Exact solution of the quantum spin chains associated with the $sp(4)$ algebra

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Abstract

The off-diagonal Bethe ansatz method is generalized to the integrable model associated with the $sp(4)$ (or $C_2$) Lie algebra. By using the fusion technique, we obtain the complete operator product identities among the fused transfer matrices. These relations, together with some asymptotic behaviors and values of the transfer matrices at certain points, enable us to determine the eigenvalues of the transfer matrices completely. For the model with the periodic boundary condition, the eigenvalues are described by homogeneous $T - Q$ relations, which coincides with those obtained by the conventional Bethe ansatz methods. For the model with the off-diagonal boundary condition, the eigenvalues are given in terms of inhomogeneous $T - Q$ relations, which is due to the fact of the $U(1)$-symmetry-broken and also has failed to be obtained by the conventional Bethe ansatz methods for many years. The method and the results in this paper can be used to study other integrable models associated with the $sp(2n)$ (i.e., $C_n$) algebra with a generic $n$.

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1 Introduction

Quantum integrable models play important roles in a variety of fields such as quantum field theory, condensed matter physics and statistical physics, because they can provide solid benchmarks for understanding the many-body effects and new physical concepts in corresponding universal classes [1, 2, 3, 4, 5].

Recently, a generic method (the off-diagonal Bethe ansatz (ODBA)), for solving the integrable models with or without obvious reference states is proposed [6]. With the help of the proposed inhomogeneous $T-Q$ relations, several typical models without $U(1)$ symmetry are solved exactly [7]. Based on the ODBA solution to the eigenvalues, the corresponding Bethe-type states are also retrieved [8, 9]. We note that the integrable models without $U(1)$ symmetry is a very hot issue and many interesting efforts have been done [10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

In order to solve the models associated with high rank algebras, the nested ODBA is proposed and the model related with $A_n$ algebra was solved exactly [20]. However, for the models related with other Lie algebras such as $B_n$, $C_n$ and $D_n$, the corresponding results are still missing. In this paper, we generalize the ODBA method to the integrable models related with the $C_2$ Lie algebra (or the $sp(4)$ algebra). We study the $C_2$ vertex model with periodic and open boundary conditions. For the present case, due to eigenvalues of transfer matrix is expressed by a polynomial where the degree is higher than the $A_n$ case, thus we need more functional relations to determine it. By using the fusion technique, we obtain the closed operator product identities of the fused transfer matrices. After analyzing the asymptotic behaviors and the values of the fused transfer matrices at certain special points, we obtain the eigenvalues which is described by the inhomogeneous $T-Q$ relations [6]. The method and the results in this paper can be used to study other integrable models associated with the $sp(2n)$ algebra with a generic $n$.

The paper is organized as follows. In section 2, we study the $C_2$ model with the periodic boundary condition. The closed functional relations are obtained by the fusion. Based on them, we obtain the exact solution of the system, which coincides with those obtained by the conventional Bethe ansatz methods [21, 22]. In section 3, we study the $C_2$ model with integrable open boundary conditions. We provide the closed operator product identities, the asymptotic behaviors and the values at certain points of the fused transfer matrices, which
enable us give the eigenvalue of the transfer matrix in terms of an inhomogeneous $T - Q$ relation. Concluding remarks are given in section 4.

2 Closed chain with the $sp(4)$ invariant

2.1 The system

Let $V$ denote a 4-dimensional linear space with an orthonormal basis $\{|i\rangle | i = 1, \ldots, 4\}$ which endows the fundamental representation of the $C_2$ algebra. The $sp(4)$-invariant $R$-matrix $R(u) \in \text{End}(V \otimes V)$ is given by its matrix elements [22]

$$R_{12}(u)_{ki}^{ij} = u(u + 3)\delta_{ik}\delta_{jl} + (u + 3)\delta_{il}\delta_{jk} - u\xi_i\xi_k\delta_{jl}\delta_{ki},$$

(2.1)

where the index $\bar{i}$ is defined by $i + \bar{i} = 5$, $\xi_i = 1$ if $i \in \{1, 2\}$ and $\xi_i = -1$ if $i \in \{3, 4\}$. Let us take the notations for simplicity

$$R_{12}(u)_{ii} = a(u) = (u + 1)(u + 3),$$

$$R_{12}(u)_{ij} = b(u) = u(u + 3), \quad i \neq j, \bar{j},$$

$$R_{12}(u)_{i\bar{i}} = c(u) = 2u + 3,$$

$$\xi_i\xi_jR_{12}(u)_{i\bar{j}} = d(u) = -u, \quad i \neq j, \bar{j},$$

$$R_{12}(u)_{i\bar{i}} = e(u) = u(u + 2),$$

$$R_{12}(u)_{ij} = g(u) = u + 3, \quad i \neq j, \bar{j}.$$  (2.2)

The $R$-matrix (2.1) enjoys the properties:

regularity : $R_{12}(0) = \rho_1(0)\frac{1}{2}P_{12},$

unitarity : $R_{12}(u)R_{21}(-u) = \rho_1(u),$

crossing − unitarity : $R_{12}(u)^{t_i}R_{21}(-u - 6)^{t_i} = \rho_1(u + 3),$

where $\rho_1(u) = a(u)a(-u)$, $P$ is the permutation operator with the elements $P_{kl}^{ij} = \delta_{il}\delta_{jk}$, and $t_i$ denotes the transposition in $i$-th space, $R_{21}(u) = P_{12}R_{12}(u)P_{12}$. Here and below we adopt the standard notation: for any matrix $A \in \text{End}(V)$, $A_j$ is an embedding operator in the tensor space $V \otimes V \otimes \cdots$, which acts as $A$ on the $j$-th space and as an identity on the other factor spaces; $R_{ij}(u)$ is an embedding operator of $R$-matrix in the tensor space, which
acts as an identity on the factor spaces except for the $i$-th and $j$-th ones. Moreover, the $R$-matrix satisfies the quantum Yang-Baxter equation (QYBE)

$$ R_{12}(u - v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v). \quad (2.3) $$

Let us introduce the “row-to-row” (or one-row) monodromy matrix $T(u)$, which is a $4 \times 4$ matrix with operator-valued elements acting on $V^{\otimes N}$,

$$ T_0(u) = R_{01}(u - \theta_1)R_{02}(u - \theta_2) \cdots R_{0N}(u - \theta_N), \quad (2.4) $$

where $\{\theta_j | j = 1, \cdots, N\}$ are arbitrary free complex parameters which are usually called as inhomogeneous parameters. The transfer matrix $t_p(u)$ of the associated spin chain with the periodic boundary condition is given by [5]

$$ t_p(u) = \text{tr}_0 T_0(u). \quad (2.5) $$

The QYBE (2.3) of the $R$-matrix implies that one-row monodromy matrix $T(u)$ satisfies the Yang-Baxter relation

$$ R_{12}(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u - v). \quad (2.6) $$

From the above relation, one can prove that the transfer matrices with different spectral parameters commute with each other, $[t_p(u), t_p(v)] = 0$. Then $t_p(u)$ serves as the generating functional of the conserved quantities, which ensures the integrability of the $sp(4)$-invariant spin chain with the periodic boundary condition, which is described by the Hamiltonian

$$ H_p = \frac{\partial \ln t_p(u)}{\partial u}|_{u=0,\{\theta_j\}=0} = \sum_{k=1}^{N} H_{kk+1}, \quad (2.7) $$

where $H_{kk+1} = \mathcal{P}_{kk+1}R'_{kk+1}(u)|_{u=0}$. The periodic boundary condition implies $H_{NN+1} = H_{N1}$.

