A Multilevel Model for Calculating the Motion of Rocks

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Abstract. The presence of a large number of loose rocks, often accompanied by giant rocks, which are highly destructive to the surface are an important factor in geological hazard risk assessment. In rocks moving along a slope, the scale of movement and the scale of deformation differ substantially in magnitude owing to their movement characteristics of large moving distance, fast rotation speed, and small deformation. A multilevel model for rock motion calculation is established to separate the rigid displacement and deformation displacement of the rock. The total rock movement displacement comprises translational displacement, rotational displacement, and deformation displacement. Three corresponding coordinate systems are set up. These include the translational coordinate system, which is used to describe the translational displacement of the rock, the rotational coordinate system, which is used to describe the rotational displacement of the rock, and the co-rotational coordinate system, which is used to describe the deformation displacement of the rock. The origin and the direction of the coordinate axis of the co-rotational coordinate system change with the movement of the rock. Lagrange's principle establishes the equations of motion, which is solved by an explicit approach in the framework of the Continuum-discontinuum element method. Several numerical cases verify that this method can accurately calculate the rotational movement of the rock.

1. Introduction
In geological disasters such as collapse and landslides, there is a large amount of solid material movement, and the rocks carried by such movements have a heavy impact and destructive force, generating a devastating effect on the ground[1-3]. The rotational inertia and movement speed of the rocks are both high, resulting in long-distance movement[4, 5]. During the movement of rocks, collisions with slopes, surrounding rocks, and surface buildings occur, and the rocks will break if the collision force is greater than the strength of the rocks. Therefore, the deformation of the rocks must be considered. Currently, the discrete element method is used to simulate rocks moving on slopes. It is controlled by the motion and rotation equations of the center of moving rigid blocks. The deformation and stress of deformable blocks can be solved using the finite difference method. Wilkins[6] used the finite difference method and finite volume method to calculate the large deformation characteristics of the elastic-plastic block; Williams[7] used the modal method to analyze the deformation of the block; Ghaboussi[8] applied a quadrilateral finite element to calculate the deformation and stresses of the block.
The Discontinuous deformation analysis (DDA) is a numerical analysis method that is based on discontinuous mechanics[9]. This method can analyze the dynamic behavior of the block system, as well as simulate the damage degree and damage range of the block[10-12]. The DDA can calculate the small deformation before and after the damage.
Continuous-discrete coupling methods, such as the numerical manifold method[13, 14] and the CDEM[15-19], combine the benefits of both the continuous and discrete models, use different criteria to consider the fracture of the block[20, 21], and can simulate the damage process of the geological body from continuous to discontinuous.

Since the magnitude difference between the movement and deformation levels is great, most discrete element methods only consider the translational movement and deformation of the rocks, making the block rotation calculation inaccurate. The use of conventional algorithms in rock motion calculations will result in inaccurate rotation and deformation results. In a multilevel model for rock motion calculation, which has been established in this paper, the total displacement of the rock movement was divided into three parts: translational displacement, rotational displacement, and deformation displacement. This model can accurately describe the motion of blocks in geological bodies. Three levels of rock motion were separated, and the calculation of deformation was performed only when there was contact and risk of cracking, thus accelerating the efficiency of the calculation.

This paper is organized as follows. Section 2 provides the theoretical basis of the multilevel element method. Section 3 determines the kinetic equations using a five-node quadrilateral multilevel element as an example. The main conclusions are drawn in Section 5.

2. Methodology

2.1. Multilevel method

The physical quantities of block displacement were differentiated using a rigid displacement term and a deformation displacement term. The rigid body displacement term was decomposed into translational and rotational displacements by taking a specific base point. Three coordinate systems are used to represent the translational, rotational, and deformational displacements. The equations of motion were established using Lagrangian equations, with the displacement of different levels as generalized variables.

The translational displacement is based on the translational coordinate system, with the origin of the translational coordinate system located at the origin of the global coordinate system and the coordinate axes parallel to the global coordinate system. The rotational displacement is based on the rotational coordinate system, with the origin of the rotational coordinate system located at the center of mass of the object, and the angle between the coordinate axis and the global coordinate system is the rotation angle of the object. The deformation displacement is expressed based on the deformation coordinate system, with the origin of the deformation coordinate system located at the center of mass of the object, and the angle between the coordinate axis and the global coordinate system is the rotation angle of the object. Within the deformation coordinate system, the shape functions are introduced. The displacement is denoted as:

\[ u = u_e + u_r + u_d \]  

where \( u_e \) is the translational displacement of the rigid body, \( u_r \) is the rotational displacement of the rigid body, and \( u_d \) is the deformation displacement.

