Few remarks on Bäcklund transformations for many-body systems

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Abstract

Using the \( n \)-particle periodic Toda lattice and the relativistic generalization due to Ruijsenaars of the elliptic Calogero-Moser system as examples, we revise the basic properties of the Bäcklund transformations (BT’s) from the Hamiltonian point of view. The analogy between BT and Baxter’s quantum \( Q \)-operator pointed out by Pasquier and Gaudin is exploited to produce a conjugated variable \( \mu \) for the parameter \( \lambda \) of the BT \( B_\lambda \) such that \( \mu \) belongs to the spectrum of the Lax operator \( L(\lambda) \). As a consequence, the generating function of the composition \( B_\lambda \circ \ldots \circ B_\lambda \) of \( n \) BT’s gives rise also to another canonical transformation separating variables for the model. For the Toda lattice the dual BT parametrized by \( \mu \) is introduced.

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1. Introduction

Bäcklund transformations (BT’s) are an important tool in the theory of integrable systems [1]. Most frequently, they are understood as special mappings between solutions of nonlinear evolution equations. The Hamiltonian properties of BT’s, as canonical transformations, are studied less well. The recent developments in the quantum integrable theories [2, 3], discrete-time dynamics [4, 5, 6] and separation of variables [7, 8] suggest, however, that the Hamiltonian aspect of BT’s deserves more attention.

The aim of the present paper is to revise the concept of BT’s from the Hamiltonian point of view and to point out some new properties of BT’s. We restrict our attention to the finite-dimensional integrable systems and illustrate our general remarks on the example of the periodic Toda lattice and the elliptic Ruijsenaars model. When elaborating our approach to BT’s, we have benefited greatly from the works of Pasquier and Gaudin [4], where a fundamental relationship between BT and Baxter’s quantum Q-operator was discovered, and of Veselov [4], who gave us the adequate mathematical language to speak about integrable mappings.

In the section 2 the main properties of Bäcklund transformations for Liouville integrable systems are enlisted and a new property of spectrality is introduced. The meaning of spectrality is elucidated by making the comparison with the Baxter’s quantum Q-operator. It is shown that spectrality of BT provides an effective solution to the problem of separation of variables. In two subsequent sections we illustrate the new property of BT’s for two families of integrable many-body systems. The concluding section 5 contains a summary and a discussion.

2. Spectrality and separation of variables

Suppose an integrable system with \( n \) degrees of freedom is described in terms of the canonical Darboux variables \( X \equiv \{X_i\}_{i=1}^n \) and \( x \equiv \{x_i\}_{i=1}^n \), with the Poisson brackets:

\[
\{X_i, X_j\} = \{x_i, x_j\} = 0, \quad \{X_i, x_j\} = \delta_{ij}, \tag{2.1}
\]

and functionally independent commuting Hamiltonians \( H_i \equiv H_i(X, x) \)

\[
\{H_i, H_j\} = 0, \quad i, j = 1, \ldots, n. \tag{2.2}
\]

For our purposes it is convenient to think of a BT as a canonical transformation \( B_\lambda \) from the canonical variables \( (X, x) \) to the canonical variables \( (Y, y) \). It is important that \( B_\lambda \) depends on a complex parameter \( \lambda \). We shall suppose that \( B_\lambda \) can be described via the generating function \( F_\lambda(y; x) \) such that

\[
X_i = \frac{\partial F_\lambda}{\partial x_i}, \quad Y_i = -\frac{\partial F_\lambda}{\partial y_i}. \tag{2.3}
\]

The list of properties defining a BT usually includes:
• *Canonicity.* See above.

• *Invariance of Hamiltonians.*

\[ H_i(X, x) = H_i(Y, y), \quad i = 1, \ldots, n. \]  

(2.4)

• *Commutativity.*

\[ B_{\lambda_1} \circ B_{\lambda_2} = B_{\lambda_2} \circ B_{\lambda_1} \]  

(2.5)

where \( \circ \) means composition of canonical transformations.

