Time evolution of a non-singular primordial black hole

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Abstract

There is growing notion that black holes may not contain curvature singularities (and that indeed nature in general may abhor such spacetime defects). This notion could have implications on our understanding of the evolution of primordial Black holes (PBHs) and possibly on their contribution to cosmic energy. In this paper we discuss the evolution of a non-singular black hole (NSBH) based on a recent model [1]. We begin with a study of the thermodynamic process the black hole in this model and demonstrate the existence of a maximum horizon temperature $T_{\text{max}}$. At this point the specific heat capacity $C$ changes signs to positive and the body loses its black hole characteristics. With no loss of generality, the model is used to discuss the time evolution of a primordial black hole (PBH), through the early radiation era of the universe to present, under the assumption that PBHs are non-singular. In particular, we track the evolution of two benchmark PBHs, namely the one radiating up to the end of the cosmic radiation domination era, and the one stopping to radiate currently, and in each case determine some useful features including the initial mass $m_f$ and the corresponding time of formation $t_f$. We find that along the evolutionary history of the universe the distribution of PBH remnant masses (PBH-RMs) follows a power law. We believe such a result can be a useful step in a study to establish current abundance of PBH-MRs.

1 Introduction

Einstein’s theory of General Relativity (GR) suggests that under rather extreme conditions on their density, matter fields can induce physical (or curvature) singularities on spacetime. Such singularities may be future-directed as is traditionally assumed to be the case following gravitational collapse, modelled by the Schwarzschild black hole [2], or they may be past-directed as in GR-based cosmological solutions [3]. With regard to the internal dynamics of the matter fields associated with it, the formation of a spacetime singularity constitutes a "dead end" (when future directed) and a "dead beginning" (when past directed). Matter just can’t wiggle its way out of a curvature singularity! Thus a "dead beginning", for example, leads to a paradox of how it is that the initial singularity predicted by GR could have given rise to the currently observed cosmic dynamics. Such paradoxes have led to the view that spacetime singularities in GR may simply be a manifestation of the theory’s breakdown at
high energy scales. This view, along with the need to unify GR with Quantum Field Theory (QFT), have led to a search of a quantum theory of gravity through new frameworks like string theory [4] and loop quantum gravity [5], (see also [6] for review). While these new frameworks are very promising, there is still yet no complete theory of quantum gravity available. In the meantime, an idea which has steadily gained popularity suggests that at high enough densities [7] regular matter fields may give way to exotic fields (possibly through phase transitions) which violate some of the traditional energy conditions so that, for example \( \rho + 3p < 0 \). Such exotic fields are usually modelled with positive definite energy density \( \rho \) and hence with a net negative pressure \( p \). It is this net negative pressure which, through its inflationary effects on spacetime, offsets the formation of curvature singularities in various non-singular black a hole models. Understanding and verification of such phase transitions, if at all, must await formulation of a quantum theory of gravity and/or observations.

While fields with exotic characteristics as above have never been observed, their application in physics has not been restricted to black holes par se. For example, exotic fields have severally been invoked to explain observations in cosmology. Such invocations range from an inflaton (sitting on a de Sitter-like potential) initially introduced to explain the horizon problem using the inflationary paradigm [8] to the various dark energy fields lately introduced [9] to explain the currently observed late-time cosmic acceleration [10]. In fact, the practice of invoking exotic fields in physics does date back quite a while. Recall for example Einstein introduced a cosmological constant (see for example [11]) in his field equations in order to keep the universe in static equilibrium from collapse against its own gravity. Later, [12] Sakharov considered a superdense fluid with an equation of state \( p = -\rho \), and Gliner [13] suggested that such a fluid could constitute the final state of gravitational collapse. Bardeen’s solution [14] was the first of non-singular black hole models. Since then several solutions of non-singular black holes have been put forward. Other models have also focused on regular stars without horizons (gravastars), first suggested by Mazur and Mottola [15], as possible end-products of gravitational collapse. A review of various solutions of non-singular spacetimes can be found in [16].

