Safe Voting: Resilience to Abstention and Sybils

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Abstract
Voting rules may implement the will of the society when all eligible voters vote, and only them. However, they may fail to do so when sybil (fake or duplicate) votes are present and when only some honest (non sybil) voters actively participate. As, unfortunately, sometimes this is the case, our aim here is to address social choice in the presence of sybils and voter abstention. To do so we build upon the framework of Reality-aware Social Choice: we assume the status-quo as an ever-present distinguished alternative, and study Status-Quo Enforcing voting rules, which add virtual votes in support of the status-quo. We characterize the tradeoff between safety and liveness (the ability of active honest voters to maintain/change the status-quo, respectively) in several domains, and show that the Status-Quo Enforcing voting rules are often optimal. We comment on the applicability of our methods and analyses to the governance of digital communities.

Keywords: computational social choice, voting theory, sybil attacks, vote abstention, vote delegation

1 Introduction
Voting procedures are a simple and widely used way to aggregate the preferences of multiple individuals. Voting, however, can truly reflect the will of the society only insofar as all eligible people in the society—and only them—vote.

*This full paper combines and extends two preliminary papers published in conferences; in particular, the work of Shahaf et al. [1] and its follow-up by Meir et al. [2]. This version provides more extensive discussions, a presentation that combines these two preliminary papers, and additional results.
Indeed, this corresponds to two different challenges: the problem of sybil votes and the problem of partial participation. These problems are particularly crucial as a single vote may tilt a majoritarian group decision and as such, sybils infiltrating a group of agents that employ egalitarian democratic group decision making literally pose an existential threat to the group; a threat that intuitively – and as we formally show here – is amplified in the presence of vote abstention.

Sybil attacks, in which fake or duplicate identities (sybils) infiltrate an online community, pose a serious threat to such communities, as they might tilt community-wide decisions in their favor. While the extensive research on sybil identification (see in Section 1.1) may help keep the fraction of sybils in such communities low, in online communities one cannot assume sybils to be perfectly identified and completely eradicated. Furthermore, the vast literature on social choice proposes many aggregation methods that, unfortunately, cannot be directly used in many online settings, in which a fraction of the electorate might consist of sybils. Thus, our goal in this context is to enhance social choice theory with effective group decision mechanisms for communities with bounded sybil penetration; put differently, to develop group decision making processes that can be safely used in online communities that are not sybil-free.

Orthogonally, the problem of partial participation in online voting is particularly acute, as online voting often exhibit very low participation rates [3, 4]. For example, in the 2006 Cambridge MA participatory budgeting program, only 7.5% out of ∼64000 eligible voters actually participated [5] (see Section 1.1). Our goal in this context is to enhance social choice theory with effective group decision mechanisms that can handle both sybil votes and vote abstention.

Our key approach is inspired by Reality-Aware Social Choice [6]: we use the status quo as the anchor of resilience, characterized by safety – the inability of sybils or abstentions to change the status quo against the will of the genuine agents, and liveness – the ability of the genuine agents to change the status quo.

Specifically, we formalize these properties and design specialized resilient voting rules for different social choice settings and under different sets of assumptions. Intuitively, such rules are ‘conservative’ with an inherent bias towards the status quo (for safety), but not too conservative (to maintain liveness).

1.1 Related Work

We discuss related work regarding sybils and partial participation.

Sybils

There is a vast literature on defending against sybil attacks, see, e.g., two surveys on this topic [7, 8]. That literature is usually concerned with graphs on which the genuine and sybil entities reside, and the focus is usually not on group decision making but on identifying the sybils. As a prominent example, Douceur [2002] describes a very general model for studying sybil resilience and presents some initial negative results in this model. Others consider leveraging graph properties such as various centrality measures to identify suspicious nodes (see, e.g., [10]). As further examples, Molavi et al. [2013] aim to shield online ranking sites from the negative effects of sybils and Chiang et al. [2013] consider sybil-resilience in the context of radio networks.
We are particularly interested in sybil-resilient group decision making. This scenario is considered by Tran et al. [2009], but with a different goal and solution: While we aim to protect democratic decisions from sybil attacks, they are considering ranking online content. Other relevant papers are the paper of Conitzer and Yokoo [2010], concentrating on axiomatic characterizations of sybil-resilient rules in a certain formal model. In essence, Conitzer et al. show that in a model without a distinguished status quo alternative, the only voting rules which are sybil-safe, in the sense that there is no incentive for an attacker to produce sybils, is of the form “if all vote unanimously for \( c \), pick \( c \), otherwise pick a winner at random”. Indeed, this negative result can also be seen as a motivation for our model of sybil-safety, which does incorporate the status quo as a distinguished alternative, as this allows for a conservative default to the status quo, rendering the negative result of Conitzer et al. inapplicable. Other relevant papers are the paper of Wagman and Conitzer [15, 16], which consider design of mechanisms to be resilient to false-name manipulation where the creation of sybils incurs some non-negligible cost. Waggoner et al. [17] study ways to evaluate the correctness of a certain election result when the number of sybils in the electorate is assumed to be known. Conitzer et al. [18] consider using connections in a social network to increase the effectiveness of sybil resilient methods. Finally, we mention the recent work of Gersbach et al. [19, 20] that consider a situation with “well-behaving” and “misbehaving” voters that share some similarities with our work, however the model is different.

**Control and bribery**

We also mention the vast literature on control and bribery in elections [21], studying malicious entities aiming at changing elections outcomes (we also mention recent work connecting bribery to robustness measures of voting rules [22, 23]). The model of election control assumes a given voting rule and a given electorate, and the question is whether an external agent, called the chair of the election, may change the election structure, e.g. by adding or removing candidates or votes, to have its preferred candidate win (or lose). The model of a sybil attack is that of an external agent that cannot change the vote structure, but has control of the actual votes of a fraction of the electorate (the sybils and their creators/perpetrators). Hence, formally, a sybil attack by a fraction \( \sigma \in [0, 1] \) of the voters is similar to election control where the chair may add up to a fraction \( \sigma \) of the voters. However, rather than studying how this specific form of control may affect existing voting rules, we design new voting rules that are resilient to sybil attacks, a notion defined below.

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Partial participation

There is extensive work in the social choice literature on the strategic justification of partial participation/abstention, going back to the “paradox of nonvoting” [24–26]. Voting with a random set of active voters has been widely considered, and boils down to problems of statistical estimation. See e.g. [27, 28]. Other works consider ways to elicit the preferences of specific voters in order to reduce communication complexity [29]. Yet we are unaware of works that consider resilience to arbitrary partial participation.

‘Resilience’ analysis in voting

It is quite common in the voting literature to assume that votes may deviate from the preference profile. However most of the literature assumes either some sort of stochastic noise; or strategic behavior; or both. A prominent example is the ‘Calculus of Voting’ where voters decide whether and how to vote based on a known type distribution [24, 30].

However such distributions are rarely known either to the center or to the voters themselves, and strategic decisions may also be quite complex and relying on unknown factors. There is therefore value in understanding when results are resilient to some deviation from the benign behavior, as long as this deviation is not too large. This was done for example in the context of aggregation accuracy [31], where the authors assume that up to a certain number of bits in the profile may be corrupted. Two similar approaches from the side of a strategic voter include ‘Local Dominance’, where a voter assumes the real profile may deviate from her point estimate by some margin [32]; and ‘Safe Manipulation’, where a voter considers other similar voters may follow their behavior, but does not know how many [33].

1.2 Structure of the Paper

As our model has several ingredients of differing complexity, our basic approach in structuring the paper is to start from the simplest setting, and then add the orthogonal concepts and ingredients as we go along. We feel that this allows to first grasp the basic ideas and then, as the paper proceeds, to identify the changes that are needed to be made in the model to encompass the different aspects. Concretely, the paper is structured as follows:

• In Section 2 we consider the simplest social choice setting in which the voters should choose between the status-quo and a single alternative proposal. We begin with the simplifying assumption of full participation and introduce the fundamental concepts of safety and liveness (Section 2.2). We then relax the assumption of full participation and formulate the adaptation of safety and liveness to this setting (Section 2.3). Following these definitions, we are able to define the general Status-Quo Enforcing mechanism, analyze its safety-liveness tradeoff and prove its optimality in Section 2.4.

• In Section 3 we move beyond the binary domain -of Section 2, generalizing our results for the social choice settings of multiple alternatives (Section 3.1), multiple referenda (Section 3.2), and to single-peaked domains (Section 3.3).

For this purpose, we first extend the formal definitions of safety and liveness, essentially accepting as safe outcomes that are anywhere ‘between’ the honest outcome and the status-quo.

• In Section 4 we introduce an approximate notion for safety—essentially meaning that we may arrive to an alternative that is not the preferred alternative of the honest voters, however not far from it in some sense (in particular, such that its margin of defeat is not
too high). We analyze the effect of varying degrees of such approximate notion to the safety-liveness tradeoff that can be achieved for the social choice domains that are treated in Sections 2 and 3.

- In all previous sections, the definitions of safety and liveness are worst case definitions in two aspects, considering both the behavior of sybils and the identity of absentee as adversarial.

  In Section 5 we relax the latter assumption, by assuming instead that the active honest voters are chosen uniformly at random, and only requiring safety to hold with high probability as population grows. As expected, the safety-liveness tradeoff that can be achieved in this model with a weaker adversary are better—we analyze this improved tradeoff and prove it.

- The final technical section (Section 6) considers another modification of the environment by allowing inactive voters to delegate their vote. We show that—under common delegation assumptions—we can completely eliminate the dependency on the turnout (i.e. the fraction of active voters), as long as the number of active voters is not too small.

- We end the paper with a discussion and an outlook (Section 7).

2 The Basic Setting: Two Alternatives

In this section we introduce and analyze the simplest possible setting, where there are two alternatives. The alternatives are not the same: one of them (denoted \( r \)) stands for the current reality, or status-quo, whereas the other alternative, \( p \), can be viewed as a proposal to replace it.

Intuitively, replacing the status-quo for a bad proposal is considered worse than keeping a bad status quo, but replacing the status quo should still be possible. Correspondingly, we define the basic concepts of safety and liveness in face of sybils and partial participation, and show how the best trade off between them can be obtained in the worst case.

2.1 Preliminaries

We consider voting situations with a set \( A = \{r, p\} \) of alternatives, with \( r \) referred to as the current reality, or status-quo and \( p \) is a competing proposition/proposal.

There is a set of \( V \) of \( n \) voters, each specifying whether she prefers \( r \) to \( p \) or vice versa. A voting rule is a function taking the \( n \) votes and returning an outcome in \( A \). Most social choice settings we consider in this paper are such that each voter votes by picking an alternative and the aggregated outcome is also an alternative; thus, a voting rule \( R \) is a sequence of functions \( R^n : A^n \rightarrow A \), for all \( n \in \mathbb{N} \).

Honest voters and sybils

The set of voters \( V \) is partitioned into a set of honest (i.e., genuine; non-sybil) voters \( H \) and a set of sybil voters \( S \); so, \( V = H \cup S \) with \( H \cap S = \emptyset \). Ideally, we would like our voting rules to reflect only the preferences of the honest voters, but without access to who is honest and who is sybil, and when not all honest voters vote.
Fig. 1 Example of a voting setting with three alternatives \( A = \{ r, p, p' \} \). There are \( |V| = 10 \) voters overall, of which \( |S| = 3 \) are sybils, and \( |H^-| = 4 \) are inactive. Therefore \( s = \frac{3}{10} \) and \( h = \frac{4}{10} \). Similarly, \( s_p = \frac{|S_p|}{|V|} = \frac{2}{10} \) as there are 2 sybils voting for \( p \). We keep using full/hollow blue circles for active / inactive voters and red squares for sybils throughout the paper.

Further notation

In many places it will be convenient to refer to the fraction of some set of voters rather than to their absolute size. For any subset of voters \( U \subseteq V \), we denote by the lowercase letter \( u := \frac{|U|}{|V|} \) the relative size of this set to the entire population.

We denote by \( U_a \subseteq U \) the subset of \( U \) voters who prefer alternative \( a \in A \), and by \( u_a = \frac{|U_a|}{|V|} \) their relative fraction. An example of the different voters’ types and their notation is in Fig. 1.

Crucially, we do not know up front how many voters are sybils. However, we assume for the purpose of analysis that the fraction of sybils voters—which we denote by \( s \)—is upper bounded by the known parameter \( \sigma \in [0,1) \) (in Section 7 we discuss how to estimate such value). Thus, higher values of \( \sigma \) allow for a wider range of instances, and more difficult ones.

2.2 Safety and Liveness (Full Participation)

Suppose we have some preferred voting rule \( \mathcal{G} \), for the “standard” setting without sybils and with full participation. This may be due to favorable axiomatic or social properties of \( \mathcal{G} \), because of its simplicity, due to legacy, or for any other reason. For the setting of two alternatives, the Majority rule is natural – see discussion below as well. Ideally, we would like to always get outcome \( \mathcal{G}(H) \), that is, the result of all honest voters voting under \( \mathcal{G} \). However, if we use \( \mathcal{G} \) in a straightforward way, then the outcome may be distorted due to the existence of sybil votes, due to the partial participation, or both.

Example 1 (Sybils). As a simple example, say that we would like to use Majority rule. Our population consists of five active honest voters, three of which vote for \( r \) and two for \( p \). However if we add two sybils voting for \( p \) then the less desired outcome \( p \) now has most votes; see left figure.

