ALMOST GLOBAL EXISTENCE FOR CUBIC NONLINEAR
SCHRÖDINGER EQUATIONS IN ONE SPACE DIMENSION

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Abstract. We consider non-gauge-invariant cubic nonlinear Schrödinger equations in one space dimension. We show that initial data of size $\varepsilon$ in a weighted Sobolev space lead to solutions with sharp $L^\infty$ decay up to time $\exp(C\varepsilon^{-2})$. We also exhibit norm growth beyond this time for a specific choice of nonlinearity.

1. Introduction

We study the initial-value problem for the following cubic nonlinear Schrödinger equation (NLS) in one space dimension:

$$\begin{cases}
(i\partial_t + \frac{1}{2}\partial_{xx})u = \lambda_1 \bar{u}^3 + \lambda_2 u^3 + \lambda_3 |u|^2 \bar{u} + \lambda_4 |u|^2 u, \\
u(1) = u_1 \in \Sigma,
\end{cases}$$

(1.1)

where $u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{C}$ is a complex-valued function of space-time, $\lambda_j \in \mathbb{C}$ for $j = 1, \ldots, 4$, and

$$\|u_1\|_{\Sigma} := \|u_1\|_{L^2_x} + \|\partial_x u_1\|_{L^2_x} + \|x u_1\|_{L^2_x}.$$

Our main result, Theorem 1.1 below, is almost global existence for small solutions to (1.1).

The most widely-studied cubic NLS is the gauge-invariant equation

$$\begin{cases}
(i\partial_t + \frac{1}{2}\partial_{xx})u = \pm |u|^2 u, \\
u(1) = u_1 \in \Sigma,
\end{cases}$$

(1.2)

Gauge invariance (that is, the symmetry $u \mapsto e^{i\theta}u$ for $\theta \in \mathbb{R}$) corresponds to the conservation of the $L^2_x$-norm. As the cubic NLS in one dimension is $L^2_x$-subcritical, this conservation law (together with Strichartz estimates) leads to a simple proof of global well-posedness of (1.2) in $L^2_x$. As for long-time behavior, equation (1.2) in one dimension is a borderline case for the $L^2_x$ scattering theory: for ‘short-range’ nonlinearities $|u|^p u$ with $p > 2$ there are positive results, while for the ‘long-range’ case $0 < p \leq 2$ there is no $L^2_x$ scattering [1, 24]. For (1.2), small data in $\Sigma$ lead to global solutions that decay in $L^\infty_x$ at the sharp rate $t^{-1/2}$ and exhibit modified scattering, that is, linear behavior up to a logarithmic phase correction as $t \to \infty$ (see for example [4, 7, 17, 14]).

For non-gauge-invariant equations like (1.1), the question of global existence is less well-understood. Hayashi–Naumkin have studied non-gauge-invariant cubic NLS in one space dimension extensively (see [8, 9, 10, 11, 12, 13, 19], for example). They have shown that under specific conditions on the initial data, small data in weighted Sobolev spaces can lead to global solutions. In these cases they are also able to describe the asymptotic behavior. In this paper, we prove almost global existence for small (but otherwise arbitrary) data in $\Sigma$. Such a result is in the spirit of the well-known works concerning quadratic wave equations in three dimensions [15, 16]. Similar results have also been established for the cubic NLS with derivative nonlinearities; see, for example, [21, 23]. However, as mentioned in [8, 19, 21], there is a sense in which cubic nonlinearities containing at least one derivative can be considered ‘short-range’, while this
is not the case for the problem without derivatives. We will discuss this in a bit more detail below in Section 1.1.

Our main theorem is the following.

**Theorem 1.1 (Almost global existence).** Let \( u_1 \in \Sigma \) and let \( \varepsilon := \| u_1 \|_{\Sigma} \). If \( \varepsilon > 0 \) is sufficiently small, then there exists a unique solution \( u \in C([1,T_\varepsilon]; \Sigma) \) to (1.1) with \( u(1) = u_1 \), where \( T_\varepsilon = \exp\left(\frac{1}{c\varepsilon^2}\right) \) for some absolute constant \( c > 0 \). Furthermore, for some \( C > 0 \),

\[
\sup_{t \in [1,T_\varepsilon]} \left\{ \| \tilde{u}(t) \|_{L^\infty_\xi} + t^{\frac{2}{3}} \| u(t) \|_{L^\infty_\xi} + t^{-\frac{2}{3}} \| (x + it\partial_x)u(t) \|_{L^2_\xi} \right\} \leq C\varepsilon.
\]

In (1.1) we have decided to set the initial time \( t_0 = 1 \) for notational convenience. One could of course take \( t_0 = 0 \) with minor modifications. Moreover, the same result also holds for negative times \( t \in [-T_\varepsilon,0] \).

It is important to observe that without imposing further conditions on the initial data or the coefficients in the nonlinearity, Theorem 1.1 is essentially sharp. To demonstrate this, we consider the particular model

\[
i\partial_t + \frac{1}{2}\partial_{xx} u = i|u|^2 u
\]

and show that solutions either blow up or exhibit norm growth after the almost global existence time. The idea is that for sufficiently small data, certain ODE dynamics will dictate the behavior of the solution. The particular model (1.4) has the following advantages: (i) solutions to the ODE blow up in finite time, and (ii) since \( |u|^2 u \) is gauge-invariant, we get better estimates for \( u \) than those appearing in Theorem 1.1 specifically, a slower growth rate for the \( L^2_\xi \)-norm of \( (x + it\partial_x)u \). Thanks to (i) we need not fine-tune the initial conditions to make the arguments work, while (ii) allows us to show in a fairly straightforward fashion that the ODE can accurately model the PDE for long times. The precise result we prove is the following.

**Theorem 1.2 (Norm growth).** There exists \( \varepsilon > 0 \) sufficiently small that the following holds. Suppose \( u_1 \in \Sigma \) satisfies

\[
\| u_1 \|_\Sigma = \varepsilon, \quad \| \tilde{u}_1 \|_{L^\infty_\xi} \geq \frac{1}{2} \varepsilon,
\]

and let \( u \in C([1,T_\varepsilon]; \Sigma) \) be the solution to (1.4) with \( u(1) = u_1 \) given by Theorem 1.1. In particular, \( T_\varepsilon = \exp\left(\frac{1}{c\varepsilon^2}\right) \) for some \( c > 0 \), and

\[
\sup_{t \in [1,T_\varepsilon]} \| \tilde{u}(t) \|_{L^\infty_\xi} \lesssim \varepsilon.
\]

Denoting by \( T_{\text{max}} \in (T_\varepsilon, \infty] \) the maximal time of existence, there exists an absolute constant \( K \gg \varepsilon^2 \) and a finite time \( T_K > T_\varepsilon \) such that either \( T_{\text{max}} \leq T_K \), or there exists \( t \in [T_\varepsilon, T_K] \) such that

\[
\| \tilde{u}(t) \|_{L^\infty_\xi}^2 \geq K.
\]

Remarks.

- The proof will show that we could take, for example, \( K = (200c)^{-1} \). This means that \( K \) is a small but fixed constant independent from \( \varepsilon \) and, in particular, large compared to \( \varepsilon^2 \).
- The time \( T_K \) is the time at which the associated ODE solution reaches size \( 4K \); see (5.10).
- By the standard local theory (see below), if \( T_{\text{max}} < \infty \) then \( \| u(t) \|_{L^2_\xi} \rightarrow \infty \) as \( t \rightarrow T_{\text{max}} \).
- With trivial modifications, our arguments apply to (1.4) with a nonlinearity of the form \( \lambda |u|^2 u \) with \( \text{Im} \lambda > 0 \).

\[\text{1}\text{The case Im} \lambda = 0 \text{ reduces to (1.2), while for Im} \lambda < 0 \text{ one can prove that small solutions exist on } [1, \infty) \text{ and have \textquote{dissipative} behavior, namely, additional logarithmic time decay [22]. This same dissipative behavior occurs for (1.4) in the negative time direction.}\]
The strategy described above, namely, deducing behavior about solutions from associated ODE dynamics, has been carried out in many previous works. In the case of NLS with the $|u|^2 u$ nonlinearity, this approach leads to a proof of modified scattering \cite{7,17,14}. In other cases, for some specific nonlinearities and well-prepared initial data one can prove global existence and describe the asymptotics \cite{8,9,10,11,12,13,19}. In our case, we pick an equation for which the ODE solutions blow up; accordingly, we can demonstrate norm growth. This example demonstrates that one cannot hope to improve on Theorem 1.1 without imposing some more specific conditions.

1.1. **Strategy of the proof of Theorem 1.1.** We begin by recalling the standard local theory for (1.1).

**Theorem 1.3** (Local well-posedness). For $u_1 \in L^2_x$ the initial-value problem
\[
\begin{cases}
  (i\partial_t + \frac{1}{2}\partial_{xx})u = \lambda_1 \tilde{u}^3 + \lambda_2 u^3 + \lambda_3 |u|^2 \tilde{u} + \lambda_4 |u|^2 u \\
u(1) = u_1
\end{cases}
\]
has a unique solution $u \in C([1, T]; L^2_x)$, with $T \sim 1 + \|u_1\|^{-1}_{L^2_x}$, obeying
\[
u(t) = e^{i(t-1)\partial_{xx}/2}u_1 - i \int_1^t e^{i(t-s)\partial_{xx}/2} [\lambda_1 \tilde{u}^3(s) + \lambda_2 u^3(s) + \lambda_3 |u|^2 \tilde{u}(s) + \lambda_4 |u|^2 u(s)] \, ds. \tag{1.6}
\]
Furthermore, if $u_1 \in \Sigma$ then $u \in C([1, T]; \Sigma)$.

The existence in $C([1, T]; L^2_x(\mathbb{R}))$ follows from the standard arguments, namely contraction mapping and Strichartz estimates. The fact that the time of existence depends only on the norm of the data is a consequence of scaling. The existence in $C([1, T]; \Sigma)$ follows from standard persistence of regularity arguments, which involve commuting the equation with $\partial_x$ and $J(t) = x + it\partial_x$. We refer the reader to the textbook \cite{2} and the references cited therein.

For a solution $u$ we define
\[
f(t) = e^{-i t \partial_{xx}/2}u(t), \quad J u(t) = (x + it\partial_x)u(t).
\]

The proof of Theorem 1.1 will be based on a bootstrap argument in a properly chosen norm. To this end we introduce the notation
\[
\|u(t)\|_{X(t)} := \frac{1}{2} \left[ \|\hat{f}(t)\|_{L^\infty} + t^{-\frac{1}{4}} \|Ju(t)\|_{L^2_x} \right].
\]
We record here two facts that we prove in Section 2.

**Lemma 1.4.** The following estimates hold:
\[
\|u(1)\|_{X(1)} \leq \|u_1\|_{\Sigma}, \tag{1.7}
\]
\[
t^{\frac{1}{4}} \|u(t)\|_{L^\infty} \lesssim \|u(t)\|_{X(t)}. \tag{1.8}
\]

The next two propositions are the main ingredients for the bootstrap argument used to prove Theorem 1.1. They constitute the heart of the paper.

**Proposition 1.5.** For $u : [1, T] \times \mathbb{R} \to \mathbb{C}$ a solution to (1.1) and $1 \leq t \leq T$, there exists an absolute constant $C > 0$ such that
\[
\|\hat{f}(t)\|_{L^\infty} \leq \|\hat{f}(1)\|_{L^\infty} + C \left[ \|u_1\|_{\Sigma}^3 + \|u(t)\|_{X(t)}^3 + \int_1^t s^{-1} (\|u(s)\|_{X(s)}^3 + \|u(s)\|_{X(s)}^5) \, ds \right]. \tag{1.9}
\]
Proof of Theorem 1.1. Let 0 < ε < 1 to be specified below and let \( \|u_1\|_\Sigma = \varepsilon \). If \( u \) solves \((1.1)\), then Proposition 1.5, Proposition 1.6, and Lemma 1.4 imply

\[
\|u(t)\|_{X(t)} \leq \varepsilon + C\left[2\|u_1\|_\Sigma^3 + \|u(t)\|_{X(t)}^3 + \int_1^t s^{-\frac{3}{4}} (\|u(s)\|_{X(s)}^3 + \|u(s)\|_{X(s)}^5) ds\right].
\]

(1.11)
for some absolute constant $C > 0$. We choose $\varepsilon = \varepsilon(C) > 0$ and define $T_\varepsilon$ so that
\begin{equation}
170C\varepsilon^2 < \frac{1}{2}, \quad T_\varepsilon := \exp\left(\frac{1}{800C\varepsilon^2}\right).
\end{equation}
We now claim that the following estimate holds:
\begin{equation}
\|u(t)\|_{X(t)} \leq 2\varepsilon \quad \text{for all} \quad t \in [1, T_\varepsilon].
\end{equation}
This holds at $t = 1$ by (1.7). By continuity, if it is not true for all $t \in [1, T_\varepsilon]$ there must be a first time $t \in (1, T_\varepsilon]$ such that $\|u(t)\|_{X(t)} = 2\varepsilon$. Applying (1.11) at this time and using (1.12) yields
\begin{align*}
2\varepsilon &\leq \varepsilon + C(2\varepsilon^3 + (2\varepsilon)^3 + [4 + \log t][(2\varepsilon)^3 + (2\varepsilon)^5]) \\
&\leq \varepsilon(1 + 10C\varepsilon^2 + C[4 + \log T_\varepsilon][40\varepsilon^2]) < 2\varepsilon,
\end{align*}
which is a contradiction. This proves (1.13).

