A sufficient condition for an orthogonally transitive G2 cylindrical spacetime to be singularity-free is shown. The condition is general enough to comprise all known geodesically complete perfect fluid cosmologies.

1 Introduction

The main goal of this talk will be to discuss the emergence of singularity-free cosmological models in the frame of orthogonally-transitive G2-spacetimes, more precisely in cylindrical spacetimes.

The issue was triggered in 1990 by Senovilla \(^1\) by means of publication of a diagonal G2 spacetime with a radiation fluid as matter content. This spacetime had regular curvature invariants. The question arose then whether this spacetime was in fact singularity-free, that is, geodesically complete. By this it is meant that every causal geodesic can be defined from \(-\infty\) to \(\infty\) in the affine parametrization. In a subsequent paper \(^2\) by Chinea, Senovilla and the author, the issue was settled. It was geodesically complete.

What about other spacetimes? Curvature invariants are easily checked and there is no problem in determining whether they are regular. Geodesics, on the other hand, are determined by a system of non-linear ODEs and it is therefore not easy to check whether they are complete. Since general relativity deals with lorentzian geometry, no riemannian metric is available and hence no Hopf-Rinow theorem relates geodesic completeness to metric completeness.

This talk is an attempt to simplify the issue in the context of cylindrical spacetimes, which has been the source of non-singular spacetimes with a perfect fluid as matter content.

We shall choose isotropic coordinates, \(t, r\), in the timelike part of the metric, \(z, \phi\) are cyclic coordinates corresponding to cylindrical symmetry,

\[
ds^2 = e^{2 g(t,r)} \left\{ -dt^2 + dr^2 \right\} + \rho^2(t,r)e^{2 f(t,r)}d\phi^2 + e^{-2 f(t,r)} \{dz + A(r,t)d\phi\}^2,
\]

and in order to ensure that the Riemann tensor is well defined, we shall impose that all metric functions are \(C^2\) in their usual ranges, although geodesic equations just involve first derivatives of the metric. Furthermore, we shall require the axis to be locally flat \(^3\). Coordinates are chosen so that it is located at \(r = 0\).

Instead of writing the whole set of geodesic equations, we shall make use of the existence of constants of motion, \(P, L\), related to translations along and rotations around the axis, and \(\delta \in \{0, 1\}\) for lightlike and timelike geodesics.
2 Results

The results can be summarized in the following theorem. Reference\textsuperscript{4} provides more details. A similar theorem is obtained for past causal geodesics changing $u$-derivatives for minus $v$-derivatives. This theorem comprises all known perfect-fluid regular spacetimes in the literature\textsuperscript{5,6}.

Theorem 1: Under the previous requirements a metric has complete future causal geodesics if the following set of conditions is fulfilled:

1. For large values of $t$ and increasing $r$,
   
   \[ \begin{align*}
   & (a) \begin{cases}
   g_u \geq 0 \\
   h_u \geq 0 \\
   q_u \geq 0,
   \end{cases} \\
   & (b) \text{Either} \begin{cases}
   g_r \geq 0 \text{ or } |g_r| \lesssim q_u \\
   h_r \geq 0 \text{ or } |h_r| \lesssim h_u \\
   q_r \geq 0 \text{ or } |q_r| \lesssim q_u.
   \end{cases}
   \end{align*} \]

2. For $L \neq 0$ and large values of $t$ and decreasing $r$,

   \[ \begin{align*}
   & (a) \delta g_v + P^2 e^{2f} q_v + \Lambda^2 \frac{e^{-2f}}{\rho^2} h_v \geq 0 \\
   & (b) \text{Either} \delta g_r + P^2 e^{2f} q_r + \Lambda^2 \frac{e^{-2f}}{\rho^2} h_r \leq 0 \text{ or } \delta g_r + P^2 e^{2f} q_r + \Lambda^2 \frac{e^{-2f}}{\rho^2} h_r \lesssim \\
   & \delta g_v + P^2 e^{2f} q_v + \Lambda^2 \frac{e^{-2f}}{\rho^2} h_v.
   \end{align*} \]

3. For large values of the time coordinate $t$, constants $a, b$ exist such that

   \[ \begin{align*}
   & 2 g(t,r) \\
   & g(t,r) + f(t,r) + \ln \rho - \ln |A| \\
   & g(t,r) - f(t,r)
   \end{align*} \geq - \ln |t + a| + b.
   \]

4. The limit $\lim \limits_{r \to 0} \frac{A}{\rho}$ exists.

where $u, v$ are the usual ingoing and outgoing light coordinates, $q = g + f$, $h = g - f - \ln |L - PA|$.

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