In case of the *algebraically integrable* systems one more property can be added to the list:

• *Algebraicity.* The equations (2.3) describing \( B_{\lambda} \) are supposed to be algebraic with respect to \( X, Y \) and properly chosen functions of \( x \) and \( y \) (say, exponential or elliptic).

In the present paper, however, we concentrate on the analytic properties of BT’s and ignore their algebraic and algebro-geometric aspects.

It is important to make clear distinction between the notion of BT and the close notions of *integrable canonical mapping* [4], or *integrable discrete-time dynamics*. The latter two are defined by the properties of canonicity and invariance only, the parameter \( \lambda \) being disregarded. The term ‘discrete-time dynamics’ refers usually to the case when the canonical transformation degenerates, in a certain limit, into an infinitesimal generator \( \{H, \cdot\} \) of a continuous Hamiltonian flow. Existence of the parameter \( \lambda \) is crucial for our definition of BT and enriches it with new properties.

Though the commutativity of BT’s is traditionally proved as an independent property, in fact it follows from the canonicity and the invariance of Hamiltonians. Indeed, as shown in [4], any integrable canonical mapping acts on the Liouville torus as a shift (or a collection of shifts, in case of multivalued mappings) of the angle variables \( \varphi_i \to \varphi_i + b_i(\lambda) \). The commutativity is then obvious.

The theory of BT’s acquires a new aspect if the integrable system in question is solvable via Inverse Scattering (or Inverse Spectral Transform) method. Suppose that the commuting Hamiltonians \( H_i \) can be obtained as the coefficients of the characteristic polynomial

\[ W(u, v; \{H_i\}) = \det(v - L(u)) \]  

(2.6)

of a matrix \( L(u) \equiv L(u; X, x) \) (Lax operator) depending on \( X, x \) and a complex parameter \( u \). Note that the invariance of \( H_i \) under \( B_{\lambda} \) is equivalent then to the invariance of the spectrum of \( L(u) \), that is there exists an invertible matrix \( M(u) \) such that

\[ M(u)L(u; X, x) = L(u; Y, y)M(u), \quad \forall u \in \mathbb{C}. \]  

(2.7)

The properties of BT’s enlisted above are well known. Now we are going to add to the list a new property which is the main contribution of the present paper.

3
• Spectrality. Let $\mu$ be defined as the variable conjugated to $\lambda$:

$$
\mu = -\frac{\partial F_\lambda}{\partial \lambda}.
$$

(2.8)

We shall say that the BT $B_\lambda$ is associated to the Lax operator $L(u)$ if for some function $f(\mu)$ the pair $(\lambda, f(\mu))$ lies on the spectral curve of the Lax matrix

$$
W(\lambda, f(\mu); \{H_i\}) \equiv \det(f(\mu) - L(\lambda)) = 0.
$$

(2.9)

This spectrality property of BT seems to be new, at least we failed to find it in the literature. We have verified it for the Toda lattice and the elliptic Ruijsenaars model for which $f(\mu) = e^{-\mu}$ (see sections 3 and 4). It seems plausible, however, that spectrality is the property shared by BT’s for a much larger class of models.

The meaning of the equality (2.9) becomes clear if we turn to the quantum case. In the pioneering paper by Pasquier and Gaudin [2], based on the earlier treatment of the classical Toda lattice by Gaudin [10], a remarkable connection has been established between the classical BT $B_\lambda$ for the Toda lattice and the famous Baxter’s $Q$-operator [11]. Pasquier and Gaudin have constructed certain integral operator $\hat{Q}_\lambda$

$$
\hat{Q}_\lambda : \Psi(x) \to \int dx Q_\lambda(y; x)\Psi(x)
$$

(2.10)

(here and below $dx \equiv dx_1 \wedge \ldots \wedge dx_n$ etc) whose properties parallel those of the classical BT $B_\lambda$. In the quantum case the canonical transformation is replaced with the similarity transformation

$$
\hat{Y}_i = \hat{Q}_\lambda \hat{X}_i \hat{Q}_\lambda^{-1}, \quad \hat{y}_i = \hat{Q}_\lambda \hat{x}_i \hat{Q}_\lambda^{-1},
$$

(2.11)

where the hat $\hat{}$ distinguishes the quantum operators from their classical counterparts. The correspondence between the kernel $Q_\lambda(y; x)$ of $\hat{Q}_\lambda$ and the generating function $F_\lambda(y; x)$ of $B_\lambda$ is given by the semiclassical relation

$$
Q_\lambda(y; x) \sim \exp\left(-\frac{i}{\hbar}F_\lambda(y; x)\right), \quad \hbar \to 0.
$$

(2.12)

After publication of [2] the $Q$-operators have been found for a number of other quantum integrable models [3].

