Thermal pure state path integral and emergent symmetry

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We investigate a thermally isolated quantum many-body system with an external control represented by a time-dependent parameter. We formulate a path integral in terms of thermal pure states and derive an effective action for trajectories in a thermodynamic state space, where the entropy appears with its conjugate variable. In particular, for quasi-static operations, the symmetry for the uniform translation of the conjugate variable emerges in the path integral. This leads to the entropy as a Noether invariant.

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Introduction. — Thermodynamics and quantum mechanics are fundamental theories in physics. The universal behavior of macroscopic objects is described by thermodynamics, while the microscopic dynamics of any system is governed ultimately by quantum mechanics. Statistical mechanics connects them in equilibrium states; however, the relation between their dynamics is not established yet although it has been studied in many contexts, such as thermodynamic processes in quantum systems [1–6] and relaxation of pure quantum states to the thermal equilibrium [7–15]. Recently, state-of-the-art experiments for these studies are realized by using ultracold atoms [16–19], nuclear magnetic resonance [20], trapped ions [21], and electronic circuits [22]. Given these backgrounds, we propose a theory connecting thermodynamical behavior to quantum mechanics.

Our strategy is to construct a thermodynamical path integral. In thermodynamics, an equilibrium state of a system is represented by a point in the thermodynamic state space. In quantum mechanics, on the other hand, the time evolution of a system is formulated in terms of a sum over all possible paths in a configuration space, weighted by the exponent of the action. In this letter, we combine these two concepts for a thermally isolated quantum many-body system under a time-dependent external control. We formulate the unitary evolution of quantum states by an integral over paths in the thermal pure state path integral, and we find an emergent symmetry.

The path integral can be obtained using the concept of thermal pure state in the standard formulation. The procedure for constructing a path integral is as follows: prepare a complete basis set at each time, insert this basis set into each step of the evolution, evaluate a one-step propagator, and take a continuum limit. The key of our derivation lies in employing microcanonical pure states as a basis set; these states represent equilibrium states [23–24] proposed along with the pioneering works on the foundation of statistical mechanics in terms of pure states [7, 25–28]. We extend these pure states in order to form an over-complete basis set at each time, and we utilize the basis set in the above procedure.

For the path integral formulation, we derive an effective action for trajectories in the thermodynamic state space, where the thermodynamic entropy appears with its conjugate variable hθ. This action connects the concepts of thermodynamics and quantum mechanics in dynamical problems. In particular, under quasi-static operations, the symmetry for θ → θ + η emerges in the path integral, leading to entropy conservation, where η is an infinitely small parameter. This provides a complementary view to the quantum adiabatic theorem [29, 30] because the operations are assumed to be slow yet so fast that transitions between different energy levels occur.

This emergent symmetry is related to the following topics. First, the Euler–Lagrange equation of θ for the effective action is expressed as dθ = dt/β, where β is the inverse temperature depending on time. This θ corresponds to a thermal time, which was introduced as a parameter of the flow determined by a statistical state [31–34]. Through the relation dθ = dt/β, the symmetry of the effective action for θ → θ + η is connected to that for t → t + η/β, which leads to entropy in classical systems [35]. Second, a similar symmetry has been phenomenologically studied for perfect fluids [36, 37] and for effective field theories [38–40]. Finally, the entropy of a stationary black hole is derived as the Noether charge for v → v + ηhβH, where v is the Killing parameter and 1/βH is the Hawking temperature [41]. Thus, our theory provides a unified perspective for studying the thermal time, perfect fluids, and black hole in terms of quantum mechanics.
Setup. — Although the theory developed in this Letter is applicable to a wide class of quantum many-body systems, we specifically consider a Hamiltonian \( \hat{H}(h) \) consisting of \( N \) spins with spin-1/2 under a uniform magnetic field \( h > 0 \) so that the argument is explicit. The eigenvalues and eigenstates satisfy \( \hat{H}(h) |n,h\rangle = E(n,h) |n,h\rangle \), where \( n = 1, 2, \ldots, 2^N \). By incorporating the magnetic moment into \( h \), we assume the dimensions of \( h \) to be energy. Then, \( h \), represents the characteristic energy scale per unit spin. We study the macroscopic behavior of the system for considering the large \( N \) limit.

We choose an energy shell \( I_E \equiv \{ E - \Delta/2 < E < E + \Delta/2 \} \) with \( \Delta = O(h/\sqrt{N}) \). The number of eigenvalues in the shell is given as \( \sum_n \chi_{I_E}(E(n,h)) \), where \( \chi_{I_E}(x) = 1 \) for \( x \in I_E \), and zero otherwise. The density of states, \( D(E,h) \), is then defined as the distribution for each \( E \), \( dE \). By setting \( \xi = \max_E D(E,h) / B \), we consider the limit \( \xi \to 0 \) before considering other limits including the large \( N \) limit. Then, through direct calculation \([40]\), we confirm that the decomposition of unity

\[
\hat{1} = \int dE D(E,h) |E,h\rangle \langle E,h| \tag{5}
\]

holds almost surely in the limit \( \xi \to 0 \), which means that \( \{|E,h\rangle\} \) is an over-complete basis set.

During the time interval \( 0 < t < \Delta t \), the state evolves following \([2]\) and eventually becomes

\[
e^{-\frac{\hbar}{\epsilon_0} \Delta t} |E_0,h_0\rangle = e^{-\frac{\hbar}{\epsilon_0} \Delta t} |E_0,h_0\rangle \Delta t, \tag{6}
\]

where \( \hat{H}_j = \hat{H}(h_j) \), and \( |E,h_j\rangle \Delta t \) represents \([3]\) with the phase \( \varphi_n(E) \) replaced by \( \varphi_n(E) - (E(n,h) - E(j)) \Delta t / h_j \). Next, at \( t = \Delta t \), we change \( h_0 \) to \( h_1 \). We then use \([4]\) for \( h_1 \) to re-express this state as

\[
\int dE_1 D(E_1,h_1) e^{-\frac{\hbar}{\epsilon_0} \Delta t} |E_1,h_1\rangle \Delta t \langle E_1,h_1|E_0,h_0\rangle \Delta t. \tag{7}
\]

By repeating this procedure \( M - 1 \) more times, the state at \( t = t_f = M \Delta t \) is expressed as

\[
|\Psi(t_f)\rangle = \prod_{j=1}^M \int dE_j D(E_j,h_j) \left| E_M,h_M \right\rangle_{t_f} \prod_{j=1}^M e^{-\frac{\hbar}{\epsilon_0} \Delta t} \langle E_j,h_j|E_{j-1},h_{j-1}\rangle \Delta t. \tag{8}
\]

We refer to this expression as a thermodynamic state path integral. It should be noted that \([3]\) holds for any sequence of \( h_j \). Indeed, for fast operations, \( |\Psi(t_f)\rangle \) does not represent an equilibrium state. The formula implies that such non-equilibrium states can be expressed by a superposition of microcanonical pure states corresponding to equilibrium states.

