Remarks on Newton’s second law for variable mass systems

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Abstract
Misinterpretations of Newton’s second law for variable mass systems found in the literature are addressed. In particular, it is shown that Newton’s second law in the form \( F = \dot{\mathbf{p}} \) is valid for variable mass systems in general, contrary to the claims by some authors that it is not applicable to such systems in general. In addition, Newton’s second law in the form \( \mathbf{F} = m \mathbf{v} \) — commonly regarded as valid only for constant mass systems — is shown to be valid for variable mass systems also. Furthermore, it is argued that \( \mathbf{F} = m \mathbf{v} \) may be considered as a fundamental law, as an alternative to \( \mathbf{F} = \dot{\mathbf{p}} \). The present work should be of direct relevance to both instructors and students who are engaged in teaching and/or learning classical mechanics in general and variable mass systems in particular at middle- to upper-level university physics.

Keywords: variable mass system, Newton’s second law, classical mechanics

(Some figures may appear in colour only in the online journal)

1. Introduction

In most standard textbooks [1, 2], Newton’s second law in the form

\[
\mathbf{\ddot{F}} = \frac{d\dot{\mathbf{p}}}{dt} = \dot{\mathbf{p}}
\]

(1)

is considered to be valid for all systems, and when the system’s mass is constant, it reduces to

\[
\mathbf{\ddot{F}} = m \frac{d\mathbf{v}}{dt} = m\mathbf{\ddot{v}}.
\]

(2)

In the above equations \( \dot{\mathbf{p}} \) stands for the (linear) momentum defined as \( \dot{\mathbf{p}} \equiv m\mathbf{v} \), with \( m \) denoting the mass of the system and \( \mathbf{v} \), its velocity.
The application of Newton’s second law to variable mass systems has been a source of some confusion and disagreement in the past, and some authors [3–9] have addressed this issue specifically. They all conclude that Newton’s second law in the form given by equation (1) does not hold for variable mass systems in general, and that it is no more general than equation (2). More specifically, the authors of [3–6, 9] state that equation (1) applies only to two special situations, while the authors of [7, 8] simply state that equation (1) is not valid for variable mass systems. The major reasons for the authors arriving at this conclusion are twofold. One is that, for variable mass systems, it is well known that the correct equation-of-motion describing such systems is [10]

$$\dot{p} = \frac{d\dot{\rho}}{dt} - \frac{dm\ddot{u}}{dt} \equiv \dot{p} - m\ddot{u},$$

as derived from momentum conservation theorem; it can be derived from other approaches as well [3, 5, 9]. In the above equation, $\ddot{u}$ denotes the velocity of the amount of mass of the system being changed at the rate $\dot{m}$. The other reason, as pointed out in [8], is that equation (1) apparently leads to a violation of the Galilean principle of relativity, for, in a primed inertial frame moving with a constant velocity $\vec{v}_0$ with respect to the unprimed inertial frame, $\vec{v}' = \vec{v} + \vec{v}_0$, equation (1) leads to

$$F' = \ddot{p}' = \frac{d}{dt}(mv') = m\ddot{v}' + m\ddot{v}' + m\ddot{v}_0.$$  

In the following—contrary to the claims found in the literature [3–9]—we show that equation (1) is valid for variable mass systems in general, and that no violation of Galilei’s principle of relativity exists. In addition, Newton’s second law in form (2)—commonly regarded as valid only for constant mass systems—is shown to be valid for variable mass systems as well. Also, it is argued that equation (2) may be considered as a fundamental law, as an alternative to equation (1) or to equation (3) for that matter. It should be made clear that our criticism to [3–9] is not about the derivations/calculations presented there, but to the statement that equation (1) is not valid in general for variable mass systems. Even though some of the aspects in the following have been discussed in the past, we include them here as an integral part of addressing the issue at hand.

The present work should be of direct relevance to both instructors and students alike who are involved in teaching and/or learning classical mechanics in general and variable mass systems in particular at middle- to upper-level university physics. By clarifying the existing issues on variable mass systems in the literature, this article should help avoid potential confusions and misinterpretations by those engaged in teaching and/or learning this topic. Furthermore, it offers a choice to treat the variable mass systems on the same footing as the constant mass systems—a feature that greatly facilitates solving problems involving variable mass systems in practice.

