Curvatures and Hyperbolic Flows for Natural Mechanical Systems in Finsler Geometry

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Abstract
We consider a natural mechanical system on a Finsler manifold and study its curvature using the intrinsic Jacobi equations (called Jacobi curves) along the extremals of the least action of the system. The curvature for such a system is expressed in terms of the Riemann curvature and the Chern curvature (involving the gradient of the potential) of the Finsler manifold and the Hessian of the potential w.r.t. a Riemannian metric induced from the Finsler metric. As an application, we give sufficient conditions for the Hamiltonian flows of the least action to be hyperbolic and show new examples of Anosov flows.

Keywords Finsler geometry · Natural mechanical systems · Jacobi equations · Curvatures · Hyperbolic flows

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1 Introduction
In 1990s A. Agrachev and R. Gamkrelidze [2] proposed the program of studying an extremal of the optimal control problems on a manifold $M$ through the intrinsic Jacobi equations (called Jacobi curves) along the extremal. The Jacobi curve is a curve in a Lagrangian Grassmannian defined up to a symplectic transformation and containing all information about the solutions of the Jacobi equations along this extremal. Based on the study of the differential geometry of the parameterized curves in Lagrangian Grassmannians (see [2, 13, 14] and the recent monograph [1]), we can apply to Jacobi curves to constructing the curvature-type invariants (called curvatures in short) for natural mechanical systems on various smooth geometric structures including (sub-)Riemannian and (sub-)Finsler manifolds. The curvatures for (sub-)Riemannian and (sub-)Finsler geodesic problems then can be taken

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as particular cases. Moreover, by using the curvatures we can derive the qualitative properties of the extremals of the optimal control problems, such as various comparison theorems and hyperbolicity.

For the simplest case of a Riemannian geodesic problem, the aforementioned curvature invariants essentially coincides with the Riemannian curvature tensor [2]. Further, for the least action problems of a natural mechanical system on a Riemannian manifold, the curvatures are expressed by the Riemannian curvature tensor and the Hessian of the potential (see [3]).

It is very natural to expect that for a Finsler geodesic problem the curvature coincides with the Riemann curvature of the Finsler manifold (at some reference vector) and it is verified from a unified Hamiltonian viewpoint [10]. Now a natural question arises: how to express the curvature for a natural mechanical system on a Finsler manifold using the geometric quantities in the Finsler manifold and the potential?

One reason to find the answer to the above question is that it can be used to studying the hyperbolicity of the Hamiltonian flows. We first of all recall the following (see, e.g., [9])

**Definition 1** Let $e^{tX}, t \in \mathbb{R}$ be the flow generated by the vector field $X$ on a manifold $P$. A compact invariant set $A \subset P$ of the flow $e^{tX}$ is called a hyperbolic set if there exists a Riemannian structure in a neighborhood of $A$, a positive constant $\delta$, and a splitting: $T_z P = E^+_z \oplus E^-_z \oplus \mathbb{R} X(z)$, $z \in A$ such that $X(z) \neq 0$ and

1. $e^{tX}_z E^+_z = e^{tX}_{e^{tX} z} E^+_z$, $e^{tX}_z E^-_z = e^{tX}_{e^{tX} z} E^-_z$,
2. $\|e^{tX}_z \xi^+\| \geq e^{\delta t} \|\xi^+\|$, $\forall t > 0$, $\forall \xi^+ \in E^+_z$,
3. $\|e^{tX}_z \xi^-\| \leq e^{-\delta t} \|\xi^-\|$, $\forall t > 0$, $\forall \xi^- \in E^-_z$.

If the entire manifold $P$ is a hyperbolic set, then the flow $e^{tX}$ is called a flow of Anosov type.

It is well known that the geodesic flows on a closed Riemannian manifold with negative sectional curvature is of Anosov type [6]. Such a result has a Finsler version [8]: the geodesic flows on a closed reversible Finsler manifold with negative flag curvature must be of Anosov type. Actually both of them can be derived from a more general criteria of hyperbolic flows from the Hamiltonian viewpoint [4] and this criteria can also be used to get some sufficient conditions for the Hamiltonian flows for a natural mechanical system on a Finsler manifold.

The purpose of the present draft is twofold. On the one hand we interpret the curvatures for the Finsler least action problems by the curvature tensors (Riemannian curvature and Chern curvature) and the Hessian of the potential in Finsler geometry by the formulas which can be plugged into the framework of calculations in [15]. On the other hand we also get the sufficient conditions for the Hamiltonian flows for Finsler least action problems to be of Anosov type on Finsler manifolds.

Note that we always use Einstein summation convention: when an index variable appears twice in a single term it implies summation.

## 2 Main Results

In this section we present the main results on curvatures and hyperbolic flows. The proofs are postponed to the next section.
2.1 Preliminaries in Finsler Geometry

We first recall various notations which are needed in the rest of the draft (see, e.g., [17] for more details).

Given a local coordinate \((x^i)_{i=1}^n\) on an open set \(\Omega\) in a smooth manifold \(M\) of dimension \(n\), we will always use the coordinate \((x^i, y^j)_{i,j=1}^n\) of \(T\Omega\) with

\[ v = y^j \frac{\partial}{\partial x^j} |_{x} \in T_x M \quad \text{for} \ x \in \Omega. \]

While we use the coordinate \((x^i, p_j)_{i,j=1}^n\) of the cotangent bundle \(T^*\Omega\) with

\[ p = p_j dx^j |_{x} \in T^*_x M \quad \text{for} \ x \in \Omega. \]

**Definition 2 (Finsler structures)** A nonnegative function \(F : TM \to [0, \infty)\) is called a smooth Finsler structure of \(M\) if the following three conditions hold.

1. (Regularity) \(F\) is smooth on \(TM \setminus 0\), where 0 stands for the zero section.
2. (Positive 1-homogeneity) \(F(cv) = cF(v)\) for all \(v \in TM\) and \(c > 0\).
3. (Strong convexity) The \(n \times n\) matrix \((g_{ij}(v))_{i,j=1}^n := \left(\frac{1}{2} \frac{\partial^2 (F^2)}{\partial y^i \partial y^j} (v)\right)_{i,j=1}^n\) is positive-definite for all \(v \in TM \setminus 0\).

We call the manifold \(M\) a (smooth) Finsler manifold with Minkowski norm \(F\).

For \(x_0, x_1 \in M\), we define the distance from \(x_0\) to \(x_1\) in a natural way by

\[ d(x_0, x_1) := \inf_{x} \int_0^1 F(\dot{x}(t))dt, \]

where the infimum is taken over all \(C^1\)-curves \(x : [0, 1] \to M\) such that \(x(0) = x_0\) and \(x(1) = x_1\). Since \(F\) is only positively homogeneous, the distance can be non-reversible, i.e., \(d(x_0, x_1) \neq d(x_1, x_0)\) for some \(x_0, x_1 \in M\). A smooth curve \(x(\cdot)\) on \(M\) is called a geodesic if it is locally minimizing and has a constant speed (i.e., \(F(\dot{x})\) is constant).

For each \(v \in T_x M \setminus \{0\}\), the positive-definite matrix \((g_{ij}(v))_{i,j=1}^n\) induces the Riemannian structure \(g_v\) of \(T_x M\) as

\[ g_v \left( a^i \frac{\partial}{\partial x^i} |_x, b^j \frac{\partial}{\partial x^j} |_x \right) := a^i b^j g_{ij}(v). \]

For later convenience, we recall a basic fact on homogeneous functions.

**Theorem 2.1** Suppose that a differentiable function \(H : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}\) is positively \(r\)-homogeneous, i.e., \(H(cv) = c^r H(v)\) for some \(r \in \mathbb{R}\) and all \(c > 0\) and \(v \in \mathbb{R}^n \setminus \{0\}\). Then we have

\[ \frac{\partial H}{\partial y^i}(v)y^i = r H(v) \quad \text{for all} \ v \in \mathbb{R}^n \setminus \{0\}. \]
The Cartan tensor
\[ C_{ijk}(v) := \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}(v) \quad \text{for } v \in TM \setminus 0 \]
is a pure Finsler quantity. Indeed, \( C_{ijk} \)'s vanish everywhere on \( TM \setminus 0 \) if and only if \( F \) comes from a Riemannian metric. As \( g_{ij} \) is positively 0-homogeneous on each \( T_xM \setminus 0 \), Theorem 2.1 yields
\[ C_{ijk}(v)y^i = C_{ijk}(v)y^j = C_{ijk}(v)y^k = 0 \quad (2.1) \]
for all \( v \in TM \setminus 0 \).

