Weak solutions of the cohomological equation on \( \mathbb{R}^2 \) for regular vector fields

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Abstract

In a recent article [De 11], we studied the global solvability of the so-called cohomological equation \( L_\xi f = g \) in \( C^\infty(\mathbb{R}^2) \), where \( \xi \) is a regular vector field on the plane and \( L_\xi \) the corresponding Lie derivative. In a joint article with T. Gramchev and A. Kirilov [DGK11], we studied the existence of global \( L_\xi^1 \) weak solutions of the cohomological equation for vector fields depending only on one coordinate. Here we generalize the results of both articles by providing explicit conditions for the existence of global weak solutions to the cohomological equation when \( \xi \) is intrinsically Hamiltonian or of finite type.

1 Introduction

The topological structure of regular (i.e., without zeros) vector fields \( \xi \) on \( \mathbb{R}^2 \) has been thoroughly investigated during the last century and is well understood (see [De 11] for detailed references). The global analytic properties of the corresponding partial differential operators \( L_\xi \) (Lie derivative in the \( \xi \) direction) are, on the contrary, much less known. Some of their most basic properties were studied in [De 11].

The main purposes of this article are to refine some of the results in [De 11], concerning the action of \( L_\xi \) on spaces of differentiable functions, and to use them to generalize to a much wider set of regular vector fields the results obtained in [DGK11], concerning weak solutions of the cohomological equation

\[
L_\xi f = g \in C^k(\mathbb{R}^2)
\]  (1)

for regular planar vector fields depending only on one variable.
2 Definitions and main results

The following definitions and notations will be used in the present article.

**Vector Fields and Foliations.** We will usually denote vector fields by $\xi$ and, to avoid ambiguities, the corresponding Lie derivative operator by $L_\xi$. We say that a $C^1$ function is regular if its differential is never zero; analogously, we say that a vector field is regular when it has no zeros. We denote by $\mathfrak{X}_r(\mathbb{R}^2)$ the set of all smooth regular vector fields on the plane. Given any $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$, $\mathcal{F}_\xi$ will denote the smooth foliation of its integral trajectories and, by abuse of notation, the space of leaves endowed with its canonical quotient smooth structure. We denote by $\pi_\xi : \mathbb{R}^2 \to \mathcal{F}_\xi$ the canonical projection that sends a point to the leaf passing through it. A saturated neighborhood of a leaf $\ell$ of $\mathcal{F}_\xi$ is a set $\pi_\xi^{-1}(U)$, where $U$ is a neighborhood of $\ell$ in $\mathcal{F}_\xi$. We say that two integral trajectories $s_1, s_2$ of $\xi$ are inseparable when they are inseparable as points in the topology of $\mathcal{F}_\xi$ (e.g. see Fig. 3). An integral trajectory $s$ which is inseparable from some other integral trajectory is said a separatrix. We denote by $\mathcal{S}_\xi$ the set of all separatrices of $\xi$.

**Definition 1.** A regular planar vector field $\xi$ (and, by extension, the foliation $\mathcal{F}_\xi$) is of finite type if the set of its separatrices is closed in $\mathcal{F}_\xi$ and every separatrix is inseparable from just finitely many other integral trajectories.

The set of vector fields of finite type is of great relevance since important categories of vector fields belong to it. For example, every regular polynomial vector field is of finite type: finite bounds for the number of separatrices of a polynomial vector field were found first by Markus [Mar72] and later improved independently by M.P. Muller [Mul76b] and S. Schechter and M.F. Singer [SS80]. It is easy to verify that also all vector fields strongly proportional to regular vector fields invariant with respect to translations in a given direction are of finite type. An important feature of vector fields of finite type is that the complement of the set of the separatrices of a vector field of finite type is the disjoint union of countably many unbounded connected open sets (named by Markus [Mar54] canonical regions) whose boundary has only a finite number of connected components.

We say that a smooth foliation is Hamiltonian if its leaves are the level sets of a regular smooth function. A vector field $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$ is Hamiltonian when $L_\xi(dx \wedge dy) = 0$.

**Definition 2.** Two smooth vector fields are strongly proportional if they are proportional through a strictly positive or strictly negative smooth function. A smooth vector field is intrinsically Hamiltonian if it is strongly proportional to a smooth Hamiltonian vector field and is transversally Hamiltonian if it is transversal to a Hamiltonian smooth vector field (equivalently, to the level sets of a regular smooth function).
It is easily seen that a smooth regular vector field $\xi$ is intrinsically Hamiltonian if and only if the partial differential equation $L_\xi f = 0$ admits a smooth regular solution and is transversally Hamiltonian if and only if the partial differential inequality $L_\xi f > 0$ has a smooth solution. It was proved in [Wei88] (resp. in [De 11]) that every Hamiltonian vector field (resp. every vector field of finite type) is transversally Hamiltonian.

**Differential Structures.** Every locally injective homeomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}$ determines a differential $C^\infty$ structure on $\mathbb{R}^2$ given by the atlas containing all charts $(U, \Phi|_U)$, where $U \subset \mathbb{R}^2$ is any open set on which $\Phi$ is injective. We denote by $\mathbb{R}^2_\Phi$ the differential manifold $\mathbb{R}^2$ with the differential structure induced by $\Phi$. In general $\mathbb{R}^2_\Phi$ is different from $\mathbb{R}$, and in particular $C^\infty(\mathbb{R}^2_\Phi) \neq C^\infty(\mathbb{R})$, but $\mathbb{R}^2_\Phi$ is always globally diffeomorphic to $\mathbb{R}^2$, e.g. because it is well-known that, in dimension smaller than four, every topological manifold admits just one differential structure modulo diffeomorphisms (see [Mo77] and [Ru01]).

**Functional Spaces.** Let $L^1_0 = [-1, 0]$ and $L^2_0 = [-1, 0] \times [-1, 1] \setminus (0, 0)$. We denote by $\mathcal{S}^k(L^1_0)$, $i = 1, 2$, the ring of left germs at the origin of functions in $C^k(L^1_0)$, i.e. the equivalence classes determined by the equivalence relation $h \sim h'$ if $h$ and $h'$ coincide in some left neighborhood of the origin. We focus our attention of the following subrings of $\mathcal{S}^k(L^1_0)$: $\mathcal{S}^{r,k}(L^1_0)$, with $r = 0, \ldots, k$, containing the left germs of all functions belonging to $C^r(L^1_0) \cap C^k(L^1_0)$, and $\mathcal{S}^{l,p,k}(L^1_0)$, with $p \geq 1$ and $l = 0, \ldots, k + 3$, containing the left germs of all functions belonging to $W^{l,p}_0(L^1_0) \cap C^k(L^1_0)$.

**Definition 3.** We call singular left germs at the origin the elements of the quotient rings $S\mathcal{S}^{r,k}(L^1_0) = \mathcal{S}^k(L^1_0)/\mathcal{S}^{r,k}(L^1_0)$ and $S\mathcal{S}^{l,p,k}(L^1_0) = \mathcal{S}^{k}(L^1_0)/\mathcal{S}^{l,p,k}(L^1_0)$. By abuse of notation, we denote by $S\mathcal{S}^{k+1,k}(L^1_0)$ the singular germs of germs of $C^k$ functions which are $C^{k+1}$ in the first variable.

**Main objectives.** We consider the partial differential operators\footnote{We will omit the upper index in the operators $L_\xi$ when there is no ambiguity.}$L_\xi^{(r)} : C^r(\mathbb{R}^2) \to C^{r-1}(\mathbb{R}^2)$, $r = 1, 2, \ldots, \infty$

and their weak extensions

$L_\xi^{(l,p)} : W^{l,p}_0(\mathbb{R}^2) \to W^{l-1,p}_0(\mathbb{R}^2)$, $p \geq 1$, $l = 1, 2, \ldots$,

where $W^{l,p}_0(\mathbb{R}^2)$ is the Sobolev space of $L^p_0$ functions whose first $l$ weak derivatives are also $L^p_0$. We endow $C^r(\mathbb{R}^2)$ with the Whitney topology. Our main aim is studying the images

$L_\xi (C^r(\mathbb{R}^2)) \cap C^k(\mathbb{R}^2)$, $L_\xi \left(W^{l,p}_0(\mathbb{R}^2)\right) \cap C^k(\mathbb{R}^2)$.

In other words, we study the existence of global $C^r$ or $W^{l,p}$ solutions of the cohomological equation when the right hand side is of class $C^k$ (notice that, as
easily shown via the method of characteristics, such solutions are at least \( C^k \) everywhere except, at most, on the separatrices. In case there is no regularity loss (i.e. \( r = k + 1 \)), the problem reduces to studying the full image \( L_\xi(C^{k+1}(\mathbb{R}^2)) \).

