A note on instabilities of extremal black holes under scalar perturbations from afar

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Received 13 December 2012, in final form 15 March 2013
Published 12 April 2013
Online at stacks.iop.org/CQG/30/095010

Abstract

In a previous work of the author, it was shown that instabilities of solutions to the wave equation develop asymptotically along the event horizon of extremal Kerr provided a certain expression $H_0$ of the initial data is non-trivial on the horizon. In this note, we remove this restriction by showing that instabilities develop even from the initial data supported arbitrarily far away from the horizon (for which, in particular, $H_0 = 0$). The latter instabilities concern one order higher derivatives compared to the case where $H_0 \neq 0$. The result also applies to extremal Reissner–Nordstrom. This note was motivated by the numerical analysis of Lucietti, Murata, Reall and Tanahashi.

PACS numbers: 02.30.Jr, 04.20.−q

1. Introduction

The wave equation on black hole backgrounds has attracted significant interest of the mathematical community in recent decades in view of its intimate relation with the stability of the spacetimes themselves in the context of the Cauchy problem for the Einstein equations. Specifically, one is interested in the decay properties of the solution $\psi$ to the wave equation

$$\Box_g \psi = 0$$

in the domain of outer communications up to and including the event horizon $\mathcal{H}^+$.
The study of the wave equation on black holes goes back to Regge and Wheeler [19] in 1957. The first complete boundedness result for the solution $\psi$ to (1), on Schwarzschild backgrounds, without any restriction of the initial data on $\mathcal{H}^+$, was due to Kay and Wald [14] and the first complete quantitative decay result for such solutions appeared in 2005 by Dafermos and Rodnianski [10]. The latter paper initiated the use of multiplier vector fields which capture the so-called redshift effect along the horizon $\mathcal{H}^+$. Since then impressive progress has been made for more general spacetimes with contributions by many authors; for a review, with many references, see [9]. This work has in some sense culminated in [11] where uniform boundedness and decay estimates for $\psi$ and all its derivatives are derived (up to and including the event horizon) for general subextremal Kerr backgrounds with parameters $|a| < M$.

The wave equation on extremal black holes (for which the redshift effect degenerates at the horizon $\mathcal{H}^+$) was much less studied until recently; see however [6, 8, 13, 17]. The general study of the wave equation on such backgrounds was initiated by the author in a series of papers [1–5] where it was shown that the solutions of (1) exhibit both stability and instability properties. Specifically, it was shown that general solutions on extremal Reissner–Nordström and axisymmetric solutions on extremal Kerr decay pointwise toward the future. On the other hand, it was also shown that, if $Y$ is a translation-invariant vector field, then for the initial data for which a certain expression is non-zero on $\mathcal{H}^+$, the following behavior is observed along $\mathcal{H}^+$ as the advanced time $v \to +\infty$.

\begin{align}
\text{Non-decay:} & \quad \sup_{S_v} |Y \psi| \not\to 0 \\
\text{Blow-up:} & \quad \sup_{S_v} |Y^k \psi| \to +\infty, \quad k \geq 2,
\end{align}

(2)

along $\mathcal{H}^+$ as the advanced time $v \to +\infty$. Here, $S_v$ denotes the (spherical) section of the horizon $\mathcal{H}^+$ at advanced time $v$. As far as the instability results are concerned, the axisymmetry assumption for solutions on the extremal Kerr can be dropped by simply projecting on the zeroth azimuthal frequency.

The origin of the instability results is the existence of conserved quantities along extremal horizons $\mathcal{H}^+$. Hence, if the conserved quantities are initially non-zero, then they are everywhere non-zero on $\mathcal{H}^+$. A question that was raised by Dain and Dotti [12] is: Whether the instabilities (2) develop from the initial data which are supported away from the horizon (and hence the conserved quantities are initially—and thus everywhere—zero on $\mathcal{H}^+$). In the latter case, the authors of [12] derived a simple proof of the pointwise boundedness of $\psi$ on extremal Reissner–Nordström by extending Wald’s argument [20]. Returning to the question of Dain and Dotti [12], Lucietti et al [15] performed the numerical analysis which suggests that instabilities develop even from the initial data supported away from the horizon of extremal Reissner–Nordström. Motivated by [15], we rigorously show the validity of this scenario which we in fact extend to extremal Kerr backgrounds.

