THE DERIVATIVE OF INFLUENCE FUNCTION,
LOCATION BREAKDOWN POINT,
GROUP INFLUENCE
AND
REGRESSION RESIDUALS’ PLOTS

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Summary

In several linear regression data sets, \( Y(\in R) \) on \( X(\in R^p) \), visual comparisons of \( L_1 \) and \( L_2 \)-residuals’ plots indicate bad leverage cases. The phenomenon is confirmed theoretically by introducing Location Breakdown Point (LBP) of a functional \( T \): any point where the derivative of \( T \)’s Influence Function either takes values at infinities or does not exist. Guidelines for the plots' visual comparisons as diagnostic are provided. The new tools used include \( E \)-matrix and suggest influence diagnostic \( RINFIN \) which measures the distance in the derivatives of \( L_2 \)-residuals at \((x, y)\) from model \( F \) and from gross-error model \( F_{e,x,y} \). The larger \( RINFIN(x, y) \) is, the larger \((x, y)\)’s influence in \( L_2 \)-regression residual is. \( RINFIN \) allows measuring group influence of \( k \) \( x \)-neighboring data cases in a size \( n \) sample using their average, \((\bar{x}_k, \bar{y}_k)\), as one case with weight \( \epsilon = k/n \). For high dimensional, simulated data, the misclassification proportion of bad leverage cases in data’s \( RINFIN \)-ordering decreases to zero as \( p \) increases, thus reconfirming the blessing of high dimensionality in the detection of remote clusters. The visual diagnostic and \( RINFIN \) are successful in applications and complement each other.

Some key words: Breakdown Point, Influence Function, Least Absolute Deviation Residuals, Least Squares Residuals, Leverage, Location Breakdown Point, Local-Shift-Sensitivity, Masking, Residual’s Influence Index (RINFIN)

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1 Introduction

Tukey (1962, p.60) wrote: “Procedures of diagnosis, and procedures to extract indications rather than extract conclusions, will have to play a large part in the future of data analyses and graphical techniques offer great possibilities in both areas.”

In linear regression of $Y$ on $X$ it is often assumed that the data follows probability model $F; Y \in R, X \in R^p$. However, “It also happens not infrequently that only part of the data obeys a different model.” (Hampel et al., 1986). Thus, in reality, data may follow gross-error model $F_{\epsilon,G} = (1 - \epsilon)F + \epsilon G$ (Huber, 1964); $G$ is gross-error probability, $0 < \epsilon < 1$. The goal of this work is to provide simple and fast procedures for extracting indications when remote cases (from $G$) affect the statistical analysis in least squares ($L_2$) regression. These procedures are particularly useful for Big Data, when the number of predictors, $p$, and the sample size, $n$, increase to infinity; for $L_2$-regression $p < n$.

The initial motivation was provided by the observation, in several data sets, that: neighboring, remote factor space cases, $(x, y)$, have least absolute deviation ($L_1$) regression residuals significantly larger in size than the corresponding $L_2$-regression residuals; see, e.g., Figures 1 and 2 in section 4.

This phenomenon is theoretically confirmed herein using new tools: $E$-matrix and derivatives of the regression coefficients’ Influence Functions. The latter allow calculating changes in $L_1$ and $L_2$-regressions residuals for small perturbations of $(x, y)$ from $F$ and also from $F_{\epsilon,G}$. The calculation of $L_1$-residuals’ changes is possible when $(x, y)$ is not $L_1$-Location Breakdown Point (LBP), thus first order linear approximation of the $L_1$-residual near $(x, y)$ is valid. LBP complements the notion of Weight Breakdown Point (Hampel, 1971). Derivatives of Influence Functions indicate a new influence diagnostic: $RINFIN$ (see below).

A simple graphical method is thus proposed to detect rapidly in linear regression data remote cases, $(x, y)$, affecting drastically $L_2$-regression coefficients. Plots of absolute regression residuals against square $x$-length provide the visual indications when $L_1$-residuals’ sizes for $x$-remote cases are larger, e.g. double in size, than the corresponding $L_2$-residuals, thus causing a larger visual gap in the $L_1$ plot. A different pattern in the residuals is used
to identify other types of outlying cases near a LBP, as described in section 4.

The regression diagnostic, *Residual’s Influence Index (RINFIN)* for \((x, y)\), is also introduced that measures the distance in the derivatives of \(L_2\)-residuals when \((x, y)\) follows either probability \(F\) (the model) or its gross-error mixture \(F_{\epsilon,x,y}\), i.e. \((1 - \epsilon)F + \epsilon\Delta_{x,y}\), \(0 < \epsilon < 1\), \(\Delta_u\) unit mass at \(u\). The larger RINFIN\((x, y)\) is, the larger \((x, y)\)'s influence in the \(L_2\)-residual is. For a group of remote \(x\)-neighboring cases from gross-error probability \(G\), with proportion \(\epsilon\) in the data, their group average \((\bar{x}, \bar{y})\) is used as one case from \(F_{\epsilon,x,y}\) to calculate the group’s influence, RINFIN\((\bar{x}, \bar{y})\), that depends also on \(\epsilon\). This is an advantage over other methods that use group deletion to determine influence and are exposed i) to masking from neighboring \(G\) cases that remain in the model, ii) to a combinatorial explosion due to the very large number of groups to exclude.

RINFIN is successful with several known data sets and in simulations, especially when the dimension \(p\) of the data is large. In simulations with normal mixtures and \(n\) fixed, the misclassification proportion of bad leverage cases in the RINFIN ordering of the data decreases to zero as \(p\) increases. The effect of increase in \(p\)-values is equivalent to larger standardized distance between the means of \(F\) and \(G\) in \(F_{\epsilon,G}\). A similar phenomenon has been observed and confirmed theoretically for mixture densities in a Projection Pursuit cluster detection method (Yatracos, 2013) due to the “separation” of the mixtures’components, measured by their Hellinger’s distance, as \(p\) increases.

In a nutshell, the justification for the visual phenomenon and the form of RINFIN are presented for simple linear regression:

i) For \(\epsilon(> 0)\) small in \(F_{\epsilon,x,y}\) residuals are compared,

\[
\frac{|r_{2,x,y}(x, y) - r_2(x, y)|}{|r_{1,x,y}(x, y) - r_1(x, y)|} \approx C|r_2(x, y)|, \tag{1}
\]

\(r_m\) and \(r_{m,x,y}\) are \(L_m\)-residuals, respectively, for \(F\)-regression and \(F_{\epsilon,x,y}\)-regression, \(m = 1, 2\); \(C\) is constant, “\(\approx\)” denotes approximation. When \((x, y)\) is gross-error and for \(L_1\) and \(L_2\) \(F\)-regressions \(r_1(x, y) \approx r_2(x, y)\), with \(|r_2(x, y)| > 1\), from (1) it follows for \(F_{\epsilon,x,y}\)-regression that \(L_2\) residual of \((x, y)\) is reduced more than its \(L_1\) residual, especially when \(|x|\) is large (because then \(|r_2(x, y)|\) is also large).
ii) $L_2$-residual’s influence index of $(x, y)$ from gross-error model $F_{\epsilon, x, y}$ is

\[ RINFIN(x, y) = \epsilon \cdot \frac{|2r_2(x, y)(x - EX) - \beta_{1,L_2}[(x - EX)^2 + Var(X)]|}{Var(X)}; \]

(2)

$L_2$-residual ($r_2$), slope ($\beta_{1,L_2}$), mean ($EX$) and variance ($VarX$) are all under $F$.

$LBP$ of a statistical functional $T$ is motivated and introduced in section 2 using $x$-perturbations of $F_{\epsilon, x}$. $LBP$ is a point where the directional or one of the partial derivatives of $T$’s Influence Function (Hampel, 1971, 1974) either take values at infinities or do not exist. Local-shift-sensitivity (Hampel, 1974) cannot replace the derivatives, as explained.

In section 3, regression coefficients’ Influence Functions and their derivatives, obtained via $E$-matrices, are used to show that: in $F_{\epsilon, x, y}$-regression, when remote $x$-case becomes slightly more extreme without reaching $L_1$ $LBP$, the size of the corresponding $L_2$-residual is drastically reduced whereas the $L_1$-residual is reduced less.

The graphical method and $RINFIN$ are supported by applications and simulations in section 4. Instead of square $x$-length on the plot’s horizontal axis, $x$-length can be used. For some data sets, plotting regression residuals rather than their absolute values may be more informative. However, for remote gross-error model with small variance, e.g. cases 1-10 in Hawkins-Bradu-Kass (1984) data, absolute residuals are informative.

Robust residual plots are accompanied with confidence ellipsoids. Otherwise, visual indications from distorted residuals lead to inaccuracies. $L_1$ and $L_2$ residuals and the square length of dependent variables are used herein because they do not cause unknown amount of distortion in relative visual distances. For example, in the Stackloss Data plot (Rousseeuw and van Zomeren, 1990, p. 636, Figure 3) relative sizes of the absolute standardized Least Median of Squares (LMS) residuals of cases 1, 3, 4 and 21 differ from those in the $L_2$-absolute residuals in Figure 2 herein.

