Cohomological aspects of gauge theories: superfield formalism

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Abstract: Some of the key cohomological features of the two (1+1)-dimensional (2D) free Abelian- and self-interacting non-Abelian gauge theories (having no interaction with matter fields) are briefly discussed first in the language of symmetry properties of the Lagrangian densities and the same issues are subsequently addressed in the framework of superfield formulation on the four (2+2)-dimensional supermanifold. Special emphasis is laid on the on-shell- and off-shell nilpotent (co-)BRST symmetries that emerge after the application of (dual) horizontality conditions on the supermanifold. The (anti-)chiral superfields play a very decisive role in the derivation of the on-shell nilpotent symmetries. The study of the present superfield formulation leads to the derivation of some new symmetries for the Lagrangian density and the symmetric energy-momentum tensor. The topological nature of the above theories is captured in the framework of superfield formulation and the geometrical interpretations are provided for some of the topologically interesting quantities.

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1 Introduction

The principle of local gauge invariance has played a notable role in the modern developments of theoretical high energy physics up to the energy scale of the order of grand unification. The theories, based on such a principle, are known as gauge theories and they are endowed with first-class constraints in the language of Dirac’s classification scheme [1,2]. The Lagrangian density of such a class of theories is always singular and respects the “classical” local gauge symmetry transformations that are generated by these first-class constraints. In addition to their aesthetic theoretical appeal, these theories (in particular, one-form gauge theories) have also shed light on the results of some of the landmark experiments in the recent past. One of the most elegant methods for the covariant canonical quantization of such type of gauge theories is the Becchi-Rouet-Stora-Tyutin (BRST) formalism which, in its own theoretical setting, maintains unitarity and “quantum” gauge invariance together at any arbitrary order of perturbation theory. In this formalism, the “classical” local symmetries of the gauge theories are traded with the supersymmetric type nilpotent “quantum” gauge symmetries that are usually called the BRST symmetries. In fact, the latter symmetries are generated by a conserved ($\dot{Q}_b = 0$) and nilpotent ($Q_b^2 = 0$) BRST charge $Q_b$. The physical states of the theory belong to a subspace of the total Hilbert space of states where the physicality condition $Q_b|\text{phys} >= 0$ is satisfied. This condition implies the annihilation of the physical states of the “quantum” gauge theory by the operator form of the first-class constraints of the original “classical” gauge theory. Thus, BRST closed $Q_b|\text{phys} >= 0$ states turn out to be consistent with the Dirac’s prescription for the quantization of systems with constraints [1,2]. This formalism, in the present scenario of the frontier areas of research in theoretical physics, is indispensable in the context of modern developments in topological field theories (TFTs) [3-5], topological superstring theories [6,7] and, in general, in the domain of (super)string theories, M-theory and D-branes, etc., (see, e.g., Ref. [8] and references therein). The range and scope of BRST formalism have been beautifully extended to include the second-class constraints in its domain of applicability [9,10]. Its mathematically elegant inclusion in the well-known Batalin-Vilkovisky formalism [10,11], its deep connection with the mathematics of differential geometry and cohomology [12-14], its clear and transparent geometrical interpretation in the framework of superfield formalism [15-19], etc., have elevated this subject of investigation to a fairly high degree of physical as well as mathematical sophistication.

The nilpotency of the BRST charge and physicality condition are the two key properties that are deeply connected with the cohomological properties of the closed (i.e. $df = 0$) differential forms $f$ w.r.t. the nilpotent $d^2 = 0$ exterior derivative $d$ (with $d = dx^{\mu} \partial_{\mu}$). In fact, two closed ($df = df' = 0$) forms $f$ and $f'$ are said to belong to the same cohomology class w.r.t. $d$ if they differ by an exact form (i.e. $f' = d + dg$). Similarly, two physical states belong to the same cohomology class w.r.t. the BRST charge $Q_b$ if they differ by a BRST exact state. Thus, it is evident that the operator $d$ of differential geometry finds its
analogue in the BRST charge $Q_b$ that generates a local, covariant, continuous and nilpotent symmetry for the Lagrangian density of a given gauge theory. In addition to $d$, there are two more operators $\delta = \pm \ast d\ast$ and $\Delta = (d + \delta)^2$ that form a set $(d, \delta, \Delta)$ of the de Rham cohomology operators of differential geometry. In terms of these operators, the celebrated Hodge decomposition theorem (HDT) is defined which states that, on a compact manifold without a boundary, any arbitrary form $f_n$ of degree $n$ ($n = 0, 1, 2, 3, \ldots$) can be uniquely written as the sum of a harmonic form $h_n$ ($\Delta h_n = dh_n = \delta h_n = 0$), an exact form ($de_{n-1}$) and a co-exact form ($\delta c_{n+1}$) as given below \[20-23\]

$$f_n = h_n + de_{n-1} + \delta c_{n+1}. \quad (1.1)$$

It has been a long-standing problem to find (i) the analogues of $\delta$, $\Delta$ in the language of symmetry properties (and corresponding generators) of the Lagrangian density of a given gauge theory in any arbitrary spacetime dimension. (ii) The clear description of and about the cohomology of quantum states in the total Hilbert space of states w.r.t. the analogues of $d, \delta, \Delta$. (iii) The analogue of HDT (cf. (1.1)) in the language of states of the quantum theory in the total quantum Hilbert space of states. Some interesting attempts were made towards this goal for the interacting (non-)Abelian gauge theories in any arbitrary spacetime dimension but the relevant symmetry transformations turned out to be non-local and non-covariant \[24-27\]. In the covariant formulation, the nilpotency was restored only for a certain specific value of the parameter of these theories \[28\].

Recently, in a set of papers \[29-33\], a possible connection between the local, continuous and covariant symmetries and their generators on the one hand and the cohomological operators $(d, \delta, \Delta)$ on the other hand, has been established in the Lagrangian formulation for the case of the 2D free- as well as interacting (non-)Abelian gauge theories. A discrete symmetry transformation has been shown to correspond to the Hodge duality $\ast$ operation. The existence of such type of local, covariant and continuous symmetries as well as a discrete symmetry has also been shown for the physical $(3+1)$-dimensional (4D) free 2-form Abelian gauge theory \[34\]. All the above examples provide a beautiful set of tractable field theoretical models for the Hodge theory (from the point of view of mathematics as well as physics). One of the physical consequences of these studies has been to establish the fact that the free 2D Abelian- and self-interacting non-Abelian gauge theories belong to a new class of TFTs in the flat spacetime (i.e., a field theory endowed with a flat spacetime metric and having no propagating degrees of freedom associated with the basic field of the theory) (see, e.g., \[5\] for details). In fact, these new TFTs capture together some of the key topological properties of the Witten- and Schwarz type TFTs in flat spacetime. For instance, the appearance of the Lagrangian densities turns out to be like Witten type TFTs (because they are equal to the sum of a BRST- and co-BRST anti-commutators) but the local symmetries of these theories are that of the Schwarz type \[35\] (as there are

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\textsuperscript{1}The operators $\delta$ and $\Delta$ are called the co-exterior derivative and Laplacian operators respectively and $\ast$ is the Hodge duality operation on the manifold. Together, all these three operators obey an algebra: $d^2 = \delta^2 = 0, \Delta = \{d, \delta\} = d\delta + \delta d, [\Delta, d] = [\Delta, \delta] = 0$ implying that $\Delta$ is the Casimir operator \[20-23\].
no topological shift symmetries in the theory whose existence is one of the characteristic features of the Witten type TFTs). Topological invariants for these theories have been computed and their recursion relations have been obtained [29,30,35] on a flat 2D manifold (which is equivalent to a 2D closed Riemann surface in its Euclidean version).

A different but interesting aspect of the above set of problems (cited after (1.1)) is to provide the geometrical origin and interpretation for the conserved and nilpotent (co-)BRST charges (and the symmetry transformations they generate) in the language of translations along the Grassmannian directions of the $(2+2)$-dimensional compact supermanifold. Generally, in the superfield approach [15-19] to BRST formalism for the $p$-form ($p = 1, 2, 3, \ldots$) gauge theories, the curvature ($(p+1)$-form) tensor is restricted to be flat along the Grassmannian directions of the $(D+2)$-dimensional supermanifold, parametrized by $D$-number of spacetime (even) coordinates and two Grassmannian (odd) coordinates. This flatness condition, popularly known as horizontality condition †, provides the origin for the existence of (anti-)BRST symmetry transformations and leads to the geometrical interpretation of the conserved and nilpotent ($Q^2_{(a)b} = 0$) (anti-)BRST charges ($Q_{(a)b}$) as the translation generators along the Grassmannian directions. In this derivation, the super exterior derivative $\tilde{d}$ and the Maurer-Cartan equation (for the definition of the curvature tensor) are exploited together for the imposition of the horizontality condition. Recently, in a couple of papers [37,38], all the three super de Rham cohomology operators ($\tilde{d}, \tilde{\delta}, \tilde{\Delta}$), defined on the $(2+2)$-dimensional supermanifold, have been exploited to show the existence of nilpotent (anti-)BRST- and (anti-)co-BRST symmetries as well as a bosonic symmetry (which is equivalent to the anti-commutators of the nilpotent (anti-)BRST and (anti-)co-BRST symmetries) in the framework of superfield formulation for the case of a free 2D Abelian gauge theory. Such kind of geometrical superfield formulation has also been carried out for the self-interacting 2D non-Abelian gauge theory [39]. In these discussions and derivations, a generalized version of the horizontality conditions w.r.t. the super cohomological operators ($\tilde{d}, \tilde{\delta}, \tilde{\Delta}$) has been used. The topological nature of these theories has also been captured in the (chiral) superfield formulation and the geometrical interpretations for some of the physically interesting quantities have been briefly discussed [40,41]. In the relevant and pertinent literature available on the subject under discussion, there are many ways to define the topological nature of a field theory. However, it should be re-emphasized at this stage that, in all our earlier works (e.g. [29],[30],[35],[40],[41]), we have taken the point of view that a theory is topological in a given flat spacetime if there are no propagating degrees of freedom associated with the basic field(s) of the theory (see. e.g., [5] for details).

The purpose of the present paper is to capture the on-shell and off-shell nilpotent symmetries, Lagrangian density, symmetric energy momentum tensor, topological invariants, etc., for the 2D free Abelian- and self-interacting non-Abelian gauge theories in the language of superfield formulation on a four $(2+2)$-dimensional supermanifold. It is essential

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†This restriction has been referred to as the “soul-flatness” condition in Ref. [36] which amounts to setting the Grassmannian components of the super curvature ($(p+1)$-form) tensor equal to zero.
and important to lay emphasis on the fact that, to the best of our knowledge, the on-shell nilpotent symmetries for the (non-)Abelian gauge theories have not yet been discussed and derived in the framework of superfield approach to BRST formalism despite a rich literature available on this subject connected with the one-form and 2-form gauge theories. In our present paper, for the discussion of the on-shell nilpotent symmetries, we invoke the (anti-)chiral superfields in terms of which the corresponding Lagrangian density, symmetric energy momentum tensor and topological invariants are expressed. This exercise leads to the geometrical interpretation for the above physically interesting quantities in the language of translations along some specific direction of the supermanifold. For the 2D free Abelian gauge theory, the choice of the (anti-)chiral superfields enables us to demonstrate that the on-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries co-exist together for a given Lagrangian density (see, e.g., eqns. (2.1) and (2.2) below). However, for the self-interacting 2D non-Abelian gauge theory, it turns out that only the on-shell nilpotent (co-)BRST symmetries exist for a given Lagrangian density when we choose chiral superfield for the description of this theory on a (2+2)-dimensional supermanifold. The on-shell nilpotent anti-BRST and anti-co-BRST symmetries do not exist for the same Lagrangian density if we choose the anti-chiral superfield on the same supermanifold. This feature is drastically different from the free 2D Abelian gauge theory. In fact, with the best of our familiarity with the relevant literature, the on-shell nilpotent anti-BRST and anti-co-BRST symmetries do not exist for the (self-)interacting non-Abelian gauge theories in any space-time dimension. We provide, in our present paper, an explanation for this discrepancy in the language of superfield formulation on the (2+2)-dimensional supermanifold. These on-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries are exploited to provide the geometrical interpretation for some of the key topological properties of the free 2D Abelian- and self-interacting non-Abelian gauge theories. For the discussion of the off-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries, we choose the most general superfields (not the (anti-)chiral) on the above supermanifold and show that the (dual) horizontality conditions w.r.t. super cohomological operators $\tilde{d}$ and $\tilde{\delta}$, lead to the geometrical interpretation for the corresponding conserved and nilpotent charges as translation generators along the Grassmannian directions of the supermanifold.

Our present study is essential on three counts. First, to the best of our knowledge, the super co-exterior derivative $\tilde{\delta}$ has not yet been exploited extensively in the context of BRST formalism (except in our recent works [37-41]). Thus, it is an interesting endeavour to tap the full potential of this operator in as many distinct and diverse ways as possible and to understand the geometry behind it in the framework of superfield formulation. In our present paper, for instance, we show that the nilpotent (anti-)co-BRST symmetries and some of the topologically interesting quantities owe their origin to this operator in the framework of superfield approach to BRST formalism. Second, the insights gained in the context of 2D one-form gauge theories might turn out to be quite useful for similar discussions in the context of 4D 2-form gauge theories where the existence of (anti-)co-BRST
and (anti-)BRST symmetries has already been shown [34]. In fact, in a recent work [42], we have been able to show the quasi-topological (see, e.g., Ref. [5] for details) nature of the free 2-form Abelian gauge theory in the Lagrangian formulation where the cohomological properties and quasi-topological nature are found to be very elegantly intertwined. Finally, the geometrical understanding of some of the topological features of 2D theories might play an important role in the context of discussion about the (super)string theories and topological 2D gravity where normally a non-trivial (spacetime dependent) metric is taken into account for the study of gauge theories in curved spacetime background.

The contents of our present paper are organized as follows. In section 2, we set up the notations (as well as conventions) and briefly recapitulate the essentials of the on-shell and off-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries in the Lagrangian formulation for the 2D free Abelian gauge theory. These symmetries are subsequently derived in the framework of superfield formulation by exploiting the (dual) horizontality conditions w.r.t. the super cohomological operators $\tilde{\delta}$ and $\tilde{d}$. The choice of (anti-)chiral superfields is shown to help in the derivation of on-shell nilpotent symmetries. Section 3 is devoted to the derivation of on-shell nilpotent (co-)BRST symmetries and off-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries for the self-interacting 2D non-Abelian gauge theory in the framework of Lagrangian- and superfield formulations. Section 4 deals with the discussion of topological aspects of the above theories in the superfield formulation. Finally, our paper ends with a few conclusions along with some future perspectives in section 5.

2 Free 2D Abelian gauge theory

We discuss here the on-shell- and off-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries for a 2D Abelian gauge theory in the Lagrangian and superfield formulations.

2.1 (Anti-)BRST- and (anti-)co-BRST symmetries: Lagrangian formulation

Let us start off with the BRST invariant Lagrangian density $L_b$ [43-46] for a non-interacting two $(1 + 1)$-dimensional Abelian gauge theory in any arbitrary gauge with a parameter $\xi$

$$L_b = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 - i \partial_\mu \bar{C} \partial^\mu C \equiv \frac{1}{2} E^2 - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 - i \partial_\mu \bar{C} \partial^\mu C,$$

(2.1)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (with $F_{01} = E$) is the field strength tensor constructed from the vector potential $A_\mu$, $\xi \neq 0$ and the (anti-)ghost fields $(\bar{C})C$ (with $\bar{C}^2 = C^2 = 0$, $C\bar{C} + \bar{C}C = 0$) are required in the theory to maintain the unitarity and “quantum” gauge invariance together at any arbitrary order of perturbation theory §. The above Lagrangian density

§We follow here the conventions and notations such that the 2D flat Minkowski metric is: $\eta_{\mu\nu} = \text{diag} (+1, -1)$ and $\Box = \eta^{\mu\nu} \partial_\mu \partial_\nu = (\partial_0)^2 - (\partial_1)^2; \varepsilon_{\mu\nu} = -\varepsilon^{\mu\nu}; \varepsilon_{01} = +1, F_{01} = F^{10} = E = -\varepsilon^{\mu\nu} \partial_\mu A_\nu$. Here the Greek indices: $\mu, \nu, \rho... = 0, 1$ correspond to the spacetime directions on the 2D ordinary manifold.
respects the following on-shell \((\Box C = \Box \bar{C} = 0)\) nilpotent \((s^2_{(a)b} = s^2_{(a)d} = 0)\) transformations

\[
\begin{align*}
s_b A_\mu &= \partial_\mu C, \quad s_b C = 0, \quad s_b = -\frac{i}{\xi}(\partial_\mu A^\mu), \quad s_b A_\mu = \partial_\mu C, \\
s_{ab} C &= 0, \quad s_{ab} C = +\frac{i}{\xi}(\partial_\mu A^\mu), \quad s_{ab} A_\mu = -\varepsilon_\mu \partial^\nu \bar{C}, \quad s_{ab} = 0, \\
s_d C &= -i E, \quad s_{ad} A_\mu = -\varepsilon_\mu \partial^\nu C, \quad s_{ad} C = 0, \quad s_{ad} = +i E,
\end{align*}
\]

where \(s_{(a)b}\) and \(s_{(a)d}\) stand for the (anti-)BRST and (anti-)co-BRST symmetry operations on the basic fields (with \(s_b s_{ab} + s_{ab} s_b = 0\) and \(s_d s_{ad} + s_{ad} s_d = 0\) \(\ddagger\)). One can linearize the kinetic energy and the gauge-fixing terms of the Lagrangian density (2.1) by invoking a couple of auxiliary fields \(B\) and \(\bar{B}\). The ensuing Lagrangian density [29,30,35]

\[
\mathcal{L}_B = B E - \frac{1}{2} B^2 + B(\partial_\mu A^\mu) + \frac{1}{2} \chi B^2 - i \partial_\mu \bar{C} \partial^\mu C,
\]

respects the off-shell nilpotent version of symmetries (2.2) as given below [35]

\[
\begin{align*}
\tilde{s}_b A_\mu &= \partial_\mu C, \quad \tilde{s}_b C = 0, \quad \tilde{s}_b \bar{C} = i B, \quad \tilde{s}_b B = 0, \quad \tilde{s}_b \bar{B} = 0, \\
\tilde{s}_{ab} A_\mu &= \partial_\mu \bar{C}, \quad \tilde{s}_{ab} \bar{C} = 0, \quad \tilde{s}_{ab} C = -i B, \quad \tilde{s}_{ab} \bar{B} = 0, \quad \tilde{s}_{ab} B = 0, \\
\tilde{s}_d A_\mu &= -\varepsilon_\mu \partial^\nu \bar{C}, \quad \tilde{s}_d \bar{C} = 0, \quad \tilde{s}_d C = -i B, \quad \tilde{s}_d \bar{B} = 0, \quad \tilde{s}_d B = 0, \\
\tilde{s}_{ad} A_\mu &= -\varepsilon_\mu \partial^\nu C, \quad \tilde{s}_{ad} C = 0, \quad \tilde{s}_{ad} \bar{C} = +i B, \quad \tilde{s}_{ad} \bar{B} = 0, \quad \tilde{s}_{ad} B = 0.
\end{align*}
\]

The Lagrangian densities (2.1) and (2.3) respect two more continuous and covariant symmetries. The anti-commutator of the two nilpotent symmetries \(s_w = \{s_b, s_d\} = \{s_{ab}, s_{ad}\}\) leads to the existence of a bosonic (i.e. \(s^2_{w} \neq 0\)) symmetry transformation \(s_w\). The other symmetry is the ghost symmetry transformation \(s_g\) under which: \(C \to e^{-\lambda} C, \bar{C} \to e^{+\lambda} \bar{C}, A_\mu \to A_\mu, B \to B, \bar{B} \to B\) where \(\lambda\) is an infinitesimal global parameter. According to the Noether’s theorem, all the above continuous symmetries lead to the derivation of conserved charges which turn out to be the generators for the above transformations. For the generic field \(\Psi\), this statement can be expressed succinctly, in the mathematical form, as

\[
s_r \Psi = -i \left[\Psi, Q_r\right]_\pm, \quad r = b, ab, d, ad, w, g,
\]

where \(Q_r\) are the conserved charges corresponding to the above symmetries and brackets \([ , ]_\pm\) stand for the (anti-)commutators for \(\Psi\) being (fermionic)bosonic in nature. The local expressions for the charges \(Q_r\), which are not required for our present discussion, are given in Refs. [29,30,35] for the case of Feynman gauge where \(\xi = 1\) in (2.1) and (2.3).

