Geometry of a two-spin quantum state in evolution

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Abstract
We study the quantum evolution of a two-spin system described by the isotropic Heisenberg Hamiltonian in the external magnetic field. It is shown that this evolution happens on a two-parametric closed manifold. The Fubini–Study metric of this manifold is obtained. It is found that this is the metric of the torus. The entanglement of the states which belong to this manifold is investigated.

Keywords: quantum evolution, two-spin system, manifold, Fubini–Study metric, entanglement

1. Introduction

It is well-known that quantum theory can be formulated in geometrical language [1–4]. The geometrical methods are successfully applied to the consideration of the unitary evolution of quantum states. This is because the states of a quantum system are represented by rays in a complex Hilbert space that in turn leads to a geometrical formulation of the postulates of quantum mechanics.

For an understanding of the dynamics of a quantum system it is useful to investigate manifolds which contain all states that can be reached during these dynamics. For instance, the whole state space of a two-level system (qubit) can be represented by a 2D sphere called the Bloch sphere. Then the trajectory of quantum evolution between the two states is a curve between two points on this sphere (see, for instance, [5–9]). Using the geometrical approach in quantum mechanics one can often find solutions to many problems in a simple way. For example, the problem of finding the Hamiltonian which provides the time-optimal evolution between two states was solved using symmetry properties of the quantum state space [10]. Another interesting problem, which was solved in a similar way, is the Zermelo navigation...
problem [11–15]. Also, the quantum brachistochrone problem for an arbitrary spin in a magnetic field was solved using geometrical properties of rotational manifolds [16].

A geometrical approach to study the evolution of a multilevel quantum system (qudit) was developed in [16–21]. In [17–20] it was shown that the problem of finding the quantum circuit of a unitary operation which provides time-optimal evolution on a system of qubits is closely related to the problem of finding the minimal distance between two points on the Riemannian metric. A similar problem was considered for the case of \( n \) qutrits in [21]. The authors of that work showed that the quantum gate complexity, which provides optimal evolution on a system of \( n \) qutrits, is equivalent to the problem of finding the shortest path between two points in a certain curved geometry of \( SU(3^n) \). The geometrical properties of some well-known coherent state manifolds were studied in [22, 23]. One can find more about geometry features of multilevel quantum systems in [24–27].

In our previous paper [28] we proposed a two-step method for the preparation of an arbitrary quantum state on a two-spin system represented by the isotropic Heisenberg Hamiltonian. In the present paper, we study the quantum evolution of a two-spin system with the isotropic Heisenberg interaction in the external magnetic field (section 2). In section 3 we show that this evolution happens on a two-parametric closed manifold and calculate its Fubini–Study metric. It is shown that this manifold is a torus. The entanglement of the states which belong to this manifold is studied (section 4). Conclusions are presented in section 5.

2. The quantum evolution of a two-spin system

We consider a two-spin system represented by the isotropic Heisenberg Hamiltonian. The system is placed in the external magnetic field directed along the \( z \)-axis. The Hamiltonian of the system is as follows

\[
H = H_{\text{int}} + H_{\text{mf}},
\]

with

\[
H_{\text{int}} = J \left( \sigma^1 \sigma^2 + 1 \right),
\]

\[
H_{\text{mf}} = h_z \left( \sigma^z_1 + \sigma^z_2 \right),
\]

where \( \sigma_i^1 = \sigma_i \otimes 1, \sigma_i^2 = 1 \otimes \sigma_i \) and \( \sigma_i \) are the Pauli matrices, \( J \) is the interaction coupling and \( h_z \) is proportional to the strength of the magnetic field. Here \( i = x, y, z \). It is worth noting that \( H_{\text{int}} \) commutes with \( H_{\text{mf}} \). This in turn means that the Hamiltonian (1) and the isotropic Heisenberg Hamiltonian (2) have a common set of eigenvectors

\[
| \uparrow \uparrow \rangle,
\]

\[
| \downarrow \downarrow \rangle,
\]

\[
\frac{1}{\sqrt{2}} (| \uparrow \downarrow \rangle + | \downarrow \uparrow \rangle),
\]

\[
\frac{1}{\sqrt{2}} (| \uparrow \downarrow \rangle - | \downarrow \uparrow \rangle).
\]

