Quantum walks can unitarily match random walks on finite graphs

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Abstract

Quantum and random walks were proven to be equivalent on finite graphs by demonstrating how to construct a time-dependent random walk sharing the exact same evolution of vertex probability of any given discrete-time coined quantum walk. Such equivalence stipulated a deep connection between the processes that is far stronger than simply considering quantum walks as quantum analogues of random walks. This article expands on the connection between quantum and random walks by demonstrating a procedure that constructs a time-dependent quantum walk matching the evolution of vertex probability of any given random walk in a unitary way. It is a trivial fact that a quantum walk measured at all time steps of its evolution degrades to a random walk. More interestingly, the method presented describes a quantum walk that matches a random walk without measurement operations, such that the unitary evolution of the quantum walk captures the probability evolution of the random walk. The construction procedure is general, covering both homogeneous and non-homogeneous random walks. For the homogeneous random walk case, the properties of unitary evolution imply that the quantum walk described is time-dependent since homogeneous quantum walks do not converge for arbitrary initial conditions.

1 Introduction

Quantum walks on graphs are one of the few known design techniques for quantum algorithms. They led descriptions of quantum algorithms with interesting speed ups in applications like searching [3, 10, 9], Monte Carlo methods [12], claw finding [17] and backtracking algorithms [11]. Albeit its simple description, quantum walks are extremely powerful since they are universal for quantum computing [4, 6, 8]. Fundamentally, there are many distinct ways of defining quantum walks and this work addresses discrete-time coined quantum walks on graphs [2, 7, 15, 16]. In an intuitive way, quantum walks on graphs extend the definition of classical random walks to unitary processes on a Hilbert space codifying the edges of a graph.

Recent work has shown that, regardless of the disparities between quantum and random walks, it is possible to construct a time-dependent random walk with the exact same evolution of vertex probability of any given discrete-time coined quantum walk [3]. When the opposite direction is considered, it is a trivial result that measuring a quantum walk at each time instant recovers the behavior of a random walk. In particular, it can be shown that the application of a quantum walk operator on the system state after a measurement yields a probability evolution that is compatible with the application of a stochastic matrix. This behavior comes from the destruction of quantum
interference caused by the collapse of the wavefunction after measurement \cite{13}. As an example, it is well known that the Hadamard walk on a cycle collapses to the usual uniform random walk on the cycle, i.e. the random walk that has a probability of 0.5 to move to the left and to the right, if the walker state is probed at every instant.

Nonetheless, it is interesting to describe quantum walk operators capable of matching random walks such that the walker system does not have to be measured consistently, intrinsically embedding the probabilities of a random walk in the unitary evolution of a quantum walk. Establishing the equivalence between the coherent evolution of quantum walks and the local evolution of random walks contributes to indicate a fundamental connection between the two processes, which was initially demonstrated in \cite{3}. In this context, the main contribution of this work is to describe a construction procedure that maps any given random walk to a unitary discrete-time coined quantum walk on the same graph. The existence of such a construction procedure establishes that the initial divergences between quantum and random walks are not leveraged by unitarity alone, depending also on the time-homogeneity of both processes \cite{1}.

The remainder of this article is structured as follows. In section 2, a description of quantum and random walks is given. The fundamental result is demonstrated in Section 3. The article is concluded in Section 4.

2 Background

Throughout this work, we consider a graph \( G = (V, E) \) to be the directed version of an undirected graph, such that, for an edge \((v, u) \in E, (u, v) \in E\). In addition, we refer to the set of neighbors of a vertex \( v \) as \( N(v) \) and the degree of \( v \) as \( \delta(v) = |N(v)| \). Essentially, the notation adopted follows the one defined in \cite{3}.