Together, the above six local, continuous and covariant symmetry transformations for the Lagrangian densities (2.1) (and (2.3)) obey the following operator algebra

\[
\begin{align*}
s^2_b &= s^2_{ab} = s^2_{ad} = 0, \quad s_w = \{s_b, s_d\} = \{s_{ab}, s_{ad}\}, \\
s_d s_{ad} + s_{ad} s_d = 0, \quad s_b s_{ab} + s_{ab} s_b = 0, \quad \left[s_w, s_r\right] = 0, \quad s_r = s_b, s_{ab}, s_d, s_{ad}, s_g, \\
i[s_g, s_b] &= +s_b, \quad i[s_g, s_d] = -s_d, \quad i[s_g, s_{ab}] = -s_{ab}, \quad i[s_g, s_{ad}] = +s_{ad},
\end{align*}
\]

\(\ddagger\)Here the conventions and notations of Ref. [46] have been adopted. The (co)-BRST transformations \(\delta_{(D)B}\) (with \(\delta^2_{(D)B} = 0\)), in their totality, are equivalent to the product of an anti-commuting \((\eta C = -C \eta, \eta \bar{C} = -\bar{C} \eta)\) parameter \(\eta\) and the transformations \(s_{(a)b}\) (i.e. \(\delta_{(D)B} = \eta s_{(a)b}\)) where \(s^2_{(a)b} = 0\).

\(\ddagger\)We use here, and in what follows until (2.7), only the notations for the on-shell nilpotent symmetries but our statements are valid for the off-shell nilpotent symmetries as well.
which is reminiscent of the algebra obeyed by the de Rham cohomological operators. Thus, we see that there is a mapping between the cohomological operators on one hand and symmetry transformations on the other hand. This mapping is \( d \leftrightarrow (s_b, s_d), \delta \leftrightarrow (s_d, s_{ab}), \Delta \leftrightarrow \{ s_b, s_d \} \). In physical language, it can be noticed that it is the kinetic energy term, gauge-fixing term and ghost term of the Lagrangian densities (2.1) and (2.3) that remain invariant under the (anti-)BRST \( s_{(a)b} \), (anti-)co-BRST \( s_{(a)d} \) and bosonic symmetry \( s_w \) transformations, respectively. It is straightforward to check that the Lagrangian densities in (2.1) and (2.3) can be expressed in terms of transformations in (2.2), (2.4) and (2.5), modulo some total derivatives, as

\[
\mathcal{L}_b = \{ Q_d, \frac{i}{2} EC \} - \{ Q_b, \frac{1}{2} (\partial_\rho A^\rho) \bar{C} \} \equiv \{ Q_{ab}, \frac{1}{2} (\partial_\rho A^\rho) C \} - \{ Q_{ad}, \frac{i}{2} E \bar{C} \}, \quad (2.7)
\]

\[
\mathcal{L}_B = BE - iB^2 + X, \quad \mathcal{L}_b = B(\partial_\rho A^\rho) + \frac{\xi}{2} B^2 + Y,
\]

\[
X = \bar{s}_b \bar{s}_a b \left( \frac{1}{2} A^2 - \frac{1}{2} \bar{C} \bar{C} \right) \equiv \bar{s}_b (-i \bar{C} \left[ \partial_\rho A^\rho + \frac{i}{2} B \right]) \equiv \bar{s}_b (+i \bar{C} \left[ \partial_\rho A^\rho + \frac{i}{2} B \right]), \quad (2.8)
\]

\[
Y = \bar{s}_d \bar{s}_a d \left( \frac{1}{2} A^2 - \frac{1}{2} \bar{C} \bar{C} \right) \equiv \bar{s}_d (-i \bar{C} \left[ E - \frac{1}{2} B \right]) \equiv \bar{s}_d (+i \bar{C} \left[ E - \frac{1}{2} B \right]).
\]

The appearance of the Lagrangian density in (2.7) is like Witten type TFTs if we assume that the vacuum as well as physical states of the theory are invariant under the (anti-)BRST and (anti-)co-BRST symmetries (i.e. \( Q_{(a)b} | \text{vac} >= 0, Q_{(a)b} | \text{phys} >= 0, Q_{(a)d} | \text{vac} >= 0, Q_{(a)d} | \text{phys} >= 0 \)). Such a requirement is satisfied when we choose the physical state (as well as the vacuum) to be the harmonic state of the Hodge decomposed version of any arbitrary state in the quantum Hilbert space \([30,35]\). In contrast to the Witten type appearance of the Lagrangian density (2.1), the local symmetries of the theory are that of Schwarz type. Thus, Lagrangian density (2.1) (or (2.3)) describes a new type of TFT [35]. The topological nature of this theory is confirmed by the following expression for the symmetric energy momentum tensor \( T_{\alpha\beta}^{(s)} \) for the Lagrangian density (2.1)

\[
T_{\alpha\beta}^{(s)} = -\frac{i}{2 \pi} (\partial_\rho A^\rho) \left[ \partial_\alpha A_\beta + \partial_\beta A_\alpha \right] - \frac{1}{2} E \left[ \varepsilon_{\alpha\rho} \partial_\beta A^\rho + \varepsilon_{\beta\rho} \partial_\alpha A^\rho \right] - i \partial_\alpha \bar{C} \partial_\beta C - i \partial_\beta \bar{C} \partial_\alpha C - \eta_{\alpha\beta} \mathcal{L}_b \quad (2.9)
\]

which turns out to be the sum of (co-)BRST anti-commutators as given below

\[
T_{\alpha\beta}^{(s)} = \{ Q_b, V_{\alpha\beta}^{(1)} \} + \{ Q_d, V_{\alpha\beta}^{(2)} \} \equiv \{ Q_{ab}, \bar{V}_{\alpha\beta}^{(1)} \} + \{ Q_{ad}, \bar{V}_{\alpha\beta}^{(2)} \}
\]

\[
V_{\alpha\beta}^{(1)} = \frac{1}{2} \left[ (\partial_\alpha \bar{C}) A_\beta + (\partial_\beta \bar{C}) A_\alpha + \eta_{\alpha\beta} (\partial_\mu A^\mu) \bar{C} \right],
\]

\[
V_{\alpha\beta}^{(2)} = \frac{1}{2} \left[ (\partial_\alpha \bar{C}) \varepsilon_{\beta\rho} A^\rho + (\partial_\beta \bar{C}) \varepsilon_{\alpha\rho} A^\rho - \eta_{\alpha\beta} \varepsilon \bar{C} \right], \quad (2.10)
\]

where \( \bar{V}' \)'s can be derived from the \( V' \)'s by the substitution \( C \leftrightarrow \bar{C}, E \leftrightarrow -\bar{E}, (\partial_\mu A^\mu) \leftrightarrow -(\partial_\mu A^\mu) \) under which \( s_b \leftrightarrow s_{ab}, s_d \leftrightarrow s_{ad} \) in (2.2). The form of \( T_{\alpha\beta}^{(s)} \) in the above equation establishes the fact that there are no energy excitations \((< \text{phys} | T_{00}^{(s)} | \text{phys}' >= 0)\) in the physical sector of the theory because the hermitian (co-)BRST charges annihilate all the physical states. On the 2D compact manifold **, there are three \((k = 0, 1, 2)\) topological invariants (i.e., zero-form, one-form and two-form) which can be generically expressed as

\[
I_k = \oint \bar{C}_k V_k, \quad J_k = \oint \bar{C}_k W_k, \quad (k = 0, 1, 2), \quad (2.11)
\]

**To be very precise, 2D Minkowski manifold is not a compact manifold. For the discussion of the topo-
where \( C_k \) are the \( k \)-dimensional homology cycles in the 2D manifold and \( I_k \) and \( J_k \) are the invariants w.r.t. BRST and co-BRST charges respectively. Similar expressions can be obtained (with \( C \leftrightarrow \bar{C}, E \leftrightarrow -E, (\partial_{\mu} A^\mu) \leftrightarrow -(\partial_{\mu} A^\mu) \)) as far as the nilpotent anti-BRST and anti-co-BRST charges are concerned. These invariants are connected with one-another by a certain specific kind of recursion relations [35]. Thus, if we know the zero-forms (that are explicitly BRST- and co-BRST invariants), we can compute the rest of the forms by exploiting the recursion relations. For the Lagrangian densities (2.1) and (2.3), these physical (anti-)BRST- and (anti-)co-BRST invariant quantities (zero-forms) are

\[
\begin{align*}
V_0 &= -\frac{i}{\xi}(\partial_{\mu} A^\mu)C, \\
\bar{V}_0 &= -\frac{i}{\xi}(\partial_{\mu} A^\mu)\bar{C}.
\end{align*}
\]

where \((\bar{V}_0)V_0\) and \((\bar{W}_0)W_0\) are on-shell \((\Box C = \Box \bar{C} = 0)\) invariant quantities for the Lagrangian density (2.1) and \((\bar{V}_0)V_0\) and \((\bar{W}_0)W_0\) are off-shell invariant quantities for (2.3).

2.2 On-shell nilpotent (co-)BRST symmetries: chiral superfield formalism

To provide the geometrical origin and interpretation for the on-shell nilpotent (co-)BRST symmetries and corresponding generators, we resort to the superfield formulation on the four \((2 + 2)\)-dimensional supermanifold. To this end in mind, first of all we generalize the generic local field \( \Psi(x) = (A_\mu(x), C(x), \bar{C}(x)) \) of the Lagrangian density (2.1) to a chiral \((\partial_\theta \bar{A}_M(x, \theta, \bar{\theta}) = 0)\) supervector superfield \( A_M(x, \theta, \bar{\theta}) = (B_\mu(x, \theta), \bar{\Phi}(x, \bar{\theta}), \bar{\Phi}(x, \bar{\theta})) \) with the following super expansions along the Grassmannian \( \bar{\theta} \)-direction of the \((2 + 2)\)-dimensional supermanifold

\[
\begin{align*}
B_\mu(x, \bar{\theta}) &= A_\mu(x) + \bar{\theta} R_\mu(x), \\
\bar{\Phi}(x, \bar{\theta}) &= C(x) - i \bar{\theta} B(x), \\
\bar{\Phi}(x, \bar{\theta}) &= \bar{C}(x) + i \bar{\theta} B(x).
\end{align*}
\]

There are a few salient points which we summon here: (i) it is obvious that in the limit \( \bar{\theta} \to 0 \), we get back the generic field \( \Psi(x) \) of the Lagrangian density (2.1). (ii) In general, a superfield on the four \((2 + 2)\)-dimensional supermanifold is parametrized by the superspace variables \( Z^M = (x^\mu, \theta, \bar{\theta}) \) where \( x^\mu(\mu = 0, 1) \) are the two even spacetime variables and \( \theta, \bar{\theta} \) are the odd variables (with \( \theta^2 = \bar{\theta}^2 = 0, \theta \bar{\theta} = -\bar{\theta} \theta \)). However, for our present discussions, we have chosen \( Z^M = (x^\mu, \bar{\theta}) \). (iii) The minus sign in the super expansion of \( \Phi(x, \bar{\theta}) \) has been taken just for the algebraic convenience. (iv) The total number of degrees of freedom for the odd fields \( (R_\mu, C, \bar{C}) \) and even fields \( (A_\mu, B, \bar{B}) \) match in the above expansion for the sake of consistency with the basic tenets of supersymmetry. (v) The auxiliary fields \( R_\mu, B, \bar{B} \) will be fixed in terms of the basic fields after the application of the (dual) horizontality

logical invariants, homology cycles, etc. and their connections with the curves in the algebraic geometry, one should consider the Euclidean version of the above manifold which turns out to be a 2D closed Riemann surface. Here the metric, unlike the case of Minkowskian manifold, carries the same diagonal signatures and \( \mu, \nu, ..., = 1, 2 \). For the sake of brevity, however, we shall continue with our earlier Minkowskian notations but shall keep in mind this important fact and very delicate issue (see, e.g., [47,48]).
conditions. Some of them can also be fixed by resorting to the equations of motion for the Lagrangian density (2.3). (vi) All the component fields, on the r.h.s. of the above expansion, are functions of the spacetime even variable $x^\mu$ alone. (vii) The super expansions in (2.13) are the chiral limit ($\theta \to 0$) of the most general expansion of the superfields on the $(2 + 2)$-dimensional compact supermanifold (see, e.g., [38])

$$
B_\mu(x, \theta, \bar{\theta}) = A_\mu(x) + \theta \bar{R}_\mu(x) + \bar{\theta} R_\mu(x) + i \theta \bar{\theta} S_\mu(x),
$$

$$
\Phi(x, \theta, \bar{\theta}) = C(x) - i \theta B(x) - i \bar{\theta} \bar{B}(x) + i \theta \bar{\theta} s(x),
$$

$$
\bar{\Phi}(x, \theta, \bar{\theta}) = \bar{C}(x) + i \theta \bar{B}(x) + i \bar{\theta} B(x) + i \theta \bar{\theta} \bar{s}(x),
$$

(2.14)

where signs in the above expansion have been chosen for the algebraic convenience and auxiliary fields $B$ and $\bar{B}$ are the ones present in the Lagrangian density (2.3). Here too, it can be seen that the fermionic ($R_\mu, \bar{R}_\mu, C, \bar{C}, s, \bar{s}$) degrees of freedom do match with that of the bosonic ($A_\mu, S_\mu, +B, -B, +\bar{B}, -\bar{B}$) degrees.

Now we exploit the horizontality condition ($\bar{F} = \bar{d}\bar{A} = dA = F$) w.r.t. (super) exterior derivatives ($\bar{d})d$ in the construction of the (super) curvature two-form. Physically, the requirement of the horizontality condition implies an imposition that the curvature two-form in the ordinary spacetime manifold remains unchanged and it is restricted not to get any contribution from the superspace variables. The explicit expressions for the l.h.s. and r.h.s. in ($\bar{d}\bar{A} = dA$) are

$$
\bar{d}\bar{A} = (dx^\mu \wedge dx^\nu)(\partial_\mu B_\nu) + (dx^\mu \wedge d\bar{\theta})(\partial_\mu \Phi - \partial_\bar{\theta} B_\mu) - (d\bar{\theta} \wedge d\bar{\theta})(\partial_\bar{\theta} \Phi),
$$

$$
dA = (dx^\mu \wedge dx^\nu)(\partial_\mu A_\nu) \equiv \frac{1}{2}(dx^\mu \wedge dx^\nu)(\partial_\mu A_\nu - \partial_\nu A_\mu),
$$

(2.15)

where the super exterior derivative $\bar{d}$ (defined in terms of the chiral superspace coordinates) and super connection one-form $\bar{A}$ (defined in terms of the chiral superfields) are

$$
\bar{d} = dZ^M \partial_M \equiv dx^\mu \partial_\mu + d\bar{\theta} \partial_{\bar{\theta}},
$$

$$
\bar{A} = dZ^M \bar{A}_M \equiv dx^\mu B_\mu(x, \bar{\theta}) + d\bar{\theta} \Phi(x, \bar{\theta}).
$$

(2.16)

The above equations are the chiral limits ($\theta \to 0, d\theta \to 0$) of the following most general definitions for the super exterior derivative and super one-form connection (see, e.g., [38])

$$
\bar{d} = dZ^M \partial_M \equiv dx^\mu \partial_\mu + d\bar{\theta} \partial_{\bar{\theta}} + d\theta \partial_\theta,
$$

$$
\bar{A} = dZ^M \bar{A}_M \equiv dx^\mu B_\mu(x, \theta, \bar{\theta}) + d\theta \Phi(x, \theta, \bar{\theta}) + d\bar{\theta} \bar{\Phi}(x, \theta, \bar{\theta}),
$$

(2.17)

on the $(2 + 2)$-dimensional compact supermanifold. It is straightforward to check that the horizontality restriction $\bar{d}\bar{A} = dA$ leads to the following relationships

$$
\partial_\theta \Phi(x, \bar{\theta}) = 0 \rightarrow \mathcal{B}(x) = 0, \quad \partial_\bar{\theta} B_\mu(x, \bar{\theta}) = \partial_\mu \Phi(x, \bar{\theta}) \rightarrow R_\mu(x) = \partial_\mu C(x),
$$

(2.18)

and the restriction $\partial_\mu B_\nu - \partial_\nu B_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$ implies $\partial_\mu R_\nu - \partial_\nu R_\mu = 0$ which is readily satisfied by $R_\mu = \partial_\mu C$. It is obvious that the condition $\bar{d}\bar{A} = dA$ does not fix the auxiliary field $B(x)$ in terms of the basic fields of the Lagrangian density (2.1). However, the equation
of motion for the Lagrangian density (2.3) comes to our rescue as: \( B(x) = -\frac{1}{4}(\partial_{\mu}A^\mu)(x) \). With these substitutions for the auxiliary fields, the super expansion (2.13) becomes:

\[
\begin{align*}
B_\mu(x, \bar{\theta}) &= A_\mu(x) + \bar{\theta} \partial_\mu C(x) \equiv A_\mu(x) + \bar{\theta} (s_b A_\mu(x)), \\
\Phi(x, \bar{\theta}) &= C(x) - i \bar{\theta} (B(x) = 0) \equiv C(x) + \bar{\theta} (s_b C(x) = 0), \\
\bar{\Phi}(x, \bar{\theta}) &= \bar{C}(x) - i \bar{\theta} (\partial_{\mu}A^\mu)(x) \equiv \bar{C}(x) + \bar{\theta} (s_b \bar{C}(x)).
\end{align*}
\]

\( (2.19) \)

In fact, now the on-shell nilpotent BRST symmetry transformations in (2.2) can be concisely written in terms of the above superfields expansions as

\[
\begin{align*}
s_b B_\mu(x, \bar{\theta}) &= \partial_\mu \Phi(x, \bar{\theta}), \quad s_b \Phi(x, \bar{\theta}) = 0, \quad s_b \bar{\Phi}(x, \bar{\theta}) = -\frac{i}{2}(\partial_{\mu}B^\mu)(x, \bar{\theta}).
\end{align*}
\]

\( (2.20) \)

One can readily check that the first transformation in the above equation leads to \( s_b A_\mu = \partial_\mu C, s_b C = 0 \); the second transformation produces \( s_b \bar{C} = -\frac{i}{2}(\partial_{\mu}A^\mu) \), \( s_b (\partial_{\mu}A^\mu) = \Box C \) in terms of the basic fields of Lagrangian density (2.1).

It is interesting to check, vis-a-vis equation (2.5), that \( A \) is dual (2,2)-dimensional supermanifold. The process of this translation generates the on-shell nilpotent BRST symmetry transformations on \( \Psi \) which correspond to (2.2). In addition, the nilpotency of \( s_b^2 = 0 \) (and \( Q_b^2 = 0 \)) is intimately connected with the property of the square of the translational generator (i.e. \( (\partial/\partial \bar{\theta})^2 = 0 \)).

