Perfect quantum state transfer of hard-core bosons on weighted path graphs

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The ability to accurately transfer quantum information through networks is an important primitive in distributed quantum systems. While perfect quantum state transfer (PST) can be effected by a single particle undergoing continuous-time quantum walks on a variety of graphs, it is not known if PST persists for many particles in the presence of interactions. We show that if single-particle PST occurs on one-dimensional weighted path graphs, then systems of hard-core bosons undergoing quantum walks on these paths also undergo PST. The analysis extends the Tonks-Girardeau ansatz to weighted graphs using techniques in algebraic graph theory. The results suggest that hard-core bosons do not generically undergo PST, even on graphs which exhibit single-particle PST.

I. INTRODUCTION

Perfect quantum state transfer (PST) was first conceived in the context of coupled spin networks [1, 2], in which quantum information encoded in a given spin (qubit) is transferred with unit fidelity to a different spin elsewhere in the network. PST was shown to occur between spins across the diameter of hypercubes, as well as on the one-dimensional (1D) spin chains whose spatially varying (but time-independent) coupling constants derive from a suitable projection of the hypercube. A complete set of coupling constants for 1D chains has been subsequently characterized [3]. The single-excitation subspace of a spin network is equivalent to a single particle undergoing a continuous-time quantum walk on a graph with the same connectivity [4], and in this context PST has been shown to occur on variety of graphs including graph quotients and joins, circulants, double cones, cubelike graphs, and signed graphs, among others [5–15]. If local control is permitted at the endpoints, then PST is possible on a wider assortment of networks [16, 17]. PST is a basic primitive in quantum communication across quantum networks, and can be a building block in the construction of approaches to universal quantum computation [18]. References [19] and [20] provide relatively recent reviews of PST on graphs and spin networks, respectively.

Any physical implementation of PST within the framework of continuous-time quantum walks on graphs will likely involve multiple particles, for example ultracold bosons in optical lattices [21–23]. Non-interacting bosons undergo PST if the single-particle graph also supports PST [24]; many-boson systems also yield a hierarchy of (generally weighted) graphs that enable single-particle PST [14, 25]. Physical particles generically interact with each another; for example, ultracold atoms in optical lattices are well-described by having effective on-site interactions. Yet few results are known about the occurrence of PST for interacting bosons, beyond an early analysis within the context of the Bose-Hubbard model [26] or for very small graphs [27].

In the limit of very strong repulsive interactions between bosons, known as the hard-core limit, at most one boson may occupy a given lattice site or graph vertex [28]. The restriction on site occupancy for hard-core bosons is reminiscent of the Pauli exclusion principle for (spinless or spin-polarized) fermions. Indeed, for one-dimensional lattices the ground state can be obtained exactly using the Tonks-Girardeau (TG) construction [29, 30]. The TG eigenstates correspond to those of non-interacting fermions which are then expressly symmetrized by the application of a ‘unit antisymmetry operator’ $\mathcal{A}$ to ensure that the total bosonic wavefunction is symmetric under particle exchange. For the ground state (which is necessarily all-positive for bosons), this is equivalent to taking the absolute value of the many-particle wavefunction, which is non-analytic. The TG state has been realized experimentally in an ultracold bosonic gas [31].

In fact, the XY model generally used to study PST on 1D spin chains maps exactly via the Jordan-Wigner transformation [32] to a model of non-interacting fermions, with the variable coupling constants mapping to the variable hopping amplitudes. Thus, truly non-interacting fermions on 1D lattices (path graphs) that exhibit PST should also undergo PST. That said, the presence of the unit antisymmetry operator that maps all fermionic eigenstates to those for hard-core bosons in the TG construction could have important consequences for PST, whose existence depends crucially on constructive phase interference. Likewise, the presence of the $\mathcal{A}$ complicates the calculation of other expectation values for hard-core bosons, compared to those for non-interacting fermions, such as the single-particle correlation function [28]. Thus, in principle hard-core bosons on path graphs could display different dynamics from those of non-interacting fermions on the same graphs.

In this work, we prove that hard-core bosons on path graphs with edge weights derived from hypercubes in-
deed undergo PST. The proofs make heavy use of algebraic graph theory as well as the TG construction for distinguishable bosons framed in a graph theory context, which is shown to remain exact for weighted paths. In order to subsequently map the system to identical bosons, we construct an extension of simple graph equitable partitioning to general weighted graphs. Because hard-core bosons are not able to pass through one another as they undergo PST, the bosons must retain their original ordering throughout their evolution. The symmetry associated with PST in these systems is found to be a combination of parity and particle re-ordering, and the existence of PST hinges on the fact that for path graphs the associated symmetry operator commutes with the unit antisymmetry operator that maps non-interacting fermions to hard-core bosons. The inference is that hard-core bosons on more general graphs are unlikely to exhibit PST, even when the underlying graphs exhibit PST for single particles.

The manuscript is organized as follows. Section II reviews the essential material that will be needed in the discussion of PST for hard-core bosons on weighted path graphs. This includes the properties of continuous-time quantum walks on graphs, equitable partitioning, PST on graphs, the theory of many-boson systems on paths (both non-interacting and hard-core bosons), and the Tonks-Girardeau construction. The main results of the work are presented in Section III: the characteristics of the Cartesian power graphs under vertex deletions (due to multiple bosonic occupation of a given vertex) are given, followed by the description of the important symmetries of the system. These ingredients are used to prove the main theorem. We conclude in Section IV with a discussion of the ramifications of these results for more general graphs.

II. BACKGROUND

A. Continuous-time quantum walk on graphs

Consider finite, connected, and undirected graphs $G = (V,E,w)$ with vertex set $V$ and edge set $E \subseteq V \times V$ connecting the $n = |V|$ vertices. The weight function is $w : E \rightarrow \mathbb{R}^+$, so that the adjacency matrix $A_G$ associated with the graph $G$ has elements $A_{G} = w(u,v) = w_{uv}$ where $(u,v) \in E$. Because the graph is undirected the adjacency matrix is symmetric, $w_{uv} = w_{vu}$. The time-evolution operator for a single particle undergoing a continuous-time quantum walk on a graph is defined as [33]

$$U_G(t) = e^{-itA_G}. \quad (1)$$

The adjacency matrix can be expressed in its spectral decomposition

$$A_G = \sum_{j=0}^{n-1} \tilde{\lambda}_j |z_j\rangle \langle z_j|, \quad (2)$$

where $|z_j\rangle$ and $\tilde{\lambda}_j$ are its orthonormal eigenvectors and eigenvalues, respectively. The time-evolution operator (1) then becomes

$$U_G(t) = \sum_{j=0}^{n-1} e^{-it\tilde{\lambda}_j} |z_j\rangle \langle z_j|. \quad (3)$$

The time evolution of an initial state $|\psi(0)\rangle$ is effected by direct application of the time evolution operator, $|\psi(t)\rangle = U_G(t)|\psi(0)\rangle$.

One can rewrite Eq. (3) as

$$U_G(t) = e^{-it\tilde{\lambda}_1} \sum_{j=0}^{n-1} e^{it(\tilde{\lambda}_1 - \tilde{\lambda}_j)} |z_j\rangle \langle z_j|, \quad (4)$$

with $\tilde{\lambda}_1$ an arbitrary eigenvalue. It is easy to verify that if the eigenvalues satisfy the ratio condition

$$\frac{\tilde{\lambda}_1 - \tilde{\lambda}_j}{\tilde{\lambda}_k - \tilde{\lambda}_l} \in \mathbb{Q}, \quad (5)$$

for all possible combinations of $\lambda_r, r \in \{i,j,k,l\}$ (except $\lambda_k = \lambda_l$), then the graph is periodic. In other words, there exists a time $t = \tau'$ where $U_G(\tau')$ is equivalent to an $n \times n$ identity matrix $I_n = \sum_j |z_j\rangle \langle z_j|$ up to an unimportant phase. Alternatively, defining the vertex state $|u\rangle$ as the unit vector corresponding to the vertex $u$, one can write $U_G(\tau')|u\rangle = \gamma'|u\rangle$ for all vertex states $|u\rangle$, where $\gamma'$ is a complex number with unit norm, i.e. $|\gamma'| = 1$.

Perfect quantum state transfer (PST) is said to be achieved if there exists a time $\tau$ such that

$$U_G(\tau)|u\rangle = \gamma|v\rangle \quad (6)$$

is satisfied for two distinct vertex states $|u\rangle$ and $|v\rangle$, where again $|\gamma| = 1$. If there exists PST between vertices $|u\rangle$ and $|v\rangle$ at time $t = \tau$, then there also exists PST from $|v\rangle$ to $|u\rangle$ at the same time. This implies that in twice the time, there is PST from $|u\rangle$ back to itself: $U_G(2\tau)|u\rangle = U_G(\tau)|u\rangle = \gamma'|u\rangle$. Thus the presence of PST ensures vertex periodicity, though it is important to underline that the converse is not generally true [19]. That said, if PST exists for a periodic graph, then it will occur in a time $\tau = \tau'/2$. The ratio condition (5) is therefore a necessary but not sufficient condition for PST to take place on a graph $G$.

The adjacency matrix for two distinguishable particles undergoing a quantum walk on the graph $G$ is given by the Cartesian product of the graph adjacency matrix with itself, $G \square G$ [14]. For two general graphs $G$ and $H$ on $n$ and $m$ vertices, respectively, the Cartesian product is defined in terms of their adjacency matrices as

$$A_{G \square H} = A_G \otimes I_m + I_n \otimes A_H, \quad (7)$$

where $\otimes$ denotes the tensor product. If both $G$ and $H$ are path graphs $P_n$ and $P_m$, their Cartesian product is
the $n \times m$ grid. The $k$-fold Cartesian power of the graph $G$ on $n$ nodes is $G^\square_k = G^\square G^\square \cdots \square G$, which is explicitly

$$A_{G^\square_k} = (A_G \otimes I_n \otimes \cdots \otimes I_n) + (I_n \otimes A_G \otimes \cdots \otimes I_n) + \cdots + (I_n \otimes \cdots \otimes I_n \otimes A_G).$$

in terms of its adjacency matrix. Because all of the terms in the above sum are mutually commuting, the time-evolution operator of the $k$-fold Cartesian power is

$$U_{G^\square_k}(t) = \bigotimes_{i=1}^k U_G(t),$$

the $k$-fold tensor product of the time-evolution operator on the original graph $G$.

Eq. (9) ensures that if there exists PST, or periodicity of any vertex, on the underlying graph, then the same property exists in the Cartesian power graph. Consider for example PST on the hypercube $Q_n$, which is defined as the $n$-dimensional analog of the cube. It is constructed by a recursive product of the path graph $P_2$ with itself, so that

$$Q_2 = P_2, \quad Q_{n>2} = P_2 \square Q_{n-1}.$$  \hfill (10)

Because there exists PST on $P_2$ in time $t = \pi/2$, then there also must exist antipodal PST on the hypercube $Q_n$ in the same time.

### B. Equitable partitioning

Equitable partitioning is a powerful tool in the analysis of graphs, in which vertices are grouped together into disjoint subsets, usually according to some symmetry of the graph [13, 34, 35]. For unweighted graphs (considered in this section), a partition of vertices $\Pi = \{C_i\}^m_{i=1}$ into $m$ disjoint vertex subsets or ‘cells’ $C_i \subseteq V(G)$ (such that $V = \cup_i C_i$ and $C_i \cap C_j = \emptyset$ for $i \neq j$) is said to be equitable if for every subset $C_i$, the number of edges $d_{ij}$ connecting $C_i$ to $C_j$ is independent of the choice of vertex in $C_i$. Thus, every vertex within a cell has the same number of neighbors in an adjacent cell, and equitable partitioning induces graph regularity within each cell and semi-regularity between cells. A cell containing only one vertex (i.e., $|C_i| = 1$) is known as a ‘singleton cell.’ Formal definitions and further details on equitable partitioning can be found in Appendix A.

