Existence of solutions for a class of nonlinear boundary value problems on the hexasilinane graph

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Abstract
Chemical graph theory is a field of mathematics that studies ramifications of chemical network interactions. Using the concept of star graphs, several investigators have looked into the solutions to certain boundary value problems. Their choice to utilize star graphs was based on including a common point connected to other nodes. Our aim is to expand the range of the method by incorporating the graph of hexasilinane compound, which has a chemical formula $\text{H}_{12}\text{Si}_6$. In this paper, we examine the existence of solutions to fractional boundary value problems on such graphs, where the fractional derivative is in the Caputo sense. Finally, we include an example to support our significant findings.

Keywords: Hexasilinane graph; Fractional calculus; Fixed points

1 Introduction
A subfield of mathematics known as chemical graph theory is concerned with the implications of the connectedness of chemical networks. Whether natural or synthetic, almost every chemical system may be represented by a chemical graph (i.e., molecular transformations in a chemical reaction). Moreover, the word “chemical” is used to highlight that, in contrast to graph theory, we may depend on scientific investigation of many ideas and theorems rather than exact mathematical proofs, which is a significant difference.

When it comes to graph theory, Lumer [1] was the first to use differential equation theory on graphs. With the use of ramification spaces and different operator specifications, he explored the solutions of extended evolution equations. Zavgorodnij [2] investigated linear differential equations in 1989 using a geometric network, with suggested boundary value problem solutions arranged at the network interior nodes. On the other hand, Gordeziani et al. [3] utilized the double-sweep method to obtain analytical results for differential equations, which they observed to be more productive on graphs.

However, utilizing fixed point methods, a limited amount of studies on star graphs (see Fig. 1) associated with the solutions of boundary value problems has emerged in the particular research (see, e.g., [4, 5]).
In 2014, Graef et al. [4] used the idea of a star graph and proposed the existence of solutions to the following fractional differential equation:

\[
\begin{cases}
-RLD_\ell^\gamma z(\gamma)(s) = h_\gamma(s)L_\gamma(s,y(s)) & (s \in (0,\tilde{\varrho}_\gamma), \gamma = 1, 2), \\
z_1(0) = z_2(0) = 0, \\
z_1(\tilde{\varrho}_1) = z_2(\tilde{\varrho}_2), \\
RLD_\nu^w w_1(\tilde{\varrho}_1) + RLD_\nu^w w_2(\tilde{\varrho}_2) = 0,
\end{cases}
\]

(1.1)

where \( \ell \in (1, 2], \nu \in (0, \ell - 1], h_\gamma : [0, \tilde{\varrho}_\gamma] \to \mathbb{R} \) are continuous functions with \( h_\gamma(s) \neq 0 \) on \([0, \tilde{\varrho}_\gamma] \), and \( L_\gamma : [0, \tilde{\varrho}_\gamma] \times \mathbb{R} \to \mathbb{R} \) are continuous functions. Also, \( RL^\ell \) and \( RL^\nu \) represent the Riemann–Liouville fractional derivatives of orders \( \ell \) and \( \nu \), respectively.

Mehandiratta et al. [5], extended the work of Graef et al. [4] by proposing the following fractional differential equation:

\[
\begin{cases}
D_\ell^\gamma z_\gamma(s) = S_\gamma(s,z_\gamma(s),D^\eta z_\gamma(s)) & (s \in (0,\tilde{\varrho}_\gamma), \gamma = 1, 2, \ldots, n), \\
z_\gamma(0) = 0, \\
z_\gamma(\tilde{\varrho}_\gamma) \neq z_\gamma(\tilde{\varrho}_\gamma) (\gamma \neq \tilde{\gamma}), \\
\sum_{\gamma=1}^n z_\gamma(\tilde{\varrho}_\gamma) = 0,
\end{cases}
\]

(1.2)

where \( \ell \in (1, 2], \nu \in (0, \ell - 1], S_\gamma : [0, \tilde{\varrho}_\gamma] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) are continuous functions, and \( D^\delta \) denotes the Caputo fractional derivative of order \( \delta \in [\ell, \nu] \).

Recently, Mophou et al. [6] investigated the solution of the following fractional Sturm–Liouville boundary value problems on a star graph:

\[
\begin{cases}
D^\ell_{b_\gamma} (\beta^\gamma D^\nu_{a_\gamma} z(\gamma))(s) + q^\gamma(s)z(\gamma)(s) = S(\gamma)(s), & s \in (a,b_\gamma), \gamma = 1, 2, \ldots, n, \\
I^\ell_{a_\gamma} z(\gamma)(a^\gamma) = I^\eta_{a_\gamma} z^\nu(\gamma)(a^\gamma), & \ell \neq \nu = 1, 2, \ldots, n, \\
\sum_{\gamma=1}^n \beta^\gamma(a)D^\nu_{a_\gamma} z^\nu(\gamma)(a^\gamma) = 0, \\
I^\ell_{a_\gamma} z^\nu(\gamma)(b_\gamma) = 0, \\
I^\ell_{a_\gamma} z^\nu(\gamma)(b_\gamma) \neq 0, & \gamma = 2, 3, \ldots, p, \\
\beta^\nu(b_\gamma)D^\nu_{a_\gamma} z^\nu(\gamma)(b_\gamma) = v_\gamma, & \gamma = p, p + 1, \ldots, n,
\end{cases}
\]

(1.3)

where \( D^\ell_{a_\gamma} \) and \( D^\ell_{b_\gamma} \), \( \gamma = 1, 2, \ldots, n \), are, respectively, the left Riemann–Liouville and right Caputo fractional derivatives of order \( \ell \in (0,1) \), \( I^\ell_{a_\gamma} \) is the Riemann–Liouville fractional
integral of order $\ell$, the real functions $\beta^\gamma$ and $q^\gamma$ are defined on $[a, b_\gamma]$ $(\gamma = 1, 2, \ldots, n)$, the functions $S^\gamma$ belong to $L^2(a, b_\gamma)$, $\gamma = 1, 2, \ldots, n$, and the controls $v_\gamma$, $\gamma = 1, 2, \ldots, n$, are real variables.

For the recent research in this area, we refer to [7–9] and the references therein.

To extend the work presented in [4–6], we use the concept of hexasilinane graphs (see Fig. 2), which are more general than star graphs. Moreover, the techniques employed in [4–6] are inadequate since the hexasilinane graphs contain more junction points than star graphs. As a result, we adopt a different method, in which the graph vertices are labeled by 0 or 1 with edge length $|\tilde{b}_k| = 1$ (see Fig. 3).
Here we examine the existence of solutions to the following system:

\[
\begin{align*}
D_\ell z_\gamma(s) &= S_\gamma(s, z_\gamma(s), D_\nu z_\gamma(s), z_\gamma'(s), z_\gamma''(s)) \quad (s \in [0, 1]), \\
\sigma_1 z_\gamma(0) &= \sigma_2 D_\ell z_\gamma(1) + \sigma_3 D_\nu z_\gamma(1), \\
\sigma_1 z_\gamma(1) &= \sigma_2 D_{\ell-1} z_\gamma(1) + \frac{1}{\sigma_3} \int_0^\theta D_{\ell-1} z_\gamma(\xi) \, d\xi,
\end{align*}
\]

(1.4)

where \(\sigma_p \, (p = 1, 2, 3)\) are real constants with \(\sigma_p \neq 0\), \(\theta \in (0, 1)\), \(D^\ell\) and \(D^\nu\) denote the Caputo fractional derivatives of orders \(2 < \ell \leq 3\) and \(\nu \in (0, 2)\), respectively. Also, \(S_\gamma : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) is a given continuously differentiable function for \(\gamma = 18\), where \(\gamma\) denotes the total number of edges of hexasilinane compound with \(|\tilde{k}_\gamma| = 1\).

