LOW ENERGY STRING: AN ARISTOTELIAN TOP?

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It is argued that a low energy string may be an Aristotelian top, i.e. a rigid body which however cannot be rotationally excited, but in an external electromagnetic field exhibits a sort of precession with a Larmor type angular velocity. On the basis of this observation a proposal is made for a new two steps derivation of the electroweak standard model from the string dynamics. In the first step a theory of Aristotelian top is formulated and studied in more detail and in the second step an attempt is made to derive the electroweak standard model from the top dynamics. Before the symmetry breaking fermions are represented by straight frozen strings - rotators, whose symmetry under the rotation around their axes is interpreted as the group $U(1)\gamma$. The emergence of $SU(2)_L$ group is somewhat less transparent and is supposed to be connected with the new degree of freedom of relativistic rotators, which leads to the up and down type fermions. The symmetry breaking is associated with the bending of the rotators under the influence of Higgs field and with their subsequent transformation into the curved frozen strings - tops. In this new picture of electroweak interaction the chirality of the theory has a simple and natural explanation, the weak isospin and hypercharge are inherent properties of relativistic rotators/tops and the superselection rule associated with electric charge is a consequence of the accepted distinction between up and down type fermions.

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1. Introduction

String theory is regarded as the most promising candidate for the unified field theory, including gravity, free of infinities and capable of solving the problem of phenomenological constants of the standard model [1]. So far this program has been realized in a modest extent in the form of string-derived models such as Calabi-Yau compactification [2] or orbifold models [3]. One can include here also string-motivated models as for instance flipped $SU(5) \times U(1)$ [4], or string no-scale supergravity [5]. The main problem on the way to a ”theory of everything” is the selection of true vacuum from among many possibilities and the determination of the corresponding expectation values of generalized Higgs fields. It is clear that in such a situation it is necessary in modelbuilding to combine deductive methods with inductive and intuitive ones.

In the present paper a proposal is made for a new two steps derivation of the electroweak standard model (EWSM) from the string dynamics (The prospect for derivation of QCD will be shortly discussed in Conclusions). In the first step (see Fig.1) it is argued that a low energy string with frozen vibrational and rotational degrees of freedom is a top which however cannot be rotationally excited and consequently has a lowest value of spin, in case of fermions equal $1/2$.

\[
\text{String} \quad M_{Pl} \to \infty \quad \text{Aristotelian top} \quad \longrightarrow \quad \text{Standard model}
\]

Fig.1

In an external e.g. electromagnetic field the top exhibits a sort of precession with Larmor type angular velocity since a precession is not connected with spin excitation. A top which can rotate in external field only is an exotic object and will be called the Aristotelian top (AT) in what follows, in view of the obvious reference to Aristotelian dynamics.

Despite the nonconventional nature of AT its dynamics can be formulated in a consistent way, as it has been shown in a series of papers [6], [7], [8], [9], [10], where a model of spinning particle realized through AT was elaborated for the purpose of path integrals. It was shown there that the theory of AT is much more simple than the theory of conventional top [11]. For instance the quantization of AT leads to Dirac equation in a straightforward manner, in contrast to the quantization procedure in case of conventional relativistic top, where one is confronted with serious problems.

In the second step the final derivation of EWSM is investigated by a transition from AT to EWSM under the assumption that nothing substantial has been lost in the transition. For instance a tacit assumption has been made that AT has retained those
properties of string which are substantial for the derivation of EWSM. This assumption seems to be satisfied. Actually, as it is well known, the canonical quantization of strings gives only the spectrum of various excitations of a free string. To get a more rigorous and complete quantization one has to apply path integrals, which automatically lead to a perturbative theory, including hopefully the standard model.

This state of affairs fully corresponds to the situation of quantization of AT. Here also the canonical quantization leads to the spectrum of free ATs and does not represent the complete and consistent quantum theory. Due to the peculiar properties of AT the canonical equations of motion for the rotation of a free AT are reducible and decouple into separate equations for canonically conjugated variables $q_i$ and $p_i$ (for the definitions of canonically conjugated variables for top see Sec. 2). As a result there is no quantum mechanical correlations between $q_i$ and $p_i$, the principle of complementarity does not work for free ATs. In order to generate the quantum dispersions $\Delta p_i$, $\Delta q_i$ it is necessary to couple AT to quantized fields and to choose the values of coupling constants in such a way as to reach the saturation of the indeterminacy relations $\Delta p_i \Delta q_i \geq \hbar$. The free AT becomes then quantized due to the interaction with the quantum fluctuations of the gauge fields. In other words a consistent quantization of AT also requires perturbative quantum field theory in order to generate quantum mechanical dispersions through the radiation corrections.

In the present paper we restrict ourselves to the fermionic ATs, i.e. fermionic strings. The gauge as well as Higgs fields will be treated here as genuine fields necessary for quantization, but their stringy nature will be ignored.

An important topic which must be discussed here concerns spin of AT/string. Already the nonrelativistic quantum theory of conventional tops [12] revealed the interesting fact that tops can have both integral and half-integral spins, provided they are elementary objects, i.e. they have no composite structure. This condition is essential, since in opposite case the rotation of the top could be reduced to orbital motion of its constituents (as in the case of molecules) and the orbital motion cannot lead to half-integral spins. Thus the condition of elementarity radically restricts candidates among subatomic objects for a top with half integral spin. In fact the only objects which fulfill the condition of elementarity and at the same time are spatially extended are strings themselves. We thus consider the tops with spin $1/2$ as fully legitimate representatives of spin $1/2$ particles.

As a result the conventional low energy superstrings do not fall into the category of ATs considered here, since they involve Grassmannian variables in their description. Neither the original bosonic strings do, because they possess only integral angular momenta. We have come to a surprising conclusion that the upper vertex of the triangle in Fig. 1 seems to be vacant, there is no string known that would fit the spin $1/2$ AT in the low energy limit. However this conclusion in fact only shows that our top-down strategy we have followed so far has failed. The reversed bottom-up strategy dictates that we have to start with AT (if it really leads to EWSM) and find the corresponding string. In the Appendix B some arguments will be given that such string does exist. These arguments are based on a well known theorem from the differential geometry stating that a curve in 3D space given by parametric equation $\mathbf{x} = \mathbf{x}(\sigma)$ can be equiv-
ently characterized by a set of equations for three orthogonal unit vectors \( \mathbf{t} = \mathbf{t}(\sigma) \), \( \mathbf{n} = \mathbf{n}(\sigma) \), \( \mathbf{b} = \mathbf{b}(\sigma) \), the tangent, normal and binormal vectors and a vector \( \mathbf{x} = \mathbf{x}(0) \) which defines the boundary point on the string. The vectors \( \mathbf{t}, \mathbf{n}, \mathbf{b} \) for any \( \sigma \) can be interpreted as unit vectors defining the body coordinate system of a top. We can then pass to Euler angles determining the configuration of the top in space: \( \theta = \theta(\sigma) \), \( \varphi = \varphi(\sigma) \), \( \psi = \psi(\sigma) \). Thus we can choose either original kinematics for the description of the string, \( \mathbf{x} = \mathbf{x}(\sigma) \), in which the string is sequence of points, or a new kinematics, in which a string is a sequence of tops. Clearly one can expect that in the second case the string would have both integral and half-integral angular momenta. In the present paper we will not try to derive the string equations of motion in the new kinematics since the more urgent task is to ascertain whether QCD follows from AT, too. Also the issue of the coupling constants calculation can be treated in the framework of AT without the explicit reference to strings.

The paper consists of three parts. In the first, introductory, part (Sec. 2 and 3) a nonrelativistic theory of AT is considered. It shows how to introduce canonically conjugate variables for a top (since Lagrangian theory of AT does not exist) and also it shows the origin of \( U(1)_Y \) symmetry. The second part (Sec. 4, 5, 6) is devoted to the relativistic theory of free massless and massive tops. It is shown that a very suitable framework for this theory represents the dynamical group \( SO(3,3) \). Namely this group contains as a subgroup the group of right-handed and left-handed rotations of the top \( SO(3) \times SO(3) \), as well as the Lorentz group \( SO(3,1) \). Note that since the top involves for its description Euler angles the Lorentz transformations of Euler angles follow from the mathematical formalism developed here. The third part (Sec. 7, 8, 9, 10) contains an attempt to derive EWSM from AT. First, the relativistic version of Sec. 3 is given. Then it is shown that there are two kinds of rotators/tops, which differ in the Lorentz transformation properties of Euler angles. It is assumed that they correspond to up and down type fermions. The assumption is based on the existence of a superselection rule associated with these two sorts of rotators/tops. In Sec. 10 the origin of the group \( SU(2)_L \times U(1)_Y \) is discussed from the point of view of Dirac equation. Finally it is argued that the simplest mechanism for breaking the gauge symmetry in our approach is the Aristotelian deformation of rotator (i.e. bending without vibration). However no definite model for the description of this process can be offered so far.

2. Rotators with integral spins

In this and the next Section a nonrelativistic model of AT will be developed in order to show its relationship to EWSM, in particular to the generation of gauge symmetries, which will be described here by \( U(1)_A \times U(1)_Z \) group. The most important property of this model is that it points to the possible origin of \( U(1)_Z \) (that is \( U(1)_Y \)) symmetry, which will be associated with the symmetry of straight string - rotator - under the rotations around its axis. This conjecture is supported by two arguments. First, it will be shown that this rotational symmetry has a gauge nature and, second, the corresponding interaction with the gauge field violates parity.
The main problem in the interpretation of fermions as rotators before symmetry breaking is the integral spin of the latter. Clearly it is necessary to modify the dynamics of rotators in such a way that half-integral spins are not forbidden. This necessity is underlined by the fact that the rotator is a particular case of a top. How is it possible that tops can possess both integral and half-integral spins, while rotators only integral ones? To answer this question we start with Lagrangian of the free nonrelativistic symmetric top
\[ L = \frac{I_1}{2} (\dot{\varphi}^2 \sin^2 \theta + \dot{\theta}^2) + \frac{I_3}{2} (\dot{\varphi} \cos \theta + \dot{\psi})^2 \] (2.1)
where \( I_1, I_3 \) are moments of inertia. Putting \( I_3 = 0 \) we obtain Lagrangian for the rotator
\[ L = \frac{I}{2} (\dot{\varphi}^2 \sin^2 \theta + \dot{\theta}^2) \] (2.2)
with \( I_1 = I \). This implies the primary constraint
\[ p_\psi = \frac{\partial L}{\partial \dot{\psi}} = 0 \] (2.3)
The canonical Hamiltonian is
\[ H_C = p_\theta \dot{\theta} + p_\varphi \dot{\varphi} + p_\psi \dot{\psi} - L = \frac{1}{2I} (p_\theta^2 + \frac{1}{\sin \theta} p_\varphi^2) \] (2.4)

There are two alternatives how to proceed further: either one suppresses the variables \( \psi, p_\psi \) completely and restricts to angles \( \theta, \varphi \) only, or one retains formally the original 3-dimensional configuration space, in which however \( \psi \) plays now the role of a gauge variable. We shall follow the second route, since we want to maintain the contact with the top dynamics as close as possible. In that case the constraint (2.3) is taken in the weak sense, i.e. as an initial condition.

