Critical and multicritical behavior of the $\pm J$ Ising model in two and three dimensions

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Abstract. We report our Monte Carlo results on the critical and multicritical behavior of the $\pm J$ Ising model [with a random-exchange probability $P(J_{xy}) = p\delta(J_{xy} - J) + (1 - p)\delta(J_{xy} + J)$], in two and three dimensions. We study the transition line between the paramagnetic and ferromagnetic phase, which extends from $p = 1$ to a multicritical (Nishimori) point. By a finite-size scaling analysis, we provide strong numerical evidence that in three dimensions the critical behavior along this line belongs to the same universality class as that of the critical transition in the randomly dilute Ising model. In two dimensions we confirm that the critical behavior is controlled by the pure Ising fixed point and that disorder is marginally irrelevant, giving rise to universal logarithmic corrections. In both two and three dimensions, we also determine the location of the multicritical Nishimori point, as well as the renormalization-group dimensions of the operators that control the renormalization-group flow close to it.

1. Introduction

The $\pm J$ Ising model is a simplified model [1] for disordered uniaxial magnetic systems showing glassy behavior in some region of their phase diagram and represents an interesting theoretical laboratory to study the effects of quenched disorder and frustration. It is defined by the lattice Hamiltonian

$$\mathcal{H} = -\sum_{\langle xy \rangle} J_{xy} \sigma_x \sigma_y,$$

where $\sigma_x = \pm 1$, the sum is over the nearest-neighbor sites of a simple cubic lattice, and the exchange interactions $J_{xy}$ are uncorrelated quenched random variables, taking values $\pm J$ with probability distribution

$$P(J_{xy}) = p\delta(J_{xy} - J) + (1 - p)\delta(J_{xy} + J).$$

In the following we set $J = 1$ without loss of generality. For $p = 1$, we recover the standard Ising model, while for $p = 1/2$, we obtain the usual bimodal Ising spin-glass model. The phase diagram in two and three dimensions is sketched in Figure [1] it is symmetric for $p \rightarrow 1 - p$ and thus here and in the following we only consider the case $1 - p < 1/2$, i.e., $p > 1/2$. 
In both $d = 2$ and $d = 3$, the model exhibits a paramagnetic-ferromagnetic (PF) phase transition in the region of low frustration. The PF transition line starts at the Ising point $X_{\text{Is}} = (T = T_{\text{Is}}, p = 1)$, where $T_{\text{Is}}$ is the critical temperature of the Ising model, and extends up to the multicritical Nishimori point (MNP) at $X_{\text{MNP}} = (T^*, p^*)$. Along this line, the critical behavior is analogous to that observed in randomly dilute Ising (RDI) models.

In two dimensions, renormalization-group (RG) and conformal field theory predict [2, 3, 4, 5] that disorder is marginally irrelevant, thus the critical behavior is controlled by the pure Ising fixed point, with universal logarithmic corrections [6]. In three dimensions disorder is relevant and the model shows a second-order phase transition in the three-dimensional (3D) RDI universality class [7], which describes transitions in generic diluted Ising systems with ferromagnetic exchange interactions [8].

As argued in [9, 10, 11], the MNP is located along the so-called Nishimori line (N line) [12, 13] defined by the equation

$$\tanh \beta = 2p - 1,$$  

where $\beta \equiv 1/T$. At the MNP the transition line is predicted to be parallel to the $T$ axis [11]. Then, it reaches the $T = 0$ axis at $X_c = (0, p_c)$. It has been proved [12] that ferromagnetism can only exist in the region $p \geq p^*$; thus, $p_c$ must satisfy the inequality $p_c \geq p^*$. Recent numerical works indicate that $p_c$ is strictly larger than $p^*$, though deviations are quite small, both in two [14] and three [15] dimensions, as well as in related models [16, 17].

The critical behavior for $p < p^*$ depends on the dimension. The 3D $\pm J$ model exhibits a paramagnetic-glassy (PG) transition line, which extends from the MNP up to $p = 1/2$. On this transition line, the critical behavior is independent on $p$ and belongs to the Ising spin-glass universality class [18]. In contrast, in two dimensions there is no evidence of a finite-temperature glassy phase. Glassy behavior is only expected for $T = 0$ and $p < p_c$: the glassy phase at $T = 0$ is unstable with respect to thermal fluctuations.

