Determining the Geometry and the Cosmological Parameters of the Universe through SZE Cluster Counts

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ABSTRACT

We study Sunyaev-Zel’dovich Effect (SZE) cluster counts in different cosmologies. It is found that even without the full knowledge of the redshift distribution of SZE clusters, one can still readily distinguish a flat universe with a cosmological constant from an open universe. We divide clusters into a low redshift group (with redshift $z \leq 0.5$) and a high redshift group (with $z \geq 1$), and compute the ratio of $r = N(z \leq 0.5)/N(z \geq 1)$, where $N(z \leq 0.5)$ is the number of flux-limited ($S_{lim}^\nu$) SZE clusters with $z \leq 0.5$ and $N(z \geq 1)$ is the number of flux-limited SZE clusters with $z \geq 1$. With about the same total number of SZE clusters $N(z \geq 0)$, the $r$ value for a flat universe with a non-zero cosmological constant and that for an open universe occupy different regions in the $S_{lim}^\nu - r$ plot for the most likely cosmological parameters $0.25 \leq \Omega_0 \leq 0.35$ and $0.2 \leq \Gamma \leq 0.3$, where $\Omega_0$ is the matter density parameter of the universe, and $\Gamma$ is the shape parameter of the power spectrum of linear density fluctuations. Thus with a deep SZE cluster survey, the ratio $r$ can reveal, independent of the normalization of the power spectrum, whether we are living in a low-density flat universe or in an open universe. Within the flat universe scenario, the SZE cluster-normalized $\sigma_8$ is studied, where $\sigma_8$ is the r.m.s. density fluctuation within the top-hat scale $8 \text{Mpc}h^{-1}$ where $h$ is the Hubble constant in units of $100 \text{km s}^{-1} \text{Mpc}^{-1}$. A functional relation $\sigma_8 \propto \Omega_0^{-0.13}$ is found. Combined with the X-ray cluster-normalized $\sigma_8 \propto \Omega_0^{-0.52+0.13\Omega_0}$, one can put constraints on both $\Omega_0$ and $\sigma_8$ simultaneously.

Subject headings: cosmology: theory— galaxy: cluster — large-scale structure of universe
Clusters of galaxies are the largest virialized objects in the universe, and contain valuable information of the universe and of the large-scale structure. There have been intensive studies on clusters from different approaches, such as strong and weak gravitational lensing, X-ray, and optical observations. With the technical advents of interferometers, the cluster’s Sunyaev-Zel’dovich Effect (SZE) (Sunyaev & Zel’dovich 1970, 1980; Birkinshaw 1999, Carlstrom et al. 1999), a spectral distortion of Cosmic Microwave Background Radiation (CMB) due to scattering of CMB photons by hot electrons within clusters, has been becoming a new probe for the cluster study. As CMB photons pass through intrachannel hot electrons, on average they gain energies through the inverse-Compton scattering, and as a result of this, the number of low energy photons decreases while the number of high energy ones increases. Thus for observations with the frequency $\nu > (\lesssim)219$ GHz, hot clusters behave like emitting sources (absorbers) of photons. The equivalent temperature increment (or decrement) $\Delta T$ of CMB photons toward clusters is proportional to $\int n_e T_{\text{gas}} dl$, where $n_e$ is the number density of electrons, $T_{\text{gas}}$ is the hot gas temperature, and $dl$ is the line element along the line of sight. The integrated SZ effect of a cluster is then directly proportional to the cluster’s gas mass if the gas is close to be isothermal. Therefore the integrated SZ effect is not sensitive to the lumpy structures of the gas and the gas fraction can be estimated relatively clean from the SZ effect in conjunction with lensing observations (e.g., Grego et al. 2000). On the other hand, because of the different dependence on $n_e$ of the cluster’s X-ray surface brightness ($S_x \propto \int n_e^2 \Lambda_{eH} dl$, where $\Lambda_{eH}$ is the X-ray cooling function) and of the SZ effect, the angular diameter distance to a cluster can be derived directly from a joint analysis of the X-ray emission and the SZ effect through modeling the gas density profile properly (e.g., Reese et al. 2000).

Apart from the SZE studies for individual clusters, statistical investigations on large
number of SZE clusters can also yield very promising results. In fact several interferometric arrays have been proposed for surveying SZE clusters, including the AMIBA (Array for Microwave BAckground) project which has been founded in Taiwan. Due to the frequency (energy) redshift dependence of the CMB photons, the SZ effect is independent of the redshift, which permits a SZE cluster survey to detect very high redshift clusters with relative ease. Therefore the cluster redshift evolution can be studied with high statistical significance. Another advantage of the SZE cluster studies over those of X-ray is that the integrated SZ effect of an individual cluster is directly proportional to the gas mass within the cluster (assuming the gas is isothermal), which is in turn related to the total mass of the cluster, and the number of SZE clusters can be predicted analytically from the Press-Schechter formula (or other similar models) in a straightforward manner with certain qualifications. By contrast, because of the $n_e^2$ dependence of the X-ray surface brightness, the gas density profile has to be modeled in the X-ray studies to estimate flux-limited X-ray cluster number counts analytically, a procedure that can introduce large uncertainties (the prediction on the number of temperature-limited X-ray clusters suffers less problem). One may investigate both aspects of clusters with numerical simulations, but finite numerical resolutions and box sizes ultimately limit their applications, and the analytical analysis can be complementary.

