GLOBAL REGULARITY FOR THE CRITICAL 2-D DISSIPATIVE QUASI-GEOSTROPHIC EQUATION WITH FORCE

SARI GHANEM

Abstract. This is a remark that by using an adaptation of the technique invented by A. Kiselev, F. Nazarov, and A. Voldberg, with a modified scaling argument, we can prove global regularity of the critical 2-D dissipative quasi-geostrophic equation with smooth periodic force, under the assumption that the initial data is smooth and periodic, and the force is $\alpha$-Hölder continuous in space, $\alpha > 0$.

1. Introduction

The problem of breakdown of solutions of the critical quasi-geostrophic equation with arbitrary smooth initial data was suggested by S. Klainerman in [Kl] as one of the most challenging problems in partial differential equations of the twenty-first century. In an elegant paper, [KNV], A. Kiselev, F. Nazarov and A. Voldberg proved global well-posedness of the critical 2-dimensional dissipative quasi-geostrophic equation with smooth periodic initial data. This note is a remark that by using an adaptation of the technique introduced by Kiselev, Nazarov and Voldberg in [KNV], with a modified scaling argument, we can immediately prove global regularity of the critical 2-dimensional dissipative quasi-geostrophic equation with smooth periodic force, under the assumption that the initial data is smooth and periodic, and the force $\alpha$-Hölder continuous in space, $\alpha > 0$.

1.1. The statement.

We consider the critical surface quasi-geostrophic equation with force, which we will write as the following:

$$\partial_t \theta(x, t) = u \cdot \nabla \theta(x, t) - (-\Delta)^{\frac{1}{2}} \theta(x, t) + f(x, t)$$  \hspace{1cm} (1)

where $x \in \mathbb{R}^2$, $u(x, t) = (-R_2 \theta, R_1 \theta)$, where $R_1$ and $R_2$ are the usual Riesz transforms in $\mathbb{R}^2$, $\theta(x, t) : \mathbb{R}^2 \to \mathbb{R}$ is a scalar function, and $f(x, t) : \mathbb{R}^2 \to \mathbb{R}$ is the force function.

We assume $f$ smooth and periodic on $\mathbb{R}^2$ (in space), and bounded in space and time, i.e.

$$\|f(x, t)\|_{L^\infty} < \infty$$  \hspace{1cm} (2)
We also assume \( f \) to be \( \alpha \)-Hölder continuous with \( \alpha > 0 \), i.e. there exist constants \( C_1 \geq 0 \) and \( \alpha > 0 \) which do not depend on \( t \), such that for all \( x, y \) in \( \mathbb{R}^2 \),

\[
|f(x, t) - f(y, t)| \leq C_1 |x - y|^\alpha
\]

The goal of section (2) is to prove the following theorem,

**Theorem 1.2.** Local solutions of the critical surface dissipative quasi-geostrophic equation with smooth periodic force, (1), with smooth periodic initial data, can be extended to global solutions in time under assumptions (2) and (3) on the force.

**Remark 1.3.** One can prove existence and uniqueness of local solutions of equation (1) under the assumptions of theorem (1.2), by adapting the argument of J. Wu in [Wu]. Thus, theorem (1.2) gives global well-posedness for the 2-dimensional critical quasi-geostrophic equation with force on the torus satisfying (2) and (3).

### 1.4. Strategy of the proof.

We will prove theorem (1.2) by proving that for \( \theta \) a solution of (1) with smooth periodic initial data \( \theta_0 \), \( ||\nabla \theta||_{L^\infty} \) is bounded by a constant depending on \( ||f||_{L^\infty} \), on \( C_1 \) and \( \alpha \) as defined in (3), on \( ||\nabla \theta_0||_{L^\infty} \), and on the period of \( \theta_0 \) and \( f \). Once this is achieved, one can show that local solutions can be extended to global solutions in time by adapting the argument shown by A. Kiselev in [K]. To prove such an estimate on \( ||\nabla \theta||_{L^\infty} \) we will use the method of modulus of continuity of A. Kiselev, F. Nazarov, and A. Volberg in [KNV], with a modified scaling argument.

