A splitting proximal point method for Nash-Cournot equilibrium models involving nonconvex cost functions

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Abstract

Unlike convex case, a local equilibrium point of a nonconvex Nash-Cournot oligopolistic equilibrium problem may not be a global one. Finding such a local equilibrium point or even a stationary point of this problem is not an easy task. This paper deals with a numerical method for Nash-Cournot equilibrium models involving nonconvex cost functions. We develop a local method to compute a stationary point of this class of problems. The convergence of the algorithm is proved and its complexity is estimated under certain assumptions. Numerical examples are implemented to illustrate the convergence behavior of the proposed algorithm.

Keywords. Nonconvex Cournot-Nash models, splitting proximal point method, local equilibria, gradient mapping.

1 Introduction

Nash-Cournot oligopolistic equilibrium models have been widely applied in economics, electricity markets, transportation, networks as well as in environments. Such a model can be formulated as a game model, where each player has a profit function which can be expressed as the difference of its price and cost functions. In classical models, the price function is affine while the cost function is assumed to be convex. In this case, a local equilibrium point is also a global one. Mathematical programming and variational inequality approaches can be used to treat this problem (see, e.g., [3, 4, 8, 9, 14]).

In practical models, the cost per a unit usually decreases as the production level increases. This situation requires a nonconvex function to represent the production cost of the model. In [15], a global optimization algorithm has been developed to find a global equilibrium point to the Nash-Cournot oligopolistic equilibrium market model involving piecewise concave cost functions. However, global algorithms only work well for the problems of moderate size, while it becomes intractable when the size of the problem increases, except for special structures are exploited.

In this paper, we continue the work in [15] by proposing a local solution method for finding a stationary point of Nash-Cournot equilibrium models with nonconvex (not necessarily concave) cost functions. We consider a Nash-Cournot model involving an affine price function and nonconvex smooth production cost functions. With this structure, the cost function of the model can be decomposed as the sum of a convex quadratic function and a nonconvex smooth function. Then, we develop a local method for finding a stationary point of such a model. The method is called a splitting proximal point algorithm. The main idea of this algorithm is to preserve the convexity of the problem while convexifies the nonconvex part by linearizing it around each iteration point.

Proximal point methods have been well developed in optimization as well as in nonlinear analysis. Myriad of research papers concerned to these methods were published (see, e.g., [11] [12] [22] [23].
and the references quoted therein). However, all these papers only deal with a class of monotone problems. Recently, Pennanen in [21] extended the proximal point method to nonmonotone cases for solving variational inequality problems. Lewis [11] further generalized this algorithm in a unified framework using the prox-regularity concept [23]. A main point of the proximal point methods is to choose the proximal parameter sequence. This affects to the performance of the algorithm as well as its global convergence behavior. G"uler in [6] investigated the rate of global convergence of the classical proximal point methods for convex programs. The worst case complexity bound of this method is $O(1/k)$, where $k$ the number of iterations. The author further accelerated the classical proximal point method to get a better complexity bound for the convex programming problems, precisely, $O(1/k^2)$ [7] by using the idea of Nesterov for gradient methods [18]. A splitting proximal point is also developed in many research papers. Such a method was applied to optimization problems by Mine and Fukushima in [13] and, recently, by Nesterov in [17].

This paper contributes a new local method for finding a stationary point of a Nash-Cournot equilibrium models involving the nonconvex cost functions. The algorithm called splitting proximal point method is presented and its convergence is investigated. The global worst case complexity bound is also provided, which is $O(1/\sqrt{k})$, where $k$ is the iteration counter. To our knowledge, this is the first estimation proposed to Nash-Cournot equilibrium model.

The rest of the paper is organized as follows. Section 2 presents a formulation of a Nash-Cournot oligopolistic equilibrium model involving nonconvex cost function. The problem is reformulated as a mixed variational inequality. In Section 3 we define three concepts including local and global equilibria, and stationary points of the Nash-Cournot equilibrium model. Section 4 deals with a gradient mapping and its properties. The splitting algorithm is described in Section 5, where its convergence is proved and the worst-case complexity is estimated. Two numerical examples are implemented in the last section.

## 2 Mixed variational inequality formulation

We consider a Nash-Cournot oligopolistic equilibrium market models with $n$-firms producing a common homogeneous commodity in a non-cooperative fashion. The price $p$ of production depends on the total quantity $\sigma := \sum_{i=1}^{n} x_i$ of the commodity. Let $h_i(x_i)$ denote the cost of the firm $i$ when its production level is $x_i$. Suppose that the profit of firm $i$ is given as

$$f_i(x_1, \ldots, x_n) := x_i p \left( \sum_{i=1}^{n} x_i \right) - h_i(x_i), \quad i = 1, \ldots, n, \quad (2.1)$$

where $h_i$ is the cost function of the firm $i$ which is assumed to only depend on its production level.

Let $C_i \subset \mathbb{R}$ ($i = 1, \ldots, n$) denote the strategy set of the firm $i$, which is assumed to be closed and convex. Each firm seeks on its strategy set to maximize the profit by choosing the corresponding production level under the presumption that the production of the other firms are parametric inputs. In this context, a Nash equilibrium is a production pattern in which no firm can increase its profit by changing its controlled variables.

Thus under this equilibrium concept, each firm determines its best response given other firms’ actions. Mathematically, a point $x^* = (x_1^*, \ldots, x_n^*)^T \in C := C_1 \times \cdots \times C_n$ is said to be a Nash equilibrium point if

$$f_i(x_1^*, \ldots, x_{i-1}^*, y_i, x_{i+1}^*, \ldots, x_n^*) \leq f_i(x_1^*, \ldots, x_n^*), \quad \forall y_i \in C_i \quad (i = 1, \ldots, n). \quad (2.2)$$

When $h_i$ is affine, this market problem can be formulated as a special Nash equilibrium problem in the $n$-person non-cooperative game model, which in turns is a strongly monotone variational inequality (see, e.g., [29]).

