LARGE DEVIATIONS PRINCIPLE FOR SOME BETA ENSEMBLES

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ABSTRACT. Let $L$ be a positive line bundle over a projective complex manifold $X$, $L^p$ its tensor power of order $p$, $H^0(X, L^p)$ the space of holomorphic sections of $L^p$ and $N_p$ the complex dimension of $H^0(X, L^p)$. The determinant of a basis of $H^0(X, L^p)$, together with some given probability measure on a weighted compact set in $X$, induces naturally a $\beta$-ensemble, i.e., a random $N_p$-point process on the compact set. Physically, this general setting corresponds to a gas of free fermions on $X$ and may admit some random matrix models. The empirical measures, associated with such $\beta$-ensembles, converge almost surely to an equilibrium measure when $p$ goes to infinity. We establish a large deviations principle (LDP) with an effective speed of convergence for these empirical measures. Our study covers the case of some $\beta$-ensembles on a compact subset of the unit sphere $S^n \subset \mathbb{R}^{n+1}$ or of the Euclidean space $\mathbb{R}^n$.

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1. INTRODUCTION

Let $K$ be a metric space and $N$ a positive integer. If $x = (x_1, \ldots, x_N)$ is a point in the $N$-fold product $K^N$, then the associated empirical measure is the probability measure

$$\mu^x := \frac{1}{N} \sum_{k=1}^N \delta_{x_k}$$

which is equidistributed on $x_1, \ldots, x_N$. Here, $\delta_x$ denotes the Dirac mass at $x$. Any probability measure $\nu$ on $K^N$ induces a random $N$-point process on $K$ and $\nu$ is the law of this random process.

Let $\{N_p\}_{p \geq 1}$ be a sequence of positive integers such that $N_p \to \infty$ as $p \to \infty$ and $\{\nu_p\}_{p \geq 1}$ a sequence of probability measures on $K^{N_p}$. In many problems from mathematics or mathematical physics, a central question is to study the eventual convergence of the sequence $\mu^{x(p)}$ to an equilibrium measure, where $x^{(p)}$ is the random $N_p$-point process on $K$ described by the law $\nu_p$. A significantly interesting setting considered in literature is the case of $\beta$-ensembles on a compact subset of the unit sphere $S^n$ in $\mathbb{R}^{n+1}$ or a compact subset of $\mathbb{R}^n$. We will obtain in this paper a LDP for such $\beta$-ensembles with an explicit rate of convergence. Our approach uses however techniques from complex analysis, and therefore we will first describe the general setting, which, physically, corresponds to a gas of free fermions and may admit some random matrix models. The reader will find in the paper of Berman [1] a detailed exposition and a list of references. The case of $\beta$-ensembles on the unit sphere or on the real Euclidean space, mentioned above, will be obtained as a corollary, see Examples 1.5 and 1.6 below.
Let $X$ be a compact Kähler manifold of dimension $n$. Let $L \to X$ be a positive line bundle endowed with a given smooth Hermitian metric $h_0$. We assume that the metric $h_0$ is positively curved, that is, the Chern form $\omega_0$ associated with $h_0$ is a Kähler form on $X$. For simplicity, we will use the Riemannian metric on $X$ induced by $\omega_0$. The space of holomorphic sections of $L^p := L \otimes \cdots \otimes L$ ($p$ times) is denoted by $H^0(X, L^p)$. Since $L$ is ample, by Kodaira-Serre vanishing and Riemann-Roch-Hirzebruch theorems (see [18, Thm 1.5.6 and 1.4.6]), we have

$$N_p := \dim H^0(X, L^p) = \frac{p^n}{n!} \|\omega_0^n\| + O(p^{n-1}).$$

Here, $\|\omega_0^n\|$ denotes the mass of the volume form $\omega_0^n$. It depends only on the Chern class of $L$.

If $L_1, L_2$ are line bundles over complex manifolds $X_1$ and $X_2$ respectively, we denote by $L_1 \boxtimes L_2$ the line bundle over the product manifold $X_1 \times X_2$ defined as $L_1 \boxtimes L_2 := \pi_1^* (L_1) \otimes \pi_2^* (L_2)$, where $\pi_1, \pi_2$ are the natural projections from $X_1 \times X_2$ to its factors. If $L_1$ and $L_2$ are endowed with some Hermitian metrics, then $L_1 \boxtimes L_2$ carries also a metric induced by those on $L_1$ and $L_2$.

Let $S_p = (s_1, \ldots, s_{N_p})$ be a basis of $H^0(X, L^p)$. We define the section $\det S_p$ of the line bundle $(L^p)^{\otimes N_p} := L^p \boxtimes \cdots \boxtimes L^p$ ($N_p$ times) over $X^{N_p}$ by the identity

$$\det S_p(x^{(p)}) := \sum_{\sigma \in \text{Sym}_{N_p}} \text{sgn}(\sigma) \prod_{i=1}^{N_p} s_\sigma (x^{(p)}) \quad \text{for} \quad x^{(p)} = (x_1, \ldots, x_{N_p}) \in X^{N_p},$$

where $\text{Sym}_{N_p}$ denotes the permutation group of $\{1, \ldots, N_p\}$. Note that when we change the basis $S_p$, this section only changes by a non-zero multiplicative constant.

Let $K$ be a compact set in $X$ and $\phi$ a continuous real-valued function on $K$. We say that the pair $(K, \phi)$ is a weighted compact set. Let $\mu$ be a probability measure on $K$.

**Definition 1.1.** Let $\beta > 0$ be a constant. A $\beta$-ensemble associated with the line bundle $L^p$, the weighted compact set $(K, \phi)$ and the probability measure $\mu$, is the random $N_p$-point process on $K$ whose joint distribution is given by

$$\nu^\beta_p := c_{p,\beta} \|\det S_p(x^{(p)})\|^\beta e^{-\beta p(\phi(x_1) + \cdots + \phi(x_{N_p}))} d\mu(x_1) \otimes \cdots \otimes d\mu(x_{N_p}),$$

where $c_{p,\beta}$ is the normalizing constant so that $\nu^\beta_p$ is a probability measure on $K^{N_p}$.

Observe that the constant $c_{p,\beta}$ depends also on $L^p, K, \phi, \mu$, but the above random point process, i.e., the measure $\nu^\beta_p$, is independent of the choice of the basis $S_p$ of $H^0(X, L^p)$. We will study these $\beta$-ensembles when $p$ goes to infinity. We need some assumptions on the regularity of $K, \phi$ and $\mu$. Under such conditions, we will see later that the sequence $\mu^{x^{(p)}}$ converges almost surely to a limit $\mu_{\text{eq}}(K, \phi)$ which is called the equilibrium measure of the weighted compact set $(K, \phi)$.

Recently, Berman [11] obtained a LDP in the spirit of Donsker and Varadhan [18] using some functionals on the space of measures. In the case where $K = X, \phi = 0$, and $\mu$ is the Lebesgue measure on $X$, Carroll, Marzo, Massaneda and Ortega-Cerdà obtained precise and optimal estimates on the expectation of the Kantorovich-Wasserstein distance between $\mu^{x^{(p)}}$ and $\mu_{\text{eq}}(K, \phi)$ when $p \to \infty$ [6]. An advantage of the latter work is that Kantorovich-Wasserstein distance gives us a very explicit information about the convergence of $\mu^{x^{(p)}}$ to $\mu_{\text{eq}}(K, \phi)$. Our aim is to establish a LDP with precise estimations in
a quite general setting and in the spirit of the work by Carroll, Marzo, Massaneda and Ortega-Cerdà. In order to state the main result, we need to introduce some more notions.

Let \( \mathcal{M}(X) \) denote the space of all (Borel) probability measures on \( X \). For \( \gamma > 0 \), define the distance \( \text{dist}_\gamma \) between two measures \( \mu \) and \( \mu' \) in \( \mathcal{M}(X) \) by

\[
 \text{dist}_\gamma(\mu, \mu') := \sup_{\|v\|_{\mathcal{C}_0} \leq 1} \big| \langle \mu - \mu', v \rangle \big|
\]

where \( v \) is a test smooth real-valued function. This distance induces the weak topology on \( \mathcal{M}(X) \). By interpolation between Banach spaces (see [10, 22]), for \( 0 < \gamma \leq \gamma' \), there exists a constant \( c > 0 \) such that

\[
(1.3) \quad \text{dist}_{\gamma'} \leq \text{dist}_\gamma \leq c[\text{dist}_{\gamma}]^{\gamma/\gamma'}.
\]

Note that \( \text{dist}_1 \) is equivalent to the classical Kantorovich-Wasserstein distance.

In Section 2 below, we will single out a very large class of compact sets \( K \) which enjoy the so-called \((\mathcal{C}^\alpha, \mathcal{C}^\alpha')\)-regularity. We will also introduce the notion of \( \delta \)-Bernstein-Markov measures which enjoy a quantified version of the Bernstein-Markov property. Here, \( \delta \) is a constant such that \( 0 < \delta < 1 \). Having in hand these natural notions, we are in the position to state the main result of the paper.

