SPARSE DOMINATION RESULTS FOR COMPACTNESS ON WEIGHTED SPACES

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Abstract. By means of appropriate sparse bounds, we deduce compactness on weighted $L^p(w)$ spaces, $1 < p < \infty$, for all Calderón-Zygmund operators having compact extensions on $L^2(\mathbb{R}^n)$. Similar methods lead to new results on boundedness and compactness of Haar multipliers on weighted spaces. In particular, we prove weighted bounds for weights in a class strictly larger than the typical $A_p$ class.

1. Introduction

Calderón-Zygmund theory is concerned with $L^2(\mathbb{R}^n)$ bounded singular integral operators, $T$, of the form

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) \, dy,$$

where $f$ is compactly supported, $x \notin \text{supp } f$, and $K$ is a kernel function defined on $(\mathbb{R}^n \times \mathbb{R}^n) \setminus \{(x, y) : x = y\}$ that, for some $C_K > 0$ and $0 < \delta \leq 1$, satisfies

$$|K(x, y)| \leq \frac{C_K}{|x - y|^n}$$

whenever $x \neq y$ and

$$|K(x, y) - K(x', y')| \leq C_K \frac{|x - x'|^\delta + |y - y'|^\delta}{|x - y|^{n+\delta}}$$

whenever $|x - x'| + |y - y'| \leq \frac{1}{2} |x - y|$. The fact that these operators, known as Calderón-Zygmund operators, extend to be bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ is of central importance in harmonic analysis.

In [3], Hunt, Muckenhoupt, and Wheeden extended the Calderón-Zygmund theory to weighted spaces when they characterized the classes of weights, $A_p$, such that the Hilbert transform is bounded on $L^p(w)$ for $1 < p < \infty$. A positive almost everywhere and locally integrable function $w$ is an $A_p$ weight if

$$[w]_{A_p} := \sup_Q \langle w \rangle_Q \langle w^{1-p'} \rangle_Q^{\frac{1}{p-1}} < \infty,$$

where $\langle w \rangle_Q := \frac{1}{|Q|} \int_Q w(x) \, dx$, $p'$ satisfies $\frac{1}{p} + \frac{1}{p'} = 1$, and the supremum is taken over all cubes $Q \subseteq \mathbb{R}^n$ with sides parallel to the coordinate axes.

Shortly later, it was shown that any Calderón-Zygmund operator $T$ is bounded on $L^p(w)$ for all $1 < p < \infty$ and all $w \in A_p$. However, determining the optimal dependence of

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∥T∥_{L^p(w)→L^p(w)} on [w]_{A_p} was a much more difficult problem. Extrapolation methods allowed for a reduction to the case \( p = 2 \), and the following optimal estimate became known as the \( A_2 \) conjecture: if \( T \) is a Calderón-Zygmund operator and \( w \in A_2 \), then
\[
∥Tf∥_{L^2(w)} \lesssim [w]_{A_2} ∥f∥_{L^2(w)}
\]
for any \( f \in L^2(w) \). This question was first solved by Hytönen in the celebrated paper [5].

In [8], Lerner pursued a different approach to the \( A_2 \) conjecture using a bound by positive and local operators, called sparse operators. A sparse operator has the form
\[
Sf := \sum_{Q \in S} \langle f \rangle_Q \mathbb{1}_Q
\]
for locally integrable \( f \), where \( S \) is a collection of cubes satisfying the sparseness condition: for every \( Q \in S \),
\[
\sum_{P \in \text{ch}_S(Q)} |P| \leq \frac{1}{2} |Q|,
\]
where \( \text{ch}_S(Q) \) is the set of maximal elements of \( S \) that are strictly contained in \( Q \). A refinement of Lerner’s result states that there exists a constant \( C > 0 \) such that for any compactly supported \( f \in L^1(\mathbb{R}^n) \), there is a sparse operator \( S \) satisfying
\[
|Tf(x)| \leq C|Sf|(x)
\]
for almost every \( x \in \text{supp} \ f \), see [2, 7, 9]. Since optimal weighted bounds for sparse operators are immediate, this method gives a different proof of the \( A_2 \) conjecture. Such “sparse domination” results have been of immense interest following [8].

It is natural and of independent interest to study compactness of singular integral operators in addition to the previously described theory concerning boundedness. In [16], the second author began this study by describing necessary and sufficient conditions for Calderón-Zygmund operators to extend compactly on \( L^p(\mathbb{R}) \) for \( 1 < p < \infty \). Since then, a complete theory for compact Calderón-Zygmund operators on \( L^p(\mathbb{R}^n) \) and the corresponding endpoints has been established, see [13, 14, 17]. As shown in these papers, if a Calderón-Zygmund operator extends compactly on \( L^p(\mathbb{R}^n) \), then the kernel \( K \) satisfies the estimates
\[
|K(x, y)| \lesssim \frac{F_K(x, y)}{|x - y|^n}
\]
whenever \( x \neq y \) and
\[
|K(x, y) - K(x', y')| \leq \frac{|x - x'|^\delta + |y - y'|^\delta}{|x - y|^{n+\delta}} F_K(x, y)
\]
for some \( 0 < \delta \leq 1 \) whenever \( |x - x'| + |y - y'| \leq \frac{1}{2} |x - y| \), where \( F_K \) is a bounded function satisfying
\[
\lim_{|x - y| \to \infty} F_K(x, y) = \lim_{|x - y| \to 0} F_K(x, y) = \lim_{|x + y| \to \infty} F_K(x, y) = 0.
\]
The main result of this theory we use here is the characterization for compactness of Calderón-Zygmund operators at the endpoint case from \( L^1(\mathbb{R}^n) \) to \( L^{1,\infty}(\mathbb{R}^n) \). The explicit statement of this result can be found in Theorem 3.2 of Section 3.

The aim of the current paper is to extend the theory of compact Calderón-Zygmund operators on \( L^p(\mathbb{R}^n) \) to weighted Lebesgue spaces using sparse domination methods.
Theorem 1.1. Let $T$ be a Calderón-Zygmund operator that extends compactly on $L^2(\mathbb{R}^n)$. If $1 < p < \infty$ and $w \in A_p$, then $T$ extends compactly on $L^p(w)$.

The proof of Theorem 1.1 involves establishing an appropriate sparse domination result which is interesting in its own right, Theorem 3.7. The details are described in Section 3.

It is worth noting that although our proof of Theorem 1.1 is direct, it is possible to achieve weighted compactness results via extrapolation methods, see for example the subsequent paper of Hytönen and Lappas [4]. The sparse technology also allows us to deduce results that are not attainable with extrapolation as we will next describe.

Motivated by results of [7,15], we also study properties of Haar multiplier operators. For a bounded sequence of real numbers indexed by the standard dyadic grid of cubes $D$ on $\mathbb{R}^n$, there exists an operator $S$ bounded with compact support, then there exists an operator $S$ satisfying

$$Tf = \sum_{Q \in D} \varepsilon_Q \langle f, h_Q \rangle h_Q,$$

where $h_Q$ is the Haar function adapted to $Q$. For generality, we work with Haar multipliers in the setting of arbitrary Radon measures on $\mathbb{R}^n$. See Section 2 for precise details.

Estimates for Haar multiplier operators are often similar to those satisfied by Calderón-Zygmund operators but are easier to establish because of a Haar multiplier’s diagonal structure. In this case, we obtain the following sparse bound.

Theorem 1.2. Let $T$ be a Haar multiplier adapted to a Radon measure $\mu$ and a bounded sequence of real numbers $\{\varepsilon_Q\}_{Q \in D}$. Assume that $\mu$ is supported in a dyadic cube $Q_0$. If $f$ is bounded with compact support, then there exists an operator $S_{\varepsilon}$ satisfying

$$|Tf(x)| \lesssim S_{\varepsilon} |f|(x) := \sum_{Q \in S} \varepsilon_Q \langle |f| \rangle_Q 1_Q(x)$$

for almost every $x \in \text{supp } f$, where $S$ is a sparse collection of cubes, $\varepsilon_Q := \sup_{Q' \in D(Q)} |\varepsilon_{Q'}|$, and $D(Q)$ is the set of dyadic cubes properly contained in $Q$.

The first consequence of Theorem 1.2 is a weighted bound for Haar multipliers with weights in a class strictly larger than $A_p$. For a bounded sequence of real numbers $\{\varepsilon_Q\}_{Q \in D}$, $0 < q < \infty$, and $1 < p < \infty$, we say that a nonnegative locally integrable function $w$ is an $\varepsilon^q A_p$ weight if

$$[w]_{\varepsilon^q A_p} := \sup_{Q \in D} \langle w \rangle_Q \langle w^{1-q'} \rangle_Q^{-1} < \infty.$$

Notice that if $\{\varepsilon_Q\}_{Q \in D}$ is a bounded sequence of real numbers, we have $[w]_{\varepsilon^q A_p} \leq \tilde{\varepsilon}^q[w]_{A_p}$, where $\tilde{\varepsilon}^q := \sup_{Q \in D} \langle \varepsilon_Q \rangle^q$, and thus $A_p \subseteq \varepsilon^q A_p$. Again, the averages above are taken with respect to a general Radon measure $\mu$.

Theorem 1.3. Let $T$ be a Haar multiplier adapted to a Radon measure $\mu$ and a bounded sequence of real numbers $\{\varepsilon_Q\}_{Q \in D}$ and let $\tilde{\varepsilon} := \sup_{Q \in D} \langle \varepsilon_Q \rangle$. If $2 \leq p < \infty$, and $w \in \tilde{\varepsilon}^p A_p$, then $T$ is bounded on $L^p(w)$ with

$$\|Tf\|_{L^p(w)} \lesssim [w]_{\tilde{\varepsilon}^p A_p} \|f\|_{L^p(w)}$$

for all $f \in L^p(w)$; if $1 < p \leq 2$ and $w \in \tilde{\varepsilon}^{p-1} A_p$, then $T$ is bounded on $L^p(w)$ with

$$\|Tf\|_{L^p(w)} \lesssim [w]_{\tilde{\varepsilon}^{p-1} A_p} \|f\|_{L^p(w)}.$$
for all \( f \in L^p(w) \).

We remark that Theorem 1.3 cannot be obtained by existing extrapolation methods since it holds for weights beyond the \( A_p \) classes.

Moreover, if the coefficients \( \varepsilon_Q \) possess extra decay, then we can deduce compactness of the associated Haar multiplier. We use Theorem 1.2 to obtain the following.

**Theorem 1.4.** Let \( T \) be a Haar multiplier adapted to a Radon measure \( \mu \) and a bounded sequence of real numbers \( \{\varepsilon_Q\}_{Q \in D} \) such that
\[
\lim_{\ell(Q) \to \infty} |\varepsilon_Q| = \lim_{\ell(Q) \to 0} |\varepsilon_Q| = \lim_{c(Q) \to \infty} |\varepsilon_Q| = 0,
\]
where \( \ell(Q) \) and \( c(Q) \) denote the side length and center of \( Q \) respectively. If \( 1 < p < \infty \) and \( w \in A_p \), then \( T \) is compact on \( L^p(w) \).

The paper is organized as follows. We prove the sparse bound (Theorem 1.2) and its applications to weighted boundedness (Theorem 1.3) and compactness (Theorem 1.4) for Haar multipliers in Section 2. We prove the sparse bound (Theorem 3.7) and weighted compactness result (Theorem 1.1) for Calderón-Zygmund operators in Section 3.

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## 2. Haar multipliers

### 2.1. Definitions and notation.

Let \( \mu \) be a Radon measure on \( \mathbb{R}^n \). Throughout this section, all of our integrals, averages, inner products, etcetera will be taken with respect to \( \mu \). This will change in Section 3 where we will instead work with Lebesgue measure.

Let \( D \) denote the standard dyadic grid on \( \mathbb{R}^n \), that is, the family of cubes of the form \( Q = \prod_{i=1}^{n} [2^k m_i, 2^k (m_i + 1)] \) for \( k, m_i \in \mathbb{Z} \). The expression \( \widehat{Q} \) denotes the parent of \( Q \), namely, the unique dyadic cube such that \( \ell(\widehat{Q}) = 2 \ell(Q) \) and \( Q \subseteq \widehat{Q} \). We denote by \( \text{ch}(Q) \) the children of \( Q \), that is, the set of cubes \( R \in D \) such that \( \ell(R) = \ell(Q)/2 \) and \( R \subseteq Q \).

