Split bounded extension algebras and Han’s conjecture
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Abstract
A main purpose of this paper is to prove that the class of finite dimensional algebras which verify Han’s conjecture is closed under split bounded extensions.

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1 Introduction
Given a finite dimensional algebra \( A \) over an algebraically closed field \( k \), Han’s conjecture relates two homological invariants associated to \( A \): its global dimension and its Hochschild homology. In the commutative case – non necessarily finite dimensional but finitely generated – the finiteness of the global dimension is equivalent to the fact that \( A \) is geometrically regular [3, 30]. In general we are going to say that an algebra with finite global dimension is smooth.

On the other hand, we consider Hochschild homology of \( A \). Let \( A^e = A \otimes A^{op} \) be the enveloping algebra. Let us recall that given an \( A \)-bimodule \( X \) – or equivalently a left or right \( A^e \)-module –, the Hochschild homology of \( A \) with coefficients in \( X \) is \( H_n(A, X) = \text{Tor}^A_n (A, X) \); it is functorial in both variables.

Han’s conjecture [20] states that for a finite dimensional, \( A \) is smooth if and only if \( HH_n(A) = 0 \) for \( n >> 0 \). The direct implication is true.

Next we recall some previous results. Well before being formulated, Han’s conjecture has been proved for commutative algebras which are finitely generated, which encompasses finite dimensional commutative algebras, see [8, 4]. Y. Han proved the conjecture for monomial algebras in [20]. P.A. Bergh and D. Madsen have shown that it holds in characteristic zero for graded finite dimensional local algebras, Koszul algebras and graded cellular algebras [6]. They have also obtained a confirmation of Han’s conjecture in [7] for trivial extensions of several sorts of algebras, by proving that their Hochschild homology is non zero in large enough degrees. P.A. Bergh and K. Erdmann proved in [5] that quantum complete intersections - at a non-root of unity- satisfy Han’s conjecture, as well as A. Solotar and...
M. Vigué-Poirrier [32] for a generalization of quantum complete intersections and for a family of algebras which are in some sense opposite to these ones. Later, A. Solotar, M. Suárez-Alvarez and Q. Vivas proved in [31] Han’s conjecture for quantum generalized Weyl algebras (out of a few exceptional cases). In [13] null-square projective algebras extensions were considered, the present paper goes further in this direction.

Concerning the commutative case, it is worth to mention that in characteristic zero, in positive degrees $HH_n(A)$ has a decomposition, called Hodge decomposition, as a direct sum of subspaces, see for example [18, 28, 33]. One of them is the $n$-th exterior power of the $A$-module of Kähler differentials, $\Omega^n_A|_k$ and another one is $D_n(A|k)$, the André-Quillen homology of the commutative $k$-algebra $A$. When $A$ is smooth, in positive degrees $HH_n(A) = \Omega^n_A|_k$ and the other summands annihilate. In fact, the main condition for smoothness is that $D_n(A|k) = 0$ for positive $n$ [23], and the Jacobi-Zariski long exact sequence for André-Quillen homology relating $D_n(A|k)$, $D_n(A|B)$ and $D_n(B|k)$ for any extension of algebras $k \subseteq B \subseteq A$ plays an important role.

In the non commutative setting André-Quillen homology does not exist, but A. Kaygun has proved recently in [24, 25] the existence of a Jacobi-Zariski long exact sequence starting in degree one for Hochschild homology for any extension of $k$-algebras $B \subseteq A$, such that $A$ is $B$-flat. It relates the ordinary Hochschild homologies of $A$ and $B$ with the relative Hochschild homology of $A$ with respect to $B$. In this paper, with different hypotheses we also obtain a long exact sequence of Jacobi-Zariski type for large enough degrees.

In this paper we consider split extension algebras in relation with Han’s conjecture. By definition, a split extension algebra over a field $k$ is a $k$-algebra of the form $A = B \oplus M$, where $B$ is a subalgebra of $A$ and $M$ is a two-sided ideal of $A$. As a consequence of our work, we prove that in some cases, adding or deleting arrows to a quiver – even adding or deleting certain relations – does not change the situation with respect to Han’s conjecture, see also [14]. Indeed, these processes are special cases of split extension algebras, see Example 2.2 (3). In a subsequent work, conditions will be given for these operations to fit within the framework of the theory we provide in this paper.

