CHARACTERIZATION OF SIMPLICES VIA THE BEZOUT INEQUALITY FOR MIXED VOLUMES

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Abstract. We consider the following Bezout inequality for mixed volumes:

\[ V(K_1, \ldots, K_r, \Delta[n-r]) V_n(\Delta)^{r-1} \leq \prod_{i=1}^{r} V(K_i, \Delta[n-1]) \quad \text{for } 2 \leq r \leq n. \]

It was shown previously that the inequality is true for any \( n \)-dimensional simplex \( \Delta \) and any convex bodies \( K_1, \ldots, K_r \) in \( \mathbb{R}^n \). It was conjectured that simplices are the only convex bodies for which the inequality holds for arbitrary bodies \( K_1, \ldots, K_r \) in \( \mathbb{R}^n \). In this paper we prove that this is indeed the case if we assume that \( \Delta \) is a convex polytope. Thus the Bezout inequality characterizes simplices in the class of convex \( n \)-polytopes. In addition, we show that if a body \( \Delta \) satisfies the Bezout inequality for all bodies \( K_1, \ldots, K_r \) then the boundary of \( \Delta \) cannot have strict points. In particular, it cannot have points with positive Gaussian curvature.

1. Introduction

It was noticed in [SZ] that the classical Bezout inequality in algebraic geometry [F, Sec. 8.4] together with the Bernstein–Kushnirenko–Khovanskii bound [B, Ku, Kh] produces a new inequality involving mixed volumes of convex bodies:

\[ V(K_1, \ldots, K_r, \Delta[n-r]) V_n(\Delta)^{r-1} \leq \prod_{i=1}^{r} V(K_i, \Delta[n-1]) \quad \text{for } 2 \leq r \leq n. \]

Here \( \Delta \) is an \( n \)-dimensional simplex and \( K_1, \ldots, K_r \) are arbitrary convex bodies in \( \mathbb{R}^n \). Throughout the paper \( V_n(K) \) denotes the \( n \)-dimensional Euclidean volume of a body \( K \) and \( V(K_1, \ldots, K_n) \) denotes the \( n \)-dimensional mixed volume of bodies \( K_1, \ldots, K_n \). Furthermore, \( K[m] \) indicates that the body \( K \) is repeated \( m \) times in the expression for the mixed volume.

In [SZ] it was conjectured that the Bezout inequality characterizes simplices, that is if \( \Delta \) is a convex body such that (1.1) holds for all convex bodies \( K_1, \ldots, K_r \) then \( \Delta \) is necessarily a simplex (see [SZ, Conjecture 1.2]). It was proved that \( \Delta \) has to be indecomposable (see [SZ, Theorem 3.3]) which, in particular, confirms the conjecture in dimension \( n = 2 \). In the present paper we prove this conjecture for the class of convex polytopes.

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Theorem 1.1. Fix $2 \leq r \leq n$. Let $\Delta$ be a convex $n$-dimensional polytope in $\mathbb{R}^n$ satisfying (1.1) for all convex bodies $K_1, \ldots, K_r$ in $\mathbb{R}^n$. Then $\Delta$ is a simplex.

Although the above theorem covers a most natural class of convex bodies, in full generality the conjecture remains open. Going outside of the class of polytopes we show that if a convex body $\Delta$ satisfies (1.1) for all convex bodies $K_1, \ldots, K_r$ in $\mathbb{R}^n$ then $\Delta$ cannot have strict points. We say a boundary point $x \in K$ is a strict point if $x$ does not belong to any segment contained in the boundary of $K$.

Theorem 1.2. Fix $2 \leq r \leq n$. Let $\Delta$ be an $n$-dimensional convex body in $\mathbb{R}^n$ satisfying (1.1) for all convex bodies $K_1, \ldots, K_r$ in $\mathbb{R}^n$. Then $\Delta$ does not contain any strict points.

In particular, we see that $\Delta$ cannot have points with positive Gaussian curvature.

