Non-Gaussian noise induced stochastic resonance in FitzHugh–Nagumo neural system with time delay

Cite as: AIP Advances 10, 025310 (2020); https://doi.org/10.1063/1.5118730
Submitted: 22 July 2019 . Accepted: 19 January 2020 . Published Online: 05 February 2020

Shenghong Li, and Jiwei Huang

ARTICLES YOU MAY BE INTERESTED IN

Influences of discharge modes and gas bubbling conditions on E. coli sterilization by pulsed underwater discharge treatments
AIP Advances 10, 025207 (2020); https://doi.org/10.1063/1.5126378

Tunable multi-resonance of terahertz metamaterial using split-disk resonators
AIP Advances 10, 025108 (2020); https://doi.org/10.1063/1.5139263

Recurrent neural networks made of magnetic tunnel junctions
AIP Advances 10, 025116 (2020); https://doi.org/10.1063/1.5143382
Non-Gaussian noise induced stochastic resonance in FitzHugh–Nagumo neural system with time delay

Cite as: AIP Advances 10, 025310 (2020); doi: 10.1063/1.5118730
Submitted: 22 July 2019 • Accepted: 19 January 2020 • Published Online: 5 February 2020

Shenghong Li1,2,a) and Jiwei Huang1

AFFILIATIONS
1 School of Science, Jiangsu University of Science and Technology, Zhenjiang 212003, China
2 School of Mathematical Sciences and Institute of Finance and Statistics, Nanjing Normal University, Nanjing 210023, China

a)Author to whom correspondence should be addressed: shenghongli07@163.com

ABSTRACT
In this paper, non-Gaussian noise induced stochastic resonance for the FitzHugh–Nagumo neural system with a time delay is investigated. Through the path integral method, the non-Gaussian noise is approximated as a colored noise, and according to the unified colored noise theory and the method of probability density approximation, a stochastic differential equation with a Markovian property is obtained. Then, by applying the two-state theory, the expression of the signal-to-noise ratio (SNR) is derived. Finally, the effects of non-Gaussian noise and time delay parameters in the neural system on the SNR are discussed with the help of analytical results.

© 2020 Author(s). All article content, except where otherwise noted, is licensed under a Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/). https://doi.org/10.1063/1.5118730

I. INTRODUCTION
Stochastic resonance was first presented to investigate the quaternary glacial problem,1,2 and then, its existence was found in the experiment of the Schmitt trigger circuit system by Fauve.3 Because stochastic resonance illustrates the beneficial aspect of noise, great attention was paid. Later, the stochastic resonance phenomenon was investigated and applied extensively in many fields, such as electronic circuits, laser, bistable system, and biology system (see Refs. 4–11).

The evolution of the nonlinear system depends not only on its current state but also on the state of a certain period in the past. Time delay exists extensively and has an important influence on the behavior of stochastic nonlinear dynamical systems. The stochastic resonance phenomenon can be promoted or suppressed by time delay,12,13 and even induce multiple stochastic resonances.14,15 Moreover, the intermittent appearance of stochastic resonance regions with increasing delays was motivated.16,17 However, the fact that the existence of time delay makes stochastic systems non-Markovian leads to difficulty in obtaining analytical results, so some approximation methods have been used to obtain analytical expressions. The approximation of little time delay was proposed and the stationary probability density solution of a stochastic system via this method was derived by Guillouzi.18,19 Subsequently, Frank presented that the Novikov theorem could still facilitate a stochastic time-delay system, which is given in Refs. 20–22, and the equivalent Itô stochastic differential equation to the original system was obtained by probability density approximation.

The FitzHugh–Nagumo (FHN) neural system describes the firing behavior of sensory neurons and is one of the Hodgkin–Huxley models that are simplified.23,24 Then, many researchers have investigated the stochastic impact of the FHN system excited by all kinds of noises, which are provided in Refs. 25–27. Particularly, it can be found that the disturbances involved in the publications are modeled by Gaussian noise. Nevertheless, a lot of evidence displays that transportation in biological, physical, and chemical processes tends to be anomalous with long correlation or memory,28–31 which just exhibits a non-Gaussian property. Therefore, in this paper, non-Gaussian noise induced stochastic resonance in the FitzHugh–Nagumo neural system with a time delay is studied.32–35 The FHN neural model is introduced in Sec. II. In Sec. III, through simplifying the non-Gaussian noise and applying the approximate method of probability
density, the equivalent Itô stochastic differential equation having a Markovian property is derived. In Sec. IV, the effects of non-Gaussian noise and time delay on the signal-to-noise ratio (SNR) are given. The results are concluded in Sec. V.

