On the possibility of the critical behavior of LGT in the area of asymptotically large $\beta$.

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Abstract

Coupling dependence on lattice spacing and size is estimated analytically at $\beta \to \infty$ region where for $a \to 0$ the critical area is shifted in accordance with Callan-Symanzik relation. In considered approximation no trace of critical behavior is found in this area.

1 Introduction

In lattice gauge theory (LGT) the non-perturbative aspects are of primary interest, but the renormalization-group technique is added, as a rule, in a perturbative way and the relation between cut-off $\Lambda_L$ and gauge coupling is given by (see e.g. [4, 5])

$$a\Lambda_L \equiv R(\beta) = \exp \left\{ -\frac{\beta}{4Nb_0} + \frac{b_1}{2b_0^2} \ln \frac{\beta}{2Nb_0} \right\},$$ (1)

where $\beta = 2Ng^{-2}$, $g$ is coupling constant, $a$ is lattice spacing, $N$ is the number of colors and

$$b_0 = \frac{11}{3} \frac{N}{16\pi^2}; \quad b_1 = \frac{34}{3} \left( \frac{N}{16\pi^2} \right)^2.$$ (2)

Although the continuum limit in asymptotically free theories corresponds to $g \to 0$, there are reasons to believe, that such theories do not become perturbative at $a \sim 0$ [3]. On the basis of today's numeric computations, it is difficult to anticipate the behavior of $R(\beta)$ in the limit of $a \to 0$, taking perturbative calculations as a guidance. Indeed, numerical studies [4] showed deviations from (4) when the correlation length begins to grow. It is especially worth to note, that these deviations are of such a pattern, as
if the theory approaches the fixed point $g_0$ at which the Callan-Symanzik $\beta$-function $\beta_{CS}(g_0)$ has $n$-order zero

$$\beta_{CS}(g) \equiv -a \frac{\partial g}{\partial a} \simeq -b'_n (g^2 - g_0^2)^n$$

and consequently the theory is not asymptotically free. Cogent arguments in favor of such behavior of $\beta_{CS}(g_0)$ were given in [4].

Data on deep inelastic scattering does not eliminate the fixed point [5], however, the available data cannot distinguish between the first and second order fixed points. One may conclude only, that $\beta_{CS}(g)$ may, indeed, turn into zero, presumably located within intervals $0.8 < g_0 < 1.2$ for $n = 1$ or $0.6 < g_0 < 1$ for $n = 2$. Such intervals may appear even wider, but beyond specified intervals the errors doesn’t allow to determine $g_0$ from the data [5].

Phenomenological analysis of available MC lattice data in the $SU(2)$-gluodynamics shows no contradiction with the first order fixed point of $\beta_{CS}(g)$ [6]. In case of $SU(2)$-gluodynamics the presumed fixed point may located at $g_0 \simeq 0.563$ [6].

On the other hand, it is quite within a reason to suggest that the lattice data deviations from (1) are the result of the finite size effects. The dependence on spatial lattice size $N_\sigma$ may be almost removed, e.g., for the $SU(3)$ gauge theory [2], by

$$\beta_c(N_\tau, N_\sigma) = \beta_c(N_\tau, \infty) - (N_\tau/N_\sigma)^3 h$$

with $h \lesssim 0.1$ [2]. To remove remnant deviation, one may assume, that (2) contain some preasymptotic terms, which cause a disagreement with the data at achieved $\beta$ and vanish when $\beta \to \infty$. Indeed, a set of sophisticated tunings, such as $R(\beta) \to \lambda(\beta)R(\beta)$ and $\Lambda_L \to \Lambda_L(\beta)$ with $\lambda(\infty) = 1$ and $\Lambda_L(\infty) = const$ [1, 2, 3] may bring (1) into sync with available MC data.