Overlap. — We study the overlap \( \langle E', h'| E, h \rangle \). Hereafter, we use notations \( \chi_n = \chi_{I_E}(E(n,h)) \), \( \chi'_n = \chi_{I_E'}(E(n,h')) \), \( D = D(E,h) \), \( D' = D(E',h') \), \( \langle n', h' \rangle = \langle n', \{n, h\} = |n\rangle \), \( \varphi_{n,n'} = \varphi_n(E') - \varphi_n(E) \), and \( d_{n,n'} = \)
By using (3), we express the overlap as
\[ \langle E', h'|E, h \rangle = J/\sqrt{DD'} \] with
\[ J = \sum n' \chi^{\dagger} n \chi_{n'} q_{n'n}, \] (9)

Then, from the randomness of \( \varphi_n(E) \) \([47]\), we can show \( \log |J|^2 = \log |\bar{J}|^2 + o(N) \). (10)

See (24) for the precise statement and the proof. Here, \( \log |J|^2 = O(N) \), and the term \( o(N) \) depends on the random phases. Using the randomness of \( \varphi_n(E) \) again, we obtain \( |\bar{J}|^2 = \sum_{n'} |q_{n'n}|^2 \) \([18]\). Thus, we have
\[ \langle E', h'|E, h \rangle = \frac{\sqrt{\sum_{n'} |q_{n'n}|^2}}{\sqrt{DD'}} e^{i\psi + o(N)}, \] (11)

where \( \psi(E', h'; E, h) \) is a random phase of \( J: J = |\bar{J}| e^{i\psi} \).

In order to evaluate the right-hand side of (11), we express \( \sum_{n'} |q_{n'n}|^2 \) using thermodynamic quantities. The key idea is to introduce a probability density \( P(E', h'|E, h) \), where \( P(E', h'|E, h) \Delta \) is the probability of finding the energy in \( I_{E'} \) when we instantaneously change the field from \( h \) to \( h' \) under the constraint that energy eigenstates satisfying \( E(n, h) \in I_E \) are prepared with equal probability \( \frac{1}{N} \).

Explicitly, this is written as
\[ P(E', h'|E, h) = \frac{\sum_{n'} |q_{n'n}|^2}{\sum_n \chi_n^2} e^{i\psi + o(N)}, \] (12)

Then, if \( P(E', h'|E, h) \) is determined, we can express
\[ \sum_{n'} |q_{n'n}|^2 = P(E', h'|E, h) D \Delta^2. \] (13)

Here, by noting \( P(E', h'|E, h) D = P(E, h|E', h') D' \), we have log \( P(E', h'|E, h) \) − log \( P(E, h|E', h') \) = \( \Delta S \), where \( \Delta S \equiv S(E', h') - S(E, h) \). When \( 1/\sqrt{N} \ll |\Delta h|/h \ll 1 \), the most probable transition, \( (E, h) \rightarrow (E', h') \), is described by thermodynamics. That is, the most probable value of \( \Delta S \) is given by \( Na(\Delta h)^2/2 \), where \( Na^2/2 \) is the adiabatic susceptibility evaluated at \( (E' + E)/2 \) and \((h' + h)/2 \) \([50]\).

By assuming this and expanding \( \log P(E', h'|E, h) \) up to the second order of \( \Delta S \), we can obtain \([51]\)
\[ P(E', h'|E, h) = e^{-\frac{1}{2Na(\Delta h)^2/2} + i\psi + o(N)}. \] (14)

By combining (13) and (14) with (11), we get
\[ \langle E', h'|E, h \rangle = e^{-\frac{1}{2Na(\Delta h)^2/2} + i\psi + o(N)}. \] (15)

**Effective action.**—Now, we construct the effective action. By substituting (15) into (8) and introducing a variable \( \theta \) through the formula
\[ e^{-\frac{1}{2Na(\Delta h)^2/2} + i\psi} = \int d\theta e^{-Na(\Delta h)^2\theta^2 - i\theta(\Delta S - Na(\Delta h)^2)/2 + o(N)}, \] (16)

we obtain
\[ |\Psi(t_f)| = \int DE \int D\theta \left| E_M, h_M \right|_{t_f} e^{\mathcal{J} + \mathcal{I}_{\text{eff}}}, \] (17)

with
\[ \mathcal{J} = \sum_{j=1}^M \left[ S(E_j, h_j)/2 - Na_j(\Delta h_j)^2\theta^2 + o(N) \right], \] (18)

\[ \mathcal{I}_{\text{eff}} = \sum_{j=1}^M \left[ -E_j - \Delta t - \hbar \theta_j (\Delta S_j - Na_j(\Delta h_j)^2/2) \right. \]
\[ \left. + \hbar \psi_j + o(N) \right], \] (19)

where \( \psi_j = \psi(E_j, h_j; E_{j-1}, h_{j-1}) \), and \( \int DE \int D\theta = \prod_{j=1}^M \int dE_j \int d\theta_j \). Here, we emphasize that the path-integral \([17]\) is derived from \([8]\) on the basis of the general assumptions that the asymptotic form of \( D(E, h) \) obeys \([11]\), and the most probable value of \( \Delta S \) is given by \( Na(\Delta h)^2/2 \) for a small but macroscopic step \( \Delta h \).

Let us consider the continuous limit of the effective action \([19]\) by considering \( M \gg 1 \) and \( \Delta t \ll \hbar \beta \) with \( M\Delta t = t_f \) fixed, where \( \beta \) is a characteristic value of the inverse temperature \([52]\). For simplicity, we assume that \( h_j \) increases monotonically, i.e., \( \Delta h_j/h_j = O(1/M) \).