Let us define

$$\Psi(x, y) := - \sum_{i=1}^{n} f_i(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n), \quad (2.3)$$

and

$$\phi(x, y) := \Psi(x, y) - \Psi(x, x). \quad (2.4)$$
Then, as proven in [9], the problem of finding an equilibrium point of this model can be reformulated as the following equilibrium problem:

\[ \text{Find } x^* \in C \text{ such that: } \phi(x^*, y) \geq 0 \text{ for all } y \in C. \] (EP)

This generalized setting was proposed by Blum and Oettli in [2] (see also in [15]).

In classical Cournot models [4, 9], the price and the cost functions for each firm are assumed to be affine and given as follows:

\[ p(\sigma) = \alpha_0 - \beta \sigma, \quad \alpha_0 \geq 0, \quad \beta > 0, \quad \text{with } \sigma = \sum_{i=1}^{n} x_i, \] \hspace{1cm} (2.5)

\[ h_i(x_i) = \mu_i x_i + \xi_i, \quad \mu_i > 0, \quad \xi_i \geq 0 \quad (i = 1, \ldots, n). \] \hspace{1cm} (2.6)

In this case, using (2.1), (2.2), (2.3) and (2.4), it is easy to check that

\[ \phi(x, y) = (\tilde{B}x + \mu - \alpha)^T (y - x) + \frac{1}{2} y^T B y - \frac{1}{2} x^T B x, \]

where

\[ B = \begin{bmatrix} 2\beta & 0 & 0 & \cdots & 0 \\ 0 & 2\beta & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2\beta \end{bmatrix}; \quad \tilde{B} = \begin{bmatrix} 0 & \beta & \beta & \cdots & \beta \\ \beta & 0 & \beta & \cdots & \beta \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta & \beta & \beta & \cdots & 0 \end{bmatrix}, \]

\[ \alpha = (\alpha_0, \ldots, \alpha_0)^T, \quad \text{and} \quad \mu = (\mu_1, \ldots, \mu_n)^T. \]

Then the problem of finding a Nash equilibrium point can be formulated as a mixed variational inequality of the form:

\[ \text{Find } x^* \in C \text{ such that: } (\tilde{B}x^* + \mu - \alpha)^T (y - x^*) + \frac{1}{2} y^T B y - \frac{1}{2} x^T B x^* \geq 0, \quad \forall y \in C. \] \hspace{1cm} (2.7)

Let \( Q := B + \tilde{B} \). Since \( \beta > 0 \) and matrices \( \tilde{B} \) and \( B \) are symmetric, it is clear that \( Q \) is symmetric and positive definite. This mixed variational inequality can be reformulated equivalently to the following strongly convex quadratic programming problem:

\[ \min_{x \in U} \left\{ \frac{1}{2} x^T Q x + (\mu - \alpha)^T x \right\}. \] \hspace{1cm} (QP)

Hence, problem (QP) has a unique optimal solution, which is also the unique equilibrium point of the classical oligopolistic equilibrium market model.

The oligopolistic market equilibrium models, where the profit functions \( f_i \) \((i = 1, \ldots, n)\) of each firm is assumed to be differentiable and convex with respect to its production level \( x_i \) while the other production levels are fixed, are studied in [4] (see also [9]). This convex model is reformulated equivalently to a monotone variational inequality.

In practical models, the production cost function \( h_i \) assumed to be affine is no longer satisfied. Since the cost per a unit of the action does decrease when the quantity of the commodity exceeds a certain amount. Taking into account this fact, in the sequel, we consider market equilibrium models where the cost function \( h \) may not be convex, whereas the price function is affine as in (2.5). Typically, the cost function \( h \) is given as:

\[ h(x) := \sum_{i=1}^{n} h_i(x_i), \] \hspace{1cm} (2.8)

where \( h_i \) \((i = 1, \ldots, n)\) is differentiable and nonconvex.
Let us denote by

\[ F(x) := \tilde{B}x - \tilde{\alpha} \]
\[ \varphi(x) := g(x) - h(x), \]
where \( \tilde{\alpha} := \alpha - \mu, \ g(x) := \frac{1}{2}x^TBx \) and \( h(x) \) defined as (2.5).

Obviously, matrix \( \tilde{B} \) is symmetric, using the same notation \( \tilde{B}, \ B \) and \( \alpha \) as in (2.7), we can formulate the nonconvex Nash-Cournot equilibrium model as a mixed variational inequality:

Find \( x^* \in C \) such that:

\[ F(x^*)^T(y - x^*) + \varphi(y) - \varphi(x^*) \geq 0 \text{ for all } y \in C. \tag{ncMVIP} \]

Note that mixed variational inequality problems of the form (ncMVIP), where \( \varphi \) is convex, i.e. \( h \) is concave, were extensively studied in the literature (see, e.g., \([1, 4, 5, 9, 10, 16, 19, 24]\)).

**Remark 2.1.** If the function \( \varphi \) is convex and differentiable then problem (ncMVIP) can be reformulated equivalently to a classical variational inequality problem. More generally, it can be converted to a generalized variational inequality problem when \( \varphi \) is convex and subdifferentiable (see, e.g. \([9]\)). However, the mixed variational inequality (ncMVIP) can not equivalently transform into a variational inequality if \( \varphi \) is nonconvex.

Note, if we define \( \phi(x, y) := F(x)^T(y - x) + \varphi(y) - \varphi(x) \) then problem (ncMVIP) coincides with a nonconvex equilibrium problem of the form (EP).

