Restarted Nonconvex Accelerated Gradient Descent:  
No More Polylogarithmic Factor in the $O(\epsilon^{-7/4})$ Complexity

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Abstract
Nonconvex optimization with great demand of fast solvers is ubiquitous in modern machine learning. This paper studies two simple accelerated gradient methods, restarted accelerated gradient descent (AGD) and restarted heavy ball (HB) method, for general nonconvex problems under the gradient Lipschitz and Hessian Lipschitz conditions. We establish that the two algorithms find an $\epsilon$-approximate first-order stationary point in $O(\epsilon^{-7/4})$ gradient computations with simple proofs. Our complexity does not hide any polylogarithmic factors, and thus it improves over the state-of-the-art one by the $O(\log \frac{1}{\epsilon})$ factor. Our algorithms are simple in the sense that they only consist of Nesterov’s classical AGD or Polyak’s HB iterations, as well as a restart mechanism. They do not need the negative curvature exploitation or the minimization of regularized surrogate functions. Our simple proofs only use very elementary analysis, and in contrast with existing analysis, we do not invoke the analysis of the strongly convex AGD or HB.

1. Introduction
Nonconvex optimization has become the foundation of training machine learning models and emerging machine learning tasks can be formulated as nonconvex problems, typical examples include matrix completion (Hardt, 2014), one bit matrix completion (Davenport et al., 2014), robust PCA (Netrapalli et al., 2014), phase retrieval (Candès et al., 2015), and deep learning (LeCun et al., 2015). In this paper, we consider the following general nonconvex problem:

$$\min_{x \in \mathbb{R}^d} f(x),$$

where $f(x)$ is bounded from below and has Lipschitz continuous gradient and Hessian. The goal is to find an $\epsilon$-approximate first-order stationary point, defined as

$$\|\nabla f(x)\| \leq \epsilon.$$

Gradient descent, a fundamental algorithm in machine learning, is commonly used due to its simplicity and practical efficiency. Theoretically, gradient descent is the optimal algorithm among the first-order methods under the gradient Lipschitz condition for nonconvex optimization (Carmon et al., 2020), which means that we cannot find a deterministic first-order method with theoretically faster convergence rate under these conditions. When additional structure is assumed, such as the Hessian Lipschitz condition, improvement is possible. On the other hand, for convex problems, gradient descent is known to be suboptimal and several accelerated gradient methods with theoretically faster convergence rate were proposed. Typical examples include Polyak’s heavy ball (HB) method (Polyak, 1964) and Nesterov’s accelerated gradient descent (AGD) (Nesterov, 1983, 1988, 2005). Motivated by the practical superiority and rich theory of AGD and HB for convex optimization, accelerated gradient methods seem to be the potential direction to accelerate gradient descent for nonconvex optimization under both the gradient and Hessian Lipschitz conditions, and they have attracted tremendous attentions in recent years. In this paper, we study the restarted AGD and HB method, variants of the original AGD and HB by employing the restart mechanism, and give a slightly faster convergence rate than the state-of-the-art accelerated methods by simple proofs.

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1.1. Literature Review

In this section, we briefly review the convergence rates of gradient descent, accelerated gradient descent, and the heavy ball method for convex optimization, as well as the state-of-the-art accelerated methods for nonconvex optimization.

1.1.1. Accelerated Gradient Methods for Convex Optimization

For convex problems, gradient descent is known to find an $\epsilon$-optimal solution in $O(\frac{1}{\epsilon^2})$ and $O\left(\frac{1}{\mu \epsilon} \log \frac{1}{\epsilon} \right)$ iterations for $L$-smooth general convex problems and $\mu$-strongly convex problems, respectively (Nesterov, 2004). Polyak’s heavy ball method (Polyak, 1964) was the first accelerated first-order method, which finds an $\epsilon$-optimal solution in $O\left(\sqrt{\frac{L}{\mu}} \log \frac{1}{\epsilon^2} \right)$ steps when the objective function is twice continuously differentiable, $L$-smooth, and $\mu$-strongly convex. When the strong convexity is absent, currently, only the $O\left(\frac{1}{\epsilon^2} \right)$ complexity is proved for the heavy ball method (Ghadimi et al., 2015), which is the same as gradient descent. In a series of celebrated works (Nesterov, 1983; 1988; 2005), Nesterov proposed several accelerated gradient descent methods. The same $O\left(\sqrt{\frac{L}{\mu}} \log \frac{1}{\epsilon^2} \right)$ complexity is established for strongly convex problems without the twice continuously differentiability assumption. Moreover, when the objective is $L$-smooth and general convex, Nesterov’s accelerated methods find an $\epsilon$-optimal solution in $O\left(\frac{L}{\mu} \log \frac{1}{\epsilon^2} \right)$ iterations, which is faster than gradient descent and the heavy ball method in theory. Nesterov’s accelerated methods are proven to be optimal among the first-order methods for convex optimization (Nesterov, 2004). For more topics on accelerated methods for convex optimization, interested readers can refer to the survey paper (Li et al., 2020), for example.

1.1.2. Accelerated Gradient Methods to Achieve Nonconvex First-order Stationary Point

For nonconvex problems, gradient descent finds an $\epsilon$-approximate first-order stationary point of problem (1) in $O(\epsilon^{-2})$ iterations (Nesterov, 2004). Enormous amount of effort has been spent on speeding up gradient descent in the last decade. Zavriev & Kostyuk (1993); Ochs et al. (2014); Ochs (2018); Liang et al. (2016) studied the convergence of the HB method. However, no faster convergence rate is proved than gradient descent under the general gradient Lipschitz assumption. Ghadimi & Lan (2016); Li & Lin (2015); Li et al. (2017) studied the nonconvex AGD under the gradient Lipschitz condition. The efficiency is verified empirically and there is also no speed improvement in theory. Carmon et al. (2017) proposed a “convex until guilty” mechanism with nested-loop under both the gradient Lipschitz and Hessian Lipschitz conditions, which finds an $\epsilon$-approximate first-order stationary point in $O(\epsilon^{-7/4} \log \frac{1}{\epsilon})$ gradient and function evaluations. Their method alternates between the minimization of a regularized surrogate function and the negative curvature exploitation, where in the former subroutine, Carmon et al. (2017) add a proximal term to reduce the nonconvex subproblem to a convex one and use the convex AGD to minimize it until the function is “guilty” of being nonconvex. When the third-order derivative of the objective is Lipschitz, the $O(\epsilon^{−5/3} \log \frac{1}{\epsilon})$ complexity can be obtained (Carmon et al., 2017).

1.1.3. Accelerated Gradient Methods to Achieve Nonconvex Second-order Stationary Point

When studying nonconvex accelerated methods, most literatures focus on the second-order stationary point (see definition in (4)). Carmon et al. (2018) combined the regularized accelerated gradient descent and the Lanczos method, where the latter is used to search the negative curvature. Agarwal et al. (2017) implemented the cubic-regularized Newton steps carefully by using accelerated method for fast approximate matrix inversion, while Carmon & Duchi (2020; 2018) employed the Krylov subspace method to approximate the cubic-regularized Newton steps. The above methods find an $\epsilon$-approximate second-order stationary point in $O(\epsilon^{-7/4} \log \frac{1}{\epsilon})$ gradient evaluations or Hessian-vector products. To avoid the Hessian-vector products, Xu et al. (2018) and Allen-Zhu & Li (2018) proposed the NEON and NEON2 first-order procedures to extract negative curvature of the Hessian, respectively. Other typical methods include the Newton-conjugate gradient (Royer et al., 2020) and the second-order line-search method (Royer & Wright, 2018), which are beyond the AGD class. The above methods are nested-loop algorithms. They either alternate between the negative curvature exploitation and the optimization of a regularized surrogate function using convex AGD (Carmon et al., 2018; 2017), or call the accelerated methods to solve a series of cubic regularized Newton steps (Agarwal et al., 2017; Carmon & Duchi, 2020; 2018). Jin et al. (2018) proposed the first single-loop accelerated method, which finds an $\epsilon$-approximate second-order stationary point in $O(\epsilon^{-7/4} \log \frac{1}{\epsilon})$ gradient and function evaluations. The method in (Jin et al., 2018) runs the classical AGD until some condition triggers, then calls the negative curvature exploitation, and continues on the classical AGD. It is, as far as we know, the simplest algorithm among the nonconvex accelerated methods with fast rate guarantees.

Although achieving second-order stationary point ensures the method not to get stuck at the saddle points, some researchers
show that gradient descent and its accelerated variants that converge to first-order stationary point always converge to local minimum. Lee et al. (2016) established that gradient descent converges to a local minimizer almost surely with random initialization. Sun et al. (2019) gave the similar result for the heavy ball method. O’Neill & Wright (2019) proved that accelerated method is unlikely to converge to strict saddle points, and diverges from the strict saddle point more rapidly than the steepest-descent method for specific quadratic objectives.

1.1.4. LOWER BOUND FOR SECOND-ORDER SMOOTH NONCONVEX PROBLEMS

As for the lower bound, Carmon et al. (2021) established that no deterministic first-order method can find $\epsilon$-approximate first-order stationary point of functions with Lipschitz continuous gradient and Hessian in less than $O(\epsilon^{-12/7})$ gradient evaluations. There exists a gap of $O(\epsilon^{-1/28} \log \frac{1}{\epsilon})$ between the lower bound and the state-of-the-art upper bound (Carmon et al., 2017; Jin et al., 2018). It remains an open problem of how to close this gap.

1.2. Contribution

All of the above accelerated methods (Carmon et al., 2017; 2018; Agarwal et al., 2017; Carmon & Duchi, 2020; Jin et al., 2018) share the $O(\epsilon^{-7/4} \log \frac{1}{\epsilon})$ complexity, which has a $O(\log \frac{1}{\epsilon})$ factor. To the best of our knowledge, even when we apply the methods designed to find second-order stationary point to the easier problem of finding first-order stationary point, the $O(\log \frac{1}{\epsilon})$ factor still cannot be removed. On the other hand, almost all the existing methods are complex with nested loops. Even the single-loop method proposed in (Jin et al., 2018) needs the negative curvature exploitation procedure.

In this paper, we propose two simple accelerated methods, restarted AGD and restarted HB, which have the following three advantages:

1. Our methods find an $\epsilon$-approximate first-order stationary point in $O(\epsilon^{-7/4})$ gradient computations under the gradient Lipschitz and Hessian Lipschitz conditions. Our complexity does not hide any polylogarithmic factors, and thus it improves over the state-of-the-art one by the $O(\log \frac{1}{\epsilon})$ factor.

2. Our methods are simple in the sense that they only consist of Nesterov’s classical AGD or Polyak’s HB iterations, as well as a restart mechanism, and they do not need the negative curvature exploitation or the optimization of regularized surrogate functions. The simple structure makes our algorithms potentially more practical in real-world machine learning applications.

3. Technically, our proof is much simpler than all those in the existing literatures. Especially, we do not invoke the analysis of the strongly convex AGD or HB, which is crucial to remove the $O(\log \frac{1}{\epsilon})$ factor. We also extend our technical tricks to the state-of-the-art method proposed in (Jin et al., 2018) and greatly simplify their proofs with slightly faster convergence rate.

1.3. Notations and Assumptions

We use lowercase bold letters to represent vectors, uppercase bold letters for matrices, and non-bold (both lowercase and uppercase) letters for scalars. Denote $x_j$ and $\nabla_j f(x)$ as the $j$th element of $x$ and $\nabla f(x)$, respectively. For the vectors produced in the iterative algorithms, for example, $x$, denote $x^k$ to be the value at the $k$th iteration. For fixed scalars and matrices, for example, $\theta$ and $A$, we denote $\theta^k$ and $A^k$ to be their $k$th power, respectively. We denote $\| \cdot \|$ to be the $\ell_2$ Euclidean norm for vectors, $\| \cdot \|_2$ as the spectral norm and $\| \cdot \|_F$ as the Frobenius norm for matrices.

We make the following standard assumptions in this paper:

**Assumption 1**

1. $f(x)$ is $L$-gradient Lipschitz: $\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|$,  
2. $f(x)$ is $\rho$-Hessian Lipschitz: $\|\nabla^2 f(x) - \nabla^2 f(y)\|_2 \leq \rho\|x - y\|$,  

which yield the following two well-known inequalities:

\[
\|f(y) - f(x) - \langle \nabla f(x), y - x \rangle \| \leq \frac{\rho}{2} \|y - x\|^2, \quad (2)
\]

\[
\|f(y) - f(x) - \langle \nabla f(x), y - x \rangle - (y - x)^T \nabla^2 f(x)(y - x) \| \leq \frac{\rho}{2} \|y - x\|^3. \quad (3)
\]

We also assume that the objective function is lower bounded, that is, $\min_x f(x) > -\infty$. 

**Restarted Nonconvex Accelerated Gradient Descent**
Algorithm 1 Restated AGD

1. Initialize $\mathbf{x}^{-1} = \mathbf{x}^0 = x_{\text{init}}, k = 0$.
2. while $k < K$ do
3.     $\mathbf{y}^k = \mathbf{x}^k + (1 - \theta)(\mathbf{x}^k - \mathbf{x}^{k-1})$
4.     $\mathbf{x}^{k+1} = \mathbf{y}^k - \eta \nabla f(\mathbf{y}^k)$
5.     $k = k + 1$
6.     if $k \sum_{t=0}^{k-1} \|\mathbf{x}^{t+1} - \mathbf{x}^t\|^2 > B^2$ then
7.         $\mathbf{x}^{-1} = \mathbf{x}^0 = \mathbf{x}^k, k = 0$
8.     end if
9. end while
10. $K_0 = \arg\min_{\frac{K}{10} \leq k \leq K-1} \|\mathbf{x}^{k+1} - \mathbf{x}^k\|$
11. Output $\hat{\mathbf{y}} = \frac{1}{K_0 + 1} \sum_{k=0}^{K_0} \mathbf{y}^k$

2. Restated Accelerated Gradient Descent

Nesterov’s classical AGD consists of the following iterations:

$$
\mathbf{y}^k = \mathbf{x}^k + (1 - \theta)(\mathbf{x}^k - \mathbf{x}^{k-1}),
$$

$$
\mathbf{x}^{k+1} = \mathbf{y}^k - \eta \nabla f(\mathbf{y}^k),
$$

where $\theta = \frac{2 \sqrt{\eta}}{\sqrt{\eta} + \sqrt{\eta}}$ for strongly convex problems and it varies as $\frac{3}{k + 2}$ at the $k$th iteration for general convex problems. When applying the above iteration to nonconvex problems, the major challenge in faster convergence analysis is that the objective function (even the Hamiltonian potential function used in (Jin et al., 2018)) does not decrease monotonically, especially when we set $\eta = \mathcal{O}\left(\frac{1}{\sqrt{t}}\right)$ and $\theta$ small. To overcome this issue, Jin et al. (2018) use the negative curvature exploitation when the objective is very nonconvex. An open problem is asked in Section 5 of (Jin et al., 2018) whether the negative curvature exploitation is necessary for the fast rate. In contrast with (Jin et al., 2018), we use the restart mechanism to ensure the decrease of the objective function, and thus avoid the negative curvature exploitation.

