Gravito-Electromagnetic coupled perturbations and quasinormal modes of the reduced Bah-Heidmann black hole

Wen-Di Guo\textsuperscript{a}\textsuperscript{b}\textsuperscript{, Qin Tan\textsuperscript{b}, and Yu-Xiao Liu\textsuperscript{ab}\textsuperscript{†}}

\textsuperscript{a}Lanzhou Center for Theoretical Physics, Key Laboratory of Theoretical Physics of Gansu Province, School of Physical Science and Technology, Lanzhou University, Lanzhou 730000, People’s Republic of China
\textsuperscript{b}Institute of Theoretical Physics & Research Center of Gravitation, Lanzhou University, Lanzhou 730000, People’s Republic of China

From the quantum point of view, singularity should not exist. Recently, Bah and Heidmann constructed a five-dimensional singularity free topology star/black hole [Phys. Rev. Lett. 126, 151101 (2021)]. By integrating the extra dimension, a four-dimensional static spherical black hole with a magnetic charge can be obtained. In this paper, we study the quasinormal modes (QNMs) of the magnetic field and gravitational field on the background of this reduced four-dimensional Bah-Heidmann black hole. The odd parity of the gravitational perturbations couples with the even parity of the magnetic field perturbations. Two coupled second-order derivative equations are obtained. Using the matrix-valued direct integration method, we obtain the fundamental QNM frequencies numerically. The effect of the magnetic charge on the QNMs is studied. The differences of the frequencies of the fundamental QNMs between the reduced Bah-Heidmann black hole and the Reissner-Norström black hole are very small for the angular number $l = 2$. However, some new interesting results are found for higher angular number.

PACS numbers:

\textbf{I. INTRODUCTION}

Black hole physics has entered a new era since the detection of the gravitational waves from a binary black hole merger by Laser Interferometer Gravitational-Wave Observatory (LIGO) and Virgo [1] and the first picture of a supermassive black hole at the center of galaxy M87 photographed by the Event Horizon Telescope (EHT) [2–7]. Recently, the picture of the black hole in our Milky way was also taken by EHT [8–13]. These breakthroughs provide us with more possibilities to test some fundamental physical problems, for example, does singularity exist [14, 15]? Usually, a spacetime singularity is located at the center of a black hole. However, from the quantum aspect, spacetime should not be singular. In order to mimic black holes classically, some ultra-compact objects have been constructed, such as, gravastars [16], boson stars [17], and wormholes [18–21]. For more details, see the review [22] and references therein. But usually they need some exotic matters and the UV origin is unclear. From the top-down point view, string theory is regarded as the candidate that can unify quantum theory and gravity. Some horizonless models constructed from string theory, such as fuzz balls [23], are similar to black holes up to the Planck scale, and they have smooth microstate geometries. However, a lot of degrees of freedom in supergravity are needed, and the astrophysical observations of these horizonless models are difficult [24–26]. Recently, a five-dimensional nonsingular topological star/black hole model was proposed based on a five-dimensional Einstein-Maxwell theory [27, 28]. The spacetime in this model has advantages both in microstate (smooth geometry) and macrostate geometries (similar to classical black holes). So it is interesting to study their astrophysical observations. Last year, Lim studied the motion of a charged particle in this nonsingular topological star/black hole model [29]. The thermodynamic stability of the solutions has been carefully analysed in Ref. [30]. Integrating the extra dimension, a four-dimensional Einstein-Maxwell-Dilaton theory can be obtained, and a static spherically symmetric solution was solved in this background [25, 26]. Shadows of this black hole were studied in Ref. [31]. In this paper, we will study the quasinormal modes (QNMs) of this model.

As the characteristic modes of a dissipative system, QNMs play important roles in a lot of aspects of our world. Due to the presence of the event horizon, black holes are nature dissipative systems. For a binary black hole merger system, there are three stages: inspiral, merger, and ringdown. In the ringdown stage, the gravitational waves are regarded as a superposition of QNMs [32]. Compared with the normal modes, the eigenfunctions of QNMs generally do not form a complete set, and they are not normalizable [33]. The frequencies of QNMs are complex, and the imaginary parts are related to the decay timescale of the perturbation. One can use the QNMs to infer the mass and angular momentum of a black hole [34] and to test the validity of the no-hair theorem [35–37]. The echoes in the ringdown signals can be used to distinguish the black hole from the ultra compact objects [15, 22, 38]. Recently, the pseudospectrum of gravitational physics showed that the QNM spectrum is unstable for the fundamental mode and the overtone modes [39, 40]. Besides, the properties of QNMs can also be used to constrain modified gravity theories [41–
The stability under perturbations of the background spacetime can also be partly revealed from the QNM frequencies [50, 51]. Except for black hole physics, QNMs are also very useful in other dissipative systems, such as leaky resonant cavities [52], and brane world theories [53–55]. So, QNMs have been studied widely [56–60].

In this paper, we are interested in the QNMs of the four-dimensional spherically Bah-Heidmann black hole with a magnetic charge. The organization of this paper is as follows. In Sec. II, we briefly review the Bah-Heidmann black hole and the KK reduction. In Sec. III, we study the linear perturbation of the electromagnetic field and gravitational field. Separating radial part of the perturbed fields from the angular part, we derive the perturbation equations. In Sec. IV, we compute the QNM frequencies using the matrix-valued direct integration method. Finally, we give our conclusions in Sec. V.

