WEBS OF STARS OR HOW TO TRIANGULATE FREE SUMS OF POINT CONFIGURATIONS

BENJAMIN ASSARF, MICHAEL JOSWIG, AND JULIAN PFEIFLE

Abstract. The triangulations of point configurations which decompose as a free sum are classified in terms of the triangulations of the summands. The methods employ two new kinds of partially ordered sets to be associated with any triangulation of a point set with one point marked, the web of stars and the stabbing poset. Triangulations of smooth Fano polytopes are discussed as a case study.

1. Introduction

The investigation of triangulations of point configurations and their secondary fans is motivated by numerous applications in many areas of mathematics. For an overview see the introductory chapter of the monograph [5] by De Loera, Rambau and Santos. The secondary fan is a complete polyhedral fan which encodes the set of all (regular) subdivisions of a fixed point configuration, partially ordered by refinement. As secondary fans form a very rich concept, general structural results are hard to obtain. There are rather few infinite families of point configurations known for which the entire set of all triangulations can be described in an explicit way; see [7] for a classification which covers very many of the cases known up to now. The purpose of the present paper is to examine the triangulations of point configurations which decompose as a free sum, and we give a full classification in terms of the triangulations of the summands. A case study on a configuration of 17 points in \( \mathbb{R}^6 \) underlines that, for point configurations which decompose, our methods significantly extend the range where explicit computations are possible.

Let \( P \subseteq \mathbb{R}^d \) and \( Q \subseteq \mathbb{R}^e \) be two finite point configurations containing the origin in their respective interiors. Their (free) sum is the point set

\[
P \oplus Q := (P \times \{0\}) \cup (\{0\} \times Q) \subseteq \mathbb{R}^{d+e},
\]

and their (affine) join is

\[
P * Q := (\{0\} \times P \times \{0\}) \cup (\{1\} \times \{0\} \times Q) \subseteq \mathbb{R}^{1+d+e}.
\]

Starting from a triangulation of the sum, the join or the Cartesian product of two point configurations, a natural question to ask is whether the triangulation can be expressed or constructed using individual triangulations of \( P \) and \( Q \).

2010 Mathematics Subject Classification. 52B11 (57Q15, 52B20).

Key words and phrases. triangulations of point configurations, polytope free sums, smooth Fano polytopes.

M. Joswig is partially supported by Einstein Foundation Berlin and Deutsche Forschungsgemeinschaft (DFG) within the Priority Program 1489 “Experimental Methods in Algebra, Geometry and Number Theory”.

arXiv:1512.08411v1 [math.CO] 28 Dec 2015
In [5] there are several results on affine joins and Cartesian products, but none for sums. A complete characterization for the affine joins is given by [5, Theorem 4.2.7]. It turns out that every subdivision of an affine join is determined by the subdivisions of the factors in an unique way. For the product the situation is much more involved. Any two subdivisions of two point sets give rise to a subdivision of the product point configuration [5, Definition 4.2.13], but not every subdivision of the product arises in this way.

The free sum is dual to the product, and hence it is a very natural construction to look at. We examine how an arbitrary triangulation $\Delta_P$ of $P$ and an arbitrary triangulation $\Delta_Q$ of $Q$ give rise to a triangulation of $P \oplus Q$. However, in order to make the construction work additional data is required. This leads us to defining webs of stars in $\Delta_P$ and $\Delta_Q$. These are families of star-shaped balls (containing the origin) in $\Delta_P$ and $\Delta_Q$, respectively, which satisfy certain compatibility conditions. This compatibility is expressed in terms of visibility from the origin. Our first main result (Theorem 5.2) says that each triangulation of $P \oplus Q$ arises in this way. We found it surprisingly difficult to show, however, that the resulting conditions on the summands always suffice to construct a triangulation. This is our second main result (Theorem 6.7), and this completes our characterization.

One good reason for considering free sums (and their subdivisions) is that interesting classes of polytopes are closed with respect to this construction. This includes the smooth Fano polytopes, which are (necessarily simplicial) lattice polytopes with an origin as an interior lattice point such that the vertices on each facet form a lattice basis. For each dimension there are only finitely many smooth Fano polytopes, up to unimodular equivalence. They are classified in dimensions up to nine; cf. [3], [9], [12], and [10]. Smooth Fano polytopes play a role in algebraic geometry and mathematical physics. Interestingly, many of these polytopes decompose as a free sum. There is a more precise general statement conjectured [1, Conjecture 9], which has partially been confirmed [2, Theorem 1]. In Section 7 we report on a case study where we apply our methods to a six-dimensional smooth Fano polytope with 16 vertices, which decomposes into a 2-dimensional and a 4-dimensional summand. With standard techniques (cf. [13] and [14]) it seems to be out of reach to compute all its triangulations up to symmetry on a standard desktop computer within several weeks. Yet, our approach solves this problem on the same hardware within ten days.

Our paper is organized as follows. The Section 2 starts out with investigating two partially ordered sets which can be associated to any triangulation $\Delta$ of a point set, in which one point is marked. Throughout the marked point will be the origin. The first poset contains the triangulated balls (of maximal dimension) which contain the origin, not necessarily as a vertex, and which are strictly star-shaped (with respect to the origin). The second poset comprises the facets of the triangulation with the partial ordering induced by visibility from the origin; this is the stabbing poset. In Section 3 we start to investigate triangulations of sums of point configurations. A key step is to analyze how a triangulation $\Delta_{P \oplus Q}$ induces triangulations on the two summands. Here we obtain a unified treatment which simultaneously covers the case where the origin is a vertex of $\Delta_{P \oplus Q}$ and the case where it is
not. Next, in Section 4 we can define webs of stars and sum-triangulations. Section 5 is devoted to proving Theorem 5.2, which says that every triangulation of the free sum of two point configurations is a sum-triangulation. Finally, the Section 6 is dedicated to the proof of the converse direction, namely that every pair of triangulations of the summands can be used to construct a triangulation of the sum. However, the correspondence is not one-to-one, meaning that different pairs of triangulations of the summands may produce the same sum-triangulation. In order to show the applicability and the usefulness of our methods, in Section 7 we analyze one specific point configuration in detail: the free sum $\text{DP}(2) \oplus \text{DP}(4)$ of two del Pezzo polytopes (of dimensions two and four). This is a smooth Fano polytope in dimension six with 16 vertices; taking the origin into account gives a total of 17 points. Using the triangulations of the summands as input (obtained via TOPCOM [14]) we compute all triangulations of $\text{DP}(2) \oplus \text{DP}(4)$ with polymake [6]. We close the paper with a conjecture about regular triangulations and an appendix on an algorithmic detail.

2. Toolbox

We start out with collecting basic facts about triangulations of point sets which are relevant for our investigation. Let $P \subset \mathbb{R}^d$ be a finite point set. An interior point of $P$ is a point in $P$ which is contained in the interior of the convex hull $\text{conv} P$. Clearly, the convex hull of $P$ needs to be full-dimensional in order to have any interior points. Now let $Q \subset \mathbb{R}^e$ be another configuration of finitely many points. Throughout the paper, we will assume that the origin 0 (in $\mathbb{R}^d$ and $\mathbb{R}^e$, respectively) is an interior point of both $P$ and $Q$. This entails that $P$ linearly spans the entire space $\mathbb{R}^d$, and $Q$ spans $\mathbb{R}^e$ likewise. The origin in $\mathbb{R}^{d+e}$ plays a special role in $P \oplus Q$, since it is the only point in the intersection $(P \times \{0\}) \cap (\{0\} \times Q)$.

We denote triangulations of $P$, $Q$ and $P \oplus Q$ by $\triangle P$, $\triangle Q$ and $\triangle P \oplus Q$, respectively. As usual, a simplex $\sigma$ of a triangulation $\Delta$ is the convex hull of its vertices, and its dimension is the dimension of their affine hull. We write $\triangle = k$ for the set of all simplices of dimension $k$ in $\Delta$, and $\partial \triangle$ for the boundary complex of $\Delta$.

Consider a full-dimensional simplex $\sigma \in \Delta_{P \oplus Q}$. Because the vertex set of $\sigma$ is affinely independent, it contains at most $d + 1$ points of $P$ and at most $e + 1$ points of $Q$. On the other hand, since $\sigma$ is a $(d + e)$-simplex, it has exactly $d + e + 1$ vertices. Therefore, $\sigma$ contains at least $d$ points of $P$ and at least $e$ points of $Q$, and we express $\sigma$ as $\sigma = \text{conv}(\sigma_P, \sigma_Q)$ with

$$\sigma_P := \text{conv} \left( \text{Vert} \sigma \cap (P \times \{0\}) \right) \quad \text{and} \quad \sigma_Q := \text{conv} \left( \text{Vert} \sigma \cap (\{0\} \times Q) \right),$$

where Vert $\sigma$ denotes the set of vertices of $\sigma$.

Observation 2.1. If $0 \notin \text{Vert} \sigma$, then exactly one of the simplices $\sigma_P$, $\sigma_Q$ is full-dimensional, and the other one has codimension 1 in the affine span of its containing polytope. On the other hand, if $0 \in \text{Vert} \sigma$ then both simplices are full-dimensional.
We will be a bit imprecise with our notation. Often we will confuse $P$ with $P \times \{0\}$, and $\sigma_P$ with its canonical projection to the linear subspace $\mathbb{R}^d$. Accordingly, instead of $\sigma = \text{conv}(\sigma_P, \sigma_Q)$ we will also write $\sigma = \sigma_P \oplus \sigma_Q$.

Collecting all simplices of $\triangle_{P \oplus Q}$ that lie in $\mathbb{R}^d \times \{0\}$ or $\{0\} \times \mathbb{R}^e$ yields simplicial complexes on the vertex sets of $P$ and $Q$ that do not necessarily cover the respective convex hulls. In Section 3 we prove that these complexes can be extended to proper triangulations. Hence there exist triangulations $\triangle_P$ of $P$ and $\triangle_Q$ of $Q$ such that every full-dimensional cell of $P \oplus Q$ is the sum of two cells of those two triangulations. We defer the obvious question of how those cells are to be combined into a decomposition of $P \oplus Q$ until we describe our main construction, the sum-triangulation, in Definition 4.4.

From any two fixed triangulations of the summands, it can produce several triangulations of the sum. Conversely, every triangulation of the sum of two polytopes is a sum triangulation and can be produced from triangulations of the summands, but not necessarily in a unique way (Construction 5.1).

The remainder of this section lays the technical foundations. To this end we first need to establish several notions.

**Definition 2.2 (star/link/restriction).** Let $\triangle$ be a triangulation of a point configuration in $\mathbb{R}^d$ and $\sigma$ be a face in $\triangle$. The (closed) star $\text{st}_\triangle(\sigma)$ of $\sigma$ is the subcomplex of $\triangle$ consisting of all simplices containing $\sigma$, and all their faces. The link of $f$ is the simplicial complex $\text{lk}_\triangle(\sigma) := \{ \tau \in \text{st}_\triangle(f) \mid \sigma \cap \tau = \emptyset \}$.

Consider a point $x$ in the set covered by $\triangle$, which, however, does not need to be a vertex, and let $\sigma$ be the minimal face containing it. We let $\text{st}_\triangle(x) = \text{st}_\triangle(\sigma)$ and $\text{lk}_\triangle(x) = \text{lk}_\triangle(\sigma)$.

For any closed set $S \subseteq \mathbb{R}^d$ we call $\triangle|_S := \{ \sigma \in \triangle \mid \sigma \subseteq S \}$ the restriction of $\triangle$ to $S$.

**Observation 2.3.** As we triangulate polytopes lying in Euclidean space, the link of any interior cell in a triangulation forms a triangulated sphere, while the link of any cell in the boundary is a triangulated ball, cf. Hudson [8, Chapter 1].

Traditionally, a set $S$ is called star shaped with respect to the point $x \in S$ if for every $y \in S$ the line segment $\overline{xy}$ is completely contained in $S$. We need a slightly stricter version of this generalization of convexity.

