Cosmological perturbations in SFT inspired non-local scalar field models

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Abstract: We study cosmological perturbations in models with a single free non-local scalar field originating from the string field theory description of the rolling tachyon dynamics. We construct equation for the energy density perturbation of the non-local scalar field and explicitly prove that it is identical to a closed system of local cosmological perturbation equations in a particular model with multiple local free scalar fields.

Keywords: Cosmological perturbations, Rolling tachyon, String Field Theory.

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1. Introduction

Recently a new class of cosmological models based on the string field theory (SFT) (for a review see [1]) and the $p$-adic string theory [2] emerged and attracted a lot of attention [3]–[17] thanks to their specific properties. It is known that the SFT and the $p$-adic string theory are UV-complete ones. Thus one can expect that resulting (effective) models should be free of pathologies. Furthermore, models originating from the SFT exhibit one general non-standard property, namely, they have terms with infinitely many derivatives, i.e. non-local terms. The higher derivative terms usually produce the well known Ostrogradski instability [18] (see also [4])\(^1\). However the Ostrogradski result is related to higher than two but finite number of derivatives. In the case of infinitely many derivatives it is possible that instabilities do not appear [13].

SFT inspired cosmological models [3] are intensively considered as models for the dark energy (DE). The way of solving of non-local Friedmann equations with

\(^1\)Additional phantom solutions, obtained by the Ostrogradski method in some models can be interpreted as non-physical ones [19, 20], which one can treat perturbatively, i.e. evaluating them using the lower order equations of motion.
a quadratic potential of the scalar field, reducing them to a system of Friedmann
equations with many non-interacting free massive local scalar fields has been pro-
posed in [6, 8]. The obtained local fields satisfy the second order linear differential
equations. The masses of all local fields are roots of an algebraic or transcendent-
al equation, which appears in the non-local model. In the representation of many
scalar fields some of them are normal and some of them are phantom (ghost) ones.
Moreover some local fields appear with complex masses squared.

It is known that the equation of state parameter $w > -1$ can be described by
quintessence models, $w = -1$ corresponds to the Cosmological Constant (CC) while
$w < -1$ is a characteristic property of models with a single phantom scalar field. The
inequality $w < -1$ means the violation of all energy conditions. In particular, the
null energy condition (NEC) is violated. Therefore, models with a phantom often
plague by the vacuum quantum instability in the ultraviolet region. Phantom fields
look harmful to the theory and a model with a phantom scalar field is not acceptable
from the general point of view. On the other hand a possibility of existence of the
DE with $w < -1$ is not excluded experimentally. Indeed, contemporary cosmological
observational data [21, 22] strongly support that the present Universe exhibits an
accelerated expansion providing thereby an evidence for a dominating DE component
(for a review see [23]). Recent results of WMAP [22] together with the data on Ia
supernovae give the following bounds for the DE state parameter $w_{DE} = -1.02^{+0.14}_{-0.16}$.
Note that the present cosmological observations do not exclude an evolving DE state
parameter $w$.

Due to the presence of phantom excitations non-local models are of interest
for the present cosmology. To construct a stable model with $w < -1$ one should
construct the effective theory with the NEC violation from the fundamental theory,
which is stable and admits quantization. Without quantum gravity one can give a
try to string theory or an effective theory admitting the UV-completion. This is a
hint towards the SFT inspired cosmological models. Among cosmological models
with $w < -1$, which have been constructed to be free of instability problem, we
can mention the Lorentz-violating dark energy model [24], the ghost condensation
model [25]-[27] and the brane-world models [28].

Also cosmological models coming out of SFT or $p$-adic string theory are con-
sidered in application to inflation [12]-[17] to explain in particular appearance of
non-gaussianities. Such models of inflation generically have the remarkable prop-
erty that slow roll inflation can proceed even with an extremely steep potential [3].
Furthermore it is shown in [17] that the non-linearity parameter which characterizes
the non-gaussianity in the cosmic microwave background can be observably large in
contrary to the standard inflation scenarios and observationally distinguishable from
Dirac-Born-Infeld inflation models.

For a more general discussion on the string cosmology and coming out of string
theory theoretical explanations of the cosmological observational data the reader is
referred to [29]. Other models obeying non-locality and their cosmological consequences are considered in [30]. Note also that linear differential equations of infinite order were studied in the mathematical literature long time ago [31]–[33] (see [15] as a review).

The purpose of this paper is to study the cosmological perturbations in linear non-local string field theory inspired models. The particular models are inspired by the fermionic SFT and the most well understood process of tachyon condensation. Namely, starting with a non-supersymmetric configuration where the tachyon of the fermionic string exists the tachyon rolls down towards the non-perturbative minimum of the tachyon potential. This process represents the non-BPS brane decay according to Sen’s conjecture (see [1] for details). From the point of view of the SFT the whole picture is not yet known and only vacuum solutions were constructed (see [34] for the bosonic SFT and [35] for the fermionic SFT). An effective field theory description explaining the rolling tachyon in contrary is known and numeric solutions describing the tachyon dynamics were obtained [36]. This effective field theory description of course captures the non-locality of the SFT. Linearizing the latter lagrangian around the true vacuum one gets a model which is of main concern in the present paper. In this paper we consider a very general form of linear non-local action for the scalar field keeping the main ingredient, the function $F(\Box)$ which in fact produces the non-localities in question, almost unrestricted. The only strong restriction we impose is the analyticity of $F(\Box)$. In this paper we also assume that all roots of $F(\Box)$ are simple.

The paper is organized as follows. In Section 2 we describe the non-local non-linear SFT model, whose linear approximation is studied in this paper. The corresponding non-local linear cosmological model is explicitly constructed. In Section 3 we sketch the construction of background solutions in such models. In Section 4 we briefly outline Bardeen’s gauge-invariant formalism for the cosmological perturbations. We write down the equations for perturbations in models with one and many local scalar fields. In Section 5 we derive gauge invariant perturbation equations for a model with a free non-local scalar field and prove that these equations are identical to equations for perturbations in a local model with many free non-interacting scalar fields\(^2\). In Section 6 perturbation equations for space-homogeneous perturbations in our model are derived and explicitly integrated out. In Section 7 we summarize the obtained results and propose directions for further investigations.

2. Model setup

We work in 1 + 3 dimensions, the coordinates are denoted by Greek indices $\mu, \nu, \ldots$ running from 0 to 3. Spatial indexes are $a, b, \ldots$ and they run from 1 to 3. The

\(^2\)For applications of other multifield cosmological models and related technical aspects see for instance [37].
four-dimensional action motivated by the string field theory is as follows \[ S = \int d^4x \sqrt{-g} \left( \frac{R}{16\pi G_N} + \frac{1}{g_0^2} \left( -\frac{1}{2} \partial_\mu T \partial^{\mu} T + \frac{1}{2\alpha'} T^2 - \frac{1}{\alpha'} V(T) \right) \right). \tag{2.1} \]

Here \( G_N \) is the Newton constant: \( 8\pi G_N = 1/M_P^2 \), where \( M_P \) is the Planck mass, \( \alpha' \) is the string length squared (we do not assume \( \alpha' = 1/M_P^2 \)). We use the signature \((- , +,+,+), \) \( g_{\mu\nu} \) is the metric tensor, \( R \) is the scalar curvature, \( T = \mathcal{G}(\alpha' \Box)T \) with

\[ \Box = D_\mu \partial_\mu = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu, \tag{2.2} \]

and \( D_\mu \) being a covariant derivative, \( T \) is a scalar field primarily associated with the open string tachyon. \( \mathcal{G} \) may have infinitely many derivatives manifestly producing thereby the non-locality. Fields are dimensionless while \( [g_0] = \text{length} \). \( V(T) \) is an open string tachyon potential. Factor \( 1/\alpha' \) in front of the tachyon potential looks unusual and can be easily removed by a rescaling of fields. For our purposes it is more convenient keeping all the fields dimensionless.