To complete the proof, it suffices to show that if $u : [1, T] \times \mathbb{R} \to \mathbb{C}$ is a solution such that $T \leq \exp\left(\frac{1}{800C\varepsilon^2}\right)$ and $\sup_{t \in [1, T]} \|u(t)\|_{X(t)} \lesssim \varepsilon$, then we may continue the solution in time. By the local theory it suffices to prove that $\|u(T)\|_{L^2_x} \lesssim \|u_1\|_{L^2_x}$. We use the Duhamel formula (1.6), Lemma 1.3 and the bound on $u$ to estimate
\begin{equation*}
\|u(T)\|_{L^2_x} \lesssim \|u_1\|_{L^2_x} + \int_1^T \|u(s)\|_{L^\infty_x}^2 \|u(s)\|_{L^2_x} ds \lesssim \|u_1\|_{L^2_x} + \varepsilon^2 \int_1^T s^{-1} \|u(s)\|_{L^2_x} ds.
\end{equation*}
Thus by Gronwall’s inequality and the bound on $T$, we deduce $\|u(T)\|_{L^2_x} \lesssim T^{C\varepsilon^2}\|u_1\|_{L^2_x} \lesssim \|u_1\|_{L^2_x}$, as was needed to show. This completes the proof of Theorem 1.1.

The rest of the paper is organized as follows: In Section 2 we set up notation and collect some useful lemmas. The main trilinear estimates that we will use repeatedly in the proofs of Proposition 1.5 and 1.6 are given in Lemma 2.3. In Section 3 we prove Proposition 1.5 and in Section 4 we prove Proposition 1.6. As shown above, these two propositions imply the main result, Theorem 1.1. Section 5 contains the proof of Theorem 1.2 in which we demonstrate norm growth for a model nonlinearity. In Appendix A we discuss the construction of some cutoffs used in Sections 3 and 4.

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2. Notation and useful lemmas

For nonnegative $X, Y$ we write $X \lesssim Y$ to denote $X \leq CY$ for some $C > 0$. We write $X \ll Y$ to denote $X \leq cY$ for some small $c \in (0, 1)$. We write $O(X)$ to denote a finite linear combination of terms that resemble $X$ up to constants, complex conjugation, and Littlewood–Paley projections. For example, the nonlinearity in (1.1) is $O(u^3)$.

The Fourier transform and its inverse are given by
\begin{equation*}
\mathcal{F}u(x) = \widehat{u}(\xi) = (2\pi)^{-\frac{1}{2}} \int_\mathbb{R} e^{-ix\xi} u(x) \, dx, \quad \mathcal{F}^{-1}u(x) = (2\pi)^{-\frac{1}{2}} \int_\mathbb{R} e^{ix\xi} u(\xi) \, d\xi.
\end{equation*}

For $s \in \mathbb{R}$ we define the fractional derivative operator $|\partial_x|^s$ as a Fourier multiplier, namely, $|\partial_x|^s = \mathcal{F}^{-1}|\xi|^s \mathcal{F}$. We define the homogeneous Sobolev space $H^s_x$ via
\begin{equation*}
\|u\|_{H^s_x} = \||\partial_x|^s u\|_{L^2_x}.
\end{equation*}

We employ the standard Littlewood–Paley theory. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an even bump supported on $[-\frac{10}{9}, \frac{10}{9}]$ and equal to one on $[-1, 1]$. For $N \in 2^\mathbb{Z}$ we define
\begin{equation*}
\widehat{P_{\leq N}}f(\xi) = \hat{f}_{\leq N}(\xi) := \phi(\xi/N)\hat{f}(\xi), \quad \widehat{P_{> N}}f(\xi) = \hat{f}_{> N}(\xi) := [1 - \phi(\xi/N)]\hat{f}(\xi),
\end{equation*}
where $\hat{f}_{\leq N} := \sum_{m=0}^{N-1} \hat{f}(\xi - 2\pi m/N)$ and $\hat{f}_{> N} := \sum_{m=N}^{\infty} \hat{f}(\xi - 2\pi m/N)$.
\[ \hat{P}_N f(\xi) = \hat{f}_N(\xi) := [\phi(\xi/N) - \phi(2\xi/N)]\hat{f}(\xi). \]

These operators commute with all other Fourier multiplier operators. They are self-adjoint and bounded on every \( L^p \) space and obey the estimate
\[ \|f\|_{L^q} \lesssim N^{-s+\frac{1}{q} - \frac{1}{r}} \|\partial_x|^s f\|_{L^r} \]
for \( 1 \leq r \leq q \leq \infty \) and \( s > \frac{1}{r} - \frac{1}{q} \).

2.1. **Linear theory.** The free Schrödinger propagator is defined as a Fourier multiplier: \( e^{it\partial_{xx}/2} = \mathcal{F}^{-1}e^{-it\xi^2/2}\mathcal{F} \). In physical space we have

\[ [e^{it\partial_{xx}/2} f](x) = (2\pi i t)^{-\frac{1}{2}} \int e^{i(x-y)^2/2t} f(y) dy. \]  

From (2.2) we can read off the following factorization:
\[ e^{it\partial_{xx}/2} = M(t)D(t)\mathcal{F}M(t), \]
where the modulation \( M(t) \) and dilation \( D(t) \) are defined by
\[ [M(t)f](x) = e^{ix^2/2t} f(x) \quad \text{and} \quad [D(t)f](x) = (it)^{-\frac{1}{2}} f(\frac{x}{t}). \]

We define the operator \( J(t) = x + it\partial_x \). By (2.3), we have \( J(t) = e^{it\partial_{xx}/2} x e^{-it\partial_{xx}/2} \). For a solution \( u(t) \) to (1.1) we write \( f(t) = e^{-it\partial_{xx}/2}u(t) \) and note that
\[ \|Ju\|_{L^2} = \|xf\|_{L^2} = \|\partial_x f\|_{L^2}. \]

2.2. **Notation and Duhamel formula.** Suppose that \( u \) is a solution to (1.1) and denote \( f(t) = e^{-it\partial_{xx}/2}u(t) \). Using the Duhamel formula (1.6) and taking the Fourier transform leads to:

\[ \hat{f}(t, \xi) = \hat{f}(1, \xi) - i\lambda_1 (2\pi)^{-1} \int_1^t \int \int e^{i\Phi(\xi, \eta, \sigma)} \hat{f}(\xi - \eta) \hat{f}(\eta - \sigma) \hat{f}(\sigma) d\sigma d\eta ds \\
- \lambda_2 (2\pi)^{-1} \int_1^t \int \int e^{i\Psi(\xi, \eta, \sigma)} \hat{f}(\xi - \eta) \hat{f}(\eta - \sigma) \hat{f}(\sigma) d\sigma d\eta ds \\
- \lambda_3 (2\pi)^{-1} \int_1^t \int \int e^{i\Omega(\xi, \eta, \sigma)} \hat{f}(\xi - \eta) \hat{f}(\eta - \sigma) \hat{f}(\sigma) d\sigma d\eta ds \\
- \lambda_4 \int_1^t \mathcal{F}(e^{-i\partial_{xx}/2}\|u\|^2u)(s, \xi) ds, \]

where the phases \( \Phi, \Psi, \Omega \) are given by
\[ \Phi = \frac{1}{2}[\xi^2 + (\xi - \eta)^2 + (\eta - \sigma)^2 + \sigma^2], \quad \Psi = \frac{1}{2}[\xi^2 - (\xi - \eta)^2 - (\eta - \sigma)^2 - \sigma^2], \]
\[ \Omega = \frac{1}{2}[\xi^2 + (\xi - \eta)^2 + (\eta - \sigma)^2 - \sigma^2]. \]

We do not write out the phase for the gauge-invariant nonlinearity \( |u|^2u \), since this term is amenable to a simpler analysis.

It is convenient to introduce the notation
\[ \xi_1 = \xi - \eta, \quad \xi_2 = \eta - \sigma, \quad \xi_3 = \sigma, \quad \bar{\xi} = (\xi_1, \xi_2, \xi_3), \quad \xi = \xi_1 + \xi_2 + \xi_3. \]

In this notation we may rewrite the phases as follows:
\[ \Phi = \frac{1}{2}[\xi_1^2 + \xi_2^2 + \xi_3^2], \quad \Psi = \xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1, \quad \Omega = \xi_1^2 + \xi_2^2 + \xi_1\xi_2 + \xi_2\xi_3 + \xi_1\xi_3. \]
We also need to consider derivatives of the phases, which we record here:

\[
\begin{align*}
\partial_t \Phi &= \xi + \xi_1, & \partial_\xi \Psi &= \xi_2 + \xi_3, & \partial_\xi \Omega &= \xi + \xi_1, \\
(\partial_\eta \Psi, \partial_\eta \Psi) &= (\xi_1 - \xi_2, \xi_2 - \xi_3), & (\partial_\eta \Omega, \partial_\eta \Omega) &= (\xi_2 - \xi_1, -\xi_2 - \xi_3).
\end{align*}
\]

(2.9)

Finally, we set up notation concerning frequency cutoffs. For a function \( f = f(s, x) \) and \( s \geq 1 \), we define \( f_{lo} \) and \( f_{hi} \) via \( \hat{f}_{lo} = \hat{P}_{lo} f := \phi_{lo} \hat{f} \) and \( \hat{f}_{hi} = \hat{P}_{hi} f := \phi_{hi} \hat{f} \), where

\[
\phi_{lo}(\xi) := \phi(s^{1/2} \xi), \quad \phi_{hi}(\xi) := 1 - \phi(s^{1/2} \xi).
\]

(2.10)

Here \( \phi \) is the standard cutoff defined earlier at the beginning of this section. We use the notation \( \phi_s \) to denote that either \( \phi_{lo} \) or \( \phi_{hi} \) may appear, and \( f_s = P_s f \) to denote that either \( f_{lo} \) or \( f_{hi} \) may appear.

2.3. Proof of Lemma 1.4. In this section we prove (1.7) and (1.8). First,

\[
\begin{align*}
\| J u_1 \|_{L^2_x} &\leq \| xu_1 \|_{L^2_x} + \| \partial_x u_1 \|_{L^2_x}, \\
\| \widehat{u_1} \|_{L^{\infty}_s} &\leq (2\pi)^{-\frac{1}{2}} \| u_1 \|_{L^1_s} \leq \pi^{-\frac{1}{2}} \| (1 + |x|) u_1 \|_{L^2_x},
\end{align*}
\]

so that (1.7) holds. For (1.8) we first use (2.3) to write

\[
u(t) = M(t) D(t) \hat{f}(t) + M(t) D(t) \mathcal{F}[M(t) - 1] f(t).
\]

It therefore suffices to show

\[
\| \mathcal{F}[M(t) - 1] f(t) \|_{L^{\infty}_s} \lesssim t^{-\frac{1}{4}} \| J u(t) \|_{L^2_x}.
\]

We split at frequency \( \sqrt{t} \), using the operators \( \hat{P}_{\leq N} := \mathcal{F} \phi(\cdot/N) \mathcal{F}^{-1} \). These share the same estimates as the usual projections, as \( \hat{P}_{\leq N} f = \hat{P}_{\leq N} \hat{f} \).

We first use (2.1), Plancherel, and (2.3) to estimate

\[
\begin{align*}
\| \hat{P}_{> \sqrt{t}} \mathcal{F}[M(t) - 1] f(t) \|_{L^2_x} &\lesssim t^{-\frac{1}{2}} \| \partial_x \mathcal{F}[M(t) - 1] f(t) \|_{L^2_x} \\
&\lesssim t^{-\frac{1}{2}} \| x f(t) \|_{L^2_x} \lesssim t^{-\frac{1}{2}} \| J u(t) \|_{L^2_x}.
\end{align*}
\]

Second, we use Hausdorff–Young, Cauchy–Schwarz, (2.4), and the bound

\[
|M(t) - 1| = |e^{ix^2/2t} - 1| \lesssim t^{-\frac{1}{4}} |x|
\]

(2.11)

to estimate

\[
\| \hat{P}_{\leq \sqrt{t}} \mathcal{F}[M(t) - 1] f(t) \|_{L^\infty_s} \lesssim \| \phi \sqrt{t} \| \| M(t) - 1 \| f(t) \|_{L^2_x} \lesssim t^\frac{1}{4} \| [M(t) - 1] f(t) \|_{L^2_x} \lesssim t^{-\frac{1}{4}} \| x f(t) \|_{L^2_x}.
\]

2.4. Useful estimates. For a function \( m : \mathbb{R}^3 \to \mathbb{R} \) we define the trilinear operator \( T_m \) as follows:

\[
\mathcal{F}(T_m[a, b, c])(\xi) = \int \int \int m(\xi - \eta, \eta - \sigma) \hat{a}(\xi - \eta) \hat{b}(\eta - \sigma) \hat{c}(\sigma) \, d\sigma \, d\eta.
\]

(2.12)

If \( a, b, c \) are functions of space-time, we employ the following notation:

\[
\mathcal{F}(T_m[a(t), b(t), c(t)])(\xi) = \mathcal{F}(T_m[a, b, c])(t, \xi).
\]

We also make use of the notation introduced in (2.7).

The following multilinear estimate due to Coifman–Meyer is one of the primary technical tools used in this paper. For the original result, see [3, Chapter 13]; for a more modern treatment, see [18].
Lemma 2.1 (Coifman–Meyer estimate \([3]\)). Let \(m \in C^\infty(\mathbb{R}^3 \setminus \{0\}; \mathbb{R})\) be a symbol satisfying
\[
\sup_{\xi \in \mathbb{R}^3 \setminus \{0\}} |\xi|^{\alpha} |\partial_\xi m(\xi)| < \infty
\] (2.13)
for all multiindices \(\alpha\) with \(|\alpha| \leq 10\). Then
\[
\|T_m[a, b, c]\|_{L_x^r} \lesssim \|a\|_{L_x^{r_1}} \|b\|_{L_x^{r_2}} \|c\|_{L_x^{r_3}}
\]
for all \(1 < r_1, r_2, r_3 \leq \infty\) and \(1 \leq r < \infty\) such that \(\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\).

