Exact integration of height probabilities in the Abelian Sandpile model

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Received 3 September 2012
Accepted 5 September 2012
Published 28 September 2012

Online at stacks.iop.org/JSTAT/2012/P09013
doi:10.1088/1742-5468/2012/09/P09013

Abstract. The height probabilities for the recurrent configurations in the Abelian Sandpile model on the square lattice have analytic expressions, in terms of multidimensional quadratures. At first, these quantities were evaluated numerically with high accuracy and conjectured to be certain cubic rational-coefficient polynomials in $\pi^{-1}$. Later their values were determined by different methods.

We revert to the direct derivation of these probabilities, by computing analytically the corresponding integrals. Once again, we confirm the predictions on the probabilities, and thus, as a corollary, the conjecture on the average height, $\langle \rho \rangle = 17/8$.

Keywords: sandpile models (theory)

ArXiv ePrint: 1207.6074
1. Introduction

The Abelian Sandpile model is a non-equilibrium system, driven at a slow steady rate with local threshold relaxation rules, which in the steady state shows relaxation events, called *avalanches*, in bursts of a wide range of sizes and critical spatio-temporal correlations, obtained without fine-tuning of any control parameters. We refer to the introductory reviews [1]–[4].

In the set of *stable* configurations in the Abelian Sandpile model on (portions of) a square lattice, at each site \(i \in \mathbb{Z}^2\), the height variable can take the values \(z_i = 0, 1, 2, 3\).\(^1\) Particles are added randomly and the addition of a particle increases the height at that site by one. If this height exceeds the critical value \(z_c = 3\), then the site *topples*. On a toppling event, its height decreases by 4 and the heights at each of its nearest neighbours increases by 1.

A very natural question is: what is the asymptotic (i.e., infinite-volume) probability \(P_i\) for the heights \(z_i\), for \(i = 0–3\), in the ensemble of *recurrent* configurations?

After some numerical studies [5]–[7], the first exact result [8] concerns with probability \(P_0\) for a site to be empty:

\[
P_0 = \frac{2}{\pi^2} - \frac{4}{\pi^3}; \quad (1a)
\]

An analytic expression for the other probabilities was obtained in [9, 10]:

\[
P_1 = \frac{1}{2} - \frac{3}{2\pi} - \frac{2}{\pi^2} + \frac{12}{\pi^3} + \frac{I_1}{4}; \quad (1b)
\]

\[
P_2 = \frac{1}{4} + \frac{3}{2\pi} + \frac{1}{\pi^2} - \frac{12}{\pi^3} - \frac{I_1}{2} + \frac{3I_2}{32}; \quad (1c)
\]

\(^1\) Some authors prefer the values \(z_i = 1, 2, 3, 4\). Results are easily translated between the two notations.
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\[ P_3 = \frac{1}{4} - \frac{1}{\pi^2} + \frac{4}{\pi^3} + \frac{I_1}{4} + \frac{3I_2}{32}; \quad (1d) \]

where \( I_1 \) and \( I_2 \) are expressed as multiple integrals. These results had been obtained by using a mapping from the set of recurrent configurations onto the set of spanning trees covering the lattice. These trees are rooted (on the boundary of the lattice, where dissipation occurs). Introduce the concept of predecessor: a vertex \( j \) precedes a vertex \( i \) if the unique path on the spanning tree from \( j \) to the root includes \( i \). Then, the probabilities \( P_k \) at the vertex \( i \) are simply related to the numbers \( X_k \) of spanning trees in which the vertex \( i \) has exactly \( k \) predecessors among its nearest neighbours. And the \( X_k \)s can be, at the end, expressed in terms of the lattice Green function.

Furthermore, an indirect argument fixes a relation between \( I_1 \) and \( I_2 \) (see equation (15) later on)\(^2\). Thus any single further linearly independent information on \( I_1 \) and \( I_2 \), or on the \( P_k \)s, would have fixed the height probabilities completely.

An extensive account on the derivation of these results is provided in [14], together with several other interesting properties\(^3\).

As a corollary, the average density in the ensemble of recurrent configurations is given by

\[ \langle \rho \rangle = \frac{3}{4} \sum_{k=0}^{3} kP_k = \frac{7}{4} + \frac{3}{2\pi} - \frac{3}{\pi^2} + \frac{3I_2}{32}; \quad (2) \]

As reported in [1], this quantity was conjectured by Grassberger to be\(^4\)

\[ \langle \rho \rangle = \frac{17}{8}. \quad (3) \]

An interesting observation is the following: the expectation value of the height probabilities does not help in understanding the conformal features of the corresponding field theory in the continuum, at least in the whole plane. But it is not so in the presence of a boundary. Indeed, in [11]–[14], the evaluation of the height probabilities in the upper half plane has been used to reveal that the continuum theory is a logarithmic conformal theory with central charge \( c = -2 \). Afterwards, also two-point correlation functions for the height variables have been computed [15] and found in agreement with the prediction of a logarithmic conformal field theory based on field identifications obtained previously.