Intuitively, this means Majority is unsafe in the presence of sybils.
We can also think about the opposite situation where the honest voters want \( p \), but the Majority rule maintains the status-quo due the presence of sybils, as in the right figure. This is not considered a violation of safety, since maintaining the status-quo is always safe.

**Base rules and the Majority rule**

Note that the Majority rule plays a double role in our examples above: it defines what is the desired outcome, and is also the voting rule being used.

In general we may use different rules for these roles; we will denote the base rule (which sets the desired outcome) by \( G \). Throughout this section, the base rule \( G \) will always be the Majority rule, as this is the only rule that is monotone, anonymous, and neutral \([34]\), and is generally the rule that makes most sense.

**Safety**

The examples above demonstrate that the Majority voting rule is unsafe (with respect to the Majority rule as the base rule, as discussed above); i.e., it may result in undesired departure from the status-quo in the presence of sybils.

We now define this property formally for the two-alternatives setting. The definition is straightforward under full participation, and we explain in Section 2.3 below how to apply it under partial participation.

**Definition 1** (Safety, two-alternatives, full participation). A voting rule \( R \) is safe with respect to the base rule \( G \) and active population \( V = S \cup H \), if \( R(V) \in \{G(H), r\} \).

That is, if the rule either selects the same outcome that the base rule would choose, if applied to honest voters only; or sticks to the status-quo.

The Majority voting rule is not safe with respect to itself and the population shown in Example 1, since \( MJ(V) = p \) whereas \( \{MJ(H), r\} = \{r\} \).

**Liveness**

Intuitively, if safety is our only requirement, then it is very easy to fulfill: just keep the status-quo \( r \) regardless of votes. However, a secondary requirement that we have is that the honest population of voters can enforce any outcome. Liveness does not depend on a base rule, nor on the structure of alternatives.

For an outcome \( a \in A \), and a set of votes \( U \), define \( U \rightarrow a \) as the same profile \( U \) where all voters vote to \( a \) (but their types remain the same).

**Definition 2** (Liveness, full participation). A voting rule \( R \) is live w.r.t active population \( V = S \cup H \), if for all \( a \in A \), \( R(S \cup H \rightarrow a) = a \).

Note that without any further restrictions, safety and liveness may be incompatible, e.g. if there are only sybils. We are therefore interested which rules are safe and live for all population that have certain limits on the fractions of problematic voters. For example, the Unanimity voting rule is safe whenever \( \sigma < 1 \) but not live for any \( \sigma > 0 \). A rule that requires a supermajority of \( 3/4 \) for \( p \), and otherwise selects \( r \), is safe (under full participation) for \( \sigma = \frac{1}{3} \) but not for \( \sigma = \frac{2}{3} \). We aim to understand the best possible tradeoff between safety and liveness.
2.3 Partial Participation

Next, we introduce to the model the possibility of voters to abstain from the vote. We need some further notation and definitions to capture this aspect first.

Active and passive voters

Recall that the set of voters $V$ is partitioned into a set of honest voters $H$ and a set of sybil voters $S$. As we assume the worst case, we assume that all sybil voters participate; but the set of honest voters, $H$, is further partitioned into $H^+ = H^+ \cup H^-$ (with $H^+ \cap H^- = \emptyset$), where $H^+$ is the non-empty set of honest voters who did cast a vote, and are thus labeled by their vote, and $H^-$ is the set of honest voters who did not cast a vote. We refer to the voters in $H^+$ as active honest voters and to the voters in $H^-$ as passive honest voters, or passive voters in short. Thus, both the active honest voters and the sybils are active. Denote $V^+ := H^+ \cup S$ and note that we now have that $V = H^+ \cup H^- \cup S$.

Further notation

As with the rate of sybils, we do not know up front how many voters are inactive. However, we assume for the purpose of analysis that the fraction of inactive voters—which we denote by $h^-$—is upper bounded by the known parameter $\mu \in [0, 1)$ (we discuss in Section 7 how to estimate this value). Thus, higher value of $\mu$ allow for a wider range of instances, and more difficult ones.

Example 2 (Abstention). Consider an example without sybils, where $|H_r| = 3$ and $|H_p| = 2$. If two of the active voters for $r$ abstain (i.e. $|H^+_r| = 1$ then $p$ would win, see left figure:

|     | unsafe | safe |
|-----|--------|------|
|     | ○ ○    | ●●   |
| $r$ |        |      |
| $p$ |        |      |

This again demonstrates how abstention, just like sybil participation, may lead to an unsafe outcome. As in Example 1, if abstention results in the selection of $r$ (as in the right figure), we should not consider this a violation of safety.

In order to turn this intuition into a formal definition, note that we determined the desired outcome $G(H)$ exactly as in Def. 1, but computed the realized outcome only on the active voters. Also, as we allow both sybils and abstentions, we assume naturally that no rule can distinguish between voters in $S$ and in $H^+$.

Definition 3 (Safety and Liveness under Partial Participation). For voting rule $R$:

- $R$ is safe with respect to $G$ and $V = S \cup H$ if $R(V^+) \in \{G(H), r\}$.
- $R$ is live with respect to $V = S \cup H$ if $R(S \cup H^+_a) = a$ for all $a \in A$.

We already saw that the Majority rule is not safe with respect to itself even if all voters are active (Example 1) or if there are no sybils (Example 2).
Example under Supermajority

Consider the \( \frac{3}{4} \)-supermajority rule \( R' \). We argue that, for the left instance in Example 1, \( R' \) is both safe and live. To see why it is safe, note first that \( R'(V^+) \) also selects \( r \), since \( p \) only has a \( \frac{4}{7} \)-majority which is less than \( \frac{3}{4} \). To see why it is live, note that if all honest voters switch to \( p \), then \( S \cup H_{+p}^+ \) has 7 votes to \( p \) vs. 0 votes to \( r \), so the \( \frac{3}{4} \)-supermajority rule will select \( p \).

It is also not hard to see that \( R' \) is safe and live for both instances in Example 2. In contrast, in the right instance in Example 1, there is nothing honest voters can do to get \( p \) selected under rule \( R' \). Therefore \( R' \) is not live for that instance. Weakening the supermajority requirement to anything strictly below \( \frac{4}{7} \) would regain liveness, since it would enable the three active honest voters to obtain any outcome.

2.4 Optimal Safety-Liveness Tradeoff

As the examples above hint, a possible way to obtain both safety and liveness is to relax neutrality, and explicitly prefer \( r \) to \( p \) unless the latter has supermajority—but not too large.

Definition 4. Suppose \( A = \{ r, p \} \). The \( \tau \)-Supermajority rule (\( \tau \)-SMJ) selects \( p \) if \( v_p > \frac{1+\tau}{2} \) voters vote for \( p \) (recall that \( v_p \) is the fraction of voters voting for \( p \)); otherwise, it selects \( r \).

Indeed, if we restrict attention to anonymous and monotone rules, then there is not much else we could do. Intuitively, as we increase the supermajority we require, we get more safety (i.e., for higher rates of sybils and abstention), but less liveness.\(^1\)

Our goal is to characterize this tradeoff.

The Status-Quo Enforcing mechanism

We next propose another way to modify the majority rule, using virtual voters. Informally, the Status-quo enforcing (SQ) mechanism is simply adding some amount of ‘virtual voters’ to the status-quo \( r \).

Definition 5 (Status-Quo Enforcing mechanism). Let \( R \) be a voting rule. Then, we define \( \tau \)-SQ-\( R(V) := R(V \cup Q) \), where \( Q \) is a set of \( \tau |V^+| \) voters voting for \( r \).

In particular, the Status-quo enforcing mechanism can be applied to Majority, in which case it is equivalent to supermajority.

Observation 1. For \( A = \{ r, p \} \) and any \( \tau \geq 0 \), the \( \tau \)-SQ-MJ rule and the \( \tau \)-SMJ rule coincide.

Proof. Consider the fraction \( v_p \) of votes for \( p \). The claim follows since \( v_p > v_r + q_r = (1 - v_p) + \tau \) (i.e. \( \tau \)-SQ-MJ selects \( p \)) iff \( v_p > \frac{1+\tau}{2} \) (i.e. \( p \) has supermajority). \( \square \)

The reason why we will be focusing on \( \tau \)-SQ-MJ rather than on \( \tau \)-SMJ is that the former naturally generalizes to other domains (see Section 3).

\(^1\)Here ‘more’ refers to the range of instances on which safety or liveness can be obtained. Later in Section 4 we propose an additional way to quantify safety of a rule on a given instance.

\(^2\)Note that \( |V^+| \) may be fractional but for most voting rules – including Majority – this is not a problem.
virtual votes

Fig. 2 Two instances from the previous examples, where Majority is unsafe with respect to itself but adding virtual voters (gray diamonds) restores safety with respect to Majority.

Safety and Liveness of the Status-Quo Enforcing Mechanism
The \(\tau\)-SQ-MJ is safer than Majority. We can see that on our running examples.

Example 3 (Status-Quo Enforcing Majority). Consider the unsafe instance from Example 1. Applying \(\frac{2}{3}\)-SQ-MJ to this instance would add \(\frac{2}{3}|V^+| = 2\) virtual voters on \(r\) (see Fig. 2, Left). Thus \(\frac{2}{3}\)-SQ-MJ\((V) = MJ(V \cup Q)\) with 5 voters on \(r\) vs. only 4 on \(p\). So \(r \in \{MJ(H), r\}\) wins and safety is restored.

Similarly, applying \(\frac{1}{3}\)-SQ-MJ to the unsafe instance from Example 2 adds \(\frac{1}{3}|V^+| = 1\) virtual voter on \(r\) (see Fig. 2, Right). By tie-breaking \(r\) wins so there is no violation of safety.

We are now ready to answer our main question regarding safety-liveness tradeoff in a binary setting. The proof for safety is omitted, as the theorem will follow as a special case from more general Theorem 13.

Theorem 2. The following hold:

* \(\tau\)-SQ-MJ is safe w.r.t \(MJ\) iff \(\tau \geq \frac{1+\sigma}{1-\mu} - 1\).
* \(\tau\)-SQ-MJ is live iff \(\tau < \frac{2(1-\sigma-\mu)}{1-\mu} - 1\).
* There is a value \(\tau\) that guarantees both iff \(3\sigma + 2\mu < 1\).

E.g., with 20% sybils and 20% abstention, or with 10% sybils and 35% abstention, we can get both safety and liveness by respecting the condition of the above theorem.

One way to visualize the safety-liveness tradeoff is in Fig. 3. We can see that when \(\sigma\) and \(\mu\) are low (meaning few sybils and low abstention), there is a wide range of mechanisms that are both safe and live, but this range diminishes as \(\sigma\) and/or \(\mu\) is increasing, becoming empty when \(3\sigma + 2\mu \geq 1\).

Proof of Theorem 2 (liveness). Suppose first that \(\tau < \frac{2(1-\sigma-\mu)}{1-\mu} - 1\). The worst case for liveness is when all voters are on \(r\).

In the profile \(H_{\rightarrow p}\) there will be \(|H^+| \geq (1 - \mu - \sigma)|V|\) active votes for \(p\), vs. at most \(\sigma|V| + \tau|V^+| = (\sigma + \tau(1 - \mu))|V|\) active votes for \(r\). We compare:

\[
v^+_p \leq \sigma + \tau(1 - \mu) < \sigma + \left(\frac{2(1-\sigma-\mu)}{1-\mu} - 1\right)(1 - \mu)
= \sigma + 2(1 - \sigma - \mu) - (1 - \mu) = 1 - \sigma - \mu \leq v^+_p,
\]

so \(p\) is selected.
In this figure (solid lines) the fraction of sybils is fixed at $\sigma = 0.1$, i.e. 10% sybils. For every value of abstention $\mu$, we color in blue the range of $\tau$-SQ-MJ mechanisms that are safe. The range of live mechanisms is in red. The dotted lines mark the ranges when there are 20% sybils rather than 10%.

In the other direction, set $s = \sigma$ and $h^- = \mu$ and then all weak inequalities become equalities, and the strict inequality flips, so $MJ(H_{\rightarrow p}) = r$.

**Lower bound**

We complement our analysis with the following lower bound, showing the optimality of $\tau$-SQ-MJ.

**Theorem 3.** There is no mechanism $R$ such that $R$ is both safe (with respect to Majority) and live when $3\sigma + 2\mu \geq 1$.

**Proof.** Assume towards a contradiction that such a mechanism $R$ exists. By liveness, there is a profile $V$ with $s_r = \sigma$ (i.e. all allowed sybils exist and are voting for $r$), and yet $p$ is selected, i.e. $R(V^+) = R(S \cup H^+) = p$. The total number of active voters for $p$ is $h^+_p$. Note that $h^+_p \leq h^+ \leq 1 - \mu - \sigma$.

Now, consider a profile $V = \overline{S} \cup \overline{H} \cup \overline{H}^-$, where $|\overline{S}| = |S|$, $|\overline{H}^+| = |H^+|$, $|\overline{H}^-| = |H^-|$, so $\sigma$ and $\mu$ are still respected in $V$. Set $\overline{\tau}_p := \min\{h^+_p, \sigma\}$ sybils to vote for $p$, as well as exactly $\overline{\tau}^+ : = h^+_p - \overline{\tau}_p$ honest voters. All other voters vote for $r$ (including all inactive honest voters). Since $\overline{\tau}^+_p = v^+_p$ and $\overline{\tau}^-_p = v^-_p$, the profiles $V$ and $\overline{V}$ are indistinguishable for $R^+$, and we have $R^+(V) = p$ as well.

