Level-$\delta$ limit linear series

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Abstract

We introduce the notion of level-$\delta$ limit linear series, which describe limits of linear series along families of smooth curves degenerating to a singular curve $X$. We treat here only the simplest case where $X$ is the union of two smooth components meeting transversely at a point $P$. The integer $\delta$ stands for the singularity degree of the total space of the degeneration at $P$. If the total space is regular, we get level-1 limit linear series, which are precisely those introduced by Osserman [10]. We construct a projective moduli space $G_{r,d,\delta}(X)$ parameterizing level-$\delta$ limit linear series of rank $r$ and degree $d$ on $X$, and show that it is a new compactification, for each $\delta$, of the moduli space of Osserman exact limit linear series, an open subscheme $G_{r,d,1}(X)$ of the space $G_{r,d,1}(X)$ already constructed by Osserman. Finally, we generalize [6] by associating to each exact level-$\delta$ limit linear series $g$ on $X$ a closed subscheme $P(g) \subseteq X^{(d)}$ of the $d$th symmetric product of $X$, and showing that $P(g)$ is the limit of the spaces of divisors associated to linear series on smooth curves degenerating to $g$ on $X$, if such degenerations exist. In particular, we describe completely limits of divisors along degenerations to such a curve $X$.

1 Introduction.

The theory of linear series, meaning spaces of sections of line bundles, has a long history and plays an important role in Algebraic Geometry. The special case of curves is particularly rich and has been investigated from several directions. Following on their proof of the Brill–Noether Theorem in the 1980’s, Eisenbud and Harris [3] introduced the notion of limit linear series, providing a powerful framework to study degenerations of linear series on families of smooth curves degenerating to curves of compact type. As a consequence of their general theory, they were able to simplify the proof of the Brill-Noether Theorem and to prove a number of other results about curves; see the introduction to [3].

The success achieved by Eisenbud and Harris in the 80’s, and further applications of the theory of limit linear series, particularly in computing divisors in the moduli space of stable curves, has motivated further study into the foundational aspects of the theory. For starters it was observed that most of the applications of the theory are obtained by considering an open

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subscheme of the projective moduli space $G^r_d(X)$, parameterizing limit linear series of rank $r$ and degree $d$ on a curve of compact type $X$, that whose points correspond to special limit linear series called refined. Not always limit linear series on a curve $X$ are limits of linear series on smooth curves degenerating to $X$. In any case, refined limit $g^1_d$ are always limits of linear series, and for $g^r_d$ with $r \geq 2$ a similar result, called Regeneration Theorem, holds; see [3], Thm. 3.4, p. 360. This is Eisenbud’s and Harris’ main theorem, the one often used in applications. Unfortunately, there are points on $G^r_d(X)$ that are not refined, but carry important information, as they correspond to limit linear series that are actually limits of linear series.

A fundamental breakthrough was made by Osserman [10] in the 2000’s, when he gave a new definition of limit linear series on a curve $X$ and constructed a projective moduli space $G^r_{d, \text{Oss}}(X)$ parameterizing those new objects. An Osserman limit linear series carries more information than a usual one, and thus there is a forgetful map $G^r_{d, \text{Oss}}(X) \to G^r_d(X)$. The map is an isomorphism over the refined locus of $G^r_d(X)$, and thus $G^r_{d, \text{Oss}}(X)$ can be viewed as a different natural compactification of this locus; see [10], Section 6, p. 1183. Among the Osserman limit linear series there are those called exact, which form an open subscheme of $G^r_{d, \text{Oss}}(X)$. Exact limit linear series are more amenable to work. And they are dense among all limit linear series, by [9], Cor. 1.4, p. 4034, at least if $X$ is general. Furthermore, all limits of linear series along families of smooth curves degenerating to $X$ are exact, as long as the total space of the family is regular; see [6], Section 5, p. 90. The space $G^r_{d, \text{Oss}}(X)$ has a defect similar to that of $G^r_d(X)$: there are non-exact limit linear series that are limits of linear series, along a nonregular smoothing of $X$.

Exact limit linear series are also special in the following sense: A linear series $g$ of rank $r$ and degree $d$ on a smooth curve $C$ corresponds to a subscheme $\mathbb{P}(g)$ of the symmetric product $C^{(d)}$ of $d$ copies of $C$, whose points parameterize the divisors of zeros of sections of $g$. This correspondence is fundamental in the theory of curves. Given a family of linear series $g$ of degree $d$ on smooth curves degenerating to a singular curve $X$, we may ask what the limit of $\mathbb{P}(g)$ in $X^{(d)}$ is. In [6] Osserman and the first author considered the corresponding subscheme $\mathbb{P}(g)$ of $X^{(d)}$ associated to an Osserman limit linear series $g$ on $X$. It is defined as for smooth curves but, as certain sections of $g$ may vanish on a whole component of $X$, the subscheme $\mathbb{P}(g)$ is actually the closure of the locus of divisors of zeros of the other sections. The two showed then that, if $g$ is exact, then $\mathbb{P}(g)$ has the expected Hilbert polynomial, as if $g$ were a limit, and in this case $\mathbb{P}(g)$ is actually the limit of the schemes corresponding to the linear series degenerating to $g$; see [6], Thms. 4.3 and 5.2. The converse is shown here, our Theorem 5.5.

Our goal in the present paper is to study limits of linear series along nonregular smoothings of a given $X$, that is, along families of smooth curves degenerating to $X$ whose total space is not regular. In principle, as $X$ is nodal, one could replace the family by its semistable reduction, thus replacing $X$ by a curve $\tilde{X}$ obtained from $X$ by replacing the nodes by chains of rational curves. However, dealing with $\tilde{X}$ instead of $X$ is substantially more difficult in the approach by Osserman. So much difficult that, though Eisenbud and Harris developed their theory for curves of compact type, only very recently ([11] and [12]) has Osserman extended his theory to curves...
X that are not simply unions of two smooth components meeting transversely at a single point P. That is to say: From the above third paragraph of this introduction on, and throughout the whole article, X stands actually for such a simple curve!

So we take a different approach: We introduce what we call level-δ limit linear series on X; see Definition 3.1. As in Osserman’s work, there are special level-δ limit linear series, also called exact. And we show that certain exact level-δ limit linear series arise as limits of linear series along smoothings of X whose total space has singularity degree δ at P; see the discussion before Definition 3.1. Level-1 limit linear series are simply Osserman limit linear series. Following Osserman, we construct a projective moduli space \( G_{r,d,\delta}(X) \) for level-δ limit linear series in Proposition 3.2.

Level-δ limit linear series carry more information than Ossermann limit linear series. There is in fact a forgetful map \( G_{r,d,\delta}(X) \to G_{r,\text{Oss}}(X) \). More generally, there are forgetful maps \( \rho_{\delta,\delta'}: G_{r,d,\delta}(X) \to G_{r,d,\delta'}(X) \) as long as \( \delta \mid \delta' \). In our first main result, Theorem 4.1, we show that \( \rho_{\delta,\delta'} \) is surjective and describe its fibers. As a consequence, we show in Proposition 4.3 that \( \rho_{\delta,\delta'} \) is an isomorphism over the open subscheme \( G_{r,\text{Oss}}(X) \) parameterizing exact limit linear series. Also, \( \rho_{\delta,\delta'}^{-1}(G_{r,\text{Oss}}(X)) \subseteq G_{r,d,\delta}(X) \) and \( \rho_{\delta,\delta}(G_{r,d,\delta'}(X)) = G_{r,d,\delta}(X) \) if \( \delta' > \delta \). It turns out that, for each \( \delta \), we may view \( G_{r,d,\delta}(X) \) as a compactification of the moduli space of Osserman exact limit linear series.

Finally, following [6], we associate to each level-δ limit linear series \( g \) a subscheme \( P(g) \) of \( X^{(d)} \). As in the level-1 case, we show that if \( g \) is exact, then \( P(g) \) has the expected Hilbert polynomial, as if \( g \) were a limit, and in this case \( P(g) \) is actually the limit of the schemes corresponding to the linear series degenerating to \( g \). This is contained in our last two main results, Theorems 5.4 and 5.5, where the converse is proved. The key to showing the converse is to show that if \( g = \rho_{\delta,\delta'}(g') \), then \( P(g) \subseteq P(g') \). Furthermore, equality holds if \( g \) is exact and does not hold if \( g \) is not exact but \( g' \) is.

In a forthcoming article [5], we will give yet another notion of limit linear series on X, and construct a moduli space which will be a sort of glueing of the exact loci \( G_{r,d,\delta}(X) \) for all \( \delta \), modulo a certain equivalence relation. The remarkable fact is that this new moduli space is projective, thus giving rise to a compactification of the locus of Osserman exact limit linear series, and thus of Eisenbud and Harris refined limit linear series by exact limit linear series, precisely those which have good properties, as for instance those found in the present article.

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2 Twists

Let X be a projective curve defined over an algebraically closed field \( k \). Assume that X has exactly two irreducible components, denoted Y and Z, that they are smooth and intersect transversally at a single point, denoted \( P \). Let \( \pi: X \to B \) be a smoothing of X, that is, a flat,
projective map to \( B := \text{Spec}(k[[t]]) \) whose generic fiber is smooth and special fiber is isomorphic to \( X \). We let \( \eta \) and \( o \) denote the generic and special points of \( B \), and \( X_\eta \) and \( X_o \) the respective fibers of \( \pi \). Notice that, by semicontinuity, not only is \( X_\eta \) smooth, but also geometrically connected. We will fix an identification of \( X_o \) with \( X \).

Since \( \pi \) is flat and \( B \) is regular, the total space \( \mathcal{X} \) is regular except possibly at the node \( P \). Furthermore, since the general fiber of \( \pi \) is smooth, there are a positive integer \( \delta \) and a \( k[[t]] \)-algebra isomorphism (see [I], pp. 104–109):

\[
\hat{\mathcal{O}}_{\mathcal{X}, P} \cong \frac{k[[t, y, z]]}{(yz - t^{\delta})}.
\]

The integer \( \delta \) is called the *singularity degree of \( \pi \) at \( P \).* (Also, we say that the singularity of \( \mathcal{X} \) at \( P \) is of type \( A_{k-1} \).) We say that \( \pi \) is a *regular smoothing* if its singularity type at \( P \) is 1, in other words, if \( \mathcal{X} \) is regular.

**Lemma 2.1.** There is a unique effective Cartier divisor of \( \mathcal{X} \) whose associated 1-cycle is \( i[Y] \) (resp. \( i[Z] \)) if and only if \( \delta | i \).

**Proof.** Since \( \mathcal{X} \) is regular off \( P \), there is a unique effective Cartier divisor \( \mathcal{Y}_i^* \) on \( \mathcal{X}^* := \mathcal{X} - P \) whose associated 1-cycle is \( i[Y - P] \). If there were an effective Cartier divisor on \( \mathcal{X} \) associated to \( i[Y] \), it would be the schematic closure \( \mathcal{Y}_i \) of \( \mathcal{Y}_i^* \), whence unique.

Now, fix an isomorphism of the form \([\mathcal{Y}_i] \), and let \( A := k[[t, y, z]]/(yz - t^{\delta}) \). Up to exchanging \( y \) with \( z \), the ideal defining \( Y \) (resp. \( Z \)) in \( \hat{\mathcal{O}}_{\mathcal{X}, P} \) corresponds to \( (y, t)A \) (resp. \( (z, t)A \)) in \( A \). Let \( I \subseteq A \) be the ideal corresponding to \( \mathcal{Y}_i \). Localizing, \( I_z = t^iA_z \), and thus \( I = I_z \cap A \).

