Nearest Common Ancestors: Universal Trees and Improved Labeling Schemes

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Abstract

We investigate the nearest common ancestor (NCA) function in rooted trees. As the main conceptual contribution, the paper introduces universal trees for the NCA function: For a given family of rooted trees, an NCA-universal tree $S$ is a rooted tree such that any tree $T$ of the family can be embedded into $S$ such that the embedding of the NCA in $T$ of two nodes of $T$ is equal to the NCA in $S$ of the embeddings of the two nodes.

As the main technical result we give explicit constructions of NCA-universal trees of size $n^{2.318}$ for the family of rooted $n$-vertex trees and of size $n^{1.894}$ for the family of rooted binary $n$-vertex trees. A direct consequence is the explicit construction of NCA-labeling schemes with labels of size $2^{2.318 \log_2 n}$ and $1.894 \log_2 n$ for the two families of rooted trees. This improves on the best known such labeling schemes established by Alstrup, Halvorsen and Larsen [SODA 2014].

Keywords: Rooted Trees, NCA, Nearest Common Ancestor, Lowest Common Ancestor, Universal Trees, Labeling Schemes, Embedding Schemes

1 Introduction

The nearest common ancestor (NCA) of two vertices $u$ and $v$ of a rooted tree $T$ is the first common vertex of the paths connecting $u$ and $v$ to the root of $T$. Finding the nearest common ancestor appears as an essential operation in many algorithms and applications (see for example the survey by Alstrup, Gavoille, Kaplan, and Rauhe [2]).

NCA-Universal Trees. The present paper introduces the notion of $NCA$-universal trees as a novel tool to study and algorithmically deal with the NCA function in rooted trees. We define an NCA-universal tree $S$ for a family of rooted trees $\mathcal{T}$ as such that every tree $T \in \mathcal{T}$ can be embedded into $S$ such that the NCA function is preserved by the embedding. More formally, an embedding of $T$ into $S$ is an injective mapping $\varphi_T$ of $V(T)$ into $V(S)$ such that the embedding function $\varphi_T$ and the NCA function commute.

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i) In the literature, the nearest common ancestor of two vertices in a rooted tree is sometimes also referred to as the lowest or least common ancestor (LCA).
NCA-Labeling Schemes. As an immediate application of an NCA-universal tree $S$ for a family $T$ of rooted trees, $S$ directly implies an NCA-labeling scheme [2] for the family $T$. Generally, a labeling scheme is a way to preprocess the structure of a graph to later allow simple and fast queries. A labeling scheme consists of an encoder and a decoder, where the encoder must be able to label a family of graphs such that the decoder can answer queries, given just the labels and no additional information about the underlying graph. More specifically, an NCA-labeling scheme assigns a unique label to each node of a rooted tree $T$ such that given the labels of two vertices $u$ and $v$ of $T$, it is possible to compute the label of the NCA of $u$ and $v$ in $T$. If an NCA-universal tree $S$ for a family $T$ of rooted trees is given, we can get an NCA-labeling scheme for $T$ as follows. Let $|S|$ be the number of vertices of $S$ and assume that the vertices of $S$ are labeled from 0 to $|S| - 1$ in an arbitrary fixed way. Given an embedding of a tree $T \in T$ into $S$, we then get the labeling of a vertex $v$ of $T$ by using the label of the vertex $x$ of $S$ to which $v$ is embedded. The size of the labels (in bits) of the labeling scheme is therefore exactly $\lceil \log |S| \rceil$. We remark that throughout the paper, all logarithms are to base 2.

Contribution. We show that the family of all rooted trees with at most $n$ vertices has an NCA-universal tree of size $O(n^{2.318})$ and that the family of all binary rooted trees with at most $n$ vertices has an NCA-universal tree of size $O(n^{1.894})$. This implies that the families of rooted $n$-vertex trees and of rooted $n$-vertex binary trees have labeling schemes with labels of size $2.318 \log n$ and $1.894 \log n$, respectively. This improves on the best previous NCA-labeling schemes that were developed by Alstrup, Halvorsen and Larsen [3] and which require labels of size $2.772 \log n$ for general rooted trees and of size $2.585 \log n$ for binary rooted trees. In [3], it is also shown that any NCA-labeling scheme for general $n$-vertex rooted trees requires labels of size at least $1.008 \log n$.