Associating every cell to a single vertex yields a new graph which can be considered as an exact coarse-grained version of the original graph (the interpretation of ‘exact’ is given below); the resulting graph on $m$ vertices is known as the ‘quotient graph’ $\Gamma = G/\Pi$ under the equitable partitioning $\Pi$. The quotient graph derives its name from the elimination of eigenvalues from the spectrum, corresponding to dividing the characteristic equation of the graph adjacency matrix by a polynomial in $\lambda$ with the deleted eigenvalues as roots. For example, if the adjacency matrix $A_G$ for a graph $G$ with $n$ vertices has the characteristic polynomial $p_G(\lambda) = \cdots (\lambda_N - \lambda)$, then at least one term is deleted in the adjacency matrix of the quotient graph, $p_{\Gamma}(\lambda) = \cdots (\lambda_N - \lambda)/(\lambda - \lambda)$. In practice, the partition process may eliminate more than one eigenvalue. It follows directly that the spectrum of the quotient graph is a subset of the spectrum of the primary graph [34]. Which eigenvalues remain under equitable partitioning depends on the details, but in all cases the maximal eigenvalue is preserved. Equivalently, all the eigenvalues of the quotient (including the maximal one, which corresponds to the ground-state of the physical system) are also eigenvalues of the original graph; the coarse-graining procedure is therefore exact for (arguably) the most physically-relevant eigenvector. Further details can be found in Lemmas 8 and 9 in Appendix B. The edges between vertices of the quotient graph are generally weighted; the edge weights are given by (recall only for unweighted graphs)

$$\omega_{ij} = \sqrt{d_{ij} d_{ji}},$$

where $d_{ij}$ is the number of edges connecting cell $C_i$ to cell $C_j$. The weightings account for the semi-regularity of the connectivity between partitions.

Equitable partitioning can be considered as an isometry effected by an operator $Q$, known as the normalized partition matrix, defined as

$$Q = \sum_{i=1}^m \frac{1}{|V(G)|} \sum_{v \in C_i} |v \rangle \langle v|,$$

where $v \in V(G)$ and $|v\rangle \in V(\Gamma)$. If $|V(G)| = n$ and $|V(\Gamma)| = m$ then $Q$ must be a $n \times m$ matrix. The normalized partition matrix satisfies the following properties [36]:

$$Q^T Q = I_m$$

$$[Q^T Q, A_G] = [QQ^T, A_G] = 0.$$  \hfill (13)

In general, $QQ^T \neq I_n$ unless every cell of $\Gamma$ is a singleton, so that $m = n$ and $Q = I_n$. While the first commutation relation in Eq. (13) is trivial, the second indicates that $QQ^T$ represents a symmetry of the graph $G$. The vertices are therefore grouped into cells according to this symmetry.

The adjacency matrix of the quotient graph is obtained via

$$A_\Gamma = Q^T A_G Q,$$

i.e. the quotient graph is $G$ with the attendant symmetry removed. The spectral decomposition

$$A_\Gamma = \sum_i \hat{\lambda}_i Q^T |z_i\rangle \langle z_i| = \sum_i \hat{\lambda}_i |\tilde{z}_i\rangle \langle \tilde{z}_i|$$  \hfill (15)

shows that the eigenvectors of the quotient graph adjacency matrix are related to those of the original graph.
This also reveals that the spectrum of the quotient is a subset of that of the original graph, as expected from the definition of an equitable partition. Given that there are \( n \) \( |z_i \rangle \) eigenvectors but only \( m \) \( |\tilde{z}_i \rangle \) eigenvectors, some comments on Eqs. (15) and (16) are in order. Because \( QQ^T \) is a symmetry of the adjacency matrix according to Eq. (13), eigenvectors \( |z_i \rangle \) are either even or odd eigenstates of \( QQ^T \). Even functions map under Eq. (16), while odd functions map under \( Q^T \) to the null vector. Thus, in practice, the sum in Eq. (15) contains \( m < n \) terms.

An example of equitable partitioning is shown in Fig. 1. One can group vertices of the 3-cube into cells based on their distance from the left-most vertex. This yields a partitioning scheme with four cells, of which the two at the ends are singletons. The normalized partition matrix can be written to clearly reveal the cell structure:

\[
Q_{3,4} = \begin{pmatrix}
J_1 & 0_1 & 0_1 & 0_1 \\
0_1 & J_3/\sqrt{3} & 0_3 & 0_3 \\
0_3 & 0_3 & J_3/\sqrt{3} & 0_3 \\
0_1 & 0_1 & 0_1 & J_1
\end{pmatrix},
\]

where \( J_k \) \((0_k)\) is the all-one (all-zero) vector of length \( k \). Applying this to the adjacency matrix for the hypercube via Eq. (14) yields the weighted path graph \( \tilde{P}_4 \) as a quotient, with successive weights \( \sqrt{3} \), and \( \sqrt{3} \). In general, the equitable partitioning of \( Q_n \) to \( \tilde{P}_{n+1} \) via the normalized partition matrix \( Q_{n,n+1} \) yields the hypercubic edge weights \( w(v, v+1) = \sqrt{v(n+1-v)} \). Just as the hypercube exhibits PST between antipodal vertices, so does the weighted path graph; this is an immediate consequence of the fact that antipodal vertices are singleton cells of the equitable partition [37].

The standard definition of an equitable partition for an unweighted graph via the normalized partition matrix (12) must be extended to the case of weighted graphs in order to prove the main results of the present work. This extension has not previously been made, to the best knowledge of the authors, and so it is outlined here with full definitions to be found in Appendix A. Define the weight of a vertex \( v \) to be

\[
\omega(v) \equiv \sqrt{\sum_{u=1}^{n} A^2_{uv}},
\]

which in the unweighted case reduces to \( \omega(v) = \sqrt{d_v} \), the square root of the degree of vertex \( v \). Defining the normalized weight of \( v \) with respect to its containing cell \( C_v \) to be

\[
\Omega(v) \equiv \frac{\omega(v)}{\sqrt{\sum_{u \in C_v} \omega^2(u)}} = \frac{\omega(v)}{\omega(C_v)},
\]

the normalized partition matrix for a weighted graph is then

\[
Q = \sum_{v=1}^{n} \Omega(v)|v\rangle\langle \tilde{v}|.
\]

Together with the constraints discussed in detail in Appendix A that ensure a partition of a weighted graph is equitable, this definition of the normalized partition matrix is entirely consistent with the previous definition in the unweighted case.

Consider the time evolution operator for a quantum walker on the quotient graph,

\[
U_T(t) \equiv e^{-iQ^TAQ^T} = \sum_{k=0}^{\infty} \frac{(-it)^k}{k!} (Q^TAQ^T)^k.
\]

The \( k = 2 \) term is

\[
(Q^TAQ^T)^2 = Q^TAQQ^TAQ = Q^TA^2Q,
\]

where the last step follows from \([A, QQ^T] = 0\) and \(Q^TQ = I\). Because all powers proceed in the same manner one obtains

\[
U_T(t) = Q^T U_G(t) Q.
\]

Result (23) is particularly useful in the case of singleton cells. Suppose vertices \( u \) and \( v \) of \( G \) are singletons in the partition \( \Pi \), in which case the maps from these vertices to their containing cells are invertible: \( Q|u\rangle = |\tilde{u}\rangle \) and \( Q|\tilde{u}\rangle = |u\rangle \). The evolution between these vertices on the quotient graph is therefore identical to that between the associated vertices on the original graph:

\[
|\tilde{u}\rangle U_T(t)|\tilde{v}\rangle = \langle\tilde{u}|Q^TU_G(t)Q|\tilde{v}\rangle = \langle u|U_G(t)|v\rangle.
\]

Equation (24) holds for any undirected graph with real edge and self-loop weights under a partition that satisfies the condition for being equitable in the weighted case, Definition 1 in Appendix A.

C. Multiboson walks on path graphs

The graph-theoretic formalism for the analysis of many-boson systems has been previously elucidated. The
dynamics of quantum particles on discrete lattices is governed by the hopping Hamiltonian, which is the negative of the graph adjacency matrix (times an unimportant constant with units of energy). Non-interacting distinguishable bosons are represented by Cartesian powers of graphs, with identical bosons following by a suitable equitable partitioning [14]. Hard-core bosons are represented by symmetric powers of graphs, corresponding to deleting diagonal vertices of Cartesian powers, followed by an equitable partitioning [38]. Because the main result of the present work extends the concept of the symmetric power to weighted graphs, these two constructions are briefly reviewed here.

1. Non-interacting bosons on weighted path graphs

The eigenvectors and eigenvalues of the hypercuically weighted path graphs (referred in this work generically as weighted path graphs) derive from those of the hypercubes. The eigenvectors are Krawtchouk polynomials $|\mathcal{K}_i\rangle$, which can be expressed in terms of hypergeometric functions or Jacobi polynomials $\left|K_{ij}^n\right\rangle$. The eigenvalues of $A_G^n$ are the $n$-fold Cartesian power of $\mathcal{K}_i$, $\lambda(A_G^n) = \{-n, n+2, \ldots, n-n\}$. All of these eigenvalues, except the extremal ones, are degenerate. Each of the vertices in $Q_n$ can be labeled by a unique bit string of length $n$. One can then construct a partition where cells are comprised of vertices whose labels share the same Hamming weight. The resulting quotient graph has the adjacency matrix $A_{P_4,n+1}$ of the weighted path graph. Its spectrum coincides with that of $A_{Q_n}$, i.e. $\lambda(A_{Q_n}) = \{-n + 2\{0, 1, \ldots, n\}\}$, but the degeneracy of $\lambda$ is completely removed. These eigenvalues obviously satisfy the ratio condition (5). Alternatively, if one substitutes this spectrum into Eq. (4) using $\lambda_i = \lambda_0 = -n$, one obtains $U(t) = e^{t\mathcal{E}} = \left(e^{t\mathcal{E}}\right)|z_j\rangle = e^{-2\lambda_i|z_j\rangle}$, which is equivalent to an identity (up to an unimportant constant) whenever $t$ is an integer multiple of $\pi$. Thus, the weighted path graphs are periodic with period $\pi' = \pi$, a property inherited from the parent hypercubes.

Path graphs (both the hypercubically weighted $P_n$ and the simple $P_n$) on $n$ vertices are symmetric about their midpoints by construction. This ensures that all eigenvectors $|z_j\rangle$ of the associated adjacency matrices are either even or odd under a reflection about the midpoint of the graph, i.e. $|z_j\rangle = (z_{j,1}, z_{j,2}, \ldots, z_{j,n-1}, z_{j,n})^T = \pm(z_{j,n}, z_{j,n-1}, \ldots, z_{j,2}, z_{j,1})^T = \pm(|z_j\rangle)$ where $z_{j,m}$ is the $m$th element of the $|z_j\rangle$ eigenvector and $|z_j\rangle$ is reflected about the midpoint. The upper (lower) sign corresponds to an even (odd) eigenstate. Alternatively, the eigenvectors are eigenstates of the graph parity (or reflection) operator $\hat{P}$, where $\hat{P}|z_j\rangle = \pm|z_j\rangle$ for even (upper sign) and odd (lower sign) functions. Thus, $[\hat{P}, A_{P_n}] = 0$. Furthermore, for path graphs the eigenvectors always satisfy $\hat{P}|z_j\rangle = (-1)^j|z_j\rangle$, so that the parity of the eigenstates oscillates between even and odd as $j$ increases.

That PST occurs on hypercubically weighted path graphs follows directly from considerations of parity. PST for one particle on the weighted path graph corresponds to $U_G(\tau)|z_j\rangle = e^{i\phi}|z_j\rangle = e^{i\phi}\hat{P}|z_j\rangle$, where $\phi$ is an unimportant global phase. Suppose that one constructs an equitable partitioning $\Pi_P$ where each vertex is grouped in a cell with its reflection symmetry-related counterpart, i.e. vertex 1 with $n$, 2 with $n-1$, etc. Because the associated normalized partition matrix $Q_{\Pi_P}$ is even, $Q_{\Pi_P}|z_j\rangle = |0\rangle$ for $j$ odd. Every second eigenvalue is thereby eliminated from the spectrum $\{-n, n+2, \ldots, n-n\}$ of the quotient graph $\Gamma_P = \hat{P}_n/\Pi_P$. The quotient graph $\Gamma_P$ exhibits periodicity in time $\tau = \pi/2$. But every vertex state $|u\rangle$ in $\Gamma_P$ is a normalized superposition of symmetry-related vertices in $\hat{P}_n$, i.e. $|u\rangle = \frac{1}{\sqrt{2}}(|i\rangle + |n-i+1\rangle)$. Mapping $|u\rangle$ to itself in the quotient is equivalent to mapping $|i\rangle \leftrightarrow |n-i+1\rangle$, i.e. interchanging vertices across the midpoint in the original path graph. Thus, vertex periodicity at time $\tau' = \pi/2$ in the quotient implies PST in the original path graph at time $\tau = \pi/2$.

