TOTAL REALITY
OF CONORMAL BUNDLES OF HYPERSURFACES
IN ALMOST COMPLEX MANIFOLDS

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Abstract. A generalization to the almost complex setting of a well-known result by S. Webster is given. Namely, we prove that if \( \Gamma \) is a strongly pseudoconvex hypersurface in an almost complex manifold \((M, J)\), then the conormal bundle of \( \Gamma \) is a totally real submanifold of \((T^* M, J)\), where \( J \) is the lifted almost complex structure on \( T^* M \) defined by Ishihara and Yano.

1. Introduction

Let \((M, J)\) be an almost complex manifold of real dimension \(2n\) and \( J \) the associated almost complex structure on \( T^* M \) defined by Ishihara and Yano in [IY] (see definition in \( \S 2 \), below). Consider a smooth real hypersurface \( \Gamma \subset M \) and denote by \( \mathcal{N}(\Gamma) \subset T^* M \) the conormal bundle of \( \Gamma \), i.e. the submanifold of \( T^* M \) defined by

\[
\mathcal{N}(\Gamma) = \bigcup_{x \in \Gamma} \mathcal{N}(\Gamma)_x , \quad \text{where} \quad \mathcal{N}(\Gamma)_x \overset{\text{def}}{=} \{ \alpha \in T^*_x M : \alpha|_{T_x \Gamma} = 0 \} .
\]

The almost complex structure \( J \) of \( M \) induces on \( \Gamma \) a possibly non-integrable CR structure \((\mathcal{D}, J)\), i.e. a distribution \( \mathcal{D} \) given by the \( J \)-invariant subspaces of the tangent spaces of \( \Gamma \) and the complex structures \( J_x \overset{\text{def}}{=} J|_{\mathcal{D}_x} \) on the subspaces of the distribution \( \mathcal{D} \).

In this short note we prove that if the (possibly non-integrable) CR structure \((\mathcal{D}, J)\) on \( \Gamma \) is strongly pseudoconvex or if the almost complex structure \( J \) is integrable and \( \mathcal{D} \) is a contact distribution, then the complement of the zero section \( \mathcal{N}(\Gamma) \setminus \{ \text{zero section} \} \) is a totally real submanifold of \((T^* M, J)\)(see Theorem 3.3, below).

This fact was first proved by S. Webster in [We] for strongly pseudoconvex hypersurfaces in a complex manifold \( M \). Later, always assuming that \( M \) is a complex manifold, the result was generalized for the conormal bundles of Levi non-degenerate hypersurfaces and for Levi non-degenerate submanifolds of codimension higher then one by A. Tumanov in [Tu]. Another proof for the Levi non-degenerate hypersurfaces in complex manifold was given by Z. M. Balogh and C. Leuenberger in [BL]. At the best of our knowledge, the result under the weaker assumption that \( J \) is a possibly non-integrable almost complex structure was not previously known.

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We have to point out that requiring the conditions $J$ integrable and $D$ contact is equivalent to assume the Levi non-degeneracy of the hypersurface $\Gamma$ in the complex manifold $(M, J)$. This means that the second part of our claim is just equivalent to Tumanov’s generalization for hypersurfaces in complex manifolds.

The question whether Webster’s result could be valid also in the almost complex setting was asked to the author by A. Sukhov, being an interesting problem related to the theory of $J$-holomorphic discs attached to the boundary of strongly pseudoconvex domains. Indeed, as for the classical case (see e.g. [We]), our result should turn out to be quite useful for proving boundary regularity properties of $J$-biholomorphisms between bounded domains in almost complex manifolds. Our result has been also used by H. Gaussier and A. Sukhov in their recent paper [GS].

The author is grateful to A. Sukhov for telling him the problem and for useful discussions on the subject.

We tried to make the paper as much as possible complete and self-contained: After giving all needed preliminaries in §2 (e.g. presentation of Ishihara and Yano’s lifted almost complex structure in “coordinate-free notation”, definition of pseudoconvexity and Levi-forms for hypersurfaces in almost complex manifolds, etc.) the proof of our main result is given in §3.