We see that both approaches have enough room for adjustment and their capacity in experimental data description will be hardly exhausted in foreseeable future. We would like to try analytical estimations to find more or less tangible difference between such approaches. To study the continuum limit it is more convenient to regard spacing $a$ as an independent parameter and $\beta = \beta(a)$. As it follows from (3) fixed pole model predicts

$$\lim_{a \to 0} g(a) = g_0 \sim 1.$$

Unfortunately, in the area $g \sim 1$ the analytical methods are not efficient enough, but the perturbative approach (1) dictates $\beta \propto -\ln a \Lambda_L$ in a limit $a \to 0$. Moreover, critical coupling $\beta_c$ corresponds to the critical temperature $T_c = 1/(N_\tau a_c)$, where $a_c$ is defined by $\beta_c = \beta(a_c)$ and determined from (1). Therefore, if one claims $T_c = T_c(N_\tau) \to const$ and $\Lambda_L \to const$ with rising temporal lattice size $N_\tau$, then for the critical
coupling we get
\[ \beta_c \simeq 4N b_0 \ln \left( N \tau T_c / \Lambda_L \right) \simeq 4N b_0 \ln N \tau. \quad (5) \]

So the critical area is steadily displaced into the region of infinite \( \beta \) in which one may expect essential simplification of analytical computations. In this paper we try to at least partly clarify some features of critical behavior in the area of \( N \tau \to \infty \) and \( \beta \to \infty \) inaccessible in MC experiment. That allows us to establish a relation between \( \beta \) and \( a \) and compare it with (1).

## 2 Partition function at extremely large \( \beta \)

Conjecturable vanishing \( N \tau \)-dependance of \( \beta_c \) in a fixed point model [6]
\[ g_c = g_0 + (3.15N \tau)^{-b'_1}; \quad b'_1 = 0.111 \quad (6) \]
looks reasonable for \( Z(N) \) LGT where critical coupling \( \beta_c(N \tau, \infty) \) for \( N = 2, 3 \) tends to finite value \( \beta_c(\infty, \infty) = (1 - 1/N) \ln \left( 1 + \sqrt{N} \right) \). However, such finite size dependance is considered as unallowable for the standard \( SU(N) \) LGT as inconsistent with (1). Moreover, vanishing of \( N \tau \)-dependance of the crossover peak may be regarded in \( SU(N) \) LGT as a reason to conclude that such crossover is not the result of an ordinary phase transition [21].

Such difference between \( SU(N) \) and \( Z(N) \) may be regarded as still another reason to presume that the center subgroup loses its significance at \( \beta \to \infty \). It is generally assumed [20] that, in the continuum formulation there is no local distinction between pure \( SU(N) \) and \( SU(N)/Z(N) \), so it is expected that such difference will disappear in LGT as the continuum is approached. Although some speculative reasons indicate that \( Z(N) \) doesn’t play any essential role when \( a \to 0 \) (and consequently \( \beta \to \infty \)), yet it is not incurious to estimate how fast the center contribution may fade out with increasing \( \beta \).

Let us consider partition function of the pure Yang-Mills theory
\[ Z = \int \exp \left\{ -S \right\} \prod_{x, \mu} d\mu \left( U_\nu \left( x \right) \right); \quad U_\nu \left( x \right) \in SU \left( N \right). \quad (7) \]