**Definition 1.5.** We say that a function \( \omega \) is a modulus of continuity if \( \omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is increasing, continuous, concave, and \( \omega(0) = 0 \).

**Definition 1.6.** We say that \( \theta \) has modulus of continuity \( \omega \), or \( \omega \) is preserved by \( \theta \), at time \( t \), if for all \( x, y \in \mathbb{R}^2 \),

\[
|\theta(x, t) - \theta(y, t)| \leq \omega(|x - y|)
\]

Observe now that if at time \( t \), \( \theta \) has \( \omega \) as modulus of continuity, then

\[
\frac{|\theta(x + h, t) - \theta(x, t)|}{|h|} \leq \frac{\omega(|h|)}{|h|}
\]

By taking the limit when \( |h| \rightarrow 0 \) in the above inequality, we obtain for all \( x \in \mathbb{R}^2 \)

\[
|\nabla \theta(x, t)| \leq \omega'(0)
\]

Therefore, by taking the supremum in space in the above inequality, we get that

\[
||\nabla \theta(x, t)||_{L^\infty} \leq \omega'(0)
\]

Consequently, if we manage to find one special function \( \omega \), modulus of continuity, such that given \( A \) large enough depending on \( ||\nabla \theta_0||_{L^\infty} \), where \( \theta_0 \) is the initial data, on \( ||f||_{L^\infty} \), and on the period of \( \theta_0 \) and of \( f \), such that

\[
\omega_A(\zeta) = \omega(A\zeta)
\]
is a modulus of continuity for $\theta_0$, and $\omega_A$ remains preserved for all time $t$ by $\theta$, a smooth solution of (1) with $\theta_0$ as initial data, in the sense of (4), then
\[ ||\nabla \theta(x, t)||_{L^\infty} \leq A \omega'(0) \]  
(7)

Let’s look for such $\omega$:

If
\[ \omega'(0) = 1 \]  
(8)
and
\[ \lim_{\zeta \to \infty} \omega(\zeta) = \infty \]  
(9)
then we notice that since any smooth periodic function $\theta_0$ is bounded, we can choose $A > 0$ large enough such that $\theta_0$ has $\omega_A(\zeta) = \omega(A\zeta)$ as modulus of continuity, with $A$ depending on $||\nabla \theta_0||_{L^\infty}$ and on the period of $\theta_0$.

If we also impose on $\omega$ to have
\[ \lim_{\zeta \to 0^+} \omega''(\zeta) = -\infty \]
then, since $\theta$ is smooth because $\theta_0$ and $f$ are smooth, the only way for $\omega$ to stop being a modulus of continuity for $\theta$ after some time is that there exists a time $T$, and $x, y \in \mathbb{R}^2$, $x \neq y$, such that
\[ \theta(x, T) - \theta(y, T) = \omega_A(|x - y|) \]  
(10)
and
\[ \partial_t(\theta(x, T) - \theta(y, T)) \geq 0 \]  
(11)
Hence, we are going to look for $\omega$ verifying (8) and (9) such that
\[ \omega''(0) = -\infty \]  
(12)
and such that at $x, y \in \mathbb{R}^2$ where (10) is verified, we have
\[ \partial_t(\theta(x, T) - \theta(y, T)) < 0 \]  
(13)
Because of (11), inequality (13) will prove that $\omega_A$ is preserved by $\theta$ for all time $t$, and consequently we will have our estimate.

Acknowledgments. The author would like to thank his advisors, Frédéric Hélein and Vincent Moncrief, for their continuous advice and support during his PhD studies. The author would also like to thank Alexander Kiselev for suggesting the problem as an exercise, yet the remark presented in this note that consists in modifying the scaling argument in his original technique with F. Nazarov and A. Voldberg, to get the hereby stated result, was not apparent to him, hence the author’s interest in posting it. This work was supported by a full tuition fellowship from Université Paris VII - Institut de Mathématiques de Jussieu.
2. Estimate for $||\nabla \theta(x,t)||_{L^\infty}$

Let $\omega$ a modulus of continuity, in the sense of (1.5), such that,

\begin{align*}
\lim_{\zeta \to \infty} \omega(\zeta) &= \infty \\
\omega'(0) &= 1 \\
\omega''(0) &= -\infty
\end{align*}  
(14)  
(15)  
(16)

