Derivation of the Euler Equations

from Quantum Dynamics

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Abstract

We derive the Euler equations from quantum dynamics for a class of fermionic many-body systems. We make two types of assumptions. The first type are physical assumptions on the solution of the Euler equations for the given initial data. The second type are a number of reasonable conjectures on the statistical mechanics and dynamics of the Fermion Hamiltonian.

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1 Main notations

\( \mu \) .... Typical index labeling the five conserved quantities.

\( j \) .... Typical index referring to time, \( j = 0 \), and space, \( j = 1, 2, 3 \), components.

\( w \) .... \( w = (w^\mu) = (w^\mu_{j,x}), \mu = 0, \ldots, 4, j = 1, 2, 3 \), are the components of the current densities. \( \mu = 0 \) is the particle current, \( \mu = 1, 2, 3 \), indexes the three components of the momentum current, \( \mu = 4 \) is the energy current. \( j = 1, 2, 3 \) refers to the three spatial directions of the current.

\( W \) .... All quantities denoted by \( W_\ast \), are understood to be given by \( l^3 w_\ast \), for any subscript \( \ast \).

\( w \) .... Underlined vectors have an additional component, referring to time, or, in the case of the currents, the conserved quantities.

\( \nabla \) .... \( \nabla = (\partial_t, \nabla) \) is the four-component gradient, including the derivative with respect to time.

\( u \) .... \( u = (u^0, \ldots, u^4) = \underline{w}_0 := w_0 \) are the five conserved quantities: particle number, three components of the momentum, and energy.

\( u^0_x \) .... \( u^0_x = n_x \) is the quantum observable for the particle number density at microscopic space point \( x \).

\( u_x \) .... \( u_x = (u^1_x, u^2_x, u^3_x) \), the quantum observables for the three components of the momentum density at \( x \).

\( u^4_x \) .... \( u^4_x = h_x \) is the quantum observables for the energy density at \( x \).

\( A \) .... \( A = (A_\mu^j) \) are the classical currents appearing the RHS of the Euler equations (\( \mu = 0, \ldots, 4, j = 1, 2, 3 \)). They are defined by

\[
\begin{align*}
A^0_j &= q^i \\
A^i_j &= \delta_{ij} P + q_i q_j / q_0 \\
A^4_j &= q^i (q_4 + P) / q_0 .
\end{align*}
\]

\( \underline{A} \) .... \( \underline{A} = (A^\mu_j) \) are the classical currents augmented with the conserved quantities (\( j = 0 \), vanishing current in the time direction): \( \underline{A}_0 = q = A_0 \).

\( q \) .... \( q = (q^0, \ldots, q^4) \): \( q^0 = \rho \) is the classical particle density, \( q^1, q^2, q^3 \) are the three components of the classical momentum density, and \( q^4 = e \) is the classical energy density. These quantities are a function of macroscopic space and time. This notation is also used for the expectation value of these quantities in a quantum Gibbs state.

\( q \) .... \( q = (q^1, q^2, q^3) \) are the three components of the classical momentum density.

\( v \) .... \( v = q / \rho \) is the classical, macroscopic mean velocity per particle.

\( \bar{e} \) .... \( \bar{e} = e / \rho \) is the classical, macroscopic mean energy per particle.
\[ P(e, \rho) \quad \text{The thermodynamic pressure as a function of the energy and particle densities. Appears in the Euler equations and is defined as the quantum statistical pressure for the Fermion system under consideration.} \]

\[ \varepsilon \quad \text{\( \varepsilon \) is the scaling parameter relating the macroscopic coordinates } X, T, \text{ with the microscopic coordinates } x, t: X = \varepsilon x, T = \varepsilon t. \text{ The hydrodynamic limit is the limit } \varepsilon \to 0. \]

\[ \mathcal{I} \quad \text{The embedding from a collection of independent subcubes of periodic boundary condition to the cube } \Lambda_{\varepsilon^{-1}}. \]

\[ I \quad \text{The embedding from a subcube of periodic boundary condition to the cube } \Lambda_{\varepsilon^{-1}}. \]

\[ \mathbf{u}_{x, \ell}^+ \quad \text{local conservative quantities in a cube of size } \ell \text{ centered at } x. \]

\[ \Lambda_\ell \quad \text{cube of width } \ell \text{ centered at the origin.} \]

\[ \lambda_\varepsilon(t, x) \quad \lambda_\varepsilon(t, x) = \lambda(\varepsilon t, \varepsilon x) \]

1.1 A convention

In the course of our arguments, we will encounter a large (but finite!) number of error terms. Therefore, we introduce a common notation that conveys all relevant information about these error terms. For \( k \geq 1, y \) any list of symbols, let \( \Omega_y(x) \) denote a real-valued function of \( x \in \mathbb{R}^k \), with the property that

\[ \lim_{x_k} \lim_{x_{k-1}} \cdots \lim_{x_1} \Omega_y(x) = 0 \]

where the limits are determined by the names \( x_1, \ldots, x_k \) of the variables. E.g., any term denoted by \( \Omega_{k, X}(\varepsilon, \ell) \) has the property

\[ \lim_{\ell \to \infty} \lim_{\varepsilon \to 0} \Omega_{k, X}(\varepsilon, \ell) = 0 \quad (1.1) \]

The limits are to be taken in the specified order and the quantities denoted by \( y = (y_1, \ldots, y_l) \) are kept fixed in the limits. The limit for \( \varepsilon \) is \( \varepsilon \to 0 \), as this is the limit we are considering and similarly the limit for \( \ell \) is unambiguously \( \ell \to \infty \). The actual value of any \( \Omega_y(x) \) may vary from occurrence to occurrence.

For quantities used in error bounds about which no claim of convergence to zero is made, we will usually use the notation \( C_y \), where \( y \) lists the relevant parameters the constant may depend on.
2 Introduction

The fundamental laws of non-relativistic microscopic physics are Newton’s and Schrödinger equations in the classical and the quantum case respectively. These equations are impossible to solve for large systems and macroscopic dynamics is therefore modeled by phenomenological equations such as the Euler or the Navier-Stokes equations. Although they were derived centuries ago from continuum considerations, they are in principle consequences of the microscopic physical laws and should be viewed as secondary equations. It was first observed by C. Morrey [12] in the fifties that the Euler equations become ‘exact’ in the Euler limit, provided that the solutions to the Newton’s equation are ‘locally’ in equilibrium. Morrey’s original work was far from rigorous and the meaning of ‘local equilibrium’ was not clear. It is nevertheless a very original idea and it contributed significantly to the later development of the hydrodynamical limits of interacting particle systems, see [20] for a review. Instead of considering general classical dynamical systems with two body interactions, a different approach is to prove as much as possible for some simplified models. Outstanding examples are the works by Boldrighini, Dobrushin, and Suhov [11], and Sinai [19] in the case of one space dimension, and the more recent work by Eyink and Spohn [5] who study a d-dimensional classical system of non-interacting particles. In terms of a rigorous proof of Morrey’s idea, however, significant progress has only been made rather recently [16]. This long delay is mostly due to a serious lack of tools for analyzing many-body dynamics, in the classical case and even more so in the quantum case.

In this paper, we derive the Euler equations from microscopic quantum dynamics, extending the relative entropy method of [22, 16] to the quantum cases. Our main result was announced in [14]. As we want to consider the genuine quantum dynamics for a system with short-range pair interactions, we cannot take a semiclassical limit. Although one-particle quantum dynamics converges to Newtonian dynamics in the semiclassical limit, this limit does not commute with the scaling limit needed for the Euler equation. This is most clearly seen in the pressure function, for which quantum corrections survive at the macroscopic scale. In fact, one of the conclusions of our work is that under rather general conditions, the pressure function is the only place where the quantum nature of the underlying system, in particular the particle statistics, survive in the Euler limit.

The Euler equations have traditionally been derived from the Boltzmann equation both in the classical case and in the quantum case, see Kadanoff and Baym [9] for the quantum case. Since the Boltzmann equation is valid only in very low density regions, these derivations are not satisfactory, especially in the quantum case where the relationship between the quantum dynamics and the Boltzmann equation is not entirely clear. There were, however, two approaches based
directly on quantum dynamics. The first was due to Born and Green [2], who used an early version of what was later called the BBGKY hierarchy, together with moment methods and some truncation assumptions. A bit later, Irving and Zwanzig [8] used the Wigner equation, moment methods and truncations to accomplish a similar result. These two approaches rely essentially on the moment method with the Boltzmann equation replaced by the Schrödinger equation. Unlike in the Boltzmann case, where one can do asymptotic analysis to justify this approach, it seems unlikely that this can be done for the Schrödinger dynamics.

One of the benefits of our approach is that we develop a general strategy applicable to all situations where a number of reasonable assumptions are satisfied. We believe that our general assumptions, which are discussed in detail in Section 2.3, hold for a large class of physical models. We regard proving the properties that we assume as an important, although rather challenging, research project in quantum statistical mechanics.

The main novelty of our work lies in the fact that, for the first time, the relative entropy method is applied to a quantum mechanical system. This requires solving a number of technical problems which, not surprisingly, all stem from the fact that the local observables corresponding to the globally conserved quantities of the dynamics, are represented by non-commuting operators. This is mainly discussed in Section 6.

Although our goal is a derivation of the Euler equations, the relative entropy method indeed constructs an approximate solution to the underlying many body dynamics based on solution of Euler equations and the concept of local Gibbs states. It thus establishes the key role played by the Euler equations: they are not just a set of conservation laws but, with the correct choice for the pressure function, they actually dictate the leading approximation to the many body classical or quantum dynamics. Thus, the Euler equations may also be used to obtain information about the solutions of the many-body Schrödinger equation.

### 2.1 Schrödinger and Euler dynamics

We begin by considering $N$ particles on $\mathbb{R}^3$, evolving according to the Schrödinger equation

$$i\partial_t \psi_t(x_1, \ldots, x_N) = H \psi_t(x_1, \ldots, x_N)$$

where the Hamiltonian is given by

$$H = \sum_{j=1}^{N} \frac{-\Delta_j}{2} + \sum_{1 \leq i < j \leq N} W(x_i - x_j).$$  \hspace{1cm} (2.1)

Here, $W$ is a two-body short-ranged stable isotropic pair interaction and $\psi_t(x_1, \ldots, x_N)$ is the wave function of particles at time $t$. We only consider Fermions (such as electrons, but for simplicity we
ignore spin) and thus the state space $\mathcal{H}^N$ is the subspace of antisymmetric functions in $L^2(\mathbb{R}^{3N})$, i.e.,

$$\psi(x_{\sigma_1}, \ldots, x_{\sigma_N}) = (-1)^\sigma \psi(x_1, \ldots, x_N),$$

for any permutation $\sigma$ of $\{1, \ldots, N\}$. It is more suitable not to fix the total number of particles and to use the second quantization terminology. In fact, it would be extremely cumbersome to work through all arguments without the second quantization formalism. The state space of the particles, called the Fermion Fock space, is the direct sum of $\mathcal{H}^N$: $\mathcal{H} := \bigoplus_{N=0}^{\infty} \mathcal{H}^N$.

Define the annihilation and creation operators $a_x$ and $a_x^+$ by

$$
(a_x \Psi)^N(x_1, \ldots, x_N) = \sqrt{N+1} \Psi^{N+1}(x, x_1, \ldots, x_N) \quad (2.2)
$$

$$
(a_x^+ \Psi)^N(x_1, \ldots, x_N) = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} (-1)^{j-1} \delta(x - x_j) \Psi^{N-1}(x_1, \ldots, \hat{x}_j, \ldots, x_N), \quad (2.3)
$$

where, as usual, $\hat{}$ means “omit”. $a_x$ and $a_x^+$ are to be interpreted as operator-valued distributions [3]. The annihilation operator $a_x$ is simply the adjoint of $a_x^+$ with respect to the standard inner product of the Fock space with Lebesgue measure $dx$. These operators satisfy the canonical anticommutation relations

$$
[a_x, a_y^+] := a_x a_y^+ + a_y a_x^+ = \delta(x - y), \quad [a_x^+, a_y^+] = [a_x, a_y]_+ = 0, \quad (2.4)
$$

where $\delta$ is the delta distribution. The derivatives of these distributions with respect to the parameter $x$ are denoted by $\nabla a_x$ and $\nabla a_x^+$. With this notation, we can express the Hamiltonian as $H = H_0 + V$ where the kinetic energy is given by

$$
H_0 = \frac{1}{2} \int \nabla a_x^+ \nabla a_x \, dx
$$

and the potential energy

$$
V = \frac{1}{2} \int \int dx dy W(x - y) a_x^+ a_y^+ a_y a_x.
$$

It is more convenient to put the Schrödinger equation into the operator form, which is sometimes called the Schrödinger-Liouville equation. Denote the density matrix of the state at time $t$ by $\gamma_t$. Only normal states, which can be represented by density matrices, will be considered in the time evolution. Then the Schrödinger equation is equivalent to

$$
i \partial_t \gamma_t = \delta_H \gamma_t \quad \text{with} \quad \delta_H \gamma_t := [H, \gamma_t]. \quad (2.5)
$$

The conserved quantities of the dynamics are the number of particles, the three components of the momentum and the energy. The local densities of these quantities are denoted by $u = (u^\mu), \mu = 1$,
0, · · · , 4, and are given by the following expressions:

\[
\begin{align*}
u^0_x &= n_x &= a_x^+ a_x \\
u^j_x &= p^j_x &= \frac{i}{2} [\nabla_j a_x^+ a_x - a_x^+ \nabla_j a_x], & j = 1, 2, 3, \\
u^4_x &= h_x &= \frac{1}{2} \nabla a_x^+ \nabla a_x + \frac{1}{2} \int dy W(x-y) a_x^+ a_y^+ a_y a_x
\end{align*}
\]

(2.6)

We also introduce the notation \( u = (u^1, u^2, u^3) \). This convention will be followed for the rest of the paper. We use the bold face for the vector of the conservative quantities and use the frac for the vector consisting only the components 1, 2, 3.

Let \( \Lambda_\ell \) denote a cube of width \( \ell \) centered at the origin. The subscript \( \ell \) may be omitted if it plays no active role. We shall adopt the convention that unbounded observables on \( \Lambda \) will be defined with periodic boundary conditions. E.g., the number of particles in \( \Lambda \), the total momentum, and the total energy of the particles in \( \Lambda \), respectively, are defined by

\[
\begin{align*}
N_\Lambda &= \int_\Lambda dx n_x \\
P^j_\Lambda &= \int_\Lambda dx p^j_x, & j = 1, 2, 3 \\
H_\Lambda &= \int_\Lambda dx h_x
\end{align*}
\]

In other words, we shall always view \( \Lambda \) as a three-dimensional torus.

We slightly generalize the definition of the grand canonical Gibbs states to include a parameter for the total momentum of the system: the Lagrange multiplier \( \alpha \). We will work under the assumption that the temperature and chemical potential are in the one-phase region of the phase diagram of the system under consideration such that the thermodynamic limit is unique. The finite volume Gibbs states are then given by the following formula:

\[
\omega^\Lambda_{\beta,\alpha,\mu}(X) = \frac{\text{Tr} X e^{-\beta (H_0,\Lambda + V_\Lambda - \alpha \cdot P_\Lambda - \mu N_\Lambda)}}{\text{Tr} e^{-\beta (H_0,\Lambda + V_\Lambda - \alpha \cdot P_\Lambda - \mu N_\Lambda)}}
\]

(2.7)

The infinite volume Gibbs states \( \omega_{\beta,\alpha,\mu} \) are the limiting points of the finite volume ones. It is convenient to denote the parameters \( (\beta, \alpha, \mu) \) by \( \lambda = (\lambda^\mu), \mu = 0, \cdots, 4 \) with \( \lambda^0 = \beta \mu, \lambda^j = \beta \alpha^j, \lambda^4 = \beta \). Define (notice the sign convention)

\[
\lambda \cdot u = \sum_{\mu=0}^3 \lambda^\mu u^\mu - \lambda^4 u^4
\]

(2.8)

and

\[
\langle \lambda, u \rangle_\Lambda = |\Lambda|^{-1} \int_\Lambda dx \lambda(x) \cdot u(x)
\]
These notations allow us to give a compact formula for the unique, translation invariant Gibbs state (defined with constant $\lambda$), as well as for the states describing local equilibrium (defined with $x$-dependent $\lambda$):

$$\omega_\Lambda^\lambda = e^{\Lambda|\langle \lambda, u \rangle_\Lambda}/Z_\Lambda(\lambda)$$

(2.9)

where $Z_\Lambda(\lambda)$ is the partition function

$$Z_\Lambda(\lambda) = \text{Tr } e^{\Lambda|\langle \lambda, u \rangle_\Lambda}$$

(2.10)

The pressure as a function of the constant vector $\lambda$, is defined by

$$\psi(\lambda) = \lim_{\Lambda \to \infty} |\Lambda|^{-1} \log Z_\Lambda(\lambda)$$

Denote the expectation value of the conservative quantities in an infinite-volume equilibrium state, $\omega_\lambda$ introduced following (2.7), by $q = (q^0, \cdots, q^4)$. The we have

$$\frac{\partial \psi}{\partial \lambda^\mu} = \omega_\lambda(u^\mu)$$

(2.11)

Explicitly,

$$\rho = \omega_\lambda(n_x) = \lim_{\Lambda \to \mathbb{R}^3} \frac{1}{|\Lambda|} \omega_\lambda^\Lambda(N_\Lambda)$$

$$q = \omega_\lambda(p_x) = \lim_{\Lambda \to \mathbb{R}^3} \frac{1}{|\Lambda|} \omega_\lambda^\Lambda(P_\Lambda)$$

$$e = \omega_\lambda(h_x) = \lim_{\Lambda \to \mathbb{R}^3} \frac{1}{|\Lambda|} \omega_\lambda^\Lambda(H_\Lambda)$$

Notice that $q$ and $e$ are momentum and energy per volume.

Again, we will work under the assumption that these parameters stay in the one-phase region, the limiting Gibbs state is unique and these definitions are unambiguous. Although momentum is preferable as a quantum observable, we also introduce the velocity in order to be able to compare with the classical case. The velocity field $v(x)$ has to be defined as a mean velocity of the particles in a neighborhood of $x$. Therefore we have $v(x) = q(x)/\rho(x)$. We also introduce the energy per particle defined by $\tilde{e} = e/\rho$. The usual Euler equations are written in terms of $\rho, v,$ and $\tilde{e}$.

In order to derive the Euler equations, we need to perform a rescaling. So we shall put all particles in a torus $\Lambda_{\varepsilon^{-1}}$ of size $\varepsilon^{-1}$ and use $(X, T) = (\varepsilon x, \varepsilon t)$ to denote the macroscopic coordinates. For all equations in this paper periodic boundary conditions are implicitly understood.

The Euler equations for the five conserved quantities, which arise in the limit $\varepsilon \to 0$, are given
by
\[ \frac{\partial \rho}{\partial T} + \sum_{j=1}^{3} \frac{\partial}{\partial X_j} (\rho v_j) = 0 \]

\[ \frac{\partial (\rho v_k)}{\partial T} + \sum_{j=1}^{3} \frac{\partial}{\partial X_j} [\rho v_j v_k] + \frac{\partial}{\partial X_k} P(e, \rho) = 0 \]  

(2.12)

\[ \frac{\partial (\rho \tilde{e})}{\partial T} + \sum_{j=1}^{3} \frac{\partial}{\partial X_j} [\rho \tilde{e} v_j + v_j P(e, \rho)] = 0 \]

These equations are in form identical to the classical ones but all physical quantities are computed quantum mechanically. In particular, \( P(e, \rho) \) is the thermodynamic pressure computed from quantum statistical mechanics for the microscopic system. It is a function of \( X \) and \( T \) only through its dependence on \( e \) and \( \rho \). If no velocity dependent forces act between the molecules of the fluid under consideration (we consider only a pair potential), the pressure is independent of the velocity.

The conservative quantities \( q = (q^0, \ldots, q^4) \), related to density, momenta and energy as follows:

\[ q^0 = \rho \ , \quad q^i = \rho v^i \ , \quad q^4 = e = \rho \tilde{e} \ , \]

(2.13)

In other words \( q^1, q^2, q^3 \), and \( q^4 \) are momenta and energy per volume instead of per particle as in the usual Euler equation (2.12). We rewrite the Euler equations in the following form

\[ \frac{\partial q^\mu}{\partial T} + \sum_{i=1}^{3} \nabla_i X [A^\mu_i(q)] = 0 \ , \quad \mu = 0, 1, 2, 3, 4 \]  

(2.14)

The matrix \( A \) is determined by comparison with the Euler equations:

\[ A^0_j = q^j \]

\[ A^i_j = \delta_{ij} P + q_i q_j / q_0 \]

\[ A^4_j = q^j (q_4 + P) / q_0 \]  

(2.15)

2.2 Local equilibrium

To proceed we need a microscopic description of local equilibrium and a microscopic prescription to compute the pressure from quantum statistical mechanics. Suppose we are given macroscopic functions \( q(X) \). We wish to find a local Gibbs state with the conserved quantities given by \( q(X) \).

The local Gibbs states are states locally in equilibrium. In other words, in a microscopic neighborhood of any point \( x \in T^3 \) the state is given by a Gibbs state. More precisely, we wish to find a local Gibbs state with the expected values of the energy, momentum, and particle number per
unit volume at $X$ given by $q(X)$. To achieve this, we only have to adjust the parameter $\lambda$ at every point $X$. More precisely, we choose $\lambda(X)$ such that the equation (2.11) holds at every point, i.e.,

$$\frac{\partial \psi(\lambda(X))}{\partial \lambda(X)} = q(X).$$

If we denote the solution to the Euler equation by $q(X, T)$, then we can choose in a similar way a local Gibbs state with given conserved quantities at the time $T$. Define the local Gibbs state

$$\omega_t^\varepsilon = \frac{1}{c_\varepsilon(t)} \exp \left[ \varepsilon^{-3} \langle \lambda(\varepsilon t, \varepsilon \cdot), u \rangle_{\Lambda_{\varepsilon-1}} \right]$$

(2.16)

where $c_\varepsilon(t)$ is the normalization constant. Clearly, we have that $\omega_t^\varepsilon(u^\mu) = q^\mu(\varepsilon x, \varepsilon t)$ to leading order in $\varepsilon$. Our construction of local equilibrium states is consistent with the abstract framework discussed in [21].