In this note we show the following.

**Theorem 1.** Consider the extremal Kerr black hole with parameters $a$ and $M$ such that $|a| = M$. Let $\Sigma$ be a spacelike hypersurface which either crosses the event horizon and satisfies the assumptions of section 2.2 or coincides with $t = 0$, where $t$ denotes the Boyer–Lindquist time coordinate and $K = \Sigma \cap \{R_1 \leq r \leq R_2\}$, where $M < R_1 < R_2$. Let $Y$ denote a translation-invariant vector field (that is, $[Y, T] = 0$, where $T$ is the stationary Killing vector field).
transversal to $\mathcal{H}^+$. Then, for the generic smooth initial data supported in $K$, we obtain that the solution $\psi$ to the wave equation (1) satisfies

\begin{align*}
\text{Non-decay:} & \quad \sup_{S_v} |Y \psi| \rightarrow 0 \\
\text{Blow-up:} & \quad \sup_{S_v} |Y^k \psi| \rightarrow +\infty, \ k \geq 3, \\
\end{align*}

along $\mathcal{H}^+$ as the advanced time $v \rightarrow +\infty$. Here, $S_v$ denotes the (spherical) section of the horizon $\mathcal{H}^+$ at the advanced time $v$.

In particular, $K$ may be chosen to be arbitrarily small and arbitrarily far away from the horizon. Note that the instabilities of theorem 1 concern one order higher derivatives compared to the previous results (see (2)) for the initial data whose support includes $\mathcal{H}^+$. On the other hand, note that our previous stability results and the vanishing of the conserved quantity $H_0[\psi] = 0$ (see [5] or section 5 below) imply that for the solution $\psi$ as in theorem 1, we have

\begin{align*}
|\psi| \rightarrow 0, \quad \int_{S_v} Y \psi \rightarrow 0,
\end{align*}

as $v \rightarrow +\infty$. Note that the above, in particular, precludes the first non-decay inequality of (2).

We remark that our argument can be readily adapted for the easier case of extremal Reissner–Nordström backgrounds. We finally mention an independent work of Bizon and Friedrich [7] which provides a nice heuristic analysis of the problem based on the existence of conformal symmetries and heuristic analysis on $I^+$.

### 2. Geometric setup

#### 2.1. The Kerr metric

The Kerr metric with respect to the Boyer–Lindquist coordinates $(t, r, \theta, \phi)$ is given by

\[ g = g_{tt} \, dt^2 + g_{rr} \, dr^2 + g_{\phi\phi} \, d\phi^2 + g_{\theta\theta} \, d\theta^2 + 2g_{t\phi} \, dt \, d\phi, \]

where

\begin{align*}
g_{tt} & = \frac{\Delta - a^2 \sin^2 \theta}{\rho^2}, \quad g_{rr} = \frac{\rho^2}{\Delta}, \quad g_{\phi\phi} = -\frac{2Mr \sin^2 \theta}{\rho^2}, \\
g_{\theta\theta} & = \frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}{\rho^2}, \quad g_{t\phi} = \frac{\rho^2}{\Delta},
\end{align*}

with

\begin{align*}
\Delta & = r^2 - 2Mr + a^2, \quad \rho^2 = r^2 + a^2 \cos^2 \theta.
\end{align*}
Schwarzschild corresponds to the case $a = 0$, subextremal Kerr to $|a| < M$ and extremal Kerr to $|a| = M$.

Note that the metric component $g_{rr}$ is singular precisely at the points where $\Delta = 0$. To overcome this coordinate singularity, we introduce the functions $r^*(r), \phi^*(\phi, r)$ and $v(t, r^*)$ such that

$$r^* = \int \frac{r^2 + a^2}{\Delta} \, dr, \quad \phi^* = \phi + \int \frac{a}{\Delta} \, dr, \quad v = t + r^*.$$ 