### 3 Local equilibria and stationary points

Unlike the convex case, if the cost function \( \varphi \) of the problem (ncMVIP) is nonconvex, it may not have a global equilibria even if \( C \) is compact, and \( F \) and \( \varphi \) are continuous. Indeed, let us consider \( C := [-1, 1] \subset \mathbb{R}, \ F(x) := x \) and \( \varphi(x) = -\frac{1}{2}x^2 \), which is concave, then \( F(x)^T(y - x) + \varphi(y) - \varphi(x) = -\frac{1}{2}(y - x)^2 \). Therefore, problem (ncMVIP) corresponding to this function has no solution.

For a given \( x \in C \), let \( B(x, r) \) be an open ball centered at \( x \) of radius \( r > 0 \) in \( \mathbb{R}^n \). Borrowing the concepts from classical optimization, we firstly propose a local equilibria and critical points (or stationary points) of the mixed variational inequality (ncMVIP).

**Definition 3.1.** A point \( x^* \in C \) is called a local solution (or local equilibria) to (ncMVIP) if there exists a ball \( B(x^*, r) \) such that

\[ F(x^*)^T(y - x^*) + \varphi(y) - \varphi(x^*) \geq 0 \text{ for all } y \in C \cap B(x^*, r). \tag{3.1} \]

If \( C \subseteq B(x^*, r) \) then \( x^* \) is called a global solution (or global equilibria) to (ncMVIP).

Let

\[ \psi(x; y) := (\tilde{B}x - \tilde{\alpha})^T(y - x) + \frac{1}{2}g^TBg - h(y). \] \tag{3.2}

We consider a function \( m : C \times \mathbb{R}_+^+ \to \mathbb{R} \) and a mapping \( S : C \times \mathbb{R}_+^+ \to 2^C \) defined as follows:

\[
\begin{align*}
m(x; r) &:= \min \{ \psi(x; y) \mid y \in C \cap \overline{B}(x, r) \}, \\
S(x; r) &:= \arg\min \{ \psi(x; y) \mid y \in C \cap \overline{B}(x, r) \},
\end{align*}
\]

where \( \overline{B}(x, r) \) stands for the closure of the open ball \( B(x, r) \). As usual, we refer to \( m \) as a local gap function for problem (ncMVIP). Obviously, if \( h \) is continuous and \( C \) is compact then the function \( m \) as well as the mapping \( S \) are well-defined. If \( h \) is concave then \( S \) is reduced to a single valued mapping due to the symmetric and positive definiteness of \( B \).

The following proposition gives a necessary and sufficient conditions for a point to be a local or global solution to (ncMVIP).
Proposition 3.1. The function $m$ defined by (3.3) satisfies $m(x) \leq 0$ for all $x \in C$. Moreover, the following statements are equivalent:

a) $x^*$ is a local solution to (ncMVIP);

b) There exists $\bar{r} > 0$ and $x^* \in C$ such that $m(x^*; \bar{r}) = 0$;

c) There exists $\bar{r} > 0$ and $x^* \in C$ such that $x^* \in S(x^*; \bar{r})$.

Proof. Note that $\psi(x, x) = 0$ for all $x \in C$, and for any $x \in C$ and $\bar{r} > 0$, $x \in C \cap B(x, \bar{r})$. Therefore, from the definition (3.3) of $m$, it is clear that, with a given $\bar{r} > 0$, $m(x; \bar{r}) = \min \{ \psi(x, y) \mid y \in C \cap B(x, \bar{r}) \} \leq \psi(x, x) = 0$ for every $x \in C$. The equivalence between b) and c) is trivial. We only prove that a) is equivalent to b).

Suppose that there exists $\bar{r} > 0$ and $x^* \in C$ such that $m(x^*, \bar{r}) = 0$. It follows from the definition of $m$ that $\psi(x^*, y) \geq \psi(x^*, x^*) = 0$ for all $x \in B(x^*, \bar{r}) \cap C$. In particular, $\psi(x^*, y) \geq 0$ for all $y \in C \cap B(x^*, \bar{r})$. Thus $x^*$ is a local equilibria of (ncMVIP). Conversely, if $x^*$ is a local equilibria of (ncMVIP) then there exists a neighbourhood $B(x^*, r)$ such that $r > 0$ and

$$F(x^*)^T (y - x^*) + \varphi(y) - \varphi(x^*) \geq F(x^*)^T (x^* - x^*) + \varphi(x^*) - \varphi(x^*) = 0, \forall y \in C \cap B(x^*, r).$$

Since $r > 0$ and $B(x^*, r) \subset \mathbb{R}^n$, there exists $0 < \bar{r} \leq r$ such that $B(x^*, \bar{r}) \subseteq B(x^*, r)$. This relation shows that the last inequality holds for all $y \in C \cap B(x^*, \bar{r})$. Therefore, $m(x^*, \bar{r}) = 0$. \qed

Clearly, if the conclusions of Proposition 3.1 hold for a given $\bar{r} > 0$ and $C \subseteq B(x^*, \bar{r})$ then $x^*$ is a global solution to (ncMVIP).

Next, let us define

$$F_C(x) := \{ d := t(y - x) \mid y \in C, t \geq 0 \},$$

(3.5)

a cone of all feasible directions of $C$ starting from $x \in C$. The dual cone of $F_C(x)$ is the normal cone of $C$ at $x$ which is defined as

$$N_C(x) := \begin{cases} \{ w \in \mathbb{R}^n \mid w^T (y - x) \geq 0, \ y \in C \}, & \text{if } x \in C, \\ \emptyset, & \text{otherwise.} \end{cases}$$

(3.6)

By Proposition 3.1, a point $x \in C$ is a local solution to (ncMVIP) if and only if it solves the following optimization problem:

$$x \in \arg \min \left\{ (\tilde{B}x - \alpha)^T (y - x) + \frac{1}{2} y^T B y - h(y) \mid y \in C \cap \overline{B(x, \bar{r})} \right\},$$

(3.7)

for some $\bar{r} > 0$. Since $h$ is not necessarily concave, finding such a point $x$ satisfying (3.7), in general, is not an easy task. In this paper, we concentrate in finding a stationary point rather than local equilibria. We develop a method to find such a point for (ncMVIP). Borrowing the concept of stationary points in optimization, we define a stationary point (or a critical point) for the mixed variational inequality (ncMVIP) as follows.

