A STUDY OF KOSTANT-KUMAR MODULES VIA LITTELMANN PATHS

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ABSTRACT. We study, by means of Littelmann’s theory of paths, Kostant-Kumar modules (KK modules for short), which by definition are certain submodules of the tensor product of two irreducible integrable highest weight representations of a symmetrizable Kac-Moody algebra. Our main result is an identification of a path model for any KK module as a subset of the well known path model for the tensor product consisting of concatenations of Lakshmibai-Seshadri paths. The technical results about extremal elements in Coxeter groups that we formulate and prove en route and the technique of their proofs should be of independent interest. We also discuss the existence of PRV components and generalised PRV components in KK modules.

Specialising to the case of the special linear Lie algebra, we record a decomposition rule for KK modules in terms of Littlewood-Richardson tableaux. In this connection, we present a new procedure to determine the permutation that is the initial element of the minimal standard lift of a semi-standard Young tableau. The appendix, necessitated by the derivation of the tableau decomposition rule, deals with standard concatenations of Lakshmibai-Seshadri paths of arbitrary shapes, of which semi-standard Young tableaux form a very special case.

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1. Description of the Results

Let \( \mathfrak{g} \) be a symmetrizable Kac Moody Lie algebra. For \( \lambda \) a dominant integral weight, let \( V_{\lambda} \) denote the irreducible integrable representation of \( \mathfrak{g} \) with highest weight \( \lambda \), and \( v_{\lambda} \) a highest weight (non-zero) vector in \( V_{\lambda} \).

Fix dominant integral weights \( \lambda, \mu \) and an element \( w \) of the Weyl group \( W \). Let \( v_{w\mu} \) denote a non-zero vector in the one-dimensional weight space of weight \( w\mu \) in \( V_{\mu} \). The cyclic \( \mathfrak{g} \)-submodule \( U_{\mathfrak{g}}(v_{\lambda} \otimes v_{w\mu}) \) of \( V_{\lambda} \otimes V_{\mu} \), where \( U_{\mathfrak{g}} \) denotes the universal enveloping algebra of \( \mathfrak{g} \), is called a Kostant-Kumar module, KK module for short, and denoted \( K(\lambda, w, \mu) \). The following facts are elementary to prove and well known (see §5):

- \( K(\lambda, 1, \mu) \) is the copy of \( V_{\lambda+\mu} \) in the tensor product (where 1 denotes the identity element of \( W \)); \( K(\lambda, w_0, \mu) \) is the whole tensor product \( V_{\lambda} \otimes V_{\mu} \), when \( \mathfrak{g} \) is of finite type and \( w_0 \) denotes the longest element of \( W \).
- \( K(\lambda, w', \mu) \subseteq K(\lambda, w, \mu) \) for \( w' \) in the Weyl group with \( w' \leq w \) in the Bruhat order.

In this paper we study KK modules by means of Littelmann’s theory of paths [15, 16]. Let \( \mathcal{P}_{\lambda} \) and \( \mathcal{P}_{\mu} \) denote respectively the sets of Lakshmibai-Seshadri paths of shapes \( \lambda \) and \( \mu \). Let \( \mathcal{P}_{\lambda} \ast \mathcal{P}_{\mu} \) be the set of paths \( \pi \ast \pi' \) with \( \pi \in \mathcal{P}_{\lambda} \) and \( \pi' \in \mathcal{P}_{\mu} \), where \( \pi \ast \pi' \) denotes the concatenation of \( \pi \) and \( \pi' \). By work of Littelmann [15, 16], it is well known that \( \mathcal{P}_{\lambda} \ast \mathcal{P}_{\mu} \) is a “path model” for \( V_{\lambda} \otimes V_{\mu} \). The technical novelty that we introduce to the study of KK modules via paths is the association (§3.1) of an element denoted \( w(\pi \ast \pi') \) of the Weyl group \( W \) to each element \( \pi \ast \pi' \) of \( \mathcal{P}_{\lambda} \ast \mathcal{P}_{\mu} \). To describe this association, let us assume for the sake of simplicity that both \( \lambda \) and \( \mu \) are regular (see §3.1 for the general definition). Let \( \tau \) and \( \varphi \) be the elements of \( W \) representing respectively the final direction of \( \pi \) and the initial direction of \( \pi' \). Consider the following subset of \( W \):

\[
\{ \tau' \varphi \mid \tau' \in W, \tau' \leq \tau^{-1} \text{ in the Bruhat order} \}
\]

This admits a unique minimal element (Corollary 2.20 (2)), which \( w(\pi \ast \pi') \) is defined to be. The key property of \( w(\pi \ast \pi') \) is that it is invariant under the action of root operators on \( \pi \ast \pi' \) (Proposition 3.3). Towards the proof of this property, we establish in §2 some general results about extremal elements in Coxeter groups. These results and especially the technique of their proofs are, we believe, of independent interest.

Our main result (Theorem 7.1) identifies, under the possibly removable hypothesis that \( \mathfrak{g} \) is either of finite type or symmetric, a subset of \( \mathcal{P}_{\lambda} \ast \mathcal{P}_{\mu} \) that is a path model for the KK module \( K(\lambda, w, \mu) \). More precisely, we have, for any element \( w \) in the Weyl group:

\[
\text{char } K(\lambda, w, \mu) = \sum_{\pi \ast \pi' \in \mathcal{P}_{\lambda} \ast \mathcal{P}_{\mu}} \exp (\pi \ast \pi')(1)
\]

where \( \mathcal{P}(\lambda, w, \mu) \) denotes the set of those paths \( \pi \ast \pi' \) in \( \mathcal{P}_{\lambda} \ast \mathcal{P}_{\mu} \) with \( w(\pi \ast \pi') \leq w \). The proof of this theorem combines the invariance of \( w(\pi \ast \pi') \) under root operators mentioned above with Joseph’s decomposition rule [7, Theorems 5.25, 5.22] for KK modules.
(which we recall below in §6 and from which the restrictive hypothesis on \( g \) is inherited) and two fundamental results of Littelmann, namely the path character formula [15, page 330] and the isomorphism theorem [16, Theorem 7.1].

Let \( w' \) be an element of the Weyl group and \( \nu' \) denote the unique dominant conjugate of \( \lambda + w' \mu \). A copy of \( V_{\nu'} \) in \( V_\lambda \otimes V_\mu \) (or any submodule thereof) is called a “PRV component”, following a conjecture of Parthasarathy, Ranga Rao, and Varadarajan [26] that was proved independently by Kumar [10, Theorem 2.15] and Mathieu [21, Corollaire 3]. The use of paths to prove the existence of PRV components is well known: see e.g. [15, §7] and [5, §2.7]. The proof of Theorem 8.1 below, which is about PRV components in KK modules, follows this beaten track. Theorem 8.3 is about “generalised PRV components” in KK modules and follows Montagard [24, Theorem 3.1].

The second part of the paper (§9–10) deals with the special case of the special linear Lie algebra. In §9 we deduce, from the general decomposition rule (Theorem 6.1), a “tableau decomposition rule” (§9.4) for KK modules in terms of LR tableaux (LR is short for Littlewood-Richardson), generalising the classical LR rule (see e.g. [3, 20]) for decomposing \( V_\lambda \otimes V_\mu \). As is well known and in any case easy to see (§9.6.1, §9.8.3), the LR tableaux that figure in the classical LR rule can be identified with semi-standard Young tableaux (SSYT for short) of shape \( \mu \) that are \( \lambda \)-dominant. Again as is well known (and recalled with proof in §A.1), any SSYT has an interpretation as a standard concatenation of LS paths (LS is short for Lakshmibai-Seshadri) and therefore gives rise to a standard tuple and corresponding minimal standard lift (in the sense of §2.5).

Fix a LR tableau \( T \) that by the classical rule makes a contribution to the decomposition of the whole tensor product \( V_\lambda \otimes V_\mu \). Let \( v \) be the permutation that is the initial element of the minimal standard lift of the SSYT corresponding to \( T \). The tableau decomposition rule—of which we state three variations ((18), (19), (20)) for ease of ready reference—says that \( T \) contributes to the decomposition of the KK module \( K(\lambda, w, \mu) \) if and only if \( v \leq w \) in the Bruhat order.

The standard concatenation of LS paths attached to \( T \) may in turn be interpreted, via the isomorphism theorem of Littelmann mentioned above (in connection with the proof of the main Theorem 7.1), as an LS path \( \pi \) of shape \( \mu \). The fact that the permutation \( v \) represents the initial direction of \( \pi \) is what is involved in the derivation (in §9.9) of the tableau decomposition rule. While this fact will come as no surprise to experts in the theory of paths—it is hinted at by Littelmann already in [15] and later stated in [17, §11] with a sketch of proof—we could not find a reference to a complete proof. The appendix below is an attempt to address this inadequacy in the literature. It develops with complete proofs the necessary results about standard concatenations of LS paths of arbitrary shapes.

Given the important role played by the initial element \( v \) of the minimal standard lift of a SSYT in the tableau decomposition rule for KK modules, we present in §9.5.1 a new procedure to produce it from the SSYT. The naivest way to produce \( v \) would perhaps be to closely follow the proof of its existence in §2.5, or in other words repeatedly apply...
the construction in Deodhar’s lemma (Proposition 2.17). Lascoux-Schützenberger [14] give an algorithm to produce \(v\) via their notion of the “right key” associated to a SSYT. Willis [29] gives a more efficient algorithm to produce the right key. Our procedure stands apart from these other ones. The justification for it (namely the proof that it produces \(v\)) is provided in §10.

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2. Generalities on Extremal Elements in Coxeter Groups

Proposition 3.3 is the main technical result of the present paper. It relies (via Lemma 3.2) on the existence and properties of unique minimal elements in a particular kind of subsets of a Weyl group. The purpose of this section is to formulate and prove the required result about these minimal elements—Corollary 2.20 below—in the more natural context of Coxeter groups. The arguments leading up to the result are all elementary. The reader willing to accept it at face value without proof may want to skip this section at a first pass.

The results in §2.3, 2.4, and 2.5 are well known (e.g., from [2,12] as specifically indicated in a few places below), but we have included them because we need them and it is easier to prove them ab initio in our set up than to refer to sources. It not only makes the paper more self-contained but also more readable with these results stated and proved rather than just quoted.

2.1. Notation for this section. Let \((W, S)\) be a Coxeter system. Let \(\leq\) denote the (strong) Bruhat order on \(W\). For a subset \(K\) of the Coxeter group \(W\), we let \(\min K\) and \(\max K\) denote respectively the unique minimal and unique maximal elements of \(K\) in the Bruhat order (if they do exist). For \(u\) in \(W\) and \(s\) in \(S\), the elements \(u\) and \(su\) (respectively \(u\) and \(us\)) are comparable. Thus \(\min \{u, su\}\), \(\max \{u, su\}\), \(\min \{u, us\}\), and \(\max \{u, us\}\) make sense. We denote these respectively by \(u \land su\), \(u \lor su\), \(u \land us\), and \(u \lor us\).

2.2. The results. The following basic fact is repeatedly applied in this section:

\((*)\) For elements \(u \leq v\) in \(W\) and \(s\) in \(S\), we have \(u \land su \leq v \land sv\) and \(u \lor su \leq v \lor sv\).

The “right analogue” of the above fact asserts: \(u \land us \leq v \land vs\) and \(u \lor us \leq v \lor vs\) (under the same hypothesis). Only the left analogues of the “one sided” results below are explicitly stated. Their right analogues hold good too.

Remark 2.1. Suppose that \(u \leq v\) is a covering relation in \(W\) (that is, \(\text{length}(u) = \text{length}(v) - 1\) and \(u = tv\) for some reflection \(t\) in \(W\)). Then, if for \(s\) in \(S\), we have \(sv < v\)
and $u < su$, then $t = s$. Indeed, it follows from (*) that $u \leq sv$, but then equality is forced since $u$ and $sv$ have the same length.

A simple application of (*) gives:

**Proposition 2.2.** Suppose that a subset $K$ of the Coxeter group $W$ has a unique minimal element $u$ under $\leq$. Then, for any $s$ in $S$, the subset $K \cup sK$ also has a unique minimal element under $\leq$, namely, $u \wedge su$. Analogously, if $K$ admits a unique maximal element $v$ under $\leq$, then $v \vee sv$ is the unique maximal element of $K \cup sK$.

**Corollary 2.3.** Let $s: s_1, s_2, \ldots$ be a (possibly infinite) sequence of simple reflections (elements of $S$). For $s': s_1, s_2, \ldots, s_m$ a finite subsequence of $s$, let $w(s')$ denote the element $s_ms_{s_{m-1}}\cdots s_1$ of the Coxeter group (note the order reversal). Let $K$ be a subset of $W$ with a unique minimal element $u$ with respect to $\leq$. Then $\cup w(s')K$, where the union runs over all finite subsequences $s'$ of $s$, has a unique minimal element $u_\infty$, the stable value of $u_j$ as $j \to \infty$, where $u_j$ is recursively defined: $u_0 = u$, and $u_{j+1} = u_j \wedge s_{j+1}u_j$ for $0 \leq j$.

**PROOF:** For $j$ a non-negative integer, let $s_j$ denote the subsequence $s_1, s_2, \ldots, s_j$ of $s$. By a repeated application of Proposition 2.2, we see that $u_j$ is the unique minimal element of $K_j := \cup w(s')K$, where the union runs over subsequences of $s_j$. Since the subsets $K_j$ increase with $j$, it follows that $u_{j+1} \leq u_j$. Since any decreasing sequence in the Bruhat order stabilizes, we conclude that $u_j$ is constant for $j$ sufficiently large.

**Remark 2.4.** What about the maximal analogue of Corollary 2.3? Let $K$ be a subset of $W$ that has a unique maximal element $v$. With notation as in the proof just above, we conclude analogously that $v_k$ is the unique maximal element of $K_k$, where $v_k$ is defined recursively as follows: $v_0 = v$, and $v_{i+1} = v \vee s_{i+1}v$ for $0 \leq i$. Since the $K_k$ increase with $k$, we have $v_{k+1} \geq v_k$. If the $v_k$ stabilize to a stable value $v_\infty$ as $k \to \infty$ (which in general need not happen), then $v_\infty$ is the unique maximal element of $\cup_{k\geq 0}K_k$. In particular, the maximal analogue holds if the sequence $s$ is finite.

We now apply Corollary 2.3 (and its right analogue) in two special cases. First, let $\sigma$ be an element of $W$ and, with notation as in the corollary, choose the sequence $s: s_1, s_2, \ldots, s_m$ of elements of $S$ to be such that $s_ms_{s_{m-1}}\cdots s_1$ is a reduced expression for $\sigma$. Then \{$w(s') \mid s'$ is a subsequence of $s$\} equals $I(\sigma) := \{\sigma' \in W \mid \sigma' \leq \sigma\}$. We conclude that $I(\sigma)K$ has a unique minimal element and further that this element is the unique minimal element in $I(\sigma)u$. Now applying the right analogue of this argument, we obtain:

**Corollary 2.5.** Let $K$ be a subset of the group $W$ that admits a unique minimal element $u$. Then, for any two elements $\sigma_1$ and $\sigma_2$ of $W$, the set $I(\sigma_1)KI(\sigma_2)$ has a unique minimal element, and this element is the unique minimal element in $I(\sigma_1)uI(\sigma_2)$.

The special case of the above result (as also its maximal analogue, namely, Corollary 2.8) when $K$ is a singleton and $\sigma_2$ is the identity element appears in [12, Lemma 11 (i)].
Corollary 2.6. Let $\sigma$ and $\varphi$ be elements of $W$, and $s$ an element of $S$. Suppose that $\sigma s < \sigma$. Then $\min I(\sigma)\varphi$ equals either $\min I(\sigma)s\varphi$ or $\min I(\sigma)s\varphi$ accordingly as $s\varphi < \varphi$ or $\varphi < s\varphi$.

Proof: Choose a sequence $s$: $s = s_1, s_2, \ldots, s_m$ such that $s_ms_{m-1}\cdots s_1$ is a reduced expression for $\sigma$. Let $\varphi_i, \varphi'_i$, and $\varphi''_i$ be sequences defined recursively as follows:

1. $\varphi_0 = \varphi$, and $\varphi_{i+1} = \varphi_i \wedge s_{i+1}\varphi_i$ for $0 \leq i < m$.
2. $\varphi'_0 = s\varphi$, and $\varphi'_{i+1} = \varphi'_i \wedge s_{i+1}\varphi'_i$ for $0 \leq i < m$.
3. $\varphi''_1 = \varphi$, and $\varphi''_{i+1} = \varphi''_i \wedge s_{i+1}\varphi''_i$ for $1 \leq i < m$.

By Corollary 2.3, $\min I(\sigma)\varphi$, $\min I(\sigma)s\varphi$, and $\min I(\sigma)s\varphi$ are equal respectively to $\varphi_m$, $\varphi'_m$, and $\varphi''_m$.

First suppose that $s\varphi < \varphi$. Then $\varphi_1 = \varphi'_1$: indeed, $\varphi_1 = \varphi \wedge s\varphi = s\varphi$, and $\varphi'_1 = s\varphi \wedge s(s\varphi)) = s\varphi$. Thus $\varphi_i = \varphi'_i$ for all $1 \leq i$, and in particular for $i = m$.

Now suppose that $\varphi < s\varphi$. Then $\varphi_1 = \varphi'_1$: indeed, $\varphi_1 = \varphi \wedge s\varphi = \varphi$, and $\varphi'_1 = \varphi$ by definition. Thus $\varphi_i = \varphi''_i$ for all $1 \leq i$, and in particular for $i = m$. \hfill \square

Towards a second application of Corollary 2.3, let $S_1$ be a subset of $S$ and $W_1$ the subgroup of $W$ generated by $S_1$. With notation as in Corollary 2.3, choose the sequence $s$: $s_1, s_2, \ldots$ to consist of elements of $S_1$ and such that every element of $W_1$ arises as $w(s')$ for some finite subsequence $s'$ of $s$. Then $\{w(s') \mid s' \text{ is a subsequence of } s\}$ equals $W_1$. We conclude that $W_1K$ has a unique minimal element. Now applying the right analogue of this argument, we obtain:

Corollary 2.7. Let $K$ be a subset of the group $W$ that admits a unique minimal element $u$. Then, for any two standard parabolic subgroups $W_1$ and $W_2$ of $W$, the set $W_1KW_2$ has a unique minimal element, and this element is the unique minimal element in $W_1uW_2$. In particular, any double coset of a pair of standard parabolic subgroups has a unique minimal element.

Of course, when the subgroups $W_1$ and $W_2$ are finite, Corollary 2.7 is a special case of Corollary 2.5. Indeed, letting $w_1$ and $w_2$ be the unique maximal elements of $W_1$ and $W_2$ respectively, we have $W_1 = I(w_1)$ and $W_2 = I(w_2)$.

As the maximal analogues of the above two corollaries, we have:

Corollary 2.8. Let $K$ be a subset of $W$ having a unique maximal element $v$. Then, for any two elements $\sigma_1$ and $\sigma_2$ of $W$, the set $I(\sigma_1)KI(\sigma_2)$ has a unique maximal element, namely, the unique such element in $I(\sigma_1)vI(\sigma_2)$. In particular, for any two finite standard parabolic subgroups $W_1$ and $W_2$ of $W$, the union $W_1KW_2$ of double cosets has a unique maximal element, namely, the unique such element in $W_1vW_2$.

Remark 2.9. (Relation to Deodhar’s $\star$ operation.) In [2, Lemma 2.4], Deodhar states: there exists a unique associative binary operation $\star$ on $W$ such that $w \star \text{id} = w$ and $w \star s = w \lor ws$ for all $w \in W$ and $s \in S$. The uniqueness is clear. For the proof of the existence, we define $w \star x := \max I(w)I(x)$ for all $w$ and $x$ in $W (\max I(w)I(x)$ exists by
Corollary 2.8). It is easy to verify, using Corollary 2.8, that this operation has the requisite properties: \( \text{max } I(w)I(id) = \text{max } I(w) = w; \text{max } I(w)I(s) = \text{max } wI(s) = w \vee w; \text{ and } \)

\[
\text{max } I(w)I(\text{max } (x)I(y)) = \text{max } I(w)\text{max } (I(x)I(y)) = \text{max } I(w)I(x)I(y)
\]

\[
= \text{max } (\text{max } I(w)(I(x))I(y) = \text{max } I(\text{max } (I(w)(I(x))))I(y),
\]

so associativity holds.

We have:

- The unique maximal element of \( I(\sigma_1)KI(\sigma_2) \) in Corollary 2.8 is \( \sigma_1 \ast v \ast \sigma_2 \).
- \( I(w)I(x) = I(w \ast x) \) for all \( w \) and \( x \) in \( W \).
- Let \( K \) be a subset of \( W \) with a unique maximal element \( v \). For any collection \( \sigma_1, \ldots, \sigma_s, \tau_1, \ldots, \tau_t \) of elements in \( W \), the set \( I(\sigma_1) \cdots I(\sigma_s)KI(\tau_1) \cdots I(\tau_t) \) equals \( I(\sigma_1 \ast \cdots \ast \sigma_s)KI(\tau_1 \ast \cdots \ast \tau_t) \) and admits a unique maximal element, namely, \( \sigma_1 \ast \cdots \ast \sigma_s \ast v \ast \tau_1 \ast \cdots \ast \tau_t \).

**Remark 2.10.** Consider the specialized Hecke algebra \( \mathcal{H} \) defined as the associative algebra with identity (over say a field \( k \)) generated by variables \( T_s, s \in S \), and subject to the relations \( T_s^2 = T_s \) (for all \( s \) in \( S \)) and the braid relations. For \( w \in W \), let \( T_w \) be the element \( T_{s_1}T_{s_2} \cdots T_{s_r} \) of \( \mathcal{H} \) where \( s_1, s_2, \ldots, s_r \) is a reduced expression for \( w \): this definition does not depend on the choice of reduced expression because the braid relations are satisfied. The algebra \( \mathcal{H} \) is just the semigroup algebra of the semigroup \( W \) with respect to the \( \ast \) operation as in Remark 2.9: \( \{ T_w \mid w \in W \} \) is a basis for \( \mathcal{H} \) and \( T_wT_x = T_{w \ast x} \) for all \( w \) and \( x \) in \( W \).

Let \( kW \) be the free \( k \)-vector space with elements of \( W \) as a basis. We can make \( kW \) to be \( \mathcal{H} \)-\( \mathcal{H} \) bimodule as follows. For \( s \) in \( S \), let \( \wedge_s \) denote the (left) operator on \( W \) defined by \( \wedge_s w := sw \wedge w \) and \( s \wedge \) the right operator on \( W \) defined by \( w \wedge_s := w \wedge ws \) (for \( w \) in \( W \)). The linear extensions of the operators \( s \wedge \) and \( \wedge_s \) to \( kW \) are denoted by the same symbols. We have, for \( s, t \) in \( S \) and \( w \) in \( W \):

- \( \wedge_s(\wedge_s w) = \wedge_s w \) and \( (w \wedge_s) \wedge_s = w \wedge_s \).
- \( (\wedge_s w)_t \wedge_s = (\wedge_s(w_t)) \).
- Let \( s_1, \ldots, s_m \) be a sequence of elements of \( S \) such that \( s_m, s_{m-1}, \ldots, s_1 \) is a reduced expression for an element \( \sigma \) of \( W \). Then \( \wedge_m \cdots \wedge_1 (w) = \min I(\sigma)w \) (see the paragraph preceding Corollary 2.5) and analogously \( w \wedge_1 \cdots \wedge_m = \min wI(\sigma^{-1}) \), where \( \wedge_j \) and \( j \wedge \) stand for \( \wedge_j \) and \( j \wedge \) respectively. Thus the operators \( s \wedge \) (respectively \( \wedge_s \), \( s \in S \), satisfy the braid relations.

Thus, letting \( T_s, s \in S \), act on \( kW \) on the left by \( \wedge_s \) and on the right by \( \cdot \wedge_s \), we get a bimodule structure on \( kW \).

2.3. **Bruhat order on double coset spaces.** Let \( W_1 \) and \( W_2 \) be standard parabolic subgroups of \( W \). It is convenient to identify the coset space \( W_1 \backslash W/W_2 \) as a subset of \( W \) via
the association \( W_1 u W_2 \mapsto \min W_1 u W_2 \). The Bruhat order on \( W_1 \setminus W / W_2 \) is the restriction to this subset of the Bruhat order on \( W \).

**Corollary 2.11.** Given two elements \( W_1 u W_2, W_1 v W_2 \) in \( W_1 \setminus W / W_2 \), we have \( W_1 u W_2 \leq W_1 v W_2 \) in Bruhat order if and only if there exist \( u' \) in \( W_1 u W_2 \) and \( v' \) in \( W_1 v W_2 \) such that \( u' \leq v' \). In particular, if \( \min W_1 u W_2 \leq v' \) for some \( v' \) in \( W_1 v W_2 \), then \( \min W_1 u W_2 \leq v'' \) for every \( v'' \in W_1 v W_2 \).

**Proof:** For the only if part, just take \( u' = \min W_1 u W_2 \) and \( v' = \min W_1 v W_2 \). For the if part, apply Corollary 2.7 with \( K = \{u', v'\} \). Since \( u' \leq v' \), we conclude that \( \min W_1 u' W_2 = \min W_1 K W_2 \). But \( \min W_1 u' W_2 = \min W_1 u W_2 \) and \( \min W_1 K W_2 \leq \min W_1 v' W_2 = \min W_1 v W_2 \) since \( v' \in K \).

**Corollary 2.12.** For \( u \) an element of \( W \) and \( s \) an element of \( S \), suppose that \( su W_1 \leq u W_1 \). Then there exists either \( v \) in \( u W_1 \), we have \( sv < v \). Conversely, if \( su \leq u \) for the unique minimal element \( u \) in \( u W_1 \), then \( su W_1 \leq u W_1 \), and \( su \) is the unique minimal element in \( su W_1 \).

**Proof:** For the first statement, observe the following: if \( v < sv \), then, by Corollary 2.11, \( u W_1 = v W_1 \leq sv W_1 = su W_1 \), a contradiction. For the first part of the converse, observe that \( su W_1 \leq u W_1 \) by Corollary 2.11, and that equality cannot hold (if \( su \) were to belong to \( u W_1 \) the minimality of \( u \) in \( u W_1 \) would be contradicted). For the second part of the converse, suppose that \( x \in su W_1 \). Then \( sx \in u W_1 \), and so \( u \leq sx \), which means \( su = u \land su \leq sx \land x \leq x \).

