On μ-Semiregular Module

Eman Mohmmed and Wasan Khalid
Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraqi

E-mail: emandj66@gmail.com

Abstract. Let R be an associative ring with identity and let M be right R-module. M is called μ-semi hollow module if every finitely generated submodule of M is μ-small submodule of M. The purpose of this paper is to give some properties of μ-semi hollow module. Also, we gives conditions under, which the direct sum of μ-semi hollow modules is μ-semi hollow. An R-module is said has a projective μ-cover if there exists an epimorphism f: P → M, Where P is a projective R-module and ker(f) ≪ M. And study some properties of Projective μ-cover of M. Moreover, An module M is μ-semiregular module if every cyclic submodule of M is μ-lying summand of M. We add some results of μ-semiregular module.

Keywords: μ-small submodule, μ-semi hollow module, projective μ-cover, μ-semiregular module.

1. Introduction
Throughout this paper, all rings are associative with identity and modules are, unital right R-module, where R denotes such a ring and M such a modules. A submodule of M is called a small submodule if whenever M = A + B for some submodules B of M, we have M = B, see [1]. A submodule of M is called μ-small submodule of M if whenever M = A + X with M/X is cosingular, implies that M = X, see [2].

An module M is called semi hollow module if every finitely generated proper submodule of M is, a small submodules of M, see [3].

We define μ-semi hollow module as a generalization of semi hollow module and give basic properties of these concepts with condition under which the direct sum of μ-semi hollow modules is μ-semi hollow module.

A submodules N of an R-modules M is called lie over a, projective summand of M if there exists a decompositions M = A ⊕ B, where A, is a projective submodule of N and M ⊆ B ≪ M. See [4].

In [5], Nichloisin introduced the following concept An R-module M is called a semiregular if every cyclic submodule of M is lying over a projective summand of M.

A submodule N of an R-module M is called μ- lie over a (projective) summand of M if there exists a decompositions M = A ⊕ B, where A is projective submodule of N ands B ∩ N ≪ M.
As a generalization of semiregular module, we introduce the concept of μ-semiregular module. A module $M$ is called μ-semiregular if every cyclic submodule of $M$ is μ-lying over summand of $M$. We study the main properties of μ-semiregular, and suppling examples and remarks for this concept.

2. μ-semi hollow module:
It is known that every small submodule of $R$-module $M$ is a μ-small, However the converse in general is not true, it was introduced in [6]. μ-hollow modules, in this section are introduced μ-semi hollow module as a generalization of μ-hollow module.

For a module $M$, $Z^*(M) = \{ m \in M : Rm \ll E(M) \}$ , where $E(M)$ is the injective hull of $M$. $M$ is called cosingular (non-cosingular) module if $Z^*(M), = M,(Z^*(M), = 0$)[6].

Lemma 2.1: [6]. Let $M$ be an $R$- Module. Then,
1- If $f : M, \rightarrow M'$, is a homomorphism of $R$- modules $M, M'$, then $(Z^*(M)), \leq Z^*(M')$.
2- Let $A$ be a submodule of $M$. Then $Z^*(A), = A \cap Z^*(M)$.
3- Let $M_i (i \in I)$ be any collection of $R$-module and let $M = \bigoplus_{i \in I} M_i$ Then$sZ^*(M), = \bigoplus_{i \in I} Z^*(M_i),.$

Lemma 2.2: [7]. Let $R$ be a ring such that $\bigoplus_{i \in I} R \rightarrow R$ = $R$, then for any $R$-module $M$, $Z^*(M) = M$.

Proposition 2.3: [6]. Let $R$ be a cosingular ring. Then any $R$-module is cosingular.
Corollary 2.4: [6]. Every Z- module is cosingular.

Proposition 2.5: [2]. Let $A \leq B \leq M$ such that $\frac{M}{A}$ is cosingulars, Then $\frac{M}{A+B}$ is cosingular.

Corollary 2.6: [2]. Let $A$, and $B$ be submodule of an $R$- module $M$. If $\frac{M}{A}$ is cosingular, then $\frac{M}{A+B}$ is cosingular.