II. FitzHugh–Nagumo NEURAL SYSTEM

The FitzHugh–Nagumo neural model is given as follows:

\[
\begin{align*}
\frac{dv}{dt} &= (a + 1)v^2 - v^3 - av - w, \\
\frac{dw}{dt} &= bv - rw,
\end{align*}
\]

where \(v\) indicates the membrane voltage and is taken as a fast variable. \(w\) is related to the time dependent conductance of the potassium channels in the membrane as a slow recovery variable. The constant \(a (0 < a < 1)\) is a threshold value essentially, and both \(b\) and \(r\) are positive constants. Based on the adiabatic elimination method, the FHN system (1) translates into a one-dimensional differential equation,

\[
\frac{dv}{dt} = (a + 1)v^2 - v^3 - av - \frac{b}{r}v. \tag{2}
\]

The corresponding potential function is written as

\[
U(v) = \frac{1}{4}v^4 - \frac{a + 1}{3}v^3 + \frac{a + b/r}{2}v^2. \tag{3}
\]

From Fig. 1(a), it can be seen there is no bifurcation at \(r = 1\), but bifurcation may occur when \(r\) is less than 0.8. In order to simplify the calculation, let \(r = 1\). Therefore, under the condition of \(b < (a - 1)^2/4\), there are two stable states in Eq. (2): \(v_1 = 0\), which represents the neuronal cells in the resting state, and \(v_2 = [a + 1 + \sqrt{(a - 1)^2 - 4b}]/4\), which represents the neuronal cells in the excited state. There is also an unstable state \(v_0 = [a + 1 - \sqrt{(a - 1)^2 - 4b}]/4\). Thus, there are two stable points \(v = 0\) and \(v = 0.98\), as shown in Fig. 1(b).

In fact, the external environment such as temperature, light, and ionic intensity affects the release intensity of membrane voltage and appears as an additive noise in the model. Similarly, the intrinsic neuronal structure, mitochondrion, or thermal fluctuation is regarded as the multiplicative noise described by the non-Gaussian noise. The quiescent neuron is activated due to appropriate time delay, and time delay in coupling between neurons also plays an important role in enhancing synchronization in the network and in almost all the biological systems. Therefore, time delay is considered in this paper. Thus, the one-dimensional Langevin equation of Eq. (2) is expressed as

\[
\frac{dv}{dt} = (a + 1)v^2 - v^3 - av(t - \tau) - bv + A\cos(\Omega t) + \eta(t) + \xi(t), \tag{4}
\]

where \(A\cos(\Omega t)\) is an external weak periodic signal with amplitude \(A\) and frequency \(\Omega\) that is regarded as a medical therapy or an external effect. \(\tau\) is a time delayed parameter and placed in the term \(-av(t - \tau)\) because we consider the influence of coefficient \(a\) that mainly controls the change in voltage; it needs some time from the voltage output to the neuron receiving the signal. \(\eta(t)\) is a multiplicative non-Gaussian noise, and \(\xi(t)\) is an additive Gaussian white noise; their statistical properties are presented as follows:

\[
\begin{align*}
\frac{d\eta(t)}{dt} &= -\frac{1}{\tau_0} \frac{d}{d\eta}[V_p(\eta)] + \frac{1}{\tau_0}\zeta(t), \\
V_p(\eta) &= \frac{P}{\tau_0(q - 1)}\ln[1 + \frac{\tau_0}{2P}(q - 1)\eta^2],
\end{align*}
\]

where \(\zeta(t)\) is also a Gaussian white noise, and \(q\) is the distance between \(\eta(t)\) and the Gaussian white noise; when \(q \rightarrow 1\), \(\eta(t)\) becomes a Gaussian colored noise with the auto-correlation time \(\tau_0\); that is, it is an Ornstein–Uhlenbeck process whose auto-correlation...
function is \( \langle \eta(t), \eta(t) \rangle = (p/t_0) \times \exp(\frac{-|t-t'|}{t_0}) \). If \( q \neq 1 \), \( \eta(t) \) is expressed as non-Gaussian noise. Here, the statistical characters of the Gaussian white noises \( \xi(t) \) and \( \zeta(t) \) are as follows:

\[
\begin{align*}
\langle \xi(t) \rangle &= < \xi(t) > = 0,
\langle \xi(t), \xi(t') \rangle &= 2P \delta(t-t'),
\langle \xi(t), \zeta(t') \rangle &= 2Q \delta(t-t'),
\langle \xi(t), \zeta(t') \rangle &= 0,
\end{align*}
\]

(6)

where \( P \) and \( Q \) are the noise intensities of \( \zeta(t) \) and \( \xi(t) \), respectively.