### 2.2 Fusion

The $R$-matrix (2.1) can degenerate into the projector operators at some special points, which make it possible for us to do the fusion procedure [23, 24, 25, 26, 27, 28]. For example, if $u = -3$, we have

$$ R_{12}(-3) = P_{12}^{(1)} \times S_1. \quad (2.8) $$
Here \( P_{12}^{(1)} \) is an one-dimensional projector operator with the form
\[
P_{12}^{(1)} = |\psi_0\rangle\langle\psi_0|,  \tag{2.9}
\]
where \( |\psi_0\rangle = \frac{1}{2}(|14\rangle + |23\rangle - |32\rangle - |41\rangle) \) is a vector in the space \( V \otimes V \) and \( S_1 \) is a constant matrix which we do not present here for simplicity. If \( u = -1 \), then
\[
R_{12}(-1) = P_{12}^{(5)} \times S_2. \tag{2.10}
\]
Here \( P_{12}^{(5)} \) is a five-dimensional projector operator with the form
\[
P_{12}^{(5)} = \sum_{i=1}^{5} |\psi_i^{(5)}\rangle\langle\psi_i^{(5)}|, \tag{2.11}
\]
where the corresponding vectors are
\[
|\psi_1^{(5)}\rangle = \frac{1}{\sqrt{2}}(|12\rangle - |21\rangle), \quad |\psi_2^{(5)}\rangle = \frac{1}{\sqrt{2}}(|13\rangle - |31\rangle),
\]
\[
|\psi_3^{(5)}\rangle = \frac{1}{2}(|14\rangle - |41\rangle + |32\rangle - |23\rangle), \quad |\psi_4^{(5)}\rangle = \frac{1}{\sqrt{2}}(|24\rangle - |42\rangle),
\]
\[
|\psi_5^{(5)}\rangle = \frac{1}{\sqrt{2}}(|34\rangle - |43\rangle),
\]
and \( S_2 \) is a constant matrix.

From the QYBE \((2.3)\), the one-dimensional fusion associated with the projector \((2.9)\) leads to
\[
P_{12}^{(1)} R_{23}(u) R_{13}(u - 3) P_{12}^{(1)} = a(u) e(u - 3) P_{12}^{(1)},
\]
\[
P_{21}^{(1)} R_{32}(u) R_{31}(u - 3) P_{21}^{(1)} = a(u) e(u - 3) P_{21}^{(1)}. \tag{2.12}
\]
From the five-dimensional fusion associated with the projector \((2.11)\), we obtain a new fused \( \bar{R} \)-matrix
\[
\bar{R}_{12}^{(5)}(u) = \left( u + \frac{3}{2} \right) \bar{p}_0(u + \frac{1}{2})^{-1} P_{12}^{(5)} R_{23}(u + \frac{1}{2}) R_{13}(u - \frac{1}{2}) P_{12}^{(5)},
\]
\[
\bar{R}_{31}^{(5)}(u) = \left( u + \frac{3}{2} \right) \bar{p}_0(u + \frac{1}{2})^{-1} P_{21}^{(5)} R_{32}(u + \frac{1}{2}) R_{31}(u - \frac{1}{2}) P_{21}^{(5)}, \tag{2.13}
\]
where \( \bar{p}_0(u) = (u - 1)(u + 3) \). For simplicity, let us denote the resulting five-dimensional fusion space by \( \bar{V}_1 \) which is spanned by \( \{|\psi_i^{(5)}\rangle| i = 1, \cdots, 5\} \). It is easy to check that the
matrix elements of the fused $R$-matrix $\bar{R}_{13}(u) \equiv \bar{R}_{(12)3}(u)$ [or $\bar{R}_{31}(u) \equiv \bar{R}_{3(12)}(u)$], as function of $u$, are polynomials of $u$ with degree one. Moreover, we have

\[
\bar{R}_{12}(u)\bar{R}_{21}(-u) = -(u + \frac{5}{2})(u - \frac{5}{2}),
\]

\[
\bar{R}_{12}(u)^{t_1}\bar{R}_{21}((-u - 6)^{t_1}) = -(u + \frac{1}{2})(u + \frac{11}{2}),
\]

\[
\bar{R}_{12}(u - v)\bar{R}_{13}(u)\bar{R}_{23}(v) = R_{23}(v)\bar{R}_{13}(u)\bar{R}_{12}(u - v). \tag{2.14}
\]

At the point of $u = -\frac{5}{2}$, the fused $\bar{R}$-matrix degenerates into the four-dimensional projector

\[
\bar{R}_{12}(-\frac{5}{2}) = P^{(4)}_{12} \times \bar{S}, \tag{2.15}
\]

where $\bar{S}$ is a constant matrix and the four-dimensional projector $P^{(4)}_{12}$ takes the form of

\[
P^{(4)}_{12} = \sum_{i=1}^{4} \langle \psi^{(4)}_i | \psi^{(4)}_i \rangle,
\]

and the corresponding vectors are

\[
|\psi^{(4)}_1\rangle = \frac{1}{\sqrt{5}}(\sqrt{2}|\psi^{(5)}_1\rangle \otimes |3\rangle - \sqrt{2}|\psi^{(5)}_2\rangle \otimes |2\rangle - |\psi^{(5)}_3\rangle \otimes |1\rangle),
\]

\[
|\psi^{(4)}_2\rangle = \frac{1}{\sqrt{5}}(-\sqrt{2}|\psi^{(5)}_1\rangle \otimes |4\rangle - \sqrt{2}|\psi^{(5)}_2\rangle \otimes |1\rangle + |\psi^{(5)}_3\rangle \otimes |2\rangle),
\]

\[
|\psi^{(4)}_3\rangle = \frac{1}{\sqrt{5}}(-\sqrt{2}|\psi^{(5)}_2\rangle \otimes |4\rangle - \sqrt{2}|\psi^{(5)}_5\rangle \otimes |1\rangle + |\psi^{(5)}_3\rangle \otimes |3\rangle),
\]

\[
|\psi^{(4)}_4\rangle = \frac{1}{\sqrt{5}}(-\sqrt{2}|\psi^{(5)}_4\rangle \otimes |3\rangle + \sqrt{2}|\psi^{(5)}_5\rangle \otimes |2\rangle - |\psi^{(5)}_3\rangle \otimes |4\rangle).
\]

According the property (2.15), we can do the fusion by the four-dimensional projector $P^{(4)}_{12}$ again, which gives

\[
R_{(12)3}(u) = (u + 5)^{-1}P^{(4)}_{12} R_{23}(u + 2)\bar{R}_{13}(u) - \frac{1}{2})P^{(4)}_{12},
\]

\[
R_{3(12)}(u) = (u + 5)^{-1}P^{(4)}_{21} R_{32}(u + 2)\bar{R}_{31}(u) - \frac{1}{2})P^{(4)}_{21}. \tag{2.17}
\]

