The non-relativistic self-gravitating gas in thermal equilibrium in the presence of a positive cosmological constant \( \Lambda \) (dark energy) is investigated. The dark energy introduces a force pushing outward all particles with strength proportional to their distance to the center of mass. We consider the statistical mechanics of the self-gravitating gas of \( N \) particles in a volume \( V \) at thermal equilibrium in the presence of \( \Lambda \). It is shown that the thermodynamic limit exists and is described by the mean field equations provided \( N, V \to \infty \) with \( N/V^{1/3} \) fixed and \( \Lambda V^{2/3} \) fixed. That is, \( \Lambda \to 0 \) for \( N, V \to \infty \). The case of \( \Lambda \) fixed and \( N, V \to \infty \) is solved too. We solve numerically the mean field equation for spherical symmetry obtaining an isothermal sphere for \( \Lambda > 0 \). The particle distribution turns to flatten compared with the \( \Lambda = 0 \) case. Moreover, the particle density increases with the distance when the cosmological constant dominates. There is a bordering case with uniform density. The density contrast between the center and the boundary may be significantly reduced by the dark energy. In addition, the critical point associated to the collapse (Jeans') phase transition is pushed towards higher values of \( N/[T V^{1/3}] \) by the presence of \( \Lambda > 0 \). The nature and the behaviour near the critical points is not affected by the presence of \( \Lambda > 0 \).

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I. INTRODUCTION

The self-gravitating gas in thermal equilibrium has been thoroughly studied since many years. As a consequence of the long range attractive Newton force, the self-gravitating gas admits a consistent thermodynamic limit $N, V \rightarrow \infty$ with $N/V$ fixed. In this limit, extensive thermodynamic quantities like energy, free energy, entropy are proportional to $N$.

We investigate in this paper how the presence of a cosmological constant affects the properties of the nonrelativistic self-gravitating gas in thermal equilibrium.

The cosmological constant modifies the Poisson equation adding a negative term in the r. h. s. This introduces an extra force pushing outwards all particles with a strength proportional to their distance to the center of mass. This outward force is the nonrelativistic version of the exponentially fast expansion in the de Sitter universe. It is an antigravitational effect.

We consider the statistical mechanics of the self-gravitating gas in the presence of a cosmological constant and show that it is described by the mean field equation in the thermodynamic limit. We derive the mean field equation in the thermodynamic limit of large number $N$ of particles and large volume $V$ with $N/V^{1/3}$ and $\Lambda V^{2/3}$ fixed. Furthermore, we consider the case $\Lambda$ fixed for $N$ and $V$ going to infinity with $N/V^{1/3}$ fixed.

As known, the usual thermodynamic limit $N, V \rightarrow \infty$ with $N/V$ fixed leads to collapse in absence of the cosmological constant, while the dilute limit keeping $N/V^{1/3}$ fixed leads to a gaseous phase. Here, we consider a vanishing $\Lambda$ for $V \rightarrow \infty$ keeping $\Lambda V^{2/3}$ fixed as well as $N/V^{1/3}$.

The meaning of the condition $\Lambda V^{2/3}$ for $N, V \rightarrow \infty$ goes as follows. The dark energy in a volume $V$ is of the order $V \Lambda$ while the thermal (kinetic) energy (as well as the interaction energy between the particles) is of the order $N T$. Since $N \sim V^{1/3}$, $\Lambda V^{2/3}$ fixed implies that the ratio of dark and thermal energies is fixed for $N \rightarrow \infty$.

When $\Lambda$ and $N/V^{1/3}$ are kept fixed for $N$, $V \rightarrow \infty$ the cosmological constant dominates over self-gravity. The result is a set of non-gravitating particles in a harmonic oscillator with negative squared frequencies. This system is exactly solvable as we show.

It is convenient to define the dimensionless parameter:

$$\xi = 2 \Lambda G m \frac{L^2}{T}. \quad (1.1)$$

All physical magnitudes are expressed in terms of $\xi$ and

$$\eta = G m^2 N/[T L],$$

the dimensionless parameter used in ref. [2].

We solve the mean field equations numerically for spherical symmetry. That is, we find the isothermal sphere in the presence of a non-zero cosmological constant.

We find that this ‘antigravitational effect’ of $\Lambda$ flattens the particle distributions in thermal equilibrium. The density contrast (ratio of the density at the center and at the surface of the sphere) decreases for increasing $\Lambda$. The particle density at the boundary is conformally translated as a function of $\eta$ by the presence of $\Lambda$ (see fig. 2).

In absence of cosmological constant the particle density is a monotonically decreasing function of the radial distance $R$. We find in the presence of a nonzero $\Lambda$ that the particle density may be either monotonically decreasing or monotonically increasing with $R$. The particle density grows with $R$ when $\Lambda$ dominates over the selfgravitational attraction.

In the special case $\eta = \xi$ the gravitational self-interaction exactly compensates the cosmological constant. As a result the gas becomes a perfect gas with uniform density.

The position of the collapse (Jeans’) phase transition is pushed by the cosmological constant to higher values of $\eta$. Namely, lower temperatures or volumes or higher number of particles. The nature of the phase transition is not affected by the presence of $\Lambda > 0$ and the behaviour of the physical magnitudes does not change far from the phase transition (see figs. 2 and 6). Far from the collapse point, the energy and the free energy are basically shifted by the presence of $\Lambda > 0$.

The thermodynamic quantities: free energy, energy and entropy turn to be proportional to the number of particles in the thermodynamic limit. We compute them as functions of $\eta$ and $\xi$. We also consider the limiting case $\xi \gg \eta$ which is solved analytically in sec. 4. This corresponds to keep $\Lambda$ fixed for $V \rightarrow \infty$.

As for $\Lambda = 0$ the gas collapses in the canonical ensemble when the compressibility diverges and becomes negative. This happens for a value of $\eta$ smaller than the one where the specific heat is singular.

This article is organized as follows: in sec. 2 we recall the equations of motion for nonrelativistic particles in the presence of a cosmological constant. In section 3 we present the self-gravitating gas in the presence of $\Lambda > 0$; first in
a hydrostatic approach, then the statistical mechanics and finally the mean field approach including the spherically symmetric case with the calculation of the physical magnitudes (energy, free energy, entropy, surface density, pressure contrast, specific heat and compressibility). In sec. 4 we present the physical results for the self-gravitating gas in the presence of \( \Lambda > 0 \): phase structure and particle density behaviour in the various cases. In sec. 5 we consider the exactly solvable case where the cosmological constant dominates overwhemly over the self-gravitation. We present our conclusions in sec. 6.

II. NONRELATIVISTIC GRAVITY IN THE PRESENCE OF THE COSMOLOGICAL CONSTANT

The energy-momentum tensor for massive particles plus dark energy (the cosmological constant) takes the form

\[
T^\alpha_\beta = \Lambda \delta^\alpha_\beta + \rho \delta^{\alpha 0} \delta_{\beta 0},
\]

where \( \Lambda \) stands for the cosmological constant, \( \rho \) for the energy density of the massive particles, \( 0 \leq \alpha, \beta \leq 3 \) and we assume nonrelativistic matter so its pressure can be neglected compared with its rest mass. The Einstein equations take the form

\[
R^\alpha_\beta = 8 \pi G \left( T^\alpha_\beta - \frac{\delta^\alpha_\beta}{2} T \right),
\]

where \( R^\alpha_\beta \) stands for the Ricci tensor and \( T \equiv T^\alpha_\alpha \). We find from eq.(2.1) that

\[
T = 4 \Lambda + \rho
\]

and

\[
T^{00} - \frac{1}{2} T = \frac{1}{2} \rho - \Lambda.
\]

We are interested in the non-relativistic limit where gravitational fields are weak and we have for the metric

\[
g_{00} = 1 + 2 V, \quad g_{ik} = -\delta_{ik},
\]

where \( V \) stands for the gravitational potential. One has for the 00 component of the Ricci tensor

\[
R^{00} = \nabla^2 V.
\]

Therefore, the 00 component of the Einstein equations becomes

\[
\nabla^2 V = 4 \pi G \rho - 8 \pi G \Lambda.
\]

For zero cosmological constant we recover the usual Poisson equation, as it must be. Eq.(2.3) can be written in an integral form,

\[
V(\vec{q}) = V_{self}(\vec{q}) + V_{dark}(\vec{q}) = -G \int \frac{\rho(\vec{q}^\prime) \, d^3q^\prime}{|\vec{q} - \vec{q}^\prime|} - \frac{4 \pi G \Lambda}{3} \vec{q}^2,
\]

where \( V_{self} \) is the contribution of the massive particles and \( V_{dark} \) is the contribution of the dark energy. Here we choose the center of mass of the particles as origin of the coordinates. Linear terms in \( \vec{q} \) of the type

\[
\vec{a} \cdot \vec{q}
\]

with \( \vec{a} \) an arbitrary constant vector, are ignored in \( V(\vec{q}) \) since they correspond to a constant external gravitational field unrelated to the cosmological constant. [Notice that such terms have zero laplacian and are not determined by eq.(2.3)]. Such external field will accelerate the particles in the direction \( \vec{a} \). Changing the origin of coordinates, terms linear in \( \vec{q} \) can be added to eq.(2.4). Additional constant terms are clearly irrelevant.