When the end-product of gravitational collapse is a black hole, the spacetime is expected to radiate according to Hawking’s prediction [17]. In the traditional model of a black hole with a curvature singularity the Hawking radiation process is described by a temperature \( T \) that is a monotonically decreasing function of the black hole mass \( T(m) \sim \frac{1}{m} \), leading to an increasingly negative specific heat capacity \( C \) and eventual run-away temperatures. Such a body is persistently out of thermodynamic equilibrium with its surroundings as can be seen (see [20] and citations therein) from its microscopic entropy \( S_{\text{mic}} \) dependence on the (negative) specific heat capacity in \( S_{\text{mic}} = S - \frac{1}{2} \ln C + ... \), with \( S \) being the uncorrected Berkenstein-Hawking entropy. On the other hand when the end-product of gravitational collapse is a non-singular black hole, the spacetime will admit two horizons, the exterior Schwarzschild-like horizon and an interior de-Sitter-like horizon. As such a spacetime radiates, the two horizons approach each other until they eventually merge [18]. At the point this merging happens the temperature has a finite maximum [18,19] that corresponds to a critical mass value. Thereafter the black hole remnant mass (BH-RM) cools with a positive specific heat capacity and is therefore not a black hole since it has no horizon. A review of non-singular black hole radiation process can be found, for example, in Hayward’s work
The possibility that nature may set a (lower bound) length scale so that black holes are non-singular is one that could have implications on our understanding of primordial black holes' (PBHs') evolution and their potential impact on the energy of the universe. PBHs were predicted several decades ago [21, 22] as topological defects that formed in the early universe. There exists, in literature, an abundance of mechanisms through which PBHs could have formed. Such mechanisms include, initial density inhomogeneities, non-linear metric perturbations, blue spectra of density fluctuations, equation of state softening, supermassive particles and/or scalar field dominance and evolution of gravitationally bound objects. A review of these processes can be found in [23] and citations therein. Note that virtually all these mechanisms for forming PBHs were specially effective in the very early universe. As it turns out, PBH formation in the early universe is constrained within a small time window by the effects of inflation, on the one hand, and on the other by the requirement to not disrupt nucleosynthesis through high energy particle emission from Hawking radiation [23]. There are other constraints on, for example, the PBH density from cosmic rays [24]. Further, causality constraints require that at the time of formation a PBH is will be no larger than the contemporary horizon size. Consequently, only small size PBHs could form.

In the traditional picture of a singularity containing black hole, there could currently be several different kinds of relics as signatures of PBH [23]. Thus while PBHs of masses \( m < 10^{15}g \) would have evaporated away, they would have left behind radiation in form of photons, stable and/or unstable particles and even naked singularities. Those PBHs with initial masses \( m > 10^{15}g \) would still be around. It is also possible that some PBHs could have survived to seed galaxy formation [25]. On the other hand, if black holes in general, and PBHs in particular, are non-singular then their evolutionary path could be somewhat different from the foregoing traditional view. As we verify, during the evaporation of a non-singular primordial black hole (NSPBH), the temperature \( T(m) \) will evolve to a finite maximum value at some critical non-zero mass. This (remnant) mass (which as we show is not a black hole) will then cool down to eventually attain thermodynamic equilibrium with the surrounding universe. In this case it follows that each and every PBH that ever formed would leave behind a non-singular primordial black hole remnant mass (NSPBH-RM) as a signature of its previous existence. This feature which is generic to NSBH models [26] sets different predictions of the evolutionary end result of PBHs from those due to traditional singular models. Observe that since PBHs formed at different times and by implication with possibly a significant spread in their formation mass spectrum, their radiation life-times will vary so that the history of the universe should be littered with creation-events of NSPBH-RMs as end-products of the NSPBHs. One of the aims of this paper is to inquire on the rate of primordial black hole remnant mass (PBH-RM) creation during the time evolution of the universe, as a useful step in the quest towards a count of NSPBH-RMs.

Finally, it is reasonable to wonder whether such remnant masses from NSPBHs could contribute any appreciable component to the total energy of the universe. The notion that PBHs could contribute to dark matter has been raised before (see for example [27]). Considerations for such candidates have, however, largely focused on the various kinds of relics associated with the traditional singularity containing black hole, mentioned above. In this paper the same question is also posed to wonder what contribution, if any, non-singular end
products of PBH evaporation could make to dark matter. Motivated by this interesting (albeit speculative) scenario we inquire into the characteristics of the radiation process of NSPBHs. Without loss of generality, we will utilize a particular model of a non-singular black hole based on the Mbonye-Kazanas (M-K) solution [1]. For completeness the discussion starts with a study of thermodynamic processes of this solution. We then apply the results to PBHs to discuss the PBHs' time evolution during the various eras of the universe, including the radiation era, and the matter dominated era. Planck units are used in the figures.

This paper is arranged as follows. In section 2 an overview of the solution is given. In section 3 we develop the thermodynamic features of the M-K solution. In section 4 the time evolution of the associated PBH is discussed both during the cosmic radiation era and also continuing into the later, (largely empty) matter dominated, universe. Section 5 concludes the paper.