After taking the correspondence

\[
|\psi^{(4)}_i\rangle \longrightarrow |i\rangle, \quad i = 1, \cdots, 4, \tag{2.18}
\]

we have the very identifications

\[
R_{(12)3}(u) = R_{13}(u), \quad R_{3(12)}(u) = R_{31}(u), \tag{2.19}
\]

where the $R$-matrices $R_{13}(u)$ and $R_{31}(u)$ are given by (2.1).
2.3 $T - Q$ relations

From the fused $\tilde{R}$-matrix, we can define the fused monodromy matrix

$$\tilde{T}_0(u) = \tilde{R}_{01}(u - \theta_1)\tilde{R}_{02}(u - \theta_2) \cdots \tilde{R}_{0N}(u - \theta_N),$$  \hfill (2.20)

which is a $5 \times 5$ matrix with operator-valued elements acting on $V^{\otimes N}$. The fused $\tilde{R}$-matrix and the fused monodromy matrix $\tilde{T}(u)$ satisfy the Yang-Baxter relation

$$\tilde{R}_{12}(u - v)\tilde{T}_1(u)T_2(v) = T_2(v)\tilde{T}_1(u)\tilde{R}_{12}(u - v).$$  \hfill (2.21)

The fused transfer matrix is given by

$$\tilde{t}_p(u) = tr_0\tilde{T}_0(u).$$  \hfill (2.22)

Using fusion relations (2.12), (2.13) and (2.17), we have

$$P_{21}^{(1)}T_1(u)T_2(u - 3)P_{21}^{(1)} = \prod_{i=1}^{N} a(u - \theta_i)e(u - \theta_i - 3)P_{21}^{(1)},$$  \hfill (2.23)

$$P_{21}^{(5)}T_1(u)T_2(u - 1)P_{21}^{(5)} = \prod_{i=1}^{N} (u - \theta_i + 1)\tilde{\rho}_0(u - \theta_i)\tilde{T}_{12}(u - \frac{1}{2}),$$  \hfill (2.24)

$$P_{12}^{(4)}T_2(u)\tilde{T}_1(u - \frac{5}{2})P_{12}^{(4)} = \prod_{i=1}^{N} (u - \theta_i + 3)T_{12}(u - 2).$$  \hfill (2.25)

Following the method developed in [20] and using the identification (2.19), we can show that the identities hold

$$T_1(\theta_j)T_2(\theta_j - 3) = P_{21}^{(1)}T_1(\theta_j)T_2(\theta_j - 3),$$  \hfill (2.26)

$$T_1(\theta_j)T_2(\theta_j - 1) = P_{21}^{(5)}T_1(\theta_j)T_2(\theta_j - 1),$$  \hfill (2.27)

$$T_2(\theta_j)\tilde{T}_1(\theta_j - \frac{5}{2}) = P_{12}^{(4)}T_2(\theta_j)\tilde{T}_1(\theta_j - \frac{5}{2}).$$  \hfill (2.28)

Considering the relations (2.19) and (2.23)-(2.28), we obtain the operator production identities among the fused transfer matrices as

$$t_p(\theta_j)t_p(\theta_j - 3) = \prod_{i=1}^{N} a(\theta_j - \theta_i)e(\theta_j - \theta_i - 3),$$  \hfill (2.29)

$$t_p(\theta_j)t_p(\theta_j - 1) = \prod_{i=1}^{N} (\theta_j - \theta_i + 1)\tilde{\rho}_0(\theta_j - \theta_i)\tilde{t}_p(\theta_j - \frac{1}{2}),$$  \hfill (2.30)

$$t_p(\theta_j)\tilde{t}_p(\theta_j - \frac{5}{2}) = \prod_{i=1}^{N} (\theta_j - \theta_i + 3)t_p(\theta_j - 2).$$  \hfill (2.31)
The commutativity of the transfer matrices $t_p(u)$ and $\bar{t}_p(u)$ with different spectral parameters implies that they have common eigenstates (namely, the common eigenstates do not depend on the spectrum parameter $u$). Let us denote the eigenvalues of the transfer matrices $t_p(u)$ and $\bar{t}_p(u)$ as $\Lambda_p(u)$ and $\bar{\Lambda}_p(u)$, respectively. From the operator production identities (2.29)-(2.31), we have

$$\Lambda_p(\theta_j)\Lambda_p(\theta_j - 3) = \prod_{i=1}^{N} a(\theta_j - \theta_i)e(\theta_j - \theta_i - 3), \quad (2.32)$$

$$\Lambda_p(\theta_j)\Lambda_p(\theta_j - 1) = \prod_{i=1}^{N} (\theta_j - \theta_i + 1)\tilde{\rho}_0(\theta_j - \theta_i)\bar{\Lambda}_p(\theta_j - \frac{1}{2}), \quad (2.33)$$

$$\Lambda_p(\theta_j)\bar{\Lambda}_p(\theta_j - \frac{5}{2}) = \prod_{i=1}^{N} (\theta_j - \theta_i + 3)\Lambda_p(\theta_j - 2). \quad (2.34)$$

The eigenvalue $\Lambda_p(u)$ of the transfer matrix $t_p(u)$ is a polynomial of $u$ with the degree $2N$, which can be completely determined by $2N + 1$ conditions. Besides the functional relations (2.32)-(2.34), we still need one condition which can be obtained by analyzing the asymptotic behavior of $t_p(u)$. Form the definition, the asymptotic behavior of $t_p(u)$ can be calculated as

$$t_p(u)|_{u\to\infty} = 4u^{2N} \times \text{id} + \cdots,$$

which leads to that

$$\Lambda_p(u)|_{u\to\infty} = 4u^{2N} + \cdots. \quad (2.35)$$

The eigenvalue $\bar{\Lambda}_p(u)$ of the fused transfer matrix $\bar{t}_p(u)$ is a polynomial of $u$ with the degree $N$, which can be completely determined by the functional relations (2.32)-(2.34) and the asymptotic behavior of $\bar{t}_p(u)$ given by

$$\bar{t}_p(u)|_{u\to\infty} = 5u^{N} \times \text{id} + \cdots,$$

giving rise to that

$$\bar{\Lambda}_p(u)|_{u\to\infty} = 5u^{N} + \cdots. \quad (2.36)$$

Then the $3N + 2$ relations (2.32)-(2.36) completely determine the eigenvalues $\Lambda_p(u)$ and $\bar{\Lambda}_p(u)$, which are given in terms of the homogeneous $T-Q$ relations:

$$\Lambda_p(u) = Z_1^{(p)}(u) + Z_2^{(p)}(u) + Z_3^{(p)}(u) + Z_4^{(p)}(u), \quad (2.37)$$
\[ \tilde{\Lambda}_p(u) = \prod_{i=1}^{N} [(u - \theta_i + \frac{3}{2})\tilde{p}_0(u - \theta_i + \frac{1}{2})]^{-1} \left\{ Z_1^{(p)}(u + \frac{1}{2})[Z_2^{(p)}(u - \frac{1}{2}) + Z_3^{(p)}(u - \frac{1}{2}) + Z_4^{(p)}(u - \frac{1}{2})] + [Z_2^{(p)}(u + \frac{1}{2}) + Z_3^{(p)}(u + \frac{1}{2})]Z_4^{(p)}(u - \frac{1}{2}) \right\}, \quad (2.38) \]

