An introduction to symmetric edge graphs

Shivani Chauhan, A. Satyanarayana Reddy

Department of Mathematics, Shiv Nadar University, India-201314
(e-mail: sc739@snu.edu.in, satya.a@snu.edu.in)

Abstract

Given a graph $X$, we define the symmetric edge graph of $X$, denoted as $\gamma(X)$ and study various properties of $\gamma(X)$ with respect to $X$. If $X''$ denotes the Kronecker double cover of $X$, we characterize all graphs which satisfy $\gamma(X) = L(X'')$, where $L(Y)$ is the line graph of $Y$.

Keywords: Line graph, edge adjacency matrix of graph, Ihara zeta function of a graph, Kronecker double cover.

Subject Classification (2020): 05C05, 05C50, 15A20, 05C99.

1 Introduction

In this paper, we restrict ourselves to finite graphs with no self loops and multiple edges. We denote the cycle graph, path graph, complete graph and star graph on $n$ vertices by $C_n, P_n, K_n$ and $K_{1,n-1}$ respectively. Let $X = (V, E)$ be a graph with $|V(X)| = n$, $|E(X)| = m$. We orient the edges arbitrarily and label them as $e_1, e_2, \ldots, e_m$ and also $e_{m+i} = e_i^{-1}$, $1 \leq i \leq m$, where $e_i^{-1}$ denotes the edge $e_i$ with direction reversed. Then the edge adjacency matrix of $X$ denoted $M(X)$ or simply $M$, is defined as

$$M_{ij} = \begin{cases} 1 & \text{if } t(e_i) = s(e_j) \text{ and } s(e_i) \neq t(e_j), \\ 0 & \text{otherwise}. \end{cases}$$

where $s(e_i)$ and $t(e_i)$ denote the starting and terminal vertex of $e_i$ respectively.

Example 1. The process of computation of edge adjacency matrix of $C_3$ is given below.

\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
$X$ & $e_1^{-1}$ & $e_1$ \\
\hline
$e_2$ & $e_3$ & \\
\hline
$e_1$ & $0$ & $0$ \\
$e_2$ & $0$ & $1$ \\
$e_3$ & $1$ & $0$ \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\begin{tabular}{|c|c|c|c|c|c|}
\hline
& $e_1$ & $e_2$ & $e_3$ & $e_1^{-1}$ & $e_2^{-1}$ & $e_3^{-1}$ \\
\hline
$e_1$ & $0$ & $1$ & $0$ & $0$ & $0$ & $0$ \\
$e_2$ & $0$ & $0$ & $1$ & $0$ & $0$ & $0$ \\
$e_3$ & $1$ & $0$ & $0$ & $0$ & $0$ & $0$ \\
$e_1^{-1}$ & $0$ & $0$ & $0$ & $0$ & $0$ & $1$ \\
$e_2^{-1}$ & $0$ & $0$ & $0$ & $1$ & $0$ & $0$ \\
$e_3^{-1}$ & $0$ & $0$ & $0$ & $0$ & $1$ & $0$ \\
\hline
\end{tabular}
\end{table}
The edge adjacency matrix of a graph $X$ is very useful in evaluating the Ihara zeta function of $X$ and in community detection on a graph [3][11]. A path $P = e_1 e_2 \cdots e_t$, where $e_i$ is an oriented edge, is said to backtrack if $e_{i+1} = e_i^{-1}$ for some $k \in \{1, 2, 3, \ldots, t-1\}$, i.e. it crosses the same edge twice in a row. A path $P$ is said to have a tail if $e_t = e_1^{-1}$, i.e. the last edge of $P$ is the reverse of the first edge. A closed path $C = e_1 e_2 \cdots e_t$ is said to be prime or primitive if it has no backtrack or tail and $C \neq D^f$ for some closed path $D$ and $f > 1$. The Ihara zeta function of a graph $X$ is defined to be

$$
\zeta_X(u) = \prod_{[C]} (1 - u^{\ell(C)})^{-1},
$$

where the product is over the primes $[C]$ of $X$ and $\ell(C)$ is the length of cycle $C$. It is clear that the computation of Ihara zeta function using the definition is difficult except for the cycle graph. The following two results by Bass [1] simplified the evaluation of the Ihara zeta function significantly.

Theorem 2. [1] Let $A$ be the adjacency matrix of $X$ and $Q$ be the diagonal matrix with $j$th diagonal entry $q_j$ such that $q_j + 1$ is the degree of the $j$th vertex of $X$. Suppose that $r$ is the rank of the fundamental group of $X$; $r - 1 = |E| - |V|$. Then

$$
\zeta_X(u)^{-1} = (1 - u^2)^{r-1} \det(I - Au + Qu^2).
$$

The main purpose of introducing edge adjacency matrix of a graph can be seen from the following result.