We point out that, since clearly

\[
L_\xi(C^{k+1+k'}(\mathbb{R}^2)) \cap C^k(\mathbb{R}^2) = L_\xi(C^{k+1+k'}(\mathbb{R}^2)) \cap C^{k+k'}(\mathbb{R}^2)
\]

it is enough to consider, for any given \( k \), just the cases \( r = 1, \ldots, k+1 \). Similarly, since \( W^{k+1,p}(\mathbb{R}^2) \subset C^k(\mathbb{R}^2) \) for \( p > 2 \) and \( W^{k+2,p}(\mathbb{R}^2) \subset C^k(\mathbb{R}^2) \) for \( 1 \leq p \leq 2 \) (e.g. see [DD12]), it is enough to consider the cases \( l = 0, \ldots, k+1 \) for \( p > 2 \) and \( l = 0, \ldots, k+2 \) for \( 1 \leq p \leq 2 \).

**Remark 1.** Note that it makes sense considering the case \( l = 0 \), namely \( W^{0,p}_{\text{loc}} = L^p_{\text{loc}} \) solutions, because, as noted above, all solutions are always of class at least \( C^k \) outside of the separatrices and, by definition, their derivative in the direction \( \xi \) is also \( C^k \). Namely, “\( L^p_{\text{loc}} \)” weak solutions of \( L_\xi f = g \in C^k(\mathbb{R}^2) \) are actually elements of \( L^p_{\text{loc}}(\mathbb{R}^2) \cap C^k(\mathbb{R}^2 \setminus S_\xi) \).

**Example 1.** Consider the regular vector field \( \xi = 2y\partial_x + (1 - y^2)\partial_y \). As a corollary of Proposition 2 in [De 11], \( 1 \notin L_\xi(C^r(\mathbb{R}^2)) \cap C^\infty(\mathbb{R}^2) \) for any \( r \). On the other side \( 1 \in L_\xi(L^1_{\text{loc}}(\mathbb{R}^2)) \cap C^\infty(\mathbb{R}^2) \) since, for example, \( L_\xi f = 1 \) for \( f(x,y) = \frac{1}{2} \ln \left| \frac{1+y}{1-y} \right| \). Note that solutions that diverge on just one of the two separatrices can be easily obtained through the \( L_\xi \)'s weak first-integral \( h(x,y) = x + \ln|1-y^2| \).

**Main Results.** In Section 3 we refine some results in [De 11] about the local geometry of Hamiltonian and finite type regular foliations on the plane. The section’s main result, contained in Theorem 4 is that for such vector fields, locally, the problem of the extension of a solution of the cohomological equation from a saturated neighborhood of a separatrix \( s_1 \) to the saturated neighborhood of an adjacent separatrix \( s_2 \) can be always reduced to the problem of the extension, from \( L^0_\xi \setminus \{0\} \times [0, \infty) \) to the whole \( L^0_\xi \), of a solution of \( \partial_y f = g \in C^k(\mathbb{R}^2) \).

This theorem generalizes Proposition 8 in [De 11] and fixes a minor mistake in its statement.

In Section 4 we study the images of \( L^{(r)}_\xi \) and \( L^{(l,p)}_\xi \) Our main results are contained in Theorem 8 where we provide explicit criteria for the solubility of the cohomological equation in the Hamiltonian case, and Theorem 10 were weaker results are provided for the finite-type non-Hamiltonian case. Moreover, in Theorem 8 we show that the solvability of the cohomological equation, in the Hamiltonian case, is stable with respect to small perturbation of the right hand side.

Finally, in Section 5 we present in some detail four concrete examples. In the first two we consider, respectively, the cases of two regular Hamiltonian and non-Hamiltonian vector fields depending only on one variable and with just a pair a separatrices and compare our results with those in [DGK11]. In the last two we consider two case not covered by the results in [DGK11]: the case of a regular Hamiltonian vector field with just a pair a separatrices and not
invariant with respect to translations in any direction and the case of a regular Hamiltonian vector field with three separatrices inseparable from each other.

3 Geometry of $\mathcal{F}_\xi$

The geometry of the set of separatrices of a regular planar foliation can be quite non-trivial. Explicit examples of $C^\infty$ (see [Waz34] and [Wei88]) and $C^\omega$ (see [Mul76a]) foliations of the plane with a set of separatrices dense on some open set are known in literature. In the example below, we show how to build an explicit instance, simpler and more natural than the ones mentioned above, of $C^\infty$ foliation whose separatrices are dense on the whole plane.

Example 2. The building blocks of the present example are the $C^\infty$ foliation $\mathcal{S}$ of the vertical half-stripe $S = [0,1] \times [0, \infty)$, shown in Fig. 3 (left), and the foliation $\mathcal{F}_0$ in vertical lines of the whole plane.

Denote by $t$ the $x$-axis, everywhere transversal to $\mathcal{F}_0$, and by $P_k$ the half-plane $y < k$ and select a sequence of vertical half-stripes $S_n = [\ell_n, \ell'_n] \times [n, \infty)$ such that $S_i \cap S_j = \emptyset$ for $i \neq j$. After replacing, on each $S_n$, the vertical foliation with a suitably rescaled version of $\mathcal{S}$, we get a $C^\infty$ foliation of the plane $\mathcal{F}_1$ coinciding with $\mathcal{F}_0$ for $y < 1$ and with a set of separatrices dense on the open set $U = \pi^{-1}(\pi \mathcal{F}_1(t)) \supset P_1$.

Now, let $t_n$ be the image of the transversal $\gamma_n$ in $S_n$, $U_n = \pi^{-1}(\pi \mathcal{F}_1(t_n))$ and let $\Phi_n = (\varphi_n, \psi_n) : U_n \to \mathbb{R}^2$ be a rectifying diffeomorphism such that: 1. $t_n$ has equation $\psi_n = 0$; 2. the leaves of $U_n$ are sent to vertical lines and, in particular, $s_n$ has equation $\varphi_n = 0$; 3. the leaves outside $U$ are those for which $\varphi_n > 0$.

With the same construction described above, we can modify this foliation in the half-plane $\varphi_n > 0$ to produce a new foliation $\mathcal{F}_2$ which is dense on the open set $U \bigcup_{n \in \mathbb{N}} \pi^{-1}(\pi \mathcal{F}_1(t_n))$ and coincides with $\mathcal{F}_1$ on $P_2$.

By repeating this construction recursively, we get a sequence $\{\mathcal{F}_n\}$ of $C^\infty$ regular foliations of the plane such that $\mathcal{F}_{n+1}|_{P_{n+1}} = \mathcal{F}_n|_{P_{n+1}}$ for every $n \in \mathbb{N}$ and the closure of the set of the separatrices of $\mathcal{F}_n$ contains $P_n$. Hence the $\mathcal{F}_n$ converge to a smooth foliation $\mathcal{F}_\infty$ with a set of separatrices dense in the whole plane.

Remark 2. The leaf space of $\mathcal{F}_\infty$ is a 1-dimensional, smooth non-Hausdorff smooth manifold with a dense set of binary branch points, namely such that at every branch point exactly two branches (or plumes) meet. It is easy to modify the foliation $\mathcal{S}$ in order to have, at every branch point, the concurrence of any finite number of plumes, or even infinitely (countably) many.

Note that the foliation $\mathcal{S}$ (see Fig. 3) is Hamiltonian, so that also every $\mathcal{F}_n$, and therefore even $\mathcal{F}_\infty$, is Hamiltonian. In particular, as a corollary of a Lemma of Weiner [Wei88] stating that the first component projection $\pi : \text{Imm}^\infty(\mathbb{R}^2, \mathbb{R}^2) \to \text{Sub}^\infty(\mathbb{R}^2, \mathbb{R})$ is surjective, this shows that the topology of immersions of the plane into itself can be quite non-trivial:
Figure 1: Two $C^\infty$ foliations $S$ (left) and $S'$ (right) of the strip $S = [0, 1] \times [0, \infty)$. [left] $S$ is Hamiltonian and it can be obtained as the level sets of the (non-regular) $C^\infty$ function

$$F(x, y) = \arctan \left( \frac{1}{f(\frac{15-x}{10})h(5)+\left(1-f(\frac{15-x}{10})\right)h(x)+y^2-1} \right) + \frac{1}{(y-1)^3} + \frac{1}{(y+1)^2} \right)^2,$$

where $f(x)$ is any $C^\infty$ function equal to 0 for $x < 0$, 1 for $x > 1$ and strictly increasing between 0 and 1, and $h(x) = \frac{30}{x+5}$. The black thick lines $s_1 = \{x = -1\}$ and $s_2 = \{y = \frac{5+\sqrt{2}}{1+x}\}$ are inseparable in the quotient topology. The black dashed line $\gamma$ is a particular choice of a globally transverse to the foliation. [right] $S'$ has the same topology as $S$ but it is non-Hamiltonian and it can be obtained as the level sets of the non-regular $C^\infty$ function

$$F(x, y) = \arctan \left( \frac{1}{f(\frac{15-x}{10})h(5)+\left(1-f(\frac{15-x}{10})\right)h(x)+y^2-1} \right) + \frac{1}{(y-1)^3} + \frac{1}{(y+1)^2} \right)^2.$$ Its two separatrices coincide with the ones of the previous foliation.

