Characterization of affine differences and related forms

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Abstract. In the present paper we consider the problem of characterization of maps which can be expressed as an affine difference i.e. a three-place map of the form

\[ tf(x) + (1 - t)f(y) - f(tx + (1 - t)y). \]

We give a general solution of a functional equation associated with this problem.

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1. Introduction

Let \( f : G \to H \), where \( G \) is an Abelian group and \( H \) is a divisible Abelian group. The two place function

\[ F(x, y) = f(x) + f(y) - f(x + y), \quad x, y \in G \]

is called the Cauchy difference of \( f \). It is easy to verify that it satisfies the so-called cocycle functional equation

\[ F(x + y, z) + F(x, y) = F(x, y + z) + F(y, z) \]

which has been studied by many authors see \([2, 10–13]\) and \([14]\). It has occurred in different fields, including homological algebra, the Dehn theory of polyhedra, statistics and information theory. The general solution of the cocycle functional equation on abelian groups has been known for about half a century (see \([2, 11, 13]\)). It turns out that a function \( F : G \to H \) is a solution to the system of functional equations

\[
\begin{align*}
F(x + y, z) + F(x, y) &= F(x, y + z) + F(y, z), \quad x, y, z \in G \\
F(x, y) &= F(y, x), \quad x, y \in G
\end{align*}
\]
if and only if the function $F$ is representable as a Cauchy difference. The above characterization was proved by J. Erdős in [11] and independently by Jessen, Karpf and Thourup in [13]. M. Hosszú in [12] relying upon the above mentioned result of Erdős proved that without any assumptions the general solution of the cocycle functional equation is

$$F(x, y) = f(x) + f(y) - f(x + y) + B(x, y),$$

where $B$ is an arbitrary antisymmetric function i.e.

$$B(x, y) = -B(y, x),$$

which is additive in its single variables.

Although the general solution of the cocycle functional equation on abelian groups is well-known, the theory concerning solutions of that equation on commutative semigroups is far from complete. The first step to the development of this theory was done by B. Ebanks in [6], for more information on this topic the reader is referred to [4,5,7,9].

A characterization for differences of the form

$$\Delta(x, y) = f(x) + f(y) - \lambda f(\mu(x + y)),$$

where $f$ is an arbitrary function and $\lambda$ and $\mu$ are given parameters was given by Ebanks in [8]. In particular, he characterized a Jensen difference

$$\Delta(x, y) = f(x) + f(y) - 2f\left(\frac{x + y}{2}\right)$$

as a special case $\lambda = 2; \mu = \frac{1}{2}$. The above form is called a Jensen difference because it vanishes exactly when $f$ is a solution of the Jensen functional equation.

In the paper [18] we considered a problem of characterization of the so-called $t$-affine difference i.e. a two place map of the form

$$tf(x) + (1 - t)f(y) - f(tx + (1 - t)y)$$

where $t \in (0, 1)$ is a fixed number. In this paper we are going to consider a similar problem for the above difference but treated as a three-place map.

2. Results

Throughout the whole paper $X$ and $Y$ stand for real linear spaces and these assumptions will not be repeated in the sequel. The aim of the present paper is to characterize a difference $a_f : X \times X \times [0, 1] \to Y$ of the form

$$a_f(x, y, t) = tf(x) + (1 - t)f(y) - f(tx + (1 - t)y),$$
where \( f : X \to Y \). Recall that a function \( f : X \to \mathbb{R} \) is said to be \textit{convex}, \textit{concave}, \textit{affine} if
\[
af(x, y, t) \geq 0, \quad af(x, y, t) \leq 0, \quad af(x, y, t) = 0, \quad x, y \in X, \ t \in [0, 1],
\]
respectively. The above difference will be called an \textit{affine difference} because it vanishes exactly when \( f \) is an affine function. Let us observe that for all \( x, y \in X \) and \( t, s \in [0, 1] \) an affine difference satisfies the following properties:

(i) \( af(x, x, t) = 0 \),
(ii) \( af(x, y, t) = af(y, x, 1 - t) \),
(iii) \[
\begin{align*}
& saf(u, x, t) + (1 - s)af(v, y, t) - af(su + (1 - s)v, sx + (1 - s)y, t) \\
& = taf(u, v, s) + (1 - t)af(x, y, s) - af(tu + (1 - t)x, tv + (1 - t)y, s).
\end{align*}
\]

