A CONVERSE TO MAZUR’S INEQUALITY
FOR SPLIT CLASSICAL GROUPS

CATHERINE LEIGH

Our goal is to prove a converse to Mazur’s inequality for split classical groups. This work stems from results in [KR02], whose notation we will follow. Let $G$ be a split connected reductive group with Borel subgroup $B$ and let $T$ be a maximal torus contained in $B$. We abbreviate $X,T$ to $X$. We denote by $a$ the real vector space $X \otimes_{\mathbb{Z}} \mathbb{R}$, and by $a_{\text{dom}}$ the cone of dominant elements in $a$. For $x,y \in a$ we say $x \leq y$ if $\langle x,\omega \rangle \leq \langle y,\omega \rangle$ for all fundamental weights $\omega$ and $y-x$ is in the linear span of the coroots of $G$. We write $X_G$ for the quotient of $X$ by the coroot lattice for $G$, and $\varphi_G : X \to X_G$ for the natural projection map.

Now let $\mu \in X$ be $G$-dominant. The Weyl group $W$ acts on $X$. We define the subset $P_\mu \subset X$ by

$$P_\mu := \{ \nu \in X : \begin{array}{l} (i) \ \varphi_G(\nu) = \varphi_G(\mu), \\
(ii) \ \nu \in \text{Conv}(W_\mu) \end{array} \}$$

where $W_\mu = \{ w(\mu) : w \in W \}$ and Conv($W_\mu$) denotes the convex hull of $W_\mu$ in $a$.

Let $P$ be a parabolic subgroup of $G$ which contains $B$ and let $M$ be the unique Levi subgroup of $P$ containing $T$. Thus $T$ is also a maximal torus in $M$. We write $X_M$ for the quotient of $X$ by the coroot lattice for $M$, and let $\varphi_M : X \to X_M$ be the natural projection map. Since the coroot lattice for $M$ is a subgroup of the coroot lattice for $G$, the map $\varphi_G$ factors through $X_M$ via $\varphi_M$.

Now let $F$ be a finite extension of $\mathbb{Q}_p$ and let $L$ be the completion of the maximal unramified extension of $F$. Denote the ring of integers in $F$, resp. $L$, by $O_F$, resp. $O_L$, and set $K = G(O_L)$. Let $\pi \in O_F$ be a uniformizer and let $\sigma$ be the relative Frobenius automorphism of $L/F$. Recall that $b_1,b_2 \in G(L)$ are said to be $\sigma$-conjugate if there exists $g \in G(L)$ such that $b_2 = g^{-1}b_1\sigma(g)$ and that $B(G)$ denotes the set of $\sigma$-conjugacy classes in $G(L)$. Finally, as in Section 6 of [Kot97], define $B(G,\mu) := \{ b \in B(G) : \kappa_G(b) = \varphi_G(\mu) \text{ and } \bar{\nu}(b) \leq \mu \}$, where $\bar{\nu}(b) \in a_{\text{dom}}$ is the image of $b$ under the Newton map (cf. [RR94], [Kot97]) and $\kappa_G$ is the map (4.9.1) of [Kot97]. We can now state our main theorem.

Converse to Mazur’s Inequality. Let $G$ be a split connected reductive group over $F$ which is weakly classical, in the sense that each irreducible component of its Dynkin diagram is of type $A_n$, $B_n$, $C_n$, or $D_n$. Let $\mu \in X$ be $G$-dominant and let $b \in B(G,\mu)$. Then the $\sigma$-conjugacy class of $b$ in $G(L)$ meets $K\pi^\mu K$.

The converse to Mazur’s inequality is proven in [KR02] (resp. [FR02]), for $G = GL_n$ and $G = GSp_{2n}$ (resp. $G = GL_n$). The reader who would like to know how

---

Date: November 19, 2002.

2000 Mathematics Subject Classification. Primary: 14L05; Secondary: 11S25, 14F30, 20G25.
Indeed, in the theorem only the Weyl group orbit of $\nu$ satisfies the desired condition, i.e. that $\varphi_M(\nu, \mu) = \nu_1$ and $(\nu, \mu)$ have the same image in $X_G$. Second, the image of $\varphi_M(\nu)$ in $a_M$ is $pr_M(\nu)$, which lies in $pr_M(\Conv(W\mu))$ since $\nu \in \Conv(W\mu)$.

To prove that the right hand side is a subset of the left hand side, we must show that given $\nu_1 \in X_M$ satisfying conditions (i) and (ii), we can find $\nu \in X$ such that $\varphi_M(\nu) = \nu_1$ and $\nu \in \Conv(W\mu)$. Note that if $\varphi_M(\nu) = \nu_1$, then $\nu$ and $\mu$ will have the same image in $X_G$ by condition (i). To show that we can find such a $\nu$ for classical groups, we will examine each type separately. Before doing so, we will make several reductions.

We claim that we may assume without loss of generality that $\nu_1$ is $G$-dominant. Indeed, in the theorem only the Weyl group orbit of $\nu$ plays a role, and therefore we are free to choose the Borel subgroup $B$ such that $\nu_1$ is $G$-dominant. We now do so, and then (cf. e.g. Lemma 2.2 (ii) in [KR96]) the condition that $\nu_1 \in \Conv(W\mu)$ is equivalent to $\nu_1 \leq \mu$.

A coweight $\nu$ is said to be $G$-minuscule if for every root $\alpha$ of $G$, we have $\langle \alpha, \nu \rangle \in \{-1, 0, 1\}$. We define $M$-minuscule analogously. Bourbaki (cf. Ch. VIII, §7 Prop. 8) shows that given $\nu_1 \in X_M$ there exists a unique $M$-dominant, $M$-minuscule coweight $\nu \in X$ such that $\varphi_M(\nu) = \nu_1$. We will show that this choice of $\nu$ satisfies the desired condition, i.e. that $\nu \in \Conv(W\mu)$.

Denote $pr_M(\nu)$ by $\nu_M$. We are reduced to proving the following: Given $\nu, \mu \in X$ such that $\mu$ is $G$-dominant, $\nu$ is $M$-dominant and $M$-minuscule, and $\varphi_G(\nu) = \varphi_G(\mu)$, and given that $\nu_M \in a^{W_M}$ is $G$-dominant and $\nu_M \leq \mu$, it follows that $\nu \in \Conv(W\mu)$.
Lemma 1.1. \( I \) is in the linear span of the coroots of \( G \) for all \( i \)

(Cf. [Bour81] Ch. VI §1 equation (1.1), this is equivalent to showing that \( P \) containing \( r \) is \( \mu \)-dominant.

Proof. Since \( I \) is in the linear span of the coroots of \( G \) for all \( i \), we have

\[ \langle \beta, \omega \rangle \geq i \]

By hypothesis, we have \( r_i \geq 0 \) for all \( i \in I_N \). It remains to show that \( r_i \geq 0 \) for all \( i \in I_M \). To do so, it is enough to show that \( \sum_{i \in I_M} r_i \alpha_i^\vee \) is \( M \)-dominant (cf. [Bour81] Ch. VI §1, 10), i.e. that \( \langle \sum_{i \in I_M} r_i \alpha_i^\vee, \alpha_j \rangle \geq 0 \) for all \( j \in I_M \). By equation (1.1), this is equivalent to showing that

\[ \langle \mu - \beta - \sum_{i \in I_N} r_i \alpha_i^\vee, \alpha_j \rangle \geq 0. \] (1.2)
For all \(j \in I_M\), we have \(\langle \mu, \alpha_j \rangle \geq 0\) since \(\mu\) is \(G\)-dominant, and we have \(\langle \beta, \alpha_j \rangle = 0\) since \(\beta \in \mathfrak{a}^{W_M}\). Finally, \(\sum_{i \in I_N} r_i \alpha_{i,j}^\vee, \alpha_j \rangle \leq 0\) since \(\langle \alpha_{i,j}^\vee, \alpha_j \rangle \leq 0\) and \(r_i \geq 0\) for all \(i \in I_N\) and \(j \in I_M\). Therefore inequality (1.2) holds for all \(j \in I_M\).

\[\text{2. The Proof for } G = GL_n\]

Let \(G = GL_n\) and let \(M\) be the Levi subgroup \(GL_{n_1} \times GL_{n_2} \times \cdots \times GL_{n_r}\) of \(G\) where \(n_1 + n_2 + \cdots + n_r = n\). Let \(T\) be diagonal matrices and \(B\) upper triangular matrices. Then \(X = \mathbb{Z}^r\). In this case, \(x \in X\) is \(G\)-dominant if \(x_1 \geq x_2 \geq \cdots \geq x_n\).

