Additivity of minimal entropy output for a class of
covariant channels

M. Fannes\textsuperscript{1}, B. Haegeman\textsuperscript{2}, M. Mosonyi\textsuperscript{3} and D. Vanpeteghem\textsuperscript{4}

\textit{Instituut voor Theoretische Fysica, K.U.Leuven, Celestijnenlaan 200D,
B-3001 Heverlee, Belgium}

Abstract: Additivity of minimal entropy output is proven for the class
of quantum channels $\Lambda_t(A) := tA^T + (1 - t)\tau(A)$ in the parameter range
$-2/(d^2 - 2) \leq t \leq 1/(d + 1)$.

1 Introduction

A quantum channel $\Phi$ is a CPTP (completely positive, trace-preserving) map
that maps states of the input Hilbert space $\mathcal{H}_{\text{in}}$ into states of the output
Hilbert space $\mathcal{H}_{\text{out}}$. The minimal entropy output of the channel is

$$S_{\text{min}}(\Phi) := \min\{S(\Phi(\rho)) : \rho \in S(\mathcal{H}_{\text{in}})\},$$

where $S(\mathcal{H}_{\text{in}})$ denotes the state space of the input system, i.e. the set of self-
adjoint matrices with trace one, and $S(\sigma) := -\text{Tr} \sigma \log \sigma$ is the von Neumann
entropy of the density matrix $\sigma$. By concavity of the entropy, $S_{\text{min}}$ can be
expressed as

$$S_{\text{min}}(\Phi) := \min\{S(\Phi(|\psi\rangle\langle\psi|)) : \psi \in \mathcal{H}_{\text{in}}, \|\psi\| = 1\}.$$ 

Hence, $S_{\text{min}}$ is a lower bound on the mixedness of the output state, measured
by entropy, when the input is a pure state. It can therefore be interpreted
as a measure of the noisiness of the channel.

\textsuperscript{1}E-mail: mark.fannes@fys.kuleuven.ac.be
\textsuperscript{2}E-mail: bart.haegeman@fys.kuleuven.ac.be, Research assistant of the Fund for Scientific Research–Flanders (Belgium) (F.W.O.–Vlaanderen)
\textsuperscript{3}E-mail: mosonyi@math.bme.hu, On leave of absence from Mathematical Institute, Budapest University of Technology and Economics, 1111 Budapest XI. Egry Jozsef u. 1., Hungary
\textsuperscript{4}E-mail: dimitri.vanpeteghem@fys.kuleuven.ac.be, Research assistant of the Fund for Scientific Research–Flanders (Belgium) (F.W.O.–Vlaanderen)
A fundamental question of quantum information theory is whether the minimal entropy output is additive for product channels, i.e., given two channels \( \Phi_1 : \mathcal{S}(\mathcal{H}_{\text{in}}^1) \rightarrow \mathcal{S}(\mathcal{H}_{\text{out}}^1) \) and \( \Phi_2 : \mathcal{S}(\mathcal{H}_{\text{in}}^2) \rightarrow \mathcal{S}(\mathcal{H}_{\text{out}}^2) \), whether the equality

\[
S_{\text{min}}(\Phi_1 \otimes \Phi_2) = S_{\text{min}}(\Phi_1) + S_{\text{min}}(\Phi_2)
\]

holds. This question was shown to be equivalent to the additivity of other quantities, like the Holevo capacity and the entanglement of formation \[12, 3, 16, 17\]. It has been solved in the special cases when one of the channels is the identical channel \[2\], the depolarising channel \[10\], a unital qubit channel \[9\] or an entanglement breaking channel \[15, 11\].

The inequality

\[
S_{\text{min}}(\Phi_1 \otimes \Phi_2) \leq S_{\text{min}}(\Phi_1) + S_{\text{min}}(\Phi_2)
\]

holds trivially, since \( S(\Phi_1 \otimes \Phi_2(|\psi_1\rangle\langle\psi_1|)\otimes |\psi_2\rangle\langle\psi_2|) = S_{\text{min}}(\Phi_1) + S_{\text{min}}(\Phi_2) \) for the states \(|\psi_1\rangle\langle\psi_1|\) and \(|\psi_2\rangle\langle\psi_2|\) at which the output entropies of \( \Phi_1 \) and \( \Phi_2 \) are minimised. Therefore, in order to show additivity, we have to show that \( S(\Phi_1 \otimes \Phi_2(\psi\langle\psi|)) \) attains its minimal value at \( \psi = \psi_1 \otimes \psi_2 \). Using the Schmidt decomposition of \( \psi \), we have to minimise

