On regular irreducible components of module varieties over string algebras

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Abstract

We determine the regular irreducible components of the variety $\text{mod}(\mathcal{A},d)$, where $\mathcal{A} = kQ/I$ is a string algebra and $I$ is generated by a set of paths of length two. Our case is among the first examples of descriptions of irreducible components, aside from hereditary, tubular (see [4]) and Gelfand-Ponomarev algebras (see [7]).

1 Introduction

Fix an algebraically closed field $k$, and let $\mathcal{A}$ be a finite dimensional associative $k$-algebra with a unit. By $\text{mod}(\mathcal{A},d)$ we denote the affine subvariety of $\text{Hom}_k(\mathcal{A}, M_d(k))$ of $k$-algebra homomorphisms $\mathcal{A} \rightarrow M_d(k)$, where $M_d(k)$ is the algebra of $d \times d$ matrices over $k$. The algebraic group $\text{GL}_d(k)$ of invertible $d \times d$ matrices acts on $\text{mod}(\mathcal{A},d)$ by conjugation. The $\text{GL}_d(k)$-orbits in $\text{mod}(\mathcal{A},d)$ are in one-to-one correspondence with the isomorphism classes of $d$-dimensional left $\mathcal{A}$-modules.

We recall the definition of a string algebra and describe its finite dimensional modules. A quiver is a tuple $Q = (Q_0, Q_1, s, t)$ consisting of a finite set of vertices $Q_0$, a finite set of arrows $Q_1$ and maps $s, t : Q_1 \rightarrow Q_0$. We call $s(\alpha)$ the source and $t(\alpha)$ the target of the arrow $\alpha \in Q_1$. A path in $Q$ is either $1_u$, where $u$ is a vertex of $Q$, or a finite sequence $\alpha_1 \ldots \alpha_n$ of arrows of $Q$, satisfying $s(\alpha_i) = t(\alpha_{i+1})$. The path algebra $kQ$ is the $k$-algebra with $k$-basis $\{p : p \text{ is a path in } Q\}$, where the product of two paths $p, q$ is the concatenation $pq$ in case $s(p) = t(q)$ and zero otherwise. A string algebra $\mathcal{A}$ over $k$ is an algebra of the form $\mathcal{A} = kQ/I$ for a quiver $Q$ and an ideal $I$ in $kQ$ satisfying the following:
• $I$ is admissible, i.e. there is an $n \in \mathbb{N}$ with $(kQ^+)^n \subseteq I \subseteq (kQ^+)^2$, where $kQ^+$ is the ideal generated by all arrows.

• $I$ is generated by a set of paths in $Q$.

• There are at most two arrows with source $u$ for any vertex $u$ of $Q$.

• There are at most two arrows with target $u$ for any vertex $u$ of $Q$.

• For any arrow $\alpha$ of $Q$ there is a most one arrow $\beta$ with $s(\alpha) = t(\beta)$ and $\alpha \beta \notin I$.

• For any arrow $\beta$ of $Q$ there is a most one arrow $\alpha$ with $s(\alpha) = t(\beta)$ and $\alpha \beta \notin I$.

Fix a string algebra $A = kQ/I$. The opposite quiver $Q^o$ is $(Q_0, Q_1^{-1}, s, t)$,
where $Q_1^{-1} = \{ \alpha^{-1} : \alpha \in Q_1 \}$, $s(\alpha^{-1}) := t(\alpha)$ and $t(\alpha^{-1}) := s(\alpha)$. A string (of $A$) is either $1_u$, where $u$ is a vertex of $Q$, or a finite sequence $\alpha_1 \cdots \alpha_n$ of arrows of $Q$ and $Q^o$, satisfying $s(\alpha_i) = t(\alpha_{i+1})$ and $\alpha_i \neq \alpha_{i+1}^{-1}$ such that none of its partial strings $\alpha_1 \cdots \alpha_j$ nor its inverse $\alpha_j^{-1} \cdots \alpha_1^{-1}$ belongs to $I$. By $W$ we denote the set of all strings. Let $c$ be a string. We set $s(c) := t(c) := u$ in case $c = 1_u$ and $s(c) := s(\alpha_n)$ and $t(c) = t(\alpha_1)$, in case $c = \alpha_1 \cdots \alpha_n$. We say that $c$ starts/ends with an (inverse) arrow if $c = \alpha_1 \cdots \alpha_n$ and $\alpha_n \in Q_1$ ($\in Q_1^{-1}$), $\alpha_1 \in Q_1$ ($\in Q_1^{-1}$), respectively. We define the inverse string $c^{-1}$ of $c$ by $c^{-1} := 1_u$ in case $c = 1_u$ and $c^{-1} := \alpha_n^{-1} \cdots \alpha_1^{-1}$ if $c = \alpha_1 \cdots \alpha_n$. The length $l(c)$ of $c$ is 0 if $c = 1_u$ and $n$ if $c = \alpha_1 \cdots \alpha_n$. We call the strings of length 0 trivial. Note that the concatenation $cd$ of two strings $c$ and $d$ is not necessarily a string.

**Remark 1.1.**

• A string $c$ is trivial if and only if $c = c^{-1}$.

• If $I$ is generated by paths of length two, then $c = \alpha_1 \cdots \alpha_n$ is a string if and only if $\alpha_i \alpha_{i+1}$ is a string for all $1 \leq i \leq n - 1$.

By $Ld(c) := \{ c' : c = c'c'' \}$ we denote the set of leftdivisors of $c$. For any string $c$ we define the string module $M(c)$ with basis $\{ e_{c'} : c' \in Ld(c) \}$ by

$$p \cdot e_{c'} = \begin{cases} e_{c'p^{-1}} & \text{if } c'p^{-1} \in Ld(c), \\ e_{c''} & \text{if } c' = c''p, \\ 0 & \text{otherwise}, \end{cases}$$

2
for any path \( p \) in \( Q \). For an arrow \( \alpha \) and a vertex \( u \) we thus have

\[
e_{\alpha_1\cdots\alpha_{i-1}} \xrightarrow{\alpha} e_{\alpha_1\cdots\alpha_i} \quad \text{if } \alpha = \alpha_i^{-1},
\]

\[
e_{\alpha_1\cdots\alpha_{i-1}} \xrightarrow{\alpha} e_{\alpha_1\cdots\alpha_i} \quad \text{if } \alpha = \alpha_i,
\]

\[
1_u \cdot e_{\alpha_1\cdots\alpha_i} = \begin{cases} e_{\alpha_1\cdots\alpha_i} & \text{if } u = s(\alpha_i), \\ 0 & \text{if } u \neq s(\alpha_i). \end{cases}
\]

By [2] string modules are indecomposable and two string modules \( M(c) \) and \( M(d) \) are isomorphic if and only if \( c = d \) or \( c = d^{-1} \). An isomorphism \( M(c) \rightarrow M(c^{-1}) \) is given by sending \( e_c \) to \( e_{c'}^{-1} \) for \( c' \in \text{Ld}(c) \) with \( c = c'c'' \). We will refer to such an isomorphism as "the isomorphism from \( M(c) \) to \( M(c^{-1}) \)."

Aside from string modules there is another type of indecomposable (finite dimensional) \( A \)-modules, the band modules. To make it easier to describe degenerations (see section [3]) we also define quasi-band modules, which are a generalization of band modules.

A quasi-band \( (b,m) \) is a map \( b: \mathbb{Z} \rightarrow Q \cup Q^{-1} \) together with an integer \( m \geq 1 \) such that \( b(i) = b(i + m) \) for all \( i \in \mathbb{Z} \) and \( b(i)b(i+1)\cdots b(i+n) \) is a string for all \( i \in \mathbb{Z} \) and all \( n \geq 0 \). Frequently we will just write \( (b,m) = b(1)\cdots b(m) \). A quasi-band \( (b,m) \) is called a band provided \( (b,m') \) is not a quasi-band for any \( 0 < m' < m \). For any quasi-band \( (b,m) \) and any \( \phi \in \text{Aut}_k(V) \), where \( V \) is a finite dimensional \( k \)-vector space, we define the quasi-band module \( M(b,m,\phi) \) in the following way. First we define an (infinite dimensional) \( A \)-module \( M(b) \) with basis \( \{e_i : i \in \mathbb{Z} \} \) by

\[
p \cdot e_i = \begin{cases} e_j & \text{if there is a } j \geq i \text{ such that } b(i)p^{-1} = b(i)\cdots b(j), \\ e_{j-1} & \text{if there is a } j \leq i + 1 \text{ such that } pb(i+1) = b(j)\cdots b(i+1), \\ 0 & \text{otherwise}, \end{cases}
\]

for any path \( p \) in \( Q \). Note that we write \( pb(i+1) = b(j)\cdots b(i+1) \) instead of \( p = b(j)\cdots b(i) \) in order to include trivial paths. For an arrow \( \alpha \) and and a vertex \( u \) we thus have

\[
e_{i-1} \xrightarrow{\alpha} e_i \quad \text{if } \alpha = b(i)^{-1},
\]

\[
e_{i-1} \xrightarrow{\alpha} e_i \quad \text{if } \alpha = b(i),
\]

\[
1_u \cdot e_i = \begin{cases} e_i & \text{if } u = s(b(i)), \\ 0 & \text{if } u \neq s(b(i)). \end{cases}
\]
We define an \( \mathcal{A} \)-module structure on \( V \otimes_k M(b) \) by setting
\[
p \cdot (v \otimes w) := v \otimes (p \cdot w)
\]
for any path \( p \) in \( Q \). Finally we set
\[
M(b, m, \phi) := V \otimes_k M(b) / \text{span}_k \{ v \otimes e_i - \phi(v) \otimes e_{i+m} : v \in V, i \in \mathbb{Z} \}.
\]
In case \( V = k \) the automorphism \( \phi \) is given by multiplication with a \( \lambda \in k^\ast \) and we set \( M(b, m, \lambda) = M(b, m, \phi) \). We call \( M(b, m, \phi) \) a band module provided \((b, m)\) is a band and the \( k[x] \)-module defined by \( \phi \) is indecomposable.

By [2] any band module is indecomposable and two band modules \( M(b, m, \phi) \) and \( M(b', m', \phi') \) are isomorphic if and only if \( m = m' \) and one of the following holds:

- There is an \( i \in \mathbb{Z} \) with \( b(j) = b'(i + j) \) for all \( j \in \mathbb{Z} \) and \( \phi \) and \( \phi' \) are isomorphic as \( k[x] \)-modules.
- There is an \( i \in \mathbb{Z} \) with \( b(j) = b'(i - j)^{-1} \) for all \( j \in \mathbb{Z} \) and \( \phi^{-1} \) and \( \phi' \) are isomorphic as \( k[x] \)-modules.

This motivates the definition of an equivalence relation \( \sim \) for quasi-bands, defined by \((b, m) \sim (b', m')\) if \( m = m' \) and one of the following holds:

- There is an \( i \in \mathbb{Z} \) with \( b(j) = b'(i + j) \) for all \( j \in \mathbb{Z} \).
- There is an \( i \in \mathbb{Z} \) with \( b(j) = b'(i - j)^{-1} \) for all \( j \in \mathbb{Z} \).

By \([b, m]\) we denote the equivalence class of \((b, m)\) with respect to \( \sim \).

It is shown in [2] that the finite-dimensional indecomposable \( \mathcal{A} \)-modules are precisely the string and band modules up to isomorphism.

