Liouville Models of Black Hole Evaporation

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Abstract

A renormalizable two-dimensional quantum field theory, containing a metric, a dilaton and $N$ massless scalar matter fields, has been proposed as a model for black hole evaporation. Essential ingredients are a dilaton-dependent cosmological constant and a Polyakov action reflecting the conformal anomaly. Previous work on this model has been done in the large-$N$ (weak coupling) approximation and clear evidence for Hawking radiation and its back-reaction on the metric has been seen. There are, however, quantum consistency questions since the original model was only designed to be a $c = 26$ conformal field theory in the weak coupling limit. In this paper we construct new theories, differing from the old only in the dilaton dependence of the cosmological constant, and reducing to it in the weak coupling limit. They are exact $c = 26$ conformal field theories and presumably consistent frameworks for discussing this problem. We also study the new theories with a change in the Polyakov action proposed by Strominger with a view to eliminating unphysical ghost Hawking radiation. The classical equations of motion of the new theories are explicitly soluble, thus permitting an exact analysis of both static solutions and dynamic scenarios. While the static solutions are, by and large, physically reasonable, the dynamical solutions include puzzling examples where wrong-sign Hawking radiation is stimulated by allowing matter to fall into a static solution. We indicate how the latter problem may be resolved in the full quantum theory.

5/92

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1. Introduction

In a recent paper [1] a renormalizable two-dimensional quantum field theory was proposed as an instructive model for the study of black hole quantum mechanics. The starting point is the following “string-inspired” classical action for a metric, a dilaton and a collection of \( N \) conformally coupled massless matter fields:

\[
S = \frac{1}{2\pi} \int d^2x \sqrt{-g} \left[ e^{-2\phi} (R + 4(\nabla \phi)^2 + 4\lambda^2) - \frac{1}{2} \sum_{i=1}^{N} (\nabla f_i)^2 \right]. \tag{1.1}
\]

As long as the cosmological constant \( \lambda \) is non-zero, this action has a number of nontrivial solutions: First, there is a vacuum solution in which the dilaton is linear in the spatial coordinate, so that the quantum coupling strength, \( e^\phi \), goes to zero on one side of the world and to infinity on the other (we refer to this as the linear dilaton vacuum, or LDV). Second, there are static black hole solutions which approach the weak coupling limit of the LDV at infinity. Finally, there are also explicit solutions describing the formation of a black hole by the infall of an arbitrary pulse of massless scalar matter.

The physics of this model is most easily analysed in conformal gauge, defined by

\[
g_{+-} = -\frac{1}{2} e^{2\rho} \quad g_{\pm\pm} = 0 \quad x^\pm = \tau \pm \sigma. \tag{1.2}
\]

In this gauge, the classical action reduces to

\[
S_N = \frac{1}{\pi} \int d^2\sigma \left[ e^{-2\phi} (2\partial_+ \partial_- \rho - 4\partial_+ \phi \partial_- \phi + \lambda^2 e^{2\rho}) + \frac{1}{2} \sum_{i=1}^{N} \partial_+ f_i \partial_- f_i \right] \tag{1.3}
\]

and the equations of motion for the fields \( \rho, \phi \) and \( f_i \) must be supplemented by two constraints (the equations of motion for the missing components of the metric \( g_{\pm\pm} \)):

\[
T_{\pm\pm} = e^{-2\phi} \left( 4\partial_+ \rho \partial_- \phi - 2\partial_\pm^2 \phi \right) + \frac{1}{2} \sum_{i=1}^{N} \partial_+ f_i \partial_- f_i = 0 \tag{1.4}
\]

Note that the \( f_i \) satisfy free field equations and influence the metric-dilaton system only through the constraints.

To study Hawking radiation one of course has to quantize the above system. The principal observation of [1] was that, since the matter fields are free, their only quantum effect is through the conformal anomaly. This is accounted for by adding the Polyakov
term (multiplied by $N$ to reflect the multiplicity of scalar fields) to the action and correspondingly modifying the constraints. The results are particularly simple in conformal gauge:

\[
S_N = \frac{1}{\pi} \int d^2 \sigma \left[ e^{-2\phi}(2\partial_+ \partial_- \rho - 4\partial_+ \phi \partial_- \phi + \lambda^2 e^{2\rho}) + \frac{1}{2} \sum_{i=1}^N \partial_+ f_i \partial_- f_i - \frac{N}{12} \partial_+ \rho \partial_- \rho \right],
\]

\[
T_{\pm \pm} = e^{-2\phi}(4\partial_+ \phi \partial_\pm \rho - 2\partial^2_\pm \phi) + \frac{1}{2} \sum_{i=1}^N \partial_\pm f_i \partial_\pm f_i
- \frac{N}{12} (\partial_\pm \rho \partial_\pm \rho - \partial^2_\pm \rho).
\]

Strictly speaking, it is also necessary to give a quantum treatment of the graviton-dilaton sector as well, and it is not obvious how to do that. However, if $N$ is large, it seems reasonable to assume that matter quantum effects dominate those of gravity and to proceed with a classical treatment of the system summarized in (1.5). This line of investigation, initiated in [1], has been pursued by many authors [2,3,4,5] and we shall assume that the reader is familiar with at least the essentials of these papers. The basic result is that the large-$N$ system correctly accounts for Hawking radiation and its backreaction on the metric so long as $e^{2\phi}$ (a measure of the strength of purely gravitational quantum corrections) is smaller than a certain critical value (itself of order $1/N$) where a singularity, signalling the breakdown of the approximation, if nothing else, must occur. Unfortunately, in the large-$N$ treatment of the formation and subsequent evaporation of a black hole, the appearance of such singularities seems to be inevitable.

Evidently, a full quantum treatment of the graviton-dilaton sector of the theory will be needed in order to make further progress. In this paper we will show how to obtain some exact results for theories of the type given in (1.5). The most important issue concerns the central charge: for quantum consistency, the theory must be a conformal field theory with $c = 26$ and there is no reason to believe that the action (1.3) does not require corrections to achieve this. By adapting familiar 2d quantum gravity techniques [6,7], we will be able to identify modifications of (1.5) (having no effect on the weak coupling behavior of the physical degrees of freedom) which turn it into a $c = 26$ conformal field theory. Since the resulting theories turn out to be exactly soluble, we will be able to address the question whether they have a satisfactory physical interpretation in terms of the formation and evaporation of black holes.
2. Transforming to Free Fields and Fixing the Central Charge

With this background in mind, let us now turn to the problem of quantizing the theory described by (1.5). This action has two pieces which we will treat separately: all the terms involving derivatives of fields, which we will collectively call $S_{\text{kin}}$ and the cosmological constant term, proportional to $\lambda^2$, which we will call $S_{\text{cos}}$. Both require some generalization: There is no reason to believe that the coefficient $N/12$ of the Polyakov term in $S_{\text{kin}}$ is exact, so we will replace it by $\kappa$, where $\kappa$ will be fixed by the $c = 26$ requirement ($\kappa$ of course should reduce to $N/12$ in the large-$N$ limit). Alternatively, one might argue à la David, Distler and Kawai [6,7] about the measure in the path integral and arrive at the same conclusion. Also, there is no reason to believe that the specific dilaton dependence of $S_{\text{cos}}$ specified in (1.5) is exact. From string theory experience we might expect it to be a more general power series in the loop coupling constant squared $e^{2\phi}$ (the dependence specified in (1.5) is appropriate for a tree-level cosmological constant and should be accurate in the weak coupling, or $e^{2\phi} \to 0$, limit only).

The generalized action of interest to us can therefore be written as $S_\kappa = S_{\text{kin}} + S_{\text{cos}}$ where

$$S_{\text{kin}} = \frac{1}{\pi} \int d^2 \sigma \left[ e^{-2\phi}(2\partial_+ \rho \partial_- \phi + 2\partial_+ \phi \partial_- \rho - 4\partial_+ \phi \partial_- \phi) \right. \left. + \frac{1}{2} \sum_{i=1}^{N} \partial_+ f_i \partial_- f_i - \kappa \partial_+ \rho \partial_- \rho \right], \quad (2.1)$$

$$S_{\text{cos}} = \frac{1}{\pi} \int d^2 \sigma e^{-2\phi} \lambda^2 D(\phi) e^{2\rho} ,$$

and all we know about $D(2\phi)$ is that it should approach unity in the weak coupling limit. Since the cosmological constant term does not affect the constraints, they are the same as in (1.3) with the replacement of $N/12$ by $\kappa$:

$$T_{\pm\pm} = e^{-2\phi}(4\partial_+ \phi \partial_- \rho - 2\partial_+^2 \phi) + \frac{1}{2} \sum_{i=1}^{N} \partial_+ f_i \partial_+ f_i \quad \text{(2.2)}$$

$$- \kappa \left( \partial_+ \rho \partial_- \rho - \partial_+^2 \rho \right) .$$

At this point it is easy to see the origin of the troubles that were found [3,4] to afflict the large-$N$ limit (and perhaps the theory as a whole!): The kinetic action density is of the form $\partial_+ \Phi \cdot M(\phi) \cdot \partial_- \Phi$ where $\Phi$ collects the $N + 2$ fields of the problem into a vector and $M(\phi)$ is an $(N + 2) \times (N + 2)$ matrix. The determinant of $M$ has the value

$$\text{det}(M) = -4e^{-4\phi}(1 - \kappa e^{2\phi}) \left(-\frac{1}{2}\right)^N \quad (2.3)$$
and something singular obviously happens at the critical value of the dilaton field, $\kappa e^{2\phi} = 1$, where the determinant changes sign. We will have more to say about how to deal with this issue later on.