\[
S\left(\sum_{ij} \sqrt{\lambda_i \lambda_j} \Phi_1(|e_i\rangle\langle e_j|) \otimes \Phi_2(|f_i\rangle\langle f_j|)\right),
\]

where \( \{e_i : i = 1, 2, \ldots, \dim \mathcal{H}_{\text{in}}^1\} \) and \( \{f_j : j = 1, 2, \ldots, \dim \mathcal{H}_{\text{in}}^2\} \) are orthonormal bases of the input spaces \( \mathcal{H}_{\text{in}}^1 \) and \( \mathcal{H}_{\text{in}}^2 \) and where \( \lambda = \{\lambda_i : i = 1, \ldots, (\dim \mathcal{H}_{\text{in}}^1) \wedge (\dim \mathcal{H}_{\text{in}}^2)\} \) is a probability distribution.

Solving this problem in full generality is rather hopeless, however, it simplifies a great deal if the channels \( \Phi_1 \) and \( \Phi_2 \) possess the additional property of unitary covariance.

1.1 Definition. A channel \( \Phi : \mathcal{S}(\mathcal{H}_{\text{in}}) \rightarrow \mathcal{S}(\mathcal{H}_{\text{out}}) \) is unitarily covariant, if for every unitary \( U \) in \( \mathcal{H}_{\text{in}} \) there exists a unitary \( V \) in \( \mathcal{H}_{\text{out}} \) such that

\[
\Phi \circ \text{Ad}_U = \text{Ad}_V \circ \Phi
\]

holds, where \( \text{Ad}_X \) is the map \( A \mapsto XAX^* \).
Obviously, for unitarily covariant channels $S(\Phi(|\psi\rangle\langle\psi|))$ is independent of the input state $|\psi\rangle\langle\psi|$ and therefore equal to $S_{\min}(\Phi)$.

A significant simplification of the additivity problem for unitarily covariant maps is that the entropy in (2) depends only on the Schmidt coefficients of the input state $\psi$. Indeed, if $\psi$ and $\varphi$ have the same Schmidt coefficients, then there exist unitaries $U_1$ and $U_2$ in $H^{1}_{in}$ and $H^{2}_{in}$ such that $\varphi = U_1 \otimes U_2 \psi$. Therefore

$$\Phi_1 \otimes \Phi_2(|\varphi\rangle\langle\varphi|) = (V_1 \otimes V_2) \Phi_1 \otimes \Phi_2(|\psi\rangle\langle\psi|)(V_1^* \otimes V_2^*)$$

for some unitaries $V_1$ and $V_2$ in $H^{1}_{out}$ and $H^{2}_{out}$, hence $\Phi_1 \otimes \Phi_2(|\varphi\rangle\langle\varphi|)$ and $\Phi_1 \otimes \Phi_2(|\psi\rangle\langle\psi|)$ have the same entropy. Now to prove additivity, it is sufficient to show that the function

$$\lambda \mapsto S\left(\sum_{ij} \sqrt{\lambda_i \lambda_j} \Phi_1(|e_i\rangle\langle e_j|) \otimes \Phi_2(|f_i\rangle\langle f_j|)\right)$$

reaches its minimal value at the vertices of the simplex $S$. Here, $\{e_i\}$ and $\{f_j\}$ are fixed orthonormal bases in $H^{1}_{in}$ and $H^{2}_{in}$, and $S$ is the simplex of probability distributions on $(\dim H^{1}_{in}) \wedge (\dim H^{2}_{in})$ points.

A trivial example of a unitarily covariant channel is the trace map

$$\tau(A) := (\text{Tr} A) \frac{1}{d_{out}} 1_{out},$$

where $1_{out}$ is the identity on $H_{out}$ and $d_{out} := \dim H_{out}$. In this case, the covariance relation (3) holds for any pair of unitaries $(U, V)$. However, if we strengthen the definition of covariance, and require uniqueness of $V$, then the map $U \mapsto V$ is easily seen to be a representation of the group $G = U(d_{in})$ on $H_{out}$. As $\text{Ad}_U$ is insensitive to phase factors, we can consider $G = SU(d_{in})$ as well. We can now turn the question around, fixing a unitary representation $\alpha_{out}$ of $SU(d_{in})$ on $H_{out}$ and looking for channels that satisfy the following property:

1.2 Definition. A channel $\Phi : S(H_{in}) \to S(H_{out})$ is $\alpha$-invariant, if

$$\Phi \circ \text{Ad}_U = \text{Ad}_{\alpha_{out}(U)} \circ \Phi$$

for all $U \in SU(d_{in})$. 

