 \) and \( \phi_2^2 \neq 0 \) or the opposite and thus it has the same ground-state degeneracy: the ground state breaks one of the two \( O(2) \) groups and the \( Z_2 \) field-interchange symmetry.

A lattice discretization of (1.6) is \([10]\)

\[
\mathcal{H}_\phi = -J \sum_{\langle ij \rangle, a} \phi_{a,i} \phi_{a,j} + \sum_{a,i} \left[ \phi_{a,i}^2 + U(\phi_{a,i}^2 - 1)^2 \right] + 2(U + D) \sum_i \phi_{1,i}^2 \phi_{2,i}^2,
\] (1.7)
where $J > 0$ (the model is ferromagnetic), $a = 1, 2$, $\phi_{a,i}^\alpha$ is a real two-component variable, the first sum goes over all nearest-neighbor pairs, and $\phi_{a,i}^\alpha \equiv \vec{\phi}_a \cdot \hat{\alpha}$. The correct symmetry-breaking pattern is obtained for $D > 0$, which corresponds to $v_0 > 0$ in (1.6). Moreover, stability requires $U > 0$. For $U \to \infty$, $\mathcal{H}_\phi$ becomes simpler and we obtain

$$
\mathcal{H} = -J \sum_{\langle ij \rangle, a} \vec{\phi}_{a,i} \cdot \vec{\phi}_{a,j} + 2D \sum_i \phi_{1,i}^2 \phi_{2,i}^2,
$$

where the fields satisfy the constraint $\phi_{1,i}^2 + \phi_{2,i}^2 = 1$. This is the 4-vector model with a spin-4 perturbation that breaks the $O(4)$ symmetry to $[O(2) \oplus O(2)] \otimes \mathbb{Z}_2$. If we additionally take $D \to \infty$ we must have $\phi_{1,i}^2 \phi_{2,i}^2 = 0$. In this case, we can parametrize

$$
\vec{\phi}_{1,i} = \frac{1}{2} (1 + \sigma_i) \vec{s}_i, \quad \vec{\phi}_{2,i} = \frac{1}{2} (1 - \sigma_i) \vec{s}_i,
$$

where $\sigma_i$ is an Ising spin and $\vec{s}_i$ is a unit two-component vector. The Hamiltonian reduces to

$$
\mathcal{H} = -\frac{J}{2} \sum_{\langle ij \rangle} (1 + \sigma_i \sigma_j) \vec{s}_i \cdot \vec{s}_j.
$$

Hamiltonian (1.10) has the same invariance as (1.7) although, in terms of the new fields, the $O(2) \oplus O(2)$ symmetry is nonlinearly realized:

$$
\vec{s}_i' = \left[ \frac{1}{2} (1 + \sigma_i) R_1 + \frac{1}{2} (1 - \sigma_i) R_2 \right] \vec{s}_i,
$$

$$
\sigma_i' = \sigma_i,
$$

where $R_1$ and $R_2$ are $O(2)$ rotation matrices. It is possible to add terms of the form $\sigma_i \sigma_j$ without breaking the symmetry of the Hamiltonian. We can thus consider the more general Hamiltonian [11]

$$
\mathcal{H}_{\text{IsXY}} = -\sum_{\langle ij \rangle} \left[ \frac{J}{2} (1 + \sigma_i \sigma_j) \vec{s}_i \cdot \vec{s}_j + C \sigma_i \sigma_j \right],
$$

We will call this model the Ising-XY (IsXY) model. For $J > 0$ and $C + J/2 > 0$ it has the same symmetry-breaking pattern as FFXY systems. Thus, it represents another example of this class of models. We should mention that there are many other systems that share the same symmetry-breaking pattern: for an extensive list of references see [12].

2. Results

Two-dimensional FFXY systems (but note that these systems have also been studied, both theoretically and experimentally, in three dimensions [13, 14, 15, 16]) have been extensively studied in the last thirty years, after the appearance of the seminal papers by Villain [6]. For an extensive list of references, see [12]. In spite of that, their critical behavior is object of debate still today. Two scenarios have been proposed for the critical behavior of models (1.2) and (1.5).