Given an arbitrary smooth periodic initial data $\theta_0$, since it is a $C^1$ function on a compact, we can choose $A$ large enough depending on $||\nabla \theta_0||_{L^\infty}$ and the period of $\theta_0$, such that $\theta_0$ has $\omega_A$ as modulus of continuity, i.e. for all $x, y \in \mathbb{R}^2$, we have

$$|\theta_0(x) - \theta_0(y)| \leq \omega_A(|x - y|)$$
(17)

This gives for all $x, y \in \mathbb{R}^2$

$$|\theta_0(x_A) - \theta_0(y_A)| \leq \omega(|x - y|)$$
(18)

**Definition 2.1.** Let $A$ such that we have (17), we define

$$\hat{\theta}(x, t) = \theta(x_A, t_A)$$
(19)

If $\theta(x, t)$ solves (1), then $\hat{\theta}(x, t)$ satisfies

$$\partial_t \hat{\theta}(x, t) = u \nabla \hat{\theta}(x, t) - (-\Delta)^{\frac{1}{2}} \hat{\theta}(x, t) + \frac{1}{A} f(x_A, t_A)$$
(20)

We would want to find $\omega$ preserved by $\hat{\theta}$ for all time $t$. For this, we will proceed as explained in (1.4):

Let $x, y \in \mathbb{R}^2$, $x \neq y$, be such that $\hat{\theta}$ has $\omega$ as modulus of continuity for all time $t \leq T$, and

$$\hat{\theta}(x, T) - \hat{\theta}(y, T) = \omega(|x - y|)$$
(21)

Let

$$\zeta = |x - y|$$
(22)

As explained in (1.4), we want to find $\omega$ such that for $x, y \in \mathbb{R}^2$ as in (21), we have

$$\partial_t (\hat{\theta}(x, T) - \hat{\theta}(y, T)) < 0$$
(23)

This will give that $\omega$ is preserved by $\hat{\theta}$ for all time $t$, and consequently $\omega_A$ is preserved by $\theta$ for all time, and therefore we will have our desired estimate (7).
QUASI-GEOSTROPHIC EQUATION WITH FORCE

Computing,
\[ \partial_t (\hat{\theta}(x, T) - \hat{\theta}(y, T)) = \hat{u} \cdot \nabla \hat{\theta}(x, T) - \frac{1}{A} f\left(\frac{x}{A}, T\right) - \frac{1}{A} f\left(\frac{y}{A}, T\right) \]
\[ + \frac{1}{A} f\left(\frac{x}{A}, T\right) - \frac{1}{A} f\left(\frac{y}{A}, T\right) \]  
(24)

**Lemma 2.2.** If the function \( \hat{\theta} \) has modulus of continuity \( \omega \), then \( \hat{u} = (-R_2 \hat{\theta}, R_1 \hat{\theta}) \) has modulus of continuity \( \Omega(\zeta) \), where
\[
\Omega(\zeta) = B \left( \int_0^\zeta \frac{\omega(\eta)}{\eta} d\eta + \zeta \int_0^{\infty} \frac{\omega(\eta)}{\eta^2} d\eta \right)
\]
with some universal constant \( B > 0 \).

**Proof**
A sketch of the proof of (2.2) is in the Appendix of [KNV].

**Lemma 2.3.** For \( x, y \) and \( T \) as in (21), and \( \zeta \) defined as in (22), we have
\[
\hat{u} \cdot \nabla \hat{\theta}(x, T) - \hat{u} \cdot \nabla \hat{\theta}(y, T) \leq \Omega(\zeta) \omega'(\zeta) \geq 0
\]
(26)
\[
-[(\Delta) \frac{\hat{\theta}}{\hat{\theta}}(x, T) - (\Delta) \frac{\hat{\theta}}{\hat{\theta}}(y, T)] \leq \frac{1}{\pi} \int_0^\zeta \frac{\omega(\eta)}{\eta^2} d\eta + \frac{1}{\pi} \int_\zeta^{\infty} \frac{\omega(\eta)}{\eta^2} d\eta
\]
\[ + \frac{1}{A} \left[ f\left(\frac{x}{A}, T\right) - f\left(\frac{y}{A}, T\right) \right] \leq C_1 \frac{1}{A^{1+\alpha}} \zeta^\alpha
\]
(27)
for some \( \alpha > 0 \) and \( C_1 \geq 0 \) as in (3).