It is possible to define a random walk on a graph \( G \) as a probability distribution over the vertices of the graph that varies on time depending on the connectivity expressed by its edges, codifying vertex position as a random variable. Consider a probability vector \( \pi(t) \in \mathbb{R}^{|V|} \), such that \( \pi_v(t) \) denotes the probability of such random variable to be vertex \( v \) at time \( t \). A random walk is any process described by the equation

\[
\pi(t + 1) = P(t)\pi(t),
\]

where \( P(t) \) is a stochastic matrix containing transition probabilities between vertices that respects the adjacency matrix of \( G \), i.e. the entry \( p_{vu}(t) \) is the probability to move from \( u \) to \( v \) at instant \( t \) and \( p_{vu} > 0 \) only if \((u, v) \in E\). Systems described by Equation \ref{eq:1} are said to be Markovian since the probabilities at a given instant \( t \) are completely determined by the probabilities at \( t - 1 \).

Precisely, the evolution of probability for a particular vertex \( v \) at instant \( t \) in a random walk is a linear combination of the probabilities for the neighbors of \( v \) at \( t - 1 \) following the equations

\[
\pi_v(t) = \sum_{u \in N(v)} p_{vu}(t - 1)\pi_u(t - 1)
\]

\[
\sum_{v \in N(u)} p_{vu}(t) = 1, \text{ for every } u \in V,
\]

implying that \( P(t) \) is column stochastic, i.e. its columns sum to one for every instant \( t \).

A discrete-time coined quantum walk on a graph \( G \) defines the evolution of a unit vector in a Hilbert space \( \mathcal{H}_v \) that codifies the edges of \( G \) \cite{14}. Let \( \mathcal{H}_v \) and \( \mathcal{H}_c \) denote Hilbert spaces with dimension \( |V| \) and \( d_{\text{max}} = \max_v d(v) \), respectively. The space \( \mathcal{H}_w \subseteq \mathcal{H}_w \otimes \mathcal{H}_c \) is spanned by unit vectors \( |u, c\rangle \), where \( u \in V \) and \( c \in \{0, \ldots, d(u) - 1\} \), that can be mapped to edges of the graph through a function \( \eta : V \times C \to V \), with \( C = \{0, \ldots, d_{\text{max}} - 1\} \). The space \( \mathcal{H}_w \) is the vertex space of the walker system, codifying elements of \( V \) as basis states, while \( \mathcal{H}_c \) is the coin space of the
walker, codifying the degrees of freedom for the walker’s movement. Essentially, the wavefunction of the quantum walker at a given time step is a superposition of edges of the graph, having form given by the equation

$$|\Psi(t)\rangle = \sum_{u \in V, c \in C_u} \psi(u, c) |u, c\rangle,$$  \hspace{1cm} (4)

where $C_u = \{0, ..., d(u) - 1\}$ is the set of degrees of freedom of vertex $u$.

The evolution of the walker state at discrete time instant $t$ is performed by two time-dependent unitary operators $S(t) : \mathcal{H}_w \rightarrow \mathcal{H}_w$ and $W(t) : \mathcal{H}_w \rightarrow \mathcal{H}_w$ on the system state vector as

$$|\Psi(t+1)\rangle = SW|\Psi(t)\rangle.$$  \hspace{1cm} (5)

$S(t)$ is known as the shift operator and performs a permutation between the edges of the graph that is only allowed to map a given edge $(u, v)$ to an edge $(v, w)$. $W(t)$ is named the coin operator, acting exclusively on $\mathcal{H}_C$ by mixing the wavefunction incident to a given edge $(u, v)$ to edges $(u, w)$. Formally, a generic coin operator is defined as

$$W = \sum_{v \in V} |v\rangle \langle v| \otimes W_v,$$  \hspace{1cm} (6)

where $W_v : \mathcal{H}_{C_v} \rightarrow \mathcal{H}_{C_v}$ is a unitary operator, where $\mathcal{H}_{C_v} \subseteq \mathcal{H}_C$ is the Hilbert space codifying the degrees of freedom of $v$. In order to define a generic shift operator precisely, it is necessary to define the function $\eta$ used to map edges to states, as well as two auxiliary functions $\sigma : V \times V \rightarrow C$ and $\sigma^{-1} : V \times V \rightarrow C$ that respectively map an inward edge of a vertex to an outward edge of vertex and an outward edge of a vertex to its inward correspondent, that are all depicted in Figure 1. Thus, consider the meaning of $\eta(v, c) = u$ to be that $u$ is the $c$-th neighbor of $v$; $\sigma(u, v) = c$ meaning that edge $(u, v)$ is mapped to the state $|v, c\rangle$; and that $\sigma^{-1}(u, v) = c$ meaning that state $|w, c\rangle$ is mapped to $(u, v)$, whoever vertex $w \in N(u)$ satisfies $\eta(w, u) = c$. Then, the action of the shift operator is expressed as