A first possibility is that these models have two continuous transitions. As temperature decreases, there is first a transition associated with the chiral degrees of freedom: at the transition there is no magnetic ordering but only chiral order. As temperature further decreases, there is an intermediate phase in which spins are disordered while chiral variables are magnetized. Then, a second transition occurs, followed by a low-temperature (LT) phase in which spin-spin correlations decay algebraically. In this scenario chiral and spin modes do not interact at the transitions and thus, if the transitions are continuous, one expects a chiral Ising transition (the order parameter is a scalar) and a spin Kosterlitz-Thouless (KT) transition. This scenario has not been confirmed numerically so far. The computed exponents at the
chiral transition do not agree with the Ising ones. For instance, one finds \( \nu \approx 0.8 \) instead of the Ising value \( \nu = 1 \). Moreover, it is not clear how much one can believe in the presence of two transitions that are very close to each other: numerically \( (T_{\text{spin}} - T_{\text{chiral}}) \approx 10^{-2}J \).

The inconsistencies of the two-transition scenario apparently favor the presence of a single critical point. In this scenario, the observed difference between the critical temperatures is interpreted as a correction-to-scaling effect. Since chiral and spin modes become critical at the same temperature, it is possible that the transition belongs to a new universality class with a new set of critical exponents. In this scenario, the result \( \nu \approx 0.8 \) would be fully acceptable. Of course, it is also possible to interpret the results for \( \nu \) in terms of crossover effects. Since the two transitions are very close, scaling corrections may be large so that the asymptotic behavior can be observed only on very large lattices [17].

We have considered again the issue [10, 12], performing extensive simulations of the FFXY model (1.5), of the \( \phi^4 \) model (1.7) with \( U = 1 \) and of the IsXY model (1.13). In all cases we have considered the square lattice. We have used a mixture of Metropolis and overrelaxation updates as well as cluster updates in the LT phase, essentially following [18] (see [12] for a detailed discussion). We have studied the critical behavior on lattices of size \( L \times L \), in some cases up to \( L \sim 10^3 \): for the FFXY model, the \( \phi^4 \) model with \( D = 1/2 \), and the IsXY model with \( C = 0 \), the largest lattice we have used at the chiral transition corresponds to \( L = 1000, L = 1200, L = 360 \) respectively. In the LT phase the Monte Carlo algorithm is much more efficient and we have been able to simulate even larger sizes: for the \( \phi^4 \) model with \( D = 1/2 \) we performed simulations for \( L = 2048 \).

The analysis of the Monte Carlo results for the square-lattice FFXY model definitely shows that this model undergoes two transitions: the chiral one belongs to the Ising universality class, while the spin one is compatible with a KT behavior. In the LT phase the critical behavior of the spin modes is controlled by the same line of Gaussian fixed points as in the standard XY model. The discrepancies from Ising behavior at the chiral transition that have been observed in previous studies are simply crossover effects. They are due to the presence of a large, albeit finite spin correlation length \( \xi_s^{(c)} \) at the chiral transition. In finite-size scaling studies the asymptotic behavior can only be observed if \( L \gg \xi_s^{(c)} \). Since \( \xi_s^{(c)} \) is quite large, \( \xi_s^{(c)} = 118(1) \), the asymptotic behavior can only be observed in simulations with \( L \) approximately greater than 500, i.e. for values of \( L \) that are much larger than those that could be used in simulations until a few years ago. In the other models we have studied, the determination of the asymptotic behavior may be even more difficult. For instance, in the \( \phi^4 \) model with \( D = 1/2 \), \( \xi_s^{(c)} \approx 380 \). Thus, even with simulations with \( L = 1200 \), we have not been able to observe the Ising behavior but only the beginning of the crossover towards the asymptotic behavior.

In the IsXY and in the \( \phi^4 \) model we have observed two transitions in a large parameter region. However, for \( D \) or \( -C \) large, we have also observed a unique first-order transition which separates the LT phase with chiral order and spin quasi-long-range order from the disordered phase, see Fig. 5. Thus, our results confirm the two-transition scenario for generic FFXY systems in the sense that we have found no evidence of a unique continuous transition where chiral and spin modes become both critical. A single transition occurs only if it is of first order.