As we show how to explicitly construct the NCA-universal trees, our labeling schemes are constructive. Note that the best NCA-labeling schemes of [3] are not constructive and that the best previous constructive NCA-labeling scheme for $n$-vertex rooted trees requires labels of size $3 \log n$. Further, our NCA-labeling schemes are efficient, the embedding of a rooted tree into the constructed NCA-universal tree can be computed efficiently and a single query can be answered in time $O(\log^2 n)$ ($O(\log n)$ for binary trees). We believe that our new NCA-labeling schemes are not only interesting because they improve upon the best existing schemes, but also because our approach leads to more intuitive and significantly simpler constructions.

Related Work. Graph labeling schemes are an elegant way to store structural information about a graph. As every vertex is only assigned a small label, the information is stored in a completely distributed way and graph labelings therefore are particularly interesting in a distributed context, where labeling schemes are used for various kinds of graph queries [10, 15]. In addition, labeling schemes can be used in a context where extremely large graphs are processed and where accessing the data is expensive. To answer a pair-wise query, only the two labels of the corresponding vertices need to be accessed.

The first labeling schemes that appear in the literature are adjacency labeling schemes (given the labels of two vertices, determine whether the vertices are adjacent). They were introduced among others by Breuer [6] and Folkman [7]. In the context of adjacency labeling schemes, it is well known that they are tightly connected to induced universal graphs. A graph $G$ is an induced universal graph for a graph family $H$ if $G$ contains every graph $H \in H$ as an induced subgraph. Induced universal graphs were first described by Rado [17] and Kannan et al. [14] noted the equivalence between adjacency labeling schemes and induced universal graphs.

When considering the family of rooted trees we are interested in different queries such as whether
a vertex is an ancestor of the other. Ancestry labeling has been studied and labeling schemes of size $\log(n) + \Theta(\log(\log(n)))$ are known to be tight [1]. If the tree has low depth, then a scheme of size $\log(n) + 2\log(d) + O(1)$ is known, where $d$ is the depth of the tree [8].

For NCA-labeling schemes, a linear-size labeling scheme that answers queries in constant time was introduced by Harel and Tarjan in [12,13]. In the following, there was a series of significant improvements in [16], [4], [9], [18], [5], [2] and most recently in [3]. In particular in [3], a lower bound for NCA labeling schemes of $1.008 \log n$ is shown, which separates NCA-labelings, that need labels of size $\log n + \Omega(\log n)$, from ancestry labeling schemes, where labels of size $\log n + O(\log \log n)$ are sufficient.

There is concurrent work by Gawrychowski and Łopuszański [11] who reach the exact same bounds for labeling schemes and construct almost identical universal trees as we do. The proof method is a bit different and in their paper they also have lowerbounds for the size of a universal tree for the NCA function.

Outline. The rest of the paper is organized as follows. In the remainder of this section, we first formally define the problems and state our results in Sections 1.1 and 1.2. In Section 2, we first prove a simpler upper bound of $O(n^2)$ on the size of NCA-universal trees for the family of binary rooted trees. We extend the construction of Section 2 to obtain the stronger and more general results stated above in Section 3. Finally, in Section 4, we sketch how to efficiently implement our labeling scheme and in Section 5, we conclude the paper and discuss some open issues.

1.1 Definitions

We next define the necessary graph-theoretic concepts and notation and we in particular formally introduce the notion of universal trees for the NCA function. In the following let $T_n$ be the family of unlabeled rooted trees with at $n$ vertices and $B_n$ the family of unlabeled rooted binary trees with $n$ vertices.

Definition 1.1. In a rooted tree $T = (V,E)$ a vertex $u$ is an ancestor of a vertex $v$ if $u$ is contained in the (unique) path from $v$ to the root of $T$.

Note that according to the above definition, a node $u$ is an ancestor of itself.

Definition 1.2. Let $T = (V,E)$ be a rooted tree. For a pair of vertices $u$ and $v$ their nearest common ancestor (NCA) $nca_T(u,v)$ is the unique common ancestor that is furthest from the root of $T$.

With this notation at hand we can define the notion of a universal tree for NCA.

Definition 1.3. A rooted tree $S$ is called an NCA-universal tree for a family of rooted trees $T$, if for every tree $T \in T$ there is an embedding function $\varphi_T : V(T) \mapsto V(S)$ such that $\varphi_T$ commutes with the NCA function, i.e., for all $u,v \in V(T)$, $\varphi_T(nca_T(u,v)) = nca_S(\varphi_T(u), \varphi_T(v))$.

Hence, the embedding has the property that the NCA of two nodes $u$ and $v$ of $T$ is mapped to the NCA of $\varphi_T(u)$ and $\varphi_T(v)$ in $S$. Note that we do not require the root of $T$ to be embedded to the root of $S$. In the following, a rooted tree that is universal for the NCA function is also called an NCA-universal tree.