**Theorem 1.2.** Let \( X \) be a complex projective manifold of dimension \( n \). Let \( L \) be a positive line bundle over \( X \) endowed with a smooth positively curved Hermitian metric \( h_0 \). Let \( \beta > 0 \) and \( 0 < \gamma \leq 2 \) be constants. Let \( K \) be a \((\mathcal{C}^\alpha, \mathcal{C}^\alpha')\)-regular compact subset of \( X \) and \( \phi \) a \( \mathcal{C}^\alpha \) real-valued function on \( K \) for some constants \( 0 < \alpha \leq 2 \) and \( 0 < \alpha' \leq 1 \). Let \( \mu \) be a probability measure on \( K \) which is \( \delta \)-Bernstein-Markov with respect to \((K, \phi)\) for some \( 0 < \delta < 1 \). Then, for every \( \lambda > 0 \), there are \( c > 0 \) and Borel sets \( E_{p} \subseteq K^{\mathbb{N}} \) such that

(a) \( \nu^\beta_p (E_{p}) \leq e^{-\lambda p^{\alpha+1-\delta}} \);

(b) if \( \mu^x_p \) denotes the empirical measure associated with \( x \in K^{\mathbb{N}} \setminus E_p \), then

\[
\text{dist}_\gamma(\mu^x_p, \mu_{\text{eq}}(K, \phi)) \leq c q^\gamma.
\]

Here, \( q := p^{-\delta/4} \) if \( \delta/4 < \alpha' \), \( q := p^{-\alpha'} (\log p)^{3a''} \) if \( \delta/4 \geq \alpha' \), and \( a'' := \alpha'/ (24 + 12\alpha') \).

If a sequence of points \( x^{(p)} \in K^{\mathbb{N}} \) satisfies \( x^{(p)} \not\in E_p \) for \( p \) large enough, then we deduce from the last theorem that \( \mu^{x^{(p)}} \rightarrow \mu_{\text{eq}}(K, \phi) \) when \( p \) goes to infinity. Therefore, \( \mu^{x^{(p)}} \) converge almost surely to \( \mu_{\text{eq}}(K, \phi) \) when \( p \) goes to infinity. More precisely, the infinite product \( \nu^\beta := \nu^\beta_1 \times \nu^\beta_2 \times \cdots \) is a probability measure on the space of all sequences \( (x^{(p)})_{p=1}^{\infty} \). With respect to this measure, the convergence \( \mu^{x^{(p)}} \rightarrow \mu_{\text{eq}}(K, \phi) \) holds for almost every sequence \( (x^{(p)})_{p=1}^{\infty} \).

The estimate on the size of \( E_p \) is a version of LDP. Our result also implies that

\[
(1.4) \quad \int_{X^{\mathbb{N}}} \text{dist}_\gamma(\mu^x, \mu_{\text{eq}}(K, \phi)) \, d\nu^\beta_p(x) = O(q^\gamma).
\]

This distance expectation estimate is similar to the one obtained by Carroll, Marzo, Massaneda and Ortega-Cerdà in [6] that we mentioned above. These authors proved for \( K = X, \phi = 0 \) and \( \mu \) the normalized Lebesgue measure on \( X \) that there is a constant \( c > 0 \) satisfying

\[
(1.5) \quad c^{-1} p^{1/2} \leq \int_{X^{\mathbb{N}}} \text{dist}_1(\mu^x, \mu_{\text{eq}}(K, \phi)) \, d\nu^\beta_p(x) \leq c p^{1/2}
\]
for all $p$.

In order to get more concrete applications of our main result, we need the following natural class of positive Borel measures.

**Definition 1.3.** We say that a positive measure $\mu$ on $X$ satisfies the mass-density condition with respect to $K$ if there are two constants $c > 0$ and $\rho > 0$ such that

$$\mu(B(x, r)) \geq cr^\rho \quad \text{for} \quad x \in W \quad \text{and} \quad 0 < r < 1.$$ 

Here, $B(x, r)$ denotes the ball in $(X, \omega_0)$ of radius $r$ and centered at the point $x$.

Assume now that $K$ is a smooth real manifold in $X$ with piecewise smooth boundary such that the tangent space of $K$ at each point is not contained in a complex hyperplane of the tangent space of $X$ at that point. It was shown in [9, 21], for $0 < \alpha < 1$, that $K$ is $(C^\alpha, C^{\alpha/2})$-regular and is $(C^\alpha, C^\alpha)$-regular when its boundary is smooth, see Theorem 2.3 below. In this case, if $\mu$ is a probability measure on $K$ satisfying the above mass-density condition for $W = K$, we will show in Corollary 2.13 below that it satisfies the $\delta$-Bernstein-Markov property required in Theorem 1.2. Therefore, the following result is a direct consequence of that theorem.

**Corollary 1.4.** Let $X, L, h_0, \beta, \gamma$ be as in Theorem 1.2. Let $K$ be a smooth real manifold in $X$ with piecewise smooth boundary such that the tangent space of $K$ at each point is not contained in a complex hyperplane of the tangent space of $X$ at that point. Let $\mu$ be a probability measure on $K$ satisfying the mass-density condition with respect to $K$. Let $\phi$ be a $C^\alpha$ real-valued function on $K$ with $0 < \alpha < 1$. Then, for every $0 < \beta < 1$, the conclusion of Theorem 1.2 holds for $\alpha'' := \alpha/(48 + 24\alpha)$. Moreover, if the boundary of $K$ is smooth, the same statement holds for $\alpha'' := \alpha/(24 + 12\alpha)$.

Of course, Corollary 1.4 holds when $\mu$ is given by the normalized volume form on $K$.

It is worthy noting that the assumption on the mass-density condition of the measure $\mu$ in this result can be weakened. In fact, we only need that $\mu$ satisfies the mass-density condition on a subset $W \subset K$ which satisfies a maximum principle, see Corollary 2.13 below.

**Example 1.5.** Let $K$ be the closure of an open set with piecewise smooth boundary in $\mathbb{R}^n$. Let $\phi$ be a $C^\alpha$ real-valued function on $K$ and $\mu$ a probability measure on $K$ which satisfies the mass-density condition with respect to $K$. It is already interesting to consider the case where $\mu$ is the normalization of the restriction to $K$ of the Lebesgue measure on $\mathbb{R}^n$. Denote by $P_p$ the set of real polynomials of degree at most $p$ and $N_p$ the dimension of $P_p$. Choose a basis $(P_1, \ldots, P_{N_p})$ of $P_p$. Define the probability measure $\nu_p^{(\beta)}$ at a point $x = (x_1, \ldots, x_{N_p})$ on $K^{N_p}$ by

$$c_{p, \beta} |\det(P_1(x_1))| \cdots |\det(P_{N_p}(x_{N_p}))| e^{-\beta \delta(p(x_1) + \cdots + p(x_{N_p}))} e^{-\frac{1}{2} \beta \delta(\log(1 + ||x_1||^2)+\cdots+\log(1+||x_{N_p}||^2))} \mu(x_1) \otimes \cdots \otimes \mu(x_{N_p}),$$

where $c_{p, \beta}$ is a normalizing constant so that $\nu_p^{(\beta)}$ is a probability measure. Here, $|\det(\cdot)|$ denotes the standard determinant of a square matrix. Then the conclusion of Theorem 1.2 holds for $\alpha'' := \alpha/(48 + 24\alpha)$. If the boundary of $K$ is smooth, we can take $\alpha'' := \alpha/(24 + 12\alpha)$. The equilibrium measure $\mu_{eq}(K, \phi)$ is a probability measure supported by $K$. Its definition is given in Section 2.

In order to obtain this result as a consequence of Theorem 1.2 and Corollary 1.4, consider $\mathbb{R}^n$ as the real part of $\mathbb{C}^n$ and $\mathbb{C}^n$ as a Zariski open set of the projective space.
Denote by $[z_0 : \cdots : z_n]$ the homogeneous coordinates of $\mathbb{P}^n$. We identify $\mathbb{C}^n$ with the open set $\{z_0 = 1\}$. Define $X := \mathbb{P}^n$. We can identify, in the natural way, the polynomials of degree $\leq p$ on $\mathbb{R}^n$ with holomorphic sections of $L^p$ with $L = \mathcal{O}(1)$ the tautological line bundle of $X = \mathbb{P}^n$. We consider the standard Hermitian metrics on these line bundles. So $\{P_1, \ldots, P_{N_p}\}$ is identified to a basis of $H^0(X, L^p)$. If a section $s$ in $H^0(X, L^p)$ is identified to a polynomial $P$ then

$$\|s(z)\| = |P(z)| e^{-\frac{1}{2} p \log(1 + \|z\|^2)} \text{ for } z \in \mathbb{C}^n.$$ 

So the factor involving $\log(1 + \|x_i\|^2)$ in the definition of $\nu^\beta_p$ is due to the standard Hermitian metric of $L^p$. We can now apply Theorem 1.2 and Corollary 1.4 and get the LDP in this case. An interesting particular situation is the case where the weight $\phi$ is equal to $-\frac{1}{2} \log(1 + \| \cdot \|^2)$.

**Example 1.6.** Let $K$ be the closure of an open set with piecewise smooth boundary in the unit sphere $S^n$ of $\mathbb{R}^{n+1}$. Let $\phi$ be a $C^\alpha$ real-valued function on $K$ and $\mu$ a probability measure on $K$ which satisfies the mass-density condition with respect to $K$. It is already interesting to consider the case where $\phi = 0$ and $\mu$ is the normalization of the restriction to $K$ of the Haar measure on $S^n$. Consider the functions which are restrictions of (real) polynomials on $\mathbb{R}^{n+1}$ to $S^n$. Denote by $\mathcal{P}_p$ the set of these functions obtained by using polynomials of degree at most $p$ and $N_p$ the dimension of $\mathcal{P}_p$. Note that $\mathcal{P}_p$ is isomorphic to the quotient of the space of polynomials of degree $\leq p$ by the subspace of polynomials divisible by $x_1^2 + \cdots + x_{n+1}^2 - 1$, where $(x_1, \ldots, x_{n+1})$ is the standard coordinate system of $\mathbb{R}^{n+1}$.

