The Radial Part of Brownian Motion with respect to $\mathcal{L}$-Distance under Ricci Flow

Li-Juan Cheng *

Abstract

Let $\{g_t\}_{t \in [0,T]}$ be a family of complete time-depending Riemannian metrics on a manifold which evolves under backwards Ricci flow. The Itô formula is established for the $\mathcal{L}$-distance of the $g_t$-Brownian motion to a fixed reference point ($\mathcal{L}$-base). Furthermore, as an application, we construct a coupling by parallel displacement which yields a new proof of some results of Topping.

Keywords: Ricci flow, $\mathcal{L}$-functional, $\mathcal{L}$-cut-locus, $g_t$-Brownian motion, coupling

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1 Introduction and main result

Let $M$ be a $d$-dimensional differentiable manifold carrying a complete backwards Ricci flow $\{g_\tau\}_{\tau \in [0,T]}$, $0 < T \leq \infty$, i.e. a smooth family of Riemannian metrics solving the nonlinear PDE

$$\frac{\partial g_\tau}{\partial \tau} = 2\text{Ric}_\tau,$$

such that $(M, g_\tau)$ is complete for all $\tau \in [0,T)$, where Ric$_\tau$ is the Ricci curvature induced by the metric $g_\tau$. According to Perelman [9], for $0 \leq \tau_1 < \tau_2 < T$, Perelman’s $\mathcal{L}$-length of a differentiable path $\gamma : [\tau_1, \tau_2] \to M$ is then defined by

$$\mathcal{L}(\gamma) := \int_{\tau_1}^{\tau_2} \sqrt{\tau} \left[ |\dot{\gamma}(\tau)|_\tau^2 + R(\tau, \gamma(\tau), \tau) \right] \, d\tau,$$

where $R(x, \tau)$ is the scalar curvature at $x \in M$ w.r.t. the metric $g_\tau$. Define the $\mathcal{L}$-distance between two points $(x, \tau_1)$ and $(y, \tau_2)$ by

$$Q(x, \tau_1; y, \tau_2) = \inf \{ \mathcal{L}(\gamma) | \gamma : [\tau_1, \tau_2] \to M \text{ is smooth and } \gamma(\tau_1) = x, \gamma(\tau_2) = y \}.$$ 

Note that the $\mathcal{L}$-distance can be negative, and it is in general not a real distance. But it reduces to the Riemannian distance in the sense that

$$\lim_{\tau_2 \downarrow \tau_1} 2(\sqrt{\tau_2} - \sqrt{\tau_1})Q(x, \tau_1; y, \tau_2) = \rho_{\tau_1}^2(x, y),$$

where $\rho_{\tau_1}$ is the Riemannian distance with respect to $g_{\tau_1}$.

In this paper, we want to use the comparison theorem to analyze the behavior of the $g_t$-Brownian motion. Let $\nabla^t$ and $\Delta_t$ be the Levi-Civita connection and the Laplace operator associated with the metric $g_t$ respectively. Let $\mathcal{F}(M)$ (resp. $\mathcal{O}(M)$) be the (resp. $g_t$-orthonormal)

*School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, The People’s Republic of China. E-mail: chenglj@mail.bnu.edu.cn (L.J. Cheng)
frame bundle. Let \( p : \mathcal{F}(M) \to M \) be the canonical projection. Set \((e_i)_{i=1}^d\) be the orthonormal basis on \( \mathbb{R}^d \). For \( t \in [0,T] \) and \( u \in \mathcal{O}_t(M) \), let \( H_i(t,u) \) be the \( \nabla^t \)-horizontal lift of \( ue_i \) and \((V_{\alpha\beta}(u))_{\alpha\beta=1}^d\) the canonical vertical vector fields. Let \((B_t)_{t\geq 0}\) be a standard \( \mathbb{R}^d \)-valued Brownian motion on a complete filtered probability space \((\Omega,\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})\). In this situation, Arnaudon, Coulibaly and Thalmaier \( [1] \) constructed the horizontal Brownian motion on \( \mathcal{F}(M) \) by solving the following Stratonovich SDE

\[
\begin{align*}
    du_t &= \sqrt{2} \sum_{i=1}^d H_i(t,u_t) \circ dB_t^i - \frac{1}{2} \sum_{\alpha,\beta=1}^d G_{\alpha\beta}(t,u_t)V_{\alpha\beta}(u_t)dt, \\
    u_{s_0} \in \mathcal{O}_{s_0}(M), \ p u_{s_0} &= x,
\end{align*}
\]

where \( G_{\alpha\beta}(t,u_t) := \partial_t g_t(u_t e_{\alpha}, u_t e_{\beta}), \ alpha, \beta = 1,2, \cdots, d \). They have shown that the drift term in the equality is essential to ensure \( u_t \in \mathcal{O}_t(M) \) for all \( t \in [0,T] \). Moreover, this process is non-explosive up to \( T \) since \( g_t \) is the complete backward Ricci flow (see \cite[Theorem 1]{[7]}). The \( g_t \)-Brownian motion is then defined by \( X_t = pu_t \). For a given reference point \((L\text{-base}) (o,0)\), \( o \in M \), define the radius function

\[
Q(x,t) := Q(o,0;x,t), \quad \text{for } x \in M
\]

as the \( L \)-distance between \((x,t)\) and \((o,0)\). If \( Q \) is smooth, then by the Itô formula, we have

\[
Q(X_t,s) = Q(x,s_0) + \sqrt{2} \int_{s_0}^t \langle \nabla^s Q(X_s), u_s dB_s \rangle_s + \int_{s_0}^t [\Delta_s Q + \partial_s Q](X_s,s)ds, \quad t \geq s_0,
\]

where \( \langle \cdot , \cdot \rangle_s := g_s(\cdot , \cdot) \). However, in general, \( Q \) is not smooth on whole manifold, so that it is even not clear whether \( Q(X_t,t) \) is a semimartingale. The purpose of this paper is to prove that \( Q(X_t,t) \) is indeed a semimartingale and establish the Itô formula for it.

We would like to indicate that when the metric is independent of \( t \), the semimartingale property for the radial part of the Brownian motion w.r.t. Riemannian distance was first proved by Kendall \cite{[6]}, which is fundamental to analyze the Brownian motion on a Riemannian manifold. Especially, Kendall’s Itô formula was applied to the construction of coupling processes on manifolds (see \cite[Chapter 2]{[14]}). For the time-inhomogeneous case, Kuwada and Philipowski \cite{[7]} shows that the radial part of Brownian motion \( \rho_t(o,X_t) \), the Riemannian distance from \( o \) to \( X_t \) w.r.t. \( g_t \), is a semimartingale, which is applied to the non-explosion of the \( g_t \)-Brownian motion. See \cite{[5]} for more discussions in this direction.

By using an approximation approach to the \( L \)-cut-locus, we are able to extend Kendall’s Itô formula to the \( g_t \)-Brownian motion as follows.

