1. INTRODUCTION

The purpose of this paper is to explore the connection between the theory of integrable discrete equations of Painlevé type and the Riemann–Hilbert problem for matrix orthogonal polynomials. In the present
work, we shall extend the Riemann–Hilbert approach to the unit circle, for a class of biorthogonal polynomials of Szegő type defined in terms of a matrix of Hölder weights. With the aid of the Riemann–Hilbert problem we will find a matrix version of the discrete Painlevé II equations that holds for the Verblunsky matrices.

The unit circle is denoted by \( T := \{ z \in \mathbb{C} : |z| = 1 \} \), while its interior, the unit disk by \( D := \{ z \in \mathbb{C} : |z| < 1 \} \) and its exterior by \( \bar{D} := \{ z \in \mathbb{C} : |z| < 1 \} \). Given complex Borel measure \( \mu \) supported in \( T \) we say that is positive definite if it maps measurable sets into non-negative numbers, that in the absolutely continuous situation, with respect to the Lebesgue measure \( d m(\zeta) = \frac{d \zeta}{2\pi i \zeta} \), has the form \( w(\zeta) \ d m(\zeta), \zeta \in T \), with the weight function \( w(\zeta) \), integrable or Hölder, depending on the context. For the positive definite situation the orthogonal polynomials in the unit circle (OPUC) or Szegő polynomials are defined as those monic polynomials \( P_n \) of degree \( n \) that satisfy \( \int_T P_n(z) z^{-k} \ d \mu(z) = 0 \), for \( k \in \{ 0, 1, \ldots, n-1 \} \). We refer the reader to Barry Simon’s books [108] and [109] for a very detailed studied of OPUC, and for a survey on matrix orthogonal polynomials we refer the reader to [39].

Orthogonal polynomials on the real line (OPRL) supported in the interval \([-1,1]\) and OPUC are deeply connected as has been shown in several papers, see [60] and [23]. From the side of an spectral approach a study of the operator of multiplication by \( z \) is required, this analysis is associated to the existence of recursion relations. For OPRL the three term recurrence laws provide a tridiagonal matrix, the so called Jacobi operator, while for OPUC one is faced with a Hessenberg matrix, being a more involved scenario that the Jacobi one (as it is not a sparse matrix with a finite number of non vanishing diagonals). In fact, A better approach are Szegő recursion relation, which need of the reciprocal or reverse Szegő polynomials \( \tilde{P}_n(z) := z^k \tilde{P}_n(z^{-1}) \) and the reflection or Verblunsky coefficients \( \alpha_1 := P_1(0) \). Szegő recursion relations for for OPUC are

\[
\begin{pmatrix}
    P_1(z) \\
    \tilde{P}_1(z)
\end{pmatrix} = \begin{pmatrix}
    z & \alpha_1 \\
    z \bar{\alpha}_1 & 1
\end{pmatrix}
\begin{pmatrix}
    P_{-1}(z) \\
    \tilde{P}_{-1}(z)
\end{pmatrix}.
\]

The study of zeroes of OPUC has been a very active area, see for example [8] [14] [21] [61] [69] [71] [92] [99], and interesting applications to signal analysis theory [80] [81] [102] [103] have been found. Let us stress that in general Szegő polynomials do not provide a dense set in the Hilbert space \( L^2(T, \mu) \) and its exterior by \( \bar{D} \). From the side of an spectral approach a study of orthogonal polynomials in the real line and its Cauchy transforms. Deift and Zhou combined these ideas with a non-linear steepest descent analysis in a series of important works [43] [44] [46] [47] which as a
byproduct generated a large activity in the field. To mention just a few relevant results let us cite the study of strong asymptotic with applications in random matrix theory, [43, 45], the analysis of determinantal point processes [40, 41, 85, 86], orthogonal Laurent polynomials [88, 89] and Painlevé equations [79, 42]. For the case of OPUC, a Riemann–Hilbert problem was discussed in [20], see also [90, 91]. An excellent introduction, in the realm of integrable systems, of the Riemann Hilbert problem is given [78]. In the monograph we can find a modern account of the Painlevé equations and the Riemann–Hilbert or isomonodromy method. For a very good account of the very close theme of the 21rst Hilbert problem see [16]. For more on integrable systems, Painlevé equations and its discrete versions see [112].

The study of equations for the recursion coefficients for OPRL or OPUC has been a subject of interest. The question of how the form of the weight and its properties, for example to satisfy a Pearson type equation, translates to the recursion coefficients has been treated in several places, a good review is [115]. It was in [59] were Géza Freud studied weights in $\mathbb{R}$ of exponential variation $w(x) = |x|^\rho \exp(-|x|^m)$, $\rho > -1$ and $m > 0$. For $m = 2, 4, 6$ he constructed relations among them as well as determined its asymptotic behavior. However, Freud did not found the role of the discrete Painlevé I, that was discovered later by Magnus [87]. For the unit circle and a weight of the form $w(\theta) = \exp(k \cos \theta)$, $k \in \mathbb{R}$, Periwal and Shevitz [104, 105], in the context of matrix models, found the discrete Painlevé II equation for the recursion relations of the corresponding orthogonal polynomials. This result was rediscovered latter and connected with the Painlevé III equation [75]. In [19] the discrete Painlevé II was found using the Riemann–Hilbert problem given in [20], see also [114]. For a nice account of the relation of these discrete Painlevé equations and integrable systems see [36]. We also mention the recent paper [35] where a discussion on the relationship between the recurrence coefficients of orthogonal polynomials with respect to a semiclassical Laguerre weight and classical solutions of the fourth Painlevé equation can be found.

Back in 1949, Krein [83, 84] used orthogonal polynomials with matrix coefficients on the real line and thereafter were studied sporadically until the last decade of the XX century, being some relevant papers [22, 68] and [17]. For a kind of discrete Sturm–Liouville operators the scattering problem is solved in [17], finding that the polynomials that satisfy a relation of the form

$$xP_k(x) = A_k P_{k+1}(x) + B_k P_k(x) + A_{k-1} P_{k-1}(x),$$

are orthogonal with respect to a positive definite measure; i.e., a matrix version of Favard’s theorem. Then, in the 1990’s and the 2000’s it was found that matrix orthogonal polynomials (MOP) satisfy in some cases properties as do the classical orthogonal polynomials. For example, Laguerre, Hermite and Jacobi polynomials, i.e., the scalar-type Rodrigues’ formula [48, 49] and a second order differential equation [50, 51, 26]. It has been proven [52] that operators of the form $D = \partial^2 F_2(t) + \partial^2 F_1(t) + \partial F_0$ have as eigenfunctions different infinite families of MOP’s. A new family of MOP’s satisfying second order differential equations whose coefficients do not behave asymptotically as the identity matrix was found in [26]; see also [30]. We have studied [9, 11] matrix extensions of the generalized polynomials studied in [5, 6]. Recently, in [12], we have extended the Christoffel transformation to MOPRL obtaining a new matrix Christoffel formula, and in [13] more general transformations –of Geronimus and Uvarov type– where also considered.

In [31] the Riemann–Hilbert problem for this matrix situation and the appearance of non-Abelian discrete versions of Painlevé I were explored, showing singularity confinement [32]. The singularity analysis for a matrix discrete version of the Painlevé I equation was performed. It was found that the singularity confinement holds generically, i.e. in the whole space of parameters except possibly for algebraic subvarieties. For an alternative discussion of the use of Riemann–Hilbert problem for MOPRL see [23]. Let us mention that in [93, 94] and [28] the MOP are expressed in terms of Schur complements that play the role of determinants in the standard scalar case. In [28] an study of matrix Szegő polynomials and the relation with a non Abelian Ablowitz–Ladik lattice is carried out, and in [18] the CMV ordering is applied to study orthogonal Laurent polynomials in the circle.

The layout of the paper is as follows. We start in §2 recalling some facts of measure theory in $\mathbb{T}$ and, in particular, we discuss matrices of measures in the unit circle. We then proceed with the construction of matrix Szegő polynomials in the quasi-definite scenario, we introduce reverse polynomials, their quasi-determinantal expressions, the Verblunsky coefficients and the Gauss–Borel factorization for the moment matrix, that leads to biorthogonal families of matrix polynomials. We also discuss some symmetry properties. In §3 following [33], we introduce in the first place the Cauchy transforms of the matrix orthogonal polynomials, and then discuss the Riemann–Hilbert problem in this more general context. Next, in §4 we
apply the Riemann–Hilbert problem to derive the Szegő recursion relations and some differential relations. We conclude the paper in [33] with the finding of a matrix discrete Painlevé II equation, which holds for the Verblunsky matrices whenever the matrix of weights has a right logarithmic derivative of certain form — of Fuchsian and non-Fuchsian singularity type — associated with a monodromy free system. This Pearson type equation for the matrix of weights allows us to avoid serious difficulties, that appear if one insists in deriving these matrix discrete Painlevé equations directly from explicit Freud’s weights. Let us notice that these discrete Painlevé systems have non local terms, involving not only near neighbors. This issue is also consider and non trivial reductions that get rid of these non local contributions are presented. We also include, in Appendix [A] some examples of matrices of weights, for the Fuchsian situation, constructed as solutions to the mentioned Pearson equations.

2. Matrix Szegő biorthogonal polynomials

In this section we consider the matrix extension of Szegő biorthogonal polynomials in the unit circle [13], see [28, 18].

2.1. Matrices of measures in the unit circle. We recall here some facts regarding measure theory on the unit circle \( T \), we follow [106, 53, 33]. The Lebesgue measure, for \( \zeta \in T \), is

\[
d\,m(\zeta) := \frac{d\zeta}{2\pi i}\zeta.
\]

We shall consider a matrix of finite complex valued Borel measures \( \mu \) supported in \( T \), we denote the set of such measures by \( \mathcal{B} \). A matrix of measures \( \mu \in \mathcal{B} \) is said absolutely continuous, with respect to the Lebesgue measure \( m \), written \( \mu \ll m \), if \( \mu(A) = 0_N \), where \( 0_N \in \mathbb{C}^{N \times N} \) denotes the zero matrix, for all Borel sets \( A \) of Lebesgue zero measure, \( m(A) = 0 \). A matrix of measures \( \mu \in \mathcal{B} \) in singular, with respect to the Lebesgue measure \( m \), written \( \mu \perp m \), if for two disjoint Borel sets \( A, B \) we have \( \mu(A) = 0_N \) and \( m(B) = 0 \). Any matrix of measures \( \mu \in \mathcal{B} \) can be decomposed uniquely \( \mu = \mu_a + \mu_s \), \( \mu_a \ll m \) and \( \mu_s \perp m \). As was proven by Radon and Nikodym a \( \mu \in \mathcal{B} \) is absolutely continuous if and only if there exist a matrix \( w : T \rightarrow \mathbb{C}^{N \times N} \) built up with \( L^1(T, m) \) weights such that \( d\mu(\zeta) = w(\zeta) \, d\,m(\zeta) \), i.e.,

\[
\mu(B) = \int_B w(\zeta) \, d\,m(\zeta);
\]

the matrix of weights \( w(\zeta) \) is called the matrix of Radon–Nikodym derivatives and we write

\[
w(\zeta) = \frac{d\mu}{d\,m}(\zeta).
\]

Therefore, according to the Lebesgue–Radon–Nikodym theorem [106, 53] for any matrix of measures \( \mu \in \mathcal{B} \) and any Borel set \( B \subset T \) we can write

\[
\mu(B) = \int_B \frac{d\mu}{d\,m}(\zeta) \, d\,m(\zeta) + \mu_s(B),
\]

where, following [33], we have introduced the matrix of Radon–Nikodym derivatives of \( \mu \), \( \mu = \mu_a + \mu_s \), with respect to the Lebesgue measure, as the matrix of Radon–Nikodym derivatives of its absolutely continuous component,

\[
\frac{d\mu}{d\,m} := \frac{d\mu_a}{d\,m}.
\]

For any Borel measure we can consider its differential [106, 53, 33], let \( I(\zeta, t) \) be the arc of the unit circle subtended by the points \( \zeta e^{it} \) and \( \zeta e^{-it} \) and consider, for \( \mu \in \mathcal{B} \),

\[
(\mathcal{D}\mu)(\zeta) := \liminf_{t \to 0^+} \frac{\mu(I(\zeta, t))}{m(I(\zeta, t))}, \quad (\overline{\mathcal{D}}\mu)(\zeta) := \limsup_{t \to 0^+} \frac{\mu(I(\zeta, t))}{m(I(\zeta, t))}.
\]

When these two matrices are bounded and equal, \( (\mathcal{D}\mu)(\zeta) = (\overline{\mathcal{D}}\mu)(\zeta) =: (\mathcal{D}\mu)(\zeta) \), we say that \( \mu \) is differentiable (with respect to the Lebesgue measure) at \( \zeta \in T \) with matrix of differentials \( \mathcal{D}\mu(\zeta) \). Then,
see [106 53 33], $\mu \in \mathcal{B}$ is differentiable $m$-almost for every $\zeta \in \mathbb{T}$, moreover its matrix of differentials $D\mu \in (L^1(\mathbb{T}, \mu))^N \times N$ is a matrix of integrable functions and for any Borel set $B$ we have

$$\mu(B) = \int_B (D\mu)(\zeta) \, d\mathcal{m}(\zeta) + \mu_s(B),$$

where $\mu_s \perp m$ and $D\mu_s(\zeta) = 0$ for $m$-almost every $\zeta \in \mathbb{T}$. In this situation, the matrix of differentials and the matrix of Radon–Nikodym derivatives coincide, $(D\mu)(\zeta) = \frac{d\mu}{dm}(\zeta)$, for $m$-almost every $\zeta \in \mathbb{T}$.

2.2. Matrix Szegő polynomials on the unit circle. Here we follow [28] and [18].

**Definition 1** (Szegő matrix polynomials). Given a matrix of measures $\mu$, the left and right monic matrix Szegő polynomials $P_{1,n}^L(z)$, $P_{1,n}^R(z)$, $P_{2,n}^L(z)$, $P_{2,n}^R(z)$ are monic polynomials

$$P_{1,n}^L(z) = p_{1,n,0}^L + \cdots + p_{1,n,n-1}^L z^{n-1} + I_N z^n,$$

$$P_{1,n}^R(z) = p_{1,n,0}^R + \cdots + p_{1,n,n-1}^R z^{n-1} + I_N z^n,$$

$$P_{2,n}^L(z) = p_{2,n,0}^L + \cdots + p_{2,n,n-1}^L z^{n-1} + I_N z^n,$$

$$P_{2,n}^R(z) = p_{2,n,0}^R + \cdots + p_{2,n,n-1}^R z^{n-1} + I_N z^n,$$

where $I_N \in \mathbb{C}^{N \times N}$ is the identity matrix and $P_{1,n,j}^L$, $P_{1,n,j}^R$, $P_{2,n,j}^L$, $P_{2,n,j}^R \in \mathbb{C}^{N \times N}$, such that the following orthogonality conditions

(1)\[ \int_T p_{1,n}^L(\zeta) \, d\mu(\zeta) \bar{\zeta}^j = 0_N, \]

(2)\[ \int_T \bar{\zeta}^j \, d\mu(\zeta) p_{1,n}^R(\zeta) = 0_N, \]

(3)\[ \int_T \zeta \, d\mu(\zeta) (P_{2,n}^L(\zeta))^\dagger = 0_N, \]

(4)\[ \int_T (P_{2,n}^R(\zeta))^\dagger \, d\mu(\zeta) \zeta^j = 0_N, \]

stand for all $j \in \{0, \ldots, n - 1\}$.

From the second families of left and right Szegő matrix polynomials $P_{1,n}^L(z)$ and $P_{2,n}^L(z)$ we construct

**Definition 2** (Reciprocal Szegő polynomials). The reciprocal (or reverse) left and right Szegő matrix polynomials $\bar{P}_{1,n}^L(z)$ and $\bar{P}_{2,n}^L(z)$ are given by

$$\bar{P}_{1,n}^L(z) := \zeta^n (P_{2,n}^L(\bar{z}^{-1}))^\dagger = I_N + (P_{2,n,n-1}^L)^\dagger z + \cdots + (P_{2,n,n}^L)^\dagger z^n,$$

$$\bar{P}_{2,n}^L(z) := z^n (P_{2,n}^L(\bar{z}^{-1}))^\dagger = I_N + (P_{2,n,n-1}^L)^\dagger z + \cdots + (P_{2,n,n}^L)^\dagger z^n.$$

**Definition 3** (Verblunsky matrices). The Verblunsky matrices are the evaluations at the origin, $z = 0$, of the Szegő polynomials

$$\alpha_{1,n}^L := p_{1,n}^L(0), \quad \alpha_{1,n}^R := p_{1,n}^R(0), \quad \alpha_{2,n}^L := p_{2,n}^L(0), \quad \alpha_{2,n}^R := p_{2,n}^R(0).$$

**Proposition 1.** In terms of the Verblunsky coefficients, the Szegő matrix polynomials of type 1 and its reciprocals can be written as follows

$$P_{1,n}^L(z) = \alpha_{1,n}^L + \cdots + \alpha_{1,n,n-1}^L z^{n-1} + I_N z^n,$$

$$\bar{P}_{1,n}^L(z) = \alpha_{1,n}^L + \cdots + \alpha_{1,n,n-1}^L z^{n-1} + I_N z^n,$$

$$\bar{P}_{2,n}^L(z) = I_N + (\alpha_{2,n}^L)^\dagger z + \cdots + (\alpha_{2,n}^L)^\dagger z^n,$$

$$\bar{P}_{2,n}^R(z) = I_N + (\alpha_{2,n}^L)^\dagger z + \cdots + (\alpha_{2,n}^L)^\dagger z^n.$$

**Proposition 2.** The reciprocal Szegő matrix polynomials $\bar{P}_{1,n}^L(z)$ and $\bar{P}_{2,n}^L(z)$ satisfy the following orthogonality relations

(5)\[ \int_T \bar{\zeta}^j \, d\mu(\zeta) \bar{P}_{1,n}^L(\zeta) = 0_N, \]

(6)\[ \int_T \bar{P}_{2,n}^L(\zeta) \, d\mu(\zeta) \bar{\zeta}^j = 0_N, \]

for all $j \in \{1, \ldots, n\}$. 


Proof. From (3) and (4) we get for 

\[
\int_T \zeta^{j-n} \, d \mu(\zeta) \bar{p}_n^L(\zeta) = 0_N, \quad \int_T \bar{p}_n^R(\zeta) \, d \mu(\zeta) \zeta^{j-n} = 0_N,
\]

and relabeling the indexes we get the stated orthogonality relations. \( \square \)

Following \cite{18} we introduce

**Definition 4.** The moments or Fourier coefficients \( \hat{\mu}(j) \in \mathbb{C}^{N \times N} \) of the matrix of measures \( \mu \) are defined by

\[
\hat{\mu}(j) := \int_T \zeta^j \, d \mu(\zeta),
\]

with \( j \in \mathbb{Z} \).

**Definition 5.** We introduce the left and right semi-infinite moment matrices

\[
\mathcal{M}^L := \begin{pmatrix}
\hat{\mu}(0) & \hat{\mu}(-1) & \hat{\mu}(-2) & \cdots & \hat{\mu}(-n+1) \\
\hat{\mu}(1) & \hat{\mu}(0) & \hat{\mu}(-1) & \cdots & \hat{\mu}(-n+2) \\
\hat{\mu}(2) & \hat{\mu}(1) & \hat{\mu}(0) & \cdots & \hat{\mu}(-n+3) \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\hat{\mu}(n-1) & \hat{\mu}(n-2) & \hat{\mu}(n-3) & \cdots & \hat{\mu}(0)
\end{pmatrix},
\]

and its truncations

\[
\mathcal{M}_{[n]}^L := \begin{pmatrix}
\hat{\mu}(0) & \hat{\mu}(-1) & \hat{\mu}(-2) & \cdots & \hat{\mu}(-n+1) \\
\hat{\mu}(1) & \hat{\mu}(0) & \hat{\mu}(-1) & \cdots & \hat{\mu}(-n+2) \\
\hat{\mu}(2) & \hat{\mu}(1) & \hat{\mu}(0) & \cdots & \hat{\mu}(-n+3) \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\hat{\mu}(n-1) & \hat{\mu}(n-2) & \hat{\mu}(n-3) & \cdots & \hat{\mu}(0)
\end{pmatrix},
\]

\[
\mathcal{M}_{[n]}^R := \begin{pmatrix}
\hat{\mu}(-2) & \hat{\mu}(-1) & \hat{\mu}(0) & \cdots & \hat{\mu}(-n+3) \\
\hat{\mu}(-1) & \hat{\mu}(0) & \hat{\mu}(-1) & \cdots & \hat{\mu}(-n+2) \\
\hat{\mu}(0) & \hat{\mu}(-1) & \hat{\mu}(0) & \cdots & \hat{\mu}(-n+1) \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\hat{\mu}(-n+1) & \hat{\mu}(-n+2) & \hat{\mu}(-n+3) & \cdots & \hat{\mu}(0)
\end{pmatrix}.
\]

The matrix of measures \( d \mu(\zeta) \) is quasi-definite if \( \det \mathcal{M}_{[n]}^L \neq 0 \) and \( \det \mathcal{M}_{[n]}^R \neq 0 \) for all \( n \in \{1, 2, \ldots \} \).

Observe that these truncated moment matrices are block Toeplitz matrices organized by block diagonals. We now need of the notion of quasi-determinant, see \cite{67,66,65,101}.

**Proposition 3.** The matrix of measures \( \mu \) is quasi-definite if the last quasi-determinants \( \Theta_s \mathcal{M}_{[n]}^L \) and \( \Theta_s \mathcal{M}_{[n]}^R \) are not singular matrices.
**Proposition 4.** The Szegő matrix polynomials exists whenever the matrix of measures $\mu$ in quasi-definite. Moreover, they can be expressed in terms of last quasi-determinants of bordered truncated moment matrices

\[
P_{L,n}(z) = \Theta_*,
\]

\[
P_{L,n}(z) = \Theta_*
\]

\[
(p_{L,n}(z))^\dagger = \Theta_*
\]

\[
(p_{R,n}(z))^\dagger = \Theta_*
\]

**Proof.** In terms of moments of the matrix of measures the orthogonality relations (1) and (2) read

\[
(p_{L,n,0}, \ldots, p_{L,n,n-1}) \left( \begin{array}{cccc}
\hat{\mu}(0) & \hat{\mu}(1) & \hat{\mu}(2) & \cdots & \hat{\mu}(n-1) \\
\hat{\mu}(-1) & \hat{\mu}(0) & \hat{\mu}(1) & \cdots & \hat{\mu}(n-2) \\
\hat{\mu}(-2) & \hat{\mu}(-1) & \hat{\mu}(0) & \cdots & \hat{\mu}(n-3) \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\hat{\mu}(-n+1) & \hat{\mu}(-n+2) & \hat{\mu}(-n+3) & \cdots & \hat{\mu}(0) \\
\end{array} \right) = -(\hat{\mu}(-n), \ldots, \hat{\mu}(-1)),
\]

\[
(p_{R,n,0}, \ldots, p_{R,n,n-1}) \left( \begin{array}{cccc}
\hat{\mu}(0) & \hat{\mu}(-1) & \hat{\mu}(-2) & \cdots & \hat{\mu}(-n) \\
\hat{\mu}(1) & \hat{\mu}(0) & \hat{\mu}(-1) & \cdots & \hat{\mu}(-n+1) \\
\hat{\mu}(2) & \hat{\mu}(1) & \hat{\mu}(0) & \cdots & \hat{\mu}(-n+2) \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\hat{\mu}(n-1) & \hat{\mu}(n-2) & \hat{\mu}(n-3) & \cdots & \hat{\mu}(0) \\
\end{array} \right) = -(\hat{\mu}(n), \ldots, \hat{\mu}(-1)),
\]
Thus, assuming the quasi-definite condition we get

\[
\begin{pmatrix}
\hat{\mu}(0) & \hat{\mu}(-1) & \hat{\mu}(-2) & \cdots & \hat{\mu}(-n+1) \\
\hat{\mu}(1) & \hat{\mu}(0) & \hat{\mu}(-1) & \cdots & \hat{\mu}(-n+2) \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\hat{\mu}(n-1) & \hat{\mu}(n-2) & \hat{\mu}(n-3) & \cdots & \hat{\mu}(0)
\end{pmatrix}
\begin{pmatrix}
(p_{2,n,0}^L)^\dagger \\
(p_{2,n,n-1}^L)^\dagger
\end{pmatrix}
= -
\begin{pmatrix}
\hat{\mu}(1) \\
\vdots \\
\hat{\mu}(n)
\end{pmatrix},
\]

and

\[
\begin{pmatrix}
\hat{\mu}(0) & \hat{\mu}(-1) & \hat{\mu}(-2) & \cdots & \hat{\mu}(-n+1) \\
\hat{\mu}(1) & \hat{\mu}(0) & \hat{\mu}(-1) & \cdots & \hat{\mu}(-n+2) \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\hat{\mu}(n-1) & \hat{\mu}(n-2) & \hat{\mu}(n-3) & \cdots & \hat{\mu}(0)
\end{pmatrix}
\begin{pmatrix}
\hat{\mu}(-n) \\
\vdots \\
\hat{\mu}(-1)
\end{pmatrix},
\]

Thus, assuming the quasi-definite condition we get

\[
\begin{pmatrix}
\hat{\mu}(0) & \hat{\mu}(1) & \hat{\mu}(2) & \cdots & \hat{\mu}(n-1) \\
\hat{\mu}(-1) & \hat{\mu}(0) & \hat{\mu}(1) & \cdots & \hat{\mu}(n-2) \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\hat{\mu}(-n+1) & \hat{\mu}(-n+2) & \hat{\mu}(-n+3) & \cdots & \hat{\mu}(0)
\end{pmatrix}
\begin{pmatrix}
(p_{2,n,0}^R)^\dagger \\
(p_{2,n,n-1}^R)^\dagger
\end{pmatrix}
= -(\hat{\mu}(1), \ldots, \hat{\mu}(n)).
\]
Therefore, for the first family of left Szegő matrix polynomials we have

\[ P_{I,n}^L(z) = I_N z^n + (P_{I,n,0}^L, \ldots, P_{I,n,n-1}^L) \begin{pmatrix} I_N \\ \vdots \\ I_N z^{n-1} \end{pmatrix} \]

\[ = I_N z^n - (\hat{\mu}(-n), \ldots, \hat{\mu}(-1)) \begin{pmatrix} \hat{\mu}(0) & \hat{\mu}(1) & \hat{\mu}(2) & \cdots & \hat{\mu}(n-1) \\ \hat{\mu}(-1) & \hat{\mu}(0) & \hat{\mu}(1) & \hat{\mu}(n-2) \\ \hat{\mu}(-2) & \hat{\mu}(-1) & \hat{\mu}(0) & \cdot & \cdot \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ \hat{\mu}(-n+1) & \hat{\mu}(-n+2) & \hat{\mu}(-n+3) & \cdots & \hat{\mu}(0) \end{pmatrix}^{-1} \begin{pmatrix} I_N \\ \vdots \\ I_N z^{n-1} \end{pmatrix}, \]

while for the first family of right Szegő polynomials we find

\[ P_{I,n}^R(z) = I_N z^n + (I_N, \ldots, I_N z^{n-1}) \begin{pmatrix} P_{I,n,0}^R \\ \vdots \\ P_{I,n,n-1}^R \end{pmatrix} \]

\[ = I_N z^n - (I_N, \ldots, I_N z^{n-1}) \begin{pmatrix} \hat{\mu}(0) & \hat{\mu}(-1) & \hat{\mu}(-2) & \cdots & \hat{\mu}(-n+1) \\ \hat{\mu}(1) & \hat{\mu}(0) & \hat{\mu}(-1) & \hat{\mu}(-n+2) \\ \hat{\mu}(2) & \hat{\mu}(1) & \hat{\mu}(0) & \cdot & \cdot \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ \hat{\mu}(n-1) & \hat{\mu}(n-2) & \hat{\mu}(n-3) & \cdots & \hat{\mu}(0) \end{pmatrix}^{-1} \begin{pmatrix} \hat{\mu}(-n) \\ \vdots \\ \hat{\mu}(-1) \end{pmatrix}. \]