We omit the simple proof of properties (i) and (ii). For the proof of property (iii) fix \( x, y, u, v \in X, \) and \( t, s \in [0, 1] \) arbitrarily. We have
\[
\begin{align*}
& saf(u, x, t) + (1 - s)af(v, y, t) - af(su + (1 - s)v, sx + (1 - s)y, t) \\
& = s[tf(u) + (1 - t)f(x) - f(tu + (1 - t)x)] + (1 - s)[tf(v) \\
& + (1 - t)f(y) - f(tv + (1 - t)y)] \\
& - tf(su + (1 - s)v) - (1 - t)f(sx + (1 - s)y) + f(t[u + (1 - s)vf] \\
& + (1 - t)[sx + (1 - s)y]] \\
& = taf(u, v, s) + (1 - t)af(x, y, s) - af(tu + (1 - t)x, tv + (1 - t)y, s).
\end{align*}
\]

In our main result we show that conditions (i)-(iii) characterize exactly those maps \( \omega : X \times X \times [0, 1] \to [0, \infty) \) which can be expressed as an affine difference \( af \) for some convex function \( f : X \to \mathbb{R} \).

We already know one result in this spirit. During the 17-th Katowice-Debrecen Winter Seminar on Functional Equations and Inequalities Z. Boros [3] (solving the problem posed by M. Adamek [1]) presented the following result which gives a necessary and sufficient condition under which a three place function can be expressed as an affine difference.

**Theorem 1.** Let \( I \subset \mathbb{R} \) be an interval and let \( G : [0, 1] \times I \times I \to \mathbb{R} \). Then the following statements are equivalent:

(i) There exists a function \( f : I \to \mathbb{R} \) satisfying the functional equation
\[
f(tx + (1 - t)y) = tf(x) + (1 - t)f(y) + G(t, x, y)
\]
for every \( t \in [0, 1] \) and \( x, y \in I \).
(ii) \( G \) satisfies the functional equation
\[
G(s, rx + (1 - r)y, tx + (1 - t)y) = G(sr + (1 - s)t, x, y) - sG(r, x, y) - (1 - s)G(t, x, y)
\]
for all \( s, r, t \in [0, 1] \) and \( x, y \in I \).

In [18] we proved the following theorem which gives a general solution of the functional equation corresponding to the equation (iii) for fixed \( t = s \in (0, 1) \) which we are going to use in the sequel.

**Theorem 2.** Let \( X \) and \( Y \) be two linear spaces over the field \( K \) such that \( Q(t) \subset K \subset \mathbb{R} \), where \( t \in (0, 1) \) and \( Q(t) \) denotes the smallest field containing a singleton \( \{ t \} \). The general solution \( \omega : X \times X \rightarrow Y \) of the functional equation
\[
t\omega(u, x) + (1 - t)\omega(v, y) - \omega(tu + (1 - t)v, tx + (1 - t)y) = t\omega(u, v) + (1 - t)\omega(x, y) - \omega(tu + (1 - t)x, tv + (1 - t)y) \tag{1}
\]
is given by
\[
\omega(x, y) = d(x) + r(y) + tn(x) + (1 - t)n(y) - n(tx + (1 - t)y) + \bar{c}, \quad x, y \in X,
\]
where \( d, r : X \rightarrow Y \) are additive functions satisfying the condition
\[
(d + r)(tx) = t(d + r)(x), \quad x \in X,
\]
\( \bar{c} \in Y \) is an arbitrary constant and \( n : X \rightarrow Y \) is an arbitrary function.

In the proof of our first theorem we use the result which gives a general form of the solution to the following functional equation
\[
f(tx + (1 - t)y) + f((1 - t)x + ty) = f(x) + f(y), \quad x, y \in D,
\]
where \( D \) is a \( t \)-convex set i.e. \( tD + (1 - t)D \subset D \). Any function \( f : D \rightarrow Y \) satisfying the above equation is called \( t \)-Wright affine. Theorem 3 below was proved by K. Lajkó in [15] for real functions defined on an interval and was extended in the paper [16] for functions defined on more general structures. We present here a very particular case of Theorem 1.8 from [16] which is sufficient for our considerations.