Among other promising aspects, the redshift distribution of SZE clusters can be used to constrain cosmological parameters. The redshift distribution of SZE clusters with $\Omega_0 = 1$ is distinctly different from those with $\Omega_0 \sim 0.3$, and the existence of several SZE clusters at redshift $z \gtrsim 1$ would strongly exclude the $\Omega_0 = 1$ model. The difference between the redshift distribution of a low-density flat universe model and of a low-density open universe model is less dramatic, and one needs a relatively large number of clusters at high redshift to falsify them. The SZE cluster surveys are suitable for this purpose. By fully using the redshift distribution and the total number of SZE clusters, Haiman, Mohr, & Holder (2000)
studied constraints on the quintessence theory from future SZE (and X-ray) cluster surveys. The redshift of a cluster with \( z \leq 1 \) can be obtained at least by using the photometric method around a characteristic spectral break. For clusters with \( z > 1 \), it however appears difficult to measure the redshifts precisely except for very large clusters. In this paper, we propose a method which can distinguish, within the parameter regime \( 0.25 \leq \Omega_0 \leq 0.35 \) and \( 0.2 \leq \Gamma \leq 0.3 \), a flat universe with a non-zero cosmological constant from an open universe even without knowing the full redshift distribution of SZE clusters. We divide clusters into two groups: a low redshift one with \( z \leq 0.5 \), and a high redshift one with \( z \geq 1 \). We study the ratio \( r = N(z \leq 0.5)/N(z \geq 1) \) for different cosmologies with different parameters, where \( N(z \leq 0.5) \) is the total flux-limited number of clusters with \( z \leq 0.5 \), and \( N(z \geq 1) \) is the total flux-limited number of clusters with \( z \geq 1 \). It is found that \( r \) can be used to disentangle a low-density flat universe from a low-density open universe. Notice that to compute \( r \), we only need to know the redshift range of a cluster (whether it is \( z \leq 0.5 \) or \( z \geq 1 \)) rather than its precise redshift.

We also investigate, within the framework of the flat universes with a non-zero cosmological constant, the SZE cluster-normalized \( \sigma_8 \). The \( \sigma_8 - \Omega_0 \) relation inferred from the SZE cluster counts is distinctly different from that from the X-ray cluster counts with the latter more steeper. This difference can be used to limit the parameter regime of \( \Omega_0 \) and \( \sigma_8 \) simultaneously.

The rest of the paper is organized as follows. Section 2 will present the formulation for the study. The results will be shown in section 3. Section 4 contains a summary.
2. Formulation

As CMB photons pass through a sea of hot electrons, their blackbody spectrum is distorted by the inverse Compton scattering. The SZ effect can be characterized by the Compton $y$ parameter,

$$y = \int n_e \sigma_T \left( \frac{kT_{\text{gas}}}{m_e c^2} \right) dl,$$

where $n_e$ is the number density of hot electrons, $\sigma_T = 6.65 \times 10^{-25}$ cm$^2$ is the Thomson cross section, $k$ is the Boltzmann’s constant, $T_{\text{gas}}$ is the temperature of the hot intracluster gas, $m_e$ is the electron mass, and $c$ is the speed of light. The integration is along the line-of-sight, and $y$ parameter is proportional to the integrated thermal pressure along the line-of-sight. When the electron temperature $T_{\text{gas}}$ is much higher than the temperature $T_{\text{CMB}}$ of the CMB photons, the CMB flux change due to the presence of a cluster can be written as

$$S_\nu = S_{\nu\text{CMB}} Q(x) Y,$$

where $x = \hbar \nu / kT_{\text{CMB}}$, $\nu$ is the frequency of the CMB photons, $\hbar$ is the Planck’s constant, the unperturbed CMB flux $S_{\nu\text{CMB}} = (2\hbar \nu^3 / c^2) / (e^x - 1)$,

$$Q(x) = \frac{xe^x}{e^x - 1} \left[ \frac{x}{\tanh(x/2)} - 4 \right],$$

and

$$Y = R_d^{-2} \int ydA,$$

where $R_d$ is the angular diameter distance of the cluster, and the integration is over the projected area of the cluster. It is seen that when $\nu \approx 219$ GHz, $Q(x) = 0$. At the lower and higher frequency parts, $Q(x) < 0$ and $Q(x) > 0$, respectively. For AMIBA, $\nu = 90$ GHz, $x \approx 1.58$, and $Q(x) \approx -3.185$.

We assume that the intracluster gas is isothermal and the gas mass fraction $f_{\text{ICM}}$ is a
constant. Then we have (e.g., Eke, Cole, & Frenk 1996)

\[
Y = \frac{\sigma_T}{2m_e m_p c^2} R_d^{-2} f_{ICM}(1 + X) kT_{gas} M, \tag{5}
\]

where \( m_p \) is the proton mass, \( X \) is the hydrogen mass fraction, and \( M \) is the total mass (including the dark matter) of the cluster. Here we have used that the intracluster gas mass is dominated by hydrogen and helium. Further the gas is assumed to be in hydrostatic equilibrium with the gravitational potential of the total mass of the cluster, then

\[
kT_{gas} = -\frac{1}{[d\ln\rho_{gas}(r)/d\ln r]_{r_{vir}}} \mu m_p \frac{GM}{r_{vir}}, \tag{6}
\]

where \( \rho_{gas}(r) \) is the radial density profile of the gas, \( r_{vir} \) stands for the virial radius of the cluster, and \( \mu = 4/(5X + 3) \) is the mean molecular weight. We have used, in equation (6), the virial mass to represent the total mass of the cluster. Let \( \Delta_c \) be the average mass density with respect to the critical density at redshift \( z \) of the cluster formation, then

\[
kT_{gas} = -\frac{7.75}{0.5[d\ln\rho_{gas}(r)/d\ln r]_{r_{vir}}} \left( \frac{6.8}{5X + 3} \right) \left( \frac{M}{10^{15} h^{-1} M_\odot} \right)^{2/3}
\times (1 + z) \left[ \frac{\Omega_0}{\Omega(z)} \right]^{1/3} \left( \frac{\Delta_c}{178} \right)^{1/3} \text{keV}, \tag{7}
\]

