Long time asymptotic behavior for the nonlocal nonlinear Schrödinger equation with weighted Sobolev initial data

Gaozhan LI$^1$, Yiling YANG$^1$ and Engui FAN$^1$*

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Abstract

In this paper, we extend $\partial$ steepest descent method to study the Cauchy problem for the nonlocal nonlinear Schrödinger (NNLS) equation with weighted Sobolev initial data

$$iq_t + q_{xx} + 2\sigma q^2(x,t)q(-x,t) = 0,$$

$$q(x,0) = q_0(x),$$

where $q_0(x) \in L^{1,1}(\mathbb{R}) \cap L^{2,1/2}(\mathbb{R})$. Based on the spectral analysis of the Lax pair, the solution of the Cauchy problem is expressed in terms of solutions of a Riemann-Hilbert problem, which is transformed into a solvable model after a series of deformations. Finally, we obtain the asymptotic expansion of the Cauchy problem for the NNLS equation in solitonic region. The leading order term is soliton solutions, the second term is the error term is the interaction between solitons and dispersion, the error term comes from the corresponding $\bar{\partial}$ equation. Compared to the asymptotic results on the classical NLS equation, the major difference is the second and third terms in asymptotic expansion for the NNLS equation were affected by a function $\text{Im} \nu(\xi)$ for the stationary phase point $\xi$.

Keywords: Nonlocal Schrödinger equation, Riemann-Hilbert problem, $\partial$ steepest descent method, soliton resolution.

Mathematics Subject Classification: 35Q51; 35Q15; 35C20; 37K15.

$^1$ School of Mathematical Sciences and Key Laboratory of Mathematics for Nonlinear Science, Fudan University, Shanghai 200433, P.R. China.

* Corresponding author and email address: faneg@fudan.edu.cn
1 Introduction

In this paper, we extend the steepest descent method to study the Cauchy problem for the nonlocal nonlinear Schrödinger (NNLS) equation with weighted Sobolev initial data

\[ \begin{align*}
  iq_t + q_{xx} + 2\sigma q^2(x,t)q(-x,t) &= 0, \\
  q(x,0) &= q_0(x),
\end{align*} \tag{1.1, 1.2} \]

where \( \sigma = \pm 1 \) and \( q_0(x) \in L^{1,1}(\mathbb{R}) \cap L^{2,1/2}(\mathbb{R}) \), which are defined by

\[ \begin{align*}
  L^{1,1}(\mathbb{R}) &= \left\{ f = f(x) | f(x) \in L^1(\mathbb{R}), xf(x) \in L^1(\mathbb{R}) \right\}, \\
  L^{2,1/2}(\mathbb{R}) &= \left\{ f = f(x) | f(x) \in L^2(\mathbb{R}), x^{1/2}f(x) \in L^2(\mathbb{R}) \right\}.
\end{align*} \]

The NNLS equation was first proposed by Ablowitz and Musslimani [1, 2] and has attracted much attention in recent years due to its distinctive properties. The NNLS equation is invariant under the joint transformations \( x \to -x, \ t \to -t \) and complex conjugation [3, 4]. PT symmetric and non-Hermitian physics has been the subject of an intense research for the last decade most notably in classical optics, quantum mechanics and topological photonics [5–11]. PT symmetry of the NNLS equation amounts to the
invariance of the so-called self-induced potential in the case of classical optics \[12\]. It also has been experimentally observed that wave propagation in PT symmetric coupled waveguides or photonic lattices. In recent years, much work on the mathematical structure and properties for the NNLS equation such as exact solutions, Hamilton structure, Darboux transformation also have been widely studied \[13–15\].

The inverse scattering transform method was first used to solve the NNLS equation by Ablowitz and Musslimani \[1\]. The IST method provides the qualitative analysis on the Cauchy problem for the NNLS equation. The long-time asymptotics for the NNLS equation with rapidly decaying initial data was first considered due to Rybalko and Shepelsky consider with nonlinear steepest decent method \[16\]. In their series of papers, they further studied long-time asymptotics for the NNLS equation with step-like initial data \[17, 18\]. The major difference from classical NLS equation is that the NNLS equation is absent of the symmetries on their Jost functions and scattering data. Also the NNLS equation admits two scattering coefficients \(r_1(k), r_2(k)\) without symmetry like NLS equation \(r_1(k) = r_2(k), k \in \mathbb{R}\). The scattering data \(\text{Im} \nu(ξ)\) will affect asymptotic results. These make more difficulties to the classify discrete spectrum and analyze asymptotics of solutions.

The nonlinear steepest descent method which was firstly developed by Deift and Zhou to investigate the long-time behavior of integrable systems \[19\]. Later this method is further generalized into nonlinear \(\bar{\partial}\) steepest approach to analyze asymptotic of orthogonal polynomials by McLaughlin-Miller \[20, 21\]. In recent years, the \(\bar{\partial}\) steepest approach has been widely used to analyze long time asymptotics of integrable systems \[22–25\]. In this paper, we would like to extend \(\bar{\partial}\) steepest approach to analyze long time asymptotics of the NNLS equation with weighted Sobolev initial data. Compared with work \[16\], we consider more weighted Sobolev initial data which also allows soliton appearance. It was that the imaginary part \(\text{Im} \nu(−ξ)\) of reflect coefficient affects asymptotic expansion of the NNLS equation. To our knowledge this is the first time that the \(\bar{\partial}\)-steepest decent method is applied to nonlocal integrable system.

This paper is arranged as follows. In Section 2 we recall some main results in the construction process of the RH problem with respect to the initial problem \[1.1–1.2\]
obtained in [16]. We further prove that the scattering coefficients $r_1(k)$ and $r_2(k)$ are in general Sobolev space $H^1(\mathbb{R})$. In Section 3, we transform the RH problem into a mixed $\overline{\partial}$-RH problem, which is solved by separating it into a pure $\overline{\partial}$-problem and a pure RH problem. The pure RH problem can be estimated by a solvable model RH problem and a soliton RH problem. The pure $\overline{\partial}$-problem is analyzed for large $t \to \infty$. Finally, long time asymptotic behavior of the NNLS equation is obtained by using the reconstruction formula in Section 4.

2 The spectral analysis and a RH problem

The NNLS equation (1.1) admits the Lax pair [16]

$$
\Phi_x = (-ik\sigma_3 + Q)\Phi, \quad \Phi_t = (-2ik^2\sigma_3 + V)\Phi,
$$

(2.1)

where

$$
Q = \begin{pmatrix} 0 & q(x,t) \\ -\sigma q(-x,t) & 0 \end{pmatrix}, \quad V = \begin{pmatrix} A & B \\ C & -A \end{pmatrix},
$$

(2.2)

with

$$
A = i\sigma q(x,t)\overline{q(-x,t)}, \quad B = 2kq(x,t) + iq_x(x,t), \quad C = -2k\sigma \overline{q(-x,t)} + i\sigma (\overline{q(-x,t)})_x.
$$

(2.3-2.5)

Here $k \in \mathbb{C}$ is spectral parameter, and $\sigma_3$ is the Pauli matrix. Under the initial value (1.2), the Lax pair (2.1) have matrix-valued Jost solutions $\Phi_j$, $j = 1, 2$. Let

$$
\Phi_j = \Psi_j e^{(-ikx - 2ik^2t)\sigma_3},
$$

then $\Psi_j$ solves following integral equation of Volterra form

$$
\Psi_1(x; t; k) = I + \int_{-\infty}^{x} e^{ik(y-x)\sigma_3} Q(y, t)\Psi_1(y, t; k)e^{-ik(y-x)\sigma_3} dy,
$$

(2.6)

$$
\Psi_2(x; t; k) = I + \int_{+\infty}^{x} e^{ik(y-x)\sigma_3} Q(y, t)\Psi_2(y, t; k)e^{-ik(y-x)\sigma_3} dy.
$$

(2.7)
It was shown that for \(q_0(x) \in L^1(\mathbb{R})\), the columns of \(\Psi_1\) and \(\Psi_2\) are analytic in \(k \in (\mathbb{C}^+, \mathbb{C}^-)\) and \(k \in (\mathbb{C}^-, \mathbb{C}^+)\), respectively \[16\]. Here, \(\mathbb{C}^\pm\) means the upper and lower half plane. There exists the scattering matrix \(S(k)\), such that

\[
\Phi_1(x, t; k) = \Phi_2(x, t; k)S(k),\quad S(k) = \begin{pmatrix}
a_1(k) & \tilde{b}(k) \\
b(k) & a_2(k)
\end{pmatrix}.
\] (2.8)

In \[16\], Rybalko and Shepelsky give the symmetry of \(\Phi_1(x, t; k), \Phi_2(x, t; k)\)

\[
\Lambda \Phi_1(-x, t, -k) \Lambda^{-1} = \Phi_2(x, t; k),\quad \Lambda = \begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix}.
\] (2.9)

Combining with the (2.8), we have

\[
\tilde{b}(k) = -\sigma b(-k),\quad k \in \mathbb{R},
\] (2.10)

\[
a_1(k) = a_1(-k),\quad k \in \mathbb{C}^+,
\] (2.11)

\[
a_2(k) = a_2(-k),\quad k \in \mathbb{C}^-.
\] (2.12)

Thus, we get the relationship of \(\Psi_j\), \(j = 1, 2\)

\[
\Psi_1(x, t; k) = \Psi_2(x, t; k)e^{-it\hat{\sigma}_3}S(k),
\] (2.13)

\[
\theta(x, t; k) = 4k\xi + 2k^2,\quad \xi = \frac{t}{4x},\quad \xi \in \mathbb{R}.
\] (2.14)

We define a matrix-valued function

\[
M(x, t; k) = \begin{cases}
\begin{pmatrix} [\Psi_1(x, t; k)]_1 & [\Psi_2(x, t; k)]_2 \end{pmatrix}, & k \in \mathbb{C}^+, \\
\begin{pmatrix} [\Psi_2(x, t; k)]_1 & [\Psi_1(x, t; k)]_2 \end{pmatrix}, & k \in \mathbb{C}^-.
\end{cases}
\] (2.15)

Moreover, \(r_1(k)\) and \(r_2(k)\) are the reflection coefficients defined by

\[
r_1(k) = \frac{b(k)}{a_1(k)},\quad r_2(k) = \frac{b(-k)}{a_2(k)}.
\] (2.16)

The zeros of \(a(z)\) on \(\mathbb{R}\) are known to occur and they correspond to spectral singularities. They are excluded from our analysis in the this paper. To deal with our following work and make \(r_1(k), r_2(k)\) have enough smoothness and decaying property, we assume our initial data is under following assumption:
Assumption 1. $a_1(k), a_2(k)$ only have simple zeros in their own analytic area and nonzero on the boundary respectively.