Field redefinition \( \tau = T \) yields

\[ S = \int d^4x \sqrt{-g} \left( \frac{R}{16\pi G_N} + \frac{1}{g_0^2} \left( -\frac{1}{2} \partial_\mu \tilde{T} \partial^{\mu} \tilde{T} + \frac{1}{2\alpha'} \tilde{T}^2 - \frac{1}{\alpha'} V(\tau) \right) \right). \tag{2.3} \]

Here \( \tilde{T} = \mathcal{G}(\alpha' \Box)^{-1} \tau \). One can introduce dimensionless coordinates \( \bar{x}_\mu = x_\mu/\sqrt{\alpha'} \). This allows us to rewrite the above action as follows (where we omit bars for simplicity)

\[ S = \int d^4x \sqrt{-g} \alpha' \left( \frac{R}{16\pi G_N} + \frac{1}{g_0^2} \left( -\frac{1}{2} \partial_\mu \tilde{T} \partial^{\mu} \tilde{T} + \frac{1}{2} \tilde{T}^2 - V(\tau) \right) \right). \tag{2.4} \]

In this paper we study models with a quadratic potential \( V(\tau) = m^2 \tau^2 \) and add the CC \( \Lambda \) to the Lagrangian. Such models appear as the linearization of the SFT inspired model in the neighborhood of extrema of the potential. In this case action \[(2.4)\] can be rewritten as

\[ S = \int d^4x \sqrt{-g} \alpha' \left( \frac{R}{16\pi G_N} + \frac{1}{2g_0^2} \tau \mathcal{F}(\Box) \tau + \Lambda \right) \tag{2.5} \]

where

\[ \mathcal{F} = (\Box + 1) \mathcal{G}^{-2} - m^2. \tag{2.6} \]

The function \( \mathcal{F} \) is assumed to be an analytic function of its argument, such that one can represent it by the convergent series expansion with real coefficients:

\[ \mathcal{F} = \sum_{n=0}^{\infty} f_n \Box^n \text{ and } f_n \in \mathbb{R}. \tag{2.7} \]
Equations of motion are
\[ G_{\mu\nu} = \frac{8\pi G}{g_o^2} T_{\mu\nu} = 4\pi G \sum_{n=1}^{\infty} \sum_{l=0}^{n-1} \left( \partial_\mu \square^{l} \partial_\nu \Box^{n-1-l} \tau + \partial_\nu \square^{l} \partial_\mu \Box^{n-1-l} \tau \right) \]
\[ - g_{\mu\nu} \left( g^{\rho\sigma} \partial_\rho \square^{l} \partial_\sigma \Box^{n-1-l} \tau + \Box^{l} \tau \Box^{n-1-l} \tau \right) \right) + 8\pi G N g_{\mu\nu} \Lambda, \tag{2.8} \]
\[ F(\Box)\tau = 0 \quad \tag{2.9} \]

where \( G_{\mu\nu} \) is the Einstein tensor, \( T_{\mu\nu} \) is the energy-momentum (stress) tensor and \( G \equiv G_N / g_o^2 \) is a dimensionless analog of the Newton constant. It is easy to check that the Bianchi identity is satisfied on-shell and in a simple case \( F = f_1 \Box + f_0 \) the usual energy-momentum tensor for the massive scalar field is reproduced. Note that equation (2.9) is an independent equation consistent with system (2.8) due to the Bianchi identity.

3. Background solutions construction

Classical solutions to equations (2.8), (2.9) were studied and analyzed in [6]-[9]. Here we just briefly notice the key points useful for the purposes of the present paper. The main idea of finding the solutions to the equations of motion is to start with equation (2.9) and to solve it, assuming the function \( \tau \) is an eigenfunction of the box operator. If \( \Box \tau = M \tau \), then such a function \( \tau \) is a solution to (2.9) if and only if
\[ F(M) = 0. \tag{3.1} \]

The latter condition is an algebraic or transcendental equation. We name it the characteristic equation. Note that \( M \) is dimensionless. In this way of solving all the information is extracted from the roots of equation (3.1). Values of roots of \( F(M) \) do not depend on the metric. Since equation (2.9) is linear one can take the following function as a solution
\[ \tau = \sum_i \tau_i \quad \text{where } \Box \tau_i = M_i \tau_i \quad \text{and } \quad F(M_i) = 0 \quad \text{for any } i = 1, 2, \ldots, N. \tag{3.2} \]

Hereafter we use \( N \) (which can be infinite as well) denoting the number of roots and omit writing explicitly the summation limits over \( i \). Without loss of generality we assume that for any \( i_1 \) and \( i_2 \neq i_1 \) condition \( M_{i_1} \neq M_{i_2} \) is satisfied. Indeed, if the sum (3.2) includes two summands \( \tau_{i_1} \) and \( \tau_{i_2} \), which correspond to one and the same \( M_i \), then we can consider them as one summand \( \tau_i \equiv \tau_{i_1} + \tau_{i_2} \), which corresponds to \( M_i \). We can consider the solution \( \tau \) as a general solutions if all roots of \( F \) are simple. The analysis is more complicated in the case of multiple roots and we skip this possibility for simplicity.
In an arbitrary metric the energy-momentum tensor in (2.8) evaluated on such a solution is
\[ T_{\mu\nu} = \sum_i \mathcal{F}_M(M_i) \left( \partial_\mu \tau_i \partial_\nu \tau_i - \frac{1}{2} g_{\mu\nu} \left( g^{\rho\sigma} \partial_\rho \tau_i \partial_\sigma \tau_i + M_i \tau_i^2 \right) \right) + g_o^2 g_{\mu\nu} \Lambda \] (3.3)
where we note the absence of cross terms \( \tau_i \tau_j \) for \( i \neq j \). The last formula is exactly the energy-momentum tensor of many free massive scalar fields. If \( \mathcal{F}_M \) has simple real roots, then positive and negative values of \( \mathcal{F}_M(M_i) \) alternate, so we can obtain phantom fields. We stress again that all the above formulae are valid for an arbitrary metric and a general solution.