The Hamiltonian \( H_{\text{int}} \) has the three-fold degenerate eigenlevel \( 2J \) with eigenvectors (4)–(6) (triplet state) and eigenlevel \(-2J\) with the singlet state (7). Due to the external magnetic field, which splits energy levels, the Hamiltonian (1) has four eigenvalues, namely, \( 2(J + h_z) \), \( 2(J - h_z) \), \( 2J \) and \(-2J\) with the corresponding eigenvectors (4)–(7).
Let us consider the quantum evolution of a two-spin system with this Hamiltonian. Taking into account that the $H_{\text{int}}$ commutes with the $H_{\text{mf}}$, we can represent the evolution operator in the following form

$$U(t) = e^{-iH_{\text{int}}t}e^{-ih_{z}\sigma_{z}^{2}t}e^{-ih_{z}\sigma_{z}^{2}t},$$

where

$$e^{-iH_{\text{int}}t} = \cos(2Jt) - i\frac{J}{2J}\sin(2Jt)H_{\text{int}}.$$  

Here we use the fact that $H_{\text{int}}^{2} = (2J)^{2}$. We set $\hbar = 1$, which means that the energy is measured in frequency units. In the basis labelled by $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$, the evolution operator $U(t)$ can be represented as

$$U(t) = \begin{pmatrix} e^{-i2(h_{z}+J)t} & 0 & 0 & 0 \\ 0 & \cos(2Jt) & -i\sin(2Jt) & 0 \\ 0 & -i\sin(2Jt) & \cos(2Jt) & 0 \\ 0 & 0 & 0 & e^{i2(h_{z}-J)t} \end{pmatrix}.$$  

Let us consider the evolution of a two-spin state having started from the initial state

$$|\psi\rangle = a|\uparrow\uparrow\rangle + b|\uparrow\downarrow\rangle + c|\downarrow\uparrow\rangle + d|\downarrow\downarrow\rangle,$$

with parameters $a = a_{1}$, $b = b_{1}$, $c = c_{1}$ and $d = d_{1}$. The normalisation condition is the following $|a|^{2} + |b|^{2} + |c|^{2} + |d|^{2} = 1$. The action of the evolution operator (8) on the state (11) is as follows

$$|\psi(\theta, \phi)\rangle = U(t)|\psi\rangle = a_{1}e^{-i(\phi+\theta)}|\uparrow\uparrow\rangle + (b_{1}\cos \theta - ic_{1}\sin \theta)|\uparrow\downarrow\rangle + (-ib_{1}\sin \theta + c_{1}\cos \theta)|\downarrow\uparrow\rangle + d_{1}e^{i(\phi-\theta)}|\downarrow\downarrow\rangle,$$

where

$$\theta = 2Jt, \quad \phi = 2h_{z}t.$$  

Note that this state is defined by two real independent parameters $\theta$ and $\phi$ which in turn are defined by the value of the magnetic field $h_{z}$ and the period of evolution $t$. For any pre-defined set of values $\theta$ and $\phi$ there exists a set of values $h_{z}$ and $t$. An arbitrary quantum state of two qubits contains six real parameters. Due to this fact, we cannot reach an arbitrary state of a two-spin system, which is represented by the Hamiltonian (1).

It is easy to see from (12) that the following equalities are satisfied

$$|\psi(\theta + \pi, \phi)\rangle = -|\psi(\theta, \phi)\rangle, \quad \text{and} \quad |\psi(\theta, \phi + 2\pi)\rangle = |\psi(\theta, \phi)\rangle.$$  

So, modulo a global phase, state $|\psi(\theta, \phi)\rangle$ is periodic with period $\pi$ for $\theta$ and with period $2\pi$ for $\phi$. This means that parameters $\theta$ and $\phi$ belong to the intervals $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$, respectively. In the next section in order to investigate the properties of the manifold, which contains all states achieved during the evolution of two spins, we consider the Fubini–Study metric.