Our method is described in Algorithm 1. It runs Nesterov’s classical AGD until the “if condition” triggers. Then we restart by setting $\mathbf{x}^0$ and $\mathbf{x}^{-1}$ equal to $\mathbf{x}^k$ and do the next round of AGD. The algorithm terminates when the “if condition” does not trigger in $K$ iterations. To simplify the description, we define one round of AGD to be one “epoch”. The restart trick is motivated by (Fang et al., 2019), who proposed a ball-mechanism as the stopping criteria to analyze SGD.

We present our main result in Theorem 1, which establishes the $\mathcal{O}(\epsilon^{-7/4})$ complexity to find an $\epsilon$-approximate first-order stationary point.

**Theorem 1** Suppose that Assumption 1 holds. Let $\eta = \frac{1}{\mathcal{L}t}, B = \sqrt{\frac{\epsilon}{\theta}}, \theta = 4(\epsilon \eta)^{1/4} < 1, K = \frac{1}{\eta}$. Then Algorithm 1 terminates in at most $\frac{\Delta f L^{1/2} \rho^{1/4}}{\epsilon^{7/4}}$ gradient computations and the output satisfies $\|\nabla f(\hat{\mathbf{y}})\| \leq 8\epsilon$, where $\Delta f = f(x_{\text{init}}) - \min_{x} f(x)$.

Among the existing methods, Carmon et al. (2017) established the $\mathcal{O}(\frac{\Delta f L^{1/2} \rho^{1/4}}{\epsilon^{7/4}} \log \frac{L \Delta f}{\epsilon})$ complexity to find an $\epsilon$-approximate first-order stationary point, which has the additional $\mathcal{O}(\log \frac{1}{\epsilon})$ factor compared with our one. The complexity given in other literatures concentrating on second-order stationary point, such as (Carmon et al., 2018; Agarwal et al., 2017; Carmon & Duchi, 2020; Jin et al., 2018), also has the additional $\mathcal{O}(\log \frac{1}{\epsilon})$ factor even for finding first-order stationary point. Take (Jin et al., 2018) as the example. Their Lemma 7 studies the first-order stationary point. Their proof in Lemmas 9 and 17 is built upon the analysis of strongly convex AGD, which generally needs $\mathcal{O}(\sqrt{\frac{\mathcal{L}}{\epsilon} \log \frac{1}{\epsilon}})$ iterations such that the gradient norm will be less than $\epsilon$, and thus the $\mathcal{O}(\log \frac{1}{\epsilon})$ factor appears.

**Remark 1**

1. The specific average is the crucial reason to improve the convergence rate. See the proof in Section 5.1.3. This phenomenon that some averaged iterate converges faster than the final iterate has also been observed in other algorithms. For example, for Lipschitz and strongly convex functions, but not necessarily differentiable, Shamir & Zhang (2013) proved the $\mathcal{O}(\log^T)_{\epsilon}$ error of the final iterate of SGD while the $\mathcal{O}(\frac{1}{\epsilon})$ one for the suffix averaged iterate.
Algorithm 2 Practical Restarted AGD

1: Initialize $x^{-1} = x^0 = x^0_{\text{cur}} = x_{\text{init}}, k = 0, B_0, c > 1$.
2: while $k < K$ or $B_0 > B$ do
3:   $y^k = x^k + (1 - \theta)(x^k - x^{k-1})$
4:   $x^{k+1} = y^k - \eta \nabla f(y^k)$
5:   $k = k + 1$
6:   if $k \sum_{t=0}^{k-1} \|x^{t+1} - x^t\|^2 > \max\{B^2, B_0^2\}$ or $k > K$ then
7:     if $f(x^k) - f(x^0) \leq -\min \left\{ \frac{\epsilon_1}{\sqrt{\rho}}, \frac{\epsilon_2}{\rho} \right\}$ then
8:       $x^{-1} = x^0 = x^k, x^0_{\text{cur}} = x^k, k = 0$
9:     else
10:        $x^{-1} = x^0, k = 0, B_0 = B_0/c$
11:    end if
12: end if
13: end while
14: $K_0 = \arg\min_{0 < t \leq K-1} \|x^{k+1} - x^k\|$
15: $\hat{y} = \frac{1}{K_0+1} \sum_{k=0}^{K_0} y^k$
16: Output $x_{\text{out}} = \arg\min_{x, \hat{y}} \{ \|\nabla f(x^K)\|, \|\nabla f(\hat{y})\| \}$

Both rates are tight matching the corresponding lower bounds (Harvey et al., 2019). For linearly constrained convex problems, Davis & Yin (2017) proved the $O(\frac{1}{\sqrt{T}})$ rate for the final iterate of ADMM while the $O(\frac{1}{T})$ one for the averaged iterate. The two rates are also tight (Davis & Yin, 2017).

On the other hand, we can also prove the gradient at the last iterate in our method is small with norm being less than $\epsilon$ by employing the proof techniques in (Jin et al., 2018), at the expense of introducing the additional $O(\log \frac{1}{\epsilon})$ factor and complicating the proof.

2. Restart plays the role of decreasing the objective function at each epoch of AGD. Intuitively, when the iterates are far from the local starting point $x^0$ or the momentum $x^k - x^{k-1}$ is large such that it may potentially increase the objective function, restart cancels the effect of the momentum by setting it to 0.

3. As discussed in Section 5.2, since our proof does not invoke the analysis of the strongly convex AGD or HB, the acceleration mechanism for nonconvex optimization seems irrelevant to the analysis of convex AGD. It is just because of the momentum and its parameter $\theta$, which may be of independent interest.

4. In Appendix C, we give a detailed implementation of the average in Algorithm 1, which determines the index $K_0$ and computes the average $\hat{y}$ easily on the fly.

2.1. Practical Implementation

In Algorithm 1, we set $B$ of the order $O(\sqrt{\epsilon})$ and the method may restart frequently in the first few iterations. In this case, Algorithm 1 almost reduces to the classical gradient descent. To make use of the practical superiority of AGD in the first few iterations, we should reduce the frequency of restart. A practical implementation is presented in Algorithm 2, which relaxes the restart condition of $k \sum_{t=0}^{k-1} \|x^{t+1} - x^t\|^2 > B^2$ in Algorithm 1 to $k \sum_{t=0}^{k-1} \|x^{t+1} - x^t\|^2 > \max\{B^2, B_0^2\},$ and $B_0$ can be initialized much larger than $B$. We decrease $B_0$ and drop the whole iterates in this epoch of AGD when the objective value does not decrease or decreases less than a threshold of the order $O(\epsilon^{3/2})$. When $B_0 \leq B$, Algorithm 2 is equivalent to Algorithm 1 since $f(x^k) - f(x^0) \leq -\min \left\{ \frac{\epsilon^{3/2}}{\sqrt{\rho}}, \frac{\epsilon_3}{\rho} \right\}$ always holds from (16). Algorithm 2 terminates when $B_0 \leq B$ and $k$ equals to $K$. On the other hand, we output the one of $x^K$ and $\hat{y}$ with the smaller gradient norm, because in practice we always use the last iterate, rather than the averaged one. We present the $O(\epsilon^{-7/4})$ complexity of Algorithm 2 in Theorem 2.

Theorem 2 Suppose that Assumption 1 holds and use the parameter settings in Theorem 1. Then Algorithm 2 terminates in at most $O \left( \frac{\epsilon^{1/2} \sqrt{\rho} \sqrt{B}}{\epsilon^{7/4} \sqrt{\rho} \sqrt{B}} \right)$ gradient computations and $O \left( \frac{\epsilon^{1/2} \sqrt{\rho} \sqrt{B}}{\epsilon^{7/4} \sqrt{\rho} \sqrt{B}} \right)$ function evaluations, and the output satisfies $\|\nabla f(x_{\text{out}})\| \leq 82\epsilon$, where $\triangle_f = f(x_{\text{init}}) - \min_x f(x)$.
Remark 2 Algorithm 2 also applies to the case when the Lipschitz constants $L$ and $\rho$ are unknown. We can initialize a small guess and replace line 10 in Algorithm 2 by the following steps:

$$x^{-1} = x^0 = x_{\text{cur}}, \quad k = 0, \quad B_0 = B_0/c_1, \quad L = Lc_2, \quad \rho = \rho c_3,$$

where $\{c_1, c_2, c_3\} > 1$, and the output also satisfies $\|\nabla f(x_{\text{out}})\| \leq O(\epsilon)$ within $O(\epsilon^{-7/4})$ gradient computations and $O(\epsilon^{-3/4})$ function evaluations. Intuitively, when $B_0 \leq B$ and the guessed Lipschitz constants $L$ and $\rho$ are larger than the true ones, we always have $f(x^k) - f(x^0) \leq -\min\{\frac{\epsilon^2}{\sqrt{\rho}}, \frac{\epsilon}{\rho}\}$ from (16). Thus, when the descent property does not hold, it is either because $B_0 > B$, or due to the two guessed Lipschitz constants are smaller than the true ones. So we only need to decrease $B_0$ and increase $L$ and $\rho$.

2.2. Extension to the Second-order Stationary Point

Our restarted AGD can also find $\epsilon$-approximate second-order stationary point, defined as

$$\|\nabla f(x)\| \leq \epsilon, \quad \lambda_{\min}(\nabla^2 f(x)) \geq -\sqrt{\epsilon \rho}, \quad (4)$$

where $\lambda_{\min}$ means the smallest eigenvalue. We follow (Jin et al., 2017; 2018) to add the perturbations generated uniformly from the ball $B_0(r)$ with radius $r$ and center 0. Specifically, we only need to replace line 7 in Algorithm 1 by the following step:

$$x^-1 = x^0 = x^k + \xi \|\nabla f(y^k+1)\| \leq \frac{\rho}{\epsilon}, \quad k = 0$$

where $\xi \sim \text{Unif}(B_0(r))$ and $1\|\nabla f(y^k-1)\| \leq \frac{\rho}{\epsilon} = \left\{ \begin{array}{ll} 1, & \text{if } \|\nabla f(y^k-1)\| \leq \frac{B}{\epsilon}, \\ 0, & \text{otherwise.} \end{array} \right.$

The complexity is given in Theorem 3. We see that the algorithm needs at most $O(\epsilon^{-7/4} \log \frac{d}{\epsilon^2})$ gradient computations to find an $\epsilon$-approximate second-order stationary point with probability at least $1 - \zeta$, where $d$ is the dimension of $x$ in problem (1). This complexity is the same with the one given in (Jin et al., 2018). Comparing with Theorem 1, we see that there is a $O(\log \frac{d}{\epsilon^2})$ term. Currently, it is unclear how to remove it, and we conjecture that the polylogarithmic factor may not be canceled.

Theorem 3 Suppose that Assumption 1 holds. Let $\chi = O(\log \frac{d}{\epsilon^2}) \geq 1$, $\eta = \frac{1}{4\epsilon L}$, $B = \frac{1}{288\epsilon^2} \sqrt{\frac{\epsilon}{\rho}}, \quad \theta = \frac{1}{2} \left( \frac{\rho}{L^2} \right)^{1/4} < 1$, $K = \frac{2\chi}{\theta}$, $r = \min\{\frac{B}{2}, \frac{\theta B}{20K}, \sqrt{\frac{\theta B^2}{2K}}\} = O(\epsilon)$. Then the algorithm terminates in at most $O\left( \frac{\chi \epsilon L^{1/2} \rho^{1/4} \sqrt{\epsilon}}{\epsilon^{7/4}} \right)$ gradient computations and the output satisfies $\|\nabla f(y)\| \leq \epsilon$, where $\triangle f = f(x_{\text{init}}) - \min_x f(x)$. It also satisfies $\lambda_{\min}(\nabla^2 f(y)) \geq -1.011 \sqrt{\epsilon \rho}$ with probability at least $1 - \zeta$.

Theorem 3 also applies to the perturbed variant of Algorithm 2. In short, the while loop in Algorithm 2 will not terminate until $B_0 \leq B$, and Algorithm 2 reduces to Algorithm 1 when $B_0 \leq B$.

3. Restarted Heavy Ball Method

Polyak’s classical heavy ball method (Polyak, 1964) iterates with the following step

$$x^{k+1} = x^k - \eta \nabla f(x^k) + (1 - \theta)(x^k - x^{k-1}),$$

where $\eta = \frac{4}{(\sqrt{\rho L} + \sqrt{\rho})^2}$ and $1 - \theta = \frac{(\sqrt{\epsilon} - \sqrt{\rho})^2}{(\sqrt{\epsilon} + \sqrt{\rho})^2}$ for strongly convex problems (Ochs et al., 2015). In the deep learning literature, people often use the following equivalent iterations empirically (Sutskever et al., 2013),

$$m^k = \beta m^{k-1} + \nabla f(x^k), \quad x^{k+1} = x^k - \eta m^k,$$

where $m^{-1} = 0$ and $\beta = 1 - \theta$ for the deterministic problems. When applying the heavy ball iteration to nonconvex optimization, people often set $\eta = O\left( \frac{\rho}{\epsilon^2} \right)$ to ensure the convergence (Ochs et al., 2014; Sun et al., 2019), which prevents us from proving faster convergence in theory and slows down the algorithm in practice when $\theta$ is small. To address this issue,
Algorithm 3 Restated HB

1: Initialize $x^{-1} = x^0 = x_{int}, k = 0.$
2: while $k < K$ do
3: $x^{k+1} = x^k - \eta \nabla f(x^k) + (1 - \theta)(x^k - x^{k-1})$
4: $k = k + 1$
5: if $k \sum_{t=0}^{k-1} \|x^{t+1} - x^t\|^2 > B^2$ then
6: $z^k = \frac{x^k + (1 - 2\theta)(1 - \theta)x^{k-1}}{1 + (1 - 2\theta)(1 - \theta)}$
7: $x^{-1} = x^0 = z^k, k = 0$
8: end if
9: end while
10: $K_0 = \text{argmin}_{\frac{K}{2}} \|x^{k+1} - x^k\|
11: Output $\hat{x} = \frac{1}{K_0 + 1} \sum_{k=0}^{K_0} x^k$

Algorithm 4 Practical Restated HB

1: Initialize $x^{-1} = x^0 = x_{cur} = x_{int}, k = 0, B_0, c > 1.$
2: while $k < K$ or $B_0 > B$ do
3: $x^{k+1} = x^k - \eta \nabla f(x^k) + (1 - \theta)(x^k - x^{k-1})$
4: $k = k + 1$
5: if $k \sum_{t=0}^{k-1} \|x^{t+1} - x^t\|^2 > \max \{B^2, B_0^2\}$ or $k > K$ then
6: $z^k = \frac{x^k + (1 - 2\theta)(1 - \theta)x^{k-1}}{1 + (1 - 2\theta)(1 - \theta)}$
7: if $f(z^k) - f(x^0) \leq -\min \left\{ \frac{\epsilon^2}{2\sqrt{\rho}}, \frac{3\epsilon L}{16\rho} \right\}$ then
8: $x^{-1} = x^0 = z^k, x_{cur}^0 = z^k, k = 0$
9: else
10: $x^{-1} = x^0 = x_{cur}, k = 0, B_0 = B_0/c$
11: end if
12: end if
13: end while
14: $K_0 = \text{argmin}_{\frac{K}{2}} \|x^{k+1} - x^k\|
15: \hat{x} = \frac{1}{K_0 + 1} \sum_{k=0}^{K_0} x^k
16: Output $x_{out} = \text{argmin}_{x} \{\|\nabla f(x^K)\|, \|\nabla f(\hat{x})\|\}$

similar to the restarted AGD, we combine the restart mechanism with the heavy ball method such that $\eta = O\left(\frac{1}{T}\right)$ while maintaining $\theta$ small. Our method is presented in Algorithm 3. It runs Polyak’s classical HB iteration until the “if condition” triggers. Then we restart from the auxiliary vector $z^k$, a convex combination of $x^k$ and $x^{k-1}$, and do the next round of HB method. Algorithm 3 shares almost the same framework as Algorithm 1, and the only difference comes from the iterate $z^k$, which is designed to fit the proof. See Remark 4 for the detailed reason.

