BASES AND SELECTORS FOR TALL FAMILIES

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Abstract. We show that the Nash-Williams theorem has a uniform version and that the Galvin theorem does not. We show that there is an $F_\sigma$ tall ideal on $\mathbb{N}$ without a Borel selector and also construct a $\Pi^1_2$ tall ideal on $\mathbb{N}$ without a tall closed subset.

1. Introduction

A family $C$ of subsets of $\mathbb{N}$ is tall if for every infinite $x \subseteq \mathbb{N}$ there is an infinite $y \in C$ such that $y \subseteq x$. We are interested in tall families $C$ which are in addition definable as subsets of $2^{\mathbb{N}}$. Take for example the set $\text{hom}(c)$ of all monochromatic subsets of $\mathbb{N}$ for some coloring $c : [\mathbb{N}]^2 \to 2$. This is, by Ramsey theorem, a tall family and moreover it is a closed subset of $2^{\mathbb{N}}$. We deal with the question when we can effectively witness that a family $C$ is tall i.e. when there is a Borel function $S : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that for every infinite $x \in 2^{\mathbb{N}}$ is $S(x) \in C$, $S(x)$ is infinite and $S(x) \subseteq x$. We call such function $S$ a Borel selector for $C$. Note that if there is a Borel selector $S$ for $C$ then $C$ contains analytic subfamily which is also tall. This leads to a natural basis problem of whether a given tall family $C$ contains simpler tall subfamily $C' \subseteq C$. By simpler we mean that $C'$ is of lower complexity (for example closed) or is of specific form (for example $\text{hom}(c)$ for some coloring $c$).

The main source of examples of tall families of subsets of $\mathbb{N}$ are tall Borel ideals on $\mathbb{N}$. Recall that an ideal $I$ is Katětov below an ideal $J$ ($I \leq_K J$) if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $f^{-1}[x] \in J$ for every $x \in I$. It is proved in [4] that having a Borel selector is closed upwards in the Katětov order and if $I$ is a tall Borel ideal with a Borel selector then there is a tall Borel ideal $J$ such that $I \not\leq_K J$. All known examples of Borel ideals so far have a Borel selector (for random ideal $R$ see [5] and for Solecki ideal $S$ see [4]). We show that there is a $F_\sigma$ tall ideal without a Borel selector. The proof of this result is based on the following facts. Every $F_\sigma$ ideal can be coded by a closed collection of sets, i.e. by an element of the hyperspace $K(2^{\mathbb{N}})$. In [4] it is proved that the set of codes of tall $F_\sigma$ ideals is a $\Pi^1_2$–complete subset of $K(2^{\mathbb{N}})$. To show that there is an $F_\sigma$ ideal without a selector we prove that the complexity of the set of codes of $F_\sigma$ ideals with a Borel selector is $\Sigma^1_2$. However, it is an open question to find a concrete example of such $F_\sigma$ ideal.

Another important source of examples are given by some well studied generalizations of $\text{hom}(c)$. Given a subset $O$ of infinite subsets of $\mathbb{N}$, a set $x \subseteq \mathbb{N}$ is called $O$-homogeneous, if either $[x]^\omega \subseteq O$ or $[x]^\omega \cap O = \emptyset$. A well known theorem of Silver says that for every analytic set $O$ the collection $\text{hom}(O)$ of $O$-homogeneous sets is tall. When $O$ is open

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(resp. clopen), the corresponding Ramsey result is called Galvin’s lemma [2] (resp. Nash-Williams’ theorem [9]). The existence of Borel selectors for families of the form $\text{hom}(\mathcal{O})$ is a consequence of the fact that the corresponding Ramsey theorem holds uniformly. For instance, the fact that the Random ideal $\mathcal{R}$ has a Borel selector is due to the fact there is uniform approach of finding an infinite monochromatic subset of a given set $x \subseteq \mathbb{N}$ (or having a Borel proof of Ramsey theorem) [5]. Analogously, we show that Nash-Williams’ theorem also has a uniform version and thus $\text{hom}(\mathcal{O})$ has a Borel selector for every clopen set $\mathcal{O}$. However, we show there is an open set $\mathcal{O}$ such that $\text{hom}(\mathcal{O})$ does not have a Borel selector and therefore Galvin’s lemma does not admit a uniform version. Ramsey type theorems have been analyzed from a related but different complexity point of view. Solovay ([10]) studied when $\text{hom}(\mathcal{O})$ contains an element which is hyperarithmetical on the code of $\mathcal{O}$ (see also [1]).

We show that the basis problem also has a negative answer. We construct a $\Pi^1_2$ tall ideal $\mathcal{I}$ such that $\text{hom}(\mathcal{O}) \not\subseteq \mathcal{I}$ for all open set $\mathcal{O} \subseteq [\mathbb{N}]^\omega$, in particular, $\mathcal{I}$ does not contain any tall closed subset. It is still an open question whether every tall Borel (analytic) ideal contains a closed tall subset.

2. Preliminaries

In this section we fix our notation, give some basic definitions and results that are later used. We consider the natural isomorphism $\mathcal{P}(\mathbb{N}) \approx 2^\mathbb{N}$ and view all relations such as $\subseteq, \cap, [.]^{<\omega}$, etc, as defined on $2^\mathbb{N}$ i.e. we use $x \subseteq y$, $x \cap y$, $[x]^{<\omega}$, etc, for $x, y \in 2^\mathbb{N}$. We use the standard descriptive set theoretic notions and notations (as in [6]). The projective classes are denoted $\Sigma^1_n$ and $\Pi^1_n$.

**Definition 2.1.** Let $C \subseteq 2^\mathbb{N}$ be a tall family. We say that $C$ has a Borel selector, if there is a Borel function $S : 2^\mathbb{N} \to 2^\mathbb{N}$ such that for every $x \in 2^\mathbb{N}$

- $S(x) \subseteq x$,
- if $|x|$ is infinite then $|S(x)|$ is infinite,
- $S(x) \in C$.

Note that we define the notion of a Borel selector only for tall families so if we say that $C$ has a Borel selector it automatically means that $C$ is tall. We say that a family $C$ is hereditary if $y \in C$ whenever there is $x \in C$ such that $y \subseteq x$. We say that $\mathcal{I} \subseteq 2^\mathbb{N}$ is an ideal on $\mathbb{N}$ if it is hereditary and it is closed under finite unions. The following characterization of $F_\sigma$ ideals on $\mathbb{N}$ was given by Mazur [5]. Recall that a map $\varphi : 2^\mathbb{N} \to [0, \infty]$ is a lower-semicontinuous submeasure (lcsms) if for all $x, y \in \mathbb{N}$

- $\varphi(\emptyset) = 0$,
- $x \subseteq y$ implies $\varphi(x) \leq \varphi(y)$,
- $\varphi(x \cup y) \leq \varphi(x) + \varphi(y)$,
- $\varphi(x) = \lim_{n \to \infty} \varphi(x \cap n)$.

Each lcsms $\varphi$ naturally corresponds to the $F_\sigma$ ideal $\text{Fin}(\varphi) := \{x : \varphi(x) < \infty\}$.