Thus the \(G\)-dominant, \(G\)-minuscule elements of \(X\) are of the form

\[
(1, \ldots, 1, 0, \ldots, 0) + k(1, \ldots, 1)
\]

where \(0 \leq t \leq n\) and \(k\) is an integer. Also \(x \leq \mu\) if the following conditions hold for \(S_i(x) = x_1 + x_2 + \cdots + x_i:\)

\[
S_i(x) \leq S_i(\mu) \quad \text{for } 1 \leq i < n,\tag{2.1}
\]

\[
S_n(x) = S_n(\mu).\tag{2.2}
\]

The vector \(\nu_M\) is obtained from \(\nu\) by averaging the entries of \(\nu\) over \(M\) batches where the first \(n_1\) entries of \(\nu\) constitute the first batch, the next \(n_2\) the second batch, and so on. Thus we have

\[
\nu_M = \left(\bar{\nu}_1, \ldots, \bar{\nu}_{n_1}, \bar{\nu}_{n_2}, \ldots, \bar{\nu}_r, \ldots, \bar{\nu}_r\right),
\]

where \(\bar{\nu}_k\) denotes the average of the entries of the \(k\)th batch of \(\nu\). We define \(\sigma(k) = n_1 + n_2 + \cdots + n_k\).

To prove the theorem, we will reorder the entries of \(\nu\) to form a new coweight \(\eta\) and show that, for the proper choice of Levi subgroup, \(\eta\) satisfies all of the hypotheses on \(\nu\) as well as being \(G\)-dominant. This reduces the problem to proving the theorem with the additional hypothesis that \(\nu\) is \(G\)-dominant. We first show that the batches of \(\nu\) satisfy a nice order property.

**Lemma 2.1.** Let \(f_k(\nu)\) denote the first entry of the \(k\)th batch of \(\nu\). Then \(f_1(\nu) \geq f_2(\nu) \geq \cdots \geq f_r(\nu)\).

**Proof.** Suppose that there exists \(k < r\) such that \(f_{k+1}(\nu) > f_k(\nu)\). Both are integers, so

\[
f_{k+1}(\nu) - 1 \geq f_k(\nu).\tag{2.3}
\]

Since \(\nu\) is \(M\)-minuscule and \(M\)-dominant and \(\bar{\nu}_k\) is the average of the \(k\)th batch of \(\nu\), we have \(f_k(\nu) \geq \bar{\nu}_k > f_k(\nu) - 1\). Similarly,

\[
f_{k+1}(\nu) \geq \bar{\nu}_{k+1} > f_{k+1}(\nu) - 1.
\]

Combining these with (2.3) gives \(\bar{\nu}_{k+1} > f_{k+1}(\nu) - 1 \geq f_k(\nu) \geq \bar{\nu}_k\), which contradicts the \(G\)-dominance of \(\nu_M\).

We will now create the new coweight \(\eta\) by reordering the entries of \(\nu\) in such a way that the inequalities in Lemma 2.1 are strict for \(\eta\). We will then show that \(\eta\) still satisfies all of the hypothesis on \(\nu\), but for a different Levi subgroup. To simplify doing so, we first prove the following lemma.
Lemma 2.2. Let $\beta \in a^W_M$ be of the form

$$\beta = (\beta_1, \ldots, \beta_1, \beta_2, \ldots, \beta_2, \ldots, \beta_r, \ldots, \beta_r).$$

To show that $\beta \leq \mu$, it is enough to show that the inequalities corresponding to the end of each batch are satisfied, i.e. that inequality (2.4) holds for $i = \sigma(k)$ for all $k < r$ and that condition (2.4) holds.

Proof. Follows from Lemma 1.1. \hfill \Box

Now we form the coweight $\eta$ by combining all batches of $\nu$ which have the same first entry into one batch and reordering each new batch in nonincreasing order. Let $L$ be the Levi subgroup corresponding to these new batches. Let $\eta_L$ be the vector obtained by averaging $\eta$ over the batches of $L$ and denote its entries by $\bar{\eta}_k$. We now check that $\eta$ is $G$-dominant and satisfies all of the hypotheses on $\nu$, but for the Levi subgroup $L$.

Lemma 2.3. The coweight $\eta$ is $L$-minuscule and $\eta_L \leq \mu$. Moreover, $\eta$ is $G$-dominant, therefore $\eta$ is $L$-dominant and $\eta_L$ is $G$-dominant.

Proof. By construction, $\eta$ is $L$-minuscule since $\nu$ is $M$-minuscule. Also, the entries of $\eta$ are nonincreasing, so $\eta$ is $G$-dominant. Therefore $\eta$ is $L$-dominant and $\eta_L$ is $G$-dominant.

Finally, for every $k$ there exists a $j$ so that the sum of the first $k$ batches of $\eta$ is equal to the sum of the first $j$ batches of $\nu$. Therefore, since $\nu_M \leq \mu$, the inequalities for $\eta_L \leq \mu$ are satisfied at the end of each batch and by Lemma 2.2 we have $\eta_L \leq \mu$. \hfill \Box

We have shown that $\eta$ satisfies all of the hypotheses on $\nu$ for the Levi subgroup $L$ and that $\eta$ is $G$-dominant. Moreover, by its construction, $\eta \in W \nu$ so it is enough to prove the theorem for $(L, \eta)$ instead of $(M, \nu)$. Thus it is enough to prove the theorem with the additional hypothesis that $\nu$ is $G$-dominant. We can now prove that $\nu \in \text{Conv}(W \mu)$ by proving that $\nu \leq \mu$.

Theorem 2.4. $\nu \in \text{Conv}(W \mu)$

Proof. We will suppose that $\nu \not\leq \mu$ and obtain a contradiction. If $\nu \not\leq \mu$, then there exists an $i$ such that

$$\nu_1 + \nu_2 + \cdots + \nu_i > \mu_1 + \mu_2 + \cdots + \mu_i. \tag{2.4}$$

Choose the smallest such $i$. Then $\nu_i > \mu_i$ and both are integers, so $\nu_i - 1 \geq \mu_i$.

Suppose $\nu_i$ is in the $k$th batch of $\nu$. We consider the $(i + 1)^{\text{th}}$ to $(\sigma(k))^{\text{th}}$ entries of $\nu$ and $\mu$. Since $\nu$ is $M$-dominant and $M$-minuscule, $\nu_{i+1}, \ldots, \nu_{\sigma(k)} \in \{\nu_i, \nu_i - 1\}$. Thus $\nu_{i+1} + \cdots + \nu_{\sigma(k)} \geq (\sigma(k) - i)(\nu_i - 1)$. Also, since $\mu$ is $G$-dominant and $\mu_i \leq \nu_i - 1$, it follows that $\mu_{i+1} + \cdots + \mu_{\sigma(k)} \leq (\sigma(k) - i)(\nu_i - 1) + 1$.