For any sequence \( S = (c_1, \ldots, c_l, (b_1, m_1), \ldots, (b_n, m_n)) \) with \( l, n \geq 0 \) consisting of strings \( c_1, \ldots, c_l \) and quasi-bands \((b_1, m_1), \ldots, (b_n, m_n)\) the family of modules \( \mathcal{F}(S) \subseteq \text{mod}(\mathcal{A}, d) \) is the image of the morphism
\[
GL_d(k) \times (k^\ast)^n \longrightarrow \text{mod}(\mathcal{A}, d)
\]
sending \((g, \lambda_1, \ldots, \lambda_n)\) to
\[
g \ast \left( \bigoplus_{i=1}^l M(c_i) \oplus \bigoplus_{j=1}^n M(b_j, m_j, \lambda_j) \right),
\]
where
\[
d = \sum_{i=1}^l \dim_k M(c_i) + \sum_{j=1}^n \dim_k M(b_j, m_j, 1).
\]
We call a subset $F$ of $\text{mod}(A, d)$ an $S$-family of strings and quasi-bands if there is a sequence $S$ of strings and quasi-bands with $F = F(S)$ and we call $F$ an $S$-family of (strings and) bands if $S$ is a sequence of (strings and) bands.

Note that a band module $M(b, \phi)$ with $\phi \in \text{GL}_p(k) = \text{Aut}_k(k^p)$ does not necessarily belong to any $S$-family of strings and quasi-bands, as $\phi$ might not be diagonalizable. But, as the set of diagonalizable matrices in $\text{GL}_p(k)$ is dense in $\text{GL}_p(k)$, we see that $M(b, \phi)$ belongs to the closure of the $S$-family $F(b, b, \ldots, b)$, which is an $S$-family of bands. Thus $\text{mod}(A, d)$ is a union of closures of $S$-families of strings and bands. As $\text{GL}_d(k)$ is irreducible, any $S$-family of strings and quasi-bands is irreducible and as there are only finitely many different $S$-families of strings and bands in $\text{mod}(A, d)$, any irreducible component of $\text{mod}(A, d)$ is the closure of an $S$-family of strings and bands.

Let $r : \text{mod}(A, d) \rightarrow \mathbb{N}$ be the function sending $X$ to

$$r(X) = \sum_{\alpha \in Q_1} \text{rank } X(\alpha).$$

We call $X \in \text{mod}(A, d)$ regular if $r(X) = d$. From the direct decomposition of $X$ into a direct sum of string and band modules we obtain that $r(X) \leq d$ for any $X \in \text{mod}(A, d)$ and that $X$ is regular if and only if $X$ is isomorphic to a direct sum of band modules. As $r$ is lower semi-continuous, we see that the regular elements of $\text{mod}(A, d)$ form an open subset of $\text{mod}(A, d)$.

Let $C$ be an irreducible component of $\text{mod}(A, d)$ such that there is a regular $X \in C$. We call such an irreducible component regular. We already know that there is a sequence of strings and bands $S$ such that the closure of $F(S)$ is $C$. Obviously $S$ has to be a sequence of bands. In order to determine the regular irreducible components of $\text{mod}(A, d)$ it suffices to solve the following problem: Given a sequence of bands $S$ with $F(S) \subseteq \text{mod}(A, d)$, determine whether the closure of $F(S)$ is an irreducible component or not.

We apply the result on decompositions of irreducible components as presented in [3] and obtain the following: Let $S = (b_1, \ldots, b_n)$ be a sequence of bands. We set $d_i := \dim_k M(b_i, 1)$ and $d = d_1 + \ldots + d_n$. The closure of $F(S)$ is an irreducible component of $\text{mod}(A, d)$ if and only if the following holds:

i) For all $i \neq j$, there are $X \in F(b_i)$ and $Y \in F(b_j)$ with $\text{Ext}^1_A(X, Y) = 0$.

ii) The closure of $F(b_i)$ is an irreducible component of $\text{mod}(A, d_i)$ for $i = 1, \ldots, n$.

Our goal is to characterize the conditions i) and ii) by combinatorial criteria on bands. For i) we have a complete solution, whereas our characterization of ii) only holds if the ideal $I$ is generated by paths of length two.
Note that the result from \cite{3} can also be applied to sequences of strings and bands in order to determine the non-regular irreducible components, but we were not able to characterize condition $ii$ for strings.

We call a pair of bands $((b, m), (c, n))$ extendable if there are $s, t \geq 1$, strings $u, v, w$ and arrows $\alpha, \beta, \gamma, \delta$ with

$$ (b, sm) = w\beta u\alpha^{-1} \quad \text{and} \quad (c, tn) = w\delta^{-1} v\gamma $$

such that

$$ (d, n + m) := (c, n)(b, m) := c(1) \cdots c(n)b(1) \cdots b(m) $$

is a quasi-band. Note that $l(w) > m, n$ is possible, which explains why $s$ and $t$ are needed. We call a pair of equivalence classes of bands $(B, C)$ extendable, if there are bands $(b, m) \in B$ and $(c, n) \in C$ such that $((b, m), (c, n))$ is extendable.

**Proposition 1.2.** Let $(b, m)$ and $(c, n)$ be bands. There are $X \in \mathcal{F}(b, m)$ and $Y \in \mathcal{F}(c, n)$ with $\text{Ext}^1_A(X, Y) = 0$ if and only if the pair $([(b, m)], [(c, n)])$ is not extendable.

Note that $\text{Ext}^1_A(X, Y) = 0$ for some $X \in \mathcal{F}(b, m)$ and $Y \in \mathcal{F}(c, n)$ implies that $\text{Ext}^1_A(\cdot, \cdot)$ vanishes generically on $\mathcal{F}(b, m) \times \mathcal{F}(c, n)$.

We call a band $(b, m)$ negligible if one of the following holds:

- There are strings $u, v, w, x, y$, arrows $\alpha, \beta, \gamma, \delta$ and an $s \geq 1$ with
  
  $$ (b, m) = w\gamma v\alpha^{-1} \quad \text{and} \quad (b, sm) = w\beta x\alpha^{-1} = w\gamma w\delta^{-1} y $$

  such that
  
  $$ (c, n) := w\gamma \quad \text{and} \quad (d, m - n) := v\alpha^{-1} $$

  are quasi-bands.

- There is a string $u$ that starts and ends with an arrow, a string $v$ that starts and ends with an inverse arrow and a string $w$ with $(b, m) = wu w^{-1} v$ such that
  
  $$ (c, m) := wu^{-1} w^{-1} v $$

  is a quasi-band.

We call an equivalence class of bands $B$ negligible if there is a band $(b, m) \in B$ which is negligible. One can show that $(B, B)$ is extendable if $B$ is negligible, but we will not use it.
Proposition 1.3. Let \((b,m)\) be a band with \(\mathcal{F}(b,m) \subseteq \text{mod}(\mathcal{A},d)\). If the closure of \(\mathcal{F}(b,m)\) is an irreducible component of \(\text{mod}(\mathcal{A},d)\), then \([(b,m)]\) is not negligible.

We do not know whether the converse holds in general, but it does in case \(I\) is generated paths of length two:

Proposition 1.4. Assume that \(I\) is generated by a set of paths of length two and let \((b,m)\) be a band with \(\mathcal{F}(b,m) \subseteq \text{mod}(\mathcal{A},d)\). If \([(b,m)]\) is not negligible, then the closure of \(\mathcal{F}(b,m)\) is an irreducible component of \(\text{mod}(\mathcal{A},d)\).

We call a sequence \(S = (b_1, \ldots, b_n)\) of bands negligible, if one of the following holds:

- \([b_i]\) is negligible for some \(1 \leq i \leq n\).
- \([(b_i), [b_j]]\) is extendable for some \(1 \leq i,j \leq n\).

Our main result is the following theorem which is just a consequence from the previous propositions.

Theorem. Let \(\mathcal{A} = kQ/I\) be a string algebra and let \(S\) be a sequence of bands with \(\mathcal{F}(S) \subseteq \text{mod}(\mathcal{A},d)\).

a) If the closure of \(\mathcal{F}(S)\) is an irreducible component of \(\text{mod}(\mathcal{A},d)\), then \(S\) is negligible.

b) If \(S\) is negligible and \(I\) is generated by paths of length two, then the closure of \(\mathcal{F}(S)\) is an irreducible component of \(\text{mod}(\mathcal{A},d)\).

If \(I\) is generated by a set of paths of length two and \(b\) is a band, then \([b]\) is negligible if and only if \([(b), [b]]\) is extendable (see Lemma 3.1 and 3.2).

From the previous theorem we thus obtain:

Corollary 1.5. Assume that \(I\) is generated by paths of length two and let \(\mathcal{F} \subseteq \text{mod}(\mathcal{A},d)\) be an \(S\)-family of bands. The closure of \(\mathcal{F}\) is an irreducible component of \(\text{mod}(\mathcal{A},d)\) if and only if there are \(X, Y \in \mathcal{F}\) with \(\text{Ext}_{\mathcal{A}}^1(X,Y) = 0\).

Note that Corollary 1.5 is not true if \(I\) is not generated by paths of length two. Indeed, consider the algebra \(\Lambda = k[\alpha, \beta]/(\alpha^3, \beta^3, \alpha\beta)\) and the band \(\alpha^{-1}\beta\). The closure of \(\mathcal{F}(\alpha^{-1}\beta)\) is an irreducible component of \(\text{mod}(\Lambda,2)\), as there are no other \(S\)-families of band modules in \(\text{mod}(\Lambda,2)\). On the other hand, \(\text{Ext}_{\Lambda}^1(X,Y)\) does not vanish for any \(X, Y \in \mathcal{F}(\alpha^{-1}\beta)\), as one can
easily construct a short exact sequence $0 \rightarrow X \rightarrow Z \rightarrow Y \rightarrow 0$ for some $Z \in \mathcal{F}(\alpha^{-2}\beta^2)$.

If $I$ is generated by paths of length two, there is a simple formula for the dimension of a regular irreducible component: We call a vertex $u \in Q_0$ gentle (w.r.t. $I$) if it satisfies the following:

- For any arrow $\alpha$ of $Q$ with $s(\alpha) = u$ there is a most one arrow $\beta$ with $s(\alpha) = t(\beta)$ and $\alpha \beta \in I$.

- For any arrow $\beta$ of $Q$ with $t(\beta) = u$ there is a most one arrow $\alpha$ with $s(\alpha) = t(\beta)$ and $\alpha \beta \in I$.

We call $A$ a gentle algebra if any vertex of $Q$ is gentle and the ideal $I$ is generated by paths of length two.

**Proposition 1.6.** Assume that $I$ is generated by paths of length two and let $S$ be a sequence of bands such that the closure $\mathcal{F}(S)$ is an irreducible component of $\text{mod}(A,d)$. The dimension of $\mathcal{F}(S)$ is given by the formula

$$\dim \mathcal{F}(S) = d^2 - \sum_{u \in Q_0 \text{ non-gentle}} \dim_k \text{Hom}_A(X, M(1_u)) \dim_k \text{Hom}_A(M(1_u), X),$$

for any $X \in \mathcal{F}(S)$. In particular, the dimension of any regular irreducible component of $\text{mod}(A,d)$ is $d^2$ provided $A$ is a gentle algebra.

The paper is organized as follows. In section 2 we recall results on homomorphisms between representations of string algebras from [5]. In section 3 we prove Proposition 1.4 and the dimension formula Proposition 1.6. Section 4 is devoted to explicit inclusions among closures of $S$-families of bands and the proof of Proposition 1.3. In section 5 we study extensions and prove Proposition 1.2.
2 Homomorphisms

We show that $S$-families of band modules can be separated by hom-conditions using string modules (see Proposition 2.5), a result we need for the proof of Proposition 1.4. We recall a basis of homomorphism spaces between representations of string algebras worked out in [5].

