A STABILIZED NONCONFORMING NITSCHE’S EXTENDED FINITE ELEMENT METHOD FOR STOKES INTERFACE PROBLEMS

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Abstract. In this paper, a stabilized extended finite element method is proposed for Stokes interface problems on unfitted triangulation elements which do not require the interface align with the triangulation. The problem is written on mixed form using nonconforming $P_1$ velocity and elementwise $P_0$ pressure. Extra stabilization terms involving velocity and pressure are added in the discrete bilinear form. An inf-sup stability result is derived, which is uniform with respect to mesh size $h$, the viscosity and the position of the interface. An optimal priori error estimates are obtained. Moreover, the errors in energy norm for velocity and in $L^2$ norm for pressure are uniform to the viscosity and the location of the interface. Results of numerical experiments are presented to support the theoretical analysis.

1. Introduction. A variety of phenomena with discontinuities exist in the real world. For example, because of the different physical parameters, the velocity has kinks and the pressure is discontinuous for the multiphase flow. Therefore, simulating such phenomena must treat the discontinuities carefully. Standard finite element methods can perform well when the interface coincides with mesh lines, known as the interface-fitted mesh. Optimal convergence orders can be obtained for interface-fitted mesh methods where every element is contained in one sub-region (see [4, 12]).
However, it is expensive to generate a good interface-fitted mesh for the complicated interface and especially for the time-dependent interface problems. Therefore, varieties of unfitted grid numerical methods have been proposed over the past decades, as they cannot deal with the position of the interface, which are very attractive due to their simplicity. That is to say, those methods do allow that the interface is not aligned with the mesh. Some special techniques incorporating the jump conditions across the interface with the unfitted grid methods are needed. One way is to employ the immersed finite element methods based on cartesian mesh where the standard finite element basis functions are locally modified for elements cut by the interface to satisfy the jump conditions across the interface exactly or approximately. We can see [17, 23, 26, 27, 28] for elliptic interface problems, [29] for parabolic interface problems and [1] for Stokes interface problems.

The other way is to use the extended finite element methods (XFEMs) based on unfitted-interface mesh, which are mainly designed to solve the problems with discontinuities, kinks and singularities within elements. For XFEMs, extra basis functions are added for elements intersected by the interface so that the discontinuities can be captured, and the jump conditions are enforced by a variance of Nitsche’s approach. This Nitsche’s XFE method (NXFEM) was originally considered in [19] to solve the elliptic interface problems. Then a large number of related methods have been developed, such as [2, 6, 9, 10, 22, 24, 32, 34, 37, 38] for elliptic interface problems, [3, 11, 20, 25, 35, 36] for Stokes interface problems and [31] for Oseen problems.

From now on, we will focus on the NXFEM schemes to solve the Stokes interface problems. In this paper we consider the following two-phase Stokes problem of two fluids with different kinematic viscosities on a bounded polygonal domain $\Omega \subset \mathbb{R}^2$. The whole domain is crossed by an interface $\Gamma$ which is assumed to have at least $C^2$-smooth and is divided into two open sets $\Omega_1$ and $\Omega_2$ (see Figure 1 for an illustration). Denote by $[v] = v|_{\Omega_1} - v|_{\Omega_2}$ the jump across the interface $\Gamma$. Then we study the problem as follows: Find a velocity $u$ and a pressure $p$ such that

$$
\begin{cases}
- \nabla \cdot (\mu \nabla u) + \nabla p = f, & \text{in } \Omega_1 \cup \Omega_2, \\
\nabla \cdot u = 0, & \text{in } \Omega_1 \cup \Omega_2, \\
[u] = 0, & \text{on } \Gamma, \\
[\mu \nabla u \cdot n - pn] = \sigma \kappa n, & \text{on } \Gamma, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
$$

(1)

where $f \in [L^2(\Omega)]^2$ and $\mu$ is a piecewise constant viscosity, namely $\mu|_{\Omega_i} = \mu_i > 0$. $\sigma$ is the surface tension coefficient, $\kappa$ is the curvature of the interface, and $n$ is the unit normal vector on $\Gamma$ pointing from $\Omega_1$ to $\Omega_2$.

![Figure 1. A sample domain $\Omega$.](image-url)
Considering the type of Stokes equations is significantly motivated by two (or multi) phase incompressible flows, which are usually modeled by Navier-Stokes equations with discontinuous density and viscosity coefficients. If two (or multi) phase incompressible flows with highly viscosities, the stokes equations with discontinuous viscosities are a reasonable model. Therefore, stokes interface problems have been embraced by the numerical analysis community. We note that the equations of incompressible elasticity written on mixed form with the pressure as an auxiliary variable is form identical to the stokes equations of creeping viscous flow.

It is well known that mixed finite elements are a typical choice to approximate a saddle point problem without interface. Therefore, the natural idea is that same finite element spaces would be adequate to solve Stokes interface problem by the NXFEM formulation. In [22], we have studied a nonconforming NXFEM to solve elliptic interface problems. Thus, we want to apply it to solve Stokes interface problems. However, since the computational mesh of the XFEMs does not fit the interface, the approximation of the pressure may be unstable near the interface even though for the inf-sup stable finite elements (see [11]). That is to say, XFEMs break the stability condition for mixed problems. Therefore, extra pressure stabilization approaches in the elements cut by the interface are chosen to ensure the inf-sup condition. Before introducing our method, we investigate the stabilization techniques used in the literatures to solve the stokes interface problems. A well-known unfitted finite element method based on Nitsche’s method for incompressible elasticity problems was firstly proposed in [3], where mixed form conforming-$P_1/P_0$ was used, numerical fluxes were adopted from [19] and a stabilization term by penalizing the pressure jumps was added. Later, the NXFEM with $P_{\text{bubble}}^1/P_1$ couple functions was proposed in [11]. Instead of stabilization techniques based on the interior penalty technique, the symmetric pressure stabilization operator based on Brezzi-Pitkäranta stability technique on the cut region was used to ensure the stability. They also considered the case of unstable $P_1/P_1$ couple and employed the Brezzi-Pitkäranta stabilization on the entire domain. Then, a NXFEM based on $P_1$-iso-$P_2/P_1$ elements to solve Stokes interface problems was proposed [20]. In the method, extra stabilization terms for normal-gradient jumps over some element faces with respect to both pressure and velocity were added. In [25], an XFEM with the $P_2/P_1$ pair as underlying spaces was studied and the same stabilization technique as in [20] was used. Recently, a Nitsche formulation for Stokes interface problems based on $P_1/P_1$ elements was developed in [36], where on a patch of elements intersected by the interface, extra penalty terms that contained the difference between the solution and an $L^2$ projection of the solution for velocity and pressure were added to ensure the stability. This extra penalty terms are called ghost penalty which was proposed in [5]. We remark that the stability techniques are also used to solve Stokes problems with the interface in the context of fictitious domain method, where the extra penalty terms for the jumps in the normal velocity and pressure gradients near the interface were adopted to stabilize pressure in [30].

Very recently, the nonconforming-$P_1/P_0$ NXFEM for a steady state Stokes interface problem was considered in [35]. The arithmetic averages were used and some stabilization terms were defined on interface edges and cut edges. It is proved that the energy error is independent of the viscosity coefficients and the position of the interface with respect to the mesh. We remark that the extended nonconforming $P_1/P_0$ for Stokes interface problems with the unfitted mesh was also considered in PhD thesis [13], where the weights dependent on the viscosity parameters and the
area of local sub-region cut by the interface were used across the interface, and the weights dependent on the area of two local elements were used on the local cut segments. The stabilization terms based on a projection operator for the velocity was added on the local cut segments. The stabilization terms based on a projection operator for the velocity was added on the local cut segments. The optimal energy error is robust with respect to the parameter and the position of the interface with respect to the mesh. And the error estimates in $L^2$-norm for velocity and pressure have been analyzed.

In this paper, we will use the nonconforming NXFEM of [22] and propose an accurate and stable extended finite element method for Stokes interface problems based on nonconforming-$P_1/P_0$ shape functions with the unfitted-interface mesh.

Although the same spaces considered in this paper (compared to [13, 35]), we mention the following contributions of this paper. Instead of the weights involving the viscosity parameters and subareas in [13] and the arithmetic averages in [35], harmonic weight fluxes only involving the viscosity parameters are used on the interface. The arithmetic averages same as that used in [35] are adopted on cut edges (the local segment of edges cut by the interface), which are different from the weights dependent on the subareas in [13]. Comparing with [35], different stabilization terms involving the jumps in the normal pressure on the edges and velocity gradients in the vicinity of the interface are added in our method. Moreover, our finite element space to approximate the pressure is different from [35]. Optimal error estimates in energy norm for velocity and in $L^2$-norm for pressure are obtained. We shows that the errors do not depend on the jump of different viscosities and the position of the interface with respect to the mesh. Finally, a series of numerical examples are discussed to illustrate our theoretical analysis.

The rest of this paper is organized as follows. In Section 2, we describe the Nitsche’s extended finite element method formulation. In Section 3, we list some preliminary lemmas. The stable inf-sup condition and error analysis are given in Section 4. Numerical tests are presented in Section 5. Finally, we make a conclusion in Section 6.

Throughout the paper, C and $C$ with a subscript are generic positive constants which are independent of $h$, the penalty parameters, the jump of the coefficient $\mu$, and location of the interface relative to the mesh. We also use the shorthand notation $A \lesssim B$ and $B \gtrsim A$ for the inequality $A \leq CB$ and $B \geq CA$. $A \approx B$ is for the statement $A \approx B$ and $B \approx A$. Moreover, denote by $H^s(\Omega_1 \cup \Omega_2) := \{v : v|_{\Omega_i} \in H^s(\Omega_i), i = 1, 2\}$ the piecewise $H^s$ space on $\Omega_1 \cup \Omega_2$ and by $|v|_{s,\Omega_1\cup\Omega_2}$ and $\|v\|_{s,\Omega_1\cup\Omega_2}$ its norm and semi-norm.

