Improving the measurement precision of temperature is very important and challenging, especially at low temperatures. Based on the existence of invariant subspaces during the polariton thermalization, a new way to enhance the measurement precision of low temperature is proposed and the Landau bound is obtained to avoid that the measurement uncertainty of the temperature diverges as the temperature approaches zero. The measurement precision of low temperature increases significantly with the number of polariton states. In order to resist the dissipation, the incoherent pumping is necessary for obtaining the information of the temperature encoded in the steady state. It should be noted that too strong incoherent pumping is wasteful due to that the quantum Fisher information of the temperature becomes less and less dependent on the total number of the polaritons.

1. Introduction

Precise estimation of temperature is significant and crucial for the fundamental natural science and the changing quantum technology.[1–11] Since quantum devices generally work at low temperature[12–14] and the development of the field of quantum thermodynamics[15–19] also needs low temperature to reserve the quantum properties, precise control and measurement of low temperature are becoming an important subject in quantum metrology[11] and quantum sensing.

Enhancing the estimation precision of temperature with quantum resources and investigating the fundamental limitations on temperature estimation have attracted a lot of attention.[20–31] Temperature measurements can generally be divided into two categories: one is to measure the temperature encoded in the thermal equilibrium state of the probe system, the other is in the non-equilibrium state. Either way, the measurement of low temperature has always been a challenging and arduous task. The uncertainty of the temperature diverges as the temperature approaches zero.[32] Recently, some works attempted to alleviate the divergence. Correa et al.[33] showed that the thermometric precision at low temperature could be significantly enhanced by the strong probe-sampling coupling. Mukherjee et al.[34] utilized the periodic modulation to obtain the low-temperature thermometry with temperature-independent relative uncertainty. Planella[35] obtained high precision thermometry by using correlations among multiple probes induced by the common bath. More importantly, Zhang et al.[36] obtained the Landau bound[37] by using a continuous-variable system to detect the temperature of a non-Markovian reservoir.

In this work, we propose a new way to obtain the optimal low temperature estimation, and the Landau bound can also be achieved. Our scheme does not require a non-Markovian reservoir as in ref. [36], leading to that the temperature of the thermal bath can be measured by the weak coupling between the probe system and the thermal bath. Therefore, the damage caused by the measurement can be reduced by using our scheme. In ref. [37], the Landau bound only be obtained with the vanishing gap. However, for any quantum system with a non-vanishing gap, the estimation precision of the temperature will diverge. Our scheme can obtain the Landau bound in the case of non-vanishing gap.

The exciton-polariton Bose–Einstein condensates (BECs) are used as the thermometer to measure the temperature of the phonons in the semiconductors or the temperature of intermolecular oscillations of the organic dyes. Different from the Bose polaron model[12] in BEC, there are invariant subspaces due to the polaron thermalization, in which the final thermal state is not unique. For the infinite states of the polaritons, the Landau bound can also be achieved at low temperatures. In addition, we find that incoherent pumping can resist the dissipation. But too strong incoherent pumping is useless in enhancing the estimation precision of the temperature due to that the temperature estimation precision will not increase with the increase of the total number of the polaritons for enough polaritons.

2. The Physical Model of Polaritons

When cavity photons strongly interact with an optical transition of active material, new eigenstates can be generated, that is, lower and upper exciton-polariton branches.[38,39] We only consider the lower branch in which the BEC occurs. Due to pair particle scattering, the exciton polaritons of the lower polariton branch can be treated as harmonic oscillators by neglecting the nonlinearity.[40]
The Hamiltonian of the polaritons can be described as

$$H = \sum_{j=0}^{M} \hbar \omega_j a_j^\dagger a_j$$  \hspace{1cm} (1)$$

where the ground state (mode) is the state with $j = 0$ and $a_j$ ($a_j^\dagger$) denotes the bosonic annihilation (creation) operator. The master equation for the density matrix of the polaritons $\rho$, subject only to the polariton thermalization, can be expressed as

