WEAK SOLUTIONS TO COMPLEX MONGE-AMPÈRE EQUATION ON HYPERCONVEX DOMAINS

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ABSTRACT. We show a very general existence theorem to the complex Monge-Ampère type equation on hyperconvex domains.

1. INTRODUCTION

Let $\Omega$ be a bounded hyperconvex domain in $\mathbb{C}^n$ and $F$ a nonnegative function defined on $\mathbb{R} \times \Omega$. In the present note, we shall consider the existence and uniqueness of weak solution of the complex Monge-Ampère type equation

$$
(dd^c u)^n = F(u, \cdot) d\mu
$$

where $u$ is plurisubharmonic on $\Omega$ and $\mu$ is a nonnegative measure. This problem has been studied extensively by various authors, see for example [2], [4], [9], [10], [12], [14], [15], [16], [19], [20], [21], [22]... and reference therein for further information about Complex Monge-Ampère equations.

It was first considered by Bedford and Taylor in [3]. In connection with the problem of finding complete Kähler-Einstein metrics on pseudoconvex domains, Cheng and Yau [16] treated the case $F(t, z) = e^{Kt} f(z)$. More recently, Czyż [17] treated the case $F$ bounded by a function independent of the first variable and $\mu$ is the Monge-Ampère of a plurisubharmonic function $v$, generalizing some results of Cegrell [13], Kołodziej [21] and Cegrell and Kołodziej [14] [15]. In this paper we will consider a more general case. With notations introduced in the next section, our main result is stated as follows.

Main Theorem. Let $\Omega$ be a bounded hyperconvex domain and $\mu$ be a nonnegative measure which vanishes on all pluripolar subsets of $\Omega$. Assume that $F : \mathbb{R} \times \Omega \to [0, +\infty)$ is a measurable function such that:

1) For all $z \in \Omega$ the function $t \mapsto F(t, z)$ is continuous and nondecreasing;

2) For all $t \in \mathbb{R}$, the function $z \mapsto F(t, z)$ belongs to $L^1_{\text{loc}}(\Omega, \mu)$;

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3) There exists a function $v_0 \in \mathcal{N}^a$ which is a subsolution to (1.1) i.e.
$$(dd^c v_0)^n \geq F(v_0, \cdot) d\mu.$$ Then for any maximal function $f \in \mathcal{E}$ there exists a uniquely determined function $u \in \mathcal{N}^a(f)$ solution to the complex Monge-Ampère equation
$$(dd^c u)^n = F(u, \cdot) d\mu.$$ Note that the solution, as we will see in the proof, is given by the following upper envelope of all subsolutions;
$$u = \sup \{ v \in \mathcal{E}(\Omega); v \leq f \text{ and } (dd^c v)^n \geq F(v, \cdot) d\mu \},$$ where $\mathcal{E}(\Omega)$ is the set of non-positive plurisubharmonic functions defined on $\Omega$ for which the complex Monge-Ampère operator is well defined as nonnegative measure (a precise definition will be given shortly).

2. Background and Definitions

Recall that $\Omega \subset \mathbb{C}^n$, $n \geq 1$ is a bounded hyperconvex domain if it is a bounded, connected, and open set such that there exists a bounded plurisubharmonic function $\rho : \Omega \to (-\infty, 0)$ such that the closure of the set \{ $z \in \Omega : \rho(z) < c$ \} is compact in $\Omega$, for every $c \in (-\infty, 0)$. We denote by $PSH(\Omega)$ the family of plurisubharmonic functions defined on $\Omega$.

We say that a bounded plurisubharmonic function $\varphi$ defined on $\Omega$ belongs to $\mathcal{E}_0$ if $\lim_{z \to \zeta} \varphi(z) = 0$, for every $\zeta \in \partial \Omega$, and $\int_{\Omega} (dd^c \varphi)^n < +\infty$. See [10] for details.

