Abstract. In this paper we study the stratifying systems, introduced by K. Erdmann and C. Saenz in [17], by using the \( \tau \)-tilting theory introduced by T. Adachi, O. Iyama and I. Reiten in [1]. We give a constructive proof that every non-zero \( \tau \)-rigid module induces at least one stratifying system for any finite dimensional algebra \( A \). Later, we show that each stratifying system found this way is a \( \tau \)-exceptional sequence as defined by A. B. Buan and R. Marsh in [5].

1. Introduction

The concept of quasi-hereditary algebra was introduced by L. L. Scott in [34] and used firstly by E. Cline, B. Parshall and L. L. Scott [8] to characterise the highest weight categories of finite length having a finite number of simple objects, a concept that arises naturally in the representation theory of Lie algebras and algebraic groups. Because of their good properties, quasi-hereditary algebras were extensively studied by different mathematicians, among others, by V. Dlab and C. M. Ringel in [15, 13, 14, 32].

One of the main features of quasi-hereditary algebras is that they are determined by a distinguished set of representations, called the standard modules. The standard modules have several homological properties, regarding their endomorphism algebras, the space of morphisms between them and their space of extensions. Aiming to generalise the class of quasi-hereditary algebras, V. Dlab defined in [12] the class of standardly stratified algebras, which at their turn are determined by the existence of a distinguished set of standard modules having mildly weaker properties than the standard modules coming from quasi-hereditary algebras.

Later on, K. Erdmann and C. Sáenz wanted to generalize the good homological properties of the standard modules in order to construct, in a general way, standardly stratified algebras that were linked with any finite dimensional algebra. That lead them to define in [17] the stratifying systems. In that paper, they showed that one can associate to every stratifying system an object whose endomorphism algebra is standardly stratified. Since the stratified systems appeared in light, they have gotten a great amount of attention, see for instance [18, 16, 21, 22, 23, 26, 25, 24, 27, 25, 23, 21, 30, 31, 33].

One of the consequences of the work on stratifying systems is that their definition have been greatly simplified. In Definition 1.1 we recall the definition of a stratifying system, as appeared in [23].

Throughout this paper, the term algebra means a non-zero finite dimensional basic \( k \)-algebra, over an algebraically closed field \( k \); and for a given algebra \( A \), we denote by \( \text{mod} (A) \) the category of finitely generated left \( A \)-modules and \( \text{proj} (A) \) the finitely generated projective left \( A \)-modules.
For each $M \in \mod(A)$, $rk(M)$ stands for the rank of $M$ which is the number (up to isomorphisms) of pairwise non-isomorphic indecomposable direct summands of $M$.

**Definition 1.1.** [23] Characterisation 1.6] Let $A$ be an algebra. A stratifying system of size $t$, in $\mod(A)$, consists of a pair $(\Theta, \leq)$, where $\Theta := \{\Theta(i)\}_{i=1}^{t}$ is a family of indecomposable objects in $\mod(A)$ and $\leq$ is a linear order on the set $[1, t] := \{1, \ldots, t\}$ satisfying the following conditions:

(SS1) $\Hom_A(\Theta(j), \Theta(i)) = 0$ if $j > i$;

(SS2) $\Ext_A(\Theta(i), \Theta(j)) = 0$ if $j \geq i$.

It worth mentioning that the notion of stratifying system, given above, generalises the standardizable systems introduced by V. Dlab and M. C. Ringel in [14]. Note that stratifying systems are defined by fixing a linear order, and this is the reason why they are also known as linear stratifying systems. For the case of any finite pre-ordered set, it can also be defined stratifying systems.

We already argued that the existence of a stratifying system $(\Theta, \leq)$, in $\mod(A)$, allow us to construct a standardly stratified algebra $B$ and a nice connection between the $\Theta$-filtered $A$-modules and the $B$-modules filtered by the standard ones in $B$. However, up to now, the existence of a non-trivial stratifying system for a given algebra was difficult to settle. There is some work done for hereditary algebras in [6]; and for the case of a hereditary path algebra of type $A_{\rho, q}$ there is a description of all the stratifying systems [7].

In this paper we use the techniques provided by $\tau$-tilting theory, firstly introduced by T. Adachi, O. Iyama and I. Reiten in [1], to attack this problem in full generality. Our first main result can be summarised as follows, for a complete version, see in Theorem 3.4, Corollary 3.6 and Corollary 3.17.

**Theorem 1.2.** Let $A$ be an algebra and $M \in \mod(A)$ be a non-zero $\tau$-rigid module. Then, the following statements hold true.

(a) Associated with $M$ there is a stratifying system $\Delta_M$, called the $M$-standard system, in $\mod(A)$ of size $rk(M)$.

(b) $\Delta_M$ can be completed to a stratifying system in $\mod(A)$ of size $rk(\Delta_A)$.

(c) If $M$ has a $\Delta_M$-filtration, then $\Lambda := \End_A(M)^{op}$ is a basic standardly stratified algebra. Moreover, the functor $\Hom_A(M, \cdot) : \mathcal{F}(\Delta_M) \to \mathcal{F}(\Delta)\Lambda$ is an equivalence of categories with a quasi-inverse given by $M \otimes_{\Lambda} - : \mathcal{F}(\Delta)\Lambda \to \mathcal{F}(\Delta_M)$.

(d) $\Fuc(M)$ is the smallest torsion class in $\mod(A)$ containing $\Delta_M$.

Note that, as a consequence of Corollary 3.6 (a), for the $\tau$-rigid module $\Delta_A$ we have that the $\Delta_A$-standard system is just the usual family $\Delta_A$ of standard $A$-modules. Thus, $\Delta_A$ form a stratifying system of size $rk(\Delta_A)$. Since any $\tau$-rigid module $M \in \mod(A)$ satisfies that $rk(M) \leq rk(\Delta_A)$, by using the above theorem we obtain stratifying systems in $\mod(A)$ of size at most $rk(\Delta_A)$. However, we make it notice that for some algebra $A$ could exists a stratifying system of rank bigger than $rk(\Delta_A)$; for example, in [29] Example 6.11] it is given an algebra $A$ and a stratifying system of size $rk(\Delta_A) + 1$. From this point of view, it seems to be that the tau-tilting theory is not enough in order to obtain all the stratifying systems in $\mod(A)$. Furthermore, it is not known if there is an upper bound for the size of the stratifying systems in $\mod(A)$.

As a consequence of the developed theory, we give a description of the $\tau$-torsion-free admissible Ext-projective stratifying systems, which were introduced in Definition 3.13 in terms of a subclass of $\tau$-rigid modules. In Theorem 3.10 we give all the details of such construction. For the sake of completeness, we state the dual of the above results and constructions which are applied for $\tau^{-1}$-rigid modules and can be seen in detail in Theorem 3.11 Corollary 3.15 and Theorem 3.19.
Recall that the family $A^\Delta$ of the usual standard modules of a quasi-hereditary algebra $A$ form a set of objects in $\text{mod}(A)$ known as exceptional sequences. Exceptional sequences were first introduced on the study of coherent sheaves over an algebraic curve and were introduced to representation theory by W. Crawley-Boevey in [9]. Later on, in [19], K. Igusa and G. Todorov generalised exceptional sequences to signed exceptional and used them to describe problems arising in the theory of cluster-tilted algebras.

One of the main features of $\tau$-tilting theory is the fact that captures some combinatorial properties of cluster algebras in the module category of any finite dimensional algebra. Building on this property, A. B. Buan and R. Marsh introduced in [5] the concept of $\tau$-exceptional sequences, a generalisation of signed exceptional sequences for every algebra. The concept of $\tau$-exceptional sequence allow them to attack problems in a great level of generality, but this never comes for free. In the case of $\tau$-exceptional sequences, the problem is that the introduction of a new category is needed for their definition. This new category is, roughly speaking, the disjoint union of $\text{mod}(A)$ with $\text{mod}(A)[1]$, where $[1]$ is the suspension functor in the bounded derived category $D^b(\text{mod}(A))$.

In the second part of the paper, we study the relationship between $\tau$-exceptional sequences and the stratifying systems $\Delta_M$ for non-zero $\tau$-rigid modules $M$. Our main result in this second part can be written in a simplified way as follows. For the complete version, see Theorem [5.1].

**Theorem 1.3.** Let $A$ be an algebra and let $M \in \text{mod}(A)$ be a non-zero $\tau$-rigid $A$-module. Then all the $M$-standard systems $\Delta_M$ associated with $M$ are $\tau$-exceptional sequences.

It is worth noticing that Theorem [5.1] give us a characterisation of all stratifying systems induced by a $\tau$-rigid object.

The structure of the article is the following. In Section 2, it is introduced the needed background in order to state and prove our first main result, which is proved in Section 3. In Section 4 we recall the definition of $\tau$-exceptional sequences as was given in [3], together with the results of [20], which are necessary to give the said definition. In Section 5 by proving our second main result: that is, we show the compatibility between the stratifying systems associated to non-zero $\tau$-rigid objects and the $\tau$-exceptional sequences. Finally, in Section 6 we quantify the number of stratifying systems and $\tau$-torsion-free admissible Ext-projective stratifying systems (up to isomorphisms) induced by a $\tau$-tilting module in one specific example.

## 2. Setting and background

In this paper, when we say that $A$ is an *algebra* we mean that $A$ is a finitely $k$-dimensional basic algebra, over a fixed algebraically closed field $k$. By $\tau$ we denote the Auslander-Reiten translation in $\text{mod}(A)$ which is the category of finitely generated left $A$-modules. If $M \in \text{mod}(A)$, we denote by $\text{rk}(M)$ the rank of $M$ which is the number (up to isomorphisms) of pairwise non-isomorphic indecomposable direct summands of $M$. We also recall the usual duality functor $D := \text{Hom}_A(-, k) : \text{mod}(A) \rightarrow \text{mod}(A^{\text{op}})$.

Given an algebra $A$, we denote by $M \rightarrow N$ an epimorphism from $M$ to $N$ in $\text{mod}(A)$; and for each $M \in \text{mod}(A)$, we consider the class

$$\text{Fac}(M) := \{X \in \text{mod}(A) : \exists M^n \rightarrow X \text{ for some } n \in \mathbb{N}\}.$$ 

Dually, the arrow $\hookrightarrow$ stands for a monomorphism in $\text{mod}(A)$, and we have the class

$$\text{Sub}(M) := \{X \in \text{mod}(A) : \exists X \hookrightarrow M^n \text{ for some } n \in \mathbb{N}\}.$$ 

For any subclass $\mathcal{X} \subseteq \text{mod}(A)$, we have the right perpendicular complement of $\mathcal{X}$

$$\mathcal{X}^\perp := \{M \in \text{mod}(A) : \text{Hom}_A(-, M)|_{\mathcal{X}} = 0\}.$$
Dually, we have the left perpendicular complement $\perp X$ of $X$. For a single element class $X = \{M\}$, we just write $M^\perp$ and $\perp M$. It is said that $N \in \text{mod}(A)$ admits an $X$-filtration if there is a chain of submodules $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_{n-1} \subseteq M_n = M$ satisfying that each quotient $M_i/M_{i-1} \cong X_i \in X$. The class of all $N \in \text{mod}(A)$ which admits an $X$-filtration will be denoted by $\mathcal{F}(X)$.