Our goal is to prove the existence of solutions to the suggested problem (1.4) by using appropriate fixed point theorems. Finally, we give an example to demonstrate the significance of our findings in light of the existing literature.

For the details about fixed point theory and its applications in different spaces, we refer to [10–18] and the references therein. Several new papers have recently been published dealing with the existence of solutions to nonlinear fractional differential equations (for details, see [19–24]).

2 Preliminaries

**Definition 2.1** ([25]) The Caputo fractional derivative of order \(\ell > 0\) for \(S \in C_\xi(0, +\infty)\) is given by

\[
D_\ell S(s) = \frac{1}{\Gamma(\xi - \ell)} \int_0^s (s - \theta)^{\xi-\ell-1} S^{(\ell)}(\theta) \, d\theta \quad (0 < \ell < \xi, \xi = [\ell] + 1),
\]

where \([\ell]\) is the integer part of \(\ell\).

For \(\ell > 0\), the general solution of \(D_\ell z(s) = 0\) is given as

\[
z(s) = b_0 + b_1 s + b_2 s^2 + \cdots + b_{n-1} s^{n-1},
\]

and

\[
|D_\ell z(s) = z(s) + b_0 + b_1 s + b_2 s^2 + \cdots + b_{n-1} s^{n-1}
\]

for \(b_k \in \mathbb{R}\) and \(k = 0, 1, \ldots, n - 1\).

**Lemma 2.2** Let \(\phi_1, \phi_2, \ldots, \phi_{18}\) be continuous real-valued functions on \([0, 1]\). Then \(z^*_\gamma\) is a solution of the problem

\[
\begin{align*}
D_\ell z_\gamma(s) &= \Upsilon_\gamma(s) \quad (s \in [0, 1], \gamma = 1, 2, \ldots, 18), \\
\sigma_1 z_\gamma(0) &= \sigma_2 D_\ell z_\gamma(1) + \sigma_3 D_\nu z_\gamma(1), \\
\sigma_1 z_\gamma(1) &= \sigma_2 D_{\ell-1} z_\gamma(1) + \frac{1}{\sigma_3} \int_0^\theta D_{\ell-1} z_\gamma(\xi) \, d\xi,
\end{align*}
\]

(2.1)

if and only if \(z^*_\gamma\) is a solution of the fractional integral equation

\[
z_\gamma(s) = \int_0^s \frac{(s - \xi)^{\xi-1}}{\Gamma(\xi)} \Upsilon_\gamma(\xi) \, d\xi + \frac{1}{V_0} \left(\frac{\sigma_2}{\sigma_3} + s\right) \int_0^1 \int_0^\theta \Upsilon_\gamma(\tau) \, d\tau \, d\xi
\]
Theorem 2.4 ([26]) Let \( V \) be a closed, bounded, convex, and nonempty subset of a Banach space \( U \), and let \( S_1, S_2 : V \rightarrow U \) be two operators such that \( S_1k + S_2k' \in V \) whenever \( k, k' \in V \). Suppose that \( S_1 \) is compact and continuous and \( S_2 \) is a contraction. Then \( S_1 + S_2 \) has a fixed point.

3 Main results

Throughout this paper, \( U_F = \{ z' : z' \in D^* z, z'' \in \mathbb{R} \} \) is a Banach space with norm

\[
\|z\|_{U_F} = \sup_{s \in [0,1]} \|z(s)\| + \sup_{s \in [0,1]} \|D^* z(s)\| + \sup_{s \in [0,1]} \|z'(s)\| + \sup_{s \in [0,1]} \|z''(s)\|
\]
for $\gamma = 1, 2, \ldots, 18$. It is obvious that the product space $U = U_1 \times U_2 \times \cdots \times U_{18}$ is a Banach space with norm

$$\|z = (z_1, z_2, \ldots, z_{18})\|_U = \sum_{\gamma=1}^{18} \|z_\gamma\|_{U_\gamma}.$$

Referring to Lemma 2.2, we introduce the operator $S : U \to U$ by

$$S(z_1, z_2, \ldots, z_{18})(s) := (S_1(z_1, z_2, \ldots, z_{18}), \ldots, S_{18}(z_1, z_2, \ldots, z_{18})(s)), \quad (3.1)$$

where

$$S_\gamma(z_1, z_2, \ldots, z_{18})(s)$$

$$= \int_0^s \frac{(s - \xi)\ell - 1}{\Gamma(\ell)} S_\gamma(\xi, z_\gamma(\xi), D^\ell z_\gamma(\xi), z_\gamma'(\xi), z_\gamma''(\xi)) d\xi$$

$$+ \frac{1}{V_0} \left( \frac{\sigma_2}{\sigma_1} + s \right)$$

$$\times \left[ \sigma_3 \int_0^\theta \int_0^\xi S_\gamma(\tau, z_\gamma(\tau), D^\ell z_\gamma(\tau), z_\gamma'(\tau), z_\gamma''(\tau)) d\tau d\xi$$

$$+ \sigma_2 \int_0^1 \sigma_3(\xi, z_\gamma(\xi), D^\ell z_\gamma(\xi), z_\gamma'(\xi), z_\gamma''(\xi)) d\xi$$

$$- \sigma_2 \int_0^1 \frac{(1 - \xi)\ell - 1}{\Gamma(\ell)} S_\gamma(\xi, z_\gamma(\xi), D^\ell z_\gamma(\xi), z_\gamma'(\xi), z_\gamma''(\xi)) d\xi$$

$$+ \frac{1}{\sigma_1 V_0} (V_0 - \sigma_2 - \sigma_3 s)$$

$$\times \left[ \sigma_2 \int_0^1 \frac{(1 - \xi)\ell - 2}{\Gamma(\ell - 1)} S_\gamma(\xi, z_\gamma(\xi), D^\ell z_\gamma(\xi), z_\gamma'(\xi), z_\gamma''(\xi)) d\xi$$

$$+ \sigma_2 \int_0^1 \frac{(1 - \xi)\ell - 3}{\Gamma(\ell - 2)} S_\gamma(\xi, z_\gamma(\xi), D^\ell z_\gamma(\xi), z_\gamma'(\xi), z_\gamma''(\xi)) d\xi \right]$$

for $s \in [0, 1]$ and $z_\gamma \in U_\gamma$.

To facilitate calculations, we use the following notation:

$$V_0 = \left[ \sigma_1 + \frac{\sigma_2(\Gamma(3 - \ell) - 1)}{\Gamma(3 - \ell)} - \frac{\sigma_3\theta^{3 - \ell}}{\Gamma(4 - \ell)} \right] \neq 0,$$

$$V_1 = \left[ |\sigma_1| + \frac{|\sigma_2|(|\Gamma(3 - \ell) - 1|)}{\Gamma(3 - \ell)} + \frac{|\sigma_3|}{\Gamma(4 - \ell)} \right] \neq 0,$$

$$I_0^* = \frac{1}{\Gamma(\ell + 1)} + \left( \frac{\sigma_2}{|\sigma_1| V_1} \left( \frac{|\sigma_3|}{2} + \frac{|\sigma_2|}{\Gamma(\ell + 1)} \right) + \frac{|\sigma_3|}{\Gamma(\ell)} \right)^2,$$

$$I_1^* = \frac{1}{\Gamma(\ell - v + 1)} + \left( \frac{1}{V_1 \Gamma(2 - v)} \right)^2 \times \left( \frac{|\sigma_2|(|\Gamma(3 - \ell) - 1|)}{2\Gamma(3 - \ell)} + \frac{|\sigma_2(1 + \Gamma(\ell))|}{\Gamma(\ell)} + \frac{|\sigma_3|}{\Gamma(4 - \ell)} \right),$$