2.1 Substring morphisms for a string

Let $c$ be a string. A substring of $c$ is a triple $(c_1, c_2, c_3)$ of strings with $c = c_1c_2c_3$ satisfying the following:

- $c_1$ is either trivial or it starts with an inverse arrow ($c_1 = c'_1 \alpha^{-1}$).
- $c_3$ is either trivial or it ends with an arrow ($c_3 = \alpha c'_3$).

By $\text{sub}(c)$ we denote the set of substrings of $c$.

For each $(c_1, c_2, c_3) \in \text{sub}(c)$ we define the homomorphism

$$\iota_{c_2,(c_1,c_2,c_3)} : M(c_2) \longrightarrow M(c)$$

by sending $e_d$ to $e_{c_1d}$ for $d \in \text{Ld}(c_2)$. We call such a morphism a substring morphism. For a string $d$ we set

$$\text{sub}(d, c) := \{(c_1, c_2, c_3) \in \text{sub}(c) : c_2 \in \{d, d^{-1}\}\}.$$

2.2 Factorstring morphisms for a string

Let $c$ be a string. A factorstring of $c$ is a triple $(c_1, c_2, c_3)$ of strings with $c = c_1c_2c_3$ satisfying the following:

- $c_1$ is either trivial or it starts with an arrow ($c_1 = c'_1 \alpha$).
- $c_3$ is either trivial or it ends with an inverse arrow ($c_3 = \alpha^{-1} c'_3$).

By $\text{fac}(c)$ we denote the set of substrings of $c$.

For each $(c_1, c_2, c_3) \in \text{fac}(c)$ we define the homomorphism

$$\pi_{c_2,(c_1,c_2,c_3)} : M(c) \longrightarrow M(c_2)$$

by sending $e_d$ to

$$\begin{cases} e_{d'} & \text{if } d = c_1d' \in \text{Ld}(c_1c_2) \\ 0 & \text{otherwise}. \end{cases}$$

We call such a morphism a factorstring morphism. For a string $d$ we set

$$\text{fac}(d, c) := \{(c_1, c_2, c_3) \in \text{fac}(c) : c_2 \in \{d, d^{-1}\}\}.$$
2.3 Winding and unwinding morphisms

Let \((b, m)\) be a quasi-band. For \(s \geq 1\) and \(\lambda \in k^*\) we define the winding morphism

\[
w_{(b,m),s,\lambda} : M(b, sm, \lambda^s) \rightarrow M(b, m, \lambda)
\]

by sending \(1 \otimes e_i\) to \(1 \otimes e_i\) for \(i = 0, \ldots, sm - 1\).

Dually, the unwinding morphism

\[
u_{(b,m),s,\lambda} : M(b, m, \lambda) \rightarrow M(b, sm, \lambda^s),
\]

sends \(1 \otimes e_i\) to

\[
\sum_{j=0}^{s-1} \lambda^j \otimes e_{i+jm}
\]

for \(i = 0, \ldots, m - 1\).

2.4 Substring morphisms for a quasi-band

Let \((b, m)\) be a quasi-band and \(c\) a string. We define the set

\[
\text{sub}_1(c, (b, m)) := \{1 \leq i \leq m : b(i) \cdots b(i + l(c)) = b(i)c, \ b(i)^{-1}, b(i + l(c) + 1) \in Q_1\}
\]

Note that we write \(b(i) \cdots b(i + l(c)) = b(i)c\) instead of \(b(i+1) \cdots b(i + l(c)) = c\) in order to include the case \(l(c) = 0\). For any \(i \in \text{sub}_1(c, (b, m))\) and any \(\lambda \in k^*\) we define a morphism

\[
\iota_{i,c,(b,m),\lambda} : M(c) \rightarrow M(b, m, \lambda)
\]

as a composition

\[
M(c) \xrightarrow{f} M(b, sm, \lambda^s) \xrightarrow{w_{(b,m),s,\lambda}} M(b, m, \lambda)
\]

for some integer \(s \geq 1\) chosen in such a way that \((b, sm) = d_1 \alpha^{-1} c \beta d_2\) for some arrows \(\alpha, \beta\) and some strings \(d_1, d_2\) with \(l(d_1) = i - 1\), where the morphism \(f\) sends \(e_d\) to \(1 \otimes e_{i+l(d)}\) for \(d \in Ld(c)\).
Note that \( \iota_{i,c,(b,m),\lambda} \) does not depend on the choice of \( s \).

We set \( \text{sub}^{-1}(c,(b,m)) := \text{sub}(c^{-1},(b,m)) \). Note that \( \text{sub}^{-1}(c,(b,m)) = \text{sub}_1(c,(b,m)) \) if \( c \) is trivial. For each \( i \in \text{sub}^{-1}(c,(b,m)) \) we define a morphism

\[
\iota_{i,c,(b,m),\lambda} : M(c) \to M(b,m,\lambda),
\]

as the composition

\[
M(c) \xrightarrow{\sim} M(c^{-1}) \xrightarrow{\iota_{i,c,(b,m),\lambda}} M(b,m,\lambda),
\]

where the first morphism is the isomorphism from \( M(c) \) to \( M(c^{-1}) \).

We have thus defined morphisms \( \iota_{i,c,(b,m),\lambda} \), called substring morphisms, for any \( \lambda \in k^* \) and any

\[
i \in \text{sub}(c,(b,m)) := \text{sub}_1(c,(b,m)) \cup \text{sub}^{-1}(c,(b,m)).
\]

Whereas substring morphisms for strings are always injective, substring morphisms for quasi-bands are not necessarily.

Dually we define the factorstring morphisms for quasi-bands:

### 2.5 Factorstring morphisms for a quasi-band

Let \((b,m)\) be a quasi-band and \(c\) a string. We define the set

\[
\text{fac}_1(c,(b,m)) := \{1 \leq i \leq m : b(i) \cdots b(i + l(c)) = b(i)c, b(i), b(i + l(c) + 1)^{-1} \in Q_1\}
\]

For any \( i \in \text{fac}_1(c,(b,m)) \) and any \( \lambda \in k^* \) we define a morphism

\[
\pi_{i,c,(b,m),\lambda} : M(b,m,\lambda) \to M(c)
\]
as a composition

\[
M(b, m, \lambda) \xrightarrow{u(b, m), \lambda} M(b, sm, \lambda^s) \xrightarrow{f} M(c)
\]

for some integer \(s \geq 1\) chosen in such a way that \((b, sm) = d_1\alpha c\beta^{-1}d_2\) for some arrows \(\alpha, \beta\) and some strings \(d_1, d_2\) with \(l(d_1) = i - 1\), where \(f\) is the morphism that sends \(1 \otimes e_j\) to 0 if either \(0 \leq j < i\) or \(i + l(c) < j < sm\), and \(1 \otimes e_j\) to \(e_{d_i}\) if \(i \leq j \leq i + l(c)\), where \(d_i\) is the leftdivisor of \(c\) of length \(j - i\).

We set \(\text{fac}_{-1}(c, (b, m)) := \text{fac}(c^{-1}, (b, m))\). For each \(i \in \text{fac}_{-1}(c, (b, m))\) we obtain a morphism

\[
\pi_{i, c, (b, m), \lambda} : M(b, m, \lambda) \rightarrow M(c)
\]

by identifying \(M(c)\) and \(M(c^{-1})\) just as above.

We have thus defined morphisms \(\pi_{i, c, (b, m), \lambda}\), called factorstring morphisms, for any \(\lambda \in k^*\) and any \(i \in \text{fac}(c, (b, m)) := \text{fac}_1(c, (b, m)) \cup \text{fac}_{-1}(c, (b, m))\).

### 2.6 Morphisms between string and band modules

In this section we will frequently use the abbreviation

\[
[X, Y] := \dim_k \text{Hom}_A(X, Y)
\]

for \(A\)-modules \(X, Y\). The following three propositions are reformulations of results from [5].

**Proposition 2.1.** Let \(M(b, m, \lambda)\) be a band module and \(M(c)\) a string module. The morphisms

\[
\iota_{d, x} \circ \pi_{i, d, (b, m), \lambda} : M(b, m, \lambda) \rightarrow M(c),
\]

where \(d\) is a string of length at most \(l(c)\), \(x \in \text{sub}(d, c)\) and \(i \in \text{fac}(d, (b, m))\), form a basis of \(\text{Hom}_A(M(b, m, \lambda), M(c))\). In particular,

\[
[M(b, m, \lambda), M(c)] = \sum_{d \in W, l(d) \leq l(c)} \sharp \text{fac}(d, (b, m)) \sharp \text{sub}(d, c).
\]

**Proposition 2.2.** Let \(M(b, m, \lambda)\) be a band module and \(M(c)\) a string module. The morphisms

\[
\iota_{i, d, (b, m), \lambda} \circ \pi_{d, x} : M(c) \rightarrow M(b, m, \lambda),
\]

for some integer \(s \geq 1\) chosen in such a way that \((b, sm) = d_1\alpha c\beta^{-1}d_2\) for some arrows \(\alpha, \beta\) and some strings \(d_1, d_2\) with \(l(d_1) = i - 1\), where \(f\) is the morphism that sends \(1 \otimes e_j\) to 0 if either \(0 \leq j < i\) or \(i + l(c) < j < sm\), and \(1 \otimes e_j\) to \(e_{d_i}\) if \(i \leq j \leq i + l(c)\), where \(d_i\) is the leftdivisor of \(c\) of length \(j - i\).
where $d$ is a string of length at most $l(c)$, $x \in \text{fac}(d, c)$ and $i \in \text{sub}(d, (b, m))$, form a basis of $\text{Hom}_A(M(c), M(b, m, \lambda))$. In particular,

$$[M(c), M(b, m, \lambda)] = \sum_{d \in W, l(d) \leq l(c)} \sharp \text{fac}(d, c) \sharp \text{sub}(d, (b, m)).$$

**Proposition 2.3.** Let $M(b, m, \lambda)$ and $M(c, n, \mu)$ be band modules. The morphisms

$$\iota_{j,d,(c,n),\mu} \circ \pi_{i,d,(b,m),\lambda} : M(b, m, \lambda) \rightarrow M(c, n, \mu),$$

where $d$ is a string, $j \in \text{sub}(d, (c, n))$ and $i \in \text{fac}(d, (b, m))$ (together with an isomorphism in case $M(b, m, \lambda)$ and $M(c, n, \mu)$ are isomorphic) form a basis of $\text{Hom}_A(M(b, m, \lambda), M(c, n, \mu))$.

As an example, we present a result which we will need in section 5.

**Lemma 2.4.** Let $M(b, m, \lambda)$ and $M(c, n, \mu)$ be band modules, $d$ a string, $j \in \text{sub}(d, (c, n))$ and $i \in \text{fac}(d, (b, m))$. The morphism

$$\iota_{j,d,(c,n),\mu} \circ \pi_{i,d,(b,m),\lambda} : M(b, m, \lambda) \rightarrow M(c, n, \mu)$$

is injective if $m \leq l(d) < n + m$ and $m < n$, and it is surjective if $n \leq l(d) < m + n$ and $n < m$.

Note that Lemma 2.4 may become wrong if we drop the condition $l(d) < m + n$.

