Cutting feedback and modularized analyses in generalized Bayesian inference

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Abstract: This work considers Bayesian inference under misspecification for complex statistical models comprised of simpler submodels, referred to as modules, that are coupled together. Such “multi-modular” models often arise when combining information from different data sources, where there is a module for each data source. When some of the modules are misspecified, the challenges of Bayesian inference under misspecification can sometimes be addressed by using “cutting feedback” methods, which modify conventional Bayesian inference by limiting the influence of unreliable modules. Here we investigate cutting feedback methods in the context of generalized posterior distributions, which are built from arbitrary loss functions, and present novel findings on their behaviour. We make three main contributions. First, we describe how cutting feedback methods can be defined in the generalized Bayes setting, and discuss the appropriate scaling of the loss functions for different modules to each other and the prior. Second, we derive a novel result about the large sample behaviour of the posterior for a given module’s parameters conditional on the parameters of other modules. This formally justifies the use of conditional Laplace approximations, which provide better approximations of conditional posterior distributions compared to conditional distributions from a Laplace approximation of the joint posterior. Our final contribution leverages the large sample approximations of our second contribution to provide convenient diagnostics for understanding the sensitivity of inference to the coupling of the modules, and to implement a new semi-modular posterior approach for conducting robust Bayesian modular inference. The usefulness of the methodology is illustrated in several benchmark examples from the literature on cut model inference.

Keywords and phrases: Cutting feedback; Model misspecification; Modularization; Semi-modular inference; Generalized Bayesian inference.

1. Introduction

Complex statistical models are sometimes composed of smaller sub-models, which we call modules, that are interconnected. This modular structure is common when integrating information from multiple data sources, where each data
source is associated with a separate sub-model. When a model with a modular structure is correctly specified, Bayesian inference has some desirable properties, regardless of the number or complexity of the modules. However, when there is misspecification, conventional Bayesian inference may need to be adapted to account for it. This paper explores some new forms of a method called “cutting feedback” for modified Bayesian inference under misspecification.

It is well-known that misspecification of an assumed model compromises the use and interpretation of Bayesian inference; see, e.g., Grünwald (2012) for examples. Nonetheless, when dealing with a multi-modular model, a researcher may suspect that only some modules are grossly misspecified. In such cases, modified Bayesian analyses can be used to preserve valid inference for parameters in the correctly specified modules. This can make model criticism easier and ensure that estimates of parameters in the misspecified modules retain a useful interpretation (Liu et al., 2009). These are some of the goals of the cutting feedback methods which are the focus of this paper, which attempt to limit the influence of unreliable modules. To understand better the wide-ranging applications of cutting feedback and modularized Bayesian inference, we recommend the papers by Jacob et al. (2017) and Liu et al. (2009), with the latter paper focusing on applications in the analysis of computer models.

The current literature on cutting feedback mainly focuses on fully specified parametric models. However, if a parametric model is misspecified, researchers can still produce useful Bayesian inferences by using a posterior based on a loss function that captures the features of the data that are most important. Such generalized Bayesian inference methods (see, for example, Bissiri et al. (2016)), have become increasingly popular in statistical inference. They recover conventional Bayesian inference as a special case when the loss function used in their construction is the negative log likelihood. This paper combines the use of cutting feedback methods with generalized Bayesian inference, resulting in an attractive approach to Bayesian modular inference. Our framework allows a targeted loss function to be used for modules which are misspecified, instead of relying on the negative log likelihood function. Meanwhile, we can continue to use the negative log likelihood function as the loss for modules that are well specified. The generalized Bayes perspective on modular inference is useful in model improvement. Starting with a flawed parametric model specification, we can replace the negative log likelihood for suspect modules with other loss functions to see whether this resolves any incompatibility between the “cut posterior” produced by cutting feedback methods and full posterior inferences.

Our work makes three main contributions to the literature on generalized Bayesian inference and cutting feedback. Firstly, we describe how to define cutting feedback in the generalized Bayesian setting, and discuss how to appropriately scale loss functions for different modules to each other and the prior. Secondly, we derive a novel large sample result that allows us to express the posterior for the parameters of a given module conditional on the parameters of the remaining modules. In contrast, the only existing result on the large sample behaviour of cut posteriors of which we are aware (Pompe and Jacob, 2021) presents a joint analysis of the cut posterior. Pompe and Jacob (2021)
also discuss a novel posterior bootstrap approach to cut posterior computation. As we argue in Section 2.3, a normal approximation to the joint cut posterior provides only limited insight into propagation of uncertainty in cutting feedback, because conditioning on a subset of variables in a multivariate normal distribution results in a conditional covariance matrix that doesn’t depend on the values of the conditioning variables. In contrast, our results justify normal approximations for conditional posterior distributions where covariance matrices change with the values for the conditioning variables, giving useful insights into uncertainty propagation in cut posteriors. Our new result is also applicable to general loss functions, and only requires weak smoothness conditions.

Finally, we use the large sample approximations provided in our second contribution to develop easily computable diagnostics for understanding the coupling of the modules, and to implement a new “semi-modular” posterior for conducting robust modular inference. Semi-modular inference (Carmona and Nicholls, 2020) partially cuts feedback, interpolating between inferences based on the cut and full posterior according to a tuning parameter. The challenges of cut posterior computation also apply to semi-modular inference, with the key difficulty being the evaluation of an intractable marginal likelihood term. Estimation of the semi-modular (and cut) posterior is often done using a computationally burdensome nested Markov chain Monte Carlo (MCMC) method, and our novel semi-modular posterior can be computed efficiently using the large sample approximations we develop, delivering similar results to the semi-modular posterior of Carmona and Nicholls (2020). See Section 3.4 for further details. We illustrate the above diagnostics and semi-modular posterior in two benchmark examples found in the literature on cutting feedback.

**Notation.** Here we define notation used in the remainder of the paper. The term $|| \cdot ||$ denotes the Euclidean norm, while $| \cdot |$ denotes the absolute value function. $C$ denotes an arbitrary positive constant that can change from line-to-line. For $x = (x_1^\top, x_2^\top)^\top \in \mathbb{R}^d$ and a function $f : \mathbb{R}^d \to \mathbb{R}$, we let $\nabla_x f(x)$ denote the gradient of $f(x)$ wrt $x$, and $\nabla_{xx}^2 f(x)$ the Hessian. Let $N\{\mu, \Sigma\}$ denote the normal distribution with mean $\mu$ and covariance matrix $\Sigma$, with $N\{x; \mu, \Sigma\}$ the corresponding normal density at the point $x$. For $\mathcal{D}$ some known distribution, and $x = (x_1^\top, x_2^\top)^\top \in \mathbb{R}^d$ a $d$-dimensional random variable, the notation $x \sim \mathcal{D}$ signifies that the law of $x$ is $\mathcal{D}$, while $x_1 \mid x_2 \sim \mathcal{D}$ signifies that the conditional law of $x_1$ given $x_2$ is $\mathcal{D}$. The measure $P_0^{(n)}$ denotes the true unknown probability measure generating the data, and $\Rightarrow$ denotes weak convergence (under $P_0^{(n)}$).

## 2. Motivation and Framework

Modifying Bayesian inference to limit the influence of a suspect module is the main idea of cutting feedback methods. But what is a module exactly, and how is cutting feedback defined for multi-modular models of arbitrary complexity? This is not a settled question in the current literature. Recent work by Liu and Goudie (2022a) has provided a first step towards clarity, where the authors define modules based on the representation of a Bayesian model in terms of a
directed acyclic graph (DAG) and a partitioning of the observable quantities. It is fair to say, however, that different general formulations of modular inference are still being explored.

In previous work, the most general approach to cutting feedback methods has involved an “implicit” definition through modification of an MCMC algorithm designed to sample the conventional posterior distribution. One implementation of this approach is through the cut function of the WinBUGS and OpenBUGS software packages (Lunn et al., 2009). If a Bayesian model is defined through a DAG, and a Gibbs sampler is considered for sampling the posterior distribution using the DAG parameter nodes as blocks, then “cuts” can be defined for some links of the graph. Each cut corresponds to leaving out a certain term in the joint model when forming the full conditional posterior density for one of the parameter nodes. Once modified full conditional distributions have been constructed, a modified Gibbs sampler iteratively samples from these, and the cut posterior distribution is defined as the stationary distribution of the resulting Markov chain. See Lunn et al. (2009) or Plummer (2015) for a more detailed description.

Lunn et al. (2009) note that the modified full conditional distributions are not the full conditional distributions of any well-defined joint distribution but argue that the use of such inconsistent conditional distributions can be sensible. If modified Gibbs steps are replaced by Metropolis-within-Gibbs updates in the sampling process, Plummer (2015) observed that the stationary distribution of the Markov chain can depend on the proposal used, and went on to define a “two-module” system where an explicit definition of the cut posterior distribution can be given, clarifying some aspects of the implicit cut approach. This two module system is general enough for many applications of Bayesian modular inference in which there might be one suspect model component of particular concern. This two module system is also fundamental to the recent work of Liu and Goudie (2022a) where multi-modular systems and cut posteriors are defined generally. Liu and Goudie (2022a) define two module systems first, based on a partitioning of the observables into two parts, and then consider recursively splitting existing modules into two in order to define more complex multi-modular representations. In what follows, we will focus our discussion on cutting feedback in two-module systems, given their usefulness in applications and their role in defining multi-modular models with more than two modules. We define modules and cutting feedback precisely in the context of this two module system, and refer the interested reader to Liu and Goudie (2022a) for a more general discussion.

2.1. Two module system

The “two module” system of Plummer (2015) considers two data sources, denoted here as $\mathbf{z}$ and $\mathbf{w}$. The data $\mathbf{z}$ consists of $n_1$ observations $\mathbf{z} = (z_1, \ldots, z_{n_1})^\top$, $z_i \in \mathcal{Z}$, and $\mathbf{w}$ consists of $n_2$ observations $\mathbf{w} = (w_1, \ldots, w_{n_2})$, $w_i \in \mathcal{W}$, and we write $n = n_1 + n_2$. Let $\mathbf{y} = (\mathbf{z}^\top, \mathbf{w}^\top)^\top$ denote the entire set of observed data. A
A potentially misspecified statistical model for $y$ is postulated that depends on parameters $\theta \in \Theta \subseteq \mathbb{R}^{d_\theta}$, where $\theta = (\varphi^\top, \eta^\top)^\top$, with $\varphi \in \Phi \subseteq \mathbb{R}^{d_\varphi}$, $\eta \in \mathcal{E} \subseteq \mathbb{R}^{d_\eta}$, and $d_\theta = d_\varphi + d_\eta$. Prior beliefs over $\Theta$ are represented by the prior density $\pi(\theta) = \pi(\varphi)\pi(\eta|\varphi)$.

Plummer (2015) considers Bayesian inference on $\theta$ in cases where the distribution of $z$ depends on $\varphi$, with density $p(z|\varphi)$, while the distribution of $w$ depends on $\eta$ and $\varphi$, with density $p(w|\eta, \varphi)$. In other words, the models for the two data sources have a shared dependence on $\varphi$ (a global parameter) whereas $\eta$ appears only in the model for $w$. We define “modules” here as subsets of terms in the joint Bayesian model. Here there are two modules, with the first consisting of the likelihood term $p(z|\varphi)$ and prior $\pi(\varphi)$, and the second consisting of the likelihood $p(w|\eta, \varphi)$ and conditional prior $\pi(\eta|\varphi)$. The structure of the model is modular in the sense that valid Bayesian inference about $\varphi$ can be obtained based on module one only (i.e. $\pi(\varphi|z) \propto \pi(\varphi)p(z|\varphi)$), while given a value of $\varphi$, valid conditional Bayesian inference for $\eta|\varphi$ can be obtained based on module two only (i.e. $\pi(\eta|\varphi, w) \propto \pi(\eta|\varphi)p(w|\eta, \varphi)$). The graphical structure of the model is given in Figure 1, with the nodes to the left of the red dashed line comprising module one, and the nodes to the right comprising module two, where it is assumed in the figure that $\pi(\eta|\varphi)$ does not depend on $\varphi$ for simplicity.

We assume that there is high confidence in the accuracy of the model for $z|\varphi$ in the first module, but it is uncertain that the model for $w|\eta, \varphi$ in the second model is adequate. Consequently, if we were to conduct standard Bayesian inference on $\theta$ using both modules, our inferences on the shared parameter $\varphi$ could be contaminated by misspecification of the second module, and any useful interpretation for our inferences about $\eta$ may also be compromised if the parameters $\varphi$ do not have their intended meaning. See Section 4 for examples.

We will discuss two methods that can guard against compromised inferences on $\varphi$ due to potential misspecification of the second module. The first method involves using a loss function rather than a parametric model to capture the important features of the data for the second module; a generalized posterior is constructed based on the loss function for the parameters of interest. The second approach is to employ cutting feedback methods. Generalized Bayesian

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Graphical structure of the two-module system. The red dashed line indicates the cut.}
\end{figure}
methods and cutting feedback are not used here as approximations to conventional Bayesian inference; they are alternative inferential approaches intended to address the issue of misspecification and having a sound statistical justification in their own right. Approximate methods for computation may be of interest, but this is discussed later in Section 3, based on the asymptotic results we develop there. In this article, we aim to combine cutting feedback and generalized Bayesian updating to produce robust Bayesian inferences on \( \theta \), and we explain these concepts next.

2.2. Generalized Posteriors

When the model is misspecified, standard Bayesian approaches can deliver inferences that are poor or unreliable (see, e.g., Grünwald and Van Ommen (2017) for specific examples, as well as Kleijn and van der Vaart (2012) for general results in parametric models). Specifying full probabilistic models for complex data can be difficult, and it would be attractive if Bayesian inference could be done only for parameters of interest appearing in a loss function. Under some mild conditions on the loss, Bissiri et al. (2016) justify a Bayesian analysis in this setting in which the likelihood in the usual Bayesian update is replaced with a “loss likelihood” with a highly constrained form. The target parameter of interest is the population minimizer of the loss.

In a standard generalized posterior analysis without modular structure, there is a parameter \( \theta \) and data \( \mathbf{x} = (x_1, \ldots, x_n)^\top \) say. The prior \( \pi(\theta) \) is to be updated into a generalized posterior \( \pi(\theta|\mathbf{x}) \), where the belief update depends on \( \mathbf{x} \) only through a loss function \( q_n(\theta) = \sum_{i=1}^n q(x_i; \theta) \), where \( q(x_i; \theta) \) is the loss for the \( i \)th observation. A remarkable argument in Bissiri et al. (2016) specifies the form that the belief update must take, under some mild conditions. They consider the requirement of order coherence, where if the data \( \mathbf{x} \) are split into two parts and an update is done sequentially, then the result should be the same as if a single update were done using all the data. Order coherence is enough to determine the form of the generalized posterior density, which is

\[
\pi(\theta|\mathbf{x}) \propto \pi(\theta) \exp\{-\nu q_n(\theta)\},
\]

where \( \nu \geq 0 \) is called the learning rate, and scales the information in the loss function appropriately relative to the information in the prior. While the generalized Bayesian update of Bissiri et al. (2016) is motivated by Bayesian notions of coherence, the choice of learning rate gives the opportunity to bring in other considerations such as information matching in the update (Holmes and Walker, 2017; Lyddon et al., 2019) or achieving good frequentist performance for estimating functionals of interest (Syring and Martin, 2018). A generalization of the arguments in Bissiri et al. (2016) relevant to the justification of parametric cutting feedback methods is discussed in Nicholls et al. (2022). Generalized Bayesian updating is also related to PAC-Bayes methods; see Alquier (2021) for an introduction.
Consider now the case of modular Bayesian inference in the two module system. The decomposition of the statistical model into two distinct modules, containing data \( z \) and \( w \) respectively, implies that we are free to choose separate loss functions for each module. Let \( \ell : Z \times \Phi \to \mathbb{R} \) denote the loss function for module one, involving a parameter \( \varphi \), and \( m : W \times E \times \Phi \to \mathbb{R} \) denote the loss function for module two, involving parameters \( \eta \) and \( \varphi \). In the following we write

\[
Q_n(\theta) = L_{n_1}(\varphi) + M_{n_2}(\eta, \varphi), \quad L_{n_1}(\varphi) = -\sum_{i=1}^{n_1} \ell(z_i, \varphi), \quad M_{n_2}(\eta, \varphi) = -\sum_{i=1}^{n_2} m(w_i, \eta, \varphi),
\]

so that \(-L_{n_1}(\varphi)\) and \(-M_{n_2}(\eta, \varphi)\) are the empirical loss functions for the first and second modules respectively. When the two sample sizes are equal, i.e., \( n_1 = n_2 \), we abuse notation and simply denote the criteria as \( L_n(\varphi) \) and \( M_n(\eta, \varphi) \).

