Interaction between two non-threshold bound states

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Abstract

A general non-threshold BPS (F, D\(_p\)) (or (D\(_{p-2}\), D\(_p\))) bound state can be described by a boundary state with a quantized world-volume electric (or magnetic) flux and is characterized by a pair of integers (m, n). With this, we calculate explicitly the interaction amplitude between two such non-threshold bound states with a separation Y when each of the states is characterized by a pair of integers (m\(_i\), n\(_i\)) with i = 1, 2. With this result, one can show that the non-degenerate (i.e., m\(_i\)n\(_i\) \neq 0) interaction is in general attractive for the case of (D\(_{p-2}\), D\(_p\)) but this is true and for certain only at large separation for the case of (F, D\(_p\)). In either case, this interaction vanishes only if m\(_1\)/n\(_1\) = m\(_2\)/n\(_2\) and n\(_1\)n\(_2\) > 0. We also study the analytic structure of the corresponding amplitude and calculate in particular the rate of pair production of open strings in the case of (F, D\(_p\)).
1 Introduction

It is well-known by now that two parallel Dp-branes separated by a distance feel no force between them, independent of their separation, when they are both at rest. This is due to the BPS nature or the preservation of certain number of space-time supersymmetries of this system, and goes by the name “no-force” condition. This was shown initially for brane supergravity configurations through a probe [1, 2, 3] and later through the string level computations as an open string one-loop annulus diagram with one end of the open string located at one D-brane and the other end at the other D-brane making use of the “usual abstruse identity” [4]. With this feature, one can easily infer that when one of branes in the above is replaced by the corresponding anti-brane, there must be a separation-dependent non-vanishing force to arise since such a system is not a BPS one and breaks all the space-time supersymmetry. The corresponding forces can easily be computed given our knowledge of computing forces between two identical branes. In general, no separation-dependent force arising is a good indication that the underlying system preserves certain number of space-time supersymmetries.

In addition to the simple strings and simple D-branes, i.e., extended objects charged under only one NS-NS potential or one R-R potential, there also exist their supersymmetry preserving bound states such as (F, Dp) [5, 6, 7, 8, 9, 10, 11, 12] and (Dp−2, Dp) [13, 14, 15], i.e., extended objects charged under more than one potential. It would be interesting to know how to compute the forces between two such bound states separated by a distance. Since each of the bound states involves at least two kinds of branes, the force structure is richer and more interesting to explore. In this paper, we will focus on the above mentioned two types of the so-called non-threshold BPS bound states, namely (F, Dp) and (Dp−2, Dp), with even p in IIA and odd p in IIB, respectively.

The non-threshold BPS bound state (F, Dp), charged under both NS-NS 2-form potential and R-R (p+1)-form potential, is formed from the fundamental strings and Dp branes by lowering the system energy through dissolving the strings in the Dp branes, turning the strings into flux. The similar picture applies to the non-threshold BPS (Dp−2, Dp) bound state charged under both R-R (p−1)-form potential and R-R (p+1)-form potential, where the initial Dp−2 branes dissolve in Dp branes, turning into flux, too. Dirac charge quantization implies that the two potentials for either bound state are characterized by their corresponding quantized charges, therefore each bound state is characterized by a pair of integers (m, n). When the pair of integers is co-prime, the system is stable (otherwise it is marginally unstable) [16]. In this paper, we will use the description of a boundary state with a quantized world-volume flux given in [15, 12, 19] for the bound state to calculate explicitly the interaction between two non-threshold (F, Dp) (or (Dp−2, Dp)) bound states.
separated by a distance. Here each state is characterized by an arbitrary pair of integers \((m_i, n_i)\) with \(i = 1, 2\). We find that the non-degenerate (i.e., \(m_i n_i \neq 0\)) force is in general attractive for the case of \((D_{p-2}, D_p)\) but this is only certain at large separation for the case of \((F, D_p)\). This interaction in either case vanishes only if \(m_1/n_1 = m_2/n_2\) and \(n_1 n_2 > 0\).

The expected vanishing interaction for the special case of two identical \((F, D_p)\) bound states was previously shown in [12].

