Inversion of the Indefinite Double Covering Map

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Abstract

Algorithmic methods for the explicit inversion of the indefinite double covering maps are proposed. These are based on either the Givens decomposition or the polar decomposition of the given matrix in the proper, indefinite orthogonal group $SO^+(p, q)$. As a by-product we establish that the preimage in the covering group, of a positive matrix in $SO^+(p, q)$, can always be chosen to be itself positive definite. Inversion amounts to solving a polynomial system. These methods solve this system by either inspection, Groebner bases or by inverting the associated Lie algebra isomorphism and computing certain exponentials explicitly. The techniques are illustrated for $(p, q) \in \{(2, 1), (2, 2), (3, 2), (4, 1)\}$.

**Keywords:** Bivectors, Givens decomposition, polar decomposition, indefinite orthogonal groups, spin groups
1 Introduction

The double covers of the definite and indefinite orthogonal groups, $SO^+(p, q)$ by the spin groups are venerable objects. They arise in a variety of applications. The covering of $SO(3)$ by $SU(2)$ is central to robotics. The group $SO^+(2,1)$ arises in polarization optics, [4]. The group $SO^+(3,2)$ arises as the dynamical symmetry group of the 2D hydrogen atom and also in the analysis of the DeSitter space time [7, 11]. The group $SO^+(4,1)$ arises as the versor representation of the three dimensional conformal group and thus has applications in robotics and computer vision, [13].

In this work we provide explicit algorithms to invert these double covering maps. More precisely, given the double covering map, $\Phi_{p,q} : \text{Spin}^+(p,q) \to SO^+(p,q)$ and an $X \in SO^+(p,q)$ (henceforth called the target), we provide algorithms to compute the matrices $\pm Y$, in the matrix algebra that the even subalgebra of $\text{Cl}(p,q)$ is isomorphic to, satisfying $\Phi_{p,q}(\pm Y) = X$.

One of our methods which works for all $(p,q)$, described in Remark (32), finds the matrices in the preimage when $\text{Spin}^+(p,q)$ is viewed as living in the matrix algebra that $\text{Cl}(p,q)$ is isomorphic to and this method trivially extends to the inversion of the abstract double covering map. Our other methods, aid in finding the preimage when $\text{Spin}^+(p,q)$ is viewed as living in the matrix algebra that the even sublagebra of $\text{Cl}(p,q)$ is isomorphic to. Since the even subalgebra typically consists of matrices of lower size than $\text{Cl}(p,q)$ itself, such results are of great interest. Naturally, such methods use the detailed knowledge of the matrix forms of the various Clifford theoretic automorphisms (such as grade, reversion), [8].

Our interest in finding $Y$ as a matrix, as opposed to an abstract element of $\text{Cl}(p,q)$, stems from wishing to use specific matrix theoretic properties of $Y$ (respectively $X$), together with an explicit matrix form of the direct map $\Phi_{p,q}$, to infer similar properties for $X$ (respectively $Y$). A classic example of this is the usage of the unit quaternions (equivalently $SU(2)$) to find axis-angle representations of $SO(3)$. In the same vein to compute the polar decomposition of a matrix in $SO^+(3,2)$, it is easier to find that of its preimage in the corresponding spin group, $Sp(4,\mathbb{R})$ and then project both factors via $\Phi_{3,2}$. It is easy to show that the projected factors constitute the polar decomposition of the original matrix. Since the polar decomposition of a $4 \times 4$ symplectic matrix can even be found in closed form, [3], this is indeed a viable strategy.
Similarly, one can find the polar decomposition of a matrix $X$ in $SO^+(4,1)$ in closed form, [2] and this can be then used to find the polar decomposition of $Y \in Spin^+(4,1)$. Since $Y$ is a $2 \times 2$ matrix with quaternionic entries and there is no method available for its polar decomposition, [15] (barring finding the polar decomposition of an associated $4 \times 4$ complex matrix), this is indeed useful. Similarly, block properties of the preimage, $Y \in Spin^+(p,q)$, viewed as a matrix, may provide useful information about $X$. Such information is typically unavailable when finding $Y$ only as an abstract element of $Cl(p,q)$. Furthermore, one of the methods to be used in this work, viz., the inversion of the linearization of the covering map, may be used to also compute exponentials in the Lie algebras $so(p,q)$.

There is literature on the problem being considered in this work. For the case $(p,q) = (0,3)$ [or $(3,0)$] this problem is classical. The case $(0,4)$ is considered in [5]. The cases $(0,5)$ and $(0,6)$ are treated in [6]. The excellent work of [16] treats the general case with extensions to the Pin groups, under a genericity assumption, but finds the preimage in the abstract Clifford algebra via a formula requiring the computation of several determinants. Section 1.1 below provides a detailed discussion of the relation between the present work and [6, 16]. In [17] an algorithm is proposed, for the inversion of the abstract covering map, but which requires the solution of a nonlinear equation in several variables for which no techniques seem to be presented.

**Remark 1** It is worth noting that even though the abstract map $\Phi_{p,q}$ is uniquely defined, as is the matrix algebra that $Cl(p,q)$ is isomorphic to, the matrix form of $\Phi_{p,q}$ very much depends on the matrices chosen to represent the one-vectors of $Cl(p,q)$. Similar statements apply to the explicit matrix forms of reversion and Clifford conjugation. Indeed, even the group of matrices which constitute $Spin^+(p,q)$ (within the matrix algebra that the even sublagebra of $Cl(p,q)$ is isomorphic to) can change with the choice of one-vectors - though, of course, all these groups are isomorphic to each other. Thus, each abstract $\Phi_{p,q}$, in reality, determines many “matrix”, $\Phi_{p,q}$’s. Thus, our techniques are especially useful if one specific matrix form of $\Phi_{p,q}$ has been chosen in advance. This is illustrated via Example 3 later in this section.

With this understood, we will write $\Phi_{p,q}$ for the covering map, with respect to the choice of one-vectors specified in advance. Then $\Phi_{p,q}(Y)$ is a quadratic map in the entries of $Y \in$
Spin\(^+\)(p, q). Given the target \(X \in \text{SO}^+(p, q)\), we have to solve this quadratic system of equations in several variables to obtain \(Y\). Solving this system is arduous in general. If \(X\) can be factored as a product of matrices \(X_j\), then it is conceivable that these systems become simpler for the individual \(X_j\). Then using the fact that \(\Phi_{p,q}\) is a group homomorphism, one can synthesize the solution for \(X\) out of those for the \(X_j\)'s. This is the technique employed in our earlier work, [6].

One standard choice of these simpler factors are standard Givens and hyperbolic Givens rotations. These are matrices which are the identity except in a \(2 \times 2\) principal submatrix, where they are either a plane rotation or a hyperbolic rotation. This decomposition of \(X\) is constructive and thus leads to a complete algorithmic procedure to find \(Y\). For this it is important that these Givens factors also belong to \(\text{SO}^+(p, q)\). When \(pq = 0\), this is straightforward to show. The general case is less obvious [since not all matrices of determinant 1 in \(\text{SO}(p, q)\) belong to \(\text{SO}^+(p, q)\)] and is demonstrated in Section 2 by displaying these Givens factors as exponentials of matrices in the Lie algebra \(\text{so}(p, q)\) (see Proposition 19 in Section 2).

As a second variant, we also invoke the polar decomposition for some \((p, q)\) to invert \(\Phi_{p,q}\). This is aided by the remarkable fact that for certain \((p, q)\) (e.g., \((p, q) \in \{(2, 1), (3, 1)\}\)) \(\Phi_{p,q}\) is inverted, essentially by inspection, when the target in \(\text{SO}^+(p, q)\) is positive definite or special orthogonal. This circumstance may be used to not only invert \(\Phi_{p,q}\) but also compute the polar decomposition of \(X\) (or for that matter that of the preimage \(\pm Y\)) without any eigencalculations. Once again, it is important to know that both factors in the polar decomposition of an \(X \in \text{SO}^+(p, q)\) belong to \(\text{SO}^+(p, q)\). It is known that for matrices in groups preserving the bilinear form defined by \(I_{p,q}\), both factors in the polar decomposition do belong to the group also. However, it does not immediately follow that the same holds for matrices in \(\text{SO}^+(p, q)\). Once again the issue at hand is that \(\text{SO}(p, q)\) is not equal to \(\text{SO}^+(p, q)\). Inspite of this, as shown in [2], both factors in the polar decomposition of a matrix in \(\text{SO}^+(p, q)\) also belong to \(\text{SO}^+(p, q)\).

A related method to invert \(\Phi_{p,q}\) is considered in this work. Namely we invert instead the linearization \(\Psi_{p,q}\) of \(\Phi_{p,q}\). The map \(\Psi_{p,q}\) is easily inverted since it is linear. For this method to be viable, however, one has to first have an explicit formula for the logarithm (within \(\text{so}(p, q)\))
of $X$ or the $X_j$. Next, one has also to be able to compute the exponential of this preimage. In Remark 32 we show that both of these steps go through, when the target is decomposed via Givens decomposition regardless of the value of $(p, q)$.

In the interests of brevity, we illustrate only the cases $(p, q) \in \{(2, 1), (2, 2), (3, 2), (4, 1)\}$. The $(2, 1)$ case arises in polarization optics, for instance. The $(2, 2)$ case is the simplest non-trivial case of the split orthogonal group. The $(3, 2)$ case is of importance in the study of the hydrogen atom among other things. The $(4, 1)$ case is important in computer graphics.

Specifically, we will address

- The $(2, 1)$ case by inspection and by invoking the polar decomposition connection aforementioned. The important $(3, 1)$ case is also easily amenable to this method, but surprisingly $(2, 2)$ is not, [2].

- The $(2, 2)$ and $(3, 2)$ cases via Groebner bases.

- The $(4, 1)$ case via linearizing $\Phi_{4,1}$. We will provide results both when $X$ is decomposed using Givens factors and when it is decomposed using polar decomposition.

### 1.1 Relation to Work in the Literature

In this section the relation of the present work with [6, 16] is discussed.

The relation between [6] and the present work is as follows. In [6] the concern is with inversion for the $(0, 5)$ and $(0, 6)$ cases. Since the polar decomposition of an orthogonal matrix is trivial, it does not help at all with the task of inversion. On the other hand, in this work it plays a significant role precisely because the polar decomposition for the $(p, q), pq \neq 0$ case is no longer trivial. Therefore, when the inversion of a positive definite target in $SO^+(p, q)$ can be carried out efficiently, it becomes even more useful than when the target is an ordinary or hyperbolic Givens matrix, since the number of Givens factor grows rapidly with $n = p + q$.

Next the relation to [16] is discussed. Significant portions of the present work were completed in late 2016, [1]. As this paper was being written up, we became aware of the 2019 paper [16]. In [16], an elegant solution is provided for inverting the abstract (as opposed to the matrix) map $\Phi_{p,q}$, under a generic condition on $X$, not required by our work. The solution in [16] is a generalization of a method proposed in [9] for the $(3, 1)$ case. This formula is as follows. Let
$X \in SO^+(p,q)$ and define the element $M$ of $Cl(p,q)$ via

$$M = \sum_{\alpha,\beta} \det(X_{\alpha,\beta})e_{\alpha}(e_{\beta})^{-1}.$$ 

Here $\alpha = \{i_1, \ldots, i_k\}$ and $\beta = \{j_1, \ldots, j_k\}$ are subsets of $\{1, 2, \ldots, n\}$ of equal cardinality (including the empty set), $X_{\alpha,\beta}$ is the square submatrix of $X$ located in rows indexed by $\alpha$ and columns indexed by $\beta$, $e_{\alpha} = e_{i_1}e_{i_2}\ldots e_{i_k}$ and similarly for $e_{\beta}$. Here $e_l$ is the $l$-th basis one-vector in the abstract Clifford algebra $Cl(p,q)$. $(e_{\beta})^{-1}$ is the inverse of $e_{\beta}$ in $Cl(p,q)$.

It is assumed that $MM^{rev} \neq 0$ in $Cl(p,q)$. This is the aforementioned genericity assumption. Then $\Phi_{p,q}(\pm Y) = X$, where

$$Y = \frac{M}{\sqrt{MM^{rev}}}.$$ 

Here $M^{rev}$ is the reversion of $M$.

**Remark 2** The following nuances, besides the genericity condition $MM^{rev} \neq 0$, of this formula need attention

i) The principal burden in implementing the formula in [16] is that one has to compute all the minors of $X$. If we ally this formula with one innovation of the current work, namely decomposing $X$ into hyperbolic and usual Givens rotations, then a significant reduction in complexity in implementing the formula in [16] can be expected. Indeed, the number of non-zero minors of a hyperbolic or standard Givens is much lower than that for a general $X$. However, from many viewpoints, it is still emphatically not true that if $X$ is a Givens rotation then only principal minors are non-zero, and thus still several determinants have to be computed. For instance, consider

$$X = \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ b & 0 & a \end{pmatrix}, a^2 - b^2 = 1.$$ 

Thus $X$ is a hyperbolic Givens rotation in $SO^+(2,1)$. Then, for instance, the following non-principal $2 \times 2$ minors are non-zero: $\{(1,2), (2,3)\}, \{(2,3), (1,2)\}$. Hence, even when the target $X$ is sparse, such as a Givens matrix, one has to calculate several determinants.

ii) Furthermore, due to the involvement of several determinants, the formula obtained for
Y often is quite elaborate and occludes the “half-angle” nature of the inversion even when X is simple - see Example 3 below for an illustration of this issue.

iii) The formula only finds Y as an element of the abstract Clifford algebra Cl(p, q). Our methods also provide such an inversion, without the need for determinants, but by using Givens decompositions - see Remark 32. Of course, by using specific matrices constituting a basis of one-vectors for Cl(p, q), Y can be recovered as a matrix. The matrix Y thereby obtained, even though an even vector, will live in Cl(p, q) which is typically an algebra of matrices of larger size than the matrix algebra constituting the even subalgebra. This is due to the very nature of the formula. Thus, for instance this formula will yield, for the case (p, q) = (3, 2), Y as a 8 × 8 matrix, even though the covering group consists of symplectic matrices of size 4. To get around this one has to know how to embed the even subalgebra in Cl(p, q), [8]. In effect, one has to compute the matrix form of the grade involution. This limitation is thus due to not having at one’s disposal an explicit matrix form for \( \Phi_{p,q} \), when using the formula in [16].

iv) Next the matrix form of ±Y very much depends on the basis of one-vectors. Without this caveat, one can find different find different matrices, Y, with ±Y projecting to the same X. This matter is illustrated in Example 3.

v) Other steps in this method such as finding the reversion of M can, in principle, be performed without having to resort to finding reversion as an explicit automorphism of the matrix algebra that Cl(p, q) is isomorphic to. However, as \( p + q \) grows, it is more convenient to work with a concrete matrix form of reversion, such as those in [8]). Indeed \( MM^{rev} \) is proportional to the identity and thus, if a matrix form of M (and \( M^{rev} \)) is available, then one needs to only compute the trace of \( MM^{rev} \). These issues are all mitigated when the methods being proposed here are used, since our methods make systematic use of the structure of the matrix form of the map \( \Phi_{p,q} \), whereas this is not the case in [16].