From now on, however, the only metric we will be interested in is the spatially flat Friedmann–Robertson–Walker (FRW) metric of the form
\[ ds^2 = -dt^2 + a^2(t) \left( dx_1^2 + dx_2^2 + dx_3^2 \right) \] (3.4)
where \( a(t) \) is the scale factor, \( t \) is the cosmic time. Some useful quantities in this metric read
\[ \Gamma_0^{ab} = H g_{ab}, \quad \Gamma_0^a = H \delta_0^a, \quad \Box = -\partial_t^2 - 3H \partial_t + \frac{1}{a^2} \partial^a \partial_a, \]
\[ R_{\mu\nu} = \begin{pmatrix} -3(\dot{H} + H^2) & 0 \\ 0 & g_{ab}(\dot{H} + 3H^2) \end{pmatrix}, \quad R = 6\dot{H} + 12H^2 \]
where \( H = \dot{a}/a \) and the dot hereafter in this paper denotes a derivative with respect to the cosmic time \( t \). Background solutions for \( \tau \) are taken to be space-homogeneous as well. The energy-momentum tensor in (2.8) in this metric can be written in the form of a perfect fluid \( T_\mu^\nu = \text{diag}(-\varrho, p, p, p) \), where
\[ \varrho = \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \left( \partial_t \Box^l \tau \partial_t \Box^{n-1-l} \tau + \Box^l \tau \Box^{n-1-l} \tau \right) + g_o^2 \Lambda, \]
\[ p = \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \left( \partial_t \Box^l \tau \partial_t \Box^{n-1-l} \tau - \Box^l \tau \Box^{n-1-l} \tau \right) - g_o^2 \Lambda. \] (3.5)
Using the above notations we can rewrite equation (2.8) in the canonical form as follows
\[ 3H^2 = 8\pi G \varrho, \quad \dot{H} = -4\pi G(\varrho + p). \] (3.6)
The consequence of (3.6) is the conservation equation:
\[ \dot{\varrho} + 3H(\varrho + p) = 0. \] (3.7)
Note that system (3.6) is a non-local and non-linear system of equation. At the same time using formulae (3.5) for the energy and pressure densities it is possible to
generate local systems out of (3.6), corresponding to particular solutions of the initial non-local system. Sometimes it gives a possibility to find exact analytic solutions to the initial non-local system [8].

Assuming the field $\tau$ is of the form (3.2), formula (3.5) gives:

$$
\rho = \frac{1}{2} \sum_i \mathcal{F}_M'(M_i) \left( \dot{\tau}_i^2 + M_i \tau_i^2 \right) + g_0^2 \Lambda, \quad p = \frac{1}{2} \sum_i \mathcal{F}_M'(M_i) \left( \dot{\tau}_i^2 - M_i \tau_i^2 \right) - g_0^2 \Lambda.
$$

(3.8)

Therefore, we can rewrite system (3.6) as follows:

$$
3H^2 = 4\pi G \sum_i \mathcal{F}_M'(M_i) \left( \dot{\tau}_i^2 + M_i \tau_i^2 \right) + 8\pi G_N \Lambda, \quad \dot{H} = -4\pi G \sum_i \mathcal{F}_M'(M_i) \dot{\tau}_i^2.
$$

(3.9)

It is easy to check that equations for $\tau_i$ in (3.2) and (3.9) coincide with the Einstein equation for the following action:

$$
S_{\text{local}} = \int d^4x \sqrt{-g} \left( \frac{R}{16\pi G_N} + \frac{1}{2g_0^2} \sum_i \mathcal{F}_M'(M_i) \left( -g^{\mu\nu} \partial_{\mu} \tau_i \partial_{\nu} \tau_i - M_i \tau_i^2 \right) + \Lambda \right)
$$

(3.10)

on the solutions with the FRW metric. Here number of local scalar fields is equal to the number of roots of the characteristic equation (3.1). This system is equivalent to the initial non-local one (2.5) because any solution to the equation of motion for the non-local field $\tau$ from action (2.5) can be written as $\tau = \sum_i \tau_i$ where $\tau_i$ are solutions to equations of motion for local fields $\tau_i$ from action (3.10) and moreover Hamiltonians are the same. It may turn out that even if there infinitely many derivatives in the non-local system it can be equivalent in the above sense to the local one with finite number of fields. As an example one can take $\mathcal{F}(M) = (M+1)\text{e}^M$ with only one root $M = -1$. Physically this statement is easy to understand: theories with only a single pole in the propagator describe only one physical degree of freedom and hence the non-local structure does not spoil the system with new spurious ghost states\footnote{If $\mathcal{F}(M)$ has no zeros like for example $\mathcal{F}(M) = \text{e}^M$ then the underlying field theory has no physical excitations at all.}. At the same time, for $\mathcal{F}(M)$ with two or more simple real roots we obtain that the non-local model with action (2.5) contains ghost-like excitations. Note that the use of truncated function

$$
\hat{\mathcal{F}}(\Box) \equiv \sum_{n=0}^{\hat{N}} f_n \Box^n
$$

(3.11)

instead of $\mathcal{F}(\Box)$ as an approximation is not correct, because $\hat{\mathcal{F}}$, which is the $\hat{N}$-th degree polynomial in $\Box$, can contain spurious zeros which are not present in $\mathcal{F}(\Box)$. Hence the corresponding solution for $\tau$ can contain modes $\tau_i$ which are not present in the full theory (see, for example, [4]). Note that the violation of the Ostrogradski
statements for infinite derivatives theories was noted in mathematical papers [38, 31]. The detailed analysis of the initial value problem for such non-local equations and deeper analysis of their mathematical properties can be found in [15].

4. Cosmological perturbations formalism

4.1 Metric perturbations

The metric perturbations can be divided into four scalar, four vector and two tensor perturbations, according to their transformation properties with respect to three-space coordinate transformations on the constant-time hypersurface [39]. Different types do not mix at linear order [40]-[42]. We are focused on the scalar perturbations, because both vector and tensor perturbations exhibit no instabilities [41].

Scalar metric perturbations are given by four arbitrary scalar functions $\alpha, \beta, \varphi, \gamma$ in the following way

$$ds^2 = a(\eta)^2 \left(-\left(1 + 2\alpha\right)d\eta^2 - 2\partial_a \beta d\eta dx^a + (g^{(3)}_{ab} (1 + 2\varphi) + 2\partial_a \partial_b \gamma) dx^a dx^b\right) \quad (4.1)$$

where $\eta$ is the conformal time related to the cosmic one as $a(\eta)d\eta = dt$. The perturbation functions are as usually Fourier transformed with respect to the spatial coordinates $x^a$ having thereby the following form: $\alpha(\eta, x^a) = \alpha(\eta, k)e^{ik_a x^a}$ and similar for $\beta, \varphi$ and $\gamma$. Although the metric perturbations are defined in the conformal time frame in the sequel the cosmic time $t$ will be used as the function argument and all the equations will be formulated with $t$ as the evolution parameter. $k = \sqrt{k_a k^a}$ is the comoving wavenumber. Appearance of just simple partial derivatives in (4.1) and exponents $e^{ik_a x^a}$ reflects the fact that the spatial curvature is zero.

It is very instructive and more intuitive to derive equations for perturbations for fluids although we consider models involving scalar fields. Moreover as we have seen above the scalar field energy-momentum tensor can be easily rewritten in fluid like quantities.