A rejuvenation of the interest in the exact determination of \( \langle \rho \rangle \) has arisen with the work of Fey et al [16, 17], in which a subtle difference has been elucidated between the uniform average on the ensemble of recurrent configurations, and the properties of the critical system with conserved mass. As the discrepancies in densities between the two regimes are numerically very small (\( \delta \rho/\rho \sim 10^{-4} \)), although the numerical determination of the integral \( I_2 \) appearing in (2) has a much higher precision, it would have been more satisfactory to have an exact result for at least one of these two quantities.

In fact, it was possibly this rejuvenated interest that led some time later to two independent proofs, methodologically similar, of the density conjecture (and, through the

\(^2\) A certain combinatorial quantity, known to be finite, is formulated as a lattice integral presenting a divergence: tuning to zero an overall factor in the divergent part gives the aforementioned relation.

\(^3\) In [14], in equation (4.1), the authors also correct a misprint in equation (32) of [10], which was wrong by a factor 2. Here we notice a misprint in their equation (4.8), where the second term on the right-hand side should be \( 7/\pi^2 \) instead of \( 7/\pi^2 \).

\(^4\) These authors use the range \( 1 \leq z_i \leq 4 \), and accordingly write \( \langle \rho \rangle = 25/8 \).
argument above, of all the height probabilities) [18, 19]. The single missing linear relation
was the intensity of the loop-erased random walk at a first neighbour of the source of
the walk, which is combinatorially related to the density (and turned out to be 5/16 on
the square lattice). The role of the loop-erased random walk in the uniform spanning tree
model (and thus in the Abelian Sandpile model) should not be surprising because of the
works on the subject culminating in the Propp and Wilson exact sampling algorithm [20].

In this paper we shall provide a different proof, conceptually simpler (although,
admittedly, theoretically less illuminating): we shall revert to the original formulation
of the problem, and evaluate exactly the integrals in question. They have the form of
two-loop Feynman integrals in a two-dimensional scalar field theory on the lattice. There
has been a long-standing effort to reduce the evaluation of lattice Feynman integrals at
the one-loop level, through simple algebraic methods, both in momentum space [21, 22],
and in coordinate space [23]. These methods have also found important applications in
two dimensions, respectively [24, 25] and [26]. In particular, in [27], there is an extension
to the triangular lattice, which can be of help to generalize our procedure to this system
(note that the study of sandpiles on the triangular and honeycomb lattices, with the aim
of height probabilities and correlation functions, has also been considered in [19, 28]).

1.1. The integrals to evaluate

We shall use the following notations of lattice momenta, which are common in lattice field
theory
\[ \overline{p}_\mu := \sin p_\mu, \]  
\[ \hat{p}_\mu := 2 \sin \frac{p_\mu}{2} \]  
where, in our two-dimensional case, the index \( \mu \) can take two values, which we choose to
be 0 and 1. Then
\[ \hat{p}^2 := \sum_{\mu=0,1} \hat{p}_\mu^2 \]  
is the quantity, invariant under the lattice symmetry, which appears in the lattice
propagator
\[ \Delta(p) := \frac{1}{\hat{p}^2 + h}. \]  
We have added the regulator of the infrared singularity \( h \) just to have well-defined
quantities in all our manipulations, but we are interested only in the limit of vanishing \( h \)
(and thus, to integrals for which this limit exists). In the following we will not mention
explicitly the regulator \( h \), and the extraction of the limit will be understood where
pertinent. Given the shorthand
\[ \int dp := \int_{-\pi}^\pi \frac{dp_0}{2\pi} \int_{-\pi}^\pi \frac{dp_1}{2\pi}; \quad \delta_2(p) := (2\pi)^2 \delta(p_0) \delta(p_1); \]  
we want to integrate polynomial expressions in the lattice momenta (4), in the measure
\[ d\mu := dp dq dk \delta_2(p + q + k) \Delta(p) \Delta(q) \Delta(k) \]  

doi:10.1088/1742-5468/2012/09/P09013
which is invariant under all the permutations of the momenta \( p, q, k \), under exchange of the indices 0 with 1, and under simultaneous inversion of all the momenta along one of the lattice axis. These invariances imply relations between the integral of different polynomials, to which we will refer generically in the following as ‘symmetry of the integration measure’. In particular, we use the symbol \( A \rightarrow B \) to denote the fact \( \int d\mu A = \int d\mu B \).

In order to define the integrands pertinent to the expressions in (1), we have to start from the matrix \( M(c_1, c_2, c_3) \), given in (equation (3.18)) [14]

\[
M(c_1, c_2, c_3) = \begin{pmatrix}
c_1 & 1 & e^{iq_1} \\
c_3 e^{ip_0+ip_0} & e^{-iq_0+iq_0} & 1 \\
c_2 e^{ip_1+iq_1} & 1 & e^{-ip_1} \\
c_2 e^{-ip_1-iq_1} & e^{2iq_1} & e^{ip_1}
\end{pmatrix}.
\] (10)

The interesting quantity is the integral [ [14], equation (3.17)]

\[
I(c_1, c_2, c_3) = \int d\mu \sin p_0 \det M(c_1, c_2, c_3).
\] (11)

It is soon realized that the integral is real, does not depend on \( c_3 \), and is, of course, linear in \( c_1 \) and \( c_2 \):

\[
I(c_1, c_2, c_3) = \frac{1}{8} (J_1 c_1 + J_2 c_2).
\] (12)