Definition 3.2. A point $x \in C$ is called a stationary point (or critical point) to the problem (ncMVIP) if

$$0 \in Qx - \tilde{\alpha} - \nabla h(x) + N_C(x),$$

(3.8)

where $N_C$ is defined by (3.6) and $Q := \tilde{B} + B$.

Since $N_C$ is a cone, for any $c > 0$, the inclusion (3.8) is equivalent to

$$0 \in c[(\tilde{B} + B)x - \tilde{\alpha} - \nabla h(x)] + N_C(x).$$

(3.9)

Let

$$D\phi(x; d) := [(\tilde{B} + B)x - \tilde{\alpha} - \nabla h(x)]^T d,$$

(3.10)
for any $x \in C$ and $d \in F_C(x)$. Then the condition (3.8) is equivalent to

$$D\phi(x^*; d) \geq 0, \quad \forall d \in F_C(x^*).$$

(3.11)

Let us denote by $S^*$ the set of stationary points of (ncMVIP). The following lemma shows that every local equilibria of problem (ncMVIP) is its stationary point. The proof is simple and short, we present here for reading convenience.

**Lemma 3.1.** Suppose that $h$ is continuously differentiable on its domain. Then, every local equilibria of (ncMVIP) is a stationary point of problem (ncMVIP).

**Proof.** Suppose that $x^*$ is a local equilibria of (ncMVIP). Then there exists a neighborhood $B(x^*, r)$ of $x^*$ such that $F(x^*)^T(y - x^*) + \varphi(y) - \varphi(x^*) \geq 0$ for all $y \in B(x^*, r) \cap C$. According to Proposition 3.1, this requirement is equivalent to $x^*$ being a local solution of

$$\min \left\{ (\tilde{B}x^* - \tilde{\alpha})^T(y - x^*) + \frac{1}{2}y^TBy - h(y) \mid y \in B(x^*, r) \cap C \right\},$$

(3.12)

for $0 < r \leq r$. Since $h$ is continuously differentiable, the function inside the brackets is also continuously differentiable. Applying the first order necessary optimality condition for the smooth optimization problem (3.12) (see, e.g., [18]), we obtain:

$$0 \in (\tilde{B} + B)x^* - \tilde{\alpha} - \nabla h(x^*) + N_{C \cap \Pi(x^*, r)}(x^*).$$

However, $N_{C \cap \Pi(x^*, r)}(x^*) = N_C(x^*) \cap N_{\Pi(x^*, r)}(x^*) = N_C(x^*)$.

Let $\partial \delta_C(x)$ denote the subdifferential of the indicator function $\delta_C$ of $C$ at $x$. One has $\partial \delta_C(x) = N_C(x)$. Since matrix $B$ is symmetric and positive definite, if we define $g_1(x) := \frac{1}{2}x^TBx + \delta_C(x)$ then $\partial g_1(x) = Bx + \partial \delta_C(x)$ and this mapping is maximal monotone. Consequently, $T_{\varepsilon}^{-1} := (I + c\partial g_1)^{-1}$ is well-defined and single valued, where $I$ is the identity mapping (see [22, 23]).

The following proposition provides a necessary and sufficient condition for a stationary point of (ncMVIP).

**Proposition 3.2.** A necessary and sufficient condition for a point $x \in C$ to be a stationary point to problem (ncMVIP) is:

$$x = (I + c\partial g_1)^{-1}\left(x - c(\tilde{B}x - \tilde{\alpha}) + c\nabla h(x)\right),$$

(3.13)

where $c > 0$ and $I$ stands for the identity mapping.

**Proof.** Since $g_1$ is proper closed convex, the inverse $(I + \partial g_1)^{-1}$ is single valued and defined everywhere [22]. Thus $x$ satisfies (3.13) if and only if $x - c(\tilde{B}x - \tilde{\alpha}) + c\nabla h(x) \in (I + c\partial g_1)(x)$. Moreover, since $N_C(x)$ is a cone and $\partial g_1(x) = Bx + \partial \delta_C(x) = Bx + N_C(x)$, the latter inclusion is equivalent to $0 \in \tilde{B}x - \tilde{\alpha} + Bx - \nabla h(x) + N_C(x)$, which shows that $x$ is a stationary point of (ncMVIP).

Now, if we define $y_c(x) := x - c(\tilde{B}x - \alpha) + c\nabla h(x)$ and

$$S_c(x) := (I + c\partial g_1)^{-1}(x - c(\tilde{B}x - \alpha) + c\nabla h(x)),$$

(3.14)

then, it follows from Proposition 3.2 that $x = S_c(x)$. Therefore, every stationary point $x$ of (ncMVIP) is a fixed-point of $S_c(\cdot)$. To compute $S_c(x)$, it requires to solve the following strongly convex quadratic problem over a convex set:

$$\min \left\{ \frac{1}{2}y^TBy + \frac{1}{2c}\|y - y_c(x)\|^2 \mid x \in C \right\}.$$

(3.15)

This problem has a unique solution for any $c > 0$.