**Definition 2.4.** A set $S$ is strictly star shaped with respect to $x \in S$ if for every $y \in S$ the line segment $\overline{xy}$ is completely contained in $\text{relint}(S) \cup \{y\}$.

Thus, the point $x$ must be contained in the relative interior of $S$, and the line segment $\overline{xy}$ is only allowed to intersect the boundary of $S$ in $y$. Another way of saying the same is that every ray starting at $x$ can intersect $\partial S$ in at most one point.

**Lemma 2.5.** Let $\triangle$ be a triangulation of a point configuration, and let $x$ be a point in its (relative) interior. Then $\text{st}_\triangle(x)$ is strictly star shaped with respect to $x$.

**Proof.** Because every simplex is convex, $\text{st}_\triangle(x)$ is star shaped with respect to $x$. The relative interior of the star is $\text{st}_\triangle(x) \setminus \text{lk}_\triangle(x)$, and it is clear that
it contains $x$. Let $F$ be a maximal cell in the boundary of $st_\Delta(x)$. Then $F$ is contained in $\text{lk}_\Delta(x)$, and it does not contain $x$. It follows that the vertices of $F$ and $x$ form an affinely independent set, and thus the intersection of $F$ and any line segment $xy$ for $y \in F$ is just the point $y$. □

Two partially ordered sets will play a crucial role in the rest of the paper. The first poset is associated to any triangulation $\Delta$ of a point configuration in $\mathbb{R}^d$, namely

$$B^k_\Delta(x) = \left\{ \text{subcomplexes of } \Delta \text{ that are } k\text{-dimensional balls and strictly star shaped with respect to } x \right\} \cup \{\emptyset\},$$

partially ordered by inclusion. For convenience, we abbreviate $B_\Delta := B^d_\Delta(0)$.

The second partial order is defined on the full-dimensional simplices in $\Delta$. We say that $\sigma$ precedes $\tau$ in the stabbing order, and write $\sigma \preceq \tau$, if $\sigma = \tau$ or (compare Figure 1)

- for every linear or affine hyperplane that separates $\sigma$ and $\tau$ (not necessarily strictly), $\sigma$ lies in the same closed half space as 0; and
- $\sigma$ and $\tau$ are separated by at least one strictly affine hyperplane $H$.

Note that the minimal elements in the $\preceq$-ordering are the simplices in $st_\Delta(0)$, as they already contain the origin. Throughout, we write $\sigma \prec \tau$ if $\sigma \preceq \tau$ and $\sigma \neq \tau$.

**Lemma 2.6.** For any two distinct, comparable simplices $\sigma \prec \tau$ in a triangulation $\Delta$ in $\mathbb{R}^d$, there exists a ray from the origin that stabs first $\sigma$ and then $\tau$, i.e., there exists an $r \in \mathbb{R}^d \setminus \{0\}$ and $0 \leq \lambda < \mu$ such that $\lambda r \in \sigma$ and $\mu r \in \tau$.

**Proof.** First we briefly discuss that if $\sigma \prec \tau$ and both are stabbed by a ray $\rho$ through the origin, we may assume that two intersection points are of the form $\lambda r \in \sigma$ and $\mu r \in \tau$ with $r \in \mathbb{R}^d \setminus \{0\}$ and $0 \leq \lambda < \mu$. To get some intuition for this, consider the separating affine hyperplane $H$ whose existence is guaranteed by the definition of $\preceq$, and choose $r := \rho \cap H$. Then $\lambda \leq \mu$ is clear because $\sigma$ is contained in the same half-space of $H$ as the origin, by the definition of $\preceq$, but is separated from $\tau$ by $H$; see Figure 1a.
Furthermore, a ray with $\lambda < \mu$ exists even if the intersection $\sigma \cap \tau$ is just one point or empty.

To formally prove the lemma, we assume that there exists no stabbing ray $\rho$, and construct a linear hyperplane $L$ that separates $\sigma$ from $\tau$, and that can be perturbed to leave the origin on either side while still separating $\sigma$ from $\tau$. The existence of such an $L$ then directly contradicts $\sigma \leq \tau$.

To construct $L$, we distinguish two cases. First, suppose that $\sigma \cap \tau = \emptyset$. As there exists no stabbing ray for $\sigma$ and $\tau$, the two polyhedral cones induced by those two cells $\text{cone}(\sigma) := \{\lambda x \mid x \in \sigma, \lambda \geq 0\}$ and $\text{cone}(\tau)$ intersect only in the origin. Therefore, we can separate these cones by a linear hyperplane $L$, which of course also separates $\sigma$ and $\tau$. It is clear that $L$ can be perturbed or moved slightly in the required way; see Figure 1b.

Finally, assume that $\sigma \cap \tau \neq \emptyset$, and note that $\text{aff}(\sigma \cap \tau)$ is contained in every hyperplane that separates $\sigma$ and $\tau$. If all such hyperplanes are linear, we have our desired contradiction to $\sigma \leq \tau$, so we may assume that there exists a strictly affine separating hyperplane $H$, so that $0 \notin H$. Then we conclude $0 \notin \text{aff}(\sigma \cap \tau)$ because $\text{aff}(\sigma \cap \tau) \subset H$. Our assumption about the non-existence of a stabbing ray $\rho$ now yields the existence of a separating linear hyperplane $L$ with $\sigma \cap \tau = \sigma \cap L = L \cap \tau$ and $\text{codim aff}(\sigma \cap \tau) \geq 1$.

So we can perturb $L$ in the required way.

Lemma 2.6 yields a necessary condition for distinct simplices to be $\prec$-comparable. However, from an algorithmic point of view, the definition of $\sigma \prec \tau$ is inconvenient, because one has to consider all separating hyperplanes. For this reason, Figure 10 in the appendix illustrates an algorithm for deciding whether $\sigma \prec \tau$ or not.

3. Triangulations of Free Sums

In this section we analyze triangulations $\Delta_{P \oplus Q}$ of the free sum of two point configurations $P$ and $Q$. For each simplex $\sigma \in \Delta_{P \oplus Q}$ we define $\sigma_P$ and $\sigma_Q$ as in (2.1), and we have $\sigma = \sigma_P \oplus \sigma_Q$. Observation 2.1 says what is known about the dimensions of $\sigma_P$ and $\sigma_Q$. Throughout the rest of the paper, we abbreviate

\begin{align*}
(3.1) \quad \Lambda(\sigma) & := \text{lk}_{\Delta_{P \oplus Q}}(\sigma) \quad \text{and} \\
(3.2) \quad \bar{\Lambda}(\sigma) & := \emptyset \ast \Lambda(\sigma) = \{\lambda x \mid x \in \sigma, \lambda \in [0,1]\} .
\end{align*}

Here “$\ast$” is the join operation. Since 0 is a single point, however, the set $\Lambda(\sigma)$ is the cone over the link $\Lambda(\sigma)$. We write “$\ast$” instead of “$\ast$” since, in contrast with the general situation in (1.2) this cone has a natural realization in $\mathbb{R}^{d+e}$, i.e., we do not need to increase the dimension. Let $\sigma = \sigma_P \oplus \sigma_Q$ and $\tau = \tau_P \oplus \tau_Q$ be two cells of $\Delta_{P \oplus Q}$.

Note that the definition of the cone in (3.2) makes sense for simplices in $\Delta_P$ or $\Delta_Q$ since they occur as simplices in $\Delta_{P \oplus Q}$. In these cases we have the trivial decompositions $\sigma_P = \sigma_P \oplus \emptyset$ and $\sigma_Q = \emptyset \oplus \sigma_Q$.

The next two sections are dedicated to the structure of $\bar{\Lambda}$. We will verify the following three properties:

- (strictly star shaped) if $\sigma_P$ is $d$-dimensional then the restriction $\Delta_{P \oplus Q}|_{\bar{\Lambda}(\sigma_P)}$ is a strictly star-shaped ball;
• (order preserving) for \(d\)-dimensional \(\sigma_P, \tau_P\) with \(\sigma_P \preceq \tau_P\) we obtain
  \[\bar{\Lambda}(\sigma_P) \subseteq \bar{\Lambda}(\tau_P) ;\]
• (complementarity) for \(d\)-dimensional \(\sigma_P\) and \(e\)-dimensional \(\tau_Q\) we have
  \[\sigma_P \subseteq \bar{\Lambda}(\tau_Q) \iff \tau_Q \nsubseteq \bar{\Lambda}(\sigma_P) .\]

The following lemma, which is a version of Pasch’s theorem, will be used extensively in the rest of this paper.

**Lemma 3.1.** Let \(x, y \in \mathbb{R}^d\) be linearly independent. For every choice of \(\lambda, \mu > 1\) the line segment between \(x\) and \(\mu y\) crosses the line segment between \(y\) and \(\lambda x\).

**Proof.** Check that
\[tx + (1 - t)\mu y = sy + (1 - s)\lambda x\]
for \(t = \frac{\lambda(\mu - 1)}{\mu \lambda - 1}\) and \(s = (1 - t)\mu\). \(\square\)

3.1. **Strictly star shaped balls and the preservation of order.** For each simplex in a summand we need to examine its link in the free sum.

**Proposition 3.2.** Let \(\sigma_P\) be a full-dimensional simplex of \(\triangle P\). Then the link \(\Lambda(\sigma_P)\) in \(\triangle P \oplus Q\) is a subcomplex of \(\triangle Q\) in \(\{0\} \times \mathbb{R}^e\) which is homeomorphic to an \((e - 1)\)-dimensional sphere. Moreover, its cone \(\bar{\Lambda}(\sigma_P)\) is a ball which is strictly star-shaped with respect to \(0\). In particular, we have
\[\triangle P \oplus Q \mid \bar{\Lambda}(\sigma_P) \in B_{\triangle P \oplus Q}(0) .\]

The analogous statements hold for full-dimensional simplices of \(\triangle Q\).

**Proof.** Since \(\dim \sigma_P = d\) the Observation 2.3 implies that the link \(\Lambda(\sigma_P)\) is an \((e - 1)\)-dimensional triangulated sphere that is completely contained in \(\{0\} \times \mathbb{R}^e\). Up to identifying \((0, y) \in \mathbb{R}^{d+e}\) with \(y \in \mathbb{R}^e\) that sphere is a subcomplex of \(\triangle Q\).

The main task is to show that the cone \(\bar{\Lambda}(\sigma_P)\) is strictly star-shaped with respect to \(0\). To this end we need to prove that each ray, emanating from the origin in \(\mathbb{R}^e\), meets the boundary sphere \(\Lambda(\sigma_P)\) only once. We may assume that \(e \geq 2\) since otherwise the claim is obvious.

Assume, to the contrary, that \(\rho\) is such a ray. We first consider the case where \(\rho\) meets the same \((e - 1)\)-face \(\sigma_Q\) of \(\Lambda(\sigma_P)\) twice. Since \(\tau\) is convex and since \(e \geq 2\) it follows that \(\rho\) meets two boundary points of the simplex \(\sigma_Q\). Now the boundary of the entire sphere \(\Lambda(\sigma_P)\) is empty. Therefore, we may assume that \(\rho\) meets two distinct maximal faces of \(\Lambda(\sigma_P) \subseteq \triangle Q\). Without loss of generality we have found a non-zero vector \(r \in \mathbb{R}^e\) which positively spans \(\rho\) with
\[r \in \sigma_Q \quad \text{and} \quad \lambda r \in \tau_Q \quad \text{for some} \quad \lambda > 1 .\]
Furthermore, because \(\dim \sigma_P = d\), we can find another ray that intersects \(\sigma_P\) in more than one interior point. Here it is safe to assume that \(d \geq 1\). So we obtain a point
\[s \in \text{relint} \sigma_P \quad \text{such that} \quad \mu s \in \text{relint} \sigma_P \quad \text{for some} \quad \mu > 1 ,\]
see Figure 2a. As \(\sigma_Q\) and \(\tau_Q\) are both cells in \(\Lambda(\sigma_P)\), we know that \(\triangle P \oplus Q\) contains the two cells \(\sigma = \sigma_P \oplus \sigma_Q\) and \(\tau = \sigma_P \oplus \tau_Q\). Applying Lemma 3.1
to $r$, $\lambda r$, $s$ and $\mu s$ yields a point $x$ in the interior of both $\sigma$ and $\tau$, because $s, \mu s \in \text{relint} \, \sigma_P$. Yet this bad intersection between $\sigma$ and $\tau$ is not possible since we started with a proper triangulation of $P \oplus Q$.