Throughout the paper, all cubes are defined by the tensor product of intervals, and thus their sides are always parallel to the coordinates axes. For \( \lambda > 0 \) and any cube \( Q \), we write \( \lambda Q \) for the unique cube that satisfies \( c(\lambda Q) = c(Q) \) and \( \ell(\lambda Q) = \lambda \ell(Q) \). Given a measurable set \( \Omega \subseteq \mathbb{R}^n \), we denote by \( D(\Omega) \) the family of dyadic cubes \( Q \) such that \( Q \subsetneq \Omega \). If \( \Omega \) is a dyadic cube, this inclusion is equivalent to \( \widehat{Q} \subseteq \Omega \).

For \( Q \in D \) such that \( \mu(Q) > 0 \), define the *Haar function adapted to \( Q \)* by
\[
h_Q := \mu(Q)^{-\frac{1}{2}} \left( \mathbb{1}_Q - \frac{\mu(Q)}{\mu(\widehat{Q})} \mathbb{1}_{\widehat{Q}} \right).
\]

We note that this notation for \( h_Q \) is not standard, but it is convenient for our purposes. Using this notation, \( h_Q \) is supported on \( \widehat{Q} \) and constant on \( Q \) and on \( \widehat{Q} \setminus Q \). As shown in [17], we have
\[
f = \sum_{Q \in D} \langle f, h_Q \rangle h_Q.
\]
with convergence in $L^2(\mu)$ norm for $f \in L^2(\mu)$ with mean zero, where we write $\langle f, g \rangle := \int_{\mathbb{R}^n} fg\,d\mu$.

**Remark 2.1.** Since

\begin{equation}
\langle h_{Q}, h_{R} \rangle = \delta(Q,R) \left( \delta(Q,R) - \frac{\mu(Q)^{\frac{1}{2}} \mu(R)^{\frac{1}{2}}}{\mu(Q)} \right),
\end{equation}

where $\delta(Q,R) = 1$ if $Q = R$ and zero otherwise, $\{h_{Q}\}_{Q \in \mathbb{D}}$ is not an orthogonal system. However, $\{h_{Q}\}_{Q \in \mathbb{D}}$ is a frame for $L^2(\mu)$, namely, there exist $0 < C_1 \leq C_2$ such that

$$C_1 \|f\|_{L^2(\mu)} \leq \left( \sum_{Q \in \mathbb{D}} \langle f, h_{Q} \rangle^2 \right)^{\frac{1}{2}} \leq C_2 \|f\|_{L^2(\mu)},$$

for all $f \in L^2(\mu)$ with mean zero, which is enough to prove our results. The two inequalities above follow directly from (2.1).

Recall that for a bounded sequence of real numbers indexed by dyadic cubes $\{\varepsilon_Q\}_{Q \in \mathbb{D}}$, the associated Haar multiplier, $T$, is given by

$$Tf = \sum_{Q \in \mathbb{D}} \varepsilon_Q \langle f, h_Q \rangle h_Q.$$

The previous equality is understood with almost everywhere pointwise convergence, meaning

$$Tf = \lim_{M \to \infty} \sum_{Q \in \mathcal{D}(\mathbb{B}_M)} \varepsilon_Q \langle f, h_Q \rangle h_Q,$$

where $\mathbb{B}_M$ is the ball centered at the origin with diameter $M$ and $\mathcal{D}(\mathbb{B}_M)$ is the finite family of dyadic cubes $Q$ such that both $Q \subseteq \mathbb{B}_M$ and $\ell(Q) > M^{-1}$.

Writing $\langle f \rangle_Q := \frac{1}{\mu(Q)} \int_Q f\,d\mu$, we note that

\begin{equation}
\langle f, h_Q \rangle h_Q = \left( \langle f \rangle_Q - \langle f \rangle_{\hat{Q}} \right) \left( 1_Q - \frac{\mu(Q)}{\mu(\hat{Q})} 1_{\hat{Q}} \right) = \langle f \rangle_Q 1_Q + a_Q,
\end{equation}

where $a_Q := -\frac{\mu(Q)}{\mu(\hat{Q})} \langle f \rangle_Q 1_{\hat{Q}} - \langle f \rangle_Q 1_Q + \frac{\mu(Q)}{\mu(\hat{Q})} \langle f \rangle_{\hat{Q}} 1_{\hat{Q}}$ satisfies the bound

$$|a_Q| \leq \left( \frac{1}{\mu(Q)} \int_Q |f|\,d\mu + 2 \langle |f| \rangle_{\hat{Q}} \right) 1_{\hat{Q}} \leq 3 \langle |f| \rangle_{\hat{Q}} 1_{\hat{Q}}.$$

### 2.2. Technical results.

We use the following auxiliary maximal function in the proof of Theorem 1.2. Let $\{\varepsilon_Q\}_{Q \in \mathbb{D}}$ be a bounded sequence of real numbers indexed by dyadic cubes and define $M_{\varepsilon}$ by

$$M_{\varepsilon} f(x) := \sup_{Q \in \mathbb{D}} \max_{Q' \in \mathcal{D}(Q)} |\varepsilon_{Q'}| \langle |f| \rangle_{Q}.$$

Since trivially

$$M_{\varepsilon} f(x) \leq \sup_{Q \in \mathbb{D}} \max_{Q' \in \mathcal{D}(Q)} |\varepsilon_{Q'}| M f(x),$$

where $M$ is the dyadic Hardy-Littlewood maximal operator defined by

$$M f(x) := \sup_{Q \in \mathbb{D}} \langle |f| \rangle_{Q},$$

we have the following property.
Lemma 2.2. If μ is a Radon measure supported in a dyadic cube Q₀ and \( \{\varepsilon_Q\}_{Q \in \mathcal{D}} \) is a bounded sequence of real numbers, then
\[
\|M_{\varepsilon}f\|_{L^1,\infty(\mu)} := \sup_{\lambda > 0} \lambda \mu(\{x \in \mathbb{R}^n : M_{\varepsilon}f(x) > \lambda\}) \lesssim \sup_{Q \in \mathcal{D}(Q_0)} \max_{Q' \in \text{ch}(Q)} |\varepsilon_{Q'}| \|f\|_{L^1(\mu)}
\]
for all \( f \in L^1(\mu) \).

We will also use the following auxiliary maximal truncation Haar multiplier in the proof of Theorem 1.2. Let \( \{\varepsilon_Q\}_{Q \in \mathcal{D}} \) be a bounded sequence of real numbers indexed by dyadic cubes and define \( T^{\max} \) by
\[
T^{\max} f := \sup_{Q \in \mathcal{D}} \left| \sum_{P \in \mathcal{D}} \varepsilon_P \langle f, h_P \rangle h_P \right|.
\]

Lemma 2.3. If μ is a Radon measure supported in a dyadic cube Q₀, \( \{\varepsilon_Q\}_{Q \in \mathcal{D}} \) is a bounded sequence of real numbers, and \( T^{\max} \) is defined as above, then
\[
\|T^{\max} f\|_{L^1,\infty(\mu)} \lesssim \sup_{Q \in \mathcal{D}(Q_0)} |\varepsilon_Q| \|f\|_{L^1(\mu)}
\]
for all \( f \in L^1(\mu) \).

We will use the following Calderón-Zygmund decomposition in the proof of Lemma 2.3. This decomposition is described in [1, Theorem 4.2] and is related to the decomposition of [11, Theorem 2.1].

Lemma 2.4. Let μ be a Radon measure. If \( f \in L^1(\mu) \) is nonnegative and \( \lambda > 0 \) (or \( \lambda > \frac{\|f\|_{L^1(\mu)}}{\|\mu\|} \) if μ is a finite measure), then we can write
\[
f = g + \sum_{j=1}^{\infty} b_j,
\]
where
(1) \( \|g\|_{L^2(\mu)}^2 \lesssim \lambda \|f\|_{L^1(\mu)} \),
(2) there exist pairwise disjoint dyadic cubes \( Q_j \) such that \( \text{supp} b_j \subseteq \tilde{Q}_j \) and
\[
\sum_{j=1}^{\infty} \mu(Q_j) \leq \frac{1}{\lambda} \|f\|_{L^1(\mu)},
\]
and
(3) \( \int_{\mathbb{R}^n} b_j \, d\mu = 0 \) for each \( j \) and \( \sum_{j=1}^{\infty} \|b_j\|_{L^1(\mu)} \lesssim \|f\|_{L^1(\mu)} \).

Proof of Lemma 2.4. We only consider the case when \( \mu(\mathbb{R}_j^n) = \infty \) for each \( j = 1, 2, \ldots, 2^n \), where the \( \mathbb{R}_j^n \) denote the \( 2^n \) \( n \)-dimensional quadrants in \( \mathbb{R}_j^n \); the case where at least one quadrant has finite measure is handled using arguments of [11, Section 3.4]. Write
\[
\Omega := \{Mf > \lambda\} = \bigcup_{j=1}^{\infty} Q_j,
\]
where the \( Q_j \) are maximal dyadic cubes in the sense that
\[
\langle f \rangle_Q \lesssim \lambda < \langle f \rangle_{Q_j}.
\]
whenever $Q \supseteq Q_j$. Set
\[ g := f \mathbb{1}_{\mathbb{R}^n \setminus \Omega} + \sum_{j=1}^{\infty} \langle f \mathbb{1}_{Q_j} \rangle \mathbb{1}_{\hat{Q}_j} \]
and
\[ b := \sum_{j=1}^{\infty} b_j, \quad \text{where} \quad b_j := f \mathbb{1}_{Q_j} - \langle f \mathbb{1}_{Q_j} \rangle \mathbb{1}_{\hat{Q}_j}. \]
Clearly,
\[ f = g + b = g + \sum_{j=1}^{\infty} b_j. \]

To prove (1), write $g = g_1 + g_2$ where
\[ g_1 := f \mathbb{1}_{\mathbb{R}^n \setminus \Omega} \quad \text{and} \quad g_2 := \sum_{j=1}^{\infty} \langle f \mathbb{1}_{Q_j} \rangle \mathbb{1}_{\hat{Q}_j}. \]

By the Lebesgue Differentiation Theorem, $\|g_1\|_{L^\infty(\mu)} \leq \lambda$, and so $\|g_1\|_{L^2(\mu)} \leq \lambda \|f\|_{L^1(\mu)}$. On the other hand,
\[ \|g_2\|_{L^2(\mu)}^2 = \sum_{i,j=1}^{\infty} \langle f \mathbb{1}_{Q_i} \rangle \mathbb{1}_{\hat{Q}_i} \langle f \mathbb{1}_{Q_j} \rangle \mathbb{1}_{\hat{Q}_j} \mu(\hat{Q}_i \cap \hat{Q}_j). \]

Since $\hat{Q}_i \cap \hat{Q}_j \in \{\hat{Q}_i, \hat{Q}_j, \emptyset\}$, by symmetry we have
\[ \|g_2\|_{L^2(\mu)}^2 \leq 2 \sum_{i=1}^{\infty} \langle f \mathbb{1}_{Q_i} \rangle \mathbb{1}_{\hat{Q}_i} \int_{\hat{Q}_i} f d\mu = 2 \sum_{i=1}^{\infty} \langle f \mathbb{1}_{Q_i} \rangle \mathbb{1}_{\hat{Q}_i} \int_{\hat{Q}_i} f d\mu \leq 2 \lambda \|f\|_{L^1(\mu)}. \]

Now, since $Q_j \subseteq \hat{Q}_j \subseteq \hat{Q}_i$ and since the cubes $Q_j$ are pairwise disjoint by maximality, we get
\[ \|g_2\|_{L^2(\mu)}^2 \leq 2 \sum_{i=1}^{\infty} \langle f \mathbb{1}_{Q_i} \rangle \mathbb{1}_{\hat{Q}_i} \int_{\hat{Q}_i} f d\mu = 2 \sum_{i=1}^{\infty} \langle f \mathbb{1}_{Q_j} \rangle \mathbb{1}_{\hat{Q}_j} \int_{\hat{Q}_j} f d\mu \leq 2 \lambda \|f\|_{L^1(\mu)}. \]

For property (2), notice that supp $b_j \subseteq \hat{Q}_j$ by definition of $b_j$. Also, the cubes $Q_j$ are pairwise disjoint by maximality. With this and the stopping condition $\lambda < \langle f \rangle_{Q_j}$ for each $j$, we have
\[ \sum_{j=1}^{\infty} \mu(Q_j) < \sum_{j=1}^{\infty} \frac{1}{\lambda} \|f \mathbb{1}_{Q_j}\|_{L^1(\mu)} \leq \frac{1}{\lambda} \|f\|_{L^1(\mu)}. \]