Let the class $\mathcal{E}(\Omega)$ be the set of plurisubharmonic functions $u$ such that for all $z_0 \in \Omega$, there exists a neighborhood $V_{z_0}$ of $z_0$ and $u_j \in \mathcal{E}_0(\Omega)$ a decreasing sequence which converges towards $u$ in $V_{z_0}$ and satisfies
$$\sup_j \int_{\Omega} (dd^c u_j)^n < +\infty.$$ U.Cegrell [10] has shown that the operator $(dd^c \cdot)^n$ is well defined on $\mathcal{E}(\Omega)$, continuous under decreasing limits and the class $\mathcal{E}(\Omega)$ is stable under taking maximum i.e. if $u \in \mathcal{E}(\Omega)$ and $v \in PSH^-(\Omega)$ then $\max(u, v) \in \mathcal{E}(\Omega)$. $\mathcal{E}(\Omega)$ is the largest class with these properties (Theorem 4.5 in [10]). The class $\mathcal{E}(\Omega)$ has been further characterized by Z.Błocki [7], [8].

The class $\mathcal{F}(\Omega)$ is the “global version” of $\mathcal{E}(\Omega)$: a function $u$ belongs to $\mathcal{F}(\Omega)$ iff there exists a decreasing sequence $u_j \in \mathcal{E}_0(\Omega)$ converging towards $u$ in all of $\Omega$, which satisfies $\sup_j \int_{\Omega} (dd^c u_j)^n < +\infty$. Furthermore characterizations are given in [5] [6].
Define $\mathcal{N}(\Omega)$ the family of all functions $u \in \mathcal{E}(\Omega)$ which satisfies: if $v \in PSH(\Omega)$ is maximal and $u \leq v$ then $v \geq 0$, i.e. the smallest maximal psh function above $u$ is null. In fact, this class is the analogous of potentials for subharmonic functions (see [9] for more details).

The class $\mathcal{F}^a(\Omega)$ (resp. $\mathcal{N}^a(\Omega)$, $\mathcal{E}^a(\Omega)$) is the set of functions $u \in \mathcal{F}(\Omega)$ (resp. $u \in \mathcal{N}(\Omega)$, $u \in \mathcal{E}(\Omega)$) whose Monge-Ampère measure $(dd^c u)^n$ is absolutely continuous with respect to capacity i.e. it does not charge pluripolar sets.

Finally, for $f \in \mathcal{E}$, we denote by $\mathcal{N}(f)$ (resp. $\mathcal{F}(f)$) the family of those $u \in PSH(\Omega)$ such that there exists a function $\varphi \in \mathcal{N}$ (resp. $\varphi \in \mathcal{F}$) satisfying the following inequality

$$\varphi(z) + f(z) \leq u(z) \leq f(z) \quad \forall z \in \Omega.$$ 

We shall use repeatedly the following well known comparison principle from [4] as well as its generalizations to the class $\mathcal{N}(f)$ (cf [1] [9]).

**Theorem 2.1** ([1] [4] [9]). Let $f \in \mathcal{E}(\Omega)$ be a maximal function and $u, v, \in \mathcal{N}(f)$ be such that $(dd^c u)^n$ vanishes on all pluripolar sets in $\Omega$. Then

$$\int_{(u<v)} (dd^c v)^n \leq \int_{(u<v)} (dd^c u)^n.$$ 

Furthermore if $(dd^c u)^n = (dd^c v)^n$ then $u = v$.

### 3. Proof of Main Theorem

**Lemma 3.1** (Stability). Let $\mu$ be a finite nonnegative measure which vanishes on all pluripolar subsets of $\Omega$ and $f \in \mathcal{E}(\Omega)$ be a maximal function. Fix a function $v_0 \in \mathcal{E}(\Omega)$. Then for any $u_j, u \in \mathcal{N}^a(f)$ solutions to

$$(dd^c u_j)^n = h_j d\mu, \quad (dd^c u)^n = h d\mu$$

such that $0 \leq h d\mu, h_j d\mu \leq (dd^c v_0)^n$ and $h_j d\mu \rightarrow h d\mu$ as measures, we have that $u_j$ converges towards $u$ weakly.

The statement of the lemma fails if no control on the complex Monge-Ampère measures is assumed (see [15]).

