Unified Analysis of the Average Gaussian Error Probability for a Class of Fading Channels

José F. Paris

Abstract

This paper focuses on the analysis of average Gaussian error probabilities in certain fading channels, i.e., we are interested in $E[Q(\sqrt{p\gamma})]$ where $Q(\cdot)$ is the Gaussian Q-function, $p$ is a positive real number and $\gamma$ is a nonnegative random variable. We present a unified analysis of the average Gaussian error probability, derive a compact expression in terms of the Lauricella $F_D^{(n)}$ function that is applicable to a broad class of fading channels, and discuss the relation of this expression and expressions of this type recently appeared in literature. As an intermediate step in our derivations, we also obtain a compact expression for the outage probability of the same class of fading channels. Finally, we show how this unified analysis allows us to obtain novel performance analytical results.

Index Terms

Performance analysis, average bit error probability, outage probability, fading channels, Lauricella functions.

I. INTRODUCTION

For many decades, communication theorists have analyzed the performance of single-channel and multichannel receivers in a fading environment. Among performance measures, the average bit error probability (BEP) is perhaps the one that is most revealing about the nature of the system behavior. The average BEP is defined as the average of the conditional BEP over the fading statistics. The conditional BEP of a fading channel is often equivalent to the BEP of an additive white Gaussian noise (AWGN) channel, e.g., systems employing M-AM or QAM under

* The author is with the Dpto. de Ingeniería de Comunicaciones, Universidad de Málaga, Spain. paris@ic.uma.es
** This work is partially supported by the Spanish Government under project TEC2007-67289/TCM and by AT4 wireless.
ideal coherent detection [1]. In such cases, the average BEP can be computed from the statistical
expectation of Gaussian error probabilities with respect to the fading distribution.

This paper focuses on the analysis of average Gaussian error probabilities in fading channels,
i.e. we are interested in $E[Q(\sqrt{p\gamma})]$ where $Q(\cdot)$ is the Gaussian $Q$-function, $p$ is a positive
real number and $\gamma$ is a nonnegative random variable. We may note that the final average BEP
expression is usually expressed as a weighted sum of average Gaussian error probabilities, e.g. see
[2] for QAM with Gray mapping; however, for clarity these details will be overlooked here. The
literature concerning average Gaussian error probability calculations is now quite voluminous.
The most complete account of this problem is found in [1]; nevertheless, a great deal of new
results have appeared after the publication of [1]. Some of these new closed-form results involve
the Lauricella function $F^{(n)}_D$, e.g. [3]-[5]. The approach identified in these new contributions is
based on applying an appropriate change of variable to the expression obtained by the well-
known moment generating function (MGF) method developed in [1]. Therefore, these results
raise the natural question of whether the involved fading distributions share a common property
that leads to this particular mathematical form.

In this paper we identify the common property of certain fading distributions that leads to
average Gaussian probabilities expressed by the Lauricella function $F^{(n)}_D$. Such property is related
to the form of the associated MGF. We derive unified expressions for the average Gaussian error
probability and the outage probability, which are applicable to a large class of fading distributions.
Moreover, this unified analysis provides a systematic method for obtaining new analytical results.

The remainder of this paper is organized as follows. The unified analysis is presented in
Section II. In Section III we apply our analysis to derive both published and novel analytical
results. Finally, some conclusions are given in Section IV.

II. UNIFIED ANALYSIS

In this section we derive the key results of this work. Some comments on notation are in
order. For an arbitrary function $\phi(x)$ we denote the Laplace transform as $L[\phi(x); s]$. As in [1],
we define the MGF of a nonnegative random variable $\gamma$ as $M_\gamma(s) = E[e^{s\gamma}] = L[f_\gamma(\gamma); -s]$, where $s \in \mathbb{C}$ and $f_\gamma(\gamma)$ is the probability density function (PDF) of $\gamma$.

The following definition will be very useful for our purposes\(^1\).