**Proof.** We may assume that $j = n$ and $i = m$. Up to duality it suffices to show that the morphism

$$f := \iota_{j,d,(c,n),\mu} \circ \pi_{i,d,(b,m),\lambda}$$

is injective if $m \leq l(d) < n + m$ and $m < n$. Let $A$ be the matrix of $f$ with respect to the bases $1 \otimes e_0, \ldots, 1 \otimes e_{m-1}$ of $M(b, m, \lambda)$ and $1 \otimes e_0, \ldots, 1 \otimes e_{n-1}$ of $M(c, n, \mu)$. To show that $f$ is injective, we list the possible forms of $A$ depending on the relation between $n$, $m$ and $l(d)$:

If $l(d) < n$, then $A$ is of the form

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix},$$

where $A_1 \in \text{Mat}(m \times m, k)$ is the identity matrix and $A_2 \in \text{Mat}(n - m \times m, k)$. From now on we assume that $n \leq l(d)$. If $2m \leq n$, then $A$ is of the form

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix},$$

13
where $A_1, A_2 \in \text{Mat}(m \times m, k)$, $A_3 \in \text{Mat}(n - 2m \times m, k)$ and $A_2 = \lambda \cdot 1_{m \times m}$ is a multiple of the identity matrix. Finally, we assume that $n < 2m$. We decompose $A = B + C$, where

\[
B = \begin{pmatrix}
1_{n-m \times n-m} & 0_{n-m \times 2m-n} \\
0_{2m-n \times n-m} & 1_{2m-n \times 2m-n} \\
\lambda \cdot 1_{n-m \times n-m} & 0_{n-m \times 2m-n}
\end{pmatrix},
\]

\[
C = \begin{pmatrix}
0_{2m-n \times n-m} & C_1 \\
C_2 & 0_{n-m \times 2m-n} \\
0_{n-m \times n-m} & 0_{n-m \times 2m-n}
\end{pmatrix}
\]

and $C_1$ and $C_2$ are diagonal matrices. Note that the sizes of the blocks in $B$ and $C$ are not necessarily the same. Now we see that $A$ is of the form

\[
\begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22} \\
A_{31} & A_{32}
\end{pmatrix},
\]

where

- $A_{11}, A_{31} \in \text{Mat}(n - m \times n - m, k)$,
- $A_{12}, A_{32} \in \text{Mat}(n - m \times 2m - n)$,
- $A_{21} \in \text{Mat}(2m - n \times n - m, k)$,
- $A_{22} \in \text{Mat}(2m - n \times 2m - n)$,
- $A_{22}$ is upper triangular and all its entries on the diagonal are 1,
- $A_{31} = \lambda \cdot 1_{n-m \times n-m}$ and
- $A_{32}$ is zero.

The following proposition shows that the functions

\[
[M(c), -], [-, M(c)]: \text{mod}(\mathcal{A}, d) \longrightarrow \mathbb{N}
\]

for $c \in \mathcal{W}$ separate $\mathcal{S}$-families of band modules.

**Proposition 2.5.** Let $S = (b_1, \ldots, b_m)$ and $T = (c_1, \ldots, c_n)$ be sequences of bands with $\mathcal{F}(S), \mathcal{F}(T) \subseteq \text{mod}(\mathcal{A}, d)$. If

\[
[M(c), X] = [M(c), Y] \text{ and } [X, M(c)] = [Y, M(c)]
\]

for any string $c$, $X \in \mathcal{F}(S)$ and $Y \in \mathcal{F}(T)$, then

\[
\mathcal{F}(S) = \mathcal{F}(T).
\]
We do not claim that Proposition 2.5 holds for sequences of quasi-bands.

Let $X, Y$ be $\mathcal{A}$-modules. Clearly the dimension vectors of $X$ and $Y$ coincide (i.e. $\dim_k \operatorname{im} X(1_u) = \dim_k \operatorname{im} Y(1_u)$ for all $u \in Q_0$) if and only if $[P, X] = [P, Y]$ for any projective $\mathcal{A}$-module $P$ if and only if $[X, J] = [Y, J]$ for any injective $\mathcal{A}$-module $J$. Recall from [1], that

$$[U, X] - [X, \tau U] = [P_0, X] - [P_1, X],$$

where $P_1 \rightarrow P_0 \rightarrow U \rightarrow 0$ is a minimal projective presentation of an $\mathcal{A}$-module $U$ and $\tau$ denotes the Auslander-Reiten translation. Dually, if $0 \rightarrow U \rightarrow J_0 \rightarrow J_1$ is a minimal injective copresentation of $U$, then

$$[X, U] - [\tau^{-1} U, X] = [X, J_0] - [X, J_1].$$

As the Auslander-Reiten translate of a string module is either 0 or a string module (see [2]) and as all projective and injective $\mathcal{A}$-modules are string modules, we obtain the following corollary.

**Corollary 2.6.** Let $S$ and $T$ be sequences of bands with $\mathcal{F}(S), \mathcal{F}(T) \subseteq \operatorname{mod}(\mathcal{A}, d)$ and let $X \in \mathcal{F}(S)$ and $Y \in \mathcal{F}(T)$. The following are equivalent:

i) $\mathcal{F}(S) = \mathcal{F}(T)$

ii) $[M(c), X] = [M(c), Y]$ for any string $c$.

iii) $[X, M(c)] = [Y, M(c)]$ for any string $c$.

Before we can prove Proposition 2.5 we need some additional definitions and a technical lemma. For any non-trivial string $c$ and any quasi-band $(b, m)$ we set

- $\text{part}_1(c, (b, m)) := \{1 \leq i \leq m : b(i) \cdots b(i + l(c) - 1) = c\}$,
- $\text{part}_{-1}(c, (b, m)) := \text{part}_1(c^{-1}, (b, m))$ and
- $\text{part}(c, (b, m)) := \text{part}_1(c, (b, m)) \cup \text{part}_{-1}(c, (b, m))$.

We extend the definition of $\text{sub}(c, -)$, $\text{fac}(c, -)$ for a string $c$ and $\text{part}(c, -)$ for a non-trivial string $c$ to sequences of quasi-bands instead of a single quasi-band. For a sequence $S = (b_1, \ldots, b_n)$ of bands, we set

- $\text{part}(c, S) := \bigcup_{i=1}^{n} (\text{part}(c, b_i) \times \{i\}) \subseteq \mathbb{N} \times \mathbb{N}$
- $\text{sub}(c, S) := \bigcup_{i=1}^{n} (\text{sub}(c, b_i) \times \{i\}) \subseteq \mathbb{N} \times \mathbb{N}$
• $\text{fac}(c, S) := \bigcup_{i=1}^{n} (\text{fac}(c, b_i) \times \{i\}) \subseteq \mathbb{N} \times \mathbb{N}$

Moreover, we define

$$[c, S] := \sum_{d \in \mathcal{W}, l(d) \leq l(c)} \sharp \text{fac}(d, c) \sharp \text{sub}(d, S)$$

and

$$[S, c] := \sum_{d \in \mathcal{W}, l(d) \leq l(c)} \sharp \text{fac}(d, S) \sharp \text{sub}(d, c).$$

As direct consequence of Proposition 2.1 and Proposition 2.2 we obtain

**Corollary 2.7.** Let $S$ be a sequence of bands and $X \in \mathcal{F}(S)$. For any string $c$ we have

$$[c, S] = [M(c), X] \quad \text{and} \quad [S, c] = [X, M(c)].$$

We come to the technical lemma.

**Lemma 2.8.** Let $S$ and $T$ be sequences of bands with rank $X(\alpha) = \text{rank} Y(\alpha)$ for any arrow $\alpha$, $X \in \mathcal{F}(S)$ and $Y \in \mathcal{F}(T)$ and let $N \in \mathbb{N}$. If $[c, S] = [c, T]$ and $[S, c] = [T, c]$ for any string $c$ of length at most $N$, then $\sharp \text{part}(d, S) = \sharp \text{part}(d, T)$ for any non-trivial string $d$ of length at most $N + 2$.

**Proof.** It follows from Corollary 2.4 that $\sharp \text{sub}(c, S) = \sharp \text{sub}(c, T)$ and $\sharp \text{fac}(c, S) = \sharp \text{fac}(c, T)$ for any string $c$ of length at most $N$. We will prove that $\sharp \text{part}(d, S) = \sharp \text{part}(d, T)$ for $1 \leq l(d) \leq N + 2$ by induction on the length of $d$. If $l(d) = 1$, we may assume that $d$ is an arrow. Let $X \in \mathcal{F}(S)$ and $Y \in \mathcal{F}(T)$. We have

$$\sharp \text{part}(d, S) = \text{rank} X(d) = \text{rank} Y(d) = \sharp \text{part}(d, T).$$

If $N + 2 \geq l(d) > 1$, then $d$ is of the form $d = d_1 c d_2$ for a (possibly trivial) string $c$ of length at most $N$ and strings $d_1, d_2$ of length one. We assume that $\sharp \text{part}(d, S) \neq \sharp \text{part}(d, T)$. By exchanging $S$ and $T$ we can assume that $\sharp \text{part}(d, S) > \sharp \text{part}(d, T)$. By the induction hypothesis we know that $\sharp \text{part}(d_1 c, S) = \sharp \text{part}(d_1 c, T)$ and thus

$$\sharp \text{part}(d_1 c, T) = \sharp \text{part}(d_1 c, S)$$

$$\geq \sharp \text{part}(d_1 c d_2, S)$$

$$> \sharp \text{part}(d_1 c d_2, T)$$

This shows that part$(d_1 c, T) - \text{part}(d_1 c d_2, T)$ is non-empty, which implies that there must be a string $d_3 \neq d_2$ of length one such that $d_1 c d_3$ is a string.
As $d_1c$ is non-trivial, there is a most one arrow $\alpha$ such that $d_1c\alpha$ is a string and at most one arrow $\beta$ such that $d_1c\beta^{-1}$ is a string. Consequently, such arrows $\alpha$ and $\beta$ exist and satisfy $\{\alpha, \beta^{-1}\} = \{d_2, d_3\}$. We obtain

\[
\sharp \text{part}(d_1 cd_2, S) + \sharp \text{part}(d_1 cd_3, S) = \sharp \text{part}(d_1 c, S) = \sharp \text{part}(d_1 cd_2, T) + \sharp \text{part}(d_1 cd_3, T)
\]

and thus $\sharp \text{part}(d_1 cd_3, S) \neq \sharp \text{part}(d_1 cd_3, T)$. If $d_1$ is an arrow, then $\sharp \text{fac}(c, S) \neq \sharp \text{sub}(c, T)$ and if $d_1^{-1}$ is an arrow, then $\sharp \text{sub}(c, S) \neq \sharp \text{sub}(c, T)$. This gives a contradiction in any case.

Proof of Proposition 2.5. By the definition of $S$-families of band modules it suffices to show that $\# \{1 \leq i \leq m : [b_i] = [b]\} = \# \{1 \leq i \leq n : [c_i] = [b]\}$ for any band $b$. We first show that $\text{part}(d, S) = \text{part}(d, T)$ for any non-trivial string $d$. In order to apply Lemma 2.8, we need to show that $\text{rank } X(\alpha) = \text{rank } Y(\alpha)$ for any arrow $\alpha$, $X \in F(S)$ and $Y \in F(T)$. Let $\alpha$ be an arrow. Let $p$ and $q$ be the paths of maximal length such that $q\alpha$ and $\alpha p^{-1}$ are strings.

