On Kernel Mengerian Orientations of Line Multigraphs

Han Xiao *

Department of Mathematics,
The University of Hong Kong,
Hong Kong, China

Abstract

We present a polyhedral description of kernels in orientations of line multigraphs. Given a directed multigraph $D$, let $\sigma(D)$ denote the linear system consisting of domination, independence and nonnegativity constraints, and let $FK(D)$ denote the set of all solutions to $\sigma(D)$. We call $D$ kernel ideal if $FK(D')$ is integral for each induced subgraph $D'$ of $D$ and kernel Mengerian if $\sigma(D')$ is TDI for each induced subgraph $D'$ of $D$. We also call $D$ good if it is clique-acyclic and each directed odd cycle has a chord or a pseudochord. We show that an orientation of a line multigraph is kernel ideal if and only if it is kernel Mengerian if and only if it is good. Our result strengthens the theorem of Borodin et al. on kernel perfect digraphs and generalizes the theorem of Király and Pap on stable matching problem.

AMS subject classifications: 90C10, 90C27, 90C57.
Key words: kernel, stable matching, line graph, integral polytope, total dual integrality.

* hxiao.math@connect.hku.hk
1 Introduction

An undirected graph is called *simple* if it contains neither loops nor parallel edges and is called a *multigraph* if parallel edges are allowed. *Simple* digraphs and *directed multigraphs* are defined analogously.

Let $G$ be an undirected graph. The *line graph* of $G$, denoted by $L(G)$, is an undirected graph that represents the adjacency of edges of $G$: each vertex of $L(G)$ corresponds to an edge of $G$, and two vertices of $L(G)$ are adjacent if and only if they are incident as edges in $G$. We call $L(G)$ the *line multigraph* of $G$ if any two vertices of $L(G)$ are connected by as many edges as the number of their common ends in $G$.

Let $D = (V, A)$ be a digraph. For $U \subseteq V$, we call $U$ an *independent* set of $D$ if no two vertices in $U$ are connected by an arc, call $U$ a *dominating* set of $D$ if for each vertex $v \notin U$, there is an arc from $v$ to $U$, and call $U$ a *kernel* of $D$ if it is both independent and dominating. We call $D$ *kernel perfect* if each of its induced subgraphs has a kernel. A *clique* of $D$ is a subset of $V$ such that any two vertices are connected by an arc. We call $D$ *clique-acyclic* if for each clique of $D$ the induced subdigraph of one-way arcs is acyclic, and call $D$ *good* if it is clique-acyclic and every directed odd cycle has a chord or a pseudochord.

**Theorem 1.1** (Borodin et al. [3]). *Let $G$ be a line multigraph. The orientation $D$ of $G$ is kernel perfect if and only if it is good.*

A subset $P$ of $\mathbb{R}^n$ is called a *polytope* if it is the convex hull of finitely many vectors in $\mathbb{R}^n$. A point $x$ in $P$ is called a *vertex* or an *extreme point* if there exist no distinct points $y$ and $z$ in $P$ such that $x = \alpha y + (1 - \alpha)z$ for $0 < \alpha < 1$. It is well known that $P$ is the convex hull of its vertices, and that there exists a linear system $Ax \leq b$ such that $P = \{x \in \mathbb{R}^n : Ax \leq b\}$. We say $P$ is $1/k$-integral if its vertices are $1/k$-integral vectors, where $k \in \mathbb{N}$. By a theorem in linear programming, $P$ is $1/k$-integral if and only if $\max\{c^T x : Ax \leq b\}$ has an optimal $1/k$-integral solution for every integral vector $c$ for which the optimum is finite. If, instead, $\max\{c^T x : Ax \leq b\}$ has a dual optimal $1/k$-integral solution, we say $Ax \leq b$ is *totally dual $1/k$-integral* (TDI/$k$). It is easy to verify that $Ax \leq b$ is TDI/$k$ if and only if $Bx \leq b$ is TDI, where $B = A/k$ and $k \in \mathbb{N}$. Thus from a theorem of Edmonds and Giles [6], we deduce that if $Ax \leq b$ is TDI/$k$ and $b$ is integral, then $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is $1/k$-integral.