As a rule, Wilson action contains the link variables \( U_\mu \left( x \right) \) in fundamental representation
\[ S = \sum_x \sum_{\mu > \nu} S_{\mu \nu} \left( x \right); \quad S_{\mu \nu} \left( x \right) = -\beta \frac{1}{N} \text{Re} \chi \left( U_{\mu \nu} \left( x \right) \right); \quad (8) \]
with $\chi(U) \equiv \text{Tr}U$ and the plaquette variable $U_{\mu\nu}(x)$ is defined as
\begin{equation}
U_{\mu\nu}(x) = U_\mu(x)U_\nu(x + \mu)U_\mu^\dagger(x + \nu)U_\nu^\dagger(x); \quad \mu, \nu = 0, 1, 2, 3.
\end{equation}
Following [22] we decompose $SU(N)$ matrices $U_{\mu\nu}$ as
\begin{equation}
U_{\mu\nu} = z_{\mu\nu}\tilde{U}_{\mu\nu};
\end{equation}
where
\begin{equation}
z_{\mu\nu}(x) = z_\mu(x)z_\nu(x + \mu)z_\mu^*(x + \nu)z_\nu^*(x) \in \mathbb{Z}(N)
\end{equation}
and
\begin{equation}
-\frac{\pi}{N} < \arg\chi\{\tilde{U}\} < \frac{\pi}{N},
\end{equation}
Now plaquette action $S_{\mu\nu}(x)$ in (8) may be rewritten as
\begin{equation}
-S_{\mu\nu}(x) = \hat{\beta}_{\mu\nu}\left(\tilde{U}(x)\right)z_{\mu\nu}(x)
\end{equation}
so we see that (13) presents the action of $\mathbb{Z}(N)$ gluodynamics with varying couplings
\begin{equation}
\hat{\beta}_{\mu\nu}\left(\tilde{U}(x)\right) = \beta\frac{1}{N}\text{Re}\chi\{\tilde{U}_{\mu\nu}(x)\}
\end{equation}
In particular for $N = 2$ gluodynamics in $(3+1)$-dimensional space we get
\begin{equation}
Z = \int d\tilde{U} \exp\left\{-\sum_{x,\mu\nu}\tilde{S}_{\mu\nu}(x) - \Xi\right\}
\end{equation}
with
\begin{equation}
-\tilde{S}_{\mu\nu}(x) = \ln\left(2\cosh\hat{\beta}_{\mu\nu}\right) - \beta
\end{equation}
and
\begin{equation}
\exp\{-\Xi\} = \sum_{(z)}\prod_{x,\mu\nu}\left(1 + z_{\mu\nu}(x)\tanh\hat{\beta}_{\mu\nu}\right).
\end{equation}

Duality transformation $z_{\mu\nu}(x) \rightarrow z'_{\rho\omega}(x')$ may be fulfilled for $\mathbb{Z}(N)$ gluodynamics with coupling being different for each plaquette [22] and one may get
\begin{equation}
\exp\{-\Xi\} = \sum_{(z')}\left(1 + z'_{\rho\omega}(x')\tanh\hat{\beta}'_{\rho\omega}\right).
\end{equation}
So the original plaquettes \( z_{\mu\nu}(x) \) carrying coupling \( \hat{\beta}_{\mu\nu}(\tilde{U}(x)) \) are transformed into dual \( Z(2) \) plaquettes \( z'_{\rho\omega}(x') \) with varying coupling \( \tilde{\beta}'_{\rho\omega} \), that are related to \( \hat{\beta}_{\mu\nu}(\tilde{U}(x)) \) by

\[
\tanh \tilde{\beta}'_{\rho\omega} = \exp \left\{ -2 \hat{\beta}_{\nu\mu}(\tilde{U}(x)) \right\}, \quad \mu \neq \nu \neq \rho \neq \omega.
\] (19)

Summing over the dual \( Z(2) \) variables \( \{z'\} \) we obtain

\[
-\Xi = \sum_A \exp \left\{ -2 \sum_{(x)_{\mu\nu} \in A} \hat{\beta}_{\nu\mu} \left( \tilde{U}(x) \right) \right\}
\] (20)

where \( \sum_A \) is taken over all closed self-avoiding\footnote{Although in \( SU(3) \) case surfaces \( A \) are not obligatory self-avoiding, this doesn’t make essential change.} connected surfaces \( A \) and equiform surfaces considered as different, if they are located at different places.