This paper is organized as follows. In the following section, we will review the boundary state with a given external field, therefore providing a possible representation for the non-threshold \((F, D_p)\) or \((D_{p-2}, D_p)\) bound state. In addition, we present the various couplings of the boundary state to bulk massless fields and set the conventions for the following sections. In section 3, we calculate the long-range interaction between two \((F, D_p)\) (or \((D_{p-2}, D_p)\)) bound states separated by a distance \(Y\) with each state characterized by an arbitrary pair of integers \((m_i, n_i)\) \((i = 1, 2)\), and study the underlying properties. In section 4, we calculate the interaction at the string level between two arbitrary \((F, D_p)\) (or \((D_{p-2}, D_p)\)) bound states placed parallel to each other with a separation \(Y\) using the closed string boundary state approach. We summarize the results in section 5.

2 The boundary state and its couplings

We in this section briefly review what we need about the boundary state of D-branes with a constant external field on the world-volume as well as its couplings to various bulk massless modes. In addition, we present the derivation of these couplings through the D-brane effective action with a constant world-volume field and set the conventions for this paper. The material of this section is largely taken from [17, 15, 12, 18, 19] and the detail is referred to those papers.

2.1 The boundary state with an external world-volume field

In the closed string operator formalism, the supersymmetric BPS D-branes of type II theories can be described by means of boundary states \(|B\rangle\) [20, 21]. For such a description, we have two sectors, namely NS-NS and R-R sectors, respectively. Both in the NS-NS and in R-R sectors, there are two possible implementations for the boundary conditions of a D-brane which correspond to two boundary states \(|B, \eta\rangle\) with \(\eta = \pm\). However, only the following combinations

\[|B\rangle_{\text{NS}} = \frac{1}{2} \left[|B, +\rangle_{\text{NS}} - |B, -\rangle_{\text{NS}}\right], \]

(1)
and

\[ |B\rangle_R = \frac{1}{2} [|B, +\rangle_R + |B, -\rangle_R] \]  \hspace{1cm} (2)

are selected by the GSO projection in the NS-NS and in the R-R sectors, respectively. The boundary state \( |B, \eta\rangle \) is the product of a matter part and a ghost part [17] as

\[ |B, \eta\rangle = \frac{c_p}{2} |B_{\text{mat}}, \eta\rangle |B, \eta\rangle, \]  \hspace{1cm} (3)

where

\[ |B_{\text{mat}}, \eta\rangle = |B_X\rangle |B_\psi, \eta\rangle, \quad |B, \eta\rangle = |B_{gh}\rangle |B_{sgh}, \eta\rangle. \]  \hspace{1cm} (4)

The overall normalization \( c_p \) can be unambiguously fixed from the factorization of amplitudes of closed strings emitted from a disk [22, 15] and is given by

\[ c_p = \sqrt{\pi} \left( \frac{2\pi \sqrt{\alpha'}}{\alpha'} \right)^{3-p}. \]  \hspace{1cm} (5)

The explicit expressions of the various components of \( |B\rangle \) as indicated above are given in [17] in the case of a static D-brane without any external field on its world-volume. However, as discussed in [12], the operator structure of the boundary state does not change even when more general configurations such as the presence of an external field on the world-volume are considered and is always of the form

\[ |B_X\rangle = \exp \left[ -\sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n} \cdot S \cdot \tilde{\alpha}_{-n} \right] |B_X\rangle^{(0)}, \]  \hspace{1cm} (6)

and

\[ |B_\psi, \eta\rangle_{\text{NS}} = -i \exp \left[ i \eta \sum_{m=1/2}^{\infty} \psi_{-m} \cdot S \cdot \tilde{\psi}_{-m} \right] |0\rangle \]  \hspace{1cm} (7)

for the NS-NS sector and

\[ |B_\psi, \eta\rangle_{\text{R}} = -\exp \left[ i \eta \sum_{m=1}^{\infty} \psi_{-m} \cdot S \cdot \tilde{\psi}_{-m} \right] |B, \eta\rangle^{(0)} \]  \hspace{1cm} (8)

for the R-R sector\(^5\). The matrix \( S \) and the zero-mode contributions \( |B_X\rangle^{(0)} \) and \( |B, \eta\rangle^{(0)}_{\text{R}} \) encode all information about the overlap equations that the string coordinates have to satisfy, which in turn depend on the boundary conditions of the open strings ending on the D-brane. Since the ghost and super-ghost fields are not affected by the type of the boundary conditions imposed, the corresponding part of the boundary state remains the same and its explicit expressions can be found in [17]. We would like to point out that the

\(^5\)The phases chosen in (7) and (8) are just for the convenience when we compute the couplings to various bulk massless modes.
boundary state must be written in the $(-1, -1)$ super-ghost picture in the NS-NS sector, and in the asymmetric $(-1/2, -3/2)$ picture in the R-R sector in order to saturate the super-ghost number anomaly of the disk [23, 17].