---

1 A pseudochord is an arc $(v_i, v_{i-1})$ in a directed cycle $v_1v_2 \ldots v_l$. 

---

2
Let $\sigma(D)$ denote the linear system consisting of the following inequalities:

$$
\begin{align*}
    x(v) + x(N_D^+(v)) & \geq 1 \quad \forall \ v \in V, \\
    x(Q) & \leq 1 \quad \forall \ Q \in Q, \\
    x(v) & \geq 0 \quad \forall \ v \in V,
\end{align*}
$$

where $x(U) = \sum_{u \in U} x(u)$ for any $U \subseteq V$, $N_D^+(v)$ denotes the set of all out-neighbors of vertex $v$, and $Q$ denotes the set of all cliques of $D$. Observe that incidence vectors of kernels of $D$ are precisely integral solutions $x \in \mathbb{Z}^A$ to $\sigma(D)$. The kernel polytope of $D$, denoted by $K(D)$, is the convex hull of incidence vectors of all kernels of $D$. The fractional kernel polytope of $D$, denoted by $FK(D)$, is the set of all solutions $x \in \mathbb{R}^A$ to $\sigma(D)$. Clearly, $K(D) \subseteq FK(D)$. We call $D$ kernel ideal if $FK(D')$ is integral for each induced subgraph $D'$ of $D$, and kernel Mengerian if $\sigma(D')$ is TDI for each induced subgraph $D'$ of $D$.

As described in Egres Open [1], the polyhedral description of kernels remains open. Chen et al. [4] attained a polyhedral characterization of kernels by replacing clique constraints $x(Q) \leq 1$ for $Q \in Q$ with independence constraints $x(u) + x(v) \leq 1$ for $(u, v) \in A$. In this paper we show that kernels in digraph $D$ can be fully characterized by polyhedral approaches if $D$ is an orientation of line multigraphs.

**Theorem 1.2.** Let $D$ be an orientation of a line multigraph. Then the following statements are equivalent:

(i) $D$ is good;

(ii) $D$ is kernel perfect;

(iii) $D$ is kernel ideal;

(iv) $D$ is kernel Mengerian.

The equivalence of (i) and (ii) was established by Borodin et al. [3] (Maffray [8] proved the case when the underlying undirected graph of $D$ is perfect). Király and Pap [7] proved the theorem when the underlying undirected graph of $D$ is the line graph of a bipartite graph, which will be described in the next section. Our result strengthens the theorem of Borodin et al. [3] and generalizes the theorem of Király and Pap [7] to line multigraphs.

## 2 Preliminaries

Let $G = (V, E)$ be an undirected graph. For $v \in V$, let $\delta(v)$ denote the set of all edges incident to $v$ and $\prec_v$ be a strict linear order on $\delta(v)$. We call $\prec_v$ the preference of $v$, and for edges $e$ and
f incident to v we say v prefers e to f or e dominates f if e ⪯ v f. Let ≺ be the set of all v≺ for v ∈ V. We call (G, ≺) a preference system and say (G, ≺) is simple if G is simple. For e ∈ E, let ϕ(e) denote the set of all edges that dominate e in (G, ≺). Let M be a matching of G. We say M is stable in (G, ≺) if every edge of G is dominated by some edge in M.

Let π(G, ≺) denote the linear system consisting of the following inequalities:

\[
\begin{align*}
  x(\varphi(e)) & \geq 1 & \forall e \in E, \\
  x(\delta(v)) & \leq 1 & \forall v \in V, \\
  x(e) & \geq 0 & \forall e \in E.
\end{align*}
\]

As observed by Abeledo and Rothblum [2], incidence vectors of stable matchings of (G, ≺) are precisely integral solutions x ∈ Z^E to π(G, ≺). The stable matching polytope, denoted by SM(G, ≺), is the convex hull of incidence vectors of all stable matchings of (G, ≺). The fractional stable matching polytope, denoted by FSM(G, ≺), is the set of all solutions x ∈ R^E to π(G, ≺).

Theorem 2.1 (Rothblum [9]). Let (G, ≺) be a simple preference system. If G is bipartite, then SM(G, ≺) = FSM(G, ≺).

Theorem 2.2 (Király and Pap [7]). Let (G, ≺) be a simple preference system. If G is bipartite, then π(G, ≺) is totally dual integral.

Let C = v_1v_2...v_l be a cycle in G, we say that C has cyclic preferences in (G, ≺) if v_{i-1}v_i ≺ v_i v_{i+1} for i = 1, 2, ..., l or v_{i-1}v_i ≻ v_i v_{i+1} for i = 1, 2, ..., l, where indices are taken modulo l. For x ∈ FSM(G, ≺), let E_α(x) denote the set of all edges with x(e) = α where α ∈ R and E_+(x) denote the set of all edges with x(e) > 0.

Theorem 2.3 (Abeledo and Rothblum [2]). Let (G, ≺) be a simple preference system. Then FSM(G, ≺) is 1/2-integral. Moreover, for each 1/2-integral point x in FSM(G, ≺), E_{1/2}(x) consists of vertex disjoint cycles with cyclic preferences.

Theorem 2.4 (Chen et al. [5]). Let (G, ≺) be a simple preference system. Then π(G, ≺) is totally dual 1/2-integral. Moreover, π(G, ≺) is totally dual integral if and only if SM(G, ≺) = FSM(G, ≺).