In the area \( \beta \sim 2N \) the critical behavior of the partition function is defined mainly by \( \Xi \) and, therefore, \( SU(N) \simeq Z(N) \) approximation, i.e.

\[
S \simeq \Xi; \quad \tilde{\beta}_{\nu\mu} \left( \tilde{U}(x) \right) \simeq \beta,
\] (21)

gives reasonable description of phase structure. Indeed, in the area \( g^2 \sim 1 \) the center elements carry most of the information about the string tension of the full theory \[23\]. Nonetheless, in the opposite extreme case \( SU(N) \simeq SU(N)/Z(N) \) one obtains a system with phase structure quite similar to the previous case. Thus, the action \( S_{\mu\nu}(x) \) is split into \( \tilde{S}_{\mu\nu}(x) \) with sharp maximum at \( \chi = N \) (\( \varphi_{\mu\nu}(x) = \varphi_{\mu\nu}^{\text{eff}} = 0 \)) and polynomial in \( \exp \left\{ -2 \hat{\beta}_{\nu\mu} \right\} \) with supremum at \( \chi = 0 \). Therefore, maximum \( S_{\mu\nu}(x) \) is located between \( \chi = N \) and \( \chi = 0 \) and for any small, but finite difference \( \pi/N - \left| \varphi_{\mu\nu}^{\text{eff}} \right| \) maximum \( S_{\mu\nu}(x) \) steadily moves to \( \chi = N \) with increasing \( \beta \). Really, it is enough to have \( \chi > 0 \) to get good grounds for discarding the terms \( \exp \left\{ -2 \hat{\beta}_{\nu\mu} \right\} \) for \( \beta \to \infty \) and in this case we obtain a version of positive plaquette action

\[
-S_{\mu\nu}(x) = -\tilde{S}_{\mu\nu}(x) + O \left( e^{-2\hat{\beta}_{\nu\mu}} \right) = \left| \tilde{\beta}_{\mu\nu} \right| - \beta + O \left( e^{-2\hat{\beta}_{\nu\mu}} \right)
\] (22)

Similar models from a more general point of view were intensively studied in recent years (see e.g. \[24\]). It was shown, that such approximation
didn’t change the continuum limit, i.e., the universality class. Moreover, the
Callan-Symansik β–function of the positive plaquette model shows no an-
noying “dip” inherent to standard Wilson action [24]. Thereby, as it can be
anticipated, for β → ∞ the center subgroup contributes only exponentially
small terms.

Now let’s consider a more general case. As it is known, the lattice act ion
of a given continuum theory is not unique and one could consider an extended
lattice theory, that includes higher representations and belongs to the same
universality class [8]. The study of the phase diagram of fundamental-adjoint
pure gauge systems revealed a non-trivial and considerably more compi-
lcated phase structure [9]. So we consider a more general case where the
plaquette action $S_{\mu \nu}$ includes an arbitrary set of irreducible representations
$j \equiv \{l_1, l_2, ..., l_{N-1}\}^{2}$

$$-S_{\mu \nu} (x) = \sum_j \beta_j (\chi_j (\varphi) - \chi_j (0)); \quad \beta_j = \beta \eta_j$$

(23)

with $\eta_j = const$ and $\chi_j (\varphi) = Tr \left\{ U_{\mu \nu}^{(j)} (x) \right\}$. Here plaquette variables $U_{\mu \nu}^{(j)} (x)$
are expressed in the same way as fundamental ones in [9] through the link
variables $U_{\nu}^{(j)} (x)$

$$U_{\nu}^{(j)} (x) = \exp \left\{ i \varphi_{\nu}^{(j)} (x) \right\}; \quad \varphi_{\nu}^{(j)} (x) \equiv \sum_{m=1}^{N^2-1} \varphi_{m,\nu} (x) \cdot T^{(j)}_m$$

(24)

where $SU(N)$ matrices $T^{(j)}_m$ are the group generators in irreducible represen-
tations $j$, that obey

$$Tr \left( T^{(j)}_n T^{(j)}_m \right) = \delta_{nm} \frac{TrI^{(j)}}{N^2 - 1} \cdot C_2 (j)$$

(25)

where $I^{(j)}$ is the unite matrix and $C_2 (j)$ is the quadratic Casimir operator.
For instance, in $SU(3)$ case Casimir operator is $C_2 (j) = (l_1^2 + l_2^2 - l_1 l_2) / 3 - 1$.