**Approximations.** Let $A$ be an algebra, $X \subseteq \text{mod}(A)$ and $M \in \text{mod}(A)$. A morphism $f : X \to M$ in $\text{mod}(A)$ is an $X$-precover (a right $X$-approximation) if $X \in X$ and $\text{Hom}_A(X', X) \to \text{Hom}_A(X', M)$ is surjective for any $X' \in X$. Moreover, if the equality $fh = f$ holds true only for an automorphism $h : X \to X$, we say that the $X$-precover is an $X$-cover. The class $X$ is precoversing and enveloping, it is said that $X$ is functorially finite. The first torsion class is $\tau = \text{precover}(X)$, the class of all $A$-modules with the property that $X$ is a subfunctor of the identity functor $1_{\text{mod}(A)}$, and the torsion free functor $f$ is the quotient functor $1/t$. Moreover, for every $M \in \text{mod}(A)$ there exists the so-called canonical short exact sequence $0 \to t(M) \to M \to f(M) \to 0$, where $t(M) \in \mathcal{T}$ and $f(M) \in \mathcal{F}$, and such exact sequence is unique up to isomorphisms.

A morphism $f : X \to M$ in $\text{mod}(A)$ is a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\text{mod}(A)$ is a pair $(\mathcal{T}, \mathcal{F})$ of classes of objects in $\text{mod}(A)$ such that $\mathcal{T} = \perp \mathcal{F}$ and $\mathcal{F} = \perp \mathcal{T}$. Given a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\text{mod}(A)$, it is said that $\mathcal{T}$ is a torsion class and $\mathcal{F}$ is a torsion free class. Moreover, $\mathcal{T}$ is closed under quotients and extensions, while $\mathcal{F}$ is closed under submodules and extensions. It is also well known that, for every subclass $\mathcal{T} \subseteq \text{mod}(A)$ which is closed under quotients and extensions, there exists a subclass $\mathcal{F} \subseteq \text{mod}(A)$ such that $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\text{mod}(A)$. In particular, if $\mathcal{T} = \text{Fac}(M)$ for some $M \in \text{mod}(A)$, then $\mathcal{F} = M^\perp$.

It is also well known that each torsion pair $(\mathcal{T}, \mathcal{F})$ in $\text{mod}(A)$ has associated two additive functors $t, f : \text{mod}(A) \to \text{mod}(A)$, where the torsion functor $t$ is a subfunctor of the identity functor $1_{\text{mod}(A)}$, and the torsion free functor $f$ is the quotient functor $1/t$. Moreover, for every $M \in \text{mod}(A)$ there exists the so-called canonical short exact sequence $0 \to t(M) \to M \to f(M) \to 0$, where $t(M) \in \mathcal{T}$ and $f(M) \in \mathcal{F}$, and such exact sequence is unique up to isomorphisms.

Let $X \subseteq \text{mod}(A)$. It is said that $X \subseteq \text{Ext-projective}$ in $\mathcal{X}$ if $\text{Ext}^1_A(X, -)|_{\mathcal{X}} = 0$. Moreover, if $X$ is functorially finite and a torsion class, it is well known that there are only finitely many (up to isomorphisms) indecomposable Ext-projective modules in $X$, and we denote by $\mathcal{P}(X)$ the direct sum of each of them. One of the main features of $\tau$-tilting theory is that all functorially finite torsion classes in $\text{mod}(A)$ can be described by using support $\tau$-tilting modules, as stated in the following result that has been taken from [1] Theorem 2.7 and [3] Theorem 5.10.

**Theorem 2.2.** For any algebra $A$, there is a well defined function

$$\Phi : \tau\text{-rig}(A) \to \text{f-tors}(A), \ M \mapsto \text{Fac}(M),$$

from $\tau$-rigid modules to functorially finite torsion classes in $\text{mod}(A)$. Moreover, $\Phi$ is a bijection if we restrict it to the class $\tau\text{-tilt}(A)$ of support $\tau$-tilting modules, and in this case $\Phi^{-1}(\mathcal{T}) = \mathcal{P}(\mathcal{T})$.

**The Bongartz completion in $\text{mod}(A)$.** A nice properties of the $\tau$-rigid modules is that they can always be completed to a $\tau$-tilting module through a process called the Bongartz completion that we shall describe.

Let $M \in \text{mod}(A)$ be a $\tau$-rigid module. Then $M$ has two different torsion classes naturally associated to it, which are functorially finite. The first torsion class is $\text{Fac}(M)$, as was already
pointed out in Theorem 2.2 and the second one is \( \frac{1}{2} (\tau M) \). Now, by applying Theorem 2.2 again, we have that the \( A \)-module
\[
\mathcal{B}_M := \mathbb{P} \left( ^{-1} (\tau M) \right)
\]
is a \( \tau \)-rigid module and it is known as the Bongartz completion of \( M \). However one can say much more about it as shown in the following result from [1, Theorem 2.10].

**Proposition 2.3.** Let \( A \) be an algebra and \( M \in \text{mod} (A) \) be a \( \tau \)-rigid module. Then \( \mathcal{B}_M \) is a \( \tau \)-tilting module having a direct summand the \( A \)-module \( M \).

**The trace and the reject.** Let \( A \) be an algebra, \( X \subseteq \text{mod} (A) \) and \( M \in \text{mod} (A) \). The trace of \( X \) in \( M \) is
\[
\text{Tr}_X (M) := \sum \{ \text{Im}(f) : f \in \text{Hom}_A (X, M), X \in X \};
\]
and the reject of \( X \) in \( M \) is
\[
\text{Rej}_X (M) := \bigcap \{ \text{Ker}(f) : f \in \text{Hom}_A (M, X), X \in X \}.\]
For a more detailed treatment of the trace and the reject, we recommend the reader to see in [2].

**Standardly stratified algebras.** Let \( A \) be an algebra. Since \( A \) is basic, we have that \( A \), as a left \( A \)-module over itself, decomposes as the direct sum \( A = \bigoplus_{i=1}^{n} P(i) \), where \( P(1), P(2), \ldots, P(n) \) is a complete list of pairwise non-isomorphic indecomposable projective \( A \)-modules. Fix some linear order \( \leq \) on the set \([1, n] := \{1, 2, \ldots, n\}\). For each \( i \in [1, n] \), we consider \( \mathcal{T}(i) := \bigoplus_{j \geq i} P(j) \), and hence, the \( i \)-th standard \( A \)-module is \( A \Delta(i) := P(i) / \text{Tr}_A (P(i)) \). The family of standard \( A \)-modules is \( A \Delta := \{ A \Delta(i) \}_{i=1}^{n} \). It is well known [14] that the standard modules \( A \Delta \), together with the linear order \( \leq \) on \([1, n] \) form a stratifying system in \( \text{mod} (A) \) of size \( n \). Observe that the canonical stratifying system \( (A \Delta, \leq) \) depends heavily on the linear order we impose on the decomposition \( A = \bigoplus_{i=1}^{n} P(i) \) of \( A \)-modules. Therefore, there are \( n! \)-sets of standard modules that can be constructed. However, depending on the characteristics of \( A \), some of them might coincide.

Consider a decomposition of the minimal injective cogenerator \( D(A_A) = \bigoplus_{i=1}^{n} I(i) \) into pairwise non-isomorphic indecomposable injective \( A \)-modules. Fix a linear order \( \leq \) on the set \([1, n] \), and let \( \mathcal{T}(i) := \bigoplus_{j \geq i} I(j) \). The \( i \)-th costandard \( A \)-module is \( A \nabla(i) := \text{Rej}_A (P(i)) \). The family of costandard \( A \)-modules is \( A \nabla := \{ A \nabla(i) \}_{i=1}^{n} \). It can be shown that the costandard modules \( A \nabla \), together with the opposite order \( \leq^\text{op} \) of the fixed linear order \( \leq \) on \([1, n] \) form a stratifying system in \( \text{mod} (A) \) of size \( n \).

**Definition 2.4.** An algebra \( A \) is standardly stratified, with respect to a decomposition \( A = \bigoplus_{i=1}^{n} P(i) \) into indecomposable pairwise non-isomorphic projective \( A \)-modules and a linear order \( \leq \) on the set \([1, n] \), if \( A = A \Delta \) is a stratifying system in \( \text{mod} (A) \). Moreover, if each \( \text{End}_A (A \Delta(i)) \) is a division ring, the standardly stratified algebra \( A \) is called quasi-hereditary.

**Ext-projective and Ext-injective stratifying systems.** Ext-projective and Ext-injective stratifying systems can be used to obtain the relative version of the existence of relative projective covers and relative injective envelopes, respectively, for the exact category \( \mathcal{F}(\Theta) \) of a given stratifying system \( (\Theta, \leq) \). Moreover, by using these systems, we can construct two standardly stratified algebras, one for Ext-projective and the other for Ext-injective, see the details in [17, 23, 25].

**Definition 2.5.** Let \( A \) be an algebra, \( \Theta = \{ \Theta(i) \}_{i=1}^{t} \) be a family of non-zero \( A \)-modules, \( Q = \{ Q(i) \}_{i=1}^{t} \) be a family of indecomposable \( A \)-modules and \( \leq \) be a linear order on \([1, t] \). The triple \( (\Theta, Q, \leq) \) is an Ext-projective stratifying system of size \( t \), in \( \text{mod} (A) \), if the following three conditions hold true

(EPSS1) \( \text{Hom}_A (\Theta(j), \Theta(i)) = 0 \) for \( j > i \).

(EPSS2) for each \( i \in [1, t] \), there is an exact sequence in \( \text{mod} (A) \)
\[
0 \rightarrow K(i) \rightarrow Q(i) \xrightarrow{\Theta(i)} \Theta(i) \rightarrow 0
\]
such that $K(i) \in \mathcal{F}(\{\Theta(j) : j > i\})$.

\[(\text{EPSS3}) \quad \text{Ext}_A^i(Q, \cdot)|_{\mathcal{F}(\Theta)} = 0 \text{ for } Q := \oplus_{i=1}^t Q(i).\]

**Definition 2.6.** \([17]\) Let $A$ be an algebra, $\Theta = \{\Theta(i)\}_{i=1}^t$ be a family of non-zero $A$-modules, $\sum = \{Y(i)\}_{i=1}^t$ be a family of indecomposable $A$-modules and $\leq$ be a linear order on $[1, t]$. The triple $(\Theta, \sum, \leq)$ is an Ext-projective stratifying system of size $t$, in mod $(A)$, if the following three conditions hold true:

1. (EISS1) $\text{Hom}_A(\Theta(j), \Theta(i)) = 0$ for $j > i$,
2. (EISS2) for each $i \in [1, t]$, there is an exact sequence in mod $(A)$
   \[0 \to \Theta(i) \xrightarrow{\lambda_i} Y(i) \to Z(i) \to 0\]
   such that $Z(i) \in \mathcal{F}(\{\Theta(j) : j < i\})$,
3. (EISS3) $\text{Ext}_A^i(-, Y)|_{\mathcal{F}(\Theta)} = 0$ for $Y := \oplus_{i=1}^t Y(i)$.

