A note on linearized stability of Schwarzschild thin-shell wormholes with variable equations of state

Victor Varela
Av. das Acacias 540, Rio de Janeiro-RJ 22776-000, Brasil

Abstract

Linearized stability of thin-shell wormholes with a phantom-like equation of state (EoS) was analyzed by Kuhfittig. As a result, thin-shell wormholes constructed with two copies of Schwarzschild spacetime turned out to be unstable. We search for variable EoS and introduce coefficients explicitly dependent on throat radius. The arising models exhibit substantially improved stability properties. In the case with regular coefficients, a sequence of semi-infinite stability regions is found such that every throat in equilibrium becomes stable for a particular subsequence. In another situation, involving a singular EoS, the second derivative of the effective potential is positive definite, so linearized stability is guaranteed for every equilibrium radius. We briefly reconsider the determination of sound velocity in throat fluids, and examine the potential relevance of non-linear, variable EoS to thin-shell wormhole stability.

1 Introduction

Garcia et al. [1] have summarized the status of the energy conditions of standard general relativity in connection with the theoretical construction of wormholes. These hypothetical tunnels in spacetime are supported by "exotic matter" which violates all the pointwise and all the averaged energy conditions. To minimize the extent of this problem, researchers have considered various approaches including stationary and axially symmetric wormholes, dynamic wormholes, non-linear electrodynamics, $f(R)$ gravity, conformally coupled scalar fields, and braneworlds. Also, procedures to alleviate these violations based on the thin-shell formalism [2] and the cut-and-paste technique [3] have been proposed.

In the context of spherical symmetry, two identical sections of Schwarzschild’s spacetime, excluding horizons and black hole interiors, can be glued to each other. As a consequence, non-zero stresses associated with exotic matter concentrate on the boundary layer between the joined spacetime regions. This hypersurface constitutes the throat of the thin-shell wormhole. Poisson and Visser [4] analyzed the linearized stability of this type of wormhole under perturbations preserving the symmetry, without assuming specific EoS for the exotic matter. A number of papers have extended this approach to include charge and/or cosmological constant, as well as other features (see, for example, references [29-32] in Garcia et al.’s paper [1]).

1Email: victor.varela.abdn@gmail.com
2Previous address: Institute of Mathematics, University of Aberdeen, King’s College, Aberdeen AB24 3UE, UK.
Attempts to reduce exotic matter usage, as well as the pursuit of cosmological contexts for wormhole solutions have motivated stability analyses with specific barotropic EoS (in which pressure depends exclusively on energy density). Lobo [5] constructed Morris-Thorne-type wormholes supported by exotic matter in the form of generalized Chaplygin gas [6]. The new wormholes were designed such that the exotic matter is confined in a finite space region around the throat. These structures may arise from density fluctuations in the cosmological background. Eiroa and Simeone [7] simplified Lobo’s model introducing a thin-shell throat, where all of the exotic matter is located. Selecting the Chaplygin EoS [8], and using techniques of dynamical systems, these authors determined the instability of the Schwarzschild thin-shell wormhole under small perturbations which preserve the symmetry. The case of linear EoS was discussed in [9]. Based on Eiroa and Simeone’s approach, Bandyopodhyay et al. [10] showed that throats supported by the modified Chaplygin gas are conditionally stable in the case of Schwarzschild wormholes. Furthermore, Kuhfittig [11] solved the energy conservation equation for spherically-symmetric thin-shell wormholes with phantom-like EoS, and considered slight perturbations of the thin-shell potential. This approach also led to the instability of the Schwarzschild thin-shell wormhole.

The barotropic EoS is a rather strong assumption usually made in studies of thin-shell dynamics. Garcia et al. [1] have argued for local thin-shell wormhole solutions with variable EoS, in which surface pressure explicitly depends on both surface energy density and throat radius.

Cosmological models based on variable EoS (with explicit dependence on the scale factor) have been already discussed in the literature. A simple relativistic model of a 3D gas of massive particles leading to non-linear, variable EoS was proposed by de Berredo-Peixoto et al. [12]. Guo and Zhang [13] substituted the proportionality constant of the original Chaplygin EoS by a function of the scale factor. Debnath [14] introduced a variable coefficient in the modified Chaplygin EoS proposed by Benaoum [15]. More recently, a cosmological model based on a pair of linear, variable EoS has been proposed by Ponce de León [16]. On the other hand, Rahaman et al. [17] and Kuhfittig [18] have constructed Morris-Thorne-type wormholes supported by fluids satisfying variable EoS, in which volumetric pressure depends explicitly on volumetric energy density and spacetime coordinates [19]. At this point, it seems natural to address the linearized stability of thin-shell wormholes with variable EoS. The existence of local solutions pointed out in [1] is essential for the construction of the new models.