Thus every ray starting at 0 and contained in the affine hull of $\{0\} \times Q$ intersects $\Lambda(\sigma_P)$ exactly once, so that the $e$-dimensional ball $\Lambda(\sigma_P)$ is strictly star shaped with respect to the origin. We conclude that the restricted complex lies in $\mathcal{R}_\Delta^{d+e}(0)$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Illustration of the proofs of Proposition 3.2, shown in (a), and Proposition 3.3, shown in (b).}
\end{figure}

**Proposition 3.3.** Consider cells $\sigma = \sigma_P \oplus \sigma_Q$ and $\tau = \tau_P \oplus \tau_Q$ in $\Delta P \oplus Q$. If $\dim \sigma_P = \dim \tau_P = d$ and $\sigma_P \preceq \tau_P$, then $\Lambda(\sigma_P) \subseteq \Lambda(\tau_P)$. The analogous statement holds if $\dim \sigma_Q = \dim \tau_Q = e$.

**Proof.** If $\sigma_P = \tau_P$ there is nothing to show. As $\sigma_P \preceq \tau_P$, we know that $0 \notin \tau_P$. Then Lemma 2.6 yields $r \in \mathbb{R}^d \times \{0\}$ and $0 \leq \lambda < \mu$ with $\lambda r \in \sigma_P$ and $\mu r \in \tau_P$. We also know that both $\Lambda(\sigma_P)$ and $\Lambda(\tau_P)$ are $e$-dimensional.

Suppose that there exists a point $x \in \Lambda(\sigma_P) \setminus \Lambda(\tau_P)$. Since $0 \in \Lambda(\tau_P)$ and $\Lambda(\tau_P)$ is full-dimensional in $\{0\} \times \mathbb{R}^e$ and strictly star-shaped with respect to 0 by Proposition 3.2, we obtain a multiple $s = \lambda x \in \Lambda(\tau_P)$, and finally a multiple $\nu s \in \Lambda(\sigma_P)$ for some $\nu > 1$. Thus, we have found a cell $\tau_Q \in \Lambda(\tau_P)$ with $s \in \tau_Q$, and a cell $\sigma_Q \in \Lambda(\sigma_P)$ with $\nu s \in \sigma_Q$; see Figure 2b.

If $0 \notin \sigma_P$ then as in the proof of Proposition 3.2 we use Lemma 3.1 to show that the cells $\sigma = \sigma_P \oplus \sigma_Q$ and $\tau = \tau_P \oplus \tau_Q$ have a non-proper intersection. As we find a non-linear hyperplane which separates $\sigma_P$ and $\tau_P$, we know that $\lambda r \notin \tau_P$ or $\mu r \notin \sigma_P$. Assume that $\lambda r \notin \tau_P$, then every proper convex combination of a point in $\sigma_P$ with $\lambda r$ should not lie in $\tau_P$. This contradicts Lemma 3.1.

On the other hand, if $0 \in \sigma_P$ then $\sigma$ and $\tau$ still do not intersect in a common face. The line segment between the origin and $\nu s$ lies completely in $\sigma$, but only a part of it lies in $\tau$: this holds since neither 0 nor $\nu s$ are contained in $\tau$ but $s \in \tau$ and $\nu > 1$. Hence, $\sigma$ and $\tau$ do not intersect properly.

**3.2. Complementarity.** To establish the complementarity property we need to compare the cone $\Lambda$ with the star of the origin in $\Delta P \oplus Q$. Note that by Equation (2.1), the simplices $\sigma_P$ and $\sigma_Q$ need not be disjoint, but their intersection $\sigma_P \cap \sigma_Q$ is either empty or just the origin.
Lemma 3.4. Let $\sigma = \sigma_P \oplus \sigma_Q$ be a full-dimensional cell in $\text{st}_{\triangle_{P \oplus Q}}(0)$.

(1) If $\sigma_P \cap \sigma_Q \neq \emptyset$ then that intersection contains the origin only, and 0 is a vertex of $\sigma$, $\sigma_P$ and $\sigma_Q$.

(2) If $\sigma_P \cap \sigma_Q = \emptyset$ then 0 is not a vertex of $\sigma$, and either $0 \in \sigma_P$ or $0 \in \sigma_Q$.

Proof. Suppose that $\sigma_P$ and $\sigma_Q$ intersect non-trivially. Then the intersection can only contain the origin as that is the only point which the linear subspaces $\mathbb{R}^d$ and $\mathbb{R}^e$ have in common. Since $\sigma_P$ and $\sigma_Q$ both are faces of the triangulation $\triangle_{P \oplus Q}$ they need to intersect properly. It follows that 0 is a vertex of both $\sigma_P$ and $\sigma_Q$. Hence it is also a vertex of $\sigma$.

Now let $\sigma_P \cap \sigma_Q = \emptyset$. Then $\sigma_P$ and $\sigma_Q$ span mutually skew affine subspaces of $\mathbb{R}^{d+e}$, and $\sigma$ is an affinely isomorphic image of the affine join of $\sigma_P$ and $\sigma_Q$. Yet $\sigma_P$ and $\sigma_Q$ are also contained in linear subspaces, $\mathbb{R}^d$ and $\mathbb{R}^e$, which are complementary. This implies that $\sigma_P$ or $\sigma_Q$ must contain the origin. They cannot both contain 0 since their intersection is empty. If 0 were a vertex of $\sigma$ it would need to be a vertex of both $\sigma_P$ and $\sigma_Q$. □

In the case (2) of Lemma 3.4 we have $0 \in \partial \sigma$, as $\sigma_P$ and $\sigma_Q$ are faces of $\sigma$.

Proposition 3.5. If $0 \in \sigma_P$ holds for one full-dimensional cell $\sigma$ in the star of the origin, then $0 \in \tau_P$ for every full-dimensional cell $\tau \in \text{st}_{\triangle_{P \oplus Q}}(0)$. Moreover, $\dim \tau_P = d$. The analogous statements hold if $0 \in \sigma_Q$.

Proof. If the origin is a vertex in $\triangle_{P \oplus Q}$, the statement is trivial because $\dim \sigma_P = d$ and $\dim \sigma_Q = e$ for every $\sigma \in \text{st}_{\triangle_{P \oplus Q}}(0)$ by Observation 2.1.

Suppose there exist full-dimensional cells $\sigma$ and $\tau$ in $\text{st}_{\triangle_{P \oplus Q}}(0)$ with $0 \in \sigma_P$, $\tau_Q$ and $0 \notin \sigma_Q$, $\tau_P$. Because 0 lies on the boundary of both $\sigma$ and $\tau$, there exist minimal faces $\sigma_0 \subseteq \sigma_P$ and $\tau_0 \subseteq \tau_Q$ which contain the origin. Because of Lemma 3.4 and the fact that $0 \notin \text{Vert } \sigma$ and $0 \notin \text{Vert } \tau$, we know that $\sigma_0 \neq \{0\}$ and $\tau_0 \neq \{0\}$. We have $\sigma_0 \cap \tau_0 = \{0\}$ as $\sigma_0$ and $\tau_0$ lie in orthogonal linear subspaces, and therefore $\sigma$ and $\tau$ do not intersect in a common face. This is absurd since we started with a triangulation $\triangle_{P \oplus Q}$.

Suppose that the second assertion does not hold. Then $0 \in \sigma_P$ but $\dim \sigma_P = d-1$. This implies that $\dim \sigma_Q = e$, and $\Lambda(\sigma_Q) \subseteq \triangle_{P \oplus Q}$ is a sphere which contains the origin. This is a contradiction to Proposition 3.2. □

Corollary 3.6. All full-dimensional simplices $\sigma = \sigma_P \oplus \sigma_Q \in \text{st}_{\triangle_{P \oplus Q}}(0)$ satisfy $0 \in \sigma_P$, or they all satisfy $0 \in \sigma_Q$. Both conditions are satisfied simultaneously if and only if the origin occurs as a vertex of $\sigma$.

The previous result says that the origin always lies in the “$\sigma_P$-part” or always in the “$\sigma_Q$-part”, independent of the choice of the cell $\sigma$. We say that the origin lies in the $P$-part or in the $Q$-part of the triangulation, respectively.

Proposition 3.7. Let $\sigma = \sigma_P \oplus \sigma_Q$ and $\tau = \tau_P \oplus \tau_Q$ both be full-dimensional cells in $\text{st}_{\triangle_{P \oplus Q}}(0)$. If $0 \in \sigma_P$, then $0 \in \tau_P$ and $\Lambda(\sigma_P) = \Lambda(\tau_P)$. The analogous statements hold if $0 \in \sigma_Q$.

Proof. Assuming $0 \in \sigma_P$, we can infer $0 \in \tau_P$ from Proposition 3.5 and Corollary 3.6. So it remains to show that $\Lambda(\sigma_P) = \Lambda(\tau_P)$.

Remember that $\triangle_{P \oplus Q}|_{\Lambda(\sigma_P)} \subseteq \text{E}_{\Delta_{P \oplus Q}}(0)$ according to Proposition 3.2. Suppose that $\Lambda(\sigma_P) \neq \Lambda(\tau_P)$. Then there exists a ray which intersects $\Lambda(\sigma_P)$
and $\Lambda(\tau_P)$ in different points, say $x$ and $y = \lambda x$ for $\lambda > 0$. This gives rise to two full-dimensional cells $\sigma', \tau' \in \text{st}_{P \oplus Q}(0)$, with $x \in \sigma'$ and $y \in \tau'$. These cells do not intersect in a common face, as the line segment between 0 and $x$ is contained in $\sigma'$ and the line segment between 0 and $\lambda x$ is contained in $\tau'$. The minimal faces which contain those line segments cannot intersect properly, since only the shorter one of those line segments is contained in both minimal faces. This contradiction allows us to refute the assumption $\Lambda(\sigma_P) \neq \Lambda(\tau_P)$, and hence the claim.

We now treat the remaining cells to finally show the complementarity property of $\Lambda$.

**Proposition 3.8.** Let $\sigma = \sigma_P \oplus \sigma_Q$ and $\tau = \tau_P \oplus \tau_Q$ be cells in $\Delta_{P \oplus Q}^{d+e}$ such that $\dim \sigma_P = d$, $\dim \tau_Q = e$, and at least one of the two simplices $\sigma$ or $\tau$ does not contain the origin. Then

$$\tau_Q \subseteq \bar{\Lambda}(\sigma_P) \iff \sigma_P \not\subseteq \bar{\Lambda}(\tau_Q).$$

**Proof.** For each $\gamma \in \{\sigma_P, \tau_Q\}$, we set

$$\alpha(\gamma) := \Delta_{P \oplus Q}|_{\bar{\Lambda}(\gamma)},$$

so that trivially $\tau_Q \subseteq \bar{\Lambda}(\sigma_P)$ if and only if $\tau_Q \in \alpha(\sigma_P)$, and $\sigma_P \subseteq \bar{\Lambda}(\tau_Q)$ if and only if $\sigma_P \in \alpha(\tau_Q)$.