Given two Ext-projective stratifying systems, with the same size and order in mod $(A)$, it is possible to compare them and to say either are or not isomorphic \([23\text{ Definition } 2.4]\). For the case of Ext-injective stratifying systems, see \([23\text{ Definition } 1.3]\).

**Definition 2.7.** \([25]\) Let $(\Theta, \{Q(i)\}_{i=1}^t, \leq)$ and $(\Theta', \{Q'(i)\}_{i=1}^t, \leq)$ be Ext-projective stratifying systems in mod $(A)$. Consider the canonical morphisms $\gamma_i : Q(i) \to \Theta(i)$ and $\gamma_i' : Q'(i) \to \Theta(i)'$ appearing, respectively, in Definition \([25]\). An isomorphism of Ext-projective stratifying systems

$f : (\Theta, \{Q(i)\}_{i=1}^t, \leq) \to (\Theta', \{Q'(i)\}_{i=1}^t, \leq)$

is a family of morphisms $f = \{f_1(i), f_2(i)\}_{i=1}^t$ in mod $(A)$, where $f_1(i) : \Theta(i) \to \Theta(i)'$ and $f_2(i) : Q(i) \to Q'(i)$ are isomorphisms such that $f_1(i)\gamma_i = \gamma_i'f_2(i)$ for all $i \in [1, t]$.

Given a stratifying system $(\Theta, \leq)$, it is well known \([17, 23, 25]\) that there exist a unique (up to isomorphism) Ext-projective stratified system $(\Theta, Q, \leq)$ and an Ext-injective stratified system $(\Theta, Q', \leq)$. Furthermore, for a given Ext-projective stratified system $(\Theta, Q, \leq)$, we have that the pair $(\Theta, \leq)$ is a stratifying system. Moreover the same statement holds true for Ext-injective stratifying systems.

### 3. Stratifying systems from $\tau$-rigid modules

The concept of stratifying system was introduced by K. Erdmann and C. Saenz in \([17]\) to generalise the usual standard objects $\Delta$ of standarly stratified algebras. In what follows, we recall how to calculate the usual standard modules of a given algebra.

Let $A$ be an algebra. Since $A$ is basic, we have that $A$, as a left $A$-module over itself, decomposes as the direct sum $A = \bigoplus_{i=1}^n P(i)$, where $P(1), P(2), \ldots, P(n)$ is a complete list of pairwise non-isomorphic indecomposable projective $A$-modules. Fix a linear order $\leq$ on the set $[1, n]$. For each $i \in [1, n]$, we consider $\overline{\mathcal{P}}(i) := \bigoplus_{j>i} P(j)$, and hence, the $i$-th standard $A$-module is $A\Delta(i) := P(i)/\text{Tr}_{\overline{\mathcal{P}}(i)}(P(i))$. Note that the standard $A$-module $\Delta(i)$ can be computed by using right approximations. Indeed, let $f_j : P'_j \to P(i)$ be an add $\overline{\mathcal{P}}(i)$-precover of $P(i)$. Then, it can be shown that $\Delta(i) = \text{CoKer}(f_j)$ for each $i \in [1, n]$.

In this section, we show that the same construction of standard modules can be generalised to all non-zero $\tau$-rigid $A$-modules. But, as for the standard modules, we need to put conditions over the imposed order on the indecomposable direct summands of the $\tau$-rigid modules. This motivates the following definition.

**Definition 3.1.** Let $A$ be an algebra and $M \in \text{mod } (A)$ be a basic non-zero $A$-module. We say that a decomposition of $M = \bigoplus_{i=1}^t M_i$ as the direct sum of indecomposable $A$-modules is *torsion free admissible* (TF-admissible, for short) if $M_i \not\subseteq \text{Fac} \left( \bigoplus_{j>i} M_j \right)$, for every $i \in [1, t]$.
Before going any further, we show that every non-trivial basic \( \tau \)-rigid object always has a TF-admissible decomposition.

**Proposition 3.2.** For an algebra \( A \), every basic non-zero \( \tau \)-rigid module in \( \text{mod} \, (A) \) admits a TF-admissible decomposition.

**Proof.** Let \( M \in \text{mod} \, (A) \) be a basic non-zero \( \tau \)-rigid \( A \)-module. We prove the above statement by induction on the number of indecomposable direct summands of \( M \).

If \( M \) is indecomposable, then \( M = M_1 \) and the claim follows immediately.

Let \( M \) be the direct sum of \( t \geq 2 \) pairwise non-isomorphic indecomposable \( A \)-modules. Given that \( M \) is a \( \tau \)-rigid \( A \)-module, Theorem 2.2 implies that \( \text{Fac}(M) \) is a functorially finite torsion class in \( \text{mod} \, (A) \). By applying Theorem 2.2 again, we have the existence of a support \( \tau \)-tilting module \( T \) such that \( \text{Fac}(T) = \text{Fac}(M) \). Moreover, [1] Proposition 2.9 implies that \( M \) is a direct summand of \( T \). In other words, \( T \cong M \oplus \tilde{M} \) for some \( \tau \)-rigid \( M \in \text{mod} \, (A) \). Note that \( \tilde{M} \) might be zero.

Now, since \( M \) is nonzero we have that \( \{0\} \subseteq \text{Fac}(M) \). Then [10] Theorem 3.1] implies the existence of a support \( \tau \)-tilting module \( N \in \text{mod} \, (A) \) which is a mutation of \( T \) over an indecomposable direct summand \( M_1 \) and such that \( \{0\} \subseteq \text{Fac}(N) \subseteq \text{Fac}(M) \).

We claim that \( M_1 \) is an indecomposable direct summand of \( M \). Suppose to the contrary that \( M_1 \) is a direct summand of \( M \). Then we have that \( M \) is a direct summand of \( N \). This implies in particular that \( \text{Fac}(M) \subseteq \text{Fac}(N) \subseteq \text{Fac}(M) \), which is a contradiction proving that \( M_1 \) is a direct summand of \( M \). Hence \( M \cong M_1 \oplus M' \), where \( M' \) has \( t-1 \) non-isomorphic indecomposable summands. Then, by the inductive hypothesis, we get that \( M' \) admits a TF-admissible decomposition \( M' = \bigoplus_{i=1}^{t} M_i \), where \( M_i \notin \text{Fac} \left( \bigoplus_{j>i} M_j \right) \) for every \( 2 \leq i \leq t \).

On the other hand, note that \( M' \) is also a direct summand of \( N \) because we only mutate over \( M_1 \). This implies in particular that \( \text{Fac}(M') \subseteq \text{Fac}(N) \). Hence, since \( M_1 \notin \text{Fac}(N) \), we can conclude that \( M_1 \notin \text{Fac}(M') = \text{Fac} \left( \bigoplus_{j>i} M_j \right) \). Therefore, we get the decomposition of \( M \)

\[
M \cong M_1 \oplus M' = M_1 \oplus \left( \bigoplus_{i=2}^{t} M_i \right) = \bigoplus_{i=1}^{t} M_i,
\]

where \( M_i \notin \text{Fac} \left( \bigoplus_{j>i} M_j \right) \) for every \( 1 \leq i \leq t \). This finishes the proof. \( \square \)

We have seen in Proposition 2.3 that every \( \tau \)-rigid module \( M \in \text{mod} \, (A) \) can be completed into a \( \tau \)-tilting \( A \)-module \( \mathcal{B}_M \) called the Bongartz completion of \( M \). In the following result, we show that the TF-admissible decomposition of any basic \( \tau \)-rigid \( A \)-module \( M \) can be extended to a TF-admissible decomposition of its Bongartz completion \( \mathcal{B}_M \).

**Proposition 3.3.** Let \( A \) be an algebra, \( n := \text{rk}(AA) \) and let \( M \in \text{mod} \, (A) \) be a basic non-zero \( \tau \)-rigid \( A \)-module with TF-admissible decomposition \( M = \bigoplus_{i=1}^{t} M_i \). Then every decomposition \( \mathcal{B}_M = \bigoplus_{j=1}^{a} N_j \) into indecomposable modules of the Bongartz completion of \( M \), such that \( M_i \cong N_{n-t+i} \forall i \in [1, n] \), is a TF-admissible decomposition.

**Proof.** Let \( \mathcal{B}_M = \tilde{M} \oplus M \) and \( \tilde{M} = \bigoplus_{i=1}^{a} \tilde{M}_i \) be any decomposition into pairwise non-isomorphic indecomposable \( A \)-modules. Consider \( N_j := \tilde{M}_j \) for \( j \in [1, a] \), and \( N_j := M_{j-a} \) for \( j \in [a+1, n] \).

Let us prove that \( \mathcal{B}_M = \bigoplus_{j=1}^{a} N_j \) is TF-admissible. In order to do that, we start by proving

\[
(*) \quad X \notin \text{Fac} \left( M \oplus \frac{\tilde{M}}{X} \right) \quad \text{for any indecomposable } X \in \text{add} \, (\tilde{M}).
\]
Indeed, by Proposition 2.3 we have \( \text{Fac}(M \oplus \tilde{M}) = \tau (\tau M) \). Consider \( T_X \in \text{mod}(A) \) which is the mutation of \( M \oplus \tilde{M} \) over an indecomposable direct summand \( X \) of \( \tilde{M} \). Then, by Theorem 2.18, it follows that \( \text{Fac}(T_X) \subseteq \tau (\tau M) \). On the other hand, since \( \text{Fac}(M \oplus \frac{M}{M'}) \subseteq \text{Fac}(T_X) \), to prove (\star), it is enough to show that \( X \not\in \text{Fac}(T_X) \). Suppose that \( X \in \text{Fac}(T_X) \). Then \( \tau (\tau M) = \text{Fac}(M \oplus \tilde{M}) \subseteq \text{Fac}(T_X) \subseteq \tau (\tau M) \), which is a contradiction and thus (\star) holds true.

Once we have that (\star) holds true, we are ready to finish the proof that \( B_M = \bigoplus_{j=1}^n N_j \) is TF-admissible. Let \( i \in [1, a] \). Then by (\star) we know that \( N_i = M_i \not\in \text{Fac}(\bigoplus_{j \neq i} N_j) \). In particular \( N_i \not\in \text{Fac}(\bigoplus_{k>j} N_j) \). If \( i \in [a+1, n] \), then by the fact that \( M = \bigoplus_{i=1}^t M_i \) is TF-admissible, we conclude that \( N_i = M_{i-a} \not\in \text{Fac}(\bigoplus_{k>i-a} M_k) = \text{Fac}(\bigoplus_{j>i} N_j) \). \( \square \)

Now that we know that every basic non-zero \( \tau \)-rigid module admits a TF-admissible decomposition, we prove the main result of this section.