We claim that \( I_z \cap A = y^q(y, t^r)A \), where \( q \) is the quotient in the Euclidean division of \( i \) by \( \delta \) and \( r \) is the remainder. Indeed, since \( yA_z = t^\delta A_z \), it follows that \( y^q(y, t^r)A_z = t^rA_z \), and thus \( I_z \cap A \supseteq y^q(y, t^r) \). On the other hand, if \( g \in I_z \cap A \) then there is an integer \( n \geq i \) such that \( z^ng \in t^A \). Thus \( z^n g \in z^q y^q t^r A \). Since \( y, z \) form a regular sequence of \( A \), it follows that \( g = y^g' \), where \( z^n - g' \in t^r A \). Since \( z \) is not a zero divisor modulo \( (y, t^r)A \), it follows that \( g' \in (y, t^r)A \), and thus \( g \in y^q(y, t^r)A \).

Finally, since \( I = y^q(y, t^r)A \), we have that \( I \) is principal if and only if \( (y, t^r)A \) is principal, thus if and only if \( r = 0 \). \( \square \)

We let \( \delta Y \) (resp. \( \delta Z \)) denote the effective Cartier divisor of \( \mathcal{X} \) whose associated 1-cycle is \( \delta[Y] \) (resp. \( \delta[Z] \)).

**Proposition 2.2.** The following statements hold:

(a) \( \delta Y \cdot Z = \delta Z \cdot Y = 1 \).

(b) \( \mathcal{O}_\mathcal{X}(\delta Y)|_Z \cong \mathcal{O}_Z(P) \) and \( \mathcal{O}_\mathcal{X}(\delta Z)|_Y \cong \mathcal{O}_Y(P) \).

(c) \( \mathcal{O}_\mathcal{X}(\delta Y)|_Y \cong \mathcal{O}_Y(-P) \) and \( \mathcal{O}_\mathcal{X}(\delta Z)|_Z \cong \mathcal{O}_Z(-P) \).
Proof. Fixing an isomorphism of the form (1), we have that $Y$ (resp. $Z$) is defined at $P$ by, say, $(y,t)$ (resp. $(z,t)$), whereas $\delta Y$ (resp. $\delta Z$) is defined by $y$ (resp. $z$). The first two statements follow. As for the last statement, it is enough to observe that $\delta Y + \delta Z = \text{div}(t^\delta)$, and thus $\mathcal{O}_X(\delta Y) \otimes \mathcal{O}_X(\delta Z) = \mathcal{O}_X$. \hfill \Box

Blowing up $X$ at $P$, and then successively at the singular points of each blowup, we end up with a regular scheme $\tilde{X}$ and a map $\psi: \tilde{X} \rightarrow X$ such that the composition $\pi := \pi\psi$ is a regular smoothing of its special fiber. Furthermore, the special fiber can be identified with the curve $\tilde{X}$ obtained from $X$ by splitting the branches of $X$ at the node $P$ and connecting them by a chain $E$ of rational smooth curves of length $\delta - 1$, in such a way that $\psi|_{\tilde{X}} : \tilde{X} \rightarrow X$ is the map collapsing $E$ to $P$. We say that $\psi$ is the semistable reduction of $\pi$.

We will also denote by $Y$ (resp. $Z$) the irreducible component of $\tilde{X}$ mapped isomorphically by $\psi$ to $Y$ (resp. $Z$) on $X$. Also, we identify the generic fiber of $\tilde{X}$ with that of $\pi$ through $\psi$.

Finally, we will order the rational components $E_1, \ldots, E_{\delta-1}$ in such a way that $E_1$ intersects $Y$, while $E_{\delta-1}$ intersects $Z$, and $E_i$ intersects $E_{i+1}$ for $i = 1, \ldots, \delta - 2$.

**Proposition 2.3.** Let $L_\eta$ be an invertible sheaf on $X_\eta$ of degree $d$. Then there is an invertible sheaf $\mathcal{L}$ on $\tilde{X}$ whose restriction to the generic fiber is $L_\eta$, whose restriction to $Y$ (resp. $Z$) has degree $d$ (resp. 0) and whose restriction to $E$ is trivial.

Proof. Since $\tilde{X}$ is regular, there is an invertible extension $\mathcal{M}$ of $L_\eta$ to $\tilde{X}$. Let $m$ and $n$ be the degrees of its restriction to $Y$ and $Z$, respectively, and $d_i$ the degree on $E_i$ for $i = 1, \ldots, \delta - 1$. Since $\tilde{X}$ is regular, $Y$, $Z$ and the $E_i$ are effective Cartier divisors of $\tilde{X}$. A simple computation shows that

$$\mathcal{L} := \mathcal{M} \otimes \mathcal{O}_{\tilde{X}} \left( nZ + \sum_{i=1}^{\delta-1} (d_i + \cdots + d_{\delta-1} + n)(Z + E_i + \cdots + E_{\delta-1}) \right)$$

has the required degrees. \hfill \Box

Let $\mathcal{I}$ be a torsion-free rank-1 sheaf on $X/B$. In other words, $\mathcal{I}$ is a coherent sheaf on $X$, flat over $B$, invertible everywhere but possibly at $P$, and such that $\mathcal{I}_P$ is isomorphic to an ideal of $\mathcal{O}_{X,P}$. Any invertible sheaf on $X$ is torsion-free rank-1 on $X/B$, for instance $\mathcal{O}_X$.

In [4], §3, a procedure was outlined to modify $\mathcal{I}$: Set $\mathcal{I}^{(0)} := \mathcal{I}$, and for each integer $i > 0$, define the $i$-th twist by $Z$ of $\mathcal{I}$ by:

$$\mathcal{I}^{(i)} := \ker \left( \frac{\mathcal{I}^{(i-1)}}{\text{tor}} \bigg|_{\tilde{X}} \right).$$

Notice that $\mathcal{I}^{(i)} \supseteq \mathcal{I}^{(i-1)} \mathcal{I}_{Z|\tilde{X}}$, with equality away from $P$. In particular, the $\mathcal{I}^{(i)}$ are all equal away from $Z$, thus on $X_\eta$. Furthermore, as pointed out in [4], p. 3063, it follows from an argument analogous to the one found in [3], Prop. 6, p. 100, that the $\mathcal{I}^{(i)}$ are torsion-free, rank-1 on $X/B$. Finally, it is stated in [4], Lemma 23, p. 3063, that there is a natural surjection of
short exact sequences:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{I}^{(i+1)} & \longrightarrow & \mathcal{I}^{(i)} & \longrightarrow & \frac{\mathcal{I}^{(i)}}{\text{torsion}} \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \frac{\mathcal{I}^{(i+1)|_Y}}{\text{torsion}} & \longrightarrow & \mathcal{I}^{(i)}|_X & \longrightarrow & \frac{\mathcal{I}^{(i)}}{\text{torsion}} \\
& & & & 0. & & \\
\end{array}
\]

Clearly, \(\mathcal{O}_X^{(1)} = \mathcal{I}|_{\mathcal{X}|_X}\). Also, \(\mathcal{O}_X^{(i)}\) is a sheaf of ideals containing \(\mathcal{I}|_{\mathcal{X}|_X}\) and equal to it away from \(P\), for each \(i \geq 0\). If \(\mathcal{I}\) is invertible then, by exactness of \(\mathcal{I} \otimes -\), we have that \(\mathcal{I}^{(i)} = \mathcal{I} \otimes \mathcal{O}_X^{(i)}\) for each \(i \geq 0\). In this case, it follows from [2], Prop. 3.1, p. 14, that there are isomorphisms

\[
\frac{\mathcal{I}^{(i+1)|_Y}}{\text{torsion}} \cong \mathcal{I}|_Y \otimes \mathcal{O}_Y(-(q+1)P) \quad \text{and} \quad \frac{\mathcal{I}^{(i)}|_Z}{\text{torsion}} \cong \mathcal{I}|_Z \otimes \mathcal{O}_Z(qP),
\]

where \(q\) is the quotient of the Euclidean division of \(i\) by \(\delta\). Furthermore, it follows from the local description given in the first paragraph of the proof of [2], Prop. 3.1, p. 14, that \(\mathcal{I}^{(i)}\) is invertible if and only if \(\delta|i\), in which case \(\mathcal{I}^{(i)} \cong \mathcal{I} \otimes \mathcal{O}_X(-q\delta Z)\).

Thus we have a complete description of the sheaves \(\mathcal{I}^{(i)}|_X\) when \(\mathcal{I}\) is invertible. Namely, for each integer \(i\), letting \(q\) and \(r\) be the quotient and the remainder of the Euclidean division of \(i\) by \(\delta\), if \(r = 0\) then \(\mathcal{I}^{(i)}|_X\) is the invertible sheaf on \(X\) whose restrictions to \(Y\) and \(Z\) are \(\mathcal{I}|_Y \otimes \mathcal{O}_Y(-qP)\) and \(\mathcal{I}|_Z \otimes \mathcal{O}_Z(qP)\), unique since \(X\) is of compact type. On the other hand, if \(r \neq 0\), then \(\mathcal{I}^{(i)}|_X\) is not invertible, whence the bottom sequence in Diagram [2] splits and we have \(\mathcal{I}^{(i)}|_X = \mathcal{I}|_Y \otimes \mathcal{O}_Y(-(q+1)P) \oplus \mathcal{I}|_Z \otimes \mathcal{O}_Z(qP)\).

There is a parallel construction on \(\mathcal{X}\), which is helpful to have in mind. Namely, if \(\mathcal{L}\) is an invertible sheaf on \(\mathcal{X}\), let \(\mathcal{L}^{(0)} := \mathcal{L}\), and for each integer \(i > 0\) let

\[
\mathcal{L}^{(i)} := \mathcal{L}^{(i-1)} \otimes \mathcal{O}_X(-E_r - E_{r+1} - \cdots - E_{\delta-1} - Z),
\]

where \(r\) is the remainder of the Euclidean division of \(i\) by \(\delta\), if \(r > 0\), and

\[
\mathcal{L}^{(i)} := \mathcal{L}^{(i-1)} \otimes \mathcal{O}_X(-Z)
\]

if \(r = 0\).

Let \(\mathcal{L}\) be an invertible sheaf on \(\mathcal{X}\) whose restriction \(\mathcal{L}|_{E_j}\) has degree 0 but for at most one \(j\), for which the degree is 1. Let \(\ell := 0\) if no such \(j\) occurs; otherwise, let \(\ell\) be that \(j\). A simple computation shows that \(\mathcal{L} = \mathcal{M}^{(\ell)}\) for a certain invertible sheaf \(\mathcal{M}\) whose restriction to \(E\) is trivial. Furthermore, \(\mathcal{M}^{(i)}|_{E_j}\) has degree 0 for each \(j \neq r\), whereas \(\mathcal{M}^{(i)}|_{E_r}\) has degree 1, where \(r\) is the remainder of the Euclidean division of \(i\) by \(\delta\). It follows from [7], Thm. 3.1, that \(R^1\psi_*\mathcal{M}^{(i)} = 0\), and \(\psi_*\mathcal{M}^{(i)}\) is a torsion-free, rank-1 sheaf on \(\mathcal{X}/B\) whose formation commutes with base change, for each \(i \geq 0\). Furthermore, \(\psi_*\mathcal{M}^{(i)}\) is invertible if and only if \(\delta|i\).

