On the quasi-invariance of the Wiener measure on path spaces and the anticipative integrals over a Riemannian manifold

Adnan Aboulalaâ∗

Abstract

Some parts of stochastic analysis on curved spaces are revisited. A concise proof of the quasi-invariance of the Wiener measure on the path spaces over a Riemannian manifold is presented. The shifts are allowed to be in the Cameron-Martin space and random. The second part of the paper presents some remarks on the anticipative integrals on Riemannian manifolds.

Key-words : Diffusions on manifolds, Quasi-invariance property, Malliavin calculus, anticipative integrals.

Mathematics Subject Classification (2000): 58J65, 60H07, 28C20.

1 Introduction

Let $d > 0$, $W = (C_0([0, 1], \mathbb{R}^d), \mathbb{H}, \mu)$ denotes the standard Wiener space: $\mathbb{H} = \{ h \in C_0([0, 1], \mathbb{R}^d) : \int_0^1 |\dot{h}_s|^2 ds < \infty \}$ and $\mu$ is the Wiener measure on $C_0([0, 1], \mathbb{R}^d)$ the space of $\mathbb{R}^d$-valued continuous functions on $[0, 1]$ starting from 0. The problem of the quasi-invariance of the Wiener measure under translations $T : w \mapsto w + k(w)$ with $k(w) \in \mathbb{H}$ a.s. was considered by Cameron and Martin [5] who have first dealt with the case where $k(w)$ is a non-random element of $\mathbb{H}$. In this case $T$ leaves the Wiener measure $\mu$ quasi-invariant i.e., the measure $T_*\mu$ induced by $T$ and $\mu$ are mutually absolutely continuous. Later the case where $k$ is random was considered [6] and has been generalized by many authors. The analogue of the Cameron-Martin theorem

∗This paper was written in 1996, when the author was at the Laboratoire de Probabilités, Université Paris VI during the period 1995-1997. Some more recent references have been added in this version. E-mail : adnan.aboulalaa@polytechnique.org
in the case of Brownian motion on a Riemannian manifold was first developed by Driver [11]. Let $M$ be a compact Riemannian manifold endowed with a connection $\nabla$ compatible with the metric, $m_0 \in M$ a fixed point and $P_{m_0}(M) := \{ p \in C([0, 1], M) : p_0 = m_0 \}$ the path space over $M$. Let $\nu$ be the Wiener measure on $P_{m_0}(M)$ i.e. the law of the Brownian motion $(p_s(w))$ on $M$ starting from $m_0$. A path space analogue of the flow $(t, w) \mapsto w + th$ which leaves the Wiener measure $\nu$ quasi-invariant was constructed in [11]. It was shown in that paper that among other possible flows, a flow of semimartingales $\sigma(t)$ which has the quasi-invariance property is the solution, in an appropriate sense, to

$$\frac{d\sigma(t, w)}{dt} = \iota^{\sigma(t)}_t h, \quad \sigma(0, w) = p(w)$$

where $\iota^{\sigma(t)}_t$ is the Itô stochastic parallel transport (with respect to $\nabla$) and $h \in \mathbb{H}$. (Note that $t \in \mathbb{R}$ is the parameter of the flow while $s$ and “.” denote the time.) Another convenient form of this equation is

$$\frac{d\sigma(t, w)}{dt} = H_s(\sigma(t, w))h,$$

(1.1)

where $s \mapsto H_s(\sigma(t))$ is the horizontal lift of the semimartingale $\sigma(t)$. The main result of [11] is that if the function $h$ is $C^1$ then the equation (1.1) has a unique solution $\sigma(t)$ in the set of “brownian semimartingales”, and if the torsion of the connection is “skew-symmetric” (see § 4 below) then, for each $t$, the transformation $p \mapsto \sigma(t)(p)$ in the path space $P_{m_0}(M)$ leaves the Wiener measure quasi-invariant. Later E. Hsu [20] removed the restriction $C^1$ on $h$ to allow it to be in the Cameron-Martin space $\mathbb{H}$. See also Enchev-Stroock [14] for another approach.

The purpose of the first part of this paper (Sections 2-4) is to prove the existence and uniqueness of solutions to (1.1) in the case where $h$ is random adapted and such that there exists a constant $C$ with $\int_0^1 |h_s|^2 ds \leq C$ a.s. While slight modifications to Hsu’s proof could give the same result, we feel that the proof given here –which is inspired by that of Driver[11], Section 7– is elementary and more direct.

Notice that in the case where $h$ is random, the transformation $t \mapsto \sigma(t)$ will not have the flow property just like the flat case where the transformation $(t, w) \mapsto w + th(w)$ has not the flow property in general. At this point one can also try the generalization to the case where $h$ is not adapted. This does not seem plausible; see remark 4.1 below.

The “intrinsic” Cameron-Martin theorem for Brownian motion on Riemannian manifolds had renewed the interest in stochastic analysis on path and loop spaces in the mid-1990s. In fact, let us recall that the quasi-invariance property plays a major role in the analysis of Wiener functionals – the flat case – based on the stochastic calculus of variations; for it allows to define a closable gradient (via an integration by parts formula) which can be applied to a wide class of Wiener functionals; this is not
possible with the usual differential calculus without probability theory (see e.g., [26], [33]). Hence, as we have a Cameron-martin theorem on the path space $P_{m_0}(M)$, we would be able to develop the same methods on $P_{m_0}(M)$. However, it should be noted that such a formalism can be constructed without using this quasi-invariance theorem, see Fang-Malliavin [17].

As in the flat case, one can seek the anticipative calculus based on the stochastic calculus of variations over the path space. The first step is the study of the anticipative integrals. One of the difficulties is that a chaotic development machinery adapted to this calculus is not available except in the case of Lie groups (see e.g., [31], [19], [34]). The second part of this paper contains some remarks about these anticipative integrals. Using the parallel transport, a gradient $D^M$ on cylindrical functions on the path space $P(M)$ can be defined ([11], [25], [17]). This gradient is related to the transformation given by (1.1). The corresponding integration by part formula gives rise to a curvature term. In order to get rid of this term, another gradient $\tilde{D}$ was introduced in [17]. Thus, following Gaveau-Trauber [18], we define two anticipative integrals $\delta^M$ and $\tilde{\delta}$ which correspond to $D^M$ and $\tilde{D}$ respectively. In [15], Fang proved that the domain of $\tilde{\delta}$ contains the Sobolev spaces $L^{1,p}(M), p > 2$ (see Section 5). In this part we prove that the domain of $\tilde{\delta}$ contains the spaces $L^{1,2}_C$, and we give an ‘explicit’ expression of $\tilde{\delta}$ and the corresponding Itô formula. Let us remark that the expression of $\delta^M$ is not as convenient as that of $\tilde{\delta}$. We end this section by some remarks on the difficulties to find satisfactory $L^p$-estimates of these stochastic integrals, which are mainly due to the rotational derivative which appear in their expressions.

Some additional references: For a general account on the subject, see [12]. For another presentation and other results see Elworthy and Li [13]. Further extensions of the quasi-invariance property on path spaces may be found in Hsu [21], Hsu and Ouyang [22], Bell [2] and Zhang and Kannan [35].

2 Preliminaries

2.1 Some geometric notations

Let $M$ be a compact connected Riemannian manifold of dimension $d$. We denote by $g$ the tensor metric and by $\mathcal{O}(M)$ the bundle of orthonormal frames i.e. the set of $r = (m, r_m)$ where $m \in M$ and $r_m$ is an orthonormal basis of $T_m M$. Throughout, we fix an element $r_0 = (m_0, r_{m_0})$, $T_{m_0} M$ will be identified to $\mathbb{R}^d$ and for each $r = (m, r_m)$, $r_m$ will be identified with an isometrie of $\mathbb{R}^d$ onto $T_m M$.

In $\mathcal{O}(M)$ the canonical $\mathbb{R}^d$-valued form $\theta$ is defined by $\theta_v(\xi) = r^{-1}d\pi(\xi)$, where $\pi : \mathcal{O}(M) \rightarrow M$ is the canonical projection. We suppose that $M$ is endowed with an affine connection $\nabla$ compatible with $g$. This connection determines an $so(d)$-valued one form $\omega$ on $\mathcal{O}(M)$ by $\omega_v(\dot{\gamma}(0)) = \gamma^{-1}(0)(\nabla(\gamma(s))/ds(s = 0))$ for every smooth curve
\(\gamma\) on \(O(M)\). Equivalently, given \(r \in O(M)\), the connection \(\nabla\) determines the horizontal subspace \(H_r O(M)\) of \(T_r O(M)\) (cf., e.g., [3], [7]) by \(\xi \in H_r (O(M))\) iff \(\omega_r (\xi) = 0\) (\(\dim (H_r (O(M))) = d\)). For \(v \in T_m M\), a vector \(\tilde{v} \in T_r O(M)\) is a horizontal lift of \(v\) iff \(\pi (r) = m, \tilde{v} \in H_r O(M)\) and \(d \pi (r) \cdot \tilde{v} = v\).

Throughout, \((e_i), i = 1, \ldots, d\) will denote the canonical basis of \(\mathbb{R}^d\). To each \(i \in \{1, \ldots, d\}\) we associate a vector field \(L_i\) on \(O(M)\) as follows: for \(r \in O(M)\), \(L_i (r)\) is the horizontal lift of \(r e_i\) (\(\omega_r (L_i (r)) = 0\) and \(\theta_r (L_i (r)) = e_i\)).