Theorem 3. [1] Let $M$ be the edge adjacency matrix of a graph $X$. Then

$$
\zeta_X(u)^{-1} = \det(I - Mu).
$$

Now we will state a few properties of the edge adjacency matrix of a graph. Many of these have been discussed in the thesis of Horton [8] and one can also find in the book by Terras [15].

$$
M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},
$$

where $A, B, C, D$ are $m \times m$ matrices with the following properties:

1. $B = B^T$, $C = C^T$.
2. $D = A^T$.
3. The diagonals of $B, C$ are zeros.
4. The diagonals of $A, D$ are zeros if the graph has no self loops.
5. If $J = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}$, where $I_m$ denotes the identity matrix of order $m$, then $M^T = JMJ$.
6. The $i^{th}$ row sum of $M$ is equal to $\deg(t(e_i)) - 1$, where $\deg(v)$ denotes the degree of vertex $v$.
7. The sum of the blocks of $M, A + B + C + D$ is the adjacency matrix of $L(X)$. It is immediate to note that Hadamard product of any two matrices from $\{A, B, C, D\}$ is the zero matrix.

Let $M$ be the edge adjacency matrix of a graph $X$. It is easy to see that $M + M^T$, where $A^T$ denotes the transpose of $A$, is a symmetric matrix with entries 0 or 1. We call $M + M^T$ as
**symmetric edge adjacency matrix** of X, and the graph whose adjacency matrix is $M + M^T$ is called as **symmetric edge graph** of X. Let $\mathcal{G}$ denote the set of all finite graphs up to isomorphism.

We define the map $\gamma : \mathcal{G} \mapsto \mathcal{G}$ as $\gamma(X) = Y$, where $Y$ is the symmetric edge graph of X. If $\mathfrak{F} = \gamma(\mathcal{G})$, the range of $\gamma$, then $\mathfrak{F}$ is proper subset of $\mathcal{G}$. Our goal in this paper is to study the properties of elements of $\mathfrak{F}$, the set of all symmetric edge graphs. With these notations in hand, we have $M + M^T = A(\gamma(X))$, where $A(Y)$ denotes the adjacency matrix of Y. We define $\gamma^k(X) = \gamma(\gamma^{k-1}(X))$, where $k \in \mathbb{N}$ with $\gamma^0(X) = X$.

**Example 4.** 1. If $X = C_n$ then $\gamma(X) = 2C_n$ and $\gamma^k(X) = 2^kC_n$.

2. If $X = K_{1,3}$ then $\gamma(X) = C_6$ and $\gamma^k(X) = 2^{k-1}C_6$.

![Figure 1: C₃, γ(C₃) and A(γ(C₃)).](image)

![Figure 2: K₁,₃, γ(K₁,₃) and A(γ(K₁,₃)).](image)

The following are few immediate observations of the graph $\gamma(X)$.

1. The number of vertices of $\gamma(X)$ is twice the number of edges of X.

2. The $(ij)^{th}$ entry of $A(\gamma(X))$ is 1 if $e_i$ feeds into $e_j$, where $j \neq m + i$ and $1 \leq i, j \leq 2m$ and is 0 otherwise.

3. We have,

$$Tr((M + M^T)^2) = 2|E(\gamma(X))| = 2e^TMe,$$

where e denotes the column vector with all entries 1′s and $J(M + M^T) = (M + M^T)J$.

iii
Let $|E(\gamma(X))| = 2|E(L(X))| = \sum_{i=1}^{V(X)} d_i^2 - 2|E(X)|,$

where $d_i$ is the degree of $i^{th}$ vertex of $X$. From Equation 3 we observe that for a given graph $X$, $\gamma(X)$ cannot be bicyclic. Recall a bicyclic graph is a connected graph with the number of edges is equal to one more than the number of vertices.

5. From Equation 2 we have

$$M + M^T = \begin{bmatrix} A + D & B + C \\ B + C & A + D \end{bmatrix}.$$  \hspace{1cm} (4)

If we denote $A + D, B + C$ by $A_0, B_0$ respectively, then as proved in (Davis [5]) that the spectrum of $\begin{bmatrix} A_0 & B_0 \\ B_0 & A_0 \end{bmatrix}$ is the union of spectra of $A_0 + B_0$ and $A_0 - B_0$. Therefore, the spectrum of $A(L(X))$ is contained in the spectrum of $A(\gamma(X))$.

6. From Equation 4 we can observe that if $(d_1, d_2, \ldots, d_n)$ is the degree sequence of $L(X)$ then $(d_1, d_1, d_2, d_2, \ldots, d_n, d_n)$ is the degree sequence of $\gamma(X)$. It is easy to see that if $X$ is Eulerian then $\gamma(X)$ is Eulerian provided $\gamma(X)$ is connected it follows from the fact that if $X$ is Eulerian then $L(X)$ is Eulerian (see Harary [7]).

In Section 2 we study various properties of $\gamma(X)$ with respect to $X$. Graphs with the property that $\gamma(X) = L(X''')$ are characterized in Section 3. For more details of the Ihara zeta function and edge adjacency matrices refer the book in [15]. For other results and proofs related to graph theory refer [7, 12]. In the rest of this section, we discuss the prove for theorem 5 first part of which has also been stated at the end of Page 3 of [10] and proved in [6, 16]. In this work, we are extending it further.