**Theorem 1.1.** Let \( X_t \) be a \( g_t \)-Brownian motion starting at time \( s_0 \in (0,T) \). Then there exists a non-decreasing continuous process \( L \) which increases only when \((X_t,t) \in L\text{Cut}((o,0)) \) such that

\[
dQ(X_t,t) = \sqrt{2} \langle \nabla^t Q(X_t,t) , dB_t \rangle_t + [\Delta_t Q + \partial_t Q](X_t,t)dt - dL_t, \quad t \in (s_0,T),
\]

where \( \nabla^t Q(\cdot , t) \) and \( \Delta_t Q(\cdot , t) \) are defined to be zero where \( Q(\cdot , t) \) fails to be differentiable. In particular, \( Q(X_t,t) \) is semimartingale.
As an application, we will construct a coupling of $g_{t_1}$- and $g_{t_2}$-Brownian motions by parallel displacement, where $0 < t_1 \leq t_2 < T$ and $0 < s \leq t < T/t_2$. It is well-known that the coupling method is a useful tool both in stochastic analysis and geometric analysis. We will use this tool to obtain the martingale property of $Q(X_{t_1 t}, \tau_1 t; \bar{X}_{t_2 t}, \bar{\tau}_2 t)$. We would like to point out that very recently, Kuwada and Philipowski [8] constructed a coupling via approximation by geodesic random work, and applied it to proving the monotonicity of the normalized $\mathcal{L}$-transportation cost between solutions of the heat equation. Here, we present an alternative construction such that a large number of estimates presented in [8] are avoided. When $g_t$ is independent of $t$, our construction is due to Wang [14].

The rest parts of the paper is organized as follows. In Section 2, we introduce the $\mathcal{L}$-cut-locus, and some properties of it. In Section 3, we prove Theorem 1.1. In the final section, we construct a coupling of $g_{t_1}$- and $g_{t_2}$-Brownian motions by parallel displacement, which leads to a new proof of the normalized $\mathcal{L}$-transportation cost inequality introduced by Topping [13].

For readers’ convenience, we will take the same notations as in [13].

## 2 Definition and properties of $\mathcal{L}$-cut-locus

Recall that $\{g_t\}_{t \in [0, T)}$ is a complete backwards Ricci flow. Let

$$\mathcal{Y} = \{(x, \tau_1; y, \tau_2) \mid x, y \in M \text{ and } 0 \leq \tau_1 < \tau_2 < T\}.$$ 

Similar to the Riemannian distance, in general, $Q$ fails to be smooth on some subset $\mathcal{LCut}$ defined as follows. For $\tau_1, \tau_2 \in [0, T)$ with $\tau_1 < \tau_2$, $x \in M$ and $Z \in T_x M$, we define the $\mathcal{L}$-exponential map $\mathcal{L}_{\tau_1, \tau_2} \exp_x : T_x M \to M$ by $\mathcal{L}_{\tau_1, \tau_2} \exp_x (Z) = \gamma (\tau_2)$, where $\gamma$ is a unique $\mathcal{L}$-geodesic (see Remark 4.2) starting from $x$ at time $\tau_1$ with the initial condition $\lim_{\tau \downarrow \tau_1} \sqrt{\gamma '} (\tau) = Z$. Note that the $\mathcal{L}$-geodesic also induces a notation of $\mathcal{L}$-Jacobi fields (see e.g. [4 Chapter 7]). It is convenient to define

$$\Omega (x, \tau_1; \tau_2) = \left\{ Z \in T_x M \left| \begin{array}{c}
\gamma : [\tau_1, \tau_2] \to M \text{ defined by } \gamma (\tau) = \mathcal{L}_{\tau_1, \tau} \exp_x (Z) \\
is a unique minimising $\mathcal{L}$-geodesic
\end{array} \right. \right\}.$$ 

Let

$$\Omega^* (x, \tau_1; T) := \bigcap_{\tau_2 \in (\tau_1, T)} \Omega (x, \tau_1; \tau_2).$$ 

For any $Z \in T_x M \setminus \Omega^* (x, \tau_1; T)$, let $\bar{\tau} (x, \tau_1; Z) = \sup \{ \tau \in (\tau_1, T) \mid Z \in \Omega (x, \tau_1; \tau) \}$. Then, the $\mathcal{L}$-cut-locus is defined as follows:

$$\mathcal{LCut} = \left\{ (x, \tau_1; y, \tau') \left| \begin{array}{c}
x \in M, \tau_1 \in [0, T) ; \\
y = \mathcal{L}_{\tau_1, \tau'} \exp_{x} (Z) \text{ for some } Z \in T_x M \setminus \Omega^* (x, \tau_1; T) ; \\
\tau' = \bar{\tau} (x, \tau_1; Z) \in [\tau_1, T)\end{array} \right. \right\}.$$ 

Let

$$\mathcal{LCut} ((x, \tau_1)) = \{(y, \tau_2) \in M \times (\tau_1, T) \mid (x, \tau_1; y, \tau_2) \in \mathcal{LCut} \}.$$ 

The set $\mathcal{LCut}$ can be decomposed into two parts: the first consists of points $(x, \tau_1; y, \tau_2)$ such that there exists more than one minimizing $\mathcal{L}$-geodesic $\gamma : [\tau_1, \tau_2] \to M$ with $\gamma (\tau_1) = x$ and $\gamma (\tau_2) = y$, ...
and the second is the set of points \((x, \tau_1; y, \tau_2)\) such that \(y\) is conjugate to \(x\) (with respect to \(\mathcal{L}\)-Jacobi fields) along a minimizing \(\mathcal{L}\)-geodesic \(\gamma : [\tau_1, \tau_2] \to M\) with \(\gamma(\tau_1) = x, \gamma(\tau_2) = y\).

The following important properties about the \(\mathcal{L}\)-cut-locus and \(Q\) can be found in [4] Lemma 7.27 and [15], Lemma 2.14.

**Proposition 2.1.** (1) The two sets \(\mathcal{L}\text{Cut} \text{ and } \mathcal{L}\text{Cut}((o,0))\) are closed of measure zero in \(\Upsilon\) and \(M \times [0,T]\) respectively. Moreover, for any \(t \in [0,T]\), the set

\[
\mathcal{L}\text{Cut}_t(o) := \{x \in M : (x,t) \in \mathcal{L}\text{Cut}((o,0))\}
\]

is of measure zero in \(M\).

(2) The function \(Q\) is smooth on \(\Upsilon \setminus \mathcal{L}\text{Cut}\).

(3) If we associate to each point \((x, \tau_1; y, \tau_2)\) in \(\Upsilon \setminus \mathcal{L}\text{Cut}\) the vector \(Z \in \Omega(x, \tau_1; \tau_2) \in T_x M\) for which \(\mathcal{L}_{\tau_1,\tau_2}\exp_z(Z) = y\), then \(Z\) depends smoothly on \((x, \tau_1, y, \tau_2)\).

(4) On \(\Upsilon \setminus \mathcal{L}\text{Cut}\), we have

\[
\frac{\partial Q}{\partial \tau_1}(x, \tau_1; y, \tau_2) = \sqrt{r_1} (|\dot{\gamma}(\tau_1)|_{\gamma_1}^2 - R(x, \tau_1)); \nabla_{\tau_1} Q(x, \tau_1; y, \tau_2) = -2\sqrt{r_1}\dot{\gamma}(\tau_1);
\]

\[
\frac{\partial Q}{\partial \tau_2}(x, \tau_1; y, \tau_2) = \sqrt{r_2} (|\dot{\gamma}(\tau_2)|_{\gamma_2}^2 - R(x, \tau_2)); \nabla_{\tau_2} Q(x, \tau_1; y, \tau_2) = 2\sqrt{r_2}\dot{\gamma}(\tau_2),
\]

where \(\gamma : [\tau_1, \tau_2] \to M\) is the minimizing \(\mathcal{L}\)-geodesic from \(x\) to \(y\) and \(\nabla_{\tau_1}^R\) (resp. \(\nabla_{\tau_2}^R\)) denotes the gradient with respect to the variable \(x\) (resp. the variable \(y\)) by using the metric \(g_{\tau_1}\) (resp. \(g_{\tau_2}\)).