The main result is given in Theorem 4, which also establishes the $O(\epsilon^{-7/4})$ complexity to find an $\epsilon$-approximate first-order stationary point. Comparing with Theorem 1, we see that the two algorithms need the same assumptions, share the same convergence rate, and have almost the same parameter settings, which indicate that no one is superior to the other in theory for nonconvex optimization. As a comparison, the heavy ball method requires more assumptions for strongly convex problems and has the slower convergence rate in theory for general convex problems than AGD.

Theorem 4 Suppose that Assumption 1 holds. Let $\eta = \frac{1}{\sqrt{T}}, B = \sqrt{\frac{7}{2\epsilon}}, \theta = 10\left(\frac{\epsilon\rho^2}{2}\right)^{1/4} \leq \frac{1}{10}, K = \frac{1}{3}$. Then Algorithm 3 terminates in at most $\frac{\Delta f(x_0)^{1/2}}{e^{1/4}}$ gradient computations and the output satisfies $\|\nabla f(\hat{x})\| \leq 242\epsilon$, where $\Delta f = f(x_{int}) - \min x f(x)$.

In practice, Algorithm 3 has the same disadvantages as Algorithm 1 when $B$ is small. Similar to Algorithm 2, we also propose a practical implementation of Algorithm 3, and present it in Algorithm 4. Theorem 5 gives the $O(\epsilon^{-7/4})$ complexity.
We follow (Jin et al., 2018) to deal with the first two cases. For the third case, by rewriting the algorithm in epochs, we can prove the gradient at the specific averaged iterate \( \hat{y} \) in the last epoch is small with norm less than \( \epsilon \), where \( \Delta f = f(x_{\text{init}}) - \min_x f(x) \).

4. Extension to Jin’s Method

In this section, we extend our analysis to the state-of-the-art algorithm proposed in (Jin et al., 2018), which is detailed in Algorithm 5. No perturbation is added since we do not consider second-order stationary point for simplicity. Except the perturbation and that we specify the stopping criteria and the output, as well as that we rewrite the algorithm in epochs, Algorithm 5 is equivalent to the one in (Jin et al., 2018). However, we give a slightly faster convergence rate by a factor with much simpler proofs.

For each epoch of AGD, we have three cases:

1. The negative curvature exploitation (NCE) is employed at the last iteration. We restart by setting the momentum to 0 in the next epoch.

2. The condition \( (k+1) \sum_{t=0}^{k} \| x^{t+1} - x^t \|^2 > B^2 \) triggers at the last iteration. Note that in this case, AGD does not restart because \( x^0 - x^{-1} = v^0 \neq 0 \).

3. None of the above two cases occurs, and the while loop breaks until \( k = K \). This is the last epoch.

We follow (Jin et al., 2018) to deal with the first two cases. For the third case, by rewriting the algorithm in epochs, we can prove the gradient at the specific averaged iterate \( \hat{y} \) in the last epoch is small with norm less than \( \epsilon \) within \( K = O(\frac{1}{\epsilon^2}) \).

**Algorithm 5** AGD-Jin

1: Initialize \( x^0 = x_{\text{init}}, v^0 = 0, k = 0. 
2: while \( k < K \) do
3: \( y^k = x^k + (1 - \theta)v^k \)
4: \( x^{k+1} = y^k - \eta \nabla f(y^k) \)
5: \( v^{k+1} = x^{k+1} - x^k \)
6: if \( f(x^k) < f(y^k) + \langle \nabla f(y^k), x^k - y^k \rangle - \frac{\eta}{2} \| x^k - y^k \|^2 \) then
7: \( x^{k+1} \leftarrow -\text{Negative Curvature Exploitation}(x^k, v^k, s) \)
8: \( x^0 = x^{k+1}, v^0 = v^{k+1} = 0, k = 0 \)
9: else if \( (k + 1) \sum_{t=0}^{k} \| x^{t+1} - x^t \|^2 > B^2 \) then
10: \( x^0 = x^{k+1}, v^0 = v^{k+1}, k = 0 \)
11: else
12: \( k = k + 1 \)
13: end if
14: end while
15: \( K_1 = \arg\min_{1 \leq k \leq \lfloor \frac{K}{2} \rfloor} \| x^k - x^{k-1} \| \)
16: \( K_2 = \arg\min_{\frac{K}{2} \leq k \leq K-1} \| x^{k+1} - x^k \| \)
17: Output \( \hat{y} = \frac{1}{K_2-K_1+1} \sum_{k=K_1}^{K_2} y^k \)

**Algorithm 6** Negative Curvature Exploitation(\( x^k, v^k, s \))

1: if \( \| v^k \| \geq s \) then
2: \( x^{k+1} = x^k \)
3: else
4: \( \delta = s \| v^k \| \)
5: \( x^{k+1} = \arg\min_{x^k + \delta, x^k - \delta} f(x) \)
6: end if
7: Return \( x^{k+1} \)

**Theorem 5** Suppose that Assumption 1 holds and use the parameter settings in Theorem 4. Then Algorithm 4 terminates in at most \( O \left( \frac{\| \Delta f \|^2}{\epsilon^2} \right) \) gradient computations and \( O \left( \frac{\| \Delta f \|^1}{\epsilon^2} \right) \) function evaluations, and the output satisfies \( \| \nabla f(x_{\text{out}}) \| \leq 242 \epsilon \), where \( \Delta f = f(x_{\text{init}}) - \min_x f(x) \).
iterations. As a comparison, Jin et al. (2018) invoke the analysis of strongly convex AGD and thus introduce the additional $O(\log \frac{1}{\epsilon})$ factor. The proof in (Jin et al., 2018), although very novel, is quite involved, especially the spectral analysis of the second-order system. Our specific average greatly simplifies the proof in (Jin et al., 2018). We present the total complexity in Theorem 6.

**Theorem 6** Suppose that Assumption 1 holds. Let $\eta = \frac{1}{4L}$, $B = \sqrt{\frac{2}{\rho^2}}$, $\theta = 4 \left(\epsilon \rho \eta^2\right)^{1/4} < 1$, $K = \frac{1}{\theta}$, $\gamma = \frac{\theta^2}{\eta}$, $s = \frac{3}{4\rho}$. Then Algorithm 5 terminates in at most $\frac{\Delta_f L^{1/2} \rho^{1/4}}{\epsilon^2}$ gradient and function evaluations and the output satisfies $\|\nabla f(\hat{y})\| \leq 267\epsilon$, where $\Delta_f = f(x_{init}) - \min_x f(x)$.

It should be noted that we measure the convergence rate at the average of the iterates. When measuring at the final iterate, which is always used in practice, we should use the proof in (Jin et al., 2018), and we conjecture that the $O(\log \frac{1}{\epsilon})$ factor in unlikely to cancel.

### 5. Proof of the Theorems

We prove Theorems 1, 2, and 4 in this section, and leave the proofs of Theorems 3 and 6 in the appendix.

#### 5.1. Proof of Theorem 1

Define $K$ to be the iteration number when the “if condition” triggers, that is,

$$K = \min_k \left\{ k \left| k \sum_{t=0}^{k-1} \|x^{t+1} - x^t\|^2 > B^2 \right\} \right..$$

Denote the iterations from $k = 0$ to $k = K$ to be one epoch. Then for each epoch except the last one, we have $1 \leq K \leq K,

$$K \sum_{t=0}^{K-1} \|x^{t+1} - x^t\|^2 > B^2, \quad (5a)$$

$$\|x^k - x^0\|^2 \leq k \sum_{t=0}^{k-1} \|x^{t+1} - x^t\|^2 \leq B^2, \forall k < K, \quad (5b)$$

$$\|y^k - x^0\| \leq \|x^k - x^0\| + \|x^k - x^{k-1}\| \leq 2B, \forall k < K. \quad (5c)$$

For the last epoch, that is, the “if condition” does not trigger and the while loop breaks until $k = K$, we have

$$\|x^k - x^0\|^2 \leq k \sum_{t=0}^{k-1} \|x^{t+1} - x^t\|^2 \leq B^2, \forall k \leq K, \quad (6a)$$

$$\|y^k - x^0\| \leq 2B, \forall k \leq K. \quad (6b)$$

We will show in Sections 5.1.1 and 5.1.2 that the function value decreases with a magnitude at least $O(\epsilon^{1.5})$ in each epoch except the last one. Thus the algorithm terminates in at most $O(\epsilon^{-1.5})$ epochs, and accordingly $O(\epsilon^{-1.75})$ total gradient computations since each epoch needs at most $O(\epsilon^{-0.25})$ iterations. In the last epoch, we will show that the gradient norm at the output iterate is less than $O(\epsilon)$, which is detailed in Section 5.1.3.

#### 5.1.1. Large Gradient of $\|\nabla f(y^{K-1})\|$

We first consider the case when $\|\nabla f(y^{K-1})\|$ is large.

**Lemma 1** Suppose that Assumption 1 holds. Let $\eta \leq \frac{1}{4L}$ and $0 \leq \theta \leq 1$. When the “if condition” triggers and $\|\nabla f(y^{K-1})\| > \frac{B^2}{\eta}$, then we have

$$f(x^K) - f(x^0) \leq -\frac{B^2}{4\eta}.$$
Proof 1 From the L-gradient Lipschitz condition (2), we have

\[ f(x^{k+1}) \leq f(y^k) + \langle \nabla f(y^k), x^{k+1} - y^k \rangle + \frac{L}{2} ||x^{k+1} - y^k||^2 \]

\[ = f(y^k) - \eta \|\nabla f(y^k)\|^2 + \frac{L\eta}{2} \|\nabla f(y^k)\|^2 \]

\[ \leq f(y^k) - \frac{7\eta}{8} \|\nabla f(y^k)\|^2, \]

where we use the AGD iteration and \( \eta \leq \frac{1}{4L} \). From the L-gradient Lipschtiz, we also have

\[ f(x^k) \geq f(y^k) + \langle \nabla f(y^k), x^k - y^k \rangle - \frac{L}{2} ||x^k - y^k||^2. \]

So we have

\[ f(x^{k+1}) - f(x^k) \]

\[ \leq \frac{1}{\eta} \langle x^{k+1} - y^k, x^k - y^k \rangle + \frac{L}{2} ||x^k - y^k||^2 - \frac{7\eta}{8} \|\nabla f(y^k)\|^2 \]

\[ = \frac{1}{2\eta} \left( ||x^{k+1} - y^k||^2 + ||x^k - y^k||^2 - ||x^{k+1} - x^k||^2 \right) + \frac{L}{2} ||x^k - y^k||^2 - \frac{7\eta}{8} \|\nabla f(y^k)\|^2 \]

\[ \leq \frac{5}{8\eta} ||x^k - y^k||^2 - \frac{1}{2\eta} ||x^{k+1} - x^k||^2 - \frac{3\eta}{8} \|\nabla f(y^k)\|^2 \]

\[ \leq \frac{5}{8\eta} ||x^k - x^{k-1}||^2 - \frac{1}{2\eta} ||x^{k+1} - x^k||^2 - \frac{3\eta}{8} \|\nabla f(y^k)\|^2, \]

where we use \( L \leq \frac{1}{\eta} \) in \( \leq \) and \( ||x^k - y^k|| = (1 - \theta)||x^k - x^{k-1}|| \leq ||x^k - x^{k-1}|| \) in \( \leq \). Summing over \( k = 0, \ldots, K - 1 \) and using \( x^0 = x^{-1} \), we have

\[ f(x^K) - f(x^0) \leq \frac{1}{8\eta} \sum_{k=0}^{K-2} ||x^{k+1} - x^k||^2 - \frac{3\eta}{8} \sum_{k=0}^{K-1} \|\nabla f(y^k)\|^2 \]

\[ \leq \frac{B^2}{8\eta} - \frac{3\eta}{8} \|\nabla f(x^{K-1})\|^2 \]

\[ \leq \frac{B^2}{8\eta} - \frac{3B^2}{8\eta} = -\frac{B^2}{4\eta}, \]

where we use (5b) in \( \leq \) and \( \|\nabla f(x^{K-1})\| > \frac{B}{\eta} \) in \( \leq \).