Let \(R, T\) be the curvature and the torsion tensors and \(\Omega, \Theta\) be the curvature \((so(d)\text{-valued}) 2\text{-form}\) and the torsion \((\mathbb{R}^d\text{-valued})\) form respectively defined on \(O(M)\). Then we have the following structural equations

\[
d\theta = -\omega \wedge \theta + \Theta, \quad d\omega = -\omega \wedge \omega + \Omega.
\]

(2.2)

Also, for \(v_1, v_2 \in \mathbb{R}^d\) we denote by \(\Omega_r (v_1, v_2)\) the matrix \(\Omega_r (r \tilde{v}_1, r \tilde{v}_2)\) and we set

\[
\text{ric}_r (v) = -\sum_{i=1}^d \Omega_r (e_i, v) e_i, \quad \text{for } v \in \mathbb{R}^d
\]

(the covariant representation of the Ricci tensor).

### 2.2 An imbedding procedure

In this paragraph we recall some results of \([11], \text{section 2}\) concerning the imbedding of the manifold \((M, g, \nabla)\) into an open neighborhood \(Y \subset \mathbb{R}^N\) for some \(N\), with a convenient extension of the covariant derivative \(\nabla\). This extension is needed to guarantee property (i) of the Proposition 2.1 below and will be used in the next section. So let \(M\) be imbedded in \(\mathbb{R}^N\) for some \(N\). Then:

There exists an open neighborhood \(Y_0 \subset \mathbb{R}^N\) of \(M\) endowed with a Riemannian \(\tilde{g}\) and a covariant derivative \(\nabla\) compatible with \(\tilde{g}\) and a map \(p: Y_0 \rightarrow M \subset \mathbb{R}^N\) such that

(i) \(p_M = id_M\) and if \(i: M \rightarrow Y_0\) is the inclusion map then the map \(j : O(M) \rightarrow \mathbb{R}^N \times L(\mathbb{R}^d, \mathbb{R}^N)\) \(r \mapsto (i \circ \pi (r), \dot{i} (\pi (r)) \circ r)\) is an imbedding of \(O(M)\).

(ii) If \(\Gamma\) is the \(N \times N\)-matrix one form on \(Y\) such that \(\nabla_X Z = d_X Z + \Gamma (X).Z\) for each vector fields \(X, Z\) on \(Y\), then for all \(y \in Y, v_y \in T_y Y\):

\[
p'(y) \Gamma (v_y) = p''(v_y) + \Gamma_y (p'(y).v_y).p'(y).
\]

(2.3)

The main consequences of this last equation is the property (i) of proposition 2.1 below and a stochastic equation of the horizontal lift (see equation (2.4) below).

Now, let \((x_s)\) be an \(M\)-valued semimartingale. A semimartingale \(X_s\) with values in \(O(M)\) is a horizontal lift of \((x_s)\) (with respect to the connection \(\nabla\)) if \(\pi \circ X_s = x_s\).
a.s. and \( \int_{0, X_s} \omega \circ dX \) vanishes a.s. (the last integral is the stochastic line integral, see Ikeda-Watanabe \cite{23} and the references therein).

For example the horizontal lift \( r(s) \) of the Brownian motion \( p_s \) on \( M \) is defined by the following SDE on \( O(M) \):

\[
dr(s) = L_i(r(s)) \circ dw_s, \quad r(0) = r_0.
\]

This is, in fact, a method for the construction of the Brownian motion on \( M \) by setting \( p_s = \pi(r(s)) \). See \cite{23}.

We have the following existence and uniqueness result: given an \( M \)-valued semimartingale, there is a unique horizontal lift \( X_s \) of \( (x_s) \) such that \( X_0 = r_0 \). Furthermore, by using the above imbedding \( X_s \) is the unique solution to the Stratonovich SDE

\[
dX_s = -\Gamma_{x_s}(\circ dx_s).X_s, \quad X_0 = r_0. \tag{2.4}
\]

Given \( (y_s) \) a \( Y_0 \)-valued semimartingale, we will denote by \( (\tilde{H}_s(x)) \) the solution \( X_s \) to the equation \( dX_s = \Gamma_{y_s}(\circ dy_s).X_s \), and if \( y_s \in M \) a.s. we denote this solution by \( (H_s(x)) \).

In the sequel, \( Y \subset Y_0 \) will denote a compact neighborhood of \( M \). So \( \Gamma \) is bounded with its derivatives on \( Y \).

**Proposition 2.1** Let \( x_s, s \in [0, 1] \) be a continuous \( Y \)-valued semimartingale such that \( x_0 \in M \). Then

(i) \( p'(x)sH_s(x) = H_s(p \circ x) \).

(ii) if \( x_s \) is of the form \( dx_s = O_s dw_s + A_s ds \) then \( X_s = H_s(x) \) is of the form

\[
dX_s = G_s(dw_s).X_s + F_sX_s ds, \tag{2.5}
\]

where \( G_s \) and \( F_s \) are such that for some constant \( K \) we have:

\[
\begin{align*}
&\sup_{s \in [0, 1]} |O_s| \leq K \sup_{s \in [0, 1]} |O_s| \quad \text{a.s.} \\
&\int_0^1 |F_s|^2 ds \leq K \int_0^1 |A_s|^2 ds + \sup_{s \in [0, 1]} |O_s|^2 + \sup_{s \in [0, 1]} |O_s|^4.
\end{align*}
\]

**Proof.** (i) is the lemma 7.1 of \cite{11}.

The stochastic equation of the horizontal lift \( X_s \) written in the Itô form is

\[
dX_s = \Gamma_{x_s}(O_s dw_s).X_s - \{\Gamma_{x_s}(A_s) \}
+ \frac{1}{2} \sum_{i=1}^d \Gamma_{x_s}(O_s e_i) \Gamma_{x_s}(O_s e_i) - \Gamma'_{x_s}(O_s e_i, O_s e_i) \}.X_s, \tag{2.6}
\]

so that (ii) follows by the boundedness of \( X_s, \Gamma, \Gamma' \).
3 Existence and uniqueness results for the transformation

First, let us introduce some notations.

The norms on vector spaces like \( \mathbb{R}^N, L(\mathbb{R}^N, \mathbb{R}^m) \) will be denoted by \( |.| \). For a random variable \( F \) we set \( \| F \|_\infty = \| F \|_{L^\infty(\Omega)} \). We denote by \( BS(\mathbb{R}^n) \) the set of Brownian semimartingales (terminology used in [11]) i.e. the semimartingales \( x_s = O_s dw_s + A_s ds \) where \( (w_s) \) is an \( \mathbb{R}^m \)-valued Brownian motion, \( (O_s) \) is a \( L(\mathbb{R}^m, \mathbb{R}^n) \)-valued adapted process and \( (A_s) \) is an \( \mathbb{R}^n \)-valued adapted process. Similarly \( BS(M) \) is the subset of \( x, \in BS(\mathbb{R}^N) \) such that \( x_s \in M \) a.s. In the sequel we denote by \( E \) the set of \( x = x_0 + \int_0^1 O_s dw_s + \int_0^1 A_s ds \in BS(\mathbb{R}^N) \) such that

\[
\| x \|_E := (E \sup_{s \in [0,1]} |O_s|^2 + E \int_0^1 |A_s|^2 ds)^{1/2} < +\infty.
\]

\((E, \| . \|_E)\) is a Banach space.

3.1 Statement of the result

Definition 3.1 Let \( \sigma^0 \) be a semimartingale on \( M \). A map \( \sigma: \mathbb{R} \rightarrow BS(M) \) is a solution of the initial value problem

\[
\mathcal{P}_1 : \begin{cases} 
\frac{d\sigma}{dt} = H(\sigma(t)) h(w) \\
\sigma(0) = \sigma^0.
\end{cases}
\]

if \( \sigma \), viewed as a map \( \mathbb{R} \rightarrow E \), satisfies

\[
\mathcal{P}_2 : \begin{cases} 
\frac{d\sigma}{dt} = \tilde{H}(\sigma(t)) h(w) \\
\sigma(0) = \sigma^0
\end{cases}
\]

\((E \text{ is equipped with the norm } \| . \|_E) \text{ and } \sigma_s(t) \in M, \ s \in [0,1], \ a.s.. \)

Remark 3.2 In this definition we restrict ourselves to the solutions \( \sigma \) such that \( \sigma(t) \in BS(M) \) for each \( t \).

Let \( x \) be a semimartingale in \( E \) with \( dx_s = O_s dw_s + A_s ds \). We will say that \( x \) satisfies the hypothesis (H) if

\[
\| \sup_s |O_s| \|_\infty + \| \int_0^1 |A_s|^2 ds \|_\infty < \infty
\]

We can now state the existence and uniqueness result concerning the problem \( \mathcal{P}_1 \).
Theorem 3.3 let \( h : [0, 1] \times W \rightarrow \mathbb{R}^d \) be such that \( h \in \mathbb{H} \), a.s. and \( \int_0^1 h_t^2(w) \leq C \) a.s. for some constant \( C \). Then for every semimartingale \( \sigma^0 \in BS(M) \cap E \) which satisfies the hypothesis (H) there exists a unique solution \( t \mapsto \sigma(t) \) defined on \( \mathbb{R} \) to the problem \( P_1 \).

Let us point out that the main problem here consists in solving an ordinary differential equation which does not satisfy the local Lipschitz condition.

### 3.2 Proof of the theorem

In this section \( \sigma^0 \in BS(M) \) and \( h \in \mathbb{H} \) are fixed and satisfy the assumptions of Theorem 3.1. We will set \( \|h\|_{\infty} := \sup_s |h_s| \) (which is \( \leq \|\int_0^1 |h_s|^2 ds\|_{1/2} \|_{\infty} \leq C \)).