2. Finite element formulation. Let $\{T_h\}$ be a family of conforming, quasi-uniform, and regular triangulations of the domain $\Omega$ independent of the location of the interface $\Gamma$. Moreover, the mesh should be fine enough to ensure that the interface is well resolved. To do this, we need to make some assumptions concerning the intersection between $\Gamma$ and the mesh (see assumptions (A1)–(A3) below). For any $K \in T_h$, define $h_K$ as diam($K$) and $h := \max_{K \in T_h} h_K$. Then $h \approx h_K$. Note that any element $K \in T_h$ is considered as closed. Let us introduce the set of cut elements $G^+_h := \{K \in T_h : K \cap \Gamma \neq \emptyset\}$ and denote $\Gamma_K = K \cap \Gamma$ for $K \in G^+_h$. Denote $T_{h,i} := \{K \in T_h : K \cap \Omega_i \neq \emptyset\}$. Then we define the elements extended and restricted sub-domains $\Omega^{+}_{h,i}$ and $\Omega^{-}_{h,i}$ respectively, as follows:

$$
\Omega^{+}_{h,i} := \bigcup_{K \in T_{h,i}} K, \quad \Omega^{-}_{h,i} := \bigcup_{K \in T_h \setminus G^+_h} K.
$$
See Figure 2 for an illustration of these definitions.

\[ \Omega^{+}_{h,1} \cap \Omega^{+}_{h,2} \]

Figure 2. Illustration of definitions of set \( G_{h}^{T} \), \( \Omega^{+}_{h,1} \), \( \Omega^{+}_{h,2} \) and \( \Omega^{+}_{h,2} \). Left figure: elements in \( G_{h}^{T} \) (magenta area), \( \Omega^{+}_{h,1} \) and \( \Omega^{+}_{h,2} \) (cobalt blue area). Center figure: elements in \( \Omega^{+}_{h,1} \) (magenta area). Right figure: elements in \( \Omega^{+}_{h,2} \) (magenta area).

Let \( F_{h,i}^{nc}, F_{h,i}^{cut} \) and \( F_{h,i}^{\Gamma} \) denote the set of all the edges of \( T_{h,i} \), the set of uncut edges of \( T_{h,i} \) and the set of cut segments contained in \( \Omega_{i} \) respectively. Here \( F_{h,i}^{nc} \) and \( F_{h,i}^{cut} \) are given by

\[ F_{h,i}^{nc} := \{ e \in F_{h,i} : e = \partial K_{l} \cap \partial K_{r}, K_{l}, K_{r} \in T_{h,i}, e \subset \Omega_{i} \}, \]

and

\[ F_{h,i}^{cut} := \{ e \in \Omega_{i} : e = \partial K_{l} \cap \partial K_{r}, K_{l}, K_{r} \in G_{h}^{T} \}. \]

Finally, the set of all the edges of \( G_{h}^{T} \) restricted to the interior of \( \Omega^{+}_{h,i} \) is defined by \( F_{h,i}^{\Gamma} := \{ e \in \Omega_{i} : e = \partial K_{l} \cap \partial K_{r}, K_{l}, K_{r} \in G_{h}^{T} \} \). See Figure 3 for an illustration of definitions of \( F_{h,1}^{nc}, F_{h,1}^{cut} \) and \( F_{h,1}^{\Gamma} \) respectively.

Figure 3. Illustration of definitions of set \( F_{h,1}^{nc}, F_{h,1}^{cut} \) and \( F_{h,1}^{\Gamma} \).

Left figure: edges in \( F_{h,1}^{nc} \) (red lines). Center figure: edges in \( F_{h,1}^{cut} \) (red lines). Right figure: edges in \( F_{h,1}^{\Gamma} \) (red lines).

In this paper, we make the following assumptions (see [20]):

(A1) It is assumed that the interface intersects the boundary of each triangle at most two points and each (open) edge at most once, or that the interface coincides with one edge of the element.

(A2) We assume that for each \( K \in G_{h}^{T} \) there exists one \( K' \subset \Omega_{i}, i = 1, 2 \) such that \( K' \) shares an edge or a vertex with \( K \). That is to say, if \( z \in \Omega_{i} \) is a vertex of \( K \) and \( \triangle z \) denotes the patch of elements sharing \( z \), i.e. \( \triangle z = \bigcup\{ K : K \in T_{h}, z \in \partial K \} \), then there exists an element \( K' \subset \Omega_{i} \) such that \( K' \in \triangle z \).
Further, we define the weak velocity space by

\[ V_i := \{ v \in L^2(\Omega_{h,i}^+) : v \in H^2(K), \forall K \in T_{h,i} \}^2, i = 1, 2, \]

and the pressure space is

\[ Q_i := \{ p \in L^2(\Omega_{p,i}^+) : p \in H^1(K), \forall K \in T_{h,i} \}, i = 1, 2. \]

Moreover, the two assumptions imply that the straight line segment, which connects the points of intersection between interface and the boundary of each triangle, divides elements into simple shapes (two triangles or a triangle and a quadrilateral).

Now we define that the velocity space is

\[ V_i := \{ v \in L^2(\Omega_{h,i}^+) : v \in H^2(K), \forall K \in T_{h,i} \}, i = 1, 2, \]

and the weak pressure space by

\[ V := \{ v = (v_1|_{\Omega_1}, v_2|_{\Omega_2}) : v_i \in V_i, i = 1, 2, v|_{\partial\Omega} = 0 \}, \]

and the weak pressure space by

\[ Q := \{ p = (p_1|_{\Omega_1}, p_2|_{\Omega_2}) : p_i \in Q_i, p \in L^2(\Omega) \}, \]

where \( L^2(\Omega) := \{ q \in L^2(\Omega) : (\mu^{-1}q, 1)_{\Omega_1 \cup \Omega_2} = 0 \} \). We now introduce the couple of inf-sup stable spaces on the extended sub-domain \( \Omega_{h,i}^+ \),

\[ V_{h,i} := \{ v \in L^2(\Omega_{h,i}^+) : v|_K \in S_h(K), K \in T_{h,i} ; \]

\[ if e = \partial K_1 \cap \partial K_r, K_1, K_r \in T_{h,i}, then \int_e [v]ds = 0; \]

\[ if e = \partial K \cap \partial \Omega, K \in T_{h,i}, then \int_e [v]ds = 0 \}^2, i = 1, 2, \]

with \( S_h(K) = \text{span}\{ \phi_l : \phi_l \in P_1(K), \frac{1}{|\partial e|} \int_{\partial e} \phi_l dm = \delta_{le}, e \in \partial K, l, m = 1, 2, 3 \} \), and

\[ Q_{h,i} := \{ p \in L^2(\Omega_{h,i}^+) : p|_K \in P_0(K), \forall K \in T_{h,i} \}. \]

Then we define a couple of finite element spaces. Let \( V_h \) be the extended velocity space of nonconforming piecewise linear polynomials defined on \( T_h \) as follows:

\[ V_h := \{ v = (v_1|_{\Omega_1}, v_2|_{\Omega_2}) : v_i \in V_{h,i}, i = 1, 2 \}, \]

and \( Q_h \) be the extended pressure space of piecewise constant functions defined on \( T_h \) as follows:

\[ Q_h := \{ p = (p_1|_{\Omega_1}, p_2|_{\Omega_2}) : p_i \in Q_{h,i}, i = 1, 2, p \in L^2(\Omega) \}. \]

The above extended finite element spaces double the degrees of freedom in the elements which are cut by the interface. Clearly, \( V_h \not\subset V \) and \( Q_h \not\subset Q \).

Recalling the definition of \( F_{h,i}^{\text{cut}} \), for each edge \( e \in \mathcal{F}_{h,i}^{\text{cut}} \), there exist two cut elements \( K_i, K_r \in G_h^i \) and \( K_j = K_j \cap \Omega, j = l, r \) such that \( e = K_l \cap K_r \). Define jumps of \( v \in V + V_h \) and \( p \in Q \), and jump of the flux of \( v \) by \( [v] = v|_{K} - v|_{K'}, [p] = p|_{K'} - p|_{K} \) and \( [\nabla v \cdot n_e] = \nabla v|_{K'} \cdot n_e - \nabla v|_{K} \cdot n_e \), respectively, provided that \( n_e \) is a unit normal vector to the edge \( e \) pointing from \( K_l \) to \( K_r \). Similarly, for \( e \in \mathcal{F}_{h,i}^{\text{p}} \), we can also define the jumps of \( v \in V + V_h \) and \( p \in Q \) on \( e \) and a unit normal vector to the edge \( e \) by \( n_e \). In particular, we note that \([v] = v|_K\) for \( e = \partial K \cap \partial \Omega \) with \( K \in T_{h,i} \). Further, we define jump \( [\nabla v] = \nabla v|_{K} - \nabla v|_{K'} \) for \( v \in V + V_h \) on each edge \( e \in \mathcal{F}_{h,i}^{\text{p}} \).
For any function $\chi$ discontinuous across $\Gamma$, we define the averages $\{\chi\}_w$ and $\{\chi\}_w^\ast$ as follows:
\[
\{\chi\}_w = w_1\chi|_\Gamma + w_2\chi_2|_\Gamma, \quad \{\chi\}_w^\ast = w_2\chi_1|_\Gamma + w_1\chi_2|_\Gamma,
\]
where $\chi_i = \chi|_{\Omega_i}$, $i = 1, 2$ and we use the so-called “harmonic weights” as adopted by [7, 8, 14, 21, 22, 24],
\[
w_1 = \frac{\mu_2}{\mu_1 + \mu_2}, \quad w_2 = \frac{\mu_1}{\mu_1 + \mu_2}.
\]
It is clear that
\[
\{\mu\}_w = 2\mu_i w_i = \frac{2\mu_1\mu_2}{\mu_1 + \mu_2}.
\]
Likewise, we denote the arithmetic averages $\{\chi\}_k$ on the cut edges $e \in F_{h,i}^{\text{cut}}$ by
\[
\{\chi\}_k = \frac{1}{2} \chi|_e + \frac{1}{2} \chi|_{\tilde{e}},
\]
where $\chi_j = \chi|_{K_j}$, $j = l, r$ provided $\bar{e} = \partial K_j^l \cap \partial K_j^r$, $K_j^l = K_j \cap \Omega_l$ for $K_l, K_r \in G^l_k$.