Then, \( M \) satisfies \( 1 \ll M \ll \sqrt{N} \) so that \( 1/\sqrt{N} \ll \Delta h_j/h_j \ll 1 \), and the sum of the third term of (19) can be neglected under the continuous limit. We then obtain an integral form of (19) as
\[ \mathcal{I}_{\text{eff}} = \int_0^{t_f} dt \left[ -E(t) - \hbar \theta(t) \frac{dS(E(t), h(t))}{dt} + o(N) \right], \] (20)

where \( \hbar \psi \) is included in the \( o(N) \) term \([53]\). For a given \( h \), \( E \) has one-to-one correspondence with \( S \) through the thermodynamic relation \( S = S(E, h) \). We thus choose \( S(t) \) as an independent variable instead of \( E(t) \). In this representation, \( \mathcal{I}_{\text{eff}} \) is expressed as
\[ \mathcal{I}_{\text{eff}} = \int_0^{t_f} dt \left[ -E(S(t), h(t)) - \hbar \theta(t) \frac{dS(E(t), h(t))}{dt} + o(N) \right], \] (21)

which is the **effective action in the thermodynamic state space** of the thermally isolated quantum many-body system. Here, one may interpret \( (S, -\hbar \theta) \) as a canonical coordinate of the thermally isolated quantum many-body system. Previously, such a variable was referred to as **thermacy** \([54]\), and effective actions for perfect fluids were constructed without microscopic derivation \([36, 37]\). Indeed, our action (21) takes the same form as the previous ones for the spatially homogeneous cases; however, in these studies, the Planck constant does not appear and \( (d\theta/dt)S \) is included instead of \( \theta(dS/dt) \).

**Quasi-static operations and emergent symmetry.**—We consider slow protocols referred to as **quasi-static operations**. First, we fix \( (h_j)_{j=1} \) and \( \Delta t \), which corresponds
to a general continuous protocol \((h(t))_{t=0}^{t_f} \) in the limit \(1 \ll M \ll \sqrt{N} \) and \(\Delta t \ll h\beta\).

We attempt to construct the quasi-static operation \(h'(t)\) for this \(h(t)\) such that \(h'(t) = h(t)\) is satisfied for \(0 \leq t \leq t_f\), where \(\epsilon\) is a small dimensionless parameter that characterizes the slowness of the operation.

We define the discrete protocol as \(h'(t) = h'_{j}\) for \(t \leq t_{j+1}\), where \(0 \leq j \leq M = M/\epsilon\), and \(h'_{j} = \begin{cases} (1 - \epsilon j + [\epsilon j])h_{[\epsilon j]} + (\epsilon j - [\epsilon j])h_{[\epsilon j]+1} \text{.} \\
\end{cases}\)

Here, \([x]\) represents the largest integer less than or equal to \(x \in \mathbb{R}\). Indeed, this \(h'(t)\) satisfies \(h'(t) = h(t)\) in the continuous limit \((17)\). Note that, in order to use the formula \((17)\), the condition \(1/\sqrt{N} \ll \Delta h'_{j}/h'_{j} \ll 1\) needs to be satisfied; this leads to \(1 \ll M' \ll \sqrt{N}\). Because of this condition and \(1 \ll M \ll \sqrt{N}\), \(\epsilon\) should be small but finite so that \(M/\sqrt{N} \ll \epsilon \ll 1\).

For such quasi-static operations, we find an emergent symmetry and the associated conservation law. We first notice that the sum of the second term of \((15)\) is estimated as \(O(\sqrt{N}/M)\), and it becomes smaller as \(\epsilon\) is decreased with \(N\) and \(M\) fixed.

We thus reasonably conjecture that the second term can be neglected in the path integral \((17)\) for the quasi-static operations. Then, under the transformation \(\theta_j \rightarrow \theta_j + \eta, \quad J\) is invariant and \(I_{\text{eff}}\) is transformed to \(I_{\text{eff}} - \eta \int dS\). As a result, we have a simple expression. By differentiating this expression with respect to \(\eta\) and setting \(\eta = 0\), we obtain \((22)\):

\[
\langle \Psi(t_f') | \hat{S}(h'_{M'}) | \Psi(t_f') \rangle = \langle \Psi(0) | \hat{S}(h'_{0}) | \Psi(0) \rangle + o(N) \tag{22}
\]

for the entropy operator \(\hat{S}(h) = \log D(\hat{H}(h), h)\). This conservation law of the expectation value of the entropy operator is the Noether theorem in quantum theory.

**Thermal time.**— We discuss the concept of thermal time \(\tau\), a dimensionless quantity that parameterizes the flow generated by \( - \log \hat{\rho} \) with a statistical state \(\hat{\rho}\) \([31]–[34]\).

In particular, \(\tau\) satisfies \(dA/d\tau = [\hat{A}, - \log \hat{\rho}]/i\) for Heisenberg operators \(\hat{A}\). When \(\hat{\rho} = e^{-\hat{H}/Z} Z, \quad d\hat{A}/d\tau = h\beta \hat{A}/d\tau\) holds because \(d\hat{A}/d\tau = [\hat{A}, \hat{H}]/ih\). On the other hand, the Euler–Lagrange equation for \((21)\) provides \(dS/d\tau = 0\) and

\[
\frac{d\theta}{d\tau} = \frac{1}{h\beta} \tag{23}
\]

This equation implies that \(\theta\) corresponds to the thermal time. Expressing \((23)\) as \(dt = h\beta d\theta\), we find that the symmetry of \(I_{\text{eff}}\) for \(\theta \rightarrow \theta + \eta\) is equivalent to that for \(t \rightarrow t + \eta h\beta\) \([35]\).

**Proof of \((17)\).**— The technical highlight of our theory is proving \((10)\). First, the precise statement of \((10)\) is expressed as a probability:

\[
\lim_{N \rightarrow \infty} \text{Prob}(\log |J|^2 - \log \overline{|J|^2}/N \geq \epsilon) = 0 \tag{24}
\]

for any \(\epsilon > 0\). To show this, we prove that \(X = \log |J|^2 + o(N)\) and \(\overline{X}^2 - X^2 = o(N^2)\), where \(X \equiv \log |J|^2\), and use Chebyshev’s inequality. The strategy is to use \((25)\):

\[
X^m = \frac{\partial^m |J|^{2K}}{\partial K^m} |_{K=0} \tag{25}
\]

Let us estimate \(\overline{|J|^2}^K\). For \(K = 2\), we have \((26)\):

\[
\overline{|J|^4} = 2 \sum_{n_1} |q_{n_1}^1|^2 + 2 \sum_{n_1,n_2} q_{n_1}^1 \chi_{n_2}^1 q_{n_2}^1 \chi_{n_2}^1 \tag{26}
\]

