RATIONAL CAYLEY INNER HERGLOTZ-AGLER FUNCTIONS: POSITIVE-KERNEL DECOMPOSITIONS AND TRANSFER-FUNCTION REALIZATIONS

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Abstract. The Bessmertnyi class consists of rational matrix-valued functions of \( d \) complex variables representable as the Schur complement of a block of a linear pencil \( A(z) = z_1 A_1 + \cdots + z_d A_d \) whose coefficients \( A_k \) are positive semidefinite matrices. We show that it coincides with the subclass of rational functions in the Herglotz–Agler class over the right poly-halfplane which are homogeneous of degree one and which are Cayley inner. The latter means that such a function is holomorphic on the right poly-halfplane and takes skew-Hermitian matrix values on \( (i\mathbb{R})^d \), or equivalently, is the double Cayley transform (over the variables and over the matrix values) of an inner function on the unit polydisk. Using Agler–Knese’s characterization of rational inner Schur–Agler functions on the polydisk, extended now to the matrix-valued case, and applying appropriate Cayley transformations, we obtain characterizations of matrix-valued rational Cayley inner Herglotz–Agler functions both in the setting of the polydisk and of the right poly-halfplane, in terms of transfer-function realizations and in terms of positive-kernel decompositions. In particular, we extend Bessmertnyi’s representation to rational Cayley inner Herglotz–Agler functions on the right poly-halfplane, where a linear pencil \( A(z) \) is now in the form \( A(z) = A_0 + z_1 A_1 + \cdots + z_d A_d \) with \( A_0 \) skew-Hermitian and the other coefficients \( A_k \) positive semidefinite matrices.

1. Introduction

In the 1980s, M. F. Bessmertnyi (see \cite{9, 10, 11, 12, 13}) studied \( n \times n \) matrix-valued rational functions of \( d \) variables which admit a so-called finite-dimensional long resolvent representation,

\[
f(z) = A_{11}(z) - A_{12}(z)A_{22}(z)^{-1}A_{21}(z), \quad z = (z_1, \ldots, z_d) \in \mathbb{C}^d. \tag{1.1}
\]

Here

\[
A(z) = A_0 + z_1 A_1 + \cdots + z_d A_d = \begin{bmatrix} A_{11}(z) & A_{12}(z) \\ A_{21}(z) & A_{22}(z) \end{bmatrix} \tag{1.2}
\]

is a linear \( \mathbb{C}^{(n+m) \times (n+m)} \)-valued function. He showed that if no additional restrictions on \( f \) are assumed, then such a representation \( \text{(1.1)} \) always exists. If, moreover, \( f \) satisfies an additional condition (a) \( f(z) = \overline{f(z)} \) (resp., (b) \( f(z) = f(z)^\top \), (c)}

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\( f(\lambda z) = \lambda f(z), \lambda \in \mathbb{C} \setminus \{0\} \), then one can choose the matrices \( A_k, k = 0, \ldots, d \), to be (a) real (resp., (b) symmetric, (c) such that \( A_0 = 0 \)).

A particular role in Bessmertnyí’s work is played by functions of the form (1.1) with \( A_0 = 0 \) and \( \overline{A_k} = A_k^\top = A_k \geq 0 \), \( k = 1, \ldots, d \) (i.e., matrices \( A_k \) in (1.2) are assumed to be real, symmetric, and positive semidefinite), with motivation coming from electrical engineering. He proved that such functions form the class (which we denote by \( \mathbb{R}B_d^{n \times n} \)) of characteristic functions of passive 2n-poles, where impedances of elements (resistances, capacitances, inductances, and ideal transformers are allowed) are considered as independent variables. (To put it in a broader context of multidimensional circuit synthesis, see [5].) It is easy to see that a function \( f \in \mathbb{R}B_d^{n \times n} \) satisfies the conditions

\[
(\lambda f)(z) = \lambda f(z), \quad \lambda \in \mathbb{C} \setminus \{0\},
\]

\[
f \text{ is holomorphic on } \Pi^d,
\]

\[
f(z) + f(z)^* \geq 0, \quad z \in \Pi^d,
\]

where \( \Pi^d = \{ z \in \mathbb{C}^d : \text{Re} \ z_k > 0, k = 1, \ldots, d \} \) is the open right poly-halfplane, and

\[
f(\bar{z}) = f(z)^* = \overline{f(z)},
\]

where \( \bar{z} = (\bar{z}_1, \ldots, \bar{z}_d) \in \mathbb{C}^d \). In other words, the class \( \mathbb{R}B_d^{n \times n} \) is a subclass of the class of rational \( n \times n \) matrix-valued homogeneous (of degree 1) positive real functions of \( d \) variables, denoted \( \mathbb{R}P_d^{n \times n} \).

We will also consider here the classes \( B_d^{n \times n} = \mathbb{C}B_d^{n \times n} \) and \( P_d^{n \times n} = \mathbb{C}P_d^{n \times n} \). The first one is obtained if we relax the condition that matrices \( A_k \) in a representation (1.1)–(1.2) have real entries and require just \( A_0 = 0 \) and \( A_k^\top = A_k \geq 0, k = 1, \ldots, d \), and the second one is obtained if we relax the condition (1.6) and require just

\[
f(\bar{z}) = f(z)^*,
\]

Thus we have \( \mathbb{R}B_d^{n \times n} \subseteq B_d^{n \times n} \) and \( \mathbb{R}P_d^{n \times n} \subseteq P_d^{n \times n} \). We also have \( B_d^{n \times n} \subseteq P_d^{n \times n} \).

The case of \( d = 1 \) is not interesting: the classes \( \mathbb{R}B_1^{n \times n} \) and \( \mathbb{R}P_1^{n \times n} \) (resp., \( B_1^{n \times n} \) and \( P_1^{n \times n} \)) coincide and consist of functions of the form \( f(z) = Az \) with a \( n \times n \) matrix \( A \) satisfying \( \overline{A} = A^\top \geq 0 \) (resp., \( A^* = A \geq 0 \)). If \( d = 2 \), then we also have the coincidence of the classes: \( \mathbb{R}B_2^{n \times n} = \mathbb{R}P_2^{n \times n} \) and \( B_2^{n \times n} = P_2^{n \times n} \); the first equality was shown by Bessmertnyí in [10], and exactly the same argument works to show the second equality. The question on whether the inclusions \( \mathbb{R}B_d^{n \times n} \subseteq \mathbb{R}P_d^{n \times n} \) and \( B_d^{n \times n} \subseteq P_d^{n \times n} \) are proper for \( d \geq 3 \) is open.

Bessmertnyí has found some necessary conditions for a function \( f \) to belong to the class \( \mathbb{R}B_d^{n \times n} \), however no necessary and sufficient conditions for that in intrinsic function-theoretical terms (as opposed to the existence of a certain representation) were established in his work.

In [10], the classes above were generalized as follows. Let \( \mathcal{U} \) be a (complex) Hilbert space. The class \( \mathcal{B}_d(\mathcal{U}) \) consists of \( L(\mathcal{U}) \)-valued functions \( f \) holomorphic on the domain

\[
\Omega_d = \bigcup_{\lambda \in \mathbb{T}} (\lambda \Pi)^d \subset \mathbb{C}^d
\]

(here, for a fixed \( \lambda \in \mathbb{T}, \) we have \( \lambda \Pi = \{ \lambda z : z \in \Pi \} \)) and representable there in the form (1.1)–(1.2) where the operators \( A_0 = 0 \) and \( A_k \in L(\mathcal{U} \oplus \mathcal{H}) \) are positive semidefinite (hence selfadjoint), with some Hilbert space \( \mathcal{H}, k = 1, \ldots, d \). Here we denote the space of bounded linear operators acting from a Hilbert space \( \mathcal{X} \) to a
Hilbert space $\mathcal{Y}$ (resp., to $\mathcal{X}$ itself) by $L(\mathcal{X}, \mathcal{Y})$ (resp., by $L(\mathcal{X})$). The class $\mathcal{P}_d(\mathcal{U})$ consists of $L(\mathcal{U})$-valued functions $f$ holomorphic on $\Omega_d$ and satisfying (1.3), (1.5), and (1.7).

Recall [10] that a mapping $\imath: \mathcal{U} \rightarrow \mathcal{U}$ is called an anti-unitary involution of a Hilbert space $\mathcal{U}$ if $i^2 = i$ and $\langle iu_1, u_2 \rangle = \langle u_2, u_1 \rangle$ for any $u_1, u_2 \in \mathcal{U}$. Such a mapping is anti-linear and bijective. We say that an operator $T \in L(\mathcal{U}, \mathcal{Y})$ is $i_{\mathcal{U}}$-real if $i_{\mathcal{U}}$ and $i_{\mathcal{Y}}$ are anti-unitary involutions of Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$ and $i_{\mathcal{Y}}T = Ti_{\mathcal{U}}$. (In the case where $\mathcal{Y} = \mathcal{U}$ and $i_{\mathcal{Y}} = i_{\mathcal{U}}$, we just say “$i_{\mathcal{U}}$-real”.) Let $\Omega \subseteq \mathbb{C}^d$ be a set invariant under (entrywise) complex conjugation, and let $i_{\mathcal{U}}$ and $i_{\mathcal{Y}}$ be anti-unitary involutions of Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$. We say that a function $f: \Omega \rightarrow L(\mathcal{U}, \mathcal{Y})$ is $(i_{\mathcal{U}}, i_{\mathcal{Y}})$-real if $f^*(z) = f(z), z \in \Omega$, where $f^*(z) = i_{\mathcal{Y}}f(z)i_{\mathcal{U}}$. (In the case where $\mathcal{Y} = \mathcal{U}$ and $i_{\mathcal{Y}} = i_{\mathcal{U}}$, we just say “$i_{\mathcal{U}}$-real”.) If $\mathcal{U} = \mathbb{C}^n$, $\mathcal{Y} = \mathbb{C}^m$, and $i_{\mathcal{U}}$, $i_{\mathcal{Y}}$ are complex conjugations, then the matrix of a $(i_{\mathcal{U}}, i_{\mathcal{Y}})$-real operator $T \in L(\mathbb{C}^n, \mathbb{C}^m)$ in the standard bases has all real entries, and a $(i_{\mathcal{U}}, i_{\mathcal{Y}})$-real function $f: \Omega \rightarrow L(\mathbb{C}^n, \mathbb{C}^m)$ satisfies $f(z) = \overline{f(z)}, z \in \Omega$ — we will call such a function real. The class $i_{\mathbb{R}}\mathcal{P}_d(\mathcal{U})$ is a subclass of $\mathcal{P}_d(\mathcal{U})$ consisting of $i$-real functions, where $i = i_{\mathcal{U}}$ is an anti-unitary involution of $\mathcal{U}$. The class $i_{\mathbb{R}}\mathcal{B}_d(\mathcal{U})$ consists of functions $f$ for which there exist a Hilbert space $\mathcal{H}$, an anti-unitary involution $i_\mathcal{H}$ of $\mathcal{H}$, and a long resolvent representation (1.1)–(1.2) of $f$ such that $A_0 = 0$ and the operators $A_k \in L(\mathcal{U} \otimes \mathcal{H}), k = 1, \ldots, d$, are (selfadjoint) positive semidefinite and $i_{\mathcal{U}} \otimes i_{\mathcal{H}}$-real.