Beside these results that confirm the two-transition scenario, we have also observed an unexpected universal crossover behavior. We find that renormalization-group invariant quantities (e.g., critical exponents, Binder parameters, ...) computed in the different models scale at the chiral and spin transitions respectively as

\[ R = f_R^{(c)}(L/l), \quad R = f_R^{(s)}(L/l), \]  

(2.1)

where \( l \) is a model-dependent scaling factor that is identical at the two transitions. At the chiral transition—the only case in which we have done a systematic investigation by varying \( C \) and \( D \)—corrections appear to increase as \( D \) or \( -C \) increases (in practice, significant deviations are observed for \( D \gtrsim 4 \) and \( C \lesssim -2 \)). Of course, it is of interest to have a renormalization-group explanation of this apparent universality. Eq. (2.1) can be explained by the presence of a multicritical point [19, 20, 21].
(or of a line of multicritical points of the same type) where chiral and spin modes become both critical. Indeed, close to a multicritical point we expect that any RG-invariant quantity behaves as

$$\mathcal{R} = f_R(L/\xi_s, L/\xi_{ch}),$$

(2.2)

where $\xi_s$ and $\xi_{ch}$ are the infinite-volume correlation lengths for spin and chiral variables. Our analysis at the Ising and KT transitions corresponds to fixing $L/\xi_{ch} = 0$ and $L/\xi_s = 0$, i.e. provides the scaling function along two particular lines. If the interpretation in terms of a multicritical point is correct, the functions $f_R(x)$ provide informations on the behavior at the multicritical point. Indeed, while Ising or KT behavior is observed for $x \to \infty$, in the opposite limit $x \to 0$ we obtain the value of the RG-invariant quantity $\mathcal{R}$ at the multicritical point.

The nature of the multicritical point is unclear. One possibility is the $O(4)$ multicritical point that is present in the $\phi^4$ theory for $D = 0$. Another possibility is the multicritical point that appears in frustrated XY systems with modulated couplings (for instance in model (1.5) for $\alpha \neq 1$) [22, 23, 24] or in generalizations of the IsXY model, in which an additional spin-spin coupling is added, breaking the $O(2) \oplus O(2)$ symmetry.

Finally, we wish to compare with field-theory (FT) approaches. Perturbative analyses [25] of model (1.6) indicate the existence of a new universality class associated with the symmetry-breaking pattern $[O(2) \oplus O(2)] \otimes Z_2 \to O(2)$. Even though we have found no evidence for it, our results do not necessarily contradict those of Ref. [25]. It is possible that the models we have considered are outside the attraction domain of the FT fixed point. If this is the case, field theory provides another candidate for the multicritical point. The models we consider could be outside, but close to the attraction domain of the fixed point—this is not unplausible since $\xi_s^{(c)}$ is large—so that the crossover behavior is controlled by the FT fixed point. We wish also to make a remark on the validity of (1.6). In the derivation of Hamiltonian (1.6) by using the standard Hubbard-Stratonovitch transformation, terms with more than four fields are neglected [7, 8]. In particular, terms of the form $(\vec{\phi}_1 \cdot \vec{\phi}_2)^n$ appear at sixth order ($n = 3$) (resp. eighth order) in the case of the triangular-lattice (resp. square-lattice) FFXY model [8]. These terms have only $Z_2 \oplus O(2)$ symmetry and thus, under renormalization-group transformations, are bound to generate a term of the form $\vec{\phi}_1 \cdot \vec{\phi}_2$, or even a quadratic term of the form $(\vec{\phi}_1 \cdot \vec{\phi}_2)$. We obtain therefore the multicritical Hamiltonian

$$\mathcal{H}_{\text{LGW},2} = \mathcal{H}_{\text{LGW}} + \int d^d x \left[ \frac{1}{2} r_2(\vec{\phi}_1 \cdot \vec{\phi}_2) + \frac{1}{4} z_0(\vec{\phi}_1 \cdot \vec{\phi}_2)^2 + \frac{1}{4} z_1(\vec{\phi}_1 \cdot \vec{\phi}_2)(\phi_1^2 + \phi_2^2) \right],$$

(2.3)

as it has been postulated for modulated systems. If this interpretation is correct, the FT fixed point may only be relevant at the multicritical point, provided it is stable under the quartic perturbations that break the $O(2) \oplus O(2)$ symmetry. This holds in three dimensions [26], but nothing is known in the two-dimensional case. It should be remarked that these considerations are only relevant for the FFXY model.

