Minimal Off-Shell Version of $N = 1$ Chiral Supergravity

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Abstract

We construct the minimal off-shell formulation of $N = 1$ chiral supergravity (SUGRA) introducing a complex antisymmetric tensor field $B_{\mu\nu}$ and a complex axial-vector field $A_\mu$ as auxiliary fields. The resulting algebra of the right- and left-handed supersymmetry (SUSY) transformations closes off shell and generates chiral gauge transformations and vector gauge transformations in addition to the transformations which appear in the case without auxiliary fields.
\( N = 1 \) chiral supergravity (SUGRA), in which only a right-handed spin-3/2 field is coupled to gravitational field, was first formulated by Jacobson [1] as the extension of Ashtekar’s canonical formulation of general relativity [2, 3]. There appear two kinds of supersymmetry (SUSY) transformations, namely, right- and left-handed SUSY transformations. After that, the first-order formulation of \( N = 1 \) chiral SUGRA was made using the method of the two-form gravity [4, 5]. However, in the two-form SUGRA, the SUSY transformation parameters are constrained and the problem of the off-shell closure of SUSY algebra is still open.

In a previous paper [6], we constructed the first-order formulation of \( N = 1 \) chiral SUGRA following the usual \( N = 1 \) SUGRA [7, 8]. In this formulation, the SUSY transformation parameters are not constrained at all in contrast with the method of the two-form gravity, although we introduce a complex tetrad and take a chiral Lagrangian as analytic in field variables. We also studied the SUSY algebra in the second-order formulation and showed that it closes only on shell as extra terms proportional to spin-3/2 field equations appear in the algebra. These results suggest that we need additional auxiliary fields in \( N = 1 \) chiral SUGRA as in the usual one.

There exist two minimal off-shell versions in the usual \( N = 1 \) SUGRA, namely, one with the set of auxiliary fields \((S, P, A_\mu)\) [9] and the other with the set of auxiliary fields \((B_{\mu\nu}, A_\mu)\) [10]. We show in this letter that it is possible to construct a minimal off-shell version of \( N = 1 \) chiral SUGRA provided one uses a complex antisymmetric tensor field \( B_{\mu\nu} \) and a complex axial-vector field \( A_\mu \) as auxiliary fields. We also note that the set of auxiliary fields \((S, P, A_\mu)\) does not suit for \( N = 1 \) chiral SUGRA.

Let us begin with the linearized off-shell theory of \( N = 1 \) chiral SUGRA. The field variables are complex gravitational field \( h_{\mu\nu} \), two independent (Majorana) Rarita-Schwinger fields \( (\psi_{R\mu}, \tilde{\psi}_{L\mu}) \), \(^1\) and the complex auxiliary fields \((B_{\mu\nu}, A_\mu)\). The two kinds of linearized SUSY transformations can be defined as follows. The right-handed rigid SUSY transformations are generated by a constant Majorana spinor parameter \( \alpha_R \) and take the form

\[
\delta_R h_{\mu\nu} = \frac{i}{2} (\alpha_L \gamma_\mu \tilde{\psi}_{L\nu} + \overline{\alpha_L} \gamma_\nu \tilde{\psi}_{L\mu}),
\]

\[
\delta_R \psi_{R\mu} = -2i S^\rho\sigma \partial_\rho h_{\sigma\mu} \alpha_R + i (A_\mu + V_\mu) \alpha_R + S_{\mu\nu} V^\nu \alpha_R,
\]

\(^1\) We assume \( \psi_\mu \) and \( \tilde{\psi}_\mu \) to be two independent (Majorana) Rarita-Schwinger fields, and define the right-handed spinor fields \( \psi_{R\mu} := (1/2)(1 + \gamma_5)\psi_\mu \) and \( \tilde{\psi}_{R\mu} := (1/2)(1 + \gamma_5)\tilde{\psi}_\mu \). The \( \psi_{R\mu} \) and \( \tilde{\psi}_{R\mu} \) are related to the left-handed spinor fields \( \psi_{L\mu} \) and \( \tilde{\psi}_{L\mu} \) respectively, because \( \psi_\mu \) and \( \tilde{\psi}_\mu \) are Majorana spinors. We shall follow the notation and convention of Ref. [3].
\[ \begin{align*}
\delta_R \tilde{\psi}_L^\mu &= 0, \\
\delta_R A^\mu &= 2i \bar{\alpha}_R \gamma^\mu S^\rho \partial^\rho \tilde{\psi}_L^\sigma, \\
\delta_R B_{\mu\nu} &= 2i (\bar{\alpha}_L \gamma^\mu \tilde{\psi}_L^\nu - \bar{\alpha}_L \gamma^\nu \tilde{\psi}_L^\mu) 
\end{align*} \]