We will show that $\overline{\tau}_r \geq \overline{\tau}_p$, which entails a violation of safety. Suppose first that $\sigma < h^+_p$. Then,

$$\overline{\tau}_r - \overline{\tau}_p = (\overline{\tau}^- + \overline{\tau}^+_r) - \overline{\tau}^-_p = (\overline{\tau}^- + \overline{\tau}^+ - \overline{\tau}^+_p) - \overline{\tau}_p$$

\(^3\)We assume that there is at least one honest voter, otherwise safety is meaningless.
\[ \bar{h}^- + \bar{h}^+ - 2\bar{h}_p^+ = \mu + (1 - \sigma - \mu) - 2\bar{h}_p^+ \\
= 1 - \sigma - 2(h_p^+ - \bar{\sigma}) = 1 - \sigma - 2(h_p^+ - \sigma) \\
= 1 + \sigma - 2h_p^+ \geq 1 + \sigma - 2(1 - \mu - \sigma) \\
= 3\sigma + 2\mu - 1 \geq 0, \]

where the last inequality is by the premise of the theorem.

If \( \sigma \geq h_p^+ \), then

\[ \bar{h}_p^+ = h_p^+ - \min\{h_p^+, \sigma\} = h_p^+ - h_p^+ = 0, \]

i.e., \( \overline{V} \) contains no honest voters for \( p \) at all, which means \( \overline{r} > \overline{h}_p \). \( \square \)

3 Beyond the Binary Domain

The modification we applied to the Majority voting rule simply added ‘virtual votes’ on the status quo. It is not hard to see that this idea easily extends to many other domains, i.e. that \( \tau \)-SQ-\( \mathcal{R} \) is well-defined for any voting rule \( \mathcal{R} \) in any domain were votes can be thought of as positions in some space.

However, our current definition of safety is too narrow. For example, suppose that \( A \) is the real line, the status quo is \( r = 0 \), and some rule \( \mathcal{G} \) is our base rule (say, Median). If the honest population prefers \( \mathcal{G}(H) = 3 \), then only ’0’ and ’3’ are considered ‘safe’. But if we are willing to accept both ’0’ and ’3’, then it makes sense to all accept all outcomes in between. Indeed this is the logic behind our general definition of \textit{between set} below.

Our definition of liveness also needs an adaptation: The space of allowed ballots may not coincide with \( A \), and thus \( H_{\rightarrow a} \) may not be well-defined.

\textit{Betweeness}

Our modeling is such that, generally, we view the alternatives as residing in a metric space (the specific metric space considered in each of our results will be clear from the context), where each vote specifies one alternative in that space. Any metric space \( (A, \delta) \) induces a natural trinary relation of \textit{betweenness}, where \( b \) is between \( a \) and \( c \) if \( \delta(a, b) + \delta(b, c) = \delta(a, c) \) (see [35, 36]).

**Definition 6** (Between set). For \( x, y \in A \), \( B(x, y) \subseteq A \) is the set of all points that are between \( x \) and \( y \), including \( \{x, y\} \). We define \( B(x; Y) := \bigcup_{y \in Y} B(x, y) \).

We can now extend Def. 3 to measure safety in any domain, with the appropriate between set \( B \). The difference from Def. 3 is colored in dark green.

**Definition 7** (Safety, general domain). \( \mathcal{R} \) is safe with respect to \( \mathcal{G} \) and \( V = S \cup H \) if \( \mathcal{R}(V^+) \in B(r; \mathcal{G}(H)) \).

We also extend the definition of liveness, by allowing honest voters to vote arbitrarily in \( H_{\rightarrow a} \). This is similar to the difference between a voting rule being \textit{unanimous} and being \textit{onto} (difference from Def. 3 is highlighted):
Fig. 4 We consider two instances with five active votes. On the left there is an instance where any \( \tau\)-SQ-PL mechanism with less than \( 3 \times 0.6 \cdot |V^+| \) virtual votes violates safety, since \( p' \) is selected. On the right there is another instance where at least 3 virtual voters mean violation of liveness since \( r \) is selected regardless of how honest voters vote.

**Definition 8** (Liveness, general domain). A voting rule \( R \) is live w.r.t population \( V = S \cup H \), if for all \( a \in A \), there is some alternative vote \( H \rightarrow a \) of the honest voters such that \( R(S \cup H^+ \rightarrow) = a \). The definitions above allow us to analyze different social choice settings; in the next subsections, we consider the following social choice settings – below we mention what the between set means for each of them:  
- Multiple alternatives: That is, a discrete unordered set \( A \). Here \( B(x, y) = \{x, y\} \) as in the binary setting;  
- Multiple referenda: with \( d \) binary issues and the Hamming distance, i.e. \( A = \{0, 1\}^d \). Then \( B(x, y) \) is the smallest box containing both \( x \) and \( y \) [37];  
- Single-peaked domains: including the real line, in which \( B(x, y) \) is the smallest interval containing both \( x \) and \( y \).

**Remark 1.** Note that in the first case with two or more unordered alternatives, the general definitions of safety and liveness collapse to the simple ones we used in the previous sections (Def. 3).

### 3.1 Multiple Alternatives

Here we consider setting in which \( A \) is the set of alternatives with \( r \in A \) being the status quo, but in which \( A > 2 \). In contrast to the binary domain, where the Majority rule is the natural base rule, when \( |A| > 2 \) there are many reasonable voting rules in the literature. We start by extending some of our results to Plurality voting, then considering other voting rules.

**Plurality**

We can naturally extend the \( \tau\)-SQ-MJ mechanism, by using the Plurality rule \( R = PL \). That is, the mechanism \( \tau\)-SQ-PL applies the Plurality rule after adding a fraction of \( \tau \) voters to \( r \).

**Observation 4.** \( \tau\)-SQ-PL cannot be both safe with respect to Plurality and live for three alternatives. This is regardless of \( \tau \), and even if there is full participation (\( \mu = 0 \)) and only \( \sigma > 0.2 \) sybils.

To see why, let \( \varepsilon \in (0, (\sigma - 0.2))/2 \). Consider candidates \( \{r, p, p'\} \) and suppose that \( h_p = 0.4 \) honest voters vote \( p \), and all other voters vote \( p' \). Thus \( h_{p'} = 1 - h_p - \sigma < 1 - 0.4 - 0.2 = h_p \) and \( p \) is the truthful outcome. A safe rule must therefore select \( \tau\)-SQ-PL\((V) \in B(r; p) = \{r, p\} \).
Since $v_{p'} = h_{p'} + \sigma = 0.6 > v_p$, we get that $p'$ is selected (which violates safety, see Fig. 4, Left), unless $\tau \geq 0.6$. However if $\tau \geq 0.6$ then in a profile where all $\sigma$ vote $r$ there are $\tau + \sigma > 0.6 + 0.2 = 0.8 > h$ so neither $p$ nor $p'$ can be selected, regardless of how honest voters vote—i.e. liveness is violated.

The bound of 0.2 is not tight, but instead of trying to characterize exactly the (deteriorated) safety-liveness tradeoff of $\tau$-SQ-PL, we return to the $\tau$-SMJ rule (see Def. 4). Its natural extension to multiple alternative is to select the unique alternative with strictly more than $\frac{1}{2} + \tau$ votes, if one exists, and otherwise return $r$.

It turns out that when there are more than 2 alternatives, the mechanism no longer coincides with $\tau$-SQ-PL. Moreover, $\tau$-SMJ inherits the same safety and liveness guarantees from the binary case, whereas the example above shows that $\tau$-SQ-PL does not.

**Theorem 5.** The $\tau$-SMJ voting rule is safe w.r.t Plurality if and only if $\tau \geq \frac{1 + \sigma}{1 - \mu} - 1$.

**Theorem 6.** The $\tau$-SMJ voting rule is live if and only if $\tau < \frac{2(1 - \sigma - \mu)}{1 - \mu} - 1$.

Note that the bounds in the theorems are identical to the bounds for $\tau$-SMJ in the binary case (Section 2.3), which are the same bounds as $\tau$-SQ-MJ. Theorem 5 follows as a special case from Theorem 14 in Section 4.3. For liveness, the number of alternatives is irrelevant so the proof of the binary case immediately applies for Theorem 6.

In particular, obtaining both safety and liveness is possible iff $2\sigma + 3\mu \leq 1$ (i.e. just as in the binary case).

Note that in the example above where $\tau$-SQ-PL fails (with 0.4 of voters on $p$ and the rest on $p'$), using e.g. 0.3-SMJ is safe, since $v_{p'} = 0.6 < 0.65 = (1 + \tau)/2$, and thus 0.3-SMJ($V$) = $r$.

Another feature of the $\tau$-SMJ rule is that it may select $r$ even if no one voted for it!

### Condorcet Conservative rules

Both Plurality and Supermajority allow only a simple ballot where every voter votes for a single alternative (plurality/1-approval ballots).

However there are many other rules that are based on ranking the alternatives (i.e., voting rules for ordinal-based elections), such as Borda and other positional scoring rules, Maximin, STV and so on. Many voting rules are guided or justified by selecting the Condorcet winner, when one exists. The outcome of these rules typically differ when there is no Condorcet winner.

A ‘conservative’ decision in the current context, would mean selecting the status-quo $r$ whenever there is no Condorcet winner. We call this rule the Condorcet Conservative rule (CC).

The $\tau$-Super Condorcet Conservative rule ($\tau$-SCC) is similar but $p_i$ only beats $p_j$ if it has a supermajority of $\frac{1 + \tau}{2}$ of the votes. That is, if there is an alternative $p$ that has a supermajority against any other alternative (including $r$) it is selected, and otherwise $r$ is selected.

**Proposition 7.** The following hold:
- $\tau$-SCC has the same liveness guarantees as $\tau$-SMJ.
- Let $\mathcal{G}$ be any Condorcet consistent rule. Then $\tau$-SCC has the same safety guarantees with respect to $\mathcal{G}$, as $\tau$-SMJ has with respect to MJ.

**Proof.** We prove each claim separately.
Liveness:

Let $\tau, \mu, \sigma \geq 0$ such that $\tau$-$SMJ$ is live, and consider some $p \in A$. Set $H_\rightarrow$ s.t. all voters rank $p$ at the top. In particular, when comparing $p$ to any other alternative $p'$ (including $r$), all honest voters vote for $p$ and thus liveness of $\tau$-$SMJ$ entails that $p$ is selected, i.e. has the required $\tau$-supermajority over $p'$. Thus $\tau$-$SCC(S \cup H_\rightarrow^+) = p$.

Safety:

Let $\tau, \mu, \sigma \geq 0$ such that $\tau$-$SMJ$ is safe. Consider any profile $V = H \cup S$ where some $p \neq r$ wins in $\tau$-$SCC(V^+)$ (otherwise safety is trivial). Then we need to show that $G(H) = p$.

Indeed, consider any $p' \neq p$ (including $r$). Since $\tau$-$SCC(V^+) = p$, we know that in the pairwise match of $p$ vs. $p'$, there is a fraction of at least $(\frac{1+\tau}{2})v^+$ voters that prefer $p$, meaning that $p$ beats $p'$ under $\tau$-$SMJ$.

By safety of $\tau$-$SMJ$ (and since $p \neq r$), this means that more than half of the honest voters prefer $p$ over $p'$. Since this holds for all $p' \neq p$, we have that $p$ is the Condorcet winner of $H$, and thus $G(H) = p$.

An immediate implication of Prop. 7 is that the bounds of Theorem 2 hold also for the $\tau$-$SCC$ rule.

3.2 Multiple Referenda

We move to the social choice setting of multiple referenda. That is, suppose that $A \ni r$ where w.l.o.g. $0$ (otherwise safety is trivial). Let $V = H \cup S$ where some $p \neq r$ wins in $\tau$-$SCC(V^+)$ (otherwise safety is trivial). Then we need to show that $G(H) = p$.

For any given profile $V = H \cup S$ where some $p \neq r$ wins in $\tau$-$SCC(V^+)$ (otherwise safety is trivial). Then we need to show that $G(H) = p$.

Indeed, consider any $p' \neq p$ (including $r$). Since $\tau$-$SCC(V^+) = p$, we know that in the pairwise match of $p$ vs. $p'$, there is a fraction of at least $(\frac{1+\tau}{2})v^+$ voters that prefer $p$, meaning that $p$ beats $p'$ under $\tau$-$SMJ$.

By safety of $\tau$-$SMJ$ (and since $p \neq r$), this means that more than half of the honest voters prefer $p$ over $p'$. Since this holds for all $p' \neq p$, we have that $p$ is the Condorcet winner of $H$, and thus $G(H) = p$.

An immediate implication of Prop. 7 is that the bounds of Theorem 2 hold also for the $\tau$-$SCC$ rule.

Proposition 8. The following hold:

- $\tau$-$IMJ$ has the same liveness guarantees as $\tau$-$SQ-MJ$.
- $\tau$-$IMJ$ has the same safety guarantees with respect to $IMJ$, as $\tau$-$SQ-MJ$ has with respect to $MJ$.

Proof. For an issue $j \leq d$ and voter set $U$, we denote by $U|_j \in \{0,1\}^{|U|}$ the projected opinions of all $U$ voters on issue $j$. We prove each claim separately.

