PIVOTAL STRUCTURES OF THE DRINFELD CENTER OF A FINITE TENSOR CATEGORY

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Abstract. We classify the pivotal structures of the Drinfeld center \( Z(C) \) of a finite tensor category \( C \). As a consequence, every pivotal structure of \( Z(C) \) can be obtained from a pair \((\beta, j)\) consisting of an invertible object \( \beta \) of \( C \) and an isomorphism \( j: \beta \otimes (\cdot) \otimes \beta^* \to (\cdot)^* \) of monoidal functors.

1. Introduction

Throughout this paper, we assume that \( k \) is an algebraically closed field of arbitrary characteristic. By a tensor category (over \( k \)), we mean a \( k \)-linear abelian rigid monoidal category satisfying natural conditions [EGNO15]. The rigidity means that every object \( X \) in a tensor category has a well-behaved ‘dual’ object \( X^* \). Although the notion of the dual object generalizes the contragradient representation in the group representation theory, the ‘double dual’ object \( X^{**} \) is no longer isomorphic to \( X \) in general.

It is known that the assignment \( X \mapsto X^{**} \) extends to a tensor functor. A pivotal structure of a tensor category is an isomorphism \( X \cong X^{**} \) of tensor functors. Such an isomorphism does not exist in general, however, we often require a pivotal structure in some applications of tensor categories to, for example, representation theory, low-dimensional topology, and conformal field theory. Thus it is interesting and important to know when a tensor category admits a pivotal structure.

A finite tensor category [EO04] is a tensor category satisfying a certain finiteness condition. In this paper, we classify the pivotal structures of the Drinfeld center of a finite tensor category. To state our main result, we introduce some notations: Let \( C \) be a finite tensor category, and let \( Z(C) \) denote the Drinfeld center of \( C \). The definition of \( Z(C) \) will be recalled in Subsection 2.7 in detail, but here we note that an object of \( Z(C) \) is a pair \( V = (V, \sigma) \) consisting of an object \( V \in C \) and a natural isomorphism \( \sigma(X) : V \otimes X \to X \otimes V \ (X \in C) \) called a half-braiding. We also note that the functor \( I^\beta(X) := \beta \otimes X \otimes \beta^* \ (X \in C) \) is a tensor functor if \( \beta \in C \) is invertible. For an invertible object \( \beta \in C \), we set

\[ \mathcal{J}_\beta := \{ (\beta, j) \mid j : I^\beta \to (-)^* \ \text{is a monoidal natural transformation} \} \]

and then define \( \mathcal{J} := \mathcal{J}_\beta_1 \sqcup \cdots \sqcup \mathcal{J}_\beta_n \), where \( \{\beta_1, \ldots, \beta_n\} \) is a complete set of representatives of the isomorphism classes of invertible objects of \( C \). Our main result is that the pivotal structures of \( Z(C) \) is in bijection with the set \( \mathcal{J} \). More precisely, we prove the following theorem:
Theorem 1.1. For an element \( \beta = (\beta, j) \in \mathcal{J} \), we define
\[
\Phi(\beta)_V : V \to V^{**} \quad (V = (V, \sigma) \in \mathcal{Z}(C))
\]
by the composition
\[
V \xrightarrow{\text{id}_V \otimes \text{coev}_\beta} V \otimes \beta \otimes \beta^* \xrightarrow{\sigma(\beta) \otimes \text{id}_{\beta^*}} \beta \otimes V \otimes \beta^* \xrightarrow{j_V} V^{**},
\]
where \( \text{coev}_\beta \) is the coevaluation (see Subsection 2.3 for our convention). The assignment \( \beta \mapsto \Phi(\beta) \) gives a bijection between the set \( \mathcal{J} \) defined in the above and the set of the pivotal structures of \( \mathcal{Z}(C) \).

The assignment \( \beta \mapsto \Phi(\beta) \) can be defined for arbitrary rigid monoidal categories, but it doesn’t seem to be a bijection in general. Techniques of Hopf (co)monads, initiated and organized in [BV07, BV12, BLV11], are essentially used to construct the inverse of \( \beta \mapsto \Phi(\beta) \).

Organization of this paper. This paper is organized as follows: In Section 2, we collect some basic results on monoidal categories from [ML98, EGN015] and fix some notations used throughout in this paper.

In Section 3, we recall the notion of Hopf comonad and the fundamental theorem for Hopf modules over a Hopf comonad. We then use it to prove some technical lemmas on the ‘adjoint algebra’ in a finite tensor category \( C \), which naturally arise from a right adjoint of the forgetful functor \( \mathcal{Z}(C) \to C \).

In Section 4, we prove Theorem 1.1 in a more general form: Let \( \{\beta_1, \ldots, \beta_n\} \) be as above. Given a tensor autoequivalences \( F \) on \( C \), we denote by \( \tilde{F} \) the braided tensor autoequivalence on \( \mathcal{Z}(C) \) induced by \( F \). For two tensor autoequivalences \( F \) and \( G \) on \( C \), we construct a map
\[
\Phi : \bigsqcup_{\beta = \beta_1, \ldots, \beta_n} \{(\beta, j) \mid j \in \text{Nat}_{\otimes}(I^\beta F, G)\} \to \text{Nat}_{\otimes}(\tilde{F}, \tilde{G}),
\]
where, for tensor functors \( S \) and \( T \), we have denoted by \( \text{Nat}_{\otimes}(S, T) \) the set of monoidal natural transformations from \( S \) to \( T \). By using techniques developed in Section 3, we show that the map \( \Phi \) is bijective (Theorem 4.1). Applying this result to \( F = \text{id}_C \) and \( G = (-)^{**} \) on \( C \), we obtain Theorem 1.1. By this result, we also see that the automorphism group of \( \text{id}_{\mathcal{Z}(C)} \) as a tensor functor is isomorphic to the group of the isomorphism classes of invertible objects of \( \mathcal{Z}(C) \).

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2. Preliminaries

2.1. Monoidal categories. A monoidal category [ML98 VII.1] is a category \( C \) endowed with a functor \( \otimes : C \times C \to C \) (called the tensor product), an object \( \mathbb{1} \in C \) (called the unit object), and natural isomorphisms
\[
(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z) \quad \text{and} \quad \mathbb{1} \otimes X \cong X \cong X \otimes \mathbb{1} \quad (X, Y, Z \in C)
\]
satisfying the pentagon and the triangle axioms. If these natural isomorphisms are identities, then \( C \) is said to be strict. In view of the Mac Lane coherence theorem, we may assume that all monoidal categories are strict. Given a monoidal category \( C \), we denote by \( C^{\text{rev}} \) the monoidal category obtained from \( C \) by reversing the order of the tensor product.
Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. A \textit{monoidal functor} \cite{ML98} X.1.2] from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ endowed with a natural transformation

$$F_2(X, Y) : F(X) \otimes F(Y) \to F(X \otimes Y) \quad (X, Y \in \mathcal{C})$$

and a morphism $F_0 : \mathbb{I} \to F(\mathbb{I})$ in $\mathcal{C}$ that is ‘associative’ and ‘unital’ in a certain sense. A monoidal functor $F$ is said to be \textit{strong} if $F_2$ and $F_0$ are invertible, and said to be \textit{strict} if $F_2$ and $F_0$ are identities. For a monoidal functor $F : \mathcal{C} \to \mathcal{D}$ and objects $X_1, \ldots, X_n \in \mathcal{C}$, we denote by

$$F_n(X_1, \ldots, X_n) : F(X_1) \otimes \cdots \otimes F(X_n) \to F(X_1 \otimes \cdots \otimes X_n)$$

the canonical morphism obtained by the iterative use of $F_2$.

Let $F$ and $G$ be monoidal functors from $\mathcal{C}$ to $\mathcal{D}$. A \textit{monoidal natural transformation} from $F$ to $G$ is a natural transformation $\xi : F \to G$ between underlying functors satisfying $\xi_1 \circ F_0 = G_0$ and $\xi_{X \otimes Y} \circ F_2(X, Y) = G_2(X, Y) \circ (\xi_X \otimes \xi_Y)$ for all objects $X, Y \in \mathcal{C}$. A monoidal natural transformation $\xi : F \to G$ satisfies

$$\xi_{X_1 \otimes \cdots \otimes X_n} \circ F_n(X_1, \ldots, X_n) = G_n(X_1, \ldots, X_n) \circ (\xi_{X_1} \otimes \cdots \otimes \xi_{X_n})$$

for all $X_1, \ldots, X_n \in \mathcal{C}$. If $\mathcal{C}$ is essentially small, then we denote by $\text{Nat}_{\otimes}(F, G)$ the set of all monoidal natural transformations from $F$ to $G$.

\[2.2. \text{Module categories.} \] Let $\mathcal{C}$ be a monoidal category. A \textit{left $\mathcal{C}$-module category} is a category $\mathcal{M}$ endowed with a functor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ (called the \textit{action}) and natural isomorphisms $(X \otimes Y) \otimes M \cong X \otimes (Y \otimes M)$ and $\mathbb{I} \otimes M \cong M$ fulfilling the axioms similar to those of monoidal categories. A \textit{left $\mathcal{C}$-module functor} is a functor between left $\mathcal{C}$-module categories preserving the actions. The notions of a right module category and functors between them are defined analogously. See \cite{EGNO15} Chapter 7 for the precise definitions.

Let $\mathcal{M}$ be a left $\mathcal{C}$-module category. If $A$ is an algebra in $\mathcal{C}$ (= a monoid in $\mathcal{C}$ \cite[VI.3]{ML98}), then the endofunctor $T = A \otimes (\cdot)$ is a monad on $\mathcal{M}$. We denote by $A \mathcal{M}$ the category of $T$-modules (= the Eilenberg-Moore category of $T$) and refer to an object of $A \mathcal{M}$ as a \textit{left $A$-module in} $\mathcal{M}$. The category $\mathcal{N}_A$ of right $A$-modules in a right $\mathcal{C}$-module category $\mathcal{N}$ is defined in a similar way.

\[2.3. \text{Rigidity.} \] Let $\mathcal{C}$ be a monoidal category, let $L$ and $R$ be objects of $\mathcal{C}$, and let $\varepsilon : L \otimes R \to \mathbb{I}$ and $\eta : \mathbb{I} \to R \otimes L$ be morphisms in $\mathcal{C}$. We say that $(L, \varepsilon, \eta)$ is a \textit{left dual object} of $R$ and $(R, \varepsilon, \eta)$ is a \textit{right dual object} of $L$ if the equations

$$(\varepsilon \otimes \text{id}_L) \circ (\text{id}_L \otimes \eta) = \text{id}_L \quad \text{and} \quad (\text{id}_R \otimes \varepsilon) \circ (\eta \otimes \text{id}_R) = \text{id}_R$$

hold. If this is the case, the morphisms $\varepsilon$ and $\eta$ are called the \textit{evaluation} and the \textit{coevaluation}, respectively.

