OPERADS WITH TRIVIAL $A$-ACTIONS

YU LI, ZIHAO QI, YONGJUN XU, JAMES J. ZHANG, ZERUI ZHANG, AND XIANGUI ZHAO

Abstract. We study operads with trivial $A$-actions and prove an equivalence between the category of $A$-trivial operads and that of pseudo-graded-Perm associative algebras. As a consequence, we show that finitely generated $A$-trivial operads are right noetherian of integral Gelfand-Kirillov dimension and that every element in a prime $A$-trivial operad is central.

0. Introduction

The notion of an operad was introduced by Boardman-Vogt [BV73] and May [Ma72] in homotopy theory in early 1970s. Since then many applications of operads have been discovered in different fields such as algebra, category theory, combinatorics, geometry, mathematical physics, topology and so on. Important developments in operad theory include the Koszul duality by Ginzburg-Kapranov [GK94] and an operadic proof of the formality theorem by Kontsevich and Tamarkin [Ko99, Ta98, Ta99]. A number of interesting questions have been proposed from different aspects of the operad theory. In this paper we study operads from an algebraic viewpoint.

In this paper an operad is always symmetric and locally finite (i.e., an operad $P$ with each $P(n)$ finite dimensional over $k$) unless otherwise stated.

Definition 0.1. Let $P$ be an operad over a base field $k$ and $n \geq 1$ be an integer.

1. An element $\lambda \in P(n)$ is called $A$-trivial if $\lambda * \sigma = \lambda$ for all $\sigma \in A_n$.
2. $P$ is called $A$-trivial if elements in $P(n)$ is $A$-trivial for all $n \geq 1$.
3. $P$ is called almost $A$-trivial if elements in $P(n)$ is $A$-trivial for all $n \gg 0$.

In this paper we study (almost) $A$-trivial operads.

0.1. Main results. Our main result is

Theorem 0.2 (Theorem 5.6). Suppose char $k \neq 2$. There is an equivalence between the category of $A$-trivial operads and that of pseudo-graded-Perm algebras (Definition 3.5).

We have two corollaries that show nice properties of $A$-trivial operads.

Let $\dim$ denote the $k$-vector space dimension. The Gelfand-Kirillov dimension (or $GK$-dimension for short) of a locally finite operad $P$ is defined to be

$$GKdim(P) := \limsup_{n \to \infty} \log_n \left( \sum_{i=0}^{n} \dim P(i) \right).$$
The GK-dimension of an operad was first defined in [KP15, p. 400] and then in [BYZ20] Definition 4.1 and studied for nonsymmetric operads in [QX20].

**Corollary 0.3** (Theorem 5.7). Suppose char \( k \neq 2 \). Let \( P \) be a finitely generated almost \( A \)-trivial operad.

1. The Hilbert series (namely, the generating series) of \( P \) is rational.
2. \( P \) is right noetherian.
3. \( \text{GKdim} \, P \) is an integer.

In general, \( P \) in Corollary 0.3 is not left noetherian [Example 6.4(1)]. We should compare part (1) of the above Corollary with [QX20, Proof of Theorem 0.7(3)] which, together with [LQ21a, Theorem 4.4], says that Corollary 0.3(1) fails for any operad of GK-dimension strictly larger than 1. Surprisingly Corollary 0.3 fails for nonsymmetric operads of Gelfand-Kirillov dimension one, see [QX20, Construction 1.3 and Example 6.2]. If char \( k = 2 \), we suspect that Corollary 0.3 is still valid. However our proof of Corollary 0.3 is heavily dependent on Theorem 0.2 that is dependent on the hypothesis that char \( k \neq 2 \).

Another surprising consequence of Theorem 0.2 is the following. An operad \( P \) is called **prime** if \( I \circ J := \left( \sum_{i} I \circ_{i} J \right) \circ S \neq 0 \) for all nonzero ideals \( I \) and \( J \) of \( P \) [Definition 1.6(1)].

**Corollary 0.4** (Corollary 5.12). Let \( P \) be an infinite dimensional almost \( A \)-trivial prime operad. Then every nonzero element \( \mu \) of \( P \) of arity \( \geq 1 \) is central, namely, for any other element \( \nu \) in \( P \),

\[
\mu \circ_{i} \nu = \nu \circ_{j} \mu
\]

for all \( 1 \leq i \leq \text{Ar(} \mu \text{)} \) and \( 1 \leq j \leq \text{Ar(} \nu \text{)} \).

The main result and the corollaries have several applications.

**0.2. Application 1:** classification of prime operads of GK-dimension 1. Let \( \text{Com} \) be the operad that encodes nonunital commutative algebras over \( k \). To state the classification result we need to recall the following operad that encodes all skew-symmetric totally associative ternary algebras [AM, Section 2.1].

**Example 0.5.** [AM] Definition 2.1 Let \( Ope \) be the symmetric operad defined by

1. \( Ope(n) = \begin{cases} 0 & n \text{ is even,} \\ \{ \mu_{n} \} & n \text{ is odd,} \end{cases} \)
2. \( \mu_{n} \ast \sigma = \text{sgn(} \sigma \text{)} \mu_{n} \) for all \( \sigma \in S_{n} \), where \( \text{sgn(} \sigma \text{)} \) is the sign of \( \sigma \),
3. when both \( n \) and \( m \) are odd, \( \mu_{n} \circ_{i} \mu_{m} = \mu_{n+m-1} \) for all \( 1 \leq i \leq n \).

By [AM Section 2.1] algebras over \( Ope \) are skew-symmetric totally associative ternary algebras.

We say \( \dim \, P(n) \) is **uniformly bounded** if there is a finite number \( N \) such that \( \dim \, P(n) \leq N \) for all \( n \). Here is one of the main results of [LQ21a].

**Theorem 0.6.** [LQ21a, Theorem 5.2] Suppose \( P \) is an infinite dimensional prime operad such that \( \dim \, P(n) \) is uniformly bounded. Then \( P \) is finitely generated. If further \( k \) is algebraically closed, then \( P \) is a suboperad of \( \text{Com} \) or \( Ope \). In this case \( P \) is also connected.

Note that the hypothesis of \( \dim \, P(n) \) being uniformly bounded is equivalent to \( \text{GKdim} \, P = 1 \) in some cases.
0.3. Application 2: Bergman’s gap theorem for operads. Bergman proved that there is no finitely generated associative algebra with GK-dimension strictly between 1 and 2 [KL00, Theorem 2.5]. In [QX20, Theorem 0.1], the authors proved that there is no finitely generated nonsymmetric operad with GK-dimension strictly between 1 and 2. Note that the Bergman’s gap theorem fails for shuffle operads [LQ21a]. See [QX20] and [LQ21a] for a list of algebraic structures that Bergman’s gap theorem holds. [QX20, Question 0.8] asks if the Bergman’s gap theorem holds for symmetric operads and we answer this question affirmatively in [LQ21a].

Theorem 0.7. [LQ21a, Theorem 4.4] Let $P$ be a finitely generated operad of GK-dimension $< 2$. Then $P$ is almost $A$-trivial and of GK-dimension $\leq 1$. If further char $k \neq 2$, then conclusions in Corollary 0.3 hold.

For every non-negative integer $d$ or any real number $d \geq 3$, there is a finitely generated operad with GK-dimension $d$ [QX20, Theorem 0.7(2)]. By the way it is still open if there is a finitely generated operad with GK-dimension strictly between 2 and 3, see [QX20, Question 0.8].

0.4. Application 3: Central elements and centralizers. Motivated by Corollary 0.4 we study central elements in an operad. Surprisingly central elements will not occur often in a prime operad.

Theorem 0.8. [LQ21b, Theorem 0.1] Let $P$ be a prime operad containing a nonzero central element of arity $\geq 2$. Then $P$ is $A$-trivial, every element in $P$ is central, and $P$ is isomorphic to either $G_{str}(A)$ [Example 3.3] or $G_{sg}(A)$ [Example 3.7 and Remark 6.1] for some commutative graded domain $A$.

The above theorem fails if $P$ has only a central element of arity 1, see [LQ21a, Example 6.2(7)].

By the above applications, $A$-trivial operads are useful for understanding several special classes of operads.

0.5. Application 4: Dotsenko’s forgetful functor. Dotsenko defined a forgetful functor $F$ from the category of operads to the category of N-graded associative algebras, see [Dot19, Definition 3.1] and Subsection 1.5. There are a lot of fundamental questions in understanding properties of this forgetful functor. For example, at what level is $P$ determined by $F(P)$ [LQ21b]? The following is an interesting result related to the above question whose proof uses $A$-trivial operads.

Theorem 0.9. [LQ21b, Corollary 0.5] Let $A$ be a N-graded prime algebra that is not commutative. If $A$ contains a nonzero central element of positive degree, then $A$ is not isomorphic to $F(P)$ for any symmetric operad $P$.

The rest of this paper is organized as follows. Section 1 recalls the partial definition of an operad and other basic material for later sections. We introduce torsion elements in operads in Section 2. In Section 3, we study pseudo-graded-Perm algebras that are associative-algebra models for $A$-trivial operads. In Section 4, we prove an equivalence between the category of $S$-trivial operads and the category of graded Perm algebras. Generalizing the results in Section 4, we prove the main result Theorem 0.2 and Corollaries 0.3 and 0.4 in Section 5. In the final section, we give some comments, remarks, examples, and questions.
1. Preliminaries

This section contains some definitions and preliminary material that will be used in later sections. Throughout let $k$ be a base field. All algebraic objects are over $k$ unless otherwise stated.

1.1. Partial definition of a symmetric operad. We recall the partial definition \cite{LV12} Section 5.3.4] of an operad below.

**Definition 1.1.** A symmetric operad consists of the following data:

1. a sequence $\{P(n)\}_{n \geq 0}$ of right $kS_n$-modules, whose elements are called $n$-ary operations,
2. the arity of an element $\nu \in P(n)$ is defined to be $\text{Ar}(\nu) := n$,
3. an element $1 \in P(1)$ called the identity, which is also denoted by $1_P$,
4. for all integers $m \geq 1$, $n \geq 0$, and $1 \leq i \leq m$, a partial composition map $-\circ_i : P(m) \otimes P(n) \to P(m + n - 1)$,

satisfying the following axioms:

(a) for $\theta \in P(n)$ and $1 \leq i \leq n$,
\[ \theta \circ_i 1_P = \theta = 1_P \circ_i \theta; \]

(b) for $\lambda \in P(l)$, $\mu \in P(m)$ and $\nu \in P(n)$,
\[ (\lambda \circ_i \mu) \circ_{i+1+j} \nu = \lambda \circ_i (\mu \circ_j \nu), \quad 1 \leq i \leq l, 1 \leq j \leq m, \]

(E1.1.1)
\[ (\lambda \circ_i \mu) \circ_{k+1+m} \nu = (\lambda \circ_k \nu) \circ_i \mu, \quad 1 \leq i < k \leq l; \]

(E1.1.2)
\[ \mu \circ_i (\nu \ast \sigma) = (\mu \circ_i \nu) \ast \sigma', \]

(E1.1.3)
\[ (\mu \ast \phi) \circ_i \nu = (\mu \circ_i \nu) \ast \phi'' , \]

where
\[ \sigma' = \vartheta_{m;1,\ldots,1,n,1,\ldots,1}(1_m,1,\ldots,1,1,1,\ldots,1), \]
\[ \phi'' = \vartheta_{m;1,\ldots,1,n,1,\ldots,1}(\phi,1,\ldots,1,1,1,1,\ldots,1). \]

(see \cite{BYZ20} E8.1.3) for the definition of $\vartheta_{m;1,\ldots,1,n,1,\ldots,1}$.

In this paper, by an operad, we usually mean a reduced (i.e., $P(0) = 0$), locally finite (i.e., $\text{dim} P(n) < \infty$ for all $n$), symmetric operad unless otherwise stated.

1.2. A sign lemma. Let $\text{sgn}(\sigma)$ denote the sign of $\sigma \in S_n$. The following lemma can be verified routinely and its proof is omitted.

**Lemma 1.2.** Retain the notation as in Definition 1.1

1. In the situation of \[E1.1.3\], $\text{sgn}(\sigma') = \text{sgn}(\sigma)$.
2. In the situation of \[E1.1.4\], $\text{sgn}(\phi'') = (-1)^{(\text{Ar}(\nu)-1)(\phi(i)-1)} \text{sgn}(\phi)$. Consequently, we have
3. (a) $\text{sgn}(\phi'') = \text{sgn}(\phi)$ if the arity $\text{Ar}(\nu)$ of $\nu$ is odd as in \[E1.1.4\].
4. (b) If $\phi(i) = i$ in \[E1.1.4\], then $\text{sgn}(\phi'') = \text{sgn}(\phi)$.
5. (c) Suppose $\text{Ar}(\nu)$ is even as in \[E1.1.4\]. If $\phi = (a,i)$ with $a < i$ (resp. $(i,b)$ with $i < b$), then $\text{sgn}(\phi'') = (-1)^{i-a-1}$ (resp. $\text{sgn}(\phi'') = (-1)^{b-i-1}$).
(2d) Suppose \( \text{Ar}(\nu) \) is even in \([E1.1.4]\). If \( \phi = (a, a+1, a+2) \) with \( i = a \) (resp. \( i = a+1 \) or \( i = a+2 \)), then \( sgn(\phi'') = -1 \) (resp. \( sgn(\phi'') = -1 \) or \( sgn(\phi'') = 1 \)).

1.3. \( \mathbb{S}_n \)-representations. In this subsection we discuss some representations of \( \mathbb{S}_n \) and their extension groups (and we believe all statements are known). It is well-known that there are only two 1-dimensional \( \mathbb{S}_n \)-modules, namely, the trivial representation, denoted by \( \text{tr} \), and the sign representation, denoted by \( \text{sg} \).

Lemma 1.3. Let \( p \) be \( \text{char} \ k \). Suppose \( n \geq 5 \).

1. If \( p \neq 2 \), then \( \text{Ext}^1_{k\mathbb{S}_n}(M, N) = 0 \) for all \( M, N \in \{ \text{tr}, \text{sg} \} \).
2. If \( p = 2 \), then \( \text{tr} = \text{sg} \) and \( \text{Ext}^1_{k\mathbb{S}_n}(\text{tr}, \text{tr}) = k \).
3. Suppose \( p = 2 \). If

\[
0 \rightarrow \text{tr} \rightarrow E \rightarrow \text{tr} \rightarrow 0
\]

is a short exact sequence, then the \( \mathbb{A}_n \)-action on \( E \) is trivial.