Let us say a few words about the idea behind the proofs of Theorems 1.1 and 1.2. First, note that it is enough to prove the theorems in the case of $r = 2$ as this implies the general statement. Thus we are going to restate (1.1) for $r = 2$ as follows

(1.2) \[ V(L, M, K[n-2])V_n(K) \leq V(L, K[n-1])V(M, K[n-1]), \]

where $L$ and $M$ are convex bodies and $K$ is a polytope. The fact that there is equality in (1.2) when $L = K$ allows us to see this as a variational problem, by fixing an appropriate body $M$ and using an appropriate deformation $L = K_t$ of $K$. In the case of Theorem 1.1, $K_t$ is obtained from $K$ by moving one of its facets along the direction of its normal unit vector. In the case of Theorem 1.2, $K_t$ is obtained from $K$ by cutting out a small cup in a neighborhood of a strict point.

2. Preliminaries

In this section we collect basic definitions and set up notation. As a general reference on the theory of convex sets and mixed volumes we use R. Schneider’s book “Convex bodies: the Brunn-Minkowski theory” [Sch].

A convex body is a non-empty convex compact set. A (convex) polytope is the convex hull of a finite set of points. An $n$-dimensional polytope is called an $n$-polytope for short. For $x, y \in \mathbb{R}^n$ we write $\langle x, y \rangle$ for the inner product of $x$ and $y$. We use $S^{n-1}$ to denote the $(n-1)$-dimensional unit sphere and $B(x, \delta)$ to denote the closed Euclidean ball of radius $\delta > 0$ centered at $x \in \mathbb{R}^n$.

For a convex body $K$ the function $h_K : S^{n-1} \rightarrow \mathbb{R}$, $h_K(u) = \max\{\langle x, u \rangle \mid x \in K\}$ is the support function of $K$. For every $u \in S^{n-1}$ we write $H_K(u)$ to denote the supporting hyperplane for $K$ with outer normal $u$

$$H_K(u) = \{x \in \mathbb{R}^n : \langle x, u \rangle = h_K(u)\}.$$ 

Furthermore, we use $K^u$ to denote the face $K \cap H_K(u)$ of $K$.

Let $\beta$ be a subset of the boundary $\partial K$ of a convex body $K$. The spherical image $\sigma(K, \beta)$ of $\beta$ with respect to $K$ is defined by

$$\sigma(K, \beta) = \{u \in S^{n-1} : \exists x \in \beta, \text{ such that } \langle x, u \rangle = h_K(u)\}.$$ 

If $\Omega$ is a subset of $S^{n-1}$ define the inverse spherical image $\tau(K, \Omega)$ of $\Omega$ with respect to $K$ by

$$\tau(K, \Omega) = \{x \in \partial K : \exists u \in \Omega, \text{ such that } \langle x, u \rangle = h_K(u)\}.$$
The surface area measure $S(K, \cdot)$ of $K$ (viewed as a measure on $\mathbb{S}^{n-1}$) is defined as

$$S(K, \Omega) = \mathcal{H}^{n-1}(\tau(K, \Omega)), \quad \text{for } \Omega \text{ a Borel subset of } \mathbb{S}^{n-1}.$$ 

Here $\mathcal{H}^{n-1}(\cdot)$ stands for the $(n-1)$-dimensional Hausdorff measure.

Let $V(K_1, \ldots, K_n)$ denote the $n$-dimensional mixed volume of $n$ convex bodies $K_1, \ldots, K_n$ in $\mathbb{R}^n$. We write $V(K_1[m_1], \ldots, K_r[m_r])$ for the mixed volume of the bodies $K_1, \ldots, K_r$ where each $K_i$ is repeated $m_i$ times and $m_1 + \cdots + m_r = n$. In particular, $V(K[n]) = V_n(K)$, the $n$-dimensional Euclidean volume of $K$.

Let $S(K_1, \ldots, K_{n-1}, \cdot)$ be the mixed area measure for bodies $K_1, \ldots, K_{n-1}$ defined by

$$V(L, K_1, \ldots, K_{n-1}) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L dS(K_1, \ldots, K_{n-1}, \cdot)$$

for any compact convex set $L$. In particular, when the $K_i$ are polytopes the mixed area measure $S(K_1, \ldots, K_{n-1}, \cdot)$ has finite support and for every $u \in \mathbb{S}^{n-1}$ we have

$$S(K_1, \ldots, K_{n-1}, u) = V(K^u_1, \ldots, K^u_{n-1}),$$

where $V(K^u_1, \ldots, K^u_{n-1})$ is the $(n-1)$-dimensional mixed volume of the faces $K_i^u$ translated the the subspace orthogonal to $u$, see [Sch, Sec 5.1].