3 Reductions

To study kernels in directed multigraphs, it suffices to work on its minimal spanning subdigraph that preserves all the connection relations of vertices. In the remainder of this section, assume
that \( D \) is an orientation of line multigraph \( L(H) \) such that any two vertices in \( H \) are joined by at most two edges and parallel edges in \( L(H) \) are orientated oppositely. Hence \( D \) is a directed multigraph such that any two distinct vertices are joined by at most one arc in each direction.

Kernels are closely related to stable matchings. When \( D \) is clique-acyclic, let \( e \prec_v f \) if \((e, f)\) is an arc in \( D \) for any two incident edges \( e \) and \( f \) with common end \( v \) in \( H \). Hence each clique-acyclic orientation \( D \) of line multigraph \( L(H) \) is associated with a preference system \((H, \prec)\). Let \( \O(H) \) denote the set of all complete subgraphs of \( H \) induced on three vertices (with parallel edges allowed).

Hence constraints \((1.2)\) can be viewed as constraints \((2.2)\) together with some extra constraints on \((H, \prec)\). Let \( \O(H) \) denote the set of all complete subgraphs of \( H \) induced on three vertices. Then \( \sigma(D) \) can be reformulated in terms of preference system \((H, \prec)\):

\[
\begin{align*}
x(\phi(e)) & \geq 1 & \forall e \in E(H), & (3.1) \\
x(\delta(v)) & \leq 1 & \forall v \in V(H), & (3.2) \\
x(S) & \leq 1 & \emptyset \subset S \subset \delta(v), & \forall v \in V(H), & (3.3) \\
x(O) & \leq 1 & \forall O \in \O(H), & (3.4) \\
x(e) & \geq 0 & \forall e \in E(H). & (3.5)
\end{align*}
\]

Notice that constraints \((3.1), (3.2)\) and \((3.5)\) constitute the Rothblum system \( \pi(H, \prec) \) which defines \( FSM(H, \prec) \). Constraints \((3.3)\) are redundant with respect to \( \pi(H, \prec) \) due to constraints \((3.2)\). As we shall see, constraints \((3.4)\) are also redundant with respect to \( \pi(H, \prec) \) when \( D \) is good. Hence \( FK(D) \) is essentially defined by Rothblum system \( \pi(H, \prec) \), or equivalently, that \( FK(D) = FSM(H, \prec) \), when \( D \) is good.

**Lemma 3.1.** For parallel edges \( e \) and \( e' \) in \( H \), there exists no edge \( f \) such that \( e \prec_v f \prec_v e' \), where \( v \) is a common end of \( e \) and \( e' \).
Proof. Since $D$ is clique-acyclic, by the construction of preference system $(H, \prec)$, the lemma follows directly. \hfill \Box

By Lemma \ref{lem:parallel-edges} parallel edges play the same role in preference system $(H, \prec)$. Hence we turn to study underlying simple preference systems of $(H, \prec)$. Let $(\hat{H}, \preceq)$ be a simple preference system, where $\hat{H}$ is a spanning subgraph of $H$ obtained by keeping one edge between every pair of adjacent vertices and $\preceq$ is the restriction of $\prec$ on $\hat{H}$. Before proceeding, we introduce a technical lemma.

Lemma 3.2. Let

\[ Ax \leq b, \ x \geq 0 \]  

and

\[ \tilde{A} \tilde{x} \leq b, \ \tilde{x} \geq 0 \]  

be two linear systems, where $\tilde{A}$ is obtained from $A$ by duplicating some columns. If (3.6) is totally dual $1/k$-integral, then so is (3.7), where $k \in \mathbb{N}$.

Proof. It suffices to prove that the theorem holds for $\tilde{A}$ with one duplicate column. Let $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_{n+1}) \in \mathbb{R}^{n+1}$ and $\tilde{A} = (\tilde{a}_1, \ldots, \tilde{a}_{n+1}) = (A, a_k)$, where $a_k$ is the $k$th column of $A$. Let $\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_{n+1}) \in \mathbb{Z}^{n+1}$ be an integral vector such that $\max\{\tilde{c}^T \tilde{x} : \tilde{A} \tilde{x} \leq b, \ \tilde{x} \geq 0\}$ is finite.

Define $\hat{c} \in \mathbb{Z}^n$ by

\[ \hat{c}_i := \begin{cases} \tilde{c}_i, & i \neq k, \\ \max\{\tilde{c}_k, \tilde{c}_{n+1}\} & i = k, \end{cases} \]

and define $\hat{x} \in \mathbb{R}^n$ by

\[ \hat{x}_i := \begin{cases} \tilde{x}_i, & i \neq k, \\ \tilde{x}_k + \tilde{x}_{n+1} & i = k. \end{cases} \]