**Remark 2.13.** For \( u \) in \( W \) and \( s \) in \( S \), it is possible that \( us W_1 \leq u W_1 \) but there exists \( v \) in \( u W_1 \) with \( vs > v \). For example, let

\[
W = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = 1, s_1 s_2 s_1 = s_2 s_1 s_2, s_2 s_3 s_2 = s_3 s_2 s_3, s_1 s_3 = s_3 s_1 \rangle
\]

\[
W_1 = \langle s_1, s_3 \rangle, u = s_1 s_2, and s = s_2. Then u W_1 = \{s_1 s_2, s_1 s_2 s_1, s_1 s_2 s_3, s_1 s_2 s_1 s_3\}, the minimal element in u W_1 is u, and us W_1 = W_1 \leq u W_1, but vs > v for v = s_1 s_2 s_3 in u W_1.
\]

**Proposition 2.14.** Suppose that \( W_1 u W_2 \leq W_1 v W_2 \). Then, given \( u' \) in \( W_1 u W_2 \), there exists \( v' \) in \( W_1 v W_2 \) with \( u' \leq v' \).

**Proof:** Proceed by induction on the length of \( u' \). Let \( u_0 \) be the minimal element in \( W_1 u W_2 \). We have \( u_0 \leq u' \). If \( u' = u_0 \), then \( u' \leq v' \) for any \( v' \) in \( W_1 v W_2 \) (since \( u_0 \leq v_0 \) by definition, where \( v_0 \) is the minimal element of \( W_1 v W_2 \)). Now suppose that \( u_0 \leq u' \). Then there exists either \( s \in S \cap W_1 \) such that \( su' < u' \), or \( t \in S \cap W_2 \) such that \( u't < u' \). Let us suppose that the former condition holds (the case when the latter holds is handled analogously). Observe that \( su' \) belongs to \( W_1 u W_2 \). By induction, there exists \( v' \) in \( W_1 v W_2 \) such that \( su' \leq v' \). By (*) at the beginning of this section, we have \( u' \leq su' \vee u' \leq v' \vee sv' \). But \( v' \vee sv' \) belongs to \( W_1 v W_2 \).
2.4. Deodhar’s Lemma. Let $W_1$ be a standard parabolic subgroup of $W$ and let $\sigma$, $w$ be elements of $W$. Set $I_{\sigma W_1}(w) := \{v \in \sigma W_1 | w \leq v\}$.

**Proposition 2.15.** Let $s$ be in $S$.

1. $I_{\sigma W_1}(w) \supseteq I_{\sigma W_1}(w')$ for $w \leq w'$.
2. $I_{\sigma W_1}(w) \subseteq sI_{\sigma W_1}(w \wedge sw)$
3. $I_{\sigma W_1}(w)$ is non-empty if and only if $wW_1 \leq \sigma W_1$.

**Proof:** Statement (1) is immediate. For (2), just observe that $w \leq x$ implies $w \wedge sw \leq x \wedge sx \leq sx$. As for (3), the only if part is trivial, and the if part follows from Proposition 2.14.

**Lemma 2.16.** Suppose that $\sigma$ is the minimal element of $\sigma W_1$, and let $s \in S$ be such that $s\sigma < \sigma$. Then

1. $sx < x$ for any $x$ in $I_{\sigma W_1}(w)$; and $v < sv$ for any $v$ in $I_{s\sigma W_1}(w \wedge sw)$.
2. $I_{\sigma W_1}(w) = sI_{s\sigma W_1}(w \wedge sw)$. In particular, $I_{\sigma W_1}(w) = I_{\sigma W_1}(w \wedge sw) = I_{\sigma W_1}(w \vee sw)$.
3. If either $I_{\sigma W_1}(w)$ or $I_{s\sigma W_1}(w \wedge sw)$ has a unique minimal element $u$, then so does the other and $su$ is that unique minimal element.

**Proof:** (1): We have $s\sigma W_1 \leq \sigma W_1$ by the second part of Corollary 2.12. From the first part of that corollary, it follows that $sx < x$ for any $x \in \sigma W_1$. Since $I_{\sigma W_1}(w) \subseteq \sigma W_1$ by definition, the first statement follows. The second statement too follows from the first assertion in Corollary 2.12.

(2): For the first assertion, given item (2) of Proposition 2.15, it is enough to show that $I_{\sigma W_1}(w) \supseteq sI_{s\sigma W_1}(w \wedge sw)$. Suppose $x$ belongs to $I_{s\sigma W_1}(w \wedge sw)$. Then evidently $sx$ belongs to $\sigma W_1$. By item (1) of the present lemma, we have $sx = x \vee sx$. Since $x \geq w \wedge sw$ by hypothesis, we have $sx = x \vee sx \geq (w \wedge sw) \vee s(w \wedge sw) \geq w$.

To see that $I_{\sigma W_1}(w) = I_{\sigma W_1}(w \wedge sw)$ (respectively, $I_{\sigma W_1}(w) = I_{\sigma W_1}(w \vee sw)$), put $v = w \wedge sw$ (respectively, $v = w \vee sw$). Then $v \wedge sv = w \wedge sw$, and so, by the first assertion, $I_{\sigma W_1}(v) = sI_{s\sigma W_1}(v \wedge sv) = sI_{s\sigma W_1}(w \wedge sw) = I_{\sigma W_1}(w)$.

(3) Suppose $u$ is the unique minimal element in $I_{s\sigma W_1}(w \wedge sw)$. Then $su$ belongs to $I_{\sigma W_1}(w)$ by (2). Let $x$ be any element in $I_{\sigma W_1}(w)$. We have $x \vee sx = x$ by (1), and $sx \in I_{s\sigma W_1}(w \wedge sw)$ by (2), so $u \leq sx$. Thus $su \leq u \vee su \leq sx \vee s(sx) = sx \vee x = x$.

Suppose $u$ is the unique minimal element in $I_{\sigma W_1}(w)$. Then $su$ belongs to $I_{s\sigma W_1}(w \wedge sw)$ by (2). Let $y$ be any element in $I_{s\sigma W_1}(w \wedge sw)$. We have $u \wedge su = su$ by (1), and $sy \in I_{\sigma W_1}(w)$ by (2), so $u \leq sy$. Thus $su = u \wedge su \leq sy \wedge s(sy) = sy \wedge y \leq y$.

**Proposition 2.17.** ("DEODHAR’S LEMMA", see e.g. [13, Lemma 5.8]) Suppose that $I_{\sigma W_1}(w)$ is not empty. Then it contains a unique minimal element. Moreover, this element can be constructed recursively as follows: let $\sigma$ be the minimal element in its coset $\sigma W_1$ and let $s_1, \ldots, s_m$ be a sequence of elements of $S$ such that $s_1 \cdots s_m$ is a reduced expression for $\sigma$; put $v_0 = w$ and
Remark 2.18. We make a few remarks regarding the construction in Proposition 2.17.

1. The element \( v_m \) in the statement of Proposition 2.17 is \( \min I(\sigma^{-1})w \) (see the third item in the list in Remark 2.10). Thus \( J_{\sigma W_1}(w) \) is non-empty if and only if \( \min I(\sigma^{-1})w \) belongs to \( W_1 \) and in this case its unique minimal element is \( \sigma(\min I(\sigma^{-1})w) \).

2. Given a double coset \( W_1 \sigma W_2 \), where \( W_2 \) is also a standard parabolic subgroup, there need not be a unique minimal element among those in the double coset that are \( \geq w \). Consider for instance the following simple example. Let \( W \) be the Weyl group of type \( A_2 \): \( S = \{s_i, s_j\} \) and \( W = \langle s_1, s_2 | s_1^2 = s_2^2 = 1, s_1s_2s_1 = s_2s_1s_2 \rangle \).

3. Let \( w' \) be an element of \( w \) such that \( w \leq w' \). Then, evidently, \( \min J_{\sigma W_1}(w) \leq \min J_{\sigma W_1}(w') \) (assuming that both sets are non-empty).

4. Let \( \sigma' \) be an element of \( W \) such that \( \sigma W_1 \leq \sigma' W_1 \). Suppose that \( J_{\sigma W_1}(w) \) is non-empty. Then \( J_{\sigma W_1}(w') \) is non-empty too: for any \( u \) in \( \sigma W_1 \) (and in particular for any \( u \) in \( J_{\sigma W_1}(w) \)), there exists, by Proposition 2.14, \( u' \) in \( \sigma' W_1 \) such that \( u \leq u' \), and evidently \( u' \) belongs to \( J_{\sigma W_1}(w') \) when \( u \) belongs to \( J_{\sigma W_1}(w) \). However, it need not be true that \( \min J_{\sigma W_1}(w) \leq \min J_{\sigma' W_1}(w) \), as the following simple example shows. Let \( W \) be the Weyl group of type \( A_3 \):

\[
\langle s_1, s_2, s_3 | s_1^2 = s_2^2 = s_3^2 = (s_1s_2)^3 = (s_2s_3)^3 = (s_1s_3)^2 = 1 \rangle
\]

Let \( W_1 \) be the parabolic subgroup \( \langle s_2 \rangle = \{1, s_2\}, w = s_2, \sigma = s_1s_3, \) and \( \sigma' = s_1s_2s_3 \).

Then \( J_{\sigma W_1}(w) \) is non-empty, and \( \min J_{\sigma W_1}(w) = s_1s_3s_2 \not\leq \min J_{\sigma' W_1}(w) = s_1s_2s_3 \).

5. (Deodhar’s functions \( f \) and \( g \) [2]) Suppose that \( \sigma \) is the least element in the coset \( \sigma W_1 \). Let \( \sigma' \) be an element of the Weyl group that is the least in its coset \( \sigma' W_1 \) and suppose that \( \sigma W_1 \leq \sigma' W_1 \) (equivalently \( \sigma \leq \sigma' \)).

Given \( x \) in \( W_1 \), there exists \( x' \) in \( W_1 \) such that \( \sigma x \leq \sigma' x' \) (see Proposition 2.14). Deodhar in [2, Lemma 2.2] states that there is a function \( f' : W_1 \to W_1 \) (depending upon \( \sigma \) and \( \sigma' \)) such that, for \( x \) and \( x' \) in \( W_1 \), \( \sigma x \leq \sigma' x' \) if and only if \( f' x \leq x' \). (Deodhar writes \( g \) for \( f' \).) In the notation of Remark 2.10, this function is given by \( f' x = \bigwedge_{\sigma' \to \sigma} \sigma x \). Indeed it follows from Proposition 2.17 that \( \bigwedge_{\sigma' \to \sigma} \sigma x \) has the required property.
Deodhar also asserts the existence of a function \( f : W_1 \to W_1 \) (depending upon \( \sigma \) and \( \sigma' \)) such that, for \( x \) and \( x' \) in \( W_1 \), \( \sigma x \leq \sigma' x' \) if and only if \( x \leq fx' \). To describe this function, let \( s_1, \ldots, s_m \) be a sequence of elements of \( S \) such that \( s_1 \cdots s_m \) is a reduced expression for \( \sigma' \). Put \( v_0 = \sigma \), and \( v_{i+1} := v_i \wedge s_i v_i \) inductively for \( 1 \leq i \leq m \). Let \( p \) be least, \( 0 \leq p \leq m \), such that \( v_p = \text{id} \): such a \( p \) exists because \( \sigma \leq \sigma' \). Let \( s_{i_1}, \ldots, s_{i_p} \) be the subsequence of \( s_{p+1}, \ldots, s_m \) consisting precisely of those elements that belong to \( W_1 \). Putting \( z := s_{i_1} \cdots s_{i_p} \), Deodhar’s function \( f \) is given by \( fx' = z * w' \). We omit the justification (which is not difficult) since we have no use in what follows for this function \( f \).

(6) (see [12, Lemma 11 (ii)]) Put \( K_{\sigma W_1}(w) := \{ v \in \sigma W_1 \mid v \leq w \} = I(w) \cap \sigma W_1 \). If \( K_{\sigma W_1}(w) \) is non-empty (or, equivalently, \( \sigma \leq w \)) then it has a unique maximal element. Indeed, writing \( w \) as \( \sigma' x' \) where \( \sigma' \) is the minimal element in \( w W_1 \) and \( x' \) is in \( W_1 \), this unique maximal element is \( \sigma fx' \), where \( f \) is Deodhar’s function of the previous item.

**Example 2.19.** Let \( W \) be the group of permutations of \([n]\) with \( S \) as the set of simple transpositions \( (1,2), \ldots, (n-1,n) \). For \( r, 1 \leq r < n \), let \( W_r \) be the standard maximal parabolic subgroup generated by all simple transpositions except \( (r,r+1) \). Let \( \sigma \) be a permutation of \([n]\) with one-line notation \( \sigma_1 \sigma_2 \cdots \sigma_n \). For \( \sigma \) to be of minimal length in its coset \( \sigma W_r \), it is necessary and sufficient that the sequences \( \sigma_1 \cdots \sigma_r \) and \( \sigma_{r+1} \cdots \sigma_n \) are both increasing. Suppose that this is the case.

Let \( w \) be another permutation. For \( J_{\sigma W_r}(w) \) to be non-empty, it is necessary and sufficient that \( w_1^r \leq \sigma_1, \ldots, w_r^r \leq \sigma_r \), where \( w_1^r < \cdots < w_r^r \) are the elements \( w_1, \ldots, w_r \) arranged in increasing order. Suppose that this is the case.

Let us suppose further that the \( w_1, \ldots, w_r \) were themselves in increasing order, so that \( w_1^r = w_1, \ldots, w_r^r = w_r \). In this case, \( \tau := \min J_{\sigma W_r}(w) \) is determined as follows. Put \( \tau_j = \sigma_j \) for \( 1 \leq j \leq r \). For \( j > r \), an induction on \( j \) determines \( \tau_j \) as follows. Let \( \tau_j^{i-1} < \cdots < \tau_j^{j-1} \) be \( \tau_1, \ldots, \tau_{j-1} \) in increasing order and \( w_1^j < \cdots < w_j^j \) be \( w_1, \ldots, w_j \) in increasing order. Then \( \tau_j = w_k^j \), where \( k \) is the largest \( 1 \leq k \leq j \), such that \( \tau_{k-1}^{j-1} < w_k^j \) (we put \( \tau_0^{j-1} = -\infty \)).

As an example, let \( n = 6, r = 3, w = 145362, \) and \( \sigma = 246135 \). Then, by the recipe above, \( \tau = 246153 \).

Justification for the recipe appears later: see Example 10.4 in §10.2.

\[ \square \]

2.5. **Standard tuples and standard lifts.** Let \( W_1, \ldots, W_m \) be a sequence of standard parabolic subgroups of \( W \) and let \( \theta = (\tau_1, \ldots, \tau_m) \in W/W_1 \times \cdots \times W/W_m \). We call \( \theta \) standard if there exists a chain \( \bar{\tau}_1 \geq \cdots \geq \bar{\tau}_m \) of elements in \( W \) such that \( \bar{\tau}_j W_j = \tau_j \) for \( 1 \leq j \leq m \). Such a chain is called a standard lift of \( \theta \).

Fix a standard tuple \( \theta \) of cosets and a standard lift of it as above. Put \( \sigma_m = \min \tau_m \). Observe that \( \bar{\tau}_{m-1} \geq \bar{\tau}_m \geq \sigma_m \). This means that \( J_{\tau_{m-1}}(\sigma_m) \) is not empty, and so it has a unique minimal element by Proposition 2.17. Put \( \sigma_{m-1} = \min J_{\tau_{m-1}}(\sigma_m) \). Proceeding this
way, choose inductively $\sigma_j = \min I_\tau(\sigma_{j+1})$ for $j$ equal to $m - 2, m - 3, \ldots, 1$. We call the chain $\sigma_1 \geq \ldots \geq \sigma_m$ the minimal standard lift of $\theta$. We denote by $\mathfrak{m}(\theta)$ the initial element $\sigma_1$ of the minimal standard lift of $\theta$.

Let $\sigma_1 \geq \ldots \geq \sigma_n$ be the minimal standard lift of $\theta$ (for some standard tuple $\theta$ of cosets). As is easily observed (by a downward induction on $j$), $\sigma_j \leq \tilde{\tau}_j$ for $1 \leq j \leq m$ for any standard lift $\tilde{\tau}_1 \geq \ldots \geq \tilde{\tau}_m$ of $\theta$. Furthermore, this property characterises the minimal standard lift.

2.6. The takeaway from this section. Finally, we isolate the takeaway from this section in its (admittedly strange and whimsical) specific form that will be invoked later.

Corollary 2.20. Let $(W, S)$ be a Coxeter system, $\tau, \varphi$ be elements of $W$, and $W_1, W_2$ be standard parabolic subgroups of $W$. Let $I(\tau^{-1})$ be the Bruhat interval $\{ w \in W \mid w \leq \tau^{-1} \}$. Then:

1. If $\tau', \varphi'$ are elements of $W$ such that $\tau W_1 = \tau' W_1$ and $\varphi W_2 = \varphi' W_2$, then:

$$W_1 I(\tau'^{-1}) \varphi' W_2 = W_1 I(\tau^{-1}) \varphi W_2$$

2. There exists a unique minimal element in $W_1 I(\tau^{-1}) \varphi W_2$, denoted $\min W_1 I(\tau^{-1}) \varphi W_2$.

3. Let $s$ be an element of $S$ such that $s \tau < \tau$ and $s \varphi W_2 \geq \varphi W_2$. Then

$$\min W_1 I(\tau^{-1}) \varphi W_2 = \min W_1 I(\tau^{-1}s) \varphi W_2$$

4. Let $s$ be an element of $S$ such that $s \varphi < \varphi$ and $s \tau W_1 \leq \tau W_1$. Then

$$\min W_1 I(\tau^{-1}) \varphi W_2 = \min W_1 I(\tau^{-1}) s \varphi W_2$$

Proof: For (1), it being evident that $W_1 I(\tau^{-1}) \varphi W_2 = W_1 I(\tau'^{-1}) \varphi' W_2$, it is enough to show that $W_1 I(\tau^{-1}) = W_1 I(\tau'^{-1})$. By Corollary 2.7, $W_1 \tau^{-1} = W_1 \tau'^{-1}$ has a unique minimal element, say $\sigma^{-1}$. It is enough to show that $W_1 I(\sigma^{-1}) = W_1 I(\tau^{-1})$. Since $\sigma^{-1} \leq \tau^{-1}$, it follows that $I(\sigma^{-1}) \subseteq I(\tau^{-1})$ and so $W_1 I(\sigma^{-1}) \subseteq W_1 I(\tau^{-1})$. To prove the other way containment, write $\tau^{-1} = u^{-1} \sigma^{-1}$ with $u \in W_1$ and $\ell(\tau) = \ell(u) + \ell(\sigma)$, where $\ell$ stands for “length”. Suppose that $\rho^{-1} \leq \tau^{-1}$. Then $\rho^{-1} = v^{-1} \rho'^{-1}$ with $v \leq u$ (hence $v \in W_1$) and $\rho' \leq \sigma$. Thus $W_1 \rho^{-1} = W_1 v^{-1} \rho'^{-1} = W_1 \rho'^{-1}$, and we are done.

Assertion (2) follows from Corollaries 2.5 and 2.7.

Proof of (3): By Corollary 2.5, $\min I(\tau^{-1}) \varphi \exists$. First suppose that $s \varphi > \varphi$. Then, by Corollary 2.6, $\min I(\tau^{-1}) \varphi = \min I(\tau^{-1}s) \varphi$, and, by Corollary 2.7, the desired equality follows. Next suppose that $s \varphi < \varphi$. Then, by Corollary 2.11, $s \varphi W_2 \leq \varphi W_2$ and, given our hypothesis that $s \varphi W_2 \geq \varphi W_2$, we conclude that $s \varphi W_2 = \varphi W_2$. Put $\varphi' := s \varphi$. Then $\varphi = s \varphi' > \varphi' = s \varphi$, and, by the first case, we have $\min W_1 I(\tau^{-1}) \varphi' W_2 = \min W_1 I(\tau^{-1}s) \varphi' W_2$. But, since $\varphi' W_2 = \varphi W_2$, this is exactly the desired equality.

Proof of (4): This is analogous to the proof of (3). By Corollary 2.5, $\min I(\tau^{-1}) \varphi \exists$. First suppose that $s \tau < \tau$. Then, by Corollary 2.6, $\min I(\tau^{-1}) \varphi = \min I(\tau^{-1}s) \varphi$, and, by Corollary 2.7, the desired equality follows. Next suppose that $s \tau > \tau$. Then, by Corollary 2.11, $s \tau W_1 \geq \tau W_1$ and, given our hypothesis that $s \tau W_1 \leq \tau W_1$, we conclude
that \( s \tau W_1 = \tau W_1 \). Put \( \tau' := s \tau \). Then \( s \tau' = \tau < s \tau = \tau' \), and, by the first case, we have
\[
\min W_1 I(\tau'^{-1}) \varphi W_2 = \min W_1 I(\tau'^{-1}) s \varphi W_2.
\]
But, since \( \tau' W_1 = \tau W_1 \), we have, by part (1), \( W_1 I(\tau'^{-1}) = W_1 I(\tau^{-1}) \), and the desired equality follows. \( \square \)

3. The KK filtration on concatenated LS paths

In this section, we introduce the two key elements that underpin this entire paper, namely:

- the definition of Kostant-Kumar (KK) sets of concatenated Lakshmibai-Seshadri (LS) paths (Equation (2)).
- the result that such a KK set is invariant under the action of root operators (Proposition 3.3)

The following notation remains fixed throughout this paper: \( g \) denotes a symmetrizable Kac-Moody algebra; \( \lambda \) and \( \mu \) are fixed dominant integral weights; \( W \) is the Weyl group and \( W_\lambda, W_\mu \) are respectively the stabilizers in \( W \) of \( \lambda, \mu \).

We assume familiarity with the basic notions and results of Littelmann’s theory [15,16] of paths. Let \( P_\lambda, P_\mu \) be respectively the sets of Lakshmibai-Seshadri (LS) paths of shape \( \lambda, \mu \). Let \( P_\lambda \ast P_\mu := \{ \pi \ast \pi' \mid \pi \in P_\lambda, \pi' \in P_\mu \} \), where \( \ast \) denotes concatenation.

Recall that a path \( \pi \) in \( P_\lambda \) consists of a sequence \( \tau_1 > \tau_2 > \ldots > \tau_r \) of elements in \( W/W_\lambda \) and a sequence \( 0 = a_0 < a_1 < \ldots < a_{r-1} < a_r = 1 \) of rational numbers (subject to some integrality conditions as in [15, §2], the details of which are not so relevant for the moment). We call \( \tau_1 \) the initial direction and \( \tau_r \) the final direction of \( \pi \).

3.1. Definition of a KK set in \( P_\lambda \ast P_\mu \). Given a path \( \pi \ast \pi' \) in \( P_\lambda \ast P_\mu \), we define, using Corollary 2.20, part (2), its associated Weyl group element \( w(\pi \ast \pi') \) by:

\[
(1) \quad w(\pi \ast \pi') := \min W_\lambda I(\tau^{-1}) \varphi W_\mu
\]

where \( \tau \) and \( \varphi \) are lifts in \( W \) respectively of the final direction of \( \pi \) and the initial direction of \( \pi' \). Part (1) of Corollary 2.20 says that \( w(\pi \ast \pi') \) is independent of the choice of the lifts \( \tau \) and \( \varphi \).

Given an element \( \varphi = W_\lambda w W_\mu \) of the double coset space \( W_\lambda \backslash W/W_\mu \), we define the associated KK set by:

\[
(2) \quad \mathcal{P}(\lambda, \varphi, \mu) := \{ \pi \ast \pi' \in P_\lambda \ast P_\mu \mid w(\pi \ast \pi') \leq \varphi \}
\]

The choice of the lift \( w \) in \( W \) of \( \varphi \) does not matter (see Corollary 2.11), and we often write \( \mathcal{P}(\lambda, \varphi, \mu) \) in place of \( \mathcal{P}(\lambda, w, \mu) \).

Clearly \( \mathcal{P}(\lambda, \varphi, \mu) \subseteq \mathcal{P}(\lambda, \varphi', \mu) \) if \( \varphi \leq \varphi' \). Thus the KK sets form an increasing filtration of the space \( P_\lambda \ast P_\lambda \) of concatenated LS paths, with underlying poset being the double coset space \( W_\lambda \backslash W/W_\mu \) with its Bruhat order. We call this the KK filtration on paths.

Remark 3.1. For this remark alone, we suppose that \( W \) is finite. Let \( w_0 \) be the longest element of \( W \). For \( w \) and \( x \) in \( W \), let \( w \otimes x := (ww_0 \ast x w_0)w_0 \), where \( \ast \) is Deodhar’s operation on the Weyl group discussed in §2.9. Let \( \pi \) in \( P_\lambda \) comprise the sequence \( \tau_1 >
\[ \tau_2 > \ldots > \tau_r \] of elements in \( W/W_\lambda \) and the sequence \( 0 = a_0 < a_1 < \ldots < a_{r-1} < a_r = 1 \) of rational numbers. Let \( \pi^\dagger \) be the path in \( P_\lambda \) comprising \( w_0\tau_r > \ldots > w_0\tau_1 \) and \( 0 = 1 - a_r < 1 - a_{r-1} < \ldots < 1 - a_1 < 1 = 1 - a_0 \). Then
\[
\mathfrak{w}(\pi \ast \pi') = \phi(\pi^\dagger)^{-1} \otimes \phi(\pi')
\]
where \( \phi(\eta) \) for an LS path \( \eta \) is the minimal lift in \( W \) of the initial direction of \( \eta \).

3.2. Stability of KK sets under root operators. For \( \alpha \) a simple root, let \( e_\alpha \) and \( f_\alpha \) be the root operators on paths as defined in [16]. Although this definition differs from the earlier one in [15], it is “backwards-compatible”: as explained in [16, Corollary 2 on page 512], the results of [15] are unaffected and we can freely quote them.