Proposition 1. For every $k = 2, 3, \ldots, \infty$ there exists a $C^\infty$ immersion $\Phi_{FG} = (F, G) : \mathbb{R}^2 \to \mathbb{R}^2$ such that the space of the foliation $F$ of the level sets of $F$ is a infinite feather of order $k$.

At the other end of the spectrum, if we use instead the foliation $S'$ in the construction presented in Example 2, we end up with a foliation $F'_\infty$ which cannot be obtained as the level set of any $C^1$ function. Indeed, by construction, any $C^1$ function giving rise to the foliation of $S'$ has differential equal to 0 on $x = -1$. Since the final foliation was built in such a way that the set of separatrices on which the differential is zero is dense on the plane, any $C^1$ function having those lines as level sets must have its differentiable null on a dense set and therefore, by continuity, must be constant.

At this regard, it is important to notice that, for a foliation in $\mathbb{R}^2$, the lack of a smooth regular first-integral is a relative, rather than absolute, property. Indeed, as a corollary of a result of Kaplan that every regular foliation of the plane has a continuous first-integral [Kap10], we can prove the following:

Theorem 1. Every regular $C^0$ foliation of the plane is a regular $C^\infty$ Hamiltonian foliation for some suitable differential structure.

Proof. Let $\mathcal{F}$ be a regular planar foliation and let $\mathcal{G}$ be any foliation everywhere
transversal to it. Assume that $F$ is not Hamiltonian, otherwise there is nothing to prove, and let $F$ and $G$ two continuous first-integrals of, respectively, $F$ and $G$. Then the map $\Phi_{FG} = (F, G) : \mathbb{R}^2 \to \mathbb{R}$ is locally injective and, in $\mathbb{R}^2 \Phi_{FG}$, both $F$ and $G$ are $C^\infty$ Hamiltonian. Indeed in every chart $(U, (\Phi_{FG})|U)$, by definition, $F$ and $G$ are, respectively, the $x$ and $y$ coordinates and therefore are $C^\infty$ and their differential is nowhere zero. Moreover, by construction, $F = \{dF = 0\}$ and $G = \{dG = 0\}$. 

Applying to this context the proof of Weiner’s Lemma in [Wei88] and of Theorem 2 in [De 11], we can prove the following more general result:

**Theorem 2.** Let $F$ be a $C^0$ foliation of $\mathbb{R}^2$. Then, if $F$ is $C^r$ and either Hamiltonian or of finite type in $\mathbb{R}^2 \Phi$, it admits a transversal $C^r$ Hamiltonian foliation in $\mathbb{R}^2 \Phi$.

Now we refine some result in [De 11] on the local geometry of regular foliations.

**Definition 4.** Given a pair of adjacent inseparable leaves $s_1, s_2$ of a foliation $F$, we say that a curve $\gamma$ separates them, or that $\gamma$ is between $s_1$ and $s_2$, if $s_1$ and $s_2$ belong to different connected components of $\mathbb{R}^2 \setminus \gamma$. We say that a foliation $G$ transversal to $F$ minimally separates $F$ if there is only one leaf of $G$ between every two adjacent separatrices of $F$.

We start with a technical Lemma:

**Lemma 1.** Consider the foliation $V$ in vertical lines of the set

$$ S = (-1, 0] \times (-2, 2) \setminus \{0\} \times [-1/2, 1/2]. $$

There exists a $C^\infty$ Hamiltonian foliation $T$ of $S$ which is everywhere transversal to $V$ and minimally separates it.
Proof. We start by replacing the rectangle $R_1 = [1/2, 1/4] \times [-1, 0]$ with a suitably rescaled copy of the rectangle $R$ shown in Fig. 3. Because of how $R$ is foliated, after the substitution we get a foliation with the same regularity of the previous one and still transversal at every point to the vertical direction. Inside $R_1$ though, all leaves crossing the line $x = -1$ at $-1/2 \leq y \leq -1/4$ are rerouted so that they cross $\ell_1$ at $-3/4 \leq y \leq -1/2$.

Now we repeat the procedure by replacing the rectangles $R_n = [2^{-n}, 2^{-n-1}] \times [-1, 0]$ with suitably rescaled copies of $R$. At the step $n$ therefore we still have a foliation $G_n$ of the same regularity of the original one and everywhere transversal to the vertical direction but this time though, among all leaves crossing $x = -1$ for $-1 \leq y \leq 0$, only those for which $-1/2^{n+1} \leq y \leq 0$ do not cross $\ell_1$.

In the limit for $n \to \infty$ we are left with a foliation $G'$ such that all of its leaves crossing $x = -1$ for $-1 \leq y \leq 0$ do cross the line $\ell_1$. By replacing the upper half of the foliation with the symmetric of the lower part with respect to the $x$ axis we found the foliation $G$ mentioned above.

The claim is proved by repeating this procedure for all pairs of adjacent separatrices of the foliation. \hfill $\Box$

Theorem 3. Let $\mathcal{F}$ be a $C^0$ foliation of $\mathbb{R}^2$. Then there exists a $C^0$ transverse foliation $\mathcal{G}$ which minimally separates $\mathcal{F}$. Moreover, if $\mathcal{F}$ is $C^\gamma$ and either Hamiltonian or of finite type in some $\mathbb{R}^2_\Phi$, then $\mathcal{G}$ can be chosen to be $C^\gamma$ and Hamiltonian with respect to $\mathbb{R}^2_\Phi$.

Proof. Let $\mathcal{A}$ be any atlas of $\mathbb{R}^2$ where $\mathcal{F}$ is of class $C^\gamma$. Then, either by Weiner’s Lemma in [Wei88] (if $\mathcal{F}$ is Hamiltonian) or by Theorem 2 in [De 11] (if it is of finite type), there exists a $C^\gamma$ locally injective map $\Phi_{\mathcal{F},\mathcal{G}} = (F, G)$ (which is an immersion if $\mathcal{F}$ is Hamiltonian) that sends $\mathcal{F}$ and $\mathcal{G}$, respectively, in vertical and horizontal lines.

By definition of inseparable leaves, for every pair of adjacent inseparable leaves $s_1, s_2 \in \mathcal{F}$ cut, respectively, by the transversals $t_1, t_2 \in \mathcal{G}$, the set $U_{12} = \pi_\mathcal{F}(\pi_\mathcal{F}(t_1)) \cap \pi_\mathcal{G}(\pi_\mathcal{G}(t_2))$ contains a saturated one-sided (left or right, in the $(F, G)$ coordinates) neighborhood of $s_1$ and $s_2$. Let $s_1 \cup s_2 \subset F^{-1}(a)$, $t_i = G^{-1}(b_i)$, with $b_1 < b_2$, and assume, for the sake of the argument, that $U_{12} \subset F^{-1}((-, a), (a, \infty))$. Then $U_{12} \supset U_{12} = (a - \epsilon, a) \times (b_1, b_2)$ for some $\epsilon > 0$.

By Lemma 1 we can replace the restriction of $\mathcal{G}$ to $R_x$ with a new foliation in such a way that the new foliation $\mathcal{G}'$ is still Hamiltonian, has the same regularity of $\mathcal{G}$ and separates minimally $s_1$ and $s_2$. The proof is concluded by repeating this process for all pairs of adjacent separatrices of $\mathcal{F}$. \hfill $\Box$

The previous result allows us to state a stronger version of Proposition 8 in [De 11]. This version also fixes a minor mistake in that Proposition’s claim.

Theorem 4. Let $\xi \in \mathcal{X}_r(\mathbb{R}^2_\Phi)$ be either Hamiltonian or locally finite and let $F \in C^\infty(\mathbb{R}^2_\Phi)$ be a generator of $\ker L_\xi$ and $G \in C^\infty(\mathbb{R}^2_\Phi)$ such that $L_\xi G > 0$ and $\mathcal{G} = \{dG = 0\}$ minimally separates $\mathcal{F}_\xi$. Then, for every pair of adjacent separatrices $s_1, s_2 \in \mathcal{F}_\xi$, with $s_1 \cup s_2 \subset F^{-1}(a)$, separated by $t \in \mathcal{G}$, with $t \subset G^{-1}(b)$, and leaves $t_1, t_2 \in \mathcal{G}$, with $t_i \subset G^{-1}(b_i)$, cutting respectively $s_1$ and $s_2$,
set $U = \pi_{\xi}^{-1}(\pi_{\xi}(t_1)) \cup \pi_{\xi}^{-1}(\pi_{\xi}(t_2))$ and $V = \pi_{\xi}^{-1}(\pi_{\xi}(t_1)) \cap \pi_{\xi}^{-1}(\pi_{\xi}(t_2))$. The map $\Phi_{FG} = (F, G): \mathbb{R}^2 \to \mathbb{R}^2$ satisfies the following conditions:

1. the restriction of $\Phi_{FG}$ to $U$ is a diffeomorphism onto $\Phi_{FG}(U)$;
2. the leaves of $\mathcal{F}_{\xi}$ and $\mathcal{G}$ are mapped, respectively, into vertical and horizontal lines;
3. $\Phi_{FG}(s_1) = \{x\} \times G(s_1)$, with $G(s_1) \cup G(s_2) = (b_1', b) \cup (b, b_2')$ for some $-\infty \leq b_1' < b$ and $b < b_2' \leq \infty$;
4. if $V \subset F_{-1}((a, \infty))$ (resp. if $V \subset F_{-1}((a, \infty))$, then $\Phi_{FG}(t) = (a_1, a) \times \{b\}$ for some $-\infty \leq a_1 < a$ (resp. $\Phi_{FG}(t) = (a_1, a) \times \{b\}$ for some $a < a_1 \leq \infty$) and $(a-\epsilon, a) \times (b_1, b_2) \subset \Phi_{FG}(V)$ (resp. $(a, a+\epsilon) \times (b_1, b_2) \subset \Phi_{FG}(V)$) for some $\epsilon > 0$.