Since the center subgroup contribution is neglected, action $S$ has single
(up to a gauge transformation) minimum. If we fix the gauge having put
$U_{\mu}^{(j)} (x) = 1$ for ‘redundant’ links, this minimum will located at $\varphi_{m,\nu} = 0$.
Since the minimum point is nondegenerate with $S$ being infinitely different-
able, $Re S > 0$ and $Im S = 0$, the conditions are fulfilled to make it possible
to apply the Laplace method for the integral (13) evaluation and the result

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2 Integer numbers $l_n$ obey $l_{n+1} < l_n$. 6
for partition function computation may be written immediately. Yet we prefer to compute it gradually, estimating the errors introduced in each step.

To begin we expand \( \chi_j(\varphi) \) in the minimum point of action (23)

\[
\chi_j(\varphi) \simeq \chi_j(0) - \frac{\chi_j(0) C_2(j)}{2(N^2 - 1)} \varphi_{\nu\mu}(x)^2 + O(\varphi^4); \quad \chi_j(0) = \text{Tr} I^{(j)}
\]

(26)

with

\[
\varphi_{\nu\mu}(x)^2 \equiv \sum_{n=1}^{N^2-1} (\varphi_{n;\mu}(x) + \varphi_{n;\nu}(x + \mu) - \varphi_{n;\mu}(x + \nu) - \varphi_{n;\nu}(x))^2
\]

(27)

and finally get

\[
-S_{\mu\nu}(x) \simeq -\frac{\kappa \beta}{2} \varphi_{\nu\mu}(x)^2
\]

(28)

with

\[
\kappa = \frac{1}{N^2 - 1} \sum_j \eta_j \chi_j(0) C_2(j)
\]

(29)

Thus, in the area of asymptotically large \( \beta \) the higher representations contribution leads to plain renormalization of coupling \( \beta \to \kappa \beta \).

Action (28), known as the Manton action [10], originally is defined as

\[
S_M = \beta d^2(U_{\mu\nu}, I)
\]

(30)

where \( d(U_1, U_2) \) is the interval between \( U_1 \) and \( U_2 \) in group space, that for \( SU(2) \) may be written as

\[
d(U_1, U_2) = \arccos \frac{1}{2} \text{Tr} \left( U_1 U_2^\dagger \right),
\]

(31)

and we get

\[
\cos \left[ d(U_{\mu\nu}, I) \right] = \frac{1}{2} \text{Tr} (U) = \frac{1}{2} \chi = \cos \varphi_{\mu\nu}
\]

(32)

finally coming to (28).

In [14] solid grounds are given to suppose that Wilson and Manton actions belong to the same universality class, so Manton action may be regarded as a suitable alternative action with correct continuum limit [11]. Furthermore, in [11] weighty arguments are presented in favor of Manton action providing an
appreciably faster approach to the continuum limit than does the Wilson’s. Moreover, Manton action violates asymptotic scaling in the same direction as does the standard one, but is significantly more weakly [11].

As it was shown in [12], Manton action violates Osterwalder-Schrader positivity condition [13] essential for the continual theory. Such violation, however, appears only in the strong coupling region \( \beta \ll 1 \). Indeed, the positivity condition may be written as

\[
\int F(U_1) \exp \left\{ -\beta d^2(U_1, U_2) \right\} G(U_2) d\mu(U_1) d\mu(U_2) \geq 0
\]  

(33)

that in a case of \( SU(2) \) \( (\varphi_1 = -\varphi_2 = \phi/2) \) is equivalent to positivity requirement of all coefficients

\[
\zeta_j^M \equiv \int e^{-\beta \varphi^2/8} \chi_j d\mu = \frac{e^{-2j^2/\beta} - e^{-2(j+1)^2/\beta}}{\sqrt{2\pi \beta}} \left( 1 + O(e^{-\pi^2/\beta^2}) \right)
\]  

(34)

and in case of asymptotically large \( \beta \) Osterwalder-Schrader positivity condition is fulfilled.

