SINGULARITIES OF ADMISSIBLE NORMAL FUNCTIONS

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ABSTRACT. In a recent paper, M. Green and P. Griffiths used R. Thomas’ work on nodal hypersurfaces to sketch a proof of the equivalence of the Hodge conjecture and the existence of certain singular admissible normal functions. Inspired by their work, we study normal functions using M. Saito’s mixed Hodge modules and prove that the existence of singularities of the type considered by Griffiths and Green is equivalent to the Hodge conjecture. Several of the intermediate results, including a relative version of the weak Lefschetz theorem for perverse sheaves, are of independent interest.

1. INTRODUCTION

Let $S$ be a complex manifold. A variation of pure Hodge structure $\mathcal{H}$ of weight $-1$ on $S$ induces a family of compact complex tori $\pi: J(\mathcal{H}) \to S$. Let $\mathcal{C}_S$ denote the sheaf of continuous functions on $S$, $\mathcal{O}^\text{an}_S$ the sheaf of holomorphic functions on $S$, and $\mathcal{J}(\mathcal{H})$ the sheaf of continuous sections of $\pi$. The exact sequence

$$0 \to \mathcal{H}_S \to \mathcal{H}_S \otimes \mathcal{C}_S / F^0 \mathcal{H} \otimes \mathcal{O}^\text{an}_S \to \mathcal{J}(\mathcal{H}) \to 0$$

of sheaves of abelian groups on $S$ induces a long exact sequence in cohomology. Writing $\text{cl}_Z: H^0(S, \mathcal{J}(\mathcal{H})) \to H^1(S, \mathcal{H}_S)$ for the first connecting homomorphism, we find that, to each continuous section $\nu$ of $\pi$, we can associate a cohomology class $\text{cl}_Z(\nu) \in H^1(S, \mathcal{H}_S)$.

Assume now that $j: S \to \overline{S}$ is an embedding of $S$ as a Zariski open subset of a complex manifold $\overline{S}$ [Sai96, Definition 1.4]. If $U$ is an (analytic) open neighborhood of a point $s \in S(C)$, we can restrict $\text{cl}_Z(\nu)$ to $U \cap S$ to obtain a class in $H^1(U \cap S, \mathcal{H}_S)$. Taking the limit over all open neighborhoods $U$ of $s$, we obtain a class

$$\sigma_{Z,s}(\nu) \in \colim_{s \in U} H^1(U \cap S, \mathcal{H}_S).$$

We call this class the singularity of $\nu$ at $s$, and we say that $\nu$ is singular on $\overline{S}$ if there exists a point $s \in S$ with a non-torsion singularity $\sigma_{Z,s}(\nu)$.

In this paper, we will study $\sigma_{Z,s}(\nu)$ when $\nu$ is a normal function; that is, a horizontal holomorphic section of $\pi$. In fact, we will restrict our attention to admissible normal functions which are normal functions satisfying a very restrictive (but, from the point of view of algebraic geometry, very natural) constraint on their local monodromy. These normal functions were systematically studied by Saito in [Sai96].

Now suppose $X$ is a projective complex variety of dimension $2n$ with $n$ an integer. Let $\mathcal{L}$ be a very ample invertible sheaf on $X$, and let $\zeta \in \text{Hodge}^{2n}(X)$ :=

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\[H^n(X) \cap H_{2n}(X, \mathbb{Z}(n))\] be a primitive Hodge class; that is, assume that \(c_1(\mathcal{L}) \cup \zeta = 0\). Recall that the map \(p: H^n_{\mathbb{Z}}(X, \mathbb{Z}(n)) \rightarrow H^n_{\mathbb{Z}}(X)\) from Deligne cohomology to Hodge classes is surjective. To any \(\omega \in H^n_{\mathbb{Z}}(X, \mathbb{Z}(n))\) such that \(p(\omega) = \zeta\), one can associate a normal function \(v(\omega, \mathcal{L})\) on the complement \(|\mathcal{L}| - X'\) of the dual variety \(X'\) in the complete linear system \(|\mathcal{L}|\). This function takes a point \(f \in |\mathcal{L}| - X'\) to the restriction of \(\omega\) to \(H^n_{\mathbb{Z}}(V(f), \mathbb{Z}(n))\) where \(V(f)\) denotes the zero locus of \(f\). Since \(\zeta\) is primitive, \(\omega|_{V(f)}\) lands in \(J(H^{2n-1}(V(f))(n))\), a subgroup of \(H^n_{\mathbb{Z}}(V(f), \mathbb{Z}(n))\). Moreover, if \(\omega'\) is another Deligne cohomology class such that \(p(\omega') = \zeta\), then \(v(\omega, \mathcal{L})\) is singular on \(|\mathcal{L}|\) if and only if \(v(\omega', \mathcal{L})\) is singular (see [4]). We say that \(\zeta\) is singular on \(|\mathcal{L}|\) if \(v(\omega, |\mathcal{L}|)\) is singular on \(|\mathcal{L}|\) for some \(\omega \in H^n_{\mathbb{Z}}(X, \mathbb{Z}(n))\) such that \(p(\omega) = \zeta\).

**Conjecture 1.2.** Let \(X\) and \(\mathcal{L}\) be as above. For every non-torsion primitive Hodge class \(\zeta\), there is an integer \(k\) such that \(\zeta\) is singular on \(|\mathcal{L}^k|\).

In this paper, we prove the following result motivated by the work of Green and Griffiths [GG07].

**Theorem 1.3.** Conjecture [1.2] holds (for every even dimensional \(X\) and every non-torsion primitive middle dimensional Hodge class \(\zeta\)) if and only if the Hodge conjecture holds (for all smooth projective algebraic varieties).

In the paper of Green and Griffiths [GG07], an analogous result is stated. The arguments of Green and Griffiths rely on R. Thomas’s paper [Tho05] which shows that the Hodge conjecture is equivalent to the statement that every non-torsion Hodge class \(\zeta\) in an even dimension smooth projective complex variety \(X\) has non-zero restriction to some divisor \(D\) in \(X\) which is smooth outside of finitely many nodes. Our proof of Theorem 1.3 does not use Thomas’ result concerning nodal hypersurfaces. It relies instead on the theory of admissible normal functions and the “Gabber decomposition theorem” in Morihiko Saito’s theory of mixed Hodge modules [Sai89]. More importantly, the argument of Green and Griffiths relies on Hironaka’s resolution of singularities to modify \(|\mathcal{L}^k|\) so that \(X'\) becomes a normal crossing divisor. This makes the argument of Green and Griffiths somewhat less explicit than one would hope.

We have two intermediate results which may be particularly interesting in their own right. The first is Lemma 2.18 which gives a criterion for the intermediate extension functor \(j^!\) of [BBDS2] to preserve the exactness of a sequence of mixed Hodge modules. The second is Theorem 5.2 which we call the “perverse weak Lefschetz.” It is a relative weak Lefschetz for families of hypersurfaces.

The organization of this paper is as follows. In [2] we study the general properties of admissible normal functions and their singularities. In particular, we show that the singularity is always a Tate class which lies in the local intersection cohomology, a subgroup of the local cohomology. In [3] we generalize Saito’s definition of absolute Hodge cohomology slightly. In [4] we introduce some notation concerning the decomposition theorem of Beilinson-Bernstein-Deligne-Gabber and Saito. In [5] we prove the perverse weak Lefschetz theorem alluded to above and use it to compute the singularity of a normal function associated to a primitive Hodge class (as in Conjecture [1.2]) in terms of restriction of the Hodge class to a hyperplane. In [6] we prove Theorem 1.3.
The last section, §7, links our work directly to that of Green and Griffiths [GG07]. Doing this involves showing that singularities of admissible normal functions do not disappear after modification of the base. Unfortunately, we have been unable to prove that this is the case for all admissible normal functions. However, by the work of Thomas’ work alluded to above, we have been able to show that this is the case for the types of singularities occurring in [GG07]. This answers a question of Green and Griffiths (see note at bottom of [GG07] p. 225).

**Notation.** A complex variety will mean an integral separated scheme $X$ of finite type over $\mathbb{C}$. Following Saito, we write $d_X$ for $\dim X$ to shorten some of the expressions. If $\mathcal{E}$ is a locally free sheaf on $X$ and $s \in \Gamma(X, \mathcal{E})$, we write $V(s)$ for the zero locus of $s$ [Har77].

By a perverse sheaf we mean a perverse sheaf for the middle perversity. If $f: X \to Y$ is a morphism between complex varieties, we write $f_*, f_!$, $f^*$, $f^!$ for the derived functors between the bounded derived categories of constructible sheaves following the convention of [BBD82] 1.4.2.3. However, we deviate slightly from this convention is §7 where we write $f_* \mathcal{F}$ (instead of $0^\mathcal{H} f_* \mathcal{F}$) for the usual push-forward of a constructible sheaf $\mathcal{F}$.

We write MHS for Deligne’s category of mixed Hodge structures. When necessary for clarity, we write $\text{MHS}_R$ for the category of mixed Hodge structures with coefficients in a ring $R$. Similarly, we write $\text{VMHS}(S)$ or $\text{VMHS}_R(S)$ for the category of variations of mixed Hodge structures with $R$ coefficients over a separated analytic space $S$.

**Remark 1.4.** The reader might guess that analogues of the results in this paper can be obtained in characteristic $p$ by replacing mixed Hodge modules by mixed perverse sheaves. Indeed this is the case. To the best of our knowledge, in proving our key intermediate results we have used no fact about mixed Hodge modules that is not the direct analogue of a corresponding fact about mixed perverse sheaves.

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2. Admissible Normal Functions and Intersection Cohomology

Let $j: S \to \overline{S}$ be an open immersion of smooth complex manifolds. If $E$ is a local system of $\mathbb{Q}$-vector spaces on $S$ and $s \in \overline{S}$ is a closed point, we set

$$H^i_s(E) := \text{colim}_{s \in U} \pi_0^i \left( S \cap U, E \right)$$

where the colimit is taken over all open neighborhoods $U$ of $s$. If $i: \{s\} \to \overline{S}$ denotes the inclusion morphism, then $H^i_s(E) = H^i(\{s\}, i^* \mathcal{F})$. (We ask the reader to distinguish between the integer $i$ and the morphism $i$ based on the context.)
2.1. Now assume that $S$ and $\overline{S}$ are both equidimensional of dimension $d$ and that $j$ is an open immersion. The local system $E$ defines a perverse sheaf $E[d]$ on $S$ (since $S$ is smooth). Moreover, by intermediate extension, it defines a perverse sheaf $j_*E_{[d]}$ on $\overline{S}$. Adopting the standard notation, we set

$$IH^i(\overline{S}, E) = H^{i-d}(\overline{S}, j_*, E[d])$$

$$IH^i(E) = H^{i-d}((s, i^*j_*E[d])].$$

Note that, $j_*E[d]$ maps to $j_*E[d]$: it is defined as a subobject of $\mathfrak{p}j_*E[d] := \mathfrak{p}H^d(j_*E[d])$ in the category of perverse sheaves and $\mathfrak{p}j_*$ is left $t$-exact. Therefore we have natural maps

$$IH^i(\overline{S}, E) \to H^i(S, E); \quad IH^i(E) \to H^i(E).$$

**Lemma 2.2.** With $E$, $S$ and $\overline{S}$ as in (2.1), we have

$$IH^0(\overline{S}, E) = H^0(S, E),$$

$$IH^0(E) = H^0(E),$$

$$IH^1(\overline{S}, E) \hookrightarrow H^1(S, E),$$

$$IH^1(E) \hookrightarrow H^1(E).$$

**Proof:** Since $\mathfrak{p}j_*$ is left $t$-exact, we have a distinguished triangle

$$\mathfrak{p}j_*E[d] \to j_*E[d] \to \mathfrak{p}\tau_{\geq 1}j_*E[d] \to \mathfrak{p}j_*E[d + 1].$$

