A DECOMPOSITION OF CALDERÓN–ZYGMUND TYPE AND SOME OBSERVATIONS ON DIFFERENTIATION OF INTEGRALS ON THE INFINITE-DIMENSIONAL TORUS

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Abstract. In this note we will show a Calderón–Zygmund decomposition associated with a function \( f \in L^1(T^\omega) \). The idea relies on an adaptation of a more general result by J. L. Rubio de Francia in the setting of locally compact groups. Some related results about differentiation of integrals on the infinite-dimensional torus are also discussed.

1. Introduction

In [28], José L. Rubio de Francia (JLR) showed a result on differentiation of integrals in the context of a locally compact group \( G \), that contained a decomposition of Calderón–Zygmund type [9, Ch. I, Lemma 1] under certain conditions. In view of the publication date, his study could be a contemporary of the one by Edwards and Gaudry in [13, Ch. 2]. In this note we revisit and adapt those results by JLR to the case of the (compact, abelian) group \( T^\omega \) (the infinite torus) defined below. Indeed, our goals are the following:

(1) Present a decomposition of Calderón–Zygmund (CZ) type in \( T^\omega \), devised by JLR.

(2) Observe some issues on differentiation of integrals in \( T^\omega \).

Let \( T = \{ z \in \mathbb{C} \mid |z| = 1 \} \) be the one-dimensional torus, identified naturally with the interval \([0, 1)\) of the real line through the group isomorphism \( e^{2\pi it} \leftrightarrow t \). We can also identify \( T \) with the additive quotient group \( \mathbb{R}/\mathbb{Z} \). We denote by \( T^\omega \) the compact group formed by the product of countably infinite many copies of \( T \) (complete direct sum, [31, B7.]). We will call it briefly the infinite torus. The operation in the group \( T^\omega \) is the sum (mod 1) of real sequences, with identity element \( \overline{0} := (0, 0, \ldots) \).

For a fixed \( n \in \mathbb{N} \) we can write the infinite torus, of points \( x = (x_1, x_2, \ldots) \), as the cartesian product \( T^n \times T^n, \omega \) of an \( n \)-dimensional torus \( T^n \) of points \( x_{(n)} = (x_1, \ldots, x_n) \) times a copy, but denoted \( T^n, \omega \), of the infinite torus itself, of points \( x^{(n)} = (x_{n+1}, x_{n+2}, \ldots) \).

We denote by \( m \), or \( dx \), the Haar measure (translation invariant) on \( T^\omega \), normalized such that \( m(T^\omega) = 1 \). This measure coincides [19, §22] with the measure product of countably infinite many copies of the Lebesgue measure \( |\cdot| \) on \( T \), so the basic \( m \)-measurable sets are the so-called intervals, i.e. subsets of \( T^\omega \) of the form

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I = \prod_{j \in \mathbb{N}} I_j, where I_j is an interval of \(\mathbb{T}\) for each \(j\), and \(\exists N \in \mathbb{N}\) such that \(I_j = \mathbb{T}\) for all \(j > N\). The measure of the interval \(I\) is then \(m(I) = \int_{\mathbb{T}} \chi_I(x) \, dx = \prod_{j=1}^{N} |I_j|\).

The space \(\mathbb{T}^\omega\) is metrizable. For instance, the function

\[
d(x, y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n} \quad (x, y \in \mathbb{T}^\omega)
\]

defines a metric in \(\mathbb{T}^\omega\) [31, p. 157]. We write \(\delta(S) := \sup_{x, y \in S} d(x, y)\) for the diameter of the set \(S \subset \mathbb{T}^\omega\).

The \(\sigma\)-algebra \(B\) of Borel sets in \(\mathbb{T}^\omega\), which is the smallest \(\sigma\)-algebra containing the open intervals, coincides with the least \(\sigma\)-algebra containing the open balls with respect to the metric [33, II §2.4].

The study of Harmonic Analysis on the infinite torus finds a motivation, on one hand, because it constitutes a logical extension of the \(n\)-dimensional setting in which estimates have to be obtained independent of the dimension \(n\). On the other hand, \(\{e^{2\pi i k x} : k = 1, 2, \ldots\}\) is a system of independent random variables uniformly distributed in the complex unit circumference (i.e., of a complexified version of Rademacher’s functions), whose natural completion in \(L^2\) is the trigonometric system on \(\mathbb{T}^\omega\). Then, the Fourier series of infinitely many variables turn out to be the complex analogue of the Walsh series. Fourier series in \(\mathbb{T}^\omega\) also have connection with the Dirichlet series [6] and with Prediction Theory [18]. All these issues were already pointed out by JLR in [29] (see also references therein), where he studied pointwise and norm convergence of Fourier series of infinite variables, although the proofs are just sketched.

There is considerable interest on the infinite torus from the point of view of Potential Theory, see [1, 2, 3, 4, 5]. Apart from this, problems of approximation theory on \(\mathbb{T}^\omega\) have been analyzed for instance in [26].

The JLR decomposition of Calderón–Zygmund type in \(\mathbb{T}^\omega\) will be shown in Section 2 (see Subsection 2.2), and the issues related to differentiation of integrals in \(\mathbb{T}^\omega\) are contained in Section 3. To be precise, we will look at three differentiation bases.

First, the \textit{Rubio de Francia restricted basis} \(R_0\), which is the family associated to the Calderón–Zygmund decomposition in Section 2 and that differentiates \(L^1(\mathbb{T}^\omega)\), see Corollary 16. Second, the \textit{Rubio de Francia basis} \(R\), which arises naturally in the light of the general results by JLR in [28, Thm. 8]. For such a basis several questions remain open concerning differentiation and the associated maximal function, see Subsection 3.1. Finally, we present a negative result of differentiation on \(\mathbb{T}^\omega\) relative to the so called \textit{extended Rubio de Francia basis} \(R^*\), see Subsection 3.3.

Acknowledgments

The original idea of the Calderón–Zygmund (CZ) decomposition presented in Section 2 was sketched in a personal communication of JLR to the first author in 1977, in Madrid.

The authors would like to thank the referees for their very careful reading and useful comments which indeed improved the presentation of the paper.