We will quantize this theory in two steps: First, we will show that $S_{\text{kin}}$ is actually a free field theory with an improvement term and that it has $c = 26$ if we choose $\kappa = (N - 24)/12$. Then we will construct $(1, 1)$ operators relative to this simple conformal field theory and so identify the correct cosmological constant function $D(\phi)$. Our first main point is that $S_{\text{kin}}$ can be reduced to a free theory by a sequence of field redefinitions. Applying

$$\omega = \frac{e^{-\phi}}{\sqrt{\kappa}}, \quad \chi = \frac{1}{2}(\rho + \omega^2)$$

(2.4)

to the action (2.1) and constraints (2.2) gives

$$S_{\text{kin}} = \frac{1}{\pi} \int d^2\sigma \left\{ -4\kappa \partial_+ \chi \partial_- \chi + 4\kappa (\omega^2 - 1) \partial_+ \omega \partial_- \omega + \frac{1}{2} \sum_{i=1}^{N} \partial_+ f_i \partial_- f_i \right\}$$

$$S_{\text{cos}} = \frac{1}{\pi} \int d^2\sigma \left[ \kappa \lambda^2 \omega^2 e^{-2\omega^2} D(1/\kappa \omega^2) e^{4\chi} \right]$$

(2.5)

$$T_{\pm \pm} = -4\kappa \partial_\pm \chi \partial_\pm \chi + 2\kappa \partial_\pm^2 \chi + 4\kappa (\omega^2 - 1) \partial_\pm \omega \partial_\pm \omega + \frac{1}{2} \sum_{i=1}^{N} \partial_\pm f_i \partial_\pm f_i .$$

A further transformation on $\omega$ alone,

$$\partial \Omega = \sqrt{\omega^2 - 1} \partial \omega \quad \Rightarrow \quad \Omega = \frac{1}{2} \omega \sqrt{\omega^2 - 1} - \frac{1}{2} \log(\omega + \sqrt{\omega^2 - 1}) ,$$

(2.6)

finally reduces $S_{\text{kin}}$ and $T_{\pm \pm}$ to free field form:

$$S_{\text{kin}} = \frac{1}{\pi} \int d^2\sigma \left\{ -4\kappa \partial_+ \chi \partial_- \chi + 4\kappa \partial_+ \Omega \partial_- \Omega + \frac{1}{2} \sum_{i=1}^{N} \partial_+ f_i \partial_- f_i \right\}$$

(2.7)

$$T_{\pm \pm} = -4\kappa \partial_\pm \chi \partial_\pm \chi + 2\kappa \partial_\pm^2 \chi + 4\kappa \partial_\pm \Omega \partial_\pm \Omega + \frac{1}{2} \sum_{i=1}^{N} \partial_\pm f_i \partial_\pm f_i .$$

There are various subtleties to discuss at this point. As is apparent from the definition (2.4) of $\omega$, we are assuming that $\kappa > 0$. We shall stay with that assumption for the moment and come back later to the question of what happens when $\kappa < 0$. The principal issue concerns the range over which the fields $\phi$, $\omega$ and $\Omega$ are supposed to vary. As $\phi$ ranges from $-\infty$ to $\infty$, $\omega$ ranges from 0 to $\infty$. But (2.3) shows that the signature of the $\omega$ kinetic term changes sign at $\omega = 1$. In the large-$N$ treatments of this problem, a curvature singularity
was always found at this critical value of the dilaton field and it has been suggested that
this is a natural boundary and that $\omega$ should be restricted to range from 1 to $\infty$ (the
corresponding range of $\Omega$ is then from 0 to $\infty$). Such restrictions are certainly not natural
from the point of view of the action (2.7) and we have no idea how to implement them
(or even if they are really necessary). To proceed, we will assume that such boundary
conditions are either not necessary, or harmless, and that all relevant information about
the conformal properties of the theory is contained in the action.

It is now a straightforward matter to analyze the central charge. The argument is
clearest if we rescale the fields $\chi$ and $\Omega$ in (2.7) by a factor of $2\sqrt{\kappa}$ so as to eliminate $\kappa$, obtaining

$$S_{\text{kin}} = \frac{1}{\pi} \int d^2\sigma \left[ - \partial_+ \hat{\chi} \partial_- \hat{\chi} + \partial_+ \hat{\Omega} \partial_- \hat{\Omega} + \frac{1}{2} \sum_{i=1}^N \partial_+ f_i \partial_- f_i \right]$$

$$T_{\pm\pm} = - \partial_\pm \hat{\chi} \partial_\pm \hat{\chi} + \sqrt{\kappa} \partial_\pm^2 \hat{\chi} + \partial_\pm \hat{\Omega} \partial_\pm \hat{\Omega} + \frac{1}{2} \sum_{i=1}^N \partial_\pm f_i \partial_\pm f_i .$$  (2.8)

The $N$ matter fields and the free $\hat{\Omega}$ field each of course contribute unity to the central
charge. The $\hat{\chi}$ field is free, but has an improvement term, so its contribution is shifted from
unity. With our normalizations, one would normally set $c_\chi = 1 + 12(\sqrt{\kappa})^2 > 1$. However,
since $\hat{\chi}$ has a ”wrong-sign” kinetic term (relative to the matter fields), its central charge is
decreased from unity by the improvement term and the correct formula is $c_\chi = 1 - 12(\sqrt{\kappa})^2$.
The net result is $c = (1 - 12\kappa) + 1 + N = N + 2 - 12\kappa$ and we can set $c = 26$ by taking

$$\kappa = (N - 24)/12 .$$  (2.9)

This behaves as expected for large $N$, but it is perhaps a bit surprising that the $1/N$
corrections stop at $O(N^0)$.

Finally, we have to ask whether the above line of argument can be carried through
for $\kappa < 0$ so that we can deal with the case $N < 24$. Since the central charge formula is
perfectly analytic in $N$, there should be no difficulty, but it will be instructive to consider
the question in detail. The version of the field redefinitions of (2.4) appropriate to the
$\kappa < 0$ case is

$$\hat{\omega} = \frac{e^{-\phi}}{\sqrt{|\kappa|}} \quad \hat{\chi} = \frac{1}{2}(\rho - \hat{\omega}^2) .$$  (2.10)
Their application to the action (2.1) and constraints (2.2) gives

\[ S_{\text{kin}} = \frac{1}{\pi} \int d^2 \sigma \left[ 4|\kappa|\partial_+ \hat{\chi} \partial_- \hat{\chi} - 4|\kappa|(\bar{\omega}^2 + 1)\partial_+ \bar{\omega} \partial_- \bar{\omega} + \frac{1}{2} \sum_{i=1}^{N} \partial_+ f_i \partial_- f_i \right] \]

\[ S_{\text{cos}} = \frac{1}{\pi} \int d^2 \sigma \left[ |\kappa| \lambda^2 \bar{\omega}^2 e^{2\bar{\omega}^2} D(1/|\kappa| \bar{\omega}^2) e^{4\chi} \right] \]

\[ T_{\pm \pm} = 4|\kappa|\partial_+ \hat{\chi} \partial_+ \hat{\chi} - 2|\kappa|\partial_+ \hat{\chi} - 4|\kappa|(\bar{\omega}^2 + 1)\partial_+ \bar{\omega} \partial_- \bar{\omega} + \frac{1}{2} \sum_{i=1}^{N} \partial_+ f_i \partial_- f_i . \]

(2.11)

Two important things have changed compared to the previous case: the signature of the \( \chi \) kinetic term has switched from time-like to space-like (so that its improvement term now increases the central charge) and the signature of the \( \bar{\omega} \) term is now definite (and timelike).

A final transformation on \( \bar{\omega} \) alone,

\[ \partial \bar{\Omega} = -\sqrt{\bar{\omega}^2 + 1} \partial \bar{\omega} \Rightarrow \bar{\Omega} = -\frac{1}{2} \sqrt{\bar{\omega}^2 + 1} - \frac{1}{2} \log(\bar{\omega} + \sqrt{\bar{\omega}^2 + 1}) , \]

(2.12)

transforms the action and constraints to free field form:

\[ S_{\text{kin}} = \frac{1}{\pi} \int d^2 \sigma \left[ \partial_+ \hat{\chi} \partial_- \hat{\chi} - \partial_+ \hat{\Omega} \partial_- \hat{\Omega} + \frac{1}{2} \sum_{i=1}^{N} \partial_+ f_i \partial_- f_i \right] \]

(2.13)

\[ T_{\pm \pm} = \partial_+ \hat{\chi} \partial_+ \hat{\chi} - \sqrt{|\kappa|}\partial_+ \hat{\chi} - \partial_+ \hat{\Omega} \partial_- \hat{\Omega} + \frac{1}{2} \sum_{i=1}^{N} \partial_+ f_i \partial_- f_i . \]

(we have rescaled the \( \chi \) and \( \Omega \) fields by a further factor of \( 2\sqrt{|\kappa|} \) in order to have standard normalization).