\(^1\)This terminology is inspired by the geometric programming theory.
**Definition 1 (Monomial and Posynomial MGF):** A nonnegative random variable $\gamma$ has a *posynomial* MGF if its MGF has the form

$$
\mathcal{M}_\gamma(s) = \sum_{k=1}^{K} c_k \prod_{i=1}^{n_k} \left(1 - \frac{s}{a_{k,i}}\right)^{-b_{k,i}},
$$

where the involved parameters satisfy the following conditions

$$
\begin{align*}
\text{Re}[a_{k,i}] &> 0, \quad i = 1, 2, \ldots, n_k \text{ and } k = 1, 2, \ldots, K, \\
\sum_{i=1}^{n_k} b_{k,i} &> 0, \quad k = 1, 2, \ldots, K, \\
\sum_{k=1}^{K} c_k &= 1.
\end{align*}
$$

For the special case $K = 1$, the MGF will be called *monomial*.

In the interest of brevity, we will say a random variable or a distribution is monomial or posynomial if its associated MGF is monomial or posynomial, respectively. In addition, the coefficients in Definition 1 will be referred as *characteristic coefficients* and the conditions given in (2) will be called *compatibility conditions*. The necessity of these conditions will be revealed along this section\(^2\). As shown in Table 1, identifying $\gamma$ with the instantaneous signal-to-noise ratio (SNR), we observe that many common fading distribution have monomial MGF.

To grasp the generality of the results derived in this section, we may define the average Gaussian error probability in a more fundamental form. Integral transforms theory provides a convenient framework for our purposes. In [11] a new integral transform, called *Shannon transform* was considered in relation to the ergodic capacity analysis. Regarding the problem treated here, we may consider the following integral transform.

**Definition 2 (Gaussian $Q$-transform):** Given a nonnegative absolutely continuous random variable $\gamma$ with PDF $f_\gamma(\gamma)$, we define

$$
Q_\gamma(p) \equiv Q[f_\gamma(\gamma); p] = \mathbb{E}[Q(\sqrt{p\gamma})],
$$

where $p$ is a real nonnegative number.

Some straightforward properties of the Gaussian $Q$-transform, including an inversion formula, are collected in the following result.

\(^2\)We anticipate that the first and second compatibility conditions will be used in the proof of Theorem 1, while the third condition is necessary to satisfy $\mathcal{M}_\gamma(0) = 1$.  

January 13, 2013 DRAFT
Lemma 1: Let us consider a nonnegative absolutely continuous random variable $\gamma$. Then, the corresponding Gaussian $Q$-transform is always defined in the set $\mathbb{R}^+ \equiv [0, \infty)$, and the following properties hold:

(i) $Q_\gamma(p)$ is a continuous and decreasing function in $\mathbb{R}^+$ such that $Q_\gamma(0) = \frac{1}{2}$ and $\lim\limits_{p \to \infty} Q_\gamma(p) = 0$.

(ii) If $Q_\gamma(p) = \Phi(p)$ then

$$f_\gamma(\gamma) = \sqrt{2\pi} \frac{d}{d\gamma} \left\{ \sqrt{\gamma} \int_{e^{-j\infty}}^{e^{+j\infty}} \frac{\Phi(p)}{\sqrt{p}} e^{\frac{\gamma}{p}} dp \right\},$$

for any $\varepsilon > 0$.

Proof: See Appendix I.

Although the Gaussian $Q$-transform is an attractive concept, it is beyond the scope of this paper to develop these matters further.

Under the previous theoretical framework, one can think in Gaussian $Q$-transform pairs. The key result of this work is conceptually summarized as follows: monomial distributions and certain expressions involving the Lauricella $F_D^{(n)}$ function are connected by the Gaussian $Q$-transform. The extension of this idea to posynomial distributions is straightforward. From an operational point of view, once the underlaying fading distribution is identified as posynomial, obtaining the Gaussian $Q$-transform reduces to extracting the characteristic coefficients from the MGF. The mathematically precise statements are given below.

It is most common to find monomial distributions in practical problems, thus, we start presenting the results for the monomial case.