Remark 2.2. Symbols with the property (2.13) will be called \textit{Coifman–Meyer symbols}.

We will now establish some trilinear estimates that will be used frequently in Sections 3 and 4. The proofs rely on Lemma 2.1, together with the following estimate, which is a consequence of Plancherel and Hölder:
\[
N^{\frac{1}{2}}\|u \geq N\|_{H_x^1} + N^{\frac{3}{2}}\|u \geq N\|_{H_x^2} + N^{-\frac{1}{2}}\|u \leq N\|_{L_x^2} \lesssim \|\hat{u}\|_{L_x^\infty}.
\] (2.14)

Lemma 2.3 (Trilinear estimates). Let \(T_m\) be an operator of the form (2.12).

(i) If \(|\xi_3|^2 m\) is a Coifman–Meyer symbol supported where \(|\xi_3| \geq \max\{|\xi_2|, |\xi_1|\}\), then
\[
\|\mathcal{F}(T_m[a, b, c_{>N}])\|_{L_x^\infty} \lesssim N^{-\frac{1}{2}}\|\hat{a}\|_{L_x^\infty} \min\left(\|b\|_{L_x^\infty}, \|c\|_{L_x^\infty}\right).
\] (2.15)

(ii) If \(|\xi_3| m\) is a Coifman–Meyer symbol supported where \(|\xi_3| \geq \max\{|\xi_2|, |\xi_1|\}\), then
\[
\|\mathcal{F}(T_m[a, b, c_{>N}])\|_{L_x^\infty} \lesssim N^{-\frac{1}{2}}\|a\|_{L_x^\infty} \min\left(\|b\|_{L_x^\infty}, \|c\|_{L_x^\infty}\right).
\] (2.16)

(iii) If \(|\xi_3| m\) is a Coifman–Meyer symbol supported where \(|\xi_3| \geq \max\{|\xi_2|, |\xi_1|\}\), then
\[
\|T_m[a, b, c_{>N}]\|_{L_x^2} \lesssim N^{-\frac{1}{2}}\|a\|_{L_x^2} \min\left(\|b\|_{L_x^2}, \|c\|_{L_x^2}\right).
\] (2.17)

In all of the above estimates, we can exchange the role of \(a\) and \(b\) on the right-hand side.

As explained above, we obtain \textit{a priori} bounds on solutions to (2.5) by analyzing the cubic terms in Fourier space. This leads us to study trilinear expressions of the form (2.12) with symbols that have some degeneracies for small frequencies. By properly dividing frequency space, we will be able to show that these singularities are always of the form \(\max(|\xi_1|, |\xi_2|, |\xi_3|)^{-\frac{1}{2}}\) or \(\max(|\xi_1|, |\xi_2|, |\xi_3|)^{-2}\). We will therefore be able to use the bounds (2.15)–(2.17) to control these expressions; in particular, we will make the choice \(N \sim s^{-1/2}\), where \(s\) is the time variable.

Proof of Lemma 2.3 The estimates (2.15)–(2.17) will all follow from Lemma 2.1 and (2.14), together with suitable high-low decompositions.

Proof of (2.15). Suppose \(|\xi_3|^2 m\) is a Coifman–Meyer symbol supported in a region where \(|\xi_3| \geq \max\{|\xi_2|, |\xi_1|\}\).

We decompose \(a = a_{\leq N} + a_{> N}\) and first estimate
\[
\|\mathcal{F}(T_m[a_{\leq N}, b, c_{>N}])\|_{L_x^\infty} \lesssim \|\mathcal{F}(T_{\xi_3 m}[a_{\leq N}, b, |\partial_x|^{-2} c_{>N}])\|_{L_x^\infty}
\]
\[
\lesssim \|a_{\leq N}\|_{L_x^2} \|b\|_{L_x^\infty} \|c_{>N}\|_{H_x^{-2}} \lesssim N^{-1}\|\hat{a}\|_{L_x^\infty} \|b\|_{L_x^\infty} \|c\|_{L_x^\infty}.
\]

Next, note that under our assumptions, \(|\xi_1 \xi_3| m\) is also a Coifman–Meyer multiplier. Thus,
\[
\|\mathcal{F}(T_{\xi_3 m}[a_{> N}, b, c_{>N}])\|_{L_x^\infty} \lesssim \|\mathcal{F}(T_{|\xi_3|^{-1} \xi_1 \xi_3 m}[|\partial_x|^{-1} a_{> N}, b, |\partial_x|^{-1} c_{>N}])\|_{L_x^\infty}
\]
\[
\lesssim \|a_{> N}\|_{H_x^{-1}} \|b\|_{L_x^\infty} \|c_{>N}\|_{H_x^{-1}} \lesssim N^{-1}\|\hat{a}\|_{L_x^\infty} \|b\|_{L_x^\infty} \|c\|_{L_x^\infty}.
\]

Combining the two estimates above yields the first estimate in (2.15).
We turn to the second inequality in (2.15). We decompose both \( a = a_{\leq N} + a_{> N} \) and \( b = b_{\leq N} + b_{> N} \). Using (2.1) as well, we first have

\[
\| \mathcal{F}(T_m[a_{\leq N}, b_{\leq N}, c_{> N}]) \|_{L^\infty} \lesssim \| \mathcal{F}(T_{\xi^2 m}[a_{\leq N}, b_{\leq N}, |\partial_x|^{-2} c_{> N}]) \|_{L^\infty}
\]

\[
\lesssim \| a_{\leq N} \|_{L^2} \| b_{\leq N} \|_{L^2} \| |\partial_x|^{-2} c_{> N} \|_{L^\infty} \lesssim N^{-1} \| \hat{a} \|_{L^\infty} \| \hat{b} \|_{L^\infty} \| c \|_{L^\infty}.
\]

Next, note that under our assumptions, \( \xi^2 m \) is also Coifman–Meyer. Thus,

\[
\| \mathcal{F}(T_m[a_{\leq N}, b_{> N}, c_{> N}]) \|_{L^\infty} \lesssim \| \mathcal{F}(T_{\xi^2 m}[a_{\leq N}, |\partial_x|^{-2} b_{> N}, c_{> N}]) \|_{L^\infty}
\]

\[
\lesssim \| a_{\leq N} \|_{L^2} \| b_{> N} \|_{H^1} \| c \|_{L^\infty} \lesssim N^{-1} \| \hat{a} \|_{L^\infty} \| \hat{b} \|_{L^\infty} \| c \|_{L^\infty}.
\]

Noting that \( \xi^2 m \) is also Coifman–Meyer, we can similarly obtain

\[
\| \mathcal{F}(T_m[a_{> N}, b_{\leq N}, c_{> N}]) \|_{L^\infty} \lesssim N^{-1} \| \hat{a} \|_{L^\infty} \| \hat{b} \|_{L^\infty} \| c \|_{L^\infty}.
\]

The remaining case can be treated similarly, as \( |\xi_1 \xi_2| m \) is also Coifman–Meyer.

Proof of (2.16). Suppose \( |\xi_3| m \) is a Coifman–Meyer symbol supported in a region where \( |\xi_3| \gtrsim \max\{ |\xi_2|, |\xi_1| \} \).

We can obtain the first estimate in (2.16) as follows:

\[
\| \mathcal{F}(T_m[a, b, c_{> N}]) \|_{L^\infty} \lesssim \| \mathcal{F}(T_{\xi_3 m}[a, b, |\partial_x|^{-1} c_{> N}]) \|_{L^\infty}
\]

\[
\lesssim \| a \|_{L^2} \| b \|_{L^2} \| c_{> N} \|_{H^1} \lesssim N^{-\frac{1}{2}} \| a \|_{L^\infty} \| b \|_{L^2} \| c \|_{L^2}.
\]

To obtain the second estimate in (2.16), we decompose \( b = b_{\leq N} + b_{> N} \). Using (2.1) as well, we first have

\[
\| \mathcal{F}(T_{\xi_3 m}[a, b_{\leq N}, |\partial_x|^{-1} c_{> N}]) \|_{L^\infty} \lesssim \| a \|_{L^\infty} \| b_{\leq N} \|_{L^2} \| c_{> N} \|_{H^1} \leq N^{-\frac{1}{2}} \| a \|_{L^\infty} \| b \|_{L^2} \| c \|_{L^2}.
\]

Next, note that under our assumptions, \( |\xi_2| m \) is also Coifman–Meyer. Thus,

\[
\| \mathcal{F}(T_{|\xi_3| m}[a, |\partial_x|^{-1} b_{> N}, c_{> N}]) \|_{L^\infty} \lesssim \| a \|_{L^\infty} \| b_{> N} \|_{H^1} \| c \|_{L^2} \lesssim N^{-\frac{1}{2}} \| a \|_{L^\infty} \| b \|_{L^2} \| c \|_{L^2}.
\]

Proof of (2.17). Suppose \( |\xi_3| m \) is a Coifman–Meyer symbol supported on a region where \( |\xi_3| \gtrsim \max\{ |\xi_1|, |\xi_2| \} \).

We can obtain the first estimate in (2.17) as follows:

\[
\| T_{|\xi_3| m}(a, b, |\partial_x|^{-1} c_{> N}) \|_{L^2} \lesssim \| a \|_{L^\infty} \| b \|_{L^\infty} \| c_{> N} \|_{H^1} \lesssim N^{-\frac{1}{2}} \| a \|_{L^\infty} \| b \|_{L^2} \| c \|_{L^2}.
\]

To obtain the second estimate in (2.17), we proceed as above and decompose \( b = b_{\leq N} + b_{> N} \). Using (2.1) as well, we first have

\[
\| T_{|\xi_3| m}(a, b_{\leq N}, |\partial_x|^{-1} c_{> N}) \|_{L^2} \lesssim \| a \|_{L^\infty} \| b_{\leq N} \|_{L^2} \| |\partial_x|^{-1} c_{> N} \|_{L^\infty} \lesssim N^{-\frac{1}{2}} \| a \|_{L^\infty} \| b \|_{L^2} \| c \|_{L^2}.
\]

As \( |\xi_2| m \) is Coifman–Meyer, we can also estimate

\[
\| T_{|\xi_2| m}(a, |\partial_x|^{-1} b_{> N}, c_{> N}) \|_{L^2} \lesssim \| a \|_{L^\infty} \| b_{> N} \|_{H^1} \| c \|_{L^2} \lesssim N^{-\frac{1}{2}} \| a \|_{L^\infty} \| b \|_{L^2} \| c \|_{L^2}.
\]

This completes the proof. \( \square \)
3. Proof of Proposition 1.5

In this section we prove the estimate (1.9) for \( \hat{f}(t) \). Using (2.5) we see that it suffices to estimate the following terms in \( L^\infty_\xi \):

\[
\begin{align*}
\int_1^t \int_{\mathbb{R}^2} e^{is\Phi} \hat{f}(\xi_1) \hat{f}(\xi_2) \hat{f}(\xi_3) \, d\sigma \, d\eta \, ds, \\
\int_1^t \int_{\mathbb{R}^2} e^{is\Psi} \hat{f}(\xi_1) \hat{f}(\xi_2) \hat{f}(\xi_3) \, d\sigma \, d\eta \, ds, \\
\int_1^t \int_{\mathbb{R}^2} e^{is\Omega} \hat{f}(\xi_1) \hat{f}(\xi_2) \hat{f}(\xi_3) \, d\sigma \, d\eta \, ds, \\
\int_1^t \mathcal{F}(e^{-is\partial_x^2/2}|u|^2u)(s, \xi) \, ds,
\end{align*}
\]  

(3.1) (3.2) (3.3) (3.4)

where the phases \( \Phi, \Psi, \Omega \) are as in (2.8) and we use the notation from (2.7).

3.1. Estimation of (3.1). We recall the notation from (2.10) and write

\[
1 = [\phi_{lo}(\xi_1) + \phi_{hi}(\xi_1)][\phi_{lo}(\xi_2) + \phi_{hi}(\xi_2)][\phi_{lo}(\xi_3) + \phi_{hi}(\xi_3)]
\]

in the integrand of (3.1). Expanding the product, we encounter two types of terms: (i) the low frequency term \( \phi_{lo}(\xi_1)\phi_{lo}(\xi_2)\phi_{lo}(\xi_3) \), (ii) terms are of the form \( \phi_s(\xi_j)\phi_s(\xi_k)\phi_{hi}(\xi_l) \), where \( j, k, \ell \in \{1, 2, 3\} \).

We estimate the contribution of the low frequency term by using volume bounds:

\[
\left\| \int_1^t \int_{\mathbb{R}^2} e^{is\Phi} \hat{f}_{lo}(\xi_1) \hat{f}_{lo}(\xi_2) \hat{f}_{lo}(\xi_3) \, d\sigma \, d\eta \, ds \right\|_{L^\infty_\xi} \lesssim \int_1^t s^{-1} \| \hat{f}(s) \|_{L^3_\xi}^4 \, ds,
\]

(3.5)

which is an acceptable contribution to the right-hand side of (1.9).