In the ingoing Eddington–Finkelstein coordinates $(v, r, \theta, \phi^*)$, the metric takes the form

$$g = g_{vv} \, dv^2 + g_{rr} \, dr^2 + g_{\phi^*\phi^*} \, (d\phi^*)^2 + g_{\theta\theta} \, d\theta^2 + 2g_{v\phi^*} \, dv \, d\phi^* + 2g_{r\phi^*} \, dr \, d\phi^*,$$

where

$$g_{vv} = - \left( 1 - \frac{2Mr}{\rho^2} \right), \quad g_{rr} = 0, \quad g_{\phi^*\phi^*} = g_{\phi\phi}, \quad g_{\theta\theta} = \rho^2,$$

$$g_{vv} = 1, \quad g_{v\phi^*} = - \frac{2Mar}{\rho^2}, \quad g_{r\phi^*} = - a \sin^2 \theta. \quad (4)$$

For convenience, we denote

$$T = \partial_v, \quad Y = \partial_r, \quad \Phi = \partial_{\phi^*}.$$ 

For completeness, we include the computation for the inverse of the metric in $(v, r, \theta, \phi^*)$ coordinates:

$$g^{vv} = \frac{a^2 \sin^2 \theta}{\rho^2}, \quad g^{rr} = \frac{\Delta}{\rho^2}, \quad g^{\phi^*\phi^*} = \frac{1}{\rho^2 \sin^2 \theta}, \quad g^{\theta\theta} = \frac{1}{\rho^2},$$

$$g^{vv} = \frac{r^2 + a^2}{\rho^2}, \quad g^{\phi^*\phi^*} = \frac{a}{\rho^2}, \quad g^{\theta\theta} = \frac{a}{\rho^2}.$$ 

Clearly, the metric expression (4) does not break down at the points where $\Delta = 0$, and in fact, it turns out that this expression is regular even for $r < 0$. On the other hand, the curvature would blow up at $\rho^2 = 0$, i.e. the equatorial points of $r = 0$.

Let $(\theta, \phi^*)$ represent standard global3 spherical coordinates on the sphere $S^2$ and $S_{eq}$ denote the equator, i.e. $S_{eq} = S^2 \cap \{ \theta = \pi/2 \}$. Let also $(v, r)$ be a global coordinate system on $\mathbb{R} \times \mathbb{R}$. We define the differential structure of the manifold $N$ to be

$$N = \{(v, r, \theta, \phi^*) \in (\mathbb{R} \times \mathbb{R} \times S^2) \setminus (\mathbb{R} \times \{0\} \times S_{eq})\}.$$ 

From now on, we restrict our attention to the extremal Kerr $|a| = M$, unless otherwise stated. The event horizon $H^+$ is defined by

$$H^+ = N \cap \{ r = M \}.$$ 

The black hole region $N_{bh}$ corresponds to

$$N_{bh} = N \cap \{ r < M \};$$

it is characterized by the fact that observers in the black hole region cannot send signals to observers located at points with $r > M$. The exterior region $D$ given by

$$D = N \cap \{ r > M \}$$

is the so-called domain of outer communications. This is precisely the region covered by the Boyer–Lindquist coordinates. Note that we shall be interested in studying the solutions to the wave equation in the region $D \cup H^+$.

3 Modulo the standard degeneration at $\theta = 0, \pi \ldots$
2.2. The initial hypersurface $\Sigma$

Let $\Sigma$ be an axisymmetric hypersurface which crosses the horizon $\mathcal{H}^+ = \{r = M\}$, terminates at $i^0$, is everywhere transversal to the Killing field $T$ and also such that

$$M^2 \sin^2 \theta - 2(r^2 + M^2)h + \Delta h^2 < 0,$$

where $h(r, \theta) = -\frac{g(Y, n_\Sigma)}{\partial T r_\Sigma^2}$ and $n_\Sigma$ denotes the future unit normal to $\Sigma$. Note that $g(T, n_\Sigma) = g(T + \omega \Phi, n_\Sigma) < 0$ and also $g(Y, n_\Sigma) = -g(-Y, n_\Sigma) > 0$. Note that the above expression is negative for a neighborhood of $\mathcal{H}^+$ and for large $r$ (for which $h \to 1$). We can hence choose $\Sigma$ such that the left-hand side of (5) is indeed negative everywhere on $\Sigma$ (note that this is a very mild assumption on $\Sigma$). For example, if $\Sigma = \{t = 0\}$, then $h = \frac{r^2 + M^2}{(r-M)^2}$ from which it immediately follows that (5) is satisfied. Let now $M < R_1 < R_2$. Consider the following pieces:

$$\Sigma_1 = \Sigma \cap \{M \leq r \leq R_1\}, \quad \Sigma_2 = \Sigma \cap \{R_1 \leq r \leq R_2\}, \quad \Sigma_3 = \Sigma \cap \{r \geq R_2\}$$

and the bit inside the black hole region

$$\Sigma_0 = \Sigma \cap \left\{\frac{3}{2}M \leq r \leq M\right\}.$$

We also define

$$\mathcal{R} = D^+ (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3) \cup \mathcal{H}^+.$$
2.3. The well-posedness of the wave equation

Recall that $T$ is smooth in $D(\Sigma)$ and also everywhere transversal to $\Sigma$. Hence, given the smooth initial data $(f_1, f_2) \in C^\infty(\Sigma) \times C^\infty(\Sigma)$, there exists a unique solution to the Cauchy problem

$$\Box g \psi = 0, \quad \psi|_{\Sigma_1} = f_1, \quad T\psi|_{\Sigma_1} = f_2,$$

which is smooth in the interior of $D^+(\Sigma)$ and so smooth in $\mathcal{R} \cup \mathcal{H}^+$.

3. Construction of the time integral $\phi$

On Schwarzschild backgrounds, Wald [20] showed that given a solution to the wave equation with compactly supported initial data on $t = 0$, one can construct its time integral (see also the discussion in section 3.2 in [9]).

In this section, we will reverse the logic. We will construct a (specific) solution $\psi$ to the wave equation by first constructing what will be its time integral $\phi$. We first prescribe the initial data for $\phi$ on $\Sigma$. Let

$$\phi|_{\Sigma_0 \cup \Sigma_1} = 1, \quad \phi|_{\Sigma_1} = 0, \quad T\phi|_{\Sigma_0 \cup \Sigma_1} = 0, \quad T\phi|_{\Sigma_1} = 0,$$

and consider axisymmetric functions

$$\phi|_{\Sigma_1}, \quad T\phi|_{\Sigma_1},$$

such that

$$\phi|_{\Sigma}, \quad T\phi|_{\Sigma}$$

are smooth and axisymmetric functions on $\Sigma$. 

\[6\]
Then, there exists a unique solution $\phi$ to
\[ \square_g \phi = 0, \]  
which is smooth in $\mathcal{R}$. By virtue of the axisymmetry of the initial data of $\phi$, the solution $\phi$ is axisymmetric.

4. Construction of $\psi$ with compactly supported initial data

We define
\[ \psi = T \phi \in C^\infty(D^+(\Sigma)) \subset C^\infty(\mathcal{R}). \]
Since $T$ is Killing, we have $[\square_g, T] = 0$, and hence
\[ \square_g \psi = 0 \]
in $D^+(\Sigma)$.

**Proposition 1.** The data of $\psi$ on the hypersurface $\Sigma$ are compactly supported (and supported away from $\mathcal{H}^+$).

**Proof.** First note that since $[T, \Phi] = 0$ the initial data of $\psi$ on $\Sigma$ are axisymmetric, and thus $\Phi \psi = 0$ in $D^+(\Sigma)$. Clearly,
\[ \psi|_{\Sigma_1} = T \phi|_{\Sigma_1} = 0, \quad \psi|_{\Sigma_1} = T \phi|_{\Sigma_1} = 0. \]
We will show that we also have
\[ T \psi|_{\Sigma_1} = TT \phi|_{\Sigma_1} = 0, \quad T \psi|_{\Sigma_1} = TT \phi|_{\Sigma_1} = 0. \]
We start with $\Sigma_1$, where $\phi$ satisfies $\phi = 1$, $T \phi = 0$ and $\square_g \phi = 0$. The wave operator in $(v, r, \theta, \phi^*)$ coordinates is (recall that $\partial_v = T$, $\partial_r = Y$ and $\partial_{\phi^*} = \Phi$):
\[ \square_g \phi = \frac{a^2}{\rho^2} \sin^2 \theta (TY \phi) + \frac{2(r^2 + M^2)}{\rho^2} (YY \phi) + \frac{\Delta}{\rho^2} (T \Phi \phi) + \frac{2a^2}{\rho^2} (Y \Phi \phi) \]
\[ + \frac{2r}{\rho^2} (T \phi) + \frac{\Delta'}{\rho^2} (Y \phi) + \frac{1}{\rho^2} \delta_{(\theta, \phi^*)} \phi, \]
where $\delta_{(\theta, \phi^*)} \psi = \frac{1}{\sin \theta} (\partial_\theta [\sin \theta \cdot \partial_\theta \psi]) + \frac{1}{\sin^2 \theta} \partial_{\phi^*} \partial_{\phi^*} \psi$ denotes the standard Laplacian on $S^2$ with respect to $(\theta, \phi^*)$ and $\rho^2 = r^2 + a^2 \cos^2 \theta$ and $\Delta = (r - M)^2$ (recall also that we consider $|a| = M$).