Define the formal Christoffel symbols
\[ \gamma^{kl}_{ij}(v) := \frac{1}{2} g^{kl}(v) \left\{ \frac{\partial g_{il}}{\partial x^j}(v) + \frac{\partial g_{lj}}{\partial x^i}(v) - \frac{\partial g_{ij}}{\partial x^l}(v) \right\} \quad \text{for } v \in TM \setminus 0, \]
where \((g^{ij}(v))\) stands for the inverse matrix of \((g_{ij}(v))\). We also introduce the geodesic spray coefficient and the nonlinear connection
\[ G^i(v) := \gamma^{jk}_{ik}(v)y^j y^k, \quad N^i_j(v) := \frac{1}{2} \frac{\partial G^i}{\partial y^j}(v) \quad \text{for } v \in TM \setminus 0, \]
and \( G^i(0) = N^i_j(0) = 0 \) by convention.

Chern connection is torsion free and almost compatible with the metric and its coefficients are given by
\[ \Gamma^i_{jk} := \gamma^{ij}_{jk} - g^{il} \left( C_{jlm} N^m_k + C_{lkml} N^m_j - C_{jkm} N^m_l \right) \quad \text{on } TM \setminus 0. \]

One can show
\[ N^i_j = \Gamma^i_{jk} y^k. \quad (2.2) \]

We recall various curvature tensors in Finsler geometry which will be needed in the draft. Let \( U = U^k \frac{\partial}{\partial x^k}, V = V^l \frac{\partial}{\partial x^l}, W = W^j \frac{\partial}{\partial x^j} \). The Chern connection gives the Riemannian curvature tensor \( \mathcal{R} \) which can be written as
\[ \mathcal{R}(U, V)W := R^i_{jkl} U^k V^l W^j \frac{\partial}{\partial x^i} \]
where
\[ R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \frac{\partial \Gamma^k_{jl}}{\partial y^m} N^m_i - \frac{\partial \Gamma^k_{jm}}{\partial y^m} N^m_l + \Gamma^m_{jl} \Gamma^i_{mk} - \Gamma^m_{jk} \Gamma^i_{ml}. \quad (2.3) \]

Another curvature is Chern curvature which is a non-Riemannian curvature defined by
\[ P_v(U, V, W) := P^i_{jkl}(v) U^k V^l W^j \frac{\partial}{\partial x^i}, \]
where \( P^i_{jkl} = -\frac{\partial \Gamma^i_{jk}}{\partial y^l} \).

Let \( P \subset T_xM \) be a tangent plane. For a vector \( v \in P \setminus \{0\} \), define
\[ K(P, v) := \frac{g_v(\mathcal{R}(w, v)v, w)}{g_v(v, v)g_v(w, w) - g_v(v, w)^2}, \quad (2.4) \]
where \( w \in P \) such that \( P = \text{span}\{v, w\} \). The number \( K(P, v) \) is called the flag curvature of the flag \( (P, v) \) in \( T_xM \).

Finally we recall the Legendre transform in Finsler geometry. Denote by \( F^* \) the dual norm on \( T^*M \), i.e.,
\[ F^*(p) := \sup_{F(v) = 1} p(v), \ p \in T^*_xM. \]
Recall that we can write

\[ F^*(p)^2 = g^*_{ji}(p)p_ip_j, \quad g^*_{ij}(p) = \frac{1}{2} \frac{\partial^2 ((F^*)^2)}{\partial p_j \partial p_i}(p). \]

Let us denote by \( L^* : T^*M \to TM \) the Legendre transform associated with \( F \) and \( F^* \). More precisely, \( L^*(p) \) is the unique vector \( v = y^i(p) \frac{\partial}{\partial x^i} \in T_xM \) such that

\[ p(v) = F^*(p)^2, \quad F(v) = F^*(p). \]

For later use, we recall the following relations.

\[ y^i(p) = g^*_{ji}(p)p_j, \quad p_i = g_{ij}(v)y^j(p). \quad (2.5) \]

In the remainder of the draft, for the reason of simplicity we adopt the following convention on the notations: denote by \( v \) the image of \( p \) via the Legendre transform \( L^* \), i.e.,

\[ v = L^*(p) \]

and write \( v = y^i \frac{\partial}{\partial x^i} \).

### 2.2 Curvatures for Least Action Problems for a Natural Mechanical System on a Finsler Manifold

On a Finsler manifold \( M \) with Minkowski norm \( F \), consider the Finsler version of the least action problem of a natural mechanical system

\[ A(x(\cdot)) = \int_0^T \frac{1}{2} F(\dot{x}(t))^2 - U(x(t)) dt \to \min \]

\[ x(0) = x_0, \quad x(T) = x_1. \quad (2.6) \]

As in Riemannian geometry, the minimizers coincide when minimizing the length and the kinetic energy. Hence, when the potential \( U \) is identical to a null function, the above problem reduces to a Finsler geodesic problem. And since a Riemannian metric is a Finsler metric satisfying quadratic condition, the Riemannian least action problem is a particular case of the Finsler least action problem.

As optimal control problems, the Finsler least action problems can be solved by [16]. Let \( \sigma \) be the canonical symplectic form on \( T^*M \), i.e., \( \sigma = dx^i \wedge dp_i \). Let \( H \) be the maximized Hamiltonian (see Lemma 2.1 below). Then Pontryagin Maximum Principle tells that the minimizers are projections to the manifold \( M \) of the Hamiltonian flows generated by the vector field \( \vec{H} \) on \( T^*M \).

**Lemma 2.1** The maximized Hamiltonian \( H \) is written as follows.

\[ H(p) = \frac{1}{2} F^*(p)^2 + U(x) = \frac{1}{2} F(v)^2 + U(x). \quad (2.7) \]

**Proof** As a result of Theorem 2.1, \( F^2(v) = g_{ij} y^i y^j \), hence

\[ H(p) = \max_{v \in T_xM} (p(v) - \frac{1}{2} F^2(v) + U(x)) \]

\[ = \max_{v \in T_xM} (p(v) - \frac{1}{2} g_{ij} y^i y^j + U(x)) \]

\[ = \frac{1}{2} F^*(p)^2 + U(x) = \frac{1}{2} F(v)^2 + U(x). \]
We will construct the curvatures for Finsler least action problem. For this let us introduce the Jacobi curves associated with an extremal of the Finsler least action problem to describe its dynamical property. Let us fix the level set of the Hamiltonian function $\mathcal{H}$:

$$\mathcal{H}_c := \{ \lambda \in T^*M | \mathcal{H}(\lambda) = c \}, \quad c > 0.$$ 

Let $\Pi_\lambda$ be the vertical subspace of $T_\lambda \mathcal{H}_c$, i.e.,

$$\Pi_\lambda = \{ \xi \in T_\lambda \mathcal{H}_c : \pi^*(\xi) = 0 \},$$

where $\pi : T^*M \to M$ is the canonical projection. The curve defined by

$$t \mapsto \tilde{J}_\lambda(t) := e^{-t\mathcal{H}} \left( \Pi_\lambda \mathcal{H}_\lambda \right) / \mathbb{R} \mathcal{H}(\lambda). \quad (2.8)$$

is called the Jacobi curve of the extremal $e^{t\mathcal{H}}$ (attached at the point $\lambda$). The curve $J_\lambda(t)$ is a curve in the Lagrange Grassmannian of the linear symplectic space $W_\lambda = T_\lambda \mathcal{H}_c / \mathbb{R} \mathcal{H}(\lambda)$ (endowed with the symplectic form induced in the obvious way by the canonical symplectic form $\sigma$ of $T^*M$).