Later we will need the following relation for the normalization constant $c_\varepsilon(t)$:

$$\frac{d}{dt} \log c_\varepsilon(t) = \varepsilon \text{Tr} \omega_t^\varepsilon [ \varepsilon^{-3} \langle \partial_t \lambda(\varepsilon t, \varepsilon \cdot), u \rangle_{\Lambda_{\varepsilon-1}} ]$$

(2.17)

Since the inner product almost exclusively taken on $\Lambda_{\varepsilon-1}$, we shall drop this subscription or replaced it by $\langle , \rangle_{\varepsilon-1}$ for the rest of this paper.

In summary, the goal is to show that, in the limit $\varepsilon \to 0$, the following diagram commutes:

$$\begin{array}{ccc}
q(X, 0) & \xrightarrow{\text{Euler}} & q(X, T) \\
\text{local equilibrium} & \downarrow & \text{limit } \varepsilon \downarrow 0 \text{ of expectation of locally averaged observables} \\
\gamma_0 & \xrightarrow{\text{Schrödinger}} & \gamma_{\varepsilon^{-1}T}
\end{array}$$

As smooth solutions of the Euler equations are guaranteed to exist only up to a finite time [10], say $T_0$, we will formulate our assumptions on the dynamics of the microscopic system for a finite time interval as well, say $t \in [0, T_0/\varepsilon]$. Note the cutoff assumptions below would hold automatically for lattice models.

### 2.3 Assumptions and the main theorem

Our main result is stated in Theorem 2.1 below. First, we state the assumptions of the theorem with some brief comments. There are three kinds of assumptions.

The first category of assumptions could be called physical assumptions on the solution of the Euler equations that we would like to obtain as a scaling limit of the underlying dynamics, and on the pair interaction potential of this system.
I. **One-phase regime:** We assume that the pair potential, $W$, is $C^1$ radial and supported in a ball of radius $R$. Furthermore, we assume that $W$ is stable in the sense that

$$W(x) = W_0(x) + W_1(x) \text{ where } W_0 \geq 0, W_0(0) > 0 \text{ and } W_1 \text{ is positive definite.} \quad (2.18)$$

Here, the positive-definiteness of $W_1$ refers too the sesquilinear form, not the function itself. I.e.,

$$\sum_{1 \leq i,j \leq n} W_1(x_i - x_j) z_i z_j, \text{ for all } n \geq 1, x_i, x_j \in \mathbb{R}^3, z_i, z_j \in \mathbb{C}.$$  

Such potentials automatically satisfy the usual super-stability property [18]. In particular, they are stable, i.e., there is a constant $B \geq 0$ such that, for all $N \geq 2, x_1, \ldots, x_N \in \mathbb{R}^3$,

$$\sum_{1 \leq i,j \leq N} W(x_i - x_j) \geq -BN$$

Of the Fermion system with potential $W$ we assume that there is an open region $D \subset \mathbb{R}^2$, which we will call the one-phase region, such that the system has a unique limiting Gibbs state and a regular pressure function for all values of particle density and energy density $(\rho, e) \in D$.

The solution of the Euler equations we consider, $q(X, T)$, will be assumed to $C^1$ in $X$ for $T \in [0, T_0]$, and have local particle and energy density in the one-phase region for all times $T \in [0, T_0]$. I.e., $(\rho(X, T), e(X, T)) \in D$, for all $X \in \Lambda_1$ and $T \in [0, T_0]$.

The next category of assumptions is on the local equilibrium states for the Fermion system that we construct and on their time-evolution under the Schrödinger equation.

II. **Cutoff assumptions:** Suppose that $\gamma_t$ is the solution to the Schrödinger equation (2.5) with a local equilibrium state as initial condition, constructed with the parameters derived from a solution of the Euler equations (with the appropriate pressure function) for times $t$ in a finite interval $[0, T_0]$, that does not leave the one-phase region. We make the following two assumptions.

1. **High-momentum cutoff assumption:** Let $N_p(t) = \text{Tr } \gamma_t a_p^+ a_p$, where $a_p^\#$ is the Fourier transform of $a_x^\#$. Then there is a constant $c > 0$ such that for all $t \leq T_0/\varepsilon$,

$$\varepsilon^d \int dpe^{\varepsilon^2 N_p(t)} \leq C_{T_0} \quad (2.19)$$

where $C_{T_0}$, is constant only depending on $T_0$.

2. **Non-implosion assumption:** There is a constant $C_{T_0}$ (not necessarily the same as $C_{T_0}$ in the previous paragraph) such that for all $t \leq T_0/\varepsilon$,

$$\text{Tr} \left\{ \gamma_t \varepsilon^d \int_{\Lambda_1} dx_n \left[ \int_{|x-y| \leq 2R} n_y dy \right]^2 \right\} \leq C_{T_0} \quad (2.20)$$
where $R$ is the range of the interaction $W$.

Finally, we have an assumption on the set of the time-invariant ergodic states of the Fermion system. To state this assumption we need the notion of relative entropy, of a normal state $\gamma$ with respect to another normal state $\omega$. Let $\gamma$ and $\omega$ denote the density matrices of these states. The relative entropy, $S(\gamma | \omega)$, is defined by

$$S(\gamma | \omega) = \begin{cases} \text{Tr} \{ \gamma (\log \gamma - \log \omega) \} & \text{if } \ker \omega \subset \ker \gamma \\ +\infty & \text{otherwise} \end{cases}$$

For a pair of translation invariant locally normal states, one can show existence of the relative entropy density [15], defined by the limit

$$s(\gamma | \omega) = \lim_{\varepsilon \downarrow 0} \varepsilon^3 S(\gamma_{\Lambda_{\varepsilon^{-1}}} | \omega_{\Lambda_{\varepsilon^{-1}}}),$$

where $\gamma_{\Lambda_{\varepsilon^{-1}}}$ and $\omega_{\Lambda_{\varepsilon^{-1}}}$ denote the density matrices of the normal states obtained by restricting $\gamma$ and $\omega$ to the observables localized in $\Lambda_{\varepsilon^{-1}} = \varepsilon^{-1} \Lambda_1$. The existence of the limit can be proved under more general conditions on the finite volumes, but this is unimportant for us.

III. Ergodicity assumption (“Boltzmann Hypothesis”): All translation invariant, ergodic with respect to space translations, stationary (i.e., time invariant) states to the Schrödinger equation with the Hamiltonian $H$ are Gibbs states with the same Hamiltonian provided they satisfy the following assumptions: 1) the density and energy is in one phase region. 2) The relative entropy density with respect to some Gibbs state is finite.

We expect that the cutoff assumptions hold for the solutions $\gamma_t$ of the Schrödinger equation that we employ, but for now there is no complete proof that it holds for Gibbs states other than the free Fermi gas. For Gibbs states in the high temperature region we expect these assumptions can be proved by using some type of cluster expansion methods. A partial result in this direction has been obtained recently, in the case of Bosons, by Gallavotti, Lebowitz, and Mastropietro in [6].

For the rest of this paper, we shall assume this cutoff assumptions for the solution to the Schrodinger equations as well as the Gibbs states in the one phase regions considered in this paper.

We wish to point out that in the treatment of the classical case in [16] the cut-off assumption 2 was not needed. There is however no proof for the cut-off assumption 1 even in the classical case. (In [16], the usual quadratic kinetic energy was replaced by one with bounded derivatives with respect to momentum. So the cut-off assumption 1 is not needed too.)

The cutoff assumptions are technical in nature. For Fermion models on a lattice instead of in the continuum, no cut-off assumptions are required. The Boltzmann hypothesis on the other hand is a fundamental problem in statistical physics. A version of it was proved to hold for a
classical ideal gas by Eyink and Spohn in [5]. Gurevich and Suhov [7] proved that a stationary Gibbs state to a classical dynamics with a Hamiltonian $H$ has to be a Gibbs state with the same Hamiltonian. Under the assumption that the stationary measures velocity distribution has no correlation (a weaker assumption than in [7]), the Boltzmann hypothesis was proved for classical gas with two-body interaction [16].

Our main result is the following Theorem. We also expect it to hold for Bosons with a super-stable interaction.

**Theorem 2.1** Suppose that $q(X,T)$ is a smooth solution to the Euler equation in one phase region up to time $T \leq T_0$. Let $\omega_\varepsilon^T$ be the local Gibbs state with conserved quantities given by $q(X,T)$. Suppose that the cutoff assumptions and the ergodicity assumption hold. Let $\gamma_\varepsilon$ be the solution to the Schrödinger equation (2.5) and $\gamma_0 = \omega_\varepsilon^0$ (Note that $\gamma_\varepsilon$ depends on $\varepsilon$). Then we have

$$\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq \varepsilon^{-1} T_0} s(\gamma_\varepsilon | \omega_\varepsilon^T) = 0$$

In other words, $\omega_\varepsilon^T$ is a solution to the Schrödinger equation (2.5) in entropy sense. In particular, for any smooth function $f$ on $\Lambda$, we have, for all $0 \leq T \leq T_0$,

$$\lim_{\varepsilon \to 0} \varepsilon^3 \int_{\Lambda_{\varepsilon^{-1}}} dx f(\varepsilon x) [\gamma_\varepsilon^{-1} T(u_x) - q(T,\varepsilon x)] = 0$$

The proof of this theorem will be given in Section 7.2.

Illustrated with a diagram, the main theorem says

$$\omega_{q_0} \xrightarrow{\text{Euler equation}} \omega_{q_T} \xrightarrow{\lim_{\varepsilon \to 0} s(\gamma_\varepsilon^{-1} T | \omega_{q_T}) = 0} \gamma_0 \xrightarrow{\gamma_\varepsilon^{-1} T}$$

Notice that we have proved more than just convergence to the Euler equation. We have shown that the local-equilibrium Gibbs state constructed from the evolution of the Euler equations solves the many-body Schrödinger equation, approximately in entropy sense.

### 2.4 Outline of the proof

The basic structure of our proof follows the relative entropy approach of [16, 22]. The aim is to derive a differential inequality for the relative entropy between the solution to the Schrödinger equation and a time-dependent local Gibbs state constructed to reproduce the solution of the Euler equations. The time derivative of the relative entropy can be expressed as an expectation of the
local currents with respect to the solution to the Schrödinger equation. Since we do not know the solution well-enough, this expectation cannot be computed.

**Step 1:** *Replace the local microscopic currents by macroscopic currents.* The basic idea in hydrodynamical limit is first to show that the local space time average of the solution is time invariant. From the Boltzmann hypothesis, ergodic time invariant states are Gibbs. For Gibbs states, we can replace the local microscopic currents by macroscopic currents. This is the first step.

In the quantum setting, there are several crucial issues we need to address.

1a: *Construct a commuting version of the local conserved quantities.* Recall macroscopic currents are functions of the local conserved quantities, i.e., density, momentum and energy. For the microscopic quantum system, the local conservative quantities are operators which commute only up to boundary terms. In order to express the macroscopic currents as functions of the local conserved quantities, we need either to prove that the non-commutativity does not affect the macroscopic currents or we need to construct some commuting version of the local conserved quantities. As the first approach seems very difficult to carry out, we follow the second one and construct a commuting version of local conservative quantities in section 4.

1b: *Restriction to the one phase region.* Since the Boltzmann hypothesis holds only in the one phase region, we have to exclude the region outside the one phase region. To perform this restriction to the one-phase region, we would normally multiply the observables by some cutoff function. In our case however, the cutoff function does not commute with the local currents. This seemingly trivial multiplication by a cutoff function illustrates the kind of technical problems we have to address in this work. Our approach to this is presented in Section 6.

1c: *Virial Theorem.* Even assuming the local ergodic states are Gibbs in the one phase region, we still have to compute the macroscopic currents from the microscopic currents. This requires a virial Theorem, which we provide in Section 10.

**Step 2:** *Estimate all errors by local conservative quantities.* As will become clear, the errors associated with the cutoff of the one phase region are difficult to control directly. We shall bound them by the local conserved quantities. This will be carried out in sections 5 and 6.

**Step 3:** *Derive a differential inequality of the entropy with error term given by a large deviation formula.* After Step 2, we have an expression of the derivative of the entropy in terms of local (commuting) conservative quantities. Since these quantities commute, by an entropy inequality, we can bound it by a large deviation expression. Notice that it is crucial that we control everything by commuting objects. There is no large deviation theory for non-commuting observables.

After this step, the standard relative entropy method provides the rest of the argument. Technically speaking, the main difficulty to study a quantum mechanical system, in comparison with a classical one, can be traced back to the non-commutativity of the algebra of observables. E.g.,
suppose $A$ and $B$ are two self-adjoint operators representing observables of the system. A simple inequality, such as $|A + B| \leq |A| + |B|$, which is used numerous times in estimates for classical systems, is false, and so is $|AB| \leq |A||B|$. Therefore, there are essentially no absolute values taken in our proof and we estimate all quantities by commuting versions of the locally conserved quantities. Of course, these inequalities hold with the absolute value replaced by the norm. However, we will frequently deal with error terms that are expectations of unbounded observables, such as, e.g., the high-momentum contributions to the energy. Clearly, norm estimates are useless in this situation.

3 Relative entropy identity and high momentum cutoff

3.1 Entropy identity

The first step in the derivation of a differential inequality for the relative entropy is the compute the derivative. Suppose $\gamma_t$ is a solution to the Schrödinger equation. Recall that one has

$$\frac{d}{dt} \text{Tr} A(t)B(t) = \text{Tr} A'(t)B(t) + A(t)B'(t), \quad \frac{d}{dt} \text{Tr} e^{A(t)} = \text{Tr} e^{A(t)}A'(t)$$

(3.1)

and

$$\frac{d}{dt} S(\gamma_t) = 0$$

(3.2)

Thus we have for any time-dependent density matrix $\rho_t$ the identity

$$\frac{d}{dt} S(\gamma_t | \rho_t) = \text{Tr} \gamma_t \{-i\delta_H - \partial_t\} \log \rho_t .$$

(3.3)

This identity replaces the relative entropy inequality in [16, 22]. Thus, by (2.16),

$$\frac{d}{dt} s(\gamma_t | w_\varepsilon^\mu) = \text{Tr} \gamma_t \{-i\delta_H - \partial_t\} \{ \langle \lambda_\varepsilon(t, \cdot), u \rangle_{\varepsilon^{-1}} - |\varepsilon|^3 \log c_\varepsilon(t) \},$$

(3.4)

where $\langle \cdot, \cdot \rangle_{\varepsilon^{-1}} = \langle \cdot, \cdot \rangle_{\lambda_{\varepsilon^{-1}}}$. Let $w$ denote the current tensor with components $w_{k,x}^\mu$ defined by

$$w_{k,x}^0 = p_x^k = \frac{i}{2}[\nabla_k a_x^+ a_x - a_x^+ \nabla_k a_x]$$

(3.5)

$$w_{k,x}^j = \frac{1}{2}[\nabla_j a_x^+ \nabla_k a_x + \nabla_k a_x^+ \nabla_j a_x]$$

$$- \frac{1}{2} \int dy \left[ W'(x-y) \frac{(x-y)_j(x-y)_k}{|x-y|} a_x^+ a_y^+ a_x a_x \right], \quad k = 1, 2, 3$$

(3.6)

$$w_{k,x}^4 = -i \left[ \nabla_k a_x^+ \Delta a_x - \Delta a_x^+ \nabla_k a_x \right] + \frac{i}{4} \int dy W(x-y) \left[ \nabla_k a_x^+ a_y^+ a_y a_x - a_x^+ a_y^+ a_y \nabla_k a_x \right]$$

$$- \frac{i}{4} \int dy \left[ W'(x-y) \frac{(x-y)_k(x-y)_j}{|x-y|} \right] \left[ a_x^+ \nabla_j a_y^+ a_y a_x - a_x^+ a_y^+ \nabla_j a_y a_x \right]$$

(3.7)
where we have used the rotation invariance of the potential to write
\[ W'(x) = \frac{dW(r)}{dr} \big|_{r=|x|}. \]

We have the following proposition. See Section 9 for the derivation of these expressions for the current.

**Proposition 3.1** Let \( \Omega_{\lambda}(\varepsilon) \) be defined by the equation

\[ i\delta H (\lambda_{\varepsilon}(t, \cdot), u)_{\varepsilon^{-1}} = \varepsilon \sum_{j=1}^{3} \langle \nabla_j \lambda_{\varepsilon}(t, \cdot), \ w_j(t) \rangle_{\varepsilon^{-1}} + \Omega_{\lambda}(\varepsilon) \quad (3.9) \]

Then \( \Omega_{\lambda}(\varepsilon) \) is an error term which satisfies the condition

\[ \lim_{\varepsilon \to 0} \text{Tr} \gamma \Omega_{\lambda}(\varepsilon) = 0 \]

These expressions of the microscopic currents seemingly bear no relationship to the macroscopic currents in the Euler equations, even when one assumes that \( \gamma_t \) is locally Gibbs. This difficulty already appears in the classical case. But by reasonably straightforward computation and application of a quantum version of the virial theorem \[10.1\] one can show that indeed these currents correspond to the standard Euler equations given in \[2.12\].

Define \( \nabla_0 = \partial_t \) and \( w_{\mu}^{\mu_0} = u_{\mu}^{\mu}. \) We have

\[ \{ -i\delta H - \partial_t \} \varepsilon^3 \langle \lambda_{\varepsilon}(t, \cdot), u \rangle_{\varepsilon^{-1}} = -\varepsilon \sum_{j=0}^{3} \langle \nabla_j \lambda_{\varepsilon}(t, \cdot), \ w_j(t) \rangle_{\varepsilon^{-1}} := \varepsilon G(\lambda_{\varepsilon}, a^{+}, a) \quad (3.10) \]

where \( \lambda_{\varepsilon}(t, x) = \lambda(t, x). \) Introduce the notations \( \nabla = (\nabla_0, \nabla), \ w = (w_0, w) \) and

\[ \mathbf{A} \cdot \mathbf{B} = \sum_{j=0}^{3} \sum_{\mu=0}^{3} A_{j}^{\mu} B_{j}^{\mu} - \sum_{j=0}^{3} A_{j}^{4} B_{j}^{4}. \quad (3.11) \]

Then we can rewrite the last expression as

\[ \{ -i\delta H - \partial_t \} \varepsilon^3 \langle \lambda_{\varepsilon}(t, \cdot), u \rangle_{\varepsilon^{-1}} = -\langle \nabla \lambda_{\varepsilon}(t, \cdot) \cdot w(t) \rangle_{\varepsilon^{-1}} \quad (3.12) \]

If we wish to emphasize the dependence on the operator, we shall write \( w_{\mu}^{\mu_j, x} = u_{\mu_j, x}^{\mu_j}(a^{+}, a). \) From now on, we shall drop the subscript \( \varepsilon^{-1} \) in \( \langle , \rangle_{\varepsilon^{-1}}. \)
3.2 High-momentum cutoff

Most of the estimates we need are obtained using bounded versions of the creation and annihilation operators, i.e., suitable so-called smeared operators. Physically, this corresponds to introducing a high-momentum cutoff. The precise form of the cutoff will be important for us, as we will have strict requirements on the behavior of the error terms for the proof to go through.

Let \( \hat{\phi}_M \) be a smooth function such that

1. \( |\hat{\phi}_M(p) - 1| \leq e^{-M^2} \) for \( |p| \leq M \) and \( |\hat{\phi}_M(p)| \leq e^{-M^2} \) for \( |p| \geq 2M \).
2. The support of \( \phi_M \) is bounded in a ball of radius \( e^{M^2} \).

To construct such a function, let \( g \) be a smooth function supported in \( |x| \leq 2 \) such that

\[
\hat{g}(p) \leq C[1 + p^2]^{-3}
\]

Define \( g_\lambda(x) = g(x/\lambda)\lambda^{-3/2} \). Let \( h_M \) be a smooth function such that

\[
\hat{h}_M(p) = 1 \text{ for } |p| \leq M \text{ and } \hat{h}_M(p) = 0, \text{ for } |p| \geq 2M .
\]

Let

\[
\phi_M = (g_\lambda * g_\lambda) h_M
\]

Notice that \( \int \phi_M = 1 \). Let \( \lambda = e^{M^2} \). Then we can check easily the properties 1 and 2. Although \( \phi_M \) is supported in a ball of radius \( e^{M^2} \), its mass is concentrated in a ball of radius \( M^{-1} \). More precisely, there is a constant \( c \) such that

\[
\int_{|x| \geq r} h_M(x) dx \leq e^{-crM}
\]

Define

\[
a^+_x,M = \int \phi_M(x - y) a^+_y = a^+_{\phi_x,M}, \quad a_x,M = \int \phi(x - y) a_y = a_{\phi_x,M}
\]

where \( \phi_x,M = \phi_M(x - y) \). In our setting \( \phi_{x,M} = \phi_{x,M} \). Define

\[
\nabla a^+_{x,M} = a^+_{\nabla \phi_{x,M}}, \quad \nabla a_{x,M} = a \nabla \phi_{x,M}
\]

Notice that \( \nabla a^\pm_{x,M} \) and \( a^\pm_{x,M} \) are bounded operators localized in a ball of radius \( e^{M^2} \).

We now perform the preliminary truncation. Denote the cutoff version of the current by

\[
w_{j,x,M} = w_{j,x}(a^+_M, a_M)
\]

Notice that \( w_{j,x,M} \) is bounded. The difference between the kinetic energy term in the energy current with and without cut-off can be calculated using

\[
\nabla a^+_x \Delta a_x - \nabla a^+_x,M \Delta a_x,M = \nabla b^+_x,M \Delta a_x + \nabla a^+_x \Delta b_x,M
\]
where
\[ b^+_{x,M} = a^+_x - a^+_{x,M} \]

**Lemma 3.2** For any state \( \gamma \) that satisfies the cutoff assumptions and for \( G \) defined by (3.10), we have
\[
\epsilon^d \text{Tr} \gamma \int dx \left[ \nabla b^+_{x,M} \Delta a_x + \nabla a^+_x \Delta b^+_{x,M} \right] \leq e^{-cM^2}
\]

**Proof:** By using the Fourier transform, we have
\[
\int dx \nabla b^+_{x,M} \Delta a_x = \int dp (1 - \hat{\phi}(p)) p^2 a^+_p a_p
\]
Let \( N_p(t) = \text{Tr} \gamma a^+_p a_p \). Then
\[
\text{Tr} \gamma \int dp (1 - \hat{\phi}_M(p)) p^2 a^+_p a_p = \int dp (1 - \hat{\phi}_M(p)) p^2 N_p
\]
The lemma is thus a simple consequence of the Chebyshev inequality and the definition of \( \phi_M \).