2. Analysis

We begin by focusing our attention on the term $\dot{m}\ddot{u}$ appearing in equation (3). The key issue here is to realize that the system in which the mass is changing cannot be a closed (or isolated) system in general\(^1\) and that $\dot{m}\ddot{u}$ is a part of the external force. It accounts for the rate at which the momentum is being transferred to or removed from the system. It is this transferred momentum that is responsible, for example, for the recoil effect. In fact, when a

\(^1\) A closed system is a system where the total external force is zero.
gun is fired, the expelled bullet exerts a force on the gun which causes it to recoil, or when a rocket expels fuel in the form of exhaust, the latter exerts a force on the rocket providing the thrust to the rocket. The fact that \( \dot{m} \ddot{u} \) belongs to external force is further discussed in appendix A. Accordingly, we may incorporate \( \dot{m} \ddot{u} \) into \( \vec{F} \), so that equation (3) becomes of form (1), with \( \vec{F} \) being the total external force including

\[
\vec{F}^{\text{mass}}_u \equiv \dot{m} \ddot{u}
\]

associated with the change in mass, i.e. \( \vec{F} + \vec{F}^{\text{mass}}_u \rightarrow \vec{F} \).

Note that the term \( \dot{m} \ddot{u} \) is a velocity-dependent force which is (inertial) frame-dependent. This is different from velocity-dependent forces like friction, where the dependence is on the velocity relative to the system. It arises directly from the considerations based on momentum conservation theorem (see, for example, [10]) or Newton’s second law (2) for constant mass systems as shown in appendix A. Expressing \( \ddot{u} \) as

\[
\ddot{u} = \ddot{v} + \dddot{v}_{\text{rel}},
\]

where \( \dddot{v}_{\text{rel}} \) denotes the relative velocity of the mass being transferred measured with respect to the system, we obtain \( \dot{m} \dddot{v}_{\text{rel}} + \dot{m} \ddot{v} \), showing that the (inertial) frame dependence arises from the last term, \( \dot{m} \ddot{v} \). Its presence is a serious issue at first sight. However, this term is exactly what is needed to cancel the same term contained in \( \dddot{b} = m \dddot{v} + \dot{m} \ddot{v} \) appearing in both equations (1) and (3); hence it is of no consequence to the resulting equation-of-motion and both equations (1) and (3) are invariant under Galilean transformation, as they should be.

From the above observation, we conclude that Newton’s second law in the form of equation (1) is valid for all systems in general, provided we account for the term \( \vec{F}^{\text{mass}}_u \) given by equation (5) in the external force. Variable mass systems are not closed systems and, as such, they are always subject to a non-vanishing external force \( \vec{F}^{\text{mass}}_u \).

Reference to the ‘spurious’ (inertial) frame-dependent force, \( \dot{m} \ddot{v} \), can be eliminated from equations (1), (3) by rewriting them in the form given by equation (2), with \( \vec{F} \) being the total external force including

\[
\vec{F}^{\text{mass}} \equiv \dot{m} \dddot{v}_{\text{rel}},
\]

instead of \( \vec{F}^{\text{mass}}_u \) which enters in equation (1). Thus, Newton’s second law in the form of equation (2) is also valid for a variable mass system, provided we include the force \( \vec{F}^{\text{mass}} \) given by equation (7) in the external force, i.e. \( \vec{F} + \vec{F}^{\text{mass}} \rightarrow \vec{F} \). In other words, variable mass systems can be treated in the same way as constant mass systems by considering an external dissipative force \( \vec{F}^{\text{mass}} \), analogous to the usual frictional force, \( \vec{F}^{\text{friction}} = -k \dddot{v}_{\text{rel}} \).

The preceding considerations reveal that Newton’s second law can be written in different forms as given by equations (1)–(3), provided one takes into account the external forces appropriately:

\[
\text{Eq. (1): } \vec{F} = \dddot{b}, \quad (\vec{F} \text{ includes } \vec{F}^{\text{mass}}_u),
\]

\[
\text{Eq. (2): } \vec{F} = m \dddot{v}, \quad (\vec{F} \text{ includes } \vec{F}^{\text{mass}}),
\]

\[
\text{Eq. (3): } \vec{F} = \dddot{b} - \dot{m} \ddot{u}, \quad (\vec{F} \text{ excludes } \vec{F}^{\text{mass}}_u).
\]

In other words, as far as for practical application purposes, they are all equivalent forms and are valid for both constant and variable mass systems in general.