Instead of canonical Hamiltonian \( H_C \) one introduces in this case the extended Hamiltonian
\[ H = H_C + a(t) p_\psi \] (2.5)
where \( a(t) \) is an arbitrary function, reflecting the fact that we have to do with a gauge system. The generating function of an infinitesimal gauge transformation is
\[ F = p_\psi \delta \alpha(t) \]
The transformed Hamiltonian reads
\[ H' = H + \frac{\partial F}{\partial t} = H_C + a'(t) p_\psi \]
where \( a'(t) = a(t) + \delta \dot{\alpha}(t) \). Note also that
\[ \psi' = \psi + \{ F, \psi \} = \psi + \delta \alpha(t) \]
In quantum theory the constraint (2.3) implies
\[ \frac{\partial}{\partial \psi} \Phi(\theta, \varphi, \psi) = 0 \] (2.6)
When taken as an initial condition (2.6) holds in fact for any \( t \), because the operator \( \frac{\partial}{\partial \psi} \) commutes with the operator form of the Hamiltonian (2.5). The wave function \( \Phi \) of the top can be expressed by means of the functions \( D_{mk}^s(\theta, \varphi, \psi) \), the matrix elements of an irreducible representation of the SO(3) group. The constraint (2.6) can be fulfilled for \( k = 0 \) only, which implies an integral spin \( s \). We conclude that the integral spin of rotators is a consequence of (2.3), which follows from the particular form of Lagrangian (2.2).

The classical model of spinning particle elaborated in [6] involves non-standard approach in that this model cannot be described by a Lagrangian. Due to the absence of the rotational kinetic energy in the Hamiltonian the spinning particle is a non-Lagrangian object, but can be described in the framework of the standard canonical formalism. The canonical Hamiltonian, including orbital degrees of freedom, is
\[ H_C = \frac{1}{2m} (p - \frac{e}{c} A)^2 + eA_0 - \mu \cdot B \] (2.7)
where \( \mu = \frac{e}{mc} s \) is the magnetic moment and the components of spin \( s \) are
\[ s_1 = \xi_2 \eta_3 - \xi_3 \eta_2 + \xi_1 \frac{\xi_1}{\xi_1^2 + \xi_2^2} \xi_i \eta_i \]
\[ s_2 = \xi_3 \eta_1 - \xi_1 \eta_3 + \xi_2 \frac{\xi_2}{\xi_1^2 + \xi_2^2} \xi_i \eta_i \]
\[ s_3 = \xi_1 \eta_2 - \xi_2 \eta_1 \] (2.8)
where \( \xi_i, \eta_i \) are canonically conjugated variables and \( \xi_i \) are defined by means of Euler angles
\[ \xi_1 = e^\psi \sin \theta \sin \varphi \]
\[ \xi_2 = -e^\psi \sin \theta \cos \varphi \]
\[ \xi_3 = e^\psi \cos \theta \]
\[ \xi = \sqrt{\xi_i^2} = e^\psi \] (2.9)
Since the Hamiltonian (2.7) does not involve rotational kinetic energy it does not contain moments of inertia. Therefore conventional transition to rotator \( (I_3 = 0) \) is no longer possible.

Despite this fact one can realize the rotator mode of the top by introducing the gauge symmetry as in (2.5)
\[ H = H_C + a(t) s_3' = \]
\[ s'_3 = \nu_3 s_i = \xi_i \eta_i \] is the projection of spin to the symmetry axis parallel to the unit vector \( \nu_3 = \hat{\xi} \). The generating function for the gauge symmetry is again \( F = \delta \alpha(t) s'_3 \) and the new Hamiltonian

\[ H' = H + \{ H, F \} + \frac{\partial F}{\partial t} = H + \delta \dot{\alpha}(t)s'_3 \]

has the same form as the old one due to the vanishing Poisson brackets

\[ \{ s_i, s'_3 \} = 0 \] (2.11)

The arbitrary function \( a(t) \), which transforms according to equation

\[ a'(t) = a(t) + \delta \dot{\alpha}(t) \] (2.12)

has formally the status of a physical variable. As a result the action \( S \) must be varied with respect to this variable and one obtains

\[ \frac{\partial H}{\partial a} = s'_3 = 0 \] (2.13)

i.e. the same constraint as in the case of the conventional rotator and with the same conclusion concerning the integral spins.

In the next section we shall show that in order to get half-integral spin it is necessary, in addition to the choice of non-conventional Hamiltonian (2.10), to change also the interpretation of \( a(t) \) and \( \alpha(t) \).

3. Rotators with integral and half-integral spins

In order to reformulate the gauge transformations for the nonrelativistic rotator it is useful to start with the corresponding action integral

\[ S = \int_{t_1}^{t_2} (\eta_i \dot{\xi}_i - R) dt - \xi_{2i} \eta_{2i} \] (3.1)

where \( R \) is the Routh function [6]

\[ R = -\frac{1}{2} m \dot{v}^2 + e A_0 - e \frac{\nabla}{c} A - \mu \cdot B + a s'_3 \] (3.2)

This function plays the role of a Hamiltonian with respect to rotational coordinates \( \xi_i, \eta_i \) and the role of a Lagrangian (with the opposite sign) with respect to orbital coordinates.
As mentioned before, our top is a non-Lagrangian object and cannot be characterized by a pure Lagrangian.

In the new approach to gauge transformations we shall assume that the particle interacts with some vector field $Z^\mu$ in addition to the electromagnetic field $A^\mu$ and that

$$a(t) = -\frac{g}{\hbar} (Z^0 - \frac{\mathbf{v}}{c} \cdot \mathbf{Z}) \quad (3.3)$$

where $Z^\mu$ is taken in the point $x = x(t)$, the position of the particle. Furthermore we put

$$\alpha(t) = -\Lambda(x(t), t)$$

where $\Lambda(x)$ is an arbitrary function of $x$. The gauge transformation (2.12) leads now to the conventional vector potential transformation

$$Z^\mu' = Z^\mu + \frac{\hbar c}{g} \frac{\partial \Lambda}{\partial x_\mu} \quad (3.4)$$

Note that the original meaning of $a(t)$ and $\alpha(t)$ as arbitrary functions of $t$ remains preserved: arbitrariness of $a(t)$ is for instance associated with the arbitrariness proper to any gauge field characterized by a vector potential.

The decisive moment of the new interpretation of gauge transformation is connected with the introduction of a new term in the action integral, namely the term

$$S_{\text{field}} = -\frac{1}{4} \int G_{\mu\nu} G^{\mu\nu} dV dt \quad (3.5)$$

where

$$G_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu$$

This term is an obvious consequence of the new interpretation of $a(t)$ as given by (3.3). The formal status of $a(t)$ as a physical variable is now transferred to $Z^\mu(x)$. Variation of $S$ with respect to this function would lead without (3.5) to inconsistencies. After inclusion of $S_{\text{field}}$ this variation leads to the field equations for $Z^\mu$. The Coulomb law following from these equations

$$\int_S G . ds = \frac{g}{\hbar} s_3' \quad (3.6)$$

is now the correct replacement of (2.13) as a gauge constraint. The new constraint (3.6), unlike (2.13), can be fulfilled in quantum theory both for integral and half-integral spins, since it is a condition on the field $G$ and not on $s_3'$.

So we come to the conclusion that half-integral spin of rotors is possible under two conditions: i. rotators do not have rotational kinetic energy, ii. the function $a(t)$ in (3.2) must be interpreted as (3.3), i.e. an interaction with a new vector field must be introduced.

Note that a finite gauge transformation reads

$$\xi'_i = e^{-\Lambda} \xi_i$$


\[ \eta'_i = e^A \eta_i \] (3.7)

The action integral (3.1) with
\[ R = -\frac{1}{2}m v^2 + eA_0 - e\frac{v}{c} A - \mu B - \frac{g}{\hbar} s'_3 (Z_0 - \frac{v}{c} Z) \] (3.8)

where \( g/\hbar \) is a coupling constant, is invariant with respect to (3.7), since
\[ \eta_i \dot{\xi}_i \rightarrow \eta_i \dot{\xi}_i - s'_3 (\frac{\partial A}{\partial t} + \frac{\partial A}{\partial x} \cdot v) \] (3.9)

and the last term on the right hand side of (3.9) cancels with the term coming from \( R \). Note that the last term in (3.8) violates parity, since \( s'_3 \) is a pseudoscalar - a projection of spin \( s \) on the symmetry axis, \( s'_3 = \nu_3 \cdot s \).

Summarizing the results of the last two Sections we can say that the fermionic nature of rotators requires that the rotational symmetry be realized through the gauge symmetry \( U(1)_Z \), assuming that rotator interacts with some vector field \( Z^\mu \). Taking into account the standard interaction with electromagnetic field \( A \) we see that the total gauge symmetry is \( U(1)_A \times U(1)_Z \).