In multiple regression, with observations from $F$, a case $(x, y)$ with factor space component, $x$, far away from the bulk of $F$’s factor space is called leverage case (Rousseeuw and Leroy, 1987, Huber, 1997). A “good” leverage case is either near or on the regression hyperplane determined by $F$. A “bad” leverage case forces the $F$-hyperplane to change drastically when $x$ becomes more remote. The suggested comparisons of $L_1$ and $L_2$ residuals’ plots and data’s $RINFIN$ values reveal “bad” leverage cases.
The abundance of high dimensional data sets from various fields, with \( p \) and \( n \) both large, created the need for new methods to detect outliers/influential cases affecting linear regression analysis. High dimensional influence measures have been proposed among others by Alphons et al. (2013), Zhao et al. (2013, 2016) and references there in. She and Owen (2011) identify influential cases using nonconvex penalized likelihood. Influence Function in outlier detection has been used by Campbell (1978) and Boente et al. (2002). The influence of observations in estimates’ values has been also studied by several authors, among others by Cook (1977), Cook and Weisberg (1980), Ruppert and Carroll (1980), Carroll and Ruppert (1985), Hampel (1985), Hampel et. al. (1986), Ronchetti (1987), Rousseeuw and van Zomeren (1990), Ellis and Morgenthaler (1992), Bradu (1997), Flores (2015) and Genton and Hall (2016).

In Genton and Ruiz-Gazen (2010) an observation is influential “whenever a change in its value leads to a radical change in the estimate” and the hair-plot is introduced to identify it. Two influence measures are proposed using partial derivative of the estimate: a) the local, with a small perturbation in one coordinate of the observation, and b) the global, using the most extreme contamination for each coordinate. Differences in our work include: i) leverage cases affecting drastically \( L_2 \)-regression residuals are visually identified combining information from \( L_1 \) and \( L_2 \) residuals’ plots, ii) the derivative of the estimate’s influence function is used instead of the estimate’s derivative, iii) RINFIN measures distance in residuals’ derivatives and can be used to evaluate group influence of neighboring cases.

Work has been done to identify “bad” leverage cases using \( L_1 \) residuals. Barrodale (1968) compared \( L_1 \) and \( L_2 \) residuals for regression function \( \sum_{j=1}^{p} a_j \phi_j(x) \) using tables for different \( x \)-values; \( x \in R^d, \phi_j \) is known, \( a_j \) is an unknown coefficient, \( j = 1, ..., p \). Barnett and Lewis (1984) present the absolute residuals as a tool in outlier detection. Narula and Wellington (1985) looked for observations that do and do not affect the analysis in \( L_1 \)-regression using the residuals. Ellis and Morgenthaler (1992) and Bradu (1997) examined the performance of the \( L_1 \) regression estimator facing outliers in the response variable. Additional results on \( L_1 \)-regression and outliers may be found in Dodge (1987).

Recent results combine also information from \( L_1 \) and \( L_2 \) regression. Giloni and Padberg
presented a lower bound on total sum of absolute $L_1$-residuals using the total sum of squared $L_2$-residuals. Flores (2015) studied for a particular regression model the behavior of $L_1$-estimates by comparing them with $L_2$-estimates, and introduced leverage constants for a design matrix to determine whether leverage cases are good or bad.

Proofs are in the Appendix.

2 Location Breakdown Point (LBP)

Hampel (1971) introduced the influence function, $IF(x; T, F)$, of a functional $T$ at probability $F$,

$$IF(x; T, F) = \lim_{\epsilon \to 0} \frac{T[(1 - \epsilon)F + \epsilon \Delta_x] - T(F)}{\epsilon},$$

when this limit exists; $x(\in R^p)$, $\Delta_x$ is the probability distribution that puts all its mass at the point $x$, $0 < \epsilon < 1$.

$IF(x; T, F)$ determines the “bias” in the value of $T$ at $F$ due to an $\epsilon$-perturbation of $F$ with $\Delta_x$:

$$T[(1 - \epsilon)F + \epsilon \Delta_x] - T(F) \approx \epsilon IF(x; T, F).$$

**Definition 2.1** (Hampel, 1971) The weight breakdown point is the upper bound on $\epsilon$ for which linear approximation (4) can be used.

Discussing further concepts related to the influence function, Hampel (1974, p. 389) introduced local-shift-sensitivity,

$$\lambda^* = \sup_{x \neq y} \frac{|IF(x; T, F) - IF(y; T, F)|}{||x - y||},$$

as “a measure for the worst (approximate) effect of wiggling the observations”; $|| \cdot ||$ is a Euclidean distance in $R^p$.

Unlike the extensive use of the weight breakdown point, local-shift-sensitivity was never fully exploited. One reason is that, in reality, it is a “global” measure as supremum over all
Thus, $\lambda^*$ cannot be used to study $T$’s bias for $x$'s small perturbation in the $\epsilon$-mixture, from $x$ to $x + h$, $||h||$ small,

$$T[(1 - \epsilon)F + \epsilon\Delta_{x+h}] - T[(1 - \epsilon)F + \epsilon\Delta_x]. \tag{6}$$

Rousseeuw and Leroy (1987) presented a physical analogy to the notion of weight breakdown point. A beam is fixed at one end and, at point $x$ on the beam, a stone with weight $\epsilon$ is attached. For small weights, the “deformation” (i.e., the bias) of the beam is linear in $\epsilon$ and one can predict the weight’s effect. As soon as $\epsilon$ takes value larger than the “breakdown value” (that depends on the location $x$), (6) cannot be used.

For the physical analogue of location breakdown, a sufficiently long beam is used and weight $\epsilon$ “travels” at different $x$-locations far away from the fixed end of the beam. There is a location $x_{0,\epsilon}$ that makes the beam “break”. The beam will break also with a small perturbation from $x_{0,\epsilon} - h$ to $x_{0,\epsilon}$, $||h||$ small. This is the reason we study $h$-perturbations for remote $x$’s.

When $F$ is defined on the real line, to express the physical analogue of location breakdown with the derivative of the influence function we evaluate (6) at neighboring points $x, x + h, x \in R, h \in R, |h|$ small.

**Lemma 2.1**

$$\lim_{h \to 0} \lim_{\epsilon \to 0} \frac{\epsilon h}{T[(1 - \epsilon)F + \epsilon\Delta_{x+h}] - T[(1 - \epsilon)F + \epsilon\Delta_x]} = IF'(x; T, F). \tag{7}$$

$IF'(x; T, F)$ is used to approximate (6) for small $\epsilon$, $|h|$: $$T[(1 - \epsilon)F + \epsilon\Delta_{x+h}] - T[(1 - \epsilon)F + \epsilon\Delta_x] \approx \epsilon h IF'(x; T, F); \tag{8}$$ (8) is the tool used to approximate $L_1$ and $L_2$ residuals and determine group influence.

In simple linear $L_1$-regression, derivatives of $IF(x, y; T, F)$ are constant where the residual does not vanish; $T$ is any of the regression coefficients. As $(x, y)$ becomes more remote in $x$, eventually there is a change of the $L_1$-regression coefficients at $(x + h, y)$, the $L_1$-residual vanishes, derivatives of $IF(x, y; T, F)$ takes values infinities and (8) is not valid.
This observation motivates the definition of location breakdown point (LBP) where the derivative of the influence function takes infinite values. In $L_1$ and $L_2$ linear regressions partial derivatives of the coefficients influence functions exist and, in addition, one remote coordinate in the factor space is enough to reach LBP. Thus, in the definition of LBP for $T$ partial derivatives are used instead of a directional derivative.

**Definition 2.2** Let $T$ be a functional defined on probabilities in $R^p$, with real values, $p \geq 1$. Then, $x \in R^p$ is Location Breakdown Point (LBP) if there is $j \leq p$:

$$\left| \frac{\partial}{\partial x_j} IF(x; T, F) \right| = \infty \text{ or does not exist;}$$

$x_j$ is $x$’s $j$-th coordinate, $F$ is probability.

**Example 2.1** Let $F$ be a probability on the real line, $T_1(F)$ is the median of $F$, $T_2(F)$ is the mean of $F$ and their influence functions are:

$$IF(x; T_1, F) = \frac{\text{sign}[x - T_1(F)]}{2f[T_1(F)]}, \quad IF(x; T_2, F) = x - T_2(F).$$

From (9), there are no LBPs on the real line for the mean, $T_2$, but for the median, $T_1$, its value is the only LBP.

**Example 2.2** Consider a simple linear regression model, $Y = \beta_0 + \beta_1 X + e$, with error $e$ having mean zero and finite second moment, $F$ is the joint distribution of $(X, Y)$ and $f_{Y|X}$ is the conditional density of $Y$ given $X$,

$$\tilde{f}_{Y|X}(x) = f_{Y|X}[\beta_{0,L_1}(F) + \beta_{1,L_1}(F)x|x].$$

The influence functions for the $L_2$-parameters $\beta_{0,L_2}(F), \beta_{1,L_2}(F)$, obtained at $F$ are

$$IF(x, y; \beta_{0,L_2}, F) = [y - \beta_{0,L_2}(F) - \beta_{1,L_2}(F)x] \frac{EX^2 - xEX}{Var(X)}, \quad (11)$$

$$IF(x, y; \beta_{1,L_2}, F) = [y - \beta_{0,L_2}(F) - \beta_{1,L_2}(F)x] \frac{x - EX}{Var(X)}; \quad (12)$$

$EU$ and $Var(U)$ denote, respectively, $U$’s mean and variance. The derivatives of influence functions (11), (12) do not satisfy (9) for $x \in R, y \in R$, thus there are no LBPs.
The influence functions for the $L_1$-parameters $\beta_{0,L_1}(F)$, $\beta_{1,L_1}(F)$, obtained at $F$ are

$$\text{IF}(x, y; \beta_{0,L_1}(F), F) = \frac{\text{sign}(y - \beta_{0,L_1}(F) - \beta_{1,L_1}(F)x)}{2} \frac{EX^2\tilde{f}_Y|X(X) - xEX\tilde{f}_Y|X(X)}{E\tilde{f}_Y|X(X)EX^2\tilde{f}_Y|X(X) - (EX\tilde{f}_Y|X(X))^2},$$

(13)

$$\text{IF}(x, y; \beta_{1,L_1}(F), F) = \frac{\text{sign}(y - \beta_{0,L_1}(F) - \beta_{1,L_1}(F)x)}{2} \frac{xEX\tilde{f}_Y|X(X) - EX\tilde{f}_Y|X(X)}{E\tilde{f}_Y|X(X)EX^2\tilde{f}_Y|X(X) - (EX\tilde{f}_Y|X(X))^2},$$

(14)

From (9), LBP's in $L_1$-regression are all $x, y$ satisfying the relation $y = \beta_{0,L_1}(F) + \beta_{1,L_1}(F)x$.