In this note we start from Kuhfittig’s treatment of energy conservation, linear EoS, and thin-shell potentials. However, instead of improving stability through the inclusion of charge or cosmological constant, we discuss the effects of variable EoS on the stability of Schwarzschild wormholes. Specifically, we present a family of EoS depending explicitly on powers of the throat radius. This feature brings about a sequence of stability regions. Additionally, we propose an EoS whose linear coefficient diverges as the throat approaches the would-be Schwarzschild horizon. Interestingly, this choice entails linearized stability of static wormholes with arbitrary throat radius.

2 Schwarzschild thin-shell wormholes

The basic ingredient for the construction of the thin-shell wormhole is the Schwarzschild metric

\[ ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \] (1)
describing a black-hole spacetime. We take two copies (±) of this manifold and remove from each the corresponding four-dimensional region \( \Omega^\pm = \{ r \leq a | a > 2M \} \). Next, we identify the time-like hypersurfaces \( \partial \Omega^\pm = \{ r = a | a > 2M \} \) and obtain the only one hypersurface \( \partial \Omega \). The arising spacetime manifold is geodesically complete \([3]\). It includes two asymptotically flat regions connected through \( \partial \Omega \), which constitutes the throat of the wormhole.

Israel’s thin shell formalism requires the same induced metric on each side of \( \partial \Omega \). Also, it relates the discontinuity in the extrinsic curvature at the throat to its intrinsic energy-momentum tensor, which describes a two-dimensional perfect fluid.

The derivation of the equations of motion of spherically-symmetric, time-like thin shells with proper time \( \tau \), radius \( a(\tau) \), surface energy density \( \sigma(\tau) \), and surface pressure \( p(\tau) \) is a standard application of Israel’s formalism. The reader is referred to \([20]\) for a pedagogical presentation. A general discussion of thin-shell wormhole construction methods is presented in \([1]\).

We choose units \( G = c = 1 \). The wormhole equations of motion take the form

\[
\sigma = -\frac{1}{2\pi a} \sqrt{1 - 2M/a + \dot{a}^2},
\]

\[
p = \frac{1}{4\pi a} \frac{1 - M/a + \dot{a}^2 + a\ddot{a}}{\sqrt{1 - 2M/a + \dot{a}^2}},
\]

where overdots denote derivatives with respect to \( \tau \).

Static equilibrium is defined by \( \dot{a} = \ddot{a} = 0 \). The corresponding energy density and pressure are given by

\[
\sigma_0 = -\frac{1}{2\pi a_0} \sqrt{1 - 2M/a_0},
\]

\[
p_0 = \frac{1}{4\pi a_0} \frac{1 - M/a_0}{\sqrt{1 - 2M/a_0}},
\]

where \( a_0 \) is the throat’s radius at equilibrium. We see that \( \sigma_0 \to 0 \) and \( p_0 \to +\infty \) as \( a_0 \to 2M \). Only static throat radii satisfying \( a_0 > 2M \) are physically meaningful.

Equations \([2]\) and \([3]\) entail the energy conservation law

\[
\dot{\sigma} + \frac{2}{a} (\sigma + p) \dot{a} = 0.
\]

Assuming the generic barotropic EoS \( p = p(\sigma) \) this differential equation leads to

\[
\int_{\sigma_0}^{\sigma} \frac{d\varsigma}{\varsigma + p(\varsigma)} = -2 \ln \left( \frac{a}{a_0} \right),
\]

where \( a_0 = a(\sigma_0) \). Thus in principle, we obtain the function \( a = a(\sigma) \) which implies \( \sigma = \sigma(a) \) over suitable domains. This inverse function necessarily satisfies

\[
\sigma' = -\frac{2}{a} [\sigma + p(\sigma)],
\]

where prime denotes differentiation with respect to \( a \).

On the other hand, to discuss the integrability of \([4]\) with variable EoS \( p = p(a, \sigma) \), we impose the restriction \( \sigma = \sigma(a) \) from the outset and obtain

\[
\sigma' = -\frac{2}{a} [\sigma + p(a, \sigma)].
\]
Proofs of existence of local solutions for this type of differential equation are available (see, for example, [21]). The existence of local solutions is sufficient for the purpose of the present stability analysis, which linearizes the dynamics around static solutions [1].