(1)

with \( V^\mu \) being defined by

\[ V^\mu := \frac{1}{4} \epsilon^{\mu\rho\sigma} \partial_\nu B_{\rho\sigma}. \]

(2)

On the other hand, the left-handed rigid SUSY transformations, which are generated by a constant Majorana spinor parameter \( \tilde{\alpha}_L \), are given by

\[ \begin{align*}
\delta_L h_{\mu\nu} &= \frac{i}{2} (\bar{\alpha}_R \gamma^\mu \psi_R^\nu + \bar{\alpha}_R \gamma^\nu \psi_R^\mu), \\
\delta_L \psi_R^\mu &= 0, \\
\delta_L \tilde{\psi}_L^\mu &= -2i S^\rho \partial_\rho h_{\sigma\mu} \bar{\alpha}_L - i (A^\mu + V^\mu) \bar{\alpha}_L - S_{\mu\nu} V^\nu \bar{\alpha}_L, \\
\delta_L A^\mu &= -2i \bar{\alpha}_R \gamma^\mu S^\rho \partial^\rho \psi_R^\sigma, \\
\delta_L B_{\mu\nu} &= 2i (\bar{\alpha}_R \gamma^\mu \tilde{\psi}_R^\nu - \bar{\alpha}_R \gamma^\nu \tilde{\psi}_R^\mu). 
\end{align*} \]

(3)

The algebra of SUSY transformations (1) and (3) closes off shell, i.e., without using the field equations for \( \tilde{\psi}_R \) and \( \tilde{\psi}_L \).

The linearized theory of \( N = 1 \) chiral SUGRA with the auxiliary fields \((B_{\mu\nu}, A^\mu)\) are also invariant under gauge transformations

\[ \begin{align*}
\delta_g h_{\mu\nu} &= \partial_\mu a_\nu + \partial_\nu a_\mu, \\
\delta_g \psi_R^\mu &= \partial_\mu \epsilon_R, \quad \delta_g \tilde{\psi}_L^\mu = \partial_\mu \bar{\epsilon}_L, \\
\delta_g A^\mu &= \partial_\mu \Lambda, \\
\delta_g B_{\mu\nu} &= \partial_\mu b_\nu - \partial_\nu b_\mu 
\end{align*} \]

(4-7)

with five independent parameters \( a_\mu, \epsilon_R, \bar{\epsilon}_L, \Lambda \) and \( b_\mu \).

The linearized chiral Lagrangian, which is invariant under the right- and left-handed SUSY transformations and the gauge transformations, can be written as

\[ L^{(+)} = -\frac{1}{2} h^{\mu\nu} G_{\mu\nu} - \epsilon^{\mu\nu\rho\sigma} \bar{\psi}_R^\rho \gamma_\sigma \partial_\sigma \psi_R^\nu + \frac{3}{4} V^\mu V^\mu + A^\mu V^\mu, \]

(8)

where \( G_{\mu\nu} \) is the linearized Einstein tensor,

\[ G_{\mu\nu} = -\{ \Box \tilde{h}_{(\mu\nu)} - \partial^j (\partial_\mu \tilde{h}_{(\nu)j} + \partial_\nu \tilde{h}_{(\mu)j}) + \eta_{\mu\nu} \partial^\rho \partial^\sigma \tilde{h}_{(\rho\sigma)} \} \]

(9)
with

\[ \bar{h}_{(\mu\nu)} := h_{(\mu\nu)} - \frac{1}{2} \eta_{\mu\nu} h, \quad h := \eta^{\mu\nu} h_{(\mu\nu)}, \]

and the d’Alembertian operator being defined by \( \Box := \partial^\mu \partial_\mu \).