5.1.2. SMALL GRADIENT OF \( \|\nabla f(x^{K-1})\| \)

If \( \|\nabla f(x^{K-1})\| \leq \frac{B}{\eta} \), then from the AGD iteration and (5c) we have

\[ ||x^k - x^0|| \leq ||y^{K-1} - x^0|| + \eta \|\nabla f(y^{K-1})\| \leq 3B. \]

For each epoch, denote \( H = \nabla^2 f(x^0) \) and \( H = U \Lambda U^T \) to be its eigenvalue decomposition with \( U, \Lambda \in \mathbb{R}^{d \times d} \). Let \( \lambda_j \) be the \( j \)th eigenvalue. Denote \( \bar{x} = U^T x, \bar{y} = U^T y, \) and \( \nabla f(y) = U^T \nabla f(y). \) From the \( \rho \)-Hessian Lipschitz condition (3), we have

\[ f(x^k) - f(x^0) \leq \langle \nabla f(x^0), x^k - x^0 \rangle + \frac{1}{2} (x^k - x^0)^T H (x^k - x^0) + \frac{\rho}{6} ||x^k - x^0||^3 \]

\[ = \langle \nabla f(x^0), \bar{x}^k - \bar{x}^0 \rangle + \frac{1}{2} (\bar{x}^k - \bar{x}^0)^T \Lambda (\bar{x}^k - \bar{x}^0) + \frac{\rho}{6} ||x^k - x^0||^3 \]

\[ \leq g(\bar{x}^k) - g(\bar{x}^0) + 4.5\rho B^3, \]
where we denote
\[ g(x) = \langle \nabla f(x^0), x - x^0 \rangle + \frac{1}{2}(x - x^0)^T \Lambda (x - x^0), \]
\[ g_j(x) = \langle \nabla_j f(x^0), x - x_j^0 \rangle + \frac{1}{2} \lambda_j (x - x_j^0)^2. \]

Denoting
\[ \bar{\delta}_j^k = \nabla_j f(y^k) - \nabla g_j(y_j^k), \quad \delta_j^k = \nabla f(y^k) - \nabla g(y^k), \]
then the AGD iterations can be rewritten as
\[ \bar{y}_j^k = \bar{x}_j^k + (1 - \theta) (\bar{x}_j^k - \bar{x}_j^{k-1}), \]
\[ x_j^{k+1} = \bar{y}_j^k - \eta \nabla_j f(y^k) = \bar{y}_j^k - \eta \nabla g_j(y_j^k) - \eta \bar{\delta}_j^k, \]  
\[ \delta_j^k = \nabla f(y^k) - \nabla g(y^k), \]
and \( \| \delta_j^k \| \) can be bounded as
\[
\| \delta_j^k \| = \| \nabla f(y^k) - \nabla f(x^0) - \Lambda (y_j^k - x^0) \|
\leq \| \nabla f(y^k) - \nabla f(x^0) - H(y_j^k - x^0) \|
\leq \left( \int_0^1 \| \nabla^2 f(x^0 + t(y_j^k - x^0)) - H \| dt \right) \| y_j^k - x^0 \|
\leq \frac{\rho}{2} \| y_j^k - x^0 \|^2 \leq 2\rho B^2
\]
for any \( k < K \), where we use the \( \rho \)-Hessian Lipschitz assumption and (5c) in the last two inequalities.

From (8), to prove the decrease from \( f(x^0) \) to \( f(x^K) \), we only need to study \( g(x^K) - g(x^0) \), that is, the decrease of \( g(x) \). Iterations (10a) and (10b) can be viewed as applying AGD to the quadratic approximation \( g(x) \) coordinately with the approximation error \( \bar{\delta}_j^k \), which can be controlled within \( O(\rho B^2) \). The quadratic function \( g(x) \) equals to the sum of \( d \) scalar functions \( g_j(x_j) \). We decompose \( g(x) \) into \( \sum_{j \in S_1} g_j(x_j) \) and \( \sum_{j \in S_2} g_j(x_j) \), where
\[ S_1 = \left\{ j : \lambda_j \geq -\frac{\theta}{\eta} \right\}, \quad S_2 = \left\{ j : \lambda_j < -\frac{\theta}{\eta} \right\}. \]

We see that \( g_j(x) \) is approximate convex when \( j \in S_1 \), and strongly concave when \( j \in S_2 \). We will prove the approximate decrease of \( g_j(x_j) \) in the two cases. We first consider \( \sum_{j \in S_1} g_j(x_j) \) in the following lemma.

**Lemma 2** Suppose that Assumption 1 holds. Let \( \eta \leq \frac{1}{4\theta} \) and \( 0 \leq \theta \leq 1 \). When the “if condition” triggers and \( \| \nabla f(y_j^{K-1}) \| \leq \frac{8K^2}{\theta} \), then we have
\[
\sum_{j \in S_1} g_j(x_j^K) - \sum_{j \in S_1} g_j(x_j^0) \leq - \sum_{j \in S_1} \frac{3\theta}{8\eta} \sum_{k=0}^{K-1} |x_j^{k+1} - x_j^k|^2 + \frac{8\eta \rho^2 B^4 K}{\theta}. \tag{12}
\]

**Proof 2** Since \( g_j(x) \) is quadratic, we have
\[
g_j(x_j^{k+1}) = g_j(x_j^k) + \langle \nabla g_j(x_j^k), x_j^{k+1} - x_j^k \rangle + \frac{\lambda_j}{2} |x_j^{k+1} - x_j^k|^2
\leq g_j(x_j^k) - \frac{1}{\eta} \left( |x_j^{k+1} - x_j^k| + \eta \bar{\delta}_j^k \right) x_j^{k+1} - x_j^k \rangle
+ \langle \nabla g_j(x_j^k), x_j^{k+1} - x_j^k \rangle + \frac{\lambda_j}{2} |x_j^{k+1} - x_j^k|^2
\]
Lemma 3 Suppose that Assumption 1 holds. Let \( \eta \leq \frac{1}{4\pi} \) and \( 0 \leq \theta \leq 1 \). When the “if condition” triggers and \( \|\nabla f(y^{k-1})\| \leq \frac{\theta}{\eta} \), then we have
\[
\sum_{j \in S_2} g_j(\tilde{x}^k_j) - \sum_{j \in S_2} g_j(\tilde{x}^0_j) \leq \sum_{j \in S_2} \frac{\theta}{2\eta} \sum_{k=0}^{K-1} \|\tilde{\delta}^k_j\|^2 + \frac{2\eta}{\theta} \sum_{k=0}^{K-1} \|\tilde{\delta}^k\|^2.
\]
Proof 3 Denoting \( v_j = x_j^0 - \frac{1}{\lambda_j} \nabla_j f(x^0) \), \( g_j(x) \) can be rewritten as
\[
g_j(x) = \frac{\lambda_j}{2} \left( x - x_j^0 + \frac{1}{\lambda_j} \nabla_j f(x^0) \right)^2 - \frac{1}{2\lambda_j} | \nabla_j f(x^0) |^2
\]
\[
= \frac{\lambda_j}{2} | x - v_j |^2 - \frac{1}{2\lambda_j} | \nabla_j f(x^0) |^2.
\]
For each \( j \in S_2 = \{ j : \lambda_j < - \frac{\theta}{\eta} \} \), we have
\[
g_j(x_j^{k+1}) - g_j(x_j^k) = \frac{\lambda_j}{2} |x_j^{k+1} - v_j|^2 - \frac{\lambda_j}{2} |x_j^k - v_j|^2
\]
\[
= \frac{\lambda_j}{2} |x_j^{k+1} - x_j^k|^2 + \lambda_j \langle x_j^{k+1} - x_j^k, x_j^k - v_j \rangle
\]
\[
\leq - \frac{\theta}{2\eta} |x_j^{k+1} - x_j^k|^2 + \lambda_j \langle x_j^{k+1} - x_j^k, x_j^k - v_j \rangle.
\]
So we only need to bound the second term. From (10b) and (10a), we have
\[
\bar{x}_j^{k+1} - \bar{x}_j^k = \bar{y}_j^k - \bar{x}_j^k - \eta \nabla g_j(\bar{y}_j^k) - \eta \bar{\delta}_j^k
\]
\[
= (1 - \theta)(\bar{x}_j^k - \bar{x}_j^{k-1}) - \eta \nabla g_j(\bar{y}_j^k) - \eta \bar{\delta}_j^k
\]
\[
= (1 - \theta)(\bar{x}_j^k - \bar{x}_j^{k-1}) - \eta \lambda_j \langle \bar{x}_j^k - v_j, x_j^k - v_j \rangle - \eta \bar{\delta}_j^k
\]
\[
= (1 - \theta)(\bar{x}_j^k - \bar{x}_j^{k-1}) - \eta \lambda_j \langle x_j^k - v_j, x_j^k - v_j \rangle + (1 - \theta)\langle x_j^k - \bar{x}_j^k, x_j^k - v_j \rangle - \eta \bar{\delta}_j^k.
\]
So for each \( j \in S_2 \), we have
\[
\lambda_j \langle \bar{x}_j^{k+1} - \bar{x}_j^k, \bar{x}_j^k - v_j \rangle
\]
\[
= (1 - \theta)\lambda_j \langle \bar{x}_j^k - \bar{x}_j^{k-1}, \bar{x}_j^k - v_j \rangle - \eta \lambda_j^2 \| \bar{x}_j^k - v_j \|^2 - \eta \lambda_j^2 (1 - \theta) \langle \bar{x}_j^k - \bar{x}_j^{k-1}, \bar{x}_j^k - v_j \rangle - \eta \lambda_j \langle \bar{\delta}_j^k, \bar{x}_j^k - v_j \rangle
\]
\[
\leq (1 - \theta)\lambda_j \langle \bar{x}_j^k - \bar{x}_j^{k-1}, \bar{x}_j^k - v_j \rangle - \eta \lambda_j^2 \| \bar{x}_j^k - v_j \|^2
\]
\[
+ \frac{\eta \lambda_j^2 (1 - \theta)}{2} \left( | \bar{x}_j^k - \bar{x}_j^{k-1} |^2 + | \bar{x}_j^k - v_j |^2 \right) + \frac{\eta}{2(1 + \theta)} | \bar{\delta}_j^k |^2
\]
\[
= (1 - \theta)\lambda_j \langle \bar{x}_j^k - \bar{x}_j^{k-1}, \bar{x}_j^k - v_j \rangle + \frac{\eta \lambda_j^2 (1 - \theta)}{2} \| \bar{x}_j^k - \bar{x}_j^{k-1} \|^2 + \frac{\eta}{2(1 + \theta)} | \bar{\delta}_j^k |^2
\]
\[
\leq (1 - \theta)\lambda_j \langle \bar{x}_j^k - \bar{x}_j^{k-1}, \bar{x}_j^{k-1} - v_j \rangle + \frac{\eta}{2} | \bar{\delta}_j^k |^2,
\]
where we use \( \left( 1 + \frac{\eta \lambda_j^2}{2} \right) (1 - \theta) \geq \left( 1 - \frac{\eta \lambda_j^2}{2} \right) (1 - \theta) \geq 0 \) and \( \lambda_j < 0 \) when \( j \in S_2 \) in (a). So we have
\[
\lambda_j \langle \bar{x}_j^{k+1} - \bar{x}_j^k, \bar{x}_j^k - v_j \rangle \leq (1 - \theta)\lambda_j \langle \bar{x}_j^k - \bar{x}_j^{k-1}, \bar{x}_j^0 - v_j \rangle + \frac{\eta}{2} \sum_{t=1}^{k} | (1 - \theta)^{k-t} | \bar{\delta}_j^t |^2
\]
\[
\leq \frac{b}{2} - (1 - \theta)\eta \lambda_j^2 \| \bar{x}_j^0 - v_j \|^2 + \frac{\eta}{2} \sum_{t=1}^{k} | (1 - \theta)^{k-t} | \bar{\delta}_j^t |^2
\]
\[
\leq \frac{\eta}{2} \sum_{t=1}^{k} (1 - \theta)^{k-t} | \bar{\delta}_j^t |^2,
\]
where we use
\[
\bar{x}_j^0 = \bar{x}_j^0 - \eta \nabla_j f(x^0) = - \eta \nabla_j f(x^0)
\]
\[
= - \eta \nabla_j g_j(x^0) = - \eta \lambda_j (x_j^0 - v_j)
\]
in \( b \). Plugging into (14), we have

\[
g_j(\tilde{x}_j^{k+1}) - g_j(\tilde{x}_j^k) \leq -\frac{\theta}{2\eta} |\tilde{x}_j^{k+1} - \tilde{x}_j^k|^2 + \frac{\eta}{2} \sum_{t=1}^{k} (1 - \theta)^{k-t} |\tilde{\theta}_j|^2.
\]

Summing over \( k = 0, 1, \ldots, K-1 \) and \( j \in S_2 \), we have

\[
\sum_{j \in S_2} g_j(\tilde{x}_j^k) - \sum_{j \in S_2} g_j(\tilde{x}_j^0) \leq -\sum_{j \in S_2} \frac{\theta}{2\eta} \sum_{k=0}^{K-1} |\tilde{x}_j^{k+1} - \tilde{x}_j^k|^2 + \frac{\eta}{2} \sum_{k=0}^{K-1} \sum_{t=1}^{k} (1 - \theta)^{k-t} |\tilde{\theta}_j|^2
\]

\[
\leq -\sum_{j \in S_2} \frac{\theta}{2\eta} \sum_{k=0}^{K-1} |\tilde{x}_j^{k+1} - \tilde{x}_j^k|^2 + 2\eta \rho^2 B^4 \sum_{k=0}^{K-1} \sum_{t=1}^{k} (1 - \theta)^{k-t}
\]

\[
\leq -\sum_{j \in S_2} \frac{\theta}{2\eta} \sum_{k=0}^{K-1} |\tilde{x}_j^{k+1} - \tilde{x}_j^k|^2 + \frac{2\eta \rho^2 B^4 K^2}{\theta},
\]

where we use (11) in \( \leq \).

Putting Lemmas 2 and 3 together, we can show the decrease of \( f(x) \) in each epoch.

**Lemma 4** Suppose that Assumption 1 holds. Under the parameter settings in Theorem 1, when the “if condition” triggers and \( \|\nabla f(y^{K-1})\| \leq \frac{B}{7} \), then we have

\[
f(x^K) - f(x^0) \leq -\frac{c^{3/2}}{\sqrt{\rho}}.
\]

**Proof 4** Summing over (12) and (13), we have

\[
g(\tilde{x}^K) - g(\tilde{x}^0) = \sum_{j \in S_1 \cup S_2} g_j(\tilde{x}_j^K) - g_j(\tilde{x}_j^0)
\]

\[
\leq -\frac{3\theta}{8\eta} \sum_{k=0}^{K-1} |\tilde{x}_j^{k+1} - \tilde{x}_j^k|^2 + \frac{10\eta \rho^2 B^4 K}{\theta}
\]

\[
= -\frac{3\theta}{8\eta} \sum_{k=0}^{K-1} |\tilde{x}_j^{k+1} - \tilde{x}_j^k|^2 + \frac{10\eta \rho^2 B^4 K}{\theta}
\]

\[
\leq -\frac{3\theta B^2}{8\eta K} + \frac{10\eta \rho^2 B^4 K}{\theta},
\]

where we use (5a) in \( \leq \). Plugging into (8) and using \( K \leq K \), we have

\[
f(x^K) - f(x^0) \leq -\frac{3\theta B^2}{8\eta K} + \frac{10\eta \rho^2 B^4 K}{2\theta} + 4.5\rho B^3
\]

\[
\leq -\frac{3\theta B^2}{8\eta K} + \frac{10\eta \rho^2 B^4 K}{2\theta} + 4.5\rho B^3 \leq -\frac{c^{3/2}}{\sqrt{\rho}}.
\]

5.1.3. SMALL GRADIENT IN THE LAST EPOCH

In this section, we prove Theorem 1. The main job is to establish \( \|\nabla f(\tilde{y})\| \leq O(\varepsilon) \) in the last epoch.