In terms of an operator \(\hat{\psi} \equiv \sum n' \hat{\chi}_{n'} \chi_n \langle n' | n \rangle \langle n \rangle\), we can express the second term of \((26)\) by \(2 \sum_n \langle n' | \hat{\psi} \rangle \langle \hat{\psi} | n' \rangle\), which is less than \(2 \sum_n \langle n' | \hat{\psi} \rangle \langle \hat{\psi} | n' \rangle\). This is equal to \(2 |\psi\rangle \langle \psi| = 2 |\langle n_1 | q_{n_1}^1 | n \rangle|^2 = 2 |\langle J |^2|^2\). Because of \((13)\), the ratio of the second term on the right-hand side of \((26)\) to the first term is \(O(e^{-cN})\), with a positive constant \(c\). A similar argument can be developed for any \(K\), and we can show

\[
\overline{|J|^2k} = k! \left( \overline{|J|^2} \right)^k (1 + O(e^{-cN})) \tag{27}
\]

for large \(N\) \([55]\). By substituting this result into \((25)\), we obtain \(X = \log \overline{|J|^2} + o(N)\) and \(\overline{X}^2 - X^2 = o(N^2)\). We thus conclude \((24)\).

**Concluding remarks.**— Before ending this Letter, we present a few remarks. First, in order to evaluate physical quantities, we have to perform the integration of \(E_j\) and \(\theta_j\) in \((17)\). Here, considering that each term of \(J\) and \(I_{\text{eff}}\) is \(O(N)\), one may employ a saddle point method with the analytic continuation of \(J + iI_{\text{eff}}/h\) for complex variables \(E_j\) and \(\theta_j\). One can then estimate the integral \((17)\) for specific models and directly confirm the symmetry. Furthermore, it is an important future problem to study how entropy is not conserved for fast protocols through the saddle point estimation of \(J + iI_{\text{eff}}/h\).

Second, we remark on the quantum adiabatic theorem: the amplitude in each energy level remains constant (and thus \(S\) is kept constant) if the operation speed is sufficiently slow \([29, 30]\). For macroscopic systems, such a speed becomes extraordinarily slow, which is \(e^{-\Omega(N)}\), because the distances of neighboring energy levels are \(e^{-\Omega(N)}\). In our theory, by contrast, the operation speed is so fast that transitions between different energy levels occur. Nevertheless, the entropy is conserved in \((22)\) under such operations. It is a natural question how to unify the two theories.

Finally, we hope that experiments will be conducted to verify our theory. In particular, if one observes an entropic effect of the effective action, the measurement result is quite interesting. In the future, we will propose a design of experiments for this observation.
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More precisely, $\Delta$ is much smaller than $hN$ but should be large so that $I_E$ contains $e^{O(N)}$ energy levels. $\Delta = O(h\sqrt{N})$ is nothing but an example, and one may choose $\Delta = O(1)$.

It should be noted here that the standard perturbation technique cannot be employed, because $1/\sqrt{N} \ll \Delta h_j/h_j$.

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See Supplement Material for the detailed argument of the derivation of (13).

When $|E' - E| \leq B^{-1}$, $\varphi_n(E')$ correlates with $\varphi_n(E)$. However, $E(n,h') - E(n,h)$ is expected to be larger than $\Delta$ because $\Delta h_j/h_j \gg 1/\sqrt{N}$, which leads to $q_{nn} \simeq 0$ when $|E' - E| \leq B^{-1}$. From this result, we neglect the contribution originating from the correlation for $|E' - E| \leq B^{-1}$ in the following evaluation.

We have $|J|^2 = \sum_{n,n',m,m'} e^{-i\Phi} q_{nn'} q_{m'm}$, where $\Phi = \varphi_n(E') - \varphi_n(E) - \varphi_{m'}(E') + \varphi_m(E)$. Considering the average of random phases, only the terms $e^{-i\Phi}$ with $n' = m'$ and $n = m$ become finite, when $|E' - E| \geq B^{-1}$. Although the additional contribution $|\sum_n q_{nn}|^2$ appears when $|E' - E| \leq B^{-1}$, this is almost zero. See (47).

We do not assume that an equilibrium state is realized immediately after the parameter change $h \rightarrow h'$. We employ $P(E',h'|E,h)$ for the estimation of the overlap $\langle E',h'|E,h \rangle$, and not for describing physical processes.

See Supplement Material for the argument of $\Delta S$ in thermodynamics.

See Supplement Material for the detailed argument of the derivation of (13).

Note that $h\beta$ provides an estimation of the minimum time scale in thermodynamic behaviors.

We assume that $\psi_j = o(N)$, because it is difficult to imagine that the random phase grows in proportion to $N$.

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Note that $h_{j+1}' - h_j' = \epsilon(h_{j+1} - h_j)$, which implies that $h_j'$ changes slower than $h_j$ by the factor $\epsilon$, that is, $h_j'(t) = h(\epsilon t)$.

See Supplement Material for the derivation of (22).

Recall $d^n(a')/dx^m = a'/(\log a)^m$ and set $a = |J|^2$ and $x = K$.

The contributions of $n_1 = n_2$ or $n_1' = n_2'$ are ignored because $2N \gg K$.

See Supplement Material for the proof of (22).

SUPPLEMENTAL MATERIAL

Derivation of (5)

In this section, we explain the decomposition of unity given by (5) in the main text. Let us start with the precise expression of (5). By using

$$\hat{Q} \equiv 1 - \int dE D(E,h) \left| E,h \right\rangle \left\langle E,h \right|,$$  \hfill (S1)

we claim

$$\|\hat{Q}\| \to 0 \hfill (S2)$$

in the limit $\zeta \to 0$, where $\|\hat{Q}\|$ is the operator norm of $\hat{Q}$ and $\zeta \equiv \max_E D(E,h)/B$. We write the explicit definition of $\|\hat{Q}\|$ as

$$\|\hat{Q}\| = \sup_{\langle c \rangle} \left| \sum_{mn} c_m^* c_n \left\langle m,h \right| \hat{Q} \left| n,h \right\rangle \right|^{1/2}, \hfill (S3)$$

where $c_n$ are assumed to satisfy $\sum_n |c_n|^2 = 1$. Below, we give a proof of (S2).