Furthermore, we consider the coordinate system $(r, \theta, \phi^*)$ for $\Sigma$ (and, in fact, for $\Sigma_r$) and let $\partial_r$ be the coordinate vector field tangential to $\Sigma$ such that $\partial_r r = 1$. Then,
\[ \partial_r = h(r, \theta) T + Y, \]
where $h$ is a (strictly) positive function that depends only on $\Sigma$ given by

$$h = -\frac{g(Y, n_{\Sigma})}{g(T, n_{\Sigma})}. $$

Hence, the wave operator becomes

$$\rho^2 \cdot (\Box g \phi) = M^2 \sin^2 \theta (TT \phi) + 2(r^2 + M^2)(\partial_\rho T \phi) - 2(r^2 + M^2)h \cdot (TT \phi)$$

$$+ \Delta \cdot (\partial_\rho \partial_\rho \phi + h^2 TT \phi - h^2 T \phi - 2h \partial_\rho T \phi) + 2M^2(T \Phi \phi) + 2M(\partial_\rho \Phi \phi)$$

$$- 2Mh T \Phi \phi + 2r(T \phi) + \Delta'(\partial_\rho \phi) - \Delta'h T \phi + \phi \phi. $$

By restricting the above on $\Sigma_0 \cup \Sigma_1$ and using (6) (and that $\partial_\rho$ is tangential to $\Sigma$), we obtain

$$(M^2 \sin^2 \theta - 2(r^2 + M^2)h + \Delta h^2) \cdot (TT \phi) = 0. $$

According to property (5) of $\Sigma$, the coefficient of $TT \phi$ above is always negative. Therefore, $T \psi = TT \phi = 0$ on $\Sigma_0 \cup \Sigma_1$. The result on $\Sigma_3$ follows by the finite speed propagation property. \hfill \Box

5. Instabilities for $\psi$ along the horizon

First, note that since the initial data of $\psi$ are smooth, axisymmetric and compactly supported, in view of our previous stability results (see theorem 5 of section 3 in [4]), we have that

$$|\psi| \to 0, \ |T \psi| \to 0, \ |TT \psi| \to 0, $$

along $\mathcal{H}^+$ as $v \to +\infty$. Since the conserved quantities $H_0[\psi] = H_0[T \psi] = 0$, we further obtain (see [5])

$$\int_{S_v} Y \psi \to 0, \quad \int_{S_v} YT \psi \to 0,$$

where $S_v$ are the $(\theta, \phi^*)$ sections of $\mathcal{H}^+$. Nonetheless, we have the following instability results.

**Proposition 2.** Let $\Sigma$ be a Cauchy hypersurface crossing $\mathcal{H}^+$ as defined in section 2 and $K = \Sigma \cap \{R_1 \leq r \leq R_2\}$, where $M < R_1 < R_2$. Then, there exists a solution $\psi$ to the wave equation whose initial data are supported only on $K$ such that

$$\sup_{S_v} |Y^2 \psi| \to 0, \quad \sup_{S_v} |Y^k \psi| \to +\infty, \ k \geq 3,$$

along the horizon as the advanced time parameter $v$ tends to infinity.