Now we propose the following Nitsche method to approximate problem (1) with assumptions (A1)-(A3): find $(u, p_h) \in V_h \times Q_h$ such that
\[
B_h[(u, p_h), (v_h, q_h)] = L_h(v_h), \forall (v_h, q_h) \in V_h \times Q_h,
\]
where
\[
B_h[(u, p_h), (v_h, q_h)] = A_h(u, v_h) + b_h(p_h, v_h) - b_h(q_h, u_h) + J_p(p_h, q_h),
\]
and
\[
A_h(u, v_h) = a_h(u, v_h) + J_u(u_h, v_h).
\]
Here, $a_h(\cdot, \cdot)$, $J_u(\cdot, \cdot)$ are the bilinear forms on $(V + V_h) \times (V + V_h)$ defined by
\[
a_h(u, v) = \sum_{i=1}^2 \sum_{K \in T_h} \int_{K \cap \Omega_i} \mu_i \nabla u \cdot \nabla v - \sum_{K \in T_h} \int_{\Gamma_K} \{\mu \nabla u \cdot n\}_w \cdot [v]
\]
\[
+ [u] \{\mu \nabla v \cdot n\}_w + \sum_{K \in T_h} \int_{\Gamma_K} \gamma_0 \{\mu\}_w \cdot [u] \cdot [v] + \sum_{i=1}^2 \sum_{\tilde{e} \in F_{h,i}^{\text{cut}}} \left( \int_{\tilde{e}} (-\{\mu_i \nabla u \cdot n_e\}_k [v] - \{\mu_i \nabla v \cdot n_e\}_k [u])
\]
\[
+ \gamma_i|\tilde{e}|^{-1} \mu_i \int_{\tilde{e}} [u] [v] \right),
\]
and
\[
J_u(u, v) = \sum_{i=1}^2 \left( \sum_{e \in F_{h,i}^\ast} |e| \mu_i \int_e [\nabla u] \cdot [\nabla v] + \sum_{\tilde{e} \in F_{h,i}^{\text{cut}}} \int_{\tilde{e}} |\tilde{e}| \mu_i [\nabla u \cdot n_e] [\nabla v \cdot n_e] \right),
\]
$b_h(\cdot, \cdot)$ is defined in $Q \times (V + V_h)$ by
Now we introduce the norms. For $v$ see that the following equality holds,

$$b_h(p, v) = -\sum_{i=1}^{2} \left( \sum_{K \in T_h} \int_{K \cap \Omega_i} p \nabla \cdot v - \sum_{\tilde{e} \in F_{h,i}^{cut}} \int_{\tilde{e}} \{p\}_k [v \cdot n_{\tilde{e}}] \right) + \sum_{K \in G_h^t} \int_{\Gamma_K} \{p\}_w [v \cdot n],$$

($6$)

$J_p(\cdot, \cdot)$ is defined in $Q \times Q$ by

$$J_p(p, q) = \sum_{i=1}^{2} \left( \sum_{e \in F_{h,i}^{cut}} |e| \int_{e} \mu^{-1}_i \{p\}_e[q] + \sum_{\tilde{e} \in F_{h,i}^{cut}} |\tilde{e}| \int_{\tilde{e}} \mu^{-1}_i \{q\}_e[p] \right),$$

($7$)

and $L_h(\cdot)$ is a linear form defined by

$$L_h(v) = \sum_{i=1}^{2} \int_{\Omega_i} f v + \sum_{K \in G_h^t} \int_{\Gamma_K} \sigma_K \{v \cdot n\}_w,$$

($8$)

where $\gamma_0$, $\gamma_1$ and $\gamma_2$ are sufficiently large, positive parameters to be chosen.

**Remark 1.** The stabilization terms $J_u$, $J_p$ are added in our method. The term $J_u(u_h, v_h)$ is added to ensure the coercivity of $A_h(\cdot, \cdot)$ and the term $J_p(p_h, q_h)$ is used to guarantee the inf-sup stability of the method.

For any $u \in [H^2(\Omega_1 \cup \Omega_2) \cap H^1_0(\Omega)]^2$ and $p \in H^1(\Omega_1 \cup \Omega_2) \cap L^2_\mu(\Omega)$, it is easy to see that the following equality holds,

$$B_h[(u - u_h, p - p_h), (v_h, q_h)] = \sum_{i=1}^{2} \sum_{e \in F_{h,i}^{cut}} \left( \int_{e} \mu_i (\nabla u \cdot n_e[v_h] - \int_{e} p [v_h \cdot n_e]) \right), \quad \forall (v_h, q_h) \in V_h \times Q_h.$$

($9$)

Now we introduce the norms. For $v \in V + V_h$, we define

$$\|v\|_V := \sum_{i=1}^{2} \sum_{K \in T_h,i} \|\mu_i \nabla v\|^2_{0,K \cap \Omega_i} + \frac{\|\mu\|_w}{h} \sum_{K \in G_h^t} \|v\|^2_{0,\Gamma_K} + \sum_{i=1}^{2} \sum_{\tilde{e} \in F_{h,i}^{cut}} |\tilde{e}|^{-1} \mu_i \|v\|^2_{0,\tilde{e}} + J_u(v, v),$$

($10$)

and

$$\|v\|_{V_h} := \|v\|_V + \frac{h}{\|\mu\|_w} \sum_{K \in G_h^t} \|\mu \nabla v \cdot n\|^2_{0,\Gamma_K} + \sum_{i=1}^{2} \sum_{\tilde{e} \in F_{h,i}^{cut}} |\tilde{e}| \mu_i \|\{\nabla v \cdot n_{\tilde{e}}\}_k\|^2_{0,\tilde{e}}.$$

($11$)

For $(v, p) \in (V + V_h) \times Q$, we define

$$\|(v, p)\|^2 := \|v\|^2 + \|\mu^{-1/2} p\|_{0,\Omega_1 \cup \Omega_2}^2 + J_p(p, p),$$

($12$)
and
\[ \| (v, p) \|^2_V := \| v \|^2_V + \| \mu^{-1/2} p \|^2_{0,\Omega_1 \cup \Omega_2} + J_p(p, p) \]
\[ + \frac{h}{\mu w} \sum_{K \in G_h^e} \| p \|^2_{0,\Gamma_K} + \sum_{i=1}^2 \sum_{\tilde{e} \in F_{h,i}^{\text{cut}}} \| e \|^2_{\partial K} \cdot \mu_i^{-1} \| p \|^2_{e,\partial K}. \tag{13} \]

3. Preliminary. In this section, we will give some preliminaries for the later error analysis. Firstly, we give the following lemma which is proved in [18].

**Lemma 3.1.** If \( \tilde{e} \in \mathcal{F}_{h,i}^{\text{cut}} \), that is to say, \( \tilde{e} = \partial K_i^l \cap \partial K_i^r \), where \( K_i, K_r \in G^i_h \) and \( K_j = K_j \cap \Omega_i, j = l, r \), and for sufficiently small \( h \), then there exists a constant \( \theta > 0 \) such that
\[ |\tilde{e}|^2 \leq \theta \max_{j=l,r} |K_j|. \]
The constant \( \theta \) depends on the \( C^2 \)-norm of the parametrization of \( \Gamma \) and the shape regularity of \( K_i \) and \( K_r \).

We also need the following trace inequality for interface edges and its proof can be found in [37].

**Lemma 3.2.** Suppose \( h \) be sufficiently small, then for each \( K \in G^i_h \) and \( v \in H^1(K) \), it holds
\[ \| v \|^2_{0,\Gamma_K} \lesssim h^{-1/2}_K \| v \|_{0,K} + \| \nabla v \|_{1/2,K}^{1/2}. \]
Further, if \( v \in P_1(K) \), then
\[ \| v \|^2_{0,\Gamma_K} \lesssim h^{-1/2}_K \| v \|_{0,K}. \]

In order to estimate the error of our method, the following trace inequality is needed for the cut segments totally contained in \( \Omega_i \). We have proved in [22].

**Lemma 3.3.** Suppose that \( v \in H^2(K) \) for \( K \in G^i_h \). For \( e \in \mathcal{F}_{h,i}^{\text{cut}} \), if \( e \subseteq \partial K \) such that \( \tilde{e} \subseteq e \). Then we have
\[ \frac{1}{|\tilde{e}|} \| v \|^2_{0,\tilde{e}} \leq C \left( \frac{1}{h^2_K} \| v \|^2_{0,K} + \| \nabla v \|^2_{0,K} + h^2_K \| \nabla v \|^2_{1,K} \right). \]

Next, we give some properties of \( A_h(\cdot, \cdot) \). The proof of Lemma 3.4 and Lemma 3.5 can be obtained from [22].