For the terms of type (ii) we write \( 1 = \chi_1(\vec{\xi}) + \chi_2(\vec{\xi}) + \chi_3(\vec{\xi}) \) for \( \vec{\xi} \in \mathbb{R}^3 \), where each \( \chi_j \) is a smooth Coifman–Meyer multiplier such that

\[
|\xi_j| \geq \max \{ \frac{|\vec{\xi}|}{|\xi_k|} : k \neq j \} \quad \text{for all } \vec{\xi} \in \text{support}(\chi_j).
\]

(3.6)

See Appendix A for the construction of such multipliers. We will show how to estimate the contribution from \( \chi_3 \). The same ideas suffice to treat the (almost symmetric) contributions from \( \chi_1 \) and \( \chi_2 \). Note that on the support of \( \chi_3 \) we need only consider the contribution of the terms containing \( \hat{f}_{hi}(\xi_3) \); indeed, if \( |\xi_3| \lesssim s^{-\frac{1}{2}} \), then \( \max_j |\xi_j| \lesssim s^{-\frac{1}{2}} \) and we can estimate with volume bounds as we did for (3.5) above. Thus, it suffices to consider the contribution of the term

\[
\int_1^t \int_{\mathbb{R}^2} e^{is\Phi} \chi_3(\vec{\xi}) \hat{f}_{lo}(\xi_1) \hat{f}_{lo}(\xi_2) \hat{f}_{hi}(\xi_3) \, d\sigma \, d\eta \, ds.
\]

(3.7)

In the region of integration in (3.7) we have \( \Phi \neq 0 \), and in particular \( |\Phi| \gtrsim |\vec{\xi}|^2 \sim \xi_3^2 \). We may therefore use the identity \( e^{i\Phi} = (i\Phi)^{-1} \partial_\xi e^{i\Phi} \) and integrate by parts to write

\[
\begin{align*}
(3.7) & = \left[ \int_{\mathbb{R}^2} e^{is\Phi} \chi_3(\vec{\xi}) \hat{f}_{lo}(\xi_1) \hat{f}_{lo}(\xi_2) \hat{f}_{hi}(\xi_3) \, d\sigma \, d\eta \right]_s^t \\
& \quad - \int_1^t \int_{\mathbb{R}^2} \frac{e^{is\Phi}}{i\Phi} \chi_3(\vec{\xi}) \partial_\xi [\phi_s(\xi_1)\phi_s(\xi_2)\phi_{hi}(\xi_3)] \hat{f}(\xi_1) \hat{f}(\xi_2) \hat{f}(\xi_3) \, d\sigma \, d\eta \, ds \\
& \quad - \int_1^t \int_{\mathbb{R}^2} \frac{e^{is\Phi}}{i\Phi} \chi_3(\vec{\xi}) \phi_s(\xi_1) [\partial_\xi \hat{f}(\xi_1)] \hat{f}_{lo}(\xi_2) \hat{f}_{hi}(\xi_3) \, d\sigma \, d\eta \, ds
\end{align*}
\]

(3.8) (3.9) (3.10)
\[- \int_{\mathbb{R}^2} \frac{e^{i\Phi}}{\Phi} \chi_3(\xi) \hat{f}_s(\xi) \hat{\phi}_s(\xi) [\partial_s \hat{f}(\xi)] \hat{f}_{hi}(\xi) \, d\sigma \, d\eta \, ds \tag{3.11} \]
\[- \int_{\mathbb{R}^2} \frac{e^{i\Phi}}{\Phi} \chi_3(\xi) \hat{f}_s(\xi) \hat{\phi}_s(\xi) \hat{\phi}_{hi}(\xi) [\partial_s \hat{f}(\xi)] \, d\sigma \, d\eta \, ds. \tag{3.12} \]

Using the notation from (2.12), we notice that we can write
\[
\| (3.8) \|_{L^\infty_\xi} \lesssim \int_{\mathbb{R}^2} s^{-\frac{1}{2}} \| \hat{f}(s) \|_{L^\infty_\xi}^2 \| u(s) \|_{L^\infty_\xi} \, ds,
\]
where \( m = \chi_3(\xi)(i\Phi)^{-1} \) is a symbol satisfying the hypotheses of Lemma 2.3(i). That is, \( m \) is supported on a region where \( |\xi_3| \gtrsim \max\{|\xi_1|, |\xi_2|\} \), and one can check that \( \xi_3^3 m \) is Coifman–Meyer. We apply (2.15) with \( (3.10) \) to obtain
\[
\| (3.8) \|_{L^\infty_\xi} \lesssim \int_{\mathbb{R}^2} s^{-\frac{1}{2}} \| \hat{f}(s) \|_{L^\infty_\xi}^2 \| u(s) \|_{L^\infty_\xi} \, ds,
\]

In view of Lemma 1.4, this is an acceptable contribution to the right-hand side of (1.9).

We next turn to (3.9). From the definition of the cutoffs \( \phi_{t_0} \) and \( \phi_{hi} \) (cf. (2.10)), we have \( \partial_s \phi_{t_0}(\xi) = \pm (1/2)s^{-1/2} \xi_j \phi'(s^{1/2} \xi_j) \). As multiplication by \( s^{-1/2} \xi_j \phi'(s^{1/2} \xi_j) \) corresponds to a bounded projection to frequencies of size \( \sim s^{-1/2} \), we can write \( \partial_s \phi_{hi}(\xi) \hat{f}(\xi) = s^{-1} \hat{f}_{med}(\xi) \), where \( f_{med} \) denotes such a projection of \( f \). Distributing the derivatives and considering all of the possibilities, one can see that to treat (3.9) it ultimately suffices to show how to estimate a term such as the following:
\[
\int_{\mathbb{R}^2} s^{-1} \frac{e^{i\Phi}}{\Phi} \chi_3(\xi) \hat{f}_s(\xi) \hat{\phi}_s(\xi) \hat{\phi}_{hi}(\xi) d\sigma \, d\eta \, ds. \tag{3.13} \]

This can be estimated as we did above for (3.8), using Lemma 2.3(i):
\[
\| (3.13) \|_{L^\infty_\xi} \lesssim \int_{\mathbb{R}^2} s^{-\frac{1}{2}} \| \hat{f}(s) \|_{L^\infty_\xi}^2 \| u(s) \|_{L^\infty_\xi} \, ds,
\]
which is acceptable in view of (1.8).

We next turn to (3.10). Noting that
\[
e^{i\sigma \partial_{xx}/2} \partial_s f = (\partial_s + \frac{i}{2} \partial_{xx}) u = O(u^3), \tag{3.14}\]
we can use Lemma 2.3(i) with \( (a, b, c) = (u_s, e^{i\sigma \partial_{xx}/2} \partial_s f, \hat{u}_{hi}) \) to get:
\[
\| (3.10) \|_{L^\infty_\xi} \lesssim \int_{\mathbb{R}^2} \| \hat{u}(s) \|_{L^\infty_\xi} \| O(u^3(s)) \|_{L^\infty_\xi} \, ds \lesssim \int_{\mathbb{R}^2} s^{-1} \| u(s) \|_{L^\infty_\xi}^3 \| \hat{f}(s) \|_{L^\infty_\xi}^2 \, ds,
\]
which is acceptable. We can treat (3.11) in the same way, as we can exchange the role of \( a \) and \( b \) in (2.15).

The term (3.12) can be treated similarly, using the second inequality in (2.15) with \( (a, b, c) = (u_s, u_s, \hat{P}_{hi} e^{i\sigma \partial_{xx}/2} \partial_s f) \):
\[
\| (3.12) \|_{L^\infty_\xi} \lesssim \int_{\mathbb{R}^2} s^{-1} \| \hat{f}(s) \|_{L^\infty_\xi}^2 \| O(u^3(s)) \|_{L^\infty_\xi} \, ds,
\]
which is acceptable (cf. (1.8)). This completes the estimation (3.1).
3.2. Estimation of \((3.2)\). As before we write

\[
1 = [\phi_{lo}(\xi_1) + \phi_{hi}(\xi_1)][\phi_{lo}(\xi_2) + \phi_{hi}(\xi_2)][\phi_{lo}(\xi_3) + \phi_{hi}(\xi_3)]
\]

in the integrand of \((3.2)\). We estimate the contribution of \(\phi_{lo}(\xi_1)\phi_{lo}(\xi_2)\phi_{lo}(\xi_3)\) by volume bounds, as in \((3.5)\).

For the remaining terms we once again write \(1 = \chi_1(\vec{\xi}) + \chi_2(\vec{\xi}) + \chi_3(\vec{\xi})\) for \(\vec{\xi} \in \mathbb{R}^3\), where each \(\chi_j\) is a smooth Coifman–Meyer multiplier such that \((3.6)\) holds. We will show how to estimate the contribution of \(\chi_2\). Similar ideas suffice to treat the contribution of \(\chi_1\) and \(\chi_3\) (see Remark \(3.1\) below for more details). We need only consider the contribution of \(\chi_2\) in terms containing \(\hat{f}_{hi}(\xi_2)\), since if \(|\xi_2| \lesssim s^{-\frac{3}{2}}\), then \(\max_j |\xi_j| \lesssim s^{-\frac{1}{2}}\) and we can simply estimate using volume bounds, as we did for the low frequency term.

On the support of \(\chi_2(\vec{\xi})\) we further decompose \(1 = \chi_1(\vec{\xi}) + \chi_2(\vec{\xi}) + \chi_3(\vec{\xi})\) and let \(\chi_{2,*} := \chi_2 \chi_3\) be smooth Coifman–Meyer multipliers such that

\[
|\xi_1 - \xi_2| \geq \frac{1}{100} |\xi_2| \quad \text{for} \quad \vec{\xi} \in \text{support}(\chi_1),
\]

\[
|\xi_3 - \xi_2| \geq \frac{1}{100} |\xi_2| \quad \text{for} \quad \vec{\xi} \in \text{support}(\chi_3),
\]

\[
|\xi_1 - \xi_2| \leq \frac{1}{100} |\xi_2| \quad \text{and} \quad |\xi_3 - \xi_2| \leq \frac{1}{100} |\xi_2| \quad \text{for} \quad \vec{\xi} \in \text{support}(\chi_3).
\]

See Appendix \(A\) for the construction of such multipliers. The subscripts indicate the variable with respect to which we will integrate by parts. According to this decomposition of the frequency space, we are faced with estimating the following three terms:

\[
\int_1^t \int_\mathbb{R}^2 \int_\mathbb{R}^2 e^{is\psi} \chi_{2,\eta}(\vec{\xi}) \hat{f}_{s}(\xi_1) \hat{\phi}_{hi}(\xi_2) \hat{f}_s(\xi_3) d\sigma d\eta ds,
\]

\[
\int_1^t \int_\mathbb{R}^2 \int_\mathbb{R}^2 e^{is\psi} \chi_{2,\sigma}(\vec{\xi}) \hat{f}_{s}(\xi_1) \hat{\phi}_{hi}(\xi_2) \hat{f}_s(\xi_3) d\sigma d\eta ds,
\]

\[
\int_1^t \int_\mathbb{R}^2 \int_\mathbb{R}^2 e^{is\psi} \chi_{2,\sigma}(\vec{\xi}) \hat{f}_{s}(\xi_1) \hat{\phi}_{hi}(\xi_2) \hat{f}_s(\xi_3) d\sigma d\eta ds.
\]

3.2.1. Estimation of \((3.18)\). Using \((2.8)\)–\((2.9)\), we see that on the support of \(\chi_{2,\eta}\) we have

\[
|\partial_\eta \Psi| = |\xi_2 - \xi_1| \gtrsim |\xi_2| \gtrsim |\vec{\xi}|.
\]

Thus we can use the identity \(e^{is\psi} = \partial_\eta e^{is\psi}(i s \partial_\eta \Psi)^{-1}\) and integrate by parts in \(\eta\) to write

\[
\int_1^t \int_\mathbb{R}^2 \int_\mathbb{R}^2 e^{is\psi} \partial_\eta \left( \frac{1}{i s \partial_\eta \Psi} \right) \chi_{2,\eta}(\vec{\xi}) \hat{f}_{s}(\xi_1) \hat{\phi}_{hi}(\xi_2) \hat{f}_s(\xi_3) d\sigma d\eta ds
\]

\[
- \int_1^t \int_\mathbb{R}^2 \int_\mathbb{R}^2 e^{is\psi} \frac{\partial_\eta \chi_{2,\eta}(\vec{\xi}) \hat{f}_{s}(\xi_1) \hat{\phi}_{hi}(\xi_2)}{i s \partial_\eta \Psi} \hat{f}(\xi_1) \hat{f}_s(\xi_3) d\sigma d\eta ds
\]

\[
- \int_1^t \int_\mathbb{R}^2 \int_\mathbb{R}^2 e^{is\psi} \chi_{2,\eta}(\vec{\xi}) \frac{\partial_\eta \hat{f}_{s}(\xi_1) \hat{\phi}_{hi}(\xi_2)}{i s \partial_\eta \Psi} \hat{f}_s(\xi_3) d\sigma d\eta ds
\]

\[
- \int_1^t \int_\mathbb{R}^2 \int_\mathbb{R}^2 e^{is\psi} \chi_{2,\eta}(\vec{\xi}) \hat{f}_{s}(\xi_1) \frac{\partial_\eta \hat{f}_{hi}(\xi_2) \hat{f}_s(\xi_3)}{i s \partial_\eta \Psi} d\sigma d\eta ds.
\]

We first estimate \((3.21)\). Using \((2.9)\), we see that \(\partial_\eta (1/\partial_\eta \Psi) = -2|\xi_2 - \xi_1|^{-2}\). Recalling \((3.21)\) and the fact that \(|\xi_2| \gtrsim \max\{|\xi_1|, |\xi_2|\}\) in the integral above, we can write

\[
\int_1^t (is)^{-1} e^{is|\xi|^2/2} \mathcal{F}(T_m[u_s, u_{hi}, u_s]) (s, \xi) ds,
\]
where the symbol \( m = \partial_\eta (1/\partial_\eta \Psi) \chi_{2,\eta} \) satisfies the hypotheses of Lemma \( 2.3(i) \). Here and throughout Section \( 3.2 \), \( \xi_2 \) plays the role of \( \xi_3 \) in the application of Lemma \( 2.3 \). Applying \( 2.15 \), we obtain an acceptable contribution:

\[
\| (3.22) \|_{L^\infty_x} \lesssim \int_1^t s^{-\frac{1}{2}} \| \hat{f}(s) \|_{L^\infty_x}^2 \| u(s) \|_{L^\infty_x} \ ds.
\]

We next turn to \( (3.23) \). Two types of terms arise, depending on where \( \partial_\eta \) lands on \( \chi_{2,\eta} \). First, if \( \partial_\eta \) lands on \( \chi_{2,\eta} \) we are led to consider the following:

\[
\int_1^t \int_{\mathbb{R}^2} s^{-\frac{1}{2}} e^{is\Phi(\xi)} \frac{\partial_\eta \chi_{2,\eta}(\xi)}{\partial_\eta \Psi} \hat{f}(\xi_1) \hat{f}(\xi_2) \hat{f}(\xi_3) \ ds \ d\eta \ ds.
\]

Second, we note that \( \partial_\eta \Phi(\xi_j) = \pm s \frac{i}{2} \phi'(s \frac{i}{2} \xi_j) \), and that multiplication by \( \phi'(s \frac{i}{2} \cdot) \) corresponds to a projection to frequencies \( \sim s^{-\frac{1}{2}} \). As before, we denote this by \( P_{med} f = f_{med} \). Considering all of the possibilities, one can see that to treat the terms that arise when \( \partial_\eta \) lands on one of the \( \phi_*(\xi_j) \), it suffices to estimate the terms

\[
\int_1^t \int_{\mathbb{R}^2} s^{-\frac{1}{2}} e^{is\Phi(\xi)} \frac{\partial_\eta \chi_{2,\eta}(\xi)}{\partial_\eta \Psi} \hat{f}(\xi_1) \hat{f}(\xi_2) \hat{f}(\xi_3) \ ds \ d\eta \ ds,
\]

\[
\int_1^t \int_{\mathbb{R}^2} s^{-\frac{1}{2}} e^{is\Phi(\xi)} \frac{\partial_\eta \chi_{2,\eta}(\xi)}{\partial_\eta \Psi} \hat{f}_{med}(\xi_1) \hat{f}(\xi_2) \hat{f}(\xi_3) \ ds \ d\eta \ ds.
\]

Thus, to treat \( (3.23) \) it suffices to estimate \( (3.26) - (3.28) \).