**Proposition 2.4.** If \(\mathcal{I} = \psi_*\mathcal{L}\) then \(\mathcal{I}^{(i)} = \psi_*\mathcal{L}^{(i)}\) for each \(i \geq 0\).

**Proof.** Clearly, \(\mathcal{I}^{(0)} = \psi_*\mathcal{L}^{(0)}\). Assume by induction that \(\mathcal{I}^{(i-1)} = \psi_*\mathcal{L}^{(i-1)}\) for a certain \(i > 0\).
Let $r$ be the remainder of the Euclidean division of $i + \ell$ by $\delta$.

Suppose first that $r > 0$. Consider the natural exact sequence defining $\mathcal{L}^{(i)}$:

$$0 \to \mathcal{L}^{(i)} \to \mathcal{L}^{(i-1)} \to \mathcal{L}^{(i-1)}|_{E_r + \cdots + E_{\delta-1} + Z} \to 0.$$ 

Since $R^1\psi_*\mathcal{L}^{(i)} = 0$, applying $\psi_*$ to it we get another natural short exact sequence:

$$0 \to \psi_*\mathcal{L}^{(i)} \to \psi_*(\mathcal{L}^{(i-1)}) \to \psi_*(\mathcal{L}^{(i-1)}|_{E_r + \cdots + E_{\delta-1} + Z}) \to 0.$$ 

It remains to show that $\psi_*(\mathcal{L}^{(i-1)}|_{E_r + \cdots + E_{\delta-1} + Z})$ is $\mathcal{I}^{(i-1)}|_Z$ modulo torsion.

To simplify the notation, set $\mathcal{N} := \mathcal{L}^{(i-1)}$ and $Z_r := E_r + \cdots + E_{\delta-1} + Z$. We need to prove that $\psi_*(\mathcal{N}|_{Z_r})$ is $\psi_*(\mathcal{N})|_Z$ modulo torsion. First observe that the degree of $\mathcal{N}|_{E_j}$ is 0 for every $j \geq r$. Thus

$$\psi_*(\mathcal{N}|_{Z_r-Z}(Q)) = 0 = R^1\psi_*((\mathcal{N}|_{Z_r-Z}(Q)) = 0,$$

where $Q$ is the intersection of $E_{\delta-1}$ with $Z$. By applying $\psi_*$ to the short exact sequence

$$0 \to \mathcal{N}|_{Z_r-Z}(Q) \to \mathcal{N}|_{Z_r} \to \mathcal{N}|_Z \to 0,$$

and considering the associated long exact sequence, it follows that $\psi_*(\mathcal{N}|_{Z_r}) \cong \psi_*(\mathcal{N}|_Z)$. In particular, $\psi_*(\mathcal{N}|_{Z_r})$ is an invertible sheaf on $Z$, isomorphic to $\mathcal{N}|_Z$.

Since $\psi(Z_r) = Z$, there is a natural map

$$h: \psi_*(\mathcal{N})|_Z \to \psi_*(\mathcal{N}|_{Z_r}).$$

Since $\psi_*(\mathcal{N}|_{Z_r})$ is invertible, the torsion is mapped to zero. Since the source is a quotient of $\mathcal{I}^{(i-1)}$, and the target is a quotient of $\psi_*(\mathcal{L}^{(i-1)})$, and both are equal by induction hypothesis, it follows that $h$ is surjective. Since both source and target are rank 1, the kernel of $h$ is torsion. So $h$ induces an isomorphism between $\psi_*(\mathcal{N})|_Z$ modulo torsion and $\psi_*(\mathcal{N}|_{Z_r})$.

If $r = 0$, the proof goes through as before, but simpler, with $Z$ replacing $Z_r$. □

Clearly, we may define in a similar way the $i$-th twist by $Y$ of $\mathcal{I}$, for every $i \geq 0$. For each $i \geq 0$, let $\mathcal{I}^{(-i)}$ be $t^{-i}$ times the $i$-th twist by $Y$ of $\mathcal{I}$. Since the first twist is a subsheaf of $\mathcal{I}$ containing $t\mathcal{I}$, it follows that $\mathcal{I} \subseteq \mathcal{I}^{(-1)} \subseteq \mathcal{I}^{(-2)} \subseteq \cdots$. Furthermore, the notation is justified, as it follows from [4], Lemma 23, p. 3063 that

$$(\mathcal{I}^{(i)})^{(j)} = \mathcal{I}^{(i+j)} \quad \text{for all } i, j \in \mathbb{Z}. \quad (4)$$

Also, by the same lemma, there is a natural surjection of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I}^{(i-1)} & \longrightarrow & \mathcal{I}^{(i)} & \longrightarrow & \mathcal{I}^{(i)}|_{\mathcal{I}^{(i)}_{\text{torsion}}} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \parallel & & \\
0 & \longrightarrow & \mathcal{I}^{(i-1)}|_\mathcal{I}^{(i)}_{\text{torsion}} & \longrightarrow & \mathcal{I}^{(i)}|_X & \longrightarrow & \mathcal{I}^{(i)}|_X|_{\mathcal{I}^{(i)}_{\text{torsion}}} & \longrightarrow & 0. 
\end{array} \quad (5)$$
Observe that, because of (4), Diagrams (2) and (5) can be considered for every \( i \in \mathbb{Z} \).

If \( \mathcal{I} \) is invertible, then isomorphisms analogous to those in (3) follow, again by (2), Prop. 3.1, p. 13. Here, we will display them in a format valid for every \( i \in \mathbb{Z} \):

\[
\frac{\mathcal{I}^{(i)}|_Y}{\text{torsion}} \cong \mathcal{I}|_Y \otimes \mathcal{O}_Y(q_-P) \quad \text{and} \quad \frac{\mathcal{I}^{(i)}|_Z}{\text{torsion}} \cong \mathcal{I}|_Z \otimes \mathcal{O}_Z(q_+P),
\]

where \( q_- \) (resp. \( q_+ \)) is the quotient of the Euclidean division of \(-i\) (resp. \( i\)) by \( \delta \). These isomorphisms describe the \( \mathcal{I}^{(i)}|_X \) completely: For \( i \in \delta\mathbb{Z} \), the sheaf \( \mathcal{I}^{(i)}|_X \) is the invertible sheaf on \( X \) whose restrictions to \( Y \) and \( Z \) are given by (6), whereas for \( i \notin \delta\mathbb{Z} \), the sheaf \( \mathcal{I}^{(i)}|_X \) is the direct sum of the sheaves in (6).

When we put together the bottom exact sequences of Diagrams (2) and (5), we obtain the following Diagram:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I}^{(i+1)}|_Y \quad \text{torsion} & \longrightarrow & \mathcal{I}^{(i)}|_X & \longrightarrow & \mathcal{I}^{(i)}|_Z \quad \text{torsion} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}^{(i+1)}|_Y & \longleftarrow & \mathcal{I}^{(i+1)}|_X & \longleftarrow & \mathcal{I}^{(i+1)}|_Z & \longleftarrow & 0
\end{array}
\]

The two maps, \( \phi_i: \mathcal{I}^{(i+1)} \rightarrow \mathcal{I}^{(i)} \) and \( \phi^i: \mathcal{I}^{(i)} \rightarrow \mathcal{I}^{(i+1)} \), both natural inclusions, the second involving a multiplication by \( t \), restrict to isomorphisms on \( X_q \), but restrict to maps on \( X \) whose compositions both ways are zero; they are the maps \( \phi_i \) and \( \phi^i \) in Diagram (7).

**Definition 2.5.** A *twist \( \delta \)-sequence* associated to an invertible sheaf \( I \) on \( X \) is a collection of sheaves \( I^{(i)} \) and maps \( \phi^i, \phi_i \) indexed by \( i \in \mathbb{Z} \) such that:

(a) The sheaves \( I^{(i)} \) are invertible with restrictions \( I|_Y \otimes \mathcal{O}_Y(q_-P) \) and \( I|_Z \otimes \mathcal{O}_Z(q_+P) \) if \( \delta|i \), and

\[
I^{(i)} = \left( I|_Y \otimes \mathcal{O}_Y(q_-P) \right) \bigoplus \left( I|_Z \otimes \mathcal{O}_Z(q_+P) \right)
\]

otherwise; here, \( q_- \) (resp. \( q_+ \)) is the quotient of the Euclidean division of \(-i\) (resp. \( i\)) by \( \delta \).

(b) The Diagram

\[
\begin{array}{cccccccc}
0 & \longrightarrow & I|_Y \otimes \mathcal{O}_Y(q'_-P) & \overset{\iota_{i,Y}}{\longrightarrow} & I^{(i)} & \overset{\rho_{i,Z}}{\longrightarrow} & I|_Z \otimes \mathcal{O}_Z(q_+P) & \longrightarrow & 0 \\
\downarrow & & \downarrow \phi_i & & \downarrow & & \downarrow \phi^i & & \\
0 & \longrightarrow & I|_Y \otimes \mathcal{O}_Y(q'_-P) & \overset{\rho_{i+1,Y}}{\longrightarrow} & I^{(i+1)} & \overset{\iota_{i+1,Z}}{\longrightarrow} & I|_Z \otimes \mathcal{O}_Z(q_+P) & \longrightarrow & 0
\end{array}
\]

commutes, where \( q'_+ \) is as before, \( q'_- \) is the quotient of the Euclidean division of \(-i+1\) by \( \delta \), the maps \( \rho_{i+1,Y} \) and \( \rho_{i,Z} \) are the natural surjections, and \( \iota_{i,Y} \) and \( \iota_{i+1,Z} \) are the natural inclusions.

Observe that \( q'_- = q_- \) if \( \delta \) does not divide \( i \), whereas \( q'_- = q_- - 1 \) otherwise. Similarly, the quotient of the Euclidean division of \( (i+1) \) by \( \delta \) is \( q_+ \) if \( \delta \) does not divide \( i+1 \), and \( q_+ + 1 \)
otherwise. Hence it follows from the description of the sheaves \( I^{(i)} \) in item (a) what the natural inclusions \( t_{i,Y} \) and \( t_{i+1,Z} \) are. Also, notice that any other twist \( \delta \)-sequence associated to \( I \) is essentially the same, modulo obvious isomorphisms.

If \( I \) is invertible, it follows from the isomorphisms \( \delta \) that the sheaves \( \mathcal{I}^{(i)}|_X \) and the maps \( \varphi^i \) and \( \varphi_i \) of Diagrams \( \delta \) form a twist \( \delta \)-sequence associated to \( \mathcal{I}|_X \).

### 3 Limit linear series

As in Section 2 we let \( X \) denote a curve defined over an algebraically closed field \( k \) with exactly two irreducible components, \( Y \) and \( Z \), that are smooth and intersect transversally at a single point \( P \). We let \( \pi : X \to B \) be a smoothing of \( X \), denote by \( \eta \) and \( o \) the generic and special points of \( B \), and by \( X_\eta \) and \( X_o \) the respective fibers of \( \pi \). We identify \( X_o \) with \( X \).

