PROJECTIVE TORIC VARIETIES OF CODIMENSION 2 WITH
MAXIMAL CASTELNUOVO–MUMFORD REGULARITY

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ABSTRACT. The Eisenbud–Goto conjecture states that $\text{reg } X \leq \deg X - \text{codim } X + 1$ for a nondegenerate irreducible projective variety $X$ over an algebraically closed field. While this conjecture is known to be false in general, it has been proven in several special cases, including when $X$ is a projective toric variety of codimension 2. We classify the projective toric varieties of codimension 2 having maximal regularity, that is, for which equality holds in the Eisenbud–Goto bound. We also give combinatorial characterizations of the arithmetically Cohen–Macaulay toric varieties of maximal regularity in characteristic 0.

1. Introduction

Let $k$ be a field. Recall that for a closed subscheme $X \hookrightarrow \mathbb{P}^n_k$ with ideal sheaf $\mathcal{I}_X$, the Castelnuovo–Mumford regularity of $X$ is defined as

$$\text{reg } X := \max \{ r > 0 \mid H^i(\mathbb{P}^n_k, \mathcal{I}_X(r-i)) = 0 \text{ for all } i > 0 \}.$$ 

Equivalently, $\text{reg } X$ may be defined in terms of the minimal free resolution of $\mathcal{I}_X$, viewed as an $\mathcal{O}_{\mathbb{P}^n_k}$-module (see §2 for this definition). In [EG84], Eisenbud and Goto made the following conjecture.

Conjecture 1.1. If $k$ is an algebraically closed field and $X \hookrightarrow \mathbb{P}^n_k$ is a nondegenerate irreducible projective variety, then

$$\text{reg } X \leq \deg X - \text{codim } X + 1.$$ 

In 2018, Peeva and McCullough proved that this conjecture is false in general by exhibiting an explicit counterexample [MP18].

While not true in full generality, Conjecture [1.1] has been proven true in several special cases. For example, in [EG84], Eisenbud and Goto proved this bound holds if $X$ is arithmetically Cohen–Macaulay. In [GLP83], Conjecture [1.1] is proven in the case where $X$ is a projective curve, using algebro-geometric methods. However, it is still open whether Conjecture [1.1] is true for general toric varieties, which are the subject of focus of the present paper. In this direction, [HH03] resolved the case where the coordinate ring of $X$ is a simplicial semigroup ring with isolated singularity, and [Nit12] resolved the case where the coordinate ring is a seminormal simplicial affine semigroup ring. A combinatorial proof of Conjecture [1.1] in the case of monomial curves was given in [Nit14]. In [PS98], a combinatorial proof was given for toric varieties of codimension 2.
In the cases of (nondegenerate, irreducible) curves, arithmetically Cohen–Macaulay varieties, and toric varieties of codimension 2 (each a large class of varieties for which Conjecture 1.1 is known to hold), we will use the phrase having maximal regularity to refer to varieties \( X \) which achieve equality in Conjecture 1.1, i.e. such that \( \text{reg} X = \deg X - \text{codim} X + 1 \).

The main question motivating the results of this paper is the following.

**Question 1.2.** Which toric varieties have maximal regularity (in the cases where the Eisenbud–Goto bound is known to hold)?

We work over an arbitrary field \( k \). Let \( S \) denote the coordinate ring of \( \mathbb{P}^{n-1} \). To a lattice \( L \subseteq \mathbb{Z}^n \) of rank \( r \) perpendicular to \( (1, \ldots, 1) \in \mathbb{Z}^n \), there is an associated homogeneous lattice ideal \( I_L \subseteq S \) and a closed projective scheme \( X_L := \text{Proj} S/I_L \hookrightarrow \mathbb{P}^{n-1} \). Also associated to \( L \) is its Gale diagram, which is a certain collection of \( n \) vectors in \( \mathbb{Z}^r \) that is defined uniquely up to the action of \( \text{GL}_r(\mathbb{Z}) \). When \( L \) is saturated (that is, \( \mathbb{Q} L \cap \mathbb{Z}^n = L \)), then \( I_L \) is prime and is a toric ideal, and \( X_L \) is a (projective) toric variety. The 1-dimensional toric varieties are precisely the monomial curves.

Our main result addresses the general case when \( \text{codim} X_L = 2 \), and is stated below.

**Theorem 1.3.** Suppose \( X_L \) is a nondegenerate toric variety of codimension 2. Let \( n' \geq 3 \) be the number of nonzero Gale vectors of \( L \). Then \( X_L \) has maximal regularity (i.e., achieves equality in the Eisenbud–Goto conjecture) if and only if \( L \) satisfies one of the following:

- After removing Gale vectors equal to 0, the lattice \( L \) is one of the 14 saturated lattices given in Table 4.1, up to permutations of the coordinates. In this case, \( 4 \leq n' \leq 8 \), and \( X_L \) is a complete intersection.
- \( n' = 4 \), and \( L \) has a Gale diagram whose nonzero vectors are equal to
  \[
  \{(1, 0), (-1, 1), (-1, -d + 1), (1, d - 2)\}
  \]
  for some \( d \geq 3 \). In this case, \( X_L \) is not a complete intersection, and \( X_L \) is arithmetically Cohen–Macaulay if and only if \( d = 3 \).
- \( n' = 5 \), and the Gale diagram of \( L \), ignoring zeros, is of the form
  \[
  \{u, v, w, -u, -v, -w\},
  \]
  where \( u, v, w \) are visible lattice points that are pairwise linearly independent and are such that \( u \neq \pm (v + w) \) and \( |\text{det}(v, w)| = 1 \). In this case, \( X_L \) is not a complete intersection.
- \( n' = 6 \), and the Gale diagram of \( L \), ignoring zeros, is of the form
  \[
  \{u, v, w, -u, -v, -w\},
  \]
  where \( u, v, w \) are visible lattice points that are pairwise linearly independent and are such that \( |\text{det}(v, w)| = 1 \). In this case, \( X_L \) is not a complete intersection.

Note that in the third case, the conditions given on the Gale diagram are indeed \( \text{GL}_2(\mathbb{Z}) \)-invariant. Moreover, the set of saturated lattices given in Table 4.1 is independent of the field \( k \).
Remark 1.4. A projective toric subvariety of $\mathbb{P}^{n-1}$ of codimension $r$ may also be specified by a linear map $A : \mathbb{Z}^n \rightarrow \mathbb{Z}^{n-r}$; we set $X_A := X_L$ where $L = \ker A$. For simplicity assume $n = n'$, i.e. that all Gale vectors are nonzero. Then the toric varieties of Theorem 1.3 which are not complete intersections can be described as the varieties $X_A$ where $A$ takes one of the following forms (for $n = 4, 5, 6$, respectively):

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & d - 1 & d
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 \\
1 & b & c & 0 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
1 & b & c & 0 & 0 & 0
\end{bmatrix}
\]

where $d \geq 3$ and $b, c$ are coprime nonzero integers (and in the case $n = 5$, $(b, c) \neq \pm(1, 1)$). This description is equivalent to the one in Theorem 1.3. It is easy to compute the degrees of the corresponding varieties to be

\[d, \quad 1 + \max\{|b|, |c|, |b-c|\}, \quad 1 + |b| + |c|,\]

respectively. (Note that $b$ and $c$ are the unique integers for which $u = bv + cw$.)

Our result shows that all examples of maximal regularity for toric varieties of codimension 2, in a suitable sense, come from examples in $\mathbb{P}^n$ for $n \leq 5$. In [Kwa99], Kwak shows that the only smooth nondegenerate complex threefolds in $\mathbb{P}^5$ with $\text{reg} X \leq \deg X - 1$ are complete intersections of two quadrics or the Segre threefold (obtained by the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5$). It may be verified explicitly that the complete intersections appearing in Theorem 1.3 are not smooth, so it follows from Kwak’s results that the only smooth example of a toric threefold in $\mathbb{P}_C^5$ having maximal regularity is given by the Segre threefold. A corollary of Theorem 1.3 is the following stronger fact.

**Theorem 1.5.** A smooth nondegenerate toric variety of codimension 2 (a priori of arbitrary dimension) having maximal regularity is, up to isomorphism, one of the following:

- a monomial curve (embedded in $\mathbb{P}^3$),
- a hyperplane section of the Segre threefold (embedded in $\mathbb{P}^4$), or
- the Segre threefold (embedded in $\mathbb{P}^5$).

This hyperplane section of the Segre threefold is obtained by substituting $(b, c) = (1, -1)$ in the $3 \times 5$ matrix of Remark 1.4, and the Segre threefold itself is obtained by substituting $(b, c) = (1, \pm 1)$ in the $4 \times 6$ matrix. We omit the proof of this theorem; the result may be deduced by elementary means from the explicit form of the matrices in Remark 1.4 (in all cases there, except those listed in Theorem 1.3, the varieties $X_A$ are not even normal) and the explicit lattices of Table 4.1.

Along the way to proving Theorem 1.3, we also characterize toric varieties $X_L$ of maximal regularity in the cases when $X_L$ is a curve, a complete intersection, or arithmetically Cohen–Macaulay in characteristic 0.

While our main results apply only to toric varieties, many of our intermediate results apply more generally to nonreduced schemes $X_L$ (or occasionally to even more general classes of schemes), and so we will state intermediate results as generally as the methods allow.
Outline. Preliminaries and notation are discussed in Section 2. In Section 3, we characterize monomial curves of maximal regularity. Specializing this result to the case of codimension 2 allows us to solve the $n' = 4$ case of Theorem 1.3. In Section 4, we focus on the case when $I_L$ is Cohen–Macaulay; that is, when $X_L$ is arithmetically Cohen–Macaulay.

In Section 5, we recall the methods of [PS98], which allow us to extract information about general schemes $X_L$ of codimension 2 by understanding the case where $X_L \hookrightarrow \mathbb{P}^3$ is a curve. The bulk of the work lies in Section 6, in which we use this framework to analyze the case when $X_L$ has codimension 2 and is not arithmetically Cohen–Macaulay. This allows us to complete the proof of Theorem 1.3 in Section 7. We suggest directions for future research in Section 8.

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2. Preliminaries

Let $k$ be a field, and let $S_n := k[x] = k[x_1, \ldots, x_n]$ be a graded polynomial ring (with the standard grading). When $n$ is fixed or clear from context, we will simply write $S$ for $S_n$. All ideals of $S$ are assumed to be homogeneous, unless stated otherwise; in the same vein, all generators of ideals are assumed to be homogeneous. Given an ideal $I \subseteq S$, we write $\deg I$ for the multiplicity of $S/I$.

For a graded $S$-module $M$, we denote the graded minimal free resolution of $M$

$$
\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0
$$

by $F_*$. Each $F_i$ is a direct sum of twists of $S$, so we can write $F_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{i,j}}$ for some nonnegative integers $\beta_{i,j}$, called the Betti numbers of $F_*$. The Castelnuovo–Mumford regularity of a finitely generated graded $S$-module $M$ is defined by

$$
\operatorname{reg} M := \max\{j \mid \beta_{i,i+j} \neq 0 \text{ for some } i\}.
$$

It does not depend on the choice of minimal free resolution. For a closed subscheme $X \hookrightarrow \mathbb{P}^{n-1}$, we write $\operatorname{reg} X$ for the Castelnuovo–Mumford regularity of its vanishing ideal; this is equivalent to the definition of $\operatorname{reg} X$ given in Section 4 but is more suited to our purposes. We will often refer to this quantity simply as the “regularity.”

Because we are primarily concerned with toric varieties, we include some background on lattice ideals. Let $L \subseteq \mathbb{Z}^n$ be a lattice of rank $r \geq 1$ that is orthogonal to the all-1’s vector $(1,1,\ldots,1) \in \mathbb{Z}^n$, so $r \leq n-1$. Then $L$ defines a homogeneous lattice ideal

$$
I_L := \langle x^u - x^v \mid u - v \in L \rangle \subseteq S.
$$

The codimension of $I_L$ equals $r$. We say that $L$ is saturated if $\mathbb{Q}L \cap \mathbb{Z}^n = L$. In this case, $I_L$ is prime and we say that $I_L$ is toric.

Let $B \in \mathbb{Z}^{n \times r}$ be a matrix whose columns form a basis for $L$, whose entry in the $(i,j)$ position is given by $b_{ij}$. The Gale diagram $G := G_B$ of $L$ is the collection of Gale vectors, which are the row vectors $b_i := (b_{i1}, \ldots, b_{ir}) \in \mathbb{Z}^r$. Note that for two such matrices $B$
and $B'$, the corresponding Gale diagrams differ only by an element of $\text{GL}_r(\mathbb{Z})$. We may therefore speak of the Gale diagram $G_L \in \mathbb{Z}^{n \times r}/\text{GL}_r(\mathbb{Z})$ of $L$, independent of the choice of $B$. One can often move back and forth between the combinatorial data contained in the Gale diagram and the algebraic properties of a lattice ideal. Note that Gale vectors should be thought of as ordered pairs $(i, b_i)$ (so that particular vectors $b_i \in \mathbb{Z}^r$ are considered “with multiplicity”), but for convenience, we will frequently treat each Gale vector $(i, b_i)$ simply as a vector $b_i \in \mathbb{Z}^r$.

