Operation of weaving partial Steiner triple systems

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Abstract

We introduce an operation of a kind of product which associates with a partial Steiner triple system another partial Steiner triple system, the starting one being a quotient of the result. We discuss relations of our product to some other product-like constructions and prove some preservation/non-preservation theorems. In particular, we show series of anti-Pasch Steiner triple systems which are obtained that way.

Key words: convolution (of a partial Steiner triple system and a group), Veblen (= Pasch) configuration, (partial) Steiner triple system, Desargues configuration, Fano configuration, (finite) affine space, affine slit space.

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Introduction

In the paper we introduce the operation of weaving, which associates with a partial Steiner triple system (shortly: with a PSTS) \( \mathcal{M} \) a “product” \( \oplus^m \mathcal{M} \) of \( \mathcal{M} \) and a cyclic group of order \( m \) in such a way that a quotient of \( \oplus^m \mathcal{M} \) wrt. to a congruence \( \approx \) is \( \mathcal{M} \), and the coimage of a line of \( \mathcal{M} \) under natural projection \( \oplus^m \mathcal{M} \longrightarrow \oplus^m \mathcal{M} / \approx \) is a generalization of the Pappus configuration. Clearly, these properties do not characterize the operation of weaving uniquely, and several constructions which have these properties can be found in the literature, just to mention the operation of convolution and the “product” defined in [4], of an STS with a parallel class distinguished and an abelian group.

The notion of convolution was introduced in [15] (in a slightly less general way than this adopted in this paper), though it was used, implicitly, e.g. in [6], [2], [8] (comp. also [7]). Both in the construction of the convolution and the construction of \( m \)-th weaved configuration applied to a partial Steiner triple system \( \mathcal{M} \) the constructed points are “weighted” points of \( \mathcal{M} \) i.e. pairs of the form \( a_i = (a, i) \), where \( a \) is a point of \( \mathcal{M} \) and \( i \) is an element of a fixed (in case of weaving – cyclic) group \( G \). The lines are sets of triples of weighted points on lines of \( \mathcal{M} \) whose weights satisfy certain conditions. One can note an analogy between these constructions and the product construction. An analogy only, since the triples of weights on lines of the constructed configuration need not yield any PSTS defined on \( G \). Examples of classical configurations that are convolutions were already quoted in [14]; these are e.g. the Veblen configuration (also called the Pasch configuration), the Reye configuration, the Pappus configuration. Some of them are also weaved configurations. Some of them are also members of another family, family of configurations...
presentable as cyclically inscribed triangles\(^1\) (comp. \[10\]). In this paper we do not study this family on its own but its members play an important role in characterizations of weaved configurations (cf. Lem. \[3.2\] and its consequences). The Pappus configuration is an important example of a configuration which is in each of these three families.

We start with establishing general properties of weaved configurations. The operation of weaving destroys most of classical configurations based on the Veblen configuration; in particular it gives Desargues-free (Prop. \[2.9\]) and Fano-free (Prop. \[2.6\]) configurations. They are also miter-free (Prop. \[2.13\]). The operation of weaving applied to a Pasch-free configuration yields a Pasch-free configuration (Cor. \[2.5\]). In particular, when applied to a Pasch-free STS it produces a Pasch-free PSTS which has a unique, Pasch-free, completion to an STS (Rem. \[4.11\]).

Still, geometry of an \(m\)-th weaved configuration is relatively easy to understand. In particular, it is easy to determine the triangles and cliques in it (Lem. \[2.2\]), to characterize some characteristic subconfigurations (being direct generalizations of the Pappus configuration, Prop. \[2.15\], the proof of \[3.2\]), and to characterize (for \(m > 3\)) its automorphism group (Thm. \[3.6\]).

After proving general properties of weaving operation, in the last section we show some applications of the obtained results. In particular, we get a method to obtain a class of STS’s with parameters of an affine space over \(GF(3)\), which are not embeddable into any affine space over \(GF(3)\) and which are Pasch-free (Prop. \[4.12\]). Each line of the resulting configuration can be extended to an affine (sub)plane, and planes are their maximal subspaces which are affine.

## 1 Definitions and representations

Let \(\mathfrak{M} = (S, \mathcal{L})\) be a partial Steiner triple system. Let \(m > 2\) be an integer. Write \(X := S \times C_m^3\), and \(\mathcal{C} = \{(i, j, k) \in C_m^3 : i = j = k - 1 \lor i = k = j - 1 \lor j = k = i - 1\}\). Finally, we define

\[
\mathcal{L}_e := \left\{ \{(a, i), (b, j), (c, k)\} : \{a, b, c\} \in \mathcal{L}, (i, j, k) \in \mathcal{C}\right\},
\]

\[
\otimes^m \mathfrak{M} := \langle X, \mathcal{L}_* \rangle.
\]

The structure \(\otimes^m \mathfrak{M}\) will be called a configuration weaved from \(\mathfrak{M}\) (more precisely, the configuration \(m\)-weaved from \(\mathfrak{M}\)).

Recall a similar construction of the convolution \(\mathfrak{M} \bowtie_e G\) of an abelian group \(G = \langle G, 0, + \rangle\) and a partial Steiner triple system \(\mathfrak{M}\) (cf. \[15\]). Let \(X := S \times G\), \(e \in G\), and \(G_e = \{ (\alpha, \beta, \gamma) \in G^3 : \alpha + \beta + \gamma = e \}\). We set

\[
\mathcal{L}_e := \left\{ \{(x, \alpha), (y, \beta), (z, \gamma)\} : \{x, y, z\} \in \mathcal{L}, (\alpha, \beta, \gamma) \in G_e\right\},
\]

\[
\mathfrak{M} \bowtie_e G := \langle X, \mathcal{L}_e \rangle.
\]

**Fact 1.1** (comp. \[14\]). Let \(e, f \in G\) and \(f \in \text{Aut}(G)\). Then \(\mathfrak{M} \bowtie_e G \cong \mathfrak{M} \bowtie_{e+3e} G \cong \mathfrak{M} \bowtie_{f(e)} G\). In particular, \(\mathfrak{M} \bowtie_0 C_2 \cong \mathfrak{M} \bowtie_1 C_2\).