Next we introduce another version of Jacobi curves $\tilde{\tilde{J}}_\lambda(\cdot)$, called non-reduced Jacobi curves, by

$$t \mapsto \tilde{\tilde{J}}_\lambda(t) := e^{-t\mathcal{H}} \left( T^*_\lambda \mathcal{H}_\lambda T^*\pi(e^{t\mathcal{H}}) M \right). \quad (2.9)$$

There is a close relation between the two kinds of Jacobi curves: $J_\lambda(\cdot)$ can be obtained from $\tilde{\tilde{J}}_\lambda(\cdot)$ after the reduction of the first integral $\mathcal{H}$ (of the Hamiltonian flow generated by $\mathcal{H}$).

Next we give a concise description of the construction of the curvature-type invariants for the parametrized curves in some Lagrangian Grassmannians. For our purpose we focus on the case of regular curves and refer the reader to the relevant references for more general cases.

Recall that the tangent space $T_\Lambda L(W)$ to the Lagrange Grassmannian $L(W)$ of a linear symplectic space $W$ (endowed with a symplectic form $\omega$) at the point $\Lambda$ can be naturally identified with the space $\text{Quad}(\Lambda)$ of all quadratic forms on linear space $\Lambda \subset W$. Namely, given $\mathcal{W} \in T_\Lambda L(W)$ take a curve $\Lambda(t) \in L(W)$ with $\Lambda(0) = \Lambda$ and $\Lambda = \mathcal{W}$. Given some vector $l \in \Lambda$, take a curve $\ell(\cdot)$ in $W$ such that $\ell(t) \in \Lambda(t)$ for all $t$ and $\ell(0) = l$. Define the quadratic form

$$Q_\mathcal{W}(l) = \omega \left( l, \frac{d}{dt} \ell(0) \right). \quad (2.10)$$

Using the fact that the spaces $\Lambda(t)$ are Lagrangian, it is easy to see that $Q_\mathcal{W}(l)$ does not depend on the choice of the curves $\ell(\cdot)$ and $\Lambda(\cdot)$ with the above properties, but depends only on $\mathcal{W}$. So, we have the linear mapping from $T_\Lambda L(W)$ to the spaces $\text{Quad}(\Lambda)$, $\mathcal{W} \mapsto Q_\mathcal{W}$. A simple counting of dimensions shows that this mapping is a bijection and it defines the required identification. A curve $\Lambda(\cdot)$ in a Lagrangian Grassmannian is called regular, if its velocity is a non-degenerate quadratic form at any time $t$. A curve $\Lambda(\cdot)$ is called monotone (monotonically nondecreasing or monotonically nonincreasing) if the velocity is sign definite (nonnegative or nonpositive) at any point.

Note that one can show that either Jacobi curves $J_\lambda(\cdot)$ or the non-reduced Jacobi curves $\tilde{\tilde{J}}_\lambda(\cdot)$ for the case of Finsler least action is regular and monotone (see, for example, [5, Proposition 1]).

The curvatures for regular curves in Lagrangian Grassmannians are constructed in earlier work [2] and can be taken as a particular case of the results in [13, 14].
Theorem 2.2 Let $\Lambda(\cdot)$ be a regular curve in the Lagrangian Grassmannian $L(W)$ of a $2n$-dimensional linear symplectic space $W$. Then there exists a moving Darboux frame $(E(t), F(t))$ of $W$:

$$E(t) = (e_1(t), ..., e_n(t)), \quad F(t) = (f_1(t), ..., f_n(t))$$

such that $\Lambda(t) = \text{span}\{E(t)\}$ and there exists a one-parametric family of symmetric matrices $R(t) : \Lambda(t) \to \Lambda(t)$ satisfying

$$\begin{cases} E'(t) = F(t), \\ F'(t) = -E(t)R(t). \end{cases} \quad (2.11)$$

The moving frame $(E(t), F(t))$ is called a normal moving frame of $\Lambda(t)$. A moving frame $(\widetilde{E}(t), \widetilde{F}(t))$ is a normal moving frame of $\Lambda(t)$ if and only there exists a constant orthogonal matrix $O$ of size $n \times n$ such that

$$\widetilde{E}(t) = E(t)O, \quad \widetilde{F}(t) = F(t)O. \quad (2.12)$$

As a matter of fact, normal moving frames define a principal $O(n)$-bundle of symplectic frame in $W$ endowed with a canonical connection. Also, relations (2.12) imply that the following $n$-dimensional subspaces

$$\Lambda^{\text{trans}}(t) = \text{span}\{F(t)\} = \text{span}\{f_1(t), ..., f_n(t)\} \quad (2.13)$$

of $W$ does not depend on the choice of the normal moving frame. It is called the canonical complement of $\Lambda(t)$ in $W$. Moreover, the subspaces $\Lambda(t)$ and $\Lambda^{\text{trans}}(t)$ are endowed with the canonical Euclidean structure such that the tuple of vectors $E(t)$ and $F(t)$ constitute an orthonormal frame w.r.t. to it, respectively. The linear map from $\Lambda(t)$ to $\Lambda(t)$ with the matrix $R(t)$ from (2.11) in the basis $\{E(t)\}$, is independent of the choice of normal moving frames and is self-adjoint with respect to the Euclidean structure in $\Lambda(t)$. It will be denoted by $\mathfrak{R}(t)$ and it is called the curvature map of the curve $\Lambda(t)$.

The construction of curvature map for curves in Lagrangian Grassmannians naturally applies to the Jacobi curves $\tilde{J}_\lambda(\cdot)$ and non-reduced Jacobi curves $\tilde{J}(\cdot)$ under consideration.

Let $\tilde{J}^{\text{trans}}(t)$ and $\tilde{J}^{\text{trans}}(t)$ be the canonical complement of $\tilde{J}(t)$ (in the linear symplectic space $W_\lambda$) and $\tilde{J}(t)$ (in the linear symplectic space $T_\lambda T^*M$), respectively. Then, $\tilde{J}_\lambda^{\text{trans}} := \tilde{J}^{\text{trans}}(0)$ and $\tilde{J}_\lambda^{\text{trans}} := \tilde{J}^{\text{trans}}(0)$ give the canonical complement of $\Pi_\lambda$ (in $W_\lambda$) and $T_\lambda T^*_\lambda M$ (in $T_\lambda T^*M$), respectively. Note here we used that $\tilde{J}(0)$ is naturally identified with $\Pi_\lambda$ and $\tilde{J}(0) = T_\lambda T^*_\lambda M$. See Section 3.2 for a more detailed discussion on the canonical complements.

Let $\mathfrak{R}_\lambda(t)$ be the curvature for the Jacobi curve $\tilde{J}_\lambda(\cdot)$ and let $\tilde{\mathfrak{R}}_\lambda(\cdot)$ be the curvature for the non-reduced Jacobi curve $\tilde{J}_\lambda(\cdot)$. Then the linear maps

$$\mathfrak{R}_\lambda := \mathfrak{R}_\lambda(t)|_{t=0} : \Pi_\lambda \to \Pi_\lambda$$

and

$$\tilde{\mathfrak{R}}_\lambda := \tilde{\mathfrak{R}}_\lambda(t)|_{t=0} : T_\lambda T^*_\lambda M \to T_\lambda T^*_\lambda M$$

are said to be the curvature (at $\lambda$) and non-reduced curvature (at $\lambda$) of the Finsler least action problem.

### 2.3 Statements of the Main Results

To show the results on curvatures we need the following notations. Let, as before, $\nu = \mathcal{L}^*(p)$ and $g_\nu$ be the Riemannian metric induced by the vector $\nu$. Note that $T_\lambda T^*_\lambda M$ is identified with $T_x^*M$. Then for any $\xi \in T_\lambda T^*_\lambda M(\sim T_x^*M)$ we associate a vector $\xi^h \in T_x M$ via the
Riemannian metric $g_v$, i.e., $g_v(\xi^h, \cdot) = \xi$. In particular $p^h = v$, which is a consequence of the identity

$$(L^a)^{-1}(v) = g_v(v, \cdot).$$

Note that $\Pi_\lambda$ is embedded in $T_xT^*_xM$. Hence, the above operator of superscript $^h$ also applies to $\xi \in \Pi_\lambda$ to get a vector $\xi^h \in T_xM$.