2. A CZ decomposition in \(\mathbb{T}^\omega\)

For completeness we will recall some concepts from Probability Theory used later in this section (see for instance [32 p. 89–94], [13 Ch. 5], [33 2.7]).
Definition 1. Let \((X, \mathcal{A}, \mu)\) be a finite measure space, and \(\mathcal{B}\) be a \(\sigma\)-algebra contained in \(\mathcal{A}\). The \textit{conditional expectation of} \(f\) \textit{given} \(\mathcal{B}\) is the (unique \(\mu\)-a.e.) \(\mathcal{B}\)-measurable function \(E^\mathcal{B}f\) (the notation is that of \cite{25}), such that

\[
\int_B f \, d\mu = \int_B (E^\mathcal{B}f) \, d\mu \quad \forall B \in \mathcal{B}.
\]

E.g., suppose that \(\{B_n\}_{n=1}^\infty\) is a countable division of \(X\) in \(\mathcal{A}\)-measurable sets of positive measure, and consider the least \(\sigma\)-algebra \(\mathcal{B}\) which contains those sets (we write \(\mathcal{B} := \sigma(\{B_n\})\)). Then,

\[
E^\mathcal{B}f(x) = \sum_n f_{B_n} \chi_{B_n}(x)
\]

(\text{where } f_B := \frac{1}{\mu(B)} \int_B f \, d\mu \text{ and } \chi_S \text{ denotes the characteristic function of the set } S),

\text{since the function } s(x) \text{ at the right hand side of (3) is } \mathcal{B}\text{-measurable and } \int_{B_n} s \, d\mu = f_{B_n} \mu(B_n) = \int_{B_n} f \, d\mu \text{ holds}.

Property 2. If \(\mathcal{B} \subseteq \mathcal{C}\) are sub-\(\sigma\)-algebras of \(\mathcal{A}\), then \(E^\mathcal{B}(E^\mathcal{C}f) = E^\mathcal{B}f\) a.e.

Definition 3. Let \((X, \mathcal{A}, m)\) be a finite measure space and let

\[B_1 \subset B_2 \subset \cdots \subset B_n \subset B_{n+1} \subset \cdots\]

be an increasing sequence of sub-\(\sigma\)-algebras of \(\mathcal{A}\).

A sequence of functions \(\{f_n\}_{n \in \mathbb{N}} \subset L^1(m)\) such that, for each \(n \geq 1\) the function \(f_n\) is \(B_n\)-measurable and \(E^{B_n} f_{n+1} = f_n\) (a.e.), is called a \textit{martingale}.

For instance, for every \(f \in L^p(\mu)\) \((1 \leq p \leq \infty)\) the sequence \(f_n := E^{B_n} f\) \((n \in \mathbb{N})\) is a martingale, since \(E^{B_n} f_{n+1} = E^{B_n} (E^{B_{n+1}} f) = E^{B_n} f\) a.e., according to Property 2. Moreover the following holds:

Theorem 4. (i) \textit{The maximal operator, associated to} \(\{B_n\}\), \textit{defined on} \(L^1(\mu)\) \textit{by} \(E^+ f(x) := \sup_n |f_n(x)|\), \textit{where} \(f_n = E^{B_n} f\) \((n \in \mathbb{N})\), \textit{is weak} \((1,1)\) \textit{(Doob’s inequality \cite{11} VII, Thm. 3.2)}, \textit{and strong} \((p,p)\), \(1 < p \leq \infty\).

(ii) \textit{Furthermore,} \(\{f_n\}\) \textit{converges almost everywhere}. \textit{Actually},

\[
\lim_{n \to \infty} f_n(x) = (E^+ f)(x) \quad \mu\text{-a.e.,}
\]

where \(B = \sigma(\bigcup_{n=1}^\infty B_n)\).

2.1. JLR on decomposition of CZ type in locally compact groups. Let \(G\) be a locally compact group with identity \(e\) and Haar measure (left invariant) \(m\), and \(H\) be a discrete subgroup of \(G\). We will first give a definition and a lemma.

Definition 5. (\cite{28} Section 1.\textsuperscript{1}) An open subset \(V\) of \(G\) is called a \textit{fundamental domain} (FD) for the quotient group \(G/H\) if these two conditions hold:

1. \(VV^{-1} \cap H = \{e\}\) (or what is the same, the restriction \(\pi|_V\) of the canonical projection \(\pi: G \to G/H\) is \(1-1\)).

2. The complement of \(VH\) in \(G\) is a locally null set.

\text{For example, the open interval} \(0, 1\) \text{is a FD for} \(\mathbb{R}/\mathbb{Z}\). \text{For each} \(n \in \mathbb{N}\), \text{the interval} \(0, \frac{1}{n}\) \text{is a FD for} \(\mathbb{T}/R_n\), \text{where} \(R_n := \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}\}\) \text{is the subgroup of the} \(n\text{th}\) \text{roots of unity in} \(\mathbb{T}\).

Lemma 6. (\cite{28} Lemma 2.\textsuperscript{1}) Assume that \(G\) contains a sequence of discrete subgroups

\[H_1 \subset H_2 \subset \cdots \subset H_n \subset \cdots \subset G\]

\text{1Cf. \cite{19} (20.11) Definition].}
such that each \( G/H_n \) is compact, and write \( k_n := \text{order}(H_{n+1}/H_n) \). Then, there is a sequence of open sets

\[
V_1 \supset V_2 \supset \cdots \supset V_n \supset \cdots,
\]

such that \( V_n \) is a FD for \( G/H_n \), and each \( V_n \) is, except for a null set, the disjoint union of \( k_n \) translates of \( V_{n+1} \) by elements of \( H_{n+1} \).

The main result of JLR on decomposition of CZ type in this context, is the following:

**Theorem 7.** ([23] Thm. 8, into the first part of the proof[].) Assume additionally that \( \cup_n H_n \) is dense in \( G \) and that

\[
\sup_n k_n = k < \infty.
\]

Then, for each \( f \in L^1(G) \) and \( a > \|f\|_1 \) there is a disjoint sequence of open sets \( S_j \) belonging to the family \( \{V_n : t \in H_n, n \in \mathbb{N}\} \), such that \( |f(x)| \leq a \) a.e. outside \( A = \cup_j S_j \), \( m(A) \leq C\|f\|_1/a \) for a constant \( C \) independent from \( f \) and \( a \), and \( a \leq |f|_{S_j} \leq ka \) \((j = 1, 2, \ldots)\).

2.2. A decomposition of CZ type in \( \mathbb{T}^\omega \).

Our subsequent Theorem will show the original proof of JLR Theorem in case of the group (compact, abelian) \( G = \mathbb{T}^\omega \). Before its statement we must establish a suitable sequence of subgroups (that JLR did actually teach us in the aforementioned personal communication), which we present below. The decomposition of CZ type in \( \mathbb{T}^\omega \) will turn out to be associated with a certain family \( \mathcal{R}_0 \) (see [6]) of “dyadic intervals”.