Theorem 1: Let $\gamma$ be a monomial random variable, i.e.

$$M_\gamma(s) = \prod_{i=1}^{n} \left(1 - \frac{s}{a_i}\right)^{-b_i},$$

(4)

where \(\{a_i\}_{i=1}^{n}\) and \(\{b_i\}_{i=1}^{n}\) satisfy the compatibility conditions given in (2). Then

(i) The cumulative distribution function (CDF) of $\gamma$ is expressed as

$$F_\gamma(\gamma) = \frac{\prod_{i=1}^{n} (a_i)^{b_i}}{\Gamma \left(1 + \sum_{i=1}^{n} b_i\right)} \sum_{i=1}^{n} b_i \Phi_2^{(n)} \left(b_1, \ldots, b_n; 1 + \sum_{i=1}^{n} b_i; -a_1\gamma, \ldots, -a_n\gamma\right),$$

(5)

where $\Gamma$ is the gamma function and $\Phi_2^{(n)}$ is the confluent Lauricella function defined in [12]-[13].

January 13, 2013 DRAFT
(ii) The Gaussian $Q$-transform of $\gamma$ is given by

$$Q_\gamma(p) = \frac{1}{2\sqrt{\pi}} \left\{ \prod_{i=1}^{n} (a_{i})^{b_{i}} \right\} \left\{ \frac{\Gamma \left( \frac{1}{2} + \sum_{i=1}^{n} b_{i} \right)}{\Gamma \left( 1 + \sum_{i=1}^{n} b_{i} \right)} \right\} \left( \frac{2}{p} \right)^{\sum_{i=1}^{n} b_{i}} \times F^{(n)}_{D} \left( \frac{1}{2} + \sum_{i=1}^{n} b_{i}, b_{1}, \ldots, b_{n}; 1 + \sum_{i=1}^{n} b_{i}; -\frac{2a_{1}}{p}, \ldots, -\frac{2a_{n}}{p} \right),$$

where $F^{(n)}_{D}$ is the Lauricella function defined in [12]-[13].

**Proof:** See Appendix II.

Extending this last result to posynomial random variables is straightforward.

**Corollary 1:** If $\gamma$ is a *posynomial* random variable such that its MGF is given by (1) then

(i) The CDF of $\gamma$ is expressed as

$$F_\gamma(\gamma) = \sum_{k=1}^{K} c_{k} \left\{ \prod_{i=1}^{n} (a_{k,i})^{b_{k,i}} \right\} \left( \frac{\Gamma \left( \frac{1}{2} + \sum_{i=1}^{n} b_{k,i} \right)}{\Gamma \left( 1 + \sum_{i=1}^{n} b_{k,i} \right)} \right) \times \Phi_{2}^{(n)} \left( b_{k,1}, \ldots, b_{k,nk}; 1 + \sum_{i=1}^{nk} b_{k,i}; -a_{k,1}\gamma, \ldots, -a_{k,nk}\gamma \right).$$

(ii) The Gaussian $Q$-transform of $\gamma$ is given by

$$Q_\gamma(p) = \frac{1}{2\sqrt{\pi}} \sum_{k=1}^{K} c_{k} \left\{ \prod_{i=1}^{nk} (a_{k,i})^{b_{k,i}} \right\} \left\{ \frac{\Gamma \left( \frac{1}{2} + \sum_{i=1}^{nk} b_{k,i} \right)}{\Gamma \left( 1 + \sum_{i=1}^{nk} b_{k,i} \right)} \right\} \left( \frac{2}{p} \right)^{\sum_{i=1}^{nk} b_{k,i}} \times F^{(nk)}_{D} \left( \frac{1}{2} + \sum_{i=1}^{nk} b_{k,i}, b_{k,1}, \ldots, b_{k,nk}; 1 + \sum_{i=1}^{nk} b_{k,i}; -\frac{2a_{k,1}}{p}, \ldots, -\frac{2a_{k,nk}}{p} \right).$$

**Proof:** Repeat the steps of the proof of Theorem 1.

Theorem 1 and Corollary 1 have important theoretical and practical consequences. Next result provides an integral representation for the Gaussian $Q$-transform of posynomial random variables that is convenient for numerical evaluation.