As it is known changing the coordinate system one can both produce fictitious perturbations and remove real ones. Thus, it is important to distinct real and fictitious perturbations. Natural way to do this is introducing gauge-invariant variables, which are free of these complications and are equal to zero for a system without perturbations. The gauge transformation includes four arbitrary parameters $\xi^\mu$:

$$\tilde{x}^\mu = x^\mu + \xi^\mu.$$  

Two of this parameters affect scalar perturbations and change therefore functions $\alpha, \beta, \varphi, \gamma$ under the gauge transformation [41]. There exist two independent gauge-invariant combinations of these functions, which fully determine the scalar perturbations of the metric tensor. These gauge-invariant functions are the Bardeen potentials [40]:

$$\Phi = \alpha - \dot{\chi}, \quad \Psi = H\chi - \varphi, \quad (4.2)$$
where \( \chi \equiv a\beta + a^2\dot{\gamma} \). The case of \( \Phi = \Psi \equiv 0 \) corresponds to fictitious perturbations. To construct the perturbation equations one can use the following elegant way \[41\]. In the longitudinal (conformal-Newtonian) gauge, defined by conditions \( \beta = \gamma = 0 \) one can construct equations for \( \alpha \) and \( \varphi \). In the obtained equations using that \( \chi = 0 \) one can just simply replace \( \alpha \) by \( \Phi \) and \( \varphi \) by \( -\Psi \), getting thereby equations for \( \Phi \) and \( \Psi \), which manifestly contain only gauge invariant variables.

Note that equations to be derived in this Section are valid only for \( k \neq 0 \) and the zero mode \( k = 0 \) will be considered separately in Section 6.

### 4.2 Single perfect fluid or scalar field

Considering the FRW background and the scalar perturbations, the energy-momentum tensor can be parameterized in the conformal time frame as follows

\[
T^{0}_{0} = -(\rho + \delta\rho), \quad T^{0}_{a} = -\frac{1}{k}(\rho + p)\partial_{a}v, \quad T^{a}_{b} = (p + \delta p)\delta^{a}_{b} + \left( \frac{\partial^{a}\partial_{b}}{k^2} + \frac{\delta^{a}_{b}}{3} \right) \pi^{s},
\]

where \( v \) is the velocity or the flux related variable and \( \pi^{s} \) is the anisotropic stress. The perturbation functions in \( T^{\mu}_{\nu} \) are as follows: \( \delta\rho(\eta, x^{a}) = \delta\rho(\eta, k)e^{ikax^{a}} \) and similar for \( \delta p, v \) and \( \pi^{s} \). The following notations will be used in the sequel:

\[
w \equiv \frac{p}{\rho}, \quad c_{s}^{2} \equiv \frac{\delta p}{\delta\rho}, \quad e \equiv \delta p - c_{s}^{2}\delta\rho, \quad \delta \equiv \delta\rho/\rho
\]

where \( w \) is the equation of state parameter, \( c_{s}^{2} \) is the speed of sound\(^4\). Constant \( w \) obviously results in \( c_{s}^{2} = w \) and \( e = 0 \). Non-zero \( e \) describes entropic perturbations.

Following the lines of Bardeen’s paper \[40\] we define gauge invariant quantities

\[
v\chi = v - \frac{k}{a}\chi, \quad \varepsilon = \delta + 3(1 + w)H\frac{a}{k}v.
\]

Starting with the Einstein equations \( G_{\mu\nu} = 8\pi GT_{\mu\nu} \) one yields

\[
\Psi - \Phi = 8\pi a^{2}G\frac{\rho^{2}}{k^{2}}\pi^{s},
\]

\[
\Psi = -4\pi G\rho a^{2}k^{2}\varepsilon,
\]

\[
\dot{v}\chi + Hv\chi = \frac{k}{a(1 + w)}\left( \frac{c}{\rho} + c_{s}^{2}\varepsilon + \Phi(1 + w) - \frac{2\pi^{s}}{3\Theta} \right),
\]

\[
\ddot{\varepsilon} - 3H\dot{\varepsilon} + \frac{k}{a}(1 + w)v\chi + 2H\frac{\pi^{s}}{\rho} = 0.
\]

\(^4\)This definition of the speed of sound \[42\] conforms with the canonical one (derivative of the pressure density w.r.t. the energy density at constant entropy) for perfect fluids but gives different result for scalar fields. It is convenient, however, keeping this notation \( c_{s}^{2} \) for scalar fields as well while it is not really the physical “speed of sound”.

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Now one can express $v_\chi$ from the latter equation, express $\Phi$ through $\varepsilon$ and $\pi^s$ using (4.5) and (4.6) and substitute all of this into (4.7). This will result in a single second order equation for $\varepsilon$ and $\pi^s$. It reads

$$
\ddot\varepsilon + \dot\varepsilon H (2 + 3c^2_s - 6w) + \varepsilon \left( \dot H (1 - 3w) - 15H^2w + 9H^2c^2_s + \frac{k^2}{a^2c^2_s} \right) =
$$

$$
= - \frac{k^2 e}{a^2 \varrho} - 2H \frac{\dot\pi^s}{\varrho} + \frac{\pi^s}{\varrho} \left( 2H^2(3w - 3c^2_s - 2) + \frac{2k^2}{3a^2} \right).
$$

(4.9)

One can check this equation against (4.9) in [40]. Our’s and Bardeen’s formulae are in a perfect agreement with each other. To do this comparison one has to account that dot in Bardeen’s paper denotes a derivative with respect to the conformal time, our $\varepsilon$ is equal to $P_0 \eta$ in [40] and our $\pi^s$ is equal to $P_0 \pi^{(0)}_T$ in [40]. Taking $\pi^s = 0$ one has

$$
\ddot\varepsilon + \dot\varepsilon H (2 + 3c^2_s - 6w) + \varepsilon \left( \dot H (1 - 3w) - 15H^2w + 9H^2c^2_s + \frac{k^2}{a^2c^2_s} \right) = - \frac{k^2 e}{a^2 \varrho}.
$$

(4.10)

In the case of a single free local massive scalar field we have an action easily read from (3.10) with only one $i$. This case corresponds to the situation when only one mode survives. Hereafter we consider only the form of scalar field Lagrangians relevant for our model. To the background order one uses (3.8) with only one field $\tau_i$. In the case of a scalar field perturbations of the energy-momentum tensor are parameterized by the perturbations of the metric and the perturbation of the scalar field $\delta \tau(t, x) = \delta \tau(t, k)e^{ika}$. To the perturbed order one has

$$
\delta \varrho = \mathcal{F}'_M(M)(\dot{\delta \tau} - \alpha \dot{\tau}^2 + M \tau \delta \tau), \quad \delta \phi = \mathcal{F}'_M(M)(\dot{\delta \tau} - \alpha \dot{\tau}^2 - M \tau \delta \tau),
$$

$$
v = \frac{k \delta \tau}{a \tau}, \quad \pi^s = 0.
$$

(4.11)

It is easy to show that $e = (1 - c^2_s)\varrho \varepsilon$ and using equation (4.10) one gets

$$
\ddot\varepsilon + \dot\varepsilon H (2 + 3c^2_s - 6w) + \varepsilon \left( \dot H (1 - 3w) - 15H^2w + 9H^2c^2_s + \frac{k^2}{a^2} \right) = 0.
$$

(4.12)

This is the equation governing perturbations of the energy density if we take as the background solution (3.2) with $N = 1$.