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Such $\alpha$-invariant channels were studied in [7], where it was also shown that for an irreducible representation $\alpha_{\text{out}}$

$$\chi(\Phi) = \log d_{\text{out}} - S_{\text{min}}(\Phi).$$

Here, $\chi(\Phi)$ is the Holevo capacity of the channel [6], given by the formula

$$\chi(\Phi) = \max \left\{ S\left( \sum_j p_j \Phi(\rho_j) \right) - \sum_j p_j S(\Phi(\rho_j)) \right\},$$

where the maximum is taken over all finitely supported measures $p$ on the state space $\mathcal{S}(\mathcal{H}_{\text{in}})$ and $p(\rho_j) := p_j$. Therefore, for such channels, if additivity holds for the minimal entropy output it also holds for the Holevo capacity.

In section 2, we deal with the description of the set $\mathcal{B}_\alpha$ of $\alpha$-invariant channels, focusing mainly on the case $\mathcal{H}_{\text{out}} = \mathcal{H}_{\text{in}}$ and $\alpha_{\text{out}}$ is either

(a) the identical representation, or

(b) the conjugate representation $U \mapsto \overline{U}$.

$\overline{U}$ is the entrywise conjugate of the matrix $U$ in some fixed base. It turns out that

- in case (a), $\mathcal{B}_\alpha$ is the family of depolarising channels, given by

$$\Delta_t(A) := tA + (1 - t)\tau(A), \quad -\frac{1}{d^2 - 1} \leq t \leq 1,$$

where $d = d_{\text{in}} = d_{\text{out}}$ and $[-1/(d^2 - 1), 1]$ is the maximal range of $t$ such that $\Delta_t$ is completely positive, while

- in case (b), we obtain the family

$$\Lambda_t(A) := tA^T + (1 - t)\tau(A), \quad -\frac{1}{d - 1} \leq t \leq \frac{1}{d + 1},$$

where $A^T$ is the transpose of $A$ in the same base in which the $U$’s are conjugated.

The elements of this second class will be called transpose depolarising channels.

Additivity of the minimal entropy output was shown for the depolarising channels in [10], even when one of the channels is completely arbitrary, and for the extreme transpose depolarising channel $\Lambda_{-1/(d-1)}$ in [13], followed by proofs using different methods in [5] and [11]. In section 3 we shall show additivity for the subset of the transpose depolarising channels determined by $-2/(d^2 - 2) \leq t \leq 1/(d + 1)$.
2 Extreme invariant maps

We start out with the more general setting of a pair $\alpha$ of unitary representations $\alpha_{\text{in}}$ and $\alpha_{\text{out}}$ of an abstract group $G$ on $\mathcal{H}_{\text{in}}$ and $\mathcal{H}_{\text{out}}$ respectively and define a map $\Phi$ to be $\alpha$-invariant, if

$$\Phi \circ \text{Ad}_{\alpha_{\text{in}}(g)} = \text{Ad}_{\alpha_{\text{out}}(g)} \circ \Phi, \quad g \in G.$$ 

Next, we fix a basis $\{e_i : i = 1, 2, \ldots, d_{\text{in}}\}$ in $\mathcal{H}_{\text{in}}$ with corresponding anti-unitary $J$ defined by $Je_i = e_i$. The map

$$T : |B\rangle\langle A| \mapsto JAJ^* \otimes B$$

is well-defined on $\mathcal{B}(\mathcal{B}(\mathcal{H}_{\text{in}}), \mathcal{B}(\mathcal{H}_{\text{out}}))$ and extends to a linear isomorphism between $\mathcal{B}(\mathcal{B}(\mathcal{H}_{\text{in}}), \mathcal{B}(\mathcal{H}_{\text{out}}))$ and $\mathcal{B}(\mathcal{H}_{\text{in}}) \otimes \mathcal{B}(\mathcal{H}_{\text{out}})$. For a general element $\Phi : \mathcal{B}(\mathcal{H}_{\text{in}}) \to \mathcal{B}(\mathcal{H}_{\text{out}})$ we obtain the formula

$$T\Phi = \sum_{ij} e_{ij} \otimes \Phi(e_{ij}). \quad (4)$$

The right-hand side of eq. 4 is called the Choi matrix of $\Phi$ and we shall denote it by $C(\Phi)$. A fundamental theorem of Choi [4] states that $\Phi$ is completely positive if and only if its Choi matrix is positive semidefinite.