Let \( \pi \) be a path in \( P_\lambda \). Recall from [15]:

(a) The straight line path \( \pi_\lambda \) from the origin to \( \lambda \) belongs to \( P_\lambda \).
(b) \( \pi \) is piece-wise linear and its end point \( \pi(1) \) is an integral weight.
(c) For a simple root \( \alpha \), if \( e_\alpha(\pi) \) (respectively \( f_\alpha(\pi) \)) does not vanish, then it belongs to \( P_\lambda \), its end point is \( \pi(1) + \alpha \) (respectively \( \pi(1) - \alpha \)), and \( f_\alpha(e_\alpha(\pi)) \) (respectively \( e_\alpha(f_\alpha(\pi)) \)) equals \( \pi \).
(d) Let \( \alpha \) be a simple root and \( \tau \) the initial (respectively, final) direction of \( \pi \). If \( e_\alpha \pi \) does not vanish, then its initial (respectively, final) direction is either \( \tau \) or \( s_\alpha \tau \). The same holds for \( f_\alpha \) in place of \( e_\alpha \).
(e) \( \pi \) is obtained from \( \pi_\lambda \) by applying a suitable finite sequence of the root operators \( f_\alpha \) as \( \alpha \) varies. In particular, the end point of \( \pi \) is of the form \( \lambda - \kappa \) where \( \kappa \) is a non-negative integral linear combination of the simple roots.
(f) Every value that is a local minimum of the function \( h^\alpha_\pi(t) := \langle \pi(t), \alpha^\vee \rangle \) on \( t \in [0,1] \) is an integer, for every simple root \( \alpha \). (A value \( h^\alpha_\pi(t_0) \) is called a local minimum if \( h^\alpha_\pi(t_0) \leq h^\alpha_\pi(t) \) for \( 0 \leq |t - t_0| < \epsilon \) for some \( \epsilon > 0 \).) This follows from the proof of [16, Lemma 4.5, part (d)] although the definition of local minimum there is less inclusive.
(g) If \( e_\alpha(\pi) \) vanishes for every simple root \( \alpha \), then \( \pi = \pi_\lambda \) [15, Corollary in §3.5]. In particular, if (the image of) \( \pi \) lies entirely in the dominant Weyl chamber, then \( \pi = \pi_\lambda \).

Lemma 3.2. Let \( \pi \ast \pi' \) be a path in \( P_\lambda \ast P_\mu \) and \( \alpha \) a simple root. Then:

1. Every local minimum value of the function \( h^\alpha_\pi \ast \pi'(t) := \langle (\pi \ast \pi')(t), \alpha^\vee \rangle \) is an integer.

Suppose that \( e_\alpha(\pi \ast \pi') \) does not vanish. Then:

2. \( e_\alpha(\pi \ast \pi') \) equals either \( e_\alpha \pi \ast \pi' \) or \( \pi \ast e_\alpha \pi' \).
3. \( \mathfrak{w}(\pi \ast \pi') = \mathfrak{w}(e_\alpha(\pi \ast \pi')) \).

Proof: Statement (1) holds because \( \pi(1) \) is an integral weight (item (b) above) and local minima of both functions \( h^\alpha_\pi \) and \( h^\alpha_\pi' \) are integers (item (f) above).
Item (2) appears as [16, Lemma 2.7]. At any rate, it follows readily from the definition of \( e_\alpha \) once we know that the absolute minima of the functions \( h^{\pi}_\alpha \) and \( h^{\pi'}_\alpha \) are both integers, which is guaranteed by item (f) above.

To prove (3), let \( \tau \) be the final direction of \( \pi \) and \( \varphi \) the initial direction of \( \pi' \). First suppose that \( e_\alpha(\pi \circ \pi') = e_\alpha \pi \circ \pi' \). By item (d) above, the final direction of \( e_\alpha \pi \) is either \( \tau \) or \( s_\alpha \tau \). If it is \( \tau \), then there is nothing for us to do. In case it is \( s_\alpha \tau \), then, from the definition of \( e_\alpha \) and properties of \( \pi \) and \( \pi' \), it follows that \( s_\alpha \tau < \tau \) and \( s_\alpha \varphi \geq \varphi \). The assertion now follows from part 3 of Corollary 2.20.

Now suppose that \( e_\alpha(\pi \circ \pi') = \pi \circ e_\alpha \pi' \). By item (d) above, the initial direction of \( e_\alpha \pi' \) is either \( \varphi \) or \( s_\alpha \varphi \). If it is \( \varphi \), then there is nothing for us to do. In case it is \( s_\alpha \varphi \), then, from the definition of \( e_\alpha \) and properties of \( \pi \) and \( \pi' \), it follows that \( s_\alpha \varphi < \varphi \) and \( s_\alpha \tau \leq \tau \). The assertion now follows from part 4 of Corollary 2.20.

\[ \square \]

3.2.1. The equivalence relation on \( P_\lambda \star P_\mu \) defined by root operators. Given \( \pi \circ \pi' \) and \( \sigma \circ \sigma' \) paths in \( P_\lambda \star P_\mu \), let us say \( \pi \circ \pi' \) is related to \( \sigma \circ \sigma' \) if \( \pi \circ \pi' \) equals either \( e_\alpha(\sigma \circ \sigma') \) or \( f_\alpha(\sigma \circ \sigma') \) for some simple root \( \alpha \). This relation is symmetric since \( \pi \circ \pi' = e_\alpha(\sigma \circ \sigma') \) if and only if \( f_\alpha(\pi \circ \pi') = \sigma \circ \sigma' \). Denote by \( \sim \) the reflexive and transitive closure of this relation (as we vary over all simple roots).

As an immediate consequence of the item (3) of Lemma 3.2, we have:

**Proposition 3.3.** The association \( \pi \circ \pi' \mapsto \varpi(\pi \circ \pi') \) is constant on equivalence classes of the equivalence relation \( \sim \). In particular, for any \( \varphi \in W_\lambda \setminus W/W_\mu \), the KK set \( P(\lambda, \varphi, \mu) \) is a union of such equivalence classes.

In other words, each KK set is stable under the root operators. We will show in §7 that a KK set provides a *path model* for the corresponding KK module.

### 4. More Preliminaries

Notation is fixed as in §3: \( g \) is a symmetrizable Kac-Moody algebra; \( \lambda, \mu \) are dominant integral weights; \( W \) is the Weyl group; and \( W_\lambda, W_\mu \) are the stabilisers in \( W \) of \( \lambda, \mu \) respectively.

4.1. **Geometric interpretation of minimal representatives in** \( W_\lambda \setminus W/W_\mu \). We now give a geometric interpretation of the unique minimal element in a given double coset in \( W_\lambda \setminus W/W_\mu \). The association \( \varpi \leftrightarrow w \mu \) (for \( w \in W \)) gives a bijection of the coset space \( W/W_\mu \) with the set \( W_\mu \) of \( W \)-conjugates of \( \mu \). We identify the sets \( W/W_\mu \) and \( W_\mu \) via this bijection. The double coset space \( W_\lambda \setminus W/W_\mu \) may then be identified with the set of \( W_\lambda \)-orbits of \( W_\mu \).

**Proposition 4.1.** Every \( W_\lambda \)-orbit of the set \( W_\mu \) of \( W \)-conjugates of \( \mu \) contains a unique element \( w \mu \) such that \( \lambda + tw \mu \) is dominant for some real number \( t > 0 \).
PROOF: Each such orbit contains a unique $w\mu$ that is $W_\lambda$-dominant. The $W_\lambda$-dominance means precisely that $\langle w\mu, \alpha^\vee \rangle \geq 0$ for every simple root $\alpha$ in $W_\lambda$. It is easily verified that $w\mu$ has the desired property. Conversely, if $w\mu$ is not $W_\lambda$-dominant, then $\langle w\mu, \alpha^\vee \rangle < 0$ for some simple root $\alpha$ in $W_\lambda$, and so $\langle \lambda + tw\mu, \alpha^\vee \rangle = t\langle w\mu, \alpha^\vee \rangle < 0$ for $t > 0$. □

The double coset space $W_\lambda \backslash W / W_\mu$ may thus be identified with the set of those Weyl conjugates $w\mu$ of $\mu$ such that $\lambda + tw\mu$ is dominant for some positive $t$. We illustrate this by means of an example.

**Example 4.2.** Let $g$ be of type $B_2$. Let $e_1$ and $e_2$ be the standard basis vectors in $\mathbb{R}^2$ with its standard inner product. We may take $\alpha_1 := e_1$ and $\alpha_2 := e_2 - e_1$ to be the simple roots. Then the set of all positive roots is $\{\alpha_2, \alpha_1, \alpha_2 + \alpha_1, \alpha_2 + 2\alpha_1\}$, and the fundamental weights are $\omega_1 = \frac{1}{2}(e_1 + e_2)$ and $\omega_2 = e_2$. The Weyl group consists of 8 elements:

$$W = \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_2s_1s_2, s_2s_1s_2s_1 = s_2s_1s_2\}$$

where $s_1$ and $s_2$ are the reflections in the hyperplanes perpendicular to $\alpha_1$ and $\alpha_2$ respectively. The shaded portion in the figure is the dominant Weyl chamber.

Take $\lambda = 2\omega_1$ and $\mu = 2\omega_1 + \omega_2$. The stabilizers of $\lambda$ and $\mu$ are respectively: $W_\lambda = \{1, s_2\}$ and $W_\mu = \{1\}$. The set of double cosets $W_\lambda \backslash W / W_\mu$ is:

$$\{\{1, s_2\}, \{s_1, s_2s_1\}, \{s_1s_2, s_2s_1s_2\}, \{s_1s_2s_1, s_2s_1s_2s_1\}\}$$

As is clear from Figure 4.1, $\mu$, $s_1\mu$, $s_1s_2\mu$, and $s_1s_2s_1\mu$ are all the conjugates of $\mu$ for which the line segment joining $\lambda$ to the conjugate lies for some positive distance in the dominant Weyl chamber. □

**Proposition 4.3.** Given a double coset in $W_\lambda \backslash W / W_\mu$, let $w$ be the unique minimal element in it with respect to the Bruhat order (as guaranteed by Corollary 2.7). Then $w\mu$ is such that $\lambda + tw\mu$ is dominant for all small positive $t$.

PROOF: From the proof of Proposition 4.1, it is enough to show that $w\mu$ is $W_\lambda$-dominant. Suppose that this is not so. Then there exists simple root $\alpha$ with $s_\alpha$ in $W_\lambda$ such that $\langle w\mu, \alpha^\vee \rangle < 0$. We then have $s_\alpha w < w$, which contradicts the hypothesis that $w$ is the minimal element in its double coset. □

4.2. **Two key propositions.** An LS path $\pi$ of shape $\mu$ is said to be $\lambda$-dominant if $\lambda + \pi(t)$ belongs to the dominant Weyl chamber for every $t \in [0, 1]$. The set of $\lambda$-dominant paths of shape $\mu$ is denoted by $\mathcal{P}_\mu^\lambda$. For $w$ an element of the Weyl group, $\mathcal{P}_\mu^\lambda(w)$ denotes the elements of $\mathcal{P}_\mu^\lambda$ whose initial direction is $\leq w\mu$.

**Proposition 4.4.** Let $\theta$ be a path in $\mathcal{P}_\lambda * \mathcal{P}_\mu$. Then there exists a unique path $\eta$ in the equivalence class (of the relation $\sim$ defined in §3.2.1) containing $\theta$ such that $e_\alpha \eta$ vanishes on $\eta$ for all simple roots $\alpha$. Moreover, $\eta$ has the following properties:

1. $\eta$ lies entirely in the dominant Weyl chamber.
(2) \( \eta = \pi_\lambda \ast \pi \) for some \( \pi \) in \( \mathcal{P}_\mu^\lambda \).

(3) \( w(\theta) = w(\eta) = v \) where \( v \) is minimal in the Weyl group such that \( v\mu \) is the initial direction of \( \pi \). In particular, if \( \theta \in \mathcal{P}(\lambda, w, \mu) \) (for some \( w \) in \( \mathcal{W} \)), then \( \pi \in \mathcal{P}_\mu^\lambda(w) \).

**Proof:** For the existence of \( \eta \), there is the following standard argument. Construct by induction a sequence \( \theta_0, \theta_1, \ldots \) of elements in the equivalence class of \( \theta \) as follows. Choose \( \theta_0 \) to be \( \theta \). Given \( \theta_i \), if \( e_\alpha \theta_i \) vanishes for all simple roots \( \alpha \), then set \( \eta = \theta_i \) and we are done. If not, then choose a simple root \( \alpha \) arbitrarily such that \( e_\alpha \theta_i \) does not vanish and put \( \theta_{i+1} = e_\alpha \theta_i \). By induction \( \theta_{i+1} \) belongs to the equivalence class of \( \theta \). We will eventually find an \( \eta \) this way, for this process must terminate at some point. In fact, the length of the sequence is bounded by the sum of the coefficients of \( \kappa \) where \( \kappa \) is the non-negative integral linear combination of the simple roots such that the end point of \( \theta \) equals \( \lambda + \mu - \kappa \).

Since \( e_\alpha \eta \) vanishes for all simple \( \alpha \) and since the absolute minimum of the function \( h^\alpha_\eta(t) \) is an integer for every simple \( \alpha \) (see item (1) in Lemma 3.2 above), it follows from the definition of \( e_\alpha \) that \( \eta \) lies entirely in the dominant Weyl chamber.

The uniqueness of \( \eta \) now follows from [16, Corollary 1 (b) of §7].

Write \( \eta = \zeta \ast \pi \) with \( \zeta \in \mathcal{P}_\lambda \) and \( \pi \in \mathcal{P}_\mu \). Since \( \eta \) lies entirely in the dominant Weyl chamber, clearly so does \( \zeta \). Thus \( \zeta = \pi_\lambda \) by item (g) in §3.2 above, and \( \pi \) belongs to \( \mathcal{P}_\mu^\lambda \).
The equality \( w(\theta) = w(\eta) \) follows from Proposition 3.3. Since \( \eta \) lies entirely in the dominant Weyl chamber, it follows that \( \lambda + tw \mu \) is dominant for sufficiently small \( t \geq 0 \). By Proposition 4.3, the unique minimal element of \( W_\lambda v W_\mu \) lies in \( v W_\mu \) and hence equals \( v \). But \( w(\eta) = \min W_\lambda v W_\mu \) by its definition.

**Proposition 4.5.** With notation as in Proposition 4.4, write \( \theta = \pi_1 \ast \pi_2 \). Then the following conditions are equivalent:

1. \( \eta = \pi_\lambda \ast \pi_\mu \) (that is, \( \pi = \pi_\mu \))
2. \( w(\theta) = \text{identity} \)
3. there exist \( \tau \) and \( \phi \) in \( W \) such that \( \tau \geq \phi \), \( \tau W_\lambda \) is the final direction of \( \pi_1 \), and \( \phi W_\mu \) is the initial direction of \( \pi_2 \).

**Proof:** Let \( v \) be as defined in item (3) of Proposition 4.4. Observe that condition (1) is equivalent to saying that \( v \) is identity. Since \( w(\theta) = v \) by Proposition 4.4(3), we have \( (1) \Leftrightarrow (2) \).

\( (2) \Rightarrow (3) \): Let \( \tau \) and \( \phi \) be arbitrary elements in \( W \) such that \( \tau W_\lambda \) is the final direction of \( \pi_1 \) and \( \phi W_\mu \) is the initial direction of \( \pi_2 \). Condition (2) says that \( \min W_\lambda I(\tau^{-1}) \phi W_\mu \) equals identity. Let \( u \in W_\lambda \), \( \sigma \leq \tau \) in \( W \), and \( v \in W_\mu \) be such that \( u^{-1} \sigma^{-1} \phi v = \text{identity} \), or \( \phi v = \sigma u \). We have \( \sigma W_\lambda \leq \tau W_\lambda \) (by Corollary 2.11). By Proposition 2.14, there exists \( \bar{\tau} \) in \( \tau W_\lambda \) such that \( \sigma u \leq \bar{\tau} \). Taking \( \bar{\phi} = \phi v = \sigma u \), (3) is proved.

\( (3) \Rightarrow (2) \): Since \( \bar{\phi} \leq \tau \), it follows that \( \bar{\phi}^{-1} \) belongs to \( I(\bar{\tau}^{-1}) \). This implies that \( \min W_\lambda I(\bar{\tau}^{-1}) \bar{\phi} W_\mu \) equals identity. But \( w(\theta) = \min W_\lambda I(\bar{\tau}^{-1}) \bar{\phi} W_\mu \) by definition.

**4.3. Extremal paths.** Let \( \theta \) be a path in \( P_\lambda \ast P_\mu \) and let \( \eta \) be as in Proposition 4.4 above. Following Montagard [24], we call \( \theta \) extremal if the dominant Weyl conjugate \( \bar{\theta}(1) \) of the end point of \( \theta \) equals the end point \( \lambda + \pi(1) \) of \( \eta \).

The following observation [24, Theorem 2.2 (i)] applied to the path \( \pi_\lambda \ast \pi_\mu \) is already used in Littelmann’s proof [15, §7] of the PRV conjecture (here \( u \) denotes an element of \( W \) and \( \pi_\mu \) the straight line path to the extremal weight \( u \mu \) in \( V_\mu \)):

**Proposition 4.6.** If a path \( \theta \in P_\lambda \ast P_\mu \) lies entirely in the dominant Weyl chamber except perhaps for a portion of its last straight line segment, then \( \theta \) is extremal in the above sense.

**5. The KK (SUB)modules of \( V_\lambda \otimes V_\mu \)**

In this section we recall the definition of Kostant-Kumar (KK) modules and two basic results about them (Propositions 5.1 and 5.2).

Let \( g \) be a symmetrizable Kac-Moody algebra. Let \( \lambda, \mu \) be dominant integral weights. Let \( V_\lambda, V_\mu \) be the irreducible integrable \( g \)-modules with respective highest weights \( \lambda, \mu \). Let \( W_\lambda, W_\mu \) be the respective stabilizers of \( \lambda, \mu \) in the Weyl group \( W \).
5.1. **Filtration by KK modules of** $V_\lambda \otimes V_\mu$. Fix an element $w$ of the Weyl group. Let $v_\lambda$ be a highest weight vector in $V_\lambda$. Let $v_{w_\mu}$ be a non-zero vector in the (one-dimensional) weight space $V_{w_\mu}$ of weight $w_\mu$ in $V_\mu$. The Kostant-Kumar module, or simply KK module, $K(\lambda, w, \mu)$ is defined to be the cyclic submodule of the tensor product $V_\lambda \otimes V_\mu$ generated by $v_\lambda \otimes v_{w_\mu}$:

\[(3) \quad K(\lambda, w, \mu) := U_\mathfrak{g} (v_\lambda \otimes v_{w_\mu})\]

where $U_\mathfrak{g}$ denotes the universal enveloping algebra of $\mathfrak{g}$.

**Proposition 5.1.** Let $u$ and $w$ be elements of the Weyl group such that $W_\lambda u W_\mu = W_\lambda w W_\mu$. Then $K(\lambda, u, \mu) = K(\lambda, w, \mu)$.

**Proof:** For the proof, we will first recall a basic result from [8, §3.8]. Let $V$ be an integrable representation of $\mathfrak{g}$. For a simple reflection $s$, there is a corresponding linear automorphism $r^V$ of $V$ (defined in [8, Lemma 3.8]) such that:

1. For $v \in V$ a weight vector of weight $\eta$, $r^V(v)$ is a weight vector of weight $s(\eta)$.
2. $r^V \otimes r^{V'} = r^V \otimes r^{V'}$ for $V'$ an integrable representation.
3. For $v \in V$, there exists $x_v \in U_\mathfrak{g}$ such that $r^V(v) = x_v(v)$.

It suffices to show that $K(\lambda, u, \mu) \subseteq K(\lambda, w, \mu)$, for then the other containment also holds by interchanging the roles of $u$ and $w$. Write $u = \tau \omega \varphi$ with $\tau \in W_\lambda$ and $\varphi \in W_\mu$. We have $u_\mu = \tau \omega \varphi \mu = \tau w_\mu$. Let $\tau = s_{i_1} \cdots s_{i_k}$ be a reduced expression for $\tau$. Note that all $s_i$ belong to $W_\lambda$. Consider the operator $r_{i_1} \cdots r_{i_k}$ on $V_\lambda \otimes V_\mu$ where $r_{i_j} = r^V_{i_j}$ is the linear automorphism corresponding to $s_{i_j}$ (as recalled above).

On the one hand, by properties (1) and (2) above, we have:

\[r_{i_k} (v_\lambda \otimes v_{w_\mu}) = r^V_{i_k} (v_\lambda) \otimes r^V_{i_k} (v_{w_\mu}) = c \cdot v_{s_{i_k} \lambda} \otimes v_{s_{i_k} w_\mu} = c \cdot v_\lambda \otimes v_{s_{i_k} w_\mu}\]

where $c$ is a non-zero scalar. By a chain of similar calculations, we get

\[(4) \quad r_{i_1} \cdots r_{i_k} (v_\lambda \otimes v_{w_\mu}) = c' \cdot v_\lambda \otimes v_{\tau w_\mu} = c' \cdot v_\lambda \otimes v_{w_\mu}\]

where $c'$ is a non-zero scalar.

On the other hand, by property (3), there exist elements $x_{i_1}, \ldots, x_{i_k}$ of $U_\mathfrak{g}$ such that

\[(5) \quad r_{i_1} \cdots r_{i_k} (v_\lambda \otimes v_{w_\mu}) = x_{i_1} \cdots x_{i_k} (v_\lambda \otimes v_{w_\mu})\]

From (4) and (5) we get

\[v_\lambda \otimes v_{w_\mu} = c' \cdot r_{i_1} \cdots r_{i_k} (v_\lambda \otimes v_{w_\mu}) = c' \cdot x_{i_1} \cdots x_{i_k} (v_\lambda \otimes v_{w_\mu})\]

and thus $K(\lambda, u, \mu) = U_\mathfrak{g} (v_\lambda \otimes v_{w_\mu}) \subseteq U_\mathfrak{g} (v_\lambda \otimes v_{w_\mu}) = K(\lambda, w, \mu)$.

**Proposition 5.2.** For elements $u$ and $w$ of the Weyl group $W$ such that $W_\lambda u W_\mu \leq W_\lambda w W_\mu$ in the Bruhat order on $W_\lambda \backslash W / W_\mu$ (see §2.3), we have $K(\lambda, u, \mu) \subseteq K(\lambda, w, \mu)$. 

\[\square\]
PROOF: By Proposition 5.1, we may assume $u = \min W \lambda u W \mu$ and $w = \min W \lambda w W \mu$, so that $u \leq w$. Let $U_b(v_{w\mu})$ be the Demazure module generated by $v_{w\mu}$. Since $u \leq w$, we have $v_{u\mu} \in U_b(v_{w\mu})$. Thus $v_{\lambda} \otimes v_{u\mu} \in U_b(v_{\lambda} \otimes v_{w\mu}) \subseteq U_g(v_{\lambda} \otimes v_{w\mu})$, and $U_g(v_{\lambda} \otimes v_{u\mu}) \subseteq U_g(v_{\lambda} \otimes v_{w\mu})$.

 REMARK 5.3. The KK module $K(\lambda, 1, \mu) = U_g(v_{\lambda} \otimes v_{\mu})$ corresponding to the identity element 1 of the Weyl group is the copy of the irreducible representation $V_{\lambda + \mu}$ in $V_{\lambda} \otimes V_{\mu}$. When $g$ is of finite type, the KK module $K(\lambda, w_0, \mu)$ corresponding to the longest element $w_0$ of the Weyl group is the whole tensor product $V_{\lambda} \otimes V_{\mu}$. Indeed, letting $b_+$ and $b_-$ denote respectively the positive and negative Borel subalgebras, we have

$$U_g(v_{\lambda} \otimes v_{w_0\mu}) = U_{b_-} \cdot U_{b_+}(v_{\lambda} \otimes v_{w_0\mu}) = U_{b_-}(v_{\lambda} \otimes U_{b_+}v_{w_0\mu}) = U_{b_-}(v_{\lambda} \otimes V_{\mu}) = (U_{b_-}v_{\lambda}) \otimes V_{\mu} = V_{\lambda} \otimes V_{\mu}$$

6. RECALL OF A DECOMPOSITION RULE FOR KOSTANT-KUMAR (KK) MODULES

The decomposition rule (Theorem 6.1 below) that gives the break up of a KK module into a direct sum of irreducibles is well known. For example, at least in the case when $g$ is symmetric (i.e., has a symmetric generalized Cartan matrix), it follows immediately from Joseph’s results [7, Theorems 5.25, 5.22]. Our purpose in this section is to state the theorem and also give, for the sake of readability and completeness, a proof in the case when $g$ is of finite type. The restrictive hypothesis on $g$ (namely that it be of finite type or symmetric) is imposed only due to the use of a positivity result of Lusztig [19, 22.1.7] by Joseph in [7] and is possibly not required: see [7, §1.4].

**Theorem 6.1.** (Joseph [7]) Let $g$ be a symmetrizable Kac-Moody Lie algebra that is either of finite type or symmetric. Let $\lambda, \mu$ be dominant integral weights and $w$ an element of the Weyl group. Then the decomposition of the KK module $K(\lambda, w, \mu)$ as a direct sum of irreducible $g$-modules is given by

$$K(\lambda, w, \mu) = \bigoplus V_{\lambda + \pi(1)}$$

where the sum is over $\pi \in P^\lambda_{\mu}(w)$ where $P^\lambda_{\mu}(w)$ denotes the set of $\lambda$-dominant LS paths of shape $\mu$ with initial direction $\leq wW_{\mu}$ ($\lambda$-dominance of a path is defined in §4.2).

In the case when $g$ is symmetric, the theorem follows from Joseph’s results as already indicated (see also Naoi [25, Remark 2.12]). In the finite type case, a proof is recorded below (§6.2). For this proof, we need a result of Lakshmibai, Littelmann, and Magyar [12], which is a combinatorial analogue of the existence of “excellent filtrations”, a la Joseph [4], Mathieu [22,23], Polo [27], and van der Kallen [28]. We first recall this result.

6.1. A result of Lakshmibai-Littelmann-Magyar. In order to state the result, we introduce some notation. The term path in this section means a piecewise linear path whose endpoint lies in the weight lattice (for instance, a concatenation of LS paths of
various shapes. Let \( \mathcal{P} \) be a set of paths. We define its character, denoted \( \text{char} \mathcal{P} \), by:

\[
\text{char} \mathcal{P} := \sum_{\eta \in \mathcal{P}} e^{\eta(1)}.
\]

If \( \pi \) is any path, we let \( \pi \ast \mathcal{P} \) denote the set of paths \( \{\pi \ast \eta : \eta \in \mathcal{P}\} \).