To reach a contradiction, suppose that $\tau_Q \in \alpha(\sigma_P)$ and $\sigma_P \in \alpha(\tau_Q)$. We choose some non-zero point $x$ in $\text{relint}(\sigma_P)$. Then the ray spanned by $x$ intersects the boundary of $\alpha(\tau_Q)$ exactly once, since $\alpha(\tau_Q)$ is a strictly star shaped ball. This intersection point can be written as $\lambda x$ for precisely one $\lambda > 1$; see Figure 3a. Next we choose some maximal cell $\tilde{\tau}_P \in \partial \alpha(\tau_Q)$ that contains $\lambda x$, and some $y \in \text{relint}(\tau_Q)$ with $\mu y \in \alpha_Q \in \partial \alpha(\sigma_P)$ for $\mu > 1$; see Figure 3b.

Note that the cell $\sigma_P \oplus \tilde{\sigma}_Q \in \Delta_{P \oplus Q}$ contains $x$ and $\mu y$. Since $x$ lies in the relative interior of $\sigma_P$, all proper convex combinations $tx + (1 - t)\mu y$ for $t \in (0, 1)$ lie in the relative interior of that cell. The same holds for $y$ and $\lambda x$ and $\tilde{\tau}_P \oplus \tau_Q$.

Applying Lemma 3.1 we find a point which lies in the interior of those two cells, which violates the intersection property of the triangulation $\Delta_{P \oplus Q}$; compare Figure 2b. A similar argument, with $\sigma_P$ outside of $\Lambda(\tau_Q)$ and $\tau_Q$ outside of $\Lambda(\sigma_P)$, works for the remaining case $\sigma_P \not\in \alpha(\tau_Q)$ and $\tau_Q \not\in \alpha(\sigma_P)$.

### 4. Webs of Stars and Sum-Triangulations

Inspired by the properties of $\bar{\Lambda}$ established in the previous section, we now define webs of stars. This key concept will tell us how to combine the cells of triangulations of the summands to construct triangulations of a free sum.

**Definition 4.1** (web of stars). For any pair of triangulations $\Delta_P$ and $\Delta_Q$ of point configurations $P \subseteq \mathbb{R}^d$ and $Q \subseteq \mathbb{R}^e$, a web of stars in $\Delta_Q$ with respect to $\Delta_P$ is an order preserving map

$$\alpha : (\Delta_P^{=d}, \leq) \to (\mathcal{B}_{\Delta_Q}, \subseteq).$$
Two webs of stars $\alpha : \triangle_{P}^{\text{sd}} \rightarrow \mathcal{B}_{\triangle_{Q}}$ and $\beta : \triangle_{Q}^{\text{sd}} \rightarrow \mathcal{B}_{\triangle_{P}}$ are compatible if
\[ \sigma \in \beta(\tau) \iff \tau \notin \alpha(\sigma) \text{ for every } \sigma \in \triangle_{P}^{\text{sd}} \text{ and } \tau \in \triangle_{Q}^{\text{sd}}. \]

**Observation 4.2.** The condition (4.1) directly shows that for each web of stars $\alpha$ there is at most one web of stars $\beta$, in the reverse direction, which is compatible. Essentially, the process of going from $\alpha$ to $\beta$ consists of transposing and complementing the incidence matrix corresponding to $\alpha$.

**Example 4.3.** Consider the following two point configurations in $\mathbb{R}^2$:
\[ P = \{ (1,0), (0,1), (-1,1), (-1,0), (1,-1), (0,0) \}, \]
\[ Q = \{ (-1,-1), (0,-1), (1,-1), (-1,0), (0,0), (1,0), (-1,1), (0,1) \}. \]

Figure 4 shows possible triangulations $\Delta_{P}$ and $\Delta_{Q}$ and the strictly star shaped ball associated to $\sigma_1$.

Here, e.g., $\sigma_1 \mapsto \langle \tau_1, \tau_2, \tau_3 \rangle_{\Delta_{Q}}$ indicates that $\alpha$ maps $\sigma_1$ to the subcomplex of $\Delta_{Q}$ induced by $\tau_1$, $\tau_2$ and $\tau_3$. We observe that $\alpha$ and $\beta$ satisfy the following:
They send faces to strictly star shaped balls with respect to the origin. Hence the name “web of stars”.

They are order preserving. For instance, \( \sigma_2 \preceq \sigma_1 \) and \( \alpha(\sigma_2) \subseteq \alpha(\sigma_1) \). On the other hand, as \( \sigma_1 \) and \( \sigma_3 \) are not comparable, and neither are their images.

As far as the compatibility condition is concerned, e.g., look at \( \tau_2 \). This face is contained in the image of each face under \( \alpha \), and this agrees with \( \beta(\tau_2) = \emptyset \).

Now we can describe our main construction.

**Definition 4.4** (sum-triangulation). A triangulation \( \Delta_{P \oplus Q} \) of the free sum \( P \oplus Q \) is called a \( P \)-sum-triangulation of \( \Delta_P \) and \( \Delta_Q \) if there exists a compatible pair of webs of stars \( \alpha : \Delta_P^\circ \to B_{\Delta_Q} \) and \( \beta : \Delta_Q^\circ \to B_{\Delta_P} \) with the following properties:

1. The \( d \)-simplices in the star of 0 of \( P \) are sent to the entire star of 0 in \( Q \):
   \[
   \alpha(\sigma_P) = \text{st}_{\Delta_Q}(0) \quad \text{for every } \sigma_P \in \text{st}_{\Delta_P}(0)^{=d}.
   \]

   This special role of \( P \) motivates the name “\( P \)-sum-triangulation”.

2. The set of all full-dimensional simplices in \( \Delta_{P \oplus Q} \) is obtained by summing each simplex \( \sigma_P \subset P \) with all simplices in the boundary of its associated star-shaped ball \( \alpha(\sigma_P) \), and (almost) vice versa. More precisely,
   \[
   \Delta_{P \oplus Q} = \bigcup_{\sigma_P \in \Delta_P^d} \{ \sigma_P \oplus \sigma_Q \mid \sigma_Q \in \partial \alpha(\sigma_P)^{=e-1} \}
   \]
   \[
   \cup \bigcup_{\tau_Q \in \Delta_Q^{=e}} \{ \tau_P \oplus \tau_Q \mid \tau_P \in \partial \beta(\tau_Q)^{=d-1} \}.
   \]

This union is not as asymmetric as it seems: Condition (1) and the fact that \( \alpha \) is an order preserving map imply that \( \alpha(\sigma_P) \neq \emptyset \) always holds.

If the roles of \( P \) and \( Q \) are switched we call \( \Delta_{P \oplus Q} \) a \( Q \)-sum-triangulation. The triangulation \( \Delta_{P \oplus Q} \) is a sum-triangulation of \( \Delta_P \) and \( \Delta_Q \) if it is a \( P \)-sum-triangulation or a \( Q \)-sum-triangulation.

**Remark 4.5.** If \( \Delta_{P \oplus Q} \) is a \( P \)-sum triangulation, then the compatibility and the preservation of the order imply that \( \beta(\tau_Q) = \emptyset \) for each \( \tau_Q \in \text{st}_{\Delta_Q}(0)^{=e} \).

**Example 4.6.** Consider Example 4.3 again. The webs of stars \( \alpha \) and \( \beta \) satisfy condition (1) of Definition 4.4, so they yield a \( P \)-sum-triangulation \( \Delta_{P \oplus Q} \) via condition (2). Some 4-dimensional simplices in \( \Delta_{P \oplus Q} \) are the convex hull of \( \sigma_2 \) and every facet of \( \tau_2 \), where \( \sigma_2 \) and \( \tau_2 \) are embedded in the appropriate linear subspaces of \( \mathbb{R}^4 \). The triangulation which arises in this way has 24 facets and 11 vertices.

**Example 4.7.** Consider the point sets \( P = \{-1, 0, 1, 2\} \) and \( Q = \{-1, 0, 1\} \) in the real line \( \mathbb{R}^1 \). Every triangulation of \( P \oplus Q \) is a sum-triangulation. Figure 5 lists them all, together with the corresponding triangulations \( \Delta_P \) and \( \Delta_Q \) as well as the compatible webs of stars \( \alpha \) and \( \beta \). From the picture we see that (a), (b), (c) and (e) are \( P \)-sum-triangulations, whereas (d) and (f) are \( Q \)-sum-triangulations.
Figure 5. All triangulations of the sum $P \oplus Q$ where the summands are $P = \{-1, 0, 1, 2\}$ and $Q = \{-1, 0, 1\}$, with a representation as a sum-triangulation. (a), (b), (c) and (e) are $P$-sum-triangulations, whereas (d) and (f) are $Q$-sum-triangulations.
A triangulation need not have a unique representation as a P- or Q-sum triangulation. Consider $\Delta_P = \langle [-1, 0], [0, 1], [1, 2] \rangle$ and $\Delta_Q = \langle [-1, 0], [0, 1] \rangle$. Then the triangulation in (a) of $P \oplus Q$ arises as a P-sum triangulation via the web of stars

$$\alpha([-1, 0]) = \Delta_Q \quad \beta([-1, 0]) = \emptyset$$
$$\alpha([0, 1]) = \Delta_Q \quad \beta([0, 1]) = \emptyset$$
$$\alpha([1, 2]) = \Delta_Q,$$

or as a Q-sum triangulation via the web of stars

$$\tilde{\alpha}([-1, 0]) = \emptyset \quad \tilde{\beta}([-1, 0]) = \text{st}_{\Delta_P}(0)$$
$$\tilde{\alpha}([0, 1]) = \emptyset \quad \tilde{\beta}([0, 1]) = \text{st}_{\Delta_P}(0)$$
$$\tilde{\alpha}([1, 2]) = \Delta_Q.$$

**Example 4.8.** The point configuration $P \oplus Q$ from the previous example can be seen as a bipyramid over $P$. More generally, suppose that one of the summands – say $Q$ – consists of the vertices of a simplex together with the origin as an interior point. Then the poset $B_{\Delta_Q}(0)$ just consists of the empty set $\emptyset$ and the complete triangulation $\Delta_Q$. In this case, for fixed $\Delta_P$ and $\Delta_Q$ there exists only one pair of compatible web of stars that produces a $P$-sum-triangulation, and only one pair that produces a $Q$-sum-triangulation. For the $P$-sum-triangulation it is

$$(\forall \sigma \in \Delta_P) \quad \alpha(\sigma) = \Delta_Q \quad (\forall \tau \in \Delta_Q) \quad \beta(\tau) = \emptyset,$$

while the $Q$-sum-triangulation corresponds to the web of stars

$$(\forall \sigma \in \text{st}_{\Delta_P}(0)) \quad \tilde{\alpha}(\sigma) = \emptyset \quad (\forall \tau \in \Delta_Q) \quad \tilde{\beta}(\tau) = \text{st}_{\Delta_P}(0)$$
$$(\forall \sigma \notin \text{st}_{\Delta_P}(0)) \quad \tilde{\alpha}(\sigma) = \Delta_Q.$$

5. Constructing the triangulations of the summands

In this section we show that any triangulation $\Delta_{P \oplus Q}$ induces (not necessarily unique) triangulations $\Delta_P$ and $\Delta_Q$ of $P$ and $Q$, as well as two webs of stars $\alpha : \Delta_P^{=d} \rightarrow B_{\Delta_Q}$ and $\beta : \Delta_Q^{=e} \rightarrow B_{\Delta_P}$ that turn $\Delta_{P \oplus Q}$ into a sum-triangulation. To this end we define the simplicial complexes

$$\tilde{\Delta}_P = \{ \sigma_P \mid \sigma = \sigma_P \oplus \sigma_Q \in \Delta_{P \oplus Q} \}$$
$$\tilde{\Delta}_Q = \{ \sigma_Q \mid \sigma = \sigma_P \oplus \sigma_Q \in \Delta_{P \oplus Q} \}.$$

**Construction 5.1** (Triangulations of $P$ and $Q$ from a triangulation of $P \oplus Q$). We consider two cases. If $0$ is a vertex of $\Delta_{P \oplus Q}$, then $\tilde{\Delta}_P$ and $\tilde{\Delta}_Q$ cover $P$ and $Q$, respectively. We can take $\Delta_P = \tilde{\Delta}_P$ and $\Delta_Q = \tilde{\Delta}_Q$.