**Theorem 2.3** The non-reduced curvature for the Finsler least action problems satisfies for $\forall \bar{\xi}, \bar{\eta} \in T_xT^*_xM$,

$g_v\left(\left(\bar{\mathcal{R}}_{\bar{\lambda}\bar{\xi}}\right)^h, \bar{\eta}^h\right) = g_v\left(\bar{\mathcal{R}}\left(\bar{\xi}^h, v\right) v, \bar{\eta}^h\right) + \text{Hess}_v U \left(\bar{\xi}^h, \bar{\eta}^h\right) + g_v\left(P_v\left(\bar{\xi}^h, \nabla_v U, \bar{\eta}^h\right), v\right),$

where $\mathcal{R}$ is the Riemannian curvature tensor, $P$ is the Chern curvature tensor, $\text{Hess}_v$ is the Hessian and $\nabla_v U$ is the gradient w.r.t. the Riemannian metric $g_v$.

**Theorem 2.4** Using the same notations as in the last theorem, the curvature $\mathcal{R}_{\lambda}$ satisfy for $\forall \xi, \eta \in \Pi_\lambda$,

$g_v\left(\left(\mathcal{R}_{\lambda}\xi\right)^h, \eta^h\right) = g_v\left(\mathcal{R}\left(\xi^h, v\right) v, \eta^h\right) + \text{Hess}_v U \left(\xi^h, \eta^h\right) + g_v\left(P_v\left(\xi^h, \nabla_v U, \eta^h\right), v\right) + \frac{3}{F(v)^2}g_v\left(\xi^h, \nabla_v U\right) g_v\left(\eta^h, \nabla_v U\right).$

**Corollary 2.1** For Finsler geodesic problems ($U = 0$), the curvatures satisfy $\forall \xi, \eta \in \Pi_\lambda$,

$g_v\left(\left(\mathcal{R}_{\lambda}\xi\right)^h, \eta^h\right) = g_v\left(\mathcal{R}\left(v, \xi^h\right) \eta^h, v\right).$

Now we turn to the study of Anosov flows on Finsler manifolds. In the present setting, the criteria for hyperbolicity and Anosov flows in [4] is written as the following

**Theorem 2.5** Let $c$ be a positive constant. Let $S$ be a compact invariant set of the flow $e^{t\vec{H}}$ contained in a fixed level of $H^{-1}(c)$. If the curvature satisfies that $g_v\left(\left(\mathcal{R}_{\lambda}\xi\right)^h, \xi^h\right) < 0$, $\forall v, \xi \in H_c$ at every point $x$ of $S$, then $S$ is a hyperbolic set of the flow $e^{t\vec{H}}|_{H^{-1}(c)}$.

Combining this theorem and Theorem 2.4 we have

**Theorem 2.6** Assume that the flag curvature of a closed reversible Finsler manifold $(M, F)$ is bounded from above by $k$. If the constant $c$ satisfies that

$$\max_{v \perp w, F(v) = F(w) = 2(c - U)} \{\text{Hess}_v U (w, w) + g_v(P_v (w, \nabla_v U, w), v) + \frac{3}{4(c - U)^2}g_v(w, \nabla_v U)^2 I\} < -4k(c - U)^2,$$

then the flow $e^{t\vec{H}}|_{H_c}$ is an Anosov flow.

When specializing to a closed Riemannian manifold $(M, g)$, we have the following
**Corollary 2.2** Assume that the sectional curvature of \((M, g)\) is bounded from above by \(k\). If the constant \(c\) satisfies that

\[
\max_{v, w : v \perp w, |v| = |w| = 1} \frac{\text{Hess} \ U(w, w)}{2(c - U)} + \frac{3}{4(c - U)^2} g(w, \nabla U)^2 < -k,
\]

then the flow \(e^{\tilde{H} |_{\tau_c}}\) is an Anosov flow. If denote by \(\|\text{Hess} \ U\|\) the operator norm of \(\text{Hess} \ U\) and \(\|\nabla U\|\) the norm operator of \(g(\nabla U, \cdot)\), then above condition can be written as

\[
\max_{x \in M} \left\{ \frac{\|\text{Hess} \ U\|}{2(c - U)} + \frac{3}{4(c - U)^2} \|\nabla U\|^2 \right\} < -k.
\]

It follows immediately that the geodesic flows on a closed reversible Finsler manifold with negative flag curvature are of Anosov type.

**Remark 2.1** There are partial results on the expressions of the curvatures for the least action problems of a natural mechanical system on a contact sub-Riemannian manifold with transverse symmetries (see [12]). And the hyperbolicity of the reduced Hamiltonian flows (reduced by the first integral from the transverse symmetries) for the least action problems of a natural mechanical system on a sub-Riemannian manifold with commutative transverse symmetries are discussed in [11].

### 3 Proofs of the Main Results

In this section we show the proofs of Theorems 2.3 and 2.4.

As before, let \(v = v^j \frac{\partial}{\partial x^j} = L^*(p)\). Note that, for the reason of simplicity we will not write \(p\) and \(v\) in the tensors. For example, we write \(g_{ij}\) and \(g^*_{ij}\) instead of \(g_{ij}(v)\) and \(g^*_{ij}(p)\), respectively. However, one should understand that such tensors in general depend on \(p\) or \(v\), which is the essential non-Riemannian phenomenon.

Note that for the rest of the draft, we consider \(y^j\) as a function of \(p_1, ..., p_n\) via the Legendre transformation \(L^*\). For simplicity again, we write \(y^j\) instead of \(y^j(p)\).

#### 3.1 Some Useful Lemmas

First of all, Theorem 2.1 implies

**Lemma 3.1** \(\frac{\partial^j}{\partial p_j} = g^*_{ij}\).

Further, let \(\nabla^v_w\) be the horizontal lift of \(w \in T_x M\) via the Chern connection with the reference vector \(v\). In local coordinates,

\[
\nabla^v_x = \frac{\partial}{\partial x^i} + \Gamma^k_{ij} p_k \frac{\partial}{\partial p_j}.
\]

Now we expression the Hamiltonian vector field \(\tilde{H}\) using the Chern connection. Note that it is also a consequence of homogeneity on the fibers of \(F^*\) (see [2] for this point).

**Lemma 3.2** \(\tilde{H}(p) = \nabla^v + \tilde{U}\).
Proof It follows from Lemma 2.1 that
\[ H(p) = \frac{1}{2} g_{ij}^* p_i p_j + U. \]

Hence,
\[
\tilde{H}(p) = \frac{\partial H}{\partial p_i} (p) \frac{\partial}{\partial x^i} - \frac{\partial H}{\partial p_i} (p) \frac{\partial}{\partial p_i} + \tilde{U} \nabla_v + \tilde{U}
= g_{ij}^* p_i \frac{\partial}{\partial x^j} + \frac{1}{2} \frac{\partial g_{ij}^*}{\partial p_k} p_i p_j \frac{\partial}{\partial x^k} \frac{1}{2} \frac{\partial g_{ij}^*}{\partial p_k} p_i p_j \frac{\partial}{\partial p_k} + \tilde{U}.
\]

As \(g_{ij}^*\) is positively 0-homogeneous in \(p\), Theorem 2.1 implies
\[
\frac{\partial g_{ij}^*}{\partial p_k} p_i p_j = 0.
\]

Therefore,
\[
\tilde{H}(p) = g_{ij}^* p_i \frac{\partial}{\partial x^j} - \frac{1}{2} \frac{\partial g_{ij}^*}{\partial x^k} p_i p_j \frac{\partial}{\partial p_k} + \tilde{U}. \tag{3.15}
\]