$$I_2^* = \frac{1}{\Gamma(\ell - v + 1)} + \left( \frac{1}{V_1 \Gamma(2 - v)} \right)^2 \times \left( \frac{|\sigma_2(2 + \Gamma(\ell - 1))|}{2\Gamma(\ell - 1)} + \frac{|\sigma_2(1 + \Gamma(\ell))|}{\Gamma(\ell)} + \frac{|\sigma_3|}{\Gamma(4 - \ell)} \right).$$
\[ T^*_a = \frac{1}{\Gamma(\ell)} + \frac{1}{\Gamma(\ell-1)} \left( \frac{|\sigma_2(2 + \Gamma(\ell - 1))|}{2\Gamma(\ell - 1)} + \frac{|\sigma_2(1 + \Gamma(\ell)|}{\Gamma(\ell)} + \frac{|\sigma_1|}{\Gamma(\ell + 1)} \right), \quad (3.7) \]

\[ T^*_a = \frac{1}{\Gamma(\ell - 1)}, \quad (3.8) \]

\[ T^*_a = \left( \frac{|\sigma_2| + |\sigma_1|}{|\sigma_1|V_1} \right) \left( \frac{|\sigma_2|}{2} + \frac{|\sigma_1|}{\Gamma(\ell + 1)} \right) \]

\[ + \left( \frac{V_1 - \sigma_2 - \sigma_1}{|\sigma_1|V_1} \right) \left( \frac{|\sigma_2|}{\Gamma(\ell)} + \frac{|\sigma_3|}{\Gamma(\ell - 1)} \right), \quad (3.9) \]

\[ T^*_a = \left( \frac{1}{V_1\Gamma(2-n)} \right) \left( \frac{|\sigma_2(2 + \Gamma(\ell - 1))|}{2\Gamma(\ell - 1)} + \frac{|\sigma_2(1 + \Gamma(\ell)|}{\Gamma(\ell)} + \frac{|\sigma_1|}{\Gamma(\ell + 1)} \right), \quad (3.10) \]

\[ T^*_a = \frac{1}{V_1} \left( \frac{|\sigma_3(2 + \Gamma(\ell - 1))|}{2\Gamma(\ell - 1)} + \frac{|\sigma_2(1 + \Gamma(\ell)|}{\Gamma(\ell)} + \frac{|\sigma_1|}{\Gamma(\ell + 1)} \right). \quad (3.11) \]

**Theorem 3.1** Let \( S_1, S_2, \ldots, S_{18} : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be continuous functions, and let there exist constants \( M_\gamma > 0, \gamma = 1, 2, \ldots, 18 \), satisfying

\[ |S_\gamma(s, z_1, z_2, z_3, z_4)| \leq M_\gamma \]

for all \( z, z_1, z_2, z_3, z_4 \in \mathbb{R} \) and \( s \in [0,1] \). Then problem (1.4) has a solution.

**Proof** It is obvious from (3.2) that the fixed points of the operator \( S \) given in (3.1) exist if and only if (1.4) has a solution. To prove this, we first show that \( S \) is completely continuous.

As \( S_1, S_2, \ldots, S_{18} \) are continuous, \( S : \mathcal{U} \to \mathcal{U} \) is continuous too. Let \( V \in \mathcal{U} \) be a bounded set, and let \( z = (z_1, z_2, \ldots, z_18) \in \mathcal{U} \). So for each \( s \in [0,1] \), we have

\[ |(S_\gamma)(s)| \]

\[ \leq \int_0^s \frac{(s - \xi)^{\ell-1}}{\Gamma(\ell)} \left| S_\gamma(\xi, z_\gamma(\xi), D^\nu z_\gamma(\xi), \xi_\gamma(\xi), \xi_\gamma''(\xi)) \right| d\xi + \frac{1}{V_1} \left( \frac{|\sigma_2|}{|\sigma_1|} + s \right) \]

\[ \times \left[ |\sigma_3| \int_0^s \frac{d\tau}{\Gamma(\ell)} \left| S_\gamma(\tau, z_\gamma(\tau), D^\nu z_\gamma(\tau), \xi_\gamma(\tau), \xi_\gamma''(\tau)) \right| d\tau \right] \]

\[ + |\sigma_2| \int_0^s \left| S_\gamma(\xi, z_\gamma(\xi), D^\nu z_\gamma(\xi), \xi_\gamma''(\xi)) \right| d\xi \]

\[ + |\sigma_1| \int_0^s \frac{(1 - \xi)^{\ell-1}}{\Gamma(\ell)} \left| S_\gamma(\xi, z_\gamma(\xi), D^\nu z_\gamma(\xi), \xi_\gamma''(\xi)) \right| d\xi \]

\[ + \frac{|V_1 - \sigma_2 - \sigma_1 s|}{|\sigma_1| V_1} \]

\[ \times \left[ |\sigma_2| \int_0^s \frac{(1 - \xi)^{\ell-2}}{\Gamma(\ell - 1)} \left| S_\gamma(\xi, z_\gamma(\xi), D^\nu z_\gamma(\xi), \xi_\gamma''(\xi)) \right| d\xi \right] \]

\[ + \sigma_3 \int_0^s \frac{(1 - \xi)^{\ell-3}}{\Gamma(\ell - 2)} \left| S_\gamma(\xi, z_\gamma(\xi), D^\nu z_\gamma(\xi), \xi_\gamma''(\xi)) \right| d\xi \]

\[ \leq M_\gamma T^*_a. \]
where $I_0^*$ is given in (3.5). Also,

$$
|\mathcal{D}^\nu S_\gamma z(s)| \\
\leq \int_0^s \frac{(s - \xi)^{\nu - 1}}{\Gamma(\nu)} |S_\gamma(\xi, z_\gamma(\xi), D^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi))| \, d\xi \\
+ \left( \frac{1}{V_1(2 - \nu)} \right) \\
\times \left[ |\sigma_3| \int_0^1 \int_0^{\xi} |S_\gamma(\tau, z_\gamma(\tau), D^\nu z_\gamma(\tau), z'_\gamma(\tau), z''_\gamma(\tau))| \, d\tau \, d\xi \\
+ |\sigma_2| \int_0^1 \frac{(1 - \xi)^{\nu - 1}}{\Gamma(\nu)} |S_\gamma(\xi, z_\gamma(\xi), D^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi))| \, d\xi \\
+ |\sigma_1| \int_0^1 \frac{(1 - \xi)^{\nu - 2}}{\Gamma(\nu - 1)} |S_\gamma(\xi, z_\gamma(\xi), D^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi))| \, d\xi \right] \\
\leq M_\gamma I_1^*,
$$

$$
|S_\gamma z(s)| \\
\leq \int_0^s \frac{(s - \xi)^{\nu - 2}}{\Gamma(\nu - 1)} |S_\gamma(\xi, z_\gamma(\xi), D^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi))| \, d\xi \\
+ \frac{1}{V_1} \left[ |\sigma_3| \int_0^1 \int_0^{\xi} |S_\gamma(\tau, z_\gamma(\tau), D^\nu z_\gamma(\tau), z'_\gamma(\tau), z''_\gamma(\tau))| \, d\tau \, d\xi \\
+ |\sigma_2| \int_0^1 |S_\gamma(\xi, z_\gamma(\xi), D^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi))| \, d\xi \\
+ |\sigma_1| \int_0^1 \frac{(1 - \xi)^{\nu - 1}}{\Gamma(\nu)} |S_\gamma(\xi, z_\gamma(\xi), D^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi))| \, d\xi \right] \\
+ \frac{1}{V_1} \left[ |\sigma_2| \int_0^1 \frac{(1 - \xi)^{\nu - 2}}{\Gamma(\nu - 1)} |S_\gamma(\xi, z_\gamma(\xi), D^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi))| \, d\xi \\
+ |\sigma_3| \int_0^1 \frac{(1 - \xi)^{\nu - 3}}{\Gamma(\nu - 2)} |S_\gamma(\xi, z_\gamma(\xi), D^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi))| \, d\xi \right] \\
\leq M_\gamma I_2^*,
$$