Thus the classes $\mathcal{B}_d(\mathcal{U}), \mathcal{P}_d(\mathcal{U}), i_{\mathbb{R}}\mathcal{B}_d(\mathcal{U})$, and $i_{\mathbb{R}}\mathcal{P}_d(\mathcal{U})$ are generalizations of the classes $\mathcal{B}_d^{n \times n}, \mathcal{P}_d^{n \times n}, i_{\mathbb{R}}\mathcal{B}_d^{n \times n}$, and $i_{\mathbb{R}}\mathcal{P}_d^{n \times n}$, respectively. For these generalized classes we also have that $\mathcal{B}_d(\mathcal{U}) \subseteq \mathcal{P}_d(\mathcal{U}), i_{\mathbb{R}}\mathcal{B}_d(\mathcal{U}) \subseteq i_{\mathbb{R}}\mathcal{P}_d(\mathcal{U})$ (of course, for the same $i$ in both classes in the last inclusion); if $d = 1, 2$, then these inclusions are equalities; the class $\mathcal{B}_1(\mathcal{U}) = \mathcal{P}_1(\mathcal{U})$ (resp., $i_{\mathbb{R}}\mathcal{B}_1(\mathcal{U}) = i_{\mathbb{R}}\mathcal{P}_1(\mathcal{U})$) consists of functions of the form $f(z) = Az$ with a positive semidefinite operator $A \in L(\mathcal{U})$ (resp., with a $i$-real positive semidefinite operator $A$); the question on whether the inclusions are proper for $d \geq 3$ is open.

In [16], several characterizations of the classes $\mathcal{B}_d(\mathcal{U})$ and $i_{\mathbb{R}}\mathcal{B}_d(\mathcal{U})$ were obtained via the double Cayley transformation which establishes the relation of these classes to the Schur–Agler class $\mathcal{S}_d(\mathcal{U})$. Let $f \in \mathcal{P}_d(\mathcal{U})$. The double Cayley transform of $f$, denoted $\mathcal{F} = \mathcal{C}(f)$, is defined as

$$
\mathcal{F}(\zeta) = \left(f(\frac{1 + \zeta_1}{1 - \zeta_1}, \ldots, \frac{1 + \zeta_d}{1 - \zeta_d}) - I_{\mathcal{U}}\right) \left(f(\frac{1 + \zeta_1}{1 - \zeta_1}, \ldots, \frac{1 + \zeta_d}{1 - \zeta_d}) + I_{\mathcal{U}}\right)^{-1}, \quad \zeta \in \mathbb{D}^d,
$$

(1.8)

where $\mathbb{D}^d = \{\zeta \in \mathbb{C}^d: |\zeta_k| < 1, k = 1, \ldots, d\}$ is the open unit polydisk. It is easy to see that the function $\mathcal{F}$ is holomorphic and contractive in $\mathbb{D}^d$ (the latter means that $\|\mathcal{F}(\zeta)\| \leq 1, \zeta \in \mathbb{D}^d$), i.e., $\mathcal{F}$ belongs to the $d$-variable Schur class $\mathcal{S}_d(\mathcal{U})$, and that $\mathcal{F}$ is inner, i.e., the boundary values of $\mathcal{F}$ are unitary operators almost everywhere on the distinguished boundary $\mathbb{T}^d = \{\zeta \in \mathbb{C}^d: |\zeta_k| = 1, k = 1, \ldots, d\}$ of the polydisk $\mathbb{D}^d$. The Schur–Agler class $\mathcal{S}_d(\mathcal{U})$ is a subclass of $\mathcal{S}_d(\mathcal{U})$ consisting of functions $\mathcal{F}(\zeta) = \sum_{\zeta \in \mathbb{Z}_d^d} \tilde{F}_\zeta \zeta^d$ (here $\zeta^d = \zeta_1^d \cdots \zeta_d^d$) satisfying $\|\mathcal{F}(T)\| \leq 1$ for every $T \in \mathcal{C}_d$, the class of $d$-tuples of commuting strict contractions on a Hilbert space, say $\mathcal{K}$, where $\mathcal{F}(T) = \sum_{\zeta \in \mathbb{Z}_d^d} \tilde{F}_\zeta \otimes T^\zeta \in L(\mathcal{U} \otimes \mathcal{K})$ and $T^\zeta = T_1^{\zeta_1} \cdots T_d^{\zeta_d}$. 


Here we say that the function \( \Theta : \Lambda \times \Lambda \to L(\mathcal{U}) \) is a \textit{positive kernel on a set} \( \Lambda \) if it holds that
\[
\sum_{i,j=1}^{N} (\Theta(\lambda_i, \lambda_j)u_j, u_i)_{\mathcal{U}} \geq 0 \quad \text{for all } \lambda_1, \ldots, \lambda_N \in \Lambda, \ u_1, \ldots, u_N \in \mathcal{U},
\]
for all \( N = 1, 2, \ldots \). An equivalent condition is that there exist a Hilbert space \( \mathcal{M} \) and a function \( \theta : \Lambda \to L(\mathcal{U}, \mathcal{M}) \) so that
\[
\Theta(\omega, \zeta) = \theta(\omega)^* \theta(\zeta) \quad \text{for all } \omega, \zeta \in \Lambda.
\]

**Theorem 1.1** ([16]). \textit{Let} \( \mathcal{F} \) \textit{be a holomorphic} \( L(\mathcal{U}) \)-\textit{valued function on} \( \mathbb{D}^d \). \textit{The following statements are equivalent:}

1. \( \mathcal{F} \in \mathcal{S}_d(\mathcal{U}) \).
2. There exist positive kernels \( \Theta_k(\omega, \zeta) \) on \( \mathbb{D}^d \), \( k = 1, \ldots, d \), \textit{holomorphic in} \( \zeta \) and \textit{anti-holomorphic in} \( \omega \), \textit{such that}
   \[
   I_{\mathcal{U}} - \mathcal{F}(\omega)^* \mathcal{F}(\zeta) = \sum_{k=1}^{d} (1 - \overline{\omega}_k \zeta_k) \Theta_k(\omega, \zeta), \quad \omega, \zeta \in \mathbb{D}^d.
   \]
3. There exist Hilbert spaces \( \mathcal{M}_k \) and holomorphic \( L(\mathcal{U}, \mathcal{M}_k) \)-\textit{valued functions} \( \theta_k \) on \( \mathbb{D}^d \), \( k = 1, \ldots, d \), \textit{such that}
   \[
   I_{\mathcal{U}} - \mathcal{F}(\omega)^* \mathcal{F}(\zeta) = \sum_{k=1}^{d} (1 - \overline{\omega}_k \zeta_k) \theta_k(\omega)^* \theta_k(\zeta), \quad \omega, \zeta \in \mathbb{D}^d.
   \]
4. There exist Hilbert spaces \( \mathcal{X}, \mathcal{X}_1, \ldots, \mathcal{X}_d \) with \( \mathcal{X} = \bigoplus_{k=1}^{d} \mathcal{X}_k \), and a unitary operator
   \[
   U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in L(\mathcal{X} \oplus \mathcal{U})
   \]
   such that
   \[
   \mathcal{F}(\zeta) = D + C(I_{\mathcal{X}} - P(\zeta)A)^{-1}P(\zeta)B, \quad \zeta \in \mathbb{D}^d,
   \]
   where \( P(\zeta) = \zeta_1 P_{\mathcal{X}_1} + \cdots + \zeta_d P_{\mathcal{X}_d} \) and \( P_{\mathcal{Y}} \) \textit{denotes} \textit{the} \textit{orthogonal projector} \textit{onto} \textit{a subspace} \( \mathcal{Y} \) \textit{of} \textit{a} Hilbert space \( \mathcal{X} \).

We notice that the representation ([14]) is a realization of \( \mathcal{F} \) as the transfer function of a conservative \( d \)-\textit{dimensional} Givone–Roesser system (see details in [8]).

In order to formulate the main result of [16], we also need the following definitions. The class \( \mathcal{A}^d \) is the class of \( d \)-\textit{tuples} \( R = (R_1, \ldots, R_d) \) \textit{of} \textit{commuting} \textit{strictly} \textit{accretive operators} \textit{on} \textit{a common} Hilbert space, say \( \mathcal{K} \), i.e., the operators \( R_k \) \textit{commute} \textit{and} \textit{there exists} \textit{a} real \textit{constant} \( s > 0 \) \textit{such that} \( R_k R_k^* \geq sI_{\mathcal{K}} \), \( k = 1, \ldots, d \).

It is easy to see that the operator Cayley transform, defined by
\[
R_k = (I_{\mathcal{K}} - T)^{-1}(I_{\mathcal{K}} + T), \quad k = 1, \ldots, d,
\]
maps the class \( \mathcal{C}^d \) onto the class \( \mathcal{A}^d \), and its inverse
\[
T_k = (R - I_{\mathcal{K}})(R + I_{\mathcal{K}})^{-1}, \quad k = 1, \ldots, d,
\]
maps \( \mathcal{A}^d \) onto \( \mathcal{C}^d \). For a function \( f \in \mathcal{P}_d(\mathcal{U}) \) and an operator \( d \)-\textit{tuple} \( R \in \mathcal{A}^d \) we define \( f(R) = \mathcal{F}(T) \), where \( \mathcal{F} = \mathcal{C}(f) \in \mathcal{S}_d(\mathcal{U}) \) is given by ([13]) and \( T \in \mathcal{C}^d \) is defined by ([10]).

**Theorem 1.2** ([16]). \textit{Let} \( f \) \textit{be a holomorphic} \( L(\mathcal{U}) \)-\textit{valued function} \textit{on} \( \Omega_d \). \textit{The following statements are equivalent:}
(0) \( f \in \mathcal{B}_d(\mathcal{U}) \).

(1) \( f \) satisfies the conditions:

\begin{align*}
& (1a) \quad f(\lambda z) = \lambda f(z), \quad \lambda \in \mathbb{C} \setminus \{0\}, \quad z \in \Omega_d. \\
& (1b) \quad f(R) + f(R)^* \geq 0, \quad R \in A^d. \\
& (1c) \quad f(z) = f(z)^*, \quad z \in \Omega_d.
\end{align*}

(2) There exist positive kernels \( \Phi_k(w, z) \) on \( \Omega_d, k = 1, \ldots, d \), holomorphic in \( z \) and anti-holomorphic in \( w \), that satisfy

\[
\Phi_k(\lambda w, \lambda z) = \Phi_k(w, z), \quad w, z \in \Omega_d, \quad \lambda \in \mathbb{C} \setminus \{0\},
\]

\[
(1.17)
\]

\[
k = 1, \ldots, d, \text{ such that}
\]

\[
f(z) = \sum_{k=1}^{d} z_k \Phi_k(w, z), \quad w, z \in \Omega_d.
\]

\[
(1.18)
\]

(2') There exist Hilbert spaces \( \mathcal{M}_k \) and holomorphic \( L(\mathcal{U}, \mathcal{M}_k) \)-valued functions \( \phi_k \) on \( \Omega_d \) that satisfy

\[
\phi_k(\lambda z) = \phi_k(z), \quad z \in \Omega_d, \quad \lambda \in \mathbb{C} \setminus \{0\},
\]

\[
(1.19)
\]

\[
k = 1, \ldots, d, \text{ such that}
\]

\[
f(z) = \sum_{k=1}^{d} z_k \phi_k(w)^* \phi_k(z), \quad w, z \in \Omega_d.
\]

\[
(1.20)
\]

(3) There exist Hilbert spaces \( \mathcal{X}, \mathcal{X}_1, \ldots, \mathcal{X}_d \) with \( \mathcal{X} = \bigoplus_{k=1}^{d} \mathcal{X}_k \), and a representation \( \| \) of a double Cayley transform of \( f, \mathcal{F} = \mathcal{C}(f) \) (which is defined by \( (1.8) \)), such that the operator \( U \) in \( (1.3) \) is not only unitary, but also selfadjoint: \( U^{-1} = U^* = U \).