### 4.3 Many non-interacting perfect fluids or scalar fields

We assume non-interacting fluids resulting in individual conservation equations

$$
T = \sum_i T_i, \quad D_{\mu}T_{i\nu} = 0.
$$

(4.13)
Energy and pressure densities also acquire index \(i\) and \(\rho = \sum_i \rho_i, \ p = \sum_i p_i\). To describe energy-momentum tensor perturbations we introduce individual quantities accompanied with index \(i\) and the following summation rules hold

\[
\delta \rho = \sum_i \delta \rho_i, \ \delta p = \sum_i \delta p_i,
\]

\[
(\rho + p) v = \sum_i (\rho_i + p_i) v_i, \ \pi^s = \sum_i \pi^s_i.
\]  \hspace{1cm} (4.14)

The following additional notations are useful:

\[
w_i \equiv p_i / \rho_i, \ c_{s_i}^2 \equiv \dot{\rho}_i / \dot{\rho}_i, \ e_i \equiv \delta p_i - c_{s_i}^2 \delta \rho_i, \ \delta_i \equiv \delta \rho_i / \rho_i.
\]

For individual fluids one can define the following gauge invariant quantities:

\[
v_{i\chi} = v_i - \frac{k}{a} \chi, \ \varepsilon_i = \delta_i + 3(1 + w_i) H \frac{a}{k} v_i
\]  \hspace{1cm} (4.15)

Then starting with the Einstein equations \(G_{\mu\nu} = 8\pi G T_{\mu\nu}\) supplemented with \(\dot{\rho}_i + 3H(\rho_i + p_i) = 0\) for all \(i\) one yields an analog of equation (4.7) for \(i\)-th fluid

\[
\dot{v}_{i\chi} + Hv_{i\chi} = \frac{k}{a(1 + w_i)} \left( \frac{e_i}{\rho_i} + c_{s_i}^2 \varepsilon_i + \Phi(1 + w_i) - \frac{2\pi^s_i}{3\rho_i} \right).
\]  \hspace{1cm} (4.16)

We note the only change is that everything that can carry a fluid index \(k\) has acquired it. An analog of equation (4.8) is not so straightforward and is given by

\[
\ddot{\varepsilon}_i - 3H w_i \varepsilon_i + \frac{k}{a(1 + w_i)} v_{i\chi} \left( 1 - 3H \frac{a^2}{k^2} \right) + 2H \frac{\pi^s_i}{\rho_i} = -\frac{k}{a} (1 + w_i) 3H \frac{a^2}{k^2} v_{i\chi}\]  \hspace{1cm} (4.17)

Further manipulations bring us to a system of equations for \(\varepsilon_i\)

\[
\ddot{\varepsilon}_i + \dot{\varepsilon}_i H (2 + 3c_{s_i}^2 - 6w_i) +
+ \varepsilon_i \left( -3H (c_{s_i}^2 + w_i) + 9H^2 c_{s_i}^2 - 15H^2 w_i + \frac{k^2}{a^2} c_{s_i}^2 \right) =
= -\frac{k^2}{a^2} \frac{e_i}{\rho_i} + \frac{12\pi G}{\rho_i} \sum_m (\rho_i + p_i) e_m - (\rho_m + p_m) e_i +
+ 4\pi G (1 + w_i) \sum_m \rho_m \varepsilon_m (1 + 3c_{s_m}^2) +
+ \frac{12\pi G H}{3H - \frac{k^2}{a^2}} \sum_m \left[ \rho_m (1 + 3c_{s_m}^2) ((1 + w_m) (\ddot{\varepsilon}_i - 3H w_i \varepsilon_i) -
- (1 + w_i) (\dot{\varepsilon}_m - 3H w_m \varepsilon_m)) \right] +
+ \frac{2k^2}{3a^2} \frac{\pi^s_i}{\rho_i} - \frac{2}{a^2} \frac{d}{dt} \left( \frac{a^2 H^2 \pi^s_i}{\rho_i (1 + w_i)} \right) +
+ \frac{24\pi G H^2}{3H - \frac{k^2}{a^2}} \sum_m \rho_m (1 + 3c_{s_m}^2) \left( (1 + w_m) \frac{\pi^s_i}{\rho_i} - (1 + w_i) \frac{\pi^s_m}{\rho_m} \right).
\]  \hspace{1cm} (4.18)
One can compare the above system with equations (61), (62) in [42]. Identification is almost straightforward (their $\mu_{(i)}$ is our $\varrho_i$ and their $\delta_{(i)\nu_{(i)}}$ is our $\varepsilon_i$). We note a disagreement, namely ours last line with $\pi_i^s$-dependent terms is absent in equations in [42]. However, in a system with scalar fields and perfect fluids without anisotropic stresses equations are in perfect agreement (see also [43] where equations without anisotropic stresses derived as well). Note that $3\dot{H} - \frac{k^2}{a^2}$ may produce a pole only if $\varrho + p < 0$. The latter means that the phantom phase is entered, or in other words $w < -1$. Taking all $\pi_i^s = 0$ one has

\[
\begin{align*}
\ddot{\varepsilon}_i + \dot{\varepsilon}_i H \left(2 + 3c_{s_i}^2 - 6w_i\right) + \\
+ \varepsilon_i \left(-3\dot{H}(c_{s_i}^2 + w_i) + 9H^2c_{s_i}^2 - 15H^2w_i + \frac{k^2}{a^2}c_{s_i}^2\right) = \\
= -\frac{k^2}{a^2}e_i + \frac{12\pi G}{\varrho_i} \sum_m (\varrho_i + p_i)e_m - (\varrho_m + p_m)e_i + \\
+ 4\pi G(1 + w_i) \sum_m \varrho_m \varepsilon_m (1 + 3c_{s_m}^2) + \\
+ \frac{12\pi GH}{3H - \frac{k^2}{a^2}} \sum_m [\varrho_m(1 + 3c_{s_m}^2)((1 + w_m)(\dot{\varepsilon}_i - 3Hw_i\varepsilon_i) - \\
- (1 + w_i)(\dot{\varepsilon}_m - 3Hw_m\varepsilon_m))] .
\end{align*}
\]

If there is the CC in the system one accounts it as an extra perfect fluid distinguished with index $\Lambda$, with $w_\Lambda = c_{s_\Lambda}^2 = -1$. Multiplying (4.16) corresponding to $\Lambda$ by $1 + w_\Lambda$ one gets $\varepsilon_\Lambda = 0$ if $\pi_\Lambda^s = 0$. As a consequence the corresponding second order equation turns out to be homogeneous and one can consistently put perturbations of the cosmological constant to zero. Thus one should have in the above system all the perfect fluids but the CC. If there is one perfect fluid and the CC one gets only one equation which is of the form (4.10) with $w$ and $c_{s}^2$ corresponding to the perfect fluid. In the case of only the CC in the energy-momentum tensor energy density perturbations are absent.