The factor 1/8 is due to our choice to maintain the usual definition of the lattice propagator. This differs from the choice in [9, 10, 14] by a factor 2, and we have three propagators in the definition of the integration measure. Then, for the quantities \( I_1, I_2 \) defined above in (1),

\[
I_1 = J_1 + \left( \frac{4}{\pi} - 1 \right) J_2; \quad (13)
\]

\[
I_2 = 8J_2 - \frac{16}{\pi} + \frac{4}{\pi^2}. \quad (14)
\]

In [14] it is shown, by an indirect compatibility argument, that the relation

\[
J_1 + J_2 = \frac{2}{\pi} - \frac{4}{\pi^2}
\] (15)

must hold. This is verified numerically to high precision (to order \( 10^{-12} \)). It is also observed numerically to high precision that

\[
J_2 = \frac{1}{2}. \quad (16)
\]

If these relations hold exactly then

\[
P_1 = \frac{1}{4} - \frac{1}{2\pi} - \frac{3}{\pi^2} + \frac{12}{\pi^3}; \quad (17a)
\]

\[
P_2 = \frac{3}{8} + \frac{1}{\pi} - \frac{12}{\pi^3}; \quad (17b)
\]

\[
P_3 = \frac{3}{8} - \frac{1}{2\pi} + \frac{1}{\pi^2} + \frac{4}{\pi^3}; \quad (17c)
\]
and also the conjecture by Grassberger on the density follows:

\[ \sum_{k=0}^{3} k P_k = \frac{17}{8}. \]  

1.2. The strategy

The details of the derivation are given in the following sections. Let us outline here the general strategy that we adopted all along the calculation. By exploiting the symmetry of the integration measure, we try to obtain, in the numerator, a factor which can cancel one of the propagators (at vanishing regularization). Say that we get a \( \hat{p}^2 \) in the numerator. Then, we write all other appearances of \( p_{\mu} \) as \( - (q_{\mu} + k_{\mu}) \). Thus, possibly through a trigonometric expansion at the numerator, the remaining integrals are factorized in independent one-loop integrals in the two other momenta. Some useful trigonometric identities used toward this goal are

\[ \sum_{\mu=0,1} p_{\mu}^2 = \hat{p}^2 - \frac{1}{4} \sum_{\mu=0,1} p_{\mu}^4; \]  
\[ \hat{k}_{\mu}^2 = q_{\mu}^2 + p_{\mu}^2 - \frac{1}{2} q_{\mu}^2 p_{\mu} + 2 p_{\mu}^2 p_{\mu}. \]  

The latter, which is valid when the sum of the momenta \( p, q \) and \( k \) vanishes, is sometimes useful also in the inverse form, in which \( q_{\mu} p_{\mu} \) is expressed in terms of the others.

We shall need the very elementary one-loop integrals:

\[ \int dq \frac{q_{0}^2}{q^2} = 2; \quad \int dq \frac{q_{0}}{q^2} = 0; \quad \int dq \frac{q_{0}^2}{q^2} = \frac{1}{2}; \quad \int dq \frac{q_{0}^4}{q^2} = \frac{4}{\pi}. \]  

From these building blocks, other integrals soon follow, for example

\[ \int dq \frac{q_{0}^2 q_{1}^2}{q^2} = \int dq \frac{q_{0}^2 (q_{1}^2 - q_{0}^2)}{q^2} = 2 - \frac{4}{\pi}; \]  
\[ \int dq \frac{q_{0}^2 q_{1}^2}{q^2} = \int dq \left( \frac{q_{0}^2}{q^2} + \frac{1}{4} \frac{q_{0}^4}{q^2} \right) = \frac{1}{2} - \frac{1}{\pi}; \]  

and the slightly more tricky

\[ \int dq \frac{q_{0}^2 q_{1}^2}{q^2} = \frac{1}{2} \int dq \frac{q_{0}^2 q_{1}^2 (q_{0}^2 + q_{1}^2)}{q^2} = \frac{1}{2} \int dq \frac{q_{0}^2 q_{1}^2}{q^2} = 2. \]  

We shall also need

\[ \int dq \frac{q_{0}^2 \cos q_{1}}{q^2} = \int dq \left( \frac{q_{0}^2}{q^2} - \frac{1}{4} \frac{q_{0}^4}{q^2} - \frac{1}{2} \frac{q_{0}^2 q_{1}^2}{q^2} + \frac{1}{8} \frac{q_{0}^4 q_{1}^2}{q^2} \right) = -\frac{1}{4} + \frac{1}{\pi}. \]  

One more trigonometric identity has been used in the appendix in order to compute a slightly more complex integral.