Next we describe the contents of this article. In the first section, in order to compute the relative Hochschild (co)homology introduced by G. Hochschild in [22], we construct a reduced relative bar resolution of a split extension algebra. We use it particularly when $M$ is $B$-tensor nilpotent, that is if there exists $n$ such that $M^{\otimes_B n} = 0$.

In Section 3 we obtain a Jacobi-Zariski long exact sequence in the following situation. A $B$-bimodule $M$ is called bounded if $M$ is $B$-tensor nilpotent, of finite projective dimension as $B$-bimodule and projective either as left or as right $B$-module. A split bounded extension $A = B \oplus M$ where $M$ is bounded. For these algebras we obtain a Jacobi-Zariski long exact sequence in large enough degrees. We set up techniques based on nearly exact sequences of complexes, see Definition 3.1. Actually the relative resolution of Section 2 provides a nearly exact sequence, which in turn gives the required Jacobi-Zariski long exact sequence in large enough degrees.

In Section 4 we prove our main result: the class $\mathcal{H}$ of finite dimensional algebras which verify Han’s conjecture is closed under split bounded extensions. More precisely if $A = B \oplus M$ is such an extension, then $A \in \mathcal{H}$ if and only if $B \in \mathcal{H}$. 

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The proofs make use of the Jacobi-Zariski long exact sequence, and of the reduced relative bar resolution.

2 A reduced relative bar resolution for split extension algebras

Let $B \subset A$ be an extension of algebras over a field $k$. In this context G. Hochschild introduced in [22] relative homological algebra, which corresponds to consider the exact category of $A$-modules with respect to $B$-split short exact sequences, see [27] [9]. More precisely, an induced module is an $A$-module of the form $A \otimes_B M$, where $M$ is a left $B$-module. An $A$-module $P$ is relative projective if any $A$-morphism $X \to P$ which has a $B$-section has an $A$-section. Equivalently, $P$ is relative projective if it is an $A$-direct summand of an induced module. There are enough relative projectives since for any $A$-module $X$ the canonical $A$-map $A \otimes_B X \to X$ has a $B$-section. Of course if $B = k$ we recover the ordinary definition, and if $B = A$ all modules are relative projective.

A relative projective resolution of an $A$-module $X$ is a sequence

$$
\cdots d_2 \to P_2 \to P_1 \to P_0 \to X \to 0
$$

where each $P_i$ is a relative projective $A$-module, the $d_i$'s are $A$-morphisms, $d^2 = 0$ and there exists a $B$-contracting homotopy, see [22] p. 250.

Two relative projective resolutions of $X$ are homotopic and the functor $A \otimes_B -$ is exact, so that for $X$ and $Y$ respectively right and left $A$-modules, the functor $\text{Tor}_A^B(X,Y)$ is well defined. For $X$ and $Y$ left $A$-modules, the functor $\text{Ext}_A^*(X,Y)$ is well defined.

Consider the extension of enveloping algebras $B^e \subset A^e$. For $X$ an $A$-bimodule, the relative Hochschild homology and cohomology vector spaces are defined in [22] respectively as follows:

$$
H_*(A|B,X) = \text{Tor}_A^*(B^e|X,A) \quad \text{and} \quad H^*(A|B,X) = \text{Ext}_A^*(A|B^e,A,X).
$$

Observe that in [22] those vector spaces are defined with respect to the extension $B \otimes A^{op} \subset A^e$. This turns out to be equivalent since the relative canonical resolution of $A$ is relative projective in both situations, and the canonical contracting homotopies agree.