If \( \iota = \iota_U \) is an anti-unitary involution on \( \mathcal{U} \) and (0) is replaced by the condition (0c) \( f \in \iota \mathcal{B}_d(\mathcal{U}) \), then one should add to (1) the condition (1i) \( f \) is a \( \iota_U \)-real function; add to (2) the condition (2i) \( \Phi_k \) are \( \iota_U \)-real functions, \( k = 1, \ldots, d \); add to (2') the condition (2'i) \( \phi_k \) are \( \iota_U(1, \mathcal{M}_k) \)-real functions for some anti-unitary involutions \( \iota_M, k = 1, \ldots, d \); and add to (3) the condition (3i) \( U \) is \( \iota_X \oplus \iota_U \)-real for some anti-unitary involution \( \iota_X \) which commutes with \( P_{\mathcal{X}_k} \) for all \( k = 1, \ldots, d \). Then the modified conditions (0)-(3) are equivalent.

**Remark 1.3.** The conventions concerning the definition of a positive kernel are actually different in [16] from those used here. Namely, in place of the equivalent conditions \( (1.9) \) or \( (1.10) \), the following alternative equivalent conditions are used [16]:

\[
\sum_{i,j=1}^{N} \langle \Theta(\lambda_i, \lambda_j) u_i, u_j \rangle_{\mathcal{U}} \geq 0 \quad \text{for all } \lambda_1, \ldots, \lambda_N \in \Lambda, \ u_1, \ldots, u_N \in \mathcal{U},
\]

\[
(1.21)
\]

\[
\Theta(\omega, \zeta) = \theta(\zeta)^* \theta(\omega) \quad \text{for all } \omega, \zeta \in \Lambda.
\]

\[
(1.22)
\]

That the condition \( (1.9) \) or \( (1.10) \) is not equivalent to \( (1.21) \) or \( (1.22) \) in general (for the matrix-valued case) can be seen as a consequence of the fact that the matrix transposition map \( A \mapsto A^\top \) is not completely positive [4, Page 144]. However the analysis in [16] was based on the work in [8] which used the convention \( (1.9) \) or \( (1.10) \) rather than \( (1.21) \) or \( (1.22) \). The resulting confusion can all be fixed by rearranging the formulas to conform to consistent conventions.
Denote by $\mathcal{B}^\text{rat}_d(\mathbb{C}^n)$ (resp., by $\mathbb{R}\mathcal{B}^\text{rat}_d(\mathbb{C}^n)$) the subclass of $\mathcal{B}_d(\mathbb{C}^n)$ (resp., of $i\mathbb{R}\mathcal{B}^\text{rat}_d(\mathbb{C}^n)$) with $i$ being the complex conjugation operator on $\mathbb{C}^n$ consisting of rational functions. It is obvious that
\[ \mathcal{B}^n_{d \times n} \subseteq \mathcal{B}^\text{rat}_d(\mathbb{C}^n), \quad \mathbb{R}\mathcal{B}^n_{d \times n} \subseteq \mathbb{R}\mathcal{B}^\text{rat}_d(\mathbb{C}^n). \] (1.23)

In the present paper, we prove that the inclusions in (1.23) are, in fact, equalities. Moreover, we obtain stronger versions of Theorem 1.2 for the classes $\mathcal{B}^n_{d \times n} = \mathcal{B}^\text{rat}_d(\mathbb{C}^n)$ and $\mathbb{R}\mathcal{B}^n_{d \times n} = \mathbb{R}\mathcal{B}^\text{rat}_d(\mathbb{C}^n)$ in Sections 3 and 4, respectively. This resolves an open problem raised in the first two paragraphs on page 257 of [16] and a question in [16] Problem 2, page 287.

We will say that the $L(\mathcal{U})$-valued function $f$ on $\mathbb{D}^d$ (on $\Pi^d$) is a Cayley inner function if $f$ is holomorphic with positive semidefinite real part there and such that its strong non-tangential boundary values $f(t)$ have zero real part:
\[ f(t) + f(t)^* = 0 \text{ for a.e. } t \in \mathbb{T}^d \text{ (for a.e. } t \in (i\mathbb{R})^d). \] (1.24)

Note that such a function is just the Cayley transform (the double Cayley transform) of an inner function on the polydisk.\footnote{In the single-variable case ($d = 1$), this is consistent with the terminology of Rosenblum and Rovnyak [20]; the parallel engineering terminology would be (continuous-time, impedance) lossless (see [22]).} When $f$ is rational matrix-valued and hence has meromorphic continuation to $\mathbb{C}^d$, uniqueness of meromorphic continuation off of $(i\mathbb{R})^d$ implies that the condition (1.24) can be replaced by
\[ f(z) = -f(-\overline{z})^* \text{ at all points of analyticity of } f. \] (1.25)

Notice that the functions from the class $\mathcal{P}_d(\mathcal{U})$ (and therefore from any of the classes $\mathcal{B}_d(\mathcal{U}), \mathcal{P}^n_{d \times n}, \mathcal{B}^n_{d \times n}$) are necessarily Cayley inner on $\Pi^d$.

In Section 2 we obtain a stronger version of Agler’s Theorem 1.1 for the class $\mathcal{B}^\text{rat}_d(\mathbb{C}^n)$ of rational inner functions from $\mathcal{S}_d(\mathbb{C}^n)$ as a straightforward extension of Knese’s result from [17] to the matrix-valued case. This result is used then in all subsequent sections. We already mentioned characterizations of complex and real rational Bessmertnyi’s classes that we obtain in Sections 3 and 4. In Section 3 we obtain several characterizations of the subclass $\mathcal{C}\mathcal{I}\mathcal{A}^\text{rat}_d(\mathbb{D}^d, \mathbb{C}^n)$ of rational Cayley inner functions from the Herglotz–Agler class $\mathcal{HA}(\mathbb{D}^d, \mathbb{C}^n)$. (We recall here that the Herglotz–Agler class on $\mathbb{D}^d$, denoted as $\mathcal{HA}(\mathbb{D}^d, \mathcal{U})$, consists of $L(\mathcal{U})$-valued functions which are holomorphic on $\mathbb{D}^d$ and whose values on any commutative $d$-tuple of strict contractions on a Hilbert space have positive semidefinite real part.) These results are a stronger version of the results in [1] where the general Herglotz–Agler class $\mathcal{HA}(\mathbb{D}^d, \mathcal{U})$ was introduced and characterized.

The Herglotz–Agler class on $\Pi^d$, denoted as $\mathcal{HA}(\Pi^d, \mathcal{U})$, consists of $L(\mathcal{U})$-valued functions which are holomorphic on $\Pi^d$ and whose values on any commutative $d$-tuple of strictly accretive operators on a Hilbert space have positive semidefinite real part. We also introduce the subclass $\mathcal{C}\mathcal{I}\mathcal{A}^\text{rat}(\Pi^d, \mathcal{U})$ of $\mathcal{HA}(\Pi^d, \mathcal{U})$ that consists of Cayley inner functions. Then it follows from Theorem 1.2 and a remark two paragraphs above that the class $\mathcal{B}_d(\mathcal{U})$ is a subclass of functions from $\mathcal{C}\mathcal{I}\mathcal{A}^\text{rat}(\Pi^d, \mathcal{U})$ satisfying the additional homogeneity condition (1.4), and that $\mathcal{B}^n_{d \times n} = \mathcal{B}_d(\mathbb{C}^n)$ is a subclass of rational functions from $\mathcal{C}\mathcal{I}\mathcal{A}^\text{rat}(\Pi^d, \mathbb{C}^n)$ satisfying (1.4). In Section 4 we obtain several characterizations of the subclass $\mathcal{C}\mathcal{I}\mathcal{A}^\text{rat}_d(\Pi^d, \mathbb{C}^n)$ that consists of rational functions. In particular, we extend Bessmertnyi’s long resolvent representation (1.4) to functions from $\mathcal{C}\mathcal{I}\mathcal{A}^\text{rat}_d(\Pi^d, \mathbb{C}^n)$
where the linear pencil \((1.2)\) has a skew-Hermitian matrix \(A_0\) and, as in the case of functions from \(B_{k,d}^{n\times n}\), the other coefficients \(A_k\) are positive semidefinite matrices.

We remark that various characterizations of the general Herglotz–Agler classes \(\mathcal{H}A(\mathbb{D}^d, \mathcal{U})\) and \(\mathcal{H}A(\mathbb{P}^d, \mathcal{U})\) appear in our paper [9] (we also mention a related recent paper [2]).

2. The rational inner Schur–Agler class

In this section, we tailor Theorem 1.1 to the case where \(\mathcal{F}\) is finite matrix-valued (so \(\mathcal{U} = \mathbb{C}^n\) for some \(n \in \mathbb{Z}\)) and \(\mathcal{F}\) is rational inner, i.e., each matrix entry \(F_{ij}\) of \(\mathcal{F}\) is a rational function and \(\mathcal{F}(z)\) is unitary at each point of analyticity \(\omega\) of \(\mathcal{F}\) on the unit torus \(\mathbb{T}^d\). We mention that a consequence of Lemma 6.3 in [7] is that the set of singularities of \(\mathcal{F}\) on \(\mathbb{T}^d\) has \(\mathbb{T}^d\)-Lebesgue measure zero. By uniqueness of analytic continuation (see [21, page 21]), we see that the rational matrix function is inner if and only if the identity

\[
\mathcal{F}(1/\omega)^* \mathcal{F}(\omega) = I_n
\]

at each nonzero nonsingular point \(\omega\) of \(\mathcal{F}\) where \(\det \mathcal{F}(z) \neq 0\) (where we set \(1/\omega = (1/\omega_1, \ldots, 1/\omega_d)\) if \(\omega = (\omega_1, \ldots, \omega_d) \in \mathbb{C}^d\)). The following result characterizes the rational inner matrix-valued Schur–Agler class \(\mathcal{ISA}_{d}^{m\times n}(\mathbb{C}^n)\). We remark that the single-variable case \((d = 1)\) is well known and has origins in the circuit theory literature (see [3]), while the bivariate case (where the Schur–Agler class coincides with the Schur class) seems to have appeared for the first time in the work of Kummert [19] (see [5, 17, 18] for additional discussion), and the scalar-valued case \((n = 1)\) for an arbitrary number \(d\) of variables appears in [17].

**Theorem 2.1.** Let \(\mathcal{F}\) be a \(\mathbb{C}^{n\times n}\)-valued function of \(d\) complex variables. The following statements are equivalent:

1. \(\mathcal{F} \in \mathcal{ISA}_{d}^{m\times n}(\mathbb{C}^n)\).
2. There exist rational \(\mathbb{C}^{N_k\times n}\)-valued functions \(\theta_k\), with some \(N_k \in \mathbb{N}\), \(k = 1, \ldots, d\), which have no singularities on \(\mathbb{D}^d\) and satisfy \((1.12)\).
3. \(\mathcal{F}\) has a finite-dimensional Givone–Roesser unitary realization, i.e., there exist \(m, m_1, \ldots, m_d \in \mathbb{Z}_+\), with \(m = m_1 + \cdots + m_d\), and a unitary matrix

\[
U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{C}^{(m+n)\times(m+n)},
\]

where

\[
A = [A_{ij}]_{i,j=1,\ldots,d}, \quad B = \text{Col}_{i=1,\ldots,d}[B_i], \quad C = \text{Row}_{j=1,\ldots,d}[C_j]
\]

are block matrices with blocks \(A_{ij} \in \mathbb{C}^{m_i \times m_j}\), \(B_i \in \mathbb{C}^{m_i \times n}\), and \(C_j \in \mathbb{C}^{n \times m_j}\), such that \(\mathcal{F}\) has a representation of the form

\[
\mathcal{F}(\zeta) = D + C(I_m - P(\zeta)A)^{-1}P(\zeta)B.
\]

Here \(P(\zeta) = \text{Diag}[\zeta_1 I_{m_1}, \ldots, \zeta_d I_{m_d}]\).