\[ \text{doi:10.1088/1742-5468/2012/09/P09013} \]
2. The integral $J_2$

The contribution proportional to $c_2$ comes from the integral of

$$2 \sin p_0 \sin q_1 \left\{ \sin p_1 \left[ (-\cos q_0 + \cos q_1) \sin p_0 - (1 + \cos p_0 + \cos q_1) \sin q_0 \right] \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \r
Using (20) in the inverse form, the expansion of the square produces 3 contributions. The first one is

$$-\frac{1}{4}(\hat{k}^2 - q^2 - \hat{q}^2)^2 \rightarrow -\frac{1}{4} \left[ 3(\hat{k}^2)^2 - 2q^2\hat{q}^2 \right]$$

$$= -\frac{3}{4} \hat{k}^2 \left( \hat{p}^2 + q^2 - \frac{1}{2} \sum_{\mu=0,1} \hat{q}^2 p_{\mu}^2 + 2 \sum_{\mu=0,1} p_{\mu} q_{\mu} \right) + \frac{1}{2} q^2 \hat{q}^2$$

$$\rightarrow -q^2\hat{p}^2 + \frac{3}{16} k^2 q^2\hat{q}^2,$$

(34)

where we used also the fact that in the integration the factor \(\hat{k}^2\) cancels all the dependence from \(k\) and the subsequent integration of \(p_{\mu}\hat{q}_{\mu}\) vanishes, while the integration of \(q_{\mu}^2\hat{q}_{\mu}^2\) does not depend on the values of \(\mu\) and \(\nu\).

The second contribution is

$$-\frac{1}{4}(\hat{k}^2 - q^2 - \hat{q}^2)^2 \sum_{\mu=0,1} \hat{q}_{\mu}^2 p_{\mu}^2 \rightarrow -\frac{1}{8} \hat{k}^2 q^2\hat{p}^2 + \frac{1}{2} q^2 \sum_{\mu=0,1} \hat{q}_{\mu}^2(q + \hat{k})^2$$

$$\rightarrow -\frac{1}{8} \hat{k}^2 q^2\hat{p}^2 + q^2 \hat{q}_{0}^2(q_{0} + \hat{k}_{0}) - \frac{1}{2} \hat{k}^2 \hat{q}_{0}^2$$

$$\rightarrow -\frac{1}{8} \hat{k}^2 q^2\hat{p}^2 + q^2 \hat{q}_{0}^2 + \frac{1}{4} \hat{k}^2 q^2\hat{p}^2 - \frac{1}{2} \hat{k}^2 q^2\hat{q}_{0}^2.$$  

(35)

We now combine the last term with the similar one in (33), that is

$$\frac{1}{16} \left[ \left( \sum_{\mu=0,1} \hat{p}_{\mu}^4 \right) \left( \sum_{\nu=0,1} \hat{q}_{\nu}^4 \right) - \left( \sum_{\mu=0,1} \hat{p}_{\mu}^2 \hat{q}_{\mu}^2 \right)^2 \right] = \frac{1}{16} (\hat{p}_{0}^2 \hat{q}_{1}^2 - \hat{p}_{1}^2 \hat{q}_{0}^2)^2$$

$$= \frac{1}{16} \left[ \hat{p}_{0}^2 (\hat{q}_{0}^2 - \hat{q}_{0}^2)^2 - (\hat{p}_{0}^2 - \hat{p}_{0}^2) \hat{q}_{0}^2 \right] = \frac{1}{16} \left[ \hat{p}_{0}^2 (\hat{q}_{0}^2 - \hat{q}_{0}^2)^2 \right]$$

$$\rightarrow \frac{1}{8} \hat{k}^2 \hat{q}_{0}^2 (\hat{q}_{0}^2 - \hat{q}_{0}^2)^2$$

$$\rightarrow \frac{1}{8} \hat{k}^2 \hat{q}_{0}^2 (\hat{p}_{0}^2 + \hat{q}_{0}^2 - \frac{1}{2} \sum_{\mu=0,1} \hat{q}_{\mu}^2 p_{\mu}^2) q_{0}^2 - \hat{q}_{0}^2 (\hat{p}_{0}^2 + \hat{q}_{0}^2 - \frac{1}{2} \hat{q}_{0}^2 \hat{p}_{0}^2)$$

$$\rightarrow \frac{1}{8} \hat{k}^2 \hat{p}_{0}^2 \hat{q}_{0}^4 - \frac{1}{16} \hat{k}^2 \hat{q}_{0}^2 \hat{q}_{0}^2 + \frac{1}{16} \hat{k}^2 \hat{p}_{0}^2 \hat{q}_{0}^2 + \frac{1}{16} \hat{k}^2 \hat{q}_{0}^2 \hat{q}_{0}^2$$

$$\rightarrow \frac{1}{8} \hat{k}^2 \hat{p}_{0}^2 \hat{q}_{0}^4 - \frac{1}{16} \hat{k}^2 \hat{p}_{0}^2 \hat{q}_{0}^2.$$  

(36)

By collecting all the pieces, and using the elementary integrals (21), we get

$$\left( \frac{3}{16} + \frac{1}{8} - \frac{1}{16} \hat{q}_{0}^2 \right) \hat{k}^2 q^2\hat{p}^2 + \left( \frac{1}{8} - \frac{1}{4} \right) \hat{k}^2 \hat{p}_{0}^2 \hat{q}_{0}^4 \rightarrow \left( \frac{3}{16} - \frac{1}{2\pi} \right) \hat{k}^2 q^2\hat{p}^2.$$  

(37)