Given an ideal $I \subseteq S_n$ (not necessarily prime), one can form for each $m \geq n$ the ideal $I_m := IS_m \subseteq S_m$ (under the natural embedding $S_n \hookrightarrow S_m$). It is easy to argue that for each $m \geq n$, $\text{reg } I_m = \text{reg } I$, $\deg I_m = \deg I$, and $\text{codim } I_m = \text{codim } I$.

If $I = I_L \subseteq S_n$ is a lattice ideal, then the ideal $I_m$ is the lattice ideal whose Gale diagram is obtained by adding $m - n$ zero vectors to the Gale diagram of $I$. Thus, as far as regularity, degree, and codimension are concerned, it suffices to consider lattice ideals $I_L$ all of whose Gale vectors are nonzero.

Given $L$, we let $\Gamma := \mathbb{Z}^n/L$ be an abelian group. Note that $S, I_L, S/I_L$, the minimal free resolution of $S/I_L$ over $S$, and the Koszul homology of $S/I_L$ are graded by $\Gamma$ \cite{PS98}. A fiber is a set of all monomials of $S$ with a fixed degree $C \in \Gamma$. The requirement that $I_L$ be orthogonal to $(1, \ldots, 1)$ ensures that all fibers are finite.

In the case that $L$ has rank $r = 2$, there are more precise results on when $I_L$ is toric, a complete intersection, or Cohen–Macaulay. In this case, $\text{codim } I_L = 2$ and a Gale diagram $G$ consists of vectors in $\mathbb{Z}^2$. Following \cite{PS98}, we say a Gale diagram is imbalanced if every Gale vector lies on the $y$-axis or has nonpositive $y$-coordinate. We have the following:

**Lemma 2.1** (\cite{PS98, Lemma 3.1}). The lattice ideal $I_L$ is a complete intersection if and only if it has an imbalanced Gale diagram.

In particular, this result implies that if $I_L$ is degenerate, i.e., if $I_L \not\subseteq \langle x_1, \ldots, x_n \rangle^2$, then $I_L$ is a complete intersection. Moreover, we have the following result.

**Proposition 2.2** (\cite{PS98, Proposition 4.1}). The ideal $I_L$ is not Cohen–Macaulay if and only if there exists a Gale diagram $G$ for $L$ which intersects each of the four open quadrants.

### 3. Monomial curves

Let $A$ be a $2 \times n$ matrix of the form

$$
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
a_1 & a_2 & \cdots & a_n
\end{bmatrix},
$$

(3.1)

where $0 = a_1 < a_2 < \cdots < a_n$ and $\text{gcd}(a_2, \ldots, a_n) = 1$. Every toric ideal of $S_n = \mathbb{k}[x_1, \ldots, x_n]$ defining a curve in $\mathbb{P}^{n-1}$ is of the form $I_{\text{ker } A}$ for some such matrix $A$, up to reordering columns. In this notation, $\deg I_{\text{ker } A} = a_n$.

The results of \cite{HHS10} in conjunction with the general results on curves from \cite{GLP83} give a characterization of which monomial curves have maximal regularity, in terms of the entries $\{a_i\}$. We record the following corollary of their results.
Proposition 3.1. Let $A$ be a matrix as in (3.1), let $d = a_n$, and let $I = \ker A$. Then $I$ has maximal regularity, i.e. $\reg I = \deg I - \codim I + 1$, if and only if one of the following holds:

- $d \leq n$,
- $d \geq n + 1$ and the integers $(a_1, a_2, \ldots, a_n)$ are either $(0, 1, 2, \ldots, n - 3, d - 1, d)$ or $(0, 1, d - n + 3, d - n + 4, \ldots, d)$.

Proof. Without loss of generality suppose $k$ is algebraically closed. By general theory, since $X_L$ is toric, $X_L$ is rational.

If $\deg I = n + 1$, then [GLP83, Theorem 3.1] implies that equality is achieved. So assume $\deg I = n + 1$. Comment 1 at the end of §2 of [GLP83] now says that a necessary condition for $X_L$ to achieve equality in EG is that $X_L$ be smooth. So henceforth assume that $X_L$ is smooth, i.e., that $a_2 = 1$ and $a_{n-1} = d - 1$.

Using the notation of [HHS10], let $\lambda(X_L)$ be the length of the longest gap (i.e., the largest value of $a_k - a_{k-1} - 1$) in $S = \{a_1, a_2, \ldots, a_{n-1}, a_n\}$ and let $\varepsilon = \max\{i \mid [0, i], [d - i, d] \subseteq S\}$. Observe that the sum of all gaps is $a_n - a_1 - n + 1 = d - n + 1$, so $\lambda(X_L) = d - n + 1$ is achieved if and only if there is only a single gap of positive length. We then see from [HHS10, Theorem 2.7] that

$$\reg I = \reg(S/I) + 1 \leq \frac{\lambda(X_L) - 1}{\varepsilon} + 3 \leq d - n + 3$$

with equality only if $\varepsilon = 1$ and $\lambda(X_L) = d - n + 1$. Since there can only be a single gap, and $\varepsilon = 1$ says that 2 and $d - 2$ cannot both appear in $S$, we conclude that the only possibilities for $S$ are $\{0, 1, 2, \ldots, n - 3, d - 1, d\}$ or $\{0, 1, d - n + 3, d - n + 4, \ldots, d\}$. Note that these are the same list up to $k \mapsto d - k$.

Finally, we need to know that these cases actually give equality. This is now exactly implied by the statement of [HHS10, Theorem 3.4] with $\varepsilon = 1$ and $p = d - n + 3$.

In the case of codimension 2, we find the following corollary.

Corollary 3.2. Suppose $\mathcal{L} \subseteq \mathbb{Z}^4$ is saturated and has rank 2, and $I_L$ is nondegenerate. Then $\reg I_L = \deg I_L - 1$ if and only if $\mathcal{L}$ satisfies one of the following cases, up to a reordering of coordinates:

- $\mathcal{L}$ has a Gale diagram equal to $\{(0, 2), (-1, 0), (-1, -1), (2, -1)\}$. In this case, $I_L$ is a complete intersection.
- $\mathcal{L}$ has a Gale diagram equal to $\{(1, 0), (-1, 1), (-1, -d+1), (1, d-2)\}$ for some $d \geq 3$. In this case, $I_L$ is not Cohen–Macaulay if $d \geq 4$, and is Cohen–Macaulay but not a complete intersection if $d = 3$.

Proof. The set of possible Gale diagrams for $\mathcal{L}$ follows directly from Proposition 3.1. The remaining assertions follow from Lemma 2.1 and Proposition 2.2.

4. Cohen–Macaulay lattice ideals

In Section 4.1, we characterize when $\reg I_L = \deg I_L - \codim I_L + 1$ for Cohen–Macaulay lattice ideals $I_L \subseteq (x_1, \ldots, x_n)^2$, under the assumption $\text{char } k = 0$. Then in Section 4.2, we briefly address the case when $I_L$ is a complete intersection (in arbitrary characteristic).
Finally, in Section 4.3, we characterize in arbitrary characteristic the lattice ideals $I_L$ of codimension 2 that are Cohen–Macaulay but not complete intersections that satisfy $\text{reg } I_L = \deg I_L - \text{codim } I_L + 1$.

4.1. The characteristic zero case. Let $k$ be a field of characteristic 0. In Section 3 of [PS08], the authors prove that $\text{reg } I \leq \deg I - \text{codim } I + 1$ for any Cohen–Macaulay homogeneous ideal $I \subseteq \langle x_1, \ldots, x_n \rangle^2$. Following their proof gives the first part of the following result.

**Proposition 4.1.** Suppose $\text{char } k = 0$. If $I \subseteq \langle x_1, \ldots, x_n \rangle^2$ is a Cohen–Macaulay homogeneous ideal with codimension $c$ and $n_2$ minimal homogeneous generators of degree 2, then $n_2 \leq \binom{c+1}{2}$. Further, $\text{reg } I = \deg I - c + 1$ if and only if $n_2 = \binom{c+1}{2} - 1$ or $n_2 = \binom{c+1}{2}$, and in this case, the Hilbert series of $S/I$ is given by

$$
\frac{1 + ct + \sum_{i=2}^{\text{reg } I-1} t^i}{(1-t)^{n-c}}.
$$

If $n_2 = \binom{c+1}{2}$, then $\text{reg } I = 2$ and all minimal generators of $I$ have degree 2. Moreover, the minimal free resolution of $I$ has the following form:

$$
0 \rightarrow S(-c-1)^{\binom{c+1}{2}} \rightarrow \cdots \rightarrow S(-3)^{\binom{c+1}{3}} \rightarrow S(-2)^{\binom{c+1}{2}} \rightarrow I \rightarrow 0.
$$

If $n_2 = \binom{c+1}{2} - 1$, then $\text{reg } I \geq 3$ and $I$ satisfies one of the following two conditions:

- All minimal generators of $I$ have degree 2, and $\text{reg } I = 3$.
- $I$ has exactly one minimal generator of degree greater than or equal to 3, which has degree exactly $\text{reg } I$.

**Proof.** Using the notation in [PS08], let $N$ be the generic initial ideal of $I$ with respect to the reverse lexicographic order. Then $N \subseteq \langle x_1, \ldots, x_n \rangle^2$ is a Cohen–Macaulay Borel ideal, and none of the variables $x_{c+1}, \ldots, x_n$ appear in the minimal monomial generators of $N$. Furthermore, $I$ and $N$ have the same Hilbert function, so it follows that $n_2 \leq \binom{c+1}{2}$. Moreover, $x_{c}^{\text{reg } I}$ is a minimal monomial generator of $N$, and $\overline{N} := N \otimes S$ is an artinian ideal in $\overline{S} := S/(x_{c+1}, \ldots, x_n)$, where $\deg I$ equals the length of $\overline{S}/\overline{N}$. It then follows that the length of $\overline{S}/\overline{N}$ equals $\text{reg } I + c - 1$ if and only if all the monomials of degree 2 supported on $x_1, \ldots, x_c$ lie in $\overline{N}$, except possibly for $x_c^2$. Since $N$ is Borel, this is equivalent to requiring $n_2 = \binom{c+1}{2} - 1$ or $n_2 = \binom{c+1}{2}$. We then see that $\text{reg } I \geq 3$ if $n_2 = \binom{c+1}{2} - 1$ and $\text{reg } I = 2$ if $n_2 = \binom{c+1}{2}$.

The Hilbert series of $S/I$ equals the Hilbert series of $S/N$. If $\text{reg } I = \deg I - c + 1$, it is straightforward to compute the Hilbert series of $S/N$ using the above information.

Now, suppose $n_2 = \binom{c+1}{2}$. Since $\text{reg } I = 2$, it follows that $I$ has no minimal generators of degree more than 2. Moreover, the minimal free resolution of $I$ is pure, and we can see that the resolution takes the desired form (say, by considering the Hilbert series of $S/I$).

Suppose $n_2 = \binom{c+1}{2} - 1$. Since the Betti numbers of $N$ are at least those of $I$, we find that $I$ has at most one minimal generator of degree at least 3, and if such a generator exists, it must have degree exactly $\text{reg } I$. If no such generator exists, then $I$ is generated in degree 2; if furthermore $\text{reg } I \geq 4$, then $N$ has no generator in degree 3, which leads to a contradiction.
by Green’s crystallization principle [Gre98, Proposition 2.28]. Thus, if \( I \) is generated in degree 2, then \( \text{reg} I \leq 3 \).

This yields the following corollary for lattice ideals.

**Corollary 4.2.** Suppose \( \text{char} \, k = 0 \), and that \( I_\mathcal{L} \subseteq \langle x_1, \ldots, x_n \rangle^2 \) is Cohen–Macaulay, where \( \mathcal{L} \subseteq \mathbb{Z}^n \) is a lattice of rank \( r \). Then \( \text{reg} I_\mathcal{L} \leq \text{deg} I_\mathcal{L} - r + 1 \), and the number of fibers of \( \mathcal{L} \) of degree 2 is at least \( \binom{n+1}{2} - \binom{r+1}{2} \). Furthermore, \( \text{reg} I_\mathcal{L} = \text{deg} I_\mathcal{L} - r + 1 \) if and only if the number of fibers of \( \mathcal{L} \) of degree 2 equals \( \binom{n+1}{2} - \binom{r+1}{2} \) or \( \binom{n+1}{2} - \binom{r+1}{2} + 1 \).

### 4.2. Complete intersections

If \( \text{char} \, k = 0 \), for complete intersection ideals \( I \), Proposition 4.1 states that \( \text{reg} I = \text{deg} I - \text{codim} I + 1 \) if and only if either \( \text{codim} I = 1 \) or \( \text{codim} I = 2 \) and \( I \) is generated by two quadratics. But a straightforward computation, using the Koszul complex, allows us to generalize this to arbitrary characteristic. We state this formally below.

**Proposition 4.3.** Let \( k \) be a field of arbitrary characteristic. Suppose \( I \subseteq \langle x_1, \ldots, x_n \rangle^2 \) is a homogeneous complete intersection ideal. Then \( \text{reg} I \leq \text{deg} I - \text{codim} I + 1 \), with equality holding if and only if \( \text{codim} I = 1 \) or \( \text{codim} I = 2 \) and \( I \) is generated by two quadratics.