Liveness: Let $\tau, \mu, \sigma \geq 0$ such that $\tau$-$SQ-MJ$ is live. Consider some position $p \in \{0,1\}^d$. For any given profile $V = H \cup S$, set $H_\rightarrow$ s.t. all honest voters vote for $p$. This means that in $S \cup H_\rightarrow$ at least $h^+$ honest voters agree with $p_j$ for every issue $j$. From liveness of $tREM$ it follows that $\tau$-$SQ-MJ((S \cup H_\rightarrow^+)|_j) = p_j$. Thus

$$\tau$-$SQ-MJ(S \cup H_\rightarrow^+) = (p_j)_{j \leq d} = p$$

Safety: Let $\tau, \mu, \sigma \geq 0$ such that $\tau$-$SQ-MJ$ is safe. Suppose that $\tau$-$SQ-MJ(V^+) = p \neq r$ (otherwise 0-safety is trivial). To show safety, we need to prove $p \in B(r; IMJ(H))^d$. This means showing $p_j \in \{r_j, IMJ(H)_j\}$ for all $j \leq d$.

This is the first nontrivial use of the “betweeness” notion in the paper, i.e. where the set contains not just $r$ and $G(H)$. See Definition 6.
By safety of $\tau$-$SQ-MJ$, we know that $\tau$-$SQ-MJ(V^+|_j) \in \{r_j, MJ(H|_j)\}$ for all $j$. To complete the proof, we observe that $\tau$-$SQ-IMJ(V^+|_j) = \tau$-$SQ-MJ(V^+|_j)$ and that $\{r_j, IMJ(H|_j)\} = \{r_j, MJ(H|_j)\}$.

As with the Condorcet Conservative rule, we can conclude that the bounds in Theorem 2 apply to $\tau$-$SQ-IMJ$.

### 3.3 Intervals and the Real Line

In this section we consider voters that pick a position on a line. The natural base rule to consider here is the Median rule ($MD$),\(^5\) which returns the position of the median voter on the real line (everything in this section also applies naturally to bounded intervals). The median rule has many desired properties such as Condorcet consistency, strategyproofness, and social optimality [39–41].

Cohensius et al. [2017] consider the case with very few active participants but with no sybils,\(^6\) and we return to their model in Section 6.

As in the previous sections, we consider the $\tau$-$SQ-MD$ rule which places $\tau|V^+|$ virtual voters on the status-quo $r$, and analyze its safety and liveness guarantees using a reduction to the binary setting.

We consider an arbitrary population $V = H^+ \cup H^- \cup S$ with partial participation and sybils and consider $\tau$-$SQ-MD$. We use the following straightforward connection between the median and majority rules.

**Lemma 9.** Let $z$ be the position of the median voter of $V$, and let $x > y \geq z$. Then $y$ has a majority in $V$ against $x$.

This is simply because the position of most voters is at $z$ or below and by definition all of them prefer $y$ to $x$.

The lemma clearly still holds if we modify the set of voters by adding votes for $r$ and/or ignoring passive voters. Thus, the lemma still applies if we replace “median” with $\tau$-$SQ-MD$ and “majority” with $\tau$-$SQ-MJ$, or replace $V$ with $V^+$. We use Lemma 9 to derive the following.

**Theorem 10.** The following hold:

* $\tau$-$SQ-MD$ has the same liveness guarantees as $\tau$-$SQ-MJ$.
* $\tau$-$SQ-MD$ has the same safety guarantees with respect to $MD$, as $\tau$-$SQ-MJ$ has with respect to $MJ$.

Clearly, if $\tau$-$SQ-MJ$ violates safety/liveness in some profile $V$, create an instance where all voters are located either on $r$ or on $p$ (according to their preference in $V$). Then, $\tau$-$SQ-MD(V) = \tau$-$SQ-MJ(V)$ so we get a violation of safety/liveness in $\tau$-$SQ-MD$ as well.

In the other direction, the construction is somewhat more involved. The proof for safety will follow from the more general Theorem 15, which also considers approximate safety.

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\(^5\)In the Appendix, in Section B we also consider the mean, generally showing its (rather expected) inferiority to the median.

\(^6\)The opposite case of sybils with full participation was considered in the two conference papers initiating the current work: in [1] a specialized voting rule that explicitly ignores the most extreme votes was suggested and analyzed; and in [43] we showed that this rule in fact coincides with $\tau$-$SQ-MD$. We therefore only consider $\tau$-$SQ-MD$ here.
Fig. 5 A demonstration of the $\alpha$-safety property in the 1-dimensional real line (left) and in some 2-dimensional metric space (right). The status-quo $r$ and the ideal point $G(H)$ are marked by *. The area inside the solid line is $\overline{G}_\alpha(H)$. The area inside in the green lines (solid or dashed) is $B(r; \overline{G}_\alpha(H))$.

**Proof for liveness.** For a profile $U$ of locations on $\mathbb{R}$ and a pair of locations $x, y \in \mathbb{R}$, we denote by $U|_{xy}$ the projection of $U$ on $A = \{x, y\}$. That is, a binary profile where each voter votes for the more preferred alternative among $x$ and $y$. In case of a tie, the voter selects $x$.

Consider any set of parameters $\mu, \sigma, \tau \geq 0$ such that $\tau$-$SQ$-$MD$ is live. Let $V = S \cup H$ be some profile of voters on the real line, and let $x$ be any position on the real line. We argue that $\tau$-$SQ$-$MD(S \cup H, x) = x$.

We show this for $x \geq r$. The other side is symmetric. Indeed, denote $V^x = S \cup H^+_x$ and assume towards a contradiction that $\tau$-$SQ$-$MD(V^x) = y \neq x$.

Note that in the binary profile $V^x|_{xy}$ all honest voters vote for $x$, thus from liveness of $\tau$-$SQ$-$MJ$ we get $\tau$-$SQ$-$MJ(V^x|_{xy}) = x$. On the other hand, since the median of $V^x$ is at $\tau$-$SQ$-$MD(V^x) = y \neq x$, then by Lemma 9, $y$ has a majority against $x$, which is a contradiction.

The above reduction allows us to easily transfer all previous results to the real line domain.

**Corollary 11.** The following hold:

- $\tau$-$SQ$-$MD$ is safe w.r.t MD as the base rule if and only if $\tau \geq \frac{1+\sigma}{1-\mu} - 1$.
- $\tau$-$SQ$-$MD$ is live iff $\tau < \frac{2(1-\sigma-\mu)}{1-\mu} - 1$.
- There is no mechanism $R$ that is both safe w.r.t MD and live when $3\sigma + 2\mu \geq 1$.

### 4 Relaxed Safety

So far we have treated safety as a dichotomy: for a given fraction of sybils, a mechanism is either safe or not. Suppose we still use Majority as our base rule.

If we think about a violation of safety as a situation in which most honest voters prefer the status-quo $r$ and the mechanism (perhaps due to sybils or abstention) selects $p$, then it should also be clear that some violations are worse than others:

- If the honest voters are almost evenly split between $r$ and $p$ then it does not matter much which alternative is selected, as both outcomes are ‘acceptable’;
- In contrast, if there is an overwhelming majority of honest voters for $r$ (meaning only $r$ is acceptable) but $p$ is selected then this is a more serious violation of safety.

Next, we introduce a formal definition of an acceptable outcome that contains a sensitivity parameter.
Yet, this approximation is measured not by the similarity of the alternatives, but rather by the when does it also contain which means \( p / \). We show this for \( \text{Proof}. \)

Observation 12. In the binary setting, \( p \in \overline{M_j}(H) \) if and only if \( h_p > h_r - 2\gamma \cdot h \); and \( r \in \overline{M_j}(H) \) if and only if \( h_r \geq h_p - 2\gamma \cdot h \).

Proof. We show this for \( p \). The proof for \( r \) is symmetric except for the tie-breaking.

Suppose \( h_p > h_r - 2\gamma \cdot h \), then either \( p \in \overline{M_j}(H) \); or \( h_r \geq 0.5h \). Let \( \gamma' \) be a set of \( 0.5, \gamma \) then \( \gamma' h \leq h_r \). Now, Let \( H' \subseteq H \), be an arbitrary set of \( r \) voters of size \( \gamma' \), and let \( H' := (H \setminus H') \cup H'' \). We then have

\[
h'_p - h'_r = h_p + \gamma' h - (h_r - \gamma' h) = h_p - h_r + 2\gamma' h
\]

\[
= \min\{h_p - h_r + 2\gamma' h, h_p - h_r + h\} \geq 0.
\]

On the other hand, if \( h_p \leq h_r - 2\gamma \cdot h \), then \( h_p + \gamma h \leq 0.5h \leq h_r - \gamma h \), and in any population \( H' \) with a majority for \( p \) we have

\[
|H' \setminus H| \geq |H'_p \setminus H_p| = |H'_p| - |H_p| = |V|(h'_p - h_p) \geq |V|(h'_p - (h_r - 2\gamma h))
\]

\[
> |V|(0.5h' - (h_r - 2\gamma h)) \geq |V|(0.5h - (h_r - 2\gamma h))
\]

\[
= |V|(0.5h - (h_r - \gamma h) + \gamma h) \geq |V| \gamma h = \gamma|H|,
\]

which means \( p \notin \overline{M_j}(H) \). \( \square \)

4.1 Quantifying Safety

Following the above discussion, we extend the definition of safety with a parameter. We highlight the difference from Def. 7 in red.

Definition 10 (Quantified safety). \( \mathcal{R} \) is \( \alpha \)-safe with respect to \( \mathcal{G} \) and \( V = S \cup H \) if \( \mathcal{R}(V^+) \in B(r, \overline{G}_\alpha(H)) \).

Note that for \( \alpha = 0 \) the definition collapses to safety, as in Def. 7.

---

\( ^7 \)This is sometimes called ‘input approximation’, in contrast to ‘output approximation’ [44]. It can be thought also as a negative measure of margin of victory, i.e., an alternative is considered acceptable also if it loses with a small margin.
Fig. 6 Visualization of relaxed safety on the same example from Fig. 3. Here the fraction of sybils is fixed at $\sigma = 0.2$ and the different curves show the range of $\alpha$-safe mechanisms for different levels of safety. Note that Majority is $\alpha$-safe w.r.t. itself whenever the curve is below the X-axis.

Fig. 5 demonstrates how the outcome range combines with the notion of betweeness in Euclidean spaces. The $\alpha$-safe area $B(r; \mathcal{G}_\alpha(H))$ includes all alternatives enclosed in either dashed or solid lines.

4.2 Relaxed Safety in the Binary Setting

Our next theorem characterises exactly the conditions in which $\tau$-SQ-MJ is $\alpha$-safe. This is also visualized (for specific values) in Fig. 6.

**Theorem 13 (Safety bound).** The $\tau$-SQ-MJ voting rule is $\alpha$-safe w.r.t Majority as the base rule if and only if

$$
\alpha \geq \frac{1+\sigma-(1+\tau)(1-\mu)}{2(1-\sigma)}.
$$

Note that the safety bound in Theorem 2 is derived by setting $\alpha = 0$.

**Proof.** Consider a given profile $V$. If $\tau$-SQ-MJ($V^+$) = $r$ or $p \in \overline{MJ}_\alpha(H)$ then there is no violation of $\alpha$-safety and we are done. Thus, assume that $\tau$-SQ-MJ($V^+$) = $p \neq r$. Recall that $h_p^+$ denotes the fraction of active honest voters voting for $p$. W.l.o.g. we may assume that all of $S$ vote for $p$, since if profile $V$ violates $\alpha$-safety, we can define a new profile $V'$, by switching all $S$ agents who vote for $r$ with $p$ voters, and we would still have $\tau$-SQ-MJ($V'^+$) = $p$ (and $\overline{MJ}_\alpha(H)$ is unaffected) and thus there is still a violation in $V'$ (so, intuitively, profiles in which all sybils vote for $p$ are the hardest case for keeping safety). Similarly, we assume w.l.o.g that all of $H^-$ vote for $r$, thus $h_r = h^- + h_r^+$, $h_p = h_p^+$ (again, profiles in which all passive voters vote for $r$ are the hardest case for keeping safety, as safety is defined w.r.t all honest voters); so, the fraction of active honest voters voting for $r$ is $h_r^+ = 1 - \sigma - \mu - h_p^+$. Since
\(\tau\text{-SQ-MJ}(V^+) = p\), we have that

\[
h_p^+ + \sigma = v_p^+ > v_p^+ + q = h_p^+ + q = h_p^+ + q = (1 - \mu - \sigma - h_p^+) + \tau(1 - \mu), \text{ and thus}
2h_p^+ > (1 + \tau)(1 - \mu) - 2\sigma.
\] (1)

To show that \(p \in M\mathbb{M}_H(H)\), which would show \(\alpha\)-safety, it is left to show that we can change the votes of \(\alpha \cdot |H|\) honest voters from \(r\) to \(p\), to create a new profile \(H'\) where \(p\) has a strict majority of honest votes. Denote

\[
\alpha' = \alpha h = \alpha(1 - \sigma) \geq \frac{1 + \sigma - (1 + \tau)(1 - \mu)}{2}.
\] (2)

Indeed, after moving \(\alpha'\) votes, \(r\) has

\[
h'_p = h_r - \alpha' = h - h_p - \alpha' = 1 - \sigma - h_p^+ - \alpha'
\]
honest votes, whereas \(p\) has \(h_p' = h_p^+ + \alpha'\) honest votes. Therefore, we have that

\[
h'_p - h'_p = (h_p^+ + \alpha') - (1 - \sigma - h_p^+ - \alpha')
= 2(h_p^+ + \alpha') - (1 - \sigma)
\geq 2h_p^+ + (1 + \sigma - (1 + \tau)(1 - \mu)) - (1 - \sigma)
= (1 - \sigma) - (1 - \sigma) = 0.
\]

So, there are strictly more honest votes for \(p\) than for \(r\).