Finally, we introduce the following concept, which will be used in the sequel. For a given tolerance $\varepsilon \geq 0$, a point $x^* \in C$ is said to be an $\varepsilon$-stationary point to (ncMVIP) if

$$D\phi(x^*; d) \geq -\varepsilon, \quad \forall d \in F_C(x^*), \quad \|d\| = 1.$$

(3.16)
4 Gradient mapping and its properties

By substituting $y_c(x)$ into (3.15), after a simple rearrangement, we can write problem (3.15) as

$$\min \left\{ \frac{1}{2} y^T B y + [\tilde{B} x - \tilde{\alpha} - \nabla h(x)]^T (y - x) + \frac{1}{2c} \| y - x \|^2 \mid y \in C \right\}. \tag{4.1}$$

Now, we consider the following mappings:

$$m_c(x; y) := \frac{1}{2} y^T B y + [\tilde{B} x - \tilde{\alpha} - \nabla h(x)]^T (y - x) - h(x) + \frac{1}{2c} \| y - x \|^2, \tag{4.2}$$

and $s_c(x) := \arg \min \left\{ m_c(x; y) \mid y \in C \right\}. \tag{4.3}$

Then, since problem (4.3) is strongly convex, $s_c(x)$ is well-defined and single-valued. Let us define

$$G_c(x) := \frac{1}{c} [x - s_c(x)]. \tag{4.4}$$

The mapping $G_c(\cdot)$ is referred as a gradient-type mapping of (3.3) \cite{7}. Applying the optimality condition for (4.3) we have

$$\left[ B s_c(x) + \tilde{B} x - \tilde{\alpha} - \nabla h(x) - G_c(x) \right]^T (y - s_c(x)) \geq 0, \forall y \in C. \tag{4.5}$$

From now on, we further suppose that the cost function $h$ is Lipschitz continuous differentiable on $C$ with a Lipschitz constant $L_h > 0$, i.e.

$$\| \nabla h(x) - \nabla h(y) \| \leq L_h \| x - y \|, \forall x, y \in C. \tag{4.6}$$

By using the mean-valued theorem, it is easy to show that the condition (4.6) implies

$$| h(y) - h(x) - \nabla h(x)^T (y - x) | \leq \frac{1}{2} L_h \| y - x \|^2, \forall x, y \in C. \tag{4.7}$$

The following lemma shows some properties of $D\phi(\cdot; \cdot)$.

**Lemma 4.1.** For any $x \in C$, we have

$$D\phi(s_c(x); x - s_c(x)) \geq \frac{1 - c(L_h + \| \tilde{B} \|)}{c^2} \| G_c(x) \|^2, \tag{4.8}$$

$$D\phi(s_c(x); y - s_c(x)) \geq - [1 + c(L_h + \| \tilde{B} \|)] \| G_c(x) \| \| y - s_c(x) \|, \forall y \in C. \tag{4.9}$$

As a consequence, for any $d \in F_C(s_c(x))$ with $\| d \| = 1$, we have

$$D\phi(s_c(x); d) \geq - [1 + c(L_h + \| \tilde{B} \|)] \| G_c(x) \|. \tag{4.10}$$

**Proof.** From the definition of $D\phi$ in (3.10), we have

$$D\phi(s_c(x); x - s_c(x)) = \left[ B s_c(x) - \tilde{\alpha} + B s_c(x) - \nabla h(s_c(x)) \right]^T (x - s_c(x))$$

$$= \left[ \tilde{B} x - \tilde{\alpha} - \nabla h(x) + B s_c(x) \right]^T (x - s_c(x))$$

$$- \left[ B s_c(x) - \tilde{\alpha} - \nabla h(s_c(x)) - \tilde{B} x - \tilde{\alpha} + \nabla h(x) \right]^T (s_c(x) - x)$$

$$\geq \left[ \tilde{B} x - \tilde{\alpha} - \nabla h(x) + B s_c(x) \right]^T (x - s_c(x)) - (L_h + \| \tilde{B} \|) \| x - s_c(x) \|^2. \tag{4.11}$$

Substituting (4.5) into (4.11) we obtain

$$D\phi(s_c(x); x - s_c(x)) \geq \left( \frac{1}{c} - [L_h + \| \tilde{B} \|] \right) \| x - s_c(x) \|^2$$

$$= \frac{1 - c(L_h + \| \tilde{B} \|)}{c^2} \| G_c(x) \|^2,$$
which proves (4.8).

Using again (4.5) and (4.6) we have

\[ D\phi(s_c(x);y-s_c(x)) = \left[ \tilde{B}s_c(x) - \tilde{\alpha} + Bs_c(x) - \nabla h(s_c(x)) \right]^T (y - s_c(x)) \]

\[ = \left[ \tilde{B}s_c(x) - \tilde{\alpha} - \nabla h(s_c(x)) \right]^T (y - s_c(x)) + (Bs_c(x))^T (y - x) \]

\[ \geq \left[ \tilde{B}s_c(x) - \tilde{\alpha} - \nabla h(s_c(x)) \right]^T (y - s_c(x)) \]

\[ + \left[ \tilde{B}x - \nabla h(x) + \frac{1}{c}(s_c(x) - x) \right]^T (s_c(x) - y) \]

\[ = \tilde{B}s_c(x) - \nabla h(s_c(x)) - (\tilde{B}x - \tilde{\alpha}) + \nabla h(x) \]

\[ \geq -(L_h + \|\tilde{B}\|)\|x - s_c(x)\|\|y - s_c(x)\| - \|G_c(x)\|\|y - s_c(x)\| \]

\[ \geq - \left[ 1 + c(L_h + \|\tilde{B}\|) \right] \|G_c(x)\|\|y - s_c(x)\|, \]

which proves (4.9).

