An update on the status of NSPT computations

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Outline of the talk

► Motivation

► Numerical Stochastic Perturbation Theory

► RI-MOM’ scheme

► Perturbative results

► Resummation of PT series

► Conclusions

Talk based on

M. B., F. Di Renzo, Eur. Phys. J. C 73 (2013) 2666
M. B., F. Di Renzo, M. Hasegawa, arXiv:1402.6581 [hep-lat]
(accepted on EPJC)
Non-perturbative computations has been the preferred choice for quite a long time, but:

- strictly speaking multiplicative renormalizability is proved only in Perturbation Theory; and
- fermion bilinears are either finite or only logarithmically divergent. Since there are no power divergences PT must work.

**Drawbacks of PT**

- perturbative series are badly convergent.
  - go to high order
- diagrammatic Lattice PT is cumbersome;
  - use an automated technique
A sketch of NSPT

Let the system evolve according Langevin dynamic in a “fictitious” time $t$

\[ \partial_t U(x, t) = \{-i\nabla S[U(x, t)] - i\eta(x, t)\} U(x, t) \]

where $\langle \eta(x, t) \rangle = 0 \quad \langle \eta(x, t)\eta(x', t') \rangle = 2\delta(x - x')\delta(t - t').$

By expanding the link in a power series one gets a system of equations to be truncated at a given order (Stochastic PT).

The differential equations can be traded for integral ones (in this way one would get diagrams); in out approach the integration is performed numerically on a computer.

Inverting the fermionic (Dirac) operator turns into inverting a series:

\[ M[U(x, t)]^{-1} = M^{-1(0)} + \beta^{-\frac{1}{2}} M^{-1(1)} + \ldots \]

\[ M^{-1(0)} = M^{(0)}^{-1}, \quad M^{-1(n)} = -M^{(0)}^{-1} \sum_{j=0}^{n-1} M^{(n-1)} M^{(j)}^{-1} \]
RI-MOM' SCHEME

Starting from Green functions (in Landau gauge)

\[ G_{\Gamma}(p) = \int dx \langle p | \bar{\psi}(x)\Gamma\psi(x) | p \rangle \]

vertex functions are obtained by amputation

\[ \Gamma_{\Gamma}(p) = S^{-1}(p) G_{\Gamma}(p) S^{-1}(p). \]

The quark field renormalization constant has to be computed from the condition

\[ Z_q(\mu, \alpha) = -i \frac{1}{12} Tr(\not{p}S^{-1}(p)) \frac{1}{p^2} \bigg|_{p^2=\mu^2}. \]

After projecting on tree-level structure

\[ O_{\Gamma}(p) = Tr \left( \hat{P}_{O_{\Gamma}} \Gamma_{\Gamma}(p) \right), \]

one enforces renormalization conditions that read

\[ Z_{O_{\Gamma}}(\mu, \alpha) Z_q^{-1}(\mu, \alpha) O_{\Gamma}(p) \big|_{p^2=\mu^2} = 1. \]
Zero quark mass and logarithmic divergencies

In order to have a mass-independent scheme, all this is defined at zero quark mass: this requires knowledge of the critical mass (known up to 2-loop, 3-loop as a byproduct).

Critical mass is computed from the propagator:

\[
\hat{S}(\hat{p}, \hat{m}_{cr}, \beta^{-1})^{-1} = i\hat{p} + \hat{m}_W(\hat{p}) - \hat{\Sigma}(\hat{p}, \hat{m}_{cr}, \beta^{-1})
\]

\[
\hat{\Sigma}(0, \hat{m}_{cr}, \beta^{-1}) = \hat{m}_{cr}
\]

\[
\hat{m}_{cr}^{(3),tls} = -3.94(4) \quad \hat{m}_{cr}^{(3),iwa} = -0.78(2)
\]

Advantage of RI-MOM' scheme: logarithmic contributions to quark bilinears can be inferred from continuum computations \((l = \log(\mu a)^2)\)

\[
\gamma_{O\Gamma} = \frac{1}{2} \frac{d}{dl} \log Z_{O\Gamma} \quad \Rightarrow \quad Z_{O\Gamma} = 1 + \alpha \left( c_1 - \gamma_{O\Gamma}^{(1)} l \right) + O(\alpha^2)
\]
LATTICE ARTIFACTS

A prototypal fitting form of ours reads:

\[ \hat{O}_\Gamma(\hat{p}, pL, \nu) = c_1 + c_2 \sum_\sigma \hat{p}_\sigma^2 + c_3 \frac{\sum_{\sigma} \hat{p}_\sigma^4}{\sum_\rho \hat{p}_\rho^2} + c_4 \hat{p}_\nu^2 + \Delta \hat{O}_\Gamma(pL) + \mathcal{O}(a^4) \]