**Proof.** Let \( I \) have codimension \( m \) and let its minimal generators have degrees \( d_1, \ldots, d_m \geq 2 \). Then \( I \) is resolved by a Koszul complex, and one directly finds that \( \text{reg} I = d_1 + \cdots + d_m - m + 1 \) while \( \text{deg} I = d_1 \cdots d_m \). The given inequality is then equivalent to \( d_1 + \cdots + d_m \leq d_1 \cdots d_m \). This is always true when \( m = 1 \) and impossible for \( m \geq 3 \). When \( m = 2 \), it holds only for \( d_1 = d_2 = 2 \).

If \( I = I_\mathcal{L} \) is a nondegenerate complete intersection lattice ideal, we see from Proposition 4.3 that \( \text{reg} I_\mathcal{L} = \text{deg} I_\mathcal{L} - \text{codim} I_\mathcal{L} + 1 \) only holds if \( r \leq 2 \), and it always holds if \( r = 1 \). If \( r = 2 \), we find the following.

**Corollary 4.4.** Fix a field \( k \) of arbitrary characteristic. There are only finitely many non-degenerate lattice ideals \( I_\mathcal{L} \) with the following properties:
\begin{itemize}
  \item all Gale vectors are nonzero,
  \item \( \text{codim} I_\mathcal{L} \geq 2 \),
  \item \( I_\mathcal{L} \) is a complete intersection, and
  \item \( \text{reg} I_\mathcal{L} = \text{deg} I_\mathcal{L} - \text{codim} I_\mathcal{L} + 1 \).
\end{itemize}

Moreover, all such lattices have rank 2, and have \( n \leq 8 \). Up to permutations of the coordinates, there are exactly 23 such lattice ideals, 14 of which are saturated (see Table 4.1 for a full list).

**Proof.** Note that any quadratic generator coming from a lattice has at most four variables in its support (it must be of the form \( x_p x_q - x_r x_s \), for not-necessarily-distinct \( p, q, r, s \in \{1, \ldots, n\} \)). If \( I_\mathcal{L} \) is a complete intersection such that \( \text{reg} I_\mathcal{L} = \text{deg} I_\mathcal{L} - \text{codim} I_\mathcal{L} + 1 \), of codimension at least 2, Proposition 4.3 implies that there can be at most 8 variables from \( \{x_1, \ldots, x_n\} \) used in its minimal generators, since \( I_\mathcal{L} \) is generated by two quadratics. There are therefore finitely many such lattices. Conducting this finite search explicitly in Macaulay2 [GS], we obtain the full list in Table 4.1.

This finite set of lattices is independent of the field \( k \), by [PS98, Remark 3.2].
4.3. Lattice ideals of codimension 2. In this section, we focus exclusively on Cohen–Macaulay lattice ideals $I_L$ of codimension 2; since the case of complete intersections is addressed in Section 4.2, we may assume that the lattice ideals in question are not complete intersections.

First, we set some notation. Given any ideal $I$ of $S$, we write the minimal free resolution of $I$ as follows:

$$0 \rightarrow \bigoplus_{j=1}^{b_p} S(-d_{pj}) \rightarrow \cdots \rightarrow \bigoplus_{j=1}^{b_1} S(-d_{1j}) \rightarrow I \rightarrow 0,$$

where $p$ equals the projective dimension of $S/I$. For all $1 \leq i \leq p$, we also let $M_i := \max\{d_{ij} | 1 \leq j \leq b_i\}$ and $m_i := \min\{d_{ij} | 1 \leq j \leq b_i\}$. We recall that the resolution is called pure if $m_i = M_i$ for all $i$. From [PS98, Remark 5.8 and Theorem 6.1], we obtain the following.

**Lemma 4.5.** Let $I_L \subseteq \mathbb{k}[x_1, \ldots, x_n]$ be a lattice ideal of codimension 2 that is Cohen–Macaulay but not a complete intersection. Then the minimal free resolution of $I_L$ is given by the Hilbert–Burch complex, and satisfies $p = 2$, $b_1 = 3$, $b_2 = 2$, $d_{11} + d_{12} + d_{13} = d_{21} + d_{22}$, and $M_1 < m_2$. 

### Table 4.1

| $n$ | Saturated $\mathcal{L}$ | Non-Saturated $\mathcal{L}$ |
|-----|--------------------------|-------------------------------|
| 3   | $\emptyset$              | $\begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix}$, $\begin{bmatrix} 0 & 2 \\ -2 & -1 \end{bmatrix}$ |
| 4   | $\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix}$, $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ |
| 5   | $\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \end{bmatrix}$ |
| 6   | $\begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 \end{bmatrix}$ |
| 7   | $\begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 \end{bmatrix}$ |
| 8   | $\begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 \end{bmatrix}$ |
We have the following lower bound for the degree of $I_L$.

**Lemma 4.6** ([MNR04 Theorem 1.2(a), Corollary 1.3]). Suppose $I$ is Cohen–Macaulay and codim $I = 2$. Then

$$\deg I \geq \frac{1}{2}m_1m_2 + \frac{1}{2}(M_2 - M_1)(M_2 - m_2 + M_1 - m_1).$$

Furthermore, the minimal free resolution of $I$ is pure if and only if $\deg I = \frac{1}{2}m_1m_2$ if and only if $\deg I = \frac{1}{2}M_1M_2$.

We can use Lemma 4.6 to give a necessary and sufficient condition for when a Cohen–Macaulay lattice ideal $I_L$ of codimension 2 that is not a complete intersection satisfies $\text{reg } I_L = \deg I_L - \text{codim } I_L + 1 = \deg I_L - 1$.

**Proposition 4.7.** Suppose the lattice ideal $I_L$ is of codimension 2 and is Cohen–Macaulay but not a complete intersection, so $I_L \subseteq \langle x_1, \ldots, x_n \rangle^2$. Then $\text{reg } I_L \leq \deg I_L - 1$, with equality if and only if $I_L$ is minimally generated by 3 quadratics. Moreover, if equality holds, then the minimal free resolution of $I_L$ has the form

$$0 \longrightarrow S(-3)^2 \longrightarrow S(-2)^3 \longrightarrow I_L \longrightarrow 0.$$

**Proof.** The assertion that $I_L \subseteq \langle x_1, \ldots, x_n \rangle^2$ follows from Lemma 4.1. This implies $m_1 \geq 2$. Now, by Lemma 4.6

$$2\deg I_L - 2M_2 \geq m_1m_2 + (M_2 - M_1)(M_2 - m_2 + M_1 - m_1) - 2M_2 = m_2(m_1 - 2) + (M_2 - M_1 - 2)(M_2 - m_2) + (M_2 - M_1)(M_1 - m_1).$$

By Lemma 4.5, $M_1 < m_2 \leq M_2$, so $\text{reg } I_L = M_2 - 1$. In addition, the quantity $(M_2 - M_1 - 2)(M_2 - m_2)$ is always nonnegative; this is clear if $M_2 - M_1 \geq 2$, and if $M_2 - M_1 = 1$ then necessarily $m_2 = M_2$, since

$$M_1 < m_2 \leq M_2 = M_1 + 1.$$ 

It follows that $\text{reg } I_L = M_2 - 1 \leq \deg I_L - 1$, with equality holding only if $m_1 = 2 = M_1$, which happens if and only if all minimal generators are of degree 2. By Lemma 4.5, $I_L$ has exactly $b_1 = 3$ minimal generators.

Conversely, if all three minimal generators are of degree 2, then by Lemma 4.5, $d_{21} + d_{22} = d_{11} + d_{12} + d_{13} = 6$ and $d_{21}, d_{22} \geq m_2 > M_1 = 2$, so $d_{21} = d_{22} = 3$. Thus, the minimal free resolution of $I_L$ is pure. Examining the degrees of the twists, we find $\text{reg } I_L = 2$, and by Lemma 4.6 we have $\deg I_L = 3$, so equality holds. The claimed Betti numbers for the minimal free resolution of $I_L$ follow as well. \qed

In fact, using the results in [PS98] (or by mimicking the argument of Corollary 4.4), it can be seen that the requirement given in Proposition 4.7 that $I_L$ be minimally generated by 3 quadratics shows that there can only be finitely many rank 2 lattices $L$ with no zero Gale vectors such that $I_L$ is nondegenerate, Cohen–Macaulay, not a complete intersection, and satisfies $\text{reg } I_L = \deg I_L - 1$. After performing this finite search using [GS], we find the following.

**Proposition 4.8.** Suppose the lattice $L$ has rank 2, and that $I_L$ is Cohen–Macaulay but not a complete intersection, which implies $I_L \subseteq \langle x_1, \ldots, x_n \rangle^2$. Then $\text{reg } I_L \leq \deg I_L - 1$, with
equality if and only if \( L \) has a Gale diagram in which the subset of nonzero vectors equals one of the following:

- \( \{(1, 1), (1, -2), (-2, 1)\} \),
- \( \{(1, 0), (0, 1), (1, -2), (-2, 1)\} \),
- \( \{(1, 1), (0, 1), (-1, 0), (-1, -1), (1, -1)\} \),
- \( \{(1, 0), (0, 1), (1, 1), (-1, 0), (0, -1), (-1, -1)\} \).

Of these four possibilities for \( L \), the first is not saturated, and the remaining possibilities are saturated.

By combining Proposition 4.8 with Corollary 4.4, we see that any nondegenerate Cohen–Macaulay lattice ideal \( I_L \) of codimension 2 satisfies \( \text{reg} I_L = \text{deg} I_L - 1 \) if and only if its subset of nonzero Gale vectors is given by one of the finitely many options from Proposition 4.8 and Table 4.1.

**Remark 4.9.** The setting of [MNR04] also allows us to determine when \( \text{reg} I_L = \text{deg} I_L - \text{codim} I_L + 1 \) for nondegenerate Gorenstein ideals \( I \) of codimension 3: from [MNR04, Theorem 1.4, Corollary 1.5], we see that \( \text{reg} I_L \leq \text{deg} I_L - 2 \) for all nondegenerate Gorenstein ideals \( I \) of codimension 3, with equality holding if and only if the minimal free resolution of \( I \) has the form

\[
0 \longrightarrow S(-5) \longrightarrow S(-3)^5 \longrightarrow S(-2)^5 \longrightarrow I \longrightarrow 0.
\]

In this case, \( \text{reg} I = 3 \) and \( \text{deg} I = 5 \). This can be rephrased to say that equality is achieved if and only if all the minimal generators of \( I \) have degree 2 and \( I \) is not a complete intersection.

### 5. Reduction to curves in \( \mathbb{P}^3 \)

The remainder of the paper is devoted to characterizing the non-Cohen–Macaulay toric ideals of codimension 2 that have maximal regularity. By Proposition 2.2, all such toric ideals have at least 4 nonzero Gale vectors, and Corollary 3.2 finishes the case when \( \mathcal{L} \) has exactly 4 nonzero Gale vectors. Thus, we are able to restrict to the case when \( \mathcal{L} \) has at least 5 nonzero Gale vectors; see Proposition 7.2. We then combine Proposition 7.2 with previous results to complete the proof of Theorem 1.3.

In this section, we recall the tools that were used in [PS98] to prove Theorem 5.1, restricting our attention to the case when \( I_\mathcal{L} \) has codimension 2, equivalently, when \( \mathcal{L} \) has rank 2. The strategy of [PS98] is to reduce to the case of curves in \( \mathbb{P}^3 \).

As usual, we assume that \( \mathcal{L} \subseteq \mathbb{Z}^n \) is orthogonal to the all-1’s vector \((1, 1, \ldots, 1) \in \mathbb{Z}^n \). If \( I_\mathcal{L} \) is not Cohen–Macaulay, then the projective dimension of the \( S \)-module \( S/I_\mathcal{L} \) equals 3. We fix a Gale diagram \( G \) for \( \mathcal{L} \). Much of our work is motivated by the following result.

**Theorem 5.1 ([PS98, Theorem 7.3 and Proposition 7.7]).** Let \( I_\mathcal{L} \subseteq \mathbb{k}[x_1, \ldots, x_n] \) be a lattice ideal of codimension 2 that is not Cohen–Macaulay. Then \( \text{reg} I_\mathcal{L} \leq \text{deg} I_\mathcal{L} \), and this inequality is strict if \( I_\mathcal{L} \) is toric. Furthermore, if equality holds, then any Gale diagram for \( \mathcal{L} \) lies on two lines through the origin in \( \mathbb{R}^2 \).

Given \( C \in \Gamma = \mathbb{Z}^n/\mathcal{L} \) and \( a \in C \), the monomials of degree \( C \) are in bijection with the lattice points in \( P_a := \text{conv}\{u \in \mathbb{Z}^2 | Bu \leq a\} \). Note that \( P_a \) and \( P_{a'} \) are lattice translates
if \( a - a' \in \mathcal{L} \), so by considering polygons up to translation, we define \( P_C := P_a \) for \( a \in C \). We say \( P_C \) is primitive if it contains no lattice points other than its vertices. For each \( C \in \Gamma \), let \( \Delta_C \) be the simplicial complex on \( \{1, \ldots, n\} \) generated by the supports of all monomials in the fiber of \( C \).

