Lie Algebra of Hamiltonian Vector Fields and the Poisson-Vlasov Equations

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Abstract: We introduce natural differential geometric structures underlying the Poisson-Vlasov equations in momentum variables. First, we decompose the space of all vector fields over particle phase space into a semi-direct product algebra of Hamiltonian vector fields and its complement. The latter is related to dual space of the Lie algebra. We identify generators of homotheties as dynamically irrelevant vector fields in the complement. Lie algebra of Hamiltonian vector fields is isomorphic to the space of all Lagrangian submanifolds with respect to Tulczyjew symplectic structure. This is obtained as tangent space at the identity of the group of canonical diffeomorphisms represented as space of sections of a trivial bundle. We obtain the momentum-Vlasov equations as vertical equivalence, or representative, of complete cotangent lift of Hamiltonian vector field generating particle motion. Vertical representatives can be described by holonomic lift from a Whitney product to a Tulczyjew symplectic space. We show that vertical representatives of complete cotangent lifts form an integrable subbundle of this Tulczyjew space. A generalization of complete cotangent lift is obtained by a Lie algebra homomorphism from the algebra of symmetric contravariant tensor fields with Schouten concomitant to the Lie algebra of Hamiltonian vector fields. Momentum maps for particular subalgebras of symmetric contravariant tensors result in plasma-to-fluid map in momentum variables of Vlasov equations. We exhibit dynamical relations between Lie algebras of Hamiltonian vector fields and of contact vector fields, in particular; infinitesimal quantomorphisms on their quantization bundle. A diagram connecting these kinetic and fluid theories is presented. Gauge symmetries of particle motion are extended to tensorial objects including complete lift of particle motion. Poisson equation is then obtained as zero value of momentum map for the Hamiltonian action of gauge symmetries for kinematical description.

1 This is an expanded (with the additions of more remarks and, sections 4 and 5.4) version of the article “Geometry of plasma dynamics II: Lie algebra of Hamiltonian vector fields” to appear in Journal of Geometric Mechanics, 2012.
1 Introduction

The dynamics of collisionless plasma is governed by the Poisson-Vlasov equations

\[ \nabla_q^2 \phi_f(q) = -e \int f(q,p) d^3p \] (1)

\[ \frac{\partial f}{\partial t} + \frac{p}{m} \cdot \nabla_q f - e \nabla_q \phi_f \cdot \nabla_p f = 0 \] (2)

where \( f(q,p) \) is the density of plasma particles and \( \phi_f(q) \) is the electric potential depending on the density through the Poisson equation (1). The underlying geometric structure of the Poisson-Vlasov system of differential equations was made available by their Hamiltonian formulation [44], [83], [45], [46], [20], [47]. The Vlasov equation was shown to be the Lie-Poisson equation on the dual of Lie algebra of group of canonical diffeomorphisms of particle phase space \( T^*Q \), identified with the space of densities [38], [39], [40]. This identification was obtained as the dual of the isomorphism between the Lie bracket algebra of Hamiltonian vector fields and the Poisson bracket algebra of functions on \( T^*Q \) defined up to addition of a constant. Based on this and with reference to the work of Van Hove in [77], it was already (foot-)noted in [38] that the correct configuration space for plasma dynamics is the group of transformations of \( T^*Q \times \mathbb{R} \) preserving the contact one-form, also known as the group of quantomorphisms.

In [24], adapting the group of canonical diffeomorphisms as configuration space, we obtained Poisson and Vlasov equations in Eulerian momentum variables which, by symmetry reduction, define the plasma density function \( f \) and give Eqs.(1) and (2) as well as their Hamiltonian structure. In this work we shall elaborate geometric structures underlying plasma dynamics in momentum variables and, we shall indicate, at infinitesimal level, connection with quantomorphisms in the framework of kinetic theories.

1.1 Preliminaries and motivation

The Lie-Poisson construction starts with the kinematical description of particle motion on the phase space \( T^*Q \) where \( Q \subseteq \mathbb{R}^3 \) is the configuration space of particles [39]. Take a curve \( \varphi_t \) in the group \( G = \text{Diff}_{can}(T^*Q) \) of all canonical diffeomorphisms of \( T^*Q \) preserving the canonical symplectic two-form \( \Omega_{T^*Q} \) (see [4], [53], [52], [4], [42] for aspects of diffeomorphism groups). Given a point \( Z \in T^*Q \) regarded as a Lagrangian label, let \( z = (q,p) = \varphi_t(Z) = \varphi(Z,t) \) denote the Eulerian coordinates of plasma particles. The phase space velocity

\[ \dot{z} = \frac{d}{dt} \varphi_t(Z) = X(z,t) = X_t(\varphi_t(Z)) \] (3)

generates the flow \( \varphi_t \). Since \( \varphi_t \) is canonical, \( X \) is locally Hamiltonian. We assume that it is globally Hamiltonian and write \( h(z,t) \) for the corresponding Hamiltonian function so that \( X = X_h = \Omega_{T^*Q}(\cdot dh) \) on \( T^*Q \).
Remark 1 For diffeomorphisms groups, exponential map assigns to each vector the time-one map of its flow. This is neither surjective nor injective around identity \([5]\). However, restricted to compact support, any smooth Hamiltonian vector field on \(T^*Q\) generates a flow in \(G\). The group \(G\) of canonical (or Hamiltonian) diffeomorphisms is a subgroup in the identity component of the group of symplectomorphisms, that is, diffeomorphisms preserving a symplectic two-form. It is normal and path-connected \([42]\). See \([4]\) for a proof that \(G\) is a simple group. In fact, groups of symplectomorphisms, diffeomorphisms, volume preserving diffeomorphisms and contactomorphisms are simple indecomposable Lie groups in Cartan’s list \([42]\).

The gauge group of particle motion is the additive group \(\mathcal{F}(Q)\) of functions on \(Q\) acting on \(T^*Q\) by fiber translations. The Lie algebra \(g = \{\mathfrak{X}_{ham}(T^*Q); [-,-]\}\) of \(G\) consists of (smooth) Hamiltonian vector fields on \(T^*Q\) and \([\ , \ ]\ denotes the standard Jacobi-Lie bracket, with conventions as in \([1]\). The dual vector space \(g^*\) of \(g\) is the non-closed one-form densities on \(T^*Q\). Equivalently, the space of one-form densities whose symplectic duals have non-vanishing divergences

\[
\mathfrak{g}^* = \{\Pi_{id} \otimes d\mu \in \Lambda^1(T^*Q) \otimes Den(T^*Q) \mid \text{div}_{T^*Q} \Pi_{id} \neq \text{constant}\}
\] (4)

so that the pairing of \(g\) and \(g^*\) is weakly nondegenerate with respect to the \(L^2\)-norm \([24], [13]\). The further requirement in Eq.(4) that they be different from constants is of dynamical origin and will be explained later in section 2.2.

Reduction of canonical bracket on \(T^*G\) by right invariant extension, that is the invariance under particle relabelling symmetry, of functions on \(T^*G\) gives the +Lie-Poisson bracket

\[
\{K(\Pi_{id}), H(\Pi_{id})\}_{LP} = \int_{T^*\mathcal{G}} \Pi_{id}(z) \cdot \left[\frac{\delta K}{\delta \Pi_{id}(z)}, \frac{\delta H}{\delta \Pi_{id}(z)}\right] d\mu(z)
\] (5)
on \(g^*\), where \(\delta K/\delta \Pi_{id}(z)\) and \(\delta H/\delta \Pi_{id}(z)\) are regarded to be elements of \(g\) \([37]\), \(d\mu(z) = d^3q d^3p\) is the Liouville volume element on \(T^*Q\) and the bracket inside the integral is the Lie algebra bracket. In particular, for the right invariant Hamiltonian functional

\[
H_{LP}(\Pi_{id}) = \int_{T^*\mathcal{G}} \langle \Pi_{id}(z), X_{h_f}(z) \rangle d\mu(z),
\] (6)

involving the particle Hamiltonian

\[
h_f(z) = \frac{p^2}{2m} + e \phi_f(q),
\] (7)

which depends on the density, and the non-closed one-form \(\Pi_{id}(z) = \Pi_q \cdot dq + \Pi_p \cdot dp\), the Lie-Poisson equations on \(g^*\) are

\[
\dot{\Pi}_q = -X_h(\Pi_q) + e (\Pi_p \cdot \nabla_q) (\nabla_q \phi_f)
\] (8)

\[
\dot{\Pi}_p = -X_h(\Pi_p) - \frac{1}{m} \Pi_q
\] (9)
which are the momentum-Vlasov equations [24]. The unconventional factor $1/2$ in the potential term of the function $h_f$ is a manifestation of the nonlinearity arising from the constraint imposed by the Poisson equation [44], [46], [27]. By definition, the momentum variables $(\Pi_q, \Pi_p)$ represent equivalence classes up to additions of the terms $\nabla_q k(z)$ and $\nabla_p k(z)$, respectively, for arbitrary function $k(z)$. Thus, the reduced dynamics on $\mathfrak{g}^*$ has a further symmetry given by the action of the additive group $\mathcal{F}(T^*Q)$ of functions on $T^*Q$. The momentum map $\mathfrak{g}^* \to \mathcal{F}^*(T^*Q) = \text{Den}(T^*Q)$ given by the differential substitution

$$f(z) = \text{div}_{T^*Q} \Pi_{id}^T = \nabla_p \cdot \Pi_q(z) - \nabla_q \cdot \Pi_p(z)$$

(10)

defines the plasma density function $f$ [24], [13]. The reduction of the Lie-Poisson structure gives the Vlasov equation (2) in density variable as well as the non-canonical Hamiltonian structure defined by the Lie-Poisson bracket

$$\{K(f), H(f)\}_{LP} = \int_{T^*Q} f(z) \cdot \left[ \frac{\delta K}{\delta f(z)} \frac{\delta H}{\delta f(z)} \right] d\mu(z)$$
on Den$(T^*Q)$ and the Hamiltonian functional

$$H_{LP}(f) = \int_{T^*Q} h_f(z) f(z) d\mu(z).$$

**Remark 2** The symmetries of the momentum-Vlasov equations were used, in [24], to cast them into a canonical Hamiltonian formalism with a quadratic Hamiltonian functional. This leads Eqs. (5) and (6) to admit a variational formulations. Namely, the Lagrangian functional

$$L_0[\Pi_p] = \int_{T^*Q} \left( \frac{m}{2} X_h(\Pi_p) + \frac{d\Pi_p}{dt} \right)^2 - \frac{e}{2} \frac{\partial^2 \phi_h}{\partial q^i \partial q^j} \Pi^i \Pi^j(z) d\mu(z)$$

involving the velocity $d\Pi_p/dt$ shifted by the term $-X_h(\Pi_p)$, gives the Euler-Lagrange equations

$$\ddot{\Pi}^i(z) + 2 X_h(\Pi^i(z)) + X^2_h(\Pi^i(z)) + \frac{e}{m} \delta^{ij} \frac{\partial^2 \phi_h(q)}{\partial q^k \partial q^j} \Pi^k(z) = 0$$

which can also be obtained from Eqs. (5) and (6) by eliminating the variables $\Pi_i$.

In [24], we presented Lie-Poisson structures in momentum and density variables and establish the relations between the two. The formulation of dynamics in density variable is obtained by further reduction of momentum-Vlasov equations by the symmetry defining gauge equivalence classes of momentum variables. The gauge algebra is, as a vector space, shown to be the same as $\mathfrak{g}$ but with an action different from the coadjoint action. The Eulerian velocity and momentum variables are elements of $\mathfrak{g} = \mathfrak{x}_{\text{ham}}(T^*Q)$ and $\mathfrak{g}^*$, respectively. These variables are complementary in the vector space $TT^*Q$. Obviously, this and other geometric properties disappear upon identification of $\mathfrak{g}$ and $\mathfrak{g}^*$ with
function spaces \( F(T^*Q) \) and \( \text{Den}(T^*Q) \), respectively. Introduction of a formulation in the variables \( \Pi_{id} \) provides a computational advantage and this also prevents us from confusion in the geometry which may arise upon identification with function spaces. For example, the function \( h_f \) and the density \( f \) appear symmetrically in the Hamiltonian functional of the Lie-Poisson structure whereas the corresponding variables \( X_{hf} \) and \( \Pi_{id} \) in \( g \) and \( g^* \) are complementary in the sense that \( \Omega_{T^*Q}^\sharp(g) \) and \( g^* \) decompose the space of one-forms on \( T^*Q \) into spaces of exact and non-closed one-forms, respectively (c.f. section 2.1). The momentum-Vlasov equations in components of \( \Pi_{id} \) expresses the evolution of a volume cell, that is the density \( f \), in the phase space \( T^*Q \) in terms of its boundaries, that is, surfaces of the momenta \( \Pi_{id} \). This interpretation was first given by Ye and Morrison in [75] for the Clebsch variables \( (\alpha, \beta) \) defined by \( \{\alpha, \beta\}_{T^*Q} = f \). In the present context, they form a non-closed one-form \( \alpha d\beta \) and can be identified with \( \Pi_{id} \). The momentum formulation clarifies the geometric relation between the motions of plasma particles and the Lie-Poisson description of dynamics. It does become necessary to investigate the plasma dynamics described by the more basic momentum-Vlasov equations.

This observation is the starting point of the present work and motivates the elaboration of geometric setting underlying the momentum-Vlasov equations \([3]\) and \([9]\). Our aim is to analyse in detail the structures of Lie algebra of Hamiltonian vector fields and its dual in order to prepare a suitable background for application of Tulczyjew construction to orbits of canonical diffeomorphisms. As will be seen in the sequel, present formulation of dynamics on higher order tangent and cotangent bundles over \( T^*Q \) constitutes a useful model for investigation of orbital dynamics.

1.2 Content of the work

In this work, we shall elaborate the central part of the following diagram

\[
\begin{align*}
&TT^*Q \xrightarrow{\Omega^\sharp_{T^*Q}, \Omega^\flat_{T^*Q}} T^*T^*Q \\
&X_{\phi} \xrightarrow{\tau_{T^*Q}} X_{h} \quad \pi_{T^*Q} \xrightarrow{\Pi_{id}} \Phi_{\phi}
\end{align*}
\]

which summarizes the mapping properties with reference to particle phase space \( T^*Q \) of elements of \( T_\phi G, T_\phi^*G, g \) and \( g^* \); namely, \( X_{\phi}, \Pi_{\phi}, X_{h} \) and \( \Pi_{id} \), respectively. This may serve as the main diagram to relate the constructions of present work for Eulerian variables to the Lagrangian variables. We shall establish the precise relation between the particle motion, its symmetries and the Poisson-Vlasov equations. We shall obtain Poisson equation as a consequence of gauge symmetries of Hamiltonian description of motions of plasma particles.
In the next section, we shall first decompose the algebra of vector fields on $T^*Q$ into a semi-direct product algebra of Hamiltonian vector fields and its complement in $\mathfrak{X}(T^*Q)$. The latter, as a vector space, is isomorphic to dual space of the Lie algebra. Further properties of this decomposition will be studied. In particular, we shall identify homotheties as non-dynamical part of the dual of Lie algebra. We shall then present, according to the diagram,

$$
\begin{array}{ccc}
T^*T^*Q & \xleftarrow{\Omega_T} & T^*Q \\
\pi_{T^*Q} \searrow & & \swarrow \pi_{T^*Q} \\
\downarrow & & \\
\phantom{T^*T^*Q} & & \\
TT^*Q & \xrightarrow{\alpha_Q} & T^*TQ \\
\pi_{T^*Q} \searrow & & \swarrow \pi_{T^*Q} \\
\downarrow & & \\
\phantom{TT^*Q} & & \\
T^*Q & \searrow & \swarrow T^*\pi_Q \\
\downarrow & & \\
\phantom{T^*Q} & & \\
\phantom{T^*Q} & & \\
\phantom{T^*T^*Q} & & \\
\end{array}
$$

two special symplectic structures on $TT^*Q$ with the underlying symplectic manifold being endowed with the Tulczyjew two-form. Considering a description of configuration space as the space of sections of a trivial bundle, we shall show that the algebra of Hamiltonian vector fields is isomorphic to the space of Lagrangian submanifolds of Tulczyjew symplectic manifold.

In section three, starting with the Hamiltonian vector field generating the particle motion, we shall obtain the momentum-Vlasov equations as the vertical equivalence of its complete cotangent lift. This will be shown to be the same as the vertical lift of coadjoint action on momentum variables. As a result, we shall realize the commutative diagram

$$
\begin{array}{ccc}
\text{canonical Hamiltonian motion of particles on } T^*Q & \xrightarrow{\text{complete cotangent lift}} & \text{Hamiltonian motion on } T^*T^*Q \\
\downarrow & & \downarrow \\
\text{vertical lift of coadjoint action} & \searrow & \swarrow \text{vertical (jet) equivalence} \\
\downarrow & & \\
\text{momentum-Vlasov equations on } VT^*T^*Q & & \\
\end{array}
$$

connecting motion of individual plasma particles to Eulerian dynamics in momentum variables. Complete cotangent lifts are Hamiltonian vector fields with a degenerate Hamiltonian function for the canonical symplectic structure. We shall point out a Lagrangian formulation for them with a Morse family on certain Whitney product. We shall give a geometric description of vertical representative of cotangent lift in terms of holonomic lift operator from this Whitney product into the Tulczyjew symplectic space $TT^*T^*Q$.

In section four, we will define a Lie algebra homomorphism from the algebra of symmetric contravariant tensor fields with Schouten concomitant to the algebra $\mathfrak{X}_{ham}(T^*Q) = \mathfrak{g}$ of Hamiltonian vector fields. This will generalize the complete cotangent lift of vector fields to symmetric contravariant tensors. We will then obtain the moments of momentum-Vlasov dynamical variables. For particular subalgebras of symmetric contravariant tensors these moments will give plasma-to-fluid map in momentum variables of $\mathfrak{g}^*$. 

(11)
In section five, we establish a correspondence between the Lie algebra of Hamiltonian vector fields on $T^\ast \mathbb{Q}$ and the Lie algebra of infinitesimal strict contact transformations, also called quantomorphisms, of quantization bundle of $T^\ast \mathbb{Q}$. Relying on our recent work [13], we first present kinetic equations, both in momentum and density variables, of particles moving according to contact transformations of standard three-dimensional contact manifold. We then restrict the group of contactomorphisms to strict contact transformations and obtain a system of kinetic equations equivalent to the momentum-Vlasov equations in one dimension. This section will be concluded with a diagram summarizing the relations between various kinetic and fluid theories.

In section six, we expand on our earlier result in [24] where we described the Poisson equation as a kinematical constraint on the dynamics of Eulerian variables. More precisely, we shall show that the Poisson equation characterizes the set of zero values of the momentum map associated with the action of additive group of functions $\mathcal{F}(\mathbb{Q})$ on the position space $\mathbb{Q}$ of particles. This is the gauge group of particle motion on the canonical phase space $T^\ast \mathbb{Q}$. Momentum map realization of the Poisson equation implies that the true configuration space for the Poisson-Vlasov dynamics must be the semi-direct product space $\mathcal{F}(\mathbb{Q}) \ltimes \text{Diff}^\text{can}(T^\ast \mathbb{Q})$ with the action of $\mathcal{F}(\mathbb{Q})$ given by fiber translation on $T^\ast \mathbb{Q}$ and, by composition on right with the canonical transformations.

Section seven will be devoted to a summary and discussion of the results as well as some future work to be addressed elsewhere. See also the introductions of each section where we summarize contents in more technical terms.

1.3 Notations

General definitions will be given with reference to an arbitrary smooth manifold $\mathcal{M}$. $\theta_\mathcal{M}$, $\Omega_\mathcal{M}$ will be used for canonical one-form and symplectic two-form defined on $\mathcal{M}$. This convention of showing the space of definition as a subscript will be extended to other objects when necessary. $\Gamma(pr)$ will usually denote the space of sections of a bundle $pr : E \rightarrow B$. For spaces of sections of tangent and cotangent bundles of a manifold $\mathcal{M}$ we will use $\mathcal{X}(\mathcal{M})$ and $\Lambda^1(\mathcal{M})$, respectively. The bracket $\langle, \rangle_\mathcal{M}$ will be used for natural pairing between differential forms and vector fields over $\mathcal{M}$. $\mathcal{F}(\mathcal{M})$ and $\text{Den}(\mathcal{M})$ will denote spaces of functions (zero-forms) and volume forms on $\mathcal{M}$. $i_X$ and $\mathcal{L}_X$ will be used for the interior product (contraction) and the Lie derivative with respect to the vector field $X$. Throughout the work $G$, $g$ and $g^\ast$ will be used frequently for $\text{Diff}^\text{can}(T^\ast \mathbb{Q})$, its Lie algebra $\mathfrak{x}_{\text{ham}}(T^\ast \mathbb{Q})$ and the dual of the latter, respectively. If $q \in \mathbb{Q}$ then, we will use

$$\begin{align*}
(q, \dot{q}) & \in T_q \mathbb{Q}, & (q, \dot{q}, \lambda_q, \lambda_{\dot{q}}) & \in T^\ast_q T_q \mathbb{Q}, \\
(z, \pi) & \in T^\ast_z T^\ast_q \mathbb{Q}, & (z, \dot{z}, \lambda_z, \lambda_{\dot{z}}) & \in T^\ast_z T^\ast_q \mathbb{Q}, \\
(\zeta, \dot{z}) & \in T^\ast_z T^\ast_q \mathbb{Q}. & (\zeta, \dot{z}, \lambda_{\zeta}, \lambda_{\dot{z}}) & \in T^\ast_z T^\ast_q \mathbb{Q}.
\end{align*}$$

(12)

The canonical one-form on $T^\ast \mathbb{Q}$ will be $\theta_{T^\ast \mathbb{Q}}(z) = p \cdot dq$ and the symplectic two-form is $\Omega_{T^\ast \mathbb{Q}}(z) = dp \wedge dq$. 7
2 Lie Algebra of Hamiltonian Vector Fields

We shall show that the algebra of all vector fields on particle phase space can be decomposed into a semi-direct product of Hamiltonian vector fields and space of symplectic duals of all non-closed one-forms on particle phase space. Among the latter, those vector fields with constant divergence are related to homotheties on particle phase space and they correspond to constant plasma densities. We then introduce special symplectic structures and present the Tulczyjew symplectic structure on $TT^*Q$ relevant to a description of particle dynamics as Lagrangian submanifolds. Finally, we shall expand on the remark in [24] that the configuration space $G$ of plasma dynamics can be represented as space of Lagrangian submanifolds in the space of sections of a trivial bundle, and show that the Lie algebra of Hamiltonian vector fields can be obtained, as tangent space over the identity, to be spaces of all Lagrangian submanifolds of Tulczyjew symplectic space for particle dynamics.