In conclusion, we find that the value of the first term is

$$J_2^{(a)} = \int d\mu \left( p_{0} q_{1} - q_{0} p_{1} \right)^2 = \frac{3}{16} - \frac{1}{2\pi}.$$  

(38)

2.2. The integral \(J_2^{(b)}\)

We now consider the evaluation of the integral in (31), and note that the integrand can be written as

$$4p_{0} \frac{p_{0}}{p_{0} + q_{0}} \left( \frac{q_{1}}{q_{1} - p_{1}} \right) \rightarrow -2 \sum_{\mu=0,1} p_{\mu} \xi_{\mu} \sum_{\nu=0,1} q_{\nu} (q_{\nu} - p_{\nu})$$  

(39)
because the terms with $\mu = \nu$ give a vanishing contribution (they are anti-symmetric under the exchange of $p$ with $q$). Repeated use of (19) and (20) gives

$$
\frac{1}{2} \left[ (\hat{q}^2 - \hat{k}^2 - \hat{p}^2) + \frac{1}{2} \sum_{\mu=0,1} \hat{p}_\mu^2 \mu^2 \right] \left( \hat{k}^2 - 3q^2 - \hat{p}^2 \right) + \frac{1}{2} \sum_{\nu=0,1} \hat{q}_\nu^4 + \frac{1}{2} \sum_{\nu=0,1} \hat{p}_\nu^2 \hat{q}_\nu^2.
$$

(40)

We split this evaluation into three terms (the following (41), (42) and (45)). A first contribution is

$$
\frac{1}{2} \left[ (\hat{q}^2 - \hat{k}^2 - \hat{p}^2) + \frac{1}{2} \sum_{\mu=0,1} \hat{p}_\mu^2 \mu^2 \right] \left( \hat{k}^2 - 3q^2 - \hat{p}^2 \right)

\rightarrow -\frac{3}{2} \left[ (\hat{q}^2 - \hat{k}^2 - \hat{p}^2) + \frac{1}{2} \sum_{\mu=0,1} \hat{p}_\mu^2 \mu^2 \right] \hat{q}^2 = -3q^2 \sum_{\mu=0,1} \hat{p}_\mu \hat{k}_\mu \rightarrow 0
$$

(41)

which vanishes in the integral. A second contribution is

$$
\frac{\hat{q}^2 - \hat{k}^2 - \hat{p}^2}{4} \left( \sum_{\nu=0,1} \hat{q}_\nu \left[ \sum_{\nu=0,1} \hat{p}_\nu^2 \hat{q}_\nu \right] \right)

\rightarrow \frac{\hat{q}^2 - 2\hat{k}^2}{4} \sum_{\nu=0,1} \hat{q}_\nu^4 - \frac{\hat{k}^2}{4} \sum_{\nu=0,1} \hat{p}_\nu^2 \hat{q}_\nu^2.
$$

(42)

We have now, in all summands, an exposed propagator. Following our general strategy, we rewrite the remaining expressions using $p + q + k = 0$, namely

$$
\frac{1}{4} \hat{q}^2 \sum_{\nu=0,1} \hat{q}_\nu^4 = \frac{1}{4} \hat{q}^2 \sum_{\nu=0,1} \left[ \hat{k}_\nu^2 \left( 1 - \frac{1}{4} \hat{p}_\nu^2 \right) + \hat{p}_\nu^2 \left( 1 - \frac{1}{4} \hat{k}_\nu^2 \right) + 2\hat{k}_\nu \hat{p}_\nu \right]^2

\rightarrow \frac{1}{2} \hat{q}^2 \sum_{\nu=0,1} \left[ \hat{k}_\nu^4 \left( 1 - \frac{1}{4} \hat{p}_\nu^2 \right)^2 + 3\hat{p}_\nu^2 \left( 1 - \frac{1}{4} \hat{k}_\nu^2 \right) \hat{k}_\nu^2 \left( 1 - \frac{1}{4} \hat{k}_\nu^2 \right) \right]

\rightarrow \frac{1}{2} \hat{q}^2 \sum_{\nu=0,1} \left( \hat{k}_\nu^4 + 3\hat{k}_\nu^2 \hat{p}_\nu^2 - 2\hat{k}_\nu^2 \hat{p}_\nu^2 + \frac{1}{4} \hat{k}_\nu^4 \hat{p}_\nu^4 \right)
$$

(43)

so that the whole contribution from (42) is

$$
\hat{q}^2 \sum_{\nu=0,1} \left( \frac{5}{4} \hat{k}_\nu^2 \hat{p}_\nu^2 - \frac{5}{4} \hat{k}_\nu^2 \hat{p}_\nu^4 + \frac{1}{8} \hat{k}_\nu^4 \hat{p}_\nu^4 \right) \rightarrow \frac{5}{8} \hat{q}^2 \hat{k}_\nu^4 \hat{p}_\nu^2 - \frac{3}{8} \hat{q}^2 \hat{p}_\nu^2 \hat{k}_\nu^4 + \frac{1}{4} \hat{q}^2 \hat{k}_\nu^4 \hat{p}_\nu^4.
$$

(44)