$$\frac{d}{dt} \rho = \mathcal{L} \rho = -i[H, \rho] + L_{\text{thermal}}(\rho)$$  \hspace{1cm} (2)$$

where the polariton thermalization is described by the Lindblad superoperator

$$L_{\text{thermal}}(\rho) = \sum_{j=0}^{M} \sum_{k=0}^{M} \Gamma_{jk} \left( a_j a_k^\dagger \rho a_k - \frac{1}{2} (a_j^\dagger a_k^\dagger a_k a_j^\dagger - \frac{1}{2} a_j a_k^\dagger a_k a_j^\dagger \rho) \right)$$  \hspace{1cm} (3)$$

where $\Gamma_{jk}$ is the transition rate from the $j$th polariton state to the $k$th state. The thermalization rates obey the Kubo–Martin–Schwinger relation$^{[41,42]} \Gamma_{jk} = \exp\left( \frac{\hbar \omega_{jk}}{k_B T} \right)|\Gamma_{ij}$, where $k_B$ is the Boltzmann constant and $T$ is the temperature of intermolecular oscillations of the organic dyes or the temperature of the phonons in the semiconductors that we want to estimate. There are different underlying mechanisms of the polariton thermalization, which is dependent on the detail system. For example, the polariton thermalization comes from the nonlinear interaction with low frequency vibrations in organic polariton systems.$^{[43-45]}

The operator of the total polariton number $\sum_{j=0}^{M} a_j^\dagger a_j$ is a constant of motion during the thermalization process. The constant of motion implies that there are invariant subspaces $|n_0, n_1, \ldots, n_M\rangle$ with the total number of polaritons equal to $\sum_{j=0}^{M} n_j = N$. When there are invariant subspaces, the stationary solution is not unique.$^{[46]}$ The Gibbs distribution over the states of a given invariant subspace is also a stationary solution, which is given by$^{[40]}

$$\rho_s = \sum_{N=0}^{\infty} P_N(0) \frac{1}{Z_N} \sum_{n_0+n_1+\ldots+n_M=N} \nu_0^{n_0} \cdots \nu_M^{n_M}$$ \times |n_0, n_1, \ldots, n_M\rangle \langle n_0, n_1, \ldots, n_M|$$  \hspace{1cm} (4)$$

where $P_N(0)$ denotes the probability that there are $N$ polaritons in total in the lower polariton branch at the initial time, which is given by $P_N(0) = \sum_{n_0+n_1+\ldots+n_M=N} \text{Tr}[\rho(0)|n_0, n_1, \ldots, n_M\rangle \langle n_0, n_1, \ldots, n_M|]$, with $\sum_{n_j=n_0+n_1+\ldots+n_M=N} \nu_j^{n_j}$ the partition function described by

$$Z_N = \sum_{n_0+n_1+\ldots+n_M=N} \nu_0^{n_0} \cdots \nu_M^{n_M}$$  \hspace{1cm} (5)$$

The estimation uncertainty of the unbiased estimator $\hat{\delta}^2 T$ is bounded by the quantum Cramér–Rao lower bound as $\delta^2 T \geq 1/F_\gamma$, where $F_\gamma$ is the quantum Fisher information (QFI)$^{[49]}$ of the temperature $T$ in the steady state $\rho_s$, which is given by

$$F_\gamma = \sum_{N=0}^{\infty} \sum_{n_0+n_1+\ldots+n_M=N} \frac{(\partial_T P_N(n_0, n_1, \ldots, n_M))^2}{P_N(n_0, n_1, \ldots, n_M)}$$  \hspace{1cm} (6)$$

where $P_N(n_0, n_1, \ldots, n_M) = \frac{\rho_0^{n_0} \cdots \nu_M^{n_M}}{Z_N}$ denotes the probability of projection onto the state $|n_0, n_1, \ldots, n_M\rangle$ with $\sum_{n_j=n_0+n_1+\ldots+n_M=N} \nu_j^{n_j} = N$, and the shorthand notation $\partial_T = \frac{\partial}{\partial T}$. Hereafter, we set $h = \hbar = 1$ for convenience.