**Definitions 8.** ([21] VII.43[].) A net \([\mathcal{M}_n]_{n \in \mathbb{N}}\) in \( \mathbb{T}^\omega \) is a countable class of disjoint measurable sets whose union is \( \mathbb{T}^\omega \) except for a set of null measure. Let \( \{\mathcal{M}_n\}_{n \in \mathbb{N}} \) be a sequence of nets. The sequence is called monotonic if for each positive integer \( n \), every set of \( \mathcal{M}_{n+1} \) is a subset of some set of \( \mathcal{M}_n \). In this case, for almost all \( x \) there exists, for each \( n \in \mathbb{N} \), an unique set \( I_x^{(n)} \in \mathcal{M}_n \) such that \( x \in I_x^{(n)} \).

Remember that \( R_k := \{0, \frac{1}{k}, \ldots, \frac{k-1}{k}\}, k \in \mathbb{N} \). In the following table we see the first terms of the increasing sequence of subgroups \( H_m \subset \mathbb{T}^\omega \) proposed by JLR, as well as some of the first terms of the associated decreasing sequence of FD:

| \( m \) | \( H_m \) | \( V_m \) |
|---|---|---|
| 1 | \( H_1 = R_2 \times \{\emptyset^1\} \) | \( V_1 = (0, \frac{1}{2}) \times \mathbb{T}^{1,\omega} \) |
| 1 + 1 | \( H_2 = R_2 \times R_2 \times \{\emptyset^2\} \) | \( V_2 = (0, \frac{1}{2})^2 \times \mathbb{T}^{2,\omega} \) |
| | \( H_3 = R_2 \times R_2 \times \{\emptyset^2\} \) | |
| 2 | \( H_4 = R_2 \times R_2 \times \{\emptyset^2\} \) | \( V_4 = (0, \frac{1}{4})^2 \times \mathbb{T}^{2,\omega} \) |
| | \( H_5 = R_4 \times R_4 \times \{\emptyset^3\} \) | |
| 2 + 2 | \( H_6 = R_4 \times R_4 \times R_4 \times \{\emptyset^3\} \) | \( V_6 = (0, \frac{1}{4})^3 \times \mathbb{T}^{3,\omega} \) |
| | \( H_7 = R_4 \times R_4 \times R_4 \times \{\emptyset^3\} \) | |
| | \( H_8 = R_4 \times R_4 \times R_4 \times \{\emptyset^3\} \) | |
| 3 | \( H_9 = R_8 \times R_8 \times \{\emptyset^3\} \) | \( V_8 = (0, \frac{1}{8})^2 \times (0, \frac{1}{8}) \times \mathbb{T}^{3,\omega} \) |
| | \( H_{10} = R_8 \times R_8 \times R_8 \times \{\emptyset^4\} \) | \( V_9 = (0, \frac{1}{8})^3 \times \mathbb{T}^{3,\omega} \) |
| \| \| |

\[\text{Do not confuse it with net in the sense [23] Ch. 2] of directed set or generalized sequence. The concept of sequence of nets, originally in the euclidean space, is due to de la Vallé Poussin [33] 10.67]. Saks [31 p. 153] generalizes it to measure metric spaces. See also [20] §6.}
Actually, after $H_1 = R_2 \times \{0^1\}$ we define, for each $n \geq 1$

$$H_{n^2 + j} = \tilde{H}_{n^2 + j} \times \{0^{(n+1)}\}, \quad (1 \leq j \leq 2n + 1),$$

and

$$\tilde{H}_{n^2 + j} := \begin{cases} 
R_{2^n} \times (\cdots \times R_{2^n} \times R_{2^j}) & \text{if } j \in \{1, \ldots, n\}, \\
R_{2^{n+1}} \times (\cdots \times R_{2^{n+1}} \times R_{2^n} \times (2^{2n+1-j} - j) \times R_{2^n}) & \text{if } j \in \{n + 1, \ldots, 2n + 1\}.
\end{cases}$$

For each positive integer $m$, $H_m$ is a discrete (finite, of order $2^m$) subgroup of $T^\omega$, $H_m \subset H_{m+1}$, and order($H_{m+1}/H_m$) = 2 for all $m \geq 1$. Moreover, each $T^\omega/H_m$ is compact, because $T^\omega$ is, and the union $\bigcup_m H_m$ is a dense subset of $T^\omega$ as is easily checked.

![Figure 1. First members of the sequence \{\tilde{V}_m\}](image)

E.g., $\tilde{V}_7 = (0, \frac{1}{2}) \times (0, \frac{1}{4})^2$. Two translations of $V_{m+1}$ by elements of $H_{m+1}$ cover (a.e.) $V_m$.

The associated decreasing sequence of open sets $\{V_m\}$, where for each $m \geq 1$, $V_m$ is a FD for $T^\omega/H_m$, is defined by $V_1 = (0, \frac{1}{2}) \times T_1^\omega$ and, for each $n \geq 1$,

$$V_{n^2 + j} = \tilde{V}_{n^2 + j} \times T^{n+1,\omega} \quad (1 \leq j \leq 2n + 1),$$

with

$$\tilde{V}_{n^2 + j} := \begin{cases} 
(0, \frac{1}{2^n}) \times (0, \frac{1}{2^n}) & \text{if } j \in \{1, \ldots, n\}, \\
(0, \frac{1}{2^n})^{(j-n)} \times (0, \frac{1}{2^n})^{2^{n+1}-j} & \text{if } j \in \{n + 1, \ldots, 2n + 1\}
\end{cases}$$

(see Figure 1).

We can consider the (finite) net $\mathcal{N}_m := \{t + V_m : t \in H_m\}$ for each $m \in \mathbb{N}$. The sequence $\{\mathcal{N}_m\}_{m \in \mathbb{N}}$ is monotonic. Our final family $R_0$ of dyadic intervals in $T^\omega$ is the union of this monotonic sequence of nets,

$$R_0 := \bigcup_m \mathcal{N}_m = \{t + V_m : m \in \mathbb{N}, t \in H_m\}.$$

The following announced result holds. The collection $\{I_j\}$ will be a Calderón-Zygmund decomposition of intervals of $R_0$ associated with the function $f$ at level $a$.

**Theorem 9** (José L. Rubio de Francia). Let $f \in L^1(T^\omega)$ and $a > \|f\|_1$. There exists a closed set $F_a$ and an open set $\Omega_a = T^\omega \setminus F_a$ such that

(i) $|f(x)| \leq a$ for almost all $x \in F_a$. 