In the case of many free local massive scalar fields we consider action (3.10). To the background order one has (3.8). To the perturbed order one has

\[
\begin{align*}
\delta \varrho_i = \mathcal{F}'_{,M}(M_i) \left(\dot{\tau}_i \delta \tau_i - \alpha \dot{\tau}_i^2 + M_i \tau_i \delta \tau_i\right), \quad \\
\delta p_i = \mathcal{F}'_{,M}(M_i) \left(\dot{\tau}_i \delta \tau_i - \alpha \dot{\tau}_i^2 - M_i \tau_i \delta \tau_i\right), \\
v_i = \frac{k \delta \tau_i}{a \dot{\tau}_i}, \quad \pi_i^s = 0 .
\end{align*}
\]

\[(4.20)\]
It is easy to show that $e_i = (1 - c_{s_i}^2)q_i \varepsilon_i$ and using equation (4.19) one gets

$$\ddot{\varepsilon}_i + \dot{\varepsilon}_i H \left(2 + 3c_{s_i}^2 - 6w_i\right) +$$

$$+ \varepsilon_i \left(-3\dot{H}(1 + w_i) - 15H^2w_i + 9H^2c_{s_i}^2 + \frac{k^2}{a^2}\right) =$$

$$= 16\pi G(1 + w_i) \sum_m q_m \varepsilon_m +$$

$$+ \frac{12\pi GH}{3H - \frac{k^2}{a^2}} \sum_m [q_m(1 + 3c_{s_m}^2)((1 + w_m)(\dot{\varepsilon}_i - 3Hw_i\varepsilon_i) -$$

$$- (1 + w_i)(\dot{\varepsilon}_m - 3Hw_m\varepsilon_m))].$$

These are the equations governing perturbations of the energy density if we take as the background solution (3.22) with arbitrary number of summands. If we have a mixture of perfect fluids and scalar fields one can easily compose a system of equations with one subset representing perturbations of perfect fluids and another subset representing the perturbations of scalar fields. For the purposes of this paper we may have plus to scalar fields only the perfect fluid of the CC. As was noted above $\varepsilon_\Lambda = 0$ if $\pi_\Lambda^a = 0$ and effectively one is left with the above system of equations for scalar fields only. If there is one scalar field and the CC one gets only one equation which is of the form (4.12) with $w$ and $c_{s_i}^2$ corresponding to the scalar field.

Another point of view is that the CC is a part of a scalar field potential. For instance, we may say that the CC is a part of the potential of the first scalar field. Then $\rho_1$ and $p_1$ get modified with the CC. Consequently, $w_1$ and $c_{s_1}^2$ become modified. $\varepsilon_1$ in this case represents perturbations of modified effective fluid quantities. At the end of the day it is a question of interpretation of the results. Our point of view in this paper that the CC is not an ingredient of any scalar field potential and is a separate perfect fluid. Moreover its own perturbations do not exist. Since there was only one scalar field in the original problem we are mainly interested in the behavior of the perturbation of the total energy-momentum tensor of scalar fields. Using a relation

$$\rho \varepsilon = \sum_m \rho_m \varepsilon_m = \rho_\tau \varepsilon_\tau$$

we see that $\varepsilon_\tau$ as well as total $\varepsilon$ can be easily extracted. Here by the subscript $\tau$ we denote the total scalar fields quantities.

Different approach is deriving a system of equations which manifestly contains one equation for $\varepsilon$. It is possible if one takes as perturbation variables $\varepsilon$ and

$$\zeta_{ij} \equiv \frac{\delta \tau_i}{\dot{\tau}_i} - \frac{\delta \tau_j}{\dot{\tau}_j}. $$

The latter variables are manifestly gauge invariant. In our case each scalar field satisfies to the background order

$$\Box \tau_i = M_i \tau_i.$$
One can perturb the latter equation with the result

\[ \ddot{\tau}_i + 3H\dot{\tau}_i + \frac{k^2}{a^2}\dot{\tau}_i + M_i\dot{\tau}_i = 2\alpha\dot{\tau}_i + (\kappa + 3H\alpha + \dot{\alpha})\dot{\tau}_i \]

where \( \kappa = 3(-\dot{\varphi} + H\alpha) + \frac{k^2}{a^2}\chi \). Subtracting two of such equations for indexes \( i \) and \( j \) one may get

\[ \ddot{\zeta}_{ij} + \left(3H + \frac{\dot{\tau}_i}{\tau_i} + \frac{\dot{\tau}_j}{\tau_j}\right)\dot{\zeta}_{ij} + \left(-3\dot{H} + \frac{k^2}{a^2}\right)\zeta_{ij} = \]

\[ = \left(\frac{M_i\tau_i}{\tau_i} - \frac{M_j\tau_j}{\tau_j}\right)\left(\sum_{m} \frac{\mathcal{F}_{,M}(M_m)\tau^2_m}{\varrho + p}(\dot{\zeta}_{im} + \dot{\zeta}_{jm}) + \frac{2}{1 + w}\varepsilon\right) \]

together with

\[ \ddot{\varepsilon} + \dot{\varepsilon}H(2 + 3c_s^2 - 6w) + \varepsilon \left(\dot{H}(1 - 3w) - 15H^2w + 9H^2c_s^2 + \frac{k^2}{a^2}\right) = \]

\[ = \frac{k^2}{a^2} \frac{1}{\varrho + p} \sum_{m,l} \mathcal{F}_{,M}^\prime(M_m)\mathcal{F}_{,M}^\prime(M_l)M_m\tau_m\tau^2_m\tau^2_l\zeta_{ml}. \]

It is worth noting that despite of the fact that \( \frac{1}{2}N(N - 1) \) nontrivial functions \( \zeta_{ij} \) can be constructed only \( N - 1 \) are truly independent thanks to the property \( \zeta_{ij} + \zeta_{jm} = \zeta_{im} \). For example, we can leave only functions \( \zeta_{ij} \). The two above equations are valid both with and without the CC. In these equations all quantities without indexes are collective for all scalar field and the CC (if present). Using (4.22) one can find \( \varepsilon = \frac{\varrho}{\varrho}\varepsilon \). The two latter equations govern perturbations of the energy density if we take as the background solution (3.2) with more than one summand (the previous approach is also valid, of course). If for some field the trivial background solution is taken, say \( \tau_i = 0 \) for some \( i \), corresponding \( i \)-s perturbation variables turn out to be trivial as well according to (4.20). In other words it is not possible at least at linear order that modes which are trivial in the background affect perturbations.

5. Cosmological perturbations with single free non-local scalar field

Now we turn to the main problem of the present paper: derivation of cosmological perturbation equations in models with free non-local scalar field.

The lagrangian in question is (2.5) and to the background order energy and
pressure densities are given by (3.5). To the perturbed order one has
\[
\delta \rho = \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \left( \partial_t \delta (\Box^l \tau) \partial_t \Box^{n-1-l} \tau + \partial_t \Box^l \tau \partial_t \delta (\Box^{n-1-l} \tau) - 2 \alpha \partial_t \Box^l \tau \partial_t \Box^{n-1-l} \tau + \delta (\Box^l \tau) \Box^{n-1-l} \tau + \Box^l \tau \delta (\Box^{n-1-l} \tau) \right),
\]
(5.1)
\[
\delta p = \frac{1}{2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \left( \partial_t \delta (\Box^l \tau) \partial_t \Box^{n-1-l} \tau + \partial_t \Box^l \tau \partial_t \delta (\Box^{n-1-l} \tau) - 2 \alpha \partial_t \Box^l \tau \partial_t \Box^{n-1-l} \tau - \delta (\Box^l \tau) \Box^{n-1-l} \tau + \Box^l \tau \delta (\Box^{n-1-l} \tau) \right),
\]
(5.2)
\[
v = \frac{k}{a(\rho + p)} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \partial_t \Box^l \tau \delta (\Box^{n-1-l} \tau),
\]
(5.3)
\[
\pi^* = 0.
\]
(5.4)