2. Basic definitions and preliminaries

2.1. Lift of an almost complex structure to the cotangent bundle. Let $M$ be a $2n$-dimensional manifold endowed with an almost complex structure $J$ and let $\pi : T^* M \to M$ be its cotangent bundle. As we mentioned in the Introduction, Ishihara and Yano proved in [IY] that there exists a natural almost complex structure $\mathcal{J}$ on $T^* M$, such that:

a) it is a “lift” of $J$ on $T^* M$, i.e. $\pi_*(\mathcal{J} V) = J_\pi(V)$ for any vector field $V \in T(T^* M)$;

b) it is invariant w.r.t. the lifted action on $T^* M$ of any $J$-preserving diffeomorphism $f : M \to M$.

In order to define such lifted almost complex structure on $T^* M$, we first have to introduce a few objects.

First of all, let us denote by $\theta$ the so-called tautological 1-form of $T^* M$, i.e. the 1-form on $T^* M$ defined by

$$\theta_\alpha(V) \overset{\text{def}}{=} \alpha(\pi_*(V))$$

for any $\alpha \in T^* M$, $V \in T_\alpha(T^* M)$. It is known that $\omega = d\theta$ is a symplectic form on $T^* M$, which is called canonical symplectic form of $T^* M$.

We may also consider the inverse tensor field $\omega^{-1} \in \Lambda^2 T(T^* M)$, i.e. the tensor field of type $(2, 0)$ so that for any $\alpha \in T^* M$, $V \in T_\alpha(T^* M)$

$$\omega^{-1}|_\alpha(\omega_\alpha(V, \cdot), \cdot) = V.$$  \hfill (2.1)

It is clear that, in any given basis, the components of $\omega^{-1}$ are the entries of the inverse of the matrix given by the components of $\omega$. It is also immediate to check that for any $\alpha \in T^* M$ and $A \in T^*_\alpha(T^* M)$

$$\omega_\alpha(\omega^{-1}|_\alpha(A, \cdot), \cdot) = A.$$  \hfill (2.2)
Let us now consider the Nijenhuis tensor \( N^J \) of \( J \), i.e. the tensor field of type \((1,2)\) on \( M \), defined as follows: for any given pair of vectors \( v, w \in T_x M \), let us denote by \( X^{(v)} \) and \( X^{(w)} \) two vector fields such that \( X^{(v)}|_x = v \) and \( X^{(w)}|_x = w \); then \( N^J_x(v, w) \) is defined by

\[
N^J_x(v, w) \overset{\text{def}}{=} [JX^{(v)}, JX^{(w)}]_x - [X^{(v)}, X^{(w)}]_x - J([X^{(v)}, JX^{(w)}]_x + [JX^{(v)}, X^{(w)}]_x) .
\]

It is easily seen that \( N^J_x(v, w) \) is independent of the choice of the vector fields \( X^{(v)} \) and \( X^{(w)} \). The relevance of the tensor field \( N^J \) is determined by the celebrated theorem of Newlander and Nirenberg ([NN]), for which an almost complex structure \( J \) is an (integrable) complex structure if and only if \( N^J \equiv 0 \).

Now, with the help of \( N^J \) and of the symplectic form \( \omega \), we may define the following tensor field \( g^J \) of type \((0,2)\) on \( T^*M \): for any \( \alpha \in T^*M \) and \( V, W \in T_\alpha T^*M \), we set

\[
g^J_\alpha(V, W) \overset{\text{def}}{=} \frac{1}{2} \alpha(N^J_\pi^*(\pi_*(V), J\pi_*(W)) =
\]

\[
= \frac{1}{2} \alpha \left(-[JX^{(v)}, X^{(w)}]_x - [X^{(v)}, JX^{(w)}]_x + J[X^{(v)}, X^{(w)}]_x - J[JX^{(v)}, JX^{(w)}]_x \right) ,
\]

where, as before, \( X^{(v)} \) and \( X^{(w)} \) are two vector fields on \( M \) such that \( X^{(v)}|_x = v = \pi_*(V) \) and \( X^{(w)}|_x = w = \pi_*(W) \). From definitions, it is clear that \( g^J(\alpha)(V, W) \) is skew-symmetric w.r.t \( V \) and \( W \) and that, if \( \pi_*(W) = J\pi_*(V) \) then \( g^J(\alpha)(V, W) = 0 \).