Definition 2.9: [2]. Let $M$ be an $R$- module and let $A$ be a submodule of $M$, We say that $A$ is μ-small submodule of $M$ (denoted by $A \ll_\mu M$), if whenever $M = \bigoplus_{i=1}^n A_i$, $\forall i \in \{1, \ldots, n\}$. Then $M = \bigoplus_{i=1}^n A_i \ll_\mu M$,

Proposition 2.10: [2]. Let $M$ be an $R$-module and let $A \leq B \leq M$. Then $A \ll_\mu M$, $B \ll_\mu M$, $A \ll_\mu B$, $B \ll_\mu M$, and $A \ll_\mu B \ll_\mu M$.

Proposition 2.11: [2]. Let $M$ be an $R$-module and let $A \leq B \leq M$. Then $A \ll_\mu B \ll_\mu M$.

Recall that a module $M$ is called μ-hollow if every proper submodule of $M$ is μ-small in $M$.[2].

Now we introduce the following:

Definition 2.13: A non- Zero module $M$ is called μ- semi hollow module, If every finitely generated submodule of $M$ is μ-small in $M$.[2].

Note: Every μ- hollow module is μ- semi hollow, but the converse is not true. For example $Q$ as $Z$-module.

Example and Remark 2.14: 1- $Q$ as $Z$- module is μ- semi hollow module.
2- $Z$ as $Z$- module is not μ- semi hollow module.
3- $Zp\infty$ as $Z$- module is μ- semi hollow module.
5. Every semi hollow is μ- semi hollow. But the converse is not true in generals. For example Z₆ as Z₆-module. 

6. Let M be a, cosingular R-module. Then M is semi hollow if M is μ- semi hollow.

The following are some basic properties of μ- Semi hollow module.

Proposition 2.15: Let M be a μ- Semi hollow. Then \( \frac{M}{N} \) is a μ- Semi hollow for all proper submodules N of M.

Proof: Let \( \frac{U}{N} \) be a proper finitely generated submodule of \( \frac{M}{N} \), with \( \frac{U}{N} + \frac{V}{N} = \frac{M}{N} \) and \( Z(\frac{M}{V}) = \frac{M}{V} \) for submodule V of M. Then \( \frac{U}{N} = R(x_1+N), R(x_2+N), \ldots , R(x_n+N) \) = \( R(x_1 + R(x_2 + \ldots + R(x_n + N), x_i \in U, \forall i = 1, 2, \ldots , n \)

Now, \( \frac{M}{N} = R(x_1 + R(x_2 + \ldots + R(x_n) + V) \)

Hence \( M = R(x_1 + R(x_2 + \ldots + R(x_n + V), \) but M is semi hollow. Then each \( R(x_i) \ll \mu M_i \), By cor (2.5) we get \( M = V \).

The converse is not true in general, for example consider Z as Z-module. Note that \( \frac{Z}{4Z} \equiv \frac{Z}{4} \) is μ- Semi hollow but Z is not μ- Semi hollow.

Proposition 2.16: A non-Zero epimorphic image of μ- semi hollow is μ- semi hollow.

Proof: Let \( h:M \rightarrow M/\mu \) be a proper finitely generated submodule of \( M \). Then \( \text{Im}(h) \) is μ- semi hollow. But the converse is not true in general. For example, Z is a submodule of the μ- semi hollow module Q, but Z is not μ- semi hollow, since \( 2Z \) is a proper submodule of Z.

Remark 2.19: The direct summand of μ- semi hollow is μ- semi hollow.

Proof: Let \( M = M_1 \oplus M_2 \) be a μ- semi hollow. Thus \( \frac{M}{M_1} \equiv M_2 \) and \( \frac{M}{M_2} \equiv M_1 \) is μ- semi hollow.

Thus \( M_1 \) and \( M_2 \) are μ- semi hollow.

By Propositions (2.10, 4).

Proposition 2.18: Let \( N \ll \mu M \) if \( \frac{M}{N} \) is μ- semi hollow. Then M is μ- semi hollow.

Proof: Let K be a proper finitely generated submodule of M with \( K+L=M \) and \( Z^*, \left( \frac{M}{L} \right) = \frac{M}{L} \). Then \( \left( \frac{K+L}{N} \right) = \left( \frac{K}{N} + \frac{L}{N} \right) \) since K is finitely generated submodule. Then \( K = R(x_1 + R(x_2 + \ldots + R(x_n + N)) \) for \( x_i \in M \) and \( i = 1, 2, \ldots , n \), and hence \( \frac{K}{N} = \frac{L}{N} \).