With \( u_p \) denoted as the position in the rotational coordinate system and \( u_d \) as the deformation displacement in the displacement coordinate system, the displacement of the block can be written as

\[ u = u_r + A \dot{u}_p + \dot{A} u_p \]  

where \( A \) is the rotation transformation matrix, define \( \phi = d\phi / dt \), the form of \( A_\phi \) and \( \dot{A}_\phi \) are

\[ A_\phi = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}, \dot{A}_\phi = \begin{bmatrix} -\sin \phi & -\cos \phi \\ -\cos \phi & -\sin \phi \end{bmatrix} \phi = A_\phi \phi. \]

Define a matrix with the form as

\[ A_\omega = \begin{bmatrix} -\sin \phi & -\cos \phi \\ \cos \phi & -\sin \phi \end{bmatrix} \]
the velocity equation can be written as

\[ \dot{u} = u_c + A_p \dot{u}_p + A_s \dot{u}_s = u_c + A_p \dot{u}_p + A_s \dot{u}_s \]  \hspace{1cm} (5)

The displacement and velocity can be written as

\[ \{u\} = \begin{bmatrix} I & [A_p] & [A_s] \end{bmatrix} \begin{bmatrix} u_c \\ u_p \\ u_s \end{bmatrix}, \quad \{\dot{u}\} = \begin{bmatrix} I & [A_p] & [A_s] \end{bmatrix} \begin{bmatrix} \dot{u}_c \\ \dot{u}_p \\ \dot{u}_s \end{bmatrix} \]  \hspace{1cm} (6)

2.2. Lagrange's equations

2.3. Basic on the CDEM theory

The CDEM establishes the control equations based on the Lagrangian energy system[22-24]. Set \( q = (q_1, q_2, \ldots, q_N) \) and \( \dot{q} = dq / dt = (\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_N) \) as generalized coordinates of the system, the extended Lagrange’s equations are[15]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = Q_k \quad (k = 1, 2, \ldots, N) \]  \hspace{1cm} (7)

where the non-conservative force of the system is given as \( Q_k \), the Lagrangian function \( L \) is

\[ L = L_m + L_e + L_f \]  \hspace{1cm} (8)

where \( L_m \) represents the kinetic energy of the system, \( L_e \) represents the elastic energy, and \( L_f \) represents the work of the conservative force. Under the Lagrangian system, the energy functional of the element is

\[ L = \frac{1}{2} \int_V \rho \dot{\mathbf{u}}^2 \, dV + \int_V \frac{1}{2} \sigma_{ij} (\varepsilon_{ij} + u_{ij}) \, dV - \int_V f_i u_i \, dV \]  \hspace{1cm} (9)

where \( \rho \) is the density of the element, \( \dot{\mathbf{u}} \) is the velocity of the node, \( u_i \) is the nodal displacement of the element, \( \sigma_{ij} \) is the stress tensor of the element, \( \varepsilon_{ij} \) is the strain tensor of the element, \( f_i \) is the body force, and \( V \) is the volume of the element. Based on the Lagrangian equation, the kinetic equation can be written as[15]

\[ M \ddot{\mathbf{u}}(t) + C \dot{\mathbf{u}}(t) + K \mathbf{u}(t) = \mathbf{F}(t) \]  \hspace{1cm} (10)

where, \( M \), \( C \), and \( K \) are the cell mass matrix, damping matrix, and stiffness matrix, respectively, and \( \dot{\mathbf{u}}(t) \), \( \ddot{\mathbf{u}}(t) \), and \( \mathbf{u}(t) \) are the acceleration, velocity, and displacement arrays, respectively.

The CDEM is solved by dynamic relaxation, which eliminates the total stiffness. The dynamic relaxation method is an explicit method of converting a static problem into a dynamic problem by introducing a damping term into the dynamic calculation.