where \( \Omega(z) \) is the density parameter at redshift \( z \), and \( h \) is the Hubble constant in units of \( 100 \text{ km s}^{-1} \text{Mpc}^{-1} \). Then

\[
S_\nu = 2.29 \times 10^4 \frac{x^3}{e^x - 1} Q(x) \times 1.70 \times 10^{-2} h \left( \frac{f_{ICM}}{0.1} \right) \left( \frac{1 + X}{1.76} \right)
\times \left\{ \frac{7.75}{0.5[d\ln\rho_{gas}(r)/d\ln r]_{r_{vir}}} \right\} \left( \frac{6.8}{5X + 3} \right) \left( \frac{R_d}{100 h^{-1} \text{Mpc}} \right)^{-2}
\times (1 + z) \left[ \frac{\Omega_0}{\Omega(z)} \right]^{1/3} \left( \frac{\Delta_c}{178} \right)^{1/3} \left( \frac{M}{10^{15} h^{-1} M_\odot} \right)^{5/3} \text{mJy}, \tag{8}
\]

where 1 mJy = \( 10^{-26} \text{erg cm}^{-2} \text{s}^{-1} \text{Hz}^{-1} \). In the following we will use the value \([d\ln\rho_{gas}(r)/d\ln r]_{r_{vir}} = 2\), which is consistent with both the observational and the numerical
simulation results. In fact, the results for different values of \( \frac{d \ln \rho_{\text{gas}}(r)}{d \ln r} \) can be obtained from our analyses by rescaling the overall flux \( S_\nu \) up or down, as can be seen from equation (8). The cosmology enters the relation (8) between \( S_\nu \) and \( M \) through the angular diameter distance \( R_d \), the density parameters \( \Omega_0 \) and \( \Omega(z) \), and the over density parameter \( \Delta_c \). We will calculate the flux-limited SZE cluster counts. Note by using equation (8) to calculate \( M_{\text{lim}} \) from a given flux limit \( S_{\nu \text{lim}} \), we have implicitly assumed that the counts are for unresolved clusters. An array of interferometers must, however, have a minimum baseline which is essentially limited by the dish diameter \( D \). Signals from angular scales larger than about \( \lambda/(2D) \) are lost where \( \lambda \) is the observing wavelength. For AMIBA there are two sets of dishes with \( D = 1.2 \) m and 0.3 m, respectively. For \( \nu = 90 \) GHz, \( \lambda \approx 0.33 \) cm, and \( \lambda/(2D) \sim 4.7 \) arcmin for \( D = 1.2 \) m and 18.9 arcmin for \( D = 0.3 \) m. For the smaller set of dishes, there should not be any loss of signals from clusters. We have estimated the mass limit for \( D = 1.2 \) m by simply cutting off any signals from \( \theta \geq 4.7 \) arcmin. The hydrostatic equilibrium gas density profile has been used by assuming that the underlying dark matter distribution has a universal density profile (Navarro, Frenk & White 1997). The mass limit estimated here is not very different from that for the unresolved clusters at \( S_{\nu \text{lim}} \sim 5 \) mJy of the AMIBA design. Thus our studies presented in the next section will only consider the unresolved cluster counts. Holder et al. (1999) determined \( M_{\text{lim}} \) by performing mock observations appropriate for a proposed interferometric array which consists of ten 2.5 m telescopes operating on \( \nu = 30 \) GHz (Mohr et al. 1999) on simulated clusters; the shape of the \( M_{\text{lim}}(z) \) is similar to that for unresolved clusters. In the work of Haiman et al. (2000), the mass limit for their fiducial model is from Holder et al. (1999), and \( M_{\text{lim}} \) for other cosmological models is obtained by using the same scaling relation as that of equation (8). They found that the mass limit from the scaling relation agrees with the mock survey results better than 10% for the two testing cosmological models. We use the Press-Schechter formalism (Press & Schechter 1974) to calculate the number of SZE
clusters. The comoving number density of clusters of mass \( M \) with width \( dM \) is,

\[
n(M) dM = \left( \frac{2}{\pi} \right)^{1/2} \frac{\rho_0 \delta_c(z) d\sigma_0}{M \sigma_0^2} \exp \left( -\frac{\delta_c^2(z)}{2\sigma_0^2} \right),
\]

where \( \rho_0 \) is the present mass density of the universe, \( \delta_c(z) \) the linear overdensity threshold for collapse at redshift \( z \), and \( \sigma_0 \) the r.m.s. linear density fluctuation on the scale corresponding to \( M \). Notice that \( \delta_c(z) \) and \( \sigma_0 \) are computed from the extrapolated-to-present linear density perturbations. Then the differential number of SZE clusters is

\[
\frac{dN}{dzd\Omega} = \frac{dV}{dzd\Omega} \int_{M_{\text{lim}}(z)} n(M) dM,
\]

where \( d\Omega \) is the solid angle element, \( dV \) is the comoving volume element which is dependent of cosmologies, and \( M_{\text{lim}}(z) \) is calculated from equation (8).

3. Analyses

3.1. Flat Models versus Open Models

For a cold-dark-matter universe, the power spectrum of the linear density fluctuation field can be written as (e.g., Efstathiou, Bond, & White 1992)

\[
P(k) = \frac{Bk^n}{\left\{1 + [ak + (bk)^{3/2} + (ck)^2]^{\nu/2}\right\}^{2/\nu}},
\]

where \( a = (6.4/\Gamma) h^{-1} \) Mpc, \( b = (3.0/\Gamma) h^{-1} \) Mpc, \( c = (1.7/\Gamma) h^{-1} \) Mpc, \( \nu = 1.13 \), \( \Gamma \) is the shape parameter of the power spectrum, which is related to the time of equal matter-radiation energy density in the universe, \( h \) is the Hubble constant in units of 100 kms\(^{-1}\) Mpc\(^{-1}\), \( B \) represents the perturbation amplitude, and \( n \) is the power index which is taken to be \( n = 1 \).