Consider first a belief update of the prior density \( \pi(\varphi) \) using \( z \) and the first module loss function \( \ell(\cdot) \). The order coherence argument of Bissiri et al. (2016) implies that the generalized posterior density \( \pi(\varphi|z) \) must take the form

\[
\pi(\varphi|z) \propto \pi(\varphi) \exp\{\nu L_{n_1}(\varphi)\}, \quad \nu \geq 0 \text{ is a learning rate for the first module that needs to be chosen. If the loss function is the negative log-likelihood, and we take } \nu = 1, \text{ this is the conventional Bayesian update.}
\]

Once \( \pi(\varphi|z) \) is obtained, suppose we now take \( \pi(\theta|z) = \pi(\varphi|z)\pi(\eta|\varphi) \) as the “prior” for a Bayesian update using the information in the second module. Again following the order coherence argument of Bissiri et al. (2016), and its extensions in Nicholls et al. (2022), the generalized posterior density \( \pi(\theta|z, w) \) given \( z \) and \( w \) must take the form

\[
\pi(\theta|z, w) \propto \pi(\theta|z) \exp\{\nu' M_{n_2}(\eta, \varphi)\}
\]

\[
\propto \pi(\varphi)\pi(\eta|\varphi) \exp\{\nu L_{n_1}(\varphi) + \nu' M_{n_2}(\eta, \varphi)\},
\]

where \( \nu' \geq 0 \) is an additional learning rate. As before, if \( \nu' = 1 \) and the loss function \( m(\cdot) \) is the log-likelihood, this is a conventional Bayesian update using the data for the second module. For the full belief update \( (2.2) \), it takes the form of Bayesian updating where the likelihood has been replaced by the loss likelihood \( \exp\{\nu L_{n_1}(\varphi) + \nu' M_{n_2}(\eta, \varphi)\} \). If the two loss terms \( L_{n_1}(\varphi) \) and \( M_{n_2}(\eta, \varphi) \) are of the same type and hence on the same scale, then it could make sense to choose \( \nu = \nu' \), and we would obtain a loss likelihood with a single learning rate \( \nu \) and of the customary form in a generalized Bayesian analysis, \( \exp\{\nu Q_n(\theta)\} \), where \(-Q_n(\theta)\) is the overall empirical loss.

In generalized Bayesian inference the choice of the learning rate is very important, and this is true in the case of modular inference considered here also. See Wu and Martin (2020) for a review and comparison of different methods. Generalizing similar ideas to Holmes and Walker (2017) and Lyddon et al. (2019), later we suggest choosing \( \nu \) and \( \nu' \) based on an information matching argument. It is often not necessary in the applications we consider to estimate the first module learning rate \( \nu \); if the first module is specified through a probabilistic model and we are confident in this specification, \( \nu = 1 \) is the natural choice.
However, we discuss the choice of learning rates in Section 3.2 in a general way, addressing the situation where it may be desired to choose both $\nu$ and $\nu'$. Our later theoretical results will be written using a loss likelihood with a single learning rate where $\nu = \nu'$. There is no loss of generality in this, or even in omitting learning rates altogether in the theoretical discussion, since any learning rates can be absorbed into the definition of the loss function.

### 2.3. Cutting Feedback with Generalized Posteriors

Our confidence in the accuracy of the first module means that the criterion $L_{n_1}(\phi)$ can be chosen as the log-likelihood. However, since we are working with generalized posteriors, we only maintain that $L_{n_1}(\phi)$ produces “reliable inferences” for $\phi$. Our lack of confidence in the specification of the second module means we are concerned that incorporating this module may contaminate our inferences for $\phi$. In such situations, cutting feedback methods (see, e.g., Plummer, 2015) can be used to mitigate the impact of misspecification.

In the two module system discussed in Section 2.1 for a probabilistic model, the first module consists of the terms $\pi(\phi)$ and $p(z|\phi)$ in the joint Bayesian model, and the second module consists of $p(\eta|\phi)$ and $p(w|\eta, \phi)$. In a generalized Bayesian analysis, module one consists of $p(\phi)$ and the loss likelihood term $\exp\{\nu L_{n_1}(\phi)\}$, and module two consists of $\pi(\eta|\phi)$ and the loss likelihood $\exp\{\nu M_{n_2}(\eta, \phi)\}$, if a single learning rate is assumed for both modules.

Generalized Bayesian analyses have been used in the context of two module system previously, but only as a justification for parametric cutting feedback methods when a probabilistic model is specified. Carmona and Nicholls (2020) considered order coherence for cut and semi-modular inference methods, and Nicholls et al. (2022) observed that the implicit loss function used in these approaches is not additive as required in the theory of Bissiri et al. (2016). Nicholls et al. (2022) generalize the existing theory to “prequentially additive” loss functions, which is enough to justify standard parametric cut inference as valid and order coherent generalized Bayesian updating. In contrast to this work, our aim is not to justify cutting feedback methods for probabilistic multi-modular models as coherent in some sense, but to consider situations where there may be no probabilistic model for the data, but only loss functions to connect module data to parameters.

To present cutting feedback for generalized posteriors, decompose $\pi(\theta|y)$ in (2.2) as the product of a marginal posterior for $\phi|z$, a conditional posterior for $\eta|w, \phi$, and a “feedback term”:

$$
\pi(\theta|y) = \pi_{\text{cut}}(\phi|z)\pi(\eta|w, \phi)\tilde{p}(w|\phi),
$$

(2.3)

where $\pi_{\text{cut}}(\phi|z) \propto \pi(\phi)\exp\{\nu L_{n_1}(\phi)\}$, $\pi(\eta|w, \phi) := \pi(\eta|\phi)\exp\{\nu M_{n_2}(\eta, \phi)\}/m_\eta(w|\phi)$, and

$$
\tilde{p}(w|\phi) \propto m_\eta(w|\phi), \quad m_\eta(w|\phi) = \int_E \pi(\eta|\phi)\exp\{\nu M_{n_2}(\eta, \phi)\}d\eta.
$$

(2.4)
The feedback term $\tilde{p}(w|\varphi)$ derives its name from representing the influence of module two on the marginal posterior for $\varphi$. To understand this better, consider integrating out $\eta$ in (2.3), to obtain $\pi(\varphi|y) = \pi_{\text{cut}}(\varphi|z)\tilde{p}(w|\varphi)$. Since $\pi_{\text{cut}}(\varphi|z)$ represents the posterior density for $\varphi$ based only on the first module data $z$, we see that $\tilde{p}(w|\varphi)$ modifies this posterior based on the second module data to give the $\varphi$ marginal of $\pi(\theta|y)$. Dropping the feedback term $\tilde{p}(w|\varphi)$ in (2.3) produces a “generalized cut posterior”:

$$\pi_{\text{cut}}(\theta|z, w) := \pi_{\text{cut}}(\varphi|z)\pi(\eta|w, \varphi).$$

In this joint cut posterior, marginal posterior inferences for $\varphi$ are obtained based on module one only, and the conditional posterior density of $\eta$ given $\varphi$ is the same as for $\pi(\theta|y)$ and based on module two only. Our discussion of cut inference is in the generalized Bayesian framework, but if we use negative log likelihood as the loss for an assumed probabilistic model, our definition of the cut posterior reduces to the conventional one in the literature.

Obtaining samples from the cut posterior $\pi_{\text{cut}}(\theta|z, w)$ can be challenging. Since

$$\pi_{\text{cut}}(\theta|z, w) \propto \pi(\varphi)\exp\left\{\nu L_{n_1}(\varphi)\right\} \frac{\pi(\eta|\varphi)\exp\left\{\nu M_{n_2}(\eta, \varphi)\right\}}{m_\eta(w|\varphi)},$$

if MCMC is used to sample from $\pi_{\text{cut}}(\theta|z, w)$, we must evaluate the term $m_\eta(w|\varphi)$. This term is similar to a “marginal likelihood” for $\eta$ conditional on a fixed $\varphi$, and is generally not available in closed form outside of toy examples. In principle, even though we are in the case of generalized posteriors, the computationally intensive methods proposed by Plummer (2015), and Jacob et al. (2017) to deal with the intractable term $m_\eta(w|\varphi)$ could be used to sample from the cut posterior.

While sampling from $\pi_{\text{cut}}(\theta|z, w)$ is difficult, draws from $\pi(\eta|w, \varphi)$ for any $\varphi$ can be made without the need to compute $m_\eta(w|\varphi)$. This suggests the following sequential algorithm to obtain draws from $\pi_{\text{cut}}(\theta|w, z)$: first, sample $\varphi' \sim \pi_{\text{cut}}(\varphi|z)$; then, sample $\eta' \sim \pi(\eta|w, \varphi')$. At the first stage, draws from $\pi_{\text{cut}}(\theta|z, w)$ could be obtained by running an MCMC chain targeting the posterior density $\pi_{\text{cut}}(\varphi|z)$. The conditional draws of $\eta$ given $\varphi$ are then performed by running a separate MCMC chain for each sample, which is computationally burdensome. The approach is reminiscent of multiple imputation algorithms, and was originally suggested by Plummer (2015), who also discussed a related tempering method of similar computational complexity. The sequential sampling approach above can also be thought of as implementing a modified Gibbs sampling algorithm with blocks $\varphi$ and $\eta$, but where the likelihood term from the second module is dropped when forming the full conditional distribution for $\varphi$. As mentioned earlier, the resulting modified conditional distributions are not the full conditional distributions of any joint distribution in general, and if we attempt to replace the usually intractable direct sampling of the modified conditional distributions with Metropolis-within-Gibbs steps, then the stationary distribution of the MCMC sampler depends on the proposal used. A number of other authors have investigated computation for cutting feedback (Jacob et al.,...
and this remains an active area of research.

The sequential definition of the cut posterior distribution in the two-module system suggests that the statistical analysis of cut procedures should study the marginal cut posterior density \( \pi_{\text{cut}}(\varphi | z) \) to understand cut inferences for \( \varphi \), and the conditional posterior of \( \pi(\eta | w, \varphi) \) to understand how uncertainty about \( \varphi \) propagates to marginal cut inferences about \( \eta \). This is the strategy we follow in the next section. Such an analysis is complicated by the fact that \( \pi_{\text{cut}}(\theta | z, w) \) does not arise as a posterior for a generative model, and therefore we must use techniques employed in the study of generalized posteriors to analyze \( \pi_{\text{cut}}(\theta | z, w) \).

3. The Behavior of \( \pi_{\text{cut}}(\theta | z, w) \)

In this section, we explore the behavior of \( \pi_{\text{cut}}(\theta | z, w) \) by separately analysing \( \pi_{\text{cut}}(\varphi | z) \), and then analysing \( \pi(\eta | w, \varphi) \), when we condition on an observed value of \( \varphi \) within the high probability region of \( \pi_{\text{cut}}(\varphi | z) \). This yields useful insights into the behavior of cut posteriors and allows us to develop new diagnostic tools for examining these posteriors. The normal approximations implied by our asymptotic results are also valuable for cut posterior computation. As discussed in Section 2.3, a common way to sample the cut joint posterior distribution involves a nested MCMC scheme where a separate MCMC chain is run to draw a sample of \( \eta \) from its posterior conditional density for each marginal cut posterior sample \( \varphi \). If this MCMC step can be replaced by a draw from a normal approximation, or the normal approximation is used to obtain a good proposal density for MCMC or importance sampling, then this can reduce the computational burden of commonly used methods for cut posterior computation.

3.1. Maintained Assumptions and Main Results

The assumptions used to obtain the following theoretical result constitutes a generalization of the assumptions often employed to analyze the behavior of generalized posteriors; see Miller (2021) for an in-depth discussion. We consider an asymptotic regime in which there is a limiting ratio for the sample sizes for the two modules, \( \zeta := \lim_{n \to \infty} n_1/n_2, 0 < \zeta < \infty \). First, we consider the cut posterior \( \pi_{\text{cut}}(\varphi | z) \), and maintain the following conditions, which are sufficient to demonstrate posterior concentration.

Assumption 1. (i) There exist \( L(\varphi) \) such that \( \sup_{\varphi \in \Phi} |n_1^{-1}L_{n_1}(\varphi) - L(\varphi)| = o_p(1) \). (ii) There is a unique \( \varphi^* \in \text{Int}(\Phi) \) such that for every \( \delta > 0 \) there exists \( \epsilon(\delta) > 0 \) so that \( \sup_{\|\varphi - \varphi^*\| \geq \delta} \{L(\varphi) - L(\varphi^*)\} \leq -\epsilon(\delta) \). (iii) \( \pi(\varphi) \) is continuous on \( \Phi \), with \( \pi(\varphi^*) > 0 \), and \( \int_{\Phi} \|\varphi\|^2 \pi(\varphi) d\varphi < \infty \). (iv) For an arbitrary \( \delta > 0 \), and \( \|\varphi - \varphi^*\| \leq \delta \), \( L(\varphi) \) and \( L_{n_1}(\varphi) \) are twice continuously differentiable, with \( \sup_{\|\varphi - \varphi^*\| \leq \delta} \|\nabla^2_{\varphi^*} L_{n_1}(\varphi)/n_1 - \nabla^2_{\varphi^*} L(\varphi)\| = o_p(1) \), and \( \nabla^2_{\varphi^*} L(\varphi^*) \) positive-definite. (v) \( \nabla_{\varphi} L_{n_1}(\varphi^*)/\sqrt{n_1} = O_p(1) \).
Remark 1. Assumption 1 is similar to the standard conditions employed to obtain posterior asymptotic normality, see, e.g., Lehmann and Casella (2006) (Ch 6.8.1) or Theorem 4 in Miller (2021), but allows $L_{n_1}(\varphi)$ to be an arbitrary criterion function. Assumptions (i)-(iii) allow for posterior concentration onto $\varphi^*$, while the smoothness conditions in (iv)-(v) ensure this concentration occurs in a Gaussian manner. Assumption 1 (iv) and (v) are maintained for simplicity, and can be replaced with ‘stochastic differentiability’ assumptions at the introduction of additional technicalities.

Define $\Sigma_{11} := \nabla_\varphi \ln(\varphi^*), \quad Z_{n_1}(\varphi^*) := -\Sigma_{11}^{-1} \nabla_\varphi L_{n_1}(\varphi^*)/\sqrt{\bar{m}_1}$, the local parameter $\phi := \sqrt{\bar{m}_1}(\varphi - \varphi^*)$ and its posterior $\pi(\phi|z) = \pi(\varphi^* + \phi/\sqrt{\bar{m}_1}|z)/\sqrt{\bar{m}_1}^{d_\varphi}$, which has support $\Phi_{n_1} := \{\phi : \sqrt{\bar{m}_1}(\varphi - \varphi^*) \in \Phi\}$. Lemma 1 states that the cut posterior $\pi_{cut}(\phi|z)$ behaves like a Gaussian density with mean $Z_{n_1}(\varphi^*)$, and covariance $[\nu \Sigma_{11}]^{-1}$.

Lemma 1. Under Assumption 1, $\int_{\Phi_{n_1}} \|\phi\| \left|\pi_{cut}(\phi|z) - N\{\phi; Z_{n_1}(\varphi^*), [\nu \Sigma_{11}]^{-1}\}\right| d\phi = o_p(1)$.

