Schwinger pair creation in constant and time-dependent fields

Adolfo Huet¹, Sang Pyo Kim², Christian Schubert³

¹Centro de Física Aplicada y Tecnología Avanzada, (CFATA) UNAM Juriquilla, Boulevard Juriquilla No. 3001, C.P. 76230, Querétaro, México.
²Department of Physics, Kunsan National University, Kunsan 573-701, Korea.
³Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Apdo. Postal 2-82, C.P. 58040, Morelia, Michoacán, Mexico.
E-mail: ahuet@fata.unam.mx, sangpyokim79@gmail.com, schubert@ifm.umich.mx

Abstract. The effect of pair creation from the vacuum in a strong electric field, predicted by Schwinger in 1951, may conceivably be confirmed experimentally with some of the ultrastrong lasers presently under construction. However, it is not easy to calculate the effect for realistic laser fields. We will shortly review here the intermediate case of a purely time-dependent but otherwise generic electric field, which is still relatively amenable to calculation.

1. Introduction

As early as 1931 Sauter [1] understood that Dirac’s theory of the electron predicts that an electric field of sufficient strength and extent can induce spontaneous creation of electron – positron pairs from the vacuum. In the vacuum tunneling picture, one imagines that, by a statistical fluctuation, a virtual pair separates out far enough to draw its rest mass energy from the field (fig. 1).

\[ \vec{E} \]

\[ e^- \quad e^+ \]

Figure 1. The tunneling picture of pair creation by an external field.

Twenty years later Schwinger [2] developed the modern approach to pair creation, which relates the total pair production probability \( P \) (probability of the decay of the vacuum) to the imaginary part of the effective action \( \Gamma[E] \):

\[ P = 1 - e^{-2\text{Im}\Gamma(E)} \approx 2\text{Im}\Gamma(E). \] (1.1)
For the constant field case, Schwinger found his well-known representation of this imaginary part in terms of an infinite sum of exponentials \cite{2},

\[
\text{Im}\mathcal{L}(E) = \frac{m^4}{32\pi^3} \left( \frac{eE}{m^2} \right)^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \exp \left[ -n\pi \frac{m^2}{eE} \right].
\] (1.2)

Here the \(n\)th term relates to coherent creation of \(n\) pairs in one Compton volume. \(\text{Im}\mathcal{L}(E)\) depends on \(E\) nonperturbatively, which confirms the tunneling picture. Schwinger also derived the corresponding formula for scalar QED \cite{2}:

\[
\text{Im}\mathcal{L}_{\text{scal}}(E) = \frac{m^4}{16\pi^3} \left( \frac{eE}{m^2} \right)^2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \exp \left[ -n\pi \frac{m^2}{eE} \right].
\] (1.3)

Starting with the work of Ritus in the seventies, also radiative corrections to the one-loop formula (1.2) have been extensively studied \cite{3, 4, 5, 6, 7, 8}. It turns out that the two-loop correction to Schwinger’s formula implies an enhancement of pair creation probability. More interestingly, Lebedev and Ritus \cite{5} showed in 1984 that for a weak (subcritical) field, this enhancement can be effectively described by a field-dependent mass,

\[
m(E) \approx m - \frac{\alpha eE}{2m}.
\] (1.4)

This again fits into the tunneling picture: a virtual pair turns real at a finite distance and thus gets born with a negative Coulomb energy that subtracts from the energy to be drawn from the field. The second term on the rhs of (1.4) is easily shown to correspond precisely to this energy. For a constant field the pair creation probability is exponentially small for

\[
E \ll E_{\text{crit}} \approx 10^{18}\text{V/m}.
\] (1.5)

Although present laser technology fails this threshold only by a few orders of magnitude, it is generally believed that, to have any chance at seeing pair creation soon, complicated laser configurations must be used to lower the pair creation threshold. The calculation of pair creation rates for such fields will usually require approximative methods. Presently, there exist essentially four methods that can be attempted for the calculation of the pair creation rate in a generic electric field:

- Until recent years, practically the only approximation method used in this context was WKB \cite{9, 10, 11}.
- A more sophisticated version of WKB is the worldline instanton formalism, which was invented in 1982 by Affleck et al. \cite{4} but applied to non-constant fields only much later \cite{12, 13, 14}.
- The quantum kinetic approach, based on a quantum Vlasov equation (“QVE”) \cite{15, 16, 17, 18, 19}.
- The Wigner formalism \cite{20, 21}.