4. Classical theory of massless tops

In [6] the classical theory of point-like massive tops was elaborated. Here we shall show that this theory can be easily extended to massless tops. The starting point is the group \( SO(3,3) \), which reflects the fact that the most general motion of a top can be regarded as right-handed and left-handed rotations. The generators of this group \( S^{AB}, A, B = 0, 1, 2, 3, 4, 5 \) obey the Poisson brackets relations
\[ \{ S^{AB}, S^{CD} \} = g^{BD} S^{AC} + g^{AC} S^{BD} - g^{BC} S^{AD} - g^{AD} S^{BC} \] (4.1)

where \( \text{diag} \ g^{AB} = 1, -1, -1, -1, 1, 1 \). The expressions for \( S^{AB} \) are given in Appendix A.

The standard approach to spinning particles is based on the representation of Poincaré group. On the classical level the generators of this group are \( p^\mu \) and \( M^{\mu \nu} = x^\mu p^\nu - x^\nu p^\mu + S^{\mu \nu} \). The Casimir invariants are \( p^2 \) and \( w^2 \), where \( w^\mu \) is Pauli-Lubanski vector
\[ w_\sigma = \frac{1}{2} \varepsilon_{\lambda \mu \nu \sigma} p^\lambda S^{\mu \nu} \] (4.2)

In our case the spinning particle is realized through top, so that besides \( p^2 \) and \( w^2 \) other invariants exist, namely
\[ U = S^{45}, \quad V = p_\mu S^{\mu 4}, \quad W = p_\mu S^{\mu 5} \] (4.3)

for which it holds
\[ \{ M^{\mu \nu}, U \} = 0, \quad \{ M^{\mu \nu}, V \} = 0, \quad \{ M^{\mu \nu}, W \} = 0 \] (4.4)
One can also show that
\[ \{w^{\mu}, U\} = 0, \quad \{w^{\mu}, V\} = 0, \quad \{w^{\mu}, W\} = 0 \quad (4.5) \]

When \( p^2 = m^2 \neq 0 \), the quantities \( U, V/m, W/m \) form the Lie algebra of left-handed rotations of the top. In case \( p^2 = 0 \) one can derive the relations
\[ \{V, W\} = 0, \quad \{V, U\} = -W, \quad \{W, U\} = V \quad (4.6) \]

From (4.6) it follows that the group of left-handed motions of the massless top is \( E(2) \), the group of motions in the Euclidean plane. The group of right-handed motions is given by the little Lorentz group, which is conventionally defined for standard momentum \( p = p(1, 0, 0, 1) \). From the definition of \( w^{\mu} \) we have
\[ w^0 = p.s, \quad w = p^0 s - p \times N \]

where
\[ s = (S^{23}, S^{31}, S^{12}) \]
\[ N = (S^{01}, S^{02}, S^{03}) \]

and one obtains
\[ w^0 = ps_3 \]
\[ w_1 = p(s_1 + N_2) \]
\[ w_2 = p(s_2 - N_1) \]
\[ w_3 = ps_3 \quad (4.7) \]

The Lie algebra of right-handed motions corresponds also to \( E(2) \)
\[ \{w_1, w_2\} = 0, \quad \{w_1, s_3\} = w_2, \quad \{w_2, s_3\} = -w_1 \quad (4.8) \]

Casimir invariants of both algebras are \( V^2 + W^2 \) and \( w_1^2 + w_2^2 \), respectively. We shall show that
\[ V^2 + W^2 = w_1^2 + w_2^2 \quad (4.9) \]

For this purpose we first introduce complex vector
\[ U^\mu = -S^{\mu4} + iS^{\mu5} \]

From the definitions of \( S^{AB} \) given in Appendix A one can derive for \( U^\mu \) and complex conjugate vector \( U^{\mu\ast} \) the following relations
\[ U^{\mu\ast} U^\nu + U^{\nu\ast} U^\mu = 2S^{\mu\sigma} S^\nu_\rho \quad (4.10) \]

Denoting \( P = p_\mu U^\mu \) one gets from (4.10)
\[ V^2 + W^2 = PP^\ast = p_\mu p_\nu S^{\mu\rho} S^\nu_\rho \quad (4.11) \]
From the definition of $w^\mu$ it follows

$$-w^2 = \frac{1}{2} S^{\mu\nu} S_{\mu\nu} p^2 + p_\mu p_\nu S^{\mu\nu} S^\rho_\rho$$  \hspace{1cm} (4.12)$$

Using the expressions for $S^{\mu\nu}$ given in Appendix A one becomes

$$S^{\mu\nu} S_{\mu\nu} = 2 U^2$$  \hspace{1cm} (4.13)$$

If $p^2 = m^2 \neq 0$, then eq. (4.11) - (4.13) imply

$$-\frac{w^2}{m^2} = U^2 + \frac{V^2}{m^2} + \frac{W^2}{m^2}$$  \hspace{1cm} (4.14)$$

The last relation reflects the fact that the laboratory and the body components of spin lead to the same absolute value of spin.

In case of massless particle one gets instead of (4.14) the equation

$$-w^2 = V^2 + W^2$$  \hspace{1cm} (4.15)$$

If one takes into account that for $p^\mu = p(1,0,0,1)$ one has $w^0 = w_3$, then from (4.15) one obtains (4.9).

5. Quantum theory of massless tops

We summarize first the quantum theory of massive tops given in [7]. Quantum mechanical generators of $SO(3,3)$ group are defined in Appendix A. They can be obtained from their classical expressions by the conventional replacement $\eta_i \to \frac{\partial}{\partial \xi_i}$ up to the ordering ambiguity (for simplicity the factor $-i$ is omitted) and they obey the commutation relations

$$[S^{AB} S^{CD}] = g^{BD} S^{AC} + g^{AC} S^{BD} - g^{BC} S^{AD} - g^{AD} S^{BC}$$  \hspace{1cm} (5.1)$$

The ordering problem was solved in [7] by means of the requirement that the representation of $SO(3,3)$ be finite dimensional (in order to get the conventional finite dimensional representations of Lorentz group). This leads to tops with some spin $s$ (integral or half-integral). Taking $s = 1/2$ one gets tops, which can assume unique spin value characteristic for conventional fermionic particles (leptons and quarks).

The generators of Poincaré group are

$$p^\mu = i \partial^\mu, \quad M^{\mu\nu} = i (x^\mu \partial^\nu - x^\nu \partial^\mu) - i S^{\mu\nu}$$

Components of Pauli-Lubanski vector of a massless particle for standard momentum $p^\mu = p(1,0,0,1)$ have the form similar to (4.7)

$$w^0 = p I_3$$
\[ w_1 = p(I_1 + N_2) \]
\[ w_2 = p(I_2 - N_1) \]
\[ w_3 = pI_3 \]  
(5.2)

where \( I_i \) can be obtained from the classical expressions (2.8) by means of the replacement \( \eta_i \rightarrow \frac{\partial}{\partial \xi_i} \) and \( N \) is defined as follows

\[ N = -\nu_3 \times I + s\nu_3 \]  
(5.3)

Here the term \( s\nu_3 \) comes from the ordering ambiguity and we assume that \( s = 1/2 \).

As in the classical case Lie algebra of \( E(2) \) group can be derived

\[ [w_1, w_2] = 0, \quad [w_1, I_3] = w_2, \quad [w_2, I_3] = -w_2 \]  
(5.4)

The quantum analogues of (4.3) are

\[ U = \nu_3 I, \quad V = -\frac{1}{2}(P + P_c), \quad W = \frac{1}{2i}(P - P_c) \]  
(5.5)

where \( P = p_\mu U^\mu; \quad P_c = p_\mu U^\mu_c \) and

\[ U^0 = \nu I \]
\[ U^0_c = \nu^* I \]
\[ U = -i\nu \times I + is\nu \]
\[ U_c = i\nu^* \times I - is\nu^* \]  
(5.6)

with \( \nu \) defined in Appendix A. In analogy with (4.4) and (4.5) it holds

\[ [M^{\mu\nu}, U] = 0, \quad [M^{\mu\nu}, V] = 0, \quad [M^{\mu\nu}, W] = 0 \]  
(5.7)

\[ [w^\mu, U] = 0, \quad [w^\mu, V] = 0, \quad [w^\mu, W] = 0 \]  
(5.8)

The left-handed motions of the massless top are given also by the \( E(2) \) group

\[ [V, W] = 0, \quad [V, U] = -W, \quad [W, U] = V \]  
(5.9)

The quantum analogues of eq. (4.10) can be derived from the definitions (5.6) and the results reads

\[ U_c^{\mu\nu} U^{\nu} + U^{\mu} U_c^{\nu} = 2S^{\mu\rho} S_{\rho}^{\nu} + 4S^{\mu\nu} - 2s(s + 2)g^{\mu\nu} \]  
(5.10)

This implies

\[ P_c P + PP_c = 2p_\mu p_\nu S^{\mu\rho} S_{\rho}^{\nu} - 2s(s + 2)p^2 \]

In virtue of eq. (4.12), which can be extended to quantum theory without any change and also due to equation

\[ S^{\mu\nu} S_{\mu\nu} = 2U^2 - 2s(s + 2) \]  
(5.11)
we obtain
\[ V^2 + W^2 + U^2 p^2 = -w^2 \]  
(5.12)
For \( p^2 = m^2 \neq 0 \) this relation is the quantum counterpart of eq. (4.14). For \( m^2 = 0 \) we get in analogy with (4.9)
\[ V^2 + W^2 = w_1^2 + w_2^2 \]  
(5.13)
Thus the generators \( w_1, w_2, I_3, V, W, U \) form the Lie algebra of \( E(2) \times E(2) \) group for which however the constraint (5.13) holds.

We have come to the conclusion that the notion of AT has a well defined physical and mathematical meaning. The object under study is a top/rotator, since it is described by the group \( SO(3) \times SO(3) \) of right- and left-handed rotations characteristic for massive tops and similar group \( E(2) \times E(2) \) for massless tops/rotators. It is an Aristotelian top/rotator since there is no term in the Routh function corresponding to the rotational kinetic energy (see (3.8) and (7.2)) and so there is no rotational excitation of the top/rotator. In quantum theory the operator ordering can be always chosen so that the top has the only spin value \( s = 1/2 \), in accordance with no excitation assumption.