2. Finite-size scaling
2.1. Paramagnetic-ferromagnetic transition line

We present here the strategy used to analyze the critical behavior on the PF transition line in the 3D $\pm J$ model. The two-dimensional (2D) case is analyzed in [6].

According to the renormalization group (RG), in the case of periodic boundary conditions and for $L \to \infty$, where $L$ is the lattice size, a generic RG invariant quantity $R$ at the critical
Temperature $1/\beta_c$ behaves as
\[
R(L, \beta = \beta_c) = R^* \left( 1 + c_{11} L^{-\omega} + c_{12} L^{-2\omega} + \cdots + c_{21} L^{-\omega_2} + \cdots \right),
\]
where $R^*$ is the universal infinite-volume limit and $\omega$ and $\omega_2$ are the leading and next-to-leading correction-to-scaling exponents. In 3D RDI systems scaling corrections play an important role, since $\omega$ is quite small; indeed, we have $\omega = 0.29(2)$ and $\omega_2 = 0.82(8)$ [19,8]. Instead of computing the various quantities at fixed Hamiltonian parameters, we keep a RG invariant quantity $R$ fixed at a given value $R_f$ [20]. This means that, for each $L$, we determine the pseudocritical inverse temperature $\beta_f(L)$ such that $R(\beta = \beta_f(L), L) = R_f$. All interesting thermodynamic quantities are then computed at $\beta = \beta_f(L)$. The pseudocritical inverse temperature $\beta_f(L)$ converges to $\beta_c$ as $L \to \infty$. The value $R_f$ can be specified at will, as long as $R_f$ is taken between the high- and low-temperature fixed-point values of $R$. The choice $R_f = R^*$ (where $R^*$ is the critical-point value) improves the convergence of $\beta_f$ to $\beta_c$ for $L \to \infty$; indeed $\beta_f - \beta_c = O(L^{-1/\nu})$ for generic values of $R_f$, while $\beta_f - \beta_c = O(L^{-1/\nu - \omega})$ for $R_f = R^*$.

We can then consider any other RG invariant quantity $R_\alpha$ at fixed $R = R_f$, i.e., $\bar{R}_\alpha(L) = R_\alpha(L, \beta_f(L))$. For $L \to \infty$, one can show that [8]
\[
\bar{R}_\alpha(L) = \tilde{R}_\alpha \left( 1 + b_{11} L^{-\omega} + b_{12} L^{-2\omega} + \cdots + b_{21} L^{-\omega_2} + \cdots \right),
\]
\[
\bar{R}_\alpha'(L) = a_0 L^{1/\nu} \left( 1 + a_{11} L^{-\omega} + a_{12} L^{-2\omega} + \cdots + a_{21} L^{-\omega_2} + \cdots \right),
\]
\[
\tilde{\chi}(L) \equiv \chi(L, \beta = \beta_f(L)) = d_0 L^{2-\eta} \left( 1 + d_{11} L^{-\omega} + d_{12} L^{-2\omega} + \cdots + d_{21} L^{-\omega_2} + \cdots \right) + d_0,
\]
where $\bar{R}_\alpha'$ is the derivative of $\bar{R}_\alpha$ with respect to $\beta$ and $\tilde{\chi}$ is the susceptibility at fixed $R = R_f$.
More details on this method can be found in [20,8].

2.2. Multicritical Nishimori point
In the absence of external fields, the critical behavior at the MNP is characterized by two relevant RG operators. The singular part of the disorder-averaged free energy in a volume $L^d$ can be written as
\[
F_{\text{sing}}(T, p, L) = L^{-d} f(u_1 L^{y_1}, u_2 L^{y_2}, \{u_i L^{y_i}\}), \quad i \geq 3,
\]
where $y_1 > y_2 > 0$, $y_i < 0$ for $i \geq 3$, $u_i$ are the corresponding scaling fields, $u_1 = u_2 = 0$ at the MNP. The scaling fields $u_i$ are analytic functions of the model parameters $T$ and $p$. Using symmetry arguments, [10,11] showed that the scaling axis corresponding to $u_2 = 0$ is along the $N$ line, so that $u_2 = \tanh \beta - 2p + 1$. As for the scaling axis $u_1 = 0$, $\epsilon \equiv 6 - d$ expansion calculations predict [11] to be parallel to the $T$ axis. The extension of this result to lower dimensions suggests $u_1 = p - p^*$. Note that, if this conjecture holds, only the scaling field $u_2$ depends on the temperature $T$.

