Supplemental Materials

prepared for "An intermittent control model of flexible human gait using a stable manifold of saddle-type unstable limit cycle dynamics"

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A Details of the model

A.1 Equations of motion

The equation of motion (Eqs. 4 and 5 in the main text) is rewritten as follows:

\[ J\ddot{q}+B+G+U_{ff}+U_{fb}, \]

Elements of each of \( J, B, K, G, U_{ff} \) and \( U_{fb} \) are detailed here as follows:

\[
J = \begin{pmatrix}
  j_{1,1} & j_{1,2} & \cdots & j_{1,9} \\
  j_{2,1} & j_{2,2} & & j_{2,9} \\
  \vdots & \ddots & \ddots & \vdots \\
  j_{9,1} & j_{9,2} & \cdots & j_{9,9}
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
b_1 \\
b_2 \\
\vdots \\
b_9
\end{pmatrix},
\]

\[
K = \begin{pmatrix}
k_1 \\
k_2 \\
\vdots \\
k_9
\end{pmatrix},
\]

\[
G = \begin{pmatrix}
g_1 \\
g_2 \\
\vdots \\
g_9
\end{pmatrix},
\]

\[
U_{ff} = (0, 0, 0, u_{ff,a}^l, u_{ff,a}^r, u_{ff,h}^l, u_{ff,h}^r, u_{ff,k}^l, u_{ff,k}^r)^T,
\]

\[
U_{fb} = (0, 0, 0, u_{fb,a}^l, u_{fb,a}^r, u_{fb,h}^l, u_{fb,h}^r, u_{fb,k}^l, u_{fb,k}^r)^T,
\]

\[
j_{1,1} = 2(I_1+m_1l_1^2)+2(I_2+m_1l_2^2+m_2l_2^2)+2(I_3+m_1l_3^2+m_2l_2^2+m_3l_3^2)
\]

\[\quad +\left(4+2(m_1+m_2+m_3)(d_4-L_4)^2\right)
\]

\[\quad +2m_1d_1\left\{L_2\cos(-\theta_a^l)+L_3\cos(-\theta_k^l-\theta_a^l)-(d_4-L_4)\cos(-\theta_h^l-\theta_k^l-\theta_a^l)\right\}
\]

\[\quad +2(m_1L_2+m_2d_2)\left\{L_3\cos(-\theta_k^l)-(d_4-L_4)\cos(-\theta_h^l-\theta_k^l)\right\}
\]

\[\quad -2(m_1L_3+m_2L_2+m_3d_3)(d_4-L_4)\cos(-\theta_h^l)
\]

\[\quad +2m_1d_1\left\{L_2\cos(-\theta_a^r)+L_3\cos(-\theta_k^r-\theta_a^r)-(d_4-L_4)\cos(-\theta_h^r-\theta_k^r-\theta_a^r)\right\}
\]

\[\quad +2(m_1L_2+m_2d_2)\left\{L_3\cos(-\theta_k^r)-(d_4-L_4)\cos(-\theta_h^r-\theta_k^r)\right\}
\]

\[\quad -2(m_1L_3+m_2L_2+m_3d_3)(d_4-L_4)\cos(-\theta_h^r)
\]

\[
j_{1,2} = 2(m_1+m_2+m_3)(d_4-L_4)\sin(\theta)
\]

\[\quad -m_1d_1\sin\left(\theta-\theta_h^l-\theta_k^l-\theta_a^l\right)
\]

\[\quad -(m_1L_2+m_2d_2)\sin\left(\theta-\theta_h^l-\theta_k^l\right),\]
\[-(m_1 L_3 + m_2 L_3 + m_3 d_3) \sin(\theta - \theta_h^i)\]
\[-m_1 d_1 \sin(\theta - \theta_h^i - \theta_k^i - \theta_a^i)\]
\[-(m_1 L_2 + m_2 d_2) \sin(\theta - \theta_h^i - \theta_k^i)\]
\[-(m_1 L_3 + m_2 L_3 + m_3 d_3) \sin(\theta - \theta_h^i)\]
\[j_{1,3} = -2(m_1 + m_2 + m_3)(d_4 - L_4) \cos(\theta)\]
\[+ m_1 d_1 \cos\left(\theta - \theta_h^i - \theta_k^i - \theta_a^i\right)\]
\[+ (m_1 L_2 + m_2 d_2) \cos\left(\theta - \theta_h^i - \theta_k^i\right)\]
\[+ (m_1 L_3 + m_2 L_3 + m_3 d_3) \cos(\theta - \theta_h^i)\]
\[+ m_1 d_1 \cos(\theta - \theta_h^i - \theta_k^i - \theta_a^i)\]
\[+ (m_1 L_2 + m_2 d_2) \cos(\theta - \theta_h^i - \theta_k^i)\]
\[+ (m_1 L_3 + m_2 L_3 + m_3 d_3) \cos(\theta - \theta_h^i)\]
\[j_{1,4} = -(I_1 + m_1 d_1^2)\]
\[-m_1 d_1 L_2 \cos(-\theta_a^i)\]
\[-m_1 d_1 L_3 \cos(-\theta_k^i - \theta_a^i)\]
\[+ m_1 d_1 (d_4 - L_4) \cos(-\theta_h^i - \theta_k^i - \theta_a^i)\]
\[j_{1,5} = -(I_1 + m_1 d_1^2) - (I_2 + m_1 L_2^2 + m_2 d_2^2)\]
\[-2 m_1 d_1 L_2 \cos(-\theta_a^i)\]
\[-m_1 d_1 L_3 \cos(-\theta_k^i - \theta_a^i)\]
\[+ m_1 d_1 (d_4 - L_4) \cos(-\theta_h^i - \theta_k^i - \theta_a^i)\]
\[-(m_1 L_2 + m_2 d_2) L_3 \cos(-\theta_k^i)\]
\[+ (m_1 L_2 + m_2 d_2)(d_4 - L_4) \cos(-\theta_h^i - \theta_k^i)\]
\[j_{1,6} = -(I_1 + m_1 d_1^2) - (I_2 + m_1 L_2^2 + m_2 d_2^2) - (I_3 + m_1 L_3^2 + m_2 L_3^2 + m_3 d_3^2)\]
\[-2 m_1 d_1 L_2 \cos(-\theta_a^i)\]
\[-2 m_1 d_1 L_3 \cos(-\theta_k^i - \theta_a^i)\]
\[+ m_1 d_1 (d_4 - L_4) \cos(-\theta_h^i - \theta_k^i - \theta_a^i)\]
\[-2(m_1 L_2 + m_2 d_2) L_3 \cos(-\theta_k^i)\]
\[+ (m_1 L_2 + m_2 d_2)(d_4 - L_4) \cos(-\theta_h^i - \theta_k^i)\]
\[+ (m_1 L_3 + m_2 L_3 + m_3 d_3)(d_4 - L_4) \cos(-\theta_h^i)\]
\[j_{1,7} = -(I_1 + m_1 d_1^2)\]
\[-m_1 d_1 L_2 \cos(-\theta_a^i)\]
\[-m_1 d_1 L_3 \cos(-\theta_k^i - \theta_a^i)\]
\[ j_{1.8} = +m_1 d_1 (d_4 - L_4) \cos(-\theta_h^r - \theta_k^r - \theta_a^r) - (I_1 + m_1 d_1^2) - (I_2 + m_1 L_2^2 + m_2 d_2^2) - 2m_1 d_1 L_2 \cos(-\theta_a^r) - m_1 d_1 L_3 \cos(-\theta_k^r - \theta_a^r) + m_1 d_1 (d_4 - L_4) \cos(-\theta_h^r - \theta_k^r - \theta_a^r) - (m_1 L_2 + m_2 d_2) L_3 \cos(-\theta_k^r) + (m_1 L_2 + m_2 d_2)(d_4 - L_4) \cos(-\theta_h^r - \theta_k^r) \]