First, for any eigenstate $|n,h\rangle$, we have

$$\hat{Q} \left| n,h \right\rangle = \left| n,h \right\rangle - \int dE \left| E,h \right\rangle D(E,h) \left\langle E,h \left| n,h \right\rangle \right|,$$

$$= \left| n,h \right\rangle - \sum_{n'} \frac{1}{\Delta} \int dE e^{i\varphi_{n'}(E) - \varphi_n(E)} \chi_{IE}(E(n',h))\chi_{IE}(E(n,h)) \left| n',h \right\rangle,$$

$$= - \sum_{n':n' \neq n} \frac{1}{\Delta} \int dE e^{i\varphi_{n'}(E) - \varphi_n(E)} \chi_{IE}(E(n',h))\chi_{IE}(E(n,h)) \left| n',h \right\rangle,$$

$$\equiv - \left| \psi_n \right\rangle . \hfill (S4)$$
Here, we define

\[ J_{n'n} = \{ E|\chi_{I_E}(E(n', h))\chi_{I_E}(E(n, h)) = 1 \} \]  \hspace{1cm}  \text{(S5)}

That is, \( J_{n'n} \) is empty or an interval in \( [E(n, h) - \Delta/2, E(n, h) + \Delta/2] \). Using the notation, \( \varphi_{n'n}(E) = \varphi_{n'}(E) - \varphi_n(E) \), we have

\[ \frac{1}{\Delta} \int dE e^{i\varphi_{n'n}(E)} \chi_{I_E}(E(n', h))\chi_{I_E}(E(n, h)) = \frac{1}{\Delta} \int_{J_{n'n}} dE e^{i\varphi_{n'n}(E)}. \]  \hspace{1cm}  \text{(S6)}

We thus can express

\[ |\psi_n'\rangle = \sum_{n' \neq n} \frac{1}{\Delta} \int_{J_{n'n}} dE e^{i\varphi_{n'n}(E)}. \]  \hspace{1cm}  \text{(S7)}

From this expression, we find that \( \langle \psi_m | \psi_n' \rangle = 0 \) when \( J_{n'n'} \cap J_{n'm} \) is empty for any \( n' \). We then define \( O_{nm} \) as \( O_{nm} = 1 \) if there exists \( n' \) such that \( J_{n'n} \cap J_{n'm} \) is not empty, \( O_{nm} = 0 \) otherwise. By using this, we have

\[ \sum_{mn} c_m^* c_n \langle m, h | \hat{Q} \hat{Q} | n, h \rangle = \sum_{mn} c_m^* c_n \langle \psi_m | \psi_n \rangle = \sum_{mn} c_m^* c_n \langle \psi_m | \psi_n \rangle O_{nm}. \]  \hspace{1cm}  \text{(S8)}

By applying the Cauchy-Schwarz inequality, we obtain

\[ \sum_{mn} c_m^* c_n \langle \psi_m | \psi_n \rangle O_{nm} \leq \sum_{mn} O_{nm} |c_m^* c_n| \langle \psi_m | \psi_m \rangle \langle \psi_n | \psi_n \rangle |^{1/2}, \]

\[ \leq \left[ \max_{n} \langle \psi_m | \psi_n \rangle \right] \sum_{mn} O_{nm} |c_m^* c_n|. \]  \hspace{1cm}  \text{(S9)}

Here, for a fixed \( n \), the number of \( m \) such that \( O_{nm} = 1 \) is \( O(D\Delta) \). We thus have

\[ \sum_{mn} |c_m^* c_n| O_{nm} = O(\max_E D(E, h)\Delta). \]  \hspace{1cm}  \text{(S10)}

From (S3), (S9), and (S10), we arrive at

\[ \|\hat{Q}\|^2 \leq \max_{n} \langle \psi_m | \psi_n \rangle O \left( \max_E D(E, h)\Delta \right), \]  \hspace{1cm}  \text{(S11)}

which means that \( \|\hat{Q}\| \to 0 \) if \( \langle \psi_m | \psi_n \rangle \to 0 \) for any \( n \) with \( N \) fixed.

From now, we show that

\[ \langle \psi_n | \psi_n \rangle = O(\zeta). \]  \hspace{1cm}  \text{(S12)}

In the argument below, we employ the following estimation: For a random sequence \( (X_l)_{l=1}^L \), where each \( X_l \) is a random variable with zero mean and finite variance, \( \sum_{l=1}^L X_l \) is a random variable with zero mean and the variance of \( O(L) \) when \( L \) is large. We thus make an estimation

\[ \sum_{l=1}^L X_l = O(\sqrt{L}) \]  \hspace{1cm}  \text{(S13)}

for typical sequences \( (X_l)_{l=1}^L \).

We start with the explicit expression:

\[ \langle \psi_n | \psi_n \rangle = \sum_{n' : n' \neq n} \frac{1}{\Delta^2} \left| \int_{J_{n'n}} dE e^{i\varphi_{n'n}(E)} \right|^2. \]  \hspace{1cm}  \text{(S14)}
We decompose the interval $J_{n'n}$ into $[E_j, E_{j+1}]$, where $E_{j+1} - E_j = B^{-1}$ and $0 \leq j \leq j_{\text{max}} = \lfloor B|J_{n'n}| \rfloor$. Here $\lfloor x \rfloor$ is the smallest integer greater than or equal to $x \in \mathbb{R}$, and $|J_{n'n}|$ represents the norm of the interval $J_{n'n}$. Then, we have

$$
\frac{1}{\Delta} \int_{J_{n'n}} dE e^{i\varphi_{n'n}(E)} = \frac{1}{B\Delta} \sum_{j=1}^{j_{\text{max}}} B^{-1} \int_{E_{j-1}}^{E_j} dE e^{i\varphi_{n'n}(E)}.
$$

(S15)

For $n' \neq n$ such that $|J_{n'n}| > 0$, we define

$$
\Psi_{n'n,j} = \frac{1}{B^{-1}} \int_{E_{j-1}}^{E_j} dE e^{i\varphi_{n'n}(E)}.
$$

(S16)

Then, for each $n'$, $(\Psi_{n'n,j})_{j=1}^{j_{\text{max}}}$ is a random sequence, because $B^{-1}$ is the energy scale at which the correlation between phases of different energies is lost. Following (S13), we therefore estimate

$$
\frac{1}{B\Delta} \sum_{j=1}^{j_{\text{max}}} \Psi_{n'n,j} = O \left( \frac{1}{\sqrt{B\Delta}} \right)
$$

(S17)

for typical sequences $\Psi_{n'n,j}$, where we have used $|J_{n'n}| = O(\Delta)$. Thus, we rewrite (S14) as

$$
\langle \psi_n | \psi_n \rangle = \sum_{n':n' \neq n|J_{n'n}| > 0} \left| \frac{1}{B\Delta} \sum_{j=1}^{j_{\text{max}}} \Psi_{n'n,j} \right|^2
$$

$$
= \sum_{n':n' \neq n|J_{n'n}| > 0} O \left( \frac{1}{B\Delta} \right)
$$

$$
= O \left( \frac{(D(E(n,h),h)\Delta)}{B\Delta} \right),
$$

(S18)

where we have used the fact that the number of $n'$ such that $|J_{n'n}| > 0$ for a given $n$ is $O(D(E(n,h),h)\Delta)$. Since $\zeta \geq D(E(n,h),h)/B$, we have (S12) in the limit $\zeta \to 0$.