Suppose that $a_1(k)$ only has $N$ simple zeros $\omega_1, ..., \omega_N$ on $\{k \in \mathbb{C} | \text{Im} k > 0, \text{Re} k > 0\}$, while $a_2(k)$ has $M$ simple zeros $\gamma_1, ..., \gamma_N$ on $\{k \in \mathbb{C} | \text{Im} k < 0, \text{Re} k > 0\}$. And assume $\{\text{Re} \omega_i | i = 1..N\}, \{\text{Re} \gamma_i | i = 1..M\}$ are different, without loss of generality we let

$$\text{Re} \omega_1 < .. < \text{Re} \omega_N, \quad \text{Re} \gamma_1 < .. < \text{Re} \gamma_M.$$ 

The symmetry in (2.11) and (2.12) imply that

$$a_1(\omega_n) = 0 \iff a_1(-\bar{\omega}_n) = 0, \quad n = 1, ..., N,$$

$$a_2(\gamma_n) = 0 \iff a_2(-\bar{\gamma}_n) = 0, \quad n = 1, ..., M.$$ 

Therefore, the discrete spectrum is

$$N_1 = \{\omega_n, -\bar{\omega}_n | a_1(\omega_n) = a_1(-\bar{\omega}_n) = 0, n = 1, ..., N\}, \quad (2.17)$$

$$N_2 = \{\gamma_m, -\bar{\gamma}_m | a_2(\gamma_m) = a_2(-\bar{\gamma}_m) = 0, m = 1, ..., M\}. \quad (2.18)$$

It is easy to compute the residue condition of $M(x, t; k)$ as follows

$$\text{Res}_{k=x \in N_1} M(x, t; k) = \lim_{k \to x} M(x, t; k) \begin{pmatrix} 0 & b(z) e^{2it\theta} \alpha' (z) e^{-2it\theta} \\ 0 & 0 \end{pmatrix},$$

$$\text{Res}_{k=x \in N_2} M(x, t; k) = \lim_{k \to x} M(x, t; k) \begin{pmatrix} 0 & -\sigma \frac{b(-\bar{\gamma})}{a_2(z)} e^{-2it\theta} \\ 0 & 0 \end{pmatrix}. \quad (2.20)$$

For Brief, denote

$$-\bar{\omega}_n = \omega_{-n}, n = 1, ..., N, \quad -\bar{\gamma}_m = \gamma_{-m}, m = 1, ..., M \quad (2.21)$$

and the norming constant $c_n$ with:

$$\frac{b(\omega_n)}{a_1'(\omega_n)} = c_n, \quad n = -N, ..., N,$$

$$-\sigma \frac{b(-\bar{\gamma}_m)}{a_2(\gamma_m)} = d_m, \quad m = -M, ..., M. \quad (2.23)$$
$M(x, t; k)$ solves the following matrix-valued Riemann-Hilbert problem:

**RHP0.** Find a matrix-valued function $M(x, t; k)$ which satisfies:

- $M(x, t; k)$ is analytic in $\mathbb{C} \setminus (\mathbb{R} \cup \mathcal{N}_1 \cup \mathcal{N}_2)$.
- Jump condition: $M(x, t; k)$ has continuous boundary values $M(x, t; k^\pm)$ on $\mathbb{R}$ with
  \[
  M(x, t; k^+) = M(x, t; k^-) J(x, t; k),
  \]  
  where $k^\pm$ means nontangential limitation of $k \in \mathbb{R}$ and
  \[
  J(x, t; k) = \begin{pmatrix}
  1 + \sigma r_1(k) r_2(k) & \sigma r_2(k) e^{-2it\theta(x,t;k)} \\
  r_1(k) e^{2it\theta(x,t;k)} & 1
  \end{pmatrix}.
  \]  
- Residue condition: The residue of $M(x, t; k)$ satisfies (2.19), (2.20).
- Asymptotic condition:
  \[
  M(x, t; k) = I + \mathcal{O}(1/k), \quad k \to \infty.
  \] (2.26)

The reconstruction formula of $q(x, t)$ is given by

\[
q(x, t) = 2i \lim_{k \to \infty} (k [M(x, t; k)]_{12}).
\] (2.27)

Observing the RHP0, we can reconstruct the solution $q(x, t)$ by a set of scattering data

\[
\left\{ r_1(k), r_2(k), \{ \omega_n, \gamma_m, c_n, d_m \} | n = 1, \ldots, 2N, m = 1, \ldots, 2M \right\}.
\] (2.28)

### 2.1 Reflection coefficients and scattering map

In this subsection, we establish map from the initial data $q_0(x)$ to scattering coefficient $r_1, r_2$.

**Proposition 1.** When the initial value $q_0(x) \in L^{1,1}(\mathbb{R}) \cap L^{2,1/2}(\mathbb{R})$ and has efficient small norm, the scattering coefficient $r_1(k), r_2(k) \in H^1(\mathbb{R})$.

**Proof.** Considering $k \in \mathbb{R}$, $t = 0$. From (2.6), (2.7), separating $\Psi_1, \Psi_2$ by column we
obtain

\[
[\Psi_1(x, 0; k)]_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{-\infty}^{x} \left( -\sigma q_0(-y) e^{-2ik(y-x)} [\Psi_1(x, 0; k)]_{11} \right) dy,
\]
\text{(2.29)}

\[
[\Psi_1(x, 0; k)]_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_{-\infty}^{x} \left( e^{2ik(y-x)} q_0(y) [\Psi_1(x, 0; k)]_{22} \right) dy,
\]
\text{(2.30)}

\[
[\Psi_2(x, 0; k)]_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{x}^{+\infty} \left( -\sigma q_0(-y) [\Psi_2(x, 0; k)]_{21} \right) dy,
\]
\text{(2.31)}

\[
[\Psi_2(x, 0; k)]_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_{-\infty}^{x} \left( e^{-2ik(y-x)} q_0(y) [\Psi_2(x, 0; k)]_{22} \right) dy.
\]
\text{(2.32)}

Take \([\Psi_1(x, 0; k)]_1\) as an example. For any \(k \in \mathbb{R}\), let \(f \in L^\infty(-\infty, 0)\) be a column vector valued function,

\[
T[f](x) = \int_{-\infty}^{x} \left( -\sigma q_0(-y) e^{-2ik(y-x)} f(y) \right) dy.
\]
\text{(2.33)}

Thus,

\[
|T[f](x)| \leq \int_{-\infty}^{x} |q_0(y)| dy \|f\|_{L^\infty(-\infty, 0)},
\]
\text{(2.34)}

which implies that \(T\) is a bounded linear operator on \(L^\infty(-\infty, 0]\). Furthermore, for \(n \in \mathbb{N}\), simply mathematical induction gives that

\[
|T^n f(x)| \leq \frac{1}{n!} \left( \int_{-\infty}^{x} |q_0(y)| dy \right)^n \|f\|_{L^\infty(-\infty, 0]}.
\]
\text{(2.35)}

Therefore, following series are uniform convergence in \(x \in (-\infty, 0]\),

\[
[\Psi_1(x, 0; k)]_1 = \sum_{n=0}^{\infty} T^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]
\text{(2.36)}

So, for any \(k \in \mathbb{R}\), \(x \in (-\infty, 0]\),

\[
\| [\Psi_1(x, 0; k)]_1 \| \leq e^{|q_0|_{L^1(\mathbb{R})}}.
\]
\text{(2.37)}

Because of the uniform convergence of the series \(2.36\), we have

\[
\partial_k ([\Psi_1(x, 0; k)]_1) = \sum_{n=0}^{\infty} \partial_k \left[ T^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]
\]
\text{(2.38)}

\[
= \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} T^m T^n T^{n-m-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\]
\text{(2.39)}

8
Through the integral form of \( \Psi \)

\[
T'[f](x) = \int_{-\infty}^{x} \begin{pmatrix} 1 & 0 \\ 0 & e^{-2ik(y-x)} \end{pmatrix} \begin{pmatrix} q_0(y)f_2 \\ 2i(y-x)\sigma q_0(-y)f_1 \end{pmatrix} dy. 
\]

(2.40)

Similarly, \( T' \) is a bounded linear operator on \( L^\infty(-\infty, 0) \) with

\[
||T'||_{\mathcal{B}(L^\infty(-\infty, 0))} \leq 2||q_0||_{L^{1,1}(\mathbb{R})}.
\]

(2.41)

Then for \( k \in \mathbb{R}, \ x \in (-\infty, 0] \),

\[
|\partial_k ([\Psi_1(x, 0; k)]_1)| \leq \sum_{n=0}^{\infty} |\partial_k \left[ T^n \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right]|
\]

\[
\leq \sum_{n=1}^{\infty} \frac{||q_0||_{L^{1,1}(\mathbb{R})}^{n-1}}{(n-1)!} \frac{2||q_0||_{L^{1,1}(\mathbb{R})}}{||q_0||_{L^{1,1}(\mathbb{R})}}
\]

\[
\leq 2||q_0||_{L^{1,1}(\mathbb{R})} e^{||q_0||_{L^{1,1}(\mathbb{R})}}.
\]

(2.42)

Similarly,

\[
|[[\Psi_1(x, 0; k)]_2]_1| \leq e^{||q_0||_{L^{1,1}(\mathbb{R})}}, \ x \in (-\infty, 0],
\]

(2.45)

\[
|[[\Psi_2(x, 0; k)]_1]_1| \leq e^{||q_0||_{L^{1,1}(\mathbb{R})}}, \ x \in [0, \infty),
\]

(2.46)

\[
|[[\Psi_2(x, 0; k)]_2]_1| \leq e^{||q_0||_{L^{1,1}(\mathbb{R})}}, \ x \in [0, \infty),
\]

(2.47)

\[
|\partial_k ([\Psi_1(x, 0; k)]_2)| \leq 2||q_0||_{L^{1,1}(\mathbb{R})} e^{||q_0||_{L^{1,1}(\mathbb{R})}}, \ x \in (-\infty, 0],
\]

(2.48)

\[
|\partial_k ([\Psi_2(x, 0; k)]_1)| \leq 2||q_0||_{L^{1,1}(\mathbb{R})} e^{||q_0||_{L^{1,1}(\mathbb{R})}}, \ x \in [0, \infty),
\]

(2.49)

\[
|\partial_k ([\Psi_2(x, 0; k)]_2)| \leq 2||q_0||_{L^{1,1}(\mathbb{R})} e^{||q_0||_{L^{1,1}(\mathbb{R})}}, \ x \in [0, \infty).
\]

(2.50)

Through the integral form of \( \Psi_1(x, 0, k), \ x \in (-\infty, 0] \), we have

\[
[\Psi_1(x, 0; k)]_{21} = \int_{-\infty}^{x} e^{-2ik(y-x)} (-\sigma q_0(-y)) [\Psi_1(y, 0; k)]_{11} dy.
\]

(2.51)

\[
\partial_k [\Psi_1(x, 0; k)]_{21} = \int_{-\infty}^{x} e^{-2ik(y-x)} (-\sigma q_0(-y)) \partial_k [\Psi_1(x, 0; k)]_{11}
\]

\[
- 2i(y-x)e^{-2ik(y-x)} (-\sigma q_0(-y)) [\Psi_1(y, 0; k)]_{11} dy.
\]

(2.52)

(2.53)

Then \( \forall \psi(k) \in L^2(\mathbb{R}), \)

\[
\left| \int_{\mathbb{R}} \psi(k) \int_{-\infty}^{x} e^{-2ik(y-x)} (-\sigma q_0(-y)) [\Psi_1(x, 0; k)]_{11} dy dk \right|
\]

\[
\leq \frac{1}{\sqrt{2}} e^{||q_0||_{L^1(\mathbb{R})}} ||q_0||_{L^2(\mathbb{R})} ||\psi||_{L^2(\mathbb{R})}.
\]
Here, the $\hat{\psi}$ denotes the Fourier transformation of $\psi$. Then above inequation implies for $x \in (-\infty, 0]$,

$$
\| [\Psi_1(x, 0; k)]_{21} \|_{L^2_k(\mathbb{R})} \leq \frac{1}{\sqrt{2}} \| q_0 \|_{L^2(\mathbb{R})} e^{\| q_0 \|_{L^1(\mathbb{R})}}.
$$

(2.54)

Using the same method to estimate, for $x \in (-\infty, 0]$, there is

$$
\| \partial_k [\Psi_1(x, 0; k)]_{21} \|_{L^2_k(\mathbb{R})} \leq \sqrt{2} \| q_0 \|_{L^{1,1}(\mathbb{R})} \| q_0 \|_{L^2(\mathbb{R})} e^{\| q_0 \|_{L^1(\mathbb{R})}}
$$

+ \sqrt{2} \| q_0 \|_{L^2(\mathbb{R})} e^{\| q_0 \|_{L^1(\mathbb{R})}}.