Since \((X_t)\) is generated by a non-degenerated operator, the following is a direct consequence of Proposition 2.1(1).

**Lemma 2.2.** Suppose \(X_t\) is a \(g_t\)-Brownian motion starting at time \(s_0 \in (0,T)\). The set \(\{t \in [s_0,T] \mid (X_t, t) \in \mathcal{L}\text{Cut}((o,0))\}\) has Lebesgue measure zero almost surely.

**Proof.** According to [7], Lemma 2, for any starting point \(x \in M\), the law of \(X_t\) under \(\mathbb{P}^x\) is absolutely continuous with respect to the \(g_t\)-Riemannian volume measure. Moreover, by Proposition 2.1(1), \(\mathcal{L}\text{Cut}_t(o)\) measures zero, we have

\[
\mathbb{E}^x \left[ \int_{s_0}^T 1_{\{(X_t, t) \in \mathcal{L}\text{Cut}(o,0))\}} dt \right] = \int_{s_0}^T \mathbb{P}^x((X_t, t) \in \mathcal{L}\text{Cut}(o,0)) dt = \int_{s_0}^T \mathbb{P}^x(X_t \in \mathcal{L}\text{Cut}_t(o)) dt = 0,
\]

it follows that \(\int_{s_0}^T 1_{\{(X_t, t) \in \mathcal{L}\text{Cut}(o,0))\}} dt = 0\), a.s.

\(\square\)

### 3 Proof of Theorem 1.1

Since \(X_t\) is non-explosive before the life time of the metric family, by a localization argument, it is sufficient to consider the case of compact \(M\). Thus, in this section, we assume that \(M\) is compact and \([0,T]\) is a finite interval. We first state the Itô formula for smooth functions.
Lemma 3.1. Suppose $X_t$ is a $g_t$-Brownian motion starting at time $s_0 \in (0,T)$. Let $f$ be a smooth function on $M \times [s_0,T]$. Then,

$$df(X_t,t) = \frac{\partial f}{\partial t}(X_t,t)dt + \Delta_t f(X_t,t)dt + \sqrt{2} \sum_{i=1}^{d} u_i e_i f(X_i,t)dB_i^t, \quad s_0 < t \leq T.$$

According to Proposition 2.1(2) and Lemma 3.1, we see that, if $(X_t,t)$ stays away from the $\mathcal{L}$-cut-locus of $(o,0)$, then

$$dQ(X_t,t) = d\beta_t + \left[ \Delta_t Q + \frac{\partial Q}{\partial t} \right] (X_t,t)dt, \quad t \in [s_0,T],$$

(3.1)

where $\beta_t$ is the martingale term given by

$$d\beta_t := \sqrt{2} \sum_{i=1}^{d} (u_i e_i)Q(X_i,t)dB_i^t.$$

By Proposition 2.1(4), the quadratic variation of the martingale $\beta_t$ is computed as follows

$$d\langle \beta \rangle_t = 2 \sum_{i=1}^{d} [(u_i e_i)Q(X_i,t)]^2 dt = 2|\nabla_1^t Q(X_i,t)|^2 dt = 8t |\dot{\gamma}(t)|^2 dt,$$

where $\gamma : [0,t] \to M$ is the minimal $\mathcal{L}$-geodesic from $o$ to $x_t$ and $|\cdot|_t := \sqrt{\langle \cdot, \gamma \rangle_t}$. Thus, $\beta_t$ can be represented by

$$2\sqrt{2t} |\dot{\gamma}(t)|_t dt,$$

(3.2)

where $b_t$ is a standard one-dimensional Brownian motion. We will explain in Remark 4.2 that the coefficient $2\sqrt{2t} |\dot{\gamma}(t)|_t$ is not a constant, which is different from the fixed metric case.

Next, to control the drift term of (3.1), we need the comparison theorem, which is a combination of the following two lemmas (see [4] Lemma 7.45 and [3] Lemma 7.13).

Lemma 3.2 ([4]). For $0 \leq \tau_1 < \tau_2 \leq T$, let $\gamma : [\tau_1, \tau_2] \to M$ be a minimal $\mathcal{L}$-geodesic from $p$ to $q$. At $(q,\tau_2)$ the $\mathcal{L}$-distance satisfies

$$\frac{\partial}{\partial \tau_2} Q(p, \tau_1; q, \tau_2) + \Delta_{\tau_2} Q(p, \tau_1; \tau_2) (q) \leq \frac{d}{\sqrt{\tau_2 - \sqrt{\tau_1}}} - \frac{1}{2(\tau_2 - \tau_1)} Q(p, \tau_1; q, \tau_2).$$

Since $M$ is compact, there exists some constant $C_0 < \infty$ such that

$$\max_{(x,\tau) \in M \times [0,T]} |\text{Rm}_\tau|(x) \vee |\text{Ric}_\tau|(x) \leq C_0.$$  

(3.3)

We obtain the lower bound for $Q$ from (3.3).

Lemma 3.3 ([4]). For $0 \leq \tau_1 < \tau_2 \leq T$, let $\gamma : [\tau_1, \tau_2] \to M$ be a minimal $\mathcal{L}$-geodesic. Then,

$$Q(\gamma(\tau_1), \tau_1; \gamma(\tau_2), \tau_2) \geq e^{-2C_0(\tau_2 - \tau_1)} \rho_{\tau_1}(\gamma(\tau_1), \gamma(\tau_2))^2 - \frac{2}{3} dC_0(\tau_2^{3/2} - \tau_1^{3/2}).$$
By Lemmas 3.2 and 3.3 we have, for \((x, t) \not\in \mathcal{LCut}(o, 0)\),
\[
\frac{\partial}{\partial t} Q(x, t) + \Delta_t Q(x, t) \leq \frac{d}{\sqrt{t}} - \frac{1}{2t} Q(x, t) \leq \frac{d}{\sqrt{t}} + \frac{1}{2t} Q(x, t) - \frac{d}{\sqrt{t}} + \frac{dC_0}{3t^{3/2}}.
\]
Define \(V(t) := \frac{4d}{\sqrt{t}} + \frac{dC_0}{2t^{3/2}}\). It is easy to see that
\[
\frac{\partial}{\partial t} Q(x, t) + \Delta_t Q(x, t) \leq V(t).
\]

(3.4)