The expression (34) allows to roughly estimate the error of considered approximation. If we compare \( \zeta_j^M \) coefficients with those computed for Wilson action

\[
\zeta_j^W \equiv \int e^{\beta (\cos^2 \phi - 1)} \chi_j d\mu = (2j + 1) e^{-\beta I_{2j+1}(\beta) / \beta}
\]  

(35)

one can easily show that \( \zeta_j^W / \zeta_j^M = 1 + O(1/\beta) \).

Now the measure, that in general is defined as

\[
d\mu = \sqrt{\text{det} (\partial_k \partial_{\phi^m})} \prod_{m=1}^{N^2-1} d\varphi_m
\]  

(36)

may be computed. The integrand in (13) has a sharp maximum at \( \varphi_{\nu\mu}(x) = 0 \), meaning that (up to gauge transformation) it has acute maximum at \( \varphi_{n\nu}(x) = 0 \). Hence computing the measure with the same accuracy as in (26), one may get

\[
d\mu \simeq C \exp \left\{ - \sum_{n=1}^{N^2-1} \varphi_n \varphi_n / (N^2 - 1) \right\} \prod_{n=1}^{N^2-1} d\varphi_n; \quad N = 2, 3.
\]  

(37)

Therefore, we may finally write for the partition function

\[
Z \simeq \int \exp \left\{ -\beta \left( 1 + \frac{1}{N^2 - 1} \right) \sum_{\nu, x} \varphi_{\nu\mu}(x)^2 / 2 \right\} \prod_{\nu, x} \prod_{n=1}^{N^2-1} d\varphi_{n\nu}(x)
\]  

(38)
since the measure contribution had been very important in the region $\beta \sim N$, in the area $\beta \gg 1$ it introduces negligible (of order $1/\beta$) correction.

As long as the integration area is compact $|\varphi| < \varphi_{\supr}$, theory remains non-trivial and, at least, on finite lattice shows critical behavior at $g \sim 1$. Since the action (28) yields a real positive-definite quadratic form, the error introduced by the extension of area integration by $\varphi_{n,\mu}(x)$ to infinity is of the order $\exp\{-\beta\varphi_{\supr}^2/2\}$. As a matter of fact, such extension is doubtful at finite $\beta$, but positively harmless for $\beta \gg 1$. So at asymptotically large $\beta$, integration in (38) may be done trivially by substitution $\varphi_{n,\mu}(x) \to \varphi_{n,\mu}(x)/\sqrt{\beta \kappa}$ and we immediately get

$$F = -\frac{1}{N_\tau N_\sigma^3} \ln Z(\beta \kappa) \approx -C \ln \frac{\beta_1}{\beta}$$

where $\beta_1$ is defined by

$$C \ln \beta_1 = \frac{1}{N_\tau N_\sigma^3} \ln Z(1)$$

Coefficient $C$ includes a factor to account for the fact that the number of integration variables is by approximately a quarter less than that of the links, because in order to fix the gauge we must freeze 'redundant' link variables in (28), i.e. put $\bar{\varphi}^{(j)}(x) = 0$ for corresponding link variables.

### 3 Fermion contribution

In perturbation theory the fermion part of action doesn’t play a leading role in computing of the Callan-Symanzik $\beta$-function $\beta_{\text{CS}}(g_0)$, nonetheless, its contribution is discernible, especially in three-loop calculation. Since the growing importance of the fermion contribution in nonperturbative calculations can’t be excluded for $\beta \to \infty$, such input should be, at least, roughly estimated.