Thus

$$\mu_{i+1} + \cdots + \mu_{\sigma(k)} \leq \nu_{i+1} + \cdots + \nu_{\sigma(k)}.$$

Combining this with inequality (2.4) yields

$$\mu_1 + \cdots + \mu_{\sigma(k)} < \nu_1 + \cdots + \nu_{\sigma(k)},$$

which contradicts the hypothesis that $\nu_M \leq \mu$ since $\nu_1 + \cdots + \nu_{\sigma(k)} = n_1 \bar{\nu}_1 + \cdots + n_r \bar{\nu}_r$. \hfill \Box
3. The proof for \( G = SO_{2n+1} \)

Let \( G = SO_{2n+1} \) and let \( M \) be the Levi subgroup \( GL_{n_1} \times GL_{n_2} \times \cdots \times GL_{n_r} \times SO_{2j+1} \) of \( G \) where \( n_1 + n_2 + \cdots + n_r + j = n \). Let \( T \) be diagonal matrices and \( B \) upper triangular matrices. Then

\[
X = \{(a_1, a_2, \ldots, a_n, 0, -a_n, \ldots, -a_2, -a_1) : a_i \in \mathbb{Z} \} \cong \mathbb{Z}^n.
\]

In this case, \( x \in X \) is \( G \)-dominant if \( x_1 \geq x_2 \geq \cdots \geq x_n \geq 0 \). Thus the \( G \)-dominant, \( G \)-minuscule elements of \( X \) are \((1,0,\ldots,0)\) and \((0,\ldots,0)\). Also \( x \leq \mu \) if the following condition holds for \( S_i(x) = x_1 + x_2 + \cdots + x_i \):

\[
S_i(x) \leq S_i(\mu) \quad \text{for} \quad 1 \leq i \leq n.
\]

The hypothesis that \( \varphi_G(x) = \varphi_G(\mu) \) is equivalent to

\[
S_n(\mu) - S_n(x) \in 2\mathbb{Z}.
\]

The vector \( \nu_M \) is obtained from \( \nu \) by averaging the entries of \( \nu \) over batches where the first \( n_1 \) entries of \( \nu \) constitute the first batch, the next \( n_2 \) the second batch, and so on until the final batch. The value for the entries of the final batch of \( \nu_M \) is obtained by averaging over the middle \( 2j + 1 \) entries of \( \nu \). Thus we have

\[
\nu_M = \left( \bar{\nu}_1, \ldots, \bar{\nu}_{n_1}, \bar{\nu}_{n_2}, \ldots, \bar{\nu}_{n_r}, 0, \ldots, 0 \right),
\]

where \( \bar{\nu}_k \) denotes the average of the entries of the \( k \)th batch of \( \nu \). We define

\[
\sigma(k) = n_1 + n_2 + \cdots + n_k \quad \text{for} \quad k \leq r \quad \text{and} \quad \sigma(r + 1) = n.
\]

To prove the theorem, we will reorder the entries of \( \nu \) to form a new coweight \( \eta \) and show that, for the proper choice of Levi subgroup, \( \eta \) satisfies all of the hypotheses on \( \nu \) as well as being \( G \)-dominant. This reduces the problem to proving the theorem with the additional hypothesis that \( \nu \) is \( G \)-dominant. We first show that all of the entries of \( \nu \) are nonnegative.

**Lemma 3.1.** \( \nu_i \geq 0 \) for all \( i \)

**Proof.** Since \( \nu \) is \( M \)-minuscule and \( M \)-dominant, \( \nu_i \geq 0 \) for \( i > n - j \). Suppose \( \nu_i < 0 \) for some \( i \leq n - j \) and that \( \nu_i \) is in the \( k \)th batch of \( \nu \). For \( \nu_m \) in the \( k \)th batch, we have \( \nu_m \leq \nu_i + 1 \) since \( \nu \) is \( M \)-minuscule, and therefore \( \nu_m \leq 0 \) since \( \nu_i \) is an negative integer. Thus \( \bar{\nu}_k < 0 \). This contradicts the \( G \)-dominance of \( \nu_M \). \( \square \)

Next we show that the batches of \( \nu \) satisfy a nice order property.

**Lemma 3.2.** Let \( f_k(\nu) \) denote the first entry of the \( k \)th batch of \( \nu \). Then \( f_1(\nu) \geq f_2(\nu) \geq \cdots \geq f_r(\nu) \).

**Proof.** The proof proceeds exactly as in Lemma 2.1. \( \square \)

We will now create a new coweight \( \eta' \) by reordering the entries of \( \nu \) in such a way that the inequalities in Lemma 3.2 are strict for \( \eta' \). We will then modify \( \eta' \) slightly to form the coweight \( \eta \) and show that \( \eta \) still satisfies all of the hypotheses on \( \nu \), but for a different Levi subgroup. To simplify doing so, we first prove the following lemma.

**Lemma 3.3.** Let \( \beta \in \mathfrak{a}^W_M \) be of the form

\[
\beta = (\beta_1, \ldots, \beta_{n_1}, \beta_2, \ldots, \beta_{n_2}, \ldots, \beta_r, \beta_{r+1}, \ldots, \beta_{r+1}),
\]
where $\beta_{r+1} = 0$. To show that $\beta \leq \mu$, it is enough to show that the inequalities corresponding to the end of each batch are satisfied, i.e. that inequality \((3.4m)\) holds for $i = \sigma(k)$ for all $k \leq r$.

**Proof.** Follows from Lemma 1.1. $\square$

Now we form the coweight $\eta$ in two steps. First, we form the coweight $\eta'$ by considering the first $r$ batches of $\nu$; we combine all batches which have the same first entry into one batch and reorder each new batch in nonincreasing order. We take the final batch of $\nu$ as the final batch of $\eta'$. Let $L = GL_{m_1} \times \cdots \times GL_{m_r} \times SO_{2j+1}$ be the Levi subgroup corresponding to these new batches. By construction, $\eta'$ is $L$-dominant, and is $L$-minuscule since $\nu$ is $M$-minuscule. Moreover $\eta'$ is $GL_{n-j} \times SO_{2j+1}$-dominant as in Lemma 2.4. To form $\eta$ we modify $\eta'$ by considering its final batch. There are two cases.

First, if the final batch is of the form $1, 0, \ldots, 0$ (so, in particular, $j \neq 0$) and the last entry of the $s^{th}$ batch of $\eta'$ is zero, then we form $\eta$ by swapping the one at the beginning of the final batch of $\eta'$ with the left most zero entry in $\eta'$. For example, let $G = SO_{13}$, let $M = GL_2 \times GL_1 \times GL_4 \times SO_5$, and let $\nu = (2, 1, 2, 0, 1, 0)$. Then $\eta' = (2, 2, 1, 0, 1, 0, 0)$, the Levi subgroup $L = GL_3 \times GL_1 \times SO_5$, and $\eta = (2, 2, 1, 1, 0, 0, 0)$. Note that the one will move into a batch consisting only of zeros and ones since $\eta'$ is $L$-minuscule and has no negative entries; hence $\eta$ is $L$-minuscule.

Otherwise, we set $\eta = \eta'$.

We obtain $\eta_L$, resp. $\eta'_L$, from $\eta$, resp. $\eta'$, in the same manner in which we obtained $\nu_M$ from $\nu$, and denote its entries by $\eta_k$, resp. $\eta'_k$. We now check that $\eta$ is $G$-dominant and satisfies all of the hypotheses on $\nu$, but for the Levi subgroup $L$.

**Lemma 3.4.** The coweight $\eta$ is $L$-minuscule and $\eta_L \leq \mu$. Moreover $\eta$ is $G$-dominant, hence $\eta$ is $L$-dominant and $\eta_L \leq \mu$.

**Proof.** We have already shown that $\eta$ is $L$-minuscule. By construction, the entries of $\eta$ are nonincreasing and, by Lemma 3.1, they are all nonnegative, so $\eta$ is $G$-dominant. Therefore $\eta$ is $L$-dominant and $\eta_L$ is $G$-dominant.

It remains to show that $\eta_L \leq \mu$. By Lemma 3.3, it is enough to check that inequality \((3.1m)\) holds for $i = \tilde{\sigma}(k)$ for all $k$, where $\tilde{\sigma}(k) = m_1 + \cdots + m_k$ for $k \leq s$ and $\tilde{\sigma}(s+1) = n$. We see that $\eta'_L \leq \mu$ as Lemma 2.3; therefore if $\eta = \eta'$, we have $\eta_L \leq \mu$. Otherwise, suppose that the left most zero entry of $\eta'$ is in its $l^{th}$ batch, i.e. that we swapped the one from the final batch with a zero from the $l^{th}$ batch. It follows that inequality \((3.1m)\) holds for $k < l$ as Lemma 2.3. Moreover, since $\eta$ is $G$-dominant and all of its entries are nonnegative, all of the entries to the right of the one must be zero. Hence, since $\mu_i \geq 0$ for all $i$ as $\mu$ is $G$-dominant, it is enough to check inequality \((3.1m)\) for $i = \tilde{\sigma}(l)$.