II. THE BAH-HEIDMANN BLACK HOLE

In this section we briefly review the black hole proposed by Bah and Heidmann [27, 28]. We start from a five-dimensional Einstein-Maxwell theory. The action is

$$S = \int d^5x \sqrt{-g} \left( \frac{1}{16\pi G_5} \hat{R} - \frac{1}{16\pi} F_{MN} \hat{F}^{MN} \right),$$

where $\hat{F}_{MN}$ is the electromagnetic field tensor and $G_5$ is the five-dimensional gravitational constant. The quantities with hat denote that they are constructed in the five-dimensional spacetime. The capital Latin letters $M, N, \ldots$ denote the five-dimensional coordinates. The metric with three-dimensional spherically symmetry can be assumed as [61]

$$ds^2 = -f_S(r)dt^2 + f_B(r)dy^2 + \frac{1}{f_S(r)f_B(r)}dr^2$$

$$+ r^2d\theta^2 + r^2\sin^2\theta d\phi^2. \quad (2)$$

The extra dimension, denoted by the coordinate $y$, is a warped circle with radius $R_y$. The field strength with a magnetic flux is

$$\hat{F} = P \sin \theta d\theta \wedge d\phi. \quad (3)$$

The solution with double Wick rotation symmetry is [61]

$$f_B(r) = 1 - \frac{r_B}{r},$$

$$f_S(r) = 1 - \frac{r_S}{r},$$

$$P = \pm \frac{1}{G_5} \sqrt{3r_Sr_B}. \quad (4)$$

That is to say, the metric (2) is invariant under rotation $(t, y, r_S, r_B) \rightarrow (iy, it, r_B, r_S)$. There are two coordinate singularities located at $r = r_S$ (corresponding to a horizon) and $r = r_B$ (corresponding to a degeneracy of the $y$-circle). Bah and Heidmann found that, after some coordinate transformations, a smooth bubble locates at $r = r_B$ [27, 28]. This provides an end of the spacetime. For $r_S \geq r_B$, the bubble is hidden behind the horizon and the metric (2) describes a black string. For $r_S < r_B$, the spacetime ends at the bubble before reaching the horizon and the metric (2) describes a topological star [27, 28].

The metric (2) can be rewritten as

$$ds_5^2 = e^{2\Phi} ds_4^2 + e^{-4\Phi} dy^2,$$

where

$$e^{2\Phi} = f_B^{-1/2},$$

$$ds_4^2 = f_B^2 \left( -f_Sdt^2 + \frac{dr^2}{f_Sf_B} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \right). \quad (7)$$

We can integrate the extra dimension $y$ (this process is called Kaluza-Klein reduction). Then, a four-dimensional Einstein-Maxwell-dilaton theory is obtained from the five-dimensional Einstein-Maxwell theory

$$S_4 = \int d^4x \sqrt{-g} \left( \frac{1}{16\pi G_4} R_4 - \frac{3}{8\pi G_4} g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi \right.$$ 

$$- \frac{e^{-2\Phi}}{16\pi e^2} F_{\mu\nu} F^{\mu\nu} \right), \quad (8)$$

where $e^2 = \sqrt{\frac{1}{16\pi G_4}}$ and $\Phi$ is a dilaton field. The Greek letters $\mu, \nu, \ldots$ denote the four-dimensional coordinates. Here, $g_{\mu\nu}$ and $F_{\mu\nu}$ are the four-dimensional metric (7) and the electromagnetic field strength, respectively. The four-dimensional Ricci scalar $R_4$ is determined by the metric $g_{\mu\nu}$, and the four-dimensional gravitational constant is defined as

$$G_4 = e^2 G_5. \quad (9)$$

Varying the action (8) with respect to the scalar field $\Phi$, the vector potential $A_\mu$, and the metric $g_{\mu\nu}$, we obtain the field equations

$$\frac{6}{G_4} \Box \Phi + \frac{e^{-2\Phi}}{e^2} F_{\mu\nu} F^{\mu\nu} = 0, \quad (10)$$

$$\nabla_\mu F^{\mu\nu} = 0, \quad (11)$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = 8\pi G_4 T_{\mu\nu}, \quad (12)$$

where $\Box$ is the four-dimensional D’Alembert operator, $T_{\mu\nu} = T^s_{\mu\nu} + T^w_{\mu\nu}$ is the energy momentum tensor containing the contributions of the scalar field and the magnetic field:

$$T^s_{\mu\nu} = \frac{3}{4\pi G_4} \nabla_\mu \Phi \nabla_\nu \Phi - \frac{3}{8\pi G_4} g_{\mu\nu} \Box \Phi, \quad (13)$$

$$T^w_{\mu\nu} = \frac{e^{-2\Phi}}{4\pi e^2} F^{\mu\alpha} F^\alpha_\nu - \frac{e^{-2\Phi}}{16\pi e^2} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}. \quad (14)$$

We can solve the vector potential corresponding to the magnetic field as

$$A_\mu = (0, 0, 0, -\frac{e}{2\sqrt{3r_Sr_B} G_4} \cos \theta). \quad (15)$$
Thus, the field strength reads as
\[
F_{\mu\nu} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{3r_B r_S}{G_4} \sin \theta \\
0 & -\frac{3r_B r_S}{G_4} \sin \theta & 0 & 0
\end{bmatrix}.
\] (16)

Note that, when \( r_B = 0 \), the metric (7) recovers to the Schwarzschild one.

The parameters \( r_S \) and \( r_B \) are related to the four-dimensional Arnowitt-Deser-Misner mass \( M \) and the magnetic charge \( Q_m \) as
\[
M = \left( \frac{2r_S + r_B}{4G_4} \right),
\] (17)
\[
Q_m = \frac{1}{2} \sqrt{\frac{3r_B r_S}{G_4}}.
\] (18)

On the other hand, in terms of \( M \) and \( Q_m \), we have
\[
\begin{align*}
(r_S^{(1)})^2 &= 2G_4(M - M_\Delta), \quad (r_B^{(1)})^2 = G_4(M + M_\Delta); \\
(r_S^{(2)})^2 &= G_4(M + M_\Delta), \quad (r_B^{(2)})^2 = 2G_4(M - M_\Delta),
\end{align*}
\] (19)
where
\[
M_\Delta^2 = M^2 - \left( \frac{\sqrt{2}Q_m}{\sqrt{3}G_4} \right)^2.
\] (20)

Note that, in four-dimensional spacetime, when \( r < r_B \), \( r_B^{1/2} \) becomes imaginary. So, \( r = r_B \) is the end of the spacetime. This is consistent with the result in five-dimensional spacetime [27, 28]. Usually, a black string scenario has the Gregory-Laflamme instability [62]. However, compact extra dimensions leading to a discrete KK mass spectrum makes it possible to avoid the Gregory-Laflamme instability. Stotyn and Mann demonstrated that, the solution (19) is unstable under perturbation, while, when \( r_B > \frac{\sqrt{2}Q_m}{\sqrt{3}G_4} \), the solution (20) is stable. That is to say, the solution (20) does not have the Gregory-Laflamme instability. Actually, the spacetime at \( r = r_B \) is singular in four-dimensional spacetime. When \( r_B \geq r_S \) the metric (7) corresponds to a naked singularity, and when \( r_B < r_S \) the metric (7) corresponds to a black hole, which is named as the reduced Bah-Heidmann black hole. In this paper, we will only focus on the case \( r_B < r_S \), i.e., the reduced Bah-Heidmann black hole.