2.1 Algebra of vector fields in $TT^*Q$

Let $\mathfrak{X}(T^*Q)$ and $\Lambda^1(T^*Q)$ denote the spaces of smooth sections of $TT^*Q \to T^*Q$ and $T^*T^*Q \to T^*Q$, respectively. The nondegeneracy of canonical symplectic form $\Omega_{T^*Q}$ on $T^*Q$ leads to the musical isomorphism $\Omega_{T^*Q}^*: \mathfrak{X}(T^*Q) \to \Lambda^1(T^*Q)$ defined, for arbitrary vector fields $X, Y \in \mathfrak{X}(T^*Q)$, by $\Omega_{T^*Q}^*(X)(Y) = \Omega_{T^*Q}(X, Y)$ or, alternatively, by $\Omega_{T^*Q}^*(X) = i_X \Omega_{T^*Q}$. The isomorphism $\Omega_{T^*Q}^*: \Lambda^1(T^*Q) \to \mathfrak{X}(T^*Q)$ is obtained by fiberwise inversion of $\Omega_{T^*Q}^*$.

If the image of a vector field $X$ by the mapping $\Omega_{T^*Q}^*$ is closed then $X$ is a locally Hamiltonian vector field, and hence, the space $\mathfrak{X}_{ham}(T^*Q)$ of Hamiltonian vector fields on $T^*Q$ is isomorphic to the space $\ker d$ of all closed one-forms on $T^*Q$. If $\Omega_{T^*Q}(X)$ is exact, we write $\Omega_{T^*Q}^*(X_h) = -dh$ and $X_h$ is said to be globally Hamiltonian [1], [37].

**Proposition 3** Let $\mathfrak{g}^* = \Omega_{T^*Q}^*(\mathfrak{g})$ and $(\mathfrak{g}^*)^\perp \subset \mathfrak{X}(T^*Q)$ denote the vector space of closed one-forms on $T^*Q$ and the image of $\mathfrak{g}^*$ under the mapping $\Omega_{T^*Q}^*$, respectively. With the isomorphism

$$
\Omega_{T^*Q}^*: \mathfrak{g} = \mathfrak{X}_{ham}(T^*Q) \leftrightarrow \mathfrak{g}^b = \ker d \cap \Lambda^1(T^*Q)
$$

we have the decompositions

$$
\Lambda^1(T^*Q) = \ker d \oplus \mathfrak{g}^* = \mathfrak{g}^b \oplus \mathfrak{g}^*
$$

$$
\mathfrak{X}(T^*Q) = \mathfrak{g} \oplus (\mathfrak{g}^*)^\perp
$$
of the spaces of one-forms and vector fields on $T^*Q$. Moreover, $\mathfrak{g}$ and $(\mathfrak{g}^* )^\sharp$ are Lie subalgebras, and $(\mathfrak{g}^* )^\sharp$ is an ideal of $\mathfrak{X}(T^* Q)$

$$[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g}, \quad [(\mathfrak{g}^* )^\sharp,(\mathfrak{g}^* )^\sharp] \subset (\mathfrak{g}^* )^\sharp, \quad [\mathfrak{g},(\mathfrak{g}^* )^\sharp] \subset (\mathfrak{g}^* )^\sharp.$$  

**Proof.** Eq. (15) follows from the definitions above. Since the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of Hamiltonian vector fields is defined to be non-closed one-form densities on $T^*Q$, the remaining elements of $\Lambda^1 (T^*Q)$, namely, closed one-forms constitute the underlying vector space of $\mathfrak{g}^\flat$. For decomposition in Eq. (16) we have that Hamiltonian vector fields are divergence-free with respect to the symplectic or Liouville volume $d\mu = \Omega^3_{T^* Q}$. Let $(\mathfrak{g}^* )^\sharp$ denote the image of dual $\mathfrak{g}^*$ of Lie algebra $\mathfrak{g}$ of Hamiltonian vector fields under the isomorphism $\Omega^3_{T^* Q}$. For a non-degenerate $L^2$-pairing of $\mathfrak{g}$ and $\mathfrak{g}^*$, $(\mathfrak{g}^* )^\sharp$ contains vector fields with non-vanishing divergences. In other words, $\alpha^\sharp \equiv \Omega^3_{T^* Q}(\alpha) \in (\mathfrak{g}^* )^\sharp$ is not Hamiltonian in any sense. $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g}$ follows from definition of (locally) Hamiltonian vector fields. For vector fields with non-constant divergences second property in the last conclusion can be obtained by direct computation. For the last property, if $X$ is locally Hamiltonian, then for the non-Hamiltonian vector field $\alpha^\sharp$ we compute

$$i_{[X,\alpha^\sharp]}\Omega_{T^* Q} = \mathcal{L}_X i_{\alpha^\sharp} \Omega_{T^* Q} = d\Omega_{T^* Q} (\alpha^\sharp, X) + i_X di_{\alpha^\sharp} \Omega_{T^* Q}$$  

which need not be closed for arbitrary choices of $X$ and $\alpha^\sharp$ and hence, not Hamiltonian. Last conclusion implies that the algebraic structure on sections of $TT^* Q$ is a semi-direct product algebra

$$\mathfrak{X}(T^* Q) = \mathfrak{g} \otimes (\mathfrak{g}^* )^\sharp$$

of vector fields with the Hamiltonian vector fields in $\mathfrak{g}$ acting on the second factor $(\mathfrak{g}^* )^\sharp$ by Lie derivative. This, of course, is a consequence of the coadjoint action of $\mathfrak{g}$ on its dual $\mathfrak{g}^*$ which, in turn, produces Lie-Poisson dynamics. □

## 2.2 Homotheties

Defining the divergence of an element of $(\mathfrak{g}^* )^\sharp$ to be a density, we obtain the identification of $(\mathfrak{g}^* )^\sharp$ with the space $\text{Den}(T^* Q)$ of densities on $T^* Q$. In particular, vector fields $\Pi^\sharp_c$ with constant divergences (with respect to Liouville volume and for $n$-dimensional plasma)

$$\mathcal{L}_{\Pi^\sharp_c} \Omega^n = (\text{div}_{\Pi^\sharp_c} \Omega^n) = c \Omega^n, \quad c = \text{constant}$$  

(18) correspond to constant plasma densities. In this case, we have either Lagrangian description of kinematics or, no dynamics in an Eulerian description. Since $\Omega$ is nondegenerate, Eq. (18) implies

$$\mathcal{L}_{\Pi^\sharp_c} \Omega = \frac{c}{n} \Omega, \quad c = \text{constant.}$$
That means, vector fields with constant divergences are infinitesimal homotheties of the symplectic form $\Omega$ [76]. Although, $\Pi^c \# c$ is not even locally Hamiltonian, it follows from the identity
\[ L_{[X,Y]} = L_X L_Y - L_Y L_X \] (19)
that the Lie bracket of two vectors with constant divergence is locally Hamiltonian. If we denote the set in $(g^*)^2$ of vector fields with constant divergences by $(g^*_c)^2$ then, straightforward computations prove

**Proposition 4** $[g, (g^*_c)^2] \subset g$, $[(g^*_c)^2, (g^*_c)^2] \subset (g^*_c)^2$, $[(g^*_c)^2, (g^*_c)^2] \subset g$.

**Proof.** For the first assertion we have
\[
i_{[X_h, \Pi^c]} \Omega = i_{X_h} di_{\Pi^c} \Omega + di_{X_h} i_{\Pi^c} \Omega = i_{X_h} \frac{c}{n} \Omega + d \Omega (\Pi^c, X_h)
= d \left(-\frac{c}{n} h + \Omega (\Pi^c, X_h)\right)
\]
where we used the identity
\[ i_{[X,Y]} = L_X i_Y - i_Y L_X. \] (20)

If we replace $X_h$ with a locally Hamiltonian vector field then a similar computation implies that the bracket is again locally Hamiltonian. For the second, we compute, from the definition of locally Hamiltonian vector fields
\[
d i_{[\Pi^c_{id}, \Pi^c_{id}]} \Omega = \mathcal{L}_{\Pi^c_{id}} di_{\Pi^c_{id}} \Omega - di_{\Pi^c_{id}} \frac{1}{n} div_{\Omega} \Pi^c_{id} \Omega
= \frac{c}{n^2} div_{\Omega} \Pi^c_{id} \Omega - d \left(\frac{1}{n} (div_{\Omega} \Pi^c_{id}) \right) \wedge i_{\Pi^c_{id}} \Omega
= -d \left(\frac{1}{n} (div_{\Omega} \Pi^c_{id}) \right) \wedge i_{\Pi^c_{id}} \Omega.
\]
This can be zero only if $\Pi^c_{id}$ is globally Hamiltonian with divergence of the arbitrary element $\Pi^c_{id}$ of $(g^*)^2$, which is not possible. 

The action of homotheties on plasma density function may be computed using the identity in Eq. (19)
\[
\mathcal{L}_{[\Pi^c_{id}, \Pi^c_{id}]} (d\mu) = \mathcal{L}_{\Pi^c_{id}} (c d\mu) = c f d\mu - df \wedge i_{\Pi^c_{id}} (d\mu) - c f d\mu
= -df \wedge i_{\Pi^c_{id}} (d\mu) = \Pi^c_{id} (f) d\mu
\]
and thus, is described by
\[
\Pi_{id} \to f d\mu, \quad \left[\Pi^c_{id}, \Pi^c_{id}\right] \to \Pi^c_{id} (f) d\mu. \] (22)
This can be neglected by a redefinition of density. We thus restrict the definition of $g^*$ to one-forms $\Pi_{id}$ for which $div_{\Omega} \Pi^c_{id} = f \neq \text{constant}$.
Remark 5 Proposition 4 opens up the possibility to apply the following Lebedev-Manin construction [33] to plasma dynamics. Assume that we have $\mathfrak{a} = \mathfrak{g} \oplus (\mathfrak{g}^*)^2$ as a vector space with $\mathfrak{g}, (\mathfrak{g}^*)^2$ and $(\mathfrak{g}^*)^2$ satisfying

$$[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}, \quad [(\mathfrak{g}^*)^2, (\mathfrak{g}^*)^2] \subset (\mathfrak{g}^*)^2, \quad [\mathfrak{g}, (\mathfrak{g}^*)^2] \subset (\mathfrak{g}^*)^2.$$

Let $\langle \cdot, \cdot \rangle$ be an invariant non-degenerate scalar product on $\mathfrak{a}$ with

$$\langle \mathfrak{g}, \mathfrak{g} \rangle = \langle (\mathfrak{g}^*)^2, (\mathfrak{g}^*)^2 \rangle = 0.$$

For $F : (\mathfrak{g}^*)^2 \rightarrow \mathbb{R}$, define $\delta F : (\mathfrak{g}^*)^2 \rightarrow \mathfrak{g}$ by

$$\langle \Pi^{\sharp}_{id}, \delta F(\Pi^{\sharp}_{id})/\delta \Pi^{\sharp}_{id} \rangle = d_{\Pi^{\sharp}_{id}} F(\Pi^{\sharp}_{id} + e \Pi^\sharp) \big|_{e=0}, \quad \forall \Pi^{\sharp}_{id}, \Pi^\sharp \in (\mathfrak{g}^*)^2.$$

Let $F$ be an invariant function on $\mathfrak{a}$, that is, $[X, \delta F(X)/\delta X] = 0$ for all $X \in \mathfrak{a}$. For $\Pi^{\sharp}_{id} \in (\mathfrak{g}^*)^2$, set $F_{\Pi^{\sharp}_{id}}(\Pi^{\sharp}_{id}) = F(\Pi^{\sharp}_{id} + \Pi^\sharp)$ for all $\Pi^{\sharp}_{id} \in (\mathfrak{g}^*)^2$. Then, for two invariant functions $F, G$ we have $\{F_{\Pi^{\sharp}_{id}}, G_{\Pi^{\sharp}_{id}}\}_{LP} = 0$ on $(\mathfrak{g}^*)^2$ (with Lie-Poisson bracket adapted from $\mathfrak{g}^*$). The Lie-Poisson equations

$$\dot{\Pi}^{\sharp}_{id} = \left[\Pi^{\sharp}_{id}, \delta F(\Pi^{\sharp}_{id})/\delta \Pi^{\sharp}_{id} \right]$$

can be written in equivalent Lax form

$$\frac{d}{dt}(\Pi^{\sharp}_{id} + \Pi^\sharp) = \left[\Pi^{\sharp}_{id} + \Pi^\sharp, \frac{\delta F(\Pi^{\sharp}_{id})}{\delta \Pi^{\sharp}_{id}} \right].$$

Remark 6 It has been argued that the physical initial conditions must satisfy $f(z, 0) > 0$ [46]. The restrictions on the definition of momentum variables may further be expanded to the physical requirement that the density function be positive. We remark that this condition is intimately related to the non-degeneracy of symplectic structure on coadjoint orbit of canonical diffeomorphisms. The condition $f(z, 0) > 0$ requires the description of density by elements $\Pi_{id} \in \mathfrak{g}^*$ with $div_{T^*\mathbb{Q}} \Pi^{\sharp}_{id} > 0$. Equivalently, in the language of differential forms, we have $d(\Pi^{\sharp}_{id} \wedge \Omega^\sharp_{T^*\mathbb{Q}}) > 0$. Consider a six dimensional domain $D$ in $T^*\mathbb{Q}$ with boundary $\partial D$. Then, the positive divergence implies

$$\int_{\partial D} \Pi_{id}(z) \wedge \Omega^{\sharp}_{T^*\mathbb{Q}}(z) > 0 \quad (23)$$

so that we have a volume element or, an orientation, for the five dimensional boundary of the region $D$. This can now be related to the nondegeneracy of the coadjoint orbit symplectic structure on $\mathfrak{g}^*$. An element of the tangent space to the coadjoint orbit through $\Pi_{id}$ will be of the form $\mathcal{L}_{X_{\Pi}}(\Pi_{id})$. By definition, the
orbit symplectic structure is

\[ \Omega_{\Pi_{id}}(\mathcal{L}_X(\Pi_{id}), \mathcal{L}_Y(\Pi_{id})) = \int_{\Omega} \Pi_{id}(z) \cdot [X_k(z), X_y(z)] \Omega_{\mathcal{T}^*Q}(z) = \int_{\Omega} \{g(z), k(z)\} \Pi_{id}(z) \wedge \Omega_{\mathcal{T}^*Q}(z) \]

which, by Eq. (23) does not vanish for arbitrary functions \( g \) and \( k \).

2.3 Lie algebra of one-forms over \( T^*Q \)

To study the algebraic structure on \( \Lambda^1(T^*Q) = \mathfrak{g}^\text{c} \oplus \mathfrak{g}^\text{t} \) we define the bracket of one-forms

\[ \{\alpha, \beta\}_{\Omega_{\mathcal{T}^*Q}} = \mathcal{L}_{\alpha\beta} - \mathcal{L}_{\beta\alpha} - d\Omega_{\mathcal{T}^*Q}(\alpha, \beta) \]

where \( \Omega_{\mathcal{T}^*Q} \) denotes the Poisson bi-vector obtained by inverting the matrix of symplectic two-form \( \Omega_{\mathcal{T}^*Q} \) and \( \alpha^g = \Omega_{\mathcal{T}^*Q}(\alpha) \) is a vector field on \( T^*Q \) corresponding to the one-form \( \alpha \) in \( \Lambda^1(T^*Q) \). If \( \alpha \) and \( \beta \) are closed forms in \( \mathfrak{g}^\text{c} \) corresponding to Hamiltonian vector fields then we have

\[ \{\alpha, \beta\}_{\Omega_{\mathcal{T}^*Q}} = d \left( i_{\alpha} \beta - i_{\beta} \alpha - \Omega_{\mathcal{T}^*Q}(\alpha, \beta) \right) \]

which is exact and hence in \( \mathfrak{g}^\text{c} \). If \( \alpha \) is closed and \( d\Pi_{id} \neq 0 \), then

\[ \{\alpha, \Pi_{id}\}_{\Omega_{\mathcal{T}^*Q}} = i_{\alpha} d\Pi_{id} + d \left( i_{\alpha} \Pi_{id} - i_{\Pi_{id}} \alpha - \Omega_{\mathcal{T}^*Q}(\alpha, \Pi_{id}) \right) \]

where the condition that the first term be closed requires the invariance relations \( d_{\alpha} d\Pi_{id} = \mathcal{L}_{\alpha} d\Pi_{id} = d\mathcal{L}_{\alpha} \Pi_{id} = 0 \) for arbitrary \( \alpha \in \mathfrak{g}^\text{c} \) and \( \Pi_{id} \in \mathfrak{g}^\text{t} \). The same argument applies for two arbitrary elements of \( \mathfrak{g}^\text{c} \). Thus we have the following proposition, which summarizes the calculations above.

**Proposition 7** \( \{\mathfrak{g}^\text{c}, \mathfrak{g}^\text{c}\}_{\Omega_{\mathcal{T}^*Q}} \subset \mathfrak{g}^\text{c}, \{\mathfrak{g}^\text{c}, \mathfrak{g}^\text{c}\}_{\Omega_{\mathcal{T}^*Q}} \subset \mathfrak{g}^\text{t}, \{\mathfrak{g}^\text{c}, \mathfrak{g}^\text{c}\}_{\Omega_{\mathcal{T}^*Q}} \subset \mathfrak{g}^\text{c} \).

According to this result, it is obvious that \( \mathfrak{g}^\text{c} \) is a subalgebra of \( \Lambda^1(T^*Q) \) with respect to the bracket \( \{,\}_{\Omega_{\mathcal{T}^*Q}} \). In particular, for locally Hamiltonian vector fields in \( \mathfrak{g}^\text{c}, \mathfrak{g}^\text{t} \) consists of closed but non-exact one forms which are elements of the first de Rham cohomology space of the particle phase space \( T^*Q \). These cohomological one-forms satisfy

\[ \{\mathfrak{g}^\text{c}, \mathfrak{g}^\text{c}\}_{\Omega_{\mathcal{T}^*Q}} \subset \mathfrak{g}^\text{c}, \quad \{\mathfrak{g}^\text{c}, \mathfrak{g}^\text{t}\}_{\Omega_{\mathcal{T}^*Q}} \subset \mathfrak{g}^\text{c} \].

**Remark 8** The equation \( \text{div} \Pi_{id}^g = f \) offers an alternative notation for the elements \( \Pi_{f}, \Pi_{g} \) in \( \mathfrak{g}^\text{c} \) satisfying \( \text{div} \Pi_{id}^f = f \) and \( \text{div} \Pi_{id}^g = g \), respectively. We can pull-back the canonical Poisson structure on \( T^*Q \) by the map \( \mathfrak{g}^\text{c} \to F(T^*Q) ; \Pi_{f} \to f \) hence define a Lie algebra structure

\[ [\Pi_{f}, \Pi_{g}] = \Pi_{\{f,g\}} \]

(24)

on \( \mathfrak{g}^\text{c} \). This is the reduced Poisson structure on \( \mathfrak{g}^\text{c} \) given in proposition 7. Eq. (24) gives also that the map \( \Pi_{f} \to f \) is a Poisson map.
2.4 Tulczyjew symplectic structure on $T^*T Q$

The space $T^*T Q$ admits a symplectic structure first described by Tulczyjew [64, 65, 66, 67, 70]. A special symplectic structure is a quintuple

$$(P, \pi^P_M, M, \theta_P, \chi)$$

where $\pi^P_M : P \rightarrow M$ is a fibre bundle, $\theta_P$ is a one-form on $P$, and $\chi : P \rightarrow T^*M$ is a fiber preserving diffeomorphism such that $\chi^*\theta_{T^*M} = \theta_P$ for $\theta_{T^*M}$ being the canonical one-form on $T^*M$. $\chi$ can be characterized uniquely by the condition $\langle \chi(p), X_M(x) \rangle = \langle \theta_P(p), X_P(p) \rangle$ for each $p \in P$, $\pi^P_M(p) = x$ and for vector fields $X_M : M \rightarrow TM$, $X_P : P \rightarrow TP$ satisfying $(\pi^P_M)_*(X_P) = X_M$. $(P, d\theta_P)$ is the underlying symplectic manifold of the special symplectic structure.

**Proposition 9** The space $T^*T Q$ is the underlying symplectic manifold for two different special symplectic structures

$$(T^*T Q, \tau_{T^*T Q}, \tau_{T^*T Q}, \vartheta_1, \Omega_{T^*T Q}), \quad (T^*T Q, T\pi_Q, TQ, \vartheta_2, \alpha_Q) \quad (25)$$

where the one-forms $\vartheta_1$ and $\vartheta_2$ are, in the adapted coordinates,

\[
\vartheta_1(z, \dot{z}) = ((\Omega^T_{T^*T Q})^*\theta_{T^*T Q})(z, \dot{z}) = \dot{p} \cdot dq - \dot{q} \cdot dp \quad (26)
\]

\[
\vartheta_2(z, \dot{z}) = \alpha^*_{\pi_Q}(\theta_{T^*T Q})(z, \dot{z}) = \dot{p} \cdot dq + p \cdot d\dot{q} \quad (27)
\]

and the Tulczyjew two-form of the underlying symplectic manifold is

$$\Omega_{T^*T Q}(z, \dot{z}) = d\vartheta_1(z, \dot{z}) = d\vartheta_2(z, \dot{z}) = d\dot{p} \wedge dq + dp \wedge d\dot{q}. \quad (28)$$

These are constructed by means of two different fibrations of $T^*T Q$ over $T^*Q$ and $TQ$ which can be represented by the diagram

\[
\begin{array}{ccc}
T^*T^*Q & \xleftarrow{\Omega_{T^*T Q}} & TT^*Q \\
\pi_{T^*T Q} \searrow & & \searrow \tau_{T^*T Q} \\
\pi_Q \nearrow \quad -dh & & \nearrow \tau_Q \\
T^*Q & \searrow & \nearrow TQ \\
\pi_Q \searrow & & \nearrow \tau_Q \\
Q & & \end{array}
\quad (29)
\]

known as the Tulczyjew triple. Here, $\tau_Q$, $\pi_Q$, $\tau_{T^*T Q}$ and $\pi_{T^*T Q}$ are natural projections, $\Omega_{T^*T Q}$ is the induced map from the symplectic two-form $\Omega_{T^*Q}$ on $T^*Q$, $\alpha_Q$ is a diffeomorphism constructed as a dual of canonical involution $\kappa_Q$ of $TT^*Q$. $\alpha_Q$ is a canonical description of the equivalence of functors $TT^*$ and $T^*T$ while $\kappa_Q$ describes the canonical flip of the first derivatives with respect to two different parametrizations for second order tangent bundle. In coordinates, we have $\alpha_Q(q, p; \dot{q}, \dot{p}) = (q, \dot{q}; \dot{p}, p)$. The triangular diagrams on left and right define special symplectic structures on $TT^*Q$ by pull-back of canonical one-forms $\theta_{T^*T Q}$ and $\theta_{T^*Q}$ on the cotangent bundles $T^*T^*Q$ and $T^*TQ$, respectively.
Hamiltonian and Lagrangian formulations can then be realized as Lagrangian submanifolds of $\mathcal{T}^*\mathcal{Q}$. A submanifold $\mathcal{S}$ of a symplectic manifold $(\mathcal{M}, \Omega)$ is a Lagrangian submanifold, if its dimension is half the dimension of $\mathcal{M}$ and the restriction of $\Omega$ on $\mathcal{S}$ vanishes, that is $\Omega|_{\mathcal{S}} = 0$ \cite{51}, \cite{52}.