Choose a basis $(P_1, \ldots, P_{N_p})$ of $\mathcal{P}_p$. Define the probability measure $\nu^\beta_p$ on $K^{N_p}$ by

$$\nu^\beta_p(x) := c_{p, \beta} |\det(P_i(x_j))| \beta^p e^{-\beta p (\phi(x_1) + \cdots + \phi(x_{N_p}))} \mu(x_1) \otimes \cdots \otimes \mu(x_{N_p}),$$

where $x = (x_1, \ldots, x_{N_p})$ is a point in $K^{N_p}$ and $c_{p, \beta}$ is a normalizing constant so that $\nu^\beta_p$ is a probability measure. Then the conclusion of Theorem 1.2 holds for $\alpha'' := \alpha/(48 + 24\alpha)$. If the boundary of $K$ is smooth, we can take $\alpha'' := \alpha/(24 + 12\alpha)$. The measure $\mu_{eq}(K, \phi)$ is supported by $K$. In the case where $K = S^n$ and $\phi = 0$, by symmetry, this measure coincides with the Haar measure on $S^n$.

In order to obtain this result as a consequence of Theorem 1.2 and Corollary 1.4, we need to complexify $S^n$. Consider $\mathbb{R}^{n+1}$ as the real part of $\mathbb{C}^{n+1}$ and $\mathbb{C}^{n+1}$ as a Zariski open set of the projective space $\mathbb{P}^{n+1}$. Denote by $[z_0 : \cdots : z_{n+1}]$ the homogeneous coordinates of $\mathbb{P}^{n+1}$. We identify $\mathbb{C}^{n+1}$ with the open set $\{z_0 = 1\}$. The sphere $S^n$ is then the intersection of $\mathbb{R}^{n+1}$ with the complex hypersurface $z_1^2 + \cdots + z_{n+1}^2 = z_0^2$ in $\mathbb{P}^{n+1}$. Denote by $X$ this hypersurface. We can identify, in the natural way, the polynomials of degree $\leq p$ on $\mathbb{R}^{n+1}$ with holomorphic sections of $L^p$ with $L = \mathcal{O}(1)$ the tautological line bundle of $\mathbb{P}^{n+1}$. As in Example 1.5 we consider the standard Hermitian metrics on these line bundles. Note that $|z_1|^2 + \cdots + |z_{n+1}|^2$ is constant on $S^n$ and therefore, the formula for $\nu^\beta_p$ is simpler than the one in Example 1.5. Observe also that a section of $L^p$ vanishes on $X$ if and only if it vanishes on $S^n$. Therefore, $\{P_1, \ldots, P_{N_p}\}$ is identified to a basis of $H^0(X, L^p)$. We can now apply Theorem 1.2 and Corollary 1.4.

The plan of the paper is as follows. In Section 2, we discuss different notions of regularity for the weighted compact set $(K, \phi)$ and the measure $\mu$. We also give criteria to check the regularity conditions used in our study. In Section 3, we prove the main
theorem (Theorem 1.2) which uses an equidistribution result for almost Fekete configurations. The last result has been obtained in collaboration with Ma in the last version of [9, Remark 3.17]. For the reader’s convenience, we provides here a detailed proof that we need in this paper. Note that the case of Fekete points can be seen as the limit case of $\beta$-ensembles when $\beta \to \infty$. We refer to [3, 9, 13, 14, 15, 21], the references therein and also the end of this paper for more results on Fekete points and other configurations.

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2. Pluri-regularity for weighted compact sets and measures

As we have seen in Introduction, our study requires some regularity properties of the weighted compact set $(K, \phi)$ and the probability measure $\mu$ on $K$. In this section, we will recall some known facts and also introduce and study new notions that will be used in the proof of our main theorem. The reader will find in [7, 10, 12, 18] basic notions and results from complex geometry and pluripotential theory.

Let $L$ be a positive (i.e., ample) holomorphic line bundle over a projective manifold $X$ of dimension $n$. Fix a smooth Hermitian metric $h_0$ on $L$ such that its first Chern form $\omega_0$ is a Kähler form on $X$. Define $\mu_0 := \|\omega_0^n\|^{-1}\omega_0^n$ the probability measure associated with the volume form $\omega_0^n$. Here, $\|\omega_0^n\|$ is the total mass of $\omega_0^n$ which is the integral of this volume form on $X$. Recall that a real-valued function on $X$ is quasi-p.s.h. if it is locally the difference between a p.s.h. function and a smooth one. Let $\text{PSH}(X, \omega_0)$ be the cone of $\omega_0$-p.s.h. functions, i.e., the quasi-p.s.h. functions $\varphi$ such that $dd^c \varphi + \omega_0 \geq 0$.

**Definition 2.1.** We call weighted compact subset of $X$ a data $(K, \phi)$, where $K$ is a non-pluripolar compact subset of $X$ and $\phi$ is a real-valued continuous function on $K$. The function $\phi$ is called a weight on $K$. The equilibrium weight associated with $(K, \phi)$ is the upper semi-continuous regularization $\phi^*_{eq}$ of the function

$$\phi_K(z) := \sup \{ \psi(z) : \psi \text{ $\omega_0$-p.s.h. such that } \psi \leq \phi \text{ on } K \}.$$ 

We also call equilibrium measure of $(K, \phi)$ the normalized Monge-Ampère measure

$$\mu_{eq}(K, \phi) := \|\omega_0^n\|^{-1}(dd^c \phi^*_{eq} + \omega_0)^n.$$ 

Note that the equilibrium measure $\mu_{eq}(K, \phi)$ is a probability measure supported by $K$ and $\phi^*_{eq} = \phi_K$ almost everywhere with respect to this measure, see e.g., [2]. The following notions are important in our study, see [2].

**Definition 2.2.** Denote by $P_K$ the projection onto $\text{PSH}(X, \omega_0)$ which associates $\phi$ with $\phi^*_{eq}$. We say that $(K, \phi)$ is regular if $\phi_K$ is upper semi-continuous, i.e., $P_K \phi = \phi_K$. Let $(E, \|\cdot\|_E)$ be a normed vector space of continuous functions on $K$ and $(F, \|\cdot\|_F)$ a normed vector space of functions on $X$. We say that $K$ is $(E, F)$-regular if $(K, \phi)$ is regular for $\phi \in E$ and if the projection $P_K$ sends bounded subsets of $E$ into bounded subsets of $F$.

We have the following result.
Theorem 2.3 ([9, 21]). Let $X$ and $L$ be as above. Let $K$ be a smooth compact real manifold in $X$ with piecewise smooth boundary. Assume that the tangent space to $K$ at any point is not contained in a complex hyperplane of the tangent space to $X$ at that point. Let $0 < \alpha < 1$ be any real number. Then $K$ is $(\mathcal{C}^{\alpha}, \mathcal{C}^{\alpha/2})$-regular. Moreover, it is $(\mathcal{C}^{\alpha}, \mathcal{C}^{\alpha})$-regular if the boundary of $K$ is smooth.

Consider now a real-valued function $\psi$ on $X$. We can associate the line bundle $L$ with a singular Hermitian metric $h := e^{-2\psi}h_0$. More precisely, if $v$ is a vector in the fiber of $L$ over a point $x \in X$, its norms with respect to the metrics $h$ and $h_0$ are related by the formula

$$|v|_h = e^{-\psi(x)}|v|_{h_0}.$$  

The metrics $h_0$ and $h$ induce in a canonical way metrics $h_0 \otimes p$ and $h \otimes p$ on the power $L^p$ of $L$. They are related by the formula $h_0 \otimes p = e^{-2\psi}h_0 \otimes p$. Recall that for simplicity, we will use the notation $| \cdot |_{p \psi}$ instead of $| \cdot |_{h_0 \otimes p}$ for the norm of a vector in $L^p$ with respect to the metric $h \otimes p$. We also drop the subscript $h_0$, e.g., $|v|$ means $|v|_{h_0}$.

Consider now a weighted compact set $(K, \phi)$ in $X$. We can, in a similar way, define the metric $h = e^{-2\phi}h_0$ on $L$ over $K$. Let $\mu$ be a probability measure with support in $K$. Consider the natural $L^\infty$ and $L^2$ semi-norms on $H^0(X, L^p)$ induced by the metric $h$ on $L$ and the measure $\mu$, which are defined for $s \in H^0(X, L^p)$ by

$$||s||_{L^\infty(K, \phi)} := \sup_K |s|_{p\phi} \quad \text{and} \quad ||s||_{L^2(\mu, p\phi)}^2 := \int_X |s|^2_{p\phi}d\mu. \quad (2.1)$$

We will only use measures $\mu$ such that the above semi-norms are norms, i.e., there is no section $s \in H^0(X, L^p) \setminus \{0\}$ which vanishes on $K$ or on the support of $\mu$. The first semi-norm is a norm when $K$ is not contained in a hypersurface of $X$. The second one is a norm when the support of $\mu$ is not contained in a hypersurface of $X$. In particular, this is the case when $\mu$ is the normalized Monge-Ampère measure with continuous potentials because such a measure has no mass on hypersurfaces of $X$.

We need the following quantified Bernstein-Markov property, see also [2, 3, 16, 19].