The properties of $\hat{Q}_\lambda$ such as the invariance of the Hamiltonians

$$
[\hat{Q}_\lambda, H_i] = 0
$$

(2.13)

and the commutativity

$$
[\hat{Q}_{\lambda_1}, \hat{Q}_{\lambda_2}] = 0
$$

(2.14)

reproduce the respective properties (2.4) and (2.5) of $B_\lambda$. The most interesting property of $\hat{Q}_\lambda$, however, is that its eigenvalues $\phi(\lambda)$ on the joint eigenvectors $\Psi_\nu$ of $H_i$ and $\hat{Q}_\lambda$ labelled with the quantum numbers $\nu$

$$
Q_\lambda \Psi_\nu = \phi_\nu(\lambda) \Psi_\nu
$$

(2.15)
satisfy the separation equation, which is a certain differential or difference equation

\[ \hat{\mathcal{W}} \left( \lambda, -i\hbar \frac{d}{d\lambda}; \{ h_i \} \right) \phi_\nu(\lambda) = 0 \]  

(2.16)

containing the eigenvalues \( h_i \) of \( H_i \). In the classical limit the equation (2.16) goes over into the spectrality equation (2.9).

An important application of the spectrality property of BT is that to the problem of separation of variables \[7, 8\]. Again, it is instructive to start with the quantum case. A separating operator \( \hat{K} \) is, by definition, an operator, transforming the joint eigenfunctions \( \Psi_\nu \) of \( H_i \) into the product

\[ \hat{K} \Psi_\nu = c_\nu \prod_{i=1}^{n} \phi_\nu(\lambda_i) \]  

(2.17)

of separated functions \( \phi_\nu(\lambda) \) of one variable \( \lambda \) satisfying the separation equation (2.16). Since the coefficients \( c_\nu \) in (2.17) can be chosen arbitrarily, abstractly speaking, there exist infinitely many separating operators \( \hat{K} \). The difficult problem, however, is to find the ones which can be described as integral operators with explicitly given kernels.

Knowing a \( Q \)-operator gives one an immediate opportunity to construct plenty of separating operators. Indeed, consider the operator product \( \hat{Q}_{\lambda_1\ldots\lambda_n} = \hat{Q}_{\lambda_1} \ldots \hat{Q}_{\lambda_n} \) having the kernel \( Q_{\lambda_1\ldots\lambda_n}(y; x) \) and for any function \( \rho(y) \) introduce the operator

\[ \hat{K}_\rho : \Psi(x) \rightarrow \int dx \int dy \rho(y) Q_{\lambda_1\ldots\lambda_n}(y; x) \Psi(x). \]  

(2.18)

It is obvious from (2.15) that \( \hat{K}_\rho \) is a separating operator, the coefficients \( c_\nu \) being

\[ c_\nu = \int dy \rho(y) \Psi_\nu(y). \]  

(2.19)

Since the eigenfunctions \( \Psi_\nu(y) \) form a basis in the corresponding Hilbert space, the formula (2.19) provides a one-to-one correspondence between reasonably chosen classes of \( c_\nu \) and \( \rho(y) \). Therefore, arguably, the formula (2.18) describes all possible separating operators. Their kernels \( K_\rho(\lambda; x) \) are given explicitly as multiple integrals

\[ K_\rho(\lambda; x) = \int dy \int d\xi^{(1)} \ldots \int d\xi^{(n-1)} \times \rho(y) Q_{\lambda_1}(y; \xi^{(1)}) Q_{\lambda_2}(\xi^{(1)}; \xi^{(2)}) \ldots Q_{\lambda_n}(\xi^{(n-1)}; x). \]  