The central charge calculation is the same as before, except that, since the \( \chi \) field now has spacelike signature, the improvement term increases its central charge. With this modification, we have \( c = (1 + 12|\kappa|) + 1 + N = N + 2 - 12\kappa \) and the \( c = 26 \) condition reduces to \( |\kappa| = (24 - N)/12 \). Since \( \kappa \) is negative, this is the same result as before. We note that the kinetic part of the action and constraints in (2.13), when written in terms of the unrescaled fields \( \hat{\chi} \) and \( \hat{\Omega} \), are exactly the same as (2.7) for \( \chi \) and \( \Omega \), and the dependence on \( \kappa \) is analytic. One interesting difference concerns the ranges of the fields. Referring back to the discussion around (2.3), we see that if \( \kappa < 0 \), there is no real value of the dilaton field at which the determinant vanishes and therefore no singularity to avoid by restricting the range of variation of any of the fields. It is true that as \( \phi \) ranges from \( -\infty \) to \( \infty \), \( \omega \) and \( \Omega \) only take on positive values. Since \( \omega \) is \( 1/g_{\text{string}}^2 \), and since all physical quantities depend only on \( g_{\text{string}}^2 \), it seems quite reasonable to let the \( \omega \) fields range over negative values as well.
3. Constructing (1, 1) Operators

The next task is to identify the cosmological constant actions, $S_{\text{cos}}$, which can consistently be added to the $c = 26$ conformal theory defined by $S_{\text{kin}}$. The infinitesimal conformal perturbations are identified by the operators of conformal weight $(1, 1)$ with respect to $S_{\text{kin}}$. Finding the exact finite conformal perturbations is a lot harder, but experience with Liouville-like theories (and the theory at hand is of that general type, as we shall see) shows that the infinitesimal perturbations are usually exact (no higher-loop corrections to the anomalous dimension operator). We don’t know for sure that this is so, but, for the purposes of this paper, we will assume that it is enough to identify the $(1, 1)$ operators.

Consider first the $\kappa > 0$ case. It is of course easiest to construct the conformal weight operator in terms of the conventionally normalized free fields $\hat{\chi}$ and $\hat{\Omega}$ (using the action and energy-momentum tensor specified in (2.8)). The condition for a scalar function $V(\hat{\chi}, \hat{\Omega})$ to be a $(1, 1)$ operator is

$$\left[ \frac{1}{2} \sqrt{\kappa} \frac{\partial}{\partial \hat{\chi}} + \frac{\partial^2}{\partial \hat{\chi}^2} - \frac{\partial^2}{\partial \hat{\Omega}^2} \right] V(\hat{\chi}, \hat{\Omega}) = V(\hat{\chi}, \hat{\Omega}).$$

(3.1)

The opposite signs of the second derivative terms of course stems from the opposite signatures of the two fields. The linear term in $\partial/\partial \hat{\chi}$ of course comes from the improvement term and its normalization can be checked in the classical limit ($\kappa \to \infty$) where the dimension operator is dominated by the $\sqrt{\kappa}$ term: The classical cosmological constant term displayed in (2.5) can be rewritten as

$$V_{\text{class}} = \kappa \omega^2 e^{-2\omega^2} e^{4\hat{\chi}} = F(\hat{\Omega}) e^{\sqrt{\kappa} \hat{\chi}}.$$

(3.2)

and it is obviously a solution of the “classical” $(1, 1)$ condition

$$\frac{1}{2} \sqrt{\kappa} \frac{\partial}{\partial \hat{\chi}} V(\hat{\chi}, \hat{\Omega}) = V(\hat{\chi}, \hat{\Omega}).$$

(3.3)

There are many exact solutions to (3.1) and a particularly simple class is given by (the normalization is chosen for later convenience)

$$V_{\text{exact}}^\pm = \frac{\kappa}{4e} e^{\pm \frac{2}{\sqrt{\kappa}}} \hat{\Omega} e^{\frac{2}{\sqrt{\kappa}} \hat{\chi}} = \frac{\kappa}{4e} e^{4\hat{\chi} \pm 4\Omega}.$$

(3.4)

We are most interested in solutions which reduce to the classical cosmological constant $e^{2(\rho - \phi)}$ in the classical limit $e^\phi \to 0$. The solution (3.4) with the minus sign in the
exponent in fact has this property. Using the transformations given in (2.4) and (2.6) we find that
\[ V_{\text{exact}} = \frac{\kappa}{4e} e^{4x-4\Omega} = e^{2\rho} e^{-2\phi} D(e^{2\phi}) \] (3.5)
where
\[ D(e^{2\phi}) = \frac{1}{4} (1 + \sqrt{1 - \kappa e^{2\phi}})^2 \exp \left[ \frac{(1 - \sqrt{1 - \kappa e^{2\phi}})}{(1 + \sqrt{1 - \kappa e^{2\phi}})} \right] . \] (3.6)

The suggestion is that with this choice of the cosmological constant function, the theory defined in (2.1) (with the appropriate value of \( \kappa \)) is an exact \( c = 26 \) conformal field theory. Though there are other choices of \( D(e^{2\phi}) \) for which the theory is conformal, (3.6) is the unique one which reduces to the original classical action in the weak coupling limit, i.e. \( D(0) = 1 \). Thus this exact conformal theory should have the same structure of asymptotic states as were encountered in the original semiclassical studies of these theories and should be suitable for the exact study of the black hole evaporation problem. Note that the cosmological constant function now has a branch point at precisely the value of the dilaton field where the determinant of the kinetic term changes sign. We will have some comments on how to deal with the associated singularities later on in the paper.

It remains to determine the function \( D(\phi) \) appropriate to the \( \kappa < 0 \) case. As the discussion of the central charge made clear, one can, for all practical purposes, take the \( \kappa > 0 \) formulas and analytically continue them to \( \kappa < 0 \). The \((1,1)\) operator with the desired weak coupling behavior is easily seen to be
\[ \tilde{V}_{\text{exact}} = \frac{|\kappa|}{4e} e^{4\bar{x}-4\bar{\Omega}} = \frac{|\kappa|}{4e} e^{4\bar{x}} (\bar{\omega} + \sqrt{1 + \bar{\omega}^2})^2 e^{2\bar{\omega}} \sqrt{\bar{\omega}^2 + 1} = e^{2(\rho - \phi)} \frac{1}{4} (1 + \sqrt{1 + |\kappa| e^{2\phi}})^2 \exp \left[ \frac{(1 - \sqrt{1 + |\kappa| e^{2\phi}})}{(1 + \sqrt{1 + |\kappa| e^{2\phi}})} \right] . \] (3.7)
An important point is that this function is well-behaved for all values of \( \phi \). Since, as we showed in the previous section, the kinetic action is free of singularity as well, it seems plausible that the \( \kappa < 0 \) theory is a completely well-defined quantum theory of black hole evaporation. We will explore this notion in a later section. Note also that, in the strong coupling limit \( (e^\phi \to \infty) \), (3.7) asymptotes to \( \frac{|\kappa|}{4e} e^{2\rho} \) and the dependence of the action (2.1) on the metric variable \( \rho \) reduces to the standard Liouville action. Since the Liouville theory is well-behaved and since the whole issue of singularities has to do with the behavior of the theory at strong coupling, this suggests that the \( \kappa < 0 \) theory should have no singularity problems at the quantum level.
4. Exact Solutions

We would now like to see what can be said about the formation and subsequent evaporation of black holes in the conformally invariant dilaton-gravity theory constructed in the preceding sections. Ideally, of course, we should develop a full quantum treatment. While this may be possible, it has yet to be done and we will limit ourselves here to a study of the classical solutions of the action (2.1) (with the special values of $\kappa$ and $D(\phi)$ described above). Apart from the “improved” choice of $D(\phi)$, this is the approach adopted in all previous work on this subject [1,2,3] and, as has been explained elsewhere, it amounts to a semiclassical treatment in which matter quantum loops are accounted for, but graviton-dilaton loops are not. Such an approach includes the basic physics of Hawking radiation, in which left-moving infalling matter creates, via the anomaly, right-moving outgoing matter radiation along with an appropriate back-reaction on the metric. The approximation will of course fail if the dilaton-gravity theory becomes strongly coupled anywhere in spacetime. We will defer discussion of the self-consistency of the approximation until after we have constructed the solutions of the classical equations. As we will now explain, with the new conformally invariant form of $D(\phi)$ ((3.6) or (3.7)), the equations of motion become exactly soluble.

Consider the $\kappa > 0$ case. In the previous sections we have shown that, with the special choice (3.6) for $D(\phi)$, the action (2.1) has a very simple form when expressed in terms of the fields $\chi$ and $\Omega$ (these fields can be expressed in terms of the original $\rho$ and $\phi$ fields using (2.4) and (2.6):

$$S^{\rho,\phi} = \frac{4\kappa}{\pi} \int d^2\sigma \left[ -\partial_+ \chi \partial_- \chi + \partial_+ \Omega \partial_- \Omega + \frac{\lambda^2}{16e} e^{4(\chi-\Omega)} \right].$$ (4.1)

Things are even simpler in terms of $\Psi_\pm = \chi \pm \Omega$:

$$S^{\rho,\phi} = \frac{4\kappa}{\pi} \int d^2\sigma \left[ -\partial_+ \Psi_+ \partial_- \Psi_- + \frac{\lambda^2}{16e} e^{4\Psi_-} \right].$$ (4.2)

The equations of motion that follow from this action are simply

$$\partial_+ \partial_- \Psi_- = 0, \quad \partial_+ \partial_- \Psi_+ = -\frac{\lambda^2}{4e} e^{4\Psi_-}.$$ (4.3)