Consider a special symplectic structure $(\mathcal{P}, \pi^\mathcal{P}_\mathcal{M}, \mathcal{M}, \vartheta_\mathcal{P}, \chi)$ and let $g : \mathcal{M} \to \mathbb{R}$ be a real valued function. Then, the set

$$\mathcal{S}_\mathcal{P} = \{ p \in \mathcal{P} : \langle dg(x), T\pi^\mathcal{P}_\mathcal{M} \circ \chi_\mathcal{P}(p) \rangle = \langle \vartheta_\mathcal{P}(p), \chi_\mathcal{P}(p) \rangle, \forall \chi_\mathcal{P} \in \mathcal{X}(\mathcal{P}) \}$$

is a Lagrangian submanifold of the underlying symplectic manifold $(\mathcal{P}, d\vartheta_\mathcal{P})$ and the function $g$ is called the generating function \cite{53}. It follows from the definition of $\mathcal{S}_\mathcal{P}$ that, the one-form $\vartheta_\mathcal{P}$ is characterized by the relation $(\pi^\mathcal{P}_\mathcal{M})^*dg = \vartheta_\mathcal{P}$. Since $\chi$ is a symplectic diffeomorphism, it maps $\mathcal{S}_\mathcal{P}$ to the space $Im\{(dg)\}$ which is a Lagrangian submanifold of $T^*\mathcal{M}$. In general, the image of a closed one-form on $\mathcal{M}$ is a Lagrangian submanifold of $T^*\mathcal{M}$ and its pull-back to $\mathcal{P}$ by $\chi$ is a Lagrangian submanifold of $\mathcal{P}$.

Let $l : T\mathcal{Q} \to \mathbb{R}$. The image of mapping $dl : T\mathcal{Q} \to T^*T\mathcal{Q}$ is described by the equations $\lambda_q = \nabla^*_q\{l(q, \dot{q})\}$ and $\lambda_q = \nabla^*_q\{q, \dot{q}\}$. Pull back of this to $T^*\mathcal{Q}$ gives the dynamical equations $(T\pi_\mathcal{Q})^*dl = \vartheta_2$ which, in coordinates, read $\nabla^*_q\{l(q, \dot{q})\} = p \cdot \nabla^*_q\{q, \dot{q}\}.$ For a Hamiltonian function $h : T^*\mathcal{Q} \to \mathbb{R}$, the image $Im\{-dh\}$ is a Lagrangian submanifold of $T^*T^*\mathcal{Q}$. The Hamilton’s equations on $T^*\mathcal{Q}$ are obtained from the relation $\vartheta_1 = \tau^\mathcal{P}_{T^*Q}(\vartheta_2) = -d(h \circ \tau^\mathcal{P}_{T^*Q})$, which, in coordinates, are expressed as

$$-dh(z) = \dot{p} \cdot dq - \dot{q} \cdot dp, \quad \dot{q} = \nabla_p h(z), \quad \dot{p} = -\nabla_q h(z).$$

(30)

Since the derivative of $\vartheta_1 = \tau^\mathcal{P}_{T^*Q}(\vartheta_2)$ vanishes, the Hamiltonian dynamics becomes a Lagrangian submanifold of $(T^*\mathcal{Q}, d\vartheta_1)$ generated by the function $-h$. If $X$ is locally Hamiltonian then the one form $i_X\Omega_{T^*\mathcal{Q}}$ is still closed by definition and $Im\{X\}$ defines a Lagrangian submanifold of $T^*\mathcal{Q}$, as well. Thus, we have the identification of the vector space $\mathcal{X}_{ham}(T^*\mathcal{Q})$ with the space of all Lagrangian submanifolds of the Tulczyjew symplectic space $(T^*\mathcal{Q}, \Omega_{T^*\mathcal{Q}})$. In the next subsection, we shall obtain this space as tangent space over identity of configuration space of plasma.

### 2.5 Spaces of Lagrangian submanifolds

As the Hamiltonian dynamics of a single particle described by a diffeomorphism $\varphi \in Diff_{can}(T^*\mathcal{Q})$ corresponds to a Lagrangian submanifold of the Tulczyjew symplectic manifold $TT^*\mathcal{Q}$, it is possible to describe all such motions, that is, each configuration of plasma by a Lagrangian submanifold in $TT^*\mathcal{Q}$ \cite{24}. We shall show that the space $Lag(TT^*\mathcal{Q}, \Omega_{T^*\mathcal{Q}})$ of all Lagrangian submanifolds can be obtained as the tangent space over identity of a suitable representation of the group $Diff_{can}(T^*\mathcal{Q})$ of all canonical transformations. We rely on the fact that the configuration space $Diff_{can}(T^*\mathcal{Q})$, as a manifold of maps \cite{44},
can also be given a description in terms of sections \( \Gamma(pr_0) \) of the trivial bundle \( pr_0 : T^*Q_0 \times T^*Q \to T^*Q_0 \) where \( T^*Q_0 \) is the particle phase space with Lagrangian coordinates \( Z \) and \( T^*Q \) carries Eulerian coordinates \( z \). The total space \( T^*Q_0 \times T^*Q \) is then symplectic with the two-form \([58],[69],[80],[7]\).

\[
\Omega_-(Z,z) = \Omega_{T^*Q_0}(Z) - \Omega_{T^*Q}(z) = dP \wedge dQ - dp \wedge dq.
\]

**Proposition 10** \( Diff_{can}(T^*Q) \) can be identified with the space \( \text{Lag}\Gamma(pr_0, \Omega_-) \) of all Lagrangian sections of the trivial bundle \((pr_0, \Omega_-)\). In this case, the Lie algebra \( X_{\text{ham}}(T^*Q) \) of Hamiltonian vector fields corresponds to the space \( \text{Lag}(TT^*Q,\Omega_{TT^*Q}) \) of all Lagrangian submanifolds of the Tulczyjew symplectic space.

**Proof.** A diffeomorphism \( \varphi : T^*Q_0 \to T^*Q \) is canonical if \( \Omega_{T^*Q_0} - \varphi^*\Omega_{T^*Q} = 0 \). It follows that \( \Omega_- \) vanishes when restricted to the graphs

\[
\text{Gr}\varphi = \{(Z, \varphi(Z)) : Z \in T^*Q_0\} \subset \Gamma(pr_0)
\]

of canonical diffeomorphisms \([80],[7]\). For a base point \( Z \in T^*Q_0 \), the total space is twelve dimensional and \( \text{Gr}\varphi \) is a six dimensional subspace. When \( \varphi \) is canonical, \( \Omega_- \) vanishes on \( \text{Gr}\varphi \) and this is a Lagrangian submanifold in \((T^*Q_0 \times T^*Q, \Omega_-)\). If we denote the space of all sections of the trivial bundle on which the restriction of \( \Omega_- \) vanishes, namely, the space of all Lagrangian sections by \( \text{Lag}\Gamma(pr_0, \Omega_-) \), then we have the bijective correspondence

\[
\text{Diff}_{can}(T^*Q) \longleftrightarrow \text{Lag}\Gamma(pr_0, \Omega_-) : \varphi \longleftrightarrow \text{Gr}\varphi.
\]

To find the tangent space over the identity mapping, we proceed as follows. Corresponding to a curve \( \varphi_t \in \text{Diff}_{can}(T^*Q) \) with \( \varphi_0(Z) = z \), we have the curve \( t \mapsto \text{Gr}\varphi_t \) in \( \text{Lag}\Gamma(pr_0, \Omega_-) \) with \( \text{Gr}\varphi_0 = \{(Z,z) : Z \in T^*Q_0\} \). The tangent space \( T_{\text{Gr}\varphi_t}\text{Lag}\Gamma(pr_0, \Omega_-) \) consists of vectors

\[
X_{\text{Gr}\varphi_t}(Z) = \frac{d}{dt}\text{Gr}\varphi_t(Z) = (Z, \varphi_t(Z); 0, \frac{d\varphi_t(Z)}{dt}) = (Z, \varphi_t(Z); 0, X_{\varphi_t}(Z)) \tag{31}
\]

tangent to \( \text{Gr}\varphi_t \). For each \( Z \in T^*Q_0 \), this is a vector tangent to the fiber \( T^*Q \) over \( Z \). That means, \( X_{\text{Gr}\varphi_0}(Z) \) is in the vertical tangent space

\[
V_{\text{Gr}\varphi_0}(Z)(T^*Q_0 \times T^*Q) \tag{32}
\]

Over the identity, \( t = 0 \), we have

\[
X_{\text{Gr}\varphi_0}(Z) = \frac{d}{dt}\text{Gr}\varphi_t(Z)|_{t=0} = (Z, z; 0, \frac{d\varphi_t(Z)}{dt}|_{t=0}) = (Z, z; 0, X_h(z))
\]

where \( X_h(z) = X_h(z) \cdot \nabla_z \) is the Hamiltonian vector field generating \( \varphi_t \in \text{Diff}_{can}(T^*Q) \). Thus, over the identity mapping we have

\[
V_{(Z,z)}(T^*Q_0 \times T^*Q) \longleftrightarrow T_zT^*Q : X_{\text{Gr}\varphi_0}(Z) \longleftrightarrow X_h(z).
\]
In fact, for each $\varphi$ the vertical tangent space in $\mathcal{Z}$ is isomorphic to a copy of $TT^*Q$. To this end, we recall the definition of pull-back bundle. Given $E \longrightarrow N$ and a continuous map $\Phi : M \longrightarrow N$, the pull-back of $E$ by $\Phi$ is the bundle $\Phi^*E \longrightarrow M$ whose fiber $(\Phi^*E)_x$ over $x \in M$ is the fiber $E_{\Phi(x)}$ of $E \longrightarrow N$ over $\Phi(x)$. In our case, $E_{\Phi(x)} = \mathcal{V}_{Gr\varphi_t}(Z)(\mathcal{T}^*Q_0 \times T^*Q)$ and hence $\mathcal{V}_{Gr\varphi_t}(Z)(\mathcal{T}^*Q_0 \times T^*Q) = Gr\varphi_t^*(\mathcal{V}(Z, z)(\mathcal{T}^*Q_0 \times T^*Q))$.

Since we are dealing with a trivial bundle, the vertical space is just the tangent space to the second factor, so we have

$$V_{Gr\varphi_t}(Z)(\mathcal{T}^*Q_0 \times T^*Q) = Gr\varphi_t^* = (id, \varphi_t)^*(TT^*Q) = \varphi_t^*(TT^*Q)$$

as the tangent space over $\varphi \in \text{Diff}_{\text{can}}(T^*Q)$. Thus, the tangent space at $Gr\varphi_t$ to the space of Lagrangian sections $\text{Lag}(\text{pr}_0, \Omega_\sim)$ is the space of sections consisting of pull-back by $\varphi$ of (Hamiltonian) vector fields on $T^*Q$. As $\varphi$ is canonical, these sections are images of Hamiltonian vector fields on $T^*Q$ and hence are Lagrangian submanifolds in $\varphi^*(TT^*Q)$. For $\varphi$ being the identity element of $G$, we obtain the space of Lagrangian submanifolds $\text{Lag}(TT^*Q)$ as the Lie algebra of the group $\text{Diff}_{\text{can}}(T^*Q)$.

**Remark 11** Having the correspondence $\text{Diff}_{\text{can}}(T^*Q) \longleftrightarrow \text{Lag}(\text{pr}_0, \Omega_\sim)$ for configuration space of plasma dynamics, one needs operations between Lagrangian submanifolds of $\text{Lag}(\text{pr}_0, \Omega_\sim)$ similar to right and left multiplications of the group $\text{Diff}_{\text{can}}(T^*Q)$ producing particle relabelling symmetry and kinematical motion, respectively. This can be achieved in the framework of symplectic relations. The Lagrangian submanifolds of symplectic spaces of the form $(\mathcal{M}_2 \times \mathcal{M}_1, \Omega_\sim = \Omega_2 - \Omega_1)$ were defined as symplectic relations and their composition rules were proved in [58].
3 From Particle Dynamics to Vlasov Equation

We shall describe a purely geometric framework in which one can find a precise relation between the individual particle motion and the Vlasov equation. In this framework, the one-form $\theta_2(z, \dot{z})$ of the special symplectic structure in Eq.(27) and the Tulczyjew symplectic two-form can be obtained as complete tangent lifts of canonical forms on particle phase space. Given the infinitesimal generator of particle motion, its complete cotangent lift describes a Lagrangian submanifold of certain special symplectic structure. We shall present a Morse family generating Legendre transformation, in the sense of Tulczyjew, for the lifted motion. Using a holonomic lift operator, we shall carry the lifted motion to vertical subspace of a Tulczyjew symplectic space. This subspace is integrable when restricted to generators lifted from the Lie algebra of Hamiltonian vector fields. Finally, introducing the vertical lifts of one-forms we shall obtain the relation between the Hamiltonian vector fields generating the particle motion and the momentum-Vlasov equations. We refer to [54], [11], [62], [29], [34], [71], [73], [60], [61], [72], [74], [36], [15], [9], [51], [28], [30], [37], [10] for definitions and various aspects of lifts of geometric objects some of which we summarize in the following subsection.

3.1 Complete lifts

Let $X$ be a vector field on $\mathcal{M}$, $\phi^t : \mathcal{M} \rightarrow \mathcal{M}$ be its flow and $\tau_\mathcal{M} : T\mathcal{M} \rightarrow \mathcal{M}$ be the tangent bundle. The tangent lift $\phi^c_t : T\mathcal{M} \rightarrow T\mathcal{M}$ of $\phi^t$ is defined as to satisfy $\tau_\mathcal{M} \circ \phi^c_t = \phi^t \circ \tau_\mathcal{M}$ and constitutes a one-parameter group of diffeomorphisms on $T\mathcal{M}$. Differentiating the defining relation we obtain $T\tau_\mathcal{M} \circ X^c = X \circ \tau_\mathcal{M}$ where $T\tau_\mathcal{M}$ is the tangent mapping of $\tau_\mathcal{M}$. This means that $X$ and $X^c$ are $\tau_\mathcal{M}$ related. In local coordinates $(x^a, v^a)$ of $T\mathcal{M}$, the complete tangent lift of $X(x) = X^a(x)\partial/\partial x^a$ is given by

$$X^c(x, v) = X^a(x) \frac{\partial}{\partial x^a} + v^b \frac{\partial X^a(x)}{\partial x^b} \frac{\partial}{\partial v^a}. \quad (33)$$

Similarly, the cotangent lift of the flow $\phi^t$ is a one-parameter group of diffeomorphisms $\phi^c_t$ on $T^*\mathcal{M}$ satisfying $\pi_\mathcal{M} \circ \phi^{c*}_t = \phi^t \circ \pi_\mathcal{M}$, where $\pi_\mathcal{M}$ is the natural projection of $T^*\mathcal{M}$ to $\mathcal{M}$. The generator $X^{c*}$ of $\phi^{c*}_t$ is the complete cotangent lift of $X$ and is obtained from the infinitesimal version of the defining relation as

$$T\pi_\mathcal{M} \circ X^{c*} = X \circ \pi_\mathcal{M}, \quad (34)$$

which means that, $X$ and $X^{c*}$ are $\pi_\mathcal{M}$ related. Since Eq.(34) is equivalent to $\pi_\mathcal{M}^*X = X^{c*}$, we have

**Proposition 12** The cotangent lift $c^* : \mathfrak{X}(\mathcal{M}) \rightarrow \mathfrak{X}(T^*\mathcal{M})$ is a Lie algebra isomorphism into $[X, Y]^{c*} = [X^{c*}, Y^{c*}]$, for all $X, Y \in \mathfrak{X}(\mathcal{M})$. 

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In fact, this is the homomorphism that leads to the so called plasma-to-fluid map (c.f section 4.4).

The complete cotangent lift \( X^c \) of a vector field \( X \) on \( \mathcal{M} \) is a Hamiltonian vector field on the canonically symplectic manifold \( T^*\mathcal{M} \). Indeed, the lifted flow \( \varphi_{t}^c \) preserves the canonical one-form \( \theta_{T^*\mathcal{M}} \). Differentiating at \( t = 0 \) gives \( \mathcal{L}_X = di_X \theta_{T^*\mathcal{M}} = 0 \). Using the identity \( L_X = di_X + i_X d \) we obtain the Hamilton’s equations

\[
\dot{X}^a(x, y) = X^a(x) \frac{\partial}{\partial x^a} - y_b \frac{\partial X^b(x)}{\partial y_a} 
\]

for \( X^c \) with the Hamiltonian function \( i_X \theta_{T^*\mathcal{M}} \). Taking \( \theta_{T^*\mathcal{M}}(x, y) = y_a dx^a \) and \( X = X^a(x) \partial/\partial x^a \) we obtain

\[
X^c(x, y) = X^a(x) \partial_x^a - y_b \frac{\partial X^b(x)}{\partial x^a} \partial_y^a.
\]

and the Hamiltonian function \( y_a X^a(x) \) is degenerate in the fiber variables \( y_a \). We observe that complete cotangent lifts can be given a variational formulation as well. Define the Whitney product

\[
T^*\mathcal{M} \times \mathcal{M} T\mathcal{M} = \{ (\alpha, X) \in T^*\mathcal{M} \times \mathcal{T}\mathcal{M} : \pi_{\mathcal{M}}(\alpha) = \tau_{\mathcal{M}}(X) \}
\]

which may be viewed as a submanifold of \( TT^*\mathcal{M} \) given by \( \dot{y} = 0 \). Then, the following result is straightforward.

**Proposition 13** Associated to the cotangent lift in Eq. (36), the first order differential equations

\[
\dot{x}^a = X^a(x), \quad \dot{y}_a = -y_b \frac{\partial X^b(x)}{\partial x^a}
\]

are the Euler-Lagrange equations for the (degenerate) Lagrangian density

\[
L(x, \dot{x}, y) = y_a (\dot{x}^a - X^a(x)) = y_a \dot{x}^a - H(x, y)
\]

defined on the Whitney product \( T^*\mathcal{M} \times \mathcal{M} T\mathcal{M} \).

**Remark 14** In [48], the Whitney product was shown to be isomorphic to the restriction of \( TT^*\mathcal{M} \) to zero section of \( T^*\mathcal{M} \). Then, a generalized tangent bundle and a generalized complex structure on \( \mathcal{M} \) were introduced as the Whitney product being a bundle over \( \mathcal{M} \) and, as a complex structure on Whitney product.

**Complete tangent lift** \( f^c \in F(T\mathcal{M}) \) of a function \( f \in F(\mathcal{M}) \) is simply the directional derivative \( f^c(x, v) = df(x) \cdot v \) and is given in coordinates as \( f^c = v^a \partial f/\partial x^a \). Let \( \omega_{\mathcal{M}} \in \Lambda^k(\mathcal{M}) \) be a differential k-form on \( \mathcal{M} \). Its complete tangent lift \( \omega^c_{\mathcal{M}} \in \Lambda^k(T\mathcal{M}) \) is a differential k-form on \( T\mathcal{M} \) and is defined by means of the lifts of vector fields and functions, namely,

\[
\omega^c_{\mathcal{M}}(X_1^c, ..., X_k^c) = (\omega_{\mathcal{M}}(X_1, ..., X_k))^c. \tag{37}
\]
For a one-form $\theta_M = \theta_a dx^a \in \Lambda^1 (M)$, we compute
\[
\theta'_M = \frac{\partial \theta_a}{\partial x^b} dx^b + \theta_a dv^a
\]  
and for a two-form $\Omega_M = (1/2)\Omega_{ab} dx^a \wedge dx^b \in \Lambda^2 (M)$, we find
\[
\Omega'_M (x, v) = \frac{1}{2} (v^a \frac{\partial \Omega_{ab}}{\partial x^d} dx^d \wedge dx^b + \Omega_{ab} dv^a \wedge dx^b + \Omega_{ab} dx^a \wedge dv^b).
\]  
For a constant matrix $\Omega_{ab}$, this reduces to
\[
\Omega'_M (x, v) = \Omega_{ab} dx^a \wedge dv^b = dx^a \wedge d(\Omega_{ab} v^b).
\]  
If $\Omega_M$ defines a constant symplectic structure on $M$, Hamiltonian vector fields are of the form $\Omega_{ab} v^b = \partial k(x)/\partial x^a$ for functions $k(x)$ on $M$. Restricting the lifted two-form to Hamiltonian vector fields we find
\[
\Omega'_M (x, v) = dx^a \wedge d\partial k(x)/\partial x^a = dx^a \wedge \frac{\partial^2 k(x)}{\partial x^a} = d^2 k(x) \equiv 0
\]
which means that, with respect to the lift $\Omega'_M$ of (constant) symplectic structure $\Omega_M$, Hamiltonian vector fields of $\Omega_M$ define Lagrangian submanifolds of $(T^*M, \Omega'_M)$. In fact, the Tulczyjew symplectic two-form is of this sort.