\(^1\)This family contains all (i.e. three) \(9_3\)-configurations.
Weaved partial STS’s

The choice of the value ‘1’ in the definition of the lines in \( L_\ast \) may seem arbitrary, and one can consider the class \( \{(a, i), (b, i), (c, i + \varepsilon)\} \) with fixed \( \varepsilon \) instead. However, the obtained configuration may stay disconnected, and its connected components are isomorphic to \( \otimes^k \mathcal{M} \) where \( k \) is the rank of \( \varepsilon \) in \( C_m \).

Geometry of the convolution \( \mathcal{M} \otimes_\varepsilon G \) may depend on \( \varepsilon \), though.

The structures of the form \( \mathcal{M} \otimes_0 G \) were studied in [15] in much detail. We write, shortly, \( \mathcal{M} \otimes G \) instead of \( \mathcal{M} \otimes_0 G \). Basic parameters of the structures defined above are easy to compute.

**Fact 1.2.** Let \( \mathcal{M} \) be a \((\nu, r, b_3)\)-configuration. Then \( \otimes^m \mathcal{M} \) is a \((m\nu, 3m\nu r, 3m\nu b_3)\)-configuration and \( \mathcal{M} \otimes_\varepsilon G \) is a \((\nu, r, \gamma^2 b_3)\)-configuration, where \( \gamma \) is the rank of \( G \) in \( C_m \).

Let us begin with some evident examples.

**Example 1.3.** Let \( \mathcal{T} \) be a single-line structure i.e. let \( |S| = 3 \) and \( \mathcal{T} = \langle S, \{S\} \rangle \). Then \( \otimes^3 \mathcal{T} \) is a series of cyclically inscribed triangles, as considered in [10]. In particular, \( \otimes^3 \mathcal{T} \) is the Pappus configuration. It is known (cf. [15]) that \( \mathcal{T} \otimes_0 C_3 \) is the Pappus configuration as well.

**Remark 1.4.** For each partial Steiner triple system \( \mathcal{T} \) we have \( \otimes^3 \mathcal{T} \cong \mathcal{T} \otimes_1 C_3 \) for each \( \mathcal{M} \).

**Example 1.6.** There is no configuration \( \mathcal{M} \) and no element \( \varepsilon \) of any group \( G \) such that \( \otimes^5 \mathcal{T} \cong \mathcal{M} \otimes_\varepsilon G \).

**Example 1.7.** \( \otimes^5 \mathcal{T} \not\cong \mathcal{T} \otimes C_3 \) and \( \otimes^5 (\mathcal{T} \otimes C_3) \) for each partial Steiner triple system \( \mathcal{M} \).

Finally, let us recall a few definitions from the general theory of (partial) Steiner triple systems. With each partial Steiner triple system \( \mathcal{M} = \langle S, \mathcal{L} \rangle \) we associate the partial binary operation \( \ominus \) defined on \( S \) by the conditions:

\[
 p \ominus q := \begin{cases} 
 p & \text{when } p = q \\
 r & \text{when } \{p, q, r\} \in \mathcal{L}.
\end{cases}
\]

Let \( \Delta = \{p, q, r\} \) be a (nondegenerate) triangle in \( \mathcal{M} \), i.e. let it be a triple of pairwise collinear points not on a line. We set

\[
\Delta' := \{p \ominus q, q \ominus r, r \ominus p\}, \quad \Delta^{(n+1)} := \Delta^{(n)}',
\]

for each integer \( n \) such that the points in \( \Delta^{(n)} \) are pairwise collinear. The structure \( \mathcal{M} \) is called Moufangian iff \( \Delta' \) is a line of \( \mathcal{M} \) for every triangle \( \Delta \). The algebraic
counterpart of this property expressed in terms of the partial algebra \((S, \circ)\) is read as the known Moufang axiom:
\[
(p \circ q) \circ (p \circ r) = q \circ r
\]
valid for every triangle \(\{p, q, r\}\). Note that the Moufang property implies the Veblen axiom.

## 2 General, subconfigurations

In most parts proofs of evident statements are omitted.

Let us fix a partial Steiner triple system \(M = \langle S, \mathcal{L} \rangle\).

**Lemma 2.1.** Distinct points \((a, i), (b, j)\) of \(\oplus^m M\) are collinear iff \(a, b\) are distinct and collinear in \(M\) and \(j \in \{i, i+1, i-1\}\).

**Lemma 2.2.** Let \(\Delta\) be a triangle in \(M = \oplus^m M\). Then one of the following holds.

(i) \(\Delta = \{(a, i), (b, i), (c, i)\}\) for \(i \in C_m\) and a triangle \(\Delta_0 := \{a, b, c\}\) of \(M\).

The set \(\Delta'\) is a triangle in \(M\) if \(\Delta'_0\) is either a triangle or a line of \(M\) (i.e. \(\Delta_0\) is Moufangian). In the second case the (periodic) series \(\Delta', \Delta'', \ldots, \Delta^{(m)}, \Delta^{(m+1)} = \Delta'\) consists of triangles distinct from \(\Delta\). In any case \(\Delta'\) is not a line of \(M\).

Let \(\Delta\) be a triangle of the form \(\text{[III]}\) and let \(\Delta_0\) be a triangle as well. Then \(\Delta, \Delta', \ldots, \Delta^{(m-1)}, \Delta^{(m)} = \Delta\) is a periodic series iff there is an integer \(m_0\) such that \(\Delta_0, \Delta'_0, \ldots, \Delta^{(m_0-1)}_0, \Delta^{(m_0)}_0 = \Delta_0\) is a periodic series of triangles in \(M\) and \(m_0\) divides \(m\).

(ii) \(\Delta = \{(a, i), (b, i), (c, i)\}\) for \(i \in C_m\) and a line \(\{a, b, c\}\) of \(M\). Then the (periodic) series \(\Delta, \Delta', \ldots, \Delta^{(m-1)}, \Delta^{(m)} = \Delta\) consists of triangles.

(iii) \(\Delta = \{(a, i), (b, i), (c, i-1)\}\) for \(i \in C_m\), where \(\{a, b, c\}\) is a line of \(M\). Then \(\Delta'\) is not a line. For \(m \neq 3\) it is not a triangle (it consists of a pair of collinear points \(p, q\) and a point \(r\) not collinear with any of \(p, q\)). For \(m = 3\), \(\Delta'\) is a triangle.