On the other hand, using (3.14), (2.1) and (2.5) we have
\[
\nabla_v = y^i \frac{\partial}{\partial x^i} + \Gamma_{ij}^k p_k y^j \frac{\partial}{\partial p_j}
= y^i \frac{\partial}{\partial x^i} + \left( y^k y^j - g^{kl} \left( C_{ilm} N_j^m + C_{ljm} N_i^m - C_{ijm} N_l^m \right) \right) p_k y^j \frac{\partial}{\partial p_j}
= y^i \frac{\partial}{\partial x^i} + y^j p_k y^j \frac{\partial}{\partial p_j} - \left( C_{ilm} N_j^m + C_{ljm} N_i^m - C_{ijm} N_l^m \right) y^j y^j \frac{\partial}{\partial p_j}
= y^i \frac{\partial}{\partial x^i} + y^j p_k y^j \frac{\partial}{\partial p_j}.
\]

The rest are actually the one for Riemannian case. Indeed, combining Lemma 3.1 with (2.1) and (2.5) we have
\[
\left( \nabla_v + \tilde{U} \right) - \tilde{H}(p) = y^k p_k y^j \frac{\partial}{\partial p_j} \frac{\partial g_{ij}^*}{\partial x^j} + \frac{1}{2} \frac{\partial g_{ij}^*}{\partial x^k} p_i p_j \frac{\partial}{\partial p_k}
= \frac{1}{2} \left\{ \frac{\partial g_{ij}^*}{\partial x^j} + \frac{\partial g_{ij}^*}{\partial x^i} - \frac{\partial g_{ij}^*}{\partial x^l} \right\} \frac{\partial g_{ij}^*}{\partial p_j} + \frac{1}{2} \frac{\partial g_{ij}^*}{\partial x^k} g_{is} y^s g_{jt} y^t \frac{\partial}{\partial p_k}
= \frac{1}{2} \frac{\partial g_{ij}^*}{\partial x^j} \frac{\partial g_{ij}^*}{\partial p_j} + \frac{1}{2} \frac{\partial g_{ij}^*}{\partial x^k} g_{is} y^s g_{jt} y^t \frac{\partial}{\partial p_k}.
\]

Using that \((g_{ij}^*)^{-1}\) is the inverse matrix of \((g_{ij})\), we conclude
\[
\left( \nabla_v + \tilde{U} \right) - \tilde{H}(p)
= \frac{1}{2} \frac{\partial g_{ij}^*}{\partial x^j} \frac{\partial g_{ij}^*}{\partial p_j} - \frac{1}{2} \frac{\partial g_{is}^*}{\partial x^k} g_{ij}^* y^s g_{jt} y^t \frac{\partial}{\partial p_k}
= \frac{1}{2} \frac{\partial g_{ij}^*}{\partial x^j} \frac{\partial g_{ij}^*}{\partial p_j} - \frac{1}{2} \frac{\partial g_{is}^*}{\partial x^k} y^s y^t \frac{\partial}{\partial p_k} = 0.
\]

It is time to introduce one more notation. For any \(w \in T_x M\) there is a unique co-vector \(w^v\) defined by \(w^v = g_v(w, \cdot)\).
The homogeneity on the fibers of $F^*$ also implies

**Lemma 3.3** \([\nabla^*_v, v^v] = -\nabla^*_v.\)

**Proof** Using Lemma 3.2 we obtain

\[
\nabla^*_v = g^*_i j p_i \frac{\partial}{\partial x^j} - \frac{1}{2} g^*_i j p_i p_j \frac{\partial}{\partial p_k}.
\]

Combining this identity with the identity $v^v = p_i \frac{\partial}{\partial p_i}$, we have

\[
[\nabla^*_v, v^v] + \nabla^*_v = \frac{1}{2} \frac{\partial^2 g^*_i j}{\partial x^k \partial p_i p_j} \frac{\partial}{\partial p_k} - \frac{\partial g^*_i j}{\partial p_a p_a p_i} \frac{\partial}{\partial x^j},
\]

which vanishes by Theorem 2.1.

The following lemma is useful when dealing the calculation of $\vec{U}$.

**Lemma 3.4** Let $U$ be the potential and $W_1, W_2$ be vertical vector fields on $TM$, i.e., $\pi_* W_1 = \pi_* W_2 = 0$. Then

1. $\tilde{U} = -(\nabla^*_v U)^v$,
2. $[\tilde{U}, v^v] = \tilde{U}$,
3. $g^v(W_1, W_2) = -\sigma (W_1^v, \nabla^*_v W_2)$.

**Proof** The first and second assertions are verified by straightforward computations.

\[
(\nabla^*_v U)^v = \left( \frac{\partial U}{\partial x^i} g^*_i j \frac{\partial}{\partial x^j} \right)^v = \frac{\partial U}{\partial x^i} g^*_i j g^v_{jk} \frac{\partial}{\partial p_k} = \frac{\partial U}{\partial x^i} \frac{\partial}{\partial p_i} = -\tilde{U}.
\]

\[
[\tilde{U}, v^v] = \left[ - \frac{\partial U}{\partial x^i} \frac{\partial}{\partial p_i}, p_j \frac{\partial}{\partial p_j} \right] = - \frac{\partial U}{\partial x^i} \frac{\partial}{\partial p_i} = \tilde{U}.
\]

For the third assertion, by linearity we can assume $W_1 = \frac{\partial}{\partial x^j}$, $W_2 = \frac{\partial}{\partial x^j}$. Then

\[
W_1^v = g^v_{ik} \frac{\partial}{\partial p_k}, \quad \nabla^*_v W_2 = \frac{\partial}{\partial x^j} + \Gamma^k_{jl} p_k \frac{\partial}{\partial p_l}.
\]

Hence,

\[
g^v(W_1, W_2) = g_{ij} = -\sigma (W_1^v, \nabla^*_v W_2).
\]

The following two lemmas provide a coordinate-free method when calculating the curvatures.

**Lemma 3.5** It holds the following identities.

\[
\sigma ([\nabla^*_v, \nabla^*_w], \nabla^*_w) = -g^v(\mathcal{R}(v, w_1)w_2, v) = g^v(\mathcal{R}(v, w_1)v, w_2) = -g^v(\mathcal{R}(w_1, v)v, w_2).
\]

**Proof** First of all, we remark that the last identity is from the antisymmetry of the Riemannian curvature in Finsler geometry w.r.t. the vectors $v, w_1$ while the second identity does not
follow from the antisymmetry of the Riemannian curvature w.r.t. the vectors \( v, w_2 \). Actually the latter antisymmetry in general doesn’t hold in Finsler geometry in contrast with the Riemannian geometry case. Rather, we can show that the sum \( g_*(\mathcal{R}(v, w_1)v, w_2) + g_*(\mathcal{R}(v, w_1)w_2, v) \)

has a factor \( C_{jbk} y^b \) (see, for instance, identity (3.4.4) in [7]). This factor vanishes from (2.1).

It remains to prove the first identity. Since both sides are linear in \( w_1 \) and \( w_2 \), it suffices to show the identity for \( w_1 = \frac{\partial}{\partial x^i} \), \( w_2 = \frac{\partial}{\partial x^j} \).

The horizontal lift w.r.t. Chern connection is Lagrangian (equivalent to the torsion freeness of Chern connection), i.e.,

\[
\sigma \left( \nabla_v^1, \nabla_v^2 \right) = 0, \quad \forall v_1, v_2 \in TM.
\]

It follows

\[
\sigma \left( \left[ \nabla_v^1, \nabla_v^2 \right], \nabla_v \right) = \sigma \left( y^a \left( \frac{\partial}{\partial x^a} + \Gamma_{ab}^c \frac{\partial}{\partial p^c} \right), \nabla_v \right) = y^a \sigma \left( \left[ \frac{\partial}{\partial x^a} + \Gamma_{ab}^c \frac{\partial}{\partial p^c}, \frac{\partial}{\partial x^i} + \Gamma_{il}^k \frac{\partial}{\partial p^l} \right], \nabla_v \right).
\]

(3.16)

Recall that \( y^a = g_{ab}^* p^b \), which gives

\[
\frac{\partial y^a}{\partial x^c} = \frac{\partial g_{ab}^*}{\partial x^c} p^b.
\]

And note that \( \Gamma_{ij}^k \) depends not only on \( x \) but also on \( v \). Now we can do the following straightforward calculations.