**Remark 2.2.** We note that the analog of condition (2) in Theorem 1.1 where the \(n \times n\) matrix-valued kernels \(\Theta_k(w, \zeta)\) are assumed to be rational in \(\mathcal{F} = (\overline{w_1}, \ldots, \overline{w_d})\) and \(\zeta = (\zeta_1, \ldots, \zeta_d)\), does not guarantee that \(\mathcal{F}\) is inner, due to the fact that \(\Theta_k\) may fail to have a Kolmogorov decomposition \(\Theta_k(w, \zeta) = \theta_k(w)^* \theta_k(\zeta)\) as in condition \((2')\) in Theorem 2.1 with \(\theta_k\) rational matrix-valued. E.g., if \(d = 1\) and \(\mathcal{F} = 0\), then
the Szegő kernel $\Theta_{\mathbb{S}}(\omega, \zeta) = \frac{1}{1 - \overline{\omega}\zeta}$ has no rational finite matrix-valued Kolmogorov decomposition, while it is rational in $\overline{\omega}$ and $\zeta$ and satisfies (1.11).

The proof of (3) $\Rightarrow$ (1) in Theorem 2.4 follows in the same way as in the bivariate case appearing in [7, Theorem 6.1]. Thus, to complete the proof of Theorem 2.4 it suffices to show (1) $\Rightarrow$ (2') $\Rightarrow$ (3). These implications follow from the following more detailed version of the result, which is just the matrix-valued extension of Knese’s Theorem 2.9 in [17].

**Theorem 2.3.** Let polynomials $p, q \in \mathbb{C}^{n \times n}[\zeta_1, \ldots, \zeta_d]$ be given with $p(\zeta)$ invertible for all $\zeta \in \mathbb{D}^d$. Consider the following statements:

1. ($F_k$) $F := qp^{-1} \in IS\mathcal{A}_d^{rat}(\mathbb{C}^n)$.
2. ($F_k$) There exist $N_k \in \mathbb{N}$ and $\psi_k \in \mathbb{C}^{N_k \times n}[\zeta_1, \ldots, \zeta_d]$, $k = 1, \ldots, d$, such that

$$p(\omega)^*p(\zeta) - q(\omega)^*q(\zeta) = \sum_{k=1}^{d} (1 - \overline{\omega}_k \zeta_k) \psi_k(\omega)^* \psi_k(\zeta).$$

(2.3)

3. ($F_k$) $F$ has a finite-dimensional Givone–Roesser unitary realization as in condition (2) of Theorem 2.4.

Then ($F_k$) $\Rightarrow$ ($F_k$) $\Rightarrow$ (3).

**Proof.** ($F_k$) $\Rightarrow$ ($F_k$): By Agler’s theorem (see Theorem 1.1), there exist Hilbert spaces $\mathcal{M}_k$ and $L(\mathbb{C}^n, \mathcal{M}_k)$-valued holomorphic functions $\theta_k$ on $\mathbb{D}^d$, $k = 1, \ldots, d$, such that (1.12) holds. Multiplying both sides of (1.12) by $p(\zeta)$ on the right and by $p(\omega)^*$ on the left, we obtain

$$p(\omega)^*p(\zeta) - q(\omega)^*q(\zeta) = \sum_{k=1}^{d} (1 - \overline{\omega}_k \zeta_k) \xi_k(\omega)^* \xi_k(\zeta), \quad \omega, \zeta \in \mathbb{D}^d,$$

(2.4)

with $L(\mathbb{C}^n, \mathcal{M}_k)$-valued holomorphic functions $\xi_k = \theta_k p$ on $\mathbb{D}^d$. Letting $\zeta = \omega = t\mu$ where $t \in \mathbb{D}$ and $\mu \in T^d$, we obtain

$$\frac{p(t\mu)^*p(t\mu) - q(t\mu)^*q(t\mu)}{1 - |t|^2} = \sum_{k=1}^{d} \xi_k(t\mu)^* \xi_k(t\mu).$$

(2.5)

Since $p(\mu)^*p(\mu) = q(\mu)^*q(\mu)$ for all $\mu \in T^d$, the numerator of the left-hand side of (2.5) is a polynomial in $t$ and $\bar{t}$ which vanishes on the variety $1 - \bar{t}t = 0$. Therefore the left-hand side of (2.5) is a polynomial in $t$ and $\bar{t}$, and also a trigonometric polynomial in $\mu$. We have

$$p(\zeta) = \sum_{\alpha} p_\alpha \zeta^\alpha, \quad q(z) = \sum_{\alpha} q_\alpha \zeta^\alpha, \quad \xi_k(\zeta) = \sum_{\alpha} \xi_{k,\alpha} \zeta^\alpha,$$

where the first two sums are finite. We also have

$$p(t\mu)^*p(t\mu) = \sum_{\alpha, \beta} p_{\alpha, \beta}^* p_{\alpha, \beta} |t|^{2|\alpha|}$$

(here we use notation $|\alpha| = \alpha_1 + \cdots + \alpha_d$), and similarly for $q$ and $\xi_k$. Therefore, the 0-th Fourier coefficients of the two sides of the equality (2.5) (as Fourier series in $\mu$) are

$$\frac{\sum_{\alpha} (p_{\alpha, \beta}^* p_{\alpha, \beta} - q_{\alpha}^* q_{\alpha}) |t|^{2|\alpha|}}{1 - |t|^2} = \sum_{k=1}^{d} \sum_{\alpha} \xi_{k,\alpha}^* \xi_{k,\alpha} |t|^{2|\alpha|}.$$
Since the left-hand side is a polynomial in $|t|^2$ of degree at most $r-1$, where $r$ is the maximum of the total degrees of $p$ and $q$, so is the right-hand side, i.e., $\xi_k^*\xi_k = 0$ when $|\alpha| > r-1$, $k = 1, \ldots, d$. This implies that $\xi_k$ is a $M_k$-valued polynomial. We have

$$\xi_k(\omega)^*\xi_k(\zeta) = \sum_{|\alpha|,|\beta|\leq r-1} \xi_{k,\beta}^*\xi_{k,\alpha}\bar{\omega}^\beta\zeta^\alpha,$$

where the positive semidefinite block matrix $X_k = [\xi_{k,\beta}^*\xi_{k,\alpha}]_{\alpha,\beta}$ (of size $\binom{r-1+d}{d}$)
can be factored as $X_k = Y_k^*Y_k$ where $Y_k$ is a matrix of size $N_k \times \binom{r-1+d}{d}n$ with

$$N_k = \text{rank}X_k \leq \binom{r-1+d}{d}n. \tag{2.6}$$

Writing $Y_k = \text{Row}_{|\alpha|\leq r-1}[Y_{k,\alpha}]$, we define $\psi_k \in \mathbb{C}^{N_k \times n}[\zeta_1, \ldots, \zeta_d]$ by

$$\psi_k(\zeta) = \sum_{|\alpha|\leq r-1} Y_{k,\alpha}\zeta^\alpha.$$

Then

$$\xi_k(\omega)^*\xi_k(\zeta) = \psi_k(\omega)^*\psi_k(\zeta), \quad \zeta, \omega \in \mathbb{C}^d, \quad k = 1, \ldots, d,$$

and (2'Kn) holds.

(2'Kn)$\Rightarrow$(3Kn): We use the so-called lurking isometry argument. Rearranging
the terms in (2.3), we obtain

$$p(\omega)^*p(\zeta) + \sum_{k=1}^d \bar{\omega}_k\zeta_k^*\psi_k(\omega)^*\psi_k(\zeta) = q(\omega)^*q(\zeta) + \sum_{k=1}^d \psi_k(\omega)^*\psi_k(\zeta).$$

Therefore the map

$$\begin{bmatrix} \zeta_1 \psi_1(\zeta) \\ \vdots \\ \zeta_d \psi_d(\zeta) \\ p(\zeta) \end{bmatrix} \mapsto \begin{bmatrix} \psi_1(\zeta) \\ \vdots \\ \psi_d(\zeta) \\ q(\zeta) \end{bmatrix}$$

is a well-defined linear and isometric map from the span of the elements on the left to the span of the elements on the right, where both spans are taken over all $\zeta \in \mathbb{C}^d$ and $h \in \mathbb{C}^n$. It may be extended (if necessary) to a unitary matrix $U$ of the required form where we set $m_k = N_k$, $k = 1, \ldots, d$, and $m = m_1 + \cdots + m_d$. Writing $\psi(\zeta) = \text{Col}_{k=1,\ldots,d}[\psi_k]$, we have by construction of $U$

$$AP(\zeta)\psi(\zeta) + Bp(\zeta) = \psi(\zeta),$$

$$CP(\zeta)\psi(\zeta) + Dp(\zeta) = q(\zeta).$$

Solving for $\psi(\zeta)$ using the first equation and then plugging the result in the second equation gives

$$F(\zeta) = q(\zeta)p(\zeta)^{-1} = D + CP(\zeta)(I_m - AP(\zeta))^{-1}B = D + C(I_m - P(\zeta)A)^{-1}P(\zeta)B$$

as desired. \qed
3. Characterizations of the class $B_d^{n×n}$

The following refinement of the main result from [16] identifies the rational subclass $B_d^{rat}(C^n)$ of the generalized Bessmertny˘ı class $B_d(C^n)$.

**Theorem 3.1.** Let $f$ be a $C^{n×n}$-valued function of $d$ complex variables. The following statements are equivalent:

1. $f \in B_d^{n×n}$.
2. $f \in B_d^{rat}(C^n)$.
3. There exist $C^{n×n}$-valued functions $\Phi_k(w, z)$, $k = 1, \ldots, d$, which are rational as functions of $z = (z_1, \ldots, z_d)$ and $w = (w_1, \ldots, w_d)$ and which are positive kernels on $\Omega_d$ that satisfy (1.17) and (1.18).
4. There exist rational $C^{n_k×n}$-valued functions $\phi_k$, with some $N_k \in \mathbb{N}$, $k = 1, \ldots, d$, with no singularities on $\Omega_d$, that satisfy (1.19) and (1.20).

$\mathcal{F} = C(f)$ (see [16]) has a finite-dimensional Givone–Roesser representation (2.2) as in part (3) of Theorem 2.1 where the colligation matrix $U$ has the additional property of being Hermitian:

$$U^{-1} = U^* = U,$$

and $1 \notin \sigma(\mathcal{F}(0))$.

**Proof of Theorem 3.1.** Our plan is to prove first $(2') \Rightarrow (2) \Rightarrow (1) \Rightarrow (2')$, and then $(2') \iff (3)$ and $(2') \iff (0)$.

$(2') \Rightarrow (2)$ is obvious: just define $\Phi_k(w, z) = \phi_k(w)\phi_k(z)$, $k = 1, \ldots, d$.

$(2) \Rightarrow (1)$: It follows from the implication $(2) \Rightarrow (0)$ of Theorem 1.2 that $f \in B_d(C^n)$. Since the kernels $\Phi_k(w, z)$ are rational matrix-valued functions in both $w$ and $z$, the matrix-valued function $f$ in (1.18) is rational.