Finally, for $u \in \mathbb{S}^{n-1}$ the orthogonal projection of a set $A \subset \mathbb{R}^n$ onto the subspace $u^\perp$ orthogonal to $u$ is denoted by $A|u^\perp$.

### 3. Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1. As mentioned in the introduction, it is enough to prove it for $r = 2$ in which case we write the Bezout inequality as

$$V(L, M, K[n-2])V_n(K) \leq V(L, K[n-1])V(M, K[n-1]).$$

We assume that $L, M$ are arbitrary convex bodies and $K$ is a polytope in $\mathbb{R}^n$.

We need to set up additional notation. Let $K$ be defined by inequalities

$$K = \bigcap_{j=1}^N \{x \in \mathbb{R}^n : \langle x, u_j \rangle \leq h_K(u_j)\},$$

where $u_j$ are the outer normals to the facets of $K$ (in some fixed order) and $N$ is the number of facets of $K$. Denote by $K_{t,i}$ the polytope obtained by moving the $i$-th facet of $K$ by $t$, that is

$$K_{t,i} = \bigcap_{j=1}^N \{x \in \mathbb{R}^n : \langle x, u_j \rangle \leq h_K(u_j)\} \cap \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leq h_K(u_i) + t\}.$$

By abuse of notation we let $K_t$ denote $K_{t,N}$.

**Lemma 3.1.** Let $K$ and $K_t$ be as above. Then there exists $\delta = \delta(K)$ such that the following supports are equal

$$\text{supp } S(K_t[r], K[n-1-r], \cdot) = \text{supp } S(K, \cdot)$$

for any $0 \leq r \leq n-1$ and any $t \in (-\delta, \delta)$. 

Proof. By (2.1) it is enough to show that \( V(K_t^u[r], K^u[n-1-r]) = 0 \) if and only if \( V_{n-1}(K^u) = 0 \), that is \( K^u \) is not a facet of \( K \). Indeed, by choosing \( \delta \) small enough we can ensure that \( K_t \) has the same facet normals as \( K \) and so \( \dim K_t^u = n-1 \) whenever \( K^u \) is a facet of \( K \). In this case \( V(K_t^u[r], K^u[n-1-r]) > 0 \).

Conversely, assume \( K^u \) is a face of \( K \) of dimension less than \( n-1 \). As before, for small enough \( t \) the face \( K_t^u \) also has dimension less than \( n-1 \). First, suppose \( K^u \) is not contained in the moving facet \( F = K \cap H_K(u_N) \). Then \( h_K(u) = h_{K_t}(u) \) and so \( K^u \subseteq K_t^u \) for \( t \geq 0 \) and \( K^u \supseteq K_t^u \) for \( t < 0 \). Then, by the monotonicity of the mixed volume, if \( t \geq 0 \) then
\[
0 \leq V(K_t^u[r], K^u[n-1-r]) \leq V_{n-1}(K_t^u) = 0,
\]
and so \( V(K_t^u[r], K^u[n-1-r]) = 0 \). The case \( t < 0 \) is similar.

Now suppose \( K^u \) is contained in the moving facet \( F \). Then \( K^u \subseteq H_K(u) \cap H_K(u_N) \) and \( K_t^u \subseteq H_{K_t}(u) \cap H_{K_t}(u_N) \). This shows that \( K^u \) and \( K_t^u \) are contained in two affine \((n-2)\)-dimensional subspaces which are translates of the same linear subspace of dimension \( n-2 \). Therefore, for any collection of line segments \((L_1, \ldots , L_{n-1})\), where \( L_i \subseteq K_t^u \) for \( 1 \leq i \leq r \) and \( L_i \subseteq K^u \) for \( r+1 \leq i \leq n-1 \), the \( L_i \) have linearly dependent directions. The latter implies that \( V(K_t^u[r], K^u[n-1-r]) = 0 \) by [Sch] Theorem 5.1.7.

\[ \square \]