The field equations at the linearized level can be derived from (10) as

\[ \mathcal{G}_{\mu\nu} = 0, \]
\[ R^\mu_R := \epsilon^{\mu\nu\rho\sigma} \gamma_\nu \partial_\rho \psi_{R\sigma} = 0, \]
\[ \tilde{R}_L^\mu := \epsilon^{\mu\nu\rho\sigma} \gamma_\nu \partial_\rho \tilde{\psi}_{L\sigma} = 0, \]
\[ V^\mu = 0, \]
\[ \epsilon^{\mu\nu\rho\sigma} (3 \partial_\rho V_\mu + 2 \partial_\rho A_\mu) = 0. \]

From the last two equations of (11), we can see that the gauge invariant part of \( B_{\mu\nu} \), namely \( V^\mu \) of (2), and that of \( A_\mu \), namely \( f_{\mu\nu}(\equiv \partial_\mu A_\nu - \partial_\nu A_\mu) \) are zero. Therefore the auxiliary fields \( (B_{\mu\nu}, A_\mu) \) do not have on-shell degrees of freedom, while they have six (complex) off-shell degrees of freedom. This number is precisely the mismatch number of the components of bosonic fields \( (h_{\mu\nu}) \) and fermionic fields \( (\psi_{R\mu}, \tilde{\psi}_{L\mu}) \). Note that if the gravitational field \( h_{\mu\nu} \) and the auxiliary fields \( (B_{\mu\nu}, A_\mu) \) are real, and if \( \tilde{\psi}_\mu = \psi_\mu \), then the linearized off-shell theory of \( N = 1 \) chiral SUGRA is reduced to that of the usual \( N = 1 \) SUGRA [10].

One method to construct the full nonlinear theory in the usual \( N = 1 \) SUGRA starts from the linearized theory and makes the rigid SUSY transformations local, adding appropriate terms to the linearized Lagrangian order by order in the gravitational constant \( \kappa \) [11]. We expect that the full nonlinear theory of \( N = 1 \) chiral SUGRA with \( (B_{\mu\nu}, A_\mu) \) can be obtained from its linearized theory in the similar manner. However we can deduce the full nonlinear form of SUSY transformations directly from the linearized transformation laws of (1) and (3), referring to the corresponding result of the usual \( N = 1 \) SUGRA.

In the full nonlinear theory, the field variables are the complex tetrad \( e^i_\mu \), two (Majorana) Rarita-Schwinger fields \( (\psi_{R\mu}, \tilde{\psi}_{L\mu}) \) and the complex auxiliary fields \( (B_{\mu\nu}, A_\mu) \). We choose the right-handed SUSY transformations generated by \( \alpha_R \) to be

\[ \delta_R e^i_\mu = i \kappa \bar{\alpha}_L \gamma_i \tilde{\psi}_{L\mu}, \]
\[ \delta_R \psi_{R\mu} = \frac{2}{\kappa} \mathcal{D}^{(-)}_\mu \alpha_R, \quad \delta_R \tilde{\psi}_{L\mu} = 0, \]

\[ \kappa^2 \text{ is the Einstein constant: } \kappa^2 = 8\pi G/c^4. \]
\[ \delta_R A_\mu = -\frac{i}{2} \bar{\alpha}_L \gamma_\mu \gamma \cdot \tilde{R}_L, \]
\[ \delta_R B_{\mu\nu} = 2i \left( \bar{\alpha}_L \gamma_\mu \tilde{\psi}_{L\nu} - \bar{\alpha}_L \gamma_\nu \tilde{\psi}_{L\mu} \right), \]
and the left-handed SUSY transformations generated by \( \tilde{\alpha}_L \) to be
\[ \delta_L \epsilon_i^\mu = i \kappa \bar{\alpha}_R \gamma^i \psi_{R\mu}, \]
\[ \delta_L \psi_{R\mu} = 0, \quad \delta_L \tilde{\psi}_{L\mu} = \frac{2}{\kappa} D_{\mu}^{(-)} \bar{\alpha}_L, \]
\[ \delta_L A_\mu = -\frac{i}{2} \bar{\alpha}_R \gamma_\mu \gamma \cdot \mathcal{R}_R, \]
\[ \delta_L B_{\mu\nu} = 2i \left( \bar{\alpha}_R \gamma_\mu \psi_{R\nu} - \bar{\alpha}_R \gamma_\nu \psi_{R\mu} \right), \]
where we define
\[ D_{\mu}^{(+)} := D_{\mu}^{(+)} + \frac{\kappa}{2} \{ i(A_\mu + V_\mu) + S_{\mu\nu} V^{\nu} \}, \]
\[ D_{\mu}^{(-)} := D_{\mu}^{(-)} - \frac{\kappa}{2} \{ i(A_\mu + V_\mu) + S_{\mu\nu} V^{\nu} \}, \]
\[ \mathcal{R}_R := \epsilon^{\mu\nu\rho\sigma} \gamma_\nu D_{\rho}^{(+)} \psi_{R\sigma}, \]
\[ \tilde{\mathcal{R}}_L := \epsilon^{\mu\nu\rho\sigma} \gamma_\nu D_{\rho}^{(-)} \tilde{\psi}_{L\sigma}, \]
and the \( V^\mu \) is now given by
\[ V^\mu := \epsilon^{\mu\nu\rho\sigma} \left( \frac{1}{4} \partial_\nu B_{\rho\sigma} - \frac{i}{2} \kappa \bar{\psi}_{R\rho} \gamma_\nu \psi_{R\sigma} \right). \]