$$|v, c\rangle \rightarrow |\eta(v, c), \sigma(v, \eta(v, c))\rangle.$$  \hspace{1cm} (7)

The probability of finding the quantum walker system at a particular state in a given time instant $t$ is codified by the function

$$\rho(u, c, t) = |\psi(u, c, t)|^2,$$  \hspace{1cm} (8)

which is extended to the probability of finding the walker in a particular vertex by summing on the degrees of freedom, yielding the function

$$\mu(u, t) = \sum_{c \in C_u} \rho(u, c, t).$$  \hspace{1cm} (9)

3 Quantum walks as non-homogeneous random walks

The unitarity of the quantum walk operators was central to guide the construction of the time-dependent random walk matching a given quantum walk [3]. Essentially, unitarity also has fundamental implications that are to be explored in order to design quantum walks capable of coherently matching random walks. Since unitary quantum walks are considered, the evolution of the walker system is performed by successive rotations on a unitary vector in $\mathcal{H}_W$ with dimension $|E|$ that can be written as product of a coin and shift operator. For random walks, the vector describing the
Figure 1: Visual depiction of auxiliary functions $\eta$ and $\sigma$. $\eta$ gives an ordering for the neighbors of $v$ such that, in this case, $\eta(v,c) = u$. $\sigma$ maps the state $|v,c\rangle$ (edge $(v,u)$) to the state $|u,\sigma(v,u)\rangle$ (edge $(u,u')$). The inverse association $\sigma^{-1}$ connects the state $|u,\sigma(v,u)\rangle$ (edge $(u,u')$) with state $|v,c\rangle$ (edge $(v,u)$).

The system lies on the positive simplex of dimension $|V|$, which is closed under applications of stochastic matrices. Thus, the task at hand is two-folded: to represent the probability vectors of random walks as state vectors of $\mathcal{H}_W$ by creating a map between $\mathbb{R}^{|V|}$ and $\mathcal{H}_W$; and to map the application of a generic stochastic matrix $P(t) : \mathbb{R}^{|V|} \to \mathbb{R}^{|V|}$ to a unitary operator $S(t)W(t) : \mathcal{H}_W \to \mathcal{H}_W$.

Since $\mathcal{H}_W$ is a complex space and $|E| \leq |V|$, there are infinitely many mappings that serve the task at hand. Precisely, let $\pi : \mathbb{N} \to [0,1]^{|V|}$ denote the probability vector of a random walk, at instant $t$, as defined in Section 2. Any state vector $|\Psi(t)\rangle$ for which

$$\sum_{u \in V, c \in C_u} |\langle u,c |\Psi(t)\rangle|^2 = \pi_v(t) \quad (10)$$

is a proper quantum state that represents $\pi(t)$, and thus mimic the evolution of the random walk. The infinitely many possibilities to choose a state for which Equation (10) holds entail that the representation of the random walk state is a matter of choice.

Furthermore, it is intuitive that there exists a time-dependent unitary operator $Q(t) : \mathcal{H}_W \to \mathcal{H}_W$ capable of matching the time evolution of states respecting Equation (10) such that

$$|\Psi(t + 1)\rangle = Q(t)|\Psi(t)\rangle, \quad (11)$$

for every instant $t$. From the geometric perspective, unitary evolution is nothing more than a self-mapping of the complex unit hyper-sphere. Unitary operators are norm preserving, performing rotations on the vectors of a given Hilbert space. Since all state vectors compliant with Equation (10) are unitary, there has to exist at least one unitary transformation $Q(t)$ satisfying Equation (11).