**Remark 2.1** The $y$-derivatives of $L_2$-influence functions (11), (12) are, respectively, $(EX^2 - xEX)/\text{Var}(X)$ and $(x - EX)/\text{Var}(X)$; those of $L_1$-influence functions (13), (14) either vanish or take values at infinities.

### 3 Influence, Residuals, Leverage Cases, RINFIN

**MULTIPLE REGRESSION MODEL**

Let $(X, Y)$ follow probability model $F$ in $R^{p+1}$,

$$Y = X^T\beta + e;$$

(15)

$X = (1, X_1, ...X_p)^T$ is the independent variable, $Y$ is the response, $\beta = (\beta_0, ..., \beta_p)^T$.

The Model Assumptions:

(A1) The error, $e$, is symmetric around zero and has finite second moment.

(A2) $X_1, ..., X_p$ are independent random variables.

(A3) Case $(x, y)$ is mixed with cases from model $F$ with probability $\epsilon$ (model $F_{\epsilon,x,y}$).

Let $(x + h, y), (x, y + h)$ be small perturbations of $(x, y)$. The goal is to compare the $(x, y)$- residual changes in $L_1$ and in $L_2$ regressions:

i) before $(x, y)$ enters model $F$ and after, i.e., under $F_{\epsilon,x,y}$,

ii) when $(x + h, y)$ replaces $(x, y)$ in the $\epsilon$-mixture, i.e., under $F_{\epsilon,x,y}$ and $F_{\epsilon,x+h,y}$ and

iii) when $(x, y + h)$ replaces $(x, y)$ in the $\epsilon$-mixture, i.e., under $F_{\epsilon,x,y}$ and $F_{\epsilon,x,y+h}$.
Let \( x \) become more extreme in the \( i \)-th coordinate, \( x_i + h, \ |h| \) small; denote by \( x_{i,h} \) this perturbation of \( x \),
\[
x_{i,h} = x + (0, \ldots, h, \ldots, 0).
\] (16)

The \( j \)-th regression coefficients obtained by \( L_m \)-minimization, respectively, at models \( F_{e,x,y} \) and \( F \) are:
\[
\beta_{j,L_m,x} = \beta_{j,L_m}([F_{e,x,y}]), \quad \beta_{j,L_m} = \beta_{j,L_m}([F]), \quad j = 0, 1, \ldots, p,
\] (17)
\[
\beta_{L_m,x} = (\beta_{0,L_m,x}, \ldots, \beta_{p,L_m,x})^T, \quad \beta_{L_m} = (\beta_{0,L_m}, \ldots, \beta_{p,L_m})^T;
\] (18)
denote the \( L_m \)- residuals for models \( F_{e,u,v} \) and \( F \), respectively,
\[
\begin{align*}
  r_{m,u} &= r_m(u, v; F_{e,u,v}) = v - \beta_{L_m,u}^T u, \quad r_m = r_m(u, v) = v - \beta_{L_m}^T u, \quad m = 1, 2.
\end{align*}
\] (19)

When indices of \( \beta \)'s and \( r \) include at least one among \( x, x_{i,h}, u, y+h \), they are determined from a gross-error model. Only \( x \) is used at \( \beta_{j,L_m,x} \) and only \( u \) is used at \( r_{m,u} \) because of interest in factor space perturbations and to avoid increasing the number of indices. The influence function of \( \beta_{j,L_m} \) is evaluated at \((x, y)\) for \( F \), thus use
\[
IF_{j,L_m} = IF(x, y; \beta_{j,L_m}, F), \quad IF'_{v,j,L_m} = \frac{\partial IF(x, y; \beta_{j,L_m}, F)}{\partial v}, \quad v = y, x_i,
\] (20)
i.e., in words, \( IF'_{v,j,L_m} \) is the derivative of \( IF_{j,L_m} \) with respect to \( v \), \( i = 1, \ldots, p, \ j = 0, 1, \ldots, p, \ m = 1, 2. \)

Influence functions of \( L_m \) regression coefficients are solutions of the equations:
\[
IF_{0,L_m} + IF_{1,L_m} EX_1 + \ldots + IF_{p,L_m} EX_p = \tilde{r}_m(x, y),
\] (21)
\[
IF_{0,L_m} EX_i + \ldots + IF_{p,L_m} EX_i X_j + \ldots + IF_{p,L_m} EX_i X_p = x_i \tilde{r}_m(x, y), \quad i = 1, \ldots, p, \ m = 1, 2,
\] (22)
with
\[
\tilde{r}_1(x, y) = \frac{\text{sign}[r_1(x, y)]}{2\tilde{f}_Y|X}}, \quad \tilde{r}_2(x, y) = r_2(x, y);
\] (23)
from the symmetry of \( e \) in assumption \((A1)\), \( \tilde{f}_Y|X \) is the common value
\[
\tilde{f}_Y|X = f_Y|X(x) = f_Y|X[\beta_{0,L_1} + \beta_{1,L_1} x_1 + \ldots + \beta_{p,L_1} x_p|X].
\] (24)
Under assumption (A2), the coefficients in the system of equations (21), (22) form a special type of matrix we call $E_p$-matrix; $p$ is the covariates’ dimension. As an illustration, for real numbers $a, b, c, A, B, C$:

$$E_4 = \begin{pmatrix}
1 & a & b & c \\
a & A & ab & ac \\
b & ba & B & bc \\
c & ca & cb & C
\end{pmatrix}.$$ 

For $E_4$, the corresponding linear regression model with independent covariates $X_1$, $X_2$, $X_3$ provides $a = EX_1$, $b = EX_2$, $c = EX_3$ and $A = EX_1^2$, $B = EX_2^2$, $C = EX_3^2$.

**Definition 3.1** $E_n$-matrix with real entries has form:

$$E_n = \begin{pmatrix}
1 & a_1 & a_2 \ldots & a_n \\
a_1 & A_1 & a_1a_2 \ldots & a_1a_n \\
a_2 & a_2a_1 & A_2 \ldots & a_2a_n \\
\vdots \\
a_n & a_na_1 & a_na_2 \ldots & A_n
\end{pmatrix}. \quad (25)$$

**Notation:** $E_{n,-k}$ denotes the matrix obtained from $E_n$ by deleting its $k$-th column and $k$-th row, $2 \leq k \leq n + 1$.

**Property of $E_n$-matrix:** Deleting the $k$-th row and the $k$-th column of $E_n$-matrix, the obtained matrix $E_{n,-k}$ is $E_{n-1}$ matrix formed by $\{1, a_1, \ldots, a_n\} - \{a_{k-1}\}$, $2 \leq k \leq n + 1$.

The cofactors of $E_n$-matrix are needed to solve (21), (22).

**Proposition 3.1** a) The determinant of $E_n$-matrix (27) is

$$|E_n| = \Pi_{m=1}^{n}(A_m - a_m^2). \quad (26)$$

b) Let $C_{i+1,j+1}$ be the cofactor of element $(i + 1, j + 1)$ in $E_n$. Then, its determinant

$$C_{i+1,j+1} = 0, \text{ if } i > 0, j > 0, i \neq j, \quad C_{1,j+1} = -a_j \Pi_{k \neq j}(A_k - a_k^2). \quad (27)$$

$$C_{i+1,1} = -a_i \Pi_{j \neq i}(A_j - a_j^2), \text{ if } i > 0, \quad C_{1,1} = |E_n| + \sum_{k=1}^{n} a_k^2 |E_{n,-k}|. \quad (28)$$
Proposition 3.2 For regression model (13) with assumptions (A1)-(A3), \( r_1(x, y) \neq 0 \), and \( \bar{r}_m \) in (23), the influence functions of \( L_m \)-regression coefficients, \( m = 1, 2 \), are:

\[
IF_{0,L_m} = \bar{r}_m[1 - p + \sum_{j=1}^{p} \frac{EX_j^2 - x_jEX_j}{\sigma_j^2}], \quad IF_{j,L_m} = \bar{r}_m \frac{x_j - EX_j}{\sigma_j^2}, \quad j = 1, \ldots, p; (29)
\]

\( \sigma_j^2 \) is the variance of \( X_j, j = 1, \ldots, p. \)

**COMPARISON OF \( L_m \)-RESIDUALS FOR \( F, F_{\epsilon,x,y}, F_{\epsilon,x_i,y}, F_{\epsilon,x,y+h} \) \( m = 1, 2 \)**

The next proposition confirms that for \( x\)-remote case \((x, y)\), the size of \( L_1 \) residual is larger than the size of its \( L_2 \) residual before \((x, y)\) reaches \( L_1 \) LBP.