By [BBD82 (2.1.2.1)], $H^i(\mathfrak{p}\tau_{\geq 1}j_*E[d]) = 0$ for $i \leq -d$. Therefore, the map $\mathfrak{p}j_*E[d] \to j_*E[d]$ induces isomorphisms

$$H^i(\overline{S}, \mathfrak{p}j_*E[d]) \to H^i(S, E[1]),$$

$$H^i_*(\mathfrak{p}j_*E[d]) \to H^i_*(j_*E[d])$$

for $i \leq -d$. Moreover, we have injections $H^{-d+1}(\overline{S}, \mathfrak{p}j_*E[d]) \to H^{-d+1}(S, E[d])$ and $H^{-d+1}_*(\mathfrak{p}j_*E[d]) \to H^{-d+1}_*(j_*E[d]).$

Similarly, there is an exact sequence

$$0 \to j_*E[d] \to \mathfrak{p}j_*E[d] \to F \to 0$$

in $\text{Perv}(S)$ where $F$ is a perverse sheaf supported on $\overline{S} \setminus S$. It follows that $H^i(F) = 0$ for $i \leq -d$. The result now follows immediately from the long exact sequence in cohomology (resp. local cohomology at $s$) induced by (2.4). \hfill $\square$

2.5. Now suppose that $j : S \to \overline{S}$ of (2.1) is an open immersion of $S$ as a Zariski open subset of $\overline{S}$ [Sai96, Definition 1.4]. Furthermore, suppose that $\mathcal{H}$ is a variation of Hodge structure of weight $-1$ on $S$. We write $\text{NF}(S, \mathcal{H})$ for the group of normal functions from $S$ into $J(\mathcal{H})$. By [Sai96], there is a canonical isomorphism $\text{NF}(S, \mathcal{H}) = \text{Ext}^1_{\text{VMHS}(S)}(\mathbb{Z}, \mathcal{H})$. Moreover, if we let $\text{VMHS}(S)^{\text{ad}}_S$ denote the subcategory of variations of mixed Hodge structure on $S$ which are admissible with respect to the open immersion $j : S \to \overline{S}$, then the group $\text{Ext}^1_{\text{VMHS}(S)^{\text{ad}}}(\mathbb{Z}, \mathcal{H})$ is a subgroup of $\text{NF}(S, \mathcal{H})$. Following [Sai96, Definition 1.4], we call these the **admissible normal functions with respect to $\overline{S}$ and write $\text{NF}(S, \mathcal{H})^{\text{ad}}_\overline{S}$ for this group.
Fact 2.6. Let \( v \in \text{NF}(S, \mathcal{H}) \) be a normal function on \( S \). Let \( \text{Shv}(S) \) denote the category of all sheaves on \( S \) and write \( r : \text{VMHS}(S) \to \text{Shv}(S) \) for the forgetful functor taking a variation of mixed Hodge structure \( \mathcal{H} \) on \( S \) to its underlying sheaf of abelian groups \( \mathcal{H}_\mathbb{Z} \). Then \( \text{cl}(v) \) is the image of \( v \) under the composition

\[
\text{NF}(S, \mathcal{H}) = \text{Ext}^1_{\text{VMHS}(S)}(\mathbb{Z}, \mathcal{H}) \xrightarrow{\pi_0} \text{Hom}_{\text{Shv}(S)}(\mathcal{H}, \mathbb{Q}) = \text{Ext}^1_{\text{Shv}(S)}(\mathbb{Z}, \mathcal{H}_\mathbb{Z}) = H^1(S, \mathcal{H}_\mathbb{Z}).
\]

We leave the (straightforward) verification of the above statement to the reader.

2.7. If \( v \in H^0(S, J(\mathcal{H})) \) is a continuous section of the complex torus \( J(\mathcal{H}) \), we write \( \text{cl}(v) \) for the image of \( \text{cl}(v) \) in \( H^1(S, \mathcal{H}_\mathbb{Q}) \). If \( s \in \mathbb{S} \) with \( \mathbb{S} \) as in (2.5), we write \( \sigma_s(v) \) for the image of \( \sigma_s(v) \) in \( H^1(S, \mathcal{H}_\mathbb{Q}) \).

The following is a type of “universal coefficient theorem” for variations of mixed Hodge structure and normal functions.

Lemma 2.8. Let \( S \) be as in (2.5)

(i) Let \( \mathcal{V} \) and \( \mathcal{W} \) be variations of mixed Hodge structure on \( S \). If \( \pi_0(S) \) is finite, then the natural map

\[
\text{Hom}_{\text{VMHS}(S)}(\mathcal{V}, \mathcal{W}) \otimes \mathbb{Q} \to \text{Hom}_{\text{VMHS}(S)}(\mathcal{V}_\mathbb{Q}, \mathcal{W}_\mathbb{Q})
\]

is an isomorphism.

(ii) If \( \pi_0(S) \) is finite and \( \pi_1(S, s) \) is finitely generated for each \( s \in S \), then the natural map

\[
\text{Ext}^1_{\text{VMHS}(S)}(\mathcal{V}, \mathcal{W}) \otimes \mathbb{Q} \to \text{Ext}^1_{\text{VMHS}(S)}(\mathcal{V}_\mathbb{Q}, \mathcal{W}_\mathbb{Q})
\]

is an isomorphism.

(iii) If the conditions of (ii) are satisfied, then, for any variation of pure Hodge structure \( \mathcal{H} \) of weight \(-1\) on \( S \), the natural map

\[
\text{NF}(S, \mathcal{H}) \otimes \mathbb{Q} = \text{Ext}^1_{\text{VMHS}(S)}(\mathcal{V}, \mathcal{W}) \otimes \mathbb{Q} \to \text{Ext}^1_{\text{VMHS}(S)}(\mathcal{V}_\mathbb{Q}, \mathcal{W}_\mathbb{Q})
\]

is an isomorphism.

Proof: (i) is obvious, and (iii) follows directly from (ii). We leave to the reader the fact that the map in (ii) is injective. To see that it is surjective, suppose

\[
0 \to \mathcal{W}_\mathbb{Q} \to \mathcal{V}_\mathbb{Q} \xrightarrow{p} \mathbb{Q} \to 0
\]

is an exact sequence of rational variations of mixed Hodge structure on \( S \). Assume first that \( S \) is connected. Then, using the fact that \( \pi_1(S, s) \) is finitely generated, we can find a lattice \( \mathcal{V}_\mathbb{Z} \subset \mathcal{V} \) such that \( \mathcal{V}_\mathbb{Z} \cap \mathcal{W}_\mathbb{Q} = \mathcal{W} \). We then have \( p(\mathcal{V}_\mathbb{Z}) = \alpha \mathcal{Z} \) for some \( \alpha \in \mathbb{Q}^\times \). Scaling by \( \alpha \) we obtain the desired result.

We leave the case where \( S \) has finitely many connected components (where we may have to scale by more than one \( \alpha \) and add up the results) to the reader.

Corollary 2.9. Under the assumptions of Lemma 2.8 and the notation of (2.5), we have

\[
\text{NF}(S, \mathcal{H})_{\mathcal{Z}} \otimes \mathbb{Q} = \text{Ext}^1_{\text{VMHS}(S)}(\mathcal{V}, \mathcal{W}_\mathbb{Q}).
\]

Proof: This follows directly from the Lemma 2.8 because admissibility of variations of a mixed Hodge structure \( \mathcal{V} \) depends only on \( \mathcal{V}_\mathbb{Q} \).
Definition 2.10. We call an element \( \nu \in \text{Ext}^1_{\text{VMHS}(\mathcal{S})}^\text{ad}(\mathbb{Q}, \mathcal{H}_Q) \) an \textit{admissible} \( \mathbb{Q} \)-normal function.

The main result of this section is the following.

Theorem 2.11. Let \( j : S \to \mathcal{S} \) be an open immersion of smooth manifolds as in (2.5) and let \( \mathcal{H} \) be a variation of pure Hodge structure of weight \(-1\) on \( S \). The group homomorphism \( \text{cl}_Q : \text{NF}(S, \mathcal{H})^\text{ad}_S(\mathbb{Q}, \mathcal{H}_Q) \to H^1(S, \mathcal{H}_Q) \) factors through \( \text{IH}^1_S(\mathcal{H}_Q) \). Similarly, for each \( s \in \mathcal{S} \), the map \( \sigma_s : \text{NF}(S, \mathcal{H})^\text{ad}_S(\mathbb{Q}, \mathcal{H}_Q) \to H^1_s(\mathcal{H}_Q) \) factors through \( \text{IH}^1_s(\mathcal{H}_Q) \).

We will use a few lemmas concerning the intermediate extensions of perverse sheaves and mixed Hodge modules on \( S \). The first concerns the fact that \( j^! \) is "End-exact" when applied to perverse sheaves on \( S \); that is, it preserves injections and surjections. In N. Katz’s book [Kat96, p. 87], this fact is stated and a proof is sketched. For completeness and the convenience of the reader, we give a proof here.

Lemma 2.12. Let \( j : S \to \mathcal{S} \) be an open immersion as in 2.11. Suppose that the sequence
\[
0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0
\]
is exact in \( \text{Perv}(S) \). Then \( j^!(f) \) is an injection and \( j^!(g) \) is a surjection in \( \text{Perv}(\mathcal{S}) \).

Proof. By [BBD82, Prop 1.4.16], \( p^j \) is right-exact and \( p^j_! \) is left-exact. From the definition of the intermediate extension functor ([BBD82, 2.1.7], we have the following commutative diagram with exact top and bottom rows.

The proposition now follows from chasing the diagram. \( \square \)

2.13. For \( "\_" \) a separated reduced analytic space, we write \( \text{MHM}(\_\_) \) for the category of mixed Hodge modules on \( "\_" \) and \( \text{MHM}(\_\_)^p \) for the category of polarizable mixed Hodge modules [Sai90, 2.17.8]. (It is understood that a left upper \( p \) stands for “perversity”, while a right upper \( p \) stands for “polarization” in this paper.) If \( j : S \to \mathcal{S} \) is an open immersion as in (2.5), then we write \( \text{MHM}(S)^p_\mathcal{S} \) for the category of polarizable mixed Hodge modules on \( S \) which are extendable to \( \mathcal{S} \). Recall that a mixed Hodge module \( M \) in \( \text{MHM}(S) \) is said to be \textit{smooth} if \( \text{rat} M \) is isomorphic to \( E[d_S] \) where \( E \) is a local system on \( S \) where \( \text{rat} : \text{MHM}(S) \to \text{Perv}(S) \) denotes the functor of [Sai90, Theorem 0.1]. By [Sai90, Theorem 3.27] we have an equivalence of categories
\[
\text{VMHS}(S)^{\text{ad}}_\mathcal{S} \cong \text{MHM}(S)^p_\mathcal{S}
\]
where the right hand denotes the full subcategory of \( \text{MHM}(S)^p_\mathcal{S} \) consisting of smooth mixed Hodge modules.
Definition 2.14. If \( a, c \in \mathbb{Z} \), then we say that an object \( M \) in \( \text{MHM}(\_\_) \) has weights in the interval \([a, c]\) if \( \text{Gr}^w_i M = 0 \) for \( i \not\in [a, c] \).