Let \( (x_s) \in E \). The following inequality, which is a consequence of the Burkholder inequalities, will be frequently used:

\[
E \sup_{s \in [0,1]} |x_s|^2 \leq C \|x\|_E^2. \tag{3.7}
\]

Let \( E \) be the space of paths on \( E \) i.e. the set of continuous maps \( \sigma : \mathbb{R} \rightarrow E \). We define the map

\[
l : \quad E \longrightarrow E
\]

\[
\sigma \mapsto \sigma_0 + \int_0^\cdot \hat{H}(\sigma(t)) hdt,
\]

where the last integral is a Riemann integral in the Banach space \( E \).

Lemma 3.1 let \( x^1, x^2 \) be in \( E \), with \( x^i = x^i_0 + \int_0^1 O_s^i dw_s + \int_0^s A_s^i ds \), then

\[
E \sup_{s \in [0,1]} |\hat{H}_s(x^1) - \hat{H}_s(x^2)|^2 \leq K_1(x^1, x^2) \|x^1 - x^2\|_E^2, \tag{3.8}
\]

where \( K_1(x^1, x^2) \) is of the a continuous function of \( \sup_{s \in [0,1]} \|O_s^i\|_{\infty}, \|\int_0^1 A_s^i \|_{\infty} \), \( i = 1, 2 \).

**Proof.** We shall use the following notation: \( X^i_s := \tilde{H}_s(x^i), \ i = 1, 2 \), \( \Delta_s := X^1_s - X^2_s \), \( \psi(t) = \sup_{t \leq s} |\Delta_t|^2 \). From the Proposition 2.1, \( X^i_s, i = 1, 2 \) satisfy equations of the form

\[
dX^i_s = G^i_s(dw_s) X^i_s + F^i_s X^i_s ds, \ i = 1, 2.
\]

Hence

\[
d\Delta_s = (G^1_s(dw_s) \Delta_s + (G^1_s - G^2_s)(dw_s) X^2_s) + (F^1_s \Delta_s + (F^1_s - F^2_s) X^2_s) ds,
\]

7
and
\[ E \psi(t) \leq 4E \sup_{\tau \in [0,t]} \int_0^\tau \left[ |G_s^1|^2 |\Delta_s|^2 ds + |X_s^2|^2 |G_s^1 - G_s^2|^2 ds \right. \\
+ \left. 4E \sup_{\tau \in [0,1]} \left| \int_0^\tau F_s^1 \Delta_s ds \right|^2 + 4E \sup_{\tau \in [0,t]} \int_0^\tau |X_s^2|^2 |F_s^1 - F_s^2|^2 ds. \]

But
\[ E \sup_{\tau \in [0,t]} \left| \int_0^\tau F_s^1 \Delta_s ds \right|^2 \leq E \sup_{\tau \in [0,t]} \left[ \int_0^\tau |F_s^1|^2 ds \int_0^\tau |\Delta_s|^2 ds \right] \]
\[ \leq \left\| \int_0^1 |F_s^1|^2 ds \right\|_\infty \int_0^t E |\Delta_s|^2 ds \]
\[ \leq \left\| \int_0^1 |F_s^1|^2 ds \right\|_\infty \int_0^t \psi(s) ds, \]
which implies that
\[ E \int_0^t |G_s^1|^2 |\Delta_s|^2 ds \leq \left\| \sup_{s \in [0,1]} |G_s^1|^2 \right\|_\infty \int_0^t E |\Delta_s|^2 ds \]
\[ \leq \left\| \sup_{s \in [0,1]} |G_s^1|^2 \right\|_\infty \int_0^t \psi(s) ds. \]

Therefore
\[ \psi(t) \leq 4 \left( \left\| \int_0^1 |F_s^1|^2 ds \right\|_\infty + \left\| \sup_{s \in [0,1]} |G_s^1|^2 \right\|_\infty \right) \int_0^t \psi(s) ds \]
\[ + \left( 4 \right) X_s^2 \int_0^1 [|G_s^1 - G_s^2|^2 + |F_s^1 - F_s^2|^2] ds. \]

On the other hand, using (2.6) and the boundedness of \( \Gamma' \) we see that there is a constant \( C_1 \) such that
\[ E \int_0^1 |G_s^1 - G_s^2|^2 ds \leq E \sup_s |G_s^1 - G_s^2|^2 \]
\[ \leq C_1 \left( \sup_s |O_s^1|^2 \right)_\infty (E \sup_s |O_s^1 - O_s^2|^2 + E \sup_s |x^1 - x^2|^2) \]
\[ \leq C_2 \left( \sup_s |O_s^1|^2 \right)_\infty \left\| x_s^1 - x_s^2 \right\|_E^2. \]

Similarly, we have
\[ E \int_0^1 |F_s^1 - F_s^2|^2 ds \leq C_3 \left( \sup_s |O_s^i| \right)_\infty \left( i = 1, 2 \right) \left\| x^1 - x^2 \right\|_E^2, \]
where \( C_3 \) is a polynomial function of \( \| \sup_s |O_i^x| \|_{\infty}, i = 1, 2 \). Therefore

\[
\psi(t) \leq C_4(x^1, x^2) \int_0^t \psi(s)ds + C_5(x^1, x^2)\|x^1 - x^2\|^2,
\]

where \( C_4, C_5 \) are (polynomial) functions of \( \| \sup_s |O_i^x| \|_{\infty}, \| \int_0^1 |F_i^x|^2|, i = 1, 2 \). Now \((3.8)\) follows from the last inequality and the Gronwall lemma.

**Lemma 3.2** For \( x^1, x^2 \in E \) such that \( x^1, x^2 \in Y \) a.s., we have

\[
\| \hat{H}(x^1)h - \hat{H}(x^2)h \|_E \leq K_2(x^1, x^2)\|x^1 - x^2\|_E,
\]

where \( K_2(x^1, x^2) \) is a continuous function of \( \| \sup_s |O_i^x| \|_{\infty}, \| \int_0^1 |A_i^x|^2ds \|_{\infty}, i = 1, 2 \). Consequently for \( \sigma^1, \sigma^2 \in \mathcal{E} \):

\[
\|l(\sigma^1)(t) - l(\sigma^2)(t)\|_E \leq \sup_{s \in [0,t]} K_2(\sigma^1(s), \sigma^2(s)) \int_0^t \|\sigma^1(t) - \sigma^2(t)\|_E. \quad (3.10)
\]

**Proof.** As usual we set \( X_s^i = \hat{H}_s(x^i), i = 1, 2 \) and \( dx_s^i = O_s^i dw_s + A_s^i ds, i = 1, 2 \). We have

\[
d(Xh)_s = -\Gamma_{x_s}(O_s dw_s).X_s h_s - [\Gamma_{x_s}(A_s).X_s]h_s
\]

\[
+ \frac{1}{2} \sum_{i=1}^N \Gamma_{x_s}(O_s^i).\Gamma_{x_s}(O_s^i).X_s h_s
\]

\[
- \frac{1}{2} \sum_{i=1}^N \Gamma_{x_s}'(O_s^i, O_s^i).X_s h_s + X_s \dot{h}_s ds.
\]

Then, if we set \( d\Delta_s := o_s dw_s + a_s ds := d(X^1 h - X^2 h)_s \) we have:

\[
E \sup_{s \in [0,1]} |o_s|^2 \leq \|h\|_\infty^2 \{|\Gamma_{x_s}\|^2 \sup_s |X_s|^2 \|_{\infty} E \sup_s |O_s^1 - O_s^2|^2
\]

\[
+ \| \sup_s |O_s^2|^2\|_{\infty} E \sup_s \{G(x_s^1, X_s^1) - G(x_s^2, X_s^2)^2\}
\]

where \( G \) is a Lipshitz function (since \( x_s \in Y \) a.s. and \( X_s, \Gamma, \Gamma' \) are bounded). Hence

\[
E \sup_s |G(x_s^1, X_s^1) - G(x_s^2, X_s^2)|^2 \leq C_1 E \sup_s (\|x_s^1 - x_s^2\|^2 + \|X_s^1 - X_s^2\|^2).
\]

Therefore, using lemma 3.1 and the Burkholder inequality (see \((3.7)\)) we find that

\[
E \sup_{s \in [0,1]} |o_s|^2 \leq C_2 \|h\|_\infty(\| \sup_s |O_s^2|^2\|_{\infty} + 1)\|x^1 - x^2\|^2_E.
\]
We turn now to control the term \( E \int_0^1 |a_s|^2 ds \). We have

\[
E \int_0^1 |a_s|^2 ds \leq \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4,
\]

with

\[
\Delta_1 = E \int_0^1 |(\Gamma_{x_1}(A^1_s)X^1_s - \Gamma_{x_2}(A^2_s)X^2_s)h_s|^2 ds
\leq C_3\|h\|_\infty \left\{ \int_0^1 |A^1_s - A^2_s|^2 ds \right\},
\]

\[
+ E \int_0^1 |G(x^1_s, X^1_s) - G(x^2_s, X^2_s)|^2 |A^2_s|^2 ds
\leq C'_3\|h\|_\infty \left\{ E \int_0^1 |A^1_s - A^2_s|^2 ds \right\},
\]

where \( G \) is (another) Lipschitz function. By the same arguments as above we get\n
\[
\Delta_3 \leq C''_3\|h\|_\infty (\int_0^1 |A^2_s|^2 ds)_{\infty} + 1)\|x^1 - x^2\|^2_E.
\]

