POSITIVITY OF HOCHSTER THETA

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Abstract. M. Hochster defines an invariant namely \( \Theta(M,N) \) associated to two finitely generated module over a hyper-surface ring \( R = P/f \), where \( P = k{x_0,\ldots,x_n} \) or \( k[x_0,\ldots,x_n] \), for \( k \) a field and \( f \) is a germ of holomorphic function or a polynomial, having isolated singularity at 0. This invariant can be lifted to the Grothendieck group \( G_0(R)_\mathbb{Q} \) and is compatible with the chern character and cycle class map, according to the works of W. Moore, G. Piepmeyer, S. Spiroff, M. Walker. They prove that it is semi-definite when \( f \) is a homogeneous polynomial, using Hodge theory on Projective varieties. It is a conjecture that the same holds for general \( f \). We give a proof of this conjecture using Hodge theory of isolated hyper-surface singularities when \( k = \mathbb{C} \).

Introduction

1. Modules over Hypersurface rings

A hyper-surface ring is a ring of the form \( R := P/(f) \), where \( P \) is an arbitrary ring and \( f \) a non-zero divisor. Localizing we may assume \( P \) is a local ring of dimension \( n + 1 \). As according to the title we assume \( P = \mathbb{C}\{x_0,\ldots,x_n\} \) and \( f \) a holomorphic germ, or \( P = \mathbb{C}[x_0,\ldots,x_n] \) and then \( f \) would be a polynomial. Then we are mainly interested to study finitely generated modules over these rings. Typical modules arise by considering sub-varieties \( Z \) lying inside the singular fiber \( X_0 = f^{-1}(0) \). As \( Z \) is supposed to be contained in \( X_0 \), one has \( f \in I = I(Z) \). The sub-variety \( Z \) determines the module \( O_Z = P/I \). A matrix factorization of \( f \) in \( P \) is a pair of matrices \( A \) and \( B \) such that \( AB = BA = f \). id. This definition orginally goes back to D. Eisenbud, [EP]. It is equivalent to the data of a pair of finitely generated free \( P \)-modules

\[
d_0 : X^0 \Rightarrow X^1 : d_1, \quad d_0d_1 = d_1d_0 = f. \ id
\]

It is a basic fact, discovered by D. Eisenbud, that the \( R \)-modules have a minimal resolution that is eventually 2-periodic. Specifically, In a free resolution of such a module \( M \), we see that after \( n \)-steps we have an exact sequence of the following form.

\[
0 \to M' \to F_{n-1} \to F_{n-2} \to \ldots \to F_0 \to M \to 0
\]

Key words and phrases. Matrix factorization, Riemann-Hodge bilinear relations, Residue pairing, Cyclic homology.
where the $F_i$ are free $R$-modules of finite rank and $\text{depth}_R(M') = n$. If $M' = 0$ then $M$ has a free resolution of finite length. If $M' \neq 0$, then $M'$ is a maximal Cohen-Macaulay module, that is $\text{depth}_R(M') = n$. So "up to free modules" any $R$-module can be replaced by a maximal Cohen-Macaulay module. If $M$ is a maximal Cohen-Macaulay $R$-module that is minimally generated by $p$ elements, its resolution as $P$-module has the form

$$0 \to P^p \to P^p \to M \to 0$$

where $A$ is some $p \times p$ matrix with $\det(A) = f^q$. The fact that multiplication by $f$ acts as 0 on $M$ produces a matrix $B$ such that $A.B = B.A = f.I$, where $I$ is the identity matrix. In other words we find a matrix factorization $(A, B)$ of $f$ determined uniquely up to base change in the free module $P^p$, by $M$. This matrix factorization not only determines $M$ but also a resolution of $M$ as $R$-module.

$$\ldots \to R^p \to R^p \to R^p \to M \to 0$$

So a minimal resolution of $M$ looks in general as follows

$$\ldots \to G \to F \to G \to F_{n-1} \to \ldots \to F_0 \to M \to 0$$

As a consequence all the homological invariants like $\text{Tor}_k^R(M, N), \text{Ext}^k_R(M, n)$ are 2-periodic, [BVS].