From the Schwarz inequality
\[
\text{Tr} \gamma \epsilon^d \int dx \nabla a^+_x \int dy W(x-y) a^+_y a_y a_x \leq \epsilon^d \left\{ \text{Tr} \gamma \int dx \nabla a^+_x \nabla a_x \right\}^{1/2} \left\{ \text{Tr} \gamma a^+_x \left[ \int dy W(x-y) n_y \right]^2 a_x \right\}^{1/2} \leq C
\]
Thus for any state \( \gamma \) satisfy the cutoff assumptions, we have
\[
\text{Tr} \gamma \epsilon^d \int dx dy W(x-y) \left[ \nabla a^+_x a^+_y a_y a_x - \nabla a^+_x a^+_y a_y a_x \right] \leq e^{-cM^2} \quad (3.14)
\]

**Lemma 3.3** For any state \( \gamma \) satisfy the cutoff assumptions, we have
\[
\text{Tr} \gamma \epsilon^d [G(\lambda, a^+, a) - G(\lambda, a^+_{M}, a_M)] \leq e^{-cM^2} \quad (3.15)
\]
Thus we have
\[
\frac{d}{dt} s(\gamma_t | \omega^\epsilon_t) = \epsilon \text{Tr} \gamma_t G(\lambda, a^+_M, a_M) - \epsilon^3 \frac{d}{dt} \log c_\epsilon(t) + \epsilon^1
\]
with \( \epsilon^1_M \leq C e^{-cM^2} \). The precise form of the last estimate is crucial as we shall see later on.

We recall the crucial relative entropy inequality [15]: for all self-adjoint observables \( h \), and for any \( \delta > 0 \),
\[
\gamma(h) \leq \delta^{-1} \log \text{Tr} e^{\delta h + \log \omega} + \delta^{-1} S(\gamma|\omega) \quad (3.17)
\]
A proof will be given in the appendix.
4 Construction of local commuting observables

The conserved quantities commute as global observables on $\Lambda$ with periodic boundary conditions. In fact, this is essential for the classical equations of motion to make sense. For any bounded quasi-local observable $X$ on the Fock space define

$$\hat{X}(q) = \text{Tr} \omega_{\lambda} X$$

where the chemical potential $\lambda$ is the dual of $q$ in the sense of (2.11). We define $\hat{X}$ only for arguments in the one phase region.

Local averages of the densities of the conserved quantities, however, do not commute due to boundary effects. Therefore, we cannot extend the functions $\hat{X}$ to functions of the operator-valued local densities of the conserved quantities, which is what we would like to do. To circumvent this difficulty, in this section we construct commuting versions of the local conserved quantities.

We have for any smooth function $J$

$$\text{Tr} \gamma_t \langle J(\varepsilon t, \varepsilon \cdot \cdot \cdot), X \rangle = \text{Tr} \gamma_t \text{Av}_{x \in \Lambda_{-1}} J(\varepsilon t, \varepsilon x) \text{Av}_{|z-x| \leq \ell/2} \tau_z X + \Omega_{\ell}(\varepsilon)$$

where $\text{Av}(\cdot)$, stands for the average of its argument over the domain indicated in the subscript, and $\lim_{\varepsilon \to 0} \Omega_{\ell}(\varepsilon) = 0$ for any $\ell$ fixed.

Denote $u_{\ell} := u_{\Lambda_{\ell}} := \int_{\Lambda_{\ell}} \tau_x u \, dx$ the local conservative quantities and we would like to replace the microscopic current $\text{Av}_{|z-x| \leq \ell/2} \tau_z X$ by certain function of the local conservative quantities $\tau_z u_{\ell}$. Unfortunately the components of $u_{\ell}$ do not commute and functions of $u_{\ell}$ are not well-defined. In fact, even the definition of $u_{\ell}$ is ambiguous since we did not specify the boundary condition. Intuitively, the components of $u_{\ell}$ actually commute up to boundary terms, and the ambiguity should be negligible in the limit $\ell \to 0$. Since it is rather difficult to control these boundary terms in a simple way, we construct in the following a commuting version of the local conservative quantities.

4.1 Construction of an isometric embedding

Let $f$ be a smooth function with

$$f(s) = \frac{1}{\sqrt{2}} \quad \text{if} \quad s \leq 0 \quad = 0 \quad \text{if} \quad s \geq 1$$

and

$$f(1) = f'(1) = f''(1) = 0 .$$

For any given $\eta, 0 < \eta < 1/2$, let

$$g(t) = \left[ 1 - f^2 \left( \frac{\frac{1}{2} - t}{\eta} \right) \right]^{1/2}, \quad 0 \leq t \leq 1/2$$
$$g(t) = f \left( \frac{t - \frac{1}{2}}{\eta} \right), \quad t \geq 1/2,$$

$$g(t) = g(-t), \quad t \in \mathbb{R}.$$ Then $g$ is smooth, supported in $|t| \leq 1/2 + \eta$ and

$$\sum_{j \in \mathbb{Z}} g^2(t + j) = 1$$

Let $\chi(x) = g(x^1)g(x^2)g(x^3)$. Then

$$\sum_{j \in \mathbb{Z}^3} \chi^2(t + j) = 1 \quad (4.2)$$

Let $\alpha^\pm := \tau_\alpha \Lambda^\pm_\ell$ be a cube of size $\ell \pm 4\ell\eta$ centered at $\alpha$. Let

$$\chi_\alpha(x) = \chi((x - \alpha)/\ell)$$

be a smooth function supported in $\tau_\alpha \Lambda_{\ell+2\ell\eta} \subset \alpha^+$ and $\chi_\alpha(x) = 1$ in $\tau_\alpha \Lambda_{\ell-2\ell\eta} \subset \alpha^-$. We collect these relations in the following:

$$\alpha^- = \tau_\alpha \Lambda_{\ell-4\ell\eta} \subset \tau_\alpha \Lambda_{\ell-2\ell\eta} \subset \{x : \chi_\alpha(x) = 1\}$$

$$\subset \{x : \chi_\alpha(x) \neq 0\} \subset \tau_\alpha \Lambda_{\ell+2\ell\eta} \subset \tau_\alpha \Lambda_{\ell+4\ell\eta} = \alpha^+ \quad (4.3)$$

There is a wide range of choices for $\eta$. The main restrictions we needed are

$$\eta \ell \to \infty, \quad \eta \to 0.$$ We shall choose, for simplicity of notation

$$\eta = \ell^{-1/2}$$

for the rest of this paper.

Recall the configuration space $S(\Lambda)$ is the space

$$S(\Lambda) = \{x := (x_1, \cdots, x_n) : n \in \{0\} \cup \mathbb{N}, x_j \in \Lambda \text{ for all } j\}$$

Denote by $\Gamma(\Lambda)$ the space of antisymmetric functions from the configuration space $S(\Lambda)$ to the complex numbers. With the standard $L^2$ inner product, $\Gamma(\Lambda)$ is a Hilbert space.

Define $I^\Lambda_\alpha$ from $\Gamma(\alpha^+)$ to $\Gamma(\Lambda)$ by

$$(I^\Lambda_\alpha \psi)(x_1, \cdots, x_n) = \left[ \prod_j \chi_\alpha(x_j) \right] \psi(x_1, \cdots, x_n)$$
Usually we shall take \( \Lambda = \Lambda_{e-1} \) and omit the labels \( \Lambda \) and \( \alpha \) whenever they are obvious or unimportant. It is crucial that \( \prod_j \chi_{\alpha_j}(x_j) \) is symmetric w.r.t. permutations of \( x \) so that \( I^\Lambda_\psi \) is antisymmetric as a function of \( x \). Define \( I^* \) to be the adjoint of \( I \), i.e., we have \( (I^* f, g) = (f, I^* g) \).

Let \( X \) be an observable \( X \) on \( \alpha \) defined by

\[
X = \int_{\alpha^+} dx_1 \cdots dx_k dy_1 \cdots dy_k f(x_1, \ldots, x_k; y_1, \ldots, y_k) a^+_{x_1, \alpha} \cdots a^+_{x_k, \alpha} a_{y_1, \alpha} \cdots a_{y_1, \alpha}
\]

where \( f \) is a distribution (kernel) with support in \((\alpha^+)^{2k}\). Here we have labelled the operators by \( \alpha \) to emphasized the cube \( \alpha \). We can check the identity:

\[
I^* X I = \int dx_1 \cdots dx_k dy_1 \cdots dy_k \chi_\alpha(x_1) \cdots \chi_\alpha(x_k) \chi_\alpha(y_1) \cdots \chi_\alpha(y_k) \times f(x_1, \ldots, x_k; y_1, \ldots, y_k) a^+_{x_1} \cdots a^+_{x_k} a_{y_1} \cdots a_{y_1} \quad (4.4)
\]

as an operator on the torus \( \Lambda_{e-1} \).

From this definition the pull-backs of all observables we need, i.e., \( I^* X I \) for a conserved quantity or current given by \( X \), can easily be computed. By using the appropriate distribution kernels \( f \), observables involving derivatives are included. E.g.,

\[
I^* \left[ \int_{\alpha^+} dx f(x) \nabla a^+_{\alpha} a_{\alpha} \right] I = I^* \left[ - \int dy' (y - x) \int_{\alpha^+} dx f(x) a^+_{y} a_y \right] I
\]

\[
= - \int dy' (y - x) \int dx f(x) \chi_\alpha(y) a^+_{y} a_y
\]

\[
= \int dx f(x) \left[ \chi_\alpha(x)^2 \nabla a^+_{\alpha} a_{\alpha} + \chi_\alpha(x) \nabla \chi_\alpha(x) a^+_{\alpha} a_{\alpha} \right] \quad (4.5)
\]

For the kinetic energy we have

\[
I^* \left[ \int_{\alpha^+} dx f(x) \nabla_j a^+_{\alpha} \nabla_j a_{\alpha} \right] I = \int dx f(x) \left[ \chi_\alpha(x)^2 \nabla_j a^+_{\alpha} \nabla_j a_{\alpha} + (\nabla_j \chi_\alpha(x))^2 a^+_{\alpha} a_{\alpha} \right. \\
\left. + \chi_\alpha(x) \nabla_j \chi_\alpha(x) (\nabla_j a^+_{\alpha} a_{\alpha} + a^+_{\alpha} \nabla_j a_{\alpha}) \right] \quad (4.6)
\]

If we take \( f = 1_{\alpha^-}(x) \), we have \( f(x) \chi_\alpha(x) = 0 \). Together with \( \chi_\alpha(x) = 1 \) in \( \tau_\alpha \Lambda_{e-2\ell_\eta} \supset \alpha^- \), we have

\[
I^* \left[ \int_{\alpha^-} dx \nabla a^+_{\alpha} a_{\alpha} \right] I = \int_{\alpha^-} dx \left[ \nabla a^+_{\alpha} a_{\alpha} \right] \quad (4.7)
\]

\[
I^* \left[ \int_{\alpha^-} dx \nabla_j a^+_{\alpha} \nabla_j a_{\alpha} \right] I = \int_{\alpha^-} dx \nabla_j a^+_{\alpha} \nabla_j a_{\alpha}
\]

### 4.2 Commuting local conserved quantities

Let \( H_{\alpha^+} \), \( P_{\alpha^+} \) be the total energy and momentum operators on \( \alpha^+ \) with periodic boundary condition. Then \( H_{\alpha^+} \) and \( P_{\alpha^+} \) commute with each other and also with the number operator \( N_{\alpha^+} \). Denote

\[
u_{\alpha^+} = \ell^{-3}(P_{\alpha^+}, N_{\alpha^+}, H_{\alpha^+})
\]

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Since the image of $I$ is in the domain of $H_{\alpha^+}$ and $P_{\alpha^+}$, the operator $I^*u_{\alpha^+}I$ is well-defined. Since the components of $u_{\alpha^+}$ commute, the function

$$I^*\hat{X}(u_{\alpha^+})I$$

is now well-defined. We shall use the notation

$$u_{x,\ell}^\pm := u_{\alpha^+}, \quad n_{x,\ell}^\pm := \ell^{-3}N_{\alpha^+}, \quad h_{x,\ell}^\pm := \ell^{-3}H_{\alpha^+}$$

(4.8) when $\alpha$ is centered at $x$. When $x = 0$, we shall omit the subscript $x$.

### 4.3 Local average of currents

Let

$$w_{\alpha^\pm} = \ell^{-3} \int_{\alpha^\pm} dx w_x, \quad W_{\alpha^\pm} = \int_{\alpha^\pm} dx w_x$$

be the average over the cube $\alpha^\pm$ of the currents $w_x$, where we have divided the integration by $\ell^3$ which is approximately the volume to the cube $\alpha^\pm$. by definition $w_x$ is an operator on the torus $\Lambda_{\ell^{-1}}$. Since $a_{x}^\pm$ can be viewed as an operator on $\alpha^+$ with periodic boundary condition for $x \in \alpha^+$, $w_{\alpha^\pm}$ can be understood as an operator on the cube $\alpha^+$ as well. We shall use the same symbol in both contexts.

Recall the cutoff version of the current $w_{x,M} = w_x(a_M^+, a_M)$ (3.13). Thus we can define the cutoff version of the current

$$w_{M,\alpha^\pm} = \ell^{-3} \int_{\alpha^\pm} dx w_{M,x}$$

(4.9) Here $w_{M,\alpha^\pm}$ can be viewed as an operator either on the torus $\Lambda_{\ell^{-1}}$ or $\alpha^+$ with periodic boundary condition.

Recall that $a_{x,M}^\pm$ are bounded operators localized in a ball of radius $e^{M^2}$ centered at $x$. Thus the support of $a_{x,M}^\pm$ is contained in $\tau_{\alpha} \Lambda_{\ell^{-2}\sqrt{7}}$ for $x \in \alpha^- = \tau_{\alpha} \Lambda_{\ell^{-4}\sqrt{7}}$, as long as $\sqrt{\ell} > e^{M^2}$, which we shall assume from now on. Since $\chi_\alpha(x) = 1$ for $x \in \tau_{\alpha} \Lambda_{\ell^{-4}\sqrt{7}}$, following the proof of (4.7) we have the identity

$$I^*w_{M,\alpha^-}I = w_{M,\alpha^-};$$

(4.10) here $w_{M,\alpha^-}$ is understood as an operator on the torus $\Lambda_{\ell^{-1}}$ on the right side and as an operator on $\alpha^+$ on the left side. Define the notation

$$w_{M,x,\ell}^- = \ell^{-3} \int_{\Lambda_\ell^-} dy w_{M,y}$$

where $\Lambda_\ell^- = \tau_x \Lambda_{\ell^{-4}\sqrt{7}} = \tau_x \Lambda_{\ell^-}^-$. 25
From (4.10), the boundedness of $a^\sharp_{M,x}$ and simple counting of the number of terms, we have the following lemma.

**Lemma 4.1** For any state $\gamma$, and any smooth function $J$, we have

$$
\operatorname{Tr} \gamma \langle J(\varepsilon \cdot), w_M \rangle = \operatorname{Tr} \gamma \operatorname{Av}_x J_\varepsilon(x) I^* w_M^{-\varepsilon,\ell} I + \Omega_M(\varepsilon, \ell) \quad (4.11)
$$

where the error term $\Omega_M(\varepsilon, \ell)$ vanishes in the sense given by (1.1) and

$$
\operatorname{Av}_x = \varepsilon^3 \int_{\Lambda_{\varepsilon^{-1}}} dx
$$

Applying this Lemma to a smooth function $J(\varepsilon t, \varepsilon x)$ and average over $t$, we have

$$
\operatorname{Av}_{t \leq T/\varepsilon} \operatorname{Tr} \gamma_t \langle J(\varepsilon \cdot, \varepsilon t), w_M \rangle = \operatorname{Av}_{t \leq T/\varepsilon} \gamma_t \langle J_\varepsilon(t, \cdot), I^* w_M^{-\varepsilon,\ell} I \rangle + \Omega_M(\varepsilon, \ell). \quad (4.12)
$$

5 **Bounds on the currents**

The aim of this section is to show that the currents with momentum cutoff, i.e., the quantities $W_{M,\alpha^\pm}$, can be bounded by a multiple of the Hamiltonian plus particle number. This is the content of the following lemma.

**Lemma 5.1** The following operator inequalities hold:

$$
W_{M,\alpha^+} \leq CM \left[ H_{\alpha^+} + N_{\alpha^+} \right] \quad (5.1)
$$

Note that the dependence on $M$ in the right hand side is linear. Similar inequality holds if $W_{M,\alpha^+}$ on the left side is replaced by $W_{M,\alpha^-}$.

As it is essential for the proof of this lemma, we first recall a standard stability result based on the superstability conditions we have assumed on the potential $W$. Stated in words, the result says that a large class of two-body quantities can be bounded in terms of the two-body interaction and the particle number. We state this result as a lemma for functions but it obviously extends to the corresponding second quantized observables.

**Lemma 5.2** Suppose $U$ is a positive bounded function with compact support on $\mathbb{R}^3$ and $W$ is a superstable potential stated in the sense of (2.18). Then there is a $\delta > 0$ such that

$$
\delta \sum_{\alpha \neq \beta}^N U(x_\alpha - x_\beta) \leq \sum_{\alpha \neq \beta}^N W(x_\alpha - x_\beta) + N.
$$

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The proof of Lemma 5.2 is contained in Ruelle’s book [18] or see [16].

**Proof of Lemma 5.1** First, we treat the one-particle (i.e., quadratic in the $a_x^\#s$) terms and show that they can be bounded by the kinetic energy term and chemical potential term of the Hamiltonian.

We will use the following inequalities several times without further reference: for any pair of bounded operators $A$ and $B$, and $c > 0$, one has

$$A^* B + B^* A \leq cA^* A + c^{-1}B^* B$$

and

$$A + B \leq |A| + |B|.$$

From the last inequality it follows that for a bounded family of self-adjoint operators $A_x$, and a real valued $L^1$-function $f$, one has

$$\int f(x)A_x dx \leq \int |f(x)||A_x|dx.$$

Note that one cannot replace the LHS of these inequalities by their absolute values unless all terms commute.

We start with bounding the momentum components of $W_M$:

$$W_{k,M,\alpha}^0 = \frac{i}{2} \int_{\alpha^+} dx \left[ \nabla_k a_{x,M} a_{x,M} - a_{x,M}^+ \nabla_k a_{x,M} \right]$$

$$\leq \int_{\alpha^+} dx \nabla_k a_{x,M}^+ a_{x,M} + a_{x,M}^+ a_{x,M}$$

(5.2)

We would like to obtain bounds by multiples the Hamiltonian and the number operator without momentum cut-off $M$. For this we use the following inequalities:

$$a_{x,M}^+ a_{x,M} \leq C|\phi_M| * a_{x}^+ a_{x}$$

(5.3)

$$\nabla a_{x,M}^+ \nabla a_{x,M} \leq CM|\phi_M| * a_{x}^+ a_{x}$$

(5.4)

$$\nabla a_{x,M}^+ \nabla a_{x,M} \leq |\phi_M| * \nabla a_{x}^+ \nabla a_{x}$$

(5.5)

$$\Delta a_{x,M} \Delta a_{x,M} \leq CM|\phi_M| * \nabla a_{x}^+ \nabla a_{x}$$

(5.6)

In the above expressions the convolutions are with respect to the variable $x$ in the RHS. The four inequalities are proved in almost identical fashion. E.g., the first inequality is obtained as follows:

$$a_{x,M}^+ a_{x,M} = \int dzdw \phi_M(x - z)a_{x}^+ \phi_M(x - w)a_{w}$$

$$\leq \frac{1}{2} \int dz \int dw |\phi_M(x - z)||\phi_M(x - w)||a_{z}^+ a_{z} + a_{w}^+ a_{w}|$$

$$\leq |\phi_M| * a_{x}^+ a_{x}$$

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where we have used that \( \int |\phi_M| = 1 \). For (5.4) and (5.6) one also has to use \( \|\nabla \phi_M\|_1 \leq CM \), for a suitable constant \( C \), but otherwise the proofs are the same. Now, we can finish the bound of \( W_{k,M,\alpha}^0 \) by using (5.3) and (5.5) in (5.2). We obtain

\[
W_{k,M,\alpha}^0 \leq \int dx \, |\phi_M|(x-y) \left[ \nabla k a_x^+ a_y^+ \nabla k a_x a_y + a_x^+ a_y^+ a_y a_y \right]
\leq C(H_0,\alpha + N_\alpha)
\]

By stability of the potential the kinetic energy term in this bound can be replaced by the full Hamiltonian, up to an adjustment to the constant \( C \). This completes the proof of the lemma for \( W_{k,M,\alpha}^0 \).

For the other one-particle terms of \( W_{M,\alpha} \) one proceeds in the same way. E.g., for the last inequality one starts from

\[
-i \left[ \nabla k a_x^+ M a_y^+ \Delta a_x a_y - \Delta a_x^+ \nabla k a_x a_y \right] \leq M \nabla k a_x^+ M \nabla k a_x a_y + M^{-1} \Delta a_x^+ \Delta a_x a_y
\]

The rest of the argument is the same. In summary, the results are

\[
\frac{i}{2} \int dx \left[ \nabla k a_x^+ a_y^+ a_x a_y - a_x^+ a_y^+ \nabla k a_x a_y \right] \leq C(H^+ + N_\alpha)
\]

\[
\int dx \nabla j a_x^+ M \nabla k a_x a_y \leq C(H^+ + N_\alpha)
\]

\[
-\frac{i}{4} \int dx \left[ \nabla k a_x^+ M \Delta a_x a_y - \Delta a_x^+ \nabla k a_x a_y \right] \leq CM(H^+ + N_\alpha)
\]

The two-particle terms (quartic in the \( a_x^\# \)'s) appearing in (3.6) and (3.8), we can follow the same procedure. E.g., to bound the middle term of (3.8), we start from

\[
\frac{i}{4} \left[ \nabla k a_x^+ a_y^+ M a_y a_x a_y - a_x^+ M a_y a_x a_y \nabla k a_x a_y \right] \leq M a_x^+ M a_y a_x a_y + M^{-1} \nabla a_x^+ a_y^+ a_y a_y a_x a_y
\]

The first term can further be bounded in a way similar to (5.8):

\[
a_x^+ M a_y^+ M a_y a_x a_y \leq \int dz |\phi_M(z-y)| a_x^+ M a_y^+ a_y a_x a_y = \int du |\phi_M(u-y)| a_y^+ a_x^+ a_y a_x a_y \leq \int du dv |\phi_M|(u-y) |\phi_M|(v-y) a_x^+ a_y^+ a_y a_x
\]

The same quantity appears in (3.6). In both cases, after integration, we get something of the form:

\[
M \int dx \int dy \int dx dy |\phi_M|(u-x) G(u-v) |\phi_M|(v-y) a_x^+ a_y^+ a_y a_x
\]

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where $G$ is a non-negative function of compact support. To estimate this term, we use Lemma 5.2 to obtain a bound of the last expression of the form $CM$ times the potential energy.