A monoidal category $\mathcal{C}$ is said to be \textit{rigid} if every object of $\mathcal{C}$ has both a left dual object and a right dual object. If this is the case, we write a left dual object of $V \in \mathcal{C}$ as $(V^*, \text{ev}_V, \text{coev}_V)$. The assignment $X \mapsto X^*$ gives rise to a strong monoidal functor from $\mathcal{C}^{\text{op}}$ to $\mathcal{C}^{\text{rev}}$. For simplicity, we assume that $(-)^*$ is a \textit{strict} monoidal functor; see \cite[Lemma 5.4]{SML5} for a discussion.
2.4. Duality transformation. Let $\mathcal{C}$ and $\mathcal{D}$ be rigid monoidal categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a strong monoidal functor. If $X \in \mathcal{C}$ is an object, then $F(X^*)$ is a left dual object of $F(X)$ with the evaluation and the coevaluation given by

\[(2.1) \quad e_X = F_0^{-1} \circ F(1_X) \circ F_2(X, X) \quad \text{and} \quad e_X = F_2(X, X)^{-1} \circ F(1_X) \circ F_0,
\]

respectively. The duality transformation \cite{NS97} Section 1 of $F$ is the natural isomorphism $\gamma^F_X : F(X^*) \to F(X)^*$ determined by either of the conditions

\[(2.2) \quad e_X = ev_{F(X)} \circ (\gamma^F_X \otimes id_{F(X)}) \quad \text{or} \quad coev_{F(X)} = (id_{F(X)} \otimes \gamma^F_X) \circ c_X
\]

for $X \in \mathcal{C}$. We note that the assignments $X \mapsto F(X^*)$ and $X \mapsto F(X)^*$ are strong monoidal functors from $\mathcal{C}^{op}$ to $\mathcal{D}^{rev}$. The duality transformation $\gamma^F$ is in fact an isomorphism of monoidal functors.

Let $G : \mathcal{C} \to \mathcal{D}$ be a strong monoidal functor, and let $j : F \to G$ be a monoidal natural transformation. Interpreting \cite{JS93} Proposition 7.1 in our notation, we see that $j$ is invertible with the inverse determined by:

**Lemma 2.1.** $(j^{-1}_X)^* \circ (\gamma^F_X) = \gamma^G_X \circ j_X^* \quad (X \in \mathcal{C})$.

2.5. Invertible objects. Let $\mathcal{C}$ be a rigid monoidal category, and let $\beta \in \mathcal{C}$ be an object. Then the endofunctor $I^\beta := \beta \otimes (-) \otimes \beta^*$ on $\mathcal{C}$ is a monoidal functor with

\[I^\beta_0 = coev_\beta \quad \text{and} \quad I^\beta_2(V, W) = id_\beta \otimes id_V \otimes ev_\beta \otimes id_W \otimes id_{\beta^*} \quad (V, W \in \mathcal{C}).\]

An object $\beta \in \mathcal{C}$ is said to be invertible if there exists an object $\overline{\beta} \in \mathcal{C}$ such that $\beta \otimes \beta$ and $\beta \otimes \overline{\beta}$ are isomorphic to the unit object $\mathbb{1}$. Let $\beta \in \mathcal{C}$ be an invertible object. Then the object $\overline{\beta}$ in the above is isomorphic to $\beta^*$ and the morphisms $ev_\beta$ and $coev_\beta$ are isomorphisms. Hence $I^\beta$ is a strong monoidal functor.

**Lemma 2.2.** The duality transformation of $I^\beta$ is given by

\[I^\beta(X^*) = \beta \otimes X^* \otimes \beta^* \xrightarrow{\circ c \otimes id} \beta^* \otimes X^* \otimes \beta^* = I^\beta(X)^* \quad (X \in \mathcal{C}),\]

where $c : \beta \to \beta^{**}$ is the isomorphism determined by either of

\[ev_\beta^* \circ (c \otimes id_{\beta^*}) = coev_\beta^{-1} \quad \text{or} \quad coev_\beta^* = (id_{\beta^*} \otimes c) \circ ev_\beta^{-1}.
\]

**Proof.** We define $e_X$ by \((2.1)\) with $F = I^\beta$. Then

\[e_X = coev_\beta^{-1} \circ (id_\beta \otimes ev_X \otimes id_{\beta^*}) \circ (id_\beta \otimes ev_X \otimes id_{\beta^*})
\]

\[= ev_\beta^* \circ (c \otimes id_{\beta^*}) \circ (id_\beta \otimes ev_{\beta \otimes X} \otimes id_{\beta^*})
\]

\[= ev_{\beta \otimes X \otimes \beta^*} \circ (c \otimes id_X \otimes id_{\beta^*} \otimes id_X \otimes id_{\beta^*}).
\]

Thus, by \((2.2)\), the duality transformation of $I^\beta$ is given as stated. \hfill \square

2.6. Finite tensor categories. Throughout this paper, we assume that $k$ is an algebraically closed field of arbitrary characteristic. Given an algebra $A$ over $k$ (= an associative and unital algebra over $k$), we denote by $A$-mod the category of finite-dimensional left $A$-modules. A finite abelian category over $k$ is a $k$-linear category that is equivalent to $A$-mod for some finite-dimensional algebra $A$ over $k$.

A finite tensor category over $k$ \cite{EO04, EGNO15} is a rigid monoidal category $\mathcal{C}$ such that $\mathcal{C}$ is a finite abelian category over $k$, the tensor product of $\mathcal{C}$ is $k$-linear in each variable, and $\text{End}_\mathcal{C}(\mathbb{1}) \cong k$. By a tensor functor, we mean a $k$-linear exact strong monoidal functor between finite tensor categories. A morphism of tensor functors is just a monoidal natural transformation.
Let $C$ be a finite tensor category over $k$. We give some remarks on invertible objects of $C$. First, an invertible object of $C$ is a simple object. Hence the set $\text{Inv}(C)$ of the isomorphism classes of invertible objects of $C$ is in fact a finite set. Second, an object $V \in C$ is invertible if and only if there exists an object $W \in C$ such that $V \otimes W \cong I$; see, e.g., [EGNO15, Chapter 4].

2.7. The Drinfeld center. The Drinfeld center of a monoidal category $C$ is the category $Z(C)$ defined as follows: An object of this category is a pair $(V, \sigma_V)$ consisting of an object $V \in C$ and a natural isomorphism

$$\sigma_V(X) : V \otimes X \to X \otimes V \quad (X \in C)$$

(called the half-braiding) such that the equation

$$\sigma_V(X,Y) = (\text{id}_X \otimes \sigma_V(Y)) \circ (\sigma_V(X) \otimes \text{id}_Y)$$

holds for all objects $X,Y \in C$. If $V = (V, \sigma_V)$ and $W = (W, \sigma_W)$ are objects of $Z(C)$, then a morphism from $V$ to $W$ is a morphism $f : V \to W$ in $C$ satisfying

$$\sigma_V(X) \circ (f \otimes \text{id}_X) = (\text{id}_X \otimes f) \circ \sigma_W(X)$$

for all objects $X \in C$. The Drinfeld center $Z(C)$ is in fact a braided monoidal category; see [EGNO15] for details.

From the viewpoint of module categories, the Drinfeld center can be described as follows: Let $C$ be a finite tensor category over $k$, and set $E := C \boxtimes C^{\text{rev}}$, where $\boxtimes$ means the Deligne tensor product of $k$-linear abelian categories. Then $C$ is a left $E$-module category by the action determined by $(X \boxtimes Y) \otimes V = X \otimes V \otimes Y$. The Drinfeld center $Z(C)$ is equivalent to the category $\text{End}_E(C)$ of $k$-linear left $E$-module endofunctors on $C$ as a tensor category [EGNO15, Proposition 7.13.7].

There is the forgetful functor $U : Z(C) \to C$: $(V, \sigma_V) \mapsto V$. Since $U$ is a strong monoidal functor, $C$ is a left $Z(C)$-module category via $U$. The following lemma is a special case of [EGNO15, Theorem 7.12.11]:

**Lemma 2.3.** Let $C$ be as above. Then the functor

$$C \boxtimes C^{\text{rev}} \to \text{End}_{Z(C)}(C); \quad V \boxtimes W \mapsto F_{V,W} := V \otimes (-) \otimes W$$

is an equivalence of tensor categories. Here, the structure morphism of the functor $F_{V,W}$ as a left $Z(C)$-module functor is given by

$$X \otimes F_{V,W}(M) = X \otimes V \otimes M \otimes W \xrightarrow{\sigma_X(V) \otimes \text{id}_M \otimes \text{id}_W} F_{V,W}(X \otimes M)$$

for $V, W, M \in C$ and $X = (X, \sigma_X) \in Z(C)$.

2.8. Graphical calculus. We often use the graphical calculus to represent morphisms in a rigid monoidal category; see, e.g., [Kas95]. Our convention is that a morphism goes from the top to the bottom of the diagram. The evaluation and the coevaluation are expressed, respectively, as

$$
\begin{array}{c}
\text{ev}_X \\
\hline
X^* X \\
\end{array} =
\begin{array}{c}
X X^* \\
\hline
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\text{coev}_X \\
\hline
X^* X \\
\end{array} =
\begin{array}{c}
X X^* \\
\hline
\end{array}
$$
If \((V, \sigma_V)\) is an object of the Drinfeld center of a rigid monoidal category \(C\), then we express the half-braiding \(\sigma_V\) by a crossing as follows:

\[
\sigma_V(X) = \begin{array}{c}
X \\
V \\
\end{array} \quad \quad (X \in C).
\]

For \((V, \sigma_V), (W, \sigma_W) \in Z(C)\) and \(X, Y \in C\), we have

\[
\sigma_V \otimes W(X) = \begin{array}{c}
X \\
W \\
V \\
\end{array} \quad \quad \text{and} \quad \quad \begin{array}{c}
X \otimes Y \\
X \otimes W \\
V \\
\end{array}
\]

by the definition of \(Z(C)\), where \((V, \sigma_V) \otimes (W, \sigma_W) = (V \otimes W, \sigma_{V \otimes W})\).

3. The central Hopf comonad

3.1. Hopf comonads. Let \(C\) be a monoidal category. A **monoidal comonad** on \(C\) is a comonad \(T\) on \(C\) such that the endofunctor \(T : C \to C\) is a monoidal functor and the comultiplication \(\delta : T \to T^2\) and the counit \(\varepsilon : T \to \text{id}_C\) of \(T\) are monoidal natural transformations.

Let \(T\) be a monoidal comonad on \(C\). A **\(T\)-comodule** is a pair \((V, \rho)\) consisting of an object \(V\) of \(C\) and a morphism \(\rho : V \to T(V)\) such that the equations

\[
\delta_V \circ \rho = T(\rho) \circ \rho \quad \text{and} \quad \varepsilon_V \circ \rho = \text{id}_V
\]

hold. We denote by \(C^T\) the category of \(T\)-comodules (= the Eilenberg-Moore category of the comonad \(T\)). This category is in fact a monoidal category with respect to the tensor product given by

\[
(M, \rho_M) \otimes (N, \rho_N) = (M \otimes N, T_2(M, N) \circ (\rho_M \otimes \rho_N))
\]

for \((M, \rho_M), (N, \rho_N) \in C^T\). The unit object of \(C^T\) is \(1^T := (1, T_0)\).