**Theorem 3.** Let \( X \) and \( Y \) be two real linear spaces and let \( f : X \rightarrow Y \). Then \( f \) is a \( t \)-Wright affine function if and only if it has the form
\[
f(x) = a_0 + a_1(x) + a_2(x, x), \quad x \in X,
\]
where \( a_0 \in Y \) is a constant, \( a_1 : X \rightarrow Y \) is an additive function and \( a_2 : X \times X \rightarrow Y \) is a bi-additive and symmetric function satisfying the condition
\[
a_2(tx, (1 - t)x) = 0, \quad x \in X.
\]

The following theorem corresponds to Theorem 4 from [18] and Theorem 2.4 from [17].
The general solution

**Theorem 4.** The general solution \( \omega : X \times X \times [0,1] \rightarrow Y \) of the functional equation

\[
s\omega(u,x,t) + (1-s)\omega(v,y,t) - \omega(su+(1-s)v,sx+(1-s)y,t) \\
= t\omega(u,v,s) + (1-t)\omega(x,y,s) - \omega(tu+(1-t)x, tv + (1-t)y,s)
\]

is given by

\[
\omega(x,y,t) = d(x,t) + \bar{r}(y,t) + af(x,y,t) + c(t),
\]

where \( \bar{d}, \bar{r} : X \times [0,1] \rightarrow Y \) are additive functions with respect to the first variable satisfying the condition

\[
td(x,s) - d(tx,s) = sd(x,t) - d(sx,t) = \bar{r}(tx,s) - t\bar{r}(x,s), \quad x \in X, \quad s,t \in [0,1],
\]

\( c : [0,1] \rightarrow Y \) and \( f : X \rightarrow Y \) are arbitrary function.

**Proof.** It is a routine matter to verify that a function of the form (4) with (5) satisfies the equation (3). Conversely, assume that \( \omega \) satisfies equation (3).

First we show that formula (4) holds for all \( x, y \in X \) and \( t \in (0,1) \). By putting \( s = t \in (0,1) \) in (3) we see that the map

\[
\omega_t(x,y) := \omega(x,y,t)
\]

satisfies equation (1) from Theorem 2. By virtue of its thesis

\[
\omega(x,y,t) = d(x,t) + r(y,t) + tf_t(x) + (1-t)f_t(y) - f_t(tx + (1-t)y) + c(t),
\]

where \( d, r : X \times (0,1) \rightarrow Y \) are additive functions with respect to the first variable satisfying the condition

\[
(d + r)(tx,t) = t(d + r)(x,t), \quad x \in X,
\]

\( c : (0,1) \rightarrow Y \) and \( f_t : X \rightarrow Y \) are arbitrary functions. Now, we substitute the above form into (3) and obtain after rearrangement

\[
\begin{align*}
&sd(u-v,t) - d(s(u-v),t) + sr(x-y,t) - r(s(x-y),t) \\
&\quad + t[sf_t(u) + (1-s)f_t(v) - f_t(su + (1-s)v)] + (1-t)[sf_t(x) \\
&\quad + (1-s)f_t(y) - f_t(sx + (1-s)y)] \\
&\quad - sf_t(tu + (1-t)x) - (1-s)f_t(tv + (1-t)y) + f_t(s[tx + (1-t)x] \\
&\quad + (1-s)[tv + (1-t)y]) \\
&= td(u-x,s) - d(t(u-x),s) + tr(v-y,s) - r(t(v-y),s) \\
&\quad + t[sf_s(u) + (1-s)f_s(v) - f_s(su + (1-s)v)] + (1-t)[sf_s(x) \\
&\quad + (1-s)f_s(y) - f_s(sx + (1-s)y)] \\
&\quad - sf_s(tu + (1-t)x) - (1-s)f_s(tv + (1-t)y) + f_s(s[tx + (1-t)x] \\
&\quad + (1-s)[tv + (1-t)y]).
\end{align*}
\]
Putting $p = p_{s,t} = f_s - f_t$ we can rewrite the previous equation as

$$sd(u - v, t) - d(s(u - v), t) - td(u - x, s) + d(t(u - x), s) + sr(x - y, t) - r(s(x - y), t) - tr(v - y, s) + r(t(v - y), s) = t[sp(u) + (1 - s)p(v) - p(su + (1 - s)v)] + (1 - t)[sp(x) + (1 - s)p(y) - p(sx + (1 - s)y)] - sp(tu + (1 - t)x) - (1 - s)p(tv + (1 - t)y) + p(s[ts + (1 - t)x] + (1 - s)[tv + (1 - t)y]).$$