and

$$
|S_\gamma^* z(s)| \leq \int_0^s \frac{(s - \xi)^{\nu - 3}}{\Gamma(\nu - 2)} |S_\gamma(\xi, z_\gamma(\xi), D^\nu z_\gamma(\xi), z'_\gamma(\xi), z''_\gamma(\xi))| \, d\xi \\
\leq M_\gamma I_3^*.$$
for all $s \in [0, 1]$, where $\mathcal{I}_1^* - \mathcal{I}_1^*$ are defined in (3.6)–(3.8), respectively. Therefore
\[
\| (S_y z)(s) \|_{U_0} \leq M_z (\mathcal{I}_0^* + \mathcal{I}_1^* + \mathcal{I}_2^* + \mathcal{I}_3^*).
\]

Hence
\[
\| (Sz)(s) \|_{U_0} = \sum_{y=1}^{18} \| (S_y z)(s) \|_{U_0}
\]
\[
\leq \sum_{y=1}^{18} M_z (\mathcal{I}_0^* + \mathcal{I}_1^* + \mathcal{I}_2^* + \mathcal{I}_3^*)
\]
\[
< \infty,
\]

which reveals that $S$ is uniformly bounded.

Now we have to show that $S$ is equicontinuous. For this purpose, let $z = (z_1, z_2, \ldots, z_{18}) \in \mathcal{V}$ and $s_1, s_2 \in [0, 1]$ with $s_1 < s_2$. Then we have
\[
\left| (S_y z)(s_2) - (S_y z)(s_1) \right|
\]
\[
\leq \int_{s_1}^{s_2} (s_2 - \theta)^{\ell-1} - (s_1 - \theta)^{\ell-1} \frac{1}{\Gamma(\ell)} \left| S_y (\xi, z_1 (\xi), \mathcal{D}^\nu z_2 (\xi), z_3' (\xi), z_3'' (\xi)) \right| d\xi
\]
\[
+ \int_{s_1}^{s_2} (s_2 - \theta)^{\ell-1} \frac{1}{\Gamma(\ell)} \left| S_y (\xi, z_1 (\xi), \mathcal{D}^\nu z_2 (\xi), z_3' (\xi), z_3'' (\xi)) \right| d\xi
\]
\[
+ \left( \frac{s_2 - s_1}{V_1} \right)
\]
\[
\times \left[ |\sigma_1| \int_0^{s_1} \int_0^{\bar{\xi}} |S_y (\tau, z_1 (\tau), \mathcal{D}^\nu z_2 (\tau), z_3' (\tau), z_3'' (\tau))| d\tau d\bar{\xi} + |\sigma_2| \int_0^{s_1} |S_y (\xi, z_1 (\xi), \mathcal{D}^\nu z_2 (\xi), z_3' (\xi), z_3'' (\xi))| d\xi + |\sigma_1| \int_0^{s_1} \frac{1}{\Gamma(\ell)} \left| S_y (\xi, z_1 (\xi), \mathcal{D}^\nu z_2 (\xi), z_3' (\xi), z_3'' (\xi)) \right| d\xi \right]
\]
\[
+ \left( \frac{s_2 - s_1}{V_1} \right)
\]
\[
\times \left[ |\sigma_2| \int_0^{s_1} \frac{1}{\Gamma(\ell - 1)} \left| S_y (\xi, z_1 (\xi), \mathcal{D}^\nu z_2 (\xi), z_3' (\xi), z_3'' (\xi)) \right| d\xi + |\sigma_1| \int_0^{s_1} \frac{1}{\Gamma(\ell - 2)} \left| S_y (\xi, z_1 (\xi), \mathcal{D}^\nu z_2 (\xi), z_3' (\xi), z_3'' (\xi)) \right| d\xi \right].
\]

We can see that if $s_1 \to s_2$, then, independently, the right-hand side of the expression converges to zero. Also,
\[
\lim_{s_1 \to s_2} \left| (\mathcal{D}^\nu S_y z)(s_2) - (\mathcal{D}^\nu S_y z)(s_1) \right| = 0,
\]
\[
\lim_{s_1 \to s_2} \left| (S_y z)(s_2) - (S_y z)(s_1) \right| = 0,
\]
\[ \lim_{s_1 \to s_2} |(S_{y} z)(s_2) - (S_{y} z)(s_1)| = 0. \]

As a result, \( \|(S z)(s_2) - (S z)(s_1)\|_{U} \to 0 \) as \( s_1 \to s_2 \). This proves that \( S \) is equicontinuous on \( U = U_1 \times U_2 \times \cdots \times U_{18} \). Now the Arzelà–Ascoli theorem implies the complete continuity of the operator.

Further, we define the subset \( \Lambda \) of \( U \) as

\[ \Lambda := \left\{ (z_1, z_2, \ldots, z_{18}) \in U : (z_1, z_2, \ldots, z_{18}) = \theta S(z_1, z_2, \ldots, z_{18}) \}, \theta \in (0, 1) \right\}. \]

We will show that \( \Lambda \) is bounded. For this, let \( (z_1, z_2, \ldots, z_{18}) \in \Lambda \). Then we can write

\[ (z_1, z_2, \ldots, z_{18}) = \theta S(z_1, z_2, \ldots, z_{18}), \]

and so

\[ z_{y}(s) = \theta S_{y}(z_1, z_2, \ldots, z_{18}) \]

for all \( s \in [0, 1] \) and \( y = 1, 2, \ldots, 18 \). Thus

\[
|z_y(s)| \leq \theta \left[ \int_0^1 \frac{(s - \xi)^{y-1}}{\Gamma(y)} \left| S_y\left( \xi, z_y(\xi), D^y z_y(\xi), z'_y(\xi), z''_y(\xi) \right) \right| d\xi 
+ \frac{1}{V_1} \left( \frac{|\sigma_2|}{|\sigma_1|} + \theta \right) \left\{ \left| \sigma_3 \right| \int_0^1 \int_0^s \left| S_y\left( \tau, z_y(\tau), D^y z_y(\tau), z'_y(\tau), z''_y(\tau) \right) \right| d\tau d\xi 
+ \left| \sigma_2 \right| \int_0^1 \left| S_y\left( \xi, z_y(\xi), D^y z_y(\xi), z'_y(\xi), z''_y(\xi) \right) \right| d\xi 
+ \left| \sigma_1 \right| \int_0^1 \frac{(1 - \xi)^{y-1}}{\Gamma(y)} \left| S_y\left( \xi, z_y(\xi), D^y z_y(\xi), z'_y(\xi), z''_y(\xi) \right) \right| d\xi 
+ \frac{|V_1 - \sigma_2 - \sigma_1|}{|\sigma_1| V_1} \right\} \right] 
\times \left\{ \left| \sigma_2 \right| \int_0^1 \frac{(1 - \xi)^{y-2}}{\Gamma(y - 1)} \left| S_y\left( \xi, z_y(\xi), D^y z_y(\xi), z'_y(\xi), z''_y(\xi) \right) \right| d\xi 
+ \left| \sigma_3 \right| \int_0^1 \frac{(1 - \xi)^{y-3}}{\Gamma(y - 2)} \left| S_y\left( \xi, z_y(\xi), D^y z_y(\xi), z'_y(\xi), z''_y(\xi) \right) \right| d\xi \right\} \right]
\leq \theta M_y \mathcal{I}^*_0,
\]

and by similar computations we have

\[ |D^y z_y(0)| \leq \theta M_y \mathcal{I}^*_0, \]
\[ |z''_y(s)| \leq \theta M_y \mathcal{I}^*_0, \]
\[ |z''_y(s)| \leq \theta M_y \mathcal{I}^*_0, \]

where \( \mathcal{I}^*_0 - \mathcal{I}^*_3 \) are given in (3.5)–(3.8). Hence

\[ \|z\|_U = \sum_{y=1}^{18} \|z_y\|_{U_y}. \]
which shows the boundedness of $\Lambda$. Now using Theorem 2.4 and Lemma 2.2, we see that $S$ has a fixed point in $U$. This demonstrates that (1.4) does indeed have a solution. \qed

We will now examine the solution of problem (1.4) by applying various conditions.