The second term in the RHS of (5.7) gives rise to

$$M^{-1} \int_{\alpha^+} \int_{\alpha^+} dx \int dy M \left[ \int du dv |\nabla \phi_M| (u - x) G(u - v) |\phi_M| (v - y) a_x^+ a_y^+ a_x a_y \right]$$

and something of the same form for the third term in (3.8), which, with another application of Lemma 5.2 can also be bounded by the $CM$ times the potential energy. This completes the proof of the lemma.

### 6 Local ergodicity

Recall that by assumption the solution up to time $t \leq T_0 / \varepsilon$ of the Euler equations has density and energy taking values in a compact set strictly contained in the one phase region of the phase diagram of the fermion systems. Let $\sigma^\kappa$ be a smooth function supported in the one phase region such that $\sigma^\kappa = 1$ on this compact set. Furthermore, we require that as $\kappa \to 0$, $\sigma^\kappa$ becomes the characteristic function of a compact neighborhood of this set contained in the one phase region. Since the phase transition region depends only on the density and energy, $\sigma^\kappa$ needs to depend only on the density and energy. We will take $\sigma^\kappa(e, n)$ of the form $\rho^\kappa (\kappa e) \rho^\kappa (\kappa n)$ where $\rho^\kappa$ and $\rho^\kappa$ are some smoothed versions of the characteristic functions on a set of sufficiently high $e$ and sufficiently low $n$, respectively.

The aim of this section is to prove the following theorem. Recall $\hat{X}$ is defined in (4.1).

**Theorem 6.1** For all smooth functions $J$, and $X$ any one of the components of $w_M$ we have

$$\begin{align*} &\text{Av}_{\varepsilon} \frac{\gamma_t}{Av_x} J(\varepsilon t, \varepsilon x) I^* X_{-t} I_x \leq \text{Av}_{\varepsilon} \text{Tr}_{\varepsilon} \frac{\gamma_t}{Av_x} J(\varepsilon t, \varepsilon x) \left\{ I^*(\hat{\sigma}^\kappa \hat{X} \hat{\sigma}^\kappa)(u_{x,t}^+) + I^*(1 - \sigma^\kappa(u_{x,t}^+))X_{-t}(1 - \sigma^\kappa(u_{x,t}^+))I \right\} \\
&+ \Omega_{J, \kappa, X}(\varepsilon, \ell, a) \end{align*}$$

where $\hat{\sigma}^\kappa = \sqrt{\sigma^\kappa(2 - \sigma^\kappa)}$.

The function $\hat{\sigma}^\kappa$ behaves essentially the same way as $\sigma^\kappa$, i.e., it is a smooth version of a characteristic function supported in the one-phase region.

As a first step towards the proof of Theorem 6.1 we partition $\Lambda_{\varepsilon^{-1}}$ into cubes of size $a \varepsilon^{-1}$, where $a$ is a sufficiently small positive constant. For any $z \in \Lambda_{\varepsilon^{-1}}$, let $Q = \Lambda_{z, a \varepsilon^{-1}}$ denote the cube
of size $a\varepsilon^{-1}$ centered at $z$. For any bounded quasi-local observable $Z$, define the average of $Z$ in the cube $Q$ by

$$Z_Q = \text{Av}_{y \in Q} \tau_y Z$$

We also divide the time interval $[0, \varepsilon^{-1}T]$, into disjoint intervals of size $2a\varepsilon^{-1}$ and label the centers by $t_1, \cdots, t_n$, $n = T/(2a)$ (the $n$-th interval is $[t_n - a, t_n + a] \cap [0, \varepsilon^{-1}T]$).

Since $J$ is a smooth function,

$$\langle J_{\varepsilon}(t, \cdot), Z \rangle = \frac{1}{n} \sum_{j=1}^{n} \text{Av}_{z} \left( J(\varepsilon t_j, \varepsilon z) \gamma_t^{-1} Z_{\Lambda_{t, a\varepsilon^{-1}}} \right) + \Omega_{Z, J}(a, \varepsilon) \quad (6.2)$$

where $\lim_{a \to 0} \lim_{\varepsilon \to 0} \Omega_{Z, J}(a, \varepsilon) = 0$ and the average is over $z \in a\varepsilon^{-1}Z^3 \cap \Lambda_{\varepsilon^{-1}}$.

For $Q, a, j$ fixed, define a family of states labelled by $\varepsilon$ consisting of the states defined by

$$\gamma_{Q,j}^{\varepsilon}(Z) = \text{Av}_{|t-t_j| \leq a/\varepsilon} \gamma_t Z_Q$$

Then $\{\gamma_{Q,j}^{\varepsilon} | \varepsilon > 0\}$ is $\text{w}^*$-precompact and, hence, has at least one limit point.

**Lemma 6.2** Let $\overline{\omega}$ be the Gibbs state on $\Lambda_{\varepsilon^{-1}}$ defined in (2.10) with $\Lambda = \Lambda_{\varepsilon^{-1}}$ and the chemical potential $\lambda := \overline{\lambda}$ is chosen to be

$$\overline{\lambda} = \text{Av}_x \lambda(0, \varepsilon x)$$

where $\lambda(0, \cdot)$ are the parameters for the initial condition defined in (2.10). Then for any $t \geq 0$ the relative entropy

$$s(\gamma_t | \overline{\omega}) \leq C$$

for some constant $C$ depending only on the initial value $\lambda(0, \cdot)$.

**Proof:** Recall the initial state is

$$\omega_0^{\varepsilon} = \frac{1}{c_\varepsilon(0)} \exp \left[ \varepsilon^{-3} \langle \lambda(0, \cdot), u \rangle \right] \quad (6.3)$$

Then we have

$$s(\omega_0^{\varepsilon} | \overline{\omega}) = \int d\omega_0^{\varepsilon} \left( \langle \lambda(0, \cdot), u \rangle - \langle \overline{\lambda}, u \rangle \right) + \varepsilon^3 \log c_\varepsilon(0) - \varepsilon^3 \log Z_{\Lambda_{\varepsilon^{-1}}} \quad (6.4)$$

Since each term on the right side is bounded, we have $s(\omega_0^{\varepsilon} | \overline{\omega}) \leq C$. From a simple direct calculation, we know that $s(\gamma_t^\varepsilon | \overline{\omega})$ is a constant of motion. This proves the Lemma. 

\[\square\]
Lemma 6.3  Fix the parameter $a$ and let $\eta$ be any limit point of $\{\gamma_{Q,j}^{\varepsilon} \mid \varepsilon > 0\}$. Then $\eta$ is a translation invariant, time invariant state of the dynamics. Furthermore, the specific relative entropy of $\eta$ with respect to the translation invariant state $\omega_{Q}^{\varepsilon}$, satisfies the bound

$$s(\eta|\omega_{Q}^{\varepsilon}) \leq C a^{-3}$$

**Proof:**  The invariance under space and time translations is an immediate consequence of the scaling by $\varepsilon^{-1}$. Since the proof for quantum case is parallel to that of the classical case, we refer the reader to [16] for a proof of the classical case. To show that the specific relative entropy with respect to $\omega_{Q}^{\varepsilon}$ is finite, we start from Lemma 6.2 stating that the relative entropy $\varepsilon^3 S(\gamma_{Q}^{\varepsilon} \mid \omega_{Q}^{\varepsilon}) \leq C$ for a suitable constant $C$.

The operations of averaging over translations in a cube $Q$ and over times in an interval $[t_j - a/\varepsilon, t_j + a/\varepsilon]$, are completely positive, therefore, by the monotonicity (or convexity) of the relative entropy (see, e.g., [15]), we have

$$\varepsilon^3 S(\text{Av}_{|t-t_j| \leq a/\varepsilon} \gamma_{Q}^{\varepsilon} \circ \tau_y \mid \omega_{Q}^{\varepsilon}) \leq C$$

The relative entropy is also monotone with respect to restriction to the algebra of observables of a subvolume. Therefore we have

$$\varepsilon^d S(\text{Av}_{|t-t_j| \leq a/\varepsilon} \gamma_{Q}^{\varepsilon} \circ \tau_y \mid Q \mid \omega_{Q}^{\varepsilon}) \leq C$$

Now, $\eta$ is a limiting point of $\{\gamma_{Q,j}^{\varepsilon} \mid \varepsilon > 0\}$, where

$$\gamma_{Q,j}^{\varepsilon} = \text{Av}_{|t-t_j| \leq a/\varepsilon} \gamma_{Q}^{\varepsilon} \circ \tau_y \mid Q .$$

Therefore, by the lower semicontinuity of the specific relative entropy, we can conclude

$$s(\eta \mid \omega_{Q}^{\varepsilon}) \leq \lim_{\varepsilon \to 0} \frac{1}{(2a\varepsilon^{-1})^3} S(\gamma_{Q,j}^{\varepsilon} \mid \omega_{Q}^{\varepsilon}) \leq (2a)^{-3} \limsup_{\varepsilon} \varepsilon^3 S(\gamma_{Q,j}^{\varepsilon} \mid \omega_{Q}^{\varepsilon}) \leq C(2a)^{-3}$$

Consider any limiting point $\eta$ of $\{\gamma_{Q,j}^{\varepsilon} \mid \varepsilon > 0\}$. Since $\eta$ is translation invariant, we can decompose it into ergodic components (with respect to space translations) and there is a probability measure $\mu$ supported on ergodic states $\omega$ such that

$$\eta = \int \mu(d\omega) .$$

The key property of $\eta$ is the following lemma.
Lemma 6.4 Let $\eta$ be as above, and $X \in A_{\Lambda_0}$. Then there is $\Omega_M(\ell, \kappa)$ such that
\[
|\eta(I^*X^- I) - \eta \left[ I^* \left( \tilde{\sigma}^\kappa \tilde{X} \tilde{\sigma}^\kappa \right) (u^+_{\ell}) I + I^* (1 - \sigma^\kappa(u^+_{\ell})) X^- (1 - \sigma^\kappa(u^+_{\ell})) I \right] | \leq \Omega_M(\ell, \kappa) \tag{6.5}
\]
where $\tilde{\sigma}^\kappa = \sqrt{\sigma^\kappa (2 - \sigma^\kappa)}$.

Proof of Theorem 6.1 assuming Lemma 6.4.

Let $Z = I^*X^- I - \left[ I^* \left( \tilde{\sigma}^\kappa \tilde{X} \tilde{\sigma}^\kappa \right) (u^+_{\ell}) I + I^* (1 - \sigma^\kappa(u^+_{\ell})) X^- (1 - \sigma^\kappa(u^+_{\ell})) I \right]$

It is crucial that $Z$ is a bounded and local observable.

Theorem 6.1 now follows immediately from (6.2) and Lemma 6.4.

The rest of this section is devoted to prove Lemma 6.4. We shall drop the labels $\pm$ on $X$ and $u$ etc for the rest of this section.

6.1 General Properties of limiting states

We now prove a number of results for the ergodic components of the limit points $\eta$. At this point $\eta$ depends on a macroscopic space point $z$, and a macroscopic time $t_j$, and in principle also on the subsequence, but we will eventually see that $\eta$ is in fact independent of the subsequence.

Lemma 6.5 Let $\gamma_n, \gamma$ be normal states on a von Neumaan algebra $A$, and $\gamma_n \to \gamma$ weakly. Suppose that $A$ is a non-negative self-adjoint operator affiliated with $A$, such that $\gamma_n(A)$ is bounded by a constant $M$, uniformly in $n$. Then, $\lim_{n \to \infty} \gamma_n(A)$ exists and satisfies
\[
\gamma(A) \leq \lim_{n \to \infty} \gamma_n(A)
\]

Proof: Let $A = \int \lambda dE_\lambda$ be the spectral resolution of $A$. As $A$ is affiliated with $A$, the projections $P_k = \int_0^k dE_\lambda$ belong to $A$, and $\gamma_n(A) = \sup_k \gamma(A_k)$, where $A_k = AP_k$. The supremum is finite by the assumptions. Therefore,
\[
\lim_n \gamma_n(A) = \limsup_n \gamma_n(A_k) \geq \sup_k \lim_n \gamma_n(A_k) = \gamma(A)
\]

Recall that $h_\ell$ and $n_\ell$ (we omit the superscripts $+$) are the average of the local conservative quantities $H_\ell$ and $N_\ell$. Let $e$ and $\rho$ be determined by
\[
e = \lim_{\ell \to \infty} \omega(I^*h_\ell I), \quad \text{and} \quad \rho = \lim_{\ell \to \infty} \omega(I^*n_\ell I) \tag{6.6}
\]

Where necessary, we will indicate the dependence on $\omega$ by $e(\omega)$, and $\rho(\omega)$. In the following lemma we prove the existence and finiteness of these limits when the parameter $a$ is fixed.
Lemma 6.6 For $\mu -$ almost all states $\omega$, the limits $e$ and $\rho$ of (6.6) are finite. For the state $\eta = \int \omega \mu(d\omega)$, we have the following bounds

$$\limsup_{\ell \to \infty} \eta(I^* h_\ell I) \leq Ca^{-3}e_0, \quad \limsup_{\ell \to \infty} \eta(I^* n_\ell I) \leq Ca^{-3}\rho_0,$$

(6.7)

Proof: We have $\text{Av}_{\text{all boxes }} A_V \gamma_\ell^{Q,J}(h_\ell) = e_0$, where $e_0$ is the initial total energy. This is a direct consequence of the fact that the energy is conserved by the dynamics. Therefore, for each box $Q$, we have

$$\text{Av}_Q \gamma_\ell^{Q,J}(h_\ell) \leq Ca^{-3}e_0.$$

By Lemma 6.5 with $A = I^* h_\ell I$, it follows that

$$\eta(I^* h_\ell I) = \lim_{\varepsilon \to 0} \text{Av}_{\text{all boxes }} A_V \gamma_\ell^{Q,J}(h_\ell) \leq \sum_Q \text{Av}_Q \gamma_\ell^{Q,J}(h_\ell) \leq Ca^{-3}e_0,$$

which implies the bound for the energy (6.7). The proof for the particle density is the same. As $\eta = \int \omega \mu(d\omega)$, it then also follows that $e(\omega)$ and $\rho(\omega)$ are finite for $\mu -$ almost all $\omega$.

Let $A$ be a bounded observable in the local algebra $A_{\Lambda_0} \subset A_{\mathbb{R}^3}$. E.g., $A = \int dx dy f(x,y) a_x^+ a_y$, where $f(x,y) = 0$ unless $x,y \in \Lambda_0$. For concreteness, we assume that $\Lambda_0$ contains the origin. We will also use the notation $\text{Av}_\Lambda(A) = \frac{1}{|\Lambda|} \int_\Lambda dx \tau_x(A)$

Lemma 6.7 Suppose $\lim_\ell \eta_\ell = 0$. For every translation invariant ergodic state $\omega$ on $A_{\mathbb{R}^3}$, any bounded local observable $A$ and any continuous function $f$, we have the limit

$$\lim_\ell \omega(I^*_\ell f(\text{Av}_\Lambda A)I_\ell) = f(\omega(A))$$

(6.8)

Proof: The proof rests on the following property of $I$: for $A \in A_{\Lambda_0}$, we have

$$I^* \tau_x(A) I = \tau_x(A), \quad \text{if } \tau_x(\Lambda_0) \subset \Lambda^-_\ell$$

(6.9)

Denote by $\Lambda^\text{int}_\ell = \{ x \in \Lambda_\ell \mid \tau_x(\Lambda_0) \subset \Lambda^-_\ell \}$. Note that

$$\frac{|\Lambda_\ell \setminus \Lambda^\text{int}_\ell|}{|\Lambda_\ell|} \leq 2\eta_\ell + \frac{\text{diam } \Lambda_0}{\ell} =: \delta_\ell$$

(6.10)

and that $\lim_\ell \delta_\ell = 0$.

First, consider the function $f(x) = x$. Then, using the property (6.9), we have

$$\omega(I^*_\ell \text{Av}_\Lambda(A) I) = \omega(\frac{1}{|\Lambda_\ell|} \int_{\Lambda^\text{int}_\ell} \tau_x(A)) + \omega(I^* \left[ \frac{1}{|\Lambda_\ell|} \int_{\Lambda_\ell \setminus \Lambda^\text{int}_\ell} \tau_x(A) \right] I)$$

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Without loss of generality we may assume $\omega(A) = 0$. Using the definition of $\delta_\ell$ (6.10), and the isometry property of $I$, we find

$$\left| \omega(I^* A_v(A) I) \right| \leq (1 - \delta_\ell) \left| \frac{1}{|\Lambda^\text{int}_\ell|} \int_{\Lambda^\text{int}_\ell} dx \omega(\tau_x(A)) \right| + \delta_\ell \|A\|$$

The two terms in the RHS tend zero, the first due to the ergodicity of $\omega$, the second because $\delta_\ell \to 0$.

Next, we prove by induction the result for $f(x) = x^n$, for all $n \geq 1$. Suppose we have the result for $f(x) = x^{n-1}$, i.e.,

$$\lim_{\ell} \omega(I^*(A_v A)^{n-1} I) = 0$$

Then, by the same arguments as above, we have the estimate

$$\lim_{\ell} \left| \omega(I^*(A_v A)^n I) \right| \leq (1 - \delta_\ell)^n \left| \omega((A_v A)^n) \right| + C(1 - (1 - \delta_\ell)^n)$$

and the result follows by the ergodicity of $\omega$. For arbitrary continuous functions $f$, (6.8) can now be obtained by approximating $f$ by polynomials, uniformly on $[-\|A\|, \|A\|]$. This proves the Lemma.

Lemma 6.7 can trivially be extended as follows:

**Corollary 6.8** For any bounded local observables $X, Y, A \in A_{\Lambda_0}$, and continuous functions $f$ and $g$, we have that

$$\lim_{\ell} \left[ \omega(I^* X f(A_v A) Y g(A_v A) I) - f(\omega(A))g(\omega(A)) \omega(I^* XY I) \right] = 0$$

### 6.2 Extension to unbounded conserved quantities

Lemmas 6.7 and Corollary 6.8 are general properties of ergodic states applied to bounded observables. We now show how the one-phase region cut-off functions, which depend on unbounded but conserved quantities, can be included. This is a difficult step and we will have to use the special forms of the conserved quantities. The key technical estimate is contained in Lemma 6.10. We remark that a naive application of Schwarz' inequality to prove Lemma 6.9 would produce expressions with six or more creation or annihilation operators about which we have no control.

**Lemma 6.9** Let $\eta$ be any limiting point of $\{\gamma^{Q,J}_\varepsilon \mid \varepsilon > 0\}$, let $X$ be one of the components of $w$, and let $X_\ell$ the averaged version of $X$. Then the following limits vanish:

$$\lim_{\ell \to \infty} \eta(I^* B_\ell X_\ell [\sigma^h_1(\varepsilon)\sigma^h_2(\varepsilon)] I) = 0$$  \hspace{1cm} (6.11)

$$\lim_{\ell \to \infty} \eta(I^* B_\ell X_\ell [\sigma^h_2(n_\ell) - \sigma^h_2(\rho)] I) = 0$$  \hspace{1cm} (6.12)
for \( B_\ell = 1 \) or

\[
B_\ell = \sigma^\kappa(h_\ell)\sigma^\omega(n_\ell)
\]

Here \( e = e(\eta) = \int e(\omega) \, d\mu(\omega) \), and similarly for \( \rho \). In particular, we have

\[
\lim_{\ell \to \infty} \eta \left( I^* \left[ \sigma^\kappa(h_\ell, n_\ell)X_\ell\sigma^\kappa(h_\ell, n_\ell) \right] - \left[ \sigma^\kappa(e, \rho)X_\ell\sigma^\kappa(e, \rho) \right] \right) = 0 \quad (6.13)
\]

and the same result holds if \( \sigma^\kappa(h_\ell, n_\ell)X_\ell\sigma^\kappa(h_\ell, n_\ell) \) is replaced by \( \sigma^\kappa(h_\ell, n_\ell)X_\ell \) or by \( X_\ell\sigma^\kappa(h_\ell, n_\ell) \).