We are left with the third term

$$
\frac{1}{8} \sum_{\mu=0,1} \hat{p}_\mu^2 \hat{k}_\mu^2 \sum_{\nu=0,1} \hat{q}_\nu^4 \left( \hat{q}_\nu^2 + \hat{p}_\nu^2 \right).
$$

(45)

One summand gives

$$
\frac{1}{8} \sum_{\mu=0,1} \hat{p}_\mu^2 \hat{k}_\mu^2 \sum_{\nu=0,1} \hat{q}_\nu^4 \rightarrow \frac{1}{8} \hat{p}_0^2 \hat{k}_0^2 \left( \hat{q}_0^4 + \hat{q}_1^4 \right) \rightarrow \frac{1}{2} \hat{p}_0^2 \hat{k}_0^2 \hat{q}_0^4
$$

(46)

because

$$
\int d\mu \hat{p}_0^2 \hat{k}_0^2 \hat{q}_1^4 = \int d\mu \hat{p}_0^2 \hat{k}_0^2 \left( \hat{q}_0^2 - \hat{q}_0^2 \right)^2 = \int d\mu \hat{p}_0^2 \hat{k}_0^2 \left[ \hat{q}_0^4 - 2\hat{q}_0^2 \hat{q}_0^2 + (\hat{q}_0^2)^2 \right]

\rightarrow \frac{1}{4} \int d\mu \hat{p}_0^2 \hat{k}_0^2 \left( \hat{q}_0^4 + \hat{q}_0^2 \left( \hat{q}_0^2 - \hat{q}_0^2 \right) \right) = \int d\mu \hat{p}_0^2 \hat{k}_0^2 \hat{q}_0^4
$$

(47)

doi:10.1088/1742-5468/2012/09/P09013
where we used the fact that
\[
\int \mu \, \rho_0^2 \hat{k}_0^2 \hat{q}_0^2 (\hat{q}_1^2 - \hat{q}_0^2) = \int dp \, \int dk \, \rho_0^2 \hat{k}_0^2 \left( 2 \hat{p}_1^2 - \frac{1}{2} \hat{p}_1^2 \hat{k}_1^2 - 2 \hat{p}_0^2 + \frac{1}{2} \hat{p}_0^2 \hat{k}_0^2 \right) \Delta(p) \Delta(k)
\]
\[
= 2 - \frac{4}{\pi} - \frac{1}{2} \left( 2 - \frac{4}{\pi} \right) \left( \frac{4}{\pi} + \frac{8}{\pi^2} \right) = 0.
\]
The second summand is
\[
\frac{1}{8} \sum_{\mu=0,1} \rho_0^2 \hat{k}_0^2 \sum_{\nu=0,1} \hat{q}_0^2 \hat{p}_\nu^2 \rightarrow \frac{1}{4} \rho_0^2 \hat{k}_0^2 \hat{q}_0^2 + \frac{1}{4} \rho_0^2 \hat{k}_0^2 \hat{q}_1^2
\]
and
\[
\frac{1}{4} \rho_0^2 \hat{k}_0^2 \hat{q}_0^2 \hat{q}_1^2 \rightarrow \frac{1}{4} \rho_0^2 \hat{k}_0^2 \hat{q}_0^2 (\hat{q}_0^2 - \hat{q}_0^2) = \frac{1}{8} \left( 2 - \frac{4}{\pi} \right) - \frac{1}{4} \rho_0^2 \hat{k}_0^2 \hat{q}_0^2
\]
while
\[
- \frac{1}{4} \rho_0^2 \hat{k}_0^2 \hat{q}_0^2 \hat{q}_0^2 \rightarrow \frac{1}{4} \rho_0^2 \hat{k}_0^2 \hat{q}_0^2 - \frac{1}{4} \rho_0^2 \hat{k}_0^2 \hat{q}_0^2
\]
and
\[
- \frac{1}{4} \int \mu \, \rho_0^2 \hat{k}_0^2 \hat{q}_0^2 = - \frac{1}{4} \int \mu \, \rho_0^2 \hat{k}_0^2 \hat{q}_0^2 = - \frac{1}{4} \int \mu \, \rho_0^2 \hat{k}_0^2 \hat{q}_0^2 = - \frac{1}{4} \int \mu \, \rho_0^2 \hat{k}_0^2 \hat{q}_0^2
\]
so that
\[
\frac{1}{8} \sum_{\mu=0,1} \rho_0^2 \hat{k}_0^2 \sum_{\nu=0,1} \hat{q}_0^2 \hat{p}_\nu^2 \rightarrow \frac{1}{2} \rho_0^4 \hat{k}_0^4 \hat{q}_0^2 + \frac{1}{4} - \frac{3}{2\pi} + \frac{2}{\pi^2}.
\]
By collecting all the pieces
\[
\frac{1}{8} \int \mu \, \rho_0^2 \hat{k}_0^2 \sum_{\nu=0,1} \hat{q}_0^2 (\hat{q}_0^2 + \hat{p}_\nu^2) = \int \mu \, \rho_0^2 \hat{k}_0^2 \hat{q}_0^2 + \frac{1}{4} - \frac{3}{2\pi} + \frac{2}{\pi^2}
\]
and using the result (A.11), computed in the appendix, the whole expression \(J_2^{(b)}\) is
\[
-2 \int \mu \, \rho_0^2 \hat{k}_0^2 \sum_{\nu=0,1} \hat{q}_0^2 (\hat{q}_0^2 + \hat{p}_\nu^2)
\]
\[
= \int \mu \left( \rho_0^2 \hat{k}_0^2 \hat{q}_0^2 + \frac{5}{8} \hat{q}_0^2 \hat{p}_0^2 - \hat{q}_0^2 \hat{p}_0^2 \hat{q}_0^2 + \frac{1}{4} \hat{q}_0^2 \hat{k}_0^4 \hat{p}_0^2 \right) \Delta(p) \Delta(k)
\]
\[
= \left( -1 + \frac{6}{\pi} - \frac{6}{\pi^2} \right) + \frac{5}{8} \frac{4}{\pi} + \frac{4}{\pi^2} + \frac{1}{4} - \frac{3}{2\pi} + \frac{2}{\pi^2} = -\frac{1}{8} + \frac{1}{2\pi}.
\]
In conclusion, by adding the first and the second result, computed respectively in (38) and (55), we get
\[
\frac{J_2}{8} = \frac{3}{16} - \frac{1}{2\pi} - \frac{1}{8} + \frac{1}{2\pi} = \frac{1}{16}
\]
in agreement with the prediction \(J_2 = 1/2\).
3. The integral $J_1$