Proposition 3.3 For regression model (13) with (A1)-(A3), perturbation (10) and \( r_1(x, y) \neq 0 \):

a) For \( \epsilon \) small:

a1) The difference of \((x, y)\)-residuals at \( F_{\epsilon,x,y} \) and \( F \) is:

\[
r_{m,x}(x, y) - r_m(x, y) \approx -\epsilon[IF_{0,L_m} + \sum_{j=1}^{p} x_jIF_{j,L_m}] = -\epsilon \bar{r}_m(x, y)[1 + \sum_{j=1}^{p} \frac{(x_j - EX_j)^2}{\sigma_j^2}]; (30)
\]

\( r_{m,x}(x, y) \) and \( r_m(x, y) \) have the same sign and \( |r_{m,x}(x, y)| < |r_m(x, y)|, m = 1, 2. \)

a2) The ratio:

\[
\frac{r_{2,x}(x, y) - r_2(x, y)}{r_{1,x}(x, y) - r_1(x, y)} \approx 2\tilde{f}_{Y|x} \frac{r_2(x, y)}{\text{sign}[r_1(x, y)]}; (31)
\]

\( \tilde{f}_{Y|x} \) is positive constant (24).

b) For \( \epsilon \) and \(|h|\) both small:

b1) The difference of \((x, y)\)-residuals at \( F_{\epsilon,x,y} \) and \( F_{\epsilon,x_i,y} \) is:

\[
r_{m,x_i,h}(x_i,h,y) - r_{m,x}(x, y) + \beta_{i,L_m}h \approx -\epsilon h[IF_{i,L_m} + IF'_{x_i,h,L_m} + \sum_{j=1}^{p} x_jIF'_{x_i,j,L_m}] - h^2 IF'_{x_i,i,L_m}. (32)
\]

Thus,

\[
r_{1,x_i,h}(x_i,h,y) - r_{1,x}(x, y) \approx -\epsilon h \frac{\text{sign}[r_1(x, y)]}{\tilde{f}_{Y|x}} \frac{x_i - EX_i}{\sigma_i^2} - \beta_{i,L_m}h - h^2 IF'_{x_i,i,L_m}, (33)
\]
From (39) and (40), the right side of the latter measures influence of \( x \)'s \( i \)-th coordinate in the residual's derivative and provides the motivation for defining influence. When \( p > 1 \), coordinates other than the \( i \)-th are involved in \( \sum_{j=1}^{p} x_j F'_{x_i,j,L_m} \) in (40), motivating the use of two influence indices.

**Proposition 3.4** For models \( F, F_{e,x,y}, F_{e,x_i,y} \) and \( L_m \) regression it holds

\[
\lim_{h \to 0} \frac{r_{m,x_i,h} - r_{m,x}}{h} + \beta_{i,L_m} = -\epsilon [IF_{i,L_m} + IF'_{x_i,0,L_m} + \sum_{j=1}^{p} x_j IF'_{x_i,j,L_m}], \quad i = 1, \ldots, p, \ m = 1, 2. 
\] (40)
Definition 3.2 For gross-error model $F_{\varepsilon,x,y}$,

a) the influence of $x$’s $i$-th coordinate in the $L_m$-residual is

$$
\varepsilon \cdot |IF_{i,L_m}(x, y) + IF'_{x_i,0,L_m}(x, y) + \sum_{j=1}^{p} x_j IF'_{x_i,j,L_m}(x, y)|, \quad m = 1, 2, \quad (41)
$$

b) the influence of $x$ in the $L_m$-residual is

$$
\varepsilon \cdot \sum_{i=1}^{p} |IF_{i,L_m}(x, y) + IF'_{x_i,0,L_m}(x, y) + \sum_{j=1}^{p} x_j IF'_{x_i,j,L_m}(x, y)|, \quad m = 1, 2. \quad (42)
$$

Influences for models $F_{\varepsilon_1,x_1,y_1}$, $F_{\varepsilon_2,x_2,y_2}$ can be compared.

Definition 3.3 Case $(x_1, y_1)$ with weight $\varepsilon_1$ is more influential for $L_m$-residuals than case $(x_2, y_2)$ with weight $\varepsilon_2$, $m = 1, 2$, if

$$
\varepsilon_2 \cdot \sum_{i=1}^{p} |IF_{i,L_m}(x_2, y_2) + IF'_{x_i,0,L_m}(x_2, y_2) + \sum_{j=1}^{p} x_{2,j} IF'_{x_i,j,L_m}(x_2, y_2)|
$$

$$\leq \varepsilon_1 \cdot \sum_{i=1}^{p} |IF_{i,L_m}(x_1, y_1) + IF'_{x_i,0,L_m}(x_1, y_1) + \sum_{j=1}^{p} x_{1,j} IF'_{x_i,j,L_m}(x_1, y_1)|. \quad (43)
$$

The $L_2$-Residual Influence Index (RINFIN): For gross-error model $F_{\varepsilon,x,y}$, (12) for $m = 2$ becomes from (56) in the Appendix,

$$
\text{RINFIN}(x, y; \varepsilon, L_2) = \varepsilon \cdot \sum_{i=1}^{p} \left\{ \frac{2 r_2(x,y)(x_i - EX_i)}{\sigma_i^2} \right\} - \beta_{i,L_2} \left[ 1 + \sum_{j=1}^{p} \frac{(x_j - EX_j)^2}{\sigma_j^2} \right]. \quad (44)
$$

Abuse of notation: Using $\text{RINFIN}(x, y)$ instead of $\text{RINFIN}(x, y; \varepsilon, L_2)$.

Remote $x$’s have large $\text{RINFIN}(x, y; \varepsilon, L_2)$.

Proposition 3.5

$$
\lim_{|x_i| \to \infty} \text{RINFIN}(x, y; \varepsilon, L_2) = \infty. \quad (45)
$$

Remark 3.1 (RINFIN*) To measure strictly the influence of $x$’s $i$-th component, which is dominant when $x_i$ is remote (see (57)), use also:

$$
\text{RINFIN}^*(x, y; \varepsilon, L_2) = \varepsilon \cdot \sum_{i=1}^{p} \left\{ \frac{2 r_2(x,y)(x_i - EX_i)}{\sigma_i^2} \right\} - \beta_{i,L_2} \left[ 1 + \frac{(x_i - EX_i)^2}{\sigma_i^2} \right]. \quad (46)
$$
Note that RINFIN dominates RINFIN* in the simulations with the normal model in section 4. However RINFIN* allows to identify the bad leverage cases in Hawkins-Bradu-Kass data.

\textit{y-Influence on Residuals}

Since \( IF'_{y,j,L_1} \) vanishes for every \( j \), influence index from \( y \)-derivatives of residuals is only presented for \( L_2 \)-regression.

For \((x, y + h)\) and \((x, y)\) both under model \( F \),

\[
\frac{r_2(x, y + h) - r_2(x, y)}{h} = 1, \quad i = 1, \ldots, p. \tag{47}
\]

**Proposition 3.6** For models \( F, F_{\epsilon,x,y}, F_{\epsilon,x,y+h} \) and \( L_2 \) regression it holds

\[
\lim_{h \to 0} \frac{r_{2,x,y+h}(x, y + h) - r_{2,x,y}(x, y)}{h} - 1 \approx -\epsilon[1 + \sum_{j=1}^{p} \frac{(x_j - EX_j)^2}{\sigma_j^2}]. \tag{48}
\]

**Remark 3.2** From (48), the \( y \)-influence index is

\[
\sum_{j=1}^{p} \frac{(x_j - EX_j)^2}{\sigma_j^2}; \tag{49}
\]

it is maximized for cases in the extremes of the \( x \)-coordinates and can be visually implemented with the proposed plot when \( x \)-coordinates have all the same sign.

4 Applications: RINFIN and Residuals’ Plots

**RINFIN’S PERFORMANCE IN SIMULATIONS**

Data \((X, Y)\) follows linear model (15) with \( \beta = (1.5, 0.5, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, \ldots, 0) \), \( p \geq 11 \), and with \( X \) obtained from \( p \)-dimensional normal distribution, \( \mathcal{N}(0, \Sigma) \), with \( \Sigma_{i,j} = 0.5|j-i|, 1 \leq i, j \leq p \), as in Alfons, Croux and Gelper (2013, p.11). When \( p < 11 \), \( \beta \)'s first \( p \) coordinates are used and \( X \) is obtained as in the case \( p \geq 11 \). Contaminated
**X** with probability **G** consists of **p** independent normal random variables with mean **\( \mu \)** and standard deviation 1. Each sample has size **n = 200** and various values for **p** and **\( \mu \)** are used, **p < n = 200**. Errors **e** for model **F** and for **G** are independent, normal random variables with mean zero and variance 1. The percentage of contaminated cases **\( \epsilon = .10 \)**. The number of simulated samples **N = 100**.

The 20 contaminated cases are the first cases in the data. **RINFIN** and **RINFIN**\(^*\) values are calculated and the case numbers of the top 20 values are recorded and compared with contaminated cases 1-20 to calculate the misclassification proportion. The results appear in Table 1. **RINFIN** misclassification proportion is only reported being uniformly better than **RINFIN**\(^*\) for the data used.

| \( p \) | \( \mu = 0 \) | \( \mu = .5 \) | \( \mu = 1 \) | \( \mu = 1.5 \) | \( \mu = 2 \) | \( \mu = 2.5 \) | \( \mu = 3 \) |
|---|---|---|---|---|---|---|---|
| 2 | 0.9095 | 0.866 | 0.7895 | 0.642 | 0.524 | 0.3805 | 0.281 |
| 5 | 0.91 | 0.857 | 0.7065 | 0.5035 | 0.2935 | 0.155 | 0.0685 |
| 10 | 0.9305 | 0.849 | 0.621 | 0.361 | 0.1365 | 0.0435 | 0.0085 |
| 15 | 0.93 | 0.8485 | 0.558 | 0.242 | 0.062 | 0.01 | 0 |
| 20 | 0.9085 | 0.8425 | 0.496 | 0.167 | 0.0335 | 0.0015 | 0 |
| 60 | 0.933 | 0.755 | 0.2025 | 0.0075 | 0 | 0 | 0 |
| 100 | 0.928 | 0.6815 | 0.1005 | 0.001 | 0 | 0 | 0 |
| 140 | 0.9305 | 0.6275 | 0.048 | 0 | 0 | 0 | 0 |
| 190 | 0.9105 | 0.7025 | 0.1535 | 0.0155 | 0.001 | 0 | 5e-04 |

Table 1: Average misclassification proportion of **RINFIN** in **N = 100** simulated samples of size **n = 200**, of which 20 are contaminated leverage cases, from **p**-dimensional, normal-mixtures data with distance of means in each coordinate **\( \mu \)** and variance 1.