**Theorem 2.2** (Mazur [5]). An ideal $\mathcal{I}$ is $F_\sigma$ if and only if there is lcsms $\varphi$ such that $\mathcal{I} = \text{Fin}(\varphi)$.
From this characterization one easily deduces (see for example [4]) the following Proposition which allows us to consider \( K(\mathbb{2}^N) \) the hyperspace of closed subsets of \( \mathbb{2}^N \) as a space of codes of \( F_\sigma \) ideals. For \( K \in K(\mathbb{2}^N) \), let \( \mathcal{I}_K \) be ideal generated by \( K \), i.e. \( x \in \mathcal{I}_K \) if and only if there is \( y_0, \ldots, y_{n-1} \in K \) such that \( \bigcup_{i<n} y_i \subseteq x \). Clearly, \( \mathcal{I}_K \) is \( F_\sigma \).

**Proposition 2.3.** For every \( F_\sigma \) ideal \( \mathcal{I} \) there is \( K \in K(\mathbb{2}^N) \) such that \( \mathcal{I} = \mathcal{I}_K \).

In [4] it is proved that the set of codes of tall \( F_\sigma \) ideals and the set of codes of tall \( F_\sigma \) ideals containing the ideal \( \mathbb{Fin} \) are \( \text{\Pi}^1_2 \)-complete.

Next we state the combinatorial theorems (as presented in [11]). Let \( s, t \in [\mathbb{N}]^{<\omega} \). We write \( s \sqsubseteq t \) when there is \( n \in \omega \) such that \( s = t \cap \{0, 1, \ldots, n\} \) and we say that \( s \) is an initial segment of \( t \).

**Theorem 2.4 (Galvin).** Let \( \mathcal{F} \subseteq [\mathbb{N}]^{<\omega} \) and an infinite \( x \in \mathbb{2}^N \). Then there is an infinite \( y \subseteq x \) such that one of the following holds

- for all \( z \in [y]^\omega \) there is \( s \in \mathcal{F} \) such that \( s \sqsubseteq z \),
- \( [y]^{<\omega} \cap \mathcal{F} = \emptyset \).

We can think of \( \mathcal{F} \) as a coloring of \( [\mathbb{N}]^{<\omega} \) and put \( \text{\textup{hom}}(\mathcal{F}) \subseteq \mathbb{2}^N \) for the family of all \( y \) that satisfy one of the conditions from the Galvin’s theorem, such sets are called \( \mathcal{F} \)-homogeneous. It is clear that \( \text{\textup{hom}}(\mathcal{F}) \) is an hereditary tall collection. Moreover, the family of sets that satisfy the second condition is closed and hereditary and the family of sets that satisfy the first condition is \( \text{\Pi}^1_1 \). We write \( \mathbb{P}_2 \) for the set of all those \( \mathcal{F} \subseteq [\mathbb{N}]^{<\omega} \) such that first condition in Galvin’s theorem is never satisfied.

A special type of coloring of \( [\mathbb{N}]^{<\omega} \) are as follows. We say that \( \mathcal{B} \subseteq [\mathbb{N}]^{<\omega} \) is a \textup{front} on an infinite \( x \in \mathbb{2}^N \) if

- every two elements of \( \mathcal{B} \) are \( \sqsubseteq \)-incomparable,
- every infinite \( y \subseteq x \) has an initial segment in \( \mathcal{B} \).

**Theorem 2.5 (Nash-Williams).** Let \( \mathcal{B} \) be a front on \( \mathbb{N} \) and \( \mathcal{F} \subseteq \mathcal{B} \) then for every infinite \( x \in \mathbb{2}^N \) there is an infinite \( y \subseteq x \) such that one of the following holds

- \( [y]^{<\omega} \cap \mathcal{B} \subseteq \mathcal{F} \),
- \( [y]^{<\omega} \cap \mathcal{F} = \emptyset \).

Let \( \mathcal{F} \subseteq \mathcal{B} \) as above, it is easy to verify that \( y \in \text{\textup{hom}}(\mathcal{F}) \) iff \( y \) satisfies one of the conditions from the Nash-Williams’ theorem. Moreover, the family \( \text{\textup{hom}}(\mathcal{F}) \) is easily seen to be closed, hereditary and tall.

**Proposition 2.6.** For every closed, tall and hereditary \( K \subseteq \mathbb{2}^N \) there is \( \mathcal{F} \subseteq [\mathbb{N}]^{<\omega} \) such that \( \text{\textup{hom}}(\mathcal{F}) = K \).

**Proof.** Define \( \mathcal{F}_K = \{ s \in [\mathbb{N}]^{<\omega} : s \not\subseteq K \} \). We claim that \( \text{\textup{hom}}(\mathcal{F}_K) \) is equal to \( \{ y \in [\omega]^{\omega} : [y]^{<\omega} \cap \mathcal{F}_K = \emptyset \} \). Let \( y \in \text{\textup{hom}}(\mathcal{F}_K) \) and suppose \( y \) satisfies the first condition in Galvin’s theorem. Since \( K \) is tall there is an infinite \( z \subseteq y \) such that \( z \in K \). As \( y \) satisfies the first condition, there is \( s \in \mathcal{F}_K \) such that \( s \subseteq z \) but since \( K \) is hereditary we have \( s \in K \) and this contradicts the definition of \( \mathcal{F}_K \).
It remains to check that \( K = \text{hom}(\mathcal{F}_K) \). Clearly \( \subseteq \) holds. For the opposite take \( x \notin K \). Since \( K \) is hereditary and closed there must be some \( n \in \mathbb{N} \) such that \( x \cap n \notin K \) then we have \( x \cap n \in \mathcal{F}_K \). Thus \( x \notin \text{hom}(\mathcal{F}_K) \).

\[ \square \]

**Proposition 2.7.** The set \( \mathbb{P}_2 \) is \( \Pi^1_2 \)-complete.

**Proof.** This is a generalization of previous argument. Let \( \mathcal{T} \) be the set of all \( K \in K(2^\mathbb{N}) \) which generates a tall \( F_n \) ideal. As it was already mentioned \( \mathcal{T} \) is \( \Pi^1_2 \)-complete (see [14]).

Consider the continuous map \( \psi : K(2^\mathbb{N}) \to \mathcal{P}(\mathbb{N}^{<\omega}) \) given by

\[ s \in \psi(K) \Leftrightarrow \forall x \in K \ s \nsubseteq x. \]

One may check that \( \mathcal{T} = \psi^{-1}(\mathbb{P}_2) \) and the desired result follows since \( \mathbb{P}_2 \) is easily seen to be \( \Pi^1_2 \).

\[ \square \]

3. Positive results

In this section we prove the uniform version of the Nash-Williams's theorem. To state our theorem in full generality we must first introduce several definitions.

### 3.1. Uniformly \( p^+ \), \( q^+ \) and selective ideals

Let \( \mathcal{I} \) be an ideal on \( \mathbb{N} \). Recall that \( \mathcal{I}^+ = 2^\mathbb{N} \setminus \mathcal{I} \). We say that \( \mathcal{I} \) is \( q^+ \) if for all \( x \in \mathcal{I}^+ \) and every partition \( \{s_n\}_{n<\omega} \) of \( x \) into finite sets there is \( y \subseteq x \) such that \( y \in \mathcal{I}^+ \) and \( |y \cap x_n| \leq 1 \) for all \( n < \omega \). It is \( p^+ \) if for every decreasing sequence \( (x_n)_{n<\omega} \) of sets in \( \mathcal{I}^+ \) there is \( x \in \mathcal{I}^+ \) such that \( x \cap x_n \) is finite for all \( n < \omega \). It is **selective**, if for every decreasing sequence \( (x_n)_{n<\omega} \) of sets in \( \mathcal{I}^+ \) there is \( x \in \mathcal{I}^+ \) such that \( x/n \subseteq x_n \) for all \( n \in x \). We are interested in the uniform versions of these notions. We say that a Borel ideal \( \mathcal{I} \) is **uniformly selective** if there is a Borel function \( F \) such that whenever \( (x_n)_{n<\omega} \) is a decreasing sequence of sets in \( \mathcal{I}^+ \), then \( F((x_n)_{n<\omega}) = x \) is in \( \mathcal{I}^+ \) and \( x/n \subseteq x_n \) for all \( n \in x \). In an analogous way, we define when an ideal is uniformly \( p^+ \) or \( q^+ \).