Note that $M(qp^{-1}) = P_{s(\alpha)}$, where $P_{s(\alpha)}$ is the indecomposable projective module corresponding to the vertex $s(\alpha)$, and that $M(p^{-1})$ is the cokernel of the morphism $P_{t(\alpha)} \rightarrow P_{s(\alpha)}$. Applying $\text{Hom}_A(\cdot, X)$, we obtain the exact sequence

\[
0 \longrightarrow \text{Hom}_A(M(p^{-1}), X) \overset{\partial}{\longrightarrow} \text{Hom}_A(P_{s(\alpha)}, X) \overset{\phi}{\longrightarrow} \text{Hom}_A(P_{t(\alpha)}, X) \overset{\text{im } X(1_{s(\alpha)})}{\longrightarrow} \text{Hom}_A(P_{t(\alpha)}, X) \overset{\text{im } X(1_{t(\alpha)})}{\longrightarrow} 0
\]

which shows that $\text{rank } X(\alpha) = [P_{s(\alpha)}, X] - [M(p^{-1}), X]$ and thus

\[
\text{rank } X(\alpha) = [P_{s(\alpha)}, X] - [M(p^{-1}), X] = [P_{s(\alpha)}, Y] - [M(p^{-1}), Y] = \text{rank } Y(\alpha).
\]

Applying Lemma 2.8, we obtain $\sharp \text{part}(d, S) = \sharp \text{part}(d, T)$ for any non-trivial string $d$, as desired.

Let $b = (b, l)$ be a band. For any $k \geq 1$ we define the string

\[
d_k := (b, kl) = b(1)b(2) \cdots b(kl).
\]
Clearly $\sharp \text{part}(d_k, (b, l)) = 1$ for any $k \geq 1$ and if $(c, j)$ is a band with $(c, j) \sim (b, l)$, then $\sharp \text{part}(d_k, (c, j)) = 0$ for sufficiently large $k$. We choose $K \in \mathbb{N}$, such that $\sharp \text{part}(d_K, x) = \begin{cases} 1 & \text{if } x \sim b \\ 0 & \text{otherwise.} \end{cases}$ for any $x \in \{b_1, \ldots, b_m, c_1, \ldots, c_n\}$. We obtain the desired equality $\sharp \{1 \leq i \leq m : [b_i] = [b]\} = \sharp \text{part}(d_K, S) = \sharp \text{part}(d_K, T) = \sharp \{1 \leq i \leq n : [c_i] = [b]\}$

3 Proof of Proposition 1.4 and 1.6

In this section we assume that $\mathcal{A} = kQ/I$ is a string algebra such that $I$ is generated by a set of paths of length two. For the proof of Proposition 1.4 we need the following characterization of negligibility.

Lemma 3.1. The equivalence class of a band $b = (b, m)$ is negligible if and only if the following holds: There are a string $c$ and arrows $\alpha, \beta, \gamma, \delta$ such that the sets $\text{part}(\alpha^{-1}c\beta, b)$ and $\text{part}(\gamma c\delta^{-1}, b)$ are non-empty and $\alpha^{-1}c\delta^{-1}$ and $\gamma c\beta$ are strings.

Proof. If $(b, m)$ is negligible, one can find a string $c$ and arrows $\alpha, \beta, \gamma, \delta$ such that the sets $\text{part}(\alpha^{-1}c\beta, b)$ and $\text{part}(\gamma c\delta^{-1}, b)$ are non-empty and $\alpha^{-1}c\delta^{-1}$ and $\gamma c\beta$ are strings, by a simple case-by-case analysis which we omit. Indeed, the choice $c = w$ will work in both cases.

We now assume that there are a string $c$ and arrows $\alpha, \beta, \gamma, \delta$ such that the sets $\text{part}(\alpha^{-1}c\beta, b)$ and $\text{part}(\gamma c\delta^{-1}, b)$ are non-empty and $\alpha^{-1}c\delta^{-1}$ and $\gamma c\beta$ are strings, and we want to show that $(b, m)$ is negligible. Up to replacing $(b, m)$ by an equivalent band, we may assume that $m \in \text{part}_1(\alpha^{-1}c\beta, (b, m))$. We choose $n \in \text{part}(\gamma c\delta^{-1}, (b, m))$. There are two cases to consider:

- $n \in \text{part}_1(\gamma c\delta^{-1}, (b, m))$
- $n \in \text{part}_-1(\gamma c\delta^{-1}, (b, m))$

18
If \( n \in \text{part}_1(\gamma c \delta^{-1}, (b, m)) \), we set
\[
w = c, u = b(1) \cdots b(n - 1), v = b(n + 1) \cdots b(m - 1).
\]
We have
\[
(b, m) = u \gamma v \alpha^{-1} \quad \text{and} \quad (b, sm) = w \beta x \alpha^{-1} = u \gamma w \delta^{-1} y
\]
for an integer \( s \geq 1 \) and strings \( x, y \). From now on we use that \( I \) is generated by paths of length two. As \( \gamma c \delta^{-1} \) is a string, we see that \( \gamma b(1) \) is a string and thus \( \gamma u \) is a string as well. As \( \gamma u \) and \( u \gamma \) are both strings, we obtain that \( u \gamma \) is a quasi-band. Similarly one can show that \( v \alpha^{-1} \) is a quasi-band.

We now assume that \( n \in \text{part}_{-1}(\gamma c \delta^{-1}, (b, m)) \). By the definition of the sets \( \text{part}_1 \) and \( \text{part}_{-1} \) we have
\[
b(i) = b(n + l(c) + 1 - i)^{-1}
\]
for \( 1 \leq i \leq l(c) \). Note that \( l(c) < n \), as otherwise
\[
\beta = b(l(c) + 1) = b(n)^{-1} = \delta^{-1}.
\]
Similarly we obtain that \( n + l(c) < m \). Thus the band \((b, m)\) is of the form
\[
(b, m) = cu c^{-1} v
\]
for some non-trivial strings \( u \) and \( v \) of length \( l(u) = n - l(c) \) and \( l(v) = m - n - l(c) \). As \( \beta = b(l(c) + 1) \) and \( \delta = b(n) \), we see that \( u \) starts and ends with an arrow. Similarly we obtain that \( v \) starts and ends with an inverse arrow. In order to show that \((b, m)\) is negligible, it remains to prove that
\[
(d, m) := cu^{-1} c^{-1} v
\]
is a quasi-band. As \( I \) is generated by paths of length two, it suffices to show that
\[
u^{-1} c^{-1} v \quad \text{and} \quad v c u^{-1}
\]
are strings. We show that \( v c u^{-1} \) is a string. We decompose \( v = v' \alpha^{-1} \) and \( u = u' \delta \). As
\[
v' \alpha^{-1} \quad \text{and} \quad \alpha^{-1} c \delta^{-1} \quad \text{and} \quad \delta^{-1} (u')^{-1}
\]
are strings, we obtain that
\[
v c u^{-1} = v' \alpha^{-1} c \delta^{-1} (u')^{-1}
\]
is a string. Similarly one can show that \( u^{-1} c^{-1} v \) is a string. \( \square \)
Proof of Proposition 1.4. Let \( b = (b, m) \) be a band such that the closure of \( F(b) \subseteq \text{mod}(A, d) \) is not an irreducible component of \( \text{mod}(A, d) \). We assume that \((b, m)\) is not negligible and want to obtain a contradiction.

As \( F(b) \) is irreducible it must be contained in an irreducible component \( C \), which is regular as it contains \( F(b) \). Thus there is a sequence \( S = (b_1, \ldots, b_n) \) of bands such that the closure of \( F(S) \) is \( C \). As the function \( X \mapsto \text{rank}(X(\alpha)) \) is lower semi-continuous on \( \text{mod}(A, d) \), we see that \( \text{rank}(X(\alpha)) \leq \text{rank}(Y(\alpha)) \) for any arrow \( \alpha \), any \( X \in F(b) \) and any \( Y \in F(S) \). On the other hand,
\[
\sum_{\alpha \in Q_1} \text{rank}(X(\alpha)) = r(X) = d = r(Y) = \sum_{\alpha \in Q_1} \text{rank}(Y(\alpha))
\]
and thus
\[
\sharp \text{part}(\alpha, b) = \text{rank}(X(\alpha)) = \text{rank}(Y(\alpha)) = \sharp \text{part}(\alpha, S).
\]
For any string \( c \) the functions \([M(c), -] \) and \([-1, M(c)] \) from \( \text{mod}(A, d) \) to \( \mathbb{N} \) are upper semi-continuous and thus
\[
[c, b] \geq [c, S] \text{ and } [b, c] \geq [S, c].
\]
As we know by Proposition 2.3 that strings separate \( S \)-families of bands, there is a string \( c \) of minimal length with the property that \([c, b] > [c, S] \) or \([b, c] > [S, c] \). We only examine the case \([c, b] > [c, S] \), as the other case is treated similarly. It follows from Corollary 2.1 that \( \sharp \text{sub}(c, b) > \sharp \text{sub}(c, S) \).

Thus there are arrows \( \alpha, \beta \) such that \( \alpha^{-1} c \beta \) is a string and
\[
\sharp \text{part}(\alpha^{-1} c \beta, b) > \sharp \text{part}(\alpha^{-1} c \beta, S).
\]
By Lemma 2.8 \( \sharp \text{part}(c \beta, b) = \sharp \text{part}(c \beta, S) \) and therefore there is an arrow \( \gamma \) such that \( \gamma c \beta \) is a string and
\[
\sharp \text{part}(\alpha^{-1} c \beta, b) + \sharp \text{part}(\gamma c \beta, b) = \sharp \text{part}(c \beta, b) = \sharp \text{part}(c \beta, S) = \sharp \text{part}(\alpha^{-1} c \beta, S) + \sharp \text{part}(\gamma c \beta, S).
\]
In particular \( \sharp \text{part}(\gamma c \beta, b) < \sharp \text{part}(\gamma c \beta, S) \). Similarly we find an arrow \( \delta \) such that \( \alpha^{-1} c \delta^{-1} \) is a string and \( \sharp \text{part}(\alpha^{-1} c \delta^{-1}, b) < \sharp \text{part}(\alpha^{-1} c \delta^{-1}, S) \). We obtain
\[
\sharp \text{part}(\gamma c, b) - \sharp \text{part}(\gamma c \beta, b) > \sharp \text{part}(\gamma c, S) - \sharp \text{part}(\gamma c \beta, S) \geq 0.
\]
Hence the word \( \gamma c \delta^{-1} \) has to be a string and satisfies
\[
\sharp \text{part}(\gamma c \delta^{-1}, b) = \sharp \text{part}(\gamma c, b) - \sharp \text{part}(\gamma c \beta, b) > 0.
\]
From the characterization of negligibility in Lemma 3.1 we obtain that \([b]\) is not negligible. \(\square\)
For the proof of the dimension formula Proposition 1.6 we need another lemma.

**Lemma 3.2.** Let \((b,m)\) and \((c,n)\) be bands. The pair \([(b,m), (c,n)]\) is extendable if and only if the following holds: There are a string \(d\) and arrows \(\alpha, \beta, \gamma, \delta\) such that the sets \(\text{part}(\alpha^{-1}d\beta, (b,m))\) and \(\text{part}(\gamma d\delta^{-1}, (c,n))\) are non-empty and \(\alpha^{-1}d\delta^{-1}\) and \(\gamma d\beta\) are strings.