The mass density takes the following form for a distribution of point particles at rest,

\[
\rho(\vec{q}) = \sum_i m_i \delta(\vec{q} - \vec{q}_i).
\]

Here, \( m_i \) stands for the mass of the particle at the point \( \vec{q}_i \). The gravitational potential thus becomes,

\[
V(\vec{q}) = -G \sum_i \frac{m_i}{|\vec{q} - \vec{q}_i|} - \frac{4 \pi G \Lambda}{3} \vec{q}^2.
\]
and the gravitational field at the point $\vec{q}$,

$$
\vec{g} = -\nabla V(\vec{q}) = -G \sum_i m_i \frac{\vec{q} - \vec{q}_i}{|\vec{q} - \vec{q}_i|^3} + \frac{8\pi G \Lambda}{3} \vec{q} \tag{2.6}
$$

The potential energy of such a system of non-relativistic self-gravitating particles in the presence of a cosmological constant $\Lambda$ thus takes the form,

$$
U = -G \sum_{i < j} \frac{m_i m_j}{|\vec{q}_i - \vec{q}_j|} - \frac{4\pi G \Lambda}{3} \sum_i m_i q_i^2 \tag{2.7}
$$

Therefore, the Hamiltonian can be written as

$$
H = \sum_i \frac{p_i^2}{2m_i^2} + U \tag{2.8}
$$

where $p_i$ stands for the momentum of the $i$-th particle. The cosmological constant contribution to the potential energy decreases for increasing values of the particle distances $r_i$ to the center of mass. As stated above, in absence of external fields, the terms like eq.(2.5) are absent in the potential energy. Anyway, in the center of mass frame such linear terms identically vanish upon summing over the particles in eq.(2.7).

Therefore, the gravitational effect of the cosmological constant on particles amount to push them outwards. Equivalently, the last term of the gravitational field eq.(2.6) points outward. It can be then said that the cosmological constant has an antigravitational effect.

For simplicity we shall consider all particles with the same mass $m$. The case of unequal masses may be worked out generalizing the work in ref.[4] to nonzero cosmological constant.

Dark energy introduces a kind of centrifugal force always pushing out. Notice that the last term in eq.(2.6) has the structure of a rotational energy but with negative sign. Also, it contains the distance squared of the particles to the origin and not the distance squared to an axis as it is the case for rotational energy.

It is useful to consider the case of just one particle in the presence of the cosmological constant in order to obtain physical insights. We obtain from eqs.(2.7) and (2.8),

$$
\ddot{q} = \frac{8\pi G \Lambda}{3} \vec{q} \tag{2.9}
$$

This is an oscillator equation with imaginary frequency and solution,

$$
\tilde{q}(t) = \tilde{q}(0) \cosh H t + \frac{1}{H} \tilde{q}(0) \sinh H t , \tag{2.9}
$$

where $H \equiv \sqrt{\frac{8\pi G \Lambda}{3}}$. The particle runs away exponentially fast in time.

It is interesting to compare this non-relativistic results with the full relativistic geometry in the presence of the cosmological constant. The exact solution of the Einstein equations (the full solution, not just the nonrelativistic limit) for the energy-momentum eq.(2.1) with $\rho = 0$ is the de Sitter universe,

$$
ds^2 = dt^2 - e^{2H t} \left(\frac{d\vec{x}}{e^H}\right)^2 .
$$

Comparing the non-relativistic trajectories eq.(2.9) with the exact relativistic geodesics in de Sitter space-time (see for example [8]) one sees that both exhibit the same exponential runaway behaviour. Therefore, the nonrelativistic approximation keeps the essential features of the particle motion in de Sitter space-time.

III. THE SELF-GRAVITATING GAS WITH $\Lambda > 0$

We first consider the hydrostatic description of the self-gravitating gas in thermal equilibrium in the presence of the cosmological constant. Next, we present the statistical mechanics of the self-gravitating gas in the presence of the cosmological constant and derive the mean field approach to it. The two approaches turn to be equivalent when the number of particles tends to infinity.
A. Hydrostatics of the self-gravitating gas with $\Lambda \neq 0$

Let us consider the hydrostatic equilibrium condition for a self-gravitating gas in the presence of the cosmological constant.

$$\nabla P(\vec{q}) = -\rho(\vec{q}) \nabla V(\vec{q}), \quad (3.1)$$

where $P(\vec{q})$ stands for the pressure at the point $\vec{q}$. Let us assume the ideal gas equation of state in local form (see [1, 2, 3])

$$P(\vec{q}) = \frac{T}{m} \rho(\vec{q}). \quad (3.2)$$

Combining eqs. (3.1) and (3.2) yields the Boltzmann law for the particle density

$$\rho(\vec{q}) = \rho_0 e^{-\frac{\Phi V(\vec{q})}{mT}}, \quad (3.3)$$

where $\rho_0$ is a constant, $V$ being the volume of the system. The normalization of density requires that

$$\int_V d^3\vec{q} \rho(\vec{q}) = mN. \quad (3.4)$$

Inserting eq. (3.3) into the Poisson equation (2.3) yields,

$$\nabla^2 V = 4\pi G \left( \rho_0 e^{-\frac{\Phi V(\vec{q})}{mT}} - 2\Lambda \right). \quad (3.5)$$

It is convenient to introduce dimensionless coordinates $\vec{r} \equiv \vec{q}/L$ where $L \equiv V^{1/3}$ stands for the linear size of the system. Eq. (3.5) takes thus the form,

$$\nabla^2 \Phi(\vec{r}) + 4\pi \left( \eta e^{\Phi(\vec{r})} - \xi \right) = 0, \quad (3.6)$$

where $\Phi(\vec{r}) \equiv -\frac{\Phi}{mT} V(\vec{q}) + \log \frac{\rho_0 V}{mN}$,

$$\eta = \frac{G m^2 N}{T L} \quad \text{and} \quad \xi = 2\Lambda G m \frac{L^2}{T}. \quad (3.7)$$

The mass density now becomes $\rho(\vec{r}) = e^{\Phi(\vec{r})}$ and the constraint (3.4) takes the form,

$$\int_{\text{unit volume}} d^3\vec{r} \ e^{\Phi(\vec{r})} = 1. \quad (3.8)$$

The limit for a large number of particles $N$ is well defined provided we choose

$$N \to \infty, \quad L \to \infty, \quad \Lambda \to 0,$$

with

$$\frac{N}{L} = \text{fixed}, \quad \Lambda L^2 = \text{fixed}.$$ 

It should be mention that an equation similar to eq. (3.6) appeared in two space dimensions describing (multi)-vortices in the Ginzburg-Landau or Higgs model in the limit between superconductivity of type I and II [11]. However, for the vortex case one has $\xi = \eta < 0$. The negative sign is associated to the fact that like charges repel each other in electrodynamics while masses attract each other in gravity.

In the limiting case $\xi = 0$ eq. (3.6) becomes the well known Lane-Emden equation in the absence of cosmological constant both in the hydrostatic [11] and statistical mechanics approaches [2, 3].

Notice that $\Lambda$ plays the same role as the subtraction of a constant density in the Jeans’ swindle [10].
B. The Partition function of the self-gravitating gas with $\Lambda \neq 0$

We present now the statistical mechanics of the self-gravitating gas in the presence of the cosmological constant in the canonical ensemble. The Hamiltonian is given by eqs. (2.7) - (2.8).

The classical partition function of the gas is then,

$$ Z(T, N, V) = \frac{1}{N!} \int \cdots \int \prod_{l=1}^{N} \left( \frac{d^3 \vec{p}_l}{(2\pi)^3} \right) e^{-\frac{H}{T}}. $$

It is convenient to introduce the dimensionless coordinates $\vec{r}_l = \frac{\vec{q}_l}{L}$. The momenta integrals are computed straightforwardly. Hence, the partition function factorizes as the partition function of a perfect gas times the coordinate integral $Z_{coor}$.

$$ Z = \frac{V^N}{N!} \left( \frac{mT}{2\pi} \right)^{3N/2} Z_{coor}, \quad (3.9) $$

where

$$ Z_{coor} = \int \cdots \int \prod_{l=1}^{N} d^3 \vec{r}_l \ e^{\eta u_P + \frac{2}{3} \xi u_N}, \quad (3.10) $$

and

$$ u_P \equiv \frac{1}{N} \sum_{1 \leq i < j \leq N} \frac{1}{|\vec{r}_i - \vec{r}_j|}, \quad u_N \equiv \sum_{i=1}^{N} r_i^2. $$

where $\eta$ and $\xi$ are defined by eq. (3.7).

The free energy takes then the form,

$$ F = -T \log Z(T, N, V) = -NT \log \left[ \frac{eV}{N} \left( \frac{mT}{2\pi} \right)^{3/2} \right] - T \log Z_{coor}, \quad (3.11) $$

We get for the external pressure of the gas,

$$ p = -\left( \frac{\partial F}{\partial V} \right)_T = \frac{NT}{V} + T \left( \frac{\partial}{\partial V} \right)_T \log Z_{coor}. \quad (3.12) $$

Evaluating the volume derivative of $Z_{coor}$ with the help of eqs. (3.7) and (3.10) yields,

$$ \frac{pV}{NT} = 1 + \frac{< U_P > - 2 < U_\Lambda >}{3NT} \quad (3.13) $$

where

$$ U_P = -G \sum_{i<j} \frac{m_i m_j}{|\vec{q}_i - \vec{q}_j|} \quad \text{and} \quad U_\Lambda = -\frac{4\pi G \Lambda}{3} \sum_i m_i q_i^2. $$

Eq. (3.13) is the virial theorem for the gas of self-gravitating particles in the presence of the cosmological constant.