(2.20)

It is a straightforward task to present the classical analog of the above argument. Consider the composition \( B_{\lambda_1\ldots\lambda_n} = B_{\lambda_1} \circ \ldots \circ B_{\lambda_n} \) of Bäcklund transformations and the corresponding generating function \( F_{\lambda_1\ldots\lambda_n}(y; x) \). Let us switch now the roles of \( y \)'s and \( \lambda \)'s treating \( \lambda \)'s as dynamical variables and \( y \)'s as parameters. Then \( F_{\lambda_1\ldots\lambda_n}(y; x) \) becomes the generating function of the \( n \)-parametric canonical transformation \( K_y \) from \((X, x)\) to \((\mu, \lambda)\) given by the equations

\[ X_i = \frac{\partial F_{\lambda_1\ldots\lambda_n}}{\partial x_i}, \quad \mu_i = -\frac{\partial F_{\lambda_1\ldots\lambda_n}}{\partial \lambda_i}. \]  

(2.21)
It follows directly from (2.9) that the pairs \((\lambda_i, \mu_i)\) satisfy the separation equations
\[ W(\lambda_i, f(\mu_i); \{H_j\}) = 0 \] (2.22)
which constitutes exactly the definition of the separating canonical transformation in the classical case [7].

The above construction corresponds in the quantum case to setting \(\rho(y) = \delta(y_1 - \bar{y}_1)\ldots\delta(y_n - \bar{y}_n)\) where \(\bar{y}_i\) are some constants. It remains an open question what could be the classical analog of the formula (2.18) for generic \(\rho(y)\).

As the last general remark before passing to the examples, we would like to stress that for the finite-dimensional systems the composition of \(n\) BT’s with \(n\) being the number of degrees of freedom is a sort of ‘universal’ BT in the sense that any other canonical transformation preserving the Hamiltonians \(H_i\) must be expressible in terms of \(B_{\lambda_1\ldots\lambda_n}\). To observe it one can use again the fact that in the angle coordinates \(B_\lambda\) acts as a shift \(\varphi_i \rightarrow \varphi_i + b_i(\lambda)\). For generic \(b_i(\lambda)\) the sum \(b_i(\lambda_1) + \ldots + b_i(\lambda_n)\) must then cover the \(n\)-dimensional Liouville torus which results in the universality of \(B_{\lambda_1\ldots\lambda_n}\).

3. Periodic Toda lattice

Our first example is the periodic Toda lattice [12, 13] for which there exist two alternative Lax operators associated, as we shall show, with two different BT’s. The standard and quite well studied BT [14, 12, 1, 10] which we denote here \(B_\lambda\) is associated, in the sense defined in the previous section, to the \(2 \times 2\) Lax matrix (or, monodromy matrix [15]) \(L(u; X, x)\) defined as the product of local L-operators
\[ L(u) = \ell_n(u)\ldots\ell_2(u)\ell_1(u), \] (3.1)
\[ \ell_i(u) \equiv \ell_i(u; X_i, x_i) = \begin{pmatrix} u + X_i & -e^{x_i} \\ e^{-x_i} & 0 \end{pmatrix}. \] (3.2)

The characteristic polynomial of \(L(u)\) is quadratic in \(v\)
\[ W(u, v) \equiv \det(v - L(u)) = v^2 - t(u)v + 1, \] (3.3)
and the commuting Hamiltonians \(H_i\) are obtained from the expansion of the only non-trivial spectral invariant \(t(u) \equiv \text{tr} L(u)\)
\[ t(u) = u^n + H_1u^{n-1} + \ldots + H_n. \] (3.4)

In particular,
\[ \frac{1}{2}H_1^2 - H_2 = \sum_{i=1}^n \left( \frac{1}{2}X_i^2 + e^{x_{i+1}-x_i} \right) \] (3.5)
is the standard periodic Toda Hamiltonian (in this section we use the periodicity convention \(i + n \equiv i\) for the indices \(i\)).
The Bäcklund transformation $B_\lambda$ is obtained from the generating function

$$F_\lambda(y; x) = \sum_{i=1}^{n}(e^{x_i-y_i} - e^{y_{i+1}-x_i} - \lambda(x_i - y_i))$$