Now, we turn to construct a closed set such that it is disjoint with \(\mathcal{LCut}\). To this end, we consider the product manifold \(M \times [0, T]\) equipped with metric \(\bar{g}\): for \(x \in M\) and \(t \in [0, T]\),
\[
\bar{g}(X, Y)(x, t) := g_t(X, Y); \quad \bar{g}\left(\frac{d}{dt}, \frac{d}{dt}\right) := 1, \quad X, Y \in T_x M.
\]
Given a path \(\gamma : [\tau_1, \tau_2] \to M\) with \([\tau_1, \tau_2] \subset [0, T]\) and \(\gamma(\tau_1) = x, \gamma(\tau_2) = y\). The length of the graph \(\tilde{\gamma} : [\tau_1, \tau_2] \to M \times [0, T]\), defined by \(\tilde{\gamma}(\tau) := (\gamma(\tau), \tau)\), is given by
\[
L_{\bar{g}}(\tilde{\gamma}) = \int_{\tau_1}^{\tau_2} \sqrt{\frac{d\tilde{\gamma}}{d\tau}}^2 + 1 \, d\tau,
\]
where \(|\cdot|_{\bar{g}}\) is the norm w.r.t. the metric \(\bar{g}\). Then the distance between \((x, \tau_1)\) and \((y, \tau_2)\) can be defined as before, namely,
\[
d_{\bar{g}}(x, \tau_1; y, \tau_2) := \inf_{\tilde{\gamma}} L_{\bar{g}}(\tilde{\gamma}).
\]
Let us define a set \(A\) by
\[
A = \left\{(y, \tau_1; z, \tau_2) \in \Upsilon \mid \begin{array}{l}
y, z \in M, \tau_2 \in [s_0, T); \\
\tau_1 = \tau_2/2; \\
Q(y, \tau_1; z, \tau_2) + Q(y, \tau_1) = Q(z, \tau_2)
\end{array} \right\}.
\]
Note that \(A\) is closed and hence compact since \(Q(y, \tau_1; z, \tau_2)\) is continuous in \((y, \tau_1, z, \tau_2)\). Moreover, for any \((y, \tau_1; z, \tau_2) \in A\), the point \((y, \tau_1)\) is on a minimal \(\mathcal{L}\)-geodesic joining \((o, 0)\) and \((z, \tau_2)\). In particular, the symmetry of the \(\mathcal{L}\)-cut-locus implies that
\[
A \cap \mathcal{LCut} = \emptyset.
\]
Combining this with Proposition 2.1 (1), we obtain
\[
\delta_1 := d_{\bar{g}} \otimes d_{\bar{g}}(A, \mathcal{LCut}) > 0,
\]
where \(d_{\bar{g}} \otimes d_{\bar{g}}\) is a metric on \(\Upsilon^2\).

The following lemma is essential to the proof of Theorem 1.1

**Lemma 3.4.** Let \((x_0, t_0) \in \mathcal{LCut}(o, 0)\) and \(\delta \in (0, \delta_1)\). Let \((X_t)\) be the \(g_t\)-Brownian motion starting from \(x_0\) at time \(t_0\). Let \(\bar{T} = T \wedge \inf\{t \geq t_0 | d_{\bar{g}}(x_0, t_0; X_t, t) = \delta\}\). Then
\[
\mathbb{E}\left[Q(X_{t \wedge \bar{T}}, t \wedge \bar{T}) - Q(x_0, t_0) - \int_{t_0}^{T \wedge \bar{T}} V(s)ds\right] \leq 0.
\]
Lemma 3.5. The process $Q(X_t, t) - \int_{s_0}^t V(s)ds$ is a supermartingale.
Proof. Due to the strong Markov property of the $g_t$-Brownian motion, it suffices to show that for all deterministic starting point $(x_0, t_0) \in M \times [s_0, T]$ and all $t \in [s_0, T]$,

$$\mathbb{E} \left[ Q(X_t, t) - Q(X_{t_0}, t_0) - \int_{t_0}^{t} V(s)ds \right] \leq 0.$$ 

We first observe from Lemma 3.4 and (3.1) that for all $n \in \mathbb{N}$,

$$\mathbb{E} \left[ Q(X_{t \wedge S_n^\delta}, t \wedge T_n^\delta) - Q(X_{t \wedge T_n^\delta}, t \wedge T_n^\delta) - \int_{t \wedge T_n^\delta}^{t \wedge S_n^\delta} V(s)ds \bigg| \mathcal{F}_{T_n^\delta} \right] \leq 0,$$

and

$$\mathbb{E} \left[ Q(X_{t \wedge T_n^\delta}, T_n^\delta) - Q(X_{t \wedge S_n^\delta}, T_n^\delta) - \int_{t \wedge S_n^\delta}^{t \wedge T_n^\delta} V(s)ds \bigg| \mathcal{F}_{S_n^\delta} \right] \leq 0.$$

It remains to show that $T_n \to T$ as $n \to \infty$. If

$$\lim_{n \to \infty} T_n =: T_\infty < T,$$

then $T_n^\delta - S_n^\delta$ converges to 0 as $n \to \infty$. In addition, $d_g(X_{S_n^\delta}, S_n^\delta, X_{T_n^\delta}, T_n^\delta) = \delta$ must hold for infinitely many $n \in \mathbb{N}$. It contradicts to the fact that $X_t$ is uniformly continuous on $[0, T]$.  

\begin{lemma}
\textbf{Lemma 3.6.} $\lim_{\delta \to 0} \sum_{n=1}^{\infty} |T_n^\delta - S_n^\delta| = 0$ almost surely.
\end{lemma}

\begin{proof}
For $\delta > 0$, define

$$E_\delta = \{ t \in [s_0, T] \mid \text{there exist } t' \in [s_0, T] \text{ satisfying } |t - t'| \leq \delta \text{ and } (X_{t'}, t') \in \mathcal{L}\text{Cut}((o, 0)) \},$$

$$E = \{ t \in [s_0, T] \mid (X_t, t) \in \mathcal{L}\text{Cut}((o, 0)) \}.$$

Since the map $t \to (X_t, t)$ is continuous and $\mathcal{L}\text{Cut}$ is closed, the set $E$ is closed and hence $E = \bigcap_{\delta > 0} E_\delta$ holds. By the definitions of $S_n^\delta$ and $T_n^\delta$,

$$E \subset \bigcup_{n=1}^{\infty} [S_n^\delta, T_n^\delta] \subset E_\delta,$$

which, together with the monotone convergence theorem, implies

$$\lim_{\delta \to 0} \sum_{n=1}^{\infty} |T_n^\delta - S_n^\delta| \leq \lim_{\delta \to 0} \int_{s_0}^{T} 1_{E_\delta}(t)dt = \int_{s_0}^{T} 1_E(t)dt = 0, \text{ a.e.},$$

where the last equality comes from Proposition 2.1(1).

\end{proof}

\begin{lemma}
\textbf{Lemma 3.7.} The martingale part of $Q(X_t, t)$ is

$$\langle \nabla_t^i Q(X_t, t), u_t dB_t \rangle_t = \sum_{i=1}^{d} u_t e_i Q(X_t, t) dB_i^t.$$
\end{lemma}
Proof. By the martingale representation theorem, there exists an \( \mathbb{R}^d \)-valued process \( \eta \) such that the martingale part of \( Q(X_t, t) \) equals to \( \int_0^t \eta_s dB_s \). Let

\[
N_t := \int_0^t \eta_s dB_s - \sum_{i=1}^d (u_t e_i) Q(X_t, t) dB^i_t.
\]

Using the stopping times \( S^d_n \) and \( T^d_n \), the quadratic variation \( \langle N \rangle_T \) of \( N \) is expressed as follows:

\[
\langle N \rangle_T = \sum_{i=1}^d \sum_{n=1}^{\infty} \left( \int_{T^d_{n-1} \wedge T}^{T^d_n \wedge T} \left| \eta^i_t - (u_t e_i) Q(X_t, t) \right|^2 dt + \int_{S^d_n \wedge T}^{T^d_n \wedge T} \left| \eta^i_t - (u_t e_i) Q(X_t, t) \right|^2 dt \right).
\]