A **Hopf comonad** on \(C\) is a monoidal comonad \(T\) on \(C\) such that the morphisms

\[
T_2(X, T(Y)) \circ (\text{id}_{T(X)} \otimes \delta_X) \quad \text{and} \quad T_2(T(X), Y) \circ (\delta_Y \otimes \text{id}_{T(Y)})
\]

are invertible for all \(X, Y \in C\). A Hopf comonad on \(C\) is just a Hopf monad on \(C^\text{op}\) in the sense of [BLV11]. Thus results on Hopf monads established in [BV07, BV12, BLV11] can be translated into results on Hopf comonads. For example, under the assumption that \(C\) is rigid, it follows from [BV07, Theorem 3.8] that the monoidal category \(C^T\) is rigid if and only if \(T\) is a Hopf monad.

3.2. Hopf modules. We consider the case where \(T\) is a \(k\)-linear exact Hopf comonad on a finite tensor category \(C\) over \(k\) (although some of our results hold in more general setting). Then the category \(C^T\) is a finite tensor category over \(k\) such that the forgetful functor

\[
U : C^T \to C; \quad (V, \rho) \mapsto V
\]

preserves and reflects exact sequences. By the definition of the tensor product of \(C^T\), the functor \(U\) is strict monoidal. The free \(T\)-comodule functor

\[
R : C \to C^T; \quad V \mapsto (T(V), \delta_V)
\]
is right adjoint to \( U \). The functor \( R \) is a \( k \)-linear exact monoidal functor with the same monoidal structure as \( T \) (i.e., \( R_0 = T_0 \) and \( R_2 = T_2 \)). The left Hopf operator \( \mathbb{H}^{(l)} \) and the right Hopf operator \( \mathbb{H}^{(r)} \) of the adjunction \( U \dashv R \) are the natural transformations defined by

\[
\mathbb{H}^{(l)}_{X,M} := R_2(X, U(M)) \circ (\text{id}_{R(X)} \otimes \rho_M) : R(X) \otimes M \to R(X \otimes M) \quad \text{and} \\
\mathbb{H}^{(r)}_{M,X} := R_2(U(M), X) \circ (\rho_M \otimes \text{id}_{R(X)}) : M \otimes R(X) \to R(M \otimes X),
\]

respectively, for \( X \in \mathcal{C} \) and \( M = (M, \rho_M) \in \mathcal{C}^T \). Since \( T \) is a Hopf comonad, the adjunction \( U \dashv R \) is a Hopf adjunction in the sense that the unit and the counit of \( U \dashv R \) are monoidal natural transformations and the Hopf operators \( \mathbb{H}^{(l)} \) and \( \mathbb{H}^{(r)} \) are invertible \cite[Theorem 2.15]{BLV11}.

**Lemma 3.1.** For all \( M = (M, \rho_M), N = (N, \rho_N) \in \mathcal{C}^T \) and \( X \in \mathcal{C} \), we have:

\begin{equation}
\mathbb{H}^{(l)}_{X, M \otimes N} = \mathbb{H}^{(l)}_{X,N} \otimes \mathbb{H}^{(l)}_{X,M}, \tag{3.5}
\end{equation}

\begin{equation}
\mathbb{H}^{(r)}_{M \otimes X} = \mathbb{H}^{(r)}_{N \otimes X} \otimes \mathbb{H}^{(r)}_{M,X}, \tag{3.6}
\end{equation}

\begin{equation}
\mathbb{H}^{(l)}_{X,-1} = \text{id}_{R(X)} = \mathbb{H}^{(r)}_{-1,X}, \tag{3.7}
\end{equation}

\begin{equation}
R(\varepsilon_X) \circ \mathbb{H}^{(l)}_{1,R(X)} = R_2(1, X), \quad R(\varepsilon_X) \circ \mathbb{H}^{(r)}_{R(X),1} = R_2(X, 1). \tag{3.8}
\end{equation}

**Proof.** We prove \((3.5)\). By the definition of \( \mathbb{H}^{(l)} \), we compute:

\[
\mathbb{H}^{(l)}_{X, M \otimes N} = T_2(X, M \otimes N) \circ (\text{id}_{T(X)} \otimes T_2(X,Y)(\rho_M \otimes \rho_N)) \\
= T_2(X \otimes M, N) \circ (T_2(X, M) \otimes \text{id}_{T(N)}) \circ (\text{id}_{T(X)} \otimes \rho_M \otimes \rho_N) \\
= \mathbb{H}^{(l)}_{X,N} \otimes \mathbb{H}^{(l)}_{X,M} \circ \text{id}_{N}.
\]

Equation \((3.6)\) is proved in a similar way. Equation \((3.7)\) follows directly from the definition of monoidal functors. Equation \((3.8)\) follows from the naturality of \( R \) and the definition of comodules.

The object \( A := R(1) \) is an algebra in \( \mathcal{C}^T \) with multiplication \( m := R_2(1, 1) \) and unit \( u := R_0 \). We define the category \( \mathcal{C}^T_A \) of *right Hopf modules over \( T \) to be the category of right \( A \)-modules in \( \mathcal{C}^T \). By \cite[(3.5), (3.8)]{BLV11}, the pair \((A, \hat{\sigma})\), where

\begin{equation}
\hat{\sigma}(M) = (\mathbb{H}^{(r)}_{M,1})^{-1} \circ \mathbb{H}^{(l)}_{1,M} : A \otimes M \to M \otimes A \quad (M \in \mathcal{C}^T), \tag{3.9}
\end{equation}

is a commutative algebra in \( \mathcal{Z}(\mathcal{C}^T) \) \cite[Theorem 6.6]{BLV11}. Thus a right \( A \)-module \((M, \mu)\) in \( \mathcal{C}^T \) can be regarded as an \( A \)-bimodule by defining the left action by

\[
A \otimes M \xrightarrow{\hat{\sigma}} M \otimes A \xrightarrow{\mu} M.
\]

The category \( \mathcal{C}^T_A \) is a monoidal category as a monoidal full subcategory of the category of \( A \)-bimodules in \( \mathcal{C}^T \).

There is a functor

\begin{equation}
H : \mathcal{C} \to \mathcal{C}^T_A; \quad V \mapsto (R(V), R_2(V, 1)). \tag{3.10}
\end{equation}

This functor is in fact a monoidal functor. The structure morphism \( H_0 \) is given by \( H_0 = \text{id}_A \). To define \( H_2 \), we first note that the equation

\[
R_2(1, V) \circ \hat{\sigma}(R(V)) = R_2(V, 1) \quad (V \in \mathcal{C})
\]
follows from (3.3) (see [BLV11 Theorem 6.6]). The left-hand side is just the left action of $A$ on $H(V)$. Thus, for $V, W \in C$, the tensor product of $H(V)$ and $H(W)$ over $A$ is given by the coequalizer diagram

$$H(V) \otimes A \otimes H(W) \xrightarrow{\frac{R_2(V,1) \otimes \mathrm{id}_R(W)}{\mathrm{id}_R(V) \otimes R_2(1,W)}} H(V) \otimes H(W) \xrightarrow{q_{V,W}} H(V) \otimes_A H(W).$$

Now we define $H_2(V, W) : H(V) \otimes_A H(W) \to H(V \otimes W)$ to be the unique morphism such that $q_{V,W} \circ H_2(V, W) = R_2(V, W)$. By interpreting [BLV11 Section 6] in our context, we have the following fundamental theorem for Hopf modules:

**Lemma 3.2.** If $T$ is conservative, then the monoidal functor $H : C \to C_T$ defined in the above is an equivalence of $k$-linear monoidal categories.

Here, a functor is said to be conservative if it reflects isomorphisms. As it is possibly well-known, an exact functor between abelian categories is conservative if and only if it is faithful [Shi14, Lemma 5.7].

### 3.3. Hopf bimodules

We keep the notation as in Subsection 3.2 and assume moreover that $T$ is conservative. We define the category $\tau C_T$ of right Hopf bimodules over $T$ to be the category of $A$-bimodules in $C_T$.

Since $U : C_T \to C$ is a strict monoidal functor, the category $C$ is a left $C_T$-module category by $M \otimes_V V = U(M) \otimes V$ ($M \in C$, $V \in C$). Lemma 3.1 implies that the functors $\pi : C \to C_T$ and $H : C \to C_T$ are $C_T$-module functors by the right Hopf operator $\mathbb{H}^{(r)}$. Now let $B$ be an algebra in $C_T$. Since $H$ is an equivalence of $C_T$-module categories by Lemma 3.2, it induces a category equivalence

$$H : bC \to b(C_T) = b(C_T)_A \quad (:= \text{the category of $B$-$A$-bimodules in } C_T).$$

If $B = A$, then $bC$ is the category of $T(1)$-modules in $C$ and $b(C_T)_A$ is the category of right Hopf bimodules over $T$. Thus we have obtained the following fundamental theorem for Hopf bimodules over a Hopf comonad:

**Lemma 3.3.** The functor $H$ induces an equivalence $\tau_{T(1)}C \approx \tau C_T$ of categories.

### 3.4. The central Hopf comonad

Let $C$ be a finite tensor category over $k$. Then there is a $k$-linear faithful exact comonad $Z$ on $C$ whose category of comodules can be identified with the Drinfeld center of $C$. The Hopf comonad $Z$ is called the central Hopf comonad in [Shi17] and helps us to analyze notions related to the Drinfeld center.

To give the definition of $Z$, we first recall the notion of ends of functors. Let $\mathcal{A}$ and $\mathcal{V}$ be categories, and let $H : \mathcal{A}^{\op} \times \mathcal{A} \to \mathcal{V}$ be a functor. Given an object $E \in \mathcal{V}$, a dinatural transformation from $E$ to $H$ is a family

$$\{ \pi(X) : E \to H(X, X) \}_{X \in \mathcal{A}}$$

of morphisms in $\mathcal{V}$ that is dinatural in the sense that the equation

$$H(f, \mathrm{id}_Y) \circ \pi(Y) = H(\mathrm{id}_X, f) \circ \pi(X)$$

holds for all morphisms $f : X \to Y$ in $\mathcal{A}$. An end of the functor $H$ is a pair $(E, p)$ consisting of an object $E \in \mathcal{V}$ and a dinatural transformation $p$ from $E$ to $H$ that is ‘universal’ among such pairs. The universal property implies that, if it exists, an end of $H$ is unique up to unique isomorphism. We denote

$$\int_{X \in \mathcal{A}} H(X, X)$$
an end of $H$. See [ML98, IX] for more detail.

Now, for $V \in \mathcal{C}$, we define $Z(V)$ to be the end

$$Z(V) = \int_{X \in \mathcal{C}} X \otimes V \otimes X^*,$$

which indeed exists by the argument of [KL01, Section 5]. We denote by

$$\pi_V(X) : Z(V) \to X \otimes V \otimes X^* \quad (X \in \mathcal{C})$$

the universal dinatural transformation of the end $Z(V)$. By the parameter theorem for ends [ML98, IX.7], the assignment $V \mapsto Z(V)$ extends to an endofunctor on $\mathcal{C}$ in such a way that $\pi_V(X)$ is natural in the variable $V$.