**Figure 5.** Sketch of the phase diagram of the $\phi^4$ model (1.7) for $U = 1$ and $D > 0$ (left) and of the IsXY model (1.13) (right). The continuous, dashed, and thick continuous lines represent Ising, KT, and first-order transition lines. The distance between the ferromagnetic Ising and KT lines is amplified; otherwise, the two transitions cannot be distinguished on the scale of the figure. The phase diagram within the circled region is unknown. In the IsXY case there is also an antiferromagnetic Ising transition (af) starting at $C = -C_{Is} = -\frac{1}{2}(1 + \sqrt{2})$, $J = 0$.
The IsXY and $\phi^4$ theories are $O(2) \oplus O(2)$ invariant and thus the correct FT Hamiltonian is clearly (1.6) and not (2.3).

3. Chiral transition

In order to determine the nature of the chiral transition, we have studied the behavior of several quantities at fixed $R_c \equiv \xi_c / L$ where $\xi_c$ is the chiral correlation length (see [12] for a precise definition in the different models). We use the method proposed in [27] and further discussed in [28]. We fix $R_c$ equal to $R_{Is}$, where $R_{Is} = 0.9050488292(4)$ is the universal value of $\xi / L$ at the critical point in the 2-d Ising universality class [29]. We stress that this choice does not bias our analysis in favor of the Ising nature of the chiral transition. For any chosen value (as long as it is positive) and whatever the universality class of the chiral transition is (it may also coincide with the spin transition), we are studying the model for $L$-dependent temperatures $T_{eff}(L)$ such that $T_{eff}(L) \to T_{ch}$ for $L \to \infty$. Note, however, that quantities like the Binder parameter depend on the chosen value for $R_c$. Indeed, in the finite-size scaling limit, $R_c = f_R[L^{1/\nu}(T - T_{ch})]$. Therefore, fixing $R_c$ is equivalent to fixing $X \equiv L^{1/\nu}(T_{eff}(L) - T_{ch})$. Since the Binder parameter satisfies an analogous relation $B_c = f_B[L^{1/\nu}(T - T_{ch})]$, at fixed $R_c$, $B_c$ converges to $f_B(X)$. By fixing $R_c$ to the critical Ising value, we will be able to perform an additional consistency check. If the chiral transition belongs to the Ising universality class, then $X = 0$ (apart from scaling corrections) and we should find that any RG-invariant quantity converges to its critical-point value in the Ising model.

We first verify that $\xi_s$ converges to a constant as $L \to \infty$. In Fig. 6 we show the numerical results. The correlation length is clearly finite in the FFXY model and in the IsXY model with $C = 0$. In the $\phi^4$ model with $D = 1/2$ we do not yet observe that $\xi_s$ is finite, although it is already clear that $\xi_s$ does not increase linearly with $L$, as it would be the case if the spin correlation length were infinite for $L = \infty$. Then, we verified that the transition, if continuous, belongs to the Ising universality class. The best evidence is provided by the Binder chiral parameter. If the transition belongs to the Ising universality class we should find [29, 30, 12]

$$B_c = B_{Is} + b L^{-7/4}, \tag{3.1}$$

where $B_{Is} = 1.167823(5)$ [29]. The results reported in Fig. 7 are fully consistent with Ising behavior for $L \gg \xi_{s}^{(c)}$. Not only do we observe the Ising asymptotic value, but also the rate of convergence is well verified. Also the critical exponents $\nu$ and $\eta$ converge to the Ising values for $L \gg \xi_{s}^{(c)}$ (results for $\nu$ will
Figure 7. Chiral Binder parameter $B_c$ at fixed $R_c = R_{16}$ (chiral transition). Plot of $\Delta B_c \equiv B_c - B_{16}$ at fixed $R_c = R_{16}$ vs $L^{-7/4}$, for the FFXY model, the $\phi^4$ model at $D = 1/2$, and the IsXY model at $C = 0$. $B_{16} = 1.167923(5)$ is the value of the Binder parameter at the critical point in the Ising model [29].

be shown below).

4. Low-temperature phase and spin transition

Since the spin correlation length is finite at the chiral transition there must be a paramagnetic phase with chiral order. Such a phase ends at a second transition which is followed by the LT phase in which chiral order and spin quasi-long-range order coexist.