III. PERTURBATION EQUATIONS

With the background solution (6), (7), and (15), we can derive the equations of motion for the perturbations. The perturbed scalar field, vector potential, and metric field can be written as
\[
\begin{align*}
\Phi &= \Phi + \varphi, \\
A_\mu &= A_\mu + a_\mu, \\
g_{\mu\nu} &= g_{\mu\nu} + h_{\mu\nu},
\end{align*}
\] (22)
(23)
(24)
where the quantities with a bar represent the background fields, \( \varphi \), \( a_\mu \), and \( h_{\mu\nu} \) denote the corresponding perturbations. Because the background spacetime is spherical symmetric, the perturbations can be divided into three parts based on their transformations under rotations on the 2-sphere: scalars, two-dimensional vectors, and two-dimensional tensors. The spherical harmonic function \( Y_{l,m}(\theta, \phi) \) behaves as a scalar under rotations, so it is the scalar base. The two-dimensional vector and tensor bases are introduced as follows [63–67]
\[
\begin{align*}
(V_1^1)_{l,a} &= \partial_a Y_{l,m}(\theta, \phi), \\
(V_2^2)_{l,a} &= \gamma^{bc} \epsilon_{ac} \partial_b Y_{l,m}(\theta, \phi),
\end{align*}
\] (25)
(26)
for the vector part, and
\[
\begin{align*}
(T_1^1)_{ab} &= (Y_{l,m})_{;ab}, \\
(T_2^2)_{ab} &= Y_{l,m} \gamma_{ab}, \\
(T_3^3)_{ab} &= \frac{1}{2} (\epsilon^a_b Y_{l,m})_{;cb} + \epsilon^b_a (Y_{l,m})_{;ca},
\end{align*}
\] (27)
(28)
(29)
for the tensor part. Here, the Latin letters \( a, b, c \) denote the angular coordinates \( \theta \) and \( \phi \), \( \gamma \) is the induced metric on the 2-sphere with radius 1, and \( \epsilon \) is the totally antisymmetric tensor in two dimensions. The semicolon denotes the covariant derivative on the 2-sphere.

The above quantities behave differently under the space inversion, i.e., \((\theta, \phi) \rightarrow (\pi - \theta, \pi + \phi)\). A quantity is called even or polar, if it acquires a factor \((-1)^l\) under space inversion. A quantity is called odd or axial, if it acquires a factor \((-1)^{l+1}\) under space inversion. So the above quantities can be divided into two classes, the even parts \( V_1^1, T_2^2 \), and the odd parts \( V_2^2, T_3^3 \). Note that, the spherical harmonic function \( Y_{l,m}(\theta, \phi) \) is even-parity. Usually, the perturbation equations will not mix polar and axial contributions. However, we can see from Eqs. (6), (7), and (15) that the background scalar field and metric field are even-parity and the background vector potential is odd-parity. So we expect that the scalar perturbation and even-parity parts of the metric perturbations couple to the odd-parity parts of the electromagnetic perturbations to the linear order (type-I coupling). And the odd-parity parts of the metric perturbations couple to the even-parity parts of the electromagnetic perturbations to the linear order (type-II coupling). Note that, the scalar perturbation only contains the even part. Actually, these coupled perturbation equations have been studied in Refs. [68, 69]. In this paper, we study the type-II coupling perturbations.

Based on the principle of general covariance, the theory should keep covariant under an infinitesimal coordinate transformation. Thus, we can choose a specific gauge to simplify the problem. In the Regge-Wheeler gauge [66], the odd parts of the perturbation \( h_{\mu\nu} \) can be written as
\[
\begin{bmatrix}
0 & 0 & h_0 \\
0 & 0 & h_1 \\
0 & 0 & 0 \\
* & * & 0
\end{bmatrix}
\begin{bmatrix}
\sin \theta \partial_\theta Y_{l,0}(\theta) \\
\end{bmatrix}.
\] (30)
The magnetic field also has a gauge freedom. Following Ref. [70], we denote
\[ \tilde{f}_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu, \] (31)
the even parts of the perturbation \( \tilde{f}_{\mu\nu} \) can be written as
\[ \tilde{f}_{\mu\nu} = \sum_i e^{-i\omega t} \left[ \begin{array}{ccc}
0 & f_{01} & f_{02} \\
0 & 0 & f_{12} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array} \right] \sin \theta \partial_\theta Y_{l0}(\theta). \] (32)