The simulations follow the spirit in Khan, Van Aelst and Zamar (2007). When there are no contaminated cases, i.e. **\( \mu = 0 \)**, the misclassification proportion is near 90% for all **\( p \)**-values. For any fixed **\( p \)**-value, the misclassification proportion decreases as **\( \mu \)** increases. For any fixed **\( \mu \)**-value the misclassification proportion decreases as **\( p \)** increases except for an anomaly when **\( p = 190 \)** due to its proximity to the sample size. By increasing **n** to 250 cases, this anomaly disappears. The **blessing** of high dimensionality is observed; for a
theoretical confirmation see Yatracos (2013, Section 8, Proposition 8.1)

**READING RESIDUALS’ PLOTS**

The goal is to identify quickly cases that do not follow the unknown model $F$ of the data’s majority, in particular bad leverage cases. “Naive” plots of absolute residuals for $L_1$ and $L_2$ regression against the sum of squares of the independent variables are used.

Look for:
- (A) remote neighboring plot-points creating visual gaps in the $L_1$-plot’s residuals but smaller gaps in the $L_2$-plot; these are bad leverage cases far from $L_1$ LBP. For a given $x$, the gap is “large” when the ratio of absolute residuals from the upper and lower gap’s borders is larger or equal to two.
- (B) a group of plot-points with neighboring horizontal axis projections, distant from the bulk of the plot, with the $L_1$-absolute residuals forming a vertical strip and at least one of them near zero; these are bad leverage cases near $L_1$ LBP.
- (C) If no unusual leverage cases are identified when plotting against the $x$’s square length, plot the absolute residuals against each explanatory variable and check whether there are remote $x$-coordinates for which (A), (B) hold.
- (D) Large absolute residuals, especially at the extremes of the $x$-values in the data, indicating bad leverage or other outlying cases.

**USING RINFIN WITH DATA**

The data

$$D_n = \{(x_1, y_1), \ldots, (x_n, y_n)\}, \quad D_{n,-m} = D_n - \{(x_m, y_m)\}. $$

To calculate sample $RINFIN(x_m, y_m)$ estimate the parameters in (44) and use $\epsilon = 1/n$: 

(a) Use $D_{n,-m}$ to obtain $L_2$-estimates $\hat{\beta}_{L_2}$ and $\hat{\sigma}_2^2(x_m, y_m)$.

(b) Estimate $EX_i$ and $\sigma_i^2$, respectively, by the sample average and sample variance $x$-data’s $i$-th coordinate in $D_{n,-m}$, $i = 1, \ldots, p$.

(c) Use $\hat{\beta}_{L_2}$’s $i$-th coordinate and replace $x_i$ with $x_m$’s $i$-th coordinate, $i = 1, \ldots, p$.
If a group $G$ of $k$ remote $x$-neighboring cases exists,

$$G = \{(x_{i_1}, y_{i_1}), \ldots, (x_{i_k}, y_{i_k})\} \subset D_n,$$

$D_n$ may follow a gross-error model. Let $\bar{g}$ be the average of the elements in $G$ and use, instead of $D_n$, new data

$$(D_n - G) \cup \{\bar{g}\}.$$ 

Calculate $RINFIN$-values following $a$)-c). For $RINFIN(\bar{g})$ use $\epsilon = k/n$; in the remaining $(n - k)$ cases weights are $1/n$.

With $J$ groups, $G_1, \ldots, G_J$, of remote $x$-neighboring cases, $G_k \cap G_l = \emptyset, k \neq l$, obtain averages $\bar{g}_1, \ldots, \bar{g}_J$, and use data set

$$(D_n - \bigcup_{j=1}^{J} G_j) \cup \{\bar{g}_1, \ldots, \bar{g}_J\}.$$ 

Proceed with $a$)-c). For $RINFIN(\bar{g}_j)$ use $\epsilon_j = k_j/n, k_j$ is the cardinality of $G_j, j = 1, \ldots, J$; in the remaining cases weights are $1/n$.

**DATA PLOTS & RINFIN VALUES**

In Figures 1 and 2, $L_1$ and $L_2$ plots of absolute regression residuals are presented for twelve, well known data sets; those without reference are in Rousseeuw and Leroy (1987). Several methods fail to determine cases from gross-error component(s). The visual comparison of regression plots, $RINFIN$ and $RINFIN^*$ are informative providing indications.

Six data sets present large visual gaps in $L_1$-plots (see (A)) and smaller gaps in $L_2$-plots. In one of the former, the Hawkins-Bradu-Kass data, residuals’ plots give the impression of more than one gross-error component but $RINFIN^*$ is successful after grouping. The remaining data sets presenting smaller $L_1$-gaps, if any, are: Hertzsprung-Russel, Hadi-Simonoff, Stack Loss, Coleman, Salinity and Modified Wood. The ultimate data set is the most challenging because there are no immediate visual indications, but $RINFIN$ is successful without grouping as well as after grouping along with $RINFIN^*$.

In Telephone data (covariates’ dimension $p = 1$, number of cases $n = 24$, all covariates positive), observations 15-20 cause a large gap in the residuals of the $L_1$-plot and no gap
in the $L_2$-plot; (D) applies. These are indeed the outliers because of the change in the recording system used.

In Kootenay river data ($p = 1$, $n = 13$, all covariates positive), case 4 is remote and causes a large gap in the $L_1$-residuals, unlike the $L_2$-residuals. Both (A) and (D) apply. $RINFIN$ values confirm the visual findings.

| DATA: Kootenay River (p=1, n=13) |
|---|---|---|---|---|---|---|
| CASE | 4 | 7 | 2 | 12 | 6 | 1 |
| $RINFIN$ | 8.906 | 0.106 | 0.052 | 0.044 | 0.030 | 0.015 |

In Brain and Body data ($p = 1$, $n = 28$, not all covariates positive), cases 6, 16, 25 are remote, obtained from 3 dinosaurs each with small brain and heavy body, and cause a large gap in the $L_1$ residuals. (A) applies. $RINFIN$ values confirm the visual findings. Case 26 in $R$ library is case 25 in Rousseeuw and Leroy (1987).

| DATA: LogBrain and LogBody (p=1, n=28) |
|---|---|---|---|---|---|---|
| CASE | 25 | 6 | 16 | 27 | 17 | 10 |
| $RINFIN$ | 0.298 | 0.183 | 0.157 | 0.051 | 0.039 | 0.029 |

In Hertzsprung-Russel star data ($p = 1$, $n = 47$, all covariates positive), cases 11, 20, 30, 34 correspond to giant stars. These are remote, $x$-neighboring cases and many of the remaining cases have either comparable or larger absolute residuals. Absolute residuals of cases 11, 20, 30, 34 form a narrow strip in the $L_1$ plot and that of case 11 is near zero, indicating its proximity to $L_1 LBP$. Thus, (B) applies. Barrodale (1968, p. 55, l. -2- p.56, l. 2) observed a similar behavior in an example for cases he called “wild”. $RINFIN$ values after grouping $x$-neighboring cases, $G_1 = \{11, 20, 30, 34\}$, $G_2 = \{7, 14\}$ support that $\{11, 20, 30, 34\}$ are bad leverage cases. Note that after grouping, the cases in the table form an “envelope” in the $L_1$ and $L_2$ plots.
**DATA: Hertzsprung-Russel stars** \((p = 1, n = 47)\)

| CASE | \(RINFIN\) | GROUP | \(RINFIN\) | GROUP | \(RINFIN\) |
|------|------------|-------|------------|-------|------------|
| 34   | 0.545      | 11,20,30,34 | 26.555     | 11,20,30,34 | 39.654     |
| 30   | 0.387      | 14     | 0.276      | 7,14   | 0.447      |
| 20   | 0.272      | 36     | 0.131      | 17     | 0.159      |
| 14   | 0.198      | 4      | 0.131      | 36     | 0.149      |
| 7    | 0.191      | 2      | 0.131      | 4      | 0.143      |
| 11   | 0.162      | 17     | 0.125      | 2      | 0.143      |

In Hawkins-Bradu-Kass (1984) artificial data \((p = 3, n = 75, \text{all covariates positive})\), cases 11-14 have the largest absolute residual in the \(L_1\)-plot and are the most distant from cases 15-75. The plot shows three separated, distant groups that could be attributed to two sources of gross errors. Using (A), cases 11-14 are bad leverage cases. Using (B), cases 1-10 are “bad” leverage cases near \(L_1\) \(LBP\). Using two groups, cases 1–10 and 11–14, \(RINFIN^*\) indicates the true “bad” leverage cases 1–10. Rousseeuw and van Zomeren (1990, Figure 5) identify cases 1-10 in the plot of standardized LMS residuals against robust distances.