3. The Fubini–Study metric

The Fubini–Study metric is defined by the infinitesimal distance $ds$ between two neighbouring pure quantum states $|\psi^{(\alpha)}\rangle$ and $|\psi(\xi^{\alpha} + d\xi^{\alpha})\rangle$ [5, 6, 29, 30]
where $\xi^\alpha$ is a set of real parameters which define the state $|\psi(\xi^\alpha)\rangle$. The components of the metric tensor $g_{\alpha\beta}$ have the form

$$g_{\alpha\beta} = \gamma^2 \Re \left( \langle \psi_\alpha \left| \psi_\beta \right\rangle - \langle \psi_\alpha \right| \langle \psi | \psi \rangle \langle \psi \left| \psi_\beta \right\rangle \right),$$

where $\gamma$ is an arbitrary factor which is often chosen to have value of 1, $\sqrt{2}$ or 2 and

$$\left| \psi_\alpha \right\rangle = \frac{\partial}{\partial \xi^\alpha} |\psi\rangle.$$

This metric can be obtained using the expression for the Fubini–Study distance between two neighbouring pure states $|\psi(\xi^\alpha)\rangle$ and $|\psi(\xi^\alpha + d\xi^\alpha)\rangle$ [5, 31].

Using expression (17) the metric of the manifold defined by a set of parameters $\xi^\alpha$ can be obtained. For example, the metric tensor of the ground state of the quantum XY chain in a transverse magnetic field was calculated in [32]. In this case the authors find the metric tensor of the ground state manifold depending on the exchange coupling and the magnetic field.

Let us calculate the metric of the manifold defined by state (12). This state is determined by two real parameters $\theta$ and $\phi$. In order to find the components of the metric tensor (17) we calculate the following derivatives

$$\left| \psi_\theta \right\rangle = -ia_1 e^{-i(\phi+\theta)} |\uparrow \uparrow \rangle + \{-b_1 \sin \theta - ic_1 \cos \theta \} |\downarrow \downarrow \rangle,$$

$$\left| \psi_\phi \right\rangle = -ia_1 e^{-i(\phi+\theta)} |\uparrow \uparrow \rangle + id_1 e^{i(\phi-\theta)} |\downarrow \downarrow \rangle,$$

and then we obtain

$$\langle \psi \left| \psi_\theta \right\rangle = -i[1 - B], \quad \langle \psi \left| \psi_\phi \right\rangle = -iD,$$

$$\langle \psi_\theta \left| \psi_\theta \right\rangle = 1, \quad \langle \psi_\phi \left| \psi_\phi \right\rangle = A, \quad \langle \psi_\theta \left| \psi_\phi \right\rangle = D,$$

where

$$A = |a_1|^2 + |d_1|^2, \quad B = |b_1 - c_1|^2, \quad D = |a_1|^2 - |d_1|^2.$$

Substituting (21) into (17), we have

$$ds^2 = \gamma^2 \left[ B(2 - B)(d\theta)^2 + (A - D^2)(d\phi)^2 + 2BDd\theta d\phi \right].$$

It is easy to show that the substitutions

$$\phi' = \phi + k\theta,$$

$$\theta' = \theta$$

transform this metric into a diagonal form

$$ds^2 = \gamma^2 \left[ \frac{B \left( 2A - 2D^2 - AB \right)}{A - D^2} (d\theta')^2 + \left( A - D^2 \right)(d\phi')^2 \right].$$
where $k = BD/(A - D^2)$. As we can see from (23) or (26), the components of the metric tensor depend only on parameters which determine the initial states. The state (12) with new parameters $\phi'$ and $\theta'$ takes the form

\[ \psi(\theta', \phi') = a_i e^{-i(\phi' + (1-k)\theta')} |\uparrow \uparrow \rangle + (b_i \cos \theta' - ic_i \sin \theta') |\uparrow \downarrow \rangle + (-ib_i \sin \theta' + c_i \cos \theta') |\downarrow \uparrow \rangle + d_i e^{i(\phi' - (1+k)\theta')} |\downarrow \downarrow \rangle. \]

(27)

This state satisfies the following periodic conditions

\[ \psi(\theta' + \pi, \phi' + k\pi) = -\psi(\theta', \phi'), \]
\[ \psi(\theta', \phi' + 2\pi) = \psi(\theta', \phi'). \]

(28)