2. Hochster Theta Function

**Definition 2.1.** (Hochster Theta pairing) The theta pairing of two $R$-modules $M$ and $N$ over a hyper-surface ring $R/(f)$ is

$$\Theta(M, N) := l(\text{Tor}_{2k}^R(M, N)) - l(\text{Tor}_{2k+1}^R(M, N)), \quad k \gg 0$$

This definition makes sense as soon as the length appearing are finite. This certainly happens if $R$ has an isolated singular point.

**Example 2.2.** [BVS] Take $f = xy - z^2, M = \mathbb{C}[[xyz]]/(x, y)$. A matrix factorization $(A, B)$ associated to $M$ is given by

$$A = \begin{pmatrix} y & -z \\ -z & x \end{pmatrix}, \quad B = \begin{pmatrix} x & z \\ z & y \end{pmatrix}$$

And $\text{Tor}_k^R(M, M)$ is the homology of the complex

$$\ldots \to \mathbb{C}[[y]]^2 \to \mathbb{C}[[y]]^2 \to \mathbb{C}[[y]] \to \ldots$$

where

$$\alpha = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}$$

So we find that $\Theta(M, M) = 0$. 
Hochster theta pairing is additive on short exact sequences in each argument, and thus determines a \(\mathbb{Z}\)-valued pairing on \(G(R)\), the Grothendieck group of finitely generated \(R\)-modules. One loses no information by tensoring with \(\mathbb{Q}\) and so often theta is interpreted as a symmetric bilinear form on the rational vector space \(G(R)\|\mathbb{Q}\).

It is basic that Theta would vanish if either \(M\) or \(N\) be Artinian or have finite projective dimension \([\text{MPSW}], \ [\text{BVS}]\).

**Remark 2.3.** : The quantity 
\[
h(M, N) := l(\text{Ext}^2_R(M, N)) - l(\text{Ext}^2_R + 1(M, N))
\]
called Herbrand difference would have similar properties. Under mild hypothesis these two invariants are essentially the same. If \(R\) is of finite type over a field, then upon tensoring with \(\mathbb{Q}\) each pairing is induced from the graded rational Chow groups, \(\text{CH}(R)\|\mathbb{Q}\). On the summand \(\text{CH}(R)\|\mathbb{Q}\) the Theta pairing and the Herbrand difference agree when \(j\) is even, and they differ by a sign when \(j\) is odd, \([\text{BVS}]\).

**Theorem 2.4.** \([\text{BVS}]\) When \(M = \mathcal{O}_Y = R/I, N = \mathcal{O}_Z = R/J\), where \(Y, Z \subseteq X_0\) are the sub-varieties defined by the ideals \(I, J\) respectively, then
\[
\Theta(\mathcal{O}_Y, \mathcal{O}_Z) = i(0; Y, Z)
\]
in case that \(Y \cap Z = 0\). Here \(i(0, \ )\) is the ordinary intersection multiplicity in \(\mathbb{C}^{n+1}\).

By additivity over short exact sequences and the fact that any module admits a finite filtration with sub-quotients of the form \(R/I\), knowing \(\Theta(\mathcal{O}_Y, \mathcal{O}_Z)\) determines \(\Theta(M, N)\) for all modules \(M, N\).

**Theorem 2.5.** \([\text{BVS}]\) Assume \(f \in \mathbb{C}[[x_1, \ldots, x_{2m+2}]]\) is a homogeneous polynomial of degree \(d\), and \(X_0 = f^{-1}(0) \subseteq \mathbb{C}^{2m+2}\) and \(T = V(f) \subseteq \mathbb{P}^{2m+1}\) the associated projective cone of degree \(d\). Let \(Y\) and \(Z\) be also co-dimension \(m\) cycles in \(T\). If \(Y, Z\) intersect transversely, then
\[
\Theta(\mathcal{O}_Y, \mathcal{O}_Z) = -\frac{1}{d}[Y].[Z]
\]
Where \([Y]\) := \(d[Y] - \text{deg}(Y).h^m\) is the primitive class of \([Y]\), with \(h \in H^1(T)\) the hyperplane class.