Otherwise, by Corollary 3.6, the origin lies in the $P$-part or in the $Q$-part of any cell. By symmetry we may assume that the origin lies in the $P$-part. In this case, for each $\sigma \in \Delta_{P \oplus Q}$, we have

$$\sigma_P = \sigma \cap (P \times \{0\}),$$

and thus $\tilde{\Delta}_P$ covers the entire convex hull of $P$. We let $\Delta_P := \tilde{\Delta}_P$.

What is left to define is $\Delta_Q$. Although $\tilde{\Delta}_Q$ is a simplicial complex, it is not necessarily a triangulation of $Q$. This is because we may have
\[ \sigma_Q \neq \sigma \cap \{0\} \times Q \] for some cell \( \sigma \) in the star of the origin in \( \triangle_{P \oplus Q} \). The region which is not covered is precisely the cone \( \Lambda(\sigma_P) \) for some full dimensional cell \( \sigma = \sigma_P \oplus \sigma_Q \) in \( \text{st}_{\triangle_{P \oplus Q}}(0) \). This is a strictly star shaped ball, which we may refine (in an arbitrary way) to obtain a triangulation \( \triangle_Q \) of \( Q \) with 0 as a vertex. One such example is a placing triangulation with respect to 0.

As the definition of a sum-triangulation Definition 4.4 is inspired by the results we showed in Section 3, we obtain the following result. Recall that the distinction between the \( P \)- and the \( Q \)-part of a free sum triangulation is the topic of Corollary 3.6.

**Theorem 5.2.** Let \( \triangle_{P \oplus Q} \) be any triangulation of \( P \oplus Q \) such that the origin lies in the \( P \)-part. Further, let \( \triangle_P \) and \( \triangle_Q \) be some triangulations obtained by Construction 5.1. Then the maps defined by

\[
\begin{align*}
\alpha(\sigma_P) &:= \triangle_{P \oplus Q}|_{\Lambda(\sigma_P)}, \\
\beta(\sigma_Q) &:= \begin{cases}
\emptyset & \text{if } 0 \in \text{Vert } \sigma_Q, \\
\triangle_{P \oplus Q}|_{\Lambda(\sigma_Q)} & \text{else}
\end{cases}
\end{align*}
\]

form a compatible pair of webs of stars. In particular, \( \triangle_{P \oplus Q} \) is the \( P \)-sum-triangulation with respect to \( \alpha \) and \( \beta \).

Clearly, if the origin lies in the \( Q \)-part, in the above the roles of \( P \) and \( Q \) are interchanged, and we have a \( Q \)-triangulation of the free sum.

**Proof.** There are three things to show: The maps \( \alpha, \beta \) are

- well-defined, i.e., their images are strictly star-shaped balls (Proposition 3.2);
- order preserving (Proposition 3.3);
- and compatible (the case distinction for \( \beta \) together with Proposition 3.8).

\[ \square \]

6. **Every sum-triangulation is a triangulation of the free sum**

From any two triangulations \( \triangle_P \) and \( \triangle_Q \) of our point configurations \( P \) and \( Q \), we wish to construct sum-triangulations of \( P \oplus Q \). In our description we will end up with a \( P \)-sum-triangulation. To obtain a \( Q \)-sum triangulation the roles of \( P \) and \( Q \) need to be interchanged.

Before we start we need to discuss a notational issue. So far we considered the situation where we already have a triangulation of the free sum. In this case, via the definition (2.1), each simplex in the triangulation of the free sum gives rise to one simplex in each summand. In Section 5 it was shown that this yields triangulations of both summands. If we want to revert this procedure it is clear, however, that not each simplex in \( \triangle_P \) can be matched with any other simplex in \( \triangle_Q \) in order to arrive at a simplex of sum triangulation. For instance, in Figure 5a the 1-simplex \([1, 2]\) in \( \triangle_P \) cannot be paired with the 1-simplex \([0, 1]\) in \( \triangle_Q \). Nonetheless, in order to avoid more cumbersome notation, in this section we use the notation \( \sigma_P \) to denote...
some simplex in $\triangle P$ and $\sigma_Q$ to denote some simplex in $\triangle Q$, even if there is no “common” pre-image $\sigma$ in the free sum triangulation from which both descend via (2.1).

Throughout this section we pick a web of stars $\alpha$ in $\triangle Q$ with respect to $\triangle P$ that satisfies the condition (1) of Definition 4.4. The map $\beta : \triangle Q^e \to \mathcal{B}_{\triangle P}$ constructed from $\alpha$ via Observation 4.2 automatically satisfies the compatibility condition with respect to $\alpha$. We assume that $\beta$ is itself a web of stars. To have a concise name for this situation we say that the web of stars $\alpha$ is proper.

The following example shows that proper webs of stars always exist.

**Example 6.1.** For any triangulations $\triangle P, \triangle Q$, we can choose

$$
\alpha(\sigma_P) := \text{st}_{\triangle Q}(0), \\
\beta(\sigma_Q) := \begin{cases} 
\emptyset & \text{if } \sigma_Q \in \text{st}_{\triangle Q}(0) \\
\triangle P & \text{else}
\end{cases}
$$

for $\sigma_P \in \triangle P$ and $\sigma_Q \in \triangle Q$. To see that $\alpha$ and $\beta$ are compatible webs of stars, first note that $\text{st}_{\triangle Q}(0)$ is a strictly star shaped ball with respect to 0 by Lemma 2.5. The same holds for the entire complex $\triangle P$, because $\bigcup \triangle P = \text{conv } P$ is convex and therefore strictly star shaped with respect to the interior point 0. Moreover, $\alpha$ is order-preserving because it is constant, and $\beta$ is order-preserving because the full dimensional simplices in $\text{st}_{\triangle Q}(0)$ are $\preceq$-minimal, and are mapped to the smallest element $\emptyset \in \mathcal{B}_{\triangle Q}$. The compatibility condition is immediate.

We define

$$
T(\triangle P, \triangle Q, \alpha) := \bigcup_{\sigma_P \in \triangle P^d} \left\{ \sigma_P \oplus \sigma_Q \mid \sigma_Q \in (\partial \alpha(\sigma_P))^{e-1} \right\} \\
\bigcup \bigcup_{\tau_Q \in \triangle Q^e} \left\{ \tau_P \oplus \tau_Q \mid \tau_P \in (\partial \beta(\tau_Q))^{d-1} \right\}
$$

and we abbreviate $\mathcal{T} = T(\triangle P, \triangle Q, \alpha)$. Recall that the condition (1) of Definition 4.4 requires that $\alpha(\sigma_P) = \text{st}_{\triangle Q}(0)$ for every $\sigma_P \in \text{st}_{\triangle P}(0)^d$. By construction each cell in $\mathcal{T}$ is a $(d + e)$-simplex. Our goal is to prove that the simplices in $\mathcal{T}$ are the maximal cells of a triangulation of $P \oplus Q$. This requires to show first that any two simplices meet in a common face (which may be empty), and second that they cover the convex hull of the free sum.

6.1. **Any two cells in $\mathcal{T}$ meet in a common face.** For every pair of distinct cells $\sigma_P \oplus \sigma_Q$ and $\tau_P \oplus \tau_Q$ in $\mathcal{T}$ we will find a weakly separating hyperplane $J$ such that

$$(\sigma_P \oplus \sigma_Q) \cap J = (\sigma_P \oplus \sigma_Q) \cap (\tau_P \oplus \tau_Q) = J \cap (\tau_P \oplus \tau_Q).$$

It follows that the cells not intersect in their relative interiors, but in a common (maybe trivial) face with supporting hyperplane $J$. We start out by examining how the stabbing orders in $\triangle P$ and $\triangle Q$ are related.

**Lemma 6.2.** Let $\sigma_P \oplus \sigma_Q$ and $\tau_P \oplus \tau_Q$ be two cells in $\mathcal{T}$. If $\sigma_P \prec \tau_P$, then either $\sigma_Q \preceq \tau_Q$ holds or $\sigma_Q$ and $\tau_Q$ are not comparable.
As usual the roles of $P$ and $Q$ may be interchanged.

Proof. Suppose that $\sigma_P \prec \tau_P$ and $\sigma_Q \succ \tau_Q$. If $\dim \sigma_P = \dim \tau_P = d$ then $\alpha(\sigma_P) \subseteq \alpha(\tau_P)$ since $\alpha$ is order preserving. But because we assume $\sigma_Q \succ \tau_Q$, Lemma 2.6 gives a ray $r \in \mathbb{R}^e$ with $\lambda r \in \tau_Q$ and $\mu r \in \sigma_Q$ for some $\mu > \lambda \geq 0$. As $\alpha(\sigma_P)$ and $\alpha(\tau_P)$ are strictly star shaped and $\tau_Q \in \partial \alpha(\tau_P)$, we know that $\mu r \notin \tau_Q$. We arrive at $\mu r \notin 0 \ast \alpha(\tau_P)$, but $\mu r \in \sigma_Q \in \partial \alpha(\sigma_P) \subset \alpha(\sigma_P)$, which contradicts $\alpha(\sigma_P) \subseteq \alpha(\tau_P)$. The same argument works if $\dim \sigma_Q = \dim \tau_Q = e$.

Now suppose that $\dim \sigma_P = d$ and $\dim \tau_Q = e$. As $\alpha(\sigma_P)$ is strictly star shaped and $\sigma_Q$ is a cell in $\alpha(\sigma_P)$, it is clear that $0 \ast \sigma_Q \subseteq 0 \ast \alpha(\sigma_P)$. From Lemma 2.6 and our assumption that $\sigma_Q \succ \tau_Q$ we get a stabbing ray that hits $\tau_Q$ first. Therefore, $\tau_Q$ has a non-empty intersection with $0 \ast \sigma_Q$. Because $\alpha(\sigma_P)$ is a strictly star shaped simplicial complex, this implies $\tau_Q \in \alpha(\sigma_P)$. With the same argument we get $\sigma_P \in \beta(\tau_Q)$, contradicting the compatibility of $\alpha$ and $\beta$. \hfill \Box

Lemma 6.3. Let $\sigma_P \oplus \sigma_Q$ and $\tau_P \oplus \tau_Q$ be two cells in $T$. If $\sigma_Q \neq \tau_Q$ and every separating hyperplane of $\sigma_P$ and $\tau_P$ is linear, there exists a linear hyperplane that separates $\sigma_Q$ and $\tau_Q$. Furthermore, $\dim \sigma_P = \dim \tau_P = d$.

The roles of $P$ and $Q$ may be interchanged.

Proof. The intersection over all separating hyperplanes of $\sigma_P$ and $\tau_P$ is precisely the affine span of $\sigma_P \cap \tau_P$. Hence if every separating hyperplane is linear, we must have $0 \in \text{aff}(\sigma_P \cap \tau_P)$. In particular, either $\sigma \cap \tau = \{0\}$ or there are two distinct points $x \neq y$ in $\sigma_P \cap \tau_P$ that are collinear with the origin.

If $\dim \sigma_P = \dim \tau_P = d$, we claim that $\alpha(\sigma_P) = \alpha(\tau_P)$. Otherwise, due to compatibility of $\alpha$ and $\beta$ (i.e., $\sigma \in \beta(\tau)$ if and only if $\tau \notin \alpha(\sigma)$), there would exist an $e$-dimensional cell $\gamma \in \Delta_Q$ such that $\beta(\gamma)$ contains exactly one of the two cells $\sigma_P, \tau_P$; compare Figure 6. But this would result in $\beta(\gamma)$ not being strictly star shaped, as $0 \in \partial \beta(\gamma)$ or $x, y \in \sigma_P \cap \tau_P \in \partial \beta(\gamma)$, depending on whether $\sigma \cap \tau = \{0\}$ or not.