Let \( \mathcal{I} \) be a torsion-free rank-1 sheaf on \( X/B \), and let \( \mathcal{I}^{(i)} \) denote the twists of \( \mathcal{I} \) for \( i \in \mathbb{Z} \), as defined in Section 2. Let \( I_\eta \) and \( I^{(i)}_\eta \) denote their restrictions to the generic fiber \( X_\eta \); they are all equal. Let \( V \subseteq \Gamma(X_\eta, I^{(i)}_\eta) \) be a vector subspace. Let \( r \) denote its projective dimension. View \( V \) as a subspace of \( \Gamma(X_\eta, I^{(i)}_\eta) \) for each \( i \in \mathbb{Z} \), and denote by \( \mathcal{V}^{(i)} \) the subsheaf of \( \pi_* \mathcal{I}^{(i)} \) consisting of the sections that restrict to sections in \( V \) on the generic fiber. The \( \mathcal{V}^{(i)} \) are free of rank \( r + 1 \). Let \( \phi_i : \mathcal{I}^{(i+1)} \to \mathcal{I}^{(i)} \) and \( \phi^i : \mathcal{I}^{(i)} \to \mathcal{I}^{(i+1)} \) be the natural inclusions. Then

\[
\begin{align*}
\pi_* \phi_i(V^{(i+1)}) & \subseteq \mathcal{V}^{(i)} \quad \text{and} \quad (\pi_* \phi_i)^{-1}(\mathcal{V}^{(i)}) = V^{(i+1)} \\
\pi_* \phi^i(V^{(i)}) & \subseteq \mathcal{V}^{(i+1)} \quad \text{and} \quad (\pi_* \phi^i)^{-1}(\mathcal{V}^{(i+1)}) = V^{(i)} 
\end{align*}
\]

(9)

For each \( i \in \mathbb{Z} \), let \( V^{(i)} \subseteq \Gamma(X, \mathcal{I}^{(i)}|_X) \) be the subspace generated by the restriction to \( X \) of \( \Gamma(B, \mathcal{V}^{(i)}) \subseteq \Gamma(X, \mathcal{I}^{(i)}) \). Let \( \varphi^i \) and \( \varphi_i \) denote the restrictions of \( \phi^i \) and \( \phi_i \) to \( X \). Then it follows from \( \delta \) that \( \varphi^i(V^{(i)}) \subseteq V^{(i+1)} \) and \( \varphi_i(V^{(i+1)}) \subseteq V^{(i)} \) for every \( i \). Moreover, since \( \text{Ker}(\phi^i) = \text{Im}(\phi_i) \) and \( \text{Ker}(\phi_i) = \text{Im}(\phi^i) \), it follows that

\[
\varphi^i(V^{(i)}) = \text{Ker}(\varphi_i|_{V^{(i+1)}}) \quad \text{and} \quad \varphi_i(V^{(i+1)}) = \text{Ker}(\varphi^i|_{V^{(i)}}).
\]

If \( I \) is invertible, then the data \( (\mathcal{I}|_X; V^{(i)}, i \in \mathbb{Z}) \) is an exact level-\( \delta \) limit linear series, as defined below.

**Definition 3.1.** A level-\( \delta \) limit linear series of \( X \) is the data \( g = (I; V^{(i)}, i \in \mathbb{Z}) \) of an invertible sheaf \( I \) on \( X \) and vector subspaces \( V^{(i)} \subseteq \Gamma(X, I^{(i)}) \) for \( i \in \mathbb{Z} \) of equal dimension such that

\[
\varphi^i(V^{(i)}) \subseteq V^{(i+1)} \quad \text{and} \quad \varphi_i(V^{(i+1)}) \subseteq V^{(i)}
\]

for every \( i \), where the sheaves \( I^{(i)} \) and the maps \( \varphi^i, \varphi_i \) form the twist \( \delta \)-sequence associated to \( I \). We say that \( g \) has degree \( d \) if \( I \) has degree \( d \), and that \( g \) has rank \( r \) if the \( V^{(i)} \) have projective dimension \( r \). We say that \( g \) is exact if moreover

\[
\varphi^i(V^{(i)}) = \text{Ker}(\varphi_i|_{V^{(i+1)}}) \quad \text{and} \quad \varphi_i(V^{(i+1)}) = \text{Ker}(\varphi^i|_{V^{(i)}}).
\]
Though it may seem that the data giving \( g \) depend on infinitely many parameters, this is not true. In fact, there are integers \( i_0 \) and \( i_d \) such that \( I^{(i_0)} \) and \( I^{(i_d)} \) are invertible sheaves, the first with degree 0 on \( Z \), the second with degree 0 on \( Y \). (Notice that \( i_d = i_0 + d\delta \).) Then, each \( s \in \Gamma(X, I^{(i)}) \) for each \( i \leq i_0 \) (resp. \( i \geq i_d \)) that vanishes on \( Y \) (resp. \( Z \)) vanishes on the whole \( X \). This means that \( \varphi|_{\gamma_{(i+1)}} \) (resp. \( \varphi'|_{\gamma_{(i)}} \)) is injective, and thus \( \varphi_i(V^{(i+1)}) = V^{(i)} \) for every \( i < i_0 \) (resp. \( \varphi'(V^{(i)}) = V^{(i+1)} \) for every \( i \geq i_d \)). To summarize, the \( V^{(i)} \) for \( i < i_0 \) and \( i > i_d \) are determined by the \( V^{(i)} \) for \( i_0 \leq i \leq i_d \). Furthermore, identifying the level-\( \delta \) limit linear series up to shifting, we may assume that \( i_0 = 0 \) and thus \( i_d = d\delta \).

If \( I \) has degree 0 on \( Z \) and \( \delta = 1 \), then the truncation \( \mathbf{\overline{I}} = (I, V^{(0)}, \ldots, V^{(d)}) \) of a level-\( \delta \) limit linear series \( g \) is precisely a limit linear series in the sense given by Osserman.

For each integer \( d \), let \( \text{Pic}^d(X) \) denote the degree-\( d \) Picard scheme of \( X \), parametrizing invertible sheaves of (total) degree \( d \) on \( X \). It decomposes as the disjoint union of the subschemes \( \text{Pic}^{d-i^j}(X) \), parameterizing invertible sheaves of bidegree \( (d-i, i) \), that is, degrees \( d-i \) on \( Y \) and \( i \) on \( Z \). Since \( X \) is of compact type, the restrictions give rise to isomorphisms \( \text{Pic}^{d-i^j}(X) \cong \text{Pic}^{d-i}(Y) \times \text{Pic}^i(Z) \).

We may now construct a moduli space for level-\( \delta \) limit linear series inside a product of relative Grassmannians over \( J := \text{Pic}^{d,0}(X) \).

**Proposition 3.2.** There exists a projective scheme \( G_{d,\delta}^r(X) \) parameterizing level-\( \delta \) limit linear series of degree \( d \) and rank \( r \) on \( X \).

**Proof.** The construction of \( G_{d,\delta}^r(X) \) follows the same argument given to [10], Thm. 5.3, p. 1178. To summarize it, let \( P \) be the Poincaré sheaf on \( X \times J \), trivialized at \( P \). Let \( D \) be an ample enough effective Cartier divisor of \( X \). For each \( i = 0, \ldots, d\delta \), let

\[
\mathcal{P}^{(i)} := p_1^* \mathcal{O}^{(i)}_X \otimes \mathcal{P}, \quad \mathcal{W}^{(i)} := p_2^*(p_1^* \mathcal{O}_X(D) \otimes \mathcal{P}^{(i)}), \quad \text{and} \quad \mathcal{W}^{(i)}_D := p_2^*(p_1^* (\mathcal{O}_X(D)|_D) \otimes \mathcal{P}^{(i)}),
\]

where \( p_1 \) and \( p_2 \) are the projections of \( X \times J \) onto the indicated factors. The \( \mathcal{O}^{(i)}_X \) are the sheaves in the twist \( \delta \)-sequence associated to \( \mathcal{O}_X \). Let \( \varphi_i, \varphi'_i \) be the associated maps. They induce maps \( h_i, h_i' \) between the \( \mathcal{W}^{(i)} \). Also, restriction to \( D \) induces maps \( w_i : \mathcal{W}^{(i)} \rightarrow \mathcal{W}^{(i)}_D \). The \( \mathcal{W}^{(i)}_D \) are locally free. If \( D \) is ample enough, so are the \( \mathcal{W}^{(i)} \). Let \( G_i := \text{Grass}_r(d+1, \mathcal{W}^{(i)}) \) and set \( G := G_0 \times \cdots \times G_{d\delta} \).

Then \( G_{d,\delta}^r(X) \) is a determinantal subscheme of \( G \). Indeed, let \( \mathcal{W}^{(i)} \subseteq \mathcal{W}^{(i)} \otimes \mathcal{O}_G \) be the pullback of the universal subbundle of \( \mathcal{W}^{(i)} \) from \( G_i \) to \( G \), for each \( i = 0, \ldots, d\delta \). The first condition we impose is that the compositions

\[
\mathcal{Y}^{(i+1)} \hookrightarrow \mathcal{W}^{(i+1)} \otimes \mathcal{O}_G \xrightarrow{h_i \otimes \mathcal{O}_G} \mathcal{W}^{(i)} \otimes \mathcal{O}_G \xrightarrow{\mathcal{W}^{(i)} \otimes \mathcal{O}_G} \mathcal{Y}^{(i)} \otimes \mathcal{O}_G
\]

be zero. The second condition is that the compositions

\[
\mathcal{Y}^{(i)} \hookrightarrow \mathcal{W}^{(i)} \otimes \mathcal{O}_G \xrightarrow{w_i \otimes \mathcal{O}_G} \mathcal{W}^{(i)}_D \otimes \mathcal{O}_G
\]

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be zero. These sets of conditions define $G_{d,\delta}(X)$.

The points of $G_{d,\delta}(X)$ correspond to level-$\delta$ limit linear series of $X$ up to shifting. More precisely, $G_{d,\delta}(X)$ represents the functor that associates to each scheme $T$ an invertible sheaf $\mathcal{I}$ on $X \times T$ of relative degree $d$ over $T$ and a collection of locally free subsheaves $\mathcal{V}^{(i)}$ of rank $r+1$ of $p_2^*(\mathcal{I} \otimes p_1^*\mathcal{O}_X^{(i)})$ such that the pairs $(\mathcal{I} \otimes p_1^*\mathcal{O}_X^{(i)}, \mathcal{V}^{(i)})$ are families of linear series on $X \times T/T$, and such that

$$p_2^*(\mathcal{I} \otimes p_1^*(\varphi_i))(\mathcal{V}^{(i+1)}) \subseteq \mathcal{V}^{(i)} \text{ and } p_2^*(\mathcal{I} \otimes p_1^*(\varphi^i))(\mathcal{V}^{(i)}) \subseteq \mathcal{V}^{(i+1)},$$

where the sheaves $\mathcal{O}_X^{(i)}$ and the maps $\varphi_i, \varphi^i$ form a twist $\delta$-sequence associated to $\mathcal{O}_X$, and $p_1$ and $p_2$ are the projections of $X \times T$ onto the indicated factors. Furthermore, either we identify the above objects up to shifting, or we assume that $\mathcal{I}|_{Y \times T}$ has relative degree $d$ over $T$. We may truncate the collection $(\mathcal{V}^{(i)}, i \in \mathbb{Z})$ in the appropriate range or not, it does not matter. Finally, we have used the following definition.