\[
\left[ \frac{\partial}{\partial x^a} + \Gamma_{ab}^c \frac{\partial}{\partial p^c}, \frac{\partial}{\partial x^i} + \Gamma_{il}^k \frac{\partial}{\partial p^l} \right] = \frac{\partial \Gamma_{ij}^k}{\partial x^i p^k} \frac{\partial}{\partial p^i} + \frac{\partial \Gamma_{ij}^k}{\partial y^b} \frac{\partial g_{bc}^*}{\partial x^a} p^c \frac{\partial}{\partial p^b} + \frac{\partial \Gamma_{ij}^k}{\partial p^i} \Gamma_{ab}^c p^c \frac{\partial}{\partial p^k} - \frac{\partial \Gamma_{ij}^k}{\partial p^i} \Gamma_{ab}^c i l p^c \frac{\partial}{\partial p^k} - \frac{\partial \Gamma_{ij}^k}{\partial p^i} \Gamma_{ab}^c \frac{\partial}{\partial p^k}.
\]

Then also straightforward calculation again gives

\[
\sigma \left( \left[ \frac{\partial}{\partial x^a} + \Gamma_{ab}^c \frac{\partial}{\partial p^c}, \frac{\partial}{\partial x^i} + \Gamma_{il}^k \frac{\partial}{\partial p^l} \right], \nabla_v \right) = -\frac{\partial \Gamma_{ij}^k}{\partial x^i} p^k - \frac{\partial \Gamma_{ij}^k}{\partial y^b} \frac{\partial g_{bc}^*}{\partial x^a} p^c p^k - \frac{\partial \Gamma_{ij}^k}{\partial p^i} \Gamma_{ab}^c p^c p^k - \frac{\partial \Gamma_{ij}^k}{\partial p^i} \Gamma_{ab}^c \frac{\partial}{\partial p^k} + \frac{\partial \Gamma_{ij}^k}{\partial p^i} \Gamma_{il}^k p^c + \Gamma_{aj}^l \Gamma_{ij}^k p^k.
\]

(3.17)
On the other hand, from (2.3) and (2.2) we have
\[
g_v \left( R \left( v, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}, v \right) \right) = \\
y^a p_b \left( \frac{\partial \Gamma_{ji}^b}{\partial x^a} - \frac{\partial \Gamma_{ja}^b}{\partial x^i} + \frac{\partial \Gamma_{ij}^a}{\partial y^i} \Gamma_{ak}^t y^k - \frac{\partial \Gamma_{ji}^b}{\partial y^t} \Gamma_{ak}^t y^k + \Gamma_{as}^b \Gamma_{ji}^t - \Gamma_{ja}^t \Gamma_{is}^b \right).
\]

Using this identity and (3.16), (3.17) we obtain
\[
\sigma \left( \left[ \nabla_v^s, \nabla_v^s \right], \nabla_v^s \right) + g_v \left( R \left( \frac{\partial}{\partial x^j}, v, \frac{\partial}{\partial x^i} \right) \right) = - \frac{\partial \Gamma_{ij}^k}{\partial y^b} p_c p_k y^a - \frac{\partial \Gamma_{ji}^b}{\partial p_b} \Gamma_{ac}^t g_{ts} + \frac{\partial \Gamma_{ja}^b}{\partial y^t} p_k p_c y^a \frac{\partial g_{lk}^s}{\partial x^i} \\
+ \frac{\partial \Gamma_{aj}^c}{\partial p_l} \Gamma_{il}^k p_k p_c y^a y^k + \frac{\partial \Gamma_{ji}^b}{\partial y^t} \Gamma_{ak}^t y^a y^k \\
\quad (3.18)
\]

The last goal is to show the right-hand side of the last identity vanishes. Indeed, since Chern connection is almost compatible with the Finsler metric (see, e.g., [17]),
\[
\frac{\partial g_{bc}^s}{\partial x^a} p_c = - \frac{\partial g_{bc}^s}{\partial x^a} g_{cs} y^s = - \frac{\partial g_{cs} y^s}{\partial x^a} g_{bc}^s y^s \\
= - g_{bc}^s y^s \left( \Gamma_{ac}^t g_{ts} + \Gamma_{as}^t g_{tc} + C_{cts} \Gamma_{ac}^t y^c \right) \\
= - \Gamma_{ac}^s g_{bc}^s y^s.
\]

Similarly,
\[
\frac{\partial g_{lk}^s}{\partial x^i} p_k = - \Gamma_{ik}^l g_{lk}^s p_i - \Gamma_{is}^l y^s.
\]

Using (3.19), (3.20) we finally verify by a straightforward calculation that the right-hand side of (3.18) vanishes, as claimed.

Lemma 3.6 Denote by Hess_v the Hessian w.r.t. the Riemannian metric g_v and by P the Chern curvature.
\[
\sigma \left( \left[ \bar{U}, \nabla_v^{w_1} \right], \nabla_v^{w_2} \right) = - \text{Hess}_v \left( U(w_1, w_2) - g_v (P_v (w_1, \nabla_v U, w_2) , v) \right).
\]

Proof Again, it suffices to prove the case when \( w_1 = \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \). A straightforward calculation shows
\[
\sigma \left( \left[ \bar{U}, \nabla_v^{w_1} \right], \nabla_v^{w_2} \right) = - \frac{\partial^2 U}{\partial x^i \partial x^j} + \frac{\partial U}{\partial x^k} \Gamma_{ij}^k + \frac{\partial \Gamma_{ij}^l}{\partial p_k} \frac{\partial U}{\partial x^k} p_l.
\]

On the other hand, we have
\[
\text{Hess}_v U \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = \frac{\partial^2 U}{\partial x^i \partial x^j} - \frac{\partial U}{\partial x^k} \Gamma_{ij}^k.
\]

Next, since \( \nabla_v U = \frac{\partial U}{\partial x^a} g_{ab} \frac{\partial}{\partial x^b} \), it follows (see, e.g., [17]) that
\[
g_v \left( P_v \left( \frac{\partial}{\partial x^i}, \nabla_v U, \frac{\partial}{\partial x^j} \right), v \right) = - \frac{\partial \Gamma_{ji}^l}{\partial y^k} y^b g_{ak} \frac{\partial U}{\partial x^a} = - \frac{\partial \Gamma_{ji}^l}{\partial p_k} p_l \frac{\partial U}{\partial x^k}.
\]

Combining this identity with (3.21), (3.22) we complete the proof of the lemma. \( \square \)
3.2 Canonical Complements

Recall that there is a canonical splitting

\[ W_\lambda = \Pi_\lambda \oplus J^\text{trans}_\lambda, \] (3.23)

where \( J^\text{trans}_\lambda \) is the canonical complement. A similar situation happens for the non-reduced case. Namely, it holds the following canonical splitting

\[ T_\lambda T^* M = T_\lambda T^*_x M \oplus \bar{J}^\text{trans}_\lambda, \] (3.24)

where \( \bar{J}^\text{trans}_\lambda \) is the canonical complement. In other words, \( J^\text{trans}_\lambda, \bar{J}^\text{trans}_\lambda \) are actually (nonlinear) Ehresmann connections of \( \Pi_\lambda \) in \( W_\lambda \) and of \( T_\lambda T^*_x M \) in \( T_\lambda T^* M \), respectively.

For the non-reduced case, a completely similar argument as in the proof of Lemma 11.1 in [10] gives the following

**Lemma 3.7** \( \bar{J}^\text{trans}_\lambda(\lambda) \) coincides with the Chern connection with the reference vector \( v \), i.e.,

\[ \bar{J}^\text{trans}_\lambda = \text{span} \left\{ \nabla_{\xi^h} v, i = 1, \ldots, n \right\}. \]

For the canonical complement \( J^\text{trans}_\lambda \), we have the following

**Lemma 3.8** \( J^\text{trans}_\lambda = \text{span} \left\{ \nabla_{\xi^h} - \frac{1}{F(v)^2} \cdot g(v(\xi^h, \nabla v U)) v^2, \xi \in \Pi_\lambda \right\}. \)

**Proof** Assume

\[ J^\text{trans}_\lambda = \text{span} \left\{ \nabla_{\xi^h} + A(\xi)v^v, \xi \in \Pi_\lambda \right\}. \]

Note that \( \sigma (\bar{H}, v^v) = g(v, v) = F(v)^2 \). Hence, from the fact that \( J^\text{trans}_\lambda \) is tangent to the Hamiltonian vector field \( \bar{H} \) and Lemma 3.4, we get easily that

\[ A(v) = -\frac{1}{F(v)^2} : g(v(\xi^h, \nabla v U)), \]

which completes the proof of the lemma.