3. Formulation of quadrilateral multilevel element

A five-node quadrilateral multilevel element is taken as an example (see Figure 1) to derive the kinetic equations. The deformation variables were set at the boundary nodes, displacement variables and rotation variables (deformation to zero) were set at the center nodes, and a rotate reference line (normal displacement is zero) was set at the center node. There were a total of 11 independent variables, each of which is a function of time. The derivation of the equation is done using the Lagrangian energy equation.
Figure 1. A five-node quadrilateral multilevel element.

3.1. Basic variables
A detailed description of the derivation of the five-node quadrilateral multilevel element is given here. At the center of mass of the quadrilateral element, an additional node is created to represent the rigid body displacement. The deformation at the center node is set as zero, and the deformation displacement at the quadrilateral nodes is defined as the deformation relative to the center. For five-node quadrilateral multilevel element, \( u_c, v_c \), and \( \varphi \) represent the displacements at the x-axis, the displacements at the y-axis, and the rotation angle of the rigid body respectively, and \( u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4 \) represent the deformation displacements of four nodes of the quadrilateral element.

3.2. Shape functions
The quadrilateral element is divided into four regions (I, II, III, and IV). The four regions are all triangular elements, and the deformation of each node is the displacement value relative to point 0. The displacement of each point in the element is a function of the coordinates \((x, y)\), with \( u \) as the x-direction displacement component and \( v \) as the y-direction displacement component. Assuming that the displacement pattern in each region is

\[
\begin{align*}
\mathbf{u} &= \beta_1 + \beta_2 x + \beta_3 y, \\
\mathbf{v} &= \beta_4 + \beta_5 x + \beta_6 y
\end{align*}
\]  

(11)

where \( \beta_1, \beta_2, \ldots, \beta_6 \) are the generalized coordinate and are constants to be determined.

If the coordinate of node 0 is \((x_0, y_0)\) and the displacement value is \( u_0 \),

\[
u_0 = \beta_1 + \beta_2 x_0 + \beta_3 y_0
\]  

(12)

If the coordinate of node \( i \) (\( i = 1, \ldots, 4 \)) is \((x_i, y_i)\) and the displacement value is \( u_i \),

\[
u_i = \beta_1 + \beta_2 x_i + \beta_3 y_i
\]  

(13)

The relative displacement, \( u \), is defined as

\[
u = \beta_4 (x_i - x_0) + \beta_5 (y_i - y_0)
\]  

(14)

Similarly,

\[
u = \beta_4 (x_i - x_0) + \beta_5 (y_i - y_0)
\]  

(15)

The displacement function value at the three nodes is equal to the value of the displacement components at these nodes. If the coordinate of node 1 is \((x_1, y_1)\) and the displacement values are \( u_1 \) and \( v_1 \), and the coordinate of node 2 is \((x_2, y_2)\) and the displacement values are \( u_2 \) and \( v_2 \), then

\[
\begin{bmatrix}
u_1 \\
v_2
\end{bmatrix} =
\begin{bmatrix}
x_1 - x_0 & y_1 - y_0 & 0 & 0 \\
x_2 - x_0 & y_2 - y_0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4
\end{bmatrix}
\]  

(16)

The generalized coordinates are derived as
\[
\begin{bmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4
\end{bmatrix} = \frac{1}{2 \Omega} \begin{bmatrix}
y_2 - y_0 \\
-x_2 - x_0 \\
0 \\
0
\end{bmatrix} - \begin{bmatrix}
y_1 - y_0 \\
x_1 - x_0 \\
0 \\
0
\end{bmatrix} \begin{bmatrix}
0 \\
0 \\
y_2 - y_0 \\
-x_2 - x_0 \\
y_1 - y_0 \\
x_1 - x_0
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
v_1 \\
v_2
\end{bmatrix}
\]  
(17)

where \( \Omega \) is the area of region I. The shape functions of the region I are represented as \( N_i^I, N_j^I \),

\[
N_i^I = \frac{1}{2 \Omega} \left( (y_2 - y_0)x + (x_0 - x_2)y \right), N_j^I = \frac{1}{2 \Omega} \left( (y_1 - y_0)x + (x_1 - x_0)y \right)
\]  
(18)

The displacement in the region I can be calculated using the shape function and displacement of node 1 and node 2, which are

\[
u = N_i^I u_1 + N_j^I u_2, v = N_i^I v_1 + N_j^I v_2
\]  
(19)