### III. APPROXIMATION OF NON-GAUSSIAN NOISE AND TIME DELAY

Time delay and non-Gaussian noise in Eq. (4) make the stochastic dynamical system non-Markovian. Next, we apply the approximation method to simplify them.

According to Eq. (3), its stationary probability density function can be resolved as

\[
P_q(\eta) = \frac{1}{z_q} \left[ 1 + \frac{t_0}{2P}(q-1)\eta^2 \right]^{-1/(q-1)},
\]

(7)

where \( z_q \) is a normalization constant, and the density function can be normalized only when \( q \) is less than 3. The correlation function is limited only if \( q < 5/3 \), that is,

\[
\langle \eta(t) \rangle = 0, \quad \langle \eta(t)\eta(t) \rangle = 2P/t_0(5-3q).
\]

(8)

Through the path integral method, the approximate value of the differential equation is obtained as follows:

\[
\frac{1}{t_0} \frac{d}{d\eta} V_q(\eta) = \frac{\eta}{t_0} \left[ 1 + \frac{t_0}{2P}(q-1)\eta^2 \right]^{-1}
\]

\[
\approx \frac{\eta}{t_1} \left[ 1 + \frac{t_0}{2P}(q-1)\eta^2 \right]^{-1} = \frac{\eta}{\tau_i},
\]

where \( \tau_i = 2(2 - q)\tau_0/(5 - 3q) \). Thus, Eq. (7) is simplified as a normalized Ornstein–Uhlenbeck and its effective correlation time and noise intensity are \( \tau_i \) and \( P_i \), respectively. Here, \( P_i = [2(2 - q)\tau_0(5 - 3q)]^2P \). Meanwhile, the non-Gaussian noise \( \eta(t) \) can be rewritten as a Gaussian colored noise, i.e.,

\[
\begin{align*}
\frac{dv}{dt} &= f(v) + g(v)\zeta(t) + h(v)\xi(t),
\end{align*}
\]

(11)

where

\[
\begin{align*}
f(v) &= \frac{(a + 1)v^2 - v^3 - (a + b)v}{\kappa(v, \tau_i)},
\frac{g(v)}{\kappa(v, \tau_i)} = \frac{v}{\kappa(v, \tau_i)},
\frac{h(v)}{\kappa(v, \tau_i)} = 1 + 2\tau_1v^2 - \tau_1(a + 1)v.
\end{align*}
\]

The sufficient and necessary condition for the existence of Eq. (11) is \( \kappa(v, \tau_i) > 0 \).

In Eq. (11), the non-Gaussian noise in Eq. (4) has been translated into the Gaussian white noise, but time delay still makes the system non-Markovian. Here, the approximation method of probability density given in Refs. 20–22 is applied, and an equivalent system to Eq. (4) with a Markovian property can be obtained, i.e.,

\[
\frac{dv}{dt} = f_k(v) + g(v)\zeta(t) + h(v)\xi(t),
\]

(12)

where

\[
f_k(v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v) P(v(t - \tau), t - \tau; v(t), t|v, t)
\times \exp(dv(t - \tau)dv(t)).
\]

(13)

\[
P(v(t - \tau), t - \tau|v, t)
= \frac{1}{\sqrt{2\pi G^2(v)}} \exp\left( -\frac{(v(t - \tau) - (v + f(v_0))\tau)^2}{2\tau G^2(v)} \right),
\]

(14)

where \( f(v_0) \) is the value of \( f(v) \) at \( t = 0 \), and \( G^2(v) = (P_1v^2 + Q)/\kappa(v, \tau_i) \).