**Definition 5.** Under the conditions of the previous theorem, we call $\Phi_{FG}: U \to \mathbb{R}^2$ a normal chart for the adjacent separatrices $s_1, s_2$.

**4 $L_{\xi}(C^r(\mathbb{R}^2)) \cap C^k(\mathbb{R}^2)$ and $L_{\xi}(W^{l,p}_{\text{loc}}(\mathbb{R}^2)) \cap C^k(\mathbb{R}^2)$**

It is well known that local solutions to the cohomological equation (1) can be built through the method of characteristics. When $\xi \in \mathcal{X}_r(\mathbb{R}^2)$ and $g \in C^k(\mathbb{R}^2)$, the only obstruction to the existence of a $C^r$ solution, $0 \leq r \leq k$, is the problem of extending a local solution across pairs of adjacent separatrices (e.g. see [De 11]). In a normal chart (see Theorem 4), this problem can always be reduced to the following:

**Problem 1.** Let $g \in C^k(\mathbb{R}^2)$ and $\varphi \in C^r(\mathbb{R}^2 \times \{1\})$ and define on $\mathbb{R}^2 \times \{1\}$ the function $\psi(x) = \int_{-1}^1 g(x, t) dt + \varphi(x)$. Under which conditions on $g$ and $\varphi$ can the function $\psi$ be extended to a $C^r$ function at $x = 0$? Similarly, assuming $\varphi \in W^{l,p}(\mathbb{R}^2 \times \{1\}) \cap C^r(\mathbb{R}^2 \times \{1\})$, under which conditions on $g$ the function $\psi$ does is also $W^{l,p}_{\text{loc}}$ at $x = 0$?

**Definition 6.** We denote, respectively, by $\theta_{r,k}: \mathcal{S}\mathcal{W}^k,\infty(\mathbb{L}_0^1) \to \mathcal{S}\mathcal{W}^k,\infty(\mathbb{L}_0^1)$ and $\theta_{l,p,k}: \mathcal{S}\mathcal{W}^{l,p,k}(\mathbb{L}_0^1) \to \mathcal{S}\mathcal{W}^{l,p,k}(\mathbb{L}_0^1)$ the homomorphisms associating, to the left singular germ at (0,0) of a function $g \in C^k(\mathbb{L}_0^1)$, the left singular germ at 0 of the function $f(x) = \int_{-1}^1 g(x, y) dy \in C^k(\mathbb{L}_0^1)$ modulo, respectively, functions of class $C^r$ and $W^{l,p}_{\text{loc}}$ at $x = 0$. Correspondingly, we set $\Theta_{r,k} = \ker \theta_{r,k}$ and $\Theta_{l,p,k} = \ker \theta_{l,p,k}$.

**Theorem 5.** The sets $\Theta_{r,k}$ and $\Theta_{l,p,k}$ satisfy the following properties for all $k = 0, 1, 2, \ldots, \infty$ and $p \geq 1$:

1. $\Theta_{r,k} \subseteq \Theta_{r-1,k}$ for all $1 \leq r \leq k$;
2. $\Theta_{k,k}$ contains the left singular germs of all $y$-odd $C^k$ functions;
3. $\mathcal{S}\mathcal{W}^{l,p,k}(\mathbb{L}_0^1) \subset \Theta_{l,p,k}$ for all $l = 0, 1, 2, \ldots$;
4. $\Theta_{r,k} \subseteq \Theta_{r,p,k}$ for all $0 \leq r \leq k$.

Proof. In [De 11] we showed that the (left singular germ of the) function

$$g(x, y) = \frac{x}{\sqrt{x^2 + y^2}}$$

provides an example of an element belonging to $\Theta_{0,\infty}$ but not to $\Theta_{1,\infty}$. After integrating $r$ times $g$ with respect to $x$, we get concrete examples of elements belonging to $\Theta_{r,\infty}$ but not to $\Theta_{r+1,\infty}$, which proves point (1). Point (2) is due to the fact that in that case the integral of $g$ in $y$ is zero on every interval symmetric with respect to zero.

To prove (3) we notice that, if $g \in W^{l,p}(L^2)$,

$$\|\psi\|_{W^{l,p}(L^2)} \leq \|g\|_{W^{l,p}(L^2)} + \|\varphi\|_{W^{l,p}(L^2)} < \infty$$

since, by hypothesis (see Problem 1), $\varphi \in W^{l,p}(L^2)$.

Consider now again the function $g$ used to prove point (1). Then $\partial_x g(x, y)$ provides an example of function in $\Theta_{0.1,\infty}$ but not in $\Theta_{0,\infty}$ and Point (4) is then proved by considering, more generally, $g^{1/p}$ and by integrating it with respect to $x$ as in point (1). Similar examples can also be obtained, for example, via the function $g(x, y) = (x^2 + y^2)^{-\alpha}$, $\alpha > 0$, which belongs to $\Theta_{t,p,\infty}$ for $0 \leq \alpha \leq \frac{1}{p} - \frac{1}{2}$. Note that, for $\alpha < \frac{1}{2}$, we have also that $g \in \Theta_{0,\infty}$.

We now go back to the global solution of the cohomological equation.

The Hamiltonian case. When $\xi$ is Hamiltonian with respect to the standard smooth structure, the projection $C^r(\mathcal{F}_\xi) \to C^r(U)$, given by the restriction of a $C^r$ function on $\mathcal{F}_\xi$ to any open set $U \subset \mathcal{F}_\xi$, is surjective (e.g. see [HR57]), so that the regularity of $\psi$ in Problem 1 only depends on the germ of $g$ at $(0, 0)$.

For every pair of adjacent separatrices $s_1, s_2 \in \mathcal{F}_\xi$ separated by $\gamma(s_1, s_2) \in \mathcal{G}$, we denote by $a$ the common value of $F$ at every point of the two separatrices and by $b$ the one of $G$ at every point of $\gamma$ and define the homomorphisms $\theta_{r,k}(a,b) = \theta_{r,k} \circ T(a,b)$ and $\theta_{t,p,k}(a,b) = \theta_{t,p,k} \circ T(a,b)$, where $(T(a,b)) (x, y) = g(x - a, y - b)$.

Theorem 6. Let $\xi$ be a planar Hamiltonian vector field, $F$ a generator of $\ker L_\xi$, $G$ a transversal foliation minimally separating $\mathcal{F}_\xi$ and $\xi_F = \xi|L_\xi G$ the Hamiltonian vector field of $F$ with respect to the symplectic form $dF \wedge dG$. Then, for all $k = 0, \ldots, \infty$, $r = 0, \ldots, k$, $l = 0, \ldots, k + 3$ and $p \geq 1$:

1. $g \in L_{\xi_F} (C^r(\mathbb{R}^2)) \cap C^k(\mathbb{R}^2)$ iff $[T(a,b)(\Phi_{FG}), g]_{S^{k,r}(L^2)} \in \Theta_{r,k}$ for all $i$;

2. $g \in L_{\xi_F} (C^{k+1}(\mathbb{R}^2))$ iff $(\Phi_{FG}) g$ is $C^{k+1}$ in the first variable and $[T(a,b)(\Phi_{FG}), g]_{S^{k+1,r}(L^2)} \in \Theta_{k+1,k+1}$ for all $i$;

3. $g \in L_{\xi_F} (W^{l,p}_{loc}(\mathbb{R}^2)) \cap C^k(\mathbb{R}^2)$ iff $[T(a,b)(\Phi_{FG}), g]_{S^{\infty,k,p}(L^2)} \in \Theta_{t,p,k}$ for all $i$. 

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Proof. 1. If $g \in L^r_{GF}$ then there is a $C^r$ solution $f$ to $Lc f = g$. In a normal chart $(x', y')$ of some neighborhood of any pair of adjacent separatrices, the function $\varphi(x') = \int_{x'}^\infty g(x', y')dy'$ of $(-\delta, 0) \rightarrow \mathbb{R}$ equals $f(x', \epsilon)$ modulo some function belonging to $C^r((-\delta, 0])$, i.e. $[\varphi]_{S\Theta^r(\mathbb{L})} = 0$. If, on the other side, $[T_{(a, b)}(\Phi_{FG}), g]_{S\Theta^r(\mathbb{L})} \in \Theta_r$ for all pairs of adjacent separatrices $s_1, s_2$ with transversals $\gamma_{11}, \gamma_{22}$, then we can define any $C^r$ function of one of the transversals and extend the solution to the whole plane with the method of the characteristics. The condition $[T_{(a, b)}(\Phi_{FG}), g]_{S\Theta^r(\mathbb{L})} \in \Theta_r$ grants that on every separatrix we can extend the solution to a $C^r$ solution across the separatrix.