where

\[ Z_1^{(p)}(u) = \prod_{j=1}^{N} a(u - \theta_j) \frac{Q_1^{(1)}(u - 1)}{Q_1^{(1)}(u)}, \]
\[ Z_2^{(p)}(u) = \prod_{j=1}^{N} b(u - \theta_j) \frac{Q_1^{(1)}(u + 1)Q_2^{(2)}(u - \frac{3}{2})}{Q_2^{(1)}(u)Q_2^{(2)}(u + \frac{1}{2})}, \]
\[ Z_3^{(p)}(u) = \prod_{j=1}^{N} b(u - \theta_j) \frac{Q_1^{(1)}(u + 1)Q_2^{(2)}(u + \frac{5}{2})}{Q_2^{(1)}(u + 2)Q_2^{(2)}(u + \frac{1}{2})}, \]
\[ Z_4^{(p)}(u) = \prod_{j=1}^{N} c(u - \theta_j) \frac{Q_1^{(1)}(u + 3)}{Q_1^{(1)}(u + 2)}, \]
\[ Q_p^{(m)}(u) = \prod_{k=1}^{L_m} (u - \mu_k^{(m)} + \frac{m}{2}), \quad m = 1, 2. \quad (2.39) \]

Because the eigenvalues \( \Lambda_p(u) \) and \( \tilde{\Lambda}_p(u) \) are the polynomials of \( u \), the residues of right hand sides of equations \( (2.37) - (2.38) \) should be zero, which gives rise to the constraints of the Bethe roots \( \{\mu_k^{(m)}\} \), namely, these parameters should satisfy the Bethe ansatz equations

\[ \frac{Q_1^{(1)}(\mu_k^{(1)} + \frac{1}{2})Q_2^{(2)}(\mu_k^{(1)} - 2)}{Q_1^{(1)}(\mu_k^{(1)} - \frac{3}{2})Q_2^{(2)}(\mu_k^{(1)})} = - \prod_{j=1}^{N} \frac{\mu_k^{(1)} + \frac{1}{2} - \theta_j}{\mu_k^{(1)} - \frac{1}{2} - \theta_j}, \quad k = 1, \ldots, L_1, \quad (2.40) \]
\[ \frac{Q_2^{(2)}(\mu_l^{(2)} + 1)Q_1^{(1)}(\mu_l^{(2)} - \frac{3}{2})}{Q_2^{(2)}(\mu_l^{(2)} - 3)Q_1^{(1)}(\mu_l^{(2)})} = -1, \quad l = 1, \ldots, L_2. \quad (2.41) \]

We note that the Bethe ansatz equations obtained from the regularity of \( \Lambda_p(u) \) are the same as those obtained from the regularity of \( \tilde{\Lambda}_p(u) \). It is easy to check that \( \Lambda_p(u) \) and \( \tilde{\Lambda}_p(u) \) satisfy the functional relations \( (2.32) - (2.34) \) and the asymptotic behaviors \( (2.35) \) and \( (2.36) \). Therefore, we conclude that \( \Lambda_p(u) \) and \( \tilde{\Lambda}_p(u) \) are the eigenvalues of the transfer matrices \( t_p(u) \) and \( \tilde{t}_p(u) \), respectively. It is remarked that the \( T - Q \) relation \( (2.37) \) and the associated Bethe ansatz equations \( (2.40) - (2.41) \) coincide with those \( [21, 22] \) obtained previously by the conventional Bethe ansatz methods.
The eigenvalues of the Hamiltonian (2.7) then can be expressed in terms of the Bethe roots as

\[ E_p = \left. \frac{\partial \ln \Lambda_p(u)}{\partial u} \right|_{u=0,\{\theta_j\}=0}. \]  

(2.42)

3 Open chain with the integrable boundary terms

3.1 Open chain

Integrable open chain can be constructed as follows [3, 4]. Let us introduce a pair of \( K^- \) matrices \( K^- (u) \) and \( K^+ (u) \). The former satisfies the reflection equation (RE)

\[ R_{12}(u-v)K^-_1 (u)R_{21}(u+v)K^-_2 (v) = K^-_2 (v)R_{12}(u+v)K^-_1 (u)R_{21}(u-v), \]  

(3.1)

and the latter satisfies the dual RE

\[ R_{12}(-u-v)K^+_1 (u)R_{21}(-u-v-6)K^+_2 (v) \]
\[ = K^+_2 (v)R_{12}(-u-v-6)K^+_1 (u)R_{21}(-u+v). \]  

(3.2)

For open spin-chains, instead of the standard “row-to-row” monodromy matrix \( T(u) \) [2.4], one needs to consider the “double-row” monodromy matrix as follows. Besides the monodromy matrix \( T_0 (u) \) given by (2.3), we also need the reflecting monodromy matrix

\[ \tilde{T}_0 (u) = R_{N0}(u + \theta_N) \cdots R_{20}(u + \theta_2)R_{10}(u + \theta_1), \]  

(3.3)

which satisfies the Yang-Baxter relation

\[ R_{12}(u-v)\tilde{T}_1 (u)\tilde{T}_2 (v) = \tilde{T}_2 (v)\tilde{T}_1 (u)R_{12}(u-v). \]  

(3.4)

The transfer matrix \( t(u) \) is defined as [4]

\[ t(u) = tr_0\{K^+_0 (u)T_0 (u)K^-_0 (u)\tilde{T}_0 (u)\}. \]  

(3.5)

From the Yang-Baxter relation, reflection equation and dual reflection equation, one can prove that the transfer matrices with different spectral parameters commute with each other, \([t(u), t(v)] = 0\). Therefore, \( t(u) \) serves as the generating function of all the conserved quantities of the system. The associated quantum spin chain with integrable boundary interactions
is given by the Hamiltonian

\[ H = \frac{1}{2} \frac{\partial \ln t(u)}{\partial u} \bigg|_{u=0,\{\theta_i\}=0} \]

\[ = \sum_{k=1}^{N-1} H_{kk+1} + \frac{K^{-}_1(0)}{2\zeta} + \frac{tr_0 \{ K^+_0(0) H_{N0} \}}{tr_0 K^+_0(0)}. \]  

(3.6)

In this paper, we consider the open chain with the off-diagonal \( K \)-matrix \( K^{-}(u) \) \[29, 30, 31\]

\[ K^{-}(u) = \zeta + M u, \quad M = \begin{pmatrix}
-1 & 0 & c_1 & 0 \\
0 & -1 & 0 & c_1 \\
c_2 & 0 & 1 & 0 \\
0 & c_2 & 0 & 1
\end{pmatrix}, \]  

(3.7)

while the dual reflection matrix \( K^{+}(u) \) is

\[ K^{+}(u) = K^{-}(-u-3)|_{\zeta, c_1 \rightarrow \tilde{\zeta}, \tilde{c}_1}. \]  

(3.8)

Here \( \zeta, c_1, c_2 \) and \( \tilde{\zeta}, \tilde{c}_1, \tilde{c}_2 \) are the boundary parameters which describe the boundary interactions. For the generic values of these parameters, one is easy to check that \([K^{-}(u), K^{+}(v)] \neq 0\), which implies that the \( K^{\pm}(u) \) matrices cannot be diagonalized simultaneously. This gives rise to that the conventional Bethe ansatz methods \[3, 7\] would fail to get the spectrum of the transfer matrix \( t(u) \) specified by the \( K \)-matrices given by (3.7) and (3.8), because of lacking the reference state. We will generalize the method developed in section 2 to get eigenvalues of the transfer matrix \( t(u) \) specified by the \( K \)-matrices (3.7) and (3.8) in the following subsections.