For $\kappa < 0$ everything works out the same way except that now

$$\partial_+ \partial_- \Psi_- = 0, \quad \partial_+ \partial_- \Psi_+ = +\frac{\lambda^2}{4e} e^{4\Psi_-},$$ (4.4)
where $\Psi_{\pm} = \overline{\chi} \pm \overline{\Omega}$. The most general solution for $\kappa > 0$ or $\kappa < 0$ is easy to write down:

$$2\Psi_- = \alpha(x^+) + \beta(x^-) + \log(2/(\lambda \sqrt{|\kappa|})) + \frac{1}{2}$$

$$2\Psi_+ = 2\gamma(x^+) - \alpha(x^+) + 2\delta(x^-) - \beta(x^-) + \log(2/(\lambda \sqrt{|\kappa|})) + \frac{1}{2}$$

$$- \frac{2}{\kappa} \int^{x^+} dy \ e^{2\alpha(y)} \int^{x^-} dz \ e^{2\beta(z)} ,$$

(4.5)

where $\alpha$ etc. are arbitrary functions of integration. Since the $\Psi_{\pm}$ are known functions of the original fields $\rho$ and $\phi$, the latter can be found by solving a pair of transcendental equations:

$$2\Omega(\phi) = \log\left(\frac{1 - \sqrt{1 - \kappa e^{2\phi}}}{\kappa}\right) - \phi + \frac{e^{-2\phi}}{\kappa} \sqrt{1 - \kappa e^{2\phi}} + \frac{1}{2} \log |\kappa| =$$

$$\gamma(x^+) + \delta(x^-) - \alpha(x^+) - \beta(x^-) - \frac{1}{\kappa} \int^{x^+} dy \ e^{2\alpha(y)} \int^{x^-} dz \ e^{2\beta(z)}$$

$$\rho + \log(\lambda) + \frac{e^{-2\phi}}{\kappa} = \gamma(x^+) + \delta(x^-) - \frac{1}{2} \log |\kappa|$$

$$+ \log(2) + \frac{1}{2} - \frac{1}{\kappa} \int^{x^+} dy \ e^{2\alpha(y)} \int^{x^-} dz \ e^{2\beta(z)} .$$

(4.6)

One may choose any sufficiently regular functions $\alpha, \ldots, \delta$ in (4.5) or (4.6) (though some restrictions follow from boundary conditions of various kinds). Now, the equations of motion are conformally invariant if $\phi$ is a scalar and $\rho$ transforms as $\tilde{\rho}(y) = \rho(x) - \frac{1}{2} \log\left(\frac{dy^+}{dy^-} - \frac{dy^-}{dy^+}\right)$. Thus, two sets of functions $\tilde{\alpha}(y), \ldots, \tilde{\delta}(y)$ and $\alpha(x), \ldots, \delta(x)$ correspond to the same solution if they are related by

$$\tilde{\alpha}(y^+) = \alpha(x^+) - \frac{1}{2} \log \frac{dy^+}{dx^+} , \quad \tilde{\beta}(y^-) = \beta(x^-) - \frac{1}{2} \log \frac{dy^-}{dx^-}$$

$$\tilde{\gamma}(y^+) = \gamma(x^+) - \frac{1}{2} \log \frac{dy^+}{dx^+} , \quad \tilde{\delta}(y^-) = \delta(x^-) - \frac{1}{2} \log \frac{dy^-}{dx^-} .$$

(4.7)

Alternatively, these are just the rules for the transformation of $\alpha, \ldots, \delta$ under a conformal coordinate transformation.

The $T_{\pm\pm}$ constraints associated with the dilaton-gravity action have a particularly simple form in terms of the $\chi$ and $\Omega$ fields (see (2.7) and (2.13)) and therefore also in terms of the $\Psi_{\pm}$ fields. Evaluated on the solution (1.5), the $\rho, \phi$-part of the constraints are

$$T_{++}^{\rho,\phi} = \kappa(\partial_+(\gamma - \alpha))^2 - \kappa(\partial_+ \gamma)^2 + \kappa \partial_+^2 \gamma$$

$$T_{--}^{\rho,\phi} = \kappa(\partial_-(\delta - \beta))^2 - \kappa(\partial_- \delta)^2 + \kappa \partial_-^2 \delta .$$

(4.8)
They of course automatically satisfy the conservation equations \( \partial_\tau T^{\rho,\phi}_{\mp \pm} = 0 \). As appropriate, we will try to interpret non-zero values for \( T^{\rho,\phi}_{\mp \pm} \) in terms of ingoing or outgoing fluxes of the \( f \)-matter fields (i.e. \( T^M_{\mp \pm} \neq 0 \)).

A significant advantage of having this exact solution, as compared to the situation in the \( D(\phi) = 1 \) theory, is that we can make analytic statements about the global spacetime structure of solutions without having to resort to numerical integration. One might wonder whether the \( D(\phi) = 1 \) equations could also be solved exactly. If so, there should be other conserved quantities besides \( T_{\mp \pm} \). We have checked that no conserved quantities of dimensions one, three or four (except for trivial ones built out of \( T \) itself) exist. This strongly suggests that the \( D(\phi) = 1 \) equations are not integrable.

### 5. Decoupling the Ghosts

We now want to show that the procedures developed above can be applied to an interesting variant of the action (2.1). This is important because, as we shall see, the solutions of the theory we have been studying are physically worrisome for a variety of reasons. In this section, we will study a generalization of (2.1) which can also be treated using the techniques presented in this paper and whose static solutions seem to be physically perfectly satisfactory (at least for \( N < 24 \)).

The starting point of our procedure is a split of the action into a kinetic part and a cosmological constant part. The kinetic part is the standard dilaton-gravity kinetic term plus a Polyakov term representing the net anomaly due to \( N \) matter fields plus the dilaton-gravity fields. We adjust constants so that, in conformal gauge, this kinetic action is a conformal field theory and then construct the cosmological constant action as a \((1,1)\) operator with respect to the kinetic action. The resulting theory differs from the original model only in the precise form of the cosmological constant (and is qualitatively different only in the strong coupling region).

Strominger [8] has observed that there is good physical reason to consider a modification of the kinetic action itself. His basic point is that the form of the Polyakov anomaly action depends on the metric used to define the path integral measure and that it is not obvious that one wants to use the same metric for the graviton-dilaton-ghost system as one uses for the matter fields. One could, as one often does in string theories, use a Weyl-transformed metric of the type \( g^{[\alpha]}_{ij} = e^{-\alpha \phi} g_{ij} \) (where \( g_{ij} \) is the metric appearing in the graviton-dilaton action and \( \alpha \) is some constant). The action used so far in this paper
corresponds to using the “true” ($\alpha = 0$) metric to define all the measures. This appears to make matter, ghosts, gravitons and dilatons all contribute to the Hawking radiation on the same footing (with the ghosts of course giving a negative flux!). This seems unphysical, since the graviton and dilaton represent non-propagating degrees of freedom and the ghosts should never appear in any on-shell process.

Strominger proposes a simple way out of the problem: Use the rescaled metric $g_{ij}^{[2]}$ to define the measure of the graviton-dilaton-ghost system. It has the nice property that it is flat in any solution of the ($D(\phi) = 1$) classical theory (1.3) (in conformal gauge $g_{ij} = e^{2\rho} \delta_{ij}$ and one can further gauge fix a solution to $\rho = \phi$), thus decoupling the unphysical fields from the geometry, at least in some leading order. In conformal gauge, this is implemented by building the graviton-dilaton-ghost Polyakov term out of $\rho - \phi$, while building the matter anomaly term out of $\rho$, as before. The resulting kinetic action is a simple modification of (2.1):

$$S'_{\text{kin}} = \frac{1}{\pi} \int \sigma^2 \sigma \left[ e^{-2\phi} (2 \partial_+ \rho \partial_- \phi + 2 \partial_+ \phi \partial_- \rho - 4 \partial_+ \phi \partial_- \phi) + \frac{1}{2} \sum_{i=1}^{N} \partial_+ f_i \partial_- f_i - \frac{N}{12} \partial_+ \rho \partial_- \rho + 2 \partial_+ (\rho - \phi) \partial_- (\rho - \phi) \right].$$

(5.1)

We could have written the two types of Polyakov term with arbitrary coefficients, to be determined by the requirement that the action generate a $c = 26$ conformal field theory, but we have anticipated the result of that calculation. We will now show that this theory can be explicitly transformed to free fields, that the $(1,1)$ cosmological constant term can be written down explicitly, and that explicit exact solutions of the classical equations of motion can be found. The argument is a simple modification of what was presented in earlier sections, so we shall be brief. (As a side remark, we note that all this could have been done for an action with the anomaly term built out of the most general linear combination of $\partial \rho \partial \rho$, $\partial \phi \partial \phi$ and $\partial \rho \partial \phi$. Since we have no physical motivation for this, we will restrict our attention to (5.1).)