Example 3 Consider \( \Phi_{1,1} \). Its inversion, is of course, trivial. However, it illustrates Remark 1 and also the caveats ii) and iv) above about the usage of the formula in [16].

Let us use the basis \( B_1 = \{ \sigma_x, i\sigma_y \} \) for the one-vectors of Cl(1, 1) \( \simeq M(2, \mathbb{R}) \). Incidentally,
this is the canonical basis that the constructive proof of the isomorphism between Cl\((p+1, q+1)\) and \(M(2, \text{Cl}(p, q))\) naturally yields. Then \(\text{Spin}^+(1, 1)\) is realized as

\[
\text{Spin}^+(1, 1) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} : \alpha \neq 0 \right\}.
\]

(1)

Now

\[
\Phi_{1,1}\left( \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} \right) = \frac{1}{2} \left( \begin{pmatrix} \frac{a^2+1}{a^2-1} & \frac{a^2-1}{a^2+1} \\ \frac{a^2-1}{a^2+1} & \frac{a^2+1}{a^2-1} \end{pmatrix} \right).
\]

Let \(X = \begin{pmatrix} a & b \\ b & a \end{pmatrix}\) be a target matrix is \(SO^+(1, 1)\). Here

\[
a = \cosh(x), \quad b = \sinh(x)
\]

Directly solving for \(\alpha\) from the quadratic system obtained from

\[
X = \Phi_{1,1}\left( \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} \right)
\]

one recovers

\[
Y = \pm \begin{pmatrix} e^{x/2} & 0 \\ 0 & e^{-x/2} \end{pmatrix}.
\]

(2)

The “half-angle” aspect of the covering map is manifest in this formula.

Only after some algebra, is this also the solution yielded by the formula of [16]. Specifically

\[
M = (2 + 2a)1 + 2b e_2 e_1 = \begin{pmatrix} 2 + 2a + 2b & 0 \\ 0 & 2 + 2a - 2b \end{pmatrix}
\]

since \(e_2 e_1 = \sigma_z\) if we use \(B_1\) as the basis of one-vectors for \(\text{Cl}(1, 1)\). Next, a calculation shows \(MM^{rev} = (8 + 8a)1\). So, it follows that

\[
Y = \begin{pmatrix} \frac{2 + 2a + 2b}{\sqrt{8 + 8a}} & 0 \\ 0 & \frac{2 + 2a - 2b}{\sqrt{8 + 8a}} \end{pmatrix}
\]

(3)

which, after further manipulations, coincides with Equation (2). Thus, even in this simple case, it is seen that if one is interested in a symbolic expression for \(Y\) as a function of the entries of \(X\), then Equation (3) is more complicated that Equation (2), even though they are equivalent. Next, we could also have used \(B_2 = \{\sigma_z, i\sigma_y\}\), as the basis of one-vectors. Naively applying the formula in [16] would then naturally lead to \(Y\) being a linear combination of \(I_2\).
and $\sigma_x$, which is inconsistent with Equation (1). The resolution is that with $B_2$ as the choice of the basis of one-vectors, $\text{Spin}^+(1,1)$ is just

$$\text{Spin}^+(1,1) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} ; a^2 - b^2 = 1 \right\}. \quad (4)$$

1.2 Organization of the Paper

The balance of the paper is organized as follows. In Section 2.1, we record notation used throughout the work, Section 2.2 records definitions from Clifford algebras, especially that of the covering map and its linearization. Section 2.3 discusses matrices with quaternionic entries and their complex representations. We draw attention to Remark 11 and Remark 12. Section 2.4 shows that inverse images of positive definite matrices can be chosen to be positive definite matrices in the covering group - see Proposition 13. This result assumes that the basis of one-vectors chosen satisfy two properties (called BP1 and BP2). In the Appendix we show that every real Clifford algebra has at least one such basis. Section 2.4 discusses the polar decomposition and the Givens decompositions. We draw attention to

- Remark 15 which discusses a constructive algorithm, Algorithm 25 in Section 3, for the polar decomposition in certain indefinite orthogonal groups.
- Remark 16.
- Proposition 19 which shows that Givens matrices indeed belong to $SO^+(p,q)$, and
- Example 20 which illustrates how any matrix in the indefinite orthogonal groups can be factored constructively into Givens factors.

Section 3 is devoted to the inversion of $\Phi_{2,1}$. In this section we use the polar decomposition and the key observation is that if $P \in SO^+(2,1)$ is positive definite, then its preimage(s) under $\Phi_{2,1}$ can be found by inspection. In sections 4 and 5 we directly invert $\Phi_{2,2}$ and $\Phi_{3,2}$ by solving the systems of quadratic equations when the target is a Givens factor. In Section 6, we invert $\Phi_{4,1}$ by first inverting the linearization $\Psi_{4,1}$ and then showing that the exponential of the matrix in the Lie algebra of the spin group, thereby obtained, admits a closed-form expression. We demonstrate this when the target is decomposed using Givens factors or when the polar decomposition is employed for the same purpose. In particular this provides an
algorithm for finding the polar decomposition of certain $2 \times 2$ quaternionic matrices without having to find that of the associated $4 \times 4$ complex matrix. We draw attention to Remark 32 of this section which shows that linearization provides a viable method to invert the abstract covering map, when used in conjunction with Givens decompositions. The last section offers conclusions.

2 Preliminaries

2.1 Notation

We use the following notation throughout

**N1** $\mathbb{H}$ is the set of quaternions. Let $K$ be an associative algebra. Then $M(n, K)$ is just the set of $n \times n$ matrices with entries in $K$. For $K = \mathbb{C}, \mathbb{H}$ we define $X^*$ as the matrix obtained by performing entrywise complex (respectively quaternionic) conjugation first, and then transposition. For $K = \mathbb{C}$, $\bar{X}$ is the matrix obtained by performing entrywise complex conjugation.

**N2** The Pauli Matrices are

$$
\sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \sigma_y = \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

2.2 Preliminary Observations

We will begin with informal definitions of the notions of one and two-vectors for a Clifford algebra, which is sufficient for the purpose of this work. The texts [12, 14] are excellent sources for more precise definitions in the theory of Clifford algebras. The same texts also contain precise definitions of the various automorphisms (such as grade, Clifford conjugation and reversion).

**Definition 4** Let $p, q$ be non-negative integers with $p + q = n$. A collection of matrices

$$
\{X_1, \ldots, X_p, X_{p+1}, \ldots, X_{p+q}\},
$$

with entries in $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ is a basis of one-vectors for the Clifford algebra $Cl(p, q)$ if

1. $X_i^2 = I$, for $i = 1, 2, \ldots, p$, where $I$ is the identity matrix of the appropriate size (this size is typically different from $n$)
2. \( X_i^2 = -I, \) for \( i = p + 1, p + 2, \ldots, p + q \)

3. \( X_iX_j = -X_jX_i, \) for \( i \neq j; \ i, j = 1, 2, \ldots, n. \)

A one-vector is just a real linear combination of the \( X_i \)’s, \( i = 1, 2, \ldots, n \). Similarly, a two-vector is a real linear combination of the matrices \( X_iX_j, \) \( i < j, \ i, j = 1, 2, \ldots, n. \) Analogously, we can define three, four, ..., \( n \)-vectors, etc. \( \text{Cl}(p, q) \) is just a real linear combination of \( I \), one-vectors, ..., \( n \)-vectors.

**Definition 5** \( \text{Spin}^+(p, q) \) is the connected component of the identity of the collection of elements \( x \) in \( \text{Cl}(p, q) \) satisfying the following requirements: i) \( x^{gr} = x \), i.e., \( x \) is even (here \( x^{gr} \) is the grade involution applied to \( x \)), ii) \( xx^{cc} = 1 \) (here \( x^{cc} \) is the Clifford conjugate of \( x \)) and iii) For all one-vectors \( v \) in \( \text{Cl}(0, n) \), \( xvx^{cc} \) is also a one-vector. The last condition, in the presence of the first two conditions, is known to be superfluous for \( p + q \leq 5, [12, 14] \).

**Definition 6** Let \( n = p + q \). Denote by \( I_{p,q} = I_p \oplus (-I_q) \). Then \( \text{SO}(p, q) = \{ X \in M(n, \mathbb{R}) : X^TI_{p,q}X = I_{p,q}; \det(X) = 1 \} \) \( \text{SO}^+(p, q) \) is the connected component of the identity in \( \text{SO}(p, q) \). Unless \( pq = 0 \), \( \text{SO}^+(p, q) \) is a proper subset of \( \text{SO}(p, q) \). The Lie algebra of \( \text{SO}^+(p, q) \) is denoted by \( \text{so}(p, q) \) and it is described by

\[
\text{so}(p, q) = \{ X : X \in M(n, \mathbb{R}), X^TI_{p,q} = -I_{p,q}X \}
\]

**Definition 7** The map \( \Phi_{p,q} : \text{Spin}^+(p, q) \to \text{SO}^+(p, q) \) sends \( x \in \text{Spin}^+(p, q) \) to the matrix of the linear map \( v \to xvx^{cc} \), where \( v \) is a one-vector with respect to a basis \( \{ X_1, \ldots, X_p, X_{p+1}, \ldots, X_{p+q} \} \) of the space of one-vectors. \( \text{spin}^+(p, q) \) is its Lie algebra and is known to equal the space of bivectors of \( \text{Cl}(p, q) \). It is further known that \( \Phi_{p,q} \) is a group homomorphism with kernel \( \{ \pm I \} \).

We denote by \( \Psi_{p,q} \) the linearization of \( \Phi_{p,q} \). Thus, \( \Psi_{p,q} \) sends an element \( y \in \text{spin}^+(p, q) \) to the matrix of the linear map \( v \to yv - vy \). \( \Psi_{p,q} \) is a Lie algebra isomorphism from \( \text{spin}^+(p, q) \) to \( \text{so}(p, q) \).

### 2.3 Quaternionic and \( \theta_H \) Matrices

Next, to a matrix with quaternion entries will be associated a complex matrix. First, if \( q \in \mathbb{H} \) is a quaternion, it can be written uniquely in the form \( q = z + wj \), for some \( z, w \in \mathbb{C} \). Note that \( j \eta = \bar{\eta}j \), for any \( \eta \in \mathbb{C} \). With this at hand, the following construction associating complex matrices to matrices with quaternionic entries is useful.
**Definition 8** Let $X \in M(n, \mathbb{H})$. By writing each entry $x_{pq}$ of $X$ as

$$x_{pq} = z_{pq} + w_{pq}j,$$

we can write $X$ uniquely as $X = Z+Wj$ with $Z,W \in M(n, \mathbb{C})$. Associate to $X$ the following matrix

$$\theta_H(X) = \begin{pmatrix} Z & W \\ -\bar{W} & \bar{Z} \end{pmatrix}.$$ 

**Remark 9** Viewing an $X \in M(n, \mathbb{C})$ as an element of $M(n, \mathbb{H})$ it is immediate that $jX = \bar{X}j$, where $\bar{X}$ is entrywise complex conjugation of $X$.

**Definition 10** A $2n \times 2n$ complex matrix of the form

$$\begin{pmatrix} Z & W \\ -\bar{W} & \bar{Z} \end{pmatrix}$$

is said to be a $\theta_H$ matrix.

Next some useful properties of the map $\theta_H : M(n, \mathbb{H}) \to M(2n, \mathbb{C})$ are collected.

**Remark 11** Properties of $\theta_H$

i) $\theta_H$ is an $\mathbb{R}$-linear map

ii) $\theta_H(XY) = \theta_H(X)\theta_H(Y)$

iii) $\theta_H(X^*) = [\theta_H(X)]^*$. Here the $*$ on the left is quaternionic Hermitian conjugation, while that on the right is complex Hermitian conjugation.

iv) $\theta_H(I_n) = I_{2n}$.

**Remark 12** In this remark we will collect some more facts concerning quaternionic matrices.

1. If $X, Y \in M(n, \mathbb{H})$ then it is not true that $\text{Tr}(XY) = \text{Tr}(YX)$. However, $\text{Re}(\text{Tr}(XY)) = \text{Re}(\text{Tr}(YX))$. Therefore, the following version of cyclic invariance of trace holds for quaternionic matrices

$$\text{Re}[\text{Tr}(XYZ)] = \text{Re}[\text{Tr}(YZX)] = \text{Re}[\text{Tr}(ZXY)]$$

2. Let $X$ and $Y$ be square quaternionic matrices. Then $\text{Tr}(X \otimes Y) = \text{Tr}(X)\text{Tr}(Y)$. Furthermore, if at least one of $X$ and $Y$ is real, then

$$\text{Re}(\text{Tr}(X \otimes Y)) = \text{Re}(\text{Tr}(X))\text{Re}(\text{Tr}(Y))$$
3. $X = Z + Wj$ is Hermitian iff $Z$ is Hermitian (as a complex matrix) and $W$ is skew-symmetric. This is, of course, equivalent to $\theta_H(X)$ being Hermitian as a complex matrix.

4. If $X$ is a square quaternionic matrix, we define

$$\text{Exp}(X) = I_n + X + \frac{X^2}{2!} + \ldots.$$ 

Then $\theta_H(\text{Exp}(X)) = \text{Exp}(\theta_H(X))$.

5. If $X$ is a square quaternionic matrix, it is positive definite if $q^*Xq > 0$, for all $q \in \mathbf{H}$. This is equivalent to $\theta_H(X)$ being a positive definite complex matrix.

6. Putting the last two items together we see that if $X \in M(n, \mathbf{H})$ is Hermitian, then $\text{Exp}(X)$ is positive definite.