Definition 2.4. Let $\delta$ be a real number with $0 < \delta < 1$ and $(K, \phi)$ a weighted compact subset of $X$. We say that a positive measure $\mu$ on $K$ is $\delta$-Bernstein-Markov with respect to $(K, \phi)$ if there is a constant $A > 0$ such that

$$||s||_{L^\infty(K, \phi)} \leq Ae^{A\delta^1-\delta}||s||_{L^2(\mu, p\phi)} \quad \text{for} \quad s \in H^0(X, L^p) \quad \text{and} \quad p \geq 1. \quad (2.2)$$

If $\mu$ is $\delta$-Bernstein-Markov with respect to $(K, \phi)$ for all $0 < \delta < 1$, then we say that $\mu$ is 1-Bernstein-Markov with respect to $(K, \phi)$.

The following lemma shows that we can use the notion for other norms $L^r$.

Lemma 2.5. Let $\delta, r$ be real numbers with $0 < \delta < 1$ and $r > 0$. Let $(K, \phi)$ be a weighted compact subset of $X$ and $\mu$ a positive measure on $K$. Then $\mu$ is $\delta$-Bernstein-Markov with respect to $(K, \phi)$ if and only if there is a constant $A' > 0$ such that

$$||s||_{L^\infty(K, \phi)} \leq A'e^{A'r^{1-\delta}}||s||_{L^r(\mu, p\phi)} \quad \text{for} \quad s \in H^0(X, L^p) \quad \text{and} \quad p \geq 1. \quad (2.2)$$

Proof. Assume that $\mu$ is $\delta$-Bernstein-Markov with respect to $(K, \phi)$. We will only show the existence of $A'$ as in the lemma because the converse property can be obtained in the same way. So we have property (2.2). Without loss of generality, we can assume that $\mu$
is a probability measure. If \( r \geq 2 \), then the \( L^r \)-norm is larger or equal to the \( L^2 \)-norm. Therefore, we can just take \( A' := A \).

Assume now that \( 0 < r < 2 \). By Hölder’s inequality, we have

\[
\|s\|_{L^2(\mu,p\phi)} \leq \|s\|_{L^r(\mu,p\phi)}^{2/r} \cdot \|s\|_{L^\infty(\mu,p\phi)}^{1-r/2}.
\]

This, together with (2.2), gives us the desired property for a suitable value of \( A' \). \( \square \)

In order to get a simple criterium for a measure to have the \( \delta \)-Bernstein-Markov property, we need the following notion.

**Definition 2.6.** A compact set \( W \) is said to satisfy the maximum principle relatively to a weighted compact set \( (K, \phi) \) if \( W \subset K \) and

\[
\sup_K (\psi - \phi) = \sup_W (\psi - \phi) \quad \text{for every} \quad \psi \in \text{PSH}(X, \omega_0).
\]

Clearly, \( W = K \) satisfies the maximum principle relatively to \( (K, \phi) \). In general, \( W \) may be much smaller than \( K \), see Remark 2.9 below.

**Proposition 2.7.** Let \( (K, \phi) \) be a weighted compact set and \( W \) a compact subset of \( K \). Define

\[
\partial_{\omega_0}^w K := \{ z \in K : P_K \phi(z) = \phi(z) \}.
\]

Then \( W \) satisfies the maximum principle relatively to \( (K, \phi) \) if and only if \( W \cap \partial_{\omega_0}^w K \) satisfies the same property. In particular, \( \partial_{\omega_0}^w K \) satisfies the maximum principle relatively to \( (K, \phi) \)

**Proof.** Observe that the second assertion is a consequence of the first one and Definition 2.6 by taking \( W = K \). We prove now the first assertion. If \( W \cap \partial_{\omega_0}^w K \) satisfies the maximum principle relatively to \( (K, \phi) \), then clearly \( W \) satisfies the same property. Assume that \( W \) satisfies this maximum principle. It remains to prove the same property for \( W \cap \partial_{\omega_0}^w K \).

Recall that \( P_K \phi \) is upper semi-continuous and \( \phi \) is continuous. Since \( P_K \phi \leq \phi \), we deduce that

\[
\partial_{\omega_0}^w K = \{ z \in K : P_K \phi(z) \geq \phi(z) \}
\]

So it is a compact set.

Let \( \psi \in \text{PSH}(X, \omega_0) \) and set \( m := \max_K (\psi - \phi) \). Note that \( \psi \) is also upper semi-continuous. Since \( W \) satisfies the maximum principle relatively to \( (K, \phi) \), there is a point \( z_0 \in W \) such that \( \psi - \phi \) attains its maximum value at \( z_0 \). We have \( \psi(z_0) - m = \phi(z_0) \) and \( \psi - m \leq \phi \) on \( K \). Since \( \psi - m \in \text{PSH}(X, \omega_0) \), the last inequality implies that \( \psi - m \leq P_K \phi \).

In particular,

\[
\psi(z_0) - m \leq (P_K \phi)(z_0).
\]

This, combined with the equality \( \psi(z_0) - m = \phi(z_0) \) and the inequality \( (P_K \phi)(z_0) \leq \phi(z_0) \), implies that \( (P_K \phi)(z_0) = \phi(z_0) \). Hence, \( z_0 \in \partial_{\omega_0}^w K \) and the proposition follows. \( \square \)

**Remark 2.8.** By [2] Prop. 2.10, Cor. 2.5], the equilibrium measure \( \mu_{\text{eq}}(K, \phi) \) is supported by \( \partial_{\omega_0}^w K \) and its support also satisfies the maximum principle.

**Remark 2.9.** Let \( X \) be the projective space \( \mathbb{P}^n \), seen as the natural compactification of \( \mathbb{C}^n \). Let \( L \) be the tautological line bundle \( O(1) \) over \( \mathbb{P}^n \). Then the holomorphic sections of \( L^p = O(p) \) can be identified to the complex polynomials of degree \( \leq p \) on \( \mathbb{C}^n \). With the
standard Fubini-Study metric on \( O(p) \), if a section \( s \) of \( L^p \) corresponds to a polynomial \( P(z) \) of degree \( \leq p \), then

\[
|s(z)| = |P(z)|(1 + \|z\|^2)^{-p/2}.
\]

Consider a compact subset \( K \) of \( \mathbb{C}^n \) and take \( \phi := -\frac{1}{2} \log(1 + \|z\|^2) \) on \( K \). It is not difficult to check that the boundary of \( K \) satisfies the maximum principle relatively to \( (K, \phi) \).

**Theorem 2.10.** Let \( X, L, h_0 \) be as above, \((K, \phi)\) a weighted compact subset of \( X \) and \( \mu \) a probability measure on \( K \). Let \( W \subset K \) be a compact set and \( 0 < \delta < 1 \) a real number. Assume in addition the following conditions:

(i) the functions \( \phi \) and \( P_K \phi \) are Hölder continuous;

(ii) \( W \) satisfies the maximum principle relatively to \((K, \phi)\);

(iii) \( \mu \) satisfies the mass-density condition with respect to \( W \), see Definition 1.3.

Then \( \mu \) is a \( \delta \)-Bernstein-Markov measure with respect to \((K, \phi)\).

**Remark 2.11.** We will see in the proof of this theorem that the condition (i) can be replaced by the following much weaker condition: there are constant \( c > 0 \) such that for \( z \in X \) and \( w \in K \)

\[
|(P_K \phi)(z) - (P_K \phi)(w)| \leq c(1 + \log^\frac{1}{\delta} \text{dist}(z, w))^\frac{1}{1 - \delta},
\]

and for \( z, w \in K \)

\[
|\phi(z) - \phi(w)| \leq c(1 + \log^\frac{1}{\delta} \text{dist}(z, w))^\frac{1}{1 - \delta},
\]

where \( \log^- := \max(0, -\log) \).

We are inspired by an idea of Bloom [5, Theorem 4.1]. Define \( \epsilon := p^{-\delta} \) and \( r := e^{c'p^{1-\delta}} \) where \( c' > 0 \) is a large enough constant independent of \( p \). It follows from Assumption (i) (see also Remark 2.11) that

\[
(P_K \phi)(z) - (P_K \phi)(z_0) \leq \epsilon \quad \text{for} \quad z \in B(z_0, 2r) \quad \text{and} \quad z_0 \in K.
\]

Fix \( p \geq 1 \) and \( s \in H^0(X, L^p) \setminus \{0\} \). We need to prove inequality (2.2) for some constant \( A > 0 \) independent of \( p \) and \( s \). Observe that

\[
d^p \frac{1}{p} \log |s| = \frac{1}{p} [s = 0] - \omega_0 \geq -\omega_0,
\]

where \( [s = 0] \) is the current of integration on the hypersurface \( \{s = 0\} \). So \( \frac{1}{p} \log |s| \) is \( \omega_0 \)-p.s.h. This, together with Assumption (ii), implies the existence of a point \( z_0 \in W \) such that

\[
|s(z_0)|_p = \max_{z \in K} |s(z)|_p.
\]

**Lemma 2.12.** We have

\[
| |s(z)| - |s(z_0)| | \leq \frac{1}{4} |s(z_0)| \quad \text{for} \quad z \in B(z_0, r^2).
\]

**Proof.** Consider a section \( s' = cs \) where the constant \( c \) is chosen so that \( \|s'\|_{L^\infty(K, \phi)} = 1 \). The last property implies the inequality \( \frac{1}{p} \log |s'| \leq \phi \) on \( K \). We have seen that \( \frac{1}{p} \log |s| \) is \( \omega_0 \)-p.s.h. So \( s' \) satisfies a similar property. Hence, \( \frac{1}{p} \log |s'| \leq P_K \phi \) on \( X \). We then deduce the following Bernstein-Walsh type inequality

\[
|s(z)| \leq \|s\|_{L^\infty(K, \phi)} e^{P_K \phi(z)} \quad \text{for} \quad z \in X.
\]
We have

\[ |s(z)| \leq |s(z_0)| e^{pK(z)} = |s(z_0)| e^{-p(\phi(z_0) - (pK\phi)(z_0))} e^{p(\phi(z) - (pK\phi)(z_0))}. \]