Without loss of generality, we first consider that the system is consisted of $M + 1$ states equidistant in frequency, that is, $\omega_j = \omega_0 + j\alpha/M$ with $\omega = \omega_M - \omega_0$.

### 2.1. Two Polariton States: $M = 1$

For $M = 1$, we can obtain the probability of the state $|n_0, n_1\rangle$ with $n_0 + n_1 = N$

$$P_{n_0, n_1} = \langle n_0, n_1 | \rho_s (M=1) | n_0, n_1 \rangle = \frac{\lambda^{n_1} (\lambda - 1)}{\lambda - \lambda^{-N}}$$  \hspace{1cm} (7)$$

where the factor $\lambda$ is defined as $\lambda = \exp(\alpha / T)$, and only one invariant subspace is considered, that is, $P_s(0) = 1$. By utilizing the formula in Equation (6), the QFI with $M = 1$ is analytically expressed as

$$F_\gamma = \frac{\omega^2}{T^4} \left[ \frac{\lambda}{(\lambda - 1)^2} - \frac{(1 + N)^2}{\lambda^{2+N} - 1} - \frac{(1 + N)^2}{(\lambda^{2+N} - 1)^2} \right]$$  \hspace{1cm} (8)$$

For $N \gg 1$, we can obtain the simplified form $F_\gamma \approx \frac{\omega^2}{T^2} \lambda^{2+N}$. This result is equivalent to measuring the temperature $T$ in the thermal equilibrium state of harmonic oscillator with the Hamiltonian $H_0 = \omega a^\dagger a$.$^{[28]}$ This does not reflect the advantage of having invariant subspaces. This is mainly due to the low number of states. Next, we investigate the QFI of the temperature with a large number of states, that is, $M > 1$.

### 2.2. Single Polariton: $N = 1$

We then consider another simple case, which can be analytically calculated for $N = 1$. In this case, the probability of projection onto the state $|0, 0, \ldots, n_j = 1, \ldots, 0\rangle$ is $P_{n_0, n_1, \ldots, n_M} = \frac{\lambda^{n_M} \nu_0^{n_0} \cdots \nu_M^{n_M}}{\sum_{n_0+n_1+\ldots+n_M=N} \nu_0^{n_0} \cdots \nu_M^{n_M}}$. The corresponding QFI is given by

$$F_\gamma = 2\omega^2 \lambda^{1+N} \times \frac{\text{M}(2 + M) - (1 + M)^2 \cos(\frac{\alpha}{M}) + \cos(1 + M) \frac{\alpha}{M^2}}{\left(\lambda^{1/M} - 1\right)^2 \left(\lambda^{1/M} + 1\right)^2}$$  \hspace{1cm} (9)$$

In the limit $M \to \infty$, the QFI of the temperature $T$ tends to be

$$F_\gamma|_{M=\infty} = \frac{1}{T^2} = \frac{\omega^2}{(\lambda - 1)^2 T^2}$$  \hspace{1cm} (10)$$

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where $\lambda = \exp(\omega / T)$. For $\omega \gg T$, we can obtain $\kappa T \approx 1 / T^2$, which is the maximal QFI of the temperature. According to the quantum Cramér-Rao lower bound, we obtain the Landau bound \[\omega T \] that is, $\delta T \geq T$. It means that we can obtain the infinite QFI for the temperature $T \rightarrow 0$, leading to that the optimal estimation precision $\delta T \approx 0$ is achieved. The result shows that we can perform very accurate low temperature measurement in the case of the invariant subspace with infinite states. As shown in Figure 1, the QFI increases with the value of $M$ for $\omega \gg T$. On the contrary, the QFI decreases with the value of $M$ at high temperatures. When $M$ is larger than a certain characteristic value $M_c$, the relationship between QFI and $M$ becomes increasingly independent. From Figure 1, we can see that the characteristic value $M_c$ decreases as the temperature $T$ increases. At low temperatures, the increasing number of polariton states $M$ promotes the estimation precision of temperature more obviously.