As anticipated in the introduction (see equation (15)), it is expected (by an indirect argument) that

$$\frac{J_1 + J_2}{8} = \frac{1}{4\pi} - \frac{1}{2\pi^2}. \quad (57)$$

Similarly to our evaluation of $J_2$ (but, as we will see, in a simpler way), we can attack directly the evaluation of $J_1 + J_2$, and produce an independent check of the relation above.

Recall that $J_1$ is the contribution to (11) proportional to $c_1$, namely

$$J_1 = \int d\mu \frac{2}{p_0 q_1}(k_0 q_1 - p_1 + q_0 p_1 - k_1 + p_0 k_1 - q_1). \quad (58)$$

Writing $p - q = p \cos q - q \cos p$, restate the integrand above as

$$2p_0 q_1 \left[ (p_0 q_1 - p_0 q_1) \cos k_1 + (p_0 k_1 - k_0 p_1) \cos q_1 + (k_0 q_1 - k_1 q_0) \cos p_1 \right] \quad (59)$$

and note that, by replacing $\cos \theta = 1 - \frac{1}{2} \theta^2$

$$\text{(60)}$$

all the contributions in which we take the 1, that is

$$2p_0 q_1 \left[ (p_0 p_1 - p_0 q_1) \cos k_1 + (p_0 k_1 - k_0 p_1) \cos q_1 \right] \quad (61)$$

are exactly $-J_2/8$, thus, if we keep only the other terms, we have

$$\frac{J_1 + J_2}{8} = -\int d\mu \frac{2}{p_0 q_1} \left[ (p_0 q_1 - p_0 q_1) \hat{k}_1^2 + (p_0 k_1 - k_0 p_1) \hat{q}_1^2 + (k_0 q_1 - k_1 q_0) \hat{p}_1^2 \right]. \quad (62)$$

Manipulate the integrand by exchanging $q$ with $p$, and index 0 with 1, to get

$$-p_0 q_1 \left[ (p_0 p_1 - p_0 q_1) \frac{1}{2} k_1^2 + (p_0 k_1 - k_0 p_1) q_1^2 \right] \rightarrow p_0^2 q_1^2 \frac{1}{2} k_1^2 + p_0^2 \cos p_1 k_1^2 q_1^2. \quad (63)$$

In conclusion

$$\frac{1}{8}(J_1 + J_2) = \frac{1}{2} \left( \frac{1}{2} - \frac{1}{\pi} \right)^2 + \left( \frac{1}{2} - \frac{1}{\pi} \right) \left( -\frac{1}{4} + \frac{1}{\pi} \right) = \frac{1}{2\pi} \left( \frac{1}{2} - \frac{1}{\pi} \right) \quad (64)$$

as it was predicted.

4. Conclusion

We have been able to analytically compute some lattice integrals that, through the work of [9, 10, 13, 14] and references therein, describe the height probabilities in the ensemble of recurrent configurations of the Abelian Sandpile model on the square lattice, in the thermodynamic limit.

The numerical values of these integrals were already known with high precision, and the exact expressions solidly conjectured, as rational-coefficient polynomials in $\pi^{-1}$. Most importantly, a recent indirect calculation of statistical properties of the loop-erased random walk, or equivalently of domino tilings with prescribed local patterns of monomers and dimers, was sufficient to determine completely these values [18, 19].

doi:10.1088/1742-5468/2012/09/P09013
Nonetheless, our direct evaluation of the original lattice integrals, with their strikingly simple results, could be of some interest, and of some use for future work in similar contexts.

Let us stress again that this result is not based on any new deeper understanding of the properties of the sandpile model, but completely relies on elementary trigonometry and symmetry considerations, mainly with the aim of reducing two-loop lattice integrals to quadratic polynomials in one-loop integrals. In particular, at some point we used results previously obtained in [24]. In principle, we do not see any obstacle to recovering similar results on other two-dimensional regular lattices.