6. Quantum mechanical states

In the beginning of Sec. 5 it was mentioned that by fixing the operator ordering in the expressions for \( N, U \) and \( U_c \), which is controlled by a parameter \( s \), one can select spin \( 1/2 \) particle as the only allowed state of the massive top. On the first sight this conclusion may seem controversial since intuitively one expects from a wave function of three variables like \( \Phi(\xi_i) \) the existence of infinite many states. Therefore it will be instructive to repeat similar arguments and as we shall see, with similar results for the massless top.

The wave function \( \Phi(x, \xi_i) \) of a free massless top must obey the equations
\[ p^2 \Phi = 0 \]  
(6.1)
\[-(V^2 + W^2)\Phi = \lambda \Phi \]  
(6.2)
(the minus sign in (6.2) reflects the fact that the true quantum mechanical operators are \(-iV \) and \(-iW \)). The solutions of (6.1) and (6.2) should be sought in the space, in which the unitary representations of the group \( E(2) \times E(2) \) are defined. Since \( E(2) \) is non-compact these representations are either infinite-dimensional or one-dimensional. The former should be excluded according to conventional wisdom [13], since otherwise they would lead to infinite degeneracy of states with a given momentum \( p \). Such degeneracy would cause e.g. inconsistencies in the description of statistical properties of systems consisting of such particles. Therefore one should restrict to one-dimensional representations, which correspond to \( \lambda = 0 \).
From the point of view of $SO(3,3)$ group this restriction corresponds to integral or half-integral $s$. The value $s = 1/2$ defines an operator ordering which selects spin $1/2$ as the only allowed spin value of the particle.

For a given value of $\lambda$ the wave function $\Phi$ is specified also by eigenvalues of the operator $-iV$ or $-iW$. Since $\lambda = 0$ one obtains instead of (6.2) two equations

$$V\Phi = 0$$
$$W\Phi = 0$$

(6.3)

In [7] matrix representations of $V$ and $W$ for $s = 1/2$ were found

$$V\Phi = \Omega^+ V_{\text{matr}} \Psi$$
$$W\Phi = \Omega^- W_{\text{matr}} \Psi$$

(6.5)

where

$$V_{\text{matr}} = \frac{1}{2} \gamma^\mu \gamma_5 p_\mu$$
$$W_{\text{matr}} = \frac{i}{2} \gamma^\mu p_\mu$$

$$\Phi(x, \xi_i) = \Omega^+ (\xi_i) \Psi(x)$$
$$\Omega^+ (\xi_i) = (\chi^+, \zeta^+)$

$$\chi^+ = \xi^{- \frac{i}{2}} (ie^{i\frac{\varphi}{2}} \sin \frac{\theta}{2}, e^{-i\frac{\varphi}{2}} \cos \frac{\theta}{2})$$
$$\zeta^+ = \xi^{\frac{i}{2}} (e^{i\frac{\varphi}{2}} \cos \frac{\theta}{2}, ie^{-i\frac{\varphi}{2}} \sin \frac{\theta}{2})$$

(6.6)

and $\Psi$ is Dirac wave function in the spinor representation. Eq. (6.3) then imply

$$\gamma^\mu p_\mu \Psi = 0$$

(6.7)

The conclusion, which can be drawn from these considerations is quite remarkable: under the accepted operator ordering the only quantum mechanical states of the massless top are those, which satisfy eq. (6.7). This is to be compared with classical states, which are restricted by the condition $p^2 = 0$ only, leaving $\xi_i, \eta_i$ arbitrary.

The explanation of this paradox will be discussed in Conclusions, where it will be indicated that the restriction to finite dimensional representations of Lorentz group has no real ground. In fact it is sufficient to choose a weaker condition, namely that the spin should be $1/2$, but the representation of Lorentz group may be both finite and infinite dimensional. In this case $\lambda^2$ is arbitrary and the quantum counterpart of the classical AT is in general described by a wave function $\Phi(x, \xi_i)$ with the transformation
properties, which correspond to the infinite \(\lambda^2 \neq 0\) and finite \(\lambda^2 = 0\) dimensional representation of Lorentz group.

7. Classical relativistic Aristotelian rotator

In this section we will give the relativistic version of the theory considered in Sec. 3. Again we start with the action

\[
S_R = \int_{\tau_1}^{\tau_2} (\eta_i \dot{\xi}_i - R) d\tau - \xi_2 \eta_2 \tag{7.1}
\]

The evolution parameter \(\tau\) is associated with the worldline \(x^\mu = x^\mu(\tau)\) of the rotator and the Routh function \(R\) is assumed to have the form

\[
R = \frac{\mu}{2} u^2 + \frac{m^2}{2\mu} + (eA_\mu + eZ Z_\mu + e' Z U Z_\mu) u^\mu + \frac{1}{2\mu} (eF_{\mu\nu} + eZ F^Z_{\mu\nu} + e' Z U F^Z_{\mu\nu}) S^{\mu\nu} \tag{7.2}
\]

where \(F^Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu\) and \(u^\mu = \frac{dx^\mu}{d\tau}\). The action (7.1) is reparametrization invariant, which is guaranteed by auxiliary variable \(\mu\) transforming under the reparametrization in the following way

\[
\tau' = f(\tau), \quad \mu' = \frac{df(\tau)}{d\tau} \mu
\]

This variable is determined by fixing the evolution parameter. The proper time interpretation of \(\tau\) corresponds to \(\mu = m\), where \(m\) is the mass of the rotator.

There is no rotational kinetic energy term in (7.2), rotator cannot be rotationally excited, it is of Aristotelian type. Moreover the action (7.1) is invariant under gauge transformations

\[
\begin{align*}
\xi_i' &= e^{-\Lambda} \xi_i \\
\eta_i' &= e^\Lambda \eta_i \\
Z'_\mu &= Z_\mu - \frac{1}{\dot{e}_Z} \frac{\partial \Lambda}{\partial x^\mu}
\end{align*}
\tag{7.3}
\]

which guarantees that the particle under consideration is really a rotator. The action (7.1) is of course invariant also with respect to the ordinary gauge transformations of electromagnetic potentials \(A^\mu\). Note that we have included in \(R\) terms proportional to the coupling constant \(e_Z\) in order to obtain the most general coupling to \(Z^\mu\). From the quantum theory of the rotator based on the Dirac equation (see Sec. 9) it follows that

\[
e'Z = \frac{\bar{g}}{2}; \quad e_Z = \frac{\bar{g}}{4} = \frac{\bar{g}^2}{2}; \quad e = \frac{g g'}{\bar{g}} \quad \text{where} \quad \bar{g} = (g^2 + g'^2)^{1/2} \tag{7.4}
\]
and \( g, \ g' \) are the conventional coupling constants of EWSM. The complete action includes also the field contributions \( S_A, S_Z \)

\[
S = S_R + S_A + S_Z
\]

\[
S_A = -\frac{1}{4} \int F_{\mu \nu} F^{\mu \nu} d^4x
\]

\[
S_Z = -\frac{1}{4} \int F^Z_{\mu \nu} F^{Z \mu \nu} d^4x
\]

The classical Hamiltonian which corresponds to a Routh function \( R \) is defined as follows

\[
H = -p_\mu u^{\mu} + R
\]  

(7.5)

where \( p_\mu = \frac{\partial R}{\partial u^{\mu}} \). With \( R \) given by (7.2) this leads to

\[
H = \frac{1}{2 \mu} [m^2 - (p_\mu - eA_\mu - e_Z Z_\mu - e'_Z U Z_\mu)^2 + (eF_{\mu \nu} + e_Z F^Z_{\mu \nu} + e'_Z U F^Z_{\mu \nu}) S^{\mu \nu}]
\]  

(7.6)

Variation of the action (7.1) with respect to \( \mu \) leads to the condition

\[
\frac{\partial R}{\partial \mu} = 0
\]

From (7.6) it then follows

\[
\frac{\partial H}{\partial \mu} = 0
\]

or taking into account (7.6)

\[
H = 0
\]  

(7.7)

The canonical equations for orbital variables are

\[
\frac{dx^\mu}{d\tau} = -\frac{\partial H}{\partial p_\mu}
\]

\[
\frac{dp_\mu}{d\tau} = \frac{\partial H}{\partial x^\mu}
\]  

(7.8)

As for the internal (rotational) variables the equations of motion can be most simply written in the Poisson bracket form

\[
\frac{dS^{AB}}{d\tau} = \{H, S^{AB}\}
\]  

(7.9)

The explicit form of this equation can be obtained using the Poisson bracket relation (4.1).
8. Two kinds of rotators and tops

In this section we will show that there are two kinds of rotators and tops and the difference between them is manifested by the different behaviour under Lorentz transformations.

To this aim let us first introduce slightly generalized canonically conjugated 4-component variables \( \xi_\mu = (\xi_0, \xi_i) \), \( \pi_\mu = (\pi_0, \pi_i) \), which are however subjected to constraints

\[
\begin{align*}
\xi_0^2 - \xi_i^2 &= 0 \\
\pi_0^2 - \pi_i^2 &= 0
\end{align*}
\]

Due to these constraints Poisson brackets should be replaced by Dirac brackets

\[
\{ \xi_\mu, \pi_\nu \} = g_{\mu\nu} - \frac{\xi_\nu \pi_\mu}{\xi_\pi} \\
\{ \xi_\mu, \xi_\nu \} = \{ \pi_\mu, \pi_\nu \} = 0
\]

where \( \xi_\pi = \xi_0 \pi_0 - \xi_i \pi_i \) and \( g_{\mu\nu} \) is the usual space-time metric tensor. One can easily verify that constraints (8.1) are consistent with Dirac bracket relations (8.2). Note that despite the vectorial form of eq. (8.1) and (8.2) \( \xi_\mu \) and \( \pi_\mu \) are not fourvectors!

The relation between \( \eta_i \) and \( \pi_\mu \) is

\[
\eta_i = \pi_i - \frac{\pi_0}{\xi_0} \xi_i
\]

The inverse one reads

\[
\pi_i = \eta_i - \frac{\eta_k}{2 \xi_i \eta_l} \xi_l; \quad \pi_0 = -\frac{\eta_k}{2 \xi_i \eta_l} \xi_0
\]

As for \( \xi_\mu \) the relation is simple

\[
\xi_\mu = (\pm \xi, \xi_i)
\]

One can show that from the Dirac bracket relations (8.2) and the eq. (8.3) the standard Poisson brackets relations follow

\[
\{ \eta_i, \xi_k \} = \delta_{ik} \\
\{ \eta_i, \eta_k \} = \{ \xi_i, \xi_k \} = 0
\]

as expected.