Since \( (X_t, t) \notin \mathcal{L}\text{Cut}((o, 0)) \) if \( t \in (T^d_{n-1}, S^d_n) \), the Itô formula yields

\[
\int_{T^d_{n-1} \wedge T}^{T^d_n \wedge T} \left| \eta^i_t - (u_t e_i) Q(X_t, t) \right|^2 dt = 0
\]

for \( n \in \mathbb{N} \) and \( i = 1, 2, \ldots, d \). For the second term on the right, since the manifold is compact, there exists a constant \( C > 0 \), such that

\[
\sum_{i=1}^d \sum_{n=1}^{\infty} \int_{S^d_n \wedge T}^{T^d_n \wedge T} \left| \eta^i_t - u_t e_i \right|^2 dt \leq \int_{\bigcup_{n=1}^{\infty} S^d_n, T^d_n} \left( |\eta^i|^2 + 4|t^\gamma(t)|^2 \right) dt \leq \int_{\bigcup_{n=1}^{\infty} S^d_n, T^d_n} \left( |\eta^i|^2 + C \right) dt.
\]

Since \( \eta_t \) is locally square-integrable on \( [s_0, T] \), almost surely, we obtain \( \langle N \rangle_T = 0 \) by Lemma 3.6, which yields the conclusion. \( \square \)

Proof of Theorem 1.1. Now, we can conclude the proof of Theorem 1.1. Set \( I_\delta := \bigcup_{n=1}^{\infty} [S^d_n, T^d_n] \).

Let

\[
L^\delta_t := -Q(X_t, t) + Q(X_{s_0}, s_0) + \sum_{i=1}^d \int_{s_0}^t (u_s e_i) Q(X_s, s) dB^i_s + \int_{[s_0, t]\cap I_\delta} \left[ \Delta_s Q + \frac{\partial Q}{\partial s} \right] (X_s, s) ds + \int_{[s_0, t]\cap I_\delta} V(s) ds.
\]

By (3.1), \( L^\delta_t \) is an increasing process which increases only when \( t \in I_\delta \). Moreover, we have

\[
Q(X_t, t) - Q(X_{s_0}, s_0) - \sum_{i=1}^d \int_{s_0}^t (u_s e_i) Q(X_s, s) dB^i_s - \int_{s_0}^t \left[ \Delta_s Q + \frac{\partial Q}{\partial s} \right] (X_s, s) ds + L^\delta_t \\
= -\int_{[s_0, t]\cap I_\delta} \left[ \Delta_s Q + \frac{\partial Q}{\partial s} \right] (X_s, s) ds - \int_{[s_0, t]\cap I_\delta} V(s) ds.
\]

From (3.1), we obtain

\[
\left| \int_{[s_0, t]\cap I_\delta} \left[ \Delta_s Q + \frac{\partial Q}{\partial s} \right] (X_s, s) ds + \int_{[s_0, t]\cap I_\delta} V(s) ds \right| \leq 2 \int_{[s_0, t]\cap I_\delta} V(s) ds,
\]

and \( V(s) \) is bounded on \( [s_0, t] \cap I_\delta \). Then by Lemma 3.6, the right hand of (3.5) converges to 0 as \( \delta \to 0 \). Thus, \( L_t := \lim_{\delta \to 0} L^\delta_t \) exists for all \( t \in [s_0, T] \) almost surely and hence (1.2) holds. Finally, it is easy to see that \( L_t \) increases only when \( (X_t, t) \in \mathcal{L}\text{Cut}((o, 0)) \) from the corresponding property of \( L^\delta_t \). \( \square \)
4 Coupling for $g_{\bar{\tau}_1 t}$- and $g_{\bar{\tau}_2 t}$-Brownian motions

First, we introduce some basic notations concerning the space-time parallel displacement.

Definition 4.1 (space-time parallel vector field). For $0 < \tau_1 < \tau_2 < T$, let \( \gamma : [\tau_1, \tau_2] \to M \) be a smooth curve. We say that a vector field \( Z \) along \( \gamma \) is space-time parallel if

\[
\nabla^t_{\dot{\gamma}(\tau)} Z(\tau) = -\text{Ric}_{\gamma}^t(Z(\tau))
\]  

(4.1)

holds for all \( \tau \in [\tau_1, \tau_2] \), where \( \text{Ric}_{\gamma}^t \) is defined by regarding the \( g_{\tau} \)-Ricci curvature as a (1,1)-
tensor.

Since (4.1) is a linear first order ODE, for any \( \xi \in T_{\gamma(\tau_1)} M \), there exists a unique space-time parallel vector field \( Z \) along \( \gamma \) with \( Z(\tau_1) = \xi \). Note that whenever \( Z \) and \( Z' \) are space-time parallel vector fields along a curve \( \gamma \), their \( g_{\tau_1} \)-inner product is constant in \( \tau \):

\[
\frac{d}{d\tau} \langle Z(\tau), Z'(\tau) \rangle_{\tau} = \frac{\partial}{\partial \tau} g_{\tau}(Z(\tau), Z'(\tau)) + \left( \nabla^t_{\dot{\gamma}(\tau)} Z(\tau), Z'(\tau) \right)_{\tau} + \left( Z(\tau), \nabla^t_{\dot{\gamma}(\tau)} Z'(\tau) \right)_{\tau} = 2\text{Ric}_{\tau}(Z(\tau), Z'(\tau)) - \text{Ric}_{\tau}(Z(\tau), Z'(\tau)) - \text{Ric}_{\tau}(Z(\tau), Z'(\tau)) = 0.
\]  

(4.2)

Remark 4.2. The minimal \( \mathcal{L} \)-geodesic \( \gamma = \gamma_{t_1,t_2}^{x,y} \) of \( Q(x, t_1; y, t_2) \) satisfies the \( \mathcal{L} \)-geodesic equation

\[
\nabla^t_{\dot{\gamma}(\tau)} \dot{\gamma}(t) = \frac{1}{2} \nabla^t R_{\tau} - 2\text{Ric}_{\gamma}^t(\dot{\gamma}(t)) - \frac{1}{2t} \dot{\gamma}(t).
\]

Therefore, \( \sqrt{t} \dot{\gamma}(t) \) is not space-time parallel to \( \gamma \) in general and their \( g_{\tau_1} \)-inner product is not a constant in \( t \), i.e. \( \sqrt{t} \dot{\gamma}(t) \) does not satisfy (4.1). Therefore the coefficient in the martingale part of (3.2) is not constant, which is different from the case when the metric is independent of \( t \).

Definition 4.3 (space-time parallel transport). For \( x,y \in M \) and \( 0 < \tau_1 < \tau_2 \leq T \), we define a map \( P_{x,y}^{\tau_1,\tau_2} : T_x M \to T_y M \) as follows: \( P_{x,y}^{\tau_1,\tau_2}(\xi) := Z(\tau_2) \), where \( Z \) is the unique space-time parallel vector field along \( \gamma_{\tau_1,\tau_2}^{x,y} \) with \( Z(\tau_1) = \xi \). As explained in (4.2), \( P_{x,y}^{\tau_1,\tau_2} \) is an isometry from \( (T_x M, g_{\tau_1}) \) to \( (T_y M, g_{\tau_2}) \). In addition, it smoothly depends on \( (x, \tau_1, y, \tau_2) \) outside the \( \mathcal{L} \)-cut locus.