Being derived functors, they can be computed using an arbitrary relative projective resolution. In particular these vector spaces are the homology and the cohomology of the following chains and cochains complexes $C_*(A|B,X)$ and $C^*(A|B,X)$:

$$
\cdots \xrightarrow{b} X \otimes_B A \otimes A^n \xrightarrow{b} \cdots \xrightarrow{b} X \otimes_B A \xrightarrow{b} X_B \to 0,
$$

$$
0 \to X^B \xrightarrow{b} \text{Hom}_B(A,X) \xrightarrow{b} \cdots \xrightarrow{b} \text{Hom}_B(A \otimes A^n,X) \xrightarrow{b} \cdots
$$

where

$$
X_B = X \otimes_B A = X/\langle bx - xb \rangle = H_0(B,X),
$$

$$
X^B = \text{Hom}_B(A,X) = \{ x \in X \mid bx = xb \text{ for all } b \in B \} = H^0(B,X),
$$

and where the formulas for the boundaries and coboundaries are the ordinary ones.
Definition 2.1  An extension of algebras $B \subset A$ is split if there is a morphism of algebras $\pi : A \to B$ which is a retraction to the inclusion, that is $\pi(b) = b$ for all $b \in B$.

Clearly $B \subset A$ is split if and only if there exists a two-sided ideal $M$ of $A$ such that $A = B \oplus M$.

Next we provide some examples of split extensions. In the last example we add arrows to the quiver of a bound quiver algebra. Note that in relation to the finitistic dimension conjecture, E.L. Green, C. Psaroudakis and Ø. Solberg [19] have considered the case of adding exactly one arrow, which leads to a trivial extension.

Examples 2.2

1. Let $B$ be an algebra, let $N$ be a $B$-bimodule and let $T$ be the tensor algebra

$$T = T_B(N) = B \oplus N \oplus N \otimes_B N \oplus \cdots.$$ 

Let $T^{>i} = N \otimes_B^{i+1} N \oplus N \otimes_B^{i+2} N \oplus \cdots$. We have $T = B \oplus T^{>0}$, that is $T$ is a split extension. Moreover, if $J \subset T^{>0}$ is a two-sided ideal of $T$, then $B \subset T/J$ is a split extension as well.

2. Let $Q$ be a finite quiver, that is $Q = (Q_0, Q_1, s, t)$ where $Q_0$ and $Q_1$ are finite sets called respectively vertices and arrows, and $s, t : Q_0 \to Q_1$ are maps called respectively source and target. Let $A = kQ/I$ be a bound quiver algebra, where $kQ$ is the path algebra of $Q$ and $I$ is an admissible two-sided ideal of $kQ$, see [15, 16, 17] and [1, 29]. The extension $B = kQ_0 \subset A$ is split.

3. Let $B = kQ/I$ be a bound quiver algebra, and let $F$ be a finite set of new arrows, that is $F$ is a finite set with two maps $s, t : F \to Q_0$. Let $Q_F$ be the quiver with the same vertices than $Q$, while its arrows are $Q_1 \cup F$.

Let $B_F = kQ_F/(I)_{kQ_F}$, where the denominator is the two-sided ideal of $kQ_F$ generated by $I$. It is easily proven that $B_F = T_B(N)$ where

$$N = \bigoplus_{a \in F} \mathrm{Bt}(a) \otimes s(a)B.$$

Let $J \subset B_F^{>0}$ be a two-sided ideal of $B_F$. The algebra

$$A = B_F/J = B \oplus (B_F^{>0}/J)$$

is also a split extension.

The first item of the next result is a generalisation of a reduced bar resolution obtained in [11, Lemma 2.1].

Theorem 2.3  Let $A = B \oplus M$ be a split extension of algebras.

1. There is a reduced relative bar resolution of $A$ as $A$-bimodule

$$\cdots \to A \otimes_B M \otimes_B \cdots \otimes_B A \to A \otimes_B M \otimes_B A \to A \otimes_B A \to A \to 0$$

where the formulas for the $d$’s are those of the ordinary bar resolution, see [21, 22].
In what follows the formulas for the (co)boundaries are the ordinary ones.