Now we illustrate the derivation of the on-shell nilpotent dual(co-)BRST symmetry transformations of (2.2) by exploiting the analogue of the horizontality condition \(^{11}\) w.r.t. (super) co-exterior derivatives (\( \delta \)) by requiring \( \delta \bar{A} = \delta A \) with the following expressions

\[
\begin{align*}
\delta \bar{A} &= \partial_\mu B^\mu + \epsilon^{\dot{\mu}\dot{\nu}}(\partial_{\dot{\mu}}\bar{\Phi}) - \epsilon_{\mu\nu}(\partial_\mu \Phi + \epsilon_{\mu\nu} \partial_\nu B^\nu), \\
\delta A &= \partial_\mu A^\mu, \quad A(x) = dx^\mu A_\mu(x), \quad \bar{A} = dZ^M \bar{A}_M,
\end{align*}
\]

\( (2.22) \)

where \( \delta = -*d\star, \quad \bar{\delta} = -*d\bar{\star} \) and \( \star \) corresponds to the Hodge duality operation on the compact \((2+2)\)-dimensional supermanifold. Physically, the dual horizontality condition implies a restriction on the zero-form gauge-fixing term \( (\partial_{\mu}A^\mu) \), defined on the ordinary spacetime manifold, to remain unchanged. In other words, the superspace contribution

---

\(^{11}\)We christen this condition as the dual horizontality condition because \( \delta \bar{d}(d) \) and \( \delta \bar{\delta} \) are dual \( \delta \) to each-other. The restriction \( \delta \bar{A} = \delta A \) amounts to setting equal to zero all the Grassmannian parts of the superscalar \( \delta \bar{A} \). On an ordinary even dimensional manifold, the operation \( \delta A = -*d\star A \) always yields the (zero-form) covariant gauge-fixing term (i.e. \( \delta A = \partial_{\mu}A^\mu \)) for the one-form \( (A(x) = dx^\mu A_\mu(x)) \) Abelian \( U(1) \) gauge theory in any arbitrary spacetime dimension.
to the gauge-fixing term is required to be zero. The operation of $\ast$ on the superspace differentials $dZ^M$ and their wedge products $(dZ^M \wedge dZ^N)$, in the most general form, are

\[
\begin{align*}
\ast (dx^\mu) &= \varepsilon^{\mu
u}(dx^\nu), & \ast (d\theta) &= (d\bar{\theta}), & \ast (d\bar{\theta}) &= (d\theta), \\
\ast (dx^\mu \wedge dx^\nu) &= \varepsilon^{\mu
u}, & \ast (dx^\mu \wedge d\theta) &= \varepsilon^{\mu\bar{\theta}}, & \ast (dx^\mu \wedge d\bar{\theta}) &= \varepsilon^{\mu\bar{\theta}}, \\
\ast (d\theta \wedge d\bar{\theta}) &= s^\theta, & \ast (d\theta \wedge d\theta) &= s^\bar{\theta}, & \ast (d\bar{\theta} \wedge d\bar{\theta}) &= s^\bar{\theta},
\end{align*}
\]

(2.23)

where $\varepsilon^{\mu\bar{\theta}} = -\varepsilon^{\bar{\theta}\mu}, \varepsilon^{\mu\theta} = -\varepsilon^{\bar{\theta}\mu}, s^\theta = s^\bar{\theta}$ etc. It will be noted that, in the derivation of $\delta \tilde{A} = -\ast \partial \ast \tilde{A}$, we have used the following expansion

\[
\ast \tilde{A} = \varepsilon^{\mu
u}(dx^\nu)B_\mu(x, \bar{\theta}) + d\bar{\theta} \bar{\Phi}(x, \bar{\theta}),
\]

(2.24)

which emerges from the chiral limit of the $\ast \tilde{A}$ derived from the most general definition of $\tilde{A} = dZ^M \tilde{A}_M$ in (2.17) and application of the $\ast$ operation (2.23) on it. The dual horizontality condition w.r.t. $(\delta)\delta$ (i.e. $\delta \tilde{A} = \delta A$) leads to

\[
\partial_\bar{\theta} \bar{\Phi} = 0 \rightarrow B(x) = 0, \quad \partial_\theta \bar{\Phi} = -\varepsilon_{\mu\nu} \partial_\theta B^\nu \rightarrow R_\mu = -\varepsilon_{\mu\nu} \partial^\nu \bar{C},
\]

(2.25)

and $\partial_\mu B^\mu = \partial_\mu A^\mu$ implies $\partial_\mu R^\mu = 0$ that is satisfied automatically by $R_\mu = -\varepsilon_{\mu\nu} \partial^\nu \bar{C}$.

It is obvious that the auxiliary field $B(x)$ is not fixed in terms of the basic fields of the Lagrangian density (2.1) by the dual horizontality condition (i.e. $\delta \tilde{A} = \delta A$). However, the equation of motion for the Lagrangian density (2.3) helps us to get $B = E$. Thus, the chiral super expansion (2.13), on the chiral supermanifold, becomes

\[
\begin{align*}
B_\mu(x, \bar{\theta}) &= A_\mu(x) - \bar{\theta} \varepsilon_{\mu\nu} \partial^\nu \bar{C}(x) \equiv A_\mu(x) + \bar{\theta} (sdA_\mu(x)), \\
\Phi(x, \bar{\theta}) &= C(x) - i \bar{\theta} E(x) \equiv C(x) + \bar{\theta} (sdC(x)), \\
\bar{\Phi}(x, \bar{\theta}) &= \bar{C}(x) + i \theta (B(x) = 0) \equiv \bar{C}(x) + \theta (sd\bar{C}(x) = 0).
\end{align*}
\]

(2.26)

In terms of the above chiral superfield expansion, the dual(co-)BRST symmetry transformations of (2.2) can be concisely expressed as

\[
\begin{align*}
s_d B_\mu(x, \bar{\theta}) &= -\varepsilon_{\mu\nu} \partial^\nu \phi(x, \bar{\theta}), & s_d \Phi(x, \bar{\theta}) &= 0, & s_b \Phi(x, \bar{\theta}) &= +i\varepsilon^{\mu\nu} \partial_\mu B_\nu.
\end{align*}
\]

(2.27)

It is now evident that

\[
\frac{\partial}{\partial \bar{\theta}} \tilde{A}_M(x, \bar{\theta}) = -i \left[ \Psi(x), Q_d \right]_\pm \equiv s_d \Psi(x), \quad \tilde{A}_M = (\Phi, \bar{\Phi}, B_\mu), \quad \Psi = (C, \bar{C}, A_\mu),
\]

(2.28)

where the brackets have the same meaning as discussed earlier. This equation shows that \textit{geometrically} the on-shell nilpotent co-BRST charge $Q_d$ is the generator of translation $\partial/\partial \bar{\theta}$ for the chiral superfield $\tilde{A}_M$ along the Grassmannian direction $\bar{\theta}$ of the $(2+2)$-dimensional supermanifold. Furthermore, the on-shell nilpotency conditions $s_d^2 = 0, Q_d^2 = 0$ are connected with the property of the square of the translational generator $(\partial/\partial \bar{\theta})^2 = 0$. The process of the translation of $\tilde{A}_M(x, \bar{\theta}) = (B_\mu, \Phi, \bar{\Phi})(x, \bar{\theta})$ along the $\theta$-direction produces the co-BRST transformation $s_d \Psi$ for the generic local field $\Psi = (A_\mu, C, \bar{C})$ (i.e. $s_d \leftrightarrow \partial/\partial \bar{\theta})$. There is a clear distinction, however, between $Q_b$ and $Q_d$ as far as translation of the fermionic
superfields (or transformations on (anti-)ghost fields ($\tilde{C}C$) along $\tilde{\theta}$-direction is concerned. For instance, the translation generated by $Q_b$ along $\tilde{\theta}$-direction results in the transformation for the anti-ghost field $\tilde{C}$ but analogous translation by $Q_d$ leads to the transformation for the ghost field $C$. In more sophisticated language, the horizontality condition entails upon the chiral superfield $\Phi$ to remain chiral but the chiral superfield $\bar{\Phi}$ becomes a local spacetime field (i.e., $\Phi(x, \tilde{\theta}) = C(x)$). On the contrary, the dual horizontality condition entails upon the chiral superfield $\Phi$ to retain its chirality but the chiral superfield $\bar{\Phi}$ becomes an ordinary local field (i.e., $\bar{\Phi}(x, \tilde{\theta}) = \bar{C}(x)$).

2.3 Anti-BRST and anti-co-BRST symmetries: anti-chiral superfields

To derive the on-shell nilpotent anti-BRST and anti-co-BRST symmetry transformations of (2.2), we resort to the anti-chiral superfields $\tilde{A}_M(x, \theta) = (B_\mu, \Phi, \bar{\Phi})(x, \theta, \theta)$ which have the following super expansions along the $\theta$-direction of the anti-chiral supermanifold

\begin{align*}
B_\mu(x, \theta) &= A_\mu(x) + \theta \bar{R}_\mu(x), \\
\Phi(x, \tilde{\theta}) &= C(x) - i \theta B(x), \\
\bar{\Phi}(x, \theta) &= \bar{C}(x) + i \theta B(x).
\end{align*}

These are, in fact, the anti-chiral limit ($\tilde{\theta} \to 0$) of the general super expansion (2.14) on the $(2 + 2)$-dimensional supermanifold. The super exterior derivative $\tilde{d}$ and super connection one-form $\tilde{A}$, for our present discussion, are

\begin{align*}
\tilde{d} &= dZ^M \partial_M \equiv dx^\mu \partial_\mu + d\theta \partial_\theta, \\
\tilde{A} &= dZ^M \tilde{A}_M \equiv dx^\nu B_\mu(x, \theta) + d\theta \tilde{\Phi}(x, \theta),
\end{align*}

which are the anti-chiral limit ($\theta \to 0, d\theta \to 0$) of the corresponding general expressions defined in (2.17). Now the imposition of the horizontality condition ($\tilde{d}\tilde{A} = dA$) implies the restriction that the curvature two-form $F = dA$, defined on the ordinary spacetime manifold, remains unchanged. In other words, the superspace contributions to the curvature two-form are set equal to zero. With the above definitions (2.30) on the anti-chiral supermanifold, the following inputs

\begin{align*}
\tilde{d}\tilde{A} &= (dx^\mu \wedge dx^\nu)(\partial_\mu B_\nu) + (dx^\mu \wedge d\theta)(\partial_\mu \tilde{\Phi} - \partial_\theta B_\mu) - (d\theta \wedge d\theta)(\partial_\theta \tilde{\Phi}), \\
dA &= (dx^\mu \wedge dx^\nu) (\partial_\mu A_\nu) \equiv \frac{1}{2} (dx^\mu \wedge dx^\nu) (\partial_\mu A_\nu - \partial_\nu A_\mu),
\end{align*}

lead to the following relationships due to $dA = \tilde{d}\tilde{A}$

\begin{align*}
\partial_\theta \tilde{\Phi}(x, \theta) &= 0 \to B(x) = 0, \\
\partial_\mu \tilde{\Phi}(x, \theta) = \partial_\theta B_\mu(x, \theta) \to \bar{R}_\mu(x) = \partial_\mu \bar{C}(x),
\end{align*}

and $\partial_\mu B_\nu - \partial_\nu B_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$ which implies $\partial_\mu \bar{R}_\nu - \partial_\nu \bar{R}_\mu = 0$. The latter requirement is automatically satisfied by $\bar{R}_\mu = \partial_\mu \bar{C}$. It is clear that the above horizontality restriction does not fix $B(x)$ in terms of the basic fields of the Lagrangian density (2.1). However, the
equation of motion \( B = -\frac{1}{2} (\partial_{\mu} A^{\mu}) \) for the Lagrangian density (2.3) comes to our help. With these insertions, the super expansion (2.29) becomes

\[
B_{\mu}(x, \theta) = A_{\mu}(x) + \theta \partial_{\mu} \bar{C}(x) \equiv A_{\mu}(x) + \theta (s_{ab} A_{\mu}(x)), \\
\Phi(x, \bar{\theta}) = C(x) + i \frac{1}{\xi} \theta (\partial_{\mu} A^{\mu})(x) \equiv C(x) + \theta (s_{ab} C(x)), \\
\bar{\Phi}(x, \theta) = \bar{C}(x) + i \theta (B(x) = 0) \equiv \bar{C}(x) + \theta (s_{ab} \bar{C}(x) = 0).
\]

(2.33)

In terms of the above anti-chiral superfield expansions, the anti-BRST symmetry transformations of (2.2) can be concisely expressed as

\[
s_{ab} B_{\mu}(x, \theta) = \partial_{\mu} \bar{\Phi}(x, \theta), \quad s_{ab} \bar{\Phi}(x, \theta) = 0, \quad s_{ab} \Phi(x, \theta) = +i \frac{1}{\xi} (\partial_{\mu} B^{\mu})(x, \theta).
\]

(2.34)

It is now straightforward to check that

\[
\frac{\partial}{\partial \theta} \tilde{A}_{M}(x, \theta) = -i [\Psi(x), Q_{ab}]_{\pm} \equiv s_{ab} \Psi(x), \quad \tilde{A}_{M} = (\Phi, \bar{\Phi}, B_{\mu}), \quad \Psi = (C, \bar{C}, A_{\mu}),
\]

(2.35)

where the above brackets have the same interpretation as discussed earlier. This equation shows that geometrically the on-shell nilpotent anti-BRST charge \( Q_{ab} \) is the generator of translation \( \frac{\partial}{\partial \theta} \) for the anti-chiral superfield \( \tilde{A}_{M}(x, \theta) = (B_{\mu}, \Phi, \bar{\Phi})(x, \theta) \) along the \( \theta \)-direction of the supermanifold. In fact, this process of translation produces the anti-BRST symmetry transformations (i.e. \( s_{ab} \Psi \)) for the local fields \( \Psi \) that are listed in equation (2.2). Thus, there is a mapping \( s_{ab} \leftrightarrow \frac{\partial}{\partial \theta} \) between the above two key operators and the nilpotency \( s_{ab}^{2} = 0(Q_{ab}^{2} = 0) \) is encoded in the square of the translation generator \( (\partial/\partial \theta)^{2} = 0 \).

Now we shall dwell a bit on the derivation of the on-shell nilpotent anti-co-BRST symmetry in the framework of anti-chiral superfield formulation. We exploit here the (super) co-exterior derivatives \( (\tilde{\delta}) \delta \) and (super) connection one-forms \( (\tilde{\delta}) A \) for the dual horizontality condition

\[
\tilde{\delta} \tilde{A} = \delta A, \quad \delta = -* d*, \quad A(x) = dx^{\mu} A_{\mu}(x), \quad \delta A = \partial_{\mu} A^{\mu}, \\
\tilde{\delta} \tilde{A} = \partial_{\mu} B^{\mu} - \epsilon^{\mu \nu}(\partial_{\mu} \Phi + \epsilon_{\mu \nu} \partial_{\nu} B^{\mu}) + s \theta \partial_{\theta} \Phi,
\]

(2.36)

where in the computation of the \( \tilde{\delta} \tilde{A} \), we have used \( \tilde{\delta} = -* d* \) and the expression for

\[
* \tilde{A} = \epsilon^{\mu \nu} (dx_{\nu}) B_{\mu}(x, \theta) + d\theta \Phi(x, \theta).
\]

(2.37)

The above equation emerges as the anti-chiral limit of the most general form of \( * \tilde{A} \)

\[
* \tilde{A} = \epsilon^{\mu \nu} (dx_{\nu}) B_{\mu}(x, \theta, \bar{\theta}) + d\theta \Phi(x, \theta, \bar{\theta}) + d\bar{\theta} \bar{\Phi}(x, \theta, \bar{\theta}).
\]

(2.38)

The restriction \( \tilde{\delta} \tilde{A} = \delta A \) (which physically implies an imposition that the zero-form gauge-fixing term \( \delta A = \partial_{\mu} A^{\mu} \), defined on the ordinary spacetime manifold, remains unchanged) leads to the following relationships

\[
\partial_{\theta} \Phi = 0 \to B(x) = 0, \quad \partial_{\bar{\theta}} \Phi = -\epsilon_{\mu \nu} \partial_{\theta} B^{\nu} \to \bar{R}_{\mu} = -\epsilon_{\mu \nu} \partial^{\nu} C,
\]

(2.39)
and \( \partial_\mu B^\mu = \partial_\mu A^\mu \) implies \( \partial_\mu \bar{R}^\mu = 0 \) which is readily satisfied by \( \bar{R}_\mu = -\varepsilon_{\mu
u} \partial^\nu C \). The dual horizontality condition \( \delta \bar{A} = \delta A \) does not fix the field \( B(x) \) in terms of the basic fields. The equation of motion \( B = E \) for the Lagrangian density (2.3), however, comes to our rescue. The super expansion with the above insertions turns out to be

\[
B_\mu(x, \theta) = A_\mu(x) - \theta \varepsilon_{\mu\nu} \partial^\nu C(x) \equiv A_\mu(x) + \theta (s_{ad} A_\mu(x)), \\
\Phi(x, \theta) = C(x) - i \theta (B(x) = 0) \equiv C(x) + \theta (s_{ad} C(x) = 0), \\
\bar{\Phi}(x, \theta) = \bar{C}(x) + i \theta E(x) \equiv \bar{C}(x) + \theta (s_{ad} \bar{C}(x)).
\]

With the above expansions as the backdrop, we can express now the anti-co-BRST transformations of (2.2) in terms of the anti-chiral superfields as

\[
s_{ad} B_\mu(x, \theta) = -\varepsilon_{\mu\nu} \partial^\nu \Phi(x, \theta), \quad s_{ad} \Phi(x, \theta) = 0, \quad s_{ad} \bar{\Phi}(x, \theta) = -i \varepsilon^{\mu\nu} \partial_\mu B_\nu(x, \theta).
\]

The geometrical interpretation for the co-BRST charge \( Q_{ad} \) is encoded in

\[
\frac{\partial}{\partial \theta} \tilde{A}_M(x, \theta) = -i [\Psi(x), Q_{ad}]_\pm \equiv s_{ad} \Psi(x), \quad \tilde{A}_M = (\Phi, \bar{\Phi}, B_\mu), \quad \Psi = (C, \bar{C}, A_\mu),
\]

where the brackets \([ , ]_\pm\) have the same interpretation as explained earlier. It is obvious to note that \( Q_{ad} \) turns out to be the translation generator \( \frac{\partial}{\partial \theta} \) for the anti-chiral superfields \( \tilde{A}_M(x, \theta) = (B_\mu, \Phi, \bar{\Phi})(x, \theta) \) along the \( \theta \)-direction of the supermanifold. The action of the on-shell nilpotent transformation operator \( s_{ad} \) on the local fields \( \Psi \) and the operation of \( \frac{\partial}{\partial \theta} \) on the anti-chiral superfields \( \tilde{A}_M(x, \theta) \) are inter-related and there exists a mapping \( s_{ad} \leftrightarrow \frac{\partial}{\partial \theta} \) with \( s_{ad}^2 = 0 \) connected to \( (\partial/\partial \theta)^2 = 0 \). Even though both the charges \( Q_{ad}, Q_{ab} \) have the similar kind of mapping with the translation generator, there is a clear distinction between them. Whereas the former generates a transformation for the ghost field \( C \) through the translation of the superfield \( \Phi \), the latter generates the corresponding transformation on the anti-ghost field \( \bar{C} \) through the translation of \( \bar{\Phi} \) superfield. The direction of translation is common for both of them (i.e. the \( \theta \)-direction of the supermanifold).

It should be emphasized that the on-shell nilpotent (anti-)BRST- and (anti-)co-BRST symmetries can be derived together if we merge systematically the (anti-)chiral superfields and have the super expansion as given below [37]

\[
B_\mu(x, \theta, \bar{\theta}) = A_\mu(x) + \theta \bar{R}_\mu(x) + \bar{\theta} R_\mu(x) + i \theta \bar{\theta} S_\mu(x), \\
\Phi(x, \theta, \bar{\theta}) = C(x) + i \frac{\bar{\theta}}{\theta} (\partial_\mu A^\mu)(x) - i \theta \bar{\theta} E(x) + i \theta \bar{\theta} s(x), \\
\bar{\Phi}(x, \theta, \bar{\theta}) = \bar{C}(x) + i \theta \bar{\theta} E(x) - i \frac{\theta}{\bar{\theta}} (\partial_\mu A^\mu)(x) + i \theta \bar{\theta} \bar{s}(x).
\]

In our work [37], these super expansions together with the definitions in (2.17) and \(*\) operation in (2.23) have been exploited in the horizontality condition \( \tilde{F} = \tilde{d} \tilde{A} = d A = F \) which leads to the derivation of the auxiliary fields in terms of the basic fields of the Lagrangian density (2.1). Ultimately, the above super expansions are expressed in terms of on-shell nilpotent (anti-)BRST symmetries (2.2) as

\[
B_\mu(x, \theta, \bar{\theta}) = A_\mu(x) + \theta (s_{ab} A_\mu(x)) + \bar{\theta} (s_b A_\mu(x)) + \theta \bar{\theta} (s_{ab} s_a A_\mu(x)), \\
\Phi(x, \theta, \bar{\theta}) = C(x) + \theta (s_{ab} C(x)) + \bar{\theta} (s_b C(x)) + \theta \bar{\theta} (s_{ab} s_a C(x)), \\
\bar{\Phi}(x, \theta, \bar{\theta}) = \bar{C}(x) + \theta (s_{ab} \bar{C}(x)) + \bar{\theta} (s_b \bar{C}(x)) + \theta \bar{\theta} (s_{ab} s_a \bar{C}(x)).
\]
In a similar fashion, the dual horizontality condition w.r.t. (super) co-exterior derivatives (i.e. $\delta \tilde{A} = \delta A$) have been performed in [37] which finally enforces (2.43) to be expressed in terms of the on-shell nilpotent (anti-)co-BRST symmetry transformations (2.2), as

\begin{align*}
B_\mu(x, \theta, \bar{\theta}) &= A_\mu(x) + \theta (s_d A_\mu(x)) + \bar{\theta} (s_d A_\mu(x)) + \theta \bar{\theta} (s_d s_d A_\mu(x)), \\
\Phi(x, \theta, \bar{\theta}) &= C(x) + \theta (s_d C(x)) + \bar{\theta} (s_d C(x)) + \theta \bar{\theta} (s_d s_d C(x)), \\
\bar{\Phi}(x, \theta, \bar{\theta}) &= \bar{C}(x) + \theta (s_d \bar{C}(x)) + \bar{\theta} (s_d \bar{C}(x)) + \theta \bar{\theta} (s_d s_d \bar{C}(x)).
\end{align*}

We would like to lay stress on the fact that it is only for the free 2D Abelian gauge theory that (anti-)chiral superfields are merged together systematically to produce the on-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries together. The same is not true for the non-Abelian gauge theory as we shall see later.