\[ j_{1.9} = - (I_1 + m_1 d_1^2) - (I_2 + m_1 L_2^2 + m_2 d_2^2) - (I_3 + m_1 L_3^2 + m_2 L_3 + m_3 d_3^2) - 2m_1 d_1 L_2 \cos(-\theta_a^r) - 2m_1 d_1 L_3 \cos(-\theta_k^r - \theta_a^r) + m_1 d_1 (d_4 - L_4) \cos(-\theta_h^r - \theta_k^r - \theta_a^r) - 2(m_1 L_2 + m_2 d_2) L_3 \cos(-\theta_k^r) + (m_1 L_2 + m_2 d_2)(d_4 - L_4) \cos(-\theta_h^r - \theta_k^r) + (m_1 L_3 + m_2 L_3 + m_3 d_3)(d_4 - L_4) \cos(-\theta_k^r) \]

\[ j_{2.1} = 2(m_1 + m_2 + m_3)(d_4 - L_4) \sin(\theta) - m_1 d_1 \sin(\theta - \theta_h^l - \theta_k^l - \theta_a^l) - (m_1 L_2 + m_2 d_2) \sin(\theta - \theta_h^l - \theta_k^l) - (m_1 L_3 + m_2 L_3 + m_3 d_3) \sin(\theta - \theta_h^l) - m_1 d_1 \sin(\theta - \theta_h^r - \theta_k^r - \theta_a^r) - (m_1 L_2 + m_2 d_2) \sin(\theta - \theta_h^r - \theta_k^r) - (m_1 L_3 + m_2 L_3 + m_3 d_3) \sin(\theta - \theta_h^r) \]

\[ j_{2.2} = 2(m_1 + m_2 + m_3) m_4 \]

\[ j_{2.3} = 0 \]

\[ j_{2.4} = m_1 d_1 \sin(\theta - \theta_h^l - \theta_k^l - \theta_a^l) \]

\[ j_{2.5} = m_1 d_1 \sin(\theta - \theta_h^l - \theta_k^l - \theta_a^l) + (m_1 L_2 + m_2 d_2) \sin(\theta - \theta_h^l - \theta_k^l) \]

\[ j_{2.6} = m_1 d_1 \sin(\theta - \theta_h^l - \theta_k^l - \theta_a^l) + (m_1 L_2 + m_2 d_2) \sin(\theta - \theta_h^l - \theta_k^l) + (m_1 L_3 + m_2 L_3 + m_3 d_3) \sin(\theta - \theta_h^l) \]

\[ j_{2.7} = m_1 d_1 \sin(\theta - \theta_h^r - \theta_k^r - \theta_a^r) \]

\[ j_{2.8} = m_1 d_1 \sin(\theta - \theta_h^r - \theta_k^r - \theta_a^r) + (m_1 L_2 + m_2 d_2) \sin(\theta - \theta_h^r - \theta_k^r) \]

\[ j_{2.9} = m_1 d_1 \sin(\theta - \theta_h^r - \theta_k^r - \theta_a^r) + (m_1 L_2 + m_2 d_2) \sin(\theta - \theta_h^r - \theta_k^r) + (m_1 L_3 + m_2 L_3 + m_3 d_3) \sin(\theta - \theta_h^r) \]
\[ j_{3,1} = -2(m_1 + m_2 + m_3)(d_4 - L_4) \cos(\theta) \\
+ m_1 d_1 \cos \left( \theta - \theta^r_h - \theta^l_k - \theta^l_a \right) \\
+ (m_1 L_2 + m_2 d_2) \cos \left( \theta - \theta^r_h - \theta^l_k \right) \\
+ (m_1 L_2 + m_2 L_3 + m_3 d_3) \cos \left( \theta - \theta^r_h \right) \\
+ m_1 d_1 \cos \left( \theta - \theta^r_h - \theta^r_k - \theta^r_a \right) \\
+ (m_1 L_2 + m_2 d_2) \cos \left( \theta - \theta^r_h - \theta^r_k \right) \\
+ (m_1 L_3 + m_2 L_3 + m_3 d_3) \cos \left( \theta - \theta^r_h \right) \]

\[ j_{3,2} = 0 \]

\[ j_{3,3} = 2(m_1 + m_2 + m_3) + m_4 \]

\[ j_{3,4} = -m_1 d_1 \cos \left( \theta - \theta^r_h - \theta^l_k - \theta^l_a \right) \]

\[ j_{3,5} = -m_1 d_1 \cos \left( \theta - \theta^r_h - \theta^l_k - \theta^l_a \right) \\
- (m_1 L_2 + m_2 d_2) \cos \left( \theta - \theta^r_h - \theta^l_k \right) \\
- (m_1 L_3 + m_2 L_3 + m_3 d_3) \cos \left( \theta - \theta^r_h \right) \]

\[ j_{3,6} = -m_1 d_1 \cos \left( \theta - \theta^r_h - \theta^l_k - \theta^l_a \right) \\
- (m_1 L_2 + m_2 d_2) \cos \left( \theta - \theta^r_h - \theta^l_k \right) \\
- (m_1 L_3 + m_2 L_3 + m_3 d_3) \cos \left( \theta - \theta^r_h \right) \]

\[ j_{3,7} = -m_1 d_1 \cos \left( \theta - \theta^r_h - \theta^r_k - \theta^r_a \right) \]

\[ j_{3,8} = -m_1 d_1 \cos \left( \theta - \theta^r_h - \theta^r_k - \theta^r_a \right) \\
- (m_1 L_2 + m_2 d_2) \cos \left( \theta - \theta^r_h - \theta^r_k \right) \\
- (m_1 L_3 + m_2 L_3 + m_3 d_3) \cos \left( \theta - \theta^r_h \right) \]

\[ j_{3,9} = -m_1 d_1 \cos \left( \theta - \theta^r_h - \theta^r_k - \theta^r_a \right) \\
- (m_1 L_2 + m_2 d_2) \cos \left( \theta - \theta^r_h - \theta^r_k \right) \\
- (m_1 L_3 + m_2 L_3 + m_3 d_3) \cos \left( \theta - \theta^r_h \right) \]

\[ j_{4,1} = - (I_1 + m_1 d_1^2) \cos \left( \theta_a \right) \\
+ m_1 d_1 L_2 \cos \left( \theta^l_a \right) \\
+ m_1 d_1 L_3 \cos \left( \theta^l_k - \theta^l_a \right) \\
+ m_1 d_1 (d_4 - L_4) \cos \left( \theta^l_h - \theta^l_k - \theta^l_a \right) \]

\[ j_{4,2} = m_1 d_1 \sin \left( \theta - \theta^l_h - \theta^l_k - \theta^l_a \right) \]

\[ j_{4,3} = -m_1 d_1 \cos \left( \theta - \theta^l_h - \theta^l_k - \theta^l_a \right) \]

\[ j_{4,4} = I_1 + m_1 d_1^2 \]

\[ j_{4,5} = (I_1 + m_1 d_1^2) \]

\[ + m_1 d_1 L_2 \cos \left( \theta^l_a \right) \]

\[ j_{4,6} = (I_1 + m_1 d_1^2) \]