**Theorem 3.4.** Let \( A \) be an algebra and \( M \in \text{mod}(A) \) be a basic non-zero \( \tau \)-rigid module with a TF-admissible decomposition \( M = \bigoplus_{i=1}^t M_i \), \( n := \text{rk}(A) \), and let \( f_k \) be the torsion free functor associated to the torsion pair \( \left( \text{Fac}(\bigoplus_{j \geq k} M_j), \left( \bigoplus_{j \geq k} M_j \right)^\perp \right) \). Then, the following statements hold true.

1. The family \( \Delta_M := \{ \Delta_M(i) := f_{i+1}(M_1) \}_{i=1}^t \) and the natural order on \( [1, t] \) form a stratifying system in \( \text{mod}(A) \) of size \( t \), which is called the \( M \)-standard system associated with the given TF-admissible decomposition of \( M \).
2. There exists at least one stratifying system \( (\Delta_M, \leq') \) of size \( n \geq t \) in \( \text{mod}(A) \), where \( \leq' \) is the natural order on \( [1, n] \), such that \( \Delta_M(i) = \Delta_M(n-t+i) \) for all \( i \in [1, t] \).

**Proof.** (a) Consider the family \( \Delta_M \) as defined in (a). Then, it follows directly from Lemma 4.6 that each \( \Delta_M(i) \) is indecomposable since \( M_i \not\in \text{Fac}(\bigoplus_{j>i+1} M_j) \).

Suppose that \( i > j \). We claim that \( \text{Hom}_A(\Delta_M(i), \Delta_M(j)) = 0 \). Indeed, on one hand we have that \( \Delta_M(j) \in (\bigoplus_{k>j} M_k)^\perp \) by definition. On the other hand \( \Delta_M(i) \) is a quotient of \( M_i \).

Moreover
\[
\Delta_M(i) \in \text{Fac}(\bigoplus_{k \geq i} M_k) \subseteq \text{Fac}(\bigoplus_{k \geq j} M_k);
\]
and thus, by using that \( (\bigoplus_{k \geq j} M_k), (\bigoplus_{k \geq j} M_k)^\perp \) is a torsion pair in \( \text{mod}(A) \), our claim follows.

Finally, to finish the proof of (a), we need to show that \( \text{Ext}^1_A(\Delta_M(i), \Delta_M(j)) = 0 \) if \( i \geq j \). In order to do that, we assume that \( i \geq j \) and consider the canonical short exact sequence
\[
0 \to t_{i+1}(M_i) \to M_i \to \Delta_M(i) \to 0
\]
of \( M_i \) with respect to the torsion pair \( \left( \text{Fac}(\bigoplus_{k \geq i+1} M_k), (\bigoplus_{k \geq i+1} M_k)^\perp \right) \). By applying the functor \( \text{Hom}_A(\cdot, \Delta_M(j)) \) to the previous canonical exact sequence, we obtain the following exact sequence
\[
\text{Hom}_A(t_{i+1}(M_i), \Delta_M(j)) \to \text{Ext}^1_A(\Delta_M(i), \Delta_M(j)) \to \text{Ext}^1_A(M_i, \Delta_M(j)).
\]
Note that \( M_i \) is Ext-projective in \( \text{Fac}(M) \) by Corollary 5.9, which implies that
\[
\text{Ext}^1_A(M_i, \Delta_M(j)) = 0.
\]
On the other hand, from \( i \geq j \), we get \( t_{i+1}(M_i) \in \text{Fac}(\bigoplus_{k \geq i+1} M_k) \subseteq \text{Fac}(\bigoplus_{k \geq j+1} M_k) \). Thus
\[
\text{Hom}_A(t_{i+1}(M_i), \Delta_M(j)) = 0,
\]
since $\Delta_M(j) \in \left( \bigoplus_{k \ge j+1} M_k \right)^\perp$ and \( \text{Fac}(\bigoplus_{k \ge j+1} M_k, \left( \bigoplus_{k \ge j+1} M_k \right)^\perp) \) is a torsion pair in \( \text{mod}(A) \). Then we have that \( \text{Ext}_A^1(\Delta_M(i), \Delta_M(j)) = 0 \) as claimed. This finishes the proof of (a).

(b) It follows directly from Proposition \ref{prop:characterisation} and (a).

\[ \blacksquare \]

Remark 3.5. It is important to notice that the size of a stratifying system induced by a \( \tau \)-rigid object is bounded by the number of non-isomorphic indecomposable simple \( A \)-modules. In [23, Remark 2.7], the authors build a stratifying system of length 5 in the module category of an algebra of rank 4. This implies that Theorem \ref{thm:characterisation} is not a characterisation of stratifying systems.

Let \( X \subseteq \text{mod}(A) \). Following \cite{10}, we denote by \( \top(X) \) the smallest torsion class in \( \text{mod}(A) \) containing \( X \). As a consequence of the previous theorem, we study the minimal torsion class containing a stratifying stratifying system induced by a \( \tau \)-rigid module.

Corollary 3.6. Let \( A \) be an algebra, \( M \in \text{mod}(A) \) be a basic non-zero \( \tau \)-rigid module with a TF-admissible decomposition \( M = \bigoplus_{i=1}^t M_i \), \( \overline{M}_i := \bigoplus_{k > i} M_k \), and let \( \Delta_M \) be the M-standard system associated with the given TF-admissible decomposition of \( M \). For each \( i \in \{1, \ldots, t\} \), consider the canonical exact sequence \( 0 \to t_{i+1}(M_i) \to M_i \xrightarrow{\beta_i} \Delta_M(i) \to 0 \) which is given by the torsion pair \( (\text{Fac}(\overline{M}_i), \overline{M}_i^\perp) \), and let \( h_i : N_i \to M_i \) be the \( \text{add}(\overline{M}_i) \)-cover of \( M_i \). Then, the following statements hold true.

(a) \( \text{Im}(h_i) = t_{i+1}(M_i) = \text{Tr}_i(\overline{M}_i)(M_i) \) and \( \Delta_M(i) \cong \frac{M_i}{\text{Fac}_i(\overline{M}_i)} \forall i \in \{1, \ldots, t\} \).

(b) \( \beta_i : M_i \to \Delta_M(i) \) is an \( \text{add}(M) \)-cover of \( \Delta_M(i) \forall i \in \{1, \ldots, t\} \).

(c) \( \mathcal{F}(\Delta_M) \subseteq \text{Fac}(M) = \top(\Delta_M) \).

Proof. (a) Let \( i \in \{1, \ldots, t\} \). By \cite{3} Lemma 2.3 it follows that \( \text{Im}(h_i) = t_{i+1}(M_i) \). Moreover, by \cite[Proposition 8.20]{2}, we get that \( t_{i+1}(M_i) = \text{Tr}_i(\overline{M}_i)(M_i) = \text{Tr}_i(\overline{M}_i)(M_i) \).

(b) Let \( i \in \{1, \ldots, t\} \). Since \( M_i \) is indecomposable, it is enough to show that \( \beta_i : M_i \to \Delta_M(i) \) is an \( \text{add}(M) \)-precover of \( \Delta_M(i) \). By applying the functor \( \text{Hom}_A(X, -) \) to the exact sequence \( 0 \to t_{i+1}(M_i) \to M_i \xrightarrow{\beta_i} \Delta_M(i) \to 0 \), for any \( X \in \text{add}(M) \), we get the exact sequence

\[
\text{Hom}_A(X, M_i) \xrightarrow{(X, \beta_i)} \text{Hom}_A(X, \Delta_M(i)) \to \text{Ext}_A^1(X, t_{i+1}(M_i)).
\]

Note that \( M \) is Ext-projective in \( \text{Fac}(M) \) by \cite[Corollary 5.9]{3}, and \( t_{i+1}(M_i) \subseteq \text{Fac}(M_i) \) without \( \text{Fac}(M) \), which implies that \( \text{Ext}_A^1(X, t_{i+1}(M_i))) = 0 \). Thus \( \beta_i : M_i \to \Delta_M(i) \) is an \( \text{add}(M) \)-precover, proving (b).

(c) By using the canonical exact sequence given above, it follows that \( \Delta_M \subseteq \text{Fac}(M) \). Therefore \( \mathcal{F}(\Delta_M) \subseteq \text{Fac}(M) \) and \( \top(\Delta_M) \subseteq \text{Fac}(M) \). For each \( i \in \{1, \ldots, t\} \), let \( M_i' := \bigoplus_{j \neq i} M_j \). Consider the torsion pair \( (\text{Fac}(M_i'), M_i'^\perp) \) in \( \text{mod}(A) \), the canonical exact sequence

\[
0 \to t'_i(M_i) \to M_i \xrightarrow{\beta'_i} f'_i(M_i) \to 0
\]

and the \( \text{add}(M_i') \)-cover \( h'_i : N_i \to M_i \). Then, by \cite[Lemma 2.3]{3}, we have that \( t'_i(M_i) = \text{Im}(h'_i) \). Hence, from \cite[Lemma 3.7]{13}, it follows that \( M \in \top(\bigoplus_{i=1}^t f'_i(M_i)) \); and thus, \( \text{Fac}(M) \subseteq \top(\bigoplus_{i=1}^t f'_i(M_i)) \). On the other hand, since \( M_i' \subseteq M_i \), we have \( f'_i(M_i) \subseteq M_i'^\perp \subseteq M_i^\perp \). Hence there is some \( \gamma_i : \Delta_M(i) \to f'_i(M_i) \) such that \( \gamma_i = \beta'_i \). Moreover \( \gamma_i \) is an epimorphism since \( \beta'_i \) is so. Then, we get that \( f'_i(M_i) \subseteq \top(\Delta_M) \) and thus \( \text{Fac}(M) \subseteq \top(\bigoplus_{i=1}^t f'_i(M_i)) \subseteq \top(\Delta_M) \).

\[ \blacksquare \]

Remark 3.7. Let \( A \) be an algebra. Note that any decomposition into indecomposables \( A = \bigoplus_{i=1}^n A P(i) \) of the \( \tau \)-rigid module \( A \) is TF-admissible. Moreover, by Corollary \ref{cor:characterisation} (a), we get that the \( A \)-standard system coincides with the usual standard \( A \)-modules. Observe that in general it is not true that \( A \in \mathcal{F}(\Delta) \) unless \( A \) be standardly stratified. Thus, for the general
Theorem 3.10. For any algebra $A$, let $M \in \text{mod}(A)$ be a basic non-zero $\tau$-rigid module and $M_1, M_2, \ldots, M_t$ be $\tau$-rigid modules with $M = \bigoplus_{i=1}^{t} M_i$. Let $\Delta_M$ be a TF-admissible decomposition of $M$. Then, $M_1, \ldots, M_t$ are $\tau$-rigid modules.

Corollary 3.8. Let $A$ be an algebra, $M \in \text{mod}(A)$ be a basic non-zero $\tau$-rigid module with a TF-admissible decomposition $M = \bigoplus_{i=1}^{t} M_i$. Let $\Delta_M$ be the $\tau$-rigid modules associated with the given TF-admissible decomposition of $M$. If $M \in \mathcal{F}(\Delta_M)$, then the following statements hold true.