(iv) \(\Delta = \{(a, i), (b, i), (c, i-1)\}\) for \(i \in C_m\), where \(\Delta_0 = \{a, b, c\}\) is a triangle of \(M\). Then \(\Delta'\) is not a line. If \(m \neq 3\) then \(\Delta'\) is not a triangle: it has either the form of \(\text{[III]}\) or consists of a triple of pairwise noncollinear points. If \(m = 3\) then \(\Delta'\) is a triangle iff the points in \(\Delta'_0\) are pairwise collinear.

(v) \(\Delta = \{(a, i), (b, i), (c, i+1)\}\) for \(i \in C_m\) and a triangle \(\Delta_0 := \{a, b, c\}\) of \(M\). Then \(\Delta'\) is a line of \(M\) iff \(\Delta'_0\) is a line of \(M\) (i.e. \(\Delta_0\) yields a Veblen figure in \(M\)). If \(\Delta_0\) is a triangle then \(\Delta'\) is a triangle as well.

(vi) \(\Delta = \{(a, i), (b, i+1), (c, i+2)\}\) for \(i \in C_m\) and a triangle \(\{a, b, c\}\) of \(M\). The sequence \(\Delta^{(j)}\) is as in \(\text{[III]}\).

(vii) \(\Delta = \{(a, i), (b, i+1), (c, i+2)\}\) for \(i \in C_m\) and a line \(\{a, b, c\}\) of \(M\). The sequence \(\Delta^{(j)}\) is as in \(\text{[III]}\).

If \(m \neq 3\) then \(M\) does not contain triangles of type \(\text{[VIII]}\) and \(\text{[VII]}\). If \(m = 3\) then triangles of these types may occur. If \(\Delta\) has type \(\text{[VII]}\) then there is in \(M\) a triangle \(\Delta_1\) of type \(\text{[III]}\) such that \(\Delta \cup \Delta' \cup \Delta'' = \Delta_1 \cup \Delta'_1 \cup \Delta''_1\).

**Remark 2.3.** Assume that \(M\) contains a triangle \(\Delta_0\) such that \(\Delta_0, \Delta'_0, \ldots, \Delta^{(m_0-1)}_0, \Delta^{(m_0)}_0 = \Delta_0\) is a periodic series of distinct triangles for some integer \(m_0\). Let \(n = \text{LCM}(m_0, m)\). Then \(\oplus^n M\) contains a triangle \(\Delta\) such that a series of distinct triangles \(\Delta, \Delta', \ldots, \Delta^{(n-1)}, \Delta^{(n)}\) exists and \(\Delta^{(n)} = \Delta\). Indeed, apply the construction of \(\text{[III]}\). The construction of \(\text{[VIII]}\) yields a series with \(n = m_0\).
The only Veblen subconfigurations of $\oplus^m M$ are determined by triangles characterized in 2.2. As a direct consequence we get

**Corollary 2.4.** A Veblen configuration contained in $\oplus^m M$ has form

$$(a, i), (b, i), (c, i + 1), (a \odot c, i), (b \odot c, i), (a \odot b, i + 1),$$

where a triangle $\{a, b, c\}$ yields a Veblen subconfiguration in $M$ and $i \in C_m$.

**Corollary 2.5.** If $M$ does not contain any Veblen subconfiguration (is Pasch-free) then $\oplus^m M$ is Pasch-free as well.

An important consequence of 2.4 is

**Proposition 2.6.** The structure $\oplus^m M$ does not contain any Fano subconfiguration. Actually, it is anti-Fano: no three diagonal points of a quadrangle contained in $\oplus^m M$ are on a line.

**Proof.** Suppose that $\Delta$ is a triangle in $\oplus^m M$ which spans a Fano configuration. In particular, $\Delta$ yields a Veblen configuration so, in view of 2.4, $\Delta = \{(a, i), (b, i), (c, i + 1)\}$ where $\{a, b, c\}$ is a triangle in $M$ and $i \in C_m$. Write $b' = a \odot c, a' = b \odot c$, and $c' = a \odot b$, Then a quadrangle which spans a Fano plane has form $Q = \{(a, i), (b, i), (a', i), (b', i)\}$, provided the points $a', b', c'$ are collinear in $M$. Two of the diagonal points of $Q$ are $(c, i + 1)$ and $(c', i + 1)$. To get the third diagonal point of $Q$ we need a common point $p$ of the lines through $a, a'$ and through $b, b'$. But then this third diagonal point of $Q$ is $(p, i + 1)$, and the points $(c, i + 1), (c', i + 1)$, and $(p, i + 1)$ are never on a line of $\oplus^m M$. □

**Remark 2.7** (ad the proof of 2.6). Note that if the quadrangle $a, b, a', b'$ yields a Fano plane then the diagonal points of the quadrangle $Q$ yield a nondegenerate triangle in $\oplus^m M$.

**Corollary 2.8.** Assume that $M$ contains a Fano subconfiguration. Then $\oplus^m M$ cannot be embedded to any projective space $PG(n, p)$ with even $p$ and $n \geq 2$.

In an analogous fashion we get

**Proposition 2.9.** The structure $\oplus^m M$ does not contain any Desargues subconfiguration. Actually, it is anti-Desarguesian: no three focuses of two perspective triangles contained in $\oplus^m M$ are collinear.

**Proof.** In view of 2.2, two triangles which have a perspective center such that their focuses exist (i.e. corresponding sides of the triangles intersect in pairs) have form $T_1 := ((b, i), (c, i), (d, i))$ and $T_2 := ((a \odot b, i), (a \odot c, i), (a \odot d, i))$, where $(b, c, d)$ and $(a \odot b, a \odot c, a \odot d)$ is a pair of triangles of $M$ with a perspective center $a$. Then $(a, i + 1)$ is the perspective center of $T_1$ and $T_2$. The focuses of $T_1$ and $T_2$ are the points $(b \odot c, i + 1), (c \odot d, i + 1)$, and $(d \odot b, i + 1)$. These points are not collinear. □

**Remark 2.10** (ad the proof of 2.9). Note that if the triangles $(b, c, d)$ and $(a \odot b, a \odot c, a \odot d)$ yield a Desargues configuration i.e. their three focuses colline then the focuses of $T_1$ and $T_2$ yield a nondegenerate triangle in $\oplus^m M$.

As a direct consequence of 2.9 and, in particular, 2.10 we get
Corollary 2.11. Assume that \( \mathfrak{M} \) contains a Desargues subconfiguration. Then \( \odot^m \mathfrak{M} \) cannot be embedded to any Desarguesian projective space.