The conditioning value of $\varphi$ in $\pi(\eta|w, \varphi)$ plays a key role in the behavior of posterior. To demonstrate this, we view $M_{n_2}(\eta, \varphi)$ as being indexed by a fixed $\varphi \in \Phi_3$, and to reinforce this perspective we use the notation $M_{n_2}(\eta|\varphi) := M_{n_2}(\eta, \varphi)$. Let $\Phi_\delta := \{\varphi \in \Phi : \|\varphi - \varphi^*\| \leq \delta\}$ denote an arbitrary $\delta$-neighborhood of $\varphi^*$, and consider the following regularity conditions on $M_{n_2}(\eta|\varphi)$.

Assumption 2. (i) There exist $M(\eta|\varphi)$ such that, for some $\delta > 0, \sup_{\varphi \in \Phi_3, \eta \in \mathcal{E}} |n_2^{-1} M_{n_2}(\eta|\varphi) - M(\eta|\varphi)| = o_p(1)$. (ii) Given $\delta_1 > 0$, for each $\varphi \in \Phi_3$, there is an $\eta^*_\varphi \in \text{Int}(\mathcal{E})$ such that for any $\delta_2 > 0$, there exist $\epsilon(\delta_1, \delta_2) > 0$, so that $\sup_{\varphi \in \Phi_3} \sup_{\|\eta - \eta^*_\varphi\| \geq \delta_2} \{M(\eta|\varphi) - M(\eta^*_\varphi|\varphi)\} \leq -\epsilon(\delta_1, \delta_2)$.

Assumption 3. (i) For some $\delta > 0$, and each $\varphi \in \Phi_3$, $\pi(\eta|\varphi)$ is continuous in $\eta$. (ii) $\sup_{\varphi \in \Phi_3} \int_{\mathcal{E}} \|\eta\| \pi(\eta|\varphi) d\eta < \infty$.

Remark 2. Assumption 2 constitutes a set of conditions on the conditional loss function $M_{n_2}(\eta|\varphi)$, which, together with the prior condition in Assumption 3, ensure that for a fixed $\varphi \in \Phi_3$ the conditional posterior concentrates onto $\eta^*_\varphi$. These conditions imply that if we study $\pi(\eta | \varphi, w)$ in a neighborhood of $\varphi^*$, the conditional posterior concentrates mass near $\eta^*_\varphi$. The form of this posterior means that this conditional interpretation of concentration is a more natural way of representing posterior behavior than the conventional joint analysis.

Technically, Assumption 2(i) assumes uniform convergence of the loss function, but where we restrict the $\varphi$ parameter to the neighbourhood $\Phi_3$. Of course, a sufficient condition for this would simply be uniform convergence of $M_{n_2}(\eta|\varphi)$ across $\mathcal{E} \times \Phi$. Assumption 2(ii) assumes that for each $\varphi \in \Phi_3$, the limit criterion $M(\eta|\varphi)$ has a unique optimum. This “conditioning on” $\varphi \in \Phi_3$ then allows us to view $M_{n_2}(\eta|\varphi)$ as being indexed by a fixed parameter value, and we can then employ similar regularity conditions to those used in the study of frequentist estimation theory but at an arbitrary $\varphi \in \Phi_3$; we refer to Portier (2016)
Assumption 4. For some \( \infty \)mentary material gives a similar result for the case of generalized posteriors.)

That depends on fixed variance that in large samples the cut posterior for useful for at least two reasons. Firstly, the only other result on the behavior of \( R \)If for some \( \varphi \)

which has support where \( \phi \rightarrow \varphi \) for a discussion and several examples. We remark that, in cases where \( M(\eta|\varphi) \) is known, the implicit function theorem could be used to obtain the mapping \( \phi \rightarrow \varphi \) either numerically or, if available, analytically, and would allow for immediate verification of Assumption 2(ii).

Assumption 3(i) is a standard regularity condition, while Assumption 3(ii) implies that the conditional prior has sufficient moments. A sufficient condition for the latter condition is that the posterior \( \pi(\eta|\varphi) = \pi(\eta) \) and \( \int \|\eta\|\pi(\eta)d\eta < \infty \).

Assumption 4. For some \( \delta_1, \delta_2 > 0 \), the following are satisfied. There exist a vector function \( \Delta_{n_2}(\eta|\varphi) \), and matrix function \( J(\eta|\varphi) \) such that

\[
M_{n_2}(\eta|\varphi) - M_{n_2}(\eta^*_\varphi|\varphi) = (\eta - \eta^*_\varphi)^\top \Delta_{n_2}(\eta^*_\varphi|\varphi) - \frac{n_2}{2} (\eta - \eta^*_\varphi)^\top J(\eta|\varphi)(\eta - \eta^*_\varphi) + R_{n_2}(\eta, \varphi).
\]

(i) for all \( \varphi \in \Phi_{\delta_1}, \Delta_{n_2}(\eta^*_\varphi|\varphi)/\sqrt{n_2} = O_p(1) \);

(ii) the map \( \eta \rightarrow J(\eta|\varphi) \) is continuous for all \( \|\eta - \eta^*_\varphi\| \leq \delta_2 \), for each \( \varphi \in \Phi_{\delta_1} \), and \( J(\eta^*_\varphi|\varphi) \) is positive-definite for each \( \varphi \in \Phi_{\delta_1} \);

(iii) for any \( \delta_2 > 0 \), \( \sup_{\varphi \in \Phi_{\delta_1}} \sup_{\|\eta - \eta^*_\varphi\| \leq \delta_2} R_{n_2}(\eta, \varphi)/(1 + n_2\|\eta - \eta^*_\varphi\|^2) = O_p(1) \).

Remark 3. Assumption 4 ensures that \( M_{n_2}(\eta|\varphi) \) admits a valid quadratic expansion around \( \eta^*_\varphi \) for each \( \varphi \in \Phi_{\delta} \); a sufficient condition for this is that, for each \( \varphi \in \Phi \), \( M_{n_2}(\eta|\varphi) \) is twice continuously differentiable in \( \eta \), and that the matrix of second derivatives \( -\nabla_\eta M_{n_2}(\eta|\varphi)/n_2 \) uniformly converges to its expected counterpart, denoted as \( J(\eta|\varphi) \). Assumption 4(i) requires that the first term in the quadratic expansion is asymptotically bounded for each pair \( (\eta^*_\varphi, \varphi) \). A sufficient condition for this is that for some \( \delta_1, \delta_2 \), the class \( \mathcal{D} := \{ z \rightarrow \Delta_{n_2}(\eta|\varphi)(z) : \|\eta - \eta^*_\varphi\| \leq \delta_1, \varphi \in \Phi_{\delta_2}, \eta \in \mathcal{E} \} \) is \( P \)-Donsker. Assumption 4(ii) requires continuity of the map \( \eta \rightarrow J(\eta|\varphi) \), which is satisfied if \( M_{n_2}(\eta|\varphi) \) is twice continuously differentiable in \( \eta \) for each \( \varphi \in \Phi_{\delta} \). Assumption 4(iii) gives control on the remainder term, and will be satisfied, for instance, when \( M_{n_2}(\eta|\varphi) \) is twice continuously differentiable.

The above assumptions allow us to study the large sample behavior of the translated posterior \( \pi(\eta - \eta^*_\varphi|\varphi) \). To present this behavior as succinctly as possible, define \( Z_{n_2}(\eta^*_\varphi|\varphi) := J(\eta^*_\varphi|\varphi)^{-1} \Delta_{n_2}(\eta^*_\varphi|\varphi)/\sqrt{n_2} \), as well as the local parameter \( t := \sqrt{n_2}(\eta - \eta^*_\varphi) \) and its posterior \( \pi(t|\varphi) = \pi(\eta^*_\varphi + t/\sqrt{n_2}|\varphi)/\sqrt{n_2} \), which has support where \( \mathcal{E}_{n_2} := \{ t = \sqrt{n_2}(\eta - \eta^*_\varphi) : \eta \in \mathcal{E}, \varphi \in \Phi_{\delta} \} \).

Theorem 1. If for some \( \delta > 0 \), Assumptions 1-4 are satisfied for \( \varphi \in \Phi_{\delta} \), then

\[
\int_{\mathcal{E}_{n_2}} \|t\| \pi(t|\varphi) - N \{ t; Z_{n_2}(\eta^*_\varphi|\varphi), [\nu J(\eta^*_\varphi|\varphi)]^{-1} \} \} dt = O_p(1).
\]

Theorem 1 demonstrates that in large samples \( \pi(\eta|\varphi) \) behaves like a Gaussian density with a mean and variance that both depend on \( \varphi \). This result is useful for at least two reasons. Firstly, the only other result on the behavior of cut posteriors of which we are aware, Pompe and Jacob (2021), demonstrates that in large samples the cut posterior for \( \pi(\varphi^*, \eta^*) \) is Gaussian with a variance that depends on fixed \( \varphi^* \) and \( \eta^* = \eta^*_\varphi^* \). (Corollary 1 in the supplementary material gives a similar result for the case of generalized posteriors.)
That is, in a conventional multivariate normal (Laplace) approximation of the joint cut posterior, the induced conditional posterior approximation results in a covariance matrix that does not depend on the conditioning value $\phi$ but only on $\phi^*$. Since in small-to-medium sample sizes the conditional posterior $\pi(\eta|\boldsymbol{w}, \phi)$ will have a mean and variance that changes with the value of $\phi$, such a global approximation is unlikely to be accurate.

Secondly, the conditional approximation in Theorem 1 can be directly used in cases where accessing $\pi(\eta|\boldsymbol{w}, \phi)$ may be difficult but where $Z_{n_2}(\eta^*|\phi)$ and $J(\eta^*|\phi)$ can be easily estimated. The latter may occur, for example, in cases where the MCMC sampler has a difficult time sampling $\pi(\eta|\boldsymbol{w}, \phi)$ at the particular value of $\phi$ on which we are conditioning. While direct use of Theorem 1 involves a computational approximation to the actual conditional posterior distribution, the normal approximation can also be useful as a proposal distribution for MCMC or importance sampling.

### 3.2. Calibration of learning rates

The uncertainty quantification of the generalized cut posterior density $\pi_{\text{cut}}(\theta|\boldsymbol{z}, \boldsymbol{w})$ depends crucially on the choice of learning rates, which we now discuss. Consider the loss likelihood term in (2.2), where $\nu$ and $\nu'$ need to be chosen. Lyddon et al. (2019), inspired by an earlier method of Holmes and Walker (2017), suggest to choose learning rates by matching the Fisher information number for the generalized Bayes update to the Fisher information number from an update based on a loss likelihood bootstrap approach, asymptotically. We do not describe here in detail the reasoning behind the method of Lyddon et al. (2019), but the key to its application here for estimation of multiple learning rates is to exploit the modular structure of the model. We set the first learning rate $\nu$ based on the prior to posterior update for $\phi$ in the first module, and set the second learning rate $\nu'$ based on the conditional prior to conditional posterior update for $\eta$ in the second module, fixing $\phi$ to an estimate based on module one.

To state the idealized learning rates we require some additional notation. Let

$$
\Sigma_{11} = -\nabla^2_{\phi\phi} L(\phi^*), \quad \Sigma_{22} = -\nabla^2_{\eta\eta} M(\eta^*|\phi^*), \quad \Sigma_{12} = \nabla^2_{\phi\eta} M(\eta^*|\phi^*),
$$

$$
\Psi_{11} = \lim_{n \to \infty} \text{Cov}(L_{n_1}(\phi^*)/\sqrt{n_1}), \quad \Psi_{22} = \lim_{n \to \infty} \text{Cov}(\Delta_{n_2}(\eta^*|\phi^*)/\sqrt{n_2}).
$$

With this notation, if we apply the method of Lyddon et al. (2019) for choosing $\nu$ based on the update for the parameter $\phi$ using the first module only, we obtain the ideal choice

$$
\nu = \frac{\text{tr}(\Sigma_{11}\Psi_{11}^{-1}\Sigma_{11})}{\text{tr}(\Sigma_{11})}.
$$

We can estimate $\Sigma_{11}$ by $n_1^{-1}\nabla^2_{\phi\phi} L_{n_1}(\hat{\phi})$, where $\hat{\phi} = \arg\max_{\phi} L_{n_1}(\phi)$. To estimate $\Psi_{11}$, we could use $n_1^{-1}\sum_{i=1}^{n_1} \nabla_{\phi\phi} \ell(z_i; \hat{\phi})\nabla_{\phi\phi} \ell(z_i; \hat{\phi})^\top$, although $\Psi_{11}$ can also be estimated in other ways.

After calibrating $\nu$ based on the first module, we can calibrate $\nu'$ by considering a conditional update of our beliefs for $\eta$ in the second module, conditional on
an estimate of $\varphi$ from the first module, $\varphi = \hat{\varphi}$ say. Matching the Fisher information number suggests choosing $\nu'$ as $\nu' = \text{tr}(\Sigma_{22} \Psi_{22}^{-1} \Sigma_{22})/\text{tr}(\Sigma_{22})$. To estimate $\Sigma_{22}$ we can use $n_2^{-1} \nabla^2 \eta M_{n_2}(\tilde{\eta}_{\varphi}|\varphi)$, where $\tilde{\eta}_{\varphi} = \arg\max_{\eta} M_{n_2}(\eta|\varphi)$, and $\Psi_{22}$ can be estimated by $n_2^{-1} \sum_{i=1}^{n_2} \nabla_{\eta}m(w_i; \tilde{\eta}_{\varphi}, \varphi)\nabla_{\eta}m(w_i; \tilde{\eta}_{\varphi}, \varphi)^\top$, or using some other method.

In a conventional generalized Bayesian analysis, there is only one learning rate to choose, but here there are two. This makes choosing learning rates more difficult, but also makes the modular generalized Bayesian approach more flexible. The way that marginal inferences about $\varphi$ and conditional inferences for $\eta$ given $\varphi$ can be done separately in a modular approach for two different loss functions makes the choice of two learning rates feasible. We thank two anonymous referees for their insight in encouraging us to explore further the choice of separate learning rates for different modules.

### 3.3. Diagnostics for $\eta|w, \varphi$: Understanding Uncertainty Propagation

Theorem 1 demonstrates that even in large samples the behavior of $\pi(\eta|w, \varphi)$ depends on the value of $\varphi$ on which we are conditioning. Moreover, for different values of $\varphi$, the resulting mean and variance can vary substantially. It is therefore useful to understand how our uncertainty about $\varphi$ propagates into our inferences for $\eta$.

Using the result of Theorem 1, this uncertainty can be viewed in many different ways. For instance, if $\eta$ is low-dimensional, we can visualise the impact of $\varphi$ on the posterior for $\eta|w, \varphi$ by viewing the kernel

$$|J(\eta^*_\varphi|\varphi)|^{1/2} \exp \left\{ -n_2 \cdot \nu \cdot (\eta - \eta^*_\varphi)^\top J(\eta^*_\varphi|\varphi)(\eta - \eta^*_\varphi)/2 \right\},$$

across a given range of values for $\varphi$. The resulting plot will demonstrate how the cut posterior for $\eta$ changes as the conditioning value of $\varphi$ changes.

The above approximation cannot be directly accessed, since $\eta^*_\varphi$ and $J(\eta^*_\varphi|\varphi)$ are unknown in practice. However, in cases where $M_{n_2}(\eta|\varphi)$ is twice continuously differentiable in $\eta$, it is simple to estimate $\eta^*_\varphi$ and $J(\eta^*_\varphi|\varphi)$ by their empirical counterparts $\hat{\eta}_{\varphi} := \arg\max_{\eta} M_{n_2}(\eta|\varphi)$, and $J_{n_2}(\hat{\eta}_{\varphi}|\varphi) := n_2^{-1} \nabla^2 \eta M_{n_2}(\hat{\eta}_{\varphi}|\varphi)$ respectively.