\[ + m_1 d_1 L_2 \cos \left( \theta^l_a \right) \]
\[\begin{align*}
\mathbf{j}_{4,7} &= 0 \\
\mathbf{j}_{4,8} &= 0 \\
\mathbf{j}_{4,9} &= 0 \\
\mathbf{j}_{5,1} &= -(I_1 + m_1d_1^2) - (I_2 + m_1L_2^2 + m_2d_2^2) \\
&\quad - 2m_1d_1L_2\cos(-\theta^l_a) \\
&\quad - m_1d_1L_3\cos(-\theta^l_k - \theta^l_a) \\
&\quad + m_1d_1(d_4 - L_4)\cos(-\theta^l_h - \theta^l_k - \theta^l_a) \\
&\quad - (m_1L_2 + m_2d_2)L_3\cos(-\theta^l_k) \\
&\quad + (m_1L_2 + m_2d_2)(d_4 - L_4)\cos(-\theta^l_h - \theta^l_k) \\
\mathbf{j}_{5,2} &= (m_1L_2 + m_2d_2)\sin(\theta - \theta^l_h - \theta^l_k) \\
&\quad - m_1d_1\sin(\theta - \theta^l_h - \theta^l_k - \theta^l_a) \\
\mathbf{j}_{5,3} &= -(m_1L_2 + m_2d_2)\sin(\theta - \theta^l_h - \theta^l_k) \\
&\quad - m_1d_1\cos(\theta - \theta^l_h - \theta^l_k - \theta^l_a) \\
\mathbf{j}_{5,4} &= (I_1 + m_1d_1^2) \\
&\quad + m_1d_1L_2\cos(-\theta^l_a) \\
\mathbf{j}_{5,5} &= (I_1 + m_1d_1^2) \\
&\quad + 2m_1d_1L_2\cos(-\theta^l_a) \\
&\quad + (I_2 + m_1L_2^2 + m_2d_2^2) \\
\mathbf{j}_{5,6} &= (I_1 + m_1d_1^2) + (I_2 + m_1L_2^2 + m_2d_2^2) \\
&\quad + 2m_1d_1L_2\cos(-\theta^l_a) \\
&\quad + m_1d_1L_3\cos(-\theta^l_k - \theta^l_a) \\
&\quad + (m_1L_2 + m_2d_2)L_3\cos(-\theta^l_k) \\
\mathbf{j}_{5,7} &= 0 \\
\mathbf{j}_{5,8} &= 0 \\
\mathbf{j}_{5,9} &= 0 \\
\mathbf{j}_{6,1} &= -(I_1 + m_1d_1^2) - (I_2 + m_1L_2^2 + m_2d_2^2) - (I_3 + m_1L_3^2 + m_2L_3^2 + m_3d_3^2) \\
&\quad - 2m_1d_1L_2\cos(-\theta^l_a) \\
&\quad - 2m_1d_1L_3\cos(-\theta^l_k - \theta^l_a) \\
&\quad + m_1d_1(d_4 - L_4)\cos(-\theta^l_h - \theta^l_k - \theta^l_a) \\
&\quad - 2(m_1L_2 + m_2d_2)L_3\cos(-\theta^l_k) \\
\end{align*}\]
\( + (m_1 L_2 + m_2 d_2)(d_4 - L_4) \cos(-\theta^t_h - \theta^r_h) \) \\
\( + (m_1 L_3 + m_2 L_3 + m_3 d_3)(d_4 - L_4) \cos(-\theta^t_h) \) \\
\( j_{6,2} = m_1 d_1 \sin(\theta - \theta^t_h - \theta^t_k - \theta^l_a) \) \\
\( + (m_1 L_2 + m_2 d_2) \sin(\theta - \theta^t_h - \theta^t_k) \) \\
\( + (m_1 L_3 + m_2 L_3 + m_3 d_3) \sin(\theta - \theta^t_h) \) \\
\( j_{6,3} = -m_1 d_1 \cos(\theta - \theta^t_h - \theta^t_k - \theta^l_a) \) \\
\( - (m_1 L_2 + m_2 d_2) \cos(\theta - \theta^t_h - \theta^t_k) \) \\
\( - (m_1 L_3 + m_2 L_3 + m_3 d_3) \cos(\theta - \theta^t_h) \) \\
\( j_{6,4} = (I_1 + m_1 d_1^2) \) \\
\( + m_1 d_1 L_2 \cos(-\theta^l_a) \) \\
\( + m_1 d_1 L_3 \cos(-\theta^l_k - \theta^l_a) \) \\
\( j_{6,5} = (I_1 + m_1 d_1^2) + (I_2 + m_1 L_2^2 + m_2 d_2^2) \) \\
\( + 2m_1 d_1 L_2 \cos(-\theta^l_a) \) \\
\( + m_1 d_1 L_3 \cos(-\theta^l_k - \theta^l_a) \) \\
\( + (m_1 L_2 + m_2 d_2) L_3 \cos(-\theta^l_k) \) \\
\( j_{6,6} = m_1 d_1 L_3 \cos(-\theta^l_k - \theta^l_a) + (m_1 L_2 L_3 + m_2 d_2 L_3) \cos(-\theta^l_k) \) \\
\( + (I_3 + m_1 L_2^3 + m_2 L_2^3 + m_3 d_3^2) + (I_1 + m_1 d_1^2) + m_1 d_1 L_2 \cos(-\theta^l_a) \) \\
\( + m_1 d_1 L_3 \cos(-\theta^l_k - \theta^l_a) + m_1 d_1 L_2 \cos(-\theta^l_a) + (I_2 + m_1 L_2^2 + m_2 d_2^2) \) \\
\( + (m_1 L_2 + m_2 d_2) L_3 \cos(-\theta^l_k) \) \\
\( j_{6,7} = 0 \) \\
\( j_{6,8} = 0 \) \\
\( j_{6,9} = 0 \) \\
\( j_{7,1} = -(I_1 + m_1 d_1^2) \) \\
\( - m_1 d_1 L_2 \cos(-\theta^r_a) \) \\
\( - m_1 d_1 L_3 \cos(-\theta^r_k - \theta^r_a) \) \\
\( + m_1 d_1 (d_4 - L_4) \cos(-\theta^r_h - \theta^r_k - \theta^r_a) \) \\
\( j_{7,2} = m_1 d_1 \sin(\theta - \theta^r_h - \theta^r_k - \theta^r_a) \) \\
\( j_{7,3} = -m_1 d_1 \cos(\theta - \theta^r_h - \theta^r_k - \theta^r_a) \) \\
\( j_{7,4} = 0 \) \\
\( j_{7,5} = 0 \) \\
\( j_{7,6} = 0 \) \\
\( j_{7,7} = I_1 + m_1 d_1^2 \)
\begin{align*}
j_{7,8} &= (I_1 + m_1 d_1^2) + m_1 d_1 L_2 \cos(-\theta_a^r) \\
j_{7,9} &= (I_1 + m_1 d_1^2) + m_1 d_1 L_2 \cos(-\theta_a^r) + m_1 d_1 L_3 \cos(-\theta_k^r - \theta_a^r) \\
j_{8,1} &= (m_1 L_2 + m_2 d_2)(d_4 - L_4) \cos(-\theta_k^r - \theta_a^r) - m_1 d_1 L_2 \cos(-\theta_a^r) - (I_2 + m_1 L_2^2 + m_2 d_2^2) - (m_1 L_2 L_3 + m_2 d_2 L_2) \cos(-\theta_k) + m_1 d_1 (d_4 - L_4) \cos(-\theta_k^r - \theta_a^r) - (I_1 + m_1 d_1^2) - m_1 d_1 L_2 \cos(-\theta_a^r) - m_1 d_1 L_3 \cos(-\theta_k^r - \theta_a^r) \\
j_{8,2} &= (m_1 L_2 + m_2 d_2) \sin(\theta - \theta_k^r - \theta_a^r) + m_1 d_1 \sin(\theta - \theta_k^r - \theta_a^r - \theta_a^r) \\
j_{8,3} &= -(m_1 L_2 + m_2 d_2) \cos(\theta - \theta_k^r - \theta_a^r) - m_1 d_1 \cos(\theta - \theta_k^r - \theta_a^r - \theta_a^r) \\
j_{8,4} &= 0 \\
j_{8,5} &= 0 \\
j_{8,6} &= 0 \\
j_{8,7} &= (I_1 + m_1 d_1^2) + m_1 d_1 L_2 \cos(-\theta_a^r) \\
j_{8,8} &= (I_1 + m_1 d_1^2) + (I_2 + m_1 L_2^2 + m_2 d_2^2) + 2 m_1 d_1 L_2 \cos(-\theta_a^r) \\
j_{8,9} &= (I_1 + m_1 d_1^2) + (I_2 + m_1 L_2^2 + m_2 d_2^2) + 2 m_1 d_1 L_2 \cos(-\theta_a^r) + m_1 d_1 L_3 \cos(-\theta_k^r - \theta_a^r) + (m_1 L_2 + m_2 d_2) L_3 \cos(-\theta_k^r) \\
j_{9,1} &= -(I_1 + m_1 d_1^2) - (I_2 + m_1 L_2^2 + m_2 d_2^2) - (I_3 + m_1 L_3^2 + m_2 L_3^2 + m_3 d_3^2) - 2 m_1 d_1 L_2 \cos(-\theta_a^r) - 2 m_1 d_1 L_3 \cos(-\theta_k^r - \theta_a^r) + m_1 d_1 (d_4 - L_4) \cos(-\theta_k^r - \theta_a^r) - 2 (m_1 L_2 + m_2 d_2) L_3 \cos(-\theta_k^r) + (m_1 L_2 + m_2 d_2)(d_4 - L_4) \cos(-\theta_k^r - \theta_a^r) + (m_1 L_3 + m_2 L_3 + m_3 d_3)(d_4 - L_4) \cos(-\theta_k^r) \\
j_{9,2} &= (m_1 L_3 + m_2 L_3 + m_3 d_3) \sin(\theta - \theta_k^r) + (m_1 L_2 + m_2 d_2) \sin(\theta - \theta_k^r - \theta_k^r) + m_1 d_1 \sin(\theta - \theta_k^r - \theta_k^r - \theta_a^r) \\
j_{9,3} &= -(m_1 L_3 + m_2 L_3 + m_3 d_3) \cos(\theta - \theta_k^r)
\[-(m_1L_2+m_2d_2)\cos(\theta-\dot{\theta}_h^r-\dot{\theta}_k^r)\]
\[-m_1d_1\cos(\theta-\dot{\theta}_h^r-\dot{\theta}_k^r-\dot{\theta}_a^r)\]