$(1) \Rightarrow (2')$: If (1) holds, then it follows that $\mathcal{F} = C(f)$ is a rational $n \times n$ matrix-valued function in the Schur–Agler class $\mathcal{S}A_d(C^n)$. The fact that $f$ is also Cayley inner then guarantees that $\mathcal{F}$ in addition is inner, i.e., $\mathcal{F} \in \mathcal{I}SA_d^{rat}(C^n)$. It follows from implication $(1) \Rightarrow (2')$ of Theorem 2.1 that (1.12) holds with rational $C^{n_k×n}$-valued functions $\theta_k$, for some $N_k \in \mathbb{N}$, $k = 1, \ldots, d$, which have no singularities on $D^d$. Set

$$\phi_k(z) = \frac{1}{z_k + 1} \theta_k \left( \frac{z_1 - 1}{z_1 + 1}, \ldots, \frac{z_d - 1}{z_d + 1} \right) (f(z) + I_n), \quad k = 1, \ldots, d. \quad (3.1)$$

It is clear that these $\phi_k$ are rational $C^{n\times n}$-valued functions which have no singularities on $D^d$. Moreover, since $f$ is homogeneous of degree 1, the argument in [16] Theorem 3.1] shows that these functions satisfy (1.19) and (1.20) and by homogeneity have no singularities on $\Omega_d$. Thus (2') follows.

$(2') \iff (3)$: Suppose that (2') holds. As we have already proved, $(2') \Rightarrow (2) \Rightarrow (1)$, so $f \in B_d^{rat}(C^n)$. Also, we have shown in the preceding paragraph that $\mathcal{F} = C(f) \in \mathcal{I}SA_d^{rat}(C^n)$. Since $\mathcal{F}(0) = (f(e) - I_n)(f(e) + I_n)^{-1}$, where $e = (1, \ldots, 1)$, is Hermitian and $I_n - \mathcal{F}(0) = 2(f(e) + I_n)^{-1}$ is positive definite, $1 \notin \sigma(\mathcal{F}(0))$. By the maximum principle, $1 \notin \sigma(\mathcal{F}(\zeta))$ for every $\zeta \in D^d$. Using the same argument as in the proof of the necessity part of Theorem 4.2 in [10], we first obtain (1.12) for $\mathcal{F}$ with rational $N_k \times n$ matrix-valued functions

$$\theta_k(\zeta) = \frac{1}{1 - \zeta_k} \phi_k \left( \frac{1 + \zeta_1}{1 - \zeta_1}, \ldots, \frac{1 + \zeta_d}{1 - \zeta_d} \right) (I_n - \mathcal{F}(\zeta))^{-1}, \quad k = 1, \ldots, d \quad (3.2)$$
(clearly, the transformation formulas (5.2) and (5.1) are the inverses of each other), and, in addition,

$$ F(\omega)^* - F(\zeta) = \sum_{k=1}^{d} (\overline{\omega} - \zeta) \theta_k(\omega)^* \theta_k(\zeta). \quad (3.3) $$

Then observing that the reproducing kernel Hilbert spaces \( \mathcal{H}(\theta_k(\omega)^* \theta_k(\zeta)) \) are finite-dimensional, that argument produces a finite-dimensional Givone–Roesser representation (2.2) of \( F \) with the colligation matrix \( U \) satisfying \( U^{-1} = U^* = U \).

Conversely, suppose that (3) holds. Then it is easy to verify (1.12) and (3.3) with

$$ \theta_k(\zeta) = P_k(I_m - AP(\zeta))^{-1}B, \quad k = 1, \ldots, d $$

(see (2.2)), where \( P_k = \text{Diag}[0, \ldots, 0, I_{m_k}, 0, \ldots, 0] \). Then, as in the proof of the sufficiency part of Theorem 4.2 in [16], we obtain the two decompositions

$$ f(w)^* \pm f(z) = \sum_{k=1}^{d} (\overline{\omega} \pm \zeta) \phi_k(w)^* \phi_k(z), \quad (3.4) $$

which together are equivalent to (1.20), with

$$ \phi_k(z) = \frac{1}{z_k + 1} P_k \left( I_m - AP\left(\frac{z_1 - 1}{z_1 + 1}, \ldots, \frac{z_d - 1}{z_d + 1}\right) \right)^{-1} B(f(z) + I_n) $$

(see (3.1)). It is clear that the functions \( \phi_k \) have no singularities on \( \Pi^k \), that they are rational, and by the argument in the sufficiency part of Theorem 3.1 in [10] they satisfy (1.19). By homogeneity, \( \phi_k \) have no singularities in \( \Omega_d \).

\((2') \Leftrightarrow (0)\) can be proved in the same way as Theorem 2.7 in [16] with taking care to use the additional assumption that all the functions involved are rational and finite matrix-valued. Notice that implication \((0) \Rightarrow (2')\) was proved in [9] (see also [10], Theorem 3.2) under an additional assumption of invertibility of \( f(z) \).

4. Characterizations of the class \( \mathbb{R}^{B_d^{n \times n}} \)

We recall that a \( \mathbb{C}^{n \times n} \)-valued function \( f \) is called real if \( f \) is \( \iota \)-real for \( \iota \) being the entrywise complex conjugation on \( \mathbb{C}^n \). For this \( \iota \), we use the notation \( \mathbb{R}^{B_d^{n \times n}}(\mathbb{C}^n) \) instead of \( \mathbb{R}^{B_d^{n \times n}}(\mathbb{C}^n) \).

**Theorem 4.1.** Let \( f \) be a \( \mathbb{C}^{n \times n} \)-valued function of \( d \) complex variables. The following statements are equivalent:

\begin{enumerate}
  \item \( f \in \mathbb{R}^{B_d^{n \times n}} \).
  \item \( f \in \mathbb{R}^{B_d^{n \times n}}(\mathbb{C}^n) \).
  \item There exist \( \mathbb{C}^{n \times n} \)-valued functions \( \Phi_k(w, z) \), \( k = 1, \ldots, d \), real rational as functions of \( z = (z_1, \ldots, z_d) \) and \( \overline{w} = (\overline{w_1}, \ldots, \overline{w_d}) \), which are positive kernels on \( \Omega_d \times \Omega_d \) satisfying (1.17) and (1.18).
  \item There exist real rational \( \mathbb{C}^{N_k \times n} \)-valued functions \( \phi_k \), with some \( N_k \in \mathbb{N} \), \( k = 1, \ldots, d \), with no singularities on \( \Omega_d \) and satisfy (1.19) and (1.20).
  \item There exist \( m, m_1, \ldots, m_d \in \mathbb{Z}_+ \), with \( m = m_1 + \cdots + m_d \), and a matrix

  \[ U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{R}^{(m+n) \times (m+n)} \]

  which is symmetric and orthogonal, i.e., \( U^{-1} = U^T = U \), where

  \[ A = [A_{ij}]_{i,j=1,\ldots,d}, \quad B = \text{Col}_{i=1,\ldots,d}[B_i], \quad C = \text{Row}_{j=1,\ldots,d}[C_j] \]
are block matrices with blocks $A_{ij} \in \mathbb{R}^{m_i \times m_j}$, $B_i \in \mathbb{R}^{m_i \times n}$, and $C_j \in \mathbb{R}^{n \times m_j}$, such that $F = C(f)$ (see (1.8)) has the representation (2.2), and $1 \notin \sigma(F(0))$.

Proof. We shall follow the same route as in the proof of Theorem 5.1, verifying the "reality" of all functions and matrices of interest.

$(2') \Rightarrow (2)$ is obvious: just define $\Phi_k(w, z) = \phi_k(w)^* \phi_k(z)$, $k = 1, \ldots, d$.

$(2) \Rightarrow (1)$: It follows from implication $(2) \Rightarrow (0)$ of Theorem 1.2 where both conditions $(0)$ and $(2)$ are modified as indicated in the last part of that theorem, that $f \in \mathbb{R}B_d(\mathbb{C})$. Since the kernels $\Phi_k(w, z)$ are rational matrix-valued functions in both $w$ and $z$, the matrix-valued function $f$ in (1.8) is rational.

$(1) \Rightarrow (2')$: If $(1)$ holds, then it follows from implication $(1) \Rightarrow (2')$ of Theorem 3.1 that there exist $N_k \in \mathbb{N}$ and rational $C^{N_k \times n}$-valued functions $\phi_k$ with no singularities on $\Omega_d$, $k = 1, \ldots, d$, such that (1.19) and (1.20) hold. Moreover, if $f$ is real, (1.19) and (1.20) also hold with $\phi_k$ replaced by $\phi_k^\ast$, $k = 1, \ldots, d$, or with $\phi_k$ replaced by $\tilde{\phi}_k = \text{Col}_{k} \left[ \frac{\phi_k + \phi_k^\ast}{\sqrt{2}}, \frac{\phi_k - \phi_k^\ast}{\sqrt{2}} \right]$. It remains to observe that rational $C^{2N_k \times n}$-valued functions $\tilde{\phi}_k$ are real.

$(2') \Leftrightarrow (3)$: Suppose that $(2')$ holds. Repeating the construction of matrix $U$ as in the proof of implication $(2') \Rightarrow (3)$ of Theorem 3.1 and observing that the constructed matrix $U$ is real, we obtain $(3)$.

Conversely, suppose that $(3)$ holds. Repeating the construction of functions $\phi_k$ as in the proof of implication $(3) \Rightarrow (2')$ of Theorem 3.1 and observing that the constructed functions $\phi_k$ are real, we obtain $(2')$.

$(2') \Rightarrow (0)$ can be proved in the same way as Theorem 2.7 in [13] with taking care to use the additional assumptions that all the functions involved are real rational matrix-valued and that matrices $A_k$ in the representation (1.1)–(1.2) for $f$ are real. Notice that implication $(0) \Rightarrow (2')$ was proved in [9] (see also [10] Theorem 3.2) under an additional assumption of invertibility of $f(z)$. □

5. Rational Cayley inner Herglotz–Agler-class functions on $\mathbb{D}^d$

We characterize the rational Cayley inner Herglotz–Agler class $\mathcal{CIHA}^{\text{rat}}(\mathbb{D}^d, \mathbb{C}^n)$ over the polydisk $\mathbb{D}^d$ in the following theorem, which parallels Theorem 2.1 for the rational inner Schur–Agler class $\mathcal{ISA}_d^{\text{rat}}(\mathbb{C}^n)$.

**Theorem 5.1.** Let $F$ be a $\mathbb{C}^{n \times n}$-valued function of $d$ complex variables. The following statements are equivalent:

1. $F \in \mathcal{CIHA}^{\text{rat}}(\mathbb{D}^d, \mathbb{C}^n)$.
2. There exist rational $\mathbb{C}^{N_k \times n}$-valued functions $\xi_k$, with some $N_k \in \mathbb{N}$, $k = 1, \ldots, d$, which have no singularities on $\mathbb{D}^d$ and satisfy

$$F(\omega)^* + F(\zeta) = \sum_{k=1}^{d} (1 - \mathbb{C}_k \xi_k(\omega))^* \xi_k(\zeta).$$

(5.1)

3. There exist $m, m_1, \ldots, m_d \in \mathbb{Z}_+$, with $m = m_1 + \cdots + m_d$, a unitary matrix $W \in \mathbb{C}^{m \times m}$, a matrix $\beta \in \mathbb{C}^{n \times n}$, and a matrix $V \in \mathbb{C}^{m \times n}$ such that

$$F(\zeta) = \beta + V^*(W - P(\zeta))^{-1}(W + P(\zeta))V,$$

(5.2)

where $\beta + \beta^* = 0$ and $P(z) = \text{Diag}[\xi_1 I_{m_1}, \ldots, \xi_d I_{m_d}]$. 