Given what has been said above, we would like to know what is the matrix $S$ when a constant gauge field $F$ is present on the world-volume. For this purpose, we consider the corresponding overlap conditions that the boundary state must satisfy [20]

$$\left[ (1 + \hat{F})^\alpha{}_\beta \alpha_n^\beta + (1 - \hat{F})^\alpha{}_\beta \bar{\alpha}_n^\beta \right] |B_X\rangle = 0$$
$$\left( q^i - y^i \right) |B_X\rangle = (\alpha_n^i - \bar{\alpha}_n^i) |B_X\rangle = 0 \quad n \neq 0 \tag{9}$$

for the bosonic part, and

$$\left[ (1 + \hat{F})^\alpha{}_\beta \psi_m^\beta - i \eta (1 - \hat{F})^\alpha{}_\beta \bar{\psi}_m^\beta \right] |B_\psi, \eta\rangle = 0$$
$$\left( \psi_m^i + i \eta \bar{\psi}_m^i \right) |B_\psi, \eta\rangle = 0 \tag{10}$$

for the fermionic part. In the above, the Greek indices $\alpha, \beta, \cdots$ label the world-volume directions $0, 1, \cdots, p$ along which the $D_p$ brane extends, while the Latin indices $i, j, \cdots$ label the directions transverse to the brane, i.e., $p + 1, \cdots, 9$. We also define $\hat{F} = 2\pi \alpha' F$.

One can check that the above equations are solved by the “coherent states” (6)-(8) with the following matrix $S$ [20, 12]

$$S = \left( [\left( \eta - \hat{F} \right) \left( \eta + \hat{F} \right)^{-1}]_{\alpha\beta}, -\delta_{ij} \right) \tag{11}$$

and with the zero-mode parts given by

$$|B_X\rangle^{(0)} = \sqrt{- \det \left( \eta + \hat{F} \right)} \delta^{9-p} \left( q^i - y^i \right) \prod_{\mu=0}^9 |k^\mu = 0\rangle \tag{12}$$

for the bosonic sector, and by

$$|B_\psi, \eta\rangle^R_{(0)} = \left( CT^{0} \Gamma^1 \cdots \Gamma^p \frac{1 + i \eta \Gamma_{11} U}{1 + i \eta} \right)_{AB} |A\rangle |\bar{B}\rangle \tag{13}$$

for the R sector. In the above, we have denoted by $y^i$ the positions of the D-brane along the transverse directions, by $C$ the charge conjugation matrix and by $U$ the following matrix

$$U = \frac{1}{\sqrt{- \det (\eta + \hat{F})}} \exp \left( -\frac{1}{2} \hat{F}_{\alpha\beta} \Gamma^\alpha \Gamma^\beta \right) ; \tag{14}$$

where the symbol $; ;$ means that one has to expand the exponential and then to anti-symmetrize the indices of the $\Gamma$-matrices. $|A\rangle |\bar{B}\rangle$ stands for the spinor vacuum of the R-R sector. We would like to point out that the $\eta$ in the above means either sign $\pm$ or the flat.
signature matrix \((-1, +1, \cdots, +1)\) on the world-volume and should not be confused from the content.

One remark follows that the overlap equations (9) and (10) do not allow to determine the overall normalization of the boundary state, and not even to get the Born-Infeld prefactor of equation (12). The latter was derived in [20]. It can also be obtained by boosting the boundary state and then performing a T-duality as explicitly shown in [24]. Notice also that this prefactor is present only in the NS-NS component of the boundary state because in the R-R sector it cancels out if we use the explicit expressions for the matrix \(U\) given in (14).

We also would like to point out that when we set the constant world-volume field \(F = 0\) in the above, everything will go over to the case of a static \(D_p\) brane without an external world-volume field [15]. When the constant world-volume field is an external electric field, the corresponding boundary state represents the BPS non-threshold \((F, D_p)\) bound state where the fundamental strings are represented by the electric flux. When the external field is a magnetic one, the boundary state is then the BPS non-threshold \((D_{p-2}, D_p)\) bound state where the lower dimensional \(D_{p-2}\) branes are represented by the magnetic flux. Each of the bound states preserves one half of the spacetime supersymmetry of the underlying string theories. These two non-threshold bound states are the focus of the present paper and we will discuss their couplings to the massless modes of the type II theories next.