C. Mean field theory

In the large $N$ limit the $3N$-uple integral in eq. (3.11) can be approximated by a functional integral over the density (see ref. [2]),

$$ Z_{coor} = \int D\rho(.) \ db \ e^{-N S[\rho(.)]} \quad (3.14) $$
with,
\[ S[\rho(\cdot)] = \left. \left( -\frac{\eta}{2} \int \frac{d^3x}{|x-y|} \rho(x) \rho(y) - \frac{2\pi}{3} \xi \int d^3x \rho(x) x^2 \right) \right|_{x=0} + \int d^3x \rho(x) \ln \rho(x) + i b \left[ 1 - \int d^3x \rho(x) \right], \] (3.15)
and \( b \) is the Lagrange multiplier enforcing the normalization of the density:
\[ \int d^3x \rho(x) = 1 . \] (3.16)

The function \( Z_{\text{coor}} \) thus becomes a functional integral over all density configurations \( \rho(x) \). Using eqs. (3.9), (3.14) and (3.15) the partition function of the system becomes
\[ Z = \int D\rho(\cdot) \, db \, e^{-\frac{1}{T} F[\rho(\cdot)]}, \]
where
\[ F[\rho(\cdot)] = F_0 + NT S[\rho(\cdot)] \] (3.17)
stands for the free energy of the gas with density \( \rho \), while
\[ F_0 = -NT \ln \left[ \frac{eV}{N} \left( \frac{mT}{2\pi} \right)^{\frac{3}{2}} \right] \]
is the free energy of the perfect gas. One recognizes the potential energy (the gravitational energy due to the self-interaction of the particles and the gravitational energy due to the interaction of the particles with the dark energy) in the first line of eq.(3.15), while the second line contains the entropy and the term fixing the total number of particles to be \( N \).

Since the free energy becomes large in the thermodynamic limit \( (N \gg 1) \), the functional integral \( Z_{\text{coor}} \) in eq.(3.14) is dominated by the extrema of \( S[\rho(\cdot)] \). Extremizing this quantity with respect to the density \( \rho(x) \) yields the saddle point equation
\[ \ln \rho(x) = a + \eta \int \frac{d^3y}{|y-x|} \rho(y) + \frac{2\pi}{3} \xi x^2 . \] (3.18)
This equation defines the mean field approach. We set \( ib = a + 1 \) and
\[ \rho(x) = e^{\Phi(x)} . \] (3.19)
We recognize in eq.(3.18) the gravitational potential
\[ V(x) = -\frac{T}{m} [\Phi(x) - a] . \] (3.20)
Inserting eq.(3.20) into eq.(3.19) yields the Boltzmann law for the density
\[ \rho(x) = e^a e^{-\frac{1}{T} V(x)} , \] (3.21)
containing the energy of a particle in the mean field gravitational potential \( V \). \( e^a \) plays the role of a normalization constant.

Applying the Laplace operator to the saddle point equation (3.18) we find the differential equation
\[ \nabla^2 \Phi(x) + 4\pi \left( \eta e^{\Phi(x)} - \xi \right) = 0 . \] (3.22)
Eq. (3.22) is identical to the hydrostatic equilibrium eq.(3.6). Therefore, hydrostatics and mean field are equivalent in the \( N \to \infty \) limit.

It must be stressed that in the hydrostatic approach one has to assume some local equation of state. Only a perfect gas equation yields eq.(3.22). In the mean field approach one can compute the local equation of state and demonstrate that this is the one of a perfect gas[2, 3, 4].
D. Spherically symmetric case

For spherically symmetric configurations the mean field equation \((3.6)\) becomes an ordinary non-linear differential equation. The various thermodynamic quantities are expressed in terms of their solutions.

![Graph showing pressure at the boundary of the self-gravitating gas without dark energy versus \(\eta\). PG is the perfect gas point, O is the point where the gas collapses \((\eta^C = 1.510\ldots)\), C is the branch point where \(c_v\) diverges \((\eta^C = 1.561\ldots)\), and MC is the collapse point in the microcanonical ensemble \((\eta^{HC} = 1.259\ldots)\).]

We consider here the spherically symmetric case where the mean field equation \((3.22)\) takes the form

\[
\frac{d^2 \Phi}{dR^2} + \frac{2}{R} \frac{d \Phi}{dR} + 4\pi \left( \eta e^{\Phi(R)} - \xi \right) = 0.
\]  
(3.23)

We work in a unit sphere, therefore the radial variable runs in the interval \(0 \leq R \leq R_{\text{max}}, R_{\text{max}} \equiv \left(\frac{3}{4\pi}\right)^{\frac{1}{3}}\). The density of particles \(\rho(R)\) has to be normalized according to eq.\((3.16)\). Integrating eq.\((3.23)\) from \(R = 0\) to \(R = R_{\text{max}}\) yields,

\[
\eta - \xi = -R_{\text{max}}^2 \Phi'(R_{\text{max}}).
\]  
(3.24)

Setting,

\[
\Phi(R) = u(x) + \ln \frac{\xi R}{\eta R}, \quad x = \sqrt{3\xi R} \frac{R}{R_{\text{max}}},
\]  
(3.25)
FIG. 2: The density at the boundary $f(\eta, \xi)$ versus $\eta$ for $\xi = 0, 1, 2.04 \ldots, 3$. We insert on each graph the point of homogeneous density $u = 0$ and the point of unstability where the isothermal compressibility diverges. The gas is stable from $\eta = 0$ till the point of unstability.

Let us find the boundary conditions for this equation. In order to have a regular solution at origin we have to impose

$$u'(0) = 0.$$  (3.27)

We find from eqs. 3.24 - 3.25 for fixed values of $\eta$ and $\xi$,

$$u'\left(\sqrt{3\xi R}\right) = -\frac{\eta R - \xi R}{\sqrt{3\xi R}}.$$  (3.28)

Eqs. 3.27 and 3.28 provide the boundary conditions for the nonlinear ordinary differential equation 3.26. In particular, they impose the dependence of $u_0 \equiv u(0)$ on $\eta R$ and $\xi R$.

Using eqs. 3.19 and 3.25 we can express the density of particles in terms of the solution of eq. 3.26

$$\rho(R) = \frac{\xi R}{\eta R} \exp\left[u(\sqrt{3\xi R} \frac{R}{R_{\text{max}}})\right].$$  (3.29)
Recalling that the local pressure is $P(R) = \frac{N T}{V} \rho(R)$ and using eq. (3.29), the contrast $C(\eta, \xi)$ between the pressure at the center and at the boundary can be written as,

$$C(\eta, \xi) \equiv \frac{P(0)}{P(R_{\text{max}})} = e^{u_0 - u(\sqrt{3\xi R})}.$$  

Using eqs. (3.19) and (3.25), the pressure at the boundary is given by

$$P(R_{\text{max}}) = \frac{N T}{V} f(\eta, \xi) = \frac{N T}{V} \rho(R_{\text{max}}) = \frac{N T}{V} \frac{\xi R}{\eta R} \exp[u(\sqrt{3\xi R})].$$  

(3.30)

while the pressure at the origin takes the value

$$P(0) = \frac{N T}{V} \frac{\xi R}{\eta R} e^{u_0}.$$  

We find from eq. (3.26) for small $x$ the expansion of $u(x)$ in powers of $x$ with the result,

$$u(x) = u_0 + (1 - e^{u_0}) \frac{x^2}{6} + e^{u_0}(e^{u_0} - 1) \frac{x^4}{120} + O(x^6).$$  

(3.31)
where we imposed eq. (3.27). The value of $u_0$ follows by imposing eq. (3.28). This power expansion is well suited near the center of the isothermal sphere [see sec IV.D].

Solving eq. (3.26) with the boundary conditions $u(0) = u_0, u'(0) = 0$ and eq. (3.28) yields the function $u(R)$ for the whole range of values of $\eta^R$ and $\xi^R$. We did that numerically using a fourth order Runge-Kutta method. We plot in fig. 4 the density at the boundary versus $\eta$ for $\xi = 0$ and in fig. 2 the density at the boundary versus $\eta$ for different values of $\xi$. We plot in fig. 3 the contrast $C(\eta, \xi)$ versus $\eta$ for $\xi = 0, 1, 2.04, ..., 3$. As we can see from fig. 3 the effect of the dark energy is to decrease the value of the contrast in agreement with the repulsive character of the dark energy. In particular, the contrast is unity for $\eta = \xi$. In this case the particle distribution is homogeneous [see eq. (4.1) and discussion there].

We compute in what follows the thermodynamic quantities as functions of the physical parameters $\eta$ and $\xi$.