We write \( j_! : \text{MHM}(S) \to \text{MHM}(\overline{S}) \) for the functor given by

\[
j_! M = \text{im}(H^0 j_0 M \to H^0 j_* M).
\]

By [Sai90] 2.18.1, both \( j_! \) and \( j_* \) preserve polarizability. Therefore, for \( M \) in \( \text{MHM}(S)^p \), \( j_! M \) is in \( \text{MHM}(\overline{S})^p \).

Lemma 2.15. If \( M \) is an object in \( \text{MHM}(S)^p \) with weights in the interval \([a, c]\), then \( j_* M \) also has weights in \([a, c]\).

Proof. By [Sai90] Proposition 2.26, \( H^0 j_! M \) has weights \( \leq c \) and \( H^0 j_* M \) has weights \( \geq a \). Since maps between polarizable mixed Hodge modules are strict with respect to the weight filtration, the functor \( \text{Gr}^w : \text{MHM}(\overline{S})^p \to \text{MHM}(\overline{S})^p \) is exact [Del71] Proposition 1.1.11) for each \( i \in \mathbb{Z} \). It follows that \( j_* M = \text{im}(H^0 j_0 M \to H^0 j_* M) \) has weights in \([a, c]\). \( \square \)

2.16. The functor \( j_* \) is not in general exact. However, for \( C, A \) pure of respective weights \( c \) and \( a \) in \( \text{MHM}(S)^p \),

\[
\text{Ext}^i(C, A) = 0 \text{ if } c < a + j.
\]

This is stated explicitly in the algebraic case in [Sai90] Eq. 4.5.3; however, the proof given there clearly applies to the polarizable analytic case.

From this and the fact that \( j_* \) commutes with finite direct sums, we see that \( j_* \) preserves the exactness of the sequence

\[
0 \to A \xrightarrow{j_!} B \xrightarrow{j_*} C \to 0
\]

provided \( A \) is pure of weight \( a \) and \( C \) is pure of weight \( c \) with \( c < a + 1 \).

Lemma 2.18. Suppose that the entries in (2.17) consist of objects in \( \text{MHM}(S)^p \) where \( A \) is pure of weight \( a \) and \( C \) is pure of weight \( c \) with \( c \leq a + 1 \). Then the sequence

\[
0 \to j_* A \xrightarrow{j_!(f)} j_* B \xrightarrow{j_!(g)} j_* C \to 0
\]

is exact in \( \text{MHM}(\overline{S})^p \).

Proof. Write \( i : Z \to \overline{S} \) for the complement of \( S \) in \( \overline{S} \). The lemma will follow mainly from [BBD82] Corollary 1.4.25] which gives the following description of the intermediate extension in our context.

(*) \( j_* B \) is the unique prolongement of \( B \) in \( \text{MHM}(\overline{S}) \) with no non-trivial sub-object or quotient object in the essential image of the functor \( i_* : \text{MHM}(Z) \to \text{MHM}(\overline{S}) \).

Here we have used the fact that \( \text{rat} : \text{MHM}(\_\_) \to \text{Perv}(\_\_) \) is faithful and exact to deduce (*) from the corresponding statement in [BBD82].

By (2.16), we already know that the theorem holds for \( c \leq a \); thus, it suffices to consider the case \( c = a + 1 \).

By Lemma 2.15 we know that \( j_* B \) has weights in the interval \([a, c]\). By Lemma 2.12 and the exactness of \( \text{Gr}^w \), we know that \( \text{Gr}^w_c j_* B = j_* C \oplus D \) for some object \( D \) in \( \text{MHM}(\overline{S})^p \) which is pure of weight \( c \). By the definition of \( j_* B \),
we know that $D$ is supported on $Z$. But, since $j_! B$ surjects onto $D$ via the composition

$$j_* B \twoheadrightarrow \text{Gr}_c^W j_! B \twoheadrightarrow D$$

this contradicts (*) unless $D = 0$.

Thus $\text{Gr}_c^W j_* B = j_! C$. By similar reasoning, we see that $\text{Gr}_d^W j_* B = j_! A$. □

**Lemma 2.20.** Let $S$ be as in Theorem 2.11 Then the functor $\text{VMHS}_Q (S)[w] \mapsto \text{MHM}(S)_S^P$ sending a variation $\mathcal{V}$ to $\mathcal{V}[d]$ induces isomorphisms

$$\text{Ext}^i_{\text{VMHS}_Q(S)[w]}(\mathcal{V}, \mathcal{W}) \cong \text{Ext}^i_{\text{MHM}(S)_S^P}((\mathcal{V}[d], \mathcal{W}[d])$$

for $i = 0, 1$.

**Proof.** For $i = 0$ this follows from [Sai90, Theorem 3.27]. For $i = 1$, this follows from the (easy) fact that an extension of smooth perverse sheaves is smooth. □

**Corollary 2.21.** Suppose $j : S \to \mathcal{N}$ and $\mathcal{H}$ are as in Theorem 2.11 Then the restriction map

$$\text{Ext}_{\text{MHM}(S)_S^P}^1(Q[d], j_* \mathcal{H}_Q[d]) \xrightarrow{j^*} \text{Ext}_{\text{MHM}(S)_S^P}^1(Q[d], j_* \mathcal{H}_Q[d]) = \text{NF}(S, \mathcal{H})[w] \otimes Q$$

is an isomorphism.

**Proof.** Lemma 2.18 shows that $j^*$ is surjective. On the other hand, suppose $\nabla \in \text{Ext}_{\text{MHM}(S)_S^P}^1(Q[d], j_* \mathcal{H}_Q[d])$ given by the sequence

$$0 \to j_! \mathcal{H}_Q[d] \to B \to Q[d] \to 0$$

is in the kernel of $j^*$. Then there is a splitting $i : Q[d] \to j^* B$. Applying $j_!$ to $s$, we obtain a splitting $Q[d] \to j_! j^* B$. But it is easy to see from Lemma 2.18 that $B = j_! j^* B$ (as both are extensions of $Q[d]$ by $j_! \mathcal{H}_Q[d]$). Therefore $\nabla = 0$. It follows that $j^*$ is injective. □

**Proof of Theorem 2.11** The diagram

$$\begin{align*}
\text{Ext}_{\text{MHM}(S)_S^P}^1(Q[d], j_* \mathcal{H}_Q[d]) & \xrightarrow{j^*} \text{Ext}_{\text{MHM}(S)_S^P}^1(Q[d], \mathcal{H}_Q[d]) \\
\text{rat} & \quad \text{rat} \\
\text{IH}^1(S, \mathcal{H}_Q) & \xrightarrow{j^*} \text{H}^1(S, \mathcal{H}_Q)
\end{align*}$$

commutes. The assertions in Theorem 2.11 are, thus, a direct consequence of the fact that the arrow on top is an isomorphism (2.21). □

2.23. Suppose $H$ is a $Q$-mixed Hodge structure. We call a class $v \in H_Q$ Tate of weight $w$ if it can be expressed as the image of 1 under a morphism $Q(-w/2) \to H$ of Hodge structures (for some even integer $w$).

**Theorem 2.24.** Let $\mathcal{H}$ be a variation of pure Hodge structure as in Theorem 2.11 Then, for $s \in \mathcal{N}$, the class $\sigma_s(v) \in \text{IH}^1(S, \mathcal{H}_Q)$ is Tate of weight 0.
To prove Theorem 2.24, we are going to use a general fact about mixed Hodge modules on reduced separated schemes of finite type over \( \mathbb{C} \); that is, we use a result from the theory of mixed Hodge modules in the algebraic case. If \( X \) is such a scheme, we write \( \text{MHM}(X) \) for the category of mixed Hodge modules on \( X \). If \( X \) is any proper scheme in which \( X \) is embedded as an open subscheme, then the category \( \text{MHM}(X) \) is equivalent to the category \( \text{MHM}(X^{an})^{\text{rat}}_{\mathbb{C}} \). Here, as in [Sai90, p. 313] where this statement is proved, \( X^{an} \) denotes the underlying analytic space associated to \( X \).

**Fact 2.25.** Let \( X \) be a reduced separated scheme of finite type over \( \mathbb{C} \), and let \( M \) and \( N \) be objects in \( \text{D}B \text{HM}(X) \). Then there is a natural Hodge structure on the group \( \text{Hom}_{\text{D}B \text{HM}(X)}(\text{rat}M, \text{rat}N) \) and the image of the natural map

\[
\text{Hom}_{\text{D}B \text{HM}(X)}(M,N) \xrightarrow{\text{rat}} \text{Hom}_{\text{D}B \text{Perv}(X)}(\text{rat}M, \text{rat}N)
\]

consists of Tate classes of weight 0.

**Sketch.** Let \( \pi : X \to \text{Spec} \mathbb{C} \) denote the structure morphism. Then

\[
(2.26) \quad \text{Hom}_{\text{D}B \text{Perv}(X)}(\text{rat}M, \text{rat}N) = \text{rat}H^B \pi_! \text{Hom}(M,N)
\]

where \( \text{Hom}(M,N) \) denotes the internal Hom in \( \text{D}B \text{HM}(X) \). Since \( \text{MHM}(\text{Spec} \mathbb{C}) \) is equivalent to the category of mixed Hodge structures with \( \text{rat} \) taking a Hodge structure to its underlying \( \mathbb{Q} \)-vector space, the above isomorphism puts a mixed Hodge structure on \( \text{Hom}_{\text{D}B \text{Perv}(X)}(\text{rat}M, \text{rat}N) \). We leave the rest of the verification to the reader. \( \square \)

**Proof of Theorem 2.24.** Given a \( \nu \in \text{NF}(\mathcal{S}, \mathcal{H})^{\text{dR}}_{\mathcal{S}} \), let \( \nabla \in \text{Ext}_{\text{MHM}(\mathcal{S})}^1(\mathbb{Q}[d], j_* \mathcal{H}_Q)[d] \) denote the unique class such that \( j^! \nabla = \nu \) (2.21). Let \( i : \{s\} \to \mathcal{S} \) denote the inclusion morphism. Then, by Theorem 2.11 \( \nu_i(\nu) \) is the image of \( \nabla \) in \( \text{IH}_i^1(\mathcal{H}_Q) = \text{Ext}_{\text{Perv}(\{s\})}^1(\mathbb{Q}[d], i^! j_* \mathcal{H}_Q[d]) \) under the composition

\[
\text{Hom}_{\text{D}B \text{HM}(\mathcal{S})}^1(\mathbb{Q}[d], (j_* \mathcal{H}_Q[d])[1]) \xrightarrow{i^*} \text{Hom}_{\text{D}B \text{HM}(\{s\})}^1(\mathbb{Q}[d], i^! (j_* \mathcal{H}_Q[d])[1]) \xrightarrow{\text{rat}} \text{Hom}_{\text{D}B \text{Perv}(\{s\})}^1(\mathbb{Q}[d], i^! (j_* \mathcal{H}_Q[d])[1]).
\]

By (2.25), the result follows. \( \square \)

### 3. Absolute Hodge Cohomology

#### 3.1. For a separated scheme \( Y \) of finite type over \( \mathbb{C} \) let \( a_Y : Y \to \text{Spec} \mathbb{C} \) denote the structure morphism and let \( \mathbb{Q}(p) \) denote the Tate object in \( \text{MHS} = \text{MHM}(\text{Spec} \mathbb{C}) \). Let \( \mathbb{Q}_Y(p) := a_Y^! \mathbb{Q}(p) \) in \( \text{D}B \text{HM}(Y) \). (To simplify notation, we write \( \mathbb{Q}(p) \) for \( \mathbb{Q}_Y(p) \) when no confusion can arise.) For an object \( M \) in \( \text{D}B \text{HM}(Y) \), set

\[
H^*_p(Y,M) = \text{Hom}_{\text{D}B \text{HM}(Y)}(\mathbb{Q},M[n]).
\]

The functor \( \text{rat} : \text{MHM}(Y) \to \text{Perv}(Y) \) induces a “cycle class map”

\[
\text{rat} : H^*_p(Y,M) \to H^*(Y,M)
\]

to the hypercohomology of \( \text{rat}M \). Note that \( H^*_p(Y,\mathbb{Q}(p)) = H^*_p(Y,\mathbb{Q}(p)) \) for \( Y \) smooth and projective and in this case \( \text{rat} \) is simply the cycle class map from
Deligne cohomology. Following Saito [Sai91], we will call $H^n_{\text{ad}}(Y, M)$ the absolute Hodge cohomology of $M$.