Equation (2) can be recast as

\[ \dot{a}^2 + V(a) = 0, \]  

where the effective thin-shell potential is given by

\[ V(a) = 1 - 2M/a - [2\pi a\sigma(a)]^2. \]  

3 Linearized stability analysis

Equation (10) implies that \( V(a) \) is not the type of external potential usually found in classical mechanics. The reason being that the "total energy" vanishes identically. Accordingly, every perturbation of the kinetic energy term in (10) will be compensated with a perturbation of the potential. Particularly, if the thin shell is in static equilibrium at \( a = a_0 \) when the perturbation occurs, the perturbed value of \( V(a_0) \) is necessarily negative.

Solutions of (9) depend on one integration constant. Assuming that the throat is in static equilibrium at \( a = a_0 \), this constant is chosen such that \( \sigma(a_0) \) equals the energy density given by (4). Thus, each static throat is characterized by a specific local solution \( \sigma = \sigma(a) \) which allows the evaluation of \( V = V(a) \) in a neighborhood of \( a = a_0 \) before perturbation. This thin-shell potential is locally described with the expansion

\[ V(a) = V(a_0) + V'(a_0)(a - a_0) + \frac{1}{2}V''(a_0)(a - a_0)^2 + O\left[(a - a_0)^3\right]. \]  

Linearized stability analysis involves the truncation of this series to second order.

As a consequence of (10), static equilibrium implies

\[ V(a_0) = 0. \]  

Besides, we impose the linearized equilibrium condition

\[ V'(a_0) = 0, \]  

and allow \( V''(a_0) \) to be generally non-zero.

The above considerations lead to the truncated expansion

\[ V(a) = \frac{1}{2}V''(a_0)(a - a_0)^2. \]  

We are interested in equilibrium configurations satisfying \( V''(a_0) > 0 \). In this case the unperturbed shape of \( V(a) \) approaches that of a convex parabola when we examine it in a sufficiently small neighborhood of \( a = a_0 \).

We have seen that every spherically symmetric perturbation of static equilibrium leads to a negative value of the perturbed potential at \( a = a_0 \). Assuming that the shape of \( V(a) \) is slightly deformed when the perturbation is very small, we expect the perturbed potential to be definite negative in some neighborhood of \( a = a_0 \). Furthermore, if the perturbation is sufficiently small,
both the sign of $V''(a_0)$ and the approximately parabolic shape of the potential remain unchanged after perturbation. Under these conditions, the deformed potential gets two separate zeros within a sufficiently small neighborhood of $a_0$. If these zeros are located at $a = a_1$ and $a = a_2$ ($a_1 < a_2$), the perturbed potential is negative definite in the interval $(a_1, a_2)$. Therefore, the assumption $V''(a_0) > 0$ entails oscillations between the slightly separated turning points $a_1$ and $a_2$ when the perturbation is sufficiently small. This situation describes linearized thin shell stability against radial perturbations. On the contrary, the case $V''(a_0) < 0$ implies instability.

The truncated expansion (15) is essential to the linearized stability analysis considered here. The present approach is not conclusive when applied to throats in static equilibrium with vanishing $V''(a_0)$.

### 4 Barotropic EoS

Following Kuhfittig [11], we plug the linear, barotropic EoS $p = \omega \sigma$ into (8), solve the arising differential equation, and get the local solution

$$\sigma(a) = \sigma_0\left(\frac{a_0}{a}\right)^{2(\omega+1)},$$

which satisfies the condition $\sigma(a_0) = \sigma_0$. Combining this result with (11) we find the corresponding potential

$$V(a) = 1 - \frac{2M}{a} - 4\pi^2a^2\sigma_0^2\left(\frac{a_0}{a}\right)^{4(\omega+1)}.$$  

Using (4) we see that condition (13) is automatically satisfied. On the other hand, the first derivative of $V(a)$ evaluated at $a = a_0$ vanishes if and only if

$$\omega = -\frac{1}{2}\frac{a_0 - M}{a_0 - 2M}.$$  

This expression can also be derived from (16) combined with (4) and (5).

From (17) we see that the local solution $\sigma(a)$ becomes constant when $\omega = -1$. According to (19) this occurs only at $a_0 = 3M$. Using (4) we find the corresponding (local) energy density function

$$\sigma(a) = \sigma_0|_{a_0=3M} = -\frac{1}{6\sqrt{3}\pi M}.$$  

We can determine the second derivative of $V(a)$ at the stationary point $a = a_0$. Using (4), (18) and (19) we get

$$V''(a_0) = \frac{1}{a_0^2(1-a_0/2M)},$$

which is a further simplified form of equation (14) in [11].