**Lemma 3.1.** A Borel ideal \( \mathcal{I} \) is uniformly selective iff it is uniformly \( p^+ \) and \( q^+ \).

**Proof.** Follow an standard proof of the fact that an ideal es selective iff it is \( p^+ \) and \( q^+ \) (see for instance [12], Lemma 7.4)).

**Theorem 3.2.** Let \( \mathcal{I} \) be a \( F_\sigma \) ideal. Then

(i) \( \mathcal{I} \) is uniformly \( p^+ \).

(ii) if \( \mathcal{I} \) is \( q^+ \), then it is uniformly \( q^+ \).

In particular, every selective \( F_\sigma \) ideal is uniformly selective.

**Proof.** Let \( \{s_k\}_{k<\omega} \) be some enumeration of \( \mathbb{N}^{<\omega} \) and let \( \mu \) be the lower semicontinuous submeasure such that \( \mathcal{I} = \{x \in 2^\mathbb{N} : \mu(x) < \infty \} \). First we claim that for each \( n \in \omega \) there is a Borel function \( G_n : 2^\mathbb{N} \to 2^\mathbb{N} \) such that for all \( x \notin \mathcal{I} \), \( G_n(x) \) is a finite subset of \( x \) and \( \mu(G_n(x)) \geq n \). Define \( G_n(x) = \emptyset \) for \( x \in \mathcal{I} \). For \( x \in \mathcal{I}^+ \) let \( G_n(x) = s_k \) where \( k \) is the minimal index such that \( s_k \subseteq x \) and \( \mu(s_k) \geq n \).

(i) Let \( (x_n)_{n<\omega} \) be a decreasing sequence of sets in \( \mathcal{I}^+ \). Define \( G((x_n)_{n<\omega}) = \bigcup_{n<\omega} G_n(x_n) \). Then \( G \) is Borel and has the required property.
(ii) We define inductively sequence of Borel functions \( \{F_n\}_{n<\omega} \) where \( F_n : 2^{\omega} \times ([N]^{<\omega})^\omega \rightarrow [N]^{<\omega} \) and for \( (x, (t_i)_{i<\omega}) \in 2^{\omega} \times ([N]^{<\omega})^\omega \) we have

- \( F_0(x, (t_i)_{i<\omega}) = \emptyset \),
- if \( n > 0 \), \( x \in \mathcal{I}^+ \) and \( (t_i)_{i<\omega} \) is a partition of \( x \) then \( F_n(x, (t_i)_{i<\omega}) = s_k \) where \( k \) is the minimal index such that \( s_k \) is a partial selector, \( \mu(s_k) \geq n \) and \( F_{n-1}(x, (t_i)_{i<\omega}) \subseteq s_k \),
- otherwise put \( F_n(x, (t_i)_{i<\omega}) = \emptyset \).

These are clearly Borel conditions and the functions are well defined since \( \mathcal{I} \) is \( q^+ \). Finally put \( F(x, (t_i)_{i<\omega}) = \bigcup_{n<\omega} F_n(x, (t_i)_{i<\omega}) \).

**Corollary 3.3.** Fin is uniformly selective.

Let \( \mathcal{A} \) be an almost disjoint family of infinite subsets of \( \mathbb{N} \) and \( \mathcal{I}(\mathcal{A}) \) be the ideal generated by \( \mathcal{A} \). By a result of Mathias \([7]\), \( \mathcal{I}(\mathcal{A}) \) is selective. It is easy to verify that when \( \mathcal{A} \) is closed (as a subset of \( 2^\mathbb{N} \)), then \( \mathcal{I}(\mathcal{A}) \) is \( F_\sigma \). Hence from Theorem 3.2 we get the following

**Corollary 3.4.** Let \( \mathcal{A} \) be a closed almost disjoint family. Then \( \mathcal{I}(\mathcal{A}) \) is uniformly selective.

A natural question is whether the previous result can be extended to any ideal of the form \( \mathcal{I}(\mathcal{A}) \) for \( \mathcal{A} \) an analytic almost disjoint family or, more generally, to any selective analytic ideal.

### 3.2. Uniform Ramsey type theorems

Recall that the lexicographic order \( <_{lex} \) on \( [N]^{<\omega} \) is defined by \( s <_{lex} t \) if \( \min(s \triangle t) \in s \). Let \( x \in 2^\mathbb{N} \) be infinite and \( \mathcal{B} \subseteq [x]^{<\omega} \) be a front on \( x \) then the restriction of \( <_{lex} \) on \( \mathcal{B} \) is a well-order and its order type is called the rank of \( \mathcal{B} \) (denoted \( \text{rank}(\mathcal{B}) \)).

For \( \mathcal{F} \subseteq [N]^{<\omega} \) we define \( \overline{\mathcal{F}} = \{ s \in [N]^{<\omega} : s \subseteq t \text{ for some } t \in \mathcal{F} \} \).

**Lemma 3.5.** Let \( \mathcal{B} \) be a front and \( \mathcal{F} \subseteq \overline{\mathcal{B}} \). Let \( \hat{\mathcal{F}} = \{ s \in [N]^{<\omega} : \exists t \in \mathcal{F}, \exists t' \in \mathcal{B}, t \subseteq s \subseteq t' \} \). Then \( x \in \text{hom}(\mathcal{F}) \) if and only if \( [x]^{<\omega} \cap \mathcal{F} = \emptyset \) or \( [x]^{<\omega} \cap \overline{\mathcal{B}} \subseteq \overline{\mathcal{F}} \cup \hat{\mathcal{F}} \).

**Proof.** Let \( x \in \text{hom}(\mathcal{F}) \). Suppose the first item in the conclusion of Theorem 2.4 holds. Let \( s \subseteq x \) with \( s \in \overline{\mathcal{B}} \) and put \( y = s \cup \{ n \in x : n > \max s \} \). Thus there is \( t \in \mathcal{F} \) such that \( t \subseteq y \). Hence \( s \subseteq t \) or \( t \subseteq s \). In either case, \( s \in \overline{\mathcal{B}} \cup \overline{\mathcal{F}} \). Conversely, suppose that \( [x]^{<\omega} \cap \overline{\mathcal{B}} \subseteq \overline{\mathcal{F}} \cup \hat{\mathcal{F}} \) and let \( y \in [x]^{<\omega} \). Since \( \mathcal{B} \) is a front, there is \( t \in \mathcal{B} \) such that \( t \subseteq y \). Then \( t \in \overline{\mathcal{F}} \cup \hat{\mathcal{F}} \). Since \( t \in \mathcal{B} \), there is \( s \subseteq t \) with \( s \in \mathcal{F} \). Hence \( x \in \text{hom}(\mathcal{F}) \). \( \square \)

**Theorem 3.6.** Let \( \mathcal{B} \) be a front on some set \( z \in [N]^\omega \) and \( \mathcal{I} \) be a uniformly selective Borel ideal on \( \omega \). There is a Borel map \( S : 2^\mathbb{R} \times (\mathcal{I}^+ \upharpoonright z) \rightarrow \mathcal{I}^+ \) such that \( S(\mathcal{F}, x) \) is a \( \mathcal{F} \)-homogeneous subset of \( x \) for all \( x \in \mathcal{I}^+ \) and \( x \subseteq z \).