**Lemma 3.4.** Assuming that \( h \) is small enough, the bilinear discrete form \( A_h(\cdot, \cdot) \) is coercive on \( V_h \) provided that \( \gamma_i, i = 0, 1, 2 \) are chosen large enough. That is,
\[ A_h(v, v) \geq \frac{1}{2} \| v \|^2, \quad \forall v \in V_h. \]

**Lemma 3.5.** There exists a positive constants \( C_{A_i} \) such that
\[ A_h(u, v) \leq C_{A_i} \| u \|_V \| v \|_V, \quad \forall u, v \in V. \]
Additionally, for \( u \in V \) and \( v \in V_h \), under assuming that \( h \) is small enough, there exist two positive constants \( C_{A_2} \) and \( C_{A_3} \) such that
\[ A_h(u, v) \leq C_{A_2} \| u \|_V \| v \|, \]
and
\[ \| v \|_V \leq C_{A_3} \| v \|. \]

Further, we give the following properties of \( b_h(\cdot, \cdot) \).
Lemma 3.6. There exist a positive constants $C_{b_1}$ such that, for any $v \in V + V_h, p \in Q$, the following inequality holds

$$b_h(p, v) \leq C_{b_1} \|v\| \left( \sum_{i=1}^{2} \sum_{K \in \mathcal{T}_{h,i}} \|\mu^{-1/2} p\|_{0, K \cap \Omega_i}^2 \right)^{1/2} + \frac{h}{\mu_i} \sum_{K \in \mathcal{G}_h^r} \|\{p\}_w\|_{0, \Gamma_K}^2 + \sum_{i=1}^{2} \sum_{v \in \mathcal{F}_{h,i}^\text{cut}} \|\{\bar{c}\}_i\| \|\{p\}_w\|_{0, \bar{c}}^2. \tag{14}$$

Additionally, suppose that $h$ is sufficiently small. For any $v \in V + V_h, p \in Q_h$, there exist two positive constants $C_{b_2}$ and $C_p$ such that

$$b_h(p, v) \leq C_{b_2} \|v\| \left( \sum_{i=1}^{2} \sum_{K \in \mathcal{T}_{h,i}} \|\mu^{-1/2} p\|_{0, K \cap \Omega_i}^2 + J_p(p, p) \right)^{1/2}, \tag{15}$$

and

$$\mu_i^{-1} \sum_{K \in \mathcal{T}_{h,i}} \|p\|_{0, K \cap \Omega_i}^2 \leq C_p \left( \mu_i^{-1} \sum_{K \in \mathcal{T}_{h,i} \cap \mathcal{G}_h^r} \|p\|_{0, K}^2 + J_p(p, p) \right). \tag{16}$$

Proof. It is easy to obtain the first inequality (14) by using Cauchy-Schwarz inequality directly. We will give the proof of (15) and (16) in details. First, using $\frac{h^2}{\mu_i} \leq \frac{h}{2\mu_i}$, we have

$$\frac{h}{\mu_i} \sum_{K \in \mathcal{G}_h^i} \|\{p\}_w\|_{0, \Gamma_K}^2 \leq \sum_{i=1}^{2} \frac{h}{\mu_i} \sum_{K \in \mathcal{G}_h^i} \|p\|_{0, \Gamma_K}^2.$$  

Then for any $K \in \mathcal{G}_h^r$, according to Assumption (A2), there exists $K' \in \Omega_i$ such that $K'$ shares an edge or a vertex with $K$. Since $p$ is piecewise constant on $\Omega_{h,i}$, from the proof of Lemma 4.1 of [22], we have

$$h \|p\|_{0, K'}^2 \lesssim \|p\|_{0, K}^2 + \sum_{e \in \mathcal{F}_{h,i}^\text{cut}} |e| \|\{p\}_w\|_{0, e}^2. \tag{17}$$

Hence

$$\frac{h}{\mu_i} \sum_{K \in \mathcal{G}_h^r} \|\{p\}_w\|_{0, \Gamma_K}^2 \lesssim \sum_{i=1}^{2} \mu_i^{-1} \left( \sum_{K \in \mathcal{T}_{h,i}} \|p\|_{0, K \cap \Omega_i}^2 + \sum_{e \in \mathcal{F}_{h,i}^\text{cut}} |e| \|\{p\}_w\|_{0, e}^2 \right). \tag{18}$$

For any $\bar{c} \in \mathcal{F}_{h,i}^\text{cut}$, there exist two elements $K_l, K_r \in \mathcal{G}_h^r$ so that $\bar{c} = \partial K_l^i \cap K_r^i$, where $K_j^i = K_j \cap \Omega_i, j = l, r$. Assume $|K_l^i| = \max_{j=l, r} |K_j^i|$. According to Lemma 3.1, we have $|\bar{c}|^2 \lesssim |K_l^i|$. Applying the fact that $p \in Q_{h,i}$ is a piecewise constant polynomial, we obtain

$$\frac{|\bar{c}|}{\mu_i} \|\{p\}_w\|_{0, \bar{c}}^2 \leq \mu_i^{-1} |\bar{c}| \sum_{j=l, r} \|p|_{K_j^i}^2 \|_{0, \bar{c}}^2 \lesssim \mu_i^{-1} |\bar{c}| \left( \|p|_{K_l^i}^2 + \|p|_{K_r^i}^2 \right). \tag{19}$$
Then (16) is yielded immediately.

Thus, the following inequality holds

\[ \mu_i^{-1} \sum_{K \in T_h,i} \| p \|_{0,K}^2 \lesssim \mu_i^{-1} \left( \sum_{K \in T_h,i} \| p \|_{0,K}^2 + \sum_{K \in G_{h,i}^\Gamma} \| p \|_{0,K}^2 \right), \]

(20)

and for \( e \subseteq \partial K, K \in G_{h,i}^\Gamma \)

\[ \| p \|_{0,K}^2 \lesssim |e| \| p \|_{0,e}^2. \]

(21)

For \( K \in G_{h,i}^\Gamma \), similar to (17), we have

\[ |e| \| p \|_{0,e}^2 \lesssim \| p \|_{0,K}^2 + \sum_{e \in F_{h,i}} |e| \| p \|_{0,e}^2. \]

Thus, the following inequality holds

\[ \sum_{K \in G_{h,i}^\Gamma} \| p_i \|_{0,K}^2 \lesssim \sum_{K \in T_h,i \setminus G_{h,i}^\Gamma} \| p \|_{0,K}^2 + \sum_{e \in F_{h,i}} |e| \| p \|_{0,e}^2. \]

Then (16) is yielded immediately.

From Lemma 3.5, Lemma 3.6 and Cauchy-Schwarz inequality, we can obtain the following lemma easy.

**Lemma 3.7.** For \((u, p) \in (V + V_h) \times Q\), the following inequality holds

\[ B_h[(u, p), (v, q)] \lesssim \| (u, p) \|_V \| (v, q) \|_V, \quad \forall (v, q) \in (V + V_h) \times Q. \]

Furthermore, for \((v, q) \in V_h \times Q_h\), under assuming that \( h \) is sufficiently small, we have

\[ B_h[(u, p), (v, q)] \lesssim \| (u, p) \|_V \| (v, q) \|, \]

and

\[ \| (v, q) \|_V \lesssim \| (v, q) \|. \]

4. **Error analysis.** In this section, we will give a priori error estimates. Firstly, we prove the stability of \( b_h(\cdot, \cdot) \). We use some of the ideas in [20, 25] and introduce the piecewise constant function

\[ \bar{p}_\mu = \begin{cases} \mu_1 |\Omega_1|^{-1} & \text{on } \Omega_1, \\ -\mu_2 |\Omega_2|^{-1} & \text{on } \Omega_2. \end{cases} \]

Let \( M_0 = \text{span}\{ \bar{p}_\mu \} \subset Q_h \). The space \( Q_h \) can be decomposed as \( Q_h = M_0 \oplus M_{h,0}^\perp \), with \( (p_{h,0}^\perp)_i |\Omega_i| = 0, i = 1, 2 \) for any \( p_{h,0}^\perp \in M_{h,0}^\perp \), see Lemma 2.1 of [25].

**Lemma 4.1.** Suppose that \( h \) is sufficiently small. For any \( p_0 \in M_0 \), there exist \( v_{h,0} \in V_h \) and positive constants \( C_{1, p_0} \) and \( C_{2, p_0} \) such that

\[ b_h(p_0, v_{h,0}) \geq C_{1, p_0} \| \mu^{-1/2} p_0 \|_{0, \Omega_1 \cup \Omega_2}^2, \quad \| v_{h,0} \| \leq C_{2, p_0} \| \mu^{-1/2} p_0 \|_{0, \Omega_1 \cup \Omega_2}. \]
Proof. Let \( \tilde{p}_0 = \mu^{-1}p_0 \), then \((\tilde{p}_0, 1)_{\Omega_1 \cup \Omega_2} = 0 \). The relation between \( \tilde{p}_0 \) and \( p_0 \) satisfies \( \|\mu^{-1/2} \tilde{p}_0\|_{0, \Omega_1 \cup \Omega_2}^2 = C(\mu, \Omega) \|\tilde{p}_0\|_{0, \Omega_1 \cup \Omega_2}^2 \), with

\[
C(\mu, \Omega) = \frac{\mu_1 |\Omega_1|^{-1} + \mu_2 |\Omega_2|^{-1}}{|\Omega_1|^{-1} + |\Omega_2|^{-1}} \geq \mu_{\text{max}} \min_{i=1,2} \frac{|\Omega_i|^{-1}}{|\Omega_1|^{-1} + |\Omega_2|^{-1}} = \tilde{C}_{\text{max}},
\]
with \( \mu_{\text{max}} = \max\{\mu_1, \mu_2\} \). Define \( I(\tilde{p}_0) \) such that

\[
I(\tilde{p}_0) = \begin{cases} 
\tilde{p}_0 & K \in T_h \setminus G_h^P, \\
\int_K \tilde{p}_0 & K \in G_h^P.
\end{cases}
\]

Let \( \alpha = \frac{1}{|\Omega_1|+|\Omega_2|}(I(\tilde{p}_0), 1)_{\Omega_1 \cup \Omega_2} \) and \( q_h = I(\tilde{p}_0) - \alpha \), then \((q_h, 1)_{\Omega_1 \cup \Omega_2} = 0 \). Define \( U_h \) by

\[
U_h := \{v; \forall K \in T_h, v|_K \in S_h(K) ; \forall e \in \partial K_1 \cap \partial K_2, K_1, K_2 \in T_h, \int_{e} [v] = 0 ; \forall e \in \partial K_1 \cap \partial \Omega, \int_{e} v = 0 \}^2.
\]