**Definition 3.3.** Let $f: M \to N$ be a proper flat map of schemes. A family of linear series on $M/N$ is the data of an invertible sheaf $\mathcal{L}$ on $M$ and a locally free subsheaf $\mathcal{V} \subseteq f_*\mathcal{L}$ such that, for each Cartesian diagram

$$\begin{array}{ccc} M' & \xrightarrow{u'} & M \\ f' \downarrow & & \downarrow f \\ N' & \xrightarrow{u} & N, \end{array}$$

the composition of natural maps $u^*\mathcal{V} \to u^*f_*\mathcal{L} \to f'_*(u')^*\mathcal{L}$ is injective.

Associated to a level-$\delta$ limit linear series, we have a collection of vector spaces $(\mathcal{V}^{(i)}, i \in \mathbb{Z})$ and maps $h_i := \Gamma(\varphi_i)$ and $h^i := \Gamma(\varphi^i)$ between them. These data constitute a linked sequence of vector spaces, according to the definition below, following Osserman.

**Definition 3.4.** A linked sequence of vector spaces is the data of a collection of vector spaces $(\mathcal{V}^{(i)}, i \in \mathbb{Z})$ of the same dimension and maps $h_i: \mathcal{V}^{(i+1)} \to \mathcal{V}^{(i)}$ and $h^i: \mathcal{V}^{(i)} \to \mathcal{V}^{(i+1)}$ satisfying the following conditions:

(a) $h_i h^i = 0$ and $h^i h_i = 0$ for every $i$.

(b) Ker$(h^i) \cap$ Ker$(h_{i-1}) = 0$ for every $i$.

(c) There are integers $i_0$ and $i_\infty$ such that $h_i$ is an isomorphism for every $i < i_0$ and $h^i$ is an isomorphism for every $i \geq i_\infty$.

The linked sequence is called exact if the complex

$$\mathcal{V}^{(i)} \xrightarrow{h^i} \mathcal{V}^{(i+1)} \xrightarrow{h_i} \mathcal{V}^{(i)} \xrightarrow{h^i} \mathcal{V}^{(i+1)}$$

is exact for every $i$. The dimension of the sequence is the dimension of the $\mathcal{V}^{(i)}$. Its lower bound (resp. upper bound) is the maximum $i_0$ (resp. minimum $i_\infty$) for which $h_i$ (resp. $h^i$) is an isomorphism for every $i < i_0$ (resp. $i \geq i_\infty$).
Notice that, if $h_i$ is an isomorphism, so is $h_{i-1}$. Indeed, since $\text{Im}(h_i) \subseteq \text{Ker}(h^i)$, if $h_i$ is an isomorphism, then $h^i = 0$; whence, since $\text{Ker}(h^i) \cap \text{Ker}(h_{i-1}) = 0$, it follows that $h_{i-1}$ is injective, thus an isomorphism. Analogously, if $h^i$ is an isomorphism, so is $h^{i+1}$. Thus the lower bound (resp. upper bound) is the maximum $i$ (resp. minimum $i$) for which $h_{i-1}$ (resp. $h^i$) is an isomorphism.

There are many ways to characterize exactness.

**Proposition 3.5.** Let $(V(i), h^i, h_i \mid i \in \mathbb{Z})$ be a sequence of linked vector spaces of dimension $n$. For each $i$, let

\[ p_i := \dim \text{Ker}(h_{i-1}), \quad q_i := \dim \text{Ker}(h^i) \text{ and } m_i := n - p_i - q_i. \]

Then

(a) $p_i + q_i + m_i = n$ for every $i$;
(b) $p_i, q_i, m_i \geq 0$ for every $i$;
(c) $(p_i, q_i, m_i) = (0, n, 0)$ for every $i << 0$ and $(p_i, q_i, m_i) = (n, 0, 0)$ for every $i >> 0$.
(d) $p_i + m_i \leq p_{i+1}$ for every $i$;
(e) $q_{i+1} + m_{i+1} \leq q_i$ for every $i$;
(f) $\sum m_i \leq n$;
(g) $\text{rk}(h^i) + \text{rk}(h_i) \leq n$ for every $i$.

Furthermore, equalities hold in (d) for every $i$ if and only if they hold in (e), if and only if they hold in (g), if and only if equality holds in (f), if and only if the sequence is exact.

**Proof.** Since $\text{Ker}(h_{i-1}) \cap \text{Ker}(h^i) = 0$, it follows that $p_i + q_i \leq n$, yielding the only nontrivial part of (a) and (b). Furthermore, (c) follows from the fact that $h_i$ is an isomorphism for $i << 0$ and $h^i$ is an isomorphism for $i >> 0$.

In addition, for each $i$,

\[ p_i + m_i = n - q_i = \dim \text{Im}(h^i) \leq \dim \text{Ker}(h_i) = p_{i+1}, \]
\[ q_{i+1} + m_{i+1} = n - p_{i+1} = \dim \text{Im}(h_i) \leq \dim \text{Ker}(h^i) = q_i. \]

Also,

\[ \text{rk}(h^i) + \text{rk}(h_i) = n - q_i + n - p_{i+1} = p_i + m_i + n - p_{i+1} = n - q_i + q_{i+1} + m_{i+1}. \]

Thus (d), (e) and (g) follow, as well as the equivalence between the equalities in (d), (e) or (g) and exactness. Finally, since $p_i = 0$ for $i << 0$ and $p_i = n$ for $i >> 0$, it follows from (d) that

\[ \sum m_i \leq \sum (p_{i+1} - p_i) = n, \]
with equality if and only if equalities hold in (d) for every $i$. 

The above proposition suggests a definition.

**Definition 3.6.** Let $f: \mathbb{Z} \rightarrow \mathbb{Z}^3$. For each $i \in \mathbb{Z}$, let $(p_i, q_i, m_i) := f(i)$. We say that $f$ is $n$-admissible if Conditions (a)-(f) in Proposition 3.5 are verified. Furthermore, we say that $f$ is exact if all the inequalities in (d)-(f) are equalities. If the $p_i, q_i, m_i$ are as in Proposition 3.5, we say that $f$ is the numerical function of the sequence of linked vector spaces.

Notice that, if $f$ is exact, then the $p_i$ and $q_i$ are determined from the $m_i$. Also, by Proposition 3.5, the numerical function of a sequence of linked vector spaces is exact if and only if the sequence is exact.

Proposition 3.5 suggests a stratification of $G_{r,d,\delta}(X)$. Indeed, to each $g \in G_{r,d,\delta}(X)$ assign the numerical function $f_g$ of the sequence of linked vector spaces arising from $g$. And, for each $(r+1)$-admissible $f: \mathbb{Z} \rightarrow \mathbb{Z}^3$, let

$$G_{r,d,\delta}(X; f) := \{ g \in G_{r,d,\delta}(X) | f_g = f \}.$$ 

Since rank is semicontinuous, $G_{r,d,\delta}(X; f)$ is a locally closed subset of $G_{r,d,\delta}(X)$.

It follows from Proposition 3.5 that $g$ is exact if and only if $f_g$ is exact. Thus the subset

$$G_{r,d,\delta}^*(X) \subseteq G_{r,d,\delta}(X),$$

parameterizing exact level-$\delta$ limit linear series, decomposes as

$$G_{r,d,\delta}^{r,*}(X) = \bigcup_{f \text{ exact}} G_{r,d,\delta}(X; f).$$

By semicontinuity, $G_{r,d,\delta}^{r,*}(X)$ is open in $G_{r,d,\delta}(X)$. Furthermore, the $G_{r,d,\delta}(X; f)$, for $f$ exact, are both open and closed in $G_{r,d,\delta}^{r,*}(X)$.

4 The forgetful maps

As in Section 2, let $X$ denote a curve defined over an algebraically closed field $k$ with exactly two irreducible components, $Y$ and $Z$, that are smooth and intersect transversally at a single point $P$. Let $d$ and $r$ be integers.

There are natural “forgetful” morphisms

$$\rho_{\delta', \delta}: G_{r,d,\delta'}(X) \rightarrow G_{r,d,\delta}(X)$$

for $\delta$ and $\delta'$ such that $\delta|\delta'$. Indeed, if $I$ is an invertible sheaf on $X$ and $(I^{(i)}, \varphi^i, \varphi_1 | i \in \mathbb{Z})$ is the twist $\delta'$-sequence associated to $I$, then $(I^{(ci)}, \varphi^{ci}_{r+1}, \varphi^{ci}_{ci+1} | i \in \mathbb{Z})$ is the twist $\delta$-sequence.
associated to $I$, where $c := \delta' / \delta$ and
\[
\varphi^j_i := \begin{cases} 
\varphi^j - \cdots - \varphi^{i+1} & \text{if } j > i, \\
\varphi_j \varphi_{j+1} \cdots \varphi_{i-1} & \text{if } j < i, \\
id & \text{if } j = i.
\end{cases}
\] (10)

Thus $\rho^\delta_{\delta'}$ is well-defined by taking a level-$\delta'$ limit linear series $g' = (I; V^{(i)}, i \in \mathbb{Z})$ to the level-$\delta$ limit linear series $g = (I; V^{(ci)}, i \in \mathbb{Z})$.

Clearly, the forgetful maps satisfy
\[
\rho^\delta_{\delta'} \rho^\delta_{\delta''} = \rho^\delta_{\delta''} \text{ if } \delta | \delta' | \delta''.
\]

Furthermore,
\[
\rho^\delta_{\delta'}(G^r_{d, \delta'}(X; f')) \subseteq G^r_{d, \delta}(X, f), \text{ where } f(i) = f'(ci) \text{ for every } i.
\] (11)

Indeed, for any $g' = (I; V^{(i)}, i \in \mathbb{Z})$, the kernel of $\varphi_{i-1}$ is the same as that of the composition $\varphi^j_i$ for every $j < i$. Likewise, the kernel of $\varphi^j_i$ is the same as that of the composition $\varphi^j_i$ for every $j > i$. Thus:

If $g = \rho^\delta_{\delta'}(g')$, then $f_g(i) = f_{g'}(ci)$ for every $i$.

(12)

Given $f, f' : \mathbb{Z} \to \mathbb{Z}^3$ and $c \in \mathbb{Z}$, we write $f = f'c$ when $f(i) = f'(ci)$ for every $i$.

**Theorem 4.1.** Let $f : \mathbb{Z} \to \mathbb{Z}^3$ be $(r + 1)$-admissible. If $\delta' = c\delta$ then
\[
\rho^{-1}_\delta(G^r_{d, \delta}(X; f)) = \bigcup_{f=f'c} G^r_{d, \delta'}(X; f').
\]

Furthermore, if $f = f'c$, and $f'$ is $(r + 1)$-admissible, then the restriction
\[
\rho^\delta_{\delta'} : G^r_{d, \delta'}(X; f') \to G^r_{d, \delta}(X; f)
\]
is surjective and $G^r_{d, \delta}(X; f)$-isomorphic to a nonempty open subscheme of a product of relative Grassmannians over a product of relative partial flag varieties over $G^r_{d, \delta}(X; f)$, and has relative dimension
\[
\sum_{i \in \mathbb{Z}} \left( \sum_{j=1}^{c-1} (q_{ci+j-1} - q_{ci+j})(p_{ci+c} - p_{ci+j} - m_{ci+j}) + \sum_{j=1}^{c-1} (p_{ci+j+1} - p_{ci+j})(q_{ci} - q_{ci+j} - m_{ci+j}) + \sum_{j=1}^{c-1} m_{ci+j}(p_{ci+j+1} - p_{ci+j-1} - m_{ci+j-1}) \right),
\]
where \((p_i, q_i, m_i) := f'(i)\) for each \(i \in \mathbb{Z}\). In particular, if \(f'\) is exact, the relative dimension is
\[
\sum_{i \in \mathbb{Z}} (m_{ci+1} + \cdots + m_{ci+c-1})^2
\]

Proof. The first statement follows from (12). We prove now the second statement.