\( D_{\mu}^{(+)} \) denotes the covariant derivative with respect to the self-dual connection \( A_{ij\mu}^{(+)} \) which satisfies \((1/2)\epsilon_{ij}^{kl} A_{kl\mu}^{(+)} = i A_{ij\mu}^{(+)}\):
\[ D_{\mu}^{(+)} := \partial_\mu + \frac{i}{2} A_{ij\mu}^{(+)} S_{ij} \]
with \( S_{ij} \) being the Lorentz generator. Here the self-dual connection \( A_{ij\mu}^{(+)} \) represents that in the second-order formulation, namely,
\[ A_{ij\mu}^{(+)} = A_{ij\mu}^{(+)}(e) + K_{ij\mu}^{(+)} , \]
where $A_{ij\mu}^+(e)$ is the self-dual part of the Ricci rotation coefficients $A_{ij\mu}(e)$, while $K_{ij\mu}^+$ is that of $K_{ij\mu}$ given by

$$K_{ij\mu} := \frac{i}{2} (e_i^\rho e_j^\sigma e_k^\mu \bar{\psi}_{R[\rho} \gamma_{ij] \psi_{R\sigma]} + e_i^\rho \bar{\psi}_{R[\rho} \gamma_{ij] \psi_{R\mu]} - e_j^\rho \bar{\psi}_{R[\rho} \gamma_{ij] \psi_{R\mu]}).$$ (18)

On the other hand, $D^{(-)}_{\mu}$ stands for the covariant derivative with respect to the anti-self-dual part of $A_{ij\mu}(e) + K_{ij\mu}$.

As for the gauge transformations of the auxiliary fields $(B_{\mu
u}, A_{\mu})$ defined by (6) and (7) in the linearized off-shell theory, we take in the full nonlinear theory as follows:

$$\delta V_{B_{\mu\nu}} = \partial_{\mu} b_{\nu} - \partial_{\nu} b_{\mu}$$ (19)

for vector gauge transformations and

$$\delta C A_{\mu} = -\frac{2}{\kappa} \partial_{\mu} \Lambda,$$

$$\delta C \psi_{R\mu} = i \Lambda \psi_{R\mu}, \quad \delta C \tilde{\psi}_{L\mu} = -i \Lambda \tilde{\psi}_{L\mu}$$ (20)

for chiral gauge transformations, where $\psi_{R\mu}$ and $\tilde{\psi}_{L\mu}$ need to be transformed as in (20) because the definition of $V^\mu$ is modified as (15).

It can now be shown that the commutator algebra of the right- and left-handed SUSY transformations (12) and (13) closes off shell as

$$[\delta_{R(1)}, \delta_{R(2)}] = 0 = [\delta_{L(1)}, \delta_{L(2)}],$$

$$[\delta_{R(1)}, \delta_{L(2)}] = \delta_G (\xi^i) + \delta_{Lorentz} (\xi^\mu A_{ij\mu} + \frac{\kappa}{2} \epsilon_{ijk} \xi^k V^l) + \delta_R (-\frac{\kappa}{2} \xi \cdot \psi_R)$$

$$+ \delta_L (-\frac{\kappa}{2} \xi \cdot \tilde{\psi}_L) + \delta_C (\frac{\kappa}{2} \xi \cdot A) + \delta_V (-\frac{2}{\kappa} \xi^\mu + B_{\mu\nu} \xi^\nu)$$ (22)

with

$$\xi^i := 2i \alpha_2 R^i \alpha_1 R.$$ (23)

Note that $\xi^i$ is complex because $\alpha_1 R$ and $\alpha_2 R$ are independent of each other. The commutator (22) generates the general coordinate transformations ($\delta_G$), the local Lorentz transformations ($\delta_{Lorentz}$), the right- and left-handed SUSY transformations ($\delta_R$, $\delta_L$), the chiral gauge transformations ($\delta_C$) and the vector gauge transformations.
The quantities in the parentheses in (22) denote the parameters of the respective transformations. The characteristic feature of the SUSY algebra in $N = 1$ chiral SUGRA is that the right- or left-handed SUSY transformations commute with each other, while the commutator of the right- and left-handed SUSY transformations generates all invariant transformations in the theory.