**Proof.** It follows from the comparison principle that $u_j \geq v_0$, $\forall j \in \mathbb{N}$. Therefore by the general properties of psh functions $(u_j)_j$ is relatively compact in $L^1_{loc}$--topology. Let $\tilde{u} \in \mathcal{N}^a(f)$ be any closter point of the sequence $u_j$. Assume that $u_j \rightarrow \tilde{u}$ pointwise $d\lambda$--almost everywhere, here $d\lambda$ denotes the Lebesgue measure. By Lemma 2.1 in [11], after extracting a subsequence if
necessary, we have \( u_j \to \tilde{u} \, d\mu \)–almost everywhere. Then
\[
\tilde{u} = (\limsup_{j \to +\infty} u_j)^* = \lim_{j \to +\infty} (\sup_{k \geq j} u_k)^*.
\]
Now, consider the following auxiliary functions
\[
\tilde{u}_j = (\sup_{k \geq j} u_k)^* = (\lim_{l \to +\infty} (\sup_{l \geq k \geq j} u_k)^*).
\]
Observe that
\[
(dd^c \max(u_j, u_k))^n \geq \min(h_j, h_k) d\mu.
\]
Therefore
\[
(dd^c \tilde{u}_j)^n = \lim_{l \to +\infty} (dd^c \tilde{u}_j)^n \geq \lim_{l \to +\infty} \min_{l \geq k \geq j} h_k d\mu.
\]
We Let \( j \) converges to \(+\infty\) to get
\[
(dd^c \tilde{u})^n \geq h d\mu.
\]
Now, for the reverse inequality, pick \( \varphi \in E_0 \) a negative psh function. For any \( j \geq 1 \) and since \( u_j \leq \tilde{u}_j \), we have by integration by parts, which is valid in \( N^a(f) \) (cf [11]), that
\[
\int_{\Omega} -\varphi(dd^c u_j)^n \geq \int_{\Omega} -\varphi(dd^c \tilde{u}_j)^n.
\]
Therefore
\[
\lim_{j \to +\infty} \int_{\Omega} \varphi h_j d\mu \leq \lim_{j \to +\infty} \int_{\Omega} \varphi(dd^c \tilde{u}_j)^n = \int_{\Omega} \varphi(dd^c \tilde{u})^n.
\]
Together with the first inequality, we get
\[
\int_{\Omega} \varphi(dd^c \tilde{u})^n = \int_{\Omega} \varphi h d\mu, \quad \forall \varphi \in E_0.
\]
The set of test functions \( D(\Omega) \subset E_0 - E_0 \) (cf Lemma 2.1 in [11]) therefore the equality holds for any \( \varphi \in D(\Omega) \). Thence
\[
(dd^c \tilde{u})^n = h d\mu = (dd^c u)^n.
\]
Uniqueness in the class \( N^a(f) \) implies that \( \tilde{u} = u \) which concludes the proof.

**Proof of Main Theorem.** Assume first that \( F(t, .) \in L^1(d\mu) \). Then \( F(f, .) \in L^1(d\mu) \). It follows from [9] and [1] that the nonnegative measure \( F(f, .)d\mu \) is the Monge-Ampère measure of a function \( u_0 \) from the class \( F^a(f) \). Then
\[
(dd^c u_0)^n = F(f, .)d\mu \geq F(u_0, .)d\mu.
\]
We denote by \( A \) the set of all \( u \in F^a(f) \) such that \( u \geq u_0 \). The set \( A \) is convex and compact with respect to \( L^1(d\lambda) \)-topology, where \( d\lambda \) denotes the
Lebesgue measure in $\mathbb{C}^n$. Once more, by [9] (see also [1]), we have for each $u \in A$, there exists a unique $\hat{u} \in F^a(f)$ such that 
\[(dd^c\hat{u})^n = F(u,.)d\mu.\]

Since $\hat{u} \leq f$ and $F$ is nondecreasing in the first variable then 
\[(dd^c\hat{u})^n = F(u,.)d\mu \leq F(f,.)d\mu = (dd^c\hat{u})^n.\]

The comparison principle yields that $\hat{u} \geq u \geq u_0$, hence $\hat{u} \in A$.

We define the map $T : A \to A$ by $u \mapsto \hat{u}$. By Schauder’s fixed point theorem, we are done as soon as we show that the map $T$ is continuous. Let $u_j \in A$ be a sequence which converges towards $u \in A$. By Lemma 3.1, it’s enough to show that $F(u_j,.)d\mu \to F(u,.)d\mu$. After extracting a subsequence, we may assume that $u_j \to u \ d\lambda$-a.e. Applying Lemma 2.1. in [11], we get $u_j \to u \ d\mu$-a.e. By Lebesgue convergence theorem we have $F(u_j,.)d\mu \to F(u,.)d\mu$.