Unfortunately, we can hardly attack the problem in its full, hence the approximations which hopefully capture some of the essential features of the physics may be considered. We attempt to study the fermionic action (see e.g. [16])

$$-S_F \equiv -\sum_x S_F(x) = n_f a^3 \sum_{x,x'} \left( \bar{\psi}_{x'} D^0_{x,x'} \psi_x + \bar{\xi}^r \bar{\psi}_{x'} \sum_{n=1}^{3} D_n^{x,x'} \psi_x \right)$$

with

$$D^0_{x,x'} = \frac{1-n_\nu}{2} U_\nu(x) \delta_{x,x'-\nu} + \frac{1+n_\nu}{2} U^\dagger_\nu(x') \delta_{x,x'+\nu} - (1 + \delta^0_\nu m a_\tau) \delta_{x',x},$$

$$D_n^{x,x'} = -\partial_n \delta_{x,x'} - \delta_{x,x'} \partial_n,$$

$$\bar{\xi}^r \bar{\psi}_{x'} = \varphi_{x'}(x) \neq 0, \quad \bar{\xi}^r \bar{\psi}_{x'} = 0, \quad \bar{\xi}^r \bar{\psi}_{x'} = 0.$$
on an extremely anisotropic lattice ($\tilde{\xi} \gg 1$) in the approximation where the terms proportional to $\tilde{\xi}^{-1}$ are discarded\footnote{17}. Here $\gamma_{\nu}$ are Dirac matrices, $n_f$ is the number of flavors and $\tilde{\xi} = \tilde{\xi} (g, \xi)$ is the 'bare' anisotropy parameter. The dependance of $\tilde{\xi}$ on coupling $g$ and 'renormalized' anisotropy parameter $\xi = a/a_\tau$ is defined by the condition of independence of physical values on spatial $a$ and temporal $a_\tau$ lattice spacings.

Fermion action $S$ doesn’t depend on $g$ explicitly, nonetheless, such dependance may be induced through $\tilde{\xi}$, because on the anisotropic lattice it enters in Yang-Mills part of action as well (temporal and spatial part is directly and inversely proportional to $\tilde{\xi}$, respectively). However, there are some reasons to believe that such dependence quickly disappears with $g \to 0$. Indeed, recent analysis\footnote{18} shows that, at least, for $1.5 \leq \xi \leq 6$ function $\tilde{\xi} (g, \xi)$ is linear in $\xi$ (with the natural condition of $\tilde{\xi} (g, 1) = 1$)

$$\tilde{\xi} (g, \xi) \approx \xi + q (g) (1 - \xi).$$  \hspace{1cm} (43)

The comparison of (43) with the data in\footnote{19} allows to conclude that $q (g)$ steadily decreases with $g \to 0$ and may be fitted 'on eye' as

$$q (g) \approx .02 - .02 g^2 + .3 g^4 + O (g^6) ;$$  \hspace{1cm} (44)

thereby the dependance $\tilde{\xi}$ on $g$ becomes inessential for small $g$. It allows us to assume that such dependance may be ignored for $g \to 0$ even in the area of large $\xi$, all the more that $S$ doesn’t depend on $g$ explicitly.

We fix diagonal static Hamiltonian gauge

$$U_0 (x, t)_{\nu \mu} = \begin{cases} 
\delta_{\nu \mu} & \text{for } t \neq 0 \\
\delta_{\nu \mu} \exp \{i \phi_{\nu} (x)\} & \text{for } t = 0 ;
\end{cases}$$ \hspace{1cm} (45)

where $\phi_\nu (x)$ e.g. for $N = 3$ is given by

$$\phi_{1,2} (x) = \pm \varphi_{3,0} (x, 0) + \varphi_{8,0} (x, 0) / \sqrt{3}; \hspace{1cm} \phi_3 (x) = -\phi_1 (x) - \phi_2 (x) = -2 \varphi_8 / \sqrt{3} \hspace{1cm} (46)$$

and after integration of partition function over fermion fields $\psi_x$ we get\footnote{17}

$$-S_F (x) = \sum_{\alpha=1}^3 \ln (\cos \phi_\alpha + \cosh \frac{m}{T}) + O (\xi^{-2}) \hspace{1cm} (47)$$

that gives (up to the additive constant)

$$-S_F (x) = - \frac{\varphi_{3,0}^2 (x, 0) + \varphi_{8,0}^2 (x, 0)}{1 + \cosh \frac{m}{T}} + O (\varphi^4) = - \frac{\varphi_0^2 (x, 0)}{1 + \cosh \frac{m}{T}} + O (\varphi^4) \hspace{1cm} (48)$$
So, we see that the fermion part includes only temporal link variables and yields a real positive-definite quadratic form as well as (28). However, unlike (28), $S_F$ does not depend on $\beta$ explicitly and plays a minor part for $\beta \to \infty$.