Note that

\[
\pi_0 \dot{\xi}_0 - \pi_i \dot{\xi}_i = -\eta_i \dot{\xi}_i \quad \text{and} \quad \pi_0 \dot{\xi}_0 - \pi_i \xi_i = -\xi_i \eta_i
\]
so that action integral (7.1) can be easily expressed by means of variables $\xi_\mu, \pi_\mu$. Also the generators of the $SO(3, 3)$ group can be expressed in new variables. First of all one gets from (2.8)

\[
s_1 = \xi_2\pi_3 - \xi_3\pi_2 - \frac{\xi_0\xi_1}{\xi_1^2 + \xi_2^2}\xi_\pi
\]

\[
s_2 = \xi_3\pi_1 - \xi_1\pi_3 - \frac{\xi_0\xi_2}{\xi_1^2 + \xi_2^2}\xi_\pi
\]

\[
s_3 = \xi_1\pi_2 - \xi_2\pi_1
\]

Here we have used relations (8.3) and the variables $\xi = \sqrt{\xi^2}$ in (2.8) was interpreted as $\xi_0$ in order that $s_1, s_2$ be single-valued functions of $\xi_\mu$. This requirement is necessary to obtain unique extension of $s_i$ from the region $\xi_0 > 0$ to the region $\xi_0 < 0$. The expression (8.7) for $s_i$ are then linear functions of $\pi_\mu$ and rational functions of $\xi_\mu$. In this way one can pass smoothly from the value $\xi_0 = \sqrt{\xi^2} \equiv \xi$ to the value $\xi_0 = -\xi$. When one returns back to the variables $\eta_i$ in (8.7) one obtains

\[
s_1 = \xi_2\eta_3 - \xi_3\eta_2 + \frac{\xi_0\xi_1}{\xi_1^2 + \xi_2^2}\xi_i\eta_i
\]

\[
s_2 = \xi_3\eta_1 - \xi_1\eta_3 + \frac{\xi_0\xi_2}{\xi_1^2 + \xi_2^2}\xi_i\eta_i
\]

\[
s_3 = \xi_1\eta_2 - \xi_2\eta_1
\]

Note that $s_i$ defined by (8.8) fulfil the standard Poisson brackets relations

\[
\{s_i, s_j\} = -\epsilon_{ijk} s_k
\]

for both values of $\xi_0 = \pm \xi$. The same is true for all other generators of $SO(3, 3)$ group. To show that let us first generalize the unit vectors $\nu_3$ and $\nu$, originally defined for $\xi_0 = \xi$ in the Appendix A. The generalized expressions read

\[
\nu_1 = -\frac{\xi}{\sqrt{\xi_1^2 + \xi_2^2}} (\frac{\xi_1\xi_3}{\xi_0} + i\xi_2)
\]

\[
\nu_2 = -\frac{\xi}{\sqrt{\xi_1^2 + \xi_2^2}} (\frac{\xi_2\xi_3}{\xi_0} - i\xi_1)
\]

\[
\nu_3 = \frac{\xi}{\xi_0} \sqrt{\xi_1^2 + \xi_2^2}
\]

\[
\nu_{3k} = \frac{\xi_k}{\xi_0}
\]

Actually, one can easily show that $\nu_3$ and $\nu$ obey the relations

\[
\nu^2 = 1 \quad \nu, \nu^* = 2 \quad \nu^2 = 0 \quad \nu_3, \nu = 0
\]

\[
\nu_3 \times \nu = i\nu \quad \nu \times \nu^* = 2i\nu_3
\]
irrespective of sgn $\xi_0$. Furthermore it holds

$$\{s_i, \nu_3\} = -\epsilon_{ijk}\nu_{3k}$$  \hspace{1cm} (8.12)$$
$$\{s_i, \nu_j\} = -\epsilon_{ijk}\nu_k$$

also for both signs of $\xi_0$.

Taking into account the definitions of the remaining $SO(3, 3)$ generators

$$N = s \times \nu_3$$
$$U^0 = s \cdot \nu, \quad U = is \times \nu$$  \hspace{1cm} (8.13)$$
$$U^0 = s \cdot \nu^*, \quad U^* = -is \times \nu^*$$
$$U = s \cdot \nu_3$$

and eq. (8.11), (8.12), one sees that the Poisson brackets of any two generators lead to the same result, regardless of the sign of $\xi_0$ and so the relations (4.1) hold for $\xi_0 = \pm \xi$. Since the action of $SO(3, 3)$ does not shift any point from the region $\xi_0 = \xi$ to $\xi_0 = -\xi$ and vice versa, $\xi_0/\xi$ behaves with respect to $S^{AB}$ like a constant and one can express this fact formally as follows

$$\{S^{AB}, \xi_0/\xi\} = 0$$  \hspace{1cm} (8.14)$$

In conclusion one can say that there are two actions (7.1) associated with two different Routh functions, denoted as $R_{u,d}$. In $R_u$ the generators $S^{AB}$ correspond conventionally to $\xi_0 = -\xi$, while in $R_d$ they correspond to $\xi_0 = +\xi$. Besides this the coupling constants: $e, e_Z, e_Z'$ are in general different in $R_u$ and $R_d$. In the next section we shall argue that these two Routh functions correspond to two fermions with different electric charges. The main argument will be based on the quantum version of eq. (8.14) indicating that there is an observable $(\xi_0/\xi)$ whose operator commutes with the operators of all other observables, associated with the system under consideration. Such an operator generates the superselection rule connected with the electric charge.

Here we shall restrict ourselves to another, more formal, distinction between the tops (rotators) corresponding to $\xi_0 = \pm \xi$ as manifested by the transformation properties under Lorentz group. An infinitesimal Lorentz transformation is characterized by the generating function $F = -N \cdot \delta V - s \cdot \delta \phi$ and the corresponding transformation of $\xi_i$ reads

$$\xi_i \rightarrow \xi_i + \{F, \xi_i\} = \xi_i + \frac{\partial F}{\partial \eta_i}$$  \hspace{1cm} (8.15)$$

Since $N, s$ have different form for $\xi_0 = \pm \xi$ the variables $\xi_i$ transform differently in these two cases. In particular the unit vector $\nu_{3i} = \frac{\xi_i}{\xi}$ transforms under rotations like an ordinary three-vector and under Lorentz boosts according to a non-linear transformation law:

$$\frac{\xi_i}{\xi} \rightarrow \frac{\xi_i}{\xi} - \frac{\xi_0}{\xi} (\delta v_i - \frac{\xi_i \xi_j}{\xi^2} \delta v_j)$$  \hspace{1cm} (8.16)$$

In either cases the quadratic form $\nu_{3i}^2$ is invariant, so that also the equation

$$\xi_0^2 - \xi_i^2 = 0$$  \hspace{1cm} (8.17)$$
is Lorentz invariant. In the special case of rotator the vector $\nu_3$ determines its symmetry axis and the eq. (8.16) gives two different transformation rules for this axis under Lorentz boosts.

9. In the search of a rational reconstruction of EWSM

In this and the next Section we will summarize and analyze the logical steps necessary for a final reconstruction of EWSM from the string dynamics via the Aristotelian top. In the first step we must realize that the string dynamics at low energies is greatly influenced by the large energy value of the first excitation level $O(E_{Pl})$. Consequently for energies $E \ll E_{Pl}$, which cover the region of validity of EWSM the vibrational and rotational degrees of freedom cannot be excited, they are frozen. This however does not mean that there cannot be any rotational motion at all. We assume that (besides the orbital motion of the string as a whole) a rotation in the external field is possible, leading to the process of spin precession. Such an assumption was tested in [6] and it was shown there that it leads to consistent theory of spinning particle. We also anticipate that in the Higgs field string can change its shape, causing in this manner the gauge symmetry breaking, without generating the vibrational excitations. We thus come to the conclusion that an Aristotelian regime sets up when we approach the low energy limit. In this regime the motion of string (rotation and shape deformation) is possible only if the string is subjected to some external cause - an external (gauge or/and Higgs) field.

As for the equation of motion for the Aristotelian string one can extend the theory elaborated in (6) for motion in an electromagnetic field to any gauge field. This was done in previous Sections. However we have no clear idea how to get the interaction with Higgs field from the first principles. In such situation one can borrow the whole Higgs sector from EWSM and assume that string is straight line - a rotator - in the false Higgs vacuum. The bending of string is associated with the transition of Higgs field to the true vacuum. However one does not see how the curvature of string is controlled by the Higgs field in such transition. From this point of view a more appealing approach is offered by the theory of elasticity, where the phenomenon of elastic instability, similar to that generated by Higgs field, is known for long time. It consists in bending of beam, whose one end is fixed and the other is loaded by a force acting parallel to the beam. It is known that when the force exceeds some critical value the initially straight beam bends and the axial rotational symmetry is violated. Note that this process is given in [14] as an example of spontaneous symmetry breaking in connection with EWSM!

In the present paper we will not develop these speculative ideas further and in what follows in constructing the Aristotelian equations of motion we will restrict ourselves to gauge fields and to the rotational degrees of freedom, only. The interaction with the Higgs field will be taken over from EWSM.

In the second step we will take into account the consequences resulting from the fact that there is no term in the Hamiltonian (7.6) corresponding to the kinetic energy of
the rotational motion. Due to this there is no relation between the angular velocity $\dot{\xi}_i$ and the canonical momenta $\eta_i$, and in virtue of this there is no quantum mechanical correlation between $\xi_i$ and $\eta_i$. In the conventional quantum theory of point particles the relation $p_i = m x_i$ guarantees the expansion of wave packet and equivalence of canonical and path integral quantization. When the rotational kinetic energy is missing in the Hamiltonian the path integration does not lead to any expansion of spin wave packet and the equivalence between the two procedures of quantization is lost.