(a) $\mathcal{D}_{TF}(M_i) \in \mathcal{F}(\Delta_M(j))$ for each $i \in [1, t]$.

(b) $(\Delta_M, \{M_i\}_{i=1}^{t})$ is an Ext-projective stratifying system in $\text{mod}(A)$ of size $t$, where $\Delta_M$ is the natural order on $[1, t]$.

(c) $\Lambda := \text{End}(M)^{opp}$ is a basic standardly stratified algebra with respect to the decomposition into indecomposables $\Lambda = \bigoplus_{i=1}^{t} \Lambda(i)$, where $\Lambda(i) := \text{Hom}(M_i, t)$, and the natural order $\lesssim$ is the natural order on $\mathcal{D}_{TF}(M_i)$.

(d) The functor $\text{Hom}(M, -) : \mathcal{F}(\Delta_M) \to \mathcal{F}(\Lambda\Delta)$ is an equivalence of categories with a quasi-inverse given by $M \otimes \Lambda - : \mathcal{F}(\Lambda\Delta) \to \mathcal{F}(\Delta_M)$.

(e) $\text{Hom}(M, \Delta_M(i)) \simeq \Lambda\Delta(i)$ and $M \otimes \Lambda\Delta(i) \simeq \Delta_M(i)$ for each $i \in [1, t]$.

Proof. Let $M \in \mathcal{F}(\Delta_M)$. By Theorem 3.10 (a), we know that $(\Delta_M, \lesssim)$ is a stratifying system of size $t$. Then, by [25] Corollary 2.5, Proposition 2.14 (b) there is an Ext-projective stratifying system $(\Delta_M, \mathcal{Q} = \{Q(i)\}_{i=1}^{t})$ and $\mathcal{P}(\Delta_M) \cap \mathcal{F}(\Delta_M) = \text{add}(Q)$, where $Q := \bigoplus_{i=1}^{t} Q(i)$ and $\mathcal{P}(\Delta_M)$ is formed by all $X \in \text{mod}(A)$ such that $\text{Ext}^A_{TF}(X, \mathcal{F}(\Delta_M)) = 0$. Since $M \in \mathcal{F}(\Delta_M)$, and $\text{Ext}^A_{TF}(M, \mathcal{F}(\Delta_M)) = 0$, we get from Corollary 3.8 (c) that $M \in \mathcal{P}(\Delta_M) \cap \mathcal{F}(\Delta_M)$ and so we get that $M \in \text{add}(Q)$. Moreover, [25] Remark 2.7 gives us $rk(Q) = t = rk(M)$, and hence $\text{add}(Q) = \text{add}(M)$.

Let $i \in [1, t]$. By Definition 3.9 and [25] Lemma 2.3, there is an exact sequence

$$0 \to K(i) \to Q(i) \to \Delta_M(i) \to 0,$$

where $K(i) \in \mathcal{F}(\Delta_M(j))$ for $j > i$ and $Q(i) \to \Delta_M(i)$ is an $\text{add}(Q)$-cover of $\Delta_M(i)$. On the other hand, we have the canonical exact sequence

$$0 \to t_{i+1}(M_i) \to M_i \xrightarrow{\beta_i} \Delta_M(i) \to 0,$$

which is given by the torsion pair $(\text{Fac}(\mathcal{M}_i), \mathcal{M}^{-1}_i)$. By Corollary 3.8 (b), we know that $\beta_i : M_i \to \Delta_M(i)$ is an $\text{add}(M)$-cover of $\Delta_M(i)$. Using now that $\text{add}(Q) = \text{add}(M)$, it follows that the above two exact sequences are isomorphic. In particular we get (b), and (a) follows by Corollary 3.8 (a).

Once we have that $(\Delta_M, \{M_i\}_{i=1}^{t})$ is an Ext-projective stratifying system in $\text{mod}(A)$, the items (c), (d) and (e) follow from [25] Theorem 3.2.

Definition 3.9. Let $A$ be an algebra. An Ext-projective stratifying system $(\Theta, \{Q(i)\}_{i=1}^{t})$ in $\text{mod}(A)$, with the usual natural order on $[1, t]$, is said to be $\tau$-torsion-free admissible ($\tau$TF-admissible, for short) if $Q := \bigoplus_{i=1}^{t} Q(i)$ is $\tau$-rigid and a TF-admissible decomposition. We denote by $\tau\text{TFepss}(A)$ the class of all Ext-projective stratifying systems which are $\tau$TF-admissible. We also consider the class TFproper($A$) of all pairs $(M, \{M_i\}_{i=1}^{t})$ such that $M \in \text{mod}(A)$ is a non-zero basic $\tau$-rigid $A$-module and $M = \bigoplus_{i=1}^{t} M_i$ is a TF-admissible decomposition satisfying that $M \in \mathcal{F}(\Delta_M)$.

Theorem 3.10. For any algebra $A$, there are well defined functions

$$\tau\text{TFepss}(A) \xrightarrow{\Psi} \text{TFproper}(A) \xrightarrow{\Phi} \tau\text{TFepss}(A),$$
where $\mathcal{T}(\Theta, \{Q(i)\}_{i=1}^{t}, \leq) := (\bigoplus_{i=1}^{t} Q(i), \{Q(i)\}_{i=1}^{t})$ and $\Psi(M, \{M_i\}_{i=1}^{t}) := (\Delta M, \{M_i\}_{i=1}^{t}, \leq)$. Moreover, for any $X \in \tau \text{TFepss}(A)$ and $Y \in \text{TFproper}(A)$, we have that $\Psi(\mathcal{T}(X)) \approx X$ and $\Psi(\mathcal{T}(Y)) = Y$.

**Proof.** Let $Y \in \text{TFproper}(A)$. By Corollary 3.13 (b), we get that $\Psi(Y) \in \tau \text{TFepss}(A)$. Moreover, it is clear that $\mathcal{T}(\Psi(Y)) = Y$.

Let $X := (\Theta, \{Q(i)\}_{i=1}^{t}, \leq)$ be an Ext-projective stratifying system in mod $(A)$. By [25, Remark 2.7], we know that all the elements of the family $\{Q(i)\}_{i=1}^{t}$ are pairwise non-isomorphic. Let $Q := \bigoplus_{i=1}^{t} Q(i)$ be $\tau$-rigid and a TF-admissible decomposition. Then, we have the Ext-projective stratifying system $(\Delta Q, \{Q(i)\}_{i=1}^{t}, \leq) = \Psi(\mathcal{T}(X))$. By Definition 3.12 and Corollary 3.6, we have the canonical exact sequences

$0 \to K(i) \to Q(i) \xrightarrow{\varepsilon_i} \Theta(i) \to 0$ and $0 \to t_{i+1}(Q(i)) \to Q(i) \xrightarrow{\beta_i} \Delta Q(i) \to 0$,

where $\beta_i$ and $\gamma_i$ are both add($Q$)-covers. Let $A := \text{End}_A(Q)^{op}$. Then, by [25, Theorem 3.2 (a)] and Corollary 3.8 (e), it follows that $\Theta(i) \simeq Q \otimes_A \Delta(i) \simeq \Delta(i)$, and thus we get an isomorphism $\varepsilon_i : \Theta(i) \xrightarrow{\sim} \Delta Q(i)$ for each $i \in [1, t]$. Therefore, $\varepsilon_i \gamma_i : Q(i) \to \Delta Q(i)$ is an add($Q$)-cover, for each $i \in [1, t]$. Hence, for each $i$, there exists an isomorphism $\tau_i : Q(i) \rightarrow Q(i)$ such that $\varepsilon_i \gamma_i = \beta_i \tau_i$, proving that $\Psi(\mathcal{T}(X)) \approx X$. \hfill \Box

The construction, given above, of a stratifying system and of a $\tau$-torsion-free admissible Ext-projective stratifying system, by using a non-zero $\tau$-rigid module, can be dualized. For the sake of completeness we do that and state the corresponding results in such a way they can be used to construct new stratifying systems in connection with $\tau^{-1}$-rigid modules and costandard modules.

**Definition 3.11.** Let $A$ be an algebra. An $A$-module $M \in \text{mod } (A)$ is said to be $\tau^{-1}$-rigid if $\text{Hom}_A(\tau^{-1} M, M) = 0$.

**Definition 3.12.** Let $A$ be an algebra and $M \in \text{mod } (A)$ be a basic non-zero $A$-module. We say that a decomposition of $M = \bigoplus_{i=1}^{t} M_i$ as the direct sum of indecomposable $A$-modules is **torsion admissible** (T-admissible, for short) if $M_i \not\subseteq \text{Sub} \left( \bigoplus_{j>i} M_j \right)$, for every $i \in [1, t]$.

As in the case of the existence of a TF-admissible decomposition, for a non-trivial basic $\tau$-rigid object, we have the following result.

**Proposition 3.13.** For an algebra $A$, every basic non-zero $\tau^{-1}$-rigid module in mod $(A)$ admits a T-admissible decomposition.

**Proof.** It follows from Proposition 3.2 by applying the usual duality functor $D : \text{mod } (A) \to \text{mod } (A^{op})$. \hfill \Box

The connection between $\tau^{-1}$-rigid modules and stratifying systems can be stated as follows.

**Theorem 3.14.** Let $A$ be an algebra and $M \in \text{mod } (A)$ be a basic non-zero $\tau^{-1}$-rigid module with a T-admissible decomposition $M = \bigoplus_{i=1}^{t} M_i$, $n := \text{rk}(A)$, and let $t_k$ be the torsion functor associated to the torsion pair \((\bigoplus_{j \leq k} M_j), \text{Sub}(\bigoplus_{j \geq k} M_j)\). Then, the following statements hold true.

(a) The family $\nabla_M := \{\nabla_M(i) := t_{i+1}(M_i)\}_{i=1}^{t}$ and the opposite order $\leq^{op}$ of the natural order $\leq$ on $[1, t]$ form a stratifying system in mod $(A)$ of size $t$, which is called the $M$-costandard system associated with the given T-admissible decomposition of $M$.

(b) There exists at least one stratifying system $(\nabla'_M, \leq')$ of size $n \geq t$ in mod $(A)$, where $\leq'$ is the opposite of the natural order on $[1, n]$, such that $\nabla_M(i) = \nabla'_M(n - t + i)$ for all $i \in [1, t]$. 


Proof. It follows from Theorem 3.14 by applying the usual duality functor $D : \text{mod } (A) \to \text{mod } (A^{\text{op}})$.

Let $\mathcal{X} \subseteq \text{mod } (A)$. We denote by $\perp(\mathcal{X})$ the smallest torsion free class in $\text{mod } (A)$ containing $\mathcal{X}$.