It has been shown in Ref. [14] that the evolution of $k$ distinguishable non-interacting particles on a graph $G$ is physically equivalent to the evolution of a single walker on the $k$-fold Cartesian power of $G$, denoted $G^\otimes k$. Figure 2 shows an example of two particles on the path graph $P_4$ with hypercubic edge weights. Figure 2(a) shows two copies of $P_4$, each with a configuration two particles. The two-particle effective graph $P_4^{\otimes 2}$ is shown in Fig. 2(b). If $|z_j\rangle$ is an eigenvector of $A_G$ with eigenvalue $\lambda_j$, then using Eq. (7) one obtains

$$A_G \otimes |z_i\rangle \otimes |z_j\rangle = (A_G \otimes I_n + I_n \otimes A_G)(|z_i\rangle \otimes |z_j\rangle)$$
$$= (A_G|z_i\rangle \otimes |z_j\rangle + |z_i\rangle \otimes A_G|z_j\rangle)$$
$$= (\lambda_i|z_i\rangle \otimes |z_j\rangle + |z_i\rangle \otimes \lambda_j|z_j\rangle)$$
$$= (\lambda_i + \lambda_j)(|z_i\rangle \otimes |z_j\rangle),$$

revealing that the eigenvalues of $k$-fold Cartesian powers of $G$ correspond to all possible additive combinations of $k$ eigenvalues. Recall from Sec. II A that if the graph $G$ exhibits PST then so does its Cartesian power, and the eigenvalues of $A_G$ automatically satisfy the ratio condition (5). Likewise, the eigenstates of Cartesian powers of path graphs remain eigenstates of the global parity operator.

When the particles are instead considered to be indistinguishable bosons, any vertices in the Cartesian power with labels that are identical under a permutation are grouped together within a single cell. The associated equitable partitioning is effected by the normalized partition matrix $Q_1$, where the ‘1’ subscript denotes ‘indistinguishability.’ Each of the $n^v$ vertices $v$ of the kth Cartesian power of $G$ is labeled by a list of length $k$, given as $v = (x_1, x_2, \ldots, x_k)$, where $x_i \in \{1, 2, \ldots, n\}$. The partition $\Pi_1$ groups together all vertices that are permutations $p(v)$ of the list $v$ into a single cell. Figure 2(b) shows the
equitable partitioning for two bosons on $\tilde{P}_4$. For example, vertices with the labels $(1, 2)$ and $(2, 1)$ constitute a unique cell. The quotient graph is shown in Fig. 2(c). Its eigenvalues are a subset of all combinations of those of $A_{\tilde{P}_4}$ by Lemmas 8 and 9 in Appendix B, and therefore still satisfy the ratio condition (5). The vertices derived from singleton cells exhibit PST via Eq. (24). All other vertices are derived from cells with exactly two vertices; each of these two vertices in the Cartesian product graph exhibits PST to one of two vertices both located in a distinct cell. Thus, all vertices in the quotient graph also exhibit PST between any two vertices with labels $(i, j, \ldots, k)$ and $(n+1-i, n+1-j, \ldots, n+1-k)$ in time $\tau = \pi/2$.

2. Hard-core bosons on path graphs

Thus far, we have ignored all particle interactions. When the repulsion between two particles is so strong that there is an infinite energy cost to occupying the same vertex, then the particles are said to have ‘hard-core’ interactions. In this case, at most one particle can occupy any given vertex at a time; this restriction is strongly reminiscent of the Pauli exclusion principle for (spinless) fermions, a correspondence that will be made explicit in Sec. II.D. Hard-core particles undergoing a quantum walk cannot pass through each other, which implies that their initial ordering on path graphs cannot change as a function of time. This behavior is illustrated in Fig. 3(a).

In the graph theory context, local (i.e. on-site) interactions are represented by self-loops on vertices whose labels have at least one repeating index (or, in the indistinguishable case, those corresponding to multiple occupancy). An example of such a vertex label is the sequence $(1, 2, 2, 3, \ldots)$. In the limit of infinitely strong interactions (either positive or negative), the hard-core limit, these vertices are deleted from the graph. Physically, particles will never occupy a vertex with a local infinite positive strength; alternatively, any particle incident on a vertex with infinite negative strength would never leave.

The vertex deletion operator $D$ is a diagonal matrix with entries of either zero or one; the zeroes are at vertices to be deleted. The hard-core adjacency matrix is obtained directly from that of the non-interacting case via

$$A_{HC} = DA_{NI}D,$$

where $A_{HC}$ and $A_{NI}$ represent the adjacency matrices of the hard-core and the non-interacting graphs respectively. Figure 3(b) depicts the effect of the deletion matrix on the $\tilde{P}_4^{\square 2}$ graph shown in Fig. 2(b). In this case, the vertex deletion yields two disconnected graph components because the full diagonal of the original Cartesian product is deleted. Importantly for the hard-core bosons on path graphs, the two graph components are in fact isomorphic. As will be shown in detail in this work, the ability of the deleted graph to support PST hinges on the fact that the deleted graph is composed of disconnected isomorphic graph components.

The adjacency matrix for $k$ identical hard-core bosons on a graph $G$ can therefore be defined as

$$A_{HC,1} = Q_1^T D A_{G,\square k} D Q_1,$$

where $Q_1$ is the normalized partition matrix effecting the map from distinguishable to identical particles, as illustrated in the map from Fig. 2(b) to 2(c) for bosons on
The definition (27) is identical to the kth symmetric power $G^{(k)} = \left((G \boxtimes k) \setminus D\right)/\Pi_1$ [38]. Because $D$ and $A_{G \boxtimes k}$ do not commute, the eigenvalues of symmetric powers are not trivially related to those of the Cartesian powers from which they are derived. In fact, as discussed further in the following, even if $G$ exhibits PST, $G^{(k)}$ generally does not.

Note that the deletion matrix need not be applied before equitably partitioning. An alternative approach is to first obtain the generalized quotient graph for identical noninteracting particles on the weighted path graph, and then delete vertices corresponding to multiple occupancy. For example, deleting vertices along the bottom diagonal in the graph shown in Fig. 2(c) yields the same weighted graph components shown in Fig. 3(b).

In Fig. 3(b), each disconnected graph corresponds to a different ordering of the bosons on the underlying path graph. This has two important implications. First, applying the equitable partitioning for indistinguishability yields exactly one of the disconnected graphs as the resulting symmetric power graph. Second, if there is PST for hard-core bosons, it must be between two vertices within a given disconnected graph. This is in marked contrast with the situation for non-interacting bosons, where the PST necessarily traverses the now-deleted vertices. Figure 3(a) illustrates this for two hard-core bosons on $P_4$. Because the two bosons cannot exchange their positions (doing so would require them to occupy the same site), the ordering of the bosons must remain the same.

Figure 3(c) shows the results of equitably partitioning the disconnected graphs depicted in Fig. 3(b), where each cell in the partition is either a singleton or given by a pair of blue circles connected by blue lines. The orange vertices in Figs. 3(b) and 3(c) represent the (equivalent) sites between which PST would occur. It is interesting to note that the quotient of the graph for two identical hard-core bosons on $P_4$, shown in Fig. 3(c), is itself equivalent to the graph for a single particle on the hypercubically weighted path graph $\tilde{P}_4$ which exhibits PST by construction. This observation was an initial motivation for the present work, as this ensures PST for two hard-core bosons on $P_4$. Unfortunately, for two hard-core bosons on larger weighted path graphs $P_m$, $m > 4$, there is no such simple equitable partition that can yield a $P_4$ quotient for some choice of $q$; likewise for larger numbers of particles. This raises the question of whether PST is generally possible for $k$ particles on $\tilde{P}_n$.

D. Tonks-Girardeau solution

An exact solution is known for a system of $k$ hard-core bosons in one dimension, known as the Tonks-Girardeau (TG) ansatz [29, 30]. The solution consists of a mapping from hard-core bosons to non-interacting spinless fermions. The deletion of vertices associated with multiple occupancy is then automatically accounted for by the Pauli exclusion principle, and so explicit application of a deletion operator is required. The Pauli exclusion principle requires that at most one particle can occupy a given vertex. The fermionic states are then explicitly symmetrized in order to be consistent with bosonic symmetry. This is accomplished through the use of a ‘unit antisymmetry operator’ $A$, a diagonal matrix with $\pm 1$ entries. The validity of the fermionic mapping hinges on the requirement that

$$\left[ A_{G \boxtimes k}, A \right] = 0, \quad (28)$$

which will be proved in Lemmas 1 and 2 below. This guarantees that symmetric versions of the fermionic eigenstates remain eigenstates of the Cartesian power, and furthermore that these states are mapped to eigenstates of the (bosonic) quotient graph via $Q^I$.

The TG construction proceeds as follows. There are $d = n!/k!(n-k)!$ arrangements of $k$ indistinguishable particles.
fermionic particles on a graph \( G \) with \( n \) vertices. Because the fermions are assumed to be non-interacting, each of the \( d \) eigenstates of the \( k \)-fermion system consists of each particle occupying a distinct eigenstate of \( A_G \). One can label each of these \( d \) states by an ordered \( k \)-tuple with elements from \( \{0, \ldots, n-1\} \) labeling the \( ( \text{ordered} \) eigenstates of \( A_G \), and define \( \ell \) the cardinality-\( d \) set of all such tuples; for example, the set \( \ell \) has elements \( \ell_1 = (0, 1, \ldots, k-2, k-1) \) corresponding to particles in the first \( k \) single-particle eigenstates of \( A_G \), \( \ell_2 = (0, 1, \ldots, k-2, k) \), and \( \ell_d = (n-k, n-k+1, \ldots, n-1) \). One can assume that \( k \leq n/2 \) because the states for \( k > n/2 \) are equivalent to those with \( k < n/2 \) by particle-hole symmetry. The fermionic states for \( k \) particles on \( P_n \) are therefore

\[
|F_{k,n,m}⟩ = \sum_{p \in S_k} \sigma [p(\ell_i)] \bigotimes_{j=1}^{k} |z_{\ell_i(j)}⟩,
\]

(29)

where \( \sigma \) returns the sign of \( p \). \( \ell_i \) represents the symmetric group of the \( k \)-tuple \( \ell_i \) of \( P_n \), and \( \ell_i(j) \) denotes the \( j \)th element of the ordered \( k \)-tuple \( \ell_i \). The \( \sigma \) function of the permutation \( p \) returns the sign of \( p \), and ensures the required antisymmetric behaviour of the fermionic state. The TG solution for the eigenvectors of hard-core bosons on \( \) (unweighted) path graphs is therefore

\[
|B_i⟩ = A|F_i⟩,
\]

(30)

where \( A \) is the unit antisymmetry operator ensuring that all fermionic eigenvectors \( |F_i⟩ \), defined in Eq. (29), are symmetric on particle interchange.