Consequently, neither \( \odot^m G_2(X) \) for any \( X \) with \( |X| \geq 5 \) nor \( \odot^m PG(n, 2) \) for any \( n \geq 3 \) can be embedded into a Desarguesian projective space for any \( m \geq 3 \).

In essence, 2.9 can be generalized to a wider class of 103-configurations.

Remark 2.12. Let \( \mathfrak{A} \) be a 103-configuration that contains a Veblen subconfiguration. The structure \( \odot^m \mathfrak{A} \) does not contain any subconfiguration isomorphic to \( \mathfrak{A} \) for any \( m > 3 \) and any partial Steiner triple system \( \mathfrak{M} \).

Proof. In accordance with \("\) there are exactly 6 configurations \( \mathfrak{A} \) of the form considered in 2.12 and each one can be presented as a "closure" of a \( K_4 \)-graph. That means \( \mathfrak{A} \) contains a \( K_4 \)-graph and "third points" on the edges of this graph yield a Veblen subconfiguration. Suppose that \( \mathfrak{B} := \odot^m \mathfrak{M} \) contains \( \mathfrak{A} \). Let \( \mathfrak{B} \) be the respective Veblen subconfiguration of \( \mathfrak{A} \) with the points \((c, i_0), (a, i_0), (b, i_0)\) (on a line) and \((a, i_0), (b, i_0), (c, i_0 + 1)\) (a triangle). The edges of \( \mathfrak{B} \) which pass through \((c, i_0 + 1)\) have form \((x, i_0), (y, i_0)\) \((1)\) or \((x, i_0 + 1), (y, i_0 + 2)\) \((2)\), and through \((r, i_0 + 1)\) have form \((z, i_0), (t, i_0)\) \((3)\) or \((z, i_0 + 1), (t, i_0 + 2)\) \((4)\) resp. Suppose \((1) \& (3)\). Without loss of generality we can assume \( x \neq t \) and then \((x, i_0) \odot (z, i_0) = (x \odot z, i_0 + 1)\) is another point of \( \mathfrak{B} \), which is impossible. Other cases are considered analogously.

Another configuration frequently considered in combinatorics of STS’s is the miter configuration (cf. e.g. \([2]\), where anti-miter STS’s were studied). A triangle \( \{a, b, c\} \) determines a miter configuration with the center \( a \) when the equality \( a \odot (b \odot c) = (a \odot b) \odot (a \odot c) \) holds, and the configuration in question consists of the points \( a, b, c, a \odot b, a \odot c, b \odot c, a \odot (b \odot c) \).

Proposition 2.13. The structure \( \odot^m \mathfrak{M} \) does not contain any miter-configuration.

Proof. Analyzing all the possibilities given in 2.2 we check that no triangle in \( \odot^m \mathfrak{M} \) may satisfy equation \([9]\).

Corollary 2.14. Neither the M"{o}bius 83-configuration (cf. \([9]\)) nor the affine plane \( AG(2, 3) \) can be embedded into a weaved configuration \( \odot^m \mathfrak{M} \).

Proof. It suffices to note that the 83-configuration results from \( AG(2, 3) \) by removing a point and all the lines through it, and the miter configuration results from the 83-configuration by omitting a point and all the lines through it.

In the sequel we need criterions which enable us to distinguish triangles of form \( 2.2(0) \) and those of form \( 2.2(1) \). To this aim we must recall a fragment of \([10]\). Let us start with a naive approach. Consider a triangle \( \Delta^{(0)} \). Inscribe a triangle \( \Delta^{(1)} \) into \( \Delta^{(0)} \). Inductively, inscribe a triangle \( \Delta^{(i+1)} \) into \( \Delta^{(i)} \). Continue this procedure \( (m-1) \) times so as a triangle \( \Delta^{(m-1)} \) is obtained. Finally, inscribe \( \Delta^{(0)} \) into \( \Delta^{(m-1)} \).
The obtained configuration is uniquely determined by a permutation \( \gamma \) of \( C_3 \) so as (up to an isomorphism) the points of the arising configuration denoted by \( \Pi^m \) are the elements of \( C_3 \times C_m \) and the lines of \( \Pi^m \) are the sets \( \{(a, i), (b, i), (c, i+1)\} \) for all the triples \( a, b, c \) such that \( C_3 = \{a, b, c\} \) and \( i = 0, 1, \ldots, m-2 \), and the sets \( \{(a, m-1), (b, m-1), (c, 0)\} \) with \( C_3 = \{a, b, c\} \) (cf. [10], slightly modified). For fixed integer \( m \) there are up to an isomorphism exactly three configurations of the form \( \Pi^m \): with \( \gamma = \text{id} \), \( \gamma = \tau_1 \), and \( \gamma = \sigma_0 \) \((\tau_1 \) is the translation on \( 1: \tau_1(a) = a+1 \), and \( \sigma_0 \) is the reflection in \( 0: \sigma_0(a) = -a \). Recall also that there are exactly three \((9_39_3)\)-configurations and these are \( \Pi^3_{\text{id}} \) (= the Pappus configuration), \( \Pi^3_{\tau_1} \), and \( \Pi^3_{\sigma_0} \). The following simple observation shows a close connection between weaved configurations and series of inscribed triangles (cf. [2.2][11]).

\[ \Pi^m_{\text{id}} \cong \oplus^m 3 \] for each integer \( m \geq 3 \). \hfill (10)

Contrary to [2.6] and [2.9] from [2.2][11] we have immediately

**Proposition 2.15.** If \( \mathcal{M} \) contains \( \Pi^k_\gamma \) for some permutation \( \gamma \) of \( C_3 \) and some integer \( k \) then \( \oplus^m \mathcal{M} \) also contains \( \Pi^k_\gamma \). In particular, if \( \mathcal{M} \) contains a Pappus subconfiguration then \( \oplus^m \mathcal{M} \) also contains a Pappus subconfiguration (comp. [2.14]).