Suppose \( \pi \) is a path such that \( \pi(t) \) belongs to the dominant Weyl chamber for all \( t \in [0,1] \). Fix a reduced word \( w = s_{\beta_1}s_{\beta_2} \cdots s_{\beta_k} \); here \( \beta_i \) are simple roots. Define

\[
C(\pi, w) := \{f_{\beta_1}^{n_1}f_{\beta_2}^{n_2} \cdots f_{\beta_k}^{n_k} \pi : n_i \geq 0 \text{ for all } i\}
\]

This set is independent of the reduced word chosen, and has character:

\[
\text{char} C(\pi, w) = \Lambda_w(e^{\pi(1)})
\]

where \( \Lambda_w \) is the Demazure operator corresponding to \( w \) (see, e.g., [15, §5.1]). Further, when \( \pi \) is the straight line path \( \pi_{\mu} \), we have \( C(\pi_{\mu}, w) = \mathcal{P}_{\mu}(w) \), the set of LS paths of shape \( \mu \) with initial direction \( \leq wW_{\mu} \). The following key result appears in [12, Proposition 12] (see also [6, Theorem in §2.11, also §3.5]).

**Proposition 6.2.** With notation as above, there exists a Weyl group valued function \( \pi \mapsto w(\pi) \) on \( \mathcal{P}_\lambda \) such that

\[
\pi_\lambda \ast C(\pi_{\mu}, w) = \bigsqcup_{\pi \in \mathcal{P}_{\mu}(w)} C(\pi_\lambda \ast \pi, w(\pi))
\]

(The precise form of the function \( \pi \mapsto w(\pi) \) is immaterial for our purposes.)

Computing characters of both sides in (7), we obtain:

\[
\text{char} \pi_\lambda \ast C(\pi_{\mu}, w) = \sum_{\pi \in \mathcal{P}_{\mu}(w)} \Lambda_w(\pi) (e^{\lambda+\pi(1)})
\]

### 6.2. Proof of Theorem 6.1 for \( g \) of finite type

By a result of Kumar [10, Theorem 2.14], the character of the KK module \( K(\lambda, w, \mu) \) is given by

\[
\text{char} K(\lambda, w, \mu) = \Lambda_{w_0}(e^{\lambda} \cdot \Lambda_w(e^{\mu}))
\]

where \( w_0 \) is the longest Weyl group element. Since \( \Lambda_w(e^{\mu}) \) is the character of \( C(\pi_{\mu}, w) \), we obtain

\[
\text{char} K(\lambda, w, \mu) = \Lambda_{w_0}(\text{char} \pi_\lambda \ast C(\pi_{\mu}, w))
\]

Substituting from (8) into (9), we obtain:

\[
\text{char} K(\lambda, w, \mu) = \sum_{\pi \in \mathcal{P}_{\mu}(w)} \Lambda_{w_0} \Lambda_w(\pi) (e^{\lambda+\pi(1)}) = \sum_{\pi \in \mathcal{P}_{\mu}(w)} \Lambda_{w_0}(e^{\lambda+\pi(1)})
\]

since \( \Lambda_{w_0} \Lambda_\sigma = \Lambda_{w_0} \) for all \( \sigma \in W \). This latter fact follows from the following well-known property of the Demazure operators: if \( \alpha \) is a simple root, then \( \Lambda_{s_\alpha} \Lambda_w \) equals \( \Lambda_{s_\alpha w} \) or \( \Lambda_w \) according as \( s_\alpha w \) is \( > w \) or \( < w \).
Figure 6.1. Decomposition of KK modules of $V_{2\omega_1} \otimes V_{2\omega_1+\omega_2}$ for $g$ of type $B_2$

But now $\Lambda_{w_0}(e^{\lambda+\pi(1)})$ is the character of the $g$-module $V_{\lambda+\pi(1)}$ (by the Demazure character formula applied to $w_0$). Thus the modules on both sides of (6) have the same character, and the proof is complete. □

Example 6.3. Consider the situation of Example 4.2 and Figure 4.1. Figure 6.1 depicts all $\lambda$-dominant paths of shape $\mu$, colour coded by their initial directions as follows:

| Initial direction | Colour coding |
|-------------------|--------------|
| identity          | violet       |
| $s_1$             | red          |
| $s_1s_2$          | cyan         |
| $s_1s_2s_1$       | orange       |

The KK modules decompose as follows:

- $K(\lambda, 1, \mu) = V_{\lambda+\mu}$.
- $K(\lambda, s_1, \mu) = V_{\lambda+\mu} \oplus V_{2\omega_1+2\omega_2} \oplus V_{3\omega_2}$.
- $K(\lambda, s_1s_2, \mu) = V_{\lambda+\mu} \oplus V_{2\omega_1+2\omega_2} \oplus V_{3\omega_2} \oplus V_{4\omega_1} \oplus V_{2\omega_1+\omega_2}$.
- $K(\lambda, s_1s_2s_1, \mu) = V_{\lambda+\mu} \oplus V_{2\omega_1+2\omega_2} \oplus V_{3\omega_2} \oplus V_{4\omega_1} \oplus V_{2\omega_1+\omega_2} \oplus V_{2\omega_1+2\omega_2} \oplus V_{4\omega_1} \oplus V_{2\omega_1+\omega_2}$.
7. A PATH MODEL FOR KOSTANT-KUMAR (KK) MODULES

We deduce a path model for KK modules by combining the decomposition rule (Theorem 6.1) with the invariance under the root operators (Proposition 3.3) of the association (1) of the Weyl group element \( \pi(\pi \ast \pi') \) to a concatenation \( \pi \ast \pi' \) of two LS paths. The restriction on \( g \) (namely, that it be of finite type or symmetric) in the theorem is only because of the use of the decomposition rule, and is possibly not required.

**Theorem 7.1.** Let \( g \) be a symmetrizable Kac-Moody algebra that is either of finite type or symmetric (as in Theorem 6.1). Let \( \lambda, \mu \) be dominant integral weights and \( w \) an element of the Weyl group. Let \( P_\lambda \) and \( P_\mu \) respectively be the sets of Lakshmibai-Seshadri paths of shapes \( \lambda \) and \( \mu \). For \( \pi \in P_\lambda \) and \( \pi' \in P_\mu \), let \( w(\pi \ast \pi') \) be the Weyl group element associated as in (1) in §3.1 to the concatenated path \( \pi \ast \pi' \). Then the KK set

\[
P(\lambda, w, \mu) = \{\pi \ast \pi' | \pi \in P_\lambda, \pi' \in P_\mu, w(\pi \ast \pi') \leq w\}
\]

is a path model for the KK module \( K(\lambda, w, \mu) \) in the sense that

\[
\text{char } K(\lambda, w, \mu) = \sum_{\eta \in P(\lambda, w, \mu)} \exp \eta(1)
\]

**Proof:** From Theorem 6.1, we have:

\[
\text{char } K(\lambda, w, \mu) = \sum_{\pi \in P_\lambda^\mu(w)} \text{char } V_{\lambda + \pi(1)}
\]

where \( P_\mu^\lambda(w) \) is the set of \( \lambda \)-dominant LS paths of shape \( \mu \) with initial direction \( \leq w \). For \( \pi \in P_\mu^\lambda(w) \), let \( P_{\pi \lambda \ast \pi} \) be the equivalence class in \( P_\lambda \ast P_\mu \) containing \( \pi_\lambda \ast \pi \) (under the equivalence relation \( \sim \) defined by the root operators—see §3.2.1), where \( \pi_\lambda \) denotes the straight line path from the origin to \( \lambda \). Since \( \pi_\lambda \ast \pi \) lies entirely in the dominant Weyl chamber (this is what it means for \( \pi \) to be \( \lambda \)-dominant), it follows from the “Isomorphism Theorem” in [16, Theorem 7.1] that

\[
\sum_{\sigma \in P_{\pi_\lambda \ast \pi}} \exp \sigma(1) = \sum_{\sigma \in P_{\lambda + \pi(1)}} \exp \sigma(1)
\]

(13)

(12)

(14)

Thus, for the proof of the theorem, it suffices to show the following:

\[
\mathcal{P}(\lambda, w, \mu) = \bigcup_{\pi \in P_\mu^\lambda(w)} P_{\pi \lambda \ast \pi} \quad \text{(disjoint union)}
\]
To prove (15), first let \( \pi \in P_\mu^\lambda(w) \). Let \( u \) be an element of the Weyl group such that \( u\mu \) is the initial direction of \( \pi \). From our assumption that \( uW_\mu \leq wW_\mu \), it follows that \( W_\lambda uW_\mu \leq W_\lambda wW_\mu \) (see Corollary 2.11) and \( w(\pi_\lambda \star \pi) \leq w \) (evidently \( w(\pi_\lambda \star \pi) \) is the minimal element in \( W_\lambda uW_\mu \)). By Proposition 3.3, it follows that the Weyl group elements associated via \( w \) to elements of \( P_\pi^\lambda \) are all the same. This proves \( P(\lambda, w, \mu) \supsetneq P(\pi_\lambda \star \pi) \).

Now let \( \varphi \) be an element in \( P(\lambda, w, \mu) \). Apply Proposition 4.4 to \( \varphi \) and let \( \eta \) be as in the conclusion. Then \( \eta = \pi_\lambda \star \pi \) for some \( \pi \in P_\mu^\lambda(w) \), and the containment \( \subseteq \) is proved.

That the union on the right hand side of (15) is disjoint follows from the uniqueness of \( \eta \) in Proposition 4.4 (which in turn rests on [16, Corollary 1 (b) of §7]): \( \pi_\lambda \star \pi \) is the unique path in \( P_\pi^\lambda \) on which \( e_\alpha \) vanishes for all simple roots \( \alpha \). □

8. PRV COMPONENTS AND GENERALISED PRV COMPONENTS IN KK MODULES

We show how the decomposition rule (Theorem 6.1) leads easily to results about the existence of PRV components (Theorem 8.1) and generalised PRV components (Theorem 8.3) in KK modules. The arguments are well known: see e.g. those by Joseph in [5, §2.7]. In fact, Theorem 8.1 for the finite case follows from items (i) and (iii) of the theorem in [5, §2.7].

Theorem 8.1 is at once a generalisation of two results: the so called refined PRV and KPRV theorems:

- Its special case when \( g \) is of finite type and \( w = w_0 \) (the longest element of the Weyl group) is due to Kumar [11, Theorem 1.2], who refers to his result as “a refinement of the PRV conjecture” and says that it was conjectured by D.-N. Verma.
- The special case when \( \sigma = w \) was proved by Kumar [10, page 117] and independently Mathieu [21, Corollaire 3]. Kumar calls it “the strengthened PRV conjecture (due to Kostant)”. We have called it “KPRV” following Khare [9].

Theorem 8.3 is a KK version of Montagard’s result [24, Theorem 3.1] about generalised PRV components.

8.1. The map \( \Phi \). Let \( g \) be a symmetrizable Kac-Moody algebra, and fix dominant integral weights \( \lambda \) and \( \mu \). Let \( W_\lambda \) and \( W_\mu \) denote respectively the stabilizers in the Weyl group \( W \) of \( \lambda \) and \( \mu \).

Consider the map from the Weyl group \( W \) to the set \( \Lambda^+ \) of dominant integral weights given by \( \sigma \mapsto \lambda + \sigma\mu \), where \( \lambda + \sigma\mu \) denotes the dominant Weyl conjugate of the weight \( \lambda + \sigma\mu \). This map factors through the natural quotient map from \( W \) to \( W_\lambda \backslash W/W_\mu \). We denote by \( \Phi \) the map \( W_\lambda \backslash W/W_\mu \to \Lambda^+ \) given by \( W_\lambda \sigma W_\mu \mapsto \lambda + \sigma\mu \).

8.2. PRV components in KK modules. The restrictive hypothesis on \( g \) in the following theorem (as also in Theorem 8.3), namely that it be either of finite type or symmetric, is inherited from the decomposition rule (Theorem 6.1) and is possibly not required.
Theorem 8.1. Let $\mathfrak{g}$ be a symmetrizable Kac Moody algebra that is either of finite type or symmetric (as in Theorem 6.1). Let $\lambda$, $\mu$ be dominant integral weights and $w$, $\sigma$ be elements of the Weyl group. Let $v$ be the dominant Weyl conjugate of the weight $\lambda + \sigma \mu$. Then the irreducible $\mathfrak{g}$-module $V_v$ occurs in the decomposition into irreducible $\mathfrak{g}$-modules of the KK module $K(\lambda, w, \mu)$ at least as many times as there are elements $\tau \in W_\lambda \setminus W/W_\mu$ such that $\tau \leq W_\lambda w W_\mu$ and $\Phi(\tau) = v$.

**Proof:** We describe a map $\tilde{\Phi}$ from $W_\lambda \setminus W/W_\mu$ to the set $P_\mu^\lambda$ of $\lambda$-dominant LS paths of shape $\mu$. Given $\tau \in W_\lambda \setminus W/W_\mu$, let $v$ be the unique minimal element in $\tau$. Consider the path $\varphi = \pi_\lambda \ast \pi_\nu \mu$ in $P \ast P_\mu$, where $\pi_\lambda$ and $\pi_\nu \mu$ are the straight line paths from the origin to $\lambda$ and $\nu \mu$ respectively. Note that $\nu(\varphi) = v$. Apply Proposition 4.4 to $\varphi$ and let $\eta$ be as in its conclusion. Then $\eta = \pi_\lambda \ast \pi$ for some $\pi \in P_\mu^\lambda$ and $\nu(\eta) = v$. We define $\tilde{\Phi}(\tau) := \pi$. Since $\pi$ determines $\eta$, this is the end point of $\tilde{\Phi}(\tau)$ shifted by $\lambda$ for any $\tau$ in $W_\lambda \setminus W/W_\mu$. Combining this fact with the injectivity of $\tilde{\Phi}$, we immediately obtain the theorem. □

8.3. **KPRV recovered.** It follows immediately from the theorem that $V_v$ occurs at least once in $K(\lambda, \sigma, \mu)$. To show that it occurs at most once, we repeat the following elementary argument from [10, §2.7]. Indeed in any $\mathfrak{g}$-homomorphism from $K(\lambda, \sigma, \mu)$ module to $V_v$, the vector $v_\lambda \otimes v_\nu \mu$ has to map to an element of weight $\lambda + \sigma \mu$. But the dimension of the $\lambda + \sigma \mu$-weight space in $V_v$ is clearly one, since $\lambda + \sigma \mu$ is a Weyl conjugate of $v$. Thus the space of $\mathfrak{g}$-homomorphisms from $K(\lambda, \sigma, \mu)$ to $V_v$ is one dimensional. We thus have:

**Corollary 8.2.** Let $\lambda$, $\mu$, $\sigma$, and $v$ be as in Theorem 8.1. Then the irreducible $\mathfrak{g}$-module $V_v$ occurs exactly once in the decomposition into irreducible $\mathfrak{g}$-modules of the KK module $K(\lambda, \sigma, \mu)$.

8.4. **Generalised PRV components in KK modules.** Importing to our context a result of Montagard [24, Theorem 3.1], we prove the following:

Theorem 8.3. Let $\mathfrak{g}$, $\lambda$, and $\mu$ be as in Theorem 8.1. Let $v, u$ be elements in the Weyl group and $\beta$ a positive root such that either $v^{-1} \beta$ or $u^{-1} \beta$ is a simple root. Let $k$ be an integer such that $0 \leq k \leq \min\{\langle \nu \lambda, \beta^\vee \rangle, \langle u \mu, \beta^\vee \rangle\}$ and the integral weight $v = \nu \lambda + u \mu - k \beta$ is dominant. Then the irreducible $\mathfrak{g}$-module $V_v$ occurs in the decomposition of the KK module $K(\lambda, w, \mu)$ into irreducibles where $w = v^{-1} s_\beta u$.

**Proof:** First suppose that $v^{-1} \beta$ is simple. Since $k \leq \langle v^{-1} u \mu, v^{-1} \beta^\vee \rangle$, it follows that $f_{v^{-1} \beta}^k \pi_{v^{-1} u \mu}$ does not vanish. Consider the path $\varphi := \pi_\lambda \ast f_{v^{-1} \beta}^k \pi_{v^{-1} u \mu}$ in $P \ast P_\mu$. As is easily verified, the dominant Weyl conjugate of $\varphi(1)$ is $v$ and $\nu(\varphi)$ is either $\min W_\lambda v^{-1} s_\beta u W_\mu$ or $\min W_\lambda v^{-1} u W_\mu$ depending upon whether $k > 0$ or $k = 0$. An easy verification (using the fact that $\langle u \mu, \beta^\vee \rangle \geq 0$ under our assumptions) shows that $w = v^{-1} s_\beta u \geq v^{-1} u$. Thus $w \geq \nu(\varphi)$ in either case, and $\varphi \in P(\lambda, w, \mu)$. 

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Apply Proposition 4.4 to the path \( \varphi \) and let \( \eta \) be as in its conclusion. Then \( \eta \) is of the form \( \pi_\lambda \ast \pi \) with \( \pi \in \mathcal{P}_\mu^*(w) \). By the decomposition rule (Theorem 6.1), \( V_{\eta(1)} \) occurs in \( K(\lambda, w, \mu) \). But Montagard [24, Proof of Theorem 3.1] shows that \( \varphi \) is extremal, which means that \( \nu = \varphi(1) = \eta(1) \), and the proof is done in this case.

Now suppose that \( u^{-1} \beta \) is simple. Then, applying the result in the previous case, we conclude that \( V_\nu \) occurs in the \( KK \) submodule \( K(\mu, w^{-1}, \lambda) \) of \( V_\mu \otimes V_\lambda \). But under the \( g \)-isomorphism \( a \otimes b \leftrightarrow b \otimes a \) of \( V_\mu \otimes V_\lambda \) with \( V_\lambda \otimes V_\mu \), the submodules \( K(\mu, w^{-1}, \lambda) \) and \( K(\lambda, w, \mu) \) map isomorphically to each other.

**Remark 8.4.** In the set up of the theorem, suppose we assume that \( \beta \) is simple rather than that either \( v^{-1} \beta \) or \( u^{-1} \beta \) is simple. In that case too [24, Theorem 3.1] says that \( V_\nu \) occurs in \( V_\lambda \otimes V_\mu \). We have not handled that case.

**Example 8.5.** We illustrate the result of Theorem 8.3 and also the idea behind its proof by means of an example borrowed from Montagard [24]. Let the root system of \( g \) be \( G_2 \). Let \( e_1 \) and \( e_2 \) be the standard basis vectors of \( \mathbb{R}^2 \) with its standard inner product. We may take \( \alpha_1 := e_1 \) and \( \alpha_2 := -\frac{3}{2}e_1 + \frac{\sqrt{3}}{2}e_2 \) to be the simple roots. The set of all positive roots is:

\[
\{ \alpha_2, \alpha_1 + \alpha_2, \alpha_2 + 2\alpha_1, \alpha_2 + 3\alpha_1, 2\alpha_2 + 3\alpha_1 \}
\]

The dominant integral weights \( \lambda, \mu, \nu \), the Weyl group elements \( u, v, w \), the root \( \beta \), and the integer \( k \) are all as shown in Figure 8.1. The path \( \eta \) ending at \( \nu \) appears in bold.

## 9. Tableau Decomposition Rule for Kostant-Kumar (KK) Modules

Fix an integer \( d \geq 2 \). Let \( g = sl_d \), the simple Lie algebra of traceless complex \( d \times d \) matrices. There is, in this special case, the classical Littlewood-Richardson (LR for short) rule (see e.g. [3, 20]) that gives, in terms of tableaux, the decomposition into irreducibles of the tensor product of two finite dimensional irreducible representations of \( g \). The multiplicities of the irreducibles in this rule are called “LR coefficients” and they count certain “LR tableaux”. Our purpose in this section is to deduce, from the general decomposition rule (Theorem 6.1), a version of this classical rule, which we call the “refined LR rule”, for decomposing as a direct sum of irreducibles any \( KK \) submodule of the tensor product: see §9.4 for the statement. We call the multiplicities of the irreducibles in this refined rule the “refined LR coefficients”.

The refined LR coefficients also count certain LR tableaux. The identification of the set of LR tableaux to be counted is based upon the association of a permutation to each LR tableau (§9.6). There is a very natural association of a semi-standard Young tableau (SSYT) to an LR tableau (§9.6.1) and the permutation is just the initial element of the minimal standard lift when the SSYT is interpreted as a standard concatenation of LS paths (§A.2).

In the light of this last mentioned fact, it is noteworthy that the procedure we give for determining the permutation (§9.5.1) from the SSYT is not a repeated application of
Deodhar’s lemma (Proposition 2.17): it seems to be more efficient than that. Lascoux and Schützenberger [14] associate to each SSYT a “right key” (which by definition is another SSYT) from which the permutation can be read off. Willis [29] gives an alternative method—“the scanning method”—for finding the right key of an SSYT. Our procedure is different from those in [14, 29].

9.1. Preliminaries. The choices involved (Cartan subalgebra $\mathfrak{h}$, Borel subalgebra $\mathfrak{b}$, etc.) are fixed as usual: the subalgebra of diagonal (respectively, upper triangular) traceless complex $d \times d$ matrices is taken to be $\mathfrak{h}$ (respectively, $\mathfrak{b}$). We denote by $\epsilon_j$ the linear functional on $\mathfrak{h}$ that maps a matrix to its entry in position $(j, j)$.

Recall that a partition is a weakly decreasing sequence $\lambda_1 \geq \lambda_2 \geq \ldots$ (sometimes also written $\lambda_1 + \lambda_2 + \cdots$) of non-negative integers that is eventually zero. The non-zero elements of the sequence are called the parts. We tacitly identify partitions with their (Young) shapes. To a partition $\lambda : \lambda_1 \geq \ldots \geq \lambda_d \geq 0 \geq \ldots$ with at most $d$ parts, we attach

**Figure 8.1.** $V_v \in K(\lambda, w, \mu)$
the dominant integral weight $\lambda_1 \epsilon_1 + \cdots + \lambda_d \epsilon_d$. A second such partition $\lambda'_1 \geq \cdots \geq \lambda'_d \geq 0 \geq \ldots$ corresponds to the same weight as $\lambda$ if and only if $\lambda_1 - \lambda'_1 = \cdots = \lambda_d - \lambda'_d$ (since $\epsilon_1 + \ldots + \epsilon_d = 0$ is evidently the only linear dependence relation up to scaling on $\epsilon_1, \ldots, \epsilon_d$). Thus partitions with less than $d$ parts are in one-to-one correspondence with dominant integral weights. We will abuse notation and use the same symbol for both a partition with less than $d$ parts and the corresponding dominant integral weight.

Let $[j] \colonequals \{1, \ldots, j\}$ for any integer $j \geq 1$. The Weyl group is identified with the group of permutations of the set $[d]$. The one line notation for a permutation $w$ of $[d]$ is $w_1 \ldots w_d$, where $w_j \colonequals w(j)$ (for $1 \leq j \leq d$).

9.2. Semi-standard Skew tableaux (SSST for short). Let $\nu$ and $\lambda$ be two partitions with the shape of $\nu$ containing the shape of $\lambda$. A (semi-standard) skew tableau, SSST for short, of shape $\nu/\lambda$ is a filling up by positive integers of those boxes that are in the shape of $\nu$ but not in the shape of $\lambda$ such that the entries in each row are weakly increasing rightward and those in each column are strictly increasing downward. Here are two examples with $\nu = 7 + 5 + 4 + 3 + 1$ and $\lambda = 4 + 4 + 1 + 1$:

\[
\begin{array}{cccc}
\cdot & \cdot & \cdot & 2 \\
\cdot & \cdot & 7 \\
3 & 3 & 5 \\
4 & 6 \\
1
\end{array}
\quad
\begin{array}{cccc}
\cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & 2 \\
1 & 2 & 3 \\
3 & 4 \\
1
\end{array}
\]

(16)

9.2.1. Reverse reading words and ballot sequences. Let $T$ be a SSST of shape $\nu/\lambda$. Its reverse reading word, denoted $w_{\text{row}}(T)$, is defined as follows: read the entries of $T$ from right to left in every row, scanning the rows from top to bottom. For the two SSSTs in the display above, the reverse reading words respectively are:

\[
\begin{align*}
3 & 2 & 2 & 7 & 5 & 3 & 3 & 6 & 4 & 1 \\
1 & 1 & 1 & 2 & 3 & 2 & 1 & 4 & 3 & 1
\end{align*}
\]

(17)

The word $w_{\text{row}}(T)$ (or, more generally, any word in the positive integers) is said to be a ballot sequence if for any integer $j \geq 1$ the number of times $j$ occurs up to any point in the word (while scanning it from left to right) is at least the number of times $j + 1$ occurs up to that point. In (17), the word on the left is not a ballot sequence but the one on the right is.

9.2.2. Type and weight of a word and of a SSST. The type of any word $w$ in the positive integers is the sequence $\mu: \mu_1, \mu_2, \ldots$, where $\mu_j$ denotes the number of occurrences of $j$ in $w$. The type of the word on the left in (17) is $1, 2, 3, 1, 1, 1, 0, 0, \ldots$. Evidently, permuting the letters of a word does not change its type. If $w$ is a word in $[d]$, then we may further associate to it the integral weight $\mu_1 \epsilon_1 + \cdots + \mu_d \epsilon_d$ of $g = sl_d$. This is called the weight of the word and denoted $\text{wt}(w)$.
The type and weight of a SSST $T$ are defined respectively to be the type and weight of its reverse reading word $\mathbf{w}_{\text{row}}(T)$.

If the word $\mathbf{w}$ is a ballot sequence, then its type is a partition: $\mu_1 \geq \mu_2 \geq \ldots$, and in this case we use the notation for partitions to denote types. For example, the type of the word on the right in (17) is $\mu = 5 + 2 + 2 + 1$. The weight of such a word in $[d]$ is dominant.

