ON INHOMOGENEOUS DIOPHANTINE APPROXIMATION
AND HAUSDORFF DIMENSION

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Abstract. Let $\Gamma = \mathbb{Z}A + \mathbb{Z}^n \subset \mathbb{R}^n$ be a dense subgroup of rank $n+1$ and let $\hat{\omega}(A)$ denote the exponent of uniform simultaneous rational approximation to the generating point $A$. For any real number $v \geq \hat{\omega}(A)$, the Hausdorff dimension of the set $B_v$ of points in $\mathbb{R}^n$ that are $v$-approximable with respect to $\Gamma$ is shown to be equal to $1/v$.

1. Inhomogeneous Approximation

We first introduce the general framework of inhomogeneous approximation, following the traditional setting employed in the book of Cassels [7] and adhering to the notations of [5] for the various exponents of approximation involved.

Let $m$ and $n$ be positive integers and let $A$ be an $(n \times m)$-matrix with real entries. The transposed matrix of $A$ is denoted by $^tA$. We consider both the subgroup $\Gamma = AZ^n + \mathbb{Z}^n \subset \mathbb{R}^n$, generated modulo $\mathbb{Z}^n$ by the $m$ columns of $A$, and its dual subgroup $\Gamma' = ^tAZ^m + \mathbb{Z}^m \subset \mathbb{R}^m$, generated modulo $\mathbb{Z}^m$ by the $n$ rows of $A$. Alternatively $\Gamma$ can be looked upon as a subgroup of classes modulo $\mathbb{Z}^n$ lying in the $n$-dimensional torus $T^n = (\mathbb{R}/\mathbb{Z})^n$. Kronecker’s theorem asserts in order that $\Gamma$ be dense in $\mathbb{R}^n$ it is necessary and sufficient that the dual group $\Gamma'$ have maximal rank $m+n$ over $\mathbb{Z}$. Henceforth we shall assume that $\text{rk}_\mathbb{Z}\Gamma' = m+n$.

In order to measure how sharp is the approximation to a given point $\beta$ in $\mathbb{R}^n$ by elements of $\Gamma$, we introduce the following exponent $\omega(A, \beta)$. For any point $\theta$ in $\mathbb{R}^n$, let $|\theta|$ be the supremum norm of $\theta$ and let $\|\theta\| = \min_{x \in \mathbb{Z}^n} |\theta - x|$ be the distance in $T^n$ between $\theta$ mod $\mathbb{Z}^n$ and 0.

Definition 1. For any $\beta \in \mathbb{R}^n$, let $\omega(A, \beta)$ be the supremum, possibly infinite, of the real numbers $\omega$ for which there exist infinitely many integer points $q \in \mathbb{Z}^m$ such that

$$\|Aq - \beta\| \leq |q|^{-\omega}.$$  

It is clear from the definition that $\omega(A, \beta) \geq 0$.

Now, in relation to the linear independence of the rows of $A$, we introduce, for any real matrix $M$, the following uniform homogeneous exponent.

Definition 2. Let $M$ be an $(m \times n)$-matrix with real entries. We denote by $\hat{\omega}(M)$ the supremum, possibly infinite, of the real numbers $\omega$ such that, for any sufficiently large positive real number $Q$, there exists a nonzero integer point $q \in \mathbb{Z}^n$ such that

$$|q| \leq Q, \quad \|Mq\| \leq Q^{-\omega}.$$  

By Dirichlet’s box principle, we have $\hat{\omega}(M) \geq n/m$. Now we are able to formulate the classical assertion about the relationship between homogeneous and inhomogeneous approximations in terms of these exponents. To do so we need the following result.

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Theorem 1 ([5]). For any \( n \)-tuple \( \beta \) of real numbers, the lower bound
\[
\omega(A, \beta) \geq \frac{1}{\hat{\omega}(A)}
\] (1)
holds. Moreover, inequality (1) becomes an equality for a.e. \( \beta \) with respect to the Lebesgue measure on \( \mathbb{R}^n \).