**Proof.** We may assume that \( \mathcal{B} \) is a front on \( \mathbb{N} \) and proceed by induction on \( \alpha = \text{rank}(\mathcal{B}) \). If \( \text{rank}(\mathcal{B}) = \omega \), then \( \mathcal{B} = [B]^1 \). Let \( S(\mathcal{F}, x) = (\cup \mathcal{F}) \cap y \), if \( (\cup \mathcal{F}) \cap x \in \mathcal{I}^+ \). Otherwise, \( S(\mathcal{F}, x) = x \setminus \cup \mathcal{F} \). Since \( \mathcal{I}^+ \) is Borel, then \( S \) is a Borel function.

Now suppose that the claim holds for all fronts on some set \( z \in [N]^\omega \) of rank less then \( \alpha \). For each \( n \in \mathbb{N} \) and \( \mathcal{F} \subseteq \mathcal{B} \), let

\[
\mathcal{F}_{n+1} = \{ t \in [N]^{<\omega} : n < \min(t) & \{ n \} \cup t \in \mathcal{F} \}.
\]
Observe that $B_{(n+1)}$ is a front on $x/(n+1) = \{m \in x : n < m\}$ with rank less than $\alpha$ and the function
\[
\Gamma : 2^B \times I^+ \to \prod_{n \in \mathbb{N}} (2^{B^{(n)}} \times I^+ \upharpoonright (\mathbb{N} \setminus n))
\]
where $\Gamma(F, x) = ((F^{(n)}, x \setminus n))_{n \in \mathbb{N}}$ is Borel. By the inductive hypothesis there is Borel function
\[
S : \prod_{n \in \mathbb{N}} (2^{B^{(n)}} \times I^+ \upharpoonright (\mathbb{N} \setminus n)) \to \prod_{n \in \mathbb{N}} (I^+ \upharpoonright (\mathbb{N} \setminus n))
\]
that satisfies the conclusion of the theorem for each coordinate. Denote the composition $I_i S$ of $\Gamma$, $S$, and projection to $n$-th coordinate as $S_n$.

We define a sequence of Borel functions $\{H_n\}_{n<\omega}$. For $(F, x) \in 2^B \times I^+$ define inductively
\[
\begin{align*}
H_0(F, x) &= x, \\
H_{n+1}(F, x) &= S_{n+1}(F, x) \text{ if } n \in x \text{ otherwise } H_{n+1}(F, x) = H_n(F, x).
\end{align*}
\]
Since $I$ is uniformly selective, we can extract, in a Borel way, from the sequence $\{H_n(F, x)\}_{n<\omega}$ a set $y \in I^+$ such that
\[
y/(n+1) \subseteq H_{n+1}(F, x) \text{ for all } n \in y.
\]
Lemma 3.5 naturally provides the notion of $i$-homogeneous for $F$ for $i = 0, 1$. Let
\[
y_i = \{n \in y : H_{n+1}(F, x) \text{ is } i\text{-homogeneous for } F^{(n+1)}\}.
\]
Then $y_i$ is $i$-homogeneous for $F$. In fact, for $i = 0$, let $t$ be a finite subset of $y_0$ and let $n = \min(t)$. Then $t/(n+1) \subseteq H_{n+1}(F, x)$ as $n \in y$. Therefore $t/(n+1) \not\subseteq F^{(n+1)}$, as $H_{n+1}(F, x)$ is 0-homogeneous. Thus $t = \{n\} \cup t/(n+1) \not\subseteq F$. Using Lemma 3.5, a similar argument works for $i = 1$.

By Lemma 3.5, being $i$-homogeneous for $F$ is a Borel property, therefore the function $y \mapsto (y_0, y_1)$ is Borel. Since $y \in I^+$, then at least one of the sets $y_0$ or $y_1$ belongs to $I^+$. Let $S(F, x) = y_0$ if $y_0 \in I^+$ and $y_1$, otherwise. As $I^+$ is Borel, we can pick in a Borel way the alternative that holds. Thus $S$ is Borel.

Since $\text{Fin}$ is uniformly selective (corollary 3.3), we get the uniform version of Nash-Williams’ theorem.

**Corollary 3.7.** Let $B$ be a front on $\mathbb{N}$. There is a Borel map $S : 2^B \times [\mathbb{N}]^{<\omega} \to [\mathbb{N}]^{<\omega}$ such that $S(F, x)$ is a $F$-homogeneous subset of $x$, for all $x \in [\mathbb{N}]^{<\omega}$ and all $F \subseteq B$.

Using the front $[\mathbb{N}]^n$, we get that the classical Ramsey’s theorem holds uniformly (the case $n = 2$ appeared in [5]).

**Corollary 3.8.** For each $n \in \mathbb{N}$, there is a Borel function $S : 2^{[\mathbb{N}]^n} \times [\mathbb{N}]^{<\omega} \to [\mathbb{N}]^{<\omega}$ such that $S(F, x)$ is an infinite subset of $x$ homogeneous for $F \subseteq [\mathbb{N}]^n$.

4. **Negative results**

In this section we show that there is a tall $F_\sigma$ ideal without a Borel selector and deduce from this fact that there is no uniform version of Galvin’s theorem. We also show that there is a $\Pi^1_2$ tall ideal $I$ such that $hom(F) \not\subseteq I$ for every $F \subseteq [\mathbb{N}]^{<\omega}$.
4.1. A $F_\sigma$ ideal without a selector and no uniform version of Galvin's theorem. Recall that the hyperspace $K(2^\mathbb{N})$ serves as a space of codes for $F_\sigma$ ideals (see Proposition 2.3). In [4] it is proved that the set of codes of tall $F_\sigma$ ideals is $\Pi^1_2$–complete. To show that there is an $F_\sigma$ ideal without a selector we prove that the complexity of the set of codes of $F_\sigma$ ideals with a Borel selector is $\Sigma^1_2$.

We start by modifying a bit the notion of tallness and Borel selector. For $K \in K(2^\mathbb{N})$, let

$$\downarrow K = \{x : \exists y \in K \ x \subseteq y\}.$$ 

**Definition 4.1.** We say that $K \in K(2^\mathbb{N})$ is pseudo-tall if for every infinite $x \in 2^\mathbb{N}$ there is infinite $y \in \downarrow K$ such that $y \subseteq x$.

One can verify that as a function $\downarrow : K(2^\mathbb{N}) \to K(2^\mathbb{N})$ is continuous and $K$ is pseudo-tall if and only if $\mathcal{I}_K$ is tall.

**Proposition 4.2.** [4] Given $K \in K(2^\mathbb{N})$, there is a Borel function $\phi : \mathcal{I}_K \to K^{<\omega}$ such that $x \subseteq \bigcup \phi(x)$.

**Proposition 4.3.** Let $K \in K(2^\mathbb{N})$ be pseudo-tall. Then $\mathcal{I}_K$ has a Borel selector $S$ if and only if there is a Borel selector $S'$ such that $\text{rng}(S') \subseteq \downarrow K$.