Following this argument, it is necessary to show that there exists an operator satisfying

$$Q(t) = S(t)W(t), \quad (12)$$

for every instant $t$, where $S$ and $W$ respect the graph under consideration.

### 3.1 Representation of random walks as quantum walker systems

Initially, it is necessary to define $|\Psi(t)\rangle$. A good representation for the system state is one that simplifies the search for the operators $S(t)$ and $W(t)$. To give intuition on the state representation chosen, consider that the system state is

$$|\Psi(t)\rangle = \sum_{u \in V, c \in C_u} g(u,c,t) e^{i\theta(u,c,t)} \sqrt{\pi_u(t)} |u,c\rangle, \quad (13)$$

where $\theta(u,c,t)$ is an arbitrary complex phase and $g$ respects

$$\sum_{c \in C_u} g(u,c,t)^2 = 1 \text{ for every } u \in V. \quad (14)$$
The compliance of the equations above with Equation (10) is a direct consequence of the definition of \( \mu \) (Equation 9). Equation (13) implies that the value of \( \rho(u, c, t) = g(u, c, t)^2 \pi_u(t) \), while Equation (13) assures that the sum over all degrees of freedom of \( u \) yields \( \mu(u, t) = \pi_u(t) \).

The diffusion behavior of the quantum walk leads to the definition of \( g \) and \( \theta \). Note that \( W \) acts by mixing the wavefunction among the edges of a vertex and \( S \) creates its flow. Consider a particular instant \( t \) of the quantum walk with \( W(t) = I \) (the identity matrix), such that the mixing behavior is “turned off” for \( t \) and \( Q(t) = S(t) \) acts only by creating the flow of the wavefunction. Assuming that \( S(t) \) is any valid shift operator and \( \eta(u, c) = \nu \), an inspection of Equation (2) indicates that a natural choice for the function \( g(u, c, t) \) is

\[
g(u, c, t) = \sqrt{\rho_{vu}(t)}.
\]  

(15)

This choice satisfies Equation (14) while simultaneously implying that \( \mu(u, t) = \pi_u(t) \) and \( \mu(u, t + 1) = \pi_u(t + 1) \) for all \( u \in V \). The first two properties stem directly from the law of total probability. The condition for \( t + 1 \) comes from the fact that, regardless of the functions \( \eta \) and \( \sigma \) chosen to define \( S(t) \), all states \( |v, c\rangle \) have an incident wavefunction that yields a proper proportion of the probabilities of the neighbors of \( v \) for every vertex \( v \in V \) at instant \( t \), such that

\[
\mu(v, t + 1) = \sum_{u \in N^-(v)} p_{vu}(t) \pi_u(t).
\]  

(16)

The given representation is powerful because it is a valid unitary representation that describes the operator \( Q(t) \) as a proper quantum walk operator in the particular scenario considered. This representation gives the intuition for a state representation compatible with the process for every \( t \). If the result of \( W(t) |\Psi(t)\rangle \) is given by the right-hand side of Equation (13) for all instant \( t \), \( Q(t) \) is properly decomposed into \( S(t)W(t) \) for any valid shift operator.

The key aspect is to define the state of the system based on the probabilities of instant \( t - 1 \), instead of using the probabilities of instant \( t \). It is known from the random walk description that the probability of a vertex at instant \( t \) is a linear combination of the probability of its neighbors at instant \( t - 1 \). Thus, the following Lemma inspired by Equation (2) formalizes the state representation of choice.