The last necessary ingredient to define Ishihara and Yano’s almost complex structure \( \mathcal{J} \) on \( T^*M \) is the fiber preserving diffeomorphism \( \mathcal{J} : T^*M \to T^*M \), given by the map which associates to any 1-form \( \alpha \in T^*_x M \), \( x \in M \), the 1-form

\[
\mathcal{J}(\alpha) \overset{\text{def}}{=} \alpha \circ J \in T^*_x M .
\]

Now we can give the definition of lifted almost complex structure \( \mathcal{J} \).

**Definition 2.1.** Let \( \varpi^J \) be the tensor field of type \((0,2)\) on \( T^*M \) defined by

\[
\varpi^J \overset{\text{def}}{=} \mathcal{J}^*\omega + g^J = d(\mathcal{J}^*\theta) + g^J .
\]

We call lifted almost complex structure on \( T^*M \) associated with \( J \) the tensor field \( \mathcal{J} \) of type \((1,1)\) defined by

\[
J(v) \overset{\text{def}}{=} \omega^{-1}(\varpi^J(V, \cdot), \cdot) , \quad \text{for any } V \in T_\alpha(T^*M) .
\]

It is proved in [IY] that (2.6) does define an almost complex structure on \( T^*M \). Moreover, being \( \mathcal{J}^*\omega \) and \( g^N \) invariant under any lifted action of a biholomorphism of \((M, J)\), it is immediate to realize that the requirement b) of lifted almost complex structures holds. To check that also the condition a) is satisfied, it suffices to write down the explicit expressions of the components \( \mathcal{J} \) in some coordinate basis.

For this, consider a coordinate chart

\[
\xi = (x^1, \ldots, x^{2n}) : U \subset M \to \mathbb{R}^{2n}
\]

and an associated chart

\[
\hat{\xi} = (x^1, \ldots, x^{2n}, p_1, \ldots, p_{2n}) : \pi^{-1}(U) \subset T^*M \to \mathbb{R}^{4n} ,
\]
where we denote by $p_i$’s the components of the forms $\alpha = p_i dx^i \in \pi^{-1}(\mathcal{U})$ w.r.t. the coordinate basis $dx^1, \ldots, dx^n$. Now, if we fix $x \in \mathcal{U} \subset M$ and $\alpha = p_a dx^a \in T^*_x M$, the tensors $\mathcal{J}^\alpha|_\alpha$ and $g^\alpha$ have the following components:

$$
\mathcal{J}^\alpha|_\alpha = J^\alpha_i(x)(dp_a \otimes dx^i - dx^i \otimes dp_a) + p_a \left(J^\alpha_{j,i}(x) - J^\alpha_{i,j}(x)\right) dx^i \otimes dx^j, \quad (2.7)
$$

$$
\mathcal{g}^\alpha|_\alpha = \frac{1}{2} p_a N^\alpha_{\ell,i}(x) J^\ell_j(x) dx^i \otimes dx^j = \frac{1}{2} p_a \left[J^\ell_m(x) J^\alpha_{j,\ell,m}(x) - J^\ell_m(x) J^\alpha_{\ell,j,m}(x) - J^\alpha_{m,j}(x) J^\ell_j(x) + J^\alpha_{i,m}(x) J^\ell_j(x)\right] dx^i \otimes dx^j = \frac{1}{2} p_a \left\{ [J^\ell_m(x) J^\ell_j(x) - J^\ell_m(x) J^\ell_j(x)] J^\alpha_{j,\ell,m}(x) + (J^\alpha_{i,j}(x) - J^\alpha_{j,i}(x)) \right\} dx^i \otimes dx^j. \quad (2.8)
$$