The object of this subsection is to distinguish the low-density flat universes from the low-density open universes. Before proceeding to this issue, let us first consider examples of
the redshift distribution of SZE clusters for some popular cosmologies. In Fig. 1, the SZE clusters’ redshift distribution is shown for (1). $\tau$-CDM (White, Gelmini, & Silk 1995) (solid line): $\Omega_0 = 1$, $h = 0.5$, $\Gamma = 0.25$, and $\sigma_8 = 0.52$ where $\sigma_8$ is the r.m.s. density fluctuation within the top-hat scale $8 \, h^{-1} \text{Mpc}$; (2). SCDM (dash-triple dotted line): $\Omega_0 = 1$, $h = 0.5$, $\Gamma = 0.5$, and $\sigma_8 = 0.52$; (3). open CDM (dash-dotted line): $\Omega_0 = 0.3$, $\Omega_\Lambda = 0$, $h = 0.83$, $\Gamma = 0.25$, and $\sigma_8 = 0.87$; (4). $\Lambda$CDM (dotted line): $\Omega_0 = 0.3$, $\Omega_\Lambda = 0.7$, $h = 0.83$, $\Gamma = 0.25$, $\sigma_8 = 0.93$. Here the normalization factor $\sigma_8$ is determined from the X-ray cluster counts (Eke et al. 1996). It is clearly seen that the redshift distribution of $\Omega_0 = 1$ models is drastically different from that of low-$\Omega$ models. Because of the continuing growth of the linear density perturbation for the $\Omega_0 = 1$ models, their cluster numbers at high redshift (e.g., $z \geq 1$) are very tiny when the perturbations are normalized to the local cluster abundance. The presence of a few clusters at $z \geq 1$ would strongly falsify the $\Omega_0 = 1$ models. In fact, the existence of high redshift optical or X-ray clusters has been used to constrain the range of $\Omega_0$ (e.g., Bahcall & Fan 1998). On the other hand, the differences of the redshift distribution between the two low-density cosmological models are not as impressive as the differences between them and the $\Omega_0 = 1$ models, but are still rather substantial at high redshifts. The number of SZE clusters drops rapidly for the $\Lambda$CMD model, while for the low-density open model its redshift distribution has a long tail at high redshifts. The different behaviors of the two models are mainly caused by the different angular diameter distances $R_d$ [see equation (8)].

In the following we will study the ratio of the low and high redshift SZE clusters for the two types of low-density cosmologies. Specifically, the ratio of $r = N(z \leq 0.5)/N(z \geq 1)$ is considered. The relative number of high and low-redshift clusters with their masses above a given threshold has been used in determining separately $\Omega_0$ and $\sigma_8$ by Fan, Bahcall and Cen (1997). Their study is different from ours in several aspects, but both analyses take the advantage of the dependence of the cluster evolution on cosmologies.
The model with $\Omega_0 = 0.3$, $\Omega_A = 0.7$, $\Gamma = 0.25$ and $\sigma_8 = 0.93$ is chosen to be the fiducial one. The particular $\sigma_8$ value is from the observed X-ray cluster counts. For other models, the cosmological density parameter is taken to be $0.25 \leq \Omega_0 \leq 0.35$. We vary $\Gamma$ in the range of $0.2 \leq \Gamma \leq 0.3$, consistent with large-scale structure studies (Peacock & Dods 1994, Dodelson & Gaztanaga 2000).

We determine $\sigma_8$ of a specific model in such a way that it gives rise to about the same total number of flux-limited SZE clusters as that of the fiducial one at $S_{\nu}^{lim} \approx 6.2$ mJy. We will refer to this as SZE-cluster-normalized $\sigma_8$. For the moment we pretend not to know the X-ray or the optical cluster normalization (except for the $\sigma_8$ of the fiducial model), and only the SZE cluster counts are used in determining both $\sigma_8$ and $r$. In fact the derived $\sigma_8$ from SZE cluster counts is consistent with that from X-ray or optical cluster counts for the parameter range we are considering. But later we will see that if we require the same total number of SZE clusters for two models with significantly different values of $\Omega_0$ (e.g., $\Omega_0 = 0.2$ versus $\Omega_0 = 0.4$), at least one of the two $\sigma_8$ will have to be out of the current X-ray cluster constraint. This inconsistency can be used in turn to falsify different models.

In Fig. 2 we show $r$ versus $S_{\nu}^{lim}$ for the fiducial cosmology (solid line), and for low-density open cosmologies. To avoid crowding, for open cosmologies only the highest and the lowest $r$ (dotted lines) for $0.25 \leq \Omega_0 \leq 0.35$ and $0.2 \leq \Gamma \leq 0.3$ and the results for $\Omega_0 = 0.3$ with $\Gamma = 0.2, 0.25$ and $0.3$ (dashed lines) are plotted. It is seen that the $r$ range for the open universes is quite separated from that of the fiducial model. At $S_{\nu}^{lim} \approx 6.2$ mJy, $dN(z \leq 0.5)/d\Omega \approx 4.0$ deg$^{-2}$ and $dN(z \geq 1)/d\Omega \approx 0.22$ deg$^{-2}$ for the low-density flat fiducial model. For a SZE cluster survey which covers 50 deg$^2$, $N(z \leq 0.5) \approx 200$ and $N(z \geq 1) \approx 11$. Consider the Poisson noise, then $\sigma_r/r \approx (1/200 + 1/11)^{1/2} \approx 0.31$, where $\sigma_r$ is the standard deviation of $r$. Thus the largest $r$ for the set of open universes is about $2\sigma$ away from the fiducial $r$. The open model with $\Omega_0 = 0.3$ and $\Gamma = 0.25$ is about $3\sigma$
away. This demonstrates that if the $r$ value and the total number of SZE clusters from observations are indeed around the values of the fiducial model, low-density open models can be excluded at the $2 \sim 3\sigma$ level depending on how well the $\Omega_0$ and $\Gamma$ parameters have been determined. With larger surveys, the exclusion can be made with higher statistical significance. We emphasize that different models are SZE cluster normalized, and thus the above conclusion is independent of the ‘real’ normalization (where ‘real’ normalization means the conventional normalization from X-ray/optical cluster observations). On the other hand, since the total number of SZE clusters is sensitive to the normalization factor, the combined analysis on X-ray cluster counts and on SZE cluster counts could falsify cosmological models if the normalizations determined separately from the X-ray clusters and from the SZE clusters disagree. This point will be elaborated in the next subsection.