Thus \( \frac{K}{N} \) is finitely generated submodule of \( \frac{M}{N} \) since \( \frac{M}{N} \) is μ- semi hollow, and \( Z^*, \left( \frac{K}{N} \right) = \left( \frac{M}{L+N} \right) = \left( \frac{M}{L+N} \right) \).

Remark 2.19: In general, a submodule of μ- semi hollow module need not be μ- semi hollow. For example, Z is a submodule of the μ- semi hollow module Q, but Z is not μ- semi hollow, since 2Z is finitely generated of Z, which is not μ- small in Z.

Remark 2.20: The direct sum of μ- semi hollow in general is not μ- semi hollow.

For example, \( Q \oplus Z \) are μ- semi hollow, where p is prime number, butsQ is not μ- semi hollow.

Proposition 2.21: Let M be a module, Then M, is μ- semi hollow if every cyclic submodule of M is μ- small of M.

Proof: It is clear.

Let M be R-module, Recall that an R-module N is called M-injective. If for each monomorphism \( f:A \rightarrow M \), where A is any R-module and any homomorphism \( g:A \rightarrow N \), There exists a homomorphism \( h:M \rightarrow N \) such that \( g = h \circ f \) [8].

A ring is called V-ring if every simple R-module is injective. Equivalently R is a V-ring if \( Z^*(M) = 0 \), for all R-module M [12].
Proposition 2.22: Let $R$ be $V$- rings, Then every non-zero $R$–module is $\mu$- semi hollow.

Proof: Let $R$ be $V$-ring sand let $M$ be $R$–modules. To show that $M$ is $\mu$- semi hollow, 
Let $A$ be a proper finitely generated submodule of $M$ such that $M = A + X$, $M \not\subseteq X$ is cosingular. Since $R$ is $V$-ring. Then $Z^*(M) = 0$, for any $R$-module. Hence $M = X$. Thus $M$ is $\mu$- semi hollow.

Proposition 2.23: Let $M_1, M_2$ be $R$-modules and $M = M_1 \oplus M_2$, such that $M$ is duo
Then $M$ is $\mu$- semi hollow if $M_1, M_2$ are $\mu$-semi hollow, provided that $A \cap M_i \neq M_i, i=1, 2, \forall A \subseteq M$.

Proof: $(\Rightarrow)$ By corollary(2.17)
$(\Leftarrow)$ Let $A$ be a proper finitely generated submodule of $M$. Since $M$ is a duo modules. Then $M \cong M_i \oplus M_j$, hence each of $A \cap M_i$ and $A \cap M_j$ is a proper finitely submodule of $M_i, M_j$ respectively. It follows that $A \cap M_i \ll \mu M_i$ and $A \cap M_j \ll \mu M_j$. Hence $A \ll \mu M$ by(2.10). Thus $M$ is $\mu$- semi hollows.

Recall that an $R$- module $M$ is called distributive if for all $A,B$ and $C \subseteq M$.
$A \cap (B+C),=(A \cap B)+(A \cap C),[10]$. By similar argument one can easily prove the following Proposition.

Proposition 2.24: Let $M_1, M_2$ be $R$-module and $M = M_1 \oplus M_2$, such that $M$ is distributive modules. Then $M$ is $\mu$-semi hollow if $M_1, M_2$ are $\mu$-semi hollow, provided that $A \cap M_i \neq M_i, i=1,2, \forall A \subseteq M$.

3. $\mu$-semiregular module:
A modules $M$ is called semiregular if every cyclic submodule of $M$ is lying over summand of $M$, more generally in this section the concept at $\mu$-small submodule was considered to introduce $\mu$-semiregular module.In this case we denoted the projective $\mu$-cover for $M$ by $(P,\pi)$. 

Definition 3.1: Recall that an epimorphism $\pi: P \rightarrow M$ is called a projective $\mu$- cover of $M$. If $P$ is a projective $R$-module and $\ker(\pi) \ll \mu M$.