Let \((p_i, q_i, m_i) := f'(i)\) for each \(i \in \mathbb{Z}\). Let \(I\) be an invertible sheaf on \(X\) and \((I^{(i)}, \varphi^i, \varphi_i | i \in \mathbb{Z})\) the twist \(\delta\)-sequence associated to \(I\). Then \((I^{(ci)}, \varphi^{ci+1}_i, \varphi^{ci}_i | i \in \mathbb{Z})\) is the twist \(\delta\)-sequence associated to \(I\). For each \(i \in \mathbb{Z}\), let \(I^{(i)}_Y\) (resp. \(I^{(i)}_Z\)) denote the restriction of \(I^{(i)}\) to \(Y\) (resp. \(Z\)) modulo torsion. Then
\[
I^{(ci+j)} = I^{(c(i+1))}_Y \oplus I^{(ci)}_Z
\]
for each \(j = 1, \ldots, c - 1\).

For each \(i \in \mathbb{Z}\), and each subspace \(V \subseteq \Gamma(X, I^{(i)})\) of dimension \(r + 1\), let \(h^V := \varphi_{i-1}|V\) and \(h^V_+ := \varphi'|_V\), set
\[
p(V) := \dim \ker(h^V_+),
q(V) := \dim \ker(h^V_-),
m(V) := r + 1 - p(V) - q(V),
\]
and let \(V_-\) (resp. \(V_+\)) denote the image of \(V\) in \(\Gamma(Y, I^{(i)}_Y)\) (resp. \(\Gamma(Z, I^{(i)}_Z)\)).

For each \(i \in \mathbb{Z}\), let \(V^{(ci)} \subseteq \Gamma(X, I^{(ci)})\) be a subspace such that \(g := (I; V^{(ci)}, i \in \mathbb{Z})\) is a level-\(\delta\) limit linear series of rank \(r\) of \(X\) with \(f'_\delta = f\). Since the kernel of \(\varphi^i\) is the same as that of \(\varphi^j_i\) for every \(j > i\), and likewise for \(\varphi^{i-1}_i\), it follows that
\[
f(i) = (p(V^{(ci)}), q(V^{(ci)}), m(V^{(ci)}))\]
for every \(i \in \mathbb{Z}\).

Also, for each \(i \in \mathbb{Z}\), let \(W^{(ci)}_+\) (resp. \(W^{(c(i+1))}_-\)) be the subspace of \(\Gamma(Z, I^{(ci)}_Z)\) (resp. \(\Gamma(Y, I^{(c(i+1))}_Y)\)) such that
\[
\varphi^{ci+c-1}(0 \oplus W^{(ci)}_+) = \ker(h^{V^{(c(i+1))}_+}) \quad \text{(resp. \(\varphi^{ci}(W^{(c(i+1))}_- \oplus 0) = \ker(h^{V^{(ci)}_+})\))}
\]
Then \(V^{(ci)}_+ \subseteq W^{(ci)}_+\) and \(V^{(c(i+1))}_- \subseteq W^{(c(i+1))}_-\).

If \(g' := (I; V^{(i)}, i \in \mathbb{Z})\) is a level-\(\delta'\) limit linear series of rank \(r\) such that \(f'_\delta = f'\) and \(g = \rho_{\delta', \delta}(g')\), then, for each \(i \in \mathbb{Z}\),
\[
\begin{align*}
V^{(ci)}_+ &\subseteq V^{(c(i+1))}_+ \subseteq \cdots \subseteq V^{(c(i+c-1))}_+ \subseteq W^{(ci)}_+, \quad (13) \\
V^{(c(i+1))}_- &\subseteq V^{(c(i+c-1))}_- \subseteq \cdots \subseteq V^{(ci+1)}_- \subseteq W^{(c(i+1))}_-. \quad (14)
\end{align*}
\]
Notice that
\[
\dim V^{(ci+j)}_+ = r + 1 - q_{ci+j} \quad \text{and} \quad \dim V^{(ci+j)}_- = r + 1 - p_{ci+j}
\]

for \( j = 1, \ldots, c - 1 \). Also,

\[
V_-(ci+j+1) + V_+(ci+j) \subseteq V'(ci+j) \subseteq V_-(ci+j) + V_+(ci+j) \subseteq \Gamma(Y, A_Y^{(c(i+1))}) + \Gamma(Z, I_Z^{(ci)}).
\]

Furthermore, the projections \( V'(ci+j) \to V'_-(ci+j) \) and \( V'(ci+j) \to V'_+(ci+j) \) are surjective.

Conversely, for each \( i \in \mathbb{Z} \), let \( A_{i,1}, \ldots, A_{i,c-1} \) and \( B_{i,1}, \ldots, B_{i,c-1} \) be subspaces of \( W_+^{(ci)} \) and \( W_-^{(ci+c)} \), respectively, with

\[
\dim A_{i,j} = r + 1 - q_{ci+j} \quad \text{and} \quad \dim B_{i,j} = r + 1 - p_{ci+j} \quad \text{for} \quad j = 1, \ldots, c - 1,
\]

such that

\[
V_+^{(ci)} \subseteq A_{i,1} \subseteq \cdots \subseteq A_{i,c-1} \subseteq W_+^{(ci)},
\]

\[
V_-^{(c(i+1))} \subseteq B_{i,c-1} \subseteq \cdots \subseteq B_{i,1} \subseteq W_-^{(c(i+1))}.
\]

This is possible since

\[
\dim V_+^{(ci)} = r + 1 - q_{ci} \leq r + 1 - q_{ci+j} \leq r + 1 - q_{ci+j+1} \leq p_{ci+c} = \dim W_+^{(ci)}
\]

\[
\dim V_-^{(c(i+1))} = r + 1 - p_{ci+c} \leq r + 1 - p_{ci+c-j} \leq r + 1 - p_{ci+c-j-1} \leq q_{ci} = \dim W_-^{(ci+c)}
\]

for \( j = 1, \ldots, c - 2 \). For each \( j = 1, \ldots, c - 1 \), let \( V'(ci+j) \subseteq B_{i,j} \oplus A_{i,j} \) be any subspace of dimension \( r+1 \) such that \( V'(ci+j) \supseteq B_{i,j+1} \oplus A_{i,j-1} \) and such that the projections \( V'(ci+j) \to A_{i,j} \) and \( V'(ci+j) \to B_{i,j} \) are surjective. This is possible, since

\[
\dim(B_{i,j+1} \oplus A_{i,j-1}) = 2(r+1) - p_{ci+j+1} - q_{ci+j-1} \leq r+1 \leq 2(r+1) - p_{ci+j} - q_{ci+j} = \dim(B_{i,j} \oplus A_{i,j}).
\]

Then \( g' := (I; V^{(i)}, i \in \mathbb{Z}) \) is a level-\( d' \) limit linear series of rank \( r \) such that \( f_{g'} = f' \) and \( g = \rho_{g', d'}(g') \).

If follows that \( F_{g', d'}(g) \cap G_{d', d'}(X; f') \) is parameterized by a nonempty open subset of a product of Grassmannians over a product of flag varieties. The flag varieties are

\[
F_+^{(i)} := \{ \mathcal{A}_{i,1} \subseteq \cdots \subseteq \mathcal{A}_{i,c-1} \subseteq \frac{W_+^{(ci)}}{V_+^{(ci)}} \mid \dim \mathcal{A}_{i,j} = q_{ci} - q_{ci+j} \quad \text{for} \quad j = 1, \ldots, c - 1 \},
\]

\[
F_-^{(i)} := \{ \mathcal{B}_{i,c-1} \subseteq \cdots \subseteq \mathcal{B}_{c,1} \subseteq \frac{W_-^{(c(i+1))}}{V_-^{(c(i+1))}} \mid \dim \mathcal{B}_{i,c-j} = p_{c(i+1)} - p_{c(i+1)-j} \quad \text{for} \quad j = 1, \ldots, c - 1 \}.
\]

And the Grassmannians over points \( (\mathcal{A}_{i,j}) \in F_+^{(i)} \) and \( (\mathcal{B}_{i,c-j}) \in F_-^{(i)} \) are

\[
\text{Grass}\left(\frac{p_{ci+j+1} + q_{ci+j-1} - (r + 1)}{B_{i,j+1} + A_{i,j-1}} \right).
\]

The relative dimension can thus be computed from the formulas for the dimensions of relative
flag varieties.

**Lemma 4.2.** Let $f: \mathbb{Z} \to \mathbb{Z}^3$ be an $n$-admissible function. Let $c > 1$ be an integer. Then there is an exact $n$-admissible function $f'$ such that $f = f'c$. If $f$ is exact, then there is a unique $n$-admissible function $f'$ such that $f = f'c$.

**Proof.** Let $(p_i, q_i, m_i) := f(i)$ for each $i \in \mathbb{Z}$. Let $\ell \in \{1, \ldots, c - 1\}$. Set $f'(ci) := f(i)$ for each $i$ and

$$f'(ci + j) := \begin{cases} (p_i + m_i, q_i, 0) & \text{if } j < \ell, \\ (p_i + m_i, q_i + 1 + m_{i+1}, p_{i+1} - p_i - m_i) & \text{if } j = \ell, \\ (p_{i+1} + 1, q_{i+1} + m_{i+1}, 0) & \text{if } j > \ell \\ \end{cases} \quad (15)$$

for each $i$ and $j = 1, \ldots, c - 1$; then $f'$ is $n$-admissible, is exact and $f = f'c$.

On the other hand, if $f$ is exact and $f'$ is an $n$-admissible function such that $f = f'c$, then, since $\sum m_i = n$, the maximum possible, it follows that $f'(ci + j) \in \mathbb{Z}^2 \times \{0\}$ for $j = 1, \ldots, c - 1$, and that $f'$ is exact. Now, since $f'$ is exact, the knowledge of the $m_i$ determines $f'$ uniquely. \(\square\)

**Proposition 4.3.** If $\delta|\delta'$ then

$$\rho_{\delta, \delta'}^{-1}(G^{r,s}_{d,\delta}(X)) \subseteq G^{r,s}_{d,\delta'}(X),$$

and the restriction

$$\rho_{\delta, \delta'}: \rho_{\delta, \delta'}^{-1}(G^{r,s}_{d,\delta}(X)) \to G^{r,s}_{d,\delta'}(X)$$

is an isomorphism. Furthermore, if $\delta > \delta'$, then

$$\rho_{\delta, \delta'}(G^{r,s}_{d,\delta}(X)) = G^{r,s}_{d,\delta'}(X).$$

**Proof.** It follows from Lemma 4.2 and the first statement of Theorem 4.1 that, if $f$ is exact, then $\rho_{\delta, \delta'}^{-1}(G^{r,s}_{d,\delta}(X; f)) = G^{r,s}_{d,\delta'}(X; f')$, where $f'$ is the exact function such that $f = f'c$, where $c := \delta'/\delta$. This is enough to conclude that $\rho_{\delta, \delta'}^{-1}(G^{r,s}_{d,\delta}(X)) \subseteq G^{r,s}_{d,\delta'}(X)$, since the $G^{r,s}_{d,\delta'}(X; f)$ for $f$ exact cover $G^{r,s}_{d,\delta'}(X)$ and the $G^{r,s}_{d,\delta'}(X; f')$ for $f'$ exact cover $G^{r,s}_{d,\delta'}(X)$.