**Proof.** Using Proposition 4.2 it is enough to realize that if $x$ is infinite then at least one set in $\phi(x)$ must have infinite intersection with $x$ and since $\phi(x)$ is finite we can pick such a set in a Borel way. □

This leads to a modified definition of a selector.

**Definition 4.4.** Let $K \in K(2^\mathbb{N})$ be a pseudo-tall. We say that $K$ has a Borel pseudo-selector if there is a Borel function $S : 2^\mathbb{N} \to 2^\mathbb{N}$ such that

- $S(x) \in \downarrow K$,
- if $|x| = \omega$ then $|S(x)| = \omega$,
- $S(x) \subseteq x$.

By the previous proposition, $K \in K(2^\mathbb{N})$ has a pseudo-selector if and only if $\mathcal{I}_K$ has a selector and therefore it suffices to consider only pseudo-selectors of closed subsets of $2^\mathbb{N}$, in other words the questions of existence of a Borel selector for $F_\sigma$ ideals and hereditary tall closed subsets of $2^\mathbb{N}$ are equivalent. Let us summarize this in the following proposition.

**Proposition 4.5.** Let $K \in K(2^\mathbb{N})$ be tall. The following are equivalent:

- there is a Borel selector for $K$,
- there is a Borel pseudo-selector for $K$,
- the $F_\sigma$ ideal $\mathcal{I}_K$ has a Borel selector,
- the smallest ideal $\mathcal{I}$ that contains $K$ and $\text{Fin}$ has a Borel selector.

**Proof.** It can be easily verified that the ideal $\mathcal{I}$ in the fourth condition is also $F_\sigma$. The only implication that is not clear from previous arguments is how to get a Borel selector from a Borel pseudo-selector.
Let $S : 2^\mathbb{N} \to \mathbb{N}$ be a Borel pseudo-selector for $K$. Define
\[ \{(x, y) : S(x) \subseteq y \subseteq x, \ y \in K\} \subseteq 2^\mathbb{N} \times \mathbb{N}. \]
This is a Borel set with each vertical section compact and therefore it has a Borel uniformization by a classical uniformization theorem (see, for instance, [6, Theorem 35.46]). The uniformizing function is a Borel selector for $K$.

4.1.1. Coding of Borel functions. Now we are going to present how to code Borel functions. For that end, first we need to code Borel sets. This coding is somewhat standard (see for instance [3, pag. 19]), but we need to present it with some detail. We define a set of labeled well-founded trees which will be the codes of Borel sets.

Definition 4.6. Let $\mathcal{LT}$ be the set of all trees on $\mathbb{N}$ where each node is labeled by an element of $\{0, 1\}$.

So, formally, every element of $\mathcal{LT}$ is a tuple $(T, f)$ where $T \subseteq \mathbb{N}^{<\omega}$ is a tree and $f : T \to 2$. However, we will always write only $T \in \mathcal{LT}$ and $(s, i) \in T$ meaning that $f(s) = i$.

One can easily check that there $\mathcal{LT}$ is a closed subset of the Polish space of all trees on $\mathbb{N} \times 2$, thus $\mathcal{LT}$ is a Polish space. Moreover, the set of all well-founded labeled trees $WF\mathcal{LT}$ is $\Pi^1_1$.

We are interested in a closed subspace of $\mathcal{LT}$ which will contain all codes for Borel subsets of $2^\mathbb{N}$.

Definition 4.7. Let $\mathcal{LT}_c \subseteq \mathcal{LT}$ be the set of all labeled trees satisfying the following condition.

- if $(s, 1) \in T$ then $(s^{-}(0), 0) \in T$ and it is the only immediate successor of $(s, 1)$.

One can easily verify that $\mathcal{LT}_c$ is a closed subspace of $\mathcal{LT}$ and the set of well-founded trees $WF\mathcal{LT}_c \subseteq \mathcal{LT}_c$ is $\Pi^1_1$.

Now we will define, for each $T \in WF\mathcal{LT}_c$, the Borel set $A_T$ coded by $T$. And conversely, for each Borel set $A \subseteq 2^\mathbb{N}$ there will be a $T \in WF\mathcal{LT}_c$ such that $A = A_T$. The definition of $A_T$ is by recursion on the rank of $T$.

Let $\{t_n : n \in \mathbb{N}\}$ be an enumeration of all basic open sets of $2^\mathbb{N}$, i.e. each $t_n$ is a finite binary sequence. Recursively define what each $(s, i) \in T$ codes:

- if $(s, 0)$ is a leaf then it codes the basic open set $t_{s(|s|-1)}$ (in the case of $s = \emptyset$, we put $t_{\emptyset(|\emptyset|-1)} = t_0$),

- if $(s, 0)$ is not a leaf, then it codes the union of the sets coded by $(s^{-}n, i)$ where $(s^{-}n, i) \in T$,

- $(s, 1)$ codes the complement of what $(s^{-}(0), 0)$ codes.

Finally, $A_T$ is the set coded by $(\emptyset, i)$.

Lemma 4.8. For every Borel set $A \subseteq 2^\mathbb{N}$ there is $T \in WF\mathcal{LT}_c$ such that $A = A_T$. And conversely, $A_T$ is Borel for each $T \in WF\mathcal{LT}_c$. 

Proof. Given \( T \in WFLT_c \), one easily shows for induction on the rank of \( T \) that \( A_T \) is Borel. Conversely, given a Borel set \( A \subseteq 2^\mathbb{N} \), by induction on the Borel complexity of \( A \) it is easy to construct a \( T \in WFLT_c \) such that \( A = A_T \)

Let \( C_i \subseteq 2^\mathbb{N} \times \mathcal{L}_T \), \( i \in 2 \), be given by

\[
(x, T) \in C_1 \text{ if and only if } T \in WFLT_c \text{ and } x \in A_T
\]

and

\[
(x, T) \in C_0 \text{ if and only if } T \in WFLT_c \text{ and } x \notin A_T.
\]

The following is a crucial result.

**Lemma 4.9.** The relation \( C_i \) is \( \Pi_1^1 \) for \( i \in 2 \).

For the proof we need some auxiliary results. We define the following subset \( G \subseteq 2^\mathbb{N} \times \mathcal{L}_T \times \mathcal{L} \).

**Definition 4.10.** A triple \((x, T, S)\) is in \( G \subseteq 2^\mathbb{N} \times \mathcal{L}_T \times \mathcal{L} \) if and only if

- \((s, i) \in T \) for some \( i \in 2 \) if and only if \((s, j) \in S \) for some \( j \in 2 \),
- \((s, 0) \in T \) if \((s, 0) \in T \) is leaf then \((s, 1) \in S \) if and only if \( t_{s(|s|-1)} \subseteq x \),
- \((s, 1) \in T \) then \((s, 1) \in S \) if and only if \((s^{-0}(0), 0) \in S \),
- \((s, 0) \in T \) if \((s, 0) \in T \) not a leaf then \((s, 1) \in S \) if and only if there is \( n \in \mathbb{N} \) such that \((s^{-n}, 1) \in S \).

Note that if \((x, T, S) \in G \) then \( S \) has the same tree structure as \( T \), it only has different labeling. Also note that if \( T \) is well-founded then the labeling of \( S \) is uniquely determined by the values on its leaves. This can be proved by induction on the rank of \( S \). Since the label of the leaves of \( S \) are uniquely determined by \((x, T)\), we can conclude that for each \( T \in WFLT_c \) and every \( x \in 2^\mathbb{N} \) there is exactly one \( S \) such that \((x, T, S) \in G \).