2. Let $X$ be an $A$-bimodule. The homology of the following chain complex $C^M_*(A|B, X)$ is $H_*(A|B, X)$.

$$C^M_*(A|B, X) : \cdots \xrightarrow{b} X \otimes B \xrightarrow{b} \cdots \xrightarrow{b} X \otimes B^n \xrightarrow{b} X_B \to 0 \quad (2.3)$$

3. The cohomology of the following cochain complex $C^*_M(A|B, X)$ is $H^*(A|B, X)$.

$$0 \to X^B \xrightarrow{b} \text{Hom}_{B^e}(M, X) \xrightarrow{b} \cdots \xrightarrow{b} \text{Hom}_{B^e}(M \otimes B^n, X) \xrightarrow{b} \cdots \quad (2.4)$$

Proof. The bimodules involved in the first item are induced bimodules, hence they are relative projective. Let $a = a_B + a_M$ be the decomposition of $a \in A = B \oplus M$, and let

$$t(a_1 \otimes m_2 \otimes \cdots \otimes m_{n+1} \otimes a_{n+2}) = 1 \otimes (a_1)_M \otimes m_2 \otimes \cdots \otimes m_{n+1} \otimes a_{n+2}.$$ 

It is easily proven that $t$ is a well defined $B^e$-morphism, which is a contracting homotopy.

The second item is obtained by applying the functor $X \otimes A^e$ to the resolution, and the following canonical isomorphism where $Z$ is a $B$-bimodule

$$X \otimes A^e (A \otimes B Z \otimes B A) = X \otimes B^e Z.$$ 

The last item is obtained analogously.

Remark 2.4 For later use, we record that the contracting homotopy $t$ in the previous proof is also a right $A$-module map.

A $B$-bimodule $M$ is $B$-tensor nilpotent if there exists $n$ such that $M \otimes B^n = 0$. For instance, let $kQ$ be the path algebra of a quiver $Q$. The $kQ_0$-bimodule $(Q_1) \subset kQ$ is $kQ_0$-tensor nilpotent if and only if there is no oriented cycle in $Q$.

Corollary 2.5 Let $A = B \oplus M$ be a split extension, where $M$ is $B$-tensor nilpotent. Let $X$ be an $A$-bimodule. In large enough degrees

$$H_*(A|B, X) = 0 \quad \text{and} \quad H^*(A|B, X) = 0.$$ 

Let $C_*(A, X)$ be the ordinary chain complex

$$C_*(A, X) : \cdots \xrightarrow{b} X \otimes A^\otimes n \xrightarrow{b} \cdots \xrightarrow{b} X \otimes A \xrightarrow{b} X \to 0 \quad (2.5)$$

whose homology is the Hochschild homology $H_*(A, X)$ of an $A$-bimodule $X$. Towards obtaining a Jacobi-Zariski long exact sequence for a split extension algebra, we observe the following.
Proposition 2.6 Let $A = B \oplus M$ be a split extension of algebras, and let $X$ be an $A$-bimodule. For $* \geq 1$, there is a sequence of chain complexes

$$0 \to C_*(B, X) \xrightarrow{\iota} C_*(A, X) \xrightarrow{\kappa} C_*^M(A|B, X) \to 0 \quad (2.6)$$

where $\iota$ is injective, $\kappa$ is surjective and $\kappa \iota = 0$.

In degree 0 we have the sequence

$$0 \to X \xrightarrow{1} X \to X_B \to 0.$$

Proof. The definition of the map $\iota$ is clear, and it is also clear that $\iota$ is an injective map of complexes. The map $\kappa$ given by

$$x \otimes a_1 \otimes \cdots \otimes a_n \mapsto x \otimes B \cdot [a_1 \otimes B \cdot \cdots \otimes B (a_n)_M]$$

is surjective, and $\kappa \iota = 0$. The verification that $\kappa$ is a map of complexes does not raise any difficulty. It uses extensively that $(aa')_M = aMa' + aB a_B'$ for $a, a' \in A$ and that the first tensor product in $C_*^M(A|B, X)$ is over $B$.