The eigenvalues \( \lambda_i \) of the adjacency matrix for \( k \) free fermions on a graph \( G \) coincide with those of the Cartesian power of \( G \), i.e. combinations of eigenvalues \( \lambda_i \) of \( A_G \) (the antisymmetrization of the eigenstates does not affect the eigenvalues). The ratio condition (5) then takes the form

\[
\frac{\sum_{\ell \in A_p} \lambda_i - \sum_{\ell \in A} \lambda_i}{\sum_{\ell \in A_p} \lambda_i - \sum_{\ell \in A} \lambda_i} \in \mathbb{Q},
\]

(31)

where each \( A \) is a subset of \( k \) eigenvalues of \( G \). Within the possible forms of Eq. (31) exists a subset for which the sums in the numerator differ in only a single term, and likewise for the denominator:

\[
\frac{\sum_{\ell \in A_p} \lambda_i - \sum_{\ell \in A} \lambda_i}{\sum_{\ell \in A_p} \lambda_i - \sum_{\ell \in A} \lambda_i} = \frac{\lambda_n - \lambda_m}{\lambda_x - \lambda_y} \in \mathbb{Q},
\]

(32)

where the remaining \( \lambda_{n,m,x,y} \) can each correspond to any eigenvalue of \( A_G \) (subject to \( \lambda_x \neq \lambda_y \)). Thus, if the ratio condition holds for \( A_G \) then it necessarily holds for \( k \) fermions on \( G \), and vice versa. Likewise, if Eq. (28) is satisfied then so is the ratio condition for hard-core bosons. Satisfying the ratio condition is only necessary for PST, however, not sufficient. That PST indeed occurs with hard-core bosons on weighted path graphs is the central result of the present work, and is proven in Sec. III.

III. RESULTS

In this section we prove that there exists PST for \( k \) distinguishable or identical hard-core bosons on the hypercubically weighted graphs \( P_n \). This is done in two ways, first through the direct application of the PST definition from Eq. (6), and second using the properties of equitable partitioning in order to draw conclusions about the characteristics of the quotient graph. Before proceeding to the main proofs, the graph structure of hard-core bosons is elucidated.

A. Disconnected graphs under deletion

An important property of the adjacency matrix for \( k \) distinguishable hard-core bosons on \( P_n \) or \( P_1 \) is that it disconnects into \( k! \) separate graph components, all of which are isomorphic to one another. This is shown in Lemma 1.

Lemma 1 The graph of \( k \) distinguishable hard-core bosons on a path graph of length \( n \) (\( P_n \) or \( P_1 \)) consists of \( k! \) isomorphic disconnected components described by identical adjacency matrices \( A_{HC,i} \): \( DA_{P^k,n} = \bigoplus_{i=1}^{k!} A_{HC,i} \)

Proof of Lemma 1 The graph for \( k \) distinguishable non-interacting particles on a path graph is \( P_n^k \). Each of the \( n^k \) vertices \( v \) is labeled by a list of length \( k \), given as \( v = (x_1, x_2, \ldots, x_k) \), where \( x_i \in \{1, 2, \ldots, n\} \). Consider two vertices \( v_1 = (x_1, \ldots, x_i, x_j, \ldots, x_k) \) and \( v_2 = (x_1, \ldots, x_j, x_i, \ldots, x_k) \), where \( x_i > x_j \). The shortest path from \( v_1 \) to \( v_2 \) corresponds to the sequence \( (x_i, x_j) \), \( (x_i-1, x_j) \), \( \ldots \), \( (x_j, x_j) \), \( (x_j, x_i+1) \), \( \ldots \), \( (x_j, x_i) \). The remaining vertex labels held constant. If \( D \) deletes all vertices whose labels have repeating indices, then the intermediate vertex \( v = \{x_1, \ldots, x_j, x_j, \ldots, x_k\} \) in this sequence is disconnected from both \( v_1 \) and \( v_2 \).

The positions of the entries \( x_i \) and \( x_j \) must interchange on the path from \( v_1 \) to \( v_2 \), but any edge increments \( x_i \to x_i \pm 1 \) for only one entry of the list (subject to \( 1 \leq x_i \leq n \)). This ensures that for any longer path at least one of the traversed vertices will have a pair of repeating indices, which is deleted by \( D \). Therefore \( v_1 \) and \( v_2 \) are disconnected. The map from \( v_1 \) to \( v_2 \) can equivalently be represented by an index permutation \( p \). For an index list of length \( k \), there are \( k! \) possible permutations of the indices. Therefore, for \( k \) hard-core bosons on a path graph of length \( n \), there will be \( k! \) induced graph components.

There are \( n!/(k!(n-k)! \) ordered lists of length \( k \) with non-repeating elements drawn from the set \( \{1, \ldots, n\} \). There are \( k! \) permutations of each of these, yielding \( n!/(n-k)! \) lists with non-repeating elements. Because all other lists have at least one repeating element, \( D \) deletes a total of \( n^k - n!/k!(n-k)! \) vertices from the Cartesian power graph. Thus, each of the \( k! \) graph components is
of size $d = n!/k!(n - k)!$. It is therefore always possible to write

$$A_{HC} = D A_{p(n)} D = \bigoplus_{i=1}^{k!} A_{HC,i}, \quad (33)$$

where $A_{HC,i}$ are the $d \times d$ adjacency matrices of the $i$-th graph component.

Furthermore, each vertex $v$ in a given graph component has a unique vertex counterpart in each of the $k! - 1$ other graph components, labeled by $p(v)$, where $p$ is a permutation of the list of vertices. The same $p$ maps all vertices in a given component to all vertices in a different component, which makes it an isomorphism. All $A_{HC,i}$ are therefore isomorphic, which implies that there exists an operator $\hat{\pi}_{i,j}$ satisfying $\hat{\pi}_{i,j} \hat{\pi}_{i,j}^T = \hat{\pi}_{i,j}^T \hat{\pi}_{i,j} = I$, derived from $p \equiv p_{ij}$, such that $\hat{\pi}_{i,j} A_{HC,j} \hat{\pi}_{i,j} = A_{HC,i}$. $i \neq j$.

With this lemma in hand, it is straightforward to prove that one can always express the wavefunction for distinguishable hard-core bosons on path graphs as the symmetrized wavefunction for non-interacting fermions.

**Lemma 2 (Tonks and Girardeau [29, 30])** The $j$-th wavefunction $|B_j\rangle$ for $k$ distinguishable hard-core bosons on a path graph of length $n$ ($P_n$ or $P_n^b$) can be expressed in terms of the wavefunction $|F_j\rangle$ for $k$ non-interacting fermions on the same graph via

$$|B_j\rangle = A |F_j\rangle, \quad (34)$$

where the unit antisymmetry operator is defined as

$$A = \bigoplus_{i=1}^{k!} \sigma(p_{i,1}) I_d. \quad (35)$$

Here $I_d$ is the identity matrix of size $d$ and $\sigma$ returns the sign of the vertex permutation $p_{i,1}$ from the graph component labeled ‘1’ to that labeled ‘i’.

**Proof of Lemma 2** The direct-sum structure of the hard-core adjacency matrix (33) implies that the eigenvectors of $A_{HC}$ follow directly from those of $A_{HC,1}$, given by

$$A_{HC,1} |b^{(1)}_b\rangle = \lambda^{(1)}_b |b^{(1)}_b\rangle, \quad (36)$$

where $b \in \{1, \ldots, d\}$. There are $k!d$ eigenvectors of $A_{HC}$; the first $d$ are

$$|B_b\rangle = |b^{(1)}_b\rangle \oplus \bigoplus_{a=2}^{k!} \bigoplus_{d} \{|0_d\rangle\}. \quad (37)$$

Because all disconnected graphs are isomorphic by Lemma 1, each of the $k!$ blocks of $d$ eigenvectors is obtained from those in (37) under a permutation operator. For $j = (a-1)d + b$, with $1 \leq a \leq k!$, labeling the block and $1 \leq b \leq d$ labeling the vector within it, there exists a $\hat{\pi}_{a,1}$ such that $\langle B_j | \hat{\pi}_{a,1} | B_b \rangle = 1$. Therefore one can write the eigenvalue equation for identical hard-core bosons on paths as $A_{HC} |B_j\rangle = \lambda_j |B_j\rangle$, where the eigenvectors are

$$|B_j\rangle = |B_{(a-1)d+b}\rangle = \hat{\pi}_{a,1} |B_b\rangle. \quad (38)$$

If the unit antisymmetry operator is defined in the same way as $A_{HC}$,

$$A = \bigoplus_{a=1}^{k!} c_a I_d, \quad (39)$$

where $I_d$ is the identity matrix of size $d$ and $c_a \in \{-1, 1\}$, then Eq. (28) and $[A_{HC}, A] = 0$ are both satisfied by construction. Thus $A \hat{\pi}_{a,1} |B_b\rangle$ is also an eigenstate of the distinguishable hard-core boson graph for any choice of the coefficients $c_a$.

The fermionic wavefunction Eq. (29) must be completely antisymmetric with respect to any permutation of occupation indices. Lemma 1 states that for hard-core bosons on paths such permutation maps one graph component to another. Consequently the wavefunction for distinguishable fermions can always be expressed as

$$|F_j\rangle = A |B_j\rangle = \sigma(p_{a,1}) \hat{\pi}_{a,1} |B_b\rangle, \quad (40)$$

where $p_{a,1}$ is the vertex permutation associated with the permutation operator $\hat{\pi}_{a,1}$. This corresponds to the application of a unit antisymmetry operator $A$ defined by $c_a = \sigma(p_{a,1})$. Alternatively, one can interpret the eigenstates of distinguishable hard-core bosons (38) as this $A$ operating on the fermionic state defined in Eq. (29), so that $|B_j\rangle = A |F_j\rangle$.

Lemma 2 implies that the $j$-th fermionic wavefunction (29) or (40) is always related (via $A$) to one of the $k!$-degenerate eigenstates of $k$ distinguishable hard-core bosons on a path graph (weighted or simple). The graph associated with the hard-core boson Hamiltonian is all positive, so the maximal eigenvector is also all positive by the Perron-Frobenius theorem (see Theorem 4 for more details); likewise for the ground state (eigenstate with most negative eigenvalue) of the many-body Hamiltonian. All coefficients of the maximal fermionic wavefunction in a given graph component are therefore guaranteed to have the same sign; this is somewhat surprising given that the state (29) is composed of (generally signed) single-particle states.

A second inference that can be drawn from Lemma 2 is that the hard-core boson wavefunctions cannot be derived from symmetrized fermionic states for more general graphs. The decomposition of the resulting graph for $k$ hard-core bosons into $k!$ graph components is only guaranteed for paths. For other graphs, the labels of the deleted vertices will be the same but two vertices $v_1 = (x_1, x_1, x_j, \ldots, x_k)$ and $v_2 = (x_1, x_j, x_i, \ldots, x_k)$ that differ by only a single permutation of labels may still be connected. Thus, there will generically be fewer
than \(k!\) graph components, which means that one cannot define \(c_n\) in the unit antisymmetry operator (39) to transform fermionic states to bosonic ones.

An important corollary of Lemma 2 is that the projections of the symmetrized fermionic states into the quotient graph obtained by equitably partitioning via \(\Pi_1\) constitute the complete set of eigenvectors for identical bosons on paths.

**Corollary 1.** The set of projected symmetrized fermionic functions \(\{Q_1^T \mathcal{A}[F_j] \mid 1 \leq j \leq d\},\) \(d = n!/k!(n-k)!\), constitutes the complete set of eigenvectors for \(k\) identical hard-core bosons on the path of length \(n\) \((P_n\) or \(\bar{P}_n\)).