### 3.2 Lift of particle dynamics

We consider the particle dynamics on $M = T^*Q$ described as the flow of Hamiltonian vector field
\[
X_h(z) = \frac{1}{m} p \cdot \nabla_q - e \nabla_q \phi_f (q) \cdot \nabla_p
\]
with respect to the symplectic two-form $\Omega_{T^*Q} (z) = dp \wedge dq$ which is exact $\Omega_{T^*Q} = d\theta_{T^*Q}$ with $\theta_{T^*Q} (z) = p \cdot dq$ and, for the Hamiltonian function $h(z) = p^2/(2m) + e\phi_f(q)$ which is the energy of a charged particle. First, we note

**Proposition 15** Complete tangent lifts of $\theta_{T^*Q}$ and $\Omega_{T^*Q}$ are
\[
\begin{align*}
\theta'_{T^*Q} (z, \dot{z}) &= \theta_2 (z, \dot{z}) = \alpha_Q (\theta_{T^*Q})(z, \dot{z}) \\
\Omega'_{T^*Q} (z, \dot{z}) &= \Omega_{TT^*Q} (z, \dot{z})
\end{align*}
\]
which are the one-form for the fibration over $TQ$ and the Tulczyjew two-form on $TT^*Q$, given in Eqs. (27) and (28), respectively.

The complete cotangent lift of $X_h(z)$ is the vector
\[
X'^*_{h} (z, \Pi_{id}) = X_h (z) + e(\Pi_p \cdot \nabla_q)\nabla_q \phi_f (q) \cdot \nabla_{\Pi_q} - \frac{1}{m} \Pi_q \cdot \nabla_{\Pi_p}
\]
on $T^*_z T^*_q Q$ which is canonically Hamiltonian
\[
i_{X'^*_h} \Omega_{T^*T^*Q} = -dH_{T^*T^*Q}
\]
with the canonical two-form
\[ \Omega_{T^\ast T^*} (z, \Pi_{id}) = d (\Pi_q \cdot dq + \Pi_p \cdot dp) = d\theta_{T^\ast T^*} (z, \Pi_{id}) \] (46)
and for the Hamiltonian function
\[ H_{T^\ast T^*} (z, \Pi_{id}) = \frac{1}{m} p \cdot \Pi_q - c\nabla_q \phi (q) \cdot \Pi_p = \langle X_h (z), \Pi_{id} (z) \rangle_{T^\ast T^*} . \] (47)
Hence, the constructions for \( X_h (z) \) can be carried over the cotangent lift \( X_h^* \) by replacing \( \Theta \) with \( T^* \Theta \) but with a degenerate Hamiltonian function. An invariant way of writing this Hamiltonian is
\[ H_{T^\ast T^*} = i_{X_h^*} \theta_{T^\ast T^*} = i_{X_h^*} \theta_{T^\ast T^*} \] (48)
where we used the fact that \( \theta_{T^\ast T^*} \) has no components along vertical directions. It follows that the canonical one-form \( \theta_{T^\ast T^*} \) is an absolute invariant of the cotangent lift
\[ \mathcal{L}_{X_h^*} \theta_{T^\ast T^*} = dH_{T^\ast T^*} + i_{X_h^*} \Omega_{T^\ast T^*} = 0 \] (49)
which can be regarded to be equivalent to the Hamilton’s equations (45).

The image of the section \( dH_{T^\ast T^*} : T^* T^* \Theta \rightarrow T^* T^* T^* \Theta \) is a Lagrangian submanifold of \( (T^* T^* \Theta, \Omega_{T^\ast T^*}) \) and the image of \( \Omega_{T^\ast T^*} (dH_{T^\ast T^*}) = X_h^* \) is a Lagrangian submanifold of \( T T^* \Theta \) with the Tulczyjew’s two-form \( \Theta_1 \). Hence, we have

**Proposition 16** The image of complete cotangent lift \( X_h^* \) of Hamiltonian vector field \( X_h \) is a Lagrangian submanifold of the special symplectic structure
\[ (T T^* \Theta, \tau_{T^\ast T^*} \Theta, T^* T^* \Theta, \Theta_1 = \left( \Omega^\ast_{T^\ast T^*} \right) \thickspace \theta_{T^\ast T^*} \thickspace \Omega^\ast_{T^\ast T^*} ) \] (50)
generated by the Hamiltonian function \( -H_{T^\ast T^*} (z, \Pi_{id}) \).

Since \( H_{T^\ast T^*} \) is an invariant of the lift \( X_h^* \) we have \( i_{X_h^*} \Theta_1 = 0 \). It follows that \( \Theta_1 \) and the Tulczyjew’s two-form are invariants of any cotangent lift. In other words, the cotangent lifts of diffeomorphisms of \( T^* \Theta \) are symmetries of the special symplectic structure (50).

We shall now apply Proposition (15) to obtain a variational formulation of the first order equations associated to the cotangent lift in Eq. (44). Since the equations under consideration have the additional property of being Lagrangian submanifolds described in above Proposition, we shall, instead, follow an approach based on this property. Tulczyjew proposed a geometric construction for a generalized Legendre transformation which also works for degenerate Lagrangians (70). We now adapt this construction to find an inverse Legendre transformation for the Hamiltonian system in Eq. (44) for which the Hamiltonian function \( H_{T^\ast T^*} \) is degenerate in momenta \( \Pi_{id} \). The construction consists of finding an alternative representation of the Lagrangian submanifold \( \text{Im} (X_h^*) \)
of the special symplectic structure \( (50) \) with respect to the special symplectic structure

\[
\left( TT^*Q, T^{*T^*Q}, TT^*Q, \Theta_2 = \alpha^*_T \Theta \right) \tag{51}
\]

underlying the Lagrangian formulation of dynamics. Following [70], we consider a fibration \( N \to TT^*Q \) and let \( (z^a, \dot{z}^b, \pi_\alpha) : a, b = 1, \ldots, 6, \alpha = 1, \ldots, m \) be adapted coordinates on \( N \). A function \( E : N \to \mathbb{R} \) can be considered to be a family of functions on the base \( TT^*Q \) parametrized by the fiber coordinates \( \pi_\alpha \).

This family is called a Morse family if the rank of the \( m \times (m + 12) \)-matrix

\[
\begin{pmatrix}
\frac{\partial^2 E}{\partial \pi_\alpha \partial \pi_\beta} & \frac{\partial^2 E}{\partial \pi_\alpha \partial z^b} & \frac{\partial^2 E}{\partial \pi_\alpha \partial \dot{z}^b}
\end{pmatrix}
\tag{52}
\]

is maximal [69], [61], [65], [80], [8], [6], [35].

**Proposition 17** The image of complete cotangent lift \( X^*_h \) of Hamiltonian vector field \( X_h \) is a Lagrangian submanifold of the special symplectic structure in the expression \( (51) \) generated by the Morse family

\[
E(z, \Pi_{id}, \dot{z}) = (\dot{z} - X_h(z)) \cdot \Pi_{id}
\tag{53}
\]

defined on the Whitney product \( T^*T^*Q \times_{T^*Q} TT^*Q \).

**Proof.** Let \( m = 6 \) and choose the total space \( N \) to be the Whitney product

\[
T^*T^*Q \times_{T^*Q} TT^*Q = \{(\Pi_{id}, X) \in T^*T^*Q \times TT^*Q : \pi_{T^*Q}(\Pi_{id}) = \tau_{T^*Q}(X)\}
\tag{54}
\]

for which the local coordinates are \((z, \Pi_{id}, \dot{z})\). The Whitney product is a submanifold of \( TT^*Q \) which can locally be described by the equations \( \dot{\Pi}_{id} = 0 \). Consider the function \( E(z, \Pi_{id}, \dot{z}) \) on \( TT^*Q \) defined by Eq. \( (53) \). The matrix in Eq. \( (52) \) becomes \( (0 - \partial X_h^a / \partial z^b \delta_h^a) \). This has rank 6, and so, \( E \) is a Morse family. The condition

\[
\alpha^*_T \Theta(\theta_{T^*T^*Q})(z, \Pi_{id}, \dot{z}, \dot{\Pi}_{id}) = dE(z, \Pi_{id}, \dot{z})
\tag{55}
\]

for \( E \) to generate a Lagrangian submanifold of the special symplectic structure \( (51) \) gives the components of the complete cotangent lift \( X^*_h \).

In the next subsection, we shall carry the dynamics on Whitney product to the Tulczyjew symplectic space \( TT^*T^*Q \) by means of holonomic lift.

### 3.3 Momentum-Vlasov equations

Define the holonomic lift operator

\[
\Gamma : T^*T^*Q \times_{T^*Q} TT^*Q \to TT^*T^*Q
\tag{56}
\]
from the Whitney product in Eq. \[\text{[31]}\] to the Tulczyjew symplectic space by \(\Gamma(\theta_{\mathcal{T}\cdot\mathcal{Q}}, X_{\mathcal{T}\cdot\mathcal{Q}}) = T \theta_{\mathcal{T}\cdot\mathcal{Q}}(X_{\mathcal{T}\cdot\mathcal{Q}})\). In coordinates, if \(X_{\mathcal{T}\cdot\mathcal{Q}}(z) = X^a(z) \frac{\partial}{\partial z^a}\) and \(\theta_{\mathcal{T}\cdot\mathcal{Q}}(z) = \pi_a(z) dz^a\) then,
\[
T\theta_{\mathcal{T}\cdot\mathcal{Q}}(X_{\mathcal{T}\cdot\mathcal{Q}})(z, \pi) = X^\text{hol}_{\mathcal{T}\cdot\mathcal{Q}}(z, \pi) = X^a(z) \left( \frac{\partial}{\partial z^a} + \frac{\partial \pi_b(z)}{\partial z^a} \frac{\partial}{\partial \pi_b} \right) \tag{57}\]

which is a generalized vector field of order one \[\text{[31]}, \text{[55]}\]. More generally, define the holonomic part \(HX_{\mathcal{T}\cdot\mathcal{Q}}\) of a projectable vector field
\[
X_{\mathcal{T}\cdot\mathcal{T}^*\mathcal{Q}}(z, \pi) = X^a(z) \frac{\partial}{\partial z^a} + X_b(z, \pi) \frac{\partial}{\partial \pi_b} \in T(z, \pi) T^*_z T^*\mathcal{Q} \tag{58}\]
on \(T^* T^*\mathcal{Q}\) as the holonomic lift of its projection, that is,
\[
HX_{\mathcal{T}\cdot\mathcal{T}^*\mathcal{Q}} = \Gamma \circ T \pi_{\mathcal{T}\cdot\mathcal{Q}} \circ X_{\mathcal{T}\cdot\mathcal{T}^*\mathcal{Q}} = ((\pi_{\mathcal{T}\cdot\mathcal{Q}})_* \ X_{\mathcal{T}\cdot\mathcal{T}^*\mathcal{Q}})^\text{hol}. \tag{59}\]
The vertical representative is the complement in \(TT^* T^*\mathcal{Q}\) of the holonomic part
\[
V X_{\mathcal{T}\cdot\mathcal{T}^*\mathcal{Q}} = X_{\mathcal{T}\cdot\mathcal{T}^*\mathcal{Q}} - HX_{\mathcal{T}\cdot\mathcal{T}^*\mathcal{Q}} = \left( X_b(z, \pi) - X^a(z) \frac{\partial \pi_b(z)}{\partial z^a} \right) \frac{\partial}{\partial \pi_b} \tag{60}\]
and is a vertical valued generalized vector field of order one as well \[\text{[31]}, \text{[55]}, \text{[68]}\].

The (first) prolongation of a (projectable) generalized vector field of order one
\[
X^g_{\mathcal{T}\cdot\mathcal{T}^*\mathcal{Q}}(z, \pi, \pi_z) = X^a(z) \frac{\partial}{\partial z^a} + X_b(z, \pi, \pi_z) \frac{\partial}{\partial \pi_b} \tag{61}\]
is defined by
\[
pr^1 X^g_{\mathcal{T}\cdot\mathcal{T}^*\mathcal{Q}} = X^g_{\mathcal{T}\cdot\mathcal{T}^*\mathcal{Q}} + \Phi_{ab} \frac{\partial}{\partial (\pi_b/\partial z^a)} \tag{62}\]
where the coefficient functions are
\[
\Phi_{ab} = \left( \frac{d}{dz^a} \left( X_b - X^d \frac{\partial \pi_b}{\partial z^d} \right) + X^d \frac{\partial \pi_b}{\partial z^a \partial z^d} \right). \tag{63}\]

Here, the set \((z, \pi, \pi_z)\) is the induced local coordinate system for the jet bundle of the fibration \(T^* T^*\mathcal{Q} \to T^* \mathcal{Q}\), and \(d/dz^a\) is the total derivative operator with respect to \(z^a\). Lie bracket of two first order generalized vector fields \(X^g_{\mathcal{T}\cdot\mathcal{T}^*\mathcal{Q}}\) and \(Y^g_{\mathcal{T}\cdot\mathcal{T}^*\mathcal{Q}}\) is the unique first order generalized vector field
\[
[X^g_{\mathcal{T}\cdot\mathcal{T}^*\mathcal{Q}}, Y^g_{\mathcal{T}\cdot\mathcal{T}^*\mathcal{Q}}]_{\text{pro}} = \left( pr^1 X^g_{\mathcal{T}\cdot\mathcal{T}^*\mathcal{Q}} (Y^a) - pr^1 Y^g_{\mathcal{T}\cdot\mathcal{T}^*\mathcal{Q}} (X^a) \right) \frac{\partial}{\partial z^a} + \left( pr^1 X^g_{\mathcal{T}\cdot\mathcal{T}^*\mathcal{Q}} (Y_b) - pr^1 Y^g_{\mathcal{T}\cdot\mathcal{T}^*\mathcal{Q}} (X_b) \right) \frac{\partial}{\partial \pi_b} \tag{64}\]
If \(X^g_{\mathcal{T}\cdot\mathcal{T}^*\mathcal{Q}}\) and \(Y^g_{\mathcal{T}\cdot\mathcal{T}^*\mathcal{Q}}\) are ordinary vector fields on \(T^* T^*\mathcal{Q}\), then \([,]\) reduces to the Jacobi-Lie bracket of vector fields \[\text{[50]}\].

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If $X_{T^*Q}$ and $Y_{T^*Q}$ are two projectable vector fields on $T^*T^*Q$, a straightforward calculation gives

$$[HX_{T^*Q}, HY_{T^*Q}]_{\text{pro}} = H[X_{T^*Q}, Y_{T^*Q}]$$

where $[\cdot, \cdot]_{\text{pro}}$ is the bracket in Eq.(64). That means, holonomic lift defines an isomorphism between subspace $HT^*T^*Q = \text{Im}(\Gamma)$ of the direct sum decomposition $TT^*T^*Q = VT^*T^*Q \oplus HT^*T^*Q$ and the space of projectable vector fields on $T^*T^*Q$ [51], [28], [31]. However, for vertical representatives there appears, in addition, a vector valued two-form.

**Proposition 18** For vertical representatives, the bracket $[\cdot, \cdot]_{\text{pro}}$ gives

$$[VX_{T^*Q}, VY_{T^*Q}]_{\text{pro}} = V[X_{T^*Q}, Y_{T^*Q}]_{\text{pro}} + \mathcal{B}(X_{T^*Q}, Y_{T^*Q}),$$

where $\mathcal{B}$ is a vertical-vector valued two-form

$$\mathcal{B}(X_{T^*Q}, Y_{T^*Q}) = [HY_{T^*Q}, VX_{T^*Q}]_{\text{pro}} - [HX_{T^*Q}, VY_{T^*Q}]_{\text{pro}}.$$  \hspace{1cm} (66)

If, on the other hand, $X_{T^*Q}$ and $Y_{T^*Q}$ are restricted to be complete cotangent lifts of vector fields on $T^*Q$, then the two-form $\mathcal{B}$ in Eq.(67) vanishes [13], and we obtain

**Proposition 19** Let $X_{T^*Q}, Y_{T^*Q} \in \mathfrak{X}(T^*Q)$ and denote by $X^c_{T^*Q}, Y^c_{T^*Q}$ their complete cotangent lifts and, by $VX^c_{T^*Q}, VY^c_{T^*Q}$ the vertical representatives of the latter. Following Lie algebra isomorphism hold

$$V[X_{T^*Q}, Y_{T^*Q}]^c = [VX^c_{T^*Q}, VY^c_{T^*Q}]_{\text{pro}}$$ \hspace{1cm} (68)

where the bracket $[\cdot, \cdot]_{\text{pro}}$ is defined in Eq.(64).

**Remark 20** Eq.(68) extends the isomorphism between vector fields on $T^*Q$ and their complete cotangent lifts to an isomorphism

$$\mathfrak{X}(T^*Q) \leftrightarrow \mathfrak{X}^c(T^*Q) \subset \mathfrak{X}(TT^*Q) \leftrightarrow V\mathfrak{X}^c(T^*Q) \subset VT^*T^*Q$$

between complete cotangent lifts and their vertical representatives. Here, $V\mathfrak{X}^c(T^*Q)$ is the space of vertical representatives of cotangent lifts and

$$VT^*T^*Q = \ker\{T \pi_{T^*Q} : TT^*T^*Q \to TT^*Q\}$$

is the space of all vertical vectors on $T^*T^*Q$.

Following diagram summarizes the preceding geometric constructions in which we are intended to obtain the momentum-Vlasov equations.
From a physical point of view, if components \((\Pi_q(z), \Pi_p(z))\) of \(\Pi_{id}(z)\) are solutions, parametrized by Eulerian coordinates \(z\) of particle motion, of the canonical Hamilton's equations on \(T^*T^*Q\) and \(X_h^{\pi}\) is tangent to these curves on fibers, vertical vector fields satisfy the tangency condition without reference or restrictions to the particle motion on base manifold \(T^*Q\). The vertical representative of \(X_h^{\pi}(z, \Pi_{id})\) is given by

\[
VX_h^{\pi}(z, \Pi_{id}) = e(\Pi_p(z) \cdot \nabla_q) (\nabla_q \phi_f(q)) - X_h(\Pi_q(z)) \cdot \nabla_{\Pi_q} - \frac{1}{m} \Pi_q(z) + X_h(\Pi_p(z)) \cdot \nabla_{\Pi_p},
\]

where we denote the action of \(X_h\) on components of \(\Pi_{id}\) by

\[
X_h(\Pi_p(z)) = \frac{1}{m} (p \cdot \nabla_q) \Pi_p(z) - e(\nabla_q \phi_f(q) \cdot \nabla_p) \Pi_p(z).
\]

The components of the vector field in Eq. (69) are precisely the momentum-Vlasov equations [24]

\[
\ddot{\Pi}_q = -X_h(\Pi_q) + e(\Pi_p \cdot \nabla_q) (\nabla_q \phi_f)
\]

\[
\ddot{\Pi}_p = -X_h(\Pi_p) - \frac{1}{m} \Pi_q
\]

given in Eqs. (8) and (9). Being an element of the dual of Lie algebra, \(\Pi_{id}\) and its time derivative are components of one-forms in momentum-Vlasov equations whereas, in Eq. (69) they appear as components of vector fields. To make the connection between Eqs. (8), (9) and (69) precise we need the concept of vertical lift of one-forms.

Consider the cotangent lift \(T^*\pi_{T^*Q} : T^*T^*Q \rightarrow T^*T^*T^*Q\) of the projection \(\pi_{T^*Q} : T^*T^*Q \rightarrow T^*Q\) and recall the musical isomorphism \(\Omega^{\pi}_{T^*T^*Q} : T^*T^*T^*Q \rightarrow TT^*T^*Q\) associated with the symplectic two-form \(\Omega_{T^*T^*Q}\) on the cotangent bundle \(T^*T^*Q\). For \(\pi \in T^*T^*Q\) define the Euler vector field

\[
X_E : T^*T^*Q \rightarrow TT^*T^*Q : \pi \rightarrow \Omega^{\pi}_{T^*T^*Q} \circ T^*\pi_{T^*Q}(\pi),
\]

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which is a vertical vector field, that is, \( \text{Im}(\mathcal{X}_E) \subset \ker(T^*\pi_{T^*Q}) \). Define the vertical lift of a one-from \( \pi \) on \( T^*Q \) to be the vertical vector field

\[
\text{ver}(\pi) = \mathcal{X}_E \circ \pi \circ \pi_{T^*Q} : T^*T^*Q \to TT^*Q
\]

on \( T^*T^*Q \). In coordinates, the vertical lift of \( \pi = \pi_q(z) \cdot dq + \pi_p(z) \cdot dp \) is given by \( \text{ver}(\pi) = \pi_q(z) \cdot \nabla \Pi_q + \pi_p(z) \cdot \nabla \Pi_p \). Now, if we take \( \pi = \Pi_{id} = \Pi_q \cdot dq + \Pi_p \cdot dp \) then the exact relation between Eqs. (8) and (9) can be expressed by

\[
\Pi_{id} = \text{ver}(\Pi_{id}) = V X h^*_\Pi_{id}(z, \Pi_{id})
\]

where the right hand side refers to Eqs. (8) and (9).

Thus, we show that the momentum-Vlasov equations are generated by the vertical representative of complete cotangent lift of particle motion on \( T^*Q \). The result in Proposition 19 shows that this process of lifting particle motion preserves algebraic structures.