- the \( a \to 0 \) limit can be obtained by means of the hypercubic expansion;
- by computing \( \hat{O}_\Gamma(\hat{p}, pL, \nu) \) on different volumes we can account for finite size corrections;
- performing a combined fit we account for the limits \( a \to 0 \) and \( L \to \infty \) simultaneously.
## Results

- **n_f=2** tree-level Symanzik [M. B., F. Di Renzo]

|                | analytical one-loop | one-loop | two-loop     | three-loop    |
|----------------|---------------------|----------|--------------|---------------|
| $Z_S$          | -0.6893             | -0.683(7)| -0.777(24)   | -1.96(14)     |
| $Z_P$          | -1.1010             | -1.098(11)| -1.299(38)  | -3.19(21)     |
| $Z_V$          | -0.8411             | -0.838(6)| -0.891(17)   | -1.870(65)    |
| $Z_A$          | -0.6352             | -0.633(4)| -0.611(16)   | -1.198(57)    |

- **n_f=4** Iwasaki [M. B., F. Di Renzo, M. Hasegawa]

|                | analytical one-loop | one-loop | two-loop     | three-loop    |
|----------------|---------------------|----------|--------------|---------------|
| $Z_S$          | -0.4488             | -0.442(6)| -0.170(11)   | -0.33(11)     |
| $Z_P$          | -0.7433             | -0.739(7)| -0.202(13)   | -0.58(11)     |
| $Z_V$          | -0.5623             | -0.561(7)| -0.067(12)   | -0.367(61)    |
| $Z_A$          | -0.4150             | -0.419(6)| -0.033(12)   | -0.236(56)    |

(results are available also for n_f=0)
Summing the series

We can sum the series and compare with non perturbative results (Symanzik $\beta = 4.05$) [M. Constantinou et al. JHEP08(2010)068]

|       | $Z_V$    | $Z_A$    | $Z_S$    | $Z_P$    |
|-------|----------|----------|----------|----------|
| NSPT  | 0.710(2)(28) | 0.788(2)(18) | 0.753(4)(30) | 0.601(5)(48) |
| ETMC(M1) | 0.659(4)     | 0.772(6)   | 0.645(6)  | 0.440(6)  |
| ETMC(M2) | 0.662(3)     | 0.758(4)   | 0.678(4)  | 0.480(4)  |

(Iwasaki $\beta = 2.10$) [arXiv:1403.4504 [hep-lat]]

|       | $Z_V$    | $Z_A$    | $Z_S$    | $Z_P$    |
|-------|----------|----------|----------|----------|
| NSPT  | 0.677(9)(39) | 0.769(9)(25) | 0.712(14)(36) | 0.538(15)(63) |
| ETMC(M1) | 0.655(03)     | 0.762(04)   | 0.700(06)  | 0.516(02)  |
| ETMC(M2) | 0.657(02)     | 0.752(02)   | 0.749(03)  | 0.545(02)  |

- thee-loop contribution is relatively important: quite large truncation errors
- fair agreement between PT and non PT for Iwasaki action and finite Symanzik
- deviation between PT and non PT in Symanzik divergent
We can assess irrelevant effects by discarding the continuum limit and finite size contributions:

$$\tilde{O}^{(i)}_\Gamma (\hat{p}, \nu) = c_{2}^{(i)} \sum_\sigma \hat{p}_\sigma^2 + c_{3}^{(i)} \sum_\sigma \frac{\hat{p}_\sigma^4}{\sum_\rho \hat{p}_\rho^2} + c_{4}^{(i)} \hat{p}_\nu^2 + \mathcal{O}(a^4)$$

The resummed quantity \( \sum_{i=1}^{3} \beta^{-i} \frac{1}{4} \sum_{\nu=1}^{4} \tilde{O}^{(i)}_\Gamma (\hat{p}, \nu) \) can be regarded as the irrelevant contributions to \( Z_\Gamma \)

Finite size effects can be reconstructed to a fair accuracy provided one fits terms compliant to the lattice symmetries.
**Boosting the resummations**

Re-express the series as expansions in different couplings:

can we find better convergence proprieties?

|         | $x_0 = \frac{\beta^{-1}}{\sqrt{P}}$ | $x_1 = -\frac{1}{P(0)} \log(P)$ | $x_2 = \frac{\beta^{-1}}{P}$ | (M1)  | (M2)  |
|---------|--------------------------------------|---------------------------------|--------------------------------|-------|-------|
| $Z_V$   | 0.686(21)                            | 0.688(17)                       | 0.661(55)                      | 0.659(4) | 0.662(3) |
| $Z_A$   | 0.773(12)                            | 0.775(9)                        | 0.763(26)                      | 0.772(6) | 0.758(4) |
| $Z_S$   | 0.727(29)                            | 0.726(27)                       | 0.705(49)                      | 0.645(6) | 0.678(4) |
| $Z_P$   | 0.558(45)                            | 0.558(41)                       | 0.526(73)                      | 0.440(6) | 0.480(4) |

where $P$ is the $1 \times 1$ plaquette.
BPT apparently solves the problem of the discrepancies for $Z_V$ and $Z_A$;

discrepancies are still there for $Z_S$ and $Z_P$:
  - should even higher order terms be included?
  - could non-perturbative computations suffer from finite volume effects (any interplay between IR and UV effects)?