For \( q_h \), from the inf-sup stability of the nonconforming-\( P_1/P_0 \) pair in [15], there exists a \( v_{h,0} \in U_h \), (that implies \( v_{h,0} \in V_h \) with \( \int_{e} v_{h,0} = 0, \forall e \in \partial K_1 \cap \partial \Omega \) and \( [v_{h,0}]_{\Gamma} = 0 \)) such that

\[
\|\nabla v_{h,0}\|_{0, \Omega_1 \cup \Omega_2} = \|\tilde{p}_0\|_{0, \Omega_1 \cup \Omega_2} ; \ b_h(q_h, v_{h,0}) \geq C_1 \|\tilde{p}_0\|_{0, \Omega_1 \cup \Omega_2} \|q_h\|_{0, \Omega_1 \cup \Omega_2},
\]
where \( \|\nabla v_{h,0}\|_{0, \Omega_1 \cup \Omega_2} := \sum_{i=1}^2 \sum_{K \in T_{h,i}} \|\nabla v\|_{0, K \cap \Omega_i} \) and we have used the fact that

\[
b_h(q_h, v_{h,0}) = -\sum_{i=1}^2 \sum_{K \in T_{h,i}} \int_{K \cap \Omega_i} q_h \nabla \cdot v_{h,0}.
\]

From the definition of \( b_h(\cdot, \cdot) \) and \( v_{h,0} \in V_h \) with \( \int_{e} v_{h,0} = 0, \forall e \in \partial K_1 \cap \partial \Omega \) and \( [v_{h,0}]_{\Gamma} = 0 \), we have

\[
b_h(\tilde{p}_0 - q_h, v_{h,0}) = -\sum_{i=1}^2 \sum_{K \in T_{h,i}} \int_{K \cap \Omega_i} (\tilde{p}_0 - q_h) \nabla \cdot v_{h,0} + \sum_{i=1}^2 \sum_{\tilde{e} \in F_{h,i}} \int_{\tilde{e}} (\tilde{p}_0 - q_h) v_{h,0} \cdot n_{\tilde{e}}.
\]

For \( \tilde{e} \subset e \subset \partial K_i \), using Lemma 3.3 we have

\[
|\tilde{e}|^{-1} \|v_{h,0}\|_{0, \tilde{e}}^2 \lesssim h^{-2} \|v_{h,0}\|_{0,K}^2 + \|\nabla v_{h,0}\|_{0,K}^2 ;
\]
then by the following Poincaré inequality on \( \Omega_1 \cup \Omega_2 \) (see Lemma 5 of [33])

\[
\|v_{h,0}\|_{0, \Omega_1 \cup \Omega_2} \lesssim h \|\nabla v_{h,0}\|_{0, \Omega_1 \cup \Omega_2},
\]
we get

\[
\sum_{i=1}^2 \sum_{\tilde{e} \in F_{h,i}} |\tilde{e}|^{-1} \|v_{h,0}\|_{0, \tilde{e}} \lesssim \|\nabla v_{h,0}\|_{0, \Omega_1 \cup \Omega_2}.
\]

For \( \tilde{e} \subset e \subset \partial K_i \), the following inequalities hold

\[
|\tilde{e}| \|\tilde{p}_0 - q_h\|_{0, \tilde{e}}^2 \leq |e| \|\tilde{p}_0 - q_h\|_{0, e}^2 \lesssim \|\tilde{p}_0 - q_h\|_{0,K}^2.
\]
then using \( \frac{|\Omega_{h}^{+}|}{|\Omega_{h}^{-}|} = 1 + \frac{|\Omega_{h}^{+} \setminus \Omega_{h}^{-}|}{|\Omega_{h}^{-}|} \lesssim 1 \), we get
\[
\sum_{\tilde{e} \in \mathcal{E}^{out}_{h,i}} \| \tilde{e} \| \| \tilde{p}_0 - q_h \|_{0,\tilde{e}}^2 \lesssim \| \tilde{p}_0 - q_h \|_{0,\Omega_{h,i}}^2 \lesssim \| \tilde{p}_0 - q_h \|_{0,\Omega_{h,i}}^2 .
\] (29)

By Cauchy-Schwarz inequality, (27) and (29), the following estimate holds
\[
b_h(\tilde{p}_0 - q_h, \mathbf{v}_{h,0}) \geq -C_2 \| \nabla \mathbf{v}_{h,0} \|_{0,\Omega_1 \cup \Omega_2} \| \tilde{p}_0 - q_h \|_{0,\Omega_1 \cup \Omega_2} .
\] (30)

Thus
\[
b_h(\tilde{p}_0, \mathbf{v}_{h,0}) = b_h(q_h, \mathbf{v}_{h,0}) + b_h(\tilde{p}_0 - q_h, \mathbf{v}_{h,0})
\geq C_1 \| \tilde{p}_0 \|_{0,\Omega_1 \cup \Omega_2} \| q_h \|_{0,\Omega_1 \cup \Omega_2} - C_2 \| \tilde{p}_0 \|_{0,\Omega_1 \cup \Omega_2} \| \tilde{p}_0 - q_h \|_{0,\Omega_1 \cup \Omega_2}
\geq \| \tilde{p}_0 \|_{0,\Omega_1 \cup \Omega_2} \left( C_1 \| \tilde{p}_0 \|_{0,\Omega_1 \cup \Omega_2} - (C_2 + C_1) \| \tilde{p}_0 - q_h \|_{0,\Omega_1 \cup \Omega_2} \right)
\geq \| \tilde{p}_0 \|_{0,\Omega_1 \cup \Omega_2}^2 \left( C_1 - (C_2 + C_1) \frac{\| \tilde{p}_0 - q_h \|_{0,\Omega_1 \cup \Omega_2}}{\| \tilde{p}_0 \|_{0,\Omega_1 \cup \Omega_2}} \right).
\] (31)

Note that
\[
|\alpha| = \frac{1}{|\Omega_1 \cup \Omega_2|} |(I(\tilde{p}_0), 1)_{\Omega_1 \cup \Omega_2}| = \frac{1}{|\Omega_1 \cup \Omega_2|} |(I(\tilde{p}_0) - \tilde{p}_0, 1)_{\Omega_1 \cup \Omega_2}|
\leq c \| I(\tilde{p}_0) - \tilde{p}_0 \|_{0,\Omega_1 \cup \Omega_2} \leq \epsilon h^{1/2},
\] (32)

which implies \( \| \tilde{p}_0 - q_h \|_{0,\Omega_1 \cup \Omega_2} \leq \epsilon h^{1/2} \).

Hence, \( b_h(\tilde{p}_0, \mathbf{v}_{h,0}) \geq (C_1 - \epsilon h^{1/2}) \| \tilde{p}_0 \|_{0,\Omega_1 \cup \Omega_2}^2 \). Since \( \mathbf{v}_{h,0} \in V_h \) and \( \int_{\Gamma} \mathbf{v}_{h,0} = 0, e \in \partial K \cap \partial \Omega, [\mathbf{v}_{h,0}]_{\Gamma} = 0 \), we have
\[
b_h(p_0, \mathbf{v}_{h,0}) = -\sum_{K \in G^+_h} \int_{\Gamma_K} [p_0] \mathbf{v}_{h,0} \cdot n, \quad b_h(\tilde{p}_0, \mathbf{v}_{h,0}) = -\sum_{K \in G^+_h} \int_{\Gamma_K} [\tilde{p}_0] \mathbf{v}_{h,0} \cdot n
\]
then \( b_h(p_0, \mathbf{v}_{h,0}) = C(\mu, \Omega)b_h(\tilde{p}_0, \mathbf{v}_{h,0}) \). Further, we obtain
\[
b_h(p_0, \mathbf{v}_{h,0}) = C(\mu, \Omega)b_h(\tilde{p}_0, \mathbf{v}_{h,0}) \geq (C_1 - \epsilon h^{1/2})C(\mu, \Omega) \| \tilde{p}_0 \|_{0,\Omega_1 \cup \Omega_2}^2
\geq C_{1,p_0} \| \mu^{-1/2} p_0 \|_{0,\Omega_1 \cup \Omega_2}^2,
\] (33)

provided \( h \) is sufficiently small. Finally, from Lemma 3.3, standard trace inequality and (26), we obtain
\[
\| \mathbf{v}_{h,0} \| \leq C \mu^{1/2} \| \nabla \mathbf{v}_{h,0} \|_{0,\Omega_1 \cup \Omega_2}
= C \mu^{1/2} \| \tilde{p}_0 \|_{0,\Omega_1 \cup \Omega_2} \leq C_{2,p_0} \| \mu^{-1/2} p_0 \|_{0,\Omega_1 \cup \Omega_2} .
\]

**Lemma 4.2.** Suppose that \( h \) is sufficiently small. For any \( p_{h,0}^+, (p_{h,0}^+)^2 \in M_{h,0}^+ \), there exist \( \tilde{\mathbf{v}}_h \in V_h \) and positive constants \( C_{1,p_0^+, b}, C_{2,p_0^+, b} \) and \( C_{3,p_0^+, b} \) such that
\[
b_h(p_{h,0}^+, \tilde{\mathbf{v}}_h) \geq C_{1,p_0^+, b} \| \mu^{-1/2} p_{h,0}^+ \|_{\Omega_1 \cup \Omega_2}^2 - C_{2,p_0^+, b} J_p(p_{h,0}, p_{h,0}^+) \]
and
\[
\| \tilde{\mathbf{v}}_h \| \leq C_{3,p_0^+, b} \| \mu^{-1/2} p_{h,0}^+ \|_{\Omega_1 \cup \Omega_2} .
\]
For each \( q \), from (35) and (36), we get

\[
|\alpha_i| = \frac{1}{|\Omega^-_{h,i}|} |(p^+_{h,i}, 1)_{\Omega^-_{h,i}}| = \frac{1}{|\Omega^-_{h,i}|} |(p^+_{h,i}, 1)_{\Omega_i} - (p^+_{h,i}, 1)_{\Omega_i \setminus \Omega^-_{h,i}}| \\
= \frac{1}{|\Omega^-_{h,i}|} |(p^+_{h,i}, 1)_{\Omega_i \setminus \Omega^-_{h,i}}| \leq \frac{\Omega_i \setminus \Omega^-_{h,i}^{1/2}}{|\Omega^-_{h,i}|} \|p^+_{h,i}\|_{0, \Omega_i} \leq Ch^{1/2} \|p^+_{h,i}\|_{0, \Omega_i}.
\]