### 2.2 The couplings with bulk massless modes

In this subsection, we will calculate the couplings of the non-threshold \((F, D_p)\) (or \((D_{p-2}, D_p)\)) bound state with the bulk massless modes of the underlying type II theories through the corresponding bound state world-volume effective action and bulk effective action of the given string theory (IIA or IIB). We will show that the couplings derived in the following agree completely with those found through the boundary state approach given in [12]. By this, we also set the conventions for the bulk fields in canonical form so the couplings can be used correctly in finding the long-distance interaction between two non-threshold bound states in the next section.

Let us first express the bulk fields in the effective action of a given string theory in canonical form and for this purpose we need only to consider the corresponding bosonic action. Since this works the same way in either IIA or IIB theory, we take IIA for illustration. The bosonic part of the IIA low-energy effective action in string frame is

\[
S_{\text{IIA}} = S_{\text{NS}} + S_{\text{R}} + S_{\text{CS}},
\]
\[ S_{\text{NS}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} e^{-2\Phi} \left[ R + 4(\nabla \Phi)^2 - \frac{1}{2} |H_3|^2 \right], \quad (16) \]
\[ S_{\text{R}} = -\frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left[ |F_2|^2 + |\tilde{F}_4|^2 \right], \quad (17) \]
\[ S_{\text{CS}} = -\frac{1}{4\kappa_{10}^2} \int B_2 \wedge F_4 \wedge F_4, \quad (18) \]

where NS-NS field \( H_3 = dB_2 \) while the R-R fields \( F_2 = dC_1, \tilde{F}_4 = dC_3 - C_1 \wedge H_3 \), respectively. In the above, we have grouped terms according to whether the fields are in the NS-NS or R-R sector except for the Chern-Simons action which contains both. The constant \( 2\kappa_{10}^2 \) appearing in the action is
\[ 2\kappa_{10}^2 = (2\pi)^7 \alpha'^4. \quad (19) \]

Since we are considering the field theory limit, it is proper to express the above action in the Einstein or canonical frame. This can be achieved through the so-called Einstein metric \( g_{\mu\nu} \) which is related to the string metric \( G_{\mu\nu} \) through the following
\[ g_{\mu\nu} = e^{-\phi/2} G_{\mu\nu} \quad (20) \]

where we have defined
\[ \phi = \Phi - \Phi_0 \quad (21) \]

with \( \Phi_0 \) the asymptotic value (or VEV) of the dilaton. In this frame, we have
\[ S_{\text{NS}} = \frac{1}{2g_s^2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left[ R - \frac{1}{4} (\nabla \phi)^2 - \frac{1}{2} e^{-\phi} |H_3|^2 \right], \quad (22) \]
\[ S_{\text{R}} = -\frac{1}{4\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left[ e^{3\phi/2} |F_2|^2 + e^{\phi/2} |\tilde{F}_4|^2 \right], \quad (23) \]

while the \( S_{\text{CS}} \) remains the same. In the above, we have introduced the string coupling \( g_s = e^{\Phi_0} \) and with this the physical gravitational coupling is
\[ 2\kappa^2 = 2g_s^2\kappa_{10}^2. \quad (24) \]

Considering small fluctuations of fields with respect to the flat Minkowski background, we have the action
\[ S_{\text{IIA}} = \frac{1}{2\kappa^2} \int d^{10}x \left[ -\frac{1}{4} \nabla h^{\mu\nu} \nabla h_{\mu\nu} - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} |H_3|^2 \right] - \frac{1}{4\kappa_{10}^2} \int d^{10}x \left[ |F_2|^2 + |F_4|^2 \right] + \cdots \quad (25) \]

where we keep only the lowest order terms and \( \cdots \) represents the higher order terms. In the above, \( F_2 \) and \( H_3 \) have their respective definitions defined earlier, \( F_4 = dA_3 \), and we have expanded
\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad (26) \]
with $\eta_{\mu\nu}$ the flat metric and the usual harmonic gauge for $h_{\mu\nu}$. The above action obviously becomes canonical with the following scalings:

$$h_{\mu\nu} \rightarrow 2\kappa h_{\mu\nu}, \quad \phi \rightarrow \sqrt{2}\kappa \phi, \quad B_{\mu\nu} \rightarrow \sqrt{2}\kappa B_{\mu\nu}$$  \hspace{1cm} (27)

for NS-NS fields and

$$C_n \rightarrow \sqrt{2}\kappa_{10} C_n$$  \hspace{1cm} (28)

for rank-n R-R potential. Note that the scaling of a NS-NS field differs from that of a R-R potential by a string coupling $g_s$ except for the graviton which has an additional factor of $\sqrt{2}$. These will help us to determine the corresponding couplings of bulk fields with the D-brane in the canonical form to which we will turn next.