Since $\eta'_L \leq \mu$, we have $S_{\tilde{\sigma}(l)}(\eta'_L) \leq S_{\tilde{\sigma}(l)}(\mu)$. If $S_{\tilde{\sigma}(l)}(\eta'_L) < S_{\tilde{\sigma}(l)}(\mu)$, then $S_{\tilde{\sigma}(l)}(\eta_L) \leq S_{\tilde{\sigma}(l)}(\mu)$ since both sides are integral and $S_{\tilde{\sigma}(l)}(\eta_L) = S_{\tilde{\sigma}(l)}(\eta'_L) + 1$. Otherwise,

$$S_{\tilde{\sigma}(l)}(\eta'_L) = S_{\tilde{\sigma}(l)}(\mu).$$ (3.3)

Since $\eta'_L \leq \mu$, we have

$$S_{\tilde{\sigma}(l)-1}(\eta'_L) \leq S_{\tilde{\sigma}(l)-1}(\mu).$$ (3.4)

Combining this with equality \((3.3m)\) yields

$$\eta'_L \geq \mu_{\tilde{\sigma}(l)}. \tag{3.5}$$
Since the \( l \)-th batch of \( \eta' \) consists only of zeros and ones and contains at least one zero, we have \( \bar{\eta}'_l < 1 \). We will obtain a contradiction by showing that \( \mu_{\tilde{\sigma}(l)} \geq 1 \). Now, since \( \eta'_L \leq \mu \), we have

\[
S_{\tilde{\sigma}(s)}(\eta'_L) + 1 \leq S_{\tilde{\sigma}(s) + 1}(\mu).
\]

Therefore \( 1 \leq \mu_{\tilde{\sigma}(l) + 1} + \cdots + \mu_{\tilde{\sigma}(l) + 1} \) by inequalities (3.3) and (3.4) and since \( \bar{\eta}'_k = 0 \) for all \( k > l \). Hence, since \( \mu \) is \( G \)-dominant, it follows that \( \mu_{\tilde{\sigma}(l)} \geq \mu_{\tilde{\sigma}(l) + 1} \geq 1 \), giving the desired contradiction. Thus \( S_{\tilde{\sigma}(l)}(\eta'_L) < S_{\tilde{\sigma}(l)}(\mu) \) and \( \eta_L \leq \mu \). \( \Box \)

We have shown that \( \eta \) satisfies all of the hypotheses on \( \nu \) for the Levi subgroup \( L \) and that \( \eta \) is \( G \)-dominant. Moreover, by its construction, \( \eta \in W\nu \) so it is enough to prove the theorem for \( (L, \eta) \) instead of \( (M, \nu) \). Thus it is enough to prove the theorem with the additional hypothesis that \( \nu \) is \( G \)-dominant. We can now prove that \( \nu \in \text{Conv}(W\mu) \) by proving that \( \nu \leq \mu \).

**Theorem 3.5.** \( \nu \in \text{Conv}(W\mu) \)

**Proof.** We will suppose \( \nu \not\leq \mu \) and obtain a contradiction. If \( \nu \not\leq \mu \), then there exists an \( i \) such that

\[
\nu_1 + \nu_2 + \cdots + \nu_i > \mu_1 + \mu_2 + \cdots + \mu_i.
\]

Choose the smallest such \( i \) and suppose that \( \nu_i \) is in the \( k \)-th batch of \( \nu \). As in Theorem 2.4 we obtain

\[
\mu_1 + \cdots + \mu_{\sigma(k)} < \nu_1 + \cdots + \nu_{\sigma(k)},
\]

which produces a contradiction as follows: If \( \sigma(k) = n \) and the first entry of the final batch is one, then \( \nu_1 + \cdots + \nu_{\sigma(k)} = n_1 \nu_1 + \cdots + n_k \nu_k + 1 \). Thus, since both values are integers, \( \mu_1 + \cdots + \mu_{\sigma(k)} \leq n_1 \nu_1 + \cdots + n_k \nu_k \). Since \( \nu_M \leq \mu \), we must have equality. This contradicts condition (3.2). Otherwise \( \nu_1 + \cdots + \nu_{\sigma(k)} = n_1 \nu_1 + \cdots + n_k \nu_k \), so inequality (3.8) contradicts the hypothesis that \( \nu_M \leq \mu \). \( \Box \)

4. THE PROOF FOR \( G = SO_{2n}/\{\pm 1\} \) (INTEGERS)

Let \( G = SO_{2n}/\{\pm 1\} \) and let \( M \) be the Levi subgroup \( GL_{n_1} \times GL_{n_2} \times \cdots \times GL_{n_r} \times SO_{2j} \) of \( G \) where \( n_1 + n_2 + \cdots + n_r + j = n \). We may assume that \( j \neq 1 \) since the case \( j = 1 \) is the same as the one in which \( j = 0 \) and \( n_r = 1 \). There are other Levi subgroups in \( G \), but it is not necessary to consider them since there exists an outer automorphism of \( SO_{2n} \) which takes each of these to a Levi subgroup which we have considered. Let \( T \) be diagonal matrices and \( B \) upper triangular matrices. Then

\[
X = \{ (a_1, a_2, \ldots, a_n, -a_n, \ldots, -a_2, -a_1) : \text{either } a_i \in \mathbb{Z} \forall i \text{ or } a_i \in \frac{1}{2}(\mathbb{Z} \setminus 2\mathbb{Z}) \forall i \}.
\]

We first consider the case in which \( a_i \in \mathbb{Z} \) for all \( i \); we will consider the other case in Section 3. In this case, \( x \in X \) is \( G \)-dominant if

\[
\begin{align*}
x_1 & \geq x_2 \geq \cdots \geq x_{n-1} \geq x_n, \\
\text{and } x_{n-1} + x_n & \geq 0.
\end{align*}
\]

It follows that if \( x \) is \( G \)-dominant, then \( x_i \geq 0 \) for all \( i \leq n-1 \). The \( G \)-dominant, \( G \)-minuscule elements of \( X \) are \((1, 0, \ldots, 0)\) and \((0, \ldots, 0)\). Also \( x \leq \mu \) if the following
conditions hold for $S_i(x) = x_1 + x_2 + \cdots + x_i$:

\begin{align*}
S_i(x) &\leq S_i(\mu) \quad \text{for } 1 \leq i \leq n-2, \\
S_{n-1}(x) - x_n &\leq S_{n-1}(\mu) - \mu_n, \\
S_n(x) &\leq S_n(\mu).
\end{align*}

(4.1)  (4.2)  (4.3)

The hypothesis that $\varphi_G(x) = \varphi_G(\mu)$ is equivalent to

\[ S_n(\mu) - S_n(x) \in 2\mathbb{Z} \]

(4.4)

The vector $\nu_M$ is obtained from $\nu$ by averaging the entries of $\nu$ over batches where the first $n_1$ entries of $\nu$ constitute the first batch, the next $n_2$ the second batch, and so on until the final batch. The value for the entries of the final batch of $\nu_M$ is obtained by averaging over the middle $2j$ entries of $\nu$. Thus we have

\[ \nu_M = (\bar{\nu}_1, \ldots, \bar{\nu}_{n_1}, \bar{\nu}_{n_2}, \ldots, \bar{\nu}_r, 0, \ldots, 0), \]

where $\bar{\nu}_k$ denotes the average of the entries of the $k^{\text{th}}$ batch of $\nu$. We define $\sigma(k) = n_1 + n_2 + \cdots + n_k$ for $k \leq r$ and $\sigma(r + 1) = n$.

