Finite Euler Products and the Riemann Hypothesis

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Graduate Workshop on Zeta functions, L-functions and their Applications
Outline

1. Approximations of $\zeta(s)$

2. A Function Related to $\zeta(s)$ and its Zeros

3. The Relation Between $\zeta(s)$ and $\zeta_X(s)$
I. Approximations of $\zeta(s)$
The Approximation of $\zeta(s)$ by Dirichlet Polynomials

We write $s = \sigma + it$ and assume $s$ is not near 1.

In the half–plane $\sigma > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$ 

If $X \geq 1$ and we estimate the tail trivially, we obtain

$$\zeta(s) = X \sum_{n=1}^{X} \frac{1}{n^s} + O\left(\frac{1}{\sigma - 1}\right).$$
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$$\zeta(s) = \sum_{n=1}^{X} n^{-s} + O\left(\frac{X^{1-\sigma}}{\sigma - 1}\right). $$
A crude form of the approximate functional equation extends this into the critical strip:

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{n^{-s}}{1-n^{-s}} + O(X^{-\sigma}) \quad (\sigma > 0). \]
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Example

When \( X = t \) we have

\[ \zeta(s) = \sum_{n \leq t} n^{-s} + O(t^{-\sigma}) \quad (\sigma > 0). \]
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**Theorem**

The Lindelöf Hypothesis is true if and only if

\[ \zeta(s) = \sum_{n \leq X} \frac{1}{n^s} + O(X^{1/2-\sigma}|t|^\varepsilon) \]

for \( \frac{1}{2} \leq \sigma \ll 1 \) and \( 1 \leq X \leq t^2 \).
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Thus, on LH even short truncations approximate $\zeta(s)$ well in $\sigma > 1/2$. 
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These are not equal if $X$ is small relative to $T$. 
The zeta-function also has an Euler product representation

\[ \zeta(s) = \prod_{\rho} \left(1 - \frac{1}{\rho^s}\right)^{-1} \quad (\sigma > 1). \]
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Can we extend this into the critical strip?
Yes, but we need to work with a weighted Euler product.

\[ \prod_{p \leq X^2} \left(1 - \frac{1}{p^s}\right) = \exp \left( \sum_{p \leq X^2} \sum_{k=1}^{\infty} \frac{1}{k} \frac{p}{p^k s} \right) \approx \exp \left( \sum_{n \leq X^2} \Lambda(n) n^s \log n \right) \]

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Specifically, we set

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$$\Lambda_X(n) = \begin{cases} 
\Lambda(n) & \text{if } n \leq X, \\
\Lambda(n) \left(2 - \frac{\log n}{\log X}\right) & \text{if } X < n \leq X^2, \\
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Remember

$$P_X(s) \approx \prod_{p \leq X^2} \left( 1 - \frac{1}{p^s} \right)^{-1}.$$
We also write

\[ Q_X(s) = \exp \left( \sum_{\rho} F_2( (s - \rho) \log X ) \right) \cdot \exp \left( \sum_{n=1}^{\infty} F_2( (s + 2n) \log X ) \right) \cdot \exp \left( F_2( (1 - s) \log X ) \right) \]
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with

\[ F_2(z) = 2 \int_{2z}^{\infty} \frac{e^{-w}}{w^2} dw - \int_{z}^{\infty} \frac{e^{-w}}{w^2} dw \quad (z \neq 0). \]
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For $z$ large $F_2(z)$ is small. For $z$ near 0

$$F_2(z) \sim \log(c z).$$
A Hybrid Formula for $\zeta(s)$

It follows that in the critical strip away from $s = 1$

$$Q_X(s) \approx \prod_{|\rho - s| \leq 1/ \log X} \left( c(s - \rho) \log X \right)$$
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Theorem (G., Hughes, Keating)

For $\sigma \geq 0$ and $X \geq 2$,

$$\zeta(s) = P_X(s) \cdot Q_X(s).$$
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**Theorem (G., Hughes, Keating)**

*For $\sigma \geq 0$ and $X \geq 2$,*

$$\zeta(s) = P_X(s) \cdot Q_X(s) .$$

Thus, in the critical strip away from $s = 1$

$$\zeta(s) \approx \prod_{p \leq X^2} \left( 1 - \frac{1}{p^s} \right)^{-1} \cdot \prod_{|\rho - s| \leq 1/\log X} \left( c (s - \rho) \log X \right)$$
Approximation by Finite Euler Products in the Strip