In the other direction (i.e. to show tightness of the bound), consider \(\tau, \sigma, \mu\) and \(\alpha < \frac{1 + \sigma - (1 + \tau)(1 - \mu)}{2(1 - \sigma)}\); First set \(\varepsilon = \frac{1 + \sigma - (1 + \tau)(1 - \mu)}{2(1 - \sigma)} - \alpha\). Next, set \(h_p^+ = \frac{(1 + \tau)(1 - \mu) - 2\sigma + \varepsilon'}{2(1 - \sigma)}\), where \(\varepsilon' \in (0, \frac{1}{1 - \sigma})\). All \(\sigma\) sybils vote for \(p\), and all \(\mu\) inactive honest voters vote for \(r\).

It is left to show that (a) \(M\mathbb{M}_H(H) = \{r\}\) (i.e. \(r\) is the only safe outcome); and that (b) \(\tau\text{-SQ-MJ}(V^+) = p\) (details omitted). For (a), consider any honest profile \(H'\) such that \(|H' \setminus H| \leq \alpha|H|\). In the best case, we have that \(h'_p \leq h_p + \alpha h\) and \(h'_r \geq h_r - \alpha h\). Indeed,

\[
h'_p - h'_p \leq h_p - h_r + 2\alpha h = h_p - (h - h_p) + 2\alpha h
= 2h_p - (1 - \sigma) + 2\alpha(1 - \sigma)
= 2h_p^+ - (1 - \sigma) + 2\alpha(1 - \sigma)
= [(1 + \tau)(1 - \mu) - 2\sigma + 2\varepsilon'] - (1 - \sigma)
+ [(1 + \sigma) - (1 + \tau)(1 - \mu) + 2\varepsilon(1 - \sigma)]
= 2\varepsilon' - 2\varepsilon(1 - \sigma) < 0,
\]

which shows that \(MJ(H') = r\) as required.
For (b), we can see that
\[
v_p^+ - (v_r^+ + q) = (h_p^+ + \sigma) - ((h^+ - h_p^+) + \tau v^+)
\]
\[
= 2h_p^+ - h^+ - v^+\tau + \sigma
\]
\[
= 2h_p^+ - (1 - \sigma - \mu) - (1 - \mu)\tau - \sigma
\]
\[
= 2h_p^+ - (1 - \mu)(1 + \tau) - 2\sigma
\]
\[
= 2\epsilon' > 0, \quad \text{(by definition of } h_p^+)\]
which shows that \( \tau\text{-SQ-MJ}(V^+) = p \) and thereby completes the proof.

**Mechanism design perspective**

The analysis of the \( \alpha \)-safety of \( \tau\text{-SQ-MJ} \) for given values of \( \sigma \) and \( \mu \) implies a different point of view: Indeed, in practical situations, the value of \( \alpha \)-safety might be decided by a user of the system (a stricter user would require smaller values); then, given some estimations of \( \sigma \) and \( \mu \) (\( \mu \) is usually known exactly since we know who is eligible to vote, while to estimate \( \sigma \) one can use, e.g., sampling techniques can be used to infer what value of \( \tau \) the user shall choose for the \( \tau\text{-SQ-MJ} \) mechanism to achieve the desired level of safety.

For example, we can see in Fig. 6 that under \( \sigma = 0.2 \) and \( \mu = 0.3 \) it would not be possible to get both liveness and full safety, but 0.2-safety can still be obtained. Refer to Section 7 for a further discussion on the mechanism design perspective.

4.3 Relaxed Safety in Other Domains

Some of the safety bounds for the domains studied in Section 3 similarly generalize to any \( \alpha \geq 0 \), as they are essentially based on a reduction to the binary domain that preserves the approximation. These include the results for multiple alternatives and for the real line. In contrast, our results from multiple referenda and Condorcet-conservative rules do not generalize to arbitrary \( \alpha \).

**Multiple alternatives**

**Theorem 14.** The \( \tau\text{-SMJ} \) voting rule is \( \alpha \)-safe w.r.t Plurality if and only if
\[
\alpha \geq \frac{1 + \sigma - (1 + \tau)(1 - \mu)}{2(1 - \sigma)}.
\]

By setting \( \alpha = 0 \) we get Theorem 5.

**Proof.** We follow the same steps as in the proof of Theorem 13: Suppose that \( \tau\text{-SMJ} \) selects \( p \), then we need to show \( p \) is \( \alpha \)-safe by making it the honest winner. That is, we need to construct a modified profile \( H' \) where \( p \) has most votes. In fact, we will show it gets a strict majority. For this, we need to provide corresponding inequalities to Eqs. (1) and (2).

For the first, we observe that in \( \tau\text{-SMJ}(V^+) \), alternative \( p \) gets more than \( (1 + \tau)/2 \) of all active votes.\(^8\) Thus
\[
h_p^+ + \sigma \geq v_p^+ > \frac{1 + \tau}{2} v^+ = \frac{1 + \tau}{2} (1 - \mu) \Rightarrow
\]

\(^8\)This is exactly where the proof would fail for \( \tau\text{-SQ-PL}^+ \), since \( p \) can win even with a lower fraction of votes.
\[2h^+_p > (1 + \tau)(1 - \mu) - 2\sigma. \tag{3}\]

Now, set
\[
\alpha' = \alpha h \geq \frac{1 + \sigma - (1 + \tau)(1 - \mu)}{2}. \tag{4}\]

Then, to construct \(H'\), we move a fraction of \(\alpha\) honest voters to \(p\), from any other alternative (not necessarily from \(r\)). We get:
\[
2h'_p - h = 2(h^+_p + \alpha') - (1 - \sigma) \\
\geq 2h^+_p + (1 + \sigma - (1 + \tau)(1 - \mu)) - (1 - \sigma) \quad \text{(By Eq. (4))} \\
> (1 - \sigma) - (1 - \sigma) = 0, \quad \text{(By Eq. (3))}
\]
so \(h'_p > 0.5h\), as required.

**The real line**

Here, we concentrate on the median rule; in Appendix B we consider the mean as well, generally showing its (rather expected) inferiority.

**Theorem 15.** \(\tau\)-SQ-MD has the same safety guarantees with respect to MD, as \(\tau\)-SQ-MJ has with respect to MJ, for any \(\alpha \geq 0\).

**Proof.** Consider any set of parameters \(\mu, \sigma, \tau, \alpha \geq 0\) such that \(\tau\)-SQ-MD is \(\alpha\)-safe. Denote \(a := \sup(MD_\alpha(H))\). If \(a = +\infty\) then no violation of safety is possible and we are done. Otherwise, \(a = \max(MD_\alpha(H))\) since it is obtained as \(a = MD(H')\) by placing \(\alpha h\) honest voters on \(a\) in \(H'\).

Denote \(z := \tau\)-SQ-MD\((V^+)\). We need to show that \(z \in B(r; a) = [r, a]\). Assume otherwise towards a contradiction, i.e. that \(z > a\). By Lemma 9, we have \(\tau\)-SQ-MJ\((V^+|_{a \leq z})\) = \(z\). On the other hand, for any \(H'\) obtained from \(H\) by moving \(\alpha|H|\) voters, we have \(MD(H') \leq a < z\) and thus by the same lemma, \(MJ(H'|_{a \leq z}) = \{a\}\). This entails
\[
B(a; MJ_\alpha(H|_{a \leq z})) = B(a; \{a\}) = \{a\},
\]
and thus \(\tau\)-SQ-MJ\((V^+|_{a \leq z}) = z \notin B(a; MJ_\alpha(H|_{a \leq z}))\), which is a contradiction to \(\alpha\)-safety of \(\tau\)-SQ-MJ.

Just as in Section 3, we get the safety properties of \(\tau\)-SQ-MD as an immediate corollary from Theorems 13 and 15:

**Corollary 16.** \(\tau\)-SQ-MD is \(\alpha\)-safe w.r.t MD as the base rule if and only if \(\alpha \geq \frac{1 + \sigma - (1 + \tau)(1 - \mu)}{2(1 - \sigma)}\).

**Multiple referenda**

The \(\tau\)-SQ-IMJ rule does not inherit the safety properties of \(\tau\)-SQ-MJ for \(\alpha > 0\). Intuitively, this is since honest voters might be split and only have weak agreement on each issue, which provides fewer sybils with enough power to thwart the decision.
Proposition 17. For $\alpha > 0$, the $\alpha$-safety guarantees of $\tau$-SQ-IMJ with respect to IMJ are strictly worse than those of $\tau$-SQ-MJ with respect to MJ.

This is true even with full participation ($\mu = 0$) and without virtual voters ($\tau = 0$).

Proof. We show via an explicit example.

Suppose that $|H| = 60$, $|S| = 21$ (i.e. $\sigma \cong 1/4$, $\tau = 0$, $\mu = 0$). Then by Thm. 13 we get $1/6$-safety of the MJ rule with respect to itself (indeed, if there are 40 honest voters on ‘0’ and 20 on ‘1’, then moving 10 = $|H|/6$ to ‘1’ is sufficient).

Now consider $A = \{0, 1\}^3$, where honest voters are dispersed as follows: 20 on $(0, 0, 0)$; 20 on $(0, 1, 0)$; 20 on $(1, 0, 0)$ and all 21 voters of $S$ are on $(1, 1, 1)$ so the outcome is $IMJ(V) = (1, 1, 1)$.

However we argue that $(1, 1, 1) \notin B(r, IMJ_\frac{1}{6}(H))$ which means a violation of $\frac{1}{6}$-safety.

Note that for this it is sufficient to show that there is no $H \rightarrow H'$ with $|H \cap H'| \geq \frac{5}{6}|H| = 50$ s.t. $IMJ(H') = (1, 1, 1)$.

Indeed, only 10 voters are allowed to vote differently in $H'$ than in $H$. Consider the original vote of an arbitrary ‘changed’ voter $i$ in $H' \setminus H$. W.l.o.g. $i$ voted $(1, 0, 0)$. This means that at most 9 voters in $H' \setminus H$ whose original vote on the first issue is ‘0’, and thus at most 9 new votes to ‘1’, meaning in particular that $IMJ(H') \neq (1, 1, 1)$.

Therefore, in $H'$ there are at least 31 votes to ‘0’ vs. at most 29 votes to ‘1’, meaning in particular that $IMJ(H') \neq (1, 1, 1)$.

However, by moving 5 voters from each location to $(1, 1, 1)$, i.e. 15 in total, $IMJ$ would select $(1, 1, 1)$. This entails that $IMJ$ is $\alpha$-safe with respect to itself for $\alpha = \frac{15}{60} = \frac{1}{4}$ (on the above profile), and it is not hard to see that this is tight.

That is, $IMJ$ is not $1/6$-safe with respect to itself, in contrast to MJ with the same parameters $\sigma$, $\mu$ and $\tau$.

A similar example can be constructed for Condorcet-conservative rules, where different sets of honest voters prefer $p$ over $p'$ for each $p'$.

4.4 Quantifying Liveness

It is possible to quantify liveness in a similar way, by requiring only that every outcome $p \in A$ is included in the outcome range of the active voters when some fraction of up to $\beta$ of honest voters change their vote. Then we would get the standard definition of liveness for $\beta = 1$, whereas lower values represent a stronger liveness requirement; and higher values than 1 represent a relaxed requirement.

Since we see quantifying liveness as less natural and less interesting than quantified safety, we defer the technical details to Appendix A.

5 Random Participation

The lower bound in Theorem 3 suggests that no mechanism can accommodate higher abstention and sybil rates than the SQ-MJ mechanism, even in a binary setting. This, however, holds in the ‘worst case’, making adversarial assumptions both on the sybils’ votes and on who chooses to abstain.
Fig. 7 Two possible realizations of the same instance with $|H_0| = 4$, $|H_r| = 6$ and the same number of active honest voters $n^+ = 5$. In the realization on the left, most active voters are on $p$ and thus $p$ wins (violating safety). On the realization on the right, exactly half of the honest voters on each alternative are active, and thus the only safe alternative $r$ wins.

A less extreme approach that might be more realistic is that the active honest voters are selected uniformly at random from the honest population, whereas sybils are still adversarial. As a result, we have that the votes of the active and the passive honest voters are similarly distributed.

The benefit of such an assumption is demonstrated in Fig. 7, where the ‘bad’ selection of active voters on the left is possible under arbitrary participation, but highly unlikely under random participation.

We argue that with this additional constraint on vote distributions, the safety-liveness tradeoff could be improved. However, since the votes are now stochastic, the outcome is a random variable, and so we must first adapt our definitions, and in particular state what distribution of outcomes is considered ‘safe’.

Alternatively, we can consider the limit case of a very large population, where the distributions of passive and active (honest) voters over alternatives are exactly the same, as any variance becomes negligible. This ‘nonatomic’ model is somewhat easier to analyze, but yields similar results and is deferred to Appendix C.

In the remainder of this section we consider finite populations. This requires a probabilistic extension of the safety and liveness properties.

5.1 Safety for Stochastic Outcomes

We next extend the definitions to allow randomness. It does not matter if the randomness is due to voters’ actions (i.e., randomly deciding whether to vote), due to the voting rule $\mathcal{R}$, or from other sources. We do assume however that the base rule $G$ is deterministic.

We denote by $n^+ := |H^+| = |V| \cdot h^+$ the number of active honest voters. Safety w.h.p. means that if this number is sufficiently large then the probability of an “unsafe” outcome is negligible. We highlight in blue the differences from Def. 10.

In particular, an ‘instance’ $V$ does not specify (in contrast to previous sections) who are the active voters, but only the partition to $H$ and $S$.