Definition 1.4. An NCA-labeling scheme for a family of rooted trees $T$ is a pair of functions called the encoder ($f$) and decoder ($g$) with $f : \{v | v \in T \in T\} \mapsto [m]$ and $g : [m] \times [m] \mapsto [m]$ satisfying the following properties.
i) for every $T \in \mathcal{T}$ and every $u, v \in V(T)$, $f(u) \neq f(v)$ and

ii) for every $T \in \mathcal{T}$ and every $u, v \in V(T)$, $g(f(u), f(v)) = f(\text{nca}_T(u, v))$.

For a node $v \in T \in \mathcal{T}$, $f(v)$ is called the label of $v$. The size of the labeling scheme defined by $f$ and $g$ is $\lceil \log m \rceil$, i.e., the number of bits required to store the largest label.

Given an NCA-universal tree $S$ for a family of rooted trees $\mathcal{T}$, we directly obtain an NCA-labeling scheme for $\mathcal{T}$.

**Observation 1.5.** Let $S$ be an $N$-vertex rooted tree that is universal for the NCA function and the family $\mathcal{T}$ of rooted trees. Then, there exists an NCA-labeling scheme of size $\lceil \log N \rceil$ for $\mathcal{T}$.

**Proof.** We assign unique names from 0 to $N - 1$ to the $N$ vertices of $S$. Consider a tree $T \in \mathcal{T}$ and let $f$ be an embedding of $T$ into $S$. The label of a vertex $v$ of $T$ is the name assigned to vertex $f(v)$ of $S$. Given the labels $x_u \in \{0, \ldots, N - 1\}$ and $x_v \in \{0, \ldots, N - 1\}$ of two nodes $u$ and $v$ of $T$, the decoder outputs the name $x_w \in \{0, \ldots, N - 1\}$ of the NCA of the vertices $u'$ and $v'$ with names $x_u$ and $x_v$ in $S$. □

1.2 Main Results

Our main result is an explicit construction of universal trees for the families of all trees and binary trees with $n$ vertices. The same bounds can be found in the concurrent work of Gawrychowski and Łopuszański [11].

**Theorem 1.6.** Let $n \in \mathbb{N}$. Then:

- There is a rooted tree $S_n$ of size less than $n^{2.318}$ that is universal for the NCA function and the set $\mathcal{T}_n$ of rooted trees of size $n$.

- There is a rooted tree $S_{n}^{\text{bin}}$ of size less than $n^{1.894}$ which is universal for the NCA function and the set $\mathcal{B}_n$ of rooted binary trees of size $n$.

The proof can be found in Section 3. The direct implication of this for labeling schemes is summarized in the following statement.

**Theorem 1.7.** For any $n \in \mathbb{N}$ there exists an NCA-labeling of size less than $2.318 \log n$ and an NCA-labeling for binary trees of size less than $1.894 \log n$.

This is an improvement of the current best known bound of $2.772 \log n$ from [3]. Further for the specific case of binary trees of particular interest is that the constant is now below 2 and therefore will likely not be an integer.

**Proof of Theorem 1.7.** This is exactly what we have shown in Observation 1.5 and therefore follows from Theorem 1.6. Although this is just an existential proof, from the construction we will see later, it is clear that a reasonably fast algorithmic implementation is possible and we will give a sketch in Section 4. □
2 Basic Universal Tree Construction

The NCA-universal trees of Theorem 1.6 are constructed recursively. Before proving the general statements of Theorem 1.6, we describe a simpler, slightly weaker construction that provides an NCA-universal tree of size $O(n^2)$ for $n$-vertex binary trees. The full constructions required to prove Theorem 1.6 appears in Section 3.

Theorem 2.1. For any $n \in \mathbb{N}$ there exists a rooted tree $S_n$ of size less than $n^2$ which is universal for the NCA function and the rooted binary trees of size at most $n$.

Our recursive universal tree construction requires two kinds of NCA-universal trees. In addition to ordinary unlabeled rooted trees, we also need to define NCA-universal trees for the family of rooted binary trees where one leaf node is distinct (marked). Recall that $B_n$ denotes the set of all $n$-vertex unlabeled rooted binary trees, so let $B'_n$ denote the family of unlabeled rooted binary trees on at most $n$ vertices and with one marked leaf.

Definition 2.2. A rooted tree $S'$ with one marked leaf vertex $w$ is called a NCA-universal tree for a family of rooted trees $T'$ with one marked leaf if for every tree $T \in T'$, there exists an embedding function $\varphi_T : V(T) \mapsto V(S')$ that maps the marked leaf of $T$ to the marked leaf $w$ of $S'$ and where $\varphi_T$ commutes with the NCA function, i.e., $\varphi_T(nca_T(u,v)) = nca_{S'}(\varphi_T(v),\varphi_T(u))$ for all $u, v \in T$.