We next prove a very useful result, Proposition 13, which ensures that one preimage in $\text{Spin}^+(p, q)$ of a positive definite matrix in $\text{SO}^+(p, q)$ is also positive definite. In light especially of Remark 3, it must be stressed that it is being assumed in Proposition 13 that the basis, $B$, of one-vectors for $\text{Cl}(p, q)$ being used satisfies the following two properties

- **BP1** If $V \in B$, the basis of one-vectors being used, then

$$V_i^* = \pm V_i$$

(5)

- **BP2** The matrices in $B$ are orthogonal with respect to the trace inner product. Specifically, if $B$ consists of real or complex matrices then

$$\text{Tr}(U^*V) = 0, \forall U, V \in B, U \neq V$$

(6)

and if $B$ contains quaternionic matrices then

$$\text{Re}(\text{Tr}(U^*V)) = 0, \forall U, V \in B, U \neq V.$$  

(7)

**Proposition 13** Let $P \in \text{SO}^+(p, q)$ be positive definite. Let $B$ be a set of matrices serving as a basis of one-vectors for $\text{Cl}(p, q)$ which satisfy both **BP1** and **BP2**. Then there is a unique positive definite $Y \in \text{Spin}^+(p, q)$ with $\Phi_{p,q}(Y) = P$. 

Proof: As shown in [2], there is a symmetric $Q \in so(p,q)$ such that $Exp(Q) = P$. Let $\Psi_{p,q}$ be the linearization of $\Phi_{p,q}$. We will show that the (unique) preimage $A$ of $Q$ with respect to $\Psi_{p,q}$ is Hermitian. Therefore, from the formula $\Phi_{p,q}[Exp(A)] = Exp[\Psi_{p,q}(A)]$, it follows that if we denote by $Y = Exp(A)$, then $\Phi_{p,q}(Y) = P$. Since $A$ is Hermitian, it follows that $Y = Exp(A)$ is positive definite (where, in the event $A$ is quaternionic we invoke the last item of Remark 12).

Let us now show that $A = \Psi_{p,q}^{-1}(Q)$ is Hermitian. First, suppose that $B$ consists of real or complex matrices. Since $B$ satisfies $BP1$ and $BP2$ we have

- 1) If $A \in spin^+(p,q)$, then $A^* \in spin^+(p,q)$ also. Indeed the typical element of $spin^+(p,q)$ is a real linear combination of the bivectors $V_k V_l$, $k < l$. Since $(V_k V_l)^* = \pm V_k V_l$ (using the fact that the $V_i$s anticommute), it follows that $A^*$ is also a real linear combination of the bivectors and is thus in $spin^+(p,q)$ also.

- 2) $\Psi_{p,q}(A^*) = [\Psi_{p,q}(A)]^*$. To see this note that the $(i,j)$th entry of the matrix $\Psi_{p,q}(A)$ equals, due the $V_i$'s being orthogonal with respect to the trace inner product

$[\Psi_{p,q}(A)]_{ij} = Tr[V_i^*(AV_j - V_j A)] = Tr[A(V_j V_i^* - V_i^* V_j)]$

(where we used the cyclic invariance of trace).

Similarly the $(j,i)$ entry of $\Psi_{p,q}(A^*)$ equals $Tr[A^*(V_j V_i^* - V_i^* V_j)]$. But this equals the complex conjugate of $Tr[(V_j V_i^* - V_i^* V_j)A]$, which again by cyclic invariance of trace, equals $\Psi_{p,q}(A)_{ij}$.

If $B$ contains quaternionic matrices then the above argument goes through verbatim if we replace $Tr$ by $Re\ Tr$ in light of item 1) of Remark 12.

So (as all $\Psi_{p,q}(A)$ are real), if $\Psi_{p,q}(A)$ is symmetric then, in light of $\Psi_{p,q}$ being a vector space isomorphism, it follows that $A = A^*$ and hence $A$ is Hermitian and $Y = Exp(A)$ positive definite. ♦.

Remark 14 The previous proof assumed that there is a basis of one-vectors, $\{V_i\}$ for $Cl(p,q)$ with the properties $BP1$ and $BP2$. For all the Clifford algebras discussed in this paper, this is true by construction. However, for the sake of completeness, we will prove that there is at least one such basis for all $Cl(p,q)$ in Theorem 35. Notwithstanding Theorem 35, it is
worth emphasizing that for the purpose of inversion, in light of Remark 1, one must verify
the veracity of both **BP1** and **BP2** for the specific basis of one-vectors that one chooses to
arrive at the matrix form of $\Phi_{p,q}$.

2.4 Polar and Givens Decomposition of $SO^+(p, q)$

In this section we collect together various results on decompositions of $SO^+(p, q)$ which will
play an important role in the remainder of this work.

**Remark 15** Constructive Polar Decomposition: Let $X \in SO^+(p, q)$. Then (see [2]), both
factors $V, P$ in its polar decomposition $X = VP$, where $V$ is real special orthogonal and $P$
is positive definite, also belong to $SO^+(p, q)$. Furthermore, as shown in [2], when either $p$
or $q$ is 1 one can find $V, P$ and the real symmetric $Q$ with $\text{Exp}(Q) = P$ by inspecting the first
column and row (or last if $p = 1$) and finding special orthogonal matrices which take the first
unit vector to a given vector of length one. See Algorithm 25 in Section 3 below for a special
case of this. For other values of $(p, q)$ these constructive procedures can be extended, except
that they involve substantially more matrix maneuvers. We will tacitly assume the contents
of this remark in Sections 3 and 5.

**Remark 16** Logarithms of Special Orthogonal Matrices of Size 4.

Let $X \in SO(4)$. Then one can find explicitly a pair of unit quaternions $u, v$ such $X = M_{u \otimes v}$,
(see, for instance, [5]). Suppose first that neither $u$ nor $v$ belong to the set \{±1\}. This means
$M \neq \pm I_4$.

Then one can further find, essentially by inspection of $u, v$, a real skew-symmetric $Y$
such that $\text{Exp}(Y) = X$. Specifically, let $\lambda \in (0, \pi)$ be such that $\Re(u) = \cos(\lambda)$. Then let
$p = p_1i + p_2j + p_3k$ be $\frac{1}{\sin(\lambda)} \Im(u)$. Similarly, find $q = q_1i + q_2j + q_3k$ from inspecting $v$. Then

$$Y = Y_1 + Y_2$$

with

$$Y_1 = \begin{pmatrix}
0 & -p_1 & -p_2 & -p_3 \\
p_1 & 0 & -p_3 & p_2 \\
p_2 & p_3 & 0 & -p_1 \\
p_3 & -p_2 & p_1 & 0
\end{pmatrix}$$

(8)
and

\[
Y_2 = \begin{pmatrix}
0 & q_1 & q_2 & q_3 \\
-q_1 & 0 & -q_3 & q_2 \\
-q_2 & q_3 & 0 & -q_1 \\
-q_3 & -q_2 & q_1 & 0
\end{pmatrix}.
\]  
\tag{9}

Furthermore, \( Y_1 \) and \( Y_2 \) commute.

Finally, if \( M = I_4 \), then \( M = \text{Exp}(0_4) \), while if \( M = -I_4 \), then
\[M = \text{Exp}(Y),\]
with
\[Y = Z \oplus Z.\]

where \( Z = \begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix} \).

Next we discuss Givens decompositions.

Define \( R = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \) where \( c^2 + s^2 = 1 \) and \( H = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \), for \( a^2 - b^2 = 1 \). Then the following facts are well known:

- Given a vector \((x, y)^T\) there is an \( R = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \) where \( c^2 + s^2 = 1 \) such that
  \[
  R \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{x^2 + y^2} \\ 0 \end{pmatrix}. 
  \]
  Similarly there is an \( R = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \) where \( c^2 + s^2 = 1 \)
  such that \( R \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{x^2 + y^2} \end{pmatrix} \).

- Next given a vector \((x, y)^T\), with \(|x| \geq |y|\), there is an \( H = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \) with \( a^2 - b^2 = 1 \),
  such that \( H \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{x^2 - y^2} \\ 0 \end{pmatrix} \).

\( R, H \) are called plane standard Givens and hyperbolic Givens respectively. Embedding \( R \), respectively \( H \) as a principal submatrix of the identity matrix \( I_n \), yields matrices known as standard Givens (respectively, Hyperbolic Givens).

**Definition 17** \( H_{ij} \), for \( i < j \), stands for the \( n \times n \) matrix which is the identity except in the principal submatrix, indexed by rows and columns \((i, j)\), wherein it is a hyperbolic Givens
matrix. Similarly, \( R_{ij} \) stands for the \( n \times n \) matrix which is the identity except in the principal submatrix, indexed by rows and columns \((i, j)\), wherein it is an ordinary Givens matrix.

**Remark 18** While \( H_{ij} \) is defined only if \( i < j \), the matrices \( R_{ij} \) make sense for all pairs \((i, j)\) with \( i \neq j \). \( R_{ij} \) is the matrix which zeroes out the \( j \)th component of a vector that it premultiplies. Thus, \( R_{ij} \) will be different, in general, if \( i < j \) from that when \( i > j \).

The next result shows that \( H_{ij} \)'s and \( R_{ij} \)'s belong to \( SO^+(p, q) \). Specifically

**Proposition 19** Let \( 1 \leq i \leq p \) and \( 1 \leq j \leq q \). Then \( H_{ij} \) belongs to \( SO^+(p, q) \). Similarly, if either \( 1 \leq i, j \leq p \) or \( 1 \leq i, j \leq q \), then \( R_{ij} \) belongs to \( SO^+(p, q) \).

**Proof:** Consider the \( H_{ij} \) case first. Define

\[
L_{ij} = \theta(e_i e_j^T + e_j e_i^T)
\]

with \( \theta \in \mathbb{R} \). Thus \( L_{ij} \) is the symmetric matrix which is zero everywhere, except in the \((i, j)\)th and \((j, i)\)th entries wherein it is \( \theta \). Due to the conditions, \( 1 \leq i \leq p \) and \( 1 \leq j \leq q \), it is easy to verify that \( L_{ij} \in so(p, q) \). A calculation shows that

\[
L_{ij}^2 = D_\theta,
\]

where \( D_\theta \) is \( n \times n \) diagonal with zeroes everywhere, except on the \( i \)th and \( j \)th diagonal entries wherein it is \( \theta^2 \). Therefore,

\[
L_{ij}^3 = \theta^2(e_i e_j^T + e_j e_i^T) = \theta^2 L_{ij}.
\]

Hence by the Euler-Rodrigues formula

\[
\text{Exp}(L_{ij}) = I_n + \frac{\sinh(\theta)}{\theta} L_{ij} + \frac{\cosh(\theta) - 1}{\theta^2} L_{ij}^2
\]

\[
= I_n + \sinh(\theta)(e_i e_j^T + e_j e_i^T) + \frac{\cosh(\theta) - 1}{\theta^2} D_\theta = H_{ij}.
\]

Thus \( H_{ij} \) being the exponential of a matrix in the Lie algebra \( so(p, q) \), belongs to \( SO^+(p, q) \).

The proof for \( R_{ij} \) is similar. ♦

The relevance of Givens rotations is that any matrix in \( SO^+(p, q) \) can be decomposed constructively into a product of Givens matrices. It will suffice to illustrate this via an example.
Example 20  Let $X \in \text{SO}^+(2, 2)$. Consider the first column of $X$

$$v_1 = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$ 

Since $X \in \text{SO}^+(2, 2), a^2 + b^2 - c^2 - d^2 = 1$. Therefore there are $R_{1, 2}, R_{3, 4}$ such that the first column of $R_{1, 2}R_{3, 4}X =$

$$\begin{pmatrix} \alpha \\ 0 \\ \beta \\ 0 \end{pmatrix},$$

where $\alpha = \sqrt{a^2 + b^2}$ and $\beta = \sqrt{c^2 + d^2}$. Since $a^2 + b^2 - c^2 - d^2 = 1 = \alpha^2 - \beta^2$, it follows that $|\alpha| > |\beta|$. Hence there is an $H_{1, 3}$ such that the first column of $H_{1, 3}R_{1, 2}R_{3, 4}X$ equals

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$ 

As $H_{1, 3}R_{1, 2}R_{3, 4}X \in \text{SO}^+(2, 2)$, it follows that the first row of $H_{1, 3}R_{1, 2}R_{3, 4}X$ is also $(1, 0, 0, 0)$.

Therefore, the second column of the product $H_{1, 3}R_{1, 2}R_{3, 4}X$ is of the form

$$\begin{pmatrix} 0 \\ b \\ c \\ d \end{pmatrix}$$

with $b^2 - c^2 - d^2 = 1$. So there is an $R_{3, 4}$ such that $R_{3, 4}(c, d)^T = (\gamma, 0)^T$, where $\gamma^2 = c^2 + d^2$. As before $b^2 - \gamma^2 = 1$, so there is an $H_{2, 3}$ such that $H_{2, 3}$ with $H_{2, 3}(b, \gamma)^T = (1, 0)$. So it follows that the first and the second column equal the first two standard unit vectors. Since $H_{2, 3}R_{3, 4}H_{1, 3}R_{1, 2}R_{3, 4}X \in \text{SO}^+(2, 2)$, it follows that it equals

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & y_{33} & y_{34} \\ 0 & 0 & y_{43} & y_{44} \end{pmatrix}.$$
Again the condition $H_{2,3}R_{3,4}H_{1,3}R_{1,2}R_{3,4}X \in SO^+(2,2)$, ensures that

$$\begin{pmatrix} y_{33} & y_{34} \\ y_{43} & y_{44} \end{pmatrix}$$

must itself be a plane standard Givens rotation. Therefore, pre-multiplying by the corresponding $R_{3,4}$ we get that

$$R_{3,4}H_{2,3}R_{3,4}H_{1,3}R_{1,2}R_{3,4}X = I_4.$$  

Since the inverse of each $R_{i,j}$ (respectively $H_{k,l}$) is itself an $R_{i,j}$ (respectively $H_{k,l}$), it follows that $X$ can be expressed constructively as a product of $R_{3,4}, H_{2,3}, H_{1,3}, R_{1,2}$.

**Remark 21** The following observations about Givens decomposition are pertinent for this work.

i) The above factorization is not the only way to factor an element of $SO^+(2,2)$ into a product of standard and hyperbolic Givens matrices. By way of illustration, we use a slightly different factorization in Section 4, which emanates from using an $R_{4,3}$ instead of an $R_{3,4}$ in one of the three usages of $R_{3,4}$ above. This will result, therefore, in the usage of an $H_{2,4}$ instead of an $H_{2,3}$. We note, however, that since $R_{3,4}$ and an $R_{4,3}$ are essentially the same matrix, differing only in the parameter $\theta$ which enters in them. Thus, their inversion will require the symbolic solution of the same system of equations.

ii) There are atmost $\binom{p + q}{2}$ Givens factors in the decomposition of a generic $X$. However, of these there are at most $2p + q - 2$ distinct such factors. This is pertinent, as it implies that we have to symbolically invert only $2p + q - 2$ targets.