(2.55)

Similarly, we can derive

$$
\| [\Psi_1(x, 0; k)]_{21} \|_{L^2_k(\mathbb{R})} \leq e^{\| q_0 \|_{L^{1,1}(\mathbb{R})}} \| q_0 \|_{L^2(\mathbb{R})},
$$

(2.57)

$$
\| [\Psi_2(x, 0; k)]_{21} \|_{L^2_k(\mathbb{R})} \leq e^{\| q_0 \|_{L^{1,1}(\mathbb{R})}} \| q_0 \|_{L^2(\mathbb{R})},
$$

(2.58)

$$
\| [\Psi_2(x, 0; k)]_{21} \|_{L^2_k(\mathbb{R})} \leq e^{\| q_0 \|_{L^{1,1}(\mathbb{R})}} \| q_0 \|_{L^2(\mathbb{R})},
$$

(2.59)

And for $x \in (-\infty, 0]$, $k \in \mathbb{R}$ as an example to prove. (2.8) implies

$$
b(k) = \det \left( [\Psi_2(x, 0; k)]_{1} [\Psi_1(x, 0; k)]_{1} \right)
$$

$$
= [\Psi_2(x, 0; k)]_{11} [\Psi_1(x, 0; k)]_{21} - [\Psi_2(x, 0; k)]_{21} [\Psi_1(x, 0; k)]_{11},
$$

$$
b'(k) = \partial_k [\Psi_2(x, 0; k)]_{11} [\Psi_1(x, 0; k)]_{21} + [\Psi_2(x, 0; k)]_{11} \partial_k [\Psi_1(x, 0; k)]_{21}
$$

$$
- \partial_k [\Psi_2(x, 0; k)]_{21} [\Psi_1(x, 0; k)]_{11} - [\Psi_2(x, 0; k)]_{21} \partial_k [\Psi_1(x, 0; k)]_{11}.
$$

Specially, by taking $x = 0$ in $\Psi_1$ and $\Psi_2$, we can apply these estimations to derive

$$
\| b \|_{L^2_k(\mathbb{R})} \leq 2 \| q_0 \|_{L^2(\mathbb{R})} e^{2\| q_0 \|_{L^1(\mathbb{R})}}
$$

$$
\| b' \|_{L^2_k(\mathbb{R})} \leq 4 \| q_0 \|_{L^2(\mathbb{R})} \| q_0 \|_{L^{1,1}(\mathbb{R})} e^{2\| q_0 \|_{L^1(\mathbb{R})}}
$$

$$
+ 2 \sqrt{2} e^{2\| q_0 \|_{L^1(\mathbb{R})}} \left( \| q_0 \|_{L^{1,1}(\mathbb{R})} \| q_0 \|_{L^2(\mathbb{R})} + \| q_0 \|_{L^2(\mathbb{R})} \right).
$$
(2.8) also implies

\[ a_1(k) - 1 = (\Psi_1(x, 0; k)_{11} - 1) (\Psi_2(x, 0; k)_{22} - 1) + (\Psi_1(x, 0; k)_{11} - 1) \Psi_2(x, 0; k)_{22} \cdot \]

\[ a'_1(k) = \partial_k [\Psi_1(x, 0; k)_{11} \Psi_2(x, 0; k)_{22} + [\Psi_1(x, 0; k)_{11} \partial_k \Psi_2(x, 0; k)]_{22} \]

\[ - \partial_k [\Psi_1(x, 0; k)_{11} \Psi_2(x, 0; k)]_{22} - [\Psi_1(x, 0; k)]_{21} \Psi_2(x, 0; k)_{12} \cdot \]

To estimate \( a_1 \), we recall from (2.6) and (2.7)

\[ \Psi_1(x, 0; k)_{11} - 1 = \int_{-\infty}^{x} q_0(y) [\Psi_1(y, 0; k)]_{21} dy, \]

(2.60)

\[ \Psi_2(x, 0; k)_{22} - 1 = \int_{-\infty}^{x} -\sigma q_0(-y) [\Psi_2(y, 0; k)]_{12} dy, \]

(2.61)

Because of the independence on \( x \) of \( a_1(k) \), we can let \( x = 0 \) and derive that

\[ |a_1(k) - 1| \leq ||q_0||^2_{L^1(\mathbb{R})} e^{2||q_0||_{L^1(\mathbb{R})}} + 2||q_0||_{L^1(\mathbb{R})} e^{||q_0||_{L^1(\mathbb{R})}} + e^{2||q_0||_{L^1(\mathbb{R})}}. \]

(2.62)

When \( ||q_0||_{L^1(\mathbb{R})} \) is efficiently small, there is

\[ |a_1(k)| \geq \frac{1}{2}, \quad k \in \mathbb{R}. \]

(2.63)

Using the previous estimations for every \( k \in \mathbb{R} \), we demonstrate

\[ |a'_1(k)| \leq 8||q_0||_{L^1(\mathbb{R})} e^{2||q_0||_{L^1(\mathbb{R})}}. \]

(2.64)

According to above proof, we have proved for efficient small \( ||q_0||_{L^1(\mathbb{R})} \)

\[ \inf_{k \in \mathbb{R}} \{|a_1(k)|\} \geq \frac{1}{2}, \]

(2.65)

\[ b(k) \in H^1(\mathbb{R}), \]

(2.66)

\[ a'(k) \in L^\infty(\mathbb{R}). \]

(2.67)

Therefore,

\[ r_1(k) = \frac{b(k)}{a_1(k)} \in H^1(\mathbb{R}). \]

(2.68)

Similarly, we can prove the same proposition for \( r_2(k) \). Then we finish the proof. \( \square \)
Figure 1: The yellow and blue region is $D_{\pm}$ respectively.

3 The deformation of the RH problem

Define

$$D_{\pm} = \{ z \mid \text{Re}(z + \xi) \text{Im}(z + \xi) \in \mathbb{R}^{\pm} \}, \quad (3.1)$$

then for $k \in D_{\pm}$, \( \text{Re}(i\theta(x,t;k)) \in \mathbb{R}^{\pm} \). Let

$$\nu(k) = -\frac{1}{2\pi} \log(1 + \sigma r_1(k)r_2(k)), \quad (3.2)$$

with assuming $-\frac{1}{4} < \text{Im}\nu(k) < \frac{1}{2}$, $k \in \mathbb{R}$, and

$$\delta(k) = \exp \left( i \int_{-\infty}^{-\xi} \frac{\nu(s)}{s - k} ds \right), \quad k \in \mathbb{C} \setminus (-\infty, -\xi]. \quad (3.3)$$

**Proposition 2.** \( \delta(k) \) is analytic in \( \mathbb{C} \setminus (-\infty, -\xi] \) and satisfies jump condition on $(-\infty, -\xi]$

$$\delta(k^+) = \delta(k^-)(1 + \sigma r_1(k)r_2(k)), \quad \delta(k) = 1 + O\left(\frac{1}{k}\right). \quad (3.4)$$

Moreover,

$$\delta(k) = 1 + O\left(\frac{1}{k}\right), \quad (3.5)$$
Noted
\[ \chi(k) = -i\nu(-\xi) \log(\xi + k + 1) + i \int_{-\xi}^{-\xi-1} \frac{\nu(s) - \nu(-\xi)}{s-k} + i \int_{-\infty}^{-\xi-1} \frac{\nu(s)}{s-k}. \] (3.6)
then simple calculation gives that:
\[ \delta(k) = (\xi + k)^{i\nu(-\xi)} \exp(\chi(k)). \] (3.7)
Thus, \( \delta(k) \) admits the following estimation:

**Proposition 3.** Let \( \xi + k = re^{i\phi}, |\phi| \leq \frac{\pi}{4}, r > 0, \) then
\[ |\delta(k) - \delta_0(\xi + k)^{i\nu(-\xi)}| \lesssim |k + \xi|^{\frac{1}{2} - \Im \nu(-\xi)}, \] (3.8)
where \( \delta_0 = \exp(\chi(-\xi)). \)

**Proof.** Simple calculation gives that
\[ |\delta(k) - \delta_0(\xi + k)^{i\nu(-\xi)}| = |(\xi + k)^{i\nu(-\xi)} \exp(\chi(k)) - \delta_0(\xi + k)^{i\nu(-\xi)}| \] (3.9)
\[ \leq r^{-\Im \nu(-\xi)} e^{-\phi \Re \nu(-\xi)} |\exp(\chi(k)) - \exp(\chi(-\xi))| \] (3.10)
\[ \lesssim r^{-\Im \nu(-\xi)} |\chi(k) - \chi(-\xi)| \] (3.11)
\[ \lesssim |\xi + k|^{\frac{1}{2} - \Im \nu(-\xi)}. \] (3.12)
The proof of last line is is similar with [23] Proposition 3.1.