Note that, we have chosen \( m = 0 \) for simplify, because the perturbation equations do not depend on the value of \( m \) [66]. The asterisks denote elements obtained by symmetry. The functions \( h_0, h_1, f_{01}, f_{02}, \) and \( f_{12} \) only depend on the coordinate \( r \). The perturbation of the vector potential can be expanded as
\[ a_t = - \sum_i e^{-i\omega t} f_{02} Y_{l0}, \] (33)
\[ a_r = - \sum_i e^{-i\omega t} f_{12} Y_{l0}, \] (34)
\[ a_q = 0, \] (35)
\[ a_\phi = 0. \] (36)
The field strength \( f_{01} \) can be derived from Eq. (31) as
\[ f_{01} = \partial_r f_{02} + i\omega f_{12}. \] (37)
Substituting Eqs. (30) and (32) into the equations of motion (11) and (12), after some algebra calculations we can obtain the following master perturbation equations
\[ \frac{d^2 \psi_g}{dr_*^2} + (\omega^2 - V_{11}) \psi_g - V_{12} \psi_m = 0, \] (38)
\[ \frac{d^2 \psi_m}{dr_*^2} + (\omega^2 - V_{22}) \psi_m - V_{21} \psi_g = 0, \] (39)
where
\[ \psi_g \equiv f_B^{1/4} f_s \frac{1}{r} h_1, \] (40)
\[ \psi_m \equiv \sqrt{f_B r^2} f_{01}, \] (41)
\( r_* \) is the tortoise coordinate defined as
\[ dr_* = \frac{1}{\sqrt{f_B}} dr, \] (42)
and
\[ V_{11} = f_s \left[ \frac{l(l + 1)}{r^2} - \frac{3r^2 B (13 r_s - 9r) + 16 r_s r^2}{16 f_B r^5} \right] \]
\[ + \frac{3r_B (2r - 7r_s)}{4 f_B r^3}, \] (43)
\[ V_{12} = \frac{2i f_s f_B^{1/4}}{\epsilon (l + 1)^3} \sqrt{3r_B s G_4 \omega}, \] (44)
\[ V_{21} = \frac{i \sqrt{3r_B s f_s}}{2 \sqrt{G_4} f_B^{1/4} } (l - 1)(l + 1)(l + 2), \] (45)
\[ V_{22} = f_s \left[ \frac{3r_B s^2}{r^4} + \frac{l(l + 1)}{r^2} \right]. \] (46)
The details of deriving the master equation (38) and (39) are shown in Appendix A.

Note that, when the magnetic charge \( Q_m \) vanishes, or \( r_B \) approaches to zero, the gravitational perturbation \( \psi_g \) and the magnetic field perturbation \( \psi_m \) will decouple. Furthermore, the potential \( V_{11} \) will reduce to the potential for the gravitational perturbation of the Schwarzschild black hole. Besides, the parameters \( e \) and \( G_4 \) do not affect the quasinormal modes. To see this, we can redefine
\[ \tilde{\psi}_m \equiv \frac{\sqrt{G_4}}{e} \psi_m \] (47)
to eliminate the parameters \( e \) and \( G_4 \) in Eqs. (38) and (39). The corresponding potentials are
\[ \tilde{V}_{12} = - \frac{2i f_s f_B^{1/4}}{l(l + 1)^3} \sqrt{3r_B s G_4 \omega}, \] (48)
\[ \tilde{V}_{21} = \frac{i \sqrt{3r_B s f_s}}{2 \sqrt{G_4} f_B^{1/4} } (l - 1)(l + 1)(l + 2). \] (49)

In the following, we use the redefined quantities but omit the tilde above them.

IV. QUASINORMAL MODES

In this section we will solve the master perturbation equations (38) and (39) to obtain the frequencies of the QNMs. We focus on the QNMs of the solution (20), because it is free of the Gregory-Laflamme instability. We know from Eq. (21) that the range of the magnetic charge \( Q_m \) is \([0, \sqrt{\frac{4}{3}G_4} M]\). Compared with the range of the electric charge of the Reissner-Norströrm (RN) black hole \([0, \sqrt{G_4} M]\), the range of the magnetic charge is larger than that of the RN black hole electric charge. Note that, we only study the reduced Bah-Heidmann black hole, that is, \( r_B < r_s \). In this situation, the range of the magnetic charge \( Q_m \) is \([0, 2 \sqrt{\frac{G_4}{3}} M]\). This range is still larger than that of the RN black hole electric charge.

The perturbation equations (38) and (39) are coupled and can be rewritten into a compact form
\[ \frac{d^2 Y}{dr_*^2} + (\omega^2 - \mathbf{V}) Y = 0, \] (50)
where
\[ \mathbf{Y} = \begin{pmatrix} \psi_g \\ \psi_m \end{pmatrix} \]
and \( \mathbf{V} \) is a \( 2 \times 2 \) matrix with components (43), (46), (48), and (49). The physical boundary conditions for the QNM problem are pure ingoing waves at the event horizon
\[ Y_n \sim b_n e^{-i\omega r_*}, \quad r_* \rightarrow -\infty, \] (51)
and pure outgoing waves at spatial infinity

$$Y_n \sim B_n e^{i\omega r_+}, \quad r_+ \to +\infty,$$

(52)

where $Y_n$ is the $n$-th component of $Y$, $b_n$ and $B_n$ are coefficients of the boundary conditions. With these boundary conditions, solving the QNM frequencies is an eigenvalue problem. In this paper, we use the matrix-valued direct integration method. More details can be seen in Ref. [71].

| $Q_m/M$ | Bah-Heidmann BH | $Q/M$ | RN BH |
|---------|-----------------|-------|-------|
| 0       | 0.37367         | -0.088962 | 0.37367 | -0.088962 |
| 0.2     | 0.37474         | -0.089081 | 0.2    | 0.37474   | -0.089075 |
| 0.4     | 0.37848         | -0.089429 | 0.4    | 0.37844   | -0.089398 |
| 0.6     | 0.38641         | -0.089962 | 0.6    | 0.38622   | -0.089811 |
| 0.8     | 0.40163         | -0.090500 | 0.8    | 0.40122   | -0.089643 |
| 1       | 0.43219         | -0.090574 | 1      | 0.43134   | -0.089460 |

TABLE I: The fundamental QNMs for the gravitational field $\psi_g$ of the reduced Bah-Heidmann black hole and the RN black hole for different values of the magnetic charge $Q_m$ and electric charge $Q$. The angular number $l$ is set to $l = 2$.

| $Q_m/M$ | Bah-Heidmann BH | $Q/M$ | RN BH |
|---------|-----------------|-------|-------|
| 0       | 0.45715         | -0.094784 | 0      | 0.45759   | -0.095004 |
| 0.2     | 0.46297         | -0.095377 | 0.2    | 0.46297   | -0.095377 |
| 0.4     | 0.47903         | -0.096462 | 0.4    | 0.47903   | -0.096442 |
| 0.6     | 0.51201         | -0.098155 | 0.6    | 0.51201   | -0.098017 |
| 0.8     | 0.56316         | -0.100008 | 0.8    | 0.56316   | -0.099969 |
| 1       | 0.66161         | -0.099678 | 1      | 0.66161   | -0.099673 |

TABLE II: The fundamental QNMs for the magnetic field $\psi_m$ of the reduced Bah-Heidmann black hole and the electric field $\psi_e$ of the RN black hole for different values of the magnetic charge $Q_m$ and electric charge $Q$. The angular number $l$ is set to $l = 2$.