### 3 Inversion of $\Phi_{2,1}$ and the Polar Decomposition

In this section, we treat the inversion of $\Phi_{2,1} : Spin^+(2,1) \to SO^+(2,1)$ by showing that it only requires inspection to find the preimage (under $\Phi_{2,1}$) of matrices in $SO^+(2,1)$ which are either positive definite or special orthogonal. Since the factors in the polar decomposition of an $X \in SO^+(2,1)$ also belong to $SO^+(2,1)$, the method below also simultaneously provides the polar decomposition of the $X$ being inverted, with minimal fuss. Alternatively, one can
also directly find the polar decomposition of $X$, by essentially inspecting the last row and some extra calculations, and use that to invert $\Phi_{2,1}$.

In principle these methods extend to all $(p, q)$ but are limited in that, besides the $(2, 1)$ and $(3, 1)$ cases (the latter is treated in [2]), finding the preimage by mere inspection seems difficult. See however, Section 4, wherein the $(4, 1)$ case is handled by a combination of the polar decomposition and inverting the associated Lie algebra isomorphism $\Psi_{4,1} : spin^+(4, 1) \to so(4, 1)$.

Let us first provide an explicit matrix form of the map $\Phi_{2,1}$. This follows, after some computations, from the material in [8]. Specifically we begin with the following basis of one-vectors for $Cl(2,1)$

$$B_{2,1} = \{Y_1, Y_2, Y_3\} = \{\sigma_z \otimes \sigma_z, \sigma_x \otimes I_2, i\sigma_y \otimes I_2\}.$$  \hspace{1cm} (10)

Thus, $Cl(2,1)$ is a matrix subalgebra of $M(4, \mathbb{R})$. The even subalgebra is isomorphic to $M(2, \mathbb{R})$, which can be embedded into the former subalgebra as follows. Specifically, given

$$Y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \in Cl(2,1)$$

embed it in $Cl(2,2)$ as follows

$$\begin{pmatrix} y_1 & 0 & y_2 & 0 \\ 0 & -y_1 & 0 & y_2 \\ y_3 & 0 & y_4 & 0 \\ 0 & y_3 & 0 & -y_4 \end{pmatrix}. \hspace{1cm} (11)$$

Then $Spin^+(2,1)$ is isomorphic to $SL(2, \mathbb{R})$ (embedded in $M(4, \mathbb{R})$ as in Equation (11) above).

It can then be shown that the map $\Phi_{2,1}$ sends an element

$$\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$$

in $SL(2, \mathbb{R})$ to the following matrix in $SO^+(2,1)$

$$\begin{pmatrix} 1 + 2y_2 y_3 & y_2 y_4 - y_1 y_3 & -(y_1 y_3 + y_2 y_4) \\ y_3 y_4 - y_1 y_2 & \frac{1}{2} (y_1^2 - y_2^2 - y_3^2 + y_4^2) & \frac{1}{2} (y_1^2 + y_2^2 - y_3^2 - y_4^2) \\ -(y_1 y_2 + y_3 y_4) & \frac{1}{2} (y_1^2 - y_2^2 + y_3^2 - y_4^2) & \frac{1}{2} (y_1^2 + y_2^2 + y_3^2 + y_4^2) \end{pmatrix}. \hspace{1cm} (12)$$
3.1 Preimages of Positive Definite Targets in $\text{SO}^+(2,1)$

First from many points of view, e.g., from Equation (12) itself, it is easily seen that if $Y \in \text{SL}(2, \mathbb{R})$, then $\Phi_{2,1}(Y^T) = [\Phi_{2,1}(Y)]^T$. Hence, in view of the fact that $\Phi_{2,1}$ is surjective with $\ker(\Phi_{2,1}) = \{\pm I\}$, we see that if $\Phi_{2,1}(Y)$ is a symmetric matrix in $\text{SO}^+(2,1)$ then necessarily $Y^T = \pm Y$.

Next, a symmetric $Y$ in $\text{SL}(2, \mathbb{R})$ cannot have its $(1,1)$ entry equal to zero. Thus, as $\det(Y) > 0$, $Y$ is either positive or negative definite. If $Y$ is antisymmetric, it is easily seen from Equation (12) that $\Phi_{2,1}(Y)$ is diagonal with two entries negative and one positive - i.e., it is indefinite.

Furthermore, if $Y \in \text{SL}(2, \mathbb{R})$ is symmetric then one can also deduce directly that $\Phi_{2,1}(Y)$ is positive definite. To that end, note that since $\det(\Phi_{2,1}(Y)) = 1$ and quite visibly the $(1,1)$ entry is positive, it suffices to check that the $(1,2)$ minor is positive to verify positive definiteness.

From Equation (12), this minor equals

$$\frac{1}{2}(1 + 2y_2^2)(y_1^2 - 2y_2^2 + y_4^2) - y_2^2(y_1 - y_4)^2.$$ 

Using $y_1y_4 - y_2^2 = 1$ we find that this minor is $\frac{1}{2}(1 + 2y_2^2)((y_1 - y_4)^2 + 2) - y_2^2(y_1 - y_4)^2 = 2y_2^2 + (1/2)(y_1 - y_4)^2 + 1 > 0$.

Summarizing the contents of the previous two paragraphs we have

**Theorem 22** Let $X \in \text{SO}^+(2,1)$ be symmetric. Then it is either positive definite or indefinite. In the former case $X = \Phi_{2,1}(\pm Y)$, with $Y \in \text{SL}(2, \mathbb{R})$ also positive definite. In the latter case $X$ is diagonal and $X = \Phi_{2,1}(\pm Y)$ with $Y \in \text{SL}(2, \mathbb{R})$ antisymmetric.

3.2 Finding $\Phi_{2,1}^{-1}(X)$ when $X > 0$ by Inspection

Suppose that $X$ is positive definite. Let us then address how a positive definite preimage $Y \in \text{SL}(2, \mathbb{R})$ is found by inspection of Equation (12). Let $Y = \begin{pmatrix} y_1 & y_2 \\ y_2 & y_4 \end{pmatrix}$. From the $(1,1)$ entry of Equation (12) we see $y_2 = \pm \frac{1}{\sqrt{2}}\sqrt{X_{11} - 1}$. Suppose $X_{11} \neq 1$ first. Then we find $y_1, y_4$ from the equations

$$y_4 - y_1 = \frac{X_{12}}{y_2}, \quad y_4 + y_1 = -\frac{X_{13}}{y_2}.$$
By Theorem 22, one choice of the sign for $y_2$ will lead to a $Y$ which is positive definite. If $X_{11} = 1$, then $y_2 = 0$. Now we look at $X_{22}$ and $X_{23}$ to find that $y_1^2$ and $y_4^2$ may be found by solving the system

$$y_1^2 + y_4^2 = 2X_{22}, \quad y_1^2 - y_4^2 = 2X_{23}.$$

We take the positive square roots of the solutions $y_1^2$ and $y_4^2$ to find a positive definite $Y$ projecting to $X$. This finishes our claim that if $X \in \text{SO}^+(2,1)$ is positive definite then we can find by inspection a positive definite $Y \in \text{SL}(2, \mathbb{R})$ projecting to $X$ under $\Phi_{2,1}$.

The above discussion is summarized in Algorithm 23 below.

**Algorithm 23** Let $X = (X_{ij}) \in \text{SO}^+(2,1)$ be positive definite. The following algorithm finds a positive definite $Y \in \text{SL}(2, \mathbb{R})$ satisfying $\Phi_{2,1}(Y) = X$.

1. Suppose $X_{11} \neq 1$. Let $y_2 = \pm \frac{1}{\sqrt{2}} \sqrt{X_{11} - 1}$, $y_1 = -\frac{X_{12} + X_{13}}{2y_2}$ and $y_4 = \frac{X_{12} - X_{13}}{2y_2}$.

2. Let $Y = \begin{pmatrix} y_1 & y_2 \\ y_2 & y_4 \end{pmatrix}$. There are two choices of $Y$ corresponding to the choice of the square root in $y_2$ in Step 1, which are negatives of each other. One of these $Y$ is positive definite. Pick this one.

3. Suppose $X_{11} = 1$. Then let $y_2 = 0$, $y_1 = \sqrt{X_{22} + X_{23}}$ and $y_4 = \sqrt{X_{22} - X_{23}}$. Then $Y = \text{diag}(y_1, y_4)$ is positive definite and is one preimage of $X$ in $\text{SL}(2, \mathbb{R})$.

### 3.3 Finding the polar decomposition in $\text{SO}^+(2,1)$

Let us now address how to find the polar decomposition in $\text{SO}^+(2,1)$ using Algorithm 23.

Let $X = VP$ be the polar decomposition of an $X \in \text{SO}^+(2,1)$. Then the orthogonal $V$ and positive definite $P$ are both in $\text{SO}^+(2,1)$. To find $P$ we proceed as follows. First find $X^TX$, which is then a positive definite element of $\text{SO}^+(2,1)$. Since its preimage $Z$ can be chosen to be positive definite, we find it using Algorithm 23. Once $Z$ has been found, we compute its unique positive definite square root, $W$. We note in passing that finding $Y$ from $Z$ can be executed in closed form, without any eigencalculations, [2].

Since $Z \in \text{SL}(2, \mathbb{R})$, $W$ is also in $\text{SL}(2, \mathbb{R})$. Then let $P = \Phi_{2,1}(W) \in \text{SO}^+(2,1)$. Then, we compute

$$P^2 = P^TP = \left(\Phi_{2,1}(Y)\right)^T \Phi_{2,1}(Y) = \Phi_{2,1}(Y^T) \Phi_{2,1}(Y)$$
\[ \Phi_{2,1}(Y^T Y) = \Phi_{2,1}(Z) = X^T X. \]

So \( P \) is the positive definite factor in \( X = VP \). Of course, \( V = XP^{-1} \). Next, finding \( P^{-1} \) is easy. One just interchanges \( y_1 \) and \( y_4 \) and replaces \( y_2 \) by \(-y_2\) and \( y_3 \) by \(-y_3\) in the formula \( \Phi_{2,1}(Y) = P \). This completes the determination of the polar decomposition of \( X \).

However, for the purpose of inversion of \( \Phi_{2,1} \), it still remains to find \( \pm S \in SL(2, \mathbb{R}) \) satisfying \( \Phi_{2,1}(\pm S) = V \).

To that end, note first that \( V \) is both orthogonal and in \( SO^+(2, 1) \). Thus, it is in \( SO(3) \). Hence it must have the following form \( V = \text{diag}(R, \pm 1) \), where \( R \) is \( 2 \times 2 \) orthogonal. However, from Equation (12) it is clear that the \((3, 3)\) entry of a matrix in \( SO^+(2, 1) \) is positive. So, \( V = \text{diag}(R, 1) \), with \( R \) in \( SO(2) \). Let \( c = \cos \theta \), \( s = \sin \theta \). Then

\[
V = \begin{pmatrix}
    c & -s & 0 \\
    s & c & 0 \\
    0 & 0 & 1
\end{pmatrix}.
\]

As before, simple considerations show that the matrix \( S \in SL(2, \mathbb{R}) \) projecting to \( R \) must itself be in \( SO(2) \). Finding \( R \)'s entries as functions of \( c \) and \( s \) is easy. First, if \( \theta \in (0, 2\pi) \), then \( \sin (\frac{\theta}{2}) > 0 \). Indeed, denoting by \( \widehat{c} = \cos (\frac{\theta}{2}) \), \( \widehat{s} = \sin (\frac{\theta}{2}) \), we have

\[
S = \pm \begin{pmatrix}
    \widehat{c} & -\widehat{s} \\
    \widehat{s} & \widehat{c}
\end{pmatrix}.
\]

For \( \theta = 0, 2\pi \) we get

\[
S = \pm \begin{pmatrix}
    -1 & 0 \\
    0 & -1
\end{pmatrix}.
\]

We summarize all of this in an algorithm

**Algorithm 24** Given \( X \in SO^+(2, 1) \), the following algorithm computes both its polar decomposition and the \( Y \in SL(2, \mathbb{R}) \) satisfying \( \Phi_{2,1}(\pm Y) = X \).

1. Find \( X^T X \) and find \( Z \in SL(2, \mathbb{R}) \) positive definite such that \( \Phi_{2,1}(Z) = X^T X \), using Algorithm 23.

2. Find the unique positive definite square root \( W \) of \( Z \). This step can be executed without any diagonalization - see [2].

3. Find \( P = \Phi_{2,1}(W) \) using Equation (12).
4. Find $P^{-1}$ by interchanging $p_{11}$ and $p_{22}$ and replacing $p_{12}, p_{21}$ by $-p_{12}, -p_{21}$ respectively in $P$ from Step 3.

5. Find $V = XP^{-1}$. Then $X = VP$ is the polar decomposition of $X$.

6. Find $S \in SO(2)$ satisfying $\Phi_{2,1}(S) = V$ using Equation (13) or Equation (14). Then $Y = WS$ satisfies $\Phi_{2,1}(\pm Y) = X$.

The above algorithm inverts the covering map by also finding the polar decomposition of the matrix in $SO^+(2,1)$. However, we can also find the polar decomposition without using the covering map. We now present a second algorithm which will produce the polar decomposition directly from $X$ itself, with inspection and finding special orthogonal matrices which rotate a plane vector into another of the same length. This is a special case of the algorithm for general $SO^+(n,1)$ from [2] mentioned earlier in Remark 15.

**Algorithm 25** The Polar Decomposition from the last row and column of $X$

1. Define $\sigma > 0$ by $X_{33} = \cosh(\sigma)$. If $X_{33} = 1$, then $X = I_3$ and the polar decomposition of $X$ is trivial.