### 3.1 A mixed \( \bar{\partial} \)-RH problem

The long-time asymptotic of RHP 0 is affected by the growth and decay of the exponential function \( e^{\pm 2i t \theta} \) appearing in both the jump relation and the residue conditions. Therefore, we define
\[ \tilde{M}(x,t;k) = M(x,t;k)\delta(k)^{-\sigma_3}. \] (3.13)
Through the Propostion [2] \( \tilde{M}(x,t;k) \) admits the following transformed RHP:

**RHP1.** Find a matrix-valued function \( \tilde{M}(x,t;k) \) which satisfies:

- \( \tilde{M}(x,t;k) \) is analytic in \( \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{N}_1 \cup \mathcal{N}_2) \).
Jump condition: \( \tilde{M}(x, t; k) \) has continuous boundary values \( \tilde{M}(x, t; k^+) = \tilde{M}(x, t; k^-)\tilde{J}(x, t; k) \), \( k \in \mathbb{R} \),

\[
\tilde{J}(x, t; k) = \begin{pmatrix}
\frac{1}{1 + \sigma r_1(k)r_2(k)} & \sigma_1 r_1(k) \delta^2(k^+) e^{2i\theta} \\
-\sigma_2 r_2(k) \delta^2(k^-) e^{-2i\theta} & 1
\end{pmatrix}, \quad \text{if } k < -\xi,
\]

\[
\tilde{J}(x, t; k) = \begin{pmatrix}
1 & \frac{\sigma_1 r_1(k) \delta^2(k^+) e^{-2i\theta}}{1 + \sigma r_1(k)r_2(k)} \\
\sigma_2 r_2(k) \delta^2(k^-) e^{2i\theta} & 1
\end{pmatrix}, \quad \text{if } k > -\xi.
\]

Residue condition: The residue of \( \tilde{M}(x, t; k) \) becomes

\[
\text{Res}_{k = z} \tilde{M}(x, t; k) = \lim_{k \to z} \tilde{M}(x, t; k) = \begin{pmatrix}
0 & 0 \\
\delta^{-2}(z) \frac{b(z)}{a_1(z)} e^{2i\theta} & 0
\end{pmatrix},
\]

\[
\text{Res}_{k = z} \tilde{M}(x, t; k) = \lim_{k \to z} \tilde{M}(x, t; k) = \begin{pmatrix}
0 & -\sigma \delta^{2}(z) \frac{b(z)}{a_2(z)} e^{-2i\theta} \\
0 & 0
\end{pmatrix}.
\]

Asymptotic condition:

\[
\tilde{M}(x, t; k) = I + O\left(\frac{1}{k}\right), \quad k \to \infty.
\]

3.2 Mixed \( \bar{\partial} \)-RHP Problem

There is a positive number

\[
\rho_0 = \frac{1}{2} \min \left\{ |x - y| > 0 \mid x, y \in (-\xi) \cup N_1 \cup N_2 \right\}.
\]

Let \( \mathcal{Y}(k), \ k \in \mathbb{C} \) is a smooth function supported in \( \{ k \mid |k + \xi| < \frac{3}{2}\rho_0 \} \) and \( \mathcal{Y} = 1 \) in \( \{ k \mid |k + \xi| < \rho_0 \} \). And \( \mathcal{X}(k), \ k \in \mathbb{C} \) is also a smooth function supported in \( \{ k \mid |k - z| < \frac{3}{2}\rho_0, \ z \in N_1 \cup N_2 \} \) and \( \mathcal{X} = 1 \) in \( \{ k \mid |k - z| < \rho_0, \ z \in N_1 \cup N_2 \} \). We hope to define the mixed \( \bar{\partial} \)-RHP, whose jump condition and \( \bar{\partial} \) derivative can be well controlled. Let

\[
\Sigma_j = \left\{ k \mid \xi + k = r e^{i\phi}, \phi = \frac{2j - 1}{4} \pi, \ r > 0 \right\}, \ j = 1, 2, 3, 4,
\]

which divide the complex plane \( \mathbb{C} \) into six regions \( \Omega_j, \ j = 1, \ldots, 6 \) which are shown in Figure 2.
On each $\Omega_j$, we define smooth functions $R_j(k)$ as follows

\[
R_1(k) = (1 - \mathcal{X}(k)) \left( r_1(\text{Re}k)\delta^2(k) + \sin(2\phi) \left( f_1(k) - r_1(\text{Re}k)\delta^{-2}(k) \right) \right),
\]
\[
R_3(k) = (1 - \mathcal{X}(k)) \left[ \frac{\sigma r_2}{1 + \sigma r_1 r_2} \left( \text{Re}k \delta^2(k) \left( 1 - \sin(2\phi) \right) + \sin(2\phi) f_3(k) \right) \right],
\]
\[
R_4(k) = (1 - \mathcal{X}(k)) \left[ \frac{r_1}{1 + \sigma r_1 r_2} \left( \text{Re}k \delta^{-2}(k) \left( 1 - \sin(2\phi) \right) + \sin(2\phi) f_4(k) \right) \right],
\]
\[
R_6(k) = (1 - \mathcal{X}(k)) \left( \sigma r_2(\text{Re}k)\delta^2(k) + \sin(2\phi) \left( f_6(k) - \sigma r_2(\text{Re}k)\delta^2(k) \right) \right),
\]

where

\[
f_1(k) = \mathcal{Y}(k) r_1(-\xi)\delta^{-2}_0(\xi + k)\delta^{2\nu(-\xi)},
\]
\[
f_3(k) = \mathcal{Y}(k) \frac{\sigma r_2}{1 + \sigma r_1 r_2}(-\xi)\delta^2_0(\xi + k)\delta^{2\nu(-\xi)},
\]
\[
f_4(k) = \mathcal{Y}(k) \frac{r_1}{1 + \sigma r_1 r_2}(-\xi)\delta^{-2}_0(\xi + k)\delta^{-2\nu(-\xi)},
\]
\[
f_6(k) = \mathcal{Y}(k) \sigma r_2(-\xi)\delta^2_0(\xi + k)\delta^{2\nu(-\xi)},
\]

and

\[
R_2(k) = R_5(k) = I.
\]
It is obvious that $R_j$ has boundary value with

\[
\begin{align*}
R_1(k) &= \begin{cases} 
  f_1(k), & k \in \Sigma_1 \\
  r_1(\Re k)\delta^{-2}(k), & k > -\xi,
\end{cases} \\
R_3(k) &= \begin{cases} 
  f_3(k), & k \in \Sigma_3 \\
  \frac{\sigma r_2}{1+\sigma r_1 r_2}(\Re k)\delta^2(k), & k < -\xi,
\end{cases} \\
R_4(k) &= \begin{cases} 
  f_4(k), & k \in \Sigma_4 \\
  \frac{r_1}{1+\sigma r_1 r_2}(\Re k)\delta^{-2}(k), & k < -\xi,
\end{cases} \\
R_6(k) &= \begin{cases} 
  f_6(k), & k \in \Sigma_6 \\
  \sigma r_2(\Re k)\delta^2(k), & k > -\xi.
\end{cases}
\end{align*}
\]

**Proposition 4.** $R_j(k)$ satisfies the following properties

\[
\begin{align*}
|\partial R_1(k)| &\lesssim |\partial X| + |\partial Y| + |r_1'(\Re k)| + |k + \xi|^{-\alpha}, & (3.20) \\
|\partial R_3(k)| &\lesssim |\partial X| + |\partial Y| + \left| \left( \frac{\sigma r_2}{1+\sigma r_1 r_2} \right)'(\Re k) \right| + |k + \xi|^{-\alpha}, & (3.21) \\
|\partial R_4(k)| &\lesssim |\partial X| + |\partial Y| + \left| \left( \frac{r_1}{1+\sigma r_1 r_2} \right)'(\Re k) \right| + |k + \xi|^{-\alpha}, & (3.22) \\
|\partial R_6(k)| &\lesssim |\partial X| + |\partial Y| + |r_2'(\Re k)| + |k + \xi|^{-\alpha}. & (3.23)
\end{align*}
\]

where

\[
\alpha = \begin{cases} 
  \frac{1}{2} + \Im \nu(-\xi), & \Im \nu(-\xi) \in \left[0, \frac{1}{2}\right), \\
  \frac{1}{2}, & \Im \nu(-\xi) \in \left[-\frac{1}{4}, 0\right).
\end{cases}
\]

**Proof.** We only give details of the proof for $R_1$. The others can be demonstrated in the
Depending on (3.8) and the same way.

\[
|\overline{\partial}R_1(k)| = -\overline{\partial}\chi R_1(k) + (1 - \chi) \frac{1}{2} r'_1(\text{Re}k) \delta^{-2}(k) (1 - \sin(2\phi))
\]
\[
+ (1 - \chi) \overline{\partial}Y r_1(-\xi) \delta^{-2}(\xi + k)^{-2i\nu(-\xi)} \sin(2\phi)
\]
\[
+ \frac{i e^{i\phi}}{\rho} (1 - \chi) \cos(2\phi) (f_1(k) - r_1(\text{Re}k) \delta^{-2}(k))
\]
\[
\lesssim |\overline{\partial}\chi| + |\overline{\partial}Y| + |r'_1(\text{Re}k)| + |\xi + k|^{-1} r_1(\text{Re}k) \delta^{-2}(k) \chi_{\{\rho > \rho_0\}}
\]
\[
+ |\xi + k|^{-1} \left( r_1(-\xi) \delta^{-2}(\xi + k)^{-2i\nu(-\xi)} - r_1(\text{Re}k) \delta^{-2}(k) \right) \chi_{\{\rho < \frac{3}{2} \rho_0\}}
\]
\[
\lesssim |\overline{\partial}\chi| + |\overline{\partial}Y| + |r'_1(\text{Re}k)| + |\xi + k|^{-1} \chi_{\{\rho > \rho_0\}}
\]
\[
+ \left( |k + \xi|^{-\frac{1}{2} - \text{Im}(-\xi)} + |k + \xi|^{-\frac{1}{2}} \right) \chi_{\{\rho < \frac{3}{2} \rho_0\}}.
\]

Depending on (3.8) and \( r_1 \in H^1(\mathbb{R}) \), then \( r_1 \) is \( \frac{1}{2} - \text{H"older} \) continuous. So

\[
\left( r_1(-\xi) \delta^{-2}(\xi + k)^{-2i\nu(-\xi)} - r_1(\text{Re}k) \delta^{-2}(k) \right) \chi_{\{\rho < \frac{3}{2} \rho_0\}}
\]
\[
\lesssim \left( |r_1(-\xi)| |\delta^{-2}(\xi + k)^{-2i\nu(-\xi)} - \delta^{-2}(k)| + |r_1(\xi) - r_1(\text{Re}k)| \delta^{-2}(k) \right) \chi_{\{\rho < \frac{3}{2} \rho_0\}}
\]
\[
\lesssim \left( |k + \xi|^{-\frac{1}{2} - \text{Im}(-\xi)} + |k + \xi|^{-\frac{1}{2}} \right) \chi_{\{\rho < \frac{3}{2} \rho_0\}}.
\]

Meanwhile, \( \overline{\partial}Y \) has bounded support on \( \{\rho_0 < |\rho| < \frac{3}{2} \rho_0\} \), so that

\[
|\overline{\partial}Y| |k + \xi|^{-2|\text{Im}(-\xi)|} \lesssim |\overline{\partial}Y|.
\]

Under the assumption \( |\text{Im}(-\xi)| < \frac{1}{2} \),

\[
|\xi + k|^{-1} \chi_{\{\rho > \rho_0\}} \lesssim ||k + \xi|^{-\frac{1}{2} - \text{Im}(-\xi)} + |k + \xi|^{-\frac{1}{2}}.
\]