To prove the theorem, we will reorder the entries of $\nu$ to form a new coweight $\eta$ and show that, for the proper choice of Levi subgroup, $\eta$ satisfies all of the hypotheses on $\nu$ as well as being $G$-dominant. This reduces the problem to proving the theorem with the additional hypothesis that $\nu$ is $G$-dominant. We first show that at most one of the entries of $\nu$ is negative.

**Lemma 4.1.** $\nu_i \geq 0$ for all $i$, unless $j = 0$ and $n_r = 1$, in which case, $\nu_i \geq 0$ for all $i \leq n - 1$

**Proof.** Suppose that $\nu_i < 0$ and that $\nu_i$ is in the $k^{\text{th}}$ batch of $\nu$. As in Lemma 3.1, we see that $k \neq r + 1$ and that $\bar{\nu}_k < 0$. Since $\nu_M$ is $G$-dominant, $\bar{\nu}_k < 0$ can occur only if $j = 0$, $n_r = 1$, and $k = r$, in which case $i = n$. \[\square\]

Next we show that the batches of $\nu$ satisfy a nice order property.

**Lemma 4.2.** Let $f_k(\nu)$ denote the first entry of the $k^{\text{th}}$ batch of $\nu$. Then $f_1(\nu) \geq f_2(\nu) \geq \cdots \geq f_r(\nu)$.

**Proof.** The proof proceeds exactly as Lemma 2.1. \[\square\]

We will now create a coweight $\eta'$ by reordering the entries of $\nu$ in such a way that the inequalities in Lemma 4.2 are strict for $\eta'$. We will then modify $\eta'$ slightly to form the coweight $\eta$ and show that $\eta$ still satisfies all of the hypothesis on $\nu$, but for a different Levi subgroup. To simplify doing so, we first prove the following lemma.

**Lemma 4.3.** Let $\beta \in a^{W_M}$ be of the form

\[ \beta = (\beta_1, \ldots, \beta_{n_1}, \beta_{n_1}, \ldots, \beta_{n_2}, \ldots, \beta_{n_r}, \beta_{n_r+1}), \]

where $\beta_{n_r+1} = 0$. To show that $\beta \leq \mu$, it is enough to show that the inequalities corresponding to the end of each batch are satisfied, i.e. that inequality (4.3) holds for $i = \sigma(k)$ for all $k$ such that $\sigma(k) \leq n - 2$, that inequality (4.4) holds, and, if $j = 0$ and $n_r = 1$, that inequality (4.5) holds.

**Proof.** Follows from Lemma 1.1. \[\square\]
Now we form the coweights \( \eta' \) and \( \eta \) as for \( G = SO_{2n+1} \). Again, we let \( L = GL_{m_1} \times \cdots \times GL_{m_r} \times SO_2 \) be the Levi subgroup corresponding to the new batches. We obtain \( \eta_L \), resp. \( \eta'_L \), from \( \eta \), resp. \( \eta' \), in the same manner in which we obtained \( \nu_M \) from \( \nu \), and denote its entries by \( \bar{\eta}_k \), resp. \( \bar{\eta}'_k \).

We now check that \( \eta \) is \( G \)-dominant and satisfies all of the hypotheses on \( \nu \), but for the Levi subgroup \( L \).

**Lemma 4.4.** The coweight \( \eta \) is \( L \)-minuscule and \( \eta_L \leq \mu \). Moreover \( \eta \) is \( G \)-dominant, hence \( \eta \) is \( L \)-dominant and \( \eta_L \leq \mu \).

**Proof.** As in the \( SO_{2n+1} \) case, we see that \( \eta \) and \( \eta' \) are \( L \)-minuscule. We claim that \( \eta \) is \( G \)-dominant. As before, the inequalities \( \eta_1 \geq \cdots \geq \eta_n \) follow from the way \( \eta \) was constructed, so we need only check that \( \eta_{n-1} + \eta_n \geq 0 \). This is automatic unless some entry of \( \nu \) is strictly negative, so by Lemma 4.3 we may now assume that \( j = 0 \) and \( n_r = 1 \). Thus we have \( \eta_n = \bar{\nu}_r \). We know that \( \bar{\nu}_{r-1} + \bar{\nu}_r \geq 0 \) since \( \nu_M \) is \( G \)-dominant. Hence \( \bar{\eta}_{n-1} + \eta_n \geq 0 \) since \( \bar{\eta}_{n-1} \geq \bar{\nu}_{r-1} \). Finally, since \( \eta \) is \( L \)-minuscule, \( \eta_{n-1} \) is the greatest integer less than or equal to \( \bar{\eta}_{n-1} \), so \( \eta_{n-1} + \eta_n \geq 0 \). Thus \( \eta \) is \( G \)-dominant. Therefore \( \eta \) is \( L \)-dominant and \( \eta_L \) is \( G \)-dominant.

It only remains to show that \( \eta_L \leq \mu \). To do so we apply Lemma 4.3. Define \( \bar{\sigma}(k) = m_1 + \cdots + m_k \) for \( k \leq s \) and \( \bar{\sigma}(s+1) = n \). The method used in Lemma 4.3 establishes inequality (4.1) for \( i = \bar{\sigma}(k) \) for \( k \) such that \( \bar{\sigma}(k) \leq n - 2 \), as well as inequality (4.3). It remains to verify inequality (4.2) under the assumption that \( j = 0 \) and \( m_s = 1 \). In this case, we have \( \eta = \eta' \), and we must have \( n_r = 1 \). Thus \( \bar{\eta}_s = \bar{\nu}_r \) and \( S_{\bar{\sigma}(s-1)}(\eta_L) = S_{\bar{\sigma}(r-1)}(\nu_M) \). Therefore, since \( \nu_M \leq \mu \), inequality (4.2) holds.

We have shown that \( \eta \) satisfies all of the hypotheses on \( \nu \) for the Levi subgroup \( L \) and that \( \eta \) is \( G \)-dominant. Moreover, by its construction, \( \eta \in W \nu \) so it is enough to prove the theorem for \( (L, \eta) \) instead of \( (M, \nu) \). Thus it is enough to prove the theorem with the additional hypothesis that \( \nu \) is \( G \)-dominant. We can now prove that \( \nu \in \text{Conv}(W \mu) \) by proving that \( \nu \leq \mu \).

**Theorem 4.5.** \( \nu \in \text{Conv}(W \mu) \)

**Proof.** We will suppose \( \nu \notin \mu \) and obtain a contradiction. If \( \nu \notin \mu \), then either there exists an \( i \neq n - 1 \) such that

\[
\nu_1 + \nu_2 + \cdots + \nu_i > \mu_1 + \mu_2 + \cdots + \mu_i \tag{4.5}
\]

or inequality (4.2) fails.

If inequality (4.5) holds for \( i = n \), then we will argue as in Theorem 3.3. If the first entry of the final batch is zero, then \( \nu_1 + \cdots + \nu_n = n_1 \bar{\nu}_1 + \cdots + n_k \bar{\nu}_k \), so inequality (4.3) contradicts the hypothesis that \( \nu_M \leq \mu \). If the first entry of the final batch is one, then \( \nu_1 + \cdots + \nu_n = n_1 \bar{\nu}_1 + \cdots + n_k \bar{\nu}_k + 1 \). Combining this with inequality (4.3) yields \( \mu_1 + \cdots + \mu_n \leq n_1 \bar{\nu}_1 + \cdots + n_k \bar{\nu}_k \) since both values are integers. Since \( \nu_M \leq \mu \), we must have equality. This contradicts condition (4.4).