**Proof 5** From Lemmas 1 and 4, we have

\[
f(x^K) - f(x^0) \leq -\min \left\{ \frac{c^{3/2}}{\sqrt{\rho}} \cdot \frac{cL}{\rho} \right\}.
\]
Note that at the beginning of each epoch in Algorithm 1, we set \( x^0 \) to be the last iterate \( x^K \) in the previous epoch. Summing (16) over all epochs, say \( N \) total epochs, and using \( \min_{x} f(x) \leq f(x^K) \), we have
\[
\min_{x} f(x) - f(x_{int}) \leq - N \min \left\{ \frac{\epsilon^{3/2}}{\sqrt{\rho}}, \frac{\epsilon L}{\rho} \right\}.
\]

So the algorithm will terminate in at most \( \frac{\Delta_{x} L^{1/2}}{\epsilon^{7/4}} \) epochs. Since each epoch needs at most \( K = \frac{1}{2} \left( \frac{L^2}{\eta^2} \right)^{1/4} \) gradient evaluations, the total number of gradient evaluations must be less than \( \frac{\Delta_{x} L^{1/2}}{\epsilon^{7/4}} \).

Now, we consider the last epoch. Denote \( \bar{y} = U^T \hat{y} = \frac{1}{K_0 + 1} \sum_{k=0}^{K_0} U^T y^k = \frac{1}{K_0 + 1} \sum_{k=0}^{K_0} \bar{y}^k \). Since \( g \) is quadratic, we have
\[
\| \nabla g(\bar{y}) \| = \left\| \frac{1}{K_0 + 1} \sum_{k=0}^{K_0} \nabla g(\bar{y}^k) \right\|
\leq \frac{a}{\eta (K_0 + 1)} \left\| \sum_{k=0}^{K_0} \left( \bar{x}^{k+1} - \bar{y}^k + \eta \bar{\delta}^k \right) \right\|
\leq \frac{1}{\eta (K_0 + 1)} \left\| \sum_{k=0}^{K_0} \left( \bar{x}^{k+1} - \bar{y}^k - (1 - \theta)(\bar{x}^k - \bar{x}^{k-1}) + \eta \bar{\delta}^k \right) \right\|
\leq \frac{b}{\eta (K_0 + 1)} \left\| \bar{x}^{K_0+1} - \bar{x}^0 - (1 - \theta)(\bar{x}^{K_0} - \bar{x}^0) + \theta \bar{x}^{K_0} - \bar{x}^0 \right\| + \eta \sum_{k=0}^{K_0} \| \bar{\delta}^k \|
\leq \frac{1}{\eta (K_0 + 1)} \left( \| \bar{x}^{K_0+1} - \bar{x}^{K_0} \| + \theta \| \bar{x}^{K_0} - \bar{x}^0 \| + \eta \sum_{k=0}^{K_0} \| \bar{\delta}^k \| \right)
\leq \frac{c}{\eta K} \| \bar{x}^{K_0+1} - \bar{x}^{K_0} \| + \frac{2 \theta B}{\eta K} + 2 \rho B^2,
\]
where we use (10b) in \( a \), \( x^{-1} = x^0 \) in \( b \), \( K_0 + 1 \geq \frac{K}{2} \), (6a), (11), and (6b) in \( c \). From \( K_0 = \arg \min_{\frac{K}{2} \leq k \leq K - 1} \| x^{k+1} - x^k \| \), we have
\[
\| x^{K_0+1} - x^{K_0} \|^2 \leq \frac{1}{K - [K/2]} \sum_{k=0}^{K-1} \| x^{k+1} - x^k \|^2
\leq \frac{1}{K - [K/2]} \sum_{k=0}^{K-1} \| x^{k+1} - x^k \|^2
\leq \frac{1}{K - [K/2]} \frac{B^2}{K} \leq \frac{2B^2}{K},
\]
where we use (6a) in \( \leq \). On the other hand, we also have
\[
\| \nabla f(\hat{y}) \| = \| \nabla f(\bar{y}) \|
\leq \| \nabla g(\bar{y}) \| + \| \bar{\nabla} f(\bar{y}) - \nabla g(\bar{y}) \|
= \| \nabla g(\bar{y}) \| + \| \bar{\nabla} f(\bar{y}) - \nabla f(x^0) - \Lambda(\bar{y} - \bar{x}^0) \|
= \| \nabla g(\bar{y}) \| + \| \nabla f(\bar{y}) - \nabla f(x^0) - H(\bar{y} - x^0) \|
\leq \| \nabla g(\bar{y}) \| + \frac{\rho}{2} \| \bar{y} - x^0 \|^2
\leq \| \nabla g(\bar{y}) \| + 2 \rho B^2,
\]
where we use \( \|\hat{y} - x^0\| \leq \frac{1}{K_{0}+1} \sum_{k=0}^{K_{0}} \|y^k - x^0\| \leq 2B \) from (6b) in \( \leq \). So we have

\[
\|\nabla f(\hat{y})\| \leq \frac{2\sqrt{2}B}{\eta K^2} + \frac{2\theta B}{\eta K} + 4\rho B^2 \leq 8\epsilon.
\]

**Remark 3** The purpose of using \( k \sum_{t=0}^{k-1} \|x^{t+1} - x^t\|^2 > B^2 \) in the “if condition”, rather than \( \|x^k - x^0\| \geq B \), and the special average as the output in Algorithm 1 is to establish (18).

### 5.2. Discussion on the Acceleration Mechanism

When we replace the AGD iterations in Algorithm 1 by the gradient descent iterations \( x^{k+1} = x^k - \eta \nabla f(x^k) \) with \( \eta = \frac{1}{4L} \), similar to (7), the descent property in each epoch becomes

\[
f(x^K) - f(x^0) \leq -\frac{7}{8\eta} \sum_{k=0}^{K-1} \|x^{k+1} - x^k\|^2 \leq -\frac{7B^2}{8\eta K},
\]

and the gradient norm at the averaged output \( \hat{x} = \frac{1}{K} \sum_{k=0}^{K-1} x^k \) is bounded as

\[
\|\nabla g(\hat{x})\| \leq \frac{1}{\eta K} \|x^K - x^0\| + 2\rho B^2 \leq \frac{B}{\eta K} + 2\rho B^2.
\]

By setting \( B = \sqrt{\frac{\epsilon}{\rho}} \) and \( K = \frac{L}{\sqrt{\rho}} \), we have the \( O(\epsilon^{-2}) \) total complexity.

Comparing with (15) and (17), respectively, we see that the momentum parameter \( \theta \) is crucial to speedup the convergence because it allows smaller \( K \), that is, \( \frac{1}{4L} \) v.s. \( \frac{1}{L} \) for AGD and GD, respectively. Accordingly, smaller \( K \) results in less total gradient computations. Thus, the acceleration mechanism for nonconvex optimization seems irrelevant to the analysis of convex AGD. It is just because of the momentum.

### 5.3. Proof of Theorem 2

**Proof 6** Denote one epoch to be valid when the if condition \( f(x^k) - f(x^0) \leq -\min \left\{ \frac{\epsilon^{3/2}}{\sqrt{\rho}}, \frac{\epsilon L}{\rho} \right\} \) holds. Otherwise, denote the epoch to be invalid. Since each valid epoch decreases the objective at least \( (L^1/\sqrt{\rho} + \epsilon L/\rho) \), we have at most \( \max \left\{ \Delta f/\sqrt{\rho}, \Delta f/\rho \right\} = O \left( \Delta f/\sqrt{\rho} + \Delta f/\rho \right) \) valid epochs. On the other hand, we only need \( \log \frac{B_0}{B} \) invalid epochs to decrease \( B_0 \) smaller than \( B \), and Algorithm 2 is equivalent to Algorithm 1 when \( B_0 \leq B \). In fact, from (16), we always have \( f(x^k) - f(x^0) \leq -\min \left\{ \frac{\epsilon^{3/2}}{\sqrt{\rho}}, \frac{\epsilon L}{\rho} \right\} \) when \( k \sum_{t=0}^{k-1} \|x^{t+1} - x^t\|^2 > B^2 \). That is, invalid epoch never appears when \( B_0 \leq B \). So we have at most \( \min \left\{ \Delta f/\sqrt{\rho}, \Delta f/\rho \right\} \) valid epochs. Putting the two cases together, we need at most \( O \left( \Delta f/\sqrt{\rho} + \log \frac{B_0}{B} \right) \) epochs, and accordingly, \( O \left( \Delta f/\sqrt{\rho} + \log \frac{B_0}{B} \right) \) function evaluations. On the other hand, each epoch, no matter valid or not, needs at most \( K + 1 \) gradient evaluations. So the total number gradient evaluations is \( O \left( \Delta f/\sqrt{\rho} + \log \frac{B_0}{B} \right) (K + 1) = O \left( \Delta f/\sqrt{\rho} + \log \frac{B_0}{B} \right) \).

In the last epoch, we have \( B_0 \leq B \) and \( k \sum_{t=0}^{k-1} \|x^{t+1} - x^t\|^2 \leq B^2 \) for all \( k \leq K \), and the while loop terminates when \( k \) equals \( K \). From the proof of Theorem 1, we have \( \|\nabla f(\hat{y})\| \leq 82\epsilon \). Since we output \( x_{out} = \arg\min_{x^K} \{ \|\nabla f(x^K)\|, \|\nabla f(\hat{y})\| \} \), we also have \( \|\nabla f(x_{out})\| \leq 82\epsilon \).

### 5.4. Proof of Theorem 4

We follow the proof sketch in Section 5.1 and use the notations therein. Specifically, (5a), (5b), and (6a) also hold for the heavy ball method.

### 5.4.1. LARGE GRADIENT OF \( \|\nabla f(x^{K-1})\| \)

Similar to Lemma 1, we first consider the case when \( \|\nabla f(x^{K-1})\| \) is large and have the following lemma for Algorithm 3.
Lemma 5 Suppose that Assumption 1 holds. Let $\eta \leq \frac{1}{4L}$ and $0 \leq \theta \leq \frac{1}{16}$. When the “if condition” triggers and $\|\nabla f(x^{k-1})\| > \frac{4B}{\eta}$, then we have

$$f(z^K) - f(x^0) \leq -\frac{3B^2}{16\eta}.$$  

Proof 7 From the L-gradient Lipschitz condition (2), we have

$$f(x^{k+1}) - f(x^k) \leq f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{L}{2} \|x^{k+1} - x^k\|^2 \leq f(x^k) - \eta \|\nabla f(x^k)\|^2 + \frac{\eta}{2} \|\nabla f(x^k)\|^2 + \frac{1}{2\eta} \|x^k - x^{k-1}\|^2 + L\|x^k - x^{k-1}\|^2 + \frac{B}{\eta} \|\nabla f(x^k)\|^2.$$  

where we use the heavy ball iteration in $\overset{a}{=}$. Summing over $k = 0, \ldots, K-1$ and using $x^0 = x^{-1}$, we have

$$f(x^K) - f(x^0) \leq \frac{3}{4\eta} \sum_{k=0}^{K-2} \|x^{k+1} - x^k\|^2 - \frac{\eta}{4} \sum_{k=0}^{K-1} \|\nabla f(x^k)\|^2 \leq \frac{3B^2}{4\eta} - \frac{\eta}{4} \|\nabla f(x^{K-1})\|^2,$$  

where we use (5b) in $\overset{c}{=}$. On the other hand, we also have

$$f(x^{K-1}) - f(x^0) \leq \frac{3}{4\eta} \sum_{k=0}^{K-3} \|x^{k+1} - x^k\|^2 - \frac{\eta}{4} \sum_{k=0}^{K-2} \|\nabla f(x^k)\|^2 \leq \frac{3B^2}{4\eta}.$$

Define $h(x) = f(x) + \frac{L}{2} \|x - x^0\|^2$. We know $h(x)$ is convex and thus we have $h(z^K) \leq \alpha h(x^K) + (1 - \alpha)h(x^{K-1})$ with $\alpha = \frac{1}{1 + (1 - 2\eta)(1 - \theta)} \in \left[\frac{1}{2}, \frac{1}{1.72}\right]$ and $z^K = \alpha x^K + (1 - \alpha)x^{K-1}$, which further yields

$$f(z^K) \leq \alpha f(x^K) + (1 - \alpha) f(x^{K-1}) + \frac{L\alpha}{2} \|x^K - x^0\|^2 + \frac{L(1 - \alpha)}{2} \|x^{K-1} - x^0\|^2 \leq \frac{L\alpha}{2} \|x^K - x^0\|^2 + \frac{L(1 - \alpha)}{2} \|x^{K-1} - x^0\|^2,$$

where we use $|ax + (1 - \alpha)y|^2 = \alpha x^2 + (1 - \alpha)y^2 - \alpha(1 - \alpha)|x - y|^2$ in $\overset{d}{=}$. Thus, we have

$$f(z^K) - f(x^0) \leq \frac{3B^2}{4\eta} + \frac{B^2}{16\eta} - \frac{\eta\alpha}{4} \|\nabla f(x^{K-1})\|^2 + \frac{\eta}{16} \|\nabla f(x^{K-1})\|^2 \leq \frac{13B^2}{16\eta} - \frac{\eta}{16} \|\nabla f(x^{K-1})\|^2 \leq \frac{3B^2}{16\eta},$$

where we use $\|\nabla f(x^{K-1})\| > \frac{4B}{\eta}$ in $\overset{e}{=}$.  

Restated Nonconvex Accelerated Gradient Descent
5.4.2. Small Gradient of \(\|∇f(x^{K-1})\|\)

If \(\|∇f(x^{K-1})\| \leq \frac{4B}{η}\), then from the heavy ball iteration and (5b) we have
\[
\|x^K - x^0\| \leq \|x^{K-1} - x^0\| + η\|∇f(x^{K-1})\| + (1 - θ)\|x^{K-1} - x^{K-2}\| \leq 6B.
\]

Similar to (8), using the definition of \(g(x)\) in (9), we have
\[
f(x^K) - f(x^0) \leq g(\bar{x}^K) - g(\bar{x}^0) + 36ρB^3.
\]

Denoting
\[
\bar{δ}_j^k = \bar{∇}_j f(x^k) - ∇g_j(\bar{x}_j^k), \quad \bar{δ}_j^k = \bar{∇}_j f(x^k) - ∇g_j(x^k),
\]
then the heavy ball iteration can be rewritten as
\[
\bar{x}^k_j = \bar{x}_j^k - η\bar{∇}_j f(x^k) + (1 - θ)(\bar{x}_j^k - \bar{x}^k_{j-1})
= \bar{x}_j^k - η∇g_j(\bar{x}_j^k) - η\bar{δ}_j^k + (1 - θ)(\bar{x}_j^k - \bar{x}_j^{k-1}).
\]

Similar to (11), \(\|\bar{δ}_j^k\|\) can also be bounded as
\[
\|\bar{δ}_j^k\| \leq \frac{ρ}{2}\|x^k - x^0\|^2 \leq \frac{ρB^2}{2}
\]
for any \(k < K\).