### 3.4 Lie-Poisson Hamiltonian operator

We shall relate the geometric construction of the previous subsection to Lie-Poisson structure. The Lie algebra \( g \) acts on \( g^* \) by coadjoint action given by the Lie derivative \( ad^*_X \in \mathfrak{g}^* \). The vertical lift \( \text{ver}(-ad^*_X) \) of generator of coadjoint action is a vector field tangent to the fiber coordinates \( (\Pi_q, \Pi_p) \) of \( g^* \subset T^*T^*Q \) and coincides with the right hand side of Eq. (73), namely,

\[
\text{ver}(-ad^*_{X h}(\Pi_{id})) = V X h^*_\Pi_{id}(z, \Pi_{id})
\]

This observation leads us to connect a Lie algebra element, that is, generator of particle motion, to corresponding generator of coadjoint action. This connection is provided by an operator associated to the Lie-Poisson structure on the dual space \( g^* \). Recall that \( g^* \) is a Poisson manifold with the Lie-Poisson bracket in Eq. (5) which may be written as

\[
\{K, H\}_{LP} = \int \frac{\delta K}{\delta \Pi_{id}} \cdot J_{LP}(\Pi_{id}) \frac{\delta H}{\delta \Pi_{id}} d\mu
\]

for the Hamiltonian operator

\[
J_{LP}(\Pi_{id}) = \begin{pmatrix}
\Pi_i \frac{\partial}{\partial q^j} + \frac{\partial}{\partial q^j} \cdot \Pi_j & \Pi_i \frac{\partial}{\partial p^j} + \frac{\partial}{\partial p^j} \cdot \Pi_j \\
\Pi_i \frac{\partial^2}{\partial q^j \partial q^l} + \frac{\partial}{\partial q^j} \cdot \Pi_j & \Pi_i \frac{\partial^2}{\partial p^j \partial q^l} + \frac{\partial}{\partial p^j} \cdot \Pi_j
\end{pmatrix}
\]

and for \( \frac{\partial}{\partial q^j} \cdot \Pi_j = \frac{\partial \Pi_j}{\partial q^j} + \Pi_j \frac{\partial}{\partial q^j} \) etc. [24]. As the derivative \( \delta K/\delta \Pi_{id} \) is in the Lie algebra \( g \), the operator \( J_{LP}(\Pi_{id}) \) may be considered to be a map

\[
J_{LP}(\Pi_{id}) : g \to (ad^*_g : g^* \to g^*) = \text{End}(g^*)
\]

\[
: X_h \to J_{LP}(\Pi_{id})(X_h) = \text{ver}(-ad^*_{X h}(\Pi_{id}))
\]

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taking a generator $X_h$ in the Lie algebra $\mathfrak{g}$ to the corresponding generator $ver(-ad^*_{X_h})$ of coadjoint action. Note that $J_{LP}(\Pi_{id})$ is a map from the tangent space $TT^*Q$ to $TT^*T^*Q$ both of which are Tulczyjew. Considering the representation

$$\hat{\Pi}_{id} = J_{LP}(\Pi_{id}) \frac{\delta H}{\delta \Pi_{id}}$$

(75)

of momentum-Vlasov equations on $\mathfrak{g}^*$ with $\delta H/\delta \Pi_{id} = X_h$, we conclude that, the whole geometric process we described to obtain the momentum-Vlasov equations from the generators of particle motion is encoded in $J_{LP}(\Pi_{id})$ as

\[
\begin{pmatrix}
\text{Lie-Poisson} \\
\text{Hamiltonian operator at } X_h
\end{pmatrix}
= \begin{pmatrix}
\text{vertical} \\
\text{equivalence}
\end{pmatrix}
\circ
\begin{pmatrix}
\text{complete} \\
\text{cotangent lift of } X_h
\end{pmatrix}
\]

and this represents a geometric decomposition of Lie-Poisson Hamiltonian operators. It also suggests a way to construct Lie-Poisson operator directly from an arbitrary Lie algebra element [13]. This is the way we shall relate the algebra of Hamiltonian vector fields to the algebra of strict contact vector fields in the next section.

**Remark 22** The Hamiltonian operator $J_{LP}(\Pi_{id})$ on $\mathfrak{g}^*$ transforms into the Hamiltonian operator

$$J_{LP}(f) = \nabla_p f \cdot \nabla_q - \nabla_q f \cdot \nabla_p$$

(76)

for the Vlasov equations in density variable on the space $\text{Den}(T^*Q)$ of densities under the correspondence in Eq. (11).
4 Moments of Momentum-Vlasov Dynamics

The space $\mathfrak{T}Q$ of all symmetric contravariant tensor fields on a manifold $Q$ carries a Lie algebra structure called Schouten concomitant (symmetric Schouten braket)\cite{25,56,30}. The dual $\mathfrak{T}^*Q$ of the algebra $\mathfrak{T}Q$ consists of the symmetric covariant tensor fields and carries Kuperschmidt-Manin Lie-Poisson structure,\cite{20}. In\cite{21} and\cite{63}, it was argued that, the kinetic moments of plasma density function $f$ may be considered as elements of $\mathfrak{T}^*Q$ and, it is established that, the operation taking the plasma density $f$ to the moments of the dynamics is a Poisson mapping which is the dual of a Lie algebra homomorphism from $\mathfrak{T}Q$ to $\mathcal{F}(T^*Q)$.

In this section, we first review the Schouten algebra of symmetric contravariant tensors. Then, we will define a Lie algebra homomorphism from the algebra $\mathfrak{T}Q$ of symmetric contravariant tensor fields to the algebra $\mathfrak{X}_{ham}(T^*Q) = \mathfrak{g}$ of Hamiltonian vector fields as a generalization of the complete cotangent lift introduced above. We will obtain the moments of momentum-Vlasov dynamics. Finally, we will consider some subalgebras of $\mathfrak{T}Q$, and derive plasma-to-fluid map in terms of momentum variables.

4.1 Schouten concomitant

The direct product $\mathfrak{T}Q = \oplus_{n=0}^{\infty} \mathfrak{T}^nQ$ of spaces $\mathfrak{T}^nQ$ of symmetric contravariant tensor fields on a manifold $Q$ of all orders constitutes a vector space. In a local coordinate system $(q^i)$, an element of $\mathfrak{T}Q$ is in form

$$X = \bigoplus_{n=0}^{\infty} X^n = \bigoplus_{n=0}^{\infty} X^{i_1i_2...i_n}(q) \partial_{q^{i_1}} \otimes ... \otimes \partial_{q^{i_n}},$$

where $X^n \in \mathfrak{T}^nQ$ is a symmetric contravariant tensor field of order $n$ and $X^{i_1i_2...i_n}(q)$ are the real valued coefficient functions. The dual $\mathfrak{T}^*Q$ of $\mathfrak{T}Q$ is the direct sum $\oplus_{n=0}^{\infty} \mathfrak{T}_nQ$ of symmetric covariant tensor fields $\mathfrak{T}_nQ$ of all orders. In coordinates $(q^i)$, an element of $\mathfrak{T}^*Q$ is given by

$$A = \bigoplus_{n=0}^{\infty} A_n = \bigoplus_{n=0}^{\infty} A_{i_1i_2...i_n}(q) dq^{i_1} \otimes ... \otimes dq^{i_n},$$

where $A_n \in \mathfrak{T}_nQ$ is a symmetric covariant tensor field of order $n$. The pairing between $\mathfrak{T}^*Q$ and $\mathfrak{T}Q$ is

$$\langle A, X \rangle = \sum_{n=0}^{\infty} \langle A_n, X_n \rangle = \sum_{n=0}^{\infty} \int A_{i_1i_2...i_n}(q) X^{i_1i_2...i_n}(q) dq. \quad (78)$$

If $X^n$, $Y^m$ and $Z^{n+m-1}$ are contravariant tensor fields of orders $n$, $m$ and $n + m - 1$, respectively, the Schouten concomitant

$$[X, Y]_{SC} = \bigoplus_{n=0}^{\infty} \bigoplus_{m=0}^{\infty} [X^n, Y^m]_{SC} = \bigoplus_{n=0}^{\infty} \bigoplus_{m=0}^{\infty} Z^{n+m-1} \quad (79)$$

27
defines a Lie algebra structure on the space \( \mathfrak{T}Q \) \cite{56, 30, 63}. The coefficient functions of \( \mathcal{Z}^{n+m-1} \) in terms of those of \( \mathcal{X}^n \) and \( \mathcal{Y}^m \) are

\[
\mathcal{Z}^{i_1 \ldots i_{n+m-1}} = n \mathcal{X}^{i_{m+1} \ldots i_{m+n-1}} \frac{\partial \mathcal{Y}^{i_1 \ldots i_{m}}}{\partial q} - m \mathcal{Y}^{i_{n+1} \ldots i_{n+m-1}} \frac{\partial \mathcal{X}^{i_1 \ldots i_{n}}}{\partial q}.
\]

### 4.2 Generalized complete cotangent lift

Let \( \mathcal{X}^n \) be a (not necessarily symmetric) contravariant tensor field of order \( n \). Due to the canonical inclusion \( \mathcal{X}^n \rightarrow \mathcal{T}(T^*Q) \), we may assume \( \mathcal{X}^n \) as a tensor on the cotangent bundle \( T^*Q \). We define a mapping

\[
\mathcal{X}^n \rightarrow F(T^*Q): \mathcal{X}^n \rightarrow H_{\mathcal{X}^n} = \theta_{T^*Q}^\mathcal{X}^n(\mathcal{X}^n)
\]

from \( \mathcal{X}^n \) to the space \( F(T^*Q) \) of smooth functions on \( T^*Q \), by contracting a contravariant tensor \( \mathcal{X}^n \in T^*Q \) with \( n \)-th tensor power \( \theta_{T^*Q}^n = \theta_{T^*Q} \otimes \cdots \otimes \theta_{T^*Q} \) of the canonical one-form \( \theta_{T^*Q} \). Then, define the generalized complete cotangent lift

\[
c^*: \mathcal{X}^n \rightarrow \mathfrak{x}_{ham}(T^*Q): \mathcal{X}^n \rightarrow (\mathcal{X}^n)^{c^*} = X_{\mathcal{X}^n}
\]
as an operation taking a contravariant tensor field \( \mathcal{X}^n \) on \( Q \) to a Hamiltonian vector field \( (\mathcal{X}^n)^{c^*} \) corresponding to the Hamiltonian function \( H_{\mathcal{X}^n} = i_{\mathcal{X}^n} \theta_{T^*Q} \), \cite{49}. In Darboux’s coordinates \((q, p)\), the Hamiltonian function \( H_{\mathcal{X}^n} \) is a polynomial in the fiber variables of \( T^*Q \)

\[
H_{\mathcal{X}^n}(q, p) = p_{i_1} p_{i_2} \ldots p_{i_n} \mathcal{X}^{i_1 \ldots i_n}(q)
\]

and the complete cotangent lift is

\[
(\mathcal{X}^n)^{c^*} = n p_{i_1} p_{i_2} \ldots p_{i_{n-1}} \mathcal{X}^{i_1 \ldots i_{n-1}} \frac{\partial}{\partial q} - p_{i_1} p_{i_2} \ldots p_{i_n} \frac{\partial \mathcal{X}^{i_1 \ldots i_n}}{\partial q} \frac{\partial}{\partial p_l}.
\]

We further enhance the operations given in Eqs.\( 80 \) and \( 81 \) to the product space \( \mathcal{T}Q \) as follows. For \( \mathcal{X} = \mathcal{X}^n \oplus \mathcal{Y}^m \in \mathcal{T}Q \) define the function \( H_{\mathcal{X}} \) on \( T^*Q \) as the sum

\[
\mathcal{T}Q \rightarrow F(T^*Q): \mathcal{X} \rightarrow H_{\mathcal{X}} = \sum_{n=0}^{\infty} H_{\mathcal{X}^n},
\]

\cite{12}. This infinite sum may be considered as the Taylor expansion of the function \( H_{\mathcal{X}} \) in terms of \( p \)-polynomials. A straight forward calculation proves the following proposition.

**Proposition 23** The map \( \mathcal{X} \rightarrow H_{\mathcal{X}} \) is a Lie algebra anti-homomorphism, that is

\[
H_{[\mathcal{X}, \mathcal{Y}]}_{SC} = - \{ H_{\mathcal{X}}, H_{\mathcal{Y}} \}_{T^*Q},
\]

where the bracket on the left hand side is the Schouten concomitant of covariant tensors and the bracket on the right hand side is the canonical Poisson bracket of functions on \( T^*Q \). In particular, one has

\[
H_{[\mathcal{X}^n, \mathcal{Y}^m]}_{SC} = - \{ H_{\mathcal{X}^n}, H_{\mathcal{Y}^m} \}_{T^*Q},
\]

where \([\mathcal{X}^n, \mathcal{Y}^m]_{SC} \in \mathcal{T}^{n+m-1}Q\).
For the generalized complete cotangent lift
\[
\mathcal{C}^*: \mathfrak{T}Q \to \mathfrak{X}_{ham}(T^*Q) : \mathfrak{X} = \bigoplus_{n=0}^{\infty} \mathfrak{X}^n \to \mathfrak{X}^{c*} = \sum_{n=0}^{\infty} (\mathfrak{X}^n)^{c*},
\]
we use the identity \([X_F, X_G] = -X_{\{F,G\}}\) and the above proposition to have
\[
[X^{c*}, Y^{c*}] = [X_{H^X}, X_{H^Y}] = X_{H_{[X,Y]}^{SC}} = [X, Y]^{SC},
\]
which enables us to state the following result.

**Proposition 24** The generalized complete cotangent lift in Eq. (84) is a Lie algebra isomorphism into
\[
\mathfrak{X} \to \mathfrak{X}^{c*} : \mathfrak{T}Q \to \mathfrak{g}
\]
where \([\cdot, \cdot]_{SC}\) is the Schouten concomitant of tensor fields in Eq. (79) and \([\cdot, \cdot]_{JL}\) is the Jacobi-Lie bracket of vector fields.

### 4.3 Moments of momentum variables

The dual map \(\Phi : \mathfrak{g}^* \to \mathfrak{T}^*Q\) of the homomorphism \(\mathfrak{X} \to \mathfrak{X}^{c*}\) is a momentum and Poisson mapping. To compute it, take \(\Pi_{id} = \Pi_q \cdot dq + \Pi_p \cdot dp \in \mathfrak{g}^*\), then the dual operation is
\[
\Phi (\Pi_{id}) = \bigoplus_{n=0}^{\infty} \int (\Theta_{T^*Q}^{n-1} \otimes \vartheta) \, d^3p,
\]
where \(\Theta_{T^*Q}^{n-1}\) is the \((n-1)\)-th tensor power of the canonical one form \(\Theta_{T^*Q}\) and \(\vartheta\) is a one-form on \(T^*Q\) given explicitly by
\[
\vartheta(q, p) = (n \Pi_q + (\nabla_q \Pi_p)) \cdot dq
\]
The image of \(\Pi_{id}\) under the dual map \(\Phi\) consists of moments of the momentum-Vlasov dynamics. Namely, the \(n\)-th moment of \(\Pi_{id}\) is given by
\[
\mathbb{A}_n = \int (\Theta_{T^*Q}^{n-1} \otimes \vartheta) \, d^3p.
\]
In particular, for one dimensional plasma, using the momentum map
\[
\mathfrak{g}^* \to \mathcal{F}(T^*Q) : \Pi_{id} \to f(q, p)
\]
in Eq. (10) we have the kinetic moments
\[
\mathbb{A}_n = \int p^n f(q, p) \, dp
\]
of the Vlasov density [40]. The following proposition argues that the moments are Poisson maps [20].
Proposition 25  The kinetic moments in Eq. (86) is a Poisson map from the Lie-Poisson bracket on $\mathfrak{g}^*$ to the Kuperschmidt-Manin bracket on $\mathbb{T}^*Q$.

To prove this, take a linear functional $F_X$ on $\mathbb{T}^*Q$ of the form

$$F_X(A) = \langle A, X \rangle = \sum_{n=0}^{\infty} \langle A_n, X_n \rangle = \sum_{n=0}^{\infty} \int A_{i_1i_2...i_n}(q) X^{i_1i_2...i_n}(q) \, dq.$$  

Its variation is $\delta F_X / \delta A = X$. The pull-back $\Phi^* \tilde{\mathfrak{X}}$ of $\tilde{\mathfrak{X}}$ to $\mathfrak{g}^*$ by the momentum map $\Phi$ in Eq. (86) gives

$$(\Phi^* \tilde{\mathfrak{X}})(\operatorname{id}) = \sum_{n=0}^{\infty} \int n p_{i_1} p_{i_2} ... p_{i_{n-1}} (\Pi_{i_n} + p_{i_n} \frac{\partial \Pi_{i_n}}{\partial q^l}) X^{i_1i_2...i_n}(q) \, dq \, dp.$$  

and the variation of this with respect to its argument $\operatorname{id}$ is

$$\delta (\Phi^* \tilde{\mathfrak{X}}) \delta \operatorname{id} = X_{H_X} = X_{c^*},$$

where $X_{H_X}$ is the Hamiltonian vector field corresponding to the Hamiltonian function $H_X$ in Eq. (83). The Lie-Poisson bracket on $\mathfrak{g}^*$ is

$$\{ \Phi^* \tilde{\mathfrak{X}}, \Phi^* \tilde{\mathfrak{Y}} \} = \int \langle A, [\delta (\Phi^* \tilde{\mathfrak{X}}), \delta (\Phi^* \tilde{\mathfrak{Y}})] \rangle d^3q d^3p.$$  

On $\mathbb{T}^*Q$, the Kuperschmidt-Manin bracket is given by

$$\{ \tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}} \}_{KM} = \int \left( \delta \tilde{\mathfrak{X}} \frac{\delta \tilde{\mathfrak{Y}}}{\delta k_l} - \delta \tilde{\mathfrak{Y}} \frac{\delta \tilde{\mathfrak{X}}}{\delta k_l} \right) d^3q.$$  

where the bracket inside the integral is the Schouten concomitant and the pairing inside the integral is the one in Eq. (78). The fact that $\Phi^*$ is a Poisson map

$$\Phi^* \{ \tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}} \}_{KM} = \{ \Phi^* \tilde{\mathfrak{X}}, \Phi^* \tilde{\mathfrak{Y}} \}_{\mathfrak{g}^*},$$

follows from direct substitutions.

4.4 Plasma-to-fluid map in momentum variables

The Lie algebra structure on $\mathbb{T}Q$ defined by the Schouten concomitant has only three subalgebras; the space of smooth functions $\mathbb{T}^0Q = \mathcal{F}(Q)$, the space of vector fields $\mathbb{T}^1Q = \mathfrak{X}(Q)$ and $\mathfrak{X}(Q) \times \mathcal{F}(Q)$. For the subalgebra $\mathcal{F}(Q)$, Schouten concomitant reduces to the trivial Poisson bracket of functions on $Q$. The Lie algebra homomorphism in Eq. (83) takes the particular form $\phi \rightarrow X_\phi$, where

$$X_\phi = \frac{\delta \phi}{\delta k_l}.$$
where \( X_\phi = - \nabla q \phi \cdot \nabla p \) is the infinitesimal generator of the action \((q, p) \to (q, p - \nabla q \phi)\) of additive group of functions \( F(Q) \) on \( T^*Q \) by momentum translations. For the subalgebra \( \mathfrak{x}(Q) \), the concomitant reduces to the Jacobi-Lie bracket of vector fields and the homomorphism in Eq. (38) reduces to the identity \([X, Y]_{JL} = [X^{cc}, Y^{cc}]_{JL}\), where \( X^{cc} = X \cdot \nabla q - \nabla q (p \cdot \nabla q \phi) \cdot \nabla p \) is the complete cotangent lift of the vector field \( X = X \cdot \nabla q \). The lift \( X^{cc} \) is an infinitesimal generator of the right action of diffeomorphism group on \( T^*Q \).

On the third subalgebra \( \mathfrak{x}(Q) \times \mathcal{F}(Q) \), the Schouten concomitant, for \( X = (X, \phi) \) and \( Y = (Y, \zeta) \in \mathfrak{x}(Q) \times \mathcal{F}(Q) \) gives

\[
[X, Y]_{SC} = ([X, Y], X(\zeta) - Y(\phi)),
\]

which turns \( \mathfrak{x}(Q) \times \mathcal{F}(Q) \) into a semi-direct product algebra \( s = \mathfrak{x}(Q) \circledast \mathcal{F}(Q) \) with the first factor acting on the second by Lie derivative. Namely, \( X(\zeta) \) is the directional derivative of \( \zeta \) in the direction of \( X \) and \([X, Y]\) is the Jacobi-Lie bracket of vector fields \( X \) and \( Y \). This semi-direct product algebra is the Lie algebra of the group \( S = Diff(Q) \circledast \mathcal{F}(Q) \) which is the configuration space of compressible fluid. The dual space \( s^* = \Lambda^1(Q) \times \mathcal{F}(Q) \) of the Lie algebra \( s \) is the product of one-forms and functions (identified with three-forms) on \( Q \). The Lie-Poisson structure on \( s^* \) is defined by

\[
\{F, G\}_{s^*}(\rho, M) = \int M \cdot [\frac{\delta F}{\delta M} \cdot \nabla q] \, dq + \int \rho \left( \frac{\delta G}{\delta M} \cdot \nabla q \frac{\delta F}{\delta \rho} - \frac{\delta F}{\delta M} \cdot \nabla q \frac{\delta G}{\delta \rho} \right) \, dq
\]

and is known as the compressible fluid bracket. Here \((M = M \cdot dq, \rho, M) \in s^* \) and it is assumed that \( \delta F/\delta M, \delta G/\delta M \in \mathfrak{x}(Q) \).

The complete cotangent lift from the semi-direct product to the Lie algebra of Hamiltonian vector fields

\[
\mathfrak{s} \to \mathfrak{g} : \mathfrak{x} = (\phi, X) \to \mathfrak{x}^{cc} = X^{cc} + X_\phi
\]

is the sum of \( X^{cc} \) and \( X_\phi \). \( \mathfrak{x}^{cc} \) is the Hamiltonian vector field

\[
\mathfrak{x}^{cc} = X^i \frac{\partial}{\partial q^i} - \left( p_i \frac{\partial X^i}{\partial q^j} + \frac{\partial \phi}{\partial q^j} \right) \frac{\partial}{\partial p_j}
\]

on \( T^*Q \) with the Hamiltonian function \( H_\mathfrak{x}(p, q) = p \cdot X + \phi(q) \). In this case, the dual map

\[
\Phi : g^* = s^* : \Pi_{id} \to (\rho, M)
\]

of generalized complete cotangent lift \( \mathfrak{x} \to \mathfrak{x}^{cc} \) is the first two moments of the momentum-Vlasov variables

\[
\rho(q) = \int (\nabla q \cdot \Pi_p) \, d^3p, \quad M = \int (\Pi_q + p (\nabla q \cdot \Pi_p)) \, d^3p
\]

and is the plasma-to-fluid map in the momentum variables \( \Pi_{id} \). By direct calculation, one can check that

\[31\]
Proposition 26 The mapping in Eq. (91) is a momentum and a Poisson map from the momentum-Vlasov dynamics on $g^*$ to the compressible fluid dynamics on $s^*$.