Attentive readers may have suspected that if one decreases $\Omega_0$ and $\Gamma$ for a flat model, its $r$ range can get closer to those of open models. In Fig.3, we show the result for $\Omega_0 = 0.25$, $\Omega_\Lambda = 0.75$, $\Gamma = 0.2$, and $\sigma_8 \approx 1.01$ (Model 1, solid line) along with some of the results for open models: $\Omega_0 = 0.35$ and $\Gamma = 0.3$ (Model 2, dotted line); $\Omega_0 = 0.25$ and $\Gamma = 0.3$ (Model 3, upper dashed line); $\Omega = 0.25$ and $\Gamma = 0.25$ (Model 4, middle dashed line); $\Omega_0 = 0.25$ and $\Gamma = 0.2$ (Model 5, lower dashed line). All other open models with $0.25 \leq \Omega_0 \leq 0.35$ and $0.2 \leq \Gamma \leq 0.3$ have the $r$ value in between the results of Model 2 and 5. The open models have been normalized so that they contain about the same total number of SZE clusters as that of Model 1 at $S^\text{lim}_\nu \approx 6.2$ mJy. At this $S^\text{lim}_\nu$, Model 1 has $r \approx 10.4$ and Model 2 has $r \approx 7.0$. For Model 1 $dN(z \leq 0.5)/d\Omega \approx 3.6$ deg$^{-2}$ and $dN(z \geq 1)/d\Omega \approx 0.35$ deg$^{-2}$, and the standard deviation of $r$ for a 50 deg$^2$ survey is then $\sigma_r \approx 2.6$. Thus Model 2 differs from Model 1 at about $1\sigma$ level. This difference itself may not be large enough to distinguish the two models. We however notice that the two parameters $\Omega_0$ and $\Gamma$ for the two models lie toward the opposite limit of our considered range: $\Omega_0 = 0.25$ and $\Gamma = 0.2$ for Model 1, and $\Omega_0 = 0.35$ and $\Gamma = 0.3$ for Model 2. Hence if $\Omega_0$ or $\Gamma$ can be constrained to better degrees
by other observations, the low-density flat model and open models can be distinguished at
a higher level of significance (for example, the Sloan Digital Sky Survey would give a better
constraint on the $\Gamma$ parameter). This can be seen from the $r$ differences at $S_{\nu}^{lim} \approx 6.2$ mJy
which are at $1.9\sigma$ and $2.3\sigma$ levels between Model 1 and Model 3, and between Model 1 and
Model 4, respectively. Most drastically Model 1 and Model 5 have the same $\Omega_0$ and $\Gamma$, and
the difference in $r$ increases to the $3\sigma$ level.

We show in Fig.4 the $\Gamma$ dependence of $r$ for $\Omega_0 = 0.3$ and $\Omega_\Lambda = 0.7$. The three curves
are for $\Gamma = 0.2, 0.25$ and $0.3$, respectively. All the models are normalized to contain about
the same SZE clusters as our fiducial model at $S_{\nu}^{lim} \approx 6.2$ mJy. It is seen that $r$ is not very
sensitive to $\Gamma$. In other words, $\Gamma$ cannot be strongly constrained by the SZE cluster counts
alone. It is interesting to note that if the three models are all normalized to $\sigma_8 = 0.93$, the
trend in $r$ is reversed from that shown in Fig.4, i.e., $r$ with $\Gamma = 0.2$ is the largest, and $r$ for
$\Gamma = 0.3$ is the smallest. But the curves occupy the same region as that in Fig.4.

To see the $\Omega_0$ dependence of $r$, we plot $r$ in Fig.5 for low-density flat models with
$\Omega_0 = 0.25, 0.3$ and $0.35$, respectively. The $\Gamma$ parameter is taken to be $0.25$ for all the three
models, and as before, they are normalized according to the total number of SZE clusters
of the fiducial model at $S_{\nu}^{lim} \approx 6.2$ mJy. With such normalizations, $r$ is not sensitive to $\Omega_0$
either.

We conclude that $r$ is a very useful quantity to differentiate the low-density flat
cosmological models from the low-density open cosmological models. Within the likely
parameter regime of $\Omega_0$ and $\Gamma$, the quantity $r$ is not sensitive to either of them if different
models are normalized consistently to the SZE cluster counts. This method is independent
of the means by using the CMB measurement, and thus provides an important test on our
understanding of the universe and of the formation of the large-scale structure.

Our conclusion above is not sensitive to the specific total number density of SZE
clusters used to normalize different models. For instance, if we increase the total SZE cluster number density from $5.6 \, \text{deg}^{-2}$ to $8.0 \, \text{deg}^{-2}$ (the corresponding change of $\sigma_8$ of the fiducial model is from 0.93 to 1.0), the ΛCDM model with $\Omega_0 = 0.3$ and $\Gamma = 0.25$ and the open model with the same $\Omega_0$ and $\Gamma$ can also be distinguished at about $3\sigma$ level for a 50 deg$^2$ survey. The operational steps to apply our method to observations are (1) to normalize different models to the observed total SZE cluster counts; (2) to calculate the $r$ value for different models; (3) to find the $r$ value from observations; (4) to compare (3) with (2).