The functor $Z$ has a structure of a monoidal comonad given as follows: By using the universal property of the end, the monoidal structure

$$Z_0 : \mathbb{1} \to Z(\mathbb{1}) \quad \text{and} \quad Z_2(V, W) : Z(V) \otimes Z(W) \to Z(V \otimes W)$$

for $V, W \in \mathcal{C}$ are defined to be the unique morphisms such that the equations

$$(3.12) \quad \pi_{V \otimes W}(X) \circ Z_2(V, W) = (\text{id}_X \otimes \text{ev}_X \otimes \text{id}_{W \otimes X^*}) \circ (\pi_V(X) \otimes \pi_W(X)),$$

$$(3.13) \quad \pi_\mathbb{1}(X) \circ Z_0 = \text{coev}_X$$

hold for all $X \in \mathcal{C}$. To define the comultiplication of $Z$, we note that, by the Fubini theorem for ends [ML98, IX.8], the object $ZZ(V)$ is the end

$$ZZ(V) = \int_{X, Y \in \mathcal{C}} X \otimes Y \otimes V \otimes Y^* \otimes X^*$$

with the universal dinatural transformation given by

$$(3.14) \quad \pi_V^{(2)}(X, Y) := (\text{id}_X \otimes \pi_V(Y) \otimes \text{id}_{X^*}) \circ \pi_{Z(V)}(X)$$

for $X, Y \in \mathcal{C}$. By the universal property, we define $\delta_V : Z(V) \to ZZ(V)$ to be the unique morphism such that the equation

$$(3.15) \quad \pi_V^{(2)}(X, Y) \circ \delta_V = \pi_V(X \otimes Y)$$

holds for all $X, Y \in \mathcal{C}$. Finally, we define $\varepsilon_V : Z(V) \to V$ for $V \in \mathcal{C}$ by

$$(3.16) \quad \varepsilon_V = \pi_V(\mathbb{1}).$$

The category $Z\mathcal{C}$ of $Z$-comodules can be identified with $Z(\mathcal{C})$ as follows: By the rigidity of $\mathcal{C}$ and basic properties of ends, we have isomorphisms

$$\text{Hom}_\mathcal{C}(V, Z(V)) \cong \int_{X \in \mathcal{C}} \text{Hom}_\mathcal{C}(V, X \otimes V \otimes X^*)$$

$$(3.17) \quad \cong \int_{X \in \mathcal{C}} \text{Hom}_\mathcal{C}(V \otimes X, X \otimes V)$$

$$\cong \text{Nat}(V \otimes (\mathbb{1}), (\mathbb{1}) \otimes V)$$

for $V \in \mathcal{C}$. Let $\rho : V \to Z(V)$ be a morphism in $\mathcal{C}$, and let $\sigma : V \otimes (\mathbb{1}) \to (\mathbb{1}) \otimes V$ be the natural transformation corresponding to $\rho$ via (3.17). Explicitly,

$$\sigma(X) = (\text{id}_X \otimes \text{id}_V \otimes \text{ev}_X) \circ (\pi_V(X) \rho \otimes \text{id}_X) \quad (X \in \mathcal{C}).$$

One can prove that the pair $(V, \rho)$ is a $Z$-comodule if and only if the pair $(V, \sigma)$ is an object of $Z(\mathcal{C})$. Hence we obtain a one-to-one correspondence between $\text{Obj}(Z(\mathcal{C}))$ and $\text{Obj}(Z\mathcal{C})$, which, in fact, gives rise to an isomorphism $Z(\mathcal{C}) \cong Z\mathcal{C}$ of monoidal categories (cf. [DS07, BV12]).

Since $Z(\mathcal{C})$ is rigid, $Z$ is a Hopf comonad. Now we define the functors $U$, $R$ and $H$ by (3.3), (3.4) and (5.10) with $T = Z$, respectively. By the results of [Shi14] on
the adjunction \( U \dashv R \), we see that \( Z \) is \( k \)-linear, exact and faithful. Thus, by the fundamental theorem for Hopf modules (Lemma \ref{lemma:homological_properties}), the functor \( H : C \to C_Z^{\mathbb{Z}} \) is an equivalence of monoidal categories.

The monoidal structure of \( C_Z^{\mathbb{Z}} \) is defined by using the natural isomorphism \( \hat{\sigma} \) given by \( \hat{\sigma} \). In terms not of the central Hopf comonad, but of the half-braiding, the natural isomorphism \( \hat{\sigma} \) is given as follows:

**Proposition 3.4.** \( \hat{\sigma}(M) = \sigma_M(Z(1))^{-1} \) for \( M = (M, \sigma_M) \in \mathcal{Z} \).

Let \( c \) and \( \hat{c} \) be the braiding of \( \mathcal{Z} \) and \( \mathcal{Z}(\mathcal{Z}) \), respectively. Let \( A \) be as above, and write \( A = (A, \sigma_A) \in \mathcal{Z}(\mathcal{C}) \). The theory of Hopf monads just says that \( \hat{A} := (A, \hat{\sigma}) \) is a commutative algebra in \( \mathcal{Z}(\mathcal{C}) \). Nevertheless, we have

\[
m \circ c_{A, A} = m \circ \sigma_A(A) = m \circ \hat{\sigma}(A)^{-1} = m \circ \hat{c}^{-1}_{A, A} = m
\]

by this proposition. Hence \( A \) is in fact a commutative algebra in \( \mathcal{Z}(\mathcal{C}) \), as has been well-known; see, e.g., [DMNO13 Lemma 3.5].

**Proof of Proposition 3.4.** The claim is equivalent to:

\[
(3.19) \quad \mathbb{H}^{(f)}_{1, M} \circ \sigma_M(Z(1)) = \mathbb{H}^{(r)}_{M, 1}.
\]

We verify \((3.19)\). Let \( \rho_M : M \to Z(M) \) be the coaction of \( Z \) on \( M \) corresponding to the half-braiding \( \sigma_M \). By \((3.12)\), we have

\[
\pi_M(X) \circ \rho_M = (\sigma_M(X) \otimes 1) \circ (1 \otimes \text{coev}_X)
\]

for all \( X \in \mathcal{C} \). Thus, on the one hand, we compute:

\[
\pi_M(X) \circ \mathbb{H}^{(f)}_{1, M} = \pi_M(X) \circ Z_2(1, M) \circ (1 \otimes \text{id}_{Z(1)} \otimes 1)
\]

\[
= (\pi_2(X) \otimes 1) \circ (\pi_1(X) \circ (1 \otimes \text{coev}_X))
\]

by \((3.12)\) and \((3.18)\). On the other hand:

\[
\pi_M(X) \circ \mathbb{H}^{(r)}_{M, 1} = \pi_M(X) \circ Z_2(M, 1) \circ (1 \otimes \text{id}_{Z(1)})
\]

\[
= (\pi_2(X) \otimes 1) \circ (\pi_1(X) \circ (1 \otimes \text{id}_M))
\]

by \((3.12)\) and \((3.18)\). Hence \( \pi_M(X) \circ \mathbb{H}^{(f)}_{1, M} \circ \sigma_M(Z(1)) \) and \( \pi_M(X) \circ \mathbb{H}^{(r)}_{M, 1} \) are expressed by the diagrams:

\[
\begin{align*}
M & \quad Z(1) & \quad M \quad Z(1) \\
\pi_1(X) & \quad X & \quad X^* \quad \pi_1(X),
\end{align*}
\]

respectively. It is an easy exercise of the graphical calculus to check that these diagrams represent the same morphism. Now \((3.19)\) follows from the universal property of \( Z(M) \). \(\square\)
3.5. The adjoint algebra. Keep the notations as in Subsection 3.4. We call the algebra $A := U(A)$ the adjoint algebra of $C$ as it generalizes the adjoint representation of a Hopf algebra (see [Shi17]). By using Hopf (co)monadic techniques, we prove some useful results about the algebras $A \in C$ and $A \in Z(C)$.

3.5.1. Modules over the adjoint algebra. For $V \in C$, we define

\begin{equation}
\rho_V = (\id_V \otimes ev_V) \circ (\pi_1(V) \otimes \id_V) : A \otimes V \rightarrow V.
\end{equation}

If we use the half-braiding $\sigma_A : A \otimes (-) \rightarrow (-) \otimes A$, then $\rho_V$ is expressed as

\begin{equation}
\rho_V = (\id_V \otimes \varepsilon_1) \circ \sigma_A(V).
\end{equation}

One can check that $(V, \rho_V)$ is a left $A$-module in $C$ (see [Shi17]), and thus we call $\rho_V$ the canonical action of $A$ on $V$. The category of left $A$-modules in $C$ is described as follows:

**Lemma 3.5.** The functor defined by

\begin{equation}
C \boxtimes C \rightarrow A C; \quad V \boxtimes W \mapsto (V \otimes W, \rho_V \otimes \id_W)
\end{equation}

is an equivalence of categories.

This result is essentially due to Lyubashenko: The coend $F = \int^{X \in C} X^* \otimes X$ has a structure of a coalgebra in $C$ dual to our algebra $A \in C$. Lyubashenko showed that the category of $F$-comodules in $C$ is equivalent to $C \boxtimes C$ [Lyu99 Corollary 2.7.2], and hence the category of $A$-modules in $C$ is also equivalent to $C \boxtimes C$. We give a different proof of this lemma to situate this result in the theory of module categories, which plays a very important role in the recent study of tensor categories.

**Proof of Lemma 3.5.** We write $\mathcal{M} = Z(C)_A$ for simplicity. Let $H : C \rightarrow \mathcal{M}$ be the equivalence given by $\boxtimes 1$ with $T = Z$, and let $\overline{H}$ be its quasi-inverse. As we have seen in Subsection 3.3, the functor $H$ is in fact an equivalence of left $Z(C)$-module categories, and hence so is $\overline{H}$. There are four category equivalences

\begin{equation}
\begin{array}{c}
C \boxtimes C \xrightarrow{\mathcal{E}_1} \text{End}_{Z(C)}(C) \xrightarrow{\mathcal{E}_2} \text{End}_{Z(C)}(\mathcal{M}) \xrightarrow{\mathcal{E}_3} _A Z(C)_A \xrightarrow{\mathcal{E}_4} _A C
\end{array}
\end{equation}

given as follows:

1. $\mathcal{E}_1$ is the equivalence given by Lemma 2.3.
2. $\mathcal{E}_2$ sends $F \in \text{End}_{Z(C)}(C)$ to $H \circ F \circ \overline{H}$.
3. $\mathcal{E}_3$ sends $F \in \text{End}_{Z(C)}(\mathcal{M})$ to $F(A) \in _A Z(C)_A$. Here, the left action of $A$ on $F(A)$ is given by

\begin{equation}
A \otimes F(A) \xrightarrow{\zeta} F(A \otimes A) \xrightarrow{F(m)} F(A),
\end{equation}

where $\zeta : (-) \otimes F(-) \rightarrow F(- \otimes -)$ is the structure morphism of $F$ as a left $Z(C)$-module functor; see [EGNO15 Proposition 7.11.1].
4. $\mathcal{E}_4$ is the equivalence given by Lemma 3.3.