**Proof.** Set \(w = d\) in the definition of extendability and observe that \((d,n+m)\) is a quasi-band if and only if \(\alpha^{-1}d\delta^{-1}\) and \(\gamma d\beta\) are strings. \(\Box\)

**Proof of Proposition 1.6** Let \(S = (b_1, \ldots, b_n)\) be a sequence of bands such that the closure of \(\mathcal{F}(S)\) is an irreducible component in \(\text{mod}(\mathcal{A}, d)\). From the first part of the main theorem we know that

- \([(b_i), [b_j]]\) is not extendable for \(i \neq j\) and
- \([b_i]\) is not negligible for all \(i\).

Let \(M = M(b_1, \lambda_1) \oplus \cdots \oplus M(b_n, \lambda_n) \in \mathcal{F}(S)\) such that \(M(b_1, \lambda_1), \ldots, M(b_n, \lambda_n)\) are pairwise non-isomorphic. The dimension of \(\mathcal{F}(S)\) is given by the formula

\[
\dim \mathcal{F}(S) = d^2 + n - [M, M],
\]

as \(\mathcal{F}(S)\) is an \(n\)-parameter family orbits of dimension \(d^2 - [M, M]\). Let \(N\) be the set of all tuples \((i, j, k, l, c)\) consisting of integers \(i, j, k, l\) and a string \(c\) such that \(i \in \text{fac}(c, b_k)\) and \(j \in \text{sub}(c, b_l)\). By Proposition 2.3 a basis of the space \(\text{Hom}_\mathcal{A}(M, M)\) is given by

- \(\#N\) morphisms corresponding tuples \((i, j, k, l, c)\) \(\in N\)
- \(n\) morphisms corresponding to the identities on \(M_i\) for \(i = 1, \ldots, n\).

We thus obtain

\[
\dim \mathcal{F}(S) = d^2 - \#N
\]

It remains to show that the cardinality of \(N\) is

\[
\#N = \sum_{u \in Q_0} [X, M(1_u)] [M(1_u), X]
\]

for any \(X \in \mathcal{F}(S)\). Let \((i, j, k, l, c) \in N\). There are arrows \(\alpha, \beta, \gamma, \delta\) such that the sets \(\text{part}(\alpha^{-1}c\beta, b_k)\) and \(\text{part}(\gamma c\delta^{-1}, b_l)\) are non-empty. Applying Lemma 3.2 in case \(k \neq l\) and Lemma 3.1 in case \(k = l\), we obtain that at least one
of the words $\alpha^{-1}c\delta^{-1}$ and $\gamma c\beta$ cannot be a string. This can only happen if $c$ is trivial. Let $u$ be the vertex of $Q$ with $c = 1_u$:

![Diagram](https://via.placeholder.com/150)

We apply the same lemmas once again: As the sets $\text{part}(\beta^{-1}\alpha, b_k)$ and $\text{part}(\gamma\delta^{-1}, b_l)$ are non-empty, we obtain that at least one of the words $\beta^{-1}\delta^{-1}$ and $\gamma\alpha$ cannot be a string. Therefore none of the pairs of words

- $(\delta\alpha, \gamma\beta)$
- $(\gamma\alpha, \delta\beta)$

can be a pair of strings. But this is only possible if the vertex $u$ is non-gentle. For the cardinality of $N$ we thus obtain

$$\#N = \sum_{\substack{u \in Q_0 \\text{non-gentle}}} \#\text{fac}(1_u, S) \#\text{sub}(1_u, S)$$

$$= \sum_{\substack{u \in Q_0 \\text{non-gentle}}} [S, 1_u] [1_u, S]$$

$$= \sum_{\substack{u \in Q_0 \\text{non-gentle}}} [X, M(1_u)] [M(1_u), X]$$

for any $X \in \mathcal{F}(S)$, which completes the proof. □
4 Regular components of indecomposable modules

An $A$-module $Y \in \text{mod}(A, d)$ is called a degeneration of $X \in \text{mod}(A, d)$ if $Y$ belongs to the closure of the $\text{GL}_d(k)$-orbit of $X$ in $\text{mod}(A, d)$. In that case we also say that $X$ degenerates to $Y$ and write $X \leq_{\text{deg}} Y$. We extend this notion to sequences of strings and quasi-bands: Let $S$ and $S'$ be finite sequences of strings and quasi-bands. We call $S$ and $S'$ equivalent, denoted by $S =_{\text{deg}} S'$, if $F(S) = F(S')$, and we say that $S$ degenerates to $S'$, in symbols $S \leq_{\text{deg}} S'$, if $F(S') \subseteq F(S)$. Note that $\leq_{\text{deg}}$ defines a partial order on the set of equivalence classes of finite sequences of strings and quasi-bands.

Two sequences of strings and bands

$$S = (c_1, \ldots, c_l, b_1, \ldots, b_n) \text{ and } S' = (c'_1, \ldots, c'_l, b'_1, \ldots, b'_n)$$

are equivalent if and only if $l = l'$, $n = n'$ and there are permutations $\sigma \in S_l$ and $\tau \in S_n$ satisfying

- $c'_{\sigma(i)} \in \{c_i, c_i^{-1}\}$ for $i = 1, \ldots, l$ and
- $b'_{\tau(j)} \sim b_j$ for $j = 1, \ldots, n$.

Note that this characterization might also hold for sequences of strings and quasi-bands, but we do not need it.

We establish two types of degenerations between sequences of bands, which yield a proof for Proposition 1.3.

**Proof of Proposition 1.3.** Let $(b, m)$ be a band such that the closure of

$$F(b, m) \subseteq \text{mod}(A, d)$$

is an irreducible component. If we assume that $(b, m)$ is negligible, we can apply one of the following degenerations and obtain that $F(b, m)$ is contained in the closure of another $S$-family of quasi-band modules, which is impossible as $\overline{F(b, m)}$ is an irreducible component.

The first degeneration can be described as follows: Cut off a piece of a suitable quasi-band, reverse the piece and reconnect it:

$$(b, m) = \begin{array}{c} \text{w}^{-1} \\ \text{w} \\ \text{w} \end{array} \quad \sim \sim \Rightarrow \quad (c, m) = \begin{array}{c} \text{w}^{-1} \\ \text{w} \end{array}$$
Proposition 4.1. Let \((b, m)\) be a quasi-band and assume that there is a string \(u\) that starts and ends with an arrow, a string \(v\) that starts and ends with an inverse arrow and a string \(w\) such that \((b, m) = wuw^{-1}v\) and

\[(c, m) := wuw^{-1}v^{-1}\]

is a quasi-band. Then \((c, m) <_{\text{deg}} (b, m)\).

Proof. Let \(\tilde{Q}\) be the quiver

\[
\begin{array}{ccc}
1 & \xrightarrow{\alpha_1} & 2 \\
& \xleftarrow{\alpha_4} & \\
& & \xrightarrow{\alpha_3} 3
\end{array}
\]

and \(\mathcal{V}\) be the variety of representations \(X = (X(\alpha_1), X(\alpha_2), X(\alpha_3), X(\alpha_4))\) of \(\tilde{Q}\) with dimension vector \((1, 2, 1)\), i.e.

\[
\mathcal{V} = \text{Mat}(2 \times 1, k) \times \text{Mat}(2 \times 1, k) \times \text{Mat}(1 \times 2, k) \times \text{Mat}(1 \times 2, k).
\]

For \(\lambda, \mu \in k, \lambda \neq 0\), consider the representations

\[
X_{\lambda, \mu} = \left( \begin{array}{c}
\lambda^{-1} \\
\mu
\end{array} \right), \left( \begin{array}{c}
0 \\
1
\end{array} \right), (0, 1), (-\lambda \mu, 1) \in \mathcal{V},
\]

\[
Y_\nu := \left( \begin{array}{c}
0 \\
\nu^{-1}
\end{array} \right), \left( \begin{array}{c}
1 \\
0
\end{array} \right), (0, 1), (1, 0) \in \mathcal{V}.
\]

The algebraic group \(G = k^* \times \text{GL}_2(k) \times k^*\) acts on \(\mathcal{V}\) in the usual way, i.e.

\[
(\varphi, \chi, \psi) \ast (X_1, X_2, X_3, X_4) := (\chi X_1 \varphi^{-1}, \chi X_2 \psi^{-1}, \psi X_3 \chi^{-1}, \varphi X_4 \chi^{-1}).
\]

For \(\lambda, \mu \in k^*\) we apply the base change

\[
g = (-\lambda^{-1} \mu^{-1}, \left( \begin{array}{c}
1 \\
-\lambda^{-1} \mu^{-1}
\end{array} \right), 1)
\]

to \(X_{\lambda, \mu}\) and obtain \(g \ast X_{\lambda, \mu} = Y_{-\lambda^{-1} \mu^{-1}}\). Thus \(X_{\lambda, \mu}\) belongs to a \(G\)-orbit of \(Y_\nu\) for some \(\nu \in k^*\), as long as \(\mu \neq 0\).

Let \(A\) be the set of all paths of \(Q\) of length at most one, i.e. \(A = Q_1 \cup \{1_x : x \in Q_0\}\). We identify the affine variety \(\text{mod}(A, m)\) with a subvariety of \(M_m(k)^A\). To show that \(M(b, m, \lambda)\) belongs to the closure of \(\mathcal{F}(c, m)\), we define a morphism

\[
\phi : \mathcal{V} \longrightarrow M_m(k)^A
\]

satisfying \(\phi(X_{\lambda, \mu}) \in \mathcal{F}(c, m)\) for \(\mu \neq 0\) and \(\phi(X_{\lambda, 0}) \simeq M(b, m, \lambda)\). Note that we will define \(\phi\) in such a way that \(\phi(\mathcal{V}) \subseteq \text{mod}(kQ, m)\).
Here is the definition of $\phi$: Let 

$$X = (X(\alpha_1), X(\alpha_2), X(\alpha_3), X(\alpha_4)) \in V$$

and let $Z$ be the $A$-module $Z = V \oplus M(w)^2 \oplus U$, where

$$V = \begin{cases} 
0 & \text{if } l(v) = 1 \\
M(v') & \text{if } v = \beta^{-1}v'\alpha^{-1} 
\end{cases}$$

and

$$U = \begin{cases} 
0 & \text{if } l(u) = 1 \\
M(u') & \text{if } u = \gamma u'\delta 
\end{cases}$$

We decompose $U, V$ and $M(w)^2$ as $k$-vector spaces:

$$M(w)^2 = \bigoplus_{d \in Ld(w)} W_d,$$

where $W_d = \text{span}_k\{(e_d, 0), (0, e_d)\}$,

$$V = M(v') = V_{1(v')} \oplus \cdots \oplus V_{v'}$$

in case $v = \beta^{-1}v'\alpha^{-1}$, where $V_d := \text{span}_k\{e_d\}$ for $d \in Ld(v')$ and

$$U = U_{1(u')} \oplus \cdots \oplus U_{u'}$$

in case $u = \gamma u'\delta$, where $U_d = \text{span}_k\{e_d\}$ for $d \in Ld(u')$.

We identify $M_m(k)$ with $\text{End}_k(Z)$, each $W_d$ with $k^2$ and each $U_d$ and $V_d$ with $k$ and set $\phi(X) = Z + Z_v + Z_u$, where the definition of $Z_v, Z_u \in \text{End}_k(Z)$ depends on the lengths of $u$ and $v$.