Proposition 3.2. Let \( K, P \) be \( n \)-polytopes with the following properties:
\begin{enumerate}
  \item \( \text{supp} \, S(P, \cdot) = \text{supp} \, S(K, \cdot) \),
  \item there exists a constant \( \lambda > 0 \) such that \( V(L, P[n-1]) \leq \lambda V(L, K[n-1]) \) for all convex bodies \( L \),
  \item \( V(K, P[n-1]) = \lambda V_n(K) \).
\end{enumerate}

Then,
\[ S(P, \cdot) = \lambda S(K, \cdot). \]

Proof. As before, let \( \{u_1, \ldots , u_N\} \) be the outer normals to the facets of \( K \). By assumption (1) they are the outer normals to the facets of \( P \) as well. Fix \( 1 \leq i \leq N \) and let \( L = K_{s,i} \) be the polytope obtained from \( K \) by moving its \( i \)-th facet by a small number \( s \in (-\delta_i, \delta_i) \) as in Lemma 3.1.

By assumption (2), for any \( s \in (-\delta_i, \delta_i) \) we have
\[ V(K_{s,i}, P[n-1]) \leq \lambda V(K_{s,i}, K[n-1]). \]
Consider the function
\[ F(s) = \lambda V(K_{s,i}, K[n-1]) - V(K_{s,i}, P[n-1]). \]
Then \( F(s) \geq 0 \) and \( F(0) = 0 \). Below we show that \( F(s) \) is, in fact, linear on \((-\delta_i, \delta_i)\).

But then \( F(s) \) is identically zero on \((-\delta_i, \delta_i)\), which implies that
\[ V(K_{s,i}, P[n-1]) = \lambda V(K_{s,i}, K[n-1]) \]
for all \( s \in (-\delta_i, \delta_i) \). We claim that this also implies that
\[ S(P, u_i) = \lambda S(K, u_i), \]
and since \( i \) is chosen arbitrarily and the supports of the two measures are equal, the statement of the proposition follows.
Now we show that $F(s)$ is linear and then prove that (3.2) implies (3.3). Since the polytopes $P$ and $K$ have the same set of facet normals $\{u_1, \ldots, u_N\}$, we obtain:

$$nV(K_{s,i}, P[n−1]) = \sum_{j=1}^{N} h_{K_{s,i}}(u_j)V_{n−1}(P^{u_j})$$

$$= \sum_{j=1}^{N} h_K(u_j)V_{n−1}(P^{u_j}) + (h_K(u_i) + s)V_{n−1}(P^{u_i})$$

$$= nV(K, P[n−1]) + sV_{n−1}(P^{u_i})$$

(3.4)

Similarly,

$$nV(K_{s,i}, K[n−1]) = nV_n(K) + sV_{n−1}(K^{u_i}).$$

(3.5)

Substituting (3.4) and (3.5) into the definition of $F(s)$ and using assumption (3), we see that $F(s) = \lambda s$ for some $\lambda$, that is $F(s)$ is linear.

It remains to show that (3.2) implies (3.3). Since $F(s)$ is identically zero we have $\lambda = 0$, which translates to

$$V_{n−1}(P^{u_i}) = \lambda V_{n−1}(K^{u_i}).$$

But that is precisely what (3.3) is stating, which completes the proof of the proposition.

□

**Lemma 3.3.** Let $K$ be an $n$-polytope satisfying (3.1) for all bodies $L$ and for all $M = K_t$ where $t \in (−δ, δ)$ as in Lemma 3.1. Then

$$S(K_t[r], K[n−1−r], \cdot) = \frac{V(K_t, K[n−1])^r}{V_n(K)^r}S(K, \cdot)$$

for all $0 \leq r \leq n−1$ and all $t \in (−δ, δ)$.