The primitive class of a cycle \(Y\) is the projection of its fundamental class \([Y]\) into \(H^m(T)\) into the orthogonal complement to \(h^m\) with respect to the intersection pairing into \(H^{2m}(T) = \mathbb{C}\). As \(h^m.h^m = d = \text{deg}(T)\) and \([Y].h^m = \text{deg}(Y)\) the description of the primitive class follows. Substituting the claim can be reformulated
\[
\Theta(\mathcal{O}_Y, \mathcal{O}_Z) = -\frac{1}{d}[Y].[Z] = -d[Y].[Z] + \text{deg}(Y)\text{deg}(Z)
\]
Where \([Y].[Z]\) denotes the intersection form on the cohomology of the projective space, \([\text{BVS}]\).

**Remark 2.6.** \([\text{MPSW}]\) For a smooth variety \(Y\), the Grothendieck Riemann-Roch isomorphism \(\text{ch} = \tau : G(Y) = K(Y) \rightarrow CH^*(Y)\) maps \(\alpha = [\mathcal{O}_X] - [\mathcal{O}_X(1)] = -[\mathcal{O}_H(1)]\)
to the class of a hyper-plane $\beta \in CH^1(X)$. This means that these maps interchange the class of hyperplane sections. Also it is an easy exercise in $K_0$ theory that $K(X)/\alpha = G(R)$. The chern character map ch induces an isomorphism $K(X)/\alpha \to CH(X)/\beta$.

When $f$ in consideration is a homogeneous polynomial of degree $d$, such that $X := \text{Proj}(R)$ is a smooth $k$-variety, the Theta pairing is induced, via chern character map, from the pairing on the primitive part of de Rham cohomology $H^{(n-1)/2}(X, \mathbb{C}) / \gamma^{(n-1)/2} \times H^{(n-1)/2}(X, \mathbb{C}) / \gamma^{(n-1)/2} \to \mathbb{C}$

where $\gamma$ is the class of a hyperplane section and theta would vanish for rings of this type having even dimensions. When $n = 1$ by $\gamma^0$ we mean $1 \in H^0(X, \mathbb{C})$, [MPSW].

**Theorem 2.7. [MPSW]** For $R$ and $X$ as above, if $n$ is odd there is a commutative diagram

$$
\begin{array}{ccc}
G(R)^{\otimes 2} & \xleftarrow{\Theta} & (K(X)_Q)^{\otimes 2} \\
\downarrow & & \downarrow (\text{ch}^{n-1/2})^{\otimes 2} \\
\mathbb{C} & \xleftarrow{\Theta} & (H^{(n-1)/2}(X, \mathbb{C}) / \gamma^{(n-1)/2})^{\otimes 2}
\end{array}
$$

If However $f$ is not quasi-homogeneous, there no longer will be a projective variety to do intersection theory on.

**Theorem 2.8. [BVS]** Let $f \in P = \mathbb{C}[[x_0, ..., x_n]]$ define an isolated singularity and let M, N be $R = P/(f)$-modules.

1. If $n$ is odd then $\Theta(M, N) = 0$
2. If $n = 2m$ is even, then $\Theta(O_Y, O_Z) = \text{lk}(\text{ch}(M), \text{ch}(N))$

Here $\text{ch} : K^0(L) \to H^{ev}(L)$ is the chern character. Only the m-component $\text{ch}^m \in H^{2m}(L, \mathbb{Q})$ contributes to linking form so, $\theta(O_Y, O_Z) = \text{lk}(\text{ch}^m(M), \text{ch}^m(N))$.

**Remark 2.9.** The linking form is a restriction of the Seifert form of the singularity, $L : H_n(X_t) \times H_n(X_t) \to \mathbb{Z}$, $(\alpha, \beta) \to l(\alpha, h_{1/2}(\beta))$

where $h_{1/2}$ is half monodromy rotation.

**Theorem 2.10. [MPSW]** For $R$ and $X$ as above and $n$ odd the restriction of the pairing $(-1)^{(n+1)/2}\Theta$ to

$$\text{im}(\text{ch}^{n-1/2}) : K(X)_Q / \alpha \to \frac{H^{(n-1)/2}(X, \mathbb{C})}{C.\gamma^{n-1/2}}$$
is positive definite. i.e. \((-1)^{(n+1)/2} \Theta(v, v) \geq 0\) with equality holding if and only if \(v = 0\). In this way \(\theta\) is semi-definite on \(G(R)\).