9.3. Littlewood-Richardson (LR for short) tableaux and coefficients. An LR tableau (LR is short for Littlewood-Richardson) is a SSST $T$ whose reverse reading word $\mathbf{w}_{\text{row}}(T)$ is a ballot sequence. Let $\lambda$ and $\mu$ be partitions. Let $\mathcal{T}^\lambda_\mu$ denote the set of LR tableau of shape $\nu/\lambda$ and type $\mu$—here $\nu$ is allowed to vary. If $T$ in $\mathcal{T}^\lambda_\mu$ has shape $\nu/\lambda$, we write $\nu(T)$ for $\nu$. As is well-known, $\mathcal{T}^\lambda_\mu$ has representation theoretic and geometric significance. For example (see e.g. [3, 20]) $s_\lambda s_\mu = \sum_{T \in \mathcal{T}^\lambda_\mu} s_{\nu(T)}$, where $s_\tau$ denotes the Schur function associated to a partition $\tau$.

For a fixed partition $\nu$, the number of $T$ in $\mathcal{T}^\lambda_\mu$ with $\nu(T) = \nu$ is usually denoted $c^\nu_{\lambda\mu}$. The numbers $c^\nu_{\lambda\mu}$ are called LR coefficients. In terms of these, we may write the the above rule for multiplication of Schur functions as $s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$.

9.3.1. Bruhat order on permutations. Any permutation $u$ of $[j]$ (for some integer $j \geq 1$) can naturally be considered as a permutation of $[k]$, for any integer $k \geq j$. Given two permutations $u$ and $u'$ of $[j]$ and $[j']$ respectively, we write $u \leq u'$ if that is so in the Bruhat order on permutations of $[k]$ for some $k \geq \max j, j'$. If $u \leq u'$ for some such $k$, then it is so for all such $k$.

9.3.2. Refined Littlewood-Richardson coefficients: their definition. In §9.6 below, we specify a procedure that assigns a permutation $u$ to a given SSST $T$.

Fix a permutation $w$ and let $\mathcal{T}^\lambda_\mu(w)$ denote the subset of $\mathcal{T}^\lambda_\mu$ consisting of those elements for which the associated permutation $u$ satisfies $u \leq w$ (in the Bruhat order as defined in §9.3.1 above). The result (18) below ascribes representation theoretic meaning to $\mathcal{T}^\lambda_\mu(w)$.

For a fixed partition $\nu$, we denote by $c^\nu_{\lambda\mu}(w)$ the number of $T$ in $\mathcal{T}^\lambda_\mu(w)$ with $\nu(T) = \nu$. We call the numbers $c^\nu_{\lambda\mu}(w)$ refined LR coefficients.

9.4. Tableau decomposition rule for KK modules. Suppose that $\lambda$, $\mu$ are partitions with less than $d$ parts (or, equivalently, dominant integral weights for $g = sl_d$) and that $w$

---

1 Later on we will introduce the “column word” $\mathbf{w}_{\text{col}}(T)$ of $T$, which being a permutation of $\mathbf{w}_{\text{row}}(T)$ shares its type and weight.

2 It is easy to associate to $T$ a SSYT $S$ of shape $\mu$—see §9.6.1. Interpreting $S$ as a standard concatenation of LS paths in the sense of Proposition A.3 in the appendix, the associated permutation $u$ is just the initial element of the minimal standard lift of $S$, as will be proved in §10. Observe that, if as in §9.4 the number of parts in $\nu$ is at most $d$, then the entries in $S$ and the number of parts in $\mu$ are also bounded above by $d$, so that the interpretation of $S$ as a concatenation of LS paths associated to $g = sl_d$ is possible, and $u$ is a permutation of $[d]$. 
is a permutation of \([d]\) (or, equivalently, an element of the Weyl group). Then the decomposition of the Kostant-Kumar module \(K(\lambda, w, \mu)\) (defined in §5.1) as a direct sum of irreducible \(g\)-modules is given by:

\[
K(\lambda, w, \mu) = \bigoplus_{T \in T^d_\mu(w)} V_{\nu(T)}
\]

where \(V_{\nu(T)}\) is interpreted to be zero in case \(\nu(T)\) has more than \(d\) parts. (Recall from §9.1 that to any partition with at most \(d\) parts there is associated a dominant integral weight of \(g\).)

Here is an alternative way to express the above decomposition rule:

\[
K(\lambda, w, \mu) = \bigoplus_{\nu} V_{\nu} \oplus_{\nu} \lambda^\mu (w) \nu
\]

where the sum runs over all partitions \(\nu\) with less than \(d\) parts, and \(\nu\) depending on \(\nu\) denotes the unique partition with at most \(d\) parts (if it exists) such that

\[
\nu_j = v_j - v_d \quad \text{for } 1 \leq j < d \quad \text{and} \quad v_1 + \cdots + v_d = (\lambda_1 + \cdots + \lambda_{d-1}) + (\mu_1 + \cdots + \mu_{d-1})
\]

The proof of (18) will be given below in §9.9.

9.4.1. The statement for polynomial representations of \(GL_d(\mathbb{C})\). For convenience of reference, we now state, without proof, a version of the decomposition rule (18) for polynomial representations of the general linear group \(GL_d(\mathbb{C})\). Suppose that \(\lambda, \mu\) are partitions with at most \(d\) parts and \(V_\lambda, V_\mu\) the corresponding irreducible polynomial representations. Let \(w\) be a permutation of \([d]\). Then the decomposition of the Kostant-Kumar module \(K(\lambda, w, \mu)\) (defined similarly as in §5.1) as a direct sum of irreducible polynomial representations is given by:

\[
K(\lambda, w, \mu) = \bigoplus_{\nu} V_{\nu} \oplus_{\nu} \lambda^\mu (w) \nu
\]

where the sum runs over all partitions \(\nu\) with at most \(d\) parts.

9.4.2. An example. Here is a simple example illustrating the rules (18) and (19). Let \(d = 3\), \(\lambda = 2 + 1\), and \(\mu = 3 + 1\). As the reader can readily verify, there are 7 elements \(T\) in \(T^3_\mu\) with \(\nu(T)\) having at most 3 parts. These are listed below along with the permutations of \([3]\) attached to them (as in §9.6):

- \(\begin{array}{ccc} \cdot & \cdot & 1 & 1 & 1 \\ \cdot & 2 \\ \end{array} \leftrightarrow 123\)
- \(\begin{array}{ccc} \cdot & \cdot & 1 & 1 \\ \cdot & 1 & 2 \\ \end{array} \leftrightarrow 213\)
- \(\begin{array}{ccc} \cdot & 1 & 1 & 1 \\ \cdot \\ 2 \\ \end{array} \leftrightarrow 132\)
And so we have:

\[ K(\lambda, 123, \mu) = V_{5+2} \]
\[ K(\lambda, 213, \mu) = V_{4+3} \oplus V_{5+2} \]
\[ K(\lambda, 132, \mu) = V_4 \oplus V_{5+2} \]
\[ K(\lambda, 231, \mu) = V_{3+1} \oplus V_4 \oplus V_{4+3} \oplus V_{5+2} \]
\[ K(\lambda, 321, \mu) = V_1 \oplus V_{2+2} \oplus V_{3+1} \oplus V_4 \oplus V_{4+3} \oplus V_{5+2} \]

9.5. **SSYT and permutations attached to them.** Let \( \mu \) be a partition. A *semi-standard Young tableau, SSYT for short,* of shape \( \mu \) is just a (semi-standard) skew tableau of shape \( \mu/\emptyset \) in the sense of §9.2. Here is an example of a SSYT of shape \( \mu = 4+2+1 \):

\[
\begin{array}{cccc}
1 & 3 & 6 & 8 \\
2 & 4 \\
7
\end{array}
\]

9.5.1. Associating a permutation to a given SSYT. Let \( S \) be a SSYT of shape \( \mu \) and let \( m \) be the largest entry of \( S \). We associate to \( S \) a permutation \( u \) of \( [m] \), as follows. Let \( \mu'_1 \) be the number of parts in \( \mu \). Observe that \( m \geq \mu'_1 \) since the entries in every column of \( S \) are strictly increasing downwards.

Let \( u_1 u_2 \ldots u_m \) be the one-line notation for \( u \). We will describe below an inductive procedure to produce the sequence \( u_1, \ldots, u_{\mu'_1} \). As for \( u_{\mu'_1+1}, \ldots, u_m \), we take these to be just the elements of \( [m] \setminus \{u_1, \ldots, u_{\mu'_1}\} \) arranged in increasing order.

It is easy to produce \( u_1 \): it is just the largest (right most) entry in the first row of \( S \). Suppose that \( u_1, u_2, \ldots, u_{p-1} \) have been produced (with \( 1 < p \leq \mu'_1 \)). We now describe a procedure to determine \( u_p \).

Let \( b \) be a box in \( S \). Suppose that a box \( b' \) in \( S \) is weakly to the Northeast of \( b \) and has an entry that is less than that of \( b \). Then we write \( b \succ b' \). For example, in the SSYT of (21), if \( b \) is the one with entry 7, then \( b' \) could be any of those containing 1, 2, 3, 4, or 6; if \( b \) is the one with entry 4, then \( b' \) could only be the one containing 3.

The \( b \)-depth of such a box \( b' \) is defined to be the largest \( \delta \) such that there is a chain \( b \succ b_1 \succ b_2 \succ \ldots \succ b_\delta = b' \). The \( b \)-depth of \( b \) itself is defined to be 0.

Let \( b^p \) denote the right most box in row \( p \). We write \( p \)-depth for \( b^p \)-depth. For \( 1 \leq j \leq p \), we let \( y_j \) be maximal possible entry in a box whose \( p \)-depth is \( p-j \). (The box in row \( j \) in the same column as \( b^p \) has \( p \)-depth \( p-j \), so \( y_j \) exists.) By definition, \( y_p \) is the entry in the box \( b^p \). As is easily seen, \( y_1 < \ldots < y_p \). We call this the \( p \)-depth sequence of \( S \).

Let \( a_1 < \ldots < a_{p-1} \) be the elements \( u_1, \ldots, u_{p-1} \) arranged in increasing order. Let \( k \), \( 1 \leq k \leq p \), be the largest such that \( a_{k-1} < y_k \). (\( a_0 = -\infty \) by convention). Take \( u_p \) to be \( y_k \).

**Proposition 9.1.** With notation as above, we evidently have:
• \( a_1 < \ldots < a_{k-1} < y_k < a_k < \ldots < a_{p-1} \).
• \( u_p \) is distinct from \( u_1, \ldots, u_{p-1} \).

\[ \square \]

**Remark 9.2.** The element \( y_j \) in the \( p \)-depth sequence of \( S \) is just the entry in the right most box of \( p \)-depth \( p - j \): “right most box” means box in the right most column; since no two boxes in the same column have the same \( p \)-depth, this is well defined. Indeed let \( b \) be that box and \( e \) its entry. Clearly \( e \leq y_j \). To show \( y_j \leq e \), first observe that no column to the right of the one containing \( b \) has a box of \( p \)-depth \( p - j \) (by choice of \( b \)); secondly that \( e \) dominates the entry in any box that is weakly to the Northwest of \( b \) (since \( S \) is a SSYT); and finally that any box of \( p \)-depth \( p - j \) strictly South and weakly West of \( b \) can only have an entry that is at most \( e \) (for otherwise the \( p \)-depth of \( b \) would exceed \( p - j \)).

9.5.2. Illustration of the procedure above. Let \( S \) be the SSYT in (21). The permutation associated to it is 83612457 in one-line notation. Evidently \( u_1 = 8 \) and \( \mu'_1 = 3 \); the 2-depth sequence is \( 3 < 4 \) and \( u_2 = 3 \); the 3-depth sequence is \( 3 < 6 < 7 \) and \( u_3 = 6 \).

9.5.3. A technical result that will be used later. The following lemma will be invoked later on, in Example 10.4.

**Lemma 9.3.** Let \( S \) be a SSYT and \( q \) the number of boxes in its right most column. Let \( S' \) be the SSYT obtained from \( S \) by deleting its last column. Fix \( p > q \). If in the procedure for producing \( u_p \) (where \( u \) is the permutation associated to \( S \)), we use the \( p \)-sequence of \( S' \) in place of that of \( S \), it makes no difference (that is, we still get the same \( u_p \)).

**Proof:** Let \( y_1 < \ldots < y_p \) and \( y'_1 < \ldots < y'_p \) be the \( p \)-depth sequences of \( S \) and \( S' \) respectively, and suppose that \( u_p = y_k \). Since the entries in the last column of \( S \) all belong to \( \{ u_1, \ldots, u_{p-1} \} \) but, by Proposition 9.1, \( y_k \) does not belong to that set, it follows that any box of \( S \) with \( y_k \) as its entry belongs to \( S' \). Thus \( y_k = y'_k \).

On the other hand, \( y'_j \leq y_j \leq a_{j-1} \) for all \( j > k \) (where \( a_1 < \ldots < a_{p-1} \) is the arrangement in increasing order of \( u_1, \ldots, u_{p-1} \)), so \( k \) is the largest such that \( a_{k-1} < y'_k \). \[ \square \]

9.6. **Association of permutations to LR tableaux.** Recall that the definition in §9.3.2 of refined LR coefficients refers to a certain association of permutations to LR tableaux. We describe this association now, after first associating SSYTs to LR tableaux.

Let \( T \) be an LR tableau of shape \( v/\lambda \) and type \( \mu \). If \( v \) has at most \( d \) parts, then so has \( \mu \), for each entry on row \( j \) of \( T \) is at most \( j \) (for all \( j \geq 1 \)).

9.6.1. The SSYT associated to \( T \). We associate to \( T \) a SSYT \( S \) of shape \( \mu \) as follows. The entries in row \( j \) of \( S \) from left to right are just the row numbers of \( T \) in which the entry \( j \) appears, counted with multiplicity and arranged in weakly increasing order. That the entries in every column of \( S \) are strictly increasing downward follows readily from the assumption that the reverse reading word of \( T \) is a ballot sequence: indeed, for integers \( k \geq 1 \) and \( j \geq 2 \), if the \( k \)-th appearance of \( j \) (as we read the reverse reading word from left to right) is in row \( r \), then the \( k \)-th appearance of \( j - 1 \) is in some row strictly above the \( r \)-th.
9.6.2. The permutation associated to $T$. Consider the permutation $u$ associated as in §9.5.1 to the SSYT $S$. We associate $u$ to $T$ itself. For example, for the skew tableau on the right in (16), the associated SSYT is the one shown below and the associated permutation is 51324:

\[
\begin{array}{cccc}
1 & 1 & 1 & 3 \\
2 & 3 \\
3 & 4 \\
4 \\
\end{array}
\]

(22)

9.7. $p$-dominance of words. Let $p$: $p_1 \geq p_2 \geq \ldots$ be a partition. We denote by $w(p)$ the word (in the positive integers) that has $p_1$ ones, $p_2$ twos, \ldots in succession: this is just the reverse reading word of the SSYT of shape $p$ all of whose entries in row $j$ are $j$ (for all $j$). Note that $w(p)$ is a ballot sequence.

A word (in the positive integers) is said to be $p$-dominant if when preceded by $w(p)$ the resulting word is a ballot sequence.

**Proposition 9.4.** For a given word $w$ there is a unique smallest partition $p_w$ such that $w$ is $p_w$-dominant ($p_w$ is the smallest in the sense that its shape is contained in the shape of any partition $p$ for which $w$ is $p$-dominant).

**Proof:** A letter $e > 1$ of the given word $w$ is said to be a violator if the number of $e - 1$ occurring before it does not exceed the number of $e$ occurring before it. For $j$ a positive integer, let $p_j$ be the number of violators in $w$ that exceed $j$. (For example, $p_1$ is the total number of violators.) It is elementary to see that the partition $p_1 \geq p_2 \geq \ldots$ is the unique smallest one for which $w$ is $p$-dominant. \[\Box\]

9.7.1. Weights of words in $[d]$. Let $w$ be a word in $[d]$. The weight of $w$, denoted $\text{wt}(w)$, is defined to be the weight.

9.7.2. The words $w_{\text{row}}$ and $w_{\text{col}}$ attached to a SSST. Let $T$ be a SSST. We have already defined its reverse reading word $w_{\text{row}}(T)$ in §9.2.1. We now define its reverse column word, denoted $w_{\text{col}}(T)$, as follows: we read the entries top to bottom in every column beginning with the right most column and ending with the left most. For the SSST in (16), the reverse column words respectively are 3227536341 and 1112324131.

For the SSST on the left in (16), the partitions $p_w$ attached (as in Proposition 9.4) to its words $w_{\text{row}}$ and $w_{\text{col}}$ turn out to be the same, namely $5 + 3 + 2 + 2 + 1 + 1$. For the SSST on the right in (16), both $w_{\text{row}}$ and $w_{\text{col}}$ are ballot sequences (so $p_w$ is empty for both). Indeed we have:

**Proposition 9.5.** Let $T$ be a SSST and $p$ a partition. Then $w_{\text{row}}(T)$ is $p$-dominant if and only if $w_{\text{col}}(T)$ is so.

**Remark 9.6.** This statement is well known at least in the case of a SSYT (see, e.g., [18, Exercise 5.2.4]). A proof from first principles is given below for the sake of completeness.
PROOF: For boxes \( b_1 \) and \( b_2 \) of \( T \), the phrase \( b_1 \) “occurs before” \( b_2 \) in \( w_{\text{col}}(T) \) (respectively \( w_{\text{row}}(T) \)) has the obvious meaning. We let \( b \) be an arbitrary box in \( T \). Its position is denoted by \((r, c)\) and entry by \( e \).

(1) Let \( b' \) be a box that occurs before \( b \) in \( w_{\text{col}}(T) \) but not in \( w_{\text{row}}(T) \). Let its position be denoted by \((r', c')\) and entry by \( e' \). Then \( r < r', c < c' \) and, since \( T \) is semi-standard, \( e < e' \).

(2) Let \( b'' \) be a box that occurs before \( b \) in \( w_{\text{row}}(T) \) but not in \( w_{\text{col}}(T) \). Let its position be denoted by \((r'', c'')\) and entry by \( e'' \). Then \( r'' < r, c'' < c \) and, since \( T \) is semi-standard, \( e'' < e \).

The following figure depicts the situation:

Suppose first that \( w_{\text{col}}(T) \) is \( p \)-dominant. Consider the contributions to the words \( w_{\text{row}}(T) \) and \( w_{\text{col}}(T) \) of an arbitrarily fixed box \( b \) in \( T \). With notation as above, observe that no box \( b' \) has \( e - 1 \) as an entry and no box \( b'' \) has \( e \) as an entry. Thus, letting \( m_r \) and \( n_r \) (respectively \( m_c \) and \( n_c \)) denote respectively the number of occurrences of \( e \) and \( e - 1 \) (strictly) before \( b \) in \( w_{\text{row}}(T) \) (respectively \( w_{\text{col}}(T) \)), we have \( m_r \leq m_c \) and \( n_c \leq n_r \). Since \( m_c \leq n_c \) by \( p \)-dominance of \( w_{\text{col}}(T) \), we have \( m_r \leq m_c \leq n_c \leq n_r \), so \( w_{\text{row}}(T) \) is \( p \)-dominant too.

Now suppose that \( w_{\text{row}}(T) \) is \( p \)-dominant. By way of contradiction, suppose that \( w_{\text{col}}(T) \) is not \( p \)-dominant. Choose a box \( b \) in \( T \) which “violates” the \( p \)-dominance of \( w_{\text{col}}(T) \), meaning that (with notation as above) \( n_c < m_c \). Since no box of type \( b' \) or \( b'' \) can have an entry equal to \( e \)—we have \( e'' < e < e' \)—it follows that \( m_c = m_r \).

Consider a box of type \( b'' \) with entry equal to \( e - 1 \). Let us denote by \( b''_1 \) any such box and suppose that there are \( k \) such boxes. Then \( n_r = n_c + k \), since \( e' > e \). The entry in the box just below a box \( b''_1 \) must be \( e \) (since such a box is weakly North and strictly West of \( b \) on the one hand, but on the other hand its entry must be strictly larger than \( e - 1 \)). Thus all the \( k \) boxes \( b''_1 \) must occur in row \( r - 1 \), and \( T \) looks like:
Region of $b''$

\[ g \quad f \quad \ldots \quad f \]

\[ e \quad \ldots \quad e \quad e \]

Region of $b'$

Now let $b_1$ be the box in $T$ that is $k$ boxes to the left of $b$. Let us count the number of entries equal to $e$ (respectively $e - 1$) that occur before $b_1$ in $w_{\text{row}}(T)$. This count equals $m_r + k = m_c + k$ (respectively, $n_r = n_c + k$). We have $n_c + k < m_c + k$ (since $n_c < m_c$ by choice of $b$). But this means that the box $b_1$ violates the $p$-dominance of $w_{\text{row}}(T)$, a contradiction.

\[ \square \]

9.8. **Deconstructing a SSST.** Let $T$ be a SSST of shape $\nu/\lambda$. As before, we think of $\lambda$ as being fixed and $\nu$ as varying. For $k$ a positive integer:

- Let $n_r(k)$ denote the number of times $k$ appears in row $r$.
- Consider the boxes of $T$ belonging to $\lambda$ and those with entries not exceeding $k$. Together they form a Young shape. Denote by $\lambda^k$ this shape as well as the corresponding partition. It is convenient to set $\lambda^0 = \lambda$. Observe that

\[ \lambda = \lambda^0 \subseteq \lambda^1 \subseteq \lambda^2 \subseteq \ldots \]

where $\subseteq$ between shapes means that the former is contained in the latter. We have $\lambda^k_r - 1 \lambda^k_r - 1 = n_r(k)$.
- Denote by $w_k(T)$ the word comprising the row numbers of $T$ in which $k$ appears, listed with multiplicity and in weakly decreasing order. In terms of the integers $n_r(k)$, we have $w_k(T) = \ldots 2^{n_r(k)} 1^{n_1(k)}$.

The hypothesis that $T$ is semistandard puts a constraint on the sequence of shapes that can possibly arise as (23). Indeed, the fact that an entry of $T$ is strictly larger than the one vertically just above it (if the latter happens to exist) means precisely that no two boxes in $\lambda^k \setminus \lambda^{k-1}$ are in the same column, or, in other words:

\[ \lambda^k_r \leq \lambda^{k-1}_{r-1} \quad \forall r > 1 \quad \forall k \geq 1 \]

In terms of $\lambda$ and $n_r(k)$, this can also be expressed as the following set of conditions:

\[ \lambda_r + n_r(1) + \cdots + n_r(k) \leq \lambda_{r-1} + n_{r-1}(1) + \cdots + n_{r-1}(k-1) \quad \forall r > 1 \quad \forall k \geq 1 \]

9.8.1. The position word $w_{\text{pos}}$ and its $\lambda$-dominance. To see what (25) translates to in terms of the words $w_k(T)$, let us define the position word of $T$, denoted $w_{\text{pos}}(T)$, to be the concatenation $w_1(k)w_2(k)\ldots$. For example, the position words of the SSST in (16) are, respectively, 511134342 and 5311132434. It is readily seen that (25) is equivalent to the $\lambda$-dominance of the word $w_{\text{pos}}(T)$ (in the sense of §9.7).
9.8.2. Recovering the SSST \( T \). Evidently the SSST \( T \) can be recovered from the collection of integers \( n_r(k) \) (presuming knowledge of the fixed partition \( \lambda \)). Thus it can be recovered either from the sequence (23) of increasing shapes or from the sequence \( w_1(T), w_2(T), \ldots \) of words. Moreover, if either the sequence (23) satisfies the constraint (24) or, equivalently, if the sequence \( w_1(T), w_2(T), \ldots \) is such that \( w_{\text{pos}}(T) \) is \( \lambda \)-dominant, then there exists a corresponding \( T \).

9.8.3. Bijection between \( T_\mu^\lambda(w)[d] \) and \( S_\mu^\lambda(w)[d] \). As preparation for the proof in §9.9 below of the tableau version of the decomposition rule (18) of KK modules, we apply the observations above to the case when \( T \) is LR.

Fix notation as in §9.4. Let \( T_\mu^\lambda[d] \) denote the subset of \( T_\mu^\lambda \) consisting of those elements \( T \) such that \( v(T) \) has at most \( d \) parts. Let \( S_\mu^\lambda \) denote those SSYT of shape \( \mu \) whose column word is \( \lambda \)-dominant (in the sense of §9.7), and let \( S_\mu^\lambda(w) \) be the subset of those elements of \( S_\mu^\lambda \) for which the associated permutation \( u \) (as in §9.5.1) satisfies \( u \leq w \). Put:

\[
S_\mu^\lambda[d] := \{ S \in S_\mu^\lambda \mid \text{no entry of } S \text{ exceeds } d \} \quad S_\mu^\lambda(w)[d] := S_\mu^\lambda(w) \cap S_\mu^\lambda[d]
\]

The weight of a SSYT with entries from \([d]\) is its weight thought of as a SSST (see 9.2.2).

**Proposition 9.7.** Let \( T \) be an element of \( T_\mu^\lambda \) and \( S \) the SSYT attached to \( T \) as in §9.6.1. The association \( T \mapsto S \) gives a bijection between \( T_\mu^\lambda \) and \( S_\mu^\lambda \), under which \( v(T) = \lambda + \text{wt}(S) \), and which also restricts to a bijection between the pairs \( (T_\mu^\lambda(w), S_\mu^\lambda(w)) \) and \( (T_\mu^\lambda(w)[d], S_\mu^\lambda(w)[d]) \).

**Proof:** We first show that \( T \mapsto S \) gives a bijection between \( T_\mu^\lambda \) and \( S_\mu^\lambda \). From Proposition 9.5 it follows that the \( \lambda \)-dominance of \( w_{\text{row}}(S) \) and \( w_{\text{col}}(S) \) are equivalent, so

\[
S_\mu^\lambda = \{ S \text{ is a SSYT of shape } \mu \mid w_{\text{row}}(S) \text{ is } \lambda \text{-dominant} \}
\]

It is easy to see from their definitions that the words \( w_{\text{row}}(S) \) and \( w_{\text{pos}}(T) \) are the same. Thus, from §9.8.2, we conclude:

- \( w_{\text{pos}}(T) = w_{\text{row}}(S) \) is \( \lambda \)-dominant, so \( S \) belongs to \( S_\mu^\lambda \).
- The sequence \( w_1(T), w_2(T), \ldots \) defined in §9.8 and hence \( T \) itself can be recovered readily from \( S \) by reading the entries in every row of \( S \) from right to left. This shows that \( T \mapsto S \) is one-to-one.
- Given \( S' \) in \( S_\mu^\lambda \), the \( \lambda \)-dominance of \( w_{\text{row}}(S') \) means that there exists a skew tableau \( T' \) of shape \( \nu/\lambda \) (for some \( \nu \)) that corresponds to it (in the sense of §9.8.2). The fact that the entries along any column of \( S' \) are strictly increasing downwards translates to the fact that the corresponding \( T' \) as above is LR, so \( T' \) belongs to \( T_\mu^\lambda \) and \( T' \mapsto S' \). This shows that \( T \mapsto S \) is surjective.