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We note that if RH holds and \( \sigma > \frac{1}{2} \), then
\[ \zeta(s) \approx \prod_{p \leq X^2} \left(1 - \frac{1}{p^s}\right)^{-1} \cdot \prod_{|s - \rho| \leq 1/\log X} \left(c(s - \rho) \log X\right) \]

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**Theorem**

Assume RH. Let \( 2 \leq X \leq t^2 \) and \( \frac{1}{2} + \frac{C \log \log t}{\log X} \leq \sigma \leq 1 \) with \( C > 1 \). Then

\[ \zeta(s) = P_X(s) \left(1 + O\left(\log^{(1-C)/2} t\right)\right). \]

Conversely, this implies \( \zeta(s) \) has at most a finite number of complex zeros in this region.
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The last estimate also shows that if $\sigma < 1/2$, then infinitely often in $t$

$$
P_X(s) \gg \exp \left( \frac{X^{1-2\sigma}}{\sqrt{\log X}} \right), \quad \text{which is very large}.
$$
II. A Function Related to \( \zeta(s) \) and its Zeros
On LH (and so on RH) we saw that for $\frac{1}{2} < \sigma \leq 1$ fixed,

$$\zeta(s) = \sum_{n \leq X} \frac{1}{n^s} + o(1),$$

even if $X$ is small.
Deficiency of the Sum Approximation on $\sigma = 1/2$

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Here

$$\chi(s) = \pi^{s-1/2} \Gamma(1/2 - s/2)/\Gamma(s/2).$$
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So essentially,

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we see that

$$\zeta\left(\frac{1}{2} + it\right) = \sum_{n \leq \sqrt{t}/2\pi} \frac{1}{n^{1/2+it}} + \chi\left(\frac{1}{2} + it\right) \sum_{n \leq \sqrt{t}/2\pi} \frac{1}{n^{1/2-it}} + o(1).$$
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$$\zeta(s) = P_X(s)(1 + o(1))$$

off by as $\sigma$ approaches $1/2$?
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whereas $\zeta(s) \ll t^\epsilon$. 

(University of Rochester)
Definition of $\zeta_X(s)$

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when $\sigma > 1/2$ is fixed. To study $\zeta_X(s)$ further we need a lemma.

Lemma

In $0 \leq \sigma \leq 1$, $|t| \geq 10$, $|\chi(s)| = 1$ if and only if $\sigma = 1/2$.

Furthermore,

$$\chi(s) = \left( \frac{t}{2\pi} \right)^{1/2-\sigma-it} e^{i t + i \pi/4} \left( 1 + O(t^{-1}) \right) .$$
The Riemann Hypothesis for $\zeta_X(s)$

**Theorem**

*All of the zeros of*

$$\zeta_X(s) = P_X(s) + \chi(s)P_X(\overline{s})$$

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**Proof.**

$$\zeta_X(s) = P_X(s) \left( 1 + \chi(s) \frac{P_X(\overline{s})}{P_X(s)} \right).$$

Also, $P_X(s)$ is never 0. Thus, if $s$ is a zero, $|\chi(\sigma + it)| = 1$. By the lemma, when $|t| \geq 10$ this implies that $\sigma = 1/2$. □
The Number of Zeros of $\zeta_X(s)$

The number of zeros of $\zeta(s)$ up to height $T$ is

$$N(T) = -\frac{1}{2\pi} \arg \chi\left(\frac{1}{2} + iT\right) + \frac{1}{2\pi} \arg \zeta\left(\frac{1}{2} + iT\right) + 1 = \frac{T}{2\pi} \log T - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right).$$

This vanishes if and only if $\frac{1}{2\pi} \arg \chi\left(\frac{1}{2} + iT\right) - \frac{1}{\pi} \arg P_X\left(\frac{1}{2} + iT\right) \equiv \frac{1}{2} \pmod{1}$. 

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Detecting Zeros of $\zeta_X(s)$

Set

$$F_X(t) = -\frac{1}{2\pi} \arg \chi(1/2 + it) + \frac{1}{\pi} \arg P_X(1/2 + it).$$
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$$\arg P_X(1/2 + it) = \text{Im} \log P_X(1/2 + it) = \text{Im} \sum_{n \leq X^2} \frac{\Lambda_X(n)}{n^{1/2+it} \log n} \sin(t \log n)$$

$$= -\sum_{n \leq X^2} \frac{\Lambda_X(n)}{n^{1/2} \log n}. $$
Lower Bound for the Number of Zeros