By the convexity of \( C \), there exists \( t \geq 0 \) such that \( s_c(x) + td \in C \), where \( \|d\| = 1 \). If we substitute \( y := s_c(x) + td \in C \) into (4.9) then we get

\[ D\phi(x;td) \geq -t(1 + cL_fh + c\|\tilde{B}\|)\|G_c(x)\|. \] (4.12)

If \( t = 0 \) then (4.9) automatically holds. If \( t > 0 \) then by the linearity of \( D \) with respect to the second argument, we divide both sides of (4.12) by \( t > 0 \) to get (4.9).

**Remark 4.1.** For a fixed \( x \in C \), if we define \( e_c(x) := \|G_c(x)\| \) and \( r_c(x) := \|x - s_c(x)\| \) then \( e_c(x) \) decreases in \( c \) and \( r_c(x) \) increases in \( c \).

Indeed, let \( q(y,c) := (\tilde{B}x - \tilde{\alpha})^T (y - x) + \frac{1}{2} y^T B y - h(x) - \nabla h(x)^T (y - x) + \frac{1}{2} \|y - x\|^2 \). Then \( q \) is convex jointly in two arguments \( y \) and \( c \). Thus \( \omega(c) := \min_{y\in C} q(y,c) \) is convex. It is easy to see that \( \omega'(c) = -\frac{1}{2}\|G_c(x)\|^2 \) increases in \( c \). Hence, \( e_c(x) \) decreases in \( c \). If we replace \( c \) by \( 1/c \) in \( q(y,c) \), this function becomes concave in \( c \), then by the same argument as \( \omega(c) \), we conclude that \( r_c(x) \) increases in \( c \).

Since \( B \) and \( \tilde{B} \) are symmetric, we consider a potential function defined as follows:

\[ \gamma(x) := \frac{1}{2} x^T B x + \frac{1}{2} x^T \tilde{B} x - \tilde{\alpha}^T x - h(x), \] (4.13)

Then, \( \gamma \) is nonconvex but Lipschitz continuous differentiable. We have the following statement.

**Lemma 4.2.** For \( x, y \in C \), we have

\[ m_c(x; s_c(x)) + x^T \tilde{B} x - \tilde{\alpha}^T x \leq \gamma(x) - \frac{c}{2} \|G_c(x)\|^2. \] (4.14)

Moreover, if \( c(L_h + \|\tilde{B}\|) \leq 1 \) then

\[ m_c(x; s_c(x)) + \frac{1}{2} x^T \tilde{B} x - \tilde{\alpha}^T x \geq \gamma(s_c(x)). \] (4.15)

**Proof.** It is obvious from the definition of \( m_c(x; x) \) that \( \phi(x) = m_c(x; x) + \frac{1}{2} x^T \tilde{B} x - \tilde{\alpha}^T x \). Since \( m_c(x; \cdot) \) is strongly convex quadratic with modulus \( \frac{1}{2c} \), using (4.13) we have

\[ \gamma(x) - m_c(x; s_c(x)) - \frac{1}{2} x^T \tilde{B} x + \tilde{\alpha}^T x = m_c(x; x) - m_c(x; s_c(x)) \]

\[ \geq \frac{1}{2c} \|x - s_c(x)\|^2 = \frac{c}{2} \|G_c(x)\|^2, \]
which proves (4.14).

To prove (4.15), from (4.14) and the definition of $\gamma$ we have

$$
\gamma(s_c(x)) - m_c(x; s_c(x)) - \frac{1}{2}\langle x^T \bar{B}x + \alpha^T x = \frac{1}{2} \left[ s_c(x)\bar{B}s_c(x) - x^T \bar{B}x - 2(\bar{B}x)^T(s_c(x) - x) \right]

- h(s_c(x)) + h(x) + \nabla h(x)^T(s_c(x) - x) - \frac{1}{2c} \| s_c(x) - x \|^2

\leq \frac{1}{2} \langle s_c(x) - x \rangle^T \bar{B}(s_c(x) - x) + \frac{(cL_h - 1)}{2c} \| s_c(x) - x \|^2

\leq - \frac{1 - c(L_h + \| \bar{B} \|)}{2c} \| s_c(x) - x \|^2.
$$

By assumption $c(L_h + \| \bar{B} \|) \leq 1$, we obtain (4.15).

If we combine the inequalities (4.15) and (4.14) in Lemma 4.2 then:

$$
\gamma(s_c(x)) \leq \gamma(x) - \frac{c}{2} \| G_c(x) \|^2.
$$

This inequality plays an important role in proving the convergence of the splitting proximal point algorithm in the section.

For a given starting point $x^0 \in C$, let us define the level set of $\gamma$ with respect to $C$ as

$$
\mathcal{L}_\gamma(\gamma(x^0)) := \{ x \in C \mid \gamma(x) \leq \gamma(x^0) \}.
$$

From (4.16), it is obvious that if $x^0 \in \mathcal{L}_\gamma(\gamma(x^0))$ then $s_c(x^0) \in \mathcal{L}_\gamma(\gamma(x^0))$ provided that $c(L_h + \| \bar{B} \|) \leq 1$.

## 5 A splitting proximal point algorithm and its convergence

Proposition 3.2 suggests that a proximal point method can be applied to find a stationary point of (ncMVIP). For the implementation purpose, the proximal mapping defined by (3.14) is extracted to the expression (3.15). The splitting proximal point algorithm constructs an iterative sequence as follows:

**Algorithm 1.** (The splitting proximal algorithm)

**Initialization:** Choose a positive number $c_0 > 0$. Find an initial point $x^0 \in C$ and set $k := 0$.

**Iteration $k$:** For a given $x^k$, execute the three steps below.

1. **Step 1:** Evaluate $\nabla h(x^k)$ and set $y_k := x^k - c_k(\bar{B}x^k - \alpha) + c_k \nabla h(x^k)$.