We solve the fundamental QNMs numerically, which dominate the ringdown waveform at late time. The values of the frequencies of fundamental QNMs for the gravitational field $\psi_g$ and the magnetic field $\psi_m$ for different values of the magnetic charge $Q_m$ with $l = 2$ are shown in Tables. I and II. When $Q_m = 0$, the metric (7) reduces to the Schwarzschild metric. The master equation (38) reduces to the odd parity gravitational perturbation of the Schwarzschild black hole in general relativity. The frequencies of the QNMs are also same as the Schwarzschild black hole case. This confirms that our numerical method is valid. Note that the charge of the reduced Bah-Heidmann black hole can be seen as a dark charge, which is very different from the U(1) charge of electromagnetism of the RN black hole. In this paper, we would like to compare our results with that of the RN black hole. Comparing the QNMs of the reduced Bah-Heidmann black hole and the RN black hole, we can see that, the differences of their numerical values are very small. So it almost can not distinguish them from the gravitational wave data. Note that, for the extreme RN black hole, the singularity structure of the perturbation equations is different from the nonextreme ones. The QNMs for the maximally charged RN black hole was studied in Ref. [72]. Our results for the RN black hole with $Q/M = 0.9999$ are taken from that paper.

The effects of the magnetic charge $Q_m$ of the reduced Bah-Heidmann black hole and the electric charge $Q$ of the RN black hole on the fundamental QNMs are shown in Figs. 1-3. From Figs. 1(a) and 1(b), it can be seen that the real parts of the QNMs for both black holes increase with the magnetic charge $Q_m$ or the electric charge $Q$. The imaginary parts of the QNMs for the RN black hole increase first then decrease as the electric charge $Q$ increases, which can be seen from Fig. 1(c). However, the situation for the imaginary parts of the reduced Bah-Heidmann black hole is different, which can be seen in Fig. 2. When $2 \leq l \leq 6$, an interesting phenomenon appears. The number of peaks of the QNMs' imaginary parts for the magnetic field $\psi_m$ (solid lines in Fig. 2) exactly equals to the value of $l$. While the number of peaks of QNMs' imaginary parts for the gravitational field $\psi_g$ (dashed lines in Fig. 2) is always less than that of the magnetic field $\psi_m$. This phenomenon does not appear in the RN black hole. When $l > 6$, we do not find such particular phenomenon anymore. We can see from Fig. 3 that, when $l = 7$ and $l = 8$, the imaginary parts for both the magnetic field $\psi_m$ and the gravitational field $\psi_g$ increase with the magnetic charge $Q_m$. When $l = 9$ and $l = 10$, the imaginary parts for the magnetic field $\psi_m$ decrease first then increase as the magnetic charge $Q_m$ increases, while the imaginary parts for the gravitational field $\psi_g$ increase with the magnetic charge $Q_m$.

V. CONCLUSIONS

In five-dimensional spacetime, based on the Einstein-Maxwell action (1), Bah and Heidmann proposed a non-singular black hole/topology star. This is similar to the classical black hole in macrostate geometries, more importantly, it can be constructed from type IIB string theory. Integrating the extra dimension $y$, the five-dimensional Einstein-Maxwell theory reduces to a four-dimensional Einstein-Maxwell-dilaton theory which supports a spherically static black hole/topological star solution with a magnetic charge.

We investigated the QNMs of the reduced Bah-Heidmann black hole by studying the linear perturbation of the gravitational field and the electromagnetic field. Because of the spherical symmetry of the background spacetime, the radial parts of the perturbed fields can be decomposed from the angular parts. The angular parts can be expanded by the spherical harmonics. The background scalar field (6) and metric field (7) are even parity under the space inversion, however, the background magnetic field (15) is odd parity. So the scalar perturbation
and even-parity parts of the metric perturbations couple to the odd-parity parts of the electromagnetic perturbations to the linear order, and the odd-parity parts of the metric perturbations couple to the even-parity parts of the electromagnetic perturbations to the linear order, which we named as type-I and type-II couplings, respectively. For simplicity, we study the type-II coupling perturbations. Finally, we obtained two coupled perturbation equations (38) and (39). Because the extra dimension radius $R_y$ can be eliminated from the master equations by a transformation of the electromagnetic field $\psi_m$, the extra dimension radius $R_y$ has no effect on the QNMs.

Using the matrix-valued direct integration method, we computed the fundamental QNMs for both the gravitational perturbation and the magnetic field perturbation, which will dominate the ringdown wave at late time. The values of the frequencies of the fundamental QNMs for the gravitational field $\psi_g$ and the magnetic field $\psi_m$ for different values of the magnetic charge $Q_m$ with $l = 2$ are shown in Tables. I and II. The differences of the frequencies of the fundamental QNMs between the reduced Bah-Heidmann black hole and the RN black hole are very small. So it almost can not distinguish them from the gravitational wave data. The effect of the magnetic charge $Q_m$ of the reduced Bah-Heidmann black hole on the fundamental QNMs for the reduced Bah-Heidmann black hole is that, when $2 \leq l \leq 6$, the number of peaks of the fundamental QNMs’ imaginary parts for the

FIG. 1: The effects of the magnetic charge $Q_m$ of the reduced Bah-Heidmann black hole and the electric charge $Q$ of the RN black hole on the fundamental QNMs. The solid and dashed lines correspond to the QNMs of the magnetic field $\psi_m$ (or the electric field $\psi_e$) and the gravitational field $\psi_g$, respectively. The black, red, and blue lines correspond to the QNMs with $l = 2$, $l = 4$, and $l = 6$, respectively. (a) The real parts of the QNMs for the reduced Bah-Heidmann black hole. (b) The real parts of the QNMs for the RN black hole. (c) The imaginary parts of the QNMs for the RN black hole.