\[j_{9,4} = 0\]
\[j_{9,5} = 0\]
\[j_{9,6} = 0\]
\[j_{9,7} = (I_1+m_1d_1^2)\]
\[+m_1d_1L_2\cos(-\theta_a^r)\]
\[+m_1d_1L_3\cos(-\theta_k^r-\theta_a^r)\]

\[j_{9,8} = (I_1+m_1d_1^2)+(I_2+m_1L_2^2+m_2d_2^2)\]
\[+2m_1d_1L_2\cos(-\theta_a^r)\]
\[+m_1d_1L_3\cos(-\theta_k^r-\theta_a^r)\]
\[+(m_1L_2+m_2d_2)L_3\cos(-\theta_k^r)\]

\[j_{9,9} = (I_1+m_1d_1^2)+(I_2+m_1L_2^2+m_2d_2^2)+(I_3+m_1L_3^2+m_2L_3^2+m_3d_3^2)\]
\[+2m_1d_1L_2\cos(-\theta_a^r)\]
\[+2m_1d_1L_3\cos(-\theta_k^r-\theta_a^r)\]
\[+2(m_1L_2+m_2d_2)L_3\cos(-\theta_k^r)\]

\[b_1 = -m_1d_1(d_4-L_4)\left\{\ddot{\theta}^2-\left(\dot{\theta}-\dot{\theta}_h^r-\dot{\theta}_k^r-\dot{\theta}_a^r\right)^2\right\}\sin(-\theta_h^r-\theta_k^r-\theta_a^r)\]
\[-m_1d_1(d_4-L_4)\left\{\ddot{\theta}^2-\left(\dot{\theta}-\dot{\theta}_h^r-\dot{\theta}_k^r-\dot{\theta}_a^r\right)^2\right\}\sin(-\theta_h^r-\theta_k^r-\theta_a^r)\]
\[-(m_1L_2+m_2d_2)(d_4-L_4)\left\{\ddot{\theta}^2-\left(\dot{\theta}-\dot{\theta}_h^r-\dot{\theta}_k^r\right)^2\right\}\sin(-\theta_h^r-\theta_k^r)\]
\[-(m_1L_2+m_2d_2)(d_4-L_4)\left\{\ddot{\theta}^2-\left(\dot{\theta}-\dot{\theta}_h^r-\dot{\theta}_k^r\right)^2\right\}\sin(-\theta_h^r-\theta_k^r)\]
\[-(m_1L_3+m_2L_3+m_3d_3)(d_4-L_4)\left\{\ddot{\theta}^2-\left(\dot{\theta}-\dot{\theta}_h^r\right)^2\right\}\sin(-\theta_h^r)\]
\[-(m_1L_3+m_2L_3+m_3d_3)(d_4-L_4)\left\{\ddot{\theta}^2-\left(\dot{\theta}-\dot{\theta}_h^r\right)^2\right\}\sin(-\theta_h^r)\]
\[-m_1d_1L_3\left\{\ddot{\theta}^2-\left(\dot{\theta}-\dot{\theta}_h^r-\dot{\theta}_k^r\right)^2\right\}\sin(-\theta_h^r-\theta_a^r)\]
\[-m_1d_1L_2\left\{\ddot{\theta}^2-\left(\dot{\theta}-\dot{\theta}_h^r-\dot{\theta}_k^r\right)^2\right\}\sin(-\theta_a^r)\]
\[-m_1d_1L_3\left\{\ddot{\theta}^2-\left(\dot{\theta}-\dot{\theta}_h^r-\dot{\theta}_k^r\right)^2\right\}\sin(-\theta_h^r)\]
\[-(m_1L_2+m_2d_2)L_3\left\{\ddot{\theta}^2-\left(\dot{\theta}-\dot{\theta}_h^r-\dot{\theta}_k^r\right)^2\right\}\sin(-\theta_h^r)\]
\[ b_2 = -m_1 d_1 \left\{ \left( \dot{\theta}_k - \dot{\theta}_h - \dot{\theta}_a \right)^2 \cos \left( \theta - \theta'_h - \theta'_k - \theta'_a \right) + \left( \dot{\theta}_h - \dot{\theta}_k - \dot{\theta}_a \right)^2 \cos \left( \theta - \theta'_h - \theta'_k - \theta'_a \right) \right\} \\
\quad - \left( m_1 L_2 + m_2 d_2 \right) \left\{ \left( \dot{\theta}_k - \dot{\theta}_h - \dot{\theta}_a \right)^2 \cos \left( \theta - \theta'_h - \theta'_k - \theta'_a \right) + \left( \dot{\theta}_h - \dot{\theta}_k - \dot{\theta}_a \right)^2 \cos \left( \theta - \theta'_h - \theta'_k - \theta'_a \right) \right\} \\
\quad - \left( m_1 L_3 + m_2 L_3 + m_3 d_3 \right) \left\{ \left( \dot{\theta}_k - \dot{\theta}_h - \dot{\theta}_a \right)^2 \cos \left( \theta - \theta'_h - \theta'_k - \theta'_a \right) + \left( \dot{\theta}_h - \dot{\theta}_k - \dot{\theta}_a \right)^2 \cos \left( \theta - \theta'_h - \theta'_k - \theta'_a \right) \right\} \\
\quad + 2 (m_1 + m_2 + m_3) (d_4 - L_4) \dot{\theta}_a^2 \cos (\theta) \]

\[ b_3 = -m_1 d_1 \left\{ \left( \dot{\theta}_k - \dot{\theta}_h - \dot{\theta}_a \right)^2 \sin \left( \theta - \theta'_h - \theta'_k - \theta'_a \right) + \left( \dot{\theta}_h - \dot{\theta}_k - \dot{\theta}_a \right)^2 \sin \left( \theta - \theta'_h - \theta'_k - \theta'_a \right) \right\} \\
\quad - \left( m_1 L_2 + m_2 d_2 \right) \left\{ \left( \dot{\theta}_k - \dot{\theta}_h - \dot{\theta}_a \right)^2 \sin \left( \theta - \theta'_h - \theta'_k - \theta'_a \right) + \left( \dot{\theta}_h - \dot{\theta}_k - \dot{\theta}_a \right)^2 \sin \left( \theta - \theta'_h - \theta'_k - \theta'_a \right) \right\} \\
\quad - \left( m_1 L_3 + m_2 L_3 + m_3 d_3 \right) \left\{ \left( \dot{\theta}_k - \dot{\theta}_h - \dot{\theta}_a \right)^2 \sin \left( \theta - \theta'_h - \theta'_k - \theta'_a \right) + \left( \dot{\theta}_h - \dot{\theta}_k - \dot{\theta}_a \right)^2 \sin \left( \theta - \theta'_h - \theta'_k - \theta'_a \right) \right\} \\
\quad + 2 (m_1 + m_2 + m_3) (d_4 - L_4) \dot{\theta}_a^2 \sin (\theta) \]