2. **Step 2:** Compute $x^{k+1}$ by solving the following convex quadratic program over a convex set:

$$
\min \left\{ \frac{1}{2} x^T \bar{B}x + \frac{1}{2c_k} \| y - y^k \|^2 \mid y \in C \right\}.
$$

3. **Step 3:** If $\| x^{k+1} - x^k \| \leq \varepsilon$ for a given tolerance $\varepsilon > 0$ then terminate, $x^k$ is an $\varepsilon$-stationary point of (ncMVIP). Otherwise, update $c_k$ and increase $k$ by 1 and go back to Step 1.

In Algorithm 1 we left unspecified the way to update $c_k$. If the Lipschitz constant $L_h$ is provided then we can choose $c_k = \frac{1}{\gamma^2}$ for all $k$, where $L_\gamma := L_h + \| \bar{B} \|$. Otherwise, a line-search procedure can be used to update $c_k$. The latter procedure is briefly described as follows. First, we choose two constants $\overline{c}$ and $\overline{\gamma}$ such that $\overline{c} > 0$ and $\frac{1}{\overline{\gamma}} \leq \varepsilon < +\infty$. Then we perform the following steps.

- Given a constant $\tau_c \in (0, 1)$. Choose an initial value of $c$ in $[\overline{c}, \overline{\gamma}]$. 

9
• Compute $s_c(x^k)$. While the decreasing condition
\[ \gamma(s_c(x^k)) \leq m_c(x^k; s_c(x^k)) + \frac{1}{2} (x^k)^T \tilde{B} x^k - \tilde{a}^T x^k. \] (5.2)
does not satisfy, increase $c$ by $c := \tau_c c$ and recompute $s_c(x^k)$.

• Set $c_{k+1} := c$.

Now, we define $\Delta x^k := x^{k+1} - x^k$ and
\[ \delta_k := \min_{0 \leq i \leq k} \frac{\| \Delta_i \|^2}{2c_i}. \tag{5.3} \]

The convergence of the splitting proximal point algorithm is stated as follows.

**Theorem 5.1.** Suppose that the function $h$ is Lipschitz continuous differentiable on $C$ with a Lipschitz constant $L_h \geq 0$. Suppose further that for a given $x^0 \in C$ the level set $\mathcal{L}_\gamma(\gamma(x^0))$ is bounded (particularly, $C$ is bounded). Then the sequence $\{x^k\}_{k \geq 0}$ generated by Algorithm 1 starting from $x^0$ satisfies:
\[ \delta_k \leq \frac{(\gamma(x^0) - \gamma)}{k + 1}, \quad \forall k \geq 0, \tag{5.4} \]

where $\gamma := \inf_{x \in \mathcal{L}_\gamma(\gamma(x^0))} \gamma(x)$. Moreover, for any $d \in \mathcal{F}_C(x^{i_k})$ with $\|d\| = 1$, we have
\[ D\phi(x^{i_k}; d) \geq -(1 + \langle L_h + \| \tilde{B} \| \rangle) \sqrt{\frac{2(\gamma(x^0) - \gamma)}{k + 1}}, \tag{5.5} \]

where $i_k$ is the index such that $c_{i_k} \| G_{c_{i_k}}(x^{i_k}) \|^2 = \Delta_k$.

As a consequence, if the sequence $\{x^k\}$ generated by (5.1) is bounded, then every limit point of this sequence is a stationary point of (ncMVIP). The set of limit points is connected and if it is finite then the whole sequence $\{x^k\}$ converges to a stationary point of (ncMVIP).

**Proof.** Since $\mathcal{L}_\gamma(\gamma(x^0))$ is bounded by assumption, we have $\gamma := \inf_{x \in \mathcal{L}_\gamma(\gamma(x^0))} \gamma(x)$ is well-defined due to the continuity of $\gamma$ and the closedness and nonemptiness of $\mathcal{L}_\gamma(\gamma(x^0))$ (since $x^0 \in \mathcal{L}_\gamma(\gamma(x^0))$). From Step 3 of Algorithm 1 if either the constant parameter $c_k = \frac{1}{\gamma}$ or the line search procedure is used then it implies
\[ \gamma(x^{k+1}) + \frac{1}{2c_k} \| x^{k+1} - x^k \|^2 \leq m_{c_k}(x^{k+1}) + (\tilde{B} x^k - \tilde{a})^T x^k \leq \gamma(x^k), \quad \forall k \geq 0. \tag{5.6} \]

Note that the whole sequence $\{x^k\}$ is contained in $\mathcal{L}_\gamma(\gamma(x^0))$. Rearrange and sum up these inequalities for $k = 0$ to $k = K$ we get
\[ \sum_{k=0}^{K} \frac{1}{2c_k} \| x^{k+1} - x^k \|^2 \leq \gamma(x^0) - \gamma(x^{K+1}) \leq \gamma(x^0) - \gamma. \tag{5.7} \]

Then the inequality (5.3) directly follows from the definition of $\delta$ in (5.3). Combining (5.3) and (5.10) in Lemma 4.1 we obtain (5.5).

To prove the remainder, taking into account Remark 4.1 and then passing to the limit as $k$ tends to $\infty$ the resulting inequality of (5.7), we get
\[ \sum_{k=0}^{\infty} \frac{1}{2c_k} \| x^{k+1} - x^k \|^2 < +\infty. \]

Since $\tilde{c} < +\infty$, this inequality implies that $\lim_{k \to \infty} \| x^{k+1} - x^k \| = 0$. Therefore, the set of limit points is connected. Combine this relation and the assumptions of boundedness of $\{ x^k \}$ it is easy to show that every limit point of $\{ x^k \}$ is a stationary point of (ncMVIP). When the set of the limit points is finite, the last statement of the theorem is proved similarly using the same technique as in [20] (Chapt. 28). \[ \square \]
Remark 5.1. For a given tolerance $\varepsilon > 0$, according to Theorem 5.1, the number of iterations $k$ to get an $\varepsilon$-stationary point is $O(\varepsilon^2)$. Consequently, the worst-case complexity of Algorithm 1 is $O(1/\sqrt{k})$.