Now suppose that there exists an \( i \leq n - 2 \) for which inequality (4.5) holds. Choose the smallest such \( i \) and suppose that \( \nu_i \) is in the \( k \)th batch of \( \nu \). As in Lemma 2.4, we obtain

\[
\mu_1 + \cdots + \mu_{\sigma(k)} < \nu_1 + \cdots + \nu_{\sigma(k)} \tag{4.6}
\]

If \( \sigma(k) \neq n - 1 \), then we obtain a contradiction as for \( SO_{2n+1} \).
If \( \sigma(k) = n - 1 \), then we have \( j = 0 \) and \( n_r = 1 \). We will first show that inequality (4.2) holds in this case. Since \( \nu_M \leq \mu \), we have
\[
S_{\sigma(r-1)}(\nu_M) - \bar{\nu}_r \leq S_{n-1}(\mu) - \mu_n. \tag{4.7}
\]
By the definition of \( \nu_M \), the partial sums are equal at the end of each batch and we are assuming that \( j = 0 \) and \( n_r = 1 \), so \( \bar{\nu}_r = \nu_n \) and \( S_{\sigma(r-1)}(\nu_M) = S_{n-1}(\nu) \). Substituting these values into inequality (4.7) gives inequality (4.2) as desired. Adding inequality (4.2) to inequality (4.3) yields \( S_{n-1}(\nu) \leq S_{n-1}(\mu) \), contradicting inequality (4.6).

Now suppose that inequality (4.2) fails. Thus we have
\[
S_{n-1}(\nu) - \nu_n > S_{n-1}(\mu) - \mu_n. \tag{4.8}
\]
By what we have already shown, we have
\[
S_{n-2}(\nu) \leq S_{n-2}(\mu). \tag{4.9}
\]
We claim that
\[
\nu_{n-1} - \nu_n > \mu_{n-1} - \mu_n \geq 0. \tag{4.10}
\]
The first inequality follows from inequalities (4.8) and (4.9), and the second holds since \( \mu \) is \( G \)-dominant. We have shown that inequality (4.2) holds if \( j = 0 \) and \( n_r = 1 \) so we may assume that this is not the case. Thus \( \nu_{n-1} \) and \( \nu_n \) are in the same batch of \( \nu \), so, since \( \nu \) is \( M \)-minuscule, \( \nu_{n-1} - \nu_n \in \{0,1\} \). Combining this with inequality (4.10) gives
\[
\nu_{n-1} - \nu_n = 1 \quad \text{and} \quad \mu_{n-1} - \mu_n = 0. \tag{4.11}
\]
Substituting these values into inequality (4.8) and combining the result with inequality (4.9) gives
\[
S_{n-2}(\nu) + 1 > S_{n-2}(\mu) \geq S_{n-2}(\nu).
\]
Both are integers, so \( S_{n-2}(\mu) = S_{n-2}(\nu) \). This contradicts condition (4.4) since it follows from the equalities in (4.11) that \( S_n(\mu) \) has the same parity as \( S_{n-2}(\mu) \) and \( S_n(\nu) \) has the opposite parity from \( S_{n-2}(\nu) \). Thus \( \nu \leq \mu \).

5. The proof for \( G = SO_{2n}/\{\pm1\} \) (Half Integers)

Recall from Section 3 that for \( G = SO_{2n}/\{\pm1\} \), we have
\[
X = \{(a_1, a_2, \ldots, a_n, -a_n, \ldots, -a_2, -a_1) : \text{either } a_i \in \mathbb{Z} \lor i \text{ or } a_i \in \frac{1}{2}(\mathbb{Z} \setminus 2\mathbb{Z}) \forall i \}.
\]
We now consider the case in which \( a_i \in \frac{1}{2}(\mathbb{Z} \setminus 2\mathbb{Z}) \) for all \( i \). As in Section 3, we let \( M \) be the Levi subgroup \( GL_{n_1} \times GL_{n_2} \times \cdots \times GL_{n_r} \times SO_{2j} \) of \( G \) where \( n_1 + n_2 + \cdots + n_r + j = n \) and assume that \( j \neq 1 \). Rather than consider \( X \), we will double the entries of \( X \) and adjust the notion of minuscule and the definition of the partial order accordingly.

First we note that all of the elements of \( X \) that we are considering in this case are now \( 2n \)-tuples of odd integers. As before, \( x \in X \) is \( G \)-dominant if
\[
x_1 \geq x_2 \geq \cdots \geq x_{n-1} \geq x_n,
\]
and \( x_{n-1} + x_n \geq 0 \)
It follows that if \( x \) is \( G \)-dominant, then \( x_i \geq 1 \) for all \( i \leq n - 1 \). Since we have multiplied \( X \) by two, we will now be concerned with elements whose pairing with any root yields two, zero, or negative two. We will refer to these elements as
2-minuscule. The $G$-dominant, $G$-2-minuscule elements of $X$ are $(1, 1, \ldots, 1)$ and $(1, 1, \ldots, 1, -1)$. Also $x \leq \mu$ if the following conditions hold for $S_i(x) = x_1 + x_2 + \cdots + x_i$:

\begin{align*}
S_i(x) &\leq S_i(\mu) \quad \text{for } 1 \leq i \leq n - 2, \quad (5.1) \\
S_{n-1}(x) - x_n &\leq S_{n-1}(\mu) - \mu_n, \quad (5.2) \\
S_n(x) &\leq S_n(\mu). \quad (5.3)
\end{align*}

The hypothesis that $\varphi_G(x) = \varphi_G(\mu)$ is equivalent to

$$S_n(\mu) - S_n(x) \in 4\mathbb{Z}. \quad (5.4)$$

As in the previous sections, the first $n_1$ entries of $\nu$ constitute the first batch, the next $n_2$ the second batch, and so on. The $M$-dominant, $M$-2-minuscule elements of $X$ will now be those such that the entries of each batch are nonincreasing and differ by either zero or two and such that the $r + 1$st batch is of the form $(1, 1, \ldots, 1)$ or $(1, 1, \ldots, 1, -1)$.

The vector $\nu_M$ is obtained from $\nu$ by averaging the entries of $\nu$ over batches with the exception that the value for the entries of the final batch of $\nu_M$ is obtained by averaging over the middle $2j$ entries of $\nu$. Thus we have

$$\nu_M = (\bar{\nu}_1, \ldots, \bar{\nu}_{n_1}, \bar{\nu}_{n_1+1}, \ldots, \bar{\nu}_{n_1+n_2}, \ldots, \bar{\nu}_{n_1+n_2+\cdots+n_r}, 0, \ldots, 0),$$

where $\bar{\nu}_k$ denotes the average of the entries of the $k$th batch of $\nu$. We define $\sigma(k) = n_1 + n_2 + \cdots + n_k$ for $k \leq r$ and $\sigma(r + 1) = n$.

To prove the theorem, we will reorder the entries of $\nu$ to form a new coweight $\eta$ and show that, for the proper choice of Levi subgroup, $\eta$ satisfies all of the hypotheses on $\nu$ as well as being $G$-dominant. This reduces the problem to proving the theorem with the additional hypothesis that $\nu$ is $G$-dominant. We first show that at most one of the entries of $\nu$ is less than negative one.

**Lemma 5.1.** $\nu_i \geq -1$ for all $i$, unless $j = 0$ and $n_r = 1$, in which case, $\nu_i \geq -1$ for all $i \leq n - 1$.

**Proof.** Since $\nu$ is $M$-2-minuscule and $M$-dominant, we have $\nu_i \geq -1$ for $i > n - j$. Suppose that $\nu_i < -1$ for some $i \leq n - j$ and that $\nu_i$ is in the $k$th batch of $\nu$. For $\nu_M$ in the $k$th batch, we have $\nu_M \leq \nu_i + 2$ since $\nu$ is $M$-2-minuscule, and therefore $\nu_M \leq -1$ since $\nu_i < -1$ is an odd integer. Thus $\bar{\nu}_k < -1$. Since $\nu_M$ is $G$-dominant, $\bar{\nu}_k < 0$ can only occur if $j = 0$, $n_r = 1$, and $k = r$, in which case $i = n$. \hfill $\square$

Next we show that the batches of $\nu$ satisfy a nice order property.

**Lemma 5.2.** Let $f_k(\nu)$ denote the first entry of the $k$th batch of $\nu$. Then $f_1(\nu) \geq f_2(\nu) \geq \cdots \geq f_r(\nu)$.