But now that we know $\alpha(\sigma_P) = \alpha(\tau_P)$, we can always find a linear separating hyperplane for $\sigma_Q$ and $\tau_Q$. Since $\alpha$ maps to strictly star shaped balls, the origin is affinely independent with respect to the vertex set of every cell in $\partial \alpha(\sigma_P)$. Since $\sigma_Q \in \partial \alpha(\sigma_P)$, we know that $\dim \sigma_Q = e - 1$, so that each facet of $\sigma_Q$ linearly spans a hyperplane in $\mathbb{R}^e$. One of those hyperplanes separates $\sigma_Q$ and $\tau_Q$. This concludes the first part of the statement.

As for the second part, we claim that $\dim \sigma_P = \dim \tau_P = d$. From the construction we know that $\dim \sigma_Q = \dim \tau_Q = e - 1$. Aiming at a contradiction, we suppose that $\dim \sigma_Q = e$. This would result in $0 \in \partial \beta(\sigma_Q)$ or $x, y \in \sigma_P \cap \tau_P \in \partial \beta(\sigma_Q)$, which would contradict that $\beta$ maps to strictly star shaped balls. Basically this is a similar argument as above, but $\sigma_Q$ now plays the role of $\gamma$. Thus, $\sigma_P$ and $\tau_P$ must have dimension $d$, as the argument can be repeated if $\tau_Q$ is $e$-dimensional. \hfill \Box
Expressed geometrically, there exist two hyperplanes, one separating
Suppose that
Proof. is (as the right hand side is the same for both hyperplanes).

\[ \sigma \]
\[ \tau \]
\[ d \]
\[ c \]
\[ \beta(\gamma) \]
\[ \alpha(\tau_p) \]
\[ \alpha(\sigma_p) \]

Figure 6. Illustrating the idea of the proof Lemma 6.3, where all separating hyperplanes of \( \sigma_p \) and \( \tau_p \) are linear, so that \( 0 \in \text{aff}(\sigma \cap \tau) \).

**Lemma 6.4.** Let \( \sigma_p \oplus \sigma_Q \) and \( \tau_p \oplus \tau_Q \) be two cells in \( T \). If \( \sigma_p \neq \tau_p \) and \( \sigma_Q \neq \tau_Q \) then there exist \( 0 \neq a \in \mathbb{R}^d \), \( 0 \neq c \in \mathbb{R}^e \) and \( b \in \mathbb{R} \) such that
\[
\sigma_p \subseteq \{ x \in \mathbb{R}^d \mid a^T x \leq b \}, \quad \tau_p \subseteq \{ x \in \mathbb{R}^d \mid a^T x \geq b \}, \\
\sigma_Q \subseteq \{ y \in \mathbb{R}^e \mid c^T y \leq b \}, \quad \tau_Q \subseteq \{ y \in \mathbb{R}^e \mid c^T y \geq b \}.
\]
Expressed geometrically, there exist two hyperplanes, one separating \( \sigma_p \) from \( \tau_p \) and the other separating \( \sigma_Q \) from \( \tau_Q \). Both are oriented in the same way, in the sense that \( \sigma_p \) is in the same half space as the origin, if and only if \( \sigma_Q \) is (as the right hand side is the same for both hyperplanes).

**Proof.** Suppose that \( \sigma_p \) and \( \tau_p \) are not comparable. If all hyperplanes separating them are linear, let one of them be given by \( a^T x = 0 \). As Lemma 6.3 gives us the existence of a linear hyperplane \( a^T y = 0 \) separating \( \sigma_Q \) and \( \tau_Q \), we just need to choose the orientation of \( c \) accordingly.

On the other hand, suppose that not every hyperplane separating \( \sigma_p \) and \( \tau_p \) is linear. As \( \sigma_Q \) and \( \tau_Q \) are faces of the triangulation \( \Delta_Q \), we know that \( \sigma_Q \) and \( \tau_Q \) can be separated by a hyperplane \( c^T y = b \) with \( \sigma_Q \subseteq \{ y \in \mathbb{R}^e \mid c^T y \leq b \} \) for some \( c \) and \( b \). As \( \sigma_p \) and \( \tau_p \) are not comparable and not every separating hyperplane is linear, we can find an affine separating hyperplane that leaves \( \sigma_p \) in the same half space as the origin, and another affine hyperplane where \( \sigma_p \) is not in the same half space as the origin. We choose the hyperplane accordingly, depending on whether \( \sigma_Q \) is in the same half space as the origin or not. Thus, for every \( b \in \mathbb{R} \) we can find an \( a \in \mathbb{R}^d \) such that \( a^T x = b \) induces a separating hyperplane with \( \sigma_p \subseteq \{ x \in \mathbb{R}^d \mid a^T x \leq b \} \).

Switching the roles of \( P \) and \( Q \) and repeating the argument above yields a proof if \( \sigma_Q \) and \( \tau_Q \) are not comparable. This leaves us with the case that \( \sigma_p \) and \( \tau_p \) as well as \( \sigma_Q \) and \( \tau_Q \) are comparable.

If \( \sigma_p \preceq \tau_p \) then we know that \( \sigma_p \) and \( \tau_p \) can be separated by a hyperplane \( a^T x = b \) with \( \sigma_p \subset \{ x \in \mathbb{R}^d \mid a^T x \leq b \} \) for some \( a \) and \( b > 0 \). From Lemma 6.2 we obtain that either \( \sigma_Q \preceq \tau_Q \) or that \( \sigma_Q \) and \( \tau_Q \) are not comparable. The latter case is already discussed by the arguments above. So assume that \( \sigma_Q \preceq \tau_Q \), this means that they can be separated by a hyperplane \( c^T y = d \) with \( \sigma_Q \subseteq \{ y \in \mathbb{R}^e \mid c^T y \leq d \} \) for some \( c \) and \( d > 0 \). We choose \( d = b \) by scaling \( c \) appropriately. \( \square \)
Lemma 6.4 shows that choosing a pair of compatible web of stars is quite restrictive. The fact that web of stars are order preserving and map to strictly star shaped balls are the main ingredients used in the proof. To make things a bit more clear: Starting with two triangulations $\triangle_P$ and $\triangle_Q$, every two cells belonging to the same triangulation can be separated by some hyperplane. But when constructing $\mathcal{T}$, like in Construction 5.1, the web of stars make sure that that something similar to Figure 2 cannot happen. Meaning, that we can always orient the two separating hyperplanes in the “same” way, in the sense that $\sigma_P \in \triangle_P$ and $\sigma_Q \in \triangle_Q$ do always end up in the same half space with respect to the origin and some other cell $\tau \in \mathcal{T}$. It can be seen as a consequence or an extension of Lemma 6.2.

With this fact we have all ingredients together to prove the intersection property of $\mathcal{T}$. The idea is to use the two separating hyperplanes from Lemma 6.4 and using that everything is oriented properly, to construct a “big” hyperplane which separates $\sigma, \tau \in \mathcal{T}$.

**Proposition 6.5.** Any two cells in $\mathcal{T}$ intersect in a common face.

**Proof.** Let $\sigma = \sigma_P \oplus \sigma_Q$ and $\tau = \tau_P \oplus \tau_Q$ be two simplices in $\mathcal{T}$.

If $\sigma = \tau$ there is nothing to show.

If $\sigma_P \neq \tau_P$ and $\sigma_Q \neq \tau_Q$, Lemma 6.4 yields $H = \{ x \in \mathbb{R}^d \mid a^T x = b \}$ and $G = \{ x \in \mathbb{R}^e \mid c^T x = b \}$ with $0 \neq a \in \mathbb{R}^d$, $0 \neq c \in \mathbb{R}^e$, and $b \in \mathbb{R}$ such that

\[
\begin{align*}
\sigma_P &\subseteq \{ x \in \mathbb{R}^d \mid a^T x \leq b \}, & \tau_P &\subseteq \{ x \in \mathbb{R}^d \mid a^T x \geq b \}, \\
\sigma_Q &\subseteq \{ y \in \mathbb{R}^e \mid c^T y \leq b \}, & \tau_Q &\subseteq \{ y \in \mathbb{R}^e \mid c^T y \geq b \}.
\end{align*}
\]

(6.2)

If $\sigma_P = \tau_P$, we choose $a = 0$. Since $\sigma_Q \neq \tau_Q$ we can find a separating hyperplane induced by the equation $c^T y = b$ for some $c \neq 0$. Without loss of generality we can assume that $\sigma_Q \subseteq \{ y \in \mathbb{R}^e \mid c^T y \leq b \}$ as we can change the sign of $c$ and $b$.

If $\sigma_Q = \tau_Q$ we choose $c = 0$ and find a separating hyperplane induced by the equation $a^T x = b$ for some $a \neq 0$. Again, without loss of generality we can assume that $\sigma_P \subseteq \{ x \in \mathbb{R}^d \mid a^T x \leq b \}$ as we can change the sign of $a$ and $b$.

In all cases, we have found – not necessarily unique – $a, c$ and $b$ that make Equation (6.2) valid.

Next, we perturb and scale $a$ and $c$ such that

\[
\begin{align*}
\sigma_P \cap H = \sigma_P \cap \tau_P = H \cap \tau_P, & & \sigma_Q \cap G = \sigma_Q \cap \tau_Q = G \cap \tau_Q.
\end{align*}
\]

(6.3)

Note that $a = 0$ if and only if $\sigma_P = \tau_P$, and $c = 0$ if and only if $\sigma_Q = \tau_Q$, and that our assumption $\sigma \neq \tau$ rules out the case $a = c = 0$. We may also assume that $b \geq 0$, otherwise we switch the roles of $\sigma$ and $\tau$. However, we do not assume that the intersections $\sigma_P \cap \tau_P$ and $\sigma_Q \cap \tau_Q$ are non-empty. If one of the intersections is empty, we perturb, scale, and translate $H$ or $G$ such that Equation (6.3) is valid.

Because $(a^T, c^T) \neq 0$, the set

\[
J := \{ x \in \mathbb{R}^{d+e} \mid (a^T, c^T) x = b \}
\]

is a hyperplane in $\mathbb{R}^{d+e}$. We claim that $J$ separates $\sigma$ from $\tau \in \mathcal{T}$, and that $\sigma \cap J = \sigma \cap \tau = J \cap \tau$. From the definition of $\sigma$ it is clear that every
point \( s \in \sigma \) can be written as a convex combination of points in \( \sigma_P \) and \( \sigma_Q \), while the definition of \( J \) implies that \((a^T, c^T)s \leq b\) for all \( s \in \sigma \), and that \((a^T, c^T)t \geq b\) for every \( t \in \tau \).

If the two intersections \( \sigma_P \cap \tau_P \) and \( \sigma_Q \cap \tau_Q \) are empty, then we get \((a^T, c^T)s < b\) for all \( s \in \sigma \), and that \((a^T, c^T)t > b\) for every \( t \in \tau \). Thus, \( J \) strictly separates \( \sigma \) and \( \tau \).

Suppose there is a point \( x \in (\sigma \cap \tau) \setminus J \). Since \( x \) is in both \( \sigma \) and \( \tau \) it can be written both as a convex combination of vertices of \( \sigma_P \) and \( \sigma_Q \), and as a convex combination of vertices of \( \tau_P \) and \( \tau_Q \),

\[
x = \sum_{v \in \text{Vert } \sigma} \lambda_v v = \sum_{w \in \text{Vert } \tau} \mu_w w.
\]

In order for \( x \notin J \) to hold, there must exist at least one vertex \( v \in \sigma_P \) or in \( \sigma_Q \) with \( \lambda_v > 0 \), and \((a^T, 0)v < b\) if \( v \in \sigma_P \times \{0\} \), or \((0, c^T)v < b\) if \( v \in \{0\} \times \sigma_Q \). In both cases, together with \((a^T, c^T)x \leq b\) and \( \sum_v \lambda_v = 1 \) this yields \((a^T, c^T)x < b\). Hence \( x \notin J \) by Equation (6.2), which contradicts \( x \in \sigma \cap \tau \).