This finishes the proof that \( T \mapsto S \) gives a bijection from \( T_\mu^\lambda \) to \( S_\mu^\lambda \).

It is clear from the description of the association \( T \mapsto S \) that \( S \) has type \( \mu \) and that \( \lambda + \nu(T) = \text{wt}(S) \).
The association of a permutation to an LR tableau proceeds via the SSYT attached to it, so it immediately follows that \( T \mapsto S \) gives a bijection from \( \mathcal{T}_{\mu}^{\lambda}(w) \) to \( \mathcal{S}_{\mu}^{\lambda}(w) \).

Finally, the number of parts of \( \nu(T) \) on the one hand and the maximum value of an entry in \( S \) on the other are upper bounds for each other under \( T \mapsto S \), so we get a bijection between \( \mathcal{T}_{\mu}^{\lambda}(w)[d] \) and \( \mathcal{S}_{\mu}^{\lambda}(w)[d] \).

\[9.9. \text{Proof of the tableau KK decomposition rule of } \S 9.4.\] The decomposition rule (18) in terms of tableaux can be derived, as we now show, from the general decomposition rule (6) for KK-modules in \( \S 6 \). The derivation consists of stringing together three bijections that preserve invariants.

The first of these is the bijection between \( \mathcal{T}_{\mu}^{\lambda}[d] \) and \( \mathcal{S}_{\mu}^{\lambda}[d] \) of Proposition 9.7. The second and third bijections are from the appendix: by Corollary A.4, we may identify \( \mathcal{S}_{\mu}[d] \), the set of SSYT of shape \( \mu \) with entries from \([d]\), with \( \mathcal{P}_{\text{std}} \), the set of standard concatenations of LS paths as in \( \S A.2 \); and, finally, there is the crystal isomorphism \( \Gamma \) of \( \S A.4 \) between the set \( \mathcal{P}_{\mu} \) of LS paths of shape \( \mu \) and \( \mathcal{P}_{\text{std}} \).

In the subsection below, the good properties required of the second bijection are established. For the first bijection, this was done in Proposition 9.7. As for the crystal isomorphism \( \Gamma \), it preserves end points and \( \lambda \)-dominance as shown in Proposition A.6; and the minimal element in the initial direction of \( \pi \) in \( \mathcal{P}_{\mu} \) is the initial element of the minimal standard lift of \( \Gamma \pi \) as shown in Proposition A.7.

The final upshot is a bijection \( T \leftrightarrow \pi \) between \( \mathcal{T}_{\mu}^{\lambda}[d] \) on the one hand and \( \mathcal{P}_{\mu}^{\lambda} \) on the other such that (a) \( \nu(T) \) equals the end point \( \pi(1) \) and (b) the permutation \( u \) attached to \( T \) as in \( \S 9.6.2 \) equals the minimal element in the initial direction of \( \pi \). This will finish the proof of the tableau decomposition rule (18).

\[9.9.1. \text{Good properties of the bijection of Corollary A.4.}\]

**Proposition 9.8.** Under the identification between \( \mathcal{S}_{\mu}[d] \) and \( \mathcal{P}_{\text{std}} \) of Corollary A.4, let \( S \) in \( \mathcal{S}_{\mu}[d] \) correspond to \( \theta \) in \( \mathcal{P}_{\text{std}} \). Then:

1. The weight \( \text{wt}(S) \) of \( S \) equals the end point \( \theta(1) \) of the path \( \theta \).
2. The permutation \( u \) associated to \( S \) by the procedure of \( \S 9.5.1 \) equals the initial element of the minimal standard lift of \( \theta \).
3. The column word \( w_{\text{col}}(S) \) of \( S \) is \( \lambda \)-dominant (in the sense of \( \S 9.7 \)) if and only if the path \( \theta \) is \( \lambda \)-dominant.

**Proof:** Item (1) is immediate from the definitions. As for item (2), the whole of \( \S 10 \) is devoted to its proof.

Turning to item (3), we first prove the “only if part”. Let \( c \) denote the number of columns in the shape of \( S \), let \( r_j \) denote the number of boxes in column \( j \) of \( S \) (for \( 1 \leq j \leq c \)), and let \( w'_j \) denote the word \( s_{1j} \ldots s_{r_jj} \) (where, as for a matrix, \( s_{ab} \) denotes the entry of \( S \) in row \( a \) and column \( b \)). The word \( w_{\text{col}}(S) \) is, by definition, \( w'_c w'_{c-1} \ldots w'_1 \). Its \( \lambda \)-dominance clearly implies that of any left subword of it, in particular that of the
subwords $w'_c, w'_c w'_{c-1}, \ldots, w'_c w'_{c-1} \cdots w'_2$, and $w'_c w'_{c-1} \cdots w'_2 w'_1 = w_{\text{col}}(S)$. This in turn implies that the weights $\lambda + \text{wt}(w'_c), \lambda + \text{wt}(w'_c w'_{c-1}), \ldots, \lambda + \text{wt}(w'_c w'_{c-1} \cdots w'_2)$, and $\lambda + \text{wt}(w'_c w'_{c-1} \cdots w'_2 w'_1) = \lambda + \text{wt}(w_{\text{col}}(S))$ are all dominant. But the dominance of these $c$ weights is, as is readily seen, precisely equivalent to the $\lambda$-dominance of $\theta$.

For the “if part”, we first make an observation (whose elementary proof we skip). Suppose that a word $w$ in $[d]$ is a concatenation $w_1 w_2$ of words $w_1$ and $w_2$ such that $w_1$ is $\lambda$-dominant, $w_2$ is weakly increasing (left to right), and $\lambda + \text{wt}(w)$ is dominant. Then $w$ is $\lambda$-dominant.

The $\lambda$-dominance of $\theta$ implies that $\lambda + \text{wt}(w'_c), \lambda + \text{wt}(w'_c w'_{c-1}), \ldots, \lambda + \text{wt}(w'_c w'_{c-1} \cdots w'_2)$, and $\lambda + \text{wt}(w'_c w'_{c-1} \cdots w'_2 w'_1) = \lambda + \text{wt}(w_{\text{col}}(S))$ are all dominant. Since each $w'_j$ is strictly increasing, we conclude using the observation that $w_{\text{col}}(S)$ is $\lambda$-dominant.

\[\square\]

10. An important property of the procedure of §9.5.1

Let $S$ be an SSYT (see §9.5) none of whose entries exceeds $d$, and $u$ the permutation of $[d]$ obtained by application to $S$ of the procedure of §9.5.1. As explained in §A.2 (see, in particular, Corollary A.4) such SSYT's may be identified as certain standard concatenations of LS paths whose shapes are fundamental weights (for $g = \mathfrak{sl}_d$). In what follows, we will use the notation for an SSYT to denote also the corresponding standard concatenation of paths. Let $v$ be the initial element of the minimal standard lift of $S$ (§2.5).

The purpose of this section is to show that $u = v$. The proof is given in §10.5 and §10.7 after preparations in the earlier subsections.

The procedure of §9.5.1 seems to be quite different from and more efficient than a repeated application of Deodhar’s lemma (Example 2.19) to compute the initial element of the minimal standard lift $v$. Besides, the justification we give in Example 10.4 of the recipe of Example 2.19 is itself based on the result of this section (that $u = v$).

10.1. Notation relating to permutations. Let $x$ be a permutation and let $x_1 x_2 \ldots$ denote its one-line notation.

We call $\{i \mid x_i > x_{i+1}\}$ the descent set of $x$. We say that $x$ has only $r$ significant elements if its descent set is contained in $[r]$, or, in other words, if the sequence $x_{r+1} x_{r+2} \ldots$ is increasing. E.g., the only permutation that has zero significant elements is the identity.

For $s$ an integer, let $x^s$ denote the sequence $x_1^s < \ldots < x_s^s$ of the first $s$ elements of $x$ (namely $x_1, \ldots, x_s$) arranged in increasing order.

10.1.1. On the tableau criterion for Bruhat order. Recall the following “tableau criterion” for comparability in Bruhat order of two permutations: $x \leq z$ if and only if $x^s \leq z^s$ for all $s$, where $x^s \leq z^s$ is short hand for $x_j^s \leq z_j^s$ for all $1 \leq j \leq s$.

Lemma 10.1. ([1, Corollary (5)]) For $x \leq z$, it suffices that $x^s \leq z^s$ holds for either (a) all $s$ in the descent set of $x$, or (b) all $s$ not in the descent set of $z$. 

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10.2. **An example.** For \( x \) a permutation, we denote by \( (r)x \) the permutation obtained from \( x \) by rearranging the first \( r \) elements in its one-line notation in increasing order. In other words, \( (r)x \) is the permutation whose one-line notation is \( x'_1 \ldots x'_r x_{r+1} x_{r+2} \ldots \).

**Lemma 10.2.** \( (s)x \leq (r)x \leq (1)x = x \) for \( s \geq r \geq 1 \).

**Proof:** Put \( y = (s)x \) and \( z = (r)x \). The descent set of \( y \) is contained in \( \{s, s+1, \ldots\} \). For any \( t \geq s \), we have \( y^t = z^t \), so it follows from Lemma 10.1 that \( y \leq z \). Observe that \( (1)x = x \).

Given a permutation \( x \) of \([n]\) and an integer \( r \leq n \), we let \( S(r, x) \) denote the SSYT constructed as follows: it has \( n - r + 1 \) columns; column \( j \) (counting from the left) has \( n + 1 - j \) boxes and its entries are the first \( n + 1 - j \) entries of \( x \) arranged in increasing order. E.g., if \( x \) is the permutation of \([5]\) with one-line notation 45312, then \( S(x, 3) \) is:

\[
\begin{array}{cccc}
1 & 1 & 3 \\
2 & 3 & 4 \\
3 & 4 & 5 \\
4 & 5 \\
5 \\
\end{array}
\]

**Lemma 10.3.** The initial element of the minimal standard lift of \( S(r, x) \) is \( (r)x \).

**Proof:** Let \( a \) be this initial element. By an induction argument, we may assume that \( (s)x \) is the initial element of the minimal standard lift of \( S(s, x) \) for \( s > r \). Thus \( z := (r+1)x \leq a \).

Since the first \( r \) elements of \( a \) match the respective ones of \( y := (r)x \), it follows in particular that \( y^r \leq a^r \). Since the descent set of \( y \) is contained in \( \{r, r + 1, \ldots\} \), and \( y^s = z^s \leq a^s \) for \( s > r \), it follows from Lemma 10.1 that \( y \leq a \).

On the other hand, evidently \( (n)x \leq (n-1)x \leq \ldots \leq (r)x \) is a standard lift of \( S(r, x) \), so \( a \leq (r)x = y \).

**Example 10.4.** Let notation be fixed as in Example 2.19. We described there a procedure for determining \( \tau := J_{cW_c}(w) \) without however providing a justification for it. We now provide such a justification as an application of the main result of this section \( (u = v) \).

Let \( S' \) denote the SSYT \( S(r, w) \) and \( S \) the SSYT obtained by attaching to \( S' \) on the right a column with \( r \) boxes whose entries from top to bottom are \( \sigma_1, \ldots, \sigma_r \). For the values \( n = 6, r = 3, \sigma = 246135 \), and \( w = 145362 \) used as an illustration in Example 2.19, \( S \) is:

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 2 \\
1 & 3 & 3 & 4 & 4 \\
3 & 4 & 4 & 5 & 6 \\
4 & 5 & 5 \\
5 & 6 \\
6 \\
\end{array}
\]
By Lemma 10.3, the initial element of the minimal standard lift of $S'$ is $(r)w = w$, so the initial element $v$ of the minimal standard lift of $S$ is the least element having the following two properties: $w \leq v$ and the first $r$ elements of $v$ (in its one-line notation) are $\sigma_1, \ldots, \sigma_r$, in that order.

Now, $\tau$ is the least element having the two properties: $w \leq \tau$ and $\tau W_r = \sigma W_r$. Evidently $(r)\tau W_r = \tau W_r = \sigma W_r$, and, by Lemma 10.2, $w = (r)w \leq (r)\tau \leq \tau$. So $(r)\tau = \tau$ and the first $r$ elements of $\tau$ are $\sigma_1, \ldots, \sigma_r$, in that order. This means $\tau = v$.

Thus, by the main result of this section, the element $u$ obtained by applying the procedure of §9.5.1 to $S$ equals $\tau$. It is easily seen that $u_j = \sigma_j$, for $1 \leq j \leq r$. For $j > r$, to determine $u_j$, we may use, by Lemma 9.3, the $j$-depth sequence $y_1^j < \ldots < y_j^j$ of $S'$ instead of that of $S$. The entries in the column with $j$ boxes of $S'$ being $w_1^j < \ldots < w_j^j$, it is clear that $w_i^j \leq y_i^j$ for $1 \leq i \leq j$. On the other hand, since each $y_i^j$ must be an entry in one of the columns of $S'$ with at most $j$ boxes, it follows that $w_i^j = y_i^j$ for every $i$.

This completes the justification of the recipe of Exercise 2.19 to compute $J_{\sigma W_r}(w)$. \hfill $\square$

10.3. **Truncations of permutations and SSYTs.** For $r$ an integer, let $x^{(r)}$ denote the permutation obtained from $x$ by rearranging its elements in position $r + 1$ and beyond in increasing order. We call $x^{(r)}$ the $r$-truncation of $x$. Evidently $x^{(r)}$ has only $r$ significant elements. As an easy consequence of Lemma 10.1, we have:

**Lemma 10.5.** Suppose that $x \leq z$. Then $x^{(r)} \leq z^{(r)} \leq z$.

For $r$ an integer, let $S^{(r)}$ denote the SSYT obtained by taking the first $r$ rows of $S$: if $S$ has at most $r$ rows, then $S^{(r)}$ is all of $S$. We call $S^{(r)}$ the $r$-truncation of $S$. Let $v^{(r)}$ denote the initial element of the minimal standard lift of $S^{(r)}$.

**Proposition 10.6.** Every permutation in the minimal standard lift of $S^{(r)}$ has only $r$ significant elements. In particular, if $S$ has at most $r$ rows, then $v$ has only $r$ significant elements: $v^{(r)} = v$.

**Proof:** We use Lemma 10.5 to observe that the $r$-truncation of any standard lift of $S^{(r)}$ continues to be a standard lift, and moreover that the $r$-truncation of the minimal standard lift is itself. \hfill $\square$

**Proposition 10.7.** $v^{(r)} = v^{(r)}$.

**Proof:** Using Lemma 10.5 again, we observe that the $r$-truncation of any standard lift of $S$ gives a standard lift of $S^{(r)}$. Thus $v^{(r)} \leq v^{(r)}$.

Let $\sigma_1 \leq \ldots \leq \sigma_k = v^{(r)}$ be the minimal standard lift of $S^{(r)}$. By Proposition 10.6, there are only $r$ significant elements in every $\sigma_j$. We will construct a standard lift $\tilde{\sigma}_1 \leq \ldots \leq \tilde{\sigma}_k$ of $S$ whose $r$-truncation is $\sigma_1 \leq \ldots \leq \sigma_k$. It will then follow that $v \leq \tilde{\sigma}_k$, and so, by Lemma 10.5, $v^{(r)} \leq \tilde{\sigma}_k^{(r)} = \sigma_k = v^{(r)}$.

To construct $\tilde{\sigma}_j$, proceed as follows. The first $r$ elements of $\tilde{\sigma}_j$ are the same as those of $\sigma_j$. The first entries of $\tilde{\sigma}_j$ also match the entries top-downwards in column $j$ of $S$ (until the
latter entries are exhausted). The remaining entries of $\varpartial_j$ are arranged in decreasing order. Criterion (b) of Lemma 10.1 is useful to verify $\varpartial_j \leq \varpartial_{j+1}$. 

10.4. The main part of the proof (that $u = v$). Let $k$ be the number of columns in $S$. For $i$, $1 \leq i \leq k$, let $S[i]$ denote the SSYT consisting only of the first $i$ columns (from the left) of $S$, and $u[i]$ the permutation obtained by running the procedure of §9.5.1 on $S[i]$. By the description of the procedure, it is clear that running the procedure on $S[i](p)$ yields $u[i](p)$.

Let $p$ denote a positive integer. We proceed by induction on $p$ to show the following three assertions.

(a) $u[1](p) \leq u[2](p) \leq \ldots \leq u[k](p) = u(p)$.

Consider a rectangular grid of boxes with $p$ boxes in every column and $k$ boxes in every row. Suppose we fill the boxes in column $i$ of this grid by the first $p$ entries of $u[i]$ in increasing order. It is clear from item (a) above that we then get a SSYT (see Lemma 10.1). Let $S_p$ be the SSYT whose first $p$ rows are this rectangular SSYT and whose rows $p+1$ and beyond are the same as the corresponding ones of $S$. For $i$, $1 \leq i \leq k$, we let $S_p[i]$ denote the SSYT consisting of only the first $i$ columns of $S_p$.

For example, on the left in the following display is shown $S_2$ for $S$ as in (21); and on the right is shown $S_4$ for $S$ as in (26):

\begin{tabular}{|c|c|c|c|}
\hline
1 & 3 & 3 & 3 \\
2 & 4 & 6 & 8 \\
\hline
\end{tabular}

\begin{tabular}{|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 & 2 \\
2 & 3 & 3 & 3 & 3 \\
3 & 4 & 4 & 4 & 4 \\
4 & 5 & 5 & 5 & 6 \\
5 & 6 & & & \\
6 & & & & \\
\hline
\end{tabular}

(b) Fix $i$, $1 \leq i \leq k$. For any $r > p$, the $r$-depth of sequence of $S[i]$ equals the $r$-depth sequence of $S_p[i]$.

(c) Fix $i$, $1 \leq i \leq k$. Let $a_1 < \ldots < a_p$ be the the first $p$ entries arranged in increasing order of $u[i]$. Let $y_1 < \ldots < y_{p+1}$ be the $(p+1)$-depth sequence of $S[i]$. Let $s$, $1 \leq s \leq p+1$, be such that $a_1 < \ldots < a_{s-1} < y_s < a_s < \ldots < a_p$ is the arrangement in increasing order of the first $p+1$ elements of $u[i]$ (see Proposition 9.1 and the sentence preceding it). Then $y_1 = a_1$, $\ldots$, $y_{s-1} = a_{s-1}$, and $y_s$ occurs in row $s$ of $S_p[i]$ and in a column weakly to the right of that in which $b^{p+1}$ occurs ($b^{p+1}$ is the right most box in row $p+1$ of $S$).

10.4.1. Base case of the induction. The assertions are easily verified in case $p = 1$. Indeed, for every $i$, $1 \leq i \leq k$, $u[i](1)$ has only one significant element and its first element is the entry in the first row in column $i$ of $S$. This proves (a). Assertion (b) is immediate since

\[^3\text{It is only assertion (a) that we are really interested in. Once we have it, it follows rather easily that } v \leq u \text{ (see §10.5). The other two assertions are technical devices that facilitate the proof of (a).}\]
10.4.3. Proof of assertion place $p$ as follows: Let us write a hypothesis, we know this to be true for $j$, so it remains to be proved only for $j = p$. Let us write $a_1 \ldots < a_{p-1}$ for $g^{p-1}$ and $b_1 \ldots < b_{p-1}$ for $h^{p-1}$.

Let $e_1 \ldots < e_p$ and $f_1 \ldots < f_p$ be the $p$-depth sequences of $S[i]$ and $S[i+1]$ respectively. We have, evidently, $e_j \leq f_j$. Let $s$ and $t$, $1 \leq s, t \leq p$, be such that

$$a_1 \ldots < a_{s-1} < e_s < a_s \ldots < a_{p-1} \quad \text{and} \quad b_1 \ldots < b_{t-1} < f_t < b_t \ldots < b_{p-1}$$

are the sequences $g^p$ and $h^p$.

In the case $s \leq t$, the desired conclusion $g^p \leq h^p$ follows rather easily from $g^{p-1} \leq h^{p-1}$. Indeed we have, in the case $s < t$:

$$g^p_j = a_j \leq b_j = h^p_j \quad \text{for } 1 \leq j \leq s - 1$$
$$g^p_s = e_s < a_s \leq b_s = h^p_s$$
$$g^p_j = a_{j-1} \leq b_{j-1} = h^p_{j-1} < h^p_j \quad \text{for } s + 1 \leq j < t$$
$$g^p_t = a_{t-1} \leq b_{t-1} < f_t = h^p_t$$
$$g^p_j = a_{j-1} \leq b_{j-1} = h^p_j \quad \text{for } t < j \leq p$$

For $s = t$, the three middle lines in the display above should be replaced by $g^p_s = e_s \leq f_s = h^p_s$.

So let us assume $s > t$. The cases $1 \leq j < t$ and $s < j \leq p$, are similar respectively to the first and last cases above. It is for $j$ in the range $t \leq j \leq s$ that we need some care. The key observation here is that $a_j = e_j$ for $j < s$. This follows from assertion (c) with $p$ replaced by $p - 1$ (which we may assume to be true by induction). Indeed, using this, we are done as follows:

$$g^p_j = a_j = e_j \leq b_j = h^p_j \quad \text{for } t < j \leq s$$
$$g^p_t = a_{t-1} = e_t < f_t = h^p_t \quad \square$$

10.4.2. Proof of assertion (a). To simplify notation, write $g$ and $h$ for $u[i]$ and $u[i+1]$ respectively. We need to prove that $g^j \leq h^j$ for all $j \leq p$ (Lemma 10.1). By the induction hypothesis, we know this to be true for $j < p$, so it remains to be proved only for $j = p$.

Let $y_1 \leq e_2$ and $s_2 \leq f_2$ be the 2-depth sequences of $S[i]$ and $S[i+1]$ respectively. We have, evidently, $e_j \leq f_j$. Let $s$ and $t$, $1 \leq s, t \leq p$, be such that

$$a_1 \ldots < a_{s-1} < e_s < a_s \ldots < a_{p-1} \quad \text{and} \quad b_1 \ldots < b_{t-1} < f_t < b_t \ldots < b_{p-1}$$

Let $y_1 \leq e_2$ and $s_2 \leq f_2$ be the 2-depth sequences of $S[i]$ and $S[i+1]$ respectively. We have, evidently, $e_j \leq f_j$. Let $s$ and $t$, $1 \leq s, t \leq p$, be such that

$$a_1 \ldots < a_{s-1} < e_s < a_s \ldots < a_{p-1} \quad \text{and} \quad b_1 \ldots < b_{t-1} < f_t < b_t \ldots < b_{p-1}$$

are the sequences $g^p$ and $h^p$.

In the case $s \leq t$, the desired conclusion $g^p \leq h^p$ follows rather easily from $g^{p-1} \leq h^{p-1}$. Indeed we have, in the case $s < t$:

For $s = t$, the three middle lines in the display above should be replaced by $g^p_s = e_s \leq f_s = h^p_s$.

So let us assume $s > t$. The cases $1 \leq j < t$ and $s < j \leq p$, are similar respectively to the first and last cases above. It is for $j$ in the range $t \leq j \leq s$ that we need some care. The key observation here is that $a_j = e_j$ for $j < s$. This follows from assertion (c) with $p$ replaced by $p - 1$ (which we may assume to be true by induction). Indeed, using this, we are done as follows:

$$g^p_j = a_j = e_j \leq b_j = h^p_j \quad \text{for } t < j \leq s$$
$$g^p_t = a_{t-1} = e_t < f_t = h^p_t \quad \square$$

10.4.3. Proof of assertion (b). Fix $r > p$. By induction, we know the statement for $p - 1$ in place $p$, so the $r$-depth sequences of $S[i]$ and $S_{p-1}[i]$ are the same. It is therefore enough to prove that the $r$-depth sequences of $S_{p-1}[i]$ and $S_p[i]$ are the same. It is convenient to omit the “$i$” and just write $S_{p-1}$ and $S_p$ for $S_{p-1}[i]$ and $S_p[i]$ respectively. Assertion (b) follows immediately from Corollaries 10.10 and 10.12 below.

---

4The case $s < t$ never actually occurs, but that does not concern us here.
We denote by $i_0$ the column number in which the right most box $b^p$ in row $p$ of $S_{p-1}$ (equivalently $S$) occurs. The entries in any column of $S_{p-1}$ are also entries in that same column of $S_p$. For every box $b$ of $S_{p-1}$, we denote by $b'$ the (unique) box of $S_p$ in the same column as $b$ and having the same entry. The association $b \mapsto b'$ is evidently one-to-one. Either $b'$ is in the same row as $b$ or in the next lower row.

We classify boxes of $S_p$ as follows:

- **Old** boxes are those that are in the image of the above map $b \mapsto b'$. **New** boxes are those that are not old.
- An **unmoved** box is an old box $b'$ that is in the same row as its preimage $b$. We write $b' = b$ in this case. A **moved** box is an old one that is not unmoved, or, in other words, an old one that is in a row one lower than its preimage.

A box $b$ of $S_{p-1}$ is moved or unmoved accordingly as its image $b'$ in $S_p$ is so.

As an illustration, shown on the left in the display below is $S_3$ and on the right is $S_4$ in a particular case: $p = 3$ and $i_0 = 2$. The entries in all the new boxes are in bold and underlined; those in unmoved boxes are in red; those in moved boxes are in blue. The 4-depth sequences for $S_3[3]$ and $S_3[4]$ respectively are 7, 5, 2, 1 and 7, 5, 4, 2; those for $S_3[l]$ for $l \geq 5$ are all 7, 6, 4, 3.

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 2 & 3 & 3 & 4 & 7 \\
2 & 2 & 2 & 4 & 6 & 7 & 8 & 8 \\
3 & 3 & 5 & 5 & 8 & 8 & 9 & 9 \\
6 & 7 \\
7 \\
9 \\
\end{array}
\quad
\begin{array}{cccccccc}
1 & 1 & 1 & 2 & 3 & 3 & 3 & 3 \\
2 & 2 & 2 & 4 & 4 & 4 & 7 \\
3 & 3 & 5 & 5 & 6 & 7 & 8 & 8 \\
6 & 7 & 7 & 8 & 8 & 9 & 9 \\
7 \\
9 \\
\end{array}
\]

Proposition 10.8.  
(1) In any column $i$ of $S_{p-1}$ (respectively $S_p$) with $i \leq i_0$, all boxes are unmoved (respectively old and unmoved). In particular, the right most box $b^r$ in row $r$ (with $r > p$) of $S_p$ is old and unmoved.