As in Definition 1.3, we do not require that the root of $T$ is mapped to the root of $S'$.

Overview of the Construction. The construction of the NCA-universal tree for binary rooted trees is done recursively as illustrated in Figure 1. The universal tree $S_n$ for binary trees of size at most $n$ consists of three NCA-universal trees for binary trees of size at most $n/2$, where one of these three universal trees needs to work for the more general family trees with one marked leaf vertex. Universal trees for the family of $n$-vertex trees with a marked leaf are constructed recursively in a similar way. They consist of two NCA-universal trees for $n/2$-vertex binary trees with a marked leaf and of a single NCA-universal tree for ordinary $n - 1$-vertex binary trees (see Figure 1).

In order to show that the recursive construction of Figure 1 results in an NCA-universal tree we need to argue that any $n$-vertex binary tree $T$ can be embedded. To achieve this, we show that any
rooted tree $T$ has a vertex $v$ such that $v$ splits $T$ into three subtrees of size at most $n/2$. Vertex $v$ is then embedded to the vertex marked in red in the left part of Figure 1. The three subtrees of $T$ induced by $v$ are then embedded recursively into the three parts of the universal tree construction. Further, we need to show that any rooted tree $T$ with a marked leaf can be partitioned in a similar way to be consistent with the recursive structure in the right part of Figure 1. In the following, we first give the basic technical lemmas required to partition $n$-vertex trees $T$ into the required smaller subtrees. Based on these partitioning results, we will analyze the recursive NCA-universal tree construction in more detail and prove Theorem 2.1. As the same partitioning lemmas will also be needed in the general NCA-universal tree constructions in Section 3, they are stated more generally than what we require for the simple construction of the present section.

**Lemma 2.3.** For every rooted $n$-vertex tree $T$ and for every parameter $\lambda \in (0, 1]$, there exists a vertex $v \in V(T)$ such that removing the edges from $v$ to its children splits the tree into components such that each component rooted at a child of $v$ has size at most $\lceil (1 - \lambda) \cdot n \rceil$ and the root of $T$ has size at most $\lfloor \lambda \cdot n \rfloor$.

**Proof.** We determine $v$ using the following simple iterative procedure. We initialize $v$ to be the root of $T$. We stop the procedure as soon as $v$ satisfies the conditions of the lemma. For some vertex $u$ of $T$, let $\text{size}(u)$ be the number of vertices in the subtree rooted at $u$. If $v$ does not split the tree as required, we let $w$ be the child vertex of $v$ that maximizes $\text{size}(w)$ and we set $v := w$. Since $v$ goes from being the root of $T$ to being a leaf of $T$ during this process, the component containing the root goes from being of size 1 to a set of size $n$. We claim that for the last vertex $v$ where the connected component of the root is still of size at most $\lfloor \lambda \cdot n \rfloor$, the lemma holds.

Clearly, the component with the root is of size at most $\lfloor \lambda \cdot n \rfloor$ and it thus suffices to show that all the subtrees of $v$ are of size at most $\lceil (1 - \lambda)n \rceil$. Assume for contradiction that $v$ has a child $w$ such that $\text{size}(w) \geq 1 + \lceil (1 - \lambda)n \rceil$. Then, removing all subtrees of $w$ from $T$ would result in a component of size at most $n - \lceil (1 - \lambda)n \rceil = \lfloor \lambda n \rfloor$ and thus $v$ would not be the last vertex for which the connected component of the root is of size at most $\lfloor \lambda n \rfloor$. \hfill \Box

For trees with a marked leaf we can get a similar tree splitting lemma.

**Lemma 2.4.** Given a rooted $n$-vertex tree $T$ with one marked leaf vertex $w$ and a parameter $\lambda \in (0, 1]$, if $n \geq \frac{1}{\lambda - 1}$, there exists a vertex $v \in V(T)$ such that when removing the edges connecting $v$ to its children, $T$ is split into components satisfying the following properties. The component containing the root of $T$ and vertex $v$ has size at most $\lceil \lambda n \rceil$, the component containing the marked leaf $w$ has size at most $\lfloor (1 - \lambda)n \rfloor$, and all other components have size at most $n - 1$.