2 Theoretical framework

In this section we give an overview of the M-K non-singular black hole solution [1] which is to be used later to discuss the evolutionary process of a PBH. The model sketched here, which is an exact solution of the Einstein field equations, describes the spacetime of a body that gravitationally collapsed to settle into the final state of a non-singular black hole with spherical symmetry. The line element takes the form,

\[ ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) . \]  

(2.1)

The traditional black hole with a curvature singularity is represented by the Schwarzschild solution, which is a vacuum solution. On the other hand an appropriate non-singular solution must cover both the exterior vacuum region and the interior matter fields region, and must do so in a way that allows for some mechanism to offset the singularity that such fields would otherwise produce. This requires that the various interfaces satisfy matching constraints, namely the Israel conditions [28]. As is known such constraints can be quite difficult to satisfy when applied in modeling non-singular black holes and this can lead to an ill-defined spacetime. Therefore the two main challenges for this model to overcome are (1) how to introduce gravitating matter inside the black hole horizon, and at the same time (2) avoid the creation of a matter-generated curvature singularity.

To deal with these challenges, the M-K model imposes both general and specific conditions on the fields. The general condition is that the energy momentum tensor is that of an anisotropic fluid \( T^2_2 = T^3_3 \neq T^1_1 \) and \( T^1_1 \neq T^0_0 \neq T^2_2 \). The spacetime is thus Petrov Type [II, (II)], where ( ) implies a degeneracy in the eigenvalues of the Weyl tensor. An important feature of the model is the specific condition imposed on the radial pressure \( p_r = T^1_1 \) as a function of the energy density \( \rho = T^0_0 \) through an equation of state \( p_r (\rho) \) that takes the form

\[ p_r (\rho) = \left[ \alpha - (\alpha + 1) \left( \frac{\rho}{\rho_{\max}} \right)^2 \right] \left( \frac{\rho}{\rho_{\max}} \right) \rho. \]  

(2.2)
Here $\rho_{\text{max}}$ corresponds to the maximum density at the core center and $\alpha$ is a parameter, which when constrained to $\alpha = 2.2135$ ensures that the sound speed is not super-luminal. It can be seen that the equation of state above has the required features (see also figure 1). This equation of state is matter-like at low densities and leads to the expected dust-like characteristics $p = 0$ as $\rho \to 0$. At the very high density end, deep in the core, the equation asymptotes to that of a de-Sitter fluid, $p_r = -\rho$. It is this outward pressure of a de-Sitter-like fluid that ultimately offsets the formation of a curvature singularity. The transition from a matter-like equation of state to eventually that of a purely de-Sitter fluid at the core is smooth (represented by a well-behaved function) and appears to suggest the existence of either an intermediary density dependent quintessential field, $p = w(\rho)\rho$, $-1 < w < 0$, or a two (matter/de-Sitter) fluid system with a varying density dependent partial pressure contribution. All in all, it is essential to emphasize that the continuous nature of the pressure function $p_r(\rho(r))$ represented by Eq. 2.2 gives the net (or average) value of the pressure due to all fields at a given relevant 2-surface $r$. Thus for example, the continuous curve AB represents matter that is growing increasingly relativistic, while CD would represent either a varying mixture of relativistic matter fields and a de Sitter field or alternatively represent a radial dependent quintessential field. For details, including how the various interfaces are smoothly matched, and the interior and exterior solutions, one can refer to the discussion in [1].

Using the conditions above to solve the Einstein field equations one obtains an exact solution given by a line element,

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right),$$

(2.3)

where $A(r) = \frac{1}{1 - \frac{2GM}{r}}$. For $r < R$, ($R$ being the size of the total mass $M$) we have (see also [29]) that $B(r) = \exp \left\{ - \int_r^\infty \frac{2G}{r^2} [m(r') + 4\pi r'^3 p(r')] \left[ \frac{1}{1 - \frac{2GM(r')}{r}} \right] \right\} dr'$. For $r \geq R$ the
solution takes the familiar form with \( B(r) = \frac{1}{A(r)} = 1 - \frac{2GM}{r} \).

The tangential pressure \( p_\perp = (T^2_2 = T^3_3) \) is given by

\[
p_\perp = p_r + \frac{r}{2} p_r' + \frac{1}{2} (p_r + \rho) \left[ \frac{Gm(r) + 4\pi Gr^3 p_r}{r - 2Gm(r)} \right],
\]

(2.4)

and is a generalization of the Tolman-Oppenheimer-Volkoff equation [30]. Note that in this model the last term in Eq. 2.4 does not vanish, in general, except at the center where \( p_r = -\rho \). This pressure is a unique feature of the model which in our view is also important. The feature allows the radial equation of state (Eq. 2.2) to be that of a multi-fluid system, consistently representing matter fields in the outskirts and a de-sitter field in the interior core. Note also (from Eqs. 2.2 and 2.4) that at the very center of the black hole \( p_\perp = p_r = -\rho_{\text{max}} \), and the spacetime is exactly de Sitter.