### E. Free energy

![Graph showing the free energy per particle divided by $T$](image)

**FIG. 4:** The free energy per particle divided by $T$, $\frac{F(\eta, \xi) - F_0}{NT}$ versus $\eta$ for selected values of $\xi$.

We start by computing the free energy. Using eqs. (3.15)–(3.17) we find

$$
\frac{F - F_0}{NT} = \frac{1}{2} \left[ a + \int d^3 x \Phi(\vec{x}) e^{\Phi(\vec{x})} - \frac{2\pi}{3} \xi \int d^3 x \frac{x^2}{e^{\Phi(\vec{x})}}  \right].
$$

(3.32)
In order to express the Lagrange multiplier $a$ as functions of the physical parameters $\eta$ and $\xi$ we use eq. (3.18) where we can now integrate over the angles with the result,

$$
\Phi(R) = a + 4\pi \eta \left[ \frac{1}{R} \int_0^R dR' R'^2 e^{\Phi(R')} + \int_R^{R_{\max}} dR' R' e^{\Phi(R')} \right] + \frac{2\pi}{3} \xi R^2
$$

(3.33)

We call $f(\eta^R, \xi^R)$ the density of particles at the boundary $R = R_{\max}$ [see eq. (3.30)],

$$
f(\eta^R, \xi^R) = e^{\Phi(R_{\max})} = \frac{\xi^R}{\eta^R} e^{u(\sqrt{3}\xi^R)}.
$$

(3.34)

Setting $R = R_{\max}$ in eq. (3.33) yields for the Lagrange multiplier

$$
a = \ln f(\eta^R, \xi^R) - \eta^R - \frac{\xi^R}{2}.
$$

(3.35)

where we used the normalization of the densities eq. (3.16).

Inserting eq. (3.35) into the free energy (3.32), we find

$$
\frac{F - F_0}{NT} = \frac{1}{2} \left[ \ln f(\eta, \xi) - \eta^R - \frac{\xi^R}{2} \right] + 2\pi \int_0^{R_{\max}} R^2 dR \left[ \Phi(R) - \frac{2\pi}{3} \xi R^2 \right] e^{\Phi(R)}.
$$

(3.36)
These integrals are computed in appendix A with the result

$$\frac{F - F_0}{NT} = 3[1 - f(\eta^R, \xi^R)] - \eta^R + \left(\frac{2\xi^R}{\eta^R} + 1\right) \ln f(\eta^R, \xi^R)$$

$$+ \frac{\xi^R}{2} \left(1 - \frac{4\xi^R}{5\eta^R}\right) - 8\pi \frac{\xi^R}{\eta^R} \int_0^{R_{max}} dR R^2 \Phi(R).$$

(3.37)

We plot in fig. 6 the free energy as a function of $\eta$ for fixed $\xi$. It exhibits two Riemann sheets. We call $\eta_C(\xi)$ the position of the branch point. The curves have the same qualitative behaviour as for $\xi = 0$ except that the branch point $\eta_C(\xi)$ increases with $\xi$. The dark energy moves the plot of the free energy to lower values. It is a consequence of the negative contribution of the dark energy to the energy (eq. (2.7)) and the free energy (eq. (3.32)).

When matter dominates over dark energy we have $N m \gg V \Lambda$ and therefore according to eq. (3.37) $\eta \gg \xi$. In such limit the free energy becomes,

$$\frac{F - F_0}{NT} \eta \gg \xi \ln f(\eta^R, \xi^R) - \eta^R + 3\left[1 - f(\eta^R, \xi^R)\right].$$

We recognize here the free energy of a self-gravitating gas with $N$ particles of mass $m$ as it must be (see ref. 2).
F. Energy and entropy

We compute here the gravitational energy of the particles. The gravitational potential is the sum of the self-gravity of the particles plus their interaction with the dark energy [see eq. (2.7)]. Equivalently, we find for potential energy from eqs. (2.4), (3.7), (3.8) and (3.20),

\[
E = \frac{1}{2} \int \rho(\vec{q}) V_{\text{self}}(\vec{q}) \, d^3\vec{q} + \int \rho(\vec{q}) V_{\text{dark}}(\vec{q}) \, d^3\vec{q} = NT \left[ \frac{a}{2} - \frac{1}{2} \int d^3\vec{x} \, \Phi(\vec{x}) e^{\Phi(\vec{x})} - \frac{2\pi}{3} \xi \int d^3\vec{x} \, x^2 \, e^{\Phi(\vec{x})} \right],
\]

where we use the dimensionless coordinates \( \vec{x} = \frac{\vec{q}}{L} \) and \( a \) is given by eq. (3.35).

We compute these integrals for the spherically symmetric case in appendix A with the result,

\[
E = NT \left[ a^2 - \frac{1}{2} \int d^3\vec{x} \, \Phi(\vec{x}) e^{\Phi(\vec{x})} - \frac{2\pi}{3} \xi \int d^3\vec{x} \, x^2 \, e^{\Phi(\vec{x})} \right],
\]

(3.38)

We plot the gravitational energy as a function of \( \eta \) for fixed values of \( \xi \) in fig. 5. We see a two-sheeted structure with the branch point at \( \eta_C(\xi) \). The dark energy pushes the potential energy to lower values. As discussed before for the free energy, it is a consequence of the repulsive character of the dark energy.

We now compute the entropy as

\[
S = \frac{E - F}{T}
\]

using eqs. (3.37) and (3.38) for the free energy and the energy, respectively,

\[
\frac{S - S_0}{N} = 6 \left[ f(\eta^R, \xi^R) - 1 \right] + \eta^R - \frac{15}{8} \xi^R + \frac{19}{20} \frac{\xi^R}{\eta^R} \xi^R - \left( 1 + \frac{19}{4} \frac{\xi^R}{\eta^R} \right) \ln f(\eta^R, \xi^R) +
\]

\[
+ 19 \pi \frac{\xi^R}{\eta^R} \int_{0}^{R_{\text{max}}} dR \, R^2 \Phi(R).
\]

(3.39)

We plot the entropy as a function of \( \eta \) for fixed values of \( \xi \) in fig. 6. We see again a two-sheeted structure with the branch point at \( \eta_C(\xi) \).

G. Stability of the gaseous phase

In the precedent paragraphs we present the mean field theory of the selfgravitating gas in the presence of dark energy. Mean field theory gives exactly the physical quantities of the gaseous phase in the thermodynamic limit

\[
N \to \infty, \quad L \to \infty, \quad \Lambda \to 0,
\]

with

\[
\frac{N}{L} = \text{fixed}, \quad \Lambda L^2 = \text{fixed}.
\]

But it does not tell anything directly about the stability of the gaseous phase.

Necessary conditions for stability in the canonical ensemble are positive specific heat \( c_v \) and compressibility \( \kappa_T \). We compute below \( c_v \) per particle and \( K_T \) starting from their definitions,

\[
c_v = \frac{T}{N} \left( \frac{\partial S}{\partial T} \right)_V, \quad K_T = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T.
\]

(3.40)

The point of instability of the self-gravitating gas in absence of dark energy is \( \eta^0 = 1.510 \ldots \) (see fig. 1) where the compressibility diverges. At this point the isothermal compressibilities has a simple pole and changes sign. It is also the point where the speed of sound at the center of the sphere becomes imaginary. Small density fluctuations grow exponentially in time instead of exhibiting oscillatory propagation. Such a behavior clearly leads to instability and collapse.
In order to compute the specific heat for nonzero $\xi$ we introduce the function

$$g(x, u_0) = \left( \frac{\partial u}{\partial u_0} \right)_x (x, u_0). \quad (3.41)$$

Using eqs. (3.26) and (3.27) we find that this function $g$ obeys to the second order differential equation

$$\left( \frac{\partial^2 g}{\partial x^2} \right)_{u_0} + \frac{2}{x} \left( \frac{\partial g}{\partial x} \right)_{u_0} + e^{u(x, u_0)} g(x, u_0) = 0,$$

with the boundary conditions

$$g(0, u_0) = 1 \quad \left( \frac{\partial g}{\partial x} \right)_{u_0}(0, u_0) = 0.$$
$c_v$ is computed in the Appendix B with the result,

$$
c_v = 3 \left( \eta^R - \xi^R \right) f - \frac{57}{8} \frac{\xi^R}{\eta^R} \ln f - \frac{3}{2} \frac{\eta^R}{\eta^R} + \frac{57}{40} \frac{\xi^R}{\eta^R} \xi^R + \frac{57}{2} \frac{\xi^R}{\eta^R} \int_0^{R_{\text{max}}} dR R^2 \Phi(R, u_0)
- \left[ \eta^R \left( \frac{\partial u_0}{\partial \eta^R} \right) + \xi^R \left( \frac{\partial u_0}{\partial \xi^R} \right) \right] \left[ \left( 6 f - 1 - \frac{19}{4} \frac{\xi^R}{\eta^R} \right) g \left( \sqrt{3} \frac{\xi^R}{\eta^R}, u_0 \right) \right]
+ \frac{57}{4} \frac{\xi^R}{\eta^R} \frac{1}{(3\xi^R)^2} \int_0^{\sqrt{3} \xi^R} dx x^2 g(x, u_0)$$

(3.42)

We see in fig. 7 that the specific heat at constant volume versus $\eta$ for a fixed $\xi$ is positive in the first branch till the branch point ($\eta = \eta^C(\xi)$), where the specific heat tends to infinite and changes its sign. $c_v$ is negative in the second branch, therefore the second branch of the gaseous phase can not be stable in the canonical ensemble.