These results give rise to the following predictions for the FSS behavior around $T^*, p^*$. Along the $N$ line, an RG invariant quantity $R$ has the following behavior for $L \to \infty$:
\[
R_N = R^* + b_{11} u_1 L^{y_1} + \cdots,
\]
where the subscript $N$ indicates that $R$ is restricted to the $N$ line and we have neglected scaling corrections. Its derivative $R'$ with respect to $\beta$ behaves as
\[
R' = b_{11} u_1' L^{y_1} + b_{21} u_2' L^{y_2} + \cdots = b_{21} u_2' L^{y_2} + \cdots,
\]
where the last equivalence holds only if $u_1$ does not depend on the temperature. This result gives us a method to verify the conjecture of [11]: once $y_1$ has been determined from the scaling behavior of a RG invariant quantity $R$ close to the MNP, it is enough to check the scaling behavior of $R'$. If $R'$ scales as $L^\zeta$ with $\zeta < y_1$, the conjecture is confirmed and $y_2 = \zeta$. Finally, along the $N$ line the susceptibility $\chi$ is expected to behave as
\[
\chi_N = d_0 L^{2-\eta} (1 + d_1 u_1 L^{y_1} + \cdots).
\]
3. Results

3.1. The 3D $\pm J$ model at the paramagnetic-to-ferromagnetic transition line

In [7] we perform MC simulations of Hamiltonian (1) at $d = 3$ close to the PF transition line. Using the method outlined in Sec. 2.1 we fit the large-$L$ limit of various RG-invariant quantities at fixed $R_L \equiv \xi/L = 0.5943$, where $\xi$ is the correlation length; this value is very close to the fixed-point value $R^{*}_L = 0.5944(7)$ [8] at $\beta_c$. Our FSS analysis provides strong evidence that the critical behavior of the 3D $\pm J$ Ising model along the PF line belongs to the 3D RDI universality class. Indeed, all the RG-invariant quantities considered are in agreement with the results reported in [8]. Moreover, we find $\nu = 0.682(3)$ and $\eta = 0.036(2)$, in good agreement with the presently most accurate estimates [8] $\nu = 0.683(2)$ and $\eta = 0.036(1)$ for the 3D RDI universality class.

We also note that the random-exchange interaction in the $\pm J$ Ising model gives rise to frustration, while the RDI universality class describes transitions in generic diluted Ising systems with ferromagnetic exchange interactions. Therefore, our results imply that frustration is irrelevant along the PF transition line. Moreover, the observed scaling corrections are consistent with the RDI correction-to-scaling exponents $\omega = 0.29(2)$ and $\omega_2 = 0.82(8)$ [19]. Thus, frustration does not introduce new irrelevant perturbations with RG dimension $|y_j| \lesssim 1$.

3.2. Multicritical Nishimori point

In [14, 15], we perform MC simulations of Hamiltonian (1) at $d = 2, 3$, along the N line defined by $\delta = 1$. In both cases, the position of the MNP as well as the RG dimension $y_1$ of the leading relevant operator $u_1$ are determined by fitting several RG-invariant quantities $R$ to (9) (in the 2D case we also consider scaling corrections). We verify the conjecture that $u_1$ does not depend on $T$ and compute the RG dimension associated with $u_1$ and $u_2$.

For the 2D model, the MNP is located at $p^* = 0.89081(7)$, $T^* = 0.9528(4)$, the RG dimensions of the operators that control the multicritical behavior are $y_1 = 0.655(15)$ and $y_2 = 0.250(2)$, and the susceptibility exponent is $\eta = 0.180(5)$. In three dimensions, we find $p^* = 0.76820(4)$, $T^* = 1.6692(3)$, $y_1 = 1.02(5)$, $y_2 = 0.61(2)$, $\eta = -0.114(3)$.

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