**Appendix. One more integral**

We need the evaluation of the integral
\[ \int d\mu \hat{k}_0^4 \hat{p}_0^2 \hat{q}_0^2. \]  
(A.1)

We first observe that
\[ (\mu_0 \mu_0)\hat{k}_0^2 = \left[ \frac{1}{2} (\hat{k}_0^2 - \hat{p}_0^2 - \hat{q}_0^2) + \frac{1}{4} \hat{p}_0^2 \hat{q}_0^2 \right] \left( \hat{k}_0^2 - \frac{1}{4} \hat{k}_0^4 \right), \]  
(A.2)

but it is also
\[ (\mu_0 \mu_0) (\hat{q}_0^2 \hat{k}_0^2) = \left[ \frac{1}{2} (\hat{q}_0^2 - \hat{p}_0^2 - \hat{k}_0^2) + \frac{1}{4} \hat{p}_0^2 \hat{k}_0^2 \right] \left[ \frac{1}{2} (\hat{q}_0^2 - \hat{q}_0^2 - \hat{k}_0^2) + \frac{1}{4} \hat{p}_0^2 \hat{k}_0^2 \right]. \]  
(A.3)

By difference of (A.2) and (A.3) we get the trigonometric identity
\[ \hat{k}_0^4 \hat{p}_0^2 \hat{q}_0^2 = 2(\hat{p}_0^4 + \hat{q}_0^4 + \hat{k}_0^4) - 4(\hat{p}_0^2 \hat{k}_0^2 + \hat{q}_0^2 \hat{q}_0^2 + \hat{q}_0^2 \hat{p}_0^2) + 4\hat{p}_0^2 \hat{q}_0^2 \hat{k}_0^2 
+ 2\hat{k}_0^2 (\hat{p}_0^2 + \hat{q}_0^2) - \hat{k}_0^2 (\hat{p}_0^2 + \hat{q}_0^2) - \hat{k}_0^6 \]  
(A.4)

so that
\[ \hat{k}_0^4 \hat{p}_0^2 \hat{q}_0^2 \rightarrow 6\hat{p}_0^4 - 12\hat{p}_0^2 \hat{q}_0^2 + 4\hat{p}_0^2 \hat{q}_0^2 \hat{k}_0^2 + 2\hat{p}_0^4 \hat{q}_0^2 - \hat{p}_0^6. \]  
(A.5)

Let us start with
\[ 6 \int d\mu (\hat{p}_0^4 - 2\hat{p}_0^2 \hat{q}_0^2) = -12 \int d\mu (2\mu_0^2 + 4\mu_0 \mu_0) = -12G_2 = -\frac{1}{4}; \]  
(A.6)

where the integral $G_2$ was defined and calculated numerically in [29] and subsequently computed in (equation (A.9)) [24]. Then,
\[ 4 \int d\mu \hat{p}_0^2 \hat{q}_0^2 \hat{k}_0^2 = 4A^{(3)} = \frac{1}{2}; \]  
(A.7)

where the integral $A^{(3)}$ had been introduced and computed in (equation (A.6)) [24]. For the evaluation of the last term we use the same trick that was used in [24] to compute $A^{(3)}$, that is use the fact that
\[ \int d\mu (2\hat{q}_0^2 - \hat{p}_0^2) = \int d\mu (2\hat{p}_1^2 \hat{q}_1^2 - \hat{p}_i^2) 
= \int d\mu (\hat{p}^2 - \hat{p}_0^2) [2\hat{q}^2 - \hat{p}^2 - (2\hat{q}_0^2 - \hat{p}_0^2)]; \]  
(A.8)
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which has the consequence that

\[
\int d\mu \hat{p}_0^4 (2\hat{q}_0^2 - \hat{p}_0^2) = \frac{1}{2} \int d\mu \left[ \hat{p}_1^4 (2\hat{q}_1^2 - \hat{p}_1^2) - \hat{p}_1^2 (\hat{p}_1^2 - \hat{p}_0^2) (2\hat{q}_0^2 - \hat{p}_0^2) \right] 
\]

\[
= \frac{1}{2} \int d\mu \hat{p}_1^2 [2\hat{q}_1^2 - \hat{p}_1^2 - (\hat{p}_1^4 - \hat{p}_0^2)(2\hat{q}_0^2 - \hat{p}_0^2)] \; ; \quad (\text{A.9})
\]

so that we are left only with elementary evaluations, that bring us to

\[
\int d\mu \hat{p}_0^4 (2\hat{q}_0^2 - \hat{p}_0^2) = -\frac{5}{4} + \frac{6}{\pi} - \frac{6}{\pi^2} .
\]

In conclusion

\[
\int d\mu \hat{k}_0^4 \hat{p}_0^2 \hat{q}_0^2 = -\frac{1}{4} + \frac{1}{2} - \frac{5}{4} + \frac{6}{\pi} - \frac{6}{\pi^2} = -1 + \frac{6}{\pi} - \frac{6}{\pi^2} . \quad (\text{A.11})
\]

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