**Proof:** We start with the case \( B_\ell = 1 \).

Recall, \( \sigma^\kappa(e, \rho) = \sigma_1(e)\sigma_2(\rho) \). There exist bounded functions \( \tilde{\sigma}_1^\kappa \) and \( \tilde{\sigma}_2^\kappa \) such that \( \sigma_i^\kappa(x) - \sigma_i^\kappa(y) = (x - y)\tilde{\sigma}_i^\kappa(x, y), \) for \( i = 1, 2 \). Using these functions we can write

\[
\sigma_1^\kappa(h_\ell)\sigma_2^\kappa(n_\ell) - \sigma_1^\kappa(e)\sigma_2^\kappa(\rho) = (\sigma_1^\kappa(h_\ell) - \sigma_1^\kappa(e))\sigma_2^\kappa(n_\ell) + \sigma_1^\kappa(e)(\sigma_2^\kappa(n_\ell) - \sigma_2^\kappa(\rho))
\]

\[
= \sigma_2^\kappa(n_\ell)\tilde{\sigma}_1^\kappa(h_\ell, e)(h_\ell - e) + \sigma_1^\kappa(e)\tilde{\sigma}_2^\kappa(n_\ell, \rho)(n_\ell - \rho)
\]

Therefore, for a suitable bounded function \( f \), for any ergodic state \( \omega \), we can write

\[
\omega(I^*\sigma^\kappa(h_\ell, n_\ell)X_\ell\sigma^\kappa(h_\ell, n_\ell)) = \omega(I^*X\sigma_2^\kappa(n_\ell)f(h_\ell, e)(h_\ell - e)I)
\]

\[
= \omega\left( IX\sigma_2^\kappa(n_\ell)f(h_\ell)[h_\ell^B - e_B]I \right) + \omega\left( I^*X\sigma_2^\kappa(n_\ell)f(h_\ell)(h_\ell - h_\ell^B)I \right) \quad (6.14)
\]

where

\[
h_\ell^B = \frac{1}{|\Lambda_\ell|}H_\Lambda_\ell^B
\]

with \( H_\Lambda_\ell^B = H_\Lambda_\ell \text{Ind}(H_\Lambda_\ell \leq B|\Lambda_\ell|) \), so that \( \|H_\Lambda_\ell^B\| \leq B|\Lambda_\ell| \), and \( H_\Lambda_\ell^B \uparrow H_\Lambda_\ell \) as \( B \to \infty \). Introduce

\[
e_B(\omega) = \lim_{\ell \to \infty} \frac{1}{|\Lambda_\ell|} \omega(I^*H_\Lambda_\ell^B I)
\]

and use Schwarz’ inequality to obtain

\[
\omega\left( I^*X\sigma_2^\kappa(n_\ell)f(h_\ell)[h_\ell^B - e]I \right) \leq \left| \omega\left( I^*X\sigma_2^\kappa(n_\ell)f(h_\ell)[h_\ell^B - e_B]I \right) \right| + (e_B(\omega) - e(\omega)) \omega(I^*\sigma_2^\kappa(n_\ell)f(h_\ell)I)
\]

\[
\leq \delta \omega\left( I^*X(\sigma_2^\kappa(n_\ell))^2f(n_\ell)^2X^*I \right) + \delta^{-1} \omega\left( I^*[h_\ell^B - e_B]^2I \right) + (e_B(\omega) - e(\omega)) \omega(I^*\sigma_2^\kappa(n_\ell)f(h_\ell)I)
\]

Now, we integrate over \( \omega \) with respect to the measure \( \mu \), and take absolute values. The first term is uniformly bounded in \( \ell \). As \( h_\ell^B \) is bounded, the integrand of the second term vanishes for each \( \omega \), in the limit \( \ell \to \infty \), by Lemma 6.3. As the integrand is bounded uniformly in \( \omega \), the
integral vanishes as well. The third term we use the argument of (6.6) to show that it vanishes in the limit $B \to \infty$:

$$
\limsup_{B} \left| \eta(h_{B}^{B} - h_{\ell}) \right| \leq \limsup_{B} \lim_{\varepsilon} \text{Av}_{Q} \gamma_{\varepsilon}^{Q,j}(h_{B}^{B} - h_{\ell}) \leq Ca^{-3} \lim_{\varepsilon} \varepsilon^{d} \left| \gamma_{\varepsilon}(H^{B} - H) \right|
$$

The RHS is independent of $t$ and $\varepsilon$, and vanishes as $B \to \infty$.

For the second term of (6.14), we first apply Schwarz’ inequality:

$$
\left| \omega \left( I^{*} X \sigma_{2}^{n}(n_{\ell}) f(h_{\ell}) [h_{\ell} - h_{B}^{B}] I \right) \right| \\
\leq \delta \omega \left( I^{*} X \sigma_{2}^{n}(n_{\ell}) f(h_{\ell}) [h_{B}^{B} - h_{\ell}] f(h_{\ell}) \sigma_{2}^{n}(n_{\ell}) XI \right) + \delta^{-1} \omega \left( I^{*} [h_{B}^{B} - h_{\ell}] I \right)
$$

As before, the last term vanishes in the limit $B \to \infty$. Since $(h_{\ell} - h_{B}^{B}) \leq h_{\ell}$, the first term is bounded by

$$
\omega \left( I^{*} X \sigma_{2}^{n}(n_{\ell}) f(h_{\ell}) h_{\ell} f(h_{\ell}) \sigma_{2}^{n}(n_{\ell}) XI \right)
$$

As $f$ is bounded, we have that $f(h_{\ell}) h_{\ell} f(h_{\ell}) \leq C h_{\ell}$. Therefore, after integration over $\omega$, and with the use of Lemma 6.10, we obtain the bound

$$
\delta (C \eta (I^{*} (h_{\ell} + n_{\ell}) I) + C)
$$

which can be shown to be bounded in terms of the corresponding expectation in $\gamma_{t,\varepsilon}$, as before.

In conclusion, as $B$ and $\delta$ are arbitrary, we have proved (6.11) for $B_{\ell} = 1$. It is straightforward to adapt the argument to prove also (6.12) and the case $B_{\ell} = \sigma_{1}(h_{\ell}) \sigma_{2}(n_{\ell})$.

### 6.3 Basic Estimate

In the previous proof the following lemma was used. It provides a bound on the Hamiltonian sandwiched by bounded operators.

**Lemma 6.10** For $\mu$-almost all translation invariant ergodic states $\omega$, and $X_{\ell}$ the averaged version of one of the components of $w_{M}$ (which are all self-adjoint), we have

$$
\omega \left( I^{*} X_{\ell} \sigma_{2}^{n}(n_{\ell}) h_{\ell} \sigma_{2}^{n}(n_{\ell}) X_{\ell} I \right) \leq C \omega \left( I^{*} [h_{\ell} + n_{\ell}] I \right) + C
$$

(6.15)

where the constant is independent of $\varepsilon, \ell$ but may depend on $a, M$.

**Proof:** Since $X_{\ell}$ is particle number preserving, $X_{\ell}$ commutes with $n_{\ell}$. Therefore, we can rewrite the quantity we need to estimate as

$$
\omega \left( I^{*} X_{\ell} \sigma_{2}^{n}(n_{\ell}) h_{\ell} \sigma_{2}^{n}(n_{\ell}) X_{\ell} I \right) = \omega \left( I^{*} \sigma_{2}^{n}(n_{\ell}) X_{\ell} h_{\ell} X_{\ell} \sigma_{2}^{n}(n_{\ell}) I \right)
$$
\( h_\ell \) is the sum of two terms, a kinetic energy and a potential energy term, which we will treat separately.

First, we consider the kinetic energy term: \( \int_{\Lambda_\ell} \nabla a_x^+ \nabla a_x \), defined with periodic boundary conditions. We start from the identity

\[
X_\ell \nabla a_x^+ \nabla a_x X_\ell = \nabla a_x^+ X_\ell \nabla a_x + \nabla a_x^+ [\nabla a_x, X_\ell] + [X_\ell, \nabla a_x^+] \nabla a_x X_\ell \quad (6.16)
\]

Note that \( X_\ell \) is a linear combination of linear and quadratic terms in \( a_{x,M}^+ a_{\ell,M} \) (see (6.16)). Therefore, commutators of the form \( [a_{x,M}^+ a_{y,M}, a_{x,M}^+ \nabla a_x^+] \), etc., are bounded operators. More precisely, there is a constant \( C_M \) such that

\[
\|[X_\ell, a_x^+]\| \leq C_M \ell^{-3}, \quad \text{and} \quad \|[X_\ell, \nabla a_x^+]\| \leq C_M \ell^{-3}. \quad (6.17)
\]

These bounds will be used repeatedly in the following estimates. E.g., applied to the first term of (6.16), they yield

\[
\nabla a_x^+ XX \nabla a_x \leq C_M \nabla a_x^+ \nabla a_x .
\]

To bound the second and third term we first apply Schwarz’ inequality:

\[
\begin{align*}
\omega & \left( I^* \sigma_2^\alpha(n_\ell) [X_\ell, \nabla a_x^+] \nabla a_x X_\ell \sigma_2^\alpha(n_\ell) I \right) \\
& \leq \delta \omega \left( I^* \sigma_2^\alpha(n_\ell) [X_\ell, \nabla a_x^+] [X_\ell, \nabla a_x^+] \sigma_2^\alpha(n_\ell) I \right) + \delta^{-1} \omega \left( I^* \sigma_2^\alpha(n_\ell) X_\ell \nabla a_x^+ \nabla a_x X_\ell \sigma_2^\alpha(n_\ell) I \right)
\end{align*}
\]

The first term of the RHS is bounded and the last term can be re-absorbed into the quantity we started out the estimate. Thus, for the kinetic energy term and any of the \( X_\ell \), we have an estimate of the form

\[
\omega \left( I^* \sigma_2^\alpha(n_\ell) X_\ell h_{0,\ell} X_\ell \sigma_2^\alpha(n_\ell) I \right) \leq C \omega \left( I^* \sigma_2^\alpha(n_\ell) h_{0,\ell} \sigma_2^\alpha(n_\ell) I \right) + C .
\]

Similarly, for the potential energy we start from the identity

\[
\begin{align*}
X_\ell a_x^+ a_y^+ a_y a_x X_\ell \\
= a_x^+ a_y^+ X_\ell a_y a_x + a_x^+ a_y^+ X_\ell [a_y a_x, X_\ell] + [X_\ell, a_x^+ a_y^+] a_y a_x X_\ell
\end{align*}
\]

and the bound

\[
a_x^+ a_y^+ X_\ell X_\ell a_y a_x \leq C a_x^+ a_y^+ a_y a_x .
\]

For the commutator terms we have

\[
a_x^+ a_y^+ X_\ell [a_y a_x, X_\ell] = a_x^+ a_y^+ X_\ell a_y [a_x, X_\ell] - a_x^+ a_y^+ X_\ell [a_y, X_\ell] a_x
\]

which can be estimated using Schwarz’ inequality:

\[
2 \Re \omega \left( I^* \sigma_2^\alpha(n_\ell) a_x^+ a_y^+ X_\ell a_y [a_x, X_\ell] \sigma_2^\alpha(n_\ell) I \right) \\
\leq \omega \left( I^* \sigma_2^\alpha(n_\ell) a_x^+ a_y^+ X_\ell^2 a_y a_x \sigma_2^\alpha(n_\ell) I \right) + \omega \left( I^* \sigma_2^\alpha(n_\ell) [a_x, X_\ell] a_y^+ a_y [a_x, X_\ell] \sigma_2^\alpha(n_\ell) I \right)
\]

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We use $a_x^+a_y^+X^2a_ya_x \leq C_M a_x^+a_y^+a_ya_x$ for the first term. The second term we use the identity

$$[a_x, X]^*a_y^+a_y[a_x, X] = a_y^+[a_x, X]^*a_y^+[a_x, X]a_y + a_y^+[a_x, X]^*[a_y, [a_x, X]] + [[a_x, X]^*, a_y^+]a_y[a_x, X]$$

The first term of the RHS is bounded by $C_M a_y^+a_y$. The other two terms can be bounded by $C_M a_y^+a_y + C_M$ by repeating the same procedure once more (first apply Schwarz’ inequality, then use (6.17)). We conclude that

$$\int_{\Lambda_\ell}dx \int_{\Lambda_\ell}dyW(x-y)\omega \left( I^*\sigma_2^\tau(n_\ell)x\sigma_2^\tau(n_\ell)I \right)$$

$$\leq C \int_{\Lambda_\ell}dx \int_{\Lambda_\ell}dyW(x-y)\left\{ \omega \left( I^*\sigma_2^\tau(n_\ell)a_x^+a_y^+a_ya_x\sigma_2^\tau(n_\ell)I \right) + C \right\}$$

Now from the super-stability estimate, we have

$$C \int_{\Lambda_\ell}dx \int_{\Lambda_\ell}dyW(x-y)a_x^+a_y^+a_ya_x \leq C \int_{\Lambda_\ell}dx \int_{\Lambda_\ell}dyW(x-y)a_x^+a_y^+a_ya_x + N_\ell$$

Thus,

$$\ell^{-3} \int_{\Lambda_\ell}dx \int_{\Lambda_\ell}dyW(x-y)\int_{\Lambda_\ell}dy\omega \left( I^*\sigma_2^\tau(n_\ell)x\sigma_2^\tau(n_\ell)I \right)$$

$$\leq C\ell^{-3} \int_{\Lambda_\ell}dx \int_{\Lambda_\ell}dyW(x-y)\int_{\Lambda_\ell}dy\omega \left( I^*\sigma_2^\tau(n_\ell)a_x^+a_y^+a_ya_x\sigma_2^\tau(n_\ell)I \right) + C \int_{\Lambda_\ell}dx \int_{\Lambda_\ell}dyW(x-y)\int_{\Lambda_\ell}dy\omega \left( I^*\sigma_2^\tau(n_\ell)n_\ell\sigma_2^\tau(n_\ell)I \right)$$

The last term is bounded. Combining these estimates, we have

$$\omega \left( I^*\sigma_2^\tau(n_\ell)x\sigma_2^\tau(n_\ell)I \right) \leq C\omega \left( I^*\sigma_2^\tau(n_\ell)h_\ell\sigma_2^\tau(n_\ell)I \right) + C$$

Since $h_\ell \leq h_\ell + Cn_\ell$, $h_\ell + Cn_\ell \geq 0$ and $[h_\ell, n_\ell] = 0$, we have

$$\omega \left( I^*\sigma_2^\tau(n_\ell)h_\ell\sigma_2^\tau(n_\ell)I \right) \leq C\omega \left( I^*[h_\ell + Cn_\ell]^{1/2}\sigma_2^\tau(n_\ell)\sigma_2^\tau(n_\ell)[h_\ell + Cn_\ell]^{1/2}I \right)$$

$$\leq C\omega \left( I^*[h_\ell + n_\ell]I \right) + C$$

We can prove that $\omega \left( I^*[h_\ell + n_\ell]I \right)$ is bounded by using Lemma 6.6.
6.4 Proof of main ergodic lemma

We can now prove Lemma 6.4.

**Proof:** Recall the decomposition of $\eta$ into its spatially ergodic components:

$$\eta = \int \mu(\omega)d\omega$$

Since $X$ is bounded, by Lemma 6.9 there is $\Omega_{\kappa, X}(\ell)$ such that

$$\omega(I^*(1 - \sigma^\kappa)X_\ell(1 - \sigma^\kappa)) = (1 - \sigma^\kappa(\omega))^2 \omega(I^*X_\ell I) + \sigma^\kappa(\omega)(2 - \sigma^\kappa(\omega))\omega(I^*\hat{X} I) + \Omega_{\kappa, X}(\ell)$$

where $\sigma^\kappa(\omega) = \sigma^\kappa(\lim_\ell \omega(h_\ell), \lim_\ell \omega(n_\ell))$. Therefore,

$$\eta(I^*X_\ell I) - \eta(I^*(1 - \sigma^\kappa)X_\ell(1 - \sigma^\kappa)) = \eta(I^*\sqrt{\sigma^\kappa(2 - \sigma^\kappa)} X \sqrt{\sigma^\kappa(2 - \sigma^\kappa)} I)$$

$$= \int \mu(\omega)[1 - \sigma^\kappa(\omega)]^2 \omega(I^*X_\ell I) + \int \mu(\omega)\sigma^\kappa(\omega)(2 - \sigma^\kappa(\omega))\omega(I^*\hat{X} I) + \Omega_{\kappa, \ell}(X)$$

As $1 - (1 - x)^2 - x(2 - x) = 0$, the first term vanishes identically. The middle term vanishes by the hypothesis that the only ergodic states of finite specific relative entropy in the one-phase region are the Gibbs states. The support of the function $\sigma^\kappa(\omega)(2 - \sigma^\kappa(\omega))$ is such that only these Gibbs states contribute to the integral. The integrand vanishes by the definition of $\hat{X}$ (4.1), since we have $\omega(X) = \hat{X}(\lim_\ell \omega(u_\ell))$. This concludes Lemma 6.4.

7 Relative entropy estimate

We now summarize the estimates on the relative entropy we have so far. For any $0 \leq T \leq T_0$, we write

$$s(\gamma_t | \omega_t^\varepsilon)|_{t=\varepsilon^{-1}T} = \varepsilon^{-1} \int_{0 \leq t \leq \varepsilon^{-1}T} \frac{d}{dt}s(\gamma_t | \omega_t^\varepsilon)$$

We compute the rate of change of entropy by (3.4), (3.10) and (3.16) to have

$$s(\gamma_t | \omega_t^\varepsilon)|_{t=\varepsilon^{-1}T} = T \text{ Av}_{0 \leq t \leq \varepsilon^{-1}T} \{ \text{Tr} \gamma_t G(\lambda, a_\ell^+, a_\ell^-) - \varepsilon^2 \partial_t \log c_\varepsilon(t) \} + \varepsilon^1_M$$

where $G$ is defined in (3.10) and $\varepsilon^1_M \leq C e^{-cM^2}$ (3.16).

Recall the meaning of the various length scales and cut-off parameters: $\varepsilon$ is the ratio of the macroscopic to microscopic length scale, $M$ is the high-momentum cut-off, $\ell$ is the length scale in the isometry $I$ employed to define commuting local versions of the conserved quantities, $a$ is
a length scale for averaging needed to make use of local ergodicity, $\kappa$ is the length scale used to smooth the characteristic function of the one-phase region, and $\delta$ is a small parameter used in applications of the entropy inequality.

Recall the convention

$$\mathbf{A} \cdot \mathbf{B} = \sum_{j=0}^{3} \sum_{\mu=0}^{3} A_j^\mu B_j^\mu - \sum_{j=0}^{3} A_j^4 B_j^4.$$ 

We now apply Theorem 6.1 to estimate $\text{Tr} \, \gamma_t G(\lambda_\varepsilon, a^+_M, a_M)$ by

$$\text{Av} \, \text{Tr} \, \gamma_t G(\lambda_\varepsilon, a^+_M, a_M) \leq T_1 + T_2$$

where $T_1$ and $T_2$ are defined as follows:

$$T_1 = -\text{Av} \, \gamma_t \left( \sum A_j (\varepsilon t, \varepsilon x) \cdot \nabla_j \lambda (\varepsilon t, \varepsilon x) \right)$$

$$T_2 = -\text{Av} \, \gamma_t \left( \sum A_j (\varepsilon t, \varepsilon x) \cdot \nabla_j \lambda (\varepsilon t, \varepsilon x) \right)$$

We need to compute $\hat{w}$ which we state as the following lemma.

**Lemma 7.1** We have the following identities

$$\hat{w}_j^\mu = A_j^\mu, \quad j = 0, \cdots, 3, \quad \mu = 0, \cdots, 4.$$ 

where the functions $A_j^\mu$ are given in (2.15).

These relations follow directly from the definition of the Gibbs states, the expressions for $A_j^\mu$ in (2.15), the calculation of the currents $w_j^\mu$ in Section 9 and the virial theorem proved in Section 10.

By construction of the $\omega_\varepsilon^t$, the time derivative of $\log c_\varepsilon(t)$ can be expressed as

$$\frac{d}{dt} \varepsilon^2 \log c_\varepsilon(t) = \varepsilon^2 \int dx (\partial_t \lambda \cdot \mathbf{q})(\varepsilon t, \varepsilon x)$$

where $\mathbf{q}$ is the solution of the Euler equations that we are considering. Recall the following identity about the Euler equations:

$$\int \sum_{j=1}^{3} A_j (\mathbf{q}(X)) \cdot \nabla_j \lambda (\mathbf{q}(X)) dX = 0.$$ 

Recall also $A_0(\mathbf{q}) = \mathbf{q}$. We can rewrite

$$\frac{d}{dt} \varepsilon^2 \log c_\varepsilon(t) = \varepsilon^2 \int dx (\partial_t \lambda \cdot \mathbf{q})(\varepsilon t, \varepsilon x) = \varepsilon^2 \int dx (\nabla \lambda \cdot \mathbf{A})(\varepsilon t, \varepsilon x)$$

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Together with Lemma 7.1, we have

\[
T_1 - \frac{d}{dt} \varepsilon^2 \log c_\varepsilon(t) = - \text{Av}_x \gamma_t \text{Av}_x \left[ \nabla \lambda(\varepsilon t, \varepsilon x) \cdot \left\{ I^* (\tilde{\sigma}_M \tilde{\sigma}_x) u_{x,\ell} \right\} - \tilde{w}(q(\varepsilon t, \varepsilon x)) \right]
\]

Denote

\[
\| \nabla \lambda \|_\infty = \| \nabla \lambda \|_\infty + \| \partial_t \lambda \|_\infty
\]

and introduce the functions

\[
\Gamma^1_M (\lambda, u) = \nabla \lambda \cdot \left[ (\sigma^t \tilde{w}_M \sigma^x) (u) - \tilde{w}(q) \right],
\]

\[
\Gamma^2 (\lambda, u) = \| \nabla \lambda \|_\infty \left[ (1 - \sigma^x)(u)(h + n)(1 - \sigma^x)(u) \right],
\]

where \(h, n\) are the energy and density components to \(u\) and \(q\) is the dual variable of \(\lambda\) defined in (2.11).

We can bound \(w_M\) in \(T_2\) by the cutoff Lemma 5.1. Thus we have

\[
T_1 + T_2 - \frac{d}{dt} \varepsilon^2 \log c_\varepsilon(t) \leq \text{Av}_x \gamma_t \text{Av}_x \left[ I^* \{ -\Gamma^1_M + M \Gamma^2 \} (\lambda(\varepsilon t, \varepsilon x), u_{x,\ell}) \right]
\]

Therefore, we have

\[
s(\gamma_t | \omega^T) \big|_{t=\varepsilon^{-1}T} \leq T \text{Av}_x \gamma_t \text{Av}_x \left[ I^* \{ -\Gamma^1_M + M \Gamma^2 \} (\lambda(\varepsilon t, \varepsilon x), u_{x,\ell}) \right] + \mathcal{E}_M^1 \quad (7.1)
\]

where \(\mathcal{E}_M^1 \leq C e^{-cM^2} \) \(\mathcal{E}_M^1\).

### 7.1 Reduction to large deviation

Recall the standard thermodynamics pressure is defined by

\[
\psi(\lambda) = \lim_{\ell \to \infty} \ell^{-3} \log Z_{\ell,\lambda}
\]

Define the entropy

\[
s(q') = \sup_{\lambda} [\lambda q' - \psi(\lambda)]
\]

and the rate function (notice we also use \(I\) for the embedding into the standard torus \(\Lambda_{\varepsilon^{-1}}\))

\[
I(q', \lambda) = s(q') + \psi(\lambda) - \lambda \cdot q'
\]

The rate function has the following property

\[
I(q', \lambda) \geq 0, \quad I(q, \lambda) = 0
\]
where \( q = \partial \psi(\lambda) / \partial \lambda \). Furthermore, if the Gibbs state with chemical potential \( \lambda \) is in the one phase region, we have

\[
\text{Hess } I(q, \lambda) \geq cI
\]

for some \( c > 0 \).

The main large deviation estimate we shall use is given in the following lemma. This lemma will be proved in Section 8.3.