**Proof.** Let \( \Delta_0 = \{a, b, c\} \) be a triangle of \( \mathcal{M} \) which determines a cyclic series \( \Delta_0^1, \ldots, \Delta_0^{(k)} = \Delta_0 \) of inscribed triangles such that \( \mathcal{X} = \bigcup \{\Delta^{(j)}_0: j = 0, \ldots, k-1\} \) yields the \( \Pi^k_\gamma \) subconfiguration of \( \mathcal{M} \). Take any \( i \in C_m \) and set \( \Delta = \{(a, i), (b, i), (c, i+1)\} \). It is seen that the triangle \( \Delta \) yields in \( \oplus^m \mathcal{M} \) a cyclic series of length \( k \) of inscribed triangles such that the set \( \bigcup \{\Delta^{(j)}: j = 0, \ldots, k-1\} \subset \mathcal{X} \times \{i, i+1\} \) yields the \( \Pi^k_\gamma \) subconfiguration of \( \oplus^m \mathcal{M} \).

With a bit more subtle analysis we can also prove

**Remark 2.16.** Assume that \( \mathcal{M} \) satisfies the projective Pappus axiom. Then every three diagonal points of a hexagon of \( \oplus^m \mathcal{M} \) inscribed into two lines, are on a line.

**Proof.** Let \( p_1, \ldots, p_6 \) be a hexagon of \( \oplus^m \mathcal{M} \) inscribed into two lines; i.e. assume that \( \{p_1, p_3, p_5\} \) and \( \{p_2, p_4, p_6\} \) are two lines. Let the corresponding diagonal points be: \( q_1 \) on \( p_1, p_2, p_4, p_5 \), \( q_2 \) on \( p_2, p_3, p_5, p_6 \), and \( q_3 \) on \( p_3, p_4, p_6, p_1 \). Then \( p_2, p_3, p_4 \) is a triangle inscribed into the triangle \( p_1, p_5, q_1 \), and \( q_2, p_6, q_3 \) are third points on the sides of the triangle \( p_2, p_3, p_4 \). Analyzing possible ways in which series of inscribed triangles may be obtained, with the help of [2.2] we note that the triples \( p_5, p_6, q_2 \) and \( p_1, p_6, q_3 \) are collinear only in case \( \bigcirc \) and in that case \( p_1 = (a_1, i), p_2 = (a_2, i), p_3 = (a_3, i+1), p_4 = (a_4, i), p_5 = (a_5, i), p_6 = (a_6, i+1) \), where \( a_1, \ldots, a_6 \) is a hexagon in \( \mathcal{M} \) with the diagonal points \( b_1, b_2, b_3 \) such that \( q_1 = (b_1, i+1), q_2 = (b_2, i), q_3 = (b_3, i) \). Now the claim is evident.

### 3 Automorphisms

**Lemma 3.1.** Let \( u \in C_m \) and \( f \in \text{Aut}(\mathcal{M}) \). Then the map \( f \times \tau_u: S \times C_m \ni (a, i) \mapsto (f(a), i + u) \) is an automorphism of \( \oplus^m \mathcal{M} \).

As a simple consequence of [2.2] we get
Lemma 3.2. Assume that no one of the following configurations is contained in \( \mathcal{M} \):

(i) \( \Pi_{\alpha_0}^{m_0} \), where \( m_0 \mid m \),

(ii) \( \Pi_{\sigma_0}^{m_0} \), where \( (2m_0) \mid m \), and

(iii) \( \Pi_{\tau_0}^{m_0} \), where \( (3m_0) \mid m \).

The family of sets \( L \times C_m \) with \( L \) ranging over the lines of \( \mathcal{M} \) is definable in terms of the geometry of \( \mathcal{M} \).

Proof. We need to provide an analysis of triangles slightly more subtle than in \( 2.2 \). With a triangle \( \delta = (a,b,c) \) (a sequence, not a set!) we associate the sequence \( \delta' = (a \circ b, b \circ c, c \circ a) \). As in \( 7 \) we introduce the symbols \( \delta^{(i)} \). We claim that the following conditions are equivalent

(a) \( \mathcal{X} = L \times C_m \) for a line \( L \) of \( \mathcal{M} \),

(b) \( \mathcal{X} = \Delta \cup \Delta' \cup \ldots \cup \Delta^{(m-1)} \) for a triangle \( \Delta = \{p,q,r\} \) of \( \mathcal{M} \) and \( \delta = (p,q,r) \) such that \( \Delta', \ldots, \Delta^{(m-1)} \) consists of distinct triangles, \( (\Delta^{(m-1)})' = \Delta^{(m)} = \Delta \), and \( \delta^{(m)} = \delta \).

for any set \( \mathcal{X} \) of points of \( \mathcal{M} \).

Implication \( (i) \Rightarrow (ii) \) is a direct consequence of \( 2.2 \). Assume \( (i) \) and suppose that \( (ii) \) is not valid; in view of \( 2.2 \) we get that one of the following holds: \( 2.2 \text{[i]}, 2.2 \text{[w]} \), or \( m = 3 \) and \( 2.2 \text{[v]}, 2.2 \text{[w]}, 2.2 \text{[x]} \), or \( 2.2 \text{[y]} \).

In the first four cases \( \Delta \) is associated with a triangle \( \Delta_0 = \{a,b,c\} \) of \( \mathcal{M} \). Write \( \delta_0 = (a,b,c) \). It is seen that \( \Delta^{(i+1)} \) can be considered as a triangle inscribed into \( \Delta^{(i)} \). From \( (i) \) we get that \( \Delta^{(m)}_0 = \Delta_0 \). Let \( m_0 \) be the least integer with \( \Delta^{(m_0)}_0 = \Delta_0 \).

Suppose, first, that \( 2.2 \text{[ii]} \) holds. Clearly, \( m_0 \mid m \) i.e. \( m = m_0 k \) for some integer \( k \). There is a permutation \( \gamma \) of \( C_3 \) such that \( \delta_0^{(m_0)} = (\gamma(a), \gamma(b), \gamma(c)) \) and then the points of \( \bigcup\{\Delta_0^{(i)} : i = 0, \ldots, m_0 - 1\} \) yield in \( \mathcal{M} \) the configuration \( \Pi_{\alpha_0}^{m_0} \). From assumption, \( \delta^{(m)}_0 = \delta_0 \) and thus \( \gamma^k = \text{id} \). If \( \gamma \) is a translation then \( 3 \mid k \) and if \( \gamma \) is a reflection then \( 2 \mid k \), which contradicts assumptions.

Next, suppose that \( 2.2 \text{[iv]} \) holds. In that case from \( (i) \) we get \( m_0 = m \), and then \( \delta^{(m)} = \delta \) yields that \( \mathcal{M} \) contains \( \Pi_{\alpha_0}^{m_0} \), which is impossible.