**Proof.** Let $r$ be the root vertex of $T$. We choose $v$ to be last vertex on the path from $r$ to $w$ such that when removing the subtrees of $v$, the remaining component has size at most $\lfloor \lambda n \rfloor$. $\lfloor \lambda n \rfloor < n$, so $v$ cannot be a leaf and thus $v \neq w$. Let $v'$ be the root of the subtre of $v$ containing $w$. To prove the lemma, it suffices to show that the subtree rooted at $v'$ has size $\text{size}(v') \leq \lfloor (1 - \lambda)n \rfloor$. For the sake of contradiction, assume that $\text{size}(v') \geq \lfloor (1 - \lambda)n \rfloor + 1$. In this case, the total size of all subtrees of $v'$ is at least $\lfloor (1 - \lambda)n \rfloor$ and thus removing all subtrees of $v'$ would leave a component of size at most $\lfloor \lambda n \rfloor$. This contradicts the assumption that $v$ is the last vertex on the path from $r$ to $w$ for which this is true. \hfill \Box

**Proof of Theorem 2.1.** For every integer $n \geq 1$, we show how to construct an NCA-universal tree $S_n$ for the family $B_n$ of $n$-vertex rooted binary trees and an NCA-universal tree $S'_n$ for the family $B'_n$ of $n$-vertex binary rooted trees with one marked leaf. We will prove by induction on $n$ that $|S_n| \leq n^2$ and that $|S'_n| \leq 2n^2 - 1$ for all $n \geq 1$. \hfill \Box
For the induction base, note that $S_1$ and $S'_1$ clearly need to only consist of a single vertex and we thus have $|S_1| = |S'_1| = 1$. Thus, the bounds on $|S_n|$ and $|S'_n|$ hold for $n = 1$. For the induction step, assume that $n \geq 2$ and that $|S_k| \leq k^2$ and $|S'_k| \leq 2k^2 - 1$ for all $1 \leq k < n$. We build the two NCA-universal trees $S_n$ and $S'_n$ by using smaller NCA-universal trees as given in Figure 1. That is, $S_n$ is composed of one copy of $S'_{\lceil n/2 \rceil}$ and two copies of $S_{\lfloor n/2 \rfloor}$ and $S'_n$ is composed of one copy of $S'_{\lfloor n/2 \rfloor}$, $S'_{\lceil n/2 \rceil}$, and $S_{n-1}$. We need to show that the constructed trees $S_n$ and $S'_n$ are in fact NCA-universal trees and that they satisfy the required size bounds. We first show that the trees are or the right size. Using the induction hypothesis, we have

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\[
|S_n| = |S'_{\lceil n/2 \rceil}| + 2 \cdot |S_{\lfloor n/2 \rfloor}| \leq 2 \cdot \left\lceil \frac{n}{2} \right\rceil^2 - 1 + 2 \cdot \left\lfloor \frac{n}{2} \right\rfloor^2 \leq n^2,
\]

and

\[
|S'_n| = |S'_{\lceil n/2 \rceil}| + |S'_{\lfloor n/2 \rfloor}| + |S_{n-1}| \leq 2 \cdot \left\lceil \frac{n}{2} \right\rceil^2 - 1 + 2 \cdot \left\lfloor \frac{n}{2} \right\rfloor^2 - 1 + (n-1)^2 \leq 2n^2 - 1.
\]

It thus remains to prove that the recursive construction of $S_n$ and $S'_n$ allows to find a proper embedding for every $T \in B_n$ into $S_n$ and every $T' \in B_n$ into $S'_n$, respectively.

To show this, we use Lemma 2.3 and Lemma 2.4. We first show how to construct an embedding $\varphi_T$ of a binary $n$-vertex tree $T \in B_n$ into $S_n$. For this purpose, we apply Lemma 2.3 with parameter $\lambda = 1/2$ to tree $T$. Let $v \in V(T)$ be the vertex of $T$ that splits the tree such that the part containing the root of $T$ and $v$ has size at most $\lfloor n/2 \rfloor$ and such that all other components have size at most $\lfloor n/2 \rfloor$. For the embedding take the splitting vertex $v \in T$ given by Lemma 2.3 which will be embedded to the marked vertex of the copy of $S'_{\lceil n/2 \rceil}$ (cf. Figure 1). Consider the three components of $T$ after splitting. By the induction hypothesis there is an embedding function to embed the child components (components below the splitting vertex $v$) into the two copies of $S_{\lfloor n/2 \rfloor}$. Then we take the component with the root and the marked vertex to get a tree with a single marked leaf of size at most $\lceil n/2 \rceil$ and the induction hypothesis again provides with an embedding function of this component into $S'_{\lceil n/2 \rceil}$. The embedding is depicted in Figure 2.