**Proof.** Suppose that there exists $k < r$ such that $f_{k+1}(\nu) > f_k(\nu)$. Both are odd integers, so

$$f_{k+1}(\nu) - 2 \geq f_k(\nu) \quad (5.5)$$

Since $\nu$ is $M$-2-minuscule and $M$-dominant and $\bar{\nu}_k$ is the average of the $k$th batch of $\nu$, we have $f_k(\nu) \geq \bar{\nu}_k > f_k(\nu) - 2$. Similarly, $f_{k+1}(\nu) \geq \bar{\nu}_{k+1} > f_{k+1}(\nu) - 2$. Combining these with inequality (5.5) gives $\bar{\nu}_{k+1} > f_{k+1}(\nu) - 2 \geq f_k(\nu) \geq \bar{\nu}_k$, which contradicts the $G$-dominance of $\nu_M$. \hfill $\square$
We will now create a coweight $\eta'$ by reordering the entries of $\nu$ in such a way that the inequalities in Lemma 5.4 are strict for $\eta'$. We will then modify $\eta'$ slightly to form the coweight $\eta$ and show that $\eta$ still satisfies all of the hypothesis on $\nu$, but for a different Levi subgroup. To simplify doing so, we first prove the following lemma.

**Lemma 5.3.** Let $\beta \in a^{W_M}$ be of the form

$$\beta = (\beta_1, \ldots, \beta_1, \beta_2, \ldots, \beta_2, \ldots, \beta_r, \ldots, \beta_r, \beta_{r+1}, \ldots, \beta_{r+1}),$$

where $\beta_{r+1} = 0$. To show that $\beta \leq \mu$, it is enough to show that the inequalities corresponding to the end of each batch are satisfied, i.e., that inequality (5.1) holds for $i = \sigma(k)$ for all $\sigma(k) \leq n - 2$, that inequality (5.3) holds, and, if $j = 0$ and $n_r = 1$, that inequality (5.3) holds.

**Proof.**follows from Lemma 1.1. □

Now we form the coweight $\eta$ in three steps. First, we form the coweight $\eta'$ as in the $SO_{2n+1}$ case. Again, we let $L = GL_{m_1} \times \cdots \times GL_{m_s} \times SO_{2j}$ be the Levi subgroup corresponding to the new batches. By construction, $\eta'$ is $L$-2-minuscule since $\nu$ is $M$-2-minuscule. It follows from Lemma 5.3 and the construction of $\eta'$ that $\eta'_r \geq -1$ for all $i \leq n - 1$.

Moreover we claim that any negative ones in $\eta'$ will be in its $s$'th or $s + 1$'st batch. Suppose that $\eta'$ contains a negative one prior to the $s$'th batch. Since the entries of $\eta'$ are nonincreasing, this implies that all of the entries of the $s$'th batch will be at most negative one. The $s$'th batch of $\eta'$ was formed by combining batches of $\nu$, so this yields that all of the entries of the $r$'th batch of $\nu$ will also be at most negative one. Thus $\bar{\nu}_r < 0$ which contradicts the $G$-dominance of $\nu_M$ unless $j = 0$ and $n_r = 1$. In this case, we have $\bar{\nu}_r = \nu_n$ and since $\nu_M$ is $G$-dominant, $\bar{\nu}_{r-1} + \bar{\nu}_r \geq 0$. Therefore $\bar{\nu}_{r-1} \geq 1$, so since $\nu_M$ is $G$-dominant, $\bar{\nu}_k \geq 1$ for all $k \leq r - 1$. Thus, since $\nu$ is $M$-2-minuscule, $\nu_i \geq 1$ for all $i \leq n - 1$ and it follows from the construction of $\eta'$ that $\eta'_r \geq 1$ for all $i \leq n - 1$. Therefore $\eta'$ cannot contain a negative one prior to the $s$'th batch.

Next we form a coweight $\eta''$ by replacing every negative one in the $s$'th batch of $\eta'$ with a positive one. Finally, if we made an odd number of sign changes in the previous step, then we change the sign of the final entry of $\eta''$ to form $\eta$; otherwise, $\eta = \eta''$.

We obtain $\eta_L$, resp. $\eta'_L$, from $\eta$, resp. $\eta'$, in the same manner as we obtained $\nu_M$ from $\nu$, and denote its entries by $\bar{\eta}_k$, resp. $\bar{\eta}'_k$. We now check that $\eta$ is $G$-dominant and satisfies all of the hypotheses on $\nu$, but for the Levi subgroup $L$.

**Lemma 5.4.** The coweight $\eta$ is $L$-2-minuscule and $\eta_L \leq \mu$. Moreover $\eta$ is $G$-dominant, hence $\eta$ is $L$-dominant and $\eta_L$ is $G$-dominant.

**Proof.** We begin by showing that $\eta$ is $L$-2-minuscule and $G$-dominant. It follows that $\eta$ is $L$-dominant and $\eta_L$ is $G$-dominant.

First, if $j = 0$ and $n_r = 1$, then $\eta = \eta'$, so we have shown that $\eta$ is $L$-2-minuscule. We claim that $\eta$ is $G$-dominant. The inequalities $\eta_1 \geq \cdots \eta_n$ follow from the way $\eta$ was constructed, so we need only check that $\eta_{n-1} + \eta_n \geq 0$. As in Section 3 we obtain $\bar{\eta}_{n-1} + \eta_n \geq 0$. Since $\eta$ is $L$-2-minuscule, $\eta_{n-1}$ is the greatest odd integer less than or equal to $\bar{\eta}_{n-1}$; therefore we have $\eta_{n-1} + \eta_n \geq 0$ since $\eta_n$ is an odd integer.
Otherwise, it is clear that the $s + 1$st batch of $\eta$ is still $L$-2-minuscule. If the $s$th batch of $\eta'$ contains any negative ones, then, since we have shown that $\eta'$ is $L$-2-minuscule and by Lemma 5.3, all other entries of the batch must be either positive or negative one. Therefore, since we only changed the signs of positive and negative ones, $\eta$ is $L$-2-minuscule. Moreover, by construction, $\eta_i \geq 1$ for all $i < n$, the entries of $\eta$ are nonincreasing, and, by Lemma 5.1, $\eta_n \geq -1$; hence it is clear that $\eta$ is $G$-dominant.

It only remains to show that $\eta_L \leq \mu$. To do so, we apply Lemma 5.3. Define $\bar{\sigma}(k) = m_1 + \cdots + m_k$ for $k \leq s$ and $\bar{\sigma}(s + 1) = n$.

First, we show that $\eta'_L \leq \mu$. The method used in Lemma 2.3 establishes inequality (5.2) for $i = \bar{\sigma}(k)$ for $k$ such that $\bar{\sigma}(k) \leq n - 2$, as well as inequality (5.3). It remains to verify inequality (5.2) under the assumption that $j = 0$ and $m_s = 1$; the proof proceeds exactly as in Lemma 4.4. Thus if $\eta = \eta'$, then we have $\eta_L \leq \mu$.

Now we consider the case in which $\eta \neq \eta'$. (In particular, it is not the case that $j = 0$ and $m_s = 1$.) Since $\eta$ and $\eta'$ do not differ prior to the $s$th batch and $\eta'_L \leq \mu$, inequality (5.3) is satisfied for $i = \bar{\sigma}(k)$ for all $k < s$. Moreover, since either the $s$th batch is the final batch (if $j = 0$) or $\eta_{n+1} = 0$ (if $j \neq 0$), and since $\mu$ is $G$-dominant, it is enough to check that inequality (5.3) holds for $i = \bar{\sigma}(s)$.