So

\[ F_X(t) = \frac{1}{2\pi} t \log \frac{t}{2\pi} - \frac{t}{2\pi} - \frac{1}{8} - \frac{1}{\pi} \sum_{n \leq X^2} \frac{\Lambda_X(n) \sin(t \log n)}{n^{1/2} \log n} + O\left(\frac{1}{t}\right). \]
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Ignoring the \( O(1/t) \), the condition that \( \zeta_X(1/2 + it) = 0 \) is that this is \( \equiv 1/2 \) (mod 1).
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**Theorem**

\[ N_X(T) \geq \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} - \frac{1}{\pi} \sum_{n \leq X^2} \frac{\Lambda_X(n) \sin(T \log n)}{n^{1/2} \log n} + O(1). \]
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How large can the sum be?
Call an increasing function $\Phi(t)$ *admissible* if

\[ |S(t)| \leq \Phi(t) \quad \text{and} \quad |\zeta(1/2 + it)| \ll \exp(\Phi(t)). \]
Admissible Functions

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$$\Phi(t) = \Omega(\sqrt{\log t / \log \log t}).$$
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- $\Phi(t) = (1/2 + \epsilon) \log t / \log \log t$ is admissible on RH.
The Sum on RH

Conjecture (Farmer, G., Hughes)

\[
\Phi(t) = \sqrt{\frac{1}{2} + \epsilon} \log t \log \log t \text{ is admissible, but}
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Theorem

Assume RH. Then

\[ \frac{1}{\pi} \sum_{n \leq X^2} \frac{\Lambda_X(n) \sin(t \log n)}{n^{1/2} \log n} \ll \Phi(t) + O \left( \frac{\log t}{\log X} \right). \]
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This is \( \ll \Phi(t) \) if \( X \geq \exp(c \log t / \Phi(t)) \) for some \( c > 0 \).

( Same bound as for \( S(t) \) ! )
If $F_X(t)$ is not monotonically increasing, there could be “extra” solutions of

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Extra Solutions

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$$F'_X(t) = \frac{1}{2\pi} \log \frac{t}{2\pi} - \frac{1}{\pi} \sum_{n \leq X^2} \frac{\Lambda_X(n) \cos(t \log n)}{n^{1/2}} + O\left(\frac{1}{t^2}\right).$$
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On RH the sum is $\ll \Phi(t) \log X$.

Thus, on RH there is a positive constant $C$, such that $F_X(t)$ is strictly increasing if

$$X < \exp \left(\frac{C \log t}{\Phi(t)}\right).$$
There are No Extra Solutions When $X$ is Small

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**Theorem**

Assume RH. There is a constant $C > 0$ such that if $X < \exp \left( \frac{C \log t}{\Phi(t)} \right)$, then

$$N_X(t) = \frac{t}{2\pi} \log \frac{t}{2\pi} - \frac{t}{2\pi} - \frac{1}{\pi} \sum_{n \leq X^2} \frac{\Lambda_X(n) \sin(t \log n)}{n^{1/2} \log n} + O(1).$$

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Unconditionally we can take $X$ larger, but then we only obtain an asymptotic estimate.
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**Theorem**

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**Theorem**

If $X \leq t^{o(1)}$, then

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1/2 + i\gamma is a simple zero of \( \zeta_X(s) \) if \( \zeta_X(1/2 + i\gamma) = 0 \), but \( \zeta_X'(1/2 + i\gamma) \neq 0 \).
Simple Zeros of $\zeta_X(s)$

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$1/2 + i\gamma$ is a simple zero of $\zeta_X(s)$ if $\zeta_X(1/2 + i\gamma) = 0$, but $\zeta'_X(1/2 + i\gamma) \neq 0$. Now

$$\zeta_X(1/2 + it) = P_X(1/2 + it) \left( 1 + \chi(1/2 + it) \frac{P_X(1/2 - it)}{P_X(1/2 + it)} \right)$$

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This vanishes at $1/2 + i\gamma$ if and only if $F_X'(\gamma) = 0$. 

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Recall that if $X$ is not too large,

$$F_X'(t) = \frac{1}{2\pi} \log \frac{t}{2\pi} - \frac{1}{\pi} \sum_{n \leq X^2} \frac{\Lambda_X(n) \cos(t \log n)}{n^{1/2}} + O\left(\frac{1}{t^2}\right) > 0.$$
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**Theorem**

Assume RH. There is a constant $C > 0$ such that if $X < \exp \left( C \log t / \Phi(t) \right)$, all the zeros of $\zeta_X(1/2 + it)$ with imaginary part $\geq 10$ are simple.
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**Theorem**

If \( X \leq \exp \left( o(\log^{1-\epsilon} t) \right) \), then \( \zeta_X(1/2 + it) \) has \( \sim T/2\pi \log (T/2\pi) \) simple zeros up to height \( T \).
Simple Zeros of $\zeta_X(s)$ When $X$ is Large

A zero $1/2 + i\gamma$ of $\zeta_X(s)$ is simple if and only if $F_X'(\gamma) \neq 0$.