Remarks and Example 3.2:
1- $Q$ as $Z$-module has projective $\mu$- cover.
2- Every projective module has a projective $\mu$- covers.
3- Every projective cover is a projective $\mu$- cover. But the converse is not true in general, consider $Z_6$ as $Z_6$ module, let $N =< 2^{\infty} >$ then there exists an epimorphism $f: Z_6 \rightarrow< 2^{\infty} >, \ker(f) =< 3^{\infty} >$ which is not small in $Z_6$ but $< 3^{\infty} >$ is $\mu$-small in $Z_6$ module.

Lemma 3.3: Let $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$ be such that $f_i: P_i \rightarrow M_i$ are projective $\mu$- cover .Let $P = P_1 \oplus P_2 \oplus \ldots \oplus P_n$. Then $f = \oplus f_i: P \rightarrow M$ is a projective $\mu$- cover.

Proof: By [proposition (20.10.2)] and $\ker(f) = \oplus_{i=1}^n \ker(f_i)$ is $\mu$-small in $M$.

Theorem 3.4: Let $(P,f)$ be a projective $\mu$- cover for an $R$-module $M$. Then the following are equivalent:
1- $M$, is $\mu$-semi hollow.
2- $P$, is $\mu$-semi hollow.

Proof: $\Rightarrow$ suppose that $f: P \rightarrow M \rightarrow 0$. Then $\frac{P}{\ker\phi} \cong M$.

Since $M$ is $\mu$- semi hollow. Thus $\frac{P}{\ker\phi} \cong M$. But $\ker(f) \ll \mu P$. Therefore $P$ is $\mu$- semi hollow . By [prop. (2.18)].

$\Rightarrow$ Suppose that $P$ is a $\mu$- semi hollow, where $f: P \rightarrow M$ is a projectives$\mu$-cover, of $M$. Since $f$ is onto and $P$ is a $\mu$-semi hollow .Then $M$ is a $\mu$-semi hollow.

Proposition 3.5: Let $(P_1, f_1), (P_2, f_2)$ projective $\mu$-cover for an $R$-module $M$, if $P_1, P_2$ are costingular. Then there exists an isomo $h: P_1 \rightarrow P_2$ such that $f_2 \circ h = f_1$.

Proof:
Since $P_1$ is projective then $\exists h: P_1 \to P_2$ such that $f_2 \circ h = f_1$. We claim $P_2 = h(P_1) + \ker(f_2)$. Let $x \in P_2$ then $f_2(x) \in M \vdash f_1$ is onto then $\exists y \in P_1 \ni f_1(y) = f_2(x) \iff f_2(y) = f_2(x - h(y)) = 0 \equiv \chi = h(y) \in \ker(f_2) \iff x \in h(P_1)$.

Let $0 \neq nZ$ be submodule of $Z$ since $Z$ is an indecomposable, then $\{0\}$ is only summand of $Z$ that of $M$.

Consider $A$ submodule of $M$.

Then $\exists h: \phi: P_1 \to P_2 = h(P_1) + \ker(f_2)$, but $\ker(f_2) \ll \mu P_2$.

Consider $\pi: h(P_1)$ the natural projection since $\pi_2(Z^*(P_2)) \subseteq Z^*(\frac{P_2}{h(P_1)})$ and $Z^*(P_2) = P_2$, then $\pi_2(P_2) \subseteq Z^*(\frac{P_2}{h(P_1)}) \leq Z^*(\frac{P_2}{h(P_1)}) = Z^*(\frac{P_2}{h(P_1)})$.

We claim $h \leq \ker(f_2)$, then $\ker(h) \ll \mu P_1$, and $Z^*(\frac{P_2}{h(P_1)}) \ll \mu P_1$.

Then $P_1 = h$, thus $h$ is 1-1 and hence $h$ is an isomorphism.

Definition 3.6: A submodule $N$ of an $R$-module $M$ is called $\mu$- lying over ad (projective) summand of $M$ if there exists a decompositions $M = A \oplus B$, where $A$ is projective submodule of $N$ ands $B \cap N \ll \mu M$.

Remarks and Example 3.7:

1- It is clear that every direct summand of a projective $R$-module is a $\mu$- lie over a projective summand of $M$.

2- Every nonsingular submodule of $Z$ as $Z$-module is not $\mu$- lying over a summand of $Z$. To show that.