We have shown that $S_{\bar{\sigma}(s-1)}(\eta_L) \leq S_{\bar{\sigma}(s-1)}(\mu)$. Since the last entry of the $s$th batch of $\eta$ is either positive or negative one and $\eta$ is $L$-2-minuscule, we also have that $\bar{\eta}_s \leq 1$. Moreover, since $\mu$ is $G$-dominant, we have $\mu_i \geq 1$ for all $i < n$. It follows that $S_{\bar{\sigma}(s)}(\eta_L) \leq S_{\bar{\sigma}(s)}(\mu) \leq 2$.

since $\mu$ is $G$-dominant. Thus $S_n(\eta_L) \leq S_n(\mu)$ unless $S_{\bar{\sigma}(s-1)}(\eta_L) = S_{\bar{\sigma}(s-1)}(\mu)$, $\bar{\eta}_s = 1$, and $\mu_{\bar{\sigma}(s-1)+1} + \cdots + \mu_n = m_s - 2$. In this case, we have $S_n(\eta_L) = S_n(\mu) + 2$. This is a contradiction to condition (5.4) since $S_n(\eta_L) = S_n(\eta)$ and we see that $S_n(\eta)$ and $S_n(\nu)$ are congruent modulo four since the difference in the two sums comes from changing an even number of negative ones to positive ones. \[\square\]

We have shown that $\eta$ satisfies all of the hypotheses on $\nu$ for the Levi subgroup $L$ and that $\eta$ is $G$-dominant. Moreover, by its construction, $\eta \in \mathcal{W} \nu$ so it is enough to prove the theorem for $(L, \eta)$ instead of $(M, \nu)$. Thus it is enough to prove the theorem with the additional hypothesis that $\nu$ is $G$-dominant. We can now prove that $\nu \in \text{Conv}(W \mu)$ by proving that $\nu \leq \mu$.

**Theorem 5.5.** $\nu \in \text{Conv}(W \mu)$

**Proof.** We will suppose $\nu \nleq \mu$ and obtain a contradiction. If $\nu \nleq \mu$, then either there exists an $i \neq n − 1$ such that

\[\nu_1 + \nu_2 + \cdots + \nu_i > \mu_1 + \mu_2 + \cdots + \mu_i\]  \hspace{1cm} (5.6)

or inequality (5.2) fails.

First note that if $j = 0$, then $S_n(\nu) = S_n(\nu_M)$, so inequality (5.6) cannot hold for $i = n$ since $\nu_M \leq \mu$.

Now suppose that there exists an $i \leq n − 2$ such that inequality (5.6) holds. Choose the smallest such $i$. Then $\nu_i > \mu_i$, and both are odd integers, so $\nu_i - 2 \geq \mu_i$. Suppose $\nu_i$ is in the $k$th batch of $\nu$. \hspace{1cm} (5.6)
We consider the \((i+1)\)th to \(\sigma(k)\)th entries of \(\nu\) and \(\mu\). Since \(\nu\) is \(M\)-dominant and \(M\)-2-minuscule, \(\nu_{i+1}, \ldots, \nu_{\sigma(k)} \in \{\nu_i, \nu_i - 2\}\). Thus
\[
\nu_{i+1} + \ldots + \nu_{\sigma(k)} \geq (\sigma(k) - i)(\nu_i - 2).
\]
Also, since \(\mu\) is \(G\)-dominant and \(\mu_i \leq \nu_i - 2\), it follows that \(\mu_{i+1} + \ldots + \mu_{\sigma(k)} \leq (\sigma(k) - i)(\nu_i - 2)\). Thus
\[
\mu_{i+1} + \ldots + \mu_{\sigma(k)} \leq \nu_{i+1} + \ldots + \nu_{\sigma(k)}.
\]
Combining this with inequality (5.6) yields
\[
\mu_1 + \ldots + \mu_{\sigma(k)} < \nu_1 + \ldots + \nu_{\sigma(k)}.
\] (5.7)

If \(j \geq 2\) and \(\nu_i\) is in the final batch, then we have \(\mu_i < \nu_i = \pm 1\), so \(\mu_i \leq -1\). Since \(\mu\) is \(G\)-dominant, it follows that \(i = n\). This contradicts our assumption that \(i \leq n - 2\). If \(j \geq 2\) and \(\nu_i\) is not in the final batch, then \(\sigma(k) \leq \sigma(r) \leq n - 2\) and inequality (5.7) contradicts \(\nu_M \leq \mu\).

If \(j = 0\) and \(\sigma(k) \neq n - 1\), then inequality (5.7) contradicts \(\nu_M \leq \mu\). If \(j = 0\) and \(\sigma(k) = n - 1\), then we have \(n_r = 1\). As in the proof of Theorem 4.5, we observe that inequality (5.2) holds, which when added to inequality (5.3) yields \(S_{n-1}(\nu) \leq S_{n-1}(\mu)\), contradicting inequality (5.7).

Now suppose that inequality (5.7) fails. Thus we have
\[
S_{n-1}(\nu) - \nu_n > S_{n-1}(\mu) - \mu_n.
\] (5.8)

By what we have already shown, we have
\[
S_{n-2}(\nu) \leq S_{n-2}(\mu).
\] (5.9)

We claim that
\[
\nu_{n-1} - \nu_n > \mu_{n-1} - \mu_n \geq 0.
\] (5.10)
The first inequality follows from inequalities (6.8) and (6.9), and the second holds since \(\mu\) is \(G\)-dominant. We have shown that inequality (5.2) holds if \(j = 0\) and \(n_r = 1\) so we may assume that this is not the case. Thus \(\nu_{n-1}\) and \(\nu_n\) are in the same batch of \(\nu\) so, since \(\nu\) is \(M\)-minuscule, \(\nu_{n-1} - \nu_n \in \{0, 2\}\). Combining this with inequality (5.10) gives
\[
\nu_{n-1} - \nu_n = 2 \quad \text{and} \quad \mu_{n-1} - \mu_n = 0.
\] (5.11)
Substituting these values into inequality (5.8) and combining the result with inequality (5.4) gives
\[
S_{n-2}(\nu) + 2 > S_{n-2}(\mu) \geq S_{n-2}(\nu).
\]
Both are integers of the same parity, so \(S_{n-2}(\mu) = S_{n-2}(\nu)\). This contradicts condition (5.4) since it follows from the equalities in (5.11) (bearing in mind that all entries of \(\mu\) and \(\nu\) are odd) that \(S_n(\mu)\) is not congruent modulo four to \(S_{n-2}(\mu)\), and \(S_n(\nu)\) is congruent modulo four to \(S_{n-2}(\nu)\).

Finally, suppose that inequality (5.6) holds for \(i = n\). We have already handled the \(j = 0\) case so we may assume that \(j \geq 2\). Therefore \(\nu_{n-1}\) and \(\nu_n\) will be in the \(r + 1\)th batch, so since \(\nu\) is \(M\)-2-minuscule, we have \(\nu_{n-1} + \nu_n \in \{0, 2\}\). Also, since \(\mu\) is \(G\)-dominant, we have \(\mu_{n-1} + \mu_n \geq 0\). We have shown that \(S_{n-2}(\nu) \leq S_{n-2}(\mu)\). It follows that \(S_n(\nu) \leq S_n(\mu)\) unless \(S_{n-2}(\nu) = S_{n-2}(\mu)\), \(\nu_{n-1} + \nu_n = 2\), and \(\mu_{n-1} + \mu_n = 0\). In this case, \(S_n(\nu) = S_n(\mu) + 2\) which contradicts condition (5.4). Thus \(\nu \not\leq \mu\). \(\square\)
REFERENCES

[Bour81] N. Bourbaki, *Groupes et algèbres de Lie*, Masson, Paris, 1981.
[FR02] J.-M. Fontaine and M. Rapoport, *Existence de filtrations admissibles sur des isocristaux*, preprint, 2002.
[Kot97] R. Kottwitz, *Isocrystals with additional structure. II*, Compositio Math. 109 (1997), 255–339.
[KR02] R. Kottwitz and M. Rapoport, *On the existence of F-isocrystals*, to appear in Comment. Math. Helv., arXiv:math.NT/0202228.
[Rap02] M. Rapoport, *A guide to the reduction modulo p of Shimura varieties*, 2002, arXiv:math.AG/0205022.
[RR96] M. Rapoport and M. Richartz, *On the classification and specialization of F-isocrystals with additional structure*, Compositio Math. 103 (1996), 153–181.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, 5734 SOUTH UNIVERSITY AVENUE, CHICAGO, IL 60637
E-mail address: cleigh@math.uchicago.edu