**Lemma 7.2** Suppose \( \lambda \) is a bounded smooth function so the Gibbs state with chemical potential \( \lambda(x) \) is in the one phase region for all \( x \). For any bound smooth function \( G \) that satisfies the condition

\[
|G(\lambda, q)| \leq C(e + \rho)
\]

(7.2)

where \( e \) is the energy and \( \rho \) is the density. Then there is a \( \delta_0 > 0 \) depending only on \( C \) and a convex functional \( \tilde{I} \) such that for all \( 0 < \delta \leq \delta_0 \)

\[
\tilde{I}(q', \lambda) = I(q', \lambda)
\]

in a small neighborhood of \( q = \partial \psi(\lambda) / \partial \lambda \) and

\[
\lim_{\ell \to \infty} \lim_{\varepsilon \to 0} \gamma \operatorname{Av}_x \left( I^* G(\lambda(\varepsilon x), u_{x,\ell}^+) I \right)
\]

\[
\leq \int dX \sup_{q'(X)} \left[ G(\lambda(X), q'(X)) - \delta^{-1} \tilde{I}(q'(X), \lambda(X)) \right] + \delta^{-1} \lim_{\varepsilon \to 0} s(\gamma | \omega_{\lambda}^\varepsilon)
\]

Here the sup is over all functions \( q'(X) \).

### 7.2 Conclusion of the relative entropy estimate and proof of the main theorem

We now apply Lemma 7.2 to estimate (3.17). Since we need the bound (7.2), we set \( G = M^{-1}\{-\Gamma^1_M + M\Gamma^2\} \). Thus we have for any \( \delta \leq \delta_0 \)

\[
- \operatorname{Av}_{0 \leq t \leq \varepsilon^{-1} T} \gamma_t \operatorname{Av}_x \left( I^* \{-\Gamma^1_M + M\Gamma^2\}(\lambda(\varepsilon t, \varepsilon x), u_{x,\ell}^+) I \right) \leq R_6 + \delta^{-1} M s(\gamma_t | \omega_{\lambda}^\varepsilon)
\]

where

\[
R_6 = \int dx \sup_{q'} \left( \{-\Gamma^1_M + M\Gamma^2\}(\lambda(x)), q'(x) \right) - \delta^{-1} M \tilde{I}(q'(x), \lambda(x))
\]

(7.3)

where \( \tilde{I} \) is related to the rate function defined in Lemma 7.2. We now estimate the dependence of \( \Gamma^1_M \) on \( M \).
Lemma 7.3 There is a constant $c > 0$ such that
\[ \Gamma_{M}^{1}(\lambda, q') = \Gamma^{1}(\lambda, q') + e^{-cM^{2}} \]
where
\[ \Gamma^{1}(\lambda, q') = \nabla_{\lambda} \cdot \left[ (\sigma^{\kappa_{i}}\hat{\omega}\sigma^{\kappa_{i}})(q') - \hat{\omega}(q) \right] \]

This lemma can be proved following the idea of the proofs of Lemmas 3.2 and 3.3. It is part of our assumptions that the Gibbs states satisfy the cutoff assumptions.

Now, we can conclude the relative entropy estimate and the proof of Theorem 2.1.

**Proof of Theorem 2.1.**

Recall $q = \partial q(\lambda) / \partial \lambda$. Clearly, $\Gamma^{1}(q(\varepsilon t, \varepsilon x), \lambda(\varepsilon t, \varepsilon x)) = 0$. The first derivative
\[ \frac{\partial \Gamma^{1}(\lambda(\varepsilon t, \varepsilon x), q(\varepsilon t, \varepsilon x))}{\partial q(\varepsilon t, \varepsilon x)} = 0 \]
is equivalent to the Euler equation as checked in [16]. Recall $\Gamma^{2}(\lambda(X), q'(X))$ is nonzero only when $q'(X)$ is away from $q(X)$. Thus we have for $|q'| \leq C$
\[ \{-\Gamma^{1} + M\Gamma^{2}\}(\lambda(X), q'(X)) \leq CM(q'(X) - q(X))^{2} \]
Furthermore, from the definition of $\Gamma_{j}$ we have for all $q'$
\[ \{-\Gamma^{1} + M\Gamma^{2}\}(\lambda(X), q'(X)) \leq CM\left(|q'(X)| + 1\right) \]
Since $\tilde{I}(q'(X), \lambda(X)) \geq 0$ and $\tilde{I}(q'(X), \lambda(X)) = 0$ only when $q'(X) = q(X)$, for $\delta$ small enough we have
\[ \sup_{q'} \left\{ \{-\Gamma^{1} + M\Gamma^{2}\}(q'(X), \lambda(X)) - \delta^{-1}M\tilde{I}(q'(X), \lambda(X)) \right\} \leq e^{-cM^{2}} \]
We thus have
\[ s(\gamma_{t}|\omega_{t}^{\varepsilon}) \leq \delta^{-1}M\varepsilon \int_{0}^{t} \left[ s(\gamma_{t'}|\omega_{t'}^{\varepsilon}) + Ce^{-cM^{2}} + \Omega_{M}(\ell, \kappa) + \Omega(\varepsilon, a) + CMe^{-c\ell^{3}} \right] dt' \]
By integrating this inequality (i.e., using Gronwall’s inequality), and using the fact that $\varepsilon t \leq T_{0}$, we arrive at the bound
\[ s(\gamma_{t}|\omega_{t}^{\varepsilon}) \leq \delta^{-1}MT_{0}e^{\delta^{-1}MT_{0}} \left[ Ce^{-cM^{2}} + \Omega_{M}(\ell, \kappa) + \Omega(\varepsilon, a) + CMe^{-c\ell^{3}} \right] . \]
Taking the limits $\lim_{\kappa \to 0} \lim_{a \to 0} \lim_{\ell \to \infty} \lim_{\varepsilon \to 0}$, we get the inequality
\[ \lim_{\varepsilon \to 0} s(\gamma_{t}|\omega_{t}^{\varepsilon}) \leq C\delta^{-1}MT_{0}e^{\delta^{-1}MT_{0}-cM^{2}} \]
We can now let $M \to \infty$ and conclude the proof of Theorem 2.1. We emphasize that we need the error term stemming from the high-momentum cutoff to be smaller than $e^{-cM}$ for any $C > 0$ in order to have our results hold for $t \leq CT_{0}$ for arbitrary $T_{0}$. This is guaranteed by the Maxwellian bound in the cutoff assumption II.1, expressed by (2.19).
8 Thermodynamics and large deviation

We now prepare the way for the proof of Lemma 7.2. Our approach to large deviations for quantum Gibbs states and local Gibbs states is quite different from the explicit analysis in [11] for the ideal gases. We first introduce the following local Gibbs state with independent subcubes.

8.1 Local Gibbs state with independent subcubes

Divide the torus \( \Lambda = \Lambda_{\epsilon-1} \) into unions of non-overlapping cubes of size \( \ell \). To fix the grid, we assume that the origin is the center of one small cube. Denote a typical cube by \( \alpha \). Recall the configuration space \( S(\alpha^+) \) and define the configuration space

\[
S(\Lambda^+) = \otimes_{\alpha \in \ell \mathbb{Z}^3 \cap \Lambda} S(\alpha^+)
\]

An element in this configuration space can be denoted by

\[
x^\sharp = (\cdots, (\alpha_j, x_j), \cdots), \quad x_j \in \alpha_j^+
\]

The (Fock) function space \( \Gamma(\Lambda^+) \) is the \( L^2 \) space of antisymmetric functions on \( S(\Lambda^+) \). Notice that \( S(\Lambda^+) \neq S(\Lambda^+) \) and \( \Gamma(\Lambda^+) \neq \Gamma(\Lambda^+) \).

Recall \( \Lambda = \Lambda_{\epsilon-1} \) is a torus. Define \( I_\Lambda^\ell \) from \( \Gamma(\Lambda) \) to \( \Gamma(\Lambda^+) \) (cf [4]) by

\[
(I_\Lambda^\ell \psi)(x^\sharp) = \left[ \prod_j \chi_{\alpha_j}(x_j) \right] \psi(x).
\]

The crucial fact is that \( I \) is an isometry.

**Lemma 8.1** \( I \) is an isometric embedding, i.e.,

\[
\| \phi \| = \| I \phi \|
\]

**Proof:** Recall from the construction of \( \chi \) the relation (4.2) implies that

\[
\sum_{\alpha \in \ell \mathbb{Z}^3} \chi^2_\alpha(x) = 1
\]

(8.1)

We can prove the isometry by the following identity: For any two wave functions \( f \) and \( g \), we have

\[
(I f, I g) = \sum_{\alpha} \prod_j \left[ \int_{\alpha_j} dx_j |\chi_{\alpha_j}(x_j)|^2 \right] \tilde{f}(x)g(x) = (f, g)
\]

For isometric embeddings, we have the following useful bound.
Lemma 8.2  Suppose $I : \mathcal{H}_1 \to \mathcal{H}_2$ is an isometric embedding. Then

$$\text{Tr } \mathcal{H}_1 e^{IAI} \leq \text{Tr } \mathcal{H}_2 e^A$$

Proof: We can assume that $\mathcal{H}_1$ is just a subspace of $\mathcal{H}_2$ and $I$ is the natural embedding. Let $\phi_j$ be the orthonormal eigenvectors of $A$ in $\mathcal{H}_1$. The claim follows from the following Peierls’ inequality: Suppose $\phi_j$ are orthonormal. Then

$$\sum_j e^{(\phi_j, A\phi_j)} \leq \text{Tr } e^A$$

The following Lemma shows that $I^*XI = I^*XI$ for a suitable class of observables.

Proposition 8.3  Suppose $X$ is the observable

$$X = \int_{\alpha^+} dx_1 \cdots dx_k dy_1 \cdots dy_k f(x_1, \ldots, x_k; y_1, \ldots, y_k) a_{x_1, \alpha}^+ \cdots a_{x_k, \alpha}^+ a_{y_k, \alpha} \cdots a_{y_1, \alpha} ,$$

Then we have $I^*XI = I^*XI$.

Proof: The following identity is a direct consequence of the definition of $I$: For $n \geq k$,

$$a_{y_k, \alpha} \cdots a_{y_1, \alpha} I\psi((z_1, \alpha_1), \ldots, (z_n, \alpha_n)) = a_{y_k, \alpha} \cdots a_{y_1, \alpha} \chi_{\alpha_1}(z_1) \cdots \chi_{\alpha_n}(z_n) \psi(z_1, \ldots, z_n)$$

$$= \delta_{\alpha_1, \alpha} \cdots \delta_{\alpha_k, \alpha} \chi_{\alpha_1}(y_1) \cdots \chi_{\alpha_k}(y_k) \chi_{\alpha_k+1}(z_{k+1}) \cdots \chi_{\alpha_n}(z_n)$$

$$\times \psi(y_1, \ldots, y_k, z_{k+1}, \ldots, z_n) .$$

It follows that, for any $\phi, \psi \in L^2(\Lambda^{\times n})$, $n \geq k$,

$$\langle \phi, I^*XI\psi \rangle = \int_{\alpha^+} dx_1 \cdots dx_k dy_1 \cdots dy_k f(x_1, \ldots, x_k; y_1, \ldots, y_k)$$

$$\times (a_{x_1, \alpha} \cdots a_{x_1, \alpha} I\phi, a_{y_1, \alpha} \cdots a_{y_1, \alpha} I\psi)$$

$$= \sum_{\alpha_1, \ldots, \alpha_n} \int dz_{k+1} \cdots dz_n \int dx_1 \cdots dx_k dy_1 \cdots dy_k \delta_{\alpha_1, \alpha} \cdots \delta_{\alpha_k, \alpha}$$

$$\times \chi_{\alpha_{k+1}}(z_{k+1})^2 \cdots \chi_{\alpha_n}(z_n)^2 \chi_\alpha(x_1) \cdots \chi_\alpha(x_k) \chi_\alpha(y_1) \cdots \chi_\alpha(y_k)$$

$$\times f(x_1, \ldots, x_k; y_1, \ldots, y_k) \phi(x_1, \ldots, x_k, z_{k+1}, \ldots, z_n) \psi(y_1, \ldots, y_k, z_{k+1}, \ldots, z_n)$$

The sum over $\alpha_1, \ldots, \alpha_n$ can be carried out using the Kronecker delta’s and \ref{eq:8.1}. As $n \geq k$, $\phi$, and $\psi$ are arbitrary, we have $I^*XI = I^*XI$ by the formula of $I$ in \ref{eq:4.14}.

We now construct a “special local Gibbs state”. Recall that $u_{\alpha^+}$ is defined in \ref{eq:4.8} and its component commute. For a smooth $\lambda$, let $\tilde{\omega}_\lambda^{\xi, \ell}$ be the state

$$\tilde{\omega}_\lambda^{\xi, \ell} = \text{Tr } \frac{1}{\tilde{c}_{\xi, \ell}(\lambda)} \exp \left[ \frac{\epsilon^{-3}}{\alpha} A_\lambda(\epsilon \alpha) \cdot I^*u_{\alpha^+} I \right]$$

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where the average of $\alpha$ is over $\alpha \in \ell \mathbb{Z}^3 \cap \Lambda$ and $\tilde{c}_{\varepsilon,\ell}(\lambda)$ is the partition function defined by

$$\tilde{c}_{\varepsilon,\ell}(\lambda) = \text{Tr} \exp \left[ \varepsilon^{-3} \sum_{\alpha} \lambda(\varepsilon \alpha) \cdot I^* u_{\alpha+} I \right]$$

Here the trace is over $\Gamma(\Lambda_{\varepsilon^{-1}})$.

Assume for the moment we can drop the $I$ and the small cubes are independent. Then $\tilde{c}_{\varepsilon,\ell}(\lambda)$ can be computed easily. The following Lemma asserts that this is essentially correct. Recall the partition function defined in (2.10).

Lemma 8.4

$$\lim_{\ell \to \infty} \lim_{\varepsilon \to 0} \left( \varepsilon^3 \log c_{\varepsilon}(\lambda) - \varepsilon^3 \log Z(\varepsilon,\alpha) \right) = \lim_{\ell \to \infty} \lim_{\varepsilon \to 0} \left( \varepsilon^3 \log \tilde{c}_{\varepsilon,\ell}(\lambda) - \varepsilon^3 \log Z(\varepsilon,\alpha) \right) = 0$$

(8.2)

Proof: Upper bound: We first state an upper bound for $\varepsilon^3 \log \tilde{c}_{\varepsilon,\ell}$. Notice that it is an inequality with no limits or other constants.

$$\varepsilon^3 \log \tilde{c}_{\varepsilon,\ell}(\lambda) \leq \sum_{\alpha} \lambda(\varepsilon \alpha) \cdot I^* u_{\alpha+} I$$

From Lemma 8.3, we have

$$\log \tilde{c}_{\varepsilon,\ell}(\lambda) = \text{Tr} \exp \left[ \varepsilon^{-3} \sum_{\alpha} \lambda(\varepsilon \alpha) \cdot I^* u_{\alpha+} I \right]$$

If we can neglect $I^*$ and $I$, then (8.3) follows from the fact that different cubes are considered independent. Lemma 8.2 shows that we can remove $I^*$ and $I$ to have a lower bound. This concludes the proof of (8.3).

Lower bound: Since $\lambda$ is fixed, we shall drop it in the subscript. Consider the entropy

$$0 \leq s(\omega^\varepsilon | \tilde{\omega}^\varepsilon) = R_2 + \varepsilon^3 \log \tilde{c}_{\varepsilon,\ell} - \varepsilon^3 \log c_{\varepsilon}$$

where

$$R_2 = \int d\varepsilon x \lambda(\varepsilon x) \cdot u_{\alpha} - \sum_{\alpha} \lambda(\varepsilon \alpha) \cdot I^* u_{\alpha+} I$$

From Lemma 4.1 we have

$$\lim_{\ell \to \infty} \lim_{\varepsilon \to 0} R_2 = 0$$

We have thus proved that

$$\lim_{\ell \to \infty} \lim_{\varepsilon \to 0} \left[ \varepsilon^3 \log \tilde{c}_{\varepsilon,\ell} - \varepsilon^3 \log c_{\varepsilon} \right] \geq 0$$

(8.4)

To conclude Lemma 8.4, we now obtain a lower bound on $c_{\varepsilon}$. This is the standard procedure on the thermodynamics and we shall give only sketch. We first divide the cube of size $\varepsilon^{-1}$ into cubes
of size $\ell(1 - \sqrt{\ell})$ with corridors of size $2\sqrt{\ell}$. Now we impose the Dirichlet boundary conditions on the boundary to obtain an upper bound on the kinetic energy. The partition function is bounded below by restricting the configurations so that there is no particle on the corridors. Now there is no interactions between different cubes and we obtain a lower bound of $c_\varepsilon$ in terms of average over Dirichlet boundary conditioned partition functions in cubes of size $\ell(1 - \sqrt{\ell})$. Since we can take $\eta \to 0$ after $\ell \to \infty$ and partition functions is independent of boundary conditions, we have thus proved that

$$\lim_{\ell \to \infty} \lim_{\varepsilon \to 0} \varepsilon^3 \log c_\varepsilon \geq A \varepsilon^{\alpha} \ell^{-3} \log Z_\ell(\lambda(\varepsilon\alpha))$$

This concludes the Lemma.

8.2 Large deviation for commuting variables

Recall that $u_{\alpha^+}$ is defined in (4.8) and its component commute. We shall take $x = 0$ and denote $u_{\alpha^+}$ by $u_\ell$. The following Lemma is a standard application of large deviation theory (or thermodynamics) to commuting variables.

**Lemma 8.5** Suppose $\lambda$ is a fixed constant so the Gibbs state with chemical potential $\lambda$ is in the one phase region. For any bound smooth function $G$ satisfies that

$$|G(\lambda, q)| \leq C(e + \rho)$$

where $e$ is the energy and $n$ is the density. Let $\omega_{\lambda, \ell}$ be the finite volume Gibbs state defined by

$$\omega_{\lambda, \ell}(X) = \frac{1}{Z_\ell(\lambda)} \text{Tr} \left[ \langle \lambda, u_\ell \rangle_{\Lambda_\ell} \right] X$$

with periodic boundary condition. Since the components of $u_\ell$ are commuting, we have

$$\frac{1}{Z_\ell(\lambda)} \text{Tr} \left[ \langle \lambda, u_\ell \rangle_{\Lambda_\ell} + \delta \ell^3 G(\lambda, u_\ell) \right] = \omega_{\lambda, \ell} \left( \exp \left[ \delta \ell^3 G(\lambda, u_\ell) \right] \right)$$

Then there is a $\delta_0 > 0$ depending only on $C$ and a convex functional $\tilde{I}$ such that for all $0 \leq \delta \leq \delta_0$

$$\tilde{I}(q', \lambda) = I(q', \lambda)$$

in a small neighborhood of $q = \partial \psi(\lambda)/\partial \lambda$ and

$$|\Lambda_\ell|^{-1} \log \omega_{\lambda, \ell} \left( \exp \left[ \delta \ell^3 G(\lambda, u_\ell) \right] \right) \leq \sup_{q'} \left[ \delta G(\lambda, q') - \tilde{I}(q', \lambda) \right]$$

Here the sup is over all constants $q'$. 47
We first sketch the idea of the proof for Lemma 8.5: The rate function \( I \) can be understood in the following way: The probability to find the \( u_\ell \) with a given value \( q' \) is given by \( \exp[-|\Lambda\ell|s(q')] \) with the entropy given by

\[
s(q') = \sup_\lambda [\lambda \cdot q' - \psi(\lambda)]
\]

We now write \( \frac{1}{Z(\lambda)} \exp [\langle \lambda, u_\ell \rangle] \) as

\[
\int dq' \exp [\Lambda|q'| \{ \lambda \cdot q' - \psi(\lambda) - s(q') \}]
\]

This gives the last variational formula.

**Proof of Lemma 8.5** We shall drop the constant parameter \( \lambda \) in \( G \) in this proof. Since the components of \( u \) commute, we can define the joint distribution \( \nu_\ell(du) \) of \( u \) w.r.t. the state \( \omega_{\lambda,\ell} \).