One can easily construct the variable \( \varepsilon \). Looking for an analogy with the equations for perturbations with single scalar field and in particular equation (4.12) one has to find \( e \). We note that in general
\[
e = (1 - c_s^2) \rho \varepsilon + \Delta, \quad \text{where} \quad \Delta = \delta p - \delta \rho + (1 - c_s^2) \frac{a}{k} \dot{\vartheta} v.
\]
(5.5)
\[\Delta = 0 \text{ in a model with a single local scalar field}. \]

For the non-local scalar field we have
\[
\Delta = - \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \left( \delta (\Box^l \tau) \Box^{n-1-l} \tau + \Box^l \tau \delta (\Box^{n-1-l} \tau) \right) +
\]
\[+ (1 - c_s^2) \frac{\dot{\vartheta}}{\vartheta + p} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \partial_t \Box^l \tau \delta (\Box^{n-1-l} \tau).
\]

This does not seem to be 0. Moreover, one should not expect any significant simplification just because the system with one non-local scalar field is equivalent to the background order to a system with many local scalar fields. In a general situations we may have infinitely many local scalar fields. This situation was considered in the previous Section and is rather complicated. It does not seem that a description with only one but a non-local scalar field may bring to us more beautiful equations for perturbations.

Using equation (4.10) one finds an analog of equation (4.12)
\[
\dot{\varepsilon} + \dot{\varepsilon} H (2 + 3 c_s^2 - 6 w) + \varepsilon \left( \dot{H} (1 - 3 w) - 15 H^2 w + 9 H^2 c_s^2 + \frac{k^2}{a^2} \right) = - \frac{k^2}{a^2} \frac{\Delta \rho}{\vartheta}.
\]
(5.6)

The presence of the CC does not change this equation because \( \Delta \) remains unchanged. Also in the presence of the CC one can replace the above equation with an analogous
one for $\varepsilon_\tau$ using $\varrho \varepsilon = \varrho_\tau \varepsilon_\tau$

$$\ddot{\varepsilon}_\tau + \dot{\varepsilon}_\tau H(2 + 3c_s^2 - 6w_\tau) + \varepsilon_\tau \left( \dot{H}(1 - 3w_\tau) - 15H^2 w_\tau + 9H^2 c_s^2 + \frac{k^2}{a^2} \right) = -\frac{k^2}{a^2} \Delta_\tau. \quad (5.7)$$

In this case we see that $\varepsilon_\tau$, $\varrho_\tau$ and $w_\tau$ appear instead of $\varepsilon$, $\varrho$ and $w$ respectively. $c_s^2 = c_s^2$ and can be kept as it is.

The further result for $\Delta$ can be obtained by straightforward summation using (3.2) as the background solution. To do this the following relations are useful

$$\delta(\Box^n \tau) = \Box^n \delta \tau + \sum_{m=0}^{n-1} \Box^m (\delta \Box) \Box^{n-1-m} \tau \quad \text{and} \quad \sum_{m=0}^{n-1} x^m = \frac{x^n - 1}{x - 1}.$$ 

Using (3.2) explicitly one has

$$\delta(\Box^n \tau) = \Box^n \delta \tau + \sum_i \frac{\Box^n - M_i^n}{\Box - M_i} (\delta \Box) \tau_i.$$ 

Perturbing the equation of motion for $\tau$ one has

$$\delta(F\tau) = \sum_{n=0}^{\infty} f_n \delta(\Box^n \tau) = 0. \quad (5.8)$$

More explicitly this equation can be written as

$$\delta(F\tau) = F \sum_i \left( \frac{1}{\Box - M_i} (\delta \Box) \tau_i + \delta \tau_i \right) = 0 \quad (5.9)$$

where we have put $\delta \tau = \sum_i \delta \tau_i$.

We stress here that such a decomposition of $\delta \tau$ is arbitrary since originally we have only one scalar field. Consequently no conditions on individual $\delta \tau_i$ arise. Since the decomposition is arbitrary we may require for all but one $i$ that

$$\delta((\Box - M_i) \tau_i) = 0.$$ 

For the remaining $i$, say $i = 2$, the most general condition would be

$$\delta((\Box - M_2) \tau_2) = (\Box - M_2) \psi \quad \text{with} \quad F\psi = 0.$$ 

Solution to the above equation on $\psi$ is already known from the investigation of the background. It is exactly equation (2.8) and its solution is of the form (3.2)

$$\psi = \sum_i \lambda_i \psi_i \quad \text{where} \quad (\Box - M_i) \psi_i = 0.$$
where $\lambda_i$ are arbitrary constants. Notice, $\psi_i \neq \tau_i$ because $\psi_i$ can depend on $x^a$ while $\tau_i$ is space homogeneous. Nevertheless, we see that the most general function $\psi$ can be absorbed in $\delta \tau_i$ and one can put all $\lambda_i = 0$ without any loss of generality.

After some algebra one can get the following expression for $\Delta$

$$\Delta = -\frac{2}{\rho + p} \sum_{m,l} F'_M(M_m)F'_M(M_l)M_m \tau_m \dot{\tau}_m \dot{\tau}_l \zeta_{ml}. \quad (5.10)$$

Juxtaposing equations (5.6) with the above derived $\Delta$ and (4.24) we see that perturbations become equivalent in the model with one non-local scalar field and in the model with many local scalar fields. This means that in order to study behavior of perturbation of the energy density one may use equation (4.24) with functions $\zeta_{ij}$ being subject to equations (4.23). The quantity which should be considered as energy density perturbation is $\varepsilon$. Functions $\zeta_{ij}$ play auxiliary role and normally should not be given an interpretation.

6. Space homogeneous perturbations, $k = 0$

As we have noted in Section 4 space homogeneous perturbations should be considered separately and the equations derived in the previous Sections are not applicable. Independence of perturbations of spatial coordinates implies $0a$ and $ab$ for $a \neq b$ components of the Einstein equation and $a$ components of the conservation equation remain unperturbed. Effectively this means $\beta = \gamma \equiv 0$ in the metric perturbation (4.1) and $v \equiv 0$ in the energy-momentum tensor perturbation (4.3) (the anisotropic stress $\pi^a$ can be put to zero as usual). Therefore starting with the Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ one gets only three equations which are as follows

$$\dot{\delta} \phi + 3H(\delta \phi + \dot{\phi}) + 3\dot{\phi}(\delta \rho + p) = 0, \quad (6.1)$$

$$-3H(-\dot{\phi} + H \alpha) = 4\pi G \delta \rho. \quad (6.2)$$

$$-\dot{\phi} + H \alpha = 4\pi G(\delta \rho + \delta p) \quad (6.3)$$

where the third equation is a consequence of the first two. It is not a problem to have one equation less since one of the perturbation functions can be gauged away. In a system with many fluids one has instead of (6.1)

$$\dot{\delta} \rho_i + 3H(\delta \rho_i + \dot{\rho}_i) + 3\dot{\rho}(\delta \rho_i + p_i) = 0. \quad (6.4)$$