We first study the nature of the LT phase and verify the breaking of the $Z_2$ invariance. Direct evidence is provided by the chiral Binder parameter $B_c$. If chiral modes are magnetized, $B_c \rightarrow 1 + O(L^{-2})$ for $L \rightarrow \infty$. This behavior is very well verified in all models. In the $\phi^4$ model, the $Z_2$ group corresponds to the field-interchange symmetry. Thus, the $Z_2$ symmetry breaking implies that only one of the two fields $\phi_1$ and $\phi_2$ is critical in the LT phase. Our numerical results fully confirm this expectation. One can distinguish the fields according to the value of $Q_a = \sum_i \phi_{2a}^2$. The field with the largest value of $Q$ is critical (for instance, the corresponding susceptibility and correlation length diverge as $L \rightarrow \infty$), while the other one is not (the susceptibility and the correlation length have a finite limit as $L \rightarrow \infty$).

Once we have checked that chiral modes are magnetized (this is of course obvious because of the presence of the chiral transition), we study the behavior of the spin variables and verify that the large-$L$ behavior is controlled by the same line of Gaussian fixed points as in the standard XY model. In the XY model one can derive universal relations among renormalization-group invariant quantities that are valid in the whole LT phase, up to the KT transition [31]. Indeed, below the KT transition the spin-wave approximation is asymptotically exact as $L \rightarrow \infty$ and allows an analytic determination of any quantity in terms of the spin-wave parameter. Such a parameter is not universal and can be eliminated by expressing a renormalization-group invariant quantity in terms of another. For instance, one can express the helicity modulus on a square lattice $L \times L$ with periodic boundary conditions in terms of the exponent $\eta$ computed from the size dependence of the magnetic susceptibility ($\chi \sim L^{2-\eta}$):

$$\Upsilon = \frac{1}{2\pi \eta} - \frac{\sum_{n=-\infty}^{\infty} n^2 \exp(-\pi n^2/\eta)}{\eta^2 \sum_{n=-\infty}^{\infty} \exp(-\pi n^2/\eta)}$$

(4.1)

where $0 < \eta \leq 1/4$. Analogously, one can express $\xi_s/L$ in terms of $\eta$. In Fig. 8 we compare the spin-wave prediction with numerical results. The agreement is quite good, confirming that FFXY systems and the standard XY model have the same LT phase, as far as the spin degrees of freedom are concerned.
Figure 8. Estimates of \( R_s \equiv \xi_s / L \) vs \( \eta \) in the LT phase. The continuous line is the prediction obtained by assuming that in the LT phase criticality is controlled by the same line of Gaussian fixed points as in the XY model.

Figure 9. Ratio \( R_s \equiv \xi_s / L \) (\( \xi_s \) is the spin correlation length) at the chiral transition vs \( L_r \equiv L / l \) for the FFXY model and \( \phi^4 \) (left) and IsXY (right) models. We set \( l = \xi_s^{(c)} \) for the FFXY model; the values of \( l \) for the other models are obtained by requiring that all data fall on a single curve. Note the logarithmic scale on both axes. For \( L_r \to \infty \), \( R_s \) converges to 0.

The above-reported results for the LT phase make it plausible that the spin transition belongs to the KT universality class. Another check is provided by our numerical results for \( \xi_s / L \) and \( \Upsilon \). In the XY model, at the KT transition [31] 
\[
\xi_s / L \approx 0.750691 + 0.212430 / \ln(L/C_1) \quad \text{and} \quad \Upsilon \approx 0.636508 + 0.318899 / \ln(L/C_2).
\]
In all models we have studied these two quantities assume the XY values approximately at the same temperature—which we identify with the spin critical temperature—thereby confirming the KT nature of the transition.

5. Crossover behavior
As we already mentioned, our data show the scaling behaviors (2.1). In this section we shall give a few details.

In order to verify the scaling behavior (2.1) at the chiral transition\(^2\) we have first considered the data for \( R_s \equiv \xi_s / L \) (\( \xi_s \) is the spin correlation length) at the chiral transition and we have investigated whether