For the general case, that is, $N > 1$ and $M > 1$, there are no analytical solutions. It can be calculated numerically to investigate the effects of the total number of polaritons $N$ and the number of polariton states $M + 1$ on the QFI of the temperature $T$. As shown in Figure 2, with the increase of the polaritons $N$, the QFI of the temperature $T$ increases when the temperature $T$ above a certain value. However, the enhancement effect becomes less and less obvious as $N$ increases. This result is consistent with the previous analytical result of $M = 1$, which is independent of the total number of polaritons $N$ for large number $N \gg 1$. By contrast, increasing $M$ can significantly improve the QFI with $N = 4$, especially at low temperatures, as shown in Figure 3. This conclusion also supports the previous analytical results.

As a summary, one of our main results is that increasing the number of polariton states can greatly enhance the measurement precision of low temperature. However, the total number of polaritons $N$ plays a smaller role in enhancing the measurement precision of the temperature as $N$ increases.

3. The Polariton Dissipation and Incoherent Pumping

We consider that there is polariton dissipation in the polariton system, which is generally unavoidable in real quantum systems. We assume that the coupling between the polariton system and the environment is weak, leading to that the Born–Markov approximation \[\text{[50,51]}\] can be utilized. Therefore, the polariton dissipation can be described by the Lindblad superoperator

$$L_{\text{dis}}(\rho) = \sum_{j=0}^{M} \gamma_j \left( a_j \rho a_j^\dagger - \frac{1}{2} a_j^\dagger a_j \rho - \frac{1}{2} \rho a_j^\dagger a_j \right) \quad (11)$$

where $\gamma_j$ is the dissipation rate of the $j$th polariton state. Without extra pumping, the total number of the polaritons in the steady state will be 0 due to the dissipation, that is, the steady state is in the vacuum state $|0, 0, \ldots, 0\rangle$. The information of temperature cannot be obtained by the steady state due to the dissipation erases all information about the temperature. In order to get the information of the temperature, it is necessary to take the measurements in advance before the system reaches the steady state. In this work, we assume that the thermalization process is much faster than the dissipation process, that is, $\Gamma_\Theta (1 + \langle a \rangle_\Theta) \gg \gamma_j$. The density matrix in Equation (4) can be obtained approximately at time $\gamma_j^{-1} \gg t \gg \Gamma_\Theta^{-1} (1 + \langle a \rangle_\Theta)^{-1}$. In this case, the optimal in-
terrogation time should be much smaller than the characteristic time of the dissipation process and much larger than that of the thermalization process.

In order not to control the interrogation time, extra pumping is required to obtain a non-vacuum steady state. We consider that there is an incoherent pumping. The energy transfers from the upper branch and uncoupled excitons to the lower branch can be treated as an effective incoherent pumping, which can be described by the Lindblad master equation

$$L_{\text{pump}}(\rho) = \sum_{j=0}^{M} \kappa_j \left( a_j \rho a_j^\dagger - \frac{1}{2} a_j^\dagger a_j \rho - \frac{1}{2} \rho a_j^\dagger a_j \right) + \sum_{j=0}^{M} \kappa_j \left( a_j^\dagger \rho a_j - \frac{1}{2} \rho a_j^\dagger a_j - \frac{1}{2} a_j^\dagger a_j \rho \right)$$

(12)

where $\kappa_j$ is the pumping rate of the $j$th polariton. Including the dissipation and the incoherent pumping, the density matrix of the polaritons $\rho$ is dominated by the master equation

$$\frac{d}{dt} \rho = -i[H, \rho] + L_{\text{thermal}}(\rho) + L_{\text{diss}}(\rho) + L_{\text{pump}}(\rho)$$