Note that $\chi$ used before is identically zero and thus cannot be used to produce gauge invariant Bardeen potentials. Introducing an analog of $\varepsilon_i$ which is now $\bar{\varepsilon}_i = \delta_i + 3\phi(1 + w_i)$ one gets out of the first equation of the above system

$$\dot{\varepsilon}_i + 3H\varepsilon_i(c_{s_i}^2 - w_i) + 3H \frac{\varepsilon_i}{\bar{\rho}_i} = 0. \quad (6.5)$$
In the latter equation all the quantities are gauge invariant. All equations are homogeneous and for any perfect fluid with \( c_s^2 = w_i = \text{const}_i \) and \( e_i = 0 \) one gets \( \bar{\varepsilon}_i = \text{const} \). For instance, for the CC one has \( \bar{\varepsilon}_\Lambda = \delta_\Lambda = \text{const} \neq 0 \). We see that perturbation of the energy density of the CC is not obligatory zero unlike the consideration in the previous Sections. This happens because equation (4.16) used before to claim it is zero is not applicable for space homogeneous perturbations. Such a situation for adiabatic perturbations was noted in [40] with further reference to [44].

Scalar field however is an example of a perfect fluid with entropic perturbations, i.e. \( e \neq 0 \). One can conveniently introduce the variable \( \varepsilon_i = \delta_i + 3H(1 + w_i)\frac{\delta \tau_i}{\dot{\tau}_i} \). In this variables equation (6.4) for the perturbation of the energy density of a scalar field in a system with many scalar fields (3.10) becomes

\[
\frac{d}{dt}(\rho_i \varepsilon_i) + 3H(\rho_i \varepsilon_i) + (\rho_i + p_i) \left( \frac{4\pi G}{H} \sum_m \rho_m \varepsilon_m + \sum_j \delta \theta_j \right) - 3H \frac{\delta \tau_i}{\dot{\tau}_i} - 12\pi G \sum_m F^i_M(M_k) \dot{\tau}_k \delta \tau_k = 0.
\]

(6.6)

Here in the second row \( \sum_m \rho_m \varepsilon_m = \rho_\tau \varepsilon_\tau \) is the summation over all scalar fields components and \( \sum_j \delta \theta_j \) is the summation over all other fluids. In the most interesting and important case when apart form scalar fields there is only the CC this summation over \( j \) is just a single constant term. Moreover, since for the CC \( \varepsilon = \bar{\varepsilon} \) one can write this constant term as \( \rho_\Lambda \varepsilon_\Lambda \) (here \( \rho_\Lambda = g^2_\phi \Lambda \)). Further, defining \( \varepsilon \) through \( \varepsilon = \rho_\tau \varepsilon_\tau + \rho_\Lambda \varepsilon_\Lambda \) and summing up all the equations (6.6) for the scalar fields one derives

\[
\dot{\varepsilon} + 3H \varepsilon \left( 1 + \frac{H}{3H^2} \right) = \frac{8\pi G \rho_\Lambda \varepsilon_\Lambda}{H}.
\]

(6.7)

The latter equation can be integrated in terms of \( a \) and \( H \) to give

\[
\varepsilon = \frac{1}{Ha^3} \left( \varepsilon_0 + 8\pi G \rho_\Lambda \varepsilon_\Lambda \int a^3 dt \right).
\]

(6.8)

The quantity of interest \( \varepsilon_\tau \) can be restored using \( \varepsilon_\tau = (\rho \varepsilon - \rho_\Lambda \varepsilon_\Lambda) / \rho_\tau \).

7. Summary and outlook

The main result of this paper is the explicit proof that the cosmological model with one free non-local scalar field is equivalent to a cosmological model with many free local scalar fields not only to the background but also to the linear perturbation order. The non-local model is described by action (2.3) and the corresponding local model is described by action (3.10). The latter local model contains \( N \) scalar fields where
$N$ is the number of roots of the characteristic equation $\mathcal{F}(M) = 0$. Masses squared of these local fields are exactly roots of the characteristic equation. Perturbation equations for this local model are (4.23) and (4.24) where only $N - 1$ functions $\zeta_{1j}$ are independent. In fact most interesting exactly known cosmological solutions incorporate as an additional ingredient a bare CC. We refer reader to [3, 6] on the discussion on how such a constant can be generated during the tachyon evolution. If we take that the CC is a separate parameter and not an ingredient of the potential of the scalar field then one can get $\varepsilon_\tau$ using $\varrho\varepsilon = \varrho_\tau\varepsilon_\tau$. In the case of only one root of the characteristic equation and consequently only one scalar field in the local model one finds that RHS of (4.24) is equal to zero and one is left with equations from Section 4.1 to analyze perturbations as they are in a local system with a single scalar field. Moreover, if some fields in the local model are taken trivial in the background related to them perturbations turn out to be trivial as well.

Thus the problem of cosmological perturbations in a non-local scalar field theory is reduced to the problem of cosmological perturbations in a local theory with many degrees of freedom. In particular perturbations in a quintom model very close to our setup with a phantom field without potential and an ordinary scalar field with quadratic potential were studied in [15]. Perturbations in models with many scalar fields were studied in literature considering various cosmological scenarios [16]. The characteristic feature of the present setup is that all the local fields in fact are not physical and play a role of auxiliary functions introduced for the reduction of the complicated non-local problem to a known one. Partly because of this artificial origin, the local counterpart is not always the one already studied. As it was noted in [3, 4] for a very wide class of SFT inspired functions $\mathcal{F}(M)$ infinite number of complex roots $M_i$ may appear. Looking strange they do not produce a problem for the model since they are not physical quantities. The corresponding $\tau_i$ also become complex but it is a matter of choice of integration constants to make the physical quantities, $\tau$, $\varrho_\tau$ and their perturbations real. The case of complex $M_i$ has not been studied and deserves deeper investigation. This and other technical questions related to the equations of the present paper will be addressed in the forthcoming paper [17].

As a more ambitious problem which is of great importance is a construction of the formalism analogous to presented in this paper for a model with self-interacting non-local scalar field. Such models play important role in the SFT. For instance, rolling tachyon dynamics is governed by action (2.4) with a polynomial potential of fourth degree. However, even background solutions are not very well understood because there is no general analytic way of solving non-local non-linear equations. On the other hand it follows from the present analysis that passing to a local system with many fields is vital for the construction of perturbation equations.

Looking a step further it is interesting to consider perturbations in other non-local models coming from the SFT. For instance, models where open and closed string modes are non-minimally coupled may be of interest in cosmology. An example of
the classical solution is presented in [18]. Furthermore it should be possible to extend
the formalism presented in this paper to other models involving non-localities like
modified gravity setups [30].

Acknowledgements

The authors are grateful to I.Ya. Aref’eva, B. Craps, B. Dragovich, and V.F. Mukhanov for useful comments and discussions. This work is supported in part by RFBR grant 08-01-00798. A.K. is supported in part by the Belgian Federal Science Policy Office through the Interuniversity Attraction Poles IAP VI/11, the European Commission FP6 RTN programme MRTN-CT-2004-005104 and by FWO-Vlaanderen through the project G.0428.06. S.V. is supported in part by the grant of Russian Ministry of Education and Science NSh-1456.2008.2.

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