Using \((2.6)\) and the choice of \(z_0\), we obtain

\[ |s(z)| \leq |s(z_0)| e^{pe} \quad \text{for} \quad z \in B(z_0, 2r). \]

Let \(\sigma\) be a holomorphic frame for \(L\) on an open neighborhood \(U\) of \(z_0\) with \(|\sigma(z_0)| = 1\). Write \(s = h\sigma^{\circ p}\) with \(h\) a holomorphic function on \(U\). Using local coordinates near \(z_0\) and shrinking \(U\) if necessary, we may identify \(U\) with the open unit ball in \(\mathbb{C}^n\). We can also assume that

\[ |\sigma(z)| - 1| \leq c|z - z_0| \]

for some constant \(c > 0\) independent of \(z \in U\). For \(z \in B(z_0, 2r)\), we have \(|z - z_0| \ll p^{-1}\) and the previous inequality implies that \(|\sigma^{\circ p}(z)|\) belongs to the interval \([7/8, 9/8]\) when \(z \in B(z_0, 2r)\). So the norm \(|s(z)|\) is bounded below and above by \(7|h(z)|/8\) and \(9|h(z)|/8\) respectively.

Consider the unit vector \(v := \frac{z - z_0}{|z - z_0|}\) in \(\mathbb{C}^n\), and the following holomorphic function of one variable

\[ f(\zeta) := h(z_0 + \zeta v), \quad \zeta \in \mathbb{D}. \]

We have for \(z \in B(z_0, r)\)

\[ |h(z) - h(z_0)| = |f(\|z - z_0\|) - f(0)| \leq |z - z_0| \sup_{|\zeta| \leq r} |f'(\zeta)|. \]

On the other hand, for \(|\zeta| \leq 2r\), we have \((z_0 + \zeta v) \in B(z_0, 2r)\), and by using the definition of \(f, h, (2.5)\) and that \(|s(z)|\) is in-between \(7|h(z)|/8\) and \(9|h(z)|/8\), we obtain

\[ \sup_{|\zeta| \leq 2r} |f'(\zeta)| \leq c|s(z_0)| e^{pe} \]

for some constant \(c > 0\). By Cauchy’s formula,

\[ \sup_{|\zeta| \leq r} |f'(\zeta)| \leq \frac{c}{r} |s(z_0)| e^{pe}. \]

This, together with \((2.6)\) and the choice of \(\epsilon, r\), implies for \(z \in B(z_0, r^2)\) that

\[ |h(z) - h(z_0)| \leq cr |s(z_0)| e^{pe} \ll |s(z_0)|. \]

Recall that \(|h(z_0)| = |s(z_0)|\) and \(|s(z)|\) is bounded by \(7|h(z)|/8\) and \(9|h(z)|/8\). So the last inequality implies the lemma.

\[ \text{End of the proof of Theorem 2.10.} \]

We only need to consider \(p\) large enough. We will prove that

\[ |s(z)|_{p\phi} \geq \frac{1}{2} \|s\|_{L^\infty(K, p\phi)} \quad \text{for} \quad z \in K \cap B(z_0, r^2). \]

We have

\[ \|s(z)\|_{p\phi} - |s(z_0)|_{p\phi} \leq \|s(z)\|_{p\phi(z_0)} - |s(z_0)|_{p\phi(z_0)} + \|s(z)\|_{p\phi(z_0)} - |s(z)|_{p\phi(z_0)} \]

Denote respectively by \(A_1\) and \(A_2\) the first and second terms in the last sum. By Lemma \(2.12\) we have

\[ A_1 \leq \frac{1}{4} |s(z_0)| e^{-p\phi(z_0)}. \]
On the other hand, by Lemma 2.12 again, we have
\[ A_2 = |s(z)||e^{-p\phi(z)} - e^{-p\phi(z_0)}| \leq 2|s(z_0)||e^{-p\phi(z_0)}(1 - e^{-p(\phi(z) - \phi(z_0))}). \]
Since \( z \in B(z_0, r^2) \), we deduce from Assumption (i) of the theorem (see also Remark 2.11) that \( |p(\phi(z) - \phi(z_0))| \leq 1/16 \). Hence, \( |1 - e^{-p(\phi(z) - \phi(z_0))}| \leq 1/8 \). Combining the above estimates for \( A_1 \) and \( A_2 \), we obtain
\[ \|s(z)\|_{p\phi} - |s(z_0)|_{p\phi} \leq \frac{1}{2}|s(z_0)|_{p\phi} \quad \text{for} \quad z \in K \cap B(z_0, r^2). \]
This, combined with (2.4), implies (2.7).

Now, using (2.7) and Assumption (iii), we get
\[
\int_K |s(z)|^2 d\mu \geq \int_{K \cap B(z_0, r^2)} |s(z)|^2 d\mu \\
\geq \left( \min_{K \cap B(z_0, r^2)} |s(z)|^2 \right) \mu(K \cap B(z_0, r^2)) \\
\geq \frac{1}{4} c r p \|s\|_{L^\infty(K, p\phi)}^2,
\]
where \( c > 0 \) is the constant in Definition 1.3. Hence,
\[
\|s\|_{L^\infty(K, p\phi)} \leq 2c^{-1/2} e^{cr p^{1-\delta}} \|s\|_{L^2(\mu, p\phi)}.
\]
So \( \mu \) is \( \delta \)-Bernstein-Markov with respect to \((K, \phi)\).\]

We have the following result where Condition (ii) is automatically satisfied for \( W = K \). It allows us to obtain Corollary 1.4 as a direct consequence of Theorem 1.2. Note that in Corollary 1.4 we only need to assume that the measure \( \mu \) satisfies the mass-density condition with respect to a compact \( W \subset K \) which satisfies the maximum principle relatively to \((K, \phi)\).

**Corollary 2.13.** Let \( X, L, h_0 \) be as above, \( K \) a compact subset of \( X \), \( W \) a compact subset of \( K \) and \( \mu \) a probability measure on \( K \). Assume in addition the following conditions:

(i) \( K \) is \((\mathcal{C}^\alpha, \mathcal{C}^{\alpha'})\)-regular for some constants \( \alpha > 0 \) and \( \alpha' > 0 \);
(ii) \( W \) satisfies the maximum principle relatively to \((K, \phi)\);
(iii) \( \mu \) satisfies the mass-density condition with respect to \( W \).

Then \( \mu \) is a 1-Bernstein-Markov measure with respect to \((K, \phi)\) for every \( \phi \in \mathcal{C}^\alpha(K) \).

**Proof.** Since \( \phi \in \mathcal{C}^\alpha(K) \) and \( K \) is \((\mathcal{C}^\alpha, \mathcal{C}^{\alpha'})\)-regular, \((K, \phi)\) satisfies the hypotheses of Theorem 2.10. According to that theorem, \( \mu \) is \( \delta \)-Bernstein-Markov with respect to \((K, \phi)\) for every \( 0 < \delta < 1 \). The corollary follows.\]

### 3. Almost-Fekete Configurations and Proof of the Main Result

In this section, we will give the proof of the main theorem. An important ingredient is the equidistribution of almost-Fekete points towards the equilibrium measure. This property is already mentioned in the last version of [9], see also [17]. For the reader's convenience, we will give here some details. We also give at the end of this section another application of this result.
Theorem 3.1 ([9]). Let \( X, L, h_0 \) be as above and \( K \) a compact subset of \( X \). Let \( 0 < \alpha \leq 2, 0 < \alpha' \leq 1 \) and \( 0 < \gamma \leq 2 \) be constants. Assume that \( K \) is \((\mathcal{C}^\alpha, \mathcal{C}^{\alpha'}\)\)-regular. Let \( \phi \) be a \( \mathcal{C}^\alpha \) real-valued function on \( K \) and \( \mu_{eq}(K, \phi) \) the equilibrium measure associated with the weighted set \((K, \phi)\). Then, there is a constant \( c > 0 \) with the following property. For every \( p \geq 1 \) and every configuration \( x = (x_1, \ldots, x_N) \in K^N_p \), denote by \( \mu^x \) the empirical measure associated with \( x \) and let \( S_p \) be any basis of \( H^0(X, L^p) \). Define

\[
\sigma_x := \frac{1}{pN_p} \log \| \det S_p \|_{L^\infty(K,p\phi)} - \frac{1}{pN_p} \log \| \det S_p(x) \|_{p\phi}.
\]

Then we have for all \( p > 1 \)

\[
dist_\gamma(\mu^x, \mu_{eq}(K, \phi)) \leq cp^{-\alpha''\gamma}(\log p)^{3\alpha''\gamma} + c\sigma_x^{3/4} \quad \text{with} \quad \alpha'' := \frac{\alpha'}{(24 + 12\alpha')}.
\]

Note that \( \det S_p \) is a section of the line bundle \((L^p)^{\otimes N_p} \) over \( X^N_p \). The given metric \( h_0 \) on \( L \) and the weight \( \phi \) induces naturally a metric and a weight for this line bundle. So \( \| \det S_p \|_{L^\infty(K,p\phi)} \) is the sup-norm of \( \det S_p \) on \( K^N_p \) and \( \| \det S_p(x) \|_{p\phi} \) is the norm of the value of this section at the point \( x \). Both of them are measured using the above natural metric and weight. Observe that \( \sigma_x \) is independent of the choice of \( S_p \) and we always have \( \sigma_x \geq 0 \). When \( \sigma_x = 0 \), the point \( x \) is called a Fekete configuration of order \( p \) of \( L \) with respect to the weighted compact set \((K, \phi)\). The theorem shows that if \( \sigma_x \) is small enough (e.g., when \( \sigma_x = 0 \)), then \( \mu^x \) tends to \( \mu_{eq}(K, \phi) \) as \( p \to \infty \).