Using the Itô formula for \( Q(X_t, t) \) presented in Theorem 1.1, we are able to construct a parallel coupling of \( g_{\bar{\tau}_1 t} \)- and \( g_{\bar{\tau}_2 t} \)-Brownian motions.

Theorem 4.4. Let \( x \neq y \) and \( 0 < \bar{\tau}_1 < \bar{\tau}_2 < T \) be fixed. For any \( s > 0 \), there exist two Brownian motions \( B_t \) and \( \tilde{B}_t \) on a completed filtered probability space \( (\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) such that for all \( t \in [s, T/\bar{\tau}_2) \),

\[
1 \{ (X_t, \bar{\tau}_1 t; X_t, \bar{\tau}_2 t) \notin \mathcal{L}\text{Cut} \} d\tilde{B}_t = 1 \{ (X_t, \bar{\tau}_1 t; X_t, \bar{\tau}_2 t) \notin \mathcal{L}\text{Cut} \} (\tilde{u}_t)^{-1} P_{X_t,X_t}^{\bar{\tau}_1 t,\bar{\tau}_2 t} u_t dB_t
\]

holds, where \( X_t \) with lift \( u_t \) and \( \tilde{X}_t \) with lift \( \tilde{u}_t \) solve the equation

\[
\begin{cases} 
\frac{d}{dt} X_t = \sqrt{2\bar{\tau}_1 u_t} \circ dB_t, & X_s = x; \\
\frac{d}{dt} \tilde{X}_t = \sqrt{2\bar{\tau}_2 \tilde{u}_t} \circ dB_t, & \tilde{X}_s = y.
\end{cases}
\]

(4.3)
Moreover,
\[
\text{d}Q(X_t, \tilde{\tau}_1 t; \tilde{X}_t, \tilde{\tau}_2 t) \leq 2\sqrt{2t} \left| \tilde{\tau}_1 P_{X_t, X_{\tilde{\tau}_1 t}}^{\tau_1, \tilde{\tau}_2 t}(\tilde{\tau}_1 t) - \tilde{\tau}_2\tilde{\gamma}(\tilde{\tau}_2 t) \right| \text{d}t
+ \left( \frac{d}{\sqrt{t}}(\sqrt{\tilde{\tau}_1} - \sqrt{\tilde{\tau}_2}) - \frac{1}{2t}Q(\tilde{\tau}_1 t, X_t; \tilde{\tau}_2 t, \tilde{X}_t) \right) \text{d}t,
\]
where \( \gamma : [\tilde{\tau}_1 t, \tilde{\tau}_2 t] \to M \) is the \( \mathcal{L} \)-geodesic from \( X_t \) to \( \tilde{X}_t \).

Proof. We denote \( Q(t, x, y) := Q(x, \tilde{\tau}_1 t; y, \tilde{\tau}_2 t) \) for simplicity. Our proof is divided into two parts.

(a) First, we give the construction of the couplings. Recall that \( u_t \), the horizontal lift of \( X_t \), satisfies the following SDE
\[
\begin{cases}
\text{d}u_t = -\frac{1}{2} t \sum_{\alpha,\beta} G_{\alpha,\beta}(\tilde{\tau}_1 t, u_t) V_{\alpha,\beta}(u_t) \text{d}t,
\end{cases}
\]
where \( u_s \in \mathcal{O}_{\tilde{\tau}_1 t}(M) \).

Now, for given \( x \neq y \) with \((x, \tilde{\tau}_1 t; y, \tilde{\tau}_2 t) \notin \mathcal{L} \text{Cut} \), let \( \gamma \) be the minimal geodesic from \( x \) to \( y \). Recall that \( P_{x,y}^{\tau_1, \tilde{\tau}_2 t} \) are the parallel operators. To get rid of the trouble that \( P_{x,y}^{\tau_1, \tilde{\tau}_2 t} \) does not exist on \( \mathcal{L} \text{Cut} \), we modify this operator so that it vanishes in a neighborhood of this set. To this end, for any \( n \geq 1 \) and \( \varepsilon \in (0, 1) \), let \( h_{n,\varepsilon} \in C^\infty([s, T/\tilde{\tau}_2) \times M \times M) \) such that
\[
0 \leq h_{n,\varepsilon} \leq (1 - \varepsilon), \quad h_{n,\varepsilon}|_{C_2} = 0, \quad h_{n,\varepsilon}|_{C_n} = 1 - \varepsilon,
\]
where \( C_n = \{(t, x, y) \in [s, T/\tilde{\tau}_2) \times M \times M : \text{d}_{\tilde{x}} \otimes \text{d}_{\tilde{y}}((x, \tilde{\tau}_1 t; y, \tilde{\tau}_2 t), \mathcal{L} \text{Cut}) \leq 1/n \} \). Let \( \tilde{u}_{n,\varepsilon} \) and \( \tilde{X}_{t,\varepsilon} := \text{p}\tilde{u}_{n,\varepsilon} \) solve the SDE
\[
\begin{cases}
\begin{align*}
\text{d}\tilde{u}_{n,\varepsilon} &= \sqrt{2\tilde{\tau}_1} h_{n,\varepsilon}(t, X_t, \tilde{X}_{t,\varepsilon}) \sum_{i=1}^d \text{d}B_i + \frac{1}{2} \sum_{\alpha,\beta} G_{\alpha,\beta}(\tilde{\tau}_1, \tilde{u}_{n,\varepsilon}) V_{\alpha,\beta}(\tilde{u}_{n,\varepsilon}) \text{d}t,
\end{align*}
\end{cases}
\]
where \( \tilde{u}_{n,\varepsilon} \) is a Brownian motion on \( \mathbb{R}^d \) independent of \( B_t \), and \( \text{d}\tilde{B}_t = (\tilde{u}_{n,\varepsilon})^{-1} P_{x,y}^{\tau_1, \tilde{\tau}_2 t} u_t \text{d}B_t \).

Since the coefficients involved in (4.5) are at least \( C^1 \), the solution \( \tilde{u}_{n,\varepsilon} \) exists uniquely.

Let us observe that \((u_t, \tilde{u}_{n,\varepsilon})\) is generated by
\[
L_{Q(M \times M)}(t)(u_t, \tilde{u}_{n,\varepsilon})
:= \tilde{\tau}_1 \Delta_{\tilde{\tau}_1 t}(M)(u_t) + \tilde{\tau}_2 \Delta_{\tilde{\tau}_2 t}(M)(\tilde{u}_{n,\varepsilon})
+ \sqrt{\tilde{\tau}_1 \tilde{\tau}_2} h_{n,\varepsilon}(t, X_t, \tilde{X}_{t,\varepsilon}) \sum_{i,j=1}^d \left( P_{X_t, X_{\tilde{\tau}_1 t}}^{\tau_1, \tilde{\tau}_2 t} u_t e_i, \tilde{u}_{n,\varepsilon} e_j \right)_{\tilde{\tau}_2} H(\tilde{\tau}_1, u_t) H(\tilde{\tau}_2, \tilde{u}_{n,\varepsilon})
- \frac{1}{2} \sum_{\alpha,\beta} G_{\alpha,\beta}(\tilde{\tau}_1, u_t) V_{\alpha,\beta}(u_t) \text{d}t.
\]