(ii) The set $\Omega_m$ is the union of a sequence \( \{I_j\}_{j=1}^\infty \) of pairwise disjoint intervals of the family $\mathcal{R}_0$ such that $a \leq \|f\|_{I_j} \leq 2a$ for all $I_j$.

(iii) It is verified $m(\Omega_m) \leq \frac{\|f\|}{a}$.

Proof. (See also [12, Thm. 2.10 and 2.11].)

For each $m \in \mathbb{N}$, let $\mathcal{B}_m := \sigma(N_m)$. Consider as well the trivial $\sigma$-algebra $\mathcal{B}_0 = \{\emptyset, \mathbb{T}^\omega\}$ (which is generated by the open set $V_0 := \mathbb{T}^\omega = [0, \mathbb{T}^\omega]$ and thus has the above general form if we also consider the trivial subgroup $H_0 = \{0\}$). We have $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots \mathcal{B}_m \subset \mathcal{B}_{m+1} \subset \cdots$.

Define, for $m = 0, 1, 2, \ldots$

\[
(7) \quad f_m(x) := \sum_{t \in H_m} |f|_{t+V_m} \cdot \chi_{t+V_m}(x).
\]

The function $f_m$ is $\mathcal{B}_m$-measurable for each $m$, since it is constant on each interval $t + V_m$ component of the net $N_m$. Moreover, $f_m \geq 0$ and

\[
\int_{\mathbb{T}^\omega} f_m = \sum_{t \in H_m} |f|_{t+V_m} \cdot m(V_m) = \sum_{t \in H_m} \int_{t+V_m} |f| = \int_{\mathbb{T}^\omega} |f| = \|f\|_1 < \infty
\]

by hypothesis (we have used that $\mathbb{T}^\omega \setminus (H_m + V_m)$ is a null set, since $V_m$ is a FD for $\mathbb{T}^\omega/H_m$ for each $m$). If we compare (7) with (3) we will see that $f_m$ is the conditional expectation of $|f|$ related to the $\sigma$-algebra $\mathcal{B}_m$.

By applying Theorem 4(ii) we deduce that $f_m(x) \to (E^B|f|)(x)$ a.e. as $m \to \infty$, $B$ being the least $\sigma$-algebra containing the union $\bigcup_{m=0}^\infty \mathcal{B}_m$. In our case, the set $\bigcup_{m=0}^\infty H_m$ is dense in $\mathbb{T}^\omega$, and thus the class $\bigcup_{m=0}^\infty \mathcal{B}_m$ is the class of the open sets in $\mathbb{T}^\omega$. Then $\mathcal{B}$ is the $\sigma$-algebra of Borel sets in $\mathbb{T}^\omega$. Therefore the operator $E^B$ of conditional expectation with respect to the $\sigma$-algebra $\mathcal{B}$ is the identity, and so, for all $g \in L^1(\mathbb{T}^\omega)$ it is verified that $E^Bg = g$ (a.e.). We conclude that

\[
(8) \quad \lim_{m \to \infty} f_m(x) = |f(x)| \quad \text{a.e. in } \mathbb{T}^\omega.
\]

Let

\[
f^*(x) = \sup_{m \in \mathbb{N}} f_m(x).
\]

If $x \in F_a := \{x : f^*(x) \leq a\}$, we will have $f_m(x) \leq a$ for all $m$, and an application of (3) yields $|f(x)| \leq a$, so part (i) is proven.

The set $\Omega_a = \mathbb{T}^\omega \setminus F_a$ from part (ii) is defined as $\Omega_a = \{x : f^*(x) > a\}$, and the weak type $(1,1)$ inequality (Theorem 3(i)) for the maximal operator $E^*: f \mapsto f^*$ gives

\[
(9) \quad m(\Omega_a) \leq \frac{A}{a} \|f\|_1,
\]

where $A$ is a constant independent of $f$ and $a$.

Finally, we have supposed that $\|f\|_1 < a$, thus $f_0(x) \leq a$ for all $x$, and we can recognise the set $\Omega_a$ as the disjoint union of the sets

\[
\Omega_a^{(n)} = \{x : f_i(x) \leq a < f_{i+1}(x), \ 0 \leq i \leq n-1\}, \quad n = 1, 2, \ldots.
\]

For each $n \geq 1$, the set $\Omega_a^{(n)}$ is obviously $\mathcal{B}_n$-measurable, therefore it is the disjoint union of intervals of the form $t + V_n$ with $t \in H_n$. If $f_j^{(n)}$ is one of these intervals

\footnote{According to Definitions 10 below, the operator $E^*$ would be denoted as $M^R_0$.}
we have, on one hand,

\[
\frac{1}{m(I_j^{(n)})} \int_{I_j^{(n)}} |f(x)| \, dx = \frac{1}{m(I_j^{(n)})} \int_{I_j^{(n)}} (E^{B_n}|f|)(x) \, dx \\
= \frac{1}{m(I_j^{(n)})} \int_{I_j^{(n)}} f_n(x) \, dx \geq a
\]

because \( f_n(x) > a \ \forall x \in I_j^{(n)} \subset \Omega_n^{(k)} \). On the other hand, \( I_j^{(n)} \) is contained in an interval of the form \( s + V_{n-1} \ (s \in H_{n-1}) \) which is not contained in \( \Omega_n^{(n-1)} \) (we make the agreement that \( \Omega_n^{(0)} = \emptyset \), so it is \( f_{n-1}(x) \leq a \) for all \( x \in s + V_{n-1} \). By using also that \( m(I_j^{(n)}) = m(V_n) = \frac{1}{2}m(V_{n-1}) \), we have

\[
\frac{1}{m(I_j^{(n)})} \int_{I_j^{(n)}} |f(x)| \, dx \leq \frac{2}{m(V_{n-1})} \int_{s+V_{n-1}} |f(x)| \, dx \\
= \frac{2}{m(V_{n-1})} \int_{s+V_{n-1}} (E^{B_{n-1}}|f|)(x) \, dx \\
= \frac{2}{m(V_{n-1})} \int_{s+V_{n-1}} f_{n-1}(x) \, dx \leq 2a,
\]

which finishes the proof of (ii). Moreover, from (10) it follows that \( A = 1 \) in (9). □

**Remarks.**

1. It is well known that the standard use of the CZ decomposition of the open set \( \Omega_n = \bigcup_j I_j \) involves \([9\) Ch. I, proof of Lemma 2\], also in \( T^\omega \), a decomposition of the function \( f \), at each level \( a \), in the sum of

\[
g(x) := f(x)\chi_{F_a}(x) + \sum_j f_j \chi_{I_j}(x) \quad \text{and} \quad b(x) := f(x) - g(x)
\]

\((f = g + b, g \text{ and } b \text{ good and bad (level } a\text{-parts of } f)\), verifying properties like the following:

\[
|g(x)| \leq 2a \quad \text{(a.e.,)} \quad \int_{I_j} b(x) \, dx = 0 \quad \text{and} \quad |b|_{I_j} \leq 4a \quad \text{for all } j, \text{ etc.,}
\]

(see \([14\) 5.3.8\]).