**Corollary 2:** If $\gamma$ is a *posynomial* random variable such that its MGF is given by (1) then
\[ Q_\gamma(p) = \frac{1}{2\pi} \sum_{k=1}^{K} c_k \left\{ \prod_{i=1}^{n_k} (a_{k,i})^{b_{k,i}} \right\} \left( \frac{2}{p} \right)^{\sum_{i=1}^{n_k} b_{k,i}} \times \int_0^1 u^{-\frac{1}{2} + \sum_{i=1}^{n_k} b_{k,i}} (1 - u)^{-\frac{1}{2}} \left( 1 + \frac{2a_{k,1}}{p} u \right)^{-b_{k,1}} \cdots \left( 1 + \frac{2a_{k,n_k}}{p} u \right)^{-b_{k,n_k}} \, du. \] (9)

*Proof:* Check that the compatibility conditions allow us to use the Euler-type integral representation given in [12, p. 283, eq. 34] for the Lauricella functions in (8).

From this last Corollary it is straightforward to obtain an asymptotic approximation for posynomial random variables.

**Corollary 3:** Let \( \gamma \) be a *posynomial* random variable such that its MGF is given by (1). Then

\[ Q_\gamma(p) \sim \frac{1}{2\sqrt{\pi}} \sum_{k=1}^{K} c_k \left\{ \prod_{i=1}^{n_k} (2a_{k,i})^{b_{k,i}} \right\} \frac{\Gamma \left( \frac{1}{2} + \sum_{i=1}^{n_k} b_{k,i} \right)}{\Gamma \left( 1 + \sum_{i=1}^{n_k} b_{k,i} \right)} \frac{1}{p^{\sum_{i=1}^{n_k} b_{k,i}}} \], (10)

for large values of \( p \).

*Proof:* Substitute the integrand in (9) by its asymptotic approximation and use the basic properties of the beta function [15, p. 898-899].

As expected, from (10) it is inferred that the underlaying diversity order of \( Q_\gamma(p) \) is

\[ \lim_{p \to \infty} -\log \left( \frac{Q_\gamma(p)}{\log(p)} \right) = \min_k \left\{ \sum_{i=1}^{n_k} b_{k,i} \right\}, \] (11)

assuming \( \gamma \) is a posynomial random variable.

**III. Applications**

The mathematical tools developed in previous section provide us a unified analytical framework for many existing results in literature; in addition, they systematically allow us to obtain new analytical results.

To calculate the average BEP, its asymptotic approximation and the outage probability we can follow four steps:

1) Check if the underlaying distribution is posynomial. In such case extract the characteristic coefficients.

2) Use the Theorem and Corollaries of previous section to obtain analytical expressions for the Gaussian \( Q \)-Transform, its asymptotic approximation and the outage probability.
3) If possible, reduce the Lauricella functions $F_D^{(n)}$ and $\Phi_2^{(n)}$ to simpler functions.

4) Use known formulas to express the average BEP in terms of the Gaussian $Q$-transform, e.g. see [2] for QAM with Gray mapping.

Next, we provide some examples to illustrate how this approach allows us to derive both published and novel results. For brevity we will mainly focus on the first step described above.

**Example 1: Nakagami-$m$ fading.** This a well-known example [1][14]. From Table I, we can extract the characteristic parameters for the Nakagami-$m$ distribution; specifically, $a_1 = m/\bar{\gamma}$ and $b_1 = m$. From (5) we obtain

$$F_\gamma (\gamma) = \left\{ \frac{\left(\frac{m}{\bar{\gamma}}\right)^m}{\Gamma (1 + m)} \right\} \gamma^m F_1 \left( m, 1 + m; -\frac{m}{\bar{\gamma}} \gamma \right). \tag{12}$$

Then, after considering [15, eq. 8.351-2], we derive the well-known formula

$$F_\gamma (\gamma) = 1 - \frac{\Gamma \left( m, \frac{m}{\bar{\gamma}} \gamma \right)}{\Gamma (m)}. \tag{13}$$

Applying (6) and considering [15, eq. 9.131-1] yields

$$Q_\gamma (p) = \frac{1}{2\sqrt{\pi}} \left\{ \frac{\Gamma \left( \frac{1}{2} + m \right)}{\Gamma (1 + m)} \right\} \left( \frac{1}{c(p)} \right)^m \left( 1 + \frac{1}{c(p)} \right)^{1/2-m} 2F_1 \left( \frac{1}{2}, 1; m + 1; -\frac{1}{c(p)} \right), \tag{14}$$

where $c(p) = p^{\bar{\gamma}}/(2m)$. This expression is very similar to that given [1, eq. 5.A.2][14, eq. A.8], which was derived by a different approach. Interestingly, it has been checked numerically that both are equivalent; however, the author has not been able to find the transformation between Gauss hypergeometric functions $2F_1$ connecting (14) and [1, eq. 5.A.2].