**SOME GENERAL REMARK**

we put forward a method to assess finite size effects: there is in principle no reason why one should not attempt the same in the non-perturbative case;

high-loop computations can provide a new handle to correct non-perturbative computations with respect to irrelevant contributions.
CONCLUSIONS

We computed 2 and 3-loop Renormalization Constants for quark bilinears in different regularizations.

- NSPT provides an approach independent w.r.t. non perturbative computations (different systematic effects);
- in principle there is no constraint on computing finite constants;
- in divergent constants we are limited to 3-loop order because of continuum computations;
- NSPT provides a new method to correct non-perturbative computations with respect to irrelevant contributions.

THANK YOU FOR YOUR ATTENTION
TAMING THE LOGS

$Z$’s expansion is in the form

$$Z(\mu, \alpha_0) = 1 + \sum_{n>0} \bar{d}_n(l) \alpha_0^n \quad \bar{d}_n(l) = \sum_{i=0}^n \bar{d}_n^{(i)} l^i.$$ 

By differentiating w.r.t $\log(\mu \alpha)^2$ one obtains the anomalous dimension

$$\gamma = \frac{1}{2} \frac{d}{dl} \log Z(\mu, \alpha) = \sum_{n>0} \gamma_n \alpha(\mu)^n$$

that depends only on the scheme.

PROCEDURE

- match the two expansion above (all log’s must cancel out);
- re-express the expansion in the bare coupling $\alpha_0$;
- subtract divergences from $Z$’s before performing fits.
Finite lattice spacing effects

Consider the case of quark field renormalization constant $Z_q$. Hypercubic symmetry fixes the (expected) form of self energy:

\[
\frac{1}{4} \sum_{\mu} \gamma_\mu \text{Tr}_{\text{spin}}(\gamma_\mu \hat{\Sigma}) = i \sum_{\mu} \gamma_\mu \hat{p}_\mu \left( \hat{\Sigma}_\gamma^{(0)}(\hat{p}) + \hat{p}_\mu \hat{\Sigma}_\gamma^{(1)}(\hat{p}) + \hat{p}_\mu^2 \hat{\Sigma}_\gamma^{(2)}(\hat{p}) + \ldots \right)
\]

\(\hat{\Sigma}_\gamma^{(i)}(\hat{p})\) can be expanded in hypercubic invariants

\[
\hat{\Sigma}_\gamma^{(i)}(\hat{p}) = c_1^{(i)} + c_2^{(i)} \sum_\nu \hat{p}_\nu^2 + c_3^{(i)} \frac{\sum_\nu \hat{p}_\nu^4}{\sum_\nu \hat{p}_\nu^2} + O(a^4).
\]

The only term surviving the $a \to 0$ limit is $c_1^{(0)}$. 
Finite volume effects

If there were no finite size effects, point with the same \( p_\mu = \frac{2\pi}{L} n_\mu \) should join in a perfectly smooth way.

On a dimensional ground we expect a \( pL \) dependance. We can rewrite

\[
\hat{\Sigma}_\gamma(\hat{p}, pL, \bar{\mu}) = \hat{\Sigma}_\gamma(\hat{p}, \infty, \bar{\mu}) + \left( \hat{\Sigma}_\gamma(\hat{p}, pL, \bar{\mu}) - \hat{\Sigma}_\gamma(\hat{p}, \infty, \bar{\mu}) \right)
\]

\[
\equiv \hat{\Sigma}_\gamma(\hat{p}, \infty, \bar{\mu}) + \Delta \hat{\Sigma}_\gamma(\hat{p}, pL, \bar{\mu})
\]

to a first approximation we neglect corrections on top of corrections:

\[
\Delta \hat{\Sigma}_\gamma(\hat{p}, pL, \bar{\mu}) \sim \Delta \hat{\Sigma}_\gamma(pL).
\]

Since \( p_\mu L = \frac{2\pi n_\mu}{L} L = 2\pi n_\mu \): at fixed \( n \)-tuple different lattice sizes are affected by the \( pL \) effects