Proof. (1) Consider a short exact sequence

\[
(E1.3.1) \quad 0 \rightarrow M \rightarrow E \rightarrow M \rightarrow 0
\]

where \( M = \text{tr} \). Let \( \{x_1, x_2\} \) be a \( k \)-linear basis of \( E \) such that \( kx_1 \) is a submodule of \( E \). Then, for every \( \sigma \in \mathbb{S}_n \),

\[
x_1 \ast \sigma = x_1,
\]

\[
x_2 \ast \sigma = x_2 + f(\sigma)x_1
\]

where \( f(\sigma) \in k \). It is easy to see that \( f(\sigma_1 \circ \sigma_2) = f(\sigma_1) + f(\sigma_2) \) for all \( \sigma_1, \sigma_2 \in \mathbb{S}_n \). In other words, \( f \) is a group homomorphism from \( \mathbb{S}_n \rightarrow (k,+) \). Since \( n \geq 5 \), \( \mathbb{A}_n \) is simple. Consequently, \( f \) factors through \( \mathbb{S}_n \rightarrow \mathbb{S}_n/\mathbb{A}_n \), namely, \( f(\sigma) = 0 \) for all \( \sigma \in \mathbb{A}_n \). Thus \([E1.3.1]\) is a short exact sequence of modules over \( k(\mathbb{S}_n/\mathbb{A}_n)(\cong k\mathbb{Z}/(2)) \). It must split as \( p \neq 2 \). So \( \text{Ext}^1_{k\mathbb{S}_n}(\text{tr}, \text{tr}) = 0 \). Similarly, \( \text{Ext}^1_{k\mathbb{S}_n}(\text{sg}, \text{sg}) = 0 \).

Next we consider a short exact sequence

\[
(E1.3.2) \quad 0 \rightarrow \text{tr} \rightarrow E \rightarrow \text{sg} \rightarrow 0
\]

where \( \text{tr} = ky_1 \) and \( \text{sg} = ky_2 \). So we can assume that \( \{y_1, y_2\} \) is a basis of \( E \), and for every \( \sigma \in \mathbb{A}_n \),

\[
y_1 \ast \sigma = y_1,
\]

\[
y_2 \ast \sigma = sgn(\sigma)y_2 + f(\sigma)y_1 = y_2 + f(\sigma)y_1.
\]

Then the group homomorphism \( f : \mathbb{A}_n \rightarrow (k,+) \) is trivial as \( \mathbb{A}_n \) is simple. Therefore the \( \mathbb{A}_n \)-action on \( E \) is trivial. Thus \([E1.3.2]\) is a short exact sequence of right modules over \( k(\mathbb{S}_n/\mathbb{A}_n)(\cong k\mathbb{Z}/(2)) \). It must split. So \( \text{Ext}^1_{k\mathbb{S}_n}(\text{tr}, \text{sg}) = 0 \). Similarly, \( \text{Ext}^1_{k\mathbb{S}_n}(\text{sg}, \text{tr}) = 0 \).

(2,3) Suppose we have a non-split short exact sequence

\[
0 \rightarrow \text{tr} \rightarrow E \rightarrow \text{tr} \rightarrow 0.
\]

Pick a basis element \( x \) for \( \text{tr} \). Then \( E \) is a 2-dimensional with basis \( \{x_1, x_2\} \) where \( kx_1 \) is the unique 1-dimensional submodule of \( E \). For every \( \sigma \in \mathbb{S}_n \), write \( \sigma(x_1) = x_1 \) and \( \sigma(x_2) = x_2 + f(\sigma)x_1 \). Then \( f(\sigma_1 \circ \sigma_2) = f(\sigma_1) + f(\sigma_2) \). This means that \( f : \mathbb{S}_n \rightarrow (k,+) \) is a group homomorphism. Since \( \mathbb{A}_n \) is simple (when \( n \geq 5 \)), \( f \) factors through \( \mathbb{S}_n \rightarrow \mathbb{S}_n/\mathbb{A}_n \). Note that when \( p = 2 \), there is a unique non-trivial group homomorphism (up to a scalar) \( f : \mathbb{S}_n \rightarrow (k,+) \). So \( \text{Ext}^1_{k\mathbb{S}_n}(\text{tr}, \text{tr}) = k \). \( \square \)
1.4. Almost isomorphisms. Let $P$ be an operad and $w$ be an integer $\geq 2$. Let $P_{(w)}$ denote the suboperad of $P$ such that

$$P_{(w)}(n) = \begin{cases} \frac{\mathbb{N}}{w} & n = 1, \\ 0 & 2 \leq n \leq w - 1, \\ P(n) & n \geq w. \end{cases}$$

We say two operads $P$ and $Q$ are almost isomorphic if there is a $w$ such that $P_{(w)}$ and $Q_{(w)}$ are isomorphic. It is clear that almost isomorphisms form an equivalence relation.

The Hilbert series (also called generating series) of $P$ is defined to be the formal power series $[\text{KPI}15]$ (0.1.2)

$$H_P(t) := \sum_{n=0}^{\infty} \dim P(n) t^n.$$ 

In $[\text{KPI}15]$ and $[\text{QX20}]$ it is denoted by $G_P(t)$. The Hilbert series of a graded vector space or a graded algebra is defined in the same way.

Let $I$ and $J$ be two subspace of an operad $P$. Then $I \circ J$ is the $S$-submodule of $P$ generated by $x \circ_i y$ for all $x \in I$, $y \in J$ and $1 \leq i \leq \text{Ar}(x)$ where $\text{Ar}(x)$ denotes the arity of $x$ [Definition $[\text{BD16}](\text{i})$]. Let $I \bullet J$ denote the $S$-submodule of $P$ generated by $x \circ (y_1, \ldots, y_{\text{Ar}(x)}$) where $x \in I$ and $y_1, \ldots, y_{\text{Ar}(x)} \in J$.

We say $P$ is left noetherian (resp. right noetherian) if the left (resp. right) ideals satisfy the ascending chain condition. Note that $P$ is left noetherian if and only if every left idea $J$ of $P$ is of the form $\sum_{i=1}^{N} P \circ (kx_i)$ for some finite subset $\{x_i\}_{i=1}^{N} \subseteq J$. Similarly, $P$ is right noetherian if and only if every right idea $I$ of $P$ is of the form $\sum_{i=1}^{N} (kx_i) \bullet P$ for some finite subset $\{x_i\}_{i=1}^{N} \subseteq I$.

Lemma 1.4. Let $P$ be an operad and let $w \geq 2$ be an integer.

1. $P$ has rational Hilbert series if and only if so does $P_{(w)}$.
2. $\text{GKdim } P = \text{GKdim } P_{(w)}$.
3. $P$ is finitely generated if and only if so is $P_{(w)}$.
4. $P_{(w)}$ is left noetherian if and only if so is $P$.
5. If $P_{(w)}$ is right noetherian, then so is $P$.

Proof. (1,2) Clear.

(3) If $P_{(w)}$ is finitely generated by $X$, then $P$ is generated by $X \cup \bigoplus_{i=0}^{w-1} P(i)$.

For the other implication, we assume that $P$ is finitely generated. By $[\text{QX20}]$ Lemma 3.2(2)], $P_{(2)}$ is finitely generated. Let $X$ be a finite set of generators of $P_{(2)}$. Let $r$ be the largest arity of an element in $X$. Let $N = w + r^{w-1}$. We claim that $P_{(w)}$ is generated by $\Phi := \bigoplus_{i=0}^{N} P(i)$. If the claim is true, then the assertion follows from the claim and the fact that $P$ is locally finite.

We prove the claim by the induction on the arity of an element $f \in P_{(2)}(n) = P_{(w)}(n)$ for $n \geq w$. Nothing needs to prove when $\text{Ar}(f) \leq N$. Now assume that $a := \text{Ar}(f) > N$. Considering $P_{(2)}$ to be generated by $X$, $f$ is a linear combination of the elements of form $g * \sigma$ where $g$ is a tree monomial (see $[\text{BD16}]$ or $[\text{QX20}]$ Subsection 1.3]) generated by $X$ and where $\sigma \in S_a$. Without loss of generality, we may assume that $f = g$ is a tree monomial generated by $X$. Since $a > N > r^{w-1}$, the height $ht(f)$ of the underlying tree of $f$ $[\text{QX20}]$ Subsection 1.2] is at least $w$. Pick an internal vertex $v$ such that there is a subtree of height $w - 1$ rooted at $v$. Let $v$ be the largest subtree monomial of $f$ rooted at $v$. Then $f = x \circ_i y$ for some $x$
and \(1 \leq i \leq \text{Ar}(x)\). By the choice of \(v\) and \(y\), we have that the height of \(y\) is \(w - 1\). Since every element in \(X\) has arity between 2 and \(r\), we have

\[
w = 1 + h(t(y)) \leq \text{Ar}(y) \leq r^{h(t(y))} = r^{w-1} < N.
\]

Hence \(y \in \Phi\). And

\[
\text{Ar}(x) = \text{Ar}(f) - \text{Ar}(y) + 1 \geq N - r^{w-1} + 1 \geq w + r^{w-1} - r^{w-1} + 1 > w
\]

which implies that \(x \in P\{w\}\). By induction hypothesis, \(x\) is generated by \(\Phi\). Therefore \(f = x \circ_i y\) is generated by \(\Phi\) as required.

(4,5) Proofs are easy and omitted. \(\square\)

1.5. **Forgetful functor sending an operad to an associative algebra.** We define a forgetful functor from the category of (symmetric) operads to the category of \(\mathbb{N}\)-graded algebras that is analogous to the Dotsenko’s forgetful functor [Dot19, Definition 3.1].

Let \(P\) be an operad (which is reduced in this paper). We define a unital \(\mathbb{N}\)-graded associative algebra \(A := (\oplus_{i=0}^{\infty} A_i, \cdot)\) by

(i) \(A_i = P(i + 1)\), for all \(i \geq 0\), and

(ii) \(x \cdot y = x \circ_i y\) for all homogeneous elements in \(A\).

By Definition 1.1(a) and (E1.1.1) with \(i = 1 = j\), \(A\) is a graded associative algebra. We denote it by \(A_P\). We now define a functor \(F\) from the category of (reduced) operads, denoted by \(\mathcal{O}p\), to the category of unital \(\mathbb{N}\)-graded associative algebras, denoted by \(\mathbb{N}_{gr Ass}\) as follows: for every \(P \in \mathcal{O}p\), \(F(P) = A_P\) and \(F\) is the canonical restriction when applied to morphisms. The following lemma is obvious.

**Lemma 1.5.** The \(F\) defined as above is a functor from \(\mathcal{O}p\) to \(\mathbb{N}_{gr Ass}\). Further, \(H_{P}(t) = tH_{A_p}(t)\). As a consequence, \(F\) preserves the GK-dimension and rationality of the Hilbert series.

It is obvious that the forgetful functor \(F\) "forgets" structures such as the right \(S\)-action and partial compositions. Hence \(P\) cannot be recovered from \(F(P)\) in general. However it is interesting to ask at what level \(P\) is determined by \(F(P)\). One such example is [LQ21b Theorem 0.4].

1.6. **(Semi)Primeness.** We define a prime (or semiprime) operad as follows.

**Definition 1.6.** Let \(P\) be an operad.

(1) We call \(P\) prime if for any nonzero ideals \(I\) and \(J\) of \(P\), \(I \circ J \neq 0\). An ideal \(I\) is called prime if \(P/I\) is a prime operad.

(2) We call \(P\) semiprime if the intersection of all prime ideals of \(P\) is 0.

The following two lemmas are easy and their proofs are omitted.

**Lemma 1.7.** Suppose \(P\) is semiprime. For every nonzero ideal \(I\) of \(P\), \(I \circ I \neq 0\).

**Lemma 1.8.** Let \(P\) be an operad. If \(A_P := F(P)\) is prime, then so is \(P\).

The converse of Lemma 1.8 is false, see Example 6.3.
2. Torsion elements

First we recall the definition of a torsion element in a graded algebra. Let $A := \oplus_{i=0}^{\infty} A_i$ be a locally finite $\mathbb{N}$-graded algebra. An element $x$ in $A$ is called right torsion if $xA_{\geq n} = 0$ for some $n \gg 0$ [AZ94, p. 233]. Similarly we define a left torsion element. The set of right (resp. left) torsion elements of $A$ is denoted by $\tau^r(A)$ (resp. $\tau^l(A)$). The left and right torsions of operads can be defined similarly, see below. For two subspaces $I$ and $J$ of an operad $P$, $I \circ J$ was introduced in Subsection 1.4.

Definition 2.1. Let $P$ be an operad and let $P_{\geq n}$ denote $\oplus_{i\geq n} P(i)$.

1. The left torsion ideal of $P$ is defined to be
   \[ \tau^l(P) = \{ x \in P \mid P_{\geq n} \circ (kx) = 0, \text{ for } n \gg 0 \}. \]
   If $\tau^l(P) = 0$, then $P$ is called left torsion free.

2. The right torsion ideal of $P$ is defined to be
   \[ \tau^r(P) = \{ x \in P \mid (kx) \circ P_{\geq n} = 0, \text{ for } n \gg 0 \}. \]
   If $\tau^r(P) = 0$, then $P$ is called right torsion free.

3. The $\bullet$-right torsion ideal of $P$ is defined to be
   \[ \tau^r(P) = \{ x \in P \mid (kx) \bullet P_{\geq n} = 0, \text{ for } n \gg 0 \}. \]
   If $\tau^r(P) = 0$, then $P$ is called $\bullet$-right torsion free.

4. $P$ is called torsion free if $\tau^l(P) = \tau^r(P) = \tau^r(P) = 0$.

Lemma 2.2. Let $P$ be an operad.

1. $\tau^l(P)$ is a 2-sided ideal of $P$ and $\tau^r(P)$ is a right ideal of $P$.
2. $\tau^r(P)$ is a 2-sided ideal of $P$ and $\tau^r(P) \subseteq \tau^r(P)$.
3. Suppose $P$ is semiprime. Then $\tau^l(P) \subseteq P(1)$. As a consequence, if $P$ is prime and infinite dimensional, then $\tau^l(P) = 0$.
4. Suppose $P$ is left (resp. right) torsion free. If $P_{\{w\}}$ is prime, then so is $P$.
5. Suppose $P$ is $\bullet$-right torsion free. If $P$ is prime, then so is $P_{\{w\}}$.

Proof. (1) By [E1.1.3] and [E1.1.4], both $\tau^l(P)$ and $\tau^r(P)$ are $S$-modules. The rest is also easy to check.