### 2.4 Off-shell nilpotent symmetries: superfield approach

We shall discuss here the bare essentials of our earlier work [38] where the off-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries have been obtained by exploiting the most general super expansion of the multiplets of supervector superfield $\tilde{A}_M(x, \theta, \bar{\theta}) = (B_\mu, \Phi, \bar{\Phi})(x, \theta, \bar{\theta})$ as given below

\begin{align*}
B_\mu(x, \theta, \bar{\theta}) &= A_\mu(x) + \theta \tilde{R}_\mu(x) + \bar{\theta} R_\mu(x) + i \theta \bar{\theta} S_\mu(x), \\
\Phi(x, \theta, \bar{\theta}) &= C(x) + i \theta \tilde{B}(x) - i \bar{\theta} B(x) + i \theta \bar{\theta} s(x), \\
\bar{\Phi}(x, \theta, \bar{\theta}) &= \bar{C}(x) - i \theta \tilde{B}(x) + i \bar{\theta} B(x) + i \theta \bar{\theta} \bar{s}(x),
\end{align*}

where signs are chosen for the algebraic convenience only. The application of the horizontality condition ($d\tilde{A} = dA$) w.r.t. the (super) exterior derivatives ($\tilde{d}$)$\tilde{d}$ and (super) connection one-forms ($\tilde{A}$)$\tilde{A}$ (cf. (2.17)) leads to the following relationships

\begin{align*}
s(x) = \bar{s}(x) = 0, \quad \tilde{B}(x) = B(x) = 0, \quad R_\mu(x) = \partial_\mu C(x), \\
\tilde{R}_\mu(x) = \partial_\mu \bar{C}(x), \quad S_\mu = \partial_\mu B(x), \quad B(x) + \bar{B}(x) = 0.
\end{align*}

Physically, the condition $d\tilde{A} = dA$ amounts to the restriction that there is no contribution of the superspace variables to the curvature 2-form. Insertions of these auxiliary fields in the above expansion leads to

\begin{align*}
B_\mu(x, \theta, \bar{\theta}) &= A_\mu(x) + \theta \partial_\mu \bar{C}(x) + \bar{\theta} \partial_\mu C(x) + i \theta \bar{\theta} \partial_\mu B(x), \\
\Phi(x, \theta, \bar{\theta}) &= C(x) - i \theta B(x), \\
\bar{\Phi}(x, \theta, \bar{\theta}) &= \bar{C}(x) + i \bar{\theta} B(x),
\end{align*}

which is the same expansion as quoted in equation (2.44). However, the transformations $s_{(a)b}$ of (2.44) now correspond to the off-shell nilpotent (anti-)BRST transformations $\tilde{s}_{(a)b}$ of (2.4) (i.e. $s_{(a)b} \rightarrow \tilde{s}_{(a)b}$). It is interesting to note that the horizontality condition ($d\tilde{A} = dA$) enforces the odd superfield $\Phi$ to become anti-chiral (i.e. $\partial_\theta \Phi = 0$) and the other odd superfield $\bar{\Phi}$ becomes chiral (i.e. $\partial_\bar{\theta} \Phi = 0$). Similarly, the dual horizontality condition
\(\delta \bar{A} = \delta A\) w.r.t. the (super) co-exterior derivatives \((\delta)\delta\) (together with the definitions in (2.17) and the \(\star\) operation in (2.23)), leads to the following relationships

\[
\begin{align*}
\tilde{s}(x) &= \bar{s}(x) = 0, \quad B(x) = \bar{B}(x) = 0, \quad R_\mu(x) = -\varepsilon_\mu \partial^\nu \bar{C}(x), \\
\tilde{R}_\mu(x) &= -\varepsilon_\mu \partial^\nu C(x), \quad S_\mu = \varepsilon_\mu \partial^\nu B(x), \quad \mathcal{B}(x) + \bar{\mathcal{B}}(x) = 0.
\end{align*}
\] (2.49)

Physically, the condition \(\delta \Bar{A} = \delta A\) amounts to no superspace contribution to the zero-form gauge-fixing term. When we substitute the above expressions in the super expansion (2.46), we get

\[
\begin{align*}
B_\mu(x, \theta, \bar{\theta}) &= A_\mu(x) - \theta \varepsilon_\mu \partial^\nu C(x) - \bar{\theta} \varepsilon_\mu \partial^\nu \bar{C}(x) + i \theta \bar{\theta} \varepsilon_\mu \partial^\nu B(x), \\
\Phi(x, \theta, \bar{\theta}) &= \mathcal{C}(x) - i \bar{\theta} \mathcal{B}(x), \\
\bar{\Phi}(x, \theta, \bar{\theta}) &= \bar{\mathcal{C}}(x) + i \theta \bar{\mathcal{B}}(x),
\end{align*}
\] (2.50)

which is in the same form as in (2.45). However, the transformations \(s_{(a)d}\) of (2.45) now correspond to the ones listed in (2.4) for the off-shell nilpotent (anti-)co-BRST symmetries \(\tilde{s}_{(a)d}\) (i.e. \(s_{(a)d} \rightarrow \tilde{s}_{(a)d}\)). It is worth pointing out that the dual horizontality condition \((\delta \bar{A} = \delta A)\) enforces the odd \(\Phi\) superfield to become chiral (i.e. \(\partial_\theta \Phi = 0\)) and the other odd superfield \(\Phi\) becomes anti-chiral (i.e. \(\partial_{\bar{\theta}} \Phi = 0\)). Thus, it is obvious that the effect of the horizontality-and dual horizontality conditions on the odd superfields is diametrically opposite as far as the chirality is concerned. The geometrical interpretation for the conserved and off-shell nilpotent charges \(Q_{(a)b}\) and \(Q_{(a)d}\), as the translation generators on the supermanifold, is the same as elaborated earlier. The mappings: \(\tilde{s}_{(d)b} \leftrightarrow \partial / \partial \theta, \tilde{s}_{ab} \leftrightarrow \partial / \partial \bar{\theta}, \tilde{s}_{ad} \leftrightarrow \partial / \partial \bar{\theta}\) are valid for the off-shell nilpotent symmetries as well. This can be shown explicitly by expressing \(X\) of (2.8), in terms of superfields, as

\[
\begin{align*}
\frac{1}{2} \tilde{s}_b \tilde{s}_{ab} \left( i A_\mu A^\mu - \xi \bar{C} \bar{C} \right) &= \frac{1}{2} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \bar{\theta}} \left( i B_\mu B^\mu - \xi \bar{\Phi} \Phi \right) \bigg|_{(\text{anti-)BRST}}, \\
\tilde{s}_b \left( -i \bar{C} \left[ \partial_\mu A^\mu + \frac{\xi}{2} B \right] \right) &= \lim_{\theta, \bar{\theta} \to 0} \frac{\partial}{\partial \theta} \left[ -i \bar{\Phi} (\partial_\mu B^\mu) + \frac{\xi}{2} \bar{\Phi} \frac{\partial \Phi}{\partial \bar{\theta}} \right] \bigg|_{(\text{anti-)BRST}}, \\
\tilde{s}_{ab} \left( i C \left[ \partial_\mu A^\mu + \frac{\xi}{2} B \right] \right) &= \lim_{\theta, \bar{\theta} \to 0} \frac{\partial}{\partial \theta} \left[ +i \Phi (\partial_\mu B^\mu) + \frac{\xi}{2} \Phi \frac{\partial \Phi}{\partial \bar{\theta}} \right] \bigg|_{(\text{anti-)BRST)}},
\end{align*}
\] (2.51)

and \(Y\) of (2.8) bears an appearance, in terms of superfields, as

\[
\begin{align*}
\frac{1}{2} \tilde{s}_d \tilde{s}_{ad} \left( i A_\mu A^\mu - \xi \bar{C} \bar{C} \right) &= \frac{1}{2} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \bar{\theta}} \left( i B_\mu B^\mu - \xi \bar{\Phi} \Phi \right) \bigg|_{(\text{anti-)co-BRST)}, \\
\tilde{s}_d \left( i C \left[ E - \frac{1}{2} \mathcal{B} \right] \right) &= \lim_{\theta, \bar{\theta} \to 0} \frac{\partial}{\partial \theta} \left[ -i \Phi (\varepsilon_\mu \varepsilon_\nu \partial_\mu B_\nu) + \frac{1}{2} \Phi \frac{\partial \Phi}{\partial \bar{\theta}} \right] \bigg|_{(\text{anti-)co-BRST)}, \\
\tilde{s}_{ab} \left( -i \bar{C} \left[ E - \frac{1}{2} \mathcal{B} \right] \right) &= \lim_{\theta, \bar{\theta} \to 0} \frac{\partial}{\partial \theta} \left[ +i \bar{\Phi} (\varepsilon_\mu \varepsilon_\nu \partial_\mu B_\nu) + \frac{1}{2} \bar{\Phi} \frac{\partial \Phi}{\partial \bar{\theta}} \right] \bigg|_{(\text{anti-)co-BRST)},
\end{align*}
\] (2.52)
where the above subscripts stand for the expansions in (2.48) and (2.50) respectively.

3 Self-interacting 2D non-Abelian gauge theory

In this section, the on-shell and off-shell nilpotent (anti-)BRST- and (anti-)co-BRST symmetries are discussed in the Lagrangian- and superfield formulations.

3.1 (Anti-)BRST and (anti-)co-BRST symmetries: Lagrangian formalism

Let us begin with the BRST invariant Lagrangian density \( \mathcal{L}_b^{(N)} \) for the self-interacting 2D non-Abelian gauge theory \(^{14}\) in an arbitrary gauge (with the gauge parameter \( \xi \neq 0 \)) \([43-46]\)

\[
\mathcal{L}_b^{(N)} = -\frac{1}{4} F_{\mu \nu} \cdot F_{\mu \nu} - \frac{1}{2 \xi} (\partial_\mu A^\mu) \cdot (\partial_\rho A^\rho) - i \partial_\mu \bar{C} \cdot D^\mu C, \\
\equiv \frac{1}{2} E \cdot E - \frac{1}{2 \xi} (\partial_\mu A^\mu) \cdot (\partial_\rho A^\rho) - i \partial_\mu \bar{C} \cdot D^\mu C, 
\]

(3.1)

where \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + (A_\mu \times A_\nu) \) is the field strength tensor derived from the one-form connection \( A = dx^\mu A_\mu \equiv dx^\mu A_\mu^a T^a \) by the Maurer-Cartan equation \( F = dA + A \wedge A \) with \( F = \frac{1}{2} dx^\mu \wedge dx^\nu F_{\mu \nu}^a T^a \). Here \( T^a \) are the generators of the compact Lie algebra \( [T^a, T^b] = f^{abc} T^c \) where \( f^{abc} \) are the structure constants that can be chosen to be totally antisymmetric in \( a, b, c \) (see, e.g., Ref. [46] for more details). The anti-commuting \( ([C^a]^2 = (\bar{C}^a)^2 = 0, C^a \bar{C}^b + \bar{C}^b C^a = 0) \) (anti-)ghost fields \( (\bar{C}^a) C^a \) (which interact with the self-interacting non-Abelian gauge fields \( A_\mu \) only in the loop diagrams) are required to be present in the theory to maintain the unitarity and gauge invariance together at any arbitrary order of perturbative computations \([49]\). These fields (even though they interact with the gauge fields \( A_\mu \)) are not the physical matter fields. The above Lagrangian density (3.1) respects the following on-shell \( (\partial_\mu D^\mu C = D_\mu \partial^\mu \bar{C} = 0) \) nilpotent \( (s_b^2 = s_d^2 = 0) \) BRST \( (s_b) \) and dual(co-)BRST \( (s_d) \) symmetry transformations \([31,35]\)

\[
\begin{align*}
\text{s}_b A_\mu &= D_\mu C, & \text{s}_b C &= -\frac{i}{2} C \times C, & \text{s}_b \bar{C} &= -\frac{i}{\xi} (\partial_\mu A^\mu), & \text{s}_b E &= E \times C, \\
\text{s}_d A_\mu &= -\varepsilon_\mu \partial^\mu \bar{C}, & \text{s}_d C &= 0, & \text{s}_d \bar{C} &= -i E, & \text{s}_d E &= D_\mu \partial^\mu \bar{C}.
\end{align*}
\]

(3.2)

It is worth pointing out that the kinetic energy term \( \frac{1}{2} (E \cdot E) \) remains invariant under the BRST transformation \( s_b \). On the other hand, it is the gauge fixing term \( (-\frac{1}{2 \xi} (\partial_\mu A^\mu) \cdot (\partial_\rho A^\rho)) \) that remains unchanged under the dual(co-)BRST transformations \( s_d \). The anti-commutator \( s_w = \{s_b, s_d\} \) of these nilpotent symmetries leads to the definition of a bosonic symmetry \( (s_w^2 \neq 0) \), under which, the ghost term \( -i \partial_\mu \bar{C} \cdot D^\mu C \) remains invariant \([35]\).

The auxiliary fields \( B \) and \( \mathcal{B} \) can be introduced to linearize the gauge-fixing term and the kinetic energy term \( \frac{1}{2} (E \cdot E) \) (because there is no magnetic component of \( F_{\mu \nu} \) for the

\[^{14}\]We follow here the notations such that \( F_{01} = E = \partial_0 A_1 - \partial_1 A_0 + A_0 \times A_1 = -\varepsilon^{01} (\partial_0 A_1 + \frac{1}{2} A_1 \times A_0) \), \( \varepsilon_{01} = \varepsilon^{10} = +1 \), \( D_\mu C = \partial_\mu C + A_\mu \times C, (\alpha \cdot \beta)^\mu = f^{abc} \alpha^a \beta^b \) where \( \alpha \) and \( \beta \) are the non-null vectors in the group space. Here the Greek indices: \( \mu, \nu, \rho, ... = 0, 1 \) correspond to the spacetime directions and Latin indices: \( a, b, c, ... = 1, 2, 3, ... \) stand for the “colour” values in the group space.
(1 + 1)-dimensional (2D) non-Abelian gauge theory) as

\[ \mathcal{L}_B^{(N)} = B \cdot E - \frac{1}{2} B \cdot B + B \cdot (\partial_\mu A^\mu) + \frac{\xi}{2} B \cdot B - i \partial_\mu \bar{C} \cdot D^\mu C. \] (3.3)

The above Lagrangian density (3.3) respects the following off-shell nilpotent \((\bar{s}_b)\)-and dual(co)-BRST \((\bar{s}_d)\) symmetry transformations [31,35,39]

\[ \begin{align*}
\bar{s}_b A_\mu &= D_\mu C, & \bar{s}_b C &= -\frac{1}{2} C \times C, & \bar{s}_b \bar{C} &= i B, \\
\bar{s}_b B &= B \times C, & \bar{s}_b B &= 0, & \bar{s}_b E &= E \times C, \\
\bar{s}_d A_\mu &= -\varepsilon_{\mu \nu} \partial^\nu \bar{C}, & \bar{s}_d \bar{C} &= 0, & \bar{s}_d C &= -i B, \\
\bar{s}_d B &= 0, & \bar{s}_d B &= 0, & \bar{s}_d E &= D_\mu \partial^\mu \bar{C}.
\end{align*} \] (3.4) (3.5)

Besides BRST- and co-BRST symmetry transformations (3.4) and (3.5), there are anti-BRST- and anti-co-BRST symmetries that are also present in the theory. To realize these, one has to introduce another auxiliary field \(\bar{B}\) (satisfying \(B + \bar{B} = i C \times \bar{C}\)) to recast the Lagrangian density (3.3) into the following forms [50]

\[ \begin{align*}
\mathcal{L}_B^{(N)} &= B \cdot E - \frac{1}{2} B \cdot B + B \cdot (\partial_\mu A^\mu) + \frac{\xi}{2} (B \cdot B + \bar{B} \cdot \bar{B}) - i \partial_\mu \bar{C} \cdot D^\mu C, \quad (3.6a) \\
\mathcal{L}_B^{(N)} &= B \cdot E - \frac{1}{2} B \cdot B - \bar{B} \cdot (\partial_\mu A^\mu) + \frac{\xi}{2} (B \cdot B + \bar{B} \cdot \bar{B}) - i D_\mu \bar{C} \cdot \partial^\mu C. \quad (3.6b)
\end{align*} \]

The Lagrangian density (3.6b) transforms to a total derivative under the following off-shell nilpotent anti-BRST \((\bar{s}_{ab})\)- and (anti-)co-BRST \((\bar{s}_{ad})\) symmetry transformations [35,39]

\[ \begin{align*}
\bar{s}_{ab} A_\mu &= D_\mu \bar{C}, & \bar{s}_{ab} \bar{C} &= -\frac{1}{2} \bar{C} \times \bar{C}, & \bar{s}_{ab} C &= i \bar{B}, & \bar{s}_{ab} \bar{B} &= 0, \\
\bar{s}_{ab} E &= E \times C, & \bar{s}_{ab} B &= B \times C, & \bar{s}_{ab} (\partial_\mu A^\mu) &= \partial_\mu D^\mu C, \\
\bar{s}_{ad} A_\mu &= -\varepsilon_{\mu \nu} \partial^\nu C, & \bar{s}_{ad} C &= 0, & \bar{s}_{ad} \bar{C} &= +i B, & \bar{s}_{ad} \bar{B} &= 0, \\
\bar{s}_{ad} E &= D_\mu \partial^\mu C, & \bar{s}_{ad} B &= 0, & \bar{s}_{ad} B &= 0, & \bar{s}_{ad} (\partial_\mu A^\mu) &= 0.
\end{align*} \] (3.7) (3.8)

All the above continuous symmetry transformations can be concisely expressed, in terms of the Noether conserved charges \(Q_r\) as quoted in (2.5).

The Lagrangian density (3.1) can be expressed, modulo some total derivatives, as the sum of terms that turn out to be (co-)BRST anti-commutators

\[ \mathcal{L}_b^{(N)} = \{ Q_d, \frac{1}{2} E \cdot C \} - \{ Q_b, \frac{1}{2} (\partial_\mu A^\mu) \cdot C \} \equiv s_d [ \frac{1}{2} E \cdot C ] - s_b [ \frac{1}{2} (\partial_\mu A^\mu) \cdot C ], \] (3.9)

which resembles very much like the Lagrangian density for the Witten type TFT if we assume that the vacuum and physical states of this theory are annihilated by the (co-)BRST charges (i.e., \(Q_{(d)b}|_{\text{phys}} = 0, Q_{(d)b}|_{\text{vac}} = 0\)). Such a situation does arise when we choose the harmonic state of the Hodge decomposed state to be the physical state (including the vacuum) in the quantum Hilbert space of states. This choice is guided by some aesthetic reasons because the harmonic states possess the maximum symmetries as they remain invariant under (co-)BRST symmetries as well as a bosonic symmetry (that is an anti-commutator of nilpotent (co-)BRST symmetries). In contrast to the appearance
of the Lagrangian density (which is like Witten type TFT), the local symmetries of the theory are just like Schwarz type TFT because there are no topological shift symmetries in the theory (see, e.g., Ref. [5] for details). The Lagrangian density in (3.3) can be also expressed as

$$  \mathcal{L}_B^{(N)} = B \cdot E - \frac{1}{2} B \cdot B + \bar{s}_b(-i\bar{C} \cdot [(\partial_\mu A^\mu) + \frac{\xi}{2}B]). $$

Similarly, the Lagrangian densities in (3.6) can be re-written as

$$  \mathcal{L}_B^{(N)} = B \cdot E - \frac{1}{2} B \cdot B + \bar{s}_b\bar{s}_{ab}(\frac{1}{2}A_\mu \cdot A^\mu - \xi \bar{C} \cdot C), \\
  \mathcal{L}_B^{(N)} = \frac{\xi}{2}(B \cdot B + \bar{B} \cdot \bar{B}) + B \cdot (\partial_\mu A^\mu) + \bar{s}_d(+iC \cdot [E - \frac{1}{2}B]), \\
  \mathcal{L}_B^{(N)} = \frac{\xi}{2}(B \cdot B + \bar{B} \cdot \bar{B}) - B \cdot (\partial_\mu A^\mu) + \bar{s}_{ad}(-i\bar{C} \cdot [E - \frac{1}{2}B]). $$