For the second family of left Szegő matrix polynomials we deduce

\[ (P_{2,n}^L(z))^\dagger = z^n I_N + (1, \ldots, z^{n-1}) \begin{pmatrix} (P_{2,n,0}^L)^\dagger \\ \vdots \\ (P_{2,n,n-1}^L)^\dagger \end{pmatrix} \]

\[ = z^n I_N - (I_N, \ldots, I_N z^{n-1}) \begin{pmatrix} \hat{\mu}(0) & \hat{\mu}(-1) & \hat{\mu}(-2) & \cdots & \hat{\mu}(-n+1) \\ \hat{\mu}(1) & \hat{\mu}(0) & \hat{\mu}(-1) & \hat{\mu}(-n+2) \\ \hat{\mu}(2) & \hat{\mu}(1) & \hat{\mu}(0) & \cdot & \cdot \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ \hat{\mu}(n-1) & \hat{\mu}(n-2) & \hat{\mu}(n-3) & \cdots & \hat{\mu}(0) \end{pmatrix}^{-1} \begin{pmatrix} \hat{\mu}(1) \\ \vdots \\ \hat{\mu}(n) \end{pmatrix}, \]

while for the second family of right Szegő matrix polynomials we get

\[ (P_{2,n}^R(z))^\dagger = z^n I_N + (P_{2,n,0}^R)^\dagger, \ldots, (P_{2,n,n-1}^R)^\dagger \begin{pmatrix} I_N \\ \vdots \\ I_N z^{n-1} \end{pmatrix} \]

\[ = z^n I_N - (\hat{\mu}(1), \ldots, \hat{\mu}(n)) \begin{pmatrix} \hat{\mu}(0) & \hat{\mu}(1) & \hat{\mu}(2) & \cdots & \hat{\mu}(n-1) \\ \hat{\mu}(-1) & \hat{\mu}(0) & \hat{\mu}(1) & \hat{\mu}(n-2) \\ \hat{\mu}(-2) & \hat{\mu}(-1) & \hat{\mu}(0) & \cdot & \cdot \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ \hat{\mu}(-n+1) & \hat{\mu}(-n+2) & \hat{\mu}(-n+3) & \cdots & \hat{\mu}(0) \end{pmatrix}^{-1} \begin{pmatrix} I_N \\ \vdots \\ I_N z^{n-1} \end{pmatrix}. \]

From these relations the quasi-determinantal expressions follow immediately. □
Observe that this result is just informing us that the Szegő matrix polynomials can be expressed as rational functions of the moments. For example,

\[
P_{1,1}^L(z) = \Theta_\ast \begin{pmatrix} \hat{\mu}(0) & I_N \\ \hat{\mu}(-1) & I_N z \end{pmatrix}
= -\hat{\mu}(-1)(\hat{\mu}(0))^{-1} + I_N z,
\]

\[
P_{1,2}^L(z) = \Theta_\ast \begin{pmatrix} \hat{\mu}(0) & \hat{\mu}(1) & I_N \\ \hat{\mu}(-1) & \hat{\mu}(0) & I_N z \\ \hat{\mu}(-2) & \hat{\mu}(-1) & I_N z^2 \end{pmatrix}
= -\hat{\mu}(-2)(\hat{\mu}(0))^{-1} + (\hat{\mu}(-1) + \hat{\mu}(2)(\hat{\mu}(0))^{-1}\hat{\mu}(1))((\hat{\mu}(0) - \hat{\mu}(-1)(\hat{\mu}(0))^{-1}\hat{\mu}(1))^{-1}\hat{\mu}(-1)(\hat{\mu}(0))^{-1}
- (\hat{\mu}(-1) + \hat{\mu}(2)(\hat{\mu}(0))^{-1}\hat{\mu}(1))(\hat{\mu}(0) - \hat{\mu}(-1)(\hat{\mu}(0))^{-1}\hat{\mu}(1))^{-1}z + I_N z^2.
\]

In fact, they are polynomials in the moments and the inverses of the last quasi-determinants \( \Theta_\ast M_{n}^L \) and \( \Theta_\ast M_{n}^R \).

These expressions allow us to find

**Proposition 5.** The reciprocal Szegő polynomials have the following quasi-determinantal expressions

\[
\hat{p}_{2,n}^L(z) = \Theta_\ast \begin{pmatrix} \hat{\mu}(0) & \hat{\mu}(-1) & \hat{\mu}(-2) & \cdots & \hat{\mu}(-n + 1) & \hat{\mu}(1) \\ \hat{\mu}(1) & \hat{\mu}(0) & \hat{\mu}(-1) & \hat{\mu}(-n + 2) & \hat{\mu}(2) \\ \hat{\mu}(2) & \hat{\mu}(1) & \hat{\mu}(0) & \cdots & \hat{\mu}(-n + 3) & \hat{\mu}(3) \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \hat{\mu}(n-1) & \hat{\mu}(n-2) & \hat{\mu}(n-3) & \cdots & \hat{\mu}(0) & \hat{\mu}(n) \\ I_N z^n & I_N z^{n-1} & I_N z^{n-2} & \cdots & I_N z & I_N \end{pmatrix},
\]

\[
\hat{p}_{2,n}^R(z) = \Theta_\ast \begin{pmatrix} \hat{\mu}(0) & \hat{\mu}(1) & \hat{\mu}(2) & \cdots & \hat{\mu}(n-1) & I_N z^n \\ \hat{\mu}(-1) & \hat{\mu}(0) & \hat{\mu}(1) & \hat{\mu}(n-2) & I_N z^{n-1} \\ \hat{\mu}(-2) & \hat{\mu}(-1) & \hat{\mu}(0) & \cdots & \hat{\mu}(n-3) & I_N z^{n-2} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \hat{\mu}(-n+1) & \hat{\mu}(-n+2) & \hat{\mu}(-n+3) & \cdots & \hat{\mu}(0) & I_N z \\ \hat{\mu}(1) & \hat{\mu}(2) & \hat{\mu}(3) & \cdots & \hat{\mu}(n) & I_N \end{pmatrix},
\]

**Definition 6.** The Gauss–Borel factorization of the moments matrices is

\[
M^L = (S_1^L)^{-1}H^L(S_2^L)^{-1}, \quad M^R = (S_1^R)^{-1}H^R(S_2^R)^{-1},
\]

where \( S_1^L, S_2^L, S_1^R, S_2^R \) are lower unitriangular block semi-infinite matrices and \( H^L \) and \( H^R \) are block diagonal matrices.

**Proposition 6.** The Gauss–Borel factorization can be performed when the the matrix of measures \( \mu \) is quasi-definite.

**Proof.** Follows the proof of Proposition 1 of [18] by replacing the moments matrices there by our moment matrices. \(\blacksquare\)

**Definition 7.** We introduce the semi-infinite vector of monomials

\[
\chi(z) = \begin{pmatrix} I_N \\ I_N z \\ I_N z^2 \\ \vdots \end{pmatrix},
\]

and the semi-infinite vectors of polynomials

\[
P_1^L(z) := S_1^L \chi(z), \quad P_2^L(z) := S_1^L \chi(z), \quad (P_1^R(z))^\top := (\chi(z))^\top (S_1^R)^\dag, \quad (P_2^R(z))^\top := (\chi(z))^\top (S_2^R)^\dag.
\]

**Proposition 7.** The moment matrices can be written as follows

\[
M^L = \int_T \chi(\zeta) d\mu(\zeta)(\chi(\zeta))^\dag, \quad M^R = \int_T ((\chi(\zeta))^\top)^\dag d\mu(\zeta)(\chi(\zeta))^\top.
\]
Proposition 8. We have the biorthogonality relations

\begin{align}
(8) \quad & \int_T \mu(\zeta) \, \mu(\zeta) \, dH_L = H_L, \\
(9) \quad & \int_T (\mu(\zeta))^\dagger \, \mu(\zeta) \, dH_R = H_R.
\end{align}

Proof. To prove (8) we just notice that

\begin{align*}
\int_T \mu(\zeta) \, \mu(\zeta) \, dH_L &= S^L_1 \int_T \chi(\zeta) \, \mu(\zeta) \, dH_L \dagger \dagger \\
&= S^L_1 M^L(S^L_2) \dagger \\
&= H_L \\
\text{use (7).}
\end{align*}

Now for (9)

\begin{align*}
\int_T (\mu(\zeta))^\dagger \, \mu(\zeta) \, dH_R &= S^R_2 \int_T (\chi(\zeta))^\dagger \, \mu(\zeta) \, dH_R \dagger \\
&= S^R_2 M^R(S^R_1) \dagger \\
&= H_R \\
\text{use (7).}
\end{align*}

\[ \square \]

Proposition 9. The components $P^L_{1,n}, P^R_{1,n}, P^L_{2,n}$ and $P^R_{2,n}$ of the semi-infinite vectors $P^L_1, P^R_1, P^L_2$ and $P^R_2$

(1) Satisfy the biorthogonality relations

\begin{align}
(10) \quad & \int_T \mu(\zeta) \, dH_L s_{m,n} = H_L \delta_{m,n}, \\
& \int_T (\mu(\zeta))^\dagger \, dH_R p_{m,n} = H_R \delta_{m,n}.
\end{align}

(2) The components $P^L_{1,n}, P^R_{1,n}, P^L_{2,n}$ and $P^R_{2,n}$ of the semi-infinite vectors $P^L_1, P^R_1, P^L_2$ and $P^R_2$ are the Szegö matrix polynomials of Definition 1.

(3) The Szegö polynomials and its reciprocals satisfy

\begin{align}
(11) \quad & \int_T \mu(\zeta) \, dH_L s_{m,n} = H_L \delta_{m,n}, \\
& \int_T (\mu(\zeta))^\dagger \, dH_R p_{m,n} = H_R \delta_{m,n}.
\end{align}

(4) The quasi-tau functions can be expressed as

\begin{align}
(12) \quad & H^L_L = \int_T \mu(\zeta) \, dH_L \zeta^n \\
& = \int_T \mu(\zeta) \, dH_L P^L_{2,n}(\zeta), \\
(13) \quad & H^R_R = \int_T \mu(\zeta) \, dH_R \zeta^n \\
& = \int_T \mu(\zeta) \, dH_R P^R_{2,n}(\zeta).
\end{align}

Proof. (1) Elementary.

(2) Observe that (10) implies the orthogonal relations (11), (12), (13) and (14).

(3) Use Definition 2.

(4) It follows from (10) and (11).

\[ \square \]

Definition 8. The matrices $H^L_n$ and $H^R_n$ are called quasi-tau matrices.
Proposition 10. The quasi-tau matrices must be not singular and have the following last quasi-determinantal expressions

\[
H_n^L = \Theta_* \begin{pmatrix}
\hat{\mu}(0) & \hat{\mu}(-1) & \hat{\mu}(-2) & \cdots & \hat{\mu}(-n + 1) \\
\hat{\mu}(1) & \hat{\mu}(0) & \hat{\mu}(-1) & \cdots & \hat{\mu}(-n + 2) \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\hat{\mu}(n-1) & \hat{\mu}(n-2) & \hat{\mu}(n-3) & \cdots & \hat{\mu}(0)
\end{pmatrix},
\]

\[
H_n^R = \Theta_* \begin{pmatrix}
\hat{\mu}(0) & \hat{\mu}(1) & \hat{\mu}(2) & \cdots & \hat{\mu}(n-1) \\
\hat{\mu}(-1) & \hat{\mu}(0) & \hat{\mu}(1) & \cdots & \hat{\mu}(n-2) \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\hat{\mu}(-2) & \hat{\mu}(-1) & \hat{\mu}(0) & \cdots & \hat{\mu}(n-3) \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\hat{\mu}(-n+1) & \hat{\mu}(-n+2) & \hat{\mu}(-n+3) & \cdots & \hat{\mu}(0)
\end{pmatrix}.
\]

Proof. It follows from the Gaussian factorization, see [101, 18, 28].

2.3. Symmetry properties.

Proposition 11. Assume that there is a matrix \( C \in \mathbb{C}^{N \times N} \) such that

\[
[C, \mu] = 0.
\]

Then, the matrix \( C \) commute with all the moments \( \hat{\mu}(n) \), i.e.,

\[
[C, \hat{\mu}(n)] = 0, \quad n \in \mathbb{Z},
\]

and with the Szegő matrix polynomials, reciprocals, corresponding Verblunsky coefficients and quasi-tau matrices

\[
[C, P_{1,n}(z)] = [C, P_{1,n}^R(z)] = [C, \tilde{P}_{2,n}(z)] = [C, \tilde{P}_{2,n}^R(z)] = 0,
\]

\[
[C, \alpha_{1,n}^L] = [C, \alpha_{1,n}^R] = [C, (\alpha_{2,n}^L)^\dagger] = [C, (\alpha_{2,n}^R)^\dagger] = 0,
\]

\[
[C, H_n^L] = [C, H_n^R].
\]

for all \( n \in \{0, 1, 2, \ldots\} \) and \( z \in \mathbb{C} \).

Proof. It is obvious that \( C \) commutes with the moments. Then, as Proposition 4 ensures that Szegő matrix polynomials, \( P_{1,n}^L(z) \) and \( P_{1,n}^R(z) \) are rational functions of the moments, and Proposition 5 ensures the same for the reciprocal polynomials \( \tilde{P}_{2,n}^L(z) \) and \( \tilde{P}_{2,n}^R(z) \), we see that \( C \) commutes with them. The result for the Verblunsky coefficients follow immediately. The property regarding the quasi-tau matrices is deduced from Proposition 10.

Proposition 12. Suppose that for each pair of Borel sets \( A, B \subset \mathbb{T} \) we have

\[
[\mu(A), \mu(B)] = 0.
\]

Then, the set of the moments \( \mathbb{C}[\hat{\mu}(n)]_{n \in \mathbb{Z}} \) is an Abelian algebra

\[
[\hat{\mu}(i), \hat{\mu}(j)] = 0, \quad i, j \in \mathbb{Z}.
\]

Moreover, the family of matrix polynomials

\[
\{ P_{1,n}^L(z_1), P_{1,n}^R(z_2), \tilde{P}_{2,n}^L(z_3), \tilde{P}_{2,n}^R(z_4) \}_{n \in \{0, 1, 2, \ldots\}, z_1, z_2, z_3, z_4 \in \mathbb{C}}
\]

is Abelian. Analogously, the set of Verblunsky matrices and quasi-tau matrices

\[
\{ \alpha_{1,n}^L, \alpha_{1,n}^R (\alpha_{2,n}^L)^\dagger, (\alpha_{2,n}^R)^\dagger, H_n^L, H_n^R \}_{n=0}^\infty
\]

is Abelian.
Properties of the Cauchy transform

Proof. For simplicity we give the proof for the absolutely continuous case, i.e., we assume that \( \mu = w \, d\, m \) with

\[
[w(u), w(v)] = 0, \quad \forall u, v \in \text{supp}(w(z)) \subset \mathbb{T}.
\]

First, we see that the moments commute among then. Indeed, according to Definition 4

\[
[\hat{\mu}(i), \hat{\mu}(j)] = \left[ \int_{\mathbb{T}} w(u)u^{-i} \frac{d\, u}{2\pi i} , \int_{\mathbb{T}} w(v)v^{-j} \frac{d\, v}{2\pi i} \right] = -\int_{\mathbb{T}^2} [w(u), w(v)]u^{-i}v^{-j} \frac{d\, u \, d\, v}{4\pi^2 uv} = 0.
\]

As in the previous proof Proposition 4 ensures that \( P^L_{1,n}(z) \) and \( P^R_{1,n}(z) \) are rational functions of the moments, and Proposition 5 the same for the reciprocal polynomials \( \tilde{P}^L_{2,n}(z) \) and \( \tilde{P}^R_{2,n}(z) \), and the commutativity property follows. The result for the Verblunsky coefficients follow immediately. From Proposition 10 we deduce the commutativity with the quasi-tau matrices.

3. THE RIEMANN-HILBERT PROBLEM

In this section, following the seminal paper [58] we find a general Riemann-Hilbert problem whose solution characterizes matrix Szegő polynomials. This problem constitute the keystone for the finding of the matrix discrete Painlevé II system.

3.1. Cauchy transforms. We began we some facts regarding Cauchy transforms for matrices of measures on the unit circle, we follow the excellent monograph [33].

Definition 9. Given a finite Borel matrix of measures \( \mu \) on the unit circle \( \mathbb{T} \) its Cauchy transform is defined by

\[
(C\mu)(z) := \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1 - \zeta z},
\]

where the integration along \( \mathbb{T} \) is taken counterclockwise. We denote by \((C\mu)_+\) the restriction to the unit disk \( \mathbb{D} \) and by \((C\mu)_-\) the restriction to the annulus \( \mathbb{D} \).

Let us review some properties of the Cauchy transform that are relevant for this paper.

Properties of the Cauchy transform.

1. The Cauchy transform \((C\mu)(z)\) is analytic on \( \bar{\mathbb{C}} \setminus \mathbb{T} \), where \( \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \), with analytic continuation across the complement of supp(\(\mu\)).

2. The restriction of the Cauchy transform \((C\mu)_- \in \cap_{0 < p < 1} H^p(\mathbb{D})\), where \( H^p(\mathbb{D}) \) is the Hardy space of the exterior of the unit circle [4].

3. We have \( \| (C\mu)_+ \|_{H^p(\mathbb{D})} = O((1 - p)^{-1}) \) for \( p \to 1^- \), and consequently has exterior non-tangential limits almost everywhere on \( \mathbb{T} \).

4. The Taylor series of \((C\mu)_+\) at \( z = 0 \) is

\[
(C\mu)_+(z) = \sum_{n=0}^{\infty} \hat{\mu}(n)z^n, \quad \forall z \in \mathbb{D},
\]

and of \((C\mu)_-\) about the point of infinity is

\[
(C\mu)_- = -\sum_{n=1}^{\infty} \hat{\mu}(-n)z^{-n}, \quad \forall z \in \mathbb{D}.
\]

5. For almost every \( \zeta \in \mathbb{T} \) the limits

\[
(C\mu)_+(\zeta) = \lim_{r \to 1^-} (C\mu)_+(r \zeta), \quad (C\mu)_-(\zeta) = \lim_{r \to 1^-} (C\mu)_+(\zeta/r),
\]

exist.

\footnote{We understand that \( f(z) \in H^p(\mathbb{D}) \) iff \( f(z^{-1}) \in H^p(\mathbb{D}) \). The Hardy space \( H^p(\mathbb{D}) \) contains all analytic functions \( f(z) \) on \( \mathbb{D} \) such that \( \sup_{0<r<1} ( \int_{\partial \mathbb{D}} |f(r\zeta)|^p d\, m(\zeta) )^{\frac{1}{p}} < \infty \).}
(6) The Fatou’s jump

\[(C\mu)_+(\zeta) - (C\mu)_-(\zeta) = \frac{d\mu}{dm}(\zeta)\]

holds m-almost every \(\zeta \in \mathbb{T}\).

(7) Privalov’s theorem: for m-almost every \(\zeta \in \mathbb{T}\) we have

\[\text{PV} \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1 - \zeta \xi} = \frac{1}{2}((C\mu)_+(\zeta) + (C\mu)_-(\zeta)).\]

Here the principal value at \(\zeta \in \mathbb{T}\) is defined as

\[\text{PV} \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1 - \zeta \xi} := \lim_{\varepsilon \to 0^+} \int_{|\zeta - \xi| > \varepsilon} \frac{d\mu(\xi)}{1 - \zeta \xi},\]

whenever the limit exists.

The results of Privalov and Fatou are extensions of the Sokhotski–Plemelj formulas, first discussed by Sokhotski in 1873 for \(d\mu(\zeta) = w(\zeta) \, dm(\zeta)\) where \(w(\zeta)\) is a Lipschitz function and then refined by Plemelj in 1908. Privalov results goes back to 1919. We are now prepared to introduce the Cauchy transform according to the matrix of measures \(\mu\) of the Szegő polynomials and their reciprocals.

**Definition 10** (Cauchy transforms). **We consider the following matrix Cauchy transforms of the Szegő matrix polynomials**

\[
Q_{1,n}^L(z) := \int_{\mathbb{T}} \frac{\zeta^n p_{1,n}(\zeta)}{1 - \zeta \xi} d\mu(\zeta), \quad Q_{1,n}^R(z) := \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1 - \zeta \xi} p_{1,n}(\zeta) \xi^n,
\]

\[
Q_{2,n}^L(z) := \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1 - \zeta \xi} \hat{p}_{2,n}(\zeta) \xi^{n+1}, \quad Q_{2,n}^R(z) := \int_{\mathbb{T}} \frac{\hat{p}_{2,n}(\zeta)}{1 - \zeta \xi} \frac{d\mu(\zeta)}{\xi^n}.
\]

Observe that

\[
Q_{1,0}^L(z) = Q_{1,0}^R(z) = \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1 - \zeta z} = \begin{cases} \sum_{n=0}^{\infty} \hat{\mu}(n) z^n, & z \in \mathbb{D}, \\ -\sum_{n=1}^{\infty} \hat{\mu}(-n) z^{-n}, & z \in \overline{\mathbb{D}}. \end{cases}
\]

in terms of the Fourier coefficients\(^2\) Now, observing that

\[
\frac{\zeta}{1 - \zeta z} = z^{-1} \left( \frac{1}{1 - \zeta z} - 1 \right)
\]

we get

\[
Q_{2,0}^L(z) = Q_{2,0}^R(z) = \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1 - \zeta z} \xi = z^{-1} \left( \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1 - \zeta z} - \mu(\mathbb{T}) \right).
\]

\(^2\)With respect to the Lebesgue measure \(m(z)\).

Notice that for m-almost every \(\zeta \in \mathbb{T}\) the Fatou’s jump take place

\[
\frac{d\mu}{dm}(\zeta) = \sum_{n=-\infty}^{\infty} \hat{\mu}(n) \zeta^n,
\]

which is the Fourier series of the measure, and the Privalov’s principal value holds:

\[\text{PV} \int_{\mathbb{T}} \frac{d\mu(\zeta)}{1 - \zeta z} = \frac{1}{2} \left( \sum_{n=0}^{\infty} \hat{\mu}(n) z^n - \sum_{n=1}^{\infty} \hat{\mu}(-n) z^{-n} \right).\]
Definition 11. We will use the following Fourier coefficients
\[
\hat{\rho}_1^{L}(j) := \int_{\mathbb{T}} \rho_1^{L}(\zeta) \, d\mu(\zeta) \zeta^{-j}, \quad \hat{\rho}_2^{L}(j) := \int_{\mathbb{T}} \zeta^{-j+1} \, d\mu(\zeta) \hat{\rho}_2^{L}(\zeta),
\]
\[
\hat{\rho}_1^{R}(j) := \int_{\mathbb{T}} \rho_1^{R}(\zeta) \, d\mu(\zeta) \zeta^{-j}, \quad \hat{\rho}_2^{R}(j) := \int_{\mathbb{T}} \zeta^{-j+1} \, d\mu(\zeta) \hat{\rho}_2^{R}(\zeta),
\]
for \( j \in \mathbb{Z} \).

Proposition 13. (1) We have the following cancellations
\[
\hat{\rho}_1^{L}(j) = \hat{\rho}_2^{L}(j) = \hat{\rho}_1^{R}(j) = \hat{\rho}_2^{R}(j) = 0, \quad j \in \{0, -1, \ldots, -n + 1\}.
\]
(2) The quasi-tau matrices are
\[
H_n^L = \hat{\rho}_1^{L}(-n) = \hat{\rho}_2^{L}(1), \quad H_n^R = \hat{\rho}_1^{R}(-n) = \hat{\rho}_2^{R}(1).
\]

Proof. (1) Observe that (1), (2), (5) and (6) are equivalent to these cancellations.
(2) A simple consequence of (14), (12), (13) and (15).

Proposition 14 (Power series of the Cauchy transforms). The following Taylor series of the Cauchy transforms
\[
Q_{1,n}^{L}(z) = H_n^L + \sum_{j=1}^{\infty} \hat{\rho}_1^{L}(-n-j)z^j, \quad Q_{2,n}^{L}(z) = \sum_{j=0}^{\infty} \hat{\rho}_2^{L}(-n-j)z^j,
\]
\[
Q_{1,n}^{R}(z) = H_n^R + \sum_{j=1}^{\infty} \hat{\rho}_1^{R}(-n-j)z^j, \quad Q_{2,n}^{R}(z) = \sum_{j=0}^{\infty} \hat{\rho}_2^{R}(-n-j)z^j,
\]
converge on \( \mathbb{D} \). The following Taylor series about infinity
\[
Q_{1,n}^{L}(z) = -\sum_{j=1}^{\infty} \hat{\rho}_1^{L}(j)z^{-n-j}, \quad Q_{2,n}^{L}(z) = -H_n^Lz^{-n-1} - \sum_{j=2}^{\infty} \hat{\rho}_2^{L}(j)z^{-n-j},
\]
\[
Q_{1,n}^{R}(z) = -\sum_{j=1}^{\infty} \hat{\rho}_1^{R}(j)z^{-n-j}, \quad Q_{2,n}^{R}(z) = -H_n^Rz^{-n-1} - \sum_{j=2}^{\infty} \hat{\rho}_2^{R}(j)z^{-n-j},
\]
converge in \( \mathbb{D} \).

Proof. First we recall that
\[
\xi^n \frac{1}{1 - \zeta z} = \xi^n + z\xi^{n+1} + z^2\xi^{n+2} + z^3\xi^{n+3} + \cdots, \quad |z| < 1, \quad \zeta \in \mathbb{T},
\]
uniformly, from where we get the Taylor series in the unit disk.