If \( S/I_C \) has a minimal \( i \)th syzygy of degree \( C \), then by Theorem 3.4 of [PS98], the simplicial complex \( \Delta_C \) is homologous to the \((i - 1)\)-sphere, and \( P_C \) is primitive. If \( i = 3 \), then \( P_C \) is a primitive parallelogram and is called a syzygy quadrangle. Furthermore, in this case, \( \Delta_C \) has the homology of the 2-sphere. The minimal free resolution of \( S/I_C \) is controlled by the syzygy quadrangles of \( I_C \), in a way that is made precise in [PS98].

Given \( v, w \in \mathbb{Z}^2 \) with \( |\det(v, w)| = 1 \), let \( [v, w] := \text{conv}(\{(0, 0), v, w, v + w\}) \) be a primitive parallelogram. Recall that a vector \( a \in [v, w] \) is supported by a vector \( b \) if \( b \cdot a > b \cdot a' \) for all \( a' \in [v, w] \setminus \{a\} \).

**Proposition 5.2** ([PS98] Corollary 4.2]. Suppose \( I_C \) is not a complete intersection. Then the parallelogram \([v, w] \) is a syzygy quadrangle if and only if each vertex of \([v, w] \) is supported by at least one vector in the Gale diagram \( G \).

In particular, if \( I_C \) is not a complete intersection, then by Proposition 2.2 \( I_C \) is not Cohen–Macaulay if and only if there exists a Gale diagram for which the unit square \([1, 0), (0, 1)\] is a syzygy quadrangle, or in other words, if and only if \( I_C \) has at least one syzygy quadrangle (in an arbitrary Gale diagram).

For convenience, we record the following two results, which are implicit in [PS98].

**Lemma 5.3.** Suppose the polygon \( P \subseteq \mathbb{R}^2 \) equals \( P_{a_0} \) for some \( a_0 \in \mathbb{Z}^n_{\geq 0} \). Then \( P \) contains 0. Moreover, let \( \mathcal{S} \) be the set of all \( C \in \Gamma \) such that \( P_C \) equals \( P \) up to lattice translations, and let \( C_0 \) be the unique member of \( \Gamma \) that contains \( v = (v_1, \ldots, v_n) \in \mathbb{Z}^n \), where \( v_i \) equals the maximum of \( b_i \cdot u \) for \( u \in P \). Then \( C_0 \in \mathcal{S} \), and \( \Delta_C \) is contractible for all \( C \in \mathcal{S} \setminus \{C_0\} \).

**Proof.** Note 0 \( \in P \) since \( a_0 \in \mathbb{Z}^n_{\geq 0} \). Also, \( P_v = P \), so \( P_{C_0} \) equals \( P \) up to lattice translations, and \( C_0 \in \mathcal{S} \). Let \( C \in \mathcal{S} \setminus \{C_0\} \), so there exists \( a = (a_1, \ldots, a_n) \in C \) such that \( P_a = P \). Note that \( v \leq a \). Since \( C \neq C_0 \), we have \( v \neq a \). Thus, \( v_i \leq a_i - 1 \) for some \( 1 \leq i \leq n \), so \( a_i - b_i \cdot u \geq 1 \) for all \( u \in P \). This means all monomials in the fiber of \( C \) are a multiple of \( x_i \). Thus, each of the generators of the simplicial complex \( \Delta_C \) contains \( i \in \{1, \ldots, n\} \), so \( \Delta_C \) is contractible, as desired.

**Corollary 5.4.** Suppose that \( C \in \Gamma \) is the degree of a minimal \( i \)th syzygy of \( S/I_C \), where \( i = 1, 2, 3 \). Thus, \( C \cap \mathbb{Z}^n_{\geq 0} \) is nonempty, and contains some \( a_0 \). Define \( a \in \mathbb{Z}^n \) by letting its \( j \)th coordinate equal \( \max\{b_j \cdot u \mid u \in P_{a_0}\} \) for all \( 1 \leq j \leq n \). Then \( a \in C \).

One particular consequence of this result is that if \( C_1, C_2 \) are each degrees of minimal syzygies, and \( P_{C_1} = P_{C_2} \) (up to translations), then \( C_1 = C_2 \).

**Proof of Corollary 5.4.** By our assumption on \( C \), it follows that \( \Delta_C \) has the homology of the \((i-1)\)-sphere, and is therefore non-contractible. The result then follows from Lemma 5.3.

A crucial insight in the proof of Theorem 5.1 is that one can reduce to the case of a curve in \( \mathbb{P}^3 \). We briefly outline the argument from [PS98] here, and we fix the following notation for the remainder of the paper.
Definition 5.5. A reduction datum is a triple \((\mathcal{L}, G, Q)\) with the following properties:

- \(\mathcal{L} \subset \mathbb{Z}^n\) is a lattice for which \(I_\mathcal{L}\) is not Cohen–Macaulay,
- \(G\) is a Gale diagram for \(\mathcal{L}\) for which the unit square is a syzygy quadrangle, and
- \(Q = (Q_1, Q_2, Q_3, Q_4)\) is a partition of \(G\) (that is, \(G = \bigsqcup_{i=1}^4 Q_i\)) such that every vector in \(Q_i\) lies in the \(i\)th closed quadrant.

Given such a reduction datum, we define the \(k\)-algebra morphism

\[
\phi_Q \colon k[x_1, \ldots, x_n] \to k[y_1, \ldots, y_4]
\]

so that \(\phi_Q(x_i) = y_j\) if and only if \(b_j \in Q_i\). Now define

\[
J_Q := \phi_Q(I_\mathcal{L}), \quad I_Q := (J_Q : (y_1y_2y_3y_4)\infty)
\]

to be ideals of \(k[y_1, \ldots, y_4]\). The ideal \(I_Q\) is a lattice ideal corresponding to a lattice \(\mathcal{L}_Q \subset \mathbb{Z}^4\).

The lattice \(\mathcal{L}_Q\) has a canonical Gale diagram \(G_Q := \{b'_1, \ldots, b'_4\}\), where \(b'_i := \sum_{b \in Q_i} b\).

Remark 5.6. Several properties of the above objects should be noted.

(a) The map \(\phi_Q\) is surjective since \(G\) contains a vector in each open quadrant. This ensures that \(J_Q\) is indeed an ideal.
(b) We have \(\text{codim } J_Q = 2\), so \(J_Q\) defines a (possibly reducible, nonreduced) curve in \(\mathbb{P}^3\).
(c) The ideal \(J_Q\) need not be a lattice ideal. However, it is a lattice ideal after saturation with respect to \(y_1y_2y_3y_4\) (hence the existence of \(\mathcal{L}_Q\) as above).
(d) It is not hard to show that \(\text{deg } J_Q = \text{deg } I_\mathcal{L}\). By general properties of saturation, it follows that \(\text{deg } I_Q \leq \text{deg } J_Q = \text{deg } I_\mathcal{L}\).

The following result allows us to control the regularity of \(I_Q\).

Lemma 5.7. Suppose that \([v, w]\) is a syzygy quadrangle for \(I_\mathcal{L}\), and that

\[
Q_1 \subseteq \{u \in \mathbb{R}^2 \mid u \cdot v \geq 0 \text{ and } u \cdot w \geq 0\}, \quad Q_2 \subseteq \{u \in \mathbb{R}^2 \mid u \cdot v \leq 0 \text{ and } u \cdot w \geq 0\},
\]
\[
Q_3 \subseteq \{u \in \mathbb{R}^2 \mid u \cdot v \leq 0 \text{ and } u \cdot w \leq 0\}, \quad Q_4 \subseteq \{u \in \mathbb{R}^2 \mid u \cdot v \geq 0 \text{ and } u \cdot w \leq 0\}.
\]

Then, \([v, w]\) is a syzygy quadrangle for \(I_Q\). Furthermore, the total degrees of the corresponding degrees in \(\Gamma = \mathbb{Z}^n/\mathcal{L}\) and \(\mathbb{Z}^4/\mathcal{L}_Q\) are equal.

Proof. The fact that \([v, w]\) is a syzygy quadrangle for \(I_Q\) follows directly from Proposition 5.2.

Now, let \(C \in \Gamma\) be such that \(P_C = [v, w]\) and \(I_\mathcal{L}\) has a minimal third syzygy in degree \(C\), and let \(C' \in \mathbb{Z}^4/\mathcal{L}_Q\) be such that \(P_{C'} = [v, w]\) and \(I_Q\) has a minimal third syzygy in degree \(C'\). By Corollary 5.4,

\[
\deg C = \sum_{j=1}^n \max\{b_j \cdot u \mid u \in [v, w]\}, \quad \deg C' = \sum_{i=1}^4 \max\{b'_i \cdot u \mid u \in [v, w]\}.
\]

It suffices to show that

\[
\max\{b'_i \cdot u \mid u \in [v, w]\} = \sum_{j \in Q_i} \max\{b_j \cdot u \mid u \in [v, w]\}
\]

for all \(1 \leq i \leq 4\), but this follows from the assumptions on the \(Q_i\). \(\square\)
Fix a reduction datum \((L, G, Q)\) for which the unit square is a syzygy quadrangle of \(I_L\) attaining the regularity. It follows from Lemma 5.7 that the unit square is a syzygy quadrangle for \(I_Q\), and the monomials corresponding to its vertices retain the same total degree. Thus, \(\text{reg } I_Q \geq \text{reg } I_L\).

Similarly, it follows from Corollary 5.4 that if \([v, w]\) is a syzygy quadrangle of both \(I_L\) and \(I_Q\), then its total degree with respect to \(I_Q\) is at most its total degree with respect to \(I_L\). Thus, the regularity can only strictly increase if the \(L \rightarrow L_Q\) reduction process introduces a new syzygy quadrangle.

In any case, since \(\deg I_Q \leq \deg I_L\) and \(\text{reg } I_L \leq \text{reg } I_Q\), if Theorem 5.1 holds for \(n = 4\), then all of the following three inequalities hold:

\[
\begin{align*}
\text{reg } I_L &\leq \text{reg } I_Q, \\
\text{reg } I_Q &\leq \deg I_Q, \\
\deg I_Q &\leq \deg I_L.
\end{align*}
\]

Therefore \(\text{reg } I_L \leq \deg I_L\) and the result holds for general \(n\). This is the strategy adopted by [PS98] to prove Theorem 5.1.

By inspecting this reduction process and the inequality chain (5.1) in more detail, we are able to classify the possible Gale diagrams of lattice ideals \(I_L\) satisfying \(\text{reg } I_L = \deg I_L - 1\). Observe that if \(\text{reg } I_L = \deg I_L - 1\), then inequality must occur in exactly one of (5.1a), (5.1b), or (5.1c). In the next section, we will analyze the various possibilities for where equality occurs in (5.1).

6. Degree and regularity after reduction to a curve

In this section we carry out a careful analysis of the inequality chain Equation (5.1). This gives the necessary tools to determine when \(\text{reg } I_L = \deg I_L - 1\) for a toric ideal of codimension 2.

6.1. Maximal regularity implies equality in (5.1a). In this subsection, we use Proposition 5.2 to compare the syzygy quadrangles of \(I_Q\) to those of \(I_L\) in the case when \(\deg I_L = \deg I_Q\), which then lets us establish that equality holds in (5.1a) if \(I_L\) is a toric ideal with maximal regularity.

Fix an arbitrary reduction datum \((L, G, Q)\).

**Lemma 6.1.** Let \(J_Q\) be as in Definition 5.5. The following two statements hold.

1. Any associated prime \(p\) of \(J_Q\) of codimension 2 that contains \(y_1y_2y_3y_4\) is of the form \(p = \langle y_i, y_j \rangle\) for some distinct \(i, j \in \{1, 2, 3, 4\}\).
2. Let \(i, j \in \{1, 2, 3, 4\}\) be distinct. Then \(\langle y_i, y_j \rangle\) is an associated prime of \(J_Q\) if and only if for all nonzero \(u \in \mathbb{Z}^2\), there is some \(b \in Q_i \cup Q_j\) such that \(b \cdot u < 0\). In particular, this can hold only if \(\{i, j\} = \{1, 3\}\) or \(\{2, 4\}\).

**Proof.** Statement (1) follows from Corollary 2.1 of [HS00] (though \(J_Q\) is not necessarily a lattice basis ideal, we only need that the saturation of \(J_Q\) with respect to \(y_1y_2y_3y_4\) is a lattice ideal to apply this result).
Statement (2) follows since \( \langle y_i, y_j \rangle \) is an associated prime of \( J_Q \) if and only if \( \langle y_i, y_j \rangle \supseteq J_Q \), since \( \text{codim} \ J_Q = 2 \). Translating this into a condition based on the Gale diagram gives the result.

Lemma 6.2. We have that \( \text{deg} \, I_L = \text{deg} \, I_Q \) if and only if there are two nonzero vectors \( u_{13}, u_{24} \) for which

\[
Q_1 \cup Q_3 \subseteq \{ v \in \mathbb{R}^2 \mid v \cdot u_{13} \geq 0 \}, \quad Q_2 \cup Q_4 \subseteq \{ v \in \mathbb{R}^2 \mid v \cdot u_{24} \geq 0 \}.
\]

Proof. Recall that \( I_Q \) is a lattice ideal that equals the saturation \( (J_Q : (y_1y_2y_3y_4)^\infty) \). It follows that \( \text{deg} \, I_Q = \text{deg} \, J_Q \) if and only if no codimension-2 associated prime of \( I_Q \) contains \( \langle y_1y_2y_3y_4 \rangle \), so by Lemma 6.1 we see that \( \text{deg} \, I_Q = \text{deg} \, J_Q \) if and only if \( J_Q \) does not have an associated prime of the form \( \langle y_i, y_j \rangle \) for distinct \( i, j \in \{1, 2, 3, 4\} \). The desired result then follows from Lemma 6.1 and the fact that \( \text{deg} \, J_Q = \text{deg} \, I_L \).