(2) It is clear that $\tau^r(P) \subseteq \tau^r(P)$. Let $x \in \tau^r(P)(n)$ with $(kx) \bullet P_{\geq w} = 0$ and $y \in P(m)$. Let $p_i \in P_{\geq w}$. Then
   \[
   (x \circ_i y) \circ (p_1, \ldots, p_{n+m-1})
   = x \circ (p_1, \ldots, p_{i-1}, y \circ (p_i, \ldots, p_{i+m-1}), p_{i+m+1}, \ldots, p_{n+m-1})
   = 0
   \]
as $y \circ (p_i, \ldots, p_{i+m-1}) \in P_{\geq w}$. So $x \circ_i y \in \tau^r(P)$. Similarly,
   \[
   (y \circ_j x) \circ (p_1, \ldots, p_{n+m-1})
   = y \circ (p_1, \ldots, p_{j-1}, x \circ (p_j, \ldots, p_{j+n-1}), p_{j+n+1}, \ldots, p_{n+m-1})
   = y \circ ((p_1, \ldots, p_{j-1}, 0, p_{j+n}, \ldots, p_{n+m-1}) = 0.
   \]
   So $y \circ_j x \in \tau^r(P)$. Therefore $\tau^r(P)$ is a two-sided ideal of $P$.

(3) It is easy to reduce to the prime case, so we assume $P$ is prime for the rest of the proof. If $P$ is finite dimensional, then $P_{\geq 2} = 0$. It remains to consider the case when $P$ is infinite dimensional.
For every $w$, let $J_w = \{ x \in P \mid P_{\geq w} \circ (kx) = 0 \}$. It is easy to check that $J_w$ is an ideal of $P$ and that $P_{\geq w} \circ J_w = 0$. Since $P$ is prime and $P_{\geq w}$ is nonzero, $J_w = 0$. The assertion follows by taking large $w$.

(4) Let $Q = P\{w\}$ in parts (4,5). Since $P$ is left or right torsionfree, it is infinite dimensional. Let $I$ and $J$ be two nonzero ideals of $P$. Let $I' = Q \cap I$ and $J' = Q \cap J$. Then $I'$ and $J'$ are ideals of $Q$. We claim that $I'$ and $J'$ are nonzero. If $I'$ is zero, then $I$ is finite dimensional. Then $P(n) \circ I = 0$ (resp. $I \circ P(n) = 0$) for all $n \gg 0$. Thus $I$ is left (resp. right) torsion. By hypothesis, $P$ is left (resp. right) torsionfree, yield a contradiction. Therefore $I'$ is nonzero. Similarly, $J'$ is nonzero. Since $Q$ is prime, $I' \circ J' = 0$. This implies that $I \circ J \neq 0$ as required.

(5) Let $I$ and $J$ be two nonzero ideals of $Q$. Let $K := I \bullet P_{\geq w}$ and $L := J \bullet P_{\geq w}$.

Then $K$ and $L$ are right ideals of $P$. Since $P$ is left torsionfree, $K$ and $L$ are nonzero. Let $K' := P \circ K$ and $K'' := P_{\geq w} \circ K'$. Since $K'$ is a nonzero ideal of $P$ and $P$ is prime, $K'' \neq 0$. Similarly, we define $L'$ and $L''$. So we also have $L'' \neq 0$. Since $P$ is prime, $K'' \circ L' \neq 0$. It is easy to see that $I \supseteq K''$ and $J \supseteq L''$. Therefore $I \circ J \neq 0$ as desired. \hfill \square

**Proposition 2.3.** Let $P$ be an operad.

(1) If $P$ is right noetherian, then $\tau^r(P)$ is finite dimensional.

(2) If $P$ is left noetherian, then $\tau^l(P)$ is finite dimensional.

(3) Every finite dimensional right ideal is a subspace of $\tau^r(P)$.

(4) Every finite dimensional left ideal is a subspace of $\tau^l(P)$.

(5) If $P$ is left noetherian, then $\tau^l(P) \subseteq \tau^r(P)$.

**Proof.** (1) Since $\tau^r(P)$ is right noetherian, it is finitely generated, say by $X := \{ x_1, \cdots , x_g \}$. By definition, each $x_i \in X$, $(kx_i) \circ P_{\geq w} = 0$ for some $w \gg 0$. Therefore $((kx_i) \circ P)_{\geq w + \text{Art} (x_i)} = 0$ and consequently, $((kx_i) \bullet P)_{\geq (w+1) \text{Art} (x_i)} = 0$ or $(kx_i) \bullet P$ is finite dimensional. Note that

$$\tau^r(P) = \sum_{x_i \in X} (kx_i) \bullet P.$$ 

Hence $\tau^r(P)$ is finite dimensional.

(2) Since $\tau^l(P)$ is left noetherian, as a left ideal, it is finitely generated, say by $X := \{ x_1, \cdots , x_g \}$. It is easy to see that $\tau^l(P) = \sum_{i=1}^g P \circ (kx_i)$. By definition, each $x_i \in X$, $P_{\geq w_i} \circ (kx_i) = 0$, which implies that $P \circ (kx_i)$ is finite dimensional. Consequently, $\tau^l(P)$ is finite dimensional.

(3,4) Clear.

(5) By Lemma 2.2(1), $\tau^l(P)$ is an ideal and by part (2) it is finite dimensional. By part (3), $\tau^l(P) \subseteq \tau^r(P)$. \hfill \square

### 3. (Pseudo-)Graded-Perm algebras

First we recall the definition of a right Perm algebra without unit (which is simply called Perm algebra in this paper). Then we introduce the notions of a unital graded Perm algebra and a unital pseudo-graded-Perm (or PGPerm for short) algebra.

Let $A$ be a vector space. We call $A$ a Perm algebra $\text{Perm}$ if

(i) there is a multiplication $\cdot : A \otimes A \to A$ such that $(A, \cdot)$ is a non-unital associative algebra, and

(ii) $a \cdot (b \cdot c) = a \cdot (c \cdot b)$ for all $a, b, c \in A$. 


In this paper a graded algebra means a unital \( \mathbb{N} \)-graded locally finite associative algebra unless otherwise stated. We need to introduce some grade \( d \) algebras that are similar to Perm algebras in the next three subsections.

### 3.1. Graded Perm algebras.

**Definition 3.1.** Let \( A = \bigoplus_{i=0}^{\infty} A_i \) be an \( \mathbb{N} \)-graded associative algebra with unit \( 1 \in A_0 \). We call \( A \) a graded Perm (or \( GPerm \)) algebra if

\[
(E3.1.1) \quad a \cdot (b \cdot c) = a \cdot (c \cdot b)
\]

for all \( a \in A_{\geq 1} := \bigoplus_{i=1}^{\infty} A_i \) and all \( b, c \in A \).

Recall that \( A \) is connected graded if \( A_0 = k \). Note that a GPerm algebra may not be Perm unless it is commutative.

**Example 3.2.** Let \( X := \{x_1, \ldots, x_n\} \) be a set of homogeneous elements of positive degree. The “free” connected GPerm algebra generated by \( X \), denoted by \( \text{GPerm}(X) \) is the unital associated algebra \( T(X)/\langle R \rangle \) where \( R \) is the set of homogeneous elements \( x_1 x_2 x_3 - x_1 x_3 x_2 \) for all \( i \in \{1, \ldots, n\} \). As a vector space \( \text{GPerm}(X) = k \oplus \bigoplus_{i=1}^{n} x_i [x_1, \ldots, x_n] \).

The Hilbert series of \( \text{GPerm}(X) \) is

\[
1 + \frac{\sum_{i=1}^{n} t^{\deg x_i}}{\prod_{i=1}^{n} (1 - t^{\deg x_i})}.
\]

If \( A \) is a connected GPerm algebra generated by \( X := \{x_1, \ldots, x_n\} \) of positive degrees, then it is a quotient algebra of \( \text{GPerm}(X) \).

**Example 3.3.** Let \( A \) be a GPerm algebra. Define an operad \( P_A \) as follows:

- (a) \( P(0) = 0 \), and \( P(n) = A_{n-1} \) for every \( n \geq 1 \),
- (b) \( x \circ \sigma = x \) for every \( x \in P(n) \) and every \( \sigma \in S_n \),
- (c) for any two elements \( x, y \) in \( P \) of arity \( \geq 1 \), define

\[
(E3.3.1) \quad x \circ_i y = xy
\]

for all \( 1 \leq i \leq \text{Ar}(x) \),

- (d) \( 1_P := 1_A \) is the identity of \( P \).

We claim that \( P_A \) is an operad. Since \( 1_P \) is the unit element of the associative algebra \( A \), it satisfies condition (a) in Definition 1.1. Equation (E1.1.1) follows from (E3.3.1) and the associativity of \( A \). Equation (E1.1.2) follows from (E3.3.1), the associativity of \( A \), and (E3.1.1). Two equations in Definition 1.1(c) follows from (E3.3.1) and the fact that the \( S \)-action is trivial. Therefore \( P_A \) is an operad. It is \( S \text{tr} \) in the sense of Definition 4.1(1). We denote this \( P_A \) by \( G_{S\text{tr}}(A) \).

The next proposition lists some basic properties of GPerm algebras.

**Proposition 3.4.** Let \( A \) be an infinite dimensional GPerm algebra.

(1) If \( A \) is semiprime, every element of positive degree is central. If further \( A \) is left torsionfree or prime, then \( A \) is commutative.

(2) If \( A \) is finitely generated, then its Hilbert series is rational. Further, \( A \) is right noetherian.
(3) Suppose $A$ is finitely generated. If $M$ is a finitely generated right graded $A$-module, then it is right noetherian of finite integral GK-dimension with rational Hilbert series.

Proof. (1) Let $I$ be the ideal of $A$ generated by commutators $[x, y] := xy - yx$ for $x, y \in A$. By \[(E3.1.1)\], $A_{\geq 1} I = 0$. Then $(I \cap A_{\geq 1})^2 = 0$. Since $A$ is semiprime, $I \cap A_{\geq 1} = 0$. This implies that every element of positive degree is central.

If $A$ is left torsionfree or prime, $A_{\geq 1} I = 0$ implies that $I = 0$ by definition. In this case, $A$ is commutative.

(2) Let $B = A/I$ where $I$ is the ideal of $A$ generated by commutators. Then $B$ is a commutative graded algebra which is finitely generated as $A$ is. As a consequence, $B$ is noetherian. Let $Y$ be a set of homogeneous elements that generates $B$. Let $X := \{x_1, \cdots, x_n\}$ be a finite subset of $Y$ after removing those elements of degree 0. Then $A_{\geq 1} I$ is a right $B$-module generated by $X$. Therefore $A_{\geq 1}$ is a right noetherian $B$-module and the Hilbert series of $A_{\geq 1}$ is rational. This implies that $A$ is right noetherian and the Hilbert series of $A$ is rational.

(3) The proof is similar to the proof of (2). \(\square\)

3.2. Pseudo-graded-Perm (PGPerm) algebras. In this subsection we introduce a complicated concept which matches up with the operads studied in Section 5.

Definition 3.5. An N-graded associative algebra $A = \oplus_{i \geq 0} A_i$ is called a pseudo-graded-Perm (or PGPerm) algebra if it satisfies the following conditions:

(i) for each $i \geq 1$, $A_i = A_{i, e} \oplus A_{i, o}$. Elements in $A_{i, e}$ (resp. $A_{i, o}$ or $A_0$) are called homogeneous of type $(i, e)$ (resp. $(i, o)$ or $(0)$). Define

$$t(x) = \begin{cases} 0 & x \in A_0 \\ 0 & x \in A_{i, e} \\ 1 & x \in A_{i, o} \end{cases}$$

(ii) $A_{\geq 1, o} := \oplus_{i \geq 1} A_{i, o}$ and $A_{\geq 1, e} := \oplus_{i \geq 1} A_{i, e}$ are left ideals of $A$.

(iii) $\oplus_{i \geq 2} A_{i, o}$ and $\oplus_{i \geq 2} A_{i, e}$ are two-sided ideals of $A$. As a consequence, $A_{i, o} A_{j, e} = 0 = A_{i, e} A_{j, o}$ for $i \geq 2$ and $j \geq 1$.

(iv) For three homogeneous elements $x \in A_{\geq 2}$ and $y, z \in A$, \[(E3.5.1)\] $x \cdot (y \cdot z) = ax \cdot (z \cdot y)$

where $a$ is either 1 or $-1$ only dependent on the types of $y$ and $z$ satisfying \[(E3.5.2)\] $a := a(y, z) = \begin{cases} -1 & y \in A_{i, o} \text{ and } z \in A_{j, o} \text{ with odd } i \text{ and } j, \\ 1 & \text{otherwise.} \end{cases}$

(v) Let $(-) \ast \Xi$ be the map $Id_{A_{1, e}} - Id_{A_{1, o}} \in \text{End}_k(A_{1, e} \oplus A_{1, o}) = \text{End}_k(A_1)$.

(va) If $x \in A_1, y, z \in A_0$, then \[(((x \cdot y) \ast \Xi) \cdot z) \ast \Xi = ((x \ast \Xi) \cdot z) \ast \Xi) \cdot y.\]

(vb) If $x \in A_1, y \in A_0, z \in A_{\geq 1, e} \cup A_{\geq 1, o}$, then \[((x \cdot y) \ast \Xi) \cdot z = ((x \ast \Xi) \cdot z) \cdot y.\]

(vb)' (This is equivalent to (vb).) If $x \in A_1, z \in A_0, y \in A_{\geq 1, e} \cup A_{\geq 1, o}$, then \[(x \cdot y) \cdot z = (((x \ast \Xi) \cdot z) \ast \Xi) \cdot y.\]
Lemma 3.6. Let $x \in A_1$, $y, z \in A_{\geq 1.5} \cup A_{\geq 1.6}$, then 
\[
(\lambda \circ_1 \mu) \circ_1 \nu = (\lambda \circ_1 \nu) (\mu \circ_1 \nu)
\]
for $\lambda \in P(l)$, $\mu \in P(m)$ and $\nu \in P(n)$.

(2) For $m \geq 2$ being the arity of first component in the partial composition, let $\bar{1} = m$. For example, $-\circ_1 -$ denotes $-\circ_{m} - : P(m) \otimes P(n) \to P(m+n-1)$.

Define $[y, z] := (y \cdot z - az \cdot y)$ where $a = a(y, z)$ is as in (E3.5.2), then (E3.5.1) is equivalent to
\[
x \cdot [y, z] = 0.
\]

Note that a PGPerm algebra may not be a GPerm algebra.