**DATA: Hawkins-Bradu-Kass** \((p = 3, n = 75)\)

| CASE | \(RINFIN\) | GROUP | \(RINFIN\) | GROUP | \(RINFIN^*\) |
|------|------------|-------|------------|-------|-------------|
| 12   | 0.523      | 11-14 | 17.848     | 1-10  | 16.648      |
| 14   | 0.442      | 1-10  | 15.983     | 11-14 | 14.426      |
| 11   | 0.356      | 43    | 0.028      | 43    | 0.028       |
| 13   | 0.351      | 68    | 0.022      | 68    | 0.019       |
| 7    | 0.174      | 47    | 0.021      | 47    | 0.018       |
| 6    | 0.156      | 27    | 0.019      | 27    | 0.017       |
| 3    | 0.136      | 52    | 0.018      | 54    | 0.015       |
| 5    | 0.133      | 60    | 0.0170     | 52    | 0.015       |

In Scottish Hill Races data \((p = 2, n = 35, \text{all covariates positive})\), cases 7 and 18 both at data’s extremes cause large gap in the \(L_1\)-residuals unlike the \(L_2\)-residuals. (A) and (D) both apply. According to Atkinson (1986, p. 399) observation 33 is masked by points
7 and 18 and an error is reported for case 18. \textit{RINFIN}-values identify these cases and in addition case 11.

| DATA: Scottish Hill Races (p=2, n=35) |
|---------------------------------------|
| CASE | 7  | 11 | 35 | 33 | 18 | 31 | 17 |
|\textit{RINFIN} | 3.577 | 3.230 | 2.492 | 1.558 | 1.067 | 0.796 | 0.508 |

In Hadi-Simonoff (1993) data \((p = 2, n = 25,\) one covariate negative \(-.116\)), remote cases 1-3 have the larger absolute residuals in the \(L_1\) plot and case 17 follows. In the \(L_2\) plot case 17 has the largest absolute residual and cases 1-3 have their absolute residuals reduced. (A) applies for cases 1-3. When group \(G = \{1, 2, 3\}\) is used, \textit{RINFIN} values indicate these are bad leverage cases. Hadi and Simonoff (1993) identify the \textit{true outliers}, cases 1-3, and report that “clean” cases 6, 11, 13, 17 and 24 have larger absolute Least Median of squares residuals than cases 1-3. Plots of standardized absolute residuals for bounded influence as well as \(M\)-estimates regression do not reveal the outliers as unusual cases. Yatracos (2013) identifies cases 1-3 as remote cluster with a projection-pursuit cluster index.

| DATA: Hadi-Simonoff (p = 2, n = 25) |
|-------------------------------------|
| CASE | \textit{RINFIN} | GROUP | \textit{RINFIN} |
| 22 | 0.677 | 1,2,3 | 1.074 |
| 4 | 0.572 | 4 | 0.645 |
| 17 | 0.527 | 17 | 0.620 |
| 12 | 0.565 | 22 | 0.607 |
| 25 | 0.374 | 12 | 0.464 |
| 1 | 0.351 | 13 | 0.399 |
| 2 | 0.346 | 25 | 0.328 |
| 3 | 0.340 | 24 | 0.298 |

In Education data \((p = 3, \ n = 50,\) all covariates positive), case 50 (Alaska) causes a large gap in the \(L_1\)-residuals, unlike the \(L_2\)-residuals. (A) applies. \textit{RINFIN} values confirm the visual findings.
DATA: Education \((p = 3, n = 50)\)

| CASE | 50 | 33 | 7 | 44 | 29 | 5 |
|------|----|----|---|----|----|---|
| RINFIN | 1.20 | 0.482 | 0.474 | 0.407 | 0.334 | 0.301 |

In Stackloss data \((p = 3, n = 21, \text{all covariates positive})\) cases 1, 3, 4, 21 form a large gap in the \(L_1\)-plot and the gap is reduced in the \(L_2\)-plot. \((\text{A})\) applies. \(L_1\)-plot indicates cases 1 and 3 can form a group with case 2 which has small absolute residual and is near \(LBP\). \((\text{B})\) applies for cases 1,2,3. \(\text{RINFIN}^{*}\) values indicate 1, 2, 3, 21 are bad leverage cases. Case 4 has \(\text{RINFIN}\) and \(\text{RINFIN}^{*}\) values before grouping at the 10-th percentile and after grouping below the 40th percentile. Rousseeuw and van Zomeren (1990) and Flores (2015) identify cases 1, 2, 3, 4, 21 as bad leverage cases using, respectively, plots of standardized LMS residuals against robust distances and leverage constants.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\textbf{CASE} & \textbf{RINFIN} & \textbf{CASE} & \textbf{RINFIN}^{*} & \textbf{GROUP} & \textbf{RINFIN} & \textbf{GROUP} & \textbf{RINFIN}^{*} \\
\hline
17 & 1.696 & 2 & 0.885 & 1,2,3 & 1.664 & 1,2,3 & 1.779 \\
\hline
2 & 1.527 & 12 & 0.428 & 17 & 1.481 & 21 & 0.565 \\
\hline
1 & 0.757 & 21 & 0.427 & 21 & 0.697 & 12 & 0.444 \\
\hline
15 & 0.557 & 17 & 0.420 & 7 & 0.642 & 7 & 0.409 \\
\hline
12 & 0.524 & 15 & 0.380 & 15 & 0.535 & 17 & 0.368 \\
\hline
18 & 0.520 & 11 & 0.317 & 8 & 0.531 & 15 & 0.358 \\
\hline
7 & 0.519 & 7 & 0.315 & 12 & 0.528 & 11 & 0.308 \\
\hline
8 & 0.440 & 16 & 0.264 & 18 & 0.455 & 8 & 0.301 \\
\hline
\end{tabular}
\caption{DATA: Stackloss \((p = 3, n = 21)\)}
\end{table}

In Coleman data \((p = 5, n = 20, \text{not all covariates positive})\) cases 3 and 18 cause a large gap in the \(L_1\)-plot and \((\text{D})\) applies. Case 18 has larger absolute residual than most of the remaining cases and lives at the \(x\)-extremes. Cases 3, 4, 9, 16 in the \(L_1\)-plot indicate a potential cluster near \(LBP\). According to Rousseeuw and Leroy (1987) “.... examining the Least Squares results, ... cases 3, 11 and 18 are furthest away from the linear model. ... The robust regression spots schools 3, 17 and 18 as outliers ...”. \(\text{RINFIN}\) and \(\text{RINFIN}^{*}\) highest four values identify cases 1,6,11, 18, 19.
In Salinity data (Ruppert and Carroll, 1980, $p = 3$, $n = 28$, all covariates positive) case 16 has the largest absolute residual, is $x$-remote and the gap caused in the $L_1$ plot is small. In the $L_2$ plot its absolute residual is reduced. Both (A) and (D) apply for case 16. $RINFIN$ values confirm the visual findings. In Carroll and Ruppert (1985) the analysis of the data shows that cases 3 and 16 are masking case 5.

In modified Wood data ($p = 5$, $n = 20$, all covariates positive) there is no visual gap in the $L_1$-plot. Since covariates are positive, cases 7, 19, 1, 4, 6, 8 live at one $x$-extreme of the data and present a pattern like that described in (B), with the residual of case 8 near 0. To determine the neighboring cases, plot $L_2$ (or $L_1$) residuals against each $x$-coordinate. In Figure 3 it is clear that cases 4, 6, 8, 19 are neighboring and remote in each coordinate. Cases 4, 6, 8, 19 form also a strip in the last two $L_2$-plots in Figure 3. (B) applies for these cases in view of the $L_1$ plot in Figure 1. This is confirmed by the four higher $RINFIN$ values and also by both $RINFIN$ and $RINFIN^*$ values when $\{4, 6, 8, 19\}$ are considered as group.
DATA: Modified Wood \((p = 5, n = 20)\)

| CASE | RINFIN | CASE | RINFIN* | GROUP | RINFIN | GROUP | RINFIN* |
|------|--------|------|---------|-------|--------|-------|---------|
| 19   | 1.579  | 11   | 0.871   | 4,6,8,19 | 34.729 | 4,6,8,19 | 29.583 |
| 8    | 1.532  | 12   | 0.508   | 11     | 1.710  | 11     | 1.366   |
| 6    | 1.452  | 1     | 0.493   | 7      | 1.460  | 7      | 0.597   |
| 4    | 1.3312 | 7     | 0.476   | 12     | 1.390  | 12     | 0.556   |
| 12   | 1.324  | 14    | 0.448   | 10     | 1.084  | 1      | 0.468   |
| 11   | 1.161  | 19    | 0.442   | 16     | 0.785  | 14     | 0.389   |
| 7    | 1.158  | 8     | 0.434   | 1      | 0.779  | 16     | 0.285   |
| 10   | 1.075  | 4     | 0.386   | 17     | 0.738  | 10     | 0.252   |

5 APPENDIX

Proof of Lemma 2.1 Equality (7) is obtained by adding and subtracting \(T(F)\) in the numerator of its left side and by taking first the limit with respect to \(\varepsilon\). \(\square\)

Proof for Proposition 3.1 a) Induction is used.

For \(n = 1\), the determinant is \(A_1 - a_1^2\).