### 3.2 Fusion of the reflection matrices

In order to obtain the closed operator production identities, we should do the fusion for the reflection matrices. The one-dimensional fusion for the reflection matrices gives

\[ P^{(1)}_{21} K^{-}_1(u) R_{21}(2u-3) K^{-}_2(u-3) P^{(1)}_{12} = (u-1)(u-3) h(u) P^{(1)}_{12}, \]  

(3.9)

\[ P^{(1)}_{12} K^{+}_2(u-3) R_{12}(-2u-3) K^{+}_1(u) P^{(1)}_{21} = (u+1)(u+3) \tilde{h}(u) P^{(1)}_{21}, \]  

(3.10)

where

\[ h(u) = 4[(1 + c_1 c_2)u^2 - \zeta^2], \]

\[ \tilde{h}(u) = 4[(1 + \tilde{c}_1 \tilde{c}_2)u^2 - \tilde{\zeta}^2]. \]
From the five-dimensional fusion, we obtain a new fused reflection matrices $\tilde{K}$ as

$$
\tilde{K}^-_{(12)}(u) = [(2u - 1)(2u + 3)]^{-1} P^{(5)}_{21} K^-_1(u + \frac{1}{2}) R_{21}(2u) K^-_2(u - \frac{1}{2}) P^{(5)}_{12},
$$

$$
\tilde{K}^+_{(12)}(u) = [(2u + 3)(2u + 7)]^{-1} P^{(5)}_{12} K^+_2(u - \frac{1}{2}) R_{12}(-2u - 6) K^+_1(u + \frac{1}{2}) P^{(5)}_{21},
$$

where the projector $P^{(5)}_{12}$ is given by (2.11). Due to the dimension of the fused space $\tilde{V}$ is 5, the corresponding $\tilde{K}^{\pm}_{(12)}(u)$ both are the $5 \times 5$ matrices. Moreover, we have checked that the matrix elements of $\tilde{K}^{\pm}_{(12)}(u)$, as function of $u$, are all polynomials of $u$ with degree two.

The fused $\tilde{R}$-matrix and the fused reflection matrix $\tilde{K}^{\pm}(u)$ satisfy the reflection equations

$$
\tilde{R}_{12}(u - v)\tilde{K}^{-}_{1}(u)\tilde{R}_{21}(u + v)\tilde{K}^{-}_{2}(v) = K^{-}_{2}(v)\tilde{R}_{12}(u + v)\tilde{K}^{-}_{1}(u)\tilde{R}_{21}(u - v),
$$

$$
\tilde{R}_{12}(-u + v)\tilde{K}^{+}_{1}(u)\tilde{R}_{21}(-u - v)\tilde{K}^{+}_{2}(v) = K^{+}_{2}(v)\tilde{R}_{12}(-u - v)\tilde{K}^{+}_{1}(u)\tilde{R}_{21}(-u + v). \tag{3.14}
$$

Next, we do the fusion between the reflection matrices $K^{\pm}(u)$ and $\tilde{K}^{\pm}(u)$ by the four-dimensional projector $P^{(4)}_{12}$ given by (2.10), which gives

$$
K^{-}_{(12)}(u) = 4[(2u - 1)h(u + 2)]^{-1} P^{(4)}_{12} K^{-}_{2}(u + 2) R_{12}(2u + \frac{3}{2}) K^{-}_{1}(u - \frac{1}{2}) P^{(4)}_{21},
$$

$$
K^{+}_{(12)}(u) = -2[(u + 5)\tilde{h}(u + 2)]^{-1} P^{(4)}_{21} K^{+}_{1}(u - \frac{1}{2}) R_{21}(-2u - \frac{15}{2}) K^{+}_{2}(u + 2) P^{(4)}_{12}.
$$

It is easy to check that the matrix elements of $K^{\pm}_{(12)}(u)$ are $4 \times 4$ matrices, whose matrix elements, as function of $u$, are all polynomials of $u$ with degree one. Moreover, keeping the correspondence (2.18) in mind, we have the identifications

$$
K^{\pm}_{(12)}(u) \equiv K^{\pm}(u), \tag{3.15}
$$

where the $K$-matrices $K^{\pm}(u)$ are given by (3.7) and (3.8).

### 3.3 Operator production identities

Again, besides the fused monodromy matrix $\hat{T}_0(u)$ given by (2.20), we also need the reflecting fused monodromy matrix $\hat{T}(u)$ given by

$$
\hat{T}_0(u) = R_{N0}(u + \theta_N) \cdots R_{20}(u + \theta_2) R_{10}(u + \theta_1), \tag{3.16}
$$
where the dimension of auxiliary space $\tilde{V}$ is 5 and the quantum space keeps unchanged. The matrix $\hat{T}_0$ satisfies the Yang-Baxter relation
\begin{equation}
\hat{R}_{12}(u-v)\hat{T}_1(u)\hat{T}_2(v) = \hat{T}_2(v)\hat{T}_1(u)\hat{R}_{12}(u-v).
\end{equation}
(3.17)
The fused transfer matrix $\bar{t}(u)$ is
\begin{equation}
\bar{t}(u) = tr_0\{\tilde{K}_0^+(u)\tilde{K}_0^-(u)\tilde{T}_0(u)\}.
\end{equation}
(3.18)

Some remarks are in order. From the definitions (3.5) and (3.18), the transfer matrix $t(u)$ (resp. $\bar{t}(u)$), as a function of $u$, is a polynomial with degree of $4N + 2$ (resp. a polynomial with degree of $2N + 4$). Hence in order to determine the eigenvalues of the transfer matrices $t(u)$ and $\bar{t}(u)$, one needs at least $6N + 8$ conditions. Using the similar method developed in previous section, we will look for the $6N + 8$ conditions.