\textbf{Proof:} (1)\(\Rightarrow\) (3): Represent \(F(0) = \beta + \gamma\), where \(\beta = -\beta^*\) and \(\gamma = \gamma^*\). Since \(F \in \mathcal{HA}(\mathbb{D}^d, \mathbb{C}^n)\), the matrix \(\gamma\) is positive semidefinite. Then \(\gamma = \delta^*\delta\) with some matrix \(\delta \in \mathbb{C}^{r \times n}\) of full row rank \(r(=\text{rank}\ \gamma)\). We also have that \(F - \beta \in \mathcal{CIHA}(\mathbb{D}^d, \mathbb{C}^r)\). By the maximum principle, \(\ker(F(\zeta) - \beta) = \ker(F(0) - \beta)(=\ker\gamma)\). Therefore, one can represent \(F\) as
\[
F(\zeta) = \beta + \delta^*F_+(\zeta)\delta,
\]
with \(F_+ \in \mathcal{CIHA}(\mathbb{D}^d, \mathbb{C}^r)\) satisfying \(F_+(0) = I_r\). Define
\[
\mathcal{F}_+(\zeta) = (F_+(\zeta) - I_r)(F_+(\zeta) + I_r)^{-1}.
\] (5.3)
We have \(\mathcal{F}_+ \in \mathcal{ISA}_{d}^{\text{rat}}(\mathbb{C}^r)\). By Theorem 2.1 there exist \(m, m_1, \ldots, m_d \in \mathbb{Z}_+\), with \(m = m_1 + \cdots + m_d\), and a unitary matrix \(U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{C}^{(m+r) \times (m+r)}\) such that (2.2) holds (with \(n\) replaced by \(r\) and \(\mathcal{F}\) replaced by \(\mathcal{F}_+\)). Notice that \(D = \mathcal{F}_+(0) = 0\). Therefore, \(U\) has the form
\[
U = \begin{bmatrix} A_0 & 0 & 0 \\ 0 & 0 & B_0 \\ 0 & C_0 & 0 \end{bmatrix} : \begin{bmatrix} \ker C \\ \text{range } C^* \end{bmatrix} \rightarrow \begin{bmatrix} \ker B^* \\ \text{range } B \end{bmatrix}. \] (5.4)
In order to obtain a representation \((5.2)\) for \(F\), we first obtain a similar representation for \(F_+\) applying the argument from [11] Pages 63–64]. We first rewrite (5.3) as
\[
\mathcal{F}_+(\zeta)(F_+(\zeta) + I_r) = F_+(\zeta) - I_r.
\]
Together with (2.2), this is equivalent to
\[
\begin{bmatrix} P(\zeta)A & P(\zeta)B \\ C & 0 \end{bmatrix} \begin{bmatrix} X(\zeta) \\ F_+(\zeta) + I_r \end{bmatrix} = \begin{bmatrix} X(\zeta) \\ F_+(\zeta) - I_r \end{bmatrix},
\]
with \(X(\zeta) = (I_r - P(\zeta)A)^{-1}P(\zeta)B(F_+(\zeta) + I_r)\), or to
\[
\begin{bmatrix} X(\zeta) \\ F_+(\zeta) \end{bmatrix} = \begin{bmatrix} I_m - P(\zeta)A & -P(\zeta)B \\ -C & I_r \end{bmatrix}^{-1} \begin{bmatrix} P(\zeta)B \\ I_r \end{bmatrix}.
\]
Using the block-matrix inversion formula (see, e.g., [11] II.5.4), we obtain
\[
\begin{bmatrix} I_m - P(\zeta)A & -P(\zeta)B \\ -C & I_r \end{bmatrix}^{-1} = \begin{bmatrix} (I_m - P(\zeta)W^*)^{-1} & (I_m - P(\zeta)W^*)^{-1}P(\zeta)B \\ C(I_m - P(\zeta)W^*)^{-1} & C(I_m - P(\zeta)W^*)^{-1}P(\zeta)B + I_r \end{bmatrix},
\]
where \(W^* = A + BC\). Taking into account (5.4), we can rewrite \(W^*\) as
\[
W^* = \begin{bmatrix} A_0 & 0 \\ 0 & B_0C_0 \end{bmatrix} : \begin{bmatrix} \ker C \\ \text{range } C^* \end{bmatrix} \rightarrow \begin{bmatrix} \ker B^* \\ \text{range } B \end{bmatrix},
\]
and since \(A_0: \ker C \rightarrow \ker B^*, \ B_0: \mathbb{C}^r \rightarrow \text{range } B, \) and \(C_0: \text{range } C_0^* \rightarrow \mathbb{C}^r\) are unitary operators, so is \(W^*\). Identifying the operator \(W: \mathbb{C}^{m+r} \rightarrow \mathbb{C}^{m+r}\) with its matrix in the standard basis, we conclude that \(W\) is a unitary matrix. We have therefore
\[
F_+(\zeta) = 2C(I_m - P(\zeta)W^*)^{-1}P(\zeta)B + I_r.
\]
Observing that $B^*B = I$, and $B^*W^* = C$, we obtain

$$F_+(\zeta) = 2B^*W^*(I_m - P(\zeta)W^*)^{-1}P(\zeta)B + B^*B$$

$$= 2B^*(I_m - W^*P(\zeta))^{-1}W^*P(\zeta)B + B^*B$$

$$= B^*(I_m - W^*P(\zeta))^{-1}(I_m + W^*P(\zeta))B$$

$$= B^*(W - P(\zeta))^{-1}(W + P(\zeta))B.$$ 

Setting $V = B\delta$, we obtain the desired representation (5.2) for $F$.

(3) $\Rightarrow$ (2') Using (5.2), one easily obtains (5.1) with functions

$$\xi_k(\zeta) = \sqrt{2}P_k(I_m - W^*P(\zeta))^{-1}V$$

(here $N_k = m_k$) having the required properties.

(2') $\Rightarrow$ (1): Using hereditary functional calculus as in [1], we obtain from (5.1) that

$$F(T)^* + F(T) = \sum_{k=1}^d (I_{\mathcal{H}} - T_k^*T_k)\xi_k(T)^*\xi_k(T)$$

is a positive semidefinite operator on $\mathbb{C}^n \otimes \mathcal{H} \cong \mathcal{H}^n$ for every $d$-tuple $T = (T_1, \ldots, T_d)$ of commuting strict contractions on a Hilbert space $\mathcal{H}$. It is also obvious from (5.1) that $F$ is rational, and has no singularities on $\mathbb{D}$. Finally, the union of singularity sets for rational matrix-valued functions $\xi_k$, $k = 1, \ldots, d$, inside the unit torus $\mathbb{T}^d$ is of measure zero (with respect to the Lebesgue measure on $\mathbb{T}^d$). Hence, $F$ is regular almost everywhere on $\mathbb{T}^d$, and we see from (5.1) that at those regular points $\zeta$ we have $F(\zeta)^* + F(\zeta) = 0$, i.e., $F$ is Cayley inner. \hfill \Box

6. Rational Cayley inner Herglotz-Agler-class functions on $\Pi^d$

In this section, we characterize the rational Cayley inner Herglotz–Agler class $\mathcal{CThA}^{rat}(\Pi^d, \mathbb{C}^n)$ over the poly-halfplane $\Pi^d$. As we mentioned in Introduction, it can be viewed as a version of the class $\mathcal{B}^{rat}_d(\mathbb{C}^n)$ with the homogeneity condition (1.3) dropped. Let us say that $f$ is in the nonhomogeneous Bessmertny˘ı class $\mathcal{B}^{rat}_d(\mathbb{C}^n)$ if $f$ has a long resolvent representation as in (1.1) and (1.2) subject to the conditions

$$A_0 = -A_0^*, \quad A_k = A_k^* \geq 0 \text{ for } k = 1, \ldots, d.$$ 

(6.1)

Then we have the following result.

**Theorem 6.1.** Let $f$ be a $\mathbb{C}^{n \times n}$-valued function of $d$ complex variables. The following statements are equivalent:

(0) $f \in \mathcal{B}^{n \times n}_d$.

(1) $f \in \mathcal{CThA}^{rat}(\Pi^d, \mathbb{C}^n)$.

(2') There exist rational $\mathbb{C}^{N_k \times n}$-valued functions $\phi_k$ with some $N_k \in \mathbb{N}$, $k = 1, \ldots, d$, so that

$$f(w)^* + f(z) = \sum_{k=1}^d (w_k + z_k)\phi_k(w)^*\phi_k(z).$$

(6.2)

(3) $\mathcal{F} = \mathcal{C}(f) \in \mathcal{ISA}^{rat}_d(\mathbb{C}^n)$. 

Therefore, single Cayley transformation over the values to $z$ changing the roles of $\phi$ with $\psi$ where $C$ are allowed, i.e., the state space $\mathbb{C}^m$ is replaced by an infinite-dimensional Hilbert space $\mathcal{X}$ (see also a similar argument for the homogeneous case in the proof of [16, Theorem 2.7]). We here tailor the argument presented there for the case of a finite-dimensional state space $\mathcal{X} = \mathbb{C}^m$. The assumption (0) gives us a long-resolvent representation for $f$:

$$f(z) = A_{11}(z) - A_{12}(z)A_{22}(z)^{-1}A_{21}(z)$$

where

$$\begin{bmatrix} A_{11}(z) & A_{12}(z) \\ A_{21}(z) & A_{22}(z) \end{bmatrix} = A(z) := A_0 + z_1 A_1 + \cdots + z_d A_d$$

and the coefficients $A_k$ satisfy (6.1). We compute

$$f(z) = \begin{bmatrix} I_m & -A_{21}(w)^* A_{22}(w)^{-1} \\ -A_{22}(w)^{-1} A_{21}(w) \end{bmatrix}^{*} \begin{bmatrix} A_{11}(z) & A_{12}(z) \\ A_{21}(z) & A_{22}(z) \end{bmatrix} \begin{bmatrix} I_m \\ -A_{22}(z)^{-1} A_{21}(z) \end{bmatrix} = \psi(w)^* A(z) \psi(z),$$

where $\psi(z) := \begin{bmatrix} I_m \\ -A_{22}(z)^{-1} A_{21}(z) \end{bmatrix}$ is a rational $\mathbb{C}^{(m+n) \times n}$-valued function. Interchanging the roles of $z$ and $w$, we obtain also

$$f(w)^* = \psi(w)^* A(w)^* \psi(z).$$

Therefore

$$f(w)^* + f(z) = \psi(w)^* (A(w)^* + A(z)) \psi(z) = \sum_{k=1}^d (\overline{w_j} + z_k) \phi_k(w)^* \phi_k(z),$$

with $\phi_k(z) = A_{k}^{1/2} \psi(z)$, $k = 1, \ldots, d$.