To see that the NCA-function for any two vertices $u_1$ and $u_2$ and the described embedding function $\varphi_T$ commute, we need that $\text{nca}_{S_n}(\varphi_T(u_1), \varphi_T(u_2)) = \varphi_T(\text{nca}_T(u_1, u_2))$. This can be verified through the following case analysis considering the three components after the splitting process. If $u_1$ and $u_2$ are in the same component, then we have embedded them into the same subtree and by the induction hypothesis, we have $\varphi_T(\text{nca}_T(u_1, u_2)) = \text{nca}_{S_n}(\varphi_T(u_1), \varphi_T(u_2))$. If $u_1$ and $u_2$ are in different child components, we have $\text{nca}_T(u_1, u_2) = v$ and the embedding is therefore also correct because is embedded to the vertex marked in red in Figure 2. Finally, if $u_1$ is from the
component containing the root and vertex \( v \) and \( u_2 \) is a vertex from a child component, we have \( \text{nca}_T(u_1, u_2) = \text{nca}_T(u_1, v) \) and similarly \( \text{nca}_{S_n}(\phi_T(u_1), \phi_T(u_2)) = \text{nca}_{S_n}(\phi_T(u_1), \phi_T(v)) \) and the embedding is therefore again correct by the induction hypothesis (applied to the partial embedding into the subtree \( S'_{\lceil n/2 \rceil} \)).

For the family of \( n \)-vertex binary trees with a marked leaf, the embedding into the recursively constructed tree \( S'_n \) (cf. Figure 1) works in similar way. Let \( T' \) be a binary tree of size at most \( n \) and with a marked leaf. We apply Lemma 2.4 with parameter \( \lambda = 1/2 \) to \( T' \) to obtain a vertex \( v \in V(T') \) that splits \( T' \) into a) a subtree of size at most \( \lceil n/2 \rceil \) that contains the root of \( T' \) and that contains \( v \) as a leaf vertex, b) a subtree size at most \( \lfloor n/2 \rfloor \) that is rooted at a child of \( v \) and contains the marked leaf of \( T' \), and c) a (possibly empty) subtree of size at most \( n - 1 \) rooted at a child of \( v \).

The tree \( T' \) is embedded into \( S'_n \) by embedding vertex \( v \) to the center red node separating the three recursive subtrees in Figure 1. The three subtrees resulting after splitting \( T' \) are embedded into the three recursively constructed subtrees \( S'_{\lceil n/2 \rceil}, S'_{\lfloor n/2 \rfloor}, \) and \( S_{n-1} \) in the natural way. The proof that the embedding is correct is done in the same way as for the embedding of \( T \) into \( S_n \). The details of the embedding are illustrated in Figure 3.

\[ \square \]

3 General Universal Tree Construction

The proof of the basic construction was handled in detail. We now adjust the construction to improve on the exponent in the size of \( S_n \) and to deal with general rooted trees. Throughout the section, we omit floor and ceiling functions. They do not change the calculations significantly, but hinder the readability of the proof.
Proof of Theorem 1.6. As suggested in Lemma 2.3 and Lemma 2.4, the adjustment of the basic construction can be made by choosing $\lambda \neq 1/2$ for the size of the splitted components.

We start with the binary tree case. We apply the same induction as in the proof of Theorem 2.1. In the general case, we prove that $|S_k| \leq k^\beta$ and $|S_k'| \leq c \cdot k^\beta \forall \ 1 \leq k < n$ and some constants $c$ and $\beta$ that will be determined later.

For the construction of $S_n$ we take $S_{\lambda n}'$ and attach copies of $S_{(1-\lambda)}n$, $S_{n/2}$, $S_{n/3}$, $S_{n/4}$, etc., up to $S_1$. Similarly, let $S_{n}'$ be composed of a copy of $S_{n/2}'$ and attached to the marked vertex of that tree copies of $S_{n/2}'$, $S_n$, $S_{n/2}$, $S_{n/3}$, $S_{n/4}$, etc., up to $S_1$. 

Figure 4: General recursive construction of an NCA-universal tree for $n$-vertex rooted binary trees.
For the embedding, note that we can sort the child components by size. Lemma 2.3 states that any child component is of size at most $(1 - \lambda) \cdot n$. In addition, the total size of all components cannot add up to more than the entire tree of size $n$. This implies that after ordering the components by size, the $i$th child component without a marked vertex is of size at most $n/i$.