Remark 2.7 Considering $C_*(A|B, X)$ instead of $C_*^M(A|B, X)$, and $\kappa'$ given by

$$x \otimes a_1 \otimes \cdots \otimes a_n \mapsto x \otimes B \cdot [a_1 \otimes B \cdot \cdots \otimes B a_n]$$

does not give in general $\kappa' \iota = 0$.

Let $A = B \oplus M$ be a split extension. In the ensuing decomposition of the vector space $A^\otimes n$, let $[M_p B_q]$ be the direct sum of the direct summands containing $p$ tensorands in $M$ and $q$ tensorands in $B$, with $p + q = n$. For instance – omitting the $\otimes$ signs – we have that

$$[M_2 B_2] = MMBB \oplus MBMB \oplus MBBM \oplus BMBM \oplus BMMB \oplus BBMM$$

which is a direct summand of $A^\otimes 4$.

We set

$$K_{n,0} = \text{Ker}(X \otimes M^\otimes n \to X \otimes B^\otimes M^\otimes mn). \quad (2.7)$$

Lemma 2.8 In the situation of Proposition 2.6

$$\text{Ker} \kappa = \bigoplus_{p+q=n, \ p>0} X \otimes [M_p B_q] \oplus K_{n,0}$$

$$\text{Im} \iota = X \otimes [M_0 B_n]$$

$$\text{Ker} \kappa / \text{Im} \iota = \bigoplus_{p+q=n, \ p>0, \ q>0} X \otimes [M_p B_q] \oplus K_{n,0}.$$

3 Nearly exact sequences and the Jacobi-Zariski long exact sequence

In this section we will prove that if a sequence as (2.6) has zero homology for large enough degrees at the second page of the associated spectral sequence, then there is a long exact sequence in homology starting at this precise degree.
Definition 3.1 A sequence of chain complexes concentrated in non negative degrees
\[ 0 \to C_* \xrightarrow{\iota} D_* \xrightarrow{\kappa} E_* \to 0 \]
is nearly exact if
- \( \iota \) is injective,
- \( \kappa \) is surjective,
- \( \kappa \iota = 0 \),
- the chain complex \( \text{Ker}\kappa/\text{Im}\iota \) with boundary induced by the boundary of \( D \), is exact in large enough degrees.

We will prove later on that under some hypotheses, the sequence of Proposition 2.6 is nearly exact.

Theorem 3.2 Let
\[ 0 \to C_* \xrightarrow{\iota} D_* \xrightarrow{\kappa} E_* \to 0 \quad (3.1) \]
be a nearly exact sequence of chain complexes. There is a long exact sequence in large enough degrees as follows:
\[ \cdots \to H_m(C) \xrightarrow{\iota} H_m(D) \xrightarrow{\kappa} H_m(E) \xrightarrow{\delta} H_{m-1}(C) \to \cdots \]

Proof. We will use standard results on spectral sequences, see for instance [26] or [34].

The double complex arising from the sequence (3.1) with the standard change of signs, has the complexes \( E, D \) and \( C \) at columns \( p = 0, 1 \) and 2 respectively. Firstly we claim that its total complex has zero homology in large enough degrees. Indeed, consider the spectral sequence given by the filtration by the rows. At the first page the columns corresponding to \( p = 0, 2 \) are zero since \( \iota \) is injective and \( \kappa \) is surjective. At column \( p = 1 \) we have the homology vector spaces of the sequence (3.1) corresponding to the complex in the middle. Since the sequence is nearly exact, at the second page the column \( p = 1 \) has zeros in large enough degrees, and zeros elsewhere. This proves the claim.

Secondly we consider the filtration by columns. In page 1 of the corresponding spectral sequence, the horizontal maps are induced by \( \iota \) and \( \kappa \) at the homology level of the complexes of the sequence (3.1). We still denote them \( \iota \) and \( \kappa \) these morphisms, they are the morphisms of the aimed long exact sequence. We assert that in large enough degrees there is exactness at the column corresponding to the homology of \( D \). Indeed, the vector spaces at the second page at column \( p = 1 \) are \( \text{Ker}\kappa/\text{Im}\iota \). At these spots the differentials \( d_2 \) come from zero and go to zero. Hence these vector spaces live forever in the subsequent pages of the spectral sequence. We proved before that the total complex has no homology in large enough degrees, hence these vector spaces vanish in large enough degrees.