Property (3) follows, first using Fubini’s theorem to see
\[ \int_{\mathbb{R}^n} b_j d\mu = \int_{Q_j} f d\mu - \int_{\hat{Q}_j} \langle f \mathbb{1}_{Q_j} \rangle \mathbb{1}_{\hat{Q}_j} d\mu = 0. \]

Therefore,
\[ \sum_{j=1}^{\infty} \|b_j\|_{L^1(\mu)} \leq \sum_{j=1}^{\infty} \left( \|f \mathbb{1}_{Q_j}\|_{L^1(\mu)} + \|\langle f \mathbb{1}_{Q_j} \rangle \mathbb{1}_{\hat{Q}_j}\|_{L^1(\mu)} \right) \leq \sum_{j=1}^{\infty} \|f \mathbb{1}_{Q_j}\|_{L^1(\mu)} \leq \|f\|_{L^1(\mu)}. \]

\[ \square \]
Proof of Lemma 2.3. Since \( \text{supp}\ \mu \subseteq Q_0 \), we have \( \mu(Q) = \mu(\tilde{Q}) \) and \( \int_Q f d\mu = \int_{\tilde{Q}} f d\mu \) for every dyadic cube \( Q \) containing \( Q_0 \) and every \( f \in L^1(\mu) \). With this, we have

\[
\langle f, h_Q \rangle = \mu(Q)^{\frac{1}{2}} (\langle f \rangle_Q - \langle f \rangle_{\tilde{Q}}) = 0
\]

for such cubes \( Q \). By (2.3) and the fact that \( h_Q \) is not defined for dyadic cubes \( Q \) such that \( \tilde{Q} \cap Q_0 = \emptyset \), we only need to work with cubes satisfying \( \tilde{Q} \subseteq Q_0 \), or equivalently, cubes in \( \mathcal{D}(Q_0) \). Furthermore, by the mean zero of \( h_P \), we have that

\[
T^{\max} \mathbb{1}_{Q_0} = \sup_{Q \in \mathcal{D}} \left| \sum_{P \in \mathcal{D}} \varepsilon_P \langle \mathbb{1}_{Q_0}, h_P \rangle h_P \right| = 0.
\]

With this \( T^{\max} f = T^{\max} (f - \langle f \rangle_{Q_0} \mathbb{1}_{Q_0}) \) and so, we can assume that \( f \) has mean zero.

Let \( \varepsilon := \sup_{Q \in \mathcal{D}(Q_0)} |\varepsilon_Q| \). We wish to show that for all \( \lambda > 0 \) and all \( f \in L^1(\mu) \), we have

\[
\mu\{T^{\max} f > \lambda\} \lesssim \frac{\varepsilon}{\lambda} \|f\|_{L^1(\mu)}.
\]

Fix \( x \in \mathbb{R}^n \) and \( Q \in \mathcal{D}(Q_0) \). If \( x \) is not in the same quadrant of \( \mathbb{R}^n \) as \( Q \), then \( h_P(x) = 0 \) for every \( P \) with \( \tilde{P} \supseteq Q \), and therefore \( T^{\max} f(x) = 0 \). If \( x \) and \( Q \) are in the same quadrant, let \( \hat{K} \) be the unique dyadic cube containing \( x \) such that \( \hat{K} \) is the smallest dyadic cube with \( \{x\} \cup Q \subseteq \hat{K} \). For all \( P \in \mathcal{D} \) such that \( Q \subseteq \tilde{P} \subseteq \hat{K} \) we have \( h_P(x) = 0 \), and so

\[
\left| \sum_{P \in \mathcal{D}(Q_0)} \varepsilon_P \langle f, h_P \rangle h_P(x) \right| = \left| \sum_{P \in \mathcal{D}(Q_0)} \varepsilon_P \langle f, h_P \rangle h_P(x) \right|
\]

\[
= \frac{1}{\mu(\hat{K})} \left| \int_{\hat{K}} \sum_{P \in \mathcal{D}(Q_0)} \varepsilon_P \langle f, h_P \rangle h_P(y) d\mu(y) \right|
\]

\[
= \frac{1}{\mu(\hat{K})} \left| \int_{\hat{K}} \sum_{P \in \mathcal{D}} \varepsilon_P \langle f, h_P \rangle h_P(y) d\mu(y) \right|
\]

\[
= |\langle Tf \rangle_{\hat{K}}| \leq M(Tf)(x),
\]

where we have used the fact that \( \int_{\hat{K}} h_P d\mu = 0 \) for \( P \in \mathcal{D} \) such that \( \tilde{P} \cap \hat{K} = \emptyset \) or \( \tilde{P} \subseteq \hat{K} \). Taking the supremum over all cubes \( Q \in \mathcal{D} \) gives

\[
T^{\max} f(x) \leq M(Tf)(x).
\]

To complete the proof, apply Lemma 2.4 to \( f \) at height \( \frac{\lambda}{\varepsilon} \) to write

\[
f = g + b = g + \sum_{j=1}^{\infty} b_j,
\]

where properties (1), (2), and (3) of the lemma hold. Moreover, since \( f \) has mean zero by assumption and \( b \) has mean zero by construction, \( g \) also has zero mean. Then

\[
\mu\{T^{\max} f > \lambda\} \leq \mu\left(\left\{ T^{\max} g > \frac{\lambda}{2} \right\} \right) + \mu\left(\bigcup_{j=1}^{\infty} Q_j \right) + \mu\left(\left\{ \mathbb{R}^n \setminus \bigcup_{j=1}^{\infty} Q_j : T^{\max} b > \frac{\lambda}{2} \right\} \right).
\]
Use Chebyshev’s inequality, the boundedness of $M$ on $L^2(\mu)$, and property (1) of Lemma 2.4 to bound the first term as follows:

$$
\mu\left(\left\{ T^\max g > \frac{\lambda}{2} \right\} \right) \lesssim \frac{1}{\lambda^2} \|M(Tg)\|_{L^2(\mu)}^2 \lesssim \frac{1}{\lambda^2} \|Tg\|_{L^2(\mu)}^2 \\
\lesssim \frac{1}{\lambda^2} \sum_{Q \in D(Q_0)} |\varepsilon_Q|^2 |\langle g, h_Q \rangle|^2 \lesssim \frac{\varepsilon^2}{\lambda^2} \|g\|_{L^2(\mu)}^2 \\
\lesssim \frac{\varepsilon}{\lambda} \|f\|_{L^1(\mu)}.
$$

The second term is controlled above by property (2) of Lemma 2.4:

$$
\mu\left(\bigcup_{j=1}^\infty Q_j \right) = \sum_{j=1}^\infty \mu(Q_j) \leq \frac{\varepsilon}{\lambda} \|f\|_{L^1(\mu)}.
$$

For the third term, we fix $x \in \mathbb{R}^n \setminus \bigcup_{j=1}^\infty Q_j$ and $Q \in D(Q_0)$. By linearity

$$
\left| \sum_{P \in D(Q_0)} \varepsilon_p(b, h_P)h_P(x) \right| \leq \sum_{j=1}^\infty \left| \sum_{P \in D(Q_0)} \varepsilon_{P'}(b_j, h_P)h_P(x) \right| \lesssim \varepsilon \sum_{j=1}^\infty \sum_{P \in D(Q_0)} |\langle b_j, h_P \rangle||h_P(x)|.
$$

For a fixed index $j$ and fixed $P \in D(Q_0)$ with $Q \subseteq \hat{P}$, we consider three cases:

(a) when $\hat{Q}_j \subseteq \hat{P}$, then $\langle b_j, h_P \rangle = 0$ since $h_P$ is constant on $\hat{Q}_j$ and $b_j$ has mean value zero on $\hat{Q}_j$,

(b) when $\hat{Q}_j \cap \hat{P} = \emptyset$, we have $\langle b_j, h_P \rangle = 0$ due to their disjoint supports, and

(c) when $\hat{P} \subset \hat{Q}_j$, it must be that $\hat{P} \subset Q'_j$ for some $Q'_j \in \text{ch}(\hat{Q}_j)$. If $Q'_j \neq Q_j$, then $\langle b_j, h_P \rangle = 0$ since $b_j$ is constant on $Q'_j$ and $h_P$ has mean value zero on $\hat{P} \subseteq Q'_j$. If $\hat{P} \subseteq Q_j$, then $h_P(x) = 0$ since $x \not\in Q_j$ and $\text{supp}(h_P) \subseteq \hat{P}$.

We are left with the case $\hat{P} = \hat{Q}_j$, so

$$
\left| \sum_{P \in D(Q_0)} \varepsilon_{P'}(b_j, h_P)h_P(x) \right| \lesssim \varepsilon \sum_{j=1}^\infty \sum_{P \in D(Q_0)} |\langle b_j, h_P \rangle||h_P(x)|.
$$

Taking the supremum over $Q \in \mathcal{D}$, we have

$$
T^\max b(x) \leq \varepsilon \sum_{j=1}^\infty \sum_{P \in \hat{Q}_j} |\langle b_j, h_P \rangle||h_P(x)|
$$

for $x \not\in \bigcup_{j=1}^\infty Q_j$. Now, by definition

$$
|\langle b_j, h_P \rangle| = \mu(P)^{-\frac{1}{2}} \left| \int_{\mathbb{R}^n} f_{\mathbb{1}_{Q_j}} \mathbb{1}_P - \langle f \mathbb{1}_{Q_j} \rangle \mathbb{1}_P - \frac{\mu(P)}{\mu(\hat{P})} f_{\mathbb{1}_{\hat{Q}_j}} \mathbb{1}_{P} \mathbb{1}_{\hat{Q}_j} \mathbb{1}_{\hat{Q}_j} \mathbb{1}_{\hat{Q}_j} d\mu \right|,
$$

and since $\hat{P} = \hat{Q}_j$, we have

$$
|\langle b_j, h_P \rangle| \lesssim \mu(P)^{-\frac{1}{2}} \|f_{\mathbb{1}_{Q_j}}\|_{L^1(\mu)}.
$$
On the other hand,
\[ \|h_P\|_{L^1(\mu)} = \mu(P)^{-\frac{1}{2}} \int_{\mathbb{R}^n} \left( \mathbb{I}_P(x) - \frac{\mu(P)}{\mu(P)} \mathbb{I}_{\bar{P}}(x) \right) d\mu(x) \leq 2\mu(P)^{\frac{1}{2}} \]

Therefore using Chebyshev’s inequality and the above estimates, we have
\[ \mu\left( \left\{ \mathbb{R}^n \setminus \bigcup_{j=1}^{\infty} Q_j : T_{\text{max}} b > \frac{\lambda}{2} \right\} \right) \leq \frac{1}{\lambda} \int_{\mathbb{R}^n \setminus \bigcup_{j=1}^{\infty} Q_j} T_{\text{max}} b(x) d\mu(x) \]
\[ \leq \frac{\varepsilon}{\lambda} \int_{\mathbb{R}^n \setminus \bigcup_{j=1}^{\infty} Q_j} \sum_{P \in D} \sum_{\bar{P} = Q_j} \langle b_j, h_P \rangle \|h_P(x)\| \]
\[ \leq \frac{\varepsilon}{\lambda} \sum_{j=1}^{\infty} \sum_{P \in D, \bar{P} = Q_j} \|\langle b_j, h_P \rangle\|_1 \mu(P)^{\frac{1}{2}} \]
\[ \leq \frac{\varepsilon}{\lambda} \|f\|_{L^1(\mu)}. \]

Combining all estimates gives
\[ \mu\{ T_{\text{max}} f > \lambda \} \lesssim \frac{\varepsilon}{\lambda} \|f\|_{L^1(\mu)}. \]

**Remark 2.5.** We note for later reference that \(|Tf| \leq T_{\text{max}} f\) pointwise. Indeed, by definition and using that \(\text{supp } \mu \subseteq Q_0\), we have
\[ |Tf(x)| = \lim_{\varepsilon(Q) \to 0} \left| \sum_{P \in D(Q_0), \bar{P} \supseteq Q} \varepsilon_P \langle f, h_P \rangle h_P(x) \right| \leq \sup_{Q \in D} \left| \sum_{P \in D(Q_0), \bar{P} \supseteq Q} \varepsilon_P \langle f, h_P \rangle h_P(x) \right| = T_{\text{max}} f(x). \]

### 2.3. Sparse domination.

We are ready to prove the sparse bound for Haar multipliers. Our proof follows closely the ideas of [7]. Differences include using the auxiliary maximal function \(M_\varepsilon\), using the Haar wavelet frame \(\{h_Q\}_{Q \in D}\), and tracking the role of the coefficients \(\varepsilon_Q\) throughout.