FIG. 2: The effect of the magnetic charge $Q_m$ on the imaginary parts of the fundamental QNMs for the reduced Bah-Heidmann black hole. The solid and dashed lines correspond to the QNMs of the magnetic field $\psi_m$ and the gravitational field $\psi_g$, respectively. The angular number $l$ is set to $l = 2, 3, 4, 5, 6$. 

(a) $l = 2$
(b) $l = 3$
(c) $l = 4$
(d) $l = 5$
(e) $l = 6$
magnetic field \( \psi_m \) exactly equals to the value of \( l \).

We only studied the type-II coupling perturbations, where the scalar field does not couple to the other two fields. So we expect that the type-I coupling perturbations will give us more information about the reduced Bah-Heidmann black hole, which will be studied in the future.

VI. ACKNOWLEDGMENTS

We thank Pierre Heidmann for pointing out some mistakes in our original manuscript. This work was supported by National Key Research and Development Program of China (Grant No. 2020YFC2201503), the National Natural Science Foundation of China (Grants No. 12147166, No. 11875151, No. 12075103, and No. 12247101), the China Postdoctoral Science Foundation (Grant No. 2021M701529), the 111 Project (Grant No. B20063), and Lanzhou City’s scientific research funding subsidy to Lanzhou University.

A. EXPLICIT PERTURBATION EQUATIONS

In this appendix we give the details of how to get the master perturbation equations (38) and (39). The nonvanishing parts of the perturbed Einstein equations are the \((t, \phi)\), \((r, \phi)\), and \((\theta, \phi)\) components

\[
2e \left( 4f_B r_S f - f_S r_B f - r_B f_S r_B f_B r^2 + 4l(l+1) + 10 f_S r_B f + 8 r_S r_f + 12 r_B r^2 \right) h_0 - 8 e f_B f_S r^2 h''_0 - 4 i e f_S r \omega \left( r f_B' + 4 f_B \right) h_1 - 8 i e f_B f_S r^2 \omega h'_1 = -16 a \sqrt{G_4} \sqrt{f_B f_0},
\]

\[
8 e f_B^2 \left( r^4 \omega - f_s \left( 2 f_B + 3 r_B \right) - 2 r_B r_S \left( l(l+1) - 2 r^2 \right) \right) h_1 + 4 i e f_B \omega \left( 4 f_B + r f_B' \right) h_0 - 8 i e r^4 f_B^2 h_0 = 16 a \sqrt{G_4} r^2 f_B f_{12},
\]

\[
2 f_S h_1 f_B f'_S + f_S^2 \left( h_1 f_B' + 2 f_B h'_1 \right) + 2 i h_0 \omega = 0,
\]

where the constant \( a \) is defined as \( a = e \sqrt{3 r_B S} \). And the nonvanishing parts of the perturbed Maxwell equations are the \( t, r \), and \( \theta \) components

\[
f_S r \left( r f_B' + 4 f_B \right) f_0 + 2 f_B f_S r^2 f_0' - 2 l(l+1) f_0 = \frac{a}{f_B r^2 \sqrt{G_4}} \sqrt{f_B f_0},
\]

\[
2 i \omega \sqrt{f_B r^4 f_0} + 2 f_S \sqrt{f_B r^2 l(l+1) f_{12}} = -\frac{a}{\sqrt{G_4}} f_S l(l+1) h_1,
\]

\[
2 f_B^{3/2} f_S k \sqrt{f_{12}} \left( f_{12} f_S f_B' + 2 i f_0 \omega \right) = \frac{a}{\sqrt{G_4}} (3 f_S h_1 - f_B f_S (f_S r h_1)' - \omega h_0).
\]
Actually, among the six perturbed equations only four of them are independent. Equation (53) can be derived from Eqs. (54), (55), and (58) with the background Einstein equation (12). Similarly, Eq. (58) can also be obtained by using Eqs. (57), (56). Therefore, we can use four independent equations (54)-(57) and an identity (37) to solve five independent variables \( h_0, h_1, f_{01}, f_{02}, \) and \( f_{12} \).

The variable \( h_0 \) can be solved from Eq. (55) as

\[
h_0 = \frac{i}{2\omega} f_S (f_S f_B' + 2f_B f_S') h_1 + 2f_S^2 f_B h_1.
\]  

(59)

Using this formula and Eqs. (56) and (57), we can obtain \( f_{02} \) and \( f_{12} \) in terms of \( h_1 \) and \( f_{01} \) as

\[
f_{02} = \frac{f_S r}{2l(l+1)} \left[ 2rf_B f_{01} + (4f_B + r f_B') f_{01} \right] - \frac{i a f_S}{4r^2 \sqrt{G_4} \omega \sqrt{f_B}} \left[ 2f_B (h_1 f_S)' + f_S f_B h_1 \right],
\]  

(60)

\[
f_{12} = -\frac{i \omega r^2}{f_S (l+1)} f_{01} - \frac{a}{2\sqrt{G_4} \sqrt{f_B}} h_1.
\]  

(61)

Substituting Eqs. (59)-(61) into Eqs. (54) and (37) we can obtain two second order differential equations in which \( h_1 \) and \( f_{01} \) are coupled

\[
-\frac{1}{2} \sqrt{f_B} f_{01} h_1'' + \left[ \frac{\sqrt{f_B}}{r f_S} (2rf_S - 3r_S) - \frac{r_B f_S}{2r^2 \sqrt{f_B}} \right] h_1' + \left[ \frac{r_B^2 f_S}{8r^4 f_B^{3/2}} - \frac{\omega^2}{2 \sqrt{f_B}} \right] f_{01} = \frac{2i e \omega \sqrt{G_4}}{c^2 l(l+1) f_S} f_{01},
\]  