\[ b_4 = -m_1 d_1 L_2 \left( \dot{\theta}_k - \dot{\theta}_h - \dot{\theta}_a \right)^2 \sin \left( -\theta'_a \right) \\
\quad -m_1 d_1 L_3 \left( \dot{\theta}_k - \dot{\theta}_h \right)^2 \sin \left( -\theta'_k - \theta'_a \right) \\
\quad + m_1 d_1 (d_4 - L_4) \dot{\theta}_a^2 \sin \left( -\theta'_h - \theta'_k - \theta'_a \right) \]

\[ b_5 = -m_1 d_1 L_2 \left\{ \left( \dot{\theta}_k - \dot{\theta}_h - \dot{\theta}_a \right)^2 - \left( \dot{\theta}_k - \dot{\theta}_h - \dot{\theta}_a - \dot{\theta}_h \right)^2 \right\} \sin \left( -\theta'_a \right) \\
\quad - \left( m_2 d_2 + m_1 L_2 \right) L_3 \left( \dot{\theta}_k - \dot{\theta}_h \right)^2 \sin \left( -\theta'_k \right) \\
\quad + \left( m_2 d_2 + m_1 L_2 \right) (d_4 - L_4) \dot{\theta}_a^2 \sin \left( -\theta'_h - \theta'_a \right) \\
\quad -m_1 d_1 L_3 \left( \dot{\theta}_k - \dot{\theta}_h \right)^2 \sin \left( -\theta'_k - \theta'_a \right) \\
\quad + m_1 d_1 (d_4 - L_4) \dot{\theta}_a^2 \sin \left( -\theta'_h - \theta'_k - \theta'_a \right) \]

\[ b_6 = -m_1 d_1 L_3 \left\{ \left( \dot{\theta}_h - \dot{\theta}_a \right)^2 - \left( \dot{\theta}_h - \dot{\theta}_k - \dot{\theta}_a \right)^2 \right\} \sin \left( -\theta'_k - \theta'_a \right) \\
\quad - \left( m_1 L_2 + m_2 d_2 \right) L_3 \left\{ \left( \dot{\theta}_h - \dot{\theta}_a \right)^2 - \left( \dot{\theta}_h - \dot{\theta}_k - \dot{\theta}_a \right)^2 \right\} \sin \left( -\theta'_k \right) \\
\quad + \left( m_1 L_3 + m_2 L_3 + m_3 d_3 \right) (d_4 - L_4) \dot{\theta}_a^2 \sin \left( -\theta'_h \right) \\
\quad -m_1 d_1 L_2 \left\{ \left( \dot{\theta}_h - \dot{\theta}_k - \dot{\theta}_a \right)^2 - \left( \dot{\theta}_h - \dot{\theta}_a \right)^2 \right\} \sin \left( -\theta'_a \right) \\
\quad + \left( m_2 d_2 + m_1 L_2 \right) (d_4 - L_4) \dot{\theta}_a^2 \sin \left( -\theta'_h - \theta'_a \right) \\
\quad + m_1 d_1 (d_4 - L_4) \dot{\theta}_a^2 \sin \left( -\theta'_h - \theta'_k - \theta'_a \right) \]

\[ b_7 = -m_1 d_1 L_2 \left( \dot{\theta}_h - \dot{\theta}_k - \dot{\theta}_a \right)^2 \sin \left( -\theta'_a \right) \\
\quad -m_1 d_1 L_3 \left( \dot{\theta}_h - \dot{\theta}_k \right)^2 \sin \left( -\theta'_k - \theta'_a \right) \\
\quad + m_1 d_1 (d_4 - L_4) \dot{\theta}_a^2 \sin \left( -\theta'_h - \theta'_k - \theta'_a \right) \]
\[ b_8 = -m_1 d_1 L_2 \left\{ \left( \dot{\theta} - \dot{\theta}_h^r - \dot{\theta}_k^r \right)^2 - \left( \ddot{\theta} - \ddot{\theta}_h^r - \ddot{\theta}_k^r \right) \right\} \sin(-\theta_a^r) \]
\[ - (m_2 d_2 + m_1 L_2) L_3 \left( \dot{\theta} - \dot{\theta}_h^r \right)^2 \sin(-\theta_k^r) \]
\[ + (m_2 d_2 + m_1 L_2) (d_4 - L_4) \dot{\theta}_k^r \sin(-\theta_h^r - \theta_k^r) \]
\[ - m_1 d_1 L_3 \left( \dot{\theta} - \dot{\theta}_h^r \right)^2 \sin(-\theta_k^r - \theta_a^r) \]
\[ + m_1 d_1 (d_4 - L_4) \dot{\theta}_k^r \sin(-\theta_h^r - \theta_k^r - \theta_a^r) \]
\[ b_9 = -m_1 d_1 L_3 \left\{ \left( \dot{\theta} - \dot{\theta}_h^r \right)^2 - \left( \ddot{\theta} - \ddot{\theta}_h^r - \ddot{\theta}_k^r \right) \right\} \sin(-\theta_u^r) \]
\[ - (m_1 L_2 + m_2 d_2) L_3 \left\{ \left( \dot{\theta} - \dot{\theta}_h^r \right)^2 - \left( \ddot{\theta} - \ddot{\theta}_h^r - \ddot{\theta}_k^r \right) \right\} \sin(-\theta_k^r) \]
\[ + (m_1 L_3 + m_2 L_3 + m_3 d_3) (d_4 - L_4) \dot{\theta}_h^r \sin(-\theta_h^r) \]
\[ - m_1 d_1 L_2 \left\{ \left( \dot{\theta} - \dot{\theta}_h^r - \dot{\theta}_k^r \right)^2 - \left( \ddot{\theta} - \ddot{\theta}_h^r - \ddot{\theta}_k^r - \ddot{\theta}_a^r \right) \right\} \sin(-\theta_a^r) \]
\[ + (m_2 d_2 + m_1 L_2)(d_4 - L_4) \dot{\theta}_h^r \sin(-\theta_h^r - \theta_a^r) \]
\[ + m_1 d_1 (d_4 - L_4) \dot{\theta}_h^r \sin(-\theta_h^r - \theta_k^r - \theta_a^r) \]
\[ k_1 = -2 (m_1 + m_2 + m_3)(d_4 - L_4) g \cos(\theta) \]
\[ + (m_3 d_3 + m_1 L_3 + m_2 L_3) g \left\{ \cos(\theta - \theta_h^l) + \cos(\theta - \theta_u^l) \right\} \]
\[ + (m_2 d_2 + m_1 L_2) g \left\{ \cos(\theta - \theta_h^l - \theta_k^l) + \cos(\theta - \theta_h^l - \theta_a^l) \right\} \]
\[ + m_1 d_1 g \left\{ \cos(\theta - \theta_h^l - \theta_k^l - \theta_a^l) + \cos(\theta - \theta_h^l - \theta_k^l - \theta_a^l) \right\} \]
\[ k_2 = 0 \]
\[ k_3 = (2 (m_1 + m_2 + m_3) + m_4) g \]
\[ k_4 = -m_1 g d_1 \cos(\theta - \theta_h^l - \theta_a^l) \]
\[ k_5 = -(m_2 d_2 + m_1 L_2) g \cos(\theta - \theta_h^l - \theta_k^l) \]
\[ - m_1 g d_1 \cos(\theta - \theta_h^l - \theta_k^l - \theta_a^l) \]
\[ k_6 = -(m_3 d_3 + m_1 L_3 + m_2 L_3) g \cos(\theta - \theta_h^l) \]
\[ - (m_2 d_2 + m_1 L_2) g \cos(\theta - \theta_h^l - \theta_k^l) \]
\[ - m_1 g d_1 \cos(\theta - \theta_h^l - \theta_k^l - \theta_a^l) \]
\[ k_7 = -m_1 d_1 g \cos(\theta - \theta_h^l - \theta_k^l - \theta_a^l) \]
\[ k_8 = -(m_2 d_2 + m_1 L_2) g \cos(\theta - \theta_k^l) \]
\[ - m_1 d_1 g \cos(\theta - \theta_k^l - \theta_a^l) \]
\[ k_9 = -(m_3 d_3 + m_1 L_3 + m_2 L_3) g \cos(\theta - \theta_k^l) \]
\[ - (m_2 d_2 + m_1 L_2) g \cos(\theta - \theta_k^l - \theta_a^l) \]
\[ - m_1 d_1 g \cos(\theta - \theta_k^l - \theta_a^l) \]
\[ g_1 = -F_{gr} \left\{ L_1 \cos(\theta - \theta_h^l - \theta_k^l - \theta_a^l) + L_2 \cos(\theta - \theta_h^l - \theta_k^l) \right\} \]
\[ + L_3 \cos \left( \theta - \theta_h^l \right) - \left( d_4 - L_4 \right) \cos (\theta) \]  