**Proof of Theorem 1.2.** Since \(\langle f, h_Q \rangle = 0\) for all \(Q \in D\) such that \(Q_0 \subseteq \bar{Q}\) or \(Q_0 \cap \bar{Q} = \emptyset\), we only need to work with cubes in \(D(Q_0)\), meaning that
\[ T f = \sum_{Q \in D(Q_0)} \varepsilon_Q \langle f, h_Q \rangle h_Q =: T_{Q_0} f. \]

We start by adding the cube \(Q_0\) to the family \(S\) and the function \(\varepsilon_{Q_0} |f| \mathbb{1}_{Q_0}\) to the sparse operator \(S_\varepsilon\). Define
\[ E_{Q_0} := \left\{ x \in Q_0 : \max \{ M_\varepsilon f(x), T_{\text{max}} f(x) \} > 2C \varepsilon_{Q_0} |f|_{Q_0} \right\}. \]
where $C > 0$ is the sum of the implicit constants in Lemma 2.2 and Lemma 2.3. By these two results, we have

$$
\mu(E_{Q_0}) \leq \frac{1}{2C\varepsilon_{Q_0} \|f\|_{Q_0}} C \max_{Q \in D(Q_0)} \|\varepsilon_Q\|_{L^1(\mu)} \leq \frac{1}{2} \mu(Q_0).
$$

Let $E_{Q_0}$ be the family of maximal dyadic cubes $P$ contained in $E_{Q_0}$. For $x \in Q_0 \setminus E_{Q_0}$ we trivially have

$$
|Tf(x)| \leq T^{\max} f(x) \leq 2C\varepsilon_{Q_0} \|f\|_{Q_0} \mathbb{1}_{Q_0}(x) \lesssim S_\varepsilon |f|(x).
$$

Otherwise, consider $x \in E_{Q_0}$. Let $P \in E_{Q_0}$ be the unique cube such that $x \in P \subseteq Q_0$. We formally decompose $Tf$ as follows:

$$
Tf = \sum_{I \in D \setminus I \supseteq P} \varepsilon_I \langle f, h_I \rangle h_I + \sum_{I \in D \setminus I \in \text{ch}(P)} \varepsilon_I \langle f, h_I \rangle h_I + \sum_{I \in D \setminus I \subseteq P \cap P = \emptyset} \varepsilon_I \langle f, h_I \rangle h_I.
$$

In the third term, $\hat{I} \subset \hat{P}$ implies $\hat{I} \subseteq \hat{P}'$ for some $P' \in \text{ch}(\hat{P})$, and so $I \in D(P')$. In the last term, $\hat{I} \cap \hat{P} = \emptyset$ implies $\hat{I} \cap P = \emptyset$. And since $x \in P$ and supp $h_I \subseteq \hat{I}$, we have $h_I(x) = 0$ for all cubes $I \in D$ with $\hat{I} \cap P = \emptyset$. This implies that the fourth term vanishes. Then

$$
Tf(x) = \sum_{I \in D \setminus I \supseteq P} \varepsilon_I \langle f, h_I \rangle h_I(x) + \sum_{I \in \text{ch}(\hat{P})} \varepsilon_I \langle f, h_I \rangle h_I(x) + \sum_{I \in D \setminus I \subseteq P} \varepsilon_I \langle f, h_I \rangle h_I(x)
$$

Now we use the decomposition in (2.2) and the facts that $h_I(x) = 0$ if $I \in D(P')$ for $P' \in \text{ch}(\hat{P}) \setminus \{P\}$, and $\mathbb{1}_I(x) = 0$ if $I \in \text{ch}(\hat{P}) \setminus \{P\}$, to write

$$
Tf(x) = \sum_{I \in D \setminus I \supseteq P} \varepsilon_I \langle f, h_I \rangle h_I(x) + \sum_{I \in \text{ch}(\hat{P})} (\varepsilon_I a_I(x) + \varepsilon_I \langle f \rangle \mathbb{1}_I(x)) + \sum_{I \in D \setminus I \subseteq P} \varepsilon_I \langle f, h_I \rangle h_I(x)
$$

(2.5)

$$
= \sum_{I \in D \setminus I \supseteq P} \varepsilon_I \langle f, h_I \rangle h_I(x) + \sum_{I \in \text{ch}(\hat{P})} \varepsilon_I a_I(x) + \varepsilon_P \langle f \rangle \mathbb{1}_P(x) + T_P f(x),
$$

where $T_P$ is as defined in (2.4).

By maximality of $P$, there exists a point $y \in \hat{P} \setminus E_{Q_0}$. The first term in (2.5) can be bounded as follows:

$$
\left| \sum_{I \in D \setminus I \supseteq P} \varepsilon_I \langle f, h_I \rangle h_I(x) \right| \leq \sum_{I \in D \setminus I \supseteq P} \varepsilon_I \langle f, h_I \rangle h_I(y) \leq T^{\max} f(y) \leq 2C\varepsilon_{Q_0} \langle f \rangle_{Q_0}.
$$

Similarly, since $|a_I(x)| \leq 3|\langle f \rangle|_{\hat{P}} \mathbb{1}_{\hat{P}}(x)$ for $I \in \text{ch}(\hat{P})$, we have for the second term

$$
\left| \sum_{I \in \text{ch}(\hat{P})} \varepsilon_I a_I(x) \right| \leq \sum_{I \in \text{ch}(\hat{P})} |\varepsilon_I| |\langle f \rangle|_{\hat{P}} \mathbb{1}_{\hat{P}}(x) \leq 2^n \max_{I \in \text{ch}(\hat{P})} |\varepsilon_I| |\langle f \rangle|_{\hat{P}} \mathbb{1}_{\hat{P}}(x)
$$

$$
= 2^n \max_{I \in \text{ch}(\hat{P})} |\varepsilon_I| |\langle f \rangle|_{\hat{P}} \mathbb{1}_{\hat{P}}(y) \leq 2^n M_\varepsilon f(y) \leq 2^{n+1} C\varepsilon_{Q_0} |\langle f \rangle|_{Q_0}.
$$

For the third term, we directly add the cubes $P \in E_{Q_0}$ to the family $S$ and the functions $\tilde{\varepsilon}_P |\langle f \rangle|_{P} \mathbb{1}_P$ to $S_\varepsilon |f|$. By disjointness, the sparseness condition holds for $Q_0$:

$$
\sum_{P \in E_{Q_0}} \mu(P) \leq \mu(E_{Q_0}) \leq \frac{1}{2} \mu(Q_0).
$$
The last term in (2.5) is treated by repeating the previous reasoning applied to \( T_p f \) instead of \( T f = T_{Q_0} f \), that is, starting the argument with

\[
E_P := \{ x \in P : \max \{ M_{\varepsilon} f(x), T^{\max}_P f(x) \} > 2C \varepsilon P \langle |f| \rangle_P \}
\]

and adding to \( S \) the family \( E_P \) of maximal dyadic cubes contained in \( E_P \).

\[\square\]

2.4. Boundedness and compactness on weighted spaces. We now study boundedness and compactness of Haar multiplier operators on weighted spaces. We first prove the boundedness result Theorem 1.3.

We will use the following dyadic maximal function adapted to weights. Given a locally integrable and positive almost everywhere function \( w \), we define \( M_w \) by

\[
M_w f(x) := \sup_{Q \in \mathcal{D} x \in Q} \frac{1}{w(Q)} \int_Q |f| w \, d\mu,
\]

where \( w(Q) := \int_Q w \, d\mu \). The following lemma is well-known, see [11,12] for example.

**Lemma 2.6.** If \( w \) is locally integrable and positive almost everywhere and \( 1 < p < \infty \), then \( M_w \) is bounded from \( L^p(w) \) to itself. Moreover, \( \| M_w \|_{L^p(w) \to L^p(w)} \) does not depend on \( w \).

**Proof of Theorem 1.3**. Our proof closely follows the argument in [12].

It is enough to consider the case where \( \mu \) is compactly supported, as long as we obtain bounds that are independent of \( \text{supp} \mu \). Assuming \( \mu \) has compact support, there exist pairwise disjoint dyadic cubes \( \{ Q_k \}_{k=1}^{2^n} \) with each \( Q_k \) in one of the quadrants of \( \mathbb{R}^n \) and such that \( \text{supp} \mu \subseteq \bigcup_{k=1}^{2^n} Q^k \). Dividing \( \mu \) into the \( 2^n \) measures \( \mu_k(A) := \mu(A \cap Q^k) \), we can further assume that \( \text{supp} \mu \) is contained in a dyadic cube.

Suppose that \( p \geq 2 \) and set \( \sigma = w^{1-p'} \). We use the equivalence

\[
\| T \|_{L^p(w) \to L^p(w)} = \| T(\cdot, \sigma) \|_{L^p(\sigma) \to L^p(w)}
\]

and proceed by duality. Let \( f \in L^p(\sigma) \) and \( g \in L^{p'}(w) \) be nonnegative functions with compact support. Apply Theorem 1.2 to obtain the estimate

\[
\langle |T(\sigma)|, gw \rangle \lesssim \langle S_{\varepsilon}(\sigma), gw \rangle = \sum_{j,k} \varepsilon_{Q^k_j} \langle f \sigma \rangle_{Q^k_j} \langle gw \rangle_{Q^k_j} \mu(Q^k_j),
\]

where we denote the cubes in the sparse collection \( S \) chosen at the step \( k \) by \( Q^k_j \). Note that, although the cubes \( Q^k_j \) and the coefficients \( \varepsilon_{Q^k_j} \) depend on \( \text{supp} \mu \), we aim for final estimates that are independent of \( \text{supp} \mu \).

Define \( E^k_j := Q^k_j \setminus \bigcup_j Q^{k+1}_j \). Notice that the sparseness property of the cubes \( Q^k_j \) implies that the sets \( E^k_j \) are pairwise disjoint and that \( \mu(Q^k_j) \leq 2 \mu(E^k_j) \). Using the latter inequality,
the \( \tilde{\varphi}_A \) condition for \( w \), and the containment \( E^k_j \subseteq Q^k_j \), we have
\[
\langle S_\varepsilon(f\sigma), gw \rangle = \sum_{j,k} \tilde{\varphi}_A^k \langle f\sigma \rangle_{Q^k_j} \int_{Q^k_j} gw \, d\mu
\]
\[
= \sum_{j,k} \varepsilon_{Q^k_j}^k \frac{w(Q^k_j)\sigma(Q^k_j)^{p-1}}{\mu(Q^k_j)^p} \mu(Q^k_j)^{p-1} \int_{Q^k_j} f\sigma \, d\mu \int_{Q^k_j} gw \, d\mu
\]
\[
\leq [w]_{\tilde{\varphi}_A} \sum_{j,k} \left( \frac{1}{\sigma(Q^k_j)} \int_{Q^k_j} f\sigma \, d\mu \right) \left( \frac{1}{w(Q^k_j)} \int_{Q^k_j} gw \, d\mu \right) \mu(Q^k_j)^{p-1} \sigma(Q^k_j)^{2-p}
\]
\[
\leq 2^{p-1} [w]_{\tilde{\varphi}_A} \sum_{j,k} \left( \frac{1}{\sigma(Q^k_j)} \int_{Q^k_j} f\sigma \, d\mu \right) \left( \frac{1}{w(Q^k_j)} \int_{Q^k_j} gw \, d\mu \right) \mu(E^k_j)^{p-1} \sigma(E^k_j)^{2-p}.
\]
By Hölder’s inequality,
\[
\mu(E^k_j) \leq w(E^k_j)^{\frac{1}{p}} \sigma(E^k_j)^{\frac{1}{p}},
\]
and so
\[
\mu(E^k_j)^{p-1} \sigma(E^k_j)^{2-p} \leq w(E^k_j)^{\frac{p-1}{p}} \sigma(E^k_j)^{\frac{p-1}{p}} \sigma(E^k_j)^{2-p} = w(E^k_j)^{\frac{1}{p}} \sigma(E^k_j)^{\frac{1}{p}},
\]
since \( \frac{p-1}{p} + 2 - p = \frac{1}{p} \). Using the estimates above, Hölder’s inequality, the disjointness of the sets \( E^k_j \), and Lemma 2.6, we bound \( \langle S_\varepsilon(f\sigma), gw \rangle \) by a constant times
\[
[w]_{\tilde{\varphi}_A} \sum_{j,k} \left( \frac{1}{\sigma(Q^k_j)} \int_{Q^k_j} f\sigma \, d\mu \right) \left( \frac{1}{w(Q^k_j)} \int_{Q^k_j} gw \, d\mu \right) w(E^k_j)^{\frac{1}{p}} \sigma(E^k_j)^{\frac{1}{p}}
\]
\[
\leq [w]_{\tilde{\varphi}_A} \left( \sum_{j,k} \left( \frac{1}{\sigma(Q^k_j)} \int_{Q^k_j} f\sigma \, d\mu \right)^p \sigma(E^k_j)^{\frac{1}{p}} \right)^{\frac{1}{p}} \left( \sum_{j,k} \left( \frac{1}{w(Q^k_j)} \int_{Q^k_j} gw \, d\mu \right)^{\frac{p}{p'}} \right)^{\frac{1}{p'}}
\]
\[
\leq [w]_{\tilde{\varphi}_A} \|Mg\|_{L^p(\sigma)} \|Mw\|_{L^{p'}(w)}
\]
\[
\lesssim [w]_{\tilde{\varphi}_A} \|f\|_{L^p(\sigma)} \|g\|_{L^{p'}(w)}.
\]
The case \( 1 < p < 2 \) follows from duality since \( w \in \tilde{\varphi}_A \) and only if \( \sigma \in \tilde{\varphi}_A \), and\[
[w]_{\tilde{\varphi}_A} = [w]_{\tilde{\varphi}_A}^{p'}.\]
Thus
\[
\|T\|_{L^p(w) \to L^p(w)} = \|T^{*}\|_{L^{p'}(\sigma) \to L^{p'}(\sigma)} \lesssim [\sigma]_{\tilde{\varphi}_A} = [w]_{\tilde{\varphi}_A}^{p'}.
\]