Now, let \( m = 3 \). Assume that \( 2.2 \text{[v]} \) or \( 2.2 \text{[vi]} \) holds. Clearly, \( m_0 \mid m \) and thus \( m_0 = 3 \). From \( \delta^{(3)} = \delta \) we get that \( \Pi_{\alpha_0}^{m_0} \) is contained in \( \mathcal{M} \), which contradicts assumptions.

Finally, consider the last two cases i.e. assume that \( 2.2 \text{[iii]} \) or \( 2.2 \text{[vii]} \) holds. In these cases \( \Delta \) arises from a line \( L \) of \( \mathcal{M} \). It is seen that \( \bigcup_{j=0}^2 \Delta^{(j)} = L \times C_3 \) i.e. \( (ii) \) holds.

A structure \( \mathcal{M} \) which satisfies the assumptions of \( 3.2 \) will be called \textit{anti-}m-polypappian. It is seen that a Moufangian configuration \( \mathcal{M} \) is anti-m-polypappian for each integer \( m \geq 3 \).

Corollary 3.3. Let \( \mathcal{M} \) be an anti-m-polypappian PSTS with point degree \( > 1 \). To every \( F \in \text{Aut}(\mathcal{M}) \) there corresponds a map \( \alpha_F \in \text{Aut}(\mathcal{M}) \) such that \( F(L \times C_m) = \alpha_F(L) \times C_m \) for every line \( L \) of \( \mathcal{M} \).
From [3.3] in view of [3.1] to characterize the group Aut( ⊕^m M) where M is anti-
m-polypappian and has the points of degree > 1 it suffices to determine the kernel of the epimorphism

\[ \alpha: \text{Aut}(⊕^m M) \rightarrow \text{Aut}(M). \]

**Lemma 3.4.** Assume that \( m > 3 \). Let \( F \in \text{Aut}(⊕^m M) \) such that \( F(\{a\} \times C_m) = \{a\} \times C_m \) for each point \( a \) of \( M \). Let \( L = \{a, b, c\} \) be a line of \( M \) and \( F(a, i_0) = (a, j_0) \) for some \( i_0, j_0 \in C_m \).

(i) Then \( F(b, i_0) = (b, j_0) \) and \( F(c, i_0) = (c, j_0) \).

(ii) Moreover, \( F(x, i) = (x, i + (j_0 - i_0)) \) for each \( x \in \{a, b, c\} \).

**Proof.** Let us write \( \Delta(L, i) = \{(a, i), (b, i), (c, i)\} \) for \( i \in C_m \) and a line \( L = \{a, b, c\} \) of \( M \).

Clearly, \( F \) preserves the class of triangles of \( ⊕^m M \) of the form \([1]\) of [2.2] and thus \( F(\Delta(L, i_0)) = \Delta(L, j) \) for some \( j \). From assumptions, \( j = j_0 \). This justifies \([1]\).

Note that \( \Delta(L, i)' = \Delta(L, i + 1) \) for each \( i \in C_m \). Therefore, \( F(\Delta(L, i_0 + 1)) = F(\Delta(L, i_0)') = \Delta(L, j_0) = \Delta(L, j_0 + 1) \), which gives \( F(a, i_0 + 1) = (a, j_0 + 1) \). Inductively, we get \( F(a, i_0 + v) = (a, j_0 + v) \) for each \( v \in C_m \). This proves \([1]\). \( \blacksquare \)

**Lemma 3.5.** Assume that \( M \) is anti-\( m \)-polypappian, connected, and with the points of degree > 1. Let \( F \in \ker(\alpha) \) and \( m > 3 \). Then there is \( u \in C_m \) such that \( F = \text{id} \times \tau_u \).

**Proof.** By [3.4][1], for each line \( L \) of \( M \) there is a bijection \( β_L^F \) of \( C_m \) such that \( F(a, i) = (a, β_L^F(i)) \) for every point \( a \) on \( L \). From the connectedness, \( β_L^F = β_L^F' \) for any two lines \( L', L'' \) of \( M \). Thus there is a bijection \( β_F \) of \( C_m \) such that \( F(a, i) = (a, β_F(i)) \) for every point \( a \) of \( M \). From [3.4][1] we get that \( β_F = τ_u \) for some \( u \in C_m \). \( \blacksquare \)

As an immediate corollary we get

**Theorem 3.6.** Let \( m > 3 \) and let \( M \) be an anti-\( m \)-polypappian connected partial Steiner triple system with the points of degree > 1. Then

\[ \text{Aut}(⊕^m M) = \{f \times τ_u : f \in \text{Aut}(M), u \in C_m\}. \quad (11) \]

Consequently, \( \text{Aut}(⊕^m M) ≅ \text{Aut}(M) ⊕ C_m \).

The case \( m = 3 \) is somehow exceptional in studying structures of the form \( ⊕^m M \). Note, first, that [3.3] and, after that, [3.6] do not remain valid for \( m = 3 \). An elementary reasoning shows the following

**Remark 3.7.** Let \( S \) be the point set of \( S \) and \( F \) be a bijection of \( S \times C_3 \) such that for each \( i \in C_3 \) and each point \( x \) of \( S \) there is \( j \) with \( F(x, i) = (x, j) \). Then \( F \in \text{Aut}(⊕^3 S) \) iff \( F = \text{id} \times τ_u \) for some \( u \in C_3 \) or \( F \) is defined by one of the following formulae:

\[ F(u, j) = (u, j'') \]

where \( j' \) is read as \( F(u, j') = (u, j'') \).
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| $F$ | $i$  | $i+1$  | $i+2$  |
|-----|-----|-------|-------|
| $x$ | $j$  | $\tau_2(j)$ | $\tau_2\tau_2(j)$ |
| $y$ | $j+1$ | $\tau_2(j+1)$ | $\tau_2\tau_2(j+1)$ |
| $z$ | $j+1$ | $\tau_2(j+1)$ | $\tau_2\tau_2(j+1)$ |

for some $i, j \in C_3$ and some labeling $x, y, z$ of the points of $\mathfrak{T}$.