Using fusion relations (2.12), (2.13) and (2.17), we have
\begin{align}
P_{12}^{(1)}\hat{R}_{12}(u)\hat{T}_1(u)\hat{T}_2(u-3)P_{12}^{(1)} &= \prod_{i=1}^{N} a(u + \theta_i)e(u + \theta_i - 3)P_{12}^{(1)},
(3.19)\\
P_{12}^{(5)}\hat{R}_{12}(u)\hat{T}_1(u)\hat{T}_2(u-1)P_{12}^{(5)} &= \prod_{i=1}^{N} (u + \theta_i + 1)\tilde{\rho}_0(u + \theta_i)\hat{T}_{(12)}(u - \frac{1}{2}),
(3.20)\\
P_{21}^{(4)}\hat{T}_2(u)\hat{R}_{12}(u)\hat{T}_1(u - \frac{5}{2})P_{21}^{(4)} &= \prod_{i=1}^{N} (u + \theta_i + 3)\hat{T}_{(12)}(u - 2).
(3.21)
\end{align}

We can show that Yang-Baxter relations (3.4) and (3.17) at certain points also give
\begin{align}
\hat{T}_1(-\theta_j)\hat{T}_2(-\theta_j - \frac{3}{2}) &= P_{12}^{(1)}\hat{T}_1(-\theta_j)\hat{T}_2(-\theta_j - \frac{3}{2}),
(3.22)\\
\hat{T}_1(-\theta_j)\hat{T}_2(-\theta_j - 1) &= P_{12}^{(5)}\hat{T}_1(-\theta_j)\hat{T}_2(-\theta_j - 1),
(3.23)\\
\hat{T}_2(-\theta_j)\hat{T}_1(-\theta_j - \frac{5}{2}) &= P_{21}^{(4)}\hat{T}_2(-\theta_j)\hat{T}_1(-\theta_j - \frac{5}{2}).
(3.24)
\end{align}

Keeping the identification (2.19) in mind and using the relations (2.23)-(2.28), (3.15) and (3.19)-(3.24), we obtain
\begin{equation}
t(\pm \theta_j) = \frac{1}{2^4} \frac{(\pm \theta_j - 1)(\pm \theta_j - 3)(\pm \theta_j + 1)(\pm \theta_j + 3)}{(\pm \theta_j - \frac{3}{2})(\pm \theta_j - \frac{1}{2})(\pm \theta_j + \frac{3}{2})(\pm \theta_j + 1)}
\end{equation}

\begin{equation}
\times h(\pm \theta_j)\tilde{h}(\pm \theta_j)\prod_{i=1}^{N} \tilde{a}(\pm \theta_j - \theta_i)a(\pm \theta_j + \theta_i)e(\pm \theta_j - \theta_i - 3)e(\pm \theta_j + \theta_i - 3),
(3.25)
\end{equation}
\begin{align*}
t(\pm \theta_j) t(\pm \theta_j - 1) &= \frac{(\pm \theta_j - 1)(\pm \theta_j + 1)(\pm \theta_j + 1)(\pm \theta_j + 3)}{(\pm \theta_j - \frac{1}{2})(\pm \theta_j + \frac{1}{2})(\pm \theta_j + \frac{3}{2})} \\
\times \prod_{i=1}^{N} (\pm \theta_j - \theta_i + 1)(\pm \theta_j + \theta_i + 1) \tilde{\rho}_0(\pm \theta_j - \theta_i) \tilde{\rho}_0(\pm \theta_j + \theta_i) \tilde{t}(\pm \theta_j - \frac{1}{2}), \quad (3.26)
\end{align*}

\begin{align*}
t(\pm \theta_j) \tilde{t}(\pm \theta_j - \frac{5}{2}) &= \frac{1}{2} \frac{(\pm \theta_j - \frac{5}{2})(\pm \theta_j + 3)}{(\pm \theta_j - 1)(\pm \theta_j + \frac{5}{2})} \tilde{h}(\pm \theta_j) \tilde{\tilde{h}}(\pm \theta_j) \\
\times \prod_{i=1}^{N} (\pm \theta_j - \theta_i + 3)(\pm \theta_j + \theta_i + 3) (\pm \theta_j - 2). \quad (3.27)
\end{align*}

Form the definition, the asymptotic behavior of \( t(u) \) can be calculated as

\begin{equation}
t(u)|_{u \to \infty} = -tr\tilde{M}M \times u^{4N+2} \times \text{id} + \cdots, \quad (3.28)
\end{equation}

where \( \tilde{M} = M|_{\zeta,c_1,c_2 \to \tilde{\zeta},\tilde{c}_1,\tilde{c}_2} \). Direct calculation gives

\begin{equation}
tr\tilde{M}M = 4 + 2c_1\tilde{c}_2 + 2c_2\tilde{c}_1. \quad (3.29)
\end{equation}

Besides, we also know the values of \( t(u) \) at the points of 0 and -3,

\begin{align*}
t(0) &= tr\{K^+(0)\} \zeta \prod_{i=1}^{N} \rho_1(-\theta_i) \times \text{id}, \quad (3.30) \\
t(-3) &= tr\{K^-(3)\} \tilde{\zeta} \prod_{i=1}^{N} \rho_1(-\theta_i) \times \text{id}. \quad (3.31)
\end{align*}

The asymptotic behavior of \( \tilde{t}(u) \) reads

\begin{equation}
\tilde{t}(u)|_{u \to \infty} = tr_{12} P_{12}^{(5)}(\tilde{M}M)_{1}(\tilde{M}M)_{2}P_{12}^{(5)} \times u^{2N+4} \times \text{id} + \cdots. \quad (3.32)
\end{equation}

Direct calculation shows

\begin{equation}
tr_{12} P_{12}^{(5)}(\tilde{M}M)_{1}(\tilde{M}M)_{2}P_{12}^{(5)} = (2 + c_1\tilde{c}_2 + c_2\tilde{c}_1)^2 + (1 + c_1c_2)(1 + \tilde{c}_1\tilde{c}_2). \quad (3.33)
\end{equation}

Using the method developed in [20], we can evaluate the values of \( \tilde{t}(u) \) at some special points as follows:

\begin{align*}
\tilde{t}(0) &= \frac{5}{24} (1 + c_1c_2 - 4\zeta^2)(1 + \tilde{c}_1\tilde{c}_2 - 4\tilde{\zeta}^2) \prod_{i=1}^{N} (\frac{5}{2} - \theta_i)(\frac{5}{2} + \theta_i) \times \text{id}, \quad (3.33) \\
\tilde{t}(-3) &= \frac{5}{24} (1 + c_1c_2 - 4\zeta^2)(1 + \tilde{c}_1\tilde{c}_2 - 4\tilde{\zeta}^2) \prod_{i=1}^{N} (\frac{5}{2} - \theta_i)(\frac{5}{2} + \theta_i) \times \text{id}, \quad (3.34)
\end{align*}
\[ \bar{t}(\frac{-1}{2}) = \frac{5}{4} \prod_{i=1}^{N} (1 - \theta_i) (1 + \theta_i) t(-1), \quad (3.35) \]
\[ \bar{t}(\frac{5}{2}) = \frac{5}{4} \prod_{i=1}^{N} (1 - \theta_i) (1 + \theta_i) t(-2). \quad (3.36) \]

### 3.4 Inhomogeneous $T - Q$ relations

So far we have obtained the $6N + 8$ conditions (3.25)-(3.36), which allow us to determine the eigenvalues of the transfer matrices $t(u)$ and $\bar{t}(u)$. Due to the $U(1)$-symmetry-broken, the eigenvalues are given in terms of inhomogeneous $T - Q$ relations as follows.