On the first sight this may signal an internal inconsistency of AT, but a closer look reveals that it is not so. In Sec. 2 and 3 we have seen that the rotator mode of the top can be realized only through the interaction with some gauge vector field which is responsible for the rotator $U(1)$ symmetry. Thus, there cannot be free rotator/top, it must interact with the gauge field. At least it must be thought as interacting with the vacuum fluctuations of the gauge fields. This interaction generates the quantum mechanical dispersions of the Euler angles, or equivalently $\xi_i$. In other words it leads to the expansion of the $\xi_i$-part of the wave function. To see that let us consider a propagation of AT from the point $x_1$ to $x_2$ and let us assume that the variables $\xi_i$ have in $x_1$ sharp values $\xi_i = \xi_{1i}$. The path connecting $x_1$ and the $x_2$ goes through the gauge field, so that $\xi_i$ is changed depending on the path. In $x_2$ the wave function will then be in general different from zero for all possible values of $\xi_i$. In [7] the propagator of AT in an external electromagnetic field was derived by means of path integrals and it was shown there that the spin wave functions $\chi^+, \zeta^+$ given by (6.6) are reproduced in the propagation of AT. As a result any spin state can be expressed as linear combination of the components of $\chi^+, \zeta^+$. But these functions were in Sec. 6 introduced as the result of canonical quantization, so that there is a strong evidence that canonical and path integral quantization can be reconciled if one takes AT as whole, i.e. including its gauge fields.

If so, one can perform in the third step canonical quantization by formal replacement $\eta_i \rightarrow \frac{\partial}{\partial \xi_i}$, as shown in Sec. 5. By means of a suitable ordering of operators in the expressions for $S^{AB}$ one can achieve that $s = \frac{1}{2}$ is the only allowed value of spin of the top, in accordance with its Aristotelian nature. The representation space of the generators $S^{AB}$ is spanned on the basis $D_1^{1/2}(\vartheta, \varphi, \psi)$, $k = \pm \frac{1}{2}$, $m = \pm \frac{1}{2}$, denoted here as $\chi^+, \zeta^+$ (see (6.6)). A massless top is described by the group $E(2) \times E(2)$ (Sec. 5). Its wave function $\Phi$ fulfils the equation (6.3)

$$V \Phi = 0, \quad W \Phi = 0 \quad (9.1)$$

where $V$ and $W$ are two commuting generators of the group $E(2)$. The solution of (9.1) can be expressed as $\Phi = \Omega^+(\xi_i) \Psi(x)$ where $\Omega^+ = (\chi^+, \zeta^+)$ and $\Psi$ is the solution of Dirac equation (see Sec. 6).

10. An attempt to derive $SU(2)_L \times U(1)_Y$ gauge symmetry

In the forth step on the basis of Sec. 8 we take first into account two kinds of...
rotators (tops). This allows us to distinguish between up type and down type fermions and to introduce the weak isospin. The configuration space consists of two subspaces parametrized by the coordinates $\xi_i$, $\xi_0 = \pm \sqrt{\xi_i^2}$. The representation space is spanned on the basis $\vartheta(\xi_0)\Omega^+(\xi_i)$, $\vartheta_0(-\xi_0)\Omega^+(\xi_i)$, where $\Omega^+(\xi_i)$ is defined by (6.6) and $\Omega^+ = (\chi'^+, \zeta'^+)$ forms the spinorials basis for $\xi_0 = -\xi$. To find this basis we first observe that the transition from $\xi_0 = \xi$ to $\xi_0 = -\xi$ in the generators $S^{AB}$ is equivalent to the transformation $\xi_i \to -\xi_i$, $\eta_i \to -\eta_i$ (see eq. (8.8)). In case of generators $U^\mu$, $U^\mu_c$, in addition, one has to change the sign: $U^\mu \to -U^\mu$, $U^\mu_c \to -U^\mu_c$, too (see the expressions (8.10) for $\nu_i$). However by a redefinition of $U^\mu$, $U^\mu_c$ one can omit this change of sign, so that the transition to region $\xi_0 \to -\xi$ is then the same for all $S^{AB}$ and is realized through the replacement $\xi_i \to -\xi_i$. Note that this redefinition is consistent with the commutation relations for $S^{AB}$ (5.1).

The basis for $\xi_0 \to -\xi$ is then defined as follows

$$\zeta'^+ = \zeta^+(-\xi_i) = \xi^{\frac{1}{2}}(ie^{i\varphi} \sin \frac{\vartheta}{2} e^{-\frac{i}{2}\varphi} \cos \frac{\vartheta}{2})$$

$$\chi'^+ = \chi^+(-\xi_i) = \xi^{-\frac{1}{2}}(-ie^{i\varphi} \cos \frac{\vartheta}{2}, -ie^{-\frac{i}{2}\varphi} \sin \frac{\vartheta}{2})$$

The extended generators have the form

$$S^{AB}_{\text{ext}} = \vartheta(\xi_0)S^{AB}(\xi_i) + \vartheta(-\xi_0)S^{AB}(-\xi_i)$$

and the corresponding wave functions are

$$\Phi_{\text{ext}}(x, \zeta_\mu) = \Omega^+(\xi_i)e(x)\vartheta(\xi_0) + \Omega^+(-\xi_i)\nu(x)\vartheta(-\xi_0)$$

where $e(x)$ and $\nu(x)$ are the wave functions of charged and neutral leptons respectively.

As indicated in Sec. 8, a superselection rule holds in the representation space of the operators $S^{AB}_{\text{ext}}$. The operator $\frac{\xi_0}{\xi}$ commutes with all $S^{AB}$

$$[S^{AB}_{\text{ext}}, \frac{\xi_0}{\xi}] = 0$$

and the relative phases of functions $\Omega^+(\xi_i)e(x)\vartheta(\xi_0)$ and $\Omega^+(-\xi_i)\nu(x)\vartheta(-\xi_0)$ are arbitrary, since their product is always zero due to $\vartheta(-\xi_0)\vartheta(\xi_0) = 0$. As a result the representation space can be decomposed into the direct sum of two incoherent subspaces labeled by two discrete values of $\frac{\xi_0}{\xi}$. The only interpretation of this superselection rule that can be considered in the present context is associated with the electric charge and this is the reason why the states belonging to different $\frac{\xi_0}{\xi}$ have been interpreted as up and down type fermions. With respect to this duplication of fermionic states the wave equations (9.1) must be replaced by

$$V_{\text{ext}}\Phi_{\text{ext}} = 0, \quad W_{\text{ext}}\Phi_{\text{ext}} = 0$$

where $V_{\text{ext}}$ and $W_{\text{ext}}$ are defined by means of $S^{AB}_{\text{ext}}$. 

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The equations (10.5) as they stand hold for genuine tops. To extend their validity to rotators one must implement the axial symmetry of rotators. This leads to next - the fifth - step, the introduction of the gauge symmetry of rotator. So far we have considered only symmetry $U(1)_A \times U(1)_Z$, because in Sec. 3 and 7 we have dealt with classical rotators, for which the transitions between the configuration subspaces with $\frac{\xi_0}{\xi} = \pm 1$ cannot occur. In quantum theory such transitions are allowed (by emission and absorption of $W^\pm$ bosons) and the definition of the rotator gauge symmetry must be generalized correspondingly. To this aim we first introduce the generators of the weak isospin

$$
T_1 = \frac{1}{2} I, \quad T_2 = \frac{i \xi_0}{2 \xi} I, \quad T_3 = -\frac{1}{2} \frac{\xi_0}{\xi}
$$

where $I$ is an inversion operator defined in the space of functions (10.3) as follows

$$
I \vartheta(\xi_0) = \vartheta(-\xi_0), \quad I \Omega^+(\xi_i) = \Omega^+(-\xi_i)
$$

One can easily show that

$$
T_i T_k + T_k T_i = \frac{1}{2} \delta_{ik}
$$

To make the derivation of $SU(2)_L \times U(1)_Y$ gauge symmetry more transparent, especially to show that it has its roots in the axial symmetry of rotator, we start with the simplest situation:

i) Two neutral gauge fields $A_\mu, Z_\mu$ and one kind of rotator. In the classical case we have seen in Sec. 3 and 7 that the symmetry is $U(1)_A \times U(1)_Z$ with the generators 1 and $U + U_0$, where $U = \nu_3, s$ is the projection of spin to symmetry axis and $U_0$ is some constant. In quantum theory the second generator becomes $-iU + U_0$, where $U = \nu_3, I = \xi_i \frac{\partial}{\partial \xi_i} = \xi_i \frac{\partial}{\partial \xi}$ (see eq. (5.5))). Since the projection of spin can assume only two values $s_3^i = \pm 1/2$ it is useful to introduce new gauge symmetry generators

$$
P_{L,R} = \frac{1}{2} \mp iU
$$

which are the projection operators to states with left-handed and right-handed chirality

$$
P_L \Omega^+ \psi = \frac{1}{2} \Omega^+(1 + \gamma_5) \psi = \Omega^+ \psi_L
$$

$$
P_R \Omega^+ \psi = \frac{1}{2} \Omega^+(1 - \gamma_5) \psi = \Omega^+ \psi_R
$$

where

$$
\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
$$

The gauge fields belonging to generators $P_L$ and $P_R$ will be denoted as $W_{3\mu}$ and $B_\mu$, respectively. The equations (6.3) in the explicit transcription read

$$
S^{\mu_4} p_\mu \Phi = 0, \quad S^{\mu_5} p_\mu \Phi
$$

(10.9)
The interaction corresponding to rotator mode is introduced by the replacement
\[ p_\mu = i \partial_\mu \rightarrow i \partial_\mu + \frac{g}{2} P_L W_{3\mu} + \frac{g'}{2} P_R B_\mu \]  
(10.10)

where \( g \) and \( g' \) are the coupling constants. The gauge symmetry is then \( U(1)_L \times U(1)_R \).