Therefore,

\[
|\overline{\partial}R_1(k)| \lesssim |\overline{\partial}\chi| + |\overline{\partial}Y| + |r'_1(\text{Re}k)| + |\xi + k|^{-\alpha}.
\]

\[\square\]

Let

\[
\mathcal{R}(z) = \begin{cases}
(1 \quad (1)^j R_j(z) e^{-2it\theta} \\ 0 & 1)
, & z \in \Omega_j, j = 3, 6;

(1 \quad (1)^j R_j(z) e^{2it\theta} \\ -1 & 0)
, & z \in \Omega_j, j = 1, 4;

I, & \text{elsewhere},
\end{cases}
\]

(3.25)

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To make continuous extension for the jump matrix $J$ to remove the jump from $\Sigma$. Besides, the new problem is hoped to take advantage of the decay/growth of $e^{2i\theta(z)}$ for $z \notin \Sigma$. We give the second transform

$$\tilde{M}(x,t;k) \mathcal{R}(k) = M^{(2)}(x,t;k). \quad (3.26)$$

Because $\mathcal{R}$ is a sectionally continuous function, $M^{(2)}(x,t;k)$ is derivable on $\mathbb{C} \setminus \Sigma$ with $\Sigma = \bigcup_{j=1}^{4} \Sigma_j$. We can derive the derivative condition

$$\overline{\partial}M^{(2)} = \overline{\partial}(\tilde{M} \mathcal{R}) = \tilde{M}\overline{\partial}\mathcal{R} = M^{(2)}\overline{\partial}\mathcal{R}. \quad (3.27)$$

Notice that $\mathcal{R} = I$ near the pole of $\tilde{M}$. The matrix valued function $M^{(2)}(x,t;k)$ satisfies following mixed RHP:

**RHP2.** Find a matrix-valued function $M^{(2)}(x,t;k)$ which satisfies:

- $M^{(2)}(x,t;k)$ is continuous in $\mathbb{C} \setminus \Sigma$ and meromorphic in $\Omega_2 \cup \Omega_5$.
- Jump condition: $M^{(2)}(x,t;k)$ has continuous boundary values $M^{(2)}(x,t;k^+) = M^{(2)}(x,t;k^-)J^{(2)}(x,t;k)$, $\quad (3.28)$

where

$$J^{(2)}(x,t;k) = \begin{cases} 
\left( \begin{array}{cc} 1 & 0 \\
0 & 1 
\end{array} \right), & k \in \Sigma_1, \\
\left( \begin{array}{cc} 1 & f_1(k)e^{2i\theta} \\
f_1(k)e^{-2i\theta} & 1 
\end{array} \right), & k \in \Sigma_2, \\
\left( \begin{array}{cc} 1 & 0 \\
f_3(k)e^{2i\theta} & 1 
\end{array} \right), & k \in \Sigma_3, \\
\left( \begin{array}{cc} 1 & f_4(k)e^{-2i\theta} \\
f_6(k)e^{2i\theta} & 1 
\end{array} \right), & k \in \Sigma_4.
\end{cases} \quad (3.29)$$

- Residue condition: The residue conditions of $M^{(2)}(x,t;k)$ satisfy $\quad (3.16), \quad (3.17)$ with $M^{(2)}(x,t;k)$ replacing $\tilde{M}(x,t;k)$.
- Derivative condition:

$$\overline{\partial}M^{(2)} = M^{(2)}\overline{\partial}\mathcal{R}. \quad (3.30)$$

- Asymptotic condition:

$$M^{(2)}(x,t;k) = I + \mathcal{O}(\frac{1}{k}), \quad k \to \infty. \quad (3.31)$$

To solve the mixed RHP, we define a function $M^{RHP}$ satisfies following model Riemann-Hilbert problem $\text{RHP}^3$ with with $\partial\mathcal{R} \equiv 0$:
RHP3. Find a matrix-valued function $M_{RHP}$ which satisfies:

- $M_{RHP}$ is meromorphic in $\mathbb{C} \setminus \Sigma$.
- Jump condition: $M_{RHP}(x,t;k)$ has continuous boundary values $M_{RHP}(x,t;k^\pm)$ on $\Sigma$ with

$$M_{RHP}(x,t;k^+) = M_{RHP}(x,t;k^-)J^{(2)}(x,t;k),$$  \hspace{1cm} (3.32)

where

$$J^{(2)}(x,t;k) = \begin{cases} 
1 & k \in \Sigma_1, \\
1 & k \in \Sigma_2, \\
1 & k \in \Sigma_3, \\
1 & k \in \Sigma_4.
\end{cases}$$  \hspace{1cm} (3.33)

- Residue condition: The residue conditions of $M_{RHP}$ satisfy (3.16), (3.17) with $M_{RHP}$ replacing $\tilde{M}(x,t;k)$.
- Asymptotic condition:

$$M_{RHP} = I + \mathcal{O}\left(\frac{1}{k}\right), \quad k \to \infty.$$  \hspace{1cm} (3.34)

To prove the existence of function $M_{RHP}$, we divide it to two parts:

$$M_{RHP}(x,t;k) = \begin{cases} 
E(x,t;k)M_{sol}(x,t;k)M^{PC}(z), & |k + \xi| < \rho_0, \\
E(x,t;k)M_{sol}(x,t;k), & |k + \xi| > \rho_0.
\end{cases}$$  \hspace{1cm} (3.35)

Here, the matrix function $M_{sol}(x,t;k)$ satisfies (3.16), (3.17) with $M_{sol}(x,t;k)$ replacing $\tilde{M}(x,t;k)$. And analytic in the elsewhere of $k \in \mathbb{C}$. And

$$M_{sol} = I + \mathcal{O}\left(\frac{1}{k}\right), \quad k \to \infty.$$  \hspace{1cm} (3.36)

Discussion of $M_{sol}$ is in section 3.3.

And $M^{PC}$ is the well known parabolic cylinder model [24] with parameters $\{-\xi, \tau_\xi, \tilde{\tau}_\xi, \nu(-\xi)\}$ where

$$r_\xi = r_1(-\xi)\delta_0^{-2}(8t)^{\nu(-\xi)}e^{-4it\xi^2}, \quad \tilde{r}_\xi = \sigma r_2(-\xi)\delta_0^2(8t)^{-\nu(-\xi)}e^{4it\xi^2}.$$  \hspace{1cm} (3.37)
The $M^{PC}$ satisfies jump condition

$$J^{PC}(x, t; k) = \begin{cases} 
1, & z \in \mathbb{C}, \arg(z) = \frac{\pi}{4}, \\
1 + z^{2}e^{\frac{4i}{z}} & 1, & z \in \mathbb{C}, \arg(z) = \frac{3\pi}{4}, \\
1 + z^{2} e^{\frac{4i}{z}} & 1, & z \in \mathbb{C}, \arg(z) = \frac{5\pi}{4}, \\
1 + z^{2} e^{-\frac{4i}{z}} & 1, & z \in \mathbb{C}, \arg(z) = \frac{7\pi}{4}.
\end{cases} \quad (3.38)$$

In order to motivate the model, let $z = z(k)$ denote the rescaled local variable

$$z = \sqrt{8}i(\xi + k). \quad (3.39)$$

**Proposition 5.** The large-$z$ asymptotic property satisfies

$$M^{PC}(x, t; z) = I + \frac{1}{z} \begin{pmatrix} 0 & -i\beta_{12} \\ i\beta_{21} & 0 \end{pmatrix} + O(z^{-2}) \quad (3.40)$$

\[
\beta_{12} = \sqrt{2}e^{-\frac{\pi}{2}(-\xi)}e^{\frac{\pi}{4}} \Delta t \text{Im}(-\xi)\beta_{12}, \quad (3.41) \\
\beta_{21} = -\sqrt{2}e^{-\frac{\pi}{2}(-\xi)}e^{-\frac{\pi}{4}} \Delta t^{-1} \text{Im}(-\xi)\beta_{21}. \quad (3.42)
\]

**Proof.** The proof of the above property is showed in [24]. \qed

Because $Y(k) = 1$ when $|\xi + k| < \rho_0$, we have

$$J^{(2)}(x, t; k) = \begin{cases} 
1, & |k + \xi| < \rho_0, k \in \Sigma_1, \\
1 + z^{2}e^{\frac{4i}{z}} & 1, & |k + \xi| < \rho_0, k \in \Sigma_2, \\
1 + z^{2} e^{\frac{4i}{z}} & 1, & |k + \xi| < \rho_0, k \in \Sigma_3, \\
1 + z^{2} e^{-\frac{4i}{z}} & 1, & |k + \xi| < \rho_0, k \in \Sigma_4.
\end{cases} \quad (3.43)$$
Comparing with the jump condition of \( M^{PC}(z) \), the error function \( E(x, t; k) \) satisfies a small norm RH problem 

**RHP4.** Find a matrix-valued function \( E(x, t; k) \) which satisfies:

- \( E(x, t; k) \) is analytic in \( \mathbb{C} \setminus \Sigma^E \), where as shown in Figure 3,

\[
\tilde{\Sigma}_j = \Sigma_j \setminus \{ k \in \mathbb{C} \mid |k + \xi| < \rho_0 \}, \quad \Sigma^E = \bigcup_{j=1}^{4} \tilde{\Sigma}_j \cup \{ k \in \mathbb{C} \mid |k + \xi| = \rho_0 \}. \tag{3.44}
\]

- Jump condition: For \( k \in \Sigma^E \),

\[
E(x, t; k^+) = E(x, t; k^-) J^E(x, t; k), \tag{3.45}
\]

where

\[
J^E(x, t; k) = \begin{cases} 
M_{sol}(x, t; k) J^{(2)}(x, t; k) M^{-1}_{sol}(x, t; k), & k \in \tilde{\Sigma}_j, \\
M_{sol}(x, t; k) M^{PC}(x, t; k) M^{-1}_{sol}(x, t; k), & k \in \{ k \in \mathbb{C} \mid |k + \xi| = \rho_0 \}.
\end{cases} \tag{3.46}
\]

- Asymptotic condition:

\[
E(x, t; k) = I + O\left(\frac{1}{k}\right), \quad k \rightarrow \infty. \tag{3.47}
\]
To prove the existence of $M_{RHP}(x, t; k)$, we only need to find $E(x, t; k)$ for RHP4. $C_\pm$ are the limitation of general Cauchy operators:

$$C_\pm(f)(s) = \lim_{z \to \Sigma_E} \frac{1}{2\pi i} \int_{\Sigma_E} \frac{f(s)}{s - z} ds. \tag{3.48}$$

Let

$$C_\omega[f] = C_-(f(J^E - I)). \tag{3.49}$$

The RHP 4 is solvable if and only if there is a function $\mu$ satisfies

$$(Id - C_\omega)(I + \mu) = I. \tag{3.50}$$

**Proposition 6.** $C_\omega$ is a bounded operator on $L^2(\Sigma^E) \to L^2(\Sigma^E)$ with:

$$||C_\omega||_{B(L^2(\Sigma^E))} = O(t^{-\frac{1}{2} + |\text{Im}(\nu(-\xi))|}). \tag{3.51}$$