In the second half of this subsection, we treat the compactness of Haar multipliers. We start with some definitions. For any positive integer \( N \), let \( \mathcal{D}_N \) denote the *lagom cubes*
\[
\mathcal{D}_N := \{ Q \in \mathcal{D} : -2^N \leq l(Q) \leq 2^N \text{ and } \text{rdist}(Q, B_{2^N}) \leq N \},
\]
where \( \text{rdist}(P, Q) := 1 + \frac{\text{dist}(P, Q)}{\max \{ l(P), l(Q) \} } \) and \( B_{2^N} \) is the ball centered at the origin with radius \( 2^N \). We write \( \mathcal{D}_N^c := \mathcal{D} \setminus \mathcal{D}_N \).

**Theorem 2.7.** Let \( 1 < p < \infty \) and \( w \in A_p \). If \( T \) is a linear operator and \( \{ T_N \}_{N=1}^\infty \) is a sequence of compact operators on \( L^p(w) \) satisfying for every \( \varepsilon > 0 \), there exists \( N_0 \in \mathbb{N} \) such
that for all \( N > N_0 \) and all bounded \( f \) with compact support, there exists a sparse collection \( S \) with
\[
\|(T - T_N)f(x)\| \leq \varepsilon \sum_{Q \in S} \langle |f| \rangle_Q \mathbb{1}_Q(x)
\]
for almost every \( x \in \text{supp} \, f \), then \( T \) extends compactly on \( L^p(w) \).

**Proof.** Since uniform limits of compact operators are compact, it suffices to show that
\[
\lim_{N \to \infty} \|T - T_N\|_{L^p(w) \to L^p(w)} = 0.
\]
Let \( \varepsilon > 0 \) and fix \( N_0 \in \mathbb{N} \) according to the hypotheses. Let \( f \in L^p(w) \) and assume without loss of generality that \( f \) is nonnegative, bounded, and has compact support. Then, for any \( N > N_0 \), there exists a sparse collection \( S \) such that
\[
\|(T - T_N)f(x)\| \leq \varepsilon \sum_{Q \in S} \langle |f| \rangle_Q \mathbb{1}_Q(x) =: \varepsilon Sf(x)
\]
for almost every \( x \in \text{supp} \, f \). Since sparse operators are bounded on \( L^p(w) \) with norm depending only on \( n, [w]_{A_p} \), and the sparseness constant, one has
\[
\|(T - T_N)f\|_{L^p(w)} \leq \varepsilon \|Sf\|_{L^p(w)} \lesssim \varepsilon \|f\|_{L^p(w)}
\]
and the result follows. \( \square \)

We can now prove the weighted compactness result Theorem 1.4

**Proof of Theorem 1.4.** Let \( T_N \) be given by
\[
T_N f = \sum_{Q \in D_N} \varepsilon_Q \langle f, h_Q \rangle h_Q
\]
and note that each \( T_N \) is of finite rank and hence compact. Applying Theorem 1.2 gives that for every bounded \( f \) with compact support, there exists a sparse collection \( S \) such that
\[
\|(T - T_N)f(x)\| = \left| \sum_{Q \in D_N} \varepsilon_Q \langle f, h_Q \rangle h_Q \right| \lesssim \left( \sup_{Q \in D_N} |\varepsilon_Q| \right) \sum_{Q \in S} \langle |f| \rangle_Q \mathbb{1}_Q(x)
\]
for almost every \( x \in \text{supp} \, f \). The result follows upon applying Theorem 2.7. \( \square \)

3. **Calderón-Zygmund Operators**

3.1. **Notation and definitions.** In this section, all of our integrals, averages, pairings, etcetera will be taken with respect to Lebesgue measure on \( \mathbb{R}^n \). We write \( m \) for Lebesgue measure and denote the Lebesgue measure of a set \( A \subseteq \mathbb{R}^n \) by \( |A| \).

We consider three bounded functions satisfying
\[
\lim_{x \to \infty} L(x) = \lim_{x \to 0} S(x) = \lim_{x \to \infty} D(x) = 0.
\]
Without loss of generality, we assume that \( L \) and \( D \) are non-increasing, while \( S \) is non-decreasing. Moreover, since any dilation of a function satisfying a limit in (3.1) also satisfies the same limit, we omit universal constants appearing in the argument of these functions.

A measurable function \( K : (\mathbb{R}^n \times \mathbb{R}^n) \setminus \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\} \to \mathbb{C} \) is a compact *Calderón-Zygmund kernel* if it is bounded on compact subsets of its domain and there exist a function \( \omega \) satisfying the Dini-type condition
\[
\int_0^1 \int_0^1 \omega(st) \frac{ds}{s} \frac{dt}{t} < \infty
\]
and bounded functions $L$, $S$, and $D$ satisfying (3.1) such that
\begin{equation} \label{eq:3.2}
|K(x, y) - K(x', y')| \lesssim \omega \left( \frac{|x - x'| + |y - y'|}{|x - y|} \right) \frac{F_K(x, y)}{|x - y|^n},
\end{equation}
whenever $|x - x'| + |y - y'| \leq \frac{1}{2}|x - y|$ with
\begin{equation} \label{eq:3.3}
F_K(x, y) = L(|x - y|) S(|x - y|) D(|x + y|).
\end{equation}
As shown in [16], inequality \eqref{eq:3.2} and $\lim_{|x - y| \to \infty} K(x, y) = 0$ imply that $K$ satisfies the following decay estimate
\begin{equation} \label{eq:3.4}
|K(x, y)| \lesssim \frac{F_K'(x, y)}{|x - y|^n}
\end{equation}
whenever $x \neq y$, where $F_K'$ may be slightly different from the function in \eqref{eq:3.3}, but it has a similar structure and it satisfies similar estimates.

For technical reasons, we will also use an alternative formulation of a compact Calderón-Zygmund kernel in which we substitute the function $F_K(x, y)$ of \eqref{eq:3.2} with
\begin{equation} \label{eq:3.5}
F_K(x, y, x', y') = L_1(|x - y|) S_1(|x - x'| + |y - y'|) D_1 \left( 1 + \frac{|x + y|}{1 + |x - y|} \right),
\end{equation}
where $L_1$, $S_1$, and $D_1$ satisfy the limits in (3.1). It was shown how this new condition can be obtained from \eqref{eq:3.2} in [16]. In general, we will omit the subindexes in the three factors of $F_K$, using the same notation as in \eqref{eq:3.3}.

We work with Calderón-Zygmund operators $T$ having compact extensions on $L^2(\mathbb{R}^n)$ and satisfying
\begin{equation} \label{eq:3.6}
T f(x) = \int_{\mathbb{R}^n} K(x, y) f(y) \, dy
\end{equation}
for compactly supported functions $f$ and $x \notin \text{supp } f$, where $K$ satisfies properties \eqref{eq:3.2}, \eqref{eq:3.4}, and \eqref{eq:3.5} above.

Given two cubes $I, J \in \mathcal{D}$ with $\ell(I) \neq \ell(J)$, we denote the smaller of $I$ and $J$ by $I \wedge J$ and the larger of $I$ and $J$ by $I \vee J$. We define $(I, J)$ to be the unique cube containing $I \cup J$ with the smallest possible side length and such that $|c(I)|$ is minimum. Notice that $(I, J)$ need not be dyadic. We also define the eccentricity and relative distance of $I$ and $J$ to be
\[
\text{ec}(I, J) := \frac{\ell(I \wedge J)}{\ell(I \vee J)} \quad \text{and} \quad \text{rdist}(I, J) := \frac{\ell((I, J))}{\ell(I \vee J)}.
\]
Note that
\begin{equation} \label{eq:3.7}
\text{rdist}(I, J) \approx 1 + \frac{|c(I) - c(J)|}{\ell(I) + \ell(J)}.
\end{equation}
Given $I \in \mathcal{D}$, we denote the boundary of $I$ by $\partial I$, and the inner boundary of $I$ by $\mathcal{D}_I := \cup_{I' \subset \text{ch}(I)} \partial I'$. When $J \subseteq 3I$, we define the inner relative distance of $J$ and $I$ by
\[
\text{inrdist}(I, J) := 1 + \frac{\text{dist}(J, \mathcal{D}_I)}{\ell(J)}.
\]
Given three cubes $I_1, I_2,$ and $I_3$, we denote
\[
F_K(I_1, I_2, I_3) := L(\ell(I_1)) S(\ell(I_2)) D(\text{rdist}(I_3, \mathbb{B})) \quad \text{and} \quad F_K(I) := F(I, I, I),
\]
where \( \mathbb{B} := [-\frac{1}{2}, \frac{1}{2}]^n \). We define
\[
\overline{L}(\ell(I)) := \int_0^1 \omega(t) L(\ell(t^{-1} I)) \frac{dt}{t} \quad \text{and} \quad \overline{D}(\text{rdist}(I, \mathbb{B})) := \int_0^1 W(t) D(\text{rdist}(t^{-1} I, \mathbb{B})) \frac{dt}{t},
\]
where \( W(t) := \int_0^t \omega(s) \frac{ds}{s} \). We also define the corresponding
\[
\tilde{F}_K(I_1, I_2, I_3) := \overline{L}(\ell(I_1)) S(\ell(I_2)) \overline{D}(\text{rdist}(I_3, \mathbb{B})) \quad \text{and} \quad \tilde{F}_K(I) := \tilde{F}_K(I, I, I).
\]

For a cube \( Q \), let \( Q^* \) be the cube such that \( c(Q^*) = c(Q) \) and \( \ell(Q^*) = 5 \ell(Q) \). For \( Q \in \mathcal{D} \), we again write \( h_Q \) for the Haar function adapted to \( Q \), but now with respect to Lebesgue measure. Specifically,
\[
h_Q := |Q|^{-\frac{1}{2}} (1_Q - 2^{-n} 1_{\tilde{Q}}).
\]
With this notation, \( h_Q \) is supported on \( \tilde{Q} \) and constant on \( Q \) and on \( \tilde{Q} \setminus Q \).

Define the notation, \( h_Q \) is supported on a dyadic cube \( Q \) as
\[
\Delta_Q f := \sum_{R \subset \text{ch}(Q)} (<f>_R - <f>_Q) 1_R,
\]
where now \(<f>_Q := \frac{1}{|Q|} \int_{\mathbb{R}^n} f \, dm \). It is shown in \([17]\) that
\[
\Delta_Q f = \sum_{R \subset \text{ch}(Q)} <f, h_R> h_R,
\]
where we write \(<f, g> := \int_{\mathbb{R}^n} f g \, dm \). Thus by summing a telescopic sum, we get
\[
\sum_{Q \in \mathcal{D}} \sum_{2^{-N} \leq \ell(Q) \leq 2^N} <f, h_R> h_R(x) = \sum_{Q \in \mathcal{D}} \Delta_Q f(x) = <f> 1_J(x) - <f> 1_I(x),
\]
where \( I, J \in \mathcal{D} \) are such that \( x \in J \subseteq I \), \( \ell(J) = 2^{-N} \) and \( \ell(I) = 2^{N+1} \).