\[ - F_{yrt} \left\{ L_0 \cos \left( \left( \theta - \theta_h^l - \theta_k^l - \theta_a^l \right) - \phi \right) + L_2 \cos \left( \theta - \theta_h^l - \theta_k^l \right) \right. \]  

\[ + L_3 \cos \left( \theta - \theta_h^l \right) - \left( d_4 - L_4 \right) \cos (\theta) \} \]  

\[ - F_{yth} \{ L_1 \cos (\theta - \theta_h^l - \theta_k^l - \theta_a^l) + L_2 \cos (\theta - \theta_h^l - \theta_k^l) \} \]  

\[ + L_3 \cos (\theta - \theta_h^l) - \left( d_4 - L_4 \right) \cos (\theta) \} \]  

\[ - F_{yrt} \{ L_1 \cos (\theta - \theta_h^l - \theta_k^l - \theta_a^l - \phi) + L_2 \cos (\theta - \theta_h^l - \theta_k^l) \} \]  

\[ + L_3 \cos (\theta - \theta_h^l) - \left( d_4 - L_4 \right) \cos (\theta) \} \]  

\[ + F_{xrt} \left\{ L_1 \sin \left( \theta - \theta_h^l - \theta_k^l - \theta_a^l \right) + L_2 \sin \left( \theta - \theta_h^l - \theta_k^l \right) \right. \]  

\[ + L_3 \sin \left( \theta - \theta_h^l \right) - \left( d_4 - L_4 \right) \sin (\theta) \} \]  

\[ + F_{xrh} \left\{ L_1 \sin \left( \theta - \theta_h^l - \theta_k^l - \theta_a^l \right) - \phi \right. + L_2 \sin \left( \theta - \theta_h^l - \theta_k^l \right) \right. \]  

\[ + L_3 \sin \left( \theta - \theta_h^l \right) - \left( d_4 - L_4 \right) \sin (\theta) \} \]  

\[ + F_{xlt} \{ L_1 \sin (\theta - \theta_h^l - \theta_k^l - \theta_a^l) + L_2 \sin (\theta - \theta_h^l - \theta_k^l) \} \]  

\[ + L_3 \sin (\theta - \theta_h^l) - \left( d_4 - L_4 \right) \sin (\theta) \} \]  

\[ + F_{xth} \{ L_1 \sin ((\theta - \theta_h^l - \theta_k^l) - \theta_a^l) + L_2 \sin (\theta - \theta_h^l - \theta_k^l) \} \]  

\[ + L_3 \sin (\theta - \theta_h^l) - \left( d_4 - L_4 \right) \sin (\theta) \} \]  

\begin{align*}
g_2 & = - F_{xrt} - F_{xrh} - F_{xlt} - F_{xth} \\
g_3 & = - F_{yrt} - F_{yrh} - F_{ytt} - F_{yth} \\
g_4 & = F_{yrt} L_1 \cos \left( \theta - \theta_h^l - \theta_k^l - \theta_a^l \right) \\
 & - F_{xrt} L_1 \sin \left( \theta - \theta_h^l - \theta_k^l - \theta_a^l \right) \\
 & + F_{yrt} L_1 \cos \left( \left( \theta - \theta_h^l - \theta_k^l - \theta_a^l \right) - \phi \right) \\
 & - F_{xrt} L_1 \sin \left( \theta - \theta_h^l - \theta_k^l - \theta_a^l \right) - \phi \} \\
 g_5 & = F_{yrt} \left\{ L_2 \cos \left( \theta - \theta_h^l - \theta_k^l \right) + L_1 \cos \left( \theta - \theta_h^l - \theta_k^l - \theta_a^l \right) \right. \} \\
 & + F_{yrt} \left\{ L_2 \cos \left( \theta - \theta_h^l - \theta_k^l \right) + L_1 \cos \left( \left( \theta - \theta_h^l - \theta_k^l - \theta_a^l \right) - \phi \right) \right. \} \\
 & - F_{xrt} \left\{ L_2 \sin \left( \theta - \theta_h^l - \theta_k^l \right) + L_1 \sin \left( \theta - \theta_h^l - \theta_k^l - \theta_a^l \right) \right. \} \\
 & - F_{xrt} \left\{ L_2 \sin \left( \theta - \theta_h^l - \theta_k^l \right) + L_1 \sin \left( \left( \theta - \theta_h^l - \theta_k^l - \theta_a^l \right) - \phi \right) \right. \} \\
 g_6 & = F_{yrt} \left\{ L_3 \cos \left( \theta - \theta_h^l \right) + L_2 \cos \left( \theta - \theta_h^l - \theta_k^l \right) + L_1 \cos \left( \theta - \theta_h^l - \theta_k^l - \theta_a^l \right) \right. \} \\
 & + F_{yrt} \left\{ L_3 \cos \left( \theta - \theta_h^l \right) + L_2 \cos \left( \theta - \theta_h^l - \theta_k^l \right) + L_1 \cos \left( \left( \theta - \theta_h^l - \theta_k^l - \theta_a^l \right) - \phi \right) \right. \} \\
 & - F_{xrt} \left\{ L_3 \sin \left( \theta - \theta_h^l \right) + L_2 \sin \left( \theta - \theta_h^l - \theta_k^l \right) + L_1 \sin \left( \theta - \theta_h^l - \theta_k^l - \theta_a^l \right) \right. \} \\
 & - F_{xrt} \left\{ L_3 \sin \left( \theta - \theta_h^l \right) + L_2 \sin \left( \theta - \theta_h^l - \theta_k^l \right) + L_1 \sin \left( \left( \theta - \theta_h^l - \theta_k^l - \theta_a^l \right) - \phi \right) \right. \} \\
 g_7 & = F_{ytl} L_1 \cos \left( \theta - \theta_h^l - \theta_k^l - \theta_a^l \right) \\
 & - F_{xhl} L_1 \sin \left( \theta - \theta_h^l - \theta_k^l - \theta_a^l \right) \]
A.2 The model of ground reaction forces

The ground reaction force is modeled using nonlinear dampers and springs. By defining the sagittal positions of left-toe, left-heel, right-toe and right-heel, respectively, as \((X_l,t;Y_l,t)\), \((X_l,h;Y_l,h)\), \((X_r,t;Y_r,t)\), \((X_r,h;Y_r,h)\), the vertical ground reaction forces acting at these four points in this order are defined as follows.