**Lemma 6** Suppose that Assumption 1 holds. Under the parameter settings in Theorem 4, when the “if condition” triggers and \(\|∇f(x^{K-1})\| \leq \frac{4B}{η}\), then we have
\[
f(z^K) - f(x^0) \leq -\frac{3\sqrt{ρ}}{\sqrt{ρ}}.
\]

**Proof 8** Since \(g_j(x)\) is quadratic, we have
\[
g_j(\bar{x}^{k+1}_j) - g_j(\bar{x}^k_j) = \frac{λ_j}{2}\|\bar{x}^{k+1}_j - \bar{x}^k_j\|^2 - \frac{λ_j}{2}\|\bar{x}^k_j - \bar{x}^0_j\|^2 + \left< \bar{∇}_j f(x^0), \bar{x}^{k+1}_j - \bar{x}^k_j \right>.
\]

From (20), we have
\[
\bar{x}^{k+1}_j - \bar{x}_j^k = (1 - θ)(\bar{x}_j^k - \bar{x}^{k-1}_j) - η∇g_j(\bar{x}_j^k) - η\bar{δ}_j^k
= (1 - θ)(\bar{x}_j^k - \bar{x}^{k-1}_j) - η\left(λ_j(\bar{x}_j^k - \bar{x}^0_j) + \bar{∇}_j f(x^0)\right) - η\bar{δ}_j^k.
\]

So we have
\[
\left< \bar{x}^{k+1}_j - \bar{x}_j^k, λ_j(\bar{x}_j^k - \bar{x}^0_j) + \bar{∇}_j f(x^0) \right>
= (1 - θ)\left< \bar{x}_j^k - \bar{x}^{k-1}_j, λ_j(\bar{x}_j^k - \bar{x}^0_j) + \bar{∇}_j f(x^0) \right> - η\left|λ_j(\bar{x}_j^k - \bar{x}^0_j) + \bar{∇}_j f(x^0)\right|^2
- η\left|\bar{δ}_j^k, λ_j(\bar{x}_j^k - \bar{x}^0_j) + \bar{∇}_j f(x^0)\right|^2
\leq (1 - θ)\left< \bar{x}_j^k - \bar{x}^{k-1}_j, λ_j(\bar{x}_j^k - \bar{x}^0_j) + \bar{∇}_j f(x^0) \right> - η\left|λ_j(\bar{x}_j^k - \bar{x}^0_j) + \bar{∇}_j f(x^0)\right|^2
+ \frac{η}{4θ}\left|\bar{δ}_j^k\right|^2 + η\left|λ_j(\bar{x}_j^k - \bar{x}^0_j) + \bar{∇}_j f(x^0)\right|^2
= (1 - θ)\left< \bar{x}_j^k - \bar{x}^{k-1}_j, λ_j(\bar{x}_j^k - \bar{x}^0_j) + \bar{∇}_j f(x^0) \right> - (1 - θ)η\left|λ_j(\bar{x}_j^k - \bar{x}^0_j) + \bar{∇}_j f(x^0)\right|^2 + \frac{η}{4θ}\left|\bar{δ}_j^k\right|^2.
Plugging into (22), we have

\[
g_j(\mathbf{x}_j^{k+1}) - g_j(\mathbf{x}_j^k) \leq \frac{\lambda_j}{2} |\mathbf{x}_j^{k+1} - \mathbf{x}_j^k|^2 + (1 - \theta) \left( \mathbf{v}_j \cdot \nabla f(\mathbf{x}_j^k) \right) \]

\[
(1 - \theta) |\mathbf{v}_j \cdot \nabla f(\mathbf{x}_j^k)|^2 + \frac{\eta}{4\theta} |\delta_j^k|^2.
\]  

(24)

Squaring both sides of (23) and using \((a + b)^2 \leq (1 + \frac{\theta}{1 - \theta})a^2 + (1 + \frac{1 - \theta}{1 - \theta})b^2 = \frac{a^2}{1 - \theta} + \frac{b^2}{\theta},\) we have

\[
|\mathbf{x}_j^{k+1} - \mathbf{x}_j^k|^2 \leq \frac{1}{1 - \theta} \left( (1 - \theta) |\mathbf{x}_j^{k+1} - \mathbf{x}_j^{k-1}|^2 - \eta \left( \lambda_j |\mathbf{x}_j^{k+1} - \mathbf{x}_j^0| + \nabla_j f(\mathbf{x}_0^k) \right) \right) \]

\[
+(1 - \theta) |\mathbf{x}_j^{k+1} - \mathbf{x}_j^{k-1}|^2 + \frac{\eta^2}{1 - \theta} |\lambda_j |\mathbf{x}_j^{k+1} - \mathbf{x}_j^0| + \nabla_j f(\mathbf{x}_0^k)|^2
\]

\[
- 2\eta \left( \mathbf{x}_j^{k+1} - \mathbf{x}_j^{k-1}, \lambda_j (\mathbf{x}_j^{k+1} - \mathbf{x}_j^0) + \nabla_j f(\mathbf{x}_0^k) \right) + \frac{\eta^2 |\delta_j^k|^2}{1 - \theta}.
\]  

(25)

Multiplying both sides of (25) by \(\frac{(1 - \theta)^2}{\eta},\) adding it to (24), and rearranging the terms, we have

\[
g_j(\mathbf{x}_j^{k+1}) - g_j(\mathbf{x}_j^k) \leq - \left( \frac{(1 - \theta)^2}{\eta} - \frac{\lambda_j}{2} \right) |\mathbf{x}_j^{k+1} - \mathbf{x}_j^k|^2 + \frac{(1 - \theta)^3}{\eta} |\mathbf{x}_j^{k+1} - \mathbf{x}_j^{k-1}|^2
\]

\[
- (1 - 2\theta)\left( (1 - \theta) \left( \lambda_j |\mathbf{x}_j^{k+1} - \mathbf{x}_j^0| + \nabla_j f(\mathbf{x}_0^k) \right) \right) + \left( \frac{\eta}{4\theta} + \frac{\eta(1 - \theta)^2}{\theta} \right) |\delta_j^k|^2
\]

\[
- \left( \frac{(1 - \theta)^2}{\eta} - \frac{\lambda_j}{2} \right) |\mathbf{x}_j^{k+1} - \mathbf{x}_j^k|^2 + \frac{(1 - \theta)^3}{\eta} |\mathbf{x}_j^{k+1} - \mathbf{x}_j^{k-1}|^2 + \frac{5\eta}{4\theta} |\delta_j^k|^2
\]

\[
- (1 - 2\theta)\left( (1 - \theta) \left( \lambda_j |\mathbf{x}_j^{k+1} - \mathbf{x}_j^0| + \nabla_j f(\mathbf{x}_0^k) \right) \right) + \left( \frac{\eta}{4\theta} + \frac{\eta(1 - \theta)^2}{\theta} \right) |\delta_j^k|^2
\]

\[
- (1 - \theta)\left( (1 - \theta) \left( \lambda_j |\mathbf{x}_j^{k+1} - \mathbf{x}_j^0| + \nabla_j f(\mathbf{x}_0^k) \right) \right) + \left( \frac{\eta}{4\theta} + \frac{\eta(1 - \theta)^2}{\theta} \right) |\delta_j^k|^2
\]

\[
+ \frac{5\eta}{4\theta} |\delta_j^k|^2 - (1 - 2\theta)(1 - \theta) \left( g_j(\mathbf{x}_j^{k+1}) - g_j(\mathbf{x}_j^{k-1}) \right).
\]

Note that

\[
\frac{(1 - \theta)^2}{\eta} - \frac{\lambda_j}{2} - \frac{(1 - \theta)^3}{\eta} + \frac{(1 - 2\theta)(1 - \theta)\lambda_j}{2}
\]

\[
= \frac{(1 - \theta)^2}{\eta} - \frac{\lambda_j}{2} - \frac{\lambda_j(3\theta - 2\theta^2)}{2} \leq \frac{(1 - \theta)^2}{\eta} - \frac{3\theta - 2\theta^2}{8\eta} \geq \frac{9\theta}{20\eta},
\]

where we use \(\lambda_j \leq L \leq \frac{1}{4\eta}\) in \(\geq\) and \(\theta \leq \frac{b}{16\eta}\) in \(\geq\). So we have

\[
g_j(\mathbf{x}_j^{k+1}) - g_j(\mathbf{x}_j^k) + (1 - 2\theta)(1 - \theta) \left( g_j(\mathbf{x}_j^{k+1}) - g_j(\mathbf{x}_j^{k-1}) \right)
\]

\[
\leq - \left( \frac{(1 - \theta)^2}{\eta} - \frac{\lambda_j}{2} - \frac{(1 - \theta)^3}{\eta} + \frac{(1 - 2\theta)(1 - \theta)\lambda_j}{2} \right) |\mathbf{x}_j^{k+1} - \mathbf{x}_j^k|^2
\]

\[
+ \left( \frac{(1 - \theta)^3}{\eta} - \frac{(1 - 2\theta)(1 - \theta)\lambda_j}{2} \right) |\mathbf{x}_j^{k+1} - \mathbf{x}_j^{k-1}|^2 + \frac{5\eta}{4\theta} |\delta_j^k|^2.
\]

(26)

Summing over \(k = 0, 1, \ldots, K - 1\) and using \(x^0 = x^{-1}\), we have

\[
g_j(\mathbf{x}_j^k) - g_j(\mathbf{x}_j^0) + (1 - 2\theta)(1 - \theta) \left( g_j(\mathbf{x}_j^{K-1}) - g_j(\mathbf{x}_j^0) \right)
\]

\[
\leq - \left( \frac{(1 - \theta)^3}{\eta} - \frac{(1 - 2\theta)(1 - \theta)\lambda_j}{2} \right) |\mathbf{x}_j^k - \mathbf{x}_j^{K-1}|^2 - \frac{9\theta}{20\eta} \sum_{k=0}^{K-1} |\mathbf{x}_j^{k+1} - \mathbf{x}_j^k|^2 + \frac{5\eta}{4\theta} \sum_{k=0}^{K-1} |\delta_j^k|^2.
\]
Denoting $\alpha = \frac{1}{1-(1-2\theta)(1-\theta)} \in \left[ \frac{1}{2}, \frac{1}{1+2\theta} \right]$ and multiplying both sides of (26) by $\alpha$, we have

$$\alpha (g_j(\overline{x}_j^K) - g_j(\overline{x}_j^0)) + (1-\alpha) (g_j(\overline{x}_j^{K-1}) - g_j(\overline{x}_j^0))$$

$$\leq - \alpha \left( \frac{(1-\theta)^3}{\eta} - \frac{(1-2\theta)(1-\theta)\lambda_j}{2} \right) |\overline{x}_j^K - \overline{x}_j^{K-1}|^2 - \frac{9\alpha\theta}{20\eta} \sum_{k=0}^{K-1} |\overline{x}_j^{k+1} - \overline{x}_j^k|^2 + \frac{5\alpha\eta}{4\theta} \sum_{k=0}^{K-1} |\delta_j^k|^2 

(27)$$

On the other hand, from $z^K = \frac{x^K+(1-2\theta)(1-\theta)x^{K-1}}{1+(1-2\theta)(1-\theta)} = \alpha x^K + (1-\alpha)x^{K-1}$, we have

$$g_j(z_j^K) - g_j(z_j^0)$$

$$= \frac{\lambda_j}{2} |\alpha(z_j^K - z_j^0) + (1-\alpha)(z_j^{K-1} - z_j^0)|^2 + \alpha \left( \nabla_j f(x^0), \overline{x}_j^K - \overline{x}_j^0 \right)$$

$$+ (1-\alpha) \left( \nabla_j f(x^0), \overline{x}_j^{K-1} - \overline{x}_j^0 \right)$$

$$= \frac{\lambda_j}{2} |\overline{x}_j^K - \overline{x}_j^0|^2 + \frac{\lambda_j}{2} |\overline{x}_j^{K-1} - \overline{x}_j^0|^2 - \frac{\lambda_j}{2} |\overline{x}_j^K - \overline{x}_j^{K-1}|^2$$

$$+ \alpha \left( \nabla_j f(x^0), \overline{x}_j^K - \overline{x}_j^0 \right) + (1-\alpha) \left( \nabla_j f(x^0), \overline{x}_j^{K-1} - \overline{x}_j^0 \right)$$

$$= \alpha (g_j(\overline{x}_j^K) - g_j(\overline{x}_j^0)) + (1-\alpha) (g_j(\overline{x}_j^{K-1}) - g_j(\overline{x}_j^0)) - \frac{\lambda_j\alpha(1-\alpha)}{2} |\overline{x}_j^K - \overline{x}_j^{K-1}|^2$$

$$\leq \alpha (g_j(\overline{x}_j^K) - g_j(\overline{x}_j^0)) + (1-\alpha) (g_j(\overline{x}_j^{K-1}) - g_j(\overline{x}_j^0)) + \frac{1}{32\eta} |\overline{x}_j^K - \overline{x}_j^{K-1}|^2$$

$$\leq - \frac{1}{2} \left( \frac{(1-\theta)^3}{\eta} - \frac{(1-2\theta)(1-\theta)\lambda_j}{2} - \frac{1}{16\eta} \right) |\overline{x}_j^K - \overline{x}_j^{K-1}|^2 - \frac{9\theta}{40\eta} \sum_{k=0}^{K-1} |\overline{x}_j^{k+1} - \overline{x}_j^k|^2 + \frac{\eta}{8\theta} \sum_{k=0}^{K-1} |\delta_j^k|^2$$

where we use $|ax + (1-\alpha)y|^2 = \alpha|x|^2 + (1-\alpha)y^2 - \alpha(1-\alpha)x - y|^2$ in $\leq$, $\lambda_j \geq -L = -\frac{1}{4\eta}$ in $\leq$, (27) in $\leq$, and

$$\frac{(1-\theta)^3}{\eta} - \frac{(1-2\theta)(1-\theta)\lambda_j}{2} - \frac{1}{16\eta} \geq \frac{(1-\theta)^3}{\eta} - \frac{1}{8\eta} - \frac{1}{16\eta} \geq 0$$

with $\theta \in [0, \frac{1}{16}]$ in $\leq$. Summing over $j$, using (21) and (5a), we have

$g(\overline{z}_K) - g(\overline{z}_0) = \sum_j g_j(\overline{z}_j^K) - g_j(\overline{z}_j^0) \leq - \frac{9\theta}{40\eta} \sum_{k=0}^{K-1} |\overline{x}_j^{k+1} - \overline{x}_j^k|^2 + \frac{\eta^2B^4K}{4\theta}$

$$\leq - \frac{9\theta B^2}{40\eta K} + \frac{\eta^2 B^4 K}{4\theta}.$$
Similar to the proof of Theorem 1, we know the algorithm will terminate in at most \( \frac{\Delta f}{\sqrt{\rho}} \) epochs and the total number of gradient evaluations must be less than \( \Delta f \frac{L}{\rho^{1/4}} \). We can also prove the small gradient in the last epoch by almost the same techniques given in Section 5.1.3, and we omit the details.

6. Experiments

We test the efficiency of the proposed methods on the matrix completion problem (Negahban & Wainwright, 2012; Hardt, 2014) and one bit matrix completion problem (Davenport et al., 2014).