(6)

Putting $v = x$ and $u = y$ into the above equation we have

$$sd(y - x, t) - d(s(y - x), t) - td(y - x, s) + d(t(y - x), s) + sr(x - y, t) - r(s(x - y), t) - tr(x - y, s) + r(t(x - y), s) = t[sp(y) + (1 - s)p(x) - p(sy + (1 - s)x)] + (1 - t)[sp(x) + (1 - s)p(y) - p(sx + (1 - s)y)] - sp(ty + (1 - t)x) - (1 - s)p(tx + (1 - t)y) + p(s[ty + (1 - t)x] + (1 - s)[tx + (1 - t)y]).$$

(7)

Now, replacing $x$ by $y$ we arrive at

$$sd(x - y, t) - d(s(x - y), t) - td(x - y, s) + d(t(x - y), s) + sr(y - x, t) - r(s(y - x), t) - tr(y - y, s) + r(t(y - x), s) = t[sp(x) + (1 - s)p(y) - p(sx + (1 - s)y)] + (1 - t)[sp(y) + (1 - s)p(x) - p(sy + (1 - s)x)] - sp(tx + (1 - t)y) - (1 - s)p(ty + (1 - t)x) + p(s[tx + (1 - t)y] + (1 - s)[ty + (1 - t)x]).$$

(8)

Adding (7) and (8) we obtain

$$0 = p(x) + p(y) - p(sx + (1 - s)y) - p((1 - s)x + sy) - p(tx + (1 - t)y) - p(ty + (1 - t)x) + p(s[ty + (1 - t)x] + (1 - s)[tx + (1 - t)y]) + p(s[tx + (1 - t)y] + (1 - s)[ty + (1 - t)x]).$$

Putting $t = \frac{1}{2}$ into the previous equation we get

$$p(x) + p(y) - p(sx + (1 - s)y) - p((1 - s)x + sy) = 2p\left(\frac{x + y}{2}\right) - 2p\left(\frac{x + y}{2}\right) = 0, \quad x, y \in X.$$

We have shown that $p = p_{s,\frac{1}{2}}$ is an $s$-Wright affine function. On account of Theorem 3 $p$ has the form

$$p(x) = p_{s,\frac{1}{2}}(x) = b_s + a_s(x) + A_s(x, x), \quad x \in X,$$

(9)
where \( b_s \in Y \) is a constant, \( a_s : X \to Y \) an additive function, \( A_s : X \times X \to Y \) is a bi-additive and symmetric function satisfying the condition

\[
A_s(sx, (1 - s)x) = 0, \quad x \in X.
\]  (10)

Now, we put (9) into equation (6) (for \( t = \frac{1}{2} \)) and obtain after rearrangement

\[
sd(u - v, \frac{1}{2}) - d(s(u - v), \frac{1}{2}) + sr(x - y, \frac{1}{2}) - r(s(x - y), \frac{1}{2})
= \frac{1}{4} \left[ sA_s(u, u) + (1 - s)A_s(v, v) - A_s(su, su) - A_s((1 - s)v, (1 - s)v) + sA_s(x, x) + (1 - s)A_s(y, y) - A_s(sx, sx) - A_s((1 - s)y, (1 - s)y) \right]
\]

\[
- \frac{1}{2} \left[ A_s(sx, (1 - s)y) + sA_s(u, x) + (1 - s)A_s(v, y) + A_s(su, (1 - s)v) - A_s(su, sx) - A_s((1 - s)v, (1 - s)y) - A_s(su, (1 - s)y) - A_s(sx, (1 - s)v) \right].
\]  (11)