From (19) we see that wormhole throats with equilibrium radii $a_0 > 2M$ get negative $\omega$ values in the range $(-\infty, -1/2)$. Moreover, (21) implies that $V''(a_0)$ is negative definite for $a_0 > 2M$, so every equilibrium configuration associated with (16) is linearly unstable.
Evaluating (21) at $a_0 = 3M$ we get $V''(3M) = -2/9M^2$. This result is shared with a large class of throats satisfying the barotropic EoS $p = p(\sigma)$. To see this, we examine equation (27) in Poisson and Visser’s paper [4], namely

$$V''(a_0) = -2a_0^{-2} \left[ \frac{2M}{a_0} + \frac{M^2/a_0^2}{1 - 2M/a_0} + (1 + 2\beta_0^2) \left( 1 - \frac{3M}{a_0} \right) \right], \quad (22)$$

where

$$\beta_0^2 \equiv \left. \frac{\partial p}{\partial \sigma} \right|_{a=a_0}. \quad (23)$$

It implies that every throat for which $\beta_0^2$ is finite or zero at $a_0 = 3M$ is characterized by $V''(3M) = -2/9M^2$. This result precludes the existence of stability regions including the throat radius $a_0 = 3M$. It also imposes a clear cut separation between possible stability regions defined for $a_0 > 3M$ or $a_0 < 3M$.

5 Variable EoS

Poisson and Visser’s derivation of (22) uses the property

$$p' = \frac{\partial p}{\partial \sigma} \sigma', \quad (24)$$

which is suitable for EoS of the form $p = p(\sigma)$. The present analysis of linearized stability with variable EoS $p = p(\sigma, a)$ requires the more general expression

$$p' = \frac{\partial p}{\partial \sigma} \sigma' + \frac{\partial p}{\partial a}, \quad (25)$$

which takes into account the explicit dependence of pressure on throat radius. Adapting Poisson and Visser’s procedure to the new conformation of $p'$ we find, after some straightforward algebra, that

$$V''(a_0) = -2a_0^{-2} \left[ \frac{2M}{a_0} + \frac{M^2/a_0^2}{1 - 2M/a_0} + (1 + 2\beta_0^2) \left( 1 - \frac{3M}{a_0} \right) \right] + 8\pi \sqrt{1 - \frac{2M}{a_0} \gamma_0}, \quad (26)$$

where $\gamma_0 \equiv -\left. \frac{\partial p}{\partial a} \right|_{a=a_0}$. Assuming that $\beta_0^2$ is bounded at $a_0 = 3M$, we get

$$V''(3M) = -\frac{2}{9M^2} + \frac{8\pi}{\sqrt{3}} \gamma_0 \big|_{a_0=3M}, \quad (27)$$

which depends on the selected EoS through $\gamma_0$.

In this work we investigate linear, variable EoS of the form

$$p = \omega(a) \sigma, \quad (28)$$

where $a$ is thin-shell radius. Our first choice is

$$p = \frac{A}{a^n} \sigma, \quad (29)$$

where $A, n$ are constants, and $n \neq 0$. 

6
Plugging (29) into (9), and transforming the differential equation for $\sigma$ into an equation for $\sigma^2$, we obtain

$$\frac{d}{da} (\sigma^2) + \frac{4}{a} \left( 1 + \frac{A}{a^n} \right) \sigma^2 = 0. (30)$$

This equation admits the local solution

$$\sigma(a)^2 = \sigma_0^2 \left( \frac{a_0}{a} \right)^4 \exp \left[ \frac{4A}{n} \left( 1 - \frac{1}{a_0^n} \right) \right], (31)$$

which satisfies the condition $\sigma^2 = \sigma_0^2$ at the static equilibrium radius $a = a_0$.

Using (31) and (11) we obtain

$$V(a) = 1 - 2M/a - 4\pi^2 a^2 \sigma_0^2 \left( \frac{a_0}{a} \right)^4 \exp \left[ \frac{4A}{n} \left( 1 - \frac{1}{a_0^n} \right) \right], (32)$$

which, combined with (4), features the required property $V(a_0) = 0$.

Differentiating $V(a)$ with respect to $a$ and evaluating the resulting expression at $a = a_0$, we find that $V'(a_0) = 0$ if and only if

$$A = -\frac{1}{2} \frac{a_0^3 M - a_0^{n+1}}{2M - a_0}. (33)$$

This form of $A$ can also be obtained by equating $A/a^n$ to the ratio $p_0/\sigma_0$ derived from (5) and (4).