**Theorem 5.** Let $M$ be the edge adjacency matrix of a graph $X$. Then $M$ is a nilpotent matrix if and only if $X$ is a forest. The index of nilpotency of $M$ is the largest diameter among the connected components of $X$.

We need a couple of results before proving Theorem 5

**Lemma 6.** [15] Let $M$ be the edge adjacency matrix of graph $X$. Then $Tr(M^k) = N_k$, where $N_k$ is the number of cycles of length $k$ with no backtracks and tails.

**Lemma 7.** Let $M$ be the edge adjacency matrix of a graph $X$. Then $M^2$ is a $(0, 1)$-matrix. Further, when $\ell \geq 1$ and $i \neq j$, the $(i, j)^{th}$ entry of $M^\ell$ represents the number of backtrackless walks of length $\ell + 1$.

**Proof.** Suppose there exists $i \neq j$ such that $(M^2)_{ij} = \sum_{k=1}^{2m} m_{ik}m_{kj} > 1$, then there exists multiple edges between $s(e_k)$ and $t(e_k)$. This contradicts the fact that $X$ is a simple graph. If $i = j$, then $(M^2)_{ii} = \sum_{k=1}^{2m} m_{ik}m_{ki} = 0$.

It is easy to see that the second part of the theorem is true for $\ell = 1, 2$. Assume that it is true for $\ell = t$. Let $\ell = t + 1$ and $i \neq j$. Then

$$(M^{t+1})_{ij} = \sum_{k=1}^{2m} (M^t)_{ik}(M)_{kj},$$

iv
where $M^t_{ik}$ is the number of backtrackless walks of length $t+1$ from $s(e_i)$ to $t(e_k)$ by induction hypothesis and $(M)_{kj}$ is the number of backtrackless walks of length 2 from $s(e_k)$ to $t(e_j)$. This implies $(M^{t+1})_{ij}$ is the number of backtrackless walks of length $t+2$ from $s(e_i)$ to $t(e_j)$. □

The following result is a direct consequence of Lemma 7.

**Corollary 8.** Let $X$ be a tree and $M$ be its edge adjacency matrix. Then $M^t$ is a $(0,1)$-matrix for every $t \geq 1$. Furthermore, for $i \neq j$ $(M^t)_{ij}$ is equal to 1 if and only if there exists a backtrackless walk of length $t+1$ in $X$ from $s(e_i)$ to $t(e_j)$.

Now, we are ready to prove Theorem 5.

**Proof.** Let $X_1, X_2, \ldots, X_k$ be the connected components of $X$ and $M_i$ be the edge adjacency matrix of $X_i$. Then it is easy to see that $M = M_1 \ 0 \ \cdots \ 0 \ 0 \ M_2 \ \cdots \ 0 \ \vdots \ \vdots \ \ddots \ \vdots \ 0 \ 0 \ \cdots \ M_k$.

In order to prove the result, it is sufficient to show that $M_i$ is a nilpotent matrix if and only if $X_i$ is a tree. Suppose $M_i$ is a nilpotent matrix. Then by Lemma 6 for $k \in \mathbb{N}$ we have $tr(M^k_i) = \sum_{e_1, \ldots, e_m} m_{e_1e_2}\cdots m_{e_me_1} = N_k = 0$.

Hence, $X_i$ is a tree. Conversely, suppose $X_i$ is a tree with $n$ vertices. Then the length of the largest possible backtrackless walk in $X_i$ is $n-1$. By Corollary 8, $M_i^n = 0$.

Let $\eta$ be the index of nilpotency of $M$ that is $M^{n-1} \neq 0$ and $M^n = 0$. Let $X_i$ be the component of $X$ with the largest diameter and $M_i$ be the corresponding edge adjacency matrix. Then by Corollary 8, $\eta$ becomes the diameter of $X_i$ if and only if $M_i^{\eta-1} \neq 0$ and $M_i^\eta = 0$. □

**Corollary 9.** Let $M$ be the edge adjacency matrix of a graph $X$. Then eigenvalues of $M$ are integers if and only if $X$ is a forest.

**Proof.** If eigenvalues of $M$ are integers, by Lemma 7 we have $Tr(M^2) = 0$ which implies that each eigenvalue of $M$ is zero. By Theorem 5 $X$ is a forest and the converse also holds. □

## 2 Properties of $\gamma(X)$

We begin this section by stating the Whitney theorem and then we present the same result for the $\gamma$ function.

**Theorem 10.** [17] Let $X$ and $Y$ be connected graphs with isomorphic line graphs. Then $X$ and $Y$ are isomorphic unless one is $K_3$ and the other is $K_{1,3}$.

By Example 4 we have $\gamma(K_3) = 2K_3 \neq C_6 = \gamma(K_{1,3})$, and hence the following result.

**Theorem 11.** Let $X$ and $Y$ be connected graphs. Then $\gamma(X) \succeq \gamma(Y)$ if and only if $X \succeq Y$.

Next we prove that $\gamma$ function is additive with respect to disjoint union.
Lemma 12. Let $X$ be a graph with connected components $X_1, X_2, \ldots, X_k$ i.e., $X = X_1 \cup X_2 \cup \ldots \cup X_k$. Then

$\gamma(X_1 \cup X_2 \cup \ldots \cup X_k) \cong \gamma(X_1) \cup \gamma(X_2) \cup \ldots \cup \gamma(X_k)$.