Similarly, the coordinate of node 3 is \((x_i, y_j)\) and the coordinate of node 4 is \((x_i, y_j)\), defining \(N_i^{II}, N_j^{II}, N_i^{III}, N_j^{III}, N_i^{IV}, N_j^{IV}\) as shape functions of regions II, III, and IV, respectively, and \(\Omega_2, \Omega_3, \Omega_4\) as the area of regions II, III, and IV, respectively.

\[
N_i^{II} = \frac{1}{2 \Delta} \left( (y_3 - y_0)x + (x_0 - x_3)y \right), N_j^{II} = \frac{1}{2 \Delta} \left( (y_0 - y_2)x + (x_2 - x_0)y \right)
\]

\[
N_i^{III} = \frac{1}{2 \Delta} \left( (y_2 - y_0)x + (x_0 - x_2)y \right), N_j^{III} = \frac{1}{2 \Delta} \left( (y_1 - y_3)x + (x_3 - x_1)y \right)
\]

\[
N_i^{IV} = \frac{1}{2 \Delta} \left( (y_1 - y_0)x + (x_0 - x_1)y \right), N_j^{IV} = \frac{1}{2 \Delta} \left( (y_0 - y_4)x + (x_4 - x_0)y \right)
\]  
(20)

Since the displacement of node 3 and node 4 are set as \(u_3\) and \(u_4\), respectively, then the displacement of regions II, III, and IV can be calculated as

\[
u = N_i^{II} u_3 + N_j^{II} u_4
\]  
(21)

\[
u = N_i^{III} u_3 + N_j^{III} u_4
\]  
(22)

\[
u = N_i^{IV} u_3 + N_j^{IV} u_4
\]  
(23)

Since \( \{\delta\}^e \) is the node displacement, the deformation displacement \(u_d\) can be written as

\[
u = [N] \{\delta\}^e
\]  
(24)

### 3.3. The potential energy of the system

The potential energy \( (V) \) of the system is

\[
V = \sum_{i=1}^{4} \int_{\Omega} \frac{1}{2} \{\sigma\}^T \{e\} d\Omega - \{\delta\}^e^T \{R\} = \sum_{i=1}^{4} \left( \frac{1}{2} \int_{\Omega} \{\delta\}^e^T [B]^T [D][B]\{\delta\}^e d\Omega - \{\delta\}^e^T \{F\} \right)
\]  
(25)

where \( \{\sigma\} \) is the stress array, \( \{e\} \) is the strain array, \( \{F\} \) is the equivalent nodal force array, \( [B] \) is the strain matrix, and \( [D] \) is the elasticity matrix.

### 3.4. The kinetic energy of the system

The velocity of any point of the element is

\[
\{\dot{u}\} = \left[I \right] \left[A_p^\prime \right] \{u_p\} \begin{bmatrix}
\dot{\phi}_x \\
\dot{\phi}_y
\end{bmatrix} = \left[I \right] \left[A_p^\prime \right] \{u_p\} \left[A_p^\prime \right] \{N\} \begin{bmatrix}
\dot{\phi}_x \\
\dot{\phi}_y
\end{bmatrix} = [R] \{\dot{\hat{q}}\}^e
\]  
(26)

The kinetic energy \( (T) \) of the system is

\[
T = \sum_{i=1}^{4} \left( \frac{1}{2} \int_{\Omega} \rho \{\dot{u}\}^T \{\dot{u}\} d\Omega \right) = \sum_{i=1}^{4} \left( \frac{1}{2} \int_{\Omega} \rho \{\dot{\hat{q}}\}^e^T [R]^T [R] \{\dot{\hat{q}}\}^e d\Omega \right)
\]  
(27)
\{ \dot{u} \} \text{ is the velocity of the element, } [R] \text{ is the transform matrix, and } \{ \dot{q} \}^e \text{ is the velocity array.}

3.5. Kinetic equations

Based on Lagrange’s equations,

\[
\sum_{i=1}^{4} \left( \int [B]^T [D] [B] \, d\Omega \right) \{ \dot{\delta} \}^e + \sum_{i=1}^{4} \left( \int [\rho [R]]^T [R] \, d\Omega \right) \{ \dot{\phi} \}^e = \{ F \}
\]

\[
[K] \{ \delta \}^e + [C] \{ \dot{\phi} \}^e + [M] \{ \ddot{\phi} \}^e = \{ F \}
\] (28)

Here, the kinetic equations of the quadrilateral multilevel element are established.