The equilibrium thin-shell radius $a_0 = 3M$ was distinguished in our analysis of barotropic EoS. To see how (29) diminishes the significance of this particular $a_0$ value, we determine $A$ at $a_0 = 3M$ and get $A|_{a_0=3M} = -(3M)^n$, which is definite negative. This result and the fact that $\sigma_0$ is definite negative for $a_0 > 2M$ entail the positivity of $\gamma_0|_{a=a_0}$ for $n > 0$. Hence the present possibility of $V''(3M)$ being positive, which is excluded in the barotropic case.

Further evidence of the weakened role of $a_0 = 3M$ in the case of variable EoS is provided by the associated local solution $\sigma(a)^2$. Inserting (33) into (31) and evaluating at $a_0 = 3M$ we obtain

$$\sigma(a)^2|_{a_0=3M} = \frac{3M^2 \exp \left[ \frac{4}{n} \left( 1 - \left( \frac{3M}{a} \right)^n \right) \right]}{4\pi^2 a^4}, (34)$$

which is non-constant when $n \neq 0$.

It is worth noting that, in virtue of the property $\ln u \approx u - 1$, valid for $u \approx 1$, equation (34) features the limiting behavior

$$\lim_{n \to 0} \left[ \sigma(a)^2|_{a_0=3M} \right] = \frac{1}{108\pi^2 M^2}. (35)$$

The fact that $\sigma(a)^2|_{a_0=3M}$ approaches a constant function (with this particular value) as $n$ tends to 0 is expected on the basis of (20), which is derived in the framework of linear, barotropic EoS.

Substituting (33) into (32), determining $V''(a)$, evaluating at $a = a_0$, and introducing the dimensionless throat radius $x_0 = a_0/2M$, we get

$$V''(x_0) = \frac{2nx_0^2 - (3n + 1)x_0 + n}{x_0^4 - x_0^3}, (36)$$
where \( V^{(2)}(x_0) \equiv 4M^2 V''(a_0) \). (In the remainder of this note \( a_0 \) and \( x_0 \) are alternatively used to label throat radius at static equilibrium.) The second (dimensionless) derivative of \( V \) takes the approximate form
\[
V^{(2)}(x_0) \approx -\frac{1}{x_0 - 1}
\] (37)
as \( x_0 \to 1 \). This result guarantees the negativity of \( V^{(2)}(x_0) \) near \( x_0 = 1 \). Therefore, linearized stability can be achieved only if the sign of \( V^{(2)}(x_0) \) turns positive at some \( x_0 > 1 \). Hence the importance of classifying the roots of \( V^{(2)}(x_0) = 0 \) for non-vanishing \( n \).

The quadratic equation
\[
2nx_0^2 - (3n + 1)x_0 + n = 0
\] admits the roots
\[
x_0^+(n) = \frac{\sqrt{n^2 + 6n + 1 + 3n + 1}}{4n}, \quad (39)
x_0^-(n) = -\frac{\sqrt{n^2 + 6n + 1 - 3n - 1}}{4n}, \quad (40)
\]
which coincide when \( n^2 + 6n + 1 = 0 \). Hence, repeated roots arise at \( n_1 = -2\frac{3}{2} - 3 \approx -5.83 \) and \( n_2 = 2\frac{3}{2} - 3 \approx -0.17 \), and take the values
\[
x_0^+(n_1) = x_0^-(n_1) = -x_0^+(n_2) = -x_0^-(n_2) = \frac{3\sqrt{2} + 4}{2^{5/2} + 6} \approx 0.71. \quad (41)
\]
Also, \( n^2 + 6n + 1 \) is negative if and only if \( n \in (n_1, n_2) \), so \( x_0^+(n) \) and \( x_0^-(n) \) have non-zero imaginary parts only in this interval.

The three branches of \( x_0^+(n) \) and the two branches of \( x_0^-(n) \) comprising the real roots of (38) are depicted in Figure 1. Branches \( x_{02}(n) \) and \( x_{03}(n) \) meet at \( M_1 \), while \( x_{01}(n) \) and \( x_{02}(n) \) meet at \( M_2 \). Branch \( x_{01}(n) \) does not meet any other branch. Both \( x_{01}(n) \) and \( x_{02}(n) \) become unbounded, with opposite signs, as \( |n| \to 0 \). A straightforward analysis shows that \( x_{02}(n) \) and \( x_{03}(n) \) approach \( x_0 = 1 \) from below and \( x_0 = 1/2 \) from above, respectively, as \( n \to -\infty \). Also, \( x_{01}(n) \) and \( x_{01}(n) \) approach the asymptote \( x_0 = 1 \) from below and above, respectively, as \( n \to +\infty \). Only branch \( x_{01}(n) \), satisfying \( x_{01}(n) > 1 \) for every \( n > 0 \), is physically meaningful and relevant to linearized stability analysis.