Finally we turn to the connecting homomorphism \( \delta \). In the second page of the spectral sequence just considered, at columns \( p = 0, 2 \) we have respectively \( \text{Coker}\kappa \) and \( \text{Ker}\iota \). We assert that the differentials \( d_2 : \text{Ker}\iota \to \text{Coker}\kappa \) are isomorphisms in large enough degrees. Indeed, \( \text{Ker}d_2 \) and \( \text{Coker}d_2 \) live forever in the spectral sequence, hence they vanish in large enough degrees by the same argument than above. We assert that composing \( d_2^{-1} \) with the inclusion of \( \text{Ker}\iota \) and the canonical
projection to Coker $\kappa$ provides the required connecting homomorphism of the long exact sequence, in large enough degrees. Indeed, by construction Ker $\delta = \operatorname{Im} \kappa$ and $\operatorname{Im} \delta = \ker i$.

\textbf{Proposition 3.3} Let $A = B \oplus M$ be a split algebra, let $X$ be an $A$-bimodule and consider the sequence (2.6) as a double complex after performing the standard change of signs. Let $E^2_{1,*}$ be the second page of the spectral sequence obtained by filtering by rows.

There is a double complex $C_{*,*}$ which total homology is $E^2_{1,*}$. The filtration by columns of $C_{*,*}$ yields a spectral sequence which terms at page 1 are

$$E^1_{p,q} = \operatorname{Tor}^{B_{ep}}_{p+q}(X, M^\otimes_{B_{ep}}) \text{ for } p, q > 0$$

and 0 otherwise.

\textbf{Proof.} By Lemma 2.8,

$$E^1_{1,n} = \bigoplus_{p+q=n \atop p>0 \atop q>0} X \otimes [M_p B_q] \oplus K_{n,0}.$$  

The differential of this column is deduced from the one of $C_{*,*}(A, X)$. Clearly this column is the total complex of the double chain complex.

\begin{itemize}
  \item $C_{p,q} = X \otimes [M_p B_q]$ for $p, q > 0$,
  \item $C_{p,0} = K_{p,0}$ for $p > 0$,
  \item 0 at other spots.
\end{itemize}

We modify momentarily $C_{*,*}$ at its bottom line as follows:

$$C'_{*,*} = X \otimes [M_p B_q] \text{ for } p > 0, q \geq 0, \text{ and } 0 \text{ at other spots},$$

with differentials still inherited from $C_{*,*}(A, X)$.

We assert that the homology of the column $p = 1$ of $C'_{*,*}$ is $\operatorname{Tor}^{B_{ep}}_{*}(X, M)$. Indeed, by Proposition 4.1 of [12], there is a specific projective resolution of $M$ as a $B_{ep}$-module, which verifies that applying the functor $X \otimes B_{ep} -$ to it yields the mentioned column.

For the column $p = 2$, consider the bar resolution of $M$ as a left $B$-module

$$\cdots \otimes B \otimes B \otimes M \rightarrow B \otimes M \rightarrow M \rightarrow 0.$$  

As it is well known there is a contracting homotopy $t$ given by $t(b_1 \otimes \cdots \otimes b_n \otimes m) = 1 \otimes b_1 \otimes \cdots \otimes b_n \otimes m$ which is actually a right $B$-module map. Hence we obtain a projective resolution of $M \otimes_B M$ by tensoring it over $B$ with the resolution of [12].

Applying the functor $X \otimes B_{ep} -$ to the latter yields the column $p = 2$. Therefore its homology is $\operatorname{Tor}^{B_{ep}}_{*}(X, M, \otimes_B M)$.

Iterating the process of tensoring by the bar resolution shows that the homology of the $p$-th column is $\operatorname{Tor}^{B_{ep}}_{*}(X, M^\otimes_{B_{ep}})$.