**Proof**
To prove (26), we compute
\[
\hat{u} \cdot \nabla \hat{\theta}(x, T) = \frac{d}{dh} \left|_{h=0} \hat{\theta}(x + hu(x), T) - \hat{\theta}(y + hu(y), T) \right|
\]
(29)
We have
\[
\hat{\theta}(x + hu(x), T) - \hat{\theta}(y + hu(y), T) \leq \omega(x + hu(x) - y - hu(y))
\]
(because \( \omega \) is preserved by \( \hat{\theta} \) at time \( T \))
\[
\leq \omega(\zeta + h|\hat{u}(x) - \hat{u}(y)|)
\]
(30)
and
\[ |\hat{u}(x) - \hat{u}(y)| \leq \Omega(\zeta) \]
(by (2.2))
Since $\omega$ is increasing, (30) and (31) give
\[ \hat{\theta}(x + hu(x)) - \hat{\theta}(y + hu(y)) \leq \omega(\zeta + h\Omega(\zeta)) \] (32)
(26) comes out after differentiation by injecting (32) in (29).

(27) is proved in [KNV].

(28) comes out from assumption (3), that $f$ is $\alpha$-Hölder continuous, with $\alpha > 0$:
\[ \frac{1}{A}[f(\frac{x}{A}, T) - f(\frac{y}{A}, T)] \leq \frac{C_1}{A} \frac{|x - y|^\alpha}{A^\alpha} = \frac{C_1}{A^{1+\alpha}} \zeta^\alpha \]

2.4. Construction of $\omega$.

Let $\delta > 0$ small enough to be chosen later, $\beta = \min\{\frac{1}{2}, \alpha\}$, where $\alpha > 0$ is defined as in (3), and $0 < \gamma \leq \frac{\delta}{2}$.

For $0 \leq \zeta \leq \delta$, let
\[ \omega(\zeta) = \zeta - \zeta^{1+\beta} \] (33)

For $\zeta > \delta$, let
\[ \omega'(\zeta) = \frac{\gamma}{\zeta(4 + \log(\frac{\zeta}{\delta}))} \] (34)

Remark 2.5. For $\delta$ small enough, and $0 < \gamma \leq \frac{\delta}{2}$, $\omega$ is a modulus of continuity verifying (14), (15), and (16).

Lemma 2.6. Let $x, y \in R^2$ be as in (21) with $\omega$ as defined in (33) and (34), and let $\zeta = |x - y| > 0$. If we choose $\delta$ and $\gamma$ small enough, with $0 < \gamma \leq \frac{\delta}{2}$, then for all $0 < \zeta \leq A.D$, where $D$ is the period of $\theta$, we have (23), i.e.
\[ \partial_t(\hat{\theta}(x, T) - \hat{\theta}(y, T)) < 0 \]

Proof

2.6.1. Checking inequality (23) for $0 < \zeta \leq \delta$.

Injecting (25) in (26), we get
\[ \hat{\mu} \nabla \hat{\theta}(x, T) - \hat{\mu} \nabla \hat{\theta}(y, T) \leq B(\int_0^\zeta \frac{\omega(\eta)}{\eta} d\eta + \zeta \int_\zeta^\infty \frac{\omega(\eta)}{\eta^2} d\eta) \omega'(\zeta) \] (35)

From (33), we have
\[ \omega(\eta) \leq \eta \]
Thus,
\[ \int_0^\zeta \frac{\omega(\eta)}{\eta} d\eta \leq \zeta \]  
(36)