(3.6)

and, according to (2.3), is implicitly described by the equations

$$X_i = e^{x_i-y_i} + e^{y_{i+1}-x_i} - \lambda, \quad Y_i = e^{x_i-y_i} + e^{y_i-x_{i-1}} - \lambda.$$  

(3.7)

The characteristic properties of the BT are verified easily. The invariance of the Hamiltonians can be established using the equality [10]

$$M_{i+1}(u, \lambda)(\ell_i(u; X_i, x_i)) = \ell_i(u; Y_i, y_i)M_i(u, \lambda),$$

(3.8)

where

$$M_i(u, \lambda) \equiv M_i(u, \lambda; x_{i-1}, y_i) = \begin{pmatrix} 1 & -e^{y_i} \\ e^{-x_{i-1}} & \lambda - u - e^{y_i-x_{i-1}} \end{pmatrix}$$

(3.9)

which one can verify directly using the equations (3.7). Due to the periodic boundary conditions, the local gauge transformation (3.8) results in the spectrum-preserving similarity transformation

$$M_1(u, \lambda)L(u; X, x) = L(u; Y, y)M_1(u, \lambda)$$

(3.10)

for $L(u)$ which proves the invariance (2.4) of the Hamiltonians.

The direct proof of the commutativity (2.5) of the BT’s can be found in [14, 12, 1].

To prove the spectrality equality (2.9) which in this case takes the form $\det(e^{-\mu} - L(\lambda)) = 0$ we shall apply a modified version of the argument used in [10, 2] for the quantum case. Note, first, that in our case

$$\mu = -\frac{\partial F_\lambda}{\partial \lambda} = \sum_{i=1}^{n}(x_i - y_i),$$

(3.11)

as follows from (3.4) and (2.8). It suffices then to show that $e^{-\mu}$ is an eigenvalue of the matrix $L(\lambda)$. We shall construct explicitly the corresponding eigenvector $\omega_1$:

$$L(\lambda; X, x)\omega_1 = e^{-\mu}\omega_1.$$  

(3.12)

From (3.9) it follows that $\det(M_i(u, \lambda)) = \lambda - u$. It is easy to see that for $u = \lambda$ the matrix $M_i(\lambda, \lambda)$ has the unique, up to a scalar factor, null-vector

$$\omega_i = \begin{pmatrix} e^{y_i} \\ 1 \end{pmatrix}, \quad M_i(\lambda, \lambda)\omega_i = 0.$$  

(3.13)

Using the identity (3.10) we conclude that

$$M_1(\lambda, \lambda)L(\lambda; X, x)\omega_1 = 0$$  

(3.14)

which, combined with the uniqueness of the null-vector $\omega_1$ of $M_1$, implies that $\omega_1$ is an eigenvector of $L(\lambda; X, x)$. To determine the corresponding eigenvalue, we apply the
same argument to the identity (3.8) obtaining the equality $M_{i+1}(\lambda; \lambda)e_i(\lambda; X_i, x_i)\omega_i = 0$ from which it follows that $e_i(\lambda; X_i, x_i)\omega_i \sim \omega_{i+1}$. The direct calculation shows that

$$e_i(\lambda; X_i, x_i)\omega_i = e^{y_i - x_i}\omega_{i+1}. \quad (3.15)$$

It remains only to use the formulae (3.1) and (3.11) to arrive finally at (3.12). Actually, we could skip the discussion of null-vectors of $M_i$ and to derive (3.12) directly from (3.17). In more complicated situations, however, it may be easier to find $\omega$ as the null-vector of $M$ and then to determine the corresponding eigenvalue of $L(\lambda)$.

Note that the vectors $\omega_i$ are the classical counterparts of Baxter’s [11] vacuum vectors.