**Lemma 3.6.** Let $P$ be a reduced $\mathbb{S}$-module. Suppose that there is a set of partial compositions
\[
-o_i - : P(m) \otimes P(n) \to P(m + n - 1) \quad (1 \leq i \leq m)
\]
such that (E1.1.3) and (E1.1.4) hold.

(1) Equation (E1.1.1) is equivalent to
\[
(\lambda \circ_1 \mu) \circ_1 \nu = (\lambda \circ_1 \nu) (\mu \circ_1 \nu)
\]
for $\lambda \in P(l)$, $\mu \in P(m)$ and $\nu \in P(n)$.

(2) Again one direction is trivial. Assume now (E3.6.2) and let $i, k$ be as in (E1.1.2). First we set $\lambda = \lambda' * (k\bar{1})$ in (E3.6.2) (if $k = l = \bar{1}$ then we take $\lambda = \lambda'$). Then, by (E1.1.4), we obtain
\[
(\lambda \circ_1 \mu) \circ_{k-1+m} \nu = (\lambda \circ_k \nu) \circ_1 \mu, \quad 1 < k \leq l
\]
which is a special case of (E1.1.2). Recall that $i < k$. Next we set $\lambda = \lambda' * (1i)$ (if $i = 1$ then we take $\lambda = \lambda'$). Then, by (E1.1.4), we obtain (E1.1.2) with general $i$ and $k$.

Recall that we only consider locally finite graded algebras $A$ although the next example does not require this.

**Example 3.7.** Let $(A, \cdot)$ be an $\mathbb{N}$-graded PGPerm algebra. We define an operad $P_A$ as follows:

(i) $P(0) = 0$, $P(1) = A_0$, and for every $n \geq 2$, let $P(n)_{tr} = A_{n-1.e}$, $P(n)_{sg} = A_{n-1.o}$ and $P(n) = P(n)_{tr} \oplus P(n)_{sg}$. Elements in $P(1)$, $P(n)_{tr}$ and $P(n)_{sg}$ are called homogeneous elements. Note that $Ar(x \in P) = \deg(x \in A) + 1$.

(ii) For $x \in P(n)_{tr}$ and $n \geq 2$, $x \cdot \sigma = x$ for $\sigma \in S_n$.

(iii) For $x \in P(n)_{sg}$ and $n \geq 2$, $x \cdot \sigma = sgn(\sigma)x$ for $\sigma \in S_n$.

(iv) $1_\pi := 1_A$ is the identity of $P$.

(v) Let $x$ and $y$ be homogeneous elements and $1 \leq i \leq Ar(x)$.
Therefore (E1.1.3) holds in this case.

We only need to consider the case the equation are 0.

(1b) If \( x \in P \), then we define

\[
\begin{align*}
  x \circ y &= \begin{cases} 
    (-1)^{i-1} x \cdot y & \text{for } x \in P(n)_{sg}, y \in P(m)_{sg} \text{ with } m \text{ even}, \\
    0 & \text{if } t(x) \neq t(y) \text{ and } y \notin P(1) \\
    x \cdot y & \text{otherwise.}
  \end{cases}
\end{align*}
\]

Or \( x \circ y = bx \cdot y \) where \( b := (-1)^{(\text{Ar}(y)-1)(t(x)-1)} \) is 1 or \(-1\) (dependent on the types of \( x \) and \( y \), etc).

(vb) If \( \text{Ar}(x) = 2 \), we define

\[
x \circ_1 y = x \cdot y
\]

and

\[
x \circ_2 y = \begin{cases} 
  ((x \cdot \Xi) \cdot y) \cdot \Xi & y \in P(1), \\
  (-1)^{\text{Ar}(y)(t(y))}(x \cdot \Xi) \cdot y & y \in \mathcal{P}_{tr} \cup \mathcal{P}_{sg}
\end{cases}
\]

(vc) If \( \text{Ar}(x) = 1 \), we define \( x \circ_1 y = x \cdot y \).

We claim that \( P \) is an operad. We need to verify all conditions \((a,b,c)\) in Definition 1.1.

Condition (a) is obvious.

For condition (c), there are two equations, namely, (E1.1.3) and (E1.1.4).

Verification of (E1.1.3): Case 1: For \( \text{Ar}(x) \geq 3 \) (or equivalently \( \deg x \geq 2 \)), we only need to consider the case \( y \in P(1) \) or \( t(x) = t(y) \), otherwise, both sides of the equation are 0.

(1a) If \( y \in P(1) \), then \( \sigma = \text{id}_1 \in S_1 \) and \( \sigma' = \text{id}_{\text{Ar}(x)} \in S_{\text{Ar}(x)} \). So (E1.1.3) is obvious.

(1b) If \( x \in P_{tr} \) and \( y \in P_{tr} \), then \( t(x) = 0 \) and \( x \cdot y \in P_{tr} \). We have

\[
x \circ_1 (y \cdot \sigma) = x \circ_1 y = x \cdot y = (x \cdot y) \cdot \sigma' = (x \circ_1 y) \cdot \sigma'.
\]

(1c) If \( x \in P_{sg} \) and \( y \in P_{sg} \), then \( t(x) = 1 \) and \( x \cdot y \in P_{sg} \). We have

\[
x \circ_1 (y \cdot \sigma) = (-1)^{(\text{Ar}(y)-1)(t(y)-1)} x \cdot (y \cdot \sigma)
\]

\[
= (-1)^{(\text{Ar}(y)-1)(t(y)-1)} \text{sgn}(\sigma) x \cdot y
\]

\[
= (-1)^{(\text{Ar}(y)-1)(t(y)-1)} \text{sgn}(\sigma') x \cdot y \quad \text{by Lemma 1.2(1)}
\]

\[
= (-1)^{(\text{Ar}(y)-1)(t(y)-1)} (x \cdot y) \cdot \sigma'
\]

\[
= (x \circ_1 y) \cdot \sigma'.
\]

Therefore (E1.1.3) holds in this case.

Case 2: For \( \text{Ar}(x) = 2 \), notice that if \( y \in P_{tr} \) (resp. \( y \in P_{sg} \)), for any \( x \in P \) we have \( x \cdot y \in P_{tr} \) (resp. \( x \cdot y \in P_{sg} \)), since \( A_{\geq 1,o} \) and \( A_{\geq 1,e} \) are both left ideal of \( A \).

(2a) For \( i = 1 \), on one hand

\[
x \circ_1 (y \cdot \sigma) = \begin{cases} 
  x \circ_1 y = x \cdot y & \text{if } y \in P(1) \cup P_{tr} \\
  \text{sgn}(\sigma) x \circ_1 y = \text{sgn}(\sigma) x \cdot y & \text{if } y \in P_{sg}
\end{cases}
\]

and on the other hand,

\[
(x \circ_1 y) \cdot \sigma' = (x \cdot y) \cdot \sigma' = \begin{cases} 
  x \cdot y & \text{if } y \in P(1) \cup P_{tr} \\
  \text{sgn}(\sigma') x \cdot y & \text{if } y \in P_{sg}
\end{cases}
\]

By Lemma 1.2(1), we have \( \text{sgn}(\sigma) = \text{sgn}(\sigma') \) and thus \( x \circ_1 (y \cdot \sigma) = (x \circ_1 y) \cdot \sigma' \).
(2b) For \( i = 2 \), on one hand
\[
x \circ_2 (y \ast \sigma) = \begin{cases} 
(x \ast \Xi) \circ_1 y \Xi & \text{if } y \in \mathcal{P}(1) \\
(x \ast \Xi) \circ_1 y \Xi & \text{if } y \in \mathcal{P}_{tr} \\
(1)_{Ar(y)} \text{sgn}(\sigma)(x \ast \Xi) \circ_1 y \Xi & \text{if } y \in \mathcal{P}_{sg}, 
\end{cases}
\]
and on the other hand,
\[
(x \circ_2 y) \ast \sigma' = \begin{cases} 
(x \ast \Xi) \circ_1 y \Xi & \text{if } y \in \mathcal{P}(1) \\
(x \ast \Xi) \circ_1 y \Xi & \text{if } y \in \mathcal{P}_{tr} \\
(1)_{Ar(y)} \text{sgn}(\sigma')(x \ast \Xi) \circ_1 y \Xi & \text{if } y \in \mathcal{P}_{sg}. 
\end{cases}
\]
Again by Lemma 1.2(1), we have \( x \circ_1 (y \ast \sigma) = (x \circ_1 y) \ast \sigma'. \)
Therefore (E1.1.3) holds in this case.

Case 3: For \( \text{Ar}(x) = 1 \), we have \( \sigma = \sigma' \) and it is trivial if \( \text{Ar}(y) = 1 \). If otherwise, \( y \in \mathcal{P}_{tr} \) or \( y \in \mathcal{P}_{sg} \), then \( t(x \cdot y) = t(y) \) and hence
\[
x \circ_1 (y \ast \sigma) = x \cdot (y \ast \sigma) = (x \cdot y) \ast \sigma = (x \circ_1 y) \ast \sigma.
\]
Therefore (E1.1.3) holds in this case.

**Verification of (E1.1.4):** Case 4: For \( \text{Ar}(x) \geq 3 \), we only need to consider the case \( y \in \mathcal{P}(1) \) or \( t(x) = t(y) \), otherwise, both sides of the equation are 0.
(4a) If \( x \in \mathcal{P}_{tr} \) and \( y \in \mathcal{P}_{tr} \cup \mathcal{P}(1) \), then \( t(x) = 0 \) and, by Definition 3.5(iii),
x \cdot y \in \mathcal{P}_{tr}. We have
\[
(x \ast \phi) \circ_1 y = x \circ_1 y = x \cdot y = (x \cdot y) \ast \phi'' = (x \circ_{\phi(1)} y) \ast \phi''.
\]
(4b) If \( x \in \mathcal{P}_{sg} \) and \( y \in \mathcal{P}_{sg} \cup \mathcal{P}(1) \), then \( t(x) = 1 \) and, by Definition 3.5(iii),
x \cdot y \in \mathcal{P}_{sg}. We have
\[
(x \ast \phi) \circ_1 y = \text{sgn}(\phi)x \circ_1 y
\]
\[
= (-1)^{(\text{Ar}(y) - 1)(i - 1)} \text{sgn}(\phi)x \cdot y
\]
\[
= (-1)^{(\text{Ar}(y) - 1)(i - 1)} \text{sgn}(\phi'')x \cdot y \quad \text{by Lemma 1.2(2)}
\]
\[
= (-1)^{(\text{Ar}(y) - 1)(i - 1)} (x \cdot y) \ast \phi''
\]
\[
= (x \circ_{\phi(1)} y) \ast \phi''.
\]
Therefore (E1.1.4) holds in this case.

Case 5: For \( \text{Ar}(x) = 2 \), it is trivial if \( \phi = \text{id}_2 \in \mathbb{S}_2 \), in fact in this case, \( \phi'' = \text{id}_{\text{Ar}(y) + 1} \in \mathbb{S}_{\text{Ar}(y) + 1} \)
and
\[
(x \ast \text{id}_2) \circ_1 y = (x \circ_{\text{id}_2(i)} y) \ast \text{id}_{\text{Ar}(y) + 1}.
\]
In the following, we assume \( \phi = (12) \in \mathbb{S}_2 \). By Lemma 1.2(2), we have
\[
\text{sgn}(\phi'') = (-1)^{(\text{Ar}(y) - 1)(i - 1)} \text{sgn}(\phi) = (-1)^{\text{Ar}(y)}.
\]
For any \( x \in \mathcal{P}(2)_{tr} \) or \( x \in \mathcal{P}(2)_{sg} \), it is easy to see \( x \ast \Xi = (-1)^{\text{tr}(x)}x \) and \( x \ast \phi = (-1)^{\text{tr}(x)}x \) by definition. So for any \( x \in \mathcal{P}(2) \), we have \( x \ast \Xi = x \ast \phi \) and
\[
x = (x \ast \phi) \ast \Xi = (x \ast \Xi) \ast \phi.
\]
(5a) For \( i = 1 \), on one hand
\[
(x \ast \phi) \circ_1 y = (x \circ_1 y) \ast (x \ast \Xi) \cdot y.
\]
On the other hand, by \[E3.7.2\],

\[x \circ \varphi(1) y = x \circ_2 y = \begin{cases} ((x \ast \Xi) \cdot y) \ast \Xi & y \in \mathcal{P}(1) \\ (-1)^{\text{Ar}(y)t(y)}((x \ast \Xi) \cdot y) & y \in \mathcal{P}_{tr} \cup \mathcal{P}_{sg}. \end{cases}\]

When \(y \in \mathcal{P}(1)\), we have \(x \circ_2 y \in \mathcal{P}(2)\) and \(\varphi'' = \varphi\). Hence

\[(x \circ \varphi(1) y) \ast \varphi'' = (((x \ast \Xi) \cdot y) \ast \Xi) \ast \varphi = (x \ast \Xi) \cdot y = (x \ast \varphi) \circ_1 y.

When \(y \in \mathcal{P}_{tr}\) (or \(y \in \mathcal{P}_{sg}\)), then so does \((x \ast \Xi) \cdot y\), so we have

\[(x \ast \Xi) \cdot y \ast \varphi'' = (\text{sgn}(\varphi''))t(y)((x \ast \Xi) \cdot y) = (-1)^{\text{Ar}(y)t(y)}((x \ast \Xi) \cdot y)

and therefore, by \[E3.7.2\],

\[(x \circ \varphi(1) y) \ast \varphi'' = (-1)^{\text{Ar}(y)t(y)}((x \ast \Xi) \cdot y) \ast \varphi'' = (x \ast \Xi) \cdot y = (x \ast \varphi) \circ_1 y.

(5b) For \(i = 2\), on one hand

\[(x \circ \varphi(2) y) \ast \varphi'' = (x \cdot y) \ast \varphi'' = \begin{cases} (x \cdot y) \ast \varphi & y \in \mathcal{P}(1) \\ (-1)^{\text{Ar}(y)t(y)}(x \cdot y) & y \in \mathcal{P}_{tr} \cup \mathcal{P}_{sg}. \end{cases}\]

On the other hand,

\[x \circ \varphi(2) y = \begin{cases} (((x \cdot \varphi) \ast \Xi) \cdot y) \ast \Xi = (x \cdot y) \ast \varphi & y \in \mathcal{P}(1) \\ (-1)^{\text{Ar}(y)t(y)}((x \cdot \varphi) \ast \Xi) \cdot y = (-1)^{\text{Ar}(y)t(y)}x \cdot y & y \in \mathcal{P}_{tr} \cup \mathcal{P}_{sg}. \end{cases}\]

Therefore \[E1.1.4\] holds in this case.