By analogy to (2.4), we define new fields $\omega$ and $\chi$ by

$$\omega = e^{-\phi} \quad \rho = 2\chi + f(\omega)$$

(note that this time we do not rescale $\omega$ by $\sqrt{\kappa}$) and try to choose $f(\omega)$ so as to diagonalize the action. One can easily show that $f(\omega) = \frac{2}{\kappa} (\log \omega - \omega^2/2)$ does the trick (where
\( \kappa = (N - 24)/12 \) as before. The diagonalized kinetic action of the graviton-dilaton sector is

\[
S'_{\text{kin}} = \frac{1}{\pi} \int d^2 \sigma \left[ \frac{4}{\kappa} \left( \omega^2 - (\kappa + 2) + \frac{\kappa + 2}{2\omega^2} \right) (\partial_+ \omega \partial_- \omega) - 4\kappa (\partial \chi \partial \chi) \right].
\]

(5.3)

If we define a new field \( \Omega \) such that

\[
(\partial \Omega)^2 = \frac{1}{\kappa^2} \left( \omega^2 - (\kappa + 2) + \frac{\kappa + 2}{2\omega^2} \right) (\partial \omega)^2 \Rightarrow \Omega = \frac{1}{\kappa} \int \omega \, \sqrt{y^2 - (\kappa + 2) + \frac{\kappa + 2}{2y^2}},
\]

(5.4)

then the kinetic action and constraints finally simplify to (we do not write the matter contributions explicitly)

\[
S'_{\text{kin}} = \frac{1}{\pi} \int d^2 \sigma \left[ -4\kappa \partial_+ \chi \partial_- \chi + 4\kappa \partial_+ \Omega \partial_- \Omega \right]
\]

(5.5)

\[
T_{\pm \pm} = -4\kappa \partial_+ \chi \partial_- \chi + 2\kappa \partial_\pm^2 \chi + 4\kappa \partial_\pm \Omega \partial_\pm \Omega.
\]

This is precisely the same as the system (2.7) which we have already studied and we can immediately draw the same conclusions: \( S'_{\text{kin}} \) defines a \( c = 26 \) conformal theory if \( \kappa = (N - 24)/12 \) (which we have already assumed) and the \((1,1)\) conformal fields are \( V_{(1,1)}^\pm = e^{4(X_{\pm} + \Omega)} \). It again turns out that the appropriately normalized \( V_{(1,1)}^- \) reduces to the original classical cosmological constant in the weak coupling (\( \phi \rightarrow -\infty \)) limit. This normalization depends on the choice of integration constant in the integral for \( \Omega \). To be specific, we choose it such that in the weak coupling limit (\( \omega \rightarrow \infty \)) \( 2\Omega \) is given by

\[
2\Omega \sim e^{-2\phi} + \frac{\kappa + 2}{2} \phi - \log 2 + \frac{1}{2} \log |\kappa| + \frac{\kappa + 2}{8} e^{2\phi} + O(e^{4\phi}).
\]

(5.6)

Then the correctly normalized conformally invariant cosmological constant reads

\[
S_{\text{cos}} = \frac{\lambda^2}{\pi} \int d^2 \sigma \frac{|\kappa|}{4e} e^{4\chi - 4\Omega},
\]

(5.7)

and the full action to be solved, the equations of motion and the general solution are exactly the same as in (4.1) through (4.5). Also, \( T_{\pm \pm}^{\rho \phi} \), when expressed in terms of \( \alpha, \ldots, \delta \), is the same as in (4.8). The relation between the free \( \chi \) and \( \Omega \) fields and the “physical” fields \( \rho \) and \( \phi \) are different, however, and we must replace (4.6) by

\[
2\Omega(\omega) = \gamma(x^+) + \delta(x^-) - \alpha(x^+) - \beta(x^-)
\]

\[
- \frac{1}{\kappa} \int_{x^+}^{x^-} dy \, e^{2\alpha(y)} \int_{x^-}^{x^+} dz \, e^{2\beta(z)} \rho + \log(\lambda) + \frac{\omega^2}{\kappa} - \frac{2}{\kappa} \log \omega = \gamma(x^+) + \delta(x^-) - \frac{1}{2} \log |\kappa| + \log(2) + \frac{1}{2}
\]

(5.8)
This formula is valid for both signs of $\kappa$. In what follows, we shall compare the behavior of the explicit solutions, (4.6) and (5.8), of the two theories we have studied. Just to distinguish them, we will call them Theory I (the original case) and Theory II (the theory with decoupled ghosts). We should perhaps warn the reader that the decoupling of the ghosts in Theory II will turn out to be less than perfect, leaving open the question whether some further generalization of the original theory is in order.

6. Static Solutions

The sort of question we want to ask of these (allegedly) exact quantum theories concerns the behavior of left-moving $f$-particle excitations sent in from infinity (where the theory is asymptotic to the weak coupling limit of the linear dilaton vacuum). A necessary preliminary to any quantum discussion of scattering questions, of course, is an identification of the vacuum state around which the scattering takes place. An important feature of the original matter-anomaly-improved action (1.3) is that the classical linear dilaton vacuum ($\rho = 0, \phi = -\lambda \sigma$) is still a solution. It merits the appellation of vacuum state because the curvature is everywhere zero and there is a pseudo-translation invariance in which the dilaton is shifted along with the spatial coordinate. It is fairly easy to see that the linear dilaton vacuum is not a solution of either of our exact theories. Indeed, as far as we can tell, there is no solution which has zero curvature everywhere and no obvious candidate for “the” vacuum state.

Fortunately for the physical interpretation of this theory, we can find a family of static solutions whose asymptotic behavior is governed by the linear dilaton vacuum. To proceed, we look for restrictions on the arbitrary functions $\alpha, \ldots, \delta$ in the general time-dependent solutions (4.6) and (5.8) such that the solution is static (i.e. a function of $\sigma = \frac{1}{2}(x_+ - x_-)$ only) and asymptotic to the linear dilaton vacuum as $\sigma \to \infty$. It is fairly easy to see that these boundary conditions imply a) that the $\alpha, \ldots, \delta$ are all linear functions of their arguments b) that the linear terms in $\alpha$ and $\beta$ ($\gamma$ and $\delta$ resp.) are equal and opposite and c) that the linear term in $\alpha$ is completely fixed by the linear dilaton vacuum behavior of $\phi$. Using our freedom to rescale the $x^\pm$-coordinate and shift its origin, we can write the most general set of $\alpha \ldots \delta$ consistent with these conditions as

$$\begin{align*}
\alpha(x^+) &= \frac{1}{2}x^+, & \gamma(x^+) &= \frac{1}{2}Sx^+ + \frac{1}{2}T + \frac{1}{2}U - \log 2 - \frac{1}{2} + \frac{1}{2} \log |\kappa| \\
\beta(x^-) &= -\frac{1}{2}x^-, & \delta(x^-) &= -\frac{1}{2}Sx^- + \frac{1}{2}T - \frac{1}{2}U
\end{align*}$$

(6.1)
where \( S, T \) and \( U \) are arbitrary constants. Since the solutions involve only \( \gamma + \delta, U \) is irrelevant and we seem to have a two-parameter family of static solutions.

The \( \rho \) and \( \phi \) fields are found, in the case of Theory I, by solving

\[
2\Omega(\phi) = (S - 1)\sigma + \frac{e^{2\sigma}}{\kappa} + T - 2T_c
\]

\[
\rho(\sigma) + \frac{e^{-2\phi}}{\kappa} + \log \lambda = S\sigma + \frac{e^{2\sigma}}{\kappa} + T
\]

and, in the case of Theory II, by solving

\[
2\Omega(\phi) = (S - 1)\sigma + \frac{e^{2\sigma}}{\kappa} + T - 2T_c
\]

\[
\rho(\sigma) + \frac{e^{-2\phi}}{\kappa} + \frac{2\phi}{\kappa} + \log \lambda = S\sigma + \frac{e^{2\sigma}}{\kappa} + T
\]

where we introduced for later convenience

\[
T_c = -\frac{1}{4} \log \left| \kappa \right| \quad .
\]

In the limit \( \sigma \to +\infty \), \( \rho \) and \( \phi \) approach the linear dilaton vacuum exponentially rapidly. For Theory I, we have

\[
\phi \sim -\sigma + \delta_I(\sigma)e^{-2\sigma} \quad , \quad \rho(\sigma) + \log \lambda \sim (\delta_I(\sigma) + \frac{\kappa}{8})e^{-2\sigma} \quad ,
\]

where \( \delta_I = -\frac{\kappa}{2}(S\sigma + T) \) and further corrections are of order \( e^{-4\sigma} \). For Theory II, we have

\[
\phi \sim -\sigma + \delta_{II}(\sigma)e^{-2\sigma} \quad , \quad \rho(\sigma) + \log \lambda \sim (\delta_{II}(\sigma) + \frac{\kappa + 2}{8})e^{-2\sigma} \quad ,
\]

where \( \delta_{II} = -\frac{\kappa}{2}((S + \frac{2}{\kappa})\sigma + T) \) and further corrections are of order \( e^{-4\sigma} \).

In a static solution, the constraints must be constant and it suffices to evaluate them at \( \sigma \sim +\infty \) where one finds (see also (4.8)), for both theories, that \( T_{\pm\pm}^{\rho,\phi} = \frac{\kappa}{4}(1 - 2S) \).

We presume that \( T_{\pm\pm}^{\rho,\phi} \neq 0 \) means that there is a non-zero flux of \( f \)-matter at infinity (or, worse yet, a flux of ghosts) sustaining the solution. Since we are looking for ground states, we want solutions with zero incident flux at infinity and we must set \( S = \frac{1}{2} \). This leaves one free parameter, \( T \), in the family of static solutions, so these “ground states” are not unique. This parameter is something like the ADM mass, but we have not yet constructed a plausible ADM mass in these new theories. (By ADM mass we mean a conserved quantity
expressible in terms of the asymptotic behavior of the fields.) However, it will turn out that

\[ M = \kappa \lambda (T - T_c) \]  

(6.7)

is the obvious candidate for the mass.

Now we turn to a description of the solutions. There are two main regimes, \( \kappa < 0 \) and \( \kappa > 0 \), for each of the two theories and the behavior is qualitatively different in each of the four cases. We will discuss them in turn. In all cases, as \( \sigma \to \infty \), the solution approaches the standard linear dilaton vacuum. The only issue is what happens as \( \sigma \to -\infty \) (provided the solution does not run into a singularity on the way).