The relations to the original fields are given by the conventional rotation
\[ W_{3\mu} = \cos \vartheta_W Z_\mu + \sin \vartheta_W A_\mu \]
\[ B_\mu = -\sin \vartheta_W Z_\mu + \cos \vartheta_W A_\mu \]

where \( \cos \vartheta_W = \frac{g}{\bar{g}} \), \( \sin \vartheta_W = \frac{g'}{\bar{g}} \), \( \bar{g} = \sqrt{g^2 + g'^2} \). From (10.10) one obtains
\[ i \partial_\mu \rightarrow i \partial_\mu - e A_\mu + \frac{\bar{g}}{2} (U_0 - i U) Z_\mu \]  
(10.11)

where \( e = -\frac{g' \bar{g}}{2g} \) is the electric charge and \( U_0 = \frac{1}{2} \cos 2 \vartheta_W \).

\[ ii) \] Two neutral gauge fields and both kinds of rotators. If we for a while assume that the two kinds of rotators have the opposite electric charge the generators of gauge transformation have slightly different form: \( \frac{\xi_0}{\zeta} P_L, \frac{\xi_0}{\zeta} P_R \). Instead of (10.10) we will have
\[ i \partial_\mu \rightarrow i \partial_\mu + \frac{g \xi_0}{2 \zeta} P_L W_{3\mu} + \frac{g' \xi_0}{2 \zeta} P_R B_\mu \]  
(10.12)

and this replacement should be done in eq. (10.5). The electric charge is then \( eQ \), where \( e = \frac{g' \bar{g}}{g} \) and \( Q = -\frac{\xi_0}{2 \zeta} \). In fact the charges of up type and down type fermions do not differ in sign only. Rather the charge of the whole doublet is shifted by amount \( Q_0 \), but this can be without problems incorporated in our gauge symmetry pattern, since each \( U(1) \) symmetry generator is fixed up to an additive constant. Taking into account that \( \frac{1}{2} \frac{\xi_0}{\zeta} = -T_3 \) we see that the gauge symmetry contained in (10.12) is \( U(1)_{T_3 L} \times U(1)_{T_3 R} \).

This leads us to the final situation with

\[ iii) \] two neutral and two charged gauge fields as well as two kinds of rotators. From the comparison with EWSM we conclude that the final fixation of gauge generators must be the following
\[ -T_3 P_L \rightarrow -T_1 P_L \]  
(10.13)

\[ \frac{\xi_0}{\zeta} P_R \begin{cases} \rightarrow \frac{\xi_0}{\zeta} P_R + 1 = -Y, \text{for leptons} \\ \rightarrow \frac{\xi_0}{\zeta} P_R - \frac{1}{3} = -Y, \text{for quarks} \end{cases} \]  
(10.14)

The replacement (10.13) can be interpreted in our opinion as the generalization of the rotator axial symmetry to the interaction with charged gauge fields, besides the neutral ones. In fact the gauge symmetry has the group character and the only generalization of \( U(1)_{T_3} \) is \( SU(2) \).

As for replacement (10.14) the problem is how to explain the values of additive constants in \( Y \). But the same problem was encountered in EWSM, where it was solved.
using the condition following from cancellation of triangular chiral gauge anomalies. In
particular one gets for $Y$ the condition

$$
\sum_{\text{leptons}} Y + \sum_{\text{quarks}} Y = 0 \quad (10.15)
$$

Since $\sum \frac{\xi_0}{\bar{\xi}} = 0$ it follows from (10.14) that condition (10.15) is satisfied ($4.1 - 3.4.13 = 0$).

Taking the replacements (10.13) and (10.14) as at least intuitively justified we obtain from (10.12)

$$
i\partial_\mu \longrightarrow i\partial_\mu - g T_i P_L W_{i\mu} - \frac{g'}{2} Y B_\mu
$$

Inserting this in (10.5) we get

$$
S_{\mu4}^{\text{ext}}(i\partial_\mu - g T_i P_L W_{i\mu} - \frac{g'}{2} Y B_\mu)\Phi_{\text{ext}} = 0 \quad (10.16)
$$

$$
S_{\mu5}^{\text{ext}}(i\partial_\mu - g T_i P_L W_{i\mu} - \frac{g'}{2} Y B_\mu)\Phi_{\text{ext}} = 0 \quad (10.17)
$$

To pass from differential operators to Dirac matrices we use the relations found in [7]

$$
S^{AB}(\xi_i)\Omega^+(\xi_i) = \Omega^+(\xi_i)S^{AB}_{\text{matr}} \quad (10.18)
$$

where

$$
S^{\mu\nu}_{\text{matr}} = -\frac{1}{4}(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu})
$$

$$
S^{\mu4}_{\text{matr}} = -\frac{1}{2}\gamma^{\mu}\gamma_5
$$

$$
S^{\mu5}_{\text{matr}} = \frac{i}{2}\gamma^{\mu}
$$

$$
S^{45}_{\text{matr}} = \frac{i}{2}\gamma_5
$$

and the $\gamma$-matrices are given in the spinor representation

$$
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}
$$

The relations (10.18) naturally hold also for $\xi_i \longrightarrow -\xi_i$, so that $S^{AB}_{\text{ext}}$ can be expressed by means of $\gamma$-matrices, too. Note that $\gamma$-matrices do not provide room for the distinction between up and down type fermions, while the differential operators $S^{AB}_{\text{ext}}$ such a room do provide. From the expressions for $S^{\mu4}_{\text{matr}}$ and $S^{\mu5}_{\text{matr}}$ it follows that the equations (10.16) and (10.17) are not independent, so we restrict ourselves to (10.17) only.

It is suitable to introduce

$$
\Omega^+_{\text{ext}}(\xi_0, \xi_i) = (\vartheta(-\xi_0)\Omega(-\xi_i), \vartheta(\xi_0)\Omega^+(\xi_i))
$$
Then
\[ T_k \Omega^+_{\text{ext}} = \frac{1}{2} \Omega^+_{\text{ext}} \tau_k \]
where \( \tau_k \) are weak isospin Pauli matrices. The conventional equations for fermionic fields of EWSM follow from (10.17) if we insert
\[ \Phi_{\text{ext}} = \Omega^+_{\text{ext}} \psi_L + \vartheta(-\xi_0) \Omega(-\xi_i) \nu_R + \vartheta(\xi_0) \Omega^+(\xi_i) e_R \]
where \( \psi_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \)
We obtain
\[ \gamma^\mu (i \partial_\mu - \frac{g}{2} \tau \cdot W_\mu + \frac{g'}{2} B_\mu) \psi_L = 0, \quad (h_e \psi^R \phi + h_\nu \nu_R \phi_c) \]
\[ \gamma^\mu (i \partial_\mu + g' B_\mu) e_R = 0, \quad (h_e \phi^x \psi) \]
\[ \gamma^\mu i \partial_\mu \nu_R = 0, \quad (h_e \phi_c^x \psi) \]
where
\[ \phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \quad \phi_c = \begin{pmatrix} \phi^0 \\ -\phi^- \end{pmatrix} \]
\[ \phi^x = (\phi^-, \phi^0), \quad \phi_c^x = (\phi^0, -\phi^+) \]
and \( h_e, h_\nu \) are the coupling constants. In the brackets on the r.h.s. of last equations are the terms with Higgs fields. These terms do not follow so far from the model presented here and have been included for completeness.

11. Conclusions

The notion of AT introduced in this paper plays a useful mediatory role in establishing the link between strings and the spin 1/2 particles. On one hand this notion represents a suitable model of Dirac particles in the sense that it allows only minimal value of spin \( s = 1/2 \) for the particle. On the other hand AT can be interpreted as a string at low energies, when vibrational and rotational degrees of freedom are frozen. AT offers also some sort of reconciliation of two opposite interpretations of spin: one as coming from the rotation of a rigid body and the other as a mere quantum effect. According to the novel interpretation, spin is associated with the rigid body, but this body is of Aristotelian nature, so that a free particle does not rotate, although it has spin.

The mediatory role of AT has been exploited in the present paper for a derivation of EWSM from the AT dynamics. The starting point was a peculiar property of AT consisting in the fact that the rotator mode of AT can be realized only through the gauge
interaction with a vectorial field with the gauge group $U(1)$. This group represents the axial symmetry of the rotator, i.e. the symmetry under the rotations around the rotator axis. The second fact relevant for the derivation of EWSM was the observation that there are two kinds of rotators/tops. Arguments based on the superselection rule indicate that they can be interpreted as up-type and down-type fermions. In view of two types of rotators intuitive arguments lead to the conclusion that in general the axial symmetry of the rotator can be realized through the group $SU(2) \times U(1)$ and for the group $SU(2)$ one selects either singlet or doublet representation (the former for right-handed, the latter for left-handed fields).

Note some analogy with Kaluza-Klein interpretation of electromagnetic $U(1)$ gauge symmetry in 5D classical field theory. The extra space dimension is compactified on a circle, which possesses $U(1)$ isometry and due to the general covariance this isometry is converted into $U(1)$ electromagnetic gauge symmetry. Here the isometry of the circle is replaced by the rotator axial symmetry and the locality of gauge transformation is a consequence of spin 1/2 of the rotator (Sec. 4).

If our representation of electroweak gauge symmetry is correct, then the spontaneous violation of the symmetry must be associated with the bending of rotator under the influence of Higgs field and subsequent transformation of rotator to top. We have not studied in more details the models for such a transformation, because it is a problem per se, obviously related with the complicated issue of lepton and quark masses.

Instead, the problem which must be studied next is the $SU(3)$ color gauge interaction of quarks. If our basic philosophy is correct, AT must include not only leptons, but also quarks. Consequently in mathematical description of AT there must be a room, which allows to introduce quarks as a new particular sort of AT, in a similar way as it has been done in case of two categories of fermions, the up-type and down-type ones. Preliminary search in this direction revealed an interesting possibility, namely the infinite dimensional representations of Lorentz and $SO(3,3)$ group. As stressed several times before the spin of AT must be $1/2$ and following the conventional wisdom this requirement alone leads to Dirac equation and thus to finite dimensional representation of Lorentz group. But this conclusion is incorrect. Actually, spin is associated with the dimensionality of the representation of the rotational group $SO(3)$ in the rest system of the particle. Fixing the $s = 1/2$ option we have still the liberty to select either finite or infinite dimensional representation for the Lorentz boost. Choosing the latter possibility we obtain a particle with properties in some respects different from those, which share Dirac particles. In progress is a study with the aim to find out whether these properties fit quarks or not.