Case 6: \[E1.1.4\] is trivial if \(\text{Ar}(x) = 1\).

Note that \[E3.6.1\] follows immediately from the definition. By Lemma 3.6, it remains to show \[E3.6.2\].

**Verification of \[E3.6.2\]:** Let \(x \in \mathcal{P}(l), y \in \mathcal{P}(m)\) and \(z \in \mathcal{P}(n)\). We will prove

\[(x \circ_1 y) \circ_{l+m-1} z = (x \circ_1 z) \circ_1 y.

Case 7: If \(l \geq 3\), by definition

\[(x \circ_1 y) \circ_{l+m-1} z = (-1)^{(l-1)(n-1)}(l+m-2)t(x)(x \cdot y) \cdot z,

and

\[(x \circ_1 z) \circ_1 y = (-1)^{(l-1)(n-1)}(l-1)t(x)(x \cdot z) \cdot y.

By \[E3.5.1\] and \[E3.5.2\], assertion \[E3.6.2\] follows.

Case 8: If \(l = 2\), by definition we have

\[(x \circ_1 y)_{o_{m+1}z} = \begin{cases} (((x \cdot y) \ast \Xi) \cdot z) \ast \Xi & y, z \in \mathcal{P}(1) \\ (-1)^{\text{Ar}(z)t(z)}((x \cdot y) \ast \Xi) \cdot z & y \in \mathcal{P}(1), z \notin \mathcal{P}(1), \\ (x \cdot y) \cdot z & y \notin \mathcal{P}(1), z \in \mathcal{P}(1), \\ (-1)^{\text{Ar}(z)-1}\text{Ar}(y)t(y)(x \cdot y) \cdot z & y, z \notin \mathcal{P}(1) \text{ and } t(y) = t(z), \\ 0 & y, z \notin \mathcal{P}(1) \text{ and } t(y) \neq t(z). \end{cases}\]

and

\[(x \circ_2 z) \circ_1 y = \begin{cases} (((x \ast \Xi) \cdot z) \ast \Xi) \cdot y & y, z \in \mathcal{P}(1) \\ (-1)^{\text{Ar}(z)t(z)}((x \ast \Xi) \cdot z) \cdot y & y \in \mathcal{P}(1), z \notin \mathcal{P}(1), \\ ((x \ast \Xi) \cdot z) \ast \Xi \cdot y & y \notin \mathcal{P}(1), z \in \mathcal{P}(1), \\ (-1)^{\text{Ar}(z)-1}\text{Ar}(y)t(y)(x \ast \Xi) \cdot z & y, z \notin \mathcal{P}(1) \text{ and } t(y) = t(z), \\ 0 & y, z \notin \mathcal{P}(1) \text{ and } t(y) \neq t(z). \end{cases}\]

By Definition 3.5, these two terms equal to each other in all cases. So we finish the proof.
Let \( w \geq 2 \). If \( A \) is an \( \mathbb{N} \)-graded algebra, the \( w \)th Veronese subring of \( A \) is defined to be

\[
A^{(w)} := \bigoplus_{i=0}^{\infty} A_{wi}.
\]

Similar to (E1.3.3), we define \( w \)th connected graded subring, denoted by \( A_{(w)} \) of \( A \) by

\[
(A_{(w)})_i = \begin{cases} 
  k & i = 0, \\
  0 & 1 \leq i \leq w - 1, \\
  A_i & i \geq w.
\end{cases}
\]

**Lemma 3.8.** Let \( A \) be a finitely generated PGPerm algebra.

1. \( A^{(2)} \) is a finitely generated GPerm algebra after forgetting the map \( (\cdot)^{\ast} \Xi \) and the decomposition into types.
2. \( A \) is a finitely generated right module over \( A^{(2)} \). As a consequence \( A \) is right noetherian.
3. The Hilbert series of \( A \) is rational.
4. \( \text{GKdim}(A) \) is an integer.
5. Every finitely generated graded right \( A \)-module \( M \) is right noetherian of finite integral GK-dimension with rational Hilbert series.

**Proof.**

1. Suppose \( A \) is generated by \( X := \{y_1, \ldots, y_b\} \cup \{z_1, \ldots, z_c\} \) which is a \( k \)-linear homogeneous basis of \( \bigoplus_{d=0}^{d} A_i \) (for some \( d \geq 3 \)) where the degrees of \( y_j \) are even, and the degrees of \( z_j \) are odd. We may assume that \( 1_A \in X \). Then \( A^{(2)} \) is generated by

\[
\{y_1, \ldots, y_b\} \cup \{z_j z_k\}_{1 \leq j, k \leq c} \cup \{z_j y_h z_k\}_{1 \leq j, k \leq c, 1 \leq h \leq b}.
\]

This follows easily from induction on the degree of an element and (E3.5.1). When we consider the product of elements in \( A^{(2)} \), \( a \) in (E3.5.2) is always 1. In this case (E3.5.1) becomes (E3.1.1). Therefore \( A^{(2)} \) is a GPerm algebra.

2. By (E3.5.1), \( A \) is a right \( A^{(2)} \)-module generated by

\[
X \cup \{yz_j\}_{1 \leq i \leq b, 1 \leq j \leq c}.
\]

(Further we can assume \( \deg y_i \geq 2 \) in the above expression.) This follows from induction on the degree of an element and (E3.5.1). By part (1), \( A^{(2)} \) is finitely generated. Then by Proposition 3.4, \( A \) is right noetherian. Hence it is right noetherian.

3. Every finitely generated graded right \( A \)-module \( M \) is right noetherian of finite integral GK-dimension with rational Hilbert series.

3.3. **Pseudo-graded-commutative algebras.** A special class of PGPerm algebras are the following.

**Definition 3.9.** An \( \mathbb{N} \)-graded associative algebra \( A = \bigoplus_{i=0}^{\infty} A_i \) is called a pseudo-graded-commutative (or PGC) algebra if it satisfies the following conditions:

1. [Definition 3.5(i)] for each \( i \geq 1 \), \( A_i = A_{i,e} \oplus A_{i,o} \). Elements in \( A_{i,e} \) (resp. \( A_{i,o} \) or \( A_0 \)) are called homogeneous of type \((i, e)\) (resp. \((i, o)\) or \((0)\)). Define

\[
t(x) = \begin{cases} 
  0 & x \in A_0 \\
  0 & x \in A_{i,e} \\
  1 & x \in A_{i,o}
\end{cases}
\]
(ii) \( A_o := \bigoplus_{i \geq 1} A_{i,o} \) and \( A_e := \bigoplus_{i \geq 1} A_{i,e} \) are two-sided ideals of \( A \).

(iii) For all homogeneous elements \( y, z \in A \),
\[ yz = azy \]
where \( a = (-1)^{\deg(y)\deg(z)}t(y)t(z) \) which agrees with \( a(y, z) \) given in (E3.5.2).

It is tedious but easy to verify that Definition \( \ref{def:4.9} \) follows from Definition \( \ref{def:3.9} \). Therefore every PGC algebra is a PGPerm algebra.

A PGPerm (or PGC) algebra \( A \) is said to be of even type (resp. odd type) if \( A_o = 0 \) (resp. \( A_e = 0 \)). It is clear that a PGPerm algebra of even type is equivalent to a GPerm algebra. The next lemma concerns left torsionfree PGPerm (PGC) algebras of odd type.

**Lemma 3.10.** Let \( A \) be a left torsionfree PGPerm algebra of odd type. Then \( A \) is a graded commutative algebra. Conversely, every graded commutative algebra is a PG algebra with \( A_{\geq 1} = A_o \).

**Proof.** Let \( x \in A_{\geq 2} \) and \( y, z \in A \) be homogeneous. By (E3.5.1), \( x(yz - azy) = 0 \). Hence \( (yz - azy) \in \tau'(A) \). Since \( A \) is left torsionfree, \( yz = azy \). By the definition of \( a \) in (E3.5.2), Definition \( \ref{def:3.9} \)(iii) holds. Note that Definition \( \ref{def:3.9} \)(ii) follows from Definition \( \ref{def:3.9} \)(iii) and the fact that \( A \) is a PGPerm algebra. Hence \( A \) is PGC.

The converse is also easy and omitted. \( \square \)

## 4. (Almost) \( S \)-trivial operads

**Definition 4.1.** Let \( \mathcal{P} \) be an operad.

1. An element \( \lambda \in \mathcal{P}(n) \) is called \( S \)-trivial (or \( \text{Str} \)) if \( \lambda \star \sigma = \lambda \) for all \( \sigma \in S_n \).
2. We call \( \mathcal{P} \) \( S \)-trivial (or \( \text{Str} \)) if every element in \( \mathcal{P}(n) \) is \( \text{Str} \) for all \( n \).
3. We call \( \mathcal{P} \) almost \( S \)-trivial (or almost \( \text{Str} \)) if \( \mathcal{P}(w) \) is \( \text{Str} \) for some \( w \geq 2 \).

Examples of \( \text{Str} \) operads are given in Example \( \ref{ex:3.3} \). We can define a functor \( G_{\text{Str}} \) from the category of GPerm algebras to the category of \( S \)-trivial operads by \( G_{\text{Str}}(A) = \mathcal{P}_A \), see Example \( \ref{ex:3.3} \). Let \( F_{\text{Str}} \) be the forgetful functor \( F \) defined in Subsection \( \ref{subsec:1.5} \) when restricted to the category of \( \text{Str} \) operads.

**Theorem 4.2.** The functors \( (G_{\text{Str}}, F_{\text{Str}}) \) induce an equivalence between the category of GPerm algebras and the category of \( \text{Str} \) operads. Furthermore, a GPerm algebra \( A \) is finitely generated if and only if \( G_{\text{Str}}(A) \) is.

**Proof.** One direction is Example \( \ref{ex:3.3} \). For the other direction, let \( \mathcal{P} \) be an \( S \)-trivial operad. We need to show that \( A := F_{\text{Str}}(\mathcal{P}) = F(\mathcal{P}) \) is a GPerm algebra. We claim that \( x \circ_i y = x \circ_1 y \) for all \( 1 \leq i \leq \text{Ar}(x) \). If \( n := \text{Ar}(x) \geq 2 \), let \( \phi \in S_n \) interchanging 1 with \( i \) when \( i \neq 1 \), then, by (E1.1.4) and \( S \)-triviality,
\[ (E4.2.1) \]
\[ x \circ_1 y = (x \circ \phi) \circ_1 y = (x \circ_{\phi(1)} y) \circ \phi'' = x \circ_i y \]
as desired.

By the definition of \( F \) given in Subsection \( \ref{subsec:1.5} \) let \( A_n = \mathcal{P}(n + 1) \). We define a multiplication on \( A := \bigoplus_{i \geq 0} A_i \) by
\[ x \cdot y := x \circ_1 y = x \circ_i y. \]

By Lemma \( \ref{lem:1.6} \), \( A := \bigoplus_{i \geq 0} A_i \) is a graded associative algebra. It remains to show that \( A \) is GPerm or that (E3.1.1) holds. For \( x \in A_{\geq 1} \) and \( y, z \in A \), \( x \) has arity
\( n \geq 2 \) in \( \mathcal{P} \). Then, by Lemma 4.2.1 and definition,
\[
x \cdot y \cdot z = (x \circ_1 y) \circ_1 z = (x \circ_n y) \circ_1 z = (x \circ_1 z) \circ (n + \text{Art}(z) - 1) y = (x \circ_1 z) \circ_1 y = x \cdot z \cdot y
\]
as desired.

By construction, it is routine to check that \( G_{\text{str}} \) and \( \mathcal{F}_{\text{str}} \) are indeed inverse to each other.

The second statement about the finite generation is easy to verify.

A right \( S_n \)-module \( V \) is called \( \mathbb{A}_n \)-trivial if, for all \( x \in V \), \( x \cdot \sigma = x \) for all \( \sigma \in \mathbb{A}_n \) [Definition 1.1].

**Lemma 4.3.** Let \( \mathcal{P} \) be an operad. Suppose \( V \subseteq \mathcal{P}(n) \) be a right \( S_n \)-submodule generated by \( \{v_1, \ldots, v_s\} \). Let \( y \in \mathcal{P}(m) \).

1. Let \( i \) be an integer with \( 1 \leq i \leq m \). Then \( y \circ_i V \) is a right \( S_{n+m-1} \)-submodule of \( \mathcal{P}(n+m-1) \) generated by \( \{y \circ_i v_1, \ldots, y \circ_i v_s\} \). If \( y \circ_i V \) is a simple right \( S_n \)-module, then either \( y \circ_i V = 0 \) or \( \dim y \circ_i V = \dim V \).

2. Suppose \( V \) is a right \( S_n \)-module such that \( V/W \) is not \( \mathbb{A}_n \)-trivial for any \( S_n \)-submodule \( W \subseteq V \). If \( \mathcal{P}(n+m-1) \) is \( \mathbb{A}_{n+m-1} \)-trivial, then \( y \circ_i V = 0 \).

3. Suppose \( V \) is a right \( S_n \)-module such that \( V/W \) is not \( \mathbb{A}_n \)-trivial for any \( S_n \)-submodule \( W \subseteq V \). If \( \mathcal{P}(n+m-1) \) is \( \mathbb{A}_{n+m-1} \)-trivial, then \( y \circ_i V = 0 \).

4. Suppose \( \mathcal{P}(n+m-1) \) is \( \text{Str} \). For every \( \nu \in \mathcal{P}(n) \) and \( \sigma \in \mathbb{A}_n \),
\[
y \circ_i (\nu \ast \sigma - \nu) = 0.
\]

5. Suppose \( \mathcal{P}(n+m-1) \) is \( \mathbb{A}_n \)-trivial. For every \( \nu \in \mathcal{P}(n) \) and \( \sigma \in \mathbb{A}_n \),
\[
y \circ_i (\nu \ast \sigma - \nu) = 0.
\]

**Proof.** (1) Since \( V = \sum_{j=1}^s v_j \ast S_n \), by (E1.1.3),
\[
y \circ_i V = \sum_{j=1}^s y \circ_i (v_j \ast S_n) \subseteq \sum_{j=1}^s (y \circ_i v_j) \ast S_{n+m-1}.
\]
The first assertion follows.

For the second assertion we assume that \( V \) is a simple right \( S_n \)-module and \( y \circ_i V \neq 0 \). It remains to show that the linear map \( y \circ_i (\cdot) : V \rightarrow y \circ_i V \) is injective. If this is not an injective map, there is a \( 0 \neq v \in V \) such that \( y \circ_i v = 0 \). Since \( V \) is generated by \( v \) as a right \( S_n \)-module. By the first assertion, \( y \circ_i V \subseteq (y \circ_i v) \ast S_{n+m-1} = 0 \), yielding a contradiction. The second assertion follows.