For the Taylor expansions at infinity we observe that
\[
\xi^n \frac{1}{1 - \zeta z} = -\xi^{n-1}z^{-1} \frac{1}{1 - z^{-1}\xi^{-1}}
\]
\[= -z^{-1}\xi^{-n} - z^{-2}\xi^{-n-2} - z^{-3}\xi^{-n-3} - \cdots, \quad |z| > 1, \quad \zeta \in \mathbb{T},
\]
uniformly. Then,
\[
Q_{1,n}^{L}(z) := -\sum_{j=1}^{\infty} z^{-j} \int_{\mathbb{T}} \rho_1^{L}(\zeta) \, d\mu(\zeta) \xi^{n-j}, \quad Q_{2,n}^{L}(z) := -\sum_{j=1}^{\infty} z^{-j} \int_{\mathbb{T}} \xi^{n-j+1} \, d\mu(\zeta) \hat{\rho}_2^{L}(\zeta),
\]
\[
Q_{1,n}^{R}(z) := -\sum_{j=1}^{\infty} z^{-j} \int_{\mathbb{T}} \rho_1^{R}(\zeta) \, d\mu(\zeta) \xi^{n-j}, \quad Q_{2,n}^{R}(z) := -\sum_{j=1}^{\infty} z^{-j} \int_{\mathbb{T}} \xi^{n-j+1} \, d\mu(\zeta) \hat{\rho}_2^{R}(\zeta),
\]
and consequently, recalling the orthogonal relations (1), (2), (5) and (6), we get
\[
Q_{1,n}^L(z) := - \sum_{j=n+1}^{\infty} z^{-j} \int T \xi_j^{n-j} d \mu(\xi) \xi^{n-j}, \quad Q_{2,n}^L(z) := - \sum_{j=n+1}^{\infty} z^{-j} \int T \xi_j^{n-j+1} d \mu(\xi) \xi^{n-j+1},
\]
\[
Q_{1,n}^R(z) := - \sum_{j=n+1}^{\infty} z^{-j} \int T \xi_j^{n-j} d \mu(\xi) \xi^{n-j}, \quad Q_{2,n}^R(z) := - \sum_{j=n+1}^{\infty} z^{-j} \int T \xi_j^{n-j+1} d \mu(\xi) \xi^{n-j+1},
\]
and the result follows.

**Proposition 15** (Fatou’s jump formulæ). The Cauchy transforms have the following jumps m-almost every \( \zeta \in \mathbb{T} \)
\[
(Q_{1,n}^L(\zeta) - (Q_{1,n}^R(\zeta)), (Q_{2,n}^L(\zeta) - (Q_{2,n}^R(\zeta)) = \frac{d \mu}{d m}(\zeta) \xi^{n-j}.
\]

3.2. **Riemann–Hilbert problems.** Inspired by the seminal paper [58] and following the implementation for scalar Szegő polynomials given in [20], see also [90, 91], here we propose a block type Riemann–Hilbert problem in the unit circle and find its solution in terms of matrix Szegő polynomials. We start with a general situation, the weak problem, with integrable jump functions and then move to the more classical scenario, the strong problem, with Hölder jump functions.

**Definition 12** (Weak and strong Riemann–Hilbert problems). The Riemann–Hilbert problem (RHP) consists in the finding of a 2N \times 2N matrix function, \( Y_n(z) \in \mathbb{C}^{2N \times 2N} \), for a given \( n \in \{0, 1, 2, \ldots \} \), such that

1. **Is analytic in** \( \mathbb{C} \setminus \mathbb{T} \).
2. **Satisfies the jump condition at** \( \zeta \in \mathbb{T} \)

\[
(Y_n)_+ (\zeta) - (Y_n)_- (\zeta) = \left( \begin{array}{cc} I_N & w(\zeta) \xi^{-n} \\ 0_N & I_N \end{array} \right),
\]

where \( w = (w_{1,j}) \) is a matrix of weights. In the weak RHP we have \( w_{1,j} \in L^1 \) and request the jump to hold m-almost every \( \zeta \in \mathbb{T} \). In the strong RHP the weights \( w_{1,j} \) are Hölder and the jump must hold for all \( \zeta \in \mathbb{T} \).
3. **About infinity has the following asymptotic**

\[
Y_n(z) = (I_{2N} + O(z^{-1})) \left( \begin{array}{cc} I_{Nz}^n & 0_N \\ 0_N & I_{Nz}^{-n} \end{array} \right), \quad |z| \to \infty.
\]

We have the following result

**Theorem 1.** A solution of the Riemann–Hilbert problem stated in Definition 12 is given by the block matrix function

\[
Y_n(z) := \left( \begin{array}{cc} P_{1,n}^l(z) & Q_{1,n}^l(z) \\ - (H_{n-1}^{-1})^{-1} P_{2,n-1}^r(z) & - (H_{n-1}^{-1})^{-1} Q_{2,n-1}^r(z) \end{array} \right), \quad n \in \{1, 2, \ldots \},
\]

and

\[
Y_0(z) := \left( \begin{array}{cc} I_N & Q_{1,0}^l(z) \\ 0_N & I_N \end{array} \right).
\]

For the weak RH problem the measure \( \mu \), involved in the Szegő polynomials and the Cauchy transform, is taken such that its Radon–Nikodym derivative is the matrix of weights \( w = \frac{d \mu}{d m} \). For the strong RH problem the measure is \( d \mu = w d m \) and the solution is unique.

**Proof.** Let us show that the matrix given in (17) is a solution of the RHP. We do it in three steps:

1. The polynomials \( P_{1,n}^l(z) \) and \( P_{2,n-1}^r(z) \) are analytic functions in \( \mathbb{C} \) and the Cauchy transforms \( Q_{1,n}^l(z) \) and \( Q_{2,n-1}^r(z) \) are analytic in \( \mathbb{C} \setminus \mathbb{T} \). Therefore, \( Y_n(z) \) is analytic in \( \mathbb{C} \setminus \mathbb{T} \).

A function \( f : \mathbb{T} \to \mathbb{C} \) is said Hölder if \( |f(\zeta_1) - f(\zeta_2)| \leq M|\zeta_1 - \zeta_2|^\alpha \) for \( M > 0 \) and \( 0 < \alpha \leq 1 \), see [96].
(2) From Proposition [44] we can deduce the following asymptotics
\[ Y_n(z) = \left( I_n z^n + O(z^{n-1}) \right) \begin{pmatrix} O(z^{-n-1}) \\ I_n z^{-n} + O(z^{-n-1}) \end{pmatrix} = (I_2 n + O(z^{-1})) \begin{pmatrix} I_n z^n & 0_n \\ 0_N & I_n z^{-n} \end{pmatrix}. \]

(3) For the weak RHP, Proposition [15] gives
\[
(Y_n)_+(\zeta) - (Y_n)_-(\zeta) = \begin{pmatrix} \left( p_{1,n}^L \right)_+ (\zeta) - \left( p_{1,n}^L \right)_- (\zeta) \\ \left( H_{n-1}^R \right)^{-1} \left( \left( p_{2,n-1}^R \right)_+ (\zeta) - \left( p_{2,n-1}^R \right)_- (\zeta) \right) \end{pmatrix} = \begin{pmatrix} \left( Q_{1,n}^L \right)_+ (\zeta) - \left( Q_{1,n}^L \right)_- (\zeta) \\ \left( H_{n-1}^R \right)^{-1} \left( \left( Q_{2,n-1}^R \right)_+ (\zeta) - \left( Q_{2,n-1}^R \right)_- (\zeta) \right) \end{pmatrix}
\]
\[ = \begin{pmatrix} 0_N & p_{1,n}^L(\zeta) w(\zeta) \zeta^n \\ 0_N & \left( H_{n-1}^R \right)^{-1} p_{2,n-1}^R(\zeta) w(\zeta) \zeta^n \end{pmatrix} = (Y_n)_-(\zeta) \begin{pmatrix} 0_N & w(\zeta) \zeta^n \\ 0_N & 0_N \end{pmatrix}, \]
for \( m \)-almost every \( \zeta \in \mathbb{T} \). When we consider the strong RH situation then the matrix of weights is Hölder and the Sokhotski–Plemelj holds for every \( \zeta \in \mathbb{T} \), see [64].

Once we have proven the existence of a solution to the Riemann–Hilbert problem, let us show its uniqueness for the strong RHP. We notice that \( \det Y_n(z) \) is an analytic function in \( \mathbb{C} \setminus \mathbb{T} \), and has jump at \( \zeta \in \mathbb{T} \), indeed
\[
\det(Y_n)_+(\zeta) = \det(Y_n)_-(\zeta) \begin{vmatrix} I_n & w(\zeta) \zeta^n \\ 0_N & I_n \end{vmatrix} = \det(Y_n)_-(\zeta).
\]

Therefore, \( \det Y_n(z) \) is analytic in \( \mathbb{C} \), and the Liouville theorem ensures that is constant, but \( \det Y_n \to 1 \) as \( |z| \to \infty \), consequently, we have that \( \det Y_n(z) = 1 \). Thus, \( (Y_n(z))^{-1} \) is a matrix of analytic functions for all \( z \in \mathbb{C} \setminus \mathbb{T} \). Given two solutions \( \bar{Y}_n, Y_n \) to the Riemann–Hilbert problem, the block matrix \( \bar{Y}_n(Y_n)^{-1} \) has no jump at \( \zeta \in \mathbb{T} \), and therefore is analytic in the whole complex plane, thus is a constant matrix, and the asymptotic implies that this constant is the identity. \( \square \)

As we have seen, the non uniqueness in the weak case is related to the weak Fatou jump corollary, that holds only almost everywhere in the circle, allowing therefore for singularities. Moreover, if the weak situation the matrix of \( L^1(\mathbb{T}, \mu) \) weights only fixes the absolutely continuous part of the measure \( \mu_a = w \ d m \ll m \), and we have the freedom of adding any singular measure \( \mu_s \perp m \) as \( D \mu_s = 0 \). Therefore, given \( Y_n \) constructed for \( w \ d m \) we may consider \( \bar{Y}_n \) associated with \( w \ d m + \mu_s \) and we will have another solution to the weak RHP. For the strong RHP we refer the reader to [64], observe that in this situation the Lebesgue integration coincides with the Riemann integration.

From hereon we consider only the strong RH problem.

**Definition 13.** We define the block matrix
\[
X_n(z) := \begin{pmatrix} p_{1,n}^L(\zeta) \\ -\left( H_{n-1}^R \right)^{-1} p_{2,n-1}^R(\zeta) \end{pmatrix} \begin{pmatrix} Q_{1,n}^L(\zeta) \\ -\left( H_{n-1}^R \right)^{-1} Q_{2,n-1}^R(\zeta) \end{pmatrix} \begin{pmatrix} I_n z^{-n} & 0_N \\ 0_N & I_n z^n \end{pmatrix}, \quad n \in \{1, 2, \ldots\},
\]
and
\[
X_0(z) := \begin{pmatrix} I_n & Q_{1,0}^L(\zeta) \\ 0_N & I_n \end{pmatrix}.
\]

Observe that \( X_0(z) = Y_0(z) \).

**Proposition 16.** For each \( n \in \{0, 1, 2, \ldots\} \), \( X_n(z) \) is the unique matrix function such that
1. \( X_n(z) \begin{pmatrix} I_n z^n \\ 0_N \end{pmatrix} \) is analytic in \( \mathbb{C} \setminus \mathbb{T} \).

\footnote{Incidentally, we observe that in the weak RHP we only know that is analytic everywhere but for a Borel set \( B \subset \mathbb{T} \) of zero Lebesgue measure, \( m(B) = 0 \), and we can not apply the Liouville theorem.}
(2) Satisfies the jump condition for \( \zeta \in \mathbb{T} \)

\[
(X_n)_{+}(\zeta) = (X_n)_{-}(\zeta) \left( \begin{array}{cc}
I_N & w(\zeta)\bar{\zeta}^{-n} \\
0_N & I_N
\end{array} \right).
\]

(3) Asymptotically behaves as \( X_n(z) = I_{2N} + O(z^{-1}) \) for \( |z| \to \infty \).

**Proposition 17 (Series for \( X_n \)).**

(1) The following Laurent series

\[
X_n(z) = \begin{pmatrix}
\alpha_{1,n} & 0_N \\
-(H^n_{n-1})^{-1}z^{-n-1} & 0_N
\end{pmatrix}
+ \begin{pmatrix}
\frac{p_l}{(H^n_{n-1})^{-1}(a^n_{2,n-1})} & 0_N \\
0_N & (H^n_{n-1})^{-1}(a^n_{2,n-1})
\end{pmatrix} z^{-1} + \begin{pmatrix}
I_N & 0_N \\
0_N & I_N
\end{pmatrix}
\]

converges in the annulus \( \mathbb{D} \setminus \{0\} \).

(2) The following Taylor series about infinity

\[
X_n(z) = I_{2N} + \begin{pmatrix}
\frac{p_l}{(H^n_{n-1})^{-1}z^{-n-1}} & 0_N \\
-(H^n_{n-1})^{-1} & 0_N
\end{pmatrix} z^{-1} + \begin{pmatrix}
I_N & 0_N \\
0_N & I_N
\end{pmatrix}
\]

converges at \( \mathbb{D} \).

We see that the Laurent expansion at the origin is rather peculiar in its block structure.

(1) The matrix \([X_n(z)]_{\text{principal}} = (X_n(z) - I_{2N})(I_N 0_N 0_N 0_N)\) is the principal part of the Laurent series of \( X_N \) at \( z = 0 \). Thus, in the principal part the second block column cancels:

\[
[X_n(z)]_{\text{principal}} = \begin{pmatrix}
z^{-n}p_l & 0_N \\
-(H^n_{n-1})^{-1} & 0_N
\end{pmatrix}.
\]

(2) The regular part of the Laurent series is \([X_n(z)]_{\text{regular}} = (I_N 0_N 0_N 0_N) + X_n(z)(0_N 0_N 0_N)\). Hence, the regular part \([X_n(z)]_{\text{regular}} - (I_N 0_N 0_N 0_N)\) has a zero first block column

\[
[X_n(z)]_{\text{regular}} = \begin{pmatrix}
I_N & 0_N \\
0_N & z^n Q_{1,n}(z)
\end{pmatrix}.
\]

**Definition 14.** We write the Taylor series of \( X_n(z) \) about infinity, which converges for \( z \in \mathbb{D} \), as follows

\[
X_n(z) = I_{2N} + X_n^{(1)} z^{-1} + X_n^{(2)} z^{-2} + \ldots
\]

where

\[
X_n^{(i)} := \begin{pmatrix}
a_n^{(i)} & b_n^{(i)} \\
c_n^{(i)} & d_n^{(i)}
\end{pmatrix}, \quad i \geq 0,
\]

\( a_n^{(i)}, b_n^{(i)}, c_n^{(i)}, d_n^{(i)} \in \mathbb{C}^{N \times N} \).

For \( n = 1 \) we use the simplified notation \( X_n^{(1)} = \begin{pmatrix} a_n^{(i)} b_n^{(i)} \\ c_n^{(i)} d_n^{(i)} \end{pmatrix} \).

Then, the matrix function \( Y_n(Z) \) for \( z \in \mathbb{D} \) can be expressed as

\[
Y_n(z) = \begin{pmatrix}
z^n I_N + a_n z^{-n-1} + O(z^{-n-2}) & b_n z^{-n-1} + b_n^{(2)} z^{-n-2} + O(z^{-n-3}) \\
0_N z^{-n-1} + c_n z^{-n-2} + O(z^{-n-3}) & z^{-n} I_N + d_n z^{-n-1} + O(z^{-n-2})
\end{pmatrix}.
\]

Where—recall (17), (18) and (20)—we have

\[
a_n^{(i)} = c_n^{(i)} = 0_N, \quad i > n,
\]

\[
d_0^{(i)} = 0_N, \quad i \geq 1.
\]
Then, Proposition 17 implies
\[
\begin{align*}
    a_n &= p_{1,n,n-1}^L, & b_n &= -\hat{p}_{1,n}^L(1), \\
    c_n &= -(H_{n-1}^R)^{-1}(\alpha_{2,n-1}^R)^\dagger, & d_n &= (H_{n-1}^R)^{-1}\hat{p}_{2,n-1}^R(2), \\
    a_n^{(2)} &= p_{1,n,n-2}^L, & b_n^{(2)} &= -\hat{p}_{1,n-1}^L(2), \\
    c_n^{(2)} &= -(H_{n-1}^R)^{-1}(\beta_{2,n-1,1}^R)^\dagger, & d_n^{(2)} &= (H_{n-1}^R)^{-1}\hat{p}_{2,n-1}^R(3).
\end{align*}
\]

4. Recursion relations and systems of matrix linear ordinary differential equations

From hereon we will assume that the matrix of weights \( w : \mathbb{T} \to \mathbb{C}^{N \times N} \) has an analytic extension to the annulus \( \mathbb{C} \setminus \{0\} \) with an analytic inverse on this annulus. This is, indeed, a strong assumption but one can imagine many examples fitting in this family, for example weights of Freud type of the form
\[
w(\zeta) = \prod_{i=1}^{M} \exp(V_i(\zeta)),
\]
where \( V_i(\zeta), i \in \{1, \ldots, M\}, \) are matrix Laurent polynomials in \( \zeta \in \mathbb{T}. \) For example \( w(\zeta) = \exp(V\zeta) \exp(V^\dagger \zeta) \) will be of this type and moreover Hermitian. This example, in the scalar case \( N = 1, \) is called in [77] modified Bessel, as its moment are connected with the modified Bessel functions.

Another scenario is to have
\[
w(z) = z^{-m} v(z)
\]
m \( \in \{0, 1, 2, \ldots\}, \) where \( v(z) \) is analytic in \( \mathbb{C} \) with Taylor series
\[
v(z) = v_0 + v_1 z + \cdots,
\]
where \( v_0 \) is non singular.

**Definition 15.** We consider
\[
Z_n(z) := \begin{pmatrix}
p_{1,n}^L(z) & Q_{1,n}^L(z) \\
-(H_{n-1}^R)^{-1}\hat{p}_{2,n-1}^R(z) & -(H_{n-1}^R)^{-1}Q_{2,n-1}^R(z)
\end{pmatrix}
\begin{pmatrix}
w(z)z^{-n} & 0_N \\
0_N & I_N
\end{pmatrix}, \quad n \in \{1, 2, \ldots\}
\]
and
\[
Z_0(z) := \begin{pmatrix}
w(z) & Q_{1,0}^L(z) \\
0_N & I_N
\end{pmatrix}.
\]

**Proposition 18.** For \( \zeta \in \mathbb{T} \) and for each \( n \in \{0, 1, 2, \ldots\} \) the block matrix function \( Z_n(z) \) satisfies the following jump condition \( f \)
\[
(Z_n)_+ (\zeta) = (Z_n)_- (\zeta) \begin{pmatrix}
I_N & I_N \\
0_N & I_N
\end{pmatrix}.
\]

Proof. For \( \zeta \in \mathbb{T} \) we have
\[
(Z_n)_+ (\zeta) = (Y_n)_+ (\zeta) \begin{pmatrix}
w(\zeta)\xi^n & 0_N \\
0_N & I_N
\end{pmatrix} = (Y_n)_- (\zeta) \begin{pmatrix}
I_N & w(\zeta)\xi^n \\
0_N & I_N
\end{pmatrix} = (Y_n)_- (\zeta) \begin{pmatrix}
w(\zeta)\xi^n & 0_N \\
0_N & I_N
\end{pmatrix} = (Y_n)_- (\zeta) \begin{pmatrix}
w(\zeta)\xi^n & I_N \\
0_N & I_N
\end{pmatrix} = (Z_n)_- (\zeta) \begin{pmatrix}
I_N & I_N \\
0_N & I_N
\end{pmatrix}.
\]

It is remarkable that the jump condition is now is expressed in terms of a constant matrix. The block matrix \( Z_n(z) \) can also be regarded as the solution of a Riemann–Hilbert problem. Precisely, the following simple result holds.

\[\text{\textsuperscript{6}}\text{Consequently, the right logarithmic derivative } \frac{d}{dz} (w(z))^{-1} \text{ is also analytic in the annulus } \mathbb{C} \setminus \{0\}.\]
**Proposition 19.** The block matrix function $Z_n(z)$ is the unique matrix such that

1. $Z_n(z) \left( \begin{smallmatrix} (w(z))^{-1} z^n & 0_N \\ 0_N & I_N \end{smallmatrix} \right)$ is analytic at $\mathbb{C} \setminus \mathbb{T}$.
2. Satisfies the jump condition \([24]\) $\forall \zeta \in \mathbb{T}$.
3. About infinity has the following asymptotic: $Z_n(z) = (I_{2N} + O(z^{-1})) \left( \begin{smallmatrix} w[z] & 0_N \\ 0_N & I_N z^{-n} \end{smallmatrix} \right)$ for $z \to \infty$.

4.1. **Recursion relations.**

**Definition 16 (Szegő matrix).** For each $n \in \{0, 1, 2, \ldots\}$ we introduce the Szegő matrices

$$R_n(z) := Z_{n+1}(z) (Z_n(z))^{-1}. $$

**Definition 17.** Given a Laurent series $L(z) = \sum_{m \in \mathbb{Z}} L_m z^m$ the expression stands for the Laurent series gotten from $L(z)$ by disregarding the powers less than $j$, where $j \in \mathbb{Z}$, i.e., $[L(z)]_{\geq j} := \sum_{m \geq j} L_m z^m$.

**Proposition 20.** The Szegő matrix $R_n(z)$ can be alternatively expressed as

$$R_n(z) = Y_{n+1}(z) \left( \begin{smallmatrix} z^{-1} I_N & 0_N \\ 0_N & I_N \end{smallmatrix} \right) (Y_n(z))^{-1}$$

$$= X_{n+1}(z) \left( \begin{smallmatrix} I_N & 0_N \\ 0_N & z^{-1} I_N \end{smallmatrix} \right) (X_n(z))^{-1}. $$

**Lemma 1.** The matrix $R_n$ is analytic at $\mathbb{C} \setminus \{0\}$.

**Proof.** The matrices $Z_{n+1}(z)$ and $(Z_n(z))^{-1}$ are analytic at $\mathbb{C} \setminus (\mathbb{T} \cup \{0\})$ while the $Z_n(z)$'s jump at $\mathbb{T}$ is a constant matrix which does not depend on $n$, i.e., $R_n$ is continuous on the unit circle and, consequently, analytic in the complex plane but not for the origin. \qed

**Proposition 21.** The Szegő matrix has the form

$$R_n(z) = R_{n,0} + R_{n,-1} z^{-1},$$

with

$$R_{n,0} := \left( \begin{smallmatrix} I_N & 0_N \\ 0_N & I_N \end{smallmatrix} \right), \quad R_{n,-1} := \left( \begin{smallmatrix} a_{n+1} - a_n & -b_n \\ c_{n+1} & I_N \end{smallmatrix} \right).$$

**Proof.** Considering that $R_n(z)$ is analytic on the complex plane but for a possible singularity at $z = 0$ and $z = \infty$, we deduce that $R_n(z)$ has a simple pole at the origin. To explicitly compute $R_n(z)$ we use \([26]\) at $z = \infty$ and, as there is no jump at the unit circle $\mathbb{T}$, analytically extend the result to the annulus $\mathbb{C} \setminus \{0\}$, which can be easily achieved if we truncate the Taylor series about infinity and keep only those terms involving $z^j$ with $j \geq -1$

$$R_n(z) = \left[ (I_{2N} + X_{n+1}^{(1)} z^{-1} + \cdots) \left( \begin{smallmatrix} I_N & 0_N \\ 0_N & I_N z^{-1} \end{smallmatrix} \right) (I_{2N} - X_n^{(1)} z^{-1} + \cdots) \right]_{\geq -1}$$

$$= \left( \begin{smallmatrix} I_N & 0_N \\ 0_N & I_N z^{-1} \end{smallmatrix} \right) - \left( \begin{smallmatrix} I_N & 0_N \\ 0_N & I_N z^{-1} \end{smallmatrix} \right) X_n^{(1)} z^{-1} + X_{n+1}^{(1)} \left( \begin{smallmatrix} I_N & 0_N \\ 0_N & 0_N \end{smallmatrix} \right) z^{-1}$$

$$= \left( \begin{smallmatrix} I_N + (a_{n+1} - a_n) z^{-1} & -b_n z^{-1} \\ c_{n+1} z^{-1} & z^{-1} I_N \end{smallmatrix} \right).$$

\qed

**Corollary 1.** The recursion equations

$$Y_{n+1}(z) = R_n(z) Y_n(z) \left( \begin{smallmatrix} I_N z & 0_N \\ 0_N & I_N \end{smallmatrix} \right),$$

$$X_{n+1}(z) = R_n(z) X_n(z) \left( \begin{smallmatrix} I_N & 0_N \\ 0_N & I_N z \end{smallmatrix} \right),$$
Proof. Equations (28) and (29) are a direct consequence of Proposition 20 and form them we derive (30) and (31).

Notice that $R_{n-1}(z) \cdots R_0(z)$ is usually called as transfer matrix, see [72]. The following recursion relations have been proved by algebraic means in [28] and [18] for the Szegő matrix polynomials. Here we give a more analytical proof based in the Riemann–Hilbert problem and add two analogous recursion relations for the Cauchy transforms, which are not an immediate consequence of the previous recursions.