We now proceed to complete the proof of the general case. Let us consider the set
\[K := \{\varphi \in \mathcal{N}^a(f); (dd^c\varphi)^n \geq F(\varphi,.)d\mu\} .\]

Claim 1. $K$ is not empty: It follows from the monotonicity of $F$
\[(dd^c\varphi_0 + f)^n \geq (dd^c\varphi_0)^n \geq F(\varphi_0,.)d\mu \geq F(\varphi_0 + f,.)d\mu .\]

Then the function $\varphi_0 := \varphi_0 + f$ belongs to $K$.

Let denote
\[K_0 := \{\varphi \in K; \varphi \geq \varphi_0\} .\]

Claim 2. $K_0$ is stable by taking the maximum: Let $\varphi_1, \varphi_2 \in K_0$. It’s clear that $\max(u_1, u_2) \geq \varphi_0$. Since $\mathcal{N}^a(f)$ is stable by taking the maximum then $\max(u_1, u_2) \in \mathcal{N}^a(f)$. On the other hand, from [18], we have
\[(dd^c\max(u_1, u_2))^n \geq \sum_{(u_1 \geq u_2)} (dd^c u_1)^n + \sum_{(u_1 < u_2)} (dd^c u_2)^n \geq \sum_{(u_1 \geq u_2)} F(u_1,.)d\mu + \sum_{(u_1 < u_2)} F(u_2,.)d\mu \geq F(\max(u_1, u_2),.)d\mu .\]

Which implies that $\max(u_1, u_2) \in K_0$.

Claim 3. $K_0$ is compact in $L^1_{loc}(\Omega)$: It’s enough to prove that it’s closed. Let $\varphi_j \in K_0$ be a sequence converging towards $\varphi \in \mathcal{N}^a(f)$. The limit function is given by $\varphi = (\limsup_{j \to \infty} \varphi_j)^*$. Then $\varphi_0 \leq \varphi \leq f$. The continuity of
the complex Monge-Ampère operator and the properties of $F$ yield
\[
(dd^c \varphi)^n = \lim_{j \to +\infty} (dd^c \sup_{k \geq j} \varphi_k)^n \\
= \lim_{j \to +\infty} \lim_{l \to +\infty} (dd^c \max_{l \geq k \geq j} \varphi_k)^n \\
\geq \lim_{j \to +\infty} \lim_{l \to +\infty} F(\max_{l \geq k \geq j} \varphi_k, \cdot) \, d\mu.
\]

Therefore $\varphi \in K_0$.

Consider the following upper envelope
\[
\phi(z) := \sup \{ \varphi(z); \varphi \in K_0 \}.
\]

Notice that in order to get a psh function $\phi$ we should a priori replace $\phi$ by its upper semi-continuous regularization $\phi^*(z) := \limsup_{\zeta \to z} \phi(\zeta)$ but since $\phi^* \in K_0$ as well $\phi^*$ contributes to the envelope (i.e. $\phi^* \in K_0$) and then $\phi = \phi^*$.

Claim 4. $\phi$ is solution to Monge-Ampère equation (1.1): It follows from Choquet’s Lemma that there exists a sequence $\phi_j \in K_0$ such that
\[
\phi = (\limsup_{j \to +\infty} \phi_j)^*.
\]

Since the class $K_0$ is stable under taking the maximum, we can assume that $\phi_j$ is nondecreasing. We use the classical balayage procedure to prove that $\phi$ is actually a solution of (1.1). Pick $B \Subset \Omega$ a ball and define the function
\[
\phi^B_j(z) := \sup \{ v(z); v^* \leq \phi_j \text{ on } \partial B, \ v \in PSH(B) \}, \ \forall z \in B.
\]

By the first case, there exists a function $\tilde{\phi}_j \in \mathcal{F}^a(\phi_j^B, B)$ solution to the following equation
\[
(dd^c \tilde{\phi}_j)^n = \mathbf{1}_B F(\tilde{\phi}_j, \cdot) \, d\mu.
\]

In fact, the function $\tilde{\phi}_j$ is given by the following upper envelope
\[
\tilde{\phi}_j := \sup \{ w \in \mathcal{E}(B); w \leq \phi_j^B \text{ and } (dd^c w)^n \geq F(w, \cdot) \, d\mu \}.
\]