Proof. We give the proof for $k = 2$ and the general case follows by induction on $k$. Let $X_1, X_2$ be graphs with $m_1, m_2$ edges respectively. Then $A(\gamma(X_1 \cup X_2))$ and $A(\gamma(X_1) \cup \gamma(X_2))$ have the following block structures respectively.

$$A(\gamma(X_1 \cup X_2)) = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ B_1 & 0 & A_1 & 0 \\ 0 & B_2 & 0 & A_2 \end{pmatrix}.$$ 

$$A(\gamma(X_1) \cup \gamma(X_2)) = \begin{pmatrix} A(\gamma(X_1)) & 0 \\ 0 & A(\gamma(X_2)) \end{pmatrix} = \begin{pmatrix} A_1 & B_1 & 0 & 0 \\ B_1 & A_1 & 0 & 0 \\ 0 & 0 & A_2 & B_2 \\ 0 & 0 & B_2 & A_2 \end{pmatrix}. $$

It is easy to see that

$$P^T A(\gamma(X_1 \cup X_2)) P = A(\gamma(X_1) \cup \gamma(X_2)),$$

where $P = \begin{pmatrix} I_{m_1} & 0 & 0 & 0 \\ 0 & 0 & I_{m_2} & 0 \\ 0 & I_{m_1} & 0 & 0 \\ 0 & 0 & 0 & I_{m_2} \end{pmatrix}$ is a permutation matrix and $I_{m_1}, I_{m_2}$ denote the identity matrices of order $m_1, m_2$ respectively. \qed

We will see a few examples to observe the pattern of graphs under the $\gamma$ function. For more examples refer the Table 2.

Example 13.  

1. If $X = P_n$ then $\gamma^{n-1}(X)$ is a null graph. The following table shows the effect of repeated application of the $\gamma$ function on the path graph.

| $X$ | $\gamma(X)$ | $\gamma^2(X)$ | $\gamma^4(X)$ |
|-----|--------------|---------------|---------------|
| $- - - - - -$ | $- - - - - -$ | $- - - - - -$ | $- - - - - -$ |

2. If $X = K_{1,n}$, then $\gamma(X)$ is a crown graph on $2n$ vertices. In particular, if $X = K_{1,4}$ then $\gamma(X)$ is cube. Recall, a crown graph on $2n$ vertices is an undirected graph with two sets of vertices $\{v_1, v_2, \ldots, v_n\}$ and $\{v'_1, v'_2, \ldots, v'_n\}$, with an edge from $v_i$ to $v'_j$ whenever $i \neq j$.

3. If $X = K_{2,3}$ then $\gamma(X)$ is a 6-prism graph, as visible in Fig. 3.

The following results provides how the $\gamma$ function preserves connectedness, regularity and bipartiteness. Unless specified otherwise, we assume that $A(\gamma(X)) = \begin{bmatrix} A + D & B + C \\ B + C & A + D \end{bmatrix}$ and $A_0 = A + D, B_0 = B + C$.

Proposition 14.  

1. Let $X$ be a connected graph. Then $\gamma(X)$ is connected if and only if $X$ is not a cycle graph or a path graph. Moreover, $\gamma(X)$ cannot be a cycle graph unless $X = K_{1,3}$. 

vi
2. Let $\gamma(X)$ be a connected graph, then $\gamma(X)$ has a cut edge if and only if $X$ contains a pendant vertex which is adjacent to a vertex of degree two.

3. The map $\gamma$ maps regular graphs to regular graphs. In particular, if $X$ is $k$-regular then $\gamma(X)$ is $2k - 2$ regular. Moreover, if $\gamma(X)$ is regular, then $X$ is either a regular graph or a semi-regular bipartite graph.

4. Let $X$ be a connected graph. Then $X$ is bipartite if and only if $\gamma(X)$ is bipartite.

Proof. Proof of Part 1. Let us suppose that $\gamma(X)$ is not a connected graph. Then, $B + C = 0$ and hence, $B, C = 0$. Thus we conclude that the degree of each vertex in $X$ is at most 2. Since $X$ is a connected graph, $X$ is a cycle graph or a path graph. From part 1 of Example 4 and 13 one can see the converse is also true.

For the second part of the Proposition, let $\gamma(X)$ be a cycle graph on $2k$ ($k \neq 3$) vertices. From the structure of the adjacency matrix of a cycle graph, we see that when we add the four blocks of $A(C_{2k})$, we obtain $2A(C_k)$. On adding all the blocks of $A(\gamma(X))$, we get $2A(L(X))$. From here we deduce that $L(X)$ is a cycle graph on $k$ vertices. However, we know from [7] that a connected graph is isomorphic to its line graph if and only if it is a cycle graph. This implies that $X$ is a cycle graph on $k$ vertices, which is a contradiction to Part 1 of Example 4.

If $X = K_{1,3}$, then from Part 2 of Example 3 we have already seen that $\gamma(X)$ is $C_6$.