Let $0 \neq nZ$ be submodule of $Z$ since $Z$ is an indecomposable, then $\{0\}$ is only summand of $Z$ that contained in $nZ$, but $Z \cap nZ = nZ$ is not $\mu$- small in $Z$, there for $nZ$ is not $\mu$- lying over summand of $Z$.

3- Eversynffinitely generated of $Q$, as submodule of $Q$ is a $\mu$- lying over a projective summand of $Q$ as $Z$-module, to show that. Let $N$ be finitely generated submodule of $Q$, since $Q$ is an indecomposable, then $\{0\}$ is only summand of $Q$. That contain in $N$, it easy to show that $Q \cap N = N \ll \mu Q$. Thus $N$ is $\mu$- lying over a (projective) summand of $Q$.

As a generalization of semiregular module we introduce the followings:

Definition 3.8: An $R$- module is called $\mu$- semiregular module if every cyclic submodule of $M$ is $\mu$- lying over summand of $M$.

Remarks and Example 3.9:

1- Every semiregular is $\mu$- semiregular but the covers is not true in general. For example $Z_4$ as $Z_4$-module is $\mu$- semiregular but not semiregular since $\{0\}$ is only summand of $Z_4$.

2- The module $Z$ as $Z$-module. $Z$ is not $\mu$- semiregular.

3- Consider the module $Q$ as $Z$- module. Then $Q$ is $\mu$- semiregular to see that: let $x \in Q$.

$Q = \{0\} \oplus Q$, where $\{0\}$ is a projective submodule of $Q$ ands $Q \cap \{0\} = \{0\} \ll \mu Q$.

4- A submodule of a $\mu$-semiregular $R$-module need not be $\mu$-semiregular (see 2) and 3).

Theorem 3.10: Let $M_1, M_2$ be $R$-modules and let $M = M_1 \oplus M_2$, such that $M$ is a duo modules. Then $M_1, M_2$ are $\mu$-semiregular if and only if $M$ is $\mu$-semiregular.

Proof: $(\Rightarrow)$ Let $R_x \leq M$, then $R_x = R_x \cap M_1 \oplus R_x \cap M_2$, $R_x \cap M_1 \leq M_1 \forall i = 1, 2$, since $M_1, M_2$ are $\mu$-semiregular, then $\exists$ projective submodules $A_i \leq M_i$ $\forall i = 1, 2$, such that $M_i = B_i \oplus A_i$ and $(R_x \cap M_i) \ll \mu M_i$.

$(B_1 \oplus B_2) \oplus (A_1 \oplus A_2)$ is a direct sum of $M_1 \oplus M_2$, $A_1 \oplus A_2$ is projective $R_x \cap (B_1 \oplus B_2) = (R_x \cap M_1) \oplus (R_x \cap M_2) \cap (B_1 \oplus B_2) = (R_x \cap M_1) \cap (B_1 \oplus B_2) \cap (B_2) \ll \mu M$ (by prop. 2.10, 3).

Conversely, Let $R_x \leq M_1$, then $R_x \leq M_1$, since $M$ is $\mu$-semiregular, then $\exists A \leq R_x$ such that $M = A \oplus B$ and $B \cap R_x \ll \mu M_1 \vdash M_1 = (A \oplus B) \cap M_1 = A \oplus (B \cap M_1)$.

$R_x \leq B \cap R_x \ll \mu M$. $(B \cap M_1) \cap R_x \ll \mu M$ (by prop. 2.10, 3).
Since \((B \cap M_1) \cap R_{x_1} \leq M_1 \leq M\) and \(M_1\) is a direct summand of \(M\) then \((B \cap M_1) \cap R_{x_1} \ll_{\mu} M_1\).

Similarly, we can prove that \(M_2\) is \(\mu\)-semiregular.

**Definition 3.11:** Let \(M\) be an \(R\)-module is called \(\mu\)-lifting module or \((D_1)\)-module. If for every submodule \(N\) of \(M\), there exists a decomposition \(M = A \oplus B\), where \(A \subseteq N\) and \(N \cap B \ll_{\mu} B\). [11].

**Note:** A \(\mu\)-lifting may be not \(\mu\)-semiregular module, consider the module \(Z_4\) as \(Z - \text{module}\) is \(\mu\)-lifting, it is clear that \(Z_4\) is not projective and indecomposable \(Z_4\) is not \(\mu\)-lying over summand. However \(Z_4\) is not \(\mu\)-semiregular.