**Claim 4.11.** The set \( G \) is Borel.

*Proof.* We verify that each condition is Borel. The first and the third conditions are independent of the first coordinate and are closed.

For the second condition. Let \( P_s := \{ T \in \mathcal{L}_T : s \text{ is a leaf of } T \} \) and \( Q_s := \{ T \in \mathcal{L}_T : (s, 1) \in T \} \) for each \( s \in \mathbb{N}^{<\omega} \). Then \( P_s \) and \( Q_s \) are easily seen to be closed. Define

\[
R_s := (2^\mathbb{N} \times (\mathcal{L}_T \setminus P_s) \times \mathcal{L}) \cup (t_{s(|s|-1)} \times P_s \times Q_s) \cup ((2^\mathbb{N} \setminus t_{s(|s|-1)}) \times P_s \times (\mathcal{L} \setminus Q_s)).
\]

Then \( \bigcap_{s \in \mathbb{N}^{<\omega}} R_s \) is the collection of all \((x, T, S)\) satisfying the second condition.

The fourth condition is also independent of the first coordinate and one can verify that \( Q_s' := \{ S \in \mathcal{L}_T : (s, 1) \in T \iff \exists n \in \mathbb{N} (s^{-n}(n), 1) \in S \} \) is Borel. Combination of \( P_s, Q_s' \) and their complements gives us the desired result. \( \square \)

For each \((s, i) \in T \), let \( T_{(s,i)} := \{(t, j) : (s^{-t}, j) \in T \} \). Consider the following continuous bijection \( \Gamma : \mathcal{L}_T \to \mathcal{L}_T \) where

- if \((\emptyset, 0) \in T \) then \( \Gamma(T) = R \) where \((\emptyset, 1) \in R \) and \( T_{(\emptyset,0)} = R_{(\emptyset,0)} \),
- if \((\emptyset, 1) \in T \) then \( \Gamma(T) = R \) where \((\emptyset, 0) \in R \) and \( T_{((\emptyset,0))} = R_{(\emptyset,0)} \).
Proof of Lemma 4.9. This follows from the discussion after the Definition 4.10 and the definition of $\Gamma$.

Claim 4.12. Let $T \in WFL_{T_c}$ and $x \in 2^\mathbb{N}$ then $|\{S : (x, T, S) \in G\}| = 1$ and for the unique $(x, T, S) \in G$ we have that $(\emptyset, 1) \in S$ if and only if $x$ is in the set coded by $T$. Moreover, let $(x, T, S), (x, \Gamma(T), S') \in G$, then $(\emptyset, 1)$ is in $S$ or $S'$ but not in both of them.

Proof. This follows from the discussion after the Definition 4.10 and the definition of $\Gamma$. □

Proof of Lemma 4.9. Let $G_i := \{(x, T, S) \in G : (\emptyset, i) \in S\}$ for $i \in 2$. One can easily see that $G = G_0 \cup G_1$ and both sets are Borel. Let $proj(G_i) := \{(x, T) : \exists S \in LT (x, T, S) \in G_i\}$. Then from Claim 4.12 we have

$$C_1 = (2^\mathbb{N} \times WFL_{T_c}) \cap proj(G_1)$$

and

$$C_0 = (2^\mathbb{N} \times WFL_{T_c}) \cap proj(G_0).$$

Finally, we show that the set $(2^\mathbb{N} \times WFL_{T_c}) \cap proj(G_i)$ is $\mathbf{\Pi}^1_1$ for $i < 2$. This follows from the classical result that if $A \subseteq X \times Y$ is Borel, then $\{x \in X : \exists y \in Y (x, y) \in A\}$ is $\mathbf{\Pi}^1_1$. But we can also give a direct proof as follows.

The sets $H_i := (2^\mathbb{N} \times LT_c) \setminus proj(G_i)$ are clearly $\mathbf{\Pi}^1_1$ and so are $M_i := WFL_{T_c} \cap H_i$ for $i < 2$. But then using the Claim 4.12 we see that $(2^\mathbb{N} \times WFL_{T_c}) \cap proj(G_i) = M_{1-i}$. □

Next we define a coding of Borel functions from $2^\mathbb{N}$ to $2^\mathbb{N}$. Let

$$C_n := \{x \in 2^\mathbb{N} : x(n) = 1\}.$$

Let $f : 2^\mathbb{N} \to 2^\mathbb{N}$ be a Borel function and let $A_n := f^{-1}(C_n)$. Then $f$ is described by the sequence $\{A_n\}_{n \in \omega}$ because $f(x)(n) = 1$ if and only if $x \in A_n$. Thus the following is the natural definition of codes for Borel functions.

Definition 4.13. Let $\mathcal{F}T = (LT_c)''$ and $WFF = (WFL_{T_c})''$.

The product topology on $\mathcal{F}T$ is Polish and $WFF \subseteq \mathcal{F}T$ is $\mathbf{\Pi}^1_1$. We denote the elements of $\mathcal{F}T$ also by $T$ and the $n$-th element of $T$ as $T(n)$.

Lemma 4.14. The set $WFF$ codes Borel functions from $2^\mathbb{N}$ to $2^\mathbb{N}$ i.e. every sequence $T \in WFF$ is a code for a function $f_T$ and for every Borel function $f$ there is a sequence $T \in WFF$ such that $f_T = f$.

Proof. As it was mentioned above, every Borel function $f$ is coded by a sequence of Borel sets $\{A_n\}_{n \in \omega}$. Let $T = (T(n))_n$ be such that $T(n) \in WFL_{T_c}$ codes $A_n$ for each $n \in \mathbb{N}$. □

4.1.2. Coding of selectors and $F_\sigma$ ideals. Now we will show that the codes for $F_\sigma$ ideals with Borel selector is $\mathbf{\Sigma}^1_2$ and then conclude with the main results of this section.

Consider the following map $\Omega : 2^\mathbb{N} \times WFF \to 2^\mathbb{N}$ by $\Omega(x, T)(n) = 1$ if and only if $x$ is in the set coded by $T(n)$. From the definitions of $C_i$, $\Omega$ and Lemma 4.9 the following is straightforward.
Lemma 4.15. Let $\mathcal{R} \subseteq 2^{\mathbb{N}} \times \mathcal{F}\mathcal{T} \times 2^{\mathbb{N}}$ be given by $(x, T, y) \in \mathcal{R}$ if and only if
$$\forall n \in \mathbb{N}[( (x, T(n)) \in \mathcal{C}_1 \rightarrow y(n) = 1) \land ( (x, T(n)) \in \mathcal{C}_0 \rightarrow y(n) = 0)].$$
Then $\mathcal{R}$ is $\Sigma^1_1$ and for all $(x, T, y) \in 2^{\mathbb{N}} \times \mathcal{W}\mathcal{F}\mathcal{T} \times 2^{\mathbb{N}}$ we have
$$\Omega(x, T) = y \iff (x, T, y) \in \mathcal{R}.$$ 

Consider the following set $\mathcal{M} \subseteq 2^{\mathbb{N}} \times \mathcal{F}\mathcal{T} \times K(2^{\mathbb{N}})$ defined by $(x, T, K) \in \mathcal{M}$ if and only if
\begin{itemize}
  \item $T \in \mathcal{W}\mathcal{F}\mathcal{T}$,
  \item $\Omega(x, T) \in \downarrow K$,
  \item $\Omega(x, T) \subseteq x$.
\end{itemize}

If $|x| = \omega$, then $|\Omega(x, T)| = |x|$.