**Theorem 2** (Recursion relations). *The following equations*

\[
\begin{align*}
& P_{1,n+1}^L(z) = zP_{1,n}^L(z) + \alpha_{1,n+1}^L \tilde{P}_{2,n}^R(z), \\
& \tilde{P}_{2,n}^R(z) = (\alpha_{2,n}^R) \tilde{P}_{2,n}^R(z) + (I_N - (\alpha_{2,n}^R)^\dagger \alpha_{1,n}^L) \tilde{P}_{2,n-1}^R(z), \\
& Q_{1,n+1}^L(z) = Q_{1,n}^L(z) + \alpha_{1,n+1} Q_{2,n}^R(z), \\
& zQ_{2,n}^R(z) = (\alpha_{2,n}^R)^\dagger Q_{1,n}^L(z) + (I_N - (\alpha_{2,n}^R)^\dagger \alpha_{1,n}^L) Q_{2,n-1}^R(z),
\end{align*}
\]

*are satisfied.*

Proof. The first step is to write

\[
\begin{pmatrix}
P_{1,n+1}^L(z) \\ Q_{1,n+1}^L(z)
\end{pmatrix}
= \begin{pmatrix}
(I_N + (a_{n+1} - a_n)z^{-1} - b_nz^{-1}) & -b_nz^{-1} \\
c_{n+1}z^{-1} & I_Nz^{-1}
\end{pmatrix}
\begin{pmatrix}
P_{1,n}^L(z) \\ Q_{1,n}^L(z)
\end{pmatrix}
\]

from where we deduce

\[
\begin{align*}
& P_{1,n+1}^L(z) = (I_Nz + (a_{n+1} - a_n)) P_{1,n}^L(z) + b_n(H_{n-1}^R)^{-1} \tilde{P}_{2,n-1}^R(z), \\
& -(H_n^R)^{-1} \tilde{P}_{2,n}^R(z) = c_{n+1} P_{1,n}^L(z) - (H_{n-1}^R)^{-1} \tilde{P}_{2,n-1}^R(z), \\
& Q_{1,n+1}^L(z) = (I_N + (a_{n+1} - a_n)z^{-1}) Q_{1,n}^L(z) + b_n(H_{n-1}^R)^{-1} Q_{2,n-1}^R(z)z^{-1}, \\
& -(H_n^R)^{-1} Q_{2,n}^R(z) = c_{n+1}z^{-1} Q_{1,n}^L(z) - z^{-1}(H_{n-1}^R)^{-1} Q_{2,n-1}^R(z),
\end{align*}
\]

and, consequently, obtain

\[
\begin{align*}
P_{1,n+1}^L(z) & = (I_Nz + (a_{n+1} - a_n + b_nc_{n+1}) P_{1,n}^L(z) + b_n(H_{n}^R)^{-1} \tilde{P}_{2,n}^R(z), \\
\tilde{P}_{2,n}^R(z) & = -H_{n}^R c_{n+1} P_{1,n}^L(z) + H_{n}^R(H_{n-1}^R)^{-1} \tilde{P}_{2,n-1}^R(z), \\
Q_{1,n+1}^L(z) & = (I_N + (a_{n+1} - a_n + b_nc_{n+1})z^{-1}) Q_{1,n}^L(z) + b_n(H_{n-1}^R)^{-1} Q_{2,n}^R(z), \\
Q_{2,n}^R(z) & = -H_{n}^R c_{n+1} Q_{1,n}^L(z)z^{-1} + H_{n}^R(H_{n-1}^R)^{-1} Q_{2,n-1}^R(z)z^{-1}.
\end{align*}
\]
Now, from (1) we conclude

\[
0_N = \int_T \hat{p}_{1,n+1}^L(\zeta) w(\zeta) \, d m(\zeta) \zeta^{n-1} \\
= \int_T (I_N \zeta + a_{n+1} - a_n + b_n c_{n+1}) p_{1,n}^L(\zeta) + b_n (H_n^R)^{-1} \hat{p}_{2,n}^R(\zeta) w(\zeta) \, d m(\zeta) \zeta^{n-1} \\
= \int_T p_{1,n}^L(\zeta) w(\zeta) \, d m(\zeta) \zeta^{n-1} \\
+ (a_{n+1} - a_n + b_n c_{n+1}) \int_T \hat{p}_{1,n}^L(\zeta) w(\zeta) \, d m(\zeta) \zeta^{n-1} + b_n (H_n^R)^{-1} \int_T \hat{p}_{2,n}^R(\zeta) w(\zeta) \, d m(\zeta) \zeta^{n-1} \\
= (a_{n+1} - a_n + b_n c_{n+1}) \mathcal{H}_n^L \\
\]

and, therefore, we infer that

\[
a_{n+1} - a_n + b_n c_{n+1} = 0_N. \tag{32}
\]

Then, the recursion relations simplifies to

\[
\begin{align*}
\hat{p}_{1,n+1}^L(z) &= \hat{p}_{1,n}^L(z) z + b_n (H_n^R)^{-1} \hat{p}_{2,n}^R(z), \\
\hat{p}_{2,n}^R(z) &= -H_n^R c_{n+1} \hat{p}_{1,n}^L(z) + H_n^R (H_{n-1}^R)^{-1} \hat{p}_{2,n-1}^R(z), \\
Q_{1,n+1}^L(z) &= Q_{1,n}^L(z) + b_n (H_n^R)^{-1} Q_{2,n}^R(z), \\
Q_{2,n}^R(z) &= -H_n^R c_{n+1} Q_{1,n}^L(z) z^{-1} + H_n^R (H_{n-1}^R)^{-1} Q_{2,n-1}^R(z) z^{-1}.
\end{align*}
\]

A further simplification is obtained by evaluating the first recursion equation at \( z = 0 \), which gives

\[
\alpha_{1,n+1}^L = b_n (H_n^R)^{-1},
\]

so that

\[
b_n = \alpha_{1,n+1}^L H_n^R. \tag{33}
\]

Moreover, from (20) and (17) we get

\[
c_n = -(H_{n-1}^R)^{-1} (\alpha_{2,n-1}^R)^{\dagger}, \tag{34}
\]

which introduced in the second recursion equation gives

\[
\hat{p}_{2,n}^R(z) = (\alpha_{2,n}^R)^{\dagger} \hat{p}_{1,n}^L(z) + H_n^R (H_{n-1}^R)^{-1} \hat{p}_{2,n-1}^R(z),
\]

that evaluated at \( z = 0 \) implies

\[
I_N = (\alpha_{2,n}^R)^{\dagger} \alpha_{1,n}^L + H_n^R (H_{n-1}^R)^{-1},
\]

so that

\[
H_n^R (H_{n-1}^R)^{-1} = I_N - (\alpha_{2,n}^R)^{\dagger} \alpha_{1,n}^L. \tag{35}
\]

\[ \square \]

**Proposition 22.** We have that the following relations

\[
H_n^R (H_{n-1}^R)^{-1} = I_N - (\alpha_{2,n}^R)^{\dagger} \alpha_{1,n}^L, \quad H_n^L (H_{n-1}^L)^{-1} = I_N - \alpha_{1,n}^L (\alpha_{2,n}^R)^{\dagger}
\]

are satisfied and, consequently, the matrices

\[
I_N - (\alpha_{2,n+1}^R)^{\dagger} \alpha_{1,n+1}^L, \quad I_N - \alpha_{1,n+1}^L (\alpha_{2,n+1}^R)^{\dagger}
\]

are not singular. At the origin we have

\[
Q_{1,n}^L(0) = H_n^L, \quad Q_{2,n}^R(0) = -(\alpha_{2,n+1}^R)^{\dagger} H_n^L = -H_n^R (\alpha_{2,n+1}^L)^{\dagger},
\]

while about infinity the behavior is

\[
\lim_{z \to \infty} (z^{n+1} Q_{1,n}^L(z)) = -\alpha_{1,n+1}^L H_n^R = -H_n^R, \quad \lim_{z \to \infty} (z^{n+1} Q_{2,n}^R(z)) = H_n^R.
\]
Proof. Part of these statements are just a recollection of some intermediate conclusions obtained in the discussion of the previous proof and also of Proposition 17 of [18]. For the other we argue as follows. As the Cauchy transforms are analytic at the origin we can evaluate the third and fourth recursion relations at \( z = 0 \) to get

\[
H_{n+1}^L = H_n^L + \alpha_{1,n+1}^L Q_{2,n}^R(0),
\]

\[
0 = (\alpha_{2,n}^R)^\dagger H_{n}^L + (I_N - (\alpha_{2,n}^R)^\dagger \alpha_{1,n}^L) Q_{2,n-1}^R(0).
\]

Now, from (35) we deduce that \( I_N - (\alpha_{2,n}^R)^\dagger \alpha_{1,n}^L \) is not a singular matrix, so that we can clean \( Q_{2,n-1}^R(0) \) and write

\[
Q_{2,n-1}^R(0) = -(I_N - (\alpha_{2,n}^R)^\dagger \alpha_{1,n}^L)^{-1} (\alpha_{2,n}^R)^\dagger H_n^L,
\]

that introduced in the first equation gives

\[
\left( I_N + \alpha_{1,n+1}^L \left( I_N - (\alpha_{2,n+1}^R)^\dagger \alpha_{1,n+1}^L \right)^{-1} (\alpha_{2,n+1}^R)^\dagger \right) H_{n+1}^L = H_n^L.
\]

But, given two matrices A and B such that \( I_N - BA \) is not a singular matrix, then the matrix \((I_N - AB)^{-1} = I_N + A(I_N - BA)^{-1}B\). Therefore, we get

\[
H_{n+1}^L(H_n^L)^{-1} = I_N - \alpha_{1,n+1}(\alpha_{2,n+1}^R)^\dagger.
\]

Finally, we look at the behavior about infinity of third recursion relation to get

\[
\lim_{z \to \infty} (z^{n+1} Q_{1,n+1}^L(z)) = \lim_{z \to \infty} (z^{n+1} Q_{1,n}^L(z)) + \alpha_{1,n+1} \lim_{z \to \infty} (z^{n+1} Q_{2,n}^R(z)),
\]

and Proposition 14 gives

\[
0 = \lim_{z \to \infty} (z^{n+1} Q_{1,n}^L(z)) + \alpha_{1,n+1} H_n^R.
\]

Now we simplify (36), which can be written as

\[
Q_{2,n}^R(0) = -H_n^R (H_{n+1}^R)^{-1} (\alpha_{2,n+1}^R)^\dagger H_{n+1}^L
= -H_n^R (\alpha_{2,n+1}^R)^\dagger
= -(\alpha_{2,n}^R)^\dagger H_n^L,
\]

where we have used Proposition 17 of [18].

\[\square\]

Corollary 2. The Szegö matrix can be written as follows

\[
Q_{2,n}^R(0) = -(H_n^R)^{-1} (\alpha_{2,n}^R)^\dagger z^{-1} - \alpha_{1,n+1} H_n^R z^{-1}.
\]

Proof. Just use (33), (34) and (32).

\[\square\]

Proposition 23 (Verblunsky parametrization of \( X_n \)). The coefficients \( X_n^{(j)} \), for each \( n \in \{0, 1, 2, \ldots\} \) can be parametrized in terms of the Verblunsky matrices \( \{ \alpha_{1,m}^L, (\alpha_{2,m}^R)^\dagger \}_{m=1}^n \), the quasi-tau matrices \( \{ H_m^R \}_{m=0}^n \) and \( X_{n=1} \). In particular,

1. For \( X_n^{(1)} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \) we have the expressions

\[
a_n = \sum_{m=0}^{n-1} \alpha_{1,m+1}^L (\alpha_{2,m}^R)^\dagger, \quad b_n = \alpha_{1,n+1}^L H_n^R,
\]

\[
c_n = -(H_{n-1}^R)^{-1} (\alpha_{2,n-1}^R)^\dagger, \quad d_n = -\sum_{m=0}^{n-1} (H_m^R)^{-1} (\alpha_{2,m}^R)^\dagger \alpha_{1,m+1}^L H_m^R,
\]

where we have introduced, for convenience, the notation \( \alpha_{2,0}^R := I_N \).
(2) For $X_n^{(2)} = \left( \begin{array}{c} a_n^{(2)} \\ c_n^{(2)} \\ d_n^{(2)} \end{array} \right)$ we have the expressions

$$a_{n+1}^{(2)} = a_1^{(2)} + \sum_{m=1}^{n} \lambda_m^{(2)},$$

$$b_{n}^{(2)} = \alpha_{l,n+1}^{L}H_{n+1}^{R} - \alpha_{l,n+1}^{L}(\alpha_{2,n}^{R})^{\dagger} \alpha_{l,n}^{L}H_{n}^{R} - \alpha_{l,n+1}^{L}H_{n}^{R}\left(\alpha_{1,1}^{R} + \sum_{m=1}^{n-1} (\alpha_{2,m}^{R})^{\dagger} \alpha_{1,m+1}^{R}\right),$$

$$c_{n+1}^{(2)} = -(H_{n}^{R})^{-1}(\alpha_{2,n}^{R})^{\dagger}\left(\alpha_{l,1}^{L} + \sum_{m=1}^{n-1} \alpha_{l,m+1}^{L}(\alpha_{2,m}^{R})^{\dagger}\right) - (H_{n-1}^{R})^{-1}(\alpha_{2,n-1}^{R})^{\dagger},$$

$$d_{n+1}^{(2)} = d_1^{(2)} + \sum_{m=0}^{n} D_m^{(2)},$$

where

\[
\lambda_n^{(2)} := \alpha_{l,n+1}^{L}(\alpha_{2,n}^{R})^{\dagger}\left(\alpha_{l,1}^{L} + \sum_{m=1}^{n-1} \alpha_{l,m+1}^{L}(\alpha_{2,m}^{R})^{\dagger}\right) + \alpha_{l,n+1}^{L}H_{n}^{R}(H_{n-1}^{R})^{-1}(\alpha_{2,n-1}^{R})^{\dagger},
\]

\[
D_n^{(2)} := -(H_{n}^{R})^{-1}(\alpha_{2,n}^{R})^{\dagger}\alpha_{l,n+1}^{L}H_{n+1}^{R} - \alpha_{l,n+1}^{L}(\alpha_{2,n}^{R})^{\dagger}\alpha_{l,n}^{L}H_{n}^{R} - \alpha_{l,n+1}^{L}H_{n}^{R}\left(\alpha_{1,1}^{R} + \sum_{m=1}^{n-1} (\alpha_{2,m}^{R})^{\dagger} \alpha_{1,m+1}^{R}\right).
\]

Proof. The expressions for $b_n$ and $c_n$ were deduced before, see (33), (34) and (32) implies

$$a_{n+1} - a_n = \alpha_{l,n+1}^{L}(\alpha_{2,n}^{R})^{\dagger}.$$

Observe that we can use a telescoping sum to get

$$a_{n+1} - a_1 = \sum_{m=1}^{n} (a_{m+1} - a_m) = \sum_{m=1}^{n} \alpha_{l,m+1}^{L}(\alpha_{2,m}^{R})^{\dagger},$$

with $a_1 = p_{1,1}^{L} = \alpha_{l,1}^{L}$. Consequently, the expression for $a_n$ follows.

Now we start the essential part of the proof. We will make a substantial use of (29) in the form

\[
X_{n+1}(z)\left( \begin{array}{cc} I_N & 0_N \\ 0_N & I_Nz^{-1} \end{array} \right) = R_n(z)X_n(z),
\]

which can be expanded, for $j \in \{1,2,\ldots\}$, as follows

\[
X_{n+1}^{(j+1)}\left( \begin{array}{cc} I_N & 0_N \\ 0_N & 0_N \end{array} \right) + X_{n+1}^{(j+1)}\left( \begin{array}{cc} 0_N & 0_N \\ 0_N & I_N \end{array} \right) = \left( \begin{array}{cc} I_N & 0_N \\ 0_N & 0_N \end{array} \right)X_n^{(j+1)} + \left( \begin{array}{cc} a_{n+1} - a_n & -b_n \\ c_{n+1} & 0_N \end{array} \right)X_n^{(j)}.
\]

In terms of the different blocks we deduce

\[
a_{n+1}^{(j+1)} - a_n^{(j+1)} = (a_{n+1} - a_n)a_n^{(j)} - b_n c_n^{(j)},
\]

\[
b_n^{(j+1)} = b_n^{(j+1)} = b_{n+1} - (a_{n+1} - a_n)b_n^{(j)} + b_n d_n^{(j)},
\]

\[
c_{n+1}^{(j+1)} = c_{n+1}a_n^{(j)} + c_n^{(j)},
\]

\[
d_{n+1}^{(j)} - d_n^{(j)} = c_{n+1}b_n^{(j)}.
\]

which for $j = 1$ reads

\[
a_{n+1}^{(2)} - a_n^{(2)} = (a_{n+1} - a_n)a_n - b_n c_n,
\]

\[
b_n^{(2)} = b_{n+1} - (a_{n+1} - a_n)b_n + b_n d_n,
\]

\[
c_{n+1}^{(2)} = c_{n+1}a_n + c_n,
\]

\[
d_{n+1} - d_n = c_{n+1}b_n.
\]

In particular, for $j = 1$ the last equation is the following telescoping relation

$$d_{n+1} - d_n = -(H_{n}^{R})^{-1}(\alpha_{2,n}^{R})^{\dagger}\alpha_{l,n+1}^{L}H_{n}^{R},$$

where we have used (33). Using the telescoping trick we obtain

$$d_{n+1} = d_1 - \sum_{m=1}^{n} (H_{m}^{R})^{-1}(\alpha_{2,m}^{R})^{\dagger}\alpha_{l,m+1}^{L}H_{m}^{R}.$$
Here, according to (20) we have $d_1 = -\left( \frac{d}{d\zeta} w(\zeta) \right)^{-1} \frac{d}{d\zeta} w(\zeta) \zeta d\mu(\zeta)$, and recalling (2) we see $d_1 = -(H^R)_{1,1}^{-1} \alpha_{1,1}^R H_0^R$. \footnote{We have $\mu(T) = H_0^R = H_0^L$ and $\frac{d}{d\zeta} w(\zeta) \zeta d\mu(\zeta) = -\alpha_{1,1}^L \mu(T) = -\mu(T) \alpha_{1,1}^R$.}

In terms of Verblunsky coefficients (40) is

\begin{equation}
(42)
\begin{aligned}
 a_{n+1}^{(j+1)} - a_n^{(j+1)} &= \alpha_{1,n+1}^L (\alpha_{2,n}^R)\dagger a_n - \alpha_{1,n+1}^L H_n c_n, \\
b_{n+1}^{(j+1)} &= b_{n+1}^{(j)} - \alpha_{1,n+1}^L (\alpha_{2,n}^R)\dagger b_n + \alpha_{1,n+1}^L H_n d_n, \\
c_{n+1}^{(j+1)} &= -(H_n^R)^{-1} (\alpha_{2,n}^R)\dagger a_n + c_n, \\
d_{n+1}^{(j+1)} &= -(H_n^R)^{-1} (\alpha_{2,n}^R)\dagger b_n + d_n,
\end{aligned}
\end{equation}

we conclude that whenever the coefficients $a_n^{(j)}$, $b_n^{(j)}$, $c_n^{(j)}$ are given for all $n \in \{0, 1, 2, \ldots\}$ we can determine $a_n^{(j+1)}$, $b_n^{(j+1)}$, $c_n^{(j+1)}$ also for all $n \in \{0, 1, 2, \ldots\}$. Which is the main statement of the Proposition. For example, let us put again $j = 1$ in (42) for the first three equations and $j = 2$ for the last equation to get

\begin{equation}
(43)
\begin{aligned}
a_{n+1}^{(2)} - a_n^{(2)} &= \alpha_{1,n+1}^L (\alpha_{2,n}^R)\dagger a_n - \alpha_{1,n+1}^L H_n c_n, \\
b_{n+1}^{(2)} &= b_{n+1}^{(2)} - \alpha_{1,n+1}^L (\alpha_{2,n}^R)\dagger b_n + \alpha_{1,n+1}^L H_n d_n, \\
c_{n+1}^{(2)} &= -(H_n^R)^{-1} (\alpha_{2,n}^R)\dagger a_n + c_n, \\
d_{n+1}^{(2)} &= -(H_n^R)^{-1} (\alpha_{2,n}^R)\dagger b_n + d_n,
\end{aligned}
\end{equation}

so that using (38) we deduce

\begin{equation}
(44)
\begin{aligned}
a_{n+1}^{(2)} - a_n^{(2)} &= \alpha_{1,n+1}^L (\alpha_{2,n}^R)\dagger \left( \alpha_{1,1}^L + \sum_{m=1}^{n-1} \alpha_{1,m+1}^L (\alpha_{2,m}^R)\dagger \right) + \alpha_{1,n+1}^L H_n^R (H_n^{-1})^{-1} (\alpha_{2,n-1}^R)\dagger, \\
b_{n+1}^{(2)} &= \alpha_{1,n+1}^L H_n^R - \alpha_{1,n+1}^L (\alpha_{2,n}^R)\dagger \alpha_{1,1}^L H_n^R - \alpha_{1,n+1}^L H_n^R \left( \alpha_{1,1}^L + \sum_{m=1}^{n-1} (\alpha_{2,m}^R)\dagger \alpha_{1,m+1}^L \right), \\
c_{n+1}^{(2)} &= -(H_n^R)^{-1} (\alpha_{2,n}^R)\dagger \left( \alpha_{1,1}^L + \sum_{m=1}^{n-1} \alpha_{1,m+1}^L (\alpha_{2,m}^R)\dagger \right) - (H_n^{-1})^{-1} (\alpha_{2,n-1}^R)\dagger, \\
d_{n+1}^{(2)} &= -(H_n^R)^{-1} (\alpha_{2,n}^R)\dagger \alpha_{1,n+1}^L H_n^R - \alpha_{1,n+1}^L (\alpha_{2,n}^R)\dagger \alpha_{1,n}^L H_n^R - \alpha_{1,n+1}^L H_n^R \left( \alpha_{1,1}^L + \sum_{m=1}^{n-1} (\alpha_{2,m}^R)\dagger \alpha_{1,m+1}^L \right).
\end{aligned}
\end{equation}

4.2. Pearson equations for the matrix of weights and some of its consequences. In this subsection we analyze an important matrix $M_n(z)$ binded to the Riemann–Hilbert problem —as well as to to the right logarithmic derivative of the matrix of measures, which is analytic in the annulus $\mathbb{C} \setminus \{0\}$ and provides a linear system of ordinary differential equations for the matrix Szegő polynomials and its Cauchy transforms. Moreover, it satisfies a compatibility condition with the Szegő matrix $R_n(z)$. One of its major virtues, that we will discuss in the next section, is that it leads to non linear difference equations of Painlevé type for the Verblunsky coefficients.

**Definition 18.** We introduce the right logarithmic derivatives of the analytic extension to the annulus $\mathbb{C} \setminus \{0\}$ of matrix of measures $w(z)$ and of $Z_n(z)$ and

\begin{equation}
(45)
\begin{aligned}
W(z) &:= \frac{d}{dz} w(z) (w(z))^{-1}, \\
M_n(z) &:= \frac{d}{dz} Z_n(z) (Z_n(z))^{-1}.
\end{aligned}
\end{equation}

Observe that for $n = 0$ we have

\begin{equation}
(46)
\begin{aligned}
M_0(z) &= \begin{pmatrix}
W(z) & d Q_{1,0}^L(z) \\
0_N & W(z) Q_{1,0}^L(z)
\end{pmatrix}.
\end{aligned}
\end{equation}

Equation (43) can be understood as a Pearson equation for the matrix of weights:
This is a linear first order differential system whose properties are determined by $W(z)$. These systems constitute a very deep and profound branch in Mathematics, with pioneering work by George Birkhoff [25], for different treatments of the subject we refer the reader to [57, 74, 116, 34, 107, 95, 15, 76]. Is relevant to remark the change of the point of view. For scalar systems with $N = 1$ we normally take the weight as an explicit Freud type weight. However, for the general matrix scenario we have avoided this approach and preferred to give $W(z)$, and consider the extension of the matrix of weights $w(z)$ as a fundamental solution of (46). The Freud approach will not lead, in a general scenario, to the matrix discrete Painlevé II systems derived later in §5.

**Proposition 24 (Differential systems).** Equation (44) can be understood as a system of differential equations and can be written in the following two alternative forms

\[
M_n(z) = \frac{d}{dz} \left( (Y_n(z))^{-1} + Y_n(z) \left( W(z) - nI_N z^{-1} \right) \begin{pmatrix} 0_N \\ 0_N \end{pmatrix} \right),
\]

\[
\frac{d}{dz} \left( X_n(z) \right)^{-1} + X_n(z) \left( W(z) - nI_N z^{-1} \right),
\]

\[
\frac{d}{dz} \left( X_n(z) \right)^{-1} + X_n(z) \left( W(z) - nI_N z^{-1} \right).
\]

**Proposition 25 (Analytic properties of $M_n$).** The matrix $M_n(z)$ is analytic at the annulus $\mathbb{C} \setminus \{0\}$.

**Proof.** It follows from (47) that $M_n(z)$ is analytic at $\mathbb{C} \setminus ((0 \cup \mathbb{T})$. From (23) we infer that there is no jump at $\mathbb{T}$ and, consequently, $M_n(z)$ is analytic at $\mathbb{C} \setminus \{0\}$. \qed

**Proposition 26.** The following differential equations are satisfied

\[
M_n(z) \left( \begin{array}{c} \frac{dP_{1,n}(z)}{dz} \\ -\frac{dP_{2,n}(z)}{dz} \end{array} \right) = \left( \begin{array}{c} \frac{d}{dz} - nz^{-1}P_{1,n}(z) + \frac{d}{dz}P_{1,n}(z)W(z) \\ -\left( \frac{d}{dz} + nz^{-1}P_{2,n}(z) + \frac{d}{dz}P_{2,n}(z)W(z) \right) \end{array} \right),
\]

\[
M_n(z) \left( \begin{array}{c} \frac{dQ_{1,n}(z)}{dz} \\ -\frac{dQ_{2,n}(z)}{dz} \end{array} \right) = \left( \begin{array}{c} \frac{d}{dz} - nz^{-1}Q_{1,n}(z) + \frac{d}{dz}Q_{1,n}(z)W(z) \\ -\left( \frac{d}{dz} + nz^{-1}Q_{2,n}(z) + \frac{d}{dz}Q_{2,n}(z)W(z) \right) \end{array} \right),
\]

for $n \in \{1, 2, \ldots\}$. For $n = 0$ we have

\[
M_0(z) \left( \begin{array}{c} I_N \\ 0_N \end{array} \right) = \left( \begin{array}{c} W(z) \\ 0_N \end{array} \right),
\]

\[
M_0(z) \left( \begin{array}{c} Q_{1,0}(z) \\ I_N \end{array} \right) = \left( \begin{array}{c} \frac{d}{dz}Q_{1,0}(z) \\ 0_N \end{array} \right).
\]

**Proof.** It follows directly from (44) and (23). \qed

Observe that (49) can be written

\[
M_n(z) \left( \begin{array}{c} \frac{dP_{1,n}(z)}{dz} \\ -\frac{dP_{2,n}(z)}{dz} \end{array} \right) = \left( \begin{array}{c} \frac{d}{dz} - nz^{-1}P_{1,n}(z) + \frac{d}{dz}P_{1,n}(z)W(z) \\ -\left( \frac{d}{dz} + nz^{-1}P_{2,n}(z) + \frac{d}{dz}P_{2,n}(z)W(z) \right) \end{array} \right),
\]

\[
M_n(z) \left( \begin{array}{c} \frac{dQ_{1,n}(z)}{dz} \\ -\frac{dQ_{2,n}(z)}{dz} \end{array} \right) = \left( \begin{array}{c} \frac{d}{dz} - nz^{-1}Q_{1,n}(z) + \frac{d}{dz}Q_{1,n}(z)W(z) \\ -\left( \frac{d}{dz} + nz^{-1}Q_{2,n}(z) + \frac{d}{dz}Q_{2,n}(z)W(z) \right) \end{array} \right),
\]

for $n \in \{1, 2, \ldots\}$. For $n = 0$ we have

\[
M_0(z) \left( \begin{array}{c} I_N \\ 0_N \end{array} \right) = \left( \begin{array}{c} W(z) \\ 0_N \end{array} \right),
\]

\[
M_0(z) \left( \begin{array}{c} Q_{1,0}(z) \\ I_N \end{array} \right) = \left( \begin{array}{c} \frac{d}{dz}Q_{1,0}(z) \\ 0_N \end{array} \right).
\]

**Definition 19.** In terms of the local behavior at $z = 0$ we distinguish three cases for the matrix of weights, two singular cases and a regular case, depending on the form of the principal part of the Laurent series of the logarithmic derivative $W(z)$ of the matrix of weights $w(z)$ in the annulus $\mathbb{C} \setminus \{0\}$:

1. **Ordinary case.** The right logarithmic derivative (43) of the matrix of weights is regular at the origin

\[
W = W_0 + W_1 z + \cdots.
\]
(2) **Fuchsian case.** Now, the right logarithmic derivative \((43)\) of the matrix of weights has a simple pole at the origin

\[ W = W_{-1}z^{-1} + W_0 + \cdots, \]

with \(W_{-1} \neq 0_N\).