It is easy to show that the transfer matrix $t(u)$ and its fused one $\bar{t}(u)$ satisfy the commutation relations

\[ [t(u), t(v)] = [\bar{t}(u), \bar{t}(v)] = [t(u), \bar{t}(v)] = 0. \]

The above commutativity of the fused transfer matrices implies that they have common eigenstates. Let $|\Psi\rangle$ be a common eigenstate of the transfer matrices with the eigenvalues $\Lambda(u)$ and $\bar{\Lambda}(u)$

\[ t(u)|\Psi\rangle = \Lambda(u)|\Psi\rangle, \quad \bar{t}(u)|\Psi\rangle = \bar{\Lambda}(u)|\Psi\rangle. \]

From the operator production identities (3.25)-(3.27), we obtain the following closed functional relations

\[ \Lambda(\pm \theta_j) \Lambda(\pm \theta_j - 3) = \frac{1}{2^4} \frac{(\pm \theta_j - 1)(\pm \theta_j - 3)(\pm \theta_j + 1)(\pm \theta_j + 3)}{(\pm \theta_j - \frac{3}{2})(\pm \theta_j - \frac{1}{2})(\pm \theta_j + \frac{3}{2})(\pm \theta_j + \frac{1}{2})} \times h(\pm \theta_j) \tilde{h}(\pm \theta_j) \prod_{i=1}^{N} a(\pm \theta_j - \theta_i) a(\pm \theta_j + \theta_i) e(\pm \theta_j - \theta_i - 3) e(\pm \theta_j + \theta_i - 3), \quad (3.37) \]

\[ \Lambda(\pm \theta_j) \Lambda(\pm \theta_j - 1) = \frac{(\pm \theta_j - 1)(\pm \theta_j + 1)(\pm \theta_j + 3)}{(\pm \theta_j - \frac{3}{2})(\pm \theta_j - \frac{1}{2})(\pm \theta_j + \frac{3}{2})(\pm \theta_j + \frac{1}{2})} \times \prod_{i=1}^{N} (\pm \theta_j - \theta_i + 1)(\pm \theta_j + \theta_i + 1) \tilde{\rho}_0(\pm \theta_j - \theta_i) \tilde{\rho}_0(\pm \theta_j + \theta_i) \Lambda(\pm \theta_j - \frac{1}{2}), \quad (3.38) \]

\[ \Lambda(\pm \theta_j) \bar{\Lambda}(\pm \theta_j - \frac{5}{2}) = \frac{1}{2^4} (\pm \theta_j - \frac{5}{2})(\pm \theta_j + 3) \tilde{h}(\pm \theta_j) \bar{\tilde{h}}(\pm \theta_j) \times \prod_{i=1}^{N} (\pm \theta_j - \theta_i + 3)(\pm \theta_j + \theta_i + 3) \Lambda(\pm \theta_j - 2), \quad (3.39) \]
we first define some functions:

\[ \Lambda(u)|_{u \to \infty} = -(4 + 2c_1\bar{c}_2 + 2c_2\bar{c}_1)u^{4N+2} + \cdots, \] (3.40)

\[ \Lambda(0) = 4\zeta\tilde{\zeta}\prod_{i=1}^{N} \rho_i(-\theta_i), \] (3.41)

\[ \Lambda(-3) = 4\zeta\tilde{\zeta}\prod_{i=1}^{N} \rho_i(-\theta_i), \] (3.42)

\[ \tilde{\Lambda}(u)|_{u \to \infty} = \{(2 + c_1\bar{c}_2 + c_2\bar{c}_1)^2 + (1 + c_1c_2)(1 + \bar{c}_1\bar{c}_2)\}u^{2N+4} + \cdots, \] (3.43)

\[ \tilde{\Lambda}(0) = \frac{5}{24}(1 + c_1c_2 - 4\zeta^2)(1 + \bar{c}_1\bar{c}_2 - 4\tilde{\zeta}^2)\prod_{i=1}^{N}(\frac{5}{2} - \theta_i)(\frac{5}{2} + \theta_i), \] (3.44)

\[ \tilde{\Lambda}(-3) = \frac{5}{24}(1 + c_1c_2 - 4\zeta^2)(1 + \bar{c}_1\bar{c}_2 - 4\tilde{\zeta}^2)\prod_{i=1}^{N}(\frac{5}{2} - \theta_i)(\frac{5}{2} + \theta_i), \] (3.45)

\[ \tilde{\Lambda}(\frac{1}{2}) = \frac{5}{4}\prod_{i=1}^{N}(1 - \theta_i)(1 + \theta_i)\Lambda(-1), \] (3.46)

\[ \tilde{\Lambda}(\frac{5}{2}) = \frac{5}{4}\prod_{i=1}^{N}(1 - \theta_i)(1 + \theta_i)\Lambda(-2). \] (3.47)

The 6N + 8 relations (3.37)-(3.47) enable us completely to determine the eigenvalues \( \Lambda(u) \) and \( \tilde{\Lambda}(u) \) which are given in terms of some inhomogeneous \( T - Q \) relations. For simplicity, we first define some functions:

\[ Z_1(u) = \frac{1}{2^2} \frac{(u + 1)(u + 3)}{(u + \frac{3}{2})(u + \frac{1}{2})} \prod_{j=1}^{N} a(u - \theta_j)a(u + \theta_j) \frac{Q^{(1)}(u - 1)}{Q^{(1)}(u)}h_1(u)\tilde{h}_1(u), \]

\[ Z_2(u) = \frac{1}{2^2} \frac{u(u + 3)}{(u + \frac{3}{2})(u + \frac{1}{2})} \prod_{j=1}^{N} b(u - \theta_j)b(u + \theta_j) \frac{Q^{(1)}(u + 1)Q^{(2)}(u - \frac{3}{2})}{Q^{(1)}(u)Q^{(2)}(u + \frac{1}{2})}h_1(u)\tilde{h}_1(u), \]

\[ Z_3(u) = \frac{1}{2^2} \frac{u(u + 3)}{(u + \frac{3}{2})(u + \frac{1}{2})} \prod_{j=1}^{N} b(u - \theta_j)b(u + \theta_j) \times \frac{Q^{(1)}(u + 1)Q^{(2)}(u + \frac{3}{2})}{Q^{(1)}(u + 2)Q^{(2)}(u + \frac{1}{2})}h_2(u + 3)\tilde{h}_2(u + 3), \]

\[ Z_4(u) = \frac{1}{2^2} \frac{u(u + 2)}{(u + \frac{3}{2})(u + \frac{1}{2})} \prod_{j=1}^{N} e(u - \theta_j)e(u + \theta_j) \frac{Q^{(1)}(u + 3)}{Q^{(1)}(u + 2)}h_2(u + 3)\tilde{h}_2(u + 3), \]

\[ Q^{(m)}(u) = \prod_{k=1}^{L_m}(u - \lambda_k^{(m)} + \frac{m}{2})(u + \lambda_k^{(m)} + \frac{m}{2}), \quad m = 1, 2, \]
\[ f_1(u) = \frac{1}{2^2} \frac{u(u+1)(u+3)}{(u+\frac{3}{2})} \prod_{j=1}^{N} b(u - \theta_j)b(u + \theta_j)(u - \theta_j + 1)(u + \theta_j + 1) \]