In the above formulae, we denoted by $J^\ell_j(x)$ and $N^\alpha_{\ell,i}(x)$ the component of $J_x$ and $N^\alpha_x$ in the coordinate frames of $M$ and by $J^\alpha_{i,j}(x)$ the partial derivatives $J^\alpha_{i,j}(x) \overset{\text{def}}{=} \frac{\partial J^\alpha_{i,j}}{\partial x^j}$. Since $\omega^{-1}|_\alpha = \frac{\partial}{\partial x^a} \otimes \frac{\partial}{\partial p_a} - \frac{\partial}{\partial p_a} \otimes \frac{\partial}{\partial x^a}$, we immediately obtain that $\mathcal{J}|_\alpha$ is of the form

$$
\mathcal{J}|_\alpha = J^\alpha_i(x) dx^i \otimes \frac{\partial}{\partial x^a} + J^\alpha_j(x) dp_a \otimes \frac{\partial}{\partial p_j} + \frac{1}{2} p_a \left\{ [J^\ell_m(x) J^\ell_j(x) - J^\ell_m(x) J^\ell_j(x)] J^\alpha_{j,\ell,m}(x) - [J^\alpha_{m,j}(x) - J^\alpha_{j,m}(x)] \right\} dx^i \otimes \frac{\partial}{\partial p_i}. \quad (2.9)
$$

From this explicit expression, one can directly check that requirement a) holds. Furthermore, it is also clear that if $V \in T(T^*M)$ is a vertical vector, then also the vector $\mathcal{J}(V)$ is vertical.

2.2. Hypersurfaces in almost complex manifolds and their Levi forms. For a submanifold $S \subset M$ of an almost complex manifold $(M, J)$, we call $J$-invariant (or $J$-holomorphic) distribution of $S$ the family of subspace $D_x \subset T_x S$, $x \in S$, defined by

$$
D_x = \{ v \in T_x S : J(v) \in T_x S \}. \quad (2.10)
$$

**Definition 2.2.** A submanifold $S \subset M$ in an almost complex manifold $(M, J)$ is called totally real if the $J$-invariant subspaces (2.10) are trivial at any point.

Assume now that $\Gamma$ is a hypersurface in $M$. Notice that, in case $\Gamma$ is (locally) defined as zero set of a smooth real valued function $\rho$ (i.e. $\Gamma = \{ x \in M : \rho(x) = 0 \}$), we may consider the 1-form

$$
\partial_x = (d \rho \circ J)|_{T_x \Gamma}. \quad (2.11)
$$

Such 1-form satisfies

$$
\ker \partial_x = D_x \quad \text{for any } x \in \Gamma. \quad (2.12)
$$
We call any 1-form $\vartheta$ satisfying (2.12) a defining 1-form for $\mathcal{D}$. The Levi form at $x \in \Gamma$ associated with a defining 1-form $\vartheta$ is the quadratic form

$$L_x(v) \overset{\text{def}}{=} -d\vartheta_x(v, Jv), \quad \text{for any } v \in T_x \Gamma.$$  

Notice that, for any vector field $X(v)$ with values in $\mathcal{D}$ such that $X(v)|_x = v$, we may write

$$L_x(v) = -d\vartheta_x(X(v), JX(v)) =$$

$$= -X(v)(\vartheta(JX(v))|_x + JX(v)(\vartheta(X(v))|_x + \vartheta([X(v), JX(v)]|_x =$$

$$= \vartheta([X(v), JX(v)]|_x,$$

where we used the fact that, by construction, $\vartheta(X(v)) \equiv \vartheta(JX(v)) \equiv 0$.

The above identity shows that, in case of an integrable complex structures $J$, the Levi form $L_x$ defined in (2.13) coincides with the Levi form as classically defined in the theory of functions of several complex variables.