(2) Let $n$ be a new box. Then, to the left of $n$ and in the same row, in a column with number $i \geq i_0$, there is an old box carrying the same entry as $n$.\(^5\)

**Proof:** Item (1) is clear. Indeed $S_{p-1}$ and $S_p$ are identical in columns $i \leq i_0$.

To prove item (2), suppose that $n$ occurs in column $c$ of $S_p$. Then $S_{p-1}[c]$ has $p - 1$ boxes in its last column. Let $a_1 < \ldots < a_{p-1}$ be the entries in that column (top to bottom). Let $y_1 < \ldots < y_p$ be the $p$-depth sequence of $S[c]$ (or, what amounts to the same by the induction hypothesis, of $S_{p-1}[c]$) and let $s$, $1 \leq s \leq p$, be such that $a_1 < \ldots < a_{s-1} < y_s < a_s < \ldots < a_{p-1}$ are the entries in the last column of $S_p[c]$. The box with $y_s$ as its entry is $n$, and $n$ occurs in row $s$ of $S_p[c]$. We may assume by induction that assertion (c) of §10.4 is true with $p - 1$ in place of $p$, and conclude that $y_s$ appears as an entry in row $s$ of $S_{p-1}[c]$ in a column with number $i \geq i_0$. \(\square\)

\(^5\)Any such box is actually unmoved, but we don’t need that bit of detail.
Proposition 10.9. Let \( b_1 \) and \( b_2 \) be boxes of \( S_{p-1} \). Then \( b_1 > b_2 \) if and only if \( b'_1 > b'_2 \).

Proof: Neither the entry nor the column number changes on passage from \( b \) to \( b' \). While the row number could increase by at most 1 on this passage, consider the facts that both \( S_{p-1} \) and \( S_p \) are SSYTs and that \( b_2 \) (respectively \( b'_2 \)) is weakly to the East of \( b_1 \) (respectively \( b'_1 \)) and carries an entry which is strictly less. Together these imply that \( b_2 \) (respectively \( b'_2 \)) occurs in a higher row than \( b_1 \) (respectively \( b'_1 \)). \( \square \)

Corollary 10.10. Fix \( r > p \). Suppose that \( b^r = b_0 > \ldots > b_\delta \) is a chain of boxes in \( S_{p-1} \). Then \( b^r = b'_0 > \ldots > b'_\delta \) is a chain of boxes in \( S_p \). In particular, the \( r \)-depth sequence of \( S_{p-1} \) is term for term dominated by the \( r \)-depth sequence of \( S_p \).

Proof: That we get a chain on passing from \( b \) to \( b' \) is clear from Proposition 10.9. That \( b^r = b'^r \) follows from Proposition 10.8 (1). It is clear from the description of the association \( b \mapsto b' \) that the entries of \( b_\delta \) and \( b'_\delta \) are the same. \( \square \)

Proposition 10.11. Fix \( r > p \). Given a chain \( b^r = \tilde{b}_0 > \ldots > \tilde{b}_\delta \) of boxes in \( S_p \), there exists a chain \( b^r = b_0 > \ldots > b_\delta \) of boxes in \( S_{p-1} \) with \( b_\delta \) having the same entry as \( \tilde{b}_\delta \) and being weakly to the Northwest of it (meaning, the row and column numbers of \( b_\delta \) each is at most that of the corresponding number of \( \tilde{b}_\delta \)).

Proof: Proceed by induction on \( \delta \). For \( \delta = 0 \), the statement is easily seen to be true since \( b^r = b'^r \) is old and unmoved (Proposition 10.8 (1)). Suppose that \( \delta \geq 1 \).

First suppose that \( \tilde{b}_\delta \) is an old box. Let \( b_\delta \) be the unique box in \( S_{p-1} \) such that \( b'_\delta = \tilde{b}_\delta \). Note that \( b_\delta \) shares its entry and column number with \( \tilde{b}_\delta \) and is weakly to the North of it. By induction, choose \( b^r = b_0 > \ldots > b_{\delta-1} \) with \( b_{\delta-1} \) being weakly to the Northwest of \( b_{\delta-1} \) and having the same entry. Since \( b_{\delta-1} \) is weakly to the West of \( b_\delta \) with a strictly larger entry, it follows that it is on a strictly lower row, and so \( b_{\delta-1} > b_\delta \).

Now suppose that \( \tilde{b}_\delta \) is a new box. Using Proposition 10.8 (2), replace it by an old and unmoved box having the same entry and being to the left in the same row. Suppose that the new \( \tilde{b}_\delta \) is in column \( c \). If any \( \tilde{b}_j \) for \( j < \delta \) has a column number higher than \( c \), replace it by the one in the same row in column number \( c \). We now get a chain with \( \tilde{b}_\delta \) being old, so we are reduced to the case settled in the previous paragraph. \( \square \)

Corollary 10.12. Fix \( r > p \). The \( r \)-depth sequence of \( S_{p-1} \) dominates term for term the \( r \)-depth sequence of \( S_p \).

10.4.4. Proof of assertion (c). By assertion (b), we may take \( y_1 < \ldots < y_{p+1} \) to be the \((p+1)\)-depth sequence of \( S_p[i] \). Fix \( j < s \). We would like to show that \( y_j = a_j \). In what follows, we write just “depth” to mean “\((p+1)\)-depth”. Recall that, by definition, \( y_i \) is the maximal entry in a box of depth \( p + 1 - j \); and \( a_j \) occurs as the entry in row \( j \) and column \( i \) of \( S_p[i] \). Any box of depth \( p + 1 - j \) occurs in row \( j \) or above, and \( a_j \) dominates all the entries in those rows. Thus it is enough to show that the box in row \( j \) and column \( i \)
of \( S_p[i] \) has depth \( p + 1 - j \). Further, since any box in row \( j \) has depth at most \( p + 1 - j \), it is enough to show that the depth of that box is at least \( p + 1 - j \). Further, it is enough to show this for \( j = s - 1 \), since it follows then for the other \( j < s \) as well.

By definition, \( y_s \) occurs as an entry in a box \( b \) of \( S_p[i] \) of depth \( p + 1 - s \). Such a \( b \) can only appear in row \( s \) or above. But since \( a_{s-1} < y_s \), it follows that \( b \) cannot occur in row \( s - 1 \) or above. So it appears in row \( s \), and so \( b \succ c \) where \( c \) is the box in row \( s - 1 \) and column \( i \) of \( S_p[i] \), which means that \( c \) has depth \( p + 2 - s \).

10.5. **Proof that** \( v \leq u \). It follows from assertion (a) that \( u[1]^{(p)} \leq \ldots \leq u[k]^{(p)} \) is a standard lift of \( S^{(p)} \). Since \( v^{(p)} = v(p) \) is the initial element of the minimal lift of \( S^{(p)} \) (by Proposition 10.7), it follows that \( v^{(p)} \leq u[k]^{(p)} = u^{(p)} \). Since \( v = v^{(p)} \) and \( u = u^{(p)} \) for large \( p \), it follows that \( v \leq u \).

10.6. **A technical lemma (that is invoked in §10.7).**

**Lemma 10.13.** Let \( \sigma_1 \leq \ldots \leq \sigma_k \) be a standard lift of \( S \). Consider any box of \( b^p \)-depth \( \delta \) in \( S \), for some positive integer \( p \). (Recall that \( b^p \) denotes the right most box in row \( p \) of \( S \).) Let \( y \) be the entry in that box and \( c \) be the box of \( S \) (from the left) of the column in which that box appears. Then, among the first \( p \) elements of \( \sigma_c \) (in its one-line notation), there are at least \( \delta + 1 \) that are at least \( y \).

**Proof:** Proceed by induction on \( \delta \). Suppose first that \( \delta = 0 \). The only box with \( b^p \)-depth 0 is the box \( b^p \) itself. Since \( b^p \) occurs on row \( p \), the conclusion is easily verified to be true.

Now suppose that \( \delta \geq 1 \). From the hypothesis (and the definition of \( b^p \)-depth), there exists, for some \( c' < c \), a box in the column \( c' \) of \( S \) with entry \( y' > y \) and of \( b^p \)-depth \( \delta - 1 \). By the induction hypothesis, there exist, among the first \( p \) elements of \( \sigma_{c'} \), \( \delta \) that are at least \( y' \). Since \( \sigma_{c'} \leq \sigma_c \), the same assertion holds with \( \sigma_{c'} \) replaced by \( \sigma_c \). Now, \( y \) too occurs in the first \( p \) elements of \( \sigma_c \). Thus there are at least \( \delta + 1 \) among the first \( p \) elements of \( \sigma_c \) that are at least \( y \).

**Corollary 10.14.** Let \( y_1 < \ldots < y_p \) be the \( p \)-depth sequence of \( S \) (this was used in the procedure in §9.5.1 to determine \( u_p \)). Then, for every \( j \), \( 1 \leq j \leq p \), among the first \( p \) elements of \( v \), there occur at least \( p + 1 - j \) elements that are at least \( y_j \).

**Proof:** By definition, \( y_j \) occurs as an entry in some box of \( S \) of \( b^p \)-depth \( p - j \). Suppose \( c \) is the column number in which such a box appears. Choose the standard lift in the lemma above to be the minimal one. Then, by the lemma, among the first \( p \) elements of \( \sigma_c \), there occur at least \( p + 1 - j \) that are at least \( y_j \). Since \( \sigma_c \leq v \), the same assertion holds with \( v \) in place of \( \sigma_c \).

10.7. **Proof that** \( u \leq v \). For \( p \) a positive integer, we prove, by induction on \( p \), that \( u(p) \leq v(p) \). Since \( u(p) = u \) and \( v(p) = v \) for large \( p \), it will follow that \( u \leq v \). First consider the case \( p = 1 \). Let the right most entry in the first row of \( S \) be \( a \). From the description of the
procedure to produce \( u \) in §9.5.1, it is clear that \( u_1 = a \). On the other hand, evidently, the initial element of any standard lift of \( S \) has \( a \) as its first element (in its one-line notation), so in particular \( v_1 = a \). This proves \( u^{(1)} = v^{(1)} \).

Now let \( p > 1 \). By the induction hypothesis, we have \( u^{(p-1)} \leq v^{(p-1)} \). It is enough therefore to prove that \( u_p \leq v_p \).

Since we have proved that \( v \leq u \) (§10.5), it follows that \( v^{(p-1)} = u^{(p-1)} \). Let \( y_1 < \ldots < y_p \) be the \( p \)-depth sequence of \( S \) and let \( j, 1 \leq j \leq p \), be such that \( u_p = y_j \). Then there are exactly \( p - j \) elements among \( u_1, \ldots, u_{p-1} \) that are at least \( y_j \). Corollary 10.14 guarantees that among the first \( p \) elements of \( v \), there are at least \( p + 1 - j \) that are at least \( y_j \). Since \( u_j = v_j \) for \( j \leq p - 1 \), it follows that \( u_p = y_j \leq v_p \), and we are done.  

APPENDIX A. MULTIPLE CONCATENATIONS OF LS PATHS

The immediate provocation for this appendix comes from the need to quote its results (Propositions A.2 and A.7) in the proof of the tableau decomposition rule for KK modules (§9.9). These results are part of folklore. They are already hinted at by Littelmann in [15]: see the “precise combinatorial criterion” alluded to in the paragraph preceding the theorem in §8.1 of that paper. They are also later stated in [17, §11] with a sketch of proofs. However, we could not find a suitable reference with complete proofs. This appendix aims to provide precisely such a reference, presupposing knowledge of (a) Littelmann’s basic definitions and results on paths as in [16] and (b) the results recalled and proved from scratch in §2 above.

A.1. Standard concatenations. Let \( g \) be symmetrizable Kac-Moody algebra. Let \( \lambda_1, \ldots, \lambda_n \) be dominant integral weights. For \( j, 1 \leq j \leq n \), let \( P_j \) denote the set of LS paths of shape \( \lambda_j \). Consider the set \( P := P_1 \star \cdots \star P_n := \{ \pi_1 \star \cdots \star \pi_n \mid \pi_j \in P_j \text{ for } 1 \leq j \leq n \} \) of paths. For paths \( \theta \) and \( \theta' \) in \( P \), let us write \( \theta \sim \theta' \) if either \( e_\alpha \theta = \theta' \) or \( f_\alpha \theta = \theta' \) for some simple root \( \alpha \). This is a symmetric relation. Let us continue to denote by \( \sim \) the reflexive and transitive closure of this relation on \( P \).

A.1.1. The path \( \eta(\theta) \). Fix a \( \theta = \pi_1 \star \cdots \star \pi_n \) in \( P \). As in Proposition 4.4, which is the special case \( n = 2 \) of the present set up, it follows that:

- In the equivalence class of \( P \) containing \( \theta \), there exists a unique path \( \eta(\theta) \) that is killed by \( e_\alpha \) for every simple root \( \alpha \).
- The \( \eta(\theta) \) as above lies entirely in the dominant chamber.

A.1.2. Standard concatenations. We want to characterize those \( \theta \) for which \( \eta(\theta) = \pi_{\lambda_1} \star \cdots \star \pi_{\lambda_n} \), where as usual \( \pi_{\lambda_j} \) denotes the straight line path from the origin to \( \lambda_j \). Towards this, put \( W_j := W_{\lambda_j} \), the stabiliser of \( \lambda_j \) in the Weyl group \( W \), and let \( \tau_{j,1} > \ldots > \tau_{j,n} \) be the chain of elements in \( W/W_j \) forming the LS path \( \pi_j \) (for \( 1 \leq j \leq n \)). Consider the tuple

\[
(\tau_{1,1}, \ldots, \tau_{1,j}, \ldots, \tau_{j,1}, \ldots, \tau_{j,j}, \ldots, \tau_{1,n}, \ldots, \tau_{n,n})
\]
which is an element of

\[(W/W_1)^{\times r_1} \times \ldots \times (W/W_j)^{\times r_j} \times \ldots \times (W/W_n)^{\times r_n}\]

We call the path $\theta$ standard if the tuple (28) is standard in the sense of §2.5. A standard lift (respectively, a minimal standard lift) in the sense of §2.5 of the tuple (28) is called a standard lift (respectively, minimal standard lift) of $\theta$. We denote by $w(\theta)$ the initial element of the minimal standard lift of $\theta$.

We denote by $P_{\text{std}}$ the subset of $P$ consisting of standard paths.

**Example A.1.** The path $\pi_{\lambda_1} \ast \ldots \ast \pi_{\lambda_n}$ is standard, for (identity, $\ldots$, identity) is its minimal standard lift. Moreover, it is the only standard path in $P$ with identity as the initial element of its minimal standard lift. Thus:

\[(30) \{ \theta \in P_{\text{std}} \mid w(\theta) \leq \text{identity} \} = \{ \pi_{\lambda_1} \ast \ldots \ast \pi_{\lambda_n} \}\]

Here is a characterization of the paths $\theta$ in $P$ for which $\eta(\theta) = \pi_{\lambda_1} \ast \ldots \ast \pi_{\lambda_n}$:

**Proposition A.2.** (see [15, §8.1]) $\eta(\theta) = \pi_{\lambda_1} \ast \ldots \ast \pi_{\lambda_n}$ if and only if $\theta$ is standard.

The proof of this proposition is given in §A.3.

**A.2. Specializing to a classical case: the case of the special linear Lie algebra.** Preserve the notation of the previous subsection and specialize to the situation of §9: an integer $d \geq 2$ is fixed, $G = \mathfrak{sl}_d$, etc. Let $\mu$ be a dominant integral weight, or, equivalently a partition with less than $d$ parts. Write $\mu$ as $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{d-1} \geq 0 \geq \ldots$. Let $\omega_1 = e_1$, $\omega_2 = e_1 + e_2$, $\ldots$, $\omega_{d-1} = e_1 + \cdots + e_{d-1}$ be the fundamental weights. Let $W_{\omega_j}$, $1 \leq j \leq d$, denote the stabiliser in $W$ of $\omega_j$.

Put $m_1 = \mu_1 - \mu_2$, $\ldots$, $m_{d-2} = \mu_{d-2} - \mu_{d-1}$, $m_{d-1} = \mu_{d-1} - \mu_d = \mu_{d-1}$, and $n = m_1 + m_2 + \cdots + m_{d-1}$ (note that $n = \mu_1$). Let $\lambda_1, \ldots, \lambda_n$ be:

\[\overline{\omega_1, \ldots, \omega_j, \ldots, \omega_j', \ldots, \omega_j, \ldots, \omega_{j'}', \ldots, \omega_{j'}}, \ldots, \omega_{j'}, \ldots, \omega_j, \ldots, \omega_j, \ldots, \omega_1}\]

so that $\mu = \lambda_1 + \cdots + \lambda_n$.

The elements of $W/W_{\omega_j}$ are parametrized by subsets of cardinality $j$ of $[d]$. Each such subset is written as $\{1 \leq i_1 < \ldots < i_j \leq d\}$. Given two such subsets $i = \{1 \leq i_1 < \ldots < i_j \leq d\}$ and $i' = \{1 \leq i_1' < \ldots < i_j' \leq d\}$, we have $i \leq i'$ in the Bruhat order on $W/W_{\omega_j}$ if and only if $i_1 \leq i_1'$, $i_j \leq i_j'$, and $i_j \leq i_j'$. For a permutation $\sigma$ of $[d]$ whose one line notation is $\sigma_1 \ldots \sigma_d$, the coset $\sigma W_{\omega_j}$ corresponds to $\{1 \leq i_1 < \ldots < i_j \leq d\}$, where $i_1, \ldots, i_j$ are the elements $\sigma_1, \ldots, \sigma_j$ arranged in increasing order.

For permutations $\sigma$ and $\tau$ of $[d]$ with respective one-line notations $\sigma_1 \ldots \sigma_d$ and $\tau_1 \ldots \tau_d$, we have $\sigma \leq \tau$ in the Bruhat order if and only if $\sigma W_{\omega_j} \leq \tau W_{\omega_j}$ for every $j$, $1 \leq j < d$: see, for example, [1].

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The LS paths of shape \( \omega_j \) are all straight lines, so they too are parametrized by elements of \( W/W_{\omega_j} \). Thus a path in \( \mathcal{P} \) can be represented by a “tableau”, where a tableau consists of \( m_{d-1} + \ldots + m_1 \) top-justified columns of boxes, where each of the first \( m_{d-1} \) columns (from the left) has \( d-1 \) boxes, each of the next \( m_{d-2} \) columns has \( d-2 \) boxes, and so on; the boxes are filled with numbers between 1 and \( d \), the entries in each column being strictly increasing downwards.\(^6\)

Let, for example, \( d = 5 \) and \( m = 6 + 3 + 3 + 2 \). Then \( m_1 = 3, m_2 = 0, m_3 = 1, m_4 = 2; n = 6, \) and \( \lambda_1, \ldots, \lambda_6 \) equals \( \omega_1, \omega_1, \omega_1, \omega_3, \omega_4, \omega_4 \). And the paths in \( \mathcal{P} \) can be identified with tableaux consisting of 6 top-justified columns of boxes, the first two columns having 4 boxes each, the next column having 3 boxes, and the last three columns having 1 box each. Here are two examples of such tableaux:

\[
\begin{array}{cccc}
1 & 1 & 2 & 1 \\
3 & 2 & 4 & \\
4 & 3 & 5 & \\
5 & 5 & \\
\end{array}
\quad
\begin{array}{cccc}
1 & 1 & 2 & 3 \\
2 & 3 & 4 & \\
3 & 4 & 5 & \\
5 & 5 & \\
\end{array}
\]

For \( \theta = \tau_1 \ast \ldots \ast \tau_6 \) in \( \mathcal{P} \), the entries in the first column of the corresponding tableau define \( \tau_6 \) (which is a path of shape \( \lambda_5 = \omega_4 \)), the entries in the second column define \( \tau_5 \), and so on, until the entries in the last column define \( \tau_1 \).

**Proposition A.3.** A path \( \theta \) in \( \mathcal{P} \) is standard as defined earlier in this section (§A.1.2) if and only if the entries in the tableau corresponding to it are weakly increasing in every row from left to right.

**Proof:** Proposition 2.15 (3) is relevant here. In particular, we could use it prove the if part, but instead we directly construct an explicit standard lift. Let \( j \) be such that \( 1 \leq j < d \), and let \( h = \{1 \leq h_1 < \ldots < h_j \leq d \} \) be an element in \( W/W_{\omega_j} \). Denote by \( \tilde{h} \) the permutation whose one line notation is \( h_1 \ldots h_j h'_1 \ldots h'_{d-j'} \) where \( h'_1, \ldots, h'_{d-j} \) are the elements of \( [d] \setminus \{h_1, \ldots, h_j\} \) arranged in decreasing order. Clearly, \( \tilde{h}W_{\omega_j} = h \). Let \( k \leq j \) and let \( i = \{1 \leq i_1 < \ldots < i_k \leq d \} \) be an element of \( W/W_{\omega_k} \) such that \( h_1 \leq i_1, \ldots, h_k \leq i_k \). Let \( \tilde{i} \) be defined from \( i \) (as \( \tilde{h} \) is from \( h \) ). Then, as is not to hard to see, \( \tilde{i} \leq \tilde{i} \). This proves the if part.

Let \( j, k \) be integers such that \( 1 \leq k \leq j < d \). Let \( \sigma, \tau \) be permutations of \([d] \) with respective one line notations \( \sigma_1 \ldots \sigma_d \) and \( \tau_1 \ldots \tau_d \). Then \( \sigma W_{\omega_j} = \{h_1 < \ldots < h_j\} = h \) where \( h_1, \ldots, h_j \) are just \( \sigma_1, \ldots, \sigma_j \) arranged in increasing order, and \( \tau W_{\omega_k} = \{i_1 < \ldots < i_k\} = i \) where \( i_1, \ldots, i_k \) are just \( \tau_1, \ldots, \tau_k \) arranged in increasing order. Suppose that \( \sigma \leq \tau \). Then \( h'_1 \leq i_1, \ldots, h'_k \leq i_k \) where \( h'_1, \ldots, h'_k \) are \( \sigma_1, \ldots, \sigma_k \) arranged in increasing order. It follows that \( h_1 \leq i_1, \ldots, h_k \leq i_k \), since evidently \( h_1 \leq h'_1, \ldots, h_k \leq h'_k \). This proves the only if part of the assertion. ∎

---

\(^6\) The reversal of order, which is admittedly annoying, is necessary to preserve entrenched conventions.
Corollary A.4. The set $\mathcal{P}_{\text{std}}$ of standard paths in $\mathcal{P}$ may be identified with the set $S_\mu[d]$ of SSYT of shape $\mu$ (in the sense of §9.5) with entries from $[d]$.

The path represented by the tableau on the left in (31) is not standard whereas the one represented by the tableau on the right is standard: the tableau on the left is not a SSYT whereas the tableau on the right is.

A.3. Proof of Proposition A.2. Towards the proof, we first prove a lemma.

Lemma A.5. If $\theta$ is standard, then so is every element in the equivalence class of $\mathcal{P}$ containing $\theta$.

Proof: Let $\theta$ be standard and $\alpha$ be a simple root. We will presently show that $f_\alpha\theta$ is standard in case it does not vanish. The proof that $e_\alpha\theta$ is also standard, which we omit, is analogous. This will suffice to prove the lemma. Let us write $W/W_{\lambda_1} \times \cdots \times W/W_{\lambda_m}$ for the Cartesian product (29), and denote by $(\tau_1, \ldots, \tau_m)$ the tuple (28).

Suppose that $f_\alpha\theta$ does not vanish. From the definition of $f_\alpha$, it follows that, by increasing $m$ and replacing $\tau_j$ by $\tau_j$, $\tau_j$ for some choices of $j$, $1 \leq j \leq m$, as necessary, we may assume that the tuple as in (28) corresponding to $f_\alpha\theta$ is $(\tau'_1, \ldots, \tau'_m)$ where for every $j$, $1 \leq j \leq m$, we have

\[(32)\] $\tau'_j$ is either $\tau_j$ or $s_\alpha \tau_j$, depending upon certain conditions.

To exploit these conditions, it is useful to introduce the following terminology. Let $j$ be an integer, $1 \leq j \leq m$. We call $j$

| Changing | if $\tau'_j = s_\alpha \tau_j \neq \tau_j$ |
|---------|---------------------------------------|
| Changeable (but not changing) | if $\tau'_j = \tau_j < s_\alpha \tau_j$ |
| Resisting | if $s_\alpha \tau_j < \tau_j$ |
| Flat | if $s_\alpha \tau_j = \tau_j$ |

(33)

Using this terminology, we record some simple observations ((34)), (35), and (38) below) that we need for the proof. All of these follow readily from the definition of $f_\alpha$ as in [16]. To begin with:

(34) $j$ is changing only if $\tau_j < s_\alpha \tau_j$, so the cases in (33) are exhaustive and mutually exclusive.

In particular this means that $\tau'_j = \tau_j$ if $j$ is resisting. So we have:

(35) If $j$ is resisting or flat or changeable (but not changing), then $\tau'_j = \tau_j$.