\(^2\) Note that our results have been obtained at fixed \( R_c \) and not at the chiral critical point. However, since we are dealing with an Ising transition and \( R_c \) has been fixed to the critical-point Ising value, this is irrelevant in the scaling limit. Had we fixed \( R_c \) to a different value, we would have obtained quantitatively different scaling curves, corresponding to the limit \( \xi_{ch}, L \to \infty \) at fixed \( L / \xi_{ch} \neq 0 \) in Eq. (2.2) (\( \xi_{ch} \) is the infinite-volume chiral correlation length).
they fall on a single curve by using a rescaled variable $L_r \equiv L/l$, where $l$ is a rescaling factor that depends on the model. The results are reported in Fig. 9. The data fall on a single curve with remarkable precision, i.e. $\xi_s / L = f_s (L/l)$ where $f_s (x)$ is model independent. Note that the rescaling factors change significantly from one model to another: for instance, $l/\xi_{FFXY} = 0.031$ (resp. 1.2) in the $\phi^4$ model with $D = 4$ (resp. $D = 1/5$) and $l/\xi_{FFXY} = 0.031$ (resp. 12) in the IsXY model with $C = 0$ (resp. $C = -2$). It is easy to realize that $l$ should be proportional to $\xi_s^{(c)}$, the infinite-volume spin correlation length at the chiral transition. Indeed, since $\xi_s \to \xi_s^{(c)}$ as $L \to \infty$, we have $f_s (x) \sim a/x$ for $x \to \infty$, where $a$ is model independent ($f_s (x)$ is model independent). Moreover, $\xi_s^{(c)} = l/a$. Therefore, if we fix $l = \xi_s^{(c)}$ in one model, then the same holds in all different models. In the figures we have chosen $l_{FFXY} = 118 \approx \xi_s^{(c)}$ and thus the plots we present are indeed in terms of $L/\xi_s^{(c)}$, even in those cases in which we have not been able to determine directly the spin correlation length for $L \to \infty$.

In order to verify the universality of the scaling Ansatz (2.1), we have considered the spin Binder parameters $B_s$ and $B_{s\phi}$ versus $L/l$. We use the rescaling factors that have been determined in the analysis of $R_s$. The agreement is quite good. Deviations appear as $D$ or $-C$ increases. In particular, the data for $D = 4$ ($\phi^4$ model) and for $C = -3$ (IsXY model) are outside the curve. They would fall on the same curve as the others, only if the rescaling factor is changed by a factor of 2 and 5 respectively in the two cases. In Fig. 11 we plot the results for the helicity modulus: again all data fall on a single curve quite precisely. It is interesting to note that, for $0.02 \lesssim L_r \lesssim 0.5$, $R_s$ and $\Upsilon$ show an approximate power-law behavior: they behave as $L_r^{-\epsilon}$ with $\epsilon \approx 0.1$ ($R_s$) and $\epsilon \approx 0.33$ ($\Upsilon$). If this behavior holds also for smaller values of $L_r$, we have $\Upsilon, R_s \to \infty$ for $L_r \to 0$.

Next we consider the chiral variables. They also show the universal behavior (2.1). In Fig. 11 we
Figure 11. Helicity modulus $\Upsilon$ (left) and chiral Binder parameter $B_c$ (right) at the chiral transition vs $L_r \equiv L/l$. We report $\Delta B_c = B_c - B_{Is}$, where $B_{Is}$ is the value of the Binder parameter at the critical point in the Ising model. The rescalings $l$ are the same as in Fig. 9.

Figure 12. Effective exponent $1/\nu_{\text{eff}}$ computed by using $R_c \equiv \xi_c/L$, the Binder chiral parameter $B_c$, and $R_s \equiv \xi_s/L$ ($\xi_c$ and $\xi_s$ are respectively the chiral and spin correlation lengths) vs $L_r \equiv L/l$ at the chiral transition. For $L_r \to \infty$, $1/\nu_{\text{eff}}$ converges to $1/\nu = 1$ for $R_c$ and $B_c$, and $-1$ for $R_s$. The rescalings $l$ are the same as in Fig. 9.

report the Binder chiral parameter, with the same scaling factors $l$ as before. The data fall again on a single curve although the curve does not change significantly as $L/l$ varies, at variance with the spin variables.

Finally, we show the effective exponent $\nu_{\text{eff}}$ that can be obtained from the derivatives of $R_c$, $B_c$, and $R_s$. The exponent obtained from the chiral variables should converge to the Ising value $\nu = 1$ as $L \to \infty$, and indeed it does. The approach is however nonmonotonic, $\nu_{\text{eff}}$ being first smaller than 1, then larger. It is interesting to note that for $L_r \lesssim 1$, i.e. $L \lesssim \xi_s^{(c)}$, $\nu_{\text{eff}}$ is approximately constant and equal to 0.8. This behavior explains previous results. Indeed, if one performs simulations only for values of $L$ such that $L_r \lesssim 1$ (this was the case in previous simulations of the FFXY model since $\xi_s^{(c)} \approx 10^2$) one would
Figure 13. Ratio $R_s \equiv \xi_s / L$ (left) and helicity modulus $\Upsilon$ (right) at the spin transition vs $L_r \equiv L / l$. The rescalings $l$ are the same as in Fig. 9.

estimate $\nu = 0.8$. Note also that, for $L \lesssim \xi_s^{(c)}$, the effective exponent $\nu_{\text{eff}}$ obtained from the spin variable $R_s$ is approximately constant and close to the value obtained by using chiral variables. In this range of values of $L$ chiral and spin variables appear as if they are both critical.