Let us examine now the alternative Lax operator [12, 13] given by the $n \times n$ matrix

$$L_{jk}(v; X, x) = -X_j\delta_{jk} + v^{-1/n}e^{x_j-x_k}\delta_{j,k+1} + v^{1/n}\delta_{j+1,k}. \quad (3.16)$$

The duality between the Lax operators $L(v)$ and $L(u)$ is expressed in the switching the roles of the parameters $u$ and $v$. The characteristic polynomial $W(v, u) \equiv \det(u - L(v))$ of the Lax operator (3.11) produces the same Hamiltonians $H_i$ and the same spectral curve as $W(u, v)$, as follows from the identity

$$\det(v - L(u)) = -v\det(u - L(v)). \quad (3.17)$$

For other examples of the similar duality, see [16].

The swapping of $u$ and $v$ corresponds to switching the roles of the parameters $\lambda$ and $\mu$ in the BT. For the new Bäcklund transformation $B_\mu$ associated with the Lax operator $L(v)$ the formulae (3.6), (3.7) and (3.11) remain the same but their interpretation changes. The BT is parametrized now by the parameter $\mu$ which becomes a numerical constant. The equality (3.11) is reinterpreted now as a constraint on the variables $x_i$ and $y_i$. The parameter $\lambda$ is reinterpreted, respectively, as the Lagrange multiplier for the constraint (3.11) and becomes a dynamical variable which can be defined from the equations (3.7).

The characteristic properties of BT are verified for $B_\mu$ in very much the same manner like for $B_\lambda$. The invariance of the Hamiltonians follows from the invariance of the spectrum of $L(v)$ which, in turn, follows from the easily verified identity

$$M(v)\mathcal{L}(v; X, x) = \mathcal{L}(v; Y, y)M(v), \quad (3.18)$$

with the matrix $M(v) \equiv M(v; x, y)$ given by its components

$$M_{jk}(v) = -\delta_{jk} + v^{-1/n}e^{y_j-x_k}\delta_{j,k+1}. \quad (3.19)$$

The commutativity, as shown in section 2 follows from the canonicity and the invariance.

To prove the spectrality equality $\det(\lambda - L(e^{-\mu})) = 0$, it suffices, similarly to the case of the Lax operator $L(u)$, to present the eigenvector $\Omega$ of the matrix $L(e^{-\mu})$ corresponding to the eigenvalue $\lambda$: $L(e^{-\mu})\Omega = \lambda\Omega. \quad (3.20)$
Again, Ω can be determined as the null-vector of $\mathcal{M}(e^{-\mu})$:

$$\mathcal{M}(e^{-\mu})\Omega = 0. \quad (3.21)$$

Note that the uniqueness of Ω follows from the easily verified identity $\det(z - \mathcal{M}(v)) = (z + 1)^n - v^{-1}e^{-\mu}$ which implies that the spectrum of $\mathcal{M}(e^{-\mu})$ consists of $n$ non-degenerate eigenvalues, the 0 being one of them. From (3.21) one easily derives the recurrence relation for the components of Ω

$$\Omega_j = \Omega_{j-1} \exp \left( y_j - x_{j-1} + \frac{\mu}{n} \right) \quad (3.22)$$

which determine Ω up to a constant factor. It remains to verify the identity (3.20) which can be done by a direct calculation using the expressions (3.16) for the matrix $L(v)$, (3.11) for $\mu$ and (3.7) for $X_i$.

4. Elliptic Ruijsenaars model

Our second example is the relativistic generalization due to Ruijsenaars [17] of the elliptic Calogero-Moser [18] many-body system. For the non-relativistic Calogero-Moser system a BT was found in [19]. In [7] a discrete-time dynamics was constructed for the elliptic Ruijsenaars model. As we show below, the discrete-time evolution transformation found in [7] has all the properties of a BT if the parameter $p$ in [7] is specified in a proper way.

We use here the notations of [7] with few exceptions: our parameter $\xi$ equals to $-\lambda$ from [7], and our $e^{-\lambda}$ corresponds to $p$ from [7]. As in the case of the Toda lattice, there exist two dual BT’s: $B_\lambda$ and $B_\mu$. The standard Lax operator for the Ruijsenaars model, as shown below, is associated with $B_\mu$. Since the dual Lax operator is so far unknown, we describe here only the transformation $B_\mu$.