Now we come to our main topic: to examine, for \( v \geq 0 \), the family of subsets
\[
B_v = \{ \beta \in \mathbb{R}^n; \omega(A, \beta) \geq v \} \subseteq \mathbb{R}^n,
\]
and their Hausdorff dimension \( \delta(v) \) as a function of \( v \). It follows immediately from Theorem 1 that \( B_v = \mathbb{R}^n \) when \( v \leq 1/\hat{\omega}(A) \), while \( B_v \) is a null set for \( v > 1/\hat{\omega}(A) \). Furthermore, these sets are relatively small on account of the following crude result, quoted as Proposition 7 in [5].

Theorem 2. For any real number \( v > 1/\hat{\omega}(A) \), the Hausdorff dimension \( \delta(v) \) is strictly less than \( n \).

In fact, the proof of Proposition 7 of [5] gives the explicit upper bound
\[
\delta(v) \leq n - 1 + \frac{1}{1 + (v\hat{\omega}(A) - 1)/(1 + v)}.
\] (2)

On the other hand, an easy application of Hausdorff–Cantelli’s lemma (see [1, 3]) provides us with the following bound.

Theorem 3. For any \( v > 0 \),
\[
\delta(v) \leq \min(n, \frac{m}{v}).
\] (3)

We refer to Theorem 5 of [4] for a proof of inequality (3). We note that (2) is certainly sharper than (3) with \( v \) lying in the interval \([1/\hat{\omega}(A), m/n]\), while the upper bound (3) is expected to be an equality for sufficiently large values of \( v \). When \( m = n = 1 \), it has been proved independently in [2] and in [11] that \( \delta(v) = \min(1, 1/v) \), so that (3) is indeed an equality for any \( v > 0 \) in that case. However, the examples displayed in Theorem 1 of [4] for \((m, n) = (2, 1)\) or \((m, n) = (3, 1)\) show that inequality (3) may well be strict for any given \( v > 1 \). Motivated by Theorem 5 below, we pose the following problem.

Problem. Assume that \( \hat{\omega}(A) \) is finite. Show that \( \delta(v) = m/v \) for all \( v \) that are sufficiently large in term of \( \hat{\omega}(A) \).

We observe that \( \hat{\omega}(A) \geq m/n \). It seems plausible that the assumption \( v \geq \hat{\omega}(A) \) should always be sufficient in order to ensure that \( \delta(v) = m/v \). This holds for \( m = 1 \) by Theorem 5 below. We also note that the lower bound \( v \geq \hat{\omega}(A) \) appears naturally in the construction of a Cantor-type set \( \mathcal{K} \) as in Sec. 4.

2. Simultaneous Approximation

Our knowledge in regard to the Hausdorff dimension \( \delta(v) \) is more profound for \( m = 1 \), that is to say when
\[
\Gamma = \mathbb{Z} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} + \mathbb{Z}^n
\]
is generated by a single vector spinning in \( \mathbb{T}^n \), for in this cases there are fine results [4] by Bugeaud and Chevallier. With regard to the above Problem, let us first quote their Theorem 3 as follows.

Theorem 4. Let \( A = (\alpha_1, \ldots, \alpha_n) \) be an \((n \times 1)\) real matrix such that \( 1, \alpha_1, \ldots, \alpha_n \) are linearly independent over \( \mathbb{Q} \). Then \( \delta(v) = 1/v \) for any \( v \geq 1 \).

Now we state our main result.

Theorem 5. Let \( A = (\alpha_1, \ldots, \alpha_n) \) be an \((n \times 1)\) real matrix such that \( 1, \alpha_1, \ldots, \alpha_n \) are linearly independent over \( \mathbb{Q} \). Then the equality \( \delta(v) = 1/v \) holds for any \( v \geq \hat{\omega}(A) \).