Proof. [MPSW] Define

\[ W = H^{n-1}(X(\mathbb{C}), \mathbb{Q}) \cap H^{n-1}_{\mathcal{Q}}(X(\mathbb{C})) \]

It is classical that the image of \(ch^{(n-1)/2}\) is contained in \(W\). Define \(e : W/\mathbb{Q}^\gamma(n-1)/2 \hookrightarrow H^{n-1}(X, \mathbb{Q})\) by

\[ e(a) = a - \int_X a \cup \gamma^{(n-1)/2} \gamma^{(n-1)/2} \in W \]

We know that \(\theta(a, b) = -d.I^{\text{coh}}(e(a), e(b))\) Now the theorem follows from the polarization properties of cup product on cohomology of projective varieties. \(\square\)

3. Hodge theory and Residue form

Suppose \(f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)\) is a holomorphic germ with an isolated critical point. The local residue \(\text{Res}_{f, 0}(\omega, \eta)\) where, \(\omega, \eta\) are differential forms; defines a symmetric bilinear pairing (Grothendieck Pairing= residue form) which is non-degenerate (Proved by Grothendieck). In fact after division by \(df\) each of the forms \(\omega\) and \(\eta\) define a middle dimensional cohomology class of every local level hypersurface of the function \(f\). In this way, the forms \(\omega\) and \(\eta\) define two sections of the vanishing cohomology bundle. The asymptotic of the polarization form on vanishing cohomology gives a meromorphic function on a neighbourhood of the critical value \(0 \in \mathbb{C}\). The residue of this function at \(0 \in \mathbb{C}\) is equal to \(\text{Res}_{0, f}(\omega, \eta)\), [V], [G3], [G2], [G1], [CIR].

Assume \(f : \mathbb{C}^{n+1} \to \mathbb{C}\) is a germ of isolated singularity. As previously mentioned suppose,

\[ H^n(X_\infty, \mathbb{C}) = \bigoplus_{p, q, \lambda} (I^{p, q})_\lambda \]

be the Deligne-Hodge \(C^{\infty}\)-splitting, and generalized eigen-spaces. Consider the isomorphism obtained by composing the two maps,

\[ \Phi : H^n(X_\infty, \mathbb{C}) \xrightarrow{\hat{\Phi}} \bigoplus_{-1 < \beta < n} Gr^\beta V H'' \xrightarrow{\bigoplus_{-1 < \beta < n} Gr^\beta V H'' / Gr^\beta V \partial_1^{-1} H''} \Omega_f \]

\[ \hat{\Phi}|_{I^{p, q}} := \partial_1^{p-n} \circ \psi_\alpha|(I^{p, q})_\lambda \]

**Theorem 3.1.** [R] Let \(f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)\), be a holomorphic germ with isolated singularity at 0. Then, the isomorphism \(\Phi\) makes the following diagram commutative up to a complex constant;
\[ \overline{\text{Res}}_{f,0} : \Omega(f) \times \Omega(f) \longrightarrow \mathbb{C} \]
\[ \downarrow (\Phi^{-1}, \Phi^{-1}) \]
\[ S : H^n(X_\infty) \times H^n(X_\infty) \longrightarrow \mathbb{C} \]

where,
\[ \overline{\text{Res}}_{f,0} = \text{res}_{f,0}(\bullet, \hat{C}\bullet) \]

and \( \hat{C} \) is defined relative to the Deligne-Hodge decomposition of \( \Omega_f \), via the isomorphism \( \Phi \).