The spacetime model described above is general enough to admit any density function \( \rho = \rho(r) \) that is a decreasing function of the radial coordinate. As a particular solution we have taken [18] a density function of the form \( \rho = \rho_{\text{max}} \exp \left[ -\frac{r^3}{r_g(r_0)^3} \right] \), where \( r_g = 2M = 2 \int_0^\infty \rho_{\text{max}} \exp \left[ -\frac{r^3}{r_g(r_0)^3} \right] r^2 dr \) and \( r_0 = \sqrt{\frac{3}{8\pi G \rho_{\text{max}}}} = \sqrt{\frac{3}{\Lambda}} \). Then the mass \( m(r) \) enclosed by a 2-sphere at the radial coordinate \( r \) is given by \( m(r) = \int_0^r \rho_{\text{max}} \exp \left[ -\frac{r^3}{r_g(r_0)^3} \right] r^2 dr' = M \left[ 1 - \exp \left( -\frac{r^3}{r_g(r_0)^3} \right) \right] \). The entire mass is essentially concentrated in a region of size \( R \simeq (r_0^3 r_g)^{1/3} \). Thus, the precise value of \( R \) depends on \( r_0 \) and hence on the value of \( \rho_{\text{max}} \). In this paper we will usually assume (without proof) the maximum density \( \rho_{\text{max}} \) to be of order of the Planck density \( \rho_{\text{pl}} \) and that \( r_0 \sim l_{\text{pl}} \), where \( l_{\text{pl}} \) is the Planck length.

It is easy to verify [1] that the solution given by the line element in Eq. 2.3 leads to the expected asymptotic solutions. Thus for \( r > R \) the metric is described by the vacuum Schwarzschild solution,

\[
ds^2 = -\left( 1 - \frac{2M}{r} \right) dt^2 + \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 (d\theta + \sin^2 \theta d\varphi)^2.
\]

(2.5)

On the other hand towards the core as \( r \to r_0 \), the spacetime becomes asymptotically de Sitter so that for \( 0 < r \leq r_0 \),

\[
ds^2 = -\left( 1 - \frac{r^2}{r_0^2} \right) dt^2 + \frac{1}{1 - \frac{r^2}{r_0^2}} dr^2 + r^2 (d\theta + \sin^2 \theta d\varphi)^2.
\]

(2.6)

Such a de-Sitter-like core produces an outward pressure \( p = -\rho_{\text{max}} \) which in turn intervenes against the formation of a curvature singularity inside the black hole. The result is a non-singular black hole (NSBH).

### 3 Thermodynamic features

The spacetime depicted in the solution (Eq. 2.3) has 2 horizons: an exterior Schwarzschild horizon \( r_+ \) and an interior de-Sitter-like horizon \( r_- \). In the neighborhood of each of the horizons, the line element can be approximated to
\[ ds^2 = - \left( 1 - \frac{\chi(r)}{r} \right) dt^2 + \frac{1}{\left( 1 - \frac{\chi(r)}{r} \right)} dr^2 + r^2 \left( d\theta + \sin^2 \theta d\varphi^2 \right), \]  

where \( \chi(r) = 2m(r) = r_g \left[ 1 - \exp \left( -\frac{r^3}{r_g(r_0)^2} \right) \right] \). The horizons are given by \( r_+ = \chi_{r\to r_g} \) and \( r_- = \chi_{r\to r_0} \).

### 3.1 Hawking temperature

The Hawking temperature \([17]\) on each of the two horizons is given by

\[ T = \hbar \kappa \pm (2\pi kc)^{-1}. \]

Here \( \kappa_+ \) and \( \kappa_- \) are the surface gravity values on the outer and inner horizons, respectively. To write down the explicit Hawking temperatures for the spacetime, one first needs an expression for the surface gravity value(s) \( \kappa \). To this end it is convenient to transform Eq. 3.1 (at the horizons) to Eddington-Finkelstein coordinates so that

\[ ds^2 = - \left( 1 - \frac{\chi(r)}{r} \right) dv^2 + 2dvdr^2 + r^2 \left( d\theta + \sin^2 \theta d\varphi^2 \right), \]  

where \( v \) is the advanced time \( v = t + r + 2m \ln |r - 2m| \). The surface gravity \( \kappa \) is then obtainable from the relation \( l^a \nabla_a \kappa^b = \kappa l^b \) evaluated on the relevant horizon, where \( l^a \) is the time-translation killing vector \( l^a \partial_a = \frac{\partial}{\partial v} \). Taking \( l^a = [1, 0, 0, 0] \), \( l_a = -\left[ 1 - \frac{\chi(r)}{r}, 1, 0, 0 \right] \) one finds that

\[ -\frac{1}{2} \frac{\partial}{\partial v} \left( -1 + \frac{\chi(r)}{r} \right) = \kappa, \]

giving

\[ \kappa_{\pm} = \frac{1}{2} \left[ \frac{\chi}{r_{\pm}} - \frac{\chi'}{r_{\pm}} \right]. \]  