By similar majorizations, we get easily\n
\[
\Delta_2 := E \int_0^1 \frac{1}{2} \sum_i \Gamma_{x_1}(O^1_se_i) \Gamma_{x_1}(O^1_se_i) X^1_s h_s -
\leq \|h\|_\infty C_4 (\sup_s |O^1_s|, i = 1, 2) \left\{ E \sup_s |O^1_s - O^2_s|^2 + \|x^1 - x^2\|^2_E \right\}
\]

and\n
\[
\Delta_3 := \int_0^1 \frac{1}{2} \sum_i \Gamma'_{x_1}(O^1_se_i, O^1_se_i) X^1_s h_s^1 -
\leq \|h\|_\infty C_5 (\sup_s |O^1_s|^2, i = 1, 2) \left\{ E \sup_s |O^1_s - O^2_s|^2 + \|x^1 - x^2\|^2_E \right\},
\]

where \( C_4, C_5 \) are (polynomial) functions of their arguments. For the last term, we have by using Lemma 3.1

\[
\Delta_4 := E \int_0^1 |X^1_s - X^2_s|^2 |\dot{h}_s|^2 ds
\]
\[
\leq \| \int_0^1 \dot{h}_s \|_\infty E \sup_s |X_s^1 - X_s^2|^2
\]
\[
\leq K_1(x^1, x^2) \| \int_0^1 \dot{h}_s \|_\infty \| x^1 - x^2 \|_E^2.
\]

The inequality (3.9) is now clear and the second inequality of the lemma is immediate. □

Now, given \( T > 0 \) we denote by \( \mathcal{E}_T \) de the space of maps \( \sigma : [-T, T] \rightarrow E \) such that if \( ds_s(t) = O_s(t) dw_s + A_s(t) ds \) we have

(i) \( \sup_{s \in [0,1]} |O_s(t)|^2 + \int_0^1 |A_s(t)|^2 ds \leq 1 + \| \sup_s |O_s|^2 \|_\infty + \| \int_0^1 |A_s|^2 ds \|_\infty \) a.s.,

(ii) \( \sigma(t) \in Y \) a.s.

**Lemma 3.3** For \( T \) sufficiently small (\( T \leq T_0 \), say), we have \( l(\mathcal{E}_T) \subset \mathcal{E}_T \). Furthermore \( T_0 \) depends only on \( \| \sup_s |O_s|^2 \|_\infty, \| \int_0^1 |A_s|^2 ds \|_\infty \) and \( \| \int_0^1 |\dot{h}_s|^2 ds \|_\infty \). Finally, there is a unique map \( \sigma : [-T_0, T_0] \rightarrow E \) which satisfies

\[
\sigma(t) = \sigma(0) + \int_0^t H(\sigma(s)) ds,
\]

for all \( t \in [-T_0, T_0] \).

**Proof.** Let \( \sigma(t) \in \mathcal{E}_T \) with \( ds_s(t) = O_s(t) dw_s + A_s(t) ds \) and \( X_s(t) = \tilde{H}_s(\sigma(t)), z_s(t) = X_s(t)\dot{h}_s, Z(t) = l(\sigma)(t) \). From the equation (2.6), we deduce easily that if we set \( dz_s(t) = o_s(t) dw_s + b_s(t) ds \) then

\[
\sup_{s \in [0,1]} |o_s(t)| \leq C_1 \sup_{s \in [0,1]} |O_s(t)|
\]
\[
\int_0^1 |a_s(t)|^2 ds \leq C_2 (\sup_{s \in [0,1]} (|O_s(t)|^2 + |O_s(t)|^4) + \int_0^1 |A_s(t)|^2 ds + \int_0^1 |\dot{h}_s|^2 ds)
\]

for some constants \( C_1, C_2 \). If we denote \( dZ_s(t) = \tilde{O}_s(t) dw_s + \tilde{A}_s(t) ds \), then using the fact that \( \sigma \in \mathcal{E}_T \), we have

\[
\sup_{s \in [0,1]} |\tilde{O}_s(t)| \leq \int_0^t C_1 dt + \| \sup_s |O_s|^2 \|_\infty,
\]
\[
\int_0^1 |\tilde{A}_s(t)|^2 ds \leq C_2 \int_0^t (2 + \int_0^1 |\dot{h}_s|^2 ds) dt + \int_0^1 |A_s^0(t)|^2 ds,
\]

and we see that, for \( T \) sufficiently small, the condition (i) is satisfied.

For condition (ii), we have

\[
\| \sup_{s \in [0,1]} |l(\sigma(t)) - \sigma_0| \|_\infty \leq K |\dot{h}|_\infty |t|,
\]

11
by the boundedness of $\tilde{H}$. Since $\sigma_s(0) \in M$ for $s \in [0,1]$ we see again that for $T$
sufficiently small we have: a.s. $l(\sigma(t))_s \in Y$ for $s \in [0,1]$. The first assertion of
the lemma is now proved.

The proof of the second assertion is standard in view of the first: we have a constant $K$ such that for all $\sigma^1, \sigma^2 \in \mathcal{E}_{T_0}$:

$$\|l(\sigma^1(t)) - l(\sigma^2(t))\|_E \leq K \int_0^t \|\sigma^1(s) - \sigma^2(s)\|_E ds.$$ 

The constant $K$ is now independent of $\sigma^1, \sigma^2$. By usual arguments one can verify
the second assertion of the lemma. Namely, for some $n_0$ sufficiently large, the map $l(n_0) : \mathcal{E}_{T} \rightarrow \mathcal{E}_{T}$ is a contraction ($\mathcal{E}_{T}$ is endowed with the norm
$\sup_{t \in [-T_0,T_0]} \|\cdot\|_E$). Hence $l(n_0)$ and then $l$ admit a unique fixed point $\sigma$. □

The next lemma garantees the boundedness of the solution to $P_1$. Its proof is a slight modification of the proof of [11, Proposition 7.1].

**Lemma 3.4** Suppose that $t \mapsto \sigma(t)$ is a solution to $P_1$, then there exists a function $\beta$
such that for all $t \in \mathbb{R}$ : 

$$\|\sigma(t)\|_E \leq \beta(t) < +\infty.$$ 

**Proof.** Let $\sigma(t) = \sigma_0(t) + \int_0^t O_s(t) dw_s + \int_0^t A_s(t) ds$ be a solution to $P_1$. By [11, Lemma 7.2 ], $O(t), A(t)$ satisfy

$$
\begin{cases}
\frac{dO(t)}{dt} = C(\sigma(t))O(t) \\
\frac{dA(t)}{dt} = C(\sigma(t))A(t) + R(\sigma(t)).
\end{cases}
$$

where the derivatives are taken w.r.t the norms $\|O\|_o := E(\sup_{s \in [0,1]} |O_s|)$ and $\|A\|_a := E(\int_0^1 |A_s|^2 ds)^{1/2}$ and $C$ is uniformly bounded (the bound, $K_1$ say, depends on $|\dot{h}|_{\infty}$) and $R$ is of the form $R(\sigma(t)) = R^1(\sigma(t)) + R^2(\sigma(t))\dot{h}$, $R^1$ is a polynomial function. Therefore, we have for some constant $\gamma$ :

$$E|\frac{dO(t)}{dt}| \leq \gamma \|O(t)\|$$

which implies that

$$E\|O(t)\| \leq \|O(0)\| e^{\gamma |t|},$$

here, as in [11], we use a “vector” version of the Gronwall lemma. Turning to $A(t)$, we have

$$E \int_0^1 \left| \frac{dA_s(t)}{dt} \right|^2 ds \leq 2K_1 E \int_0^1 |A_s(t)|^2 ds + 4E \int_0^1 |R^1_s(O(t))|^2 ds + K_2 E \int_0^1 \dot{h}^2 ds,$$
hence for $\tau \leq t$

$$\left\| \frac{dA(\tau)}{d\tau} \right\| \leq K_3(\|A(\tau)\| + e^{\gamma|t|} + E\int_0^1 \hat{h}_s^2 ds)^{1/2},$$

and

$$\|A(\tau)\| \leq e^{K_3|t|}[\|A(0)\| + K_3\|\tau\|(E(\int_0^1 |\hat{h}_s|^2)^{1/2} + e^{\gamma|t|})].$$

We have used again the “vector” version of the Gronwall lemma. □

**Lemma 3.5** Let $t \mapsto \sigma(t)$ be the solution to $\mathcal{P}_2$ in the interval $[-T_0, T_0]$ (as constructed in lemma 3.3). Then there is a version of $\sigma$ such that a.s. the maps $(t, s) \mapsto \sigma_s(t)$ is continuously differentiable in the $t$ variable.

**Proof.** We will denote by $\dot{\sigma}(t)$ the derivative of $\sigma$ with respect to the norm $\|\cdot\|_E$. For $t, t' \in [-T_0, T_0]$ we have the estimates

$$E \sup_{s \in [0,1]} |\dot{\sigma}_s(t) - \dot{\sigma}_s(t')|^2 \leq \|\dot{h}_s\|_\infty E \sup_{s \in [0,1]} |H_s(\sigma(t)) - H_s(\sigma(t'))|^2$$

$$\leq \|\dot{h}_s\|_\infty K(\sigma(t))\|\sigma(t) - \sigma(t')\|_E$$

$$\leq \|\dot{h}_s\|_\infty K(\sigma(t)) \int_t^{t'} \|H(\sigma(\tau))h\|_E d\tau$$

$$\leq K'(\sigma(t))|t - t'|^2,$$

where we have used lemma 3.1 in the second inequality. The last inequality is clear in view of the definition of $\mathcal{E}_{T_0}$; furthermore this definition implies that the constants $K(\sigma(t)), K'(\sigma(t))$ are independent of $t \in [-T_0, T_0]$, see lemma 3.1. We are now able to use a consequence of the Kolmogorov lemma [[11], lemma 4.5] which gives the desired result. □

**End of proof of the theorem:**

- Since we have a local solution to the problem $\mathcal{P}_2$ let us denote by $\dot{\sigma}$ the maximal (unique) solution to this problem. By the boundedness lemma 3.4 this solution is defined on $\mathbb{R}$.