(62)

\[
-\frac{r^2 f_S f_B}{l(l+1)} f_{01} + 2f_B (r s + 4f_s) - 3rf_B f_S f_{01}' + \left[ 1 - \frac{r^2 \omega^2}{l(l+1) f_S} + \frac{f_S^2 (4rf_B + r B)}{2l(l+1)} - \frac{f_S (3rf_B + 4rf_B f_B)}{2l(l+1)} \right] f_{01} = -\frac{i a \sqrt{f_B} f_S^2}{2r^2 \omega \sqrt{G_4}} h_1' - \frac{i a f_S r r f_s + f_B (3r s - 2r f_s)}{2r^4 \omega \sqrt{G_4} f_B} h_1' - \frac{f_B^2 f_S}{8r^6 \omega \sqrt{G_4} f_B^{3/2}} h_1 + \left[ \frac{i a \sqrt{f_B} (r_S - 3rf_S)}{2r^6 \omega \sqrt{G_4}} - \frac{i a (3rr f_S + 2r^4 \omega^2 - 3r_B r f_S)}{4r^6 \omega \sqrt{G_4} \sqrt{f_B}} \right] h_1.
\]  

(63)

In order to get the Schrödinger-like form, we need to define the following master variables

\[
\psi_e \equiv f_B^{1/4} f_S r^{-1} h_1,
\]  

(64)

\[
\psi_m \equiv \sqrt{f_B} f_{01}.
\]  

(65)

In the tortoise coordinate \( r_* \), Eqs. (62) and (63) can be rewritten into the form of Eqs. (38) and (39).

[1] LIGO Collaboration and Virgo Collaboration, Observation of gravitational waves from a binary black hole merger, Phys. Rev. Lett. 116, 061102 (2016), [1602.03837].
[2] EHT Collaboration, First M87 event horizon telescope results. I. the shadow of the supermassive black hole, Astrophys. J. Lett. 875, L1 (2019), [1906.11238].
[3] EHT Collaboration, First M87 event horizon telescope results. II. array and instrumentation, Astrophys. J. Lett. 875, L2 (2019), [1906.11239].
[4] EHT Collaboration, First M87 event horizon telescope results. III. data processing and calibration, Astrophys. J. Lett. 875, L3 (2019), [1906.11240].
[5] EHT Collaboration, First M87 event horizon telescope results. IV. imaging the central supermassive black hole, Astrophys. J. Lett. 875, L4 (2019), [1906.11241].
[6] EHT Collaboration, First M87 event horizon telescope results. V. physical origin of the asymmetric ring, Astrophys. J. Lett. 875, L5 (2019), [1906.11242].
[7] EHT Collaboration, First M87 event horizon telescope results. VI. the shadow and mass of the central black hole, Astrophys. J. Lett. 875, L6 (2019), [1906.11243].
[8] EHT Collaboration, First Sagittarius A* Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole in the Center of the Milky Way, Astrophys. J. Lett. 930, L12 (2022).
[9] EHT Collaboration, First Sagittarius A* Event Horizon Telescope Results. II. EHT and Multimessenger Observations, Data Processing, and Calibration, Astrophys. J. Lett. 930, L13 (2022).

[10] EHT Collaboration, First Sagittarius A* Event Horizon Telescope Results. III. Imaging of the Galactic Center Supermassive Black Hole, Astrophys. J. Lett. 930, L14 (2022).

[11] EHT Collaboration, First Sagittarius A* Event Horizon Telescope Results. IV. Variability, Morphology, and Black Hole Mass, Astrophys. J. Lett. 930, L15 (2022).

[12] EHT Collaboration, First Sagittarius A* Event Horizon Telescope Results. V. Testing Astrophysical Models of the Galactic Center Black Hole, Astrophys. J. Lett. 930, L16 (2022).

[13] EHT Collaboration, First Sagittarius A* Event Horizon Telescope Results. VI. Testing the Black Hole Metric, Astrophys. J. Lett. 930, L17 (2022).

[14] E. Berti, E. Barausse, V. Cardoso, L. Gualtieri, P. Pani, U. Sperhake et al., Testing general relativity with present and future astrophysical observations, Class. Quantum Grav. 32, 243001 (2015), [1501.07274] .

[15] V. Cardoso, E. Franzin, and P. Pani, Is the gravitational-wave ringdown a probe of the event horizon? Phys. Rev. Lett. 116, 171101 (2016), [1602.07309] .

[16] P. O. Mazur and E. Mottola, Gravitational condensate stars: An alternative to black holes, [gr-qc/0109035] .

[17] F. E. Schunck and E. W. Mielke, Topical review: General relativistic boson stars, Class. Quant. Grav. 20, R301 (2003), [0801.0307] .

[18] S. N. Solodukhin, Restoring unitarity in BTZ black hole, Phys. Rev. D 71, 064006 (2005), [hep-th/0501053] .

[19] D. -C. Dai and D. Stojkovic, Observing a Wormhole, Phys. Rev. D 100, 083513 (2019), [1910.00429] .

[20] J. H. Simonetti, M. J. Kavic, D. Minic, D. Stojkovic, and D.-C. Da, Sensitive searches for wormholes, Phys. Rev. D 104, L081502 (2021), [2007.12184] .

[21] C. Bambi and D. Stojkovic, Astrophysical Wormholes, Universe 7, 136 (2021), [2105.00881] .

[22] V. Cardoso and P. Pani, Testing the nature of dark compact objects: a status report, Living Relativ. 22, 4 (2019), [1904.05363] .

[23] G. W. Gibbons and N. P. Warner, Global structure of five-dimensional BPS fuzzballs, Class. Quant. Grav. 31, 025016 (2014), [1305.0957] .

[24] I. Bena, F. Eperon, P. Heidmann, and N. P. Warner, The great escape: Tunneling out of microstate geometries, JHEP 04, 112 (2021), [2005.11323] .

[25] I. Bena and D. R. Mayerson, A new window into black holes, Phys. Rev. Lett. 125, 221602 (2020), [2006.10750] .

[26] I. Bena and D. R. Mayerson, Black holes lessons from multipole ratios, JHEP 03, 114 (2021), [2007.09152] .

[27] I. Bah and P. Heidmann, Topological stars and black holes, Phys. Rev. Lett. 126, 151101 (2021), [2011.08851] .

[28] I. Bah and P. Heidmann, Topological stars, black holes and generalized charged weyl solutions, [2012.13407] .