6.1. Matrix completion

The goal of matrix completion (Negahban & Wainwright, 2012; Hardt, 2014) is to recover the low rank matrix from a set of randomly observed entries, which can be formulated as follows:

\[
\min_{X \in \mathbb{R}^{m \times n}} \frac{1}{2N} \sum_{(i,j) \in O} (X_{i,j} - X^*_{i,j})^2, \quad s.t. \quad \text{rank}(X) \leq r,
\]

where \( O \) is the set of observed entries, \( N \) is the size of \( O \), and \( X^* \) is the true low rank matrix. We reformulate the above problem in the following matrix factorization form:

\[
\min_{U \in \mathbb{R}^{m \times r}, V \in \mathbb{R}^{n \times r}} \frac{1}{2N} \sum_{(i,j) \in O} ((UV^T)_{i,j} - X^*_{i,j})^2 + \frac{1}{2N} \|U^T U - V^T V\|_F^2,
\]

where \( r \) is the rank of \( X^* \) and the regularization is used to balance \( U \) and \( V \).

We test the performance on the Movielens-10M, Movielens-20M and Netflix data sets, where the corresponding observed matrices are of size 69878 \times 10677, 138493 \times 26744, and 480189 \times 17770, respectively. We set \( r = 10 \) and compare...
We set $B=10$ for simplicity. For restarted AGD (Algorithm 2) and restarted HB (Algorithm 4) with Jin’s AGD (Jin et al., 2018), the ‘convex until guilty’ method (Carmon et al., 2017), and gradient descent (GD). Denote $X_0$ to be the observed data and $\Lambda \Sigma B^2$ to be its SVD. We initialize $U = A_{:,1:r}\sqrt{\Sigma_{1,1:r}}$ and $V = B_{:,1:r}\sqrt{\Sigma_{1,1:r}}$ for all the compared methods. It is efficient to compute the maximal $r$ singular values and the corresponding singular vectors of sparse matrices, for example, by Lanczos. We tune the best stepsize $\eta$ for each compared method on each dataset. Since the Hessian Lipschitz constant $\rho$ is unknown, we set it as 1 for simplicity. For restarted AGD, we follow Theorem 1 to set $\epsilon = 10^{-16}$, $B = \sqrt{\frac{2}{\rho}}$, $\theta = 4(\epsilon \rho \eta^2)^{1/4}$, and $K = 1/\theta$. We set $B_0 = 100$ and $c = 2$ in Algorithm 2. For restarted HB, we use the same parameter settings (except the stepsize) as those of restarted AGD to illustrate their performance comparison. For Jin’s AGD (see Algorithm 5 for example), we set $\theta = 4(\epsilon \rho \eta^2)^{1/4}$, $\gamma = \frac{2\rho}{\eta}$, and $s = \frac{2}{\eta}$. For the ‘convex until guilty’ method, we follow the theory in (Carmon et al., 2017) to set the parameters except that we terminate the inner loop after 100 iterations to improve its practical performance. We run each method for 1000 total iterations.

Figure 1 plots the results. We measure the objective function value and gradient norm at each iterate $x^k$ for restarted AGD and Jin’s AGD. We observed that the figures are almost the same when measured at $x^k$ and $y^k$ when preparing the experiments. For GD and restarted HB, we test on each iterate $x^k$. We see that all the accelerated methods perform better than GD, which verifies the efficiency of acceleration in nonconvex optimization. We also observe that our restart based methods decrease the objective value and gradient norm fastest. GD and our restarted AGD and HB only perform one gradient computation at each iteration, while Jin’s AGD and the ‘convex until guilty’ method need to evaluate additional objective function values. Thus, Their methods need more total running time when we run all the methods for 1000 iterations. On the other hand, due to the specification of the matrix completion problem, we observe that Jin’s AGD and the ‘convex until guilty’ method seldom run the negative curvature exploitation.

### 6.2. One bit matrix completion

In one bit matrix completion (Davenport et al., 2014), the sign of a random subset of entries is observed, rather than observing the actual entries. Given a probability density function, for example, the logistic function $f(x) = \frac{\exp(x)}{1 + \exp(x)}$, we
observe the sign of entry \( X_{i,j} \) as \( Y_{i,j} = 1 \) with probability \( f(X_{i,j}) \), and observe the sign as \(-1\) with probability \( 1 - f(X_{i,j}) \).

The training model is to minimize the following negative log-likelihood:

\[
\min_{X \in \mathbb{R}^{m \times n}} - \frac{1}{N} \sum_{(i,j) \in O} \left\{ 1_{Y_{i,j}=1} \log(f(X_{i,j})) + 1_{Y_{i,j}=-1} \log(1 - f(X_{i,j})) \right\},
\]

\[
\text{s.t.} \quad \text{rank}(X) \leq r,
\]

where \( 1_{Y_{i,j}=1} = \begin{cases} 1, & \text{if } Y_{i,j} = 1, \\ 0, & \text{otherwise.} \end{cases} \)

We solve the following reformulated matrix factorization model:

\[
\min_{U, V} - \frac{1}{N} \sum_{(i,j) \in O} \left\{ 1_{Y_{i,j}=1} \log(f((UV^T)_{i,j})) + 1_{Y_{i,j}=-1} \log(1 - f((UV^T)_{i,j})) \right\} + \frac{1}{2N} \|U^T U - V^T V\|_F^2,
\]

where \( U \in \mathbb{R}^{m \times r} \) and \( V \in \mathbb{R}^{n \times r} \). We compare restarted AGD (Algorithm 2) and restarted HB (Algorithm 4) with Jin’s AGD (Jin et al., 2018), the ‘convex until guilty’ method (Carmon et al., 2017), and gradient descent (GD). The best stepsize is tuned for each method on each data set. We use the same initialization and set the same parameters as those in Section 6.1, and also run each method for 1000 iterations. Figure 2 plots the results. We see that acceleration also takes effect in nonconvex optimization and our restarted AGD and HB perform the best.

7. Conclusion

This paper proposes two simple accelerated gradient methods, restarted AGD and restarted HB, for general nonconvex problems under the gradient Lipschitz and Hessian Lipschitz assumptions. Our simple methods find an \( \epsilon \)-approximate first-order stationary point in \( O(\epsilon^{-7/4}) \) gradient computations, which improves over the state-of-the-art complexity by the \( O(\log \frac{1}{\epsilon}) \) factor. Our simple proofs only use very elementary analysis. We hope our analysis may lead to a better understanding of the acceleration mechanism for nonconvex optimization.

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**A. Proof of Theorem 6**

**Proof 9** Define the potential function \( \ell_k = f(x^k) + \frac{1 - \theta}{2\eta} \| v_k \|^2 \). We need the following two lemmas, which can be adapted slightly from Lemmas 4 and 5 in (Jin et al., 2018).

**Lemma 7** Suppose that Assumption 1 holds. Let \( \eta \leq \frac{1}{2L} \) and \( \theta \in [2\eta\gamma, \frac{1}{2}] \). If NCE is not performed at iteration \( k \), then for Algorithm 5 we have \( \ell_{k+1} \leq \ell_k - \frac{\theta}{2\eta} \| x^{k+1} - x^k \|^2 \).

**Lemma 8** Suppose that Assumption 1 holds. Let \( \theta \leq \frac{1}{2} \). If NCE is performed at iteration \( k \), then for Algorithm 5 we have \( \ell_{k+1} \leq \ell_k - \min \left\{ \frac{(1-\theta)s^2}{2\eta}, \frac{(\gamma - 2\rho s)^2}{2} \right\} \).

Define \( K = k + 1 \) when \( k \) resets to 0. Denote the iterations from \( k = 0 \) to \( k = K \) to be one epoch. Recall the three cases in Section 4. In Case 1, we know from Lemma 8 that the potential function decreases with a magnitude at least \( \min \left\{ \frac{(1-\theta)s^2}{2\eta}, \frac{(\gamma - 2\rho s)^2}{2} \right\} \) at the last iteration, and it does not increase in the previous iterations from Lemma 7. Setting \( \gamma = \frac{\rho}{\eta}, s = \frac{\rho}{4\eta} \), and the other parameters the same as those in Theorem 1, we have

\[
\ell_K \leq \ell_0 - \min \left\{ \frac{64\epsilon^{1.5}}{\sqrt{\rho}}, \frac{16\epsilon L}{\rho} \right\}.
\]

In Case 2, we have

\[
\ell_K - \ell_0 \leq -\frac{\theta}{2\eta} \sum_{k=0}^{K-1} \| x^{k+1} - x^k \|^2 \leq -\frac{\theta B^2}{2\eta K} \leq -\frac{\theta B^2}{2\eta K} = \frac{8\epsilon^{1.5}}{\sqrt{\rho}}.
\]
where we use $K \sum_{t=0}^{K-1} \|x^{t+1} - x^t\|^2 > B^2$. So the algorithm will terminate in at most $\frac{\Delta I}{4\beta}^2$ epochs, and each epoch needs at most $K = \frac{1}{2} \left( \frac{L}{\epsilon} \right)^{1/4}$ gradient and function evaluations. So the total number of gradient and function evaluations must be less than $\frac{\Delta I}{4\beta}^2 \frac{L^{1/4}}{\epsilon^{1/4}}$. We only need to prove $\|\nabla f(\hat{y})\| \leq O(\epsilon)$ in the last epoch. Denote

\[ h(x) = \langle \nabla f(x^0), x - x^0 \rangle + \frac{1}{2}(x - x^0)^T H(x - x^0), \]

\[ \delta^k = \nabla f(y^k) - \nabla h(x^k). \]

Similar to the deduction in Section 5.1.2, we have

\[ x^{k+1} = y^k - \eta \nabla h(y^k) - \eta \delta^k, \]

\[ \|\delta^k\| \leq \frac{\rho}{2} \|y^k - x^0\|^2 \leq 2\rho B^2, \tag{28a} \]

where we use

\[ \|x^k - x^0\|^2 \leq k \sum_{t=0}^{k-1} \|x^{t+1} - x^t\|^2 \leq B^2, \forall k \leq K, \tag{29a} \]

\[ \|y^k - x^0\| \leq 2B, \forall k \leq K, \tag{29b} \]

in the last epoch. Similar to the proof of Theorem 1, we have

\[ \|\nabla h(\hat{y})\| = \left\| \frac{1}{K_2 - K_1 + 1} \sum_{k=K_1}^{K_2} \nabla h(y^k) \right\| = \frac{1}{\eta(K_2 - K_1 + 1)} \left\| \sum_{k=K_1}^{K_2} (x^{k+1} - y^k + \eta \delta^k) \right\|, \]

and

\[ \left\| \sum_{k=K_1}^{K_2} (x^{k+1} - y^k + \eta \delta^k) \right\| = \left\| \sum_{k=K_1}^{K_2} (x^{k+1} - x^k - (1 - \theta)(x^k - x^{k-1}) + \eta \delta^k) \right\| = \left\| x^{K_2+1} - x^{K_1} - (1 - \theta)(x^{K_2} - x^{K_1-1}) + \eta \sum_{k=K_1}^{K_2} \delta^k \right\| \leq \|x^{K_2+1} - x^{K_2}\| + \|x^{K_1} - x^{K_1-1}\| + \theta \|x^{K_2} - x^{0}\| + \theta \|x^{K_1-1} - x^0\| + \eta \sum_{k=K_1}^{K_2} \|\delta^k\|. \]

From $K_2 - K_1 + 1 \geq \frac{K}{3}$, (29a), and (28a), we have

\[ \|\nabla h(\hat{y})\| \leq \frac{3}{\eta K} \|x^{K_2+1} - x^{K_2}\| + \frac{3}{\eta K} \|x^{K_1} - x^{K_1-1}\| + \frac{6\theta B}{\eta K} + 2\rho B^2. \]

On the other hand, from the definitions of $K_1$ and $K_2$, we have

\[ \|x^{K_2+1} - x^{K_2}\|^2 \leq \frac{1}{K - \lfloor 2K/3 \rfloor} \sum_{k=\lfloor 2K/3 \rfloor}^{K-1} \|x^{k+1} - x^k\|^2 \leq \frac{1}{K - \lfloor 2K/3 \rfloor} \sum_{k=0}^{K-1} \|x^{k+1} - x^k\|^2 \leq \frac{3B^2}{K^2}, \]

\[ \leq \frac{1}{K - \lfloor 2K/3 \rfloor} \sum_{k=0}^{K-1} \|x^{k+1} - x^k\|^2 \leq \frac{3B^2}{K^2}. \]
and

\[ \|x^{k_1} - x^{k_1-1}\|^2 \leq \frac{1}{\frac{K}{3}} \sum_{k=1}^{\frac{K}{3}} \|x^k - x^{k-1}\|^2 \]

\[ \leq \frac{1}{\frac{K}{3}} \sum_{k=0}^{K-1} \|x^{k+1} - x^k\|^2 \leq \frac{3B^2}{K^2}. \]

So we have

\[ \|\nabla h(y)\| \leq 6\sqrt{3B} + 6\theta B + 2\rho B^2, \]

and

\[ \|\nabla f(y)\| \leq \|\nabla h(y)\| + \|\nabla f(y) - \nabla h(y)\| \]

\[ \leq \|\nabla h(y)\| + \frac{\rho}{2} \|\hat{y} - x^0\|^2 \]

\[ \leq 6\sqrt{3B} + 6\theta B + 4\rho B^2 \leq 267\epsilon. \]

\[ \]

**B. Proof of Theorem 3**

**Proof**

Denote \( x^{t,k} \) to be the iterate and \( \xi^t \) to be the perturbation in the \( t \)th epoch, and \( \hat{y}^t = \frac{1}{K_0+1} \sum_{k=0}^{K_0} y^{t,k} \). When \( \|\nabla f(y^{t,K-1})\| > \frac{B}{\eta} \) and the “if condition” triggers, we have from Lemma 1 that

\[ f(x^{t,K}) - f(x^{t,0}) \leq -\frac{B^2}{4\eta}. \]

Since \( x^{t+1,0} = x^{t,K} \), we have

\[ f(x^{t+1,0}) - f(x^{t,0}) \leq -\frac{B^2}{4\eta}. \]

When \( \|\nabla f(y^{t,K-1})\| \leq \frac{B}{\eta} \) and the “if condition” triggers, we have from (15) that

\[ f(x^{t,K}) - f(x^{t,0}) \leq -\frac{3\theta B^2}{8\eta K} + \frac{10\rho^2 B^4 \eta K}{2\theta} + 4.5\rho B^3. \]

From the \( L \)-gradient Lipschitz, we have

\[ f(x^{t+1,0}) - f(x^{t,K}) \leq \langle \nabla f(x^{t,K}), x^{t+1,0} - x^{t,K} \rangle + \frac{L}{2} \|x^{t+1,0} - x^{t,K}\|^2 \]

\[ = \langle \nabla f(x^{t,K}), \xi^t \rangle + \frac{L}{2} \|\xi^t\|^2 \]

\[ \leq \frac{5Br}{4\eta} + \frac{Lr^2}{2} \leq \frac{\theta B^2}{8\eta K}, \]

where we use

\[ \|\nabla f(x^{t,K})\| \leq \|\nabla f(y^{t,K-1})\| + \|\nabla f(x^{t,K}) - \nabla f(y^{t,K-1})\| \leq \|\nabla f(y^{t,K-1})\| + L\|x^{t,K} - y^{t,K-1}\| \\
\]

\[ = \|\nabla f(y^{t,K-1})\| + L\|\nabla f(y^{t,K-1})\| \leq \frac{B}{\eta} + LB \leq \frac{5B}{4\eta} \]
and \( \|\xi^t\| \leq r \leq \min\{\frac{\theta B}{3\tilde{K}}, \sqrt{\frac{\theta B^2}{2\tilde{K}}}\} \). So we have
\[
f(x^{t+1,0}) - f(x^{t,0}) \leq -\frac{\theta B^2}{4\eta K} + \frac{10\rho^2 B^4\eta^2 K}{2\theta} + 4.5\rho B^3
\leq -\frac{\epsilon^{1.5}}{700000\sqrt{\rho}}.
\]

So the algorithm will terminate in at most \( \mathcal{O}\left(\frac{\Delta f}{\epsilon^{1.5}}\right) \) epochs. Since each epoch needs at most \( K = 4\chi \left(\frac{L^2}{\epsilon}\right)^{1/4} \) gradient evaluations, the total number of gradient evaluations must be less than \( \mathcal{O}(\frac{\Delta f}{\epsilon^{1.5}}\frac{1}{\epsilon^{1.5}}\chi^2) \).