On the other hand, the relatively large separation of the $r$ value between the ΛCDM and the open CDM models shown above is restricted to our considered parameter intervals of $\Omega_0$ and $\Gamma$. If the $\Omega_0$ range is increased to $0.2 \leq \Omega_0 \leq 0.4$, there will have some overlaps in $r$ for the two types of models. For example, the $r$ value of the ΛCDM model with $\Omega_0 = 0.2$ falls into about the same range as that of the open CDM model with $\Omega_0 = 0.4$. Thus solely with the SZE total counts and the $r$ value, the two models cannot be differentiated clearly. However, one would find that in order to have the same total number of SZE clusters, at least one of the two $\sigma_8$ must be out of the range allowed by the observed X-ray cluster counts. Therefore an exclusion is possible based on both the SZE cluster observations and the X-ray observations. Stated somewhat differently, in order to make a relatively clean distinction between the ΛCDM and the open CDM models by using $r$ alone, the parameters $\Omega_0$ and $\Gamma$ must be pre-determined by other observations to a relatively fine degree. In conjunction with X-ray or optical cluster observations, the SZE cluster counts can be used to distinguish the two types of models for wider parameter regimes.
3.2. Constraints on $\Omega_0$ and $\sigma_8$ for a Flat Universe

Results from the recent Boomerang Cosmic Microwave Background (CMB) radiation observation show the first Doppler peak at $l \approx 200$ beautifully (de Bernardis et al. 2000), and make the flat universe be widely accepted (Hu 2000, Tegmark & Zaldarriaga 2000). If we indeed live in a flat universe with a non-zero cosmological constant, then, as we will describe below, the combined analyses of X-ray cluster counts and SZE cluster counts can give rise to constraints on $\Omega_0$ and on $\sigma_8$ even without the cluster redshift information.

The key here is that the dependence of $\sigma_8$ on $\Omega_0$ inferred from X-ray cluster counts is different from that from SZE cluster counts. The X-ray cluster counts yielded the relation $\sigma_8 = (0.52 \pm 0.04)\Omega_0^{-0.52 \pm 0.13} \Omega_0^{0.52+0.13}$ for $\Omega_0 + \Omega_\Lambda = 1$ (Eke et al. 1996). Here we study the expected $\sigma_8 - \Omega_0$ correlation from SZE cluster counts.

The formulation presented in Sec. 2 are used to study the SZE $\sigma_8 - \Omega_0$ relation. Or more clearly, the following assumptions are employed: (1). The intracluster gas is in hydrostatic equilibrium in the gravitational potential well of the total cluster mass; (2). The gas is isothermal and the gas mass fraction is a constant among different clusters; (3). The collapse is approximately spherical; (4). The Press-Schechter formula is approximately correct in predicting the number of clusters. We would like to point out that same approximations have been used in deriving the $\sigma_8 - \Omega_0$ correlation from X-ray cluster observations (Eke et al. 1996).

The model with $\Omega_0 = 0.3$, $\Omega_\Lambda = 0.7$, $\Gamma = 0.25$, and $\sigma_8 \approx 0.93$ is taken to be the fiducial one. We first compute the total surface number density of SZE clusters at $S_{\nu}^\text{lim} = 6.2$ mJy for the fiducial model, and then find $\sigma_8$ values for other $\Lambda$CDM models with different $\Omega_0$ so that they have the same total surface number density of SZE clusters at $S_{\nu}^\text{lim} = 6.2$ mJy as the fiducial model.
The results are shown in Fig. 6. It is found that the relation can be nearly perfectly described by $\sigma_8 = A_S \Omega_0^{0.13}$ (the solid line in Fig. 6), where $A_S$ is a numerical factor which is equal to 0.794 in our analysis. The dependence of the SZE cluster-normalized $\sigma_8$ on $\Omega_0$ is much weaker than that of the X-ray cluster-normalized $\sigma_8$, which can be understood from Eqns (7) and (8). The X-ray cluster-normalized $\sigma_8$ is calculated from temperature-limited cluster counts (Eke et al. 1996). The cluster mass limit $M_{lim}$ derived from a given cluster gas temperature threshold is proportional to $[\Omega_0/\Omega(z)]^{-1/2} \Delta_c^{-1/2}$ [Eqn. (7)]. By contrast, given a SZ flux limit $S_{\nu, lim}$, we have $M_{lim} \propto [\Omega_0/\Omega(z)]^{-1/5} \Delta_c^{-1/5}$ [Eqn. (8)]. Since the cluster number counts from the Press-Schechter formalism is sensitive to the mass limit, the inferred $\sigma_8$ by comparing results from X-ray observations with the predictions of the Press-Schechter calculations depends on $\Omega_0$ differently from that from SZE cluster ‘observations’.