We have just seen that on RH $F_X'(t) > 0$ if $X < \exp(C \log t / \Phi(t))$ (for some $C$), so all zeros are simple. But even when $X$ is very large, the odds that $F_X'(\gamma) = 0$ are quite small.
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III. The Relation Between $\zeta(s)$ and $\zeta_X(s)$
Comparing $\zeta(s)$ and $\zeta_X(s)$
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Here are graphs of $2|\zeta(1/2 + it)|$ and $|\zeta_X(1/2 + it)|$: 
Comparing $\zeta(s)$ and $\zeta_X(s)$

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![Graphs](image)

**Figure:** Graphs of $2|\zeta(1/2 + it)|$ (solid) and $|\zeta_X(1/2 + it)|$ (dotted) near $t = 114$ for $X = 10$ and $X = 300$, respectively.
Figure: Graphs of $2|\zeta(\frac{1}{2} + it)|$ (solid) and $|\zeta_X(\frac{1}{2} + it)|$ (dotted) near $t = 2000$ for $X = 10$ and $X = 300$, respectively.
Comparing $\zeta(s)$ and $\zeta_X(s)$

There are two striking features:

Zeros of $\zeta_X(1/2 + it)$ and $\zeta(1/2 + it)$ are close, even for small values of $X$.

$|\zeta_X(1/2 + it)|$ seems to approach $2|\zeta(1/2 + it)|$ as $X$ increases.

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Why?
The Heuristic Reason Why

$$|\zeta_X(1/2 + it)| \approx 2 |\zeta(1/2 + it)|$$
The Heuristic Reason Why
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\[ P_X(s) \] approximates \( \zeta(s) \) in \( \sigma > 1/2 \).

Since \( \chi(s) \) is small in \( \sigma > 1/2 \), \( \zeta_X(s) = P_X(s) + \chi(s)P_X(\overline{s}) \) also approximates \( \zeta(s) \).
The Heuristic Reason Why

$$|\zeta_X(1/2 + it)| \approx 2 |\zeta(1/2 + it)|$$

$P_X(s)$ approximates $\zeta(s)$ in $\sigma > 1/2$.

Since $\chi(s)$ is small in $\sigma > 1/2$, $\zeta_X(s) = P_X(s) + \chi(s)P_X(\overline{s})$ also approximates $\zeta(s)$.

But $\zeta_X(s)$ approximates $\mathcal{F}(s) = \zeta(s) + \chi(s)\zeta(\overline{s})$ even better.
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On \( \sigma = 1/2 \)
The Heuristic Reason Why
\[ |\zeta_X(1/2 + it)| \approx 2 |\zeta(1/2 + it)| \]

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\[ \mathcal{F}(1/2 + it) = \zeta(1/2 + it) + \chi(1/2 + it)\zeta(1/2 - it) \]
The Heuristic Reason Why
\[ |ζ_X(1/2 + it)| \approx 2 |ζ(1/2 + it)| \]

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Since \( χ(s) \) is small in \( σ > 1/2 \), \( ζ_X(s) = P_X(s) + χ(s)P_X(\overline{s}) \) also approximates \( ζ(s) \).

But \( ζ_X(s) \) approximates \( F(s) = ζ(s) + χ(s)ζ(\overline{s}) \) even better.

On \( σ = 1/2 \)

\[ F(1/2 + it) = ζ(1/2 + it) + χ(1/2 + it)ζ(1/2 - it) \]
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Since \( \chi(s) \) is small in \( \sigma > 1/2 \), \( \zeta_X(s) = P_X(s) + \chi(s)P_X(\bar{s}) \) also approximates \( \zeta(s) \).

But \( \zeta_X(s) \) approximates \( F(s) = \zeta(s) + \chi(s)\zeta(\bar{s}) \) even better.

On \( \sigma = 1/2 \)

\[
F(1/2 + it) = \zeta(1/2 + it) + \chi(1/2 + it)\zeta(1/2 - it) \\
= \zeta(1/2 + it) + \zeta(1/2 + it) \\
= 2\zeta(1/2 + it).
\]
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\(P_X(s)\) approximates \(\zeta(s)\) in \(\sigma > 1/2\).