**Example 2: Maximal ratio combining (MRC) with independent but nonidentically distributed branches.** A formula for the Gaussian $Q$-transform over Hoyt fading channels is derived in [4] in terms of the Lauricella $F_D^{(n)}$ function. Now we show how to perform a unified analysis for MRC including Hoyt distributed branches. Let us consider receive MRC with $L$ independent but nonidentical distributed branches. We assume that $L_1$ branches exhibit Nakagami-$q$ (Hoyt) fading and the remainder of the $L - L_1$ branches are better modelled by the Nakagami-$m$ fading model. The Nakagami-$m$ fading model is used for both line-of-sight (LOS) and NLOS scenarios, while the Nakagami-$q$ distribution is an alternative to model NLOS channels. We assume a general scenario where each branch has arbitrary parameters, i.e. $\{q_j, \gamma_j\}_{j=1}^{L_1}$ for the first $L_1$ branches and
\{m_j, \bar{\gamma}_j\}_{j=L_1+1}^L \text{ for the remainder of the branches. Analyzing such scenario is straightforward with the results derived in this work, after observing that the associated MGF is monomial}

\[ \mathcal{M}_\gamma(s) = \prod_{i=1}^{L_1} \left(1 - \frac{2q_i^2}{1 + q_i^2} \bar{\gamma}_is\right)^{-1/2} \prod_{i=L_1+1}^{L_1+1} \left(1 - \frac{2}{1 + q_i^2} \bar{\gamma}_is\right)^{-1/2} \prod_{i=L_1+2}^{L_1+L} \left(1 - \frac{\bar{\gamma}_is}{m_i}\right)^{-m_i} \tag{15} \]

This scenario can be further generalized to consider large-scale time variations. Let us assume that the number of branches with Nakagami-\(q\) fading \(L_1\) is also random. Then, if \(Pr[l]\) represents the probability of having \(l\) branches with Nakagami-\(q\) fading, the associated MGF is now posynomial

\[ \mathcal{M}_\gamma(s) = \sum_{l=1}^{L} \Pr[l] \prod_{i=1}^{l} \left(1 - \frac{2q_{i,l}^2}{1 + q_{i,l}^2} \bar{\gamma}_{i,l}is\right)^{-1/2} \times \prod_{i=l+1}^{2l} \left(1 - \frac{2}{1 + q_{i,l}^2} \bar{\gamma}_{i,l}is\right)^{-1/2} \prod_{i=2l+1}^{L} \left(1 - \frac{\bar{\gamma}_{i,l}is}{m_{i,l}}\right)^{-m_{i,l}} \tag{16} \]

A system employing MRC with independent \(\eta-\mu\) distributed branches has been recently analyzed in [5]. This system generalizes the Nakagami-\(q\)/Nakagami-\(m\) scenario described above and it requires to estimate the set of \(3L\) parameters \(\{\eta_j, \mu_j, \bar{\gamma}_j\}_{j=1}^L\). Since the Nakagami-\(q\)/Nakagami-\(m\) scenario requires \(2L\) channel parameters, it is a reasonable alternative for certain applications. Interestingly, if we apply Theorem 1 to the monomial MGF considered in [5, eq. 2], we can extend the performance analysis in [5] with a closed-form expression for the outage probability and an asymptotic approximation for the average BEP.