Following [7, 8], we introduce the Lax operator $L(v; X, x)$ for the $n$-particle ($A_{n-1}$ type) Ruijsenaars system as the $n \times n$ matrix with the entries

$$L_{ij}(v) = -e^{x_i} \frac{\sigma(x + x_i - x_j - \xi)}{\sigma(x - x_j - \xi)} \prod_{k \neq i} \frac{\sigma(x_i - x_k + \xi)}{\sigma(x_i - x_k)}, \quad (4.1)$$

where $\sigma(x)$ is the Weierstrass sigma function and $\xi$ is a constant.

The commuting Hamiltonians

$$H_i = \sum_{J \subseteq \{1, \ldots, n\}, |J| = i} \exp \left( \sum_{j \in J} X_j \right) \prod_{j \in J} \frac{\sigma(x_j - x_k + \xi)}{\sigma(x_j - x_k)} \prod_{k \not\in J} \frac{\sigma(x_i - x_k + \xi)}{\sigma(x_i - x_k)}, \quad i = 1, \ldots, n \quad (4.2)$$

are generated from the characteristic polynomial of the matrix $L(v)$ (4.1)

$$\det(L(v) - u) = \sum_{j=0}^n (-u)^{n-j} H_j \frac{\sigma(v - j\xi)}{\sigma(v)} \quad (4.3)$$
where we assume $H_0 \equiv 1$.

The Bäcklund transformation $B_\mu$ is given by the equations

$$e^{x_i} = e^{-\lambda} \prod_{j \neq i} \frac{\sigma(x_i - x_j - \xi)}{\sigma(x_i - x_j + \xi)} \prod_{k=1}^{n} \frac{\sigma(x_i - y_k + \xi)}{\sigma(x_i - y_k)}$$

$$e^{y_i} = e^{-\lambda} \prod_{k=1}^{n} \frac{\sigma(x_k - y_i + \xi)}{\sigma(x_k - y_i)}$$

where $\lambda$ is considered as the Lagrange multiplier corresponding to the constraint

$$\mu = n \xi + \sum_{k=1}^{n} (x_k - y_k).$$

Note here that the variable $\lambda$ in formulas (4.4)–(4.5), describing the discrete-time dynamics, appeared already as $p$ in [5], but the conjugated variable $\mu$ did not. Notice also that $\lambda$ was treated in [5] as an extra parameter, not as a Lagrange multiplier corresponding to a constraint.

The generating function of the canonical transformation $B_\mu$ is expressed in terms of the function

$$S(x) = \int x \ln \sigma(y)dy$$

as follows:

$$F_\lambda(y; x) = -\lambda \sum_{i=1}^{n} (x_i - y_i + \xi) + \sum_{i<j} (S(x_i - x_j - \xi) - S(x_i - x_j + \xi))$$

$$+ \sum_{i,j=1}^{n} (S(x_i - y_j + \xi) - S(x_i - y_j)).$$

The verification of the characteristic properties of BT for $B_\mu$ proceeds in the same way as in the case of the Toda lattice.

The invariance of the Hamiltonians $H_i$ follows from the identity (see [5] for the proof)

$$M(v) L(v; X, x) = L(v; Y, y) M(v)$$

where the matrix $M(v) \equiv M(v; x, y)$ is defined as

$$M_{ij}(v) = \frac{\sigma(v + y_i - x_j - \xi)}{\sigma(y_i - x_j - \xi)} \prod_{k \neq i} \frac{\sigma(y_i - y_k + \xi)}{\sigma(y_i - y_k)} \prod_{k} \frac{\sigma(x_k - y_i + \xi)}{\sigma(x_k - y_i)}.$$

The commutativity, as usual, is a consequence of canonicity and invariance (see section 2).