(3)⇒(0): Applying the single Cayley transformation over the variables to $f$ or the single Cayley transformation over the values to $F$, we obtain $F \in \mathcal{CIHA}^{\mathbb{C}^m}(\mathbb{D}^d, \mathbb{C}^n)$:

$$F(\zeta) = f \left( \frac{1 + \zeta_1}{1 - \zeta_1}, \ldots, \frac{1 + \zeta_d}{1 - \zeta_d} \right) = \left( F(\zeta) - I_n \right)^{-1} \left( F(\zeta) + I_n \right). \quad (6.3)$$

By Theorem 5.1, $F$ admits a representation (5.2). Then

$$f(z) = F \left( \frac{z_1 - 1}{z_1 + 1}, \ldots, \frac{z_d - 1}{z_d + 1} \right) = \beta + V^* M(z) V, \quad (6.4)$$
where

\[
M(z) = \left( W - P\left(\frac{z_1 - 1}{z_1 + 1}, \ldots, \frac{z_d - 1}{z_d + 1}\right) \right)^{-1} \left( W + P\left(\frac{z_1 - 1}{z_1 + 1}, \ldots, \frac{z_d - 1}{z_d + 1}\right) \right)
\]

\[
= \left( W - (P(z) + I_m)^{-1}(P(z) - I_m) \right)^{-1} \left( W + (P(z) + I_m)^{-1}(P(z) - I_m) \right)
\]

\[
= \left( (P(z) + I_m)W - (P(z) - I_m) \right)^{-1} \left( (P(z) + I_m)W + (P(z) - I_m) \right)
\]

\[
= \left( P(z)(W - I_m) + (W + I_m) \right)^{-1} \left( P(z)(W + I_m) + (W - I_m) \right)
\]

In order to obtain a more detailed representation for \( M \) (and eventually, for \( f \)), we adopt the idea from [2] of using a partial Cayley transform of \( W \) as follows (the paper [2] deals with the Nevanlinna–Agler class which can be obtained from the Herglotz–Agler class by multiplying all the variables and the function values by \( i \)).

Let \( \mathcal{H} \subset \mathbb{C}^m \) be the eigenspace of \( W \) corresponding to the eigenvalue 1. Then, with respect to the orthogonal decomposition \( \mathbb{C}^m = \mathcal{H} \oplus \mathcal{H}^\perp \), one has

\[
W = \begin{bmatrix} I_\mathcal{H} & 0 \\ 0 & W_0 \end{bmatrix}, \quad W - I_m = \begin{bmatrix} 0 & 0 \\ 0 & W_0 - I_{\mathcal{H}^\perp} \end{bmatrix}, \quad W + I_m = \begin{bmatrix} 2I_\mathcal{H} & 0 \\ 0 & W_0 + I_{\mathcal{H}^\perp} \end{bmatrix},
\]

where \( W_0 - I_{\mathcal{H}^\perp} \) is invertible. We thus obtain

\[
M(z) = \left( P(z) \begin{bmatrix} 0 & 0 \\ 0 & W_0 - I_{\mathcal{H}^\perp} \end{bmatrix} + \begin{bmatrix} 2I_\mathcal{H} & 0 \\ 0 & W_0 + I_{\mathcal{H}^\perp} \end{bmatrix} \right)^{-1}
\]

\[
\cdot \left( P(z) \begin{bmatrix} 2I_\mathcal{H} & 0 \\ 0 & W_0 + I_{\mathcal{H}^\perp} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & W_0 - I_{\mathcal{H}^\perp} \end{bmatrix} \right).
\]

Set

\[
\alpha := (I_{\mathcal{H}^\perp} - W_0)^{-1}(I_{\mathcal{H}^\perp} + W_0).
\]

Then

\[
W_0 = (\alpha - I_{\mathcal{H}^\perp})(\alpha + I_{\mathcal{H}^\perp})^{-1} = I_{\mathcal{H}^\perp} - 2(\alpha + I_{\mathcal{H}^\perp})^{-1},
\]

and we can rewrite \( M(z) \) first as

\[
M(z) = \begin{bmatrix} 2I_\mathcal{H} & 0 \\ 0 & (W_0 - I_{\mathcal{H}^\perp})^{-1} \end{bmatrix}
\]

\[
\cdot \left( P(z) \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathcal{H}^\perp} \end{bmatrix} + \begin{bmatrix} I_\mathcal{H} & 0 \\ 0 & (W_0 + I_{\mathcal{H}^\perp})(W_0 - I_{\mathcal{H}^\perp})^{-1} \end{bmatrix} \right)^{-1}
\]

\[
\cdot \left( P(z) \begin{bmatrix} I_\mathcal{H} & 0 \\ 0 & (W_0 + I_{\mathcal{H}^\perp})(W_0 - I_{\mathcal{H}^\perp})^{-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathcal{H}^\perp} \end{bmatrix} \right)
\]

\[
\cdot \begin{bmatrix} 2I_\mathcal{H} & 0 \\ 0 & W_0 - I_{\mathcal{H}^\perp} \end{bmatrix},
\]

and then as

\[
M(z) = \begin{bmatrix} I_\mathcal{H} & 0 \\ 0 & -(\alpha + I_{\mathcal{H}^\perp}) \end{bmatrix} \left( P(z) \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathcal{H}^\perp} \end{bmatrix} + \begin{bmatrix} I_\mathcal{H} & 0 \\ 0 & -\alpha \end{bmatrix} \right)^{-1}
\]

\[
\cdot \left( P(z) \begin{bmatrix} I_\mathcal{H} & 0 \\ 0 & -\alpha \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathcal{H}^\perp} \end{bmatrix} \right) \begin{bmatrix} I_\mathcal{H} & 0 \\ 0 & -(\alpha + I_{\mathcal{H}^\perp})^{-1} \end{bmatrix}.
\]
We observe that since the operator $W_0$ is unitary, its Cayley transform $\alpha$ is skew-adjoint, i.e., $\alpha^* = -\alpha$. Therefore, we can rewrite (6.5) as

\[
M(z) = \begin{bmatrix}
I_{\mathcal{H}} & 0 \\
0 & -(\alpha + I_{\mathcal{H}^\perp})
\end{bmatrix} \begin{bmatrix}
I_{\mathcal{H}} & 0 \\
0 & -(\alpha + I_{\mathcal{H}^\perp})^*
\end{bmatrix},
\]

where

\[
N(z) = \left( P(z) \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{\mathcal{H}^\perp} & -\alpha \end{bmatrix} + \begin{bmatrix} I_{\mathcal{H}} & 0 & 0 \\ 0 & I_{\mathcal{H}^\perp} & -\alpha \end{bmatrix} \right)^{-1} \left( P(z) \begin{bmatrix} I_{\mathcal{H}} & 0 & 0 \\ 0 & I_{\mathcal{H}^\perp} & -\alpha \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{\mathcal{H}^\perp} & (I_{\mathcal{H}^\perp} - \alpha^2)^{-1} \end{bmatrix} \right)
\]

and we use that $-(\alpha + I_{\mathcal{H}^\perp})^* = \alpha - I_{\mathcal{H}^\perp}$. Writing

\[
P(z) = \begin{bmatrix} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{bmatrix},
\]

we compute

\[
\left( P(z) \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{\mathcal{H}^\perp} & -\alpha \end{bmatrix} + \begin{bmatrix} I_{\mathcal{H}} & 0 & 0 \\ 0 & I_{\mathcal{H}^\perp} & -\alpha \end{bmatrix} \right)^{-1} = \begin{bmatrix} I_{\mathcal{H}} & P_{12}(z) \\ 0 & P_{22}(z) - \alpha \end{bmatrix}^{-1} = \begin{bmatrix} I_{\mathcal{H}} & -P_{12}(z)(P_{22}(z) - \alpha)^{-1} \\ 0 & (P_{22}(z) - \alpha)^{-1} \end{bmatrix},
\]

\[
\left( P(z) \begin{bmatrix} I_{\mathcal{H}} & 0 & 0 \\ 0 & I_{\mathcal{H}^\perp} & -\alpha \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_{\mathcal{H}^\perp} & (I_{\mathcal{H}^\perp} - \alpha^2)^{-1} \end{bmatrix} \right) = \begin{bmatrix} P_{11}(z) & P_{12}(z)J \\ P_{21}(z) & (I_{\mathcal{H}^\perp} - P_{22}(z)\alpha)(I_{\mathcal{H}^\perp} - \alpha^2)^{-1} \end{bmatrix},
\]

where $J := -\alpha(I_{\mathcal{H}^\perp} - \alpha^2)^{-1}$. (Notice that $J^* = -J$.) Next, writing

\[
I_{\mathcal{H}^\perp} - P_{22}(z)\alpha = I_{\mathcal{H}^\perp} - \alpha^2 + \alpha^2 - P_{22}(z)\alpha,
\]

we obtain

\[
(I_{\mathcal{H}^\perp} - P_{22}(z)\alpha)(I_{\mathcal{H}^\perp} - \alpha^2)^{-1} = I_{\mathcal{H}^\perp} - (P_{22}(z) - \alpha)\alpha(I_{\mathcal{H}^\perp} - \alpha^2)^{-1} = I_{\mathcal{H}^\perp} + (P_{22}(z) - \alpha).J.
\]

Thus

\[
N(z) = \begin{bmatrix} I_{\mathcal{H}} & -(P_{12}(z)(P_{22}(z) - \alpha)^{-1}) \\ 0 & (P_{22}(z) - \alpha)^{-1} \end{bmatrix} \begin{bmatrix} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & I_{\mathcal{H}^\perp} + (P_{22}(z) - \alpha)J \end{bmatrix}
\]

\[
= \begin{bmatrix} P_{11}(z) - P_{12}(z)(P_{22}(z) - \alpha)^{-1}P_{21}(z) & P_{12}(z)(P_{22}(z) - \alpha)^{-1} \end{bmatrix} \begin{bmatrix} I_{\mathcal{H}^\perp} & P_{12}(z) \end{bmatrix}
\]

\[
= \begin{bmatrix} P_{11}(z) & 0 \\ 0 & J \end{bmatrix} - \left[ P_{12}(z)(P_{22}(z) - \alpha)^{-1} \right] \begin{bmatrix} P_{21}(z) & I_{\mathcal{H}^\perp} \end{bmatrix}.
\]

Consequently,

\[
M(z) = \begin{bmatrix} P_{11}(z) & 0 \\ 0 & (\alpha + I_{\mathcal{H}^\perp})J(\alpha + I_{\mathcal{H}^\perp})^* \end{bmatrix} - \left[ P_{12}(z)(P_{22}(z) - \alpha)^{-1} \right] \begin{bmatrix} P_{21}(z) & -(\alpha + I_{\mathcal{H}^\perp})^* \end{bmatrix}, \tag{6.6}
\]
The linear pencil

\[ f(z) = \beta + V^* \begin{bmatrix} P_{11}(z) & 0 \\ 0 & (\alpha + I_{H^+})J(\alpha + I_{H^+})^* \end{bmatrix} V - V^* \begin{bmatrix} P_{22}(z) \\ (\alpha + I_{H^+}) \end{bmatrix} (P_{22}(z) - \alpha)^{-1} \begin{bmatrix} P_{21}(z) \\ -(\alpha + I_{H^+})^* \end{bmatrix} V. \]

Representing

\[ V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} : \mathbb{C}^n \to H \oplus H^\perp = \mathbb{C}^n, \]

we obtain

\[ f(z) = \beta + V_1^* P_{11}(z) V_1 + V_2^* (\alpha + I_{H^+}) J(\alpha + I_{H^+})^* V_2 - (V_1^* P_{12}(z) + V_2^* (\alpha + I_{H^+}))(P_{22}(z) - \alpha)^{-1}(P_{21}(z)V_1 - (\alpha + I_{H^+})^* V_2). \]

(6.7)

In other words, we have obtained a long resolvent representation (6.1) for \( f \), with

\[ \begin{align*}
A_{11}(z) &= \beta + V_2^* (\alpha + I_{H^+}) J(\alpha + I_{H^+})^* V_2 + V_1^* P_{11}(z)V_1, \\
A_{12}(z) &= V_2^* (\alpha + I_{H^+}) + V_1^* P_{12}(z), \\
A_{21}(z) &= -(\alpha + I_{H^+})^* V_2 + P_{21}(z)V_1, \\
A_{22}(z) &= -\alpha + P_{22}(z).
\end{align*} \]