Put \( x = y = v = 0 \) in (11) to get

\[
\frac{1}{4} sA_s(u, u) - \frac{1}{4} A_s(su, su) = sd(u, \frac{1}{2}) - d(su, \frac{1}{2}), \quad u \in X, \ s \in (0, 1).
\]

Since \( A_s \) is bi-additive, \( d \) is additive with respect to the first variable and having property (10) in mind we infer that

\[
A_s(u, su) = A_s(su, su) = sA_s(u, u), \quad u \in X,
\]

and clearly also

\[
A_s((1 - s)u, (1 - s)u) = (1 - s)A_s(u, u), \quad u \in X.
\]

Analogously putting \( u = v = x = 0 \) into (11) we obtain

\[
0 = \frac{1}{4} (1 - s)A_s(y, y) - \frac{1}{4} A_s((1 - s)y, (1 - s)y) = r(sy, \frac{1}{2}) - sr(y, \frac{1}{2}),
\]

and consequently the left hand side of (11) is equal to zero. Using these properties and putting \( u = v = 0 \) into (11) we obtain

\[
A_s(sx, (1 - s)y) = 0, \quad x, y \in X,
\]

therefore, \( A_s \equiv 0 \) and \( p \) takes the form

\[
p(x) = p_{s, \frac{1}{2}}(x) = b_s + a_s(x), \quad x \in X.
\]

On the other hand

\[
p(x) = p_{s, \frac{1}{2}}(x) = f_s(x) - f_{\frac{1}{2}}(x), \quad x \in X,
\]

and hence

\[
f_s(x) = f_{\frac{1}{2}}(x) + b_s + a_s(x), \quad x \in X.
\]

Putting \( f := f_{\frac{1}{2}} \) we have

\[
f_s(x) = f(x) + b_s + a_s(x), \quad x \in X, \ s \in (0, 1).
\]
Now, substituting the above form back into the representation of $\omega$ we obtain

$$\omega(x, y, t) = d(x, t) + r(y, t) + c(t) + ta_t(x - y) - a_t(t(x - y)) + af(x, y, t).$$

Defining $\bar{d}, \bar{r} : X \times (0, 1) \to Y$ by

$$\bar{d}(x, t) := d(x, t) + ta_t(x) - a_t(tx), \quad \bar{r}(x, t) = r(x, t) + a_t(tx) - ta_t(x)$$

we see that $\bar{d}, \bar{r}$ are additive with respect to the first variable, and we can rewrite the formula for $\omega$ as

$$\omega(x, y, t) = \bar{d}(x, t) + \bar{r}(y, t) + c(t) + af(x, y, t), \quad x, y \in X, \; t \in (0, 1).$$

Now, we show that the above formula is also true for all $x, y \in X$ and $t \in \{0, 1\}$. To see it, put $s = 0$ in (3) to get

$$0 = t\omega(u, v, 0) + (1 - t)\omega(x, y, 0) - \omega(tu + (1 - t)x, tv + (1 - t)y, 0),$$

for all $x, y, u, v \in X$ and $t \in [0, 1]$, or equivalently,

$$\omega(t(u, v) + (1 - t)(x, y), 0) = t\omega(u, v, 0) + (1 - t)\omega(x, y, 0),$$

for all $x, y, u, v \in X$ and $t \in [0, 1]$ which means that the function

$$(x, y) \mapsto \omega(x, y, 0)$$

is affine. Therefore there exist a constant $c(0) \in Y$ and a linear function $a : X^2 \to Y$ such that

$$\omega(x, y, 0) = a(x, y) + c(0), \quad x, y \in X.$$ 

Let us define the functions $\bar{d}(:, 0), \bar{r}(:, 0) : X \to Y$ via the formulas

$$\bar{d}(x, 0) := a(x, 0), \quad \bar{r}(x, 0) := a(0, x), \quad x \in X.$$ 

It follows directly from the definition that both functions $\bar{d}(:, 0)$ and $\bar{r}(:, 0)$ are linear, moreover,

$$a(x, y) = \bar{d}(x, 0) + \bar{r}(y, 0), \quad x, y \in X,$$

hence

$$\omega(x, y, 0) = \bar{d}(x, 0) + \bar{r}(y, 0) + c(0), \quad x, y \in X.$$ 

Analogously, by putting $s = 1$ in (3) we see that the function

$$(x, y) \mapsto \omega(x, y, 1)$$

is affine and repeating the previous reasoning one can prove that

$$\omega(x, y, 1) = \bar{d}(x, 1) + \bar{r}(y, 1) + c(1), \quad x, y \in X,$$

where $c(1) \in Y$ is a constant and $\bar{d}(:, 1), \bar{r}(:, 1) : X \to Y$ are linear functions. Finally we get the desired form

$$\omega(x, y, t) = \bar{d}(x, t) + \bar{r}(y, t) + c(t) + af(x, y, t), \quad x, y \in X, \; t \in [0, 1].$$