We define the wavelet father adapted to \( Q \) as \( \varphi_Q := |Q|^{-\frac{1}{2}} 1_Q \). Given a function \( b \in \text{BMO} \), the paraproduct operators associated with \( b \) are defined as follows:
\[
\Pi_b(f) := \sum_{I \in \mathcal{D}} <b, h_I> <f, \varphi_I> h_I \quad \text{and} \quad \Pi_b^*(f) := \sum_{I \in \mathcal{D}} <b, h_I> <f, h_I> \varphi_I.
\]
Note that the operator
\[
\Pi_{P_M(b)}^*(f) := \sum_{I \in \mathcal{D}_M} <b, h_I> <f, h_I> \varphi_I
\]
is of finite rank.

A linear operator \( T \) satisfies the weak compactness condition if there exists a bounded function \( F_W \) satisfying
\[
\lim_{\ell(Q) \to \infty} F_W(Q) = \lim_{\ell(Q) \to 0} F_W(Q) = \lim_{\ell(Q) \to \infty} F_W(Q) = 0 \quad \text{such that}
\]
\[
|\langle T 1_Q, 1_Q \rangle| \lesssim |Q| F_W(Q)
\]
for all \( Q \in \mathcal{D} \).

We define \( \text{CMO}(\mathbb{R}^n) \) as the closure in \( \text{BMO}(\mathbb{R}^n) \) of the space of continuous functions vanishing at infinity.

For positive integers \( N \), define the projection operator \( P_N \) on lagom cubes by
\[
P_N f := \sum_{Q \in \mathcal{D}_N} <f, h_Q> h_Q
We note that the definition of used in \[17, Proposition 8.2\] is just the classical argument functions. As deduced from the proof of \[17, Proposition 8.2\] terms \(\tilde{Q}\) where Remark 3.1. To show that a linear operator \(T\) is compact on \(L^2(\mathbb{R}^n)\), for instance, one can equivalently show that for every \(\varepsilon > 0\), there exists \(N_0 > 0\) so that
\[
\|P_N^\perp T f\|_{L^2(\mathbb{R}^n)} \lesssim \varepsilon\|f\|_{L^2(\mathbb{R}^n)}
\]
for all \(N > N_0\) and all \(f \in L^2(\mathbb{R}^n)\).

3.2. Technical results. The following result is proved in \[17\] in the particular case of \(\omega(t) = t^\delta\) with \(0 < \delta \leq 1\). The proof of this lemma is a straightforward modification of that contained in \[17\] Proposition 8.2.

Lemma 3.2. Let \(T\) be a linear operator associated with a compact Calderón-Zygmund kernel satisfying the weak compactness condition (3.9) and such that \(T1, T^*1 \in \text{CMO}\). If \(\tilde{T} := T - \Pi_{T^*1}\), then
\[
\|P_N^\perp \tilde{T} f\|_{L^1(\mathbb{R}^n)} \lesssim \sup_{Q \in D_N^c} \varepsilon_Q \|f\|_{L^1(\mathbb{R}^n)}
\]
for all \(f \in L^1(\mathbb{R}^n)\). The coefficients \(\varepsilon_Q\) are defined for \(Q \in D_N^c\) by
\[
\varepsilon_Q := \sum_{e \in \mathbb{Z}, m \in \mathbb{N}} \omega(2^{-|e|}) \frac{\omega(m^{-1})}{m} \max_{R \in D} \max_{i=1,2,3} F_i(Q, R),
\]
where \(Q_{e,m} := \{R \in D : \ell(Q) = 2^e \ell(R)\ \text{and}\ m \leq \text{rdist}(Q, R) < m + 1\}, D_N^c(Q) := D(Q) \cap D_N^c,\) and
i) when \(\text{rdist}(Q, R) > 3,\)
\[
F_1(Q, R) := F_K((Q), Q \wedge R, (Q), R),
\]
ii) \(\text{rdist}(Q, R) \leq 3\) and \(\text{inrdist}(Q, R) > 1,\)
\[
F_2(Q, R) := \tilde{F}_K(Q \wedge R, Q \wedge R, (Q), R),
\]
iii) and when \(\text{rdist}(Q, R) \leq 3\) and \(\text{inrdist}(Q, R) = 1,\)
\[
F_3(Q, R) := F_2(Q, R) + \tilde{F}_K(Q \wedge R) + \delta(Q, R)F_W(Q)
\]
with \(\delta(Q, R) = 1\) if \(Q = R\) and zero otherwise, while
\[
F_W(Q) = \sup_Q (|T1 - \langle T1\rangle_Q| + \langle |T^*1 - \langle T^*1\rangle_Q|\rangle_Q),
\]
and the supremum is taken over all cubes with sides parallel to the coordinate axes.

Remark 3.3. We note that the definition of \(\varepsilon_Q\) in Lemma 3.2 follows from the reasoning used in \[17\] Proposition 8.2 after incorporating the Dini-type function \(w\), which in that paper is just the classical \(w(t) = t^\delta\).

The expression \(\varepsilon_Q\) gathers the contributions of all terms in the wavelet decomposition of the argument functions. As deduced from the proof of \[17\] Proposition 8.2, all these terms are positive and decrease with eccentricity \(e\) and relative distance \(m\) so that the contribution of all terms \(\tilde{F}_K(I_1, I_2, I_3)\) with distinct cubes is comparable to the contribution of the corresponding terms with equal cubes. Then one can see that \(\tilde{F}_K(Q) \lesssim \varepsilon_Q\) and \(\lim_{N \to \infty} \sup_{Q \in D_N^c} \varepsilon_Q = 0.\)
Finally, we also note that $Q \in \mathcal{D}_N^c$ with $\ell(Q) \leq 2^{-N}$ implies $\mathcal{D}(Q) \subseteq \mathcal{D}_N^c$, and also that

$$\sup_Q \langle |T1 - T1_Q| \rangle_Q \approx \left( |Q|^{-1} \sum_{R \in \mathcal{D}_N^c(Q)} \langle T1, h_R \rangle^2 \right)^{1/2} \leq \|P_N^\perp(T1)\|_{BMO}.$$ 

**Lemma 3.4.** If $T$ is a linear operator associated to a compact Calderón-Zygmund kernel, $Q \in \mathcal{D}$, and $N > 1$, then

$$|P_N^\perp T(\mathbb{1}_{\mathbb{R}^n \setminus Q^*})(x) - P_N^\perp T(\mathbb{1}_{\mathbb{R}^n \setminus Q^*})(x')| \leq \varepsilon_Q M f(x),$$

for all $f \in L^1(\mathbb{R}^n)$ and all $x, x' \in Q$, where $\varepsilon_Q := L(\ell(Q)) S(\ell(Q)) D(\text{dist}(Q, B)) \leq \tilde{F}_K(Q) \leq \varepsilon_Q$ with $\varepsilon_Q$ as in Lemma 3.2.

**Proof.** By definition

$$|P_N^\perp T(\mathbb{1}_{\mathbb{R}^n \setminus Q^*})(x) - P_N^\perp T(\mathbb{1}_{\mathbb{R}^n \setminus Q^*})(x')| \leq \sum_{R \in \mathcal{D}_N^c} |\langle T(\mathbb{1}_{\mathbb{R}^n \setminus Q^*}), h_R \rangle| |h_R(x) - h_R(x')|.$$

For $R \in \mathcal{D}$ such that $\hat{R} \cap Q = \emptyset$, we have $h_R(x) = h_R(x') = 0$, while if $Q \subseteq \hat{R}$ we have $h_R(x) = h_R(x')$, and so the corresponding terms in (3.10) are zero. On the other hand, for $\hat{R} \subset Q$, we have that $h_R(x) - h_R(x') \neq 0$ implies $x \in \hat{R}$ or $x' \in \hat{R}$. Moreover, in that case we have $|h_R(x) - h_R(x')| \lesssim |R|^{-\frac{1}{2}}$.

Now, since $\hat{R} \subseteq Q$ implies that $\hat{R}$ does not intersect $\mathbb{R}^n \setminus Q^*$, we can use the integral representation of $T$ and the mean zero property of $h_R$ to write

$$\langle T(\mathbb{1}_{\mathbb{R}^n \setminus Q^*}), h_R \rangle = \int_{\hat{R}} \int_{\mathbb{R}^n \setminus Q^*} f(y)h_R(z)(K(z, y) - K(c(\hat{R}), y))dydz.$$ 

Since $|x - x'| \leq \ell(Q) \leq \frac{1}{2}|x - y|$ for all $y \in \mathbb{R}^n \setminus Q^*$, we can use the smoothness condition of the kernel to write

$$|\langle T(\mathbb{1}_{\mathbb{R}^n \setminus Q^*}), h_R \rangle| \leq \int_{\hat{R}} \int_{\mathbb{R}^n \setminus Q^*} |f(y)||h_R(z)||K(z, y) - K(c(\hat{R}), y)|dydz$$

$$\leq \int_{\hat{R}} |h_R(z)| \sum_{k=0}^\infty \int_{2^{k+1}Q^*} \omega \left( \frac{|z - c(\hat{R})|}{|z - y|} \right) |F_K(z, c(\hat{R}), y)|dydz,$$

where

$$F_K(z, c(\hat{R}), y) := L(|z - y|) S(|z - c(\hat{R})|) D \left( 1 + \frac{|z + y|}{1 + |z - y|} \right).$$

Since $\ell(Q) \leq 2^{k-1}\ell(Q^*) \leq |z - y|$ and $|z - c(\hat{R})| \leq \ell(\hat{R})/2 = \ell(R) \leq \ell(Q)$, we have $L(|z - y|) \leq L(\ell(Q))$ and $S(|z - c(\hat{R})|) \leq S(\ell(Q))$.

To deal with $L$, we first note that

$$2^k \ell(Q) \leq 2^{k-1}\ell(Q^*) \leq |z - y| \leq 2^k\ell(Q^*) = 2^k \ell(Q),$$

that is, $|z - y| \approx 2^k \ell(Q)$. Using this and $|z| \leq \frac{1}{2}(|z - y| + |z + y|)$, we have

$$1 + \frac{|z|}{1 + 2^k\ell(Q)} \lesssim 1 + \frac{|z|}{1 + |z - y|} \leq \frac{3}{2} \left( 1 + \frac{|z + y|}{1 + |z - y|} \right).$$
Moreover, since $|z - c(Q)| \leq \ell(Q)/2$, we also have $1 + \frac{|c(Q)|}{1+2^k\ell(Q)} \leq \frac{5}{4}(1 + \frac{|z|}{1+2^k\ell(Q)})$. Using this and (3.7), we have

$$1 + \frac{|z|}{1+2^k\ell(Q)} \geq 1 + \frac{|c(2^kQ)|}{1+2^k\ell(Q)} \gtrsim \text{rdist}(2^kQ, \mathbb{B}).$$

Then

$$F_K(z, c(\mathcal{R}), y) \leq \ell(Q)S(\ell(Q))D(\text{rdist}(2^kQ, \mathbb{B})) = F_K(Q, Q, 2^kQ).$$

Using previous estimates together with the facts that $|z - c(\mathcal{R})| \leq \ell(\mathcal{R})$, $2^k\ell(Q) \lesssim |z - y|$, and $\|h_R\|_{L^1(\mathbb{R}^n)} \lesssim |R|^{\frac{1}{2}}$, we get

$$\|T(f1_{\mathbb{R}^n \setminus Q^*}, h_R)\| \lesssim \ell(Q)S(\ell(Q)) \int_R |h_R(z)|dz \left(\sum_{k=0}^{\infty} \omega\left(\frac{\ell(\mathcal{R})}{2^k\ell(Q)}\right)D(\text{rdist}(2^kQ, \mathbb{B})) \right) \frac{1}{|2^k+1Q^*|} \int_{2^k+1Q^*} |f(y)|dy$$

$$\lesssim L(\ell(Q))S(\ell(Q))|R|^{\frac{1}{2}} \sum_{k=0}^{\infty} \omega\left(\frac{2^{-k}\ell(\mathcal{R})}{\ell(Q)}\right)D(\text{rdist}(2^kQ, \mathbb{B}))Mf(x)$$

$$\lesssim |R|^{\frac{1}{2}}L(\ell(Q))S(\ell(Q)) \int_0^1 \omega\left(\frac{t\ell(\mathcal{R})}{\ell(Q)}\right)D(\text{rdist}(t^{-1}Q, \mathbb{B})) \frac{dt}{t}Mf(x).$$

Now we parametrize all dyadic cubes $R \in \mathcal{D}_N$ such that $\mathcal{R} \subseteq Q$ and $x \in \mathcal{R}$ or $x' \in \mathcal{R}$ by length $\ell(R_j) = 2^{-j}\ell(Q)$. We note that there are at most two such cubes for each fixed $j$, one containing $x$ and another one containing $x'$. By summing over all these cubes, we finally get

$$|P_N^1T(f1_{\mathbb{R}^n \setminus Q^*})(x) - P_N^1T(f1_{\mathbb{R}^n \setminus Q^*})(x')| \lesssim \sum_{j=0}^{\infty} \|T(f1_{\mathbb{R}^n \setminus Q^*}, h_{R_j})\||R_j|^{-\frac{1}{2}}$$

$$\lesssim L(\ell(Q))S(\ell(Q)) \sum_{j=0}^{\infty} \int_0^1 \omega(t2^{-j})D(\text{rdist}(t^{-1}Q, \mathbb{B})) \frac{dt}{t}Mf(x)$$

$$\lesssim L(\ell(Q))S(\ell(Q)) \int_0^1 \int_0^1 \omega(ts)D(\text{rdist}(t^{-1}Q, \mathbb{B})) \frac{dt}{t} \frac{ds}{s}Mf(x)$$

$$\lesssim \varepsilon_QMf(x).$$

We can also prove the following result using similar ideas.