The above forms of the Lagrangian densities are just the analogues of the corresponding forms for the Abelian gauge theories in (2.8). In fact, modulo some total derivatives, the expression $\bar{s}_b\bar{s}_{ab}(\frac{1}{2}A_\mu \cdot A^\mu - \xi \bar{C} \cdot C) = B \cdot (\partial_\mu A^\mu) - i\partial_\mu \bar{C} \cdot D^\mu C$. However, it can be shown that the same expression is also equivalent to: $-\bar{B} \cdot (\partial_\mu A^\mu) = iD_\mu \bar{C} \cdot \partial^\mu \bar{C} + \partial_\mu X^\mu$ (with $X^\mu = -i \bar{C} \cdot A^\mu \times C$) by exploiting the relationship $B + \bar{B} = i(C \times \bar{C})$ [50]. To confirm the topological nature of the above theory, it can be seen that the symmetric energy momentum for the Lagrangian density (3.1) is [35,39]

$$  T_{\alpha\beta}^{(N)} = -\frac{1}{2\xi} (\partial_\beta A^\mu) \cdot [\partial_\alpha A_\beta + \partial_\beta A_\alpha] - \frac{1}{2} E \cdot [\varepsilon_{\alpha\rho} \partial_\beta A^\rho + \varepsilon_{\beta\rho} \partial_\alpha A^\rho] \\
  - \frac{i}{2} \partial_\alpha \bar{C} \cdot (\partial_\beta C + D_\beta C) - \frac{1}{2} \partial_\beta \bar{C} \cdot (\partial_\alpha C + D_\alpha C) - \eta_{\alpha\beta}\mathcal{L}_b^{(N)}, $$

which turns out to be the sum of (co-)BRST anti-commutators as given below

$$  T_{\alpha\beta}^{(N)} = \{Q_b, L_{\alpha\beta}^{(1)}\} + \{Q_d, L_{\alpha\beta}^{(2)}\} \equiv s_b[iL_{\alpha\beta}^{(1)}] + s_d[iL_{\alpha\beta}^{(2)}], \\
  L_{\alpha\beta}^{(1)} = \frac{1}{2} \left[(\partial_\alpha \bar{C}) \cdot A_\beta + (\partial_\beta \bar{C}) \cdot A_\alpha + \eta_{\alpha\beta}(\partial_\rho A^\rho) \cdot \bar{C}\right], \\
  L_{\alpha\beta}^{(2)} = \frac{1}{2} \left[(\partial_\alpha C) \cdot \varepsilon_{\beta\rho} A^\rho + (\partial_\beta C) \cdot \varepsilon_{\alpha\rho} A^\rho - \eta_{\alpha\beta}E \cdot C\right]. $$

The form of the above symmetric energy momentum tensor for the non-Abelian gauge theory ensures that the VEV of the energy density $\langle vac|T_{00}^{(N)}|vac\rangle$ is zero and there are no energy excitation in the physical sector of the theory (i.e. $\langle phys|T_{00}^{(N)}|phys\rangle > 0$) because of the fact that the harmonic (physical) states satisfy: $Q_{d(b)}|vac\rangle > 0, Q_{d(b)}|phys\rangle = 0$ w.r.t. the (co-)BRST charges $Q_{d(b)}$ which are conserved, nilpotent, metric independent and hermitian. In fact, there are four conserved and nilpotent $(Q_{(a)b}^2 = 0, Q_{(a)d}^2 = 0)$ charges in the theory. As a consequence, there are four sets of topological invariants which obey a specific kind of recursion relations. On a 2D compact manifold, these invariants have been computed [35] in the same generic form as illustrated in (2.11). The most important quantities in this connection are the explicitly (anti-)BRST- and (anti-)co-BRST invariant zero-forms because, all the rest of the higher degree forms, can be computed from them by exploiting the recursion relations. For the on-shell $(\partial_\mu D^\mu C = D_\mu \partial^\mu C = 0)$ nilpotent (co-)BRST charges, these zero forms $(W_0)V_0$ are [31,35]

$$  W_0 = E \cdot \bar{C}, \quad V_0 = -\frac{1}{\xi}(\partial_\mu A^\mu) \cdot C - \frac{i}{2} \bar{C} \cdot (C \times C). $$
The analogous zero-forms $A_0^{(ab)}, B_0^{(b)}$ and $C_0^{(ad)}, D_0^{(d)}$ with respect to the off-shell nilpotent (anti-)BRST ($Q_{(ab)}$) and (anti-)co-BRST ($Q_{(ad)}$) charges are

$$
A_0^{(ab)} = -\bar{B} \cdot \bar{C} + \frac{1}{2} C \cdot (\bar{C} \times C), \quad C_0^{(ad)} = B \cdot C,
$$

$$
A_0^{(b)} = B \cdot C - \frac{1}{2} \bar{C} \cdot (C \times C), \quad D_0^{(d)} = B \cdot \bar{C}.
$$

(3.15)

### 3.2 On-shell nilpotent (co-)BRST symmetries: chiral superfields

In this subsection, we shall discuss some of the key features of our work in [41] for the derivation of the on-shell nilpotent (co-)BRST symmetries (3.2) in the framework of chiral superfield formulation. First of all, we generalize the generic local field $\Psi = (A_\mu, C, \bar{C})$ (for the basic local fields of the Lagrangian density (3.1)) to a chiral ($\partial_\theta \bar{A}_M = 0$) supervector superfield $\bar{A}_M(x, \theta) = (B_\mu, \Phi, \bar{\Phi})(x, \theta)$ with the following super expansion for all the component superfields along the $\theta$-direction of the chiral supermanifold

$$
(B_\mu^a T^a)(x, \theta) = (A_\mu^a T^a)(x) + \bar{\theta} (R_\mu^a T^a)(x),
$$

$$
(\Phi^a T^a)(x, \theta) = (C^a T^a)(x) - i \bar{\theta} (B^a T^a)(x),
$$

$$
(\bar{\Phi}^a T^a)(x, \theta) = (\bar{C}^a T^a)(x) + i \bar{\theta} (B^a T^a)(x),
$$

(3.16)

which is the chiral limit ($\theta \to 0$) of the super expansion (2.46) (or (2.14)) on the (2 + 2)-dimensional supermanifold. Note that we have taken here a minus sign in the expansion of $\Phi = \Phi^a T^a$ only for the algebraic convenience. The horizontality condition ($\bar{F} = \bar{D} \bar{A} = DA = F$) with the following definitions on the chiral supermanifold

$$
\bar{d} = dZ^M \partial_M \equiv dx^\mu \partial_\mu + d\bar{\theta} \partial_\theta, \quad \bar{D} = \bar{d} + \bar{A}, \quad D = d + A,
$$

$$
\bar{A} = dZ^M (\bar{A}_M^a T^a) \equiv dx^\mu (B_\mu^a T^a)(x, \bar{\theta}) + d\bar{\theta} (\Phi^a T^a)(x, \bar{\theta}),
$$

(3.17a)

provides a specific kind of relationship between the auxiliary fields and the basic fields of the Lagrangian density (3.1) as given below [41]

$$
R_\mu(x) = D_\mu C(x), \quad B(x) = -\frac{i}{2} (C \times C)(x), \quad [B, C] = 0 \rightarrow B \times C = 0.
$$

(3.17b)

The above horizontality condition does not fix the auxiliary field $B(x) = (B^a T^a)(x)$ in terms of the basic fields of (3.1). However, the equation of motion ($B(x) = -\frac{1}{2} (\partial_\mu A^\mu)(x)$) for the Lagrangian density (3.3) comes to our rescue. Now the expansion (3.16), in the concise notation $B_\mu(x, \bar{\theta}) = (B_\mu^a T^a)(x, \bar{\theta})$ etc., becomes

$$
B_\mu(x, \bar{\theta}) = A_\mu(x) + \bar{\theta} D_\mu C(x) \equiv A_\mu(x) + \bar{\theta} (s_b A_\mu(x)),
$$

$$
\Phi(x, \bar{\theta}) = C(x) - \bar{\theta} \frac{1}{2} (C \times C)(x) \equiv C(x) + \bar{\theta} (s_b C(x)),
$$

$$
\bar{\Phi}(x, \bar{\theta}) = \bar{C}(x) - \frac{1}{2} \bar{\theta} (\partial_\mu A^\mu)(x) \equiv \bar{C}(x) + \bar{\theta} (s_b \bar{C}(x)).
$$

(3.18)

We note here, vis-a-vis equation (2.5), the following interesting relationship

$$
\frac{\partial}{\partial \theta} \bar{A}_M(x, \bar{\theta}) = -i [\Psi(x), Q_b]_\pm \equiv s_b \Psi(x), \quad \bar{A}_M = (\Phi, \bar{\Phi}, B_\mu), \quad \Psi = (C, \bar{C}, A_\mu),
$$

(3.19)

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which establishes the fact that $Q_b$ is the generator of translation $(\partial/\partial \bar{\theta})$ along the Grassmannian direction $\bar{\theta}$ of the supermanifold for the superfield $\tilde{A}_M(x, \bar{\theta})$. This process of translation produces an internal transformation $(s_b \Psi(x))$ for the local generic field. Hence there is a mapping $s_b \leftrightarrow \partial/\partial \bar{\theta}$ which shows that there is a deep inter-relationship between the internal transformation on the local field $\Psi$ and the translation of the superfield $\tilde{A}_M$ along the $\bar{\theta}$-direction of the supermanifold. Furthermore, the nilpotency of $s_b^2 = 0$ is connected with $(\partial/\partial \bar{\theta})^2 = 0$. In a similar fashion, the dual horizontality condition ($\delta \tilde{A} = \delta A$) (with $\delta = - \ast d \ast$, $\delta = - \ast d \ast$) leads to
\[\partial_\mu B^\mu + s^\bar{\theta}((\partial_\theta \Phi) - \varepsilon_{\mu\nu}(\partial_\mu \Phi + \varepsilon_{\mu\nu} \partial_\nu B^\nu) = \partial_\mu A^\mu,\] (3.20)
where $s^\bar{\theta}$ and $\varepsilon_{\mu\nu}$ are defined in equation (2.23). The above equality yields
\[R_\mu(x) = - \varepsilon_{\mu\nu} \partial^\nu C(x), \quad B(x) = 0, \quad \partial_\mu B^\mu(x, \theta) = \partial_\mu A^\mu(x) \to \partial_\mu R^\mu(x) = 0.\] (3.21)
It is evident that the condition $\partial_\mu R^\mu(x) = 0$ is satisfied automatically with $R_\mu = - \varepsilon_{\mu\nu} \partial^\nu C$. The above restriction $\delta \tilde{A} = \delta A$ does not fix the auxiliary field $B(x)$ in terms of the basic local fields $\Psi(x)$. However, the equation of motion $B(x) = E(x)$ for the Lagrangian density (3.3) comes in handy. With the above values, the super expansion (3.16) now becomes
\[
\begin{align*}
B_\mu(x, \bar{\theta}) &= A_\mu(x) - \bar{\theta} \varepsilon_{\mu\nu} \partial^\nu C(x) \equiv A_\mu(x) + \bar{\theta} (s_d A_\mu(x)), \\
\Phi(x, \bar{\theta}) &= C(x) - i \bar{\theta} E(x) \equiv C(x) + \bar{\theta} (s_d C(x)), \\
\bar{\Phi}(x, \bar{\theta}) &= C(x) + i \bar{\theta} (B(x) = 0) \equiv \bar{C}(x) + \bar{\theta} (s_d \bar{C}(x) = 0).
\end{align*}\] (3.22)
This super expansion implies the following
\[\frac{\partial}{\partial \bar{\theta}} \tilde{A}_M(x, \bar{\theta}) = - i [\Psi(x), Q_d]_{+} \equiv s_d \Psi(x), \quad \tilde{A}_M = (\Phi, \bar{\Phi}, B_\mu), \quad \Psi = (C, \bar{C}, A_\mu),\] (3.23)
which shows that $Q_d$ is equivalent to a translation generator $\partial/\partial \bar{\theta}$ along the $\bar{\theta}$-direction of the supermanifold. The process of the translation of the superfield $\tilde{A}_M(x, \bar{\theta})$ along the $\bar{\theta}$-direction leads to the co-BRST transformation $s_d \Psi$ on the local field $\Psi = (A_\mu, C, \bar{C})$. Thus, we have a mapping $s_d \leftrightarrow \partial/\partial \bar{\theta}$ which establishes an inter-relationship between the internal transformation $s_d$ on the local field $\Psi$ and the translation of the superfield $\tilde{A}_M$ along the $\bar{\theta}$-direction of the supermanifold. In addition, the nilpotency $s_d^2 = 0$ property of the internal transformation $s_d$ on the local field $\Psi$ is connected with the property of the square of the translation generator $(\partial/\partial \bar{\theta})^2 = 0$ (i.e. a couple of successive translations) of the supervector superfield $\tilde{A}_M$ along $\bar{\theta}$-direction of the supermanifold. Even though $s_{(d)b} \leftrightarrow \partial/\partial \theta$, there is a clear distinction between them. Under $s_b$, both the (anti-)ghost fields $(C)C$ transform. However, under $s_d$ only the ghost field $C$ transforms but the anti-ghost fields remains intact. In more sophisticated language, it can be seen that the superfields $\Phi$ and $\bar{\Phi}$ remain chiral superfield under transformations generated by $s_b$. However, under transformations generated by $s_d$, only the superfield $\Phi$ remains chiral but $\bar{\Phi}$ becomes only a spacetime dependent local field. In other words, the local anti-ghost field $\bar{C}(x)$ does not transform under $s_d$. 

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3.3 Anti-chiral superfield formalism: no symmetries

Unlike the case of Abelian gauge theory, we show here that the choice of the anti-chiral superfields \((\partial_\theta \bar{A}_M = 0)\) does not lead to any important symmetries in spite of the fact that we exploit the (dual) horizontality conditions. The super expansions for the component superfields of the supervector anti-chiral superfield \(\bar{A}_M(x, \theta) = (B_\mu, \Phi, \bar{\Phi})(x, \theta)\) along the \(\theta\)-direction of the chiral supermanifold are

\[
\begin{align*}
(B_\mu^a T^a)(x, \theta) &= (A_\mu^a T^a)(x) + \theta \left( \bar{R}_\mu^a T^a \right)(x), \\
(\Phi^a T^a)(x, \theta) &= (\bar{C}^a T^a)(x) - i \theta \left( \bar{H}^a T^a \right)(x), \\
(\bar{\Phi}^a T^a)(x, \theta) &= (\bar{C}^a T^a)(x) + i \theta \left( B^a T^a \right)(x),
\end{align*}
\]  

which is the anti-chiral limit \((\bar{\theta} \to 0)\) of the expansion on \((2+2)\)-dimensional supermanifold in \((2.14)\) with the group valued basic- as well as auxiliary fields. We exploit now the horizontality condition \((\bar{F} = \bar{D} \bar{A} = D A = F)\) with the following definitions on the anti-chiral supermanifold

\[
\begin{align*}
\bar{d} &= dZ^M \partial_M \equiv dx^\mu \partial_\mu + d\theta \partial_\theta, \quad \bar{D} = \bar{d} + \bar{A}, \quad D = d + A, \\
\bar{A} &= dZ^M \left( \bar{A}_M^a T^a \right) \equiv dx^\mu \left( B_\mu^a T^a \right)(x, \theta) + d\theta \left( \bar{\Phi}^a T^a \right)(x, \theta).
\end{align*}
\]  

The explicit expressions for the individual terms in \(\bar{D} \bar{A} = \bar{d} \bar{A} + \bar{A} \wedge \bar{A}\) are

\[
\begin{align*}
\bar{d} \bar{A} &= (dx^\mu \wedge dx^\nu)(\partial_\mu B_\nu) + (dx^\mu \wedge d\theta)(\partial_\mu \bar{\Phi} - \partial_\theta B_\mu) - (d\theta \wedge d\theta)(\partial_\theta \bar{\Phi}), \\
\bar{A} \wedge \bar{A} &= (dx^\mu \wedge dx^\nu)(B_\mu B_\nu) + (dx^\mu \wedge d\theta)([B_\mu, \bar{\Phi}]) - (d\theta \wedge d\theta)(\bar{\Phi} \bar{\Phi}).
\end{align*}
\]  

The horizontality restrictions result in the following relationships

\[
\begin{align*}
\partial_\theta \bar{\Phi} + \frac{1}{2} \{\bar{\Phi}, \bar{\Phi}\} &= 0, \quad \partial_\mu \bar{\Phi} - \partial_\theta B_\mu + [B_\mu, \bar{\Phi}] = 0, \\
\bar{F}_{\mu \nu} &\equiv \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \equiv F_{\mu \nu},
\end{align*}
\]  

which lead to

\[
\bar{R}_\mu(x) = D_\mu \bar{C}(x), \quad B(x) = \frac{i}{2}(\bar{C} \times \bar{C})(x), \quad [B, \bar{C}] = 0 \to B \times \bar{C} = 0. \tag{3.28}
\]

The l.h.s. of the last relationship in \((3.27)\) becomes \(\bar{F}_{\mu \nu} = F_{\mu \nu} + \theta[D_\mu, D_\nu] \bar{C} = F_{\mu \nu} + \theta F_{\mu \nu} \times \bar{C}\) which is consistent, in the sense that, the kinetic energy term of the Lagrangian density \((3.1)\) remains invariant \((i.e. -\frac{1}{2} \bar{F}_{\mu \nu} \cdot \bar{F}_{\mu \nu} = -\frac{1}{2} F_{\mu \nu} \cdot F_{\mu \nu})\). The horizontality condition does not fix \(B(x)\) in terms of the basic fields of the Lagrangian density \((3.1)\). The equation of motion for the Lagrangian density \((3.3)\), however, yields a connection \(B(x) = -\frac{1}{\xi}(\partial_\mu A^\mu)(x)\). Thus, expansion \((3.24)\) becomes

\[
\begin{align*}
B_\mu(x, \theta) &= A_\mu(x) + \theta D_\mu \bar{C}(x) \equiv A_\mu(x) + \theta (\bar{s}_b A_\mu(x)), \\
\Phi(x, \theta) &= C(x) + \frac{i}{2} \theta (\partial_\mu A^\mu)(x) \equiv C(x) + \theta (\bar{s}_b C(x)), \\
\bar{\Phi}(x, \theta) &= \bar{C}(x) - \frac{1}{2} \theta (\bar{C} \times \bar{C})(x) \equiv \bar{C}(x) + \theta (\bar{s}_b \bar{C}(x)),
\end{align*}
\]  

for some transformation \(\bar{s}_b\) (which correspond to translation along \(\theta\)-direction of the anti-chiral supermanifold). However, there are a few problems. First, it can be seen that \(\bar{s}_b\) is not
the symmetry transformation for (3.1) (i.e. $\bar{s}_b\mathcal{L}^{(N)}_b \neq 0$, $\bar{s}_b\mathcal{L}^{(N)}_b \neq \partial_\mu X^\mu$ where $X^\mu$ is in terms of some local fields). Second, it is not a nilpotent transformation (e.g., $\bar{s}_b^2\mathcal{C} \sim \partial_\mu D^\mu \mathcal{C} \neq 0$ because the equation of motion for (3.1) is $D_\mu \partial^\mu \mathcal{C} = 0$ (and $\partial_\mu D^\mu \mathcal{C} \neq 0$)). Finally, the interpretation in terms of the translation along Grassmannian $\theta$-direction becomes problematic because of the fact that, even though $(\partial/\partial \theta)^2 = 0$, the internal transformation $\bar{s}_b$ under consideration is not a nilpotent transformation (i.e. $\bar{s}_b^2 \neq 0$). Similarly, it can be checked that, for the application of dual horizontality condition ($\tilde{\delta} \tilde{A} = \delta A$), we have the following expressions

\[
\begin{align*}
\star \tilde{A} &= \varepsilon^{\mu \nu}(dx_\nu)B_\mu(x, \theta) + (d\theta) \Phi(x, \theta), \\
\tilde{\delta} \tilde{A} &= - * \star \tilde{A} = \partial_\mu B^\mu + s^\theta \theta (\partial_\theta \Phi) - \varepsilon^{\mu \theta}(\partial_\mu \Phi + \varepsilon_{\mu \nu} \partial_\nu B^\nu), \\
\delta A &= - * d * A = \partial_\mu A^\mu,
\end{align*}
\]

where $s^\theta \theta$ and $\varepsilon^{\mu \theta}$ are defined in equation (2.23). The equality $\tilde{\delta} \tilde{A} = \delta A$ produces the following relationships between auxiliary fields and basic fields of (3.1)

\[
\partial_\theta \Phi = 0 \rightarrow B(x) = 0, \quad \partial_\mu \Phi + \varepsilon_{\mu \nu} \partial_\nu B^\nu = 0 \rightarrow \bar{R}_\mu = - \varepsilon_{\mu \nu} \partial_\nu \mathcal{C}.
\]

The additional restriction $\partial_\mu B^\mu = \partial_\mu A^\mu$ implies $\partial_\mu \bar{R}_\mu = 0$ which is trivially satisfied by $\bar{R}_\mu = - \varepsilon_{\mu \nu} \partial_\nu \mathcal{C}$.