We now turn our attention to a more conceptual issue involving equations (8a)–(8c). When we study a dynamical system, we consider the problem in a way whereby the system under study does not accelerate when the net force acting on it vanishes. Newton’s second law
given by equations (8a), (8b) is then formulated to describe the dynamics of such a system, with \( \mathcal{F} \) appearing in those equations as the net force acting on the system. This scheme leads to the definition of the system under consideration and the total external force by the following two conditions:

(i) The total external force is the net force acting on the system under consideration.
(ii) A system under consideration is the one which does not accelerate when the total external force acting on it vanishes.

The first condition is the usual definition of external force. The second condition specifies what exactly constitutes a system under consideration. We note that condition (ii) is a redundancy for the case of constant mass systems, which is the reason why we do not usually see this condition stated explicitly.

Much of the confusion and misinterpretation related to variable mass systems arises due to the lack of an explicit statement regarding what constitutes the system under consideration in particular. We see that in equation (8c), \( \mathcal{F} \) cannot be the total external force according to (i) and (ii), for the system may accelerate even in its absence. But, the form given by equation (8c) is obtained by calculating the variation in momentum using momentum conservation theorem and, consequently, \( \mathcal{F} \) appearing in that equation should be understood as the total external force apart from that already taken into account explicitly when calculating the variation in momentum. On the other hand, \( \mathcal{F} \) in Newton’s second law in the forms given by equations (8a), (8b) is meant to be the total external force. Then, in equation (8a), \( \mathcal{F} \) can never vanish for variable mass systems (as has been pointed out already); otherwise the system accelerates in its absence. In equation (8b), the system will not accelerate when \( \mathcal{F} = 0 \) even for variable mass systems. Thus, in the form given by equation (8b), the total external force is free from any constraints. Furthermore, and perhaps more importantly, in contrast to equation (8a), the alternative form given by equation (8b) is free from the spurious frame-dependent force \( \dot{m} \tilde{v} \). For these reasons, we may choose to adopt this latter form as a fundamental law, as an alternative to equation (8a) (or to equation (8c) for that matter).

At this point, a remark is in order. In textbooks, it is common to find the statement that Newton’s second law (1) is also a statement of momentum conservation of a closed system. However, as we have seen above, equation (1) (or equation (8a)) does not allow a mass varying system to be a closed system. In the context of that equation, to obtain momentum conservation \( (\dot{p} = 0) \) for variable mass systems, one should first cancel the Galilean-invariance-violating term \( \dot{m} \tilde{v} \) on the right-hand side of that equation by the same term present in the external force \( \mathcal{F} \), and then, set \( \dot{m} \tilde{v}_{\text{rel}} = -\dot{m} \tilde{v} \). This is most clearly seen from equation (8b) which leads to momentum conservation only when \( \mathcal{F} = \mathcal{F}^{\text{mass}} = \dot{m} \tilde{v}_{\text{rel}} = -\dot{m} \tilde{v} \), i.e. when there is an external force acting on the system.

For completeness, in appendix B, we treat two special cases we encounter in problems involving variable mass systems.

3. Lagrangian formalism for variable mass systems

Variable mass systems can also be treated most straightforwardly in Lagrangian formalism if one realizes that the force associated with varying mass is \( \mathcal{F}^{\text{mass}} \) given by equation (7). It is true that since \( \mathcal{F}^{\text{mass}} \) is (relative) velocity-dependent, as the frictional force is, it cannot be represented by a potential and, thereby, it cannot be included in a Lagrangian. However, it can be included in Lagrangian formalism as a non-conservative force. The resulting Euler–Lagrange equation is, then, given by
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = \mathcal{F}_{\text{mass}}, \quad (9) \]

where \( L = m \ddot{\mathbf{v}}^2/2 - U(q) \) is the Lagrangian exactly as for the constant mass system, i.e. with the mass entering in it (both in the kinetic and potential energies) treated as constant. The generalized force \( \mathcal{F}_{\text{mass}} \) associated with \( F_{\text{mass}} \) is given by

\[ \mathcal{F}_{\text{mass}} = F_{\text{mass}} \left( \frac{\partial \mathbf{r}}{\partial q} \right) = \dot{m} \tilde{v}_{\text{rel}} \cdot \left( \frac{\partial \mathbf{r}}{\partial q} \right). \quad (10) \]

The above result is in accordance with Newton’s second law in the form given by equation (8b).