Using eqs. (3.30), the dimensionless isothermal compressibility takes the form,

$$
\kappa_T = \frac{N T}{V} K_T = \frac{1}{f(\eta^R, \xi^R) + \frac{\eta^R}{3} \frac{\partial f}{\partial \eta^R} - \frac{2 \xi^R}{3} \frac{\partial f}{\partial \xi^R}}
$$

(3.43)

We plot in fig. 8 the inverse of the isothermal compressibility versus $\eta$ for fixed $\xi = 0, 1$ and 2.
Table 1). Thus the positivity of the isothermal compressibility is a stronger condition of stability than the positivity of the specific heat at constant volume. For $\xi = 0$ Monte Carlo simulations shows that the gas collapse in a very dense phase at the point of divergence of the isothermal compressibility. Mean field indicates that the gas in the presence of dark energy is stable from $\eta = 0$ till the point $\eta = \eta^0(\xi)$. 

| $\xi$ | $\eta^0(\xi)$ | $\eta^C(\xi)$ |
|-------|----------------|----------------|
| 0     | 1.510...       | 1.561...       |
| 1     | 1.76...        | 1.938...       |
| 1.7   | 1.88...        | 2.282...       |
| 2.04  | 2.04...        | 2.476...       |
| 3     | 2.58...        | 3.133...       |
| 4.171 | 3.46...        | 4.171...       |
| 6     | 4.92...        | 6.307...       |

TABLE 1. $\eta^0(\xi)$ and $\eta^C(\xi)$ for different values of $\xi$

IV. PHYSICAL PICTURE

Let us first recall the properties of the self-gravitating gas in absence of dark energy ($\xi = 0$) [2]. The self-gravitating system can be in one of two phases: gaseous or highly condensed. The pressure at the boundary $f(\eta, \xi = 0)$ as a function of $\eta$ is given in fig. 1 in units of $\frac{N}{L}$. The function $f(\eta, \xi = 0)$ is calculated from mean field theory and describes the gaseous phase. $f(\eta, \xi = 0)$ has two Riemann sheets in the $\eta$-plane. The mean field theory gives exactly the physical quantities in the thermodynamic limit. The stability of the gas phase follows by studying the speed of sound inside the sphere and was moreover investigated by Monte Carlo simulations [2]. The gaseous phase is stable only in one piece of the curve $f(\eta, \xi = 0)$. This zone of stability depends of the statistical ensemble. In the first branch (PG-C) $f(\eta, 0)$ monotonically decreases for increasing $\eta$ as shown in fig. 1. That is, for decreasing temperature or increasing density $N/L$. The specific heat at constant volume is positive and the gaseous phase can be stable in the microcanonical and in the canonical ensemble. In fact, the gaseous phase is stable in the canonical ensemble from the point PG ($\eta = 0$, perfect gas) till the point O ($\eta^O = 1.510...$) where the speed of sound at the center becomes imaginary and the isothermal compressibility tends to infinite and changes its sign [2]. Monte-Carlo simulations show that the gas collapses at this point to a extremely condensed phase [2]. This is before the point C ($\eta^C = 1.561...$) where $f(\eta, 0)$ exhibits a vertical slope. C is the branch point of the function $f(\eta, 0)$. The determinant of small fluctuations vanishes and the specific heat at constant volume becomes infinite at C [2]. In the second branch (C-MC) the specific heat at constant volume is negative. Then, the gaseous phase cannot be stable in the canonical ensemble in this second branch but it can be stable in the microcanonical ensemble in (C-MC). In the microcanonical ensemble the gaseous phase is stable from PG to MC and collapses in a condensed phase at the point MC ($\eta^{MC} = 1.259...$). Beyond the point MC $f(\eta, 0)$ develops a nice spiral form, but the gas is unstable in all three statistical ensembles.

That is, solving just the mean field equation does not tell us whether the gas is stable. One has to compute in addition the isothermal compressibility (or the local speed of sound) and the specific heat at constant volume.

In the gaseous phase the free energy has a minimum for the density $\rho(\vec{x})$, solution of the saddle point equation [2]. This density is the most probable distribution which becomes absolutely certain in the infinite $N$ limit. All thermodynamic quantities follow from this density $\rho(\vec{x})$. In the condensed phase, the system becomes extremely dense with all particles on the top of each other. The solution of the saddle point equation ceases to be a minimum of the free energy and the mean field approach then fails to describe the condensed phase. The condensed phase has been found by Monte-Carlo methods in ref. [2].

Let us discuss the properties of the self-gravitating gas in the presence of dark energy.

A. The Triple Point

The effects of self-gravitation and dark energy are opposed. Self-gravitating forces are attractive while dark energy produces repulsion. If the parameters $\eta$ and $\xi$, which determine the strength of the gravitation and dark energy,
respectively, are equal, a exactly homogeneous sphere $\rho(R) = 1$ is a solution (see below). In such special case the self-gravitating gas behaves as a perfect gas (with $PV = NT$ everywhere).

Such homogeneous solution is present for all $\eta = \xi \geq 0$. There is a peculiar point where the homogeneous solution coincides with the point of instability $\eta = \eta^0(\xi)$. The point $\eta = \eta^0(\xi)$ is defined as the point where the isothermal compressibility diverges and changes its sign. That is, the two conditions

$$f(\eta = \xi, \xi) = 1 \quad \text{and} \quad \eta^0(\xi) = \xi$$

define an unique ‘triple point’ $\eta = \xi = \xi^0 = 2.04 \ldots$.

**B. The critical point as a function of $\xi$**

In fig. 2 we plot the pressure at the boundary versus $\eta$ for different values of $\xi$. We see that the plots for non zero $\xi$ exhibit a form analogous to the plot for $\xi = 0$. The effect of the dark energy is to conformally translate the curve to higher pressure and higher $\eta$. Notice that the value of the point of instability $\eta^0(\xi)$ increases for increasing $\xi$. For $\xi > 0$ the gas collapses at a higher density $\frac{N}{A}$ for a given temperature and for a lower temperature for a given density $\frac{N}{A}$ than for $\xi = 0$. That is, the domain of stability of the gas increases for increasing $\xi$. The dark energy has an antigravity effect that disfavours the collapse pushing the particles towards the boundary of the sphere.
FIG. 10: The densities $\rho(R)$ vs. the radial coordinate $R$ for $\xi = 2.04\ldots$. The density increases with $R$ in the first plot. The density is uniform in the second plot and coincides with the point of instability $\eta = \eta^0(\xi)$.

C. The particle density $\rho(R)$

We study the density $\rho(R)$ as a function of the radial coordinate $R$. The density $\rho(R)$ of the self-gravitating gas in absence of dark energy ($\xi = 0$) is always a decreasing function of $R$. This follows from the attractive character of gravitation. At the point $\text{PG}(\eta = 0)$ [see fig. 1] the gas is ideal and homogeneous, and $\rho(R) = 1$. The farther we move from the point $\text{PG}(\eta = 0)$ in the $\xi = 0$ case, the more inhomogeneous and faster decreasing is the density $\rho(R)$.

The $R$-dependence of the density $\rho(R)$ of a self-gravitating gas in presence of dark energy is more involved because the dark energy opposes to the attraction by gravity. The functions $\rho(R)$ and $u(x)$ have the same qualitative dependence in $R$ since they are related by exponentiation [see eq. (3.29)]

The behaviour at the center of the sphere ($R = 0$) is governed by the sign of $u_0$ since $u'(0) = 0$ [eq. (3.27)] and we find from eq. (3.31) that $u''(0) = \frac{1}{3}(1 - e^{u_0})$. Hence,

$$\text{sign}[u''(0)] = -\text{sign}[u_0].$$

The behaviour at the boundary of the sphere ($R = R_{\max}$) is governed by the sign of $\eta - \xi$ since according to eq. (3.28)

$$\text{sign}[u'(R_{\max})] = -\text{sign}[\eta^R - \xi^R].$$

The particle density exhibits for the stable solutions (namely, for positive $\kappa_T$) one of the following behaviours:
The density $\rho(R)$ vs. the radial coordinate $R$ is an increasing function for $\xi = 3$.

- **decreasing**: $u_0 > 0$ and $\eta > \xi$. The density $\rho(R)$ decreases from the center of the sphere till the boundary. This is the only behaviour present when $\xi = 0$.

- **increasing**: $u_0 < 0$ and $\eta < \xi$. The density $\rho(R)$ increases from the center of the sphere till the boundary.

More precisely, we find for the stable solutions the following properties in the interval $0 \leq x \leq \sqrt{3\xi R}$:

- when $\eta > \xi$, $u(x)$ is positive and $u'(x) < 0$.
- when $\eta < \xi$, $u(x)$ is negative and $u'(x) > 0$.