The above dual horizontality condition does not fix $\mathcal{B}(x)$ in terms of the basic fields of (3.1). However, for the Lagrangian density (3.3), the equation of motion is: $\mathcal{B} = \mathcal{E}$. Thus, the expansion in (3.24) can be expressed in terms of a transformation $\bar{s}_d$ as

\[
\begin{align*}
B_\mu(x, \theta) &= A_\mu(x) - \theta \varepsilon_{\mu \nu} \partial^\nu \mathcal{C}(x) \equiv A_\mu(x) + \theta (\bar{s}_d A_\mu(x)), \\
\Phi(x, \theta) &= C(x) - i \theta (B(x) = 0) \equiv C(x) + \theta (\bar{s}_d C(x) = 0), \\
\Phi(x, \theta) &= C(x) + i \theta \mathcal{E}(x) \equiv C(x) + \theta (\bar{s}_d C(x)).
\end{align*}
\]

However, as it turns out, the transformations $\bar{s}_d$ are not the symmetry transformation for the Lagrangian density (3.1) (i.e. $\bar{s}_d\mathcal{L}^{(N)}_b \neq 0$, $\bar{s}_d\mathcal{L}^{(N)}_b \neq \partial_\mu Y^\mu$ where $Y^\mu$ stands for an expression in terms of some local fields). Furthermore, the transformation $\bar{s}_d$ is not an on-shell nilpotent symmetry transformation as is evident from $\bar{s}_d^2 \bar{\mathcal{C}} \sim D_\mu \partial^\mu \mathcal{C} \neq 0$. In fact, the equation of motion resulting from the Lagrangian density (3.1) is $\partial_\mu D^\mu \mathcal{C} = 0$ (and $D_\mu \partial^\mu \mathcal{C} \neq 0$). We conclude, therefore, that the on-shell nilpotent anti-BRST and anti-co-BRST symmetries do not exist for any of the Lagrangian densities quoted above for the 2D non-Abelian gauge theory. The same conclusion emerges even if we start with the anti-chiral limit of the expansion in (2.46). Thus, the exercise performed in this subsection provides, in some sense, a concrete proof of the reason behind the non-existence of the on-shell nilpotent version of the anti-BRST and anti-co-BRST symmetries for the 2D non-Abelian gauge theory. In fact, to the best of our knowledge, such on-shell nilpotent symmetries do not exist for the non-Abelian gauge theories in any arbitrary dimension of spacetime. As discussed in subsection 3.1, only the off-shell nilpotent version of the anti-BRST and anti-co-BRST symmetries exist for the self-interacting non-Abelian gauge theory in two-dimensions of spacetime.
3.4 Off-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries: general superfield formulation

We shall capture in this subsection some of the key points of our work [39] where the off-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries have been found in superfield formalism. We start off with the super expansion (2.46) but all the superfields $\tilde{A}_\alpha^a T^a(x, \theta, \bar{\theta}) = (B_\alpha^a T^a, \Phi^a T^a, \bar{\Phi}^a T^a)(x, \theta, \bar{\theta})$ as well as local fields (e.g. $A_\mu = A_\mu^a T^a, C = C^a T^a$ etc.) are group valued. It should be also noticed that the degrees of freedom of the fermionic (odd) fields $R_\mu, \tilde{R}_\mu, C, \bar{C}, s, \bar{s}$ match with that of the bosonic (even) fields $A_\mu, S_\mu, B, \bar{B}, \bar{B}$ so that the theory can be consistent with the basic requirements of supersymmetry. The horizontality restriction $\tilde{F} = \tilde{D} \tilde{A} = DA = F$ (where $\tilde{D} \tilde{A} = \tilde{d} \tilde{A} + \tilde{A} \wedge \tilde{A}$) so that the theory can be consistent with the bosonic $F$.

![Image](image.png)

where $S_\mu(x)$ can be equivalently written as: $S_\mu(x) = -D_\mu \tilde{B}(x) - (D_\mu \tilde{C} \times C)(x)$ and the individual terms in $\tilde{F} = \tilde{d} \tilde{A} + \tilde{A} \wedge \tilde{A}$ have been computed as

$$
\tilde{d} \tilde{A} = (dx^\mu \wedge dx^\nu)(\partial_\mu B_\nu) + (dx^\mu \wedge d\theta)(\partial_\mu \bar{\Phi} - \partial_\theta B_\mu) - (d\theta \wedge d\bar{\theta})(\partial_\theta \bar{\Phi})
+ (dx^\mu \wedge d\bar{\theta})(\partial_\mu \bar{\Phi} - \partial_\theta B_\mu) - (d\bar{\theta} \wedge d\theta)(\partial_\theta \Phi + \partial_\bar{\theta} \bar{\Phi}) - (d\theta \wedge d\bar{\theta})(\Phi \bar{\Phi}),
$$

and the auxiliary fields in (3.33) and basic fields, as

$$
B_\mu(x, \theta, \bar{\theta}) = A_\mu(x) + \theta \partial_\mu \bar{C}(x) + \bar{\theta} \partial_\theta \bar{C}(x) + i \theta \bar{\theta} (D_\mu B - i(D_\mu C \times C))(x),
$$
$$
\Phi(x, \theta, \bar{\theta}) = C(x) + i \theta \bar{B}(x) - \frac{1}{2} (C \times C)(x) + i \theta \bar{\theta} (\bar{B} \times C)(x),
$$
$$
\bar{\Phi}(x, \theta, \bar{\theta}) = \bar{C}(x) - \frac{1}{2} (\bar{C} \times \bar{C})(x) + i \theta B(x) - i \theta \bar{\theta} (B \times \bar{C})(x),
$$

which can be recast in exactly the same form as (2.44) with $s_{(a)b} \rightarrow \tilde{s}_{(a)b}$ and $\tilde{s}_{(a)b}$ standing for the off-shell nilpotent (anti-)BRST symmetries ((3.7) and (3.4)) for the non-Abelian gauge theories. This form of super expansion provides the geometrical interpretation for the off-shell nilpotent (anti-)BRST charges $Q_{(a)b}$ as the translation generators along the Grassmannian direction $(\theta)\bar{\theta}$ of the supermanifold. Similarly, the dual horizontality condition $\tilde{F} = \tilde{d} \tilde{A} = D \tilde{A} = F$ with $\tilde{D} \tilde{A} = \tilde{d} \tilde{A} + \tilde{A} \wedge \tilde{A}$ leads to the following relationships [39]

$$
R_\mu(x) = \partial_\mu B^\mu + s^{\theta\bar{\theta}}(\partial_\theta \bar{\Phi}) - e^{\theta\bar{\theta}}(\partial_\mu \Phi + \varepsilon_{\mu\nu} \partial_\nu B^\nu) + s^{\theta\bar{\theta}}(\partial_\bar{\theta} \bar{\Phi}),
$$

In the above computation, we have used the following

$$
\tilde{d} \tilde{A} = \partial_\mu B^\mu + s^{\theta\bar{\theta}}(\partial_\theta \bar{\Phi}) - e^{\theta\bar{\theta}}(\partial_\mu \Phi + \varepsilon_{\mu\nu} \partial_\nu B^\nu) + s^{\theta\bar{\theta}}(\partial_\bar{\theta} \bar{\Phi}),
$$

(3.37)
Now the super expansion, after the application of the dual horizontality condition, looks in terms of the (anti-)co-BRST transformations \( \tilde{s}_{(a)d} \) for non-Abelian gauge theory as

\[
B_\mu(x, \theta, \bar{\theta}) = A_\mu(x) - \theta \varepsilon_{\mu
u} \partial^\nu C(x) + \bar{\theta} \varepsilon_{\mu
u} \partial^\nu \bar{C}(x) + i \theta \varepsilon_{\mu
u} \partial^\nu \mathcal{B}(x),
\]

\[
\Phi(x, \theta, \bar{\theta}) = C(x) - i \theta \mathcal{B}(x) \equiv C(x) + \bar{\theta} (\tilde{s}_d C(x)),
\]

\[
\Phi(x, \theta, \bar{\theta}) = \bar{C}(x) + i \theta \mathcal{B}(x) \equiv \bar{C}(x) + \theta (\tilde{s}_d \bar{C}(x)),
\]

which can be, finally, recast in exactly the same form as (2.45). This happens here, unlike our earlier work [39], because of our choice of the signs in (2.46). It is very interesting to note that after the application of horizontality condition for the derivation of the (anti-)BRST symmetries, the odd superfield \( \Phi \) note that after the application of horizontality condition for the derivation of the (anti-)BRST symmetries, the odd superfield \( \Phi \) remain general (not (anti-)chiral) superfields. However, after the application of dual horizontality condition for the derivation of the (anti-)co-BRST symmetries, the odd super fields (\( \tilde{\Phi} \))\( \Phi \) become (anti-)chiral. It is also straightforward to check that

\[
\lim_{\theta, \bar{\theta} \to 0} \frac{\partial}{\partial \theta} A_M(x, \theta, \bar{\theta}) = -i [\Psi(x), Q_b]_\pm \equiv \tilde{s}_b \Psi(x),
\]

\[
\lim_{\theta, \bar{\theta} \to 0} \frac{\partial}{\partial \bar{\theta}} A_M(x, \theta, \bar{\theta}) = -i [\Psi(x), Q_d]_\pm \equiv \tilde{s}_d \Psi(x).
\]

which imply that there are mappings between (anti-)BRST and (anti-)co-BRST transformations of the local fields on the one hand and the translations of superfields along Grassmannian directions of the (2 + 2)-dimensional supermanifold on the other hand, as

\[
\tilde{s}_b \leftrightarrow \lim_{\theta, \bar{\theta} \to 0} \frac{\partial}{\partial \theta} \tilde{s}_ab \leftrightarrow \lim_{\theta, \bar{\theta} \to 0} \frac{\partial}{\partial \bar{\theta}} \tilde{s}_ad \leftrightarrow \lim_{\theta, \bar{\theta} \to 0} \frac{\partial}{\partial \bar{\theta}}.
\]

In spite the above similarity in the mapping, there is a clear-cut difference between \( \tilde{s}_b \) and \( \tilde{s}_d \) on the one hand and between \( \tilde{s}_ab \) and \( \tilde{s}_ad \) on the other hand as far as transformations on the (anti-)ghost fields (\( \bar{C} \))\( C \) are concerned. To check the sanctity of (3.40), it is straightforward to note that in (3.10) we have the following equality

\[
\tilde{s}_b \left(-i \bar{C} \cdot [\partial_\mu A^\mu + \frac{1}{2} \xi B]\right) = \lim_{\theta, \bar{\theta} \to 0} \frac{\partial}{\partial \theta} \left[-i \tilde{\Phi} \cdot \partial_\mu B^\mu + \frac{1}{2} \xi \tilde{\Phi} \cdot \frac{\partial \tilde{\Phi}}{\partial \theta}\right]|_{(\text{anti-})\text{BRST}},
\]

where the subscript stands for the expansion of the superfields in (3.35). Similarly, it can be seen that the BRST exact quantity in (3.11) is

\[
\tilde{s}_b \tilde{s}_b \left(\frac{i}{2} A_\mu \cdot A^\mu - \xi \bar{C} \cdot \bar{C}\right) = \frac{\partial}{\partial \bar{\theta}} \frac{\partial}{\partial \theta} \left(\frac{i}{2} B_\mu \cdot B^\mu - \xi \bar{\Phi} \cdot \bar{\Phi}\right)|_{(\text{anti-})\text{BRST}},
\]

and BRST co-exact- and anti-co-exact quantities of (3.11) are

\[
\tilde{s}_d \left(i C \cdot [\mathcal{E} - \frac{1}{2} \mathcal{B}]\right) = \lim_{\theta, \bar{\theta} \to 0} \frac{\partial}{\partial \theta} \left[-i \Phi \cdot \varepsilon^{\mu \nu} (\partial_\mu B_\nu + \frac{1}{2} B_\mu \times B_\nu) + \frac{1}{2} \Phi \cdot \frac{\partial \Phi}{\partial \theta}\right]|_{(\text{anti-})\text{co-BRST}}
\]

\[
\equiv \lim_{\theta, \bar{\theta} \to 0} \left[-i \Phi \cdot \varepsilon^{\mu \nu} (\partial_\mu B_\nu + \frac{1}{2} B_\mu \times B_\nu) - \frac{1}{2} \Phi \cdot \frac{\partial \Phi}{\partial \theta}\right]|_{(\text{anti-})\text{co-BRST}},
\]

\[
\equiv \lim_{\theta, \bar{\theta} \to 0} \left[-i \Phi \cdot \varepsilon^{\mu \nu} (\partial_\mu B_\nu + \frac{1}{2} B_\mu \times B_\nu) + \frac{1}{2} \Phi \cdot \frac{\partial \Phi}{\partial \theta}\right]|_{(\text{anti-})\text{co-BRST}}.
\]
\[
\tilde{s}_{ad} \left( -i \bar{C} \cdot [ E - \frac{1}{2} B ] \right) = \lim_{\theta, \bar{\theta} \to 0} \frac{\partial}{\partial \theta} \left[ +i \bar{\Phi} \cdot \varepsilon^{\mu \nu} (\partial_\mu B_\nu + \frac{1}{2} B_\mu \times B_\nu) 
- \frac{1}{2} \bar{\Phi} \cdot \frac{\partial \bar{\Phi}}{\partial \bar{\theta}} \right] \quad (\text{anti-)co-BRST}
\]
\[
\equiv \lim_{\theta, \bar{\theta} \to 0} \frac{\partial}{\partial \theta} \left[ +i \bar{\Phi} \cdot \varepsilon^{\mu \nu} (\partial_\mu B_\nu + \frac{1}{2} B_\mu \times B_\nu) 
+ \frac{1}{2} \Phi \cdot \frac{\partial \Phi}{\partial \theta} \right] \quad (\text{anti-)BRST},
\]

where the subscripts stand for the expansions in (3.38).

4 Topological aspects: superfield formulation

We deal here with the topological features of the free 2D Abelian- and self-interacting non-Abelian gauge theories in the framework of superfield formalism.

4.1 Topological features of free 2D Abelian theory: superfield approach

We explore here the possibilities of expressing some of the topological aspects of the free 2D Abelian gauge theory in terms of (anti-)chiral superfields as well as general superfields which have been responsible for our derivation of the on-shell and off-shell nilpotent symmetries. We also provide here geometrical interpretation for some of the topologically interesting quantities on the supermanifold. Let us first concentrate on the appearance of the Lagrangian density (2.1) (in its new form (2.7)) which respects the on-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries. With the help of the (anti-)chiral superfields, it can be checked that (2.7) can be expressed as

\[
\mathcal{L}_b + \partial_\mu X^\mu = - \frac{i}{2} \frac{\partial}{\partial \bar{\theta}} \left[ \{ (\varepsilon^{\mu \nu} \partial_\mu B_\nu) \bar{\Phi} \} |_{\text{co-BRST}} + \{ (\partial_\mu B^\mu) \Phi \} |_{\text{BRST}} \right],
\]
\[
\mathcal{L}_b + \partial_\mu X^\mu = + \frac{i}{2} \frac{\partial}{\partial \bar{\theta}} \left[ \{ (\varepsilon^{\mu \nu} \partial_\mu B_\nu) \bar{\Phi} \} |_{\text{anti-co-BRST}} + \{ (\partial_\mu B^\mu) \Phi \} |_{\text{anti-BRST}} \right],
\]

where the subscripts stand for the substitution of the super expansions in (2.19), (2.26), (2.33) and (2.40) and \( X^\mu = \frac{i}{2} (\partial^\mu \bar{C} + \bar{C} \partial^\mu C) \equiv \frac{i}{2} \partial^\mu (\bar{C} C) \). There is a symmetry that relates (4.1) and (4.2). In fact, these equations exchange with each other under: \( \theta \leftrightarrow -\bar{\theta}, \Phi \leftrightarrow \bar{\Phi} \) while the subscripts ‘BRST’ and ‘anti-BRST’ as well as ‘co-BRST’ and ‘anti-co-BRST’ are exchanged with each other. Mathematically, the above equation (4.1) implies that the Lagrangian density (2.1) is the \( \bar{\theta} \)-component of the composite chiral superfields \( (\varepsilon^{\mu \nu} \partial_\mu B_\nu) \bar{\Phi} \) and \( (\partial_\mu B^\mu) \Phi \) with the substitution of expansions derived after the application of (dual) horizontality conditions. The same (2.1) is also equivalent to the \( \theta \)-component of the similar kind of anti-chiral superfields (with the exchange of the fermionic superfields \( \Phi \leftrightarrow \bar{\Phi} \)) and for these superfields, the expansions are the ones derived after the application of (dual) horizontality conditions. In the language of the mapping: \( s_{(d)b} \leftrightarrow \partial/\partial \bar{\theta} \), it is clear that geometrically the Lagrangian density \( \mathcal{L}_b \) is equivalent to the sum of two terms that correspond to the translations of the composite chiral superfields \( (\varepsilon^{\mu \nu} \partial_\mu B_\nu) \Phi \) and \( (\partial_\mu B^\mu) \Phi \).
along the $\bar{\theta}$-direction of the supermanifold. This translation is generated by the on-shell nilpotent (co-)BRST charges. Similarly, in view of the mapping $s_{ab} \leftrightarrow \partial/\partial \theta$, $s_{ad} \leftrightarrow \partial/\partial \bar{\theta}$, it is straightforward to check that (2.1) is also equivalent to the translation of the anti-chiral composite superfields $(\varepsilon^{\mu \nu} \partial_{\mu} B_{\nu}) \Phi$ and $(\partial_{\mu} B^{\mu}) \Phi$ along the $\theta$-direction of the supermanifold. The corresponding translations are generated by the on-shell nilpotent anti-BRST and anti-co-BRST charges. There is another way to look geometrically at the Lagrangian density (2.1) in the light of super expansions in (2.43). In fact, in addition to (4.2) and (4.1), there is a completely new way to express (2.1)

$$\mathcal{L}_b + \partial_{\mu} X^\mu = + \frac{i}{4} \frac{\partial}{\partial \bar{\theta}} \frac{\partial}{\partial \theta} \left[ (B_{\mu} B^{\mu})(\text{anti-co-BRST}) + (B_{\mu} B^{\mu})(\text{anti-BRST}) \right],$$

because the super expansion for the bosonic field $B_{\mu}$, vis-a-vis (2.43), are

$$B_{\mu}(x, \theta, \bar{\theta})(\text{anti-co-BRST}) = A_{\mu}(x) - \theta \varepsilon_{\mu \nu} \partial^\nu C(x) - \bar{\theta} \varepsilon_{\mu \nu} \partial^\nu \bar{C}(x) + i \theta \bar{\theta} \varepsilon_{\mu \nu} \partial^\nu E(x),$$

$$B_{\mu}(x, \theta, \bar{\theta})(\text{anti-BRST}) = A_{\mu}(x) + \theta \partial_{\mu} \bar{C}(x) + \bar{\theta} \partial_{\mu} C(x) - \frac{i}{\xi} \bar{\theta} \theta \partial_{\mu}(\partial_{\mu} A^\mu)(x).$$

In the language of the (anti-)BRST $s_{(a)b}$ and (anti-)co-BRST $s_{(a)d}$ transformations on the basic fields of the Lagrangian density (2.1), the above superfield formulation implies

$$\mathcal{L}_b + \partial_{\mu} X^\mu = + \frac{i}{4} \left[ s_{b} s_{ab}(A_{\mu} A^\mu) + s_{d} s_{ad}(A_{\mu} A^\mu) \right],$$

where $X^\mu = \frac{1}{2} \left[ \frac{1}{2} A^\mu(\partial_{\mu} A^\mu) + \varepsilon^{\mu \nu} A_{\nu} E \right]$ for both (4.3) and (4.5). Thus, it can be emphasized, at this stage, that the superfield formulation provides a new insight into the topological nature of the Lagrangian density (2.1) as it can be expressed in a way that was not known hitherto in our previous works [29-35]. The equations (4.3) and (4.5) are endowed with symmetries that are inter-related. The invariance of (4.3) under $\theta \leftrightarrow -\bar{\theta}$ is reflected in the invariance of (4.5) under: $s_{b} \leftrightarrow -s_{ab}$, $s_{d} \leftrightarrow -s_{ad}$. Mathematically, the Lagrangian density (2.1) can be thought of as the $\theta \bar{\theta}$-component of the scalar superfield $(B_{\mu} B^{\mu})$ constructed by the bosonic superfields $B_{\mu}$ for which the expansions in (4.4) have to be plugged in. In the language of geometry on the supermanifold, the Lagrangian density (2.1) is equal to the sum of two terms that correspond to a couple of successive translations of the Lorentz superscalar $(B_{\mu} B^{\mu})$ along the $\theta$- and $\bar{\theta}$-directions of the supermanifold. These translations of the superscalar are generated by the (anti-)BRST and (anti-)co-BRST charges which ultimately correspond to the sum of on-shell nilpotent (anti-)BRST and (anti-)co-BRST transformations on the Lorentz scalar $(A_{\mu} A^\mu)$ as is evident from (4.5).