In the following we will discuss the worldline instanton and QVE approaches.
2. The worldline instanton approach

The worldline instanton method is based on Feynman’s worldline representation of the one-loop effective action in Scalar [22] and Spinor QED [23]. Since spin will not be important in the following, let us write down this path integral just for the scalar case:

$$\Gamma_{\text{scal}}[A] = \int_{0}^{\infty} \frac{dT}{T} e^{-m^2 T} \int D\mathbf{x} S[\mathbf{x}^{(\tau)}],$$

$$S[\mathbf{x}^{(\tau)}] = \int_{0}^{T} d\tau \left( \frac{\dot{x}^2}{4} + ie A \cdot \dot{x} \right).$$

(2.1)

Here $m$ and $T$ are the mass and proper time of the loop scalar, and the path integral $\int D\mathbf{x}$ runs over closed trajectories in (Euclidean) spacetime. In the instanton approach of [4] this path integral is replaced by a single stationary trajectory, which is simply a solution of the Lorentz force equation with periodic boundary conditions. For a constant electric field field pointing in the $x_3$ direction the $n$th worldline instanton is a circle in the $x_3 - x_4$ plane, of radius $m/eE$ and winding number $n$:

$$x_{\text{inst}}(u) = \frac{m}{eE} \left( x_1, x_2, \cos(2n\pi u), \sin(2n\pi u) \right).$$

(2.2)

The worldline action evaluated on the instanton with winding number $n$ is

$$S[x_{\text{inst}}] = \frac{n\pi m^2}{eE}$$

(2.3)

and thus reproduces the $n$th exponent in Schwinger’s representations (1.2),(1.3). Affleck et al. [4] moreover showed how to calculate the global prefactor in the scalar formula (1.3) in terms of the fluctuation determinant around the stationary trajectory [4]. In [12] it was further shown how to adapt all this to the spinor loop case. Thus for the constant field the instanton approximation yields, in both Scalar and Spinor QED, the exact imaginary part of the effective action. This exactness is due to the fact that the worldline path integral for a constant field is gaussian [24]. For generic fields there will be corrections, and the instanton approximation can a priori be expected to be good only in the limit where the mass is much larger than all mass scales contained in the external field.

Proceeding to non-constant fields, a well-studied case is the one of the “spatial bump”, a field defined by

$$E(x_3) = \frac{E}{\cosh^2(kx_3)}$$

(2.4)

and thus localized in the $x_3$ direction. Nikishov [25] in 1970 was able to derive for this case an exact formula for the pair production rate as an integral over momenta of a known function. The worldline instantons for this case can still be given in terms of trigonometric functions [12],

$$x_{3,\text{inst}}(u) = \frac{m}{eE \gamma} \arcsinh \left( \frac{\gamma}{\sqrt{1 - \gamma^2}} \sin(2n\pi u) \right),$$

$$x_{4,\text{inst}}(u) = \frac{m}{eE \gamma \sqrt{1 - \gamma^2}} \arcsin(\gamma \cos(2n\pi u)).$$

(2.5)
where $\tilde{\gamma}$ is the “inhomogeneity parameter”

$$\tilde{\gamma} \equiv \frac{mk}{eE}. \quad (2.6)$$

The corresponding stationary action is

$$S[x_{\text{inst}}] = \frac{n m^2 \pi}{eE} \left( \frac{2}{1 + \sqrt{1 - \tilde{\gamma}^2}} \right). \quad (2.7)$$

It increases with decreasing inhomogeneity, so that the pair creation rate decreases. Its form suggests that something special should happen for $\tilde{\gamma} = 1$, and indeed it turns out that real instanton solutions exist only for $\tilde{\gamma} < 1$. The form of these solutions is shown in fig. 2.

![Figure 2. Plot of instanton paths for the spacelike bump.](image)

Since for this case our the worldline instanton approximation has no reason to be exact, from the absence of instantons we cannot yet conclude that there is no pair creation at all for $\tilde{\gamma} > 1$. However, the total absence of pair creation for $\tilde{\gamma} > 1$ is supported also by a numerical evaluation of Nikishov’s formula mentioned above, as well as by a direct Monte Carlo evaluation [26] of Feynman’s path integral (2.1). This fits again into the vacuum tunneling picture - it is easy to check that for $\tilde{\gamma} > 1$ the field has insufficient extent to provide a virtual particle with its rest mass energy.