We prove this lemma by showing that the functor (3.22) is isomorphic to the composition $\mathcal{E}_4 \mathcal{E}_3 \mathcal{E}_2 \mathcal{E}_1$. We regard $1 \in C$ as a left $A$-module by the counit $\varepsilon_1 : A \rightarrow 1$. Then, by (3.3), $H(1)$ is the $A$-bimodule $A$. Thus, for $V, W \in C$, we have

\begin{equation}
\mathcal{E}_3 \mathcal{E}_2 \mathcal{E}_1(V \boxtimes W) = H(V \otimes \overline{H}(A) \otimes W) \cong H(V \otimes W) \in _A Z(C)_A.
\end{equation}
with the left $A$-module structure given by the composition

$$A \otimes H(V \otimes W) \xrightarrow{H(r \otimes 1)} H(A \otimes V \otimes W) \xrightarrow{H(\otimes_{1 \otimes \text{id}_W})} H(V \otimes A \otimes W) \xrightarrow{H(\text{id}_V \otimes \nu_1 \otimes \text{id}_W)} H(V \otimes W).$$

Thus, by (3.21), $\mathcal{E}_3 \mathcal{E}_2 \mathcal{E}_1(V \boxtimes W)$ corresponds to $(V \otimes W, g_V \otimes \text{id}_W)$ via the equivalence given by Lemma 3.3. Therefore $\mathcal{E}_3 \mathcal{E}_2 \mathcal{E}_1$ is isomorphic to (3.22).

### 3.5.2. Multiplicative characters of the adjoint algebra

For an invertible object $\beta \in \mathcal{C}$, we define the character of $\beta$ by

$$\chi_{\beta} := \text{coev}^{-1}_{\beta} \circ \pi_1(\beta) \in \text{Hom}_\mathcal{C}(A, \mathbb{1})$$

(cf. [Shi17]). By the dinaturality of $\pi$, we see that the character of $\beta$ depends only on the isomorphism class of $\beta$. Moreover, the following lemma shows that the character of an invertible object is ‘multiplicative’:

**Lemma 3.6.** $\chi_{\beta} : A \to \mathbb{1}$ is a morphism of algebras in $\mathcal{C}$.

**Proof.** An object of the form $X \otimes X^*$ for some $X \in \mathcal{C}$ is an algebra in $\mathcal{C}$ with multiplication $\text{id}_X \otimes \text{ev}_X \otimes \text{id}_{X^*}$. It is easy to see that $\pi_1(X) : A \to X \otimes X^*$ and $\text{coev}_X : \mathbb{1} \to X \otimes X^*$ are morphisms of algebras in $\mathcal{C}$. Thus $\chi_{\beta}$ is a morphism of algebras in $\mathcal{C}$ as the composition of such morphisms.

For two algebras $P$ and $Q$ in $\mathcal{C}$, we denote by $\text{Alg}_\mathcal{C}(P, Q)$ the set of algebra morphisms from $P$ to $Q$. We also denote by $\text{Inv}(\mathcal{C})$ the set of isomorphism classes of invertible objects of $\mathcal{C}$. By the above argument, we obtain the map

$$\chi : \text{Inv}(\mathcal{C}) \to \text{Alg}_\mathcal{C}(A, \mathbb{1}); \quad [\beta] \mapsto \chi_{\beta}.$$

**Lemma 3.7.** The above map is bijective.

**Proof.** We first prove that the map (3.24) is surjective. Let $\chi : A \to \mathbb{1}$ be a morphism of algebras in $\mathcal{C}$. Since $\mathbb{1} \in \mathcal{C}$ has no proper subobject, $(\mathbb{1}, \chi)$ is a simple object of $\mathcal{C}$. By the representation theory of finite-dimensional algebras, a simple object of $\mathcal{C} \boxtimes \mathcal{C}$ is of the form $V \boxtimes W$ for some simple objects $V$ and $W$ of $\mathcal{C}$. Thus, by Lemma 3.5, there are simple objects $V$ and $W$ of $\mathcal{C}$ such that

$$(\mathbb{1}, \chi) \cong (V \otimes W, g_V \otimes \text{id}_W)$$

as left $A$-modules. In particular, $V \otimes W \cong \mathbb{1}$. Thus we may assume that $\beta := V$ is an invertible object and $W = \beta^*$. Since $\text{Hom}_\mathcal{C}(\mathbb{1}, \beta \otimes \beta^*)$ is the one-dimensional vector space spanned by $\text{coev}_\beta$, we see that

$$\text{coev}_\beta : (\mathbb{1}, \chi) \to (\beta \otimes \beta^*, g_\beta \otimes \text{id}_{\beta^*})$$

must be a morphism of left $A$-modules. This means that the equation

$$\text{coev}_\beta \circ \chi = (g_\beta \otimes \text{id}_{\beta^*}) \circ (\text{id}_A \otimes \text{coev}_\beta)$$

holds. Therefore $\chi = \chi_{\beta}$. This proves the surjectivity. For the later discussion, we also note that the above argument shows that $(\mathbb{1}, \chi_{\beta})$ corresponds to $\beta \boxtimes \beta^*$ via the equivalence $\mathcal{A} \mathcal{C} \cong \mathcal{C} \boxtimes \mathcal{C}$ given in Lemma 3.3.

Now we check the injectivity of (3.24). Let $\alpha$ and $\beta$ be invertible objects of $\mathcal{C}$ such that $\chi_{\alpha} = \chi_{\beta}$. Then $(\mathbb{1}, \chi_{\alpha}) = (\mathbb{1}, \chi_{\beta})$ as left $A$-modules. By Lemma 3.5 and the above argument, we have $\alpha \boxtimes \alpha^* \cong \beta \boxtimes \beta^*$ in $\mathcal{C} \boxtimes \mathcal{C}$. This implies $\alpha \cong \beta$ in $\mathcal{C}$. Hence the map (3.24) is injective.
If, conversely to the above, \( \tilde{\delta} \) and the adjunction isomorphism

\[
\text{Proof.}
\]

\( C \) is a submonoid of \( CF(C) \).\(^{3.10}\) Remark Compose the bijections obtained by the previous two lemmas. 

Thus \( \tilde{f} \) is a morphism of algebras. The proof is completed. \( \square \)

The following lemma is one of key observations for the proof of the main result of this paper.

\[ \text{Lemma 3.9. We have a bijective map} \]

\[ \text{Inv}(C) \rightarrow \text{Alg}_{Z(C)}(A, A); \quad [\beta] \mapsto Z(\text{ch}_\beta) \circ \delta_1. \]

\[ \text{Proof.} \] Compose the bijections obtained by the previous two lemmas. \( \square \)

\[ \text{Remark 3.10.} \] The sets \( \text{Inv}(C) \) and \( \text{Alg}_{Z(C)}(A, A) \) are monoids with respect to the tensor product and the composition, respectively. One can check that \( \text{Alg}_C(A, \mathbb{1}) \) is a submonoid of \( CF(C) \). In this section, we have obtained the following bijections:

\[ \text{Inv}(C) \xrightarrow{\text{Lemma 3.7}} \text{Alg}_C(A, \mathbb{1}) \xrightarrow{\text{Lemma 3.8}} \text{Alg}_{Z(C)}(A, A). \]
These bijections are in fact isomorphisms of monoids. Since \( \text{Inv}(\mathcal{C}) \) is a group (with the inverse given by the dual), the monoids \( \text{Alg}_\mathcal{C}(A, 1) \) and \( \text{Alg}_{\mathcal{Z}(\mathcal{C})}(A, A) \) are also groups.

Remark 3.11. It may be worth to note that the set \( \text{Alg}_\mathcal{C}(A, 1) \) is linearly independent in \( \text{Hom}_\mathcal{C}(A, 1) \). To see this, we consider the end \( A^{\text{ss}} := \int_{X \in \mathcal{S}} X \otimes X^* \) in \( \mathcal{C} \), where \( \mathcal{S} \) is the full subcategory of \( \mathcal{C} \) consisting of all semisimple objects. Let \( \{V_1, \ldots, V_m\} \) be the complete set of representatives of isomorphism classes of simple objects of \( \mathcal{C} \). Then there is an isomorphism \( A^{\text{ss}} \cong \bigoplus_{i=1}^m V_i \otimes V_i^* \) and an epimorphism \( q : A \to A^{\text{ss}} \) such that the composition

\[
A \xrightarrow{q} A^{\text{ss}} \cong \bigoplus_{i=1}^m V_i \otimes V_i^* \xrightarrow{\text{projection}} V_i \otimes V_i^*
\]

is equal to \( \pi_1(V_i) \) for all \( i = 1, \ldots, m \), and hence the map

\[
\bigoplus_{i=1}^m \text{Hom}_\mathcal{C}(V_i \otimes V_i^*, 1) \cong \text{Hom}_\mathcal{C}(A^{\text{ss}}, 1) \to \text{Hom}_\mathcal{C}(A, 1)
\]

is injective [Shi17 Section 4]. Now let \( \beta_1, \ldots, \beta_n \) be the complete set of representatives of \( \text{Inv}(\mathcal{C}) \). We may assume that \( \{\beta_1, \ldots, \beta_n\} \) is a subset of \( \{V_1, \ldots, V_m\} \). Since \( \text{ch}_{\beta_i} \) corresponds to \( \text{coev}_{\beta_i}^{-1} \) via (3.25), we conclude that the subset

\[
\text{Alg}_\mathcal{C}(A, 1) = \{\text{ch}_{\beta_1}, \ldots, \text{ch}_{\beta_n}\} \subset \text{Hom}_\mathcal{C}(A, 1)
\]

is linearly independent (cf. [Shi17 Theorem 4.1]).

4. Pivotal structures of the Drinfeld center

4.1. Main result. Let \( \mathcal{C} \) be a finite tensor category, and let \( \text{Aut}_{\otimes}(\mathcal{C}) \) be the category of tensor autoequivalences of \( \mathcal{C} \). Given \( F \in \text{Aut}_{\otimes}(\mathcal{C}) \), we denote by

\[
\tilde{F} : \mathcal{Z}(\mathcal{C}) \to \mathcal{Z}(\mathcal{C})
\]

the braided tensor autoequivalence induced by \( F \). If we write \( \tilde{F}(V) = (F(V), \sigma'_V) \) for \( V = (V, \sigma_V) \in \mathcal{Z}(\mathcal{C}) \), then \( \sigma'_V : F(V) \otimes (-) \to (-) \otimes F(V) \) is the natural isomorphism uniquely determined by the property that the diagram

\[
\begin{array}{ccc}
F(V) \otimes F(X) & \xrightarrow{F_2} & F(V \otimes X) \\
\downarrow{\sigma'_V} & & \downarrow{F(\sigma_V)} \\
F(X) \otimes F(V) & \xrightarrow{F_2} & F(X \otimes V)
\end{array}
\]

commutes for all \( X \in \mathcal{C} \).