The large sample approximation can also be used to visualize the behavior of specific functionals of interest, e.g., moments of $\eta|w, \varphi$. As an example, suppose that $\eta$ is a scalar for simplicity and that we are interested in understanding how the variance of its cut posterior depends on $\varphi$. Using the law of total variance, we can write

$$\text{Var}(\eta) = E(\text{Var}(\eta|\varphi)) + \text{Var}(E(\eta|\varphi)),$$

(where expectations in this expression are with respect to the cut posterior) and for draws $\varphi^{(s)} \sim \pi_{\text{cut}}(\varphi|z)$, $s = 1, \ldots, S$, we can plot histograms of $\text{Var}(\eta|\varphi^{(s)})$ and $E(\eta|\varphi^{(s)})$ to understand how variability in $\eta$ relates to $\varphi$. The conditional means and variances can be approximated by the normal approximations obtained from Theorem 1. In an example in Section 4.1, we discuss diagnostics.
of this type, as well as methods for understanding posterior skewness in the parameter in the second module.

Credible sets of $\pi(\eta|w, \phi)$ are also of particular interest. For $\alpha \in (0, 1)$, let $C_{\alpha}^\eta(\phi)$ be the set of $\eta$ such that $\int_{C_{\alpha}^\eta(\phi)} \pi(\eta|w, \phi) d\eta = 1 - \alpha$. Then, we can visualize $C_{\alpha}^\eta(\phi)$ across several values of $\phi \in \Phi$ to understand how the shape of credible sets change as $\phi$ varies. In the case of credible sets, the normal approximation can be directly used to obtain an estimate of $C_{\alpha}^\eta(\phi)$, and an algorithm for this is given in Appendix C.

Without Theorem 1, constructing functionals of $\eta|w, \phi$ at different values of $\phi$ usually requires running an MCMC sampling algorithm to obtain draws of $\eta|w, \phi$. Theorem 1 gives a computationally cheap alternative: we simply replace $\pi(\eta|w, \phi)$ in the definition of the functional by the normal approximation in Theorem 1, for which samples can be drawn directly.

3.4. Incorporating Feedback via Tempering

Recently, Carmona and Nicholls (2020) have proposed the use of “semi-modular” posterior distributions as an extension of cut-model inference; Nicholls et al. (2022) extend this construction to prequentially additive loss functions, and Carmona and Nicholls (2022) investigate the use of normalizing flows for their computation. In this section we explain semi-modular inference and introduce a new type of semi-modular posterior, for which fast computation is possible using the asymptotic approximations developed in Section 3.1.

Consider once again the two module system, and point estimation for the shared parameter $\phi$ based on full and cut posterior distributions. The intuition behind semi-modular inference is that if the degree of misspecification is not severe, then the bias of the full posterior estimator may only be moderate, while its variance might be greatly reduced compared to the cut posterior estimator. In this case, full posterior estimates may have better frequentist performance in managing a bias-variance trade-off. If the misspecification is serious, however, full posterior estimation may have a large bias, and estimation based on the cut model may be preferred. Instead of making a binary choice between the full and cut posterior density, it might be better to modulate the influence of the misspecified module in a more continuous way, using an “influence parameter” denoted here as $\gamma \in [0, 1]$. In the proposal of Carmona and Nicholls (2020), the choice $\gamma = 0$ results in the cut posterior, whereas $\gamma = 1$ corresponds to the full posterior, so that the semi-modular posterior interpolates between cut and full posterior based on the influence parameter. Nicholls et al. (2022) also explore some more Bayesian properties of validity and order-coherence of semi-modular posteriors for their original approach and some alternatives.

The semi-modular method of Carmona and Nicholls (2020) proceeds in two stages. First, an auxiliary parameter $\tilde{\eta}$ is introduced that replicates the role of $\eta$ in the second module. Extending the discussion of Carmona and Nicholls (2020) to the generalized Bayes setting, they would construct a posterior density for
Indeed, (φ, β) as
\[ \pi_{\text{pow}, \gamma}(φ, \eta|z, w) \propto \pi(φ)\pi(\eta|φ)\exp\{\nu L_n(φ)\}\exp\{\nu M_\eta(φ)\}\gamma, \]  
(3.1)
where γ ∈ [0, 1] is an influence parameter which controls how much of the information in the second module is used in making marginal inferences about φ. A joint density for (φ, β, η) is then constructed by multiplying (3.1) by the conditional posterior density for η|φ, followed by integrating out β, to obtain the semi-modular posterior:
\[ \pi_{\text{marg}, \gamma}(φ, z, w) = \int \pi_{\text{pow}, \gamma}(φ, β|z, w)dβ \propto \pi(φ|z, w). \]  
(3.2)
It is easy to see that if γ = 0, (3.2) is the cut posterior density, whereas γ = 1 gives the conventional joint posterior density.

### 3.4.1. Marginal semi-modular inference

We now introduce an alternative semi-modular approach, where no auxiliary parameter β is introduced and computation can be conveniently done using the large sample approximations developed in Section 3.1. Recall that the marginal generalized posterior for φ can be written as \( \pi(φ|z, w) = \int_\Phi \pi(φ, η|y) dη \). Using the decomposition of \( \pi(φ, η|y) \) in equation (2.3), the fact that \( \int_\Phi \pi(η|w, φ) dη = 1 \) for each φ ∈ Φ, we can rewrite \( \pi(φ|z, w) \) as
\[ \pi(φ|z, w) = \pi_{\text{cut}}(φ|z)p(φ|φ) \propto \pi_{\text{cut}}(φ|z)m_ν(φ|φ), \]

where \( m_ν(φ|φ) \) was defined in equation (2.4).

The above suggests that a type of marginal semi-modular inference for φ can proceed via the tempered marginal posterior
\[ \pi^M_\gamma(φ|z, w) \propto \pi_{\text{cut}}(φ|z)p(φ|φ)^\gamma \propto \pi_{\text{cut}}(φ|z)m_ν(φ|φ)^\gamma, \quad γ \in [0, 1], \]  
(3.3)
where the notation ‘M’ makes clear that we are only considering a marginal semi-modular posterior. The marginal semi-modular posterior \( \pi^M_\gamma(φ|z, w) \) attenuates the impact of the feedback term \( p(φ|φ) \) by tempering its contribution in the marginal posterior, interpolating between the cut posterior marginal \( \pi_{\text{cut}}(φ|z) \) at \( γ = 0 \), and the conventional marginal posterior \( \pi(φ|z, w) \) at \( γ = 1 \).

### 3.4.2. Computation for marginal semi-modular inference

The difficulty in computing \( \pi^M_\gamma(φ|z, w) \) lies in calculating \( p(φ|φ) \), which is intractable in cases where \( m_ν(φ|φ) \) is intractable. However, observe that the form of \( m_ν(φ|φ) \) is akin to that of a “marginal likelihood” conditioned on a fixed φ. Indeed, \( m_ν(φ|φ) \) satisfies the following tautological relationship:
\[ m_ν(φ|φ) = \pi(φ|φ) \exp\{\nu M_ν(φ|φ)\}/\pi(φ|w, φ), \]  
(3.4)
which holds for any values of $\eta$ and $\varphi$. The above equation resembles the “basic marginal likelihood identity” used in Chib (1995) to estimate the marginal log-likelihood, and is related to the “candidate’s formula” presented in Besag (1989).

Following Chib (1995), taking logarithms of (3.4) and considering a chosen value $\eta^*$ of $\eta$ in the high probability region for $\pi(\eta|w, \varphi)$, we obtain

$$\ln m_\eta(w|\varphi) = \ln \pi(\eta^*|\varphi) + \nu M_{n_2}(\eta^*|\varphi) - \ln \pi(\eta^*|w, \varphi).$$  \hspace{1cm} (3.5)

Given a choice of $\eta^*$, replacing $\ln \pi(\eta^*|w, \varphi)$ with an estimate of it in (3.5), results in an estimate $\ln \hat{m}_{\eta}(w|\varphi)$ of $\ln m_\eta(w|\varphi)$.

If we have a known form for $\pi(\eta|w, \varphi)$, then this form can be directly used to obtain the estimate $\ln \hat{m}_{\eta}(w|\varphi)$. However, in general, $\pi(\eta|w, \varphi)$ is not available in closed form, and there are two obvious approaches: one, we use posterior draws and kernel density estimation to estimate $\pi(\eta^*|w, \varphi)$; two, we use the large sample approximation to $\pi(\eta|w, \varphi)$ obtained in Theorem 1 at the point $\eta^*$.

The latter approach is simple to implement when $M_{n_2}(\eta|\varphi)$ is twice-continuously differentiable in $\eta$, for all $\varphi \in \Phi$. In this case, we obtain from (3.5) the estimate:

$$\ln \hat{m}_{\eta}(w|\varphi) = \ln \pi(\eta^*|\varphi) + \nu M_{n_2}(\eta^*|\varphi) - \ln N\{\eta^*; \hat{\eta}_\varphi, [n_2\nu J_{n_2}(\hat{\eta}_\varphi)|\varphi]\}^{-1},$$  \hspace{1cm} (3.6)

where we recall that $\hat{\eta}_\varphi := \arg\max_{\eta \in \mathcal{E}} M_{n_2}(\eta|\varphi)$, and $J_{n_2}(\hat{\eta}_\varphi|\varphi) := n_2^{-1}\nabla_{\eta}^2 M_{n_2}(\hat{\eta}_\varphi|\varphi)$. Critically, unlike $m_\eta(w|\varphi)$, up to the calculation of $\hat{\eta}_\varphi$ and $J_{n_2}(\hat{\eta}_\varphi|\varphi)$, the estimator in (3.6) is known in closed-form.

Given $\ln \hat{m}_{\eta}(w|\varphi)$, and fixed $\gamma \in [0, 1]$, we can use

$$\hat{\pi}_{\gamma}^M(\varphi|z, w) \propto \pi_{\text{cut}}(\varphi|z) \exp\{\ln \hat{m}_{\eta}(w|\varphi)\}^\gamma,$$

as an approximation to the semi-modular posterior $\pi_{\gamma}^M(\varphi|z, w)$. For fixed $\gamma$, the approximate semi-modular posterior $\hat{\pi}_{\gamma}^M(\varphi|z, w)$ can be sampled using a Metropolis-Hastings MCMC (MH-MCMC) algorithm. However, since $\ln \hat{m}_{\eta}(w|\varphi)$ is only an estimate of $m_{n_2}(w|\varphi)$, the resulting MH-MCMC algorithm will not deliver draws from $\pi_{\gamma}^M(\varphi|z, w)$. Nonetheless, in large samples, by Theorem 1, we can expect these draws to yield an accurate approximation to $\pi_{\gamma}^M(\varphi|z, w)$. Similar to the method of Carmona and Nicholls (2020), the choice of the influence parameter $\gamma$ in our approach can be carried out using predictive approaches similar to those in Carmona and Nicholls (2020), or using the conflict checks, see Chakraborty et al. (2023).

It is interesting to compare our new approach to semi-modular inference based on (3.3) with the method of Carmona and Nicholls (2020), and we give an empirical comparison for the example of Section 4.1 in the supplementary material, finding they give similar results. Statistically there seems no clear reason to prefer one approach over the other in the examples we have considered, and a more detailed theoretical study is left for future work. However, the computational approximations based on Theorem 1 are helpful for implementing both of these methods. The method of Carmona and Nicholls (2020) is often implemented using a nested MCMC approach similar to that used in cut posterior computation. So repeated sampling of $\eta$ given $\varphi$ for samples $\varphi$ from the SMI
marginal posterior for ϕ can be done cheaply using the normal approximations justified by Theorem 1, or these approximations can be used as a good proposal to accelerate MCMC or importance sampling. In the supplementary material, Algorithm 2 describes how to sample from the proposed semi-modular posterior (3.3) when ln m_η(w|ϕ) is obtained using (3.6).

4. Examples

In this section we consider two examples. The first example illustrates our large sample approximations for cut posterior computation, and for implementing diagnostics for understanding uncertainty propagation between modules. We consider both probabilistic model specifications as well as a generalized Bayesian analysis using a quasi-likelihood. Our second example also considers a generalized Bayesian analysis, for which the learning rate for the second module needs to be carefully chosen. We illustrate a situation where an appropriate choice of the loss function can resolve conflict between cut and full posterior inferences, giving insight into how an initially flawed parametric model may need to be improved.

4.1. HPV prevalence

Our first example was discussed in Plummer (2015), and is based on a real epidemiological study (Maucort-Boulch et al., 2008). The model consists of two modules. Module 1 incorporates survey data from 13 countries on high-risk human papillomavirus (HPV) prevalence for women in a certain age group. Denote by z_i the number of women with high-risk HPV in country i in a survey of N_i individuals, i = 1, . . . , 13, and assume that z_i ∼ Binomial(N_i, ϕ_i), where ϕ_i ∈ [0, 1] is a country-specific prevalence probability. The parameters ϕ_i are assumed independent in their prior, with ϕ_i ∼ U[0, 1]. Write ϕ = (ϕ_1, . . . , ϕ_13)^T.

Module 2 incorporates cervical cancer incidence data w, with w_i the number of cervical cancer cases in T_i woman years of follow-up in country i, i = 1, . . . , 13. The relationship between cervical cancer incidence and HPV prevalence is described by a Poisson regression model, w_i ∼ Poisson(T_iρ_i), where log ρ_i = η_1 + η_2ϕ_i. For these data the Poisson regression model is misspecified, and because ϕ_i is appearing as a covariate in the Poisson regression, inference about ϕ_i is influenced by the misspecification in the second module. Estimation of ϕ adapts to the misspecification, distorting inference about these parameters, which also results in uninterpretable inference about the regression parameters η used to summarize the relationship between HPV prevalence and the rate of cancer incidence. The main interest of the analysis lies in understanding this relationship.

4.1.1. Cut posterior computation with large sample approximation

When cutting feedback in this example, it is straightforward to obtain posterior samples from π_{cut}(ϕ|z). This is because the likelihood for each z_i is binomial,
and the priors for the parameters $\varphi_i$ are conjugate. In $\pi_{\text{cut}}(\varphi|z)$ the $\varphi_i$ are independent, with $\pi_{\text{cut}}(\varphi_i|z)$ a beta density, $\text{Beta}(z_i + 1, n_i - z_i + 1)$. We generate samples $\varphi^{(s)}$, $s = 1, \ldots, S = 1000$, from $\pi_{\text{cut}}(\varphi|z)$ by direct Monte Carlo sampling. To generate samples $\eta^{(s)}$ so that $(\varphi^{(s)}, \eta^{(s)})$ is a draw from the joint cut posterior density, we do the following. By Theorem 1, we can approximate the conditional posterior density of $\eta$ given $\varphi^{(s)}$, $w$ and $z$ by a normal density with mean $\mu(\varphi^{(s)}) = \tilde{\eta}_{\varphi^{(s)}}$ and covariance matrix $\Sigma(\varphi^{(s)}) = n^{-1} \nabla_{\eta|\varphi^{(s)}} \text{M}_n(\tilde{\eta}_{\varphi^{(s)}}|\varphi^{(s)})$.

For each $\varphi^{(s)}$, we generate 1,000 proposal samples for $\eta$ from a multivariate $t$-distribution with mean $\mu(\varphi^{(s)})$, scale matrix $\Sigma(\varphi^{(s)})$, and 5 degrees of freedom, and draw a single sample $\eta^{(s)}$ from these proposals using sampling importance resampling (SIR).

For comparison, we can also draw an approximate sample $\tilde{\eta}^{(s)}$ say from the conditional normal approximation directly. For practical purposes the SIR samples can be considered near-exact, and Figure 2 (top row) shows the marginal posterior samples for $\eta = (\eta_1, \eta_2)$ for the two approaches. A sample based estimate of the 1-Wasserstein distance between the posterior marginal cut distributions estimated by the exact SIR and approximate conditional normal methods is 0.004 and 0.062 for $\eta_1$ and $\eta_2$ respectively, showing that our large sample conditional normal approximations result in accurate cut posterior computation. We can see that the marginal cut posterior density for $\eta$ is non-Gaussian, but this is captured very well in the approximate sampling approach where the conditional posterior density for $\eta$ is close to normal. It is the uncertainty about $\varphi$ that is propagated in making marginal inferences about $\eta$ that results in the non-Gaussian structure in the marginal posterior distribution for $\eta$. Also shown in Figure 2 are samples from the usual Bayesian posterior distribution, obtained via MCMC using the rstan package (Carpenter et al., 2017). The full and cut posterior inferences differ substantially, demonstrating how much the misspecification of the second module changes the inference about $\eta$ here. The bottom row of the figure compares the univariate marginals for the cut and full posterior densities for $\eta_1$ and $\eta_2$. 