(5) \( \Omega_f = \bigoplus_{p,q} J^{p,q} \quad \hat{C}|_{J^{p,q}} = (-1)^p \)

In other words;

(6) \( S(\frac{\omega}{df}, \frac{\eta}{df}) = \text{Const} \times \text{res}_{f,0}(\omega, \hat{C}\eta), \)

**Theorem 3.2.** \([R]\)

Assume \( f : \mathbb{C}^{n+1} \to \mathbb{C} \) is a holomorphic isolated singularity germ. The modified Grothendieck residue provides a polarization for asymptotic fiber \( \Omega_f \), via the aforementioned isomorphism \( \Phi \). Moreover, there exists a set of forms \( \{\text{Res}_k\} \) giving a graded polarization for \( \Omega_f \).

**Remark 3.3.** Let \( \mathcal{G} \) be the Gauss-Manin system associated to a polarized variation of Hodge structure \((\mathcal{L}_\mathbb{Q}, \nabla, F, S)\) of weight \( n \), with \( S : \mathcal{L}_\mathbb{Q} \otimes \mathcal{L}_\mathbb{Q} \to \mathbb{Q}(-n) \) the polarization. Then we have the isomorphism

(7) \( \bigoplus_{k \in \mathbb{Z}} \text{Gr}_{F}^k \mathcal{G} \cong \bigoplus_{k \in \mathbb{Z}} \text{Hom}_{\mathcal{O}_X}(\text{Gr}_{F}^{n-k} \mathcal{G}, \mathcal{O}_X) \)

given by (up to a sign factor) \( \lambda \to S(\lambda, -) \), for \( \lambda \in \text{Gr}_{F}^k \mathcal{G} \).

4. **Hochschild homology and Matrix factorization**

A matrix factorization of \( f \) in \( P \) is a a pair of matrices \( A \) and \( B \) such that \( AB = BA = f. \text{id} \). This definition orginally goes back to D. Eisenbud, [EP]. It is equivalent to a pair of finitely generated \( P \)-modules

\[ d_0 : X^0 \xhookrightarrow{} X^1 : d_1, \quad d_0d_1 = d_1d_0 = f. \text{id} \]

The category of matrix factorizations of \( f \) over \( R \), namely \( MF(R, f) \); is defined to be the differential \( \mathbb{Z}/2 \)-graded category, whose objects are pairs \((X, d)\), where
The Hochschild homology of $MF(R, f)$ is equivalent to the Hochschild homology of the commutative ring $R = P/f$. The Hochschild chain complex of $MF(R, f)$ is quasi-isomorphic to the Koszul complex of the regular sequence $\partial_0 f, ..., \partial_n f$. In particular the Hochschild homology (and also the Hochschild cohomology) of 2-periodic dg-category $MF(R, f)$ is isomorphic to the module of relative differentials or the Jacobi ring of $f$, [D].

**Theorem 4.2.** (T. Dyckerhoff) [D], [PV] The canonical bilinear form on the Hochschild homology of category of matrix factorizations $\mathcal{C} = MF(P, f)$ of $f$, after the identification 

$$HH_* MF(P, f) \cong Af \otimes dx[n]$$

coincides with 

$$\langle f \otimes dx, g \otimes dx \rangle = (-1)^{n(n-1)/2} \text{res}(f, g)$$