For \(n = 2\), the determinant is

\[
(A_1 A_2 - a_2^2 a_2^2) - a_1 \cdot (a_1 A_2 - a_1 a_2^2) + a_2 \cdot (a_1^2 a_2 - A_1 a_2) = A_1 A_2 - a_1^2 A_2 + a_1^2 a_2 - A_1 a_2^2
\]

\[
= A_2 (A_1 - a_1^2) - a_2^2 (A_1 - a_1^2) = (A_1 - a_1^2)(A_2 - a_2^2).
\]

Assume that (26) holds for \(E_n\). To show it holds for \(E_{n+1}\) consider the matrix \(E_{n+1}\):

\[
E_{n+1} = \begin{pmatrix}
1 & a_1 & a_2 & \ldots & a_n & a_{n+1} \\
 a_1 & A_1 & a_1 a_2 & a_1 a_n & a_1 a_{n+1} \\
a_2 & a_2 a_1 & A_2 & a_2 a_n & a_2 a_{n+1} \\
\vdots \\
a_n & a_n a_1 & a_n a_2 & A_n & a_n a_{n+1} \\
a_{n+1} & a_{n+1} a_1 & a_{n+1} a_2 & a_{n+1} a_n & A_{n+1}
\end{pmatrix}
\]

\(|E_{n+1}|\) is obtained using line \((n + 1)\) and its cofactors \(C_{n+1,1}, \ldots, C_{n+1,n+1}\):

\[
|E_{n+1}| = a_{n+1} C_{n+1,1} + a_{n+1} a_1 C_{n+1,2} + \ldots + a_{n+1} a_n C_{n+1,n} + A_{n+1} C_{n+1,n+1}.
\]
Observe that for $2 \leq j \leq n$, cofactor $C_{n+1,j}$ is obtained from a matrix where the last column is a multiple of its first column by $a_{n+1}$, thus,

$$C_{n+1,j} = 0, \quad j = 2, \ldots, n. \quad (51)$$

For the matrix in cofactor $C_{n+1,1}$, observe that in its last column $a_{n+1}$ is common factor and if taken out of the determinant the remaining column is the vector generating $E_n$, i.e. $\{1, a_1, \ldots, a_n\}$. With $n - 1$ successive interchanges to the left, this column becomes first and $E_n$ appears. Thus,

$$C_{n+1,1} = (-1)^{n+2}(-1)^{n-1} \cdot a_{n+1}|E_n| = -a_{n+1}|E_n|. \quad (52)$$

In cofactor $C_{n+1,n+1}$, the determinant is that of $E_n$,

$$C_{n+1,n+1} = (-1)^{2(n+1)}|E_n| = |E_n|. \quad (53)$$

From (50)-(53) it follows that

$$|E_{n+1}| = -a_{n+1}^2|E_n| + A_{n+1}|E_n| = \Pi_{m=1}^{n+1}(A_m - a_m^2).$$

b) We now work with $E_n$. For $i > 0, j > 0, i \neq j$, after deleting row $(j + 1)$ the remaining of column $(j + 1)$ in the cofactor is a multiple of column 1, thus $|C_{i+1,j+1}|$ vanishes.

For $C_{1,j+1}$, using column $j + 1$ to calculate $E_n$, it holds:

$$a_jC_{1,j+1} + A_jC_{j+1,j+1} = |E_n| \rightarrow a_jC_{1,j+1} = -a_j^2\Pi_{k \neq j}(A_k - a_k^2) \rightarrow C_{1,j+1} = -a_j\Pi_{k \neq j}(A_k - a_k^2).$$

For $C_{i+1,1}, i > 0$, after deletion of row $(i + 1)$ in $E_n$ the remaining of column $(i + 1)$ in the cofactor’s matrix is multiple of $a_i$ and the basic vector creating $E_{n-i}$. Column 1 of $E_n$ is also deleted and for column $(i + 1)$ in the cofactor’s matrix to become first column $(i - 1)$ exchanges of columns are needed. Thus,

$$C_{i+1,1} = (-1)^{i+2} \cdot a_i \cdot (-1)^{i-1}\Pi_{k \neq i}(A_k - a_k^2) = -a_i \cdot \Pi_{k \neq i}(A_k - a_k^2).$$

For $C_{1,1}$ we express $|E_n|$ as sum of cofactors along the first row of $E_n$,

$$C_{1,1} + a_1C_{1,2} + \ldots + a_nC_{1,n} = |E_n|$$
\[ C_{1,1} = \prod_{k=1}^{n}(A_k - a_k^2) + a_1^2 \prod_{k \neq 1} (A_k - a_k^2) + \ldots + a_n^2 \prod_{k \neq n} (A_k - a_k^2). \] \[ \square \]

**Proof of Proposition 3.2:** For system of equations (21), (22) and matrix \( E_p \) with \( a_j = EX_j, A_j = EX_j^2, j = 1, \ldots, p \), from Proposition 3.1

\[ IF_{j,L_m} = \frac{C_{1,j+1} \tilde{r}_m + C_{1,j+1} \tilde{r}_m x_j}{|E_p|} = \tilde{r}_m \frac{-EX_j \prod_{k \neq j} \sigma_k^2 + x_j \prod_{k \neq j} \sigma_k^2}{\prod_{k=1}^{p} \sigma_k^2} = \tilde{r}_m \frac{x_j - EX_j}{\sigma_j^2}, j = 1, \ldots, p. \]

\[ IF_{0,L_m} = \frac{C_{1,1} \tilde{r}_m + \sum_{j=1}^{p} C_{1,j+1} \tilde{r}_m x_j}{|E_p|} = \tilde{r}_m \frac{\prod_{k=1}^{p} \sigma_k^2 + \sum_{j=1}^{p} (EX_j)^2 \prod_{k \neq j} \sigma_k^2 - \sum_{j=1}^{p} x_j EX_j \prod_{k \neq j} \sigma_k^2}{\prod_{k=1}^{p} \sigma_k^2} = \tilde{r}_m [1 + \sum_{j=1}^{p} \frac{EX_j^2 - x_j EX_j}{\sigma_j^2}]. \]

**Lemma 5.1** For the influence functions (29) it holds:

a) \( IF_{0,L_m} + \sum_{j=1}^{p} x_j IF_{j,L_m} = \tilde{r}_m [1 + \sum_{j=1}^{p} \frac{(x_j - EX_j)^2}{\sigma_j^2}], m = 1, 2, \) \[ (54) \]

b) \( IF_{i,L_1} + IF_{x_i,0,L_1} + \sum_{j=1}^{p} x_j IF_{x_i,j,L_1} = \frac{\text{sign}(r_1(x,y)) x_i - EX_i}{\beta_i \sigma_i^2} \]

\[ (55) \]

c) \( IF_{i,L_2} + IF_{x_i,0,L_2} + \sum_{j=1}^{p} x_j IF_{x_i,j,L_2} = 2 \frac{r_2(x,y)(x_i - EX_i)}{\sigma_i^2} - \beta_i \frac{1 + \sum_{j=1}^{p} (x_j - EX_j)^2}{\sigma_j^2} \]
\[ (56) \]
\[ \approx -3\beta_i \frac{(x_i - EX_i)^2}{\sigma_i^2}, \text{ if } |x_i - EX_i| \text{ is very large}, \]

\[ (57) \]

d) \( IF_{y,0,L_2} + \sum_{j=1}^{p} x_j IF_{y,j,L_2} = 1 + \sum_{j=1}^{p} \frac{(x_j - EX_j)^2}{\sigma_j^2} \]
\[ (58) \]

**Proof of Lemma 5.1:** a) From (29),

\[ IF_{0,L_m} + \sum_{j=1}^{p} x_j IF_{j,L_m} = \tilde{r}_m [1 - p + \sum_{j=1}^{p} \frac{EX_j - x_j EX_j}{\sigma_j^2}] + \sum_{j=1}^{p} x_j \tilde{r}_m (x_j - EX_j) \]
\[ = \tilde{r}_m [1 - p + \sum_{j=1}^{p} \frac{EX_j^2 - 2x_j EX_j + x_j^2}{\sigma_j^2}] = \tilde{r}_m [1 + \sum_{j=1}^{p} \frac{(x_j - EX_j)^2}{\sigma_j^2}], m = 1, 2. \]
b) Proof is provided for \( i = 1 \). If the residual of \((x, y)\) does not vanish, since

\[
IF_{0,L_1} = \frac{\text{sign}[r_1(x, y)]}{2f_Y(x)}\left[1-p + \sum_{j=1}^{p} \frac{EX_j^2 - x_jEX_j}{\sigma_j^2}\right], \quad IF_{j,L_1} = \frac{\text{sign}[r_1(x, y)]}{2f_Y(x)}\frac{x_j - EX_j}{\sigma_j^2}, \quad j = 1, \ldots, p,
\]

\[
IF'_{x_1,0,L_1} = -\frac{\text{sign}[r_1(x, y)]}{2f_Y(x)}\frac{EX_1}{\sigma_1^2}, \quad IF'_{x_1,1,L_1} = \frac{\text{sign}[r_1(x, y)]}{2f_Y(x)}\frac{x_1 - EX_1}{\sigma_1^2}, \quad IF'_{x_1,j,L_1} = 0, \quad j \neq 1.
\]