(2) Suppose \( y \circ_i V \neq 0 \). Let \( W = \{w \in V \mid y \circ_i w = 0\} \). Then \( W \) is an \( S_n \)-submodule of \( V \) and \( y \circ_i (\cdot) : V/W \rightarrow \mathcal{P}(n+m-1) \) is injective. It follows from the hypothesis that \( V/W \) is not \( \mathbb{A}_n \)-trivial. Let \( v \) be any element in \( V \) such that \( v \ast \sigma \neq v \) in \( V/W \) for some \( \sigma \in \mathbb{A}_n \). By (E1.1.3) and the hypothesis that the \( \sigma' \)-action on \( y \circ_i v \) is trivial, we have
\[
y \circ_i v \neq y \circ_i (v \ast \sigma) = (y \circ_i v) \ast \sigma' = y \circ_i v,
\]
yielding a contradiction.

(3) The proof is omitted since it is similar to the proof of part (2).

(4) By (E1.1.3) and the \( \mathbb{S} \)-triviality of \( \mathcal{P}(n+m-1) \), we have
\[
y \circ_i (\nu \ast \sigma) = (y \circ_i \nu) \ast \sigma' = y \circ_i \nu
\]
which implies the assertion.

(5) The proof is similar to the proof of part (4).
\textbf{Proposition 4.4.} Let $\mathcal{P}$ be an almost \text{Str} operad.

(1) Suppose $V$ is a right $S_n$-submodule of $\mathcal{P}(n)$ such that $V/W$ is not isomorphic to $\mathbf{tr}$ for any submodule $W \subseteq V$. Then $V \subseteq \tau^l(\mathcal{P})$.

(2) If $\tau^l(\mathcal{P})_{\geq 2} = 0$, then $\mathcal{P}$ is $\mathbf{Str}$.

(3) If $\mathcal{P}$ is semiprime, then $\mathcal{P}$ is $\mathbf{Str}$.

\textbf{Proof.} (1) Suppose $w$ is an integer such that $\mathcal{P}_{\{w\}}$ is $\mathbf{Str}$. For every $y \in \mathcal{P}(m)$ with $m \geq w$, $y \circ_i V \subseteq \mathcal{P}_{\{w\}}$. By Lemma 1.3(2), $y \circ_i V = 0$. Hence $\mathcal{P}_{\geq w} \circ V = 0$ as required.

(2) Let $\nu \in \mathcal{P}(n)$ for $n \geq 2$. Let $y \in \mathcal{P}(m)$ with $m \geq w$. Then $\mathcal{P}(n+m-1)$ is $\mathbf{Str}$. By Lemma 4.3(4), $\nu * \sigma - \nu \in \tau^l(\mathcal{P})_{\geq 2}$ for all $\sigma \in \mathcal{P}(n)$. Since $\tau^l(\mathcal{P})_{\geq 2} = 0$, $\nu * \sigma - \nu = 0$ for all $\sigma \in \mathcal{P}(n)$. The assertion follows.

(3) The assertion follows from Lemma 2.2(3) and part (2). \hfill $\Box$

The following is an easy corollary of Theorem 4.2.

\textbf{Corollary 4.5.} The functors $(G_{\mathbf{Str}}, F_{\mathbf{Str}})$ restrict to

(1) an equivalence between the category of torsionfree commutative $\mathbb{N}$-graded algebras and that of $\mathbf{Str}$ left torsionfree operads;

(2) an equivalence between the category of prime commutative $\mathbb{N}$-graded algebras and that of $\mathbf{Str}$ prime operads.

\textbf{Proof.} (1) Suppose $A$ is a torsionfree graded commutative algebra and let $\mathcal{P} = G_{\mathbf{Str}}(A)$. By Example 3.3 $\mathcal{P}$ is $\mathbf{Str}$. In this proof we identify elements in $A$ with elements in $\mathcal{P}$ ($A_i = \mathcal{P}(i+1)$ for all $i \geq 0$). By (3.3.1), it follows from the definition that $\tau^l(\mathcal{P}) = \tau^l(A) = \tau^l(A)_{\geq 2}$. Since $A$ is torsionfree, $\mathcal{P}$ is left torsionfree.

Conversely, let $\mathcal{P}$ be an $\mathbf{Str}$ left torsionfree operad. By the computation in the proof of Theorem 4.2

\[ x \circ_1 (y \circ_1 z) = (x \circ_1 y) \circ_1 z = (x \circ_1 z) \circ_1 y = x \circ_1 (z \circ_1 y) \]

for $x \in \mathcal{P}_{\geq 2}$ and $y, z \in \mathcal{P}$. Or $x \circ_1 (y \circ_1 z - z \circ_1 y) = 0$. By (4.2.1), $x \circ_1 (y \circ_1 z - z \circ_1 y) = 0$. Therefore $\mathcal{P}_{\geq 2} \circ (k \circ_1 y - z \circ_1 y) = 0$. Since $\mathcal{P}$ is left torsionfree, $y \circ_1 z - z \circ_1 y = 0$ for all $y, z \in \mathcal{P}$. Therefore $A := F_{\mathbf{Str}}(\mathcal{P})$ is commutative. It follows from (4.2.1) that $\tau^r(A) = \tau^l(A) = \tau^l(\mathcal{P})$. Thus $A$ is torsionfree. Therefore the assertion follows from Theorem 4.2.

(2) Proofs are similar and omitted. \hfill $\Box$

5. (Almost) $A$-trivial operads and proofs of the main results

The goal of this section is to prove the main results. First we recall Definition 4.1.

\textbf{Definition 5.1.} Let $\mathcal{P}$ be an operad.

(1) An element $\lambda$ in $\mathcal{P}(n)$ is called $A$-trivial or simply $A\mathbf{tr}$ if $\lambda * \sigma = \lambda$ for all $\sigma \in A_n$.

(2) $\mathcal{P}$ is called $A$-trivial or simply $A\mathbf{tr}$ (resp. almost $A$-trivial or simply almost $A\mathbf{tr}$) if all elements in $\mathcal{P}(n)$ are $A\mathbf{tr}$ for all $n$ (resp. for all $n \gg 0$).

(3) An element $\lambda$ in $\mathcal{P}(n)$ is called $S\mathbf{sg}$ if $\lambda * \sigma = sgn(\sigma)\lambda$ for all $\sigma \in S_n$.

(4) $\mathcal{P}$ is called $S\mathbf{sg}$ (resp. almost $S\mathbf{sg}$) if for every element in $\mathcal{P}(n)$ is $S\mathbf{sg}$ for all $n$ (resp. for all $n \gg 0$).
By definition, $\mathcal{P}$ is almost $\mathbb{A}tr$ (resp. almost $\mathbb{S}sg$) if and only if $\mathcal{P}_w$ is $\mathbb{A}tr$ (resp. $\mathbb{S}sg$) for some $w \geq 2$.

In the rest of this section we assume that $\mathcal{P}$ is $\mathbb{A}$-trivial and $p := \text{char } \mathbb{k} \neq 2$. For every $n \geq 2$, $\mathcal{P}(n)$ is a $\mathbb{k}(S_n/\mathbb{A}_n)$-module. Since $p \neq 2$, it can be decomposed as

\[ \mathcal{P}(n) = \mathcal{P}(n)_{\text{tr}} \oplus \mathcal{P}(n)_{\text{sg}} \]

where elements in $\mathcal{P}(n)_{\text{tr}}$ are $\mathbb{A}tr$ and elements in $\mathcal{P}(n)_{\text{sg}}$ are $\mathbb{S}sg$. One should compare (E5.1.1) with the even-odd decomposition given in Definition 3.5(i).

Write

\[ t(x) = \begin{cases} 0 & x \in \mathcal{P}(1) \\ 0 & x \in \mathcal{P}_{\text{tr}} := \oplus_{n \geq 2} \mathcal{P}(n)_{\text{tr}} \\ 1 & x \in \mathcal{P}_{\text{sg}} := \oplus_{n \geq 2} \mathcal{P}(n)_{\text{sg}} \end{cases} \]

which is similar to definition of $t$ in Definition (E5.1.3). Let

\[ A := A_p = \mathcal{F}(\mathcal{P}) \]

be the graded associative algebra as defined in Subsection 1.5. Write

\[ A_{i,e} := \mathcal{P}(i+1)_{\text{tr}} \quad \text{and} \quad A_{i,o} := \mathcal{P}(i+1)_{\text{sg}}. \]

Similarly, write $A_e := \oplus_{i \geq 1} A_{i,e}$ and $A_o := \oplus_{i \geq 1} A_{i,o}$.

The next four lemmas provide some facts about $\mathbb{A}tr$ operads. Firstly we consider (E1.1.3):

\[ \mu \circ_i (\nu * \sigma) = (\mu \circ_i \nu) * \sigma' \]

for $\mu \in \mathcal{P}(m)$, $\nu \in \mathcal{P}(n)$ and $\sigma \in S_n$.

**Lemma 5.2.** Suppose $\text{char } \mathbb{k} \neq 2$. Let $\mathcal{P}$ be $\mathbb{A}tr$. Retain the above notation.

1. (E1.1.3) holds trivially for $\nu \in \mathcal{P}(1)$.
2. If $\nu \in \mathcal{P}(n)_{\text{tr}}$ for $n \geq 2$, then $\mu \circ_i \nu \in \mathcal{P}(m+n-1)_{\text{tr}}$ for all $\mu \in \mathcal{P}(m)$. As a consequence, $\mathcal{P}_{\text{tr}}$ is a left ideal of $\mathcal{P}$. In terms of $A$ via (E5.1.2), $A_e$ is a left ideal of $A$.
3. If $\nu \in \mathcal{P}(n)_{\text{sg}}$ for $n \geq 2$, then $\mu \circ_i \nu \in \mathcal{P}(m+n-1)_{\text{sg}}$ for all $\mu \in \mathcal{P}(m)$. As a consequence, $\mathcal{P}_{\text{sg}}$ is a left ideal of $\mathcal{P}$. In terms of $A$ via (E5.1.2), $A_o$ is a left ideal of $A$.

**Proof.** (2) Let $\sigma \in S_n$ with $\text{sgn} (\sigma) = -1$. By Lemma (E1.2.1), $\text{sgn} (\sigma') = -1$. Now

\[ (\mu \circ_i \nu) * \sigma' = \mu \circ_i (\nu * \sigma) = \mu \circ_i \nu. \]

Then $\mu \circ_i \nu \in \mathcal{P}(m+n+1)_{\text{tr}}$. The consequence follows easily.

(3) Use a similar argument as in the proof of (2). \( \square \)

Secondly we consider (E1.1.3):

\[ (\mu * \phi) \circ_i \nu = (\mu \circ_{\phi(i)} \nu) \]

for $\mu \in \mathcal{P}(m)$, $\phi \in S_m$, and $\nu \in \mathcal{P}(n)$.

**Lemma 5.3.** Retain the hypotheses as Lemma 5.2.

1. (E1.1.4) holds trivially for $\mu \in \mathcal{P}(1)$.
2. (2a) Suppose $\mu \in \mathcal{P}(2)_{\text{tr}}$. Then

\[ \mu \circ_2 \nu = \begin{cases} (\mu \circ_1 \nu) * (12) \in \mathcal{P}(2) & \text{if } \nu \in \mathcal{P}(1), \\ \mu \circ_1 \nu \in \mathcal{P}(m+n-1)_{\text{tr}} & \text{if } \nu \in \mathcal{P}(n)_{\text{tr}}, \\ (-1)^{\Lambda(\nu)} \mu \circ_1 \nu \in \mathcal{P}(m+n-1)_{\text{sg}} & \text{if } \nu \in \mathcal{P}(n)_{\text{sg}}. \end{cases} \]}
(2b) $\mathcal{P}(2)_{tr} \circ \mathcal{P}(1) \subseteq \mathcal{P}(2)$, $\mathcal{P}(2)_{tr} \circ \mathcal{P}_{sg} \subseteq \mathcal{P}_{sg}$, and $\mathcal{P}(2)_{tr} \circ \mathcal{P}_{tr} \subseteq \mathcal{P}_{tr}$.

(3) (3a) Suppose $\mu \in \mathcal{P}(2)_{sg}$. Then

$$\mu \circ_2 \nu = \begin{cases} 
-(\mu \circ_1 \nu) \ast (12) & \text{if } \nu \in \mathcal{P}(1), \\
-\mu \circ_1 \nu \in \mathcal{P}(m+n-1)_{tr} & \text{if } \nu \in \mathcal{P}(n)_{tr}, \\
(-1)^{\Delta(tr) - 1} \mu \circ_1 \nu \in \mathcal{P}(m+n-1)_{sg} & \text{if } \nu \in \mathcal{P}(n)_{sg}.
\end{cases}$$

(3b) $\mathcal{P}(2)_{sg} \circ \mathcal{P}(1) \subseteq \mathcal{P}(2)$, $\mathcal{P}(2)_{sg} \circ \mathcal{P}_{sg} \subseteq \mathcal{P}_{sg}$, and $\mathcal{P}(2)_{sg} \circ \mathcal{P}_{tr} \subseteq \mathcal{P}_{tr}$.

(4) Suppose $\mu \in \mathcal{P}(m)_{tr}$ for $m \geq 3$.

(4a) $\mu \circ_1 \nu \in \mathcal{P}(m+n-1)_{tr}$. As a consequence, $\oplus_{n \geq 3} \mathcal{P}(n)_{tr}$ is a two-sided ideal of $\mathcal{P}$.

(4b) $\oplus_{i \geq 2} A_i, e$ is a two-sided ideal of $A$.

(4c) $\mu \circ_1 \nu = \mu \circ_1 \nu$ for all $1 \leq i \leq m$.

(4d) $\mathcal{P}(m)_{tr} \circ \mathcal{P}(n)_{sg} = 0$ when $m \geq 3$ and $n \geq 2$.

(5) Suppose $\mu \in \mathcal{P}(m)_{sg}$ for $m \geq 3$.

(5a) $\mu \circ_1 \nu \in \mathcal{P}(m+n-1)_{sg}$. As a consequence, $\oplus_{n \geq 3} \mathcal{P}(n)_{sg}$ is a two-sided ideal of $\mathcal{P}$.

(5b) $\oplus_{i \geq 2} A_i, o$ is a two-sided ideal of $A$.