Remark 4.1. There are several points to be explained here. First is that there exists an exotic versions of this definition of matrix factorization that allows the free modules to be of infinite ranks, denoted $MF^\infty(R, f)$. Another is this definition may be shifted to be stated for sheaves and schemes rather than rings. Then we would begin with a scheme $X$, noetherian, separated of finite dimension, and $f$ would be a function in the structure sheaf $\mathcal{O}$ locally a non-zero divisor, and $X^0, X^1$ would be coherent sheaves (or analogously in other set up be quasi-coherent sheaves). Also there exists the derived version of the definitions that is taking the object up to homotopy equivalences, to study the categories of matrix factorization. We will not enter to details of this discussion, and the interested reader may look at the standard references as [D], [PV], [EP]. Finally is to allow derived categories modulo short exact sequences of matrix factorizations.
Remark 4.3. [BVS] Given a matrix factorization $(A, B)$ for a maximal Cohen-Macaulay $M$, one can find de Rham representatives for the chern classes. Consider $\mathbb{C}[[x_0, ..., x_n]]$ as a $\mathbb{C}[[t]]$-module with $t$ acting as multiplication by $f$. Denote by $\Omega^p$ the module of germs of $p$-forms on $\mathbb{C}^{n+1}$, and let $\Omega^p_f = \Omega^p/(df \wedge \Omega^{p-1})$. One puts $\omega(M) = dA \wedge dB$. The components of the chern character
\[ ch_M := tr(exp(\omega(M))) = \sum_i \frac{1}{i!} \omega^i(M) \]
are well-defined classes
\[ \omega^i(M) = tr((dA \wedge dB)^i) \in \Omega^{2i}_f = \Omega^p/(df \wedge \Omega^{2i-1}) \]
There are however odd degree classes
\[ \eta^i(M) := tr(AdB(dA \wedge dB)^i) \in \Omega^{2i+1}_f/\Omega^{2i}_f \]
The group $\Omega^{2i+1}_f/\Omega^{2i}_f$ can be identified with the cyclic homology $HC_i(P/\mathbb{C}\{t\})$. They fit into the following short exact sequence such that $d\eta^{i-1} = \omega^i(M)$.
\[ 0 \to \Omega^{2i-1}_f/\Omega^{2i-2}_f \to \Omega^{2i}_f/(df \wedge \Omega^{2i-1}) \to \Omega^{2i}_f/\Omega^{2i-1}_f \to 0 \]
If the number of variables $n+1$ is even, then a top degree form sits in the Brieskorn module
\[ \mathcal{H}^{(0)}_f = \Omega^n/(df \wedge d\Omega^{n-1}) \]
a free $\mathbb{C}[[t]]$-module of rank $\mu$. The higher residue pairing
\[ K : \mathcal{H}^{(0)}_f \times \mathcal{H}^{(0)}_f \to \mathbb{C}\{\partial_t^{-1}\} \]
of K. Saito can be seen as the de Rham realization of the Seifert form of the singularity.

Polishchuk and Vaintrob in [PV], establish a Riemann-Roch formula of the form
\[ h(M, N) = \langle ch(M), ch(N) \rangle \]
for maximal Cohen-Macaulay modules $M, N$, where the product in the right hand side is one in Hochschild homology, and $h$ is the herbrand difference. The chern character is a ring homomorphism
\[ ch : K_0(X) \to HH_0(X) \cong \Omega_f \]
Now using the fact that
\[ h(M, N) = \pm \Theta(M, N) \]
Then by the theorem 4.1, we obtain a formula of the form
\[ (12) \quad \Theta(M, N) = \pm Res_{f,0}(ch(M), ch(N)) \]
for maximal Cohen-Macaulay modules $M, N$; where $ch(M)$ is a certain element of Jacobi algebra
Proposition 4.4. [PV], [CW] The Hochster $\Theta$-pairing of two maximal Cohen-Macaulay modules $M, N$ is given up to a sign by the local residue of their chern classes as elements in $\Omega_f$. That is

$$\Theta(M, N) = (-1)^{n(n-1)} res_{f,0}(ch(M), ch(N))$$

only for $M, N$ maximal Cohen-Macaulay.

5. Positivity of Theta pairing—Main Result

The following theorem was conjectured by [MPSW].

Theorem 5.1. Let $S$ be an isolated hypersurface singularity of dimension $n$. If $n$ is odd, then $(-1)^{(n+1)/2}\Theta$ is positive semi-definite on $G(R)_Q$.

Proof. When $n$ is odd we have

$$\Theta(M, N) = lk(ch(M), ch(N)) = Res_{f,0}(ch(M), ch(N)).$$

By additivity of $\Theta$ on each variable, we may replace $M, N$ by maximal Cohen-Macaulay modules. According to theorem 3.2 and compatibility of chern characters in Hochschild and deRham cohomology of commutative rings; it suffices to prove that the polarization form is definite on chern classes. The chern class is a smooth form which belongs to the lowest $W$-weight, i.e. with no logarithmic or meromorphic component, thus the same is true for chern character. They belong to the image $PD : H_n \to H^n$, where $PD$ is the Poincare dual map. Note that in the isolated singularity case we only have the middle cohomology, $gr^W H^n_{\neq 1} = \oplus_{p+q=n} H^{p,q}$, and similar for $H_1$, because the action of monodromy or its logarithm never changes the $(p, p)$-types. The only non-zero component of any Hodge class is of type $(n/2, n/2)$ which is also primitive. The polarization form is evidently definite in this part.