Lemma 4.16. $\mathcal{M}$ is a $\Pi^1_1$ subset of $2^{\mathbb{N}} \times \mathcal{F}\mathcal{T} \times K(2^{\mathbb{N}})$.

Proof. It follows from Lemma 4.15. For instance, the second condition can be expressed as follows:
$$T \in \mathcal{W}\mathcal{F}\mathcal{T} \land \Omega(x, T) \in \downarrow K \iff T \in \mathcal{W}\mathcal{F}\mathcal{T} \land \forall y \in 2^{\mathbb{N}} ((x, T, y) \in \mathcal{R} \rightarrow y \in \downarrow K).$$

Theorem 4.17. The set of all $K \in K(2^{\mathbb{N}})$ that have a Borel pseudo-selector is $\Sigma^1_2$.

Proof. This set may be described as
$$\{K \in K(2^{\omega}) : \exists T \in \mathcal{F}\mathcal{T} \forall x \in 2^{\omega}(x, T, K) \in \mathcal{M}\}$$
which is $\Sigma^1_2$.

Theorem 4.18. There is a $F_\sigma$ tall ideal without a Borel selector.

Proof. The codes of $F_\sigma$ ideals with a Borel selector are clearly a subset of all tall $F_\sigma$ ideals and the former set is $\Sigma^1_2$—complete. 

Corollary 4.19. There is a closed subset of $A \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ such that $\mathbb{N}^\mathbb{N} = \text{proj}(A) = \{x \in \mathbb{N}^\mathbb{N} : \exists y \in \mathbb{N}^\mathbb{N} \text{ s.t. } (x, y) \in A\}$ and it does not have a Borel uniformization.

Proof. The space $X := 2^{\mathbb{N}} \setminus \{x : \exists n \text{ s.t. } \forall m > n x(m) = 0\}$ is homeomorphic to $\mathbb{N}^\mathbb{N}$. The restriction of the relation $S = \{(x, y) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} : x \supseteq y\}$ to $X$ is closed in $X$. By our theorem there is a tall $K \in K(2^{\mathbb{N}})$ without Borel selector. Then $K \cap X$ is closed in $X$ and the closed set $A := S \upharpoonright (X \times X) \cap (X \times (K \cap X))$ has no Borel uniformization.

4.1.3. Galvin’s theorem. Now we use the previous result to simply observe that there is no uniform version of Galvin’s theorem.

Theorem 4.20. There is $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$ such that there is no Borel function $S : 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$ satisfies $S(x) \in \text{hom}(\mathcal{F})$, $S(x) \subseteq x$ and $|S(x)| = \omega$ for every infinite $x \in 2^{\mathbb{N}}$.

Proof. Combine Theorem 4.18 and Proposition 2.6.
4.2. A \( \Pi^1_2 \) tall ideal without a closed tall subset. We construct a \( \Pi^1_2 \) tall ideal which does not contain \( \text{hom}(F) \) for every \( F \subseteq [\mathbb{N}]^{<\omega} \). Recall that \( \text{hom}(F) \) is \( \Pi^1_1 \) for every \( F \subseteq [\mathbb{N}]^{<\omega} \) and therefore we have the following.

Observation 4.21. Let \( R \subseteq 2^{[\mathbb{N}]^{<\omega}} \times [\mathbb{N}]^{\omega} \times [\mathbb{N}]^{\omega} \) be defined by

\[
R(F, x, y) \Leftrightarrow y \subseteq x \& y \in \text{hom}(F).
\]

Then \( R \) is \( \Pi^1_1 \).

Lemma 4.22. \( \{5, \text{Lemma 4.6}\} \) There is a continuous function \( \psi : [\mathbb{N}]^{\omega} \times 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}} \) such that for every infinite \( x \in [\mathbb{N}]^{\omega} \), the collection \( \{\psi(x, y) : y \in 2^{\mathbb{N}}\} \) is an almost disjoint family of infinite subsets of \( x \). Moreover, for all infinite \( x \) there is an infinite \( z \subseteq x \) such that \( z \cap \psi(x, y) = \emptyset \) for all \( y \in 2^{\mathbb{N}} \).

Theorem 4.23. There is a \( \Pi^1_2 \) tall ideal \( I \) such that for all \( x \in I^+ \) and all \( F \subseteq [\mathbb{N}]^{<\omega} \) there is \( y \subseteq x \) with \( y \in \text{hom}(F) \cap I^+ \). In particular, \( I \) does not contain any closed hereditary tall set.

Proof. The construction is similar to that presented in \( \{5, \text{Theorem 4.7}\} \). We will sketch the argument below. Let \( \varphi : 2^{\mathbb{N}} \rightarrow 2^{[\mathbb{N}]^{<\omega}} \) be a continuous surjection. By the classical uniformization theorem \( \{6\} \), let \( R^* \subseteq R \) be a \( \Pi^1_1 \) uniformization for the relation \( R \) given by 4.21. Let \( \psi \) be given by Lemma 4.22. Let

\[
C_1 = \{y \in [\mathbb{N}]^{\omega} : \exists x \in 2^{\mathbb{N}}, R^*(\varphi(x), \psi(x, x), y)\},
\]

\[
C_{n+1} = \{y \in [\mathbb{N}]^{\omega} : \exists x \in 2^{\mathbb{N}}, \exists z \in C_n, R^*(\varphi(x), \psi(z, x), y)\}.
\]

Then each \( C_n \) is \( \Sigma^1_2 \). Finally, let

\[
x \in H \Leftrightarrow (\exists n \in \mathbb{N}) (\exists y \in C_n) y \subseteq^* x.
\]

The proof of Theorem 4.7 in \( \{5\} \) shows that \( I = \mathcal{P}(\mathbb{N}) \setminus H \) is a tall ideal. We will show that it satisfies the other requirements. It is clearly \( \Pi^1_2 \). Let \( F \subseteq [\mathbb{N}]^{<\omega} \) and \( y \notin I \). Then there is \( x \in 2^{\mathbb{N}} \) such that \( F = \varphi(x) \). There is also \( n \in \mathbb{N} \) and \( z \in C_n \) so that \( z \subseteq^* y \). Let \( w \) be such that \( R^*(\varphi(x), \psi(z, x), w) \). Then \( w \subseteq z \) and is \( F \)-homogeneous. By definition, \( w \in H \). Then \( w \cap y \) is infinite and \( F \)-homogeneous.

The last claim follows from Lemma 2.6. \( \square \)

A corollary of the proof of the previous theorem provides a more general construction of co-analytic tall ideals as in \( \{5\} \).

Theorem 4.24. Let \( B \) be a front over \( \mathbb{N} \). There is a co-analytic tall ideal \( I \) such that \( \text{hom}(F) \notin I \) for all \( F \subseteq B \).

Proof. From the proof of Theorem 4.23 and using Corollary 3.7 instead of the co-analytic uniformizing set \( R^* \), we define the sets \( C_n \), which now are analytic. Thus the ideal constructed is co-analytic. \( \square \)

Question 4.25. Does every analytic tall ideal \( I \) contain a \( F_\sigma \) tall ideal? (M. Hrusak).