**Corollary 3.5.** If $b \in \text{CMO}(\mathbb{R}^n)$, $Q \in \mathcal{D}$, and $N > 1$, then

$$|P_N^1\Pi_b^*(f1_{\mathbb{R}^n \setminus Q^*})(x) - P_N^1\Pi_b^*(f1_{\mathbb{R}^n \setminus Q^*})(x')| \leq \varepsilon_QMf(x),$$

for all $f \in L^1(\mathbb{R}^n)$ and all $x, x' \in \mathbb{R}^n$, where $\varepsilon_Q := L(\ell(Q))S(\ell(Q))\tilde{D}(\text{rdist}(Q, \mathbb{B})) \leq \varepsilon$ with $L(t) = \|P_\mathcal{D}_t^c b\|_{\text{BMO}}$, $S(t) = \|P_\mathcal{D}_t^c b\|_{\text{BMO}} + (1 + \|b\|_{\text{BMO}})^{\frac{1}{2}}(\frac{1}{1+t})^{\frac{1}{2}}$, and $D(t) = \|P_\mathcal{D}_t^{log\ast} b\|_{\text{BMO}}$. We note that, using the ceiling function notation, $\mathcal{D}_t$ actually denotes $\mathcal{D}_{[t]}$. 
Proof. It was shown in \[16\] that if \( b \in \text{CMO} \), then the paraproduct operator \( \Pi_b^* \) is associated to a compact Calderón-Zygmund kernel with constant given by

\[
\|P_{D_{|x-y|}} b\|_{\text{BMO}}(\|P_{D_{|x-y|}} b\|_{\text{BMO}} + (1 + \|b\|_{\text{BMO}})\min(1, |x-y|^{\frac{3}{2}}))\|P_{D_{\log|x+y|}} b\|_{\text{BMO}} = L(|x-y|)S(|x-y|)D(|x+y|).
\]

Similar reasoning to that developed in Lemma \[3.4\] yields the result. \( \square \)

As seen in \[14\], \( P_N \) is bounded on \( \text{CMO} \). Then the hypothesis \( T^{*1} \in \text{CMO} \) implies that also \( P_N T^{*1} \in \text{CMO} \). In fact, \( \|P_N T^{*1}\|_{\text{BMO}} \leq \|T^{*1}\|_{\text{BMO}} \) and \( \|P_N^* (P_N T^{*1})\|_{\text{BMO}} = 0 \) for all \( M > N \). This justifies the expression \( \Pi^*_{P_N(T^{*1})} \) in the next result, which follows from Lemma \[3.4\] and Corollary \[3.5\].

**Corollary 3.6.** If \( T \) is a linear operator associated with a compact Calderón-Zygmund kernel, \( \tilde{T} := T - \Pi^*_{P_N(T^{*1})} \), and \( Q \in \mathcal{D} \), then

\[
|P_N^\perp \tilde{T}(f\mathbb{1}_{\mathbb{R}^n\setminus Q^*})(x) - P_N^\perp \tilde{T}(f\mathbb{1}_{\mathbb{R}^n\setminus Q^*})(x')| \leq \varepsilon_Q M f(x)
\]

for all \( f \in L^1(\mathbb{R}^n) \) and all \( x, x' \in Q \), where \( \varepsilon_Q := L(\ell(Q))S(\ell(Q))\bar{D}(\text{rdist}(Q, \mathbb{R})) \leq \varepsilon_Q \).

A consequence of the work in Lemma \(3.4\) and Corollary \(3.5\) is that the kernels of both \( T \) and \( \Pi^*_{P_N(T^{*1})} \) share similar estimates. In the next section we denote by \( K \) the kernel of \( \tilde{T} \), which satisfies the properties of a compact Calderón-Zygmund kernel \(3.2\), \(3.4\), and \(3.5\).

### 3.3. Sparse domination for compact Calderón-Zygmund operators.

**Theorem 3.7.** Let \( T \) be a linear operator associated to a compact Calderón-Zygmund kernel satisfying the weak compactness condition \(3.9\) and \( T_1, T^{*1} \in \text{CMO} \) and let \( \tilde{T} := T - \Pi^*_{P_N(T^{*1})} \). For every \( \varepsilon > 0 \) there exists \( N_0 > 0 \) such that for all \( N > N_0 \) and every compactly supported \( f \in L^1(\mathbb{R}^n) \), there is a sparse family of cubes \( \mathcal{S} \) such that

\[
|P_N^\perp \tilde{T} f(x)| \leq \varepsilon \sum_{R \in \mathcal{S}} |f|_{R^*} \mathbb{1}_R(x) =: \varepsilon_S |f|(x)
\]

for almost every \( x \in \mathbb{R}^n \).

**Proof.** Without loss of generality, suppose there is a dyadic cube \( B \) such that \( \ell(B) > 1 \) and \( \text{supp} f \subseteq B \). Let \( \{\varepsilon_Q\}_{Q \in \mathcal{D}} \) be the sequence in the statement of Lemma \(3.2\) which satisfies

\[
\lim_{N \to \infty} \sup_{Q \in \mathcal{D}_N} \varepsilon_Q = 0.
\]

Given \( \varepsilon > 0 \), let \( N_0 > 0 \) be such that \( \sup Q \in \mathcal{D}_N N > N_0 \) and let \( Q_0 \) be a cube such that \( B \subseteq Q_0 \) and \( \text{dist}(B, Q_0) \geq 2^{N+3} \ell(B) \).

We first establish the sparse estimate outside of \( Q_0 \). For \( j \geq 0 \), we write \( Q_j := 2^j Q_0 \) and for \( j \geq 1 \) we define \( P_j := Q_j \setminus Q_{j-1} \). Note that the family \( \{Q_j\}_{j \geq 0} \) is sparse by construction.

Let \( x \in P_j \). By definition,

\[
P_N^\perp \tilde{T} f(x) = \tilde{T} f(x) - \sum_{R \in \mathcal{D}_N} \langle \tilde{T} f, h_R \rangle h_R(x);
\]

we will bound each term separately. For the first term, since \( x \notin \text{supp} f \), we can write

\[
|\tilde{T} f(x)| = \left| \int_B K(x, y)f(y) \, dy \right| \leq \int_B \frac{F_K(x, y)}{|x-y|^a} f(y) \, dy
\]
where $K$ denotes the kernel of $\tilde{T}$ and
\[
F_K(x, y) = L(|x - y|)S(|x - y|)D \left(1 + \frac{|x + y|}{1 + |x - y|}\right).
\]
Since $|x - y| \approx \ell(Q_y)$ and $|y - c(Q_0)| \lesssim \ell(Q_0)$ for $y \in B$, we have by the same reasoning used in Lemma 3.4 that $L(|x - y|) \leq L(\ell(Q_j))$, $S(|x - y|) \leq S(\ell(Q_j))$, and $D(1 + \frac{|x + y|}{1 + |x - y|}) \lesssim D(\text{rdist}(Q_j, B))$. Then
\[
F_K(x, y) \leq L(\ell(Q_j))S(\ell(Q_j))D(\text{rdist}(Q_j, B)) = F_K(Q_j) \lesssim \epsilon,
\]
where in the last inequality we used that $\ell(Q_j) \geq \ell(Q_0) > 2^{N+3}\ell(B) \geq 2^{N+3}$, and thus $Q_j \in D_N$. With this and the fact that $|x - y| \approx \ell(Q_j)$, we have
\[
|\tilde{T}f(x)| \lesssim \frac{\epsilon}{|Q_j|} \|f\|_{L^1(\mathbb{R}^n)} = \epsilon(|f|)_{Q_j} 1_{Q_j}(x).
\]

On the other hand, we note that the second term in (3.11) is defined by a telescopic sum such that, for fixed $x \in P_j$, the collection of cubes $R \in D_N$ with $x \in \tilde{R}$ form a convex chain. With this we mean that if $R_1, R_2 \in D_N$ with $R_1 \subseteq R_2$, then any other cube $R' \in D$ with $R_1 \subseteq R' \subseteq R_2$ satisfies $R' \in D_N$ and so, $R'$ is also in the sum. Moreover, we can obviously assume that the sum is non-empty. In that case, we have
\[
\sum_{R \in D_N} \langle \tilde{T}f, h_R \rangle h_R(x) = \langle \tilde{T}f \rangle_{I+j} I_j(x) - \langle \tilde{T}f \rangle_{I+j} I_j(x),
\]
where $I, J \in D$ are such that $x \in J \subseteq I$, and $\ell(J), \ell(I) \leq 2^{N+1}$. We can now apply the same ideas to bound both terms, and so we only write the estimates for the second term. Since $x \in I \cap P_j$, we have as before $2^{j-2} \ell(Q_0) < |x - y|$ for all $y \in B \subseteq Q_0$. On the other hand,
\[
2^{j-2} \ell(Q_0) \geq \ell(Q_0)/2 \geq 2^{N+2} \ell(B) \geq 2\ell(I),
\]
which implies $\ell(I) \leq 2^j \ell(Q_0)$. With this and $|t - x| \leq \ell(I)$ for all $t \in I$, we get
\[
|t - y| \geq |x - y| - |t - x| \geq 2^{j-2} \ell(Q_0) - \ell(I) \geq 2^{j-3} \ell(Q_0)
\]
and
\[
|t - y| \leq |x - y| + |t - x| \leq 2^j \ell(Q_0) + \ell(I) \leq 2^{j+1} \ell(Q_0).
\]
Therefore, $|t - y| \approx \ell(Q_j)$. We also have $|y - c(Q_0)| \leq \ell(Q_0)/2$. Write
\[
|\langle \tilde{T}f \rangle_{I+j} I_j(x)| = \frac{1}{|I|} \left| \int_I \int_B K(t, y) f(y) dy dt \right|
\leq \frac{1}{|I|} \left| \int_I \int_B \frac{F_K(t, y)}{|x - y|^n} |f(y)| dy dt \right|
\leq \frac{F_K(Q_j)}{|Q_j|} \|f\|_{L^1(\mathbb{R}^n)} \leq \epsilon(|f|)_{Q_j} 1_{Q_j}(x),
\]
where the last inequality follows from (3.12).

We now work to establish the sparse bound inside $Q_0$. For this local piece, we follow the ideas from [10] to define recursively the desired sparse family $\mathcal{S}$ and sparse operator $S$. Let $D_N(Q_0) := D(Q_0) \cap D_N$ and $Q := \{ Q \in D_N(Q_0) : \ell(Q) = 2^{-(N+2)} \}$. We decompose $\tilde{T}f$ as
\[
\tilde{T}f = \sum_{Q \in Q} \tilde{T}(f 1_Q).
\]
If we assume the desired sparse domination result holds for $P_N^+ T(f 1_Q)$, then by disjointness of the cubes $Q$, we can deduce a similar sparse estimate for $P_N^+ \tilde{T}$:

$$|P_N^+ \tilde{T} f| \leq \sum_{Q \in \mathcal{Q}} |P_N^+ \tilde{T} (f 1_Q)| \lesssim \varepsilon \sum_{Q \in \mathcal{Q}} S |f 1_Q|$$

$$= \varepsilon \sum_{Q \in \mathcal{Q}} \sum_{R \in \mathcal{S}(Q)} \langle |f| \rangle_{R^1} \leq \varepsilon \sum_{R \in \mathcal{S}(Q_0)} \langle |f| \rangle_{R^1}.$$

Therefore, we will only prove the sparse estimate for each $P_N^+ \tilde{T} (f 1_Q)$.