The substitution $g^* \rightarrow \mathcal{F}(T^*Q) : \Pi_{id} \rightarrow f(q,p)$ in Eq. (10) gives the usual plasma-to-fluid map (for $M = \rho v$)

$$f(q,p) \mapsto (\rho(q) = \int f(q,p) \, d^3p, \ v(q) = \int pf(q,p) \, d^3p)$$

in density variable as described in [40].
5 Quantomorphisms for 1D Plasma

Recall that Hamiltonian function of a Hamiltonian vector field is only determined up to an additive constant. Based on this and with reference to the work of Van Hove in [77], it was already (foot-)noted in [38] that the correct configuration space for the Maxwell-Vlasov equations is the group of transformations of $\mathbb{R}^6 \times \mathbb{R}$ preserving the one-form $\mathbf{p} \cdot d\mathbf{q} + ds$. This is the group of strict contact transformations. In [57] it was shown how certain geometric constructions underlying Vlasov-type equations require the use of strict contact transformations, also known as quantomorphisms. In particular, group of quantomorphisms was used in [19] in order to cast Euler’s fluid equations on a geometric footing. In this section, we shall indicate, for the simplest case of one-dimensional plasma, relations between Vlasov dynamics and coadjoint motion on dual of Lie algebra of group of quantomorphisms. In particular, group of quantomorphisms was used in [19] in order to cast Euler’s fluid equations on a geometric footing. In this section, we shall indicate, for the simplest case of one-dimensional plasma, relations between Vlasov dynamics and coadjoint motion on dual of Lie algebra of group of quantomorphisms. In particular, group of quantomorphisms was used in [19] in order to cast Euler’s fluid equations on a geometric footing. In this section, we shall indicate, for the simplest case of one-dimensional plasma, relations between Vlasov dynamics and coadjoint motion on dual of Lie algebra of group of quantomorphisms.

5.1 Lie algebra of infinitesimal quantomorphisms

Let $\mathcal{P}$ be a three dimensional manifold with a contact form $\sigma \in \Lambda^1(\mathcal{P})$ satisfying $d\sigma \wedge \sigma \neq 0$. The kernel of $\sigma$ determines a contact structure on $\mathcal{P}$. A diffeomorphism on $\mathcal{P}$ is called a contact diffeomorphism if it preserves the contact structure. We denote the group of contact diffeomorphisms by $\text{Diff}_{\text{con}}(\mathcal{P}) = \{ \varphi \in \text{Diff}(\mathcal{P}) : \varphi^* \sigma = \lambda \sigma, \lambda \in \mathcal{F}(\mathcal{P}) \}$.

In Darboux’s coordinates $(q,p,s)$ on $\mathcal{P}$, we take the contact form to be $\sigma = pdq - ds$. A vector field on the contact manifold $(\mathcal{P}, \sigma)$ is a contact vector field if it generates one-parameter group of contact diffeomorphisms. We denote the space of contact vector fields by $\mathcal{X}_{\text{con}}(\mathcal{P}) = \{ X \in \mathcal{X}(\mathcal{P}) : \mathcal{L}_X \sigma = \tilde{\lambda} \sigma, \tilde{\lambda} \in \mathcal{F}(\mathcal{P}) \}$. (92)

For each real valued function $H = H(q,p,s)$ on $\mathcal{P}$, there corresponds a contact vector field

$$X_H = \frac{\partial H}{\partial p} \frac{\partial}{\partial q} - \left( \frac{\partial H}{\partial q} + p \frac{\partial H}{\partial s} \right) \frac{\partial}{\partial p} + \left( p \frac{\partial H}{\partial p} - H \right) \frac{\partial}{\partial s} \tag{93}$$

on $\mathcal{P}$ satisfying the identities

$$i_{X_H} \sigma = H \quad i_{X_H} d\sigma = (i_{R_\sigma} dH) \sigma - dH, \tag{94}$$

where $R_\sigma = -\partial / \partial s$ is the Reeb vector field of $\sigma$. $R_\sigma$ is the unique vector field satisfying $i_{R_\sigma} \sigma = 1$ and $i_{R_\sigma} d\sigma = 0$. The divergence $\text{div}_{dq} X_H$ of $X_H$ with respect
to the contact volume form \( d\mu = d\sigma \wedge \sigma \) can be computed to be \( \text{div}_d\mu X_H = 2R_\sigma H \). Cartan’s formula \( \mathcal{L}_{X_H} = di_{X_H} + i_{X_H}d \) and Eq. (94) imply that the coefficient functions \( \lambda \) in Eq. (92) must be of the form \( \lambda(q, p, s) = i_{R_\sigma} dH \), that is,

\[
\mathcal{L}_{X_H} \sigma = (i_{R_\sigma} dH) \sigma.
\]

Contact Poisson (or Lagrange) bracket

\[
\{ K, H \}_c = \frac{\partial H}{\partial p} \frac{\partial K}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial K}{\partial p} + \frac{\partial H}{\partial s} \left( p \frac{\partial H}{\partial p} - H \right) + \frac{\partial H}{\partial s} \left( K - p \frac{\partial K}{\partial p} \right),
\]

of two smooth functions \( H \) and \( K \) on \( \mathcal{P} \) induces a Lie algebra structure on \( \mathcal{F}(\mathcal{P}) \). The identity \(-[X_K, X_H]_{\text{JL}} = X_{\{K, H\}_c} \) establishes the isomorphism

\[
(\mathcal{X}_{\text{con}}(\mathcal{P}), [-, ]_{\text{JL}}) \leftarrow (\mathcal{F}(\mathcal{P}), \{\ , \ \}_c)
\]

between the Lie algebras of functions and contact vector fields. Here, \([\ , \ ]_{\text{JL}}\) is the Jacobi-Lie bracket of vector fields.

An element \( \varphi \in Diff_{\text{con}}(\mathcal{P}) \) is called a strict contact transformation or a quantomorphism if \( \varphi^* \sigma = \sigma \). We denote the group of quantomorphisms by \( Diff_{\text{st}}(\mathcal{P}) \). The Lie algebra of \( Diff_{\text{st}}(\mathcal{P}) \) is

\[
\mathcal{X}_{\text{st}}(\mathcal{P}) = \{ X_H \in \mathcal{X}_{\text{con}}(\mathcal{P}) : \mathcal{L}_{X_H} \sigma = 0 \}
\]

which requires, for elements of \( \mathcal{X}_{\text{st}}(\mathcal{P}) \), the condition \( \lambda = i_{R_\sigma} dH = 0 \) on the function \( H \). This gives \( \partial H/\partial s = 0 \). Hence, functions associated to elements of \( \mathcal{X}_{\text{st}}(\mathcal{P}) \) are independent of the last coordinate in \((q, p, s)\). This condition reduces the Lagrange bracket in Eq. (95) to the nondegenerate Poisson bracket

\[
\{ K, H \}_{T^*\mathcal{Q}} = \frac{\partial H}{\partial p} \frac{\partial K}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial K}{\partial p}
\]

on a two dimensional cotangent bundle \( T^*\mathcal{Q} \) with local coordinates \((q, p)\).

Thus, we are led to consider the principal circle bundle

\[
S^1 \hookrightarrow (\mathcal{P}, \sigma) \xrightarrow{pr} (T^*\mathcal{Q}, \Omega_{T^*\mathcal{Q}}),
\]

which is the so called quantization bundle of the symplectic manifold \((T^*\mathcal{Q}, \Omega_{T^*\mathcal{Q}})\), with \( \sigma = pr^*\theta_{T^*\mathcal{Q}} - ds \) where \( \theta_{T^*\mathcal{Q}} = pdq \) is the Liouville one-form on \( T^*\mathcal{Q} \) and \( pr^*\Omega_{T^*\mathcal{Q}} = d\sigma \). Since \( \text{dim} \mathcal{Q} = 1 \), \( T^*\mathcal{Q} \) is the phase space of one-dimensional plasma particles. The contact form \( \sigma \) may be regarded as a connection one-form on \( \mathcal{P} \rightarrow T^*\mathcal{Q} \). With respect to this, the horizontal lift of a vector field \( X \) on \( T^*\mathcal{Q} \) to a vector field on \( \mathcal{P} \) is locally given by

\[
X \rightarrow X^\text{hor} = X + \sigma(X) \frac{\partial}{\partial s}.
\]

For a Hamiltonian vector field \( X_h \), the vector field

\[
X_h^\text{st} = X^\text{hor} + (h \circ pr) R_\sigma
\]

(99)
is an infinitesimal quantomorphism, that is, an element of $\mathfrak{x}^\text{st}_{\text{con}}(\mathcal{P})$. Here, we regard $h$ to be restriction of a function $H$ on $\mathcal{P}$ associated to a strict contact vector field. In Darboux’s coordinate, we have

$$X_h^\text{st} = \frac{\partial h}{\partial p} \frac{\partial}{\partial q} - \frac{\partial h}{\partial q} \frac{\partial}{\partial p} + \left( p \frac{\partial h}{\partial p} - h \right) \frac{\partial}{\partial s}$$

which can be obtained from the contact vector field in Eq. (93) by imposing the condition $h(q,p) = H(q,p,s = 0)$. The equality

$$[X_h^\text{st}, X_k^\text{st}]_{JL} = [X_h, X_k]_{JL} = -X_{\{h,k\}^\text{st}}$$

reduces the isomorphism in (96) to

$$(\mathfrak{x}^\text{st}_{\text{con}}(\mathcal{P}), -[\ , \ ]_{JL}) \longleftrightarrow (\mathcal{F}(T^*Q), \{ \ , \ }_{T^*Q}) \longleftrightarrow (\mathfrak{x}_{\text{ham}}(T^*Q) \times \mathbb{R}, -[\ , \ ]_{JL})$$

establishing, at the first step, the isomorphism between the Lie algebra of infinitesimal quantomorphisms on $\mathcal{P}$ and the Poisson bracket algebra of functions on $T^*Q$ and, at the second step, the isomorphism of Poisson bracket algebra to the Lie algebra of central extension of Hamiltonian vector fields [26], [19], [17].

**Remark 27** The group of quantomorphisms coincide with the automorphism group of the quantization bundle [35]. An automorphism of a principal $S$–bundle $\mathcal{P} \rightarrow \mathcal{M}$ is a diffeomorphism of $\mathcal{P}$ equivariant with respect to the action of the structure group $S$. If $\Phi : \mathcal{P} \rightarrow \mathcal{P}$ is such a diffeomorphism, then it induces a diffeomorphism $\varphi : \mathcal{M} \rightarrow \mathcal{M}$ of the base manifold via $pr \circ \Phi = \varphi \circ pr$. If we take $\mathcal{P}$ to be the trivial bundle $S \times \mathcal{M}$ then the group $\text{Aut}(S \times \mathcal{M})$ of automorphisms is isomorphic to $\text{Diff}(\mathcal{M}) \ast \text{Gau}(S \times \mathcal{M})$ and the automorphism algebra is $\mathfrak{aut}(S \times \mathcal{M}) \cong \mathfrak{x}(\mathcal{M}) \ast \mathcal{F}(\mathcal{M},s)$ where $\mathcal{F}(\mathcal{M},s)$ is the space of $s$–valued functions on $\mathcal{M}$ [16]. Restricting to contact automorphisms of the circle bundle above one arrives at the central extension $\mathfrak{x}_{\text{ham}}(T^*Q) \ast \mathbb{R}$.

### 5.2 Kinetic equations of contact particles

**Proposition 28** The dual space $\mathfrak{x}^*_{\text{con}}(\mathcal{P})$ of the algebra $\mathfrak{x}_{\text{con}}(\mathcal{P})$ of contact vector fields is

$$\mathfrak{x}^*_{\text{con}}(\mathcal{P}) = \{ \Gamma \otimes d\mu \in \Lambda^1(\mathcal{P}) \otimes \text{Den}(\mathcal{P}) : d\Gamma \wedge \sigma - 2\Gamma \wedge d\sigma \neq 0 \}$$

where $\sigma$ is the contact form and $d\mu = d\sigma \wedge \sigma$ is the contact volume form on $\mathcal{P}$.

**Proof.** This follows from the requirement that the pairing between $\mathfrak{x}_{\text{con}}(\mathcal{P})$ and $\mathfrak{x}^*_{\text{con}}(\mathcal{P})$ be nondegenerate. We compute

$$\int_{\mathcal{P}} (\Gamma, X_H)_{\mathcal{P}} \ d\mu = \int_{\mathcal{P}} \Gamma \wedge i_{X_H} d\mu$$

$$= \int_{\mathcal{P}} \Gamma \wedge (i_{X_H} d\sigma) \wedge \sigma + \int_{\mathcal{P}} (i_{X_H} \sigma) \Gamma \wedge d\sigma$$

$$= \int_{\mathcal{P}} \Gamma \wedge ((i_{R_\sigma}dH) \sigma - dH) \wedge \sigma + \int_{\mathcal{P}} H \Gamma \wedge d\sigma$$

$$= \int_{\mathcal{P}} H (2\Gamma \wedge d\sigma - d\Gamma \wedge \sigma)$$

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where we used identities in Eq.(97) and integration by parts and, omit divergence terms.

The coadjoint action of $X_{\text{con}}(P)$ on $X^*_{\text{con}}(P)$ is given by

$$ad_{\star}^* X_H (\Gamma \otimes d\mu) = (L_{X_H} \Gamma + (\text{div}_d X_K) \Gamma) \otimes d\mu$$  \hspace{1cm} (103)

and the Lie-Poisson bracket on $X^*_{\text{con}}(P)$ is

$$\{K, H\}(\Gamma) = - \int_P \langle \Gamma, \left[ X_K, X_H \right]_{JL} \rangle d\mu - \int_P \langle [X_K, X_H]_{JL}, \Gamma \rangle d\mu,$$  \hspace{1cm} (104)

where $K, H$ are functionals on $X^*_{\text{con}}(P)$ and we assume the reflexivity condition $\delta K/\delta \Gamma = X_K, \delta H/\delta \Gamma = X_H \in X_{\text{con}}(P)$. The Hamiltonian operator $J_{LP}(\Gamma)$ associated to the Lie-Poisson bracket in Eq.(104) is defined as

$$J_{LP}(\Gamma) = \int_P \langle X_K, J_{LP}(\Gamma) X_H \rangle d\mu.$$  \hspace{1cm} (105)

and the Lie-Poisson equations on $X^*_{\text{con}}(P)$ take the form

$$\dot{\Gamma} = J_{LP}(\Gamma) X_H = -ad_{\star} X_H \Gamma = -L_{X_H} \Gamma - (\text{div}_d X_H) \Gamma.$$  \hspace{1cm} (106)

Proposition 29 The Hamiltonian differential operator associated to the Lie-Poisson bracket in Eq.(104) is

$$J_{LP}(\Gamma) = - \left( \begin{array}{cccc}
\Gamma_q \frac{\partial}{\partial q} + \frac{\partial}{\partial q} \Gamma_q & \Gamma_p \frac{\partial}{\partial q} + \frac{\partial}{\partial p} \cdot \Gamma_q & \Gamma_s \frac{\partial}{\partial q} + \frac{\partial}{\partial s} \cdot \Gamma_q \\
\Gamma_p \frac{\partial}{\partial q} & \Gamma_p \frac{\partial}{\partial q} + \frac{\partial}{\partial p} \cdot \Gamma_p & \Gamma_s \frac{\partial}{\partial q} + \frac{\partial}{\partial s} \cdot \Gamma_p \\
\Gamma_s \frac{\partial}{\partial q} & \Gamma_s \frac{\partial}{\partial q} + \frac{\partial}{\partial p} \cdot \Gamma_s & \Gamma_s \frac{\partial}{\partial q} + \frac{\partial}{\partial s} \cdot \Gamma_s 
\end{array} \right),$$  \hspace{1cm} (107)

where $\partial/\partial q \cdot \Gamma_p = \Gamma_p \partial/\partial q + \partial \Gamma_p/\partial q$, etc.

In local coordinates $(q, p, s, \Gamma_q, \Gamma_p, \Gamma_s)$ on $T^*P$, the kinetic equations (105) of contact particles take the form

$$\dot{\Gamma}_q = -X_H(\Gamma_q) + (A + \Gamma_s) \frac{\partial H}{\partial q} + 2 \frac{\partial H}{\partial s} \Gamma_q$$
$$\dot{\Gamma}_p = -X_H(\Gamma_p) + A \frac{\partial H}{\partial p} + 3 \frac{\partial H}{\partial s} \Gamma_p$$
$$\dot{\Gamma}_s = -X_H(\Gamma_s) + A \frac{\partial H}{\partial s} + 3 \frac{\partial H}{\partial s} \Gamma_s,$$  \hspace{1cm} (108)

where $A$ stands for the linear differential operator

$$A = \Gamma_q \frac{\partial}{\partial q} - \Gamma_q \frac{\partial}{\partial p} + p \Gamma_q \frac{\partial}{\partial s} - p \Gamma_q \frac{\partial}{\partial p}.$$  \hspace{1cm} (109)
With respect to $L^2$-pairing, the dual space of the Lie algebra $(\mathcal{F}(\mathcal{P}),\{\cdot,\cdot\})$ is the space of densities $\text{Den}(\mathcal{P})$. Using the Lie algebra isomorphism
\[
\mathcal{F}(\mathcal{P}) \to \mathfrak{x}^\ast_{\text{con}}(\mathcal{P}) : F \to X_F, \tag{110}
\]
the definition of an element of $\text{Den}(\mathcal{P})$ can be obtained from the dual map
\[
\mathfrak{x}^\ast_{\text{con}}(\mathcal{P}) \to \text{Den}(\mathcal{P}) : \Gamma \otimes d\mu \to 2\Gamma \wedge d\sigma - d\Gamma \wedge \sigma \tag{111}
\]
and defines a real valued function $F$ on $\mathcal{P}$ by
\[
F d\mu = 2\Gamma \wedge d\sigma - d\Gamma \wedge \sigma. \tag{112}
\]
This is the density of particles moving individually with the right action of contact diffeomorphisms. In coordinates, we let $\Gamma = \Gamma_q dq + \Gamma_p dp + \Gamma_s ds$ and recall that $d\mu = dq \wedge dp \wedge ds$. Then, the density function of contact particles with momentum coordinates $\Gamma \otimes d\mu \in \mathfrak{x}^\ast_{\text{con}}(\mathcal{P})$ becomes
\[
F(q,p,s) = \partial \Gamma_p \partial q - \partial \Gamma_q \partial p + p \partial \Gamma_s \partial p + p \partial \Gamma_p \partial s - 2\Gamma_s. \tag{113}
\]
With this definition, the Lie-Poisson bracket in Eq.(104) reduces to the one
\[
\{\mathcal{K},\mathcal{H}\}(F) = \int_{\mathcal{P}} F \left\{ \frac{\delta \mathcal{K}}{\delta F}, \frac{\delta \mathcal{H}}{\delta F} \right\} c d\mu = \int_{\mathcal{P}} J_{\text{LP}}(F) H d\mu, \tag{114}
\]
on $\text{Den}(\mathcal{P})$, where we assume $\delta \mathcal{H}/\delta F = H, \delta \mathcal{K}/\delta F = K \in \mathcal{F}(\mathcal{P})$. Here, $J_{\text{LP}}(F)$ is the Hamiltonian operator for the Lie-Poisson bracket.

**Proposition 30** The Lie-Poisson equation on $\text{Den}(\mathcal{P}) \simeq \mathfrak{x}^\ast_{\text{con}}(\mathcal{P})$, as kinetic equation of contact particles, is
\[
\dot{F} = -\text{ad}^\ast_H F = J_{\text{LP}}(F) H = \{H,F\}_c - 2\text{div}_d\mu(X_H)F \tag{115}
\]
where the Hamiltonian operator $J_{\text{LP}}(F)$ is
\[
J_{\text{LP}}(F) = X_F + 4F \frac{\partial}{\partial s} + \frac{\partial F}{\partial s}. \tag{116}
\]
**Proof:** The coadjoint action is computed from
\[
\langle \text{ad}^\ast_H F, K \rangle = \langle F, \text{ad}_H K \rangle = \langle F, \{H,K\}_c \rangle = \int_{\mathcal{P}} F \left\{ H, K \right\}_c d\mu = -\int_{\mathcal{P}} F \left( X_H(K) + \frac{\partial H}{\partial s} K \right) d\mu \tag{117}
\]
\[
= \int_{\mathcal{P}} \left( X_H(F) + \text{div}_d\mu(X_H)F - \frac{\partial H}{\partial s} F \right) K d\mu \\
= \int_{\mathcal{P}} \left( \{F,H\}_c + \text{div}_d\mu(X_H)F - 2\frac{\partial H}{\partial s} F \right) K d\mu \\
= \int_{\mathcal{P}} \left( \{F,H\}_c + 2\text{div}_d\mu(X_H)F \right) K d\mu, \tag{118}
\]
37
where we used integration by parts at the third line and the identities
\[ \{ H, K \}_c = X_K (H) + \frac{\partial K}{\partial s} H = -X_H (K) - \frac{\partial H}{\partial s} K \] (119)
at the second and fourth lines. We single out \( H \) in the expression
\[ ad_H^* F = \{ F, H \}_c + 2 \text{div}_\mu (X_H) F, \] (120)
to find the Hamiltonian operator \( J_{LP}^* (F) \). The verification of Hamilton’s equation Eq.(115) is a straightforward calculation which follows directly from the Lie-Poisson in Eq.(108) together with the definition (113) of \( F \).

5.3 Quantomorphisms and momentum-Vlasov equations

We shall obtain kinetic equations of a continuum consisting of particles in \( P \subset \mathbb{R}^3 \) moving under the right action of quantomorphisms. We shall do this by restricting the Lie-Poisson equations for contact particles to strict contact transformations. We shall establish, with a proper choice of Hamiltonian functional, the equivalence of kinetic equations of quantomorphic particles in momentum and density variables to the momentum-Vlasov and the Vlasov equations for one-dimensional plasma.