To prove the remaining of the first statement, since the $G^{r,s}_{d,\delta'}(X; f)$ for $f$ exact decompose $G^{r,s}_{d,\delta'}(X)$ into open subsets, it is enough to show that the restriction

$$\rho_{\delta, \delta'}: G^{r,s}_{d,\delta}(X; f') \to G^{r,s}_{d,\delta}(X; f)$$

is an isomorphism for every exact $f$, where $f'$ is the unique exact function such that $f = f'c$. By Theorem 4.1 it is enough to show that the relative dimension of the latter map is zero in this case. This relative dimension is computed in Theorem 4.1 and turns out to be zero because $f$ is exact.

To prove the second statement, by Theorem 4.1 it is enough to show that for each $(r + 1)$-admissible function $f: \mathbb{Z} \to \mathbb{Z}^3$ there is an exact $(r + 1)$-admissible function $f'$ such that $f = f'c$, where $c := \delta'/\delta$. But this is exactly what Lemma 4.2 claims. \(\square\)
5 Abel maps

Definition 5.1. Let \( \mathfrak{V} := (V^{(i)}, h^i, h_i \mid i \in \mathbb{Z}) \) be a sequence of linked vector spaces of dimension \( n \). Let \( m \in \mathbb{Z} \). The elementary truncation at \( m \) of \( \mathfrak{V} \) is the sequence of linked vector spaces \( \mathfrak{W} := (W^{(i)}, f^i, f_i \mid i \in \mathbb{Z}) \) where

(a) \( W^{(i)} = V^{(i)} \) for \( i \leq m \) and \( W^{(i)} = V^{(i+1)} \) for \( i > m \);

(b) \( f^i = h^i \) for \( i < m \), \( f^m = h^{m+1} h^m \) and \( f^i = h^{i+1} \) for \( i > m \); 

(c) \( f_i = h_i \) for \( i < m \), \( f_m = h_m h_{m+1} \) and \( f_i = h_{i+1} \) for \( i > m \).

A truncation is the sequence of linked vector spaces obtained after a finite sequence of elementary truncations. On the other hand, we say that a sequence of linked vector spaces is an expansion of another, if the latter is a truncation of the former.

Let \( \mathfrak{V} := (V^{(i)}, h^i, h_i \mid i \in \mathbb{Z}) \) be a sequence of linked vector spaces of dimension \( n \). Let \( i_0 \) be its lower bound and \( i_\infty \) its upper bound. Let \( i \leq i_0 \) and \( j \geq i_\infty \). Put \( \mathbb{P}^{1,1}(\mathfrak{V}) := \mathbb{P}(V^{(i)}) \times \mathbb{P}(V^{(j)}) \).

In principle, \( \mathbb{P}^{1,1}(\mathfrak{V}) \) is defined up to the choice of \( i \) and \( j \). However, since \( h^\ell \) is an isomorphism for \( \ell \geq i_\infty \) and \( h_\ell \) is an isomorphism for \( \ell \leq i_0 - 1 \), the scheme is the same up to natural isomorphism.

Let

\[
\mathbb{P}(\mathfrak{V}) := \bigcup_{\ell=i}^j \mathbb{P}(\mathfrak{V}_\ell) \subseteq \mathbb{P}^{1,1}(\mathfrak{V}),
\]

where

\[
\mathbb{P}(\mathfrak{V}_\ell) := \{(h^\ell_i(v), h^\ell_j(v)) \in \mathbb{P}^{1,1}(\mathfrak{V}) \mid v \in V^{(\ell)} - (\ker(h^\ell) \cup \ker(h_{\ell-1}))\}.
\]

Here,

\[
h^\ell_i := \begin{cases} 
  h_i h_{i+1} \cdots h_{\ell-1} & \text{if } i < \ell, \\
  h^i h^{i-1} \cdots h^{\ell+1} h^\ell & \text{if } i > \ell, \\
  \text{id} & \text{if } i = \ell.
\end{cases}
\]

Observe that \( \ker(h^\ell_i) = \ker(h_{\ell-1}) \) for \( \ell > i \) and \( \ker(h^j_j) = \ker(h^\ell) \) for \( \ell < j \). Thus \( h^\ell_i(v) \) and \( h^j_j(v) \) are nonzero for \( v \in V^{(\ell)} - (\ker(h^\ell) \cup \ker(h_{\ell-1})) \) and define points in \( \mathbb{P}(V^{(i)}) \) and \( \mathbb{P}(V^{(j)}) \).

Notice that \( \mathbb{P}(\mathfrak{V}) \) comes with natural invertible sheaves \( \mathcal{O}_{\mathbb{P}(\mathfrak{V})}(i,j) \), obtained by restriction of the sheaves \( \mathcal{O}(i,j) \) on \( \mathbb{P}^{1,1}(\mathfrak{V}) \), which do not depend of the particular \( \mathbb{P}^{1,1}(\mathfrak{V}) \) chosen. Also, if \( \mathfrak{W} \) is a truncation of \( \mathfrak{V} \), then \( \mathbb{P}(\mathfrak{W}) \subseteq \mathbb{P}(\mathfrak{V}) \) naturally, with \( \mathcal{O}_{\mathbb{P}(\mathfrak{W})}(i,j) = \mathcal{O}_{\mathbb{P}(\mathfrak{V})}(i,j)|_{\mathbb{P}(\mathfrak{W})} \) for all \( i, j \).

As in Section 2 let \( X \) denote a curve defined over an algebraically closed field \( k \) with exactly two irreducible components, \( Y \) and \( Z \), that are smooth and intersect transversally at a single point \( P \). Let \( d \) and \( r \) be integers. Let \( J \) be the connected component of the Picard scheme of \( X \) parameterizing invertible sheaves of degree \( d \) on \( Y \) and 0 on \( Z \). Recall that \( T^{(i)} \) denotes the \( i \)th symmetric product of a scheme \( T \).
Definition 5.2. The degree-$d$ Abel map $A_d: X(d) \to J$ associates to each Weil divisor $D$ of $X$ the invertible sheaf $\mathcal{O}_X(D)$ defined as that having restrictions $\mathcal{O}_Y(D_1 + d_2 P)$ and $\mathcal{O}_Z(D_2 - d_2 P)$, where $D = D_1 + D_2$, with $D_1$ supported in $Y$ of degree $d_1$ and $D_2$ supported in $Z$ of degree $d_2$.

By its very definition, $\mathcal{O}_X(D)$ does not depend on the decomposition $D = D_1 + D_2$ chosen; see [4], §3, p. 82.

Let $I$ be an invertible sheaf on $X$ and $(I^{(i)}, \varphi^i, \varphi_1 | i \in \mathbb{Z})$ an associated twist $\delta$-sequence. For each $i, j$, let $\varphi^i_j$ be as in [10]. Let $i_0$ be the integer for which $I^{(i_0)}$ is invertible of degree $d$ on $Y$ and $i_\infty$ that for which $I^{(i_\infty)}$ is invertible of degree $d$ on $Z$. Then $i_\infty = i_0 + d\delta$. We may assume, without loss of generality, that $i_0 = 0$.

Let $\mathfrak{g} = (I, V^{(i)}, i \in \mathbb{Z})$ be a level-$\delta$ limit linear series. Let $\mathfrak{W} := (V^{(i)}, h^i, h_1 | i \in \mathbb{Z})$ be the associated sequence of linked vector spaces. Then $i_0$ is at most its lower bound and $i_\infty$ is at least its upper bound. In particular, we may view $\mathbb{P}(\mathfrak{W})$ inside $\mathbb{P}(V^{(0)}) \times \mathbb{P}(V^{(d\delta}))$.

Let $h^i_j := \varphi^i_j|_{V^{(i)}}$ for every $i$ and $j$. To each $i \in \mathbb{Z}$ and $s \in V^{(i)}$ such that $0 \leq i \leq d\delta$ and $s \not\in \text{Ker}(h^i) \cup \text{Ker}(h_{i-1})$, we associate the point $[D(s)]$ on $X^{(d)}$ given by

$$D(s) := \text{div}(h^i_j(s)|_Y) + \text{div}(h^m_i(s)|_Z) - \begin{cases} P & \text{if } \ell < m, \\ 0 & \text{if } \ell = m, \end{cases}$$

where $\ell$ (resp. $m$) is the maximum (resp. minimum) integer not greater (resp. not smaller) than $i$ such that $I^{(\ell)}$ (resp. $I^{(m)}$) is invertible.

Notice that $D(s)$ has degree $d$. Indeed, $\text{div}(h^i_j(s)|_Y)$ has degree $d - \ell/\delta$, while $\text{div}(h^m_i(s)|_Z)$ has degree $m/\delta$. The sum has degree $d + (m - \ell)/\delta$, from which follows that $D(s)$ has degree $d$. Also, $D(s)$ is effective. This is clear if $\ell = m$. On the other hand, if $\ell < m$, then $I^{(\ell)} = I^{(m)}|_Y \oplus I^{(\ell)}|_Z$, and thus both $h^i_j(s)|_Y$ and $h^m_i(s)|_Z$ vanish at $P$. At any rate, it follows from Definition 5.2 that $\mathcal{O}_X(D(s))$ restricts to

$$I^{(\ell)}|_Y \otimes \mathcal{O}_Y(\ell P/\delta) \text{ and } I^{(m)}|_Z \otimes \mathcal{O}_Z(d - \ell P/\delta),$$

which are the same restrictions as those of $I^{(0)}$. Thus $[D(s)] \in A^{(-1)}_d([I^{(0)}])$.

(Another way of viewing $D(s)$ is by considering the dual $(I^{(i)})^* \to \mathcal{O}_X$ of the homomorphism induced by $s$; the image is the sheaf of ideals of a finite subscheme of $X$ whose 0-cycle is $D(s)$.)

Let

$$\mathbb{P}(\mathfrak{g}) := \bigcup_{i=0}^{d\delta} \mathbb{P}(\mathfrak{g}_i) \subseteq X^{(d)}, \text{ where } \mathbb{P}(\mathfrak{g}_i) := \{[D(s)] | s \in V^{(i)} - (\text{Ker}(h^i) \cup \text{Ker}(h_{i-1}))\}.$$ 

Then $\mathbb{P}(\mathfrak{g}) \subseteq A^{(-1)}_d([I^{(0)}])$.