2. We have seen that JLR \([28\) uses Theorem \(4\) in his proof. The a.e. convergence part (ii) of this theorem plays the role which is played by the differentiation theorem (DT) in a standard proof of the classic result of this type. But here this is not just a style option, we believe, because DT is not a priori assured. These issues are the main content of the next section.

3. **ON DIFFERENTIATION OF INTEGRALS IN \( T^\omega \)**

We will start establishing the concepts of differentiation basis and differentiation of integrals adapted to the infinite torus space, which we will adopt in this section.

**Definitions 10.** (\([7\ Section 6.1\], \(15\ Ch. 2\)) For every \( y \in T^\omega \) let \( \mathcal{B}(y) \) be a collection of measurable sets of positive measure that contain (or whose topological closures contain) the point \( y \).

If \( \{S_n\}_n \subset \mathcal{B}(y) \) and \( \delta(S_n) \to 0 \), we say that the sequence \( S_n \) “contracts to” \( y \), and write \( S_n \Rightarrow y \).

Suppose that there exists at least a sequence \( \{S_n\} \subset \mathcal{B}(y) \) such that \( S_n \Rightarrow y \).
Let $\mathcal{B} := \bigcup_{y \in \mathbb{T}^\omega} \mathcal{B}(y)$, and suppose that $\mathcal{B}$ covers (a.e.) $\mathbb{T}^\omega$. We call $(\mathcal{B}, \Rightarrow)$ a differentiation basis.\footnote{If every $B \in \mathcal{B}$ is an open set and if $x \in B \in \mathcal{B}$ then $B \in \mathcal{B}(x)$, $\mathcal{B}$ is called a Busemann-Feller basis.}

**Examples** (The names are ours):

\[ \mathcal{R}_0 := \{ t + V_m : m \in \mathbb{N}, t \in H_m \} \quad \text{(restricted Rubio de Francia basis)}, \]
\[ \mathcal{R} := \{ y + V_m : m \in \mathbb{N}, y \in \mathbb{T}^\omega \} \quad \text{(Rubio de Francia basis)}, \]
\[ \mathcal{J} := \{ J \subset \mathbb{T}^\omega : J \text{ is an interval} \} \quad \text{(Jessen basis), \cite{21, 22}.} \]

Let $(\mathcal{B}, \Rightarrow)$ be a differentiation basis in $\mathbb{T}^\omega$. Given $f \in L^1(\mathbb{T}^\omega)$, we define the *upper and lower derivative* of $f$ with respect to $\mathcal{B}$ (and the Haar measure $m$) in the point $x \in \mathbb{T}^\omega$ by (without loss of generality we assume here that $f$ is a real function)

\[
\mathcal{D}(f, x) = \sup_{\{ B_n \} \subset \mathcal{B}} \{ \limsup_{n} f_{B_n} \} \quad \text{and} \quad \mathcal{D}(f, x) = \inf_{\{ B_n \} \subset \mathcal{B}} \{ \liminf_{n} f_{B_n} \},
\]

respectively. When

\[
\mathcal{D}(f, x) = \mathcal{D}(f, x) = f(x) \quad \text{a.e.}
\]

holds, we write $D(f, x) = f(x)$ and say that the basis $\mathcal{B}$ differentiates $f$ and that the derivative of $f$ is $f$. A necessary condition for \[(11) \] is that

\[
\lim_{n \in \mathbb{N}} f_{B_n} = f(x) \quad \text{a.e.}
\]

holds, for every sequence $\{ B_n \}_{n \in \mathbb{N}} \subset \mathcal{B}$ such that $B_n \Rightarrow x$.

When $(11)$ is satisfied for all $f \in L^\infty$ (resp. $f \in L^1(\mathbb{T}^\omega)$), we say that $\mathcal{B}$ differentiates $L^\infty(\mathbb{T}^\omega)$ (resp. $L^1(\mathbb{T}^\omega)$). Note that $L^\infty(\mathbb{T}^\omega) \subset L^1(\mathbb{T}^\omega)$ and thus, if the basis $\mathcal{B}$ does not differentiate $L^\infty(\mathbb{T}^\omega)$, then also does not differentiate $L^1(\mathbb{T}^\omega)$.

Let $\mathcal{B}$ be a DB in $\mathbb{T}^\omega$. Suppose that the *maximal operator* associated with $\mathcal{B}$ given by

\[
M^\mathcal{B}f(x) = \sup_{x \in B \in \mathcal{B}} |f|_B, \quad f \in L^1(m),
\]

is well defined (i.e., we suppose that for each $f \in L^1(m)$, $M^\mathcal{B}f$ is measurable). The following result (due to de Guzmán and Welland) holds, its proof is standard.

**Theorem 11.** (\cite{16} Thm. 1.1(a)). *If the operator $M^\mathcal{B}$ is of weak type $(1, 1)$, then the basis $\mathcal{B}$ does differentiate $L^1(\mathbb{T}^\omega)$.***

### 3.1. Differentiation Theorem on locally compact groups

#### Theorem 12. (\cite{28} Thm. 8). With the hypothesis of Theorem 7, let $\mathcal{R}$ be the family formed by all sets of the form $yV_n$ with $y \in G, n = 1, 2, \ldots$. If

\[
\sup_{n} \frac{m(V_n V_n^{-1} V_n)}{m(V_n)} < \infty
\]

holds, then $M^\mathcal{R}$ is weak type $(1, 1)$ and strong $(p, p)$ for $1 < p \leq \infty$.

As an immediate consequence JLR gives the following result, which establishes a sufficient condition for the basis $\mathcal{R}$ to differentiate $L^1_{loc}(G)$ (in this setting, the notion of contraction of a sequence $(S_n) \subset \mathcal{R}$ to a point involves $m(S_n) \to 0$).
Corollary 13. \((28\text{ Corol. } 5).\) Suppose, in addition to the hypothesis of Theorem \([12]\) that \(V_n \subset U_n\) for a basis \(\{U_n\}_{n \in \mathbb{N}}\) of neighbourhoods of \(e\). Then:

\[
\lim_{m(R) \to 0} \left| \int_R f \right| = f(x) \quad \text{ (a.e.)}
\]

for any locally integrable function \(f\).