(3) **Non-Fuchsian case.** The right logarithmic derivative \((43)\) of the matrix of weights is the following Laurent polynomial

\[ W = W_{-r}z^{-r} + W_{-r+1}z^{-r+1} + \cdots, \]

with \(r > 1\) and \(W_{-r} \neq 0_N\).

We introduce the notation

\[ W_n^{[0]} := \begin{cases} -nI_N, & \text{ordinary case}, \\ W_{-1} - nI_N, & \text{Fuchsian case}, \\ W_{-r}, & \text{non-Fuchsian case, } r > 1. \end{cases} \]

The equation \((54)\) tell us that at \(z = 0\) we have an regular point, or a Fuchsian singularity or a non-Fuchsian singularity of rank \(r > 1\).

Following \[13\] we say that the point \(z = 0\) is a regular singular point for \((46)\) if there exists a constant \(k\) such that all its solutions in every sector in the complex plane with \(z = 0\) as a vertex, grow no faster than \(|z|^k\) as \(z \to 0\) within the sector. Fuchsian singularities are regular singularities but the converse is not always true.

**Proposition 27.** In order to have a Hölder matrix of weights \(w(\zeta), \zeta \in \mathbb{T}\), the matrix \(W(z)\) must be such that the Pearson system \((46)\) has trivial monodromy.

**Proof.**

Given the the monodromy matrix \(M = \exp(2\pi i R), R \in \mathbb{C}^{N \times N}\), of the system \((46)\), a fundamental solution is of the form \(S(z) = P(z)z^R\) where \(P : \mathbb{C} \setminus \{0\} \to \text{GL}(N, \mathbb{C})\) is analytic; i.e., the solutions of \((46)\) are multivalued of the form \(z^\alpha F(z)\), where \(exp(2\pi i \alpha)\) is an eigenvalue of the monodromy matrix and \(F(z)\) is analytic at the annulus \(\mathbb{C} \setminus \{0\}\). Thus, if we want a Hölder restriction on \(\mathbb{T}\) — and, consequently, single valued functions — the only possible matrices \(R\) are those with integer eigenvalues, and therefore the monodromy matrix must be the identity. Hence the Pearson system \((46)\) has trivial monodromy. \(\square\)

Let us notice that any equivalent system, and, therefore with the same trivial monodromy has a corresponding matrix of the form

\[ \tilde{W}(z) = \frac{d \Theta(z)}{dz} (\Theta(z))^{-1} + \Theta(z) W(z) (\Theta(z))^{-1} \]

where \(\Theta : \mathbb{C} \setminus \{0\} \to \text{GL}(N, \mathbb{C})\) is analytic in the annulus with at most a pole at \(z = 0\). This triviality of the monodromy could be avoided if we relax the Hölder conditions on the weight, and just request piecewise Hölder weights on \(\mathbb{T}\), allowing at the discontinuities space for the branches of multivalued functions that non-trivial monodromy implies. Then, we still have a RH problem but in the weak sense and uniqueness is not ensure, and the jump on \(\mathbb{T}\) is only ensured almost everywhere. A much more detailed analysis will be need for the analytic properties of the \(R_n(z)\) and \(M_n(z)\). This could be connected with non trivial monodromy problems as there only piecewise Hölderity is required, see \[57\].

For piecewise continuous jump functions see \[34\] \[96\]. Moreover, in Lemma 7.12 in \[43\] we read that as long \(f \in H^1(\mathbb{T})\), i.e. \(\int_{\mathbb{T}} (|f(\zeta)|^2 + |f'(\zeta)|^2) d\zeta < \infty\), where \(f'\) denotes a weak derivative, the jump of its Cauchy transform satisfies \(f(\zeta) = (Cf)_+(\zeta) - (Cf)_-(\zeta)\) pointwise in the unit circle \(\mathbb{T}\), and not only almost everywhere. This, together with the proof of Theorem 7.18 in \[43\], could indicate that the RH could be generalize to more general \(H^1(\mathbb{T})\)-matrix of weights, and that instead of analytic extensions to the annulus \(\mathbb{C} \setminus \{0\}\) we could deal with analytic functions on the universal cover of the annulus; i.e., with multivalued functions.

**Theorem 3.** Let us assume \(W\) as prescribed in in Definition \[19\] Then, the Laurent series of \(M_n(z)\) is

\[ M_n(z) = \begin{cases} M_n^{[0]}z^{-1} + M_{n,0} + M_{n,1}z + \cdots, & \text{ordinary and Fuchsian cases}, \\ M_n^{[0]}z^{-r} + M_{n,-r+1}z^{-r+1} + M_{n,-r+2}z^{-r+2} + \cdots, & \text{non-Fuchsian cases}, \end{cases} \]
where the leading coefficient is
\[
M_n^{[0]} = \begin{pmatrix}
\alpha_{1,n}^L W_n^{[0]} (\alpha_{2,n}^R)^\dagger & -(\alpha_{1,n}^L W_n^{[0]} H_n^R) \\
-(H_{n-1}^R)^{-1} W_n^{[0]} (\alpha_{2,n}^R)^\dagger & (H_{n-1}^R)^{-1} W_n^{[0]} H_n^R
\end{pmatrix},
\]
for \( n \in \{1, 2, \ldots \} \) and for \( n = 0 \) we have in both singular cases (regular and irregular)
\[
M_n^{[0]} = \begin{pmatrix}
W_0^{[0]} & -W_0^{[0]} H_0^R \\
0_N & 0_N
\end{pmatrix}.
\]

**Proof.** To prove (56) we introduce (54) into (51) and (50) and then look at the leading part to obtain
\[
M_n^{[0]} \begin{pmatrix}
\alpha_{1,n}^L \\
-(H_{n-1}^R)^{-1}
\end{pmatrix} = \begin{pmatrix}
\alpha_{1,n}^L W_n^{[0]} \\
-(H_{n-1}^R)^{-1} W_n^{[0]}
\end{pmatrix},
\]
\[
M_n^{[0]} \begin{pmatrix}
Q_{1,n}^{L}(0) \\
-(H_{n-1}^R)^{-1} Q_{2,n-1}^{R}(0)
\end{pmatrix} = \begin{pmatrix}
0_N \\
0_N
\end{pmatrix}.
\]

With the notation
\[
M_n^{[0]} = \begin{pmatrix}
A_n & B_n \\
C_n & D_n
\end{pmatrix},
\]
\( A_n, B_n, C_n, D_n \in \mathbb{C}^{N \times N} \), a component-wise form of the system of matrix equations (58) and (59) is
\[
A_n \alpha_{1,n}^L - B_n (H_{n-1}^R)^{-1} = \alpha_{1,n} W_n^{[0]},
\]
\[
A_n H_n^L + B_n (H_{n-1}^R)^{-1} (\alpha_{2,n}^R)^\dagger H_{n-1}^L = 0_N,
\]
\[
C_n \alpha_{1,n}^L - D_n (H_{n-1}^R)^{-1} = -(H_{n-1}^R)^{-1} W_n^{[0]},
\]
\[
C_n H_n^L + D_n (H_{n-1}^R)^{-1} (\alpha_{2,n}^R)^\dagger H_{n-1}^L = 0_N.
\]
Here we have used that, see Proposition\textsuperscript{22}
\[
Q_{1,n}^{L}(0) = H_n^L,
\]
\[
Q_{2,n-1}^{R}(0) = -(\alpha_{2,n}^R)^\dagger H_{n-1}^L.
\]
Consequently, cleaning \( A \) and \( C \) in the second equations in each of the two systems we get
\[
A_n = -(H_{n-1}^R)^{-1} (\alpha_{2,n}^R)^\dagger H_{n-1}^L (H_n^L)^{-1},
\]
\[
C_n = -(H_{n-1}^R)^{-1} (\alpha_{2,n}^R)^\dagger H_{n-1}^L (H_n^L)^{-1},
\]
that we insert in the first system of each equation to get
\[
B_n (H_{n-1}^R)^{-1} \left( I_n + (\alpha_{2,n}^R)^\dagger H_{n-1}^L (H_n^L)^{-1} \alpha_{1,n}^L \right) = -\alpha_{1,n} W_n^{[0]},
\]
\[
D_n (H_{n-1}^R)^{-1} \left( I_n + (\alpha_{2,n}^R)^\dagger H_{n-1}^L (H_n^L)^{-1} \alpha_{1,n}^L \right) = (H_{n-1}^R)^{-1} W_n^{[0]}.
\]
Let us notice that, see Proposition\textsuperscript{22}
\[
H_n^L (H_{n-1}^R)^{-1} = I_n - \alpha_{1,n}^L (\alpha_{2,n}^R)^\dagger,
\]
that implies
\[
H_{n-1}^L (H_n^R)^{-1} = \left( I_n - \alpha_{1,n}^L (\alpha_{2,n}^R)^\dagger \right)^{-1}.
\]
That means
\[
I_n + (\alpha_{2,n}^R)^\dagger H_{n-1}^L (H_n^L)^{-1} \alpha_{1,n}^L = I_n + (\alpha_{2,n}^R)^\dagger (I_n - \alpha_{1,n}^L (\alpha_{2,n}^R)^\dagger)^{-1} \alpha_{1,n}^L
\]
\[
= (I_n - (\alpha_{2,n}^R)^\dagger \alpha_{1,n}^L)^{-1}.
\]
Here we have used the following fact: given any two matrices \( R, S \in \mathbb{C}^{N \times N} \) with \( I_n - RS \) not singular, then
\[
I_n + S (I_n - RS)^{-1} R = (I_n - SR)^{-1}.
\]
We clean B and D in the first step and, using (60) and (61), also A and C. The final result is

\[
A_n = \alpha_{1,n}^l W_n^{[0]} (I_N - (\alpha_{2,n}^R)^\dagger \alpha_{1,n}^l) (\alpha_{2,n}^R)^\dagger (I_N - \alpha_{1,n}^l (\alpha_{2,n}^R)^\dagger)^{-1},
\]

\[
B_n = -\alpha_{1,n}^l W_n^{[0]} (I_N - (\alpha_{2,n}^R)^\dagger \alpha_{1,n}^l) H_{n-1}^R,
\]

\[
C_n = -(H_{n-1}^R)^{-1} W_n^{[0]} (I_N - (\alpha_{2,n}^R)^\dagger \alpha_{1,n}^l) (\alpha_{2,n}^R)^\dagger (I_N - \alpha_{1,n}^l (\alpha_{2,n}^R)^\dagger)^{-1},
\]

\[
D_n = (H_{n-1}^R)^{-1} W_n^{[0]} (I_N - (\alpha_{2,n}^R)^\dagger \alpha_{1,n}^l) H_{n-1}^R.
\]

which taking into account that for pair of matrices \( R, S \in \mathbb{C}^{N \times N} \) with \( \det(I_N - SR) \neq 0 \) we have \( (I_N - RS)R(I_N - SR)^{-1} = R \) and that \( (I_N - (\alpha_{2,n}^R)^\dagger \alpha_{1,n}^l) H_{n-1}^R = H_n^R \) gives the desired result. Finally, let us notice that (57) follows at once from (45).

\[\square\]

**Proposition 28.** The compatibility equation

\[
\frac{d R_n(z)}{d z} = M_{n+1}(z)R_n(z) - R_n(z)M_n(z)
\]

is fulfill.

**Proof.** From \( Z_{n+1} = R_n Z_n \) we have

\[
\frac{d R_n(z)}{d z} Z_n(z) + R_n(z) \frac{d Z_n(z)}{d z} = \frac{d Z_{n+1}(z)}{d z}
\]

so that

\[
\frac{d R_n(z)}{d z} + R_n \frac{d Z_n(z)}{d z} (Z_n(z))^{-1} = \frac{d Z_{n+1}(z)}{d z} (Z_{n+1}(z))^{-1} Z_{n+1}(z) (Z_n(z))^{-1}
\]

and the result follows. \[\square\]

**Proposition 29.** The following conditions are satisfied by the leading coefficients of the Szegö matrix \( R_n(z) \) and the matrix \( M_n(z) \)

\[
M_{n+1}^{[0]} R_{n-1} - R_{n-1} M_n^{[0]} = \begin{cases} -R_{n-1}, & \text{ordinary and Fuchsian cases}, \\ 0_{2N}, & \text{non Fuchsian cases}. \end{cases}
\]

**Proof.** We insert in (63) the Laurent series (57) of \( R_n(z) \) and the Laurent series (55) of \( M_n(z) \) and compute the leading coefficient in \( z^{-\tau-1} \). \[\square\]

**Lemma 2.** The dyadic or tensor type representations

\[
R_{n-1} = \begin{pmatrix} \alpha_{1,n+1}^l & \alpha_{2,n}^R \end{pmatrix} \begin{pmatrix} -H_{n-1}^R \end{pmatrix}, \quad M_n^{[0]} = \begin{pmatrix} \alpha_{1,n}^l \end{pmatrix} \begin{pmatrix} -H_{n-1}^R \end{pmatrix} \begin{pmatrix} W_n^{[0]} \end{pmatrix} \]

hold.

**Proposition 30.** Compatibility conditions (64) are identically satisfied upon (37) and (56).

**Proof.** We first observe that

\[
\begin{pmatrix} \alpha_{2,n}^R \end{pmatrix} \begin{pmatrix} -H_{n-1}^R \end{pmatrix} = \begin{pmatrix} \alpha_{2,n}^R \end{pmatrix} \begin{pmatrix} \alpha_{1,n}^l \end{pmatrix} + H_n^R (H_{n-1}^R)^{-1} = I_N.
\]
Then, we calculate

$$M_{n+1}^{[0]} R_{n-1} - R_{n-1} M_{n}^{[0]} = \begin{pmatrix} \alpha_{1,n+1}^{L} & \vdots \\ \vdots & \vdots \\ \alpha_{1,n+1}^{L} \end{pmatrix} W_{n+1}^{[0]} \begin{pmatrix} (\alpha_{2,n+1}^{R})^{\dagger} & -H_{n+1}^{R} \\ \vdots & \vdots \\ (\alpha_{2,n}^{R})^{\dagger} & -H_{n}^{R} \end{pmatrix} \begin{pmatrix} \alpha_{1,n+1}^{L} & \vdots \\ \vdots & \vdots \\ \alpha_{1,n+1}^{L} \end{pmatrix} W_{n}^{[0]} \begin{pmatrix} (\alpha_{2,n}^{R})^{\dagger} & -H_{n}^{R} \\ \vdots & \vdots \\ (\alpha_{2,n}^{R})^{\dagger} & -H_{n}^{R} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_{1,n+1}^{L} \end{pmatrix} W_{n+1}^{[0]} \begin{pmatrix} (\alpha_{2,n}^{R})^{\dagger} & -H_{n}^{R} \end{pmatrix} - \begin{pmatrix} \alpha_{1,n+1}^{L} \end{pmatrix} W_{n}^{[0]} \begin{pmatrix} (\alpha_{2,n}^{R})^{\dagger} & -H_{n}^{R} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_{1,n+1}^{L} \end{pmatrix} (W_{n+1}^{[0]} - W_{n}^{[0]}) \begin{pmatrix} (\alpha_{2,n}^{R})^{\dagger} & -H_{n}^{R} \end{pmatrix}$$

$$= -R_{n-1}, \text{ ordinary and Fuchsian cases},$$

$$0_{2N}, \text{ non-Fuchsian cases},$$

where we have used that

$$W_{n+1}^{[0]} - W_{n}^{[0]} = \begin{cases} -I_{N}, & \text{ordinary and regular singular cases}, \\ 0_{N}, & \text{non-Fuchsian cases}. \end{cases}$$

We now consider the behavior about infinity and assume a descending Laurent series for the right logarithmic derivative

**Definition 20.** Let us assume that

$$W(z) = W_{s} z^{s} + W_{s-1} z^{s-1} + \cdots,$$

where $W_{s}$ is a non zero matrix and with the Laurent series converging in the annulus $\mathbb{C} \setminus \{0\}$. Then, we distinguish three cases when $s \leq -2$, $s = -1$ and $s \geq 0$ and introduce the corresponding matrices

$$W_{n}^{[\infty]} := \begin{cases} -nI_{N}, & \text{case with } s < -1, \\ W_{s} - nI_{N}, & \text{case with } s = -1, \\ W_{s}, & \text{case with } s > -1. \end{cases}$$

**Proposition 31.** Assuming that the right logarithmic derivative of the matrix of weights is as in Definition 20, we have a corresponding Laurent series

$$M_{n}(z) = \begin{cases} M_{n}^{[\infty]} z^{-1} + M_{s-2,n} z^{-2} + \cdots, & \text{cases with } s \leq -1, \\ M_{n}^{[\infty]} z^{s} + M_{s-1,n} z^{s-1} + \cdots, & \text{cases with } s \geq 0, \end{cases}$$

where

$$M_{n}^{[\infty]} := \begin{pmatrix} W_{n}^{[\infty]} & 0_{N} \\ 0_{N} & 0_{N} \end{pmatrix}.$$ 

**Proof.** It follows from (47) and the analicity of $Y_{n}(z)$ about infinity and its behavior normalized by the identity there, $Y_{n}(z) = I_{2N} + O(z^{-1})$, when $z \to \infty$.  

5. The matrix discrete Painlevé II system

In this section we apply the Riemann–Hilbert problem and the properties of matrix $M_{n}(z)$ to derive matrix nonlinear difference systems of equations satisfied by the Verblunsky coefficients.

From (48) and the form of $W(z)$ prescribed in Definition 19, we get

$$M_{n}(z) = \frac{d X_{n}(z)}{dz} (X_{n}(z))^{-1} + X_{n}(z) \begin{pmatrix} W_{s-1} z^{-r} + W_{s-1} z^{-r+1} + \cdots + W_{s} z^{s} & 0_{N} \\ 0_{N} & -nI_{N} z^{-1} \end{pmatrix} (X_{n}(z))^{-1}.$$
Recall that we have the Taylor series about infinity \((19)\) and
\[
\frac{d X_n(z)}{d z} = -X_n^{(1)} z^{-2} - 2X_n^{(2)} z^{-3} + \cdots, \quad (X_n(z))^{-1} = I_{2N} - X_n^{(1)} z^{-1} - \left(X_n^{(2)} - (X_n^{(1)})^2\right) z^{-2} + \cdots
\]
which converges on the exterior of the unit circle \(\mathbb{D}\). Consequently, in the annulus \(\mathbb{D}\), we can write
\[
\frac{d X_n(z)}{d z} (X_n(z))^{-1} = -X_n^{(1)} z^{-2} - (2X_n^{(2)} - (X_n^{(1)})^2) z^{-3} + \cdots.
\]

Therefore, in the non-Fuchsian case to compute the coefficient \(M_n^{[r]}\) of \(M_n(z)\) we will require of the concourse of the matrices \(\{X_n^{(j)}\}_{j=1}^{r+1}\). In this case the contribution of the derivative term \(\frac{d X_n(z)}{d z} (X_n(z))^{-1}\) involves the coefficients \(\{X_n^{(j)}\}_{j=1}^{r-1}\). For the ordinary and Fuchsian case, the derivative term do not contribute at all, and we only need the concourse of \(\{X_n^{(j)}\}_{j=1}^{r+1}\). As we have seen in Proposition 23 all the coefficients can be parametrized in terms of the Verblunsky matrices \(\{\alpha_{1,m}^L, \alpha_{2,m}^R\}\), the quasi-tau matrices \(\{H_m^R\}\) and the initial condition \(X_{m=1}^j\). The idea is to compare this expression, obtained about infinity, with the result obtained in Theorem 3 and, given that \(M_n(z)\) is analytic in the annulus \(\mathbb{C} \setminus \{0\}\), equate both results. As a consequence, we will have a set of four, in general nonlinear, discrete matrix equations for \(\{\alpha_{1,m}^L, \alpha_{2,m}^R\}\) and \(\{H_m^R\}\).

Notice that the term \(-nNz^{-1}\) appearing in \((65)\) will always contribute non trivially to the leading term computed about infinity with, for example and among many others, terms of the form
\[
\left[X_n^{(r-1)}, \left(W_{-1} - 0_N \begin{pmatrix} 0_N & 0_N \\ -nN \end{pmatrix}\right)\right], \quad \frac{1}{2} \left[X_n^{(1)}, \left[X_n^{(r-2)}, \left(W_{-1} - 0_N \begin{pmatrix} 0_N & 0_N \\ -nN \end{pmatrix}\right)\right]\right], \quad \frac{1}{2} \left[X_n^{(r-2)}, \left[X_n^{(1)}, \left(W_{-1} - 0_N \begin{pmatrix} 0_N & 0_N \\ -nN \end{pmatrix}\right)\right]\right], \quad \cdots
\]
even when \(W_{-1}\) does cancel. Thus, the most simple situation with nonlinear contributions (cubic of the Verblunsky matrices, appears when \(W(z)\) has only three consecutive powers in \(z\) with \(z^{-1}\) among them.

Namely, we are dealing with one of the following three cases
\begin{align*}
(66) & \quad W(z) = W_{-1}z^{-1} + W_0 + W_1z, \\
(67) & \quad W(z) = W_{-2}z^{-2} + W_{-1}z^{-1} + W_0, \\
(68) & \quad W(z) = W_{-3}z^{-3} + W_{-2}z^{-2} + W_{-1}z^{-1},
\end{align*}

While in the first case \((66)\) we deal with a Fuchsian singularity at \(z=0\) in the two remaining cases \((67)\) and \((68)\) we have non-Fuchsian singularities. As we will see \((66)\) and \((67)\) lead to interesting matrix extensions of a discrete Painlevé II system for the Verblunsky coefficients of Szegő biorthogonal polynomials on the unit circle. Despite these nonlinear equations do have non local terms, they cancel when the corresponding leading terms about infinity of \(W(z)\) are proportional to the identity matrix. For the third case \((68)\) we have found that even in the scalar situation, \(N = 1\), there are non local terms. But not only, apart from the Verblunsky matrices \(\{(\alpha_{2,m}^R)^+, \alpha_{1,m}^L\}\) this case requires the concourse of the quasi-tau matrices \(H_n^R\). We have chosen to constrain our treatment just to he more interesting first two cases.

In both situations we need the following technical result

**Proposition 32.** For any matrix \(A \in \mathbb{C}^{N \times N}\) it holds that
\[
-Ab_n^{(2)} + A(a_n b_n + b_n d_n) - a_n Ab_n = -A(b_{n+1} + b_n c_{n+1} b_n) + [A, a_n] b_n, \\
-Ac_n^{(2)} + c_n Aa_n = (c_{n-1} + c_n b_{n-1} c_n) A - c_n [A, a_n].
\]

**Proof.** It follows from \((32)\) and \((41)\). \(\square\)

5.1. The Fuchsian case.
5.1.1. Monodromy free condition. Let us consider the choice
\[(69) W(z) = W_{-1}z^{-1} + W_0 + W_1z.\]

**Proposition 33.** For the Fuchsian case the matrix Hölder condition on the matrix of weights \(w(\zeta), \zeta \in \mathbb{T}\), requires \(W_{-1}\) to be a diagonalizable matrix with integer eigenvalues.