For \( (3.26) \) we use \( 3.21 \) and the fact that \( \xi_2 \partial_\eta \chi_{2,\eta}(\xi) \) is Coifman–Meyer to write

\[
(3.26) = \int_1^t \int_{\mathbb{R}^2} s^{-\frac{1}{2}} e^{is\Phi(\xi)} \mathcal{F}(T_m[u_* u_{hi}, u])(s, \xi) \ ds, \]

where \( m \) satisfies the hypotheses of Lemma \( 2.3(i) \). The estimate \( 2.15 \) then gives

\[
\| (3.26) \|_{L^\infty_x} \lesssim \int_1^t s^{-\frac{1}{2}} \| \hat{f}(s) \|_{L^\infty_x}^2 \| u(s) \|_{L^\infty_x} \ ds,
\]

which is acceptable.

Next, recalling once again \( 3.21 \) and \( 3.15 \), we can write

\[
(3.27) = \int_1^t \int_{\mathbb{R}^2} s^{-\frac{1}{2}} e^{is\Phi(\xi)} \mathcal{F}(T_m[u_* u_{med}, u_*])(s, \xi) \ ds, \]

where \( m \) is a symbol that satisfies the hypotheses of Lemma \( 2.3(ii) \); that is, \( \xi_2 | m \) is a Coifman–Meyer symbol supported on a region where \( \xi_2 \) is the largest frequency (up to a constant). We apply \( 2.16 \) and \( 2.14 \) to get the following acceptable estimate:

\[
\| (3.27) \|_{L^\infty_x} \lesssim \int_1^t s^{-\frac{1}{2}} \| u(s) \|_{L^\infty_x} \| u_{med} \|_{L^2_x} \| \hat{f}(s) \|_{L^\infty_x} \ ds \lesssim \int_1^t s^{-\frac{1}{2}} \| u(s) \|_{L^\infty_x} \| \hat{f}(s) \|_{L^\infty_x}^2 \ ds.
\]

As the term \( (3.28) \) can be estimated in the same way, this completes the treatment of \( (3.23) \).

To estimate the term \( (3.24) \) we proceed similarly. We can write

\[
(3.24) = \int_1^t s^{-1} e^{is\Phi(\xi)} \mathcal{F}(T_m[P_* J u, u_{hi}, u_*])(s, \xi) \ ds, \]

where \( m \) is a symbol satisfying the hypotheses of Lemma \( 2.3(ii) \). Using \( 2.16 \), we obtain:

\[
\| (3.24) \|_{L^\infty_x} \lesssim \int_1^t s^{-\frac{1}{2}} \| u(s) \|_{L^\infty_x} \| J u(s) \|_{L^2_x} \| \hat{f}(s) \|_{L^\infty_x} \ ds,
\]
which is an acceptable contribution to the right-hand side of \((1.9)\). The last term \((3.25)\) can be estimated in the same way.

3.2.2. **Estimation of \((3.19)\).** This term is very similar to \((3.18)\). In particular, we note that on the support of \(\chi_{2,s}\) we have \(|\partial_t \Psi| = |\xi_2 - \xi_3| \gtrsim |\xi_2| \gtrsim |\xi_3|\). Thus we can use the identity \(e^{is\Psi} = (is\partial_\sigma \Psi)^{-1}\partial_\sigma e^{is\Psi}\) and integrate by parts in \(\sigma\). The ideas used to estimate \((3.18)\) then suffice to handle the resulting terms.

3.2.3. **Estimation of \((3.20)\).** Using \((2.8)\) and \((3.17)\), we note that on the support of \(\chi_{2,s}\), we have

\[
|\Psi| = |(\xi_1 - \xi_2)\xi_2 + \xi_2(\xi_3 - \xi_2) + \xi_1\xi_3 + 2\xi^2_2| \\
\geq 2|\xi_2|^2 - |\xi_2|(|\xi_1 - \xi_2| + |\xi_3 - \xi_2|) - |\xi_1||\xi_3| \geq |\xi_2|^2 \sim |\xi_1|^2 \sim |\xi_3|^2 \sim |\xi|^2.
\]

(3.29)

We integrate by parts in \(s\) using the identity \(e^{is\Psi} = (i\Psi)^{-1}\partial_s e^{is\Psi}\). This yields

\[
(3.20) = \left[ \int_\mathbb{R}^2 \frac{e^{is\Psi} \chi_{2,s}(\xi)}{i\Psi} \tilde{f}_s(\xi) \tilde{f}_h(\xi) \tilde{f}_s(\xi) \, d\sigma \, d\eta \right]_{s=1}^t \]

(3.30)

\[
- \int_1^t \int_\mathbb{R}^2 \frac{e^{is\Psi} \chi_{2,s}(\xi)}{i\Psi} \partial_s[\phi_s(\xi) \phi_h(\xi) \phi_s(\xi)] \tilde{f}(\xi) \tilde{f}(\xi) \tilde{f}(\xi) \, d\sigma \, d\eta \, ds
\]

(3.31)

\[
- \int_1^t \int_\mathbb{R}^2 \frac{e^{is\Psi} \chi_{2,s}(\xi)}{i\Psi} \partial_s[\tilde{f}(\xi) \tilde{f}(\xi) \tilde{f}(\xi)] \, d\sigma \, d\eta \, ds.
\]

(3.32)

In light of the lower bound \((3.29)\), these terms are similar to those in \((3.8)\)–\((3.12)\).

First, using \((3.29)\), we can write \((3.30)\) as

\[
\left[ e^{is|\xi|^2/2} \mathcal{F}(T_m[u_\eta, u_{hh}, u_\xi]) (s, \xi) \right]_{s=1}^t,
\]

where the symbol \(m = \chi_2(i\Psi)^{-1}\) satisfies the hypotheses of Lemma \((2.3)\) (up to exchanging the role of \(\xi_3\) and \(\xi_2\), as above). The estimate \((2.15)\) (applied, as always, with \(N \sim s^{-1/2}\)) yields

\[
\| (3.30) \|_{L^\infty_s} \lesssim \| u(1) \|_{L^\infty_t} \| \tilde{f}(1) \|_{L^\infty_t} + t^{3/2} \| u(t) \|_{L^\infty_t} \| \tilde{f}(t) \|_{L^\infty_t},
\]

which is an acceptable contribution to the right-hand side of \((1.9)\).

We next turn to \((3.31)\). As observed earlier, we can write \(\partial_s \phi_s(\xi) \tilde{f}(\xi) = s^{-1}\tilde{f}_{med}(\xi)\), where \(f_{med}\) denotes the projection of \(f\) to frequencies \(\sim s^{-1/2}\). Considering all of the possibilities, one can see that to treat \((3.31)\) it ultimately suffices to show how to bound a term such as

\[
\int_1^t \int_\mathbb{R}^2 s^{-1} \frac{e^{is\Psi} \chi_{2,s}(\xi)}{\Psi} \tilde{f}_s(\xi) \tilde{f}_{med}(\xi) \tilde{f}_s(\xi) \, d\sigma \, d\eta \, ds.
\]

(3.33)

We can estimate this term as we did \((3.13)\), using \((2.15)\).

For the term \((3.32)\), we can proceed as we did for \((3.10)\)–\((3.12)\): the hypotheses of Lemma \((2.3)\) hold, and we can use \((2.15)\) and \((3.14)\) to estimate

\[
\| (3.32) \|_{L^\infty_s} \lesssim \int_1^t s^{3/2} \| \tilde{f} \|_{L^\infty_s}^2 \| O(u^3(s)) \|_{L^\infty_s} \, ds,
\]

which is acceptable.

**Remark 3.1.** We have now estimated \((3.18)\)–\((3.20)\). This handles the contribution to \((3.2)\) associated with the cutoff \(\chi_2\), that is, the region where \(|\xi_2| \gtrsim \max\{|\xi_1|,|\xi_3|\}\) (cf. \((3.6)\)). We now briefly discuss how to treat the terms containing \(\chi_1\) or \(\chi_3\).
In estimating the contribution from \( \chi_2 \), the key idea was to decompose frequency space into regions such that at least one of \( |\partial_\eta \Psi| \), \( |\partial_\sigma \Psi| \), or \( |\Psi| \) was suitably bounded below. In the support of \( \chi_1 \), using (2.3), we can achieve such a decomposition as follows:

- First, if \(|\xi_1 - \xi_2| \geq \frac{100}{101}|\xi_1| \) then \( |\partial_\eta \Psi| \gtrsim |\xi_1| \gtrsim |\bar{\xi}| \).
- Next, if \(|\xi_1 - \xi_2| \leq \frac{100}{101}|\xi_1| \) and \(|\xi_3 - \xi_2| \geq \frac{100}{101}|\xi_2| \), then \( |\partial_\sigma \Psi| \gtrsim |\xi_2| \gtrsim |\bar{\xi}| \).
- Finally, if \(|\xi_1 - \xi_2| \leq \frac{100}{101}|\xi_1| \) and \(|\xi_3 - \xi_2| \leq \frac{100}{101}|\xi_2| \) then \( |\Psi| \gtrsim |\bar{\xi}|^2 \gtrsim |\bar{\xi}|^2 \).

Thus, we can use arguments similar to the ones above to handle the contribution of \( \chi_1 \). Similar ideas also suffice to treat the contribution of \( \chi_3 \).

### 3.3. Estimation of (3.3)

We can estimate (3.3) in a very similar manner to (3.2). To wit, we split each function into low and high frequency pieces, and we handle the term containing all low frequencies with volume bounds. For the remaining terms, we decompose frequency space into regions where one of \(|\xi_1|\), \(|\xi_2|\), \(|\xi_3|\) is (almost) the maximum, according to (3.6). On each such region, we decompose into regions where we have suitable lower bounds on either the phase \( \Omega \) or its derivatives. Consider for example the region where \(|\xi_2| \geq \max\{\frac{1}{10}|\xi_1|, \frac{1}{10}|\xi_3|\}\). Then we can define cutoffs \( \chi_\eta, \chi_\sigma \), and \( \chi_s \) so that \( 1 = \chi_\eta + \chi_\sigma + \chi_s \) and

\[
|\xi_2 - \xi_1| \geq \frac{100}{101}|\xi_2| \quad \text{for} \quad \bar{\xi} \in \text{support}(\chi_\eta),
\]

\[
|\xi_2 + \xi_3| \geq \frac{100}{101}|\xi_2| \quad \text{for} \quad \bar{\xi} \in \text{support}(\chi_\sigma),
\]

\[
|\xi_2 - \xi_1| \leq \frac{100}{101}|\xi_2| \quad \text{and} \quad |\xi_2 + \xi_3| \leq \frac{100}{101}|\xi_2| \quad \text{for} \quad \bar{\xi} \in \text{support}(\chi_s).
\]

From the formulas (2.8)–(2.9) it is easy to see that we have suitable lower bounds for \( \partial_\eta \Omega \) and \( \partial_\sigma \Omega \) in the support of \( \chi_\eta \) and \( \chi_\sigma \), respectively. Furthermore, we claim that \( |\Omega| \gtrsim |\xi| \gtrsim |\bar{\xi}|^2 \) for \( \bar{\xi} \) in the support of \( \chi_s \). Indeed, we can write

\[
\Omega = \xi_2^2 + \xi_1(\xi_1 - \xi_2) + \xi_1(\xi_2 + \xi_3) + \xi_2(\xi_1 - \xi_2) + \xi_2(\xi_2 + \xi_3)
\]

and note that in the support of \( \chi_s \), we have

\[
|\xi_1(\xi_1 - \xi_2) + \xi_1(\xi_2 + \xi_3) + \xi_2(\xi_1 - \xi_2) + \xi_2(\xi_2 + \xi_3)| \leq \frac{3}{50}|\xi_2|^2.
\]

Thus, proceeding as in the case of (3.2), we can deal with the term (3.3). As the analysis is quite similar, we omit the details.