(13)

In general, the above equation is difficult to be solved numerically and analytically. It can be approximately solved by assuming that the thermalization process is much faster than the dissipation and the incoherent pumping, that is, $\Gamma_{\text{th}}(1 + |a_j^\dagger a_j|) \gg \gamma_j, \kappa_j$.[52,53] The general expression for the density matrix in the steady state $\rho_s$ is also described by

$$\rho_s = \sum_{N=0}^{\infty} P_N \frac{1}{Z_N} \sum_{n_0+\cdots+n_M=N} v_{n_0}^0 \cdots v_{n_M}^M \chi|n_0, n_1, \ldots, n_M\rangle \langle n_0, n_1, \ldots, n_M|$$

(14)

where $P_N$ denotes the probability that there are $N$ polaritons in total in the steady state. In this case, $P_N$ is independent of the initial value $P_N(0)$ in Equation (4). Without loss of generality, we consider $\kappa_j = \kappa$ and $\gamma_j = \gamma$. For only two polariton states ($M = 1$), we can obtain the general form of the probabilities (see Appendix A for details)

$$P_N = \left( \frac{\kappa}{\kappa + \gamma} \right)^N (N + 1) P_0$$

(15)

$$P_0 = \frac{1}{1 + \frac{\kappa (N + 2)}{\gamma}}$$

(16)

In the steady state, the probability of the state $|n_0, n_1\rangle$ with $n_0 + n_1 = N$ is given by

$$P_{n_0, n_1} = P_N \lambda^{-N} (\lambda - 1)/(\lambda - \lambda^{-N})$$

(17)

where $n_1$ ranges from 0 to $N$. The corresponding QFI can be calculated by combing Equation (6) and Equation (17), and the results are directly shown in Figure 4. The QFI can increase with the pumping rate $\kappa$ for different temperatures, but not all the time. When the pumping rate $\kappa$ is larger than a certain value, the QFI of the temperature $T$ is independent of the pumping rate $\kappa$.

It is because that the number of polaritons $N$ is significantly increased by the strong enough pumping, that is, $\sum_{N=1}^{\infty} P_N \approx 1$. As shown in Equation (8), the QFI is independent of the number of polaritons $N$ in the case of $N \gg 1$. Therefore, too strong pumping is wasteful in improving the measurement precision of the temperature $T$. Only the appropriate strength of the incoherent pumping is required to resist the effects of dissipation.

4. M Degenerate States

In order to deal with the case of $N \gg 1$ and $M \gg 1$, we consider that there are $M$ degenerate states (modes), that is, $\omega_j = \omega_0 + \omega$ for $j \geq 1$. The general form of the partition function $Z_N^d$ in the case of $M$ degenerate states can be obtained analytically (see Appendix B for details)

$$Z_N^d = \frac{\lambda^M}{(\lambda - 1)^M} \left( \frac{M + N}{M} \right)^N \int \prod_{k=1}^{M+1} \left[ \frac{F_1(1, 1 + M + N, 2 + N; 1)}{(M - 1)! (N + 1)!} \right]$$

(18)

where $F_1(a, b, c; d) = \sum_{i=0}^{\infty} \frac{(a)_i (b)_i}{(c)_i} \frac{r^i}{i!}$ denotes the generalized hypergeometric function with $(r)_i = \Gamma(r + k)/\Gamma(r)$. The general form of the probability of the state $|n_0, \ldots, n_M\rangle$ with $n_0 + \cdots + n_M = N$ is given by

$$P_{d, n_0, \ldots, n_M} = \frac{Z_N^d}{Z_N} P_N^{d, n_0, \ldots, n_M}$$

(19)