With the induction hypothesis that $|S_k| \leq k^\beta$ and $|S_k| \leq c \cdot k^\beta$, the recursion gives

$$|S_n| \leq |S_{(1-\lambda)n}| + |S'_{\lambda n}| + \sum_{i=2}^{\infty} |S_n^i| \leq (1 - \lambda)n^\beta + c(\lambda n)^\beta + (\zeta(\beta) - 1)n^{\beta - 1} \leq n^\beta$$

$$|S'_n| \leq |S'_{\lambda n}| + |S'_{\lambda n}| + \sum_{i=1}^{\infty} |S_n^i| \leq c\left(\frac{n}{2}\right)^\beta + c\left(\frac{n}{2}\right)^\beta + \zeta(\beta)n^\beta \leq n^\beta,$$

where $\zeta(\beta)$ is the Riemann zeta function ($\zeta(\beta) = \sum_{i \geq 1} i^{-\beta}$). Again we can deduce from the second inequality that $c \geq \frac{\zeta(\beta)}{1-2^{\beta-1}}$. By using $\lambda = 0.343195...$, we get that $\beta \leq 2.31757...$.

In both constructions, the fact that the NCA function and the embedding function commute follows in the same way as in the proof of Theorem 2.1. This concludes the proof of Theorem 1.6. \qed

### 4 Implementation of the NCA-Labeling Scheme

**Theorem 4.1.** The labeling schemes described in Section 3 can be constructed efficiently. Further, given two labels, the label of the nearest common ancestor can be determined in $O(\log^2 n)$ time (in $O(\log n)$ time in the binary tree case).

**Proof Sketch.** For $R \in \mathbb{R}$ we write $[R] := \{x : 1 \leq x \leq R\}$. In order to assign labels to the vertices of $S_n$ we proceed as follows. Set $s(n) = n^\beta$, and recall that $|S_n| \leq s(n)$ and $|S'_n| \leq cs(n)$. Moreover, as depicted in Figure 5 the tree $S_n$ is composed out of the $n + 1$ trees

$$T_0 = S'_{\lambda n}$$
$$T_1 = S_{(1-\lambda)n}$$
$$T_\ell = S_{n/\ell}, \text{ where } 2 \leq \ell \leq n.$$  \hfill (1)

Define the corresponding counting sequence

$$t_{-1} = 0$$
$$t_0 = cs(\lambda n)$$
$$t_1 = t_0 + s((1 - \lambda)n)$$
$$t_\ell = t_{\ell-1} + s(n/\ell), \text{ where } 2 \leq \ell \leq n.$$  \hfill (2)
For $S'_n$ we proceed similarly. As depicted in Figure 5 $S'_n$ is composed out of the $n + 2$ trees

$$
\begin{align*}
T'_{-1} &= S'_{n/2} \\
T'_0 &= S'_{n/2} \\
T'_\ell &= S'_{n/\ell}, \text{ where } 1 \leq \ell \leq n.
\end{align*}
$$

and the corresponding counting sequence is given by

$$
\begin{align*}
t'_{-2} &= 0 \\
t'_{-1} &= cs(n/2) \\
t'_0 &= t_{-1} + cs(n/2) \\
t'_\ell &= t_{\ell-1} + s(n/\ell), \text{ where } 1 \leq \ell \leq n.
\end{align*}
$$

Given these sequences, in order to assign labels to the vertices in $S_n$ we assign to the vertices of $T_i, 0 \leq i \leq n$ the labels in $[t_i] \setminus [t_{i-1}]$. The assignment is performed recursively, in the sense that as soon the labels in $T_i, 0 \leq i \leq n$ are assigned, they are translated by an additive $[t_{i-1}]$, so that they all lie (with room to spare) in the required set $[t_i] \setminus [t_{i-1}]$. The assignment is performed analogously for $S'_n$, where we use the corresponding counting sequence instead.

Given the label of a vertex in $S_n$, its location in the tree can be found with this preprocessing in $O(\log^2 n)$ time. Indeed, in every step we have to decide in which of the at most $n+2$ subtrees we have to branch to; however, this can be decided with binary search on the sequences $(t_i)_{0 \leq i \leq n}$ or $(t'_i)_{-1 \leq i \leq n}$. As the depth of the recursive construction of $S_n$ is $O(\log n)$, the claim follows.

\[ \square \]

5 Conclusion

We introduced NCA-universal trees and gave simple recursive constructions of such trees that in particular lead to improved NCA-labeling schemes for rooted trees. The paper leaves several interesting open questions. The current upper bound of 2.318 log $n$ bits per label is still quite far from the 1.008 log $n$-bit lower bound proven in [3] and it remains an intriguing open problem to close this gap. In addition, given that NCA-universal trees provide an intuitive way to argue about NCA-labeling schemes, it is natural to ask whether the approach can lead to optimal NCA-labeling schemes or whether every NCA-labeling scheme for a given tree family can be turned into an equivalent one that can be characterized by an NCA-universal tree for the tree family. The following observation shows that NCA-universal trees are equivalent to a certain well-structured class of NCA-labeling schemes.