Coadjoint action of \( \mathfrak{x}_{\text{con}}^* (P) \) on \( (\mathfrak{x}_{\text{con}}^* (P))^* \) is
\[ ad_{\mathfrak{x}_{\text{con}}^*} (\Gamma \otimes d\mu) = L_{\mathfrak{x}_{\text{con}}^*} \Gamma \otimes d\mu \]
because \( \text{div}_\mu X_h^* = 0, \forall X_h^* \in \mathfrak{x}_{\text{con}}^* (P) \). Accordingly, the Lie-Poisson equations for momentum variables \( \Gamma \in (\mathfrak{x}_{\text{con}}^* (P))^* \) become
\[
\begin{align*}
\dot{\Gamma}_q &= -X_h^* (\Gamma_q) + (\tilde{A} + \Gamma_s) \frac{\partial h}{\partial q} \\
\dot{\Gamma}_p &= -X_h^* (\Gamma_p) + \tilde{A} \frac{\partial h}{\partial p} \\
\dot{\Gamma}_s &= -X_h^* (\Gamma_s),
\end{align*}
\]
(121)
where \( \tilde{A} \) stands for the linear differential operator
\[ \tilde{A} = \Gamma_p \frac{\partial}{\partial q} - \Gamma_q \frac{\partial}{\partial p} - p\Gamma_s \frac{\partial}{\partial p}. \]
Using techniques of previous sections the following can readily be verified

**Proposition 31** The complete cotangent lift \( (X_h^*)^{cs} \) of an infinitesimal quantomorphism is
\[ (X_h^*)^{cs} = X_h^* + (\tilde{A} + \Gamma_s) \frac{\partial h}{\partial q} \frac{\partial}{\partial \Gamma_q} + \tilde{A} \frac{\partial h}{\partial p} \frac{\partial}{\partial \Gamma_p} \]
and the kinetic equation (121) of quantomorphic particles can be written as
\[ \text{ver} \left( \dot{\Gamma} \right) = V (X_h^*)^{cs}. \]
Remark 32 Eq. (121) is a system of first order pde for three unknown functions of essentially two variables \((q, p)\) because \(s\) dependence of the one-form \(\Gamma\) cannot be determined from these equations. That means, suppressing \(s\) dependence, the flow defined by Eq. (121) is actually two dimensional.

The density function of quantomorphic particles may be obtained as for density of contact particles. In this case, since components of \(X^* \) is independent of fiber variable \(s\) we get

\[
f(q, p) = \int F(q, p, s) \, dsanumber{(122)}
\]

where \(F(q, p, s)\) is given by Eq. (113).

To relate the quantomorphisms to plasma motion, we will use the Lie algebra isomorphism into \(X_{\text{ham}}(T^*Q) \rightarrow X_{\text{con}}^*(P)\) given in Eq. (99). The dual of this is the momentum map

\[
J_q : (X_{\text{con}}^*(P))^* \rightarrow X_{\text{ham}}(T^*Q) : \Gamma_q dq + \Gamma_p dp + \Gamma_s ds \rightarrow \Pi_q dq + \Pi_p dp
\]
defined as \(\langle J_q(\Gamma), X_h \rangle = \langle \Gamma, X^*_h \rangle\) and is given by

\[
\Pi_q(q, p) = \int \Gamma_q(q, p, s) \, ds, \quad \Pi_p(q, p) = \int \Gamma_p(q, p, s) \, ds, \quad \Gamma_s = 0 \quad (123)
\]

for which the Lie-Poisson equations (121) reduces to

\[
\begin{align*}
\dot{\Pi}_q &= -X_h(\Pi_q) + \Pi_p \frac{\partial^2 h}{\partial q^2} - \Pi_q \frac{\partial^2 h}{\partial q \partial p} \\
\dot{\Pi}_p &= -X_h(\Pi_p) + \Pi_p \frac{\partial^2 h}{\partial q \partial p} - \Pi_q \frac{\partial^2 h}{\partial p^2}
\end{align*}anumber{(124)}
\]

In particular, choosing \(h(q, p) = p^2/2m + e\phi(q)\) we obtain the momentum-Vlasov equations for one-dimensional plasma. The density variable defined by Eq. (122) reduces the Lie-Poisson equation (106) to one-dimensional Vlasov equation.

5.4 Hierarachy of Eulerian equations

One can now expand on the relation between the Poisson-Vlasov equations and the Euler equations of compressible fluid given by the plasma-to-fluid map. Combining this with the results of previous sections, we have the following diagram relating various kinetic and fluid theories...
| Configuration Space | Lie Algebra | Dual Space |
|---------------------|-------------|------------|
| $\text{Diff}(\mathcal{Q}) \otimes \mathcal{F}(\mathcal{Q})$ | $\mathfrak{X}(\mathcal{Q}) \otimes \mathcal{F}(\mathcal{Q})$ | $\Lambda^1(\mathcal{Q}) \times \mathcal{F}(\mathcal{Q})$ |
| $\downarrow$ cotangent lift | $\downarrow$ cotangent lift | $\uparrow$ plasma to fluid |
| $\text{Diff}_{\text{can}}(T^*\mathcal{Q})$ | $\mathfrak{X}_{\text{ham}}(T^*\mathcal{Q}) = \mathfrak{g}$ | $\Lambda^1(T^*\mathcal{Q})/d\mathcal{F}(T^*\mathcal{Q})$ |
| $\downarrow$ horizontal lift | $\downarrow$ homomorphism | $\uparrow$ quanto to plasma |
| $\text{Diff}_{\text{st}}(T^*\mathcal{Q} \times \mathbb{R})$ | $\mathfrak{X}_{\text{st}}(T^*\mathcal{Q} \times \mathbb{R})$ | $\Lambda^1_{\text{st}}(T^*\mathcal{Q} \times \mathbb{R})$ |
| $\downarrow$ inclusion | $\downarrow$ inclusion | $\uparrow$ contacto to quanto |
| $\text{Diff}_{\text{con}}(T^*\mathcal{Q} \times \mathbb{R})$ | $\mathfrak{X}_{\text{con}}(T^*\mathcal{Q} \times \mathbb{R})$ | $\Lambda_{\text{con}}(T^*\mathcal{Q} \times \mathbb{R})$ |

In addition, the Poisson map generated by the action of semi-direct product $\text{Diff}(\mathcal{Q}) \otimes \mathcal{F}(\mathcal{Q})$ reduces the Maxwell-Vlasov equations to the Euler-Maxwell equations. In the limit that the speed of light tends to infinity, these equations become the Poisson-Vlasov and compressible fluid equations, respectively. Elimination of the electric field in Euler-Maxwell equations results in the magnetohydrodynamics equations \[40\].
6 Gauge Symmetries and Poisson Equation

The general theory of reduction implies that constraints on Eulerian dynamics can be described as momentum map associated to some gauge symmetries of the underlying geometric structure. For the Maxwell-Vlasov system the non-evolutionary Maxwell equations come out as constraints resulting from the gauge symmetries of the electromagnetic field [38]. In this section, following the reference [24], we describe the Poisson equation as a momentum map associated with the gauge symmetry \( F(Q) \) of Hamiltonian dynamics on phase space \( T^\ast Q \) of particle motion. Such a description is possible only if we consider the semi-direct product space \( F(Q) \circledast \text{Diff}_{\text{can}}(T^\ast Q) \), where \( F(Q) \) acts on \( \text{Diff}_{\text{can}}(T^\ast Q) \) by composition on right. In obtaining Poisson equation, we rely necessarily on the fact that the dual of Lie algebra isomorphism into is a momentum map [22], [37] because the Lie algebra bracket on \( \text{d} F(Q) \), or equivalently, the Lie-Poisson bracket on the dual \( \Lambda^2(Q) \) is trivial.

6.1 Actions of \( F(Q) \)

The canonical symplectic structure on \( T^\ast Q \) is invariant under the translation of fiber variable by an exact one-form over \( Q \). This is the gauge transformation of canonical Hamiltonian formalism. More precisely, let \( F(Q) \) be the additive group of functions on \( Q \). Define the Lie algebra of \( F(Q) \) to be the space \( \text{d} F(Q) \) of exact one-forms on \( Q \). \( F(Q) \) acts on \( T^\ast Q \) by momentum translation \( p \rightarrow p - \nabla_q \phi(q) \) for \( \phi \in F(Q) \).

The generator is given by the vertical lift
\[
X_\phi(q, p) = -\nabla_q \phi(q) \cdot \nabla_p = \text{ver}(\phi, (q, p))
\] (125)
of the one-form \( d\phi \). This is a Hamiltonian vector on \( T^\ast_q Q \) with the Hamiltonian function \( \phi \) regarded as an element of \( F(T^\ast Q) \).

\( F(Q) \) acts on \( TT^\ast Q \) by the tangent lift of the fiber translation on \( T^\ast Q \). In coordinates, this action is given by

\[
(z, \dot{z}) = (q, p, \dot{q}, \dot{p}) \mapsto (q, p - \nabla_q \phi(q), \dot{q}, \dot{p} - \nabla_q \dot{\phi}(q))
\] (126)

and the generator is the complete tangent lift

\[
X_\phi^c(z, \dot{z}) = -\nabla_q \phi(q) \cdot \nabla_p - (\dot{q} \cdot \nabla_q) (\nabla_q \phi(q)) \cdot \nabla_p
\] (127)
of \( X_\phi \). This is also a Hamiltonian vector field on \( T_z T^\ast_q Q \) with respect to the Tulczyjew symplectic two-form \( \Omega_{TT^\ast Q} \) and for the Hamiltonian function

\[
H_{TT^\ast Q}(z, \dot{z}) = \dot{q} \cdot \nabla_q \phi(q).
\] (128)

Moreover, the one forms \( \vartheta_1 \) and \( \vartheta_2 \) in Eqs. (26) and (27) of the special symplectic structures on \( TT^\ast Q \) yield

\[
i_{X_\phi} \vartheta_1 = H_{TT^\ast Q}, \quad i_{X_\phi} \vartheta_2 = 0
\] (129)

upon contraction with the generator \( X_\phi \) of the lifted action.
Proposition 33 The additive group $\mathcal{F}(\mathcal{Q})$ of functions on $\mathcal{Q}$ acts symplectically on $T^*\mathcal{Q}$ and tensorial objects over $T^*\mathcal{Q}$.

The induced action of $\mathcal{F}(\mathcal{Q})$ on tensorial objects over $T^*\mathcal{Q}$ includes, in particular, the Tulczyjew symplectic space $TT^*\mathcal{Q}$. For another example, the action on $T^*T^*\mathcal{Q}$ is given by

$$(q, p, \pi_q, \pi_p) \rightarrow (q, p - \nabla_q \phi(q), \pi_q + (\pi_p \cdot \nabla_q) \nabla_q \phi(q), \pi_p),$$

(130)

whose infinitesimal generator is the complete cotangent lift

$$X_{\phi}(z, \pi_z) = -\nabla_q \phi(q) \cdot \nabla_p + (\pi_p \cdot \nabla_q) \nabla_q \phi(q) \cdot \nabla_{\pi_q}$$

(131)

of $X_{\phi}$. The lift $X_{\phi}^c$ is a Hamiltonian vector field on $T^*T^*\mathcal{Q}$ with the Hamiltonian function $H_{TT^*\mathcal{Q}}(z, \pi) = -\pi_p \cdot \nabla_q \phi(q)$ with respect to the canonical symplectic two-form $\Omega_{T^*T^*\mathcal{Q}}$.

The action on zero-forms, that is, on space $\mathcal{F}(T^*\mathcal{Q})$ of functions is obtained by composition and, the action on top forms, or equivalently, the space $\text{Den}(T^*\mathcal{Q})$ of densities on $T^*\mathcal{Q}$ will be given by composition of the volume density function with the fiber translation once we choose the Liouville volume $d\mu = \Omega^3_{T^*\mathcal{Q}}$ as a basis for the space of six-forms.

$X_{\phi}$ and $X_{\phi}^c$ are generators of the action of $\mathcal{F}(\mathcal{Q})$ on $g = \mathfrak{x}_{\text{ham}}(T^*\mathcal{Q})$ and $g^* \subset \Lambda^1(T^*\mathcal{Q})$, respectively. As we identified $g^*$ as the subspace $(g^*)^2$ of $TT^*\mathcal{Q}$, it will be convenient to consider the corresponding generator on this subspace. Since $\pi^2 = (-\pi_p, \pi_q)$, we find

$$X_{\phi}(z, \dot{z})|_{g^*} = -\nabla_q \phi(q) \cdot \nabla_p + (\pi_p \cdot \nabla_q) (\nabla_q \phi(q)) \cdot \nabla_{\pi_q}$$

(132)

and this is Hamiltonian with $-\pi_p \cdot \nabla_q \phi(q)$ with respect to the Tulczyjew symplectic structure.

6.2 Poisson Equations

Having the Hamiltonian actions of gauge group $\mathcal{F}(\mathcal{Q})$ on various spaces over $T^*\mathcal{Q}$, we can now compute the momentum maps into the dual space $\text{Den}(\mathcal{Q})$ of densities (three-forms) on $\mathcal{Q}$. To this end, we recall that the true configuration space of the Poisson-Vlasov dynamics is the semi-direct product space $\mathcal{F}(\mathcal{Q}) \oplus \text{Diff}_{\text{can}}(T^*\mathcal{Q})$ with the action of $\mathcal{F}(\mathcal{Q})$ on second factor given by fiber translation. On the other hand, the Lie algebra of vector fields generating the Hamiltonian action of $\mathcal{F}(\mathcal{Q})$ is commutative. That means, we have a trivial Lie-Poisson structure for the first factor. So, we first consider a convenient framework for the adjoint action of $\mathcal{F}(\mathcal{Q})$ on its algebra and the corresponding momentum map [24].

Let $\Lambda^1(\mathcal{Q})$ be the space of one-forms on $\mathcal{Q}$. We regard the algebra $d\mathcal{F}(\mathcal{Q})$ as a subspace of $\Lambda^1(\mathcal{Q})$. We obtain the action on $\Lambda^1(\mathcal{Q})$ by identifying it with $T^*\mathcal{Q}$ and take the action on $d\mathcal{F}(\mathcal{Q})$ to be the one induced from $\Lambda^1(\mathcal{Q})$. Thus, we have

$$\Lambda^0(\mathcal{Q}) \times \Lambda^1(\mathcal{Q}) \to \Lambda^1(\mathcal{Q}) : (\phi \cdot q, p \cdot dq) \mapsto p \cdot dq - d\phi(q)$$

(133)
where we denote $\Lambda^0(Q) \equiv \mathcal{F}(Q)$. From an algebraic point of view, the exterior derivative $d : \Lambda^0(Q) \to \Lambda^1(Q)$ can be interpreted as a map describing a Lie algebra isomorphism (up to addition of constants) of the additive algebra of functions $\mathcal{F}(Q)$ into the additive algebra of one-forms $\Lambda^1(Q)$ \cite{23}. We define the dual spaces of $d\mathcal{F}(Q) \subset \Lambda^1(Q)$ and $\Lambda^0(Q)$ to be the space of two forms $\Lambda^2(Q)$ and the space $\text{Den}(Q)$ of densities (three-forms), respectively. The additive algebras $d\mathcal{F}(Q)$ and $\Lambda^0(Q)$ can be identified with their duals by the Hodge duality operator $\ast$ associated to a Riemannian metric on $Q$. Then, the $L^2-$pairing between them becomes

$$\langle \ast d\phi, d\phi \rangle = -\int \phi \ast d\phi, \quad \langle \ast \phi, \phi \rangle = \int \phi^2 \ast 1$$

the first of which is non-degenerate for functions satisfying $d \ast d\phi \neq 0$. We can now compute the momentum map

$$\mathbb{J}_{\mathcal{F}(Q)} : \Lambda^2(Q) \to \text{Den}(Q)$$

for the action of gauge group $\mathcal{F}(Q)$ on its Lie algebra from

$$\langle \mathbb{J}_{\mathcal{F}(Q)}(\ast d\phi(q)), \phi(q) \rangle = -\int_Q \phi(q) \, d \ast d\phi(q)$$

which, for the Euclidean metric on $Q$, gives $d \ast d\phi(q) = \nabla_q^2 \phi(q) \, dq$.

For the momentum map $g^* \to d\mathcal{F}(Q)^*$ we first recall a property of the vertical lift of one-forms. The vertical lift of an exact one-form $d\phi(q) = \nabla_q \phi \cdot dq$ is given by $\text{ver}(d\phi) = \nabla_q \phi \cdot \nabla p$. For any function $\phi : Q \to \mathbb{R}$, $\text{ver}(d\phi)$ is a Hamiltonian vector field with respect to the canonical two-form $\Omega_{T^*Q}$ for the Hamiltonian function $-\phi$. Conversely, any Hamiltonian vector field which is also vertical can be identified with its Hamiltonian function on $Q$. Therefore, we have the identification

$$\mathcal{F}(Q) \leftrightarrow \text{ver}(d\mathcal{F}(Q)) = \mathfrak{X}_{\text{ham}}(T^*Q) \cap V T^*Q.$$  

Obviously, the algebra of vertical Hamiltonian vector fields is commutative. So, we have the commutative subalgebra

$$[\text{ver}(d\mathcal{F}(Q)), \text{ver}(d\mathcal{F}(Q))] = 0$$

in the algebra $\mathfrak{X}_{\text{ham}}(T^*Q)$ of all Hamiltonian vector fields. If we regard $d\mathcal{F}(Q)$ as a Poisson algebra with zero Poisson bracket, then the map $\text{ver} : d\mathcal{F}(Q) \longrightarrow \mathfrak{X}_{\text{ham}}(T^*Q)$ may be interpreted as a Lie algebra isomorphism-into. Hence, the dual map

$$\text{ver}^* : g^* = \mathfrak{X}_{\text{ham}}^*(T^*Q) \to d\mathcal{F}(Q)^* = \Lambda^2(Q)$$

is a Poisson map into a trivial Lie-Poisson structure. More conveniently, we take $\text{ver} \circ d : \mathcal{F}(Q) \to g$ and the dual map $(\text{ver} \circ d)^* : g^* \to \text{Den}(Q)$ is a momentum
map given by
\[
\langle (\text{ver} \circ d)^* (\Pi_{id}) , \phi \rangle = \langle \Pi_{id}, \text{ver} (d\phi) \rangle = \langle \Pi_{id}(z), \nabla_q \phi (q) \cdot \nabla_p \rangle
\]
\[
= \int_{T^* Q} \Pi_p (z) \cdot \nabla_q \phi (q) \, d\mu(z)
\]
\[
= \int_{T^* Q} -\phi (q) \nabla_q \cdot \Pi_p (z) \, d\mu(z).
\]
(140)

Combining this with the momentum map in Eq. (136) with \( * \) being defined by the Euclidean metric, we have
\[
\mathbb{J}_P : *dF(Q) \times g^* \rightarrow \text{Den}(Q)
\]
\[
\mathbb{J}_P (*d\phi(q) , \Pi_{id}(z)) = (\nabla^2_q \phi (q) - \int \nabla_q \cdot \Pi_p (z) \, d^3p) \, d^3q
\]
(141)

whose zero value gives the Poisson equation
\[
\nabla^2_q \phi (q) = \int \nabla_q \cdot \Pi_p (z) \, d^3p.
\]
(142)

**Proposition 34** The zero value of momentum map \( \mathbb{J}_P : *dF(Q) \times g^* \rightarrow \text{Den}(Q) \) for the action of gauge group \( F(Q) \) constrains the dynamics of the momentum-Vlasov equations (8), (9) on \( g^* \). Similarly, the zero value of \( *dF(Q) \times \text{Den}(T^* Q) \rightarrow \text{Den}(Q) \) constrains the dynamics of the Vlasov equation (2) on \( \text{Den}(T^* Q) \).

To obtain the usual Poisson equation as given by Eq. (1), we think of \( F(T^* Q) \) equipped with the Poisson bracket to be an algebra isomorphic to the Lie algebra of Hamiltonian vector fields. Then, \( F(Q) \) is a commutative subalgebra of \( (F(T^* Q), \{ , \})_{T^* Q} \) corresponding to the generators of action of \( F(Q) \). Thus, we have the Lie algebra isomorphism from the additive algebra of functions \( F(Q) \) into the Poisson bracket algebra on \( F(T^* Q) \). Using dualization, the momentum map \( \mathbb{J}_{\text{den}} : \text{Den}(T^* Q) \rightarrow \text{Den}(Q) \) is
\[
\langle \mathbb{J}_{\text{den}} (f \, d\mu), \phi \rangle = -\int_{T^* Q} f (z) \phi (q) \, d\mu(z).
\]
(143)

Combining with \( \mathbb{J}_{F(Q)} \) in Eq. (136), we have
\[
\mathbb{J}_P : *dF(Q) \times \text{Den}(T^* Q) \rightarrow \text{Den}(Q)
\]
\[
\mathbb{J}_P (*d\phi, e \, f \, d\mu) = -(\nabla^2_q \phi (q) + e \int_{T^* Q} f (z) \, d^3p) \, d^3q
\]
(144)

whose zero value results in Eq. (1). To show that this is equivalent to the Poisson equation (142) one can use the definition of the density in Eq. (10) and omit the divergence terms.
Remark 35  The Poisson part of the Poisson-Vlasov system involves as a kinematical constraint into the variation of the Hamiltonian functional \([24]\). The dynamical Vlasov part arises as a non-canonical Hamiltonian system in Eulerian variables. The constraint imposed by the Poisson equation is essentially obtained from the action of \(\mathcal{F}(Q)\) on the cotangent bundle \(T^*Q\). In this case, the zero level sets of momentum mappings are coisotropic \([2]\). In the language of Dirac formalism, the constraint is first class and hence does not affect the Poisson bracket on the reduced space. Thus, in obtaining equivalent dynamical formulations in alternative Eulerian variables we must use the same constraint.
7 Discussion and Conclusions

We outline the results of the present work and summarize them diagrammatically. We comment on implications for other kinetic theories of the way we obtain the Poisson equation. Finally, we conclude with a discussion on how the ingredient of this paper may be used to study the orbital dynamics of plasma.