3.3 Calculations of the Curvatures

It is convenient to introduce the notation of parallel transport of a vector field along the Hamiltonian flows.

Let \( \lambda \in T^* M \) and let \( \lambda(t) = e^{t\bar{H}} \lambda \). Assume that \( (E^\lambda(t), F^\lambda(t)) \) is a normal moving frame of the Jacobi curve \( J_\lambda(t) \) attached at point \( \lambda \). Let \( \mathcal{E} \) be the Euler field on \( T^* M \), i.e.,

the infinitesimal generator of the homotheties on its fibers. Clearly,

\[ T_\lambda(T^* M) = T_\lambda \mathcal{H} \oplus \mathbb{R} \mathcal{E}(\lambda). \]

The flow \( e^{t\bar{H}} \) on \( T^* M \) induces the push-forward maps \( e^t\bar{H} \) between the corresponding tangent spaces \( T_\lambda T^* M \) and \( T_{\lambda(t)} T^* M \), which in turn induce naturally the maps between the spaces \( T_\lambda(T^* M)/\text{span}[\bar{H}(\lambda)] \) and \( T_{\lambda(t)}(T^* M)/\text{span}[\bar{H}(\lambda(t))] \). The map \( K^t \) between \( T_{\lambda(t)}(T^* M)/\text{span}[\bar{H}(\lambda)] \) and \( T_{\lambda(t)}(T^* M)/\text{span}[\bar{H}(\lambda(t))] \), sending \( E^\lambda(0) \) to \( e^{t\bar{H}} E^\lambda(t), F^\lambda(0) \) to \( e^{t\bar{H}} F^\lambda(t), \) and the equivalence class of \( \mathcal{E}(\lambda) \) to the equivalence class of \( \mathcal{E}(e^{t\bar{H}} \lambda) \), is independent of the choice of normal moving frames. The map \( K^t \) is called the parallel trans...
transport along the extremal $e^t\vec{H}_\lambda$ at time $t$. For any $v \in T_\lambda(T^*M)/\text{span}\{\vec{H}_\lambda\}$, its image $v(t) = \mathcal{K}^t(v)$ is called the parallel transport of $v$ at time $t$.

Note that from the definition of the non-reduced Jacobi curves and the construction of normal moving frames it follows that the restriction of the parallel transport $\mathcal{K}^t$ to the vertical subspace $T_\lambda(T^*M)$ of $T_\lambda(T^*M)$ can be considered as a map onto the vertical subspace $T_{\lambda(t)}(T^*\pi_{\lambda(t)})M$ of $T_{\lambda(t)}(T^*M)$. A vertical vector field $V$ is called parallel if $V(e^t\vec{H}_\lambda) = \mathcal{K}^t(V(\lambda))$.

The rest of the draft is devoted to the proof of Theorems 2.3 and 2.4.

**Proof of Theorem 2.3** Let $\vec{W}_1, \vec{W}_2$ be parallel vertical vector fields such that $\vec{W}_1(\lambda) = \vec{\xi}, \vec{W}_2(\lambda) = \vec{\eta}$. From Lemma 3.7 the lift of $\vec{W}_i$ in $\mathfrak{g}^{\xi\text{trans}}_\lambda$ is $\nabla^\vec{W}_i$. Then it follows from Lemma 3.4 that

$$g_v(\langle \vec{\mathfrak{R}}_{\lambda}\vec{\xi}, \vec{\eta} \rangle^h, \vec{\eta}^h) = -\sigma(\vec{\mathfrak{R}}_{\lambda}\vec{\xi}, \nabla^\vec{W}_1) - \sigma(\vec{\mathfrak{H}}_\lambda \nabla^\vec{W}_1, \nabla^\vec{W}_2) = -\sigma(\vec{\mathfrak{H}}_\lambda \nabla^\vec{W}_1, \nabla^\vec{W}_2). \tag{3.25}$$

Combining this with Lemmas 3.2, 3.5 and 3.6, we complete the proof of the present theorem.

**Proof of Theorem 2.4** Let $W_1, W_2$ be vertical parallel vector fields such that $W_1(\lambda) = \xi, W_2(\lambda) = \eta$. From Lemmas 3.2, 3.4 and 3.8 we have

$$g_v(\langle \mathfrak{R}^\lambda\vec{\xi}^h, \vec{\eta}^h \rangle, \vec{\eta}^h) = -\sigma\left(\nabla^{\vec{W}_1} + \vec{U}, \nabla^{\vec{W}_2} - \frac{g_v(W^h_1, \nabla^vU)}{F(v)^2}v^v, \nabla^{\vec{W}_2} - \frac{g_v(W^h_2, \nabla^vU)}{F(v)^2}v^v\right).$$

Next we write the right-hand side of the last identity as the sum of the following $T_i$s and calculate each $T_i$ in turn. Note that when dealing with $T_5$ and $T_6$ we used the identity

$$\sigma\left(\nabla^{\vec{W}_1}, \nabla^{\vec{W}_2}\right) = -g_v(\vec{v}, W^h_2) = 0.$$

$$T_1 = -\sigma\left(\nabla^{\vec{W}_1}, \nabla^{\vec{W}_2}\right),$$

$$T_2 = \frac{g_v(W^h_2, \nabla^vU)}{F(v)^2}\sigma\left(\nabla^{\vec{W}_1}, \nabla^{\vec{W}_2}\right),$$

$$T_3 = -\sigma\left(\vec{U}, \nabla^{\vec{W}_1}\right),$$

$$T_4 = \frac{g_v(W^h_1, \nabla^vU)}{F(v)^2}\sigma\left(\vec{U}, \nabla^{\vec{W}_1}\right),$$

$$T_5 = \frac{g_v(W^h_1, \nabla^vU)}{F(v)^2}\sigma\left(\nabla^{\vec{W}_1}, v^v\right),$$

$$T_6 = \frac{g_v(W^h_2, \nabla^vU)}{F(v)^2}\sigma\left(\nabla^{\vec{W}_1}, \nabla^{\vec{W}_2}\right),$$

$$T_7 = -\frac{g_v(W^h_1, \nabla^vU)}{F(v)^2}\cdot \frac{g_v(W^h_2, \nabla^vU)}{F(v)^2}-\sigma\left(\nabla^{\vec{W}_1}, v^v\right).$$

