TWO-VALUED $\sigma$-MAXITIVE MEASURES
AND MESIAR’S HYPOTHESIS

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ABSTRACT. We reformulate Mesiar’s hypothesis [Possibility measures, integration and fuzzy possibility measures, Fuzzy Sets and Systems 92 (1997) 191196], which as such was shown to be untrue by Murofushi [Two-valued possibility measures induced by $\sigma$-finite $\sigma$-additive measures, Fuzzy Sets and Systems 126 (2002) 265268]. We prove that a two-valued $\sigma$-maxitive measure can be induced by a $\sigma$-additive measure under the additional condition that it is $\sigma$-principal.

1. INTRODUCTION

In the sequel, $(E, B)$ denotes a measurable space. Recall that a $\sigma$-maxitive measure on $B$ is a map $\tau : B \to \mathbb{R}_+$ such that $\tau(\emptyset) = 0$ and

$$\tau\left(\bigcup_{j \in \mathbb{N}} B_j\right) = \sup_{j \in \mathbb{N}} \tau(B_j),$$

for every sequence $(B_j)_{j \in \mathbb{N}}$ of elements of $\mathcal{B}$. A $\sigma$-maxitive measure $\tau$ on $\mathcal{B}$ is normed if $\tau(E) = 1$; it is two-valued if $\tau(\mathcal{B}) = \{0, 1\}$, in which case it is normed. A family $\mathcal{C}$ of sets is said to satisfy CCC (the countable chain condition) if every disjoint subfamily of $\mathcal{C}$ is countable. A measure (either $\sigma$-maxitive or $\sigma$-additive) $\mu$ on $\mathcal{B}$ is then CCC if $\{B \in \mathcal{B} : \mu(B) > 0\}$ satisfies CCC.

The following hypothesis was proposed by Mesiar [2].

**Hypothesis 1.1.** Let $\tau$ be a CCC (“countable chain condition”) two-valued $\sigma$-maxitive measure on $\mathcal{B}$. Then $\tau$ is induced by some $\sigma$-finite $\sigma$-additive measure $m$ on $\mathcal{B}$, i.e. $\tau = \delta_m$, where $\delta_m(B) = 1$ if $m(B) > 0$ and $\delta_m(B) = 0$ otherwise.

Murofushi [3] provided a counterexample and hence showed that this hypothesis as such is wrong. He focused on finding a necessary and sufficient condition for such a $\tau$ to be induced by some $\sigma$-finite $\sigma$-additive measure.
In this article we propose to give up the constraint “σ-finite” and to replace it by a more adequate condition, namely the σ-principality property.

**Definition 1.2.** A measure (either σ-maxitive or σ-additive) \( \mu \) on \( \mathcal{B} \) is σ-principal if for each σ-ideal \( \mathcal{I} \) of \( \mathcal{B} \), there exists some \( L \in \mathcal{I} \) such that \( \mu(S \setminus L) = 0 \) for all \( S \in \mathcal{I} \).

**Example 1.3.** Let \( E \) be a set endowed with its power set \( (\mathcal{B} = 2^E) \). The σ-additive measure \( \# : \mathcal{B} \to \mathbb{N} \), where \( \#B \) is the cardinality of \( B \), is σ-principal if and only if \( E \) is countable.

A σ-principal measure is always CCC. See Sugeno and Murofushi [7] for a proof of the converse statement using Zorn’s Lemma. Note also that every finite (or σ-finite) σ-additive measure is σ-principal.

2. **Modified Mesiar’s Hypothesis**

Mesiar [2] noted that, if his hypothesis were true, then every CCC σ-maxitive measure could be represented as an essential supremum with respect to a σ-additive measure. We show first that such a representation holds, then prove a modified version of Mesiar’s hypothesis.

**Theorem 2.1.** Any σ-principal (resp. CCC) σ-maxitive measure can be expressed as an essential supremum with respect to a σ-principal (resp. CCC) σ-additive measure.