Through the upper integral calculation, Eq. (11) becomes

\[
f_k(v) = 1 + \tau_i \left[ (a + 1)v^2 - v^3 - (a + b)v \right]/\kappa(v, \tau_i).
\]

Therefore, the stochastic differential equation equivalent to Eq. (11) is obtained as

\[
\frac{dv}{dt} = F_v(v) + G_v(v)\Gamma(t),
\]

(15)

where

\[
\Gamma(t)\Gamma(t') = 2\delta(t - t'),
F_v(v) = f_k(v), \quad G_v(v) = \Gamma(v).
\]

(16)

### IV. STOCHASTIC RESONANCE AND NUMERICAL RESULTS

We make the frequency \( \Omega \) of the input signal small enough so that there is enough time for the signal to arrive at the equilibrium state of the system. Solving Eq. (16), the stationary probability density function is given as

\[
P(v) = \frac{N}{G_v(v)} \exp\left( -\frac{\Phi(v)}{P_1} \right),
\]

(17)

where

\[
\Phi(v) = an + a_2v^2 - (1 + a)v^3 - \frac{1}{2}\tau_1v^4 + b_1\ln(P_1v^2 + Q) + c_1\arctan\left[ \frac{v}{\tau_1} \right] + \left[ w_1\arctan\left[ \frac{\tau_1}{v} \right] - \tau_1v \right] \times \cos(\Omega t),
\]

(18)


\[ a_1 = (1 + a)[3 \tau_1 z_1^2 - (1 + a \tau_1 + b \tau_1)], \]

\[ a_2 = (1 + \tau_1 + a^2 \tau_1)/2 + (2a + b - z_1^2) \tau_1, \]

\[ b_1 = \{(a + b) - [(a^2 + 4a + 2b + 1) \tau_1 - 1]z_1^2\}/2 + \tau_1 z_1^4, \]

\[ c_1 = (1 + a) [(a + b) \tau_1 - 3(1 + a) \tau_1 z_1^2] z_1, \]

\[ w_1 = [1 + (a + b) \tau_1]/z_1 + \tau_1 z_1, \quad \zeta = \sqrt{Q/P_1}. \]

Simultaneously, we assume that the signal intensity \( A \) is weak and the frequency \( \Omega \) of the weak periodic signal is less than the system transition time needed by the signal from the stable state to equilibrium. Thus, by using the steepest-descent method, \(^{46,47}\) the Kramers escape rates \( W_+ \) from \( v_1 \) to \( v_u \) and \( W_- \) from \( v_2 \) to \( v_u \) can be derived as

\[
W_+ = \frac{\sqrt{U''(v_1) U''(v_u)}}{2\pi} \exp \left[ \frac{\Phi(v_1) - \Phi(v_u)}{D_1} \right],
\]

\[
W_- = \frac{\sqrt{U''(v_2) U''(v_u)}}{2\pi} \exp \left[ \frac{\Phi(v_2) - \Phi(v_u)}{D_1} \right].
\]

Subsequently, the signal-to-noise ratio expression for the FHN neural system in the light of the output signal power spectrum is obtained by Refs. \(^{46,47}\)

\[
\text{SNR} = \frac{\pi A^2 (\mu_1 \beta_2 + \mu_2 \beta_1)^2}{4 \mu_1 \mu_2 (\mu_1 + \mu_2)},
\]

where \( \mu_1 = W_+ |_{\lambda \cos(\tau_1) = 0} \), \( \mu_2 = W_- |_{\lambda \cos(\tau_1) = 0} \), \( \beta_1 = -\frac{dW}{d(\lambda \cos(\tau_1))} |_{\lambda \cos(\tau_1) = 0} \), and \( \beta_2 = -\frac{dW}{d(\lambda \cos(\tau_1))} |_{\lambda \cos(\tau_1) = 0} \).

In order to observe the phenomenon of stochastic resonance directly, the figures of the SNR are delayed as a function of different parameters in the system, as shown in Figs. 2–7, respectively.

**Figure 2** gives the curves of the SNR as a function of the multiplicative noise intensity \( P \) for the varied distance \( q \). It can be seen that all four curves of the SNR are non-monotonic and the peaks appear, so the non-Gaussian noise \( q(t) \) induces the occurrence of stochastic resonance. Meanwhile, the maximal value of the SNR moves from the right to the left as \( q \) is increased, which illustrates that the multiplicative noise intensity \( P \) needed is decreased when stochastic resonance happens. However, the height of peaks is almost invariable with the increase in \( q \), which means that the distance \( q \) has little effect on the strength of stochastic resonance.