We fix a complete set \( \mathfrak{B} = \{\beta_1, \ldots, \beta_n\} \) of representatives of the set of isomorphism classes of invertible objects of \( \mathcal{C} \) and then define

\[
\mathfrak{I}(F, G) = \bigsqcup_{\beta \in \mathfrak{B}} \left\{ (\beta, j) \big| j \in \text{Nat}_{\otimes}(I^\beta F, G) \right\}
\]

for \( F, G \in \text{Aut}_{\otimes}(\mathcal{C}) \), where \( I^\beta(X) = \beta \otimes X \otimes \beta^* \) (\( X \in \mathcal{C} \)) is the tensor autoequivalence introduced in Subsection 2.5. The main result of this section is:
Theorem 4.1. Given \( \beta = (\beta, j) \in \mathfrak{J}(F, G) \), we define
\[
(4.2) \quad \Phi(\beta)_V : \widetilde{F}(V) \to \widetilde{G}(V) \quad (V \in \mathcal{Z}(C))
\]
by the composition
\[
(4.3) \quad F(V) \xrightarrow{id \otimes \text{coev}_\beta} F(V) \otimes \beta \otimes \beta^* \xrightarrow{\sigma'_V \otimes \text{id}} \beta \otimes E(V) \otimes \beta^* \xrightarrow{j_V} G(V)
\]
for \( V = (V, \sigma_V) \in \mathcal{Z}(C) \) such that \( \widetilde{F}(V) = (F(V), \sigma'_V) \). Then the map
\[
(4.4) \quad \Phi_{F, G} : \mathfrak{J}(F, G) \to \text{Nat}_{\otimes}(\widetilde{F}, \widetilde{G}), \quad \beta \mapsto \Phi(\beta)
\]
is a bijection.

Theorem 1.1 is the special case where \( F = \text{id}_C \) and \( G = (-)^* \). Applying this theorem to \( F = G = \text{id}_C \), we also obtain the following interesting corollary:

Corollary 4.2. \( \text{Nat}_{\otimes}(\text{id}_C, \text{id}_C) \cong \text{Inv}(\mathcal{Z}(C)) \) as groups.

Proof. The set \( \mathfrak{J}(\text{id}_C, \text{id}_C) \) is identified with \( \text{Inv}(\mathcal{Z}(C)) \). One can check that the bijection given by Theorem 1.1 is in fact an isomorphism of groups.

The rest of this section is devoted to prove Theorem 1.1. The proof is outlined as follows: In Subsection 4.2, we show that the map \( \Phi_{F, G} \) is well-defined, that is, \( \Phi(\beta) \) defined by (4.3) is indeed a monoidal natural transformation. In Subsection 4.3, we explain that it is enough to consider the case where \( F = \text{id}_C \). For this reason, from Subsection 4.4, we fix a tensor autoequivalence \( F \) of \( C \) and concentrate on proving the bijectivity of the map \( \Phi_0 := \Phi_{\text{id}_C, F} \). After preparing several technical lemmas in Subsection 4.4, we construct the map
\[
\Psi_0 : \text{Nat}(\text{id}_C, \widetilde{F}) \to \mathfrak{J}(\text{id}_C, F)
\]
in Subsection 4.5. We show that \( \Psi_0 \Phi_0 \) and \( \Phi_0 \Psi_0 \) are the identity maps in Subsections 4.6 and 4.7, respectively.

Notation. Till the end of this section, we fix a finite tensor category \( C \). We identify \( \mathcal{Z}(C) \) with the category \( C^Z \) of comodules over the central Hopf comonad \( Z \), and then define \( U \) and \( R \) by (3.3) and (3.4) with \( T = Z \), respectively. The symbols
\[
\pi_V(X) : Z(V) \to X \otimes V \otimes X^*, \quad \delta_V : Z(V) \to ZZ(V), \quad \varepsilon_V : Z(V) \to V, \quad Z_2(V, W) : Z(V) \otimes Z(W) \to Z(V \otimes W) \quad \text{and} \quad Z_0 : 1 \to Z(1)
\]
have the same meaning as in Section 3.11 see (3.11)-(3.16).

4.2. Well-definedness of \( \Phi \). We first check that the map (4.3) is well-defined, that is, the natural isomorphism \( \Phi(\beta) \) of Theorem 1.1 is indeed a monoidal natural transformation.

Let \( \beta \) be an invertible object of \( C \). We consider the braided tensor autoequivalence of \( \mathcal{Z}(C) \) induced by \( I^\beta \). For an object \( V = (V, \sigma_V) \in \mathcal{Z}(C) \), we define
\[
(4.5) \quad \chi^\beta_V : V \to \widetilde{I}^\beta(V)
\]
by the composition
\[
V \xrightarrow{id_V \otimes \text{coev}_\beta} V \otimes \beta \otimes \beta^* \xrightarrow{\sigma_V \otimes \beta^*} \beta \otimes V \otimes \beta^* = V^\beta.
\]

Lemma 4.3. \( \chi^\beta \) is a monoidal natural transformation from \( \text{id}_{\mathcal{Z}(C)} \) to \( \widetilde{I}^\beta \).
Proof. Let \((V, \sigma_V) \in \mathcal{Z}(\mathcal{C})\) be an object. We first show that \(\chi^\beta_V\) is indeed a morphism in \(\mathcal{Z}(\mathcal{C})\), that is, the diagram
\[
\begin{array}{c}
V \otimes X^\beta \\
\downarrow \sigma_V(X^\beta) \\
X \otimes V^\beta
\end{array}
\begin{array}{c}
\xrightarrow{\chi^\beta_V \otimes \text{id}} \\
\xrightarrow{\text{id} \otimes \chi^\beta_V} \\
\xrightarrow{\chi^\beta_V \otimes \text{id}}
\end{array}
\begin{array}{c}
V^\beta \otimes X^\beta \\
\xrightarrow{I^\beta_2(V,X)} \\
(X \otimes V)^\beta
\end{array}
\]
commutes for all \(X \in \mathcal{C}\). The paths going from \(V \otimes X^\beta\) to \((V \otimes X)^\beta\) in the clockwise and the counter-clockwise directions along the diagram are expressed by
\[
\begin{array}{c}
\begin{array}{c}
V \\
\beta \ X \\
\beta^* \ V^*
\end{array}
\begin{array}{c}
\begin{array}{c}
\beta \ V \\
\beta^* \ V^*
\end{array}
\begin{array}{c}
\begin{array}{c}
\beta \ X \\
\beta^* \ V^*
\end{array}
\begin{array}{c}
\beta \ V^*
\end{array}
\end{array}
\end{array},
\end{array}
\]
respectively. These diagrams express the same morphism. This shows that \(\chi^\beta_V\) is a morphism in \(\mathcal{Z}(\mathcal{C})\).

By the definition of morphisms of \(\mathcal{Z}(\mathcal{C})\), it is easy to see that the morphism \(\chi^\beta_V\) is natural in \(V\). We also have \(\chi^\beta_{(1, \text{id})} = I^\beta_0\) and
\[
I^\beta_2(V,W) \circ (\chi^\beta_V \otimes \chi^\beta_W) = \begin{array}{c}
\begin{array}{c}
\beta \ V \\
\beta \ W^*
\end{array}
\begin{array}{c}
\beta \ W \\
\beta \ W^*
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
\beta \ V \ W \\
\beta \ V \ W^*
\end{array}
\end{array}
= \chi^\beta_{V \otimes W}
\]
for all objects \(V = (V, \sigma_V), W = (W, \sigma_W) \in \mathcal{Z}(\mathcal{C})\). Hence \(\chi^\beta\) is monoidal. \(\square\)

The category \(\mathbf{Aut}_\otimes(\mathcal{C})\) is a strict monoidal category with the tensor product given by the composition. The category \(\mathbf{Aut}^{\text{br}}(\mathcal{Z}(\mathcal{C}))\) of braided tensor autoequivalences of \(\mathcal{Z}(\mathcal{C})\) is also a strict monoidal category in the same way. It is easy to see that the assignment \(F \mapsto \widetilde{F}\) extends to a strict monoidal functor
\[
\tilde{\cdot} : \mathbf{Aut}_\otimes(\mathcal{C}) \to \mathbf{Aut}^{\text{br}}(\mathcal{Z}(\mathcal{C})).
\]

Now let \(F, G \in \mathbf{Aut}_\otimes(\mathcal{C})\) and \(\beta = (\beta, j) \in \mathfrak{I}(F, G)\). Then the natural transformation \(\Phi(\beta)\) of Theorem 4.1 is written as the composition
\[
\Phi(\beta) = \left( \widetilde{F} = \text{id} \circ \widetilde{F} \xrightarrow{\chi^\beta} \widetilde{I}^\beta \circ \widetilde{F} = \widetilde{I}^\beta \circ F \xrightarrow{\tilde{j}} \widetilde{G} \right)
\]
of monoidal natural transformations. Hence \(\Phi(\beta) : \widetilde{F} \to \widetilde{G}\) is indeed a monoidal natural transformation as a composition of such ones.

4.3. **Reduction to the case where \(F\) is the identity.** We now explain that it is sufficient to consider the case where \(F = \text{id}_\mathcal{C}\) to Theorem 4.1. Let \(F\) and \(F'\) be tensor autoequivalences of \(\mathcal{C}\), and let \(E\) be a quasi-inverse of \(F\). By definition, there is an isomorphism \(\eta : \text{id}_\mathcal{C} \to EF\) of monoidal functors. Since \(E\) is an equivalence,
it induces a permutation $\sigma$ on $n$ letters such that $E(\beta_i) \cong \beta_{\sigma(i)}$ for all $i = 1, \ldots, n$. We fix an isomorphism $c_i : E(\sigma_i) \to \beta_{\sigma(i)}$ in $C$ for each $i$. There is the map
\[ j : F, F' \to \mathcal{J}(\text{id}_C, EF'), \quad (\beta, j) \to (\beta_{\sigma(i)}, j'), \]
where $j'_{X_0}$ for $X_0 \in C$ is the composition
\[ j'_{X_0} = E(j_X) \circ E(\beta_0, X, \beta^{*}) \circ (\text{id}_\beta \otimes \text{id}_{\mathcal{F}}(X) \otimes (\gamma^E_{\beta})^{-1}) \circ (c^{-1} \otimes \eta \otimes c^{*}) \]
(see Subsection 2.4 for the definition of $\gamma^E$). We also have a map
\[ \text{Nat}_C((\beta), (\beta') \to \text{Nat}_C((\text{id}_C), \tilde{E}(\beta)), \quad \xi \mapsto \tilde{E}(\xi) \circ \tilde{\eta}. \]
It is easy to check that the diagram
\[
\begin{array}{ccc}
\mathcal{J}(F,F') & \xrightarrow{\Phi} & \text{Nat}_C((\beta), (\beta')) \\
\downarrow{(4.7)} & & \downarrow{(4.8)} \\
\mathcal{J}(\text{id}_C, EF') & \xrightarrow{\Phi} & \text{Nat}_C((\text{id}_C), \tilde{E}(\beta'))
\end{array}
\]
commutes. Since the maps $\mathcal{J}(\beta)$ and $\text{Nat}_C((\beta))$ are bijective, the bijectivity of $\Phi_{(F,F')}$ is equivalent to the bijectivity of $\Phi_{\text{id}_C, EF'}$. Hence the proof of Theorem 4.1 is reduced to the case where $F = \text{id}_C$.