On the other hand,
\[
\int_\zeta^\infty \frac{\omega(\eta)}{\eta^2} d\eta = \int_\zeta^\delta \frac{\omega(\eta)}{\eta^2} d\eta + \int_\delta^\infty \frac{\omega(\eta)}{\eta^2} d\eta \\
= \int_\zeta^\delta \left( \frac{1}{\eta} - \eta^{-1+\beta} \right) d\eta + \left[ \frac{\omega(\delta)}{\delta} \right]_\infty^\delta - \int_\delta^\infty \frac{\omega'(\eta)}{\eta} d\eta \\
(\text{by integrating by parts in the second integral}) \\
= \ln(\frac{\delta}{\zeta}) + \left[ \frac{\omega(\delta)}{\delta} \right] + \int_\delta^\infty \frac{1}{\eta^2(4 + \ln(\frac{\eta}{\delta}))} d\eta \\
\leq \ln(\frac{\delta}{\zeta}) + 1 + \gamma \int_\delta^\infty \frac{1}{\eta^2} d\eta \\
\leq \ln(\frac{\delta}{\zeta}) + 1 + \frac{\gamma}{\delta}
\]

If we choose \( \gamma \leq \delta \), we obtain
\[ \int_\zeta^\infty \frac{\omega(\eta)}{\eta^2} d\eta \leq 2 + \ln(\frac{\delta}{\zeta}) \]  
(37)

We also have from (33),
\[ \omega'(\zeta) \leq 1 \]  
(38)

Injecting (36), (37), and (38) in (35), we get
\[ \hat{u}.\nabla \hat{\theta}(x, T) - \hat{u}.\nabla \hat{\theta}(y, T) \leq B[\zeta + \zeta(2 + \ln(\frac{\delta}{\zeta}))].1 \]
\[ \leq B[3\zeta + \zeta \ln(\frac{\delta}{\zeta})] \]  
(39)

On the other hand, (27) has two terms and they are both negative due to the concavity of \( \omega \). Indeed, the first term in (27) is
\[ \frac{1}{\pi} \int_0^\delta \frac{\omega(\zeta + 2\eta) + \omega(\zeta - 2\eta) - 2\omega(\zeta)}{\eta^2} d\eta \]

If we choose \( \delta \) small enough, then \( \omega \) is concave. In addition, \( \omega''(\zeta) > 0 \) due to the choice of \( \beta \). Hence, using the Taylor series, we can estimate
\[ \omega(\zeta + 2\eta) \leq \omega(\zeta) + 2\omega'(\zeta)\eta \]
\[ \omega(\zeta - 2\eta) \leq \omega(\zeta) - 2\omega'(\zeta)\eta + 2\omega''(\zeta)\eta^2 \]
Therefore,
\[
\frac{1}{\pi} \int_0^{\tilde{\zeta}} \frac{\omega(\zeta + 2\eta) + \omega(\zeta - 2\eta) - 2\omega(\zeta)}{\eta^2} d\eta \leq \frac{1}{\pi} \int_0^{\tilde{\zeta}} \frac{2\omega(\zeta) + 2\omega''(\zeta)\eta^2 - 2\omega(\zeta)}{\eta^2} d\eta
\]
\[
\leq \frac{1}{\pi} \int_0^{\tilde{\zeta}} 2\omega''(\zeta) d\eta
\]
\[
\leq \frac{\zeta}{\pi} \omega''(\zeta)
\]
\[
\leq -\frac{\beta(1 + \beta)}{\pi} \zeta^\beta
\]  
(40)

Whereas to the second term in (27),
\[
\frac{1}{\pi} \int_{\tilde{\zeta}}^\infty \frac{\omega(2\eta + \zeta) - \omega(2\eta - \zeta) - 2\omega(\zeta)}{\eta^2} d\eta
\]

since \(\omega\) is concave, we have
\[
\omega(2\eta + \zeta) = \omega(2\eta - \zeta + \zeta + \zeta)
\]
\[
\leq \omega(2\eta - \zeta) + \omega(\zeta + \zeta)
\]
\[
\leq \omega(2\eta - \zeta) + 2\omega(\zeta)
\]

Hence,
\[
\frac{1}{\pi} \int_{\tilde{\zeta}}^\infty \frac{\omega(2\eta + \zeta) - \omega(2\eta - \zeta) - 2\omega(\zeta)}{\eta^2} d\eta \leq 0
\]  
(41)

Injecting (40) and (41) in (27), we obtain
\[
-[(\Delta)^{\frac{1}{2}} \dot{\theta}(x, T) - (\Delta)^{\frac{1}{2}} \dot{\theta}(y, T)] \leq -\frac{\beta(1 + \beta)}{\pi} \zeta^\beta
\]  
(42)