6 Numerical test

In this section, we consider to numerical examples involving concave cost functions. The aim of these examples is to estimate the number of iterations of Algorithm 1 in a certain case compared to the worst-case complexity given in Theorem 5.1. In addition, we also test the time profile of the algorithm when the size of problem increases.

The algorithm is implemented in Matlab 7.8.0 (R2009a) running on a Pentium IV PC desktop with 2.6GHz and 512Mb RAM. We assume that the feasible set $C$ of (ncMVIP) is a box in $\mathbb{R}^n$. Therefore, the convex problem (3.15) reduces to quadratic programming. We solve this problem by using the quadprog solver (a built-in Matlab solver).

Example 1. Suppose that the cost function $h_i(x_i)$ of the firm $i$ is given as $h_i(x_i) = c_i^0 + c_i \ln(1+r_ix_i)$, where $c_i^0 \geq 0$ is the ceiling cost, $c_i > 0$ and $r_i > 0$ are given. The function $h$ becomes

$$h(x) = c^0 + \sum_{i=1}^{n} c_i \ln(1+r_ix_i) = c^0 + \ln\left(\prod_{i=1}^{n} (1+r_ix_i)^{c_i}\right), \quad (6.1)$$

where $c^0 = \sum_{i=1}^{n} c_i^0$. It is obvious that $h_i$ is well-defined if $x_i \geq 0$ and $h_i'(x_i) = \frac{c_i}{1+r_ix_i}$, which implies that $h$ is differentiable on $C = \mathbb{R}^n_+$ and

$$\nabla h(x) = \left(\frac{c_1r_1}{1+r_1x_1}, \ldots, \frac{c_nr_n}{1+r_nx_n}\right)^T. \quad (6.2)$$

Since $h_i''(x_i) = -c_i(r_i^2/(1+r_ix_i)^2)$, we have $|h_i''(x_i)| \leq c_i r_i^2$ and $h$ is concave. Moreover, $\nabla h$ is Lipschitz continuous with the Lipschitz constant $L_h := \max\{c_i r_i^2 \mid i = 1, \ldots, n\}$.

In this example, we choose $\beta = 0.1 > 0$, $\alpha = 10$, $c_i^0 = 2$, $c_i = 1.5$ for all $i = 1, \ldots, n$, and $r_i = 1 + \omega_i$, where $\omega_i$ is randomly generated in $(0, 1) \ (i = 1, \ldots, n)$. The strategy set of the firm $i$ is defined by $C_i := [0, 10]$ for all $i = 1, \ldots, n$.

We test Algorithm 1 for problem (ncMVIP) with the size increasing from 10 to 1000. The tolerance $\varepsilon$ is $10^{-3}$. The number of iterations as well as the CPU time with respect to the size of problem is visualized in Fig1. and Fig2., respectively.

![Fig1. Number of iterations depending on n [Ex. 1]](image1)

![Fig2. CPU time depending on n [Ex. 1]](image2)

From (6.5) of Theorem 5.1, it implies that the number of iterations $k$ to reach an $\varepsilon$-stationary point depends on the structure of the function $h$ and the value $\gamma(x^0) - \gamma$, $L_h$ and $\|\tilde{B}\|$. Since $h$ is a logarithm function, the value of $h$ slowly increases in $n$, while the Lipschitz constant $L_h \leq 1.5 \times 2^2 = 6$ for all $n$ and the norm $\|\tilde{B}\| = (n-1)\beta$. Consequently, the worst-case complexity bound increases almost linearly in $n$. As can be seen from the first figure, the number of iterations increases with a small slope when the size of problem grows up. The curvature of this graph stays below a linear line generated by the worst-case complexity bound. The CPU time also increases almost linearly in the size of problem.
**Example 2.** In this example, we choose the cost function $h_i$ as $h_i(x_i) = c_i^0 - c_i e^{-r_i x_i}$, where $c_i^0 \geq c_i > 0$ and $r_i > 0$ given. It is easy to see that $h_i''(x_i) = -c_i r_i^2 e^{-r_i x_i} < 0$, then $h_i$ is concave. Since $h_i$ is differentiable on $\mathbb{R}$, it means that $h$ is differentiable on $\mathbb{R}^n$ and $\nabla h$ is expressed by

$$\nabla h(x) = (c_1 r_1 e^{-r_1 x_1}, \ldots, c_n r_n e^{-r_n x_n})^T.$$ (6.3)

We have $|h_i''(x_i)| \leq c_i r_i^2$ for all $i = 1, \ldots, n$, thus $\nabla h$ is Lipschitz continuous on $\mathbb{R}^n$ with the Lipschitz constant $L_h := \max\{c_i r_i^2 | 1 \leq i \leq n\}$.

To compare with the previous example, we choose the value of the parameters $\alpha$ and $\beta$ as in Example 1. The parameters $c_i^0$ and $c_i$ are given by $c_i^0 = 4$ and $c_i = 2$ for all $i = 1, \ldots, n$. The parameter $r_i := 0.1 + 0.1\text{rand}_i$, where rand$_i$ is generated randomly in $(0, 1)$.

We also test Algorithm 1 for the problem size from 10 to 1000. The number of iterations and the CPU time are plotted in Fig3. and Fig4., respectively. Since the function $\gamma$ rapidly increases in $n$ compared to the previous case, the number of iteration also increases. Consequently, the CPU time respectively increases.

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