A weaker version of the previous question is for which tall families \( C \) there is \( F \subseteq [\mathbb{N}]^{<\omega} \) such that \( \text{hom}(F) \subseteq C \) (here \( \text{hom}(F) \) is not necessarily closed).
5. Examples of tall families with a Borel selector

We present some examples showing that the search for a Borel selector can be reduced, in some instances, to find an appropriated coloring.

Example 5.1. Let $e : [N]^\omega \to N^N$ be the increasing enumeration function, i.e. $e(x)(n)$ is the nth element of $x$ in its natural order. Notice that $e$ is continuous. Let $\gamma : [N]^\omega \times [N]^\omega \to [N]^\omega$ be given by

$$\gamma(x, y) = \{e(x)(n) : n \in y\}.$$  

Notice that $\gamma(x, y) \subseteq x$ and $\gamma$ is continuous. For each $y \in [N]^\omega$, let

$$C_y = \{\gamma(x, y) : x \in [N]^\omega\}.$$  

Then $C_y$ is a tall family and obviously $S(x) = \gamma(x, y)$ is a Borel selector for $C_y$.

We will show that $C_y$ contains $\text{hom}(c)$ for some coloring of pairs $c$. Let $(y_n)_n$ be the increasing enumeration of $y$. We assume that $y_0 \geq 1$. If $(z_n)_n$ is the increasing enumeration of an infinite set $z$, then

$$z \in C_y \iff (\forall n)(y_{n+1} - y_n \leq z_{n+1} - z_n) \& y_0 \leq z_0.$$  

Consider the following coloring:

$$c\{k, l\} = 0 \iff l - k \geq y_k \& k \geq y_0.$$  

Then any $c$-homogeneous infinite set is necessarily $0$-homogeneous. Let $h = \{h_n : n \in N\} \in \text{hom}(c)$. Then $h_0 \geq b_0 \geq 1$ and $h_k \geq h_{k+1}$. Hence $h_{k+1} - h_k \geq b_{h_k} \geq b_{h_{k+1}} \geq b_k - b_k$. Thus $h \in C_y$.

Question 5.2. Let $K \subseteq [N]^\omega \times [N]^\omega$ be a closed set without a Borel uniformization (see [6]). Consider the following family:

$$C = \{\gamma(x, y) : (x, y) \in K\}.$$  

Since the projection of $K$ is $[N]^\omega$, then $C$ is tall and, by definition, is analytic. We do not know if $C$ has a Borel selector.

Example 5.3. Let $WO(Q)$ be the collection of all well-ordered subsets of $Q$ respect the usual order. Let $WO(Q)^*$ the collection of well ordered subsets of $(Q, <^*)$ where $<^*$ is the reversed order of the usual order of $Q$. Let $C = WO(Q) \cup WO(Q)^*$. Notice that $C$ is a complete co-analytic set. To see that $C$ has a Borel selector, let $c : [\omega]^2 \to 2$ given by $c\{r_n, r_m\} = 0$ iff $n < m$, where $(r_n)_n$ is any fixed enumeration of $Q$. Then $\text{hom}(c) \subseteq C$ and the result follows from corollary 3.8.

Let $K$ be a sequentially compact space and $(x_n)_n$ be a sequence on $K$. Let

$$C(x_n)_n = \{y \subseteq N : (x_n)_n \in y \text{ is convergent}\}.$$  

Then $C(x_n)_n$ is tall.

Theorem 5.4. Let $(x_n)_n$ be dense in a compact metric space $X$. There is a coloring $c$ such that $\text{hom}(c) \subseteq C(x_n)_n$, thus $C(x_n)_n$ has a Borel selector.
Proof. Let $f : 2^\mathbb{N} \to X$ be a continuous surjection. Let $y_n \in 2^\mathbb{N}$ such that $f(y_n) = x_n$ for each $n \in \omega$. Let $\leq$ be the usual lexicographic order on $2^\mathbb{N}$. Consider the Sierpinsky coloring $c\{n, m\}_< = 1$ iff $y_n \preceq y_m$. Then for any $h \in \text{hom}(c)$, $(y_n)_{n \in h}$ is a monotone sequence in $2^\mathbb{N}$ and therefore it is convergent and so is $(x_n)_{n \in h}$. Hence $\text{hom}(c) \subseteq C(x_n)$.

Now we will look at the ideal of nowhere sets in countable spaces.

**Theorem 5.5.** Let $(X, \tau)$ be a regular space without isolated points over a countable set $X$. There is a coloring $c : [X]^2 \to 2$ such that $\text{hom}(c) \subseteq \text{nwd}(X, \tau)$ and thus $\text{nwd}(X, \tau)$ has a Borel selector.

Proof. The Sierpinski coloring $c$ on $[\mathbb{Q}]^2$ satisfies that $\text{hom}(c) \subseteq \text{nwd}(\mathbb{Q})$. Notice that every $c$-homogeneous set is a discrete subset of $\mathbb{Q}$. Let $(V_n)_n$ be a countable collection of $\tau$-open sets that separates points. Let $\rho$ be the topology generated by the $V_n$’s. Then $(X, \rho)$ is homeomorphic to $\mathbb{Q}$. Therefore the Sierpinski coloring on $\mathbb{Q}$ can be defined on $[X]^2$ such that every $c$-homogeneous set is a $\rho$-discrete subset of $X$. Since $\rho \subseteq \tau$, then $\text{hom}(c) \subseteq \text{nwd}(X, \tau)$.

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**References**

[1] J. Avigad. An effective proof that open sets are Ramsey. *Arch. Math. Logic*, 37(4):235–240, 1998.

[2] F. Galvin and K. Prikry. Borel sets and Ramsey’s theorem. *J. Symbolic Logic*, 38:193–198, 1973.

[3] S. Gao. Invariant Descriptive Set Theory. *Pure and Applied Mathematics*, 293. Taylor & Francis Group, 2009. ISBN-13: 978-1-58488-793-5.

[4] J. Grebík and M. Hrušák. No minimal tall Borel ideal in the Katětov order. arxiv.org/pdf/1708.05322.pdf, 2017.

[5] M. Hrušák, D. Meza-Alcántara, E. Thümmel, and C. Uzcátegui. Ramsey type properties of ideals. *Annals of Pure and Applied Logic*, 168(11):2022–2049, 2017.

[6] A. S. Kechris. *Classical Descriptive Set Theory*. Springer-Verlag, 1994.

[7] A.R.D. Mathias. Happy families. *Ann. Math. Log.*, 12:59–11, 1977.

[8] K. Mazur, $F_\tau$-ideals and $\omega_1\omega_1^*$-gaps in the Boolean algebras $P(\omega)/I$, *Fundamenta Mathematicae*, 138(2): 103–111, 1991.

[9] C. St. J. A. Nash-Williams. On better-quasi-ordering transfinite sequences. *Mathematical Proceedings of the Cambridge Philosophical Society*, 64(2):273–290, 1968.

[10] R. Solovay. Hyperarithmetically encodable sets. *Trans. Amer. Math. Soc.*, 239:99–122, 1978.

[11] S. Todorčević. Higher dimensional Ramsey theory. In *Ramsey methods in analysis*, Advanced courses in mathematics, CRM Barcelona. Birkhäuser, 2005.

[12] S. Todorčević. *Introduction to Ramsey spaces*. Annals of Mathematical Studies 174. Princeton University Press, 2010.
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