**Theorem 3.2** Suppose that $S_1, S_2, \ldots, S_{18} : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions and that there exist bounded continuous functions $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{18} : [0, 1] \to \mathbb{R}$, $Z_1, Z_2, \ldots, Z_{18} : [0, 1] \to [0, \infty)$ and nondecreasing continuous functions $L_1, L_2, \ldots, L_{18} : [0,1] \to [0,\infty)$ such that

$$|S_\gamma(s,z_1,z_2,z_3,z_4)| \leq \mathcal{Z}_\gamma(s)L_\gamma(|z_1|+|z_2|+|z_3|+|z_4|)$$

and

$$|S_\gamma(s,z_1,z_2,z_3,z_4) - S_\gamma(s,\bar{z}_1,\bar{z}_2,\bar{z}_3,\bar{z}_4)| \leq \mathcal{G}_\gamma(s)(|z_1 - \bar{z}_1| + |z_2 - \bar{z}_2| + |z_3 - \bar{z}_3| + |z_4 - \bar{z}_4|)$$

for all $s \in [0,1]$, $z_1,z_2,z_3,z_4,\bar{z}_1,\bar{z}_2,\bar{z}_3,\bar{z}_4 \in \mathbb{R}$, and $\gamma = 1,2,\ldots,18$. If

$$\Delta := \left(\mathcal{I}_0^* + \mathcal{I}_1^* + \mathcal{I}_2^* + \mathcal{I}_3^*\right)\sum_{\gamma=1}^{18} \|\mathcal{G}_\gamma\| < 1,$$

then (1.4) has a solution, where $\|\mathcal{G}_\gamma\| = \sup_{s \in [0,1]} |\mathcal{G}_\gamma(s)|$, and the constants $\mathcal{I}_0^* - \mathcal{I}_3^*$ are given in (3.9)–(3.11), respectively.

**Proof** Let $\|Z_\gamma\| = \sup_{s \in [0,1]} |Z_\gamma(s)|$. Suppose that for suitable constants $\varepsilon_\gamma$, we have

$$\varepsilon_\gamma \geq \sum_{\gamma=1}^{18} L_\gamma(\|z\|_U)\|Z_\gamma\|\left[\mathcal{I}_0^* + \mathcal{I}_1^* + \mathcal{I}_2^* + \mathcal{I}_3^*\right], \quad (3.12)$$

where $\mathcal{I}_0^* - \mathcal{I}_3^*$ are given in (3.5)–(3.8). We define the set

$$\mathcal{V}_{\varepsilon_\gamma} := \{z = (z_1,z_2,\ldots,z_{18}) \in U : \|z\|_U \leq \varepsilon_\gamma\},$$

where $\varepsilon_\gamma$ is defined in (3.12). It is obvious that $\mathcal{V}_{\varepsilon_\gamma}$ is a nonempty, closed, bounded, and convex subset of $U = U_1 \times U_2 \times \cdots \times U_{18}$. Now we define $S_1$ and $S_2$ on $O_{\varepsilon_\gamma}$ by

$$S_1(z_1,z_2,\ldots,z_{18})(s) := (S_1^{(1)}(z_1,z_2,\ldots,z_{18})(s), \ldots, S_1^{(18)}(z_1,z_2,\ldots,z_{18})(s)),$$

$$S_2(z_1,z_2,\ldots,z_{18})(s) := (S_2^{(1)}(z_1,z_2,\ldots,z_{18})(s), \ldots, S_2^{(18)}(z_1,z_2,\ldots,z_{18})(s)),$$
where

\[
(S_1^{(r)}z)(s) = \int_0^s \frac{(s - \xi)^{\ell - 1}}{\Gamma(\ell)} S_\gamma(\xi, z_\gamma(\xi), D^\alpha z_\gamma(\xi), z_\gamma'(\xi), z_\gamma''(\xi)) d\xi,
\]

and

\[
(S_2^{(r)}z)(s) = \frac{1}{V_0} \left( \frac{\sigma_2}{\sigma_1} + s \right) \left[ \frac{\sigma_2}{\sigma_1} \int_0^1 \int_0^s S_\gamma(\xi, z_\gamma(\xi), D^\alpha z_\gamma(\xi), z_\gamma'(\xi), z_\gamma''(\xi)) d\xi d\xi' \right. \\
+ \frac{\sigma_2}{\sigma_1} \int_0^1 \int_0^s \frac{(1 - \xi)^{\ell - 1}}{\Gamma(\ell)} S_\gamma(\xi, z_\gamma(\xi), D^\alpha z_\gamma(\xi), z_\gamma'(\xi), z_\gamma''(\xi)) d\xi d\xi' \\
+ \left. \frac{1}{|\sigma_2|} \left( \frac{V_0 - \sigma_2 - \sigma_1 s}{\sigma_1 V_0} \right) \right] \\
\times \left[ \frac{\sigma_2}{\sigma_1} \int_0^1 \int_0^s \frac{(1 - \xi)^{\ell - 2}}{\Gamma(\ell - 1)} S_\gamma(\xi, z_\gamma(\xi), D^\alpha z_\gamma(\xi), z_\gamma'(\xi), z_\gamma''(\xi)) d\xi d\xi' \\
+ \frac{1}{|\sigma_1|} \left( \frac{V_1 - \sigma_2 - \sigma_1 s}{\sigma_1 V_1} \right) \right]
\]

for all \( s \in [0, 1] \) and \( z = (z_1, z_2, \ldots, z_{18}) \in V_\gamma \).

Let \( \tilde{\gamma} = \sup_{\tau \in [0, 1]} \gamma'(\|z_t\|_{u_0}) \). For all \( \tilde{z} = (\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_{18}), z = (z_1, z_2, \ldots, z_{18}) \in V_\gamma \), we have

\[
| (S_1^{(r)}\tilde{z} + S_2^{(r)}z)(s) |
\leq \int_0^s \frac{(s - \xi)^{\ell - 1}}{\Gamma(\ell)} | S_\gamma(\xi, \tilde{z}_\gamma(\xi), D^\alpha \tilde{z}_\gamma(\xi), \tilde{z}_\gamma'(\xi), \tilde{z}_\gamma''(\xi)) | d\xi \\
+ \frac{1}{V_0} \left( \frac{|\sigma_2|}{|\sigma_1|} + s \right) \\
\times \left[ \frac{|\sigma_2|}{|\sigma_1|} \int_0^1 \int_0^s | S_\gamma(\xi, z_\gamma(\xi), D^\alpha z_\gamma(\xi), z_\gamma'(\xi), z_\gamma''(\xi)) | d\xi d\xi' \\
+ \frac{1}{|\sigma_2|} \left( \frac{V_0 - \sigma_2 - \sigma_1 s}{\sigma_2 V_0} \right) \right] \\
\times \left[ \frac{|\sigma_2|}{|\sigma_1|} \int_0^1 \int_0^s \frac{(1 - \xi)^{\ell - 2}}{\Gamma(\ell - 1)} | S_\gamma(\xi, z_\gamma(\xi), D^\alpha z_\gamma(\xi), z_\gamma'(\xi), z_\gamma''(\xi)) | d\xi d\xi' \\
+ \frac{1}{|\sigma_1|} \left( \frac{V_1 - \sigma_2 - \sigma_1 s}{\sigma_1 V_1} \right) \right]
\]