2. Find $U \in SO(2)$ so that $(X_{31}, X_{32}) = (\sinh(\sigma), 0)U^T$.

3. Find $Z \in SO(2)$ so that $ZU \begin{pmatrix} \sinh(\sigma) \\ 0 \end{pmatrix} = \begin{pmatrix} X_{13} \\ X_{23} \end{pmatrix}$

4. Define i) $C_{1 \times 1} = (\cosh(\sigma))$, ii) $\tilde{C} = C \oplus I_1$ and iii) $S_{2 \times 1} = \begin{pmatrix} \sinh(\sigma) \\ 0 \end{pmatrix}$. Then the polar decomposition of $X$ is $X = VP$, with $V = Z \oplus I_1$ and

$$P = \begin{pmatrix} U\tilde{C}U^T & US \\ S^T U^T & \cosh(\sigma) \end{pmatrix}$$

5. Finally the symmetric $Q \in so(2,1)$ satisfying $\exp(Q) = P$ is

$$Q = \begin{pmatrix} 0_{2 \times 2} & UE \\ E^T U^T & 0_{1 \times 1} \end{pmatrix}$$

with $E = \begin{pmatrix} \sigma \\ 0 \end{pmatrix}$.
4 Inversion of $\Phi_{2,2}$

In this and the next section, we will use Groebner bases to invert $\Phi_{2,2}$ and $\Phi_{2,3}$. We begin with a few observations, which are pertinent to this and the next section, collected as remarks for ease of future reference.

**Remark 26** The task of inversion of the covering maps in question, will be addressed by solving directly the system of equations $\Phi_{p,q}(Y) = G$, where $G$ is either a standard Givens matrix $G(c, s)$ or a hyperbolic Givens $H(a, b)$. In the former case $c^2 + s^2 = 1$ and in the latter $a^2 - b^2 = 1$. For each fixed $c, s$ (respectively $a, b$) this is a system of $n^2$ equations in the variables $y_1, \ldots, y_l$, where $n = p + q$ and the number $l$ is determined by the structure of the corresponding spin group. For instance, for $(p, q) = (2, 2)$, $l$ is 8, while for $(p, q) = (3, 2)$, $l$ is 16. Furthermore, these equations have degree at most one in $c$ and $s$ (respectively $a$ and $b$), thereby ensuring that the dependence of these equations in $c$ and $s$ (respectively $a$ and $b$) is rational $c$ and $s$ (respectively $a$ and $b$). Next, this system of equations has precisely two solutions. This is guaranteed by Proposition 19, since that result confirms that such Givens matrices reside in $SO^+(p, q)$. To take advantage of this finiteness of the solution set, we view the associated ideal as an ideal in the polynomial ring $K[x_1, \ldots, x_n]$, where $K$ is the field of rational functions, with complex coefficients, in the variables $c$ and $s$ (respectively $a$ and $b$). Then this ideal becomes zero dimensional and the standard wisdom suggests that we should find Groebner bases with respect to some lexicographic order in the variables $y_i$ (in the polynomial ring $K[y_1, \ldots, y_l]$). To this Groebner basis, we finally append formally the equations $c = \cos(\theta); s = \sin(\theta)$ (respectively, $a = \cosh(\beta); b = \sinh(\beta)$). This last output is then used to solve for the $y_j$’s parametrically in $\theta$ (respectively $\beta$). It is noted in passing that the equations $\Phi_{p,q}(Y) = X$ are all quadratic (and most of them actually homogeneous quadratic) and there are techniques, besides Groebner bases (such as those related to Sylvester matrices), for analyzing such equations.

**Remark 27** It is worth pointing out that Groebner bases play a twin role in the methods of this and the next section. The first has been already outlined above. The second, equally important, is to arrive at the matrix form of the covering maps $\Phi_{p,q}$. Indeed, the entries of the matrix representing $\Phi_{p,q}(Y)$ were arrived at by computing the matrix of the linear map
$V \to V Y V^{-1}$, where $Y$ is an element of the spin group and $V$ is a one-vector, with respect to the basis of one-vectors $\{V_i\}$ chosen for this purpose. So to compute $\Phi_{p,q}(Y)$, we compute the traces of $Y V_i Y^{-1} V_j^*$ modulo the defining relations for the spin group. In all instances, these defining relations are some quadratic relations in the entries of the typical element of the matrix algebra that $Cl(p,q)$ is. For instance, for $(p,q) = (2,2)$, these relations are two equations representing the fact that the determinants of a pair of matrices, representing the typical element of $Spin^+(2,2)$ both equal 1.

Now let us turn to the inversion of $\Phi_{2,2}$.

Let $X \in SO^+(2,2)$. First, following Example 20 and with the caveat in Remark 21, it is noted that $X$ can be represented as a product of ordinary and hyperbolic Givens rotations (non-uniquely) as follows

$$X = R_{1,2} R_{3,4} H_{1,3} R_{4,3} H_{2,4} R_{3,4}. \quad (15)$$

So it suffices to find the preimages of these Givens matrices in $SL(2,R) \times SL(2,R)$, the relevant spin group. As mentioned in Remark 21, the systems of equations for the inversion of $R_{3,4}$ and $R_{4,3}$ are essentially the same. Furthermore by ii) of Remark 21 we have to only solve four such systems symbolically.

To get to that goal, we begin first with an explicit description of the entries of the matrix $\Phi_{2,2}(Y), Y \in SL(2,R) \times SL(2,R)$.

Following [8], the following quartet of matrices is used for a basis of one-vectors for $Cl(2,2)$

$$B_{2,2} = \{ X_1 = \sigma_z \otimes \sigma_x, X_2 = \sigma_x \otimes I, X_3 = \sigma_z \otimes (i\sigma_y), X_4 = (i\sigma_y) \otimes I \}. \quad (16)$$

Again, as described in [8], we can view a pair of matrices

$$\left\{ \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}, \begin{pmatrix} y_5 & y_6 \\ y_7 & y_8 \end{pmatrix} \right\}$$

as embedded in $Cl(2,2) = M(4,R)$ as follows

$$Y = \begin{pmatrix} y_1 & 0 & 0 & y_2 \\ 0 & y_3 & y_4 & 0 \\ 0 & y_5 & y_6 & 0 \\ y_7 & 0 & 0 & y_8 \end{pmatrix}. \quad (17)$$
The variables $y_1, \ldots, y_8$ satisfy the quadratic relations

$$y_1 y_4 - y_2 y_3 - 1 = 0, \quad y_5 y_8 - y_6 y_7 - 1 = 0. \quad (18)$$

The entries of the matrix $\Phi_{2,2}(Y)$ are then found by calculating the matrix of the linear map $V \rightarrow Y V Y^{-1}$, with $Y$ as in Equation (17), with respect to the basis of one-vectors in Equation (16), modulo the relations in Equation (18) (cf., Remark 27. Then a Groebner basis aided calculation shows that $\Phi_{2,2}$ sends $Y$ to

$$D \in SO^+(2,2),$$

where the entries of $D$ are given by

$$d_{11} = \frac{1}{2} (y_2 y_5 + y_1 y_6 + y_4 y_7 + y_3 y_8), \quad d_{12} = \frac{1}{2} (y_2 y_6 - y_1 y_5 - y_3 y_7 + y_4 y_8),$$

$$d_{21} = \frac{1}{2} (y_6 y_7 + y_3 y_8 - y_2 y_3 - y_1 y_4), \quad d_{22} = \frac{1}{2} (y_1 y_3 - y_2 y_4 - y_5 y_7 + y_6 y_8),$$

$$d_{31} = \frac{1}{2} (y_3 y_5 + y_1 y_6 - y_4 y_7 - y_3 y_8), \quad d_{32} = \frac{1}{2} (y_2 y_6 - y_1 y_5 + y_3 y_7 - y_4 y_8),$$

$$d_{41} = -\frac{1}{2} (y_2 y_3 + y_1 y_4 + y_6 y_7 + y_5 y_8), \quad d_{42} = \frac{1}{2} (y_1 y_3 - y_2 y_4 + y_5 y_7 - y_6 y_8),$$

$$d_{13} = \frac{1}{2} (y_1 y_6 - y_2 y_5 + y_4 y_7 - y_3 y_8), \quad d_{14} = -\frac{1}{2} (y_1 y_5 + y_2 y_6 + y_3 y_7 + y_4 y_8),$$

$$d_{23} = \frac{1}{2} (y_2 y_3 - y_1 y_4 + y_6 y_7 - y_3 y_8), \quad d_{24} = \frac{1}{2} (y_1 y_3 + y_2 y_4 - y_5 y_7 - y_6 y_8),$$

$$d_{33} = \frac{1}{2} (y_1 y_6 - y_2 y_5 - y_4 y_7 + y_3 y_8), \quad d_{34} = \frac{1}{2} (y_3 y_7 + y_4 y_8 - y_1 y_5 - y_2 y_6),$$

$$d_{43} = \frac{1}{2} (y_2 y_3 - y_1 y_4 - y_6 y_7 + y_5 y_8), \quad d_{44} = \frac{1}{2} (y_1 y_3 + y_2 y_4 + y_5 y_7 + y_6 y_8).$$

To invert $\Phi_{2,2}$ we equate $D$ to each distinct $R_{ij}(c, s)$ or $H_{kl}(a, b)$ which appears in Equation (15). and solve the corresponding system of quadratic equations in the variables $y_1, \ldots, y_8$. The $R_{ij}$’s depend on entries $c$ and $s$, while the $H_{ij}$’s depend on entries $a$ and $b$, where $c = \cos \theta$, $s = \sin \theta$, $a = \cosh (\theta)$ and $b = \sinh (\theta)$.

The following result records $\Phi_{2,2}^{-1}(D)$, where $D$ is one of the Givens matrices in Equation (15). For brevity we list the matrices as living in $M(4, \mathbb{R})$. The corresponding pair in $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ can easily be read off by inspection by invoking Equation (17).

The results are then given by

**Theorem 28** Let $c = \cos \theta$, $s = \sin \theta$, $a = \cosh \beta$, $b = \sinh \beta$. Correspondingly, let $\widehat{c} = \cos \frac{\theta}{2}$, $\widehat{s} = \sin \frac{\theta}{2}$. Then, the following list provides $Y \in SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$, viewed as matrices in $M(4, \mathbb{R})$ as embedded by Equation (17), such that $\Phi_{2,2}(\pm Y) = D$, where $D \in G$, with $G$ given by Equation (15)
Here the system of equations that have to be solved are given by equating each member of $D$.

$D$ is the set of equations provided by the procedure in Remark 26. The ordering of $2, 4, 2$.

For $\theta = 0, 2\pi$, $\Phi_{2,2}^{-1}(R_{1,2}) = \Phi_{2,2}^{-1}(I_4) = \pm I_4$.

For $\theta = 0, 2\pi$, $\Phi_{2,2}^{-1}(R_{3,4}) = \Phi_{2,2}^{-1}(I_4) = \mp I_4$.

**Proof:** We just provide the details for the $H_{1,3}$ case. Following the procedure outlined in Remark 26, we equate $\Phi_{2,2}(Y)$’s entries to those of $H_{1,2}$. After a Groebner basis calculation, the system of equations that have to be solved are given by equating each member of $B$, as in the equation below, to zero.

$$B = \begin{cases}
-1 + \cosh(\beta) - \sinh(\beta), y_5^2 - (\cosh(\beta) - \sinh(\beta)), y_7, y_6 - y_8, y_5, \\
y_4, y_3 - (\cosh(\beta) + \sinh(\beta))y_8, y_2, y_1 - (\cosh(\beta) + \sinh(\beta))y_8
\end{cases}.$$  

Here $B$ is the set of equations provided by the procedure in Remark 26. The ordering of variables for the lexicographic order is $y_1 > y_2 > \ldots > y_8$. Therefore, we have $y_2 = \ldots = y_8 = 0$ and $y_1$.
\[ y_4 = y_5 = y_7 = 0, \quad y_1 = (\cosh \beta + \sinh \beta) y_8, \quad y_3 = (\cosh \beta + \sinh \beta) y_8, \quad y_6 = y_8, \quad y_8^2 = (\cosh (\beta) - \sinh (\beta)). \] Solving this system, we get

\[ y_8 = \sqrt{\cosh (\beta) - \sinh (\beta)} \quad \text{or} \quad y_8 = -\sqrt{\cosh (\beta) - \sinh (\beta)}. \]

Since

\[ \cosh (\beta) - \sinh (\beta) = e^{-\beta} \quad \text{and} \quad \cosh (\beta) + \sinh (\beta) = e^{\beta}, \]

we have

\[ y_8 = \sqrt{\cosh (\beta) - \sinh (\beta)} = e^{-\beta/2}, \quad y_1 = (\cosh \beta + \sinh \beta) y_8 = e^{\beta/2} \]
\[ y_3 = (\cosh \beta + \sinh \beta) y_8 = e^{\beta/2}, \quad y_6 = y_8 = e^{-\beta/2} \]

and analogously

\[ y_8 = -\sqrt{\cosh (\beta) - \sinh (\beta)} = -e^{-\beta/2}, \quad y_1 = (\cosh \beta + \sinh \beta) y_8 = -e^{\beta/2} \]
\[ y_3 = (\cosh \beta + \sinh \beta) y_8 = -e^{\beta/2}, \quad y_6 = y_8 = -e^{-\beta/2}. \]

Therefore, we have

\[
\Phi_{2,2}^{-1}(H_{1,3}) = \pm \begin{pmatrix}
    e^{\beta/2} & 0 & 0 & 0 \\
    0 & e^{-\beta/2} & 0 & 0 \\
    0 & 0 & e^{\beta/2} & 0 \\
    0 & 0 & 0 & e^{-\beta/2}
\end{pmatrix}.
\]

\[ \diamond \]

5 Inversion of \( \Phi_{3,2} \)

The contents of Remark 26 and Remark 27 continue to be of pertinence to this section as well.

As before, by following the procedure in Example 20, every matrix \( X \in SO^+(3, 2) \) can be represented non-uniquely as a product of ordinary and hyperbolic Givens rotations as follows

\[
D = R_{2,3}R_{1,2}R_{4,5}H_{1,4}R_{2,3}R_{4,5}H_{2,5}R_{4,5}R_{3,4}R_{4,5}.
\]

As in the previous section, we begin by detailing the entries of \( \Phi_{3,2}(Y) \), for any \( Y \in Spin^+(3, 2) \).