### Appendix A

For the sake of completeness we summarize here some kinematical definitions as well as the definitions of $SO(3,3)$ group generators $S^{AB}$. We also present some useful relations for $S^{AB}$. The body coordinate system of the rotator/top is defined by means
of unit vectors

\[ \mathbf{\nu}_1 = (\cos \psi \cos \varphi - \sin \psi \sin \varphi \cos \vartheta, \cos \psi \sin \varphi + \sin \psi \cos \varphi \cos \vartheta, \sin \psi \sin \vartheta) \]

\[ \mathbf{\nu}_2 = (- \sin \psi \cos \varphi - \cos \psi \sin \varphi \cos \vartheta, - \sin \psi \sin \varphi + \cos \psi \cos \varphi \cos \vartheta, \cos \psi \sin \vartheta) \]

\[ \mathbf{\nu}_3 = (\sin \vartheta \sin \varphi, - \sin \vartheta \cos \varphi, \cos \vartheta) \quad (A1) \]

where \( \vartheta, \varphi, \psi \) are Euler angles. A state of rotator/top is characterized by the canonically conjugated variables \( \xi_i, \eta_i \), where \( \xi_i \) are related to Euler angles

\[ \xi_1 = e^\psi \sin \vartheta \sin \varphi \]
\[ \xi_2 = -e^\psi \sin \vartheta \cos \varphi \]
\[ \xi_3 = e^\psi \cos \vartheta \quad (A2) \]

and \( \eta_i \) to spin components

\[ s_1 = \xi_2 \eta_3 - \xi_3 \eta_2 + \xi_1 \frac{\xi_1}{\xi_1^2 + \xi_2^2} \xi_i \eta_i \]
\[ s_2 = \xi_3 \eta_1 - \xi_1 \eta_3 + \xi_2 \frac{\xi_2}{\xi_1^2 + \xi_2^2} \xi_i \eta_i \]
\[ s_3 = \xi_1 \eta_2 - \xi_2 \eta_1 \quad (A3) \]

From the basic Poisson brackets relations

\[ \{\eta_i, \xi_k\} = \delta_{ik} \]
\[ \{\xi_i, \xi_k\} = \{\eta_i, \eta_k\} = 0 \]

one can derive the relations

\[ \{s_i, s_k\} = -\varepsilon_{ijk} s_j \]

The generators \( S^{AB} = -S^{BA} \) are defined as follows

\[ (S^{23}, S^{31}, S^{12}) = (s_1, s_2, s_3) \]
\[ (S^{01}, S^{02}, S^{03}) = (N_1, N_2, N_3) \]

where \( \mathbf{N} = \mathbf{s} \times \mathbf{\nu}_3 \). Furthermore

\[ S^{\mu 4} = -\frac{1}{2} (U^\mu + U^\mu*) \]
\[ S^{\mu 5} = \frac{1}{2i} (U^\mu - U^\mu*) \]
where \( U^0 = \nu \cdot s, U = -i\nu \times s, \nu = \nu_2 + i\nu_1 \) The vectors \( \nu \) and \( \nu_3 \) can be expressed by means of \( \xi_i \):

\[
\begin{align*}
\nu_1 &= -\frac{\xi_i}{\sqrt{\xi_1^2 + \xi_2^2}} (\xi_1 \frac{\xi_3}{\xi} + i \xi_2) \\
\nu_2 &= -\frac{\xi_i}{\sqrt{\xi_1^2 + \xi_2^2}} (\xi_2 \frac{\xi_3}{\xi} - i \xi_1) \\
\nu_3 &= \frac{\xi_i}{\xi} \sqrt{\xi_1^2 + \xi_2^2} \\
\nu_3 i &= \frac{\xi_i}{\xi}, \xi = \sqrt{\xi_i^2}
\end{align*}
\]

Finally

\[
S_{45} = U = \nu_3 \cdot s = \xi_i \eta_i
\]

The \( SO(3,3) \) Lie algebra is given by the Poisson brackets relations

\[
\{S^{AB}, S^{CD}\} = g^{BD} S^{AC} + g^{AC} S^{BD} - g^{BC} S^{AD} - g^{AD} S^{BC}
\]

where

\[
g^{00} = g^{44} = g^{55} = 1, \quad g^{11} = g^{22} = g^{33} = -1, \quad g^{AB} = 0 \text{ for } A \neq B.
\]

The following relations are direct consequences of the corresponding definitions

\[
\begin{align*}
S_{\mu\nu} S^{\mu\nu} &= 2U^2 \\
\tilde{S}^{\mu\nu} S_{\mu\nu} &= 0 \\
U_\mu U^\mu &= 0 \\
U^{\mu*} U^\nu - U^{\nu*} U^\mu &= -2iUS^{\mu\nu} \\
U^{\mu*} U^\nu + U^{\nu*} U^\mu &= 2S^{\mu\rho} S_{\rho\nu}
\end{align*}
\]

where

\[
\tilde{S}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} S_{\rho\sigma}
\]

Canonical quantization consists in the replacement \( \eta_i \to \frac{\partial}{\partial \xi_i} \). From \( s_i \) we then obtain

\[
\begin{align*}
I_1 &= \xi_2 \frac{\partial}{\partial \xi_3} - \xi_3 \frac{\partial}{\partial \xi_2} + \xi_1 \frac{\xi_2}{\xi_1^2 + \xi_2^2} \frac{\partial}{\partial \xi_i} \\
I_2 &= \xi_3 \frac{\partial}{\partial \xi_1} - \xi_1 \frac{\partial}{\partial \xi_3} + \xi_2 \frac{\xi_2}{\xi_1^2 + \xi_2^2} \frac{\partial}{\partial \xi_i}
\end{align*}
\]
\[ I_3 = \xi_1 \frac{\partial}{\partial \xi_2} - \xi_2 \frac{\partial}{\partial \xi_1} \]

Other \( S^{AB} \) generators are

\[ \mathbf{N} = -\mathbf{\nu}_3 \times \mathbf{I} + s\mathbf{\nu}_3 \]

\[ U^0 = -\mathbf{\nu}_1 \mathbf{I} \]

\[ \mathbf{U} = -i\mathbf{\nu} \times \mathbf{I} + is\mathbf{\nu} \]

\[ U = \mathbf{\nu}_3 \mathbf{I} = \xi_i \frac{\partial}{\partial \xi_i} \]

where \( s \) is an arbitrary constant. Additive terms in \( \mathbf{N} \) and \( \mathbf{U} \) are due to ordering ambiguity (\( \mathbf{I} \) does not commute with \( \mathbf{\nu}_3 \) and \( \mathbf{\nu} \)). \( U^0, U \) are complex conjugates of \( U^0, U \). With the same definitions of \( S^{AB} \) as in the classical case we obtain

\[ [S^{AB}, S^{CD}] = g^{BD} S^{AC} + g^{AC} S^{BD} - g^{BC} S^{AD} - g^{AD} S^{BC} \]  

(A8)

The quantum analogues of (A6) are

\[ S_{\mu\nu} S^{\mu\nu} = 2U^2 - 2s(s + 2) \]

\[ \tilde{S}^{\mu\nu} S_{\mu\nu} = -4(1 + s)U \]

\[ U_\mu U^\mu = 0 \]

\[ \frac{1}{2i} (U^\mu U^\nu - U^\mu U^\nu_c) = -US^{\mu\nu} + (1 + s)\tilde{S}^{\mu\nu} - g^{\mu\nu} U \]  

(A9)

\[ \frac{1}{2} (U^\mu U^\nu + U^\mu U^\nu_c) = S^{\mu\rho} S_\rho^{\nu} + 2S^{\mu\nu} - s(s + 2)g^{\mu\nu} \]

\[ S^{\mu\nu} U_\nu = i(U + i)U^\mu \]

**Appendix B**

Conventional configuration space of strings is a set of 3D curves given by the parametric equation \( \mathbf{x} = \mathbf{x}(\sigma) \), where \( \sigma \) is the length along the string. By means of Frenet’s equations

\[ \frac{d\mathbf{t}}{d\sigma} = k\mathbf{n} \]

\[ \frac{d\mathbf{n}}{d\sigma} = -k\mathbf{t} + \lambda\mathbf{b} \]  

\[ \frac{d\mathbf{b}}{d\sigma} = -\lambda\mathbf{n} \]  

(B1)
one can obtain two other configuration spaces. Here \( t, n, b \) are tangent, normal and binormal unit vectors, respectively, \( k(\sigma), \kappa(\sigma) \) are the curvature and the torsion of the string. Thus a curve can be equivalently characterized by \((k(\sigma), \kappa(\sigma), t(0), n(0), b(0), x)\), where \( x \) is the position vector of one of the two end points of the string. Solving the eq. (B1) one obtains another description of the curve: \((t(\sigma), n(\sigma), b(\sigma), x)\). If one identifies \( t = \nu_3, n = \nu_1, b = \nu_2 \), then a straight line given by \( k(\sigma) = 0, \kappa(\sigma) = 0 \) corresponds to a rotator. It is well known that in this case the unit vectors \( n, b \) can be chosen arbitrarily in a plane orthogonal to \( t \). This freedom corresponds just to the axial symmetry group \( U(1) \) of the rotator.

In quantum theory we expect that the state of a string (at least in a simplified nonrelativistic version) is characterized by the wave functions

\[
\Phi_s = \Phi_s[\nu_1(\sigma), \nu_2(\sigma), \nu_3(\sigma), x] \quad \text{or} \quad \Phi_s = \Phi_s[k(\sigma), \kappa(\sigma), \nu_1(0), \nu_2(0), \nu_3(0), x]
\]

In case of a frozen string \( k(\sigma), \kappa(\sigma) \) do not depend on time and ceased to be real degrees of freedom. Then the wave function reduces to

\[
\Phi_s = \Phi_s[\nu_1(0), \nu_2(0), \nu_3(0), x] = \Phi(\xi, x)
\]

which is the wave function considered in this paper. We have seen that \( \Phi(\xi, x) \) can describe spin 1/2 particle, so we expect that also in general situation corresponding to \( \Phi_s \), half - integral spin is allowed.

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