Proof of Part 2. Let $\gamma(X)$ have a cut edge and no pendant vertex. From the structure of $A(\gamma(X))$, we can observe that $\gamma(X)$ has two copies of a graph, each of whose adjacency matrix is $A_0$. Since $\gamma(X)$ is connected, the edges corresponding to the matrix $B_0$ connects the two copies of the graph given by $A_0$. As $B_0$ is symmetric, no edge given by the matrix $B_0$ can be a cut edge. Also, note that no edge in the two copies given by $A_0$ in $A(\gamma(X))$ can be a cut edge. Hence, $\gamma(X)$ has a pendant vertex which implies that $X$ has a pendant vertex which is adjacent to a vertex of degree 2. The converse is easy to see as well.

Proof of Part 3. The first part follows from the fact that if $X$ is regular, then $L(X)$ is regular. For the second part, let us assume that $\gamma(X)$ is regular. This implies that $L(X)$ is regular and hence from Lemma 6.2 in [13], we obtain that $X$ is either a regular graph or a semi-regular bipartite graph.

Proof of Part 4. Suppose that $X$ is bipartite with vertex partitions $\{v_1, v_2, \ldots, v_n\}$ and $\{v'_1, v'_2, \ldots, v'_k\}$. Choose an orientation in such a way that $e_i$'s are the directed edges from $v_i$ to
$v'_j$ for all $1 \leq i \leq n, 1 \leq j \leq k$. Then observe that $M = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ which implies

$$M + M^T = \begin{bmatrix} 0 & B_0 \\ B_0 & 0 \end{bmatrix}. \quad (5)$$

Hence, $\gamma(X)$ is bipartite. The converse is easy to see.

From the proof of Part 4 of Proposition 13 one can see that if $X$ is bipartite, spectrum of $\gamma(X)$ is given by the union of spectra of $A(L(X))$ and $-A(L(X))$. Now using Theorem 2, we provide an expression for the zeta function of $\gamma(X)$ in terms of the zeta function of $L(X)$, where $X$ is a bipartite graph.

**Theorem 15.** Let $X$ be a connected graph. Then $\zeta_{L(X)}^{-1}(u)$ divides $\zeta_{\gamma(X)}^{-1}(u)$. In particular, if $X$ is bipartite then

$$\zeta_{\gamma(X)}^{-1}(u) = \zeta_{L(X)}^{-1}(u) \zeta_{L(X)}^{-1}(-u).$$

**Proof.** From Equation 11

$$\zeta_{\gamma(X)}^{-1} = (1 - u^2)^{E(\gamma(X)) - |V(\gamma(X))|} \det(I_{\gamma(X)} - A(\gamma(X))u + Q_{\gamma(X)}u^2),$$

and note that $(1 - u^2)^{E(L(X)) - |V(L(X))|}$ divides $(1 - u^2)^{E(\gamma(X)) - |V(\gamma(X))|}$. Consider

$$I_{\gamma(X)} - A(\gamma(X))u + Q_{\gamma(X)}u^2 = \begin{pmatrix} I_{L(X)} & 0 \\ 0 & I_{L(X)} \end{pmatrix} - \begin{pmatrix} A_0 & B_0 \\ B_0 & A_0 \end{pmatrix}u + \begin{pmatrix} Q_{L(X)} & 0 \\ 0 & Q_{L(X)} \end{pmatrix}u^2 \quad (6)$$

$$= \begin{pmatrix} I_{L(X)} - A_0u + Q_{L(X)}u^2 \\ -B_0u \end{pmatrix} - \begin{pmatrix} A_0u + Q_{L(X)}u^2 \\ I_{L(X)} - A_0u + Q_{L(X)}u^2 \end{pmatrix}. \quad (7)$$

On adding the second column to the first column of matrix 7 and using the fact that $A_0 + B_0 = A(L(X))$, we have the desired result. For the second part of the theorem, observe that if $X$ is bipartite, then $A_0 = 0$ and $B_0 = A(L(X))$. Hence the result follows.

It is possible to know the number of triangles in $\gamma^k(X)$ once we know the number of triangles in $X$. Consequently, if $X$ is triangle-free then so is $\gamma^k(X)$ for every $k \in \mathbb{N}$.

**Proposition 16.** Let $t_i$ be the number of triangles in $\gamma^{i-1}(X)$, where $i \geq 1$. Then $t_i = 2^{i-1}t_1$.