We call an NCA-labeling scheme consistent if any three labels can occur together in some tree. More formally, an NCA-labeling scheme is called consistent if it satisfies the following three properties for any 3 possible labels $x, y, \text{ and } z$. In the following, $g$ is the decoder function.

(I) If $g(x, y) = z$, then $g(x, z) = z$ and $g(y, z) = z$

(i.e., if $z$ is the NCA of $x$ and $y$, then $z$ is an ancestor of $x$ and $y$)

(II) If $g(x, y) = y$ and $g(y, z) = z$, then $g(x, z) = z$

(i.e., if $y$ is an ancestor of $x$ and $z$ an ancestor of $y$, then also $z$ is an ancestor of $x$)

(III) If $g(x, y) = y$ and $g(x, z) = z$, then $g(y, z) \in \{y, z\}$

(i.e., if $y$ and $z$ are ancestors of $x$, then $z$ is an ancestor of $y$ or $y$ is an ancestor of $z$)
**Theorem 5.1.** Every NCA-universal tree $S$ for a given family $\mathcal{T}$ of trees leads to a consistent NCA-labeling scheme for $\mathcal{T}$ with labels of size $\lceil \log |S| \rceil$. Conversely, every consistent NCA-labeling scheme for $\mathcal{T}$ and with $\ell$-bit labels induces an NCA-universal tree of size $2^\ell$ for $\mathcal{T}$.

**Proof.** The first claim of the lemma is immediate because $S$ is a tree and therefore any three vertices of $S$ (i.e., any three labels) are consistent.

For the second claim, define a directed graph $G = (V, E)$ as follows. The vertex set $V$ of $G$ is the set of labels of the given NCA-labeling scheme. Assume that $g$ is the decoder function of the labeling scheme. We add a directed edge from $u \in V$ to $v \in V$ if $g(u, v) = v$ and there is no vertex $w$ such that $g(u, w) = w$ and $g(w, v) = v$ (i.e., if $v$ is the parent of $u$). We claim that $G$ is a rooted tree.

First observe that $G$ is acyclic. Otherwise, by using Property (II) several times, we can find three vertices $u$, $v$, and $w$ such that $g(u, v) = v$, $g(v, w) = w$, and $g(w, u) = u$. However from Property (II) we then also have $g(u, w) = w$, a contradiction.

Second, we show that the out-degree of each vertex of $G$ is at most 1. For contradiction, assume that there exists a vertex $u$ that has out-going edges to $v$ and $w$. We then have $g(u, v) = v$ and $g(u, w) = w$ and by Property (III) of consistent labeling schemes, we thus also have $g(v, w) = w$ or $g(w, v) = v$. Thus, one of the two edges $(u, v)$ and $(u, w)$ cannot be in $G$.

Finally, we show that there can be at most one vertex with out-degree 0. For the sake of contradiction assume that $u$ and $v$ both have out-degree 0 and let $g(u, v) = w$. By Property (I), we then also have $g(u, w) = w$ and $g(v, w) = w$. If $w \neq u$, this implies that $u$ has out-degree at least 1 and if $w = v$, it implies that $v$ has out-degree at least 1.

Hence, $G$ is a rooted tree on the set of labels of the labeling scheme. Because the ancestry relationship of $G$ is consistent with the labeling scheme, $G$ is an NCA-universal tree for the family $\mathcal{T}$. 

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A Appendix

A.1 Induction Basis

For \( n \) equals one through four it is simple to find a universal tree of size at most \( n^2 \). For example:

\[
S_1 \quad S_2 \quad S_3 \quad S_4
\]

\[
T_1 \quad T_2 \quad T_3 \quad T_4
\]

Figure 6: The root is always the top most vertex but the root of \( T \in B_n \) does not have to be embedded to the root of \( S_n \)

\( S_n' \) shall have size at most \( 2n^2 \). We want for any \( n \) a large tree \( S_n' \) with a marked leaf such that we can embed any tree on \( n \) vertices with a marked leaf with the nca-function commuting. Again as example and induction basis:

\[
S_1' \quad S_2' \quad S_3' \quad S_4'
\]

\[
T_1' \quad T_2' \quad T_3' \quad T_4'
\]

Figure 7: The larger red vertex is the marked leaf.

It is easy to check that \( \forall T \in B_i \) and \( \forall T' \in B_i' \) the NCA-query is equivalent to the NCA-query in \( S_i \) or \( S_i' \) \( \forall i \in \{1, \ldots, 4\} \) resp.