The case $\eta = \xi$ is particularly interesting, because we have $u'(0) = 0$ and $u'\left(\sqrt{3\xi R}\right) = 0$. This implies that

$$u(x) = 0, \quad (4.1)$$

is a solution of eq. (3.26). However, there exist another solution $u \neq 0$ for $\eta = \xi \geq 1.638 \ldots$, but this solution is never stable since $\kappa_T < 0$ for it. For the $u = 0$ solution, $\rho(R) = 1$ and the gas is homogeneous. The self-gravity of particles is exactly compensated by their interaction with the dark energy. We notice that the $u = 0$ homogeneous solution is stable only for $0 \leq \xi < 2.04 \ldots$.

We recall that in the vortex case $\eta = \xi < 0$ there are nontrivial solutions for $u(R)$ [11].
Let us now study the particle density $\rho(R)$ as a function of $\xi$. There are three stable cases ($\kappa_T > 0$). We illustrate these cases plotting $f(\eta, \xi)$, the surface density in units of $\frac{\kappa_T}{R^2}$ in fig. 2. That is, $f(\eta, \xi)$ vs. $\eta$ for $\xi = 1, 2.04\ldots$, and 3. The densities $\rho(R)$ are plotted as functions of $R$ for $\xi = 1, 2.04\ldots$, and 3 and chosen values of $\eta$ in figs. 9 to 11.

- **First case** $0 \leq \xi < 2.04\ldots$

  We illustrate this case choosing the value $\xi = 1$. As we see in fig. 2 the line $\eta = 1$ intercepts $f(\eta, \xi = 1)$ just in one point. This intersection corresponds to the solution $u(x) = 0$. At this point $\xi = \eta$ we also have $u_0 = 0$. We want to notice that when we move over the curve $f(\eta, \xi = \text{fixed})$ increasing $\eta$, $u_0$ **always increase**. Therefore, $u_0 < 0$ or point where $u(x) = 0$ while $u_0 > 0$ below the point where $u(x) = 0$. This property holds for all values of $\xi$ (for $\xi > 1.638\ldots$ the line $\eta = \xi$ intercepts $f(\eta, \xi)$ vs. $\eta$ in more than two points, for $1.638\ldots < \xi < 2.04\ldots$ the point $u = 0$ is in the upper branch).

  We have for this case $\eta^0(\xi) > \xi$. There are two zones of stability in fig. 2 for $f(\eta, \xi = 1)$:
  
  -the zone between the point $\eta = 0$ and the point of homogeneous density ($\eta = \xi, u = 0$), where $u_0 < 0$ and $\eta < \xi (< \eta^0(\xi))$, which corresponds to a density increasing with $R$.
  
  -the zone between the point of homogeneous density ($\eta = \xi, u = 0$) and the point of unstability $\eta = \eta^0(\xi)$, where $u_0 > 0$ and $\xi < \eta < \eta^0(\xi)$, which corresponds to a density decreasing with $R$.

  The gas is unstable in the zone beyond the critical point of unstability $\eta = \eta^0(\xi)$.

  We plot in fig. 2 the densities $\rho(R)$ vs. $R$ for $\xi = 1$ and $\eta = 0.67\ldots$, $\eta = 1$ and $\eta = 1.41\ldots$. The first plot corresponds to an increasing density. The dark energy dominates over gravitational attraction. The second plot corresponds to the uniform density associated to $u(x) = 0$. The density is homogeneous and dark energy exactly compensates gravitational attraction. The third plot corresponds to a decreasing density. The gravitational attraction dominates over dark energy.

- **Second case** $\xi = \xi^0 = 2.04\ldots$

  This is the value of $\xi$ where $\eta^0(\xi) = \xi$. This is a ‘triple point’ since it is defined by two conditions

  $$f(\eta = \xi, \xi) = 1 \quad \text{and} \quad \eta^0(\xi) = \xi$$

  In this case the point of unstability $\eta = \eta^0(\xi)$ coincides with the point of homogeneous density ($\eta = \xi^0, u = 0$). In fig. 2 $f(\eta, \xi = 2.04\ldots)$ the gas is stable in the zone between the point $\eta = 0$ and the point of homogeneous density ($\eta = \eta^0(\xi^0) = \xi^0, u = 0$), where $u_0 < 0$ and $\eta < \eta^0(\xi^0) = \xi^0$, which corresponds to an increasing density. The gas is unstable in the zone below the critical point of unstability $\eta = \eta^0(\xi)$. We plot in fig. 3 the densities $\rho(R)$ vs. $R$ for $\xi = 2.04\ldots$ and $\eta = 1, \eta = 2.04\ldots$. The first plot exhibits a density increasing with $R$. The dark energy dominates over the gravitational attraction. The second plot corresponds to the point of homogeneous density $u(x) = 0$, which coincides with the point of unstability $\eta = \eta^0(\xi)$.

- **Third case** $\xi > 2.04\ldots$

  We illustrate this case choosing the value $\xi = 3$. We have $\eta^0(\xi) < \xi$ for this case. The homogeneous solution $u = 0$ is unstable because $\kappa_T < 0$ for it. In the graph $f(\eta, \xi = 3)$ the gas is stable in the zone between the point $\eta = 0$ and the point of unstability $\eta = \eta^0(\xi)$, where $u_0 < 0$ and $\eta < \eta^0(\xi)(< \xi)$, which corresponds to an increasing density.

  We plot in fig. 4 the densities $\rho(R)$ vs. $R$ for $\xi = 3$ and $\eta = 1.379\ldots$. The plot corresponds to an increasing density. The dark energy dominates over gravitational attraction.

  It could be mentioned that the homogeneous solution $u(x) \equiv 0$ that we have for $\eta = \xi$ is the nonrelativistic analogue of the static Einstein universe. It was precisely thanks to the introduction of the cosmological constant that Einstein found such static and homogeneous universe.

### D. The Diluted expansion

We can obtain power expansions in $\eta$ and $\xi$ for the various physical quantities from eqs. (3.28) and (3.31). These analytic expressions are valid for $\eta \ll 1$ and $\xi \ll 1$. We find,

$$e^{u_0} = \frac{\eta}{\xi}\left[1 + \frac{3}{10}(\eta - \xi) + \frac{51}{1400}\xi^2 - \frac{183}{1400}\eta\xi + \frac{33}{350}\eta^2 + O(\eta^3, \eta^2 \xi, \eta \xi^2, \xi^3)\right].$$

(4.2)
Using now eqs. (3.30), (3.31) and (4.2), we obtain

$$f(\eta, \xi) = 1 - \frac{\eta - \xi}{5} - \frac{\eta - \xi}{175} (\eta + 2 \xi) + O(\eta^3, \eta^2 \xi, \eta \xi^2, \xi^3)$$

(4.3)

Eq. (4.3) reduces for $\xi = 0$ to the corresponding expression in absence of dark energy (see [2]). For $\eta = \xi$ we retrieve the solution $u = 0$ with $f = 1$. We see comparing with fig. 2 that the first order approximation $f(\eta, \xi) = 1 - \frac{\eta - \xi}{5}$ is valid even for large $\eta$ and $\xi$ till the vicinity of the point where the slope becomes vertical. This unexpectedly large domain of validity may be explained by the smallness ($\frac{1}{175}$) of the second order coefficient.

We clearly see that the cosmological constant translates as a whole the plot of $f(\eta, \xi = 0)$ by an amount $+\xi$. That is,

$$f(\eta, \xi) \simeq f(\eta - \xi, \xi = 0)$$

for $\eta \lesssim \eta_C(\xi)$.

V. THE LIMITING CASE $\xi \gg \eta$

We consider here the limit where the dark matter dominates over the ordinary matter ($\xi \gg \eta$). We have a gas of particles where their self-gravitational interactions are dominated by the interaction with the dark energy.

The mean field equation (3.22) becomes linear

$$\nabla^2 \Phi(x) = 4\pi \xi .$$

The exact solution for spherical symmetry takes the form,

$$\Phi(R) = \alpha + \frac{\xi R}{2} \left( \frac{R}{R_{\text{max}}} \right)^2 ,$$

(5.1)

where $\alpha$ is independent of $R$. We recall that $R$ runs in $0 \leq R \leq R_{\text{max}}$ with $R_{\text{max}} = \left( \frac{3}{4\pi} \right)^{\frac{1}{3}}$.

The density of matter follows from eqs. (3.19) and (5.1),

$$\rho(R) = e^\alpha e^{\frac{\xi R}{2}} \left( \frac{R}{R_{\text{max}}} \right)^2$$

(5.2)

Notice that this density increases monotonically with $R$ [see sec. IV.C].

The contrast is given by

$$C(0, \xi) = \frac{\rho(0)}{\rho(R_{\text{max}})} = e^{-\frac{\xi R}{2}} .$$

The density normalization (3.16) imposes that

$$e^{-\alpha(\xi)} = 3 \int_0^1 dy \ y^2 e^{\frac{\xi R}{2}} y^2 .$$

(5.3)

This integral can be expressed in terms of the probability integral $\Phi(x)$ as follows

$$e^{-\alpha(\xi)} = \frac{3}{\xi^2} \left[ i \sqrt{\frac{\pi}{2}} \Phi \left( i \sqrt{\frac{\xi}{2}} \right) + \sqrt{\xi} e^{\frac{\xi}{2}} \right] ,$$

where

$$\Phi(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x dy \ e^{-y^2} .$$

It is instructive to derive these results dropping the particle self-interaction in the partition function (3.9). The Hamiltonian is in this limit,

$$H = \sum_i \frac{p_i^2}{2m_i^2} - \frac{4\pi \Lambda}{3} \sum_i m_i q_i^2$$
The coordinate partition function \( Z_{\text{coor}} \) then **factorizes** and we find from eq. (3.9)

\[
Z = \frac{V^N}{N!} \left( \frac{mT}{2\pi} \right)^{3N/2} e^{-N\alpha(\xi^R)}
\]

where \( \alpha(\xi^R) \) is given by eq. (5.3) after using the dimensionless coordinates \( \bar{r} \equiv \bar{q}/L \).