**Proof.**

\[
||C_\omega[f]||_{L^2(\Sigma^E)} \lesssim ||f(J^E - I)||_{L^2(\Sigma^E)} \\
\leq ||f||_{L^2(\Sigma^E)} ||J^E - I||_{L^\infty(\Sigma^E)},
\]

\[
||C_\omega||_{B(L^2(\Sigma^E))} \lesssim ||J^E - I||_{L^\infty(\Sigma^E)} \\
\lesssim ||J^E - I||_{L^\infty(\bigcup_{j=1}^4 \Sigma_j)} + ||J^E - I||_{L^\infty(\{|k+\xi| = \rho_0\})},
\]

\[
\leq O(t^{\frac{1}{2} + |\text{Im}(\nu(-\xi))|}) + O(t^{\frac{1}{2} + |\text{Im}(\nu(-\xi))|})
\]

Then for efficient large $t$, $Id - C_\omega$ becomes a bijection in $L^2(\Sigma^E)$. Specially, there exists a function $\mu \in L^2(\Sigma^E)$ satisfying

$$(Id - C_\omega)\mu = C_\omega I, \tag{3.52}$$

and

$$||\mu||_{L^2(\Sigma^E)} \lesssim ||C_\omega I||_{L^2(\Sigma^E)} \lesssim O(t^{-\frac{1}{2} + |\text{Im}(\nu(-\xi))|}). \tag{3.53}$$

According to [?], RHP4 can be solved by

$$E(x, t; k) = I + \frac{1}{2\pi i} \int_{\Sigma^E} (I + \mu)(J^E - I)(s) \frac{ds}{s - k}, \tag{3.54}$$

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Proposition 7. $E(x, t; k)$ satisfies large $k$ asymptotic condition

$$E(x, t; k) = I + \frac{E^{(1)}(x, t)}{k} + O(k^{-2}), \quad (3.55)$$

$$E^{(1)}(x, t) = \frac{1}{\sqrt{8t}} \begin{pmatrix} 0 & -it\text{Im}\nu(-\xi)\tilde{\beta}_{21} \\ it^{-\text{Im}\nu(-\xi)}\tilde{\beta}_{12} & 0 \end{pmatrix} + O(t^{-\frac{1}{2}+|\text{Im}\nu(-\xi)|}) \left( O(t^{-\frac{1}{2}-\text{Im}\nu(-\xi)}), O(t^{-\frac{1}{2}+\text{Im}\nu(-\xi)}) \right). \quad (3.56)$$

**Proof.** According to Proposition 5 and Cauchy integral formula, we deduce

$$E^{(1)}(x, t) = \frac{1}{2\pi i} \int_{\Sigma_E} (I + \mu)(J^E - I)(s) ds$$

$$= \frac{1}{2\pi i} \int_{\Sigma_E} (J^E - I)(s) ds + \frac{1}{2\pi i} \int_{\Sigma_E} \mu(J^E - I)(s) ds$$

$$= \frac{1}{\sqrt{8t}} \begin{pmatrix} 0 & -it\text{Im}\nu(-\xi)\tilde{\beta}_{21} \\ it^{-\text{Im}\nu(-\xi)}\tilde{\beta}_{12} & 0 \end{pmatrix} + O(\exp(-4\rho_0^2t))$$

$$+ O(t^{-\frac{1}{2}+|\text{Im}\nu(-\xi)|}) \left( O(t^{-\frac{1}{2}-\text{Im}\nu(-\xi)}), O(t^{-\frac{1}{2}+\text{Im}\nu(-\xi)}) \right),$$

where

$$\frac{1}{2\pi i} \int_{\Sigma_E} \mu(J^E - I)(s) ds \leq ||\mu||_{L^2(\Sigma_E)}||J^E - I||_{L^2(\Sigma_E)}$$

$$= O(t^{-\frac{1}{2}+|\text{Im}\nu(-\xi)|}) \left( O(t^{-\frac{1}{2}-\text{Im}\nu(-\xi)}), O(t^{-\frac{1}{2}+\text{Im}\nu(-\xi)}) \right).$$

\[\square\]

3.3 Reflectionless RH problem $M_{sol}$

Let $\alpha^{(1)}_n(x, t), \alpha^{(2)}_n(x, t), \beta^{(1)}_m(x, t)$ and $\beta^{(2)}_m(x, t)$ are complex function of $x, t$ with

$$\text{Res}_{k=\omega_n \in \mathcal{N}_1} M_{sol} = \begin{pmatrix} \alpha^{(1)}_n & 0 \\ \alpha^{(2)}_n & 0 \end{pmatrix}, \quad \text{Res}_{k=\gamma_m \in \mathcal{N}_2} M_{sol} = \begin{pmatrix} 0 & \beta^{(1)}_m \\ 0 & \beta^{(2)}_m \end{pmatrix}. \quad (3.57)$$

Then $M_{sol}$ can be written as

$$M_{sol}(x, t; k) = I + \sum_{\omega_n \in \mathcal{N}_1} \frac{1}{k - \omega_n} \begin{pmatrix} \alpha^{(1)}_n & 0 \\ \alpha^{(2)}_n & 0 \end{pmatrix} + \sum_{\gamma_m \in \mathcal{N}_2} \frac{1}{k - \gamma_m} \begin{pmatrix} 0 & \beta^{(1)}_m \\ 0 & \beta^{(2)}_m \end{pmatrix}. \quad (3.58)$$

For convenience, let

$$M_{sol} = I + \frac{M^{(1)}_{sol}}{k} + O\left(\frac{1}{k^2}\right), \quad k \to \infty. \quad (3.59)$$
Hence,
\[ M_{sol}^{(1)} = \sum_{\omega_n \in \mathcal{N}_1} \begin{pmatrix} \alpha_n^{(1)} & 0 \\ \alpha_n^{(2)} & 0 \end{pmatrix} + \sum_{\gamma_m \in \mathcal{N}_2} \begin{pmatrix} 0 & \beta_m^{(1)} \\ 0 & \beta_m^{(2)} \end{pmatrix}. \] (3.60)

Through substituting above expression of \( M_{sol}^{(1)} \) into the residue condition in (3.16), (3.17) of \( M_{sol}^{(1)} \), we deduce that the \( \alpha_n \) and \( \beta_m \) are the solution of the following linear equations:

\[
\begin{bmatrix}
I + \sum_{\gamma_m \in \mathcal{N}_2} \frac{1}{\omega_p - \gamma_m} \begin{pmatrix} 0 & \beta_m^{(1)} \\ 0 & \beta_m^{(2)} \end{pmatrix}
\end{bmatrix} \begin{pmatrix}
c_p \delta(\omega_p) - 2 e^{2it\theta} & 0 \\
0 & 0
\end{pmatrix} = \begin{pmatrix} \alpha_p^{(1)} \\ \alpha_p^{(2)} \end{pmatrix}, 
\]

\( p = 1, \ldots, 2N, \)

\[
\begin{bmatrix}
I + \sum_{\omega_n \in \mathcal{N}_1} \frac{1}{\gamma_q - \omega_n} \begin{pmatrix} \alpha_n^{(1)} & 0 \\ \alpha_n^{(2)} & 0 \end{pmatrix}
\end{bmatrix} \begin{pmatrix} 0 & \beta_q^{(1)} \\ 0 & \beta_q^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, 
\]

\( q = 1, \ldots, 2M. \)

The existence of \( M_{sol}(x,t;k) \) is proved in the Appendix of [23]. Let

\[
\Delta_1 = \{ z \in \mathcal{N}_1 | \text{Re}(z) > -\xi \}, 
\]

\[
\Delta_2 = \{ z \in \mathcal{N}_2 | \text{Re}(z) > -\xi \}. 
\]

Because of the symmetry of the discrete spectrum, we know

\[
\Delta_1 = \{ \omega_n \in \mathcal{N}_1 \}, \forall m < n, \text{Re}(\omega_m) < \text{Re}(\omega_n), 
\]

\[
\Delta_2 = \{ \gamma_m \in \mathcal{N}_2 \}, \forall m < n, \text{Re}(\gamma_m) < \text{Re}(\gamma_n), 
\]

\[
\Delta_1^- = \Delta_1 \cap \{ \text{Re} < 0 \}, \Delta_1^+ = \Delta_1 \cap \{ \text{Re} > 0 \}, 
\]

\[
\Delta_2^- = \Delta_2 \cap \{ \text{Re} < 0 \}, \Delta_2^+ = \Delta_2 \cap \{ \text{Re} > 0 \}. 
\]

Without loss of generality, let \( |\Delta_1^+| - |\Delta_1^-| > |\Delta_2^+| - |\Delta_2^-| \) and \( |\Delta_1^-| > |\Delta_2^-| \). And let

\[
h_1 = \min \{ \text{Re}i\theta(\omega) | \omega \in \mathcal{N}_1 \setminus \Delta_1 \}, 
\]

\[
h_2 = \min \{ -\text{Re}i\theta(\gamma) | \gamma \in \mathcal{N}_2 \setminus \Delta_2 \}, 
\]

\[
h_3 = \min \{ -\text{Re}i\theta(\omega_n), \text{Re}i\theta(\gamma_m) | n, m = 1, \ldots, |\Delta_2| \}, 
\]

\[
h = \min \{ h_1, h_2, h_3 \}. 
\]

Let

\[
T(z) = \prod_{n=0}^{(|\Delta_1^+| - |\Delta_1^-|)-1} \frac{z - s^{(|\Delta_1^+| - n)}}{z - t^{(|\Delta_1^+| - n)}} \prod_{n=1}^{(|\Delta_2^-| - n)} \frac{(z - \omega_n)(z + \bar{\omega}_n)}{(z - \gamma_m)(z + \bar{\gamma}_m)}. 
\]

(3.67)

\( M_{sol}(x,t;k) \) exist if and only if

\[
\tilde{M}_{sol} = M_{sol}(x,t;k) \begin{pmatrix} T(k) & 0 \\ 0 & T^{-1}(k) \end{pmatrix} 
\]

(3.68)
exists. Here $\tilde{M}^{sol}$ satisfies that for $n = \pm 1, \ldots, \pm |\Delta^-_2|, |\Delta^+_1| - |\Delta^-_2| + 1, \ldots, |\Delta^+_1|$, $\Delta^+_2, \ldots, |\Delta^+_2|$,  

\[
\text{Res} \tilde{M}^{sol} = \lim_{k \to \omega_n} \tilde{M}^{sol}(x, t; k) = \left( \begin{array}{cc}
0 & \frac{T(k)}{k - \omega_n} - 2\delta^2(\omega_n)a'(\omega_n)b(\omega_n) e^{-2it\theta(\omega_n)} \\
0 & 0
\end{array} \right),
\]

(3.69)

\[
\text{Res} \tilde{M}^{sol} = \lim_{k \to \gamma_m} \tilde{M}^{sol}(x, t; k) = \left( \begin{array}{cc}
0 & -\sigma(\frac{T(k)}{k - \gamma_m})\delta^2(\gamma_m)\frac{a'(-\gamma_m)}{b(\gamma_m)} e^{2it\theta(\gamma_m)} \\
0 & 0
\end{array} \right).
\]