where $P_N^{d, n_0, \ldots, n_M}$ denotes the probability that there are $N$ polaritons in total in the steady state: $P_N^{d, n_0, \ldots, n_M} = P_N(0)$ in the absence of the dissipation and the incoherent pumping; $P_N^d = P_N$ with the dissipation and the incoherent pumping. We note that there are $(N - n_0 + M - 1)!/(N - n_0)! (M - 1)!$ different states $|n_0, \ldots, n_M\rangle$ with the same probability $P_{d, n_0, \ldots, n_M}$ due to the $M$ degenerate modes. This simplifies the calculation somewhat. We numerically calculate the corresponding QFI $F_T^d$ with large $N$ and $M$.

As shown in Figure 5, the QFI increases with the number of states $M$ with the fixed number of the polaritons $N = 100$. 
that is, the number of polariton states, can better distinguish the low-temperature thermal steady states. It will be interesting to apply our mechanism to different systems beyond polariton BECs by looking for invariant subspaces with many modes during the thermalization.

When the polaritons suffer from the dissipation, the incoherent pumping can be used to enhance the estimation precision of the temperature. Due to that the QFI of the temperature becomes less and less dependent on the total number of the polaritons, too strong incoherent pumping is a waste of energy. Whether the periodic coherent modulation and nonlinearity can be used to enhance the estimation precision of low temperature deserves further study.

Fast thermalization can be realized in the state-of-the-art technique of BEC experiments, such as, in strongly coupled organic systems made of exciton polaritons.\textsuperscript{[43,52,53]}

Appendix A

We consider that there are the polariton dissipation and the incoherent pumping of lower polariton states, which are described by the Lindblad superoperators

\begin{equation}
L_{\text{diss}}(\rho) = \sum_{j=0}^{+\infty} \gamma_j (\rho a_j a_j^\dagger - \frac{1}{2} a_j^\dagger a_j \rho - \frac{1}{2} \rho a_j^\dagger a_j)
\end{equation}

\begin{equation}
L_{\text{pump}}(\rho) = \sum_{j=0}^{+\infty} \kappa_j (a_j^\dagger \rho a_j - \frac{1}{2} a_j^\dagger \rho a_j - \frac{1}{2} \rho a_j^\dagger a_j)
\end{equation}

where $\gamma_j$ $(\kappa_j)$ is the dissipation (the incoherent pumping) rate of the $j$th state. The general solution is difficult and we also consider that the thermalization process is very rapid, that is, $\Gamma_0 \gg t$ $\approx \Gamma_j \gg t$ $\Gamma_0^{-1}(1 + (a_j^\dagger a_j))^{-1}$. The density matrix $\rho$ obeys the approximate differential equation by ignoring the effect of the dissipation and the incoherent pumping.

\begin{equation}
\frac{d}{dt} \rho = -i[\mathcal{H}, \rho] + L_{\text{thermal}}(\rho)
\end{equation}

which leads to the general expression for the density matrix $\rho$ \textsuperscript{[40]}

\begin{equation}
\rho = \sum_{N=0}^{\infty} P_N(t) \frac{1}{N!} \sum_{n_0, n_1, \ldots, n_M} \delta_{n_0 N} v_{n_0} \cdots v_{n_M} \langle n_0, n_1, \ldots, n_M | \rho | n_0, n_1, \ldots, n_M \rangle
\end{equation}

where $P_N(t) \equiv P_N(0)$ for $\gamma_j^{-1}$, $\gamma_j^{-1} \gg t \gg \Gamma_0^{-1}(1 + (a_j^\dagger a_j))^{-1}$.