The linear pencil

\[ A(z) = \begin{bmatrix} A_{11}(z) & A_{12}(z) \\ A_{21}(z) & A_{22}(z) \end{bmatrix} = A_0 + z_1 A_1 + \cdots + z_d A_d \]

has the coefficients

\[ \begin{align*}
A_0 &= \begin{bmatrix} \beta + V_2^* (\alpha + I_{H^+}) J(\alpha + I_{H^+})^* V_2 & V_2^* (\alpha + I_{H^+}) \\
-(\alpha + I_{H^+})^* V_2 & -\alpha \end{bmatrix}, \\
A_k &= \begin{bmatrix} V_1^* (P_k)_{11} V_1 & V_1^* (P_k)_{12} \\ (P_k)_{21} V_1 & (P_k)_{22} \end{bmatrix},
\end{align*} \]

(6.8)

(6.9)

where \( P_k = \text{Diag}[0, \ldots, 0, I_{m_k}, 0, \ldots, 0] \), \( k = 1, \ldots, d \). It is easy to check that the matrices \( A_0, \ldots, A_d \) satisfy (6.1). We conclude that \( f \in B_d^{n \times n} \).

\[ \square \]

Remark 6.2. It is not obvious from (6.5) that \( M(z) \) and, therefore, \( f \) as in (6.4) are Herglotz–Agler functions on \( \Pi^d \); see (3.1) and (4.3) in [2] for the Nevanlinna–Agler-class version of \( M \) and \( f \). It takes several pages in [2] (see Propositions 3.4 and 3.5 in there) to show that \( M \) is a Nevanlinna–Agler function over the upper poly-halfplane. Our representation (6.6) for \( M \) and then our representation (6.7) for \( f \) allow us to see the inclusion of these functions to the corresponding matrix-valued Herglotz–Agler classes over \( \Pi^d \) immediately. Indeed, the direction \((0) \Rightarrow (1)\) in Theorem 6.1 follows from the fact that a linear pencil \( A(z) \) as in (4.1) with \( A_0^* = -A_0 \) and \( A_k = A_k^* \geq 0, k = 1, \ldots, d \), is a rational Cayley inner Herglotz–Agler function, and taking a Schur complement of \( A(z) \) preserves this property (it just changes the matrix size for the function values).

Remark 6.3. Theorem 6.1 is a generalization of Theorem 3.1 or we can view Theorem 3.1 as a specialization of Theorem 6.1 to the Bessmertnyi class \( B_d^{n \times n} \). Thus, \( B_d^{n \times n} \) can be characterized as

\[ \begin{align*}
(0) & \text{ the subclass in } B_d^{n \times n} \text{ consisting of functions with a long resolvent representation satisfying } A_0 = 0, \\
& \text{or}
\end{align*} \]
(1) (in view of Theorem 5.2) the subclass in $\mathcal{CAHA}^{\text{rat}}(\mathbb{D}, \mathbb{C}^n)$ satisfying the homogeneity condition (1.3), or
(2') the subclass of functions satisfying condition (2') of Theorem 6.1 with the additional property that, along with the decomposition (6.2), they have the decomposition obtained from (6.2) by replacing pluses by minuses, i.e., that (6.3) holds, or
(3) the subclass of functions satisfying condition (3) of Theorem 6.1 with the additional property that the matrix $U$ is Hermitian.

Applying the single Cayley transformation over the variables to a function $f \in \mathcal{CAHA}^{\text{rat}}(\mathbb{D}, \mathbb{C}^n)$ or the single Cayley transformation over the values to a function $f \in \mathcal{CAHA}^{\text{rat}}(\mathbb{D}, \mathbb{C}^n)$. By Theorem 6.1 $F$ admits a representation (5.2). The following theorem specializes (5.2) to the case where $F$ is the single Cayley transform of a function $f \in \mathcal{B}_d^{n \times n}$.

**Theorem 6.4.** Let $f$ be a $\mathbb{C}^{n \times n}$-valued function of $d$ complex variables. Then $f \in \mathcal{B}_d^{n \times n}$ if and only if the function

$$F(\zeta) = f \left( \frac{1 + \zeta_1}{1 - \zeta_1}, \ldots, \frac{1 + \zeta_d}{1 - \zeta_d} \right)$$

satisfies the condition (3) of Theorem 5.1 with the additional properties

(i) $\beta = 0$;

(ii) $W = W^*$;

(iii) $\text{range} V \subseteq \mathcal{H}$, where $\mathcal{H} \subseteq \mathbb{C}^n$ is the eigenspace of $W$ corresponding to the eigenvalue 1.

For the proof of Theorem 6.4 we will need the following lemma.

**Lemma 6.5.** Let $f$ be a $\mathbb{C}^{n \times n}$-valued function of $d$ complex variables. Then $f \in \mathcal{B}_d^{n \times n}$ if and only if there exist a matrix $\delta \in \mathbb{C}^{r \times n}$ of full row rank $r (= \text{rank} f(e))$, where $e = (1, \ldots, 1)$ and a function $f_+ \in \mathcal{B}_d^{r \times r}$ satisfying $f_+(e) = I_r$, such that

$$f(z) = \delta^* f_+(z) \delta.$$ 

**Proof.** Suppose that $f_+ \in \mathcal{B}_d^{r \times r}$, $f_+(e) = I_r$, and $f(z) = \delta^* f_+(z) \delta$ for some $\delta \in \mathbb{C}^{r \times n}$. If $f_+$ has a long resolvent representation (1.1)–(1.2) (with $n$ replaced by $r$) with the coefficients $A_0 = 0$ and $A_k = A_k \geq 0$, $k = 1, \ldots, d$, then $f$ has a long resolvent representation with the coefficients $A_0 = 0$,

$$\begin{bmatrix} \delta^* & 0 \\ 0 & I_m \end{bmatrix} A_k \begin{bmatrix} \delta & 0 \\ 0 & I_m \end{bmatrix} \geq 0, \quad k = 1, \ldots, d,$$

i.e., $f \in \mathcal{B}_d^{n \times n}$.

Conversely, suppose that $f \in \mathcal{B}_d^{n \times n}$. Let $\mathcal{X} := \ker f(e) \subseteq \mathbb{C}^n$, so that $\mathbb{C}^n = \mathcal{X} \oplus \mathcal{X}^\perp$. By the maximum principle, $\ker f(z) = \ker f(e) = \mathcal{X}$ for every $z \in \Omega_d$ and, thus, for every $z$ a regular point of $f$. Let $r := \dim \mathcal{X}^\perp$ and let $\kappa : \mathbb{C}^r \to \mathbb{C}^n$ be an isometry with range $\kappa = \mathcal{X}^\perp$. Applying the argument as in the preceding paragraph, we obtain that

$$\tilde{f} := \kappa^* f(z) \kappa \in \mathcal{B}_d^{r \times r}.$$

Clearly $\tilde{f}(e) = \tilde{f}(e)^* > 0$. Define

$$f_+(z) = \tilde{f}(e)^{-1/2} \tilde{f}(z) \tilde{f}(e)^{-1/2}.$$
By the same argument as above, $f_+ \in B_d^{r \times r}$. We then have $f(z) = \delta^* f_+(z) \delta$ as required, with
\[ \delta = \bar{f}(e)^{1/2} \kappa^*. \]

Proof of Theorem 6.4. Suppose that $F$ satisfies the condition (3) of Theorem 5.1 with the additional properties (i)–(iii). Arguing as in the proof of (3) $\Rightarrow$ (0) of Theorem 6.1, we obtain the following. First, we have $\beta = 0$. Second, since the matrix $W$ is unitary and Hermitian, it has only two eigenvalues, 1 and $-1$, therefore $W_0 = -I_{\mathcal{H}_r}$ and $\alpha = J = 0$. Third, since range $V \subseteq \mathcal{H}$, we have $V_2 = 0$. Therefore, $f$ has a long resolvent representation (1.1) with the coefficient matrices $A_0 = 0$ and $A_k \geq 0$, $k = 1, \ldots, d$ (see (6.8)–(6.9)), i.e., $f \in B_d^{n \times n}$.

Conversely, suppose that $f \in B_d^{n \times n}$. Then by Lemma 6.5 we have that $f(z) = \delta^* f_+(z) \delta$, with some $\delta \in \mathbb{C}^{r \times n}$ of full row rank $r$ ($= \text{rank } f(e)$) and a function $f_+ \in B_d^{r \times r}$ satisfying $f_+(e) = I_r$, such that $f(z) = \delta^* f_+(z) \delta$. Applying the Cayley transform over variables as in (6.10), we obtain
\[ F(\zeta) = \delta^* F_+(\zeta) \delta, \]
with $F_+ \in CIHA(\mathbb{D}_d^d, \mathbb{C}^r)$ satisfying $F(0) = I_r$. Arguing as in the proof of (1) $\Rightarrow$ (3) of Theorem 5.1 we obtain the following. First, we have $\beta = 0$. Second, the function $F_+$ in (5.3) can be represented as $F_+ = C(f_+)$, so by Theorem 3.1 the unitary matrix $U$ in a transfer-function realization (2.2) for $F_+$ can be chosen Hermitian. Therefore, $C = B^*$ and we have (5.4) with $A_0 = A_0^*$ and $C_0 = B_0^*$. This, in turn, implies that
\[ W = W^* = W^{-1} = \begin{bmatrix} A_0 & 0 \\ 0 & I_{\text{range } B} \end{bmatrix} \]
with respect to the decomposition $\mathbb{C}^m = \ker B^* \oplus \text{range } B$. Clearly, we have that range $B \subseteq \mathcal{H}$, where $\mathcal{H}$ is the eigenspace of $W$ corresponding to the eigenvalue 1. Then range $V \subseteq \mathcal{H}$, where $V = B \delta$. Finally, we obtain the representation (5.2) for $F$ as desired. \qed

7. Some remarks

7.1. The results of Section 4 can be easily extended to the real rational Cayley inner Heerglotz–Agler class over $\Pi^d$, in view of possible applications in electrical engineering. The techniques developed in this paper would suffice for a proof, which we leave to the reader as an exercise.

7.2. In addition to the rational inner / Cayley inner Schur–Agler and Heerglotz–Agler classes over the polydisk $\mathbb{D}_d^d$ and the Heerglotz–Agler class over the right polyhalfplane $\Pi^d$, one can also consider the rational inner Schur–Agler class over $\Pi^d$ and obtain analogues of Theorems 2.1, 5.1 and 6.1. It is also possible to describe the image of the Bessmertny˘ı class $B_d^{n \times n}$ in the latter class under the Cayley transform over the function values, similarly to part (3) of Theorem 3.1 and to Theorem 6.4. These ingredients were unnecessary in our analysis and appeared somewhat isolated, so we left them aside.
7.3. In [15, Example 5.1], explicit examples of rational inner functions on $D^d$ which are not in the Schur–Agler class over $D^d$ were constructed. After applying appropriate linear-fractional changes of variable to domains and ranges and combinations thereof to those examples, one can obtain explicit examples of rational Cayley inner functions on $D^d$ which are not in the Herglotz–Agler class over $D^d$, of rational inner functions on $\Pi^d$ which are not in the Schur–Agler class over $\Pi^d$, and of rational Cayley inner functions on $\Pi^d$ which are not in the Herglotz–Agler class over $\Pi^d$. However, the corresponding question for the Bessmertny\text{"i} class remains unresolved, as already mentioned in the Introduction: can there be a function $f \in P_{d \times n}$ which is not in $B_{d \times n}^{n \times n}$?

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