Lemma 6.3. If \( \text{deg} \, I_L = \text{deg} \, I_Q \), then every syzygy quadrangle of \( I_Q \) (with respect to \( G_Q \)) is a syzygy quadrangle of \( I_L \) (with respect to \( G \)).

Proof. Suppose that \( I_Q \) has a syzygy quadrangle that is not a syzygy quadrangle for \( I_L \). Without loss of generality, after translating we may assume that this syzygy quadrangle equals \( [v, w] \) for some vectors \( v, w \) with nonnegative \( y \)-coordinates. Since \( |\text{det}(v, w)| = 1 \), after translating again if necessary, we may assume that \( v, w \) both lie in the first closed quadrant or both lie in the second closed quadrant. Suppose without loss of generality that \( \text{det}(v, w) = 1 \).

Assume that \( v, w \) both lie in the first closed quadrant. The case where \( v, w \) both lie in the second closed quadrant is similar. By Proposition 5.2, each vertex of the syzygy quadrangle is supported by some vector in \( G_Q \). Thus there exist vectors \( c'_1, c'_2, c'_3, c'_4 \in G_Q \) such that \( c'_1 \cdot v, -c'_2 \cdot v, -c'_3 \cdot v, c'_4 \cdot v > 0 \) and \( c'_1 \cdot w, c'_2 \cdot w, -c'_3 \cdot w, -c'_4 \cdot w > 0 \). Since \( G_Q \) contains exactly four vectors, in each open quadrant, we necessarily have that \( c'_i \) is the unique vector of \( G_Q \) that lies in the \( i \)-th open quadrant, that is, \( c'_i = b'_i \). Since \( [v, w] \) is not a syzygy quadrangle for \( I_L \), the Gale diagram \( G \) does not contain an analogous collection of four vectors \( c_1, c_2, c_3, c_4 \) satisfying the same inequalities as above. Since \( G \) contains a vector in each open quadrant, we see that there exist \( c_1, c_3 \) satisfying the conditions. Thus, there either does not exist \( c_2 \in G \) such that \( -c_2 \cdot v, c_2 \cdot w > 0 \), or there does not exist \( c_4 \in G \) such that \( c_4 \cdot v, -c_4 \cdot w > 0 \). Assume the first case holds; the second case is similar. This implies that each \( b \in Q_2 \) either satisfies \( b \cdot v \geq 0 \) or \( b \cdot w \leq 0 \). Since \( \sum_{b \in Q_2} b = b'_2 \), we see that there must exist some nonzero \( c_2 \in Q_2 \) such that \( c \cdot v \geq 0 \), and there exists some nonzero \( d \in Q_2 \) such that \( d \cdot w \leq 0 \). But these conditions prevent the nonzero vectors \( c, d, b'_4 \) from all lying in a single closed half-plane with boundary passing through the origin, contradicting Lemma 6.2, which asserts the existence of some nonzero \( u \in \mathbb{Z}^2 \) such that \( 0 \leq u \cdot c, u \cdot d \) and \( 0 \leq u \cdot \sum_{b \in Q_2} b = u \cdot b'_4 \).

We can now prove our first key result.

Corollary 6.4. If \( \text{reg} \, I_L \geq \text{deg} \, I_L - 1 \), then \( \text{reg} \, I_Q = \text{reg} \, I_L \).
Proof. If \( \text{reg } I_Q > \text{reg } I_L \), then as stated in Section 5 there is a syzygy quadrangle of \( I_Q \) that is not a syzygy quadrangle of \( I_L \). Then Lemma 6.3 implies that \( \text{deg } I_Q < \text{deg } I_L \). But then \( \text{reg } I_L \leq \text{deg } I_L - 2 \), a contradiction. \( \square \)

6.2. Existence of a reduction achieving equality in \( \text{reg } I_L \). The following lemma allows us to prove our next result, Proposition 6.6, which says that if \( \text{reg } I_L = \text{deg } I_L - 1 \) and certain nonrestrictive technical conditions are satisfied, then there exists some partition \( Q \) for which \( \text{reg } I_Q = \text{deg } I_Q \).

Lemma 6.5. Suppose that \( L \subseteq \mathbb{Z}^4 \) with \( \text{reg } I_L = \text{deg } I_L - 1 \), and let \( G \) be a Gale diagram for \( L \) such that the unit square is a syzygy quadrangle attaining the regularity. Then \( G \) either lies on two lines or has the following form, up to dihedral symmetries:

\[
G = \{(1, a), (-1, d - 1), (-1, 1 - a), (1, -d)\}
\]

for some \( a, d > 1 \).

Proof. Let \( C \in \mathbb{Z}^4/L \) be the multidegree for which \( P_C \) is the unit square. Then \( \text{reg } I_L = \text{deg } C - 2 \) since syzygy quadrangles correspond to third syzygies [PS98, Theorem 3.4]. We start in the same spirit as [PS98], Proposition 7.7. Suppose the Gale vectors are given by

\[
\begin{align*}
a &= (a_1, a_2), & b &= (-b_1, b_2), & c &= (-c_1, -c_2), & d &= (d_1, -d_2),
\end{align*}
\]

where \( a_i, b_i, c_i, d_i > 0 \) for all \( i = 1, 2 \). By rotating \( 180^\circ \) if necessary, we can without loss of generality assume that \( b_2 \leq d_2 \).

We now have the following chain of comparisons:

\[
\begin{align*}
\text{deg } I_L - 1 &= \text{reg } I_L = \text{deg } C - 2 \\
&= a_1 + b_2 + a_2 + d_1 - 2 \\
&\leq (a_1 + d_2 - 1) + (a_2 + d_1 - 1) \\
&\leq a_1 d_2 + a_2 d_1 \\
&= |\det(d, a)| \\
&\leq \text{deg } I_L.
\end{align*}
\]

There are three inequalities, and hence three possibilities for where the jump by 1 can occur. We treat these case by case.

**Case 1**: Inequality occurs at (1). In this case we have the following:

\[
d_2 = b_2 + 1, \quad (a_1 - 1)(d_2 - 1) = (a_2 - 1)(d_1 - 1) = 0, \quad |\det(d, a)| = \text{deg } I_L.
\]

Since \( d_2 = b_2 + 1 \), we also have \( c_2 = a_2 - 1 \). If \( a_2 = 1 \) then \( c_2 = 0 \), so we conclude that \( d_1 = 1 \) and \( a_2 > 1 \). Similarly if \( d_2 = 1 \) then \( b_2 = 0 \) so we conclude that \( a_1 = 1 \) and \( b_2 > 1 \). Since \( b_1 + c_1 = a_1 + d_1 = 2 \), we conclude that \( b_1 = c_1 = 1 \). Thus the Gale vectors have the form

\[
a = (1, a_2), \quad b = (-1, d_2 - 1), \quad c = (-1, 1 - a_2), \quad d = (1, -d_2).
\]

**Case 2**: Inequality occurs at (2). In this case we have the following:

\[
d_2 = b_2, \quad (a_1 - 1)(d_2 - 1) + (a_2 - 1)(d_1 - 1) = 1, \quad |\det(d, a)| = \text{deg } I_L.
\]

Now we have \( c_2 = a_2 \).
• **Case 2.1**: $a_1 = 1$ and $a_2 = d_1 = 2$. Then the Gale vectors have the form
\[ a = (1, 2), \quad b = (-b_1, d_2), \quad c = (-c_1, -2), \quad d = (2, -d_2). \]
We then also have
\[ 4 + c_1 d_2 = |\det(c, d)| \leq \deg I_L = |\det(a, d)| = 4 + d_2, \]
forcing $c_1 = 1$. We conclude that $b_1 = 2$, so $b + d = a + c = 0$.

• **Case 2.2**: $d_2 = 1$ and $a_2 = d_1 = 2$. Then the Gale vectors have the form
\[ a = (a_1, 2), \quad b = (-b_1, 1), \quad c = (-c_1, -2), \quad d = (2, -1). \]
Then we again have
\[ 4 + c_1 = |\det(c, d)| \leq \deg I_L = |\det(a, d)| = 4 + a_1, \]
so $c_1 \leq a_1$. By the same inequality with $|\det(a, b)|$, we obtain $b_1 \leq 2$. Then
\[ 2 + a_1 = b_1 + c_1 \leq 2 + c_1, \]
so $a_1 \leq c_1$. We conclude that $a_1 = c_1$. So $b_1 = 2$ and again $b + d = a + c$.

• **Case 2.3**: $a_1 = d_2 = 2$ and $a_2 = 1$. Proceed as in Case 2.1 to conclude that $G$ lies on two lines.

• **Case 2.4**: $a_1 = d_2 = 2$ and $d_1 = 1$. Proceed as in Case 2.2 to conclude that $G$ lies on two lines.

**Case 3**: Inequality occurs at (3). In this case we have the following:
\[ d_2 = b_2, \quad (a_1 - 1)(d_2 - 1) = (a_2 - 1)(d_1 - 1) = 0, \quad |\det(d, a)| = \deg I_L - 1. \]
We have $a_2 = c_2$ once again.

• **Case 3.1**: $a_1 = a_2 = 1$. Then the Gale vectors have the form
\[ a = (1, 1), \quad b = (-b_1, d_2), \quad c = (-c_1, -1), \quad d = (d_1, -d_2). \]
We now have
\[ c_1 d_1 + d_2 = |\det(c, d)| \leq \deg I_L = |\det(d, a)| + 1 = d_1 + d_2 + 1. \]
Then either $c_1 = 1$ or $(c_1, d_1) = (2, 1)$. In the former case, $d_1 = b_1$ and the Gale diagram lies on two lines. In the latter case, $b_1 = 2$ the Gale diagram must be
\[ a = (1, 1), \quad b = (-2, d_2), \quad c = (-2, -1), \quad d = (1, -d_2). \]

But now we see that
\[ 2d_2 + 2 = |\det(b, c)| \leq \deg I_L = |\det(d, a)| + 1 = d_2 + 2, \]
which is absurd since $d_2 > 0$, so $(c_1, d_1) = (2, 1)$ does not occur.

• **Case 3.2**: $a_1 = d_1 = 1$. Since $b_1 + c_1 = a_1 + d_1 = 2$, we conclude that $b_1 = c_1 = 1$.

Then the Gale vectors have the form
\[ a = (1, a_2), \quad b = (-1, d_2), \quad c = (-1, -a_2), \quad d = (1, -d_2), \]
and so lie on two lines.

• **Case 3.3**: $a_2 = d_2 = 1$. Proceed as in Case 3.2 to conclude that $G$ lies on two lines.

• **Case 3.4**: $d_1 = d_2 = 1$. Proceed as in Case 3.1 to conclude that $G$ lies on two lines.

□
We now prove our second key result.

**Proposition 6.6.** Suppose \( n \geq 5 \), and fix a reduction datum \((L, G, Q)\) with the following properties:

1. \( \text{reg } I_L = \deg I_L - 1 \),
2. \( \text{reg } I_Q = \deg I_Q - 1 \),
3. the unit square is a syzygy quadrangle attaining the regularity of \( I_L \),
4. \( G \) consists only of nonzero vectors, and
5. \( G \) is not contained in two lines.

Then, there is a different choice of partition \( R \) of \( G \) for which \( \text{reg } I_R = \deg I_R \).

**Proof.** Recall that the unit square is a syzygy quadrangle of \( I_Q \), with the same total degree as that of \( I_L \). Since \( \text{reg } I_Q = \text{reg } I_L \) by Corollary 6.3, it follows that the unit square still attains the regularity. We therefore know that \( G_Q = \{b_1', b_2', b_3', b_4'\} \) is of one of the forms described in Lemma 6.5. Suppose that \( G_Q \) lies on two lines \( \ell_1 \supseteq \{b_1', b_3'\} \) and \( \ell_2 \supseteq \{b_2', b_4'\} \) passing through the origin. Since \( \text{reg } I_Q = \text{reg } I_L \), we see that \( \deg I_L = \deg I_Q \), so Lemma 6.2 implies that there is a closed half-plane \( H_{13} \) containing \( Q_1 \cup Q_3 \). Since \( H_{13} \) is closed under addition, this implies that \( b_1', b_3' \in H_{13} \), so we conclude that \( \partial H_{13} = \ell_1 \). But then every vector in \( Q_1 \) must lie on \( \ell_1 \), otherwise \( b_1' \in H_{13} \setminus \ell_1 \). Likewise, \( Q_3 \subseteq \ell_1 \). Analogously, \( Q_2, Q_4 \subseteq \ell_2 \). This contradicts that \( G \) is not contained in two lines. So henceforth suppose \( G_Q \) is not contained in two lines.

Let \( \ell^+ \) and \( \ell^- \) denote the positive and negative \( y \)-axis, respectively. If \( G \cap \ell^+ \) and \( G \cap \ell^- \) are both nonempty, then Lemma 6.2 implies that there is a partition \( R \) with \( \deg I_R < \deg I_L \) (choose \( R \) so that \( R_1 \) contains vectors from \( G \cap \ell^+ \) and \( R_3 \) contains vectors from \( G \cap \ell^- \)). This would imply that \( \text{reg } I_R = \deg I_R \). So henceforth, without loss of generality, assume that \( G \cap \ell^- = \emptyset \).