First of all, from Lemma 3.5 we have

$$T_1 = -g_v\left(\mathcal{R} \left(\nabla^{\vec{W}_1} v, W^h_2\right)\right). \tag{3.26}$$
For $T_2$, we use that the symplectic form $\sigma$ is closed to show
\[
0 = d\sigma \left( \nabla_\sigma^v, \nabla_\sigma^{v^h}, v^v \right) \\
= \nabla_\sigma^v \left( \sigma \left( \nabla_\sigma^{v^h}, v^v \right) \right) - \nabla_\sigma^{v^h} \left( \sigma \left( \nabla_\sigma^v, v^v \right) \right) \\
+ v^v \left( \sigma \left( \nabla_\sigma^v, \nabla_\sigma^{v^h} \right) \right) - \sigma \left( \left[ \nabla_\sigma^v, \nabla_\sigma^{v^h} \right], v^v \right) \\
+ \sigma \left( \left[ \nabla_\sigma^v, v^v \right], \nabla_\sigma^{v^h} \right) - \sigma \left( \left[ \nabla_\sigma^{v^h}, v^v \right], \nabla_\sigma^v \right). 
\]
Then we make the following calculations. Using Lemma 3.4 we have
\[
\sigma \left( \nabla_\sigma^{v^h}, v^v \right) = g_v \left( W_1^h, v \right) = 0
\]
and
\[
\nabla_\sigma^{v^h} \left( \sigma \left( \nabla_\sigma^v, v^v \right) \right) = \nabla_\sigma^{v^h} \left( g_v(v, v) \right) = \nabla_\sigma^{v^h} \left( F(v)^2 \right) \\
= \sigma \left( 2\tilde{H} - 2\tilde{U}, \nabla_\sigma^{v^h} \right) = 2\sigma \left( \nabla_\sigma^v, \nabla_\sigma^{v^h} \right) = 0.
\]
Note that in the last equality we used Lemma 2.1, Lemma 3.2 and the fact that Chern connection is torsion free. Next we have
\[
v^v \left( \sigma \left( \nabla_\sigma^v, \nabla_\sigma^{v^h} \right) \right) = v^v(0) = 0.
\]
Similarly, we apply Lemma 3.3 to get
\[
\sigma \left( \left[ \nabla_\sigma^v, v^v \right], \nabla_\sigma^{v^h} \right) = -\sigma \left( \nabla_\sigma^v, \nabla_\sigma^{v^h} \right) = 0.
\]
And we also have
\[
\sigma \left( \left[ \nabla_\sigma^{v^h}, v^v \right], \nabla_\sigma^v \right) = -\left[ \nabla_\sigma^{v^h}, \left( v^v \right)^{(H - U)} \right] = -\nabla_\sigma^{v^h} \left( v^v(H - U) \right) + v^v \left( \nabla_\sigma^{v^h}(H - U) \right) \\
= -\nabla_\sigma^{v^h} \left( 2(H - U) \right) + v^v \left( \sigma \left( \tilde{H} - \tilde{U}, \nabla_\sigma^{v^h} \right) \right) \\
= -2\sigma \left( \nabla_\sigma^v, \nabla_\sigma^{v^h} \right) + v^v \left( \sigma \left( \nabla_\sigma^v, \nabla_\sigma^{v^h} \right) \right) = 0.
\]
Summarizing above we have
\[
T_2 = 0. \tag{3.27}
\]
For $T_3$, from Lemma 3.6 we have
\[
T_3 = \text{Hess}_v U \left( W_1^h, W_2^h \right) + g_v \left( P_v \left( W_1^h, v U, W_2^h \right), v \right).
\]
The case of $T_4$ is similar to that of $T_2$, we first of all have
\[
0 = d\sigma \left( \tilde{U}, \nabla_\sigma^{v^h}, v^v \right) \\
= v^v \left( \sigma \left( \tilde{U}, \nabla_\sigma^v \right) \right) - \sigma \left( \left[ \tilde{U}, \nabla_\sigma^{v^h} \right], v^v \right) \\
+ \sigma \left( \left[ \tilde{U}, v^v \right], \nabla_\sigma^{v^h} \right) - \sigma \left( \left[ \nabla_\sigma^{v^h}, v^v \right], \tilde{U} \right).
\]
Then we make the following calculations. Using Lemma 3.4 we have
\[
\sigma \left( \left[ \tilde{U}, v^v \right], \nabla_\sigma^{v^h} \right) = g_v \left( \nabla_v U, W_1^h \right)
\]
and
\[ \sigma \left( \left[ \nabla_{W_i^h}^v, v^v \right], \vec{U} \right) = - \left[ \nabla_{W_i^h}^v, v^v \right] (U) \]
\[ = v^v \left( \nabla_{W_i^h}^v (U) \right) \]
\[ = v^v \left( \sigma \left( \vec{U}, \nabla_{W_i^h}^v \right) \right). \]

And using Lemma 3.4 again we obtain
\[ \sigma \left( \left[ \vec{U}, v^v \right], \nabla_{W_i^h}^v \right) = \sigma \left( \vec{U}, \nabla_{W_i^h}^v \right) = g_v \left( \nabla_v U, W_i^h \right). \]

Summarizing, we have
\[ T_4 = \frac{1}{F(v)^2} g_v \left( W_i^h, \nabla_v U \right) g_v \left( W_j^h, \nabla_v U \right). \tag{3.28} \]

Using Lemma 3.3 and the fact that the Chern connection is torsion free, we get
\[ \sigma \left( \left[ \nabla_v^v, v^v \right], \nabla_{W_i^h}^v \right) = - \sigma \left( \nabla_v^v, \nabla_{W_i^h}^v \right) = 0. \tag{3.29} \]

Hence \( T_5 = 0 \).

For \( T_6 \), since
\[ \sigma \left( \left[ \vec{U}, v^v \right], \nabla_{W_i^h}^v \right) = \sigma \left( \vec{U}, \nabla_{W_i^h}^v \right) = g_v \left( W_i^h, \nabla_v U \right), \]
then
\[ T_6 = \frac{1}{F(v)^2} g_v \left( W_i^h, \nabla_v U \right) g_v \left( W_j^h, \nabla_v U \right). \tag{3.30} \]

For \( T_7 \), we make the following calculations.
\[ \sigma \left( \left[ \nabla_v^v, v^v \right], v^v \right) = - \sigma \left( \nabla_v^v, v^v \right) = -F(v)^2. \]

This gives
\[ T_7 = \frac{1}{F(v)^2} g_v \left( W_i^h, \nabla_v U \right) g_v \left( W_j^h, \nabla_v W \right). \tag{3.31} \]

Summarizing the above calculations for \( T_i \left( 1 \leq i \leq 7 \right) \), we complete the proof of the theorem.

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**References**

1. Agrachev A, Boscain U, Barilari D. A comprehensive introduction to Sub-Riemannian geometry. Cambridge studies in advanced mathematics. 2019: Cambridge University Press, Cambridge.
2. Agrachev AA, Gamkrelidze RV. Feedback-invariant optimal control theory - I. Regular extremals. J Dynam Control Sys. 1997;3:343–389.
3. Agrachev A, Chtcherbakova N, Zelenko I. On curvatures and focal points of dynamical lagrangian distributions and their reductions by first integrals. J Dynam Control Sys. 2005;11:297–327.
4. Agrachev A, Chtcherbakova N. Hamiltonian systems of negative curvature are hyperbolic. Russian Math Dokl. 2005;400:295–298.
5. Agrachev A, Zelenko I. Geometry of Jacobi curves I. J Dynam Control Sys. 2002;8(1):93–140.
6. Anosov DV. Geodesic flows on the closed Riemannian manifold of negative curvature. In: Proceedings of the steklov institute of mathematics, AMS, Providence, RI, 90, pp 3–209; 1967.
7. Bao D, Chern SS, Shen Z. An introduction to Riemann-Finsler geometry. New York: Springer-Verlag; 2000.
8. Foulon P. Estimation de l’entropie des systèmes lagrangiens sans points conjugués. Ann Inst Henri Poincaré. 1992;57(2):117–146.
9. Katok A, Hasselblatt B. Introduction to the modern theory of dynamical systems, encyclopedia of mathematics and its applications(54). 1995: Cambridge Univ Press, Cambridge.
10. Lee PWY. Displacement interpolations from a Hamiltonian point of view. J Func Anal. 2013;265(12):3163–3203.
11. Li C. A note on hyperbolic flow in sub-Riemannian structure with transverse symmetries. Acta App Math. 2012;117(1):71–91.
12. Li C. On curvature-type invariants for natural mechanical systems on sub-Riemannian structures associated with a principle G-bundle. Geometric control theory and sub-Riemannian geometry. In: Stefani G, Boscaín U, Sigalotti M, Gauthier J-P, and Sarychev A, editors; 2013. Springer. INdAM series, Vol. 5.
13. Li C, Zelenko I. Differential geometry of curves in Lagrange Grassmannians with given Young diagram. Differ Geom Appl. 2009;27(6):723–742.
14. Li C, Zelenko I. Parametrized curves in Lagrange Grassmannians. CR Acad Sci Paris, Ser I. 2007;345(11):647–652.
15. Li C, Zelenko I. Jacobi equations and comparison theorems for corank 1 sub-Riemannian structures with symmetries. J Geom Phys. 2011;61:781–807.
16. Pontryagin LS, Boltyanskii VG, Gamkrelidze RV, Mischenko EF. The mathematical theory of optimal processes. New York: Wiley; 1962.
17. Shen Z. Lectures on Finsler geometry. Singapore: World Scientific; 2001.

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