3.2. Suppose $j : S \rightarrow \overline{S}$ is the inclusion of a Zariski open subset of a smooth complex algebraic variety and $s \in \overline{S}(\mathbb{C})$. Let $i : \{s\} \rightarrow \overline{S}$ denote the inclusion. If $\mathcal{H}$ is an admissible variation of mixed Hodge structure on $S$, we adopt the notation of (2.1) and write

$$IH^n_{\text{ad}}(S, \mathcal{H}) = \text{Hom}_{\text{D}^b\text{MHM}(S)}(\mathbb{Q}[d_{S} - n], j_! \mathcal{H}[d_{S}])$$

$$IH^n_{\text{ad}, s}(\mathcal{H}) = \text{Hom}_{\text{D}^b\text{MHS}}(\mathbb{Q}[d_{S} - n], i^* j_! \mathcal{H}[d_{S}]).$$

We can now amplify Theorem 2.11.

Proposition 3.3. Let $j : S \rightarrow \overline{S}$ be an open immersion of smooth complex varieties and let $\mathcal{H}$ be a variation of pure Hodge structure of weight $-1$ on $S$. Then, for $i : \{s\} \rightarrow S$ the inclusion of a closed point, the diagram

$$\xymatrix{ NF(S, \mathcal{H}) \otimes \mathbb{Q} \ar[r]^-{\sigma} \ar[d]_{\sigma} & IH^1_{\text{ad}}(\overline{S}, \mathcal{H}) \ar[r]^-{i_*} & IH^1_{\text{ad}}(S, \mathcal{H}) \ar[d]^{i^*} \\
H^1_*(\mathcal{H}) \ar[r] & IH^1_*(\mathcal{H}) & }$$

commutes.

Proof. This is consequence of (2.22), Corollary 2.21 and the notation of (3.1) which converts the top line of (2.22) into absolute Hodge cohomology groups.

\[\square\]

Remark 3.4. Since the map $IH^1_*(\mathcal{H}) \rightarrow H^1_*(\mathcal{H})$ is an injection by Lemma 2.2 and the map $\sigma_{p} : NF(S, \mathcal{H}) \rightarrow H^1_*(\mathcal{H})$ factors through $IH^1_*(\mathcal{H})$, we can write $\sigma_{p}(\nu)$ for the class of an admissible normal function $\nu$ in $IH^1_*(\mathcal{H})$.

4. THE DECOMPOSITION OF BEILINSON-BERNSTEIN-DELINE-GABBER & SAITO

Let $\pi : \mathcal{X} \rightarrow P$ denote a projective morphism between smooth complex algebraic varieties. The fundamental theorem alluded to in the title of this section states that there is a direct sum decomposition

$$\pi_* \mathbb{Q}[d_{\mathcal{X}}] = \oplus H^i(\pi_* \mathbb{Q}[d_{\mathcal{X}}])[-i]$$

in $\text{MHM}(P)$ [Sai89, Corollary 1.11]. Moreover, the object $\pi_* \mathbb{Q}[d_{\mathcal{X}}]$ in $\text{D}^b\text{MHM}(P)$ is pure of weight $d_{\mathcal{X}}$; equivalently, the mixed Hodge modules $H^i(\pi_* \mathbb{Q}[d_{\mathcal{X}}])$ occurring in the decomposition are pure of weight $d_{\mathcal{X}} + i$ [Sai88, Theorem 1].

Remark 4.2. The decomposition of (4.1) is not unique. However, we can (and do) require that it induces the identity map on the $H^i(\pi_* \mathbb{Q}[d_{\mathcal{X}}])$. In fact, there is a preferred choice of decomposition [Del68, Remark 1.8]. To fix ideas we will choose the preferred one.

4.3. The summands appearing in (4.1) can be further decomposed by codimension of strict support [Sai89, 3.2.6]: we can write

$$H^i(\pi_* \mathbb{Q}[d_{\mathcal{X}}]) = \oplus E_i(\pi)$$
where \( Z \) is a closed subscheme of \( P \) and \( E_{i,j}(\pi) \) is a Hodge module supported on \( Z \) with no sub or quotient object supported in a proper subscheme of \( Z \).

Let us set \( E_{ij}(\pi) = \oplus_{\text{codim}_P Z = j} E_{i,j}(\pi) \). We then have a decomposition

\[
\pi_* Q[d_x] = \oplus E_{ij}(\pi)[-i].
\]

We write \( E_{i,j} \) (resp. \( E_{ij} \)) for \( E_{i,j}(\pi) \) (resp. \( E_{ij}(\pi) \)) when there is no chance of confusion. We write \( \Pi_{ij} : \pi_* Q[d_x] \to E_{ij}[-i] \) for the projection map and \( S_{ij} : E_{ij}[-i] \to \pi_* Q[d_x] \) for the inclusion. (We suppress the indices and write \( \Pi \) and \( S \) instead of \( \Pi_{ij} \) and \( S_{ij} \) when no confusion can arise.)

**Observation 4.6.** Let \( p \in P(C) \) and form the pullback diagram

\[
\begin{array}{ccc}
\mathcal{X}_p & \xrightarrow{\iota_p} & \mathcal{X} \\
\downarrow{\pi_p} & & \downarrow{\pi} \\
\{p\} & \xrightarrow{\iota} & P.
\end{array}
\]

The decomposition in (4.5) gives decompositions

\[
\oplus \Pi_{ij} : H^s_{\mathcal{X}}(\mathcal{X}, Q[d_x]) \xrightarrow{\cong} \oplus \Pi H^{s-i}(P, E_{ij});
\]

\[
\oplus \Pi_{ij} : H^s_{\mathcal{X}}(\mathcal{X}, Q[d_x]) \xrightarrow{\cong} \oplus \Pi H_i^{s-i}(t_0^* E_{ij});
\]

\[
\Pi_{ij} : H^s(\mathcal{X}, Q[d_x]) \xrightarrow{\cong} \oplus \Pi H^{s-i}(P, E_{ij});
\]

\[
\Pi_{ij} : H^s(\mathcal{X}, Q[d_x]) \xrightarrow{\cong} \oplus \Pi H_i^{s-i}(E_{ij}).
\]

The restriction morphisms on cohomology \( H^s(\mathcal{X}, Q[d_x]) \to H^s(\mathcal{X}, Q[d_x]) \) and \( H_i^{s-i}(E_{ij}) \) are the direct sums of the morphisms

\[
H^{s-i}(P, E_{ij}) \to H_i^{s-i}(E_{ij})\text{ and }
\]

\[
H_i^{s-i}(P, E_{ij}) \to H_i^{s-i}(t_0^* E_{ij}).
\]

Furthermore, the morphism \( \iota \) commutes with restriction from \( \mathcal{X} \) to \( \mathcal{X}_P \). The above assertions follow from proper base change [Sai88 4.4.3] for the cartesian diagram (4.7) and the commutativity of \( \iota \) with the six functors of Grothendieck.

**Proposition 4.8.** With the notation of (4.5), let \( j : P^m \to P \) denote the largest Zariski open subset of \( P \) over which \( \pi \) is smooth, and let \( \pi^m : \mathcal{X}^m \to P^m \) denote the pull-back of \( \pi \) to \( P^m \). Then

\[
E_{i,j} = j_*((R^{i+p} d_x - d_p \pi^m Q)[d_P]).
\]

**Proof.** Set \( F = j_*((R^{i+p} d_x - d_p \pi^m Q)[d_P]) \). Clearly \( j^* E_{i,j} = (R^{i+p} d_x - d_p \pi^m Q)[d_P] \). Since \( E_{i,j} \) is pure, it follows that \( E_{i,j} \) contains \( F \) as a direct factor. Since any complement of \( F \) in \( E_{i,j} \) would have to be supported on a proper subscheme of \( P \), the proposition follows from the definition of \( E_{i,j} \). \( \square \)

**Corollary 4.9.** With the notation as in (4.8), set \( \mathcal{H} := R^s \pi^m Q \), a variation of Hodge structures of weight \( s \) on \( P^m \). Then

(i) The group \( H^s(P, \mathcal{H}) \) (resp. \( H^s_{\mathcal{X}}(P, \mathcal{H}_P) \)) is a direct factor in \( H^{s+t}(\mathcal{X}, Q) \) (resp. \( H_i^{s+t}(\mathcal{X}_P, Q) \));

(ii) for \( p \in P \), \( H^s_{\mathcal{X}_P}(\mathcal{H}_P) \) (resp. \( H^s_{\mathcal{X}_P}(\mathcal{H}_P) \)) is a direct factor in \( H^{s+t}(\mathcal{X}_P, Q) \) (resp. \( H_i^{s+t}(\mathcal{X}_P, Q) \)).
(iii) Moreover the morphism $\text{rat}$ is compatible with the morphisms $\Pi$ and $S$ inducing the direct factors.

Proof. This follows from directly from Observation 4.6. □

4.10. Using the notation of (4.4), write $Z_j(\pi) = \text{supp} E_{ij}(\pi)$ (and write $Z_i$ for $Z_i(\pi)$). Then $Z_j$ is a reduced closed subscheme of $P$ of codimension $j$. There exists an open dense subscheme $g_j : U_{ij} \rightarrow Z_j$ and a variation of pure Hodge structures $\mathcal{H}_j$ on $U_{ij}$ such that $E_{ij} = (g_{ij})_*\mathcal{H}_j[d_P - j]$. Clearly we can take $U_{i0} = P^{sm}$ and $\mathcal{H}_{i0} = \mathcal{H}_{i+\dim \mathfrak{X} - d_P}$.