**Proof.** We choose the constants $R_1$ and $R_2$ of section 2 such that $\Sigma_2 = K$ and consider the above construction for $\psi$. Recall that $\psi = T \phi$ in $\mathcal{R}$ and that the initial data for $\phi$ are trivial in $\Sigma_3$ (and so $\phi$ is trivial in $D^+(\Sigma_3)$) and smooth and axisymmetric everywhere on $\Sigma$ (up to and including $\mathcal{H}^+$). Hence, we can apply our previous stability results to deduce that

$$|\phi| \to 0, \quad |T \phi| \to 0, \quad |TT \phi| \to 0$$
along \( \mathcal{H}^+ \) as \( v \to +\infty \). Recall that \( \phi \) is axisymmetric and its conserved quantity is given by

\[
H_0(\phi) = 4M \int_{S_0} Y \phi + 2 \int_{S_0} \phi + M \int_{S_0} \sin^2 \theta (T \phi) = 2 \neq 0,
\]

by simply computing it at \( \Sigma \cap \mathcal{H}^+ \). Hence, in view of the above stability results, we obtain

\[
\int_{S_0} Y \phi \to \frac{1}{2M} \neq 0.
\]

By restricting \( Y(\rho^2 \Box \phi) = 0 \) on the horizon \( \mathcal{H}^+ \), we obtain

\[
4M^2 \cdot YYT \phi = -M^2 \sin^2 \theta (YTT \phi) - 6M(YT \phi) - 2T \phi - 2Y \phi - \Box \phi.
\]  

(8)

Furthermore, by restricting \( \Box \phi = 0 \) on \( \mathcal{H}^+ \) and multiplying with \( \sin^2 \theta \), we obtain

\[
-M^2 \int_{S_0} \sin^2 \theta (YT \phi) \to 0
\]

as \( v \to +\infty \). Therefore,

\[
\int_{S_0} 4M^2 \cdot YY \phi = -M^2 \int_{S_0} \sin^2 \theta (YTT \phi) - 6M \int_{S_0} Y \phi - 2 \int_{S_0} T \phi - 2 \int_{S_0} Y \phi \to -\frac{1}{M}
\]

as \( v \to +\infty \), which shows that \( \sup_{S_v} |Y^2 \phi| \geq q > 0 \) asymptotically as \( v \to +\infty \).

We next look at \( Y^3 \psi \). By restricting \( Y^2 (\rho^2 \Box \psi) = 0 \) on \( \mathcal{H}^+ \), we obtain that there exist smooth functions \( \lambda_i \) such that

\[
TY^3 \psi = \lambda_0(M, \theta) T^2 Y^2 \psi + \lambda_1(M) TY \psi + \lambda_2(M) TY^2 \psi + \lambda_3(M) T \psi + \lambda_4(M) YT \phi
\]

\[
+ \Box Y^2 \psi - \lambda_5(M) Y^2 \psi
\]

(9)

where \( \lambda_5(M) > 0 \). Note that the integrals

\[
\int_{S_v} Y \psi, \int_{S_v} YY \psi, \int_{S_v} \psi, \int_{S_v} Y \phi, \int_{S_v} \lambda_0(M, \theta) TY^2 \psi
\]

are uniformly bounded in \( v \). The boundedness of the latter integral, in particular, follows by multiplying (8) (where \( \phi \) is replaced by \( \psi \)) with \( \lambda_0(M, \theta) \) and using that the integrals of the form

\[
\int_{S_v} \lambda(\theta) \cdot Y \psi
\]

(10)

can be bounded as follows. By the wave equation \( \Box \psi = 0 \) restricted on \( \mathcal{H}^+ \) (and multiplied by \( \lambda(\theta) \)), we obtain

\[
T(\lambda(\theta) T \psi + \lambda(\theta) Y \psi + \lambda(\theta) \psi) + \lambda(\theta) \cdot \Box \psi = 0,
\]

(11)

along the horizon. The boundedness of (10) follows by integrating (11) along \( \mathcal{H}^+ \) and using that

\[
\int_{S_v} \lambda(\theta) \cdot \Box \psi = \int_{S_v} \Box \lambda(\theta) \cdot \psi
\]

and that \( \psi = T \phi \) and the previous stability results. Note also that the integral \( \int_{S_v} \lambda(\theta) \cdot \Box \psi \) reduces to the previous integrals by applying Green’s identity on \( S_v \).