### 3.4. Estimation of (3.4)

We can handle the term (3.4) in a relatively simple manner due to the gauge-invariance of \( |u|^2 u \). Using (2.3) we can first rewrite

\[
(3.34) = \int_1^t s^{-1} \mathcal{F} \tilde{M}(s) \mathcal{F}^{-1}(|\mathcal{F}Mf|^2 \mathcal{F}Mf)(s, \xi) ds.
\]

Writing \( \mathcal{F} \tilde{M}(s) \mathcal{F}^{-1} = 1 + \mathcal{F} \tilde{M}(s) - 1 \mathcal{F}^{-1} \), it suffices to estimate the following:

\[
\int_1^t s^{-1}(|\mathcal{F}Mf|^2 \mathcal{F}Mf)(s, \xi) ds, \quad \text{(3.34)}
\]

\[
\int_1^t s^{-1} \mathcal{F} \tilde{M}(s) - 1 \mathcal{F}^{-1}(|\mathcal{F}Mf|^2 \mathcal{F}Mf)(s, \xi) ds. \quad \text{(3.35)}
\]

Arguing as in the proof of Lemma 1.4 (see Section 2.3), we find

\[
\| \mathcal{F} M(s) f(s) \|_{L_\xi^\infty} \lesssim \| u(s) \|_{X(s)}, \quad \text{(3.36)}
\]

and hence \( \| (3.34) \|_{L_\xi^\infty} \lesssim \int_1^t s^{-1} \| u(s) \|_{X(s)}^3 ds \), which is acceptable.
For (3.35), we again argue as in the proof of Lemma 1.4. We split at frequency $\sqrt{s}$, using the operators $\hat{P}_{\leq N} = \mathcal{F}\hat{\phi}(-N)\mathcal{F}^{-1}$. For the high frequencies, we use (2.1), Plancherel, the chain rule, and (2.4) to estimate
\[
\|\hat{P}_{\leq \sqrt{s}}\mathcal{F}[\hat{M}(s) - 1]F^{-1}(\|FMf\|^2FMf)(s)\|_{L^\infty_x} \lesssim s^{-\frac{5}{4}}\|\partial_x\mathcal{F}[\hat{M}(s) - 1]F^{-1}(\|FMf\|^2FMf)(s)\|_{L^4_x} \lesssim s^{-\frac{5}{4}}\|FMf(s)\|_{L^4_x}\|\partial_xFMf(s)\|_{L^2_x} \lesssim \|u(s)\|_X^3.
\]

For the low frequencies we use the pointwise bound (2.11), Plancherel, and (2.4) to estimate
\[
\|\hat{P}_{\leq \sqrt{s}}\mathcal{F}[\hat{M}(s) - 1]F^{-1}(\|FMf\|^2FMf)(s)\|_{L^\infty_x} \lesssim \|\phi(\frac{s}{\sqrt{s}})[\hat{M}(s) - 1]F^{-1}(\|FMf\|^2FMf)(s)\|_{L^4_x} \lesssim s^\frac{5}{4}\|\hat{M}(s) - 1]F^{-1}(\|FMf\|^2FMf)(s)\|_{L^2_x} \lesssim s^{-\frac{5}{4}}\|FMf(s)\|_{L^2_x}\|\partial_xFMf(s)\|_{L^2_x} \lesssim \|u(s)\|_X^3.
\]

Thus $(3.35)$ holds. This completes the proof of Proposition 1.5.

4. Proof of Proposition 1.6

In this section we prove the estimate (1.10) for $\hat{u}(t)$. Equivalently we will estimate $\partial_\xi \hat{f}(t)$ in $L^2_\xi$, cf. (2.4). Using (2.5) and recalling the notation from (2.7) and (2.8), we can write
\[
\partial_\xi \hat{f}(t) = \partial_\xi \hat{f}(1) - i\lambda_1(2\pi)^{-1}\int_1^t \iint \mathcal{F}[\partial_\xi \hat{f}(1)](\xi_1, \xi_2, \xi_3) d\sigma d\eta ds + \iota \lambda_2(2\pi)^{-1}\int_1^t \iint \mathcal{F}[\partial_\xi \hat{f}(1)](\xi_1, \xi_2, \xi_3) d\sigma d\eta ds + \iota \lambda_3(2\pi)^{-1}\int_1^t \iint \mathcal{F}[\partial_\xi \hat{f}(1)](\xi_1, \xi_2, \xi_3) d\sigma d\eta ds
\]

For the terms (4.1), (4.2), and (4.3) we recall that $e^{is\partial_\xi x/2}f = \hat{u}(s)$. Thus
\[
(4.1) + (4.2) + (4.3) \|L^2_\xi \lesssim \int_1^t \|\hat{u}(s)\|_{L^2_x} ds \lesssim \int_1^t \|u(s)\|_{L^\infty_x} ds,
\]

which (in light of Lemma 1.4) is bounded by the right-hand side of (1.10). It remains to estimate (4.4) through (4.7).
4.1. Estimation of (4.4). We recall the notation from (2.10) and write
\[ 1 = [\phi_{i0}(\xi_1) + \phi_{hi}(\xi_1)](\phi_{i0}(\xi_2) + \phi_{hi}(\xi_2))(\phi_{i0}(\xi_3) + \phi_{hi}(\xi_3)) \]
in the integrand of (4.4). We expand the product and encounter two types of terms: (i) 
\( \phi_{i0}(\xi_1)\phi_{i0}(\xi_2)\phi_{i0}(\xi_3), \) (ii) \( \phi(\xi_j)\phi(\xi_k)\phi_{hi}(\xi_l), \) where \( j, k, l \in \{1, 2, 3\} \).

We estimate the contribution of term (i) by volume bounds:
\[
\left\| \int_1^t \int_{\mathbb{R}^d} e^{is\Phi} [s\partial_s \Phi] \hat{f}_{i0}(\xi_1) \hat{f}_{i0}(\xi_2) \hat{f}_{i0}(\xi_3) \, ds \, d\eta \right\|_{L_x^2} \lesssim \int_1^t s^{-\frac{4}{5}} \| \hat{f}(s) \|_{L_{\xi}^2}^2 \, ds, \tag{4.8}
\]
which is acceptable.

We now turn to the terms of type (ii). As in Section 3, we write \( 1 = \chi_1(\tilde{\xi}) + \chi_2(\tilde{\xi}) + \chi_3(\tilde{\xi}) \) for \( \tilde{\xi} \in \mathbb{R}^3 \) so that (3.6) holds, and note that it suffices to show how to estimate
\[
\int_1^t \int_{\mathbb{R}^d} e^{is\Phi} [is\partial_s \Phi] \chi_3(\tilde{\xi}) \hat{f}_s(\xi_1) \hat{f}_s(\xi_2) \hat{f}_{hi}(\xi_3) \, ds \, d\eta \, ds. \tag{4.9}
\]
In the region of integration in (4.9) we have \( \Phi \neq 0; \) in fact, \( |\Phi| \gtrsim |\tilde{\xi}|^2 \). We may therefore use the identity \( e^{is\Phi} = (i\Phi)^{-1} \partial_s e^{is\Phi} \) and integrate by parts to write
\[
\int_1^t \int_{\mathbb{R}^d} e^{is\Phi} [is\partial_s \Phi] \chi_3(\tilde{\xi}) \hat{f}_s(\xi_1) \hat{f}_s(\xi_2) \hat{f}_{hi}(\xi_3) \, ds \, d\eta \, ds = \left[ \int_1^t \int_{\mathbb{R}^d} e^{is\Phi} s\partial_s \Phi \chi_3(\tilde{\xi}) \hat{f}_s(\xi_1) \hat{f}_s(\xi_2) \hat{f}_{hi}(\xi_3) \, ds \, d\eta \right]_{s=1}^t \tag{4.10}
\]
\[
- \int_1^t \int_{\mathbb{R}^d} e^{is\Phi} s\partial_s \Phi \chi_3(\tilde{\xi}) \partial_s [\phi_s(\xi_1)\phi_s(\xi_2)\phi_{hi}(\xi_3)] \hat{f}(\xi_1) \hat{f}(\xi_2) \hat{f}(\xi_3) \, ds \, d\eta \, ds \tag{4.11}
\]
\[
- \int_1^t \int_{\mathbb{R}^d} e^{is\Phi} \partial_s \Phi \chi_3(\tilde{\xi}) \hat{f}_s(\xi_1) \hat{f}_s(\xi_2) \hat{f}_{hi}(\xi_3) \, ds \, d\eta \, ds \tag{4.12}
\]
\[
- \int_1^t \int_{\mathbb{R}^d} e^{is\Phi} s\partial_s \Phi \chi_3(\tilde{\xi}) \phi_s(\xi_1)\phi_s(\xi_2)\phi_{hi}(\xi_3) \partial_s [\hat{f}(\xi_1)\hat{f}(\xi_2)\hat{f}(\xi_3)] \, ds \, d\eta \, ds. \tag{4.13}
\]

We turn to (4.10) and fix \( s \in \{1, t\} \). In the support of the integral we have \( |\xi_3| \gtrsim \max\{|\xi_1|, |\xi_2|\}; \) thus, recalling (2.8) and (2.9), we can write
\[
\int_{\mathbb{R}^d} e^{is\Phi} s\partial_s \Phi \chi_3(\tilde{\xi}) \hat{f}_s(\xi_1) \hat{f}_s(\xi_2) \hat{f}_{hi}(\xi_3) \, ds \, d\eta = e^{is\frac{\xi_3^2}{2}} \mathcal{F}(T_m[u_s, u_s, u_{hi}]) (s, \xi),
\]
where \( m \) is a symbol satisfying the hypotheses of Lemma (2.3)(iii); that is, \( |\xi_3|m \) is a Coifman–Meyer symbol. Applying (2.17) with \( N \sim s^{-1/2} \) as usual, we get
\[
\| (4.10) \|_{L_x^2} \lesssim \| u(1) \|_{L_{\xi}^2}^2 \| \hat{f}(1) \|_{L_{\xi}^\infty} + t^{\frac{5}{2}} \| u(t) \|_{L_{\xi}^2}^2 \| \hat{f}(t) \|_{L_{\xi}^\infty}.
\]
In view of (1.8), this is an acceptable contribution to the right-hand side of (1.10).

To estimate (4.11), we recall that we can write \( \partial_s \phi_s(\xi_j) \hat{f}(\xi_j) = s^{-1} f_{med} \phi(\xi_j), \) where \( f_{med} \) denotes the projection of \( f \) to frequencies \( \sim s^{-1/2} \). Considering all of the possibilities, one can see that to treat (4.11) it ultimately suffices to show how to estimate a term such as
\[
\int_1^t \int_{\mathbb{R}^d} e^{is \Phi} \partial_s \Phi \chi_3(\tilde{\xi}) \hat{f}_s(\xi_1) \hat{f}_s(\xi_2) \hat{f}_{med}(\xi_3) \, ds \, d\eta \, ds. \tag{4.14}
\]
To this end, we write (4.14) = \( \int_1^t e^{is \frac{\xi_3^2}{2}} \mathcal{F}(T_m[u_s, u_s, u_{hi}]) (s, \xi) \, ds \), where \( m \) is a symbol satisfying the hypotheses of Lemma (2.3)(iii). We now estimate using (2.17), as we did for (4.10) above:
\[
\| (4.14) \|_{L_x^2} \lesssim \int_1^t s^{\frac{5}{2}} \| u(s) \|_{L_{\xi}^2}^2 \| \hat{f}(s) \|_{L_{\xi}^\infty} \, ds.
\]
which is an acceptable contribution to the right-hand side of (1.10). Note that (1.12) is a term of the same type and can be estimated similarly. We skip the details.

To estimate (1.13), we once again use the fact that $\partial_t \Psi(\xi)^{-1} \chi_3$ satisfies the hypotheses of Lemma 2.3(iii), together with the identity $e^{is\partial_x}$ of the same type and can be estimated similarly. We skip the details.

Once again we employ the notation $\|\|_{L^2}$, as in (4.8).


de
d\psi
\chi
\xi
\end{array}
\right|_{\xi_1 = \xi_2 = \xi_3 = \tilde{\xi}}
\end{equation}

which is acceptable in light of (1.8).

4.2. Estimation of (4.5). As before we write

\begin{equation}
1 = [\phi_{lo}(\xi_1) + \phi_{hi}(\xi_1)][\phi_{lo}(\xi_2) + \phi_{hi}(\xi_2)][\phi_{lo}(\xi_3) + \phi_{hi}(\xi_3)]
\end{equation}

in the integrand of (4.5). We estimate the contribution of the term $\phi_{lo}(\xi_1)\phi_{lo}(\xi_2)\phi_{lo}(\xi_3)$ by volume bounds, as in (4.8).

For the remaining terms we proceed as we did in Section 3.2 and write $1 = \chi_1(\xi) + \chi_2(\tilde{\xi}) + \chi_3(\tilde{\xi})$ for $\xi \in \mathbb{R}^3$, where each $\chi_j$ is a smooth Coifman–Meyer multiplier such that (3.6) holds. As before, we will show how to estimate the contribution of $\chi_2$; similar ideas suffice to treat the contribution of $\chi_1$ and $\chi_3$ (see Remark 4.1 below). As before, we only need to consider the contribution of $\chi_2$ in terms containing $\hat{f}_{hi}(\xi_2)$, since if $\max_j |\xi_j| \leq s^{-\frac{1}{2}}$, then we can simply estimate by volume bounds, as we did for (4.8).