In the argument above, we do not use the large $N$ limit. Therefore, this decomposition of unity can be applied to two spins, a single harmonic oscillator, or other systems that are not interpreted as macroscopic systems, where $\Delta$ can be chosen arbitrarily.

**Thermodynamics**

In this section, we review a formula in thermodynamics. Let $\Delta S = S(E', h') - S(E, h)$ and expand it in $\Delta E = E' - E$ and $\Delta h = h' - h$. We then have

$$
\Delta S = \left( \frac{\partial S}{\partial E} \right)_h \Delta E + \left( \frac{\partial S}{\partial h} \right)_E \Delta h
$$

$$
+ \frac{1}{2} \left[ \left( \frac{\partial^2 S}{\partial E^2} \right)_h \Delta E + \frac{\partial^2 S}{\partial E \partial h} \Delta E \Delta h + \left( \frac{\partial^2 S}{\partial h^2} \right)_E \Delta h \right].
$$

(S19)

up to $O((\Delta h)^2)$. The thermodynamic value of $\Delta E$ for the adiabatic process with small $\Delta h$ is determined from $\Delta S = 0$. That is,

$$
\Delta E = - \left( \frac{\partial S}{\partial E} \right)^{-1}_h \left( \frac{\partial S}{\partial h} \right)_E \Delta h = \left( \frac{\partial E}{\partial h} \right)_S \Delta h = -M \Delta h,
$$

(S20)

where we have used the fundamental relation in thermodynamics

$$
dE = TdS - Mdh.
$$

(S21)
By substituting (S20) into (S19), we obtain
\[ \Delta S = \frac{1}{2} Na(\Delta h)^2, \]  
where
\[ Na = M^2 \left( \frac{\partial^2 S}{\partial E^2} \right)_h - 2M \frac{\partial^2 S}{\partial Eh} + \left( \frac{\partial^2 S}{\partial h^2} \right)_E. \]  
(S23)

From now, we express \( a \) in terms of experimentally measurable quantities. We start with the identity
\[ \beta \left( \frac{\partial M}{\partial h} \right)_S = \beta \frac{\partial (M,S)}{\partial (h,E)} \frac{\partial (h,E)}{\partial (h,S)} = \left( \frac{\partial M}{\partial h} \right)_E \left( \frac{\partial S}{\partial E} \right)_h - \left( \frac{\partial S}{\partial h} \right)_E \left( \frac{\partial M}{\partial E} \right)_h. \]  
(S24)

Here, we notice
\[ \beta \left( \frac{\partial M}{\partial h} \right)_E = \left( \frac{\partial^2 M}{\partial h^2} \right)_E - M \left( \frac{\partial \beta}{\partial h} \right)_E = \left( \frac{\partial^2 S}{\partial h^2} \right)_E - M \frac{\partial^2 S}{\partial h \partial E}, \]  
(S25)

and
\[ \beta \left( \frac{\partial M}{\partial E} \right)_h = \left( \frac{\partial^2 M}{\partial E^2} \right)_h - M \left( \frac{\partial \beta}{\partial E} \right)_h = \frac{\partial^2 S}{\partial E \partial h} - M \left( \frac{\partial^2 S}{\partial E^2} \right)_h, \]  
(S26)

where we have used \( dS = \beta dE + \beta M dh \). We substitute (S25) and (S26) into (S24), and compare the result with (S23). We then find
\[ Na = \beta \left( \frac{\partial M}{\partial h} \right)_S, \]  
(S27)

which means that \( Na \beta^{-1} \) is the adiabatic susceptibility. By using (S21), we also obtain
\[ Na = -\beta \left( \frac{\partial^2 E}{\partial h^2} \right)_S. \]  
(S28)

We assume that Hamiltonians we study lead to the concavity of \( E(S,h) \) in \( h \), which is a standard assumption for statistical mechanical models. Then, we conclude that \( a \geq 0 \).

**Derivation of (14)**

We consider cases where
\[ \frac{h}{\sqrt{N}} \ll |h' - h| \ll h. \]  
(S29)

The typical energy change caused by this parameter change is much smaller than the energy but much larger than energy fluctuations in small subsystems. Although the standard perturbation technique breaks down due to the last property, we can conjecture a reasonable form of \( P(E',h'|E,h) \) by employing the definition (12).

We first decompose \( \log P(E',h'|E,h) \) into
\[ \log P(E',h'|E,h) = \phi_S(E',h'|E,h) + \phi_A(E',h'|E,h), \]  
(S30)
where \( \phi_S(E', h'| E, h) = \phi_S(E, h| E', h') \) and \( \phi_A(E', h'| E, h) = -\phi_A(E, h| E', h') \). From the symmetry property
\[
P(E', h'| E, h)D(E, h) = P(E, h| E', h')D(E', h'),
\] (S31)
which can be confirmed directly by the definition (12), we can determine
\[
\phi_A(E', h'| E, h) = \frac{\Delta S}{2},
\] (S32)
where \( \Delta S \equiv S(E', h') - S(E, h) \).

Next we consider \( \phi_S(E', h'| E, h) \). From (S29) and the physical interpretation of (12), we find that the probability of large \( |E' - E| \) is small. Noting that for a given \( h \) \( E \) has one-to-one correspondence with \( S \) through the thermodynamic relation \( S = S(E, h) \), and seeing (S32), we expand \( \phi_S(E', h'| E, h) \) with respect to \( \Delta S \), instead of \( \Delta E \equiv E' - E \). Therefore, we ignore contribution of \( (\Delta S)^2 \) and higher order terms and write
\[
\phi_S(E', h'| E, h) = N f_0(\Delta h; E_M, h_M) + \frac{1}{N} f_2(\Delta h; E_M, h_M)(\Delta S)^2 + o(N),
\] (S33)
for large \( N \). Here \( f_0 \) and \( f_2 \) are \( O(N^0) \) functions of \( \Delta h \equiv h' - h \), \( E_M \equiv (E + E')/2 \) and \( h_M \equiv (h + h')/2 \) which are even in \( \Delta h \). The mid-point values \( E_M \) and \( h_M \) have been introduced so that \( \phi_S(E', h'| E, h) = \phi_S(E, h| E', h') \) is respected.