4. Numerical examples

4.1. Constant force test

The first example is the constant force movement of a block. Owing to the constant force, the block moves along the force direction. The model is established as shown in Figure 2.

The mechanical parameters of the block are set as density \( \rho = 2500 \text{ kg/m}^3 \), elastic modulus \( E = 1 \text{ GPa} \), Poisson’s ratio \( \nu = 0.3 \), and friction angle \( \varphi = 30^\circ \). The numerical simulation time step is 10 \( \mu \text{s} \) and the total simulation time is 10 s. The force is applied at the corner point and is set as \( F = 10 \text{ kN} \), with the force directed toward the negative x-axis direction. The block rotates around the center point while moving in the negative direction of the x-axis.

After comparing the numerical simulation results with theoretical solutions (see Figure 3) the numerical calculation results were found to be consistent with the theoretical solution.

4.2. Constant moment of force test

The second example is the movement of a block with a constant moment of force. The block rotates around the centroid point due to the moment of force. The model is established as shown in Figure 4.

The mechanical parameters of blocks are set as density \( \rho = 2500 \text{ kg/m}^3 \), elastic modulus \( E = 1 \text{ GPa} \), Poisson’s ratio \( \nu = 0.3 \), and friction angle \( \varphi = 30^\circ \). The numerical simulation time step is 10 \( \mu \text{s} \), and the total simulation time is 10 s. The moment of force is set as \( M = 1000 \text{ kN} \cdot \text{m} \). The moment direction is counterclockwise and the block rotates counterclockwise around the center point.

The obtained time-displacement curves of the upper right point are shown in Figure 5. After comparing the numerical simulation results with theoretical solutions, the numerical calculation results were found to be consistent with the theoretical solution.
4.3. Compound pendulum test

The third example is a simulation of a compound pendulum. A compound pendulum is a dynamic motion system in which an object makes a small swing around a fixed horizontal axis under the force of gravity. Point C was set as the center of mass of this object and point O was the fulcrum or suspension point of the compound pendulum. The intersection point between the axis of the compound pendulum and the plane passing through point C and perpendicular to the axis of rotation is point O.

During the swing process, the compound pendulum is only subjected to the reaction force of gravity and the rotating shaft, and the gravitational moment serves as the restoring moment. If the mass of the rigid body is m, the distance between the center of mass (C) and the axis of rotation (O) is I, and the moment of inertia around point (O) is I, then the low amplitude vibration period of the compound pendulum is

\[ T = \frac{2\pi}{\sqrt{I/mg}} \]  \hspace{1cm} (29)

The model is established as shown in Figure 6.

The mechanical parameters of blocks are set as density \( \rho = 2500 \text{ kg/m}^3 \), elastic modulus \( E = 1 \text{ GPa} \), Poisson’s ratio \( \nu = 0.3 \), and friction angle \( \varphi = 30^\circ \). The numerical simulation time step is 10 \( \mu \text{s} \), and the total simulation time is 7 s.

The obtained time-angular acceleration curve is shown in Figure 7. After comparing the numerical simulation results with theoretical solutions, the numerical calculation results were found to be consistent with the theoretical solution. According to the numerical result, the period of the compound pendulum is 6.16 s, which coincides with the result calculated by the theoretical formulation.

5. Conclusion

This paper presented a new model for calculating the motion of rocks, which differentiates between rigid and deformation displacement. The total displacement of the rock movement was divided into translational, rotational, and deformation displacements. Three corresponding coordinate systems were set up for the calculation of the three parts of displacement. The motion equations were established using
Lagrange's principle. In the CDEM framework, an explicit approach was applied in solving the motion equations. Simulation results were used to validate the accuracy of this model. The movement of the block under constant force and constant torque, as well as the compound pendulum case, were simulated. After comparing the numerical simulation results with theoretical solutions, the numerical calculation results were found to be consistent with the theoretical solution. The accuracy of this model has been verified whereas the simulation of fracture and crack process in blocks using a multilevel model is still in development.

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