**Hodge classes and normal functions.** The variation $\mathcal{H}_{2k-1}(k)$ on $P^{sm}$ is an admissible VMHS of weight $-1$ with respect to $P$ for each integer $k$. Then by Corollary 2.21

$$\text{IH}^j_{\text{Id}}(P, \mathcal{H}_{2k-1}(k)) = \text{NF}(P^{sm}, \mathcal{H}_{2k-1}(k))_{\text{prim}}^{\text{ad}}.$$  

By Corollary 4.9, the above is a direct factor in $H^{2k}_{\text{Id}}(\mathfrak{X}, Q(k))$. Therefore, the composition

$$N_k : H^{2k}_{\text{Id}}(\mathfrak{X}, Q(k)) \xrightarrow{\Pi} \text{IH}^j_{\text{Id}}(P, \mathcal{H}_{2k-1}(k)) = \text{NF}(P^{sm}, \mathcal{H}_{2k-1}(k))_{\text{prim}}^{\text{ad}}$$

associates an admissible $Q$-normal function to every absolute Hodge cohomology class.

For $k \in \mathbb{Z}$, write $H^{2k}(\mathfrak{X}, Q(k))_{\text{prim}}$ for the kernel of the composition

$$H^{2k}(\mathfrak{X}, Q(k)) \rightarrow H^{2k}(\mathfrak{X}^{sm}, Q(k)) \rightarrow H^0(P^{sm}, R^{2k}\pi, Q(k)).$$

In other words, $H^{2k}(\mathfrak{X}, Q(k))_{\text{prim}}$ consists of those classes $\alpha$ such that $\alpha|_{\mathfrak{X}_p} = 0$ for $p \in \mathfrak{X}(\mathbb{C})$ a point over which $\pi$ is smooth. Write

$$H^{2k}_{\text{Id}}(\mathfrak{X}, Q(k))_{\text{prim}} := \text{rat}^{-1} H^{2k}(\mathfrak{X}, Q(k))_{\text{prim}}.$$

Note that, for $p \in P^{sm}(\mathbb{C})$, the kernel of the map

$$\text{rat} : H^{2k}_{\text{Id}}(\mathfrak{X}_p, Q(k)) \rightarrow H^{2k}(\mathfrak{X}_p, Q(k))$$

consists of the intermediate Jacobian $J(\mathcal{H}_{2k-1}(k))_p = \text{Ext}^1_{\text{MHIS}}(Q, H^{2k-1}(\mathfrak{X}_p, Q(k)))$.

It follows that, for $\alpha \in H^{2k}_{\text{Id}}(\mathfrak{X}, Q(k))_{\text{prim}}$ and $p \in P^{sm}(\mathbb{C})$, $\alpha|_{\mathfrak{X}_p} = J(\mathcal{H}_{2k-1}(k))_p$.

**Fact 4.11.** For $\alpha \in H^{2k}_{\text{Id}}(\mathfrak{X}, Q(k))_{\text{prim}}$, $N_k(\alpha)(p) = \alpha|_{\mathfrak{X}_p}$.

**Sketch.** This is not hard to see using (2.3) and Remark 4.2, i.e. the fact that (4.1) induces the identity on cohomology. □

**Proposition 4.12.** Let $Z_k := \ker(\text{rat} : H^{2k}_{\text{Id}}(\mathfrak{X}, Q(k)) \rightarrow H^{2k}(\mathfrak{X}, Q(k)))$. Then, for each $p \in P$ and each $\alpha \in Z_k$, $\sigma_p(N_k(\alpha)) = 0$.

**Proof.** This follows from the commutativity of the diagram

$$\begin{array}{cccccc}
H^{2k}_{\text{Id}}(\mathfrak{X}, Q(k)) & \xrightarrow{\Pi} & \text{IH}^j_{\text{Id}}(P, \mathcal{H}_{2k-1}(k)) & = & \text{NF}(P^{sm}, \mathcal{H}_{2k-1}(k))_{\text{prim}}^{\text{ad}} & \text{rat}

\downarrow \text{rat} & & \downarrow \text{rat} & & \downarrow \sigma_p

H^{2k}(\mathfrak{X}, Q(k)) & \xrightarrow{\Pi} & \text{IH}^1(P, \mathcal{H}_{2k-1}(k)) & \rightarrow & \text{IH}^1(\mathcal{H}_{2k-1}(k)). &
\end{array}$$
□
4.13. Now suppose that $\mathcal{X}$ is projective. Then the image of $\text{rat} : H^{2k}(\mathcal{X}, \mathbb{Q}(k)) \to H^{2k}(\mathcal{X}, \mathbb{Q}(k))$ is exactly the subgroup Hodge$^k(\mathcal{X}) := H^{k,k}(\mathcal{X}) \cap H^{2k}(\mathcal{X}, \mathbb{Q}(k))$ of Hodge classes in $\mathcal{X}$. By Proposition [4,12] if $\alpha_1, \alpha_2$ are two classes in $H^{2k}(\mathcal{X}, \mathbb{Q}(k))$ such that $\text{rat}(\alpha_1) = \text{rat}(\alpha_2) \in H^{2k}(\mathcal{X}, \mathbb{Q}(k))$, then $\sigma_p(\alpha_1) = \sigma_p(\alpha_2)$ for each $p \in P$. In other words, the group homomorphism $\sigma_p : H^{2k}(\mathcal{X}, \mathbb{Q}(k)) \to H^{1}((\mathcal{H}_{2k-1}(k))$ factors through the quotient Hodge$^k(\mathcal{X})$ of $H^{2k}(\mathcal{X}, \mathbb{Q}(k))$. We, thus, obtain a group homomorphism

$$\sigma_p : \text{Hodge}^k(\mathcal{X}) \to H^1((\mathcal{H}_{2k-1}(k))$$

for every $p \in P$. In fact, it is easy to see that this group homomorphism is simply the restriction to Hodge$^k(\mathcal{X})$ of the composition of the arrows in the lower half of the diagram used in the proof of Proposition [4.12]

5. Vanishing

We begin this section by formalizing some notation.

5.1. Let $X$ be a smooth projective complex variety of dimension $2n$ with $n$ an integer and let $\mathcal{L}$ be a very ample line bundle on $X$. Set $P := |\mathcal{L}|$ and let

$$\mathcal{X} := \{(x, f) \in X \times P | f(x) = 0\}.$$ 

We call $\mathcal{X}$ the incidence variety associated to the pair $(X, \mathcal{L})$. Let $pr : \mathcal{X} \to X$ denote the first projection and $\pi : \mathcal{X} \to P$ denote the second projection. Let $d := dp$. Then $\mathcal{X}$ is smooth of dimension $r := 2n + d - 1$ because $pr$ is a Zariski local fibration with fiber $\mathbb{P}^{d-1}$. The map $\pi : \mathcal{X} \to P$ is smooth over the complement of the dual variety $X^d \subset P$.

We now state an analogue of the Weak Lefschetz theorem for the map $\pi$.

**Theorem 5.2 (Perverse Weak Lefschetz).** Let $\pi : \mathcal{X} \to P$ be as in (5.1), and let $E_{ij} = E_{ij}(\pi)$ be as in (5.5). Then

(i) $E_{ij} = 0$ unless $i = 0$ or $j = 0$.

(ii) $E_0 = H^r(X, \mathbb{Q}(2n - 1)) \otimes \mathbb{Q}[d]$ for $i < 0$.

**Proof:** Let $pr_X : X \times P \to P$ denote the projection on the second factor and let $g : U \to X \times P$ denote the complement of $\mathcal{X}$ in $X \times P$. We then have a commutative diagram

$$\begin{array}{ccc}
\mathcal{X} & \xrightarrow{i} & X \times P \\
\pi \downarrow & & \downarrow p \\
U & \xrightarrow{g} & P
\end{array}$$

where we write $p : U \to P$ for $pr_{2U}$.

Note that $g : U \to X \times P$ is an affine open immersion. Therefore $g_*\mathbb{Q}[2n+d]$ is perverse and we have an exact sequence

$$0 \to i_*\mathbb{Q}[2n+d-1] \to g_*\mathbb{Q}[2n+d] \to \mathbb{Q}[2n+d] \to 0$$

in $\text{MHM}(X \times P)$ [BBD82, Corollaire 4.1.3].

Applying $pr_2$ to (5.3) gives a distinguished triangle

$$\begin{array}{ccc}
\pi_*\mathbb{Q}[2n+d-1] & \xrightarrow{pr_2_*} & \mathbb{Q}[2n+d] \\
\downarrow & & \downarrow \\
(p_*\mathbb{Q}[2n+d] & \to & \mathbb{Q}[2n+d] \to \mathbb{Q}[2n+d-1])[1]
\end{array}$$

in $\text{Db} \text{MHM}(P)$. Since $p$ is affine, $p_*$ is left $r$-exact [BBD82, Corollaire 4.1.2]. Thus, $H^i(p_*\mathbb{Q}[2n+d]) = 0$ in $\text{MHM}(P)$ for $i < 0$. It follows then that $H^{i-1}(pr_2, \mathbb{Q}[2n+d]) = 0$ for $i < 0$.
However, by Lemma 5.6, follows from [BBD82, Proposition 2.1.11].

Let Corollary 5.7. $H \negmedspace^{-d}(E_{ij}) = E_{-i,j}(-i)$.

Therefore $E_{ij} = 0$ for $i > 0$ unless $j = 0$.

**Lemma 5.6.** Let $p \in P(C)$. Then $H^k_p(E_{ij}) = 0$ for $k < d - j$.

**Proof:** We have $E_{ij} = (g_{ij})_* H_{i,j} [d - j]$ with the notation as in (4.10). The result follows from [BBD82, Proposition 2.1.11].

**Corollary 5.7.** Let $p \in P(C)$, then

$$H^2_n(\mathcal{L}_p, \mathbb{Q}) = H^{-d}_p(E_{10}) \oplus H^{-d+1}_p(E_{00}) \oplus H^{-d+1}_p(E_{01}).$$

**Proof:** By (4.6),

$$H^2_n(\mathcal{L}_p, \mathbb{Q}) = H^{1-d}(\mathcal{L}_p, \mathbb{Q}[d_x]) = \bigoplus_i H^{1-d-i}(E_{ij}).$$

By Theorem 5.2 and (5.5), we see that, for $i \neq 0$,

$$H^k_p(E_{0i}) = \begin{cases} H^{1-d}(\mathcal{L}_p, \mathbb{Q}[2n-1]) & k = -d \\ 0 & \text{else.} \end{cases}$$

Therefore, the only summand $H^{1-d-i}_p(E_{ij})$ contributing to $H^2_n(\mathcal{L}_p, \mathbb{Q})$ with $i \neq 0$ is $H^{1-d}_p(E_{10})$. Thus

$$H^2_n(\mathcal{L}_p, \mathbb{Q}) = H^{-d}_p(E_{10}) \oplus \bigoplus_j H^{-d}_p(E_{0j}).$$

However, by Lemma 5.6, $H^{-d}_p(E_{0j}) = 0$ for $j > 1$.

In fact, the term $E_{01}$ is not difficult to compute and often trivial. It is governed by Lefschetz pencils.

**Definition 5.9.** Let $P(L)$ be a property of ample line bundles. We say that $P$ holds for $L \gg 0$ if for every ample line bundle $L$ there is an integer $N$ such that $P(L^n)$ holds for $n > N$.