Lemma 6.3 implies that up to reflections over the axes, we have

\[
G_Q = \{(1, a), (-1, d - 1), (-1, 1 - a), (1, -d)\}
\]

for some \( a, d > 1 \). Combining this with \( G \cap \ell^- = \emptyset \), we have

\[
G = \{(1, a'), (-1, b'), (-1, 1-a), (1, -d)\} \cup Q_1^{x=0} \cup Q_2^{x=0},
\]

where \( Q_1^{x=0} \) is the intersection of \( Q_1 \) with the \( y \)-axis, and \( 0 < a' < a \) and \( 0 < b' \leq d - 1 \). Suppose that \( Q_2^{x=0} \) is nonempty, that is, there is some \((0, \varepsilon) \in Q_2 \). Since \( \deg I_L = \deg I_Q \), we can choose a half-plane \( H_{24} = \{v \in \mathbb{R}^2 \mid v \cdot u \geq 0\} \) containing \( Q_2 \cup Q_4 \). If \( u = (u_1, u_2) \), then

\[
-u_1 + u_2 b' \geq 0, \quad u_1 - u_2 d \geq 0, \quad u_2 \geq 0,
\]

which quickly implies \( u_1 = u_2 = 0 \), a contradiction. So \( Q_2^{x=0} = \emptyset \) and \( b' = d - 1 \). If \( a' < a - 1 \), then the same argument shows that \( Q_1^{x=0} = \emptyset \). But this is impossible, since \( n \geq 5 \). So \( a' \geq a - 1 \). Since \( Q_1^{x=0} \) contains some nonzero vector, we also see that \( a' \leq a - 1 \). So \( a' = a - 1 \) and \( Q_1^{x=0} = \{(0, 1)\} \) (and also, evidently, \( n = 5 \)). To finish the argument, consider the partition \( R = (R_i)_{i=1}^4 \) obtained by taking \( Q \) and moving \((0, 1)\) from \( Q_1 \) to \( Q_2 \). The resulting lattice \( L_R \subseteq \mathbb{Z}^4 \) has Gale diagram

\[
G_R = \{(1, a - 1), (-1, d), (-1, 1 - a), (1, -d)\},
\]

which lies on two lines. Now, as in the first paragraph, \( \deg I_R \neq \deg I_L \). \( \square \)
6.3. When equality holds in (5.1b). In this subsection, we establish which lattices \( \mathcal{L}_0 \) with \( n = 4 \) satisfy \( \reg I_{\mathcal{L}_0} = \deg I_{\mathcal{L}_0} \). One direction is provided by [PS98] Remark 7.9, which states that if \( \reg I_{\mathcal{L}_0} = \deg I_{\mathcal{L}_0} \) and \( G_0 \) is a Gale diagram for \( \mathcal{L}_0 \) chosen so that the unit square is a syzygy quadrangle attaining the regularity, then up to dihedral symmetries, \( G_0 \) equals either
\[
\{(1,1), (a,-b), (-1,-1), (-a,b)\}, \quad \text{or} \quad \{(1,a), (1,-b), (-1,-a), (-1,b)\} \tag{6.1}
\]
for \( a, b \geq 1 \). In particular, if \( \reg I_{\mathcal{L}_0} = \deg I_{\mathcal{L}_0} \), then there exists some Gale diagram of one of those two forms. We prove the converse to this statement (also implicit in [PS98]), as stated in Lemma 6.7.

**Lemma 6.7.** Let \( \mathcal{L}_0 \subseteq \mathbb{Z}^4 \) be a lattice such that there exists a \( G_0 \) as in (6.1). Then \( \deg I_{\mathcal{L}_0} = a+b \), \( \reg I_{\mathcal{L}_0} = \deg I_{\mathcal{L}_0} \), and the unit square is a syzygy quadrangle attaining the regularity.

**Proof.** Let \( \mathcal{L}_0 \) be one of the listed lattices. The fact that \( \deg I_{\mathcal{L}_0} = a+b \) follows immediately from the formula in [OPVV14, Theorem 4.6]. By Proposition 5.2, the unit square is a syzygy quadrangle of \( I_{\mathcal{L}_0} \). Let \( C \) be the multidegree for which the unit square is \( P_C \). Then \( \deg C = a+b+2 \) in both cases of (6.1), so \( \reg I_{\mathcal{L}_0} \geq a+b \). On the other hand, \( \deg I_{\mathcal{L}_0} = a+b \), so \( \reg I_{\mathcal{L}_0} \geq \deg I_{\mathcal{L}_0} \). By Theorem 5.1, we are done. \( \square \)

This result also has implications for lattice ideals of maximal regularity in arbitrary dimension.

**Definition 6.8.** A reduction datum \((\mathcal{L}, G, Q)\) is called *perfectly balanced* if
\[
\sum_{b \in Q_1 \cup Q_3} b = \sum_{b \in Q_2 \cup Q_4} b = 0.
\]

Then Lemma 6.7 has the following corollary.

**Corollary 6.9.** Suppose \( I_{\mathcal{L}} \) is not Cohen–Macaulay and \( \reg I_{\mathcal{L}} = \deg I_{\mathcal{L}} - 1 \). Then there exists a perfectly balanced reduction datum \((\mathcal{L}, G, Q)\) for which the unit square attains the regularity.

6.4. Analyzing inequality (5.1c). In this subsection we prove Proposition 6.13, which characterizes when \( \deg I_Q = \deg I_{\mathcal{L}} - 1 \) and will help prove Theorem 1.3.

Fix a reduction datum \((\mathcal{L}, G, Q)\). Throughout this section, \( i \) and \( j \) are indices such that \( \{i,j\} \) is \{1,3\} or \{2,4\}.

**Definition 6.10.** Define \( A_i \) (resp. \( B_i \)) to be the set of \( u \in \mathbb{Z}^2 \) for which there exist \( v_1, v_{-1} \in Q_i \) (resp. \( v_1 \in Q_i, v_{-1} \in Q_j \)) such that \( v_1 \cdot u = -v_{-1} \cdot u = 1 \), and \( b \cdot u = 0 \) for all \( b \in (Q_i \cup Q_j) \setminus \{v_1, v_{-1}\} \). Also define \( C_i := A_i \cup B_i \).

**Definition 6.11.** Suppose that \( \sum_{b \in Q_1 \cup Q_j} b = 0 \). We say the reduction datum \((\mathcal{L}, G, Q)\) is \( i \)-simple if there are vectors \( v, w \in \mathbb{Z}^2 \) such that \( \det(v, w) = 1 \), the set of nonzero vectors in \( Q_i \) is \( \{v, w\} \), and the set of nonzero vectors in \( Q_j \) is either \( \{-v, -w\} \) or \( \{-v - w\} \). We say \((\mathcal{L}, G, Q)\) is \( \{i,j\} \)-simple if it is \( i \)-simple or \( j \)-simple. If the reduction datum is understood, we also refer to \( G \) as being \( i \)-simple or \( \{i,j\} \)-simple.

**Proposition 6.12.** Suppose \( \sum_{b \in Q_1 \cup Q_j} b = 0 \) and \( p = \langle y_i, y_j \rangle \) is an associated prime of \( J_Q \) (see Definition 5.3). Then the following are equivalent:
\begin{enumerate}
\item $p$ is the $p$-primary part of $J_Q$.
\item there are distinct $f, g \in \Phi$ and distinct $f', g' \in \Phi$ for which $y_i f - y_i g, y_j f' - y_j g' \in J_Q$.
\item there are $u \in C_i$ and $u' \in C_j$ with $u \neq \pm u'$.
\item $B_i$ contains at least two distinct vectors.
\item $(L, G, Q)$ is $\{i, j\}$-simple.
\end{enumerate}

**Proof.** Without loss of generality, suppose $i = 2, j = 4$. Let $\Phi = \{y_a^r y_b^s \mid a, b \geq 0\}$ be a multiplicatively closed subset of $k[y_1, \ldots, y_4]$, and let $\mathcal{M}$ denote the full set of monomials in $k[y_1, \ldots, y_4]$.

\((1) \iff (2):\) It is clear by [DMM10] Lemma 2.9, Theorem 2.14 that (2) $\implies$ (1), since in this case, $y_2 g / f \sim_{(J_Q)\Phi} y_2$ and similarly for $y_4$, so $y_2$ and $y_4$ both appear in the $p$-primary part of $J_Q$. Also by [DMM10] Lemma 2.9, Theorem 2.14, (1) implies that there is some $w = (w_1, w_2, w_3, w_4) \neq (0, 1, 0, 0)$ such that $y_2 - y_4 w \in (J_Q)\Phi$, $w_2 \geq 1, w_4 \geq 0$. The condition that $\sum_{b \in Q \cup Q_4} b = 0$ implies that $(w_2 - 1) + w_4 = 0$, so $w_2 = 1$ and $w_4 = 0$; in particular $y^w = y_2 g / f$ for some $g, f \in \Phi$. For some $p \in \Phi$, then, we have $y_2 p f - y_2 p g \in J_Q$. The same argument applies to $y_4$.

\((2) \implies (3):\) Since $y_2 f - y_2 g \in J_Q$, it follows that $C_2$ is nonempty. Similarly $C_4$ is nonempty. There is some minimal $m > 0$, monomials $\beta_1, \ldots, \beta_m \in \mathcal{M}$, and vectors $u_1, \ldots, u_m \in \mathbb{Z}^2$ for which

$$y_2 f - y_2 g = \sum_{\ell=1}^m \beta_\ell \cdot \phi_Q(x^{(Bu_\ell)^+} - x^{(Bu_\ell)^-}).$$

Suppose towards a contradiction that $|C_2 \cup (-C_4)| = 1$, so that there is some vector $u \in \mathbb{Z}^2$ for which $C_2 = \{u\}$, $C_4 = \{-u\}$. By the minimality of $m$, every $u_\ell$ must be in $C_2$ or $C_4$, so $u_\ell = \pm u$ for each $\ell$. Then $y_2 f - y_2 g = P \cdot \phi_Q(x^{(Bu)^+} - x^{(Bu)^-})$ for some polynomial $P$ (namely $\sum \beta_\ell$), which is absurd since $-u \in C_4$ but $y_2 f - y_2 g$ has no monomial terms divisible by $y_4$.

It follows that $|C_2 \cup (-C_4)| \geq 2$, so there exist $u \in C_2$ and $u' \in C_4$ with $u \neq \pm u'$.

\((3) \implies (4):\) If $u \in B_2$ and $u' \in B_4$, we are done. Up to symmetries, there are two remaining cases:

- **$u \in A_2, u' \in A_4$:** Let $v_1, v_{-1} \in Q_2$ with $v_1 \cdot u = -v_{-1} \cdot u = 1$. The condition $u' \in A_4$ implies that $v_1, v_{-1}$ are both orthogonal to $u'$, and so $v_1, v_{-1}$ are collinear, but this is impossible since both are in $Q_2$ and lie on opposite sides of $u$. So this case does not occur.

- **$u \in A_2, u' \in B_4$:** Then there are vectors $v_1, v_{-1} \in Q_2$ with $v_1 \cdot u = -v_{-1} \cdot u = 1$ with $u$ orthogonal to all other vectors of $Q_2 \cup Q_4$. Similarly there are vectors $w_{-1} \in Q_2, w_1 \in Q_4$ for which $w_1 \cdot u' = -w_{-1} \cdot u' = 1$ and $u'$ is orthogonal to all other vectors of $Q_2 \cup Q_4$. Thus $\{v_{\pm 1}, w_{\pm 1}\}$ are the only nonzero vectors in $Q_2 \cup Q_4$, since all other vectors are orthogonal to both $u$ and $u'$.

If $w_{-1}$ is distinct from both $v_1$ and $v_{-1}$, then $v_1$ and $v_{-1}$ are both orthogonal to $u'$ and so are collinear, but this is impossible since $v_1$ and $v_{-1}$ lie on opposite sides
of \( u \). So \( w_{-1} = v_1 \) or \( w_{-1} = v_1 \). If \( w_{-1} = v_1 \), it is straightforward to verify that
\[ -u - u', -u' \in B_2. \]
If \( w_{-1} = v_{-1} \), one similarly verifies that \( u - u', -u' \in B_2. \)

So in all cases, there are two distinct vectors in \( B_2 \).

(4) \( \implies \) (2): Since there are at least two distinct vectors in \( B_2 \), there are monomials \( p, q, r, s \in \Phi \) for which \( y_2p - y_4q, y_2r - y_4s \in J_Q \) and \( p/r \neq q/s \). Writing
\[
\begin{align*}
y_2ps - y_2qr &= s(y_2p - y_4q) - q(y_2r - y_4s), \\
y_4ps - y_4qr &= r(y_2p - y_4q) - p(y_2r - y_4s)
\end{align*}
\]
gives the desired elements of \( J_Q \).