We call $j$, $1 \leq j \leq m$, unobstructed if there exists $k$, $j \leq k \leq m$, such that $k$ is changing and there does not exist $j'$ with $j'$ resisting and $j \leq j' < k$. We call $j$ obstructed if it is not unobstructed. Evidently:

(36) $j$ is unobstructed if it is changing, and $j$ is obstructed if it is resisting.
So, from (35):

\[(37) \quad \text{If } j \text{ is obstructed, then } \tau'_j = \tau_j.\]

We also have (from the definition of the operator \( f_\alpha \)):

\[(38) \quad \text{If } j \text{ is changeable (but not changing), then } j \text{ is obstructed.}\]

Let now \( \tilde{\tau}_1 \geq \ldots \geq \tilde{\tau}_m \) be a standard lift of \( \theta \). For \( j, 1 \leq j \leq m \), define \( \tilde{\tau}'_j \) by:

\[(39) \quad \tilde{\tau}'_j = \begin{cases} 
\ s_\alpha \tilde{\tau}_j & \text{if } j \text{ is changing} \\
\ \tilde{\tau}_j & \text{if } j \text{ is obstructed}
\end{cases}\]

and, when \( j \) is flat and unobstructed, by a downward induction as required:

\[(40) \quad \tilde{\tau}'_{j+1} := \text{the smaller of } \tilde{\tau}_j \text{ and } s_\alpha \tilde{\tau}_j \text{ that is larger than or equal to } \tilde{\tau}'_j \]

Since \( \tilde{\tau}'_{j+1} \) is either \( \tilde{\tau}_{j+1} \) or \( s_\alpha \tilde{\tau}_{j+1} \) (by downward induction), it follows (by an application of the basic observation (*) in §2.2 applied to the hypothesis that \( \tilde{\tau}_j \geq \tilde{\tau}_{j+1} \)) that

\[(41) \quad \tilde{\tau}_j \lor s_\alpha \tilde{\tau}_j \geq \tilde{\tau}'_{j+1}\]

so at least one of \( \tilde{\tau}_j \) and \( s_\alpha \tilde{\tau}_j \) is larger than or equal to \( \tilde{\tau}'_{j+1} \) and (40) makes sense.

We now argue that \( \tilde{\tau}'_j \geq \tilde{\tau}'_{j+1} \) for all \( 1 \leq j < m \). If \( j \) is flat and unobstructed, then this follows from the definition (40) of \( \tilde{\tau}'_j \). If \( j \) is either changing or resisting, then \( \tilde{\tau}'_j = \tilde{\tau}_j \lor s_\alpha \tilde{\tau}_j \) (from (33), (34), and (35)), so it follows from (41) that \( \tilde{\tau}'_j \geq \tilde{\tau}'_{j+1} \). By the mutual exclusivity of the cases in (33) (which follows from (34) as already remarked) and (38), we may assume that \( j \) is obstructed but not resisting. But then \( j+1 \) is also obstructed, and so \( \tilde{\tau}'_{j+1} = \tilde{\tau}_{j+1} \) by (39), and \( \tilde{\tau}'_j = \tilde{\tau}_j \geq \tilde{\tau}_{j+1} = \tilde{\tau}'_j \).

We claim that \( \tilde{\tau}'_1 \geq \ldots \geq \tilde{\tau}'_m \) is a standard lift of \( f_\alpha \theta \). It remains only to verify that \( \tilde{\tau}'_j W_{\lambda_j} = \tilde{\tau}'_j \) for every \( j, 1 \leq j \leq m \). This is easily done, as follows:

- \( j \) changing: \( \tilde{\tau}'_j = s_\alpha \tilde{\tau}_j \) by (39), so \( \tilde{\tau}'_j W_{\lambda_j} = s_\alpha \tilde{\tau}_j W_{\lambda_j} = s_\alpha \tau_j \). But \( s_\alpha \tau_j = \tau'_j \) by (33).
- \( j \) obstructed: \( \tilde{\tau}'_j = \tilde{\tau}_j \) by (39), so \( \tilde{\tau}'_j W_{\lambda_j} = \tilde{\tau}_j W_{\lambda_j} = \tau_j \). But \( \tau_j = \tau'_j \) by (37).
- \( j \) flat: \( \tilde{\tau}_j \) is either \( \tilde{\tau}_j \) or \( s_\alpha \tilde{\tau}_j \), so \( \tilde{\tau}'_j W_{\lambda_j} \) is either \( \tau_j \) or \( s_\alpha \tau_j \). But \( \tau_j = s_\alpha \tau_j = \tau'_j \) by (35). \( \Box \)

**Proof (of Proposition A.2):** Write \( \eta \) for \( \eta(\theta) \). If \( \eta = \pi_{\lambda_1} \times \ldots \times \pi_{\lambda_m} \), then \( \eta \) is standard and so \( \theta \) is standard by the previous lemma. Now suppose that \( \theta \) is standard. Then so is \( \eta \) by the lemma. Let us write \( W/W_{\lambda_{i_1}} \times \ldots \times W/W_{\lambda_{i_m}} \) for the Cartesian product (29), and denote by \( (\sigma_1, \ldots, \sigma_m) \) the minimal standard lift of \( \eta \).

Let \( \alpha \) be any simple root. We claim that there cannot exist \( k, 1 \leq k \leq m \), such that:

\[(42) \quad s_\alpha \sigma_k < \sigma_k \quad \text{and} \quad s_\alpha \sigma_j W_{\lambda_j} = \sigma_j W_{\lambda_j} \text{ for all } 1 \leq j < k\]
To prove the claim, we suppose such a \( k \) exists and arrive at a contradiction. We have 
\[ s_\alpha \sigma_k W_{\lambda_{ik}} \leq \sigma_k W_{\lambda_{ik}}. \]
If strict inequality holds here, then \( e_\alpha \eta \) does not vanish, a contradiction, so equality holds. If \( s_\alpha \sigma_{k+1} > \sigma_{k+1} \), then \( s_\alpha \sigma_k = \sigma_k \land s_\alpha \sigma_k > \sigma_{k+1} \land s_\alpha \sigma_{k+1} = \sigma_{k+1} \), a contradiction to the hypothesis that \((\sigma_1, \ldots, \sigma_m)\) is a minimal standard lift of \( \eta \) (because then \( s_\alpha \sigma_k \) would work as a lift in place of \( \sigma_k \)). Thus (42) holds with \( k \) replaced by \( k + 1 \). Repeating these arguments sufficiently many times, we conclude that \( s_\alpha \sigma_m < \sigma_m \) and 
\[ s_\alpha \sigma_j W_{\lambda_{ij}} = \sigma_j W_{\lambda_{ij}} \]
for all \( 1 \leq j \leq m \). But then \( \sigma_m \) is not the minimal element in the coset \( \sigma_m W_{\lambda_{im}} \), which contradicts the hypothesis that \((\sigma_1, \ldots, \sigma_m)\) is the minimal standard lift of \( \eta \).

To show that \( \eta = \pi_{\lambda_1} \cdots \pi_{\lambda_n} \), it suffices to show that \( \sigma_1 \) is the identity element of the Weyl group \( W \). If \( \sigma_1 \) is not the identity element, let \( \alpha \) be a simple root such that \( s_\alpha \sigma_1 < \sigma_1 \). Then (42) holds with \( k = 1 \), a contradiction.

A.4. The crystal isomorphism. Fix notation as in the beginning of §A.1. Let \( P_{\text{std}} \) denote the set of all standard paths in \( P \). By Proposition A.2, \( P_{\text{std}} \) is precisely the set of paths \( \theta \) in \( P \) for which \( \eta(\theta) = \pi_{\lambda_1} \cdots \pi_{\lambda_n} \). Thus, by [16, Theorem 7.1], there is a (unique) crystal isomorphism\(^7\) \( \Gamma : P_\lambda \to P_{\text{std}} \), where \( P_\lambda \) denotes the set of LS paths of shape \( \lambda = \lambda_1 + \cdots + \lambda_n \).

**Proposition A.6.** The isomorphism \( \Gamma \) has the following properties:

- The straight line path \( \pi_\lambda \) (from the origin to \( \lambda \)) is mapped under \( \Gamma \) to \( \pi_{\lambda_1} \cdots \pi_{\lambda_n} \).
- The end point of \( \pi \) in \( P_\lambda \) is the same as that of its image \( \Gamma \pi \).
- \( \pi \) is \( \lambda \)-dominant if and only \( \Gamma \pi \) is so.

**Proof:** The first item is because \( \pi_\lambda \) (respectively \( \pi_{\lambda_1} \cdots \pi_{\lambda_n} \)) is the unique path in \( P_\lambda \) (respectively \( P_{\text{std}} \)) on which \( e_\alpha \) vanishes for every simple \( \alpha \). The second is because (a) \( \pi_\lambda \) and \( \pi_{\lambda_1} \cdots \pi_{\lambda_n} \) both have \( \lambda \) as end point, (b) every path in \( P_\lambda \) (respectively \( P_{\text{std}} \)) can be obtained by acting a sequence of \( f_\alpha \) operators on \( \pi_\lambda \) (respectively \( \pi_{\lambda_1} \cdots \pi_{\lambda_n} \)), (c) the first item, and finally (d) if \( f_\alpha \) does not vanish on any path \( \sigma \) (in either \( P_\lambda \) or \( P_{\text{std}} \)) then \( f_\alpha \sigma(1) = \sigma(1) - \alpha \). As for the third item, we make two observations from which it follows that \( \Gamma \) preserves \( \lambda \)-dominance:

- a path \( \sigma \) in \( P_\lambda \cup P_{\text{std}} \) is \( \lambda \)-dominant if and only if \( e_\alpha (\pi_\lambda \star \sigma) \) vanishes for all simple roots \( \alpha \).
- For paths \( \pi_1 \) and \( \pi_2 \) in \( P_\lambda \cup P_{\text{std}} \), \( e_\alpha (\pi_1 \star \pi_2) \) equals either \( e_\alpha \pi_1 \star \pi_2 \) or \( \pi_1 \star e_\alpha \pi_2 \) depending precisely upon whether or not \( i \geq j \) where \( i \) (respectively \( j \)) is the maximum non-negative integer \( k \) such that \( f_\alpha^k \pi_1 \) (respectively \( e_\alpha^k \pi_2 \)) does not vanish. \( \square \)

\(^7\)“Crystal isomorphism” just means a bijection that commutes with the action of the root operators \( f_\alpha \) and \( e_\alpha \).

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Proposition A.7. For an LS path \( \pi \) of shape \( \lambda \), the minimal element in the initial direction of \( \pi \) equals the initial element \( w(\pi) \) of the standard minimal lift of \( \pi \).

The proposition follows by combining Corollary A.10 with Lemma A.11.

A.4.1. A useful observation (Corollary A.10). Let \( \mu \) be a dominant integral weight. For a Weyl group valued function \( \tilde{\mathcal{F}} : \mathcal{P}_{\mu} \to W \) on the set \( \mathcal{P}_{\mu} \) of LS paths of shape \( \mu \), and \( v \) an element of \( W \), put \( \mathcal{P}_{\mu,v}(\tilde{\mathcal{F}}) := \{ \pi \in \mathcal{P}_{\mu} \mid \tilde{\mathcal{F}}(\pi) \leq v \} \).

Lemma A.8. Suppose that the following conditions hold for \( \pi \) an arbitrary path in \( \mathcal{P}_{\mu} \) and \( w := \tilde{\mathcal{F}}(\pi) \):

1. If \( \alpha \) a simple root with \( s_{\alpha}w < w \), then \( e_{\alpha}\pi \) does not vanish.
2. Suppose \( f_{\alpha}\pi \) does not vanish. Then either (a) \( \tilde{\mathcal{F}}(f_{\alpha}\pi) = w \) or (b) \( \tilde{\mathcal{F}}(f_{\alpha}\pi) = s_{\alpha}w > w \) and \( e_{\alpha}\pi \) vanishes.

Then, for \( v \in W \) and \( \beta \) simple such that \( s_{\beta}v < v \):

\[ \mathcal{P}_{\mu,v}(\tilde{\mathcal{F}}) = \{ f_{\beta}^k\pi \mid \pi \in \mathcal{P}_{\mu,s_{\beta}v}(\tilde{\mathcal{F}}), k \geq 0, f_{\beta}^k\pi \text{ does not vanish} \} \]

Proof: Since \( \mathcal{P}_{\mu,s_{\beta}v}(\tilde{\mathcal{F}}) \subseteq \mathcal{P}_{\mu,v}(\tilde{\mathcal{F}}) \) and, by (2), \( \mathcal{P}_{\mu,v}(\tilde{\mathcal{F}}) \) is closed under the action of \( f_{\alpha} \), it follows that the right hand side is contained in \( \mathcal{P}_{\mu,v}(\tilde{\mathcal{F}}) \). To prove the other containment, let \( \sigma \) be in \( \mathcal{P}_{\mu,v}(\tilde{\mathcal{F}}) \). Let \( k \geq 0 \) be maximal such that \( e_{\beta}^k\sigma \) does not vanish, and put \( \pi := e_{\beta}^k\sigma \). Then \( f_{\beta}^k\pi = \sigma \), so it is enough to show that \( \pi \) is in \( \mathcal{P}_{\mu,s_{\beta}v}(\tilde{\mathcal{F}}) \).

Put \( w := \tilde{\mathcal{F}}(\pi) \). On the one hand, since \( f_{\beta}^k\pi = \sigma \), it follows from (2) that \( \tilde{\mathcal{F}}(\sigma) \) equals either \( w \) or \( s_{\beta}w \), so that \( w \leq w \lor s_{\beta}w = \tilde{\mathcal{F}}(\sigma) \lor s_{\beta}\tilde{\mathcal{F}}(\sigma) \leq v \lor s_{\beta}v = v \). But, on the other, if \( s_{\beta}v < w \), then \( e_{\beta}\pi \) does not vanish by (1), a contradiction to the maximality of \( k \). Thus we have \( w < s_{\beta}w \) and \( w = w \lor s_{\beta}w \leq v \lor s_{\beta}v = s_{\beta}v \). \( \square \)

Corollary A.9. Let \( i : \mathcal{P}_{\mu} \to W \) be the function that maps each path to the minimal element in its initial direction. Then, for \( v \) in \( W \) and \( \beta \) simple such that \( s_{\beta}v < v \):

\[ \mathcal{P}_{\mu,v}(i) = \{ f_{\beta}^k\pi \mid \pi \in \mathcal{P}_{\mu,s_{\beta}v}(i), k \geq 0, f_{\beta}^k\pi \text{ does not vanish} \} \]

Proof: It follows easily from the definition of the operators \( e_{\alpha} \) and \( f_{\alpha} \) that the hypothesis of the lemma are satisfied for the function \( i \). (See also [15, Lemma in §5.3].) \( \square \)

Corollary A.10. Suppose in addition to the conditions (1) and (2) of Lemma A.8 the function \( \tilde{\mathcal{F}} \) satisfies the following: \( \mathcal{P}_{\mu,\text{identity}}(\tilde{\mathcal{F}}) = \{ \pi_\lambda \} \). Then \( \tilde{\mathcal{F}} = i \).

Proof: We proceed by induction on \( v \) to show that \( \mathcal{P}_{\mu,v}(\tilde{\mathcal{F}}) = \mathcal{P}_{\mu,v}(i) \). This will suffice. If \( v = \text{identity} \), then both sets are equal to \( \{ \pi_\lambda \} \) and the result holds. So suppose that \( v > \text{identity} \). Choose a simple root \( \alpha \) such that \( w := s_{\alpha}v < v \). By the induction hypothesis, \( \mathcal{P}_{\mu,w}(\tilde{\mathcal{F}}) = \mathcal{P}_{\mu,w}(i) \). But, by Lemma A.8, we have

\[ \mathcal{P}_{\mu,v}(\tilde{\mathcal{F}}) = \{ f_{\alpha}^k\pi \mid \pi \in \mathcal{P}_{\mu,w}(\tilde{\mathcal{F}}), k \geq 0, f_{\alpha}^k\pi \text{ does not vanish} \} \]
and, by Corollary A.9, we have
\[ \mathcal{P}_{\mu,v}(i) = \{ f_k^p \pi \mid \pi \in \mathcal{P}_{\mu,v}(i), k \geq 0, f_k^p \pi \text{ does not vanish} \}, \]
so it is clear that \( \mathcal{P}_{\mu,v}(\mathfrak{g}) = \mathcal{P}_{\mu,v}(i) \).

The above corollary together with the following lemma proves Proposition A.7. The proof of the lemma occupies §A.5.

**Lemma A.11.** Fix notation as in the first paragraph of §A.4. Let \( \mathfrak{g} : \mathcal{P}_{\lambda} \to W \) be the Weyl group valued function on \( \mathcal{P}_{\lambda} \) given by \( \mathfrak{g}(\pi) := \mathfrak{w}(\Gamma(\pi)) \). Then \( \mathcal{P}_{\lambda,\text{identity}}(\mathfrak{g}) = \{ \pi_{\lambda} \} \) and \( \mathfrak{g} \) satisfies the conditions (1) and (2) of Lemma A.8.

**A.5. Completion of the proof of Proposition A.7: Proof of Lemma A.11.** We first prove:

**Lemma A.12.** With notation as in the statement and proof of Lemma A.5, suppose that \( \bar{\tau}_1 \geq \ldots \geq \bar{\tau}_m \) be the minimal standard lift of \( \theta \). Then

1. Suppose that \( \bar{\tau}_p > s_\alpha \bar{\tau}_p \). Then \( p \) is either resisting or flat. If \( p \) is flat, then there exists \( r, p < r \leq m \), with \( r \) resisting and every \( q \) such that \( p < q < r \) is flat.
2. \( \bar{\tau}_j' \geq \bar{\tau}_j \) for all \( j, 1 \leq j \leq m \).
3. Suppose that \( j \) is changing and \( j < m \). Then \( \bar{\tau}_j' \land s_\alpha \bar{\tau}_j' = \bar{\tau}_{j+1} \).
4. \( \bar{\tau}_1' \geq \ldots \geq \bar{\tau}_m' \) is the minimal standard lift of \( f_\alpha \theta \).

**Proof:** (1) If \( p \) is changing or changeable (but not changing), then \( \bar{\tau}_p < s_\alpha \bar{\tau}_p \), so it would mean that \( \bar{\tau}_p < s_\alpha \bar{\tau}_p \) (Corollary 2.12). This proves that \( p \) can only be either flat or resisting. Suppose now that \( p \) is flat. Let \( r \) be the least integer, \( p < r \leq m \), (if it exists) such that \( r \) is not flat. If such an \( r \) doesn’t exist, put \( r = m + 1 \). For all \( q, p < q < r \), put \( \sigma_q := \bar{\tau}_q \land s_\alpha \bar{\tau}_q \). Then \( \sigma_p = s_\alpha \bar{\tau}_p \). We have \( \sigma_p \geq \ldots \geq \sigma_{r-1} \) (by the basic fact (*) in §2.2). If \( r \) is not resisting, then \( \bar{\tau}_r = \bar{\tau}_r \land s_\alpha \bar{\tau}_r \), so that \( \sigma_{r-1} \geq \bar{\tau}_r \). Thus \( \sigma_{r-1} \geq \bar{\tau}_r \geq \ldots \geq \bar{\tau}_m \) would be a standard lift of \((\bar{\tau}_p, \ldots, \bar{\tau}_m)\), which we could complete to a standard lift of \( \theta \). But then \( \sigma_p = s_\alpha \bar{\tau}_p < \bar{\tau}_p \), which contradicts the hypothesis that \( \bar{\tau}_1 \geq \ldots \geq \bar{\tau}_m \) is the minimal standard lift.

(2) Proceed by downward induction on \( j \). Since \( \bar{\tau}_j' = \bar{\tau}_j \) in case \( j \) is obstructed, and \( \bar{\tau}_j' = s_\alpha \bar{\tau}_j > \bar{\tau}_j \) in case \( j \) is changing, we may assume that \( j \) is flat and unobstructed, so \( j < m \) and \( \bar{\tau}_j' = \bar{\tau}_j \). We have, by the induction hypothesis, \( \bar{\tau}_j' \land s_\alpha \bar{\tau}_j' = \bar{\tau}_{j+1} \), and so by (3) of Remark 2.18:

\[ \bar{\tau}_j' \geq \min J_{\bar{\tau}_j'}(\bar{\tau}_{j+1}'(1)) = \min J_{\bar{\tau}_j}(\bar{\tau}_{j+1}') \geq \min J_{\bar{\tau}_j}(\bar{\tau}_{j+1}) = \bar{\tau}_j. \]

(3) Since by definition \( \bar{\tau}_{j+1}' \) is either \( \bar{\tau}_{j+1}' \lor s_\alpha \bar{\tau}_{j+1}' \), and \( \bar{\tau}_j' \) by item (2), it is enough to show that \( \bar{\tau}_j' \lor \bar{\tau}_{j+1}' \geq \bar{\tau}_{j+1}' \). If not, then, by item (1), there exists \( r \) such that \( j < r \leq m \) with \( r \) resisting and every \( q \) such that \( j < q < r \) is flat. But this cannot happen since \( j \) is changing, by the definition of the operator \( f_\alpha \).
(4) Let \( \tilde{\tau}'_j'' \geq \ldots \geq \tilde{\tau}'_m'' \) be another standard lift of \( f_\alpha \theta \). It suffices to show that \( \tilde{\tau}_j'' \geq \tilde{\tau}_j' \) for every \( j, 1 \leq j \leq m \). Proceed by downward induction on \( j \). It is convenient to put \( \tilde{\tau}_{m+1}' = \tilde{\tau}_{m+1}' = \text{identity} \). By the induction hypothesis, \( \tilde{\tau}_j'' \geq \tilde{\tau}_{j+1}' \).

In case \( j \) is obstructed, \( \tau_j' = \tau_j \) by (37), and we have

\[ \tilde{\tau}_j'' \geq \min J_{\tilde{\tau}_j'}(\tilde{\tau}_{j+1}') = \min J_{\tau_j}(\tilde{\tau}_{j+1}) = \tilde{\tau}_j = \tilde{\tau}_j' \]

Suppose now that \( j \) is changing. Then \( \tau_j < s_\alpha \tau_j = \tau_j' \) by (33) and (34). By (3) of Remark 2.18, Corollary 2.12 and Lemma 2.16 (2), and item (3) above:

\[ \tilde{\tau}_j'' \geq \min J_{s_\alpha \tau_j}(\tilde{\tau}_{j+1}') \geq \min J_{s_\alpha \tau_j}(\tilde{\tau}_{j+1}) = s_\alpha \min J_{\tau_j}(\tilde{\tau}_{j+1} + s_\alpha \tau_j') \]

The only remaining case is when \( j \) is flat and unobstructed. We then have \( j < m \) and \( \tau_j' = \tau_j \). By the induction hypothesis and item (2) above, we have \( \tilde{\tau}_j'' \geq \tilde{\tau}_{j+1}' \geq \tilde{\tau}_{j+1} \), so by (3) of Remark 2.18:

\[ \tilde{\tau}_j'' \geq \min J_{\tilde{\tau}_j'}(\tilde{\tau}_{j+1}') = \min J_{\tau_j}(\tilde{\tau}_{j+1}) \geq \min J_{\tau_j}(\tilde{\tau}_{j+1}) = \tilde{\tau}_j \]

This means we would be done in case \( \tilde{\tau}_j \geq \tilde{\tau}_j' \) (which by item (2) is equivalent to \( \tilde{\tau}_j = \tilde{\tau}_j' \)). But, \( \tilde{\tau}_j' \) is by definition the smaller of \( \tilde{\tau}_j \) and \( s_\alpha \tilde{\tau}_j \) that is larger than \( \tilde{\tau}_j' \). So it only remains to consider the case when \( \tilde{\tau}_j < s_\alpha \tilde{\tau}_j \) and \( \tilde{\tau}_j \geq \tilde{\tau}_j' \). In this situation, \( \tilde{\tau}_{j+1} < s_\alpha \tilde{\tau}_{j+1} = \tilde{\tau}_{j+1}' \) (for \( \tilde{\tau}_{j+1}' \) is by definition either \( s_\alpha \tilde{\tau}_{j+1} \) or \( \tilde{\tau}_{j+1} \), and \( \tilde{\tau}_j \geq \tilde{\tau}_{j+1} \)). This implies by item (1) that \( j + 1 \) is obstructed and therefore \( j \) is also obstructed, a contradiction.

Proof of Lemma A.11: That \( P_{\lambda, \text{identity}}(\pi) = \pi_\lambda \) follows from (30).

Now put \( \theta = \Gamma \pi \) and \( w(\theta) = w \). Let \( \theta = (\tau_1, \ldots, \tau_m) \) and let \( (\tilde{\tau}_1, \ldots, \tilde{\tau}_m) \) be the minimal standard lift of \( \theta \) (so that \( w = \tilde{\tau}_1 \)).

Proof of condition (1) of Lemma A.8: Let \( \alpha \) be a simple root such that \( s_\alpha \pi < \pi \). To show that \( e_\alpha \pi \) does not vanish, it is enough to show that \( e_\alpha \theta \) does not vanish, and for this it is enough to show that there exists \( r, 1 \leq r \leq m \), such that \( s_\alpha \tau_r = \tau_r \), and \( s_\alpha \tau_j = \tau_j \) for all \( j, 1 \leq j < r \). By way of contradiction, suppose that \( s_\alpha \tau_r > \tau_r \) for the least \( r \) such that \( s_\alpha \tau_r \neq \tau_r \) (the case when \( s_\alpha \tau_j = \tau_j \) for all \( 1 \leq j \leq m \) is included in the consideration: we put \( r = m + 1 \) in this case). For \( j, 1 \leq j < r \), set \( \sigma_j' := \tilde{\tau}_j \wedge s_\alpha \tilde{\tau}_j \). Observe that \( (\sigma_1', \ldots, \sigma_{r-1}', \tilde{\tau}_r, \ldots, \tilde{\tau}_m) \) is also a standard lift of \( \theta \). But then \( \sigma_1' = s_\alpha \pi < \pi = \tilde{\tau}_1 \), which contradicts the choice of \( (\tilde{\tau}_1, \ldots, \tilde{\tau}_m) \) as the minimal standard lift of \( \theta \).

Proof of condition (2) of Lemma A.8: Suppose that \( f_\alpha \pi \) does not vanish. Then \( f_\alpha \theta \) does not vanish either. By Lemma A.5, \( f_\alpha \theta \) is standard. Moreover, by Lemma A.12 \( (\tilde{\tau}_1', \ldots, \tilde{\tau}_m') \) is the minimal standard lift of \( \theta \). Since \( \tilde{\tau}_1' \) is either \( \tilde{\tau}_1 \) or \( s_\alpha \tilde{\tau}_1 \) by its definition, it follows that \( \tilde{\tau}_1' \) is either \( \tilde{\tau}_1 \) or \( s_\alpha \tilde{\tau}_1 \). So suppose that \( \tilde{\tau}_1' \neq \tilde{\tau}_1 \). Then, since \( \tilde{\tau}_1' \geq \tau_1 \) by item (2) of Lemma A.12, it follows that \( \tilde{\tau}_1' = s_\alpha \pi > \pi \). Moreover, this happens only if \( 1 \) is unobstructed, which means that minimum is 0 of the function \( t \mapsto (\pi(t), \alpha^\vee) \) on the interval \([0, 1]\), and so \( e_\alpha \pi \) vanishes. \( \square \)
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