In accordance with [8], the basis of one-vectors for \( Cl(3, 2) \) is given by

\[
B_{3,2} = \{X_1, X_2, X_3, X_4, X_5\}
\]
where
\[ X_1 = \sigma_x \otimes I_4, \quad X_2 = \sigma_z \otimes \sigma_x \otimes I_2, \quad X_3 = \sigma_z \otimes \sigma_z \otimes \sigma_z, \]
\[ X_4 = (i\sigma_y) \otimes I_4, \quad X_5 = \sigma_z \otimes (i\sigma_y) \otimes I_2. \]
Then, as shown in [8], \( \text{Spin}^+(3, 2) \) is the group
\[ \text{Spin}^+(3, 2) = \{ Y \in M(4, \mathbb{R}); Y^TMY = M \} \tag{21} \]
where
\[ M = \begin{pmatrix} 0 & 2 & J_2 \\ J_2 & 0_2 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]
viewed as living in \( M(8, \mathbb{R}) \) via the embedding
\[ Y \rightarrow \hat{Y} \tag{22} \]
where
\[
Y = \begin{pmatrix} y_1 & y_2 & y_3 & y_4 \\ y_5 & y_6 & y_7 & y_8 \\ y_9 & y_{10} & y_{11} & y_{12} \\ y_{13} & y_{14} & y_{15} & y_{16} \end{pmatrix}
\]
and
\[
\hat{Y} = \begin{pmatrix} y_1 & 0 & y_2 & 0 & y_3 & 0 & y_4 & 0 \\ 0 & y_1 & 0 & -y_2 & 0 & -y_3 & 0 & y_4 \\ y_5 & 0 & y_6 & 0 & y_7 & 0 & y_8 & 0 \\ 0 & -y_5 & 0 & y_6 & 0 & y_7 & 0 & -y_8 \\ y_9 & 0 & y_{10} & 0 & y_{11} & 0 & y_{12} & 0 \\ 0 & -y_9 & 0 & y_{10} & 0 & y_{11} & 0 & -y_{12} \\ y_{13} & 0 & y_{14} & 0 & y_{15} & 0 & y_{16} & 0 \\ 0 & y_{13} & 0 & -y_{14} & 0 & -y_{15} & 0 & y_{16} \end{pmatrix}
\]

**Remark 29** The group defined by Equation (21) is explicitly conjugate to the standard representation of the real symplectic group \( Sp(4, \mathbb{R}) \). This conjugation can be found without any eigencalculations. By this we mean that the \( 4 \times 4 \) skew-symmetric matrix \( M \) of Equation (21) defining this version of \( Sp(4, \mathbb{R}) \) can be rendered explicitly, conjugate to \( J_4 \), without any eigencalculations. We omit the details.
Next, entries of $Y$ (and thus, also those of $\hat{Y}$) satisfy quadratic relations emanating from Equation (21) which read as follows

$$f_1 = y_1 y_{16} - y_4 y_{13} - y_3 y_{12} + y_6 y_9 - 1 = 0$$
$$f_2 = y_2 y_{16} - y_4 y_{14} - y_6 y_{12} + y_8 y_{10} = 0$$
$$f_3 = y_3 y_{16} - y_4 y_{15} - y_7 y_{12} + y_8 y_{11} = 0$$
$$f_4 = y_4 y_{15} - y_3 y_{13} - y_5 y_{11} + y_7 y_9 = 0$$
$$f_5 = y_2 y_{15} - y_3 y_{14} - y_6 y_{11} + y_7 y_{10} + 1 = 0$$
$$f_6 = y_1 y_{14} - y_2 y_{13} - y_5 y_{10} + y_6 y_9 = 0.$$

As outlined in Remark 27, we use a Groebner basis for the ideal spanned by the polynomials \{f_1, \ldots, f_6\} with respect to the lexicographic order, to calculate the matrix of the linear map $V \to Y V Y^{-1}$, where $V$ is a one-vector and $Y \in \text{Spin}^+(3, 2)$, with respect to the basis $B = \{X_1, X_2, X_3, X_4, X_5\}$, from Equation (20), Then this matrix equals $M = (m_{i,j}), i, j = 1, 2, 3, 4, 5$. The $m_{i,j}$ are given by

$$m_{1,1} = \frac{1}{2}(-y_1 y_{14} - y_2 y_{15} + y_11 y_{16} - y_2 y_5 + y_1 y_6 - y_4 y_7 + y_3 y_8 + y_14 y_9)$$
$$m_{2,1} = \frac{1}{2}(-y_1 y_{10} - y_2 y_3 + y_11 y_4 + y_14 y_5 - y_1 y_6 + y_16 y_7 - y_1 y_8 + y_2 y_9)$$
$$m_{3,1} = y_1 y_5 + y_12 y_7 - y_11 y_8 - y_6 y_9$$
$$m_{4,1} = \frac{1}{2}(y_10 y_{13} + y_12 y_{15} - y_11 y_{16} - y_2 y_5 + y_1 y_6 - y_4 y_7 + y_3 y_8 - y_14 y_9)$$
$$m_{5,1} = \frac{1}{2}(-y_1 y_{10} - y_2 y_3 + y_11 y_4 - y_14 y_5 + y_1 y_6 - y_16 y_7 + y_15 y_8 + y_2 y_9)$$

$$m_{1,2} = \frac{1}{2}(y_11 y_{13} - y_12 y_{14} + y_10 y_{16} + y_3 y_5 - y_4 y_6 - y_1 y_7 + y_2 y_8 - y_15 y_9)$$
$$m_{2,2} = \frac{1}{2}(y_1 y_{11} - y_2 y_2 + y_1 y_4 + y_15 y_5 + y_16 y_6 + y_13 y_7 - y_4 y_8 - y_3 y_9)$$
$$m_{3,2} = -y_11 y_5 + y_12 y_6 - y_10 y_8 + y_7 y_9$$
$$m_{4,2} = \frac{1}{2}(-y_1 y_{13} + y_12 y_{14} - y_10 y_{16} + y_3 y_5 - y_4 y_6 - y_1 y_7 + y_2 y_8 + y_15 y_9)$$
$$m_{5,2} = \frac{1}{2}(y_1 y_{11} - y_12 y_2 + y_10 y_4 + y_15 y_5 - y_16 y_6 - y_13 y_7 + y_4 y_8 - y_3 y_9)$$

$$m_{1,3} = -y_11 y_4 + y_10 y_{15} - y_3 y_6 + y_2 y_7$$
$$m_{2,3} = -y_11 y_2 + y_10 y_3 + y_15 y_6 - y_14 y_7$$
$$m_{3,3} = -1 + 2y_11 y_6 - 2y_10 y_7$$
$$m_{4,3} = y_11 y_{14} - y_10 y_{15} - y_3 y_6 + y_2 y_7$$
$$m_{5,3} = -y_11 y_2 + y_10 y_3 - y_15 y_6 + y_14 y_7$$
The factors in Equation (19) are \( \sin \theta \) and \( \cosh \beta \). Explicit formulae for \( y_1, \ldots, y_{16} \) from the 25 equations \( m_{ij} = x_{ij} \), where \( x_{ij} \) are the entries of \( X \).

Thus, finding the preimage of \( X \in SO^+(3,2,\mathbb{R}) \), means solving for the 16 unknowns \( y_1, \ldots, y_{16} \) from the 25 equations \( m_{ij} = x_{ij} \), where \( x_{ij} \) are the entries of \( X \).

We now let \( X \) be one of the six distinct \( R_{ij}(c, s) \) or \( H_{kl}(a, b) \) where the \( R_{ij} \) and \( H_{kl} \) are as in Equation (19). The next result gives explicit formulae for \( y_1, y_2, \ldots, y_{16} \) in terms of \( s \), \( c, a, b, \).

**Theorem 30** Let \( c = \cos \theta \), \( s = \sin \theta \), \( a = \cosh \beta \), \( b = \sinh \beta \). Correspondingly, let \( \hat{c} = \cos \frac{\theta}{2} \), \( \hat{s} = \sin \frac{\theta}{2}, \hat{c}h = \cosh \frac{\beta}{2}, \hat{s}h = \sinh \frac{\beta}{2} \). Then, the preimages \( Y \in Spin^+(3,2) \) of the Givens factors in Equation (19) are

| \( M = R_{1,2} \) | \( \theta \in (0, \pi) \) | \( \theta = (\pi, 2\pi) \) | \( \theta = \pi \) |
|-----------------|----------------|----------------|-----------|
| \( \Phi_{3,2}^{-1}(M) \) | \( \hat{c} \ 0 \ 0 \ \hat{s} \) | \( \hat{c} \ 0 \ \hat{s} \) | \( 0 \ 0 \ 0 \ -1 \) |
| | \( 0 \ \hat{c} \ \hat{s} \) | \( 0 \ \hat{c} \ 0 \) | \( 0 \ 0 \ -1 \ 0 \) |
| | \( 0 \ -\hat{s} \ \hat{c} \) | \( 0 \ -\hat{s} \ 0 \) | \( 0 \ 1 \ 0 \ 0 \) |
| | \(-\hat{s} \ 0 \ 0 \ \hat{c} \) | \(-\hat{s} \ 0 \ \hat{c} \) | \( 1 \ 0 \ 0 \ 0 \) |

For \( \theta = 0, 2\pi \) \( \Phi_{3,2}^{-1}(R_{1,2}) = \Phi_{3,2}^{-1}(I_4) = \pm I_4 \).

| \( M = R_{1,3} \) | \( \theta \in (0, \pi) \) | \( \theta = (\pi, 2\pi) \) | \( \theta = \pi \) |
|-----------------|----------------|----------------|-----------|
| \( \Phi_{3,2}^{-1}(M) \) | \( \hat{c} \ 0 \ \hat{s} \ 0 \) | \( \hat{c} \ 0 \ \hat{s} \) | \( 0 \ 0 \ 1 \ 0 \) |
| | \( 0 \ \hat{c} \ 0 \ -\hat{c} \) | \( 0 \ \hat{c} \ 0 \) | \( 0 \ 0 \ 0 \ -1 \) |
| | \(-\hat{s} \ 0 \ \hat{c} \) | \(-\hat{s} \ 0 \ \hat{c} \) | \(-1 \ 0 \ 0 \ 0 \) |
| | \(0 \ \hat{s} \ 0 \ \hat{c} \) | \(0 \ \hat{s} \ 0 \ \hat{c} \) | \( 0 \ 1 \ 0 \ 0 \) |

For \( \theta = 0, 2\pi \) \( \Phi_{3,2}^{-1}(R_{1,3}) = \Phi_{3,2}^{-1}(I_4) = \pm I_4 \).
### System of Equations

**Proof:** We provide the details for $H_{1,4}$. The procedure in Remark 26 yields the following system of equations

\[
B = \{-1 + \cosh^2(\beta) - \sinh^2(\beta), x_{15}, x_{14}, x_{13}\}
\]
\[ x_{12}, x_{11} - x_{16}, x_{10}, x_9, x_8, x_7, x_6 - (\cosh(\beta) + \sinh(\beta))x_{16}, x_5, x_4, x_3, \]
\[ x_2, x_1 - (\cosh(\beta) + \sinh(\beta))x_{16}. \]

Therefore, we have
\[ x_2 = x_3 = x_4 = x_5 = x_7 = x_8 = x_9 = x_{10} = x_{15} = x_{14} = x_{13} = x_{12} = 0 \]

and
\[ x_{11} = x_{16}, \quad x_6 = (\cosh(\beta) + \sinh(\beta))x_{16}, \]
\[ x_1 = (\cosh(\beta) + \sinh(\beta))x_{16}, \quad x_{16}^2 = \cosh(\beta) - \sinh(\beta). \]

Solving this system, we get
\[ x_{16} = \sqrt{\cosh(\beta) - \sinh(\beta)} \text{ or } x_{16} = -\sqrt{\cosh(\beta) - \sinh(\beta)}. \]

Since
\[ \cosh(\beta) - \sinh(\beta) = e^{-\beta} \text{ and } \cosh(\beta) + \sinh(\beta) = e^{\beta} \]

and if \( x_{16} = \sqrt{\cosh(\beta) - \sinh(\beta)} = e^{-\beta/2} \), we get \( x_1 = e^{\beta/2}, x_6 = e^{\beta/2} \) and \( x_{11} = x_{16} = e^{-\beta/2} \).

Analogously, if \( x_{16} = -\sqrt{\cosh(\beta) - \sinh(\beta)} - e^{-\beta/2} \), then \( x_1 = -e^{\beta/2}, x_6 = -e^{\beta/2} \) and \( x_{11} = x_{16} = -e^{-\beta/2} \). Therefore, we have
\[
\Phi_{3,2}^{-1}(H_{1,4}) = \pm \begin{pmatrix}
  e^{\beta/2} & 0 & 0 & 0 \\
  0 & e^{\beta/2} & 0 & 0 \\
  0 & 0 & e^{-\beta/2} & 0 \\
  0 & 0 & 0 & e^{-\beta/2}
\end{pmatrix}.
\]

\[ \diamond \]

6 Inversion of \( \Phi_{4,1} \) via the Inversion of \( \Psi_{4,1} \)

In this section the map \( \Phi_{4,1} : \text{Spin}^+(4,1) \to \text{SO}^+(4,1) \) is inverted by linearizing \( \Phi_{4,1} \). We will see that this modus operandi works elegantly for both the case where the target matrix in \( \text{SO}^+(4,1) \) is assumed to be given by its Givens factors and the case wherein we assume that the target matrix is given by its polar decomposition. In particular, we will see that the latter provides a constructive technique to find the polar decomposition of a matrix in \( \text{Spin}^+(4,1) \).

Since this a group of certain \( 2 \times 2 \) quaternionic matrices, we have thus a technique to compute
the polar decomposition of such quaternionic matrices, without passage to the associated \( \Theta_H \) image in \( M(4, \mathbb{C}) \) and, in particular, without any eigencalculations.

As usual we begin with a basis of one-vectors for \( \text{Cl}(4, 1) = M(4, \mathbb{C}) \)

\[
V_1 = \sigma_z \otimes \sigma_x, V_2 = \sigma_y \otimes I_2, V_3 = \sigma_z \otimes \sigma_z, V_4 = \sigma_x \otimes I_2, V_5 = -\sigma_z \otimes (i\sigma_y).
\]

As shown in [8], with respect to this basis,

\[
\text{Spin}^+(4, 1) = \{ X \in M(4, \mathbb{C}) \cap \text{Im}(\Theta_H); X^* M X = M \}
\]

where

\[
M = (i\sigma_y) \oplus (-i\sigma_y).
\]

Since these matrices are \( \Theta_H \) matrices it is convenient to identify them with the corresponding matrices in \( M(2, \mathbb{H}) \). Note, however, \( M \) itself is not a \( \Theta_H \) matrix.