**Proof.** We shall prove only for the case $i = 2$ i.e., $t_2 = 2t_1$ and the general case shall follow from induction. It is easy to see that

$$6t_2 = Tr((M + M^T)^3) = 2Tr(M^3) + 3Tr(M^2M^T) + 3Tr(M^2M^T) = 0.$$ We now claim that $Tr(M^2M^T) = Tr(M(M^T)^2) = 0$. Since $M$ is a nonnegative matrix, $Tr(M^2M^T) = 0$ if and only if $(M^2M^T)_{ii} = 0$ for all $i$. We have

$$(M^2M^T)_{ii} = \sum_{k=1}^{2m}(M^2)_{ik}M^T_{ki} = \sum_{k=1}^{2m} \sum_{j=1}^{2m} M_{ij}M_{jk}M_{ik}.$$ If each of $M_{ij}, M_{jk}$ and $M_{ik}$ are nonzero, then $e_k = e_k^{-1}$. Consequently, $X$ has multiple edges, which is a contradiction. Similarly, one can show that $(M^2M^T)^2)_{ii} = 0$. Thus, $3t_2 = Tr(M^3)$. From lemma 6, we have another identity $t_2 = \frac{N_1}{3}$. Hence, the result follows from the fact that $t_1 = \frac{N_4}{6}$, as each vertex of a triangle can be an initial vertex and two directions.
Next we will present a characterization of a symmetric edge graph similar to that of a line graph, as given by Krausz in [9]. The approach used in proof of Theorem 18 is motivated from the proof of Theorem 8.4 in [7].

Theorem 17. [9] A graph is a line graph if and only if its edges can be partitioned into complete subgraphs with the property that no vertex lies in more than two of the subgraphs.

Theorem 18. A graph is a symmetric edge graph if and only if its edges can be partitioned into crown subgraphs in such a way that each vertex lies in at most two of the subgraphs.

Proof. Let \( Y \) be the symmetric edge graph of \( X \). Without loss of generality, \( X \) is connected. Let \( v \) be any vertex of \( X \), then by Part 2 of Example 13 we see that the star graph at \( v \) induces a crown subgraph of \( Y \). The vertices of \( Y \) lies in exactly one of the subgraphs. Each edge of \( X \) belongs to the star of exactly two vertices of \( X \), which shows that no vertex of \( Y \) is in more than two of the subgraphs.

Let \( H_1, H_2, \ldots, H_n \) be the partition of the graph \( Y \) satisfying the hypothesis. We explain the construction of \( X \) from \( Y \), where \( Y = \gamma(X) \).

Corollary 19. Let \( X \) be a connected graph. Then \( \gamma(X) \) is unicyclic if and only if \( X \) is a tree with \( \Delta(X) = 3 \), where \( \Delta(X) \) denotes the maximum degree of \( X \) and there is exactly one vertex of degree three.

Proof. Suppose that \( \gamma(X) \) is unicyclic, which implies that \( X \) does not contain a cycle. By Theorem 18 it is clear that \( \exists \) a vertex in \( X \) with degree greater than equal to 4. If \( \exists \) more than one vertex of degree 3, then we get a contradiction to the hypothesis. The converse is easy to follow by Theorem 18.

3 Characterization of \( X \) for which \( \gamma(X) = L(X'') \)

A graph \( Y \) is a covering graph of another graph \( X \) if there is a covering map from the vertex set of \( Y \) to the vertex set of \( X \). A covering map \( f \) is a surjection and a local isomorphism such that the neighbourhood of a vertex \( v \) in \( Y \) is mapped bijectively onto the neighbourhood of \( f(v) \) in \( X \). Additional information on covering graphs can be found in [13, 14]. In particular, we are interested in the Kronecker double cover. It is also known as the bipartite double cover or the canonical double cover. The Kronecker double cover of a graph \( X \) is the Kronecker product of \( X \) by \( K_2 \). The Kronecker product \( X_1 \otimes X_2 \) of graphs \( X_1 \) and \( X_2 \) is defined as follows. The vertex set of \( X_1 \otimes X_2 \) is \( V(X_1) \times V(X_2) \), and vertices \((x_1, x_2) \) and \((x'_1, x'_2) \) are adjacent if and only if \( x_1 \) is adjacent to \( x'_1 \) and \( x_2 \) is adjacent to \( x'_2 \). The adjacency matrix of the Kronecker double cover of \( X \) is the Kronecker product of the adjacency matrix of \( X \) and \( K_2 \). In particular, if \( X \) is a bipartite graph, then

\[
A(X'') = \begin{bmatrix} A(X) & 0 \\ 0 & A(X) \end{bmatrix}.
\]
Let \( X \) be a connected graph with \( n \) vertices and \( m \) edges. Let \( \{v_1, v_2, \ldots, v_n\} \) be the vertex set of \( X \). Clearly, \( X'' \) has \( 2n \) vertices and \( 2m \) edges. Let \( V(X'') \) have partitions \( \{v'_1, v'_2, \ldots, v'_n\} \) and \( \{v''_1, v''_2, \ldots, v''_n\} \), and \( E(X'') \) be given by \( \{e_1, e_2, \ldots, e_m, e_{m+1}, e_{m+2}, \ldots, e_{2m}\} \). We define a map \( \phi : V(X'') \rightarrow V(X) \) such that \( \phi(v'_i) \) and \( \phi(v''_i) \) are mapped to \( v_i \) for all \( 1 \leq i \leq n \). We label the edges of \( X'' \) such that \( e_k \) is an edge from \( v'_i \) to \( v''_{n+i} \) \((i \neq j)\) if and only if \( e_{m+k} \) is an edge from \( v'_j \) to \( v''_{n+i} \) \((i \neq j)\). The rows and columns of \( A(L(X'')) \) are indexed by \( E(X'') \). It is easy to see that \( A(L(X'')) \) has the following structure

\[
\begin{bmatrix}
P & Q \\
Q^T & R
\end{bmatrix},
\]

where \( P, Q, R \) are \( m \times m \) matrices with the following properties:

1. \( P = R \). Since \( P_{ij} = 1 \) implies \( e_i \) is adjacent to \( e_j \), the labelling defined above shows that \( e_{m+i} \) is adjacent to \( e_{m+j} \).