Of course, if we wish, we could let the mass entering in the Lagrangian in equation (9) be varying in time too, i.e. \( m = m(t) \) everywhere. This is in accordance with Newton’s second law in its form given by equation (8a). In this case, the Euler–Lagrange equation becomes

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = \mathcal{F}_{u}^\text{mass}, \quad (11) \]

where the generalized force \( \mathcal{F}_{u}^\text{mass} \) is now associated with \( F_{u}^\text{mass} \) given by equation (5), i.e.

\[ \mathcal{F}_{u}^\text{mass} = F_{u}^\text{mass} \left( \frac{\partial \mathbf{r}}{\partial q} \right) = \dot{m} \mathbf{u} \cdot \left( \frac{\partial \mathbf{r}}{\partial q} \right), \quad (\mathbf{u} = \mathbf{\dot{v}} + \mathbf{\tilde{v}}_{\text{rel}}). \quad (12) \]

The contribution to the generalized force due to \( \mathbf{\tilde{v}} \) will be cancelled exactly by the contribution arising from the time dependence of the mass in the Lagrangian in the left-hand side of equation (11). The latter form (equations (11), (12)) is precisely how the Lagrangian approach for variable mass systems has been presented in [8], for example.

4. Conclusion

In conclusion, for practical applications, the equation for describing the variable mass systems can be written in equivalent forms as given by equations (8a)–(8e). In particular, contrary to many claims in the literature, Newton’s second law in its form given by equation (8a) is applicable to both the constant and variable mass systems in general. In addition, we have also shown that Newton’s second law in the form given by equation (8b)—commonly regarded as applicable only to constant mass systems—is, in fact, valid for both constant and variable mass systems. Moreover, from a conceptual point of view, equation (8b) may be considered as a fundamental law, as an alternative to equation (8a), in accordance with the definitions of the system under consideration and the (total) external force given by items (i) and (ii).

Treating variable mass systems on the same footing with constant mass systems greatly facilitates solving problems involving such systems in practice, since all that is required is to consider, in equation (8b), the dissipative external force \( \mathcal{F}_{\text{mass}} \) given by equation (7). From a pedagogical point of view, this offers the possibility of introducing variable mass systems even to first year physics students.

Finally, we mention that, according to Truesdell [11], Euler has presented the form given by equation (2) for the first time ‘as general, explicit equation for mechanical problems of all kinds’ in a paper published in 1752, ‘Discovery of a new principle of Mechanics.’ In this regard, according to [4], form (2) is not to be found in Newton’s ‘Principia’. Perhaps, if Euler...
meant to include the variable mass systems in his ‘mechanical systems of all kinds’, we have just rediscovered his work of over two and a half centuries ago.

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**Appendix A**

Here, we re-derive equation (3)—usually obtained [10] from momentum conservation theorem—from Newton’s second law in form (2) applied to a constant mass system. This allows us to better address the issue of external force. To this end, we closely follow [9]². Consider a system consisting of two subsystems. We identify one of them (subsystem-1) as body-1 and the other (subsystem-2) as body-2. The two subsystems undergo incremental mass changes, increasing or decreasing. The mass of body-1 (body-2) and its velocity—measured with respect to an inertial reference frame—at an arbitrary instant \( t \) is \( m_1 \) (\( m_2 \)) and \( v_1 \) (\( v_2 \)), respectively. To be specific, we assume that body-2 consists of many tiny masses \( dm_2 \), all of which have a velocity \( v_2 \) (see figure A1). As the motion progresses, the two subsystems coalesce and subsystem-2 transfers a part of its mass, \( dm_2 \), to subsystem-1. The total system has a constant mass \( M = m_1 + m_2 \).

Figure A1. A system consisting of subsystems 1 and 2. The former has mass \( m_1 \) and velocity \( v_1 \). The latter consists of many tiny masses \( dm_2 \), all of which have velocity \( v_2 \). The total mass of subsystem-2 is \( m_2 \) and its velocity is \( v_2 \). As the motion progresses, the two subsystems coalesce and subsystem-2 transfers a part of its mass, \( dm_2 \), to subsystem-1. The total system has a constant mass \( M = m_1 + m_2 \).