The configuration space is $G = Diff_{can}(T^*Q)$. The individual motion of particles is generated by the Hamiltonian vector field $X_h \in \mathfrak{g}$. The space of sections of the tangent space $TT^*Q$ of particle phase space admits the direct sum decomposition $\mathfrak{g} \oplus (\mathfrak{g}^*)^2$. $TT^*Q$ is symplectic with the Tulczyjew’s symplectic two-form $\Omega_{TT^*Q}$. Since any Hamiltonian vector field defines a Lagrangian submanifold of $TT^*Q$, $\mathfrak{g}$ is isomorphic to the space of all Lagrangian submanifolds of $(TT^*Q, \Omega_{TT^*Q})$. Complete cotangent lift $X^c_h$ of $X_h$ is canonically Hamiltonian on $T^*T^*Q$ with the Hamiltonian function $i_{X^c_h} \Pi_{id}$. Its vertical representative $V X^c_h$ gives momentum-Vlasov equation. Both the complete lifts and their vertical representatives are Lie algebra isomorphisms into.

The configuration space can be described as the space of Lagrangian submanifolds of sections of a trivial bundle

$$G = Diff_{can}(T^*Q) \simeq Lag(\Pi_0, \Omega_-)$$

and the corresponding representation of its Lie algebra is by the space of Lagrangian submanifolds of Tulczyjew symplectic space

$$\mathfrak{g} = (\mathfrak{x}_{ham}(T^*Q); -[,] \mid ) \simeq Lag(TT^*Q, \Omega_{TT^*Q}).$$

This Lie algebra of Hamiltonian vector fields is isomorphic to the algebra of non-constant functions

$$(\mathfrak{x}_{ham}(T^*Q); -[,] \mid ) \simeq (\mathcal{F}(T^*Q)/\text{constants}, \{ , \}_{T^*Q})$$

with canonical Poisson bracket.

The generalized complete cotangent lift for symmetric contravariant tensor fields gives

$$[\mathfrak{x}, \mathfrak{y}]_{SC}^{\mathfrak{c}^s} = [\mathfrak{x}^{\mathfrak{c}^s}, \mathfrak{y}^{\mathfrak{c}^s}]_{JL},$$

which is a Lie algebra isomorphism into $\mathfrak{x} \to \mathfrak{x}^{\mathfrak{c}^s} : \mathfrak{T}Q \to \mathfrak{g}$ with $[,]_{SC}$ being the Schouten concomitant of tensor fields. The dual map gives kinetic moments in momentum variables and is a Poisson map from the Lie-Poisson bracket on $\mathfrak{g}^*$ to the Kuperschmidt-Manin bracket on $\mathfrak{T}^*Q$. For the subalgebra $\mathfrak{x}(Q) \times \mathcal{F}(Q)$ in $\mathfrak{T}Q$ this construction results in plasma-to-fluid map in momentum variables.

$\mathfrak{g}$ admits a Lie algebra isomorphism into the Lie algebra of infinitesimal quantomorphisms, or strict contact transformations

$$\mathfrak{x}_{con}^{st}(\mathcal{P}) = \{ X_H \in \mathfrak{x}_{con}(\mathcal{P}) : \mathcal{L}_{X_H} \sigma = 0 \},$$

of the quantization bundle $S^1 \to (\mathcal{P}, \sigma) \xrightarrow{pr_1} (T^*Q, \Omega_{T^*Q})$ over particle phase space. This relates the kinetic theory of plasma particles to that of particles.
moving with contact diffeomorphisms. The Lie algebra of infinitesimal quantomorphisms is included in the Lie algebra of contact vector fields on $(\mathcal{P}, \sigma)$ which, in turn, is isomorphic to the algebra of functions on $\mathcal{P}$ with Lagrange bracket

$$(x_{\text{con}}(\mathcal{P}), - [\ , \ ]_{\mathcal{L}}) \longleftrightarrow (\mathcal{F}(\mathcal{P}), \{\ , \}_c).$$

Dualizing these relations one obtains the hierarchy of kinetic theories for contact particles, quantomorphic particles which can be considered to be plasma particles and, compressible fluid. As is well-known, the last relation with fluids follows from the fact that the semi-direct product of diffeomorphisms and functions on $Q$ can be lifted to canonical diffeomorphisms on $T^*Q$.

Formulation of Vlasov dynamics with the quantomorphism group includes also particle phase space translations which is missing, at the infinitesimal level, in present treatment. Infinitesimal quantomorphisms, as central extension of the algebra of Hamiltonian vector fields, arises naturally in the dual pair construction of [17] from the requirement that the action of $x_{\text{ham}}(T^*Q)$ on certain space of embeddings be equivariant. The problem caused by constant functions on the Lie algebra side (no particle dynamics) also arises for the dual space of densities. In this case, the quotient in the space of densities

$$(\mathcal{F}(T^*Q)/\mathbb{R}) \otimes \Lambda^6(T^*Q) \simeq x_{\text{ham}}(T^*Q)$$

is identified with homotheties of particle phase space.

Hamiltonian dynamics of particles has gauge symmetries $\mathcal{F}(Q)$. The algebra of these symmetries can be realized as a subalgebra of $\mathfrak{g}$. It, thus, acts on $TT^*Q$, $T^*T^*Q$ and $\mathcal{F}(T^*Q)$ by Hamiltonian actions. Combined with the coadjoint action on the dual space $\Lambda^2(Q)$, these actions give Poisson equations as zero values of momentum maps into $\text{Den}(Q)$. Accordingly, the relations between particle motion and its symmetries with the Eulerian dynamical equations may be summarized by the diagrams

\[\begin{array}{c}
F(Q): \text{gauge symmetry} \\
\text{ver}(dF(Q)): \text{algebra} \\
\downarrow \text{isomorphism into} \\
x_{\text{ham}}(T^*Q) \approx \mathcal{F}(T^*Q) \\
\downarrow \text{dualize} \\
\Lambda^2(Q) \times \mathfrak{g}^* \\
\downarrow \text{zero-values of} \\
Poisson \text{equation in } \Pi_{id} \\
\downarrow \text{divergence} \\
Poisson \text{equation in } f \\
\end{array}\]

\[\begin{array}{c}
X_h(z) \in x_{\text{ham}}(T^*Q) \\
\downarrow \text{cotangent lift} \\
X_h^c(z, \Pi_{id}): \text{canonical motion on } \mathfrak{g}^* \\
\downarrow \text{vertical representative} \\
VX_h^c(\Pi_{id}): \text{Momentum – Vlasov equations on } \mathfrak{g}^* \\
\downarrow \text{divergence} \\
\text{Vlasov equation on } \text{Den}(T^*Q). \\
\end{array}\]

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The Poisson equations (1) and (142) constrain the regions in the product spaces $\Lambda^2(Q) \times Den(T^*Q)$ and $\Lambda^2(Q) \times \mathfrak{g}^*$ for consideration of the plasma dynamics in the Eulerian variables $(\phi_f, f)$ and $(\phi_{\Pi}, \Pi_{id})$, respectively. More generally, one can consider canonical Hamiltonian motions of an ensemble of mutually interacting identical particles. The potential energy acting on individual particles will be a function of density of particles. Gauge symmetries of canonical Hamiltonian formulation will then have an action on particle density. Reduction of Eulerian dynamics of density by gauge symmetries of individual particle motion will result in Poisson-like equation. The Vlasov equation is the collisionless limit for one-particle density function of the more general BBGKY hierarchy of equations governing the evolution of many-particle density functions [41]. It will be interesting to see implications, if any, of constraints arising from gauge symmetries of particle motions to this hierarchy.

The geometric treatment, on higher order tangent and cotangent bundles over $T^*Q$, of the momentum-Vlasov equations will act as a model for an application of Tulczyjew construction for motions on coadjoint orbits. The coadjoint orbit $O_{\Pi_{id}}^*$ through $\Pi_{id} \in \mathfrak{g}^*$ admits a symplectic structure induced from the Lie-Poisson structure. Let $O_{X_k}$ denote the adjoint orbit in $\mathfrak{g}$ through $X_k \in \mathfrak{g}$. Being cotangent spaces, $T^*O_{\Pi_{id}}^*$ and $T^*O_{X_k}$ are canonically symplectic as well. The situation is similar to the case of particle motion treated in section 2.3 once we replace the dual spaces $T^*Q$ and $TQ$ with $O_{\Pi_{id}}^*$ and $O_{X_k}$, respectively. So, we expect the tangent space $TO_{\Pi_{id}}^*$ to the coadjoint orbit to admit Tulczyjew symplectic structure. As $\mathfrak{g}$ and $\mathfrak{g}^*$ are vector subspaces of $TT^*Q$ and $T^*T^*Q$, respectively, Tulczyjew symplectic structure of $TO_{\Pi_{id}}^*$ will be the one related to $TT^*Q$ described in section 3.3. To make these ideas precise, we shall aim, in our next publication [14], to construct the Tulczyjew triple

$$T^*O_{\Pi_{id}}^* \xleftarrow{} TO_{\Pi_{id}}^* \xrightarrow{} T^*O_{X_k}$$

for orbits of canonical diffeomorphisms.

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References

[1] R. Abraham, J. E. Marsden and T. Ratiu, Manifolds, Tensor Analysis, and Applications, Springer-Verlag, 2nd edition New York, (1988).

[2] J. M. Arms, M. J. Gotay and D. C. Wilbour, Zero levels of momentum mappings for cotangent actions, Nuclear Physics B (Proc. Suppl.) 6 (1989) 384-389.

[3] V. I. Arnold, Mathematical Methods of Classical Mechanics, second ed., Graduate Texts in Mathematics 60, Springer-Verlag, 1989.

[4] A. Banyaga, Sur la structure du groupe des difféomorphismes qui preservent une forme symplectique, Comment. Math. Helvetici 53 (1978) 174-227.

[5] A. Banyaga, The Structure of Classical Diffeomorphism Groups, Kluwer, Dortrecht, 1997.

[6] S. Benenti, Hamiltonian Structures and Generating Families, Universitext v223. Springer, 2011.

[7] S. Benenti and W. M. Tulczyjew, The geometrical meaning and globalization of the Hamilton-Jacobi method, Lecture Notes in Math. 863 Springer (1980)

[8] M. Chaperon, On generating families. The Floer Memorial Volume, H. Hofer, C. H. Taubes, A. Weinstein, E. Zehnder Eds., Progress in Mathematics 133, 283–296, Birkauser 1995.

[9] M. Crampin and F. A. E. Pirani, Applicable Differential Geometry, Cambridge University Press, Cambridge, (1986).

[10] M. De León and P.R. Rodrigues, Methods of Differential Geometry in Analytical Mechanics, North-Holland Mathematics Studies, 158, North-Holland, Amsterdam, (1989).

[11] P. Dombrowski, On the geometry of tangent bundles, J. Reine Angew. Math. 210 (1962) 73-88.

[12] M. Dubois-Violette, P.W. Michor, A common generalization of the Frolicher-Nijenhuis bracket and the Schouten bracket for symmetric multivector fields, Indagationes Math. N.S. 6 (1995), 51-66.

[13] O. Esen and H. Gümral, Lifts, jets and reduced dynamics, Int. J. of Geom. Meth. in Modern Phys. Vol. 8, No. 2 (2011) 331–344.

[14] O. Esen and H. Gümral, Geometry of plasma dynamics III: Orbits of canonical diffeomorphisms, in preparation

[15] J. Gancarzewicz, Liftings of functions and vector fields to natural bundles, Proc. of the Conference (CSSR-GDR-Poland) on Diff. Geom. and its Appl., Nove Mesto na Morave, Sep. 1980, Univ. Praha, (1981) 89-102.

49
[16] F. Gay-Balmaz and T. S. Ratiu, Reduced Lagrangian and Hamiltonian formulations of Euler-Yang-Mills fluid, Journal Of Symplectic Geometry 6 (2008) 189-237.

[17] F. Gay-Balmaz, C. Tronci and C. Vizman, Geometric dynamics on the automorphism group of principal bundles: geodesic flows, dual pairs and chromomorphism groups, math.SG:1006.0650v2, 2011.

[18] F. Gay-Balmaz and C. Tronci, Vlasov moment flows and geodesics on the Jacobi group, arXiv:1105.1734v1

[19] F. Gay-Balmaz and C. Vizman, Dual pairs in fluid dynamics, arXiv:1007.1347v1

[20] J. Gibbons, Collisionless Boltzmann equations and integrable moment equations, Phys. D3:3 (1981) 503-511.

[21] J. Gibbons, D. D. Holm, C. Tronci, Geometry of Vlasov kinetic moments: A bosonic Fock space for the symmetric Schouten bracket, Phys. Lett. A, Vol:372, (2008), 4184-4196,

[22] V. Guillemin and S. Sternberg, The moment map and collective motion, Ann. of Phys. 127 (1980) 220-253.

[23] V. Guillemin and S. Sternberg, Symplectic Techniques in Physics, Cambridge University Press (Cambridge) (1984).

[24] H. Gümrül, Geometry of plasma dynamics I: Group of canonical diffeomorphisms, J. Math. Phys. 51, 083501, (2010) 23 pp.

[25] D. D. Holm and C. Tronci, Geodesic Vlasov equations and their integrable moment closures, J. Geom. Mech. 1 (2009) 181-208.

[26] R. S. Ismagilov, M. Losik and P. W. Michor [2006], A 2-cocycle on a group of symplectomorphisms, Moscow Math. J. 6 (2006) 307–315.

[27] A. N. Kaufman and R. L. Dewar, Canonical derivation of the Vlasov-Coulomb noncanonical Poisson structure, Cont. Math. AMS 28 (1984) 51-54.

[28] P. Kobak, Natural liftings of vector fields to tangent bundles of bundles of 1-forms, Math. Boch. 116 (1991) 319-326.

[29] S. Kobayashi and K. Nomizu, Foundations of differential geometry, Interscience Tract, No. 15, (1963).

[30] I. Kolar, P.W. Michor and J Slovak, Natural Operations in Differential Geometry, Springer-Verlag, Berlin Heidelberg (1993).
[31] Y. Kosmann-Schwarzbach, Vector fields and generalized vector fields on fibered manifolds, Geometry and differential geometry (Proc. Conf. Univ. Haifa, Israël, 1979), eds., R. Artzy and I. Vaisman, Lecture Notes in Math., 792, Springer-Verlag, Heidelberg, (1980) 307-355.

[32] V. V. Kozlov, The generalized Vlasov kinetic equations, Russian Math. Surveys 63 (2008) 691-726.

[33] D. R. Lebedev, Yu. I. Manin, The Benny equations of long waves II. The Lax representation and the conservation laws, Zap. Nauchn. Sem. LOMI, 96 (1980), 169–178

[34] A. J. Ledger and K. Yano, The tangent bundle of a locally symmetric space, J. London Math. Soc., 40 (1963) 487-492.

[35] P. Libermann and C. M. Marle, Symplectic Geometry and Analytic Mechanics, D. Reidel Publishing Company, Kluwer Academic Publishers Group, 1987.

[36] J. E. Marsden, A correspondence principle for momentum operators, Can. Math. Bull. 10 (1967) 247–250.

[37] J. E. Marsden and T. Ratiu, Introduction to Symmetry and Mechanics Springer, Berlin (1994).

[38] J. E. Marsden and A. Weinstein, The Hamiltonian structure of the Maxwell-Vlasov equations, Physica D 4 (1982) 394-406.

[39] J. E. Marsden, A group theoretical approach to the equations of plasma physics, Canad. Math. Bull. Vol. 25(2) (1982) 129-142.

[40] J. E. Marsden, A. Weinstein, T. Ratiu, R. Schmid, R. G. Spencer, Hamiltonian systems with symmetry, coadjoint orbits and plasma physics, Proc. IUTAM-ISIMM Symposium on Modern Developments in Analytical Mechanics, Atti della Academia della Scienze di Torino 117 (1983) 289-340.

[41] J. E. Marsden, P. J. Morrison and A. Weinstein, Hamiltonian structure of the BBGKY hierarchy, Comtemp. Math., AMS, 28 (1984) 115-124.

[42] D. McDuff, D. Salamon, Introduction to Symplectic Topology, Clarendon Press, Oxford, 1998.

[43] Peter W. Michor, Manifolds of smooth maps, Cahiers Top. Geo. Diff. 19 (1978), 47–78.

[44] P. J. Morrison, The Maxwell-Vlasov equations as a continuous Hamiltonian system, Phys. Lett. 80A (1980) 383-386.

[45] P. J. Morrison and J. M. Greene, Noncanonical Hamiltonian density formulation of hydrodynamics and magnetohydrodynamics, Phys. Rev. Lett. 45 (1980) 790-794.
[46] P. J. Morrison, Hamiltonian field description of one-dimensional Poisson-Vlasov equations, PPPL-1788, 1981.

[47] P. J. Morrison, Poisson brackets for fluids and plasmas, in Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems, (La Jolla Institute, 1981) AIP Conf. Proc. 88, edited by M. Taber and Y. Treve (AIP, New York) (1982) 13-46.

[48] A. Nannicini, Calibrated complex structures on the generalized tangent bundle of a Riemannian manifold, J. of Geom. and Phys. 56 (2006) 903–916.

[49] L. K. Norris, Generalized symplectic geometry on the frame bundle of a manifold, Proceedings of Symposia in Pure Mathematics, 54, Part 2, (1993), 435–465.

[50] P. J. Olver, Applications of Lie Groups to Differential Equations, Springer, New York, (1986).

[51] Z. Pogoda, Horizontal lifts and foliations, Suppl. ai Rendiconti del Circolo Mat. di Palermo, 21 (1989) 279-283.

[52] T. Ratiu, R. Schmid, M.R. Adams, The Lie group structure of diffeomorphism groups and invertible Fourier integral operators with applications, Infinite dimensional Groups with Applications, ed. V. Kac, Springer-Verlag, New York, 1985, pp.1-69.

[53] T. Ratiu, R. Schmid, The differentiable structure of three remarkable diffeomorphism groups, Mathematische Zeitschrift, 177 (1981) 81-100.

[54] S. Sasaki, On the differential geometry of tangent bundles of Riemannian manifolds, Tohoku Math. J., 10 (1958) 338-354.

[55] D.J. Saunders, The Geometry of Jet Bundles, London Math. Soc., Lecture Notes Series, V.142, Cambridge Univ. Press, (1989).

[56] J. A. Schouten, Über Differentialkonkomitanten zweier kontravarianter Größen, Indagationes Math. 2 (1940), 449-452.

[57] C. Scovel, A. Weinstein, Finite-dimensional Lie-Poisson approximations to Vlasov-Poisson equations. Comm. Pure Appl. Math. 47 (1994) 683-709.

[58] J. Sniatycki and W. Tulczyjew, Generating forms of Lagrangian submanifolds, Indiana Univ. Math. J. 22 (1972) 267-275.

[59] T. Swift, A note on the space of lagrangian submanifolds of a symplectic 4-manifold, Journal of Geometry and Physics 35 (2000) 183–192.

[60] S. Sternberg, Lectures on Differential Geometry, Prentice Hall, N.J. (1964).
[61] S. Tanno, An almost complex structure of the tangent bundle of an almost contact manifold, Tohoku Math. J., 17 (1965) 7-15.

[62] P. Tondeur, Structure presque kählérienne naturelle sur la fibré des vecteurs covariants d’une variété riemannienne, C. R. Acad. Sci. Paris, 254 (1962) 407-408.

[63] C. Tronci, Geometric Dynamics of Vlasov Kinetic Theory and Its Moments, PhD Thesis, Imperial College, London (2008).

[64] W. M. Tulczyjew, The Legendre transformation, Ann. Inst. Henri Poincaré Sec. A: Phys. Théor. Vol. XXVII (1977) 101-114.

[65] W. M. Tulczyjew, A symplectic formulation of relativistic particle dynamics, Acta Physica Polonica B8 (1977) 431-447.

[66] W. M. Tulczyjew, A symplectic formulation of particle dynamics, in Differential Geometric Methods in Mathematical Physics, Lect. Notes in Math. Vol 570 (1977) 457-463.

[67] W. M. Tulczyjew, A symplectic framework of linear field theories, Annali di Mathematica (1981) 177-195.

[68] W.M. Tulczyjew, The Euler-Lagrange Resolution, Part I Proceedings Of The International Colloquium Of The C.N.R.S. Aix-en-Provence, Edited By J.M. Souriau (1979).

[69] W. M. Tulczyjew, Hamiltonian systems, Lagrangian systems and the Legendre transformation, Istituto Nazionale di Alta Matematica, Symposia Mathematica 14 (1974) 247-258.

[70] W.M. Tulczyjew and E. Urbanski, A slow and careful Legendre transformation for singular Lagrangians, Acta Phys. Polon. B 30 (1999) 2909-2978.

[71] K. Yano and E. T. Davies, On the tangent bundle of Finsler and Riemannian manifolds, Rend. Circ. Mat. Palermo, 12 (1963) 211-228.

[72] K. Yano and S. Kobayashi, Prolongations of tensor fields and connections to tangent bundles I-General Theory-, J. Math. Soc. Japan 18 (1966) 194-210.

[73] K. Yano and A. J. Ledger, Linear connections on tangent bundles, J. London Math. Soc., 39 (1964), 495-500.

[74] K. Yano and E.M. Patterson, Vertical and complete lifts from a manifold to its cotangent bundle, J. Math. Soc. Japan 19, (1967) 91-113.

[75] H. Ye and P. J. Morrison, Action principles for the Vlasov equation, Phys. Fluids B 4 (1992) 771-777.

[76] I. Vaisman, Locally conformal symplectic manifolds, Int. J. Math. and Math. Sci. 8 (1985) 521-536.
[77] L. Van Hove, Sur le problème des relations entre les transformations unitaires de la mécanique quantique et les transformations canonique de la mécanique classique, Acad. Roy. Belgique Bull. Cl. Sci. 37 (1951) 610-620.

[78] C. Vizman, Some remarks on the quantomorphism group, Proc. of the Third International Workshop on Diff. Geom., Sibiu, Romania, 1997, 393-399

[79] C. Vizman, Abelian extensions via prequantization, Annals of Global Analysis and Geometry 39. 4 (2011) pp. 361-386.

[80] A. Weinstein, Lectures on symplectic manifolds. C.B.M.S. Conf. Series in Math., A.M.S. 29 (1977).

[81] A. Weinstein, Symplectic manifolds and their Lagrangian submanifolds, Advan. in Math. 6 (1971) 329-346.

[82] A. Weinstein, Lagrangian submanifolds and Hamiltonian systems, Annals of Math. 2 (1973) 377-410.

[83] A. Weinstein and P. Morrison, Comments on: The Maxwell-Vlasov equation as a continuous Hamiltonian system, Phys. Lett. A. 86 (1981) 235-236.