Now, we consider the last epoch. Denote it to be the \( T \)-th epoch. Similar to the proof of Theorem 1, we also have
\[
\|\nabla f(y^T)\| \leq \frac{2\sqrt{2}B}{\eta K^2} + \frac{28B}{\eta K} + 4\rho B^2 \leq \frac{\epsilon}{\chi^3} \leq \epsilon.
\]

On the other hand, we have
\[
\|\nabla f(y^{T-1,K-1})\| \leq \|\nabla f(y^T)\| + \|\nabla f(y^T) - \nabla f(x^{T-1,K})\| + \|\nabla f(x^{T-1,K}) - \nabla f(y^{T-1,K-1})\|
\leq \|\nabla f(y^T)\| + L\|y^T - x^{T,0}\| + L\|x^{T,0} - x^{T-1,K}\| + L\|x^{T-1,K} - y^{T-1,K-1}\|
\leq \frac{\epsilon}{\chi^2} + 2LB + Lr + \rho L\|\nabla f(y^{T-1,K-1})\|
\leq \frac{\epsilon}{\chi^2} + 2.5LB + \frac{1}{4}\|\nabla f(y^{T-1,K-1})\|,
\]
where we use \( \|y^T - x^{T,0}\| \leq \frac{1}{K\rho_0}\sum_{k=0}^{K_0}\|y^{T,k} - x^{T,0}\| \leq 2B \) in \( \frac{\epsilon}{\chi^2} \). Thus, we have \( \|\nabla f(y^{T-1,K-1})\| \leq \frac{4\epsilon}{\chi^2} + 4\rho B/3 \leq \frac{4\epsilon}{\chi^2} + \frac{2\rho B}{\eta} \leq \frac{2L}{\sqrt{\epsilon}} \) by letting \( \epsilon \leq \frac{L^2}{4\sqrt{\rho}} \). In fact, when \( \epsilon > \frac{L^2}{4\sqrt{\rho}} \), we have \( \lambda_{\min}(\nabla^2 f(x)) \geq -L > -576\sqrt{\rho} \) for any \( x \). Thus, in the last epoch, we have \( \|\nabla f(y^{T-1,K-1})\| \leq \frac{B}{\eta} \). This is the reason why perturbation is not needed when \( \|\nabla f(y^{T,K-1})\| > \frac{B}{\eta} \).

If \( \lambda_{\min}(\nabla^2 f(x^{T-1,K})) \geq -\sqrt{\epsilon\rho} \), from the perturbation theory of eigenvalues (Hoffman & Wielandt, 1953), we have for any \( j \),
\[
|\lambda_j(\nabla^2 f(y^T)) - \lambda_j(\nabla^2 f(x^{T-1,K}))| \leq \|\nabla^2 f(y^T) - \nabla^2 f(x^{T-1,K})\|_2
\leq \rho\|y^T - x^{T-1,K}\|
\leq \rho\|y^T - x^{T,0}\| + \rho r \leq 3\rho B,
\]
and
\[
\lambda_j(\nabla^2 f(y^T)) \geq \lambda_j(\nabla^2 f(x^{T-1,K})) - |\lambda_j(\nabla^2 f(y^T)) - \lambda_j(\nabla^2 f(x^{T-1,K}))|
\geq -\sqrt{\epsilon\rho} - 3\rho B \geq -1.011\sqrt{\epsilon\rho},
\]
where we use \( \|y^T - x^{T,0}\| \leq 2B \) in \( \frac{b}{\sqrt{\epsilon\rho}} \).

Now, we consider \( \lambda_{\min}(\nabla^2 f(x^{T-1,K})) < -\sqrt{\epsilon\rho} \). Define the stuck region within the perturbation ball \( B_{x^{T-1,K}}(r) \) to be the set of points starting from which the “if condition” does not trigger in \( K \) iterations, that is,
\[
\mathcal{X} = \left\{ x \in B_{x^{T-1,K}}(r) \mid \{x^{T,k}\} \text{ is the RAGD iterate with} \right\}
\left\{ x^{T,0} = x \text{ and } \sum_{k=0}^{K-1} \|x^{T,k-1} - x^{T,k}\|^2 \leq B^2 \right\}
\cup \left\{ x \mid \lambda_{\min}(\nabla^2 f(x^{T-1,K})) < -\sqrt{\epsilon\rho}, \text{ otherwise} \right\}.
\]
As pointed out in (Jin et al., 2017; 2018), the shape of the stuck region can be very complicated, but its width along the e_1 direction is thin. Similar to Lemma 8 in (Jin et al., 2018), we know from Lemma 9 that the probability of the starting point...
\[ x^{T,0} = x^{T-1,\mathcal{X}} + \xi^t \] located in the stuck region \( \mathcal{X} \) is less than
\[
\frac{r_0 V_{d-1}(r)}{V_d(r)} \leq r_0 \frac{\sqrt{d}}{r} = \zeta,
\]
where we let \( r_0 = \frac{\zeta r}{\sqrt{d}} \).

Denote \( \mathcal{H} \) to be the random event of \( x^{T,0} \notin \mathcal{X} \) (the location of \( x^{T,0} \) only depends on \( x^{T-1,\mathcal{X}} \) and the random variable \( \xi^t \)). When the random event \( \mathcal{H} \) happens, we know that if \( \lambda_{\min}(\nabla^2 f(x^{T-1,\mathcal{X}})) < -\sqrt{\epsilon\rho} \), the “if condition” must trigger. Thus, with probability at least \( 1 - \zeta \) (the random event \( \mathcal{H} \) happens), when the “if condition” does not trigger, we have \( \lambda_{\min}(\nabla^2 f(x^{T-1,\mathcal{X}})) \geq -\sqrt{\epsilon\rho} \). Thus, the output \( \check{y} \) satisfies \( \lambda_{\min}(\nabla^2 f(\check{y})) \geq -1.011\sqrt{\epsilon\rho} \) with probability at least \( 1 - \zeta \).

**Lemma 9** Suppose that \( \lambda_{\min}(\mathcal{H}) < -\sqrt{\epsilon\rho} \), where \( \mathcal{H} = \nabla^2 f(x) \). Let \( x^0 \) and \( x^{r_0} \) be at distance at most \( r \) from \( x \). Let \( x^{-1} = x^0 \), \( x^{r-1} = x^{r_0} \), and \( x^0 - x^{r_0} = r_0 e_1 \), where \( e_1 \) is the minimum eigen-direction of \( \mathcal{H} \). Under the parameter settings in Theorem 3, running AGD starting at \( x^0 \) and \( x^{r_0} \), respectively, we have
\[
\max \left\{ K \sum_{k=0}^{K-1} \| x^{k+1} - x^k \|^2, K \sum_{k=0}^{K-1} \| x^{r_k+1} - x^{r_k} \|^2 \right\} \geq B^2.
\]
that is, at least one of the iterates triggers the “if condition”.

The proof of this lemma is almost the same as that of Lemma 18 in (Jin et al., 2018). We only list the sketch and the details can be found in (Jin et al., 2018).

**Proof 11** Denote \( w^k = x^k - x^{r_k} \). From the update of AGD, we have
\[
\begin{bmatrix}
    w^{k+1} \\
    w^k
\end{bmatrix} = \begin{bmatrix}
    (2 - \theta)(I - \eta\mathcal{H}) & -(1 - \theta)(I - \eta\mathcal{H}) \\
    0 & I
\end{bmatrix} \begin{bmatrix}
    w^{k} \\
    w^{k-1}
\end{bmatrix} - \eta \begin{bmatrix}
    (2 - \theta)\triangle^k w^k - (1 - \theta)\triangle^k w^{k-1} \\
    0
\end{bmatrix},
\]
and
\[
w^k = [I, 0]A^k [w_0^r] - \eta [I, 0] \sum_{r=0}^{k-1} A^{k-1-r} [\phi^r].
\]
where \( \triangle^k = \int_0^1 \left( \nabla^2 f(ty^k + (1 - t)y^{r_k}) - \mathcal{H} \right) dt \) and \( \phi^k = (2 - \theta)\triangle^k w^k - (1 - \theta)\triangle^k w^{k-1} \).

Assume that none of the iterates \( (x^0, x^1, \ldots, x^K) \) and \( (x^{r_0}, x^{r_1}, \ldots, x^{r_K}) \) trigger the “if condition”, which yield
\[
\| x^k - x^0 \| \leq B, \| y^k - x^0 \| \leq 2B, \forall k \leq K,
\]
\[
\| x^{r_k} - x^{r_0} \| \leq B, \| y^{r_k} - x^{r_0} \| \leq 2B, \forall k \leq K.
\]
We have
\[
\| \triangle^k \| \leq \rho \max \{ \| y^k - x^k \|, \| y^{r_k} - x^{r_k} \| \}
\]
\[
\leq \rho \max \{ \| y^k - x^0 \|, \| y^{r_k} - x^{r_0} \| \} + \rho r \leq 3\rho B,
\]
\[
\| \phi^k \| \leq 6\rho B(\| w^k \| + \| w^{k-1} \|).
\]
We can show the following inequality for all \( k \leq K \) by induction:
\[
\left\| \eta [I, 0] \sum_{r=0}^{k-1} A^{k-1-r} [\phi^r] \right\| \leq \frac{1}{2} \left\| [I, 0]A^k [w_0^r] \right\|.
\]
It is easy to check the base case holds for \( k = 0 \). Assume the inequality holds for all steps equal to or less than \( k \). Then we have

\[
\| w^k \| \leq \frac{3}{2} \left\| [I, 0] A^k \left[ w^0 \right] \right\| ,
\]

\[
\| \phi^k \| \leq 18 \rho B \left\| [I, 0] A^k \left[ w^0 \right] \right\| ,
\]

by the monotonicity of \( \left\| [I, 0] A^k \left[ w^0 \right] \right\| \) in \( k \) (Lemma 33 in (Jin et al., 2018)). We also have

\[
\left\| \eta [I, 0] \sum_{r=0}^{k} A^{k-r} \left[ \phi^r \right] \right\| \leq \eta \sum_{r=0}^{k} \left\| [I, 0] A^{k-r} \left[ I \right] \right\| \left\| \phi^r \right\|
\]

\[
\leq 18 \rho B \eta \sum_{r=0}^{k} \left\| [I, 0] A^{k-r} \left[ I \right] \right\| \left\| [I, 0] A^r \left[ w^0 \right] \right\|
\]

\[
= 18 \rho B \eta \sum_{r=0}^{k} |a_{k-r}| |a_r - b_r| r_0
\]

\[
\leq 18 \rho B \eta K \left( \frac{2}{\theta} + K \right) \left\| [I, 0] A^{k+1} \left[ w^0 \right] \right\| ,
\]

where we define \( [a_{k-r} - b_k] = [I, 0] A_{min} \) and \( A_{min} = \left[ \begin{array}{c} (2 - \theta)(1 - \eta \lambda_{min}) \\ 1 \\ 0 \end{array} \right] \), uses the fact that \( w^0 = r_0 e_1 \) is along the minimum eigenvector direction of \( H \), \( b \), uses Lemma 31 in (Jin et al., 2018). From the parameter settings, we have \( 18 \rho B \eta K \left( \frac{2}{\theta} + K \right) \leq \frac{1}{2} \). Therefore, the induction is proved, which yields

\[
\| w^K \| \geq \frac{1}{2} \left\| [I, 0] A^K \left[ w^0 \right] \right\| - \left\| \eta [I, 0] \sum_{r=0}^{K-1} A^{K-1-r} \left[ \phi^r \right] \right\|
\]

\[
\geq \frac{1}{2} \left\| [I, 0] A^K \left[ w^0 \right] \right\| - \frac{r_0}{2} |a_K - b_K|
\]

\[
\geq \frac{\theta r_0}{4} \left( 1 + \frac{\theta}{2} \right) \geq 5B,
\]

where \( \geq \) uses Lemma 33 in (Jin et al., 2018) and \( \eta \lambda_{min} \leq -\theta^2 \), \( \geq \) uses \( K = \frac{2}{\theta} \log \frac{20B}{\theta r_0} \). However, (30) yields

\[
\| w^K \| \leq \| x^K - x^0 \| + \| x'' - x''^0 \| + \| x - x^0 \| + \| x - x''^0 \|
\]

\[
\leq 2B + 2r \leq 4B,
\]

which makes a contradiction. Thus the assumption is wrong and we conclude that at least one of the iterates trigger the “if condition”.

C. Efficient Implementation of the Average

Given \( x^0, x^1, \cdots, x^K \) and \( y^0, y^1, \cdots, y^K \) sequentially, we want to find \( \hat{y} = \frac{1}{K_0 + 1} \sum_{k=0}^{K_0} y^k \) efficiently, where \( K_0 = \arg \min_{k \geq 2} \sum_{k=0}^{K_0} x^{k+1} - x^k \). We present the implementation in Algorithm 7.
Algorithm 7 Implementation of the Average

Initialize $S_1 = S_2 = 0$, $K_0 = 0$

for $k = 0, 1, \ldots, K - 1$ do
  if $k \leq \lfloor \frac{K}{2} \rfloor$ then
    $S_1 = S_1 + y^k$, $K_0 = k$
  else
    if $\|x^{K_0+1} - x^{K_0}\| < \|x^{k+1} - x^k\|$ then
      $S_2 = S_2 + y^k$
    else
      $S_1 = S_1 + S_2 + y^k$, $S_2 = 0$, $K_0 = k$
    end if
  end if
end for

Output $\frac{S_1}{K_0 + 1}$