By integrating (9) along the horizon \( \mathcal{H}^+ \) and using that \( \int_{S_v} YY \psi \to -\frac{1}{4M} \neq 0 \), we obtain

\[
\sup_{S_v} |Y^3 \psi| \to +\infty,
\]

as \( v \to +\infty \) along \( \mathcal{H}^+ \). We can similarly show that

\[
\sup_{S_v} |Y^k \psi| \to +\infty,
\]

as \( v \to +\infty \) along \( \mathcal{H}^+, k \geq 3 \). \( \square \)
6. Remarks

6.1. Genericity of instabilities

By linearity one can immediately see that the instabilities can be inferred from the generic initial data when genericity can be understood as follows.

Let \( K = \Sigma \cap \{ R_1 \leq r \leq R_2 \} \), where \( M < R_1 < R_2 \). Let \( \xi \) be a general smooth (not necessarily axisymmetric) solution of (1) which does not develop instabilities along the horizon \( \mathcal{H}^+ \), i.e.

\[
|\xi| \to 0, \quad |T\xi| \to 0, \quad |Y\xi| \to 0,
\]

along \( \mathcal{H}^+ \). Consider the solution \( \psi \) constructed in section 4 such that \( \Sigma = K \). For any constant \( \epsilon > 0 \), the function \( \xi + \epsilon \psi \) is a smooth solution for which higher order transversal to \( \mathcal{H}^+ \) derivatives blow up asymptotically along \( \mathcal{H}^+ \). In particular, if \( \xi \) is not axisymmetric and is initially supported on \( K \), then \( \xi + \epsilon \psi \) is also not axisymmetric and is initially supported on \( K \). This completes the argument\(^4\).

6.2. Initial perturbations on \( t = 0 \)

Consider the spacelike (complete) hypersurface \( t = 0 \). Consider now \( \Sigma \), to be a hypersurface which crosses \( \mathcal{H}^+ \), is such that \( \Sigma \cap \{ r \geq R_1 \} = \{ t = 0 \} \cap \{ r \geq R_1 \} \) and satisfies condition (5). We also consider the subsets \( \Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3 \) of \( \Sigma \) as defined above (see also the figure below). Let \( \mathcal{R} \) be the domain of dependence of \( \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \).

Let also \( \Sigma_1' = \{ t = 0 \} \cap \{ r \leq R_1 \} \). Consider now the compactly supported initial data on \( t = 0 \) such that they are trivial on \( \Sigma_1' \) and \( \Sigma_3 \) and coincide with the data of proposition 1 on \( \Sigma_2 \). Let \( \psi_1 \) be the unique solution that arises from such data. By the domain of dependence property, the solution \( \psi_1 \) is trivial in the domain of dependence of \( \Sigma_1' \). Hence, the data of \( \psi_1 \) on \( \Sigma_1 \) are also trivial.

\(^4\) Note that the above argument effectively shows that the co-dimension of the set of the ‘stable initial data’ is strictly positive.
Therefore, the solutions $\psi_1$, $\psi$ must necessarily coincide in the region $\mathcal{R}$. By virtue of the smoothness of $\psi$ in $D^+(\Sigma)$, we have that $\psi_1$ extend smoothly to $\mathcal{R} \subset D^+(\Sigma)$ and the higher order transversal derivatives of $\psi_1$ blow up asymptotically along $\mathcal{H}^+$.

Hence, by the above and the discussion in remark 6.1, we deduce that for the generic compactly supported initial data on $t = 0$, the higher order transversal derivatives of $\psi_1$ blow up asymptotically along $\mathcal{H}^+$.

6.3. Extremal Reissner–Nordström

Finally, let us mention that our method applies also for extremal Reissner–Nordström black holes, i.e. we can replace the extremal Kerr with extremal Reissner–Nordström in theorem 1. In fact, in this case, one only requires $\Sigma_0$ to be spherically symmetric (without any additional restrictions such as (5)). Note also that one needs to prescribe the spherically symmetric initial data for $\phi$ on $\Sigma_0$.

Acknowledgments

I would like to thank Mihalis Dafermos for his constant encouragement and help. I would also like to thank Sergio Dain and Gustavo Dotti for suggesting the problem and Harvey Reall for several helpful discussions.

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