Clearly, $\max\{\hat{c}^T \hat{x} : A\hat{x} \leq b, \ \hat{x} \geq 0\} \geq \max\{\tilde{c}^T \tilde{x} : \tilde{A} \tilde{x} \leq b, \ \tilde{x} \geq 0\}$. We claim that equality always holds in this inequality. Given an optimal solution $\hat{x}^*$ to $\max\{\hat{c}^T \hat{x} : A\hat{x} \leq b, \ \hat{x} \geq 0\}$, define $\tilde{x}^*$ by $\tilde{x}^*_i := \hat{x}^*_i$ for $i \neq k, n + 1$; if $\tilde{c}_k > \tilde{c}_{n+1}$, $\tilde{x}^*_k := \hat{x}^*_k$ and $\tilde{x}^*_{n+1} := 0$, otherwise $\tilde{x}^*_k := 0$ and $\tilde{x}^*_{n+1} := \hat{x}^*_k$. It is easy to verify that $\tilde{x}^*$ is a feasible solution to $\max\{\tilde{c}^T \tilde{x} : \tilde{A} \tilde{x} \leq b, \ \tilde{x} \geq 0\}$. Moreover, $\hat{c}^T \hat{x}^* = \tilde{c}^T \tilde{x}^*$ implies $\max\{\hat{c}^T \hat{x} : A\hat{x} \leq b, \ \hat{x} \geq 0\} \leq \max\{\tilde{c}^T \tilde{x} : \tilde{A} \tilde{x} \leq b, \ \tilde{x} \geq 0\}$. Hence equality follows.

Since $Ax \leq b$ is TDI/$k$, there exists a dual optimal $1/k$-integral solution $y^*$ to $\max\{\tilde{c}^T \tilde{x} : A\tilde{x} \leq b, \ \tilde{x} \geq 0\}$ such that $(y^*)^T b = \max\{\tilde{c}^T \tilde{x} : A\tilde{x} \leq b, \ \tilde{x} \geq 0\}$, $(y^*)^T A \geq \tilde{c}^T$ and $y^* \geq 0$. It follows that

\[ (y^*)^T \tilde{A} = (y^*)^T (A, a_k) = ((y^*)^T A, (y^*)^T a_k) \geq (\tilde{c}^T, \hat{c}_k) \geq \tilde{c}^T, \]
implying that $y^*$ is a feasible dual solution to $\max\{\bar{c}^T \bar{x} : \bar{A} \bar{x} \leq b, \ \bar{x} \geq 0\}$. By the following inequalities

$$(y^*)^T b = \max\{\bar{c}^T \bar{x} : \bar{A} \bar{x} \leq b, \ \bar{x} \geq 0\} = \max\{\bar{c}^T \bar{x} : \bar{A} \bar{x} \leq b, \ \bar{x} \geq 0\} \leq (y^*)^T b,$$

$y^*$ is also a dual optimal solution to $\max\{\bar{c}^T \bar{x} : \bar{A} \bar{x} \leq b, \ \bar{x} \geq 0\}$, where the last inequality is from the weak duality theorem. Hence the lemma follows.

**Lemma 3.3.** If $\pi(\hat{H}, \prec)$ is totally dual $1/k$-integral, then so is $\pi(H, \prec)$, where $k \in \mathbb{N}$.

**Proof.** By Lemma 3.1, columns corresponding to parallel edges in the left hand side matrix of $\pi(H, \prec)$ are identical. Hence the left hand side matrix of $\pi(H, \prec)$ can be obtained from that of $\pi(\hat{H}, \prec)$ by duplicating columns corresponding to parallels edges. Then the lemma follows from Lemma 3.2.

By Theorem 2.4 and Lemma 3.3, we deduce that $\pi(H, \prec)$ is TDI/2. By the construction of $(H, \prec)$, when $D$ is good, $(H, \prec)$ admits no odd cycles with cyclic preferences. Hence constraints (3.4) are redundant in $\pi(D)$ with respect to $\pi(H, \prec)$ when $D$ is good.

**Lemma 3.4.** $FSM(H, \prec)$ is integral if and only if $FSM(\hat{H}, \prec)$ is integral.

**Proof.** For simplicity, assume that $H = (V, E)$ and $\hat{H} = (V, \hat{E})$.

We prove the “only if” part by showing that $FSM(\hat{H}, \prec)$ is a projection of $FSM(H, \prec)$. Let $\hat{x}$ be a point in $FSM(\hat{H}, \prec)$ and define $x := (\hat{x}, 0) \in \mathbb{R}^{\hat{E}} \times \mathbb{R}^{E - \hat{E}}$. To show that $x \in FSM(H, \prec)$, it suffices to prove that $x(\varphi(e)) \geq 1$ for $e \in E - \hat{E}$. Let $e' \in \hat{E}$ be the edge parallel with $e \in E - \hat{E}$. By Lemma 3.1, $x(\varphi(e)) = x(\varphi(e')) = \hat{x}(\varphi(e')) \geq 1$. Hence $FSM(\hat{H}, \prec)$ is a projection of $FSM(H, \prec)$.