We shall concentrate now on the topological invariants (in particular the zero-forms) of the theory in the language of the superfield formulation. It can be checked that, for the choice of the (anti-)chiral superfields which lead to the derivation of the on-shell nilpotent (co-)BRST symmetries in (2.26) and (2.19) (and anti-BRST and anti-co-BRST symmetries in (2.33) and (2.40)), we have the following

$$(\Phi \bar{\Phi})|_{\text{BRST}} = C \bar{C} + \frac{i}{\xi} \bar{\theta} \theta C(\partial_{\mu} A^\mu), \quad (\Phi \bar{\Phi})|_{\text{anti-BRST}} = C \bar{C} + \frac{i}{\xi} \theta \bar{\theta} \bar{C}(\partial_{\mu} A^\mu),$$

$$(\Phi \bar{\Phi})|_{\text{co-BRST}} = C \bar{C} - i \theta \bar{\theta} E C, \quad (\Phi \bar{\Phi})|_{\text{anti-co-BRST}} = C \bar{C} - i \theta \bar{\theta} E C,$$

28
where the subscripts have the same meaning as explained earlier. From the above equations, it can be seen that we obtain the following zero-forms quoted in (2.12)

\[ \frac{i}{\partial \theta} (\Phi \bar{\Phi})|_{\text{BRST}} = -\frac{1}{\xi} (\partial_{\mu} A^{\mu}) C \equiv V_0, \quad \frac{i}{\partial \theta} (\Phi \bar{\Phi})|_{\text{co-BRST}} = E \bar{C} \equiv W_0, \]
\[ \frac{i}{\partial \bar{\theta}} (\Phi \bar{\Phi})|_{\text{anti-BRST}} = -\frac{1}{\xi} (\partial_{\mu} A^{\mu}) \bar{C} \equiv \bar{V}_0, \quad \frac{i}{\partial \bar{\theta}} (\Phi \bar{\Phi})|_{\text{anti-co-BRST}} = E \bar{C} \equiv \bar{W}_0. \]

(4.7)

In the language of transformations on the basic fields of (2.1), we have

\[ V_0 = s_0(iC \bar{C}), \quad \bar{V}_0 = s_{ad}(iC \bar{C}), \quad W_0 = s_d(iC \bar{C}), \quad \bar{W}_0 = s_{ad}(iC \bar{C}), \]

(4.8)

which clearly and explicitly show that \((\bar{V}_0)V_0\) are the (anti-)BRST invariant and \((\bar{W}_0)W_0\) are the (anti-)co-BRST invariant quantities due to the on-shell nilpotency \(s^2 = 0\) of these charges. In the language of the superspace variables, the above nilpotency is encoded in the nilpotency of the derivatives \((\partial / \partial \theta)^2 = 0, (\partial / \partial \bar{\theta})^2 = 0\) w.r.t. the Grassmannian variables \(\theta\) and \(\bar{\theta}\). The above zero-forms, that have been computed by taking into account separately (anti-)chiral superfields, can be obtained together by considering the super expansion (2.43) where (anti-)chiral superfields are merged together consistently. This expansion, after the application of (dual) horizontality conditions, yields the followings

\[ (\Phi \bar{\Phi})|_{\text{(anti-)co-BRST}} = CC - i\theta CE - i\bar{\theta} E\bar{C} - \theta \bar{\theta} E^2, \]
\[ (\Phi \bar{\Phi})|_{\text{(anti-)BRST}} = CC + \frac{i}{\xi} \theta (\partial_{\mu} A^{\mu}) \bar{C} + \frac{i}{\xi} \bar{\theta} (\partial_{\mu} A^{\mu}) C + \frac{1}{\xi^2} \theta \bar{\theta} (\partial_{\mu} A^{\mu})^2. \]

(4.9)

From the above expansions, we obtain the invariant quantities of (2.12), as

\[ \text{Lim}_{\theta, \bar{\theta} \rightarrow 0} i \frac{\partial}{\partial \theta} (\Phi \bar{\Phi})|_{\text{(anti-)BRST}} = -\frac{1}{\xi} (\partial_{\mu} A^{\mu}) C \equiv V_0, \]
\[ \text{Lim}_{\theta, \bar{\theta} \rightarrow 0} i \frac{\partial}{\partial \bar{\theta}} (\Phi \bar{\Phi})|_{\text{(anti-)BRST}} = -\frac{1}{\xi} (\partial_{\mu} A^{\mu}) \bar{C} \equiv \bar{V}_0, \]
\[ \text{Lim}_{\theta, \bar{\theta} \rightarrow 0} i \frac{\partial}{\partial \theta} (\Phi \bar{\Phi})|_{\text{(anti-)co-BRST}} = E \bar{C} \equiv W_0, \]
\[ \text{Lim}_{\theta, \bar{\theta} \rightarrow 0} i \frac{\partial}{\partial \bar{\theta}} (\Phi \bar{\Phi})|_{\text{(anti-)co-BRST}} = E \bar{C} \equiv \bar{W}_0. \]

(4.10)

It is clear from equations (4.7) and (4.10) that, mathematically, the zero-forms of the topological invariants w.r.t. on-shell nilpotent (anti-)BRST and (anti-)co-BRST charges are \((\theta)\bar{\theta}\) components of a bosonic superfield, constituted by merging a couple of fermionic superfields \(\Phi\) and \(\bar{\Phi}\) in the composite form \(\Phi \bar{\Phi}\). In the language of geometry on the supermanifold, the above zero-forms are nothing but the translations of the composite superfield \(\Phi \bar{\Phi}\) along the \((\theta)\bar{\theta}\)-directions of the supermanifold by the on-shell nilpotent (anti-)BRST and (anti-)co-BRST charges. As far as the topological invariants w.r.t. off-shell nilpotent (anti-)BRST and (anti-)co-BRST charges are concerned, we notice from (2.48) and (2.50)

\[ (\Phi \bar{\Phi})|_{\text{(anti-)co-BRST}} = CC - i\theta BC - i\bar{\theta} \bar{B} \bar{C} - \theta \bar{\theta} B^2, \]
\[ (\Phi \bar{\Phi})|_{\text{(anti-)BRST}} = CC - i\theta B \bar{C} - i\bar{\theta} BC + \theta \bar{\theta} B^2. \]

(4.11)
It is straightforward to check that the invariant quantities of (2.12) are

\[
\begin{align*}
\lim_{\theta, \bar{\theta} \to 0} i \frac{\partial}{\partial \theta} (\Phi \bar{\Phi}) \big|_{\text{(anti-)BRST}} &= BC \equiv V_0, \\
\lim_{\theta, \bar{\theta} \to 0} i \frac{\partial}{\partial \bar{\theta}} (\Phi \bar{\Phi}) \big|_{\text{(anti-)BRST}} &= B\bar{C} \equiv \bar{V}_0, \\
\lim_{\theta, \bar{\theta} \to 0} i \frac{\partial}{\partial \theta} (\Phi \bar{\Phi}) \big|_{\text{(anti-)co-BRST}} &= \bar{B}C \equiv W_0, \\
\lim_{\theta, \bar{\theta} \to 0} i \frac{\partial}{\partial \bar{\theta}} (\Phi \bar{\Phi}) \big|_{\text{(anti-)co-BRST}} &= BC \equiv \bar{W}_0.
\end{align*}
\]

(4.12)

The mathematical as well as the geometrical interpretations for the above zero-forms can be given in a similar manner as has been given for equations (4.7) and (4.8). Only the difference here is that the generators for the internal transformations ($\tilde{s}_{(a)b}, \tilde{s}_{(a)d}$) on the basic fields $\Psi$ of the Lagrangian density (2.3) and corresponding translation generators ($\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \bar{\theta}}$) on the supermanifold are the off-shell nilpotent (anti-)BRST and (anti-)co-BRST charges. It is very interesting to note that (i) the zero-form(s) of the topological invariants and the Lagrangian density (cf. (4.3)) correspond to the Grassmannian translations on the local but composite even superfields, and (ii) the composite even superfields ($\Phi \bar{\Phi}$) in the former and ($B^{\mu} B_{\mu}$) (i.e., a Lorentz scalar) in the latter are constituted by the odd scalar superfields and the even vector superfields of the theory.

We shall focus on now the symmetric energy momentum tensor ($T^{(s)}_{\mu\nu}$) for the above theory. In terms of the chiral superfield expansions in (2.19) and (2.26), we have the following form, modulo some total derivatives, for $T^{(s)}_{\mu\nu}$ in (2.9)

\[
T^{(s)}_{\mu\nu} = \frac{i}{2} \frac{\partial}{\partial \theta} \left[ P^{(s)}_{\mu\nu}\big|_{\text{BRST}} + Q^{(s)}_{\mu\nu}\big|_{\text{co-BRST}} \right],
\]

(4.13)

\[
P^{(s)}_{\mu\nu} = \partial_{\mu} \Phi B_{\nu} + \partial_{\nu} \Phi B_{\mu} + \eta_{\mu\nu} (\partial_{\rho} B^{\rho}) \Phi,
\]

\[
Q^{(s)}_{\mu\nu} = \partial_{\mu} \Phi \varepsilon_{\nu\rho} B^{\rho} + \partial_{\nu} \Phi \varepsilon_{\mu\rho} B^{\rho} + \eta_{\mu\nu} \varepsilon^{\rho\sigma} \partial_{\rho} B_{\sigma} \Phi.
\]

The above energy-momentum tensor, modulo some total derivatives, can also be expressed in terms of the anti-chiral super expansions in (2.33) and (2.40), as

\[
T^{(s)}_{\mu\nu} = -\frac{i}{2} \frac{\partial}{\partial \bar{\theta}} \left[ R^{(s)}_{\mu\nu}\big|_{\text{anti-BRST}} + S^{(s)}_{\mu\nu}\big|_{\text{anti-co-BRST}} \right],
\]

(4.14)

\[
R^{(s)}_{\mu\nu} = \partial_{\mu} \Phi B_{\nu} + \partial_{\nu} \Phi B_{\mu} + \eta_{\mu\nu} (\partial_{\rho} B^{\rho}) \Phi,
\]

\[
S^{(s)}_{\mu\nu} = \partial_{\mu} \Phi \varepsilon_{\nu\rho} B^{\rho} + \partial_{\nu} \Phi \varepsilon_{\mu\rho} B^{\rho} + \eta_{\mu\nu} \varepsilon^{\rho\sigma} \partial_{\rho} B_{\sigma} \Phi.
\]

Mathematically, the above equations (4.13) and (4.14) imply that, modulo some total derivatives, $T^{(s)}_{\mu\nu}$ corresponds to the $\theta(\bar{\theta})$ components of the sum of (anti-)chiral superfields $(R)P$ and $(S)Q$ respectively. Geometrically, the symmetric energy momentum tensor corresponds to the translations of the sum of (anti-)chiral superfields $(R)P$ and $(S)Q$, along the $\theta(\bar{\theta})$-directions of the $(2 + 2)$-dimensional supermanifold. These translations are generated by the on-shell nilpotent (anti-)BRST and (anti-)co-BRST charges. In view of the expansions in (2.43) where the (anti-)chiral super expansions have been merged together
in a consistent way, it can be checked that, besides expressions in (4.13) and (4.14), there is a novel way to express the symmetric energy momentum tensor $T_{\mu\nu}^{(s)}$ of (2.9), as

\begin{equation}
T_{\mu\nu}^{(s)} = \frac{i}{2} \frac{\partial}{\partial \bar{\theta}} \partial_{\bar{\theta}} \left[ (B_\mu B^\nu)(x, \theta, \bar{\theta}) - \frac{1}{2} \eta_{\mu\nu} (B_\rho B^\rho)(x, \theta, \bar{\theta}) \right] \bigg|_{\text{(anti-)BRST}} \tag{4.15}
\end{equation}

where the subscripts correspond to the modified expansion of (2.43) after the application of the (dual) horizontality conditions (see, e.g., Ref. [40] for more details). As far as the transformations on the basic fields of the Lagrangian density (2.1) are concerned, the above symmetric energy-momentum (4.15), modulo some total derivatives, can be expressed as

\begin{equation}
T_{\mu\nu}^{(s)} = \frac{i}{2} s_b s_{ab} (A_\mu A^\mu - \frac{1}{2} \eta_{\mu\nu} A_\rho A^\rho) - \frac{i}{2} s_d s_{ad} (\varepsilon_{\mu\rho\varepsilon_{\nu\sigma}} A^\rho A^\sigma + \frac{1}{2} \eta_{\mu\nu} A_\rho A^\rho), \tag{4.16}
\end{equation}

which encodes the topological nature of the theory in a grand and illuminating manner in view of the mapping $s_{(a)\bar{b}} \leftrightarrow Q_{(a)\bar{b}}$ and $s_{(a)d} \leftrightarrow Q_{(a)d}$. In other words, the above expression for $T_{\mu\nu}^{(s)}$ shows that it is equal to the sum of two (anti-)commutators. It is worth pointing out that in all our previous works [29-35], we were unable to express the energy momentum tensor in the form presented in (4.16). Thus, the study of the theory in the superfield formulation does lead to the discovery of some new symmetries of the $T_{\mu\nu}^{(s)}$. As explained for the equations (4.13) and (4.14), we can provide the mathematical as well as geometrical interpretation for the equation (4.15). Mathematically, the $\theta \bar{\theta}$-component of the superfields $B_\mu B_\nu - \frac{1}{2} \eta_{\mu\nu} B_\rho B^\rho$ and $\varepsilon_{\mu\rho\varepsilon_{\nu\sigma}} B_\rho B^\rho + \frac{1}{2} \eta_{\mu\nu} B_\rho B^\rho$ correspond to $T_{\mu\nu}^{(s)}$ of the theory when we substitute for the bosonic superfield $B_\mu$, the super expansions of (4.4) (that are obtained after the imposition of the (dual) horizontality conditions). In the language of the geometry on the supermanifold, the equation (4.15) implies that $T_{\mu\nu}^{(s)}$ is equivalent to a couple of successive translations of the superfields $B_\mu B_\nu - \frac{1}{2} \eta_{\mu\nu} B_\rho B^\rho$ and $\varepsilon_{\mu\rho\varepsilon_{\nu\sigma}} B_\rho B^\rho + \frac{1}{2} \eta_{\mu\nu} B_\rho B^\rho$ along $\theta$ and $\bar{\theta}$-directions of the supermanifold. These translations are generated by the on-shell nilpotent (anti-)BRST and (anti-)co-BRST charges.

### 4.2 Topological features of non-Abelian theory: superfield formalism

We shall discuss here some of the key topological features of the 2D self-interacting non-Abelian gauge theory. To start with, it is evident from (3.9) that the Lagrangian density (3.1) turns out to be the sum of two anti-commutators with the on-shell nilpotent (co-)BRST charges $Q_{(d)\bar{b}}$. This fact can be captured in terms of the chiral superfield expansions in (3.18) and (3.22) that have been obtained after the imposition of the (dual) horizontality conditions. In fact, in terms of the composite chiral superfields, the Lagrangian density (3.9) can be expressed, modulo some total derivative $\partial_\mu X^\mu$, as

\begin{equation}
L_{b}^{(N)} = -\frac{i}{2} \frac{\partial}{\partial \bar{\theta}} \left( \left[ (\partial_\mu B^\mu) \cdot \bar{\Phi} \right]_{\text{BRST}} + \left[ \varepsilon^{\mu\nu} (\partial_\mu B_\nu + \frac{1}{2} [B_\mu, B_\nu] \cdot \Phi) \right]_{\text{co-BRST}} \right), \tag{4.18}
\end{equation}
where $X^\mu = \frac{i}{2}(\bar{C} \cdot D^\mu C + \partial^\mu \bar{C} \cdot C)$ and subscripts correspond to the insertions of super expansions in (3.18) and (3.22) respectively. Mathematically, the Lagrangian density (3.1) (or its analogue (3.9)) is nothing but the $\bar{\theta}$ component of the local but composite chiral superfields $(\partial_{\mu} B^\mu) \cdot \bar{\Phi}$ and $\varepsilon^{\mu\nu}(\partial_{\mu} B_{\nu} + \frac{1}{2} B_{\mu} \times B_{\nu}) \cdot \Phi$ where the insertions from (3.18) and (3.22) are made. Recalling our result of the analogy $s_{(d)b} \leftrightarrow \partial / \partial \bar{\theta}$, it is very clear that (4.18) is nothing but (3.9) where the basic fields have been replaced by the corresponding superfields. In the language of geometry on the supermanifold, it can be noticed that the translation of the local (but composite) chiral superfields $(\partial_{\mu} B^\mu) \cdot \bar{\Phi}$ and $\varepsilon^{\mu\nu}(\partial_{\mu} B_{\nu} + \frac{1}{2} B_{\mu} \times B_{\nu}) \cdot \Phi$ along the $\bar{\theta}$-direction of the supermanifold corresponds to the sum of on-shell nilpotent transformations $s_b$ and $s_d$ on the local (but composite) fields $-\frac{1}{2} (\partial_{\mu} A^\mu) \cdot \bar{C}$ and $\frac{1}{2} E \cdot C$, respectively. This observation implies the topological nature of the theory. In fact, in the language of chiral superfields, the Lagrangian density (4.18) of the 2D self-interacting non-Abelian gauge theory is nothing but the total Grassmannian derivative w.r.t. $\bar{\theta}$ on the composite chiral superfields (cf. (4.18)). Here the physical states (as well as the vacuum) of the theory are supposed to be annihilated by the conserved and nilpotent (co-)BRST charges because they are the harmonic state of any arbitrary Hodge decomposed state of the quantum Hilbert space.

Let us now concentrate on the topological invariants of the theory. In particular, we shall provide the geometrical interpretation (in the language of translations on the supermanifold) for the on-shell $(D_{\mu} \bar{\partial}^\mu \bar{C} = \partial_{\mu} D^\mu C = 0)$ (co-)BRST invariant quantities connected with the zero-forms of the topological invariants, defined on the 2D manifold. The higher degree forms (i.e. one- and two-forms) can be computed from the zero-form by exploiting the recursion relations [35]. Towards this goal in mind, let us note the following

$$
\begin{align*}
(\Phi \cdot \bar{\Phi})|_{\text{BRST}} & = C \cdot \bar{C} + \frac{i}{2} \bar{\theta} C \cdot (\partial_{\mu} A^\mu) - \frac{1}{2} \bar{\theta} \bar{C} \cdot (C \times C), \\
(\Phi \cdot \bar{\Phi})|_{\text{co-BRST}} & = C \cdot \bar{C} - i \bar{\theta} E \cdot \bar{C},
\end{align*}
$$

(4.19)

where the subscripts stand for the insertions of the chiral superfield expansions in (3.18) and (3.22) that have been obtained after the imposition of the (dual) horizontality conditions. It is straightforward to check that the invariant quantities in (3.14) are

$$
\begin{align*}
i \frac{\partial}{\partial \bar{\theta}} (\Phi \cdot \bar{\Phi})|_{\text{BRST}} & = -\frac{1}{2} C \cdot (\partial_{\mu} A^\mu) - \frac{i}{2} \bar{C} \cdot (C \times C) \equiv V_0, \\
i \frac{\partial}{\partial \bar{\theta}} (\Phi \cdot \bar{\Phi})|_{\text{co-BRST}} & = E \cdot \bar{C} \equiv W_0,
\end{align*}
$$

(4.20)

which remain on-shell $(D_{\mu} \bar{\partial}^\mu \bar{C} = \partial_{\mu} D^\mu C = 0)$ BRST and co-BRST invariant, respectively. Mathematically, the $\theta$- and $\bar{\theta}$- components of the chiral superfields $(i\Phi \cdot \bar{\Phi})$ lead to the derivation of zero-forms which are the on-shell (co-)BRST invariant quantities $(W_0)V_0$. Geometrically, the zero-forms $(W_0)V_0$ w.r.t. the on-shell nilpotent (co-)BRST charges are nothing but the translations of the chiral superfields $(i\bar{\Phi} \cdot \Phi)$ along the $(\theta)\bar{\theta}$-directions of the supermanifold. For the computation of the off-shell invariant (anti-)BRST and (anti-)co-BRST quantities, we shall concentrate on the expansion (2.46) (where all the fields are
group valued e.g. \( A_\mu = A_\mu^a T^a \) etc.) and compute the analogues of (4.20) from the general expansions in (3.35) and (3.38). Thus, the invariant quantities in (3.15) are

\[
\begin{align*}
  i \frac{\partial}{\partial \bar{\theta}} (\Phi \cdot \bar{\Phi})|_{\text{(anti-)BRST}} &= B \cdot C - \frac{i}{2} C \cdot (C \times C) \equiv V_0, \\
  i \frac{\partial}{\partial \bar{\theta}} (\Phi \cdot \bar{\Phi})|_{\text{(anti-)co-BRST}} &= -\bar{B} \cdot \bar{C} + \frac{i}{2} \bar{C} \cdot (\bar{C} \times \bar{C}) \equiv \bar{V}_0, \\
  i \frac{\partial}{\partial \theta} (\Phi \cdot \bar{\Phi})|_{\text{(anti-)co-BRST}} &= B \cdot \bar{C} \equiv W_0, \\
  i \frac{\partial}{\partial \theta} (\Phi \cdot \bar{\Phi})|_{\text{(anti-)BRST}} &= B \cdot C \equiv \bar{W}_0.
\end{align*}
\] (4.21)

The mathematical as well as geometrical interpretations of the above zero-forms will go along the same lines as that for (4.20).