The argument works similarly for point 3 mutatis mutandis.

About point 2, the argument is the same but we must first prove that the property that $(\Phi_{FG})_g$ is $C^{k+1}$ in the first variable does not depend on the particular choice of $F$ and $G$. The reason for this is that every other first-integral $F'$ of $\xi$ only depends on $F$, so that any other pair $(F', G')$, where $F'$ is a first-integral and $G'$ a transversal Hamiltonian for $J_\xi$, is such that $(F', G') = (F'(F), G'(F, G))$. Hence, if $(\Phi_{FG})_g$ is $C^{k+1}$ in the first argument for one particular choice of $F$ and $G$, it is so for every other choice.

**Corollary 1.** Under the hypotheses of Theorem 6, let $P$ be the set of all points $(a, b)$ that separate pairs of adjacent separatrices of $(\Phi_{FG})_\xi$ in $\Phi_{FG}(\mathbb{R}^2)$. Then the following inclusions hold:

1. $Lc'_c(C^r(\mathbb{R}^2)) \cap \mathcal{C}^k(\mathbb{R}^2) \supset \Phi^{*}_{FG}(C^r(\mathbb{R}^2) \cap \mathcal{C}^k(\mathbb{R}^2 \setminus P))$;
2. $Lc'_c(C^{k+1}(\mathbb{R}^2)) \supset \Phi^{*}_{FG}(C^{k+1}(\mathbb{R}^2))$;
3. $Lc'_c(W^l_{loc}(\mathbb{R}^2)) \cap \mathcal{C}^k(\mathbb{R}^2) \supset \Phi^{*}_{FG}(W^l_{loc}(\mathbb{R}^2) \cap \mathcal{C}^k(\mathbb{R}^2 \setminus P))$.

Next theorem shows in particular that the solvability of the cohomological equation is stable under small perturbations of its right hand side:

**Theorem 7.** Let $\xi \in \mathfrak{X}_r(\mathbb{R}^2)$ be Hamiltonian. Then:

1. $Lc_{\xi}(C^r(\mathbb{R}^2)) \cap \mathcal{C}^k(\mathbb{R}^2)$ is a clopen subset of $\mathcal{C}^k(\mathbb{R}^2)$ for all $r = 0, \ldots, k$ and $k = 0, \ldots, \infty$. In particular, $Lc_{\xi}(C^\infty(\mathbb{R}^2))$ is clopen in $C^\infty(\mathbb{R}^2)$;
2. $Lc_{\xi}(C^{k+1}(\mathbb{R}^2))$, is neither open or closed in $C^k(\mathbb{R}^2)$ for all $k = 0, 1, \ldots$;
3. $Lc_{\xi}(W^{l,p}_{loc}(\mathbb{R}^2)) \cap \mathcal{C}^k(\mathbb{R}^2)$ is a clopen subset of $\mathcal{C}^k(\mathbb{R}^2)$ for all $l = 0, \ldots, k + 1$, if $p > 2$, and for all $l = 0, \ldots, k + 2$, if $1 \leq p \leq 2$, for all $k = 0, 1, \ldots$.

**Proof.** 1. Set $A = Lc_{\xi}(C^r(\mathbb{R}^2)) \cap \mathcal{C}^k(\mathbb{R}^2)$ and let $g \in A$. Every positive function $\epsilon \in C^0(\mathbb{R}^2)$ defines a neighborhood $U_\epsilon$ of $g$ in the strong $C^k$ topology as the set of all $C^k$ functions $g'$ such that

$$|g'(x, y) - g(x, y)| + \|D(x,y)(g'-g)\| + \cdots + \|D_{(x,y)}^{k}(g'-g)\| \leq \epsilon(x, y)$$

for every $(x, y) \in \mathbb{R}^2$. If $\eta > 0$ is bounded then, in any normal chart,

$$\lim_{x \to -\infty} \left| \int_{-\eta}^{\eta} \partial_x^k g(x-a, y-b) dy \right| < \infty \iff \lim_{x \to -\infty} \left| \int_{-\eta}^{\eta} \partial_x^k g(x-a, y-b) + \epsilon(x, y) dy \right| < \infty,$$
where \((a, b)\) are the coordinates of the point that separates the two separatrices in the normal chart. Hence, in all normal charts, \(\theta_{r,k} \circ T_{(a,b)}([g]) = \theta_{r,k} \circ T_{(a,b)}([g'])\) for all \(g' \in U_e\), namely \(U_e \subset A\), namely \(A\) is open.

Now, let \(\{g_n\}\) a sequence of elements of \(A\) converging to \(g \in C^r(\mathbb{R}^2)\) in the strong topology. Then, almost all the \(g_n\) coincide with \(g\) outside of some compact set and, therefore, in any normal chart,

\[
\lim_{x \to 0^-} \int_{-\epsilon}^{\epsilon} \partial_x^k g(x, y) dy = \lim_{x \to 0^-} \int_{-\epsilon}^{\epsilon} \partial_x^k g_n(x, y) dy,
\]

i.e. \(\theta_{r}([g]) = \theta_{r}([g_n])\), for almost all \(n\), namely \(A\) is closed.

2. In case of \(L^2(C^{k+1}(\mathbb{R}^2))\), it is enough to observe that the property of being \(C^{k+1}\) in the first variable in every normal chart is clearly destroyed by a generic \(C^k\) small perturbation and is not preserved by \(C^k\) convergence unless \(k + 1 = k\), namely unless \(k = \infty\).

3. The proof is the same as in point 1. The limits to the values of \(l\) are due to the fact that, for larger values, \(W_{10}^{k,\beta}\) functions are at least \(C^{k+1}\) and therefore their set is neither open or closed in the \(C^k\) topology. \(\square\)

**The non-Hamiltonian case.** The method we developed for Hamiltonian vector fields is much less powerful when \(\xi\) is not Hamiltonian. The main reason for this is that, in this case, the projection \(C^r(\mathcal{F}_\xi) \to C^r(U)\) that sends \(C^r\) functions \(f\) to their restriction \(f|_U\) to an open set \(U \subset \mathcal{F}_\xi\) is not always surjective when \(U\) contains separatrices \([HR57]\). Hence there are constraints to the choice of the function \(\varphi\) of Problem 1 since, if \(\varphi\) does not extends to a global \(C^r\) first integral of \(\xi\), then the extension of the solution via the method of characteristics will sooner or later diverge on some of the separatrices.

We start by assuming that \(\xi\) is of finite type and recall the following property:

**Proposition 2.** If \(\xi \in \mathfrak{X}_r(\mathbb{R}^2)\) is not Hamiltonian, the differential of any generator of \(\ker L^r(\xi)\), \(r \geq 1\), is zero on some of the separatrices of \(\xi\). Similarly, the first derivative of every solution of \(L^r(\xi) f = g\) on some of the separatrices is determined by \(g\) modulo constants.

**Proof.** Since \(\xi\) is non-Hamiltonian, then the foliation \(\mathcal{F}_\xi\) has the following property: there exist two adjacent separatrices \(s_1\) and \(s_2\) such that, taken any two corresponding transversal segments \(t_1\) and \(t_2\), parametrized by the natural parameters \(\eta_1, \eta_2\) with respect to the Euclidean metric in such a way that \(\eta_i = 0\) is the coordinate of \(s_i \cap t_i\) and that both coordinates are positive for the points of \(t_1\) and \(t_2\) inside \(\pi_{\mathcal{F}_\xi}^1(\pi_{\mathcal{F}_\xi}^{-1}(t_1) \cap \pi_{\mathcal{F}_\xi}^{-1}(t_2))\), then \(\eta_1(\eta_2) = \eta_2^\beta + O(\eta_2^{\alpha})\), with \(\beta > \alpha\) and \(\alpha \neq 1\). In other words, the leaves of \(\mathcal{F}_\xi\) approach the two separatrices at different rates. Assume now, for the argument’s sake, that \(\alpha < 1\), and define a germ of a function \(\varphi(\eta_1)\) on \(t_1\). Then, on \(t_2\), this function becomes \(\psi(\eta_2) = \varphi(\eta_1(\eta_2)) = \varphi(\eta_2^\beta + O(\eta_2^{\alpha}))\), so that

\[
\frac{d\psi}{d\eta_2} \bigg|_{\eta_2} = \alpha \frac{d\varphi}{d\eta_1} \bigg|_{\eta_2^\beta + O(\eta_2^{\alpha})} + O(\beta - 1)
\]
and therefore we must have \( \frac{dx}{d\eta_1} \bigg|_{\eta_1=0} = 0 \) in order to be able to extend \( \varphi \) to a \( C^1 \) function beyond \( t_2 \).