Thus

\[
\omega_{\lambda,\ell} \left( \exp [\delta q^3 G(\lambda, u_\ell)] \right) = \int d\mu_\ell(u) \left( \exp [\delta q^3 G(\lambda, u_\ell)] \right)
\]

We now approximate the integral by the summation so that

\[
\int d\mu_\ell(u) \exp [\delta q^3 G(\lambda, u)] \leq \sum_{m \in \mathbb{Z}^d} P_{\lambda,\ell}([u - \xi m] \leq \varepsilon) \exp [\delta q^3 G(\varepsilon m)]
\]

where

\[
G_\varepsilon(y) = \sup_{|x-y| \leq \varepsilon} G(x)
\]

and \( P_{\lambda,\ell} \) denotes the probability of the event described in its argument, with respect to the state \( \omega_{\lambda,\ell} \). We can bound the summation by

\[
\varepsilon^{-5} \int dx P_{\lambda,\ell}([u - x] \leq \varepsilon) \exp [\delta q^3 G_\varepsilon(x)]
\]

We have

\[
P_{\lambda,\ell}([u - x] \leq \varepsilon) \leq P_{\lambda,\ell}([\xi \cdot u \geq \xi \cdot x - |\xi|\varepsilon])
\]

for all \( \xi \). Notice that from the Chebyshev inequality we have

\[
P_{\lambda,\ell}([\xi \cdot u \geq \xi \cdot x - |\xi|\varepsilon]) \leq e^{-\delta |\xi|x^\varepsilon + \delta |\xi|\varepsilon} \int d\mu_\ell(u) e^{\delta |\xi|u}
\]

Let \( \psi_\ell(\lambda) \) be the pressure defined by

\[
\psi_\ell(\lambda) = \ell^{-3} \log \text{Tr} (e^{\delta \lambda u})
\]

so that

\[
\int d\mu_\ell(u) e^{\delta |\xi|u} = \exp \left\{ e^{\delta \left[ \psi_\ell(\xi) - \psi_\ell(\lambda) \right]} \right\}
\]
Thus
\[ P_{\lambda, \ell}(\xi \cdot u \geq \xi \cdot x - |\xi| \epsilon) \leq \exp \left\{ -\ell^3 \left[ \xi \cdot x - \psi_\ell(\xi + \lambda) - \epsilon |\xi| + \psi_\ell(\lambda) \right] \right\} \]
for all
\[ \lambda_4 + \xi_4 > 0. \]

In particular, we have
\[ P_{\lambda, \ell}(|u - x| \leq \epsilon) \leq \exp \left\{ -\ell^3 \sup_{-\eta^{-1} \leq \xi_j + \lambda_j \leq \eta^{-1}, j = 0, \ldots, 3} \left( \xi \cdot x - \psi_\ell(\xi + \lambda) - \epsilon |\xi| + \psi_\ell(\lambda) \right) \right\} \]

The existence of thermodynamics states that
\[ \lim_{\ell \to \infty} \left| \psi_\ell(\lambda) - \psi(\lambda) \right| = 0 \]
uniformly in compact interval away from \( \lambda_4 = 0 \). Fix a small constant \( \eta > 0 \). Define
\[ \tilde{s}_\eta(x) = \sup_{-\eta^{-1} \leq \xi_j + \lambda_j \leq \eta^{-1}, j = 0, \ldots, 3} (\xi \cdot x - \psi(\xi)) \]

We have
\[ \sup_{-\eta^{-1} \leq \xi_j + \lambda_j \leq \eta^{-1}, j = 0, \ldots, 3} (\xi \cdot x - \psi(\xi)) \leq \tilde{s}_\eta(x) - \lambda \cdot x - C(\eta, \lambda) \epsilon + C_\ell \]
where
\[ \lim_{\ell \to \infty} C_\ell = 0 \]
Define
\[ \tilde{I}_\eta(x) = \tilde{s}_\eta(x) - \lambda \cdot x + \psi(\lambda) \]
Thus we have
\[ P_{\lambda, \ell}(|u - x| \leq \epsilon) \leq \exp \left\{ -\ell^3 \left( \tilde{I}_\eta(x) - C_\epsilon - C(\eta, \lambda) \epsilon \right) \right\} \]

We now have the estimate
\[ \ell^{-3} \log \int d\mu(x) \exp \left[ \delta \ell G(\lambda, u) \right] \leq \ell^{-3} \log \left\{ \epsilon^{-5} \int \exp \left[ \delta \ell G_\epsilon(x) \right] \right\} \]
\[ \leq \ell^{-3} \log \int \exp \left[ -\ell^3 \left( \tilde{I}_\eta(x) - \delta G_\epsilon(x) \right) \right] + C_\epsilon + C(\eta, \lambda) \epsilon + \ell^{-3} |\log \epsilon| \]
The error vanishes in the limit \( \lim_{\epsilon \to 0} \lim_{\ell \to \infty} \). The integration can be calculated using the Laplace method to give
\[ \lim_{\ell \to \infty} \ell^{-3} \log \int \exp \left[ -\ell^3 \left( \tilde{I}_\eta(x) - \delta G_\epsilon(x) \right) \right] \leq \sup_{q'} \left[ \delta G_\epsilon(\lambda, q') - \tilde{I}_\eta(q', \lambda) \right] \]
Clearly, we have
\[
\lim_{\varepsilon \to 0} \sup_{q'} \left[ \delta G_\varepsilon(\lambda, q') - \tilde{I}_\eta(q', \lambda) \right] = \sup_{q'} \left[ \delta G(\lambda, q') - \tilde{I}_\eta(q', \lambda) \right]
\]

We now collect some properties for \( \tilde{s}_\eta \) and \( \tilde{I}_\eta \). Notice that, by definitions, \( \tilde{s}_\eta \) and \( \tilde{I}_\eta \) are still convex. Furthermore, we can check that if \( \eta \) is small then \( \tilde{I}_\eta(x) = I(x) \) in a small neighborhood of \( q \). This proves the Lemma.

\[\blacksquare\]

8.3 Proof of Lemma 7.2

Recall \( \alpha \) indices disjoint subcubes of width \( \ell \). By the entropy inequality (5.17), we have
\[
- \operatorname{Av}_{0 \leq t \leq \varepsilon^{-1} T} \gamma_\varepsilon \operatorname{Av}_\alpha \left( I^* G(\lambda(\varepsilon \alpha), u_{\alpha^+}) I \right) \leq R + \delta^{-1} M s(\gamma_\varepsilon \mid \tilde{\omega}_{\ell}^{\varepsilon})
\]
where, for any \( \delta > 0, \)
\[
R = \varepsilon^d \delta^{-1} \log \operatorname{Tr} \frac{1}{\tilde{c}_{\varepsilon, \ell}(\lambda)} \exp \left[ -3 \operatorname{Av}_\alpha \lambda(\varepsilon \alpha) \cdot u_{\alpha^+} - 3 \log \tilde{c}_{\varepsilon, \ell}(\lambda) \right] I
\]
(8.5)

We first control the last term \( s(\gamma_\varepsilon \mid \tilde{\omega}_{\ell}^{\varepsilon}) \) by writing it as
\[
s(\gamma_\varepsilon \mid \tilde{\omega}_{\ell}^{\varepsilon}) = s(\gamma_\varepsilon \mid \omega_{\ell}^{\varepsilon}) + \varepsilon^3 \log \tilde{c}_{\varepsilon}(\lambda) - 3 \log \tilde{c}_{\varepsilon, \ell}(\lambda) + R_4
\]
where
\[
R_4 = \gamma_\varepsilon (\varepsilon^3 \int dx \lambda(\varepsilon x) \cdot u_x - \operatorname{Av}_\alpha \lambda(\varepsilon \alpha) \cdot I^* u_{\alpha^+} I)
\]
From the Lemma 4.1 we have
\[
\lim_{\ell \to \infty} \lim_{\varepsilon \to 0} R_4 = 0
\]

We now estimate \( R \). Using the argument in the proof of Lemma 8.3 we can drop the operator \( I \) in (8.5) to have an upper bound. Since the cubes indexed by \( \alpha \) are independent, we have
\[
R \leq \delta^{-1} \operatorname{Av}_{\alpha} Q_{\alpha} - \left( \varepsilon^3 \log \tilde{c}_{\varepsilon, \ell}(\lambda) - 3 \ell^{-3} \log Z_\ell(\lambda(\varepsilon \alpha)) \right)
\]
where
\[
Q_{\alpha} = \ell^{-3} \log \operatorname{Tr} Z_\ell(\lambda(\varepsilon \alpha))^{-1} \exp \left[ -3 \ell \left\{ \lambda(\varepsilon \alpha) \cdot u_{\alpha^+} - \delta G(\lambda(\varepsilon \alpha), u_{\alpha^+}) \right\} \right]
\]
The last term \( \left( \varepsilon^3 \log \tilde{c}_{\varepsilon, \ell}(\lambda) - 3 \ell^{-3} \log Z_\ell(\lambda(\varepsilon \alpha)) \right) \) vanishes by Lemma 8.4. Notice that the components of \( u_{\alpha^+} \) commute. Thus the trace is over functional of commuting operators and we
are essentially the same as in the classical theory. Thus we can apply Lemma 8.5 to estimate $Q_\alpha$. Summarizing, we have

$$\lim_{\ell \to \infty} \lim_{\varepsilon \to 0} \gamma \text{Av}_\alpha \left( I^\alpha G(\varepsilon \alpha), u_{\alpha+} \right)$$

$$\leq \int dX \sup_{q'(X)} \left[ G(\lambda(X)), q'(X) \right] - \delta^{-1} \tilde{I}(q'(X), \lambda(X)) ] + \delta^{-1} \lim_{\varepsilon \to 0} s(\gamma | \omega^\varepsilon_{\lambda}) + R_4$$

Notice that the right side of the inequality is independent of the location of the grid. If we average the grid over the cube of size $\ell$, we can replace the left side of the inequality from averaging over $\alpha$ to averaging over all points $x$ on the torus. This proves Lemma 7.2.

We now state a corollary to Lemma 7.2.

**Corollary 8.6** Suppose $\lambda$ is a bounded smooth function so the Gibbs state with chemical potential $\lambda(X)$ is in the one phase region for all $X$. Suppose $\gamma_\varepsilon$ is a sequence of states such that the specific entropy $s(\gamma_\varepsilon | \omega^\varepsilon_{\lambda})$ satisfies

$$\lim_{\varepsilon \to 0} s(\gamma_\varepsilon | \omega^\varepsilon_{\lambda}) = 0$$

For any bound smooth function $J$ on the unit torus, we have

$$\lim_{\varepsilon \to 0} \gamma_\varepsilon \varepsilon^3 \int dx \ J(\varepsilon x) \cdot u(\varepsilon x) = \int dX \ J(X) \cdot q(X)$$

**Proof:** Since $J$ is bounded smooth, from Lemma 4.4 we have

$$\lim_{\ell \to \infty} \lim_{\varepsilon \to 0} \gamma \left( \varepsilon^3 \int dx \ J(\varepsilon x) \cdot u(\varepsilon x) - \text{Av}_x J(\varepsilon x) \cdot u^+_x \right) = 0$$

We now apply Lemma 7.2 to have

$$\lim_{\ell \to \infty} \lim_{\varepsilon \to 0} \gamma_\varepsilon \text{Av}_x J(\varepsilon x) \cdot \left( u^+_x - q(\varepsilon x) \right)$$

$$\leq \int dX \sup_{q'(X)} \left[ J(X) \cdot (q'(X) - q(X)) - \delta^{-1} \tilde{I}(q'(X), \lambda(X)) \right] + \delta^{-1} \lim_{\varepsilon \to 0} s(\gamma_\varepsilon | \omega_{\lambda})$$

Since $\tilde{I} \geq 0$ and $\tilde{I}(q'(X), \lambda(X)) = 0$ only when $q'(X) = q(X)$, the sup is bounded by $C\delta$. To see this, consider the model problem

$$\sup_x x - \delta^{-1} x^2 \leq \delta$$

Recall the assumption $\lim_{\varepsilon \to 0} s(\gamma_\varepsilon | \omega_{\lambda}) = 0$. Since we can choose $\delta$ arbitrarily small, we prove the corollary.
9 Calculation of the currents

The current density operators \( w^\mu_{k,x} \) are implicitly defined by (3.9), i.e., for any test functions \( J = (J^\mu) \), \( \mu = 0, \ldots, 4 \), they should satisfy

\[
i\delta_H \left( \int dx \langle J, u_x \rangle \right) - \sum_{k=1}^{3} \langle \nabla_k J, w_{k,x} \rangle = \text{terms containing second and higher derivatives of the } J^\mu \text{ integrated with densities of bounded expectation.}
\]

If we apply the same sign convention for dot products with \( w \) as the convention adopted in (2.8) for \( u \), this means we are looking for the definition of \( w^\mu_{k,x} \), such that, for any test function \( J \), the following formal identity holds:

\[
i \int dx J(x) [H, w^\mu_{k,x}] = \sum_{k=1}^{3} \int dx \nabla_k J(x) w^\mu_{k,x} + \text{integrals with higher derivatives of } J \ , \quad (9.1)
\]

where \( H \) is the formal Hamiltonian \( H = \int dx h_x \), with \( h_x \) as defined in (2.6).

In order to compute the commutators we use the canonical anticommutation relations (2.4) and integration by parts. The commutation relations involving derivatives such as \( \nabla_k a_x \) etc., are most easily derived by taking derivatives of the appropriate commutation relations without derivatives. E.g., the identity

\[
\left[ a^+_u a_v, a^+_x a_y \right] = \delta(x-v) a^+_u a_y - \delta(y-u) a^+_x a_v
\]

follows directly from (2.4) and, by taking derivatives with respect to \( u \), also leads to

\[
\left[ \nabla_k a^+_u a_v, a^+_x a_y \right] = \delta(x-v) \nabla_k a^+_u a_y + \delta_k(y-u) a^+_x a_v
\]

where \( \delta_k \) is the derivative of the delta distribution with respect to the \( k \)th component. It is straightforward to derive all other necessary relations in the same way. E.g.,

\[
[\nabla_k a^+_y \nabla_k a_y, a^+_x a_x] = -\delta_k(x-y) \nabla_k a^+_y a_x + \delta_k(x-y) \nabla_k a_y a^+_x \ . \quad (9.2)
\]

There are essentially three cases to consider: i) \( \mu = 0 \), ii) \( \mu = 1, 2, 3 \), and iii) \( \mu = 4 \).

i) \( \mu = 0 \): calculation of \( w^0_{k,x} \):

As \( n_x \) commutes with the potential part of the Hamiltonian we only have to consider the kinetic energy term, which can be computed using (9.2). After integrating by parts, we get

\[
i \int dxdy J(x) \frac{1}{2} \nabla a^+_y \nabla a_y, a^+_x a_x = \int dx \nabla J(x) \frac{1}{2} i [\nabla_k a^+_x a_x - a^+_x \nabla a_x]
\]

By comparing this result and (9.1) we find agreement with the definition of \( w^0_{k,x} \) as given in (3.5).

Note that, in this case, no higher order derivatives of \( J \) appear.

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ii) $\mu = j = 1, 2, 3$: calculation of $w^j_{k,x}$.

Now, both the kinetic energy term and potential energy term both yield non-trivial contributions. First, we compute the kinetic energy term.

$$i \int dxdy J(x) \left[ \frac{1}{2} \nabla_j^+ a^+_y \nabla a_y \right] = \frac{1}{4} i \int dxdy J(x) \left[ \nabla_j^+ a_x^+ a_x, \nabla a_y^+ \nabla a_y \right] + \text{h.c.}$$

where here and in the following h.c. stands for the adjoint of the preceding term(s). By using the commutation relations and integration by parts we find the following expression for this quantity:

$$i \int dxdy J(x) \left[p^j_x, \frac{1}{2} \sum_{k=1}^3 \nabla_k^+ a^+_y \nabla_k a_y \right]$$

$$= - \int dx J(x) \left[ \nabla_j^+ a_x^+ \Delta a_x + \Delta a_x \nabla_j a_x^+ \right] - \sum_{k=1}^3 \nabla_k J(x) \Delta a_x^+ a_x + \text{h.c.}$$

$$= \sum_{k=1}^3 \int dx J(x) \left[ \nabla_k \nabla_j a_x^+ \nabla_k a_x + \nabla_k a_x^+ \nabla_k \nabla_j a_x \right]$$

$$+ 2 \sum_{k=1}^3 \int dx \nabla_k J(x) \nabla_j a_x^+ \nabla_k a_x - \int dx \nabla_j J(x) \Delta a_x^+ a_x + \text{h.c.}$$

After further integration by parts and reorganization the result can be written as

$$i \int dxdy J(x) \left[p^j_x, \frac{1}{2} \sum_{k=1}^3 \nabla_k^+ a^+_y \nabla_k a_y, p^j_x \right]$$

$$= \sum_{k=1}^3 \int dx \nabla_k J(x) \left( \frac{1}{2} \nabla_j a_x^+ \nabla_k a_x + \nabla_k a_x^+ \nabla_j a_x \right) + \frac{1}{4} \nabla_k \nabla_j J(x) \left[ \nabla_k a_x^+ a_x + a_x^+ \nabla_k a_x \right]$$

The first term of the RHS in this expression determines the first term (3.6). Note that this time higher derivative terms appear that are not included in the definition of $w^j_{k,x}$, but they contribute to the error terms.

To calculate the contribution from the potential energy term in the Hamiltonian, we start from the identity

$$\left[ \nabla_j a_u^+ a_u, a_x^+ a_y^+ a_y a_x \right] = \delta(x - u) \nabla_j a_u^+ a_y^+ a_y a_x + \delta(y - u) a_x^+ \nabla_j a_u^+ a_y a_x$$

$$+ \nabla_j \delta(y - u) a_x^+ a_y^+ a_y a_x + \nabla_j \delta(x - u) a_x^+ a_y^+ a_y a_u$$

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which leads to
\[
\int dx dy W(x - y) \left[ \int du J(u) \frac{1}{2} [\nabla_j a^+_u a_u - a^+_u \nabla_j a_u, a^+_x a^+_y a_y a_x] \right]
\]
(9.3)
\[
= \int dx dy W(x - y) \left[ J(x) [\nabla_j a^+_x a^+_y a_y a_x + \text{h.c.}] + J(y) [a^+_x \nabla_j a^+_y a_y a_x + \text{h.c.}] \right]
+ [\nabla_j J(x) + \nabla_j J(y)] a^+_x a^+_y a_y a_x
\]
\[
= - \int dx dy \left[ J(x) \nabla_j W(x - y) + J(y) \nabla_j W(x - y) \right] a^+_x a^+_y a_y a_x
\]

Due to the spherical symmetry of the potential we have
\[
\nabla_j W(x - y) = W'(x - y) \frac{(x - y)_j}{|x - y|}
\]
(9.4)

Using this identity we can write (9.3) in the form
\[
- \int \left[ (J(x) - J(y)) W'(x - y) \frac{(x - y)_j}{|x - y|} \right] a^+_x a^+_y a_y a_x
\]

As the range of $W$ is finite by assumption, we can Taylor expand $J(x) - J(y)$ to rewrite this quantity in the following form:
\[
- \sum_{k=1}^3 \int dx dy \nabla_k J(x) W'(x - y) \frac{(x - y)_k (x - y)_j}{|x - y|} a^+_x a^+_y a_y a_x + \text{higher order derivatives of } J .
\]
(9.5)

Recall that, by definition, only the coefficients of the first order derivatives of $J$ are included in the $w$ tensor. Therefore, combining (9.2) and (9.5) and also including the appropriate factors $1/2$ and $-\text{signs}$, we find the expression for $w^j_{k,x}$ claimed in (3.6).

iii) $\mu = 4$: calculation of $w^4_{k,x}$.

The calculation of the energy current proceeds in the same way as the previous cases, but there are more terms and terms with higher derivatives. The contribution from the kinetic energy in the Hamiltonian to the kinetic energy current is, up to a trivial constant, given by
\[
i \int du dx J(u) \left[ \nabla a^+_u \nabla a_u, \nabla a^+_x \nabla a_x \right] = i \sum_{k,l=1}^3 \int du dx J(u) \delta_{k,l}(u - x) \left[ \nabla_k a^+_u \nabla_l a_u - \nabla_l a^+_u \nabla_k a_u \right]
\]
\[
= i \sum_{k=1}^3 \int dx \nabla_k J(x) \left[ \nabla_k a^+_x \Delta a_x - \Delta a^+_x \nabla_k a_x \right]
\]
where $\delta_{k,l}$ is shorthand for $\nabla_k \nabla_l \delta$. This yields the first term of the energy current.

The potential energy term in the Hamiltonian does not contribute to the potential energy portion of the energy current due to the fact that the following commutators vanish:
\[
\left[ a^+_u a^+_v a_v a_u, a^+_x a^+_y a_y a_x \right] = 0
\]

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The last four terms become the last two terms of the energy current:

\[ i \int dx J(x)[h_x, H] \]

\[ = \frac{i}{4} \sum_{k=1}^{3} \int \Delta k J(x) [\nabla k a_x^+ a_x, \nabla k a_x] \]

\[- \frac{i}{4} \sum_{k=1}^{3} \int (\nabla k) J(x) W(x-y) [\nabla k a_x^+ a_y a_x - a_x^+ a_y^+ a_y \nabla k a_x] \]

\[ + \frac{i}{4} \sum_{k=1}^{3} \int (J(y) - J(x)) \nabla k W(x-y) [a_x^+ a_y a_x a_y - a_x^+ a_y^+ a_y \nabla k a_y a_x] \]

By multiplying this expression by \( W(x - y) \), and integrating over \( x \) and \( y \), and integrating by parts, we find

\[ \int dx dy du W(x - y) J(u) \left[ \nabla a_u^+ \nabla a_u, a_x^+ a_y a_x \right] \]

\[ = \sum_{k=1}^{3} \int J(x) \nabla k, x W(x - y) \left[ \nabla k a_x^+ a_y a_x a_x^+ a_y a_x - a_x^+ a_y^+ a_y \nabla k a_x \right] \]

\[ + \sum_{k=1}^{3} \int J(y) \nabla k, y W(x - y) \left[ a_x^+ \nabla k a_y a_x - a_x^+ a_y^+ \nabla k a_y a_x \right] \]

The result is

\[ \int dx dy du W(x - y) J(x) \left[ a_x^+ a_y a_x , \nabla a_x^+ \nabla a_u \right] \]

\[ = - \sum_{k=1}^{3} \int \nabla k, x (J(x) W(x - y)) \left[ \nabla k a_x^+ a_y a_x a_x^+ a_y a_y \nabla k a_x \right] \]

\[ - \sum_{k=1}^{3} \int J(y) \nabla k, y W(x - y) \left[ a_x^+ \nabla k a_y a_x a_y - a_x^+ a_y a_y a_x \right] \]
By the same argument as for (9.5), the last term can be rewritten in the form
\[ +i \sum_{k=1}^{3} \int \nabla_k J(x) \left[ W'(x-y) \frac{(x-y) \otimes (x-y)}{|x-y|} \right] \left[ a_x^+ \nabla a_y^+ a_y a_x - a_x^+ a_y^+ \nabla a_y a_x \right] + O(J'') \]

10 The virial Theorem

The purpose of this section is to relate the expectation values of the RHS of the dynamical equations (3.9) in a Gibbs state with specified values of the densities of the conserved quantities (local equilibrium), to these quantities themselves in order to obtain a closed set of equations. To achieve this we will make use of canonical transformations relating Gibbs states with respect to reference frames with different velocities. This will allow us to use reflection symmetry of Gibbs states at zero total momentum. A second element we will need is the Virial Theorem to relate the so-called virial to the thermodynamic pressure. We start with the latter. For the convenience of the reader we first recall the main definitions.

Consider a system of particles in a finite volume \( \Lambda \subset \mathbb{R}^d \), interacting via a pair potential \( W \). The pressure at inverse temperature \( \beta \) and chemical potential \( \mu \), \( P(\beta, \mu) \), is defined by
\[ P(\beta, \mu) = \lim_{\Lambda \to \mathbb{R}^d} \frac{1}{\beta |\Lambda|} \log \text{Tr} \, e^{-\beta (H_{0,\Lambda} + V_{\Lambda} - \mu N_{\Lambda})} \quad (10.1) \]
where
\[
H_{0,\Lambda} = \frac{1}{2} \int_{\Lambda} dx \, \nabla a_x^+ \nabla a_x \\
V_{\Lambda} = \frac{1}{2} \int_{\Lambda} \int_{\Lambda} dx dy \, W(x-y) a_x^+ a_y^+ a_y a_x \\
N_{\Lambda} = \int_{\Lambda} dx \, a_x^+ a_x
\]
The trace is taken over the Fermion Fock space with one-particle space \( L^2(\Lambda) \). For our purposes, we can simply consider \( \Lambda \) to be a cube of side \( L \), and define the operators with periodic boundary conditions. We will write \( V_{\Lambda}(W) \) when we wish to indicate the pair potential function explicitly. By our general assumptions, the limit (10.1) exists and we will restrict ourselves to the one-phase region of the phase diagram. In particular we assume that the pressure is continuously differentiable.