Now we dwell a bit on the form of the symmetric energy momentum tensor. In terms of the on-shell nilpotent (co-)BRST symmetries and corresponding chiral super expansion of the superfields in (3.18) and (3.22), it can be seen that

\[
T^{(N)}_{\alpha\beta} + X^{(N)}_{\alpha\beta} = i \frac{\partial}{2 \partial \theta} \left[ Y^{(N)}_{\alpha\beta} \right|_{\text{BRST}} + Z^{(N)}_{\alpha\beta} \right|_{\text{co-BRST}},
\]

\[
Y^{(N)}_{\alpha\beta} = \partial_\alpha \bar{\Phi} \cdot B_\beta + \partial_\beta \bar{\Phi} \cdot B_\alpha + \eta_{\alpha\beta} (\partial_\rho A^\rho) \cdot \bar{\Phi},
\]

\[
Z^{(N)}_{\alpha\beta} = \partial_\alpha \Phi \cdot \varepsilon_{\beta\rho} B^\rho + \partial_\beta \Phi \cdot \varepsilon_{\alpha\rho} B^\rho + \eta_{\alpha\beta} \varepsilon_{\rho\sigma} (\partial_\rho B_\sigma + \frac{1}{2} B_\rho \times B_\sigma) \cdot \Phi,
\] (4.22)

where the subscripts have the same meaning as explained earlier, and

\[
X^{(N)}_{\alpha\beta} = \frac{1}{2} \partial_\alpha \left[ \frac{1}{2} (\partial_\rho A^\rho) \cdot A_\beta + E \cdot \varepsilon_{\beta\rho} A^\rho \right] + \frac{1}{2} \partial_\beta \left[ \frac{1}{2} (\partial_\rho A^\rho) \cdot A_\alpha + E \cdot \varepsilon_{\alpha\rho} A^\rho \right] - \frac{1}{2} \eta_{\alpha\beta} \partial_\rho [\partial^\rho \bar{C} \cdot C + \bar{C} \cdot D^\rho C].
\] (4.23)

It is clear from (4.22) that the symmetric energy momentum tensor mathematically corresponds to the \( \bar{\theta} \)-component of the local (but composite) chiral superfields \( Y^{(N)}_{\alpha\beta} \) and \( Z^{(N)}_{\alpha\beta} \). Here we substitute for the superfields, the expansions in (3.35) and (3.38), which were derived after the application of (dual) horizontality conditions. In fact, equation (4.22) captures precisely the (anti-)commutators of (3.13) for the energy momentum tensor \( T^{(N)}_{\alpha\beta} \) in the language of chiral superfields. As far as the geometry on the supermanifold is concerned, it is obvious that the symmetric energy momentum tensor for the theory corresponds to the translations of the composite superfields \( Y^{(N)}_{\alpha\beta} \) and \( Z^{(N)}_{\alpha\beta} \) along the \( \bar{\theta} \)-direction of the supermanifold. This process of translation, generated by the (co-)BRST charges, corresponds to the (co-)BRST transformations \( s^{(a)}_{(d)b} \) on the ordinary (spacetime dependent) local (but composite) fields \( L^{(1)}_{\alpha\beta} \) and \( L^{(2)}_{\alpha\beta} \) of equation (3.13).

5 Conclusions

In the present investigation, we have elucidated the derivation of on-shell and off-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries for the 2D (non-)Abelian gauge theories
in the superfield formulation by exploiting the (dual) horizontality conditions w.r.t. the super cohomological operators $\tilde{d}$ and $\tilde{\delta}$, defined on the $(2+2)$-dimensional supermanifold. For the derivation of the on-shell nilpotent symmetries, we have invoked the (anti-)chiral superfields which turn out to be quite handy and helpful. In fact, we have derived a mapping between the translations generators $\partial/\partial \theta, \partial/\partial \bar{\theta}$ and the internal nilpotent transformations of the on-shell variety $s_{(a)b}, s_{(a)d}$ as well as the off-shell variety $\tilde{s}_{(a)b}, \tilde{s}_{(a)d}$ as

$$\frac{\partial}{\partial \bar{\theta}} \leftrightarrow s_{(d)b}, \quad \frac{\partial}{\partial \theta} \leftrightarrow s_{ab}, \quad \frac{\partial}{\partial \bar{\theta}} \leftrightarrow s_{ad},$$

$$\tilde{s}_{(d)b} \leftrightarrow \lim_{\theta, \bar{\theta} \to 0} \frac{\partial}{\partial \bar{\theta}}, \quad \tilde{s}_{ab} \leftrightarrow \lim_{\theta, \bar{\theta} \to 0} \frac{\partial}{\partial \theta}, \quad \tilde{s}_{ad} \leftrightarrow \lim_{\theta, \bar{\theta} \to 0} \frac{\partial}{\partial \theta}.$$ 

This mapping enables us to provide the geometrical interpretation for the nilpotent (anti-)BRST and (anti-)co-BRST charges (and transformations they generate), topological invariants (zero-forms), Lagrangian density and symmetric energy momentum tensor for the 2D (non-)Abelian gauge theories. In the language of superfield formulation, it turns out that the topological nature of the 2D free Abelian- and self-interacting non-Abelian gauge theories is encoded in the form of the Lagrangian density and symmetric energy momentum tensor which are found to be the total derivatives w.r.t. Grassmannian variables. In view of the above mapping, these physically and topologically interesting quantities are equal to the sum of (co-)BRST anti-commutators. This property establishes the fact that there are no energy excitations in the theory because $< \text{phys}|T^{00}|\text{phys}' > = 0$ due to the fact that $Q_{(d)b}|\text{phys} > = 0$. The nilpotency ($Q_{(a)b}^2 = Q_{(a)d}^2 = 0$) of the (anti-)BRST and (anti-)co-BRST charges is also encoded in the mapping (5.1) because $(\partial/\partial \bar{\theta})^2 = 0, (\partial/\partial \theta)^2 = 0$.

We have discussed in detail the geometrical aspects of the above topologically interesting quantities in the language of translations along some specific Grassmannian direction(s) of a four $(2+2)$-dimensional supermanifold.

One of the interesting features of our investigation is the observation (and its proof) that the (dual) horizontality conditions on the (anti-)chiral superfields lead to the derivation of the on-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries that co-exist for the Lagrangian density of a 2D free Abelian gauge theory. The same does not happen in the case of self-interacting 2D non-Abelian gauge theory. In fact, in some sense, the chiral superfield formulation provides an explanation for the non-existence of the on-shell nilpotent anti-BRST and anti-co-BRST symmetries for the Lagrangian density (3.1) which respects the on-shell nilpotent (co-)BRST symmetries. We have been able to establish the above fact by taking into account the anti-chiral superfield and showing that no logically consistent nilpotent symmetries emerge when we exploit the (dual) horizontality conditions. Furthermore, we have shown that (anti-)chiral superfields can merge consistently in the case of Abelian gauge theories and they shed some new light on the symmetries of the Lagrangian density and symmetric energy momentum tensor. For instance, it can be emphasized that the derivation of equations (4.5) and (4.16) from their superfield versions (4.3) and (4.15) is a completely new observation which was not seen, so far, in our earlier works [29,30,35]. This consistent merging of the (anti-)chiral superfields for the case of
the 2D non-Abelian gauge theory is impossible. This is the key reason behind the fact that the anti-chiral superfields, on their own, do not lead to any logically correct nilpotent symmetries even though we exploit the (dual) horizontality conditions correctly w.r.t. the super cohomological operators $\tilde{d}$ and $\tilde{\delta}$. Thus, the on-shell nilpotent anti-BRST and anti-co-BRST symmetries do not exist for the non-Abelian gauge theory. In other words, the Lagrangian density for the 2D self-interacting non-Abelian gauge theory that respects the on-shell nilpotent (co-)BRST symmetries does not respect the on-shell nilpotent anti-BRST and anti-co-BRST symmetries. For the derivation of the off-shell version of these symmetries one has to invoke the most general superfield expansions as in (2.46).

In our earlier works [29-35], we were able to obtain the analogues of the de Rham cohomology operators $(d, \delta, \Delta)$ in the language of symmetry properties (and corresponding generators) for a given Lagrangian density of a gauge theory. In fact, in our present paper, we considered the symmetries of the Lagrangian density for the 2D free- as well as self-interacting (non-)Abelian gauge theories and we demonstrated their existence in the (chiral) superfield formulation too. We found a two-to-one mapping: $Q_{b(ad)} \rightarrow d, Q_{d(ab)} \rightarrow \delta, \{Q_b, Q_d\} = \{Q_{ab}, Q_{ad}\} \rightarrow \Delta$ between the nilpotent and conserved charges of the theory and the de Rham cohomological operators. The algebra of these charges (cf. eqn. (2.6)) with the ghost charge is such that (see, e.g., Ref. [35])

$$
\begin{align*}
    iQ_g Q_b(ad)|\chi >_n &= (n + 1) Q_b(ad)|\chi >_n, \\
    iQ_g Q_d(ab)|\chi >_n &= (n - 1) Q_d(ab)|\chi >_n, \\
    iQ_g Q_w|\chi >_n &= (n) Q_w|\chi >_n,
\end{align*}
$$

(5.2)

where $|\chi >_n$ is an arbitrary state in the quantum Hilbert space with the ghost number $n$ (i.e. $iQ_g|\chi >_n = n|\chi >_n$). Thus, the analogue of the HDT (cf. (1.1)) can be easily defined in the quantum Hilbert space of states as given below [30,35]

$$
|\chi >_n = |\omega >_n + Q_{b(ad)}|\theta >_{n-1} + Q_{d(ab)}|\phi >_{n+1},
$$

(5.3)

where $|\omega >_n$ is the harmonic state (i.e. $Q_{b(ad)}|\omega >_n = 0, Q_{d(ab)}|\omega >_n = 0$), $Q_b|\theta >_{n-1}$ is a BRST exact state (which is equivalent to an anti-co-BRST exact state $Q_{ad}|\theta >_{n-1}$) and $Q_d|\phi >_{n+1}$ is a co-BRST exact state (which is equivalent to an anti-BRST exact state $Q_{ab}|\phi >_{n+1}$). However, these considerations do not explain the reason behind the existence of the two-to-one mapping which we discussed above. It is primarily the geometrical superfield approach to BRST formalism that clarifies the existence of a two-to-one mapping between the conserved charges and the cohomological operators. In fact, the (dual) horizontality conditions w.r.t. $\tilde{d}$ and $\tilde{\delta}$ imply that $\tilde{d} \rightarrow Q_{(a)b}, \tilde{\delta} \rightarrow Q_{(a)d}$. The ghost number consideration (cf. (5.2)), however, clinches the issue and establishes the fact that there is a two-to-one mapping between (anti-)BRST and (anti-)co-BRST charges on the one hand and (co-)exterior derivatives of the differential geometry on the other hand as: $Q_{b(ad)} \rightarrow d, Q_{d(ab)} \rightarrow \delta$. It will be noticed that we did not lay any emphasis on the (super) Laplacian operators $(\tilde{\Delta})\Delta$ in our present paper. This is because of the fact that the symmetry generated by this operator is derivable from the nilpotent symmetries generated by
the (super) cohomological operators $(\tilde{d})d$ and $(\tilde{\delta})\delta$. Furthermore, geometrically, the (super) \Laplacian operators $(\tilde{\Delta})\Delta$ do not lead to any impressive results. For the 2D free Abelian gauge theory, this exercise with $(\tilde{\Delta})\Delta$ was performed (together with the analogue of the horizontality condition w.r.t. these operators) in [38]. This attempt, with the help of certain specific discrete symmetries of the theory, led to the derivation of a bosonic symmetry only for the Lagrangian density (2.1) \textit{but not for (2.3)}.

We concentrated, in our present work, only on the 2D free Abelian- and self-interacting non-Abelian (one-form) gauge theories. However, our hope is to extend our understanding and insight of 2D one-form gauge theories to the physical 4D two-form gauge theories where we have been able to show the existence of (anti-)BRST and (anti-)co-BRST symmetries in the Lagrangian formulation [34]. The study of the geometrical aspects of the 4D two-form gauge theories in the superfield formulation (together with the understanding of the new symmetries) might turn out to be useful in the context of (super)string theories where one considers a non-trivial (spacetime dependent) metric. The analogue of the new symmetries might shed some new light on the proof of renormalizability of the interacting two-form gauge theories where matter fields are coupled to the two-form potentials. Another direction that can be pursued, following the approach adopted in [51], is to try for the superfield formulation of the Batalin-Vilkovisky formalism applied to the one-form and two-form gauge theories with an extended set of BRST operators. This extended set will include, in addition to the nilpotent (anti-)BRST charges [51], the nilpotent (anti-)co-BRST charges and a bosonic conserved charge, too. The discussion of the new dual BRST symmetries in the context of three dimensional BF system (see, e.g., Ref. [52] for details) and its application to the study of topological properties of this system, etc., are yet another few novel directions that can be taken up for considerations as far as our future endeavours connected with the superfield formalism are concerned. We have enumerated here some of the issues and pointed out a few directions that are under investigation and our results would be reported in our future publications [53].

References

[1] P. A. M. Dirac, \textit{Lectures on Quantum Mechanics}, (Yeshiva University Press, New York, 1964).

[2] For a review, see, e.g., K. Sundermeyer, \textit{Constrained Dynamics (Lecture Notes in Physics)}, Vol. 169, (Springer-Verlag, Berlin, 1982).

[3] A. S. Schwarz, Lett. Math. Phys. 2 (1978) 247.

[4] E. Witten, Commun. Math. Phys. 117 (1988) 353; E. Witten, Commun. Math. Phys. 121 (1989) 351.

[5] For a review, see, e.g., D. Bermingham, M. Blau, M. Rakowski, G. Thompson, Phys. Rep. 20 (1991) 129.

[6] R. Gopakumar, C. Vafa, \textit{M-Theory and Topological Strings-I}, hep-th/9809187.
[7] R. Gopakumar, C. Vafa, *M-Theory and Topological Strings-II*, hep-th/9812127.

[8] M. B. Green, J. H. Schwarz, E. Witten, *Superstring Theory*, Vols. 1 and 2, (Cambridge University Press, Cambridge, 1987), J. Polchinski, *String Theory*, Vols. 1 and 2, (Cambridge University Press, 1998), C. Vafa, *Lectures on strings and dualities*, hep-th/9702201, J. Maharana, *Recent developments in string theory*, hep-th/9911200.

[9] I. A. Batalin, I. V. Tyutin, *Int. J. Mod. Phys. A* 6 (1991) 3255.

[10] See, e.g., J. Gomis, J. Paris, S. Samuel, *Phys. Rep.* 259 (1995) 1.

[11] E. Witten, *Mod. Phys. Lett. A* 5 (1990) 487.

[12] K. Nishijima, *Prog. Theo. Phys.* 80 (1988) 897; 905.

[13] H. Aratyn, *J. Math. Phys.* 31 (1990) 1240.

[14] J. W. van Holten, *Phys. Rev. Lett.* 64 (1990) 2863, J. W. van Holten, *Nucl. Phys. B* 339 (1990) 258.

[15] J. Thierry-Mieg, *J. Math. Phys.* 21 (1980) 2834; *Nuovo Cimento* 56 A (1980) 396.

[16] M. Quiros, F. J. De Urries, J. Hoyos, M. L. Mazou, E. Rodrigues, *J. Math. Phys.* 22 (1981) 767.

[17] L. Bonora, M. Tonin, *Phys. Lett. B* 98 (1981) 48; L. Bonora, P. Pasti, M. Tonin, *Nuovo Cimento* 63 A (1981) 353.

[18] L. Baulieu, J. Thierry-Mieg, *Nucl. Phys. B* 197 (1982) 477, L. Baulieu, J. Thierry-Mieg, *Nucl. Phys. B* 228, 259 (1982) 259, L. Alvarez-Gaumé, L. Baulieu, *Nucl. Phys. B* 212 (1983) 255.

[19] D. S. Huang, C.-Y. Lee, *J. Math. Phys.* 38 (1997) 30.

[20] T. Eguchi, P. B. Gilkey, A. J. Hanson, *Phys. Rep.* 66 (1980) 213.

[21] S. Mukhi, N. Mukunda, *Introduction to Topology, Differential Geometry and Group Theory for Physicists*, (Wiley Eastern Pvt. Ltd., New Delhi, 1990).

[22] W. Kalau, J. W. van Holten, *Nucl. Phys. B* 361 (1991) 233.

[23] G. Fülöp, R. Marnelius, *Nucl. Phys.* 456 (1995) 442.

[24] D. McMullan, M. Lavelle, *Phys. Rev. Lett.* 71 (1993) 3757, D. McMulan, M. Lavelle, *Phys. Rev. Lett.* 75 (1995) 4151.

[25] V. O. Rivelles, *Phys. Rev. Lett.* 75 (1995) 4150, V. O. Rivelles, *Phys. Rev. D* 53 (1996) 3257.

[26] R. Marnelius, *Nucl. Phys. B* 494 (1997) 346.

[27] H. S. Yang, B. -H. Lee, *J. Math. Phys.* 37 (1996) 6106.

[28] T. Zhong, D. Finkelstein, *Phys. Rev. Lett.* 73 (1994) 3055, T. Zhong, D. Finkelstein, *Phys. Rev. Lett.* 75 (1995) 4152.

[29] R. P. Malik, *J. Phys. A: Math. Gen.* 33 (2000) 2437, hep-th/9902146.
[30] R. P. Malik, Int. J. Mod. Phys. A 15, 1685 (2000) 1685, hep-th/9808040.
[31] R. P Malik, Mod. Phys. Lett. A 14 (1999) 1937, hep-th/9903121.
[32] R. P. Malik, Mod. Phys. Lett. A 15 (2000) 2079, hep-th/0003128.
[33] R. P. Malik, Mod. Phys. Lett. A 16 (2001) 477, hep-th/9711056.
[34] E. Harikumar, R. P. Malik, M. Sivakumar, J. Phys. A: Math. Gen. 33 (2000) 7149, hep-th/0004145.
[35] R. P. Malik, J. Phys. A: Math. Gen. 34 (2001) 4167, hep-th/0012085.
[36] N. Nakanishi, I. Ojima, Covariant Operator Formalism of Gauge Theories and Quantum Gravity (World Scientific, Singapore, 1990).
[37] R. P. Malik, Phys. Lett. B 521 (2001) 409, hep-th/0108105.
[38] R. P. Malik, J. Phys. A: Math Gen 35 (2002) 3711, hep-th/0106215.
[39] R. P. Malik, Mod. Phys. Lett. A 17 (2002) 185, hep-th/0111253.
[40] R. P. Malik, J. Phys. A: Math Gen 35 (2002) 6919, hep-th/0112260.
[41] R. P. Malik, J. Phys. A: Math Gen 35 (2002) 8817, hep-th/0204015.
[42] R. P. Malik, Abelian two-form gauge theory: special features, hep-th/0209136.
[43] K. Nishijima, in: Progress in Quantum Field Theory, Eds. H. Ezawa and S. Kamefuchi (North-Holland, Amsterdam, 1986).
[44] M. Henneaux, C. Teitelboim, Quantization of Gauge Systems (Princeton University Press, New Jersey, Princeton, 1992).
[45] D. M. Gitman, I. V. Tyutin, Quantization of Fields with Constraints (Springer, 1990).
[46] S. Weinberg, The Quantum Theory of Fields: Modern Applications, Vol. 2 (Cambridge University Press, Cambridge, 1996).
[47] J. Soda, Phys. Lett. B 267 (1991) 214, M. Abe, N. Nakanishi, Prog. Theor. Phys. 89 (1993) 501.
[48] A. Hosoya, J. Soda, Mod. Phys. Lett. A 4 (1989) 2539.
[49] See, e.g., I. J. R. Aitchison, A. J. G. Hey, Gauge Theories in Particle Physics: A Practical Introduction, (Bristol, 1982).
[50] G. Curci, R. Ferrari, Phys. Lett. B 63 (1976) 51, G. Curci, R. Ferrari, Nuovo Cimento A 32 (1976) 151.
[51] N. R. F. Braga, S. M. de Souza, Phys. Rev. D 53 (1996) 916.
[52] N. Maggiore, S. P. Sorella, Nucl. Phys. B 377 (1992) 236.
[53] R. P. Malik, in preparation.