In order to return to $C_{*,*}$, note that by (2.7) we have

$$(X \otimes M^\otimes_{B_{ep}})/K_{p,0} = X \otimes B_{ep} M^\otimes_{B_{ep}} = \operatorname{Tor}^{B_{ep}}_{p}(X, M, \otimes_{B_{ep}}).$$
Hence replacing the bottom row of $C'$ by $K_{*,0}$ yields surjective maps at the bottom stage of each column, therefore we have zero homology at spots of the bottom row of $C$.

Next we provide sufficient conditions to ensure that the sequence (2.6) of Proposition 2.6 is nearly exact.

**Definition 3.4** Let $B$ be an algebra. A $B$-bimodule $M$ is bounded if

- $M$ is $B$-tensor nilpotent,
- $M$ is of finite projective dimension as a $B^e$-module,
- $M$ is either a left or a right projective $B$-module.

**Remark 3.5** Let $B$ be an algebra, and let $M$ be a $B$-bimodule with a $B$-associative structure, that is an associative map of $B$-bimodules $M \otimes_B M \to M$. Then $B \oplus M$ is a split extension algebra. Of course all split extensions occur this way.

We underline that in the requirement that $M$ is bounded, the $B$-associative structure of $M$ is not involved.

**Definition 3.6** A split bounded extension $B \oplus M$ is a split extension where $M$ is bounded.

**Proposition 3.7** Given a split bounded extension $A = B \oplus M$ and an $A$-bimodule $X$, the sequence (2.6) is nearly exact.

**Proof.** We consider the spaces $F^1_{p,q} = \Tor^B_{p+q}(M \otimes^p, X)$ for $p, q > 0$ of Proposition 3.3. There is a $v$ such that $M \otimes^v = 0$ since $M$ is $B$-tensor nilpotent. On the one hand $F^1_{p,q} = 0$ for $p \geq v$.

On the other hand, from [10, Chapter IX, Proposition 2.6] we infer that if $M$ is projective either as left or as right $B$-module, and it is of projective dimension less than $u$ as $B$-bimodule, then $M \otimes^u = 0$ is of projective dimension less than $pu$. Hence $F^1_{p,q} = 0$ for $p + q \geq pu$.

As a consequence, if $p + q \geq vu$, then $F^1_{p,q} = 0$. By Proposition 3.3 we obtain that if $n \geq vu$ then $E^1_{1,n} = 0$, which means that the column of homologies from the middle of the sequence (2.6) has in turn no homology in large enough degrees, that is the sequence is nearly exact.

The previous result and Theorem 3.2 prove the following.

**Theorem 3.8** Let $A = B \oplus M$ be a split bounded extension and let $X$ be an $A$-bimodule. There is a Jacobi-Zariski long exact sequence ending at some $n$ as follows.

$$
\cdots \to H_m(B, X) \xrightarrow{\delta} H_m(A, X) \xrightarrow{\delta} H_m(A|B, X) \xrightarrow{\delta} H_{m-1}(B, X) \to \cdots
$$

## 4 Han’s conjecture

A finite dimensional algebra is called **smooth** if it is of finite global dimension. As it is mentioned in the Introduction, the word smooth is originated in commutative
algebra and is useful here for brevity. Note that for noetherian rings, the left and right global dimensions are equal, see [2].

Han’s conjecture states that for \( A \) a finite dimensional algebra, \( H_*(A, A) \) vanishes in large enough degrees if and only if \( A \) is smooth. Let \( \mathcal{H} \) be the class of finite dimensional algebras which verify Han’s conjecture. Our aim is to prove the following.

**Theorem 4.1** Let \( A = B \oplus M \) be a split bounded extension of finite dimensional algebras.

\( A \in \mathcal{H} \) if and only if \( B \in \mathcal{H} \).

The proof relies on the next result.

**Proposition 4.2** Let \( A = B \oplus M \) be a split bounded extension of finite dimensional algebras.

1. \( H_*(A, A) \) vanishes in large enough degrees if and only if \( H_*(B, B) \) vanishes in large enough degrees.