D.T. Frazier, and D.J. Nott/Generalized Cut Posteriors
4.1.2. Generalized posterior analysis

The middle row of Figure 2 shows samples from the generalized cut posterior distribution obtained when the Poisson likelihood is replaced by a quasi-likelihood (Wedderburn, 1974), which allows for overdispersion with respect to the Poisson model. When using the negative log quasi-likelihood as the loss for the second module, it is sensible to choose a learning rate $\nu' = 1$. For the first module we use the same parametric model as before. The overdispersion parameter in the quasi-likelihood is denoted by $\lambda$, and instead of making the Poisson assumption that the mean and variance are equal, it is assumed that the variance is $\lambda$ times the mean for each $w_i$. The left plot in the middle row is for $\lambda = 75$, and the right
plot is for \( \lambda = 150 \). We can see that even if we assume a standard deviation for the \( w_i \) that is more than 10 times that implied by a Poisson mean-variance relationship, the full posterior samples do not become plausible under the cut distribution. Yu et al. (2023) have elaborated on the comparison of the cut and full posterior distributions as a kind of conflict check, and the lack of consistency of the cut and full posterior inferences here suggests that altering the parametric Poisson regression to another parametric model incorporating multiplicative overdispersion will not result in an adequate generative model for the data unless the degree of overdispersion is very large. The samples in the quasi-likelihood analysis were generated using the conditional normal approximation for the density of \( \eta \) given \( \varphi \).

4.1.3. Uncertainty propagation

Figure 3 shows, for 5 samples from the marginal cut posterior distribution of \( \varphi \), a 95\% probability ellipsoid of minimal volume for the conditional normal approximations of \( p(\eta|\varphi, y) \). The 5 \( \varphi \) samples are selected from 1,000 cut posterior samples according to the 0.1, 0.3, 0.5, 0.7 and 0.9 quantiles of the determinant of the estimated conditional covariance matrix of \( \eta \) given \( \varphi \). The variation in the shape of these ellipsoids is substantial as \( \varphi \) changes.

We can also use the normal approximation to the conditional posterior density as a diagnostic to understand the way that the uncertainty in \( \varphi \) propagates into the second module, for both the cut and full posterior density. Noting that

\[
\text{Var}(\eta_j) = E(\text{Var}(\eta_j|\varphi)) + \text{Var}(E(\eta_j|\varphi)),
\]  

(4.1)
we could plot histograms of the values $\mu(\varphi(s))_j$, $s = 1, \ldots, S$ and $\Sigma(\varphi(s))_{jj}$, $s = 1, \ldots, S$ for $j = 1, 2$ to understand how uncertainty in $\varphi$ propagates into $\eta$. In (4.1) the expectations can be defined as with respect to either the full posterior distribution or with respect to the cut posterior distribution. The mean of the samples in a histogram of $\Sigma(\varphi(s))_{jj}$ relates to the first term on the right-hand side of (4.1). The variability of the samples in a histogram of $\mu(\varphi(s))_j$ assesses variability propagated to $\eta_j$ from the second term on the right-hand side of (4.1).

Generalizing (4.1) to third central moments using the law of total cumulants (Brillinger, 1969), we can also write

$$E((\eta_j - E(\eta_j))^3) = E(E((\eta_j - E(\eta_j|\varphi))^3|\varphi)) + E((E(\eta_j|\varphi) - E(\eta_j))^3) + 3\text{Cov}(E(\eta_j|\varphi), \text{Var}(\eta_j|\varphi)).$$

(4.2)

Once again, the expectations in the above expression can be defined as with respect to either the full posterior distribution or with respect to the cut posterior distribution. If the conditional posterior for $\eta_j$ given $\varphi$ is approximately symmetric, then the first term on the right-hand side of (4.2) can be neglected. Then the posterior skewness of $\eta_j$ depends on the second and third terms. These terms relate to the skewness of the conditional expectation $E(\eta_j|\varphi)$ (considered as a function of $\varphi$) and the covariance between the conditional mean and conditional variance. The skewness of the conditional expectation can be assessed from looking at the skewness in a histogram of $\mu(\varphi(s))_j$, while plotting the samples $(\mu(\varphi(s))_j, \Sigma(\varphi(s))_{jj})$, $s = 1, \ldots, S$, is helpful for assessing the $\text{Cov}(E(\eta|\varphi), \text{Var}(\eta|\varphi))$ term in (4.2).

Figure 4 shows a scatterplot of $(\mu(\varphi(s))_1, \Sigma(\varphi(s))_{11})$, $s = 1, \ldots, S$, with histograms of each variable on the axes, for $\eta_1$. The plot on the left is for the cut posterior density, and the plot on the right is for the full posterior density. There is a strong negative relationship between the conditional posterior mean of $\varphi$ and its conditional variance, as well as negative skewness in the histogram of $\mu(\varphi(s))_1$, which by (4.2) explains the negative skew in the marginal distribution for $\eta_1$ evident in Figure 2. This is so for both the cut and full posterior densities.

Figure 5 shows a similar plot to Figure 4 for the parameter $\eta_2$. In this case, there is a strong positive relationship between the conditional posterior mean of $\varphi$ and its conditional variance, and positive skewness in the histogram of $\mu(\varphi(s))_2$, which explains the positive skew in the marginal distribution of $\eta_2$, in both the cut and full posterior densities, as shown in Figure 2. The dependence between $\mu(\varphi)_j$ and $\Sigma(\varphi)_{jj}$ in Figures 4 and 5 relates directly to the way the conditional variance of $\eta$ depends on $\varphi$, which is exactly what is being captured in the conditional perspective taken in the theory of Section 3.1. Understanding this dependence is particularly useful for explaining the marginal posterior shape for $\eta$ in the full and cut posterior distributions. A comparison of the marginal SMI approach of Section 3.4 with the SMI of Carmona and Nicholls (2020) is given in the supplementary material. The two methods give similar results in this example.
4.2. A random effects model

Our second example, discussed in Liu et al. (2009), considers a random effects model. The data are denoted by \( Y_{ij} \), \( i = 1, \ldots, N \), \( j = 1, \ldots, J \), where \( i \) indexes groups, and \( j \) indexes observations within groups. The data for group \( i \) is modelled as

\[
Y_{ij} | \beta_i, \varphi_i \sim \text{iid } N(\beta_i, \varphi_i^2), \quad j = 1, \ldots, J,
\]

where \( \beta_i \) is a random effect, and \( \varphi_i \) is a group standard deviation. The prior density for \( \beta \) is

\[
\beta_i | \psi \sim \text{iid } N(0, \psi^2),
\]

\( i = 1, \ldots, N \), where \( \psi \) is the random effects standard deviation. Liu et al. (2009) consider this example to demonstrate a problem that can occur for some hierarchical models, in which there is a model for the random effects with thin tails, such as Gaussian. In the model above, if there is an outlying value for one of the random effects, this can lead to poor inference for the corresponding group.
standard deviation, and overshrinkage in estimating the random effect. The difficulty is most pronounced when the number of replicates \( J \) is small compared to \( N \). Liu et al. (2009) give an insightful discussion that exploits the simple form of the model to do analytic calculations. We do not repeat their analysis here, but demonstrate the problem numerically and illustrate the utility of our generalized Bayes approaches to modular inference.

First, we will set up the model so that it takes the form of a two module system. Write \( \beta = (\beta_1, \ldots, \beta_N)^T \) and \( \phi = (\varphi_1, \ldots, \varphi_N)^T \). Let \( \eta = (\beta^T, \psi)^T \). We use similar priors to Liu et al. (2009), although we parametrize our model in terms of standard deviations rather than variances and transform priors appropriately. Components of \( \phi \) are independent in the prior, with marginal densities \( \pi(\phi_i) \propto \phi_i^{-1} \). For the prior on \( \psi \), we use \( \pi(\psi|\phi_i) \propto (\bar{\phi}_i^2/J + \psi^2)^{-1} \psi \), where \( \bar{\phi}_i^2 = N^{-1} \sum_{j=1}^N \phi_i^2 \).

We will reduce the full data down to sufficient statistics. Let \( w_i = J^{-1} \sum_{j=1}^J Y_{ij} \), \( z_i = \sum_{j=1}^J (Y_{ij} - z)^2, i = 1, \ldots, N \), and write \( z = (z_1, \ldots, z_n)^T \), \( w = (w_1, \ldots, w_n)^T \). It is easily seen that \( z \) and \( w \) are sufficient for \( \theta = (\phi^T, \eta^T)^T \), with \( z \) and \( w \) being independent of each other. The density of \( z|\phi \), written \( p(z|\phi) \), depends only on \( \phi \), with

\[
z_i|\phi_i \sim \text{Gamma} \left( \frac{J-1}{2}, \frac{1}{2\phi_i^2} \right),
\]

independently for \( i = 1, \ldots, N \). Similarly, write \( p(w|\phi, \eta) \) for the density of \( w \), and

\[
w_i|\beta_i, \varphi_i \sim \mathcal{N} \left( \beta_i, \frac{\varphi_i^2}{J} \right),
\]

independently, for \( i = 1, \ldots, N \). The model for the sufficient statistics is a two-module system. The first module consists of \( p(z|\phi) \) and \( p(\phi) \), and the second module comprises \( p(w|\phi, \eta) \) and \( p(\eta|\phi) \).

We simulate a dataset from the model, with \( N = 100 \), \( J = 10 \), \( \psi = 1 \) and \( \varphi_i = 0.5, i = 1, \ldots, N \). The random effects vector \( \beta \) is simulated from its prior, except for \( \beta_1 \), which is fixed at 10. Since \( \beta_1 \) is inconsistent with the hierarchical prior, this leads to poor estimation of \( \varphi_1 \) when \( J \) is small compared to \( N \), and poor estimation of \( \beta_1 \). Figure 6 (left) compares the posterior distributions of \( \varphi_1 \) from the conventional parametric and the cut posterior distributions. The boxplots are for 1,000 posterior samples in each case. The horizontal line shows the true value. The accuracy of the conventional posterior is poor, and inconsistent with the cut posterior inferences which are more accurate.

4.2.1. Generalized posterior analysis

It is interesting in this example to replace the normal model for \( w_i \) in module 2 with a loss likelihood, to see whether this resolves the inconsistency between the cut and full generalized posterior inferences. Here we consider Tukey’s loss (Beaton and Tukey, 1974), which was recently used for a generalized Bayesian analysis by Jewson and Rossell (2022). As pointed out by Jewson and Rossell
Tukey’s loss can be useful when an analyst knows the distribution of the data has heavy tails, but a precise knowledge of the tail behaviour is difficult to formalize. Writing $w_i' = (w_i - \beta_i)/(\phi_i/\sqrt{J})$, in our generalized Bayesian analysis we replace the Gaussian negative log-likelihood terms

$$- \log p(w_i|\varphi_i, \beta_i) = \frac{1}{2} \log \left(\frac{2\pi \varphi_i^2}{J}\right) - \frac{1}{2} w_i'^2,$$

with Tukey’s loss terms

$$m(w_i; \eta, \varphi) = \begin{cases} 
\frac{1}{2} \log \frac{2\pi \varphi_i^2}{J} + \frac{w_i'^2}{2\kappa^2} - \frac{w_i'^4}{8\kappa^4} - \frac{w_i'^6}{6\kappa^6} & \text{if } |w_i'| \leq \kappa \\
\frac{1}{2} \log \frac{2\pi \varphi_i^2}{J} + \frac{\kappa^2}{6}, & \text{otherwise}
\end{cases}$$

for $i = 1, \ldots, N$, where $\kappa$ is a tuning parameter controlling the degree of robustness to departures from normality. As $\kappa \to \infty$, Tukey’s loss approaches the Gaussian negative log-likelihood, whereas small values for $\kappa$ result in greater robustness to outliers. There are a variety of ways to choose $\kappa$, but here we fix $\kappa = 5$. Jewson and Rossell (2022) describe a way of choosing $\kappa$ and other loss parameters using a so-called $H$-posterior based on the Hyvärinen score, and also consider model choice for loss functions, but these directions are not pursued here. For Tukey’s loss, the corresponding loss likelihood is not integrable in $w_i$, so it does not correspond to any probabilistic model.

Our generalized Bayesian analysis requires a choice of the learning rates $\nu$ and $\nu'$ as discussed in Section 3.2. Recall that $\nu$ calibrates the module 1 loss to the prior, and $\nu'$ can be thought of as calibrating the module 2 loss to the conditional prior for $\eta|\varphi$. Since we use the original probabilistic specification for module 1, we choose the learning rate $\nu$ to be 1, and the generalized Bayes and conventional cut posterior densities for $\varphi$ are the same. To choose $\nu'$, we use the method discussed in Section 3.2. However, noting that only the parameters...
β appear in the loss function and not the prior hyperparameter ψ, we calibrate ν′ by considering matching the Fisher information number for updates for β asymptotically with ψ fixed, for loss likelihood bootstrap and generalized Bayes. Since the matching is done asymptotically, the choice of ψ makes no difference to the value of ν′ obtained. To estimate the matrix Ψ_{22} in estimating ν′ in Section 3.2, we used a Bayesian bootstrap applied to the original data groups, since it is not possible otherwise to estimate Ψ_{22} from the data sufficient statistics. This is because there is no replication that can be used, with β_i appearing only in the model for w_i. The learning rate obtained for the second module for the analysis was ν′ = 3.3.

Figure 6 (right) compares the posterior distributions of φ_1 for the generalized Bayes posterior and the cut posterior distributions. Once again, the boxplots are for 1,000 posterior samples, and the horizontal line shows the true value. The cut posterior is the same as for the conventional posterior for the parametric model, as we are still using the negative log-likelihood as the loss for module 1. We see that now the cut and full posterior inferences are consistent with each other, so that the Tukey’s loss, which accommodates heavy-tailed data, resolves the conflict between different parts of the model. Although we have reduced the full data to sufficient statistics for inference, the non-sufficient information in the replicates is useful for model checking - using the replication we may distinguish between model failure due to outliers in the sampling density and model failure due to an inappropriate prior on the random effects. An outlying random effect for a group will influence all replicates in the group.

For computations in this example, we used the rstan package (Carpenter et al., 2017) for both the conventional and generalized posterior densities. We ran four chains with 1000 iterations burn-in and 4000 sampling iterations, thinning the output so that 1000 samples are retained. The cut posterior density for φ_1 is inverse gamma, and was sampled directly to get 1,000 cut posterior samples for φ_1.

5. Discussion

This paper combines generalized posterior inference with cutting feedback methods for flexible Bayesian modular inference. Starting out with a parametric model specification, we can replace the negative log likelihood for unreliable modules with different choices of a loss function to resolve any incompatibility between cut and full posterior inferences. We have also studied the large sample behaviour of the generalized cut posterior distribution, taking a conditional perspective. Our main result describes the asymptotic behaviour of the conditional posterior distribution of a module’s parameters given parameters in other modules, formally justifying conditional Laplace approximations. These provide more accurate approximations of conditional posterior distributions than those obtained from Laplace approximations of the joint posterior density. Our large sample approximations are useful for computing diagnostics describing uncertainty propagation between modules, as well as for the efficient implementation of a new approach to semi-modular inference.
In the framework for modular inference that we have developed, the loss function is a sum of loss functions associated with different modules. We considered calibrating the different component loss functions in one example, but more research is needed on the best way to do this for different purposes. With a single loss function, there are different methods of calibrating the loss to the prior, and the best method to use may depend on the goals of the analysis. A similar remark applies in generalized Bayesian modular inference. An anonymous referee has also asked about the connections with the “restricted likelihood” approach to dealing with misspecification, discussed recently in Lewis et al. (2021). Restricted likelihood reduces the data to an insufficient summary statistic, to discard information that cannot be matched under the assumed model. The method can be implemented computationally using likelihood-free inference algorithms, and modular inference has been considered in this context by Chakraborty et al. (2023).