Next, let
\[
L_{M \times M}^{n,\varepsilon}(t)(x, y) := \tilde{\tau}_1 \Delta_{\tilde{\tau}_1 t}(x) + \tilde{\tau}_2 \Delta_{\tilde{\tau}_2 t}(y) + \sqrt{\tilde{\tau}_1 \tilde{\tau}_2} h_{n,\varepsilon}(t, x, y) \sum_{i,j=1}^d \left( P_{x,y}^{\tau_1, \tilde{\tau}_2 t} V_{\tau_1 t} W_i, W_j \right)_{\tilde{\tau}_2 t} V_{\tau_1 t} W_j,
\]
where \{V_i\} and \{W_i\} are orthonormal bases at \(x\) and \(y\) with respect to the metrics \(g_{\tau_1t}\) and \(g_{\tau_2t}\) respectively. It is easy to see that \((X_t, \dot{X}_t^{n,\varepsilon}):=(p_{ut}, p_{ut}^{n,\varepsilon})\) is generated by \(L^{n,\varepsilon}_{M\times M}(t)\) and hence is a coupling of \(g_{\tau_1t}\)- and \(g_{\tau_2t}\)-Brownian motions, as the marginal operators of \(L^{n,\varepsilon}_{M\times M}(t)\) coincide with \(\bar{\tau}_1\Delta_{\tau_1t}\) and \(\bar{\tau}_2\Delta_{\tau_2t}\) respectively.

Since in some neighborhood of \(\mathcal{LCut}\), the coupling is independent and hence behaves as a Brownian motion on \(M \times M\), we obtain from Theorem [1] that

\[
\begin{align*}
\text{d}Q(t, X_t, \dot{X}_t^{n,\varepsilon}) &= \sqrt{8t} \left[ \bar{\tau}_1 \left( P^{\tau_1t, \tau_2t}_{X_t, X_t^{n,\varepsilon}} \right) \gamma_{n,\varepsilon}(\bar{\tau}_1 t) - \bar{\tau}_2 \left( P^{\tau_2t, \tau_1t}_{X_t^{n,\varepsilon}, X_t} \right) \gamma_{n,\varepsilon}(\bar{\tau}_2 t) \right]^2 \, \text{d}b_t^{n,\varepsilon} \\
&+ \left\{ \frac{\partial Q}{\partial t} + 1_{(\mathcal{Y}\setminus\mathcal{LCut})}[h_{n,\varepsilon}] + (1 - h_{n,\varepsilon})S \right\} (t, X_t, \dot{X}_t^{n,\varepsilon}) \, \text{d}t - \text{d}l_t^{n,\varepsilon},
\end{align*}
\]

where \(\gamma_{n,\varepsilon}: [\bar{\tau}_1 t, \bar{\tau}_2 t] \to \mathcal{L}\) is the \(\mathcal{L}\)-geodesic from \(X_t\) to \(\dot{X}_t^{n,\varepsilon}\), \(b_t^{n,\varepsilon}\) is an one-dimensional Brownian motion, \(l_t^{n,\varepsilon}\) is an increasing process which increases only when \((X_t, \bar{\tau}_1 t; \bar{\tau}_2 t) \in \mathcal{LCut}, and

\[
S(t, x, y) := \bar{\tau}_1 \Delta_{\tau_1t} Q(t, \cdot, y)(x) + \bar{\tau}_2 \Delta_{\tau_2t} Q(t, x, \cdot)(y);
\]

\[
I(t, x, y) := \sum_{i=1}^{d} \left( \sqrt{\bar{\tau}_1} V_i + \sqrt{\bar{\tau}_2} P^{\bar{\tau}_1t, \bar{\tau}_2t}_{x,y} V_i \right)^2 Q(t, x, y).
\]

Then, let \(\mathbb{P}^{(x,y)}_{n,\varepsilon}\) be the distribution of \((X_t, \dot{X}_t^{n,\varepsilon})\), which is a probability measure on the path space \(M^T_x \times M^T_y\), where

\[
M^T_x := \{ \gamma \in C([s, T/\bar{\tau}_2], M) : \gamma_s = x \}
\]

is equipped with the \(\sigma\)-field \(\mathcal{F}^T_x\) induced by all measurable cylindric functions. Note that \((M^T_x, \mathcal{F}^T_x)\) is metrizable with the distance

\[
\tilde{d}(\xi, \eta) := \sum_{n=1}^{\infty} 2^{-n} \left( 1 \wedge \sup_{t \in [n, (n+1)\wedge T/\bar{\tau}_2]} \mathbf{d}(\bar{\tau}_1 t, \xi_t; \bar{\tau}_2 t, \eta_t) \right).
\]

Furthermore, \((M^T_x, \tilde{d})\) is a Polish space. Then \(M^T_x \times M^T_y\) is a Polish space too. It is easy to see that \(\{\mathbb{P}^{x,y}_{n,\varepsilon} : n \geq 1, \varepsilon > 0\}\) is tight (see [12] Lemma 4), since they are the couplings of \(\mathbb{P}^x\) and \(\mathbb{P}^y\). We take \(n_k \to \infty\) and \(\varepsilon_l \to 0\) such that \(\mathbb{P}^{x,y}_{n_k,\varepsilon_l}\) converges weakly to some \(\mathbb{P}^{x,y}_{l}\) \((l \geq 1)\) as \(k \to \infty\) while \(\mathbb{P}^{x,y}_{\varepsilon_l}\) converges weakly to some \(\mathbb{P}^{x,y}\) as \(l \to \infty\). Then \(\mathbb{P}^{x,y}\) is also a coupling of \(\mathbb{P}^x\) and \(\mathbb{P}^y\).

Now, let \((X_t, \dot{X}_t)\) be the coordinate process in \((M^T_x \times M^T_y, \mathcal{F}^T_x \times \mathcal{F}^T_y)\) and \(\{\mathcal{F}_t\}_{t \geq s}\) be the natural filtration. Define

\[
\tilde{L}(t)(x, y) := \bar{\tau}_1 \Delta_{\tau_1t}(x) + \bar{\tau}_2 \Delta_{\tau_2t}(y) + 1_{(\mathcal{Y}\setminus\mathcal{LCut})}(x, \bar{\tau}_1 t; y, \bar{\tau}_2 t) \sum_{i,j=1}^{d} \left( P^{\bar{\tau}_1t, \bar{\tau}_2t}_{x,y} V_i, W_j \right) \bar{\tau}_j t V_i W_j,
\]

It is trivial to see that \(\mathbb{P}^{x,y}\) solves the martingale problem for \(\tilde{L}(t)\) up to \(T/\bar{\tau}_2\) (see [2] Theorem 2]), i.e.

\[
f(X_t, \dot{X}_t) - \int_0^t \tilde{L}(s)f(X_s, \dot{X}_s) \, ds
\]

is a \(\mathbb{P}^{x,y}\)-martingale w.r.t. \(\mathcal{F}\). Then \((X_t, \dot{X}_t)\) under \(\mathbb{P}^{x,y}\) is a coupling of the \(g_{\tau_1t}\) and \(g_{\tau_2t}\)-Brownian motions starting from \((x, y)\), i.e. the solution of (4.3).
(b) We first claim that the two sets
\[ \{ t \in [s, T/\bar{\tau}_2] \mid (X_t, \bar{\tau}_1 t; \bar{X}_t^n, \bar{\tau}_2 t) \in \mathcal{L} \text{Cut} \} \text{ and } \{ t \in [s, T/\bar{\tau}_2] \mid (X_t, \bar{\tau}_1 t; \bar{X}_t^\varepsilon, \bar{\tau}_2 t) \in \mathcal{L} \text{Cut} \} \]
have Lebesgue measure zero almost surely. This assertion can be checked similarly as in Lemma 2.2 by observing that \( L_M^{t,\bar{\tau}_1}(x, y, z, w) \) is strictly elliptic and all the coefficients are \( C^\infty \), then \( \mathbb{P}^x(y)(X_t, X_t^n, \bar{\tau}_1 t) \in A \) has a density \( p_t^{x, y}(x, y, z, w) \) with respect to the product volume measure \( \text{dvol}_{\bar{\tau}_1} \otimes \text{dvol}_{\bar{\tau}_2} \) (see [5, Theorem 3.16]), where \( \text{dvol}_t \) is the volume measure w.r.t. the metric \( g_t \).