Let us determine \( f_0 \) and \( f_2 \). We note that \( P(E', h'| E, h) \) is the probability that in thermally isolated macroscopic systems an equilibrium state with \( E \) becomes one with \( E' \) by the macroscopic perturbation (S29). Therefore, from the argument of thermodynamics (see (S22)), we expect that the most probable value \( E'_* \) for given \( E, h \) and \( h' \) satisfies
\[
\Delta S_* = \frac{1}{2} N a(E_M, h_M)(\Delta h)^2,
\] (S34)
where \( a(E_M, h_M) \) is a given positive function that is \( O(N^0) \) (see (S27)). Then, \( E'_* \) is characterized by
\[
\frac{\partial \log P(E', h'| E, h)}{\partial E'} \bigg|_{E'=E'_*} = 0.
\] (S35)
Through (S32) and (S33), we obtain
\[
\frac{\beta'_2}{2} \left[ 1 + \frac{4\Delta S_*}{N} f_2 \right] + \frac{N}{2} \left[ \frac{\partial f_0}{\partial E_M} \bigg|_* + \frac{(\Delta S_*)^2}{N^2} \frac{\partial f_2}{\partial E_M} \bigg|_* \right] = 0,
\] (S36)
where \( \beta' = \beta(E', h') \) and \( \big|_* \) represents the evaluation at \( E' = E'_* \). Here, suppose that \( f_0 = O((\Delta h)^{\alpha_0}) \) and \( f_2 = O((\Delta h)^{\alpha_2}) \) for small \( \Delta h/h \). Then, the first, second, third, and fourth term of (S36) have the \( \Delta h \) dependence as \( (\Delta h)^{\alpha_0} \), \( (\Delta h)^{\alpha_2} \), \( (\Delta h)^{\alpha_0} \), and \( (\Delta h)^{4+\alpha_2} \), respectively. By assuming \( \alpha_0 \geq 0 \) (otherwise (S33) would become singular when \( \Delta h \to 0 \)), we obtain \( \alpha_0 = 2 \) and \( \alpha_2 = -2 \). This leads that each bracket in (S36) vanishes, respectively:
\[
\frac{N}{4(\Delta S_*)} \frac{\partial f_0}{\partial E_M} \bigg|_* = -\frac{1}{2 a_* (\Delta h)^2},
\] (S37)
\[
\frac{(\Delta S_*)^2}{N^2} \frac{\partial f_2}{\partial E_M} \bigg|_* = -\frac{1}{8 a_* (\Delta h)^2},
\] (S38)
where (S34) has been used. We thus set
\[
f_2(\Delta h; E_M, h_M) = -\frac{1}{2 a(E_M, h_M)(\Delta h)^2},
\] (S39)
\[
f_0(\Delta h; E_M, h_M) = -\frac{1}{8 a(E_M, h_M)(\Delta h)^2}.
\] (S40)

To sum, we obtain
\[
P(E', h'| E, h) = e^{-\frac{1}{2N a(\Delta h)^2}(\Delta S - Na(\Delta h)^2/2) + o(N)},
\] (S41)
which is (14) in the main text.
Proof of (22)

We first express \( \hat{S}(h) \equiv \log D(\hat{H}(h), h) \) as

\[
\hat{S}(h) = \sum_n \log D(E(n, h), h) \langle n, h \rangle |n, h|.
\]

(S42)

We then calculate

\[
\hat{S}(h) | E, h \rangle = \sum_{n,m} \log D(E(n, h), h) |n, h\rangle \langle n, h| \frac{1}{\sqrt{D(E, h)\Delta}} e^{i\varphi_{n}\chi_{E}}(E(m, h)) |m, h\rangle,
\]

\[
= \sum_n \log D(E(n, h), h) \frac{1}{\sqrt{D(E, h)\Delta}} e^{i\varphi_{n}\chi_{E}}(E(n, h)) |n, h\rangle,
\]

\[
= \log D(E, h) \sum_n \frac{1}{\sqrt{D(E, h)\Delta}} e^{i\varphi_{n}\chi_{E}}(E(n, h)) |n, h\rangle + o(N),
\]

\[
= S(E, h) | E, h \rangle + o(N).
\]

(S43)

Now, we perform a transformation of the integral variable \( \theta_j \to \theta_j + \eta \) in (17). Since the second term in (18) can be neglected in the path integral for the quasi-static operations \( h' \), \( \mathcal{J} \) is invariant and \( \mathcal{I}_{\text{eff}} \) is transformed to \( \mathcal{I}_{\text{eff}} - \eta \int dS \). Differentiating this expression with respect to \( \eta \) and setting \( \eta = 0 \), we obtain

\[
\int D\mathcal{E} \int \mathcal{D}\theta \left| E_{M^*}, h_{M^*}^\prime \right\rangle \left\langle S_{M^*} - S_0 \right| e^{\mathcal{J} + \mathcal{I}_{\text{eff}}} = 0.
\]

(S44)

This leads to

\[
\langle \Psi(t_f^\prime) | \int D\mathcal{E} \int \mathcal{D}\theta \left| E_{M^*}, h_{M^*}^\prime \right\rangle \left\langle S_{M^*} - S_0 \right| e^{\mathcal{J} + \mathcal{I}_{\text{eff}}} = S_0.
\]

(S45)

where we have used a fact that \( S_0 \) is independent of the integration. By using (S44) and noting \( |\Psi(0)\rangle = |E_0, h_0^\prime\rangle \), we rewrite (S45) as

\[
\langle \Psi(t_f^\prime) | \hat{S}(h_{M^*}^\prime) | \Psi(t_f^\prime) \rangle = \langle \Psi(0) | \hat{S}(h_0^\prime) | \Psi(0) \rangle + o(N).
\]

(S46)

Proof of (27)

For any positive integer \( k \), we define

\[
Q_k \equiv \sum_{(n^{'})_{(n)}} q_{n^{'},n_1} q_{n^{'},n_2} \cdots q_{n^{'},n_k} q_{n_1,n_k} q_{n_2,n_1} \cdots q_{n_k,n_{k-1}},
\]