Finally, injecting (28), (39), and (42) in (24), we obtain for \(0 < \zeta \leq \delta\)
\[
\partial_t(\dot{\theta}(x, T) - \dot{\theta}(y, T)) \leq B[3\zeta + \zeta \ln(\delta)] - \frac{\beta(1 + \beta)}{\pi} \zeta^\beta + \frac{C_1}{A^{1+\alpha}} \zeta^\alpha
\]
\[
\leq 3B\zeta + B\zeta \ln(\delta) - B\zeta \ln(\zeta) - \frac{\beta(1 + \beta)}{\pi} \zeta^\beta + \frac{C_1}{A^{1+\alpha}} \zeta^\alpha
\]

Choosing \(\delta \leq 1\) and \(A \geq 1\), we have
\[
\zeta^\alpha \leq \zeta^\beta
\]
\[
\frac{1}{A^{1+\beta}} \leq \frac{1}{A^{1+\alpha}}
\]

Consequently,
\[
\partial_t(\dot{\theta}(x, T) - \dot{\theta}(y, T)) \leq B(3\zeta + \zeta \ln(\delta)) - B\zeta \ln(\zeta) - \zeta^\beta \left(\frac{\beta(1 + \beta)}{\pi} + \frac{C_1}{A^{1+\beta}}\right)
\]  
(43)
Choosing $\delta$ small enough, and $A$ large enough depending on $C_1$ and on $\beta$, and therefore on $f$, then (43) would lead to

$$\partial_t(\hat{\theta}(x, T) - \hat{\theta}(y, T)) < 0$$

2.6.2. Checking inequality (23) for $\delta \leq \zeta \leq A.D$, where $D$ is the period of $\theta$.

From (24), (25), (26) and (27), we have

$$\partial_t(\hat{\theta}(x, T) - \hat{\theta}(y, T)) \leq \omega'(\zeta)B\left(\int_0^\zeta \frac{\omega(\eta)}{\eta} d\eta + \zeta \int_\zeta^\infty \frac{\omega(\eta)}{\eta^2} d\eta\right)$$

$$+ \frac{1}{\pi} \int_0^{\zeta} \frac{\omega(\zeta + 2\eta) + \omega(\zeta - 2\eta) - 2\omega(\zeta)}{\eta^2} d\eta$$

$$+ \frac{1}{\pi} \int_\zeta^{\infty} \frac{\omega(2\eta + \zeta) - \omega(2\eta - \zeta) - 2\omega(\zeta)}{\eta^2} d\eta$$

$$+ \frac{1}{A}(f\left(\frac{x}{A}, \frac{T}{A}\right) - f\left(\frac{y}{A}, \frac{T}{A}\right))$$

(44)

We have

$$\frac{1}{A}(f\left(\frac{x}{A}, \frac{T}{A}\right) - f\left(\frac{y}{A}, \frac{T}{A}\right)) \leq \frac{2}{A}\|f\|_{L^\infty}$$

(45)

(from assumption (2) on the force).

Whereas to the term

$$\frac{1}{\pi} \int_0^{\zeta} \frac{\omega(\zeta + 2\eta) + \omega(\zeta - 2\eta) - 2\omega(\zeta)}{\eta^2} d\eta$$

since $\omega$ is concave, using the Taylor series, we can estimate

$$\omega(\zeta - 2\eta) \leq \omega(\zeta) - 2\eta \omega'\left(\zeta\right)$$

$$\omega(\zeta + 2\eta) \leq \omega(\zeta) + 2\eta \omega'\left(\zeta\right)$$

Therefore,

$$\frac{1}{\pi} \int_0^{\zeta} \frac{\omega(\zeta + 2\eta) + \omega(\zeta - 2\eta) - 2\omega(\zeta)}{\eta^2} d\eta \leq 0$$

(46)

Now, we want to evaluate the term

$$\frac{1}{\pi} \int_\zeta^{\infty} \frac{\omega(2\eta + \zeta) - \omega(2\eta - \zeta) - 2\omega(\zeta)}{\eta^2} d\eta$$