**Lemma 1** (Quantum representation of random walks). Let \( P(t) \) be a stochastic matrix that defines a random walk on \( G \) such that \( \pi(t + 1) = P(t)\pi(t) \). Let \( \sigma^{-1} : V \times V \to C \) be any function that associates an inward edge of a vertex to one of its outward edges defining a valid shift operator for \( G \). For \( t > 0 \), The probability vector \( \pi(t) \) can be represented by a discrete-time coined quantum walk state

\[
|\Psi(t)\rangle = \sum_{v \in V, c \in C_v} e^{i\theta(u, \sigma^{-1}(u,v), t-1)} \sqrt{p_{vu}(t - 1)} \sqrt{\pi_u(t - 1)} |v, c\rangle.
\]  

(17)

defined on \( G \), such that \( \theta(u, c, t) \) is an arbitrary complex phase, for all \( u \in V, c \in C_u \), \( t \in \{1, 2, \ldots\} \), and with

\[
|\Psi(0)\rangle = \sum_{v \in V, c \in C_v} \sqrt{\pi_v(0)} \frac{1}{d(v)} |v, c\rangle.
\]  

(18)

**Proof.** The proof for \( t = 0 \) is trivial. Hence, it suffices to show that, for \( t > 0 \), \( |\Psi(t)\rangle \) is a unitary vector and that \( \mu(v, t) = \pi_v(t) \), for all \( v \in V \). The measurement of state \( |v, c\rangle \) yields that

\[
\rho(v, c, t) = e^{i\theta(u, \sigma^{-1}(u,v), t-1)} \sqrt{p_{vu}(t - 1)} \sqrt{\pi_u(t - 1)} |v, c\rangle |v, c\rangle^\dagger.
\]  

(19)
Since $p_{vu}(t)$ and $\pi_u(t)$ are positive reals, $\rho(v, c, t) = p_{vu}(t)\pi_u(t)$. The definition of $\mu(v, t)$ gives

$$
\mu(v, t) = \sum_{u \in N^+(v)} p_{vu}(t-1)\pi_u(t-1)
$$

(20)

Equation (20) implies that $\mu(v, t) = \pi_u(t)$, for all $v \in V$. Since $||\Psi(t)|| = \sum_{v \in V} \mu(v, t)$, $|\Psi(t)|$ is clearly unitary and the claim is proved.

3.2 The complete description of time-evolution

From Lemma [1] any valid shift operator for $G$ can be used to determine the edge maps $\sigma$ and $\sigma^{-1}$ to represent the random walk state. In order to simplify both the analysis and the notation used, consider the following shift operator $S_{RW}$, defined in terms of its auxiliary functions (see Section [2]). Let $\eta : V \times C \rightarrow V$ be defined such that the $c$-th neighbor of $v$ is the neighbor of $v$ with the $c$-th smallest label. Formally, for all $u \in V$, all $c, c' \in C_v, c \neq c'$, it holds

$$
\eta(u, c) < \eta(u, c') \iff c < c'.
$$

Note that, for each $(u, v) \in E$, there exists a pair $c \in C_u, c' \in C_v$ such that $\eta(u, c) = v$ and $\eta(v, c') = u$. Thus, let $\sigma(u, v) = c'$ and $\sigma(v, u) = c$. Furthermore, let $\sigma^{-1}(u, v) = \sigma(v, u)$ and $\sigma^{-1}(v, u) = \sigma(u, v)$. Precisely, $S_{RW}$ is the flip-flop shift operator that maps $(u, v)$ to $(v, u)$ and is well-defined for any graph $G$ of interest. The definitions of $\eta$ and $\sigma$ for $S_{RW}$ will be used throughout this section.

The $S_{RW}$ operator yields describing the state of a vertex $v$ at time $t$ as the vector

$$
|\Psi(v, t)\rangle = \sum_{u \in N^+} e^{i\theta(u, c, t-1)}\sqrt{p_{vu}(t-1)}\sqrt{\pi_u(t-1)}|v, c\rangle,
$$

(21)

where the dependency of $c$ and $c'$ on $u$ and $v$ is omitted, i.e $\eta(u, c) = v$ and $\eta(v, c') = u$. From the analysis of the state representation on the previous section, it is enough to ensure that, for every instant $t$, the action of $W(t)$ maps

$$
|\Psi(v, t)\rangle \rightarrow |\Phi(v, t)\rangle,
$$

(22)

where

$$
|\Phi(v, t)\rangle = \sum_{c' \in C_v} e^{i\theta(v, c', t)}\sqrt{p_{vv}(t)}\sqrt{\pi_v(t)}|v, c\rangle.
$$

(23)

The definition of the coin operator (Equation [0]) implies that each vertex $v$ has its own independent unitary mixing behavior $W_v$. In addition, it is a well known result from linear algebra that any operator that changes orthonormal basis is unitary. Thus, the following Lemmas respectively provide formal constructions for a set of linearly independent vectors on the coin subspace of a vertex and the coin operator $W(t)$ itself.