(29)

with respect to parameters $\theta'$ and $\phi'$. Also, it is worth noting that the components of the metric tensor $g_{\phi'\phi'}$ and $g_{\phi'\phi'}$ have positive values. Indeed, using notations (22) in (26), we obtain that

\[ B(2A - 2D^2 - AB) = \left( |a_i|^2 + |d_i|^2 \right) \left( b_i^2 - c_i^2 \right)^2 + 8 |a_i|^2 |d_i|^2 |b_i - c_i|^2 \geq 0, \]

\[ A - D^2 = |a_i|^2 \left( 1 - |a_i|^2 \right) + |d_i|^2 \left( 1 - |d_i|^2 \right) + 2 |a_i|^2 |d_i|^2 \geq 0. \]

(30)

So, we conclude that expression (23) or (26) defines the Euclidean metric. The fact that components of the metric tensor do not depend on the parameters $\theta$ and $\phi$ means that expression (23) or (26) defines the metric of a flat manifold. Using this fact and the fact that parameters $\theta$ and $\phi$ are periodical, we conclude that this manifold is a torus.

At the end of this section it is worth noting that the geometry of the manifold defined by expression (23) or (26) depends on the parameters which determine the initial state. So, we obtain that the evolution of the two-spin system happens on the one-dimensional manifold if the parameters of the initial state satisfy the condition $B = 0$ or $A - D^2 = 0$. This condition is obtained from expression (23), when $g_{0\theta} = 0$ or $g_{0\phi} = 0$. Let us study the entanglement of the states which belong to this manifold.

4. Entanglement on the torus

The implementation of different algorithms in quantum computation demands the preparation of maximally entangled states. For example, the realization of the simplest scheme of the quantum teleportation of one qubit state requires the preparation of an EPR channel [33].

In [34] the entanglement of a multipartite system using the geometry property of the Hilbert space was considered. Also the authors considered the case of bipartite entanglement. They gave the definition of the concurrence for an arbitrary mixed state using geometry language. In this section we investigate the degree of concurrence of the states which belong to the manifold defined by metric (23).

The degree of entanglement of a two-spin system can be determined by the concurrence [35, 36]

\[ C = 2 |ad - bc|, \]

(31)

where parameters $a$, $b$, $c$ and $d$ are defined by expression (11). Using this definition, we obtain that the concurrence in state (12) is as follows
It is easy to see that the entanglement of state (12) depends only on parameter $\theta$ which contains the interaction between spins and is independent of parameter $\phi$ which contains the value of the magnetic field. This is because the action of the magnetic field is given by unitary operators that define the evolution of each spin separately and do not change the entanglement of the spin system. The interaction between spins is defined by the unitary operator which describes the evolution of the two spins together. This, in turn, leads to a change in the entanglement of the two spins. From equation (32) it follows that for a particular value of $\theta$ we can select the curves on our manifold with a constant entanglement. From expression (23) or (26) we obtain that these curves are circles with radii depending on the parameters of initial states as follows

$$R = \gamma \sqrt{A - D^2},$$

where $A$ and $D$ are defined by (22). Also, it is worth noting if the initial state is disentangled ($C = 0$) then expression (32) takes the form

$$C = |b_i - c_i|^2 |\sin 2\theta|.$$

This is due to the fact that $a_i d_i = b_i c_i$. As we can see in the case of $\theta = \pi/4$ we obtain the value of the maximal entanglement. This value depends on the parameters $b_i$ and $c_i$ that determine the initial state. For instance, in the case of $b_i = 1$ and $c_i = 0$, which corresponds to the initial state $|\uparrow \downarrow\rangle$, we obtain the maximally entangled state with $C = 1$. Let us consider another example when parameters $b_i$ and $c_i$ are the same, $b_i = c_i$. In this case we never achieve an entangled state. This is because the following initial state $|\psi_i\rangle = a_i |\uparrow \uparrow\rangle + b_i (|\uparrow \downarrow\rangle + |\downarrow \uparrow\rangle) + d_i |\downarrow \downarrow\rangle$ is an eigenstate of the two-spin system (2).