In the case in which \(G\) is the compact group \(T^\omega\) and \(R\) is our Rubio de Francia basis, the condition \([12]\) does not hold, because e.g. for \(n \geq 1\),

\[
V_{n^2} = \left(0, \frac{1}{2^n}\right)^n \times T^{n, \omega}
\]

and

\[
V_{n^2} - V_{n^2} + V_{n^2} = \left(0, \frac{1}{2^{n+1}}\right) \cup \left(\frac{2^n - 1}{2^n}, 1\right) \times T^{n, \omega},
\]

so that

\[
\frac{m(V_{n^2} - V_{n^2} + V_{n^2})}{m(V_{n^2})} = \frac{3/2^n}{(1/2^n)^n} = 3^n, \quad \text{and} \quad \sup_{n} \frac{m(V_{n^2} - V_{n^2} + V_{n^2})}{m(V_{n^2})} = \infty.
\]

Therefore, we can not guarantee (in principle) the result of Theorem \([12]\) for the Rubio de Francia basis \(R\). The additional sufficient condition of the Corollary \([13]\) is satisfied because, for instance, the family \(\{V_n - V_k\}_{n \in \mathbb{N}}\) is a basis of (symmetric) neighbourhoods of \(0\) in \(T^\omega\). On the Rubio de Francia basis \(R\) the following questions remain open:

- Does the converse of Theorem \([11]\) for the basis \(R\) in \(T^\omega\) hold? (in de Guzmán and Welland theorem in \(R^n\) \([16\text{ Thm. } 1.1(b)]\) the BD \(B\) is required to be homothecy invariant).
- Is the operator \(M^R\) weak type \((1,1)\)?
- Does \(R\) differentiate \(L^\infty(T^\omega)\)?

Customizing Jessen’s proof for the basis \(J\), we will prove below (Subsection 3.3) that a certain basis \(R^*\) slightly wider than \(R\) does not differentiate \(L^\infty(T^\omega)\).

3.2. Bases \(R_0\) and \(J\).

Definition 14. \((31\text{ p. } 153\text{, }24\text{ VII.43}).\) Let \(\{M_n\}_{n \in \mathbb{N}}\) be a monotonic sequence of nets in \(T^\omega\), and \(M := \bigcup_n M_n\). For each \(y \in T^\omega\) and each \(k\), write \(I_y^{(k)}\) for the unique element of the net \(M_k\) which contains \(y\). We say that the sequence is indefinitely fine if for each \(x \in T^\omega\) and each \(\varepsilon > 0\) there is \(n_0 \in \mathbb{N}\) such that \(\delta(I_x^{(n_0)}) < \varepsilon\).

Then, \(I_y^{(n)} \Rightarrow x\), and \(M\) is a differentiation basis. The following result holds.

Theorem 15. \((24\text{ 43.7}, \text{ cf. also } 20\text{ §9, } 31\text{ 15.7}).\) If \(\{M_n\}_{n \in \mathbb{N}}\) is a monotonic sequence of nets indefinitely fine, then the basis \(M\) differentiates \(L^1(T^\omega)\).

At the end of \(28\), JLR pointed out, in the setting of the locally compact group \(G\), that if \(R\) is defined to be only consisting of the sets \(U_n\) \((t \in H_n)\), \(n = 1, 2, \ldots\), then Theorem \([12]\) is valid without assumptions \(4\) and \(12\). In order to corroborate this statement with an example, we provide an immediate consequence of Theorem \([15]\) It is, on the other hand, an immediate consequence of Theorem \([11]\) because the maximal operator \(M^{R_0}\) is of weak type \((1,1)\).

Corollary 16. The basis \(R_0 = \{t + V_m : m \in \mathbb{N}, t \in H_m\}\) does differentiate \(L^1(T^\omega)\).
Proof. The basis $R_0$ is the union, for $m \in \mathbb{N}$, of the monotonic sequence of nets \( \{N_m\} \), where $N_m = \{ t + V_m : t \in H_m \}$. This sequence is indefinitely fine, because if $I \in N_m$ and $(n - 1)^2 < m \leq n^2$ ($n \geq 2$), it is easily seen that
\[
\delta(I) \leq \sum_{j=1}^{n-1} \frac{1}{2^{n-1+j}} + \frac{1}{2^{n+1}} + \sum_{j=n+1}^{\infty} \frac{1}{2^j} < \frac{7}{2^{n+1}}.
\]
\[\square\]

Any subfamily of a basis that differentiates $L^1$ and which is in turn a basis of differentiation, also differentiates $L^1$. In particular, the subfamily of cubic intervals of the base $R_0$ that Saks already considered [31, p. 158] (see [7, p. 28])
\[
S := \bigcup_{m=1}^{\infty} S_m \quad \text{where} \quad S_m := \{ t + V_{m,z} : t \in H_{m,z} \},
\]
also differentiates $L^1(\mathbb{T}^\omega)$.

The question (posed by A. Zygmund, see [21, p. 55]) about the differentiation of integrals in $L^1(\mathbb{T}^\omega)$ with respect to this basis $J$ of all the intervals of $\mathbb{T}^\omega$ was answered negatively around 1950 by Jessen [21, 22]. The counterexample proposed by Jessen refers to the characteristic function of certain measurable set of positive measure, so in fact he proves that the basis $J$ does not differentiate even $L^\infty(\mathbb{T}^\omega)$. It is indeed a curious phenomenon, since the basis formed by the intervals (i.e., the parallelepipeds of edges parallel to the coordinate axes) of $\mathbb{T}^n$ does differentiate $L^\infty(\mathbb{T}^n)$ for all $n \in \mathbb{N}$ [15, p. 74].

3.3. A negative result of differentiation on $\mathbb{T}^\omega$: The basis $\mathcal{R}^*$. We begin with questions of nomenclature and notation to briefly represent some dyadic sets in $\mathbb{T}^\omega$.

Definitions 17. For $m, q \in \mathbb{N}$, $m \geq 2$, and $q \leq m$, write $\tilde{\square}_{m,q} := \{0, \frac{1}{2^m}\}^q$ and $\square_{m,q} := \tilde{\square}_{m,q} \times \mathbb{T}^\omega$.

Call $\square_{m,q}$ the $(m,q)$-cube. E.g., $V_{m^2} = \square_{m,m} = \tilde{\square}_{m,q} \times (0, \frac{1}{2^m})^{m-q} \times \mathbb{T}^\omega$, for all $m \geq q$.