Remark 5.2. The conjecture has already been proved in homogeneous case by (S. Spiroff, M. Walker, G. Piepmeyer, W. Moore). Our work proves it generally over $\mathbb{C}$. They have also established other properties of $\Theta$ for homogeneous polynomials.

Remark 5.3. The above conjecture has already been proved in homogeneous case by the persons it is assigned to. Our work proves it generally over $\mathbb{C}$. They have also established other properties of $\Theta$ for homogeneous polynomials, [MPSW].
6. APPLICATION TO SERRE MULTIPlicITY CONJECTURE

The intersection multiplicity of finitely generated modules over a regular ring has been defined by J. P. Serre, as

\[
\chi_R(M, N) := \sum_{i=0}^{\dim R} (-1)^i l(Tor_R^i(M, N))
\]

where \( R \) is a regular ring, and \( M, N \) are finitely generated \( R \)-modules. He conjectured that this invariant is 0 in non-proper intersections, and positive when the intersection is proper, i.e \( \dim M + \dim N = \dim R \). This conjecture is still open for positivity.

Theorems 5.1 and 2.4 together recover a very special case of Serre multiplicity conjecture, for intersections at the isolated singularity (proved in general by O. Gabber).

Theorem 6.1. The Serre intersection multiplicity \( \chi(M, N) \) is always non-negative for proper intersections at 0, where \( M \) and \( N \) are regular \( \mathbb{R} = \mathbb{P}/(f) \)-modules essentially of finite type over \( \mathbb{C} \).

7. MUKAI PAIRING AND POSITIVITY IN THE SINGULAR CATEGORY \( MF(P, f) \)

The philosophy of Mukai-pairing is to modify \( \text{ch} \), by cup product with a class \( \sqrt{\text{td} X} \) such that we obtain a homomorphism in Riemann-Roch theorem. We have

\[
\text{td}^{1/2}_X = \bigoplus_i H^i(X, \Omega^i_X).
\]

Let \( X, Y \) be complex manifolds, and let \( E \in D^b(X \times Y) \). Let \( \pi_X, \pi_Y \) be the projections. Define the integral transform with kernel \( E \) by;

\[
\Phi^E_{X \to Y} : D^b(X) \to D^b(Y), \quad \Phi^E_{X \to Y}(\cdot) = \pi_{Y,*}(\pi_X^*(\cdot) \otimes E)
\]

Similarly for \( \mu \in H^*(X \times Y, \mathbb{Q}) \)

\[
\Phi^\mu_{X \to Y} : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q}), \quad \Phi^\mu_{X \to Y}(\cdot) = \pi_{Y,*}(\pi_X^*(\cdot) \otimes \mu)
\]

called the integral transform in cohomology associated to \( \mu \). The association between objects of \( D^b(X \times Y) \) or \( H^*(X \times Y) \) is functorial. In order to relate the above two functors we use chern character and Riemann-Roch theorem. The Riemann-Roch theorem states that, if \( \pi : X \to Y \) is a local complete intersection morphism;

\[
\pi_*(\text{ch}(\cdot) \text{td}(X)) = \text{ch}(\pi_*(\cdot)) \text{td}(Y)
\]

This suggest to define the Mukai vector of \( E \) as follows,

\[
v : D^b(X) \to H^*(X, \mathbb{Q}), \quad v(\cdot) = \text{ch}(\cdot) \sqrt{\text{td}(X)}
\]
Then the commutativity of the following diagram is straight-forward;

\[
\begin{array}{ccc}
D^b(X) & \xrightarrow{\Phi_{X \to Y}} & D^b(Y) \\
\downarrow v & & \downarrow v \\
H^*(X, \mathbb{Q}) & \xrightarrow{\Phi_{v(E)}} & H^*(Y, \mathbb{C})
\end{array}
\]

We will denote \( \Phi_* = \Phi_{v(E)} \), where \( \Phi = \Phi_{X \to Y} \), and it satisfies all associativity and functorial properties naturally. In case \( \Phi = \Phi_{X \to Y} \) be an equivalence of categories, \( \Phi_* = \Phi_{v(E)} \) would be an isomorphism, [CA1], [CA2], [CA3].