We prove the “if” part by showing that each vertex of $FSM(H, \prec)$ can be obtained from some vertex of $FSM(\hat{H}, \prec)$ by adding some zero entries. Let $x$ be a vertex of $FSM(H, \prec)$. We first show that $x(e) \cdot x(e') = 0$ for parallel edges $e$ and $e'$. Assume to the contrary that $x(e) \cdot x(e') \neq 0$, then $x(e) = x(e') = 1/2$ follows. Let $y$ and $z$ be duplicates of $x$, and further set $y(e) := z(e') := 1$ and $y(e') := z(e) := 0$. It follows that $x = (y + z)/2$. Clearly, $y, z \in FSM(H, \prec)$, contradicting to the fact that $x$ is a vertex. Now define $\hat{x} \in \mathbb{R}^{\hat{E}}$ by, for $e \in \hat{E}$, $\hat{x}(e) := x(e) + x(e')$, where $e' \in E - \hat{E}$ is parallel with $e$. By Lemma 3.1, it is easy to verify that $\hat{x} \in FSM(\hat{H}, \prec)$. We claim that $\hat{x}$ is a vertex of $FSM(\hat{H}, \prec)$. Assume to the contrary that there exist $\hat{x}_1, \hat{x}_2 \in FSM(\hat{H}, \prec)$ such that $\hat{x} = \alpha \hat{x}_1 + (1 - \alpha) \hat{x}_2$, where $0 < \alpha < 1$. For $i = 1, 2$, we extend $\hat{x}_i \in \mathbb{R}^{\hat{E}}$ to $x_i \in \mathbb{R}^E$ by, for $e \in \hat{E}$ without parallel edges in $H$, $x_i(e) := \hat{x}_i(e)$; for $e \in \hat{E}$ and its parallel edge $e' \in E - \hat{E}$, if $x(e) > x(e')$, $x_i(e) := \hat{x}_i(e)$ and $x_i(e') := 0$, otherwise $x_i(e) := 0$ and $x_i(e') := \hat{x}_i(e)$. By
Lemma 3.1 it is easy to see that $x_1, x_2 \in FSM(H, \prec)$. Since $x(e) \cdot x(e') = 0$ for parallel edges $e$ and $e'$, $x = \alpha x_1 + (1 - \alpha)x_2$ follows, a contradiction. Hence $x$ is vertex of $FSM(\hat{H}, \hat{\prec})$. Therefore each vertex $x$ of $FSM(H, \prec)$ is associated with a vertex $\hat{x}$ of $FSM(\hat{H}, \hat{\prec})$, and $x$ can be obtained from $\hat{x}$ by, for $e \in \hat{E}$ without parallel edges in $H$, $x(e) := \hat{x}(e)$; for $e \in \hat{E}$ and its parallel edge $e' \in E - \hat{E}$, if $x(e) > x(e')$, $x(e) := \hat{x}(e)$ and $x(e') := 0$, otherwise $x(e) := 0$ and $x(e') := \hat{x}(e)$.

We end this section with a summary. When $D$ is clique-acyclic, it is associated with a preference system $(H, \prec)$ and a simple preference system $(\hat{H}, \hat{\prec})$, where $\hat{H}$ is a simple spanning subgraph of $H$ maximizing the edge set and $\hat{\prec}$ is the restriction of $\prec$ on $\hat{H}$. Hence constraints (3.3) and (3.4) are redundant in $\sigma(D)$ with respect to $\pi(H, \prec)$ and $FK(D) = FSM(H, \prec)$ follows. To show $FK(D)$ is integral, by Lemma 3.1 it suffices to show that $FSM(\hat{H}, \hat{\prec})$ is integral. To show $\sigma(D)$ is TDI, by Lemma 3.3 it suffices to show $\pi(\hat{H}, \hat{\prec})$ is TDI. Moreover, when $D$ is good both $(H, \prec)$ and $(\hat{H}, \hat{\prec})$ admit no odd cycles with cyclic preferences.

4 Proofs

Before presenting our proof of the main theorem, we exhibit some properties of simple preference systems admitting no odd cycles with cyclic preferences.

**Lemma 4.1.** Let $(G, \prec)$ be a simple preference system. If $(G, \prec)$ admits no odd cycles with cyclic preferences, then $SM(G, \prec) = FSM(G, \prec)$.

As observed by Chen et al. [5], integrality of $FSM(G, \prec)$ is equivalent to total dual integrality of $\pi(G, \prec)$, where $(G, \prec)$ is a simple preference system. A corollary follows directly.

**Corollary 4.2.** Let $(G, \prec)$ be a simple preference system. If $(G, \prec)$ admits no odd cycles with cyclic preferences, then $\pi(G, \prec)$ is totally dual integral.