(4) \( \implies \) (5): Let \( u, u' \in B_2 \) be distinct. By definition of \( B_2 \), there exist \( v_1, w_1 \in Q_2 \) and \( v_{-1}, w_{-1} \in Q_4 \) such that \( v_{\varepsilon} \cdot u = w_{\varepsilon} \cdot u' = \varepsilon \) for \( \varepsilon = \pm 1 \), and \( b \cdot u = 0 \) for all \( b \in Q_2 \cup Q_4 \) \( \setminus \{v_{\pm 1}\} \), and \( b \cdot u' = 0 \) for all \( b \in Q_2 \cup Q_4 \) \( \setminus \{w_{\pm 1}\} \). Note that \( v_{\pm 1}, w_{\pm 1} \) are the only nonzero vectors in \( Q_2 \cup Q_4 \), since all other vectors equal \( [0, 0] = 0 \).

Note that \( u, u' \) are \( Q \)-linearly independent, so we can uniquely represent vectors \( v \in \mathbb{Z}^2 \) as ordered pairs \( [v \cdot u, v \cdot u'] \). If \( v \neq w_1 \), then \( v_1 = [1, 0] \) and \( w_1 = [0, 1] \), and if \( v_1 = w_1 \), then \( v_1 = w_1 = [1, 1] \). Similarly, if \( v_{-1} \neq w_{-1} \), then \( v_{-1} = [-1, 0] \) and \( w_{-1} = [0, -1] \), and if \( v_{-1} = w_{-1} \), then \( v_{-1} = w_{-1} = [-1, -1] \). It is straightforward to check that in three of these cases, the Gale diagram \( G \) is \{2, 4\}-simple. The remaining case, where \( v_1 = w_1 \) and \( v_{-1} = w_{-1} \), cannot occur if \( p \) is an associated prime of \( J_Q \), by Lemma 6.11.

(5) \( \implies \) (4): Suppose without loss of generality that \( Q_2 \) contains exactly two nonzero vectors \( v = (v_1, v_2) \) and \( w = (w_1, w_2) \), and suppose without loss of generality that \( \det(v, w) = 1 \), so \( v_1 w_2 - v_2 w_1 = 1 \). Let \( u_1 = (w_2, -v_1) \) and \( u_2 = (-v_2, w_1) \) be nonzero vectors. Note that \( v \cdot u_1 = 1 \), \( w \cdot u_1 = 0 \), \( v \cdot u_2 = 0 \), and \( w \cdot u_2 = 1 \). Furthermore, either \( -v - w \) is the only nonzero vector in \( Q_4 \), or \( -v, -w \) are the only nonzero vectors in \( Q_4 \). In either case, we see that \( u_1, u_2 \in B_2 \).

We can now prove our main result comparing \( \deg I_Q \) to \( \deg I_L \).

**Proposition 6.13.** Suppose that \( (L, G, Q) \) is perfectly balanced. Then \( \deg I_Q = \deg I_L - 1 \) if and only if for either \( \{i, j\} = \{1, 3\} \) or \( \{i, j\} = \{2, 4\} \), the vectors in \( G \setminus (Q_i \cup Q_j) \) all lie on a single line passing through the origin, and \( (L, G, Q) \) is \( \{i, j\} \)-simple.

**Proof.** Recall that \( \deg I_L = \deg J_Q \). We see that \( \deg I_Q = \deg J_Q - 1 \) if and only if there is exactly one primary component \( q \) of \( J_Q \) for which \( \text{codim } q = 2 \) and \( y_1 y_2 y_3 y_4 \in p := \sqrt{q} \), and furthermore \( \deg q = 1 \). By Lemma 6.11, this is equivalent to having \( p = (y_i, y_j) \) and \( G \setminus (Q_i \cup Q_j) \) contained in a closed half-space passing through the origin for some \( \{i, j\} \). Without loss of generality take \( \{i, j\} = \{2, 4\} \). The condition \( \deg q = 1 \) is equivalent to \( q = p \), which by Proposition 6.12 is equivalent to \( G \) being \{2, 4\}-simple. The condition that \( Q_1 \cup Q_3 \) is contained in a closed half-space is equivalent to \( Q_1 \cup Q_3 \) being contained in a line, since \( \sum_{b \in Q_1 \cup Q_3} b = 0 \). \( \square \)
6.5. When equality holds in \((5.1a)\). In this subsection, we use Corollary \([5.4]\) to describe the syzygy quadrangles (and their corresponding total degrees) of \(I_Q\) in terms of those of \(I_L\), which allows us to determine when equality holds in \((5.1a)\).

Fix a reduction datum \((L, G, Q)\).

**Proposition 6.14.** Suppose that \((L, G, Q)\) is perfectly balanced, and that for either \(\{i, j\} = \{1, 3\}\) or \(\{i, j\} = \{2, 4\}\), the following two properties hold:

- the vectors in \(G \setminus (Q_1 \cup Q_3)\) all lie on a single line passing through the origin, and
- \((L, G, Q)\) is \(\{i, j\}\)-simple in the sense of Definition \(6.11\).

Then the following are true:

1. Each syzygy quadrangle of \(I_L\) is a syzygy quadrangle of \(I_Q\), with the same total degree.
2. There exists exactly one syzygy quadrangle \(P_G\) of \(I_Q\) that is not a syzygy quadrangle of \(I_L\).
3. The total degree of \(P_G\) is at least the total degree of the unit square. Equality holds if and only if up to dihedral symmetries, \(G_Q\) is of one of the forms in \((6.1)\).
4. If up to dihedral symmetries, \(G_Q\) is of one of the forms in \((6.1)\), then the unit square is a syzygy quadrangle of \(I_L\) that attains the regularity of \(I_L\).

**Proof of Proposition \(6.14(1)\).** Let \([v, w]\) be an arbitrary syzygy quadrangle of \(I_L\), where as in the proof of Lemma \(6.3\) we may assume that \(v, w\) either both lie in the first closed quadrant or both lie in the second closed quadrant, and that \(\det(v, w) = 1\). Without loss of generality, we assume that \(v, w\) both lie in the first closed quadrant; the argument when they both lie in the second closed quadrant is similar. If \(\{i, j\} = \{1, 3\}\), then all the vectors in \(Q_2 \cup Q_4\) lie on a line passing through the origin, so it follows from Lemma \(5.7\) that \([v, w]\) is also a syzygy quadrangle with respect to \(I_Q\), with the same total degree.

Now, suppose that \(\{i, j\} = \{2, 4\}\), so \(G\) is \(\{2, 4\}\)-simple, and either \(Q_2\) or \(Q_4\) contains exactly two nonzero vectors. Without loss of generality, suppose that \(Q_2\) contains exactly two nonzero vectors, which we denote \(b\) and \(c\), so that \(b_2 = b + c\) and the set of nonzero vectors in \(Q_4\) equals either \(\{-b, -c\}\) or \(\{-b, c\}\). Since \([v, w]\) is a syzygy quadrangle of \(I_L\), by Proposition \(5.2\) either \(-b \cdot v, b \cdot w > 0\) or \(-c \cdot v, c \cdot w > 0\). Without loss of generality, suppose \(-b \cdot v, b \cdot w > 0\). By Lemma \(5.7\) it suffices to show that

\[
Q_2 \subseteq \{u \in \mathbb{R}^2 \mid u \cdot v \leq 0 \text{ and } u \cdot w \geq 0\}, \quad Q_4 \subseteq \{u \in \mathbb{R}^2 \mid u \cdot v \geq 0 \text{ and } u \cdot w \leq 0\}.
\]

Assume otherwise, so that either \(c \cdot v > 0\) or \(c \cdot w < 0\). If \(c \cdot v > 0\), then we must also have \(c \cdot w > 0\). However, this contradicts that \(\det(v, w) = |\det(b, c)| = 1\). Similarly, if \(c \cdot w < 0\), then \(c \cdot v < 0\), so again we obtain a contradiction. \(\square\)

**Proof of Proposition \(6.14(2)\).** Let \(\ell \in \{i, j\}\) be such that \(Q_\ell\) contains exactly two nonzero vectors, which are denoted \(v\) and \(w\). Let \(v', w' \in \mathbb{Z}^2\) be the images of \(v, w\) when rotated \(\pi/2\) radians counterclockwise about the origin, respectively. Let \(P_G\) denote the primitive parallelogram \([v', w']\), considered up to lattice translations. Note that if \(Q_i\) and \(Q_j\) each contains exactly two nonzero vectors, the parallelogram \(P_G\) does not depend on whether \(\ell = i\) or \(\ell = j\).

Without loss of generality, suppose \(\{i, j\} = \{1, 3\}\), and that \(Q_1\) contains exactly two nonzero vectors \(b, c\), where \(|\det(b, c)| = 1\). Let \([v, w]\) be an arbitrary syzygy quadrangle of
$I_Q$, where $\mathbf{v}, \mathbf{w}$ either both lie in the first closed quadrant or both lie in the second closed quadrant, and $\det(\mathbf{v}, \mathbf{w}) = 1$. Suppose $[\mathbf{v}, \mathbf{w}]$ is not a syzygy quadrangle of $I_L$. If $\mathbf{v}, \mathbf{w}$ both lie in the first closed quadrant, then it follows from Proposition 5.2 that $[\mathbf{v}, \mathbf{w}]$ is also a syzygy quadrangle of $I_L$, a contradiction. Thus, $\mathbf{v}, \mathbf{w}$ both lie in the second closed quadrant. Since $[\mathbf{v}, \mathbf{w}]$ is a syzygy quadrangle of $I_Q$, we must have $(\mathbf{b} + \mathbf{c}) \cdot \mathbf{v} > 0$ and $(\mathbf{b} + \mathbf{c}) \cdot \mathbf{w} < 0$, and since $[\mathbf{v}, \mathbf{w}]$ is not a syzygy quadrangle of $I_L$, we have that either $\mathbf{b} \cdot \mathbf{v} \leq 0$ or $\mathbf{b} \cdot \mathbf{w} \geq 0$, and similarly for $\mathbf{c}$. This implies that either $\mathbf{b} \cdot \mathbf{v} \leq 0$ and $\mathbf{c} \cdot \mathbf{w} \geq 0$, or $\mathbf{b} \cdot \mathbf{w} \geq 0$ and $\mathbf{c} \cdot \mathbf{v} \leq 0$. Without loss of generality, suppose $\mathbf{b} \cdot \mathbf{v} \leq 0$ and $\mathbf{c} \cdot \mathbf{w} \geq 0$.

Writing $\mathbf{b}, \mathbf{c}, \mathbf{v}, \mathbf{w}$ as $2 \times 1$ column vectors, let

$$
\begin{bmatrix}
\mathbf{b}^T & \mathbf{c}^T
\end{bmatrix}
\begin{bmatrix}
\mathbf{v} & \mathbf{w}
\end{bmatrix} = A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{Z}^{2 \times 2}.
$$

We have $|\det A| = 1$, and $-a_{11}, a_{22} \geq 0$ and $a_{11} + a_{21}, -a_{12} - a_{22} \geq 1$. Thus,

$$
\pm 1 = \det A = a_{11}a_{22} - a_{12}a_{21}
= a_{11}a_{22} + ((-a_{12} - a_{22}) + a_{22})((a_{11} + a_{21}) - a_{11})
= (-a_{12} - a_{22})a_{11} + (a_{11})(-a_{12} - a_{22}) + a_{22}(a_{11} + a_{21})
\geq 1 + 0 + 0.
$$

Thus, we have $a_{11} + a_{21} = -a_{12} - a_{22} = 1$ and $a_{11} = a_{22} = 0$. Letting $\mathbf{b}', \mathbf{c}'$ denote the images of $\mathbf{b}, \mathbf{c}$ after a rotation of $\pi/2$ about the origin, it follows that $\mathbf{b}' = \mathbf{v}$ and $\mathbf{c}' = \mathbf{w}$. Thus, $[\mathbf{v}, \mathbf{w}] = [\mathbf{b}', \mathbf{c}'] = P_G$. Conversely, it is easy to see that $P_G$ is a syzygy quadrangle of $I_Q$ but not of $I_L$. □

**Proof of Proposition 6.14(3).** Without loss of generality, suppose $\{i, j\} = \{1, 3\}$, and that $Q_1$ contains exactly two nonzero vectors $\mathbf{b}, \mathbf{c}$, where $\det(\mathbf{b}, \mathbf{c}) = 1$. Let $\mathbf{b} = (b_1, b_2)$ and $\mathbf{c} = (c_1, c_2)$. Furthermore, let the sum of the vectors in $Q_2$ equal $(-a_1, a_2)$, where $a_1, a_2 \geq 1$. Using Corollary 5.4, it is easy to see that for $I_Q$, the total degree corresponding to the unit square equals $a_1 + a_2 + b_1 + b_2 + c_1 + c_2$. Furthermore, again by Corollary 5.4, we see that the total degree corresponding to the syzygy quadrangle $P_G = [(-b_2, b_1), (-c_2, c_1)]$ of $I_Q$ equals

$$(\mathbf{b} + \mathbf{c}) \cdot (-b_2, b_1) + (-a_1, a_2) \cdot (-b_2 - c_2, b_1 + c_1) + (-\mathbf{b} - \mathbf{c}) \cdot (-c_2, c_1) + (a_1, -a_2) \cdot 0,
$$

which equals $-b_2c_1 + b_1c_2 + a_1(b_2 + c_2) + a_2(b_1 + c_1) + b_1c_2 - b_2c_1 = a_1(b_2 + c_2) + a_2(b_1 + c_1) + 2$. The desired inequality is

$$a_1 + a_2 + (b_1 + c_1) + (b_2 + c_2) \leq a_1(b_2 + c_2) + a_2(b_1 + c_1) + 2,
$$

which can be rearranged as

$$0 \leq (a_1 - 1)(b_2 + c_2 - 1) + (a_2 - 1)(b_1 + c_1 - 1).
$$

This inequality holds since either $\mathbf{b}$ or $\mathbf{c}$ lies in the first open quadrant. Equality holds if and only if $a_1 = 1$ or $b_2 + c_2 = 1$, and $a_2 = 1$ or $b_1 + c_1 = 1$, as desired. □

**Proof of Proposition 6.14(4).** By part (3), the total degree corresponding to the syzygy quadrangle $P_G$ of $I_Q$ equals the total degree corresponding to the unit square. Combining this fact with parts (1) and (2) of Proposition 6.14, we see that $\text{reg } I_L = \text{reg } I_Q$, since from [PS98], we know that the regularity of $I_L$ equals $-2$ plus the maximum total degree corresponding to a syzygy quadrangle of $I_L$, and similarly for $I_Q$. By Lemma 6.7, the unit square is a
syzygy quadrangle of $I_Q$ that attains the regularity of $I_Q$. Since the unit square is a syzygy quadrangle of both $I_L$ and $I_Q$, with the same total degree, the unit square also attains the regularity of $I_L$.