Eqs. 3.3 applied on \( T = \hbar \kappa_{\pm} (2\pi kc)^{-1} \) gives the Hawking temperature \( T_\pm = \hbar \kappa_{\pm} (2\pi kc)^{-1} \) on the respective horizons as

\[ T_- = \hbar (4\pi kr_0)^{-1} \left[ \frac{r_0}{2m} \left( 1 - e^{-4\left( \frac{m}{r_0} \right)^2} \right) - \frac{m}{r_0} e^{-4\left( \frac{m}{r_0} \right)^2} \right] \]  

and

\[ T_+ = \hbar (4\pi kr_0)^{-1} \left[ \frac{2m}{r_0} - \left( 3 + \frac{2m}{r_0} \right) e^{-\left( \frac{m}{r_0} \right)} \right]. \]

The dependence of the (outer/inner) horizon temperature \( T(m) \) on the mass in this model is displayed in Figures 2 and 3, respectively. From an observer’s vantage point only the temperature \( T_+ \) of the external horizon will give readily observable effects. In Figure 2 the solid line shows the evaporation curve of the non-singular black hole while the dashed line (added for comparison) shows the evaporation curve of the traditional singularity-containing black hole. From Figure 2, one can infer that for a large part of the mass parameter space (down to \( \frac{M}{r_0} \gtrsim 1.5 \)) the temperature follows that of the ”traditional” singularity-containing black hole. In this region the specific heat capacity \( C \sim \left( \frac{\partial T}{\partial m} \right)^{-1} \) is (as expected) negative.
Figure 2: Hawking Temperature $T_+$ on the outer horizon as a function of the mass. There is a maximum temperature $T_{\text{max}}$ corresponding to a mass $M_{\text{cr}1}$. Below this mass the system has a positive specific heat capacity (compare with singular bh, dashed line).

Figure 3: Hawking Temperature $T_-$ on the inner horizon. This is hidden from the external observer.
and the black hole, which is out of equilibrium with its environment, gets hotter as it sheds energy.

Eventually the temperature curve begins to depart from that of a black hole and at some critical mass value \( m_{cr1} \) it reaches a maximum \( T_{max} \). In our model this critical mass value is estimated as \( m_{cr1} \approx 1.13r_0 \). Here the specific heat capacity \( C \) vanishes and the microscopic entropy \( S_{mic} = S - \frac{1}{2} \ln C + ... \) is no longer well defined. This point defines a transition in the thermodynamic behavior of the body. Thereafter the object follows a cooling curve that takes it towards thermodynamic equilibrium with its environment.

The changes in thermodynamic behavior depicted in Figure 2 are intimately linked with ongoing changes in gravitational behavior. As the body mass evaporates towards \( m_{cr1} \), the two horizons \( r_+ \) and \( r_- \) approach each other. At \( m_{cr1} \) the two horizons which have opposite curvatures merge [18], weakening each others’ curvatures in the process and leaving a body with no horizon. Specifically, while the interior of this remnant mass contains a de Sitter vacuum, the exterior constitutes matter fields. The repulsive nature of the interior de Sitter vacuum balances with the attractive nature of the outside gravitating matter fields to produce a gravitationally stable object. Such an object is still gravitationally bound, akin to the G-lump first suggested by Coleman [31] and later discussed by Dymnikova [18]. In fact, to the extent that the three treatments produce asymptotically de Sitter interior spacetimes the G-lump and remnant mass should essentially describe the same object. The only difference then lies in how each of the treatments approaches this asymptote. In our treatment the pressure of the fields is initially anisotropic. The stability of the spacetime due to such a G-lump with an asymptotically de Sitter interior has previously been discussed [32].

One can reasonably speculate that at about the point of transition of the black hole to the black hole remnant \( m_{cr1} \) the matter density \( \rho_{cr} \) and temperature \( T_{max} \) are high enough that symmetry restorations of some of the fundamental forces (at least the electroweak one) has already occurred. It is therefore natural to wonder whether this, along with the presumption that the core is de-Sitter-field rich, may observationally obscure the body’s baryonic origins. If so could a non-singular black hole remnant mass (NSBH-RM) be a viable candidate for dark matter?

### 3.2 Time evolution

We now discuss the radiation process of the non-singular black hole in the model, as set in an empty background. An understanding of this radiation process is important for two reasons. First, there is the need for completeness in the evolutionary analysis of the spacetime previously modeled in [1]. Secondly, the analysis will provide a useful framework in modeling the time evolution of a PBH in the early radiation dominated universe, represented by Friedman -Robertson-Walker (FRW) spacetime.