We start by adding all cubes $Q \in \mathcal{Q}$ to the family $\mathcal{S}$ and functions $\langle |f| \rangle_Q 1_Q$ to the sparse operator $S |f|$. These cubes are pairwise disjoint and satisfy $\sum_{Q \in \mathcal{Q}} |Q| = |Q_0|$. This family does not satisfy the sparseness condition, but we can divide the family into two disjoint subfamilies $Q_1, Q_2$ containing exactly half of the cubes, each satisfying the sparseness condition $\sum_{Q \in \mathcal{Q}} |Q| = |Q_0|/2$. This leads to a domination by at most two sparse operators, which is acceptable. To simplify notation, we still denote each subfamily by $Q$.

Fix $Q \in \mathcal{Q}$ and define

$$E_Q := \{ x \in Q : M(f 1_Q)(x) > c' \langle |f| \rangle_Q \} \cup \{ x \in Q : |P_N^+ \tilde{T} (f 1_Q)(x)| > c' \varepsilon \langle |f| \rangle_Q \},$$

where $c' > 0$ is chosen so that

$$|E_Q| \leq \frac{1}{2^{n+2}} |Q|.$$

To show that $c' > 0$ is independent of $\varepsilon$, from Lemma 3.2 we have

$$|E_Q| \leq \frac{C}{c' \langle |f| \rangle_Q} \| f 1_Q \|_{L^1(\mathbb{R}^n)} + \frac{C \sup_{Q \in D_N} \| f 1_Q \|_{L^1(\mathbb{R}^n)}}{c' \varepsilon \langle |f| \rangle_Q} \| f 1_Q \|_{L^1(\mathbb{R}^n)} \leq \frac{2C}{c'} |Q| \leq \frac{1}{2^{n+2}} |Q|$$

by choosing $c' > 2^{n+3}$. We note that the constant $C > 0$ may depend on the dimension $n$ but not on $\varepsilon > 0$.

We define another exceptional set

$$\hat{E}_Q := \{ x \in Q : M(1_{E_Q})(x) > 2^{-(n+1)} \},$$

and define $\mathcal{E}_Q$ to be the family of maximal (with respect to inclusion) dyadic cubes $P$ contained in $\hat{E}_Q$. Note that for each $Q \in \mathcal{Q}$, the containment $\mathcal{E}_Q \subseteq D_N$ holds. Moreover, due to maximality, the cubes $P \in \mathcal{E}_Q$ are pairwise disjoint, and thus $\mathcal{E}_Q$ is a sparse collection:

$$\sum_{P \in \mathcal{E}_Q} |P| \leq |E_Q| \leq \frac{1}{2^{n+2}} |Q|.$$

We see now that

$$|P \cap E_Q| \leq \frac{1}{2} |P|.$$

By maximality, $2P \cap (Q \setminus E_Q) \neq \emptyset$, and so there exists $x \in 2P$ such that $M(1_{E_Q})(x) \leq 2^{-(n+1)}$. Then

$$\frac{|E_Q \cap 2P|}{|2P|} \leq 2^{-(n+1)},$$

which proves the upper inequality. Note that the inequality in (3.15) implies $|P \setminus E_Q| > \frac{1}{2} |P|$. We can now estimate $|P_N^+ \tilde{T} (f 1_Q)(x)|$ for $x \in Q$. First, for $x \in Q \setminus E_Q$ we trivially have

$$|P_N^+ \tilde{T} (f 1_Q)(x)| \leq c' \varepsilon \langle |f| \rangle_Q = c' \varepsilon \langle |f| \rangle_Q 1_Q(x).$$
Second, to obtain an estimate on $E_Q$, we note that $\left| E_Q \setminus \bigcup_{P \in \mathcal{E}_Q} P \right| \leq \left| \tilde{E}_Q \setminus \bigcup_{P \in \mathcal{E}_Q} P \right| = 0$, and so, we do not need to bound $|P_N^\perp \tilde{T}(f \mathbb{1}_Q)(x)|$ for $x \in E_Q \setminus \bigcup_{P \in \mathcal{E}_Q} P$.

It only remains to control $|P_N^\perp \tilde{T}(f \mathbb{1}_Q)(x)|$ for $x \in \bigcup_{P \in \mathcal{E}_Q} P$. For any $P \in \mathcal{E}_Q$, any $x \in P$, and any $x' \in P \setminus E_Q$, we decompose $P_N^\perp \tilde{T}f(x)$ as follows:

\[ |P_N^\perp \tilde{T}(f \mathbb{1}_Q)(x)| \leq |P_N^\perp \tilde{T}(f \mathbb{1}_{Q \cap P_\ast})(x)| + |P_N^\perp \tilde{T}(f \mathbb{1}_{P_\ast})(x)| \]

\[ \leq |P_N^\perp \tilde{T}(f \mathbb{1}_{Q \cap P_\ast})(x) - P_N^\perp \tilde{T}(f \mathbb{1}_{Q \setminus P_\ast})(x')| + |P_N^\perp \tilde{T}(f \mathbb{1}_{Q \setminus P_\ast})(x')| \]

\[ + |P_N^\perp \tilde{T}(f \mathbb{1}_{P_\ast})(x)| \]

\[ \leq |P_N^\perp \tilde{T}(f \mathbb{1}_{Q \setminus P_\ast})(x) - P_N^\perp \tilde{T}(f \mathbb{1}_{Q \setminus P_\ast})(x')| + |P_N^\perp \tilde{T}(f \mathbb{1}_{Q \setminus P_\ast})(x')| \]

\[ + |P_N^\perp \tilde{T}(f \mathbb{1}_{P_\ast})(x)| \]

\[ := I + II + III + IV. \]

The second term is easily controlled since $x' \not\in E_Q$ implies $|P_N^\perp \tilde{T}(f \mathbb{1}_Q)(x')| \leq c'\varepsilon(|f|)_Q$, and so

\[ II \leq c'\varepsilon(|f|)_Q \mathbb{1}_Q(x). \]

For the first and third terms, define

\[ E'_P := \{ x \in P : |P_N^\perp \tilde{T}(f \mathbb{1}_{P_\ast})(x)| > c'\varepsilon(|f|)_{P_\ast} \}. \]

By Lemma 3.2

\[ |E'_P| \leq \frac{C\varepsilon}{c'\varepsilon(|f|)_{P_\ast}} \| f \mathbb{1}_{P_\ast} \|_{L^1(\mathbb{R}^n)} \leq \frac{1}{2^{n+2}} |P|. \]

Then $|P \setminus E'_P| > \frac{1}{2} |P|$. This, together with $|P \setminus E_Q| > \frac{1}{2} |P|$, implies that $(P \setminus E_Q) \cap (P \setminus E'_P) \neq \emptyset$. Therefore, there exists $x' \in P$ such that $M(f \mathbb{1}_Q)(x') \leq c'\varepsilon(|f|)_Q$ and $|P_N^\perp \tilde{T}(f \mathbb{1}_{P_\ast})(x')| \leq c'\varepsilon(|f|)_{P_\ast}$. Then, since $(f \mathbb{1}_Q)\mathbb{1}_{\mathbb{R}^n \setminus P_\ast} = f \mathbb{1}_{Q \setminus P_\ast}$, we can apply Corollary 3.6 to obtain

\[ I \leq \varepsilon M(f \mathbb{1}_Q)(x') \leq c'\varepsilon(|f|)_Q \mathbb{1}_Q(x). \]

Moreover,

\[ III \leq c'\varepsilon(|f|)_{P_\ast} = c'\varepsilon(|f|)_{P_\ast} \mathbb{1}_P(x). \]

We add the cubes $P \in \mathcal{E}_Q$ to the family $\mathcal{S}$ and the functions $(|f|)_{P_\ast} \mathbb{1}_P$ into $S[f]$. The family $\mathcal{E}_Q$ is sparse by (3.14).

The fourth term is controlled by iterating the above argument, starting at (3.13) but replacing $Q$ with $P$, and so defining

\[ E_P := \{ x \in P : M(f \mathbb{1}_P)(x) > c'\varepsilon(|f|)_P \} \cup \{ x \in P : |P_N^\perp \tilde{T}(f \mathbb{1}_P)(x)| > c'\varepsilon(|f|)_P \}. \]

\[ \square \]

3.4. Compactness on weighted spaces. We can now prove the compactness of Calderón-Zygmund operators on weighted spaces.

Proof of Theorem 1.1. Let $\tilde{T} = T - \Pi^*_{P_N(T^*)}$. Since $\Pi^*_{P_N(T^*)}$ is of finite rank, showing that $T$ is compact on $L^2(w)$ is equivalent to showing that $\tilde{T}$ is compact on $L^2(w)$. In particular, we argue that for each $\varepsilon > 0$, there exists $N_0 > 0$ such that

\[ \| P_N^\perp \tilde{T} f \|_{L^p(w)} \lesssim \varepsilon[w]_{A_p}^{\max\{1, \frac{p'}{p}\}} \| f \|_{L^p(w)} \]

for all $N > N_0$ and all $f \in L^p(w)$.\[ \]
We provide a sketch of the proof using the reasoning of Theorem 1.3. By Theorem 3.7, there exist \( N_0 > 0 \) and a sparse family of cubes \( \mathcal{S} \) such that

\[
|P_N T f(x)| \lesssim \varepsilon \sum_{R \in \mathcal{S}} (|f|)_{R^*} 1_R(x) =: \varepsilon |Sf|(x)
\]

for all \( N > N_0 \) and almost every \( x \in \mathbb{R}^n \).

Let first \( p \geq 2 \) and set \( \sigma = w^{1-p} \). We use again \( \|T\|_{L^p(w) \to L^p(w)} = \|T(\cdot)\|_{L^p(\sigma) \to L^p(\sigma)} \) and proceed by duality. Let \( f \in L^p(\sigma) \) and \( g \in L^p'(w) \) be nonnegative.

For each \( R \in \mathcal{S} \) we denote by \( E(R) \) the set described in Theorem 1.3 that satisfies \( E(R) \subseteq R, |R| \leq 2 |E(R)| \), and such that given \( R, R' \in \mathcal{S} \) with \( R \neq R' \), the corresponding sets \( E(R) \) and \( E(R') \) are disjoint. We use these properties, the \( A_p \) condition for \( w \), the containment \( R \subset R' \), the inequality \( |R^*| \lesssim |R| \), and boundedness of the maximal functions \( M_{\sigma} \) and \( M_w \) from Lemma 2.6 to obtain

\[
\varepsilon \langle S(f \sigma), gw \rangle = \varepsilon \sum_{R \in \mathcal{S}} \langle f \sigma \rangle_{R^*} \int_R gw \, dm \\
\leq \varepsilon \sum_{R \in \mathcal{S}} w(R^*) \sigma(R^*)^{p-1} \frac{|R^*|^{p-1}}{w(R) \sigma(R^*)^{p-1}} \int_{R^*} f \, d\sigma \int_{R} gw \, dm \\
\lesssim \varepsilon \|[w]_{A_p} \sum_{R \in \mathcal{S}} \left( \frac{1}{\sigma(R^*)} \int_{R^*} f \, d\sigma \right) \left( \frac{1}{w(R)} \int_{R} gw \, dm \right) |R|^{p-1} \sigma(R^*)^{2-p} \\
\lesssim \varepsilon 2^{p-1}[w]_{A_p} \sum_{R \in \mathcal{S}} \langle f \rangle_{R^*, d\sigma} \langle g \rangle_{R, dw} |E(R)|^{p-1} \sigma(E(R))^{2-p} \\
\lesssim \varepsilon [w]_{A_p} \sum_{R \in \mathcal{S}} \langle f \rangle_{R^*, d\sigma} \langle g \rangle_{R, dw} w(E(R))^{\frac{1}{p}} \sigma(E(R))^{\frac{1}{p}} \\
\leq \varepsilon [w]_{A_p} \left( \sum_{R \in \mathcal{S}} \langle f \rangle_{R^*, d\sigma} \sigma(E(R)) \right)^{\frac{1}{p}} \left( \sum_{R} \langle g \rangle_{R, dw} w(E(R)) \right)^{\frac{1}{p}} \\
\lesssim \varepsilon [w]_{A_p} \|M_{\sigma} f\|_{L^p(\sigma)} \|M_w g\|_{L^p'(w)} \\
\lesssim \varepsilon [w]_{A_p} \|f\|_{L^p(\sigma)} \|g\|_{L^p'(w)}.
\]

The case \( 1 < p < 2 \) follows from duality exactly as in the proof of Theorem 1.3.

\[\square\]

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