Case $v = \beta^{-1}v'\alpha^{-1}$:

$$\text{Z}_v(\alpha) = X(\alpha_1) \quad \text{W}_{1_w} \quad \text{W}_w \quad \text{U}$$

Case $v = \alpha^{-1}$:

$$\text{Z}_v(\alpha) = X(\alpha_1) \circ X(\alpha_4) \quad \text{W}_{1_w} \quad \text{W}_w \quad \text{U}$$
Case \( u = \gamma u' \delta \):

By the definition of \( \phi \) we have:

- \( \phi(X_{\lambda,0}) \simeq M(b, m, \lambda) \in \mathcal{F}(b, m) \) for any \( \lambda \in k^* \).
- \( \phi(Y_{\nu}) \in \mathcal{F}(c, m) \) for any \( \nu \in k^* \).

The morphism \( \phi \) is \( G \)-equivariant with respect to the morphism of algebraic groups \( G \rightarrow \text{GL}_m(k) \) sending \( (\varphi, \chi, \psi) \) to

\[
\begin{pmatrix}
\varphi \cdot 1_V \\
\chi \\
\vdots \\
\chi \\
\psi \cdot 1_U
\end{pmatrix}.
\]

Therefore \( \phi(X_{\lambda,\mu}) \) belongs to \( \mathcal{F}(c, m) \) for \( \mu \neq 0 \) and thus \( M(b, m, \lambda) \simeq \phi(X_{\lambda,0}) \) belongs to the closure of \( \mathcal{F}(c, m) \) for any \( \lambda \in k^* \).

To complete the proof we show that

\[
\mathcal{F}(b, m) \neq \mathcal{F}(c, m).
\]

As \( \sharp \text{sub}(w, (b, m)) \neq \sharp \text{sub}(w, (c, m)) \), there is a string \( a \) with \( [a, (b, m)] \neq [a, (c, n)] \) and thus minimum of the function

\[
[M(a), -] : \text{mod}(\mathcal{A}, d) \rightarrow \mathbb{N}
\]

on \( \mathcal{F}(b, m) \) differs from the minimum on \( \mathcal{F}(c, m) \), which implies that these two sets cannot be equal. \( \square \)
The second degeneration can be described as follows: Cut a suitable quasi-band into two pieces, and close each piece to separate quasi-bands.

\[(b, m) \sim\sim (c, n) \sim\sim (d, m - n)\]

\[
\begin{array}{cc}
\tau & \\
\alpha^{-1} & v \\
\uparrow & \\
\downarrow & \\
\gamma & \alpha^{-1} \\
\end{array}
\]

\[
\begin{array}{c}
u \\
\end{array}
\]

**Proposition 4.2.** Let \((b, m)\) be a quasi-band and assume that there are strings \(u, v, w, x, y\), arrows \(\alpha, \beta, \gamma, \delta\) and an \(s \geq 1\) with

\[(b, m) = u\gamma v\alpha^{-1} \text{ and } (b, sm) = w\beta x\alpha^{-1} = u\gamma w\delta^{-1}y\]

such that \((c, n) := u\gamma\) and \((d, m - n) := v\alpha^{-1}\) are quasi-bands. Then \(((c, n), (d, m - n)) <_{\deg} (b, m)\).

**Proof.** We only show that \(((c, n), (d, m - n)) \leq_{\deg} (b, m)\) and leave the proof of the inequality \(((c, n), (d, m - n)) \neq_{\deg} (b, m)\) to the reader, as it is nearly the same as in the proof of Proposition 4.1.

Let \(\mu, \nu \in k^*\) and set \(M := M(d, m - n, \mu)\) and \(N := M(c, n, \nu)\). For any \(h \in \text{Hom}_k(M, N)\) there is a unique \(A\)-module structure \(X_{h, \mu, \nu}\) on the vector space \(N \oplus M\) such that

\[
\left(\begin{array}{cc}
1_N & h \\
0 & 1_M
\end{array}\right): X_{h, \mu, \nu} \longrightarrow N \oplus M
\]

is an \(A\)-isomorphism. By definition, we know that

\[
X_{h, \mu, \nu}(a) = \left(\begin{array}{c}
N(a) \\
0
\end{array}\begin{array}{c}
\zeta(a) \\
M(a)
\end{array}\right)
\]

with \(\zeta(a) = N(a)h - hM(a)\) for any \(a \in A\). Note that \(\zeta(a) = 0\) for all \(a \in A\) in case \(h\) is \(A\)-linear.

We will construct an \(h \in \text{Hom}_k(M, N)\) in such way that

\[
\zeta(\epsilon)(1 \otimes e_i) = \begin{cases} 
-\nu^{-1}\mu^{-1}1 \otimes e_0 & \text{if } i = m - n - 1 \text{ and } \epsilon = \alpha, \\
1 \otimes e_{n-1} & \text{if } i = 0 \text{ and } \epsilon = \gamma, \\
0 & \text{otherwise},
\end{cases}
\]

for \(0 \leq i < m - n\) and for any path \(\epsilon\) of length at most one. Here is a picture of the module \(X_{\mu, \nu} := X_{h, \mu, \nu}\).
\[
\begin{align*}
\{0\} \times M(d, m - n, \mu) &\rightarrow (0, 1 \otimes e_0) \\
M(c, n, \nu) \times \{0\} &\rightarrow (1 \otimes e_0, 0)
\end{align*}
\]

(An arrow from \(x\) to \(y\) labeled by \(\alpha \simeq \lambda\) indicates that \(X_{h, \mu, \nu}(\alpha)(x) = \lambda y\).)

We postpone the construction of \(h\) and show how to complete the proof, once \(h\) is defined: For a fixed \(\lambda \in k^*\) the module \(M(b, m, \lambda)\) belongs to the closure of the one-parameter family \(Y_{\mu} := X_{h, \mu, \nu} - \lambda\mu, \mu \in k^*\). Consequently,

\[
((c, n), (d, m - n)) \leq_{\text{deg}} (b, m).
\]

In order to define \(h\), we have to show that

\[
c(1) \cdots c(l(w) + 1) = w\beta = b(1) \cdots b(l(w) + 1).
\]

Let \(0 < i \leq l(w) + 1\). Obviously \(c(i) = b(i)\) if \(i \leq n\). If \(n < i\), we may assume inductively that \(c(i - n) = b(i - n)\) and thus

\[
c(i) = c(i - n) = b(i - n) = b(i).
\]

Similarly, one can show that

\[
d(1) \cdots d(l(w) + 1) = w\delta^{-1}.
\]

There are integers \(t, r \geq 0\) and strings \(z, z'\) such that

\[
(c, tn) = w\beta z\gamma \quad \text{and} \quad (d, r(m - n)) = w\delta^{-1} z'\alpha^{-1}.
\]

Let \(g : M(w) \rightarrow M(c, tn, \nu')\) be the \(k\)-linear map sending \(e_d\) to \(1 \otimes e_{l(d)}\) for \(d \in Ld(w)\). We have

\[
(g \circ \epsilon - \epsilon \circ g)(e_d) = \begin{cases} 
-\nu'1 \otimes e_{m - 1} & \text{if } d = 1_{t(w)} \text{ and } \epsilon = \gamma, \\
0 & \text{otherwise},
\end{cases}
\]

28
for any path $\epsilon$ of length at most one and any $d \in Ld(w)$. Similarly, let $f : M(d, r(m - n), \mu') \rightarrow M(w)$ be the $k$-linear map sending $1 \otimes e_i$ to

$$
\begin{cases}
e_d & \text{if } 0 \leq i \leq l(w), d \in Ld(w), l(d) = i \\
0 & \text{if } l(w) < i < r(m - n)
\end{cases}
$$

for $i = 0, \ldots, r(m - n) - 1$. The map $f$ satisfies

$$(f \circ \epsilon - \epsilon \circ f)(1 \otimes e_i) = \begin{cases}
\mu^{-r}e_{1i(w)} & \text{if } i = tn - 1 \text{ and } \epsilon = \alpha, \\
0 & \text{otherwise},
\end{cases}$$

for any path $\epsilon$ of length at most one and $0 \leq i < tn$. For the composition of $f$ and $g$, we have

$$(g \circ f \circ \epsilon - \epsilon \circ g \circ f)(1 \otimes e_i) = \begin{cases}
\mu^{-r'}1 \otimes e_{0} & \text{if } i = tn - 1 \text{ and } \epsilon = \alpha, \\
-\nu 1 \otimes e_{r(m-n)-1} & \text{if } i = 0 \text{ and } \epsilon = \gamma, \\
0 & \text{otherwise},
\end{cases}$$

for any path $\epsilon$ of length at most one and $0 \leq i < r(m - n)$. Let $h'$ be the composition

$$M(d, m - n, \mu) \xrightarrow{g \circ f} M(c, tn, \nu') \xrightarrow{g \circ f} M(c, n, \nu),$$

where the first map is an unwinding morphism and the last map is a winding morphism. We have

$$(h' \circ \epsilon - \epsilon \circ h')(1 \otimes e_i) = \begin{cases}
\mu^{-1}1 \otimes e_{0} & \text{if } i = tn - 1 \text{ and } \epsilon = \alpha, \\
-\nu 1 \otimes e_{m-n-1} & \text{if } i = 0 \text{ and } \epsilon = \gamma, \\
0 & \text{otherwise},
\end{cases}$$

for any path $\epsilon$ of length at most one and any $0 \leq i < m - n$. Finally, we set $h := -\nu^{-1}h'$. 

$\square$
5 Extensions

Let \((b, m)\) and \((c, n)\) be bands. Proposition 1.2 will follow from the following two lemmas.

**Lemma 5.1.** If the pair \([[b, m]], [[c, n]]\) is not extendable, then

\[
\text{Ext}^1_A(M(b, m, \lambda), M(c, n, \mu)) = 0
\]

for any \(\lambda, \mu \in k^*\) with \(\lambda \neq \mu, \mu^{-1}\).

**Lemma 5.2.** If the pair \(((b, m), (c, n))\) is extendable, there is a non-split short exact sequence of \(A\)-modules

\[
0 \rightarrow X \rightarrow M_{X,Y} \rightarrow Y \rightarrow 0
\]

with \(M_{X,Y} \in \mathcal{F}(d, n + m)\) for any \(X \in \mathcal{F}(c, n), Y \in \mathcal{F}(b, m)\), where

\[(d, n + m) = c(1) \cdots c(n)b(1) \cdots b(m).\]

Recall from [6] that an \(A\)-module \(A\) degenerates to a direct sum of \(A\)-modules \(B \oplus C\) whenever there is a short exact sequence

\[
0 \rightarrow B \rightarrow A \rightarrow C \rightarrow 0.
\]

Combining this result with Lemma 5.2, we obtain

**Corollary 5.3.** If the pair \(((b, m), (c, n))\) is extendable, then

\[
\mathcal{F}((c, n), (b, m)) \subseteq \mathcal{F}(d, n + m),
\]

where

\[(d, n + m) = c(1) \cdots c(n)b(1) \cdots b(m).\]