The same analysis can be repeated at the spin transition. In Fig. 13 we report $\xi_s / L$ and $\Upsilon$ versus $L / l$ where the rescaling factors $l$ are those determined at the chiral transition. The agreement is again quite good. The existence of scaling at the two transitions with the same rescaling factors is another piece of evidence in favor of the multicritical origin of the universality we observe.

The crossover curves we have computed can give us some hints on the nature of the multicritical point, if it really exists. Indeed, the behavior at the multicritical point is simply obtained by considering the limit $L_r \to 0$. First, let us notice that the data at the chiral transition apparently exclude the possibility of a decoupled multicritical point in which spin and chiral modes have XY and Ising behavior. Indeed, the helicity modulus is very much different from the KT value, $\Upsilon_{KT} = 0.63650817819 \ldots$[31] and $B_c$ is apparently smaller than the Ising value for $L_r \to 0$. O(4) behavior is possible, since, as we already discussed, our data are compatible with $\Upsilon, R_s \to \infty$ for $L_r \to 0$. Also the data for the Binder parameters $B_s$ and $B_s\phi$ are compatible with the O(4) value, $B_s = B_{s0} = 1$. For $L_r \to 0$, the crossover curves at the spin transition should converge to the same values as those at the chiral critical point. This is not evident from the results plotted in Fig. 13. This can be easily explained. At the spin transition the natural scale is the infinite-volume chiral correlation length at the transition $\xi_s^{(c)}$. We expect XY behavior for $L / \xi_s^{(c)} \gg 1$ and multicritical behavior in the opposite case. Numerically, we find $\xi_s^{(c)} / \xi_c^{(s)} \approx 15$, so that $L / \xi_s^{(c)} = 1$ corresponds to $L_r = L / \xi_s^{(c)} \approx 0.07$. As it can be seen from the figure, none of our data satisfies the condition $L_r \ll 0.07$, so that we are unable to observe multicritical behavior at the spin transition.