4.4. Technical lemmas. We fix $F \in \text{Aut}_C((\beta))$ and write
\[ \mathcal{J}_0(F) = \mathcal{J}(\text{id}_C, F) \quad \text{and} \quad \Phi_0 := \Phi_{\text{id}_C, F} \]
till the end of this paper. By the discussion of Subsection 4.3, it is sufficient to show that $\Phi_0$ is bijective to prove Theorem 4.1. We now prepare several technical lemmas for this purpose.

4.4.1. Remarks on $\chi^\beta$. Let $\beta \in C$ be an invertible object. We first express the natural isomorphism $\chi^\beta$ of Lemma 4.3 in terms of $Z$-comodules.

Lemma 4.4. For a $Z$-comodule $V = (V, \rho)$, we have $\chi^\beta = \pi_V(\beta) \circ \rho$.

Proof. Let $\sigma$ be the half-braiding of $V$ corresponding to $\rho$. Then we have
\[
\chi^\beta_V = (\sigma(\beta) \otimes \text{id}_\beta) \circ (\text{id}_V \otimes \text{coev}_\beta) \\
= (\text{id}_\beta \otimes \text{id}_V \otimes \text{ev}_\beta \otimes \text{id}_\beta) \circ (\text{id}_V \otimes \text{coev}_\beta) \quad \text{(by (3.18))} \\
= \pi_V(\beta) \circ \rho.
\]

This lemma is used to prove:

Lemma 4.5. For $V, X \in C$ and $\beta = (\beta, j)$, we have
\[ F(\pi_V(X)) \circ \Phi_0(\beta) = j_X \otimes V \otimes X \circ \pi_V(\beta \otimes X). \]

Proof. By the above lemma and the definition of $R$, we have
\[
(\text{id}_\beta \otimes \pi_V(X) \otimes \text{id}_\beta) \circ \chi^\beta_{R(V)} = (\text{id}_\beta \otimes \pi_V(X) \otimes \text{id}_\beta) \circ \pi_V(\beta) \circ \delta_V \\
= \pi_V(\beta \otimes X).
\]
Lemma 4.7. \[ \xi \]

Hence (4.10) follows. \[ \square \]

Let

\[ \rho \]

be the natural transformations corresponding to \( ZG \) via the bijection (3.17). This lemma claims that the equation

\[ \rho \]

holds for all \( X \in C \).

Proof. (4.10)

\[ \xi \]

\[ \xi \]

commutes for all \( V, X \in C \), where

\[ \xi \]

is an isomorphism

\[ \xi \]

Since \( G \) is an equivalence, and since (4.13) is invertible, the natural transformation \( \xi^G \) is in fact an isomorphism.

Lemma 4.6. If we regard the autoequivalence \( \tilde{G} : Z(C) \rightarrow Z(C) \) as an autoequivalence on the category \( C^Z \) of \( Z \)-comodules, then it is expressed by

\[ \tilde{G}(V) = (G(V), \xi^G_V G(\rho)) \]

for a \( Z \)-comodule \( V = (V, \rho) \).

Proof. Let \( V = (V, \rho) \) be a \( Z \)-comodule, and let

\[ \sigma(X) : V \otimes X \rightarrow V \otimes X \]

and

\[ \tau(X) : G(V) \otimes X \rightarrow X \otimes G(V) \]

be the natural transformations corresponding to \( \rho \) and \( \rho' := \xi^G_V G(\rho_V) \), respectively, via the bijection (3.17). This lemma claims that the equation

\[ \xi \]

holds for all \( X \in C \). We compute:

\[ \xi \]

Hence (4.10) follows. \[ \square \]

Lemma 4.7. \( \xi^G \) is an isomorphism \( \xi^G : G\mathcal{R} \rightarrow \mathcal{R} \) of monoidal functors.
Let $\eta$ and $\varepsilon$ be the unit and the counit of the adjunction $U \dashv R$. For functors $P : \mathcal{C} \to \mathcal{C}$ and $Q : \mathcal{Z}(\mathcal{C}) \to \mathcal{Z}(\mathcal{C})$, the map

$$\text{Nat}(UQ, PU) \to \text{Nat}(QR, RP); \quad \theta \mapsto RP(\varepsilon(\theta))R(\theta_{R(-)})\eta_{QR(-)}$$

is a bijection. Since $U \dashv R$ is a monoidal adjunction, this map restricts to

$$(4.11) \quad \text{Nat}(U \otimes \text{id}, PU \otimes \text{id})$$

when both $P$ and $Q$ are monoidal functors.

Now we consider the case where $P = G$ and $Q = \tilde{G}$. Then, since $UG = GU$ as monoidal functors, the identity natural transformation $\text{id}_{GU}$ lives in the source of the map (4.11). Let $\xi'$ be the monoidal natural transformation corresponding to $\text{id}_{GU}$ via (4.11). By the definition of $R$ and the previous lemma, we have

$$\tilde{G}R(V) = (GZ(V), \xi'_{\tilde{G}Z(V)}G(\delta_V))$$

for all $V \in \mathcal{C}$ as $Z$-comodules. Thus $\xi'_V$ for $V \in \mathcal{C}$ is computed as follows:

$$\xi'_V = ZG(\varepsilon_V) \circ \xi'_{\tilde{G}Z(V)} \circ G(\delta_V) = \xi'_{\tilde{G}Z(V)} \circ G(\delta_V) = \xi'_{\tilde{G}Z(V)} \circ G(\text{id}_{Z(V)}) = \xi'_{\tilde{G}Z(V)}.$$

Since $\xi'$ is a monoidal natural transformation, so is $\xi'_{\tilde{G}}$.

**4.4.3. Natural isomorphism $\zeta^\beta$.** If $\beta \in \mathcal{C}$ is an invertible object, then we can define $\zeta^\beta_V : ZI^\beta(V) \to Z(V)$ to be the unique morphism such that the diagram

$$\begin{array}{ccc}
Z(V^\beta) & \xrightarrow{\pi_\beta(X)} & X \otimes \beta \otimes V \otimes \beta^* \otimes X \\
\zeta^\beta_V \downarrow & & \downarrow \\
Z(V) & \xrightarrow{\pi_V(X \otimes \beta)} & (X \otimes \beta) \otimes V \otimes (X \otimes \beta)^*
\end{array}$$

commutes for all $X \in \mathcal{C}$. It is obvious that $\zeta^\beta_V$ is an isomorphism in $\mathcal{C}$ and natural in the variable $V$.

**Lemma 4.8.** $\zeta^\beta : RI^\beta \to R$ is an isomorphism of monoidal functors.

**Proof.** By abuse of notation, we write $\xi' G$ for $G = I^\beta$ as $\xi' \beta$. Then the diagram

$$\begin{array}{ccc}
Z(V) & \xrightarrow{\pi_V(X \otimes \beta)} & \beta \otimes X \otimes V \otimes (\beta \otimes X)^* \\
\chi^\beta_{R(V)} \downarrow & \xrightarrow{\text{(by Lemma 4.5)}} & \beta \otimes X \otimes V \otimes (\beta \otimes X)^* \\
Z(V^\beta) & \xrightarrow{\pi_\beta(X^\beta)} & \beta \otimes X \otimes \beta^* \otimes \beta \otimes V \otimes \beta^* \otimes (X \otimes \beta)^*
\end{array}$$

commutes for all $V, X \in \mathcal{C}$ (see Lemma 2.2 for the duality transformation of $I^\beta$). Thus, by the dinaturality of $\pi$ and $\text{coev}_{\beta^*} = (\text{ev}_\beta)^*$, we have

$$\pi_\beta(X^\beta) \circ \xi'_{\tilde{G}Z(V)} = \pi_V(\beta \otimes X \otimes \beta^* \otimes \beta) = \pi_V(\beta^* \otimes X \otimes \beta)$$

for all $V, X \in \mathcal{C}$. Since $I^\beta$ is an equivalence, we have

$$\pi_\beta(X^\beta) \circ \xi'_{\tilde{G}Z(V)} = \pi_V(\beta \otimes X \otimes \beta^* \otimes \beta) = \pi_V(X \otimes \beta)$$
for all \( V, X \in \mathcal{C} \). This means that \( \zeta^V = (\xi_V^* \circ \chi_{R(V)})^{-1} \) for all \( V \in \mathcal{C} \). Hence \( \zeta^V \) is a monoidal natural transformation from \( RI^\beta \) to \( R \) as the inverse of a monoidal natural transformation. \( \square \)

4.5. **Construction of the inverse.** Let \( h : \text{id}_{\mathcal{Z}(\mathcal{C})} \to \tilde{F} \) be a monoidal natural transformation. We now associate an element \( (\beta, j) \in J_0(F) \) to \( h \).

4.5.1. **Definition of \( \beta \).** We first associate an invertible object \( \beta \in \mathcal{C} \) to \( h \). For this purpose, we consider the composition

\[
A = R(\mathbb{1}) \xrightarrow{h_{R(\mathbb{1})}} \tilde{F} R(\mathbb{1}) \xrightarrow{\xi^F} RF(\mathbb{1}) \xrightarrow{R(F_0^{-1})} R(\mathbb{1}) = A
\]

of morphisms in \( \mathcal{Z}(\mathcal{C}) \). Since \( h \) and \( \xi^F \) are monoidal natural transformations, it is a morphism of algebras in \( \mathcal{Z}(\mathcal{C}) \). Thus, by Lemma 4.3, there exists a unique (up to isomorphisms) invertible object \( \beta \in \mathcal{C} \) such that the equation

\[
(4.12) \quad Z(F_0^{-1}) \circ \xi^F \circ h_{R(\mathbb{1})} = Z(\text{ch}_\beta) \circ \delta_1
\]

holds, or, equivalently, the equation

\[
(4.13) \quad \epsilon_1 \circ Z(F_0^{-1}) \circ \xi^F \circ h_{R(\mathbb{1})} = \text{ch}_\beta
\]

holds. We may assume that \( \beta \) is one of the representatives \( \beta_1, \ldots, \beta_n \) of the isomorphism classes of invertible objects of \( \mathcal{C} \).

4.5.2. **Definition of \( j \).** By using \( \beta \) defined in the above, we define \( \phi_V \) by

\[
\phi_V : R(V^\beta) \xrightarrow{\zeta^V} R(V) \xrightarrow{h_{R(V)}} \tilde{F} R(V) \xrightarrow{\xi^F} RF(V)
\]

for \( V \in \mathcal{C} \). We recall that an object of the form \( R(W) \) for some \( W \in \mathcal{C} \) is a right \( A \)-module (i.e., a right Hopf module over \( Z \)) in a natural way.