---

9For a given instance, we treat the number of active voters as fixed, i.e. they are selected from the honest voters without repetition. We can also think of a model where each honest voter is active with some fixed probability. The results would be similar, but the definitions of both safety and liveness would slightly change.
**Definition 11** (Safety w.h.p.). An aggregation rule $\mathcal{R}$ is $\alpha$-safe with high probability w.r.t $G$, if for any $\alpha' > \alpha$, there is a constant $C$, such that for all $V$ with $n^+$ active voters,

$$
Pr_{x \sim \mathcal{R}(V)}[x \in B(r; G_{\alpha'}(H))] > 1 - \exp(-C \cdot n^+).
$$

The constant $C$ may depend on the instance parameters ($\sigma, \mu, \alpha$) and as specified also on $\alpha'$. Note that the requirement of safety w.h.p. is no longer for a given instance (as it is asymptotic), but on all instances with given parameters.

We could similarly define liveness w.h.p., and this would make sense for various sources of uncertainty, but for our particular model this is not required: since there are exactly $(1 - \mu - \sigma)|V|$ active honest voters, and since in the worst case for liveness, all voters vote for $r$, all realizations are identical. The probability that there is a violation of liveness is thus either 0 or 1.

**5.2 The Binary Case**

We show an improved bound compared to the arbitrary participation case (Thm. 14).

**Theorem 18.** Under random participation, the $\tau$-SQ-MJ voting rule is $\alpha$-safe w.h.p. with respect to Majority, iff

$$
\alpha \geq \frac{(\sigma - \tau(1 - \mu))(1 - \sigma)}{2(1 - \mu - \sigma)} \frac{1}{2} + c \leq \alpha + c = \alpha' \leq \frac{h_r - h_p}{2}.
$$

**Proof.** Recall we denote by $u_r, u_p$ the fraction of voters for $r$ and $p$, respectively, in a voter set $U$.

Consider any $\alpha' > \alpha$. In the case where $h_p > h_r - 2\alpha'$, we have

$$p \in MJ_{\alpha'}(H) \subseteq B(r; MJ_{\alpha'}(H)),
$$

which means $\alpha$-safety holds regardless of the realization of active voters.

Therefore, assume that $h_p \leq h_r - 2\alpha'$. Intuitively, this means that the gap $h_r - h_p$ is large, and thus the gap $h_r^+ - h_p^+$ is likely to be large as well, leading to $v_r^+ > v_p^+$ w.h.p. E.g. in the ‘common’ realization on Fig. 7, we have $h_r^+ - h_p^+ = \frac{3}{12} - \frac{2}{12} = \frac{1}{12}$ (Right figure).

This is the main difference from the arbitrary participation case where must also consider highly skewed realizations (E.g. in the Left of Fig. 7 the gap is $-\frac{1}{4}$ and $v_p^+$ is indeed strictly higher than $v_r^+$).

The remainder of the proof is for showing, using the Hoeffding inequality, that w.h.p the gap $h_r^+ - h_p^+$ is larger than $s - q$, and hence $r$ has more active votes overall, and safety is not violated. We now turn to prove this formally.

To show safety w.h.p., we need to upper-bound the probability that $\tau$-SQ-MJ will select $p$. Denote $c := \alpha' - \alpha > 0$. Since $h_p \leq h_r - 2\alpha'$, and by the premise of the theorem, we have:

$$
\frac{(\sigma - \tau(1 - \mu))(1 - \sigma)}{2(1 - \mu - \sigma)} + c \leq \alpha + c = \alpha' \leq \frac{h_r - h_p}{2}.
$$

(5)
Every honest worker is active with probability \( \phi := \frac{|H^+|}{|H|} = \frac{1-\mu-\sigma}{1-\sigma} \) (though not i.i.d.). Alternatively, every active honest voter is a \( p \) voter with probability \( \psi := |H_p|/|H| \).

We sample \( n^+ = |H^+| \) active voters from the set \( H = H_p \cup H_r \), without replacement. Consider \( n^+ \) samples \( X_1, \ldots, X_{n^+} \in \{0, 1\} \) where \( X_i = 1 \) if the \( i \)’th active agent is a \( p \) voter, and 0 otherwise. Thus \( n_p^+ := |H_p^+| = \sum_{i \leq n^+} X_i \) and \( n_r^+ = n^+ - n_p^+ \).

Observe that \( n_p^+ \) is a random variable, whose expected value is

\[
n_p^+ = E[X_i] = n^+ \psi = |H^+| \frac{|H_p|}{|H|} = \frac{|H^+|}{|H|} |H_p| = \phi |H_p|.
\]

Recall that \( e = \alpha' - \alpha \) and let \( \varepsilon \in (0, \frac{\varepsilon}{2-\sigma}) \). Denote the event \( [n_p^+ < (\psi + \varepsilon)n^+] \) by \( I \).

By applying Hoeffding inequality,\(^1\)

\[
Pr[I] = Pr[n_p^+ \geq (\psi + \varepsilon)n^+] < \exp(-2\varepsilon^2n^+) = \exp(-c^2n^+/(1-\sigma)^2) = \exp(-C \cdot n^+),
\]

for \( C = \left( \frac{\alpha' - \alpha}{\sigma} \right)^2 \). It thus remains to show that whenever \( I \) occurs, \( r \) is selected.

For the remainder of the proof, we fix a realization where event \( I \) occurs, thus \( n_p^+ < (\psi + \varepsilon)n^+ = \phi |H_p| + e n^+ \), and \( n_r^+ = n^+ - n_p^+ > (1 - \psi - e)n^+ = \phi |H_r| - e n^+ \) (intuitively, \( n_p^+, n_r^+ \) are close to their expected values). Therefore:

\[
h_r^+ - h_p^+ = \frac{1}{n} (n_r^+ - n_p^+) \geq \frac{1}{n} (\phi (|H_r| - |H_p|)) - 2\varepsilon n^+ = \phi (h_r - h_p) - 2\varepsilon (1 - \sigma - \mu). \tag{6}
\]

By definition, \( \tau \cdot SQ-MJ(V^+) = MJ(H^+ \cup S \cup Q) \) where \( Q \) contains \( \tau |V^+| = \tau (1 - \mu)n^+ \) voters for \( r \).

Thus the total fraction of active \( r \) voters is at least \( h_r^+ + \tau (1 - \mu) \). As in the previous proofs, w.l.o.g. all sybils vote for \( p \) as this is the worst case for safety. We get that

\[
v_r^+ - v_p^+ \geq (h_r^+ + \tau (1 - \mu)) - (h_p^+ + \sigma)
\]

\[= (h_r^+ - h_p^+) - (\sigma - \tau(1 - \mu))\]

\[\geq \phi (h_r - h_p) - 2\varepsilon (1 - \sigma - \mu) - (\sigma - \tau(1 - \mu)) \tag{by Eq. (6)}
\]

\[\geq \frac{1 - \mu - \sigma}{1 - \sigma} \left( \frac{\sigma - \tau(1 - \mu)}{1 - \mu - \sigma} \right) + 2\varepsilon \]

\[\geq \frac{1 - \mu - \sigma}{1 - \sigma} \cdot \frac{\sigma - \tau(1 - \mu)(1 - \sigma)}{1 - \mu - \sigma} \tag{by Eq. (5)}
\]

\[+ 2(1 - \mu - \sigma)\left( \frac{c}{1 - \sigma} - \varepsilon \right) - (\sigma - \tau (1 - \mu))
\]

\[= (\sigma - \tau (1 - \mu)) + 2(1 - \mu - \sigma)(\frac{c}{1 - \sigma} - \varepsilon) - (\sigma - \tau(1 - \mu))
\]

\(^1\)The Hoeffding inequality applies for sampling either with or without replacement. Without replacement it is possible to get somewhat better bounds [45] but this is immaterial for our argument.
as required.

Tightness follows from the same construction used in the nonatomic case. Then there are strictly more active voters (in expectation) for the unsafe alternative $p$, and the probability of selecting $p$ is at least 0.5.

We can therefore trace the improved tradeoff between safety and liveness as follows:

**Corollary 19.** Under random participation, the following holds:
- $\tau$-SQ-MJ is 0-safe w.h.p w.r.t MJ iff $\tau \geq \frac{3\sigma}{1-\mu}$.
- $\tau$-SQ-MJ is live iff $\tau < \frac{2(1-\sigma-\mu)}{\mu-1} - 1$.
- We can get both if $3\sigma + \mu < 1$.

This is compared to $3\sigma + 2\mu < 1$ requirement in the arbitrary participation model (Thm. 2).

### 5.3 Extensions Beyond the Binary Case

Note that our definition for 'safety w.h.p' is general and applies to any domain.

All of our positive results use reductions to the binary case: either to Thm. 2 (if restricted to $\alpha = 0$); or to Thm. 13 (when apply to any $\alpha \geq 0$). The same reductions would apply for the random participation model, using Cor. 19 or Thm. 18, respectively.

Thus all of our previous results extend to the random participation model, with the improved bound. This applies to:
- Multiple alternatives (Thm. 5, Thm. 14, Prop. 8, Prop. 7);
- Multiple referenda (Prop. 8);
- Median voting (Thm. 10, Thm. 15).

### 6 Voting with Delegation

While the results above allow for partial participation, they also imply that to obtain both safety and liveness, the fraction of passive voters cannot be too large; this might be problematic in some situations. As our lower bound means that this is unavoidable, we therefore wish to relax the model to analyze other possibilities; in particular, we adopt the standard model of proxy voting, where only a small number of voters are active, and any passive voter delegates her vote to the nearest active voter [42, 46]

Voting with a constant number of alternatives

There is no reason in doing a separate analysis for delegation in the binary (or any categorical) domain, as, in this domain there is no difference between delegating to a proxy and actively voting (provided that every alternative has at least one active voter); sybils may still interfere, but the safety-liveness tradeoff of Majority with proxy delegation is just as in Thm. 2 with full participation ($\mu = 0$).

In contrast, in continuous (or any sufficiently large) domains, an inactive voter will rarely find an active voter that completely agree, and thus the effect of delegation becomes nontrivial.
\[
V = H^+ \cup H^- \cup S
\]

\[
\rho = H + \cup H - \cup S
\]

\[
z^* = MD(H)
\]

\[
\hat{h} = \tau - SQ-MD(V) = MD(V \cup Q)
\]

\[
x = \tau - SQ-MD(V^+; \bar{w}) = MD^P(H^+ \cup S \cup Q; \bar{w})
\]

Fig. 8 A demonstration of several definitions used in the proof of Theorem 20 on an example profile. The honest voters are blue circles (filled circles are active voters \(H^+\)). Sybils are marked by red squares. The full gray diamonds are the virtual voters \(Q\) added by the mechanism. In the bottom figure, hollow blue diamonds mark the followers of each active voter (their real positions are as in Fig. (c)).

### 6.1 Median with Delegation

For a finite population \(U\) and a vector of vote weights \(\bar{w} = (w_i)_{i \in U}\), we denote by \(MD(U; \bar{w})\) the weighted median, where each \(i \in U\) has weight \(w_i \in \mathbb{N}\). Formally,

\[
MD(U; \bar{w}) := \min\{u_i : i \in U, \sum_{j \leq i} w_j \geq \sum_{j > i} w_j\}.
\]

Following Section 5 we denote \(n^+ := |H^+| \geq (1 - \mu - \sigma)|V|\), and assume that active voters are sampled uniformly at random from \(H\). As we will see later, the fraction of active voters itself will not matter and can be arbitrarily close to 0.

The votes of inactive voters affect the outcome indirectly via delegation: for each \(i \in V^+\), let \(w_i = 1 + |\{j \in V^- : i = \text{argmin}_{i' \in V^+} |s_i - s_{i'}|\}|\) be the number of voters for which \(i\) is the closest active voter (their "proxy"). Indeed, this follows from our strong assumption, namely that passive votes are always delegated to the closest active voter (either honest or sybil). We leave the study of alternative delegation models for future research.

The rule \(MD^P\) (P for Proxy) takes population \(V = H \cup S\) as input together with the implicit parameter \(n^+\), samples \(n^+\) active voters from \(H\), and returns \(MD(V^+; \bar{w})\), where weights are set as above, according to the number of “followers” (i.e., delegatees) of each \(i \in V^+ \cup \{r\}\). Since \(V^+ = S \cup H^+\) and \(\bar{w}\) are random variables, so is \(MD^P(V)\).
The rule $\tau$-$SQ$-$MD^P$ is the same, except adding $\tau|V|$ virtual voters on $r$ first, i.e.

$$\tau$-$SQ$-$MD^P(V) = MD^P(V \cup Q).$$

**Remark 2.** If, for some passive voter $i$, the status-quo $r$ is closer than all active voters, then we assume that $i$ delegates to $r$ (see, e.g., Fig. 8(d)).

**Analysis**

**Theorem 20.** Under random participation, $\tau$-$SQ$-$MD^P$ is safe w.h.p.

if and only if $r \geq \sigma$.

Let us use the following notation:

- $X := \tau$-$SQ$-$MD^P(V)$ is the returned position (which is a random variable);
- $z^* := MD(H)$ is the honest outcome. We assume w.l.o.g. that $z^* \geq r$, so that the 0-safe range is $[r, z^*]$;
- $y := \tau$-$SQ$-$MD(V)$, i.e. the median with sibs and virtual voters, but with full participation.

Note that $z^*$ and $y$ are fixed positions that do not depend on realization.

In addition, we define by $z^-$ and $z^+$, respectively, the ends of the closed interval $\overline{MD}_{\mu}(H)$. Thus the $\alpha$-safe range is $[r, z^+]$. Still, $z^-$, $z^+$ are fixed positions.