**Corollary 3.15.** Let $A$ be an algebra, $M \in \text{mod } (A)$ be a basic non-zero $\tau^{-1}$-rigid module with a $T$-admissible decomposition $M = \bigoplus_{i=1}^{t} M_i$, $\overline{M}_i := \bigoplus_{k>i} M_k$, and let $\nabla_M$ be the $M$-costandard system associated with the given $T$-admissible decomposition of $M$. For each $i \in [1, t]$, consider the canonical exact sequence $0 \to \nabla_M(i) \xrightarrow{\alpha_i} M_i \to h_i(M(i)) \to 0$ which is given by the torsion pair $(\perp(\overline{M}_i), \text{Sub}(\overline{M}_i))$, and let $h_i : M_i \to N_i$ be the add$(\overline{M}_i)$-envelope of $M_i$. Then, the following statements hold true.

(a) $\text{Rej}_{\nabla_M}(M_i) = \text{Ker}(h_i) \simeq \nabla_M(i) \forall i \in [1, t]$.

(b) $\alpha_i : \nabla_M(i) \rightarrow M_i$ is an add$(M)$-envelope of $\nabla_M(i) \forall i \in [1, t]$.

(c) $\mathcal{F}(\nabla_M) \subseteq \text{Sub}(M) = \perp(\nabla_M)$.

Proof. It follows from Corollary 3.14 by applying the usual duality functor $D : \text{mod } (A) \to \text{mod } (A^{\text{op}})$.

**Remark 3.16.** Let $A$ be an algebra. Note that any decomposition into indecomposables of the $\tau^{-1}$-rigid module $D(A_A) = \bigoplus_{i=1}^{n} A(i)$ is $T$-admissible. Moreover, by Corollary 3.14(a), we get that the $D(A_A)$-costandard system coincides with the usual costandard $A$-modules.

**Corollary 3.17.** Let $A$ be an algebra, $M \in \text{mod } (A)$ be a basic non-zero $\tau^{-1}$-rigid module with a $T$-admissible decomposition $M = \bigoplus_{i=1}^{t} M_i$, $\overline{M}_i := \bigoplus_{k>i} M_k$, and let $\nabla_M$ be the $M$-costandard system associated with the given $T$-admissible decomposition of $M$. If $M \in \mathcal{F}(\nabla_M)$, then the following statements hold true.

(a) $M_i/\text{Rej}_{\nabla_M}(M_i) \in \mathcal{F}(\nabla_M(j)) : j > i$ for each $i \in [1, t]$.

(b) $(\nabla_M, \{M_i\}_{i=1}^{t}, \leq^{\text{op}})$ is an Ext-injective stratifying system in $\text{mod } (A)$ of size $t$, where $\leq^{\text{op}}$ is the opposite of the natural order $\leq$ on $[1, t]$.

(c) $\Gamma := \text{End}_A(M)$ is a basic standardly stratified algebra with respect to the decomposition into indecomposables $\Gamma = \bigoplus_{i=1}^{t} \Gamma P(i)$, where $\Gamma P(i) := \text{Hom}_A(M_i, M)$, and the opposite order $\leq^{\text{op}}$ on $[1, t]$.

(d) The functor $\text{Hom}_A(-, M) : \mathcal{F}(\nabla_M) \to \mathcal{F}(\Gamma \Delta)$ is a duality of categories with a quasi-inverse given by $\text{Hom}_A(-, M) : \mathcal{F}(\Gamma \Delta) \to \mathcal{F}(\nabla_M)$.

(e) $\text{Hom}_A(\nabla_M(i), i) \simeq \Gamma \Delta(i)$ for each $i \in [1, t]$.

Proof. It can be proved by using Corollary 3.18 and the usual duality functor $D : \text{mod } (A) \to \text{mod } (A^{\text{op}})$.

**Definition 3.18.** Let $A$ be an algebra. An Ext-injective stratifying system $(\Theta, \{Y(i)\}_{i=1}^{t}, \leq)$ in $\text{mod } (A)$, with the usual natural order on $[1, t]$, is said to be $\tau^{-1}$-torsion admissible ($\tau^{-1}$-$T$-admissible, for short) if $Y := \bigoplus_{i=1}^{t} Y(i)$ is $\tau^{-1}$-rigid and a $T$-admissible decomposition. We denote by $\tau^{-1}\text{Teiss}(A)$ the class of all Ext-injective stratifying systems which are $\tau^{-1}$-$T$-admissible. We also consider the class $\text{Teiss}(A)$ of all pairs $(M, \{M_i\}_{i=1}^{t})$ such that $M \in \text{mod } (A)$ is a non-zero basic $\tau^{-1}$-rigid $A$-module and $M = \bigoplus_{i=1}^{t} M_i$ is a $T$-admissible decomposition satisfying that $M \in \mathcal{F}(\nabla_M)$.

**Theorem 3.19.** For any algebra $A$, there are well defined functions

$$\tau^{-1}\text{Teiss}(A) \xrightarrow{\Gamma} \text{Teiss}(A) \xrightarrow{\Psi} \tau^{-1}\text{Teiss}(A),$$
is an equivalence of categories with a quasi-inverse given by
\[ A \mapsto \text{Hom}_A(B_M, -) \] and
\[ \Psi(M, \{M_i\}_i) \mapsto (\nabla M, \{M_i\}_i, \preceq). \]
Moreover, for any \( X \in \tau^{-1}\text{Teiss}(A) \) and \( Y \in \text{Teproper}(A) \), we have that \( \Psi(\Upsilon(X)) \simeq X \) and \( \Upsilon(\Psi(Y)) = Y \).

Proof. It follows from Theorem 3.10 by applying the usual duality functor \( D : \text{mod}(A) \to \text{mod}(A^{op}) \).

4. Perpendicular categories and \( \tau \)-exceptional sequences

In this second section of background, we recall the process of \( \tau \)-tilting reduction introduced by G. Jasso in [20] and the definition of \( \tau \)-exceptional sequences introduced by A. B. Buan and R. Marsh in [3]. This will allow us to compare \( \tau \)-exceptional sequences with the stratifying systems we found from Theorem 3.4 in Section 5.

In what follows, we start by given a brief summary of a technique, given by G. Jasso in [20], which is known as the \( \tau \)-tilting reduction.

Let \( M \in \text{mod}(A) \) be a basic \( \tau \)-rigid module and let \( B_M \in \text{mod}(A) \) be the Borgartz completion of \( M \). Following G. Jasso, we consider the algebras

\[ B_M := \text{End}_A(B_M)^{op} \quad \text{and} \quad C_M := B_M/(e_M), \]

where \( e_M \) is the idempotent associated to the \( B_M \)-projective module \( \text{Hom}_A(B_M, M) \). We regard \( \text{mod}(C_M) \) as a full subcategory of \( \text{mod}(B_M) \) via the canonical embedding which is given by the change of rings functor. The Jasso’s subcategory \( J(M) \) associated with the \( \tau \)-rigid module \( M \) is

\[ J(M) := M^\perp \cap \perp(\tau M). \]

Now we are able to state one of the main results of [20] that is given in [20] Theorem 3.8.

Theorem 4.1. Let \( M \) be a basic \( \tau \)-rigid module in \( \text{mod}(A) \). Then the functor

\[ F := \text{Hom}_A(B_M, -) : J(M) \to \text{mod}(C_M) \]

is an equivalence of categories with a quasi-inverse given by

\[ G := B_M \otimes_{B_M} - : \text{mod}(C_M) \to J(M). \]

The previous result says that there are relative \( \tau \)-rigid objects in the Jasso’s subcategory \( J(M) \). The following result, which appears in [20] Proposition 3.15, shows how to find them all.

Proposition 4.2. Let \( M \) and \( M' \) be two compatible \( \tau \)-rigid modules in \( \text{mod}(A) \), \( (\text{Fac}(M), M^\perp) \) be the torsion pair associated to \( M \), and let

\[ 0 \to t(M') \to M' \to f(M') \to 0 \]

be the canonical short exact sequence of \( M' \) with respect to \( (\text{Fac}(M), M^\perp) \). Then \( f(M') \) is a \( \tau \)-rigid object in \( J(M) \). Moreover, every \( \tau \)-rigid object in \( J(M) \) arises this way.

We have said already that \( \tau \)-tilting theory describes all the functorially finite torsion classes in \( \text{mod}(A) \) (cf. Theorem 2.2). But at the same time, \( \tau \)-tilting modules are not enough to complete that task and one needs to consider support \( \tau \)-tilting modules instead.

Now, when one considers a support \( \tau \)-tilting module, its associated idempotent is completely determined by the module. But sometimes it is important to consider the idempotent as an object on its own right. Then the notation of support \( \tau \)-tilting modules is not adequate.

This was already noticed by T. Adachi, O. Iyama and I. Reiten in [1] and they solved it by introducing the notation of \( \tau \)-tilting pairs. Using this notation, instead of having a module \( M \) and a "phantom" idempotent, one has a pair \( (M, P) \) where \( M \) is a \( \tau \)-rigid module and \( P \) is a projective \( A \)-module such that \( \text{Hom}_A(P, M) = 0 \).
Even that independence between the module and the idempotent is achieved with this new notation, it was not good enough for the study of exceptional sequences coming from \( \tau \)-tilting theory, since \( M \) and \( P \) of a \( \tau \)-tilting pair \((M, P)\) have different nature. That is why A. B. Buan and R. Marsh introduced in \cite{5} yet another way to denote the \( \tau \)-tilting, but this is done to the expense of working outside the realm of module categories. Before describing the notation used by A. B. Buan and R. Marsh, let us say that the perfect notation for \( \tau \)-tilting theory is yet to be defined.

**Definition 4.3.** \cite{5} Definitions 1.1] Let \( A \) be a finite dimensional algebra and consider its bounded derived category \( D^b(\mod A) \). Define \( C(A) \) to be the full subcategory of \( D^b(\mod A) \)
\[
\mathcal{C}(A) := \mod A \cup \mod A[1]
\]
whose objects are the disjoint union of the objects of the module category \( \mod A \) and its shift \( \mod A[1] \). Likewise, for every subcategory \( \mathcal{B} \) of \( \mod A \), define \( \mathcal{C}(\mathcal{B}) \) to be \( \mathcal{C}(\mathcal{B}) := \mathcal{B} \cup \mathcal{B}[1] \). An object \( \mathcal{M} = M \cup P[1] \) in \( \mathcal{C}(A) \) is said to be \( \tau \)-rigid if \( M \) is \( \tau \)-rigid in \( \mod A \), \( P \in \proj A \) and \( \Hom_A(P, M) = 0 \). If in addition we have that \( \rk(M) + \rk(P) = \rk(A) \), we say that \( \mathcal{M} \) is support \( \tau \)-tilting.

**Remark 4.4.** let \( A \) be an algebra and \( \mathcal{M} = M \cup P[1] \in \mathcal{C}(A) \) be a basic \( \tau \)-rigid object. The Jasso’s subcategory for \( \mathcal{M} \) is \( J(\mathcal{M}) := J(M) \cap J(P) \). Note that Jasso’s Theorem \cite{11} can be extended to this case, and thus \( J(\mathcal{M}) \) is a wide subcategory of \( \mod A \) which is equivalent to a module category of an algebra \cite{11} Theorem 4.12].