Gibbs states at non-vanishing total momentum are defined by introducing an additional Lagrange multiplier for the momentum as follows.
\[ \omega_{\Lambda}(X) = \lim_{\Lambda \to \mathbb{R}^d} \frac{1}{Z(\lambda)} \text{Tr} \, X e^{-\beta (H_{0,\Lambda} + V_{\Lambda} - \alpha P_{\Lambda} - \mu N_{\Lambda})} \quad (10.2) \]

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where \( \lambda = (\beta, \alpha, \mu) \), \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \), are constants, and \( P_{\Lambda} \) is the total momentum operator in the volume \( \Lambda \) defined by

\[
P_{\Lambda} = \frac{i}{2} \int_{\Lambda} dx \nabla a_{x}^+ a_x - a_{x}^+ \nabla a_x.
\]

and

\[
Z(\lambda) = \text{Tr} \ e^{-\beta (H_{0, \Lambda} + V_{\Lambda} - \alpha P_{\Lambda} - \mu N_{\Lambda})}
\]

is the partition function.

The kinetic energy density is defined by

\[
e_{\text{kin}}(\beta, \alpha, \mu) = \lim_{\Lambda \to \mathbb{R}^3} \frac{1}{|\Lambda|} \omega_{\beta, \pi, \mu}(H_{0, \Lambda})
\]

The limits \( \Lambda \to \mathbb{R}^3 \) exist and are independent of the boundary conditions under general stability assumptions [18]. We will use the abbreviations \( \omega_{\beta, \mu} = \omega_{\beta, 0, \mu} \) and \( e_{\text{kin}}(\beta, \mu) = e_{\text{kin}}(\beta, 0, \mu) \).

The virial of the potential \( W \) in the volume \( \Lambda \) is denoted by \( V_{\Lambda}(W) \) and is defined by

\[
V_{\Lambda}(W) = \frac{1}{2} \int_{\Lambda} \int_{\Lambda} dx dy \nabla W(x - y) \cdot (x - y) a_{x}^+ a_{y}^+ a_y a_x
\] (10.3)

and the density of the local density virial is given by

\[
\nu_{x} = \frac{1}{2} \int_{\mathbb{R}^3} dy \nabla W(x - y) \cdot (x - y) a_{x}^+ a_{y}^+ a_y a_x
\] (10.4)

As we have assumed that \( W \) has compact support, \( \nu_{x} \) is well-defined.

Due to Galileo invariance, the Gibbs states for different values of \( \alpha \) are related by a canonical transformation, which is why in the statistical mechanics of global equilibrium situations the total momentum is usually assumed to vanish.

The canonical transformations relating \( \omega_{\beta, \mu} \) and the states \( \omega_{\beta, \alpha, \mu} \), are defined as follows. Let \( s \in \mathbb{R}^d \), and consider the unitary \( U_s \) on \( L^2(\mathbb{R}^d, dx) \) defined by \( (U_s \psi(x)) = e^{i s \cdot x} \psi(x) \). The second quantization of \( U_s \) implements an automorphism \( \gamma_s \) on the Fermion algebra given by

\[
\gamma_s(a(f)) = a(U_s f), \quad \gamma_s(a^+(f)) = a^+(U_s f).
\]

One can easily verify that the action of \( \gamma_s \) on the operator-valued distributions \( a_x, \nabla a_x \), and their adjoints, is given by:

\[
\gamma_s(a_x) = e^{-is \cdot x} a_x, \quad \gamma_s(a_x^+) = e^{is \cdot x} a_x^+
\]
\[
\gamma_s(\nabla a_x) = e^{-is \cdot x} \nabla a_x - ise^{-is \cdot x} a_x, \quad \gamma_s(\nabla a_x^+) = e^{is \cdot x} \nabla a_x^+ + ise^{is \cdot x} a_x^+
\]

Clearly, \( \gamma_{-s}^{-1} = \gamma_{-s} \).
With these relations it is easy to check that
\[
\gamma_s(\nabla a^+ \nabla a_x) = \nabla a^+_x \nabla a_x + |s|^2 a^+_x a_x + is \cdot (a^+_x \nabla a_x - \nabla a^+_x) .
\]

Hence, the kinetic energy transforms as follows:
\[
\gamma_s(H_{\Lambda,0}) = H_{\Lambda,0} + \frac{1}{2} |s|^2 N_\Lambda - s \cdot P_\Lambda \tag{10.5}
\]

In the same way we see that
\[
\begin{align*}
\gamma_s(N_\Lambda) &= N_\Lambda \tag{10.6} \\
\gamma_s(P_\Lambda) &= P_\Lambda - s N_\Lambda \tag{10.7} \\
\gamma_s(V_\Lambda) &= V_\Lambda \tag{10.8} \\
\gamma_s(V_\Lambda(W)) &= V_\Lambda(W) \tag{10.9}
\end{align*}
\]

It follows that
\[
\gamma_s(e^{-\beta(H_\Lambda - \mu N_\Lambda)}) = e^{-\beta(H_\Lambda - s P_\Lambda - (\mu - \frac{1}{2}|\alpha|^2) N_\Lambda)}
\]

By putting \( s = \alpha \), replacing \( \mu \) by \( \mu - \frac{1}{2}|\alpha|^2 \), we obtain
\[
e^{-\beta(H_\Lambda - \alpha P_\Lambda - \mu N_\Lambda)} = \gamma_\alpha(e^{-\beta(H_\Lambda - (\mu + \frac{1}{2}|\alpha|^2) N_\Lambda)})
\]

As the trace is invariant under \( \gamma_\alpha \), this implies
\[
Z(\lambda) = \text{Tr} \ e^{-\beta(H_\Lambda - \alpha P_\Lambda - \mu N_\Lambda)} = \text{Tr} \ e^{-\beta(H_\Lambda - (\mu + \frac{1}{2}|\alpha|^2) N_\Lambda)} = Z(\bar{\lambda}) \tag{10.10}
\]

where, for \( \lambda = (\beta, \alpha, \mu) \), we define \( \bar{\lambda} = (\beta, 0, \mu + \frac{1}{2}|\alpha|^2) \). Using this relation between partition functions and the invariance of the trace under canonical transformations, we immediately get
\[
\omega_\lambda(\gamma_\alpha(X)) = \frac{1}{Z(\lambda)} \text{Tr} \ \gamma_\alpha(X) e^{-\beta(H_\Lambda - \alpha P_\Lambda - \mu N_\Lambda)}
\]
\[
= \frac{1}{Z(\lambda)} \text{Tr} \ X e^{-\beta(H_\Lambda - (\mu + \frac{1}{2}|\alpha|^2) N_\Lambda)} = \omega_{\bar{\lambda}}(X) \tag{10.11}
\]

In combination with (10.6) this implies
\[
\omega_\lambda(N_\Lambda) = \omega_{\bar{\lambda}}(N_\Lambda),
\]

and hence \( \rho(\lambda) = \rho(\bar{\lambda}) \), and we will simply write \( \rho \). Also
\[
\omega_{\bar{\lambda}}(P_\Lambda) = \alpha \omega_\lambda(N_\Lambda).
\]
If we apply this to $X = H_{\Lambda,0}$ and combine this with (10.9), to relate the kinetic energy densities of $\omega_\lambda$ and $\omega_\tilde{\lambda}$, we find
\[
e_{\text{kin}}(\beta, \alpha, \mu) = e_{\text{kin}}(\beta, \mu + \frac{1}{2}|\alpha|^2) + \frac{1}{2}|\alpha|^2 \rho, \tag{10.12}
\]
where we have also used $\omega_\tilde{\lambda}(P_\Lambda) = 0$. The relation (10.10) between partition functions immediately implies the following property of the pressure:
\[
P(\beta, \alpha, \mu) = P(\beta, 0, \mu + \frac{1}{2}|\alpha|^2) \tag{10.13}
\]
One interpretation of this relation is that the chemical potentials at different values of $\alpha$, when regarded as a function of the particle density $\rho$, satisfy
\[
\mu(\rho) = \mu(\rho) - \frac{1}{2}|\alpha|^2.
\]

We can now prove the virial theorem in the form we need.

**Theorem 10.1 (Virial Theorem)** For a three-dimensional translation invariant system with a continuously differentiable pressure function, one has
\[
2 \left[ e_{\text{kin}}(\beta, \alpha, \mu) - \frac{1}{2}|\alpha|^2 \rho \right] - \lim_{\Lambda \to \mathbb{R}^d} \frac{1}{|\Lambda|} \omega_{\beta,\alpha,\mu}(V_\Lambda(W)) = dP(\beta, \alpha, \mu)
\]
The quantity between square brackets can be considered as the gauge invariant kinetic energy.

**Proof:** Suppose that the theorem holds for $\alpha = 0$. We can then use a canonical transformation to obtain the result for arbitrary $\alpha$ as a consequence of (10.11):
\[
\omega_\lambda(V(W)) = \omega_{\lambda}(\gamma_{-1}(V(W))) = \omega_{\tilde{\lambda}}(V(W)) = 2e_{\text{kin}}(\beta, \mu + \frac{1}{2}|\alpha|^2) - dP(\beta, \mu + \frac{1}{2}|\alpha|^2)
\]
By using (10.12) and (10.13), this is equivalent to the statement of the theorem.

We now prove the theorem for $\alpha = 0$. As the pressure is independent of the boundary conditions, we can use periodic boundary conditions to compute it, i.e., we choose $\Lambda$ to be a $d$-dimensional torus. For $t > 0$, let $t\Lambda$ be the torus rescaled by $t$. Then $|t\Lambda| = t^d|\Lambda|$, and
\[
U_t : L^2(t\Lambda, dx) \to L^2(\Lambda, dx) : (U_t\psi)(x) = t^{d/2}\psi(tx)
\]
is unitary. The Laplacians on $\Lambda$ and $t\Lambda$ are related as follows:
\[
U_t \Delta_{t\Lambda} U_t^* = t^{-2} \Delta_{\Lambda}.
\]
This relation carries over to the kinetic energy in second quantization:
\[
U_t H_{0,t\Lambda} U_t^* = t^{-2} H_{0,\Lambda}.
\]
where we have used the same notation for the corresponding unitary on the Fock space $\mathcal{F}(L^2(\Lambda))$ with one-particle space $L^2(\Lambda)$. Similarly, one easily finds that the scaling behavior of the potential energy terms in the Hamiltonians is as follows:

$$U_t V_{t\Lambda}(W) U_t^* = V_{\Lambda}((W(t)),$$

and for the particle number we have

$$U_t N_{t\Lambda} U_t^* = N_{\Lambda}.$$

By using these unitary equivalences we obtain

$$P(\beta, \mu) = \lim_{\Lambda \to \mathbb{R}^d} \frac{1}{|t\Lambda|} \log \text{Tr} \mathcal{F}(L^2(t\Lambda)) e^{-\beta(H_{0,L\Lambda} + V_{t\Lambda}(W) - \mu N_{t\Lambda})}$$

$$= \lim_{\Lambda \to \mathbb{R}^d} \frac{1}{|t\Lambda|} \log \text{Tr} \mathcal{F}(L^2(\Lambda)) e^{-\beta((t^2 H_{0,\Lambda} + V_{\Lambda}(W(t)) - \mu N_{\Lambda})}$$

This shows that the last expression is independent of $t$. Setting its derivative in $t = 1$ equal to zero yields the following equation

$$2e_{\text{kin}}(\beta, \mu) - \lim_{\Lambda \to \mathbb{R}^d} \frac{1}{|\Lambda|} \omega_{\beta, \mu}(V_{\Lambda}(W)) - dP(\beta, \mu) = 0.$$

In order to close the dynamical equations, we need to express the expectation values of the currents $w^j_k$, given in (3.5-3.8), in the states $\omega_{\lambda}$ in terms of the expectations of the conserved quantities $u^j$ of (2.6).

**Proposition 10.2** The expectations of the local currents $w^j_k$ in a Gibbs state $\omega_{\lambda}$ are given by

$$\omega_{\lambda}(w^0_{k,x}) = \omega_{\lambda}(u^k_x)$$

$$\omega_{\lambda}(w^j_{k,x}) = \alpha_j \alpha_k \omega_{\lambda}(u^0_x) + \delta_{k,j} P(\lambda)$$

$$\omega_{\lambda}(w^4_{k,x}) = \alpha_k \omega_{\lambda}(u^4_x) + \alpha_k P(\lambda)$$

where $P(\lambda)$ is the pressure defined in (10.1) and $u^j_x$ are the local densities of the five conserved quantities defined in (2.6). With the definitions of (2.15) and (4.1), this is equivalent to $A = \hat{w}$. 

Explicitly: $q^0 = \rho$, and

$$\omega_{\lambda}(w^0_{k,x}) = \alpha_k \rho = q^k$$

$$\omega_{\lambda}(w^j_{k,x}) = \alpha_j \alpha_k \rho + \delta_j \delta_k P = q^j q^k / q^0 + \delta_j \delta_k P$$

$$\omega_{\lambda}(w^4_{k,x}) = \alpha_k (q^4 + P) = q^k (q^4 + P) / q^0$$

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Proof: The first equation, \( j = 0 \), follows directly from (3.5), (10.7), and (10.11). The expressions for \( w^j_{k,x} \), \( j = 1, 2, 3 \), contain the virial of the potential \( W \), which we can relate to the thermodynamic pressure by using the virial theorem, Theorem 10.1:

\[
w^j_{k,x} = \nabla_j a_x^+ \nabla_k a_x - \frac{1}{2} \left[ W'(x-y)(x-y)_j(x-y)_k \right] a_x^+ a_y^+ a_y a_x
\]

For \( j \neq k \), the expectation of the second term vanishes as it changes sign under rotation over \( \pi \) about the \( j \)th axis, which is a symmetry of the potential and the Gibbs states. Due to the rotation invariance of the potential, we also have \( W'(x)/|x| = \nabla W(x) \). Therefore, the expectation of the second term in a Gibbs state \( \omega_\lambda \) is given by

\[-\frac{1}{3} \delta_{j,k} \omega_\lambda(\nu_x)\]

To treat the first term of \( w^j_{k,x} \), as well as the first two terms of \( w^4_{k,x} \), we will transform these terms to a frame where the Gibbs state has zero total moment, so that we can more easily use invariance under reflections in space. E.g., from (10.11) we get

\[
\omega_\lambda(\nabla_j a_x^+ \nabla_k a_x) = \omega_\lambda(\gamma a(\nabla_j a_x^+ \nabla_k a_x)) = \omega_\lambda(\nabla_j a_x^+ \nabla_k a_x + \alpha_j \alpha_k \omega_\lambda(a_x^+ a_x) + i \omega_\lambda(\alpha_j a_x^+ \nabla_k a_x - \alpha_k \nabla_j a_x^+ a_x)
\]

As the total momentum has zero expectation in \( \omega_\lambda \), the last term vanishes for all \( j, k = 1, 2, 3 \). By reflection symmetry and the definition of the kinetic energy we have

\[
\omega_\lambda(\nabla_j a_x^+ \nabla_k a_x) = \frac{2}{3} e_{\text{kin}}(\beta, \mu + \frac{1}{2}|\alpha|^2)
\]

By combining the above relations we obtain

\[
\omega_\lambda(w^j_{k,x}) = \alpha_j \alpha_k \omega_\lambda(a_x^0) + \frac{1}{3} \delta_{j,k} \left[ 2 e_{\text{kin}}(\beta, \mu + \frac{1}{2}|\alpha|^2) - \omega_\lambda(\nu_x) \right] = \alpha_j \alpha_k \omega_\lambda(a_x^0) + \frac{1}{\beta} P(\beta, \alpha, \mu)
\]

where, for the last equality, we have used the virial theorem and (10.13).

To compute the energy current, \( \omega_\lambda(w^4_{k,x}) \), we need to consider the following expectations:

\[
\begin{align*}
i \omega_\lambda(\nabla_k a_x^+ a_y^+ a_y a_x - a_x^+ a_y^+ a_y \nabla_k a_x) \\
i \omega_\lambda(a_x^+ \nabla_j a_y^+ a_y a_x - a_x^+ a_y^+ \nabla_j a_y a_x) \\
i \omega_\lambda(\nabla_k a_x^+ \Delta a_x - \Delta a_x^+ \nabla_k a_x)
\end{align*}
\]
Again, we use (10.11) to relate these expectation to expectations in $\omega_\lambda$. The first expectation becomes:

$$i\omega_\lambda(\nabla_k a^+ a_y a_y a_x - a^+_x a^+_y a_y \nabla_k a_x) + 2\alpha_k \omega_\lambda(a^+_x a^+_y a_y a_x)$$  \hspace{1cm} (10.14)

The first term of this expression vanishes by symmetry. In the same way we find

$$i\omega_\lambda(a^+_x \nabla_j a^+_y a_y a_x - a^+_x a^+_y \nabla_j a_y a_x) = 2\alpha_j \omega_\lambda(a^+_x a^+_y a_y a_x) .$$  \hspace{1cm} (10.15)

We treat the third expression with similar arguments:

$$i\omega_\lambda(\nabla_k a^+_x \Delta a_x - \Delta a^+_x \nabla_k a_x) = i\omega_\lambda(\gamma_\alpha((\nabla_k a^+_x \Delta a_x - \Delta a^+_x \nabla_k a_x))$$

$$= i\omega_\lambda(\nabla_k a^+_x \Delta a_x + 2i\nabla_k a^+_x (\alpha \cdot \nabla) a_x - |\alpha|^2 \nabla_k a^+_x a_x$$

$$- i\alpha_k a^+_x \Delta a_x + 2\alpha_k a^+_x (\alpha \cdot \nabla) a_x + i\alpha_k |\alpha|^2 a^+_x a_x)$$

+ complex conjugate

$$= -\alpha_k \omega_\lambda(2|\alpha|^2 a^+_x a_x + 4\nabla_k a^+_x \nabla_k a_x - 2a^+_x \Delta a_x)$$

Then, by using integration by parts and reflection symmetry we get the following expression:

$$i\omega_\lambda(\nabla_k a^+_x \Delta a_x - \Delta a^+_x \nabla_k a_x) = -4\alpha_k \left[ \frac{1}{2}|\alpha|^2 \omega_\lambda(u^0_x) + \frac{5}{3} e_{\text{kin}}(\lambda) \right]$$  \hspace{1cm} (10.16)

Recall the expression for the energy current:

$$w^d_{k,x}(t) = -\frac{i}{4} \left[ \nabla_k a^+_x \Delta a_x - \Delta a^+_x \nabla_k a_x \right] + \frac{i}{4} \int dy W(x-y) \left[ \nabla_k a^+_x a^+_y a_y a_x - a^+_x a^+_y a_y \nabla_k a_x \right]$$

$$- \frac{i}{4} \int \left[ W'(x-y) \frac{(x-y)k(x-y)j}{|x-y|} \left[ a^+_x \nabla_j a^+_y a_y a_x - a^+_x a^+_y \nabla_j a_y a_x \right] \right]$$

Using (10.16), we see that the expectation of the first term in $\omega_\lambda$ equals

$$\alpha_k \left[ \frac{5}{3} e_{\text{kin}}(\lambda) + \frac{1}{2}|\alpha|^2 \omega_\lambda(u^0_x) \right]$$

For middle term we use (10.15) and find

$$\alpha_k \omega_\lambda \left( \frac{1}{2} \int dy W(x-y) a^+_x a^+_y a_y a_x \right)$$

Similarly, for the last term we get

$$-\frac{1}{2} \alpha_k \omega_\lambda \left( \int dy \left[ W'(x-y) \frac{(x-y)k(x-y)j}{|x-y|} \right] a^+_x a^+_y a_y a_x \right) = -\frac{1}{3} \alpha_k \omega_\lambda(\nu_x)$$

$$= \alpha_k \left[ P(\lambda) - \frac{2}{3} e_{\text{kin}}(\lambda) \right]$$

where we have used the definition of $\nu_x$ (10.3) and the virial theorem (Theorem 10.1).

By combining the three terms and applying the relation (10.12) one obtains the expression for $\omega_\lambda(w^d_{k,x}^2)$ given in the statement of this proposition.
11 Appendix. The entropy inequality

Our arguments rely in a crucial way on the following entropy inequality (3.17): For any pair of density matrices $\gamma$ and $\omega$, and for all self-adjoint $h$, and any $\delta > 0$, one has

$$\gamma(h) \leq \delta^{-1} \log \text{Tr} e^{h+\log \omega} + \delta^{-1} S(\gamma|\omega)$$

(11.1)

The inequality holds in the more general context of normal faithful states on a von Neumann algebra [17]. Here we give a proof for density matrices that emphasizes the connection with the variational principle of statistical mechanics.

**Proof:** Let $h$ be self-adjoint, and $\beta > 0$. The variational principle of statistical mechanics [18] states that

$$-\frac{1}{\beta} \text{Tr} e^{-\beta H} = \inf_{\gamma} \left[ \text{Tr} \gamma H - \beta^{-1} S(\gamma) \right]$$

where the infimum is taken over density matrices $\gamma$, and $S(\gamma) := -\text{Tr} \gamma \log \gamma$, is the von Neumann entropy of $\gamma$. For any non-singular density matrix $\omega$, define $H = -(\beta^{-1}(h+\log \omega))$, and take $\beta = \delta$, use

$$S(\gamma|\omega) = \text{Tr} \gamma (\log \gamma - \log \omega) = -\text{Tr} \gamma \log(\omega) - S(\gamma)$$

and rearrange the resulting inequality to obtain (11.1).

Equality in (11.1) holds if and only if

$$\gamma = \frac{e^{h+\log \omega}}{\text{Tr} e^{h+\log \omega}}.$$

The inequality (11.1) can also be turned around:

$$S(\gamma|\omega) \leq \gamma(h) - \log \text{Tr} e^{h+\log \omega},$$

(11.2)

and one can then take the sup over $h$ to obtain a characterization of the relative entropy (as was done [17]):

$$S(\gamma|\omega) = \sup_h \left[ \gamma(h) - \log \text{Tr} e^{h+\log \omega} \right]$$

(11.3)

with equality iff $\omega = e^{-h}/\text{Tr} e^{-h}$, i.e., iff $h = \log D_\omega + \text{constant} \times \mathbb{1}$.

In contrast to the classical case, if $\log \omega$ and $h$ do not commute, we generally have

$$\log \text{Tr} e^{h+\log \omega} \neq \log \text{Tr} \omega e^h.$$

However, due to the Golden-Thompson inequality, i.e., for any pair of self-adjoint $A$ and $B$,

$$Tr e^{A+B} \leq Tr e^A e^B,$$

we still have

$$Tr \gamma h - \log \text{Tr} \omega e^h \leq S(\gamma|\omega).$$

Whenever $\omega$ and $\gamma$ do not commute, the equality will be strict for all $h$. 
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