On the other hand, for a point \( x \in \sigma \cap J \) that satisfies Equation (6.3), a similar argument shows that every vertex \( v \in \sigma \) with \( \lambda_v > 0 \) must satisfy \((a^T, 0)v = b\) if \( v \in \sigma_P \times \{0\} \), or \((0, c^T)v = b\) if \( v \in \{0\} \times \sigma_Q \) and therefore has to be in \( \tau \). \qed

6.2. The cells in \( \mathcal{T} \) cover the free sum. We continue with the same notation.

**Proposition 6.6.** We have \( \bigcup_{\sigma \in \mathcal{T}} \sigma = \text{conv}(P \oplus Q) \).

**Proof.** We will show that every facet \( F \) of any full-dimensional simplex \( \sigma = \sigma_P \oplus \sigma_Q \in \mathcal{T} \) is covered by another simplex in \( \mathcal{T} \), unless \( F \subset \partial \mathcal{T} \).

Without loss of generality we may assume that \( \dim \sigma_P = d \), and that \( F = \text{conv}(\text{Vert } \sigma \setminus \{x\}) \). We distinguish whether the vertex \( x \) we removed is a vertex of \( \sigma_P \) or of \( \sigma_Q \).

Suppose \( x \in \text{Vert } \sigma_Q \). By the definition of \( \mathcal{T} \) we know that \( \sigma_Q \in \partial \alpha(\sigma_P) \). As the boundary complex of \( \alpha(\sigma_P) \) is a triangulated sphere, there exists a neighboring simplex

\[
\tilde{\sigma}_Q = \text{conv} \left( \text{Vert } \sigma_Q \setminus \{x\} \cup \{y\} \right) \in \partial \alpha(\sigma_P)
\]

of \( \sigma_Q \) in \( \partial \alpha(\sigma_P) \) with respect to \( x \). Thus, \( \sigma_P \oplus \tilde{\sigma}_Q \) is a simplex in \( \mathcal{T} \), which also covers the facet \( F \).

So let \( x \in \text{Vert } \sigma_P \). The two cells \( f := \text{conv}(\text{Vert } \sigma_P \setminus \{x\}) \) and \( \sigma_Q \) are both codimension 1 faces in the triangulations \( \Delta_P \) and \( \Delta_Q \) respectively. Since \( F \notin \partial \mathcal{T} \), at most one of the cells \( f, \sigma_Q \) can lie on \( \partial \Delta_P \) or \( \partial \Delta_Q \). We distinguish several cases; see Figure 7 for an illustration of the basic setup.

1. Suppose \( f \in \partial \Delta_P \), so that \( f \in \partial \mathcal{T} \). The codimension 1 cell \( \sigma_Q \) is contained in exactly two adjacent full-dimensional cells in \( \Delta_Q \), and one of these, say \( O \), satisfies \( O \notin \alpha(\sigma_P) \). By the compatibility of \( \alpha \) and \( \beta \), we conclude that \( \sigma_P \in \beta(O) \). This and the fact that \( f \in \partial \mathcal{T} \) implies that \( f \in \partial \beta(O) \). Hence \( f \oplus O \) is a full-dimensional cell of \( \mathcal{T} \) which contains \( F \) as a facet.
(2) If \( f \notin \partial \Delta_P \), we can find a neighboring simplex
\[
\tau = \text{conv} \left( \text{Vert } \sigma_P \setminus \{x\} \cup \{v\} \right) \in \Delta_P
\]
which shares the facet \( f \) with \( \sigma_P \). Depending on whether or not \( \sigma_Q \in \partial \Delta_Q \), we know that \( \sigma_Q \) is contained in at least one and at most two adjacent full-dimensional cells of \( \Delta_Q \). One of these (the inner cell \( I \)) is contained in \( \alpha(\sigma_P) \) and must exist; the other one (the outer cell \( O \)) is not contained in \( \alpha(\sigma_P) \) and might exist or not.

(a) Let us assume that \( O \) exists. By the compatibility of \( \alpha \) and \( \beta \), from \( I \in \alpha(\sigma_P) \) we conclude \( \sigma_P \notin \beta(I) \), and from \( O \notin \alpha(\sigma_P) \) that \( \sigma_P \in \beta(O) \).

**Figure 8** illustrates the four possible cases:

(i) If \( I \notin \alpha(\tau) \), compatibility yields \( \tau \in \beta(I) \). Since moreover \( \sigma_P \notin \beta(I) \), the facet \( f = \sigma_P \cap \tau \) must be part of \( \partial \beta(I) \). Hence, the simplex \( f \oplus I \in \mathcal{T} \) contains \( F \). See Figure 8a.

(ii) If \( I,O \in \alpha(\tau) \), compatibility and \( O \in \alpha(\tau) \) yield \( \tau \notin \beta(O) \). Since moreover \( \sigma_P \in \beta(O) \), the facet \( f = \sigma_P \cap \tau \) must be part of \( \partial \beta(O) \). Hence, the simplex \( f \oplus O \in \mathcal{T} \) contains \( F \). See Figure 8b.

(iii) If \( I \in \alpha(\tau) \) and \( O \notin \alpha(\tau) \), the shared facet \( \sigma_Q \) of \( I \) and \( O \) must be part of both \( \partial \alpha(\tau) \) and \( \partial \alpha(\sigma_P) \). Therefore, \( \tau \oplus \sigma_Q \) is a simplex in \( \mathcal{T} \) and contains \( F \). See Figure 8c.

(b) If \( O \) does not exist, we proceed as in 2(a)i and 2(a)iii. For the latter case, we use that \( \sigma_Q \in \partial \Delta_Q \) and therefore \( \partial \alpha(\tau) \) and \( \partial \alpha(\sigma_P) \) both contain \( \sigma_Q \).

We are finally ready to establish our second main result.

**Theorem 6.7.** Let \( \Delta_P \) and \( \Delta_Q \) be triangulations of the point configurations \( P \) and \( Q \), respectively. Further, let \( \alpha : \Delta_P^d \to B_{\Delta_Q} \) be a web of stars which is proper. Then the \((d+e)\)-simplices in \( \mathcal{T}(\Delta_P, \Delta_Q, \alpha) \), as defined in (6.1), generate a \( P \)-sum-triangulation of the free sum \( P \oplus Q \).

**Proof.** Proposition 6.5 shows that the \( \mathcal{T} = \mathcal{T}(\Delta_P, \Delta_Q, \alpha) \) generates a finite simplicial complex. By Proposition 6.6 this simplicial complex covers the entire convex hull of the points in \( P \oplus Q \). □
Figure 8. Illustrating the case distinction in the proof of Proposition 6.6, where the summand $P$ is always on the left and $Q$ is on the right.

7. Application: Fano polytopes

The theory of toric varieties is an area within algebraic geometry which is especially amenable to combinatorial techniques; see [4]. This is due to the fact that toric varieties can be described in terms of face fans (or, dually, normal fans) of lattice polytopes, i.e., polytopes whose vertices have integral coordinates. Of particular interest are the (smooth) Fano varieties. A full-dimensional lattice polytope $P$ is a smooth Fano polytope if it contains the origin 0 as an interior point, and the vertex set of each facet forms a lattice basis. In particular, smooth Fano polytopes are simplicial. Note that in the literature smooth Fano polytopes are sometimes simple; in that case the polytope $P$ needs to be exchanged with its polar $P^\ast$. In the sequel we will identify a smooth Fano polytope $P$ with its canonical point configuration.
which consists of the vertices of $P$ plus the origin. This is precisely the set of lattice points in $P$.

If $P$ and $Q$ are the canonical point configurations of two smooth Fano polytopes, say of dimensions $d$ and $e$, then the free sum $P \oplus Q$ is the canonical point configuration of a smooth Fano polytope of dimension $d + e$. This is an easy consequence of the fact that each facet of the sum is the affine join of facets, one from each summand. Our results on sum triangulations allow to combine the triangulations of the summands $P$ and $Q$ into a description of all triangulations of the sum $P \oplus Q$. This is particularly interesting since it is conjectured that many smooth Fano polytopes admit a non-trivial splitting [1, Conjecture 9]; a partial solution has been obtained in [2, Theorem 1]. In Table 1 we list the number of Fano polytopes (up to unimodular equivalence) of dimension at most six split by their numbers of free summands. The percentage of decomposable ones goes down with the dimension, but slowly. The diagonal entries in Table 1, reporting one class each of $d$-dimensional Fano polytopes which decompose into $d$ summands, correspond to the cross polytopes $\text{cross}(d)$.

| $d$ | total | 1 | 2 | 3 | 4 | 5 | 6 | decomposable |
|-----|-------|---|---|---|---|---|---|-------------|
| 2   | 5     | 4 | 1 |   |   |   |   | 20%         |
| 3   | 18    | 13| 4 | 1 |   |   |   | 28%         |
| 4   | 124   | 96| 23| 4 | 1 |   |   | 23%         |
| 5   | 866   | 690| 148| 23| 4 | 1 |   | 20%         |
| 6   | 7622  | 6261| 1165| 168| 23| 4 | 1 | 18%         |

We will devote the rest of this section to studying the following example.

**Example 7.1.** The $d$-dimensional del Pezzo polytope $\text{DP}(d)$ is defined as the convex hull of the $d$-dimensional cross polytope and the two additional points $\pm 1$. That is, it has exactly $2d + 2$ vertices, which read $\pm e_1, \pm e_2, \dotsc, \pm e_d, \pm 1$.

By construction the del Pezzo polytopes are centrally symmetric. The *pseudo del Pezzo polytope* $\text{DP}^-(d)$ is the subpolytope of $\text{DP}(d)$ which arises as the convex hull of all its vertices except for $1$. Both, $\text{DP}(d)$ and $\text{DP}^-(d)$, are smooth Fano polytopes, provided that $d$ is even.

In the sequel we will primarily address the canonical point configuration of the free sum $\text{DP}(2) \oplus \text{DP}(4)$. This comprises $10 + 6 + 1 = 17$ lattice points in $\mathbb{R}^6$. We will also consider (the canonical point configuration of) $\text{DP}(2) \oplus \text{DP}^-(4)$, which has only one point less.

Before we continue let us recall the state of the art in the enumeration of triangulations of some point set $P$. The most successful general method starts out with computing one triangulation, say $\Delta$, of $P$, e.g., a *placing triangulation*, obtained from the beneath-and-beyond convex hull algorithm. The second step, which is much more demanding, is to apply a purely
combinatorial procedure to obtain all triangulations of $P$ which are connected to $\Delta$ by a sequence of local modifications known as (bistellar) flips. The algorithm is described in [13] and implemented in TOPCOM [14]. In general, this method will not enumerate all triangulations of $P$ but rather only the regular ones along with those connected to a regular triangulation by a sequence of flips. Except for the naive, i.e., computationally intractable, approach by combinatorial enumeration from all subsets of maximal simplices there is no general method known which produces the entire set of all triangulations of any given point configuration, including the non-regular ones.

It is important that TOPCOM's flip algorithm can take symmetries into account. The group of invertible linear maps which fixes a finite point configuration also acts on the set of all its triangulations. TOPCOM can be given a set of generators of a group as input in addition to the point set; it then produces only one triangulation per orbit of the induced action. Our example is highly symmetric: the group of linear symmetries of $DP(4)$ has order 240, while the group of linear symmetries of $DP(2)$, which is a dihedral group, has order 12. It follows that our point configuration $DP(2) \oplus DP(4)$ of 17 points in dimension six admits a group of order $12 \cdot 240 = 2880$, and it turns out that this is also the entire group of linear symmetries. Yet, with standard hardware of today, it seems to be next to impossible to determine the set of triangulations of $DP(2) \oplus DP(4)$, even just up to symmetry: After nine days worth of CPU time our TOPCOM computation stopped since it reached the imposed memory limit of 26 GB, without arriving at the result.

Table 2. Regular triangulations and homomorphisms encoding compatible pairs of webs of stars for $DP(2) \oplus DP(4)$.