**5.10.** By [SGA7, Theorem 2.5], the projective embedding of $X$ via the complete linear system $|L|$ is a Lefschetz embedding. Therefore, we can find a Lefschetz pencil $\Lambda \subset P$. To each $p \in \Lambda \cap X^\vee$ one has vanishing cycles $\delta_p \in H^{2n-1}_*(\mathcal{L}_\eta, \mathbb{Q})$ where $\eta$ denotes a point of $\Lambda(C)$ such that $\mathcal{L}_\eta$ is smooth. We say that the vanishing cycles are non-trivial if $\delta_p \neq 0$ for some $p \in \Lambda \cap X^\vee$. Note that this property depends only on $L$: it is independent of the choice of $\Lambda \subset P$. By the well-known fact that the vanishing cycles are conjugates of each other by the global monodromy of the Lefschetz fibration, it is equivalent to saying that $\Lambda \cap X^\vee \neq \emptyset$ and $\delta_p \neq 0$ for all $p \in \Lambda \cap X^\vee$.

**Proposition 5.11.** For $L \gg 0$, the vanishing cycles are non-trivial.
**Proof:** If the vanishing cycles are trivial, then the global monodromy of the Lefschetz pencil is trivial. It follows from the invariant cycle theorem that \(H^{2n-1}(X)\) surjects onto \(H^{2n-1}(\mathcal{X}_\eta)\) with \(\eta\) as in (5.10). However, it is easy to see that, by taking \(n > 0\), and considering Lefschetz pencils for the complete linear system \(|\mathcal{L}|\), we can make \(\dim H^{2n-1}(\mathcal{X}_\eta)\) tend to infinity. \(\square\)

**Theorem 5.12.** If the vanishing cycles are non-trivial, we have \(E_{01} = 0\); otherwise, \(\mathcal{H}_{01}\) is a rank 1 variation of pure Hodge structure supported on a dense open subset of \(X^\vee\).

**Proof:** Suppose \(\mathcal{H}_{01} \neq 0\). Then clearly it is supported on a Zariski open subset \(U_{01}\) of \(X^\vee\) and, since \(X^\vee\) is irreducible this subset must be dense. Suppose \(p \in U_{01}(\mathbb{C})\). Then \(H_{p}^{d+1}(E_{01}) = (\mathcal{H}_{01})_{p}\). It follows from Corollary 5.7 that \(H^{2n}(\mathcal{X}_p, \mathbb{Q}) \neq H^{2n}(\mathcal{X}_0)\). There is a dense open subset \(U \subset U_{01}(\mathbb{C})\) such that, if \(p \in U(\mathbb{C})\), then there is a Lefschetz pencil \(\Lambda\) through \(p\). By the vanishing cycles exact sequence (see [SGA7, Theorem 3.4 (ii)]), this implies that all the \(\delta_p\) are zero.

Now suppose that the \(\delta_p\) are zero. Using the vanishing cycles exact sequence again, we see that \(\dim H^{2n}(\mathcal{X}_p) = \dim H^{2n}(\mathcal{X}_0) + 1\). Now, note that, since \(p\) is a smooth point of the discriminant locus \(X^\vee\),

\[
H_{p}^{1-d}(E_{00}) = IH_{p}^{1}(\mathcal{H}_{2n-1}) = 0.
\]

(5.13)

(This follows from the fact that the local intersection cohomology of a local system ramified along a smooth divisor at a point \(p\) in that divisor is trivial.) Since \(H^{1-d}(E_{01}) \cong H^{2n}(\mathcal{X}_0)\), (5.13) implies that \(\dim H_{p}^{1-d}(E_{01}) = 1\). It follows that \(\dim (\mathcal{H}_{01})_p = 1\). \(\square\)

**Remark 5.14.** In fact, N. Fakhruddin has shown us that, if \(\mathcal{L} \gg 0\), we have \(E_{ij} = 0\) for all \(i\) and for all \(j > 0\). The proof, whose details will appear elsewhere, relies on the fact that, for \(\mathcal{L} \gg 0\), the locus of hypersurfaces in \(|\mathcal{L}|\) with non-isolated singularities has codimension larger than the dimension of the hypersurfaces.

**Corollary 5.15.** Let \(\zeta \in H^{2n}(X, \mathbb{Z}(n))\) be a primitive Hodge class, let \(\omega \in H^{2n}(X, \mathbb{Q}(n))\) be a Deligne cohomology class such that \(p(\omega) = \zeta\) where \(p : H^{2n}_D(X, \mathbb{Q}(n)) \to H^{2n}_D(X, \mathbb{Q}(n))\) is the natural map (from the introduction). Suppose that the \(\mathcal{L}\) is a very ample line bundle on \(X\) such that the vanishing cycles of \(P = |\mathcal{L}|\) are non-trivial. Let \(\nu\) be the normal function on \(P \setminus X^\vee\) given by \(p \mapsto \nu_{|\mathcal{X}_p}\). Then, for \(p \in P\), we have

\[
\sigma_p(\nu) = \zeta_{|\mathcal{X}_p}
\]

in \(H^{2n}(\mathcal{X}_p, \mathbb{Q}(n))\).

**Proof:** Since the vanishing cycles in \(\mathcal{L}\) are non-trivial, proper base change shows that

\[
H^{2n}(\mathcal{X}_p, \mathbb{Q}(n)) = \text{IH}_{p}^{0}(\mathcal{H}_{2n}(n)) \oplus \text{IH}_{p}^{1}(\mathcal{H}_{2n-1}(n)).
\]

As in Proposition 4.12 write \(\Pi\) for the projection on the second factor.
Since $\zeta$ is primitive, we have $\Pi(pr^* \zeta) = pr^* \zeta$. Therefore,
\[
\sigma_p(v) = \sigma_p(pr^* \zeta) = (\Pi(pr^* \zeta))|_{X_p} = (pr^* \zeta)|_{X_p} = \zeta|_{X_p}.
\]

**Example 5.16.** Let $X \equiv \mathbb{P}^2$ and set $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(2)$. Then $\dim \mathcal{X} = 6$ and $\dim \mathcal{P} = 5$. We have $E_{-1,0} = \mathbb{Q}[5], E_{0,0} = 0$ and $E_{1,0} = \mathbb{Q}(-1)[5]$. Since the vanishing cycles are trivial ($H^1(\mathcal{X}_0) = 0$ and any Lefschetz pencil contains a singular conic), $\mathcal{H}_0$ is non-zero. In fact, let $V$ denote the locus of point $p \in \mathcal{P}$ such that $\mathcal{X}_p$ is the union of two distinct lines. Note that $V$ is a dense open subset of $X^r$ and set $\pi_1(V) \cong \mathbb{Z}/2$. It is easy to see that $\mathcal{H}_0$ is the unique non-trivial rank 1 variation of Hodge structure of weight 2 on $V$. Moreover, it is not difficult to see that $E_0j = 0$ for $j > 1$.

6. Hodge Conjecture

In this section, we complete the proof of Theorem \[1.3\]

Let $Y$ be a smooth projective complex variety and let $k \in \mathbb{Z}$. We write $\text{Alg}^k Y$ for the subspace of $H^k Y$ consisting of algebraic cycles. The Hodge conjecture for $Y$ is the statement that $\text{Alg}^k Y = H^k Y$ for all $k$. By Poincaré duality and the Hodge-Riemann bilinear relations, the cup product
\[
\cup : H^{2k}(Y, \mathbb{Q}(k)) \otimes H^{2(d_k - k)}(Y, \mathbb{Q}(d_k - k)) \to H^{2d_k}(Y, \mathbb{Q}(d_k)) = \mathbb{Q}
\]
restricts to a give a perfect pairing
\[
(6.1) \quad H^k Y \otimes H^{d_k - k} Y \to \mathbb{Q}.
\]
Therefore, the Hodge conjecture for $Y$ is equivalent to the assertion that the perpendicular subspace $(\text{Alg}^k Y)^{\perp} \subset H^{d_k - k} Y$ is zero.

**Lemma 6.2.** The following two statements are equivalent:

(i) The Hodge conjecture holds for all smooth projective complex varieties $Y$.

(ii) For every smooth projective complex variety $X$ of dimension $2n$ with $n \in \mathbb{Z}$, $(\text{Alg}^n X)^{\perp} = 0$.

**Proof:** We have already seen that the first statement implies the second. Suppose then that the second statement holds. Let $Y$ be a smooth projective variety. Suppose $\alpha \in H^k Y$ is perpendicular to $\text{Alg}^{d_k - k} Y$. To prove the Hodge conjecture, we need to show that $\alpha = 0$. If $d_k = 2k$ then we are already done.

Suppose then that $d_k < 2k$. In this case, set $X = Y \times \mathbb{P}^{2k-d_k}$ and let $\beta = pr_1^* \alpha$. Suppose $\beta \cup [Z] \neq 0$ for some $[Z] \in \text{Alg}^k X$. Then, by the projection formula, $\alpha \cup pr_{1*} [Z] \neq 0$. Since this would contradict the assumption that $\alpha \in (\text{Alg}^{d_k - k}(Y))^{\perp}$, we must have $\beta \in (\text{Alg}^k X)^{\perp}$. But then $\beta = 0$. Since the map $pr_1^*: H^{2k}(Y, \mathbb{Q}(k)) \to H^{2k}(X, \mathbb{Q}(k))$ is injective, it follows that $\alpha = 0$.

Finally, suppose that $d_k > 2k$. Since $Y$ is projective, we can use Bertini to find a smooth subvariety $i: X \hookrightarrow Y$ which is the intersection of $d_k - 2k$ hyperplane sections. By weak Lefschetz, the restriction map $i^*: H^k(Y, \mathbb{Q}(k)) \to H^k(X, \mathbb{Q}(k))$
is injective. Suppose $\alpha \neq 0$. Then $0 \neq i^*\alpha \in \text{Hodge}^d X$. Therefore, by our assumption, there exists a closed $k$-dimensional subvariety $Z \subset X$ such that $i^*(\alpha) \cup [Z] \neq 0$. Again, by the projection formula, it follows that $\alpha \cup i_*[Z] \neq 0$. Since this contradicts our assumption that $\alpha$ is perpendicular to the algebraic classes, we see that $\alpha = 0$. □

The following lemma is well-known.

**Lemma 6.3.** Let $X$ be a smooth projective complex variety. Let $L$ be an ample line bundle on $X$ and let $Z \subset X$ be a closed subvariety. Then there exists an integer $N$ such that, for all $m \geq N$, there exists a divisor $D \in |L^m|$ such that $Z \subset D$.

**Proof:** This follows from the definition of ample. □

Let $(X, L)$ be a pair as in (5.1). For each positive integer $m$, let $\mathcal{Y}_m$ denote the incidence variety associated to the pair $(\mathcal{X}, L^m)$. Write $\text{pr}_m : \mathcal{X} \to X$ and $\pi_m : \mathcal{X} \to \mathcal{P}_m := |L^m|$ for the respective projections as in (5.1).

**Lemma 6.4.** Suppose the Hodge conjecture holds for $X$, then for every non-zero Hodge class $\zeta \in \text{Hodge}^{2n}(X)$, there exists a non-zero primitive integer $m$ and a point $p \in \mathcal{P}_m(\mathbb{C})$ such that $\zeta_{|\mathcal{P}_m} \neq 0$.

**Proof:** Let $\zeta$ be a non-zero class in $\text{Hodge}^{2n}(X)$. By Poincaré duality and the Hodge-Riemann relations, there exists a class $\alpha \in \text{Hodge}^{2n}(X)$ such that $0 \neq \alpha \cup \zeta \in \text{Hodge}^{2n}(X) \cong \mathbb{Q}(2n)$.