Equation (29) completely removes the special character of throat radius \( a_0 = 3M \) (\( x_0 = 3/2 \)) which arises in the barotropic case. Using (36) to determine \( V^{(2)}(3/2) \) as a function of \( n \) we find
\[
V^{(2)}(3/2) = \frac{16n - 24}{27}. \quad (42)
\]
Therefore \( V^{(2)}(3/2) \) is positive whenever \( n > 3/2 \), so static throats with dimensionless radius \( x_0 = 3/2 \) can be linearly stable.

Figure 2 shows graphs of \( V^{(2)}(x_0) \) for different values of \( n \). In agreement with the shape of \( x_{01}(n) \) depicted in Figure 1 the zeros of these functions are shifted toward \( x_0 = 1 \) as \( n \) increases. On the other hand, the decay of these curves as \( x_0 \to +\infty \) is dictated by the asymptotic behavior of (36), namely \( V^{(2)}(x_0) \sim 2n/x_0^2 \). Each of the arising zeros turns out to be the boundary of a semi-infinite stability region. The monotonic stretching of these regions implies that every static throat in equilibrium, characterized by \( x_0 > 1 \), is linearly stable for sufficiently large values of \( n \).
6 Singular EoS

In the cases of barotropic EoS \( p = \omega \sigma \) and variable EoS \( p = (A/a^n)\sigma \), it is found that the second and higher derivatives of \( V \) diverge as the static equilibrium radius approaches \( 2M \). It is worth investigating if some of these irregularities can be mitigated by choosing a different type of variable EoS. We presently explore this possibility in the context of a linear EoS with singular coefficient.

From (5) and (4) we see that the pressure \( p_0 \) and the energy density \( \sigma_0 \) of a throat in equilibrium respectively diverge and tend to zero as \( a_0 \to 2M \). In this limit the pressure-density ratio takes the approximate form

\[
\frac{p_0}{\sigma_0} \approx -\frac{M}{2(a_0 - 2M)},
\]

which is unbounded. This result is totally independent of the selected EoS.

The above limit expression is compatible with the choice

\[
p = \frac{\psi a}{a - 2M} \sigma,
\]

which is another particular case of (28). The constant parameter \( \psi \) is to be determined at each static equilibrium radius. This EoS implies that the dynamical pressure-density ratio \( p/\sigma \) becomes unbounded as \( a \to 2M \). Also, (44) takes the approximate form \( p \approx \psi \sigma \) when \( a \gg 2M \). The simple dependence of \( p \) on \( a \) and \( \sigma \) displayed in (44), as well as the associated limiting behaviors make this EoS an interesting candidate for linearized stability analysis.

Evaluating (44) at \( a = a_0 \) and combining the result with the general expression for \( p_0/\sigma_0 \), derived from (4) and (5), we obtain

\[
\psi = -\frac{1}{2} \left( 1 - \frac{M}{a_0} \right),
\]

which is negative definite when \( a_0 > 2M \). It tends to \(-1/4\) as \( a_0 \to 2M \), and takes the approximate form \( \psi \approx -1/2 \) when \( a_0 \gg 2M \).

The fact that the equilibrium values of \( \omega \) and \( \psi \) tend to the same limit as \( a_0 \) becomes arbitrarily large motivates a comparison of linearized stability properties of (16) and (44). The following analysis shows that the approximate identification of these EoS for \( a_0 \gg 2M \) is misleading, as the variability of (44) dramatically modifies the values of \( V''(a_0) \) in the whole interval \((2M, +\infty)\).

Following a previous procedure, we insert (44) into (9) and find the differential equation

\[
\frac{d}{da} \left( \sigma^2 \right) + \frac{4}{a} \left( 1 + \frac{\psi}{1 - 2M/a} \right) \sigma^2 = 0,
\]

which admits the local solution

\[
\sigma^2 = \sigma_0^2 \left( \frac{a_0}{a} \right)^4 \left( \frac{a_0/2M - 1}{a/2M - 1} \right)^{4\psi}.
\]

The above expression fulfills the condition \( \sigma^2 = \sigma_0^2 \) at the equilibrium radius \( a = a_0 \).