\[ \times \frac{Q^{(2)}(u - \frac{1}{2})Q^{(2)}(u - \frac{3}{2})}{Q^{(1)}(u)} h_1(u) \bar{h}_1(u)x, \]

\[ f_2(u) = \frac{1}{2^2} \frac{u(u+2)(u+3)}{(u+\frac{3}{2})} \prod_{j=1}^{N} b(u - \theta_j)b(u + \theta_j)(u - \theta_j + 2)(u + \theta_j + 2) \]

\[ \times \frac{Q^{(2)}(u + \frac{3}{2})Q^{(2)}(u + \frac{5}{2})}{Q^{(1)}(u+2)} h_2(u+3) \bar{h}_2(u+3)x, \] (3.48)

where \( x \) is a constant which is related with the boundary parameters (see below (3.54)) and \( \{h_i(u), \bar{h}(u)|i = 1, 2\} \) are some functions given by

\[ h_1(u) = 2(\sqrt{1 + c_1c_2u + \zeta}), \quad h_2(u) = 2(\sqrt{1 + c_1c_2u - \zeta}), \]

\[ \bar{h}_1(u) = -2(\sqrt{1 + c_1c_2u - \bar{\zeta}}), \quad \bar{h}_2(u) = -2(\sqrt{1 + c_1c_2u + \bar{\zeta}}). \] (3.49)

Then the eigenvalues \( \Lambda(u) \) and \( \bar{\Lambda}(u) \) can be written as the form of inhomogeneous \( T - Q \) relation

\[ \Lambda(u) = Z_1(u) + Z_2(u) + Z_3(u) + Z_4(u) + f_1(u) + f_2(u), \] (3.50)

\[ \bar{\Lambda}(u) = \prod_{i=1}^{N} [(u - \theta_i + \frac{3}{2})(u + \theta_i + \frac{3}{2})\bar{\rho}_0(u - \theta_i + \frac{1}{2})\bar{\rho}_0(u + \theta_i + \frac{1}{2})]^{-1} \]

\[ \times \frac{1}{2^2} \rho_1(2u+3)(u - \frac{1}{2})^{-1}(u + \frac{3}{2})^{-2}(u + \frac{7}{2})^{-1} \]

\[ \times \{Z_1(u + \frac{1}{2})[Z_2(u - \frac{1}{2}) + Z_3(u - \frac{1}{2}) + Z_4(u - \frac{1}{2}) + f_2(u - \frac{1}{2})] \]

\[ + [Z_2(u + \frac{1}{2}) + Z_3(u + \frac{1}{2}) + f_1(u + \frac{1}{2})]Z_4(u - \frac{1}{2}) + Z_2(u + \frac{1}{2})f_2(u - \frac{1}{2}) \]

\[ + f_1(u + \frac{1}{2})Z_3(u - \frac{1}{2}) + f_1(u + \frac{1}{2})f_2(u - \frac{1}{2}) \}, \] (3.51)

where the non-negative integers \( L_1 \) and \( L_2 \) satisfy

\[ L_1 = 2L_2 + N + 1. \]

Because the eigenvalues \( \Lambda(u) \) and \( \bar{\Lambda}(u) \) are the polynomials, the residues of right hand sides of Eqs. (3.50) and (3.51) should be zero, which gives the constraints of Bethe roots

\[ \frac{(\lambda_k^{(1)} + \frac{1}{2})}{\lambda_k^{(1)} \prod_{j=1}^{N} (\lambda_k^{(1)} - \theta_j - \frac{1}{2})(\lambda_k^{(1)} + \theta_j - \frac{1}{2})} \frac{1}{Q^{(1)}(\lambda_k^{(1)} - \frac{3}{2})} \]

\[ \frac{Q^{(2)}(\lambda_k^{(1)} - \frac{3}{2})}{Q^{(2)}(\lambda_k^{(1)} - 2)} \]
\[
+ \frac{(\lambda_k^{(1)} - \frac{1}{2})}{\lambda_k^{(1)}} \frac{1}{\prod_{j=1}^{N}(\lambda_k^{(1)} - \theta_j + \frac{1}{2})(\lambda_k^{(1)} + \theta_j + \frac{1}{2})} Q^{(1)}(\lambda_k^{(1)} + \frac{1}{2}) \\
+ x(\lambda_k^{(1)} + \frac{1}{2}) Q^{(2)}(\lambda_k^{(1)} - 1) = 0, \quad k = 1, 2, \ldots, L, \quad (3.52)
\]
\[
\frac{1}{(\lambda_l^{(2)} - 1)} \frac{Q^{(2)}(\lambda_l^{(2)} - \frac{3}{2})}{Q^{(1)}(\lambda_l^{(2)} - \frac{3}{2})} h_1(\lambda_l^{(2)} - \frac{3}{2}) \bar{h}_1(\lambda_l^{(2)} - \frac{3}{2}) + \frac{1}{(\lambda_l^{(2)} + 1)} \frac{Q^{(2)}(\lambda_l^{(2)} + \frac{3}{2})}{Q^{(1)}(\lambda_l^{(2)} + \frac{3}{2})} h_2(\lambda_l^{(2)} + \frac{3}{2}) \bar{h}_2(\lambda_l^{(2)} + \frac{3}{2}) = 0, \quad l = 1, 2, \ldots, L. \quad (3.53)
\]

We note that the Bethe ansatz equations obtained from the regularity of \(\Lambda(u)\) are the same as those obtained from the regularity of \(\bar{\Lambda}(u)\). The function \(Q^{(m)}(u)\) has two zero points, namely, \(\lambda_k^{(m)} - \frac{m}{2}\) and \(-\lambda_k^{(m)} - \frac{m}{2}\). We have checked that the Bethe ansatz equations obtained from these two points also are the same. Considering the asymptotic behaviors of \(\Lambda(u)\) and \(\bar{\Lambda}(u)\), we fix the value of parameter \(x\) in (3.48)

\[
x = 2 + c_1 \bar{c}_2 + c_2 \bar{c}_1 \sqrt{(1 + c_1 c_2)(1 + \bar{c}_1 \bar{c}_2)} - 2. \quad (3.54)
\]

If \(c_1 = c_2 = \bar{c}_1 = \bar{c}_2 = 0\), then the boundary reflection matrices degenerate into the diagonal ones (e.g. the parameter \(x\) in (3.48) and (3.54) also vanishes) and our results agree with that obtained by the algebraic Bethe method [32]. Finally, the eigenvalue \(E\) of Hamiltonian (3.6) can be expressed in terms of the Bethe roots as

\[
E = \frac{1}{2} \frac{\partial \ln \Lambda(u)}{\partial u}|_{u=0,(\theta_j)=0}.
\]

4 Discussion

In this paper, we generalize the ODBA method to the integrable models related with the \(sp(4)\) Lie algebra. By using the fusion technique, we obtain the closed operator product identities of the fused transfer matrices. Based on them and the asymptotic behaviors as well as the special values, we obtain the exact solution of the system with the periodic and off-diagonal open boundary conditions. The method and the results in this paper can be generalized to the high rank \(C_n\) (i.e., the \(sp(2n)\)) case directly.
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