**Proof of Corollary 1** All of the eigenvectors for hard-core bosons are necessarily symmetric under the interchange of particle occupation labels. The equitable partition \(\Pi_1\) groups together all vertices whose labels are equivalent under a permutation. Recall that all of the \(k!\) graph components of the deleted \(k\)-fold Cartesian powers of \(P_n\) (or \(\bar{P}_n\)) are isomorphic under a vertex permutation operator \(p\). The normalized partition matrix \(Q_1^T\) therefore projects all eigenvectors that are symmetric under the application of the associated permutation operator \(\pi\) to the eigenvector of a single graph component. Likewise, \(Q_1^T\) maps non-symmetric eigenvectors to null vectors, \(Q_1^T[F_j] = 0\). The cardinality of the set of symmetrized fermionic wavefunctions of the form (34) is \(d = n!/k!(n-k)!\), and the size of each graph component is \(d\). Furthermore,

\[
\lambda_j Q_1^T |B_j\rangle = Q_1^T D A_{\mathcal{G}^{\text{cl}}} D |B_j\rangle = Q_1^T Q_1 Q_1^T D A_{\mathcal{G}^{\text{cl}}} D |B_j\rangle = Q_1^T D A_{\mathcal{G}^{\text{cl}}} D Q_1^T |B_j\rangle = A_{\mathcal{G}^{\text{cl}}} Q_1^T A[F_j] = \lambda_j Q_1^T A[F_j], \tag{41}
\]

where on the second line we have used \(Q_1^T Q_1 = I\) and on the third line \(Q_1^T D A_{\mathcal{G}^{\text{cl}}} D = 0\). Thus, the \(Q_1^T A[F_j]\) constitute a complete set of eigenvectors for the graph adjacency matrix \(A_{\mathcal{G}^{\text{cl}}} \leftrightarrow Q_1^T D A_{\mathcal{G}^{\text{cl}}} D Q_1\), corresponding to \(k\) identical hard-core bosons on \(P_n\) or \(\bar{P}_n\).

An immediate consequence of Corollary 1 is that the entire spectrum for \(k\) identical hard-core bosons on a path graph is given by the spectrum of non-interacting distinguishable fermions on the same path. Of course, this result also follows directly from the TG construction. By the ratio condition (32), one can also state that symmetric powers of hypercubically weighted path graphs are periodic. The following two sections address whether these graphs also exhibit PST.

**B. Generalized mirror symmetry for hard-core bosons**

As discussed in Sec. II C 1, non-interacting bosons (either distinguishable or indistinguishable) exhibit PST on weighted path graphs. This is because two conditions are satisfied: the eigenvalues of the attendant adjacency matrix satisfy the ratio condition (5), and the eigenvectors of the underlying path graphs are eigenstates of the parity operator. Recall that PST occurs between any two vertices of the Cartesian power graphs (i.e. for distinguishable bosons) with labels \((i,j,k)\) and \((n+1-i,n+1-j,n+1-k)\) in time \(\tau = \pi/2\).

Suppose for concreteness that one has \(k = 3\) particles on \(P_7\). An example of a vertex pair connected via PST would be \(v_1 = (1,2,3)\) and \(v_2 = (7,6,5)\). The labels of \(v_1\) are in ascending order while those of \(v_2\) are in descending order, i.e. the ordering of the labels is permuted. The particles must pass through each other during the quantum walk, which is possible because they are non-interacting.

Hard-core bosons, in contrast, cannot pass through each other during the quantum walk. The initial ordering of the vertex labels must be preserved for any pair of vertices connected via PST. In the previous example for instance, vertices \(v_1\) and \(v_2\) are located in different graph components and are disconnected. Rather, hard-core bosons require PST-connected pairs of vertices \((i,j,k)\) and \((p,q,r)\) where \(i < j < k\) and \(p < q < r\). Thus, satisfying the ratio condition (5) and ensuring the eigenstates of the underlying graph are eigenfunctions of the parity operator is not sufficient to ensure PST with hard-core bosons.

Consider the composite operator \(\hat{C}\),

\[
\hat{C} = \hat{\pi}_\perp \hat{P}, \tag{42}
\]

defined as the product of the global parity operator \(\hat{P}\) and the permutation operator \(\hat{\pi}_\perp(v)\) effecting the permutation \(p\) of the list corresponding to vertex label \(v\) to yield its mirror-symmetric ordering. For example, for a vertex \(v = (1,3,5)\) with three distinguishable bosons, \(p(1,3,5) = (5,3,1)\). The mapping of vertex \(v_1\) to \(v_2\) via \(\hat{C}\) can be considered as a reflection from \(v_1\) to \(v_2\) across a generalized mirror plane \(M\). The \(\hat{\pi}_\perp\) operator ensures that the ordering of vertex labeling is restored after having been reversed under the application of \(\hat{P}\).

Figure 4 shows the operation of \(\hat{C}\) in occupation space on a state of the system consisting of two hard-core bosons...
Consider a vertex state \( |v\rangle \) in one of the \( k! \) graph components of a system of \( k \) hard-core bosons on \( P_n \) or \( \tilde{P}_n \). Given its expansion in terms of the \( d \) orthogonal hard-core boson eigenstates with support on the component \( |\psi\rangle = \sum_{\alpha} \alpha |\beta_{\alpha}\rangle \), the vertex state \( |u\rangle = \tilde{C}|v\rangle \) has the same expansion up to sign differences in the expansion coefficients, \( \alpha \to \pm \alpha \).

**Proof of Lemma 3** Let the set \( \{|\beta_{\alpha}\rangle\} \) denote the orthonormal eigenvectors of \( k \) distinguishable hard-core bosons on \( P_n \) or \( \tilde{P}_n \), given in Lemma 2 as \( |\beta_{\alpha}\rangle = A|\beta_i\rangle \). One can express a vertex state \( |v\rangle \) on the first of the \( k! \) isomorphic graph components in terms of this eigenbasis:

\[
|v\rangle = \sum_{i=1}^{k!} |\beta_{\alpha_i}\rangle = \sum_{i=1}^{k!} \alpha_i |\beta_i\rangle,
\]

where \( \alpha_i \) is \( \langle \beta_{\alpha_i} | \beta_{\alpha_j} \rangle \) for \( 1 \leq i \leq d \), and \( \langle \beta_{\alpha_i} | \beta_{\alpha_j} \rangle = 0 \) for \( d < i \leq k!d \). That is, because the eigenvectors can be decomposed into \( k! \) cardinality-d contributions according to Eq. (38), the sum over \( k!d \) terms can be restricted to only those \( d \) terms that have non-zero overlap with \( |v\rangle \).

Under the operation of \( \tilde{C} \), any vertex is mapped to its mirror-symmetric (about the plane \( M \)) state

\[
|u\rangle = \tilde{C}|v\rangle = \sum_{i=1}^{k!} \alpha_i \hat{\tilde{P}} A|\beta_i\rangle
= \sum_{i=1}^{k!} \alpha_i \hat{\tilde{P}} A|\beta_i\rangle,
\]

where the fermionic functions are given by Eq. (29) in terms of the eigenstates of the path graphs \( |z_i\rangle \) (\( |K_i\rangle \) for hypercubically weighted path graphs). The result makes use of the fact that \( [A, \tilde{C}] = 0 \), which follows from the fact that \( \tilde{C} \) maps a vertex to another in the same graph component and \( A \) is block-diagonal in graph components, c.f. Eq. (35). Note also that we have included the sign of the effect of \( \hat{\tilde{P}}_\perp \) in the sign operator \( \sigma \) because \( \hat{\tilde{P}}_\perp \) will exchange fermionic particles. As discussed in Sec. II C 1, the parity operator on each path graph eigenvector returns \( \tilde{P}|z_i\rangle = (-1)^{n_i}|z_i\rangle \). Suppose that the fermionic state \( |\beta_i\rangle \) has \( n_i \) particles in even-j states and \( n_i \) particles in odd-j states, where \( n_i + n_i = k \) for all \( i \); that is, there are \( n_i \) odd integers and \( n_i \) even integers in \( \ell_i \).

Then \( \hat{\tilde{P}} \) on the full state returns \( (-1)^{n_i} \):

\[
|u\rangle = \sum_{i=1}^{k!} \alpha_i (-1)^{n_i} \hat{\tilde{P}} A|\beta_i\rangle
= \sum_{i=1}^{k!} \alpha_i \hat{\tilde{P}} A|\beta_i\rangle.
\]

Meanwhile, \( \hat{\tilde{P}}_\perp \) completely reverses the particle ordering. This is equivalent to

\[
\hat{\tilde{P}}_\perp \bigotimes_{j=1}^{k} |z_{\ell_{i}(j)}\rangle = |z_{\ell_{c}(1)}\rangle \otimes |z_{\ell_{c}(2)}\rangle \cdots \otimes |z_{\ell_{c}(k)}\rangle
= |z_{\ell_{c}(k)}\rangle \otimes |z_{\ell_{c}(k-1)}\rangle \cdots \otimes |z_{\ell_{c}(1)}\rangle.
\]

This is a completely antisymmetric permutation of the original ordering. Whether this changes the sign of the coefficient depends on the number of particles. If \( k \)
is even, then \( p(\hat{\pi}_\perp) = (-1)^{k/2} \) while for odd \( k \) then \( p(\hat{\pi}_\perp) = (-1)^{(k-1)/2} \); more concisely \( p(\hat{\pi}_\perp) = (-1)^k \).

The result is

\[
|u\rangle = \sum_{i=1}^{d} \alpha_i (-1)^{n_o(i)+k/2} A \prod_{\ell \in S_k} \sigma(p(\ell_i \hat{\pi}_\perp)) \prod_{j=1}^{k} |\ell_i(j)\rangle
\]

\[
= \sum_{i=1}^{d} \alpha_i (-1)^{n_o(i)+k/2} |B_i\rangle. \tag{46}
\]

The expansion coefficients \( \alpha_i' = \alpha_i (-1)^{n_o(i)+k/2} \) for the state \(|u\rangle = \hat{C}|v\rangle\) are therefore equal to the expansion coefficients \( \alpha_i \) for \(|v\rangle\), up to a sign depending on \( n_o(i) \) and the number of particles \( k \).

One can relate the sign of the expansion coefficients in Eq. (46) to the eigenvalues of \( k \) hard-core bosons on the hypercubically weighted path graph \( \tilde{P}_n \). Recall that the eigenvalues of \( \tilde{P}_n \) are \( \lambda_i = -(n - 1) + 2(0, 1, \ldots, n - 1) \).

The eigenvalues for \( k \) identical hard-core bosons on \( \tilde{P}_n \) are therefore \( \lambda_i = -k(n - 1) + 2 \sum_j \ell_i(j) \). Because one can factor out a common \(-k(n - 1)\) term, the eigenvalues are determined by a positive integer \( a = \sum_j \ell_i(j) \), the sum of \( k \) unique non-negative integers (the unique compositions) corresponding to the elements of the site occupations \( \ell_i \).

Suppose that \( a \) is even. If \( k = 2 \) then there are \((a/2) + 1\) different choices for the \( \ell_i \), each of which contains either two odd or two even integers; in both cases one obtains \((-1)^{n_o} = 1 \).

The maximum eigenvalue is given by

\[
\lambda_{\text{max}} = -k(n - 1) + 2 \sum_{j=n-k}^{n-1} j
\]

\[
= -k(n - 1) + k(2n - k - 1) = k(n - k). \tag{47}
\]

so the \( k \)-boson eigenvalues are

\[
\lambda_i = k(n - k) + 2(i - 1), \quad i \in \{1, 2, \ldots, k(n - k)\}. \tag{48}
\]

The minimum eigenvalue for \( k \) hard-core bosons on \( \tilde{P}_n \) is

\[
\lambda_{\text{min}} = -k(n - 1) + 2 \sum_{j=0}^{k-1} j
\]

\[
= -k(n - 1) + k(k - 1) = k(n - k), \tag{49}
\]

so that \( a_{\text{min}} = k(k - 1)/2 \). One can make use of the fact that \((-1)^{k(k-1)/2} = (-1)^{k^2/2}\) for all \( k \), so that \((-1)^{a_{\text{min}} + k^2/2} = 1 \).

The first expansion coefficient, the overlap of the site \( u \) with the lowest bosonic eigenstate \(|B_1\rangle = |\alpha_1(1)^{-n_o(1)+k/2}\rangle\) therefore coincides with \( \alpha_1 = \langle B_1 | v \rangle \). For the second expansion coefficient, the only quantity that can change is \( a \). The second-lowest eigenvalue corresponds to a single particle in a higher single-particle state, i.e. an energy \( k(k - n) + 2 \).

If \( a \) increases by unity, then by the argument above \((-1)^{n_o}\) changes sign relative to that for the lowest state; necessarily then \( \alpha_2(-1)^{n_o(2)+k/2} = -\alpha_2 \) for \(|u\rangle\) while it is \( \alpha_2 \) for \(|v\rangle\).

Because \( a \) is always composed of sum of integers, the expansion coefficient for \(|u\rangle\) will change sign at each successive eigenvalue.