(12)
To finish the proof, let us put (12) into equation (3) and having in mind that the map $a_f$ satisfies equation (3) we obtain
\[
\begin{align*}
    t\dd(u - x, s) - \dd(t(u - x), s) + t\rr(v - y, s) - \rr(t(v - y), s) & = s\dd(u - v, t) - \dd(s(u - v), t) + s\rr(x - y, t) - \rr(s(x - y), t),
    
\end{align*}
\]
for all $x, y, u, v \in X, s, t \in [0, 1]$. Setting $x = y = v$ in the above equation we get
\[
\begin{align*}
    t\dd(u - x, s) - \dd(t(u - x), s) & = s\dd(u - x, t) - \dd(s(u - x), t) & u, x & \in X, s, t \in [0, 1],
    
\end{align*}
\]
and analogously, by putting in (13) $u = v = y$ we obtain
\[
\begin{align*}
    \dd(t(u - x), s) - t\dd(u - x, s) & = s\rr(u - x, t) - \rr(s(u - x), t) & u, x & \in X, s, t \in [0, 1],
    
\end{align*}
\]
consequently (5) holds, which finishes the proof of the theorem. \hfill \Box

For maps satisfying equation (3) and a counterpart of condition (i) for $\omega$, as a consequence of the above theorem, we obtain the following result.

**Corollary 1.** A map $\omega : X \times X \times [0, 1] \to Y$ satisfies functional equation (3) and the condition
\[
\omega(x, x, t) = 0, \quad x \in X, \ t \in [0, 1]
\]
if and only if it has the form
\[
\begin{align*}
    \omega(x, y, t) & = d(x - y, t) + a_f(x, y, t), \quad x, y \in X, s, t \in [0, 1],
    
\end{align*}
\]
where $f : X \to Y$ is an arbitrary function and $d : X \times [0, 1] \to Y$ is a map which is additive with respect to the first variable, moreover,
\[
\begin{align*}
    td(x, s) - d(tx, s) & = sd(x, t) - d(sx, t), \quad x \in X, s, t \in [0, 1].
    
\end{align*}
\]
**Proof.** Suppose that $\omega$ is a solution of equation (3) and satisfies condition (14). From Theorem 4 we know that
\[
\begin{align*}
    \omega(x, y, t) & = d(x, t) + r(y, t) + a_f(x, y, t) + c(t), \quad x, y \in X, t \in [0, 1],
    
\end{align*}
\]
where $c : [0, 1] \to Y, f : X \to Y$ are arbitrary functions and $d, r : X \times [0, 1] \to Y$ are additive functions with respect to the first variable satisfying condition (5). Since
\[
\begin{align*}
    a_f(x, x, t) & = 0, \quad x \in X, \ t \in [0, 1],
    
\end{align*}
\]
then by (14) we get
\[
\begin{align*}
    d(x, t) + r(x, t) & = -c(t), \quad x \in X, \ t \in [0, 1].
    
\end{align*}
\]
Making use of the additivity of $d(\cdot, t) + r(\cdot, t)$ by putting $x = 0$ we obtain $c(t) = 0$ and hence
\[
\begin{align*}
    r(x, t) & = -d(x, t), \quad x \in X, \ t \in [0, 1].
    