Using (47) and (11) we get

\[
V(a) = 1 - 2M/a - 4\pi^2 \sigma_0^2 \left( \frac{a_0^4}{a^4} \right) \left( \frac{a_0/2M - 1}{a/2M - 1} \right)^{4\psi}.
\]
Combined with (4), this expression satisfies the equilibrium condition $V(a_0) = 0$. On the other hand, we find that $V'(a_0)$ vanishes if and only if $\psi$ is given by (45). Using this result we evaluate $V''(a_0)$ and obtain

$$V''(a_0) = \frac{2M}{a_0^3},$$

which is positive definite on the interval $(2M, +\infty)$ and remains finite as the throat equilibrium radius $a_0$ approaches the Schwarzschild radius. The values of the higher derivatives $V^{(i)}(a)$ ($i = 3, 4, 5, ...$) of (48) at $a = a_0$ can also be determined. These quantities become unbounded as $a_0 \to 2M$. Therefore, only $V''(a_0)$ is well-behaved in this limit as a consequence of (44).

7 Discussion

We have proposed two types of variable EoS leading to linearly stable Schwarzschild thin-shell wormholes. In the case of EoS (29) we find semi-infinite stability regions determined by positive values of $V^{(2)}(x_0)$. The corresponding boundaries get arbitrarily close to the would-be horizon ($x_0 = 1$) as $n \to +\infty$. It turns out that $V^{(2)}$ and the higher derivatives of $V$ evaluated at $x_0$ become unbounded as $x_0 \to 1$. The case of EoS (44) is substantially different since $V''(a_0)$ is definite positive throughout the entire interval $(2M, +\infty)$, and remains bounded as $a_0 \to 2M$. However, the higher derivatives of $V$ also diverge as $a_0 \to 2M$.

To identify linearized stability regions we have employed standard criteria [11] and implicitly assumed that 1) the truncated expansion (15) is applicable at every $a_0 \in (2M, +\infty)$; 2) the approximately parabolic (convex) shape of $V(a)$ in a neighborhood of $a_0$ remains undistorted under very small perturbations, regardless of the value of $a_0 \in (2M, +\infty)$. Remarkably, these assumptions have been used despite the divergent derivatives of $V$ as $a_0 \to 2M$.

To go beyond the assumption of thin shell potentials with approximately parabolic shapes undergoing small deformations, we need to undertake non-linear stability analysis. The possibility that certain types of thin-shell throats are stable under finite kinetic energy perturbations offers interesting perspectives. Particularly, the question of whether EoS (44) also implies non-linear stability at every throat radius $a_0 > 2M$ is appealing and should motivate further research work.

Poisson and Visser [4] pointed out a puzzling feature of Schwarzschild thin-shell wormholes. They found that in cases of throat fluids satisfying barotropic EoS the parameter $\beta_0^2$, defined in (23), is negative definite for linearly stable throats with radii $a_0 > 3M$. This result complicates the physical interpretation of the solution since $\beta_0^2$ is generally identified with squared sound speed, which is assumed to take values in the interval $[0, 1)$. Poisson and Visser indicated that the same anomaly arises in studies of Casimir and false vacua. These authors suggested that the adequate interpretation of $\beta_0^2$ would require a microphysical model of the exotic throat fluid.

The analysis of sound speed in 2D fluids with variable EoS is beyond the scope of this work. However, we expect that the use of variable EoS entails new results for this classical parameter. Various papers discuss squared sound speed in the context of cosmological models with variable EoS (see, for example, [15] and [22]). It would be interesting to elaborate thermodynamical models of spherically-symmetric thin-shell fluids with variable EoS leading to specific sound speed formulae [23], and see if the interpretation problem of $\beta_0^2$ is alleviated in the case of exotic fluids restricted to wormhole throats.

The Morris-Thorne-type wormhole discussed by Lobo [5] involves a finite volume layer of exotic fluid satisfying the Chaplygin EoS. This accumulation of fluid could arise from a density fluctuation...
in the cosmological background. Similarly, in the wormhole model of Eiroa and Simeone [7] the Chaplygin gas concentrates in a thin-shell. On the other hand, the cosmological justification of thin-shell throats with variable EoS seems to require a link between throat radius and background scale factor [24]. Without understanding how the thin-shell wormhole inherits a variable EoS depending on throat radius from the cosmological background, our motivations for picking specific types of variable EoS for throat fluids should be drawn from other sources.