\[
\leq \int_0^s \frac{(s - \xi)^{\ell - 1}}{\Gamma(\ell)} | \mathcal{L}_\gamma(\xi) | \mathcal{L}_\gamma(\|\tilde{z}_\gamma(\xi)\|_{u_0}) + | D^\alpha \tilde{z}_\gamma(\xi) | + | \tilde{z}_\gamma'(\xi) | + | \tilde{z}_\gamma''(\xi) | d\xi
\]
By using similar computations we have

\[
\left| (\mathcal{D}^\nu S_1^{(\nu)} \tilde{z})(s) + (\mathcal{D}^\nu S_2^{(\nu)} z)(s) \right| \leq \|Z\| \|\tilde{L}_r I_1^*\, \\
\left| (S_1^{(\nu)} \tilde{z})'(s) + (S_2^{(\nu)} z)'(s) \right| \leq \|Z\| \|\tilde{L}_r I_2^*\, \\
\text{and} \\
\left| (S_1^{(\nu)} \tilde{z})''(s) + (S_2^{(\nu)} z)''(s) \right| \leq \|Z\| \|\tilde{L}_r I_3^*\.
\]

This yields that

\[
\|S_1 \tilde{z} + S_2 z\|_{\mathcal{U}_r} = \sum_{\nu=1}^{18} \left\| S_1^{(\nu)} \tilde{z} + S_2^{(\nu)} z \right\|_{\mathcal{U}_r} \\
\leq \|Z\| \|\tilde{L}_r (I_0^* + I_1^* + I_2^* + I_3^*)\| \\
\leq \varepsilon_r,
\]

and so \(S_1 \tilde{z} + S_2 z \in \mathcal{V}_{\gamma} \). Furthermore, the continuity of \(S_1 \) is implied by the continuity of the operator \(S_r\).

We will now demonstrate that \(S_1 \) is uniformly bounded. For this, we have

\[
\left| (S_1^{(\nu)} \tilde{z})(s) \right| \leq \int_0^s (s - \xi)^{\nu-1} \frac{1}{\Gamma(\nu)} |S_r(\xi, z_r(\xi), \mathcal{D}^\nu z_r(\xi), z_r'(\xi), z_r''(\xi))| d\xi \\
\leq \frac{1}{\Gamma(\nu + 1)} \|Z\| \|L_r\| \left| (z_r(\xi)) + |\mathcal{D}^\nu z_r(\xi)| + |z_r'(\xi)| + |z_r''(\xi)| \right|
\]
for all \( z \in V_{v'} \). Also,

\[
|\mathcal{D}^\nu S_1^{(v')}(z)(s)| \leq \int_0^s \frac{(s - \theta)^{\ell - 1}}{\Gamma(\ell - v)} |\mathcal{S}_v(\xi, z_v(\xi), \mathcal{D}^\nu z_v(\xi), z_v'(\xi), z_v''(\xi))| \, d\xi \\
\leq \frac{1}{\Gamma(\ell - v + 1)} \|Z_v\| \|L_v\| \bigl( |\mathcal{D}^\nu z_v(\xi)| + |z_v'(\xi)| + |z_v''(\xi)| \bigr),
\]

and

\[
|\left(S_1^{(v')}\right)'(z)(s)| \leq \frac{1}{\Gamma(\ell)} \|Z_v\| \|L_v\| \bigl( |\mathcal{D}^\nu z_v(\xi)| + |z_v'(\xi)| + |z_v''(\xi)| \bigr),
\]

\[
|\left(S_1^{(v')}\right)''(z)(s)| \leq \frac{1}{\Gamma(\ell - 1)} \|Z_v\| \|L_v\| \bigl( |\mathcal{D}^\nu z_v(\xi)| + |z_v'(\xi)| + |z_v''(\xi)| \bigr)
\]

for all \( z \in V_{v'} \). Thus

\[
\|S_1 z\|_{\ell^p} = \sum_{\nu=1}^{18} \|S_1^{(v')}(z)\|_{\ell^p} \\
\leq \left\{ \frac{\ell^2}{\Gamma(\ell + 1)} + \frac{1}{\Gamma(\ell - v + 1)} \right\} \sum_{\nu=1}^{18} \|Z_v\| \|L_v\| \bigl( \|z_v\|_{\ell^p} \bigr),
\]

which shows that \( S_1 \) is uniformly bounded on \( V_{v'} \).

Now we will prove that \( S_1 \) is compact on \( V_{v'} \). For this, let \( s_1, s_2 \in [0, 1] \) with \( s_1 < s_2 \). Then we have

\[
|\left(S_1^{(v')}(z)\right)(s_2) - \left(S_1^{(v')}(z)\right)(s_1)| \\
\leq \left| \int_{s_1}^{s_2} \frac{(s_2 - \theta)^{\ell - 1}}{\Gamma(\ell)} \mathcal{S}_v(\xi, z_v(\xi), \mathcal{D}^\nu z_v(\xi), z_v'(\xi), z_v''(\xi)) \, d\xi \right| \\
- \left| \int_0^{s_1} \frac{(s_1 - \theta)^{\ell - 1}}{\Gamma(\ell)} \mathcal{S}_v(\xi, z_v(\xi), \mathcal{D}^\nu z_v(\xi), z_v'(\xi), z_v''(\xi)) \, d\xi \right| \\
+ \left| \int_{s_1}^{s_2} \frac{(s_2 - \theta)^{\ell - 1} - (s_1 - \theta)^{\ell - 1}}{\Gamma(\ell)} \mathcal{S}_v(\xi, z_v(\xi), \mathcal{D}^\nu z_v(\xi), z_v'(\xi), z_v''(\xi)) \, d\xi \right|
\]

\[
\leq \left| \mathcal{S}_v(\xi, z_v(\xi), \mathcal{D}^\nu z_v(\xi), z_v'(\xi), z_v''(\xi)) \right| \left| \int_0^{s_1} \frac{(s_1 - \theta)^{\ell - 1} - (s_1 - \theta)^{\ell - 1}}{\Gamma(\ell)} \, d\xi \right|
\]

\[
+ \left| \mathcal{S}_v(\xi, z_v(\xi), \mathcal{D}^\nu z_v(\xi), z_v'(\xi), z_v''(\xi)) \right| \left| \int_{s_1}^{s_2} \frac{(s_2 - \theta)^{\ell - 1} - (s_1 - \theta)^{\ell - 1}}{\Gamma(\ell)} \, d\xi \right|
\]

\[
\leq \left\{ \frac{s_2 - s_1}{\Gamma(\ell + 1)} + \frac{(s_2 - s_1)^2}{\Gamma(\ell + 1)} \right\} \|Z_v\| \|L_v\| \bigl( \|z_v\|_{\ell^p} \bigr).
\]
Hence \(|(S^{(2)}_1(s) - S^{(2)}_1(s_t)| \to 0\) as \(s_t \to s_2\). Also, we have

\[
\begin{align*}
\lim_{s_1 \to s_2} \| (\mathcal{D}^1 S^{(2)}_1)(s) - (\mathcal{D}^1 S^{(2)}_1)(s_1) \| &= 0, \\
\lim_{s_1 \to s_2} \| (S^{(2)}_1(s) - S^{(2)}_1(s_1)) \| &= 0, \\
\lim_{s_1 \to s_2} \| (S^{(2)}_1(s) - S^{(2)}_1(s_1)) \| &= 0.
\end{align*}
\]

Hence \(\|S(s) - S(s_1)\|\) tends to zero as \(s_1 \to s_2\). Thus \(S_1\) is equicontinuous, and therefore \(S_1\) is relatively compact operator on \(V_{c,\gamma}\). So \(S_1\) is compact on \(V_{c,\gamma}\) by the Arzelà–Ascoli theorem.