To prove the spectrality equality which takes the form $\det(e^{-\lambda} - L(\mu)) = 0$ it is sufficient, like in the case of the Toda lattice, to find the eigenvector $\Omega$ of the matrix $L(\mu)$ corresponding to the eigenvalue $e^{-\lambda}$. Let us show that, up to a constant multiplier, the components of the eigenvector $\Omega$ are

$$\Omega_i = \prod_{k=1}^{n} \frac{\sigma(x_i - y_k + \xi)}{\prod_{k \neq i} \sigma(x_i - x_k)}.$$
The equality
\[ \mathcal{L}(\mu) \Omega = e^{-\lambda} \Omega \] (4.12)
or
\[ \sum_{j=1}^{n} \mathcal{L}_{ij}(\mu) \Omega_j = e^{-\lambda} \Omega_i \] (4.13)
after the substitutions (4.1) for \( \mathcal{L}_{ij} \), (4.6) for \( \mu \) and (4.4) for \( e^{X_i} \) is reduced to the following identity for sigma functions
\[ \sum_{j=1}^{n} \sigma(\mu + x_i - x_j - \xi) \prod_{k=1}^{n} \sigma(x_j - y_k + \xi) \prod_{k \neq j} \frac{\sigma(x_i - x_k - \xi)}{\sigma(x_j - x_k)} = \sigma(\mu) \prod_{k=1}^{n} \sigma(x_i - y_k). \] (4.14)

Due to the symmetry, it is sufficient to prove (4.14) only for \( i = 1 \). Let \( i = 1 \) and \( n \geq 2 \). Consider the both sides of the equality (4.14) as functions of \( x_n \). It is easy to see that they are holomorphic in \( x_n \) (the apparent poles in the left-hand-side being cancelled) and have the same quasiperiodicity properties. From the basic properties of sigma functions \([20]\) it follows that it is sufficient to verify the equality of LHS and RHS only in one arbitrary point \( x_n = \bar{x} \) with the only condition \( \bar{x} \neq \mu - x_n \). Choosing \( \bar{x} = y_n - \xi \) we observe that (4.14) is reduced to the similar identity of order \( n - 1 \). The proof follows then by induction in \( n \) since the case \( n = 1 \) is trivial.

As in the section 3, the vector \( \Omega \) (4.11) is again the null-vector of the matrix \( \mathcal{M}(\mu) \), i.e.
\[ \mathcal{M}(\mu) \Omega = 0. \] (4.15)

The corresponding identity for sigma functions
\[ \sum_{j=1}^{n} \sigma(\mu + y_i - x_j - \xi) \prod_{k \neq j=1}^{n} \frac{\sigma(x_j - y_k + \xi)}{\sigma(x_j - x_k)} = 0 \] (4.16)
follows from the identity
\[ \sum_{j=1}^{n} \frac{\prod_{k=1}^{n} \sigma(x_j - x_k)}{\prod_{k \neq j=1}^{n} \sigma(x_j - x_k)} = \prod_{j=1}^{n} \frac{\sigma(x_j - x_k)}{\sigma(x_j - x_k)} = 0 \] if \( \sum_{k=1}^{n} (z_k - x_k) = 0 \), (4.17)
(cf. \([20]\), p. 451) when one substitutes \( z_k = y_k - \xi \) for \( k \neq i \) and \( z_i = \mu + y_i - \xi \).

5. Discussion

We have studied three new aspects of Bäcklund transformations. Those are spectral-ity, dual BT’s and application of BT’s to the problem of separation of variables. As demonstrated in the section 2, the composition of \( n \) BT’s, being an ‘universal’ \((n\text{-parametric})\) BT, provides a separation of variables which has \( n \) arbitrary parameters, and thereby defines an ‘universal’ \((n\text{-parametric})\) family of separating transformations. The connection between the ‘universal’ BT and the ‘universal’ SoV is intriguing and has yet to be studied in detail.
Though we have discussed in the present paper only the classical case, our primary motivation comes from the quantum case. The main problem in the quantum case is to construct Baxter’s $Q$-operator which is a quantum analog of the Bäcklund transformation. For the trigonometric case of the Ruijsenaars system, i.e. for the case of multivariable ($A_{n-1}$-type) Macdonald polynomials, we have succeeded to describe explicitly such a quantum analog of the transformation $B_n$ introduced in the section 4. The results will be reported elsewhere.

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