A straightforward computation shows that $\Phi$ is trace-preserving if and only if $\text{Tr}_2 C(\Phi) = 1_{\text{in}}$ and, moreover,

(i) $C(\text{Ad}_V \circ \Phi) = \text{Ad}_{1 \otimes V}(C(\Phi)) = (1 \otimes V) \ C(\Phi) \ (1 \otimes V^*)$

(ii) $C(\Phi \circ \text{Ad}_U) = \text{Ad}_{JU^*J^* \otimes 1}(C(\Phi))$

$$= (JU^*J^* \otimes 1) \ C(\Phi) \ (JUJ \otimes 1).$$

Properties (i) and (ii) imply that $\Phi$ is $\alpha$-invariant if and only if

$$[C(\Phi), J\alpha_{\text{in}}(g)J^* \otimes \alpha_{\text{out}}(g)] = 0, \quad g \in G.$$ 

The map $g \mapsto \kappa(g) := J\alpha_{\text{in}}(g)J^* \otimes \alpha_{\text{out}}(g)$ is again a unitary representation of $G$, which decomposes in irreducible components

$$\kappa(g) \cong \bigoplus_k \kappa_k(g) \otimes 1_{m(k)},$$
where \( k \) labels inequivalent irreducible representations of \( \mathcal{G} \) on a \( d(k) \)-dimensional Hilbert space and where \( m(k) \) is the multiplicity of the \( k \)-th representation. A general element of the commutant of \( \kappa (\text{SU}(d_{\text{in}})) \) is of the form

\[
C = \bigoplus_k 1_{d(k)} \otimes A_k,
\]

and it is the Choi matrix of a CP map if and only if all \( A_k \geq 0 \). In the special case \( m(k) = 1 \) \( \forall k \), we obtain that the map \( \Phi \) is \( \alpha \)-invariant if and only if its Choi matrix has the form

\[
C(\Phi) = \sum_k c_k P_k,
\]

(5)

where \( P_k \) is the projection onto the Hilbert space of the \( k \)-th irreducible component and where the \( c_k \)'s are arbitrary complex numbers. Obviously, \( \Phi \) is CP if and only if all the coefficients \( c_k \) are nonnegative. The maps \( T^* P_k \) span the extreme rays of the convex cone of \( \alpha \)-invariant CP maps. However, \( T^* P_k \) need not be trace-preserving, therefore formula (5) is in general insufficient to describe the convex set \( \mathcal{B}_\alpha \) of \( \alpha \)-invariant CPTP maps.

Note that the map

\[
\frac{1}{d_{\text{in}}} T : \Phi \mapsto \frac{1}{d_{\text{in}}} C(\Phi)
\]

gives a proper embedding of the convex set of CPTP maps into the state space of \( \mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{out}} \), its range, however, does not coincide with the whole of the state space because of the restriction \( \text{Tr}_2 C(\Phi) = 1_{\text{in}} \). Thus \( \alpha \)-invariant CPTP maps can be identified with a subset of the set \( \mathcal{S}_\alpha \) of \( \alpha \)-invariant states, i.e. states that satisfy \( [\rho, J_{\alpha_{\text{in}}}(g) J^* \otimes \alpha_{\text{out}}(g)] = 0 \ \forall g \in \mathcal{G} \). The characterisation of \( \mathcal{S}_\alpha \) for different symmetry groups was studied in [18]. Even if in general \( \mathcal{S}_\alpha \) is strictly larger then the embedded image of \( \mathcal{B}_\alpha \), both sets coincide when \( \alpha_{\text{in}} \) is irreducible since the invariance

\[
\text{Tr}_2 \rho = \text{Tr}_2 \left( J_{\alpha_{\text{in}}}(g) J^* \otimes \alpha_{\text{out}}(g) \right) \rho \left( J_{\alpha_{\text{in}}}(g)^* J^* \otimes \alpha_{\text{out}}(g)^* \right)
= \left( J_{\alpha_{\text{in}}}(g) J^* \right) \left( \text{Tr}_2 \rho \right) \left( J_{\alpha_{\text{in}}}(g)^* J^* \right)
\]

implies \( \text{Tr}_2 \rho = \frac{1}{d_{\text{in}}} 1_{\text{in}} \).