On the support of $\chi_2(\tilde{\xi})$ we further decompose $1 = \chi_2(\tilde{\xi}) + \chi_\sigma(\tilde{\xi}) + \chi_\tau(\tilde{\xi})$, as we did in Section 3.2 (see (3.15)–(3.17)). Once again we employ the notation $\chi_{2,\tau} = \chi_2\chi_\tau$ and find ourselves faced with estimating the following:

\begin{equation}
\int_1^t \int_{\mathbb{R}^3} e^{is\psi} \chi_{2,\eta}(\xi)[is\partial_x] \hat{f}_{hi}(\xi_2) \hat{f}_{s}(\xi_3) d\sigma d\eta ds,
\end{equation}

4.2.1. Estimation of (4.15). On the support of $\chi_{2,\eta}$ we have

\begin{equation}
|\partial_\eta \xi| = \xi_2 - \xi_1 \geq |\xi_2| \geq |\tilde{\xi}|.
\end{equation}

Thus we can use the identity $e^{is\psi} = (is\partial_\eta \Psi)^{-1}\partial_\eta e^{is\psi}$ and integrate by parts:

\begin{equation}
= -\int_1^t \int_{\mathbb{R}^3} e^{is\psi} \partial_\eta \left[\chi_{2,\eta}(\xi)[is\partial_x] \hat{f}_{hi}(\xi_2) \hat{f}_{s}(\xi_3)\right] d\sigma d\eta ds
\end{equation}

\begin{equation}
-\int_1^t \int_{\mathbb{R}^3} e^{is\psi} \chi_{2,\eta}(\xi)[is\partial_x] \frac{\partial_\eta \left[\chi_{2,\eta}(\xi)[is\partial_x] \hat{f}_{hi}(\xi_2) \hat{f}_{s}(\xi_3)\right]}{\partial_\eta \Psi} d\sigma d\eta ds
\end{equation}

\begin{equation}
-\int_1^t \int_{\mathbb{R}^3} e^{is\psi} \chi_{2,\eta}(\xi)[is\partial_x] \frac{\partial_\eta \left[\chi_{2,\eta}(\xi)[is\partial_x] \hat{f}_{hi}(\xi_2) \hat{f}_{s}(\xi_3)\right]}{\partial_\eta \Psi} d\sigma d\eta ds
\end{equation}

\begin{equation}
-\int_1^t \int_{\mathbb{R}^3} e^{is\psi} \chi_{2,\eta}(\xi)[is\partial_x] \frac{\partial_\eta \left[\chi_{2,\eta}(\xi)[is\partial_x] \hat{f}_{hi}(\xi_2) \hat{f}_{s}(\xi_3)\right]}{\partial_\eta \Psi} d\sigma d\eta ds
\end{equation}

We first consider (4.19). In the support of the integral, we have $|\xi_2| \geq \max\{|\xi_1|, |\xi_3|\}$; thus, recalling (2.8), (2.9), and (4.18), we can write (4.19) = $\int_1^t e^{is\xi_2^2} \mathcal{F}(T_m[u\xi_2, u\xi_2, u\xi_2]) (s, \xi) ds$, where $m = \partial_\eta (\partial_x \Psi/\partial_\eta \Psi)\chi_{2,\eta}$ satisfies the hypotheses of Lemma 2.3(iii). We can then apply (2.17), as
we did for the term \((4.14)\) above, to obtain an acceptable bound. As in Section 3.2, throughout Section 4.2 we apply Lemma 2.3 with \(\xi_2\) playing the role of \(\xi_3\).

We next turn to \((4.20)\). As before, \(\partial_\eta \phi_\xi(\xi_j) = \pm \bar{s}^2 \phi'(s^{\frac{1}{2}} \xi_j)\), and multiplication by \(\phi'(s^{\frac{1}{2}} \cdot)\) corresponds to a projection to frequencies \(\sim s^{-\frac{1}{2}}\), which we denote by \(P_{medf} = f_{med}\). Considering all of the possibilities, one finds that to treat \((4.20)\), it suffices to estimate

\[
\int_1^t \int \int \mathbb{R}^2 e^{is\Psi} \frac{\partial \chi(\xi_2, \eta)}{\partial \eta \Psi} \tilde{f}_s(\xi_1) \tilde{f}_s(\xi_2) \tilde{f}_s(\xi_3) d\sigma d\eta ds, \tag{4.23}
\]

\[
\int_1^t \int \int \mathbb{R}^2 s^{\frac{1}{2}} e^{is\Psi} \frac{\partial \chi(\xi_2, \eta)}{\partial \eta \Psi} \tilde{f}_s(\xi_1) \tilde{f}_s(\xi_2) \tilde{f}_s(\xi_3) d\sigma d\eta ds. \tag{4.24}
\]

Using \((4.18)\) and the fact that \(\xi_2 \partial_\eta \chi(\xi_2, \eta)\) is Coifman–Meyer, we can write

\[
\int_1^t e^{is^2 / 2} F(T_m[u_s, u_{hi}, u_s]) (s, \xi) ds,
\]

where \(m\) satisfies the hypotheses of Lemma 2.3(iii). In particular, this term can be treated as \((4.19)\) above. We next write \((4.24)\) as

\[
\int_1^t s^{\frac{1}{2}} e^{is^2 / 2} F(T_m[u_s, u_{med}, u_s]) (s, \xi) ds,
\]

where \(m\) is a Coifman–Meyer symbol (cf. \((4.18)\)). Using Lemma 2.1 and (2.11), we get an acceptable bound:

\[
\| (4.24) \|_{L^2_x} \lesssim \int_1^t s^{\frac{1}{2}} \| u \|_{L^\infty_x} \| u_{med} \|_{L^2_x} ds \lesssim \int_1^t s^{\frac{1}{2}} \| u \|_{L^\infty_x} \| f \|_{L^2_x} ds.
\]

We next write \((4.21)\) as

\[
\int_1^t e^{is^2 / 2} F(T_m[P_j, u_{hi}, u_s]) (s, \xi) ds,
\]

where \(m = \chi_{\xi_2, \eta} \chi(\xi_2, \eta)\). Recalling \((2.9)\) and \((4.18)\), we see that \(m\) is a Coifman–Meyer; thus, Lemma 2.1 gives

\[
\| (4.21) \|_{L^2_x} \lesssim \int_1^t \| u(s) \|_{L^\infty_x} \| J u(s) \|_{L^2_x} ds,
\]

which is acceptable. As \((4.22)\) can be estimated similarly, we complete the treatment of \((4.15)\).

4.2.2. Estimation of \((4.16)\). This term is very similar to \((4.15)\). In particular, on the support of \(\chi(\xi_2, \eta)\) we have \(|\partial_\eta \Psi| = |\xi_2 - \xi_3| \gtrsim |\xi_2| \gtrsim |\xi|\). Thus we can use the identity \(e^{is\Psi} = (is \partial_\eta \Psi)^{-1} \partial_\eta e^{is\Psi}\) to integrate by parts in \(\sigma\), and the same ideas used to estimate \((4.15)\) then suffice to handle the resulting terms.

4.2.3. Estimation of \((4.17)\). As in \((3.29)\) we note that on the support of \(\chi(\xi_2, \eta)\) we have

\[
|\Psi| \gtrsim |\xi_2|^2 \sim |\xi|^2 \sim |\xi_1|^2 \sim |\xi|^2.
\]

Thus, we can use the identity \(e^{is\Psi} = \partial_\eta e^{is\Psi(i)\Psi^{-1}}\) and integrate by parts in \(s\) to get

\[
\int_1^t \int \int \mathbb{R}^2 e^{is\Psi} \chi_{\xi_2, \eta} \tilde{f}_s(\xi_1) \tilde{f}_s(\xi_2) \tilde{f}_s(\xi_3) d\sigma d\eta ds \tag{4.25}
\]

\[
- \int_1^t \int \int \mathbb{R}^2 s \frac{e^{is\Psi} \chi_{\xi_2, \eta}(\xi)}{\Psi} \tilde{f}_s(\xi_1) \tilde{f}_s(\xi_2) \tilde{f}_s(\xi_3) \tilde{f}_s(\xi_3) d\sigma d\eta ds \tag{4.26}
\]

\[
- \int_1^t \int \int \mathbb{R}^2 e^{is\Psi} \chi_{\xi_2, \eta}(\xi) \tilde{f}_s(\xi_1) \tilde{f}_s(\xi_2) \tilde{f}_s(\xi_3) d\sigma d\eta ds \tag{4.27}
\]

Thanks to the lower bound on \(\Psi\), these terms are similar to the ones in \((4.10)\)–\((4.13)\). In fact, \((4.25)\) can be estimated exactly like the term \((4.10)\). For the terms \((4.26)\), we can argue as in the estimate of \((4.11)\) (see also \((4.14)\)). Furthermore, the term \((4.27)\) is similar to \((4.12)\), while
(4.28) is similar to (4.13). In particular, applying the trilinear estimate (2.17) in each case leads to acceptable contributions.

Remark 4.1. We have estimated (4.15)–(4.17), which completes the estimation of the contribution of $\chi_2$ to (4.5). As in Remark 3.1, we can also decompose the support of $\chi_1$ and $\chi_3$ so that we have suitable lower bounds for $\partial_y \Psi$, $\partial_y \Psi$, or $\Psi$. Thus, we can use similar ideas as above to estimate the contribution of $\chi_1$ and $\chi_3$.

4.3. Estimation of (4.6). We can estimate (4.6) in a very similar manner to (4.5). Once again the heart of matter is to decompose frequency space (away from the origin) into regions where one has suitable lower bounds on either the phase $\Omega$ or its derivatives. See Section 3.3 for a detailed discussion of this decomposition.

4.4. Estimation of $\Delta$. We can handle (4.7) quite simply thanks to the gauge-invariance of $|u|^2 u$. Indeed, using (2.3) we can rewrite

\begin{equation}
\tag{4.7}
(\partial_t + \frac{1}{2} \partial_{xx}) u = i |u|^2 u
\end{equation}

and prove Theorem 1.2. Throughout the section, we suppose $u$ is a solution to (1.4) as in the statement of Theorem 1.2, with $f(t) = \exp(-i t \partial_{xx}) u(t)$. In particular, $\|u_1\|_\Sigma = \varepsilon$, $u$ is defined at least up to time $T_\varepsilon = \exp(\frac{1}{c \varepsilon^2})$ for some $c > 0$, and $u$ satisfies the bounds given in (1.3). We write $T_{\max} \in (T_\varepsilon, \infty]$ for the maximal time of existence of $u$.

The plan is to exhibit growth in time of $|f(t, \xi)|^2$ by comparing it to a (growing) solution to an ODE (cf. (5.5) and (5.3) below). To prove that the ODE accurately models the PDE requires good bounds for the solution. One of the benefits of working with (1.4) is that we can prove a better estimate for the $L^2_x$-norm of $Ju$ than the one given in (1.3). (Recall that the bound in (1.3) holds with an arbitrary cubic nonlinearity.) In particular, we have the following.

Lemma 5.1 (Improved bounds). If $\varepsilon > 0$ is sufficiently small, then

\begin{equation}
\sup_{t \in [1, T_\varepsilon]} \left\{ \|f(t)\|_{L^\infty_x} + t^\frac{1}{4} \|u(t)\|_{L^2_x} + t^{- \frac{1}{4}} \|u(t)\|_{L^2_x} + t^{- \frac{1}{4}} \|Ju(t)\|_{L^2_x} \right\} \lesssim \varepsilon. \tag{5.1}
\end{equation}

Proof. Comparing with (1.3), we need only consider the $L^2_x$-norms.

Direct computation shows $J(t) := x + it \partial_x = M(t)it \partial_x \bar{M}(t)$, where $M(t) = e^{i x^2 / 2 t}$. Thus,

\[ \|J(|u|^2 u)\|_{L^2_x} \sim \|t \partial_x (\bar{M} u^2 \bar{M} u)\|_{L^2_x} \lesssim \|u\|_{L^\infty_x}^2 \|Ju\|_{L^2_x}, \]

and hence by the Duhamel formula and (1.3),

\[ \|Ju(t)\|_{L^2_x} \leq C \varepsilon + |C\varepsilon|^2 \int_1^t \|Ju(s)\|_{L^2_x} \frac{ds}{s}. \]

Thus, by Gronwall’s inequality, we have $\|Ju(t)\|_{L^2_x} \lesssim t (C\varepsilon)^2 \varepsilon$ for $1 \leq t \leq T_\varepsilon$, which suffices if $\varepsilon$ is small enough. The same argument treats the $L^2_x$-norm of $u$. \qed
Next, we prove that we can propagate bounds for $u$ as long as we can control $\hat{f}$ in $L^\infty_\xi$.