We have

$$\omega(2\eta + \zeta) = \omega(2\eta - \zeta + 2\zeta)$$

$$\leq \omega(2\eta - \zeta) + \omega(2\zeta)$$

(by concavity).
Hence,
\[
\frac{1}{\pi} \int_{\frac{\pi}{2}}^{\infty} \frac{\omega(2\eta + \zeta) - \omega(2\eta - \zeta) - 2\omega(\zeta)}{\eta^2} d\eta \leq \frac{1}{\pi} \int_{\frac{\pi}{2}}^{\infty} \frac{\omega(2\zeta) - 2\omega(\zeta)}{\eta^2} d\eta \tag{47}
\]

Since \(\omega\) is concave, we also have
\[
\omega(2\zeta) \leq \omega(\zeta) + \zeta \omega' (\zeta) \\
\leq \omega(\zeta) + \frac{\gamma}{4 + \ln\left(\frac{\delta}{\eta}\right)} \\
\leq \omega(\zeta) + \frac{\gamma}{4} \tag{48}
\]

If we choose \(\gamma < \frac{\delta}{2}\), (48) will lead to
\[
\omega(2\zeta) \leq \omega(\zeta) + \frac{\delta}{8} \tag{49}
\]

If we choose \(\delta\) small enough, we will have
\[
\delta^{1+\beta} \leq \frac{\delta}{2} \tag{50}
\]

then, from (33) and (34) we will get
\[
\frac{\delta}{2} \leq \omega(\delta) \leq \omega(\zeta) \tag{51}
\]

Injecting (51) in (49), we obtain
\[
\omega(2\zeta) \leq \frac{3}{2} \omega(\zeta)
\]

Consequently,
\[
\frac{1}{\pi} \int_{\frac{\pi}{2}}^{\infty} \frac{\omega(2\eta + \zeta) - \omega(2\eta - \zeta) - 2\omega(\zeta)}{\eta^2} d\eta \leq -\frac{1}{2\pi} \int_{\frac{\pi}{2}}^{\infty} \frac{\omega(\zeta)}{\eta^2} d\eta = -\frac{1}{\pi} \frac{\omega(\zeta)}{\zeta} \tag{52}
\]

(from (47)).

Now, we would want to evaluate the term
\[
\left( \int_{0}^{\zeta} \frac{\omega(\eta)}{\eta} d\eta + \zeta \int_{\zeta}^{\infty} \frac{\omega(\eta)}{\eta^2} d\eta \right)
\]