- By lemma 3.5 we have a version of $\dot{\sigma}$ such that $(t, s) \mapsto \dot{\sigma}$ is continuously differentiable in the $t$ variable for $t \in [-T_0, T_0]$. Since $\mathbb{R}$ is a countable union of such intervals we get a continuously differentiable version on the $t$ variable on $\mathbb{R}$. we set $\sigma_s(t) = \pi(\dot{\sigma}_s(t))$ and using (i) of Proposition 2.1 and the fact that $\dot{\sigma}_s(0) \in M$ a.s., we see that $\sigma$ is a solution to the problem $\mathcal{P}_1$.

- By the uniqueness of the solution $\dot{\sigma}$ to $\mathcal{P}_2$ we have $\sigma = \hat{\sigma}$ which shows the uniqueness of the solution to $\mathcal{P}_1$ and completes the proof of the theorem. □
4 Quasi-invariance of the Wiener measure on the path space

Let us denote by $\Phi$ the development map, that is the map which associates to an $M$-valued semimartingale $x_s = (\xi_s^1, ..., \xi_s^d)$ the $\mathbb{R}^d$-valued semimartingale $\xi_s$ given by

$$\xi_s = \int_{X[0,s]} \theta \circ dX_s,$$

where $X_s$ is the horizontal lift of $x_s$, see Shigekawa [32].

In order to get the quasi-invariance property for the family of transformations $\sigma(t)$, it is necessary impose a condition to the torsion $T$ of the connection. Namely, following [11], we say that the torsion is “skew symmetric” or that the connection $X$ is skew symmetric for all $u \in O(M)$.

**Theorem 4.1** Let $\phi_s(t) = (\Phi(\sigma(t)))_s$ where $\sigma$ is the solution to $\mathcal{P}_1$ with the initial condition $\sigma_s(0) = p_s$ with $(p_s)$ being the Brownian motion on $M$. Then $\phi(t) \in BS(\mathbb{R}^d)$ and $t \mapsto \phi(t)$ is a solution to

$$\frac{d\phi(t)(w)}{dt} = h_s(w) - \int_0^s \int_0^t \Omega_{H_s} (\frac{d\phi(t)}{dt}, h_r) \circ d\phi_s(t) - \int_0^t \Theta_{H_r} (\frac{d\phi(t)}{dt}, h_r)$$

(4.11)

where the derivative is taken in $(BS(\mathbb{R}^d), \|\cdot\|_{BS})$, $\|\cdot\|_{BS}$ is the analogue of $\|\cdot\|_E$ in $\mathbb{R}^d$ and $H(t)$ is the horizontal lift of $\sigma(t)$. Furthermore for each $t$, $\phi_s(t)$ is given by $d\phi_s(t) = o_s(t)dw_s + a_s(t)ds$ where $o_s(.)$ is an adapted $O(d)$-valued process and $a_s(.)$ is an $\mathbb{R}^d$-valued process such that for each $t$, there is a constant $C(t)$ s.t. $\int_0^1 |a_s(t)|^2 ds \leq C(t)$ a.s. Therefore the law $\mu_t$ of the process $s \mapsto \phi_s(t)$ is equivalent to $\mu$ and the law $\nu_t$ is equivalent to $\nu$ with the same Radon-Nikodým derivative $d\nu_t/d\nu = d\mu_t/d\mu$.

**Proof.** The proof of this theorem is an easy adaptation of the corresponding parts in [11] or [20], taking into account the fact that $h$ is in $\mathcal{H}$ and random and dealing with the appropriate norm. First, using the structure equations (2.2), we prove that $\phi(t)$ satisfies (4.11). Next, write $d\phi_s(t) = o_s(t)dw_s + a_s(t)ds$ (indeed, the development of a Brownian semimartingale is a Brownian semimartingale), then using (4.11), one can show that $o(t), a(t)$ are solution to

$$\frac{do(t)}{dt} = c.(\phi(t))o(t)$$

$$\frac{da(t)}{dt} = c.(\phi(t))a(t) + \dot{h} + \frac{1}{2} (\text{ric}_{H(t)} h_r + \sum_{i=1}^d \Theta'_{H_r}(e_i, h_r, e_i)),$$
where the derivatives are taken w.r.t. the norms indicated in the theorem and 
\( c_s(\phi(t)) \)
is the matrix given by
\[
c_s(\phi(t))v = \int_0^s \Omega_{H_s(t)}(\circ d\phi_r(t), h_r) v + \Theta_{H_s(t)}(h_s, v).
\]

Hence, under the assumption on the torsion, \( c_s(\phi(t)) \) is skew-symmetric and \( o_s(t) \) is orthogonal. We omit the details and refer to Theorem 5.1, Proposition 6.1 and Section 8 of [11]. □

Remark 4.2 The anticipative case.

As for the flat Wiener space, we can consider the case when \( h \) is non-adapted (but, of course, with other restrictions) and ask whether the problem \( P_1 \) has a solution \( \sigma(t) \) and if it has the quasi-invariance property. First, there are some difficulties in the definition of the horizontal lift of an anticipative process (which requires to solve an anticipative SDE) and the proof of the existence of the solution to \( P_1 \) has to be modified; for instance, we have not an analogue to Burkholder inequalities. Second and most important, even if one has succeeded to prove the existence of a solution \( \sigma(t) \) to \( P_1 \), we expect that the pullback \( \xi(t) \) of \( \sigma(t) \) to the flat Wiener space is of the form
\[
d\xi_s(t) = o_s(t)\delta w_s + a_s(t)ds,
\]
where \( \delta \) is the Skorohod integral; of course, one has to prove an existence result for such equations. But the Wiener measure could hardly be quasi-invariant under a transformation like \( \xi(t) \).

5 Anticipative integrals on a Riemannian manifold

For the sake of simplicity, in this section \( M \) is endowed with the Levi-Civita connection.

5.1 Preliminaries

5.1.1 Notations: In this section \( W = (C_0([0,1],\mathbb{R}^d),\mathbb{H},\mu) \) will denote the Wiener space on \( \mathbb{R}^d \), \( D \) the usual Malliavin derivative on \( W \) and \( \delta \) its adjoint. We denote by \( \mathcal{S} \) the set of smooth functionals \( F \) on \( W \), i.e. \( F(w) = f(w_{s_1},...,w_{s_n}) \) where \( f : (\mathbb{R}^d)^n \to \mathbb{R} \) is smooth. Recall that \( p \) is the Brownian motion on \( M \) determined by \( p_s(w) = \pi(r_s(w)) \) where \( r_s \) satisfies \( dr_s(w) = L_i(r_s(w)) \circ dw_s \); \( r_0 \) is given and the Ito stochastic parallel transport is given by \( t_{s_1\leftarrow s_2}^s = r_{s_1} \circ r_{s_2}^{-1} \). The Itô map is defined by \( I : W \to P(M) \) \( w \mapsto p(w) = \pi \circ r(w) \). A tangent vector field on \( P(M) \) is a process \( u(s) \) such \( u(s) \in T_{p(s)}M \) for \( s \in [0,1] \).

5.1.2 The gradient on the path space ([11], [25], [17]):

(a) Let \( F : P(M) \to \mathbb{R} \) be a cylindrical function i.e. \( F(p) = f(p(s_1),...,p(s_n)) \) where \( f : M^n \to \mathbb{R} \) is a smooth function. Then, the gradient of \( F \) is the element...
$D^M F \in T(P)$ defined by

$$D^M_s F = \sum_{i=1}^{d} t^p_{s-e_i} \nabla_i f 1_{s<s_i},$$

where $\nabla_i f$ is the gradient w.r.t. the component $i$ of $f$ (defined via the scalar product on $T_{p(s_i)}M$). To each $h \in H$ we associate $h^p \in T(P)$ by $h^p(s) = t^p_{s-e_0} h(s)$ and we put

$$D^M_{h} F = \int_0^1 <D^M_s F, \dot{h}^p(s)>_{T_{p(s)}M} ds.$$

(b) This gradient is related to the transformation discussed in the above paragraph as follows. For $h \in H$ let $\sigma^h(t), t \in \mathbb{R}$ be the family of transformations defined in §3. Then for a cylindrical function $F$ we have

$$D^M_{h} F = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} F(\sigma^h(1)),$$

in $L^2(W)$. Indeed, it suffices to show that for $s \in [0,1]$, we have

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \sigma^h(\epsilon) = t^p_{s-e_0} h(s). \quad (5.12)$$

For this, let $q_t = (d/d\epsilon)_{\epsilon=0}(\sigma^h(\epsilon))$. Clearly, $q_0 = 0$ and using (1.1) we get $(d/dt)q_t = H_s(p)h(s)$ which implies (5.12).