In an empty, asymptotically flat, background the radiation rate of the black hole can be estimated with use of the Stefan-Boltzmann equation,

\[
\frac{dm}{dt} = -\frac{4\pi r_+^2 \sigma T^4}{c^2}.
\]
Here $\sigma$ is the Stefan-Boltzmann constant. Applying the temperature function in Eq. 3.5 leads to a mass loss rate of

$$\frac{dm}{dt} = \frac{-\hbar^4 G^2 \sigma m^2}{16\pi^3 k_B^4 c^{10}} \left[ \frac{c^2}{2Gm} \left( 1 - e^{-\left(\frac{m}{m_c}\right)^2} \right) + \frac{6Gm}{r_0^2 c^2} e^{-\left(\frac{m}{m_c}\right)^2} \right]^4. \quad (3.7)$$

This equation is separable and, as such, one is tempted to seek analytic, albeit approximate solutions. However, in order to connect later with the upcoming analysis of time evolution of PBHs for which the differential equation will not be separable (during the early radiation era of the universe), we find it useful to proceed with a numerical approach.

It is convenient to set Eq. 3.7 in a dimensionless form. This is done by introducing dimensionless quantities $\tilde{m}$ and $\tilde{t}$, respectively given by $m = m_c \tilde{m} = \left(\frac{r_0^2 c^2}{2G}\right) \tilde{m}$ and $t = t_c \tilde{t} = \left(\frac{32\pi^3 h^4 k_B^4}{\hbar^4 G^3}\right) \tilde{t}$. For the purposes of the present calculation, and as previously mentioned, we will take $r_0 \sim l_{pl}$. With these transformations, Eq. 3.7 takes the form

$$\frac{d\tilde{m}}{d\tilde{t}} = -\tilde{m}^2 \left[ \frac{1}{\tilde{m}} \left( 1 - e^{-\tilde{m}^2} \right) - 3\tilde{m} e^{-\tilde{m}^2} \right]^4. \quad (3.8)$$

Figure 4 shows the results of integrating Eq. 3.8. for four different initial black hole masses. It is seen that the time evolution of each of the black hole masses ends with a non-radiating mass remnant $m_r$. This remnant mass, which is expectedly independent of the initial black hole mass, corresponds to $m_{cr1}$ in the earlier discussion for $T(m)$. Note that the time evolution of the two types of black holes is virtually identical, for the most part. However, at the very end the two models become different when, unlike in the case for NSBH, the radiation evolution in the singularity-containing black hole ends in an explosive process with run-away temperatures.

## 4 A non-singular primordial black hole (NSPBH)

It is usually believed that several processes existed [23] in the early universe that could have led to the formation of primordial black holes (PBHs) [21,22]. At the time of formation $t_f$ a PBH is constrained by causality to be no larger than the contemporary horizon size of the universe $a_h(t_f)$. As a result such black holes were generally small and hot. On the other hand, the reheating period at the end of cosmic inflation is known to have produced a large amount of radiation which was to dominate the energy density of the universe for about $3 \times 10^5$ years. This cosmic radiation domination era (RDE) does affect both the formation and time evolution of PBHs in the following respects. First, during the RDE the dynamics of the universe and hence the evolution of the cosmic horizon size $a_h$ is determined (largely) by the available cosmic background radiation energy (CBR) density $\rho_{rad}$. In turn, as already pointed out, the horizon size will set the upper limit of the PBH formation mass $m_f$ at the formation time $t_f$ through the causality constraint. Secondly, because the PBH is bathed in the CBR it will, from its formation time, accrete this energy.

It follows then that the evolution of a PBH in the early universe is driven by two competing processes: the loss of mass through Hawking radiation and the gain of mass through...
accretion of the CBR. Consequently, a realistic model of the time evolution of a PBH must take into account the two competing effects. This implies that the PBH time evolution will generally take the form,

\[
\left(\frac{dm}{dt}\right)_{PBH} = \left(\frac{dm}{dt}\right)_{Hawking} + \left(\frac{dm}{dt}\right)_{accretion} \tag{4.1}
\]

Where \(\left(\frac{dm}{dt}\right)_{Hawking} < 0\) and \(\left(\frac{dm}{dt}\right)_{accretion} > 0\). Classically, the accretion term is given by

\[
\left(\frac{dm}{dt}\right)_{accretion} = \sigma_g f_{rad}, \tag{4.2}
\]

where \(\sigma_g = \frac{27\pi}{4} r_g^2\) is the gravitational accretion cross-section and \(f_{rad} = c\rho_{rad}\) is the accreting radiation flux. With this Eq. 4.1 can be written as

\[
\left(\frac{dm}{dt}\right)_{PBH} = f(m) + \frac{27\pi G^2}{c^3} \rho_{rad} m^2, \tag{4.3}
\]

where \(f(m) = \left(\frac{dm}{dt}\right)_{Hawking}\) is the Hawking term in Eq. 3.7.