The density at the boundary (3.34) takes here the form

\[
f(0, \xi^R) = \rho(R_{\text{max}}) = e^{\alpha(\xi^R)} e^{\xi^R / x^R}
\]

We plot in fig. 12 the density at the boundary versus \( \xi^R \). We give below the thermodynamic quantities in terms of \( f(0, \xi^R) \) and \( \xi^R \). That is, free energy, potential energy, pressure at the boundary, entropy, specific heat at constant volume and isothermal compressibility of the gas:

\[
\begin{align*}
\frac{F - F_0}{NT} &= \alpha(\xi^R) = \ln f(0, \xi^R) - \frac{\xi^R}{2}, \\
\frac{E_P}{NT} &= \frac{3}{2} \left[ 1 - f(0, \xi^R) \right] \\
\frac{PV}{NT} &= f(0, \xi^R), \\
\frac{S - S_0}{N} &= \frac{3}{2} \left[ 1 - f(0, \xi^R) \right] - \ln f(0, \xi^R) + \frac{\xi^R}{2}
\end{align*}
\]

**FIG. 12:** The density at the boundary \( f(\eta^R = 0, \xi^R) \) versus \( \xi^R \) when the effects of the dark matter dominate over the selfgravitation.
\[ c_v = \frac{3}{2} + \frac{3}{4} \xi R f(0, \xi R) + \frac{3}{2} \left(1 - f(0, \xi R)\right) \left[1 + \frac{3}{2} f(0, \xi R)\right] \]

\[ \frac{1}{\kappa T} = f(0, \xi R) \left[f(0, \xi R) - \frac{\xi R}{3}\right] \quad (5.4) \]

\[ F_0 \text{ and } S_0 \text{ are respectively the free energy and the entropy of the perfect gas. The second and third relations in eqs. (5.4) yield virial theorem,} \]

\[ \frac{PV}{NT} = 1 - \frac{2}{3} E_P \cdot \]

which is the limiting case of eq. (1.13) for \( \eta = 0 \).

We checked that \( \rho \text{ and } \kappa T \) are positive for all \( \xi R \geq 0 \). The gaseous phase is thus stable in this limit.

We compute the asymptotic behavior of the thermodynamic functions for \( \xi R \to \infty \)

\[ \frac{PV}{NT} = f(0, \xi R) = \frac{\xi R}{3} + \frac{1}{3} + \frac{2}{3} \xi R + \frac{4}{3} \xi R^2 + O\left(\frac{1}{\xi R^3}\right) \]

\[ \frac{F - F_0}{NT} = -\frac{\xi R}{2} + \ln \frac{\xi R}{3} + \frac{1}{3} \xi R + \frac{3}{2} \xi R^2 + O\left(\frac{1}{\xi R^3}\right) \]

\[ \frac{E_P}{NT} = -\frac{\xi R}{2} + 1 - \frac{1}{2} \xi R - 6 \xi R^2 + O\left(\frac{1}{\xi R^3}\right) \quad , \quad \frac{S - S_0}{N} = -\ln \frac{\xi R}{3} + 1 - \frac{2}{\xi R} - \frac{15}{2} \xi R^2 + O\left(\frac{1}{\xi R^3}\right) \]

\[ c_v = \frac{5}{2} - \frac{7}{2} \xi R + O\left(\frac{1}{\xi R^2}\right) \quad , \quad \kappa T = \frac{9}{\xi R} - \frac{27}{\xi R^2} - \frac{63}{\xi R^3} + O\left(\frac{1}{\xi R^4}\right) \quad (5.5) \]

and for \( \xi R \to 0 \)

\[ \frac{PV}{NT} = f(0, \xi R) = 1 + \frac{\xi R}{5} + \frac{2 \xi R^2}{175} + O(\xi R^3) \quad , \quad \frac{F - F_0}{NT} = -\frac{3 \xi R}{10} - \frac{3 \xi R^2}{350} + O(\xi R^3) \]

\[ \frac{E_P}{NT} = -\frac{3 \xi R}{10} - \frac{3 \xi R^2}{175} + O(\xi R^3) \quad , \quad \frac{S - S_0}{N} = -\frac{3 \xi R^2}{350} + O(\xi R^3) \]

\[ c_v = \frac{3}{2} + \frac{3 \xi R^2}{175} + O(\xi R^3) \quad , \quad \kappa T = 1 - \frac{\xi R}{15} + \frac{13}{1575} \xi R^2 + O(\xi R^3) \quad (5.6) \]

The particle density for \( \xi R \to \infty \) takes the form,

\[ \rho(R) = \frac{\xi R + 1}{3} e^{-\frac{\xi R}{\theta}} \left[1 - \left(\frac{\theta}{\rho_{m \text{max}}^2}\right)^2\right]. \]

Notice that the \( \xi R \to \infty \) corresponds to keeping \( \Lambda \) (and \( \eta \)) fixed in the thermodynamic limit [see eq. (1.14)].

VI. DISCUSSION AND CONCLUSIONS

The behaviour of the self-gravitating gas is significantly influenced by the cosmological constant (or not) depending on the value of the ratio

\[ R_\Lambda = \frac{\xi R}{\eta} = \frac{2 \Lambda V}{m N} = 2 \left(\frac{\text{dark energy}}{\text{mass}}\right)_V = 2 \frac{\Lambda}{\rho V} \]

where \( \rho V = N m/V \) is the average mass density in the volume \( V \) and where the subscript \((\_)_V\) indicates mass and dark energy inside the volume \( V \).

We display in Table 2 the typical value of \( R_\Lambda \) for some relevant astrophysical objects. Except for the universe as a whole \( R_\Lambda \) takes small values. For such values of \( R_\Lambda \) the effect of the cosmological constant is negligible as we see from the previous sections. The smallness of \( R_\Lambda \) stems from the tiny value of \( \Lambda = 6.63 \times 10^{-29} \text{g/cm}^3 \text{K}^2 \). The particle mass density is much larger than \( \Lambda \) in all situations except for the universe as a whole. However, the nonrelativistic and equilibrium treatment does not apply for the universe as a whole.
TABLE 2. The ratio $R_\Lambda = \frac{\xi}{\eta}$ for some astrophysical objects

For the cold clouds in the interstellar medium (ISM) considered in refs. [2, 3, 4] it is more convenient to express $\xi$ as

$$\xi = 1.2 \times 10^{-6} \frac{m(A)}{T(K)} [L(pc)]^2$$

where the mass of the particles $m$ is expressed in units of the hydrogen mass and $T$ in Kelvins. For typical cold clouds in the ISM we have $1 < L(pc) < 100$ and $T \sim 5$. This yields

$$\xi \sim 10^{-3} \text{ to } 10^{-7}.$$  

while $\eta \lesssim 1$. Again, we find $R_\Lambda \ll 1$.

The isothermal sphere in the presence of a cosmological constant has been considered in refs. [14]. The authors in ref. [14] restrict themselves to the case where the gravitational force points inward in all points. This happens when $\eta > \xi$ (using our notation). As showed here, when the dark energy dominates, there are stable solutions for $\xi > \eta$ where $\rho(R)$ increases with $R$. In these situations where the dark energy dominates the gravitational force points outward in all points.

It must be noticed that the mean field equation (3.26) admits solutions for which the density is not a monotonous function of $R$. However, such solutions are unstable since they correspond to negative values for the compressibility $\kappa_T$.

The effect produced by the cosmological constant is of an opposite nature to the Jeans' or gravothermal instabilities where the self-gravitating gas collapses due to the mutual attraction. Namely, the points C and MC in figs. 1 and 2.