Consider in $\mathbb{T}^\omega$, for $j \in \mathbb{N}$, the translation $\tau_j^m$ which adds $\frac{1}{2^m}$ to the coordinate $x_j$. Define the sets (we call them sacks of the corresponding cubes)
\[
S(\square_{m,q}) := \square_{m,q} \cup \{ \tau_j^m \mid j \in \mathbb{N}, \forall \square_{m,q} \}.
\]

and, for $y \in \mathbb{T}^\omega$, $S(y + \square_{m,q}) = y + S(\square_{m,q})$.

On the other hand, write $W_{m,r} := \square_{m,m} \cup \tau_r^m(\square_{m,m})$ ($1 \leq r \leq m$). Call $W_{m,r}$ a double $(m,r)$-cube. We have $W_{m,r} = \tilde{W}_{m,r} \times \mathbb{T}^\omega$, where
\[
\tilde{W}_{m,r} := \prod_{j=1}^{m} (1 + \delta_{r,j}) \cdot \{0, \frac{1}{2^m}\} \quad \text{(Kronecker's delta)}.
\]

E.g., $W_{m,m} = V_{m^2-1}$, but $W_{m,j} \notin \mathcal{R}$ when $1 \leq j \leq m - 1$ (see Figure 2).

We define the extended Rubio de Francia basis to be the collection
\[
\mathcal{R}^* := \mathcal{R} \cup \{ y + W_{m,r} : y \in \mathbb{T}^\omega, \ m \in \mathbb{N}, \ m \geq 2, \ 1 \leq r \leq m \}.
\]

Lemma 18. Let $Q \in \{ y + \square_{m,q} : y \in \mathbb{T}^\omega \}$. For each point $x \in S(Q)$ there is an interval $I_x \in \mathcal{R}^*$ such that
\[
\frac{m(I_x \cap Q)}{m(I_x)} \geq \frac{1}{2}.
\]
Proof. First consider any case in which \( y = 0 \), \( Q = \Box_{m,q} \). Then, if \( x \in Q \) we can take the interval \( I^0_x = \Box_{m,q} \times (0, \frac{1}{2^m})^{m-q} \times T^m \omega = V_{m^2} \in \mathcal{R} \). We have \( I^0_x \subset Q \), and
\[
\frac{m(I^0_x \cap Q)}{m(I^0_x)} = 1.
\]
Otherwise, if \( x \in \tau_r(Q) \) for any \( r, 1 \leq r \leq q \), define (\( \delta_{r,j} \) is Kronecker’s delta)
\[
D^q_{m,r} := \prod_{j=1}^{q} (1 + \delta_{r,j}) \cdot (0, \frac{1}{2^m}) \subset T^q.
\]
Then, let us take \( I^0_x = D^q_{m,r} \times (0, \frac{1}{2^m})^{m-q} \times T^m \omega \). We have \( I^0_x = W_{m,r} \in \mathcal{R}^* \), \( I^0_x \cap Q = \Box_{m,q} \times (0, \frac{1}{2^m})^{m-q} \times T^m \omega = \Box_{m,m} \), and
\[
\frac{m(I^0_x \cap Q)}{m(I^0_x)} = \frac{m(\Box_{m,m})}{m(W_{m,r})} = \frac{1}{2^2}.
\]
In a general case in which \( y \neq 0 \), take \( I_x = y + I^0_x \). The lemma follows. \( \square \)

**Lemma 19.** Let \( n \geq 2 \) be a fixed integer. It is possible to find in \( T^\omega \) an enumerable family of pairwise disjoint intervals \( \{Q_\alpha\}_{\alpha \in A_n} \), \( Q_\alpha = y(\alpha) + \Box_{m(\alpha),n} \) \( (y(\alpha) \in T^\omega, m(\alpha) \geq n) \), with the sets \( S(Q_\alpha) \) also pairwise disjoint, and:

1. If \( C_n := \bigcup_\alpha Q_\alpha \), and \( N_n := T^\omega \setminus (\bigcup_\alpha S(Q_\alpha)) \), then \( m(C_n) = \frac{1}{n+1} \), and \( m(N_n) = 0 \).
(2) For every $x \notin N_n$ there exists an interval $I^x_n \in \mathcal{R}^*$ (whose first $n$ edges are $\leq 1/2^{n-1}$; in fact, $I^x_n$ is a translate either of a cube $V_{n,x}$, or of a double cube $W_{n,x-1}$) such that

$$m(I^x_n \cap C_n) \geq \frac{1}{2}.$$ 

Proof. (In what follows, we write $t_1, \ldots, t_n \bar{0}$ for the point $(t_1, \ldots, t_n, 0^{(n)}) \in \mathbb{T}^\omega$.)

In general, for each $n \geq 2$, we consider the division of $\mathbb{T}^\omega$ into $2^{n-1}$ open cubes (call them 0-cells) of edge $1/2^{n-1}$ by the “hyperplanes”

$$x_i = \frac{j}{2^{n-1}} \quad (i = 1, \ldots, n; \; j = 0, 1, \ldots, 2^{n-1} - 1).$$

The 0-cells have the form

$$I^n_{i_1, \ldots, i_n} = \frac{i_1}{2^{n-1}} \cdots \frac{i_n}{2^{n-1}} \bar{0} + (0, \frac{1}{2^{n-1}})^n \times \mathbb{T}^n, \omega,$$

where $(i_1, \ldots, i_n) \in \{0, 1, \ldots, 2^{n-1} - 1\}^n$.

Each one of the 0-cells is firstly subdivided in $2^n$ open cubic intervals of edge length $1/2^n$ (1-cells). As an example, the 1-cells of the $I^n_{i_1, \ldots, i_n}$ 0-cell have the form

$$\left(\frac{j_1}{2^n} \cdots \frac{j_n}{2^n} \bar{0} + \Box_{n,n}\right),$$

where $(j_1, \ldots, j_n) \in \{0, 1\}^n$.

Among these 1-cells, we call $0 \cdots 0 \bar{0} + \Box_{n,n}$ the principal 1-cell and denote it by $Q_{0,0,1}^{(n)}$ (abbreviated by $Q^{(n)}_1$). The sack of $Q^{(n)}_1$ is

$$S(Q^{(n)}_1) = Q^{(n)}_1 \cup \left(\bigcup_{j=1}^n \left(\frac{\delta_{1j}}{2^n} \cdots \frac{\delta_{nj}}{2^n} \bar{0} + Q^{(n)}_1\right)\right)$$

(Kronecker’s $\delta_{ij}$),

and we have $m(S(Q^{(n)}_1)) = (n + 1) \cdot m(Q^{(n)}_1)$.