**Lemma 2** (Linear independent set construction). Let $\mathcal{H}_{w,d(v)} \subset \mathcal{H}_W$ be the subspace that represents the coin space $\mathcal{H}_{d(v)}$ of a vertex $v$. Let $|a\rangle \in \mathcal{H}_{w,d(v)}$ be any vector in the subspace. Let $\beta$ be the basis of edges $\{|u, c\rangle\}$ for $\mathcal{H}_{w,d(v)}$. The set

$$
A = \{|a\rangle\} \cup \zeta(a, v) \cup B
$$

(24)

is linearly independent, where $\zeta(a, v) = \{|v, c\rangle : \langle a|v, c\rangle = 0\}$ and $B \subset \beta \setminus \zeta(a, v)$ is any subset of $\beta \setminus \zeta(a, v)$ with cardinality $|B| = |\beta \setminus \zeta(a, v)| - 1$. 

6
Proof. It is clear that all vectors from $A \setminus \{a\}$ are orthogonal, since they are a subset of the basis $\beta$. In the case where $\zeta(a,v) = \emptyset$, $\langle a|v,c\rangle \neq 0$ for all $|v,c\rangle \in \beta$, what implies that exists a $c'$ such that $\langle a|v,c'\rangle > 0$ while $\langle v,c|v,c'\rangle = 0$ for all $|v,c\rangle \in B$. Hence, it is impossible to write $|a\rangle$ as a linear combination of vectors in $B$, and $A$ is a set of linearly independent vectors.

In the case where $\zeta(a,v) \neq \emptyset$ the construction of $A$ implies that the condition of the existence of $c'$ holds because the vectors from $\zeta(a,v)$ are orthogonal to $|a\rangle$ and exactly one of the vectors of the set $\beta \setminus B$ is not a member of $B \cup \zeta(a,v)$. $\square$

**Lemma 3 (Coin operators for random walks).** Let $\mathcal{H}_{d(v)}$ denote the Hilbert space defined by the degrees of freedom of a vertex $v$. Let the sets of vectors $\alpha$ and $\beta$ be two orthonormal basis for $\mathcal{H}_{d(v)}$, where $\alpha_k$ and $\beta_k$ are, respectively, the $k$-th vectors of $\alpha$ and $\beta$. Let

$$
\alpha_0 = \frac{1}{\sqrt{\langle \Phi(v,t)|\Phi(v,t) \rangle}} |\Phi(v,t)\rangle ,
$$

$$
\beta_0 = \frac{1}{\sqrt{\langle \Psi(v,t)|\Psi(v,t) \rangle}} |\Phi(v,t)\rangle ,
$$

with $\Psi(v,t)$ and $\Phi(v,t)$ given by Equations (21) and (22) respectively. The operator

$$
W_v(t) = \sum_{k=0}^{d(v)-1} |\alpha_k\rangle \langle \beta_k| ,
$$

is unitary, inducing a unitary operator $W(t) = \sum_{v \in V} |v\rangle \langle v| \bigotimes W_v(t)$ on $\mathcal{H}_w$.

Proof. Note that $\langle \Psi(u,t)|\Psi(u,t) \rangle = \langle \Phi(u,t)|\Phi(u,t) \rangle$. It follows trivially from the completeness relation that

$$
W_v^\dagger(t)W_v(t) = W_v(t)W_v^\dagger(t) = I,
$$

and $W(t)$ is unitary. $\square$

Finally, the following Theorem states that, for any given random walk, a statistically equivalent quantum walk in terms of vertex probabilities can be constructed assuming time-dependent coin operators.