We would stress that fermion contribution into the action depends on $N_{\tau}$ and $a$ only in $N_{\tau}a = 1/T$ combination and one may hope that this property will survive in exact solution on isotropic lattices. In this case the fermion part, that doesn’t depend on $\beta$ explicitly, becomes negligible in comparison with (28) for $\beta \to \infty$.

4 Conclusions

We must conclude that in the area of extremely large $\beta$ the theory becomes trivial and partition function doesn’t show any sign of critical behavior. As it follows from (39), the condition of finiteness of free energy density

$$F = a^{-4}TF = -\frac{T}{aV} \ln Z \propto a^{-4} T \ln \frac{\beta}{\beta_1}; \quad V = (aN_\sigma)^3$$

in continuum limit leads to

$$\ln \frac{\beta}{\beta_1} \propto a^4$$

which strongly contradicts to (1), but doesn’t disagree in substance with fixed point model predictions.

To complete the picture let us consider partition function behavior for $\beta \to \infty$ in $SU(N) \simeq Z(N)$ approximation. Since on the dual lattice only surfaces of small area survive, the first non-trivial contribution comes from a six-plaquette surface (a cube) so, making allowance for (21), (19) and (20), we may write for the partition function

$$\ln Z \simeq -\Xi = N_\sigma^3 N_{\tau} \exp \{-12\beta\}$$

Although free energy density also behaves trivially

$$F = -V^{-1} T \ln Z \propto a^{-4} T \exp \{-12\beta\},$$

this, however, unexpectedly leads to passable agreement with (1) in continuum limit

$$g^{-2} \simeq 0.1 \ln \frac{1}{\Lambda_0 a}; \quad \Lambda_0 = \text{const.}$$
On the other hand, the presence of the critical point $\beta_c < \infty$ at finite $N_\tau$ is well-established in MC experiment. So, we have to conclude, that with $N_\tau \to \infty$ the $\beta_c$ critical point either expires, or gradually approaches some finite value.

Equation (1) is obtained in continuum field theory, that differs from LGT, at least, in two essential points: it is noncompact and there is no local distinction between pure $SU(N)$ and $SU(N)/Z(N)$ [20]. The arguments presented above allow us to think, that the last-mentioned difference quickly enough disappears in the region of asymptotically large $\beta$. Compact and noncompact formulations of the theory differ by the term of $O(\exp\{-\beta\})$ order.

In addition, as it can be seen from (29), coefficient $\kappa$ may be incorporated in $\beta$ by simple constant renormalization, so for $\beta \to \infty$ the action (28) is insensible to any set of irreducible representations entered in original action (28).

Unfortunately, as it follows from (28), with increasing $\beta$ the action looses its sensitivity to nonabelian properties of the gauge group, that with $d\mu \simeq d\varphi$ approximation reduces the gauge group $SU(N)$ to $U(1)^{N_2-1}$. One can hardly expect such ‘simplification’ in field theory even for $\beta \to \infty$.

Let us finally list the approximations considered in the area $\beta \to \infty$.

1. Center group contribution is neglected as exponentially small.

2. Yang-Mills part of the action is expanded into power series at the maximum and only quadratic terms are preserved.

3. Measure introduces into effective action the corrections of order $1/\beta$, that may be neglected for $\beta \to \infty$.

4. Fermion term is neglected. It may be partially justified by the fact, that on an extremely anisotropic lattice such term doesn’t depend on $\beta$ explicitly and therefore is actually negligible for $\beta \to \infty$.

5. Integration area is extended to infinity, that introduces an error of $\exp(-\beta)$ order.

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