5.1.3 Integration by parts ([4], [11], [17], [25], [1]):

The integration by parts formula associated to $D^M$ is the following (Bismut formula):

$$ED^M_{h} F = EF \int_0^1 (\dot{h}(s) + \frac{1}{2} \text{ric}_s h(s)) dw_s. \quad (5.13)$$

As a consequence of this formula, the operator $D^M$ is closable in $L^2(P) \equiv L^2(W)$; we denote by $D^{1,2}(M)$ its domain which is endowed with the norm

$$\|F\|_{1,2}^M = \|F\|_{L^2} + (E \int_0^1 |D^M_s F|^2 ds)^{1/2}. \quad (5.14)$$

5.1.4 The damped gradient [17]:

The damped gradient, denoted by $\tilde{D}$, is introduced in order to have an ‘ordinary’ integration by parts formula instead of (5.13) i.e.

$$ED\tilde{h} F = EF \int_0^1 \dot{h}(s) dw_s. \quad (5.15)$$
Equivalently, $\tilde{D}$ must satisfy for every $F, h : E \tilde{D}_h F = E D^M_h F$ where
\[ \dot{h}_1(s) + (1/2) \text{ric}_{\gamma(s)} h_1(s) = \dot{h}. \]
After some calculations, one is led to the following definition
\[ \tilde{D}_s F = \sum_{i=1}^n r_s Q_{s,s_i}^{-1} \nabla_i f 1_{s< s_i}, \tag{5.16} \]
where $Q_{s,s'} : \mathbb{R}^d \to \mathbb{R}^d$ is the solution of
\[ \frac{dQ_{s,s'}}{ds} = -\frac{1}{2} \text{ric}_{s,s'} Q_{s,s'}, \quad Q_{s',s'} = \text{Id}. \]
We then define an associated norm for $F$ by a formula like (5.14), which, in turn, is equivalent to $\| \cdot \|^{M}_{1/2}$, hence $D^M$ and $\tilde{D}$ have the same domain.

5.1.5 Strong differentiability and the intertwining formula [10]:
A tangent process on $W$ is a process $\xi(s)$ which satisfies $\xi(0) = 0$ and $d\xi(s) = A(s)dw_s + \dot{h}ds$ where an adapted $\text{so}(d)$-valued process and $\dot{h}$ is an adapted process in $L^2([0, 1] \times W)$. For such a process and a cylindrical functional $F(w) = f(w_{s_1}, ..., w_{s_n})$ we define the derivative
\[ D_\xi F := \sum_{i=1}^n \partial_i f \xi_i(s_i). \]
Observe that for such functionals $F \in S$ we have
\[ D_\xi F = \frac{d}{de}|_{e=0} F(\int_0^1 e^{tA_s} dw_s + eh_s) \quad \text{in} \quad L^2(W). \tag{5.17} \]
Notation. For $d\eta(s) = A_s dw_s$ we set $D^R_\eta F := D_\eta F$, which means that the derivative corresponds to a rotation. Notice that
\[ D_\xi F = D_h F + D^R_\xi F. \]
The following integration by parts formula is an immediate consequence of (5.17) and the invariance of the Wiener measure by rotation:
\[ ED_\xi F = EF \int_0^1 \dot{h}(s) dw_s. \]
From that we deduce that $D_\xi$ is closable in $L^2(W)$. Denoting by $\mathbb{D}_\xi^{1,2}$ its domain in $L^2(W)$, a random variable $F$ is said strongly differentiable iif $F \in \mathbb{D}_\xi^{1,2}$ for all $\xi$.

Similarly, given a cylindrical functional on $P(M) : F(p) = f(p(s_1), ..., p(s_n))$ and $\xi$ a tangent process, we define
\[ D^M_\xi F := \sum_{i=1}^n \partial_i f \xi^p_i(s_i), \quad \text{with} \quad \xi^p(s) = t^p_{s<0} \xi(s). \]
We define in the same way as above $\mathbb{D}^{1,2}_\xi(M)$ and the strong differentiability for $F : P(M) \to \mathbb{R}$. Then, for such a functional, we have: $F$ strongly differentiable if $F \circ I$ is strongly differentiable and the following (interwining formula) holds

$$D^M \xi F = D^M (F \circ I),$$

with

$$d\xi^*(s) = d\xi(s) + \gamma(s) \circ dw, \quad \gamma(s) = \int_0^s \Omega_{\tau(\theta)}(\circ dw(\xi(\theta))).$$

### 5.2 The anticipative integrals

**Convention.** In this section, every process $u(s) \in T_{p(s)} M$ will be identified to the process $t\to u(s)$ also denoted by $u(s)$. In particular, the gradients $D^M_s F, \tilde{D}^M_s F$ will designate $t\to D^M_s F$ and $t\to \tilde{D}^M_s F$ respectively. Finally every random variable $F : P(M) \to \mathbb{R}$ will be identified with $F \circ I$.

**Notation.** For a smooth functional $F$ and $i = 1, \ldots, d$ we define $\tilde{D}^i F$ by

$$\int_0^1 \tilde{D}^i_s F : h_s ds = \int_0^1 D^i_s F : h_s e_s ds$$

for every $h \in L^2([0, 1])$ and if $F = (F_1, \ldots, F_d) \in (\tilde{D}^{1,2}(M))^d$ then we set:

$$\tilde{D} F := \sum_{i=1}^d \tilde{D}^i F_i.$$

We define in the same way $D^i F$.

In the following $L^{1,p}$ will denote the set of $\mathbb{R}^d$-valued processes $u$ such that $u(t) \in \mathbb{D}^{1,2}$ for almost all $t$ and

$$\|u\|_{1,p} := E(\int_0^1 |u_s|^2)^{p/2} + E(\int_0^1 \int_0^1 |D_s u(t)|^2 ds dt)^{p/2} < \infty.$$

Similarly, we denote by $L^{1,p}(M)$ the set of the processes $u(s)$ which satisfy the same condition when replacing $D$ by $D^M$ (or $\tilde{D}$).

**Definition 5.1** The anticipative integral associated to $D^M$ is the operator $\delta^M$ whose domain is

$$\text{Dom}(\delta^M) = \{ u \in L^2([0, 1] \times W) : (D^M)^* u \in L^2(W) \},$$
and defined by \( \delta^M(u) = (D^M)^* u \), where \( (D^M)^* \) is the formal adjoint of \( D^M \). Equivalently, we have

\[
\text{Dom}(\delta^M) = \{ u \in L^2([0,1] \times W) : \exists C(u), \forall \phi \in \mathbb{D}^{2,1}_M : |E(\hat{D}_u \phi)| \leq C(u)\|\phi\|_2 \},
\]

and for \( u \in \text{Dom}(\delta^M) \), \( \delta^M \) is the unique random variable in \( L^2(W) \) which verifies

\[
E \int_0^1 D^M_s \phi.u_s ds = E\phi \delta^M(u), \text{ for all } \phi \in \mathbb{D}^{2,1}_M.
\]

Similarly, we define the anticipative integral \( \tilde{\delta} \) associated to the damped gradient \( \tilde{D} \) by replacing in the above formula \( D^M \) by \( \tilde{D} \).

The main result of Fang [15], is that if \( p > 2 \) then \( L^{1,p}(M) \subset \text{Dom}(\tilde{\delta}) \).

First we state the following proposition which is inspired by the Ogawa method for defining noncausal integrals (see [30], [29]):

**Proposition 5.2** Let \( u \in L^{1,2}(\mathbb{R}^d) \cap L^{1,2}(M) \) be a process such that a.s. the kernels \( D_su_t \) and \( \tilde{D}_s u_t \) are of trace class. Then \( u \in \text{Dom}(\tilde{\delta}) \) and

\[
\tilde{\delta}(u) = \delta(u) + \text{Trace}(D.u) - \text{Trace}(\tilde{D}.u). \tag{5.19}
\]

**Proof.** Let \( (\hat{h}_i), i \geq 1 \) be an orthonormal basis of \( L^2([0,1])) \). Without loss of generality, we suppose that \( u(s) = u(s)e \) with \( e \in \mathbb{R}^d \). Then, we have \( u = \sum_1^\infty \alpha_i(w) \hat{h}_i \) in \( L^2([0,1] \times W) \), with \( \alpha_i = \int_0^1 u_s \hat{h}_i(s)ds \). By the assumptions \( \alpha_i \in \mathbb{D}^{1,2} \cap \mathbb{D}^{1,2}(M) \) and for \( \phi \in \mathbb{D}^{2,1}(M) \), the integration by part formula yields (we identify \( \hat{h}_i \) with \( \hat{h}_i e)\):

\[
E\alpha_i \int_0^1 \tilde{D}_s \phi.\hat{h}_i(s)ds = E\phi(\alpha_i \int_0^1 \hat{h}_i(s)dw_s - \tilde{D}_h.\alpha_i) \nonumber
\]

\[
= E\phi(\int_0^1 \alpha_i \hat{h}_i(s)\delta w_s + \int_0^1 D_s \alpha_i.\hat{h}_i(s)ds - \int_0^1 \tilde{D}_s \alpha_i.\hat{h}_i(s)ds) \nonumber
\]

\[
= E\phi(\int_0^1 \alpha_i \hat{h}_i(s)\delta w_s + \int_0^1 \int_0^1 D_s u(t).\hat{h}_i(t).\hat{h}_i(s)ds - \int_0^1 \tilde{D}_s u(t).\hat{h}_i(t).\hat{h}_i(s)ds). \nonumber
\]

Taking the sum on \( i \), the first member converge to \( E < \tilde{D}\phi, u >_H \), and, under the assumptions of the proposition, the r.h.s. converges to \( \delta(u) + \text{Trace}(D.u) - \text{Trace}(\tilde{D}.u) \). \( \square \)
In [24], Kazumi follows the above method to calculate the Ornstein-Uhlenbeck operator associated to $\tilde{D}$ ($\tilde{L} := -\tilde{\delta} \circ \tilde{D}$).