**Theorem 1.** Let $P(t)$ be a stochastic matrix that defines the evolution of a random walk on a graph $G$, such that, for all $t$, $\pi(t+1) = P(t)\pi(t)$. For every instant $t$, the quantum walk with state $|\Psi(t)\rangle$ given by Lemma 2 with fixed shift operator $S(t) = S_{RW}$ and coin operator $W(t)$ given by Lemma 3 evolves according to

$$
|\Psi(t+1)\rangle = S_{RW}W(t)|\Psi(t)\rangle ,
$$

such that $\mu(u,t) = \pi_u(t)$ and $\mu(u,t+1) = \pi_u(t+1)$ for all $u \in V$.

Proof. For every $u \in V$, the conditions for $\mu(u,t) = \pi_u(t)$ and $\mu(u,t+1) = \pi_u(t+1)$ are ensured by Lemma 2. As a valid shift operator, $S_{RW}$ is unitary. At instant $t$, construct two sets of linearly independent vectors $A_{v,1}$ and $A_{v,2}$ by respectively applying Lemma 2 to $|\Phi(v,t)\rangle$ and $|\Psi(v,t)\rangle$ for every vertex $v$. Use the Gram-Schmidt procedure on the sets $A_{1,v}$ and $A_{2,v}$ to generate the orthonormal basis $\alpha_{k,v}$ and $\beta_{k,v}$ respectively. Take $W(t)$ as the unitary operator defined by Lemma 3 using all basis $\alpha_{v,k}$ and $\beta_{v,k}$. Hence, $S_{RW}$ and $W(t)$ are unitary and well defined for every instant $t$ and the claim is proved. $\square$

It is essential to note that the procedure to construct the quantum walk of Theorem 1 is not unique. In addition to the infinite possibilities of representation that lead to distinct definitions for $S(t)$ and $W(t)$, the operator $W(t)$ can also be defined differently. In fact, the Gram-Schmidt
procedure is just one convenient way to define $W(t)$. Nonetheless, there may exist alternative definitions that could be more efficient under specific conditions, such as particular random walks definitions and graphs.

Together with the procedure that construct random walks matching quantum walks, Theorem reveals that unitary discrete-time coined quantum walks and non-homogeneous random walks are intrinsically related. Knowing the time-dependent stochastic matrix $P(t)$ and the probability vector $\pi(t)$ allows for the construction of a quantum walk operator $S(t)W(t)$ and the state vector $|\Psi(t)\rangle$, and vice-versa.

The random walk considered by Theorem is general and non-homogeneous. Nonetheless, the Theorem can be used to construct a statistically equivalent quantum walk for a time-homogeneous random walk. In this case, the convergence of the vertex probability vector $\pi(t)$ is assured when the random walk is irreducible and aperiodic. In spite of the absence of convergence for the wavefunction caused by unitarity, the convergence of the vertex probability does not harm the construction of the equivalent quantum walk. To illustrate, consider a quantum walk where the wavefunction is permuted among the edges of a vertex perpetually, such that $\Psi(v, c, t) = \Psi(v, c', 0)$ for $t > 0$ and $c' \in C_v$. The vertex probability is the same for all $t$ while the wavefunction keeps alternating forever on the edges, and thus does not converge. Note also that a time-homogeneous stochastic matrix $P(t) = P$ does not implies on a time-independent coin operator $W(t) = W$.

4 Conclusion

In this work, we demonstrated a procedure that constructs a time-dependent discrete-time coined quantum walk capable of simulating the evolution of vertex probability of any given random walk on the graph. The quantum walk obtained through this procedure is general and it is, in fact, one of many possible constructions. Strikingly, creating quantum walks capable of matching the evolution of homogeneous random walks imply that time-dependent quantum walks can converge in terms of its vertex distributions for arbitrary initial conditions. The results presented here add to previous results that demonstrate the equivalence in the opposite direction. Together, they imply that quantum and random walks are distinct ways of expressing the evolution of vertex probabilities on graphs, tied to be stochastically equivalent.

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