It remains to prove that \(S_2\) is a contraction. To show this, letting \(z, \tilde{z} \in V_{c,\gamma}\), we obtain

\[
\begin{align*}
\| (S^{(2)}_2(s) - S^{(2)}_2(s)) \| &\leq \frac{1}{V_1} \left( \frac{\|s\|}{\|s_1\|} + s \right) \\
&\times \left[ |\mathcal{S}_1| \int_0^1 \int_0^{\xi} \left| S_1 \left( t, z(t), \mathcal{D}^1 z(t), z(t), z(t) \right) \right| dt \, d\xi \\
+ |\mathcal{S}_2| \int_0^1 \int_0^{\xi} \left| S_2 \left( t, z(t), \mathcal{D}^1 z(t), z(t), z(t) \right) \right| dt \, d\xi \\
+ |\mathcal{S}_3| \int_0^1 \int_0^{\xi} \left| S_3 \left( t, z(t), \mathcal{D}^1 z(t), z(t), z(t) \right) \right| dt \, d\xi \\
&\leq \frac{1}{V_1} \left( \frac{\|s\|}{\|s_1\|} + s \right) \left[ |\mathcal{S}_1| \int_0^1 \int_0^{\xi} G_1(s)(|\tilde{z}(t) - z(t)| \\
+ |\mathcal{D}^1 \tilde{z}(t) - \mathcal{D}^1 z(t)| + |\tilde{z}(t) - z(t)| \\
+ |\mathcal{D}^1 \tilde{z}(t) - \mathcal{D}^1 z(t)| + |\tilde{z}(t) - z(t)| \\
+ |\mathcal{D}^1 \tilde{z}(t) - \mathcal{D}^1 z(t)| + |\tilde{z}(t) - z(t)| \right) dt \, d\xi \\
+ |\mathcal{S}_2| \int_0^1 \int_0^{\xi} G_1(s)(|\tilde{z}(t) - z(t)| \\
+ |\mathcal{D}^1 \tilde{z}(t) - \mathcal{D}^1 z(t)| + |\tilde{z}(t) - z(t)| \\
+ |\mathcal{D}^1 \tilde{z}(t) - \mathcal{D}^1 z(t)| + |\tilde{z}(t) - z(t)| \right) dt \, d\xi \\
+ |\mathcal{S}_3| \int_0^1 \int_0^{\xi} G_1(s)(|\tilde{z}(t) - z(t)| \\
+ |\mathcal{D}^1 \tilde{z}(t) - \mathcal{D}^1 z(t)| + |\tilde{z}(t) - z(t)| \\
+ |\mathcal{D}^1 \tilde{z}(t) - \mathcal{D}^1 z(t)| + |\tilde{z}(t) - z(t)| \right) dt \, d\xi.
\end{align*}
\]
3.3 Consider the differential equations:

\[
\begin{align*}
\mathcal{D}^{2.01} z_1(s) &= \frac{4s}{1000} |\arcsin z_2(s)| + \frac{8s(\mathcal{D}^{0.2} z_2(s))}{2000 + 2000(\mathcal{D}^{0.2} z_2(s))} + 0.004s |\arcsin z'_1(s)| \\
&\quad + \frac{12s |\sin z'_2(s)|}{3000(1 + |\sin z'_2(s)|)} + 0.004s |\arcsin z'_2(s)|, \\
\mathcal{D}^{2.01} z_2(s) &= \frac{21e^s |\sin z_2(s)|}{1000(1 + |\sin z_2(s)|)} + \frac{7e^s}{1000} |\sin(\mathcal{D}^{0.2} z_2(s))| + \frac{14|\arctan z'_2(s)| e^s}{2000 + 2000(\arctan z'_2(s))} \\
&\quad + 0.007e^s |\arcsin z'(s)|, \\
\mathcal{D}^{2.01} z_3(s) &= 0.011 |\arctan z_3(s)| s + \frac{44e^{100} z_2(s)}{4000(1 + |\arctan z_3(s)|)} + \frac{11}{1000} |\arcsin z'_3(s)| s \\
&\quad + \frac{22|\sin z'_3(s)| s}{2000 + 2000 |\sin z'_3(s)|}.
\end{align*}
\]

(3.15)

for each \( r = 1, 2, \ldots, 18 \), where \( I^*_4 \) is given in (3.9). Also, by similar computations we have

\[
\sup_{s \in [0,1]} \left| (\mathcal{D}^r S^{(r)}_2 z) - (\mathcal{D}^r S^{(r)}_2 z) \right| \leq \|G_r\| \|\tilde{z} - z\|_{\ell_r},
\]

\[
\sup_{s \in [0,1]} \left| (S^{(r)}_2 z) - (S^{(r)}_2 z) \right| \leq \|G_r\| \|\tilde{z} - z\|_{\ell_r},
\]

\[
\sup_{s \in [0,1]} \left| (S^{(r)}_2 z) - (S^{(r)}_2 z) \right| \leq 0,
\]

where \( I^*_5 \) and \( I^*_6 \) are given in (3.10) and (3.11), respectively. Thus we have

\[
\|S_2 \tilde{z} - S_2 z\|_{\ell_r} = \sum_{r=1}^{18} \|S^{(r)}_2 \tilde{z} - S^{(r)}_2 z\|_{\ell_r} \leq (I^*_4 + I^*_5 + I^*_6) \sum_{r=1}^{18} \|G_r\| \|\tilde{z} - z\|_{\ell_r},
\]

and so

\[
\|S_2 \tilde{z} - S_2 z\|_{\ell_r} \leq \Delta \|\tilde{z} - z\|_{\ell_r}.
\]

Since \( \Delta < 1 \), \( S_2 \) is a contraction on \( V_{\ell_r} \). As a result of Theorem 2.3, we infer that \( S \) contains a fixed point, which is a solution to problem (1.4). \( \square \)

To illustrate the significance of our results, we provide the following example.

**Example 3.3** Consider the differential equations:

\[
\begin{align*}
\mathcal{D}^{2.01} z_1(s) &= \frac{4s}{1000} |\arcsin z_2(s)| + \frac{8s(\mathcal{D}^{0.2} z_2(s))}{2000 + 2000(\mathcal{D}^{0.2} z_2(s))} + 0.004s |\arcsin z'_1(s)| \\
&\quad + \frac{12s |\sin z'_2(s)|}{3000(1 + |\sin z'_2(s)|)} + 0.004s |\arcsin z'_2(s)|, \\
\mathcal{D}^{2.01} z_2(s) &= \frac{21e^s |\sin z_2(s)|}{1000(1 + |\sin z_2(s)|)} + \frac{7e^s}{1000} |\sin(\mathcal{D}^{0.2} z_2(s))| + \frac{14|\arctan z'_2(s)| e^s}{2000 + 2000(\arctan z'_2(s))} \\
&\quad + 0.007e^s |\arcsin z'(s)|, \\
\mathcal{D}^{2.01} z_3(s) &= 0.011 |\arctan z_3(s)| s + \frac{44e^{100} z_2(s)}{4000(1 + |\arctan z_3(s)|)} + \frac{11}{1000} |\arcsin z'_3(s)| s \\
&\quad + \frac{22|\sin z'_3(s)| s}{2000 + 2000 |\sin z'_3(s)|}.
\end{align*}
\]
associated with the boundary conditions