**Proposition 3.12:** Let \(M\) be \(\mu\)-semiregular. Then \(M\) is \(\mu\)-lifting module.

**Proof:** Let \(N \leq M\). Since \(M\) is \(\mu\)-semiregular, then there exists a decompositions \(M = A \oplus B\), where \(A\) is a projective submodule of \(N\) and \(N \cap B \ll_{\mu} M\), but \(N \cap B \leq B\) and \(B\) is summand of \(M\), therefore \(N \cap B \ll_{\mu} B\) by (prop 2.11).

Thus \(M\) is \(\mu\)-lifting.

**Remark 3.13:** Every projective \(\mu\)-lifting \(M\) is \(\mu\)-semiregular.

**Proof:** let \(R_x\) be submodule of \(M\), since \(M\) is \(\mu\)-lifting, then \(M = A \oplus B\), where \(A \leq R_x\) and \(R_x \cap B \ll_{\mu} M\), since \(M\) is projective and \(A\) is summand of \(M\). Then \(A\) is projective. But \(B \cap R_x \ll_{\mu} M\), therefore \(B \cap R_x \ll_{\mu} M\), and hence \(R_x\) is \(\mu\)-lie over projective summand of \(M\), thus \(M\) is \(\mu\)-semiregular.

**Proposition 3.14:** Let \(M\) be an indecomposable \(R\)-module, if \(M\) is \(\mu\)-semiregular. Then \(M\) is \(\mu\)-semi hollow. The converse is true if \(M\) is projective.

**Proof:** Let \(R_x\) be proper submodule of \(M\), since \(M\) is \(\mu\)-semiregular, then \(M = A \oplus B\), where \(A\) is projective submodule of \(R_x\) and \(B \cap R_x \ll_{\mu} M\), by modular law. \(R_x = R_x \cap M = R_x \cap (A \oplus B) = A \oplus (R_x \cap B)\). But \(M\) is indecomposable and \(A\) is proper summand of \(M\). Therefore \(A = 0\), and hence \(R_x = R_x \cap B \ll_{\mu} M\), thus every cyclic submodule is \(\mu\)-small. Thus \(M\) is \(\mu\)-semi hollow.

Conversely, assume that \(M\) is projective, let \(R_y\) be cyclic submodule of \(M\), assume that \(R_y\) is proper submodule of \(M\) and \(\text{then} R_y\) is \(\mu\)-small. So, \(M = \{0\} \oplus M\), where \(\{0\}\) is projective of \(R_y\) and \(M \cap R_y = R_y \ll_{\mu} M\). Now if \(R_y = M\), \(M = \{0\} \oplus M\), is projective and \(\{0\} \ll_{\mu} M\). Thus \(M\) is \(\mu\)-semiregular.

**Proposition 3.15:** Let \(M\) be a projective \(R\)-module and \(M\) be \(\mu\)-semiregular. Then \(M\) has a projective \(\mu\)-cover for every cyclic submodule \(N\) of \(M\).

**Proof:** Let \(N\) be a cyclic submodule of \(M\), since \(M\) is \(\mu\)-semiregular, then there exists a projective submodule \(A\) of \(N\) such that \(M = A \oplus B\) and \(N \cap B \ll_{\mu} M\). Now consider the following short exact sequence:

\[
0 \to (N \cap B) \to B \xrightarrow{\pi} (N \cap B) \to 0,
\]

where \(i\) is the inclusion map and \(\pi\) is the naturals epimorphism. By the second isomorphism theorem

\[
\frac{M}{N} \cong \frac{(N+\oplus)}{N \oplus B} \cong \frac{B}{N \oplus B}.
\]

Since \(M\) is projective and \(B\) is a summand of \(M\), then \(B\) is projective. But \(N \cap B \ll_{\mu} M\). And \(N \cap B \leq M\), \(B\) is direct summand of \(M\), then \(\ker \pi = N \cap B \ll_{\mu} B\), therefore \((B, \pi)\) is a \(\mu\)-projective cover for \(\frac{B}{N \cap B}\), hence \(\frac{M}{N}\) has a projective \(\mu\)-cover.

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