**Proof.** For $0 \leq r \leq n−1$, set $P_r$ to be the polytope whose surface area measure equals $S(K_t[r], K[n−1−r], \cdot)$ and let $\lambda := V(K_t, K[n−1])/V_n(K)$. For each $r$ the existence and uniqueness of $P_r$ is ensured by the Minkowski Existence and Uniqueness Theorem (see [Sch, Sections 7.1, 7.2]). We need to prove that

(3.6) $$S(P_r, \cdot) = \lambda^r S(K, \cdot), \quad r = 0, 1, \ldots, n−1.$$  

Note that by Lemma 3.1 we have:

(3.7) $$\text{supp } S(P_r, \cdot) = \text{supp } S(K, \cdot), \quad r = 1, \ldots, n−1.$$  

We prove (3.6) by induction on $r$. The case $r = 0$ is trivial. For the case $r = 1$ we apply Proposition 3.2 with $P = P_1$. Indeed, by our assumption, (3.1) is satisfied for $M = K_t$ and becomes equality when $L = K$. Thus the conditions (1)–(3) of Proposition 3.2 hold and so $S(P_1, \cdot) = \lambda S(K, \cdot)$, as required.

Now assume (3.6) holds for $1 \leq m \leq r−1$. This is equivalent to the following:

(3.8) $$V(L, P_m[n−1]) = \lambda^m V(L, K[n−1]),$$
Thus \( K_v \) chosen arbitrarily, for every vertex \( m \) which, by (3.8) with \( \lambda K \) vertex of \( R \). Therefore, the origin is the only vertex of \( K \).

Now we are ready to prove the main theorem which implies Theorem 1.1.

**Theorem 3.4.** Let \( K \) be an \( n \)-polytope in \( \mathbb{R}^n \). Suppose that

\[
V(L, M, K[n - 2])V_n(K) \leq V(L, K[n - 1])V(M, K[n - 1])
\]

holds for all convex bodies \( L \) and \( M \) in \( \mathbb{R}^n \). Then \( K \) is a simplex.

**Proof.** Let \( K_t \) be the polytope obtained by moving one of the facets of \( K \) for \( t \) small enough. Then Lemma 3.3 with \( r = n - 1 \) implies that the surface area measures of \( K_t \), Proposition 3.2, show that \( S(P_r, \cdot) = \lambda^r S(K, \cdot) \), which completes the proof of the lemma.

Now we are ready to prove the main theorem which implies Theorem 1.1.
4. Proof of Theorem 1.2

Recall that a boundary point $y \in \partial K$ is strict if it does not belong to any segment contained in $\partial K$. Note that points with positive Gaussian curvature and, more generally, regular exposed points are strict points (see [Sch] for the definitions). Clearly the boundary of a polytope does not contain any strict points, but there are other convex bodies having this property (for example, a cylinder).

As before it is enough to prove Theorem 1.2 in the case of $r = 2$. It follows from the theorem below.

**Theorem 4.1.** Let $K$ be a convex body whose boundary contains at least one strict point. Then there exist convex bodies $L$ and $M$ such that

$$V(L, M, K[n - 2])V_n(K) > V(L, K[n - 1])V(M, K[n - 1]).$$

**Proof.** First let us fix some notation. For $a > 0$ and $u \in \mathbb{S}^{n-1}$, define the closed half-spaces:

$$H^+_a(u) = \{x \in \mathbb{R}^n : \langle x, u \rangle \geq a\} \quad \text{and} \quad H^-_a(u) = \{x \in \mathbb{R}^n : \langle x, u \rangle \leq a\}.$$  

Also set $H_a(u) := H^+_a(u) \cap H^-_a(u)$. With this notation, the supporting hyperplane of $K$ whose unit normal vector is $u$, can be written as $H_{h_K(u)}(u)$.

Let $y$ be a strict point of $\partial K$ and $u$ be a normal vector of $K$ at $y$. Choose $v \in \mathbb{S}^{n-1}$, such that $y|v^+ \in \text{relint}(K|v^+)$, where $\text{relint}(K|v^+)$ denotes the relative interior of the body $K|v^+$ in $v^+$. We claim that there exists $\varepsilon > 0$, such that