(5c) $\mu \circ_1 \nu = (-1)^{\Delta(tr) - 1} (i - 1) \mu \circ_1 \nu$ for all $1 \leq i \leq m$.

(5d) $\mathcal{P}(m)_{sg} \circ \mathcal{P}(n)_{tr} = 0$ when $m \geq 3$ and $n \geq 2$.

Proof. (2) (2a) Let $\phi = (12)$ (which is the only non-trivial element in $S_2$). If $\nu \in \mathcal{P}(1)$,

$$\mu \circ_2 \nu = (\mu * \phi) \circ_2 \nu = (\mu \circ_1 \nu) * \phi$$

as desired. If $\nu \in \mathcal{P}(n)_{tr}$, by Lemma 5.2(2), $\mu \circ_1 \nu \in \mathcal{P}(n+m-1)_{tr}$, and

$$\mu \circ_2 \nu = (\mu * \phi) \circ_2 \nu = (\mu \circ_1 \nu) * \phi'' = \mu \circ_1 \nu$$

where the last equation follows from the fact that $\mu \circ_1 \nu \in \mathcal{P}(n+m-1)_{tr}$. If $\nu \in \mathcal{P}(n)_{sg}$, then by Lemma 5.2(3), $\mu \circ_1 \nu \in \mathcal{P}(n+m-1)_{sg}$, and

$$\mu \circ_2 \nu = (\mu * \phi) \circ_2 \nu = (\mu \circ_1 \nu) * \phi'' = sgn(\phi'') \mu \circ_1 \nu$$

where the last equation follows from the fact that $\mu \circ_1 \nu \in \mathcal{P}(n+m-1)_{sg}$. Note that, by Lemma 5.2(2),

$$sgn(\phi'') = (-1)^{\Delta(tr) - 1} (2 - 1) sgn(\phi) = (-1)^{\Delta(tr) \nu}.$$ 

Therefore the assertion follows.

(2b) This follows from part (2a).

(3) Similar to the proof of (2).

(4) (4a) Let $\phi \in S_m$ with $\phi(i) = i$ and $sgn(\phi) = -1$. Such a $\phi$ exists as $m \geq 3$.

By Lemma 5.2(3),

$$\mu \circ_1 \nu = (\mu * \phi) \circ_1 \nu = (\mu \circ_1 \nu) * \phi''.$$ 

By Lemma 5.2(2b), $sgn(\phi'') = sgn(\phi) = -1$. Therefore $\mu \circ_1 \nu \in \mathcal{P}(m+n-1)_{tr}$. Combining with Lemma 5.2(2), $\oplus_{n \geq 3} \mathcal{P}(n)_{tr}$ is a two-sided ideal of $\mathcal{P}$.

(4b) This is a consequence of (4a).

(4c) Let $\phi$ be the transposition $(1i)$. Then

$$\mu \circ_1 \nu = (\mu * \phi) \circ_1 \nu = (\mu \circ_1 \nu) * \phi'' = \mu \circ_1 \nu$$

where the last equation follows from part (4a).

(4d) This follows from part (4a) and Lemma 5.2(3).

(5) Similar to the proof of (4).
Lemma 5.4. Retain the hypotheses as Lemma 5.2.

(1) \((E1.1.2)\) is vacant when \(\lambda \in \mathcal{P}(1)\).

(2) Suppose \(\lambda \in \mathcal{P}(2)_{\text{tr}}\) or \(\lambda \in \mathcal{P}(2)_{\text{sg}}\). Then the following hold

(2a) If \(\mu, \nu \in \mathcal{P}(1)\), then

\[
((\lambda \circ_1 \mu) \ast (12)) \circ_1 \nu = ((\lambda \ast (12)) \circ_1 \nu) \ast (12) \circ_1 \mu.
\]

(2b) If \(\mu, \nu \in \mathcal{P}_{\text{tr}} \cup \mathcal{P}_{\text{sg}}\), then

\[
((\lambda \circ_1 \mu) \ast (12)) \circ_1 \nu = ((\lambda \ast (12)) \circ_1 \nu) \circ_1 \mu.
\]

(2c) If \(\mu, \nu \in \mathcal{P}_{\text{tr}} \cup \mathcal{P}_{\text{sg}}\), then

\[
(-1)^{\text{Ar}(\nu) - 1} \text{Ar}(\nu)(\mu)((\lambda \circ_1 \mu) \circ_1 \nu) = (-1)^{\text{Ar}(\nu)(\mu)}((\lambda \ast (12)) \circ_1 \nu) \circ_1 \mu.
\]

(3) Suppose \(\lambda \in \mathcal{P}(l)_{\text{tr}}\) for \(l \geq 3\). Then

\[
\lambda \circ_1 (\mu \circ_1 \nu) = \begin{cases}
(\lambda \circ_1 (\nu \circ_1 \mu)) & \mu, \nu \notin \mathcal{P}_{\text{sg}}, \\
\lambda \circ_1 (\nu \circ_1 \mu) & \text{otherwise}.
\end{cases}
\]

(4) Suppose \(\lambda \in \mathcal{P}(l)_{\text{sg}}\) for \(l \geq 3\). Then

\[
\lambda \circ_1 (\mu \circ_1 \nu) = \begin{cases}
(-1)^{\text{Ar}(\mu) - 1} (\lambda \circ_1 (\nu \circ_1 \mu)) & \mu, \nu \notin \mathcal{P}_{\text{tr}}, \\
\lambda \circ_1 (\nu \circ_1 \mu) & \text{otherwise}.
\end{cases}
\]

Proof. As always we only consider homogeneous elements.

(2) Since \(l = 2, i = 1\) and \(k = 2\). We need to consider several cases. By Lemma 5.3(2,3), for \(\mu, \nu \in \mathcal{P}\),

\[
(E5.4.1) \quad \mu \circ_2 \nu = \begin{cases}
((\mu \ast (12)) \circ_1 \nu) \ast (12) & \nu \in \mathcal{P}(1), \\
(-1)^{\text{Ar}(\nu)(\mu)} ((\mu \ast (12)) \circ_1 \nu) & \nu \in \mathcal{P}_{\text{tr}} \cup \mathcal{P}_{\text{sg}}
\end{cases}
\]

where \(t(\nu) = \begin{cases} 1 & \nu \in \mathcal{P}_{\text{sg}} \\
0 & \text{otherwise} \end{cases}\). By Lemma 5.3(4,5), for \(\mu \in \mathcal{P}(n)_{\text{tr}} \cup \mathcal{P}(n)_{\text{sg}}\) for \(n \geq 3\), and \(\nu \in \mathcal{P}\),

\[
(E5.4.2) \quad \mu \circ_4 \nu = (-1)^{\text{Ar}(\nu) - 1} (\lambda \circ (\mu)) \mu \circ_1 \nu.
\]

Now we divide the proof into three cases.

Case (2a): \(\mu, \nu \in \mathcal{P}(1)\). Then, by \((E5.4.1)\),

LHS of \((E1.1.2)\) = \((\lambda \circ_1 \mu) \circ_2 \nu = (((\lambda \circ_1 \mu) \ast (12)) \circ_1 \nu) \ast (12)\)

RHS of \((E1.1.2)\) = \((\lambda \circ_2 \nu) \circ_1 \mu = ((\lambda \ast (12)) \circ_1 \nu) \ast (12) \circ_1 \mu\).

So the assertion in (2a) follows.

Case (2b): \(\mu \in \mathcal{P}(1)\) and \(\nu \in \mathcal{P}(n)_{\text{tr}} \cup \mathcal{P}(n)_{\text{sg}}\). Then, by Lemma 5.3 and \((E5.4.2)\),

LHS of \((E1.1.2)\) = \((\lambda \circ_1 \mu) \circ_2 \nu = (-1)^{\text{Ar}(\nu)(\mu)} ((\lambda \circ_1 \mu) \ast (12)) \circ_1 \nu\)

RHS of \((E1.1.2)\) = \((\lambda \circ_2 \nu) \circ_1 \mu = (-1)^{\text{Ar}(\nu)(\mu)} ((\lambda \ast (12)) \circ_1 \nu) \circ_1 \mu\).

So the assertion in (2b) follows.
Case (2c): \( \mu, \nu \in P_{\text{tr}} \cup P_{\text{sg}} \). Then, by Lemma 5.3, (E5.4.2) and (E5.4.1),

\[
\text{LHS of (E1.1.2)} = (\lambda \circ \mu) \circ (m+1) \nu = (-1)^{(\lambda \circ \mu)(\lambda \circ \mu)}((\lambda \circ \mu) \circ \nu) = (-1)^{(\lambda \circ \mu)(\lambda \circ \mu)}((\lambda \circ \mu) \circ \nu).
\]

\[
\text{RHS of (E1.1.2)} = (\lambda \circ \mu) \circ (m+1) \nu = (-1)^{(\lambda \circ \mu)(\lambda \circ \mu)}((\lambda \circ \mu) \circ \nu) \circ \nu.
\]

So the assertion in (2c) follows.

(3) By Lemma 5.3, both sides of (E1.1.2) are zero unless \( \mu, \nu \notin P_{\text{sg}} \). When \( \mu, \nu \notin P_{\text{sg}} \), (E1.1.2) together with Lemma 5.3 implies that

\[
\lambda \circ \mu (\mu \circ \nu - \nu \circ \mu) = 0.
\]

(4) Similar to the proof of part (3).

□

Lastly we consider (E1.1.1): for \( \lambda \in P(l), \mu \in P(m) \) and \( \nu \in P(n) \),

\[
(\lambda \circ \nu) \circ (\mu \circ \nu) = (\lambda \circ \nu) \circ (\mu \circ \nu),
\]

1 \leq i \leq l, 1 \leq j \leq m.

By Lemma 3.6(1), up to relations (E1.1.3) and (E1.1.4), Equation (E1.1.1) is equivalent to

\[
(\lambda \circ \mu) \circ \nu = (\lambda \circ \mu) \circ \nu
\]

for \( \lambda \in P(l), \mu \in P(m) \) and \( \nu \in P(n) \).

Lemma 5.5. Retain the hypotheses as Lemma 5.2. Let \( A_P \) be the graded algebra defined by

(a) \( A_i = P(i+1) \) for all \( i \geq 0 \),

(b) for homogeneous elements \( x, y \in A \), the product of \( x \) and \( y \) is defined to be \( x \cdot y := x \circ y \),

(c) set \( A_{i,e} := P(i+1)_{\text{tr}} \) and \( A_{i,o} := P(i+1)_{\text{sg}} \).

Then \( A \) is a PGPerm algebra.

Proof. The axiom (i) in Definition 3.5 is obvious. By Lemma 5.2(2,3), we have (ii) and by Lemma 5.3(4b,5b), we have (iii). The axioms (iv) (resp. (va), (vb), (vc)) follows from Lemma 5.3(3,4) (resp. (2a),(2b),(2c)). So we finish the proof. □

The PGPerm algebra \( A \) in Lemma 5.5 is denoted by \( F_{\text{Atr}}(P) \) which is just \( F(P) \) if we only consider it as an associative graded algebra. For every PGPerm algebra \( A \), we can construct an \( \text{Atr} \) operad, denoted by \( G_{\text{Atr}}(A) \), as in Example 3.7.

Theorem 5.6. Suppose \( \text{char } k \neq 2 \). The functors \( (G_{\text{Atr}}, F_{\text{Atr}}) \) induce an equivalence between the category of PGPerm algebras and that of \( \text{Atr} \) operads. Furthermore, a PGPerm algebra \( A \) is finitely generated if and only if \( G_{\text{Atr}}(A) \) is.

Proof. Note that \( G_{\text{Atr}} \) is constructed in Example 3.7 and that \( F_{\text{Atr}} \) is constructed in Lemma 5.5. By these explicit constructions, it is straightforward to check that \( G_{\text{Atr}} \) and \( F_{\text{Atr}} \) are inverse to each other.

The second statement on finite generation is easy to verify with details omitted. □

We are now ready to prove a version of Corollary 0.3.

Theorem 5.7. Suppose \( \text{char } k \neq 2 \). Let \( P \) be a finitely generated almost \( \text{Atr} \) operad.

(1) The Hilbert series of \( P \) is rational.
Proof. (1) The assertion follows from Theorem \( \text{5.6} \) and Lemma \( \text{3.8(3)} \).

(2) The assertion follows from Theorem \( \text{5.6} \) and Lemma \( \text{3.8(2)} \).

(3) The assertion follows from Theorem \( \text{5.6} \) and Lemma \( \text{3.8(4)} \).

Lemma 5.8. Let \( \mathcal{P} \) be a left torsionfree operad.

(1) If \( \mathcal{P} \) is almost \( \text{Str} \), then it is \( \text{Str} \).

(2) If \( \mathcal{P} \) is almost \( \text{Atr} \), then it is \( \text{Atr} \).

Proof. The proofs of parts (1) and (2) are very similar. We only prove part (2). By definition \( \mathcal{P}_w \) is \( \text{Atr} \) for some \( w \geq 2 \). For all \( m \geq w \) and \( y \in \mathcal{P}(m) \), all \( \nu \in \mathcal{P}(n) \) and \( \sigma \in \Lambda_n \), Lemma \( \text{5.3(5)} \) implies that \( y_{11}(\nu \star \sigma - \nu) = 0 \). Hence \( \nu \star \sigma - \nu \in \tau^i(\mathcal{P}) \). Since \( \mathcal{P} \) is left torsionfree, \( \nu \star \sigma = \nu \) for all \( \sigma \in \Lambda_n \). So \( \mathcal{P} \) is \( \text{Atr} \).

\( \square \)

Proposition 5.9. Suppose \( \text{char } \mathbb{k} \neq 2 \). Let \( \mathcal{P} \) be a left torsionfree almost \( \text{Atr} \) operad.

(1) For \( \mu \in \mathcal{P}(m), \nu \in \mathcal{P}(l) \), then
\[
\mu \circ_1 \nu = (-1)^{\left(\text{Ar}(\mu)-1\right)\left(\text{Ar}(\nu)-1\right)\text{tr}(\mu)\text{tr}(\nu)} \nu \circ_1 \mu.
\]

(2) For \( \mu \in \mathcal{P}(m), \nu \in \mathcal{P}(l), 1 \leq i \leq \text{Ar}(\mu) \) we have
\[
\mu \circ_i \nu = (-1)^{i-1}(-1)^{\left(\text{Ar}(\mu)-1\right)\left(\text{Ar}(\nu)-1\right)\text{tr}(\mu)\text{tr}(\nu)} \mu \circ_1 \nu.
\]

(3) If \( \mu \in \mathcal{P}_{\text{tr}} \) and \( \nu \in \mathcal{P}_{\text{sg}} \), then \( \mu \circ_i \nu = 0 \) for all \( 1 \leq i \leq \text{Ar}(\mu) \).