Notice that

$$\text{div}(\varphi^0_i(s)|_Y) + \text{div}(\varphi^d_i(s)|_Z) = D(s) + dP = \tau_{dP}([D(s)])$$ (16)
for each $i$ and $s$, where $\tau_{dp} : X^{(d)} \to X^{(2d)}$ is the embedding taking $[D]$ to $[D + dP]$. Furthermore, there are natural inclusions $\iota_0 : \mathbb{P}(V^{(0)}) \hookrightarrow Y^{(d)}$ and $\iota_{dP} : \mathbb{P}(V^{(d)}) \hookrightarrow Z^{(d)}$, the first taking $[s]$ to $\text{div}(s|_Y)$, the second taking $[s]$ to $\text{div}(s|_Z)$. Composing with the sum embedding $\sigma : Y^{(d)} \times Z^{(d)} \to X^{(2d)}$, it follows from (16) that

$$\sigma(\iota_0 \times \iota_{dP})(\mathbb{P}(\mathcal{V})) = \tau_{dp}(\mathbb{P}(\mathfrak{g})).$$

Hence, we may naturally identify $\mathbb{P}(\mathcal{V})$ with $\mathbb{P}(\mathfrak{g})$ inside $X^{(2d)}$. Furthermore, if we let $Q_Y \subseteq Y - P$ and $Q_Z \subseteq Z - P$ be any two points, and set $H_{i,j}^d := iH_Y^d + jH_Z^d$, where

$$H_Y^d := \{ [D] \in X^{(d)} \mid D \geq Q_Y \} \quad \text{and} \quad H_Z^d := \{ [D] \in X^{(d)} \mid D \geq Q_Z \}$$

for all $i, j, \ell$, then $O_{\mathbb{P}(\mathcal{V})}(i, j)$ and $O_{X^{(d)}}(H_{i,j}^d)|_{\mathbb{P}(\mathfrak{g})}$ coincide, both being restrictions of $O_{X^{(2d)}}(H_{i,j}^{2d})$.

**Lemma 5.3.** Each sequence of linked vector spaces is a truncation of an exact sequence.

**Proof.** Let $\mathcal{V} = (V^{(i)}, h^i, h_i \mid i \in \mathbb{Z})$ be a sequence of linked vector spaces of dimension $n$. Recall the notation:

$$p_i := \dim \text{Ker}(h_{i-1}), \quad q_i := \dim \text{Ker}(h^i) \quad \text{and} \quad m_i := n - p_i - q_i.$$

Then $\sum m_i \leq n$ with equality if and only if $\mathcal{V}$ is exact, by Proposition 5.3. So, let $m(\mathcal{V}) := \sum m_i$.

If $\mathcal{W}$ is an expansion of $\mathcal{V}$, then $m(\mathcal{W}) \geq m(\mathcal{V})$. Thus, to prove the lemma, we need only show that, if $\mathcal{V}$ is not exact, then there is an expansion $\mathcal{W}$ of $\mathcal{V}$ with $m(\mathcal{W}) > m(\mathcal{V})$.

Thus, suppose $\mathcal{V}$ is not exact. Let $i \in \mathbb{Z}$ for which $\text{rk}(h_i) + \text{rk}(h^i) < n$. Let $W \subseteq V^{(i)} \oplus V^{(i+1)}$ be a $n$-dimensional subspace containing $\text{Im}(h_i) \oplus \text{Im}(h^i)$ such that the projection maps $W \to V^{(i)}$ and $W \to V^{(i+1)}$ have images $\text{Ker}(h^i)$ and $\text{Ker}(h_i)$, respectively. This is possible: $W = (\text{Im}(h_i) \oplus \text{Im}(h^i)) + K$, where $K$ is the graph of an isomorphism between a subspace of $\text{Ker}(h^i)$ complementary to $\text{Im}(h_i)$ and a subspace of $\text{Ker}(h_i)$ complementary to $\text{Im}(h^i)$. In particular, it follows that

$$W \cap (V^{(i)} \oplus 0) = \text{Im}(h_i) \oplus 0 \quad \text{and} \quad W \cap (0 \oplus V^{(i+1)}) = 0 \oplus \text{Im}(h^i).$$

Thus, inserting $W$ between $V^{(i)}$ and $V^{(i+1)}$, we obtain a sequence of linked vector spaces $\mathcal{W}$ expanding $\mathcal{V}$ with

$$m(\mathcal{W}) = m(\mathcal{V}) + (n - \text{rk}(h_i) - \text{rk}(h^i)) > m(\mathcal{V}).$$

**Proposition 5.4.** Let $\mathcal{V} = (V^{(i)}, h^i, h_i \mid i \in \mathbb{Z})$ be a sequence of linked vector spaces of dimension $n$. If $\mathcal{V}$ is exact, then $\mathbb{P}(\mathcal{V})$ is a connected, Cohen–Macaulay subscheme of $\mathbb{P}^{1,1}(\mathcal{V})$ of pure dimension $n - 1$ and bivariate Hilbert polynomial $h^0(\mathbb{P}(\mathcal{V}), O_{\mathbb{P}(\mathcal{V})}(i, j)) = (i+j+n-1)_{n-1}$ for $i, j >> 0$. Conversely, if $\mathbb{P}(\mathcal{V})$ has bivariate Hilbert polynomial $(i+j+n-1)_{n-1}$, then $\mathcal{V}$ is exact.
Proof. The proof of the first statement follows step-by-step the same proof given to [6], Thm. 4.3. In particular, it follows from that proof that, if \( \mathfrak{U} \) is exact, then
\[
\mathbb{P}(\mathfrak{U}) = \bigcup_{i \in S_{\mathfrak{U}}} \mathbb{P}(\mathfrak{U}_i),
\]
(17)
where
\[
S_{\mathfrak{U}} := \{ i \in \mathbb{Z} \mid V^{(i)} \neq \text{Ker}(h^i) \oplus \text{Ker}(h_{i-1}) \} = \{ i \in \mathbb{Z} \mid m_i > 0 \};
\]
see [9], Rmk. 4.9, p. 89. Furthermore, it follows from Lemmas 4.4 and 4.8 in loc. cit. that the \( \mathbb{P}(\mathfrak{U}_i) \) are irreducible of dimension \( n - 1 \) if \( i \in S_{\mathfrak{U}} \). In other words, (17) is the expression of \( \mathbb{P}(\mathfrak{U}) \) as the union of its irreducible components; there are \( |S_{\mathfrak{U}}| \) of them.

Conversely, let \( \mathfrak{W} \) be an exact sequence of linked vector spaces expanding \( \mathfrak{U} \). Then \( \mathbb{P}(\mathfrak{U}) \subseteq \mathbb{P}(\mathfrak{W}) \). Furthermore, if \( \mathfrak{W} \) is not exact then \( S_{\mathfrak{U}} \not\subseteq S_{\mathfrak{W}} \), and thus there is \( i \in \mathbb{Z} \) such that \( \mathbb{P}(\mathfrak{U}_{i}) \not\subseteq \mathbb{P}(\mathfrak{W}) \). So the bivariate Hilbert polynomial of \( \mathbb{P}(\mathfrak{W}) \) must be different from that of \( \mathbb{P}(\mathfrak{U}) \), thus different from \( \binom{1+j+n-1}{n-1} \) by the first statement of the proposition.

**Theorem 5.5.** Let \( r, d, \delta, \delta' \) be nonnegative integers, with \( \delta | \delta' \).

1. For each \( g \in G_{d, \delta}(X) \), if \( g \) is exact then the subscheme \( \mathbb{P}(g) \subseteq X^{(d)} \) is connected, Cohen–Macaulay, of pure dimension \( r \) and bivariate Hilbert polynomial \( \binom{i+j+r}{r} \). Conversely, if \( \mathbb{P}(g) \) has bivariate Hilbert polynomial \( \binom{i+j+r}{r} \), then \( g \) is exact.

2. For each \( g' \in G_{d, \delta'}(X) \), letting \( g := \rho_{\delta', \delta}(g') \), we have \( \mathbb{P}(g) \subseteq \mathbb{P}(g') \). If \( g \) is exact, then so is \( g' \), and equality holds. Conversely, if equality holds and \( g' \) is exact, then so is \( g \).

**Proof.** Statement 1 follows from Proposition 5.1 since \( \mathbb{P}(\mathfrak{U}) \) and \( \mathbb{P}(g) \) coincide, and \( \mathfrak{U} \) is exact if and only if \( g \) is, where \( \mathfrak{U} \) is the sequence of linked vector spaces associated to \( g \).

The first assertion of the second statement follows from the fact that \( \mathfrak{W} \) is an expansion of \( \mathfrak{U} \), where \( \mathfrak{W} \) is the sequence of linked vector spaces associated to \( g' \). The remaining assertions follow from Proposition 4.3 and Statement 1.

**Theorem 5.6.** Let \( \pi : X \to B \) be a smoothing of \( X \) with singularity degree \( \delta \). Let \( I \) be a torsion-free, rank-1 sheaf on \( X/B \) and \( V \subseteq \Gamma(X_{\eta}, I_{\eta}) \) a vector subspace. Let \( g \) be the level-\( \delta \) limit linear series on \( X \) that arises from \( g_{\eta} := (I_{\eta}, V) \). Then \( \mathbb{P}(V) \), viewed as a subscheme of the fiber over \( \eta \) of the relative symmetric product \( X^{(d)}_B \), has closure intersecting \( X^{(d)} \) in \( \mathbb{P}(g) \).

**Proof.** As in the proof to [6], Thm. 5.2, p. 90, since \( g \) is exact, and thus \( \mathbb{P}(g) \) has the bivariate Hilbert polynomial of the limit of \( \mathbb{P}(V) \), we need only show that the closure of \( \mathbb{P}(V) \) contains \( \mathbb{P}(g) \).

Recall the notation used in Section 3. Let \( \mathfrak{I}^{(i)} \) denote the twists of \( \mathfrak{I} \), and for each \( i \in \mathbb{Z} \), let \( \mathfrak{V}^{(i)} \) be the subsheaf of \( \pi_* \mathfrak{I}^{(i)} \) consisting of the sections that restrict to sections in \( V \) on the generic fiber. For each \( i \in \mathbb{Z} \), consider on the product \( X \times B \mathbb{P}(\mathfrak{V}^{(i)}) \) the composition
\[
\mathcal{O}_{\mathbb{P}(\mathfrak{V}^{(i)})}(-1) \to \tilde{\mathfrak{V}}^{(i)} \to \mathfrak{I}^{(i)}
\]
21
where the first map is the tautological map of $\mathbb{P}(V^{(i)})$ and the second is the evaluation map, all sheaves and maps being viewed on the product under the appropriate pullbacks. Taking its dual and twisting by $\mathcal{O}_{\mathbb{P}(V^{(i)})}(-1)$, we obtain a map to $\mathcal{O}_{\mathbb{P}(V^{(i)})}$ whose image is the sheaf of ideals of a subscheme $F_i \subseteq \mathcal{X} \times_B \mathbb{P}(V^{(i)})$. Moreover, $F_i$ is a flat subscheme of relative length $d$ over $\mathbb{P}(V^{(i)}) - (\mathbb{P}(\text{Ker}(h^i)) \cup \mathbb{P}(\text{Ker}(h_{i-1})))$, where $(V^{(i)}, h^i, h_i \mid i \in \mathbb{Z})$ is the sequence of linked vector spaces associated to $\mathfrak{g}$. Thus we obtain a map

$$\mathbb{P}(V^{(i)}) - (\mathbb{P}(\text{Ker}(h^i)) \cup \mathbb{P}(\text{Ker}(h_{i-1}))) \to \mathcal{X}^{(d)}_B$$

whose image contains $\mathbb{P}(V)$ and all points of $\mathcal{X}^{(d)}$ of the form $\text{div}(s|_Y) + \text{div}(s|_Z)$ for $s \in V^{(i)} - (\text{Ker}(h^i) \cup \text{Ker}(h_{i-1}))$. Since $\mathbb{P}(V^{(i)})$ is flat over $B$, it follows that the closure of $\mathbb{P}(V)$ in $\mathcal{X}^{(d)}_B$ contains all points of the above form. As we let $i$ vary, we get that the closure of $\mathbb{P}(V)$ contains $\mathbb{P}(\mathfrak{g})$.

\[\blacksquare\]

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