**Remark 7.1.** The map \( \Phi_* \) does respects the columns of Hodge diamond;

\[
\Phi_* = \phi_{X \to Y} : \bigoplus_{p-q=i} H^{p,q}(X) \to \bigoplus_{p-q=i} H^{p,q}(X)
\]

This is for, the class \( v(E) \) is a Hodge class.

**Remark 7.2.** Let's define \( \tau : H^*(X, \mathbb{C}) \to H^*(X, \mathbb{C}) \) by

\[
\tau(v_0, v_1, ..., v_{2n}) = (v_0, iv_1, -v_2, ..., i^{2n}v_{2n})
\]

and set,

\[
\cdot : H^*(X, \mathbb{C}) \to H^*(X, \mathbb{C}), \quad v^\cdot = \tau(v) \cdot \frac{1}{\sqrt{\text{ch}(\omega_X)}}
\]

For

\[
\text{td}(T^\vee_X) = \text{td}(T_X). \exp(-c_1(T_X)) = \text{td}(T_X). \text{ch}(\omega_X)
\]

The Mukai Pairing can be generalized to Hochschild homology as

\[
\langle ., . \rangle_M : HH_* (X) \otimes HH_*(X) \to \mathbb{C}
\]

called generalized Mukai pairing.

This generalization can be easily written using the isomorphism

\[
D : R\text{Hom}(\Delta_! O_X, \Delta_* O_X) \cong R\text{Hom}(\Delta_* O_X, \Delta_! \omega_X^{-1})
\]

where \( \Delta_! O_X \cong \Delta_* \omega_X^{-1} \) and \( \omega_X \) is the dualizing sheaf. Then, the Mukai pairing is

\[
v \otimes w \to \text{tr}_{X \times X}(D(v) \circ w)
\]

where tr is Serre duality trace. If

\[
\cdot : HH_* (X) \to HH_*(X)
\]

is the involution induced through H-K-R isomorphism by the similar one to be \((-1)^p\) on \( H^q(X, \Omega_p) \), as defined before. Then we would have
Theorem 7.3. \[\text{RA1}\] Suppose $X$ is smooth, then

\[
\langle b^\vee, a \rangle_M = \langle a, b \rangle, \quad a, b \in HH_\bullet(X)
\]

Theorem 7.4. \[\text{RA1}\] If $X$ is smooth, the generalized Mukai pairing on the Hochschild homology of $X$ satisfies

\[
\langle a, b \rangle_M = \int_X I(a)^\vee I(b) \cdot \text{td}_X, \quad a, b \in HH_\bullet(X)
\]

where $I$ is the H-K-R isomorphism.

The major point in the proof of theorem 7.4 is the isomorphic property of the Hochschild-Kostant-Rosenberg homomorphism in order to define $\vee$. In the singular varieties H-K-R is not an isomorphism any longer. Thus, theorem 7.4 fails in general for singular varieties. However according to theorem 4.1 we may identify the Hochschild homology with $\Omega_f$. Then, the chern characters take value in $\Omega_f$. Using the discussion used on the mixed Hodge structure of $\Omega_f$ via the map $\Phi$ in section 4, the chern characters would belong to $H^n(X_\infty, \mathbb{C}) \neq 1$ and even to $H^{n/2,n/2}$. In this situation we may also replace $f$ by its homogeneous compactification. On the other hand an application of theorems 7.3 with 4.1 to the canonical fiber again give the identity

\[
\Theta(M, N) = \pm \int_{X_\infty} I(ch(M)) I(ch(N)) \cdot \text{td}_{X_\infty} = v(ch(M)) \cup v(ch(N)) \geq 0
\]

for maximal Cohen-Macaulay modules, subject to the condition that H-K-R is an isomorphism (In order to apply the operation $\vee : HH_\bullet(X_0) \to HH_\bullet(X_0)$), and the cup product is the Poincare cup product, which is definite by Hodge theory of projective varieties.

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