The remaining eigenvalues \( \lambda_i \) are degenerate, corresponding to common values of \( \alpha_i = \sum_m \ell_i(m) \).

Defining the set \( \pi_i \) with elements \( \ell_i \) such that \( \sum_j \ell_i(j) = \alpha_i \), and the vectors \(|B_i^{(j)}\rangle\) corresponding to each degenerate \(|B_i\rangle\) eigenvector, one can formalize this observation by defining the spectral decomposition of the distinguishable hard-core adjacency matrix as

\[
A_{\text{HC}} = \sum_{i=1}^{k(n-k)} \lambda_i \sum_j |\pi_i| |B_i^{(j)}\rangle \langle B_i^{(j)}|, \tag{50}
\]

This allows one to rewrite Eq. (44) as

\[
|u\rangle = \sum_{i=1}^{k(n-k)} \pi_i |(\Pi_i)^{(j)}| \langle B_i^{(j)}| \langle B_i | v \rangle |B_i^{(j)}| |v\rangle, \tag{51}
\]

where \( \pi_i = |(\Pi_i)| \langle B_i | v \rangle \), and Eq. (46) as

\[
|u\rangle = \hat{C}|v\rangle = \sum_{i=1}^{k(n-k)} (-1)^{a_i} \sum_j |\pi_i| |B_i^{(j)}\rangle. \tag{52}
\]

These alternate expressions readily yield the following lemma.

**Lemma 4** Equitable partitioning of the configuration space graph representing \( k \) hard-core bosons on \( \tilde{P}_n \) with respect to the operator \( \hat{C} \) in each disconnected graph component will preserve the lowest eigenvector but annihilate every second higher eigenvector, in all degeneracy, from the distinguishable spectrum.

**Proof of Lemma 4** Consider the quotient graph \( \Gamma_C = (\tilde{P}_n^\perp \setminus D) / \Pi_C \), where \( \Pi_C \) is an equitable partitioning about \( \hat{C} \). The associated normalized partition matrix \( Q_C \) groups vertices connected by \( \hat{C} \) into cells. Because \( Q_C \) is even, the vertex states \(|v\rangle\) in \( A_{\Gamma_C} \) are superpositions of
The eigenvalues for a system of 5 vertices in the configuration-space representation is that the indistinguishability equitable partition set \( Q \) is given by Eq. (51) and (52). The effective expansion coefficient is zero for all even \( i \) in the above sum, corresponding to every second (degenerate) eigenvalue. All odd \( i \) terms are preserved, including the lowest eigenstate. The primed sum in the last line (i.e., “\( \sum' \)”) denotes the absence of even- \( i \)-terms in the sum. Because the eigenvalue for every second eigenstate is missing from the expansion of an arbitrary vertex state in the quotient graph, the spectral decomposition of the quotient adjacency matrix can be written:

\[
Q^T_{C,AHC,\tilde{Q}} = \sum_{i=1}^{k(n-k)/2} \tilde{\lambda}_i \sum_j |\tilde{B}_i^{(j)}⟩⟨\tilde{B}_i^{(j)}|Q_C
\]

Thus, while the eigenvalues of \( A_{HC} \) are \( \lambda_i = k(k-n) + 2(i-1), i = \{1, 2, \ldots, k(n-k)\} \), the eigenvalues of \( Q^T_{C,AHC,\tilde{Q}} \) are \( \tilde{\lambda}_i = k(k-n) + 4(i-1), i = \{1, 2, \ldots, k(n-k)/2\} \).

Recall from the proof of Corollary 1 that the indistinguishability equitable partition \( Q_1 \) is a \( k \)-to-one map that groups all vertices with indices belonging to the same symmetric group into one cell in the quotient graph. The eigenvalues \( \lambda_i \) in Eq. (54) are then \( k \) degenerate as a result of Lemma 1. The effect of applying \( Q_1 \) will therefore be to eliminate this degeneracy in the spectrum of the quotient. It follows that the adjacency matrix for identical hard-core bosons under \( \Pi_C \) will be

\[
Q^T_{C,AHC,\tilde{Q}} = \sum_{i=1}^{k(n-k)/2} \tilde{\lambda}_i \sum_j |\tilde{B}_i^{(j)}⟩⟨\tilde{B}_i^{(j)}|Q_C
\]

where \( |\tilde{B}_i^{(j)}⟩ = Q^T_Q|\tilde{B}_i^{(j)}⟩, A_{HC,\tilde{Q}} = Q^T_TA_{HC}Q_T, \) and \( \tilde{\lambda}_i \) are eigenvalues defined in Lemma 4. Note that the eigenvalue degeneracy corresponding to the cardinality of the set \( \tilde{\omega} \) is unchanged under \( \Pi_C \): the number of linearly independent eigenvectors is of order \( d/2 \), while the number of unique eigenvalues is the (generally lower) value \( k(n-k)/2 \). The increase in the eigenvalue spacing from 2 to 4 in the quotient graph under the equitable partitioning \( \Pi_C \) will have important consequences for PST in identical hard-core bosons, as discussed in the next section.

### C. PST for hard-core bosons on weighted path graphs

This section presents the primary result of this work, that hard-core bosons on the weighted path graph \( \tilde{P}_n \) undergo perfect quantum state transfer. The result are proven in two ways. The first proof makes direct use of the time evolution operator, while the second infers the presence of PST from the vertex periodicity of the quotient graph under \( \Pi_C \). Before stating the proofs, it is useful to introduce the following lemma.

**Lemma 5** Vertices in the configuration-space representation of \( k \) hard-core identical bosons on \( \tilde{P}_n \) are periodic in time \( t = \pi \).

**Proof of Lemma 5** The eigenvalues for \( k \) bosons on the weighted path graph \( \tilde{P}_n \) are given by Eq. (48). The spectral decomposition of the associated adjacency matrix immediately yields the time evolution operator

\[
U_{HC,\tilde{Q}}(t) = \sum_{j=1}^{k(n-k)} e^{-it\lambda_j} \sum_{m=1}^{|\tilde{Q}|} |\tilde{B}_j^{(m)}⟩⟨\tilde{B}_j^{(m)}|. \]

If \( t = \pi \) then the sums over \( j \) and \( m \) constitute a resolution of the identity and

\[
U_{HC,\tilde{Q}}(\pi) = e^{-i\pi k(n-k)}I_d. \]

Therefore every vertex in the configuration space representation of \( k \) identical hard-core bosons on \( \tilde{P}_n \) is periodic in time \( t = \pi \).

1. **PST by application of the time evolution operator**

**Theorem 1** A system of \( k \) identical hard-core bosons on the weighted path graph \( \tilde{P}_n \) undergoes PST in time \( t = \pi/2 \) between two configuration-space vertex states connected by the operation of \( \widetilde{C} = \pi_\perp \tilde{P} \).

**Proof of Theorem 1 (PST of Hardcore Bosons I)** PST between vertex states \( |v⟩ \) and \( |u⟩ \) in the configuration space of identical hard-core bosons is defined by Eq. (6). One can use Eqs. (14) and (16) to write the criterion for PST as

\[
⟨u|U_{HC}\tilde{Q}|v⟩ = (u|Q_1Q_T^T_{HC}U_{HC}Q_1Q_T^T_{HC}|v⟩ = (\tilde{u}|U_{HC,\tilde{Q}}(\pi)|\tilde{v}⟩ = \gamma, \]

where \( |\gamma| = 1 \); the above makes use of the first of the two properties (13). From Lemma 5, all vertices are periodic in time \( t = \pi \), so if PST occurs between two vertices then the PST time must be an even fraction of \( \pi \), i.e. \( \pi/2n \) for some positive integer \( n \).

Consider \( t_P = \pi/2 \). Making use of Lemma 3 and Eqs. (51) and (52), the expanded form of the above equation can be expressed as
\[
\langle \tilde{u} | U_{\text{HC},1}(t) | \tilde{v} \rangle = \sum_{l=1}^{k(n-k)} (-1)^{l-1} \prod_{j=1}^{m_l} \alpha_i^{(j)} \langle \tilde{B}_i^{(j)} | \sum_{j'=1}^{k(n-k)} e^{-2itj'} \prod_{m=1}^{M |} \langle \tilde{B}_j^{(m)} | e^{-it[k(k-n)-2]} \prod_{l'=1}^{M |} \alpha_{l'}^{(j')} \rangle. \]

If \( t = t_P = \pi/2 \) then \( e^{-2it\rho l} = (-1)^l \), and one obtains

\[
\langle \tilde{u} | U_{\text{HC},1}(t_P) | \tilde{v} \rangle = \gamma \sum_{l=1}^{k(n-k)} \prod_{j=1}^{M |} \left( \alpha_i^{(j)} \right)^2, \]

where \( \gamma = e^{-i\pi k(k-n)/2} \). The summation is the resolution of unity, reflecting the orthonormality of the bosonic eigenvectors. Consider the last term:

\[
\left( \alpha_i^{(j)} \right)^2 = \langle \tilde{v} | \tilde{B}_i^{(j)} \rangle \langle \tilde{B}_i^{(j)} | \tilde{v} \rangle = \langle \tilde{v} | \tilde{B}_i^{(j)} \rangle \langle \tilde{B}_i^{(j)} | \tilde{v} \rangle \]

Inserting into Eq. (59) gives

\[
\langle \tilde{u} | U_{\text{HC},1}(t_P) | \tilde{v} \rangle = \gamma \langle \tilde{v} | \sum_{l=1}^{k(n-k)} \prod_{j=1}^{M |} \langle \tilde{B}_i^{(j)} | \tilde{B}_i^{(j)} | \tilde{v} \rangle \]

so that PST between \( |\tilde{v}\rangle \) and \( |\tilde{u}\rangle = \tilde{C} |\tilde{v}\rangle \) occurs in time \( t_P = \pi/2 \) yielding a phase \( \gamma = e^{-i\pi k(k-n)/2} \).

2. PST by vertex periodicity in quotient graph under \( \Pi_C \)

**Lemma 6** The quotient graph resulting from the application of the equitable partition \( \Pi_C \) to the graph corresponding to \( k \) identical hard-core bosons is periodic in time \( t_P = \pi/2 \).

**Proof of Lemma 6** The proof of this Lemma follows that of Lemma 5, with some minor modifications. The time evolution operator after equitably partitioning the graph for \( k \) identical hard-core bosons under \( \Pi_C \) is

\[
Q^T_C U_{\text{HC},1}(t) Q_C = \sum_{j=1}^{k(n-k)/2} e^{-it\tilde{\lambda}_j} \sum_{m=1}^{k(n-k)/2} Q^T_C | \tilde{B}_j^{(m)} \rangle \langle \tilde{B}_j^{(m)} | Q_C
\]

\[
e^{-it[k(k-n)/4]} \sum_{j=1}^{k(n-k)/2} e^{-4itj} \times Q^T_C \left( \sum_{m=1}^{M |} | \tilde{B}_j^{(m)} \rangle \langle \tilde{B}_j^{(m)} | \right) Q_C,
\]

where the eigenvalues \( \tilde{\lambda}_j \) are given in Lemma 4. If \( t_P = \pi/2 \) then the sums over \( j \) and \( m \) constitute a resolution of the identity and

\[
Q^T_C U_{\text{HC},1}(\pi/2) Q_C = e^{-i\pi k(k-n)/2} I_d,
\]

where \( d' \approx d/2 \). Therefore every vertex in the quotient graph of \( k \) identical hard-core bosons on \( \tilde{P}_n \) under \( \Pi_C \) is periodic in time \( t_P = \pi/2 \).

One can now employ a modified version of the discussion in Sec. II C 1 to prove that vertex periodicity in the quotient graph discussed in Lemma 6 implies PST for hard-core identical bosons.

**Theorem 2** A system of \( k \) identical hard-core bosons on the weighted path \( \tilde{P}_n \) undergoes PST in time \( t_P = \pi/2 \) between two vertex states connected by the operation of \( \tilde{C} = \pi_1 \tilde{P} \) if the graph quotient under the equitable partition \( \Pi_C \) is periodic in time \( t_P = \pi/2 \).