Smaller point configurations are added for comparison

|                  | #triangulations | time | #homomorphisms | time |
|------------------|-----------------|------|----------------|------|
| $DP(2) \oplus DP(2)$ | 204             | 2s   | 1157           | 8s   |
| $DP(2) \oplus cross(4)$ | 13              | 20s  | 16             | 11s  |
| $DP(2) \oplus DP^{-}(4)$ | 250 594         | 2.5h | 1 581 647      | 27s  |
| $DP(2) \oplus DP(4)$ | ?               |      | 1 677 949 075  | 10d  |

Our techniques for triangulations of free sums are constructive, and we made a proof-of-concept implementation in polymake [6], which takes the triangulations of both summands as input. For our example, TOPCOM returns seven triangulations of $DP(2)$ and 128 triangulations of $DP(4)$; where these and all subsequent counts are up to symmetry. The time for this computation is about one minute, of which almost everything is spent on the 4-dimensional point configuration. Then, for each triangulation $\Delta_P$ of the hexagon $DP(2)$, our code computes the stabbing order among the triangles. This takes almost no time. Slightly more costly, with about 20 minutes, is the computation of the web of star poset of a triangulation $\Delta_Q$ of $DP(4)$. The final step is to compute all admissible homomorphisms from the stabbing poset of $\Delta_P$ into the web of sphere poset of $\Delta_Q$, again up to symmetry. This took us 10 days. The total number of triangulations of $DP(2) \oplus DP(4)$ seems to be huge (we feel pretty safe in guessing that it exceeds 100 million). Hence, for lack of
memory, we refrained from explicitly constructing the triangulations. Since one triangulation can be obtained from more than one homomorphism, we arrive at an overcount this way.

Interestingly, to compute the same for the subpolytope $DP(2) \oplus DP^{-1}(4)$, with only one vertexless, is fairly easy. Phrased differently, the smooth Fano polytope $DP(2) \oplus DP(4)$ is an example for which standard techniques fail by a small margin only. It is this realm where our specialized approach seems to be most useful.

Moreover, we believe that the data shown underestimates the potential of our methods for computing triangulations of smooth Fano polytopes. One reason is that very many polytopes which are listed as indecomposable in Table 1 are (possibly iterated) skew bipyramids over lower-dimensional smooth Fano polytopes; see [1, Lemma 3]. We expect that webs of stars and stabbing orders can be applied to their triangulations, too. However, this is beyond our scope here. Moreover, as far as timings are concerned, our proof-of-concept implementation in polymake leaves room for improvements. For instance, not even straightforward parallelization is employed.

8. Odds and ends

8.1. More than two summands. Until now we only considered the free sum of two summands. But since $\oplus$ is associative, we can generalize our results to the free sum of finitely many polytopes

$$P_1 \oplus P_2 \oplus P_3 \oplus \cdots \oplus P_k = (\cdots ((P_1 \oplus P_2) \oplus P_3) \oplus \cdots \oplus P_k).$$

By a repeated application of our characterization the triangulations of the summands $P_1$ up to $P_k$ contain enough information to describe every triangulation of the multiple sum.

8.2. Subfree sum. In [11] McMullen introduced the subfree sum, which generalizes the free sum by allowing the origin to lie on the boundary of the participating polytopes. The results in this paper should largely translate to the subfree sum, but there are some pitfalls. For example, combining a simplex $\sigma_P$ in $\Delta_P$ with the boundary of the web of stars $\alpha(\sigma_P)$ can yield non-proper intersections. One should only combine $\sigma_P$ with some faces on the boundary; using only those cells which “face away” from the origin should yield a correct choice.

But more subtle changes are needed to generalize our results. We suspect that, with some effort, the concepts introduced here can be extended to the case when the origin is not contained in the summands, and to arbitrary subdivisions.

8.3. Regularity. A triangulation is regular (or coherent) if it is induced be a height function. For applications in algebraic geometry such triangulations are the most interesting ones. Therefore, it is of major interest to characterize those triangulations of a free sum which are regular.

Conjecture 8.1 (regularity). A sum-triangulation $\Delta_{P\oplus Q}$ of $P \oplus Q$ determined by $\Delta_P, \Delta_Q$ and compatible webs of stars $\alpha : \Delta_P^{d} \rightarrow B_{\Delta_Q}$ and $\beta : \Delta_Q^{e} \rightarrow B_{\Delta_P}$ is regular if and only if the triangulations of the summands are regular and the images of $\alpha$ and $\beta$ are totally ordered. In other words:
for every pair of cells $\sigma_P, \tau_P \in \triangle_P$, one of the conditions $\alpha(\sigma_P) \subseteq \alpha(\tau_P)$ or $\alpha(\sigma_P) \supseteq \alpha(\tau_P)$ must hold.

Notice that the conjecture deduces a geometric property from a purely combinatorial condition. In fact, what the conjecture implicitly states is that the free sum is geometrically restrictive enough to warrant this, the reason being that the summands are embedded in \textit{mutually orthogonal} linear subspaces. To give an intuition as for why the images of $\alpha$ and $\beta$ should be totally ordered, consider the following example.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{example82.png}
\caption{The non-regular sum triangulation from Example 8.2.}
\end{figure}

\textbf{Example 8.2.} Let $P$ and $Q$ be the same point configuration

$$P = Q = \{-2, -1, 0, 1, 2\},$$

with the same triangulation

$$\triangle_P = \triangle_Q = \langle [-2, -1], [-1, 0], [0, 1], [1, 2] \rangle.$$

Furthermore we define two compatible webs of stars via

\begin{align*}
\alpha : \triangle_P &= \{ [-2, -1], [-1, 0], [0, 1], [1, 2] \} \\
\beta : \triangle_Q &= \{ [-2, -1], [-1, 0], [0, 1], [1, 2] \} \\
\end{align*}

\[(\begin{array}{ll}
[-2, -1] & \mapsto \langle [-2, -1], [0, 1], [1, 2] \rangle \\
([-1, 0], [0, 1], [1, 2]) & \mapsto \langle [-2, -1], [-1, 0], [0, 1], [1, 2] \rangle \\
([-1, 0], [0, 1], [1, 2]) & \mapsto \langle [-2, -1], [-1, 0], [0, 1], [1, 2] \rangle \\
([0, 1], [1, 2]) & \mapsto \langle [-2, -1], [-1, 0], [0, 1], [1, 2] \rangle \\
\end{array})\]

Clearly they do not meet the condition in \textbf{Conjecture 8.1}, because $\alpha([-2, -1])$ and $\alpha([1, 2])$ are not $\subseteq$-comparable. The corresponding sum-triangulation $\triangle_{P \oplus Q}$ (cf. Figure 9) is not regular.

In a more complex situation in higher dimension we might get something similar. Maybe one needs to intersect the simplicial complex with an appropriate lower dimensional subspace to prove non-regularity.
Figure 10. Decision diagram to decide whether or not $\sigma \prec \tau$.

Appendix A. Deciding the comparability in the stabbing order

The flowchart in Figure 10 gives an algorithm to determine whether $\sigma \prec \tau$ holds for two simplices in a simplicial complex. We prove its correctness step by step:

1. Cells containing the origin are $\preceq$-minimal, as they always lie in the same half space as the origin.
2. If $0 \in \sigma \cap \tau$, then $\sigma \not\prec \tau$ since every separating hyperplane is linear.
3. If $0 \in \text{aff}(\sigma \cap \tau)$, then again every separating hyperplane is linear, since it must contain the affine hull of the intersection, and so $\sigma \not\prec \tau$.
4. If $\text{cone} \sigma \cap \tau = \emptyset$ there exists no stabbing ray, as the set of all rays that stab both $\sigma$ and $\tau$ is $\text{cone} \sigma \cap \text{cone} \tau$. But the existence of such a ray is a necessary condition according to Lemma 2.6, and therefore $\sigma \not\prec \tau$.
5. Let $r \in \text{relint}(\text{cone} \sigma \cap \tau)$. The dimension of the intersection of $\sigma$ and the line segment $\overline{0r}$ can either be $-1$, $0$ or $1$.
(a) If \( \dim \overrightarrow{r} \cap \sigma = -1 \), the intersection is empty. But the ray spanned by \( r \) must intersect \( \sigma \), and \( \overrightarrow{r} \cap \sigma = \emptyset \) implies that \( \lambda \sigma \not\subseteq \sigma \) for some \( \lambda > 1 \). Every separating hyperplane must separate \( r \) from \( \lambda \sigma \), implying that \( \tau \) lies on the same half space as the origin, and therefore \( \sigma \) does not precede \( \tau \).

(b) If \( \dim \overrightarrow{r} \cap \sigma = 1 \), there exists \( \lambda < 1 \) with \( \lambda \sigma \subseteq \sigma \). As every separating hyperplane must separate \( r \) from \( \lambda \sigma \) we get that \( \sigma \) always lies in the same half space as the origin. And since \( 0 \not\in \text{aff}(\sigma \cap \tau) \) we also find at least one non-linear hyperplane separating \( \sigma \) and \( \tau \); hence \( \sigma \prec \tau \).

(c) If \( \overrightarrow{r} \cap \sigma = \{x\} \) and \( r \neq x \), then \( x = \lambda r \) for some \( \lambda < 1 \), and the same argument as above yields that \( \sigma \prec \tau \).

(6) For the last step assume that \( r = x \). Then \( r \in \partial \sigma \) and \( \sigma \cap \tau = \{r\} \), and we can find a linear supporting hyperplane of \( \sigma \) which separates \( \sigma \) and \( \tau \). Small perturbations of that hyperplane produce a separating hyperplane with \( \tau \) in the same half space as the origin, and hence \( \sigma \not\prec \tau \).

References

[1] B. Assarf, M. Joswig, and A. Paffenholz. Smooth Fano polytopes with many vertices. Discrete Comput. Geom., 52(2):153–194, 2014.
[2] B. Assarf and B. Nill. Toric Fano manifolds with large Picard number, 2014. preprint arXiv:1408.7303.
[3] V. V. Batyrev. On the classification of toric Fano 4-folds. J. Math. Sci. (New York), 94(1):1021–1050, 1999. doi: 10.1007/BF02367245.
[4] D. A. Cox, J. B. Little, and H. K. Schenck. Toric varieties, volume 124 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2011.
[5] J. De Loera, J. Rambau, and F. Santos. Triangulations: Structures for Algorithms and Applications. Algorithms and Computation in Mathematics. Springer-Verlag, 2010.
[6] E. Gawrilow and M. Joswig. polymake: a framework for analyzing convex polytopes. In Polytopes—combinatorics and computation (Oberwolfach, 1997), volume 29 of DMV Sem., pages 43–73. Birkhäuser, Basel, 2000.
[7] S. Herrmann and M. Joswig. Totally splittable polytopes. Discrete Comput. Geom., 44(1):149–166, 2010.
[8] J. F. P. Hudson. Piecewise linear topology. Mathematics lecture note series. W.A. Benjamin, 1 edition, 1969.
[9] M. Kreuzer and B. Nill. Classification of toric Fano 5-folds. Adv. Geom., 9(1):85–97, 2009. doi: 10.1515/ADVgeom.2009.005.
[10] B. Lorenz and A. Paffenholz. Smooth reflexive polytopes up to dimension 9, 2008.
[11] P. McMullen. Constructions for projectively unique polytopes. Discrete Mathematics, 14(4):347–358, 1976.
[12] M. Øbro. Classification of smooth Fano polytopes. PhD thesis, University of Aarhus, 2007. available at https://pure.au.dk/portal/files/41742384/imf_phd_2008_moe.pdf.
[13] J. Pfeifle and J. Rambau. Computing triangulations using oriented matroids. In Algebra, geometry, and software systems, pages 49–75. Springer, Berlin, 2003.
[14] J. Rambau. TOPCOM, version 0.17.5. Available at http://www.rambau.uni-bayreuth.de/TOPCOM/, 2015.