2. \( Q = Q^T \). Since \( Q_{ij} = 1 \) implies \( e_i \) is adjacent to \( e_{m+j} \), the labelling defined above shows that \( e_j \) is adjacent to \( e_{m+i} \).

3. \( P + Q = A(L(X)) \). Note that if \( P_{ij} = 1 \) then \( Q_{ij} = 0 \) and vice-versa. If \( (P+Q)_{ij} = P_{ij} + Q_{ij} = 1 \), then from the definition of covering graph we have \( (A(L(X)))_{ij} = 1 \).

We have \( |V(L(X''))| = |V(\gamma(X))| = |V(L(X''))| = 2m \) and the degree sequence of \( L(X''), \gamma(X) \) and \( L(X'') \) is same. Also, from point 3 mentioned above it is clear that spectrum of \( A(L(X)) \) is contained in the spectrum of \( A(L(X'')) \). To proceed with the proof of Theorem 21 we need to define claw free graphs. Recall, a claw is another name for the complete bipartite graph \( K_{1,3} \). In contrast, a claw-free graph is a graph in which no induced subgraph is a claw. It is proved by Beineke in [2] that line graph of any graph is claw-free.

**Proposition 20.** Let \( X \) be a connected graph. Then

1. \( L(X'') \) is disconnected if and only if \( X \) is bipartite.

2. \( 2t' = t_2 + t_3 \), where \( t', t_2, t_3 \) denotes the number of triangles in \( L(X), \gamma(X) \) and \( L(X'') \) respectively.

3. \( \zeta^{-1}_{\gamma(X)}(u) \) divides \( \zeta^{-1}_{L(X'')} (u) \). In particular, if \( X \) is bipartite, then \( \zeta^{-1}_{L(X'')} (u) = (\zeta^{-1}_{\gamma(X)}(u))^2 \).

**Proof.** Proof of Part 1 is easy to follow from the result proved in [4], which discusses that a Kronecker double cover of a graph \( X \) is connected if and only if \( X \) is connected and non-bipartite.

Proof of Part 2 We know from the definition of a line graph that

\[
t' = t_1 + \sum_i \binom{d_i}{3},
\]
where \( t_1 \) denotes the number of triangles in \( X \) and \( d_i \) is the degree of the \( i^{th} \) vertex in \( X \). From Proposition 16 we know that \( 2t_1 = t_2 \). Since \( X'' \) is bipartite we have \( t_3 = 2 \sum_i \left( \frac{d_i}{3} \right) \). Hence, \( 2t' = t_2 + t_3 \).

Proof of Part 3 is similar to the proof of Theorem 15.

We are now interested to see the relationship among \( \gamma(X) \), \( L(X'') \) and \( L(X)'' \) for a connected graph \( X \). We begin with an example.

In Figure 5, we see that \( \gamma(X) = L(X)' \) and \( \gamma(L(X)) = L(L(X)'') \), but it is not true in general, one can check with \( X = C_3 \). In the next theorem we characterize all those graphs which satisfy this property.

**Theorem 21.** Let \( X \) be a connected graph. Then

1. \( \gamma(X) = L(X)' \) if and only if \( X \) is bipartite.
2. \( \gamma(X) = L(X)' \) if and only if one of the following is true:
   - \( X \) is a path graph.
   - \( X \) is a cycle graph on even vertices.
   - \( X = K_4, K_4 - \{e\} \), a triangle with an edge on any one of its vertices.
3. \( L(X') = L(X)' \) if and only if \( X \) is either a cycle graph or a path graph.
Proof. Proof of Part 1. If $X$ is bipartite, then by Part 2 of Proposition 13, $\gamma(X)$ is bipartite which shows

$$A(\gamma(X)) = A(L(X)^{n^2}) = \begin{bmatrix} 0 & A(L(X)) \\ A(L(X)) & 0 \end{bmatrix}.$$  

Proof of Part 2. In order to prove this, we first prove that $\gamma(X)$ is a line graph if and only if $|V(X)| \leq 4$ or $X$ is either a cycle graph or a path graph.

Suppose that $\gamma(X)$ is a line graph. Let $X$ be a graph with $\Delta(X) \leq 3$. If any vertex in $X$ has $\Delta(X) \geq 4$, then by Theorem 18 the vertex with maximum degree induces a crown graph on at least 8 vertices. Hence, $\gamma(X)$ cannot be a claw-free graph.

Case 1: If $\Delta(X) \leq 2$, then $X$ is either a cycle graph or a path graph. From Part 1 of Example 4 and 13, it is clear that $\gamma(X)$ is a line graph of $2X$.