VII. APPENDIX

A. Appendix A

The goal of this appendix is to compute in the spherically symmetric case the following integrals

$$A = \int d^3 \vec{x} \, e^{\Phi(\vec{x})} \left[ \Phi(\vec{x}) - \frac{2}{3} \xi x^2 \right], \quad B = -\int d^3 \vec{x} \, e^{\Phi(\vec{x})} \left[ \Phi(\vec{x}) + \frac{\pi}{3} \xi x^2 \right].$$

Using eq. (3.6), we obtain

$$A = \frac{1}{4\pi \eta} \int d^3 \vec{x} \, \Phi(\vec{x}) \, \nabla^2 \Phi(\vec{x}) + \frac{1}{6} \frac{\xi}{\eta} \int d^3 \vec{x} \, x^2 \, \nabla^2 \Phi(\vec{x})$$

$$+ \frac{\xi}{\eta} \int d^3 \vec{x} \, \Phi(\vec{x}) - \frac{2\pi \xi}{3} \frac{\xi}{\eta} \int d^3 \vec{x} \, x^2$$

$$B = \frac{1}{4\pi \eta} \int d^3 \vec{x} \, \Phi(\vec{x}) \, \nabla^2 \Phi(\vec{x}) + \frac{1}{12} \frac{\xi}{\eta} \int d^3 \vec{x} \, x^2 \, \nabla^2 \Phi(\vec{x})$$

$$- \frac{\xi}{\eta} \int d^3 \vec{x} \, \Phi(\vec{x}) - \pi \frac{\xi}{3} \frac{\xi}{\eta} \int d^3 \vec{x} \, x^2$$

| Object                  | $R_\Lambda$    |
|-------------------------|----------------|
| Stars                   | $\sim 10^{-29}$|
| Supergiants             | $\sim 10^{-22}$|
| Galaxies                | $\sim 10^{-6} - 10^{-5}$|
| Clusters of Galaxies    | $\sim 10^{-2}$|
| The Universe            | $\sim 4$       |
The terms in the first lines of $A$ and $B$ can be integrated twice by parts with the following result in the spherically symmetric case,

$$A = \left(1 - 2 \frac{\xi}{\eta}\right) \ln f(\eta, \xi) - \frac{\xi R}{2} \left(1 - \frac{2 \xi R}{3 \eta R}\right) + I + 2J$$

$$B = - \left(1 - \frac{\xi}{2 \eta}\right) \ln f(\eta, \xi) - \frac{\xi R}{4} \left(1 - \frac{2 \xi R}{5 \eta R}\right) - I - \frac{J}{2},$$  \hspace{1cm} (7.1)

where $f(\eta, \xi)$ is defined by eqs.(3.34) and

$$I \equiv \frac{1}{\eta} \int_0^{R_{\text{max}}} dR R^2 |\Phi'(R)|^2, \quad J \equiv \frac{4 \pi \xi}{\eta} \int_0^{R_{\text{max}}} dR R^2 \Phi(R).$$  \hspace{1cm} (7.2)

We compute now $I$ in terms of $J$, $f$, $\eta$, and $\xi$. Let us consider the function

$$B(R) = R^3 e^{\phi(R)}.$$  \hspace{1cm} (7.3)

We take the derivative with respect to $R$ and use eqs.(3.23),

$$B'(R) = 3 R^2 e^{\phi(R)} - \frac{1}{4 \pi \eta} R^3 \Phi'(R) \Phi''(R) - \frac{1}{2 \pi \eta} R^2 |\Phi'(R)|^2 + R^3 \frac{\xi}{\eta} \Phi'(R).$$  \hspace{1cm} (7.4)

We now integrate $B'(R)$ between 0 and $R_{\text{max}}$. Integrating by parts the second and the fourth terms and using eqs.(3.38) and (7.2) yields,

$$I = -6 J + 6 \left[1 - f(\eta, \xi)\right] + 6 \frac{\xi}{\eta} \ln f(\eta, \xi) - \eta R + \xi R \left(2 - \frac{\xi R}{\eta R}\right).$$  \hspace{1cm} (7.5)

We finally get for $A$ and $B$ using eqs.(7.1) and (7.3),

$$A = 6 \left[1 - f(\eta, \xi)\right] + \left(1 + 4 \frac{\xi}{\eta}\right) \ln f(\eta, \xi) - \eta R + \xi R \left(\frac{3}{2} - \frac{4 \xi}{5 \eta}\right) - 4 J$$

$$B = 6 \left[f(\eta, \xi) - 1\right] - \left(1 + \frac{11 \xi}{2 \eta}\right) \ln f(\eta, \xi) + \eta R - \xi R \left(\frac{9}{4} - \frac{11 \xi}{10 \eta}\right) + \frac{11}{2} J.$$  \hspace{1cm} (7.6)

## B. Appendix B

In this appendix, we will compute the specific heat at constant volume. Using eqs.(3.37), (3.39) and (3.40), we find

$$c_v = 6 T \left(\frac{\partial f}{\partial T}\right)_V - \frac{1}{\eta R} \left(\eta R - \xi R\right)^2 - \left(1 + 5 \frac{\xi R}{\eta R}\right) \frac{T}{f} \left(\frac{\partial f}{\partial T}\right)_V +$$

$$+ 20 \pi \frac{\xi R}{\eta R} T \left(\frac{\partial}{\partial T}\right)_V \int_0^{R_{\text{max}}} dR R^2 \Phi(R)$$  \hspace{1cm} (7.7)

Using eqs.(3.30) we calculate the partial derivative of $f$ with respect to $T$, \hspace{1cm}

$$T \left(\frac{\partial f}{\partial T}\right)_V = T \left(\frac{\partial u}{\partial T}\right)_V (x = \sqrt{3 \xi R}, u_0) f.$$  \hspace{1cm} (7.8)

We obtain from eqs.(3.37), (3.29) and (3.41),

$$T \left(\frac{\partial f}{\partial T}\right)_V = \left[\frac{1}{2} (\eta R - \xi R) + T \left(\frac{\partial u_0}{\partial T}\right)_V \right] g(x = \sqrt{3 \xi R}, u_0) f.$$  \hspace{1cm} (7.9)

Using eqs.(3.37), (3.29) and (3.41) we find

$$T \left(\frac{\partial}{\partial T}\right)_V \int_0^{R_{\text{max}}} dR R^2 \Phi(R) = -\frac{3}{8 \pi (3 \xi R)^{3/2}} \int_0^{\sqrt{3 \xi R}} dx x^3 \left(\frac{\partial u}{\partial x}\right)(x, u_0)$$
\[
+ \frac{3}{4 \pi} T \left( \frac{\partial u_0}{\partial T} \right)_V \frac{1}{(3 \xi R)^{3/2}} \int_0^{\sqrt{3} \xi R} \frac{dx}{x^2} g(x, u_0). \tag{7.7}
\]

Integrating by parts the first integral of the right member of the last equation, we obtain
\[
T \left. \frac{\partial}{\partial T} \int_0^{R_{\text{max}}} R^2 \Phi(R) \frac{dR}{V} \right|_{V} = \frac{9}{8 \pi} \frac{1}{(3 \xi R)^{3/2}} \int_0^{\sqrt{3} \xi R} dx \frac{x^2}{u(x, u_0)} \frac{\partial u_0}{\partial T} \frac{1}{V} \frac{1}{(3 \xi R)^{3/2}} \int_0^{\sqrt{3} \xi R} dx \frac{x^2}{g(x, u_0)} \tag{7.8}
\]

Using eqs. (3.25), (7.6) and (7.8) we express \( c_v \) as in eq. (3.42).

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[1] R. Emden, Gaskugeln, Teubner, Leipzig und Berlin, 1907.
S. Chandrasekhar, 'An introduction to the Study of Stellar Structure', Chicago Univ. Press, 1939.
W. B. Bonnor, Mon. Not. R. astr. Soc. 116, 351 (1956).
R. Ebert, Z. Astrophys. 37, 217 (1955).
V. A. Antonov, Vest. Leningrad Univ. 7, 135 (1962).
D. Lynden-Bell and R Wood, Mon. Not. R. astr. Soc. 138, 495 (1968).
G. Horvitz and J. Katz, Ap. J. 211, 226 (1977) and 222, 941 (1978).
T. Padmanabh, Phys. Rep. 188, 285 (1990).
W. C. Saslaw, 'Gravitational Physics of stellar and galactic systems', Cambridge Univ. Press, 1987.

[2] H. J. de Vega, N. Sánchez, Phys. Lett. B490, 180 (2000).
H. J. de Vega, N. Sánchez, Nucl. Phys. B 625, 409 (2002).
[3] H. J. de Vega, N. Sánchez, Nucl. Phys. B 625, 460 (2002).
[4] H. J. de Vega, J. A. Siebert, Phys. Rev. E 66, 016112 (2002).

[5] E. V. Votyakov, A. De Martino, D. H. E. Gross, Eur. Phys. J. B29, 593 (2002) and Nucl. Phys. B654, 427 (2003).
F. H. Chavanis, I. Isoplatov, Phys. Rev. E56, 036109 (2002).

[6] L. D. Landau and E. M. Lifchitz, The Classical Theory of Fields, vol. 2, Pergamon Press, 1962.

[7] P. J. E. Peebles, Principles of Physical Cosmology, Princeton (1993).
[8] P. J. E. Peebles, B. Ratra, Revs. Mod. Phys. 75, 559 (2003).

[9] H. J. de Vega and N. Sánchez, Phys. Lett. B 197, 320 (1987).

[10] L. D. Landau and E. M. Lifchitz, Physique Statistique, 4ème édition, Mir-Ellipses, 1996.
L. D. Landau and E. M. Lifchitz, Mécanique des Fluides, Éditions Mir, Moscou 1971.

[11] See for example, J. Binney and S. Tremaine, Galactic Dynamics, Princeton Univ. Press.

[12] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series and Products, Academic Press, New York, 1980.

[13] S. Perlmutter et al., ApJ, 517, 565 (1999).
A. G. Riess et al., Astron. J. 116, 1009 (1998).

[14] I. T. Iliev and P. R. Shapiro, MNRAS, 325, 468 (2001).
R. A. Sussman and X. Hernandez, astro-ph/0304385.