If $u(s) = \alpha(w)e(s)$ with $\alpha \in \mathbb{D}^{1,2} \cap \mathbb{D}^{1,2}(M)$ and $e \in \mathbb{H}$ then $u \in \text{Dom}(\tilde{\delta})$. For example, if $u$ is a step process i.e.

$$u(s) = \sum_{j=1}^{d} \sum_{j=1}^{n} \alpha_{i,j}(w) 1_{[s_i, s_{i+1}]} e_i,$$  (5.20)

with $\alpha_{i,j} \in \mathbb{D}^{1,2} \cap \mathbb{D}^{1,2}(M)$ and $(s_i), i = 1, ..., n$ a subdivision of $[0,1]$, then $u \in \text{Dom}(\tilde{\delta}) \cap L^{1,2} \cap L^{1,2}(M)$. For notational simplicity let $u(s) = \alpha(w)1_{[s_1, s_2]}e$ with $\alpha \in \mathbb{D}^{1,2} \cap \mathbb{D}^{1,2}(M)$ and $e \in \mathbb{R}^d$. Then, by the integration by parts formula (5.15), we have for $\phi \in \mathcal{S}$:

$$E \int_0^1 \tilde{D}_s \phi . u(s) ds = E\phi (\int_0^1 \alpha 1_{[s_1, s_2]}(s)e.\delta w_s) + \int_0^1 D_s \alpha . e 1_{[s_1, s_2]}(s) ds - \int_0^1 \tilde{D}_s . u_s ds,$$

(In this case, the random variables $D_s . u(s), \tilde{D}_s . u(s)$ are clearly well defined in $L^2(W)$ for $s \in [0, 1]$). The above formula for $\tilde{\delta}$ holds for every step process of the form (5.20).

The next theorem extends this property to other processes.

Following [28], we say that a process $u$ belongs to $\mathbb{L}^{1,2}$ if there is a version of $\tilde{D}u$ such that

(i) The maps $s \mapsto \tilde{D}^+_s u$ and $s \mapsto \tilde{D}^-_s u$ are uniformly continuous (w.r.t $t$) from $[0, 1]$ onto $L^2(W, \mathbb{R}^d \times \mathbb{R}^d)$.

(ii) $\sup_{s, t} (|\tilde{D}_s u_t|^2) < +\infty$.

Similarly, we define $\mathbb{L}^{1,2}$ by replacing $\tilde{D}$ by $D$ in the above definition. Then if $u \in \mathbb{L}^{1,2}$, the processes

$$\tilde{D}^+_t . u := \lim_{\epsilon \to 0, \epsilon > 0} \sum_{i=1}^{d} \tilde{D}^+_t u^{i}_{t+\epsilon},$$

$$\tilde{D}^-_t . u := \lim_{\epsilon \to 0, \epsilon > 0} \sum_{i=1}^{d} \tilde{D}^-_t u^{i}_{t-\epsilon},$$

exist in $L^2([0,1] \times W)$ and we can define similar processes for $u \in \mathbb{L}^{1,2}$.
Theorem 5.3 Let \( u \in L_{C_1}^{1,2} \cap L_{C_2}^{1,2}(M) \). Then \( u \in \text{Dom}(\delta) \) and
\[
\delta(u) = \delta(u) + \frac{1}{2} \int_0^1 (D_+ t . u_t + D_- t . u_t) dt - \frac{1}{2} \int_0^1 (\tilde{D}_+ t . u_t + \tilde{D}_- t . u_t) dt. \tag{5.21}
\]

**Proof.** Let \( u \in L_{C_1}^{1,2} \cap L_{C_2}^{1,2}(M) \). By definition this implies that the r.h.s of (5.21) is an \( L^2(W) \) random variable; we will designate by \( \tilde{\delta}(u) \) this random variable. It remains to prove that the following integration by parts formula holds
\[
E \int_0^1 \tilde{D}_s \phi . u(s) ds = E\phi \tilde{\delta}(u) \quad \text{for all } \phi \in \mathcal{S}. \tag{5.22}
\]

Let \((s_{i,n} = i/n), i = 1, \ldots, n\) be a subdivision of \([0,1]\). Define the step process
\[
u^n(s) = \frac{1}{n} \sum_{i=1}^n u_{s_{i,n}} 1_{[s_{i,n}, s_{i+1,n}]}.
\]

Then \( u^n \in \text{Dom}(\delta) \cap \text{Dom}(\tilde{\delta}) \) and we have
\[
\tilde{\delta}(u^n) = \delta(u^n) + A_n + \tilde{A}_n,
\]

with
\[
A_n = \frac{1}{n} \sum_{i=1}^n \int_{s_{i,n}}^{s_{i+1,n}} D_s . u_t dsdt, \quad \tilde{A}_n = \frac{1}{n} \sum_{i=1}^n \int_{s_{i,n}}^{s_{i+1,n}} \tilde{D}_s . u_t dsdt.
\]

This means that for every \( \phi \in \mathcal{S} \) we have
\[
E \int_0^1 \tilde{D}_s \phi . u^n(s) ds = E\phi(\delta(u^n) + A_n + \tilde{A}_n). \tag{5.23}
\]

Since \( u \in L_{C_1}^{1,2} \), we have \( u^n \rightarrow u \) in \( L_{C_1}^{1,2} \) and \( \delta(u^n) \rightarrow \delta(u) \) in \( L^2(W) \) by Nualart-Pardoux [28]. Moreover we have
\[
E|A_n - \frac{1}{2} \int_0^1 (D_+ s . u(s) + D_- s . u(s)) ds| \rightarrow 0,
\]

see the proof of Theorem 3.1.1 p. 151 in Nualart [27]. Similarly we have
\[
E|\tilde{A}_n - \frac{1}{2} \int_0^1 (\tilde{D}_+ s . u(s) + \tilde{D}_- s . u(s)) ds| \rightarrow 0.
\]

Then, using (5.23) it follows that (5.22) holds first for each bounded \( \phi \in \mathcal{S} \) and then for each \( \phi \in \mathcal{S} \). \( \Box \)
Let us denote by $L^{k,p}$ the space $L^p([0,1], \mathbb{D}^{k,p})$. For the next proposition we assume that $u$ satisfies the following: $u \in L_C^{1,2}(M)$ and $u \in L_C^{2,4}$ i.e. $u \in L^{2,4}$ and

$$s \mapsto D_s^{\delta} u_{s\land t}, s \mapsto D_s u_{s\lor t}$$

are uniformly (in $t$) continuous from $[0,1]$ onto $L^4(W)$ and such that $\sup_{s,t} E(|D_s u_t|^4) < \infty$. We also assume that the processes $D_t^+ u_t + D_t^- u_t$, $\tilde{D}_t^+ u_t + \tilde{D}_t^- u_t$ are in $L^{1,4}$.

**Proposition 5.4 (Itô formula)** Let $u$ be a process which satisfies the above assumptions and $\phi \in C^2_2(\mathbb{R})$. Let $X_t := \int_0^t u_s \tilde{\delta} p_s := \tilde{\delta}(1_{[0,t]} u)$. Then

$$\phi(X_t) = \phi(X_0) + \int_0^t \phi'(X_s) u_s \tilde{\delta} p_s - \frac{1}{2} \int_0^t \phi''(X_s) (\tilde{D}_s^+ X_s + \tilde{D}_s^- X_s) u_s ds.$$  

($\tilde{D}_s^+ X_s$ is the vector $(\tilde{D}_s^i X_s, i = 1, ..., d)$, with an obvious notation.)

**Proof.** Remark that

$$X_t = \int_0^t u^i_s \circ dw^i_s - \frac{1}{2} \int_0^t (\tilde{D}_s^+ u_s(s) + \tilde{D}_s^- u_s(s)) ds,$$  

(5.24)

where the first integrals in the r.h.s are Stratonovich anticipative integrals. The proposition follows easily from (5.24) and the Itô formula in the flat case, see Nualart [27], Theorem 3.2.3. $\square$

**5.3 Remarks**

5.3.1. From 5.1.4. we see that a process $u$ belongs to $\text{Dom}(\delta^M)$ if $\tilde{u} \in \text{Dom}(\tilde{\delta})$ where $\tilde{u}$ satisfy

$$\tilde{u}_s + \frac{1}{2} \text{ric}(s) \int_0^s \tilde{u}_d d\theta = u(s),$$

and we have $\delta^M(u) = \tilde{\delta}(u)$.

5.3.2. Let $u(s) = \alpha(w) \dot{h}(s)$ where $\alpha$ is strongly differentiable and $\dot{h} \in L^2([0,1], \mathbb{R}^d)$. Then, by the integration by parts formula, it follows that $u \in \text{Dom}(\delta^M)$ and

$$\delta^M(u) = \alpha \int_0^1 (\dot{h}(s) + \frac{1}{2} \text{ric}(s) h(s)) dw_s - \int_0^1 D_s^M \alpha \dot{h}_s ds$$

$$= \int_0^1 \alpha \dot{h}(s) + \frac{1}{2} \text{ric}(s) h(s) \delta w_s + \int_0^1 D_s \alpha \dot{h}(s) + \frac{1}{2} \text{ric}(s) h(s)) ds$$

$$- \int_0^1 D_s^M \alpha \dot{h}_s ds.$$
But the interwining formula yields
\[ \int_0^1 D_s^M \alpha \dot{h} ds = \int_0^1 D_s \alpha (h(s) + \frac{1}{2} \text{ric}_r h(s)) ds + D^R_{\gamma_n} \alpha, \]
where \( \gamma_n(s) = \int_0^s \Omega_{\tau(s)}(\sigma dw_s, h(s)). \) Consequently
\[ \delta^M(u) = \int_0^1 (u(s) + \frac{1}{2} \text{ric}_r (\int_0^s u(\theta) d\theta)) \delta w_s - \sum_{i=1}^n D^R_{\gamma_{\alpha_i}}. \] (5.25)

5.3.3. Now let \( u(s) \) be a process such that \( u \in L^{1,2} \) and \( u(s) \) is strongly differentiable for \( s \in [0,1] \). We denote by \( \dot{h}_n \) an orthonormal basis of \( L^2([0,1]) \) and we assume without loss of generality that \( u(s) = u(s)e, \ e \in \mathbb{R}^d \). Let \( u_n = \sum_1^n \alpha_i \dot{h}_i \) with \( \alpha_i = <u, \dot{h}_i>_{L^2([0,1])} \). We have \( u_n \to u \) in \( L^{1,2} \) as \( n \to \infty \). Also \( u_n \in \text{Dom}(\delta^M) \) and
\[ \delta^M(u_n) = \int_0^1 (u_n(s) + \frac{1}{2} \text{ric}_r (\int_0^s u_n(\theta) d\theta)) \delta w_s - \sum_{i=1}^n D^R_{\gamma_{\alpha_i}}. \]

Then we can prove the following

**Proposition 5.5**

(i) There exists a random variable in \( L^1(W) \) which we denote also by \( \delta^M(u) \) such that the integration by parts formula \( E < D^M \phi, u > = E \phi \delta^M(u) \) holds for every smooth and bounded functional \( \phi \).