\[
F_{ylt} = \begin{cases} 
-\kappa Y^{l,t} - \lambda_v \dot{Y}^{l,t}, & \text{if } Y^{l,t} < 0 \text{ and } -\kappa Y^{l,t} - \lambda_v \dot{Y}^{r,t} > 0 \\
0, & \text{otherwise}
\end{cases}
\]

\[
F_{ylh} = \begin{cases} 
-\kappa Y^{l,h} - \lambda_v \dot{Y}^{l,h}, & \text{if } Y^{l,h} < 0 \text{ and } -\kappa Y^{l,h} - \lambda_v \dot{Y}^{r,h} > 0 \\
0, & \text{otherwise}
\end{cases}
\]

\[
F_{yrt} = \begin{cases} 
-\kappa Y^{r,t} - \lambda_v \dot{Y}^{r,t}, & \text{if } Y^{r,t} < 0 \text{ and } -\kappa Y^{r,t} - \lambda_v \dot{Y}^{r,t} > 0 \\
0, & \text{otherwise}
\end{cases}
\]

\[
F_{yrh} = \begin{cases} 
-\kappa Y^{r,h} - \lambda_v \dot{Y}^{r,h}, & \text{if } Y^{r,h} < 0 \text{ and } -\kappa Y^{r,h} - \lambda_v \dot{Y}^{r,h} > 0 \\
0, & \text{otherwise}
\end{cases}
\]

The horizontal ground reaction forces acting at the four points in the order of listed above are defined as follows.

\[
F_{xlt} = -\frac{2\nu F_{ylt}}{\pi} \tan^{-1} \left( \frac{\lambda_h \pi \dot{X}^{l,t}}{2\nu F_{ylt}} \right)
\]

\[
F_{xlh} = -\frac{2\nu F_{ylh}}{\pi} \tan^{-1} \left( \frac{\lambda_h \pi \dot{X}^{l,h}}{2\nu F_{ylh}} \right)
\]

\[
F_{xrt} = -\frac{2\nu F_{yrt}}{\pi} \tan^{-1} \left( \frac{\lambda_h \pi \dot{X}^{r,t}}{2\nu F_{yrt}} \right)
\]

\[
F_{xrh} = -\frac{2\nu F_{yrh}}{\pi} \tan^{-1} \left( \frac{\lambda_h \pi \dot{X}^{r,h}}{2\nu F_{yrh}} \right)
\]

The parameters used in the model of ground reaction forces are summarized in the following Table.

B Numerical evaluation of Jacobian matrix and Floquet multipliers

The state space representation of the biped model as a non-autonomous dynamical system with the vector field \(f(x,t)\) is defined by Eq. 10 in the main text, and its the Jacobian \(D_{\phi_0}(t)\) for its linearized equation
Table 1: Parameter values used in the model of ground reaction forces

| Symbol | Description                        | Value  |
|--------|------------------------------------|--------|
| \( \kappa \) | Ground reaction force parameter | 20000  |
| \( \lambda_v \) | Ground reaction force parameter | 300    |
| \( \lambda_h \) | Ground reaction force parameter | 2000   |
| \( \nu \) | Ground reaction force parameter | 0.3    |

is defined by Eq. 14. The numerical evaluation of \( D_{\phi_0}(t) \) was performed by numerical partial derivative of \( f(x,t) \) using the double side finite difference method. That is, the \( i\)-\( j \) element of \( D_{\phi_0}(t) \) was obtained as

\[
\frac{\partial f_i}{\partial x_j}(x_r(t)) \approx \frac{f_i(x_r(t)+\Delta x_j,t)-f_i(x_r(t)-\Delta x_j,t)}{2\Delta x_j}
\]

where \( i=1,\ldots,19 \) and \( \Delta x_j \) is the difference of the \( i \)-th element of the state vector \( x \). Throughout this study, we decided to use \( \Delta x_j=10^{-3} \). The size of \( \Delta x_j \) should be determined with a care, because it should balance with the time step \( \Delta t \), which was \( 10^{-5} \) in this study. Our choice of \( \Delta x_j \) was based on the fact that changes in \( x_r(t) \) for the short duration of time \( \Delta t \) along the one gait cycle is about between \( 10^{-3} \) and \( 10^{-6} \). Hence the value of \( \Delta x_j \) was the lower band of this variation.

We validated the use of \( \Delta x_j=10^{-3} \) by examining that the numerical evaluation of \( D_{\phi_0}(t) \) using \( \Delta x_j=10^{-3} \) and loci of FMs as the function of the PD-gains, as in Fig.4 of the main text, for various values of \( \Delta x_j \) ranging from \( 10^{-1} \) to \( 10^{-6} \). The result of this examination showed that was the loci of FMs were qualitatively and quantitatively the same quite robust for a wide range of \( \Delta x_j \) between \( 10^{-2} \) and \( 5\times10^{-6} \), and \( \Delta x_j=10^{-3} \) is the middle of this valid range.

### C Impedance and dynamic impedance

Here we summarize several definitions of dynamic impedance (stiffness and viscosity). Remind the biped motion equation as

\[
J(q)\ddot{q}+B(q,\dot{q})+K(q)+G(q,\omega)=U_{\text{ff}}(q(t),\dot{q}(t),\ddot{q}(t))+U_{\text{fb}}(q,\omega,\dot{q}(t),\ddot{q}(t))
\]

where

\[
U_{\text{fb}}=P(q(t)-q)+D(\dot{q}(t)-\omega)
\]

with

\[
P=\text{diag}\{0,0,0,P_a,P_k,P_a,P_k,P_h\}
\]

\[
D=\text{diag}\{0,0,0,D_a,D_k,D_a,D_k,D_h\},
\]

and denote the total joint torque as \( U=U_{\text{ff}}+U_{\text{fb}} \). The joint stiffness \( K_d \) and viscosity \( B_d \) are usually defined, respectively, by the derivatives of the total joint torque with respect to the position and the velocity, which are equal to \( P \) and \( D \) in our biped model. That is,

\[
K_d \equiv \frac{\partial U}{\partial q} = P,
\]

\[
B_d \equiv -\frac{\partial U}{\partial \omega} = D.
\]
Thus, we considered simply the PD-gains of the feedback controller as the joint impedance in this study. It is also worthwhile to consider a different type of joint impedance, we call it total impedance, which is more directly related to stability of the steady state solution $\ddot{q}, \dot{q}$ as the limit cycle. Considering a perturbed solution as $q=\ddot{q}+\dot{q}$, $\omega=\dot{q}+\ddot{q}$, $\ddot{q}=\ddot{q}+\dot{q}$, we have

$$J(\ddot{q}+\dot{q})(\ddot{q}+\dot{q})+B(\ddot{q}+\dot{q}+\ddot{q})+K(\ddot{q}+\dot{q})+G(\ddot{q}+\dot{q}+\ddot{q})=U(\ddot{q}, \dot{q}, +\ddot{q}, \dot{q}).$$

Using first order Taylor expansion,

$$\begin{align*}
[J(\ddot{q}) &+ \{ \frac{\partial J}{\partial q}(\ddot{q}, \dot{q}) \}] (\ddot{q}+\dot{q}) + B(\ddot{q}+\dot{q}+\ddot{q}) + K(\ddot{q}+\dot{q}) + G(\ddot{q}+\dot{q}+\ddot{q}) \\
&= U(\ddot{q}, \dot{q}+\ddot{q}, \dot{q}+\ddot{q}, \dot{q}+\ddot{q})
\end{align*}$$

where

$$\begin{align*}
\left\{ \frac{\partial J}{\partial q}(\ddot{q}, \dot{q}, \Delta q) \right\} = \sum_{i=1}^{9} \frac{\partial J}{\partial q_i}(\ddot{q}, \dot{q}) \Delta q_i.
\end{align*}$$