In Fig. 3, the SNRs as functions of the multiplicative noise intensity \( P \) and the distance \( q \) for an increased time delay \( \tau \) are displayed, respectively. Like Fig. 2, both \( P \) and \( q \) induce stochastic resonance. Briefly, with the increase in \( \tau \), the maxima of the SNR quickly increase, so the time delay \( \tau \) enhances the strength of stochastic resonance. However, the peaks do not move between left and right slightly, that is, the values of \( P \) and \( q \) are almost invariant as stochastic resonance occurs. Moreover, it is interesting that stochastic resonance happens twice in the interval \( q \in [0, 1.8] \), as shown in Fig. 3(b); particularly, the resonance intensity is very strong in
In addition, as shown from Fig. 4, the time delay $\tau$ also causes stochastic resonance. However, it occurs only when the value of $\tau$ is very large, which is opposite to the original intention of little time delay, and it has no practical significance.

The SNR as a function of multiplicative noise intensity $P$ for its varied auto-correlation time $\tau_0$ is shown in Fig. 5. Stochastic resonance occurs because all the curves have peaks. However, it is contrary to what is shown in Fig. 3 that the peak of the SNR declines and the peak value shifts from right to left when $\tau_0$ is increased. Therefore, the auto-correlation time $\tau_0$ weakens the intensity of stochastic resonance.

Figure 6 depicts the curves of the SNR as a function of the additive noise intensity $Q$ for the varied distance $q$. All four curves of the SNR are monotonic and decreased with the increase in $Q$, which means that the phenomenon of stochastic resonance does not happen. Moreover, the value of the SNR slightly rises with the increase in $q$.

These resonances can be seen from Fig. 7, and it can be observed that the multiplicative non-Gaussian noise in the system induces the occurrence of stochastic resonance, but the additive white noise does not. Viewed from the coordinate SNR and $P$ of Fig. 7, the surface is high in the middle and low in both sides, which is just the performance of stochastic resonance. However, the curved surface declines as the intensity $Q$ is increased, which explains that stochastic resonance does not happen; furthermore, the increase in $Q$ cripples the resonance intensity.
V. CONCLUSION

In this paper, stochastic resonance in the FHN neural system with a time delay driven by the non-Gaussian noise is investigated. Through the path integral method, the non-Gaussian noise is approximated to a colored noise and is transformed to white noise by the unified colored noise approximation. Then, applying the probability density approximation method, the differential equation having a Markovian property is obtained. Finally, based on the Fox method and two-state theory, the expression of the SNR is derived. The results conclude as follows: the non-Gaussian noise $\eta(t)$ in system (4) induces stochastic resonance, and the increase in the time delay $\tau$ can enhance the effect of stochastic resonance, but the effect of the auto-correlation time $\tau_0$ is opposed. Moreover, time delay induces the phenomenon of stochastic resonance as its value is very large. In addition, with the increase in the additive noise intensity $Q$, the value of the SNR becomes small, that is, the additive noise $\xi(t)$ does not evoke stochastic resonance.

ACKNOWLEDGMENTS

This work was jointly supported by the National Natural Science Foundation of China (Grant No. 11602098), Jiangsu Government Scholarship for Overseas Studies (Grant No. JS-2019-231), and the Postdoctoral Foundation of Jiangsu Province (Grant No. 1501069B).