Consider first Theory I with \( N > 24 \). Then by (6.2), as a function of \( \sigma \), \( \Omega \) has a minimum at \( \sigma = \frac{1}{2} \log \frac{4}{\kappa} \). At this minimum, \( 2\Omega = T - T_c \). On the other hand, as a function of \( \phi \) (\( \leq \phi_{\text{crit}} = -\frac{1}{2} \log \kappa \)), one has \( \Omega \geq 0 \), and \( \Omega \) has its minimum at \( \phi = \phi_{\text{crit}} \) where \( \Omega = 0 \). (\( \Omega \) is complex for \( \phi > \phi_{\text{crit}} \).) As a consequence, \( \phi \) is well determined for all \( \sigma \) if \( T > T_c \), while for \( T < T_c \) it is not. Hence the following picture emerges. Let first \( T > T_c \). Then, as \( \sigma \) decreases from \( +\infty \), \( \phi \) increases, reaches some maximum \( \phi_{\text{max}} \), and then decreases to \( -\infty \) as \( \sigma \) runs off to \( -\infty \). The curvature blows up at \( \sigma = -\infty \), and this point is a finite proper distance away from any finite \( \sigma \): \( \sigma = -\infty \) is a singular horizon. The maximum value of \( \phi \) depends on the free parameter \( T \) in the general static solution. As \( T \) decreases, \( \phi_{\text{max}} \) increases, and eventually at \( T = T_c \) passes through the critical value \( \phi_{\text{crit}} = -\frac{1}{2} \log \kappa \), at which point a (naked) curvature singularity forms on the timelike line where \( \phi = \phi_{\text{crit}} \). These solutions are very similar to the finite mass static solutions which were found in the study of the original action (2.1) with \( D(e^{2\phi}) = 1 \). The fact that a naked singularity is present precisely for \( M < 0 \) confirms our interpretation of \( M \) as the mass of the solution.

Now let \( N < 24 \). As has been noted before, there is no critical value of \( \phi \) at which a singularity is bound to occur, so these solutions should be in some sense better-behaved. The equation for \( \phi \) is best regarded as an equation for \( \omega = e^{-\phi} \) and the solution is such that, as \( \sigma \) runs from \( +\infty \) to \( -\infty \), \( \omega \) runs monotonically from \( +\infty \) to \( -\infty \). Since \( \omega \) is the inverse of the coupling constant of the theory, the \( \sigma = -\infty \) side of the world has weak negative coupling. It also has vanishing curvature and is an infinite proper distance away from finite \( \sigma \). Since the equations of motion only involve \( \omega^2 \), negative coupling is probably indistinguishable from positive coupling, and we will provisionally regard the negative \( \omega \) half of the world as just as physical as the positive half. So, this world has two types
of weak coupling vacuum, one for positive coupling and one for negative coupling, and the static solutions glue the two together in a non-unique way characterised by the free parameter $T$. In all these solutions, there is a point where $\omega \rightarrow 0$ and $\phi \rightarrow \infty$, but nothing singular happens to any other quantity.

Now consider Theory II for $N < 24$. This time no excuses are needed: as $\sigma$ ranges from $+\infty$ to $-\infty$, $\phi$ runs monotonically from $-\infty$ (weak coupling) to $+\infty$ (strong coupling). Curvature is everywhere finite and vanishes asymptotically in both directions. The $\sigma = -\infty$ side of the world is an infinite proper distance away from any finite $\sigma$. This world has a weak-coupling and a strong-coupling vacuum (in both of which curvature vanishes) and the static solutions glue the two together in a non-unique way characterised by the free parameter $T$. This class of solution seems to have a fairly straightforward physical interpretation.

The $N > 24$ solutions of Theory II seem to be rather sick: the equation to solve for $\phi(\sigma)$ is not invertible and there is a range of sigma where several values of $\phi$ satisfy (6.3). The situation is reminiscent of a first-order phase transition and a possible interpretation is that $\phi$ undergoes a discontinuous jump at a critical value of $\sigma$. This jump might be smoothed out by quantum fluctuations but, as we don’t have a concrete idea of how this works, we will not consider this case further. The static solution behavior described above has also been obtained, in broad outline, by Strominger in his somewhat different version of theory II [8].

7. Infalling Matter Solutions

The static solutions, by construction, do not Hawking radiate. We must now ask if they begin to radiate when they are subjected to a perturbation. With that aim in mind, we consider the effect of an incoming (left-moving) shock wave of massless $f$-matter on a static solution. Such a shock wave is described by a matter stress tensor

$$T^M_{++} \equiv \frac{1}{2} \partial_+ f_i \partial_+ f_i = a \delta(x^+ - x^+_0) .$$  (7.1)

This is clearly not a static process but, since we have the general time-dependent solutions, we can solve this problem exactly at the classical level. All one has to do is to take a general solution as given by (1.6) or (5.8) and choose the functions $\alpha, \ldots, \delta$ such that the constraint of vanishing total stress tensor, including (7.1), is satisfied. Determining the
functions $\alpha, \ldots, \delta$ only relies on the form (4.8) of $T^{\rho,\phi}_{++}$ and does not depend on the detailed form of the function $\Omega$ appearing on the r.h.s of equation (4.6) ((5.8)).

Naively one is tempted to impose the constraint $T_{++} \equiv T^{\rho,\phi}_{++} + T^M_{++} = 0$. However, this cannot be a valid statement since $T_{++}$ so defined does not transform as a tensor but rather as a projective connection, which means that under a conformal change of coordinates it picks up an extra term equal to $\kappa/2$ times the Schwarzian derivative of the transition function. This problem is cured if we remember that the $(\partial_+ \rho)^2 - \partial_+^2 \rho$ part of $T^{\rho,\phi}_{++}$ arises from the Polyakov-term $\sim \int \sqrt{g} R \Box^{-1} R$ which is non-local in a general gauge. The non-locality implies that we may add an arbitrary function (more precisely, a projective connection) $t_+$ to $T^{\rho,\phi}_{++}$. Alternatively, $T^{\rho,\phi}_{++}$ is obtained by integrating the conservation equation for the stress tensor, which allows for an arbitrary function of integration $t_+$. This function $t_+$ should be determined by boundary conditions. Under a conformal change of coordinates $t_+$ will pick up $-\kappa/2$ times the Schwarzian derivative, so that $T_{++} + t_+$ is a true tensor*, and the correct constraint is

$$T_{++} + t_+ = 0 \, . \quad (7.2)$$

Typically we do not want to have any incoming stress-energy besides the one specified by $T^M_{++}$, hence $t_+ = 0$. As we just stressed, the latter is a coordinate-dependent statement, and it is reasonable to impose $t_+ = 0$ in those coordinates that are asymptotically Minkowskian (linear coordinate transformations that do preserve the Minkowskian system have vanishing Schwarzian derivative). In the following we will assume that such a coordinate system has been adopted **. Of course, everything said also applies to the minus components of the stress tensor.

Combining eqs (4.8) and (7.1), we have to solve

$$\begin{aligned}
(\partial_+ \gamma)^2 - (\partial_+ (\gamma - \alpha))^2 - \partial_+^2 \gamma &= \frac{a}{\kappa} \delta(x^+) = \frac{a}{\kappa} \partial_+^2 (x^+ \theta(x^+)) \\
(\partial_- \delta)^2 - (\partial_- (\delta - \beta))^2 - \partial_-^2 \delta &= 0
\end{aligned} \quad (7.3)$$

---

* In the full quantum mechanical treatment $T_{++}$ transforms with an anomaly equal to $\kappa/2 - (2 + N)/24 = -26/24$ times the Schwarzian derivative, and one could interpret $t_+$ as the ghost stress tensor cancelling exactly the $-26/24$.

** In fact, we use rescaled asymptotically Minkowskian coordinates such that $\rho + \log \lambda \sim 0$, so that the truely asymptotically Minkowskian coordinates are $x^\pm/\lambda$. Note that refs. [1]-[3] call $\sigma^\pm$ the asymptotically Minkowskian coordinates. What is called $x^\pm$ there are Kruskal-type coordinates corresponding to our $\pm e^{\pm x^\pm}$. 

18
where we shifted $x^+$ in order to set $x^+_0 = 0$. For $x^+ \neq 0$ the r.h.s. of both equations vanish and we know that the static solutions discussed above solve these constraints if the parameter $S$ is set equal to one half.

To be definite, suppose that in the far past ($\tau \to -\infty$), i.e. $x^+ < 0$ almost everywhere, the $\rho, \phi$-system is described by one of the static solutions with $S = \frac{1}{2}$ but $T$ arbitrary. Now we have to patch this particular solution to a solution for $x^+ > 0$, in such a way that $T_{++}^{\rho, \phi}$ has the correct singularity at $x^+ = 0$, as specified by (7.3), and that $\rho$ and $\phi$ are continuous across the line $x^+ = 0$. In general, the solution for $x^+ > 0$ need not be static. Also, eq. (7.3) does not specify the solution for $x^+ > 0$ uniquely. Indeed, given a solution $\alpha, \ldots, \delta$ that satisfies (7.3) we can always change $\alpha(x^+) \to \alpha(x^+) + \theta(x^+) g(x^+)$, $\gamma(x^+) \to \gamma(x^+) + \theta(x^+) f(x^+)$, where e.g. $f$ and its first and second derivatives vanish at $x^+ = 0$, and $g(x^+)$ is obtained by solving $\frac{1}{2} g'' + (\partial_+ \alpha - \partial_+ \gamma - f') g' + \frac{1}{2} f'' - \partial_+ \alpha f' = 0$ (choosing the solution with $g'(0) = 0$ and the integration constant such that $g(0) = 0$).