By neglecting the second and higher order terms, with the consideration of Eq.1, this can be simplified as

$$\begin{align*}
J(\ddot{q}) \ddot{q} &+ \{ \frac{\partial J}{\partial q}(\ddot{q}, \dot{q}) \} \ddot{q} + \frac{\partial B}{\partial q}(\ddot{q}, \dot{q}) \dot{q} + \frac{\partial B}{\partial \omega}(\ddot{q}, \dot{q}) \ddot{q} \\
&+ \frac{\partial K}{\partial q}(\ddot{q}) \ddot{q} + \frac{\partial G}{\partial q}(\ddot{q}, \dot{q}) \dot{q} + \frac{\partial G}{\partial \omega}(\ddot{q}, \dot{q}) \ddot{q} \\
&= \frac{\partial U}{\partial q}(\ddot{q}, \dot{q}) \ddot{q} + \frac{\partial U}{\partial \omega}(\ddot{q}, \dot{q}) \ddot{q}.
\end{align*}$$

where the second term of the left-hand side is defined as

$$\begin{align*}
\left\{ \frac{\partial J}{\partial q}(\ddot{q}, \dot{q}) \right\} \ddot{q} & = \left( \frac{\partial J}{\partial q_1}(\ddot{q}) \ddot{q_1} + \frac{\partial J}{\partial q_2}(\ddot{q}) \ddot{q_2} + \frac{\partial J}{\partial q_3}(\ddot{q}) \ddot{q_3} + \ldots + \frac{\partial J}{\partial q_9}(\ddot{q}) \ddot{q_9} \right) \ddot{q} \\
&= \frac{\partial J}{\partial q_1}(\ddot{q}) \ddot{q_1} \ddot{q} + \frac{\partial J}{\partial q_2}(\ddot{q}) \ddot{q_2} \ddot{q} + \frac{\partial J}{\partial q_3}(\ddot{q}) \ddot{q_3} \ddot{q} + \ldots + \frac{\partial J}{\partial q_9}(\ddot{q}) \ddot{q_9} \ddot{q} \\
&= \frac{\partial J}{\partial q_1}(\ddot{q}) \ddot{q} \ddot{q_1} + \frac{\partial J}{\partial q_2}(\ddot{q}) \ddot{q} \ddot{q_2} + \frac{\partial J}{\partial q_3}(\ddot{q}) \ddot{q} \ddot{q_3} + \ldots + \frac{\partial J}{\partial q_9}(\ddot{q}) \ddot{q} \ddot{q_9} \\
&= \left[ \frac{\partial J}{\partial q_1}(\ddot{q}) \ddot{q_1} \frac{\partial J}{\partial q_2}(\ddot{q}) \ddot{q_2} \frac{\partial J}{\partial q_3}(\ddot{q}) \ddot{q_3} \ldots \frac{\partial J}{\partial q_9}(\ddot{q}) \ddot{q_9} \right] \ddot{q} \\
&= \frac{\partial J}{\partial q} \ddot{q} \ddot{q}.
\end{align*}$$

Note that, in the last line of this equation, we used the following notation:

$$\frac{\partial J}{\partial q} \ddot{q} = \left[ \frac{\partial J}{\partial q_1}(\ddot{q}) \ddot{q} \frac{\partial J}{\partial q_2}(\ddot{q}) \ddot{q} \frac{\partial J}{\partial q_3}(\ddot{q}) \ddot{q} \ldots \frac{\partial J}{\partial q_9}(\ddot{q}) \ddot{q} \right].$$

Collecting the terms with respect to $\ddot{q}$ and its derivatives, we have the following linearized equation, which describes the dynamic evolution of perturbation, in another phrase, error dynamics around the limit cycle.

$$J(q) \dddot{q} + B_{\text{total}} \dddot{q} + K_{\text{total}} \dddot{q} = 0$$

(6)
where
\[
K_{\text{total}} = -\frac{\partial U(q, \dot{q})}{\partial q} + \frac{\partial B(q, \dot{q})}{\partial q} + \frac{\partial K(q)}{\partial q} + \frac{\partial G(q, \dot{q})}{\partial q} + \frac{\partial J(q)}{\partial q},
\] (7)
and
\[
B_{\text{total}} = -\frac{\partial U(q, \dot{q})}{\partial \omega} + \frac{\partial B(q, \dot{q})}{\partial \omega} + \frac{\partial G(q, \dot{q})}{\partial \omega}.
\] (8)

We call \(K_{\text{total}}\) and \(B_{\text{total}}\) total stiffness and total dynamic viscosity, respectively. This can also be interpreted as the dynamic balance on each timing on the limit cycle.

\[
J(q) \dot{q} = -B_{\text{total}} \omega - K_{\text{total}} \dot{q}
\] (9)

In Eq.9, if the perturbation intends to cause the deviation away from the limit cycle, \(K_{\text{total}}\) and \(B_{\text{total}}\) will counteract the diverging torque and drive the trajectory back to the limit cycle. So it is also natural to define \(K_{\text{total}}\) and \(B_{\text{total}}\) as joint stiffness and viscosity.

Furthermore, \(K_{\text{total}}\) and \(B_{\text{total}}\) can be conveniently related to the Jacobian matrix around limit cycle solution. This is because of the following derivation. The state space representation is also rewritten here.

\[
\frac{d}{dt}(\begin{bmatrix} q \\ \omega \end{bmatrix}) = \begin{bmatrix} J^{-1}(q)(U(q, \omega, \dot{q}) - B(q, \omega) - K(q) - G(q, \omega)) \end{bmatrix} = \begin{bmatrix} F_1(\omega) \\ F_2(q, \omega, U) \end{bmatrix}. \] (10)

Jacobian matrix could be obtained by differentiating the vector field of Eq.10 as follows:

\[
\frac{\partial F_1}{\partial q} = O
\]
\[
\frac{\partial F_1}{\partial \omega} = I
\]
\[
\frac{\partial F_2}{\partial q} = \frac{\partial J^{-1}}{\partial q} (U - B - K - G) + J^{-1} \left( \frac{\partial U}{\partial q} - \frac{\partial B}{\partial q} - \frac{\partial K}{\partial q} - \frac{\partial G}{\partial q} \right)
\]
\[
= -J^{-1} \frac{\partial J}{\partial q} J^{-1} (U - B - K - G) + J^{-1} \left( \frac{\partial U}{\partial q} - \frac{\partial B}{\partial q} - \frac{\partial K}{\partial q} - \frac{\partial G}{\partial q} \right)
\]
\[
= -J^{-1} \frac{\partial J}{\partial q} \dot{q} + J^{-1} \left( \frac{\partial U}{\partial q} - \frac{\partial B}{\partial q} - \frac{\partial K}{\partial q} - \frac{\partial G}{\partial q} \right)
\]
\[
= J^{-1} \left( \frac{\partial U}{\partial q} - \frac{\partial B}{\partial q} - \frac{\partial K}{\partial q} - \frac{\partial G}{\partial q} \right)
\]
\[
\frac{\partial F_2}{\partial \omega} = J^{-1} \left( \frac{\partial U}{\partial \omega} - \frac{\partial B}{\partial \omega} - \frac{\partial G}{\partial \omega} \right)
\]

For the Jacobian matrix evaluated around the limit cycle, which is denoted by \(D_{\phi_0}\) in the main text, from the comparison, we can easily see that

\[
\frac{\partial F_2}{\partial q} = -J^{-1} K_{\text{total}}
\]
\[
\frac{\partial F_2}{\partial \omega} = -J^{-1} B_{\text{total}}
\]

Thus \(D_{\phi_0}\) can be written as

\[
D_{\phi_0}(t) = \begin{pmatrix}
\frac{\partial F_1}{\partial q} & \frac{\partial F_1}{\partial \omega} \\
\frac{\partial F_2}{\partial q} & \frac{\partial F_2}{\partial \omega}
\end{pmatrix} = \begin{pmatrix}
0 & I \\
-J^{-1} K_{\text{total}} & -J^{-1} B_{\text{total}}
\end{pmatrix}
\] (12)
In the calculation of Floquet matrix, the $D_{\phi_0}(t)$ has already been calculated numerically. So it is very prompt to obtain the dynamic stiffness and dynamic viscosity by multiply $-J$ to $\partial F_2/\partial q$ and $\partial F_2/\partial \omega$ block of $D_{\phi_0}$. If we only care about the leg joints and are not interested in the correlated influence between the joints, we select only the diagonals of $K_{\text{total}}$ and $B_{\text{total}}$, and consider them as dynamic stiffness and dynamic viscosity of leg joint. The non-diagonal element could be interpreted as inter-joint dynamic impedance. Dynamic stiffness and dynamic viscosity of leg joint during steady-state gait with the feedback PD gains of 1500 and 10 are illustrated in Fig.1 and Fig.2. It is noted that the peaks are due to the foot impact.

Figure 1: Dynamic total stiffness and viscosity during one gait cycle
Figure 2: Dynamic total viscosity during one gait cycle