Regarding the second part, through the method of characteristics we have that
\[
\psi(\eta_2) = \int_0^T \left( (\Phi^\xi_\tau)^*g\right)(x(\eta_2), y(\eta_2)) dt + \varphi(\eta_1(\eta_2)),
\]
where \( \Phi^\xi_\tau \) is the flow of \( \xi \). Hence
\[
\frac{d\psi}{d\eta_2} \bigg|_{\eta_2=0} = \left[ \alpha \frac{\partial_1 T(\eta_1(\eta_2), \eta_2) + \partial_2 T(\eta_1(\eta_2), \eta_2)}{\eta_2} \right] \left( (\Phi^\xi_\tau)^*g\right)(x(\eta_2), y(\eta_2)) + \frac{\alpha}{\eta_2^\delta} \frac{d\varphi}{d\eta_1} \bigg|_{\eta_1=O(\eta_2^\alpha)} + O(\beta - 1)
\]
and therefore we must have
\[
\frac{d\varphi}{d\eta_1} \bigg|_{\eta_1=0} + \lim_{\eta_2 \to 0} \partial_1 T(\eta_1(\eta_2), \eta_2)\left( (\Phi^\xi_\tau)^*g\right)(x(\eta_2), y(\eta_2)) = 0
\]
in order to be able to extend \( \varphi \) to a \( C^1 \) function beyond \( t_2 \).

**Remark 3.** Note that the derivative of \( \psi(\eta_2) \) is not necessarily null at \( \eta_2 = 0 \).

Let \( F \in \ker L^{(r)}_\xi \) and let \( G = \{dG = 0\} \) be any Hamiltonian transversal foliation which minimally separates \( F_\xi \). Then \( \Phi^G = (F,G) : \mathbb{R}^2 \to \mathbb{R}^2 \) is a \( C^r \) locally injective map whose rank is 1 on some of the separatrices. In order to make \( F \) and \( G \) both regular, we switch to the \( \mathbb{R}^2_{FG} \) differential structure of the plane. Since both \( F \) and \( G \) are \( C^r \), then \( C^r(\mathbb{R}^2_{FG}) \subseteq C^r(\mathbb{R}^2) \). By repeating all steps as in the previous section, we get the following, weaker, result

**Theorem 8.** Let \( \xi \) be a planar vector field of finite type, \( F \) a generator of \( \ker L^{(r)}_\xi \), \( G \) a \( C^r \) transversal foliation minimally separating \( F_\xi \) and \( \xi' = \xi/L_\xi G \) the Hamiltonian vector field of \( F \) with respect to the symplectic form \( dF \wedge dG \). Then, for all \( k = 0, \ldots, \infty \), \( r = 0, \ldots, k \), \( l = 0, \ldots, k + 3 \) and \( p \geq 1 \):

1. \( g \in L^{k}_{ \xi'} \left(C^r(\mathbb{R}^2) \right) \cap C^k(\mathbb{R}^2) \) if \( [T(a,b_i)(\Phi^G) + g]_{\mathcal{S}^{r-1,k}_{\xi'}(\mathbb{R}^2)} \in \Theta_r,k \) for all \( i \);
2. \( g \in L^{k}_{ \xi'} \left(C^r(\mathbb{R}^2) \right) \) if \( (\Phi^G)_{a,b_i} g \) is \( C^{k+1} \) in the first variable
   and \( [T(a,b_i)(\Phi^G) + g]_{\mathcal{S}^{r-1,k}_{\xi'}(\mathbb{R}^2)} \in \Theta_{k+1,k} \) for all \( i \);
3. \( g \in L^{k}_{ \xi'} \left(W^{l,p}_{\text{loc}}(\mathbb{R}^2) \right) \cap C^k(\mathbb{R}^2) \) if \( [T(a,b_i)(\Phi^G) + g]_{\mathcal{S}^{r-1,k}_{\xi'}(\mathbb{R}^2)} \in \Theta_{l,r,k} \) for all \( i \).

In this case the conditions are sufficient but not necessary because the cohomological equation can have \( C^r(\mathbb{R}^2) \) solutions which do not belong to \( C^r(\mathbb{R}^2_{FG}) \). In \( \mathbb{R}^2_{FG} \), such solution look like \( C^r \) functions whose derivatives of order \( r \) diverge on some of the separatrices where \( dF = 0 \) (see Proposition 1). Nevertheless, Theorem 8 is enough to extend Corollary 1 to vector fields of finite type.
Corollary 2. Under the hypotheses of Theorem 8 let \( P \) be the set of all points \((a,b)\) that separate pairs of adjacent separatrices of \((\Phi_{FG})_\xi \) in \( \Phi_{FG}(\mathbb{R}^2) \). Then the following inclusions hold:

1. \( L_{\xi}^F(C^r(\mathbb{R}^2)) \cap C^k(\mathbb{R}^2) \supset \Phi_{FG}^*(C^r(\mathbb{R}^2) \cap C^k(\mathbb{R}^2 \setminus P)) \);
2. \( L_{\xi}^F(C^{k+1}(\mathbb{R}^2)) \supset \Phi_{FG}^*(C^{k+1}(\mathbb{R}^2)) \);
3. \( L_{\xi}^F(W^{l,p}_{loc}(\mathbb{R}^2)) \cap C^k(\mathbb{R}^2) \supset \Phi_{FG}^*(W^{l,p}_{loc}(\mathbb{R}^2) \cap C^k(\mathbb{R}^2 \setminus P)) \).

The case when separatrices are not isolated is more pathological. We briefly discuss here only the limit case, when separatrices are dense on the plane in such a way that \( C^1(\mathcal{F}_\xi) \) contains only the constant functions. For such a \( \xi \), every \( g \in L_{\xi}^F(C^r(\mathbb{R}^2)) \) determines uniquely, modulo constants, the solution of the cohomological equation (and, therefore, its restriction to any line). In particular, the method of characteristics in this context can be applied only to \( C^0 \) functions since, given a transversal \( t \), a generic \( \varphi \in C^r(t) \), \( r \geq 1 \), will lead eventually to some ineliminable divergence. In \( \mathbb{R}^2 \Phi_{FG} \), though, both \( F \) and \( G \) are regular and the method of characteristic can be used to study the solvability of the cohomological equation in \( C^r(\mathbb{R}^2 \Phi_{FG}) \) for all values of \( r \).

Regarding the existence of more regular solutions with respect to the standard differential structure, we recall that, as shown in Proposition 2, \( g \) determines the derivative of the restriction of the solution of the equation \( L_\xi^{(1)} f = g \) on a dense set \( A_t \) of any transversal \( t \).

Definition 7. We say that a \( C^r \) function \( \varphi \) on \( t \) is \( \xi \)-compatible with \( g \) if its derivatives coincide with those induced by \( g \), via \( \xi \), in all points of \( A_t \).

We are lead therefore to the following result:

Theorem 9. Let \( \xi \in \mathcal{X}_r(\mathbb{R}^2) \) be such that \( \dim C^1(\mathcal{F}_\xi) = 1 \) and for each \( s_i \in S_\xi \) select a transversal \( t_i \). Then \( g \in L_{\xi}^F(C^r(\mathbb{R}^2)) \cap C^k(\mathbb{R}^2) \) iff, for each transversal \( t_i \), there exists a \( C^r \) function \( \varphi_i \) on \( t_i \) \( \xi \)-compatible with \( g \).

5 Examples

In this section we present four model examples.

5.1 \( \xi = 2y \partial_x + (1 - y^2) \partial_y \)

This vector field is Hamiltonian and invariant with respect to horizontal translations. It is easy to see that its only separatrices are the straight lines \( y = \pm 1 \).

A regular first-integral of \( L_\xi \) is the function \( F(x, y) = (y^2 - 1)e^x \) and a solution to the partial differential inequality \( L_\xi G \neq 0 \) is given by \( G(x, y) = -2ye^x \). It is easy to verify that \( \mathcal{G} = \{dG = 0\} \) separates minimally \( \mathcal{F}_\xi \). Note that, in this particular case, \( \Phi_{FG} \) is globally injective. In the normal chart \((\mathbb{R}^2, \Phi_{FG})\),
Figure 3: [left] Foliations of the integral trajectories of $\xi = 2y\partial_x + (1 - y^2)\partial_y$ (continuous lines) and $\eta = \partial_x - y\partial_y$ (dashed lines). These are tangent, respectively, to the level sets of the regular functions $F(x, y) = (y^2 - 1)e^x$ and $G(x, y) = -2ye^x$. The leaves $y = \pm 1$ are the only pair of inseparable leaves in $\mathcal{F}_\xi$, while $\mathcal{F}_\eta \simeq \mathbb{R}$ has no inseparable leaves. Note that, since $\xi$ is intrinsically Hamiltonian, $C^1(\mathcal{F}_\xi)$ contains regular functions. [right] Image of $\mathcal{F}_\xi$ and $\mathcal{F}_\eta$ via $\Phi_{FG}$. The leaves of $\mathcal{F}_\xi$ become vertical lines, those of $\mathcal{F}_\eta$ horizontal ones. The image, under $\Phi_{FG}$, of the sets $|y| < 1$, $y = 1$, $y = -1$ and $y = 0$ in the left picture are respectively represented, in the right one, by the sets $(-\infty, 0) \times \mathbb{R}$, $\{0\} \times (-\infty, 0)$, $\{0\} \times (0, \infty)$ and $(-\infty, 0) \times \{0\}$.

the point that separates the two separatrices has coordinates $(0, 0)$. A straight calculation shows that

$$\Omega_{FG} = dF \wedge dG = 2(1 + y^2)e^{2x} \, dx \wedge dy$$

and, correspondingly,

$$\xi'_{FG} = \Omega_{FG}^{-1}(dF) = \frac{e^{-x}}{2(1 + y^2)} \xi, \quad \xi'_{FG} = \Omega_{FG}^{-1}(dG) = \frac{e^{-x}}{y^2 + 1} (\partial_x - y\partial_y).$$

By Proposition 6 in [De 11], $(\Phi_{FG})_*\xi = \partial_{y'}$ and $(\Phi_{FG})_*\xi = \partial_{x'}$, where we set $(x', y') = (F, G)$, and the cohomological equation writes, in the normal chart $(\mathbb{R}^2 \setminus [0, \infty) \times \{0\}, \Phi_{FG})$, as $\partial_y f = \hat{g}$.