With this notation fixed, we can now recall the definition of \( \tau \)-exceptional sequences.

**Definition 4.5.** \cite{5} Definition 1.3] Let \( A \) be an algebra. A \( t \)-tuple of indecomposable objects \((U_1, \ldots, U_t)\) in \( \mathcal{C}(A) \) is a \( \tau \)-exceptional sequence if \( U_t \) is a \( \tau \)-rigid object in \( \mathcal{C}(A) \) and the tuple \((U_1, \ldots, U_{t-1})\) is a \( \tau \)-exceptional sequence in \( \mathcal{C}(J(U_t)) \).

**Remark 4.6.** For a \( \tau \)-exceptional sequence \((U_1, \ldots, U_t)\) in \( \mathcal{C}(A) \), we have that \( J(U_t) \) is equivalent to the module category of an algebra \( A_t \), allowing the inductive nature of the definition. Now, \( U_{t-1} \) is a \( \tau \)-rigid object inside \( \mathcal{C}(J(U_t)) = J(U_t) \cup J(U_t)[1] \). Hence one can calculate the perpendicular category of \( U_{t-1} \) inside \( \mathcal{C}(J(U_t)) \), and thus we get \( J(U_{t-1}) \). Proceeding inductively, we have that every \( \tau \)-exceptional sequence \((U_1, \ldots, U_t)\) induces a set \( \{J(U_1), \ldots, J(U_t)\} \) of nested abelian subcategories of \( \mathcal{C}(A) \).

**Definition 4.7.** \cite{5} Definition 1.2] Let \( A \) be an algebra. A \( t \)-tuple of indecomposable objects \((T_1, \ldots, T_t)\) in \( \mathcal{C}(A) \) is an ordered \( \tau \)-rigid object if \( \bigoplus_{i=1}^t T_i \) is a \( \tau \)-rigid object.

The main result of \cite{5} is the following.

**Theorem 4.8.** \cite{5} Theorem 5.4] Let \( A \) be an algebra and \( n := \rk(A) \). Then, for every \( t \in [1, n] \) there is a bijection between the set of ordered \( \tau \)-rigid objects in \( \mathcal{C}(A) \) having \( t \) non-isomorphic indecomposable summands and the set of \( \tau \)-exceptional sequences of length \( t \).

Moreover, they define a function \( E \) that is used to calculate the \( \tau \)-exceptional sequence associated to a \( \tau \)-rigid object with a given order. The reader is encouraged to go to \cite{5} Section 5] in order to see the exact definitions and constructions.

5. STRATIFYING SYSTEMS AND \( \tau \)-EXCEPTIONAL SEQUENCES

In this section we use the notation and the results which were introduced in Section 4 We show that every stratifying system, that is produced by using Theorem 3.4, can be seen as a \( \tau \)-exceptional sequence. Moreover, we characterise all the \( \tau \)-exceptional sequences that come from such stratifying systems. Our main result of this section is the following.
Theorem 5.1. Let $A$ be an algebra and $M \in \mod (A)$ be a non-zero basic $\tau$-rigid module with a TF-admissible decomposition $M = \bigoplus_{i=1}^{t} M_i$, and let $\Delta_M$ be the $M$-standard system associated with the given TF-admissible decomposition of $M$. Then, the sequence $\mathcal{U}_M := (\mathcal{U}_M(1), \ldots, \mathcal{U}_M(t))$, where $\mathcal{U}_M(i) := \Delta_M(i) \cup 0$ for $i \in [1, t]$, is a $\tau$-exceptional sequence of length $t$ in $\mathcal{C}(A)$. Moreover, every $\tau$-exceptional sequence in $\mathcal{C}(A)$ of the form $\mathcal{U} = (\mathcal{U}(1), \ldots, \mathcal{U}(t))$, where $\mathcal{U}(i) = (U(i) \cup 0$ and $U(i) \in \mod (A)$ for $i \in [1, t]$, arises this way.

The main difference between the two constructions is that $\tau$-exceptional sequences in $\mathcal{C}(A)$ are constructed inductively, while the construction of the stratifying system $\Delta_M$ is direct. Therefore, the first step in order to show the compatibility between both constructions, we need to show that the perpendicular categories that we find are the same. That is what we accomplish in the following results.

Lemma 5.2. Let $A$ be an algebra, $M \oplus M' \in \mod (A)$ be a basic $\tau$-rigid module and let $f : \mod (A) \to \mod (A)$ be the torsion free functor associated to the torsion pair $(\Fac(M), M^\perp)$. Then, $\Fac(M \oplus M') \cap M^\perp = \Fac(f(M'))$ in $\mathcal{C}(M)$.

Proof. By [20, Proposition 3.15] it follows that $f(M \oplus M')$ is Ext-projective in $\Fac(M \oplus M') \cap M^\perp$. In particular $f(M') \in \Fac(M \oplus M') \cap M^\perp$ and thus $\Fac(f(M')) \subseteq \Fac(M \oplus M') \cap M^\perp$ because it is a torsion class in $\mathcal{C}(M)$ by [20, Theorem 3.12].

Now, take $N \in \Fac(M \oplus M') \cap M^\perp$. Since $N \in \Fac(M \oplus M')$, there exist some $k \in \mathbb{N}$ and an epimorphism $p : (M \oplus M')^k \to N$. Consider the canonical short exact sequences

$$
0 \longrightarrow (M \oplus M')^k \overset{f}{\longrightarrow} (M \oplus M')^k \overset{g}{\longrightarrow} f((M \oplus M')^k) \longrightarrow 0
$$

and

$$
0 \longrightarrow t(N) \overset{f'}{\longrightarrow} N \overset{g'}{\longrightarrow} f(N) \longrightarrow 0
$$

of $(M \oplus M')^k$ and $N$ with respect to the torsion pair $(\Fac(M), M^\perp)$ and $p : (M \oplus M')^k \to N$ as before. Note that the image of the morphism $pf : t((M \oplus M')^k) \to N$ is a torsion submodule of $N$. Then, the properties of the canonical short exact sequence of a torsion pair imply that $pf$ factors through $f'$. Then we can construct the following exact and commutative diagram

$$
0 \longrightarrow (M \oplus M')^k \overset{f}{\longrightarrow} (M \oplus M')^k \overset{g}{\longrightarrow} f((M \oplus M')^k) \longrightarrow 0
$$

of $(M \oplus M')^k$ and $N$ with respect to the torsion pair $(\Fac(M), M^\perp)$ and $p : (M \oplus M')^k \to N$ as before. Note that the image of the morphism $pf : t((M \oplus M')^k) \to N$ is a torsion submodule of $N$. Then, the properties of the canonical short exact sequence of a torsion pair imply that $pf$ factors through $f'$. Then we can construct the following exact and commutative diagram

$$
0 \longrightarrow (M \oplus M')^k \overset{f}{\longrightarrow} (M \oplus M')^k \overset{g}{\longrightarrow} f((M \oplus M')^k) \longrightarrow 0
$$

where $\overline{p} : f((M \oplus M')^k) \to f(N)$ is the induced cokernel morphism of $p$.

There are several noteworthy facts coming from the previous diagram. The first of them is the fact that snake's lemma imply that $\overline{p}$ is an epimorphism because $p$ is an epimorphism. Secondly, we have by hypothesis that $N \in \Fac(M \oplus M') \cap M^\perp \subseteq M^\perp$ and thus $N$ is torsion free. Therefore, $g'$ is an isomorphism. Finally, the linearity of the functor $f$ implies that $f((M \oplus M')^k) \cong f((M')^k)$. Combining these three facts we have an epimorphism $\overline{p} : f((M')^k) \to N$, which implies that $\Fac(M \oplus M') \cap M^\perp \subseteq \Fac(f((M'))$. This finishes the proof.

As a direct consequence of the previous lemma is the following result.

Corollary 5.3. Let $A$ be an algebra, $M \in \mod (A)$ be a non-zero basic $\tau$-rigid module, $M = \bigoplus_{i=1}^{t} M_i$ be a TF-admissible decomposition of $M$, and let $f_i : \mod (A) \to \mod (A)$ be the torsion free functor associated to the torsion pair $(\Fac(M_i), M_i^\perp)$. Then $f_i(M)$ is a $\tau$-rigid object in $\mathcal{C}(M_i)$ and $f(M) = \bigoplus_{i=1}^{t} f_i(M_i)$ is a TF-admissible decomposition of $f(M)$.
Proof. The fact that \( \mathfrak{t}_i(M) \) is a \( \tau \)-rigid object in \( J(M_t) \) follows from Proposition 4.2. Similarly, the fact that \( \mathfrak{t}_i(M) \) is a \( \tau \)-rigid object in \( J(M_t) \) for all \( 1 \leq i \leq t - 1 \) follows from Proposition 4.2. Moreover, for every \( 1 \leq i \leq t - 1 \), the object \( \mathfrak{t}_i(M) \) is indecomposable by [5, Lemma 4.6]. Finally, these objects are pairwise non-isomorphic by [5, Lemma 4.7]. Therefore we have that

\[
\mathfrak{t}_i(M) = \bigoplus_{i=1}^{t-1} \mathfrak{t}_i(M_i)
\]

is a decomposition of \( \mathfrak{t}_i(M) \), since \( \mathfrak{t}_i(M_i) = 0 \). The fact that the previous decomposition is TF-admissible follows from Lemma 5.2.

Now we are ready to prove the main theorem of this section.

Proof of Theorem 5.7. Let \( M \in \text{mod} \ (A) \) be a \( \tau \)-rigid module with TF-admissible decomposition \( M = \bigoplus_{i=1}^{t-1} M_i \) as in the statement. We first show that

\[
\mathcal{U}_M := (\mathcal{U}_M(1), \ldots, \mathcal{U}_M(t)),
\]

where \( \mathcal{U}_M(i) := \Delta_M(i) \cup 0 \forall i \in [1, t] \), is a \( \tau \)-exceptional sequence. We prove this by induction.

First, by construction of \( \Delta_M \), we have that \( \Delta_M(t) = M_t \). Then \( \mathcal{U}_t := \Delta_M(t) \cup 0 \) is a support \( \tau \)-rigid object in \( \mathcal{C}(A) \) because \( \Delta_M(t) \) is a \( \tau \)-rigid indecomposable \( A \)-module.

Now, Corollary 5.3 implies that

\[
\mathfrak{t}_i(M) = \bigoplus_{i=1}^{t-1} \mathfrak{t}_i(M_i)
\]

is a TF-admissible decomposition of \( \mathfrak{t}_i(M) \), where \( \mathfrak{t}_i \) is the torsion free functor associated to the torsion pair \( (\text{Fac}(M_i), M_i^{\bot}) \). Then we have that \( \Delta_M(t-1) = \mathfrak{t}_i(M_{t-1}) \) is an indecomposable \( \tau \)-rigid module in \( J(M_t) \) by [5, Lemma 4.6] and [5, Lemma 4.7], implying that \( \mathcal{U}_{t-1} := \Delta_M(t-1) \cup 0 \) is an indecomposable support \( \tau \)-rigid in \( \mathcal{C}(\mathcal{U}_t) \). Hence \( \{\mathcal{U}_{t-1}, \mathcal{U}_t\} \) is a \( \tau \)-exceptional sequence of length 2.