As in the classical case, up to multiplication by a non-zero real number, the Levi form $L_x$ does not depend on the choice of the defining 1-form $\vartheta$. Moreover, by polarization, we may say that $L_x$ is the quadratic form associated with the symmetrized bilinear form $(L_x)^s$, where $L_x$ is

$$L_x : \mathcal{D}_x \times \mathcal{D}_x \to \mathbb{R}, \quad L_x(v, w) = -d\vartheta_x(v, Jw)$$

and $(L_x)^s$ is the symmetric part of $L_x$, i.e.

$$(L_x)^s(v, w) = -\frac{1}{2} (d\vartheta_x(v, Jw) + d\vartheta_x(w, Jv)).$$

If $J$ is an integrable complex structure (i.e. $N^J \equiv 0$), it is not difficult to check that the bilinear form $L_x$ is symmetric and $J$-invariant and that $L_x$ is the quadratic form associated with $L_x$.

**Definition 2.3.** We say that a hypersurface $\Gamma \subset M$ is strongly pseudoconvex if, for some choice of the defining 1-form $\vartheta$, the Levi form $L_x$ is positive definite at any point $x \in \Gamma$.

We say that $\Gamma$ is Levi non-degenerate if, for some choice of the defining 1-form $\vartheta$, the symmetric form $(L_x)^s$ is non-degenerate at any point $x \in \Gamma$.

It is clear that if $\Gamma$ is strongly pseudoconvex (resp. Levi non-degenerate), for any choice of the defining form $\vartheta$, the corresponding Levi form $L_x$ is either positive definite or negative definite (resp. the symmetric form $(L_x)^s$ is non-degenerate).

**Remark 2.4.** In case $J$ is an integrable complex structure, the Levi non-degeneracy condition is equivalent to the non-degeneracy of $d\vartheta|_{\mathcal{D} \times \mathcal{D}}$, i.e. to the hypothesis that $\mathcal{D}$ is a contact distribution. On the other hand, the reader should be aware that such equivalence is no longer valid if $J$ is not integrable, because, in general, the non-degeneracy of $(L_x)^s$ does not give any information on the non-degeneracy of the bilinear form $L$ (and hence on the non-degeneracy of $d\vartheta|_{\mathcal{D} \times \mathcal{D}}$). If $J$ is not integrable, it is possible to infer that $\mathcal{D}$ is contact, only if $\Gamma$ is strongly pseudoconvex or under more explicit conditions on $\mathcal{D}$.
3. **Total reality of the conormal bundles of strongly pseudoconvex hypersurfaces**

In what follows, $(M, J)$ will denote an almost complex manifold of real dimension $2n$, $\Gamma$ a hypersurface in $M$, endowed with the $J$-invariant distribution $\mathcal{D}$, and $\mathcal{N}^\ast(\Gamma) \subset T^\ast M$ the conormal bundle of $\Gamma$ (see definition in the Introduction). We will also denote by $N$ the Nijenhuis tensor of $J$, by $\hat{\mathcal{D}}$ the lifted almost complex structure of $T^\ast M$ and by $\hat{\mathcal{D}}_\alpha$ the $J$-invariant distribution of the submanifold $\mathcal{N}^\ast(\Gamma)$.

**Lemma 3.1.** For any $\alpha \in \mathcal{N}^\ast(\Gamma)$, the projection $\pi_\ast : T_\alpha(\mathcal{N}^\ast(\Gamma)) \to T_{\pi(\alpha)}\Gamma$ maps injectively the $J$-invariant subspace $\mathcal{D}_\alpha \subset T_\alpha(\mathcal{N}^\ast(\Gamma))$ onto a subspace of the $J$-invariant subspace $\mathcal{D}_{\pi(\alpha)} \subset T_{\pi(\alpha)}\Gamma$.