We now sketch the proof of Theorem 3.1. Recall that the Monge-Ampère energy functional \( \mathcal{E} \), defined on bounded weights in \( \text{PSH}(X, \omega_0) \), is characterized by

\[
\frac{d}{dt} \bigg|_{t=0} \mathcal{E}((1-t)\phi_1 + t\phi_2) = \|\omega_0^n\|^{-1} \int_X (\phi_2 - \phi_1)(ddc\phi_1 + \omega_0)^n.
\]

So \( \mathcal{E} \) is only defined up to an additive constant, but the differences such as \( \mathcal{E}(\phi_1) - \mathcal{E}(\phi_2) \) are well-defined, see [2] and also (3.2) below.

Consider a non-pluripolar compact set \( K \subset X \) and a continuous weight \( \phi \) on \( K \). Define the energy at the equilibrium weight of \((K, \phi)\) as

\[
\mathcal{E}_{eq}(K, \phi) := \mathcal{E}(P_K \phi).
\]

This functional is also well-defined up to an additive constant. We have the following property.

Lemma 3.2 ([2], Th. B). The map \( \phi \mapsto \mathcal{E}_{eq}(K, \phi) \), defined on the affine space of continuous weights on \( K \), is concave and Gâteaux differentiable, with directional derivatives given by integration against the equilibrium measure:

\[
\frac{d}{dt} \bigg|_{t=0} \mathcal{E}_{eq}(K, \phi + tv) = \langle v, \mu_{eq}(K, \phi) \rangle \quad \text{for every continuous function} \ v \ \text{on} \ K.
\]

Let \( \mu \) be a probability measure on \( X \) and \( \phi \) a continuous function on the support of \( \mu \). The semi-norm \( \| \cdot \|_{L^2(\mu, p\phi)} \) on \( H^0(X, L^p) \) is defined as in (2.1) and recall that we only consider measures \( \mu \) for which this semi-norm is a norm. Let \( B^2_p(\mu, \phi) \) denote the unit ball in \( H^0(X, L^p) \) with respect to this norm and \( N_p := \dim H^0(X, L^p) \). Consider the following \( L_p \)-functional

\[
L_p(\mu, \phi) := \frac{1}{2pN_p} \log \text{vol} B^2_p(\mu, \phi).
\]
Here, \( \text{vol} \) denotes the Lebesgue measure on the vector space \( H^0(X, L^p) \) which depends on the choice of an Euclidean norm on \( H^0(X, L^p) \). So the volume is only defined up to a multiplicative constant. Nevertheless, the differences such as \( \mathcal{L}_p(\mu_1, \phi_1) - \mathcal{L}_p(\mu_2, \phi_2) \) are well-defined and do not depend on the choice of \( \text{vol} \) for any probability measures \( \mu_1 \) and \( \mu_2 \), see [2] and also (3.2) below.

Consider the norm \( \| \cdot \|_{L^\infty(K, \mu, \phi)} \) on \( H^0(X, L^p) \) defined in (2.1). Let \( \mathcal{B}_p^\infty(K, \phi) \) denote the unit ball in \( H^0(X, L^p) \) with respect to this norm. Define

\[
\mathcal{L}_p(K, \phi) := \frac{1}{2pN_p} \log \text{vol} \mathcal{B}_p^\infty(K, \phi).
\]

Let \( \{s_1, \ldots, s_{N_p}\} \) be an orthonormal basis of \( H^0(X, L^p) \) with respect to the above \( L^2 \)-norm, see (2.1).

**Definition 3.3.** We call Bergman function of \( L^p \), associated with \((\mu, \phi)\), the function \( \rho_p(\mu, \phi) \) on the support of \( \mu \) given by

\[
\rho_p(\mu, \phi)(x) := \sup \left\{ \|s(x)\|_{p\phi}^2 : s \in H^0(X, L^p), \|s\|_{L^2(\mu, \phi)} = 1 \right\} = \sum_{j=1}^{N_p} |s_j(x)|_{p\phi}^2
\]

and we define the Bergman measure associated with \((\mu, \phi)\) by

\[
\mathcal{B}_p(\mu, \phi) := N_p^{-1} \rho_p(\mu, \phi) \mu.
\]

It is not difficult to obtain the identity in the definition of \( \rho_p(\mu, \phi) \) and to check that \( \mathcal{B}_p(\mu, \phi) \) is a probability measure. Note also that when \( \mu \) is the average of \( N_p \) Dirac masses at generic points, one can easily deduce from Definition 3.3 that \( \mathcal{B}_p(\mu, \phi) = \mu \), by considering sections vanishing on \( \text{supp}(\mu) \) except at a point. Such sections exist because \( N_p = \dim H^0(X, L^p) \). In fact, this property holds for all points \( x_1, \ldots, x_{N_p} \) such that the section \( \det S_p \) considered in Introduction does not vanish at \( (x_1, \ldots, x_{N_p}) \).

**Lemma 3.4.**

(a) The functional \( \phi \mapsto \mathcal{L}_p(\mu, \phi) \) is concave on the space of all continuous weights on the support of \( \mu \).

(b) The directional derivative of \( \mathcal{L}_p(\mu, \cdot) \) at a continuous weight \( \phi \) on the support of \( \mu \) is given by the integration against the Bergman measure \( \mathcal{B}_p(\mu, \phi) \), that is,

\[
\frac{d}{dt} \mathcal{L}_p(\mu, \phi + tv) \bigg|_{t=0} = \langle v, \mathcal{B}_p(\mu, \phi) \rangle, \quad \text{with } v, \phi \text{ continuous on the support of } \mu.
\]

(c) Let \( \mu \) be a probability measure with \( \text{supp}(\mu) \subset K \) such that the \( L^2 \)-semi-norm in (2.1) is a norm. Assume also that \((K, \phi)\) is a regular weighted compact set. Then \( \mathcal{L}_p(K, \phi) = \mathcal{L}_p(X, P_K \phi) \) and \( \mathcal{L}_p(K, \phi) \leq \mathcal{L}_p(\mu, \phi) \).

**Proof.** The concavity property of the functional \( \mathcal{L}_p \) in Part (a) has been established in [3, Proposition 2.4]. Part (b) has been established in [2, Lemma 5.1]. The property was stated there for smooth \( \phi \) but the proof also works for continuous functions, see also [4, Lemma 3.1] and [11, Lemma 2]. For Part (c), see [9, Proposition 2.5, Lemma 3.4]. \( \square \)

From now on, in order to simplify the notation, we use the following normalization

\[
E_{eq}(X, 0) = 0, \quad \mathcal{L}_p(X, 0) = 0 \quad \text{and} \quad \mathcal{L}_p(\mu^0, 0) = 0 \quad \text{for} \quad p \geq 1.
\]

Here, the function identically 0 is used as a smooth strictly \( \omega_0 \)-p.s.h. weight. Recall also that \( \mu^0 = \|\omega_0^n\|^{-1} \omega_0^n \) is the probability measure associated with the volume form \( \omega_0^n \).
The following result is an immediate consequence of [9, Proposition 3.10]. Recall that \( \mathcal{C}^{k,\alpha} = \mathcal{C}^{k+\alpha} \) for \( 0 < \alpha < 1 \) and \( \mathcal{C}^{k,1} \) is the space of \( \mathcal{C}^k \) functions whose partial derivatives of order \( k \) are Lipschitz.

**Proposition 3.5.** Let \( 0 < \alpha \leq 1 \) and \( A > 0 \) be constants. Let \( \phi \) be an \( \omega_0 \)-p.s.h. weight of class \( \mathcal{C}^{0,\alpha} \) on \( X \) such that \( \|\phi\|_{\mathcal{C}^{0,\alpha}} \leq A \). Then, there is a constant \( c_{A,\alpha} > 0 \) depending only on \( X, L, \omega_0, A \) and \( \alpha \) such that we have for all \( p > 1 \)

\[
|\mathcal{L}_p(\mu^0, \phi) - \mathcal{E}_{eq}(X, \phi)| \leq c_{A,\alpha}(\log p)^{3\beta_\alpha} p^{-\beta_\alpha}
\]

and

\[
|\left(\mathcal{L}_p(X, \phi) - \mathcal{E}_{eq}(X, \phi)\right)| \leq c_{A,\alpha}(\log p)^{3\beta_\alpha} p^{-\beta_\alpha},
\]

where \( \beta_\alpha : = \alpha/(6 + 3\alpha) \).

For the following proposition, we refer to the discussion after Theorem [3.1] for the notation.