**Proof of Theorem 2 (PST of Hardcore Bosons II)** Following Theorem 1, one can write

\[
\langle u | U_{\text{HC}}(t) | v \rangle = \langle \tilde{u} | U_{\text{HC},1}(t) | \tilde{v} \rangle = \langle \tilde{u} | Q_C Q^T_C U_{\text{HC},1}(t) Q_C | \tilde{v} \rangle. \]

Lemma 4 gives

\[
Q^T_C | \tilde{v} \rangle \equiv | \tilde{v}' \rangle = \frac{1}{\sqrt{2}} \left( | \tilde{v} \rangle + \tilde{C} | \tilde{v} \rangle \right) = \frac{1}{\sqrt{2}} (| \tilde{v} \rangle + | \tilde{u} \rangle)
\]

\[
Q^T_C | \tilde{u} \rangle \equiv | \tilde{u}' \rangle = \frac{1}{\sqrt{2}} \left( | \tilde{u} \rangle + \tilde{C} | \tilde{u} \rangle \right) = \frac{1}{\sqrt{2}} (| \tilde{u} \rangle + | \tilde{v} \rangle),
\]

where in the last line \( \tilde{C} | \tilde{u} \rangle = \tilde{C}^2 | \tilde{v} \rangle = | \tilde{v} \rangle \) has been employed; note that \( \tilde{C} \) is the identity. Equations (64) and (65) reveal that \( | \tilde{u}' \rangle = | \tilde{v}' \rangle \), so that Eq. (63) becomes

\[
\langle u | U_{\text{HC}}(t) | v \rangle = \langle \tilde{v}' | Q^T_C U_{\text{HC},1}(t) Q_C | \tilde{v}' \rangle. \]

This equation represents vertex periodicity in the adjacency matrix corresponding to the quotient graph under \( \Pi_C \). If \( t = t_P = \pi/2 \) then using the results from Lemma 4, Eq. (66) becomes

\[
\langle u | U_{\text{HC}}(t) | v \rangle = \gamma | \tilde{v}' \rangle | \tilde{v}' \rangle = \gamma,
\]

where \( \gamma = e^{-i\pi k(k-n)/2} \).

IV. DISCUSSION AND CONCLUSIONS

In the preceding sections, we have shown that \( k \) hard-core bosons on hypercubically-weighted \( n \)-path graphs \( \tilde{P}_n \) undergo perfect quantum state transfer. The proofs of the main results, Theorem 1 and Theorem 2, made use both of the celebrated Tonks-Girardeau solution (30)
for unweighted path graphs and of techniques from algebraic graph theory, such as Cartesian powers, graph isomorphisms, as well as equitable partitioning and its generalization to weighted graphs. The occurrence of PST was found to hinge on: (i) the preservation of the linear spectrum of the weighted path graph adjacency matrices in the graph for identical hard-core bosons; and (ii) the commutation of the underlying path graph and are generally signed, even for the ground state (maximal eigenvector). Applying the indistinguishability equitable partitioning operator \( Q^I_k \) would map all these states to null vectors because of the alternating signs. The unit antisymmetry operator \( \mathcal{A} \), which symmetrizes the states, is therefore essential in order to obtain an explicit representation of the identical hard-core boson eigenstates.

Alternatively, one can consider that the adjacency matrix for \( k \) distinguishable fermions on \( P_n \) (or \( P_n^k \)) should not be considered to be that for the Cartesian power \( A_{P_n^k} \); rather, it should be a signed graph with the same connectivity but where each vertex is switched (in the sense of Ref. [40]) according to the antisymmetric ordering of its label \( \ell_i \). The quotient of this graph under the equitable partition \( \Pi_f \) would again be the null graph. The presence of \( \mathcal{A} \) is therefore crucial, and it is fortunate that \([\mathcal{A}, C] = 0 \) so that PST on weighted path graphs persists for hard-core bosons as it does for non-interacting fermions. In short, \( \mathcal{A} \) in the TG theory is the switching operator that maps the signed (but balanced) graph for non-interacting fermions to that for distinguishable unsigned hard-core bosons. This interpretation also helps explain why the TG solution is specific to path and cycle graphs: no other graphs yield balanced antisymmetrized Cartesian powers.

This raises the important question: do hard-core bosons exhibit PST on more general graphs? The above discussion rules out a general mapping between hard-core bosons and non-interacting fermions. The proofs of Theorems 1 and 2 hinge on \([\mathcal{A}, D_{A_{P_n^k}, D}] = 0 \), which is no longer the case if the vertex-deleted Cartesian-power adjacency matrix can no longer be expressed in terms of \( k \) disconnected graph components. Furthermore, for more general graphs it is known that the ground state energy (maximal eigenvalue of the associated adjacency matrix) for hard-core bosons is always lower (higher) than that for non-interacting fermions [41]. Likewise, other eigenvalues are not likely to have any relationship with those of non-interacting fermions; satisfying Eq. (5) is therefore unlikely. While one might be able to find graphs which exhibit PST for some specific number of particles, we conjecture that very few graphs (but most likely no graphs other than the weighted paths described here) will support PST for arbitrary numbers of hard-core bosons, even if they support PST for single particles. Unfortunately a proof of this conjecture is beyond the scope of the current work.

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Appendix A: Equitable partitioning of weighted graphs

For a partition of a graph $G$ to be equitable, the following theorem must be satisfied.

**Theorem 3** Suppose $\Pi = \{C_i\}_{i=1}^m$ is a partition into $m$ disjoint cells of the vertices of a connected undirected weighted graph $G$, and $Q$ is its normalized partition matrix as defined by Eq. (20). Let $A$ be the adjacency matrix of $G$. Then the following are equivalent:

1. $\Pi$ is equitable.
2. The column space of $Q$ is $A$-invariant.
3. $A$ and $QQ^T$ commute.
4. There is an $m \times m$ matrix $B$ such that $AQ = QB$.

Theorem 3 provides the main result that extends the notion of equitable partitioning to weighted graphs. Its proof makes use of the following lemma.

**Lemma 7** Let $Q$ be the normalized partition matrix describing a partition $\Pi = \{C_i\}_{i=1}^m$ of a weighted undirected connected graph $G$ on $n$ vertices. Then $Q^TQ = I_m$, the $m \times m$ identity.

**Proof of Lemma 7** By the definition of $Q$, Eq. (20),

$$Q^TQ = \sum_{v=1}^{n} \sum_{w=1}^{n} \Omega(v)\Omega(w)\langle v|w\rangle\langle \tilde{v}|\tilde{w}\rangle = \sum_{v=1}^{n} \Omega^2(v)\langle \tilde{v}|\tilde{v}\rangle. \tag{A1}$$

This is a sum of $n$ elements from a set of $m$ distinct matrices, with $n \geq m$, so some of them have contributions from multiple vertex indices. Specifically, if two vertices $u$ and $v$ belong to the same cell then $\tilde{u} = \tilde{v}$, so it is natural to sum only over the cells, instead of over all vertices. Doing so and inserting Eq. (19), the definition of $\Omega$, yields

$$Q^TQ = \sum_{i=1}^{m} \sum_{v \in C_i} \omega^2(v) \langle \tilde{v}|\tilde{v}\rangle = \sum_{i=1}^{m} \frac{1}{\omega^2(C_i)} \left( \sum_{v \in C_i} \omega^2(v) \right) |i\rangle\langle i|. \tag{A2}$$

The parenthetical term is simply the square of the weight of cell $C_i$ by definition, leaving

$$Q^TQ = \sum_{i=1}^{m} |i\rangle\langle i| = I_m, \tag{A3}$$

which completes the proof.

Lemma 8.1 of Reference [19] provides three statements related to $A$, $Q$, and $\pi$, that are mutually equivalent, as well as equivalent to the statement that $\Pi$ is equitable, in the case of simple graphs. The same set of four statements can be shown to be equivalent when $A$ describes a weighted graph, and the equitableness of a partition on a weighted graph is defined as follows.

**Definition 1** A partition $\Pi$ of a weighted undirected connected graph $G$ is equitable if, for any pair of cells $C_i, C_j \in \Pi$, including $j = i$, and for any vertex $u \in C_i$, the quantity

$$b_{ij}(u) = \sum_{v \in C_j} \frac{\omega(v)}{\omega(u)} \tag{A4}$$

is a constant $b_{ij}$, independent of the choice of $u$.

The above definition is justified by the following theorem, in which a set of vectors is said to be $A$-invariant if a matrix $A$ maps each of the vectors to a linear combination of one or more of them.

**Proof of Theorem 3** The proof consists in showing that each statement implies its successor, cyclically.

(1 $\implies$ 2) To show that 1 implies 2, suppose that $\Pi$ is equitable and consider column $i$ of $Q$, denoted

$$|\phi_i\rangle = \sum_{v \in C_i} \Omega(v)|v\rangle. \tag{A5}$$

The action of $A$ on this column vector is

$$A|\phi_i\rangle = \sum_{v \in C_i} \Omega(v)A|v\rangle = \sum_{v \in C_i} \Omega(v) \sum_{u=1}^{n} A_{uv}|u\rangle = \frac{1}{\omega(C_i)} \sum_{u=1}^{n} \omega(u)A_{uv}|u\rangle, \tag{A6}$$

where the final equality follows from the definition of $\Omega(v)$, Eq. (19), and the fact that the denominator is constant over the cell $C_i$. Since $\Pi$ is equitable by assumption, there exist constants $b_{\bar{i}i}$ such that

$$\sum_{v \in C_i} \omega(v)A_{uv} = b_{\bar{i}i}\omega(u), \tag{A7}$$

so the string of equalities continues as

$$A|\phi_i\rangle = \frac{1}{\omega(C_i)} \sum_{u=1}^{n} b_{\bar{i}i}\omega(u)|u\rangle = \sum_{j=1}^{m} \frac{b_{ji}}{\omega(C_j)} \sum_{u \in C_j} \omega(u)|u\rangle. \tag{A8}$$

Finally, defining

$$a_{ij} = \frac{b_{ji}}{\omega(C_j)} \tag{A9}$$

leads to

$$A|\phi_i\rangle = \sum_{j=1}^{m} a_{ij} \sum_{u \in C_j} \Omega(u)|u\rangle = \sum_{j=1}^{m} a_{ij} |\phi_j\rangle. \tag{A10}$$
That is, $A$ takes each column of $Q$ to a linear combination of the columns of $Q$; the column space of $Q$ is $A$-invariant.

**(2 $\implies$ 3)** Now assume that the column space of $Q$ is $A$-invariant. With the definition (A5) of the $i$th column of $Q$, the partition matrix can be rewritten as

$$Q = \sum_{i=1}^{m} |\phi_i\rangle|i|,$$

(A11)

and therefore

$$QQ^T = \sum_{i=1}^{m} |\phi_i\rangle|i| \sum_{j=1}^{m} |j\rangle\langle\phi_j| = \sum_{i=1}^{m} |\phi_i\rangle\langle\phi_i|.$$  

(A12)

The column space of $Q$ is $A$-invariant by assumption so for each $i \in \{1, \ldots, m\}$,

$$A|\phi_i\rangle = \sum_{j=1}^{m} a_{ij} |\phi_j\rangle \implies |\phi_i\rangle = \sum_{j=1}^{m} a_{ij} |\phi_j\rangle,$$

(A13)

since $A$ is real and symmetric, because $G$ is assumed to be undirected. This is a non-trivial statement even though the $|\phi_i\rangle$ form an orthonormal set, since there are only $m$ of them yet they are $n$-component vectors, and therefore do not form a basis for $\mathbb{R}^n$. The commutator of $A$ and $QQ^T$ can be written as

$$AQQ^T - QQ^TA = \sum_{i=1}^{m} \sum_{j=1}^{m} [a_{ij} |\phi_j\rangle\langle\phi_i| - a_{ij} |\phi_i\rangle\langle\phi_j|]$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} (a_{ij} - a_{ji}) |\phi_i\rangle\langle\phi_j|,$$

(A14)

but since the elements of $A$ and the $|\phi_i\rangle$ are real, and $A$ is symmetric,

$$a_{ij} = \langle\phi_j|A|\phi_i\rangle = \langle\phi_i|A|\phi_j\rangle = a_{ji}.$$

(A15)

Therefore the commutator vanishes, as required.