**Proof.** We follow [15]. Let us take \(W(z) = W_{-1}z^{-1}\) the monodromy matrix is \(M = \exp(2\pi i W_{-1})\), and the fundamental solution is \(S(z) = z^{W_{-1}}S_0\) where \(S_0 \in \text{GL}(N, \mathbb{C})\) is a non-singular matrix, see §2.3 of [15]. Hence, we have \(w(z) = z^{W_{-1}}S_0w_0\), with \(w_0 \in \mathbb{C}^N\). Here, \(z^{W_{-1}} := \exp(W_{-1} \log z)\) in a multivalued sense as we have that its matrix coefficients are linear combinations of multivalued functions \(z^{i\lambda} \log^k z\), where \(k \in \{0, 1, \ldots\}\) and \(\lambda\) runs through the eigenvalue of \(W_{-1}\). To ensure that the corresponding matrix of weights \(w(z)\) is single valued we require trivial monodromy and, therefore, \(W_{-1}\) should be diagonalizable with integer eigenvalues,
\[W_{-1} = P\Lambda_{-1}P^{-1}, \quad \Lambda_{-1} = \text{diag}(\lambda_1, \ldots, \lambda_N), \quad P \in \text{GL}(N, \mathbb{C}),\]
with \(\lambda_i \in \mathbb{Z}, i \in \{1, \ldots, N\}\). Indeed, if this is the case we have
\[\exp(W_{-1} \log z) = P \exp(\Lambda_{-1} \log z)P^{-1} = P \text{diag}(z^{\lambda_1}, \ldots, z^{\lambda_N})P^{-1},\]
which is not multivalued because the eigenvalues \(\lambda_i\) are integers. Any meromorphically equivalent system corresponds to gauge transformations of the following type
\[(70) W(z) = \frac{d\Phi(z)}{dz} (\Phi(z))^{-1} + \Phi(z)W_{-1}z^{-1}(\Phi(z))^{-1},\]
where \(\Phi : \mathbb{C} \setminus \{0\} \to \text{GL}(N, \mathbb{C})\) is analytic and has at most a pole at the origin [15]. All equivalent systems have equivalent monodromies: \(\tilde{M} = \Phi(z_0)M(\Phi(z_0))^{-1}\). In particular, trivial monodromy \(M = I_N\) is preserved after equivalence transformations. Thus, all the equivalent systems (70) have trivial monodromy. Following Theorem in §2.3 in [15] we know that all systems with a regular singularity at the origin are equivalent to a system with \(W(z) = W_{-1}z^{-1}\). Consequently, all the cases we consider will be of the form \(W_{-1}z^{-1} + W_0 + W_1z + \cdots\), where \(W_{-1}\) is diagonalizable with integer eigenvalues. \(\Box\)

5.1.2. Derivation of the matrix discrete Painlevé system. From
\[M_n(z) = \frac{dX_n(z)}{dz} (X_n(z))^{-1} + X_n(z) \begin{pmatrix} W_{-1}z^{-1} + W_0 + W_1z & 0_N \\ 0_N & -nI_Nz^{-1} \end{pmatrix} (X_n(z))^{-1},\]
we deduce
\[(71) M_n^{(0)} = \begin{pmatrix} W_{-1} & 0_N \\ 0_N & -nI_N \end{pmatrix} + X_n^{(1)} \begin{pmatrix} W_0 & 0_N \\ 0_N & 0_N \end{pmatrix} + X_n^{(2)} \begin{pmatrix} W_1 & 0_N \\ 0_N & 0_N \end{pmatrix} (X_n^{(1)})^2 - X_n^{(1)} \begin{pmatrix} W_1 & 0_N \\ 0_N & 0_N \end{pmatrix} X_n^{(1)},\]
which gives
\[(72) W_{-1} + [a_n, W_0] + [a_n^{(2)}, W_1] + W_1([a_n]^2 + b_n c_n) - a_n W_1 a_n = \alpha_{n,n}^1 (W_{-1} - nI_N)(\alpha_{2,n}^R)^\dagger,\]
\[(73) -W_0 b_n - W_1 b_n^{(2)} + W_1 (a_n b_n + b_n d_n) - a_n W_1 b_n = -\alpha_{n,n}^1 (W_{-1} - nI_N)H_R,\]
\[(74) c_n W_0 + c_n^{(2)} W_1 - c_n W_1 a_n = -(H_{n-1}^R)^{-1}(W_{-1} - nI_N)(\alpha_{2,n}^R)^\dagger,\]
\[(75) -nI_N - c_n W_1 b_n = (H_{n-1}^R)^{-1}(W_{-1} - nI_N)H_R.\]

\(^8\) Note that its analytic continuation will have a logarithmic ramification at the origin, see [15]
Theorem 4 (Fuchsian matrix discrete Painlevé II system). When the right logarithmic derivative of the matrix of measures is \( W(z) = W_{-1}z^{-1} + W_0 + W_0z \), with \( W_{-1} \) a diagonalizable matrix with entire eigenvalues so that (46) is monodromy free, the corresponding Verblunsky coefficients solve to the following nonlinear matrix difference equations

\[
\begin{align*}
(76) \quad W_0 & \alpha_{1,n}^L + W_1 \alpha_{1,n+1}^L - \alpha_{1,n-1}^L (W_{-1} - (n - 1)I_N), \\
& = W_1 \left( \alpha_{1,n+1}^L (\alpha_{2,n}^R)^\dagger + \alpha_{1,n}^L (\alpha_{2,n-1}^R)^\dagger \alpha_{1,n}^L \right) + \left[ W_1, \sum_{m=1}^{n-1} \alpha_{1,m}^L (\alpha_{2,m-1}^R)^\dagger \right] \alpha_{1,n}^L,
\end{align*}
\]

\[
(77) \quad (\alpha_{2,n}^R)^\dagger W_0 + (\alpha_{2,n}^R)^\dagger W_1 - (W_{-1} - (n + 1)I_N)(\alpha_{2,n}^R)^\dagger
\]

\[
= \left( (\alpha_{2,n}^R)^\dagger \alpha_{1,n}^L (\alpha_{2,n-1}^R)^\dagger + (\alpha_{2,n}^R)^\dagger \alpha_{1,n+1}^L (\alpha_{2,n}^R)^\dagger \right) W_1 + (\alpha_{2,n}^R)^\dagger \left[ W_1, \sum_{m=1}^{n+1} \alpha_{1,m}^L (\alpha_{2,m-1}^R)^\dagger \right].
\]

where \( n \in \{1, 2, \ldots \} \).

Proof. Equations (73) and (74) can be rewritten as follows

\[
-W_0 b_n - W_1 (b_{n+1} + b_n c_{n+1} b_n) + [W_1, a_n] b_n = -\alpha_{1,n}^L (W_{-1} - nI_N) H_n^R,
\]

\[
c_n W_0 + (c_{n-1} + c_n b_n c_{n-1}) W_1 - c_n W_1, a_n = -(H_{n-1}^R)^{-1} (W_{-1} - nI_N) (\alpha_{2,n}^R)^\dagger.
\]

Using Proposition 32 and equation (38) we find that

\[
-W_0 \alpha_n - W_1 \left( \alpha_{n+1}^L H_n^R (H_{n-1}^R)^{-1} - \alpha_n^L (\alpha_{2,n}^R)^\dagger \alpha_n^L \right) + \left[ W_1, \sum_{m=1}^{n-1} \alpha_{1,m}^L (\alpha_{2,m}^R)^\dagger \right] \alpha_n^L
\]

\[
= -\alpha_{1,n-1}^L (W_{-1} - (n - 1)I_N),
\]

\[
-(\alpha_{2,n}^R)^\dagger W_0 - \left( H_n^R (H_{n-1}^R)^{-1} (\alpha_{2,n}^R)^\dagger - (\alpha_{2,n}^R)^\dagger \alpha_{1,n+1}^L (\alpha_{2,n}^R)^\dagger \right) W_1 + (\alpha_{2,n}^R)^\dagger \left[ W_1, \sum_{m=1}^{n+1} \alpha_{1,m}^L (\alpha_{2,m}^R)^\dagger \right]
\]

\[
= -(W_{-1} - (n + 1)I_N) (\alpha_{2,n+1}^R)^\dagger.
\]

That reads

\[
-W_0 \alpha_n - W_1 \left( \alpha_{n+1}^L - \alpha_n^L (\alpha_{2,n}^R)^\dagger \alpha_n^L - \alpha_n^L (\alpha_{2,n-1}^R)^\dagger \alpha_n^L \right) + \left[ W_1, \sum_{m=1}^{n-1} \alpha_{1,m}^L (\alpha_{2,m}^R)^\dagger \right] \alpha_n^L
\]

\[
= -\alpha_{1,n-1}^L (W_{-1} - (n - 1)I_N),
\]

\[
-(\alpha_{2,n}^R)^\dagger W_0 - \left( (\alpha_{2,n}^R)^\dagger - (\alpha_{2,n}^R)^\dagger \alpha_n^L (\alpha_{2,n-1}^R)^\dagger - (\alpha_{2,n}^R)^\dagger \alpha_{1,n+1}^L (\alpha_{2,n}^R)^\dagger \right) W_1 + (\alpha_{2,n}^R)^\dagger \left[ W_1, \sum_{m=1}^{n+1} \alpha_{1,m}^L (\alpha_{2,m}^R)^\dagger \right]
\]

\[
= -(W_{-1} - (n + 1)I_N) (\alpha_{2,n+1}^R)^\dagger,
\]

and the result follows. \(\square\)

Proposition 34. The matrix discrete Painlevé II system, given by equations (76) and (76), imply the equations obtained from (72) and (75) by a discrete derivative in \( n \).

Proof. Consider the matrix discrete Painlevé II system (73) and (74). Let us show that (73) \& (74) \(\Rightarrow (72)'\), where (72)' refers to the difference of the equation (72) at sites \( n + 1 \) and \( n \),

\[
-[a_{n+1} - a_n, W_0] - ((a_{n+1} - a_n)a_n - b_n c_n) W_1
\]

\[
- W_1 ((a_{n+1})^2 + b_{n+1} c_{n+1} - (a_n)^2 - b_n c_n - (a_{n+1} - a_n)a_n - b_n c_n) + a_{n+1} W_1 a_{n+1} - a_n W_1 a_n
\]

\[
= \alpha_{1,n}^L (W_{-1} - nI_N) (\alpha_{2,n}^R)^\dagger - \alpha_{1,n+1}^L (W_{-1} - (n + 1)I_N) (\alpha_{2,n}^R)^\dagger,
\]
where \([41]\) have been used. If we introduce \([32]\) in the previous relation we get

\[
[b_n c_{n+1}, W_0] + (b_n c_{n+1} a_n + b_n c_n) W_1
\]

\((78)\)

\[-W_1(-a_{n+1} b_n c_{n+1} + b_{n+1} c_{n+1}) + a_{n+1} W_1 a_{n+1} - a_n W_1 a_n
\]

\[= \alpha^L_{1,n} (W_{-1} - n I_N) (\alpha^R_{2,n})^\dagger - \alpha^L_{1,n+1} (W_{-1} - (n+1) I_N) (\alpha^R_{2,n+1})^\dagger.\]

The difference of equation \((73)\), multiplied on its right by \(c_{n+1}\), with \((74)\), evaluated at \(n+1\) and left multiplied by \(b_n\) gives, once \((41)\) is used again,

\[-W_1(b_{n+1} c_{n+1} - a_{n+1} a_n) c_{n+1} + [W_1, a_n] b_n c_{n+1} + [b_n c_{n+1}, W_0] + b_n (c_n - c_{n+1}(a_{n+1} - a_n)) W_1 - b_n c_{n+1} [W_1, a_{n+1}]
\]

\[= \alpha^L_{1,n} (W_{-1} - n I_N) (\alpha^R_{2,n})^\dagger - \alpha^L_{1,n+1} (W_{-1} - (n+1) I_N) (\alpha^R_{2,n+1})^\dagger,
\]

and recalling \((32)\) we conveniently express it as

\[-W_1(b_{n+1} c_{n+1} - a_{n+1} b_n) c_{n+1} + [W_1, a_n] b_n c_{n+1} + [b_n c_{n+1}, W_0] + b_n (c_n - c_{n+1}(a_{n+1} - a_n)) W_1 - b_n c_{n+1} [W_1, a_{n+1}]
\]

\((79)\)

\[= \alpha^L_{1,n} (W_{-1} - n I_N) (\alpha^R_{2,n})^\dagger - \alpha^L_{1,n+1} (W_{-1} - (n+1) I_N) (\alpha^R_{2,n+1})^\dagger.
\]

Therefore, and comparison of \((78)\) with \((79)\) gives the desired result.

Let us show that \((73)\) & \((74)\) \(\Rightarrow\) \((75)\), where \((75)\)' is the discrete derivative of \((75)\):

\[I_N + c_{n+1} W_1 b_{n+1} - c_n W_1 b_n = (H_{n-1}^R)^{-1} (W_{-1} - n I_N) H_n^R - (H_n^R)^{-1} (W_{-1} - (n+1) I_N) H_{n+1}^R.
\]

Using \((38)\) it can be written as follows

\[I_N - (H_{n-1}^R)^{-1} (\alpha^R_{2,n})^\dagger W_1 \alpha^L_{1,n+2} H_{n+1}^R + (H_{n-1}^R)^{-1} (\alpha^R_{2,n-1})^\dagger W_1 \alpha^L_{1,n+1} H_{n+1}^R
\]

\[= (H_{n-1}^R)^{-1} (W_{-1} - n I_N) H_n^R - (H_n^R)^{-1} (W_{-1} - (n+1) I_N) H_{n+1}^R,
\]

so that

\[I_N - (\alpha^R_{2,n})^\dagger W_1 \alpha^L_{1,n+2} H_{n+1}^R (H_{n-1}^R)^{-1} + H_n^R (H_{n-1}^R)^{-1} (\alpha^R_{2,n-1})^\dagger W_1 \alpha^L_{1,n+1}
\]

\[= H_n^R (H_{n-1}^R)^{-1} (W_{-1} - n I_N) - (W_{-1} - (n+1) I_N) H_{n+1}^R (H_{n-1}^R)^{-1},
\]

and simplifying we arrive to

\[-(\alpha^R_{2,n})^\dagger W_1 \alpha^L_{1,n+2} (I_N - (\alpha^R_{2,n+1})^\dagger \alpha^L_{1,n+1}) + (I_N - (\alpha^R_{2,n})^\dagger \alpha^L_{1,n}) (\alpha^R_{2,n-1})^\dagger W_1 \alpha^L_{1,n+1}
\]

\[= -(\alpha^R_{2,n})^\dagger \alpha^L_{1,n} (W_{-1} - n I_N) + (W_{-1} - (n+1) I_N) (\alpha^R_{2,n+1})^\dagger \alpha^L_{1,n+1}.
\]

Equation \((80)\) is gotten by multiplication on the left of \((76)\), evaluated at site \(n+1\), by \((\alpha^R_{2,n})^\dagger\) and on the right \((77)\) by \(\alpha^L_{1,n+1}\), and then taking its difference. \(\square\)

5.1.3. Reduction to a linear system. A simplification, that leads to a linear system, is to take \(W_1 = 0_N\), so that

**Proposition 35.** For a matrix of weights with right logarithmic derivative given by \(W(z) = W_{-1} z^{-1} + W_0\), where \(W_{-1}\) is a diagonalizable matrix with eigenvalues being strictly negative integers and \(W_0\) is not singular, the Verblunsky coefficients are subject to the system of equations

\[\alpha^L_{1,n} (\alpha^R_{2,n-1})^\dagger, W_0 = \alpha^L_{1,n} (W_{-1} - n I_N) (\alpha^R_{2,n})^\dagger - \alpha^L_{1,n-1} (W_{-1} - (n-1) I_N) (\alpha^R_{2,n-1})^\dagger,
\]

\[W_0 \alpha^L_{1,n+1} = \alpha^L_{1,n} (W_{-1} - n I_N),
\]

\[(\alpha^R_{2,n-1})^\dagger W_0 = (W_{-1} - n I_N) (\alpha^R_{2,n})^\dagger,
\]

\[-n (I_N - (\alpha^R_{2,n})^\dagger \alpha^L_{1,n})^{-1} = W_{-1} - n I_N.
\]
The solution of which is
\[ \alpha_{i,n}^+ = (W_0)^{-n+1} \alpha_{i,1}^+ (W_{-1} - \Lambda I_N) \cdots (W_{-1} - (n-1)\Lambda I_N), \]
\[ (\alpha_{2,n}^R) = (W_{-1} - n\Lambda I_N)^{-1} \cdots (W_{-1} - 2\Lambda I_N)^{-1} (\alpha_{2,1}^R)^\dagger (W_0)^{n-1}, \]
with initial values constrained by
\[ (86) \]
\[ (\alpha_{2,1}^R)^\dagger \alpha_{1,1}^+ = W_{-1} (W_{-1} - \Lambda I_N)^{-1}. \]

Proof. As preliminary condition we need to ensure that \((43)\) with \(W_{-1}z^{-1} + W_0\) gives a single valued weight, from Proposition \(33\) we see that is indeed the case. To proceed, we see that \((81)\) is a consequence of \((82)\) and \((83)\). The linear system given by \((82)\) and \((83)\) can be written as
\[ \alpha_{i,n}^+ = (W_0)^{-n+1} \alpha_{i,1}^+ (W_{-1} - (n-1)\Lambda I_N), \]
\[ (\alpha_{2,n}^R) = (W_{-1} - n\Lambda I_N)^{-1} (\alpha_{2,n-1}^R)^\dagger W_0. \]

Then, a complete iteration leads to the solution \((85)\). Observe that \((84)\) is formalized that
\[ \begin{bmatrix} \alpha_{2,n}^R \alpha_{1,n}^+ \end{bmatrix} = 0_N \]
and, consequently, we find
\[ (\alpha_{2,n}^R)^\dagger \alpha_{1,n}^+ = (W_{-1} - n\Lambda I_N)^{-1} \cdots (W_{-1} - 2\Lambda I_N)^{-1} (\alpha_{2,1}^R)^\dagger (W_{-1} - \Lambda I_N) \cdots (W_{-1} - (n-1)\Lambda I_N) \]
\[ = (W_{-1} - n\Lambda I_N)^{-1} (W_{-1} - \Lambda I_N) (\alpha_{2,1}^R)^\dagger \alpha_{1,1}^+ \]
so that
\[ (W_{-1} - n\Lambda I_N) (\Lambda I_N - (\alpha_{2,n}^R)^\dagger \alpha_{1,n}^+) = W_{-1} - n\Lambda I_N - (W_{-1} - n\Lambda I_N) (\alpha_{2,n}^R)^\dagger \alpha_{1,n}^+ \]
\[ = -n\Lambda I_N + W_{-1} - (W_{-1} - \Lambda I_N) (\alpha_{2,1}^R)^\dagger \alpha_{1,1}^+. \]

Hence we derive the constraint \((86)\) for the first Verblunsky coefficients. \(\square\)

5.2. The non-Fuchsian case. A more involved example is given by a logarithmic derivative of the matrix of weights of the following type
\[ W(z) = W_{-2}z^{-2} + W_{-1}z^{-1} + W_0. \]

5.2.1. Monodromy free and Stokes phenomena. According to \([57]\), Proposition 1.1, when \(W_{-2}\) is diagonalizable with \(N\) different eigenvalues
\[ W_{-2} = P \Lambda_{-2} P^{-1}, \quad P \in \text{GL}(N, \mathbb{C}), \quad \Lambda_{-2} = \text{diag}(\alpha_1, \ldots, \alpha_N), \quad \alpha_i \neq \alpha_j, \quad i \neq j, \]
the unique formal fundamental solution of \((43)\) is
\[ S(z) = P \left( \sum_{k=0}^{\infty} \sigma_k z^k \right) e^{\Delta(z)}, \quad \Delta(z) := -\Lambda_{-1}z^{-1} + \Lambda_0 \log z + \Lambda_1 + \Lambda_2 z + \ldots \]
with \(\Lambda_k\) and \(\sigma_k\) diagonal and off-diagonal matrices, respectively. These matrices \(\sigma_n\) are determined by the equations
\[ \Lambda_n + [\sigma_{n+1}, \Lambda_{-1}] = F_n, \quad n \in \{0, 1, 2, \ldots\} \]
where
\[ F_0 = P^{-1}W_{-1}P, \]
\[ F_1 = P^{-1}W_0P + P^{-1}W_{-1}P \sigma_1 - \sigma_1 W_{-1} - \sigma_1, \]
\[ F_n = P^{-1}W_0P \sigma_{n-1} - \sigma_{n-1} \Lambda_1 + P^{-1}W_{-1}P \sigma_n - \sigma_n \Lambda_0 - n \sigma_n, \quad n \in \{2, 3, \ldots\}. \]
From \((87)\) we can get uniquely and recursively all terms in the formal expansion, as the operator \(\text{ad}_{\Lambda_{-1}}\) is invertible in the space of off-diagonal matrices, in fact a polynomial in \((\text{ad}_{\Lambda_{-1}})^{-1} = P (\text{ad}_{\Lambda_{-1}})^{-1} P\). For example, \(\Lambda_0 = (P^{-1}W_{-1}P)^\text{diag}\) and \(\sigma_1 = P (\text{ad}_{\Lambda_{-1}})^{-1} (P^{-1}W_{-1}P)^\text{off}\), where we are projecting in the spaces of diagonal and off-diagonal matrices. Thus, multivaluedness of the weight \(w(z)\) is formally avoided when
the diagonal elements are integers, \((P^{-1}W_{-1}P)_{i,i} \in \mathbb{Z}\), for \(i \in \{1, \ldots, N\}\). However, the Stokes phenomena, i.e., the existence of sectors (delimited by the Stokes rays which are determined by the conditions \(\text{Re}(\alpha_j - \alpha_i z^{-1}) = 0\)) where the formal solution is asymptotic to the genuine fundamental matrix can not be avoided and a further study is needed.

5.2.2. The non-Fuchsian matrix discrete Painlevé system and the Heisenberg algebra. We have seen that matrix discrete Painlevé II system \((93)\) and \((94)\) emerges naturally when the logarithmic derivative of the matrix of measures is \(W(z) = W_{-2} z^{-2} + W_{-1} z^{-1} + W_0\). Let us mention an explicit example of a matrix of measures whose right logarithmic derivative is of the mentioned type. However, in opposition with the previous discussion \(W_{-2}\) is not diagonalizable. The matrix of weights is of Freud type and has the form

\[
w(z) = \exp(V(z)), \quad V(z) = -W_{-2}z^{-1} + W_0z,
\]

where \(W_{-2}, W_0 \in \mathbb{C}^{N \times N}\) are matrices such that with \(W_{-1} = [W_{-2}, W_0]\) form a Heisenberg algebra

\[
[W_{-2}, W_{-1}] = 0_N, \quad [W_0, W_{-1}] = 0_N, \quad [W_{-2}, W_0] = W_{-1}.
\]

Indeed, we can compute the logarithmic right derivative with the aid of the formula

\[
\frac{d}{dz} (w(z))^{-1} = \sum_{j=0}^{\infty} \frac{1}{(j+1)!} (\text{ad}_{V(z)})^j \left( \frac{d}{dz} V(z) \right),
\]

where \(\text{ad}_A(B) = [A, B]\) and get

\[
\frac{d}{dz} (w(z))^{-1} = W_{-2}z^{-2} + W_{-1}z^{-1} + W_0.
\]

Observe that in this case the matrices \(W_{-2}\) and \(W_0\) must be nilpotent, and therefore not diagonalizable.

5.2.3. Derivation of the matrix discrete Painlevé II system. We have

\[
M_n(z) = \frac{dX_n(z)}{dz} (X_n(z))^{-1} + X_n(z) \left( \frac{W_{-2}z^{-2} + W_{-1}z^{-1} + W_0}{0_N - nI_Nz^{-1}} \right) (X_n(z))^{-1},
\]

and we deduce

\[
M_n^{[0]} = -X_n^{(1)} + \left( \begin{array}{cc}
W_{-2} & 0_N \\
0_N & 0_N
\end{array} \right) + X_n^{(1)} \left( \begin{array}{cc}
W_{-1} & 0_N \\
0_N & -nI_N
\end{array} \right) - \left( \begin{array}{cc}
W_{-1} & 0_N \\
0_N & -nI_N
\end{array} \right) X_n^{(1)}
\]

\[
+ X_n^{(2)} \left( \begin{array}{cc}
W_0 & 0_N \\
0_N & 0_N
\end{array} \right) - \left( \begin{array}{cc}
W_0 & 0_N \\
0_N & 0_N
\end{array} \right) X_n^{(2)} + \left( \begin{array}{cc}
W_0 & 0_N \\
0_N & 0_N
\end{array} \right) (X_n^{(1)})^2 - X_n^{(1)} \left( \begin{array}{cc}
W_0 & 0_N \\
0_N & 0_N
\end{array} \right) X_n^{(1)}.
\]

Consequently,

\[
-a_n + W_{-2} + [a_n, W_{-1}] + [a_n^{(2)}, W_0] + W_0((a_n)^2 + b_n c_n) - a_n W_0 a_n = \alpha_{1,n}^R W_{-2}(\alpha_{2,n}^R)^{\dagger},
\]

\[
-b_n - (W_{-1} + nI_N)b_n - W_0(b_n^{(2)} + W_0(a_n b_n + b_n d_n)) - a_n W_0 b_n = -\alpha_{1,n}^R W_{-2}H_{1,n}^R,
\]

\[
-c_n + c_n(W_{-1} + nI_N) + c_n^{(2)} W_0 - c_n W_0 a_n = -(H_{1,n}^R)^{-1} W_{-2}(\alpha_{2,n}^R)^{\dagger},
\]

\[
-d_n - c_n W_0 b_n = (H_{1,n}^R)^{-1} W_{-2} H_{1,n}^R.
\]

**Theorem 5** (A non-Fuchsian matrix discrete Painlevé II system). When the right logarithmic derivative of the matrix of measures is \(W(z) = W_{-2} z^{-2} + W_{-1} z^{-1} + W_0\), with \(W(z)\) such that \((46)\) is monodromy free, the corresponding Verblunsky coefficients provide solutions to the following nonlinear matrix difference equations

\[
(W_{-1} + nI_N) \alpha_{1,n}^I + W_0 \alpha_{1,n+1}^I - \alpha_{1,n-1}^I W_{-2}
\]

\[
= W_0 \left( \alpha_{1,n+1}^I (\alpha_{2,n}^R)^{\dagger} \alpha_{1,n}^I + \alpha_{1,n}^I (\alpha_{2,n-1}^R)^{\dagger} \alpha_{1,n}^I \right) + \left[ W_0, \sum_{m=1}^{n-1} \alpha_{1,m}^I (\alpha_{2,m-1}^R)^{\dagger} \right] \alpha_{1,n}^I,
\]

\[
(\alpha_{2,n}^R)^{\dagger} (W_{-1} + nI_N) + (\alpha_{2,n-1}^R)^{\dagger} W_0 - W_{-2} (\alpha_{2,n+1}^R)^{\dagger}
\]

\[
= \left( (\alpha_{2,n}^R)^{\dagger} \alpha_{1,n}^I (\alpha_{2,n-1}^R)^{\dagger} + (\alpha_{2,n}^R)^{\dagger} \alpha_{1,n+1}^I (\alpha_{2,n}^R)^{\dagger} \right) W_0 + (\alpha_{2,n}^R)^{\dagger} \left[ W_0, \sum_{m=1}^{n+1} \alpha_{1,m}^I (\alpha_{2,m-1}^R)^{\dagger} \right],
\]
where \( n \in \{1, 2, \ldots \} \).

**Proof.** Proposition 32 allows us for expressing (90) and (91) as follows

\[
(W_{-1} + nI_N) b_{n-1} + W_0(b_n + b_{n-1}c_nb_{n-1}) = \alpha_{1,n-1}^L W_{-2} H_{n-1}^R + [W_0, a_{n-1}] b_{n-1},
\]

\[
c_{n+1}(W_{-1} + nI_N) + (c_n + c_{n+1}b_{n+1}) W_0 = - (H_{n}^R)^{-1} W_{-2} (\alpha_{2,n+1}^R)^\dagger + c_{n+1}[W_0, a_{n+1}].
\]

Then,

\[
(W_{-1} + nI_N) \alpha_{1,n}^L + W_0 \alpha_{1,n-1}^L H_{n}^R (H_{n-1}^R)^{-1} - W_0 \alpha_{1,n}^L (\alpha_{2,n-1}^R)^\dagger + \alpha_{1,n}^L = \alpha_{1,n-1}^L W_{-2} + [W_0, a_{n-1}] \alpha_{1,n},
\]

\[
(\alpha_{2,n}^R)^\dagger (W_{-1} + nI_N) + H_{n}^R (H_{n-1}^R)^{-1} (\alpha_{2,n}^R)^\dagger W_0 - (\alpha_{2,n}^R)^\dagger \alpha_{1,n+1}^L (\alpha_{2,n}^R)^\dagger W_0 = W_2(\alpha_{2,n+1}^R)^\dagger + (\alpha_{2,n}^R)^\dagger [W_0, a_{n+1}],
\]

and we get the result. \( \Box \)

**Proposition 36.** The matrix discrete Painlevé II system, given by equations (93) and (94), imply the equations obtained from (89) and (92) by a discrete derivative in \( n \).