(3.70)

Denote  

\[
\Delta = \{ \omega_n, -\omega_n | n = |\Delta^-_2| + 1, \ldots, |\Delta^+_1| - |\Delta^-_2| + |\Delta^+_2| \}.
\]

(3.71)

For $z \in \Delta \cup (N_1 \setminus \Delta_1) \cup (N_2 \setminus \Delta_2)$, we have  

\[
\text{Res} \tilde{M}^{sol} = \lim_{k \to z} \tilde{M}^{sol}(x, t; k) = \left( \begin{array}{cc}
T^2(k)\delta^{-2}(k)\frac{b(z)}{a'(z)} e^{2it\theta} & 0 \\
0 & 0
\end{array} \right).
\]

(3.72)

\[
\text{Res} \tilde{M}^{sol} = \lim_{k \to z} \tilde{M}^{sol}(x, t; k) = \left( \begin{array}{cc}
0 & -\sigma T^{-2}(k)\delta^2(k)\frac{b(-z)}{a'(z)} e^{-2it\theta} \\
0 & 0
\end{array} \right).
\]

(3.73)

Briefly, we denote  

\[
\text{Res} \tilde{M}^{sol} = \lim_{k \to \omega_n} \tilde{M}^{sol}(x, t; k) N_z(k),
\]

(3.74)

for $z \in N_1 \cup N_2$. Make transform  

\[
N^{sol}(x, t; k) = \tilde{M}^{sol}(x, t; k)(I - \frac{N_z(k)}{k - z}), |k - z| < \min \left\{ \frac{h}{2}, \rho_0 \right\}
\]

(3.75)

for $z \in N_1 \cup N_2 \setminus \Delta$. Then $N^{sol}$ can be expressed as  

\[
N^{sol} = E^{sol} M^\Delta,
\]

(3.76)

where $M^\Delta$ is a meromorphic matrix valued function satisfies $|z| \leq 3.75$ in $z \in \Delta$ and $M^\Delta = I + O(\frac{1}{k^2})$. Therefore, by reconstruction formula,  

\[
q^\Delta = \lim_{k \to \infty} (2ikM^\Delta)_{12}
\]

(3.77)

is a solution of $\Delta$. And $E^{sol}$ is a solution of $\text{RHP} E^{sol}$. Find a matrix-valued function $E^{sol}$ which satisfies:

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\( E^{\text{sol}} \) is uniform bounded in \( \mathbb{C} \) and
\[
E^{\text{sol}} = I + \mathcal{O}(\frac{1}{k}). \tag{3.78}
\]

For \( |k - z| = \rho_0, z \in \mathcal{N}_1 \cup \mathcal{N}_2 \setminus \Delta \),
\[
E^{\text{sol}}(x, t; k^+) = E^{\text{sol}}(x, t; k^-)M^\Delta(x, t; k)(I + \frac{N_z(k)}{k - z})(M^\Delta(x, t; k))^{-1}. \tag{3.79}
\]

By Beals-Coifman method, we can derive
\[
\lim_{k \to \infty} [2ikE^{\text{sol}}]_{12} = \mathcal{O}(\exp(-ht)). \tag{3.80}
\]

Then
\[
q^{\text{sol}} = \lim_{k \to \infty} [2ikM^{\text{sol}}]_{12} = \lim_{k \to \infty} [2ik\tilde{M}^{\text{sol}}]_{12} = \lim_{k \to \infty} [2ikN^{\text{sol}}]_{12} = q^\Delta + \mathcal{O}(\exp(-ht)).
\]

\( q^\Delta \) is a solution of (1.1) whose scattering data \( \{0, 0, \{\omega_n, 0, c_nT^2(\omega_n)\delta(\omega_n)^{-2}, 0 | \omega_n \in \Delta \}\} \)
is nonreflecting and only depends on \( \Delta \).
\[
q^{\text{sol}} = q^\Delta + \mathcal{O}(\exp(-ht)). \tag{3.81}
\]

### 3.4 Analysis on the pure \( \bar{\partial} \)-Problem

To demonstrate the existence of \( M_{RHP} \), we define a new matrix-valued function
\[
W(x, t; k) = W^{(2)}(x, t; k)M_{RHP}(x, t; k)^{-1}. \tag{3.82}
\]
which removes analytic component of \( M^{(2)}(z) \) to get a pure \( \bar{\partial} \)-problem. According to (3.30) we can deduce derivative condition.

**Proposition 8.** \( W(x, t; k) \) satisfies following properties.

\( W(x, t; k) \) is continues where \( k \in \mathbb{C} \).

**Derivative condition:**
\[
\overline{\partial}W = WM_{RHP}\overline{\partial}RM_{RHP}^{-1}. \tag{3.83}
\]

**Asymptotic condition:**
\[
W(x, t; k) = I + \mathcal{O}(\frac{1}{k}), \ k \to \infty. \tag{3.84}
\]
Namely, \( W(x,t;k) \) can be solved by a pure \( \overline{\partial} \) problem, and it is a solution of corresponding integral equation.

**Proposition 9.** The Proposition 8 of \( W(x,t;k) \) is equivalent to the integral equation

\[
W(x,t;k) = I - \frac{1}{\pi} \int_{C} \frac{W(x,t;s)(M_{RHP\overline{\partial}RM_{RHP}^{-1}})(x,t;s)}{s-k} dA(s), \tag{3.85}
\]

where \( dA(s) \) means integral of general Lebesgue measure on \( C \).

To solve the (3.85), define \( S \) as the left Cauchy-Green integral operator with

\[
S[f] = -\frac{1}{\pi} \int_{C} f(s)(M_{RHP\overline{\partial}RM_{RHP}^{-1}})(x,t;s) \frac{dA(s)}{s-z} \tag{3.86}
\]

Therefore, (3.85) is equivalent to

\[
(Id - S)W(x,t;k) = I. \tag{3.87}
\]

To target on existence of \( W(x,t;k) \), we demonstrate the following proposition.

**Proposition 10.** The norm of the integral operator \( S \) decays to zero as \( t \to \infty \):

\[
||S||_{B(L^{\infty})} \lesssim O(t^{-\frac{1}{4}}) + O(t^{-\frac{1}{2}+\frac{\alpha}{2}}). \tag{3.88}
\]

where \( \alpha \) is defined in (3.24).

**Proof.** According to Proposition 4,

\[
||S||_{B(L^{\infty})} \leq \sum_{j=1}^{4} \frac{1}{\pi} \int_{\Omega_j} |M_{RHP}| |\overline{\partial}R||M_{RHP}^{-1}| \frac{dA(s)}{|s-z|} \leq \sum_{\Omega_1,\Omega_2,\Omega_3,\Omega_4} \left( \int_{\Omega_1} |\overline{\partial}\chi| + |r'_1(\text{Re}k)| \frac{e^{-2|\text{Re}(i\theta)|}}{|s-z|} dA(s) \right) \nonumber
\]

\[
+ \int_{\Omega_4} \frac{|s+\xi|^{-\alpha}}{|s-z|} e^{-2|\text{Re}(i\theta)|} dA(s) \tag{3.89}
\]

Here we use that \( M_{RHP} \) is bounded in the support of \( 1 - \chi \). The first two functions in the last line of above inequations, in another word, are

\[
|\overline{\partial}\chi|, |r'_1(\text{Re}k)|, |r'_2(\text{Re}k)|, \left| \left( \frac{\sigma r_2}{1 + \sigma r_2} \right)'(\text{Re}k) \right|, \left| \left( \frac{r_1}{1 + \sigma r_2} \right)'(\text{Re}k) \right|. \]

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All of above function are in $L^2(\mathbb{R})$ when the imaginary variable is fixed, and $L^2(\mathbb{R})$ norm is independent on the imaginary variable. Therefore, it is valid to use $g(k), \|g(u + iv)\|_{L^2(\mathbb{R})} \leq C_0$ denoting them. In addition, we use $\Omega_1$ as an example to prove

$$
\int_{\Omega_1} \frac{|g(s)|}{|s - z|} e^{-2t|\text{Re}(i\theta)|} dA(s) \leq \int_0^\infty \int_v^\infty \frac{|g(s)|}{|s - z|} e^{-8tuv} du dv
$$

$$
\leq \int_0^\infty \|g\|_{L^2(\mathbb{R})} e^{-8tuv} \left\| \frac{1}{(|u|^2 + |v - \text{Im}(z)|^2)^{\frac{q}{2}}} \right\|_{L^2(\mathbb{R})} dv
$$

$$
\leq \int_0^\infty \frac{e^{-8tv^2}}{|v - \text{Im}(z)|^\frac{q}{p}} dv
$$

$$
\leq \int_\mathbb{R} \frac{e^{-8t(v+\text{Im}(z))^2}}{|v|^\frac{1}{q}} dv = O(t^{-\frac{1}{4}}).
$$

(3.90)

On the other hand, we demonstrate for $\alpha > 0, 2 > \alpha p > 1, p + q = pq$

$$
\int_{\Omega_1} \frac{|s + \xi|^{-\alpha}}{|s - z|} e^{-2t|\text{Re}(i\theta)|} dA(s)
$$

$$
= \int_0^\infty \int_v^\infty \frac{(u^2 + v^2)^{-\frac{1}{2}p} e^{-8tuv}}{((u - \xi - \text{Re}(z))^2 + (v - \text{Im}(z))^2)^{\frac{1}{2}}} du dv
$$

$$
\leq \int_0^\infty e^{-8tv^2} \left( \int_v^\infty (u^2 + v^2)^{-\frac{1}{2}p} du \right)^{\frac{1}{p}} dv
$$

$$
\left( \int_v^\infty ((u - \xi - \text{Re}(z))^2 + (v - \text{Im}(z))^2)^{-\frac{1}{2}q} du \right)^{\frac{1}{q}} dv
$$

$$
\leq \int_0^\infty e^{-8tv^2} v^{\frac{1}{p} - \alpha} |v - \text{Im}(z)|^{\frac{1}{q} - 1} dv
$$

To estimate above integral, we divide the integrating range to two parts:

$$
\int_0^\infty e^{-8tv^2} v^{\frac{1}{p} - \alpha} |v - \text{Im}(z)|^{\frac{1}{q} - 1} dv
$$

$$
= \int_0^{\text{Im}(z)} e^{-8tv^2} v^{\frac{1}{p} - \alpha} (1 - v)^{\frac{1}{q} - 1} dv + \int_\infty^{\text{Im}(z)} e^{-8tv^2} v^{\frac{1}{p} - \alpha} (v - \text{Im}(z))^{\frac{1}{q} - 1} dv
$$

$$
= \text{Im}(z)^{1 - \alpha} \int_0^1 e^{-8t\text{Im}(z)^2v^2} v^{\frac{1}{p} - \alpha} (1 - v)^{\frac{1}{q} - 1} dv + \int_0^\infty e^{-8tv^2} v^{\frac{1}{p} - \alpha} (v + \text{Im}(z))^{\frac{1}{q} - 1} dv
$$

$$
\leq \text{Im}(z)^{1 - \alpha} \int_0^1 (t^\frac{1}{2}\text{Im}(z) v)^{\alpha - 1} v^{\frac{1}{p} - \alpha} (1 - v)^{\frac{1}{q} - 1} dv + \int_0^\infty e^{-8tv^2} v^{-\alpha} dv
$$

$$
\leq O(t^{-\frac{1}{2} + \frac{\alpha}{2}}) + O(t^{-\frac{1}{2} + \frac{q}{2}}) = O(t^{-\frac{1}{2} + \frac{q}{2}}).
$$

(3.91)
According to Proposition 4, substitute $\alpha$ in (3.24) into (3.89) paralleling (3.90). Then (3.89) becomes

$$||S||_{B(L^{\infty})} \lesssim O(t^{-\frac{1}{4}}) + O(t^{-\frac{1}{2} + \frac{\alpha}{2}})$$

Therefore, for efficient large $t$, as Proposition 8 describing, $W(x,t;k)$ exists. According to Proposition 9, following proposition can be proved.