**Proof.** Let \( \tau \) be a σ-principal σ-maxitive measure on \( \mathcal{B} \) and \( m = \bar{\tau} \) be the map defined on \( \mathcal{B} \) by

\[
m(B) = \sup_{\pi} \sum_{B' \in \pi} \tau(B \cap B'),
\]

where the supremum is taken over the set of finite \( \mathcal{B} \)-partitions \( \pi \) of \( E \). It is not difficult to show that \( m \), called the disjoint variation of \( \tau \), is the least σ-additive measure greater than \( \tau \) (see e.g. [4, Theorem 3.2]). Let us show that \( m \) is σ-principal. If \( \mathcal{I} \) is a σ-ideal of \( \mathcal{B} \), there exists some \( L \in \mathcal{I} \) such that \( \tau(B \setminus L) = 0 \) for all \( B \in \mathcal{I} \) (because \( \tau \) is σ-principal). If \( B \in \mathcal{I} \), then \( \tau(B \cap B' \setminus L) = 0 \) for all \( B' \in \mathcal{B} \), since \( B \cap B' \in \mathcal{I} \). Hence we have \( m(B \setminus L) = 0 \). Moreover, \( m(B) > 0 \) implies \( \tau(B) > 0 \), so that \( m \) is CCC if \( \tau \) is CCC. With the Sugeno–Murofushi theorem (see a reminder in the appendix, Theorem A.1) and the fact that \( \tau \) is absolutely continuous with respect to \( \delta_m \), one can write \( \tau(B) = \int \! c \, d\delta_m = m \sup_{x \in B} c(x) \), where \( c : E \to \mathbb{R}_+ \) is a \( \mathcal{B} \)-measurable map. \( \square \)

**Corollary 2.2.** A two-valued σ-maxitive measure is σ-principal (resp. CCC) if and only if it is induced by a σ-principal (resp. CCC) σ-additive measure.
Proof. Let $\tau$ be a $\sigma$-principal $\sigma$-maxitive measure on $\mathcal{B}$. The above construction of $m$ shows that $\tau(B) > 0 \iff m(B) > 0$, which implies that $\tau = \delta_m$ if $\tau$ is two-valued. \hfill \Box

APPENDIX A.

Sugeno and Murofushi \cite{7} proved a Radon–Nikodym like theorem for the Shilkret integral in the case where the dominating $\sigma$-maxitive measure is CCC. Actually their proof remains valid if one just assumes $\sigma$-principality (this is straightforward since they showed under Zorn’s Lemma that every CCC measure is $\sigma$-principal).

**Theorem A.1** (Sugeno–Murofushi). Let $\tau, \nu$ be $\sigma$-maxitive measures on $\mathcal{B}$. Assume that $\nu$ is $\sigma$-finite and $\sigma$-principal. The following are equivalent:

1. $\tau$ is absolutely continuous with respect to $\nu$, i.e. $\nu(B) = 0$ implies $\tau(B) = 0$, for all $B \in \mathcal{B}$,
2. there exists some $\mathcal{B}$-measurable map $c : E \to \mathbb{R}_+$ such that, for all $B \in \mathcal{B}$,

$$
\tau(B) = \int_B c \, d\nu.
$$

If (1) or (2) holds, then $c$ is unique $\nu$-almost everywhere.

Here $\int_B^\infty c \, d\nu := \sup_{t \in \mathbb{R}_+} t \nu(B \cap \{c > t\})$ denotes the Shilkret integral, see Shilkret \cite{6}; see also Poncet \cite{5} Chapter I. The superscript $\infty$ in the notation of the Shilkret integral finds its justification in Gerritse \cite{1}, who stated that the Shilkret integral can be viewed as a limit of Choquet integrals.

REFERENCES

[1] Bart Gerritse. Varadhan’s theorem for capacities. Comment. Math. Univ. Carolin., 37(4):667–690, 1996.
[2] Radko Mesiar. Possibility measures, integration and fuzzy possibility measures. Fuzzy Sets and Systems, 92(2):191–196, 1997.
[3] Toshiaki Murofushi. Two-valued possibility measures induced by $\sigma$-finite $\sigma$-additive measures. Fuzzy Sets and Systems, 126(2):265–268, 2002.
[4] Endre Pap. Null-additive set functions, volume 337 of Mathematics and its Applications. Kluwer Academic Publishers Group, Dordrecht, 1995.
[5] Paul Poncet. Infinite-dimensional idempotent analysis: the role of continuous posets. PhD thesis, École Polytechnique, Palaiseau, France, 2011.
[6] Niel Shilkret. Maxitive measure and integration. Nederl. Akad. Wetensch. Proc. Ser. A 74 = Indag. Math., 33:109–116, 1971.
[7] Michio Sugeno and Toshiaki Murofushi. Pseudo-additive measures and integrals. J. Math. Anal. Appl., 122(1):197–222, 1987.

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