**Lemma 4.9.** \( \phi_V \) is an isomorphism of Hopf modules.

**Proof.** We set \( p_V = \xi^F \circ h_{R(V)} \) and \( q_V = \zeta^V \) for \( V \in \mathcal{C} \). Since \( p : R \to RF \) and \( q : RI^\beta \to R \) are monoidal natural transformations, the diagram

\[
\begin{array}{ccc}
R(V^\beta) \otimes A & \xrightarrow{\text{id} \otimes R(\text{coev}_A)} & R(V^\beta) \otimes R(\mathbb{1})^\beta \\
q_V \otimes q' & \downarrow & q_V \otimes q_1 \\
R(V) \otimes A & \xrightarrow{(q' := q_1 \circ R(\text{coev}_A))} & R(V) \otimes R(\mathbb{1}) \\
p_V \otimes p' & \downarrow & p_V \otimes p_1 \\
RF(V) \otimes A & \xrightarrow{\text{id} \otimes R(F_0)} & RF(V) \otimes RF(\mathbb{1}) \\
\end{array}
\]

commutes. By the definition of monoidal functors, the composite morphisms along the top row and the bottom row are \( R_2(V^\beta, \mathbb{1}) \) and \( R_2(F(V), \mathbb{1}) \), respectively. Thus we get the following commutative diagram:

\[
\begin{array}{ccc}
R(V^\beta) \otimes A & \xrightarrow{\phi_V} & R(V^\beta) \\
\phi_V \otimes p' & \downarrow & \phi_V \\
RF(V) \otimes A & \xrightarrow{R_2} & RF(V)
\end{array}
\]
To show that $\phi_V : R(V^\beta) \to RF(V)$ is an isomorphism of right Hopf modules, it suffices to show that $p'q'$ is the identity. By the definition of $\beta$, we have

$$p' = R(F_0^{-1}) \circ \xi^F \circ h_{R(1)} = Z(ch_\beta) \circ \delta_1 = Z(coev_\beta^{-1}) \circ Z(\pi_1(\beta)) \circ \delta_1.$$  

Hence we compute

$$\pi_1(X) \circ p' \circ q' = \pi_1(X) \circ Z(coev_\beta^{-1}) \circ Z(\pi_1(\beta)) \circ \delta_1 \circ \xi_1^F \circ Z(coev_\beta)$$

$$= (id_X \otimes coev_\beta^{-1} \otimes id_X) \circ \pi_1(X) \circ \delta_1 \circ \xi_1^F \circ Z(coev_\beta)$$

$$= (id_X \otimes coev_\beta^{-1} \otimes id_X) \circ \pi_1(X) \circ \delta_1 \circ \xi_1^F \circ Z(coev_\beta)$$

$$= \pi_1(X) \circ Z(coev_\beta^{-1}) \circ Z(coev_\beta) = \pi_1(X)$$

for $X \in C$. This implies that $p'q' = \text{id}_Z(1)$. \hfill \Box

By Lemma 4.2 we obtain a unique morphism $j_V : V^\beta \to F(V)$ such that

$$Z(j_V) = \phi_V = \xi_1^F \circ h_{R(V)} \circ \xi_1^F.$$  

The family $j = \{j_V\}_{V \in C}$ is in fact a natural isomorphism $j : I^\beta \to F$, since $\phi_V$ is invertible for all $V \in C$ and natural in $V \in C$. Moreover, we have:

**Lemma 4.10.** $j \in \text{Nat}(I^\beta, F)$.

**Proof.** Let $H : C \to Z(C)_A$ be the monoidal equivalence given by Lemma 3.2. We set $G = I^\beta$ for simplicity. As we have seen, $\phi : RG \to RF$ is a monoidal natural transformation such that $\phi_V$ is an isomorphism of Hopf modules for all $V \in C$. From this, we see that $\phi : HG \to HF$ is in fact a monoidal natural transformation. Hence $j : F \to G$ is also a monoidal natural transformation. \hfill \Box

4.5.3. **Definition of $\Psi_0$.** We finally define the map

$$\Psi_0 : \text{Nat}(id_Z(1), F) \to J_0(F)$$

by $\Psi_0(h) = (\beta, j)$, where $\beta$ and $j$ are constructed from $h$ in the above.

4.6. **Computation of $\Psi_0\Phi_0$.** We prove that $\Psi_0\Phi_0$ is the identity map. Let $\beta = (\beta, j)$ be an element of $J_0(F)$, and set

$$p = \Phi_0(\beta) \quad \text{and} \quad (\beta', j') = \Psi_0(p).$$

Our aim is to show that $\beta = \beta'$ and $j = j'$.

4.6.1. **Proof of $\beta = \beta'$.** We first show $\beta = \beta'$. By Lemma 4.7, it is enough to show that their characters are the same. This is proved as follows:

$$\text{ch}_{\beta'} = \pi_1(1) \circ Z(F_0^{-1}) \circ \xi_1^F \circ h_{R(1)}$$

$$= F_0^{-1} \circ F(\pi_1(1)) \circ Z(1) \circ \pi_Z(1)(\beta) \circ \delta_1$$

$$= coev_\beta^{-1} \circ (id_\beta \otimes \pi_1(1) \otimes id_\beta) \circ \pi_Z(1)(\beta) \circ \delta_1$$

$$= coev_\beta^{-1} \circ \pi_1(\beta \otimes 1) = \text{ch}_\beta.$$
Here, the second equality follows from the commutative diagram

\[
\begin{array}{ccccccccc}
FZ(1) & \xrightarrow{F(\pi_1(1))} & F(1 \otimes 1 \otimes 1^*) & \xrightarrow{F(1)} & FZ(1) \\
\xi^F_1 & \downarrow & \pi_{F(1)}(F(1)) & \downarrow & \pi_{F(1)}(F(1)) \\
ZF(1) & \xrightarrow{ZF(1)} & F(1) \otimes F(1) \otimes F(1)^* & \xrightarrow{ZF(1)} & F(1) \\
Z(F_0^{-1}) & \downarrow & (\circ \text{ by the definition of } \xi^F) & \downarrow & (\circ \text{ by the (di)naturality of } \pi) \\
Z(1) & \xrightarrow{\pi_1(1)} & 1 \otimes 1 \otimes 1^* & \xrightarrow{\pi_1(1)} & 1.
\end{array}
\]

4.6.2. Proof of $j = j'$. We fix objects $V, X \in \mathcal{C}$. The diagram

\[
\begin{array}{ccccccccc}
Z(V^\beta) & \xrightarrow{\pi_{V^\beta}(X^\beta)} & X^\beta \otimes V^\beta \otimes (X^\beta)^* \\
\zeta^\beta_V & \downarrow & \beta \otimes X \otimes (X^\beta)^* & \downarrow & \beta \otimes X \otimes (X^\beta)^* \\
Z(V) & \xrightarrow{\pi_V(\beta X)} & (X \otimes V \otimes X^*)^\beta \\
h_{R(V)} & \downarrow & \pi_{F(V)}(X) & \downarrow & \pi_{F(V)}(X) \\
FZ(V) & \xrightarrow{F(\pi_V(X))} & F(X \otimes V \otimes X^*) \\
\xi^F_V & \downarrow & (\circ \text{ byLemma} 4.5) & \downarrow & (\circ \text{ byLemma} 4.5) \\
ZF(V) & \xrightarrow{\pi_{F(V)}(F(X))} & F(X) \otimes F(V) \otimes F(X)^* \\
\end{array}
\]

commutes. By the definition of $j'$, the composition along the first column is $Z(j'_V)$. By Lemma 2.12, the composition along the second column is $j_X \otimes j_V \otimes (j_X^{-1})^*$. Thus we have

\[
\pi_{F(V)}(F(X)) \circ Z(j'_V) = (j_X \otimes j_V \otimes (j_X^{-1})^*) \circ \pi_{V^\beta}(X^\beta) = (\text{id}_{F(X)} \otimes j_V \otimes \text{id}_{F(X)}) \circ \pi_{V^\beta}(F(X)) = \pi_{F(V)}(F(X)) \circ Z(j_V).
\]

Hence, $Z(j_V) = Z(j'_V)$. Since $Z$ is faithful, we have $j_V = j'_V$. Therefore $\Psi_0 \Phi_0$ is the identity map.

4.7. Computation of $\Phi_0 \Psi_0$. Finally, we show that the composition $\Phi_0 \Psi_0$ is the identity map. Let $h : \text{id}_{Z(C)} \to F$ be a monoidal natural transformation, and set

\[
\beta = (\beta, j) = \Psi_0(h) \quad \text{and} \quad h' = \Phi_0(\beta).
\]

Since every component of the unit of the adjunction $U \dashv R$ is a monomorphism, every object $X \in Z(\mathcal{C})$ fits into an exact sequence of the form $0 \to X \to R(V) \to R(W)$ in $Z(\mathcal{C})$. By the naturality of $h$ and $h'$, we have:

\[
(4.15) \quad h = h' \iff h_{R(V)} = h'_{R(V)} \text{ for all } V \in \mathcal{C}.
\]

We fix $V \in \mathcal{C}$. For simplicity, we set

\[
p_X := (j_X \otimes \text{id}_\beta) \circ (\text{id}_\beta \otimes \text{id}_X \otimes \text{ev}_\beta^{-1})
\]
for $X \in \mathcal{C}$. Then we have the following commutative diagram:

\[
\begin{array}{ccc}
Z(V) & \xrightarrow{\pi_V(\beta \otimes X)} & \beta \otimes X \otimes V \otimes X^* \otimes \beta^* \\
& (\triangledown \text{by the dinaturality of } \pi_V(X) \text{ in } X) & \rho \otimes \text{id}_V \otimes (\rho^{-1})^* \\
Z(V) & \xrightarrow{\pi_V(F(X) \otimes \beta)} & F(X) \otimes \beta \otimes V \otimes \beta^* \otimes F(X)^* \\
& (\triangledown \text{by the definition of } \zeta^\beta) & \\
Z(V^\beta) & \xrightarrow{\pi_V(V^\beta)(F(X))} & F(X) \otimes \beta \otimes V \otimes \beta^* \otimes F(X)^* \\
& (\triangledown \text{by the naturality of } \pi_V(X) \text{ in } V) & \text{id} \otimes j_{jV} \otimes \text{id} \\
ZF(V) & \xrightarrow{\pi_{F(V)}(F(X))} & F(X) \otimes F(V) \otimes F(X)^* \\
& (\triangledown \text{by the definition of } \xi^F) & \\
FZ(V) & \xrightarrow{F(\pi_V(X))} & F(X \otimes V \otimes X^*) \\
& \text{(4.10) with } G = F & \\
\end{array}
\]

By the defining equation (4.14) of $j$, the composite morphism along the first column is $h_{R(V)}$. On the other hand, by Lemma 2.1, the composite morphism along the second column is $j_{X \otimes V \otimes X^*}$. Thus we have

\[
F(\pi_V(X)) \circ h_{R(V)} = j_{X \otimes V \otimes X^*} \circ \pi_V(\beta \otimes X)
\]

for all $X \in \mathcal{C}$. By Lemma 4.5, we also have

\[
F(\pi_V(X)) \circ h'_{R(V)} = j_{X \otimes V \otimes X^*} \circ \pi_V(\beta \otimes X)
\]

for all $X \in \mathcal{C}$. Thus we have $h'_{R(V)} = h''_{R(V)}$. By (4.15), we have $h = h'$. Therefore $\Phi_0 \Psi_0$ is the identity map. We have completed the proof of Theorem 4.1.

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