(4) If \( \mu \in \mathcal{P}_{\text{sg}} \) and \( \nu \in \mathcal{P}_{\text{tr}} \), then \( \mu \circ_i \nu = 0 \) for all \( 1 \leq i \leq \text{Ar}(\mu) \).

Proof. By Lemma \( \text{5.3(2)} \), \( \mathcal{P} \) is \( \text{Atr} \).

(1) By Lemma \( \text{5.4(3,4)} \), we have, for every \( \lambda \in \mathcal{P}_{\geq 3} \),
\[
\lambda \circ_1 Y = 0
\]
where \( Y = \mu \circ_1 \nu - (-1)^{\left(\text{Ar}(\mu)-1\right)\left(\text{Ar}(\nu)-1\right)\text{tr}(\mu)\text{tr}(\nu)} \nu \circ_1 \mu \). By Lemma \( \text{5.3(4c,5c)} \), \( \lambda \circ_i Y = 0 \) for all \( 1 \leq i \leq \text{Ar}(\lambda) \). Hence \( Y \in \tau^l(\mathcal{P}) \). Since \( \mathcal{P} \) is left torsionfree, \( Y = 0 \) which is equivalent to the assertion.

(2,3,4) Nothing needs to prove if \( m = 1 \). If \( m \geq 3 \), the assertions in parts (2,3,4) follow from Lemma \( \text{5.3(4c,4d, 5c,5d)} \). If \( m = 2 \) and \( n \geq 3 \) (or \( n = 2 \) and \( l(\mu) = \text{tr}(\nu) \)), the assertions in parts (2,3,4) follow from part (1) and Lemma \( \text{5.3(2a,3a)} \).

Next we consider the case \( m = n = 2 \) and \( l(\mu) = 0 \) and \( l(\nu) = 1 \). For \( \lambda \in \mathcal{P}(l)_{\text{sg}} \) with \( l \geq 3 \), \( \lambda \circ_1 (\mu \circ_2 \nu) = (\lambda \circ_1 \mu) \circ_2 \nu = 0 \) by Lemma \( \text{5.3(5d)} \). If \( \lambda \in \mathcal{P}(l)_{\text{tr}} \), we also have \( \lambda \circ_1 (\mu \circ_2 \nu) = (\lambda \circ_1 \mu) \circ_2 \nu = 0 \) by Lemma \( \text{5.3(4d)} \). Therefore \( \mu \circ_1 \nu \in \tau^l(\mathcal{P}) = 0 \).

By Lemma \( \text{5.3(2a)} \), both sides of \( (\text{5.9.2}) \) are 0, so assertions in parts (2,3) hold. Similarly, if \( m = n = 2 \) and \( l(\mu) = 1 \) and \( l(\nu) = 0 \), assertions in parts (2,4) hold.

It remains to consider when \( m = 2 \) and \( n = 1 \). In this case parts (3,4) are vacate. So we will consider the assertion in part (2). If \( \mu \in \mathcal{P}(2)_{\text{sg}} \) and \( \lambda \in \mathcal{P}(l)_{\text{sg}} \) with \( l \geq 3 \),
\[
\lambda \circ_1 (\mu \circ_2 \nu) = \lambda \circ_1 (- (\mu \circ_1 \nu) * (12)) \quad \text{Lemma } \text{5.3(3a)}
\]
\[
= - (\lambda \circ_1 (\mu \circ_1 \nu)) * (12)
\]
\[
= \lambda \circ_1 (\mu \circ_1 \nu) \quad \text{Lemma } \text{5.3(5a)},
\]
and when $\lambda \in P(l)_{tr}$ with $l \geq 3$,
\[
\lambda \circ_1 (\mu \circ_2 \nu) = \lambda \circ_1 (- (\mu \circ_1 \nu) \ast (12)) \quad \text{Lemma } \text{5.3(3a)}
\]
\[
= - (\lambda \circ_1 (\mu \circ_1 \nu)) \ast (12)
\]
\[
= - (\lambda \circ_1 (\mu) \circ_1 \nu) \ast (12) = 0 \quad \text{Lemma } \text{5.3(4d)}.
\]
So $\lambda \circ_1 (\mu \circ_2 \nu - \mu \circ_1 \nu) = 0$ for all $\lambda \in P_{\geq 3}$. Since $P$ is left torsionfree, we have $\mu \circ_2 \nu = \mu \circ_1 \nu$. Similarly, if $\mu \in P(2)_{tr}$, we can show that $\mu \circ_2 \nu = \mu \circ_1 \nu$. This finishes the proof of part (2).

**Corollary 5.10.** Suppose $\text{char } \mathbb{k} \neq 2$. The functors $(G_{Atr}, F_{Atr})$ restrict to

1. an equivalence between the category of torsionfree PGC algebras and that of $Atr$ left torsionfree operads;
2. an equivalence between the category of prime PGC algebras and that of $Atr$ prime operads.

**Proof.** (1) Suppose $A$ is a torsionfree PGC algebra and let $P = G_{Atr}(A)$. In this proof we identify elements in $A$ with elements in $P$. By [2.3.7.1], it follows from the definition that $\tau^l(P) = \tau^l(A)$. Since $A$ is torsionfree, $P$ is left torsionfree.

Conversely, let $P$ be a $Atr$ left torsionfree operad. By Proposition [5.9(1,2)], $\tau^r(A) = \tau^r(P)$ and $A = F_{Atr}(P)$ is PGC. Thus $A$ is torsionfree. Therefore the assertion follows from Theorem 5.6.

(2) The proof is similar and omitted.

**Lemma 5.11.** Suppose $\text{char } \mathbb{k} = 2$. Let $P$ be a semiprime almost $Atr$ operad. Then it is $Str$.

**Proof.** First of all it is easy to reduce to the prime case. So we assume that $A$ is prime. If $P$ is finite dimensional, the primeness of $P$ implies that $P = P(1)$. So it is $Str$. For the rest of the proof, we assume that $P$ is prime and infinite dimensional.

Since $P$ is prime and infinite dimensional, it is left torsionfree by Lemma [2.2(3)]. By Lemma [2.6(2)], $P$ is $Atr$.

For $n \geq 3$, let $I(n) = \{ \mu + \mu \ast \sigma \mid \mu \in P(n), \sigma \in S_n \}$ and $I = \oplus_{n\geq 3} I(3)$. Since $P$ is $Atr$, when $\sigma \in A_n$, $\mu + \mu \ast \sigma = 2\mu = 0$. Further for any two $\sigma, \phi \in S_n \setminus A_n$, $\mu \ast \sigma = \mu \ast \phi$ and hence $\mu + \mu \ast \sigma = \mu + \mu \ast \phi$. We claim:

(i) $I$ is a two-sided ideal of $P$.
(ii) If $x \in I(n)$, then $x + x \ast \sigma = 0$ for all $\sigma \in S_n$.
(iii) $I \circ I = 0$.

**Proof of (i):** For every $\lambda \in P(m)$ for $m \geq 1$ and $1 \leq i \leq m$, and for every $x = \mu + \mu \ast \sigma \in I(n)$, by [3.1.1.3], we have,
\[
\lambda \circ_i x = \lambda \circ_i (\mu + \mu \ast \sigma) = (\lambda \circ_i \mu) + (\lambda \circ_i \mu) \ast \sigma' \in I(n + m - 1).
\]

For $1 \leq j \leq n$, we can write $x = \mu + \mu \ast \sigma$ as $\mu + \mu \ast \phi$ where $\phi \in S_n \setminus A_n$ and $\phi(j) = j$ (which is possible when $n \geq 3$). Then, by [3.1.1.4],
\[
x \circ_j \lambda = (\mu + \mu \ast \phi) \circ_j \lambda = (\mu \circ_j \lambda) + (\mu \circ_j \lambda) \ast \phi'' \in I(n + m - 1).
\]

Therefore $I$ is a two-sided ideal of $P$.

**Proof of (ii):** Write $x$ as $\mu + \mu \ast \phi$ where $\phi \in S_n \setminus A_n$. If $\sigma \in A_n$, then $x + x \ast \sigma = 2x = 0$. Now assume that $\sigma \in S_n \setminus A_n$. Then
\[
x + x \ast \sigma = (\mu + \mu \ast \phi) + (\mu + \mu \ast \phi) \ast \sigma = (\mu + \mu \ast \phi \ast \sigma) + \mu \ast \phi + \mu \ast \sigma
\]
\[
= 2\mu + 2\mu \ast \phi = 0.
\]
elements of degree 2. Let $A$, Example 6.2. The next example is of this kind.

\[ x \circ_j y = (\mu + \mu \ast \phi) \circ_j (\nu + \nu \ast \sigma) \]
\[ = \mu \circ_j \nu + (\mu \circ_j \nu) \ast \sigma' + (\mu \circ_j \nu) \ast \phi'' + (\mu \circ_j \nu) \ast \phi'' \ast \sigma' \]
\[ = 2\mu \circ_j \nu + 2(\mu \circ_j \nu) \ast \sigma' = 0 \]

where the second last equation follows from the fact that $\text{sgn}(\phi'') = \text{sgn}(\phi)$ and $\text{sgn}(\sigma') = \text{sgn}(\sigma)$ [Lemma 1.2]. Therefore $I \circ I = 0$.

Since $P$ is prime, $I = 0$. This implies that $\mu + \mu \ast \sigma = 0$ or $\mu = \mu \ast \sigma$ for all $\sigma \in S_n$ when $n \geq 3$. This means that $P$ is almost $\text{Str}$. By Lemma 5.11 and the fact that $P$ is left torsionfree, we have that $P$ is $\text{Str}$.

**Corollary 5.12.** Let $P$ be an infinite dimensional prime almost $\text{At}r$ operad. Then $P = G_{\text{At}r}(A)$ for a PGC and commutative graded algebra $A$. As a consequence, every element in $P$ is central.

**Proof.** By Lemma 2.3(3), $P$ is left torsionfree and by Lemma 5.8(2), $P$ is $\text{At}r$.

Case 1: $P$ is $\text{Str}$. The assertion follows from Corollary 4.5(2). In this case $P = G_{\text{Str}}(A)$ for a commutative prime graded algebra $A$. So every element in $P$ is central.

Case 2: char $k = 2$. By Lemma 5.11 $P$ is $\text{Str}$. The assertion follows from Case 1.

Case 3: $P$ is not $\text{Str}$ and char $k \neq 2$. By Corollary 5.10(2), $P = G_{\text{At}r}(A)$ for a prime PGC algebra $A$. Since $P$ is not $\text{Str}$, $A$ is of odd type. Since $A$ is prime, as an algebra, $A$ is commutative and $A_i = 0$ for all odd $i$. So every element in $P$ is central. □

6. Remarks and examples

In this section we will give some remarks and examples.

**Remark 6.1.** Theorem 4.2 provides a convenient way to construct many $\text{At}r$ operads. We are wondering if there are natural topological operads that are analogue to $\text{At}r$ operads.

If $A$ is a graded commutative algebra viewed as a PGC of odd type, then $G_{\text{At}r}(A)$ is also denoted by $G_{\text{Seg}}(A)$. We can construct a lot of operads with $P_{\geq 2} = P_{\text{Seg}}$ this way. The next example is of this kind.

**Example 6.2.** Let $\{x_1, \ldots, x_a\}$ be elements of degree 1 and $\{y_1, \ldots, y_b\}$ be elements of degree 2. Let $A$ be the graded commutative algebra generated

\[ \{x_1, \ldots, x_a, y_1, \ldots, y_b\} \]

subject to relations

\[ x_i^2 = 0, x_i x_j = -x_j x_i, y_i y_j = y_j y_i, x_i y_j = y_j x_i \]

for all possible $i$ and $j$. Let $A_{i,e} = 0$ and $A_{i,o} = A_i$ for all $i \geq 1$. By Lemma 8.10 it is a PGPerm and PGC algebra of odd type. The operad $G_{\text{Seg}}(A)$ is denoted by $\mathcal{Mas}^2$, and called a Massey operad with parameters $(a, b)$.

We give a couple of more examples.
Example 6.3. Let $A$ be the commutative polynomial ring $k[x]$. Let $S_A$ be the operad provided by [QX20] Construction 7.1. Then $F(S_A)$ is a connected graded algebra $B$ with

$$B_i = k x_{i,1} \oplus \cdots \oplus k x_{i,i+1}$$

with associative multiplication $\cdot$ determined by

$$x_{i,s} \cdot x_{j,t} = \begin{cases} x_{i+j,t} & s = 1, \\ 0 & s \neq 1. \end{cases}$$

Then $B$ is not prime as the ideal generated by $x_{1,2}$ is nilpotent. Therefore the converse of Lemma [LS] is false.

Example 6.4. Let $A$ be the algebra $k\langle x, y \rangle / (xy, y^2)$ with $\deg x = \deg y = 1$. Then $A$ is a connected $G\Perm$ algebra. Its Hilbert series is $1 + 2t - t^2$, so $A$ has GK-dimension 1. By Theorem [12] $P := G_{\Str}(A)$ is a finitely generated connected $\Str$ operad of GK-dimension 1. The following facts are easy to verify

1. $P$ is right noetherian, but not left noetherian.
2. $\tau^r(P) = \oplus_{t=0}^{\infty} kyx^t$ which is infinite dimensional.
3. $\tau^r(P) = \tau^r(\tau^l(P)) = 0$.
4. The center of $A$ is $k$. As a consequence, $G_{\Str}(A)$ does not have any central element of arity $\geq 2$.

Remark 6.5. There are still many questions about $Atr$ operads. For example, can we classify all $Atr$ Hopf operads (resp. cyclic operads, etc.)? Note that the operad $G_{\Str}(A)$ for every commutative graded algebra $A$ is cyclic in the sense of [Me04] Definition 3.3.

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School of Mathematics and Statistics, Huizhou University, Huizhou, Guangdong 516007, China

Email address: liyu820615@126.com

Department of Mathematics, Fudan University, Shanghai 200433, China

Email address: qizihao@foxmail.com

School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China

Email address: yjxu2002@163.com

Department of Mathematics, Box 354350, University of Washington, Seattle, Washington 98195, USA

Email address: zhang@math.washington.edu

School of Mathematical Sciences, South China Normal University, Guangzhou 51063, China

Email address: 295841340@qq.com

School of Mathematics and Statistics, Huizhou University, Huizhou, Guangdong 516007, China

Email address: zhaoxg@hzu.edu.cn