Then, for \( f \in C^2(\mathbb{R}) \) with \( f' \geq 0 \) be fixed, let
\[
dN_t(f) = df \circ Q(t, X_t, \bar{X}_t) - \left[ \left( I + \frac{\partial Q}{\partial t} \right) f' \circ Q(t, X_t, \bar{X}_t) \right. \\
\left. + 4t \left( \bar{\tau}_1 P_{\bar{X}_t, \bar{X}_t}^{\bar{\tau}_1, \bar{\tau}_2} \gamma(\bar{\tau}_1 t) - \bar{\tau}_2 \gamma(\bar{\tau}_2 t) \right) t^{\bar{\tau}_2} \circ Q(t, X_t, \bar{X}_t) \right] dt, \quad t \in [s, T/\bar{\tau}_2],
\]
With a similar discussion as in the proof of [3](b), we obtain \( N_{t\wedge(T/\bar{\tau}_2)} \) is a \( \mathbb{P}^{\bar{\tau}_1, \bar{\tau}_2} \)-supmartingale.

In particular, by taking explicit \( f(r) = r \), we have
\[
dQ(t, X_t, \bar{X}_t) = dM_t + \left( I + \frac{\partial Q}{\partial t} \right)(t, X_t, \bar{X}_t)dt - dl_t, \quad t \in [s, T/\bar{\tau}_2],
\]
where \( M_t \) is a local martingale and \( l_t \) is a predictable increasing process. By the second variation formula of the \( L \)-functional (see [1] Lemma 7.37 and Lemma 7.40) for instance,
\[
\frac{\partial}{\partial t} Q(t, x, y) + \sum_{i=1}^d (\sqrt{\bar{\tau}_1} V_i + \sqrt{\bar{\tau}_2} P_{x,y}^{\bar{\tau}_1, \bar{\tau}_2} V_i)^2 Q(t, x, y) \leq \frac{d}{\sqrt{t}}(\sqrt{\bar{\tau}_1} - \sqrt{\bar{\tau}_2}) - \frac{1}{2t} Q(t, x, y) =: J(t, x, y).
\]
It follows that
\[
dQ(t, X_t, \bar{X}_t) = dM_t + J(t, X_t, \bar{X}_t)dt - dl_t, \quad t \in [s, T/\bar{\tau}_2],
\]
where \( \tilde{l}_t \) is a larger predictable increasing process. Moreover, with a similar discussion as in the proof of [14] Theorem 2.1.1(d), we further obtain
\[
dM_t = 2\sqrt{2t} \left\| \bar{\tau}_1 P_{\bar{X}_t, \bar{X}_t}^{\bar{\tau}_1, \bar{\tau}_2} \gamma(\bar{\tau}_1 t) - \bar{\tau}_2 \gamma(\bar{\tau}_2 t) \right\|_{\bar{\tau}_2} db_t,
\]
which leads to complete the proof.

\[ \square \]

As an important application, we give a new proof of Topping’s result [13], i.e. the contraction in the normalized \( L \)-transportation cost. We point out that this result recovers the monotonicity of Perelman’s monotonic quantities (involving both \( W \)-entropy and \( L \)-length), which are central in his work on Ricci flow (see [9, 10, 11]).

Suppose that \( \{ P_{s,t} \}_{0 < s < t < T/\bar{\tau}_2} \) and \( \{ T_{s,t} \}_{0 < s < t < T/\bar{\tau}_2} \) be the Markov inhomogeneous semigroup of the \( g_{\bar{\tau}_1} \)-Brownian motion and \( g_{\bar{\tau}_2} \)-Brownian motion respectively. To the \( L \)-distance \( Q \), we associate the Monge-Kantorovich minimization between two probability measures on \( M \),
\[
W^L(\mu, t_1; \nu, t_2) = \inf_{\eta \in C^L(\mu, \nu)} \int_{M \times M} Q(x, t_1; y, t_2) \text{d}\eta(x, y),
\]
where \( C^L(\mu, \nu) \) is the set of all probability measures on \( M \times M \) with marginal \( \mu \) and \( \nu \). Then, using the coupling constructed in Theorem 4.4, we have
Theorem 4.5. Assume that $M$ has bounded curvature tensor, i.e.

$$
\sup_{x \in M, t \in [0, T)} |Rm_t|(x) < \infty.
$$

Then for $0 < \bar{\tau}_1 < \bar{\tau}_2 < T$ the normalized $L$-transportation cost

$$
\Theta(t, \delta_x P_{s,t}, \delta_y T_{s,t}) := 2(\sqrt{\bar{\tau}_2 t} - \sqrt{\bar{\tau}_1 t}) W^L(\delta_x P_{s,t}, \bar{\tau}_1 t; \delta_y T_{s,t}, \bar{\tau}_2 t) - 2d(\sqrt{\bar{\tau}_2 t} - \sqrt{\bar{\tau}_1 t})^2
$$

is a non-increasing function of $t \in [s, T/\bar{\tau}_2)$, that is

$$
\Theta(t, \delta_x P_{s,t}, \delta_y T_{s,t}) \leq 2(\sqrt{\bar{\tau}_2 s} - \sqrt{\bar{\tau}_1 s}) Q(x, \bar{\tau}_1 s; y, \bar{\tau}_2 s) - 2d(\sqrt{\bar{\tau}_2 s} - \sqrt{\bar{\tau}_1 s})^2.
$$

Proof. By \cite{4.3}, there exist two coupled $g_{\bar{\tau}_1 t}$- and $g_{\bar{\tau}_2 t}$-Brownian motions $(X_t)_{t \in [s, T/\bar{\tau}_2)}$ and $(\tilde{X}_t)_{t \in [s, T/\bar{\tau}_2)}$ with initial values $X_s = x$ and $\tilde{X}_s = y$ such that the process $(\Theta(t, X_{\bar{\tau}_1 t}, \tilde{X}_{\bar{\tau}_2 t}))_{t \in [s, T/\bar{\tau}_2)}$ is a supermartingale. Taking the expectation of this supermartingale leads to complete the proof.

Denote $P_{s,t}(\cdot, dx) := u(\bar{\tau}_1 t, x) d\nu_{\bar{\tau}_1 t}$ and $T_{s,t}(\cdot, dx) := u(\bar{\tau}_2 t, x) d\nu_{\bar{\tau}_2 t}$. This density $u$ solves the following heat equation

$$
\frac{\partial u}{\partial \tau} = \Delta_{\bar{\tau}} u - R_{\bar{\tau}} u,
$$

where $R_{\bar{\tau}}$ is the scalar curvature w.r.t. the metric $g_{\bar{\tau}}$. Hence, the conclusion presented in Theorem 4.5 is consistent with that in \cite{13}.

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