Now we turn to the case \( \mathcal{H}_{\text{in}} = \mathcal{H}_{\text{out}} = \mathcal{H}, \alpha_{\text{in}} \) the identical representation of \( \mathcal{G} = \text{SU}(d) \) and either \( \alpha_{\text{out}}(U) = U \), case (a), or \( \alpha_{\text{out}}(U) = U = JUJ^* \), case (b).
In case (a), the invariant projections of $JUJ^* \otimes U$ are $P := |\psi\rangle\langle\psi|$ and $P^\perp$, where $\psi = \frac{1}{\sqrt{d}} \sum_i e_i \otimes e_i$. The corresponding extreme maps are

$$T^*(dP) = \Delta_1 = \text{id} \quad \text{and} \quad T^*\left(\frac{d}{d^2 - 1} P^\perp\right) = \Delta_{-1/(d^2-1)}.$$  

In case (b), the invariant projections are $P_s$ and $P_a$, the projections onto the symmetric and the antisymmetric subspaces of $\mathcal{H} \otimes \mathcal{H}$. The corresponding extreme maps are

$$T^*\left(\frac{2}{d+1} P_s\right) = \Lambda_{1/(d+1)} \quad \text{and} \quad T^*\left(\frac{2}{d-1} P_a\right) = \Lambda_{-1/(d-1)}.$$  

For details on decomposing tensor products of representations of $SU(d)$, see e.g. [8].

3 Minimal entropy output for the transpose depolarising channel

In this section, we shall prove that

$$S_{\min}(\Lambda_t \otimes \Lambda_t) = 2 S_{\min}(\Lambda_t), \quad -\frac{2}{d^2 - 2} \leq t \leq \frac{1}{d+1}. \quad (6)$$

First note that the partial transpose of the state $\frac{1}{d} C(\Lambda_t)$ is $\frac{1}{d} C(\Delta_t)$, hence it is positive if and only if $t \in \left[-1/(d^2 - 1), 1/(d+1)\right] =: R_1$. Having positive partial transpose is a necessary condition for a bipartite state to be separable [14] and was shown in [18] to be also sufficient for unitarily covariant states. Quantum channels with separable Choi matrix are called entanglement-breaking, and additivity of their minimal entropy output was proven in [15]. It holds even in the stronger form where one of the factors in (1) is entanglement-breaking and the other is completely arbitrary. Hence, additivity of minimal entropy output holds in the strong form

$$S_{\min}(\Lambda_t \otimes \Phi) = S_{\min}(\Lambda_t) + S_{\min}(\Phi), \quad (7)$$

when $t \in R_1$ and $\Phi$ is an arbitrary quantum channel. We can therefore assume $t$ to be negative for the rest of the argument, even though the proof applies with a trivial modification to positive $t$'s as well.
Let \( \rho(\lambda) := |\psi(\lambda)\rangle\langle\psi(\lambda)| = \sum_{ij} \sqrt{\lambda_i \lambda_j} |e_i\rangle\langle e_j| \otimes |e_i\rangle\langle e_j| \) denote the input state and \( X(\lambda) := \Lambda_t \otimes \Lambda_t (\rho(\lambda)) \) the output state for some fixed \( t \leq 0 \). An easy computation shows that

\[
X(\lambda) = t^2 \rho(\lambda) + \frac{t(1-t)}{d} \rho'(\lambda) \otimes 1 + 1 \otimes \rho'(\lambda) + \frac{(1-t)^2}{2d} 1 \otimes 1,
\]

where \( \rho'(\lambda) := \text{Tr}_2 \rho(\lambda) = \text{Tr}_1 \rho(\lambda) \). The basis \( \{e_i\} \) is the one in which the transpose is taken.

The vectors \( \{e_i \otimes e_j : i \neq j\} \) are eigenvectors of \( X(\lambda) \) with corresponding eigenvalues \( \eta_{ij} = t(1-t)(\lambda_i + \lambda_j)/d + ((1-t)/d)^2 \). The contribution of this part of the spectrum to the entropy is \( S_1(\lambda) = -\sum_{i \neq j} \eta_{ij} \ln \eta_{ij} \), which, being a concave function of \( \lambda \), takes its minimal value at the vertices of \( S \).