(S47)

where \( (n^{'}) = (n_1^{'}, \ldots, n_k^{'}) \) and \( (n) = (n_1, \ldots, n_k) \). Note that \( Q_1 = \sum_{n^{'},n} |q_{n^{'},n}|^2 \). Then, all \( Q_k \) for \( k \geq 2 \) satisfy

\[
Q_k \leq Q_{k-1}.
\]

(S48)

The proof is as follows. By substituting \( q_{n^{'},n} = \chi_{n^{'},n} \langle n^{'},n \rangle \) to \( Q_k \), we have

\[
Q_k = \sum_{(n^{'})_{(n)}} \chi_{n_1} \cdots \chi_{n_k} \langle n^{'},n_1 \rangle \cdots \langle n^{'},n_k \rangle \langle n_1 \rangle \cdots \langle n_k \rangle \langle n_{k-1} \rangle \cdots \langle n_{k-1} \rangle.
\]

(S49)

Here, we define \( \hat{P} \equiv \sum_n \chi_n |n\rangle \langle n| \) and \( \hat{P}' \equiv \sum_n \chi_n^\prime |n^{'},n \rangle \langle n^{'},n | \), which satisfy \( \hat{P}^2 = \hat{P} \) and \( \hat{P}'^2 = \hat{P}' \). We then express \( Q_k \) as

\[
Q_k = \text{Tr}[\hat{P}\hat{P}' \cdots \hat{P}'] = \text{Tr}[(\hat{P}\hat{P}')^k].
\]

(S50)

The trick for the estimation of \( Q_k \) is to use the expression

\[
(\hat{P}\hat{P}')^{k-1} \hat{P} = \hat{\psi}^\dagger \hat{\psi},
\]

(S51)
where \( \hat{\psi} \) is defined by

\[
\hat{\psi} \equiv \begin{cases} 
(\hat{P}' \hat{P})^{k/2} & \text{for even } k, \\
(\hat{P} \hat{P}')^{(k-1)/2} \hat{P} & \text{for odd } k. 
\end{cases}
\] (S52)

We then obtain

\[
Q_k = \sum_{n'} \chi_{n'} \langle n' | (\hat{P} \hat{P}')^{k-1} \hat{P} | n' \rangle
= \sum_{n'} \chi_{n'} \langle n' | \hat{\psi}^\dagger \hat{\psi} | n' \rangle
\leq \sum_{n'} \langle n' | \hat{\psi}^\dagger \hat{\psi} | n' \rangle
= \text{Tr}[\hat{\psi}^\dagger \hat{\psi}]
= \text{Tr}[(\hat{P} \hat{P}')^{k-1} \hat{P}]
= \text{Tr}[(\hat{P} \hat{P}')^{k-1}]
= Q_{k-1}
\] (S53)

This is the result given in (S48). From (S48), we also have

\[
Q_k \leq Q_1 = \sum_{n'n'} |q_{n'n'}|^2.
\] (S54)

for any \( k \geq 2 \).

Now, we consider

\[
|J|^{2K} = \sum_{\sigma \in \mathcal{S}} \sum_{\sigma' \in \mathcal{S}'} \sum_{(n'),(n)} q_{n_1'n_1} q_{n_2'n_2} \cdots q_{n_K'n_K} q_{n_1'n_1}^* q_{n_2'n_2}^* \cdots q_{n_K'n_K}^* q_{\sigma'(n_1')} q_{\sigma'(n_2')} \cdots q_{\sigma'(n_K')},
\] (S55)

where the contribution with \( n_i = n_j \) and \( n_i' = n_j' \) for \( i \neq j \) is ignored because \( 2^N \gg K \), and \( \mathcal{S} \) and \( \mathcal{S}' \) represent the symmetric group acting on \( (n_1, \ldots, n_K) \) and \( (n_1', \ldots, n_K') \), respectively. We express (S55) as

\[
|J|^{2K} = K! \sum_{\sigma \in \mathcal{S}} R(\sigma)
\] (S56)

with

\[
R(\sigma) = \sum_{(n'),(n)} q_{n_1'n_1} q_{n_2'n_2} \cdots q_{n_K'n_K} q_{n_1'n_1}^* q_{n_2'n_2}^* \cdots q_{n_K'n_K}^* q_{\sigma(n_1')} q_{\sigma(n_2')} \cdots q_{\sigma(n_K')}.
\] (S57)

It is obvious that

\[
R(1) = \left( \sum_{n'n} |q_{n'n}|^2 \right)^K,
\] (S58)

and we show that other contributions \( R(\sigma) \) with \( \sigma \neq 1 \) are substantially small for large \( N \). As an example, let us take \( \sigma \) such that \( \sigma(n_1) = n_2, \sigma(n_2) = n_1 \) and \( \sigma(n_i) = n_i \) for \( i \geq 3 \). In this case, for \( K \geq 2 \), we obtain

\[
R(\sigma) = Q_2 \left( \sum_{n'n} |q_{n'n}|^2 \right)^{K-2} \leq \left( \sum_{n'n} |q_{n'n}|^2 \right)^{K-1},
\] (S59)
where we have used the inequality (S54). We then have

$$
\frac{R(\sigma)}{R(1)} \leq \left( \sum_{n'n} |q_{n'n}|^2 \right)^{-1}
= \frac{1}{|J|^2}
= \frac{1}{P(E', h'|E, h)D \Delta^2}
\leq e^{-cN},
$$

(S61)

where \( c \) is a positive constant. Generally, each \( \sigma \in \mathfrak{S} \) can be written as a product of disjoint cycles in an essentially unique manner, \( \sigma_1 \cdots \sigma_K \), where the order of \( \sigma_k \) is denoted by \( d_k \). We then have

$$
R(\sigma) = \prod_{k=1}^{K} Q_{d_k}
\leq \left( \sum_{n'n} |q_{n'n}|^2 \right)^K.
$$

(S63)

where we have used the inequality (S54). We thus obtain

$$
\frac{R(\sigma)}{R(1)} \leq e^{-c(K-\tilde{K})N}.
$$

(S64)

Noting that \( \tilde{K} \leq K - 1 \) for \( \sigma \neq 1 \), we have arrived at

$$
|J|^{2\tilde{K}} = K! \left( \sum_{n'n} |q_{n'n}|^2 \right)^K (1 + O(e^{-cN})).
$$

(S65)