In the second stage, one can substitute the above equation into the whole evolution equation of the density matrix $\rho$, which is described as

\begin{equation}
\frac{d}{dt} \rho = -i[\mathcal{H}, \rho] + L_{\text{thermal}}(\rho) + L_{\text{diss}}(\rho) + L_{\text{pump}}(\rho)
\end{equation}

Substituting Equation (A4) into Equation (A5) and using the steady conditions $\frac{d}{dt} P_N(t \to \infty) = 0$, the steady-state solutions of the probabilities

5. Discussion and Conclusion

We have proposed a new mechanism to improve the measurement precision of low temperature and obtained the Landau bound $\delta T \approx T$. It is based on the existence of invariant subspaces during the polariton thermalization. The measurement precision of low temperature increases significantly with the number of polariton states. The intuitive explanation is that the final thermal steady state is not unique due to the existence of invariant subspaces. Increasing the dimension of the subspace, especially at low temperatures. It means that the number of polariton states $M$ is an important resource for enhancing the estimation precision of the temperature, which is consistent with the previous results by using the smaller $N$ and $M$ in the case of states equidistant in frequency.

As shown in Figure 6, in the case of large $M$, the line $N = 100$ is coincident with the line $N = 200$. It implies that the QFI of the temperature is independent of the number of the polaritons for large $N$ in the case of large $M$. The value of $M$ cannot change the relation between the QFI and the number of the polaritons $N$. 

![Diagram of the QFI](image1.png)

**Figure 5.** Diagram of the QFI $F_T$ as the function of the temperature $T$ with three different values of $M$: $M = 100, 200, 300$. The number of the polaritons is fixed, that is, $P_N^0 = 1$. Here, the dimensionless parameters are given by $\omega = 10$ and $N = 100$.

![Diagram of the QFI](image2.png)

**Figure 6.** Diagram of the QFI $F_T$ as the function of the temperature $T$ with three different numbers of the polaritons: $N = 10, 200, 300$. Here, the dimensionless parameters are given by $\omega = 10, P_N^0 = 1$, and $M = 100$. 

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\( P_N = P_N(t \to \infty) \) are achieved \(^{40}\)

\[
P_{N+1} = \frac{(d_{N+1} + \beta_N)Z_{N+1}}{d_N Z_N} P_N - \frac{\beta_{N+1} Z_{N+1}}{d_N Z_N} P_{N-1}
\]

(A6)

\[ P_1 = \frac{\beta_0 Z_1}{d_0 Z_0} P_0 \]

(A7)

where \( d_N = \sum_{n=0}^{N} \sum_{\nu=0}^{N-M} (1 + \kappa_\nu) \gamma^{n+1} Z_{N-n} \) and \( \beta_N = \sum_{n=0}^{N} \sum_{\nu=0}^{N-M} \kappa_\nu \gamma^n Z_{N-n} \).

Utilizing the above equations, we further obtain that

\[ P_N = \frac{\beta_{N+1} Z_{N+1}}{d_{N+1} Z_{N+1}} P_{N-1} \]

(A12)

\[ \frac{\prod_{n=0}^{N-1} \beta_n Z_n P_0}{\prod_{n=0}^{N-1} \beta_n Z_n} \]

(A13)

Without loss of generality, we consider \( \kappa_\nu = \kappa \) and \( \gamma_\nu = \gamma \). In the case of two states \( (N = 1) \), we can obtain the analytical results about the distribution

\[
d_N = \frac{\gamma^{(1+M)}(\gamma^{2N} - 1)}{(\gamma - 1)^N} (1 + N)(\kappa + \gamma)
\]

(A14)

\[
\beta_N = \frac{\gamma^{(1-N)}(\gamma^{1+M} - 1)}{(\gamma - 1)} (2 + N) \kappa
\]

(A15)

\[
Z_N = \sum_{n=0}^{N} \exp[-n \omega / T] = \frac{\lambda - \gamma^{N}}{\lambda - 1}
\]

(A16)

Substituting the above equations into Equation (A13), we can derive a simplified form of the probability \( P_N \), which is described as

\[ P_N = \left( \frac{\kappa}{\kappa + \gamma} \right)^N (N + 1) P_0 \]

(A17)