From (35) and (36), we get

\[
C_p \left( C^{-1} b_h(p^+_{h,0}, \bar{v}_{h,i}) + J_p(p^+_{h,0}, p^+_{h,0}) \right) \geq \left\| \mu_i^{-1/2} q_{h,i} \right\|_{0, \Omega_i}^2 \\
\geq \left\| \mu_i^{-1/2} p^+_{h,i} \right\|_{0, \Omega_i}^2 - \mu_i^{-1} |\alpha_i|^2 \geq (1 - C \mu_i) \left\| \mu_i^{-1/2} p^+_{h,i} \right\|_{0, \Omega_i}^2,
\]

and

\[
\left\| \bar{v}_{h,i} \right\| \leq C \left\| \mu_i^{-1/2} p^+_{h,i} \right\|_{0, \Omega_i}.
\]

Finally, taking \( \bar{v}_h = \bar{v}_{h,1} + \bar{v}_{h,2} \), we have complete the proof for \( h \) sufficiently small.

\[\Box\]

**Lemma 4.3.** Suppose that \( h \) is sufficiently small. For any \( p_h \in Q_h \), there exist \( \bar{v}_h \in V_h \) and positive constants \( C_1, C_2 \) and \( C_3 \) such that

\[
b_h(p_h, \bar{v}_h) \geq C_1 \left\| \mu^{-1/2} p_h \right\|_{\Omega_1 \cup \Omega_2}^2 - C_2 J_p(p_h, p_h), \quad \left\| \bar{v}_h \right\| \leq C_3 \left\| \mu^{-1/2} p_h \right\|_{\Omega_1 \cup \Omega_2}^2.
\]
Proof. For any $p_h \in \mathcal{Q}_h$, we have $p_h = p_0 + p_h^\perp$, where $p_0 \in \mathcal{P}_0$ and $p_h^\perp \in \mathcal{M}_h$. Let $v_h$ be such that Lemma 4.1 is satisfied and $\bar{v}_h$ be such that Lemma 4.2 is satisfied. Note that $v_h = 0$ on $G_h^e$, $\int v_h = 0$ for $e \in \partial K \cap \partial \Omega_{h,i}^e$ and for $e \in \partial K \cap \partial \Omega$ and $p_0$ is constant on $\Omega_{h,i}^e$, hence $b_h(p_0, v_h) = 0$. For $\zeta > 0$, define $v_h = v_{h,0} + \zeta \bar{v}_h$. We then obtain

$$b_h(p_h, v_h) = b_h(p_0, v_{h,0}) + b_h(p_h^\perp, v_{h,0}) + \zeta b_h(p_h^\perp, \bar{v}_h)$$

$$\geq C_{1,p_0} \left\| \mu^{-1/2} p_0 \right\|_{0, \Omega_{1,3}; 2} \, b_h(p_h^\perp, v_{h,0})$$

$$+ \zeta \left( C_{1,p_h^\perp} \left\| \mu^{-1/2} p_h^\perp \right\|_{0, \Omega_{1,3}; 2}^2 - C_{2,p_h^\perp} J_p(p_h^\perp, p_h^\perp) \right).$$

Since $p_0$ is constant on each $\Omega_{h,i}^+$, we have $J_p(p_h^\perp, p_h^\perp) = J_p(p_h, p_h)$. Note that $v_h \in V_h$ with $\int v_h = 0$, $e \in \partial K \cap \partial \Omega$ and $[v_{h,0}]_\Gamma = 0$. Further, similar to the estimates of (27) and (29), the following estimates hold

$$b_h(p_h^\perp, v_{h,0}) = \sum_{i=1}^2 \left( \int_{\Omega_{1,3}} p_h^\perp \nabla \cdot v_{h,0} + \sum_{e \in \partial K_{h,i}^e} \int_{\partial K_{h,i}^e} p_h^\perp \nabla \cdot [v_{h,0} \cdot n_e] \right)$$

$$\geq - C \|v_{h,0}\| \left\| \mu^{-1/2} p_h^\perp \right\|_{0, \Omega_{1,3}; 2}. $$

Thus, (39) can be estimated by

$$b_h(p_h, v_h) \geq C_{1,p_0} \left\| \mu^{-1/2} p_0 \right\|_{0, \Omega_{1,3}; 2}^2$$

$$- C C_{2,p_0} \left\| \mu^{-1/2} p_0 \right\|_{0, \Omega_{1,3}; 2} \left\| \mu^{-1/2} p_h^\perp \right\|_{0, \Omega_{1,3}; 2}$$

$$+ \zeta \left( C_{1,p_h^\perp} \left\| \mu^{-1/2} p_h^\perp \right\|_{0, \Omega_{1,3}; 2}^2 - C_{2,p_h^\perp} J_p(p_h^\perp, p_h^\perp) \right). $$

(41)

where $\varepsilon = \frac{C_{2,p_0}}{C_{2,p_0}}$ and $\zeta = \frac{C^2 C_{2,p_0}^2}{C_{1,p_h^\perp} C_{1,p_h^\perp}}$. Finally, we have

$$\|v_h\| \leq \|v_{h,0}\| + \zeta \|\bar{v}_h\| \leq C_3 \left\| \mu^{-1/2} p_h \right\|_{0, \Omega_{1,3}; 2}, $$

and combining this with (41) completes the proof. \qed

Now we prove the inf-sup stability of our scheme.

**Theorem 4.4.** Suppose that $\gamma_i, i = 0, 1, 2$ are large enough and $h$ is small enough. Let $(u_h, p_h) \in V_h \times Q_h$, then there exist a constant $C_s$ such that

$$\sup_{0 \neq (v_h, q_h) \in V_h \times Q_h} \frac{B_h[(u_h, p_h), (v_h, q_h)]}{\|(v_h, q_h)\|} \geq C_s \|(u_h, p_h)\|. $$

(42)
Proof. First, we start by choosing \((v_h, q_h) = (u_h, p_h)\) to obtain
\[
B_h[(u_h, p_h), (u_h, p_h)] = A_h(u_h, u_h) + J_p(p_h, p_h).
\]
(43)

Using Lemma 3.4, we get
\[
B_h[(u_h, p_h), (u_h, p_h)] \geq \frac{1}{2} \|u_h\|^2 + J_p(p_h, p_h).
\]
(44)

Next, from Lemma 4.3, we know that for \(p_h \in Q_h\), there exist \(w_h \in V_h\) such that
\[
\|w_h\| \leq C_3 \|\mu^{-1/2} p_h\|_{0, \Omega_1 \cup \Omega_2},
\]
(45)

and
\[
b_h(p_h, w_h) \geq C_1 \|\mu^{-1/2} p_h\|^2_{0, \Omega_1 \cup \Omega_2} - C_2 J_p(p_h, p_h).
\]
(46)

From the definition of \(B_h\), we have
\[
B_h[(u_h, p_h), (w_h, 0)] = A_h(u_h, w_h) + b_h(p_h, w_h).
\]
(47)

Using Lemma 3.5 and (45), we infer that
\[
A_h(u_h, w_h) \geq -C_A C_A^2 \|u_h\| \|w_h\|
\]
(48)

where \(C_A = C_A^2 \). Further, let \((v_h, q_h) = (u_h + \eta w_h, p_h)\), using (44),(46), (47), (48) and Young’s inequality, we get
\[
B_h[(u_h, p_h), (u_h + \eta w_h, p_h)]
\]
\[
= B_h[(u_h, p_h), (u_h, p_h)] + \eta B_h[(u_h, p_h), (w_h, 0)]
\]
\[
\geq \frac{1}{2} \|u_h\|^2 + J_p(p_h, p_h) + C_1 \|\mu^{-1/2} p_h\|^2_{0, \Omega_1 \cup \Omega_2}
\]
\[
- \eta C_2 J_p(p_h, p_h) - \eta C_A C_3 \|u_h\| \|\mu^{-1/2} p_h\|_{0, \Omega_1 \cup \Omega_2}
\]
(49)

\[
\geq \left(1 - \frac{C_A C_A^2 \eta}{2}\right) \|u_h\|^2 + (1 - \eta C_2) J_p(p_h, p_h)
\]
\[
+ \left(C_1 \eta - \frac{C_A C_A^2 \eta^2}{2}\right) \|\mu^{-1/2} p_h\|^2_{0, \Omega_1 \cup \Omega_2}
\]
\[
\geq C_{1, \eta} \|(u_h, p_h)\|^2,
\]

the last inequality holds by choosing \(\eta = \frac{1}{2C_A C_A^2}\) and \(0 < \eta < \min\{\frac{C_1}{C_A C_A^2}, \frac{1}{2C_2}\}\).

Finally, the proof follows by employing
\[
\|(u_h + \eta w_h, p_h)\| \leq \|(u_h, p_h)\| + \eta \|w_h\| \leq C_{2, \eta} \|(u_h, p_h)\|.
\]

To obtain a priori error estimates, we need the interpolation operators and their approximation errors. To show these, we need construct the extension operators \(E_2^2 : [H^2(\Omega)]^2 \to [H^2(\Omega)]^2\), and \(E_1^i : H^1(\Omega_i) \to H^1(\Omega), i = 1, 2\) such that
\[
(E_j^2 w_i, E_j^2 r_i)_{\Omega_i} = (w_i, r_i),
\]
\[
\|E_j^2 w_i\|_{s, \Omega_i} \leq C \|w_i\|_{s, \Omega_i}, \forall w_i \in [H^s(\Omega_i)]^2, s = 0, 1, 2,
\]
and
\[
\|E_j^1 r_i\|_{t, \Omega_i} \leq C \|r_i\|_{t, \Omega_i}, \forall r_i \in H^t(\Omega_i), t = 0, 1.
\]
Similarly, using triangle and trace inequalities, we obtain

\[ (I_h \mathbf{v})|_{\Omega_i}, (R_h q)|_{\Omega_i}) := ((I_h \mathbf{v}_i)|_{\Omega_i}, (R_h q_i)|_{\Omega_i}), i = 1, 2, \]  

(50)

where \( I_h \mathbf{v}_i = \Pi_1^h \mathbf{v}_i \) and \( R_h q_i = \Pi_0^h q_i \). Then, using the interpolation operators \( I_h \) and \( R_h \) defined above, we analyze the approximation properties of the proposed finite element space.