**Proof.** It suffices to show that the vertical subspace

$$\mathcal{V}_\alpha = \{ V \in T_\alpha(\mathcal{N}^\ast(\Gamma)) : \pi_\ast(V) = 0 \}$$

has trivial intersection with $\hat{\mathcal{D}}_\alpha$. But this is a direct consequence of the following two facts that: 1) $J$ maps vertical vectors into vertical vectors and hence $\mathcal{V}_\alpha \cap \hat{\mathcal{D}}_\alpha$ is an even dimensional $J$-invariant subspace of $\mathcal{V}_\alpha$; 2) $\mathcal{V}_\alpha$ has dimension one. □

From the previous lemma, we have that for any $0 \neq V \in \hat{\mathcal{D}}_\alpha$, the projection $\pi_\ast(V)$ must be a non-trivial element of $\mathcal{D}_{\pi(\alpha)}$.

**Lemma 3.2.** $\mathcal{N}^\ast(\Gamma)$ is a Lagrangian submanifold of $(T^\ast M, \omega)$, i.e. for any $\alpha \in \mathcal{N}^\ast(\Gamma)$ and any $V, W \in T_\alpha(\mathcal{N}^\ast(\Gamma))$

$$\omega_\alpha(V, W) = 0 .$$

**Proof.** Consider two vector fields $X^{(V)}, X^{(W)}$ in $\mathcal{N}^\ast(\Gamma)$ such that $X^{(V)}|_\alpha = V$ and $X^{(W)}|_\alpha = W$. By definitions,

$$\omega_\alpha(V, W) = d\theta_\alpha(X^{(V)}, X^{(W)}) =$$

$$= X^{(V)}(\theta(X^{(W)}))|_\alpha - X^{(W)}(\theta(X^{(V)}))|_\alpha - \theta_\alpha([X^{(V)}, X^{(W)}]) = 0 ,$$

because for any vector field $Z$ on $\mathcal{N}^\ast(\Gamma)$, $\pi_\ast(Z) \in TT$ and hence $\theta_\beta(Z) = \beta(\pi_\ast(Z)) = 0$ for any $\beta \in \mathcal{N}^\ast(\Gamma)$. □

Now, we can state and prove our main theorem.

**Theorem 3.3.** Let $\Gamma \subset M$ be a hypersurface in an almost complex manifold $(M, J)$ and assume that at least one of the following hypothesis is satisfied:

i) $J$ is integrable and the $J$-invariant distribution is contact (i.e. $\Gamma$ is Levi non-degenerate);

ii) $\Gamma$ is strongly pseudo-convex.

Then $\mathcal{N}^\ast(\Gamma) \setminus \{ \text{zero section} \}$ is a totally real submanifold of $T^\ast M$.

**Proof.** Suppose not and assume that, for some $\alpha \in \mathcal{N}^\ast(\Gamma) \setminus \{ \text{zero section} \}$, there exists a non-trivial vector $V$, which belongs to the $J$-invariant subspace $\mathcal{D}_\alpha$, i.e. $0 \neq J(V) \in T_\alpha \mathcal{N}^\ast(\Gamma)$. By Lemma 3.2 we also have that for any $W \in T_\alpha \mathcal{N}^\ast(\Gamma)$

$$\omega_\alpha(J(V), W) = 0 .$$

(3.1)
On the other hand, by (2.2) and the definition of $J$
\[
\omega_\alpha(J(V), W) = d(\hat{J}\theta)_\alpha(V, W) + g^J_\alpha(V, W) .
\] Formulae (3.1) and (3.2) imply that, for any vector fields $X(V)$ and $X(W)$ on $N^*(\Gamma)$ such that
\[
X(V)|_\alpha = V , \quad X(W)|_\alpha = W ,
\]
we have that
\[
X(V)(\hat{J}\theta(X(W)))|_\alpha - X(W)(\hat{J}\theta(X(V)))|_\alpha - \hat{J}\theta_\alpha([X(V), X(W)]) = \frac{1}{2}\alpha(N(\pi_*(X(V)), J\pi*(X(W))|_{\pi(\alpha)}) .
\] (3.3)