The cosmological EoS derived by de Berredo-Peixoto et al. [12] is a particular case of the variable modified Chaplygin gas EoS subsequently proposed by Debnath [14], namely

$$P = A\rho - \frac{B(a)}{\rho^\alpha},$$

(50)

where $P$ and $\rho$ are volumetric pressure and volumetric energy density, respectively; $a$ is the scale factor; $A > 0$ and $\alpha \in [0, 1]$ are constants; and $B(a)$ takes the simplified form $B(a) = B_0 a^n$, where $B_0 > 0$ and $n$ are constants.

If a variant of the approach presented in [12] were also applicable to a 2D gas with negative-definite energy density confined in a spherical thin shell, a presumably non-linear, variable EoS relating $p$ and $\sigma$ could be derived. This procedure would be relevant to Schwarzschild thin-shell wormholes, where the joined spacetime regions are empty and asymptotically flat (so no variable EoS inheritance mechanism can be invoked).

It is not clear whether thin shell EoS arising from modified relativistic gas models would be compatible with the linear form (28) in certain limiting situations. Also, the motivation for including the singular coefficient in (44) may be questioned [25, 26]. However, the emergence of linearized stability in the context of Schwarzschild thin-shell wormholes makes (28) an interesting choice. Particularly, the achieved stabilization at every throat radius is a remarkable consequence of (44) which deserves further attention.

Before any derivation of variable EoS for thin shells based on relativistic gas models becomes available, we can explore wormhole stability properties induced by the EoS

$$p = A\sigma - \frac{B_0 a^n}{\sigma^\alpha},$$

(51)

where $a$ is throat radius. Constant $B_0$ can be determined at every equilibrium radius $a = a_0$ from the condition $V'(a_0) = 0$, for different choices of the free, constant, dimensionless parameters $A$, $n$ and $\alpha$. Even if we restrict our attention to the case $\alpha = 1$, this combination of linear and non-linear terms may lead to novel features, and open new perspectives for Schwarzschild thin-shell wormholes with variable EoS.

Both linear and non-linear forms of $p = p(\sigma, a)$ could be characterized in terms of classical, self-interacting scalar fields restricted to wormhole throats. Similar procedures have been implemented in cosmology (see, for example, [8], [14], [15] and [27]). Sharif and Abbas [28] have recently considered expanding and collapsing scalar field thin shells. In the case of thin-shell wormholes with variable EoS, it would be interesting to link the arising stability properties with scalar field self-interaction.

Finally, we point out that variable EoS could have a significant impact on the stability of other gravitational sources incorporating thin shells e.g. lower-dimensional [29], cylindrical [30], intra-galactic [31], higher-dimensional [32], oblate [33], rotating [34], and Einstein-Maxwell-Gauss-Bonnet [35] wormholes, as well as stars and circumstellar shells [36], gravastars [37], and multi-layer spherical systems [38].
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Figure 1: Real roots of \( (38) \) as functions of \( n \). Roots \( x_0^+(n) \) split into branches \( x_{01}^+(n) \), \( x_{02}^+(n) \), \( x_{03}^+(n) \). Also, roots \( x_0^-(n) \) divide into branches \( x_{01}^-(n) \), \( x_{02}^-(n) \). Branches \( x_{03}^-(n) \) and \( x_{02}^-(n) \) meet at \( M_1 \), while \( x_{02}^+(n) \) and \( x_{01}^-(n) \) meet at \( M_2 \). The approximate locations of \( M_1 \) and \( M_2 \) are \((-5.83, 0.71)\) and \((-0.17, -0.71)\), respectively. Branch \( x_{01}^+(n) \) satisfies \( x_{01}^+(n) > 1 \) for \( n > 0 \). Since the line \( x = 1 \) represents the would-be Schwarzschild horizon, only \( x_{01}^+(n) \) is physically meaningful and relevant to linearized stability analysis. To increase the accuracy of the graph near \( M_2 \), we have chosen the plotting interval \( n \in [n_2 - 10, n_2 + 6] \), where \( n_2 = 2^{3/2} - 3 \approx -0.17 \).
Figure 2: Curves $a, b, c, d, e, f$ describe $V^{(2)}(x_0)$ for $n = 1, 3, 5, 7, 9, 11$, respectively. The corresponding roots take the approximate values 1.71, 1.27, 1.17, 1.13, 1.10, 1.08. Each of these zeros define the boundary of a semi-infinite linearized stability region with positive $V^{(2)}(x_0)$. These regions tend to cover the whole dimensionless radius interval $(1, +\infty)$ as the $n$ value increases.