**Proof of Lemma 4.1.** By Theorem 2.3, $FSM(G, \prec)$ is $1/2$-integral as $(G, \prec)$ is a simple preference system. Let $x$ be a $1/2$-integral point in $FSM(G, \prec)$. Since $(G, \prec)$ admits no odd cycles with cyclic preferences, $E_{1/2}(x)$ consists of even cycles $C_1, C_2, \ldots, C_r$ with cyclic preferences. For $i = 1, 2, \ldots, r$, label vertices and edges of $C_i \in E_{1/2}(x)$ such that $C_i = v_1^i v_2^i \ldots v_l^i$ and $e_k^i \prec v_{k+1}^i$ for $k = 1, 2, \ldots, l$, where $e_k^i = v_k^i v_{k+1}^i$ and indices are taken modulo $l$. We remark that the parity of vertices and edges refers to the parity of their indices. Define $z \in \mathbb{R}^{E(G)}$ by

$$z(e) := \begin{cases} 
1 & \text{if } e \text{ is an even edge in some } C \in E_{1/2}(x), \\
-1 & \text{if } e \text{ is an odd edge in some } C \in E_{1/2}(x), \\
0 & \text{otherwise}.
\end{cases}$$
We are going to exclude $x$ from vertices of $FSM(G, \prec)$ by adding perturbation $\varepsilon z$ for small $\varepsilon$ to $x$ and showing that $x \pm \varepsilon z \in FSM(G, \prec)$. Tight constraints in (2.1)-(2.3) under perturbation $\varepsilon z$ play a key role here. Observe that tight constraints in (2.2) and (2.3) are invariant under perturbation $\varepsilon z$. It remains to show that perturbation $\varepsilon z$ does not affect tight constraints in (2.1) either. Let $e$ be an edge with $x(\varphi(e)) = 1$. Clearly, $|\varphi(e) \cap E_+(x)| \in \{1, 2\}$. When $|\varphi(e) \cap E_+(x)| = 1$, $x(e) = 1$ follows, which is trivial. When $|\varphi(e) \cap E_+(x)| = 2$, we claim that the parity of dominating edges in $E_{1/2}(x)$ of $e$ does not agree (relabeling vertices and edges in $E_{1/2}(x)$ if necessary). Hence corresponding tight constraints in (2.1) are also invariant under perturbation $\varepsilon z$. To justify this claim, we distinguish four cases.

**Case 1.** Edge $e$ is an edge from some $C \in E_{1/2}(x)$. This case is trivial since $C$ admits cyclic preferences.

**Case 2.** Edge $e$ is a chord in some $C \in E_{1/2}(x)$. We first show that endpoints of $e$ have different parity in $C$. We prove it by contradiction. Without loss of generality, let $e = v_{2i}v_{2j}$.

If $e_{2i} \prec e$, then $e_{2i-1} \prec e$. Since $x(\varphi(e)) = 1$, it follows that $e \prec e_{2i-1}$ and $e \prec e_{2j}$. However, $v_{2i}e_{2j}v_{2j+1} \ldots e_{2i-1}v_{2i}$ constitute an odd cycle with cyclic preferences, a contradiction. Hence $e \prec e_{2i}$.

Similarly, if $e_{2j} \prec e$, then $e_{2j-1} \prec e$. Equally, $x(\varphi(e)) = 1$ implies that $e \prec e_{2i}$ and $e \prec e_{2i-1}$. However, $v_{2i}e_{2i+1}v_{2i+1} \ldots e_{2j-1}v_{2j}e_{2j}$ constitute an odd cycle with cyclic preferences, a contradiction. Hence $e \prec e_{2j}$.

Now $e \prec e_{2i}$ and $e \prec e_{2j}$, it follows that $e_{2i-1} \prec e$ and $e_{2j-1} \prec e$ since $x(\varphi(e)) = 1$. But in this case two odd cycles with cyclic preferences mentioned above occur at the same time. Therefore, endpoints of $e$ have different parity in $C$. Hence let $e = v_{2i}v_{2j+1}$. If $e_{2i} \prec e$ (resp. $e_{2j+1} \prec e$), it follows that $e_{2i-1} \prec e$ (resp. $e_{2j} \prec e$). Then $e$ is dominated by two consecutive
edges from $C$, which is trivial. So assume that $e \prec e_{2i}$ and $e \prec e_{2j+1}$. Since $x(\varphi(e)) = 1$, it follows that $e_{2i+1} \prec e$ and $e_{2j} \prec e$. Therefore $e$ is dominated by two edges with different parity.

**Case 3.** Edge $e$ is a hanging edge of some $C \in E_{1/2}(x)$ and dominated by two edges from $C$. This case is trivial.

**Case 4.** Edge $e$ is a connecting edge between $C_i$ and $C_j$ and dominated by one edge from $C_i$ and one edge from $C_j$ respectively, where $C_i, C_j \in E_{1/2}(x)$. For $k = 1, 2, \ldots, r$, let $F_k$ be the subset of edges in this case and incident to $C_k$. Then $\bigcup_{i=1}^{r} F_i \cup C_i$ induces a subgraph of $G$. It suffices to work on a component of the induced subgraph. We apply induction on the number $\alpha$ of cycles from $E_{1/2}(x)$ in a component.