**Proof of Lemma 5.1.** By [2] the Auslander-Reiten translate \(\tau^{-1}M\) of a band module \(M\) is isomorphic to \(M\). It is well known that

\[
\text{Ext}^1_A(N, M) \simeq \text{Hom}_A(\tau^{-1}M, N)
\]

for any finite dimensional \(A\)-modules \(M, N\), where

\[
\text{Hom}_A(\tau^{-1}M, N) = \text{Hom}_A(\tau^{-1}M, N) / \mathcal{P}(\tau^{-1}M, N)
\]

and \(\mathcal{P}(\tau^{-1}M, N)\) is the subspace of \(\text{Hom}_A(\tau^{-1}M, N)\) consisting of all homomorphisms that factor through a projective \(A\)-module. Therefore it suffices to show that any morphism \(M(c, n, \mu) \rightarrow M(b, m, \lambda)\) factors through a
projective $A$-module. The condition $\lambda \neq \mu, \mu^{-1}$ asserts that $M(c, n, \mu)$ and $M(b, m, \lambda)$ are non-isomorphic and this allows us to apply Proposition 2.3, which yields a basis for

$$\text{Hom}_A(M(c, n, \mu), M(b, m, \lambda)).$$

We show that any morphism of the form

$$\iota_{j, w, (b, m), \lambda} \circ \pi_{i, w, (c, n), \mu} : M(c, n, \mu) \rightarrow M(b, m, \lambda)$$

with $j \in \text{sub}(w, (b, m))$ and $i \in \text{fac}(w, (c, n))$ factors through a projective $A$-module. Fix a string $w$, $j \in \text{sub}(w, (b, m))$ and $i \in \text{fac}(w, (c, n))$. Up to equivalence of bands, we may assume that $i = n$ and $j = m$. Moreover, we can assume that $m \in \text{sub}_1(w, (b, m))$ and $n \in \text{sub}_1(w, (c, n))$ up to replacing $\lambda$ by $\lambda^{-1}$ and $\mu$ by $\mu^{-1}$ if necessary. By the definition of the sets $\text{sub}_1$ and $\text{fac}_1$ there are $s, t \geq 1$, strings $u, v$ and arrows $\alpha, \beta, \gamma, \delta$ with

$$(b, sm) = w\beta u \alpha^{-1}$$

and

$$(c, tn) = w\delta^{-1} v \gamma.$$ 

As the pair $((b, m), (c, n))$ is not extendable,

$$(d, n + m) = c(1) \cdots c(n)b(1) \cdots b(m)$$

cannot be a quasi-band. As $c(n) = \gamma$ and $b(m) = \alpha^{-1}$ this can only happen if one of the words

$$c(1) \cdots c(n)b(1) \cdots b(m - 1)$$

and

$$b(1) \cdots b(m)c(1) \cdots c(n - 1)$$

is not a string. We just consider the case, where the word

$$x := c(1) \cdots c(n)b(1) \cdots b(m - 1)$$

is not a string, as the other case is treated similarly. Let $1 \leq i \leq n$ be minimal with the property that $c(i), c(i + 1), \ldots, c(n)$ are arrows and let $1 \leq j \leq m - 1$ be maximal, such that $b(1), \ldots, b(j)$ are arrows. As $x$ is not a string, we see that the path

$$c(i)c(i + 1) \cdots c(n)b(1) \cdots b(j)$$

belongs to the ideal $I$. We set

$$q = c(i)c(i + 1) \cdots c(n)$$

and

$$r = b(1) \cdots b(j).$$

Obviously $w\beta \in \text{Ld}(r)$, because otherwise $r \in \text{Ld}(w)$, which implies that $qr$ is a string. We may thus decompose $r = wr'$ for some non-trivial path
Let $P = P_{s(r)}$ be the projective $\mathcal{A}$-module corresponding to the vertex $s(r)$. Obviously $P$ is isomorphic to $M(q'r(p^{-1})$ for some paths $p$ and $q'$ with $q = q''q'$ for a non-trivial path $q''$. We obtain the sequence of morphisms

$$M(c, n, \mu) \overset{h}{\longrightarrow} M(q'w) \overset{g}{\longrightarrow} P = M(q'w'r(p^{-1}) \overset{f}{\longrightarrow} M(b, m, \lambda),$$

where $h$ is the factorstring morphism corresponding to $n - l(q') \in \text{fac}_1(q'w, (c, n))$, $g$ is the substring morphism corresponding to the decomposition $q'r(p^{-1}) = (1_{l(q')}) (q'w) (r'p^{-1})$ and $f$ is the morphism sending $e_{q'r}$ to $\frac{1}{\lambda} \otimes e_{l(r)}$. As $f \circ g \circ h = \iota_{j, w, (b, m), \lambda} \circ \pi_{n, w, (c, n), \mu}$, the proof is complete.

**Proof of Lemma** As $((b, m), (c, n))$ is extendable, there are $s, t \geq 1$, strings $u, v, w$ and arrows $\alpha, \beta, \gamma, \delta$ with

$$(b, sm) = w\beta u\alpha^{-1} \text{ and } (c, tn) = w\delta^{-1}v\gamma$$

such that

$$(d, n + m) := (c, n)(b, m) := c(1) \cdots c(n)b(1) \cdots b(m)$$

is a quasi-band.

Let $x = c(1) \cdots c(n)$ and $y = b(1) \cdots b(m)$ as strings. We divide the proof into four steps:

a) $l(w) < n + m$,

b) $n + m \in \text{sub}(xw, (d, n + m))$,

c) $n \in \text{fac}(yw, (d, n + m))$ and

d) for any $\lambda, \mu \in k^*$ the sequence

$$0 \longrightarrow M(c, n, \mu) \overset{f}{\longrightarrow} M(d, n + m - \lambda \mu) \overset{g}{\longrightarrow} M(b, m, \lambda) \longrightarrow 0,$$

where

$$f = \iota_{n+m, xw, (d, n+m), -\lambda \mu} \circ \pi_{n, xw, (c, n), \mu}$$

and

$$g = \iota_{m, yw, (b, m), \lambda} \circ \pi_{n, yw, (d, n+m), -\lambda \mu}$$

is exact and does not split.
Proof of a): If we assume that \( l(w) \geq n + m \), we obtain the contradiction
\[
\alpha^{-1} = b(m) = c(m) = c(n + m) = b(n + m) = b(n) = c(n) = \gamma.
\]

Proof of b): As \( d(n + m) = b(m) = \alpha^{-1} \) is an inverse arrow and \( d(i) = c(i) \) for \( i = 1, \ldots, n \), it suffices to show that \( d(i + n) = b(i) \) for \( i = 1, \ldots, l(w) + 1 \), as \( b(l(w) + 1) = \beta \) is an arrow. This is obvious if \( i \leq m \). We may thus assume that \( i > m \). As \( l(w) < n + m \) by a), we see that \( 0 < i - m \leq n \) and \( i - m \leq l(w) \) and thus
\[
d(i + n) = d(i - m) = c(i - m) = b(i - m) = b(i).
\]

Statement c) follows dually.

Proof of d): We decompose \( f \) as the sum \( \sum_{i=0}^{l(w) + n} f_i \) of \( k \)-linear maps
\[
f_i : M(c, n, \mu) \rightarrow M(d, n + m, -\lambda\mu).
\]

Note that, in order to keep the coefficients combinatorially simple, we adapt the basis of \( M(c, n, \mu) \) to \( i \) for the definition of \( f_i \): For \( 0 \leq i \leq l(w) + n \) and \( i \leq l < i + n \) we set
\[
f_i(1 \otimes e_l) := \begin{cases} 1 \otimes e_i & \text{if } i = l \\ 0 & \text{otherwise.} \end{cases}
\]

Similarly, we decompose \( g \) as the sum \( \sum_{k=n}^{l(w) + m + n} g_k \) of \( k \)-linear maps
\[
g_k : M(d, n + m, -\lambda\mu) \rightarrow M(b, m, \lambda),
\]
where \( g_k \) is defined as follows: For \( n \leq k \leq l(w) + n + m \) and \( k \leq l < k + n + m \) we set
\[
g_k(1 \otimes e_l) := \begin{cases} 1 \otimes e_{l-n} & \text{if } k = l \\ 0 & \text{otherwise.} \end{cases}
\]

Applying Lemma 2.4, we see that \( f \) is injective and \( g \) surjective. In order to prove that the sequence is exact, it remains to show that \( g \circ f = 0 \). We have
\[
g \circ f = \sum_{i=0}^{l(w) + n} \sum_{k=n}^{l(w) + m + n} g_k \circ f_i.
\]

We claim that \( g_k \circ f_i = 0 \) unless \( (i, k) \) belongs to one of the disjoint sets
\[
A := \{(i, i + n + m) : 0 \leq i \leq l(w)\}
\]
and

\[ B := \{(i, i) : n \leq i \leq l(w) + n\}. \]

Assume that there are \(0 \leq i \leq l(w) + n, \ n \leq k \leq n + m + l(w)\) and \(0 \leq l < n\) such that \(g_k \circ f_i(1 \otimes e_i) \neq 0\). From the definition of \(f_i\) we obtain \(l - i \in n\mathbb{Z}\) and \(f_i(1 \otimes e_i) = \xi(1 \otimes e_i)\) for a \(\xi \in k^*\) and from the definition of \(g_k\) we see that \(k - i \in (n + m)\mathbb{Z}\). As \(-l(w) \leq k - i \leq n + m + l(w)\) and \(l(w) < n + m\), we obtain either \(k - i = 0\) and thus \((i, k) \in B\) or \(k - i = n + m\) and hence \((i, k) \in A\). We have

\[
 g \circ f = \sum_{(i,k) \in A} g_k \circ f_i + \sum_{(i,k) \in B} g_k \circ f_i = \sum_{(i,k) \in A} (g_k \circ f_i + g_{k-m} \circ f_{i+n})
\]

and suffices to show that \((g_k \circ f_i + g_{k-m} \circ f_{i+n}) = 0\) for \((i, k) \in A\). Let \((i, k) \in A\). For any \(i \leq l < n + i\) we have

\[
 f_i(1 \otimes e_i) = f_{i+n}(1 \otimes e_i) = 0
\]

and thus

\[
 g_k \circ f_i(1 \otimes e_i) + g_{k-m} \circ f_{i+n}(1 \otimes e_i) = 0,
\]

unless \(l = i\). In case \(l = i\) we obtain

\[
 g_k f_i(1 \otimes e_i) = g_k(1 \otimes e_i) = g_k(-\lambda \mu \cdot 1 \otimes e_{i+n+m}) = g_k(-\lambda \mu \cdot 1 \otimes e_k) = -\lambda \mu \cdot 1 \otimes e_{k-n}
\]

and

\[
 g_{k-m} f_{i+n}(1 \otimes e_i) = g_{k-m} f_{i+n}(\mu \cdot 1 \otimes e_{i+n}) = g_{k-m}(\mu \cdot 1 \otimes e_{i+n}) = g_{k-m}(\mu \cdot 1 \otimes e_{k-m}) = \mu \cdot 1 \otimes e_{k-m-n} = \lambda \mu \cdot 1 \otimes e_{k-n}
\]

and thus \(g \circ f = 0\).

To complete the proof, it remains to show that the short exact sequence

\[
 0 \longrightarrow M(c, n, \mu) \xrightarrow{f} M(d, n + m, -\lambda \mu) \xrightarrow{g} M(b, m, \lambda) \longrightarrow 0,
\]
does not split. It suffices to show that $M(d, n + m, -\lambda \mu)$ is not isomorphic to $M(c, n, \mu) \oplus M(b, m, \lambda)$. As 

$$\sharp \text{sub}(w, (c, n)) + \sharp \text{sub}(w, (b, m)) > \sharp \text{sub}(w, (d, n + m)),$$

there is an $\mathcal{A}$-module $U$, such that

$$[U, M(d, n + m, -\lambda \mu)] \neq [U, M(c, n, \mu) \oplus M(b, m, \lambda)],$$

by Proposition 2.2 about the homomorphism spaces between string and band modules. This shows that $M(d, n + m, -\lambda \mu)$ and $M(c, n, \mu) \oplus M(b, m, \lambda)$ cannot be isomorphic. \qed
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