References
[1] Wannier G H 1950 Antiferromagnetism. The triangular Ising net Phys. Rev. 79 357–364; erratum 1973 Phys. Rev. B 7 5017
[2] Houappel R M F 1950 Order-disorder in hexagonal lattices Physica 16 425–455
[3] Nagai O, Miyashita S and Horiguchi T 1993 Ground state of the antiferromagnetic Ising model of general spin $S$ on a triangular lattice Phys. Rev. B 47 202–205
[4] Lipowski A, Horiguchi T and Lipowska D 1995 Critical behavior of spin $S$ antiferromagnetic Ising model on triangular lattice Phys. Rev. Lett. 74 3888–3891
[5] Zeng C and Henley C L 1997 Zero-temperature phase transitions of an antiferromagnetic Ising model of general spin on a triangular lattice Phys. Rev. B 55 14935–14947
[6] Villain J 1977 Spin glass with non-random interactions J. Phys. C: Solid State Phys. 10 1717–1734; Villain J 1977 Two-level systems in a spin-glass model. I. General formalism and two-dimensional model J. Phys. C: Solid State Phys. 10 4793–4803
[7] Choi M Y and Doniach S 1985 Phase transitions in uniformly frustrated XY models Phys. Rev. B 31 4516–4526
[8] Yosefin M and Domany E 1985 Phase transitions in fully frustrated spin systems Phys. Rev. B 32 1778–1795
[9] Lee J, Granato E and Kosterlitz J M 1991 Nonuniversal critical behavior and first-order transitions in a coupled XY-Ising model Phys. Rev. B 44 4819–4831
[10] Hasenbusch M, Pelissetto A and Vicari E 2005 Transitions and crossover phenomena in fully frustrated XY systems Phys. Rev. B 72 184502 (5 pages) Preprint arXiv:cond-mat/0506345
[11] Granato E, Kosterlitz J M, Lee J and Nightingale M P 1991 Phase transitions in coupled XY-Ising systems Phys. Rev. Lett. 66 1090–1093
[12] Hasenbusch M, Pelissetto A and Vicari E 2005 Multicritical behavior in the fully frustrated XY model and related systems J. Stat. Mech.: Th. Exp. P12002 (62 pages) Preprint arXiv:cond-mat/0509682
[13] Kawamura H 1998 Universality of phase transitions of frustrated antiferromagnets J. Phys.: Condens. Matter 10 4707–4754 Preprint arXiv:cond-mat/9805134
[14] Pelissetto A and Vicari E 2002 Critical phenomena and renormalization-group theory Phys. Rep. 368 549–727 Preprint arXiv:cond-mat/0012164
[15] Delamotte B, Mouhanna D and Tissier M 2004 Nonperturbative renormalization-group approach to frustrated magnets Phys. Rev. B 69 134413 (53 pages) Preprint arXiv:cond-mat/0309101
[16] Calabrese P, Parruccini P, Pelissetto A and Vicari E 2004 Critical behavior of O(2)⊗O(N)-symmetric models Phys. Rev. B 70 174439 (23 pages) Preprint arXiv:cond-mat/0405667
[17] Olsson P 1995 Two phase transitions in the fully frustrated XY model Phys. Rev. Lett. 75 2758–2761
[18] Große Pawig S and Pinn K 1998 Monte Carlo algorithms for the fully frustrated XY model Int. J. Mod. Phys. C 9 727 Preprint arXiv:cond-mat/9807137
[19] Amit D J and Martín-Mayor V 2005 Field Theory, the Renormalization Group, and Critical Phenomena, third edition (Singapore: World Scientific)
[20] Lawrie D and Sarbach S 1984 Theory of tricritical points Phase Transitions and Critical Phenomena vol 9 eds C Domb and J L Lebowitz (London: Academic Press)
[21] Nelson D R, Kosterlitz J M and Fisher M E 1974 Renormalization-group analysis of bicritical and tetracritical points Phys. Rev. Lett. 33 813–817
[22] Berge B, Diep H T, Ghazali A and Lallemand P 1986 Phase transitions in two-dimensional uniformly frustrated XY spin systems Phys. Rev. B 34 3177–3184
[23] Eikmans H, van Himbergen J E, Knops H J F and Thijssen J M 1989 Critical behavior of an array of Josephson junctions with variable couplings Phys. Rev. B 39 11759–11768
[24] Granato E, Kosterlitz J M and Simkin M V 1998 Edge effects in a frustrated Josephson-junction array with modulated couplings Phys. Rev. B 57 3602–3608 Preprint arXiv:cond-mat/9710242
[25] Calabrese P and Parruccini P 2001 Critical behavior of two-dimensional frustrated spin models with noncollinear order Phys. Rev. B 64 184408 (11 pages) Preprint arXiv:cond-mat/0105551
[26] Pelissetto A and Vicari E 2005 Interacting N-vector order parameters with O(N) symmetry Cond. Matter Phys. (Ukraine) 8 87–101 Preprint hep-th/0409214
[27] Hasenbusch M 1999 A Monte Carlo study of leading order scaling corrections of $\phi^4$ theory on a three-dimensional lattice J. Phys. A: Math. Gen. 32 4851–4866 Preprint hep-lat/9902026
[28] Campostrini M, Hasenbusch M, Pelissetto A, Rossi P and Vicari E 2001 Critical behavior of the XY universality class Phys. Rev. B 63 214503 (28 pages) Preprint arXiv:cond-mat/0103060
[29] Salas J and Sokal A D 2000 Universal amplitude ratios in the critical two-dimensional Ising model on a torus J. Stat. Phys. 98 551–588 Preprint arXiv:cond-mat/9904038
[30] Caselle M, Hasenbusch M, Pelissetto A and Vicari E 2002 Irrelevant operators in the two-dimensional Ising model J. Phys. A: Math. Gen. 35 4861–4888 Preprint arXiv:cond-mat/0106372
[31] Hasenbusch M 2005 The two dimensional XY model at the transition temperature: a high precision Monte Carlo study J. Phys. A: Math. Gen. 38 5869–5884 Preprint arXiv:cond-mat/0502556