By Theorem 6, the equation $L_{\xi'_{FG}} f = g \in C^k(\mathbb{R}^2)$ has a $C^r$ solution, $r = 1, \ldots, k$, if and only if $|(\Phi_{FG})_* g|_{S^{k, r+1}(L^2)} \in \Theta_{r,k}$, namely if and only if the $C^{k+1}$ function $\varphi(x') = \int_{-1}^1 \hat{g}(x', y') \, dy' : [-\infty, 0] \to \mathbb{R}$ can be extended to a $C^r$ function at $x' = 0$. The solution is, instead, $W^{l,p}_{loc}$ if and only if $|(\Phi_{FG})_* g|_{S^{k+1, r+1}(L^2)} \in \Theta_{l,p,k}$, namely if $\varphi(x')$ has a $W^{l,p}$ singularity at $x' = 0$. Below we discuss in some detail a few concrete cases.

As shown in the proof of Theorem 5, the condition

$$|\hat{g}(x', y')| \leq C \left[(x')^2 + (y')^2\right]^{-\alpha}, \quad C > 0$$

is enough to grant the existence of $C^0$ solutions for $\alpha < 1/2$ and of $L^1_{loc}$ solutions for $1/2 \leq \alpha < 1$. 

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Consider, for example, the function
\[
\hat{g}(x', y') = \left[(x')^2 + (y')^2\right]^{-1/4} \in C^\infty(\mathbb{R}^2 \setminus (0, 0)),
\]
so that
\[
g(x, y) = \Phi^x \hat{g}(x, y) = \frac{e^{-x/2}}{\sqrt{1 + y^2}} \in C^\infty(\mathbb{R}^2).
\]
Then a solution to \(L_{\xi,f} g = f\) is given, in the normal chart \((x', y')\), by the function
\[
\hat{f}(x', y') = y' \left[1 + \frac{(y')^2}{(x')^2}\right]^{\frac{3}{4}} \cdot 2F_1 \left(1, \frac{5}{4}, \frac{3}{2}; -\frac{(y')^2}{(x')^2}\right),
\]
where \(2F_1(a, b, c; z)\) is the Gaussian hypergeometric function, which writes as
\[
f(x, y) = -ye^x \left[1 + \frac{4y^2}{(y^2 - 1)^2}\right]^{\frac{3}{4}} \cdot 2F_1 \left(1, \frac{5}{4}, \frac{3}{2}; -\frac{4y^2}{(y^2 - 1)^2}\right) \in C^0(\mathbb{R}^2)
\]
in the \((x, y)\) coordinates.

On the other side, for
\[
\hat{g}(x', y') = \frac{1}{\sqrt{(x')^2 + (y')^2}} \in C^\infty(\mathbb{R}^2 \setminus (0, 0)),
\]
amely
\[
g(x, y) = \frac{e^{-x}}{1 + y^2} \in C^\infty(\mathbb{R}^2),
\]
we get
\[
\hat{f}(x', y') = \ln \left(\sqrt{(x')^2 + (y')^2} + y'\right),
\]
amely
\[
f(x, y) = x + 2 \ln |1 - y| \in L^1_{loc}(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \setminus S_\xi).
\]
Note that the equation \(L_{\xi,f} f(x, y) = \frac{e^{-x}}{1 + y^2}\) is equivalent to \(L_{\xi} f(x, y) = 2\), which is why we found exactly the solution we already discussed in Example 1.

More generally, the condition
\[
|\hat{g}(x', y')| \leq C \left[(x')^2 + (y')^2\right]^{-\alpha}, \quad (x', y') \in U_0, C > 0
\]
where \(U_0\) is some left neighborhood of the origin, writes down, in the original coordinates \((x, y)\), as
\[
|g(x, y)| \leq Ce^{2\alpha x} \left[1 + y^2\right]^{-2\alpha}, \quad (x, y) \in S_M
\]
where \(S_M = (-\infty, -M) \times (-1, 1)\), for some \(M > 0\). Since \(1 + y^2\) is bounded and larger than 1 for every \(y \in S_M\), this means that \(g\) will give rise to \(C^0\) solutions of \(L_{\xi} f = g\) when \(|g(x, y)| \leq e^{-\alpha x}\) for \(\alpha < 1/2\) and to \(L^1_{loc}\) solutions for \(\alpha < 1\).

The latter is the same condition given in [DGK], Proposition 3.1, for the existence of \(L^1_{loc}\) solutions to \(L_{\xi} f = g\). This approach though shows that it is enough that this inequality be satisfied by \(g\) on some neighborhood of \(x = -\infty\) within the strip \(|y| < 1\) rather than on the whole plane.
5.2 $\xi = 2(2y - 1) \partial_x + (1 - y^2) \partial_y$

This vector field has the same separatrices as the previous one but it is non-Hamiltonian. A smooth generator of $\ker L_\xi$ is given by $F(x,y) = (y + 1)^3(y - 1)e^x$ and a regular Hamiltonian function for a transverse foliation that minimally separates $\mathcal{F}_\xi$ is given by $G(x,y) = (2y - 1)e^x$. In this case

$$\Omega_{FG} = dF \wedge dG = 2(1 + y)^2(2 - 4y + 3y^2)e^{2x}dx \wedge dy$$

is degenerate on the separatrix $y = -1$. Correspondingly,

$$\xi'_F = \Omega_{FG}^{-1}(dF) = \frac{e^{-x}}{2(2 - 4y + 3y^2)} \xi$$

is regular, while

$$\xi'_G = \Omega_{FG}^{-1}(dG) = \frac{e^{-x}}{2(1 + y)^2(2 - 4y + 3y^2)} \left(2\partial_x + (1 - 2y)\partial_y\right).$$

diverges on $y = -1$. Similarly, $\Phi_{FG}$ maps, like in the previous example, the whole plane injectively into $\mathbb{R}^2 \setminus [0, \infty) \times \{0\}$, but this time $\Phi_{FG}$ is not an immersion since its differential is zero on the separatrix $y = -1$.

When we restrict $\Phi_{FG}$ to the set $\{|y| < 1\}$, the local coordinates $(x', y') = (F, G)$ are well-defined and smooth we can repeat verbatim all calculation for the explicit solutions shown in the previous example. This corresponds to switching differential structure in $\mathbb{R}^2$ and looking for solutions in $\mathbb{R}^2_{\Phi_{FG}}$. We recall that,
which is the image of $\xi$ since $F$ behaves as $\xi$ of class $C^{\min}$ minimally separated by it. In particular, $g$ is everywhere transversal to the foliation in horizontal straight lines and it is $\pm$ everywhere transversal to the foliation and therefore are not covered by the theorems in [DGK11]. A direct calculation shows that this vector field and the next one are not invariant with respect to any translation and therefore are not covered by the theorems in [DGK11].

Unlike the Hamiltonian case though, now the solvability conditions based on the germs of $\hat{g}$ in a neighborhood of the separatrices are only sufficient. New solutions, not covered by the theorems in [DGK11], can be found by letting $\hat{g}$, in the normal chart coordinates, diverge on the separatrices, as long as $\Phi_{FG} \hat{g}$ has the required differentiability. Consider, for example, the case of

$$\hat{g}(x', y') = 3\sqrt{x'} \left[ y' - \sqrt{(x')^2 + (y')^2} \right]^2$$

This function behaves as $4\sqrt{x'} (y')^2$, of class $C^0$, in a neighborhood of the separatrix $\{0\} \times (0, \infty)$, which is the image of $y = -1$, and more regularly, as $4 \sqrt{x'} (y')^2$, of class $C^4$, in a neighborhood of the separatrix $\{0\} \times (-\infty, 0)$, which is the image of $y = 1$. In $\mathbb{R}^2$, therefore, $\hat{g}$ is of class $C^0$ and so it gives rise to a globally $C^0$ solution

$$\hat{f}(x', y') = 3\sqrt{x'} \left[ (x')^2 y' + \frac{2}{3} (y')^3 - \frac{2}{3} (x')^2 + (y')^2 \right]^{3/2}$$

In $\mathbb{R}^2$, instead, the solution is more regular:

$$\left( \Phi_{FG}^* \hat{f} \right)(x, y) = (1 + y)(1 - y)^{1/3} \left[ F^2 + \frac{2}{3} G^3 + \frac{2}{3} \left( F^2 + G^2 \right)^{3/2} \right] e^{x/3}$$

behaves as $(y - 1)^{13/3} e^{4x/3}$ close to $y = 1$, i.e. of class $C^1$, and is smooth close to $y = 1$, so we have a globally $C^4$ solution.