Let \( \tilde{\mathfrak{t}}_{t-1} \) be the torsion free functor associated to the torsion pair induced by \( \mathfrak{t}_i(M_{t-1}) \) within \( J(\mathcal{U}_t) = M_t^{\bot} \cap \bot (\tau M_t) \) and \( \tilde{\mathfrak{f}}_{t-1} \) be the torsion free functor associated to the torsion pair

\[
(\text{Fac}(\oplus_{i=t-1}^t M_i), (\oplus_{i=t-1}^t M_i)^{\bot})
\]

generated by \( M_{t-1} \oplus M_t \). We claim that

\[
\tilde{\mathfrak{f}}_{t-1}(\mathfrak{t}_i(M_i)) \cong \mathfrak{t}_{i-1}(M_i)
\]

for all \( 1 \leq i \leq t - 2 \).

In order to show that, we first need to remark that Lemma 5.2 implies that

\[
\text{Fac} \left( \mathfrak{t}_i(M_{t-1}) \right) = \text{Fac} \left( M_{t-1} \oplus M_t \right) \cap M_t^{\bot}.
\]

Moreover \( \text{Fac} \left( M_{t-1} \oplus M_t \right) \subseteq \bot (\tau M_t) \). Hence

\[
\text{Fac} \left( \mathfrak{t}_i(M_{t-1}) \right) = \text{Fac} \left( M_{t-1} \oplus M_t \right) \cap J(\mathcal{U}_t).
\]

Now, let \( 1 \leq i \leq t - 2 \) and consider the following exact diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & t_t(M) & \longrightarrow & M_i & \longrightarrow & \mathfrak{t}_i(M) & \longrightarrow & 0 \\
0 & \longrightarrow & t_{t-1}(M) & \longrightarrow & M_i & \longrightarrow & \mathfrak{t}_{t-1}(M) & \longrightarrow & 0,
\end{array}
\]

where the first and the second row are the canonical short exact sequences of \( M_i \) with respect to the torsion pairs \( (\text{Fac}(M_i), (M_i)^{\bot}) \) and \( (\text{Fac}(M_{t-1} \oplus M_t), (M_{t-1} \oplus M_t)^{\bot}) \), respectively. Given
that \( \text{Fac}(M_t) \subseteq \text{Fac}(M_{t-1} \oplus M_t) \), the properties of torsion pairs imply the existence a monomorphism \( h : t_t(M_t) \rightarrow t_{t-1}(M_t) \) and an epimorphism \( \rho : f_t(M_t) \rightarrow f_{t-1}(M_t) \) making commutative the following diagram

\[
\begin{array}{c}
0 \rightarrow t_t(M_t) \rightarrow M_t \rightarrow f_t(M_t) \rightarrow 0 \\
\downarrow h \downarrow \downarrow \downarrow \downarrow \rho \\
0 \rightarrow t_{t-1}(M_t) \rightarrow M_t \rightarrow f_{t-1}(M_t) \rightarrow 0.
\end{array}
\]

Moreover, the Snake's Lemma implies that \( \text{Ker}(\rho) \cong \text{CoKer}(h) \). Then

\[
\text{Ker}(\rho) \in \text{Fac}(M_{t-1} \oplus M_t) \cap J(U_t).
\]

By Lemma 5.2 this is equivalent to say that

\[
\text{Ker}(\rho) \in \text{Fac}(f_t(M_{t-1})).
\]

Consider now the diagram below, where the bottom sequence is the canonical short exact sequence of \( f_t(M_t) \) with respect to the torsion pair \( (\text{Fac}(f_t(M_{t-1})), (\text{Fac}(M_{t-1})^\perp) \) inside \( J(U_t) \).

\[
\begin{array}{c}
0 \rightarrow \text{Ker}(\rho) \rightarrow f_t(M_t) \rightarrow f_{t-1}(M_t) \rightarrow 0 \\
\downarrow h' \downarrow \downarrow \downarrow \downarrow \rho' \\
0 \rightarrow \tilde{t}_{t-1}(f_t(M_t)) \rightarrow f_t(M_t) \rightarrow \tilde{f}_{t-1}(f_t(M_t)) \rightarrow 0.
\end{array}
\]

Again, the properties of the canonical short exact sequences allow us to complete the commutative diagram with a monomorphism \( h' : \text{Ker}(\rho) \rightarrow \tilde{t}_{t-1}(f_t(M_t)) \) and an epimorphism \( \rho' : f_{t-1}(M_t) \rightarrow \tilde{f}_{t-1}(f_t(M_t)) \) as follows

\[
\begin{array}{c}
0 \rightarrow \text{Ker}(\rho) \rightarrow f_t(M_t) \rightarrow f_{t-1}(M_t) \rightarrow 0 \\
\downarrow h' \downarrow \downarrow \downarrow \downarrow \rho' \\
0 \rightarrow \tilde{t}_{t-1}(f_t(M_t)) \rightarrow f_t(M_t) \rightarrow \tilde{f}_{t-1}(f_t(M_t)) \rightarrow 0.
\end{array}
\]

On the other hand, we have that \( \tilde{f}_{t-1}(f_t(M_t)) \in (M_{t-1} \oplus M_t)^\perp \) by Lemma 5.2. Moreover \( \tilde{f}_{t-1}(f_t(M_t)) \) is a quotient of \( M_t \). Therefore we have an epimorphism

\[
q' : \tilde{f}_{t-1}(f_t(M_t)) \rightarrow f_{t-1}(M_t).
\]

Then, given that we have an epimorphism \( \rho' : f_{t-1}(M_t) \rightarrow \tilde{f}_{t-1}(f_t(M_t)) \) and another epimorphism \( q' : \tilde{f}_{t-1}(f_t(M_t)) \rightarrow f_{t-1}(M_t) \), we can conclude that \( \tilde{f}_{t-1}(f_t(M_t)) \cong f_{t-1}(M_t) \) for all \( 1 \leq i \leq t-2 \) as claimed. Thus, again by [5 Lemma 4.6] and [5 Lemma 4.7], we have that \( \Delta_M(t-2) = f_{t-1}(M_{t-2}) \) is an indecomposable \( \tau \)-rigid object in \( J(U_{t-1}) \), implying that

\[
\{U_{t-2} := \Delta_M(t-2) \cup 0, U_{t-1} := \Delta_M(t-1) \cup 0, U_t := \Delta_M(t) \cup 0\}
\]

form a \( \tau \)-exceptional sequence of length three.

Now we make the inductive step. Suppose that

\[
\{U_k := \Delta_M(k) \cup 0, \ldots, U_{t-1} := \Delta_M(t-1) \cup 0, U_t := \Delta_M(t) \cup 0\}
\]

is a \( \tau \)-exceptional sequence of length \( k \). Repeating the same argument as before we can show that \( \tilde{f}_k(f_{k-1}(M_k)) \cong f_k(M_k) \) for all \( 1 \leq i \leq k-1 \), where \( \tilde{f}_k \) is the torsion free functor associated to the torsion pair induced by \( f_{k-1}(M_k) \) within \( J(U_{k-1}) \). Then one can apply one more time [5 Lemma 4.6] and [5 Lemma 4.7] to conclude that

\[
\{U_{k+1} := \Delta_M(k+1) \cup 0, U_k := \Delta_M(k) \cup 0, \ldots, U_t := \Delta_M(t) \cup 0\}
\]
is a $\tau$-exceptional sequence of length $k+1$. This finishes the proof of the first part of the theorem.

For the moreover part, let

$$\{ U_i = N_i \sqcup 0 : N_i \in \text{mod} (A) \text{ for all } 1 \leq i \leq t \}$$

be a $\tau$-exceptional sequence as in the statement. Then there exists an ordered support $\tau$-rigid object $M$ in $\mathcal{C}(A)$ inducing it by [5, Theorem 1.4]. Now, from the fact that every element in the $\tau$-exceptional sequence is of the form $N_i \sqcup 0$, we have a decomposition

$$M = \bigoplus_{i=1}^{t} M_i = \bigoplus_{i=1}^{t} M_i \sqcup 0,$$

where $M = \bigoplus_{i=1}^{t} M_i$ is a $\tau$-rigid module in mod ($A$). Moreover [5, Remark 5.12] implies that $f_{i+1}(M_i) \neq 0$ for all $1 \leq i \leq t$. This is equivalent to the fact that

$$M_i \not\in \text{Fac} \left( \bigoplus_{j \neq i} M_j \right).$$

In other words, the order of $M$ is TF-admissible. This finishes the proof. \hfill \Box

Remark 5.4. We argued in Remark [5.3] that Theorem 3.4 is not a characterisation of all stratifying systems. In exchange, Theorem 5.1 give us a characterisation of all stratifying systems induced by a $\tau$-rigid object.

6. Example

We finish the paper studying the number of stratifying systems induced by the $\tau$-tilting modules of an algebra.

Example 6.1. Let $A$ be the path algebra of the quiver

1 \rightarrow 2 \rightarrow 3

quotiented by the third power of its Jacobson radical. The Auslander-Reiten quiver of $A$ can be seen in Figure 1. Note that every module is represented by its Loewy series and both copies of $2 \rightarrow 3$ should be identified, so the Auslander-Reiten quiver of $A$ has the shape of a cylinder. One can see that there are 10 $\tau$-tilting modules in mod ($A$). In table 1 we give a complete list of
STRATIFYING SYSTEMS THROUGH $\tau$-TILTING THEORY

Table 1. The number of stratifying systems and $\tau$TF-admissible Ext-projective stratifying systems (up to isomorphisms) induced by each $\tau$-tilting module in $\text{mod}(A)$

| $\tau$-tilting module | ind. strat. syst. | $\tau$TFepss up to isom. |
|------------------------|-------------------|---------------------------|
| $1 \oplus 2 \oplus 3$  | 6                 | 0                         |
| $2 \oplus 3 \oplus 1$  | 2                 | 0                         |
| $2 \oplus 1 \oplus 3$  | 2                 | 0                         |
| $1 \oplus 2 \oplus 3$  | 2                 | 0                         |
| $1 \oplus 2 \oplus 1$  | 3                 | 3                         |
| $1 \oplus 3 \oplus 2$  | 1                 | 1                         |
| $3 \oplus 1 \oplus 3$  | 3                 | 3                         |
| $1 \oplus 3 \oplus 1$  | 3                 | 3                         |
| $2 \oplus 1 \oplus 3$  | 1                 | 1                         |

them together and we indicate how many stratifying systems and $\tau$TF-admissible Ext-projective stratifying systems (up to isomorphisms) they induce. In particular, from the first line of the table, we get that this algebra is not standardly stratified under any linear order on the set $\{1, 2, 3\}$.

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