It is appropriate here to restate the motivating factors for this work. When one takes the traditional assumption that black holes are singular, one infers that most of such PBHs \((m < 10^{15}g)\) would have evaporated away by now, to leave behind radiation and probably naked singularities. On the other hand, if black holes are non-singular so that each PBH leaves behind a finite mass remnant \(m_r\) at the end of its radiation process, then all PBH remnants (PBH-RMs) should still be abundant. In the remaining part of this paper we solve Eq. 4.3 and discuss the results and implications subject to the view that PBHs are non-singular objects.

### 4.1 Background geometry

Two pieces of information will be needed to solve Eq. 4.3. First, in order to construct the accretion term \(\left(\frac{dm}{dt}\right)_{accretion}\) explicitly, one needs the functional dependence of the radiation density on time, \(\rho_{rad}(t)\). Secondly, to perform the integration one must set up initial
conditions in the form of the formation mass $m_f$ as a function of the formation time $t_f$. As we show below, these two pieces of information are non-trivially linked with the black hole background environment and hence with the background geometry of the contemporary universe. We briefly review this geometry, which is to be utilized, based on the Friedman-Robertson-Walker (FRW) model and then utilize it.

In the FRW model, the universe is dynamic and its evolution obeys the Einstein field equations. Under the assumption that the universe is homogeneous and isotropic, the Einstein equations reduce to the Friedman equations,

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8 \pi G}{3} \rho - \frac{k}{a^2},
\]

\[
\frac{\ddot{a}}{a} = -\frac{4 \pi G}{3} \left[ \rho + \frac{3p}{c^2} \right], \tag{4.4}
\]

where $a(t)$ is the scale factor, $\rho$ and $p$ are respectively the density and pressure of the fields, and $k = [1, 0, -1]$ is the spacetime curvature parameter. The early universe is characterized by a nearly flat ($k = 0$) radiation dominated spacetime. On solving the Friedman equations under these conditions one infers that during the RDE, when $p = \frac{1}{3} c^2 \rho$, the scale factor evolves as $a(t) = a_0 t^{\frac{k}{2}} = t^{\frac{1}{2}}$, where we take as the current value $a_0 = 1$. It follows then that the radiation density $\rho_{rad} \sim a^{-4}$ evolves as $\rho_{rad} \sim t^{-2}$. More explicitly $\rho_{rad} = \frac{\alpha}{(t/s)^2} \text{gcm}^{-3}$, where $\alpha \sim 8.4 \times 10^4$.

### 4.2 NSPBH time evolution

With $\rho_{rad} \sim \frac{\alpha}{t^2}$ Eq. 4.3 can now be written as

\[
\frac{dm}{dt} = f(m) + \frac{27 \pi G^2 \alpha}{c^5} \left( \frac{m}{t} \right)^2. \tag{4.5}
\]
On using the same transformations previously used in Eq. 3.8, a dimensionless form of Eq. 4.5 takes the form

\[ \frac{d\tilde{m}}{dt} = -\tilde{m}^2 \left[ \frac{1}{\tilde{m}} \left( 1 - e^{-\tilde{m}^2} \right) - 3\tilde{m}e^{-\tilde{m}^2} \right]^4 + 4.031 \times 10^{-5} \left( \frac{\tilde{m}}{t} \right)^2. \] (4.6)

This equation is clearly not separable and is best solved numerically. To this end we first set up initial conditions. Recall causality constraints imply that at its formation time \( t_f \) a PBH can not be larger than the horizon size \( a_h(t) \), defined as \( a_h(t) = a(t) \lim_{t \to t_f} \int_{t_f}^{t} \frac{dt'}{a(t')} \). During the radiation era when \( a(t) \sim t^{\frac{1}{2}} \), the horizon is given by \( a_h(t) = 2ct \). The total mass \( M_h \) of the radiation energy contained in the horizon can then be estimated as \( M_h \approx \frac{4}{3} \pi (2ct)^3 \rho_{\text{rad}} = \frac{32}{3} \pi \alpha c^3 t \). Based on this we will take the upper bound of the PBH formation mass \( m_f \) as

\[ m_f = M_h \approx \frac{32}{3} \pi \alpha c^3 t_f. \] (4.7)

Eq. 4.7 provides the initial conditions required to solve Eq. 4.7. Note the linear relationship between each initial PBH mass and the time of its formation.

As a working example, Figure 6 shows the time evolution of a 1g NSPBH in the radiation era (solid upper curve). For perspective this is compared with same initial mass black hole in an empty universe (lower curve). In the model the growth of the black hole due to the external radiation is initially fast but not instantaneous (as is seen from a magnification of that part of the curve). The end of the radiation process is not instantaneous, either (see also figure 3).