\end{align*}
\]
\hfill \Box
Corollary 2. A map \( \omega : X \times X \times [0, 1] \to Y \) satisfies functional equation (3), condition (14) and is symmetric in the following sense

\[
\omega(x, y, t) = \omega(y, x, 1-t), \quad x, y \in X, \ t \in [0, 1]
\]  

if and only if it has the form \( \omega(x, y, t) = d(x-y, t) + a f(x, y, t) \), \( x, y \in X, \ s, t \in [0, 1] \), where \( f : X \to Y \) is an arbitrary function and \( d : X \times [0, 1] \to Y \) is a map which is additive with respect to the first variable and satisfies the following two conditions

\[
\begin{align*}
td(x, s) - d(tx, s) &= sd(x, t) - d(sx, t), \quad x \in X, \ s, t \in [0, 1], \\
d(x, t) + d(x, 1-t) &= 0, \quad x \in X, \ t \in [0, 1].
\end{align*}
\]

Proof. By Corollary 1

\[
\omega(x, y, t) = d(x-y, t) + a f(x, y, t), \quad x, y \in X, \ s, t \in [0, 1],
\]

where \( f : X \to Y \) is an arbitrary function and \( d : X \times [0, 1] \to Y \) is additive with respect to the first variable and satisfies condition (15). Since, as we mentioned at the beginning, any affine difference is symmetric we have

\[
d(x-y, t) = d(y-x, 1-t), \quad x, y \in X, \ t \in [0, 1],
\]
or, equivalently,

\[
d(x, t) + d(x, 1-t) = 0, \quad x \in X, \ t \in [0, 1].
\]

\( \square \)

In the proof of the main result of the paper we use the following particular case of Theorem 9 from [19].

Theorem 5. Assume that for some point \( y \in X \) a map \( \omega : X \times X \times [0, 1] \to [0, \infty) \) satisfies the following conditions:

\[(a) \ \omega(y, y, t) = 0, \]
\[(b) \ \omega(x, z, t) = \omega(z, x, 1-t), \]
\[(c) \ s\omega(u, z, t) + (1-s)\omega(v, z, t) - \omega(su + (1-s)v, z, t) \leq t\omega(u, v, s) - \omega(tu + (1-t)z, tv + (1-t)z, s), \]

for all \( u, v, x, z \in X \) and \( s, t \in [0, 1] \). Then for arbitrary \( c \in \mathbb{R} \) there exists a concave function \( g_y : X \to \mathbb{R} \) such that \( g_y(y) = c, g_y(x) \leq c, \ x \in X, \) and

\[
\omega(x, z, t) = g_y(tx + (1-t)z) - tg_y(x) - (1-t)g_y(z), \quad x, z \in X, \ t \in [0, 1].
\]

The main result of the paper reads as follows.

Theorem 6. A map \( \omega : X \times X \times [0, 1] \to [0, \infty) \) satisfies (14), (16) and functional equation (3) if and only if there exists a convex function \( f : X \to \mathbb{R} \) such that

\[
\omega(x, y, t) = a f(x, y, t), \quad x, y \in X, \ t \in [0, 1].
\]
Proof. As we have already seen an affine difference $a_f$ satisfies conditions (14), (16) and (3). Conversely, assume that $\omega$ satisfies conditions (14), (16) and (3). It follows from Corollary 2 that $\omega$ has the form

$$\omega(x, y, t) = d(x - y, t) + a_f(x, y, t), \quad x, y \in X, \ t \in [0, 1],$$

where $f : X \to \mathbb{R}$ is an arbitrary function, $d : X \times [0, 1] \to \mathbb{R}$ is an additive function with respect to the first variable satisfying conditions (17) and (18).

To finish the proof it remains to observe that $\omega$ satisfies conditions (a)-(c) from Theorem 5. It is a routine matter to verify that an affine difference satisfies these conditions. On the other hand the map $(x, y, t) \to d(x - y, t)$ satisfies condition (a) on account of the additivity of $d$ with respect to the first variable, condition (b) follows from condition (18) and finally (c) holds due to the additivity of $d(\cdot, t)$ and condition (17). By Theorem 5 there exists a concave function $g : X \to \mathbb{R}$ such that

$$\omega(x, y, t) = g(tx + (1 - t)y) - tg(x) - (1 - t)g(y), \quad x, y \in X, \ t \in [0, 1].$$

Finally, by putting $f := -g$ we see that $f$ is convex and

$$\omega(x, y, t) = a_f(x, y, t), \quad x, y \in X, \ t \in [0, 1].$$

\[\square\]

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