**(3 $\implies$ 4)** To prove that 3 implies 4, assume now that $[A, QQ^T] = 0$ and consider the matrix $B = QQ^TA$, from which one sees

$$QB = QQ^TAQ = AQQ^TQ.$$  

(A16)

But Lemma 7 shows that $Q^TQ$ is the identity, leaving $QB = AQ$. Therefore, $B$ is in fact the required $m \times m$ matrix $B$.

**(4 $\implies$ 1)** Finally, to see that 4 implies 1, which will complete the proof, suppose there exists an $m \times m$ matrix $B$ such that $QB = AQ$. Then by definition,

$$\sum_{v=1}^{m} \Omega(v)\langle v|\otimes |\phi_i\rangle B = \sum_{v=1}^{m} \Omega(v)A(v)\langle v|.$$  

(A17)

Multiplying each side by $\langle u|$, on the left and $|j|$ on the right leads to

$$\Omega(u)B_{\tilde{u}j} = \sum_{v=1}^{m} \Omega(v)A_{uv}\delta_{\tilde{u}j} = \sum_{v \in C_i} \Omega(v)A_{uv}.$$  

(A18)

From the definition of $\Omega$ one obtains

$$\frac{\omega(u)}{\omega(C_j)} B_{\tilde{u}j} = \frac{1}{\omega(C_j)} \sum_{v \in C_j} \omega(v)A_{uv},$$  

(A19)

which, with $i = \tilde{u}$, in turn implies that

$$\sum_{v \in C_j} A_{uv} \frac{\omega(v)}{\omega(u)} = B_{ij} \frac{\omega(C_j)}{\omega(C_i)} = b_{ij},$$  

(A20)

from which the conclusion immediately follows since the final right-hand side is independent of the choice of $u$ from $C_i$, and thus $Q$ encodes an equitable partition, $\Pi$.

Statement 3 of Theorem 3, that $[A, QQ^T] = 0$, leads to the following useful result.

**Corollary 2** Suppose $\Pi$ is an equitable partition of a connected, weighted, undirected graph $G$. Let $A$ be the adjacency matrix of $G$, and $Q$ the normalized partition matrix of $\Pi$. Then $QQ^T$ has two distinct eigenvalues, 0 and 1.

**Proof of Corollary 2** Since $\Pi$ is equitable, Theorem 3 shows that $A$ and $QQ^T$ commute. Both are real and symmetric, and therefore diagonalizable, thus they are simultaneously diagonalizable and share an eigenspace. Let the eigenvalues of $A$ be $\lambda_i$, with corresponding eigenvectors $|\lambda_i\rangle$ where for each $i$, $j$ runs from 1 to the multiplicity of $\lambda_i$. Let $q_{ij}$ be the eigenvalues of $QQ^T$, so that

$$QQ^T|\lambda_i\rangle = q_{ij}|\lambda_i\rangle.$$  

(A21)

Left-multiplying this expression by $Q^T$ yields $Q^T|\lambda_i\rangle = q_{ij}Q^T|\lambda_i\rangle$, from which it can be concluded that either $q_{ij} = 1$ or $Q^T|\lambda_i\rangle = 0$. Clearly if the latter is the case, then it is also true that $QQ^T|\lambda_i\rangle = 0$ which must still equal $q_{ij}|\lambda_i\rangle$. But the $|\lambda_i\rangle$ are non-zero vectors, so in this case $q_{ij}$ must vanish. Therefore either $q_{ij} = 1$ or $q_{ij} = 0$.

Given a weighted, connected, undirected graph $G$ and an equitable partition $\Pi$ of its vertices into $m$ cells, with adjacency and normalized partition matrices $A$ and $Q$ respectively, Theorem 3 guarantees that there exists an $m \times m$ matrix $B$ such that $AQ = QB$. Lemma 7 can then be used to show that

$$B = Q^TA.$$  

(A22)

$B$ is also a real, symmetric matrix, since

$$\langle i|B|j \rangle = \langle \phi_i|A|\phi_j \rangle = \langle \phi_i|A^T|\phi_j \rangle = \langle \phi_j|A|\phi_i \rangle = \langle j|B|i \rangle.$$  

(A23)

Therefore, $B$ can be interpreted as the adjacency matrix of a graph on $m$ vertices; this graph is called the quotient graph of $G$ with respect to $\Pi$ and is denoted $G/\Pi$. 

Appendix B: Eigenvalues of graphs under equitable partitioning

A remarkable property of the quotient graph is that it shares all of its eigenvalues with the original one. The following lemma makes this statement concrete, and is subsequently used to prove that for all non-negative graphs (including the multi-boson graphs considered in the present work) the maximal eigenvalue is always preserved in the quotient.

**Lemma 8** If $B = Q^T AQ$ is the adjacency matrix of a quotient graph $G/\Pi$ under an equitable partition $\Pi$ from a weighted, undirected, connected graph $G$, then every eigenvalue of $B$ is also an eigenvalue of $A$.

**Proof of Lemma 8** Let $\beta$ be an eigenvalue of $B$, with corresponding eigenvector $|\beta\rangle$. Since $\Pi$ is equitable, $AQ = QB$ by Theorem 3. Therefore

$$AQ|\beta\rangle = QB|\beta\rangle = \beta Q|\beta\rangle,$$

and $\beta$ is seen to be an eigenvalue of $A$, as required.

The proof of Lemma 8 additionally shows that the eigenvectors $|\beta\rangle$ of $B$ are related to a subset of the eigenvectors $|\alpha\rangle$ of $A$ by $|\beta\rangle = Q^T|\alpha\rangle$ and $|\alpha\rangle = Q|\beta\rangle$. The determination of which eigenvectors belong to this subset in general remains an open question, though Theorem 5 below shows that the eigenvector corresponding to the maximal eigenvalue is preserved under collapse whenever every entry of $A$ is non-negative in the vertex basis.

The following lemma shows that not only does every eigenvector of $B$ yield an eigenvector of $A$ under the action of $Q$, but also that every eigenvector of $A$ that does not vanish under the action of $Q^T$ yields an eigenvector of $B$.

**Lemma 9** Let $G$ be a weighted, undirected, connected graph with adjacency matrix $A$, and $\Pi$ an equitable partition with normalized partition matrix $Q$ that generates the quotient graph $G/\Pi$ with adjacency matrix $B = Q^T AQ$. Suppose $|\lambda_{i,j}\rangle$ is an eigenvector of $A$ as defined in the proof of Corollary 2. Then either $Q^T|\lambda_{i,j}\rangle = 0$ or $Q^T|\lambda_{i,j}\rangle$ is an eigenvector of $B$.

**Proof of Lemma 9** Corollary 2 shows that the eigenvalues of $Q^T AQ$ are $\lambda_{i,j}$ with eigenvectors $|\lambda_{i,j}\rangle$ are $q_{i,j} \in \{0,1\}$. Suppose that $Q^T|\lambda_{i,j}\rangle = 0$. Then since $Q^T Q = I$, left-multiplication by $Q^T$ yields $Q^T|\lambda_{i,j}\rangle = 0$. On the other hand, in the case $Q^T|\lambda_{i,j}\rangle = |\lambda_{i,j}\rangle$, one obtains

$$A (Q^T|\lambda_{i,j}\rangle) = A|\lambda_{i,j}\rangle = \lambda_{i,j} |\lambda_{i,j}\rangle.$$  

Left-multiplying the initial and final expressions by $Q^T$ yields

$$(Q^T AQ)Q^T|\lambda_{i,j}\rangle = B (Q^T|\lambda_{i,j}\rangle) = \lambda_{i,j} (Q^T|\lambda_{i,j}\rangle),$$

showing that $Q^T|\lambda_{i,j}\rangle$ is an eigenvector of $B$, with eigenvalue $\lambda_{i,j}$.

Some further definitions and results from the fields of linear algebra and graph theory that are useful during the proof of the upcoming Theorem 5 are stated here without proof. A treatment can be found, for example, in Ref. [35].

**Definition 2** An $n \times n$ matrix $T$ is irreducible if for each $i,j \in \{1, \ldots, n\}$ there exists a positive integer $k$ such that $(T^k)_{i,j} > 0$. A graph $G$ is said to be irreducible if its adjacency matrix $A(G)$ is irreducible.

The following lemma relates the irreducibility of a matrix to connectedness of a corresponding unweighted graph.

**Lemma 10** An $n \times n$ matrix $T$ with elements $T_{i,j}$ is irreducible if and only if the unweighted directed graph $\Gamma_T$, defined on the vertex set $\{1, \ldots, n\}$ with an edge from $i$ to $j$ whenever $T_{i,j} > 0$, is strongly connected.

A graph is strongly connected if there is a (directed) path from each vertex to every other vertex. Therefore an undirected graph is strongly connected if and only if it is connected.

Another useful result is the Perron-Frobenius theorem.

**Theorem 4** (Perron-Frobenius) Let $T$ be a non-negative irreducible square matrix. Then there exists a real number $\lambda_1 > 0$ with the following properties:

1. There exists a real vector $|\lambda_1\rangle$ with each entry strictly positive, and such that $T|\lambda_1\rangle = \lambda_1 |\lambda_1\rangle$.

2. The algebraic and geometric multiplicities of $\lambda_1$ are both equal to 1. That is, its associated eigenspace is one-dimensional.

3. For each eigenvalue $\lambda_i$ of $T$, $|\lambda_i| \leq \lambda_1$.

This completes the prerequisites for the proof of the following theorem.

**Theorem 5** Let $G$ be a connected undirected graph with non-negative weights and adjacency matrix $A$, and let $\Pi$ be an equitable partition of its vertices with corresponding normalized partition matrix $Q$. Then the largest eigenvalue of $A$ is unique, and equal to the unique largest eigenvalue of the collapsed graph with adjacency matrix $B = Q^T AQ$.

**Proof of Theorem 4** Let the eigenvalues of $A$ be $\lambda_i$, with corresponding eigenvectors $|\lambda_{i,j}\rangle$, where for each $i$, $j$ runs from 1 to the algebraic multiplicity of $\lambda_i$. According to Lemma 8, each eigenvector of $B$ is given by $Q^T|\lambda_{i,j}\rangle$ for some $i$ and $j$. The eigenvalues of $B$ are therefore those $\lambda_i$ for which there exists at least one $j$ such that $Q^T|\lambda_{i,j}\rangle \neq 0$. Consider then,

$$Q^T|\lambda_{i,j}\rangle = \sum_{v=1}^n \Omega(v)|v\rangle\langle v|\lambda_{i,j}\rangle$$

$$= \sum_{k=1}^m \left( \sum_{v \in C_k} \Omega(v)|v\rangle\langle v|\lambda_{i,j}\rangle \right) |k\rangle,$$
which shows that $Q^T|\lambda_{1,1}\rangle$ is non-zero if there exists at least one $k$ for which the parenthetic coefficient does not vanish.

Since $\mathcal{G}$ is assumed to be connected and undirected, it is strongly connected; it contains no negative edge weights, satisfying the requirements of Lemma 10 and the Perron-Frobenius theorem. Therefore there is a unique largest eigenvalue $\lambda_1$ of $A$, with a single corresponding eigenvector $|\lambda_{1,1}\rangle$. The elements $\langle v |\lambda_{i,j}\rangle$ of this vector are strictly positive, and by definition $\Omega(v) > 0$ for any vertex with at least one neighbour. Since $\mathcal{G}$ is connected, it contains no isolated vertices. As such, for every k the coefficient in Equation (B4) is a sum of one or more strictly positive numbers and therefore cannot vanish. Thus $Q^T|\lambda_{1,1}\rangle \neq 0$. So by Lemma 9, $Q^T|\lambda_{1,1}\rangle$ is an eigenvector of $B$, with eigenvalue $\lambda_1$. Furthermore, it can be seen from the proof of Lemma 9 that $\lambda_1$ is the unique largest eigenvalue of $B$ as well as of $A$.

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