5.3 $\xi = 2x^2 y \partial_x - \partial_y$

This vector field and the next one are not invariant with respect to any translation and therefore are not covered by the theorems in [DGK11]. A direct calculation shows that $L_\xi F(x, y) = 0$ for the regular function

$$F(x, y) = \begin{cases} \tan^{-1} \left[ y^2 - \frac{1}{x} \right], & x < 0; \\ \pi/2, & x = 0; \\ \tan^{-1} \left[ y^2 - \frac{1}{x} \right] + \pi, & x > 0, \end{cases}$$

namely $\mathcal{F}_\xi$ is Hamiltonian. The set $F^{-1}(c)$ has two connected components for $c \geq \pi$ and only one for $c < \pi$, so the only inseparable integral trajectories of $\xi$ are the two connected components of $F^{-1}(\pi)$, namely the curves $y = \pm 1/\sqrt{x}$ (see Fig. 5, left). Since the $y$ component of $\xi$ is always non-zero, $\mathcal{F}_\xi$ is everywhere transversal to the foliation in horizontal straight lines and it is minimally separated by it. In particular, $L_\xi G(x, y) > 0$ for $G(x, y) = y$ (see Fig. 5). In this case

$$\Omega_{FG} = dF \wedge dG = \frac{dx \wedge dy}{(1 - xy^2)^2 + x^2}.$$

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Figure 5: [left] Foliations of the integral trajectories of $\xi = 2x^2y\partial_x - \partial_y$ (continuous lines) and $\eta = \partial_x$ (dashed lines). These are tangent, respectively, to the level sets of the regular functions $F(x, y) = y^2 - e^{-x}$ and $G(x, y) = y$. The leaves $y^2 = e^x$ are the only pair of inseparable leaves in $\mathcal{F}_\xi$, while $\mathcal{F}_\eta \simeq \mathbb{R}$ has no inseparable leaves. Note that, since $\xi$ is intrinsically Hamiltonian, $C^1(\mathcal{F}_\xi)$ contains regular functions. [right] Image of $\mathcal{F}_\xi$ and $\mathcal{F}_\eta$ via $\Phi_{FG}$. The leaves of $\mathcal{F}_\xi$ become vertical lines, those of $\mathcal{F}_\eta$ horizontal ones. The images of the two separatrices are the sets $\{\pi\} \times (-\infty, 0)$ and $\{\pi\} \times (0, \infty)$, the image of the line $y = 0$ is the set $(0, \pi) \times \{0\}$.

and we get

$$\xi' = -((1 - xy^2)^2 + x^2)\xi, \quad \xi' = ((1 - xy^2)^2 + x^2)\partial_x.$$ 

The image of the plane via the map $\Phi_{FG}$ is the set (see Fig. [right])

$$\tan^{-1}(y')^2 < x' < \tan^{-1}(y')^2 + \pi.$$ 

All explicit calculations shown in the first example can be repeated here. For example, this time the condition

$$|\hat{g}(x', y')| \leq C [(x')^2 + (y')^2]^{-\alpha}, \quad (x', y') \in U_0,$$

for the existence of regular and weak solutions of the cohomological equation translates, in $(x, y)$ coordinates, into

$$|g(x, y)| \leq Cx^{2\alpha}, \quad x \in S_M,$$

where in this case $S_M, M > 0$, is the portion of the set $y^2 - 1/x < 0$ contained in the half-plane $x > M$. The corresponding condition for solutions of $L_\xi f = g$ is

$$|g(x, y)| \leq C \frac{x^{2\alpha}}{(1 - xy^2)^2 + x^2} \leq C' x^{2(\alpha - 1)}, \quad x \in S_M,$$

namely $L_\xi f = g$ admits $C^0$ solutions when $|g(x, y)| \leq Cx^{-1-\epsilon}$ for some $\epsilon > 0$ and $L^1_{\text{loc}}$ solutions when $|g(x, y)| \leq Cx^{-\epsilon}$ for some $\epsilon > 0$. 

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Figure 6: [left] Foliations of the integral trajectories of $\xi = 3(1 + e^x y^2)^2 - e^x(6 - 19e^x + 22e^x y^2)/3 + 2e^x(5 + 3y^2) + 3e^{2x}(1 - y^2)^2 \partial_x + \partial_y$ (continuous lines) and $\eta = \partial_x$ (dashed lines). These are tangent, respectively, to the level sets of the regular functions $F(x, y) = (1 - (y - 2)^2 - e^{-x})(1 - y^2 - e^{-x})$ and $G(x, y) = y$. The three leaves $(y \pm 2)^2 = 1 - e^{-x}$ and $y^2 = 1 - e^{-x}$ are the only separatrices of $F_\xi$ and are all inseparable from each other, while $F_\eta \simeq \mathbb{R}$ has no inseparable leaves. Note that, since $\xi$ is intrinsically Hamiltonian, $C^1(F_\xi)$ contains regular functions. [right] Image of $F_\xi$ and $F_\eta$ via $\Phi_{FG}$. The leaves of $F_\xi$ become vertical lines, those of $F_\eta$ horizontal ones. The image, under $\Phi_{FG}$, of the three separatrices are the sets $\{0\} \times (3, 1)$, $\{0\} \times (1, -1)$ and $\{0\} \times (-1, -3)$; the images of the lines $y = \pm 1$ are the sets $(-\infty, 0) \times \{1\}$ and $(-\infty, 0) \times \{-1\}$.

5.4 $\xi = 3(1 + e^x y^2)^2 - e^x(6 - 19e^x + 22e^x y^2)/3 + 2e^x(5 + 3y^2) + 3e^{2x}(1 - y^2)^2 \partial_x + \partial_y$

Also this last vector field gives rise to a Hamiltonian foliation. For example, a generator of $\ker L_\xi$ is given by the smooth regular function

$F(x, y) = (1 - (y - 2)^2 - e^{-x})(1 - y^2 - e^{-x})(1 - (y + 2)^2 - e^{-x})$.

Using the Descartes’ rule of signs it is easy to verify that, for $c > 0$, the level sets $F^{-1}(c)$ are connected while, for $c < 0$, each level set consists of three disjoint lines. The three curves in the level set $F^{-1}(0)$ are therefore inseparable from each other. Since the $y$ component of $\xi$ is always different from 0, $\xi$ is transversal to every horizontal line and it is easily seen that the foliation in horizontal lines minimally separates it (see Fig. 6). In particular, $L_\xi G = 1 > 0$ for $G(x, y) = y$. The image of the plane via $\Phi_{FG}$ is the set

$x' < (1 - (y' - 2)^2)(1 - (y')^2)(1 - (y' + 2)^2)$

(see Fig. 6 right).

In this case,

$\Omega_{FG} = dF \wedge dG = dx \wedge dy$,

so that

$\xi_F = \Omega_{FG}^{-1}(dF) = \xi, \quad \xi_G = \Omega_{FG}^{-1}(dG) = \partial_x$
The condition for the existence of solutions of $L_\xi f = g$

$$|g(x', y')| \leq C \left[ (x')^2 + (y')^2 \right]^{-\alpha}, \quad (x', y') \in U_{\pm 1},$$

where $U_1$ and $U_{-1}$ are left neighborhoods of, respectively, $(0, 1)$ and $(0, -1)$, translates now, in $(x, y)$ coordinates, into

$$|g(x, y)| \leq Ce^{3x}, \quad (x, y) \in S_{M, \pm 1},$$

where in this case $S_{M,1}$ (resp. $S_{M,2}$), $M > 0$, is the portion of the set $F(x, y) < 0$ contained in the half-plane $x > M$. Hence, for example, $L_\xi f = g$ admits $C^0$ solutions if $|g(x, y)| \leq Ce^{(3/2-\epsilon)x}$, $(x, y) \in S_{M, \pm 1}$, for some $\epsilon > 0$ and $L_\xi^1$ solutions if $|g(x, y)| \leq Ce^{(3-\epsilon)x}$, $(x, y) \in S_{M, \pm 1}$, for some $\epsilon > 0$.

By modifying suitably this particular $F(x, y)$, it is easy to obtain examples of intrinsically Hamiltonian and intrinsically non-Hamiltonian regular vector fields whose foliation has a single node at which concur any number of inseparable separatrices and which is minimally separated by the horizontal foliation.

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