In Figure 7 we display the time evolution of a NSPBH that radiates up to the end of the universe’s radiation era. In Figure 8 we display the time evolution of a NSPBH that would be forming a NSPBH-RM now. From the plots we can infer the following benchmarks: the initial mass \( m_f \) and time of formation \( t_f \) (1) for the PBH that stops radiating to the end of the radiation era, and (2) for the one that stops radiating now. These results are tabulated below.
Further, we find that the total radiation energy \( \delta m \) accreted onto the PBH during the radiation era is not a significant fraction of the initial mass at formation \( m_f \), being of the order of \( \delta m \approx 0.04 m_f \). This is consistent with previous results for singular PBH evolution and evaporation [33].

It is also worthwhile to investigate on the rate at which PBHs die, as a function of their initial mass \( m_f \). This information is particularly useful in a non-singular model such as under discussion which considers the end-product (NSPBH-RM) to be a finite non-black hole mass. One would like to know how such remnant mass formation events were distributed in time as the universe evolved. Since each and every PBH that ever existed would have left behind its remnant mass, it is expected that an integration over such a mass distribution would lead

\[
\begin{align*}
\text{Initial mass } m_f \text{ (gm)} & \quad \text{Time of form } t_f/s & \quad \text{PBH - RM form time } t_r/s \\
5.07 \times 10^{12} & \quad 1.52 \times 10^{-25} & \quad 1.26 \times 10^{13} \\
1.65 \times 10^{14} & \quad 4.95 \times 10^{-24} & \quad 4.32 \times 10^{17}
\end{align*}
\]
Figure 9: Remnant mass formation time $\Delta t_r (m_f)$ as a function of the formation mass $m_f$ for a spectrum of NSBHs on the large $m_f$ end. The power law behavior is same as seen in the low mass end.

Figure 10: Time interval $\Delta t_r = t_r - t_f$ taken to form a black hole remnant mass (NSPBH-RM) as a function of the initial formation mass $m_f$ for a spectrum of low masses.

to a useful description of the current NSPBH-RM abundances. In turn, modeling NSPBH-RM abundances is a necessary step in establishing whether NSPBH-RMs could constitute a significant component of the total energy in the universe. Figure 9 shows the time interval $\Delta t = t_f - t_r$ taken for a given PBH to evaporate and form a NSBH-RM as a function of the initial mass $m_f$, for a spectrum of low masses. We find that this dependence is a power law in the initial mass, $\Delta t \sim m_f^\gamma$, where $\gamma \approx 3$. Further, we have verified (see Figure 10) that this same power law behavior persists even for larger initial masses.

Time interval $\Delta t_r = t_r - t_f$ taken to form a black hole remnant mass (NSPBH-RM) as a function of the initial formation mass $m_f$ for a spectrum of low masses.
5 Conclusion

In this paper we have discussed the thermodynamic features and time evolution of a non-singular black hole (NSBH) based on the Mbonye-Kazanas solution. The spacetime initially radiates with an increasing temperature driven by a negative specific heat capacity, reminiscent of a traditional black hole with a singularity. Eventually, however, the temperature maximizes to $T_{\text{max}}$ at which point the specific heat capacity $C$ drops to zero and the entropy $S_{\text{mic}} = S - \frac{1}{2} \ln C + \ldots$ of the spacetime is not well defined. It is here that the spacetime loses its black hole characteristics, including its horizon. Thereafter the specific heat capacity changes sign becoming positive and the body cools down as a regular body. We have discussed the time evolution of this spacetime. Again, here it is shown that the spacetime radiates to leave a remnant mass (NSBH-RM).

With no loss of generality, we have used the model to investigate the radiation process of a primordial black hole (PBH), under the assumption that such a black hole is non-singular. Within this framework, we constructed a differential equation governing the time evolution of a NSPBH through the radiation era of the early universe. The equation which is not separable was integrated numerically to study the time evolution of a PBH. In particular, we have tracked the evolution of two benchmark PBHs. These include the PBH radiating up to the end of the radiation domination era and the one stopping to radiate now. We determined the formation mass $m_f$ and the corresponding formation time $t_f$ for each. It is found that the total accreted CBR does not constitute a significant fraction of the initial PBH mass $m_f$, being of the order of $0.04m_f$.

Finally, we investigated the rate at which PBHs die, as a function of their initial mass $m_f$. It is found that the rate of primordial black hole remnant mass (PBH-RM) creation during the time evolution of the universe, follows a power law of the $\Delta t \sim m_f^\gamma$, where $\gamma \approx 3$. We believe this result to be a useful spring-point for discussing current PBH-RM abundances, based on the presumption that black holes are non-singular. In turn, modeling NSPBH-MR abundances is a necessary step in establishing whether PBH-remnants could constitute a significant component of the energy in the universe.

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