In the 0-cell $I^n_{0,0,0}$ apart from the $(n+1)$ 1-cells which form the sack $S(Q^{(n)}_1)$, there remain other $2^n - (n+1)$ 1-cells. In each one of these is carried out a subdivision into $2^n$ open cubes of edge $1/2^n+1$ (2-cells), one of which is the principal 2-cell $Q^{(n)}_{2,0} = 0 \bar{0} + \Box_{n+1,n}$, with an appropriate $y_3 \in \mathbb{T}^\omega$. There are $2^n - (n+1)$ principal 2-cells for each principal 1-cell. Now, we forget the sacks of the principal 2-cells, and in all the remaining 2-cells we proceed inductively.

The family $\{Q^{(n)}_{\alpha}\}_{\alpha \in A_n}$ of the statement of the Lemma is formed by all the principal $k$-cells ($k \geq 1$) in this construction. The indicial set $A_n$ can be defined.
Lemma 18, there exists an interval greater than 1.

Consider the characteristic function $\chi_{\bigcup_n C(p_n)}$ for each $p$. The Rubio de Francia extended basis $\mathcal{R}^*$ does not differentiate $L^\infty(T^\omega)$.

Proof. (This argumentation is taken from Jessen [22].)

Choose an increasing sequence of positive integers $(n_p)_{p=1}^\infty$ such that $\sum_p 1/(n_p + 1) \leq 3/4$. Then, the union $C := \bigcup_{p=1}^\infty C(p)$ is measurable and has measure $0 < \frac{1}{n_1 + 1} \leq m(C) \leq 3/4$, and the union $N := \bigcup_{p=1}^\infty N(p)$ is a null measurable set, since for each $p$, the measure of $N(p)$ is measurable and $m(N(p)) = 0$.

If $x \notin N$, then $x \in \bigcup_p S(Q^{(n_p)})$ for every $p$, and thus there exists a sequence of indexes $(n_p)_{p=1}^\infty$ such that $x \in S(Q^{(n_p)})$ for each $p$. Then, applying Lemma 18(2), for each $p$ there exists an interval $I(x) \in \mathcal{R}^*$ such that $\delta(I(x)) \leq \delta(W_{n,p}) < 3/2^{n_p}$ (consequently these intervals $I(x)$ form a sequence of $\mathcal{R}^*$ contracting to the point $x$), and

$$\frac{m(C(x) \cap I(x))}{m(I(x))} \geq \frac{1}{2}.$$ 

Consider the characteristic function $\chi_C$. For all $x \in T^\omega \setminus N$ (i.e., a.e. in $T^\omega$), we have

$$\limsup_{p \to \infty} \frac{1}{m(I(x))} \int_{I(x)} \chi_C(y) \, dy = \limsup_{p \to \infty} \frac{m(C(x) \cap I(x))}{m(I(x))} \geq \frac{1}{2},$$

which immediately implies that $\mathcal{D}(\int \chi_C, x) \geq \frac{1}{2}$ for almost all $x \in T^\omega$. But $\chi_C(x) = 0 < \frac{1}{2}$ for all $x \notin C$, a set of measure $\geq 1/4$. It follows that $\mathcal{R}^*$ does not differentiate $L^\infty(T^\omega)$.

Remarks.

1. The proof of Theorem 20 in fact shows that the subfamily extracted from $\mathcal{R}^*$ which is formed by the cubes $\{y + V_{m_2}: y \in T^\omega, m \geq 2\}$ and the double cubes $\{y + W_{m,r}: y \in T^\omega, m \geq 2, 1 \leq r \leq m\}$ (this subfamily is not contained in the Rubio de Francia basis $\mathcal{R}$) does not differentiate $L^\infty(T^\omega)$. 

CZ Decomposition and Differentiation of Integrals on $T^\omega$
The question whether the DB formed only by the cubes \( \{ y + V_{n^2} : y \in T^\omega, \; m \geq 2 \} \) does differentiate \( L^\infty(T^\omega) \) (with our notion of contraction of a sequence to a point) remains open for us at the moment.

Dieudonné [10] also proves that the basis of intervals in \( T^\omega \) (in fact, the subfamily of cubic intervals in \( [0,1[^\omega] \), does not differentiate \( L^\infty(T^\omega) \) (see [7, p. 28]). But Dieudonné works with the notion of contraction to a point for generalized subsequences in the Moore-Smith sense \( \{ S_n \}_{n \in D} \subset B(y) \), being \( D \) a directed set [23, p. 81-86], as we explain next:

Let \( F \) be the set of finite subsets of \( \mathbb{N} \). For each \( J \in F \) we consider

\[
T^\omega = T^J \times T^{J,\omega}
\]

in such a way that, if \( x \in T^\omega \), \( x = (x_J, x_{J'}) \) with \( x_J \in T^J \) and \( x_{J'} \in T^{J,\omega} \).

Dieudonné deals with the DB \( D = \bigcup_{x \in T^\omega} D(x) \) where \( D(x) \) is the net (according to the set \( \mathbb{N} \times F \) directed by the order relation \((n_1, J_1) \leq (n_2, J_2) \) if and only if \( n_1 \leq n_2 \) and \( J_1 \subseteq J_2 \)) that consists of the cubic intervals

\[
(V_{n,J}(x)) = V_{n,J}(x) \times T^{J,\omega}, \quad (n \in \mathbb{N}; \; J \in F),
\]

where \( V_{n,J}(x) \) is the cube of center \( x_J \) and side \( 1/n \).

Dieudonné defines a measurable set for whose characteristic function \( f \), the means \( f_{V_{n,J}(x)} \) cannot converge a.e. to \( f(x) \) according to the directed set \( \mathbb{N} \times F \).

A differentiation basis \( B \) satisfies the density property if \( B \) differentiates \( \chi_E \) for any measurable set \( E \), i.e. for almost every \( x \in T^\omega \) we have, if \( \{I_k\} \) is any arbitrary sequence of \( B(x) \) contracting to \( x \),

\[
\lim_{k \to \infty} \frac{m(E \cap I_k)}{m(I_k)} = \chi_E(x)
\]

([8], p. 227), [17], p. 30), [15], III.1). From our proof of Theorem 20 it follows that the basis \( R^* \) does not satisfies the density property.

In fact, it holds the following result (which for the space \( \mathbb{R}^n \) can be found, for instance, in [15], III, Thm. 1.4): The basis \( B \) differentiates \( L^\infty(T^\omega) \) if and only if satisfies the density property [24, Num. 11, C<\infty>D].

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CZ Decomposition and Differentiation of Integrals on $T^m$

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