[29] Y.-K. Lim, Motion of charged particles around a magnetic black hole/topological star with a compact extra dimension, Phys. Rev. D 103, 084044 (2021), [2102.08531] .

[30] I. Bah, A. Dey, and P. Heidmann Stability of topological solitons, and black string to bubble transition, JHEP 04, 168 (2022), [2112.11474] .

[31] W.-D. Guo, S. -W. Wei, and Y.-X. Liu Shadow of a non-singular black hole, [2203.13477] .

[32] E. Berti, V. Cardoso, J. A. Gonzalez, and U. Sperhake, Mining information from binary black hole mergers: A Comparison of estimation methods for complex exponentials in noise, Phys. Rev. D 75, 124017 (2007), [gr-qc/0701086] .

[33] H. P. Nollert and R. H. Price, Quantifying excitations of quasinormal mode systems, J. Math. Phys. 40, 980 (1999), [gr-qc/9810074] .

[34] F. Echeverria, Gravitational Wave Measurements of the Mass and Angular Momentum of a Black Hole, Phys. Rev. D 40, 3194 (1989).

[35] E. Berti, V. Cardoso, and C. M. Will, On gravitational-wave spectroscopy of massive black holes with the space interferometer LISA, Phys. Rev. D 73, 064030 (2006), [gr-qc/0512160] .

[36] E. Berti, J. Cardoso, V. Cardoso, and M. Cavagliu, Matched-filtering and parameter estimation of ringdown waveforms, Phys. Rev. D 76, 104044 (2007), [0707.1202] .

[37] M. Isi, M. Giesler, W. M. Farr, M. A. Scheel, and S. A. Teukolsky, Testing the no-hair theorem with GW150914, Phys. Rev. Lett. 123, 111102 (2019), [1905.00869] .

[38] V. Cardoso and P. Pani, Tests for the existence of black holes through gravitational wave echoes, Nature Astron. 1, 586 (2017), [1709.01525] .

[39] J. Jaramillo, R. P. Macedo, and L. A. Sheikh, Pseudospectrum and Black Hole Quasinormal Mode Instability, Phys. Rev. X 11, 031003 (2021), [2004.06434] .

[40] M. H. Cheung, K. Destounis, R. P. Macedo, E. Berti, and V. Cardoso, Destabilizing the Fundamental Mode of Black Holes: The Elephant and the Flea, Phys. Rev. Lett. 128, 111103 (2022), [2111.05415] .

[41] B. Wang, C.-Y. Lin, and C. Molina, Quasinormal behavior of massless scalar field perturbation in Reissner-Nordstrom anti-de Sitter spacetimes, Phys. Rev. D 70, 064025 (2004), [hep-th/0407024] .

[42] J. L. Blázquez-Salcedo, C. F. B. Macedo, V. Cardoso, V. Ferrari, and L. Gualtieri, Perturbed black holes in Einstein-dilaton-Gauss-Bonnet gravity: Stability, ringdown, and gravitational-wave emission, Phys. Rev. D 94, 104024 (2016), [1609.01286] .

[43] G. Franciolini, L. Hui, R. Penco, L. Santoni, and E. Trinchieri, Effective Field Theory of Black Hole Quasinormal Modes in Scalar-Tensor Theories, JHEP 02, 127 (2019), [1810.07706] .

[44] A. Aragón, P. A. González, E. Papantonopoulos, V. Ferrari, and Y. Vásquez, Quasinormal modes and their anomalous behavior for black holes in f(R) gravity, Eur. Phys. J. C 81, 407 (2021), [2005.11179] .

[45] H. Liu, P. Liu, Y.-Q. Liu, B. Wang, and J.-P. Wu, Echoes from phantom wormholes, Phys. Rev. D 103, 024006 (2021), [2007.09078] .

[46] T. Karakasis, E. Papantonopoulos, and C. Vlachos, f(R) gravity wormholes sourced by a phantom scalar field, Phys. Rev. D 105, 024006 (2022), [2107.09713] .

[47] P. A. Cano, K. Fransen, T. Hertog, and S. Maenart, Gravitational ringing of rotating black holes in higher-derivative gravity, Phys. Rev. D 105, 024064 (2022), [2110.11378] .

[48] P. A. González, E. Papantonopoulos, J. Saavedra, and...
Y. Vásquez, Quasinormal modes for massive charged scalar fields in Reissner-Nordström dS black holes: anomalous decay rate, [2204.01570].

[49] Y. Zhao, R. Xin, A. Ilyas, E. N. Saridakis, and Y.-F. Cai, Quasinormal modes of black holes in $f(T)$ gravity, [2204.11169].

[50] A. Ishibashi and H. Kodama, Stability of higher dimensional Schwarzschild black holes, Prog. Theor. Phys. 110, 901 (2003), [hep-th/0305185].

[51] A. Chowdhury, S. Devi, and S. Chakrabarti, Naked singularity in 4D Einstein-Gauss-Bonnet novel gravity: Echoes and (in)-stability, [2202.13698].

[52] K. Kristensen, R.-C. Ge, and S. Hughes, Normalization of quasinormal modes in leaky optical cavities and plasmonic resonators, Phys. Rev. A 92, 053810 (2015), [1501.05938].

[53] S. S. Seahra, Metastable massive gravitons from an infinite extra dimension, Int. J. Mod. Phys. D 14, 2279 (2005), [hep-th/0505196].

[54] Q. Tan, W.-D. Guo, and Y.-X. Liu, Sound from extra dimension: quasinormal modes of thick brane, [2205.05255].

[55] V. Cardoso, M. Kimura, A. Maselli, E. Berti, and C. F. B. Macedo, Parametrized black hole quasinormal ringdown: Decoupled equations for nonrotating black holes, Phys. Rev. D 99, 104077 (2019), [1901.01265].

[56] R. McManus, E. Berti, C. F. B. Macedo, M. Kimura, A. Maselli, and V. Cardoso, Parametrized black hole quasinormal ringdown. II. Coupled equations and quadratic corrections for nonrotating black holes, Phys. Rev. D 100, 044061 (2019), [1906.05155].