Coming to our main topic, the purely time-dependent field $E(t) = (0, 0, E(t))$, here the Lorentz force equation reduces to the following first-order ordinary DGL [12], where

$$\dot{x}_3 = -\frac{iea}{m} A_3(x_4),$$
$$\dot{x}_4 = a \sqrt{1 + \left( \frac{e A_3(x_4)}{m} \right)^2}, \quad (2.8)$$

with

$$a = \frac{m}{eE} 2n\pi. \quad (2.9)$$
At least numerically, the DGL (2.8) can always be solved. Then

\[ \text{Im} \Gamma(E) \overset{m \text{large}}{\approx} N e^{-S[x^{\text{inst}}]} \]  

(2.10)

provides a large-mass approximation for the imaginary part, where a general method for the computation of the prefactor determinant \( N \) was found in [13]. Again the simplest non-constant field to consider is the time-like analogue of the bump (2.4),

\[ E(t) = E \cosh^{2}(\omega t) \]  

(2.11)

The corresponding worldline instantons are related to the ones for the spatial bump (2.5) by the analytical continuation \( \gamma \rightarrow -i\gamma \), where \( \gamma \equiv \frac{m\omega}{eE} \), with the consequence that, unlike the spatial case, for the time-like bump real instanton solutions exist for all values of the inhomogeneity parameter \( \gamma \). And the stationary action,

\[ S[x^{\text{inst}}] = n \frac{m^{2}\pi}{eE} \left( \frac{2}{1 + \sqrt{1 + \gamma^{2}}} \right) \]  

(2.12)

decreases with increasing inhomogeneity, thus the pair creation rate increases. Thus, the worldline instanton formalism suggests the following general rule:

- Inhomogeneity in space tends to reduce the pair creation rate.
- Inhomogeneity in time tends to enhance the pair creation rate.

This also leads one to expect that a purely time-dependent field should always give a non-zero pair creation rate.

3. The quantum Vlasov equation

A very different approach to the purely time-dependent electric field case is based on the observation that for it the spatial momentum \( k \) is a good quantum number. Thus one can fix it, and use a Bogoliubov transformation between the vacua at initial time and time \( t \) to derive an evolution equation for the density of pairs \( \mathcal{N}(t) \) with fixed momenta. This is the QVE. Restricting ourselves to the scalar QED case for definiteness, here it takes the form [15, 16, 18, 17, 19]

\[ \dot{\mathcal{N}}(t) = \frac{\dot{\omega}(t)}{2\omega(t)} \int_{-\infty}^{t} dt' \frac{\dot{\omega}(t')}{\omega(t')} \left( 1 + 2\mathcal{N}(t') \right) \cos \left[ 2 \int_{t'}^{t} dt'' \omega(t'') \right] \]  

(3.1)

where

\[ \omega^{2}(t) = (k_{\parallel} - qA_{\parallel}(t))^{2} + k_{\perp}^{2} + m^{2}. \]  

(3.2)

Here the gauge \( \dot{A}(t) = -\dot{E}(t) \) has been used, and \( k_{\parallel} \) denotes the momentum component along the field. \( \dot{\mathcal{N}}(t) \) has to be set to zero at \( t = -\infty \), and for \( t \rightarrow \infty \) turns into the density of created pairs with fixed momentum \( k \); no direct physical meaning should be ascribed to this quantity at finite time, as has been emphasized in [18, 27]. For the constant field case, the QVE (3.1) and its spinor QED analogue can be solved in closed form [27], leading to the behaviour of \( \dot{\mathcal{N}}(t) \) shown in fig. 3 (this figure is taken from [27]).
Note the strongly oscillatory behaviour which is typical for solutions of the QVE.

In [28] two of the authors derived an alternative form of (3.1) which is equivalent to it (the precise relation between the two forms of the QVE will be discussed in a forthcoming publication [29]), but more amenable to an analytic treatment. This led to the discovery of an infinite set of purely time-dependent electric field configurations for which the QVE can be solved in terms of the well-known soliton solutions of the Korteweg-de Vries equation [28, 30]. The simplest one of these field configurations is given by

\[ qA(t) = k_\parallel - \sqrt{k_\perp^2 + \frac{2\omega_0^2}{\cosh^2(\omega_0 t)}} \]  

(\(\omega_0 = \sqrt{k^2 + m^2}\)). The solitonic character of the associated solutions shows itself in the property that \(\lim_{t\to\infty} N(t) = 0\) for the \(k\) used in the definition of the field (3.3). This does not mean that such an electric field does not pair-create at all - as was already mentioned pair-creation seems inevitable for purely time-dependent fields - but it can be tuned not to pair-create at a given momentum. In fig. 4 we show a plot of \(N(t)\) for the field (3.3).

\[ N(t) \] for the constant field case (solid line = fermion, dotted line = scalar).

\[ N(t) \] in Scalar QED for the simplest solitonic gauge field.

Note the symmetry with respect to \(t = 0\) and the absence of oscillations. More extensive applications of the alternative Vlasov equation of [28] will be given elsewhere.

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