**Proposition 11.** $W(x,t;k)$ satisfies large $z$ asymptotic condition

$$W(x,t;k) = I + \frac{W^{(1)}(x,t)}{k} + o(k^{-1}),$$

with

$$W^{(1)}(x,t) = \frac{1}{\pi} \iint_{C} W(x,t;s)(M_{RHP}\overline{\partial}R_{RHP}^{-1})(x,t;s)dA(s).$$

As $t \to \infty$,

$$W^{(1)}(x,t) \lesssim O(t^{-\frac{\alpha}{2}}) + O(t^{-1 + \frac{\alpha}{2}}),$$

where $\alpha$ is defined in (3.24).

**Proof.** As Proposition 9 shows, $W(x,t;k)$ has integral form

$$W(x,t;k) = I - \frac{1}{\pi} \iint_{C} \frac{W(x,t;s)(M_{RHP}\overline{\partial}R_{RHP}^{-1})(x,t;s)}{s - k}dA(s).$$

To compute the large $k$ asymptotic property, we write $W(x,t;k)$ into

$$W(x,t;k) = I + \frac{1}{k} \left[ \frac{1}{\pi} \iint_{C} W(x,t;s)(M_{RHP}\overline{\partial}R_{RHP}^{-1})(x,t;s)dA(s) \right]$$

$$- \frac{1}{\pi} \iint_{C} sW(x,t;s)(M_{RHP}\overline{\partial}R_{RHP}^{-1})(x,t;s)\frac{dA(s)}{k(s - k)}.$$

Similar to [23], (3.92) is proved. According to Proposition 4, we have

$$W^{(1)}(x,t) = \frac{1}{\pi} \iint_{C} W(x,t;s)(M_{RHP}\overline{\partial}R_{RHP}^{-1})(x,t;s)dA(s)$$

$$\lesssim ||W||_{L^{\infty}} \sum_{j=1}^{4} \iint_{\Omega_j} |M_{RHP}|\overline{\partial}R_{RHP}^{-1}|dA(s)$$

$$\lesssim \sum_{\Omega_1,\Omega_2,\Omega_3,\Omega_4} \left[ \iint_{\Omega_1} \left( |\mathcal{X}| + |\mathcal{Y}| + |\nu_2'(\text{Re}k)| \right) e^{-2|\text{Re}(i\theta)|}dA(s) ight.$$  

$$+ \iint_{\Omega_1} \left( |s + \xi|^{-\frac{1}{2}}1^{-\text{Im}(-\xi)} + |s + \xi|^{-\frac{1}{2}} \right) e^{-2|\text{Re}(i\theta)|}dA(s) \right].$$

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Similar with the proof of Proposition 10 for \( g(k), \|g(u + iv)\|_{L^2_\Omega(\mathbb{R})} \leq C_0 \), there is
\[
\int \int_{\Omega_1} |s| e^{-2t |\text{Re}(i\theta)|} dA(s) \leq \int_0^\infty \int_v^\infty |g(s)| e^{-8tuv} dudv \\
\leq \int_0^\infty \|g(u + iv)\|_{L^2_\Omega(\mathbb{R})} \int_v^\infty |g(s)| e^{-16tuv} dudv \\
\lesssim t^{-\frac{1}{2}} \int_0^\infty e^{-8tv^2} dv \\
\lesssim \mathcal{O}(t^{-\frac{3}{4}}).
\]

Meanwhile, for \( \alpha > 0, \ 2 > \alpha p > 1, \ p + q = pq \)
\[
\int \int_{\Omega_1} |s + \xi|^{-\alpha} e^{-2t |\text{Re}(i\theta)|} dA(s) \leq \int_0^\infty \int_v^\infty (u^2 + v^2)^{-\frac{1}{2} \alpha} e^{-8tuv} dudv \\
\leq \int_0^\infty \left( \int_v^\infty e^{-8qtuv} du \right)^{\frac{1}{q}} \left( \int_v^\infty (u^2 + v^2)^{-\frac{1}{2} \alpha p} du \right)^{\frac{1}{p}} dv \\
\lesssim \int_0^\infty (tv)^{-\frac{1}{4} \alpha} e^{-8tv^2} v^{-\frac{1}{p} - \alpha} dv \\
= t^{-\frac{1}{q}} \int_0^\infty v^{\frac{1}{p} - \frac{1}{q} - \alpha} e^{-8tv^2} dv \\
= \mathcal{O}(t^{-1 + \frac{\alpha}{2}}).
\]

By substituting \( \alpha \) in (3.24) into the above result, we deduce
\[
W^{(1)}(x,t) \lesssim \mathcal{O}(t^{-\frac{3}{4}}) + \mathcal{O}(t^{-1 + \frac{\alpha}{2}}). \tag{3.95}
\]

\[\square\]

4 Long time asymptotic behavior

By summarizing the transformation above, we have
\[
M(x, t; k) = W(x, t; k) E(x, t; k) M_{\text{sol}}(x, t; k) \mathcal{R}(x, t; k) \delta_{\Omega_3}(k). \tag{4.1}
\]
From (3.59), (3.56), (3.92), and $R = I$ on the imaginary axis, large $k$ along the imaginary axis expansion of $M(x, t; k)$ can be deduced as

$$M(x, t; k) = \left( I + \frac{W^{(1)}(x, t)}{k} + o(k^{-1}) \right) \left( I + \frac{E^{(1)}(x, t)}{k} + O(k^{-2}) \right)$$

$$\left( I + \frac{M^{(1)}_{sol}}{k} + O(k^{-2}) \right) \left( I + \frac{\delta^{(1)}\sigma_3}{k} + O(k^{-2}) \right)$$

$$= I + \frac{1}{k} \left( W^{(1)}(x, t) + E^{(1)}(x, t) + M^{(1)}_{sol} + \delta^{(1)}\sigma_3 \right) + o(k^{-1})$$

Denote $M^{(1)}(x, t)$ as the coefficient of $1/k$ term in above expansion of $M(x, t; k)$, then

$$M^{(1)}(x, t) = W^{(1)}(x, t) + E^{(1)}(x, t) + M^{(1)}_{sol} + \delta^{(1)}\sigma_3$$

$$= M^{(1)}_{sol} + \delta^{(1)}\sigma_3 + \frac{1}{\sqrt{8t}} \begin{pmatrix} 0 & -i t \text{Im}\nu(-\xi) \tilde{\beta}_{21} \\ it^{-\frac{1}{2}} - i t \text{Im}\nu(-\xi) \tilde{\beta}_{21} & 0 \end{pmatrix}$$

$$+ O(t^{-\frac{3}{2}} + 2^M \nu(-\xi)) \begin{pmatrix} O(t^{-\frac{1}{2}} - i t \nu(-\xi)), & O(t^{-\frac{1}{2}} + i t \nu(-\xi)) \end{pmatrix}$$

$$+ O(t^{-\frac{3}{2}}) + O(t^{-1 + \frac{\nu}{2}}).$$

Using the (2.27), we can reconstruct a solution of (1.1) denote as $q_{sol}$ where the scattering data $\{0, 0, \{\omega_n, \gamma_m, c_n \delta(\omega_n)^{-2}, d_m \delta(\gamma_m)^2 | n = 1, ..., 2N, m = 1, ..., 2M \}\}$. On this occasion, the solution of RHP0 is $M_{sol}$. Therefore, according to 3.60

$$q_{sol} = 2i \lim_{k \to \infty} (k[M_{sol}(x, t; k)]_{12}) = 2i \sum_{\gamma_m \in \mathbb{N}_2} \beta_m^{(1)}.$$ (4.2)

Then we achieve main result of this paper.

**Theorem 1.** Let $q(x, t)$ be the solution for the initial-value problem (1.1) with generic data $q_0(x)$ admitting assumption 1 and scattering data be used to reconstruct RHP0 \(\{r_1(k), r_2(k), \{\omega_n, \gamma_m, c_n, d_m| n = 1, ..., 2N, m = 1, ..., 2M \}\}. \) Let $\xi = \frac{2}{M} \in \mathbb{R}$ and $q_{sol}(x, t)$ be the solution corresponding to reflectionless scattering data

$$\{0, 0, \{\omega_n, \gamma_m, c_n \delta(\omega_n)^{-2}, d_m \delta(\gamma_m)^2 | n = 1, ..., 2N, m = 1, ..., 2M \}\}.$$

There exists a large constant $T_1 = T_1(\xi)$, for all $t > T_1$,

$$q(x, t) = q_{sol}(x, t) + \frac{\mu \nu(-\xi) \beta_{21}}{\sqrt{2t}} + \begin{cases} O(t^{1 + 2\nu(-\xi)}), & \text{Im}\nu(-\xi) \in \left( \frac{1}{6}, \frac{1}{2} \right) \\ O(t^{\frac{3}{2} + \frac{1}{2} \nu(-\xi)}), & \text{Im}\nu(-\xi) \in \left( 0, \frac{1}{6} \right) \\ O(t^{-\frac{3}{2}}), & \text{Im}\nu(-\xi) \in \left( -\frac{1}{4}, 0 \right). \end{cases} \quad (4.3)$$

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\( q_{\text{sol}}, \nu \) and \( \tilde{\beta}_{12} \) are defined in (4.2), (3.2) and (3.41), respectively. Moreover, the discrete spectrum in \( \Delta \) has main contribution as \( t \to \infty \), where \( \Delta \) is defined in (3.71), respectively. For \( q^\Delta(x,t) \) is a soliton solution with reflectionless scattering data

\[
\{0,0,\{\omega_n,0,c_nT^2(\omega_n)\delta(\omega_n)^{-2},0 \mid \omega_n \in \Delta\}\}
\]

with \( T(z) \) defined in (3.67). Then by (3.81), we can replace the term \( q_{\text{sol}}(x,t) \) in (4.3) with the term \( q^\Delta(x,t) \).

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