**Theorem 4.5.** Suppose that \( \mathbf{v} \in [H^2(\Omega_1 \cup \Omega_2)]^2 \) and \( q \in H^1(\Omega_1 \cup \Omega_2) \) and \((I_h \mathbf{v}, R_h q)\) be a pair of interpolant operators defined as in (50). Then

\[
\|(\mathbf{v} - I_h \mathbf{v}, q - R_h q)\|_V \lesssim h \left( \mu_1^{-1/2} |\mathbf{v}|_{2, \Omega_1 \cup \Omega_2} + \mu_i^{-1/2} |q|_{1, \Omega_1 \cup \Omega_2} \right).
\]

**Proof.** Denote by \( \mathbf{w}_i = \mathbf{v}_i - I_h \mathbf{v}_i, \ z_i = q_i - R_h q_i, i = 1, 2 \) and \( \mathbf{w} = \mathbf{v} - I_h \mathbf{v} \), \( \zeta = q - R_h q \). Clearly, \( \mathbf{w}|_{\Omega_i} = \mathbf{w}_i|_{\Omega_i}, \zeta|_{\Omega_i} = z_i|_{\Omega_i}, i = 1, 2 \). By Lemma 4.3 of [22], we know that

\[
\|\mathbf{w}\|^2_V \lesssim \sum_{i=1}^2 \mu_i h^2 |\mathbf{w}|^2_{1, \Omega_i}.
\]

(51)

From the standard finite element interpolation theory in [15], for \( l = 0, 1 \)

\[
\|z_i|_{l,K} \lesssim h^{m-l} |q_i|_{m,K}, l \leq m \leq 1.
\]

(52)

Further, collecting the property of extension operator, we have

\[
\sum_{K \in T_h,i} \|z_i|^2_{l, K} \lesssim h^{2-2l} |q_i|^2_{1, \Omega_1, \Omega_2} \lesssim h^{2-2l} |q_i|^2_{1, \Omega_i}, l = 0, 1.
\]

(53)

Next we estimate each term with \( \zeta \) of \( \|(|, \zeta)|_V \). Clearly

\[
\sum_{K \in T_h,i} \left\| \mu_i^{-1/2} \zeta \right\|^2_{0, K \cap \Omega_i} \lesssim \mu_i^{-1} h^2 |q_i|^2_{1, \Omega_i}.
\]

From \( \frac{w^2}{|\mu|_w} \leq \frac{1}{2\mu}, i = 1, 2 \) and Lemma 3.2, we infer that

\[
\frac{h}{|\mu|_w} \sum_{K \in G_h^{\text{int}}} \|\zeta\|^2_{\mathbf{w}, 0, K} \lesssim \sum_{i=1}^2 \sum_{K \in G_h^i} \mu_i \|\zeta_i\|^2_{0, K} \lesssim \mu_i^{-1} h^2 |q_i|^2_{1, \Omega_i}.
\]

For any \( \bar{e} \in F_{h,i}^{\text{int}} \), we assume that \( e = \partial K_i \cap \partial K_r, K_i, K_r \in T_h,i \) such that \( \bar{e} \subseteq e \). Applying the triangle and standard trace inequalities, we have

\[
\mu_i^{-1} |\bar{e}| \|\zeta_i\|^2_{\bar{e}} \lesssim \mu_i^{-1} |e| \|\zeta_i\|^2_{e} \lesssim \mu_i^{-1} |e| \sum_{j=l,r} \left( |e|^{-1} \|\zeta_j\|^2_{0, K_j} + |e| \|\zeta_j\|^2_{1, K_j} \right) \lesssim \mu_i^{-1} h^2 \sum_{j=l,r} |q_j|^2_{1, K_j}.
\]

Similarly, using triangle and trace inequalities, we obtain

\[
\sum_{e \in F_{h,i}^{\text{int}}} \mu_i^{-1} |e| \|\zeta_i\|^2_{0, e} \lesssim \mu_i^{-1} h^2 |q_i|^2_{1, \Omega_i}.
\]
and
\[
\sum_{\tilde{e} \in \mathcal{F}_{h,i}^{nc}} \mu_{\tilde{e}}^{-1} |\tilde{e}| \| \{ \zeta \}_h \|_{0,\tilde{e}}^2 \lesssim \mu_{\tilde{e}}^{-1} h^2 |q|_{1,\Omega}^2.
\]

The theorem follows by combining above estimates and the definition of \( \| (\cdot, \cdot) \|_V \).

Theorem 4.6. Let \((u, p)\) be the weak solution of (1) and \((u_h, p_h)\) be the solution of the finite element formulation (3) respectively. Suppose that the solution \((u, p)\) \(\in \) \([H^2(\Omega_1 \cup \Omega_2) \cap H^1(\Omega)]^2 \times H^1(\Omega_1 \cup \Omega_2) \cap L^2_0(\Omega) \) and \( h \) is sufficiently small. If \( \gamma_i, i = 0, 1, 2 \) are large enough, then the following error estimate holds

\[
\|(u - u_h, p - p_h)\| \lesssim h \left( \| \mu^{1/2} u \|_{2,\Omega_1 \cup \Omega_2} + \| \mu^{-1/2} p \|_{1,\Omega_1 \cup \Omega_2} \right).
\]

Proof. Adding and subtracting the interpolations \( I_h u \) and \( R_h p \) to \( \| \cdot \| \) and using the triangle inequality, we get

\[
\|(u - u_h, p - p_h)\| \leq \|(u - I_h u, u - R_h p)\|_V + \|(I_h u - u_h, R_h p - p_h)\|. \tag{54}
\]

For the second term of the right hand side of (54), using the inf-sup condition and Lemma 3.7, we get

\[
\|(I_h u - u_h, R_h p - p_h)\| \lesssim \sup_{0 \neq (v_h, q_h) \in V_h \times Q_h} \frac{B_h[(I_h u - u_h, R_h p - p_h), (v_h, q_h)]}{\| (v_h, q_h) \|}
\]

\[
= \sup_{0 \neq (v_h, q_h) \in V_h \times Q_h} \frac{B_h[(u - u_h, p - p_h), (v_h, q_h)] + B_h[(I_h u - u, R_h p - p), (v_h, q_h)]}{\| (v_h, q_h) \|}
\]

\[
\lesssim \sup_{0 \neq (v_h, q_h) \in V_h \times Q_h} \frac{B_h[(u - u_h, p - p_h), (v_h, q_h)]}{\| (v_h, q_h) \|} + \|(u - I_h u, p - R_h p)\|_V. \tag{55}
\]

From (9), it follows that

\[
B_h[(u - u_h, p - p_h), (v_h, q_h)] = \sum_{i=1}^2 \sum_{e \in \mathcal{F}_{h,i}^{nc}} \left( \int_e \mu_i \nabla u \cdot n_e [v_h] - \int_e p [v_h \cdot n_e] \right). \tag{56}
\]

Let \( \varpi = \frac{1}{|\tilde{e}|} \int_{\tilde{e}} v \). Applying the error estimate for polynomial projection and the standard error estimate on interpolation of Sobolev spaces (see [16]), the following inequality holds

\[
\| v - \varpi \|_{0, e} \lesssim h^{1/2} \| v \|_{1/2, e}. \tag{57}
\]

Further, from the Poincaré inequality, we have

\[
\| v - \varpi \|_{0, e} \lesssim |e| \| \nabla v \|_{0, e}.
\]

Since \( e \in \mathcal{F}_{h,i}^{nc} \) is the non-cut edge, there are three cases.

**Case 1:** \( e = \partial K_i \cap \partial K_r, K_i, K_r \in T_{h,i} \) are totally contained in \( \Omega_i \). We have

\[
\int_e \mu_i \nabla u \cdot n_e [v_h] = \int_e \mu_i \left( \nabla u \cdot n_e - \nabla u \cdot n_e \right) [v_h - v_{\tilde{e}}].
\]
Furthermore, using Cauchy-Schwarz, interpolation and trace inequalities, we get
\[
\int_e \mu_i \nabla u \cdot n_e [\mathbf{v}_h] \leq \mu_i \|\nabla u \cdot n_e - \nabla u \cdot n_e\|_{0,e} \sum_{j=l,r} |e| \|\nabla v_h|_{K_j}\|_{0,e}
\]
\[
\lesssim \mu_i h^{1/2} \|\nabla u \cdot n_e\|_{\frac{1}{2},e} \left(\sum_{j=l,r} |e| \|\nabla v_h|_{K_j}\|_{0,e}\right)
\]
\[
\lesssim \sqrt{\mu_i} h \|\nabla u\|_{1,K_i \cup K_r} \left(\sum_{j=l,r} |e|^{1/2} \|\sqrt{\mu_i} \nabla v_h\|_{K_j}\|_{0,e}\right)
\]
\[
\leq \mu_i^{1/2} h \|u\|_{2,K_i \cup K_r} \|\sqrt{\mu_i} \nabla v_h\|_{0,K_i \cup K_r}.
\] (58)

where we have used the fact that triangulations is conforming, quasi-uniform and regular for the last inequality. Similarly, for \(\int_e p [v_h \cdot n_e]\) we have
\[
\int_e p [v_h \cdot n_e] = \int_e (p - \bar{p}) [v_h \cdot n_e - \bar{v}_h \cdot n_e]
\]
\[
\lesssim \mu_i^{-1/2} h \|p\|_{1,K_i \cup K_r} \|\sqrt{\mu_i} \nabla v_h\|_{0,K_i \cup K_r}.
\] (59)