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Appendix A: Proofs of Main Results

Proof of Lemma 1. We prove the result by verifying the sufficient conditions in Theorem 1 of Chernozhukov and Hong (2003) for the criteria $\nu L_n(\phi)$. Assumption 1 satisfies the sufficient conditions in Lemmas 1 and 2 in Chernozhukov and Hong (2003), which together with the prior condition in Assumption 1, verifies the sufficient conditions in Theorem 1 of Chernozhukov and Hong (2003). The result follows.

Remark 4. The proof of Theorem 1 follows by generalising the arguments in Chernozhukov and Hong (2003). This is a novel generalization for at least two reasons. Firstly, the separability of the criterion functions allow us to maintain different conditions for each portion, e.g., different levels of smoothness, without requiring conditions on the joint criterion, $Q_n(\theta)$. Secondly, by focusing on $\phi \in$
\[ \Phi, \text{ the resulting posterior concentration is not directly impacted by the rate at which the posterior for } \pi_{\text{cut}}(\varphi | z) \text{ concentrates onto } \varphi^*. \] That is, the result of Theorem 1 remains valid when the posterior for \( \varphi \) concentrates at rates slower than the parametric \( \sqrt{n} \)-rate, so long as Assumption 4 remains valid.

**Proof of Theorem 1.** To simplify the proof of the result, let us abuse notation and write \( n = n_2 \). Define \( J(\varphi) := J(\eta^*_\varphi | \varphi) \), and

\[ t = \sqrt{n}(\eta - \eta^*_\varphi) - Z_n(\varphi)/\sqrt{n}, \text{ where } Z_n(\varphi) := J(\varphi)^{-1} \Delta_n(\eta^*_\varphi | \varphi). \]

From the quadratic approximation in Assumption 4, and the above definitions, we have the identity

\[ M_n(\eta | \varphi) - M_n(\eta^*_\varphi | \varphi) = -\frac{1}{2} t^T J(\varphi) t + \frac{1}{2} \frac{1}{\sqrt{n}} Z_n(\varphi)^T J(\varphi) \frac{1}{\sqrt{n}} Z_n(\varphi) + R_n(\eta, \varphi), \tag{A.1} \]

for some remainder term \( R_n(\eta, \varphi) \). Now, define \( T_n(\varphi) := Z_n(\varphi)/n + \eta^*_\varphi \), and let

\[ \omega(t) := M_n\{T_n(\varphi) + t/\sqrt{n} | \varphi\} - M_n(\eta^*_\varphi | \varphi) - \frac{1}{2} \frac{1}{\sqrt{n}} Z_n(\varphi)^T J(\varphi) \frac{1}{\sqrt{n}} Z_n(\varphi), \]

which, by (A.1), is equal to

\[ \omega(t) = -\frac{1}{2} t^T J(\varphi) t + R_n(T_n(\varphi) + t/\sqrt{n}, \varphi). \tag{A.2} \]

Using (A.2), the posterior can be stated as

\[
\pi(t | w, \varphi) := \frac{\pi\{t/\sqrt{n} + T_n(\varphi) | \varphi\} \exp\{\nu\{M_n\{T_n(\varphi) + t/\sqrt{n} | \varphi\} - M_n(\eta^*_\varphi | \varphi)\}\}}{\int_{\mathcal{E}_n} \pi\{t/\sqrt{n} + T_n(\varphi) | \varphi\} \exp\{\nu\{M_n\{T_n(\varphi) + t/\sqrt{n} | \varphi\} - M_n(\eta^*_\varphi | \varphi)\}\} dt} 
\]

\[ = \pi\{t/\sqrt{n} + T_n(\varphi) | \varphi\} \exp\{\nu \omega(t)\} / C_n, \]

for

\[ C_n := \int_{\mathcal{E}_n} \pi\{t/\sqrt{n} + T_n(\varphi) | \varphi\} \exp\{\nu \omega(t)\} dt. \]

The stated result follows if

\[ \int_{\mathcal{E}_n} ||t|| \pi(t | w, \varphi) - N\{t, 0, [\nu J(\varphi)]^{-1}\} dt = C_n^{-1} J_n = o_p(1), \]

where

\[ J_n = \int_{\mathcal{E}_n} ||t|| \exp\{\nu \omega(t)\} \pi\{T_n(\varphi) + t/\sqrt{n} | \varphi\} - C_n N\{t, 0, [\nu J(\varphi)]^{-1}\} dt. \]

However, \( J_n \leq J_{1n} + J_{2n} \), where

\[ J_{1n} := \int_{\mathcal{E}_n} ||t|| \left| \exp\left\{-\frac{1}{2} t^T [\nu J(\varphi)] t + \nu R_n(T_n(\varphi) + t/\sqrt{n}, \varphi)\right\} \pi\{T_n(\varphi) + t/\sqrt{n} | \varphi\} \right. \]

\[ - \pi(\eta^*_\varphi | \varphi) \exp\left\{-\frac{1}{2} t^T [\nu J(\varphi)] t\right\} \right| dt, \]

\[ J_{2n} := \left| C_n \frac{[\nu J(\varphi)]^{1/2}}{(2\pi)^{d_n/2}} - \pi(\eta^*_\varphi | \varphi) \right| \int_{\mathcal{E}_n} ||t|| \exp\left\{-\frac{1}{2} t^T [\nu J(\varphi)] t\right\} dt. \]
and where we have used equation (A.2) in the definition of $J_{1n}$. Further, if $J_{1n} = o_p(1)$, then

$$C_n = \pi(\eta_p^*|\varphi) \int_{\mathbb{R}^n} \exp \left\{ -\frac{1}{2} t^T [\nu J(\varphi)] t \right\} dt + o_p(1) = \pi(\eta_p^*|\varphi) \left( 2\pi \right)^{d_n/2} (\nu J(\varphi))^{1/2} + o_p(1),$$

and $J_{2n} = o_p(1)$ since for each $\varphi \in \Phi_\delta$ the matrix $J(\varphi)$ is positive-definite, by Assumption 4(ii), so that $\int_{\mathbb{R}^n} ||t|| \exp \left\{ -\frac{1}{2} t^T [\nu J(\varphi)] t \right\} dt < \infty$.

Consequently, the result follows if $J_{1n} = o_p(1)$. Inspecting $J_{1n}$ it is clear that the specific value of $\nu$ will not impact whether or not $J_{1n} = o_p(1)$, so long as $\nu > 0$. Since $\nu$ is fixed it is without loss of generality to take $\nu = 1$ in the remainder.

To demonstrate that $J_{1n} = o_p(1)$ we split $\mathcal{E}_n$ into three regions and analyze $J_{1n}$ over each region. For some $0 \leq h < \infty$ and $\gamma > 0$, with $\gamma = o(1)$, the regions are defined as follows:

- **Region 1:** $||t|| \leq h$.
- **Region 2:** $h < ||t|| \leq \gamma \sqrt{n}$.
- **Region 3:** $||t|| \geq \gamma \sqrt{n}$.

The remainder of the proof follows by extending similar arguments in the literature, e.g., Theorem 8.2 in Lehmann and Casella (2006) (pg 489), and Theorem 1 in Chernozhukov and Hong (2003), to accommodate the conditional nature of the result.

**Region 1:** Over this region $||t||$ can be neglected and the result follows if

$$\sup_{||t|| \leq h} \left| \exp \{ \omega(t) \} \left\{ \pi \left( T_n(\varphi) + t/\sqrt{n} | \varphi \right) - \pi(\eta_p^*|\varphi) \exp \{ -t^T J(\varphi) t/2 \} \right\} \right| = o_p(1).$$

Now,

$$\left| \exp \{ \omega(t) \} \left\{ \pi \left( T_n(\varphi) + t/\sqrt{n} | \varphi \right) - \pi(\eta_p^*|\varphi) \exp \{ -t^T J(\varphi) t/2 \} \right\} \right| \leq \exp \{ \omega(t) \} \left| \pi \left( T_n(\varphi) + t/\sqrt{n} | \varphi \right) - \pi(\eta_p^*|\varphi) \right| + \pi(\eta_p^*|\varphi) \exp \{ \omega(t) \} - \exp \{ -t^T J(\varphi) t/2 \} \right|.$$

First, note that by Assumption 4(i-ii), $Z_n(\varphi)/\sqrt{n} = O_p(1)$ for each $\varphi \in \Phi_\delta$; hence, from the definition $T_n(\varphi) = Z_n(\varphi)/n + \eta_p^*$, for each $\varphi \in \Phi_\delta$,

$$\sup_{||t|| \leq h} \left| T_n(\varphi) + t/\sqrt{n} - \eta_p^* \right| = O_p(1/\sqrt{n}). \quad \text{(A.3)}$$

From Assumption 3, $\pi(\cdot|\varphi)$ is continuous in the first argument, so that by (A.3),

$$\sup_{||t|| \leq h} \left| \pi \left( T_n(\varphi) + t/\sqrt{n} | \varphi \right) - \pi(\eta_p^*|\varphi) \right| = o_p(1).$$

Also, from (A.3) and Assumption 4(iii), for each $\varphi \in \Phi_\delta$,

$$\sup_{||t|| \leq h} \left| R_n \left( T_n(\varphi) + t/\sqrt{n}, \varphi \right) \right| = o_p(1);$$
using the equation for $\omega(t)$ in (A.2), we then have
\[
\sup_{\|t\| \leq h} |\exp\{\omega(t)\} - \exp\{-t^T J(\varphi)t/2\}| = o_p(1).
\]
Further, since $\sup_{\|t\| \leq h} |R_n\{T_n(\varphi) + t/\sqrt{n}, \varphi\}| = o_p(1)$, we have $\exp\{\omega(t)\} \leq \{1 + o_p(1)\}$ over $\|t\| \leq h$; since $\pi(\eta|\varphi)$ is continuous in $\eta$ for all $\varphi \in \Phi_8$, it follows that $\pi(\eta|\varphi)$ is bounded for $\eta \in \{\|\eta - \eta_0^*\| \leq h/\sqrt{n}\}$. Hence, $J_{1n} = o_p(1)$ over Region 1.

**Region 2:** For $h$ large enough and $\gamma = o(1)$, $J_{1n} \leq C_{1n} + C_{2n} + C_{3n}$ where
\[
C_{1n} := C \sup_{h \leq \|t\| \leq \gamma \sqrt{n}} \exp\{|R_n\{T_n(\varphi) + t/\sqrt{n}, \varphi\}\| \pi\{T_n(\varphi) + t/\sqrt{n}|\varphi\} - \pi(\eta_0^*|\varphi)\|
\]
\[
\times \int_{h \leq \|t\| \leq \gamma \sqrt{n}} \|t\| \exp\{-t^T J(\varphi)t/2\} dt
\]
\[
C_{2n} := C \int_{h \leq \|t\| \leq \gamma \sqrt{n}} \|t\| \exp\{-t^T J(\varphi)t/2\} \exp\{|R_n\{T_n(\varphi) + t/\sqrt{n}, \varphi\}\| \pi\{T_n(\varphi) + t/\sqrt{n}|\varphi\} dt
\]
\[
C_{3n} := C \int_{h \leq \|t\| \leq \gamma \sqrt{n}} \|t\| \exp\{-t^T J(\varphi)t/2\} dt.
\]

The first term satisfies $C_{1n} = o_p(1)$ for any fixed $h$, so that $C_{1n} = o_p(1)$ for $h \to \infty$, by the dominated convergence theorem. For $C_{3n}$, from the continuity and positive definiteness of $J(\varphi)$, for each $\varphi \in \Phi_8$, there exists $h'$ large enough such that for all $h > h'$, and $\|t\| \geq h$
\[
\|t\| \exp\{-t^T J(\varphi)t/2\} \leq \|t\| \exp[-\|t\|^2 \lambda_{\min}\{J(\varphi)\}] = O(1/h),
\]
where $\lambda_{\min}(M)$ denotes the minimum eigenvalue of the matrix $M$. Hence, $C_{3n}$ can be made arbitrarily small by taking $h$ large enough and $\gamma$ small enough.

To demonstrate that $C_{2n} = o_p(1)$, we show that
\[
\exp\{-t^T J(\varphi)t/2\} \exp\{|R_n\{T_n(\varphi) + t/\sqrt{n}, \varphi\}\| \pi\{T_n(\varphi) + t/\sqrt{n}|\varphi\} \leq C \exp\{-t^T J(\varphi)t/4\},
\]
with probability converging to one (wpc1), so that $C_{2n}$ can be bounded above by
\[
C_{2n} \leq C \int_{h \leq \|t\| \leq \gamma \sqrt{n}} \|t\| \exp\{-t^T J(\varphi)t/4\} dt.
\]
Similar to $C_{1n}$ and $C_{3n}$, the RHS of the above can be made arbitrarily small for some $h$ large and $\gamma$ small.

To demonstrate equation (A.4), first note that by continuity of $\pi(\eta|\varphi)$, Assumption 3, $\pi\{T_n(\varphi) + t/\sqrt{n}|\varphi\}$ is bounded over $\{t : h \leq \|t\| \leq \gamma \sqrt{n}\}$ for each $\varphi \in \Phi_8$ and can be dropped from the analysis. Now, since $\|T_n(\varphi) - \eta_0^*\| = o_p(1)$, for any $\gamma > 0$, $\|T_n(\varphi) + t/\sqrt{n} - \eta_0^*\| < 2\gamma$ for all $\|t\| \leq \gamma \sqrt{n}$ and $n$ large enough.
Therefore, by Assumption 4(iii), there exists some $\gamma' > 0$ and $h$ large enough so that
\[
\sup_{h \leq ||t|| \leq \gamma' \sqrt{n}} |R_n(T_n(\varphi) + t/\sqrt{n}, \varphi)| \leq 4\lambda_{\min}\{J(\varphi)\}\{1 + ||t + Z_n(\varphi)/\sqrt{n}||^2\}
\leq 4\lambda_{\min}\{J(\varphi)\}||t||^2 + O_p(1),
\]
where the last inequality follows since $||Z_n(\varphi)/\sqrt{n}|| = O_p(1)$, for each $\varphi \in \Phi_\delta$ by Assumption 4(i). Thus, for some $C > 0$, w.p.1,$$
\exp(\omega(t)) \leq \exp\left\{ -\frac{1}{2} t^T J(\varphi) t + |R_n(T_n(\varphi) + t/\sqrt{n}, \varphi)| \right\} \leq C \exp\left\{ -t^T J(\varphi) t/4 \right\}.
$$
Since the result holds for arbitrary and fixed $\varphi$, it holds for each $\varphi \in \Phi_\delta$.

**Region 3:** For $\gamma \sqrt{n}$ large, $\int_{||t|| \geq \gamma \sqrt{n}} |t| \exp\{Z_n(\varphi)/n + t/\sqrt{n} - \eta^*_\varphi|\varphi\} dt$, can be made arbitrarily small and is therefore dropped from the analysis. Using the definition of $\omega(t)$, and the identity $\eta = Z_n(\varphi)/n + t/\sqrt{n} - \eta^*_\varphi$, consider
\[
J_{1n} := \int_{||t|| \geq \gamma \sqrt{n}} ||t|| \exp(\omega(t)) \pi\{Z_n(\varphi)/n + t/\sqrt{n} - \eta^*_\varphi|\varphi\} dt
\leq \sqrt{n} \sup_{||t|| \geq \gamma \sqrt{n}} \exp\left\{ M_n(\eta|\varphi) - M_n(\eta^*_\varphi|\varphi) - \frac{1}{2n} Z_n(\varphi)^T J(\varphi)^{-1} Z_n(\varphi) \right\} \pi(\eta|\varphi) d\eta
\leq O_p(1) \sqrt{n} \sup_{||t|| \geq \gamma \sqrt{n}} \exp\left\{ M_n(\eta|\varphi) - M_n(\eta^*_\varphi|\varphi) \right\} \pi(\eta|\varphi) d\eta,
\]
since $n^{-1} Z_n(\varphi)^T J(\varphi)^{-1} Z_n(\varphi) = O_p(1)$ under Assumption 4(i) for each $\varphi \in \Phi_\delta$.