Now, recall that, by Lemma \ref{lem:3.1}, $v \equiv \pi_*(V) \neq 0$ and belongs to $D_{\pi(\alpha)}$. Assume now that the vector $W$ is so that $w \equiv \pi_*(W) \in D_{\pi(\alpha)}$ and let us denote by $\hat{X}(V)$ and $\hat{X}(W)$ the two vector fields in $TT$ defined by
\[
\hat{X}(V) \equiv \pi_*(X(V)) , \quad \hat{X}(W) \equiv \pi_*(X(W)) .
\]
Clearly, $\hat{X}(V)|_{\pi(\alpha)} = v$ and $\hat{X}(W)|_{\pi(\alpha)} = w$.

Now, notice that if $\rho$ is a local defining function for $\Gamma$, then any $\beta \in N^*(\Gamma) \setminus \{\text{zero section}\}$ is of the form $\beta = \lambda \cdot d\rho$, for some $\lambda \in \mathbb{R} \setminus \{0\}$. Hence, by definition of the tautological form $\theta$ and since $\hat{J}$ is fiber preserving, for any $\beta \in N^*(\Gamma) \setminus \{\text{zero section}\}$ and any vector field on $N^*(\Gamma)$
\[
\hat{J}\theta_\beta(Z) = \beta(J\pi_*(Z)) = \lambda \cdot d\rho(J\pi_*(Z)) = \lambda \cdot \theta_\beta(Z)
\]
where $\theta = d\rho \circ J$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Recall that $\theta$ satisfies (2.11). Now, if $\alpha = \lambda_0 \cdot d\rho|_{\pi(\alpha)}$ for some fixed $\lambda_0 \neq 0$, then the left hand side of (3.3) can be written as
\[
X(V)
\left(\lambda \cdot \theta(\hat{X}(W)) \right)|_{\pi(\alpha)} - X(W)
\left(\lambda \cdot \theta(\hat{X}(V)) \right)|_{\pi(\alpha)} - \lambda_0 \cdot \theta_\pi(\alpha)([\hat{X}(V), \hat{X}(W)]) =
X(V)(\lambda)|_{\alpha} \cdot \theta_\pi(\alpha)(w) - X(W)(\lambda)|_{\alpha} \cdot \theta_\pi(\alpha)(v) +
\lambda_0 \cdot \theta(\hat{X}(W))|_{\pi(\alpha)} - \hat{X}(W)(\theta(\hat{X}(V))|_{\pi(\alpha)} - \theta_\pi(\alpha)([\hat{X}(V), \hat{X}(W)]) =
X(V)(\lambda)|_{\alpha} \cdot \theta_\pi(\alpha)(w) - X(W)(\lambda)|_{\alpha} \cdot \theta_\pi(\alpha)(v) + \lambda_0 \cdot d\theta_\pi(\alpha)(v, w) .
\] (3.4)

Since we are assuming that $v$ and $w$ are both in $D$, we have that $\partial_\pi(\alpha)(v) = \partial_\pi(\alpha)(w) = 0$ and hence (3.3) reduces to
\[
\lambda_0 \cdot d\theta_\pi(\alpha)(v, w) = -\frac{1}{2}\alpha(N(v, Jw)) .
\] (3.5)

This should be true for any choice of $w \in D_{\pi(\alpha)}$. This gives a contradiction if $N \equiv 0$ (i.e. $J$ is integrable) and $D$ is contact, because in this case there should exist a vector $w \in D_{\pi(\alpha)}$ so that $d\theta_\pi(\alpha)(v, w) \neq 0$. In case $\Gamma$ is strongly pseudo-convex and $N$ is not necessarily equal to $0$, in order to get a contradiction it suffices to assume that $w = Jv$. In fact, in this case, $N(v, Jw) = -N(v, v) = 0$, while, by strong pseudoconvexity $\lambda_0 \cdot d\theta_\pi(\alpha)(v, Jv) = -\lambda_0 \cdot \mathcal{L}_\pi(\alpha)(v) \neq 0$. □
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