**Proposition 3.6.** Let \( K \) be a compact subset of \( X \). Let \( 0 < \alpha \leq 2 \) and \( 0 < \alpha' \leq 1 \) be constants. Assume that \( (K, \phi) \) is a weighted compact set with \( \phi \in \mathcal{C}^{\alpha}(K) \) such that \( K \) is \( (\mathcal{C}^{\alpha}, \mathcal{C}^{\alpha'}) \)-regular. Then there is a constant \( c > 0 \) with the following property. For \( p \geq 1 \) and \( x = (x_1, \ldots, x_N) \in K^N \), denote by \( \mu^x \) the empirical measure associated with \( x \) and let \( S_p \) be a basis of \( H^0(X, L^p) \). Define

\[
\sigma_x := \frac{1}{pN_p} \log \| \det S_p \|_{L^\infty(K, \mu^x)} - \frac{1}{pN_p} \log \| \det S_p(x) \|_{p\phi}.
\]

We have for all \( p > 1 \)

\[
|\mathcal{L}_p(\mu^x, \phi) - \mathcal{E}_{eq}(K, \phi)| \leq c\left(p^{-1} \log p + \sigma_x + |\mathcal{L}_p(\mu^0, P_K \phi) - \mathcal{E}_{eq}(K, \phi)|\right).
\]

**Proof.** Observe that \( \sigma_x \) does not depend on the choice of \( S_p \). So choose \( S_p \) which is an orthonormal basis of \( H^0(X, L^p) \) with respect to the \( L^2 \)-norm without weight. Let \( \mu_p \) be the empirical measure associated with a Fekete configuration of order \( p \). Using identity [3] (2.4)], we get

\[
\frac{1}{2pN_p} \log \frac{\text{vol } B_2^2(\mu^0, 0)}{\text{vol } B_2^2(\mu_p, \phi)} = \frac{1}{pN_p} \log \| \det S_p \|_{L^\infty(K, \mu^0)} - \frac{1}{2p} \log N_p
\]

and

\[
\frac{1}{2pN_p} \log \frac{\text{vol } B_2^2(\mu^0, 0)}{\text{vol } B_2^2(\mu^x, \phi)} = \frac{1}{pN_p} \log \| \det S_p(x) \|_{p\phi} - \frac{1}{2p} \log N_p.
\]

Subtracting the last line from the previous one and using (3.1), we obtain

\[
\sigma_x = \mathcal{L}_p(\mu^x, \phi) - \mathcal{L}_p(\mu_p, \phi).
\]

On the other hand, with the normalization (3.2), [9, Proposition 3.12] tells us that there is a constant \( c > 0 \) satisfying

\[
|\mathcal{L}_p(\mu_p, \phi) - \mathcal{E}_{eq}(K, \phi)| \leq c\left(p^{-1} \log p + |\mathcal{L}_p(\mu^0, P_K \phi) - \mathcal{E}_{eq}(K, \phi)|\right) \quad \text{for } p > 1.
\]

This, combined with the previous identity, implies the proposition. \( \square \)

The following two lemmas were obtained in [9, Lemmas 3.13 and 3.14].
Lemma 3.7. There is a constant $c > 0$ such that for every continuous weight $\phi$ on $K$ and every function $v$ of class $\mathcal{C}^{1,1}$ on $X$, we have
\[
|\langle \mu_{eq}(K, \phi + tv) - \mu_{eq}(K, \phi) , v \rangle| \leq c|t||v||_{L^{\infty}(K)}||dFv||_{\infty} \quad \text{for} \quad t \in \mathbb{R}.
\]

Lemma 3.8. Let $\epsilon > 0$ and $M > 0$ be constants. Let $F$ and $G$ be functions defined on $[-\epsilon^{1/2}, \epsilon^{1/2}]$ such that
(i) $F(t) \geq G(t) - \epsilon$ and $|F(0) - G(0)| \leq \epsilon$;
(ii) $F$ is concave on $[-\epsilon^{1/2}, \epsilon^{1/2}]$ and differentiable at 0;
(iii) $G$ is differentiable in $[-\epsilon^{1/2}, \epsilon^{1/2}]$, and its derivative $G'$ satisfies $|G'(t) - G'(0)| \leq M\epsilon^{1/2}$ for $t \in [-\epsilon^{1/2}, \epsilon^{1/2}]$. The last inequality holds when $|G'(t) - G'(0)| \leq M|t|$.
Then we have
\[
|F'(0) - G'(0)| \leq (2 + M)\epsilon^{1/2}.
\]

End of the proof of Theorem 3.1 By (1.3), we only need to consider the case $\gamma = 2$, i.e., to prove that
\[
|\langle \mu^{X} - \mu_{eq}(K, \phi) , v \rangle| \lesssim p^{-2\alpha''}(\log p)^{6\alpha''} + \sigma_{x}^{1/2}
\]
for every test $\mathcal{C}^{2}$ function $v$ such that $\|v\|_{\mathcal{C}^{2}} \leq 1$. Recall that $\alpha'' := \alpha/(24 + 12\alpha')$.

Define
\[
F(t) := \mathcal{L}_{p}(\mu^{X}, \phi + tv) \quad \text{and} \quad G(t) := \mathcal{E}_{eq}(K, \phi + tv) = \mathcal{E}_{eq}(X, P_{K}(\phi + tv))
\]
for $t$ in a neighborhood of $0 \in \mathbb{R}$. By Lemma 3.4(c),
\[
\mathcal{L}_{p}(\mu^{X}, \phi + tv) \geq \mathcal{L}_{p}(K, \phi + tv) = \mathcal{L}_{p}(X, P_{K}(\phi + tv)).
\]
As $0 < \alpha \leq 2$, we infer $\phi + tv \in \mathcal{C}^{\alpha}(K)$. Since $K$ is $(\mathcal{C}^{\alpha}, \mathcal{C}^{\alpha'})$-regular, we deduce that $P_{K}(\phi + tv)$ is a $\omega_{0}$-p.s.h. weight on $X$ with bounded $\mathcal{C}^{\alpha'}$-norm. This, coupled with the second inequality in Proposition 3.5, applied to $P_{K}(\phi + tv)$ and $\alpha'$ instead of $\alpha$, shows that
\[
|F(t) - G(t)| \gtrsim -p^{-4\alpha''}(\log p)^{12\alpha''}.
\]
An application of the first inequality in Proposition 3.5 for $\alpha'$ instead of $\alpha$ gives
\[
|\mathcal{L}_{p}(\mu^{0}, P_{K}\phi) - \mathcal{E}_{eq}(K, \phi)| \lesssim p^{-4\alpha''}(\log p)^{12\alpha''}.
\]
Consequently, applying Proposition 3.6 yields
\[
|F(0) - G(0)| \lesssim p^{-4\alpha''}(\log p)^{12\alpha''} + \sigma_{x}.
\]
Recall from Lemma 3.4(a) that $F$ is concave. Moreover, by Lemma 3.4(b), we have
\[
F'(0) = \langle v, \mathcal{B}_{p}(\mu^{X}, \phi) \rangle.
\]
On the other hand, by Lemma 3.2 $G$ is differentiable with
\[
G'(t) = \langle v, \mu_{eq}(K, \phi + tv) \rangle.
\]
Finally, by Lemma 3.7 condition (iii) in Lemma 3.8 is satisfied for a suitable constant $M > 0$. Combining this and the discussion between (3.3)-(3.4), we are in the position to apply Lemma 3.8 to a constant $\epsilon$ of order $p^{-4\alpha''}(\log p)^{12\alpha''} + \sigma_{x}$. Using the above expression for $F'(0)$ and $G'(0)$, we get
\[
|\langle \mathcal{B}_{p}(\mu^{X}, \phi) , v \rangle - \langle \mu_{eq}(K, \phi) , v \rangle| \lesssim p^{-2\alpha''}(\log p)^{6\alpha''} + \sigma_{x}^{1/2}.
\]
Recall from the discussion before Lemma 3.4 that $\mathcal{B}_p(\mu^x, \phi) = \mu^x$. Hence, the desired estimate follows immediately.

We continue the proof of the main theorem. We need the following result which is a consequence of [2, Lemma 5.3].

**Lemma 3.9.** Consider a probability measure $\mu$ supported on a compact set $K \subset X$ such that the $L^2$-semi-norm in (2.1) is a norm. If $S_p$ is an orthonormal basis of $H^0(X, L^p)$ with respect to this norm, then the positive measure $\| \det S_p \|_{p^0}^2 \preceq N_p^{-1}$ is of mass $N_p!$.

**End of the proof of Theorem 1.2.** Fix a constant $0 < \delta < 1$ and an orthonormal basis $S_p$ of $H^0(X, L^p)$ with respect to the $L^2$-norm induced by $\mu$ and $\phi$. We first show that there is a constant $c > 0$ such that for $p \geq 1$,

(3.5) \[ 0 \leq \log \| \det S_p \|_{L^\infty(K, p\phi)} - \log \| \det S_p \|_{L^2(\mu, p\phi)} \leq c N_p p^{-\delta}. \]

Here, similar to the discussion after Theorem 3.1, the norm $\| \det S_p \|_{L^2(\mu, p\phi)}$ is defined using the product probability measure $\mu^{\otimes N_p}$ on $K^{N_p} \subset X^{N_p}$ together with the metric and weight for $(L^p)^{\otimes N_p}$, naturally induced by $h_0$ and $\phi$.