**Proof.** We will consider the matrix discrete Painlevé II system written in the equivalent form (90) and (91). We first show the implication: (90) & (91) \( \Rightarrow \) (89). The difference of the equations (89) at \( n + 1 \) and \( n \) gives

\[
(a_{n+1} - a_n) - [a_{n+1} - a_n, W_{-1}] - ((a_{n+1} - a_n) a_n - b_n c_n) W_0
\]

\[
- W_0((a_{n+1})^2 + b_{n+1}c_{n+1} - (a_n)^2 - b_n c_n - (a_{n+1} - a_n) a_n + b_n c_n) + a_{n+1} W_0 a_{n+1} - a_n W_0 a_n
\]

\[
= \alpha_{1,n}^L W_{-2} (\alpha_{2,n}^R)^\dagger - \alpha_{1,n+1}^L W_{-2} (\alpha_{2,n+1}^R)^\dagger,
\]

where we have used (41). If we introduce (32) in the previous relation we get

\[
-b_n c_n + [b_n c_n + W_{-1}] + (b_n + b_n c_n) W_0
\]

\[
- W_0(-a_{n+1} b_n c_{n+1} + b_n c_{n+1}) + a_{n+1} W_0 a_{n+1} - a_n W_0 a_n = \alpha_{1,n}^L W_{-2} (\alpha_{2,n}^R)^\dagger - \alpha_{1,n+1}^L W_{-2} (\alpha_{2,n+1}^R)^\dagger.
\]

The difference of equation (90), multiplied on its right by \( c_{n+1} \), with (91), evaluated at \( n + 1 \) and left multiplied by \( b_n \), gives

\[
- W_0(\alpha_{n+1} - (a_{n+1} - a_n) b_n) c_{n+1} + [W_0, a_n] b_n c_{n+1}
\]

\[
- W_0((a_{n+1})^2 + b_{n+1}c_{n+1} - (a_n)^2 - b_n c_n - (a_{n+1} - a_n) a_n + b_n c_n) + a_{n+1} W_0 a_{n+1} - a_n W_0 a_n
\]

\[
= \alpha_{1,n}^L W_{-2} (\alpha_{2,n}^R)^\dagger - \alpha_{1,n+1}^L W_{-2} (\alpha_{2,n+1}^R)^\dagger,
\]

which after some cleaning and the use of (32) leads to

\[
- W_0(\alpha_{n+1} - (a_{n+1} - a_n) b_n) c_{n+1} + [W_0, a_n] b_n c_{n+1}
\]

\[
= \alpha_{1,n}^L W_{-2} (\alpha_{2,n}^R)^\dagger - \alpha_{1,n+1}^L W_{-2} (\alpha_{2,n+1}^R)^\dagger.
\]

Therefore, and comparison of (95) with (96) gives the desired result.

We now prove the implication: (90) & (91) \( \Rightarrow \) (92). The discrete derivative of (92) gives

\[
d_{n+1} - d_n + c_{n+1} W_0 b_{n+1} - c_n W_0 b_n = (H_{n}^R)^{-1} W_{-2} H_{n+1}^R - (H_{n-1}^R)^{-1} W_{-2} H_{n+1}^R.
\]

Now, using (41) and using (40), let us write this equation in an equivalent form,

\[
- (H_{n}^R)^{-1} (\alpha_{2,n}^R)^\dagger \alpha_{1,n+1}^L H_{n}^R - (H_{n-1}^R)^{-1} (\alpha_{2,n}^R)^\dagger W_0 \alpha_{1,n+2}^L H_{n+1}^R + (H_{n}^R)^{-1} (\alpha_{2,n}^R)^\dagger W_0 \alpha_{1,n+1}^L H_{n}^R
\]

\[
= (H_{n}^R)^{-1} W_{-2} H_{n+1}^R - (H_{n-1}^R)^{-1} W_{-2} H_{n+1}^R,
\]

that is

\[
- (\alpha_{2,n}^R)^\dagger \alpha_{1,n+1}^L W_0 \alpha_{1,n+2}^L H_{n+1}^R (H_{n}^R)^{-1} + H_{n}^R (H_{n-1}^R)^{-1} (\alpha_{2,n}^R)^\dagger W_0 \alpha_{1,n+1}^L
\]

\[
= H_{n}^R (H_{n-1}^R)^{-1} W_{-2} - W_{-2} H_{n+1}^R (H_{n}^R)^{-1},
\]
and after some cleaning reads

\[
\begin{align*}
(97) \quad - (\alpha_{2,n}^R)^\dagger \alpha_{1,n+1}^L + (\alpha_{2,n-1}^R)^\dagger W_0 \alpha_{1,n+1}^L - (\alpha_{2,n}^R)^\dagger \alpha_{1,n}^L \alpha_{2,n-1}^R \dagger W_0 \alpha_{1,n+1}^L + (\alpha_{2,n}^R)^\dagger \alpha_{1,n}^L W_{-2}.
\end{align*}
\]

\[
= (\alpha_{2,n}^R)^\dagger W_0 \alpha_{1,n+2}^L - (\alpha_{2,n}^R)^\dagger W_0 \alpha_{1,n+2}^L + W_{-2}(\alpha_{2,n+1}^R)^\dagger \alpha_{1,n+1}^L.
\]

Now, we manipulate the matrix discrete Painlevé system. Let us multiply on the left of (93) by \((\alpha_{2,n}^R)^\dagger\) and on the right (94) by \(\alpha_{1,n+1}^L\), and then takes its difference to get (97).

5.2.4. Reduction to a linear system. A simplification is to take \(W_0 = 0_N\), and then a new linear system for the Verblunsky coefficients appears

**Proposition 37.** When \(W(z) = W_{-2} z^{-2} + W_{-1} z^{-1}\) the Verblunsky coefficients are subject to

\[
\begin{align*}
(98) \quad - \alpha_{1,n}^L (\alpha_{2,n-1}^R)^\dagger + [\alpha_{1,n}^L (\alpha_{2,n-1}^R)^\dagger W_{-1}] &= \alpha_{1,n}^L W_{-2} (\alpha_{2,n}^R)^\dagger - \alpha_{1,n-1}^L W_{-2} (\alpha_{2,n-1}^R)^\dagger, \\
(99) \quad (W_{-1} + (n + 1) I_N) \alpha_{1,n+1}^L &= \alpha_{1,n}^L W_{-2}, \\
(100) \quad (\alpha_{2,n-1}^R)^\dagger (W_{-1} + (n - 1) I_N) &= W_{-2} (\alpha_{2,n}^R)^\dagger, \\
(101) \quad (\alpha_{2,n-1}^R)^\dagger \alpha_{1,n}^L &= - W_{-2} (\alpha_{2,n}^R)^\dagger \alpha_{1,n}^L + (\alpha_{2,n-1}^R)^\dagger \alpha_{1,n-1}^L W_{-2}.
\end{align*}
\]

whose general solution, when \(W_{-1} + n I_N\), for \(n \in \{1, 2, \ldots\}\) and \(W_{-2}\) are non singular matrices, is given by

\[
\begin{align*}
\alpha_{1,n}^L &= (W_{-1} + n I_N)^{-1} \cdots (W_{-1} + 2I_N)^{-1} \alpha_{1,n}^L (W_{-2})^{-n}_{n-1}, \\
(\alpha_{2,n}^R)^\dagger &= (W_{-2})^{-n+1} (\alpha_{2,n}^R)^\dagger (W_{-1} + I_N) \cdots (W_{-1} + (n - 1) I_N),
\end{align*}
\]

in terms of the first Verblunsky coefficients \((\alpha_{2,1}^R)^\dagger\) and \(\alpha_{1,1}^L\).

**Proof.** The system (89), (90), (91) and (92) simplifies to

\[
\begin{align*}
- a_n + W_{-2} + [a_n, W_{-1}] &= \alpha_{1,n}^L W_{-2} (\alpha_{2,n}^R)^\dagger, \\
- (W_{-1} + (n + 1) I_N) b_n &= - \alpha_{1,n}^L W_{-2} H_{n}^R, \\
c_n (W_{-1} + (n - 1) I_N) &= - (H_{n-1}^R)^{-1} W_{-2} (\alpha_{2,n}^R)^\dagger, \\
d_n - d_{n-1} &= (H_{n-1}^R)^{-1} W_{-2} H_{n}^R - (H_{n-2}^R)^{-1} W_{-2} H_{n-1}^R,
\end{align*}
\]

when \(W_0 = 0_N\). In this more simple case, we can rewrite the system by performing a discrete derivation (a difference)

\[
\begin{align*}
- (a_n - a_{n-1}) + [a_n - a_{n-1}, W_{-1}] &= \alpha_{1,n}^L W_{-2} (\alpha_{2,n}^R)^\dagger - \alpha_{1,n-1}^L W_{-2} (\alpha_{2,n-1}^R)^\dagger, \\
(W_{-1} + (n + 1) I_N) b_n &= \alpha_{1,n}^L W_{-2} H_{n}^R, \\
c_n (W_{-1} + (n - 1) I_N) &= - (H_{n-1}^R)^{-1} W_{-2} (\alpha_{2,n}^R)^\dagger, \\
- (d_n - d_{n-1}) &= (H_{n-1}^R)^{-1} W_{-2} H_{n}^R - (H_{n-2}^R)^{-1} W_{-2} H_{n-1}^R.
\end{align*}
\]

Now, we use

\[
\begin{align*}
a_n - a_{n-1} &= \alpha_{1,n}^L (\alpha_{2,n-1}^R)^\dagger, \\
b_n &= \alpha_{1,n+1}^L H_{n}^R, \\
c_n &= -(H_{n-1}^R)^{-1} (\alpha_{2,n-1}^R)^\dagger, \\
d_n - d_{n-1} &= -(H_{n-1}^R)^{-1} (\alpha_{2,n-1}^R)^\dagger \alpha_{1,n}^L H_{n-1}^R,
\end{align*}
\]

to get

\[
\begin{align*}
- \alpha_{1,n}^L (\alpha_{2,n-1}^R)^\dagger + [\alpha_{1,n}^L (\alpha_{2,n-1}^R)^\dagger W_{-1}] &= \alpha_{1,n}^L W_{-2} (\alpha_{2,n}^R)^\dagger - \alpha_{1,n-1}^L W_{-2} (\alpha_{2,n-1}^R)^\dagger, \\
(W_{-1} + (n + 1) I_N) \alpha_{1,n+1}^L &= \alpha_{1,n}^L W_{-2}, \\
(\alpha_{2,n-1}^R)^\dagger (W_{-1} + (n - 1) I_N) &= W_{-2} (\alpha_{2,n}^R)^\dagger, \\
(H_{n-1}^R)^{-1} (\alpha_{2,n-1}^R)^\dagger \alpha_{1,n}^L H_{n-1}^R &= (H_{n-1}^R)^{-1} W_{-2} H_{n}^R - (H_{n-2}^R)^{-1} W_{-2} H_{n-1}^R,
\end{align*}
\]

and (98), (99), (100) and (101) follow.
Then, we can write (99) and (100) as
\[
\alpha_{1,n}^L = (W_{-1} + nI_N)^{-1}\alpha_{1,n-1}^L W_{-2},
\]
\[
(\alpha_{2,n}^R)^\dagger = (W_{-2})^{-1}(\alpha_{2,n-1}^R)^\dagger (W_{-1} + (n-1)I_N),
\]
which iterated leads to the solution (102). From here we deduce that
\[
\alpha_{1,n}^L W_{-2}(\alpha_{2,n}^R)^\dagger = (W_{-1} + nI_N)^{-1} \cdots (W_{-1} + 2I_N)^{-1} \alpha_{1,1}^L W_{-2}(\alpha_{2,1}^R)^\dagger (W_{-1} + I_N) \cdots (W_{-1} + (n-1)I_N),
\]
\[
\alpha_{1,n}^L (\alpha_{2,n-1}^R)^\dagger = (W_{-1} + nI_N)^{-1} \cdots (W_{-1} + 2I_N)^{-1} \alpha_{1,1}^L W_{-2}(\alpha_{2,1}^R)^\dagger (W_{-1} + I_N) \cdots (W_{-1} + (n-2)I_N)
\]
and, consequently,
\[
\alpha_{1,n}^L (\alpha_{2,n-1}^R)^\dagger (W_{-1} + (n-1)I_N)) = \alpha_{1,n}^L W_{-2}(\alpha_{2,n}^R)^\dagger,
\]
\[
(W_{-1} + nI_N)\alpha_{1,n}^L (\alpha_{2,n-1}^R)^\dagger = \alpha_{1,n-1}^L W_{-2}(\alpha_{2,n-1}^R)^\dagger,
\]
so that
\[
\alpha_{1,n}^L (\alpha_{2,n-1}^R)^\dagger (W_{-1} + (n-1)I_N)) - (W_{-1} + nI_N)\alpha_{1,n}^L (\alpha_{2,n-1}^R)^\dagger = \alpha_{1,n}^L W_{-2}(\alpha_{2,n}^R)^\dagger - \alpha_{1,n-1}^L W_{-2}(\alpha_{2,n-1}^R)^\dagger
\]
and (98) is identically satisfied.

Now, from (102) we find
\[
(\alpha_{2,n-1}^R)^\dagger \alpha_{1,n}^L = (W_{-2})^{-n+2}(\alpha_{2,1}^R)^\dagger (W_{-1} + I_N)(W_{-1} + (n-1)I_N)^{-1}(W_{-1} + nI_N)^{-1}\alpha_{1,1}^L (W_{-2})^{n-1},
\]
\[
W_{-2}(\alpha_{2,n}^R)^\dagger \alpha_{1,n}^L = (W_{-2})^{-n+2}(\alpha_{2,1}^R)^\dagger (W_{-1} + I_N)(W_{-1} + nI_N)^{-1}\alpha_{1,1}^L (W_{-2})^{n-1},
\]
\[
(\alpha_{2,n-1}^R)^\dagger \alpha_{1,n-1}^L W_{-2} = (W_{-2})^{-n+2}(\alpha_{2,1}^R)^\dagger (W_{-1} + I_N)(W_{-1} + (n-1)I_N)^{-1}\alpha_{1,1}^L (W_{-2})^{n-1}.
\]
Therefore,
\[
(\alpha_{2,n-1}^R)^\dagger \alpha_{1,n}^L + W_{-2}(\alpha_{2,n}^R)^\dagger \alpha_{1,n-1}^L W_{-2} = (W_{-2})^{-n+2}(\alpha_{2,1}^R)^\dagger (W_{-1} + I_N)W_n\alpha_{1,1}^L (W_{-2})^{n-1},
\]
with
\[
W_n := (W_{-1} + (n-1)I_N)^{-1}(W_{-1} + nI_N)^{-1} + (W_{-1} + nI_N)^{-1} - (W_{-1} + (n-1)I_N)^{-1}.
\]
But, let us notice that we have
\[
W_n = (W_{-1} + (n-1)I_N)^{-1}(W_{-1} + nI_N)^{-1}(I_N + (W_{-1} + (n-1)I_N) - (W_{-1} + nI_N)) = 0,
\]
and therefore (101) is also identically satisfied for Verblunsky coefficients as in (102).

5.3. Discussion on the matrix discrete Painlevé II systems. Locality. We now compare the two matrix discrete Painlevé II systems we have obtained. For the reader convenience we write the equations again. First, for the Fuchsian case \( W(z) = W_{-1}z^{-1} + W_0 + W_1z, \) the matrix discrete Painlevé II system, given in (76) and (77), is
\[
W_0\alpha_{1,n}^L + W_1\alpha_{1,n+1}^L - \alpha_{1,n-1}^L(W_{-1} - (n-1)I_N)
\]
\[
= W_1\left(\alpha_{1,n+1}^L(\alpha_{2,n}^R)^\dagger \alpha_{1,n}^L + \alpha_{1,n}^L(\alpha_{2,n-1}^R)^\dagger \alpha_{1,n}^L\right) + \left[W_1, \sum_{m=1}^{n-1} \alpha_{1,m}^L (\alpha_{2,m-1}^R)^\dagger \right] \alpha_{1,n}^L,
\]
\[
(\alpha_{2,n}^R)^\dagger W_0 + (\alpha_{2,n-1}^R)^\dagger W_1 - (W_{-1} + (n+1)I_N)(\alpha_{2,n+1}^R)^\dagger
\]
\[
= \left((\alpha_{2,n}^R)^\dagger \alpha_{1,n}^L(\alpha_{2,n-1}^R)^\dagger + (\alpha_{2,n}^R)^\dagger \alpha_{1,n+1}^L(\alpha_{2,n}^R)^\dagger \right) W_1 + (\alpha_{2,n}^R)^\dagger \left[W_1, \sum_{m=1}^{n+1} \alpha_{1,m}^L (\alpha_{2,m-1}^R)^\dagger \right].
\]
Second, for the non-Fuchsian case $W(z) = W_2z^{-2} + W_1z^{-1} + W_0$, the matrix discrete Painlevé II system, given in (92) and (94), reads
\[
(W_{-1} + nI_N)\alpha^l_{1,n} + W_0\alpha^l_{1,n+1} - \alpha^l_{1,n-1}W_{-2} = W_0\left(\alpha^l_{1,n+1}(\alpha^R_{2,n})^\dagger\alpha^l_{1,n} + \alpha^l_{1,n}(\alpha^R_{2,n-1})^\dagger\alpha^l_{1,n}\right) + \left[W_0, \sum_{m=1}^{n} \alpha^l_{1,m}(\alpha^R_{2,m-1})^\dagger\right]\alpha^l_{1,n},
\]
\[
(\alpha^R_{2,n})^\dagger(W_{-1} + nI_N) + (\alpha^R_{2,n-1})^\dagger W_0 - W_{-2}(\alpha^R_{2,n})^\dagger = \left((\alpha^R_{2,n})^\dagger\alpha^l_{1,n}(\alpha^R_{2,n-1})^\dagger + (\alpha^R_{2,n})^\dagger\alpha^l_{1,n+1}(\alpha^R_{2,n})^\dagger\right)W_0 + (\alpha^R_{2,n})^\dagger\left[W_0, \sum_{m=1}^{n} \alpha^l_{1,m}(\alpha^R_{2,m-1})^\dagger\right].
\]

We see that they are almost the same system. In fact, the nonlinear term are in complete correspondence by $W_1 \to W_0$. However, the linear terms are not. For example, we have terms like $(n - 1)\alpha^l_{1,n-1}$ for the Fuchsian scenario and of the form $n\alpha^l_{1,n}$ for the non-Fuchsian one, spoiling a complete correspondence.

5.3.1. Local matrix discrete Painlevé II systems. These matrix discrete Painlevé II systems present cubic terms in the Verblunsky coefficients $(\alpha^R_{2,n})^\dagger$ and $\alpha^l_{1,n}$, being all these terms local, in the sense that they involve nearby neighbors (the Verblunsky matrices at the sites $n - 1, n$ and $n + 1$), but for the last commutator in the RHS.

**Definition 21.** In the matrix discrete Painlevé II systems given in (76) and (77), or in (93) and (94) we call non local terms those terms of the form
\[
\sum_{m=1}^{n+1} \alpha^l_{1,m}(\alpha^R_{2,m-1})^\dagger.
\]

When these terms are absent we say that we have local matrix discrete Painlevé II systems. These systems are
\[
(103) \quad W_0\alpha^l_{1,n} + W_1\alpha^l_{1,n+1} - \alpha^l_{1,n-1}(W_{-1} - (n - 1)I_N) = W_1\left(\alpha^l_{1,n+1}(\alpha^R_{2,n})^\dagger\alpha^l_{1,n} + \alpha^l_{1,n}(\alpha^R_{2,n-1})^\dagger\alpha^l_{1,n}\right),
\]
\[
(104) \quad (\alpha^R_{2,n})^\dagger W_0 + (\alpha^R_{2,n-1})^\dagger W_1 - (W_{-1} - (n + 1)I_N)(\alpha^R_{2,n})^\dagger = \left((\alpha^R_{2,n})^\dagger\alpha^l_{1,n}(\alpha^R_{2,n-1})^\dagger + (\alpha^R_{2,n})^\dagger\alpha^l_{1,n+1}(\alpha^R_{2,n})^\dagger\right)W_1,
\]
in the Fuchsian case, while for the non-Fuchsian case they are
\[
(105) \quad (W_{-1} + nI_N)\alpha^l_{1,n} + W_0\alpha^l_{1,n+1} - \alpha^l_{1,n-1}W_{-2} = W_0\left(\alpha^l_{1,n+1}(\alpha^R_{2,n})^\dagger\alpha^l_{1,n} + \alpha^l_{1,n}(\alpha^R_{2,n-1})^\dagger\alpha^l_{1,n}\right),
\]
\[
(106) \quad (\alpha^R_{2,n})^\dagger(W_{-1} + nI_N) + (\alpha^R_{2,n-1})^\dagger W_0 - W_{-2}(\alpha^R_{2,n})^\dagger = \left((\alpha^R_{2,n})^\dagger\alpha^l_{1,n}(\alpha^R_{2,n-1})^\dagger + (\alpha^R_{2,n})^\dagger\alpha^l_{1,n+1}(\alpha^R_{2,n})^\dagger\right)W_0.
\]

**Examples of local matrix discrete Painlevé II systems.** We now discuss some cases where we find local matrix discrete Painlevé II systems:

1. If we take, in each case, $W_1 = k_1I_N$ (Fuchsian) or $W_0 = k_0I_N$ (non-Fuchsian) with $k_1, k_0 \in \mathbb{C}$ the systems are local and read

\[
W_0\alpha^l_{1,n} + k_1\alpha^l_{1,n+1} - \alpha^l_{1,n-1}(W_{-1} - (n - 1)I_N) = k_1\left(\alpha^l_{1,n+1}(\alpha^R_{2,n})^\dagger\alpha^l_{1,n} + \alpha^l_{1,n}(\alpha^R_{2,n-1})^\dagger\alpha^l_{1,n}\right),
\]
\[
(\alpha^R_{2,n})^\dagger W_0 + k_1(\alpha^R_{2,n-1})^\dagger - (W_{-1} - (n + 1)I_N)(\alpha^R_{2,n})^\dagger = k_1\left((\alpha^R_{2,n})^\dagger\alpha^l_{1,n}(\alpha^R_{2,n-1})^\dagger + (\alpha^R_{2,n})^\dagger\alpha^l_{1,n+1}(\alpha^R_{2,n})^\dagger\right),
\]
and
\[
(W_{-1} + nI_N)\alpha^l_{1,n} + k_0\alpha^l_{1,n+1} - \alpha^l_{1,n-1}W_{-2} = k_0\left(\alpha^l_{1,n+1}(\alpha^R_{2,n})^\dagger\alpha^l_{1,n} + \alpha^l_{1,n}(\alpha^R_{2,n-1})^\dagger\alpha^l_{1,n}\right),
\]
\[
(\alpha^R_{2,n})^\dagger(W_{-1} + nI_N) + k_0(\alpha^R_{2,n-1})^\dagger - W_{-2}(\alpha^R_{2,n})^\dagger = k_0\left((\alpha^R_{2,n})^\dagger\alpha^l_{1,n}(\alpha^R_{2,n-1})^\dagger + (\alpha^R_{2,n})^\dagger\alpha^l_{1,n+1}(\alpha^R_{2,n})^\dagger\right),
\]
respectively.
Two examples with locality, and in which we can ensure that we have an appropriate matrix of measures follow.

(a) A first one is to take $W_{-1} = k_{-1}I_N$, $k_{-1} \in \mathbb{Z}$, then the matrix of weights

$$w(z) = z^{k_{-1}} \exp(k_0 z) \exp(-W_{-2} z^{-1})$$

leads to the following matrix discrete Painlevé II system

$$(k_{-1} + n)\alpha_{l,n}^1 + k_0 \alpha_{l,n+1}^1 - \alpha_{l,n-1}^1 W_{-2} = k_0 \left( \alpha_{l,n+1}^R \alpha_{l,n}^1 + \alpha_{l,n}^1 \alpha_{l,n-1}^R \right),$$

$$(k_{-1} + n)\left( \alpha_{2,n}^R \right)^\dagger + k_0 \left( \alpha_{2,n-1}^R \right)^\dagger - W_{-2} \left( \alpha_{2,n+1}^R \right)^\dagger = k_0 \left( \left( \alpha_{2,n}^R \right)^\dagger \alpha_{l,n}^1 \left( \alpha_{2,n-1}^R \right)^\dagger + \left( \alpha_{2,n}^R \right)^\dagger \alpha_{l,n+1}^1 \left( \alpha_{2,n-1}^R \right)^\dagger \right).$$

(b) A second one is

$$w(z) = z^{k_{-1}} \exp(k_1 z^2/2) \exp(W_0 z)$$

with $k_{-1} \in \mathbb{Z}$, and the corresponding matrix discrete Painlevé II equations are

$$W_0 \alpha_{l,n}^1 + k_1 \alpha_{l,n+1}^1 - \alpha_{l,n-1}^1 (k_{-1} - (n - 1)I_N) = k_1 \left( \alpha_{l,n+1}^R \alpha_{l,n}^1 + \alpha_{l,n}^1 \alpha_{l,n-1}^R \right),$$

$$\left( \alpha_{2,n}^R \right)^\dagger W_0 + k_1 \left( \alpha_{2,n-1}^R \right)^\dagger - (k_{-1} - (n + 1)I_N) \left( \alpha_{2,n+1}^R \right)^\dagger = k_1 \left( \left( \alpha_{2,n}^R \right)^\dagger \alpha_{l,n}^1 \left( \alpha_{2,n-1}^R \right)^\dagger + \left( \alpha_{2,n}^R \right)^\dagger \alpha_{l,n+1}^1 \left( \alpha_{2,n-1}^R \right)^\dagger \right).$$

(3) There are more possibilities for local matrix discrete Painlevé II systems. For example, take the following matrices of weights

$$w(z) = z^{k_{-1}} \exp(-k_{-2} z^{-1}) \exp(W_0 z), \quad k_{-1} \in \mathbb{Z}, \quad k_{-2} \in \mathbb{C}, \quad W_0 \in \mathbb{C}^{N \times N},$$

$$w(z) = z^{k_{-1}} \exp(k_0 z) \exp(W_1 z^2/2), \quad k_{-1} \in \mathbb{Z}, \quad k_0 \in \mathbb{C}, \quad W_1 \in \mathbb{C}^{N \times N},$$

$$w(z) = z^{k_{-1}} \exp(W_0 (az^{-1} + bz)), \quad k_{-1} \in \mathbb{Z}, \quad a, b \in \mathbb{C}, \quad W_0 \in \mathbb{C}^{N \times N},$$

$$w(z) = z^{k_{-1}} \exp(W_1 (az + bz^2)), \quad k_{-1} \in \mathbb{Z}, \quad a, b \in \mathbb{C}, \quad W_1 \in \mathbb{C}^{N \times N}.$$ 

This is so because in these four cases the commutativity $[w(z_1), w(z_2)] = 0_N$, for all $z_1, z_2 \in \mathbb{C}$, holds and, consequently, Proposition 12 is applicable. Hence, the non local terms disappear.