By the Hodge conjecture for $X$, we can write $\alpha = \sum_{i=1}^n a_i[Z_i]$ for $a_i \in \mathbb{Q}$ and $Z_i$ closed subvarieties of $X$. Since $\zeta \cup \alpha \neq 0$, $\zeta \cup [Z_i] \neq 0$ for some index $i$. Equivalently, $0 \neq \zeta_{|Z_i} \in \text{Hodge}^{2n}(Z_i, \mathbb{Q}(n))$. The lemma then follows from Lemma 6.3 □

As in the introduction, a class $\zeta \in \text{Hodge}^{2n}(X)$ is said to be primitive if $\zeta \cup c_1(L) = 0$. To each primitive Hodge class $\alpha$ and every positive integer $m$, we can associate a Hodge class $\text{pr}_m^*(\alpha) \in \text{Hodge}^2(\mathcal{X}, \mathbb{Q}(k))_{\text{prim}}$.

**Theorem 6.5.** Assume that Hodge conjecture holds and let $(X, L)$ be a pair as in (5.1). Then for every non-zero primitive Hodge class $\zeta \in \text{Hodge}^{2n}(X, \mathbb{Q}(n))$, there exists a positive integer $m$ and a $p \in \mathcal{P}_m$ such that $\sigma_p(\text{pr}_m^*(\zeta)) \neq 0$.

**Proof:** Let $\zeta \in \text{Hodge}^{2n}(X, \mathbb{Q}(n))$ be a non-zero primitive Hodge class. By Lemma 6.4 there exists an integer $N$ such that, for every $m \geq N$, there exists $p \in |L^m|$ such that $\zeta_{|\mathcal{P}_m} \neq 0$. By Proposition 5.11 we can assume that the vanishing cycles of Lefschetz pencils in $|L^m|$ are non-zero for $m \geq N$. Therefore, if $m \geq N$ and $p \in \mathcal{P}_m$, Corollary 5.15 shows that

$$
\sigma_p(\text{pr}_m^*\zeta) = \zeta_{|\mathcal{P}_m} \neq 0.
$$

□

**Theorem 6.6.** Suppose that for every pair $(X, L)$ as in (5.1) and every primitive Hodge class $\zeta \in \text{Hodge}^{2n}(X, \mathbb{Q}(n))$, there exists an $m \in \mathbb{Z}$ and a $p \in \mathcal{P}_m$ such that $\sigma_p(\text{pr}_m^*\zeta) \neq 0$. Then the Hodge conjecture holds.
Proof: By Lemma 6.2 we only need to show that no middle dimensional primitive Hodge class is perpendicular to the algebraic cycles. If $\zeta$ is a primitive Hodge class, then $\sigma_p(\text{pr}^*_m \zeta) \neq 0 \Rightarrow \Pi_1(\zeta_{\mathcal{F}_p}) \neq 0 \Rightarrow \zeta_{\mathcal{F}_p} \neq 0$.

We then resolve singularity of $\mathcal{F}_p$ and apply Deligne’s mixed Hodge theory to finish the proof by induction. This step is similar to the remark (attributed to B. Totaro) on the bottom of page 181 of Thomas’ paper [Tho95].

Let $\rho : \mathcal{F}_p \rightarrow \mathcal{F}_q$ be a desingularization. Then $\rho^*(\zeta_{\mathcal{F}_q}) \in H^{2n}(\mathcal{F}_q)$ is clearly a Hodge class. We now prove that it is non-zero.

$H^{2n}(\mathcal{F}_q)$ has a mixed Hodge structure whose weights range from 0 to $2n$. We have the following factorization

$$\rho^* : H^{2n}(\mathcal{F}_q) \rightarrow Gr_W H^{2n}(\mathcal{F}_q) \hookrightarrow H^{2n}(\mathcal{F}_p),$$

where the “$\rightarrow$” above the first map stands for projection onto to the top graded quotient and the second map is an injection by standard mixed Hodge theory. By the strictness of morphisms between mixed Hodge structures, we have $\zeta_{\mathcal{F}_q} \neq 0 \in Gr_W H^{2n}(\mathcal{F}_q)$. Therefore $p^*(\zeta_{\mathcal{F}_q}) \neq 0 \in H^{2n}(\mathcal{F}_p)$.

By induction on dimension, there is an algebraic cycle $W$ on $\mathcal{F}_p$ of codimension $n - 1$ (hence of dimension $n$) which pairs non trivially with $\rho^*(\zeta_{\mathcal{F}_q})$. Therefore its pushforward to $X$ pairs non trivially with $\zeta$. Then the Hodge conjecture follows by Lemma 6.2.

This completes the proof of Theorem 1.3.

7. Singularities and rational maps

Suppose $S$ is a smooth complex algebraic variety and $\mathcal{H}$ is a $\mathbb{Q}$-variation of pure Hodge structure of weight $-1$ on $S$. To simplify notation, we write $NF(S, \mathcal{H})^{\text{ad}}$ for the group $\text{Ext}^1_{\text{VMHS}(S)^{\text{ad}}}(\mathcal{Q}, \mathcal{H})$. If $\mathcal{H}$ is a variation of pure Hodge structure with integer coefficients of weight $-1$ on $S$, then $NF(S, \mathcal{H})^{\text{ad}} \otimes \mathbb{Q} = NF(S, \mathcal{H}^1)^{\text{ad}}$ by Corollary 2.9.

Lemma 7.1. Let $S$ be smooth complex algebraic variety, let $\mathcal{H}$ be a variation of $\mathbb{Q}$-Hodge structure of weight $-1$ on $S$ and let $U \subset S$ be a non-empty Zariski open subset. Then the restriction map

$$NF(S, \mathcal{H})^{\text{ad}} \rightarrow NF(U, \mathcal{H}_U^{\text{ad}})$$

is an isomorphism.

Proof: Using resolution of singularities, find an open immersion $j : S \rightarrow \overline{S}$ with $\overline{S}$ proper. Let $j_U : U \rightarrow \overline{S}$ denote the inclusion. then $j_U^*, \mathcal{H}[d_S] = j_U, \mathcal{H}[d_S]$. Therefore, by Corollary 2.9,

$$NF(S, \mathcal{H})^{\text{ad}} = NF(S, \mathcal{H})^{\text{ad}}_{\overline{S}} = \text{Ext}^1_{\text{MHMS}(\overline{S})^{\text{ad}}}(\mathcal{Q}[d_S], j^*_{\overline{S}}, \mathcal{H}[d_S]) = \text{Ext}^1_{\text{MHMS}(\overline{S})^{\text{ad}}}(\mathcal{Q}[d_S], j_U^* \mathcal{H}[d_S]) = NF(U, \mathcal{H}_U)^{\text{ad}}_{\overline{S}} = NF(U, \mathcal{H}_U)^{\text{ad}}.$$
**Definition 7.2.** Let $S$ be a smooth complex algebraic variety. We define a category $G_S$ as follows: Objects of $G_S$ are weight $-1$ variations of $\mathbb{Q}$-Hodge structure defined on some non-empty Zariski open subset $U$ of $S$. If $\mathcal{H}$ and $\mathcal{K}$ are objects in $G_S$ defined on open sets $U$ and $V$ respectively, then a morphism $\phi : \mathcal{H} \rightarrow \mathcal{K}$ is a morphism of variations of Hodge structure from $\mathcal{H}|_{U \cap V}$ to $\mathcal{K}|_{U \cap V}$. We call $G_S$ the category of variations of Hodge structure over the generic point of $S$. Note that, if we let $\text{MHM}(S)_{a,b}$ denote the full subcategory of $\text{MHM}(S)$ consisting of pure objects of weight $a$ with support of pure codimension $b$, then $G_S$ is equivalent to $\text{MHM}(S)_{a-1,0}$. This equivalence is brought about by the functor sending $\mathcal{H}$ supported on a Zariski open $j : U \hookrightarrow S$ to the mixed Hodge module $j_*\mathcal{H}$.

7.3. Let $\mathcal{H}$ and $\mathcal{K}$ be two objects in $G_S$ with $\mathcal{H}$ defined on a Zariski open subset $U \subset S$ and $\mathcal{K}$ defined on a Zariski open subset $V \subset S$. A morphism $\phi : \mathcal{H} \rightarrow \mathcal{K}$ in $G_S$ induces a morphism $\phi_* : \text{NF}(U, \mathcal{H})^{\text{ad}} \rightarrow \text{NF}(V, \mathcal{K})^{\text{ad}}$ via the composition

$$\text{NF}(U, \mathcal{H})^{\text{ad}} \cong \text{NF}(U \cap V, \mathcal{H})^{\text{ad}} \cong \text{NF}(U \cap V \cap K, \mathcal{K})^{\text{ad}} \cong \text{NF}(V, \mathcal{K})^{\text{ad}}.$$ 

It follows that the group $\text{NF}(\mathcal{K})^{\text{ad}}_{\mathbb{Q}}$ of admissible $\mathbb{Q}$-normal functions of an object in $G_S$ is an isomorphism invariant.

7.4. Let $f : S \dashrightarrow P$ be a dominant rational map between smooth projective varieties. Then $f$ induces a functor $f^* : G_P \rightarrow G_S$ defined as follows. Given $\mathcal{H}$ defined on a Zariski open subset $U$ of $P$, let $V$ denote the largest Zariski open subset of $U$ over which $f$ is defined. The functor sends $\mathcal{H}$ to $f^*\mathcal{H}_{|V}$. A similar construction defines $f^*$ of a morphism. Note that we have a natural map

$$f^* : \text{NF}(\mathcal{H})^{\text{ad}} \rightarrow \text{NF}(f^*\mathcal{H})^{\text{ad}}.$$

defined by pulling back the extension classes. In particular, if $f$ is a birational map, $\text{NF}(\mathcal{H})^{\text{ad}}_{\mathbb{Q}} \cong \text{NF}(f^*\mathcal{H})^{\text{ad}}_{\mathbb{Q}}$.

**Conjecture 7.5.** Let $f : S \dashrightarrow P$ be a birational map between smooth projective varieties, let $\mathcal{H}$ be a weight $-1$ variation of Hodge structure over the generic point of $P$ and let $\nu \in \text{NF}(\mathcal{H})^{\text{ad}}$ be an admissible normal function over the generic point of $P$. If $\nu$ is singular on $P$, then $f^*\nu$ is singular on $S$.

Our initial motivation for making this conjecture was the the comparison of our result [1.5] with the analogous assertions made in [GG07].

To explain this motivation, we briefly recall the assertions of [GG07]. Let $X, P$ and $\mathcal{K}$ be as in [5.1] and let $X^\vee \subset P$ denote the dual variety (i.e. discriminant locus) of $X$. In [GG07], the authors apply resolution of singularities to produce a projective variety $S$ equipped with a birational morphism $f : S \rightarrow P$ such that $f^{-1}X^\vee$ is a divisor with normal crossings. Let us call such an $S$ a resolution of the discriminant locus. The authors then make the following conjecture.

**Conjecture 7.6.** For every non-torsion primitive Hodge class $\zeta$, there is an integer $k$ and a resolution $f : S \rightarrow P = |\mathcal{L}^k|$ of the discriminant locus such that, for any Deligne cohomology class $\omega$ mapping to $\zeta$, $f^*\nu(\omega, \mathcal{L}^k)$ is singular on $S$. 
One of the main assertions of \cite{GG07} is that Conjecture 7.6 holds (for all even dimensional $X$) if and only if the Hodge conjecture holds (for all smooth projective algebraic varieties). In fact, we will now prove this assertion by proving Conjecture 7.5 in a special case, but we find this approach unsatisfying. Knowing conjecture 7.5 would give a more satisfying and direct proof.