The total number density of SZE clusters for the fiducial model at $S_{\nu, lim} \approx 6.2$ mJy is $dN(z \geq 0)/d\Omega \approx 5.6$ deg$^{-2}$. With a 50 deg$^2$ survey, the total number of SZE clusters is expected to be about 200, and the standard deviation given by the Poisson statistics is then about 16.7. Thus the 3$\sigma$ number density is $5.6 \pm 1.0$ deg$^{-2}$. The corresponding $\sigma_8$ range is calculated, which can be well approximated by $\sigma_8 = (0.794 \pm 0.025) \Omega_0^{-0.13}$. In Fig. 7, we plot the $\sigma_8 - \Omega_0$ relations expected from both the X-ray observations (dashed lines) and from our analysis on the SZE clusters (solid lines). Based on the current X-ray cluster results and our proposed 50 deg$^2$ SZE cluster survey, the fiducial $\Omega_0$ can be determined to $\Omega_0 = 0.3 \pm 0.08$ or $\Omega_0 = (1 \pm 27\%) \times 0.3$. From Fig. 7, it can be seen that the X-ray cluster-normalization constrains more tightly the value of $\Omega_0$ as it is much more sensitive to $\Omega_0$. With future X-ray cluster surveys such as XMM, the normalization can be determined to a much better degree. At a 2% precision on $\sigma_8$ from X-ray surveys, $\Omega_0$ can be constrained to $0.3 \pm 0.04$ ($\sim 13\%$ precision) with the allowance of a 3$\sigma$ deviation of SZE cluster counts. In comparison with the results of Haiman et al. (2000, Fig. 7 in their paper) for the $w = -1$ case, their constraint on $\Omega_0$ is more stringent. But in their analyses, they normalize all
models to give rise to the local cluster number density with \( M \geq 10^{14}h^{-1} \text{M}_{\odot} \), i.e., for each model, the normalization is fixed. They then compare both the total number of SZE clusters and the redshift distribution of a model with those of their fiducial one to constrain the parameter regimes. Note that for the \( w = -1 \) case, the total number of SZE clusters plays the dominant role in constraining \( \Omega_0 \). If we focus on the single middle dashed line in our Fig. 7, the 3\( \sigma \) determination of \( \Omega_0 \) from the SZE total number counts is similar to that of Haiman et al. (2000). Be aware however the different cosmological parameters, the normalizations and the survey parameters used in their analyses and in our studies.

The normalization factor \( \sigma_8 \) can also be constrained at the same time. In contrary to the determination of \( \Omega_0 \), the \( \sigma_8 \) value is constrained more tightly by the SZE cluster counts. At \( \Omega_0 = 0.3 \), the 3\( \sigma \) determination of \( \sigma_8 \) is \( 0.93 \pm 0.03 \) or \((1 \pm 3.2\%) \times 0.93\) in comparison with \((1 \pm 7.5\%) \times 0.93\) from the current X-ray studies.

We emphasize that although we chose \( \Gamma = 0.25 \) in the above analysis, calculations have been done for other \( \Gamma \) values. It is found that the functional relation \( \sigma_8 \propto \Omega_0^{-0.13} \) is very insensitive to the \( \Gamma \) value. Moreover the \( S_\nu^{lim} \) value has almost no effect on this functional relation.

4. Summary

In our analyses, we used \( f_{ICM} = 0.1 \), \( X = 0.76 \) and \([d\ln \rho_{gas}(r)/d\ln r]_{vir} = 2\). To change these parameters however, is equivalent to change the overall flux limit \( S_\nu^{lim} \). From the figures we showed, it is easy to see that all our conclusions remain qualitatively unchanged for different choices of these parameters. We have assumed the hydrodynamic equilibrium and isothermality for the intracluster gas. The Press-Schechter formalism has been adopted to calculate the cluster counts.
We studied the $r$ quantity for two types of cosmologies: the low-density flat models and the low-density open models. Within the studied parameter regimes $0.25 \leq \Omega_0 \leq 0.35$ and $0.2 \leq \Gamma \leq 0.3$, the $r$ value for the two sets of cosmologies are well separated. The flat model with $\Omega_0 = 0.3$ and $\Gamma = 0.25$ and the open model with the same $\Omega_0$ and $\Gamma$ can be differentiated at the $3\sigma$ level for a 50 deg$^2$ survey. Since we normalize different models in a way such that they give rise to the same total number of SZE clusters, our analyses naturally take into account the total number of SZE clusters. For wider parameter ranges, the information from other observations, such as X-ray or optical cluster surveys has to be used to differentiate the two types of models.

There are other ways to determine the geometry of the universe. Measurements on fluctuations of the CMB radiation provide a clean test on this aspect (e.g., Hu 2000). The horizon size of the universe at decoupling separates large-scale and small-scale CMB fluctuations. Large-scale fluctuations were outside the horizon when photons escaped while small-scale perturbations were within the horizon at decoupling and therefore sustained acoustic oscillations. The position of the primary Doppler peak of the CMB fluctuation power spectrum is determined by the angular size of the horizon at decoupling. For a flat universe (low-density with a non-zero cosmological constant or high density), the peak located at $l \approx 200$, and the peak is shifted to smaller angular scale or higher $l$ for a low-density open universe, where $l$ represents the two-dimensional angular wavenumber. Observations have seen the rising and the declining of CMB fluctuations around $l \sim 200$ (e.g., Miller et al. 1999, de Bernardis et al. 2000), which constrains convincingly that $\Omega_{\text{tot}}$ is close to 1, where $\Omega_{\text{tot}}$ is the total density parameter. Combining with other information, e.g., supernova measurements (Perlmutter et al. 1999, Schmidt et al. 1998), leads to the conclusion that we are living in a low-density flat universe. If the results from SZE cluster surveys are in agreement with the CMB results, we will be in a more solid position to say that the universe is flat and the simple structure formation theory we adopted here is
reasonably correct. On the other hand, any inconsistency between the SZE cluster results and the results from CMB measurements would pose challenges to our understanding of the universe. For example, alternate structure formation theories, such as non-Gaussian initial fluctuation models, may need to be considered.

Should the universe be flat, SZE cluster surveys can also provide constraints on cosmological parameters. We studied $\sigma_8(\Omega_0)$ inferred from the total SZE cluster counts. A functional relation $\sigma_8 \propto \Omega^{-0.13}$ is found. Combined with the current X-ray cluster-normalized $\sigma_8$, the parameters can be determined (take $\Omega_0 = 0.3$ and $\sigma_8 = 0.93$ as the central values) to $\Omega_0 = (1 \pm 27\%) \times 0.3$ and $\sigma_8 = (1 \pm 3.2\%) \times 0.93$ for the SZE cluster counts confined to the $3\sigma$ level in a 50 deg$^2$ survey. Note that these constrains are from the total number of clusters only, and no redshift information is needed.