**Lemma 21** (Cohensius et al. [42]). For any $U = (U^+, U^-)$, it holds that $MD(U^+; \vec{w})$ with proxy weights is the voter in $U^+$ which is closest to MD($U$).

Our argument is as follows: we show that $y \leq z^- \leq z^+$, then use the lemma to argue that in every realization $x$ of $X$, the selected $x$ is the active voter closest to $y$. Finally, we show that w.h.p. there is some active voter in $[y, z^+]$ and thus $x \leq z^+$.

**Proof of Theorem 20.** By the premise of the theorem, $\tau \geq \sigma$. Since $y = \tau$-$SQ$-$MD^P(V)$ corresponds to an instance with full participation, we get from Cor. 11 with $\tau \geq \sigma$ and $\mu = 0$ that $\tau$-$SQ$-$MD^P(V)$ is safe. Thus $r \leq y \leq z^+$.

Now consider Lemma 21, where $U := Q \cup H^+ \cup H^- \cup S$ is any realized partition of $V$ into active and inactive voters (with the added virtual voters $Q$). We get that the realized outcome $x = MD(U^+; \vec{w})$ is the position of the voter in $U = Q \cup H^+ \cup S$ which is closest to $MD(U) = \tau$-$SQ$-$MD(V) = y$. In other words, if $i^* = \arg\min_{i \in H^+ \cup S} |s_i - y|$ is the closest active voter to $y$ (in some realization), then $x = s_{i^*}$.

Since the virtual voters are active, we know $x \geq r$. It is left to show that with high probability there is an active voter between $y$ and $z^+$: We consider $n^+ > 2/\alpha'$.

Indeed, the range $\overline{MD}_{\alpha'}(H)$ contains $[\alpha'|H|]$ honest voters to each side of $z^* = MD(H)$. Since

$$|\alpha'|H| - 1 = \frac{1}{2} \alpha'|H| + \frac{1}{2} \alpha'|H| - 1 \geq \frac{1}{2} \alpha'|H| + \frac{1}{2} \alpha' n^+-1 > \frac{1}{2} \alpha'|H| + \frac{1}{2} 2-1 = \frac{1}{2} \alpha'|H|,$$

there are at least $\frac{1}{2} \alpha'|H|$ voters in $[z^*, z^+]$. Denote these voters by $\hat{H}$. Now, $H^+$ is a random sample of $n^+$ voters from $H$, so each voters $i \in H^+$ has a probability of at most $1 - \frac{1}{n^+}$ to be outside $\hat{H}$. Since we sample without repetition, the probability that all active voters are
outside (i.e. that $H^+ \cap \hat{H}$ is empty) is at most $(1 - \frac{1}{2} \alpha')^n = \exp(-C \cdot n^+)$ for some positive constant $C$ that depends only on $\alpha'$. Finally,

$$\Pr_{x \sim X} [x \in B(r; \overline{MD}_\alpha(H))] = \Pr_{x \sim X} [x \in [r, r^+]] \geq \Pr[H^+ \cap \hat{H} \neq \emptyset] > 1 - \exp(-C \cdot n^+),$$

as required.

In the other direction, if $\tau < \sigma$ then consider profiles where all voters are either on $r$ or on some other point $p$. By Cor. 11 this is unsafe even with full participation, i.e. there is an instance where most honest voters are on $r$ and yet $p$ is selected, meaning a majority (with some constant margin $\varepsilon$) of voters from $V \cup Q$ are on $p$. Set $\alpha' := \varepsilon/2$, then $p \notin \overline{MD}_\alpha(H)$. The probability that $p$ still wins when we sample the active voters is at least $1/2$ regardless of $n^+$, which means a violation of safety w.h.p.

Delegation does not affect liveness: the $\tau$-SQ-MD$^P$ is live iff $\tau < 1 - 2\sigma$, as this follows from the full participation case of Cor. 11.

**Corollary 22.** By setting $\tau = \sigma$, the $\tau$-SQ-MD$^P$ mechanism is both safe w.h.p. and live, as long as $\sigma < \frac{1}{3}$.

This shows that delegation allows us to almost completely eliminate the drawbacks of partial participation, and get the same safety level against sybils as with full participation, provided that the number of active voters is sufficiently large (but without any requirement on their fraction).

## 7 Discussion and Outlook

We have analyzed different social choice settings in which sybil entities have infiltrated the voting community and, on top of this, not all honest voters participate. We have provided a formal model to reason about such situations, developed techniques to tackle this challenge, and analyzed them.

In particular, motivated by governance and mutual decision mechanisms for online communities, we have considered the common situation in which representation is threatened both by the presence of sybils, and by partial participation of the honest voters. We have defined a general mechanism, $\tau$-RE-$R$, and analyzed its safety/liveness tradeoff for several social choice settings. For a fraction $\sigma$ of sybils and a fraction $\mu$ of passives in the population, we showed that, for voting on one proposal against the status-quo and voting in an interval domain, the $RE$ mechanism can obtain maximal safety and liveness together as long as $3\sigma + 2\mu < 1$. Furthermore, we showed: that the same tradeoff applies to categorical decisions and to multiple referenda; that no mechanism can do better than $\tau$-RE-$MJ$; that we can be satisfied with a somewhat lower participation rate $(3\sigma + \mu < 1)$ when participation is random; and that delegation allows the same level of safety with a negligible fraction of active honest voters.

To set the parameter $\tau$ (the bias towards the status-quo) effectively, after deciding upon the desired tradeoff of safety and liveness, one has to estimate $\sigma$ and $\mu$ in the population. While $\mu$ can be estimated quite accurately (as an election organizer may define the set of eligible voters), this is not the case for $\sigma$. The fraction of sybils can be approximated by sampling voters (see Remark 7) or by techniques that upper bound $\sigma$ [47]. Note that over-estimating $\sigma$ or $\mu$ always results in a mechanism that is more safe, and thus our bounds still hold.
Together with state-of-the-art mechanisms for identifying and eliminating sybils [7], our results set the foundation for reliable and practical online governance tools. Note also that, since the preliminary, conference version of this paper was published, it was identified as a crucial piece in the design of a democratic metaverse [48].

Before we discuss some avenues for future research, we wish to comment on the practicality of our methods in the context of the estimation of the different parameters.

**Estimating the sybil fraction**

How to estimate the sybil penetration $\sigma$ is an important question. While in some cases there might be other techniques available (some works on this topic – including such in which $\sigma$ can theoretically be upper-bounded – exist [47, 49]), usually it is natural to assume that by sampling a voter one can estimate the probability that the voter is genuine or fake (e.g., looking at her Facebook profile). Thus, the main general technique we suggest is to sample voters uniformly at random and, given the sampling results, estimate $\sigma$. Note that using such sampling it is then possible to compute, for a given value $p$, a value $z$, such that the probability that $\sigma$ is greater than $z$ is at most $p$. Alternatively, one can compute the mean $m$ of the sample and take an $\epsilon$ margin of safety, i.e., use $m + \epsilon$ as the estimate for $\sigma$.

Finally, below we discuss several avenues for future research:

- **Further social choice settings**: In particular, generalizing some of our results to general metric spaces seems natural. In this context, we conjecture that $\tau$-$RE$-$R$, when applied to other metric spaces (with suitable base rules), would guarantee similar safety/liveness tradeoffs.

- **Further delegation models**: Relaxing the proxy voting assumption of Cohensius et al. [2017] is a natural direction. In particular, considering more general and realistic delegation models, in particular such that relate to some underlying social network and take into account voter affinity as it relates to the network seems promising.

- **Practical considerations**: We feel that our theoretical framework and results are quite ready for being applied in the wild. However, to do so one may first go through performing extensive simulations, and then developing practical tools for communities to utilize the results presented here in a user-friendly, convenient, and robust way.

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A Quantifying Liveness

Recall the original definition of liveness (Def. 8), stating that \( R \) is live w.r.t. \( V = S \cup H \) if \( R(S \cup H_{\to a}) = a \) for all \( a \in A \).

We now relax this definition with a parameter \( \beta \).

**Definition 12 (\( \beta \)-Liveness).** An aggregation rule \( R \) is \( \beta \)-live w.r.t. population \( V \), if for all \( a \in A \), it holds that \( a \in R_{\beta}(V) \).

I.e., a rule is live w.r.t some population if any outcome can be reached by modifying not-too-many (in particular, \( \beta \)-fraction of) honest voters.

For any monotone rule, 1-liveness coincides with liveness. To see why, note that for \( a \) to belong in the outcome range \( R_{1}(V) \), there must be some honest profile \( H' \) (with same size as \( H \)) s.t. \( R(S \cup H') = a \). For a monotone rule, we can assume w.l.o.g. that all voters in \( H' \) voter \( a \) and thus the definitions coincide.

Values \( \beta < 1 \) correspond to a stronger liveness requirement, whereas \( \beta > 1 \) is relaxing liveness.

Note that values \( \gamma > 1 \) in the definition of the outcome range (Def. 9) effectively mean that we may replace all honest voters and, furthermore, add additional \( (1 - \gamma)\frac{|H|}{V} \) voters.

**Theorem 23 (Liveness).** The \( \tau \)-SQ-MJ voting rule is \( \beta \)-live if and only if \( \beta > \frac{(1-\mu)(1+\tau)}{2(1-\sigma-\mu)} \).

**Proof.** Since any vote for \( r \) reduces liveness, w.l.o.g all voters vote for \( r \). There are \( h^+ = 1 - \mu - \sigma \) active honest voters (all vote for \( r \)). Suppose we create a new profile \( V' \) by moving a fraction of \( \beta \) votes from \( r \) to \( p \), then \( p \) has \( v^+ = h^+ = \beta(1 - \mu - \sigma) \) votes.

In contrast, \( r \) has \( h^+_r = h^+ - h^+_p = 1 - \mu - \sigma - h^+_p \) active honest votes remaining, plus \( \sigma \) sybils. The \( \tau \)-SQ-MJ mechanism adds \( \tau(1 - \mu) \) votes so the total support for \( r \) is

\[
\pi^+_r = (1 - \mu - \sigma - h^+_p) + \sigma + \tau(1 - \mu) = (1 + \tau)(1 - \mu) - h^+_p.
\]

Since liveness requires \( \pi^+_p > \pi^+_r \), we get a tight bound of \( 2h^+_p > (1 + \tau)(1 - \mu) \), or, equivalently,

\[
\beta = \frac{h^+_p}{1 - \mu - \sigma} > \frac{(1 + \tau)(1 - \mu)}{2(1 - \mu - \sigma)},
\]

as required.

B Mean function

One natural aggregation function in \( \mathbb{R}^d \) is the mean function \( G(V) = \frac{1}{|V|} \sum_{i \in V} s_i \).

If we assume the domain is unbounded then the questions of safety and liveness are moot, because every single voter (honest or sybil) can arbitrarily determine the location of the mean, regardless of the profile.

Let us assume then that the domain is \([0, 1]^d \). Note that it matters where we set \( r \).
Since there are already many parameters, we will consider the questions of sybils and partial participation separately. First, $\tau$-SQ-MN cannot guarantee 0-safety even in the presence of a small fraction of sybils.

**Proposition 24.** $\tau$-SQ-MN is not $\beta$-safe w.r.t. the mean for any $\beta < \frac{\sigma}{1+\tau}$. This is true regardless of $r$.

**Proof.** It is enough to consider a single dimension, where all honest voters are on $r$, and all sybils are on 1. Since we will not use negative locations, we normalize the interval so that $r = 0$. Then $\tau$-SQ-MN(V) = $\frac{2}{1+\tau} > 0 = MN(H)$. The highest we can push the outcome in $H'$ is by moving $\beta$ voters from $r = 0$ to 1, but

$$MN(H') \leq (1-\beta)0 + \beta1 = \beta < \frac{\sigma}{1+\tau} = \tau$-SQ-MN(V),$$

So $\tau$-SQ-MN(V) $\notin B(r; MN(\beta(H)))$.

With partial participation, even without sybils, the situation is more nuanced and depends on the location of the status quo $r$.

**Proposition 25.** Suppose $\sigma = 0$, $\mu > 0$. If $r \geq 1 - \mu$, $\tau$-SQ-MN is not $\beta$-safe w.r.t. the mean for any $\beta < \frac{\mu - \mu^2 + \tau(r - (1 - \mu))}{1 + \tau - \mu}$ (in particular never 0-safe). Otherwise, $\tau$-SQ-MN is 0-safe if and only if $r \geq \frac{\mu - \mu^2}{1 - \mu - \tau}$.

**Proof.** For the first case, consider a profile where all active voters are on 1 and inactive voters are on 0. Then

$$\tau$-SQ-MN(V+) = $\frac{1 - |H^+| + r\tau}{|H^+| + \tau} = \frac{1 - \mu + r\tau}{1 - \mu + \tau} \in [r, 1].$$

To be $\beta$-safe, we need to move $\beta$ honest voters from 0 to 1 in $H'$ and obtain $MN(H') \geq \tau$-SQ-MN(V+). Writing explicitly, $\beta$ should satisfy:

$$MN(H') = 1(|H^+| + \beta) = 1 - \mu + \beta \geq \frac{1 - \mu + r\tau}{1 - \mu + \tau},$$

rearranging, we get

$$\beta(1 - \mu + \tau) \geq [\mu - \mu^2] + \tau[r - (1 - \mu)],$$

where both expressions in square brackets are non-negative, and the first one is strictly positive. This gives us the bound on $\beta$.

For the second case, we argue first that the instance described above is the worst possible.