When $\alpha = 1$, it is trivial. Hence assume the claim holds for components with $\alpha \geq 1$ cycles from $E_{1/2}(x)$. We consider a component with $\alpha + 1$ cycles $C_1, \ldots, C_{\alpha}, C_{\alpha+1}$ from $E_{1/2}(x)$. Without loss of generality, assume that deleting $C_{\alpha+1}$ yields a new component with $\alpha$ cycles. By induction hypothesis, the claim holds for the resulting component. It remains to check edges in $F_{\alpha+1}$. If there exists an edge in $F_{\alpha+1}$ violating the claim, relabel vertices and edges in $C_{\alpha+1}$. After at most one relabeling, all edges in $F_{\alpha+1}$ satisfy the claim. We prove it by contradiction. Let $f_1, f_2 \in F_{\alpha+1}$ be edges such that $f_1$ satisfies the claim but $f_2$ violates the claim. For $i = 1, 2$, let $f_i = u_iw_i$, where $u_i$ is the endpoint in the resulting component and $w_i$ is the endpoint in $C_{\alpha+1}$. By assumption, $u_1$ and $w_1$ have different parity and $u_2$ and $w_2$ have the same parity. Analogous to the definition of cycles with cyclic preferences, we call path $P = v_1v_2\ldots v_l$ a $v_1v_l$-path with linear preferences if $v_iv_{i+1} \prec v_{i+1}v_{i+2}$ for $i = 1, 2, \ldots, l-2$. Clearly, for any two vertices in the same component, there exists a path with linear preferences between them. Hence there exist a $u_1u_2$-path $P_\alpha$ and a $w_2w_1$-path $P_{\alpha+1}$, both of which admit linear preferences. Moreover, $u_1P_\alpha u_2w_2P_{\alpha+1}w_1f_1u_1$ constitute a cycle with cyclic preferences. We justify this cycle is odd by showing that the $u_1u_2$-path $P_\alpha$ is even (resp. odd) if $u_1$ and $u_2$ have the same (resp. different) parity.

If $u_1$ and $u_2$ belong to the same cycle from $E_{1/2}(x)$, it is trivial. Hence assume $u_1 \in C_s$ and $u_2 \in C_t$, where $s, t \in \{1, 2, \ldots, \alpha\}$ and $s \neq t$. We apply induction on the number $\tau$ of cycles
from $E_{1/2}(x)$ involved in $P_\alpha$. Clearly, $\tau \geq 2$. When $\tau = 2$. Take $v^s v^t \in F_s \cap F_t$ on $P_\alpha$. Let $P_s$ be the part of $P_\alpha$ from $u_1$ to $v^s$ in $C_s$ and $P_t$ be the part of $P_\alpha$ from $v^t$ to $u_2$ in $C_t$. It follows that $u_1 P_s v^s v^t P_t u_2$ constitute $P_\alpha$. By primary induction hypothesis, $v^s$ and $v^t$ have different parity since $v^s v^t \in F_s \cap F_t$. If $u_1$ and $u_2$ have the same parity, then $P_s$ and $P_t$ have different parity, implying that $P_\alpha$ is even; if $u_1$ and $u_2$ have different parity, then $P_s$ and $P_t$ have the same parity, implying that $P_\alpha$ is odd. Now assume $\tau \geq 2$. Let $C_{k_1}, \ldots, C_{k_\tau}, C_{k_{\tau+1}}$ be cycles from $E_{1/2}(x)$ involved along $P_\alpha$. Take $v^{k_\tau} v^{k_{\tau+1}} \in F_{k_\tau} \cap F_{k_{\tau+1}}$ on $P_\alpha$. Let $P_{s, k_\tau}$ denote the part of $\alpha$ from $u_1$ to $v^{k_\tau}$ and $P_{k_\tau, t}$ denote the part of $\alpha$ from $v^{k_{\tau}}$ to $u_2$. Clearly, $P_\alpha = u_1 P_{s, k_\tau} v^{k_\tau} P_{k_\tau, t} u_2$. Since $P_{s, k_\tau}$ involves $\tau$ cycles and $P_{k_\tau, t}$ involves two cycles, both length depend on the parity of endpoints. It follows that $P_\alpha$ is even when $u_1$ and $u_2$ have the same parity, and $P_\alpha$ is odd when $u_1$ and $u_2$ have different parity.

Hence when $u_1$ and $u_2$ have the same parity, $w_1$ and $w_2$ have different parity, implying that $P_\alpha$ is even and $P_{\alpha+1}$ is odd; when $u_1$ and $u_2$ have different parity, $w_1$ and $w_2$ have the same parity, implying that $P_\alpha$ is odd and $P_{\alpha+1}$ is even. Either case yields an odd cycle with cyclic preferences, a contradiction.