**Lemma 5.2** (Propagating bounds). Suppose $T_1 < T_2 < T_{\text{max}}$ and
\[
\|u(T_1)\|_{L^2_2} + \|Ju(T_1)\|_{L^2_2} + \sup_{t \in [T_1, T_2]} \|\hat{f}(t)\|_{L^\infty_\xi} \leq \mu
\]
for some $\mu > 0$. If $\mu$ is sufficiently small, then
\[
\sup_{t \in [T_1, T_2]} \left\{ t^{-\frac{1}{10}} \|u(t)\|_{L^2_2} + t^{-\frac{1}{10}} \|Ju(t)\|_{L^2_2} \right\} \lesssim \mu.
\]

**Proof.** The proof is similar to the arguments above. Define the set
\[
S = \{ t \in [T_1, T_2] : t^{-\frac{1}{10}} \|Ju(t)\|_{L^2_2} < C\mu \}.
\]
By assumption, $T_1 \in S$ for some appropriate choice of $C$. Suppose toward a contradiction that $S \neq [T_1, T_2]$. By continuity, we can find a first time $T \in (T_1, T_2)$ so that
\[
\|Ju(T)\|_{L^2_2} \geq C\mu T^{\frac{1}{10}}.
\]
Using Lemma 1.4, we find that
\[
\sup_{t \in [T_1, T]} t^{\frac{1}{2}} \|u(t)\|_{L^\infty_2} \leq \tilde{C} \cdot C\mu
\]
for some absolute constant $\tilde{C}$. Arguing as in Lemma 5.1 we deduce
\[
\|Ju(T)\|_{L^2_2} \leq C\mu T^{\frac{1}{10}}[\tilde{C} \cdot C\mu]^2
\]
However, this contradicts (5.2) for $\mu$ small enough. Thus $S = [T_1, T_2]$. A similar Gronwall argument yields the bounds for the $L^2$-norm of $u$. \(\square\)

We turn to estimating the size of $\hat{f}$. We define
\[
A(t, \xi) := 2|\hat{f}(t, \xi)|^2
\]
and observe that for each $\xi \in \mathbb{R}$, the function $A(t, \xi)$ satisfies an ODE in $t$. Indeed, rewriting the equation (1.4) as $\partial_t f = e^{-it\partial_{xx}/2}(|u|^2u)$ and using (2.3), we deduce
\[
\partial_t A = t^{-1} A^2 + t^{-1} R,
\]
where the remainder $R$ is given by
\[
R = 4 \text{Re}\{ \mathcal{F}f \left[ (\mathcal{F}M\mathcal{F}^{-1} - 1)\mathcal{F}Mf \mathcal{F}Mf + |\mathcal{F}Mf|^2 \mathcal{F}Mf - |\mathcal{F}f|^2 \mathcal{F}f \right] \}. \tag{5.4}
\]
We expect that as long as $u$ obeys good estimates, the remainder $R$ will decay in time. Thus the behavior of $A$ should be governed by a (growing) solution to the ODE
\[
\partial_t B = t^{-1} B^2. \tag{5.5}
\]

We first consider the issue of controlling the remainder.

**Lemma 5.3** (Controlling the remainder). Suppose $1 \leq T_1 < T_2 < T_{\text{max}}$ and
\[
\sup_{t \in [T_1, T_2]} \left\{ \|\hat{f}(t)\|_{L^\infty_\xi} + t^{-\frac{1}{10}} \|Ju(t)\|_{L^2_2} + t^{-\frac{1}{10}} \|u(t)\|_{L^2_2} \right\} \leq \mu \tag{5.6}
\]
for some $\mu > 0$. Then
\[
\sup_{t \in [T_1, T_2]} t^{\frac{1}{10}} \|R(t)\|_{L^\infty_\xi} \lesssim \mu^4.
\]
Proof. The main ideas appear already in the proof of Lemma 1.4, but we include the details for completeness. First, note the pointwise bound $|M(t) - 1| \lesssim t^{-\delta}|x|^{2\delta}$ for any $0 < \delta \leq \frac{1}{2}$. Taking $\delta = \frac{1}{5}$, this together with Hausdorff–Young, Cauchy–Schwarz, (2.4) and (5.6) implies

$$
\|F[M - 1]f\|_{L^\infty_x} \lesssim t^{-\frac{1}{5}}\|\langle x \rangle f\|_{L^2_x} \lesssim t^{-\frac{1}{5}}\mu.
$$

Using (5.6) we also have $\|FMf\|_{L^\infty_x} \lesssim \mu$.

Estimating as above and using Plancherel, we obtain

$$
\|F[M - 1]F^{-1}(|FMf|^2FMf)\|_{L^\infty_x} \lesssim t^{-\frac{1}{5}}\|\langle x \rangle F^{-1}(|FMf|^2FMf)\|_{L^2_x} \lesssim t^{-\frac{1}{5}}\|FMf\|_{L^2_x}^2\|\langle x \rangle FMf\|_{L^2_x} \lesssim t^{-\frac{1}{5}}\mu^3.
$$

Furthermore,

$$
\|FMf|^2FMf - |\hat{f}|^2\hat{f}\|_{L^\infty_x} \lesssim \mu^2\|F[M - 1]f\|_{L^\infty_x} \lesssim t^{-\frac{1}{5}}\mu^3,
$$

and the result follows. \hfill \square

We next look for a point $\xi_0 \in \mathbb{R}$ such that $A(t, \xi_0) \gtrsim \varepsilon^2$ at $t = T_\varepsilon$. Using (5.3),

$$
\partial_t[A(t)\exp(-\int_1^t A(s)\frac{ds}{s})] = t^{-1}\exp(-\int_1^t A(s)\frac{ds}{s})R(t).
$$

Thus, exploiting $A \geq 0$, Lemma 5.1 and Lemma 5.3 (with $\mu \sim \varepsilon$), we can deduce

$$
A(T_\varepsilon, \xi) \geq A(1, \xi) - \int_1^{T_\varepsilon} \exp\left(-\int_1^s A(\sigma)\frac{ds}{s}\right)R(s)\frac{ds}{s} \geq A(1, \xi) - O(\varepsilon^4).
$$

Recalling (1.5) and taking $\varepsilon$ small, we can find $\xi_0 \in \mathbb{R}$ so that

$$
A_0 := A(T_\varepsilon, \xi_0) \geq \frac{1}{5}\varepsilon^2. \quad (5.7)
$$

We now define $B(t)$ to be the solution to (5.5) that agrees with $A(t, \xi_0)$ at time $t = T_\varepsilon$:

$$
B(t) := \frac{A_0}{1 - A_0 \log(\frac{T_\varepsilon}{A_0})}, \quad B : [T_\varepsilon, T_\varepsilon \exp(\frac{1}{A_0})] \to [A_0, \infty). \quad (5.8)
$$

Note that $B(t) \to \infty$ as $t \to T_\varepsilon \exp(\frac{1}{A_0})$. To show that $B(t)$ is good approximation to $A(t, \xi_0)$ for $t \geq T_\varepsilon$ we introduce

$$
D(t) := A(t, \xi_0) - B(t), \quad D : [T_\varepsilon, T_{\max} \wedge T_\varepsilon \exp(\frac{1}{A_0})] \to \mathbb{R}.
$$

Here and below, $a \wedge b$ denotes $\min\{a, b\}$. We now consider the issue of controlling this difference.

Lemma 5.4 (Controlling the difference). Suppose $T_\varepsilon \leq T \leq T_{\max} \wedge T_\varepsilon \exp(\frac{1}{A_0})$ and

$$
\sup_{[T_\varepsilon, T]} \|\hat{f}(t)\|_{L^\infty_x} \leq \mu \quad (5.9)
$$

for some $0 < \varepsilon \ll \mu \ll 1$. Then

$$
|D(T)| \lesssim \frac{1}{A_0} \cdot (\frac{T}{T_\varepsilon})^{2\mu^2} \cdot \mu^4T_\varepsilon^{-\frac{1}{5}} \cdot B(T).
$$

Proof. Note that $D$ solves

$$
\partial_t D(t) = t^{-1}(A(t, \xi_0) + B(t))D(t) + t^{-1}R(t, \xi_0), \quad D(T_\varepsilon) = 0.
$$

Using the integrating factor

$$
\rho(t) := \int_{T_\varepsilon}^t [A(s, \xi_0) + B(s)]\frac{ds}{s},
$$

we have
we find
\[ D(t) = e^{\varrho(t)} \int_{T_\varepsilon}^t e^{-\varrho(s)} R(s, \xi_0) \frac{ds}{\sigma} = \int_{T_\varepsilon}^t \exp \left( \int_s^t [A(\sigma, \xi_0) + B(\sigma)] \frac{ds}{\sigma} \right) R(s, \xi_0) \frac{ds}{\sigma}. \]

An explicit computation using (5.8) shows
\[ \exp \left( \int_s^T B(\sigma) \frac{ds}{\sigma} \right) \leq B(T) \text{ for all } T_\varepsilon \leq s < T < T_\varepsilon \exp \left( \frac{1}{\varepsilon^2} \right). \]

Using (5.9), we can also estimate
\[ \exp \left( \int_s^T A(\sigma, \xi_0) \frac{ds}{\sigma} \right) \leq \left( \frac{T}{T_\varepsilon} \right)^{2 \mu^2} \text{ for all } T_\varepsilon \leq s < T < T_{\max}. \]

In view of (5.9), Lemma 5.2 and Lemma 5.3 we have
\[ \sup_{T_\varepsilon \leq t \leq T} \| \hat{f}(t) \|_{L^\infty_\xi} \lesssim \mu^4, \text{ whence } \int_{T_\varepsilon}^T |R(s, \xi_0)| \frac{ds}{\sigma} \lesssim T^{-\frac{1}{10}} \mu^4. \]

Combining these estimates yields the desired conclusion. \( \square \)

We now complete the proof of the theorem.

**Proof of Theorem 1.2.** Let \( K \gg \varepsilon^2 \) be a constant to be determined below. We define \( T_K \) to be the time such that \( B(T_K) = 4K \):
\[ T_K = T_\varepsilon \exp \left( \frac{1}{A_0} - \frac{K}{T_\varepsilon} \right) \leq T_\varepsilon \exp \left( \frac{5}{\varepsilon^2} - \frac{K}{T_\varepsilon} \right). \]

(5.10)

If \( T_{\max} \leq T_K \), the conclusion of the theorem holds. Thus it remains to consider the case \( T_{\max} > T_K \), in which case it suffices to show
\[ \| \hat{f}(t) \|_{L^\infty_\xi} \geq K^{\frac{1}{2}} \text{ for some } t \in [T_\varepsilon, T_K]. \]

We proceed by contradiction and suppose that
\[ \sup_{t \in [T_\varepsilon, T_K]} \| \hat{f}(t) \|_{L^\infty_\xi} \leq K^{\frac{1}{2}}. \]

(5.11)

Applying Lemma 5.3 (with \( \mu = K^{\frac{1}{2}} \)), we deduce that
\[ |D(T_K)| \lesssim \frac{1}{A_0} \cdot \left( \frac{T_K}{T_\varepsilon} \right)^{2K} \cdot K^2 T^{-\frac{1}{10}} B(T_K). \]

Recalling \( A_0 \geq \frac{1}{\varepsilon^2} \) and (5.10) and rearranging, we find
\[ |D(T_K)| \lesssim \frac{1}{\varepsilon^2} \exp \left( - \left[ \frac{15K}{10} - T_{\varepsilon} \right] \frac{1}{2} \right) K^2 B(T_K). \]

We now choose \( K = \frac{1}{200 \varepsilon^2 c} \), so that the above becomes
\[ |D(T_K)| \lesssim \frac{1}{\varepsilon^2} \exp \left( - \frac{1}{200 \varepsilon^2 c} \right) B(T_K). \]

For \( \varepsilon \) sufficiently small (depending only on the absolute constant \( c \)), this yields
\[ |D(T_K)| < \frac{1}{2} B(T_K), \text{ whence } |\hat{f}(T_K, \xi_0)|^2 = \frac{1}{2} A(T_K, \xi_0) > \frac{1}{4} B(T_K) = K. \]

This contradicts (5.11) and completes the proof of Theorem 1.2. \( \square \)
Appendix A. Construction of cutoffs

In this section we construct the cutoff functions used in Sections 3 and 4. Recall the notation from (2.7), and let us first describe how to write \( 1 = \chi_1 + \chi_2 + \chi_3 \) as in (3.6), that is, in such a way that

\[
|\xi_j| \geq \max \left\{ \frac{a_j}{10} |\xi_k|, \ k = 1, 2, 3 \right\} \quad \text{for all} \quad \xi \in \text{support}(\chi_j).
\]

We let \( a \) denote a smooth even function such that \( a(x) = 1 \) for \( |x| \leq 1 \) and \( a(x) = 0 \) for \( |x| > 1 + \delta \) for some small \( \delta > 0 \). Denoting \( a' = 1 - a \), which is a function supported on \( |x| \geq 1 \), we let

\[
\chi_1(\xi) := a'(\frac{\xi}{\xi_2})a'(\frac{\xi}{\xi_3}), \quad \chi_2(\xi) := a(\frac{\xi}{\xi_2})a(\frac{\xi}{\xi_3}), \quad \chi_3(\xi) := a'(\frac{\xi}{\xi_2})a(\frac{\xi}{\xi_3}) + a(\frac{\xi}{\xi_2})a(\frac{\xi}{\xi_3}).
\]

It is clear that these satisfy the desired property (A.1). Furthermore, as the derivatives of \( a \) are supported near \( |x| = 1 \), all of the multipliers appearing above are Coifman–Meyer multipliers satisfying (2.13). Also notice that \( \xi_j \partial_\eta \chi_j(\xi) \) and \( \xi_j \partial_\sigma \chi_j(\xi) \), \( j = 1, 2, 3 \), are Coifman–Meyer multipliers as well.

Next we describe how to write \( 1 = \chi_\eta + \chi_\sigma + \chi_\delta \) as in (3.15)–(3.17) on the support of \( \chi_2 \). We let \( b \) be a smooth even function such that \( b(x) = 1 \) for \( |x| \leq \frac{1}{100} \) and \( b(x) = 0 \) for \( |x| > \frac{1}{100} \), and define

\[
\chi_\eta = 1 - b(\frac{\xi_2 - \xi_3}{\xi_2}), \quad \chi_\sigma = b(\frac{\xi_2 - \xi_3}{\xi_2}) \left[ 1 - b(\frac{\xi_2 - \xi_3}{\xi_2}) \right], \quad \chi_\delta = b(\frac{\xi_2 - \xi_3}{\xi_2})b(\frac{\xi_2 - \xi_3}{\xi_2}).
\]

Then the inequalities (3.15)–(3.17) clearly hold, and furthermore one can check that the functions \( \chi_2 \chi_\eta \) define Coifman–Meyer multipliers. Finally notice that in view of the support properties of \( \chi_2 \), the multipliers \( \xi_2 \partial_\eta [\chi_2 \chi_\eta(\xi)] \) and \( \xi_2 \partial_\sigma [\chi_2 \chi_\eta(\xi)] \) are Coifman–Meyer, as well.

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