REFERENCES

1. R. Benzi, A. Sutera, and A. Vulpiani, J. Phys. A: Math. Gen. 14, L453–L457 (1981).
2. C. Nicolis and G. Nicolis, Tellus 33, 225–234 (1981).
3. Š. Fauve and F. Heldrot, Phys. Lett. A 97, 5 (1983).
4. B. C. Bag, K. G. Petrosyan, and C. K. Hu, Phys. Rev. E 76, 056210 (2007).
5. B. C. Bag and C. K. Hu, Phys. Rev. E 75, 042101 (2007).
6. D. Y. Chen and L. Zhang, Chin. Phys. B 18, 1755 (2009).
7. J. Wang, X. Y. Ma, L. Cao, and D. J. Wu, Physica A 359, 98 (2006).
8. P. M. Shi, Q. Li, D. Qun, and Y. Han, Chin. J. Phys. 54, 526–532 (2016).
9. Y. C. Hung and C. K. Hu, Comput. Phys. Commun. 182, 249–250 (2011).
10. S. H. Li and J. C. Wu, Fluctuation Noise Lett. 14, 1550019 (2015).
11. S. H. Li and Q. X. Zhu, Chin. J. Phys. 56, 346–354 (2018).
12. R. H. Shao and Y. Chen, Physica A 388, 977–983 (2009).
13. Q. L. Han, T. Yang, C. H. Zeng, H. Wang, Z. Liu, Y. Fu, C. Zhang, and D. Tian, Physica A 408, 96–105 (2014).
14. Q. Y. Wang, P. Matijaš, Z. S. Duan, and G. R. Chen, Chaos 19, 023112 (2009).
15. C. B. Gan, P. Matijaš, and Q. Y. Wang, Chin. Phys. B 19, 040508 (2010).
16. C. Liu, J. Wang, and H. T. Yu, Commun. Nonlinear Sci. Numer. Simul. 19, 1088–1096 (2014).
17. H. T. Yu, J. Wang, and J. W. Du, Phys. Rev. E 87, 052917 (2013).
18. S. Guilouzic, I. L. Heureux, and A. Longtin, Phys. Rev. E 59, 3970–3982 (1999).
19. S. Guilouzic, I. L. Heureux, and A. Longtin, Phys. Rev. E 61, 4906–4914 (2000).
20. T. D. Frank, Phys. Rev. E 69, 061104 (2004).
21. T. D. Frank, Phys. Rev. E 71, 031106 (2005).
22. T. D. Frank, Phys. Rev. E 72, 011122 (2005).
23. A. L. Hodgkin and A. F. Huxley, J. Physiol. 117, 500–544 (1952).
24. Z. S. Lv, C. P. Zhu, P. Nie, Z. Zhao, H. J. Yang, Y. J. Wang, and C. K. Hu, Front. Phys. 12, 128902 (2017).
25. D. Wu and S. Zhu, Phys. Lett. A 372, 5399–5404 (2008).
26. H. Q. Zhang, T. T. Yang, and Y. Xu, Eur. Phys. J. B 88, 1–5 (2015).
27. X. J. Sun and Q. S. Lu, Chin. Phys. Lett. 31, 020502 (2014).
28. K. Wiesenfeld, D. Pierson, E. Pantazelou, C. Dames, and F. Moss, Phys. Rev. Lett. 72, 2125–2129 (1994).
29. D. Nozaki, D. J. Mar, P. Grigg, and J. J. Collins, Phys. Rev. Lett. 82, 2402–2405 (1999).
30. R. Metzler and J. Klafter, Phys. Rep. 339, 1–77 (2000).
31. R. Metzler and J. Klafter, J. Phys. A 37, R161–R208 (2004).
32. M. A. Fuentes, H. S. Wio, and R. Toral, Physica A 303, 91–104 (2002).
33. H. Hasegawa, Physica A 384, 241–258 (2007).
34. H. Zhang, W. Xu, and Y. Xu, Physica A 388, 781–788 (2009).
35. K. K. Wang, D. C. Zong, H. Ye, and Y. J. Wang, Fluctuation Noise Lett. 18, 1950027 (2019).
36. N. B. Janson, A. Balanov, and E. Schöll, Phys. Rev. Lett. 93, 010601 (2004).
37. T. Alarcón, A. Pérez-Madrid, and J. M. Rubí, Phys. Rev. E 57, 4979–4985 (1998).
38. H. X. Qin, J. M. Jun, W. Y. Jin, and C. N. Wu, Sci. China: Technol. Sci. 57, 936–946 (2014).
39. C. Zeng, Q. Han, T. Yang, H. Wang, and Z. Jia, J. Stat. Mech.: Theory Exp. 2013, P10017.
40. T. Yang, C. Zhang, Q. Han, C. Zeng, H. Wang, D. Tian, and F. Long, Eur. Phys. J. B 87, 136 (2014).
41. P. Jung and P. Hänggi, Phys. Rev. A 35, 4464–4466 (1987).
42. P. Jung and P. Hänggi, J. Opt. Soc. Am. A 5, 979–986 (1988).
43. H. Risken, The Fokker-Planck Equation (Springer-Verlag, Berlin, 1989).
44. G. Hu, Stochastic Force and Nonlinear Systems (Shanghai Scientific and Technological Education Publishing House, Shanghai, 1994).
45. R. F. Fox, Phys. Rev. A 33, 467–476 (1986).
46. S. Bouzat and H. S. Wio, Phys. Rev. E 59, 5142–5149 (1999).
47. H. S. Wio and S. Bouzat, Braz. J. Phys. 29, 136–143 (1999).