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Fig. 1.— Redshift distributions of SZE clusters with $S_{\nu}^{lim} \approx 6.2$ mJy. The solid line is for $\tau$CDM model with $\Omega_0 = 1$, $h = 0.5$, $\Gamma = 0.25$ and $\sigma_8 = 0.52$. The dash-triple-dotted line is for the SCDM model with $\Omega_0 = 1$, $h = 0.5$, $\Gamma = 0.5$ and $\sigma_8 = 0.52$. The dash-dotted line is for the open CDM model with $\Omega_0 = 0.3$, $\Gamma = \Omega_0 h = 0.25$ and $\sigma_8 = 0.87$. The dotted line is for the $\Lambda$CDM model with $\Omega_0 = 0.3$, $\Omega_\Lambda = 0.7$, $\Gamma = \Omega_0 h = 0.25$ and $\sigma_8 = 0.93$.

Fig. 2.— The ratio $r$ against the flux limit $S_{\nu}^{lim}$. The solid line is for the fiducial model with $\Omega_0 = 0.3$, $\Omega_\Lambda = 0.7$, $\Gamma = \Omega_0 h = 0.25$ and $\sigma_8 = 0.93$. The upper dotted line is for the open model with $\Omega_0 = 0.35$, $\Gamma = \Omega_0 h = 0.3$, and $\sigma_8 = 0.785$, and the lower dotted line is for the open model with $\Omega_0 = 0.25$, $\Gamma = \Omega_0 h = 0.2$, and $\sigma_8 = 0.895$. The three dash-dotted lines are for $\Omega_0 = 0.3$ open models with $\Gamma = \Omega_0 h = 0.3, 0.25$ and 0.2, from top to bottom respectively, and the corresponding $\sigma_8 = 0.79, 0.835$ and 0.89.

Fig. 3.— $r$ versus $S_{\nu}^{lim}$. The solid line is for the $\Lambda$CDM model with $\Omega_0 = 0.25$, $\Omega_\Lambda = 0.75$, $\Gamma = \Omega_0 h = 0.2$, and $\sigma_8 = 1.01$ (Model 1). The dotted line is for the open model with $\Omega_0 = 0.35$, $\Gamma = \Omega_0 h = 0.3$ and $\sigma_8 = 0.785$ (Model 2). The three dashed lines are for the open models with $\Omega_0 = 0.25$, and from top to bottom $\Gamma = \Omega_0 h = 0.3, 0.25$ and 0.2, respectively (Model 3, 4, 5). The respective $\sigma_8$ for the open models with $\Omega_0 = 0.25$ are 0.795, 0.84 and 0.895.

Fig. 4.— $r$ versus $S_{\nu}^{lim}$ for $\Omega_0 = 0.3$ and $\Omega_\Lambda = 0.7$. The solid line is for $\Gamma = \Omega_0 h = 0.25$, and $\sigma_8 = 0.93$. The dotted line is for $\Gamma = \Omega_0 h = 0.3$ and $\sigma_8 = 0.88$. The dashed line is for $\Gamma = \Omega_0 h = 0.2$ and $\sigma_8 = 0.99$.

Fig. 5.— $r$ versus $S_{\nu}^{lim}$ for $\Lambda$CDM models with $\Gamma = \Omega_0 h = 0.25$. The solid line is for $\Omega_0 = 0.3$, $\Omega_\Lambda = 0.7$, and $\sigma_8 = 0.93$. The dotted line is for $\Omega_0 = 0.35$, $\Omega_\Lambda = 0.65$, and $\sigma_8 = 0.91$. The dashed line is for $\Omega_0 = 0.25$, $\Omega_\Lambda = 0.75$ and $\sigma_8 = 0.95$.

Fig. 6.— SZE cluster-normalized $\sigma_8$ as a function of $\Omega_0$ for flat universes with a non-zero
cosmological constant. The stars are the numerical results calculated by requiring that models with different $\Omega_0$ contain the same number of SZE clusters at $S_{\nu}^{\text{lim}} \approx 6.2$ mJy as that of the fiducial model. The fitting relation $\sigma_8 = 0.794\Omega_0^{-0.13}$ is shown as the solid line.

Fig. 7.— $\sigma_8$ versus $\Omega_0$ for flat universes. The solid lines are from the SZE cluster counts, and from top to bottom, the respective $\sigma_8$ are $\sigma_8 = 0.819\Omega_0^{-0.13}$, $\sigma_8 = 0.794\Omega_0^{-0.13}$, and $\sigma_8 = 0.769\Omega_0^{-0.13}$. The dashed lines are from X-ray temperature limited cluster counts, and $\sigma_8 = 0.56\Omega_0^{-0.52+0.13\Omega_0}$, $0.52\Omega_0^{-0.52+0.13\Omega_0}$, and $0.48\Omega_0^{-0.52+0.13\Omega_0}$, respectively, from top to bottom.
Fig. 4

$\Lambda$CDM: $\Omega_\Lambda=0.3$

$\Gamma=0.3$ --

$\Gamma=0.25$ --

$\Gamma=0.2$ --

$S_{\nu}^{\nu m} (\text{mJy})$
$\Lambda$CDM: $\Gamma = 0.25$

$\Omega_0 = 0.35$

$\Omega_0 = 0.3$

$\Omega_0 = 0.25$
Fig. 6

SZE-normalized $\sigma_8 - \Omega_0$ relation

$\sigma_8 = 0.794 \Omega_0^{-0.13}$
Fig. 7

$\sigma_8 = (0.52 \pm 0.04) \Omega_0^{-0.52 \pm 0.13 \Omega_0}$

$\sigma_8 = (0.794 \pm 0.025) \Omega_0^{-0.13}$