(ii) There is a constant \( C > 0 \) such that
\[ ||\delta^M(u)||_{L^1} \leq ||u||_{L^{1,2}} + C(||\Omega||_{L^1} + ||\Omega'||_{L^1})(||u||_{L^{1,2}} + ||u||^2_{L^{1,2}}) \] (5.26)

where \( ||\Omega|| = \sup_r ||\Omega_r||, ||\Omega'|| = \sup_r ||\Omega'_r||. \)

We omit the proof of these facts here, which is not difficult (we begin by the case where the \( \alpha_i \) are cylindrical).

5.3.4. We can do the same remark for the integral \( \tilde{\delta} \): As in 5.3.2, let \( u(s)(w) = \alpha(w) \dot{h}_s \), then we have :
\[ \tilde{\delta}(u) = \delta(u) + < D\alpha, \dot{h} > - < \tilde{D}\alpha, \dot{h} > . \]

But
\[ < \tilde{D}\alpha, \dot{h} >= < D^M \alpha, \dot{u} > \]

with the notation of 5.3.1, then the interwining formula yields
\[ < \tilde{D}\alpha, u > = < D\alpha, u > - D^R_{\gamma_n} \alpha \]
and
\[ \tilde{\delta}(u) = \delta(u) + D_{\gamma_h}^R \alpha. \]

and we have the same estimate as \([5.26]\). In view of the above expressions of \(\tilde{\delta}, \delta^M\), it seems not possible to obtain good estimates on the anticipative integrals by using the Sobolev norms associated to the gradient in the flat Wiener space. This is essentially due to the rotational derivatives \(D_{\gamma_i}^R \alpha\) which appear in the expressions. For if \(\gamma(s)\) is an adapted \(so(d)\)-valued process, then one can check easily that
\[ D_{\gamma_i}^R \alpha = \int_0^1 D_s \alpha \cdot \gamma(s) \delta w(s) + \int_0^1 \gamma_{ij}(s) D_i^s D_j^s \alpha ds. \]

Hence the expressions of the anticipative integrals of a process \(u\) involve the second derivatives of \(u(t)\). We have similar difficulties with the norms associated to \(\tilde{\alpha}, D^M\).

Let us finally mention that Cruzeiro and Fang \([8, 9]\) have constructed a norm of the type \(\|\cdot\|_{1,2}\) in the space of tangent processes for which we have an estimate of the form \(\|\delta^M(u)\|_{L^2} \leq c\|u\|_{1,2}\). Another attempt in the case of Lie groups, to obtain an energy equality, has been made by Fang and Franchi \([16]\).
References

[1] H. Airault, P. Malliavin, Semimartingales with values in a Euclidian vector bundle and Ocone’s formula on a Riemannian Manifold in M. Cranston and M. Pinsky (eds) Proc. AMS Symp. Pure Math. 57 (1995) 175-192.

[2] D. Bell, Quasi-invariant measures on the path space of a diffusion, Comptes Rendus Math. Vol. 343, 197-200 (2006). arXiv:math/0606365.

[3] R. L. Bishop and R.J. Crittenden, Geometry of Manifolds, Academic Press 1964.

[4] J. M. Bismut, Large Deviations and the Malliavin Calculus, Birkhauser 1984.

[5] R. H. Cameron and W. T. Martin, Transformations of Wiener integrals under translations, Ann. of Math. (2) 45 (1944), 386396.

[6] R.H. Cameron, W.T. Martin, The transformation of Wiener integrals by non-linear transformations, Trans. Amer. Math. Soc. 66 (1949) 253-283.

[7] Y. Choquet-Bruhat, C. DeWitt-Morrette, (with M. Dillard), Analysis, Manifolds and Physics, 2d ed North-Holland 1982.

[8] A.-B. Cruzeiro, S. Fang, Une inégalité $L^2$ pour des intégrales stochastiques anticipatives sur une variété riemannienne, C. R. Acad. Sc. Paris 321 (1995) 1245-1250.

[9] A.B. Cruzeiro, S.Fang, An $L^2$ Estimate for Riemannian Anticipative Stochastic Integrals, J. Func. Anal. 143 (1997) 40414.

[10] A.-B. Cruzeiro, P.Malliavin, [1] Renormalized differential geometry, structural equation, curvature, J. Funct. Analysis, 139 (1996) 119-181. [2] Repère mobile et géométrie riemannienne sur les espaces de chemins C. R. Acad. Sc. Paris 319 (1994) 859-864.[3] Courbure de l’espace de probabilité d’un mouvement brownien riemannien, C. R. Acad. Sc. Paris 320 (1995) 603-607.

[11] B.K. Driver, A Cameron-Martin type quasi-invariance theorem for Brownian motion on a compact Riemannian manifold, J. Funct. Analysis 110 (1992) 272-376.

[12] B.K. Driver, Curved Wiener space analysis, (2004), arXiv:math/0403073.

[13] K. D. Elworthy, Xue-Mei Li, Geometric stochastic analysis on path spaces, Proc. Int. Congr. Math 575-594 (2006).

[14] O. Enchev, D.W. Stroock, Towards a Riemannian geometry on the path space of a Riemannian Manifold, J. Funct. Analysis 134 (1995) 392-416.

[15] S. Fang, Stochastic anticipative integrals on a Riemannian manifold, J. Funct. Analysis, 131 (1995) 228-253.

[16] S. Fang, J. Franchi, Platitude de la structure riemannienne sur le groupe des chemins et identité d’énergie pour les intégrales stochastiques, C. R. Acad. Sc. Paris 321 (1995) 1371-1376.
[17] S. Fang, P. Malliavin, Stochastic analysis on the path space of a Riemannian manifold I. Markovian stochastic calculus, *J. Funct. Analysis*, 118 (1993), 249-274.

[18] B. Gaveau, P. Trauber, L’intégrale anticipante comme opérateur de divergence sur l’espace de Fock, *J. Funct. Analysis* 46 (1982) 230-238.

[19] L. Gross, The homogeneous chaos over compact Lie groups, in *Fest. of G. Kallianpur* (S. Cambanis et al. eds), pp.117-123, Springer 1993.

[20] E. P. Hsu, Quasi-invariance of the Wiener measure on the path space over a compact Riemannian manifold, *J. Funct. Analysis* 134 (1995), 417-450.

[21] E.P. Hsu, Quasi-invariance of the Wiener measure on path spaces: noncompact case, *J. Func. Anal.* 193 (2002), 278–290.

[22] E.P. Hsu, Cheng Ouyang, Cameron-Martin theorem for complete Riemannian manifolds, *J. Func. Anal.* 257, no. 5 (2009).

[23] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, 2nd ed., North Holland, 1989.

[24] T. Kazumi, Le générateur amorti d’Ornstein-Uhlenbeck sur l’espace de chemin riemanniens, *Preprint* 1996.

[25] R. Léandre, Integration by parts and rotationally invariant Sobolev calculus on free loop spaces, *J. Geom. Phys.* 11 (1993) 517-528.

[26] P. Malliavin, *Stochastic Analysis*, Springer 2002.

[27] D. Nualart, *The Malliavin Calculus and Related Topics*, Springer Verlag 1995.

[28] D. Nualart, E. Pardoux, Stochastic calculus with anticipating integrands, *Probab. Th. Rel. Fields* 78 (1988) 535-581.

[29] D. Nualart, M. Zakai, Generalized stochastic integrals and the Malliavin Calculus, *Prob. Th. Rel. Fields* 73 (1986) 255-280.

[30] S. Ogawa, Quelques propriétés de l’intégrale stochastique du type non causal, Jap. J. Appl. Math. 1 (1984) 405-416.

[31] M. Pontier, A.S. Ustünel, Analyse stochastique sur l’espace de Lie-Wiener, *C. R. Acad. Sc. Paris* 313 (1991) 313-316.

[32] I. Shigekawa, On stochastic horizontal lifts, *Prob. Th. Rel. Fields* 59 (1982) 211-221.

[33] A.S. Ustunel, *An Introduction to Analysis on Wiener Space*, Lect. Notes Math. 1610, 1995. *Analysis on Wiener Space and Applications* (2010), [arXiv:1003.1649](http://arxiv.org/abs/1003.1649).

[34] A.S. Ustünel, Stochastic analysis on Lie groups, *Preprint* 1996.

[35] J. Zhang, D. Kannan, A Girsanov type theorem on the path space over a compact Riemannian manifold, *Stochastic analysis and applications* 25 (2007) 667-678.