Thus,

\[
IF_{1,L_1} + IF'_{x_1,0,L_1} + x_1IF'_{x_1,1,L_1} + x_2IF'_{x_1,2,L_1} + \ldots + x_pIF'_{x_1,p,L_1}
\]

\[
= \frac{\text{sign}[r_1(x, y)]}{2f_Y(x)}\frac{x_1 - EX_1}{\sigma_1^2} + \frac{\text{sign}[r_1(x, y)]}{2f_Y(x)}\frac{EX_1}{\sigma_1^2} + x_1 - \frac{\text{sign}[r_1(x, y)]}{2f_Y(x)}\frac{x_1 - EX_1}{\sigma_1^2} = \frac{\text{sign}[r_1(x, y)]}{2f_Y(x)}\frac{x_1 - EX_1}{\sigma_1^2}
\]

c) Proof is provided for \( i = 1 \). Since

\[
IF_{0,L_2} = r_2\left[1-p + \sum_{j=1}^{p} \frac{EX_j^2 - x_jEX_j}{\sigma_j^2}\right], \quad IF_{j,L_2} = r_2 \frac{x_j - EX_j}{\sigma_j^2}, \quad j = 1, \ldots, p,
\]

\[
IF'_{x_1,0,L_2} = -\beta_{1,L_2}\left[1-p + \sum_{j=1}^{p} \frac{EX_j^2 - x_jEX_j}{\sigma_j^2}\right] - r_2 \frac{EX_1}{\sigma_1^2},
\]

\[
IF'_{x_1,1,L_2} = -\beta_{1,L_2} \frac{x_1 - EX_1}{\sigma_1^2} + \frac{r_2}{\sigma_1^2} \rightarrow x_1IF'_{x_1,1,L_2} = -\beta_{1,L_2} \frac{x_1^2 - x_1EX_1}{\sigma_1^2} + r_2 \frac{x_1}{\sigma_1^2},
\]

\[
IF'_{x_1,j,L_2} = -\beta_{1,L_2} \frac{x_j - EX_j}{\sigma_j^2} \rightarrow x_jIF'_{x_1,j,L_2} = -\beta_{1,L_2} \frac{x_j^2 - x_jEX_j}{\sigma_j^2}, \quad j \neq 1.
\]

Thus,

\[
IF_{1,L_2} + IF'_{x_1,0,L_2} + x_1IF'_{x_1,1,L_2} + x_2IF'_{x_1,2,L_2} + \ldots + x_pIF'_{x_1,p,L_2}
\]

\[
= 2 \frac{r_2(x_1 - EX_1)}{\sigma_1^2} - \beta_{1,L_2}\left[1-p + \sum_{j=1}^{p} \frac{x_j^2 - 2x_jEX_j + EX_j^2}{\sigma_j^2}\right]
\]

\[
= 2 \frac{r_2(x_1 - EX_1)}{\sigma_1^2} - \beta_{1,L_2}\left[1 + \sum_{j=1}^{p} \frac{(x_j - EX_j)^2}{\sigma_j^2}\right].
\]

Since

\[
r_2(x_1 - EX_1) = y(x_1 - EX_1) - \beta_{1,L_2}x_1(x_1 - EX_1) - (x_1 - EX_1) \sum_{j=2}^{p} \beta_{j,L_2}x_j
\]

\[
= y(x_1 - EX_1) - \beta_{1,L_2}(x_1 - EX_1)^2 - \beta_{1,L_2}EX_1(x_1 - EX_1) - (x_1 - EX_1) \sum_{j=2}^{p} \beta_{j,L_2}x_j,
\]
Lemma 5.2 For regression model (15) with assumptions (r) if

\[ IF_{1,L_2} + IF'_{x_1,0,L_2} + x_1 IF'_{x,1,L_2} + x_2 IF'_{x_1,1,L_2} + \ldots + x_p IF'_{p,x_1,L_2} \approx -3\beta_{1,L_2} \frac{(x_1 - EX_1)^2}{\sigma_1^2}. \]

\( d) \) From (29),

\[ IF'_{y,0,L_2} = 1 - p + \sum_{j=1}^{p} \frac{EX_j^2 - x_j EX_j}{\sigma_j^2}, \quad IF'_{y,j,L_2} = \frac{x_j - EX_j}{\sigma_j^2}, \quad j = 1, \ldots, p. \]

Thus,

\[ IF'_{y,0,L_2} + \sum_{j=1}^{p} x_j IF'_{y,j,L_2} = 1 - p + \sum_{j=1}^{p} \frac{EX_j^2 - x_j EX_j + x_j^2 - x_j EX_j}{\sigma_j^2} = 1 + \sum_{j=1}^{p} \frac{(x_j - EX_j)^2}{\sigma_j^2}. \]

A Lemma used repeatedly to calculate residuals’ differences is due to (4), (8).

**Lemma 5.2** For regression model (15) with assumptions (A1), (A3), perturbation (10), \( r_1(x,y) \neq 0 \), and \( \epsilon, |h| \) both small:

\[ \beta_{j,L,m,x} \approx \beta_{j,L,m} + \epsilon IF_{j,L,m}, \quad \beta_{j,L,m,x_i,h} \approx \beta_{j,L,m,x} + \epsilon h IF'_{x_i,j,L,m}. \quad (59) \]

**Proof of Lemma 5.2**: Use approximations (1), (8). \( \square \)

**Proof of Proposition 3.3** a) For \( a_1 \), from Lemma 5.2

\[ r_{m,x} = y - \beta_{0,L,m,x} - \beta_{1,L,m,x_1} - \ldots - \beta_{p,L,m,x_p} \]

\[ \approx y - (\beta_{0,L,m} + \epsilon IF_{0,L,m}) - (\beta_{1,L,m} + \epsilon IF_{1,L,m} x_1) - \ldots - (\beta_{p,L,m} + \epsilon IF_{p,L,m} x_p) \]

\[ = r_m - \epsilon (IF_{0,L,m} + IF_{1,L,m} + \ldots + IF_{p,L,m}). \]

(30) follows from (54). Since \( \tilde{r}_m(x,y) \) has the same sign with \( r_m(x,y) \), for \( \epsilon \) small \( r_{m,x}(x,y) \) will also have the same sign and reduced size because \( -\epsilon \tilde{r}_m(x,y) \) has opposite sign from \( r_m(x,y) \).

For \( a_2 \), (31) follows from (23).

\( b) \) Provided for \( i = 1 \) using Lemma 5.2

\[ r_{m,x_1,h} = y - \beta_{0,L,m,x_1,h} - \beta_{1,L,m,x_1,h} (x_1 + h) - \ldots - \beta_{p,L,m,x_1,h} x_p \]

\[ \approx y - [\beta_{0,L,m,x} + \epsilon h IF'_{x_1,0,L,m}] - [\beta_{1,L,m,x} + \epsilon h IF'_{x_1,1,L,m}](x_1 + h) - \ldots - [\beta_{p,L,m,x} + \epsilon h IF'_{x_1,p,L,m}] x_p \]

29
\[ r_{m,x} = \beta_{1,L,m,x} h - \epsilon h [IF'_{x_1,0,L,m} + x_1 IF'_{x_1,1,L,m} + x_2 IF'_{x_1,2,L,m} + \ldots + x_p IF'_{x_1,p,L,m}] - \epsilon h^2 IF'_{x_1,1,L,m} \]
\[ = r_{m,x} - \beta_{1,L,m} h - \epsilon h [IF'_{x_1,0,L,m} + IF'_{x_1,0,L,m} + x_1 IF'_{x_1,1,L,m} + x_2 IF'_{x_1,2,L,m} + \ldots + x_p IF'_{x_1,p,L,m}] - \epsilon h^2 IF'_{x_1,1,L,m}. \]

(33), (34) follow from (55), (56).

For \( b_2 \), if \( |x_i| \) is large and \( |h| \) is small, \( \beta_{i,L,m} h \) and \( \epsilon h^2 IF'_{x_i,i,L,m} \) are of smaller order than the remaining terms and (35) follows, in addition, (57) implies (36) and (37) follows also.

c) \[ r_{2,x,y+h}(x,y+h) = y + h - \beta_{0,L,2,x,y+h} - \beta_{1,L,2,x,y+h} x_1 - \ldots - \beta_{p,L,2,x,y+h} x_p \]
\[ \approx y + h - \beta_{0,L,2,x,y} + \epsilon h IF'_{y,0,L,2} - \beta_{1,L,2,x,y} + \epsilon h IF'_{y,1,L,2} x_1 - \ldots - \beta_{p,L,2,x,y} + \epsilon h IF'_{y,p,L,2} x_p \]
\[ = r_{2,x,y}(x,y) + h - \epsilon h [IF'_{y,0,L,2} + \sum_{j=1}^{p} x_j IF'_{y,j,L,2}] = r_{2,x,y}(x,y) + h - \epsilon h [1 + \sum_{j=1}^{p} \frac{(x_j - EX_j)^2}{\sigma_j^2}], \]
with the last equality obtained from (58).

\[ \text{Proof of Proposition 3.4:} \] Follows from (32) dividing both its sides by \( h \) and taking the limit with \( h \) converging to zero. \( \square \)

\[ \text{Proof of Proposition 3.5:} \]
\[ \lim_{|x_i| \to \infty} RINFIN(x,y;\epsilon,L_2) \geq \epsilon \cdot \lim_{|x_i| \to \infty} \left| 2 r_2(x,y)(x_i - EX_i) \frac{1}{\sigma_i^2} - \beta_{i,L,2} [1 + \sum_{j=1}^{p} \frac{(x_j - EX_j)^2}{\sigma_j^2}] \right| \]
\[ \approx \lim_{|x_i| \to \infty} 3 \beta_{i,L,2} \frac{(x_i - EX_i)^2}{\sigma_i^2} = \infty; \]
last approximation follows from (57). \( \square. \)

\[ \text{Proof of Proposition 3.6:} \] Follows from (38) dividing both its sides by \( h \) and taking the limit with \( h \) converging to zero. \( \square \)

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NAIVE L1-RESIDUAL PLOTS

Telephone

Kootenay river

Log.brain anf Log.weight

Hertzsprung-Russel

Hawkins-Bradu-Kass

Scottish Hill races

Hadi-Simonoff

Education

Stackloss

Coleman

Salinity

Modified Wood

Figure 1
NAIVE L2-RESIDUAL PLOTS

Figure 2
MODIFIED WOOD DATA
ABS. RESIDUALS AGAINST INDEP. VARIABLES

Figure 3

Observations 4, 6, 8, 19 are extreme in all the x-coordinates 1-5