This non-uniqueness does not, of course, contradict the fact that the equations of motion uniquely determine all quantities in the future once the initial data are specified. The point is that giving $\phi$ and $\rho$ or $\alpha, \ldots, \gamma$ at any finite $\tau$, for $x^+ < 0$ only, is not a complete set of initial data, and there are many different solutions. We will have to impose further physical conditions for $x^+ > 0$ to single out certain solutions.

We will work out in some detail one solution that asymptotes to the one considered in refs. [1]-[5]. Consider the following choice for the functions $\alpha, \ldots, \delta$:

$$\begin{align*}
\alpha(x^+) &= \frac{1}{2} x^+, \quad \gamma(x^+) = \frac{1}{4} x^+ - \frac{a}{\kappa} \left( e^{x^+} - 1 \right) \theta(x^+) + \frac{1}{2} T - \frac{1}{2} T_c \\
\beta(x^-) &= -\frac{1}{2} x^-, \quad \delta(x^-) = -\frac{1}{4} x^- + \frac{1}{2} T
\end{align*}$$

(7.4)

For $x^+ < 0$ this reproduces correctly our well-known static solutions. It is easy to verify that the discontinuity of $\partial_+ \gamma$ is exactly such that (7.3) is satisfied. From (4.6) or (5.8) we obtain for $x^+ > 0$

$$\begin{align*}
2\Omega(\phi) &= \frac{1}{\kappa} e^{x^+} \left( e^{-x^-} - a \right) - \frac{1}{4} x^+ + \frac{1}{4} x^- + T + \frac{a}{\kappa} - \frac{1}{2} T_c \\
\rho(x) + \log \lambda + \frac{1}{\kappa} e^{-2\phi} + \left( \frac{2}{\kappa} \phi \right) &= \frac{1}{\kappa} e^{x^+} \left( e^{-x^-} - a \right) + \frac{1}{4} x^+ - \frac{1}{4} x^- + T + \frac{a}{\kappa}
\end{align*}$$

(7.5)

where the term $\left( \frac{2}{\kappa} \phi \right)$ is present only in Theory II. It is obvious that $\phi$ and $\rho$ are not static in the $x^\pm$ coordinates (i.e. not dependent only on $\sigma = \frac{1}{2} (x^+ - x^-)$). Moreover, one can prove that for $a \neq 0$ there is no conformal coordinate transformation that makes $\phi$ and $\rho$
static for \( x^+ > 0 \). The best thing we can do is to make \( \phi \) and \( \rho \) quasi-static in a certain asymptotic region. Equations (7.5) suggest to introduce new coordinates \( \hat{x}^\pm \) by

\[
\hat{x}^+ = x^+, \quad \hat{x}^- = -\log \left( e^{-x^-} - a \right) \quad \text{for} \quad x^- < -\log a.
\]  

Note that the line \( x^- = -\log a \) corresponds to \( \hat{x}^- = +\infty \) and thus is a horizon (at finite proper distance as one can verify from the formula below). These \( \hat{x}^\pm \) coordinates are such that for \( x^+ = \hat{x}^+ \to \infty \), the leading term on the r.h.s. of (7.5) is \( \frac{1}{\kappa} e^{\hat{x}^+ - \hat{x}^-} = \frac{1}{\kappa} e^{2\hat{\sigma}} \) and \( \phi \) and \( \rho \) are quasi-static. However, the next to leading terms are not static. We have for \( \hat{x}^+ \to \infty \) or \( \hat{\sigma} \to \infty \) with finite \( \hat{x}^- \)

\[
\phi(\hat{x}) \sim -\hat{\sigma} + \delta_I e^{-2\hat{\sigma}}
\]

\[
\rho(\hat{x}) + \log \lambda \sim \left( \delta_I + \frac{\kappa}{8} \right) e^{-2\hat{\sigma}}
\]

in Theory I and

\[
\phi(\hat{x}) \sim -\hat{\sigma} + \delta_{II} e^{-2\hat{\sigma}}
\]

\[
\rho(\hat{x}) + \log \lambda \sim \left( \delta_{II} + \frac{\kappa + 2}{8} \right) e^{-2\hat{\sigma}}
\]

in Theory II, where

\[
\delta_I = -\frac{\kappa}{2} \left( \frac{1}{2} \hat{\sigma} + T + \frac{a}{\kappa} - \frac{1}{4} \log \left( 1 + ae^{\hat{x}^-} \right) \right)
\]

\[
\delta_{II} = -\frac{\kappa}{2} \left( \left( \frac{1}{2} + \frac{2}{\kappa} \right) \hat{\sigma} + T + \frac{a}{\kappa} - \frac{1}{4} \log \left( 1 + ae^{\hat{x}^-} \right) \right).
\]

In general we wish to measure Hawking radiation, if any, in asymptotically Minkowskian coordinates. As we see from (7.7) and (7.8), the \( \hat{x}^\pm \) coordinates, or more precisely the rescaled \( \hat{x}^\pm / \lambda \) coordinates, are asymptotically Minkowskian. The rate of Hawking radiation is given by the stress-energy \( t_-(\hat{x}^-/\lambda) \) flowing out to infinity. As discussed above, we have the following transformation rule for \( t_- \):

\[
t_-(\hat{x}^-) = \left( \frac{\partial \hat{x}^-}{\partial x^-} \right)^{-2} t_-(x^-) + \frac{\kappa}{2} D^S_{x^-}[\hat{x}^-]
\]

where \( D^S \) denotes the Schwarzian derivative of \( \hat{x}^- \) with respect to \( x^- \):

\[
D^S_{x^-}[\hat{x}^-] = \frac{(\hat{x}^-)'''}{(\hat{x}^-)'} - \frac{3}{2} \left( \frac{(\hat{x}^-)''}{(\hat{x}^-)'} \right)^2 = -\frac{1}{2} \left[ 1 - \left( 1 + ae^{\hat{x}^-} \right)^2 \right].
\]
Since \( t_-(x^-) = 0 \) and \( t_-(\hat{x}^-/\lambda) = \lambda^2 t_-(\hat{x}^-) \) we arrive at
\[
t_-(\hat{x}^-/\lambda) = \frac{1}{4}\lambda^2 \kappa \left[ 1 - \frac{1}{(1 + ae^{\hat{x}^-})^2} \right] . \tag{7.12}
\]
This is the rate of Hawking radiation in the asymptotically Minkowskian coordinates \( \hat{x}^-/\lambda \).

The reader will recognize that this is exactly the same result as found by CGHS \[\text{[3]},\] except for the replacement \( \frac{N}{12} \to \kappa \equiv \frac{N-24}{12} \). This is not surprising since the computation only depends, except for the prefactor \( \kappa \), on the transformation \( (7.6) \) which is the same as in ref \[\text{[3]}\]. What is different is that in ref. \[\text{[3]}\] the transformation \( (7.6) \) was sufficient to make \( \phi \) and \( \rho \) truly static, while in our case it is not, thanks to the inclusion of backreaction. As \( \hat{x}^- \to \infty \), the Hawking radiation is emitted at a constant rate \( \frac{1}{4}\lambda^2 \kappa \) (in \( \hat{x}^\pm/\lambda \) coordinates), and as \( \hat{x}^- \) gets large, the total amount of energy \( E(\hat{x}^-) \) radiated away grows like \( E(\hat{x}^-) \sim \frac{1}{4}\lambda \kappa \hat{x}^- \). The same result was found in the CGHS scenario, but there it was nonsensical since the solution was static. At present, however, the solutions for \( x^+ > 0 \) are only quasi-static and comparing eqs. \( (7.9) \) with \( (6.5) \) and \( (6.6) \) we see that we can define an effective (or adiabatic) \( T \)-parameter \( T_{\text{eff}}(\hat{x}^-) \) by
\[
T_{\text{eff}}(\hat{x}^-) = T + \frac{a}{\kappa} - \frac{1}{4} \log \left( 1 + ae^{\hat{x}^-} \right) . \tag{7.13}
\]
Recall that \( M = \lambda \kappa (T - T_c) \) is the appropriate mass parameter of the static solutions. Thus, as \( \hat{x}^- \) gets large,
\[
M_{\text{eff}}(\hat{x}^-) = \lambda \kappa (T_{\text{eff}}(\hat{x}^-) - T_c) \sim \lambda \kappa \left( T - T_c + \frac{a}{\kappa} - \frac{1}{4} \log a \right) - \frac{1}{4} \lambda \kappa \hat{x}^- \quad \text{as} \quad \hat{x}^- \to \infty \tag{7.14}
\]
and the “effective mass” decreases at the same rate as the Hawking radiation carries away the energy. Thus Hawking radiation is accompanied by the right backreaction.*

* We note that the approach to computing the Hawking radiation as taken e.g. in refs \[\text{[3]},\text{[4]}\] is much the same. There, an apparent horizon (a line where \( \partial_+ \phi = 0 \)) is determined. Then the value of \( \phi \) on that horizon is related to the mass of the solution and it is found that, for \( N > 0 \) (corresponding in our case to \( \kappa > 0 \)) the apparent horizon recedes, hence \( \phi \) increases and the mass decreases. Decreasing mass then is interpreted as due to outgoing Hawking radiation. In this spirit eq. \( (7.12) \) just appears as a consequence of \( (7.14) \). Note that we derived both equations independently from each other. For \( \kappa < 0 \) the situation is just reversed. (Note that the apparent horizon now is on the other side of the global event horizon at \( x^- = - \log a \) and approaches it as \( x^+ \to \infty \).)
There are still two unsatisfactory points. First, the Hawking radiation goes on forever (as $\hat{x}^{-} \to \infty$) at a rate independent of the initial state (parametrized by $T$) and of the strength $a$ of the shock wave. Second, and more annoying, the Hawking radiation is proportional to $N - 24$, even in Theory II, which was designed not to produce negative energy Hawking radiation from the ghosts. For $N < 24$ the Hawking radiation has the wrong sign! Clearly, all is not well with the dynamical solutions, but whether that is a defect of the underlying theory or the (classical) approximation is not yet known, see however the “Note Added” below.