Now we consider the subspace spanned by \( \{e_i \otimes e_i : i = 1, \ldots, d\} \). The restricted output density has the form

\[
\hat{X}(\lambda) = t^2 |\sqrt{\lambda}\rangle\langle \sqrt{\lambda}| + \frac{2t(1-t)}{d} \text{diag}(\lambda) + \left( \frac{1-t}{d} \right)^2 1 \otimes 1,
\]

where \( |\sqrt{\lambda}\rangle = \sum_j \sqrt{\lambda_j} e_j \) and where \( \text{diag}(\lambda) \) is the diagonal matrix with diagonal \( \lambda \). Note that, at the vertices of \( S \), \( \lambda_j = \delta_{jk} \) for some \( k \) and that in this case \( \hat{X}(\lambda) \) is diagonal in the basis \( \{e_i \otimes e_i\} \). Let \( S_2(\lambda) \) denote the entropy contribution of the eigenvalues of \( \hat{X}(\lambda) \). In order to show that it takes its minimal value at the vertices of \( S \) it is sufficient to show that \( \hat{X}(\lambda) \) is more mixed than \( \hat{X}(\lambda^*) \) for any \( \lambda \in S \), where \( \lambda^*_j = \delta_{j,1} \).

Recall that, for hermitian matrices, \( A_1 \) is called more mixed than \( A_2 \), in notation \( A_1 \succ A_2 \), if

\[
\kappa_1(A_1) \leq \kappa_2(A_2) \\
\kappa_1(A_1) + \kappa_2(A_1) \leq \kappa_1(A_2) + \kappa_2(A_2) \\
\vdots \\
\text{Tr} A_1 = \text{Tr} A_2.
\]

Here, \( \kappa_1(A_k) \geq \kappa_2(A_k) \geq \ldots \) are the eigenvalues of \( A_k \) in decreasing order. Note that \( \sum_{i \neq j} \eta_{ij} = (1-t^2)(d-1)/2d \) for any \( \lambda \) and that therefore \( \text{Tr} \hat{X}(\lambda) = 1 - \sum_{i \neq j} \eta_{ij} \) is independent of \( \lambda \). It is also clear that the
\(\lambda\)-independent diagonal term \(\left((1-t)/d\right)^2 \mathbb{1} \otimes \mathbb{1}\) doesn’t influence the more mixedness relation. The eigenvalue vector of \(\hat{X}(\lambda^*) - \left((1-t)/d\right)^2 \mathbb{1} \otimes \mathbb{1}\), arranged in decreasing order, is \((0, 0, \ldots, 0, t(2+(d-2)t)/d)\). We have therefore to show that \(\hat{X}(\lambda) - \left((1-t)/d\right)^2 \mathbb{1} \otimes \mathbb{1}\) is negative semidefinite. Writing out explicitly the corresponding quadratic form, this criterion becomes

\[
\left| \sum_j \sqrt{\lambda_j} x_j \right|^2 \leq -\frac{2(1-t)}{td} \sum_j \lambda_j |x_j|^2, \quad (x_1, \ldots, x_d) \in \mathbb{C}^d,
\]

which, by the Schwarz inequality, holds if and only if \(d \leq -2(1-t)/td\), that is, \(-2/(d^2 - 2) \leq t\).

The case \(d = 2\) is special in the sense that the conjugate representation of \(SU(d)\) is unitarily equivalent to the identical representation. As a consequence, the family of transpose depolarising channels coincides with the family of depolarising channels and additivity of the minimal entropy output holds in the stronger form \((7)\) for all \(t\) allowed by complete positivity. Also, the irreducibility of \(\alpha_{\text{out}}\) implies that an \(\alpha\)-invariant channel is bistochastic when \(d_{\text{in}} = d_{\text{out}}\) and therefore \((7)\) follows also from \((9)\) when \(d = 2\). Yet another proof of the weak version \((6)\) of additivity can be given in this case by showing that

\[
\lambda' \succ \lambda \quad \text{implies} \quad X(\lambda') \succ X(\lambda), \quad (8)
\]

for any \(\lambda, \lambda' \in \mathcal{S}\), which is a matter of straightforward computation and immediately yields \((6)\). Numerical computations suggest that \((8)\) holds for \(d > 2\) and arbitrary value of \(t\) as well, which would clearly be sufficient to prove \((9)\) for the missing domain \(t \in \left[-1/(d-1), -2/(d^2 - 2)\right]\).

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