The chiral Lagrangian density invariant under all the transformations generated by the commutator (22) has the following form:

$$
\mathcal{L}^{(+)} = -\frac{i}{2\kappa^2} \epsilon^{\mu\nu\rho\sigma} \epsilon^i_{\mu} e_j^j R_{ij\rho\sigma}^{(+)} - e \epsilon^{\mu\nu\rho\sigma} \bar{\psi}_{R\mu} \gamma_{\rho} D_{\sigma}^{(+)} \psi_{R\nu} + \frac{3}{4} e V_{\mu} V^\mu + e A_{\mu} V_{\mu},
$$

where $e$ denotes $\det(\epsilon_{\mu}^i)$ and the curvature of the self-dual connection $R^{(+)}_{ij\mu\nu}$ is

$$
R^{(+)}_{ij\mu\nu} := 2(\partial_{\mu} A^{(+)}_{ij\nu} + A^{(+)}_{kl\mu} A^{(+)}_{ij\nu}).
$$

The field equation for $A^{(+)}_{ij\mu}$ gives (17) and is used when we prove the invariance of the chiral Lagrangian density under the right- and left-handed SUSY transformations. It is noted that the linearized limit of (24) just becomes the linearized chiral Lagrangian of (8).

Brief remarks are in order about the field equations derived from the chiral Lagrangian density (24). The field equations for the auxiliary fields $(B_{\mu\nu}, A_{\mu})$ are given by

$$
V_{\mu} = 0 \quad \text{for} \quad A_{\mu},
$$

$$
e\epsilon^{\mu\nu\rho\sigma} (3\partial_{\nu} V_{\mu} - f_{\mu\nu}) = 0 \quad \text{for} \quad B_{\mu\nu}
$$

with $f_{\mu\nu} := \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$, which shows that the gauge invariant part of $A_{\mu}$, namely $f_{\mu\nu}$, is zero, and that the gauge invariant part of $B_{\mu\nu}$ can be expressed in the bilinear form of spin-3/2 fields, i.e., $\bar{\psi}_{R\mu} \gamma_{\rho} \psi_{R\sigma}$. Secondly the field equations for all the field variables are reduced to those of the usual $N = 1$ SUGRA with $(B_{\mu\nu}, A_{\mu})$ provided that the following conditions are satisfied:

(a) The tetrad $e_{\mu}^i$ and the auxiliary fields $(B_{\mu\nu}, A_{\mu})$ are real.

(b) $\bar{\psi}_{\mu} = \psi_{\mu}$.

(c) The self-dual connection $A^{(+)}_{ij\mu}$ satisfies its equation of motion.
We have constructed the minimal off-shell formulation of $N = 1$ chiral SUGRA by using the complex auxiliary fields $(B_{\mu\nu}, A_\mu)$. However we cannot use the set of $(S, P, A_\mu)$ as auxiliary fields in $N = 1$ chiral SUGRA. In order to see this, we remind of the extra terms which are proportional to spin-3/2 field equations in the SUSY algebra without auxiliary fields. If the set of $(S, P, A_\mu)$ is used as auxiliary fields, the right-handed SUSY transformations of $S$ and $P$, for example, involve the term of $\overline{\sigma}_L \gamma \cdot R_R (R_R^\mu := e^{\mu\nu\rho\sigma} \gamma_\nu D_\rho^{(+)} \psi_{R\sigma})$. Since $\overline{\sigma}_L$ is paired with $\psi_{R\mu}$, this form cannot produce the appropriate counter terms to cancel the extra terms in the SUSY algebra. Therefore the $(S, P, A_\mu)$ does not work as auxiliary fields.

Finally we comment on future problems in chiral SUGRA. In the usual $N = 1$ SUGRA, there is the conformal theory [12] and its close relationship to the various $N = 1$ Poincaré SUGRA with auxiliary fields has been clarified [13]. Similarly if there exists a conformal version of $N = 1$ chiral SUGRA, then the structure of chiral SUGRA will become clear and the matter-coupling problem will be simplified. The construction of such a conformal theory of $N = 1$ chiral SUGRA is now being investigated.

The second problem is the formulation of extended chiral SUGRA. As for $N = 2$ case, the linearized theory can be constructed by means of the SUSY transformation parameters without constraints. We are now trying to construct the full nonlinear theory.

We would like to thank Professor Y. Tanii and other members of Physics Department at Saitama University for discussions and encouragements.
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