Next, the Lie algebra of the spin group equals

\[
\text{spin}^+(4, 1) = \{ \Lambda \in M(4, \mathbb{C}) \cap \text{Im}(\Theta_H); \Lambda^* M = -M \Lambda \}
\]

Since \( \Lambda \) is in the image of \( \Theta_H \) it is of the form

\[
\begin{pmatrix}
Z & W \\
-W & \bar{Z}
\end{pmatrix}
\]

with \( Z + Wj \in M(2, \mathbb{H}) \).

The condition \( \Lambda^* M = -M \Lambda \) forces

\[
Z = \begin{pmatrix}
a_1 + ia_2 & b \\
c & -a_1 + ia_2
\end{pmatrix}
\]

and

\[
W = \begin{pmatrix}
a_1 + i\alpha_2 & \beta_1 + i\beta_2 \\
\gamma_1 + i\gamma_2 & -\alpha_1 - i\alpha_2
\end{pmatrix}.
\]

So

\[
\Lambda = \begin{pmatrix}
a_1 + ia_2 & b & \alpha_1 + i\alpha_2 & \beta_1 + i\beta_2 \\
c & -a_1 + ia_2 & \gamma_1 + i\gamma_2 & -\alpha_1 - i\alpha_2 \\
-\alpha_1 + i\alpha_2 & -\beta_1 + i\beta_2 & a_1 - ia_2 & b \\
-\gamma_1 + i\gamma_2 & a_1 - i\alpha_2 & c & -a_1 - ia_2
\end{pmatrix}.
\]

The linearization of \( \Phi_{4,1} \) associates to \( \Lambda \in \text{spin}^+(4, 1) \) the matrix of the linear map which sends a one-vector \( V \) to the one-vector \( YV - VY \), with respect to the basis \( \{V_1, \ldots, V_5\} \) above.
We then get

$$\Psi_{4,1}(\Lambda) = \begin{pmatrix}
0 & \beta_2 + \gamma_2 & -b + c & \beta_1 + \gamma_1 & -2a_1 \\
-\beta_2 - \gamma_2 & 0 & -2\alpha_2 & 2a_2 & -\beta_2 + \gamma_2 \\
b - c & 2\alpha_2 & 0 & 2\alpha_1 & b + c \\
-\beta_1 - \gamma_1 & -2\alpha_2 & -2\alpha_1 & 0 & -\beta_1 + \gamma_1 \\
-2a_1 & -\beta_2 + \gamma_2 & b + c & -\beta_1 + \gamma_1 & 0 \\
\end{pmatrix}.$$ (30)

### 6.1 Inversion via Givens Factors

Following Example 20 every matrix in $SO^+(4,1)$ can be decomposed non-unique as

$$X = R_{14}R_{13}R_{12}H_{15}R_{24}R_{23}H_{25}R_{34}H_{35}H_{45}.$$ We then have the following result.

**Proposition 31** The table immediately below describes the $Y \in Spin^+(4,1) \subseteq M(2,\mathbf{H})$ satisfying $\Phi_{4,1}(\pm Y) = X$, where $X$ is one of the Givens matrices in the last equation.

**Proof:** The proof proceeds by expressing each $R_{ij}$ or $H_{ij}$ as the exponential of an $L_{ij} \in so(5,1)$, finding the $2 \times 2$ quaternionic matrix $K_{ij} = \Psi_{4,1}^{-1}(L_{ij})$ and then exponentiating $K_{ij}$ explicitly. This last matrix is $Y$. For brevity only the details for $H_{25}$ are displayed.

We begin by noting that $H_{25} = \text{Exp}[\theta(e_2e_5^T + e_5e_2^T)]$. By inspecting, Equation (30), it is seen that its preimage in $spin^+(4,1)$ is

$$\Theta_{\mathbf{H}}[\frac{\theta}{2} \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}] = \begin{pmatrix} \cosh(\theta/2) & -\sinh(\theta/2)k \\ \sinh(\theta/2)k & \cosh(\theta/2) \end{pmatrix}.$$
| $R_{ij}$ or $H_{ij}$ | $Y$ |
|---------------------|-----|
| $R_{14}$            | $\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2)j \\ -\sin(\theta/2)j & \cos(\theta/2) \end{pmatrix}$ |
| $R_{13}$            | $\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$ |
| $R_{1,2}$           | $\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2)k \\ -\sin(\theta/2)k & \cos(\theta/2) \end{pmatrix}$ |
| $H_{1,5}$           | $\begin{pmatrix} e^{-\theta/2} & 0 \\ 0 & e^{-\theta/2} \end{pmatrix}$ |
| $R_{4,2}$           | $\begin{pmatrix} \cos(\theta/2) + \sin(\theta/2)i & 0 \\ 0 & \cos(\theta/2) - \sin(\theta/2)i \end{pmatrix}$ |
| $R_{3,2}$           | $\begin{pmatrix} \cos(\theta/2) - \sin(\theta/2)k & 0 \\ 0 & \cos(\theta/2) + \sin(\theta/2)k \end{pmatrix}$ |
| $H_{2,5}$           | $\begin{pmatrix} \cosh(\theta/2) & -\sinh(\theta/2)k \\ \sinh(\theta/2)k & \cosh(\theta/2) \end{pmatrix}$ |
| $H_{3,5}$           | $\begin{pmatrix} \cosh(\theta/2) & \sinh(\theta/2)k \\ \sinh(\theta/2)k & \cosh(\theta/2) \end{pmatrix}$ |
| $H_{4,5}$           | $\begin{pmatrix} \cosh(\theta/2) & -\sinh(\theta/2)j \\ \sinh(\theta/2)j & \cosh(\theta/2) \end{pmatrix}$ |

**Remark 32 Agnostic Inversion - Linearization and Givens in General:** The fact that the preimages of the logarithms of the Givens factors in the spin Lie algebra always had a quadratic minimal polynomial holds for general $(p, q)$ case. This provides us with a method to invert both the abstract $\Phi_{p,q}$ and the matrix $\Phi_{p,q}$ without having to find a concrete form of $\Phi_{p,q}$. We dub the latter as agnostic inversion. Thus, the method of [16] as enhanced by iii) of Remark 2 is agnostic inversion. We will now justify our claim and thereby display a second method for agnostic inversion which uses Givens decompositions instead of calculating minors of $X$.

Specifically, the spin Lie algebra is also the space of bivectors. Let $L_{ij} = \theta(e_i e_j^T + e_j e_i^T)$ be the logarithm of a hyperbolic Givens $H_{ij}$. Its preimage in the space of bivectors is $\frac{\theta}{2} X_i X_j$, where $1 \leq p \leq p$ and $p + 1 \leq j \leq q$. Indeed the abstract $\Psi_{p,q}$ sends an element $\Lambda$ in the space of bivectors to the matrix of the linear map which sends a one-vector $V$ to $\Lambda V - VA$. From
the form of \( L_{ij} \) it then follows that if \( \Lambda \) is the preimage of \( L_{ij} \), then \( \Lambda \) commutes with all one-vectors in the basis of one-vectors \( \{ X_1, \ldots, X_p, X_{p+1}, \ldots, X_q \} \) except \( X_i \) and \( X_j \). This observation plus a few calculations show that \( \Lambda = \frac{\theta}{2} X_i X_j \). Quite clearly, \( \Lambda^2 \) is a positive multiple of the identity of \( Cl(p, q) \). So \( \text{Exp}(\Lambda) = \cosh(\frac{\theta}{2}) I + \sinh(\frac{\theta}{2}) [X_i X_j] \) is the preimage of \( H_{ij} \) in the spin Lie group. Similar comments apply to \( R_{ij} \). This provides the inversion of the abstract covering map and also the agnostic inversion of \( \Phi_{p,q} \) via iii) of Remark 2.

For the inversion of the concrete \( \Phi_{p,q} \) via linearization we need, of course, an explicit matrix form of \( \Psi_{p,q} \). However, since the embedding of the even sublagebra in the full \( Cl(p, q) \) is an algebra isomorphism onto its image, it is guaranteed that the \( \Psi_{p,q}^{-1}(L_{ij}) \) also satisfies the same quadratic annihilating polynomial and hence its exponential is easily found.

### 6.2 Inversion of \( \Phi_{4,1} \) via the polar decomposition

Let \( X \in SO^+(4, 1) \). Then in view of Remark 15, one can find constructively both its polar decomposition

\[
X = VP
\]

and the \( \hat{X} \in so(4, 1) \), such that it is symmetric and \( \text{Exp}(\hat{X}) = P \). Furthermore, by invoking Remark 16 plus a little work, we can also find a skew-symmetric, \( 5 \times 5 \), real matrix whose exponential equals \( V \).

We will presently see that it is possible to exponentiate in closed form the preimage, under \( \Psi_{4,1} \), of a symmetric matrix or a skew-symmetric matrix in \( so(4, 1) \). Therefore, using the polar decomposition to invert \( \Phi_{4,1} \) is a viable option.

To that end let \( \hat{X} \in so(4, 1) \) be symmetric. Then its preimage in \( \text{spin}^+(4, 1) \) is the \( \Theta_H \) image of the following \( 2 \times 2 \) quaternionic matrix

\[
\Lambda = \begin{pmatrix}
a_1 & b + \beta_1 j + \beta_2 k \\
-\beta_1 j - \beta_2 k & -a_1
\end{pmatrix}
\]

\[\text{(31)}\]

A quick calculation \( \Lambda^2 = \lambda^2 I_2 \) where

\[
\lambda^2 = a_1^2 + |q|^2
\]

\[\text{(32)}\]
wherein \( q \) is the quaternion \( b + \beta_1 j + \beta_2 k \). Therefore
\[
\text{Exp}(Y) = \cosh(\lambda)I_2 + \frac{\sinh(\lambda)}{\lambda} \Lambda.
\]
Hence, the preimage of \( P = \text{Exp}(\hat{X}) \) is \( \Theta_H[\cosh(\lambda)I_2 + \frac{\sinh(\lambda)}{\lambda} \Lambda] \). Note also that
\[
[\cosh(\lambda)I_2 + \sinh(\lambda)\Lambda]^{-1} = \cosh(\lambda)I_2 - \sinh(\lambda)\Lambda.
\]
Next, \( V \) is both special orthogonal and in \( \text{SO}^+(4,1) \). Therefore, it is of the form \( W \oplus 1 \), where \( W \) is \( 4 \times 4 \) special orthogonal. Hence the matrix \( \hat{Y} \in \text{so}(4,1) \) with
\[
\text{Exp}(\hat{Y}) = V
\]
is of the form
\[
\hat{Y} = Y \oplus 0_{1 \times 1}
\]
with \( Y \) that is \( 4 \times 4 \) real antisymmetric.

In view of Remark 16
\[
Y = Y_1 + Y_2
\]
with \([Y_1, Y_2] = 0\). Thus \( \hat{Y} = \hat{Y}_1 + \hat{Y}_2 \), where \( \hat{Y}_l = Y_l \oplus 0_{1 \times 1}, l = 1, 2 \). Clearly \( \hat{Y}_1 \) and \( \hat{Y}_2 \) also commute. Therefore
\[
\Psi^{-1}_{4,1}(\hat{Y}) = \Psi^{-1}_{4,1}(\hat{Y}_1) + \Psi^{-1}_{4,1}(\hat{Y}_2)
\]
and as \( \Psi_{4,1} \) is a Lie algebra isomorphism we find that the two summands on the right hand side of the last equation also commute. Thus
\[
\text{Exp}[^{-1}_{4,1}(\hat{Y})] = \text{Exp}[\Psi^{-1}_{4,1}(\hat{Y}_1)]\text{Exp}[\Psi^{-1}_{4,1}(\hat{Y}_2)].
\]
Now, the preimage of \( V = \text{Exp}(\hat{Y}) \) under \( \Phi_{4,1} \) is \( \pm \text{Exp}[\Psi^{-1}_{4,1}(\hat{Y})] \), which in turn is \( \pm \) the product of the \( \text{Exp}[\Psi^{-1}_{4,1}(\hat{Y}_l)], \ l = 1, 2 \).

Let us write \( \Psi^{-1}_{4,1}(\hat{Y}_l) = \Theta_H(Z_l + W_{lj}), \ l = 1, 2 \). Then, evidently
\[
\text{Exp}[\Psi_{4,1}(\hat{Y}_l)] = \Theta_H[\text{Exp}(Z + W_{lj})], \ l = 1, 2.
\]
Now both \( Z_l + W_{lj} \) for \( l = 1, 2 \) satisfy a cubic polynomial
\[
(Z_l + W_{lj})^3 = -\kappa^2_l(Z_l + W_{lj}), \ l = 1, 2.
\]
where \( \kappa_l \) are real (as will be shown presently). Hence

\[
\text{Exp}(Z_l + W_{lj}) = I_2 + \frac{\sin(\kappa_l)}{\kappa_l} (Z_l + W_{lj}) + \frac{1 - \cos(\kappa_l)}{\kappa_l^2} (Z_l + W_{lj})^2
\]  

(33)

and hence finding \( \Phi_{4,1}^{-1}(V) \) (thus, \( \Phi_{4,1}(X) \)) is complete.

We will next justify the claim that \( Z_1 + W_{1j} \) is indeed annihilated by a real cubic polynomial. First, inspecting Equation (8) and Equation (30), it is evident that we must also impose

\[
a_1 = 0, \beta_2 = \gamma_2 = \alpha_1, b = -c = a_2, \beta_1 = \gamma_1 = -\alpha_2
\]

in Equation (28) and Equation (29) to obtain \( Z_1 + W_{1j} \). This then yields

\[
Z_1 = a_2 \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}
\]

and

\[
W_1 = \begin{pmatrix} \alpha_1 + \alpha_2 & -\alpha_2 + \alpha_1 \\ -\alpha_2 + \alpha_1 & -\alpha_1 - \alpha_2 \end{pmatrix}
\]

Next, a direct calculation yields

\[
(Z_1 + W_{1j})^2 = (Z_1^2 - W_1W_1) + (Z_1W_1 + W_1Z_1).j.
\]

A quick calculation then shows \( Z_1W_1 + W_1Z_1 = 0 \) and

\[
Z_1^2 = 2a_2^2 \begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix}
\]

while

\[
W_1W_1 = 2(\alpha_1^2 + \alpha_2^2) \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}
\]

Thus \( Z^2 - W_2W_2 = 2(a_2^2 + \alpha_1^2 + \alpha_2^2) \begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix} \). Hence

\[
(Z_1 + W_{1j})^3 = -4(a_2^2 + \alpha_1^2 + \alpha_2^2)(Z_1 + W_{1j})
\]

In other words

\[
\kappa_1 = 2(a_2^2 + \alpha_1^2 + \alpha_2^2)^{1/2}
\]

Similarly, \( Z_2 + W_{2j} \) is expressible solely in terms of \( a_2, \alpha_1, \alpha_2 \) (but, of course, the triple \( (a_2, \alpha_1, \alpha_2) \) for \( Z_2 + W_{2j} \) is different from that for \( Z_1 + W_{1j} \)). Once again \( (Z_1 + W_{1j})^3 = -\kappa_2^2(Z_2 + W_{2j}) \), with \( \kappa_2 = 2(a_2^2 + \alpha_1^2 + \alpha_2^2)^{1/2} \).
This completes the inversion of $\Phi_{4,1}$ via the polar decomposition which we present as an algorithm below.