² The derivation can be performed more sophisticatedly in terms of momentum flux by considering a velocity field associated with a continuous changing-mass distribution [3, 5]. However, for the present purpose, it suffices to be treated in a simplified way. One arrives at the same conclusion either way.
The above force, acting on the total system, may be decomposed into the forces acting on body-1 ($\mathbf{F}_1$) and body-2 ($\mathbf{F}_2$), separately, i.e. $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$, with
\[
\mathbf{F}_1 = m_1 \dot{v}_1 + m_1 (\ddot{v}_1 - \ddot{v}_2), \\
\mathbf{F}_2 = m_2 \dot{v}_2.
\] (A.2)

We note that the above identification of the forces should be the correct one, which is in accordance with the limiting case of body-2 losing its mass to body-1 completely. In this case, one should have only body-1 remaining in the system with mass $m_1 + m_2$. In particular, $\mathbf{F}_2 = 0$ when $m_2 = 0$. From a more physical point of view, $m_1 \ddot{v}_1 = -m_2 \ddot{v}_2$ is the rate at which the momentum is being transferred from body-2 to body-1 due to the transfer of the amount of mass $dm_2$ from body-2 to body-1. It is this transferred momentum that is responsible, for example, for the recoil effect when a gun is fired. Of course, in this case the gun loses a part of its mass (the bullet is expelled) instead of gaining it, i.e. $m_1 < 0$. The recoil force is an external force. Equation (A.2) can be rewritten as
\[
\mathbf{F}_1 = \frac{d}{dt}(m_1 \dot{v}_1) - \dot{m}_1 \dot{v}_2, \\
\mathbf{F}_2 = \frac{d}{dt}(m_2 \dot{v}_2) - \dot{m}_2 \dot{v}_2,
\] (A.3)
which reveals that the force acting upon each subsystem, where the mass is varying in time, takes the generic form as given by equation (3), i.e. we reproduce the result obtained from momentum conservation theorem.

**Appendix B**

There are two special cases in equation (8b) that deserve close attention:

(a) Mass is transferred at a relative velocity $\ddot{v}_{\text{rel}} = 0$ (or, equivalently, $\ddot{u} = \ddot{v}$ in equations (8a), (8c)). In this case, the external force $\mathbf{F}^{\text{mass}}$—which is a part of the total external force $\mathbf{F}$ in equation (8b)—vanishes, and the equation-of-motion becomes exactly the same as the case of a system with constant mass, even though the mass is changing. This is the situation we encounter, for example, when a star loses a part of its mass in the form of radiation/heat with an isotropic distribution with respect to the star or in the problem of a stretched rope sliding over the edge of a table.

(b) Mass is transferred at $\ddot{v}_{\text{rel}} = -\ddot{v}$ (or $\ddot{u} = 0$ in equations (8a), (8c)), in which case, equation (8b) becomes $\ddot{F} = m \ddot{v} + \dot{m} \ddot{v} = \ddot{p}$. Here, at first sight, we might think that this equation violates Galilean invariance because of the presence of the term $\dot{m} \ddot{v}$. However, this term is due to the particular value taken by $\ddot{v}_{\text{rel}}$; thus, it does not cause any problems. This corresponds to the situation, for example, of the accretion of a water drop falling through a stationary cloud or a conveyor belt on the ground transporting objects which are dropped on the belt vertically from above.

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3 Note that in the example discussed here, the velocity of the mass being removed from body-2 is the same as that of body-2 itself ($\ddot{u} = \ddot{v}_2$). Of course, in general, these velocities may differ from each other, as is the case for body-1 in the example considered.
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References

[1] Goldstein H, Poole C and Safko J 2001 Classical Mechanics 3rd edn (Reading, MA: Addison-Wesley)
[2] Thornton S T and Marion J B 2003 Classical Dynamics of Particles and Systems 5th edn (Belmont, CA: Thomson Brooks)
[3] Thorpe J F 1962 On the momentum theorem for a continuous system of variable mass Am. J. Phys. 30 637
[4] Arons A B and Bork A M 1964 Newton’s laws of motion and the 17th century laws of impact Am. J. Phys. 32 313
[5] Tiersten M S 1969 Force, momentum change, and motion Am. J. Phys. 37 82
[6] Siegel S 1972 More about variable mass systems Am. J. Phys. 40 183
[7] Kleppner D and Kolenkow R 2004 An Introduction to Mechanics 2nd edn (Cambridge: Cambridge University Press)
[8] Plastino A R and Muzzio J C 1992 On the use and abuse of Newton’s second law for variable mass problems CeMDA 53 227
[9] de Sousa C A and Rodrigues V H 2004 Mass redistribution in variable mass systems Eur. J. Phys. 25 41
[10] Sommerfeld A 1952 Mechanics, Lectures on Theoretical Physics vol 1 (New York: Academic)
[11] Truesdel C 1960 A program toward rediscovering the rational mechanics of the age of reason Arch. Hist. Exact Sci. 1 3