From this we get

**Example 3.8.** Let $\mathfrak{V}$ be the Veblen configuration with the points $\{a, b, c, d, p, q\}$; assume that the points $p, q$ are noncollinear in $\mathfrak{V}$. The bijection $F$ of the points of $\mathfrak{V}$ defined by the formula

$$
\begin{array}{c|c|c|c|c}
F & 0 & 1 & 2 \\
\hline
x & 0 & 2 & 1 \\
y & 1 & 0 & 2
\end{array}
$$

for $x \in \{p, q\}$ and $y \in \{a, b, c, d\}$ is an automorphism of $\mathfrak{V}$. It is seen that $\alpha_F = \text{id}$ but $F$ does not have form required in [3.6].

4 Weaving and other product constructions: interrelations and applications

In many cases a 3-weaved configuration is a convolution with the $C_3$-group.

**Proposition 4.1.** Assume that a partial Steiner triple system $\mathfrak{M}$ contains a hyperplane which is an anti-clique. Let $\varepsilon_1, \varepsilon_2$ be any two elements of a group $G$. Then $\mathfrak{M} \bowtie_{\varepsilon_1} G \cong \mathfrak{M} \bowtie_{\varepsilon_2} G$.

**Proof.** Let $\mathcal{H}$ be a hyperplane of $\mathfrak{M} = \langle S, \mathcal{L} \rangle$ that is an anti-clique. This means the following:

- each line of $\mathfrak{M}$ has exactly one point in common with $\mathcal{H}$.

Set $\varepsilon_0 = \varepsilon_2 - \varepsilon_1$ (computed in $G = \langle G, 0, + \rangle$) and define the map

$$
\vartheta : S \times G \rightarrow S \times G, \quad \vartheta(a, i) = \begin{cases} 
(a, i + \varepsilon_0) & \text{when } a \in \mathcal{H} \\
(a, i) & \text{when } a \notin \mathcal{H}
\end{cases}
$$

for $a \in S, i \in G$.

It is seen that $\vartheta$ maps the elements of $\mathcal{L}_{c_1}$ onto the elements of $\mathcal{L}_{c_2}$ and thus it is an isomorphism of $\mathfrak{M} \bowtie_{\varepsilon_1} G$ onto $\mathfrak{M} \bowtie_{\varepsilon_2} G$. □

**Corollary 4.2.** Let a partial Steiner triple system $\mathfrak{M}$ contain a hyperplane which is an anti-clique. Then $\mathfrak{M} \cong \mathfrak{M} \bowtie C_3$.

**Fact 4.3.** Let $\mathcal{H}$ be an anti-clique of a $(\nu, b_3)$-configuration $\mathfrak{M}$. Then $\mathcal{H}$ is a hyperplane of $\mathfrak{M}$ iff $\nu \cdot |\mathcal{H}| = b$ (equivalently: iff $3 \cdot |\mathcal{H}| = \nu$).

**Corollary 4.4.** Let $\mathfrak{M}$ be one of the following partial Steiner triple systems:

(a) the Veblen configuration;
(b) the Pappus Configuration or, more generally, an affine slit space (cf. [12], [3]) over $GF(3)$ i.e. an affine space $AG(n, 3)$ with the lines parallel to a fixed affine hyperplane deleted;

(c) the configurations $\Pi^m_{10}$ and $\Pi^m_{13}$ for arbitrary $m \geq 3$.

Then $\ast^3 \mathfrak{M} \cong \mathfrak{M} \bowtie C_3$.

**Proof.** In case (a) each pair of noncollinear points of the Veblen configuration is an anti-clique and a hyperplane. In case (b) we let $\mathcal{H}$ be any hyperplane such that $\mathfrak{M}$ does not contain lines parallel to it. Then $\mathcal{H}$ is an anti-clique and a hyperplane in $\mathfrak{M}$. In case (c) the set $\mathcal{H} = \{(0, i): i = 0, ..., m - 1\}$ is an $m$-element anti-clique and, by 4.3, it is a hyperplane as well. In each case we apply 4.2 to get the claim. □

**FACT 4.5.** Assume that a partial Steiner triple system $\mathfrak{M}$ contains a hyperplane that is an anti-clique. Then $\ast^m \mathfrak{M}$ also contains such a hyperplane.

**Proof.** Let $\mathcal{H}$ be a respective hyperplane in $\mathfrak{M}$. From 2.1 we get that $\mathcal{H} \times C_m$ is an anti-clique and from 4.3 it is a required hyperplane. □

It is not the case that each 3-weaved configuration is a convolution, though. Recall (cf. [13], [9]) that $G_2(\mathcal{X})$ with $|\mathcal{X}| = 4$ is the Veblen configuration.

**Remark 4.6.** Let $|\mathcal{X}| > 4$. Then (cf. footnote 2) $\ast^3 G_2(X) \not\cong G_2(\mathcal{X}) \bowtie C_3$.

**Proof.** Let $|\mathcal{X}| > 4$. It is known that $G_2(\mathcal{X})$ contains a Desargues subconfiguration (cf. [12]). From 1.5, $G_2(\mathcal{X}) \bowtie 0 C_3$ contains a Desargues subconfiguration, while (cf. 2.9) $\ast^3 G_2(X)$ does not contain any Desargues subconfiguration. □

**Remark 4.7.** Clearly, each point of $\mathcal{T}$ is a hyperplane. By 1.1 $\mathcal{T} \bowtie 0 C_4 \cong \mathcal{T} \bowtie 0 C_4$ for each $\varepsilon \in C_4$. The anti-Reye configuration $\mathcal{T} \bowtie C_4$ (cf. [13], [6]) is a $(12_4 12_3)$-configuration and $\ast^4 \mathcal{T}$ is a $(12_4 12_3)$-configuration, and thus $\mathcal{T} \bowtie C_4 \not\cong \ast^3 \mathcal{T}$. They are also distinct in a bit stronger meaning: It is impossible to embed $\ast^3 \mathcal{T}$ into $\mathcal{T} \bowtie C_4$. Indeed, let $a$ be a point of $\mathcal{T}$. Two disjoint pairs of lines of $\mathcal{T} \bowtie C_4$ through $(a, 0)$ yield two Veblen configurations. Suppose $\ast^4 \mathcal{T}$ is embedded. Then two of its lines through $(a, 0)$ should be a pair which yields a Veblen configuration, which is impossible, as $\ast^4 \mathcal{T}$ does not contain any Veblen configuration (cf. 2.4). Consequently, the statement “under assumptions of 4.5 the structure $\ast^m \mathfrak{M}$ is embeddable into $\mathfrak{M} \bowtie C_m$” is not valid for $m > 3$. □

As a by-product of 1.4 we get an embedding theorem

**Proposition 4.8.** Let $\mathfrak{M}$ be an affine slit space over $GF(3)$. Then $\ast^3 \mathfrak{M}$ can be embedded into the affine space over $GF(3)$.