Since \(\chi(s)\) is small in \(\sigma > 1/2\), \(\zeta_X(s) = P_X(s) + \chi(s)P_X(\bar{s})\) also approximates \(\zeta(s)\).

But \(\zeta_X(s)\) approximates \(\mathcal{F}(s) = \zeta(s) + \chi(s)\zeta(\bar{s})\) even better.

On \(\sigma = 1/2\)

\[
\mathcal{F}(1/2 + it) = \zeta(1/2 + it) + \chi(1/2 + it)\zeta(1/2 - it) \\
= \zeta(1/2 + it) + \zeta(1/2 + it) \\
= 2\zeta(1/2 + it).
\]

In fact, this suggests that \(\zeta_X(1/2 + it) \approx 2\zeta(1/2 + it)\).
Why Zeros of $\zeta_X(s)$ and $\zeta(s)$ are Close

$$F_X(t) = -\frac{1}{2\pi} \arg \chi(1/2 + it) + S(t) - \frac{1}{\pi} \text{Im} \sum_{\gamma} F_2(i(t - \gamma) \log X) + E$$
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Zeros of $\zeta_X(1/2 + it)$ occur when $F_X(t) \equiv 1/2 \pmod{1}$. 
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\[ \ll \mathcal{I} \frac{1}{\log^2 X} \sum_{\gamma} \frac{1}{(t - \gamma)^2} \to 0 \quad \text{as} \quad X \to \infty. \]
Theorem Relating $\zeta_X(s)$ and $\zeta(s)$

A similar argument shows that $\zeta_X(1/2 + it) \to 2 \zeta(1/2 + it)$.

Theorem
Assume RH. Let $I$ be a closed interval between two consecutive zeros of $\zeta(s)$ and let $t \in I$. Then $\zeta_X(1/2 + it) \to 2 \zeta(1/2 + it)$ as $X \to \infty$, and $\zeta_X(1/2 + it)$ has no zeros in $I$ for $X$ sufficiently large.
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A similar argument shows that \( \zeta_X(1/2 + it) \to 2\zeta(1/2 + it) \).

**Theorem**

Assume RH. Let \( \mathcal{I} \) be a closed interval between two consecutive zeros of \( \zeta(s) \) and let \( t \in \mathcal{I} \). Then

1. \( \zeta_X(1/2 + it) \to 2\zeta(1/2 + it) \) as \( X \to \infty \), and
2. \( \zeta_X(1/2 + it) \) has no zeros in \( \mathcal{I} \) for \( X \) sufficiently large.
Jon Keating studied $\zeta_{t/2\pi}(s)$ restricted to the one-half line in the early 90’s.
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Later Jon Keating and Eugene Bogomolny used $\zeta_{t/2\pi}(1/2 + it)$ as a heuristic tool for calculating the pair correlation function of the zeros of $\zeta(s)$. 
The general problem is to see what further insights we can gain into the behavior of $\zeta(s)$ and other $L$-functions from these models. Study the number of zeros of $\zeta_X(s)$ and the number of simple zeros when $X$ is large, say $X = t^{\alpha}$. $\zeta_X(s)$ approximates $F = \zeta(s) + \chi(s)\zeta(s)$ well in $\sigma > 1/2 + \log \log t/\log X$ and on $\sigma = 1/2$ when $X$ is large. What about in between?
Other Questions

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Other Questions

- The general problem is to see what further insights we can gain into the behavior of $\zeta(s)$ and other $L$-functions from these models.

- Study the number of zeros of $\zeta_X(s)$ and the number of simple zeros when $X$ is large, say $X = t^\alpha$.

- $\zeta_X(s)$ approximates $\mathcal{F} = \zeta(s) + \chi(s)\zeta(\overline{s})$ well in
  $\sigma > 1/2 + \log \log t / \log X$ and on $\sigma = 1/2$ when $X$ is large. What about in between?
Finite Euler products like \[ \prod_{p \leq X} \left( 1 - p^{-s} \right)^{-1} \] play a prominent role here and also in the hybrid Euler-Hadamard product representation of \( \zeta(s) \).

Very little is known analytically about the behavior of such products. For instance, how large is

\[ \int_0^T \left| \prod_{p \leq X} \left( 1 - p^{-s} \right)^{-1} \right|^2 \, dt \]?

Together with Jon Keating, we are beginning to determine the outlines of a theory of such moments, even when \( X \) is much larger than \( T \).
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