\[
\begin{aligned}
\begin{cases}
\begin{aligned}
\frac{9}{14}z_1(0) &= \frac{4}{14} D^1 z_1(1) + \frac{6}{19} D^2 z_1(1), \\
\frac{9}{14}z_1(1) &= \frac{4}{14} D^1 z_1(1) + \frac{6}{19} D^2 z_1(0), \\
\frac{9}{14}z_2(0) &= \frac{4}{14} D^1 z_2(1) + \frac{6}{19} D^2 z_2(1), \\
\frac{9}{14}z_2(1) &= \frac{4}{14} D^1 z_2(1) + \frac{6}{19} D^2 z_2(0), \\
\frac{9}{14}z_3(0) &= \frac{4}{14} D^1 z_3(1) + \frac{6}{19} D^2 z_3(1), \\
\frac{9}{14}z_3(1) &= \frac{4}{14} D^1 z_3(1) + \frac{6}{19} D^2 z_3(0), \\
\end{aligned}
\end{cases}
\end{aligned}
\tag{3.16}
\]

where \( \ell = 2.01, \nu = 0.2, \sigma_1 = \frac{9}{14}, \sigma_2 = \frac{3}{14}, \sigma_3 = \frac{6}{19}, \) and \( D^\ell, D^\nu \) represent the Caputo fractional derivatives of orders \( \ell \) and \( \nu \), respectively. Let \( S_1, S_2, S_3 : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be continuous functions given as

\[
\begin{aligned}
S_1(s, z_1(s), z_2(s), z_3(s), z_4(s)) &= \frac{4 \nu}{1000} |\arcsin z_1(s)| + \frac{4 \nu |\sin z_2(s)|}{2000 + 2000 |\sin z_2(s)|} + \frac{0.004 \nu}{12} |\arcsin z_3(s)| + \frac{0.004 \nu}{3000 |1 + \sin z_4(s)|} |z_3(s)|, \\
S_2(s, z_1(s), z_2(s), z_3(s), z_4(s)) &= \frac{2 \nu e^{|\sin z_1(s)|}}{3000 |1 + |\sin z_1(s)||} + \frac{7 e^{|\sin z_1(s)|}}{1000} |\arctan z_1(s)\arcsin z_1(s)| + \frac{0.007 \nu}{2000 + 2000 |\arctan z_3(s)|} |\arcsin z_4(s)|, \\
S_3(s, z_1(s), z_2(s), z_3(s), z_4(s)) &= \frac{0.011}{1000} |\arctan z_1(s)| + \frac{4 \nu |\arcsin z_4(s)|}{4000 |1 + |\arcsin z_4(s)||} + \frac{0.011}{22} |\arcsin z_3(s)| s + \frac{11}{1000} |\arctan z_1(s)| + \frac{4 \nu |\arcsin z_4(s)|}{22000 + 2000 |\arcsin z_4(s)|} |\arcsin z_3(s)| s.
\end{aligned}
\]

Let \( z_1, z_2, z_3, z_4, \tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4 \in \mathbb{R} \). Then we have

\[
\begin{aligned}
|S_1(s, z_1(s), z_2(s), z_3(s), z_4(s))| - |S_1(s, \tilde{z}_1(s), \tilde{z}_2(s), \tilde{z}_3(s), \tilde{z}_4(s))| &\leq \frac{4 \nu}{1000} \left( |\arcsin z_1(s) - \arcsin \tilde{z}_1(s)| + |z_2(s) - \tilde{z}_2(s)| \right. \\
&\quad + |\sin z_3(s) - \sin \tilde{z}_3(s)| + |\sin z_4(s) - \sin \tilde{z}_4(s)|, \\
|S_2(s, z_1(s), z_2(s), z_3(s), z_4(s))| - |S_2(s, \tilde{z}_1(s), \tilde{z}_2(s), \tilde{z}_3(s), \tilde{z}_4(s))| &\leq \frac{7 e^{|\sin z_1(s)|}}{1000} \left( |\arctan z_3(s) - \arctan \tilde{z}_3(s)| + |\arcsin z_4(s) - \arcsin \tilde{z}_4(s)| \right), \\
|S_3(s, z_1(s), z_2(s), z_3(s), z_4(s))| - |S_3(s, \tilde{z}_1(s), \tilde{z}_2(s), \tilde{z}_3(s), \tilde{z}_4(s))| &\leq \frac{11 \nu}{1000} \left( |\arctan z_1(s) - \arctan \tilde{z}_1(s)| + |z_2(s) - \tilde{z}_2(s)| \right. \\
&\quad + |\arcsin z_3(s) - \arcsin \tilde{z}_3(s)| + |\sin z_4(s) - \sin \tilde{z}_4(s)|.
\end{aligned}
\]
Here \( G_1(s) = \frac{4s}{1000}, \ G_2(s) = \frac{7e^s}{1000}, \ G_3(s) = \frac{11s}{1000} \), where \( \|G_1\| = \frac{4}{1000}, \|G_2\| = \frac{7}{1000}, \|G_3\| = \frac{11}{1000} \). Let \( L_1, L_2, L_3 : [0, \infty) \to \mathbb{R} \) be the identity functions. Then we obtain

\[
|S_1(s, z(s), D^{0.2}z(s), z'(s), z''(s))| \leq \frac{4s}{1000} \left( |\arcsin z| + |Dz| + |\sin z'| + |\sin z''| \right)
\]

\[
\leq \frac{4s}{1000} \left( |z| + |Dz| + |z'| + |z''| \right).
\]

Also,

\[
|S_2(s, z(s), D^{0.2}z(s), z'(s), z''(s))| \leq \frac{7e^s}{1000} \left( |\sin z| + |\sin(Dz)| + |\arctan z'| + |\arcsin z''| \right)
\]

\[
\leq \frac{7e^s}{1000} \left( |z| + |Dz| + |z'| + |z''| \right).
\]

and

\[
|S_3(s, z(s), D^{0.2}z(s), z'(s), z''(s))| \leq \frac{11s}{1000} \left( |\arctan z| + |Dz| + |\arcsin z'| + |\sin z''| \right)
\]

\[
\leq \frac{11s}{1000} \left( |z| + |Dz| + |z'| + |z''| \right),
\]

where the continuous functions \( Z_1, Z_2, Z_3 : [0, 1] \to \mathbb{R} \) are defined by

\[
Z_1(s) = \frac{4s}{1000}, \quad Z_2(s) = \frac{7e^s}{1000}, \quad Z_3(s) = \frac{11s}{1000}.
\]

Also,

\[
I^*_4 \simeq 0.9227, \quad I^*_5 \simeq 1.2360 \quad \text{and} \quad I^*_6 \simeq 1.1512,
\]

and so

\[
\Delta := (I^*_4 + I^*_5 + I^*_6) (\|G_1\| + \|G_2\| + \|G_3\|) \simeq 0.0728 < 1.
\]

Hence by Theorem 3.2 problem (3.15)–(3.16) has a solution.

### 4 Conclusion

Chemical graph theory is a broad area of research, which uses theoretical and practical techniques to analyze the molecular structure of a chemical substance on graphs while considering particular mathematical challenges. As a result of the fast growth of this area over the last few decades, many new concepts and techniques for conducting such research have emerged. In this paper, we used the graph of a hexasilinane compound and defined the Caputo fractional boundary value problem on each of its edges. We utilized the fixed point theorems of Krasnoselskii and Schaefer to prove the existence of solutions to the proposed boundary value problem. Our approach is easy to implement and can be used in a wide range of graphs, particularly digraphs, which are often used in medical technology for protein networks.
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