8. Conclusions

What does this all mean for the usual Hawking radiation puzzles? The initial proposal of [1] was that by adding a matter anomaly term to the dilaton-gravity action, and then solving the resulting effective action classically, one could expect to see the back-reaction on the metric of the matter Hawking radiation and even hope to address the question of the final state of black hole evolution. This proposition seemed to be borne out by subsequent analytic and numerical analysis of the CGHS equations with the exception that localized regions of strong coupling, where the underlying approximation breaks down, appeared in all “interesting” time evolutions.

We have attempted to improve these calculations by making the underlying action be a $c = 26$ conformal field theory, even in the strong coupling regions, and therefore, according to the conventional wisdom, a consistent 2D quantum gravity. To achieve this, we only had to modify the dilaton dependence of the cosmological constant in a way that left its weak-coupling behavior intact (which was all we had any right to believe we understood anyway). As a bonus, we were able to construct exact classical solutions of the modified theory. We also found that Strominger’s modification of the anomaly action, with its more sensible treatment of the ghosts, could be handled in a similar way.

These modifications of the theory, and especially Strominger’s suggestion for decoupling the ghosts (at least for $\kappa < 0$), significantly improve the behavior and physical interpretation of the static solutions of the classical equations of motion. The dynamical solutions are another matter entirely. Rather disappointingly, one can find solutions of the “decoupled ghost” theory in which the Hawking radiation rate is proportional to $\kappa = \frac{N-24}{12}$ (which is to say that the ghosts have not really decoupled after all: by choosing $N < 24$,
one gets radiation of the wrong sign). This problem might be solved by including graviton-dilaton quantum loops in the calculation: all that is needed is to replace $\kappa$ in appropriate places by $\kappa + 2$, an effect that could be generated in one-loop order (see “Note Added” below). It might also be that some more subtle version of Strominger’s modification of the Polyakov action is called for: that action decouples the ghosts from classical solutions of the original classical action, but not necessarily from the solutions of the modified actions we have been studying.

More generally, to obtain truly reliable results, a full quantum treatment of the underlying field theory should be given. The fact that we could construct an exact classical solution gives hope that an exact quantum solution can be found, but this line of inquiry has yet to be pursued.

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**Note Added**

While preparing this paper, we received a preprint by S. de Alwis in which similar observations to ours are made. After submitting this paper to hep-th@xxx, we learned that S. Giddings and A. Strominger also made similar progress towards an exact $c = 26$ conformal theory, and that the same idea was also contained in ref. [9] by J. Russo and A. Tseytlin without however giving the explicit free field realization or writing the general solution in closed form. Also, two-dimensional dilaton gravities, although with a somewhat different action, have been studied in the past by A. Chamseddine [10], and more recently by T. Burwick and A. Chamseddine [11].

For completeness, we would like to include here two observations made after submitting this paper for publication:

First, we would like to argue that a full quantum treatment most probably replaces the $\kappa$ prefactor in the Hawking radiation rate by $N/12$, thus making it manifestly positive for all $N$. The basic ingredients in the computation of the Hawking radiation rate were the $T_{\pm\pm} = 0$ constraints

$$T_{\pm\pm} \equiv T^{\rho,\phi}_{\pm\pm} + T^{M}_{\pm\pm} + T^{gh}_{\pm\pm} + t_{\pm} = 0$$
(where we now explicitly include the ghost contribution) and the anomalous transformation law of $T^{\rho,\phi}$. Indeed, in the classical theory (based on our quantum-improved action) only $T^{\rho,\phi}_{\pm\pm}$ transforms anomalously. Its transformation law is

$$T^{\rho,\phi}_{\pm\pm}(y^\pm) = \left(\frac{\partial y^\pm}{\partial x^\pm}\right)^{-2} \left( T^{\rho,\phi}_{\pm\pm}(x^\pm) + \frac{c_{\text{cl}}^{\rho,\phi}}{24} D^S_{x^\pm}[y^\pm] \right)$$

where the anomaly is proportional to the classical central charge (i.e. the one obtained by a Poisson bracket computation) of the $\phi, \rho$ -system: $c^{\rho,\phi}_{\text{cl}} = -12\kappa$. This makes it necessary to introduce $t_{\pm}$ to cancel this anomaly (cf (7.10)) in order that $T_{\pm\pm} = 0$ be a coordinate-invariant statement.

In the quantum theory however, $c^{\rho,\phi} = 2 - 12\kappa$, and $T^{M}_{\pm\pm}$ and $T^{gh}_{\pm\pm}$ transform with anomalies given by $c^{M} = N$ and $c^{gh} = -26$ so that the total stress tensor $T_{\pm\pm} - t_{\pm}$ transforms without anomaly : $c^{\text{tot}} = c^{\rho,\phi} + c^{M} + c^{gh} = 0$. There is now no need for a $t_{\pm}$ term and we do not include one: the full energy-momentum tensor is now defined to be $T^{\rho,\phi}_{\pm\pm} + T^{M}_{\pm\pm} + T^{gh}_{\pm\pm}$. We also make the physically very reasonable assumption that all energy flowing in or out from infinity is described by the matter energy-momentum tensor $T^{M}_{\pm\pm}$ (the ghosts are unphysical and the graviton/dilaton degrees of freedom don’t propagate: only matter remains). In particular, outgoing Hawking radiation should be signalled by a non-vanishing matter stress energy tensor $T^{M}_{-\_}$.

Now, the matter energy-momentum tensor, taken by itself, does have an anomalous transformation law. According to the above discussion, in a transformation between coordinates $x$ and $\hat{x}$ we have

$$T^{M}_{-\_}(\hat{x}^-) = \left(\frac{\partial \hat{x}^-}{\partial x^-}\right)^{-2} \left( T^{M}_{-\_}(x^-) + \frac{N}{24} D^S_{x^-}[\hat{x}^-] \right).$$

We can apply this to the infall solution described in Sect. 7 and in particular to the coordinate transformation of (7.11). The value of $T^{M}_{-\_}$ in the $x$ coordinates is clearly zero, but we would like to measure outgoing matter energy in the asymptotically Minkowskian coordinates $\hat{x}$. The Schwarzian derivative $D^S$ is still given by (7.11) and we find

$$T^{M}_{-\_}(\hat{x}^-) = \frac{N}{48} \left[ 1 - \frac{1}{(1 + a e^{\hat{x}^-})^2} \right].$$

(Recall that the truly asymptotically Minkowskian coordinates are $\hat{x}^\pm/\lambda$ and that $T_{-\_}(\hat{x}^-/\lambda) = \lambda^2 T_{-\_}(\hat{x}^-).$) Thus we find that the Hawking radiation rate is proportional to $N$ as it should and that it is given by exactly the same expression as in CGHS [1].
Second, we would like to comment on the issue of eternally continuing Hawking radiation and negative mass solutions. At the level of solutions of the equations of motion of our action (4.1) we found that, for $\kappa > 0$, as the Hawking radiation proceeds the effective mass $M_{\text{eff}}$ eventually becomes negative. This precisely occurs when $T_{\text{eff}} = T_c$. Now, we have seen for the static solutions that at this point a naked time-like singularity occurs. This means that the subsequent evolution is not well defined unless new boundary conditions are imposed on the line of singularity. This situation has been discussed in detail by Russo, Susskind and Thorlacius [11] who consider the same action (4.1) in terms of $\Omega$ and $\chi$ but with a slightly modified relation between $\Omega$, $\chi$ and $\phi$, $\rho$. Their analysis carries over to our case, word by word. One finds that at the point where the apparent horizon (defined by $\partial_+ \Omega = 0$) intersects the line of singularity (defined by $\Omega = 0$), the $\Omega$, $\chi$-solution relevant to the infalling shock wave scenario can be matched continuously onto the static $M = 0$ (i.e. $T = T_c$) solution. More precisely, before the emergence of the naked singularity, the effective mass is given by (cf (7.13))

$$M_{\text{eff}} = M + a\lambda - \frac{1}{4} \kappa \lambda \log \left(1 + ae^{\hat{x}^-}\right)$$

with $\hat{x}^-$ given by (7.4), and where $M = \kappa \lambda (T - T_c)$ is the mass of the the “initial” (static) solution. At $\hat{x}^- = \hat{x}_i^-$ where the line of singularity intersects the apparent horizon, $M_{\text{eff}}$ is zero, and by the above choice, $M_{\text{eff}}$ remains zero for $\hat{x}^- > \hat{x}_i^-$. Since we have incorporated backreaction consistently, the Hawking radiation must stop when $M_{\text{eff}} = 0$ is reached. (Of course, at $\hat{x}^- = \hat{x}_i^-$ there is also the thunderpop [11].) We see that the occurrence of negative mass solutions from a positive (or zero) mass initial state is eliminated by appropriate boundary conditions. This is very satisfactory at the level of the equations of motion, although, at present, we do not know how to deal with this at the quantum level.
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