Let us apply the above results to the disentangled initial state which has the following form

$$|\psi_i\rangle = |+ -\rangle,$$

where $|+\rangle = \cos \frac{\chi}{2} |\uparrow\rangle + \sin \frac{\chi}{2} e^{i\gamma} |\downarrow\rangle$, $|-\rangle = -\sin \frac{\chi}{2} |\uparrow\rangle + \cos \frac{\chi}{2} e^{i\gamma} |\downarrow\rangle$ are the eigenstates of the operator of the projection of spin-\frac{1}{2} on the direction defined by the unit vector $n$. The vector $n$ is represented by the spherical coordinates as follows $n = (\sin \chi \cos \gamma, \sin \chi \sin \gamma, \cos \chi)$, where $\chi \in [0, \pi]$ and $\gamma \in [0, 2\pi]$ are the polar and azimuthal angles, respectively. It is also important to note that state (35) is the eigenstate of the system of two spins in the magnetic field directed along the unit vector $n$. Substituting the parameters of states (35) into (23) we obtain that in this case the evolution happens on a torus with the metric defined by

$$ds^2 = \gamma^2 \left((d\theta)^2 + \frac{1}{2} \sin^2 \chi (d\phi)^2\right).$$

From equation (34) we obtain that the concurrence of the states on the manifold defined by (36) depends on the parameter $\theta$ as follows

$$C = |\sin 2\theta|.$$
It is easy to see that the condition $\theta = \frac{\pi}{4}$ corresponds to the maximally entangled state

\[ |\psi(\theta, \phi)\rangle = -\cos \frac{\chi}{2} \sin \frac{\chi}{2} e^{-i(\phi+\frac{\pi}{2})} |\uparrow \uparrow\rangle + \frac{1}{\sqrt{2}} \left( \cos^2 \frac{\chi}{2} + i \sin^2 \frac{\chi}{2} \right) e^{i\gamma} |\uparrow \downarrow\rangle + \frac{1}{\sqrt{2}} \left( i \cos^2 \frac{\chi}{2} + \sin^2 \frac{\chi}{2} \right) e^{i(\phi-\frac{\pi}{2})} |\downarrow \uparrow\rangle + \cos \frac{\chi}{2} \sin \frac{\chi}{2} e^{i2\gamma} e^{i(\phi-\frac{\pi}{2})} |\downarrow \downarrow\rangle. \] (38)

Taking into account (13), we conclude that the system having started from the initial state (35) can achieve state (38) during the period of time

\[ t = \frac{\pi}{8J}. \] (39)

If the parameter $\chi = 0$, which modulo a global phase corresponds to the case of initial state $|\uparrow \downarrow\rangle$, then we obtain the following final state

\[ |\psi(\theta, \phi)\rangle = \frac{1}{\sqrt{2}} |\uparrow \downarrow\rangle - \frac{i}{\sqrt{2}} |\uparrow \uparrow\rangle. \] (40)

The evolution between the initial state $|\uparrow \downarrow\rangle$ and the final one (40) happens along the curve which is a circle with the radius $\gamma$. It is easy to see if we put $\chi = 0$ into expression (36).

Finally, it should be noted that in the case of the initial state $|++\rangle$ or $|--\rangle$ the evolution happens on the manifold which is the circle with the radius $\gamma \sin \chi/\sqrt{2}$. Also, it is easy to see that all the states belonging to this circle are disentangled because the initial state is an eigenstate of the Hamiltonian (2).

5. Conclusion

We considered the quantum system of two spins represented by the isotropic Heisenberg Hamiltonian. The quantum evolution of such a system placed in an external magnetic field which is directed along the $z$-axis was studied. This evolution is defined by two real parameters, namely, the period of time of the evolution and the value of the magnetic field. The evolution of the two-spin system is periodic with respect to these parameters. Therefore, we concluded that this evolution happens on a two-parametric closed manifold. We calculated the Fubini–Study metric of this manifold and showed that it describes a flat manifold. Using this fact and the fact that parameters which define this manifold are periodic, we concluded that it is a torus. Finally, the entanglement of the states belonging to this manifold was investigated. We found that the curves of constant entanglement on the manifold are circles. Also we showed that the evolution between the disentangled and maximally entangled states happens on a torus.

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