**Proof.** From 4.4 $\mathfrak{B} := \ast^3 \mathfrak{M}$ is isomorphic to $\mathfrak{M} \bowtie 0 C_3$. On the other hand, clearly, $\mathfrak{M}$ is a substructure of an affine space $AG(n, 3)$ for some integer $n$ and thus $\mathfrak{B}$ is a substructure of $AG(n, 3) \bowtie C_3$. From 1.5, $AG(n, 3) \bowtie C_3$ is an affine slit space, embeddable into $AG(n + 1, 3)$, which proves our claim. □

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4 Analogous statement is valid for any convolution $\mathfrak{M} \bowtie 0 G$.

5 Analyzing possible series of cyclically inscribed triangles contained in $\mathcal{T} \bowtie C_m$ we can prove that $\ast^m \mathcal{T}$ is not embeddable into $\mathcal{T} \bowtie C_m$ for any integer $m > 3$. 

Let $\mathcal{M} = \langle S, \mathcal{L} \rangle$ be a partial Steiner triple system. The relation defined for $(a, i), (b, j) \in S \times C_m$ by the condition

$$(a, i) \approx (b, j) \iff a = b$$

is a congruence in $\mathcal{M}$ (cf. e.g. [15]). Note also that $\mathcal{M}/\approx \cong \mathcal{M}$.

**Fact 4.9.** Let $\mathcal{M} = \langle S, \mathcal{L} \rangle$ be a Steiner triple system. The greatest maximal anticliques in $\mathcal{M}$ are the sets $\{a\} \times C_m$ with a ranging over $S$, i.e. the equivalence classes of the relation $\approx$. In case $m = 3$ these are the only maximal cliques, and $\approx$ coincides with the binary non-collinearity relation.

**Corollary 4.10.** Let $\mathcal{M} = \langle S, \mathcal{L} \rangle$ be a Steiner triple system. There exists the unique linear completion of $\mathcal{M}$ of $\mathcal{M}$. Its line set is the union of the family $\mathcal{L}$ and the family (a parallel class) $\{\{a\} \times C_3 : a \in S\}$.

**Remark 4.11.** If an STS $\mathcal{M}$ does not contain any Veblen subconfiguration (is anti-Pasch, or Pasch-free), then (by 2.9) $\mathcal{M}$ is also anti-Pasch. It is straightforward that $\mathcal{M}$ is anti-Pasch as well. However, $\mathcal{M}$ contains a miter-configuration for each $\mathcal{M}$.

The construction of a weaved configuration has some connections with old known constructions of anti-Pasch Steiner triple systems. Namely, consider the group $C_3^n$ endowed with the family of blocks $\{u, v, 2u + 2v\}$ with $u, v$ ranging over pairs of distinct elements of $C_3^n$. It is seen that that way we present the affine space $AG(n, 3)$ simply. Let a 3-set $L = \{a, b, c\}$ be ordered with $a < b < c$ and consider the triples $(a, a, b)$, $(b, b, c)$, and $(c, c, a)$. Equivalently, we can take the $C_3$-group and the family $\mathcal{L}$ defined at the beginning of Section 1 with $m = 3$. The Bose construction (presented after [1], see [8], [1]) applied to the group $C_3^n$ and the set $L$ yields simply $AG(n, 3)$.

Generally, the modification of the Bose construction given in [1] and applied to the group $C_3^k$ and the affine space $AG(n, 3)$ yields a Steiner triple system of the parameters of the affine space $AG(n + k, 3)$. Actually, it is the affine space $AG(n + k, 3)$ with the lines in one direction replaced by some other family of blocks. The structure obtained by $k$-fold applying the operation of the form $\mathcal{M} \mapsto \mathcal{M}$ starting from $AG(n, 3)$ is another (for $k > 1$) example of an STS with parameters of an affine space. Since $AG(n, 3)$ is anti-Pasch, the obtained structure is anti-Pasch as well. It is worth to point out that it is not an affine space, though.

**Proposition 4.12.** Let $n > 1$. The structure $AG(n, 3)$ is not embeddable to any affine space $AG(N, 3)$. Consequently, $AG(n, 3)$ is not embeddable to any affine space $AG(N, 3)$. Even more generally, no structure obtained by $k$-fold applying the operation of the form $\mathcal{M} \mapsto \mathcal{M}$ starting from $AG(n, 3)$ is embeddable as well. Its maximal affine subspaces are affine planes.

**Proof.** Let $\mathcal{B} = AG(n, 3)$. Suppose that $\mathcal{B}$ is embeddable into an affine space $\mathcal{A} = AG(N, 3)$ ($N > n$) and consider a triangle $\Delta = \{(\theta, 0), (b, 0), (c, 0)\}$ of $\mathcal{B}$, where $b \neq c, 2c$, $GF(3)^n \ni b, c \neq \theta$ and $\theta$ is the zero vector of $GF(3)^n$. From i.e. a linear space with the point set $S$, extending $\mathcal{M}$, and with the lines of the size of the lines in $\mathcal{M}$, comp. [12] (or [14]).
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assumption, Δ spans a plane π in A. We compute Δ’ = {(2b, 1), (2c, 1), (2b + 2c, 1)} and Δ” = {(b + c, 2), (b + 2c, 2), (2b + c, 2)}, and thus π and X := Δ ∪ Δ’ ∪ Δ” coincide. On the other hand the set X is not a subspace in B; indeed, (c, 0) ⊙ (2b, 1) = (2c + b, 0) and (2c + b, 0) / ∈ X. Thus π is not a subspace of A, which is a contradiction. So, B is not embeddable, as required.

It is straightforward that analogous reasoning applied to the triangle (θ, 0_k), (b, 0_k), (c, 0_k) with 0_k = 0, ...., 0 justifies the third nonembeddability statement. Computing the subspaces spanned in the considered structures by (all the possible) triangles we obtain our last claim.

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7The subspace spanned by Δ i.e. the smallest subset containing Δ and closed under ⊙ is the set π_0 × C_3, where π_0 is the plane spanned by the triangle {θ, a, b} in AG(n, 3).

Analogous remark remains true for arbitrary simpleks of AG(n, 3).
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