Case 2: Let $\Delta(X) = 3$ and $|V(X)| > 4$. In particular, let $v$ be a vertex of degree 3 and vertices adjacent to $v$ be $x, y, z$. Since $|V(X)| > 4$, if we add an edge on any of the vertices $x, y, z$, then the graph $\gamma(X)$ is not a claw-free graph, which is clear from Figure 5. Conversely, if $X = C_n$ or $P_n$ then $\gamma(X)$ is a line graph of two copies of $C_n$ (or $P_n$). If $X = K_{1,3}$ then by Part 2 of Example 4, $\gamma(X)$ is $C_6$ which is a line graph of $C_6$. $\gamma(X)$ for other non-isomorphic graphs with $|V(X)| = 4$ are described in Table 1 and Figure 5.

Suppose that $\gamma(X) = L(X'')$. Then by the above statement, it can be noted that $|V(X)| \leq 4$ or $X = C_n$ or $P_n$. If $X = C_n$ and $n$ is odd, then $L(X'') = C_{2n} - 2C_n = \gamma(X)$. If $X = C_n$ and $n$ is even, then $L(X'') = 2C_n = \gamma(X)$. If $X = P_n$, then $L(X'') = 2P_{n-1} = \gamma(X)$. For $X = K_4$ or $K_4 - \{e\}$ or a triangle with an edge on any one of its vertices, we can see from Table 1 and Figure 5 that $\gamma(X) = L(X'')$. If $X = K_{1,3}$ then $L(X'') = 2C_3 - C_6 = \gamma(X)$. The converse part of the same is easy to follow.

Proof of Part 3. Assume that $L(X'') = L(X'')$. From here we can observe that $L(X'')$ is bipartite. From the definition of a line graph, it is clear that the degree of each vertex of $X$ is less than equal to two. Hence, $X$ is either a cycle graph or a path graph. Conversely, if $X = C_k$ and $k$ is odd then $X'' = C_{2k}$, $L(X'') = L(X)' = C_{2k}$. If $X = C_k$ and $k$ is even or $X = P_k$ then $X'' = 2X$, the result follows.

Corollary 22. Let $X$ be a connected graph. Then $\gamma(X) = L(X'') = L(X)'$ if and only if $X$ is either a cycle graph on even vertices or a path graph.

Proof. It follows from Part 2 and 3 of Theorem 21.

References

[1] H. Bass. The ihara-selberg zeta function of a tree lattice. *International Journal of Mathematics*, 3(06):717–797, 1992.

[2] L. W. Beineke. Characterizations of derived graphs. *Journal of Combinatorial Theory*, 9(2):129–135, 1970.

[3] C. Bordenave, M. Lelarge, and L. Massoulié. Non-backtracking spectrum of random graphs: community detection and non-regular ramanujan graphs. In 2015 IEEE 56th Annual Symposium on Foundations of Computer Science, pages 1347–1357. IEEE, 2015.

[4] R. A. Brualdi, F. Harary, and Z. Miller. Bigraphs versus digraphs via matrices. *Journal of Graph Theory*, 4(1):51–73, 1980.

[5] P. J. Davis. *Circulant matrices*. American Mathematical Soc., 2013.
[6] C. Glover and M. Kempton. Spectral properties of the non-backtracking matrix of a graph. arXiv preprint arXiv:2011.09385, 2020.

[7] F. Harary. Graph Theory. Addison-Wesley, 1969.

[8] M. D. Horton. Ihara zeta functions of irregular graphs. PhD thesis, UC San Diego, 2006.

[9] J. Krausz. Démonstration nouvelle d’une théoreme de whitney sur les réseaux. Mat. Fiz. Lapok, 50(1):75–85, 1943.

[10] F. Krzakala, C. Moore, E. Mossel, J. Neeman, A. Sly, L. Zdeborová, and P. Zhang. Spectral redemption in clustering sparse networks. Proceedings of the National Academy of Sciences, 110(52):20935–20940, 2013.

[11] M. Newman. Spectral community detection in sparse networks. arXiv preprint arXiv:1308.6494, 2013.

[12] N. L. Biggs. Algebraic graph theory, volume 67. Cambridge university press, 1993.

[13] D. Ray-Chaudhuri. Characterization of line graphs. Journal of Combinatorial Theory, 3(3):201–214, 1967.

[14] H. M. Stark and A. A. Terras. Zeta functions of finite graphs and coverings. Advances in Mathematics, 121(1):124–165, 1996.

[15] A. Terras. Zeta functions of graphs: a stroll through the garden, volume 128. Cambridge University Press, 2010.

[16] L. Torres. Non-backtracking spectrum: Unitary eigenvalues and diagonalizability. arXiv preprint arXiv:2007.13611, 2020.

[17] H. Whitney. Congruent graphs and the connectivity of graphs. In Hassler Whitney Collected Papers, pages 61–79. Springer, 1992.
| $X$ | $\gamma(X)$ |
|-----|------------|
| ![Graph](image1) | ![Graph](image2) |
| ![Graph](image3) | ![Graph](image4) |
| ![Graph](image5) | ![Graph](image6) |
| ![Graph](image7) | ![Graph](image8) |
| ![Graph](image9) | ![Graph](image10) |

Table 2

xv