Due to the normalization condition \( \sum_{N=0}^{\infty} P_N = 1 \), we can obtain that

\[ P_N = \left( \frac{\kappa}{\kappa + \gamma} \right)^N (N + 1) P_0 \]

(A18)

\[ P_0 = \frac{1}{1 + \frac{\kappa + \gamma}{\gamma}} \]

(A19)

**Appendix B**

In the case of \( M \) degenerate states, \( \omega_j = \omega_0 + \omega_j \) for \( j \geq 1 \), the corresponding density matrix in the steady state \( \rho^d = \rho^d (t \to \infty) \) is also written as

\[ \rho^d = \sum_{N=0}^{\infty} \sum_{n_0=0}^{\infty} \sum_{n_1}^{n_0} \cdots \sum_{n_M}^{n_{M-1}} \rho \]

\[ \times |n_0, n_1, \ldots, n_M \rangle \langle n_0, n_1, \ldots, n_M| \]

(B1)

The partition function can be derived

\[ Z_N^d = \sum_{n_0 + \cdots + n_M = N} \frac{\lambda^{-n_1 - \cdots - n_M}}{N} \frac{\mu(n_0) \lambda^{n_0 - N}}{n_0}
\]

(B2)

\[ \frac{\lambda^N}{(\lambda - 1)^N} - \frac{M + N + 1}{(M - 1)(N + 1) \lambda^{1+N}}
\]

(B3)

where the degeneration coefficient \( \mu(n_0) = (N - n_0 + M - 1)! / ([N - n_0]! [M - 1]!) \) comes from the \( M \) degenerate modes, and \( F_r(a, b, c, d) = \sum_{0}^{\infty} \frac{(a)_r (b)_r}{(c)_r (d)_r} \) denotes the generalized hypergeometric function with \( (r)_a = \Gamma(r + 1) / \Gamma(r) \). The general form of the probability of the state \( |n_0, n_1, \ldots, n_M \rangle \) with the total number \( n_0 + n_1 + \cdots + n_M = N \) is given by

\[ p_{n_0, n_1, \ldots, n_M}^{d, N} = \frac{\mu(n_0)}{Z_N^d} \]

(B4)

where \( p_{n_0, n_1, \ldots, n_M}^{d, N} \) denotes the probability that there are \( N \) polaritons in total in the steady state: \( P_0^d = P_N(0) \) in the absence of the dissipation and the incoherent pumping; \( P_0^d = P_N \) with the dissipation and the incoherent pumping.

In the case of \( M \) degenerate states, the general formula of QFI \( F^2_{\rho} \) can be described as

\[ F^2_{\rho} = \sum_{N=0}^{\infty} \sum_{n_0=0}^{\infty} \cdots \sum_{n_M}^{\infty} \frac{(d_T p_{n_0, n_1, \ldots, n_M})^2}{p_{n_0, n_1, \ldots, n_M}^{d, N}}
\]

(B5)

\[ = \sum_{N=0}^{\infty} \sum_{n_0=0}^{\infty} \cdots \sum_{n_M}^{\infty} \mu(n_0) \frac{(d_T p_{n_0, n_1, \ldots, n_M})^2}{Z_N^d}
\]

(B6)

\[ = \sum_{N=0}^{\infty} \sum_{n_0=0}^{\infty} \cdots \sum_{n_M}^{\infty} \mu(N - n_0) \frac{(d_T p_{n_0, n_1, \ldots, n_M})^2}{Z_N^d}
\]

(B7)

For the fixed number of the polaritons, that is, \( P_0^d = 1 \), the QFI can be calculated by

\[ F^2_{\rho} = \sum_{n_0=0}^{\infty} \mu(N - n_0) \frac{(d_T p_{n_0})^2}{Z_N^d}
\]

(B8)

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Conflict of Interest
The authors declare no conflict of interest.

Data Availability Statement
The data that support the findings of this study are available from the corresponding author upon reasonable request.

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