\(\square\)

\textbf{Example 3: Orthogonal space-time block codes (OSTBC) in spatially correlated multiple-input multiple-output (MIMO) channels.} In [16] the underlying MGF was derived for different systems using OSTBC in shadowed Rician MIMO fading channels. Such MGFs are posynomial, thus, all results derived in previous section are applicable, leading to novel analytical results. To illustrate this idea, we consider the particular MGF for correlated LOS component and spatially white scattered component [16, eq. 23]

\[ \mathcal{M}_\gamma(s) = \frac{1}{(1 + as)^{n_r/n_t}} \prod_{i=1}^{n_r/n_t} \left(1 + \frac{bs}{1 + as}\right)^{-m} = (1 + as)^{-n_r/n_t} \prod_{i=1}^{n_r/n_t} \left(1 + (a + b)s\right)^{-m} \prod_{i=1}^{n_r/n_t} (1 + as)^m, \tag{17} \]

where \(n_t, n_r, a, b\) and \(m\) are channel and system parameters defined in [16]. After rewriting the MGF given in [16, eq. 23], we clearly observe in (17) that it is a monomial MGF. \(\square\)
IV. CONCLUSIONS

A solid analytical framework for the computation of average Gaussian error probabilities has been derived in this paper. This quantity allows us to obtain the average BEP in a variety of wireless communications systems. It has been shown that if the MGF of the underlaying fading distribution has certain posynomial structure, the average Gaussian error probabilities and the outage probability can be expressed in terms of Lauricella functions. Finally, this analytical framework has been used to derive known and novel results in a simple and systematic way.

APPENDIX A

PROOF OF LEMMA 1

The existence of the Gaussian Q-transform is clear after taking into account that $Q(\sqrt{p\gamma}) \leq \frac{1}{2}$ for every $p > 0$.

(i) Continuity follows from the fact that the kernel $Q(\sqrt{p\gamma})$ is continuous and bounded in $\mathbb{R}^+$. If $p_1 < p_2$ then $Q(\sqrt{p_1\gamma}) > Q(\sqrt{p_2\gamma})$ for all $\gamma$ in $\mathbb{R}^+$, thus, $Q_\gamma(p_1) > Q_\gamma(p_2)$. The property $\lim_{p \to \infty} Q_\gamma(p) = 0$ follows after applying the monotone convergence theorem.

(ii) After integrating by parts $Q_\gamma(p)$ we can write

$$\Phi(p) = \frac{\sqrt{p}}{2\sqrt{2\pi}} \int_0^\infty e^{-\frac{t}{2}} F_\gamma(t) t^{-1/2} dt.$$ 

(18)

The integral in (18) is recognized as a Laplace transform converging for $\Re[p] > 0$. Thus, after applying the inverse Laplace transform to this equality, the desired result is obtained.

APPENDIX B

PROOF OF THEOREM 1

(i) Since $\mathcal{L}[f_\gamma(t);s] = \prod_{i=1}^n \left(1 + \frac{s}{a_i}\right)^{-b_i}$, we can write

$$\mathcal{L}[F_\gamma(t);s] = \frac{1}{s} \mathcal{L}[f_\gamma(t);s] = \prod_{i=1}^n \left(1 - \frac{(a_i)^{-b_i}}{s}\right),$$

(19)

for $\Re[s] > 0$. Taking into account the first and second compatibility conditions given in (2) and identifying [13, p. 222, eq. 5] from (19), the expression (5) is obtained.
(ii) Introducing (5) in (18) yields

\[
Q_{\gamma}(p) = \frac{\sqrt{p}}{2\sqrt{2\pi}} \left( \prod_{i=1}^{n} (a_i)^{b_i} \right) \left\{ \Gamma \left( \sum_{i=1}^{n} b_i \right) \right\} \int_{0}^{\infty} e^{-\frac{p t}{2}} t^{-1/2+\sum_{i=1}^{n} b_i} \Phi_2(n) \left( b_1, \ldots, b_n, \sum_{i=1}^{n} b_i; -a_1 t, \ldots, -a_n t \right) dt.
\]

(20)

Taking into account the first and second compatibility conditions given in (2) and identifying [12, p. 286, eq. 43] from (20), the expression (6) is obtained.

REFERENCES

[1] M. K. Simon and M-S Alouini, Digital Communications over Fading Channels, 2nd ed., John Wiley, 2005.

[2] K. Cho and D. Yoon, “On the general BER expression of one- and two- dimensional amplitude modulations,” IEEE Trans. Commun., vol. 50, no. 7, pp. 1074-1080, July 2002.

[3] F. Xu, D.-W. Yue, F. C. M. Lau and Q. F. Zhou, “Closed-form expressions for symbol error probability of orthogonal spacetime block codes over RicianNakagami channels”, IET Commun., no 4, vol. 1, pp. 655-661, Aug. 2007.