For our last result, we assume that the unit square is a syzygy quadrangle of $I_L$ that attains the regularity of $I_L$. Strictly speaking, it is possible to circumvent Proposition 6.15 in the proof of Proposition 7.2, but we include Proposition 6.15 anyway since it fits in naturally with the flow of our results.

**Proposition 6.15.** Suppose $(L, G, Q)$ satisfies the hypotheses of Proposition 6.14, and furthermore that

- the unit square is a syzygy quadrangle attaining the regularity of $I_L$, and
- $\deg I_Q = \deg I_L - 1$.

Then, $\reg I_Q = \reg I_L$ if and only if up to dihedral symmetries, $G_Q$ is of one of the forms in (6.1).

**Proof.** Since $G$ is chosen so that the unit square is a syzygy quadrangle attaining the regularity of $I_L$, it follows from Proposition 6.14 that $P_G$ is a syzygy quadrangle of $I_Q$ attaining the regularity (of $I_Q$), and that $\reg I_Q = \reg I_L$ if and only if up to dihedral symmetries, $G_Q$ is of one of the desired forms. □

7. Non-Cohen–Macaulay lattice ideals of codimension 2: Proof of Theorem 1.3

In this section we specialize the results of Section 6 to the case of toric ideals with maximal regularity to prove Theorem 1.3.

**Proposition 7.1.** Suppose that $L$ is saturated with Gale diagram containing at least 5 nonzero vectors. The toric ideal $I_L$ has maximal regularity if and only if there exists a reduction datum $(L, G, Q)$, satisfying the following properties for either $\{i, j\} = \{1, 3\}$ or $\{i, j\} = \{2, 4\}$:

- the vectors in $G \setminus (Q_i \cup Q_j)$ all lie on a single line passing through the origin,
- $(L, G, Q)$ is $\{i, j\}$-simple, and
- up to dihedral symmetries, the set $G_Q$ is of one of the forms in (6.1).

**Proof.** As mentioned in Section 2, it suffices to prove the statement under the assumption that all the Gale vectors are nonzero, so we assume that $G$ consists only of nonzero vectors. First, suppose that $I_L$ has maximal regularity. Since $I_L$ is not Cohen–Macaulay, and is therefore not a complete intersection, and $I_L$ is toric, [PS98, Proposition 7.10] implies that $G$ does not lie on two lines. Applying Corollary 6.14 and Proposition 6.6, we conclude that we can choose a reduction datum $(L, G, Q)$ in such a way that $\reg I_L = \reg I_Q = \deg I_Q = \deg I_L - 1$. Then the unit square is a syzygy quadrangle of $I_Q$ that attains the regularity of $I_Q$, so [PS98, Remark 7.9] implies that $G_Q$ is of one of the forms in (6.1). In particular, this means that $(L, G, Q)$ is perfectly balanced. Then, the remaining desired properties of $G$ follow from Proposition 6.13.

Now, suppose that there exists a reduction datum that satisfies the given properties. Per (5.1), we have $\reg I_L \leq \reg I_Q \leq \deg I_Q \leq \deg I_L$. By Lemma 6.7, we have $\reg I_Q = \deg I_Q$,
and the unit square is a syzygy quadrangle of \(I_Q\) that attains the regularity of \(I_Q\). The third condition implies that \((\mathcal{L}, G, Q)\) is perfectly balanced, so by Proposition 6.13 and the first two conditions, \(\deg I_Q = \deg I_{\mathcal{L}} - 1\). Finally, \(\reg I_{\mathcal{L}} = \reg I_Q\) by Proposition 6.15. Another way to see that \(\reg I_{\mathcal{L}} = \reg I_Q\) is to use the fact the unit square is a syzygy quadrangle that attains the regularity of \(I_{\mathcal{L}}\), and a syzygy quadrangle of \(I_Q\) that attains the regularity of \(I_Q\) but has the same corresponding total degree with respect to \(I_{\mathcal{L}}\) and \(I_Q\). \(\square\)

For the next result, we use Proposition 6.14 to reformulate the previous proposition in a way that does not require the underlying Gale diagram to be chosen so that the unit square attains the regularity. In particular, it also tells us that if \(I_{\mathcal{L}}\) is not Cohen–Macaulay and has maximal regularity, then \(\mathcal{L}\) has at most 6 nonzero Gale vectors.

**Proposition 7.2.** Suppose that \(\mathcal{L}\) is saturated, has rank 2, and the Gale diagram of \(\mathcal{L}\) contains at least 5 nonzero vectors. Then, \(I_{\mathcal{L}}\) is a non-Cohen–Macaulay ideal of maximal regularity if and only if there exists a \(\{2, 4\}\)-simple reduction datum \((\mathcal{L}, G, Q)\) such that \(Q_1 = \{(1, 1)\}\) and \(Q_3 = \{(-1, -1)\}\).

**Proof.** Suppose first that such a datum exists. It then follows from Lemma 2.1 and Proposition 2.2 that \(I_{\mathcal{L}}\) is not Cohen–Macaulay. Furthermore, by Proposition 6.14, the unit square is a syzygy quadrangle of \(I_{\mathcal{L}}\) that attains the regularity of \(I_{\mathcal{L}}\). So the desired result follows from Proposition 7.1.

For the other direction, suppose \(I_{\mathcal{L}}\) is non-Cohen–Macaulay with maximal regularity. Then there is a reduction datum \((\mathcal{L}, G, Q)\) satisfying the conditions in Proposition 7.1 and that \(I_{\mathcal{L}}\) has maximal regularity. Suppose first that \(G_Q\) equals \(\{(1, 1), (a, -b), (-1, -1), (-a, b)\}\) up to dihedral symmetries for some \(a, b > 0\); after applying a suitable dihedral symmetry (which preserves the fact that the unit square is a syzygy quadrangle attaining the regularity, as well as the conditions in Proposition 7.1), we may suppose that \(G_Q = \{(1, 1), (a, -b), (-1, -1), (-a, b)\}\). Since \(G\) intersects each open quadrant, \(Q_1\) must consist of \((1, 1)\) and some number of zero vectors, and \(Q_3\) must consist of \((-1, -1)\) and some number of zero vectors. Thus, \(G\) is not \(\{1, 3\}\)-simple, so \(\{i, j\} = \{2, 4\}\) and \(G\) is \(\{2, 4\}\)-simple. Moving all zero vectors from \(Q_1\) and \(Q_3\) to either \(Q_2\) or \(Q_4\), we obtain a partition \(Q'\) for which \((\mathcal{L}, G, Q')\) has the desired properties.

Now, suppose that \(G_Q\) equals \(\{(1, a), (1, -b), (-1, a), (-1, b)\}\) up to dihedral symmetries for some \(a, b > 0\); again, we may assume that \(G_Q = \{(1, a), (1, -b), (-1, -a), (-1, b)\}\). After reflecting if necessary, we may also suppose \(\{i, j\} = \{2, 4\}\). This forces \(Q_1\) to consist of \((1, a)\) and some number of zero vectors, and \(Q_3\) to consist of \((-1, -a)\) and some number of zero vectors. We can move all the zero vectors to \(Q_2\) or \(Q_4\). We now multiply every vector in \(G\) by the matrix

\[
\begin{bmatrix}
1 & 0 \\
-a + 1 & 1
\end{bmatrix} \in \text{GL}_2(\mathbb{Z})
\]

to obtain a Gale diagram \(G'\) of \(\mathcal{L}\). For all \(1 \leq m \leq 4\), let \(Q'_m\) denote the set of images of the Gale vectors in \(Q_m \subseteq G\), so that \(G = Q'_1 \cup Q'_2 \cup Q'_3 \cup Q'_4\), where \(Q'_1 = \{(1, 1)\}\), and \(Q'_3 = \{(-1, -1)\}\). Note that each \(Q'_m\) is contained in the \(m\)th closed quadrant and contains a vector in the \(m\)th open quadrant. Using the fact that \((\mathcal{L}, G, Q)\) is \(\{2, 4\}\)-simple, it is straightforward to check that \((\mathcal{L}, G', Q')\) is also \(\{2, 4\}\)-simple. The result follows. \(\square\)
We are finally able to give a proof of Theorem 1.3.

**Proof of Theorem 1.3.** The case when $I_L$ is a complete intersection is covered by the saturated lattices given in Table 4.1 (up to permutations), as discussed in Corollary 4.4. So, we suppose from now on that $I_L$ is not a complete intersection.

If $n' = 3$, then $I_L$ is Cohen–Macaulay by Proposition 2.2, but by Proposition 4.8, there are no such saturated $L$ that give $\text{reg } I_L = \deg I_L - 1$.

If $n' = 4$, we are done by Corollary 3.2.

Suppose $n' = 5$. By combining Proposition 4.8 and Proposition 7.2, we see that $I_L$ has maximal regularity if and only if there exists a Gale diagram $G$ of $L$ of the form

$$\{(−1, 1), (1, −1), v, w, −v − w\},$$

where $v, w$ are vectors in the closed first quadrant such that $|\det(v, w)| = 1$. Moreover, for all such Gale diagrams, the corresponding lattice ideal is toric and not a complete intersection. If $u$ is an arbitrary visible lattice point (i.e., a lattice point with relatively prime coordinates) in the open second quadrant, there exists some $M \in \text{GL}_2(\mathbb{Z})$ with nonnegative such that $M u = [−1]$. It follows that $I_L$ has maximal regularity if and only if there exists a Gale diagram $G$ of the form

$$\{u, −u, (1, 0), (0, 1), (−1, −1)\},$$

where $u$ is a visible lattice point in the open second quadrant. After applying additional $\text{GL}_2(\mathbb{Z})$ transforms (which may permute $(1, 0), (0, 1), (−1, −1)$), we may only require $u$ to be a visible lattice point not lying on the coordinate axes or on the line $y = x$. The result follows.

The case where $n' = 6$ follows by an argument similar to the one for $n' = 5$. Finally, by Proposition 4.8 and Proposition 7.2, the assumption that $I_L$ is not a complete intersection means that equality is never achieved if $n' > 6$. \hfill \Box

### 8. Future work

The methods of Section 4.3 gave a complete classification of the lattices $L$ of rank 2 for which $I_L$ is a Cohen–Macaulay lattice ideal such that $\text{reg } I_L = \deg I_L − \text{codim } I_L + 1$. We believe with some more work, a similar classification could be obtained for lattices of higher rank. Empirical evidence suggests the following.

**Conjecture 8.1.** Suppose $\text{char } \mathbb{k} = 0$, and let $I_L$ be a nondegenerate Cohen–Macaulay lattice ideal such that $\text{reg } I_L = \deg I_L − \text{codim } I_L + 1$. If $\text{codim } I_L \geq 2$, then all minimal generators of $I_L$ have degree 2.

This says that the case of Proposition 4.1 with a generator of degree $\text{reg } I \geq 3$ does not occur in the case of lattice ideals. As in Corollary 4.4 and Proposition 4.8, this would imply that up to the inclusion of nonzero Gale vectors and permuting coordinates, there are only finitely many possibilities for $L$. In fact, we have the following stronger conjecture.

**Conjecture 8.2.** Suppose $\text{char } \mathbb{k} = 0$, and let $I_L$ be a nondegenerate Cohen–Macaulay lattice ideal such that $\text{reg } I_L = \deg I_L − \text{codim } I_L + 1$. If $\text{reg } I_L \geq 3$ and $r = \text{codim } I_L \geq 2$, then the minimal free resolution of $I_L$ is pure, with the form

$$0 \to S(−r − 2)^1 \to S(−r)^{(r+1)(r+2)} \to \cdots \to S(−3)^{(r+1)(r+2)} \to S(−2)^{(r+1)(r+2)} \to I \to 0.$$
Note that these conjectures are confirmed for complete intersections by Proposition 4.3 for ideals of codimension 2 that are not complete intersections by Proposition 4.7 and for Gorenstein ideals of codimension 3 by Remark 4.9.

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