On the other hand, a mechanism that removes the $\tau$-most extreme voters (similarly to $\tau$-SOM) is 0-safe if $\tau \geq \sigma$.

Next we consider the case where there are sybils but participation is partial.
Theorem 26. Suppose \( \sigma = 0 \). The \( \tau \)-SQ-MN mechanism on an interval \([0, 1]\) is \( \frac{\sigma}{\tau \mu + \tau} \)-safe.

Note that we cannot guarantee 0-safety: if the honest voters are on \( r \) and the sybils are not, then any number of virtual voters on \( r \) will not cancel out the sybils.

Proof. Denote \( h = MN(V^+) \) and \( h^+ = MN(H^+) \). W.l.o.g. the mean \( h^+ \) is above \( r \). There are two ways in which the vote can be unsafe: either the sybils move it further above \( h^+ \), or lower below \( r \).

In the first case, the worst instance w.l.o.g. is when all of the sybils are on the upper bound \( b \) of the interval. Let us normalize the interval so that \( r = 0 \) and \( b = 1 \), then \( h^+, h \in [0, 1] \).

Considering that there are also \( \tau |V^+| \) voters on \( r \), we get

\[
h = \frac{MN(H^+)|H^+| + MN(S)|S| + r \cdot \tau |V^+|}{|H^+| + |S| + \tau |V^+|} = \frac{(1 - \sigma - \mu)h^+ + \sigma}{(1 + \tau)(1 - \mu)}.
\]

We define \( H' \) by taking the \( \beta |H| \) lowest voters and placing them on \( b = 1 \). Suppose \( \beta |H| \geq |H^-| = \mu \), then

\[
MN(H') \geq \frac{\beta |H| \cdot 1 + (1 - \beta)|H|h^+}{|H|} = \beta + (1 - \beta)h^+.
\]

divide into cases: the sybils can try to push the results up (above the mean) or down (below \( r \)). In any case they go as far as possible in the worst case. Also in the worst case \( H^+ \) are around \( r \).

However if we focus on abstention and assume participation is random, and set \( \tau = \mu \) then by linearity of expectation the expected location is

\[
E[\tau \cdot SQ-MN(H^+)] = E[MN(H^+ \cup \{\tau \cdot r\})] = (1 - mu)MN(H) + \mu r,
\]

which is indeed between \( r \) and \( MN(H) \) and thus 0-safe. In fact it is 0-safe w.h.p. but this requires a proof.

C Nonatomic Population

We consider a nonatomic population of voters, which can be thought of as the limit case of a large population. In this case, we only care about the fraction of voters for each alternative, and we can assume this fraction is exactly the same among passive and active honest voters.

We can see this in Figure 9, where the distribution of honest voters (in blue) under random participation is much more balanced than under arbitrary participation. This will allow us to show an improved safety-liveness tradeoff.

Fig. 9(a) shows an example where there is a large majority of honest voters for \( r \), and yet \( \tau \)-SQ-MJ selects \( p \). Thus, this profile implies a violation of \( \alpha \)-safety whenever \( \alpha < \frac{h - h_p}{2} \).

Otherwise, we can define an profile \( H' \) where \( \alpha |H| \) honest voters switch from \( r \) to \( p \) and get \( MJ(H' \cup S) = p \).

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Fig. 9 An example of voting profiles with the same $\sigma, \mu$ parameters under arbitrary partial participation (a), and under random partial participation (b). The thick arrows show the total amount of active votes for each alternative.

**Theorem 27.** For a nonatomic population with random participation, the $\tau$-SQ-MJ voting rule is $\alpha$-safe w.r.t Majority as the base rule if and only if

$$\alpha \geq \frac{(\sigma - \tau(1 - \mu))(1 - \sigma)}{2(1 - \mu - \sigma)}.$$

**Proof.** Recall we denote by $u_r, u_p$ the fraction of voters for $r$ and $p$, respectively, in a voter set $U$.

Suppose first that $h_p > h_r - 2\alpha$. This means that there is a profile $H'$ where we move only $\alpha h$ voters from $r$ to $p$, and $MJ(H') = p$. Thus $p \in MJ_{\alpha}(H) \subseteq B(r; MJ_{\alpha}(H))$, which means $\alpha$-safety holds.

The fraction of active voters among $H$ is denoted by $\phi := |H^+|/|H| = 1 - \mu - \sigma$.

Therefore:

$$h_r^+ - h_p^+ = \phi(h_r - h_p)$$

By definition, $\tau$-SQ-MJ($V^+$) = $MJ(H^+ \cup S \cup Q)$ where $Q$ contains $\tau(1 - \mu)$ voters for $r$.

Thus the total fraction of active $r$ voters is at least $h_r^+ + \tau(1 - \mu)$ (see Fig. 3(b)). As in the previous proofs, w.l.o.g. all sybils vote for $p$ as this is the worst case for safety. We get that

$$v_r^+ - v_p^+ \geq (h_r^+ + \tau(1 - \mu)) - (h_p^+ + \sigma)$$

(equality when $s_p = \sigma$)

$$= (h_r^+ - h_p^+) - (\sigma - \tau(1 - \mu))$$

$$= \phi(h_r - h_p) - (\sigma - \tau(1 - \mu))$$

(by Eq. (7))

$$> \frac{1 - \mu - \sigma}{1 - \sigma} \left( \frac{(\sigma - \tau(1 - \mu))(1 - \sigma)}{1 - \mu - \sigma} \right) - (\sigma - \tau(1 - \mu))$$

$$= (\sigma - \tau(1 - \mu)) - (\sigma - \tau(1 - \mu))$$

$$= 0,$$

as required.

In the other direction (i.e. to show tightness of the bound), consider any profile where all sybils vote $p$, and we set $h_p$ such that the equation holds with reversed inequality. That is,
\[
\frac{(\sigma - \tau (1 - \mu))(1 - \sigma)}{2(1 - \mu - \sigma)} > \frac{h_r - h_p}{2}.
\] (8)

Then the inequalities in the last block of equations are reversed and we get that \(v_r^+ - v_p^+ < 0\), meaning \(\tau - SQ-MJ(V^+) = p\).

On the other hand, for any \(\alpha > \frac{h_r - h_p}{2}\), we have that \(\mathcal{M}_{\alpha}(H) = \{r\}\).

Joining both observations, \(\tau - SQ-MJ\) is not \(\alpha\)-safe for any value of \(\alpha\) in the range \((\frac{h_r - h_p}{2}, \frac{(\sigma - \tau (1 - \mu))(1 - \sigma)}{2(1 - \mu - \sigma)})\).

Random participation does not allow us to improve the bound on liveness beyond Theorem 23, which is still tight.

As a result of Theorem 27, we get a better safety-liveness tradeoff under random participation:

**Corollary 28.** Under a nonatomic population with random participation:

- \(\tau - SQ-MJ\) is safe w.r.t MJ iff \(\tau \geq \frac{\sigma}{1 - \mu}\).
- \(\tau - SQ-MJ\) is live iff \(\tau < \frac{2(1 - \sigma - \mu)}{1 - \mu} - 1\).
- We can get both if \(3\sigma + \mu < 1\).

As with arbitrary participation, we show that the \(\tau - SQ-MJ\) mechanism obtains the best possible tradeoff.

**Theorem 29** (Lower bound for random participation). Under random participation and nonatomic population, there is no rule \(R\) such that \(R^+\) is both 0-safe and 1-live when \(3\sigma + \mu \geq 1\).

**Proof.** We denote by \(\phi := \frac{|H^+|}{|H|} = \frac{1 - \mu - \sigma}{1 - \mu}\) the fraction of active honest voters.

Suppose the mechanism is 1-live. By 1-liveness, there is a profile \(V\) s.t. all sybils are voting for \(r\), and \(R^+(V) = R(S \cup H^+) = p\).

For a nonatomic population, \(h_p^+ = \phi h_p\) exactly.

Now, consider a profile \(\overline{V} = S \cup \overline{H}^+ \cup \overline{H}^-,\) where \(|S| = |S|, |\overline{H}^+| = |H^+|, |\overline{H}^-| = |H^-|,\) so \(\sigma\) and \(\mu\) are the same as in \(V\). As in the proof of Thm 3, set \(\overline{\pi}_p := \min\{h_p^+, \sigma\}\) sybils to vote for \(p\). The difference from Thm. 3 is that we cannot directly (since they are selected at random), only \(\overline{h}_p\). We set

\[
\overline{h}_p := h_p - \frac{\overline{\pi}_p}{\phi}.
\] (9)

All other voters vote for \(r\).

Now, note that the total amount of active \(p\) voters is

\[
\overline{\pi}_p^+ = \overline{\pi}_p + \overline{h}_p = \overline{\pi}_p + \phi \overline{h}_p = \overline{\pi}_p + \phi (h_p - \frac{\overline{\pi}_p}{\phi}) = \overline{\pi}_p - \overline{\pi}_p + \phi h_p = \phi h_p = h_p^+ = v_p^+.
\]

This means that (as in Thm. 3), profiles \(V\) and \(\overline{V}\) are indistinguishable, and \(R^+(V) = R^+(\overline{V}) = p\).
We still need to show that $\overline{T}_r \geq \overline{T}_p$, which entails a violation of $0$-safety. Assume first that $\overline{s}_p < h^+_p$. Then $\overline{s}_p = \overline{\sigma}$ and:

$$\phi(\overline{T}_r - \overline{T}_p) = -\phi(\overline{h} - 2\overline{T}_p) = \phi(1 - \overline{\sigma} - 2\overline{T}_p)$$

$$= \phi(1 - \overline{\sigma} - 2(h_p - \overline{s}_p)) = \phi(1 - \overline{\sigma}) - 2\phi h_p + 2\overline{s}_p \quad \text{(By Eq. 9)}$$

$$= \frac{1 - \overline{\sigma} - \mu}{1 - \overline{\sigma}}(1 - \overline{\sigma}) - 2h^+_p + 2\overline{s}_p \quad \text{(By def. of $\phi$)}$$

$$\geq 1 + \overline{\sigma} - \mu - 2h^+_p \geq 1 + \overline{\sigma} - \mu - 2(1 - \overline{\sigma} - \mu) \geq 3\overline{\sigma} + \mu - 1 \geq 0,$$

where the last inequality is by the premise of the theorem. Since $\phi > 0$, this entails $\overline{T}_r - \overline{T}_p \geq 0$ as well.

If $\overline{s}_p = h^+_p$ then

$$\overline{T}_p = h_p - \frac{\overline{s}_p}{\phi} = h_p - \frac{h^+_p}{\phi} = h_p - h_p = 0,$$

meaning that in $V$ there are no honest voters for $p$. In particular $\overline{T}_r > 0 = \overline{T}_p$.

\[\square\]

### C.1 A General Result about Homogeneous Rules

A voting rule $\mathcal{R}$ is \textit{homogeneous} if $\mathcal{R}(\alpha V) = \mathcal{R}(V)$ for all $\alpha > 0$. Note that majority, mean, median, etc. all homogeneous.

**Proposition 30.** With continuous population, every homogeneous rule $\mathcal{G}$ is $\max\{\frac{\mu}{1 - \sigma}, \frac{\sigma}{1 - \mu}\}$-safe with respect to itself.

**Proof.** Suppose first that $\frac{\sigma(1 - \sigma)}{1 - \mu} \geq \mu$, and let $\mu' := \frac{\sigma(1 - \sigma)}{1 - \mu} - \mu$.

We define $H'$ follows: Selecting all of $H^-$, and additional $\mu'$ voters from $H^+$. These are $\frac{\sigma(1 - \sigma)}{1 - \mu}$ selected voters in total. Assign all of them uniformly to the locations of $S$. Denote the new locations by $S'$ and the unchanged part of the profile by $H_H$.

By construction, $H'_S = xS$ and $H'_H = yH^+$ for some $x, y$. We need to verify that $x = y$.

Indeed,

$$x = \frac{|H'_S|}{|S|} = \frac{\sigma(1 - \sigma)}{1 - \mu} \frac{1 - \sigma}{1 - \mu},$$

whereas

$$y = \frac{|H'_H|}{|H^+|} = \frac{|H| - |H'_S|}{1 - \mu - \sigma} = 1 - \sigma - \frac{\sigma(1 - \sigma)}{1 - \mu} = 1 - \sigma - \frac{(1 - \sigma)(1 - \frac{\sigma}{1 - \mu})}{1 - \sigma - \mu} = 1 - \sigma - \frac{(1 - \sigma)\frac{\mu - \sigma}{1 - \mu}}{1 - \sigma - \mu} = \frac{1 - \sigma}{1 - \mu}.$$
Therefore $H' = \frac{1-\sigma}{1-\mu} V$, and due to homogeneity

$$G(V) = G(H') \in B(\sigma; G_\sigma(H)).$$

The relative fraction of voters we moved is

$$\beta = \frac{|H'_S|}{|H|} = \frac{\sigma(1-\sigma)}{1-\mu} = \frac{\sigma}{1-\mu}.$$

If $\frac{\sigma(1-\sigma)}{1-\mu} < \mu$, then we reassign the selected voters $H'_S \subseteq H^-$ in the same way over $S$. Then we reassign the remaining $\mu - \frac{\sigma(1-\sigma)}{1-\mu}$ voters of $H^-$ over $H^+$. One can check that $H' = \frac{1-\sigma}{1-\mu} V$ as in the previous case. The difference is that we moved $\mu = \frac{\mu}{1-\sigma} |H|$ voters so $\beta = \frac{\mu}{1-\sigma}$. \qed