Since $\mu$ is a probability measure, we have

\[ \| \det S_p \|_{L^\infty(K, p\phi)} \geq \| \det S_p \|_{L^2(\mu, p\phi)}. \]

Now, to complete the proof of (3.5), we only need to show that

(3.6) \[ \log \| \det S_p \|_{L^\infty(K, p\phi)} \leq \log \| \det S_p \|_{L^2(\mu, p\phi)} + O(N_p p^{-\delta}). \]

By (2.2), we get

\[ |s(x)|_{p\phi}^2 \leq A^2 e^{-2A p^{-\delta}} \| s \|_{L^2(\mu, p\phi)}^2 \]

for every section $s \in H^0(X, L^p)$, $p \geq 1$, and $x \in X$. If $x_1, \ldots, x_{N_p}$ are points in $X$, then for each $j$

\[ x \mapsto \det S_p(x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_{N_p}) \]

is a holomorphic section in $H^0(X, L^p)$. A successive application of the last inequality for $j = 1, 2, \ldots, N_p$ and Fubini’s theorem yield

\[ \| \det S_p \|_{L^\infty(K, p\phi)}^2 \leq A^{2N_p} e^{2A N_p p^{-\delta}} \| \det S_p \|_{L^2(\mu, p\phi)}^2. \]

Estimates (3.6) and (3.5) follow.

Recall that $N_p = O(p^n)$ and by Stirling’s formula $N_p! \approx (N_p/e)^{N_p} \sqrt{2\pi N_p}$. Therefore, Lemma 3.9 implies

\[ \frac{1}{p N_p} \log \| \det S_p \|_{L^2(\mu, p\phi)} = \frac{1}{p N_p} \log \sqrt{N_p} = O(p^{-1} \log p). \]

It follows from (3.5) that

(3.7) \[ 0 \leq \frac{1}{p N_p} \log \| \det S_p \|_{L^\infty(K, p\phi)} \leq c_1 p^{-\delta} \quad \text{with some constant} \quad c_1 > 0. \]

Let $\lambda_0 > 0$ be a constant whose value will be determined later. For every $p \geq 1$, consider the set

\[ E_p := \left\{ x \in K^{N_p} : \frac{1}{p N_p} \log \| \det S_p(x) \|_{p\phi} \leq -\lambda_0 p^{-\delta} \right\}. \]
So for $x \in K^{N_p} \setminus E_p$, using (3.7), we obtain $\sigma_x \leq (e_1 + \lambda_0)p^{-\delta}$, where as above

$$\sigma_x := \frac{1}{pN_p} \log \| \det S_p \|_{L^\infty(K, \nu_0)} - \frac{1}{pN_p} \log \| \det S_p(x) \|_{\nu_0}.$$ 

Hence, applying Theorem 3.1 yields

$$\text{dist}_1(\mu^x, \mu_{\text{eq}}(K, \phi)) \leq cp^{-\alpha''(\log p)\beta\alpha''+cp^{-\gamma/4}},$$

for some constant $c > 0$.

To complete the proof of the theorem, it remains to bound the size of $E_p$. Fix a constant $\lambda$ as in Theorem 1.2. Consider two different cases according to the value of $\beta$.

**Case 1.** Assume that $\beta \geq 2$. Choose $\lambda_0 = \lambda/\beta$. We first bound the mass of $\| \det S_p \|_{p_0, \mu^\otimes N_p}$ from below. Recall that $\mu^\otimes N_p$ is a probability measure. Applying Hölder’s inequality and using Lemma 3.9, we obtain

$$\int \| \det S_p \|_{p_0, \mu^\otimes N_p}^{\beta} \geq \left( \int \| \det S_p \|_{p_0, \mu^\otimes N_p}^{2} \right)^{\beta/2} = (N_p!)^{\beta/2}.$$ 

Consequently, $\nu_p^{\beta} \leq \| \det S_p \|_{p_0, \mu^\otimes N_p}^{\beta}$. Hence, by definition of $E_p$, we get

$$\nu_p^{\beta}(E_p) \leq \int_{E_p} \| \det S_p(x) \|_{p_0, \mu^\otimes N_p}^{\beta} \leq \int_{E_p} e^{-\beta\lambda p^{1-\delta} N_p} d\mu \leq e^{-\beta \lambda p^{1-\delta} N_p}.$$ 

This completes the proof for the case $\beta \geq 2$.

**Case 2.** Assume that $0 < \beta \leq 2$. Combining (3.7) and Lemma 3.9, we get

$$\int_{K} \| \det (S) \|_{p_0, \mu^\otimes N_p}^{\beta} \geq e^{-(2-\beta)c_1 p^{1-\delta} N_p} \int_{K} \| \det (S) \|_{p_0, \mu^\otimes N_p}^{2} \geq e^{-(2-\beta)c_1 p^{1-\delta} N_p}.$$ 

Consequently,

$$\nu_p^{\beta} \leq e^{-(2-\beta)c_1 p^{1-\delta} N_p} \| \det S_p \|_{p_0, \mu^\otimes N_p}^{\beta}.$$ 

Hence, we infer

$$\nu_p^{\beta}(E_p) \leq e^{(2-\beta)c_1 p^{1-\delta} N_p} \int_{E_p} \| \det S_p(x) \|_{p_0, \mu^\otimes N_p}^{\beta} \leq e^{(2-\beta)c_1 p^{1-\delta} N_p} \int_{E_p} e^{-\beta \lambda_0 p^{1-\delta} N_p} d\mu \leq e^{p^{1-\delta} N_p \lambda_0}.$$ 

Choose $\lambda_0 \gg c_1$ and the result follows. This ends the proof of our main theorem. \qed

As mentioned above, Theorem 3.1 can be applied to other situations. We present now one more application. Consider the same setting as in Theorem 3.1 and a probability measure $\mu$ on $K$. Recall the following notion, see [3].

**Definition 3.10.** Let $0 < r \leq \infty$ and $0 < r' \leq \infty$. We say that $y \in K^{N_p}$ is an $(r, r')$-optimal configuration of order $p$ if the following function in $x \in K^{N_p}$

$$\tau_x := \sup_{s \in H^0(X, L^p) \setminus \{0\}} \| s \|_{L^r(\mu, \nu_0)}$$

attains its minimum at $y$.

We have the following elementary property, see also [3, Proposition 2.10].
Lemma 3.11. If \( y \in K^{N_p} \) is \((r, r')\)-optimal, then \( \tau_y \leq N^{1+1/r'}_p \).

Proof. Let \( x = (x_1, \ldots, x_{N_p}) \) be a Fekete configuration of order \( p \). We only need to check that \( \tau_x \leq N^{1+1/r'}_p \). Choose a basis \( S_p = (s_1, \ldots, s_{N_p}) \) of \( H^0(X, L^p) \) such that \( s_i(x_j) = 0 \) when \( i \neq j \) and \( \|s_i(x_i)\|_{\mathcal{P}^0} = 1 \). Since \( x \) is a Fekete configuration, we have \( \| \det S_p(\cdot) \|_{\mathcal{P}^0} \leq 1 \) on \( K^{N_p} \). This inequality on \( K_i := \{ x_1 \} \times \cdots \times \{ x_{i-1} \} \times K \times \{ x_{i+1} \} \times \cdots \times \{ x_{N_p} \} \) implies that \( \|s_i(\cdot)\|_{\mathcal{P}^0} \leq 1 \) on \( K \). Finally, if \( s \) is a section in \( H^0(X, L^p) \setminus \{0\} \), write \( s = \lambda_1 s_1 + \cdots + \lambda_{N_p} s_{N_p} \) and we have

\[
\frac{\|s\|_{L^r(\mu, \mathcal{P}^0)}}{\|s\|_{L^r(\mu^{(s)}, \mathcal{P}^0)}} \leq \frac{\sum |\lambda_i|}{(N_p - 1) \max |\lambda_i|^{1/r'}} \leq \frac{N_p \max |\lambda_i|}{(N_p - 1) |\lambda_i|^{1/r'}} = N^{1+1/r'}_p.
\]

The lemma follows. \( \square \)

We deduce from Theorem 3.1 the following result, where the simple convergence of \( \mu^\gamma \) when \( p \to \infty \) was established in [3].

Corollary 3.12. In the setting of Theorem 1.2, consider two numbers \( 0 < r, r' \leq \infty \). There is a constant \( c > 0 \) such that if \( y \) is an \((r, r')\)-optimal configuration of order \( p \) for some \( p > 1 \), then

\[
\operatorname{dist}_x(\mu^\gamma, \mu_{\alpha^\gamma}(K, \phi)) \leq c q^\gamma.
\]

Proof. We only have to check that

\[
\sigma_x \leq c(p^{-1} \log \tau_x + p^{-\delta}) \quad \text{for} \quad x \in K^{N_p}
\]

for some constant \( c > 0 \). Then, Theorem 3.1, Lemma 3.11 and the estimate \( N_p = O(p^n) \) imply the result.

We can assume that \( \det S_p(x) \neq 0 \) because the case \( \det S_p(x) = 0 \) is trivial. So we can choose \( S_p = (s_1, \ldots, s_{N_p}) \) as in the proof of Lemma 3.11 but here \( x \) is no more a Fekete configuration. By definition of \( \tau_x \), we have

\[
\|s_i\|_{L^r(\mu, \mathcal{P}^0)} \leq \tau_x \|s_i\|_{L^r(\mu^{(s)}, \mathcal{P}^0)} = N_p^{1/r'} \tau_x \leq \tau_x.
\]

Hence, it follows from Lemma 2.5 that

\[
\|s_i\|_{L^\infty(K, \mathcal{P}^0)} \leq A e^{A \tau_x^{1-\delta}} \tau_x.
\]

Therefore, we get

\[
\|\det S_p(\cdot)\|_{\mathcal{P}^0} \leq N_p! \left( A e^{A \tau_x^{1-\delta}} \tau_x \right)^{N_p} \quad \text{on} \quad K^{N_p}.
\]

We then deduce the desired estimate using the definition of \( \sigma_x \) and that \( \|\det S_p(x)\|_{\mathcal{P}^0} = 1 \) by the choice of \( S_p \). \( \square \)

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