From Assumption 2(iii), for fixed $\delta_1 > 0$, and any $\delta_2 > 0$, there exists an $\epsilon = \epsilon(\delta_1, \delta_2) > 0$ such that
\[
\sup_{\varphi \in \Phi_{\delta_1}} \sup_{||\eta - \eta^*_\varphi|| \geq \delta_2} \{M(\eta|\varphi) - M(\eta^*_\varphi|\varphi)\} \leq -\epsilon.
\]
Therefore, the above and the uniform convergence in Assumption 2(i) together imply that
\[
\lim_{n \to \infty} P_{0}^{(n)}\left[ \sup_{\varphi \in \Phi_{\delta_1}} \sup_{||\eta - \eta^*_\varphi|| \geq \delta_2} \exp\left\{ M_n(\eta|\varphi) - M_n(\eta^*_\varphi|\varphi) \right\} \leq \exp(-\epsilon n^2) \right] = 1.
\]
(A.5)
Since for each $\varphi \in \Phi_\delta$, $Z_n(\varphi)/\sqrt{n} = O_p(1)$, by Assumption 4(i), from equation...
By Assumption 3(ii),

\[ \theta \]

ing joint inference on differentiability at the cost of additional technicalities.

Assumption 5.

For \(|\eta|\) we abuse notation and write terms that depend on both \(\eta, \phi\) for each \(\eta \in \Phi_{\delta_1}\). To obtain such a result, we require smoothness conditions, in \(\phi\), for the functions \(\Delta_{n_2}(\eta|\phi)\) and \(J(\eta|\phi)\) in Assumption 4. Further, we assume \(\Delta_{n_2}(\eta|\phi)\) is differentiable in \(\phi\), but this can be weakened to stochastic differentiability at the cost of additional technicalities.

Throughout the remainder of this section, to make clear that we are considering joint inference on \(\theta = (\phi^T, \eta^T)^T\), rather than conditional inference for \(\eta | \phi\), we abuse notation and write terms that depend on both \(\eta, \phi\) as \((\eta, \phi)\) and not \(\eta \mid \phi\); e.g., we write \(\Delta_{n_2}(\eta, \phi)\) and \(J(\eta, \phi)\), rather than using the conditioning notation.

Assumption 5. For \(\phi \in \Phi_\delta\), and \(\Delta_{n_2}(\eta, \phi)\), \(J(\eta, \phi)\) as in Assumption 4, the following are satisfied: (i) \(\nabla^2_{\phi\phi}M(\eta^*, \phi)\) and \(J(\eta, \phi)\) are both continuous in \(\phi\); (ii) \(\nabla_{\phi} \Delta_{n_2}(\eta^*, \phi)\) exists and satisfies \(\sup_{\phi \in \Phi_\delta} \| \nabla_{\phi} \Delta_{n_2}(\eta^*, \phi) - \nabla_{\phi} M(\eta^*, \phi) \| = o_p(1)\).

Appendix B: Joint Behavior of cut posterior

While we argue that the conditional view of the posterior for \(\eta\) presented in Theorem 1 is most appropriate, it is feasible to obtain a large sample result for the joint cut posterior. To obtain such a result, we require smoothness conditions, in \(\phi\), for the functions \(\Delta_{n_2}(\eta|\phi)\) and \(J(\eta|\phi)\) in Assumption 4. Further, we assume \(\Delta_{n_2}(\eta|\phi)\) is differentiable in \(\phi\), but this can be weakened to stochastic differentiability at the cost of additional technicalities.

Throughout the remainder of this section, to make clear that we are considering joint inference on \(\theta = (\phi^T, \eta^T)^T\), rather than conditional inference for \(\eta | \phi\), we abuse notation and write terms that depend on both \(\eta, \phi\) as \((\eta, \phi)\) and not \(\eta \mid \phi\); e.g., we write \(\Delta_{n_2}(\eta, \phi)\) and \(J(\eta, \phi)\), rather than using the conditioning notation.

Assumption 5. For \(\phi \in \Phi_\delta\), and \(\Delta_{n_2}(\eta, \phi)\), \(J(\eta, \phi)\) as in Assumption 4, the following are satisfied: (i) \(\nabla^2_{\phi\phi}M(\eta^*, \phi)\) and \(J(\eta, \phi)\) are both continuous in \(\phi\); (ii) \(\nabla_{\phi} \Delta_{n_2}(\eta^*, \phi)\) exists and satisfies \(\sup_{\phi \in \Phi_\delta} \| \nabla_{\phi} \Delta_{n_2}(\eta^*, \phi) - \nabla_{\phi} M(\eta^*, \phi) \| = o_p(1)\).
To present the joint distribution of the cut posterior, we require a few additional notations. Define

\[
\Sigma := \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} = \nu \begin{pmatrix} -\nabla^2_{\varphi\varphi} \mathbb{L}(\varphi^*) & \nabla^2_{\varphi\varphi} M(\eta^*, \varphi^*) \\ -\nabla^2_{\eta\eta} M(\eta^*, \varphi^*) & -\nabla^2_{\eta\eta} M(\eta^*, \varphi^*) \end{pmatrix}
\]

and recall that \( \zeta = \lim_{n_1, n_2 \to \infty} n_1/n_2 \), with \( 0 < \zeta < \infty \), and let \( \vartheta := \zeta^{-1/2} \). Define

\[
V := \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} = \begin{pmatrix} -\vartheta \cdot \Sigma_{11}^{-1} \Sigma_{12}^{-1} & -\vartheta \cdot \Sigma_{11}^{-1} \Sigma_{12}^{-1} \\ -\vartheta \cdot \Sigma_{21}^{-1} \Sigma_{12}^{-1} & -\vartheta^2 \Sigma_{22}^{-1} \Sigma_{12}^{-1} \Sigma_{22}^{-1} \Sigma_{12}^{-1} \end{pmatrix},
\]

and note that by block matrix inversion we have

\[
V^{-1} := \begin{pmatrix} \Sigma_{11} + \vartheta^2 \cdot \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} & \vartheta \cdot \Sigma_{12} \\ \vartheta \cdot \Sigma_{21} & \Sigma_{22} \end{pmatrix}
\]

In addition, define

\[
D_n = \begin{pmatrix} n_1 \cdot I_{d_\varphi} & 0 \\ 0 & n_2 \cdot I_{d_\eta} \end{pmatrix}
\]

and \( Z_n = (Z_{n_1}^T, Z_{n_2}^T)^T \), where

\[
\begin{pmatrix} Z_{n_1} \\ Z_{n_2} \end{pmatrix} := \begin{pmatrix} -\vartheta \cdot \Sigma_{11}^{-1} \Sigma_{12}^{-1} & 0 \\ -\vartheta \cdot \Sigma_{21}^{-1} \Sigma_{12}^{-1} \Sigma_{11}^{-1} \Sigma_{22}^{-1} \end{pmatrix} D_n^{-1/2} \begin{pmatrix} \nabla^2_{\varphi} L_{n_1}(\varphi^*) \\ \nabla^2_{\eta} L_{n_1}(\eta^*, \varphi^*) \end{pmatrix}
\]

\[
= \begin{pmatrix} \Sigma_{11}^{-1} \nabla^2_{\varphi} L_{n_1}(\varphi^*)/\sqrt{n_1} \\ \Sigma_{22}^{-1} \nabla^2_{\eta} L_{n_1}(\eta^*, \varphi^*)/\sqrt{n_1} - \vartheta \cdot \Sigma_{22}^{-1} \Sigma_{12} \Sigma_{11} \nabla^2_{\varphi} L_{n_1}(\varphi^*)/\sqrt{n_1} \end{pmatrix},
\]

and define

\[
\phi := \sqrt{n_1}(\varphi - \varphi^*) - Z_{n_1}, \quad \xi := \sqrt{n_2}(\eta - \eta^*) - Z_{n_2}, \quad t := (\phi^T, \xi^T)^T, \quad T_n := D_n^{-1/2} Z_n + \theta^*.
\]

The cut posterior for \( t \) is then given by \( \pi_{\text{cut}}(t|z, w) = |D_n|^{-1/2} \pi_{\text{cut}}(D_n^{-1/2} t + T_n | z, w) \), which has support \( T_n := \{ t = D_n^{1/2}(\theta - \theta^*) - Z_n : \theta \in \Theta \} \).

**Corollary 1.** Under Assumptions 1-4 and 5, \( \int_{T_n} |\pi_{\text{cut}}(t|z, w) - N\{t; 0, V\}| \, dt = o_p(1) \).

Corollary 1 extends the results obtained by Pompe and Jacob (2021) to cut posterior densities based on arbitrary criterion functions. Using boundedness and differentiability assumptions, and a Taylor series approximation, Pompe and Jacob (2021) derive a Laplace approximation to the cut posterior via an expansion of the log joint cut posterior. Our results extend theirs in several ways: 1) our smoothness conditions imposed on \( M_\varphi(\eta, \varphi) \) and \( L_n(\varphi) \) are weaker than those used in Pompe and Jacob (2021); and 2) our results are valid for a wide range of criterion functions one may wish to choose, including quasi-likelihoods, tempered likelihoods, or any other M-estimation criterion.

Corollary 1 is presented in a slightly different manner from Proposition 3 in Pompe and Jacob (2021). Our result considers the posterior behavior of \( t = \)
\[ D_{n}^{1/2}(\theta - \theta^*) - Z_n, \] while the analysis of Pompe and Jacob (2021) considers the posterior behavior of \( \sqrt{n_2}(\theta - \theta^*) \). In this way, the scaling constants in Proposition 3 of Pompe and Jacob (2021) differ from those in Corollary 1. Since the rates of convergence for the two components, \( \varphi \) and \( \eta \), are different, we believe it more direct to consider \( t \), which cleanly disentangles the two rates, rather than to bundle the two rates together as in the result of Pompe and Jacob (2021). A result for \( \sqrt{n_2}(\theta - \theta^*) \) can be obtained by instead considering the behavior of the random variable \( n_2^{-1/2}D_{n}^{1/2}(\theta - \theta^*) - Z_n/\sqrt{n_2} \).

The following result follows immediately from Corollary 1 using standard arguments (see, e.g., Theorem 8.3 on page 490 of Lehmann and Casella, 2006), and the proof is therefore omitted for brevity.

**Corollary 2.** If \( D_{n}^{-1/2}(\nabla_\varphi L_n(\varphi^*)^\top, \Delta_n(\varphi^*, \varphi^*)^\top) = N(0, \Omega) \), then for \( \tilde{\theta}_n := \int_\Theta \theta \pi_{\text{cut}}(\theta|z, w)\,d\theta \), we have that

\[
D_{n}^{1/2}(\tilde{\theta}_n - \theta^*) \Rightarrow N \left( 0, \begin{pmatrix} \Sigma^{-1}_{11} & 0 & 0 \\ -\vartheta \cdot \Sigma^{-1}_{11} \Sigma^{-1}_{12} & -\vartheta \cdot \Sigma^{-1}_{11} \Sigma^{-1}_{22} & 0 \\ 0 & 0 & \Sigma^{-1}_{22} \end{pmatrix} \Omega \begin{pmatrix} \Sigma^{-1}_{11} & 0 & 0 \\ -\vartheta \cdot \Sigma^{-1}_{11} \Sigma^{-1}_{12} & -\vartheta \cdot \Sigma^{-1}_{11} \Sigma^{-1}_{22} & 0 \\ 0 & 0 & \Sigma^{-1}_{22} \end{pmatrix} \right).
\]

Taken together, Corollaries 1 and 2 demonstrate that the cut posterior does not correctly quantify uncertainty for the posterior mean. In the case of a correctly specified likelihood criterion, Corollary 2 demonstrates that the posterior mean will not have the same asymptotic variance as the maximum likelihood estimator since it neglects the term \( \nabla_\varphi^2 M(\eta^*, \varphi^*) \). Therefore, the posterior mean of the cut posterior will be inefficient if the model is correctly specified.

**Proof of Corollary 1.** To simplify the proof we take \( \nu = 1 \) in what follows. Use Assumption 5 to expand \( \Delta_n(\eta^*, \varphi) / \sqrt{n_2} \) as

\[
\frac{\Delta_n(\eta^*, \varphi)}{\sqrt{n_2}} = \frac{1}{\sqrt{n_2}} \Delta_n(\eta^*, \varphi) + \frac{\nabla_\eta^2 M(\eta^*, \varphi^*)}{\sqrt{n_2}}(\varphi - \varphi^*) + \frac{\{\nabla_\varphi \Delta_n(\eta^*, \varphi) - \nabla_\eta^2 M(\eta^*, \varphi)\}}{\sqrt{n_2}}(\varphi - \varphi^*) + \frac{\{\nabla_\varphi \Delta_n(\eta^*, \varphi) - \nabla_\eta^2 M(\eta^*, \varphi)\}}{\sqrt{n_2}}(\varphi - \varphi^*) + o_p(1)
\]

\[
= \frac{\Delta_n(\eta^*, \varphi)}{\sqrt{n_2}} + \frac{\nabla_\eta^2 M(\eta^*, \varphi^*)}{\sqrt{n_2}}(\varphi - \varphi^*) + o_p(1)
\]

\[
+ \frac{\{\nabla_\varphi \Delta_n(\eta^*, \varphi) - \nabla_\eta^2 M(\eta^*, \varphi)\}}{\sqrt{n_2}}(\varphi - \varphi^*) + \frac{\{\nabla_\varphi \Delta_n(\eta^*, \varphi) - \nabla_\eta^2 M(\eta^*, \varphi)\}}{\sqrt{n_2}}(\varphi - \varphi^*) + o_p(1)
\]

for some intermediate value satisfying \( ||\varphi - \varphi^*|| \leq ||\varphi - \varphi^*|| \), and where the \( o_p(1) \) term follows by applying Assumption 5(ii). From equation A.1 in the proof of Theorem 1, we have that, for

\[
M_n(\eta, \varphi) - M_n(\eta^*, \varphi^*) = -\frac{1}{2}J(\varphi^*)^\top J(\varphi^*) - \frac{1}{2}Z_n(\varphi^*)^\top J(\varphi^*)Z_n(\varphi^*) + R_n(\eta, \varphi),
\]

(B.2)
by Lemma 1, we see that
\[ M = \sqrt{n_2} \eta^* - J(\varphi)^{-1} \Delta_n(\eta^*, \varphi) \]
\[ = \sqrt{n_2}(\eta - \eta^*) - J(\varphi)^{-1} \left\{ \frac{\Delta_n(\eta^*, \varphi^*)}{\sqrt{n_2}} + \nabla^2_{\eta \varphi} M(\eta^*, \varphi^*) \dot{\varphi} \cdot \sqrt{n_1}(\varphi - \varphi^*) \right\} + o_p(1) \]
\[ + J(\varphi)^{-1} \{ \nabla^2_{\eta \varphi} M(\eta^*, \varphi) - \nabla^2_{\eta \varphi} M(\eta^*, \varphi^*) \} \dot{\varphi} \cdot \sqrt{n_1}(\varphi - \varphi^*). \]

Since \( J(\varphi)^{-1} \) is continuous in \( \varphi \), we have that
\[ t_\varphi = \sqrt{n_2}(\eta - \eta^*) - J(\varphi)^{-1} \Delta_n(\eta^*, \varphi) \]
\[ = \sqrt{n_2}(\eta - \eta^*) - J(\varphi)^{-1} \left\{ \frac{\Delta_n(\eta^*, \varphi^*)}{\sqrt{n_2}} + \nabla^2_{\eta \varphi} M(\eta^*, \varphi^*) \dot{\varphi} \cdot \sqrt{n_1}(\varphi - \varphi^*) \right\} + o_p(1). \]