We begin by establishing an easy case of Conjecture 7.5.

**Proposition 7.7.** Let $P$ be a smooth projective variety, $\mathcal{H}$ a variation of pure Hodge structure of weight $-1$ on the generic point of $P$ and $f : S \to P$ a dominant morphism. Let $\nu \in \text{NF}(\mathcal{H})^{\text{ad}}$. If $f^*\nu$ is singular on $S$, then $\nu$ is singular on $P$.

**Remark 7.8.** In the following proof and the rest of this section, we will work with constructible sheaves as opposed to perverse sheaves. To ease the notation, when $F$ is a constructible sheaf and $f$ is a morphism of complex schemes, we will write $f^*F$ for the usual (not derived) operation on constructible sheaves and $R^if_*\mathcal{F}$ for the constructible higher direct image.

**Proof.** Suppose that $\mathcal{H}$ is smooth over a dense Zariski open subset $j : U \hookrightarrow P$. The Leray spectral sequence for $R^ij_*\mathcal{H}$ gives an exact sequence
\begin{equation}
0 \to H^1(P, R^0j_*\mathcal{H}) \to H^1(U, \mathcal{H}) \xrightarrow{s_j} H^0(P, R^1j_*\mathcal{H})
\end{equation}
and $\nu$ is singular on $P$ if and only if $s_j(\text{cl}\nu) \neq 0$. The proposition follows by functoriality of the Leray spectral sequence applied to the pullback diagram (7.10)

\begin{equation}
\begin{array}{ccc}
\mathcal{V} & \xrightarrow{i} & S \\
\downarrow j & & \downarrow f \\
U & \xrightarrow{j} & P
\end{array}
\end{equation}

**Corollary 7.11.** Conjecture 7.6 implies Conjecture 1.2.

We now begin the proof of the reverse implication.

**Lemma 7.12.** Let $f : S \to P$ be a morphism of smooth, complex algebraic varieties. Let $U$ be a non-empty Zariski open subset of $P$ such that $V := f^{-1}U$ is Zariski dense in $S$, and let $\mathcal{V}$ be a $\mathbb{Q}$-local system on $U$. Form the cartesian diagram

\begin{equation}
\begin{array}{ccc}
V & \xrightarrow{i} & S \\
\downarrow g & & \downarrow f \\
U & \xrightarrow{j} & P
\end{array}
\end{equation}

using the letters on the arrows as the names for the obvious maps. Then the base change map $f^*i_*\mathcal{V} \to i_*g^*\mathcal{V}$ is an injection of constructible sheaves.

**Proof.** Suppose that $s \in S(\mathbb{C})$ and that $p = f(s) \in P(\mathbb{C})$. We can find a small ball $B$ about $p \in P$ such that $B \cap U$ is connected, and, for $z \in B \cap U$, $(f^*j_*\mathcal{V})_z = \mathcal{V}_{\pi_1(B \cap U, z)}$. We can then find a small ball $D \subset f^{-1}B$ containing $s$ such that $D \cap V$ is connected, and then for $w \in D \cap V$, $(i_*g^*\mathcal{V})_s = \mathcal{V}_{\pi_1(B \cap U, z)}$. Without loss of generality, we can assume that $f(w) = z$. Since the action of $\pi_1(D \cap V, w)$ on $\mathcal{V}_w$ then factors through $\pi_1(B \cap U, z)$, it follows that the base-change map $f^*j_*\mathcal{V} \to i_*g^*\mathcal{V}$ is injective. \qed
Lemma 7.13. Let $C$ be a smooth curve and $c \in C(\mathbb{C})$ and set $C' = C \setminus \{c\}$. Let $\pi : X \to C$ be a flat, projective morphism from a complex algebraic scheme $X$, and let $\pi'$ denote the restriction of $\pi$ to $\mathcal{X} := \pi^{-1}C'$. Suppose that $\pi'$ is smooth of relative dimension $2k - 1$ for $k$ an integer and that $\mathcal{X}$ has at worst ODP singularities. Set $\mathcal{Y} = R^{2k-1}i_*\mathcal{H}$ and let $j : C' \to C$ denote the open immersion including $C'$ in $C$. Then
\[ H^{2k-1}X_c \cong (j_*\mathcal{H})_c \]
via the natural morphism coming from the Clemens-Schmid exact sequence.

Proof. This follows from the Picard-Lefschetz formula of [SGA7, Théorem 3.4, Exposé XV]: one uses the fact that the relative dimension is odd and the vanishing cycles are orthogonal. \(\square\)

We now consider a situation where we can show that the base change morphism of Lemma 7.12 induces an isomorphism.

Lemma 7.14. Let $h : X \to P$ be a proper, flat morphism of relative dimension $2j - 1$ between smooth complex varieties such that $h$ is smooth over a dense Zariski open subset $U \subset P$ and, for all $p \in P$, $X_p$ presents at worst ODP singularities. Set $\mathcal{Y} = R^{2k-1}h_*\mathcal{H}|_U$. Let $f : S \to P$ be a morphism from a smooth variety such that $V := f^{-1}U$ is dense in $S$. Form the cartesian diagram

\[
\begin{array}{ccc}
V & \xrightarrow{i} & S \\
\downarrow{g} & & \downarrow{f} \\
U & \xrightarrow{j} & P
\end{array}
\]

using the letters on the arrows as the names for the obvious maps. Then the base change morphism induces an isomorphism $f^*j_*\mathcal{Y} \to i_*g^*\mathcal{Y}$ of sheaves.

Proof. We have already shown that the map is an injection. To prove surjectivity, we are going to use the local invariant cycle theorem of [BBD82].

Pick $s \in S(\mathbb{C})$. We can find a smooth curve $C$ passing through $s$ such that $C' := C \cap V$ is dense in $C$. Since $h : X \to P$ is flat, $h_C : X_C \to C$ is also flat. It follows that
\[ ((i'_C)_*\mathcal{H}|_C)_c \cong H^{2k-1}X_c. \]
On the other hand, since $\mathcal{X}$ is smooth, the local invariant cycle theorem shows that
\[ H^{2k-1}X_c \to (j_*\mathcal{H})_{f(c)}. \]
Therefore we have a sequence
\[ H^{2k-1}X_c \to (f_*\mathcal{H})_{f(c)} \cong (i'_C)_*\mathcal{H}|_C \cong H^{2k-1}X_c. \]
Since the composition is the identity, the maps in the sequence are all isomorphisms. \(\square\)

Lemma 7.15. Let $f : X \to Y$ be a projective birational morphism between smooth complex varieties. Let $\mathcal{F}$ be a constructible sheaf of $\mathbb{Q}$-vector spaces on $P$. Then
(i) the map $\mathcal{F} \to f_*f^*\mathcal{F}$ is an isomorphism of constructible sheaves;
(ii) we have $R^1f_*f^*\mathcal{F} = 0$. 

Proof: It suffices to check both statements on the stalks. By using proper base change, we see that the first statement follows from Zariski’s main theorem. Similarly, the second statement follows from the fact that the fibers of a projective birational morphism between separated schemes of finite type over \( \mathbb{C} \) are simply connected. \( \square \)

**Theorem 7.16.** Let \( h : X \rightarrow P \) be as in Lemma 7.14 and let \( f : S \rightarrow P \) be a projective birational morphism. Let \( \mathcal{H} \) and \( U \) be as in Lemma 7.14 and suppose that \( \nu \in NF(U, \mathcal{H})_P \). Then \( \nu \) has a non-torsion singularity on \( P \) if and only if \( f^* \nu \) has a non-torsion singularity on \( S \).

**Proof:** The “if” part follows from Proposition 7.7. To prove the “only if” direction, we can assume without loss of generality that \( f : f^{-1}U \rightarrow U \) is an isomorphism. In other words, we may replace the diagram (7.10) in the proof of Proposition 7.7 with the following diagram

\[
\begin{array}{ccc}
S & \xrightarrow{j} & U \\
\downarrow f & & \downarrow j \\
P & \xrightarrow{\nu} & P
\end{array}
\]

By the functoriality of the sequence (7.9), we have a diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & H^1(P, R^0 j_* \mathcal{H}) \\
\downarrow & & \downarrow \simeq \\
0 & \longrightarrow & H^1(S, R^0 j_* \mathcal{H}) \\
\downarrow & & \downarrow \\
H^1(U, \mathcal{H}) & \longrightarrow & H^0(P, R^1 j_* \mathcal{H}) \\
\end{array}
\]

It suffices then to show that the map \( H^1(P, R^0 j_* \mathcal{H}) \rightarrow H^1(S, R^0 j_* \mathcal{H}) \) is an isomorphism. For this, we apply the Leray spectral sequence coming from the map \( f : S \rightarrow P \). We have an exact sequence

\[
0 \longrightarrow H^1(P, j_* \mathcal{H}) \longrightarrow H^1(S, j_* \mathcal{H}) \longrightarrow H^0(P, R^1 f_* (j_* \mathcal{H})).
\]

By Lemma 7.14, \( j_* \mathcal{H} = f^* j_* \mathcal{H} \). Therefore, by Lemma 7.15 it follows that \( R^1 f_* (j_* \mathcal{H}) = R^1 f_* f^* j_* \mathcal{H} = 0 \).

From the exactness of (7.19), it follows that the map \( H^1(P, j_* \mathcal{H}) \rightarrow H^1(S, j_* \mathcal{H}) \) is an isomorphism. \( \square \)

**Corollary 7.20.** Conjectures 7.6 and 1.2 are equivalent.

**Proof:** We have already shown that Conjecture 7.6 implies Conjecture 1.2. To prove the converse, we are going to use the result of Thomas alluded to in the introduction.

Let \( X \subset P^n \) be a projective complex variety of dimension \( 2n \) with \( n \) an integer and let \( \xi \) denote a primitive Hodge class on \( X \).

Since Conjecture 1.2 holds, the Hodge conjecture also holds. Therefore, \( \xi \) is algebraic. By Thomas’ result, it follows that, for \( k \gg 0 \), we can find a hyperplane section \( s \in H^0(X, \mathcal{O}_X(k)) \) such that

1. \( \xi|_{V(s)} \) is non-zero in \( H^r(V(s), \mathbb{Q}) \);
2. \( V(s) \) has only ODP singularities.
By choosing $k \gg 0$, we can assume that the vanishing cycles of Lefschetz pencils in $|\mathcal{O}_X(k)|$ are non-trivial. Then set $\mathcal{L} = \mathcal{O}_X(k)$ and let $P, \mathcal{X}$ and $\pi$ be the incidence scheme in (5.1).

Let $\omega$ denote a lift of $p^* \zeta$ to the Deligne cohomology of $\mathcal{X}$ and $\nu = \nu(\omega, \mathcal{L})$. By Corollary 5.15 we see that $\nu$ has a non-torsion singularity at a the point $[s] \in P$. Now suppose $f : S \to P$ is any proper birational morphism. By restricting the locus in $P$ of hyperplane sections intersecting $X$ with only ODP singularities, we see from that $f^* \nu$ has a non-torsion singularity on $S$ as well. □

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