**Algorithm 33**

1. Let $X \in SO^+(4,1)$. Compute using Remark 16 both the polar decomposition $X = VP$ and the “logarithms” $Q \in so(4,1)$ of $P$ and the logarithm $\dot{Y} = (Y_1 + Y_2) \oplus 0_{1 \times 1}$ in $so(4,1)$, of $V$, where $Y_1, Y_2$ are as in Equation (8) and Equation (9).

2. Find $\Lambda = \Psi_{4,1}^{-1}(Q)$ and $\lambda$ as given by Equation (31) and Equation (32). Then $\Phi_{4,1}^{-1}(P) = \pm \Theta_H[\cosh(\lambda)I_2 + \sinh(\lambda)\Lambda]$.

3. Next find $Z_i + W_{ij} \in M(2, H)$ and $\kappa_i \in \mathbb{R}$ for $i = 1, 2$, from the entries of $Y_1, Y_2$. Then

$$\Phi_{4,1}^{-1}(V) = \pm \Theta_H\{[I_2 + \frac{\sin(\kappa_1)}{\kappa_1}(Z_1 + W_{1j})]^2[I_2 + \frac{\sin(\kappa_2)}{\kappa_2}(Z_2 + W_{2j}) + \frac{1 - \cos(\kappa_2)}{\kappa_2^2}(Z_2 + W_{2j})^2]\}.$$

**Remark 34**

As mentioned at the beginning of this section, the above considerations can be used to compute the polar decomposition of a matrix $Y$ in $Spin^+(4,1)$, without computing that of the associated $4 \times 4$ complex matrix that is the $\Theta_H$ image of it. Indeed, all that one has to do is to compute $X = \Phi_{4,1}(Y)$ and apply the previous algorithm to $X$.

### 7 Conclusions

Explicit algorithms for inverting the double covering maps $\Phi_{p,q}$, for $(p, q) \in \{(2, 1), (2, 2), (3, 2), (4, 1)\}$ were provided. These methods extend for the general $(p, q)$ case, at the cost of more computation.

A brief, and necessarily incomplete, comparison of the methods proposed here and also the formula in [16] follows. Both our methods and the method in [16] will require considerable computation if $n = p + q$ is large. Our methods require that a matrix form of $\Phi_{p,q}$ be available first. This is, in any case, a necessity if the principal aim of inversion is to relate matrix theoretic properties of an element in the indefinite orthogonal group to those of its preimage(s) in the spin group. On the other hand, this aim is, in general, impossible to execute and achieve when viewing $\Phi_{p,q}$ only as an abstract map. Next, as $n$ grows the matrix entries of $\Phi_{p,q}(Y)$ will be quadratic entries in a large number of variables. On the other hand, the formula in [16] will require the computation of a prohibitive number of determinants. Next,
among our methods, it is more direct to use Groebner bases for inversion - if the matrix form of $\Phi_{p,q}$ has been already calculated. The systems of equations and the attendant the Groebner basis calculations become cumbersome if the polar decomposition is used, instead of the Givens decompositions. This why in Section 3, we did not use Groebner basis techniques. On the other hand, the number of systems to be solved, when the Givens decomposition is employed, is larger than when the polar decomposition is deployed. Note, however, the number of such systems to be solved symbolically, in the Givens case, is typically lower than the $\begin{pmatrix} p + q \\ 2 \end{pmatrix}$ Givens factors, since there is repetition of these different factors (albeit with different $\theta$ or $\beta$) - see ii) of Remark 21 in Section 2.4. Finally, the Lie algebraic methods proposed require first that $\Psi_{p,q}$ be calculated. This is no harder than finding the entries of $\Phi_{p,q}$, but it is nevertheless a requisite. The inversion of $\Psi_{p,q}$ is, of course, orders of magnitude simpler than that of $\Phi_{p,q}$. However, to be able to use it effectively for the inversion of $\Phi_{p,q}$, one needs to be able to compute exponentials of matrices in the spin Lie algebra easily. This factor is the basic tradeoff between Groebner basis methods and the Lie algebraic method proposed here. For $n \leq 6$, in most cases, there are explicit formulae for the exponential. The more this can be extended to larger $n$, the Lie algebraic method becomes more competitive. On the other hand, for any $n$ the exponentiation is always elementary when Givens factors are used as Remark 32 shows. Finally, the combination of linearization and Givens factors provides an alternative for the inversion of the abstract covering map and also what is dubbed agnostic inversion of the concrete covering map, for any $n$.

8 Appendix I: Special Bases for Clifford Algebras

In this appendix we show that every $Cl(p, q)$ possesses a basis of one-vectors satisfying $BP1$ and $BP2$ of Section 2.3. We note that the work, [10], also provides special bases of one-vectors for real Clifford algebras, but the properties of these special bases are neither $BP1$ nor $BP2$.

We begin by recalling three iterative constructions for Clifford algebras, [12, 14] and show that these constructions inherit $BP1$ and $BP2$. 
• **IC1** If \( \{V_1, \ldots, V_p, W_1, \ldots, W_q\} \) is a basis of one-vectors for \( Cl(p, q) \) then

\[
\sigma_z \otimes V_j, \begin{pmatrix}
0 & I \\
I & 0
\end{pmatrix}, \quad \sigma_z \otimes W_k, \begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix}
\]

is a basis of one-vectors for \( Cl(p+1, q+1) \). Here \( I \) is the identity element of \( Cl(p, q) \) and \( 0 \) is the zero element of \( Cl(p, q) \).

Let \( X \in \{V_1, \ldots, V_p, W_1, \ldots, W_q\} \). Then note that

\[
\begin{pmatrix}
0 & I \\
I & 0
\end{pmatrix}^* [\sigma_z \otimes X]
\]

is a 2 \( \times \) 2 block matrix with zeroes on its diagonal block. Similarly, \( \begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix}^* [\sigma_z \otimes X] \)

also has zero diagonal blocks. Similarly, the trace (respectively real part of trace) of

\[
\begin{pmatrix}
0 & I \\
I & 0
\end{pmatrix}^* \begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix}
\]

is also zero. Finally, if \( X^*Y \) has zero trace (respectively zero real part of trace) then the same holds for \( (\sigma_z \otimes X)^* (\sigma_z \otimes Y) \). So property **BP2** is inherited by the iteration **IC1**. That property **BP1** is also inherited by the iteration **IC1** is evident.

• **IC2** If \( \{E_1, \ldots, E_m\} \) is a basis of one-vectors for \( Cl(m, 0) \) then the following set is a basis of one-vectors for \( Cl(m + 8, 0) \)

\[
\{I \otimes V_1, \ldots, I \otimes V_8, E_1 \otimes L, \ldots, E \otimes L\}
\]

where

- \( I \) is the identity on \( Cl(m, 0) \).

- \( \{V_1, \ldots, V_8\} \) is the basis of one-vectors for \( Cl(8, 0) \) used in Theorem 35 below.

- \( L = \sigma_x \otimes \sigma_x \otimes i\sigma_y \otimes i\sigma_y \).

Note that \( L \) is a real symmetric matrix. The \( V_i \)'s are also real and either symmetric or antisymmetric. Therefore, by item ii) of Remark 12 the reality of \( L \) and the \( V_i \) ensures that **BP2** is inherited by **IC2**. Since \( L \) is real symmetric and \( V_i^T = \pm V_i \), we also see that **BP1** is also inherited by **IC2**.

• **IC3** If \( \{F_1, \ldots, F_m\} \) is a basis of one-vectors for \( Cl(0, m) \) then the following is a basis of one-vectors for \( Cl(0, m + 8) \)

\[
\{I \otimes V_1, \ldots, I \otimes V_8, F_1 \otimes K, \ldots, F_m \otimes K\}
\]
where

- $I$ is the identity on $\text{Cl}(0, m)$.
- $\{V_1, \ldots, V_8\}$ is the basis of one-vectors for $\text{Cl}(0, 8)$ used in Theorem 35 below.
- $K = i\sigma_y \otimes \sigma_y \otimes \sigma_z \otimes \sigma_z$.

As in the previous case $K$ is real symmetric, while each $V_i$ is real and either symmetric or antisymmetric. Therefore, both $\text{BP1}$ and $\text{BP2}$ are inherited by $\text{IC3}$.

We are now in a position to prove the main result of this appendix.

**Theorem 35** Every real Clifford algebra has a basis of one-vectors with the properties $\text{BP1}$ and $\text{BP2}$.

**Proof:** As observed above both $\text{BP1}$ and $\text{BP2}$ are inherited by each of $\text{IC1}$, $\text{IC2}$ and $\text{IC3}$. Since every $\text{Cl}(r, s)$ can be obtained by repeatedly applying $\text{IC1}$ to either some $\text{Cl}(n, 0)$ or $\text{Cl}(0, n)$, and every $\text{Cl}(n, 0)$ (respectively $\text{Cl}(0, n)$) is obtained by applying $\text{IC2}$ (respectively $\text{IC3}$) to $\text{Cl}(m, 0)$, $m = 0, \ldots, 8$ (respectively $\text{Cl}(0, m)$, $m = 0, \ldots, 8$) it suffices to verify the theorem for $\text{Cl}(m, 0)$ and $\text{Cl}(0, m)$ for $m = 0, \ldots, 8$.

Let us begin with $\text{Cl}(m, 0)$. The following is the list of bases of one-vectors that will be used for this purpose

- $B_{0,0} = \Phi$.
- $B_{1,0} = \{\sigma_x\}$.
- $B_{2,0} = \{\sigma_z, \sigma_x\}$.
- $B_{3,0} = \{\sigma_z, \sigma_x, i\sigma_y\}$.
- $B_{4,0} = \{\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}, \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}, \sigma_z\}$.
- $B_{5,0} = \{\sigma_x \otimes \sigma_z \otimes (i), \sigma_x \otimes \sigma_z \otimes (j), \sigma_x \otimes \sigma_z \otimes (k), \sigma_z \otimes \sigma_z, \sigma_x \otimes \sigma_z \otimes \sigma_z\}$.
- $B_{6,0} = \{I_2 \otimes \sigma_z, I_2 \otimes \sigma_x, iI_2 \otimes (i\sigma_y), jI_2 \otimes (i\sigma_y), k\sigma_x \otimes (i\sigma_y), k\sigma_z \otimes (i\sigma_y)\}$.
- $B_{7,0} = \{I_4 \otimes \sigma_z, I_4 \otimes \sigma_x, -i\sigma_z \otimes I_2 \otimes (i\sigma_y), i\sigma_y \otimes I_2 \otimes (i\sigma_y), -i\sigma_x \otimes \sigma_z \otimes (i\sigma_y), -i\sigma_x \otimes \sigma_z \otimes (i\sigma_y), -i\sigma_x \otimes \sigma_z \otimes (i\sigma_y), \sigma_x \otimes -i\sigma_y \otimes (i\sigma_y)\}$.
• $B_{8,0} = \{I_8 \otimes \sigma_z, I_8 \otimes \sigma_x, -\sigma_x \otimes i\sigma_y \otimes I_2 \otimes (i\sigma_y),$
$-\sigma_x \otimes i\sigma_y \otimes I_2 \otimes (i\sigma_y), -\sigma_z \otimes i\sigma_y \otimes \sigma_z \otimes (i\sigma_y), -\sigma_z \otimes i\sigma_y \otimes \sigma_z \otimes (i\sigma_y),$
$-\sigma_z \otimes i\sigma_y \otimes (i\sigma_y), \} = \{V_1, \ldots, V_8\}.$

By construction $\text{BP1}$ and $\text{BP2}$ hold for these eight bases. Next we verify the assertion for $\text{Cl}(0, m)$. We work the following sets of one-vectors for $m \leq 8$

• $B_{0,1} = \{i\}$.

• $B_{0,2} = \{i, j\}$.

• $B_{0,3} = \{\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}, \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\}.$

• $B_{0,4} = \{\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}, \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}\}.$

• $B_{0,5} = \{i\sigma_x \otimes I_2, i\sigma_y \otimes I_2, i\sigma_x \otimes \sigma_x, i\sigma_x \otimes \sigma_z, \sigma_z \otimes (i\sigma_y)\}.$

• $B_{0,6} = \{\sigma_z \otimes (i\sigma_y) \otimes I_2, i\sigma_y \otimes I_2, I_4, i\sigma_y \otimes I_2, i\sigma_y \otimes \sigma_x, i\sigma_y \otimes \sigma_z, \sigma_z \otimes \sigma_z \otimes (i\sigma_y)\} = \{Z_1, \ldots, Z_6\}.$

• $B_{0,7} = \{Z_1 \oplus Z_1, \ldots, Z_6 \oplus Z_6, \sigma_z \otimes \sigma_z \otimes (i\sigma_y) \otimes \sigma_z \otimes \sigma_z \otimes (i\sigma_y)\}$, where the $Z_j$'s are defined as in the previous item.

• $B_{0,8} = \{I_4 \otimes \sigma_z \otimes (i\sigma_y), I_4 \otimes (i\sigma_y) \otimes I_2, I_2 \otimes \sigma_z \otimes \sigma_z \otimes (i\sigma_y), I_2 \otimes \sigma_z \otimes \sigma_z \otimes (i\sigma_y), I_2 \otimes (i\sigma_y) \otimes \sigma_z \otimes I_2, I_2 \otimes (i\sigma_y) \otimes \sigma_z \otimes \sigma_z, I_2 \otimes (i\sigma_y) \otimes \sigma_z \otimes \sigma_z \otimes (i\sigma_y), I_2 \otimes (i\sigma_y) \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes (i\sigma_y)\} = \{V_1, \ldots, V_8\}.$

Again, by construction $\text{BP1}$ and $\text{BP2}$ hold for these bases also. This concludes the proof.

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