Indeed, if we denote by $g$ the right hand side function, then $\tilde{\phi}_j \leq g \leq \phi_j^B$. Hence $g \in \mathcal{F}^a(\phi_j^B, B)$. It follows by Lemma 3.3 in [1] that
\[
\int_{\Omega} \chi(dd^c \tilde{\phi}_j)^n \leq \int_{\Omega} \chi(dd^c g)^n, \ \forall \chi \in \mathcal{E}_0.
\]

On the other hand, as before, we have $g = (\lim g_k)^*$ where $g_k \in \mathcal{E}(B)$ is a nondecreasing sequence satisfying $\phi_j^B \geq g_k \geq \phi_j$ and $(dd^c g_k)^n \geq F(g_k, \cdot) \, d\mu$. Therefore $(dd^c g)^n \geq F(g, \cdot) \, d\mu$. Then
\[
(dd^c \tilde{\phi}_j)^n = F(\tilde{\phi}_j, \cdot) \, d\mu \leq F(g, \cdot) \, d\mu \leq (dd^c g)^n.
\]

Combining (3.1) and (3.2), one get
\[
(dd^c \tilde{\phi}_j)^n = (dd^c g)^n,
\]
therefore, by the comparison principle, we have $\tilde{\phi}_j = g$.

Now, for $j \in \mathbb{N}$, let consider the function $\psi_j$ defined on $\Omega$ by

$$
\psi_j(z) = \begin{cases} 
\tilde{\phi}_j(z) & \text{if } z \in B \\
\phi_j(z) & \text{if } z \not\in B
\end{cases}.
$$

On $B$ we have $\phi_j \leq \tilde{\phi}_j \leq \phi_j^B \leq f$ and on $\Omega \setminus B$ we have $\tilde{\phi}_j = \phi_j \leq f$. Hence $\psi_j \in \mathcal{N}^a(f)$. From the definition of $\psi_j$, we deduce that $(dd^c\psi_j)^n \geq F(\psi_j, \cdot) d\mu$. Therefore $\psi_j \in \mathcal{K}_0$ and

$$
\phi = (\lim_{j \to +\infty} \psi_j)^*.
$$

Since the complex Monge-Ampère operator is continuous under monotonic sequences and $B$ is arbitrarily chosen, to conclude the proof of the claim it’s enough to observe that the sequence $\psi_j$ is nondecreasing.

Uniqueness follows in a classical way from the comparison principle and the monotonicity of the function $F$. Indeed, assume that there exist two solutions $\varphi_1$ and $\varphi_2$ in $\mathcal{N}^a(f)$ such that

$$(dd^c\varphi_i)^n = F(\varphi_i, \cdot) d\mu, \quad i = 1, 2.$$ 

Since the function $F$ is nondecreasing in the first variable, then

$$
F(\varphi_1, \cdot) d\mu \leq F(\varphi_2, \cdot) d\mu \quad \text{on} \quad (\varphi_1 < \varphi_2).
$$

On the other hand, by the comparison principle we have

$$
\int_{(\varphi_1 < \varphi_2)} F(\varphi_2, \cdot) d\mu = \int_{(\varphi_1 < \varphi_2)} (dd^c\varphi_2)^n \leq \int_{(\varphi_1 < \varphi_2)} (dd^c\varphi_1)^n = \int_{(\varphi_1 < \varphi_2)} F(\varphi_2, \cdot) d\mu.
$$

Therefore

$$
F(\varphi_1, \cdot) d\mu = F(\varphi_2, \cdot) d\mu \quad \text{on} \quad (\varphi_1 < \varphi_2).
$$

In the same way, we get the equality on $(\varphi_1 > \varphi_2)$ and then on $\Omega$. Hence $(dd^c\varphi_1)^n = (dd^c\varphi_2)^n$ on $\Omega$. Therefore uniqueness in the class $\mathcal{N}^a(f)$ yields $\varphi_1 = \varphi_2$ and the proof is completed.

Remarks. 1– We have no precise knowledge when the subsolution of the equation (1.1) exists. However, if there exists a negative function $\psi \in PSH(\Omega)$ such that

$$
\int_{\Omega} -\psi F(0, \cdot) d\mu < \infty,
$$

then (1.1) admits a subsolution $v \in \mathcal{N}^a$. This is an immediate consequence of Proposition 5.2 in [9].

2– The condition 2 in the Theorem is necessary.
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