**Case 2:** \(e = \partial K_i \cap \partial K_r, K_i, K_r \in T_{h,i}\) where only one of the two elements is the cut element. Without loss of the generality, we assume that \(K_i\) is totally contained in \(\Omega_i\) and \(K_r \in G^i_h\). Similar to (58), we have the following estimates
\[
\int_e \mu_i \nabla u \cdot n_e [\mathbf{v}_h] \lesssim \mu_i h \|\nabla u\|_{1,K_i} \left(\sum_{j=l,r} |e|^{1/2} \|\nabla v_h|_{K_j}\|_{0,e}\right)
\]
\[
\leq \mu_i h \|u\|_{2,K_i} |e|^{1/2} \left(\|\nabla v_h|_{K_i}\|_{0,e} + \|\nabla v_h|_{K_i} \pm |\nabla v_h|\|_{0,e}\right)
\]
\[
\lesssim \sqrt{\mu_i} h \|u\|_{2,K_i} \left(\|\sqrt{\mu_i} \nabla v_h\|_{0,K_i} + |e|^{1/2} \sqrt{\mu_i} \|\nabla v_h\|_{0,e}\right).
\] (60)

Likewise, for \(\int_e p [v_h \cdot n_e]\) we have
\[
\int_e p [v_h \cdot n_e] \lesssim \mu_i^{-1/2} h \|p\|_{1,K_i} \left(\|\sqrt{\mu_i} \nabla v_h\|_{0,K_i} + |e|^{1/2} \sqrt{\mu_i} \|\nabla v_h\|_{0,e}\right).
\] (61)

**Case 3:** \(e \in \mathcal{F}_{h,i}^{nc}\) and \(e \in \partial K \cap \partial \Omega\) for \(K \in T_{h,i}\). Similarly, we have
\[
\int_e \mu_i \nabla u \cdot n_e [\mathbf{v}_h] \lesssim \mu_i^{1/2} h \|u\|_{2,K} \|\sqrt{\mu_i} \nabla v_h\|_{0,K},
\] (62)

and
\[
\int_e p [v_h \cdot n_e] \lesssim \mu_i^{-1/2} h \|p\|_{1,K} \|\sqrt{\mu_i} \nabla v_h\|_{0,K},
\] (63)

where we have used \(\int_e v_h = 0\) for \(v_h \in V_h\).

Hence, (66) is estimated by
\[
B_h([u - u_h, p - p_h], (v_h, q_h)) \lesssim h \left(\|\mu^{1/2} u\|_{2,\Omega_1 \cup \Omega_2} + \|\mu^{-1/2} p\|_{1,\Omega_1 \cup \Omega_2}\right) \|v_h\|.
\] (64)

Finally, the result follows by combining (54), (55), (64) and Theorem 4.5. □
5. **Numerical examples.** In the above section, we have shown that the proposed finite element method with nonconforming-$P_1/P_0$ pair is of optimal convergence order. In this section we investigate results for numerical experiments in two dimension space for the Stokes interface problem. We present the convergence rate of $H^1$, $L^2$ errors for velocity and $L^2$ error for pressure from two examples. Let $\gamma \cdot 1_{h,\Omega}$ be the piecewise $H^1$ semi-norm. Then we denote the errors as follows:

$$e_{h,u}^0 := \|u - u_h\|_{0,\Omega}, e_{h,p}^0 := \|\mu^{-1/2}(p - p_h)\|_{0,\Omega}, e_{h,u}^1 := \|\mu^{1/2}(u - u_h)\|_{1,h,\Omega},$$

All the following results are obtained by our method with penalty parameters $\gamma_0 = 100$ and $\gamma_1 = \gamma_2 = 100$.

5.1. **Example 1: A continuous problem.** We consider a continuous problem presented in [3]. The computational domain is $\Omega = [-1,1] \times [-1,1]$, the interface is a circle centered in $(0,0)$ with radius 0.5 and $\mu = 1$. The Dirichlet boundary conditions on $\partial \Omega$ are chosen such that the exact solution satisfies $u = (20xy^3, 5x^4 - 5y^4)$ and $p = 60x^2y - 20y^3$.

**Table 1.** Errors for a continuous problem with $\mu = 1$.

| $h$   | $e_{h,u}^0$ rate | $e_{h,u}^1$ rate | $e_{h,p}^0$ rate |
|-------|-----------------|-----------------|-----------------|
| 1/4   | 0.5040          | 0.2726          | 0.5597          |
| 1/8   | 0.2816          | 0.8389          | 0.9200          | 0.7900          |
| 1/16  | 0.1458          | 0.9497          | 0.0262          | 1.8124          | 0.1439          | 1.1696          |
| 1/32  | 0.0737          | 0.9843          | 0.0066          | 0.1980          | 0.0615          | 1.2264          |
| 1/64  | 0.0372          | 0.9864          | 0.0016          | 2.0444          | 0.0300          | 1.0356          |

We test our theoretical results with the convergence of errors $e_{h,u}^0$, $e_{h,u}^1$, and $e_{h,p}^0$. Five kinds of mesh size are chosen as $h = 1/4, 1/8, 1/16, 1/32, 1/64$. The errors and their convergence orders for the velocity in $L^2$ and $H^1$ norms and the pressure in $L^2$ norm are shown in Table 1. We can see that the convergence orders of the errors are optimal. Namely, the second order for $e_{h,u}^1$, and the first order for $e_{h,u}^0$ and $e_{h,p}^0$. These results support our theoretical results.

5.2. **Example 2: An interface problem.** We now consider a problem where the pressure is continuous and the velocity field is discontinuous on the interface due to different fluid viscosities. Let $\Omega = [-1,1] \times [-1,1]$, the interface is a circle centered in $(0,0)$ with the radius 0.5. The interface separates domain $\Omega$ into two regions $\Omega_1 = \{(x,y) : x^2 + y^2 > 0.25\}$ and $\Omega_2 = \{(x,y) : x^2 + y^2 < 0.25\}$. The Dirichlet boundary conditions on $\partial \Omega$ are chosen such that the exact solution of the Stokes equation is given by

$$u = \begin{cases} (y(x^2+y^2-0.25), -x(x^2+y^2-0.25)) & (x,y) \in \Omega_1, \\ (y(x^2+y^2-0.25), -x(x^2+y^2-0.25)) & (x,y) \in \Omega_2, \end{cases}$$

and

$$p = 4(y^2 - x^2),$$

then the right hand side $f = (-8x-8y, 8x+8y)^T$ and the jump conditions $[u] = 0, [\mu n - \mu \nabla u \cdot n] = 0$ on the interface. The viscosity is taken by $\mu_1 = 1000$ and $\mu_2 = 1$. Five kinds of mesh size are chosen as $h = 1/4, 1/8, 1/16, 1/32, 1/64$. The results are
shown in Table 2. It is observed that the convergence orders for \(e_{h,u}^1, e_{h,u}^0\) and \(e_{h,p}^0\) are optimal, which demonstrate the theoretical results.

### Table 2. Errors for an interface problem with \(\mu_1 = 1000\) and \(\mu_2 = 1\).

| \(h\)  | \(e_{h,u}^1\) | rate | \(e_{h,u}^0\) | rate | \(e_{h,p}^0\) | rate |
|--------|---------------|------|---------------|------|---------------|------|
| 1/4    | 0.5115        |      | 0.2754        |      | 0.5438        |      |
| 1/8    | 0.2850        | 0.8438 | 0.0913        | 1.5928 | 0.2976        | 0.8697 |
| 1/16   | 0.1463        | 0.9620 | 0.0253        | 1.8515 | 0.1503        | 0.9855 |
| 1/32   | 0.0738        | 0.9872 | 0.0063        | 2.0057 | 0.0641        | 1.2294 |
| 1/64   | 0.0373        | 0.9844 | 0.0016        | 1.9773 | 0.0302        | 1.0858 |

### Table 3. Errors for an interface problem with \((\mu_1, \mu_2) = (10, 1), (10^2, 1), \cdots, (10^5, 1)\) and fixed mesh \(h = 1/32\).

| \(\mu_1\) | \(\mu_2\) | \(e_{h,u}^1\) | \(e_{h,u}^0\) | \(e_{h,p}^0\) |
|-----------|-----------|---------------|---------------|---------------|
| 1E + 01   | 1         | 0.0738        | 0.0063        | 0.0598        |
| 1E + 02   | 1         | 0.0737        | 0.0066        | 0.0612        |
| 1E + 03   | 1         | 0.0737        | 0.0066        | 0.0615        |
| 1E + 04   | 1         | 0.0737        | 0.0066        | 0.0615        |
| 1E + 05   | 1         | 0.0737        | 0.0066        | 0.0615        |

For the above interface problem, the second numerical test is designed to investigate the influence of the jump of the different viscosities on the errors. To do this, we fix the mesh size \(h = 1/32\). The errors for velocity and pressure are listed in Table 3 with \((\mu_1, \mu_2) = (10, 1), (10^2, 1), \cdots, (10^5, 1)\). It indicates that the errors converge as \(\frac{\mu_{\max}}{\mu_{\min}} \to \infty\), which means that they are all independent of the jump of the viscosities.

6. **Conclusions.** In this paper, we have introduced a nonconforming Nitsche’s extended finite element method which gives a way to accurately solve the Stokes interface problems with different viscosities. The method allows for discontinuities across the interface, namely, the interface can be intersected by the mesh. Harmonic weighted averages and arithmetic averages are used. Furthermore, the extra stabilization terms for both velocity and pressure are added such that the inf-sup condition holds for the nonconforming-\(P_1/P_0\) pair. It is shown that the convergence orders of errors are optimal. Moreover, the errors do not depend on the jump of the viscosities and the position of the interface with respect to the mesh. Numerical results for both the continuous problem and the interface problem in two dimensions have been given to support our theoretical results.

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