We have,
\[
\int_{0}^{\zeta} \frac{\omega(\eta)}{\eta} d\eta \leq \int_{0}^{\delta} \frac{\omega(\eta)}{\eta} d\eta + \int_{\delta}^{\zeta} \frac{\omega(\eta)}{\eta} d\eta \\
\leq \delta + \omega(\zeta) \ln\left(\frac{\zeta}{\delta}\right)
\]
If we choose $\delta$ small enough as before in (50), so that $\omega(\zeta) \geq \omega(\delta) \geq \frac{\delta}{2}$, we obtain
\[
\int_0^\zeta \frac{\omega(\eta)}{\eta} d\eta \leq 2\omega(\zeta) + \omega(\zeta) \ln(\zeta) \\
\leq \omega(\zeta)(2 + \ln(\zeta)) \tag{53}
\]
On the other hand, integrating by parts and using (34), we can evaluate
\[
\int_\zeta^\infty \frac{\omega(\eta)}{\eta^2} d\eta = \frac{\omega(\zeta)}{\zeta} + \gamma \int_\zeta^\infty \frac{1}{\eta^2(4 + \ln(\eta))} d\eta \\
\leq \frac{\omega(\zeta)}{\zeta} + \frac{\gamma}{\zeta}
\]
Consequently, if we choose $\gamma \leq \frac{\delta}{2}$, with $\delta$ small enough as in (50), then from (51) we get
\[
\int_\zeta^\infty \frac{\omega(\eta)}{\eta^2} d\eta \leq \frac{2\omega(\zeta)}{\zeta} \tag{54}
\]
Hence, from (53) and (54), we get
\[
(\int_0^\zeta \frac{\omega(\eta)}{\eta} d\eta + \zeta \int_\zeta^\infty \frac{\omega(\eta)}{\eta^2} d\eta) \leq \omega(\zeta)(2 + \ln(\zeta)) + 2\omega(\zeta) \\
\leq \omega(\zeta)(4 + \ln(\zeta)) \tag{55}
\]
Finally, injecting (45), (46), (52), and (55) in (44), we obtain
\[
\partial_t(\hat{\theta}(x, T) - \hat{\theta}(y, T)) \leq B\omega(\zeta)(4 + \ln(\frac{\zeta}{\delta}))\omega'(\zeta) - \frac{1}{\pi} \frac{\omega(\zeta)}{\zeta} + 2\|f\|_{L^\infty} A
\]
Therefore, from (34) we have
\[
\partial_t(\hat{\theta}(x, T) - \hat{\theta}(y, T)) \leq B\gamma \frac{\omega(\zeta)}{\zeta} - \frac{1}{\pi} \frac{\omega(\zeta)}{\zeta} + 2\|f\|_{L^\infty} A \\
\leq \frac{\omega(\zeta)}{\zeta} (B\gamma - \frac{1}{\pi}) + 2\|f\|_{L^\infty} A
\]
If we choose $\gamma$ small enough, we get
\[
B\gamma - \frac{1}{\pi} < 0
\]
then, we get for all $\delta \leq \zeta \leq A.D$, where $D$ is the period of $\theta$,
\[
\partial_t(\hat{\theta}(x, T) - \hat{\theta}(y, T)) \leq \frac{\omega(A.D)}{A.D} (B\gamma - \frac{1}{\pi}) + 2\|f\|_{L^\infty} A
\]
Since $\omega$ is increasing, we can choose $A$ large enough depending on $D$ and $\|f\|_{L^\infty}$, such that
\[
\frac{\omega(A.D)}{A.D}(B\gamma - \frac{1}{\pi}) + 2\frac{\|f\|_{L^\infty}}{A} < 0
\]

Remark (2.5) and lemma (2.6) show that for $\delta$ and $\gamma$ chosen small enough, with $0 < \gamma \leq \frac{\delta}{2}$, $\omega$ is preserved by $\hat{\theta}$ for $0 \leq \zeta \leq A.D$, where $D$ is the period of $\theta$, for all time $t$. Since $\theta$ is periodic of period $D$ depending on the period of $\theta_0$ and of $f$, then $\hat{\theta}$ is periodic of period $A.D$, and since $\omega$ is increasing, we have by then that for all $\zeta \geq A.D$ and for all time $t$,
\[
\hat{\theta}(x, t) - \hat{\theta}(y, t) \leq \omega(\zeta)
\]
Therefore, $\omega_A$ is preserved by $\theta$ for all time. Consequently, from (7) we have
\[
\|\nabla \theta\|_{L^\infty} \leq A\omega'(0) \leq A
\]
where $A$ depends only on $\|f\|_{L^\infty}$, on $C_1$ and $\beta = \min\{\frac{1}{2}, \alpha\}$, on $\|\nabla \theta_0\|_{L^\infty}$, on the period of $\theta_0$, and on the period $D$ of $\theta$ (which is given by the period of $\theta_0$ and the period of $f$). If $A$ is finite, this gives that local solutions of (1) can be extended globally in time.

References

[K] A. Kiselev, Some recent results on the critical surface quasi-geostrophic equation: A review, Proceedings of Symposia in Applied Mathematics, 67.1 (1952), 2009.

[Kl] S. Klainerman, Great problems in Nonlinear Evolution Equations, the AMS Millenium Conference in Los Angeles, August, 2000.

[KNV] A. Kiselev, F. Nazarov, A. Volberg, Global well-posedness for the critical 2D dissipative quasi-geostrophic equation, Invent. math. 167, 445-453 (2007).

[Wu] J. Wu, The quasi-geostrophic equation and its two regularizations, Comm. Partial Differential Equations 27 (2002), 1161-1181

Institut de Mathématiques de Jussieu, Université Paris Diderot - Paris VII, 75205 Paris Cedex 13, France

E-mail address: ghanem@math.jussieu.fr