[4] R. M. Radaydeh, “Average error performance of M-ary modulations schemes in Nakagami-q (Hoyt) fading channels,” IEEE Commun. Lett., vol. 11, pp. 255–257, March 2007.

[5] K. Peppas, F. Lazarakis and A. Alexandridis and K. Dangakis, “Error Performance of Digital Modulation Schemes with MRC diversity reception over \( \eta-\mu \) fading channels,” IEEE Trans. Wireless Commun., vol. 8, no. 10, pp. 4974–4980, Oct 2009.

[6] L. Rayleigh, “On the resultant of a large number of vibrations of the same pitch and of arbitrary phase,” Phil. Mag., vol. 27, pp. 460-469, June 1889.

[7] R. S. Hoyt, “Probability Functions for the Modulus and Angle of the Normal Complex Variate,” Bell. Syst. Tech. J., vol. 26, pp. 318-359, April 1947.

[8] M. Nakagami, “The m-Distribution- A General Formula of Intensity Distribution of Rapid Fading,” In Statistical Methods in Radio Wave Propagation, Oxford, U.K.: Pergamon Press, pp. 3-36, 1960.

[9] A. Abdí, C. Lau, M.-S. Alouini, and M. Kaveh, “A new simple model for land mobile satellite channels: first- and second-order statistics,” IEEE Trans. Wireless Commun., vol. 2, pp. 611-615, May 2003.

[10] M. D. Yacoub, “The \( \kappa-\mu \) and the \( \eta-\mu \) distribution,” IEEE Antennas and Propagation Magazine, vol. 49, pp. 68-81, Feb. 2007.

[11] A. M. Tulino and S. Verdú, Random matrix theory and wireless communications, now Publishers Inc., 2003.

[12] H. M. Srivastava and P. W. Karlsson, Multiple Gaussian Hypergeometric Series, John Wiley, 1985.

[13] A. Erdelyi, Tables of Integrals Transforms, vol. I, McGraw Hill, New York, 1954.

[14] T. Eng and L. B. Milstein, “Coherent DS-CDMA performance in Nakagami multipath fading,” IEEE Trans. Commun., vol. 43, pp. 1134–1143, Feb/May/April 1995.

[15] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, 6 ed. San Diego, Academic Press, 2000.

[16] I.-M. Kim, “Exact BER Analysis of OSTBCs in Spatially Correlated MIMO Channels,” IEEE Trans. Commun., vol. 54, no 8, pp. 1365-1373, Aug. 2006.
### Table I

**Some Monomial MGFs.**

| Fading Distribution          | MGF $\mathcal{M}_\gamma(s)$                                                                 | References |
|------------------------------|--------------------------------------------------------------------------------------------|------------|
| Rayleigh                     | $(1 - \bar{\gamma}s)^{-1}; 0 \leq \bar{\gamma}.$                                         | [6],[1]    |
| Nakagami-$q$ (Hoyt)          | $\left(1 - \frac{2q^2}{1 + q^2\bar{\gamma}s}\right)^{-1/2} \left(1 - \frac{2}{1 + q^2\bar{\gamma}s}\right)^{-1/2}$; $0 \leq \bar{\gamma}, 0 < q \leq 1.$ | [7],[1]    |
| Nakagami-$m$                 | $\left(1 - \frac{\bar{\gamma}}{m}s\right)^{-m}; \frac{1}{2} \leq m.$                      | [8],[1]    |
| Rician Shadowed*             | $\left(1 - \frac{\bar{\gamma}}{1 + K}s\right)^{m-1} \left(1 - \left(1 + \frac{Km}{1 + K}\right)^{-1}\right)^{-m}$; $0 \leq \bar{\gamma}, 0 \leq K, 0 < m.$ | [9],[1]    |
| $\eta$-$\mu$ Physical Model | $\left(1 - s\frac{\bar{\gamma}}{n(h + H)}\right)^{-n/2} \left(1 - s\frac{\bar{\gamma}}{n(h - H)}\right)^{-n/2}$ | [10]       |

*Note that $K = \frac{\Omega}{2\sigma^2}$, using the same notation as in [9],[1].*