2. \( A \) is smooth if and only if \( B \) is smooth.

**Remark 4.3** For a split extension \( A = B \oplus M \), it is trivial that if \( H_*(A, A) \) vanishes in large enough degrees, then the same happens for \( H_*(B, B) \). Indeed, \( H_*(-, -) \) is a functor on the category of algebras. Hence \( H_*(B, B) \) is a direct summand of \( H_*(A, A) \).

**Proof.**

1. We claim that \( H_*(A, A) \) and \( H_*(B, B) \) are actually isomorphic in large enough degrees.

Recall that by Corollary [2.5] we have that \( H_*(A|B, A) \) vanishes in large enough degrees. Hence the Jacobi-Zariski long exact sequence of Theorem [3.8] shows that \( H_*(B, A) \) and \( H_*(A, A) \) are isomorphic in large enough degrees.

On the other hand \( H_*(B, A) = H_*(B, B) \oplus H_*(B, M) \). Moreover,

\[
H_*(B, M) = \text{Tor}^{B^e}_* (B, M).
\]

Since \( M \) is of finite projective dimension as a \( B^e \)-module, \( H_*(B, M) \) vanishes in large enough degrees.

2. The bimodule \( M \) is projective from at least one side, without loss of generality we will suppose that \( M \) is right projective. If \( M \) is left projective, the proof is deduced from what follows by reversing sides.

First we prove that if \( A \) is smooth then \( B \) is smooth. Let \( \pi : A \to B \) be the retraction algebra map of \( B \subset A \), with kernel \( M \). Let \( Y \) be a right \( B \)-module, and let \( \overline{Y} \) be the \( A \)-module obtained by restricting scalars through \( \pi \). We have \( Y = \overline{Y} \) as right \( B \)-modules and \( \overline{Y}M = 0 \).

Let \( P_* \to \overline{Y} \) be a finite right \( A \)-projective resolution of \( \overline{Y} \). It remains of course exact when considering it as an exact sequence of right \( B \)-modules. Moreover, if \( P \) is a right projective \( A \)-module then it is also projective as a right \( B \)-module. Indeed, this is true for the free rank one \( A \)-module \( B \oplus M \). Then the standard arguments enable to conclude.
To prove that if \( B \) is smooth then so is \( A \), we begin by proving that any induced \( A \)-module \( Z = A \otimes_B Y \) is of finite projective dimension. Let \( Q_* \to Y \) be a finite left \( B \)-projective resolution of \( Y \). The functor \( A \otimes_B - \) is exact since \( A \) is right projective. Moreover if \( Q \) is a left projective \( B \)-module, then \( A \otimes_B Q \) is a left projective \( A \)-module, this follows from the fact that this is true for \( P = B \). Therefore \( A \otimes_B Q_* \to A \otimes_B Y \) is a finite left \( A \)-projective resolution of the induced module \( Z \).

Let \( X \) be a left \( A \)-module. We claim that there exists an exact sequence of \( A \)-modules

\[
0 \to Z_n \to Z_{n-1} \to \cdots \to Z_0 \to X \to 0
\]

where the \( A \)-modules \( Z_i \) are induced. This claim ends the proof, indeed each \( Z_i \) is of finite projective dimension by the previous assertion, hence \( X \) is of finite projective dimension.

To prove the claim, consider the relative reduced bar resolution \( 2.2 \) which is finite since \( M \) is \( B \)-tensor nilpotent. Moreover its contracting homotopy is a right \( A \)-module map, see Remark \( 2.4 \). Consequently the relative reduced bar resolution remains exact by applying the functor \( - \otimes_A X \). For some \( n \) we obtain

\[
0 \to A \otimes_B M \otimes^{\mathbb{L}} M \to A \otimes_B X \to \cdots \to A \otimes_B X \to X \to 0.
\]

Note that all the \( A \)-modules except \( X \) are induced \( A \)-modules. \( \diamond \)

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