Further, since \( \nabla^2_{\eta \varphi} M(\eta^*, \varphi) \) is continuous in \( \varphi \), and since \( \sqrt{n}(\varphi - \varphi^*) = O_p(1) \) by Lemma 1, we see that
\[ t_\varphi = \sqrt{n_2}(\eta - \eta^*) - J(\varphi)^{-1} \left\{ \frac{\Delta_n(\eta^*, \varphi^*)}{\sqrt{n_2}} + \nabla^2_{\eta \varphi} M(\eta^*, \varphi^*) \dot{\varphi} \cdot \sqrt{n_1}(\varphi - \varphi^*) \right\} + o_p(1). \]

Now, use the fact that \( J(\varphi^*) = \Sigma_{22}, \Sigma_{12} = \nabla^2_{\eta \varphi} M(\theta^*) \), and re-arrange the first term in \( \nu \) as
\[ t_\varphi = \sqrt{n_2}(\eta - \eta^*) - \Sigma_{22}^{-1} \left\{ \Delta_n(\eta^*, \varphi)/\sqrt{n_2} + \Sigma_{12} \cdot \dot{\varphi} \cdot Z_{n_1} \right\} - \Sigma_{22}^{-1} \Sigma_{21} \cdot \dot{\varphi} \cdot \left\{ \sqrt{n_1}(\varphi - \varphi^*) - Z_{n_1} \right\} + o_p(1) \]
\[ = \sqrt{n_2}(\eta - \eta^*) - Z_{n_2} - \Sigma_{22}^{-1} \Sigma_{21} \cdot \dot{\varphi} \cdot \left\{ \sqrt{n_1}(\varphi - \varphi^*) - Z_{n_1} \right\} + o_p(1). \]

Recalling
\[ \xi := \sqrt{n_2}(\eta - \eta^*) - Z_{n_2}, \quad \phi := \sqrt{n_1}(\varphi - \varphi^*) - Z_{n_1}, \]
we then see that
\[ t_\varphi = \xi - \dot{\phi} \cdot \Sigma_{12} \phi + o_p(1). \]

Applying this into equation (B.2) then yields
\[ M_n(\eta, \varphi) - M_n(\eta^*, \varphi^*) = -\frac{1}{2} (\xi - \dot{\xi} \cdot \Sigma_{12} \phi)^\top \Sigma_{22} (\xi - \dot{\xi} \cdot \Sigma_{12} \phi) + \frac{1}{2} Z_{n_2}(\varphi^*)^\top \Sigma_{22} Z_{n_2}(\varphi^*) + R_n(\eta, \varphi), \]

Similarly, from Assumption 1, we have the following expansion for \( L_{n_1}(\varphi) - L_{n_1}(\varphi^*) \):
\[ L_{n_1}(\varphi) - L_{n_1}(\varphi^*) = \sqrt{n_1}(\varphi - \varphi^*)^\top \nabla_\varphi L_{n_1}(\varphi^*)/\sqrt{n_1} - \frac{n_1}{2} (\varphi - \varphi^*)^\top [-\nabla_\varphi L(\varphi^*)](\varphi - \varphi^*) + R_{4n}(\varphi) \]
\[ = -\frac{1}{2} \left\{ \sqrt{n_1}(\varphi - \varphi^*) - Z_{n_1} \right\}^\top \Sigma_{11} \left\{ \sqrt{n_1}(\varphi - \varphi^*) - Z_{n_1} \right\} + \frac{1}{2} Z_{n_1}^\top \Sigma_{11}^{-1} Z_{n_1} R_{4n}(\varphi) \]
\[ = -\frac{1}{2} \phi^\top \Sigma_{11} \phi + \frac{1}{2} Z_{n_1}^\top \Sigma_{11}^{-1} Z_{n_1} R_{4n}(\varphi). \]
where, by Assumption 1, the remainder term $R_{4n}(\varphi)$ satisfies $R_{4n}(\varphi)/[1 + n_1||\varphi - \varphi^*||^2] = o_p(1)$.

Recalling that $Q_n(\theta) = L_{n_1}(\varphi) + M_{n_2}(\eta, \varphi)$, and adding the two expansions together yields, for $T_n = D_n^{-1/2}Z_n + \theta^*$,

$$Q_n(D_n^{-1/2}t + T_n) - Q_n(\theta^*) = -\frac{1}{2} \phi^\top \Sigma_{11} \phi - \frac{1}{2} (\xi - \varphi^* \cdot \Sigma_{22}^{-1} \Sigma_{21} \phi) \Sigma_{22}^{-1} (\xi - \varphi^* \cdot \Sigma_{22}^{-1} \Sigma_{21} \phi)$$

$$+ \frac{1}{2} Z_{n_1}^\top \Sigma_{11}^{-1} Z_{n_1} + \frac{1}{2} Z_{n_2}^\top \Sigma_{22}^{-1} Z_{n_2} + \sum_{j=1}^{4} R_{jn}(D_n^{-1/2}t + T_n).$$

Lastly, we can rewrite the above equation in the following form:

$$Q_n(D_n^{-1/2}t + T_n) - Q_n(\theta^*) = -\frac{1}{2} (\phi^\top, \xi^\top)^\top \left( \begin{array} {cc} \Sigma_{11} + \varphi^2 \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} & \varphi^* \Sigma_{12} \\ \varphi^* \Sigma_{12} & \Sigma_{22} \end{array} \right) \left( \begin{array} {c} \phi \\ \xi \end{array} \right) + \frac{1}{2} Z_{n_1}^\top \Sigma_{11}^{-1} Z_{n_1}$$

$$+ \frac{1}{2} Z_{n_2}^\top \Sigma_{22}^{-1} Z_{n_2} + \sum_{j=1}^{4} R_{jn}(D_n^{-1/2}t + T_n).$$

Recalling the definition of $V^{-1}$ given in equation (B.1), for $t = (\phi^\top, \xi^\top)^\top$ we have that

$$Q_n(D_n^{-1/2}t + T_n) - Q_n(\theta^*) = -\frac{1}{2} t^\top V^{-1} t + \frac{1}{2} Z_{n_1}^\top \Sigma_{11}^{-1} Z_{n_1} + \frac{1}{2} Z_{n_2}^\top \Sigma_{22}^{-1} Z_{n_2} + \sum_{j=1}^{4} R_{jn}(D_n^{-1/2}t + T_n),$$

and the cut posterior $\pi(t \mid w, z)$ can be restated as

$$\pi(t \mid w, z) = \frac{\{D_n^{-1/2}t + T_n\} \exp\{Q_n(D_n^{-1/2}t + T_n) - Q_n(\theta^*)\}}{\int_{T_n} \{D_n^{-1/2}t + T_n\} \exp\{Q_n(D_n^{-1/2}t + T_n) - Q_n(\theta^*)\} dt} = \frac{\pi(D_n^{-1/2}t + T_n) \exp\{\omega(t)\}}{C_n},$$

where

$$\omega(t) = -\frac{1}{2} t^\top V^{-1} t + \sum_{j=1}^{4} R_{jn}(D_n^{-1/2}t + T_n),$$

and

$$C_n := \int_{T_n} \pi(D_n^{-1/2}t + T_n) \exp\{Q_n(D_n^{-1/2}t + T_n) - Q_n(\theta^*)\} dt.$$
The above equation takes precisely the same form as in the proof of Theorem 1, but where the remainder term is now \( R_{1n}(\theta) + R_{2n}(\theta) + R_{3n}(\theta) + R_{4n}(\varphi) \). Hence, so long as this new remainder satisfies Assumption 4(iii), the proof follows the same arguments used in Theorem 1. A sufficient condition for this new remainder term to satisfy Assumption 4(iii) is that Assumption 4(iii) is satisfied for each term. We note that \( R_{1n}(\theta) \) and \( R_{4n}(\varphi) \) both satisfy the condition by hypothesis, while Lemma 2 verifies Assumption 4(iii) for \( R_{2n}(\theta) \) and \( R_{3n}(\theta) \).

The remainder of the proof follows the same arguments as those used to prove Theorem 1 and is omitted for the sake of brevity.

\[ \square \]

B.1. Lemmas

**Lemma 2.** Under the assumptions of Corollary 1, Assumption 4(iii) is satisfied for \( R_{2n}(\theta) \), and \( R_{3n}(\theta) \).

**Proof.** For \( j = 2, 3 \), Assumption 4(iii) is equivalent to the following condition: for any \( \delta_n = o(1) \),

\[
\sup_{\|\theta - \theta^*\| \leq \delta_n} \frac{|R_jn(\theta)|}{1 + n\|\theta - \theta^*\|^2} = o_p(1). \tag{B.3}
\]

We verify (B.3) separately for \( j = 2 \) and \( j = 3 \).

**Term** \( R_{2n}(\theta) \): Define the matrix function \( V(\varphi, \varphi^*) := \{ \nabla_{\eta\varphi} M(\eta^{*}, \varphi) - \nabla_{\eta\varphi} M(\eta^{*}, \varphi^*) \} \)
and consider

\[
R_{2n}(\theta) = n(\eta - \eta^*)^\top V(\bar{\varphi}, \varphi^*)(\varphi - \varphi^*),
\]
where \( \bar{\varphi} \) is some intermediate value satisfying \( \|\bar{\varphi} - \varphi^*\| \leq \|\varphi - \varphi^*\| \). We then see that

\[
|R_{2n}(\theta)| \leq \|\sqrt{n}(\theta - \theta^*)\|^2 \|V(\bar{\varphi}, \varphi^*)\|^2,
\]
and

\[
\sup_{\|\theta - \theta^*\| \leq \delta_n} \frac{|R_{2n}(\theta)|}{1 + n\|\theta - \theta^*\|^2} \leq \sup_{\|\theta - \theta^*\| \leq \delta_n} \|V(\bar{\varphi}, \varphi^*)\|^2 \frac{\|\sqrt{n}(\theta - \theta^*)\|^2}{1 + \|\sqrt{n}(\theta - \theta^*)\|^2} \leq \|V(\bar{\varphi}, \varphi^*)\|^2,
\]

since \( \sup_{\|\theta - \theta^*\| \leq \delta_n} \frac{\|\sqrt{n}(\theta - \theta^*)\|^2}{1 + \|\sqrt{n}(\theta - \theta^*)\|^2} \leq 1 \) for any \( \delta_n = o(1) \).

From the definition of the intermediate value \( \bar{\varphi} \), we have that \( \|\bar{\varphi} - \varphi^*\| \leq \|\varphi - \varphi^*\| \leq \delta_n \). From Assumption 5, \( V(\varphi, \varphi^*) \) is continuous in \( \varphi \) for all \( \varphi \) in a neighborhood of \( \varphi^* \). Conclude from this continuity that \( \|V(\varphi, \varphi^*)\|^2 = o(1) \) when \( \|\theta - \theta^*\| \leq \delta_n \). Equation (B.3) is satisfied for \( R_{2n}(\theta) \).

**Term** \( R_{3n}(\theta) \): Now, let \( V(\varphi, \varphi^*) := [J(\eta^{*}, \varphi) - J(\eta^{*}, \varphi^*)] \). Similar to the proof of the \( R_{2n}(\theta) \) term,

\[
R_{3n}(\theta) = -\frac{n}{2}(\eta - \eta^*)^\top V(\varphi, \varphi^*)(\eta - \eta^*) = -\sqrt{n}(\theta - \theta^*)^\top \begin{pmatrix} 0 & 0 \\ V(\bar{\varphi}, \varphi^*) \end{pmatrix} \sqrt{n}(\theta - \theta^*),
\]
so that \(|R_{3n}(\theta)| \leq \|\sqrt{n}(\theta - \theta^*)\|^2\|V(\varphi, \varphi^*)\|^2\). Repeating the same argument as used in the first part of the result, we have that

\[
\sup_{\|\theta - \theta^*\| \leq \delta_n} \frac{|R_{3n}(\theta)|}{1 + \alpha\|\theta - \theta^*\|^2} \leq \sup_{\|\theta - \theta^*\| \leq \delta_n} \frac{\|\sqrt{n}(\theta - \theta^*)\|^2}{1 + \sqrt{n}(\theta - \theta^*)}\|
\]

\[
\leq \sup_{\|\theta - \theta^*\| \leq \delta_n} \|V(\varphi, \varphi^*)\|^2 = o(1).
\]

\(\square\)

Appendix C: Credible sets for understanding uncertainty propagation in the cut posterior

The following algorithm describes the construction of credible sets for \(\eta\) repeatedly for samples of \(\varphi\) from the cut posterior, using the large sample approximation of Theorem 1.

Algorithm 1 Diagnostic for \(\eta|w, \varphi\)

Inputs: A sequence \(\varphi_1, \ldots, \varphi_M\), and quantile \(\alpha\).

Output: A sequence of approximate confidence sets \(\{C_{\eta}(\varphi_j) : j \leq M\}\)

for \(j = 1, \ldots, M\) and \(\varphi = \varphi_j\) do

Estimate \(\eta^*_\varphi, J(\eta^*_\varphi|\varphi)\) by \(\hat{\eta}_\varphi, J_n(\hat{\eta}_\varphi|\varphi)\).

Draw: \(Z_k \sim N(\hat{\eta}_\varphi, [nJ_n(\hat{\eta}_\varphi|\varphi)]^{-1})\), for \(k = 1, \ldots, K\)

Calculate \(Y_k = [nJ_n(\hat{\eta}_\varphi|\varphi)]^{-1/2}\{Z_k - \hat{\eta}_\varphi\}\)

Retain all \(Y_k\) such that \(\|Y_k\|^2 \leq \chi^2_{2\alpha}(1 - \alpha)\).

end for

Appendix D: Marginal semi-modular posterior

Algorithm 2 describes how to draw MCMC samples from our proposed semi-modular posterior density introduced in Section 3.4.
Algorithm 2 Semi-modular posterior

Inputs: a value of $\gamma$, and a transition kernel $q(\varphi|\varphi')$.
Output: Draws from the approximate semi-modular cut posterior $\pi^\gamma_{\text{cut}}(\varphi|z, w)$.

Initialize $\varphi^{(0)}$

for $j = 1, \ldots, M$ and $\varphi = \varphi_j$

1. Draw $\bar{\varphi} \sim q(\varphi|\varphi_{i-1})$
2. Estimate $\eta_{\bar{\varphi}}, J(\eta_{\bar{\varphi}})$ by $\eta_{\bar{\varphi}}, J_n^2(\eta_{\bar{\varphi}})$
3. Choose $\eta^*$ in the HPD region of $N\{\eta; \eta_{\bar{\varphi}}, [n^2 J_n^2(\pi_{\text{cut}}(\varphi|z))]^{-1}\}$
4. Compute $\ln b_m(\varphi|w)$ via (3.6).
5. Compute $L^i = \pi_{\text{cut}}(\varphi|z) \exp\{\ln \pi_{\text{cut}}(w|\varphi)\}^\gamma$, $L^{i-1} = \pi_{\text{cut}}(\varphi^{i-1}|z) \exp\{\ln \pi_{\text{cut}}(w|\varphi^{i-1})\}^\gamma$,
6. and the Metropolis-Hastings ratio: $r = L^i \pi(\varphi) q(\varphi^{i-1}|\varphi) / L^{i-1} \pi(\varphi^{i-1}) q(\varphi|\varphi^{i-1})$
7. if $U(0, 1) < r$ then
8. Set $\varphi^i = \bar{\varphi}$
9. else
10. Set $\varphi^i = \varphi^{i-1}$
11. end if
12. end for

Figure 7 shows semi-modular posterior densities for $\eta_1$ and $\eta_2$ for the method of Carmona and Nicholls (2020) (top) and marginal semi-modular approach of Section 3.3 (bottom) for $\gamma \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$ for the epidemiological example of Section 4.1. We can see that for the same value of $\gamma$ for the two methods, the semi-modular posterior densities are similar. For both approaches, after the SMI samples for $\varphi$ are drawn, we obtained samples for $\eta$ for each $\varphi$ sample using the SIR approach described in Section 4.1. However, using the normal approximation directly makes little difference to the result (results not shown).
Fig 7: Semi-modular posterior densities for method of Carmona and Nicholls (2020) (top row) and marginal semi-modular method (bottom row) for $\eta_1$ (left) and $\eta_2$ (right) and $\gamma \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$