Therefore 1/2-integral points are not vertices of $FSM(G, \prec)$ as they can be perturbed by $\epsilon z$ for small $\epsilon$ without leaving $FSM(G, \prec)$. By Theorem 2.3 $SM(G, \prec) = FSM(G, \prec)$ follows.

Now we are ready to present a proof of our main theorem.

**Proof of Theorem 4.3.** It suffices to show the equivalence of (i), (iii) and (iv). Without loss of generality, let $D$ be an orientation of line multigraph $L(H)$ such that any two vertices in $H$ are joined by at most two edges and parallel edges in $L(H)$ are orientated oppositely. When $D$ is good, $D$ is associated with a preference system $(H, \prec)$ and a simple preference system $(\hat{H}, \hat{\prec})$, both of which admit no odd cycles with cyclic preferences, where $\hat{H}$ is a simple spanning subgraph of $H$ maximizing the edge set and $\hat{\prec}$ is the restriction of $\prec$ on $\hat{H}$. Hence $\sigma(D)$ can be viewed as a linear system defined on preference system $(H, \prec)$ and consisting of constraints (3.1)-(3.5), where constraints (3.1), (3.2) and (3.3) constitute the Rothblum system $\pi(H, \prec)$.

By Lemma 4.1 $FSM(H, \hat{\prec})$ is integral. Integrality of $FSM(H, \prec)$ follows from Lemma 3.4. Hence constraints (3.3) and (3.4) are both redundant in $\sigma(D)$ with respect to $\pi(H, \prec)$. Therefore $FK(D) = FSM(H, \prec)$, implying that $FK(D)$ is integral. Similar arguments apply to any induced subgraphs of $D$. Hence (i) $\implies$ (iii).

By Corollary 4.2 $\pi(\hat{H}, \hat{\prec})$ is TDI. Total dual integrality of $\pi(H, \prec)$ follows from Lemma 3.3. Since $\pi(H, \prec)$ is part of $\sigma(D)$ and the other constraints (3.3)-(3.4) are redundant in $\sigma(D)$ with respect to $\pi(H, \prec)$, total dual integrality of $\sigma(D)$ follows. Similar arguments apply to any induced subgraphs of $D$. Hence (iii) $\implies$ (iv).
By a theorem of Edmonds and Giles [6], implication $(iv) \implies (iii)$ follows directly.

To prove implication $(iii) \implies (i)$, we assume the contrary. Observe that strong kernel idealness of $D$ implies the existence of kernels for any induced subgraphs of $D$. Let $D$ be a digraph such that $D$ is kernel ideal but not good. Then there exists either a clique containing directed cycles or a directed odd cycle without chords nor pseudochords in $D$. We show that neither case is possible. If $D$ has a clique containing directed cycles, we consider the subgraph induced on this clique. There is no kernel for this induced subgraph, a contradiction. If $D$ contains a directed odd cycle without chords nor pseudochords, we restrict ourselves to the subgraph induced on this directed odd cycle. There is no kernel for this induced subgraph either, a contradiction.

\begin{acknowledgments}
\end{acknowledgments}

The author would like to thank Prof. Wenan Zang for his invaluable suggestions.

\begin{references}
[1] Egres Open. Available from $http://lemon.cs.elte.hu/egres/open$.

[2] H. G. Abeledo and U. G. Rothblum, Stable matchings and linear inequalities, Discrete Appl. Math., 54 (1994), pp. 1–27.

[3] O. V. Borodin, A. V. Kostochka, and D. R. Woodall, On kernel-perfect orientations of line graphs, Discrete Math., 191 (1998), pp. 45–49.

[4] Q. Chen, X. Chen, and W. Zang, A polyhedral description of kernels. Manuscript.

[5] X. Chen, G. Ding, X. Hu, and W. Zang, The maximum-weight stable matching problem: duality and efficiency, SIAM J. Discrete Math., 26 (2012), pp. 1346–1360.

[6] J. Edmonds and R. Giles, A min-max relation for submodular functions on graphs, Ann. of Discrete Math., 1 (1977), pp. 185–204.

[7] T. Király and J. Pap, Total dual integrality of Rothblum’s description of the stable-marriage polyhedron, Math. Oper. Res., 33 (2008), pp. 283–290.

[8] F. Maffray, Kernels in perfect line-graphs, J. Combin. Theory Ser. B, 55 (1992), pp. 1–8.

[9] U. G. Rothblum, Characterization of stable matchings as extreme points of a polytope, Math. Programming, 54 (1992), pp. 57–67.
\end{references}
[10] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley-Interscience Series in Discrete Mathematics, John Wiley & Sons, Ltd., Chichester, 1986.