$$(K \cap H^-_{h_K(u)-\varepsilon}(u))|v^+ = K|v^+.$$  

To see this, assume that (4.2) is not true for all $\varepsilon > 0$. This means that for any $\varepsilon > 0$, there exists a point $x_\varepsilon \in \partial K$, such that $x_\varepsilon|v^+ \in \partial(K|v^+)$ and $x_\varepsilon \in H^+_{h_K(u)-\varepsilon}(u)$. Let $x_0$ be an accumulation point of the set $\{x_\varepsilon : \varepsilon > 0\}$. Then, by compactness, $x_0 \in \partial K$, $x_0|v^+ \in \partial(K|v^+)$, and $x_0 \in H_{h_K(u)}(u)$ (because $x_0 \in H^+_{h_K(u)}(u)$ and $x_0 \in K$). Note that, since $x_0|v^+ \in \partial(K|v^+)$ and $y|v^+ \in \text{relint}(K|v^+)$, we have $x_0 \neq y$. It follows that the segment $[x_0, y]$ is contained in a supporting hyperplane of $K$, thus $[x_0, y] \subseteq \partial K$, which contradicts the assumption that $y$ is strict. Hence, (4.2) holds for some $\varepsilon > 0$.

Next, set $K_\varepsilon := K \cap H^-_{h_K(u)-\varepsilon}(u)$. Clearly, $h_{K_\varepsilon} \leq h_K$. We claim that there exists an open subset $\beta \subset \partial K \setminus \partial K_\varepsilon$, such that $y \in \beta$ and

$$h_{K_\varepsilon}(u) < h_K(u), \quad \text{for all } u \in \sigma(K, \beta).$$  

Suppose not. Then for any $\delta$-neighborhood $\beta_\delta = (\partial K \setminus \partial K_\varepsilon) \cap B(y, \delta)$ of $y$ there exists a unit vector $u_\delta \in \sigma(K, \beta_\delta)$ such that $h_K(u_\delta) = h_{K_\varepsilon}(u_\delta)$. In other words, there exist points $y_0 \in \beta_\delta$ and $x_\delta \in \partial K_\varepsilon$ lying in the same hyperplane $H_K(u_\delta)$. But then, by compactness, there exist a point $x \in \partial K_\varepsilon$ and a unit vector $u$, which is normal for $K$ at $y$ and at $x$. This shows again that the points $y$ and $x$ of $K$ lie in the same supporting hyperplane $H_K(u)$, thus $[y, x]$ is a boundary segment of $K$, which contradicts our assumption. Therefore, (4.3) holds for some open set $\beta \subseteq \partial K \setminus \partial K_\varepsilon$.

Note, furthermore, that $\tau(K, \sigma(K, \beta)) \geq \beta$, thus $\mathcal{H}^{n-1}(\tau(K, \sigma(K, \beta))) > 0$, which shows that

$$(4.4) \quad S(K, \sigma(K, \beta)) > 0.$$
Now we are ready to exhibit examples of compact convex sets $L$ and $M$ satisfying (4.1). Set $L = [-v, v]$ and $M = K_\varepsilon$. Then, by (5.3.23) in [Sch, p. 294] and applying (4.2) we obtain

$$V(L, M, K[n-2]) = V(K_\varepsilon|v^\perp, K|v^\perp[n-2]) = V_{n-1}(K|v^\perp) = V(L, K[n-1]).$$

On the other hand, by (4.3) and (4.4), we have:

$$V(M, K[n-1]) = V(K_\varepsilon, K[n-1]) = \frac{1}{n} \int_{S^{n-1}} h_{K_\varepsilon}dS(K, \cdot)$$

$$< \frac{1}{n} \int_{S^{n-1}} h_KdS(K, \cdot) = V_n(K).$$

This shows that

$$V(L, M, K[n-2])V_n(K) > V(L, K[n-1])V(M, K[n-1]),$$

as asserted. □

**Remark 4.2.** One might ask the following: If $K$ is a convex body whose boundary contains at least one strict point $x$, is it true that $\partial K$ has an open neighborhood that does not contain any line segments, i.e. $K$ is strictly convex in a neighborhood of $x$? If yes, this would simplify the proof of Theorem 4.1 considerably. The following simple 3-dimensional example shows, however, that this is not the case. Take $K$ equal to

$$\{x \in \mathbb{R}^3 : x_3 \leq 1\} \bigcap \text{conv}\left(\{(0, x_2, x_3) \in \mathbb{R}^3 : x_3 = x_2\} \cup \{(x_1, 0, x_3) \in \mathbb{R}^3 : x_3 = x_1^2\}\right).$$

Then the origin is a strict point of the boundary of $K$, but no neighborhood of the origin is strictly convex.

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