(4) A more general scenario were these symmetry considerations are applicable, and the non local terms are set off, are

(a) For the Fuchsian case we take the triple $W_{-1}, W_0, W_1 \in \mathbb{C}^{N \times N}$ in an Abelian algebra, and $W_{-1}$ a diagonalizable matrix with integer eigenvalues.

(b) Similarly, for the non-Fuchsian case we choose triple $W_{-2}, W_{-1}, W_0 \in \mathbb{C}^{N \times N}$ in an Abelian algebra and $W_{-1}$ a diagonalizable matrix with integer eigenvalues.

**Theorem 6** (Local matrix discrete Painlevé II systems). The matrix discrete Painlevé II systems are local whenever

(1) **Fuchsian case:** we choose the triple of matrices $\{W_{-1}, W_0, W_1\}$ such that $|W_i, W_0| = |W_1, W_{-1}| = 0_N$, and such that for some nonzero complex number $z_0$ we have the commutativity $[W_1, w(z_0)] = 0_N$. The matrix of weights will have the form

$$w(z) = \exp(W_1 z^2/2) \tilde{w}(z)$$

where the associated matrix of weights $\tilde{w}(z)$ is the unique solution to

$$\frac{d \tilde{w}}{dz} = (W_{-1} z^{-1} + W_0) \tilde{w}, \quad \tilde{w}(z) = w(z_0).$$

(2) **Non-Fuchsian case:** we take the triple of matrices $\{W_{-2}, W_{-1}, W_0\}$ such that $|W_0, W_{-1}| = |W_1, W_{-2}| = 0_N$, and such that for some nonzero complex number $z_0$ we have the commutativity $[W_0, w(z_0)] = 0_N$. The matrix of weights will have the form

$$w(z) = \exp(W_0 z) \tilde{w}(z)$$

$^9$We can argue also saying that $[w(z), W_0] = 0_N, \forall z \in \mathbb{C}$, in the first and third cases, and $[w(z), W_1] = 0_N, \forall z \in \mathbb{C}$, for the second and fourth cases, and we can apply Proposition 11.
where the associated matrix of weights \( \tilde{\omega}(z) \) is the unique solution to

\[
\frac{d \tilde{\omega}}{dz} = (W_{-2} z^{-2} + W_{-1} z^{-1}) \tilde{\omega}, \quad \tilde{\omega}(z) = \omega(z_0).
\]

Proof. We prove the Proposition just for the Fuchsian case; in the non-Fuchsian case the proof goes analogously. If \( W_1 \) commutes with the matrix of weights \( \omega(z) \), Proposition 11 ensures that it commutes with the Verblunsky matrices, and the locality is achieved. But, \( [W_1, \omega_0] = [W_1, \omega_{-1}] = 0_N \) are equivalent to \( [W_1, \omega(z)] = 0_N \) for all \( z \in \mathbb{C} \). Now, the Picard’s method of successive approximations completes the argument, see [74]. Indeed, for a non singular point \( z_0 \), i.e., any non zero complex number, the matrix of weights \( \omega(z) \) which satisfies (46) with \( \omega(z_0) = \omega_0 \) is the solution to the integral equation

\[
\omega(z) = \omega_0 + \int_{z_0}^{z} W(s) \omega(s) \, ds.
\]

Then, this solution can be obtained by the Picard iteration method

\[
\omega_n(z) := \omega_0 + \int_{z_0}^{z} W(s) \omega_{n-1}(s) \, ds, \quad n \in \{1, 2, \ldots \}
\]

as the limit

\[
\omega_n(z) \xrightarrow[n \to \infty]{} \omega(z).
\]

Consequently, if a matrix \( M \) commutes with \( W(z) \) and the initial condition \( \omega_0 \) it also commutes with \( \omega_n(z) \), for all \( n \), and, therefore, with the limit \( \omega(z) = \lim_{n \to \infty} \omega_n(z) \). Finally, the gauge transformation \( \omega(z) \to \tilde{\omega}(z) := \exp(-W_1 z^2/2) \omega(z) \), reads

\[
\frac{d \tilde{\omega}(z)}{dz} (\tilde{\omega}(z))^{-1} = -W_1 z + \exp(-W_1 z^2/2) \frac{d \omega(z)}{dz} (\omega(z))^{-1} \exp(W_1 z^2/2)
\]

\[
= -W_1 z + \exp(-W_1 z^2/2) \omega(z) \exp(W_1 z^2/2),
\]

\[
= \omega_0 + W_1 z.
\]

One could think that this condition leads to the Abelian triple discussed in the preliminary examples of local systems. But \( [W_1, \omega_0] = [W_1, \omega_{-1}] = 0_N \) do not imply that \( [\omega_0, \omega_{-1}] = 0_N \) that in turn gives, for the Fuchsian situation, the following form for the measure \( \omega(z) = \exp(W_{-1} \log z + W_0 z + W_1 z^2/2) \). Now, we give two examples that are not an Abelian triple for the Fuchsian case (the non-Fuchsian case goes similarly).

1. For \( N = 2 \) we choose

\[
W_{-1} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad W_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad W_1 = k_1 I_2
\]

where \( m_1, m_2 \in \mathbb{Z} \) and \( a, b, c, d, k \in \mathbb{C} \). This example fits in the first family we considered with \( W_1 \) proportional to the identity.

2. Much more interesting is the following choice for \( N = 3 \). We pick

\[
W_{-1} = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_1 & 0 \\ 0 & 0 & m_2 \end{pmatrix}, \quad W_0 = \begin{pmatrix} e & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}, \quad W_1 = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_2 \end{pmatrix},
\]

with \( m_1, m_2 \in \mathbb{Z} \) and \( a, b, c, d, e, k_1, k_2 \in \mathbb{C} \). Here \( W_1 \) is not proportional to the identity but commutes with \( W_{-1} \) and \( W_0 \), and \( [W_{-1}, W_0] \neq 0 \). Obviously, this example can be generalized. These type of choices are relevant because then the commutativity \( \{w(z), w(z')\} = 0_N \) for all \( z, z' \in \mathbb{C} \setminus \{0\} \) does not necessarily holds. Hence, the matrix Szegő polynomials and Verblunsky matrices do not necessarily form an Abelian set, which indeed happens if \( [W_0, W_{-1}] = 0_N \).
5.3.2. The scalar case. Finally, let us mention that in the scalar case, when \( N = 1 \), right and left indexes disappear and we get the following systems of equations

\[
\begin{align*}
    k_0 \alpha_{1,n} + k_1 \alpha_{1,n+1} - \alpha_{1,n-1} (k_1 - (n - 1) I_N) &= k_1 \left( \alpha_{1,n+1} (\alpha_{2,n})^\dagger + \alpha_{1,n} (\alpha_{2,n-1})^\dagger \alpha_{1,n} \right), \\
    k_0 (\alpha_{2,n})^\dagger + k_1 (\alpha_{2,n-1})^\dagger - (k_1 - (n + 1) I_N)(\alpha_{2,n+1})^\dagger &= k_1 \left( (\alpha_{2,n})^\dagger \alpha_{1,n} (\alpha_{2,n-1})^\dagger + (\alpha_{2,n})^\dagger \alpha_{1,n+1} (\alpha_{2,n})^\dagger \right),
\end{align*}
\]

for the Fuchsian case, which correspond to the complex weight \( W \) to the general situation for the constraints ensuring single-valuedness. Using the techniques of \([116, 57]\) we can apply these ideas to the monodromy. This construction works for larger \( k \) in certain algebraic hypersurface of degree \( k \) in \( \mathbb{C}^8 \), the corresponding linear ODE system \((16)\) has trivial monodromy. This construction works for larger \( k \); we will obtain higher degree algebraic hypersurfaces for the constraints ensuring single-valuedness. Using the techniques of \([116, 57]\) we can apply these ideas to the general situation \( N \geq 3 \).

### Appendix A. Fuchsian examples of matrices of weights for \( N = 2 \)

We will study some examples for the Fuchsian case and \( N = 2 \) given by

\[
W_{-1} = \begin{pmatrix} p & 0 \\ 0 & p + k \end{pmatrix}
\]

with \( p \in \mathbb{Z} \) and \( k \in \{0, 1, \ldots\}\)\(^{10}\). Observe that for \( k \in \{1, 2, \ldots\} \) we are dealing with resonant cases. With the aid of \([116]\) we explore the cases \( k = 1, 2, 3 \) and get explicit examples of matrices of weights linked to the original assumption for the right logarithmic derivative. We obtain that if the coefficients of \( W(z) \) lay in certain algebraic hypersurface of degree \( k \) in \( \mathbb{C}^8 \), the corresponding linear ODE system \((16)\) has trivial monodromy. This construction works for larger \( k \); we will obtain higher degree algebraic hypersurfaces for the constraints ensuring single-valuedness. Using the techniques of \([116, 57]\) we can apply these ideas to the general situation \( N \geq 3 \).

#### A.1. \( N = 2 \) and \( k = 1 \)

Let us assume that

\[
W_{-1} = \begin{pmatrix} p & 0 \\ 0 & p + 1 \end{pmatrix}
\]

with \( p \in \mathbb{Z} \), so that

\[
W(z) = \begin{pmatrix} p & 0 \\ 0 & p + 1 \end{pmatrix} z^{-1} + \begin{pmatrix} a_0 + a_1 z & b_0 + b_1 z \\ c_0 + c_1 z & d_0 + d_1 z \end{pmatrix}
\]

\(^{10}\) For \( W_{-1} = \begin{pmatrix} p+k & 0 \\ 0 & p \end{pmatrix} \) a similar discussion can be carried out.
for some \((a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1)^\top \in \mathbb{C}^8\). We will study the corresponding matrix of weights \(w(z)\), which is a solution of \((46)\). An equivalent system, obtained by a shearing transformation generated by \((108)\) is given by
\[
W^{(1)}(z) = \frac{dS(z)}{dz}(S(z))^{-1} + S(z)W(z)(S(z))^{-1}
\]
(108)

This case is not resonant anymore, we have get rid of it using the shearing transformation. Therefore, a fundamental solution to the associated differential system
\[
\frac{dw^{(1)}(z)}{dz} = W^{(1)}(z)w^{(1)}(z)
\]
(109)
is of the form \(w^{(1)}(z) = \Phi(z)z^p\) where \(\Phi\) is analytic with a Taylor series convergent at the annulus \(\mathbb{C} \setminus \{0\}\)
\[
\Phi(z) = I_\mathbb{N} + \Phi_1z + \Phi_2z^2 + \cdots.
\]

Thus, to avoid multivalued functions we require
\[
c_0 = 0,
\]
so that the fundamental solution will have the form \(w^{(1)}(z) = z^p\Phi(z)\)\(^{[1]}\) Then, from \((109)\) we get
\[
\Phi_1 = W^{(1)}_0,
\]
\[
\Phi_2 = \frac{1}{2}(W^{(1)}_0\Phi_1 + W^{(1)}_1),
\]
\[
\Phi_3 = \frac{1}{3}(W^{(1)}_0\Phi_2 + W^{(1)}_1\Phi_1 + W^{(1)}_2),
\]
\[
\Phi_4 = \frac{1}{4}(W^{(1)}_0\Phi_3 + W^{(1)}_1\Phi_2 + W^{(1)}_2\Phi_1),
\]
\[\vdots\]
\[
\Phi_{n+3} = \frac{1}{n+3}(W^{(1)}_0\Phi_{n+2} + W^{(1)}_1\Phi_{n+1} + W^{(1)}_2\Phi_n), \quad n \in \{1, 2, \ldots\}
\]

We see that all coefficients are obtained recursively, and the first three coefficients are
\[
\Phi_1 = W^{(1)}_0,
\]
\[
\Phi_2 = \frac{1}{2}(W^{(1)}_0)^2 + W^{(1)}_1),
\]
\[
\Phi_3 = \frac{1}{6}(W^{(1)}_0)^3 + W^{(1)}_0W^{(1)}_1W^{(1)}_1 + 2W^{(1)}_1W^{(1)}_0 + 2W^{(1)}_2).
\]

Finally, for the matrix of weights we get
\[
w(z) = (S(z))^{-1}w^{(1)}(z)
\]
\[
= \begin{pmatrix} z^p & 0 \\ 0 & z^{p+1} \end{pmatrix} (I_2 + \Phi_1z + \Phi_2z^2 + \cdots)
\]

\(^{[1]}\)This condition also appears naturally when one considers the meromorphic equivalent systems to \((p \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ p+1)^\top \cdot z^{-1}\). Indeed, if the equivalence is realized by \(I_2z^{-m} + \Phi_1z^{-m+1} + \Phi_2z^{-m+2} + \cdots\) we get that the equivalent, free monodromy system, have the form
\[
(\begin{pmatrix} p-m & 0 \\ 0 & p+1-m \end{pmatrix} \cdot z^{-1} + \begin{pmatrix} a_1 & (p+2)b_1 \\ 0 & d_1 \end{pmatrix}) + O(z), \quad \text{where } \Phi_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}.
\]
A.2. \( N = 2 \) and \( k = 2 \). Let us assume that
\[
W_{-1} = \begin{pmatrix} p & 0 \\ 0 & p + 2 \end{pmatrix}
\]
with \( p \in \mathbb{Z} \), so that
\[
W(z) = \begin{pmatrix} p & 0 \\ 0 & p + 2 \end{pmatrix} z^{-1} + \begin{pmatrix} a_0 + a_1 z & b_0 + b_1 z \\ c_0 + c_1 z & d_0 + d_1 z \end{pmatrix}
\]
for \( (a_0, b_0, c_0, d_0, a_1, c_1, d_1)^T \in \mathbb{C}^8 \). As for \( k = 1 \), we perform a shearing transformation generated by \( S(z) \), see (107), and obtain
\[
W^{(1)} = \begin{pmatrix} p & 0 \\ 0 & p + 1 \end{pmatrix} z^{-1} + W_0^{(1)} + W_1^{(1)} z + W_2^{(1)} z^2,
\]
where \( W_0^{(1)}, W_1^{(1)}, W_2^{(1)} \) are given in (108). To continue with the simplification, we now perform the following diagonalization
\[
(110) \quad \begin{pmatrix} p & 0 \\ 0 & p + 1 \end{pmatrix} = \mathcal{T} \begin{pmatrix} p & 0 \\ 0 & p + 1 \end{pmatrix} \mathcal{T}^{-1}, \quad \mathcal{T} := \begin{pmatrix} 1 & 0 \\ c_0 & 1 \end{pmatrix}.
\]
Consequently, the new matrix is
\[
\tilde{W}^{(1)} = \mathcal{T} W^{(1)} \mathcal{T}^{-1} = \begin{pmatrix} p & 0 \\ 0 & p + 1 \end{pmatrix} z^{-1} + \tilde{W}_0^{(1)} + \tilde{W}_1^{(1)} z + \tilde{W}_2^{(1)} z^2,
\]
where
\[
\tilde{W}_0^{(1)} := \mathcal{T} W_0^{(1)} \mathcal{T}^{-1}, \quad \tilde{W}_1^{(1)} := \mathcal{T} W_1^{(1)} \mathcal{T}^{-1}, \quad \tilde{W}_2^{(1)} := \mathcal{T} W_2^{(1)} \mathcal{T}^{-1},
\]
\[
= \begin{pmatrix} a_0 & 0 \\ c_0(a_0 - d_0) + c_1 & d_0 \end{pmatrix}, \quad = \begin{pmatrix} a_1 - b_0 c_0 & b_0 \\ c_0(a_1 - d_1) - b_0(c_0)^2 & b_0 c_0 + d_1 \end{pmatrix}, \quad = \begin{pmatrix} -b_1 c_0 & b_1 \\ -b_1(c_0)^2 & b_1 c_0 \end{pmatrix}.
\]
After a second shearing transformation generated by \( S(z) \) as in (107) we obtain
\[
W^{(2)}(z) = \frac{dS(z)}{dz} (S(z))^{-1} + S(z) \tilde{W}^{(1)}(z) (S(z))^{-1}
\]
\[
= \begin{pmatrix} p & 0 \\ c_0(a_0 - d_0) + c_1 & 1 \end{pmatrix} z^{-1} + W_0^{(2)} + W_1^{(2)} z + W_2^{(2)} z^2 + W_3^{(2)} z^3,
\]
with
\[
W_0^{(2)} := \begin{pmatrix} a_0 & 0 \\ c_0(a_1 - d_1) - b_0(c_0)^2 & d_0 \end{pmatrix}, \quad W_1^{(2)} := \begin{pmatrix} a_1 - b_0 c_0 & 0 \\ -b_1(c_0)^2 & b_0 c_0 + d_1 \end{pmatrix},
\]
\[
W_2^{(2)} := \begin{pmatrix} -b_1 c_0 & 0 \\ 0 & b_1 c_0 \end{pmatrix}, \quad W_3^{(2)} := \begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix}.
\]
Thus, the matrix
\[
w^{(2)}(z) = S(z) \mathcal{T} S(z) W(z)
\]
satisfies an ODE of the form
\[
\frac{d w^{(2)}(z)}{dz} = W^{(2)}(z) w^{(2)}(z),
\]
which is resonance free. Thus, a fundamental solution is
\[
w^{(2)}(z) = (I_2 + \Phi_1 z + \Phi_2 z^2 + \cdots) z^\left(\begin{pmatrix} p \\ 0 \end{pmatrix} \right), \quad \Phi_2 := c_0(a_0 - d_0) + c_1.
\]
Furthermore, to avoid a multivalued situation we impose to the coefficients of \( W(z) \) to belong to the quadric in \( \mathbb{C}^8 \) determined by the equation
\[
\partial_2 = c_0(a_0 - d_0) + c_1 = 0,
\]
and the solution will be
\[ w^{(2)}(z) = z^p (I_2 + \Phi_1 z + \Phi_2 z^2 + \cdots). \]

The corresponding matrix of weights is
\[ w(z) = (S(z))^{-1} f^{-1} (S(z))^{-1} w^{(2)}(z) \]
\[ = \begin{pmatrix} z^p & 0 \\ -c_0 z^{p+1} & z^{p+2} \end{pmatrix} (I_2 + \Phi_1 z + \Phi_2 z^2 + \cdots), \]
where the series converge at the annulus \( \mathbb{C} \setminus \{0\} \) and

\[ \Phi_1 = W_0^{(2)}, \]
\[ \Phi_2 = \frac{1}{2} (W_0^{(2)} \Phi_1 + W_1^{(2)}), \]
\[ \Phi_3 = \frac{1}{3} (W_0^{(2)} \Phi_2 + W_1^{(2)} \Phi_1 + W_2^{(2)}), \]
\[ \Phi_4 = \frac{1}{4} (W_0^{(2)} \Phi_3 + W_1^{(2)} \Phi_2 + W_2^{(2)} \Phi_1 + W_3^{(2)}), \]
\[ \Phi_5 = \frac{1}{5} (W_0^{(2)} \Phi_4 + W_1^{(2)} \Phi_3 + W_2^{(2)} \Phi_2 + W_3^{(2)} \Phi_1), \]
\[ \vdots \]
\[ \Phi_{n+4} = \frac{1}{n+4} (W_0^{(2)} \Phi_{n+3} + W_1^{(2)} \Phi_{n+2} + W_2^{(2)} \Phi_{n+1} + W_3^{(2)} \Phi_n), \quad n \in \{1, 2, \ldots\}. \]

A.3. \( N = 2 \) and \( k = 3 \). Finally, let us assume that
\[ W_{-1} = \begin{pmatrix} p & 0 \\ 0 & p + 3 \end{pmatrix} \]
with \( p \in \mathbb{Z} \), and
\[ W(z) = \begin{pmatrix} p & 0 \\ 0 & p + 3 \end{pmatrix} z^{-1} + \begin{pmatrix} a_0 + a_1 z & b_0 + b_1 z \\ c_0 + c_1 z & d_0 + d_1 z \end{pmatrix} \]
for \((a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1)^T \in \mathbb{C}^8\). As for \( k = 2 \), we perform a transformation generated by \( S(z)TS(z) \), see (107) and (110), and get
\[ W^{(2)}(z) = \begin{pmatrix} p & 0 \\ \tilde{\vartheta}_2 & p + 1 \end{pmatrix} z^{-1} + W_0^{(2)} + W_1^{(2)} z + W_2^{(2)} z^2 + W_3^{(2)} z^3 \]
with coefficients given in (111). A further diagonalization
\[ \begin{pmatrix} p & 0 \\ 0 & p + 1 \end{pmatrix} = \vartheta^{(2)} \begin{pmatrix} p & 0 \\ \tilde{\vartheta}_2 & p + 1 \end{pmatrix} (\vartheta^{(2)})^{-1}, \quad \vartheta^{(2)} := \begin{pmatrix} 1 & 0 \\ \tilde{\vartheta}_2 & 1 \end{pmatrix}, \]
gives
\[ \tilde{W}^{(2)} = T^{(2)} W^{(2)} (T^{(2)})^{-1} \]
\[ = \begin{pmatrix} p & 0 \\ 0 & p + 1 \end{pmatrix} z^{-1} + \tilde{W}_0^{(2)} + \tilde{W}_1^{(2)} z + \tilde{W}_2^{(2)} z^2 + \tilde{W}_3^{(2)} z^3, \]
where
\[
\begin{align*}
    \hat{W}_0^{(2)} &= \mathcal{T}^{(2)} A_0^{(2)} (\mathcal{T}^{(2)})^{-1} = \begin{pmatrix} c_0 (a_1 - d_1) - b_0 (c_0)^2 + \vartheta_2 (a_0 - d_0) & a_0 \\ a_1 - b_0 c_0 & b_0 \\ \vartheta_2 & d_0 \end{pmatrix}, \\
    \hat{W}_1^{(2)} &= \mathcal{T}^{(2)} W_1^{(2)} (\mathcal{T}^{(2)})^{-1} = \begin{pmatrix} -b_1 (c_0)^2 + \vartheta_2 (a_1 - d_1 - 2 b_0 c_0) & -b_1 c_0 - \vartheta_2 b_0 \\ a_1 - b_0 c_0 & b_0 \\ -2 \vartheta_2 b_1 c_0 - (\vartheta_2)^2 b_0 & b_1 c_0 + \vartheta_2 b_0 \end{pmatrix}, \\
    \hat{W}_2^{(2)} &= \mathcal{T}^{(2)} W_2^{(2)} (\mathcal{T}^{(2)})^{-1} = \begin{pmatrix} -b_1 c_0 - \vartheta_2 b_0 & b_0 \\ -2 \vartheta_2 b_1 c_0 - (\vartheta_2)^2 b_0 & b_1 c_0 + \vartheta_2 b_0 \end{pmatrix}, \\
    \hat{W}_3^{(2)} &= \mathcal{T}^{(2)} A_3^{(2)} (\mathcal{T}^{(2)})^{-1} = \begin{pmatrix} -\vartheta_2 b_1 & b_1 \\ -(\vartheta_2)^2 b_1 & \vartheta_2 b_1 \end{pmatrix}.
\end{align*}
\]

Now, as a final step, we perform a third shearing transformation generated by $S(z)$, see [107], to get
\[
W^{(3)}(z) = \frac{d S(z)}{d z} (S(z))^{-1} + S(z) \hat{W}^{(2)}(z) (S(z))^{-1}
= \begin{pmatrix} p & 0 \\ \vartheta_3 & p \end{pmatrix} z^{-1} + W_0^{(3)} z + W_1^{(3)} z^2 + W_2^{(3)} z^3 + W_3^{(3)} z^4,
\]
with
\[
\vartheta_3 := c_0 (a_1 - d_1) - b_0 (c_0)^2 + \vartheta_2 (a_0 - d_0)
= c_0 (a_1 - d_1) + (a_0 - d_0)^2 - b_0 (c_0)^2 + c_1 (a_0 - d_0)
\]
(112)

Hence, the matrix $w^{(3)}(z) = S(z) \mathcal{T}^{(2)} S(z) T S(z) w(z)$ satisfies the following non-resonant linear system of ODE
\[
\frac{d w^{(3)}(z)}{d z} = W^{(3)}(z) w^{(3)}(z)
\]
with a fundamental solution given by
\[
w^{(2)}(z) = (I_2 + \Phi_1 z + \Phi_2 z^2 + \cdots) z \begin{pmatrix} p & 0 \\ \vartheta_3 & p \end{pmatrix}.
\]
Furthermore, to avoid a multivalued situation we impose to the coefficients of $W(z)$ to lay in the cubic hypersurface
\[
\vartheta_3 = c_0 (a_1 - d_1) + (a_0 - d_0)^2 - b_0 (c_0)^2 + c_1 (a_0 - d_0) = 0
\]
and the solution is simplifies to $w^{(2)}(z) = z^p (I_2 + \Phi_1 z + \Phi_2 z^2 + \cdots)$. Hence, the corresponding matrix of weights is
\[
w(z) = (S(z))^{-1} \mathcal{T}^{-1} (S(z))^{-1} (\mathcal{T}^{(2)})^{-1} (S(z))^{-1} w^{(3)}(z)
= \begin{pmatrix} z^p & 0 \\ -c_0 z^{p+1} - \vartheta_2 z^{p+2} & z^{p+3} \end{pmatrix} (I_2 + \Phi_1 z + \Phi_2 z^2 + \cdots),
\]
where
\[ \Phi_1 = W_0^{(3)}, \]
\[ \Phi_2 = \frac{1}{2} (W_0^{(3)} \Phi_1 + W_1^{(3)}), \]
\[ \Phi_3 = \frac{1}{3} (W_0^{(3)} \Phi_2 + W_1^{(3)} \Phi_1 + W_2^{(3)}), \]
\[ \Phi_4 = \frac{1}{4} (W_0^{(3)} \Phi_3 + W_1^{(3)} \Phi_2 + W_2^{(3)} \Phi_1 + W_3^{(3)}), \]
\[ \Phi_5 = \frac{1}{5} (W_0^{(3)} \Phi_4 + W_1^{(3)} \Phi_3 + W_2^{(3)} \Phi_2 + W_3^{(3)} \Phi_1 + W_4^{(3)}), \]
\[ \Phi_6 = \frac{1}{6} (W_0^{(3)} \Phi_5 + W_1^{(3)} \Phi_4 + W_2^{(3)} \Phi_3 + W_3^{(3)} \Phi_2 + W_4^{(3)} \Phi_1), \]
\[ \vdots \]
\[ \Phi_{n+5} = \frac{1}{n+5} (W_0^{(3)} \Phi_{n+4} + W_1^{(3)} \Phi_{n+3} + W_2^{(3)} \Phi_{n+2} + W_3^{(3)} \Phi_{n+1} + W_4^{(3)} \Phi_n), \quad n \in \{1, 2, \ldots \}. \]

Notice that the series converge at the annulus \( C \setminus \{0\} \).

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