For this, let us consider the bosonic world-volume action of a $D_p$ brane with a constant world-volume field $\hat{F}$ in string frame which is

$$S = -T_p \int d^{1+p}\sigma e^{-\Phi} \sqrt{-\det(G + B + \hat{F})} + T_p \int \left[ e^{B + \hat{F}} \wedge \sum_l C_{p+1-2l} \right]_{p+1},$$  \hspace{1cm} (29)

where the metric $G$, the NS-NS rank-2 potential $B$ and the R-R potential $C_{p+1-2l}$ are the pullbacks of the corresponding bulk fields to the world-volume, an n-form potential is defined as

$$A_n = \frac{1}{n!} A_{\alpha_1 \cdots \alpha_n} d\sigma^{\alpha_1} \wedge \cdots \wedge d\sigma^{\alpha_n},$$  \hspace{1cm} (30)

and

$$T_p = \frac{2\pi}{(4\pi^2\alpha')^{p+1/2}}.$$  \hspace{1cm} (31)

The square bracket in the above Wess-Zumino term means that in expanding the exponential form one picks up only terms of total degree of $(p+1)$. We now express the above action in Einstein frame using equation (20) as

$$S = -\frac{T_p}{g_s} \int d^{1+p}\sigma e^{(p-3)\phi/4} \sqrt{-\det(g + (B + \hat{F})e^{-\phi/2})} + T_p \int \left[ e^{B + \hat{F}} \wedge \sum_l C_{p+1-2l} \right]_{p+1},$$  \hspace{1cm} (32)

where we have also used eq.(21). By the same token, we expand the above action with fixed $\hat{F}$ to the leading order in small fluctuations of background as before and we end up with

$$S = -\frac{T_p}{g_s} \int d^{1+p}\sigma \sqrt{-\det(\eta + \hat{F})} \left\{ 1 + \frac{1}{2}[(\eta + \hat{F})^{-1}]^{\alpha\beta} (h_{\beta\alpha} + B_{\beta\alpha}) + \frac{1}{4} \left[ p - 3 - \text{Tr}(\hat{F}(\eta + \hat{F})^{-1}) \right] \phi \right\} + T_p \int \left(C_{p+1} + \hat{F} \wedge C_{p-1} + \cdots \right),$$  \hspace{1cm} (33)
where \( \cdots \) means terms with the lower rank of R-R potentials wedged with more \( \hat{F} \)'s. Now we use the scalings in (27) for NS-NS fields and in (28) for R-R fields to replace the background fluctuations in the above action and have

\[
S = - \frac{T_p \kappa}{g_s} \int d^{1+p} \sigma \sqrt{-\det(\eta + \hat{F})} \left\{ 1 + \frac{1}{\sqrt{2}} [(\eta + \hat{F})^{-1}]^{\alpha \beta} (\sqrt{2} h_{\beta \alpha} + B_{\beta \alpha}) \\
+ \frac{1}{2 \sqrt{2}} [p - 3 - \text{Tr}(\hat{F}(\eta + \hat{F})^{-1})] \phi \right\} + \sqrt{2} T_p \kappa_{10} \int \left( C_{p+1} + \hat{F} \wedge C_{p-1} + \cdots \right) (34)
\]

From the above action we can read the respective coupling in the canonical form

\[
J_h = -c_p V_{p+1} \sqrt{-\det(\eta + \hat{F})} \left[ (\eta + \hat{F})^{-1} \right]^{\alpha \beta} h_{\beta \alpha} (35)
\]

for the graviton,

\[
J_\phi = \frac{c_p}{2 \sqrt{2}} V_{p+1} \sqrt{-\det(\eta + \hat{F})} \left[ 3 - p + \text{Tr}(\hat{F}(\eta + \hat{F})^{-1}) \right] \phi (36)
\]

for the dilaton,

\[
J_B = -\frac{c_p}{\sqrt{2}} V_{p+1} \sqrt{-\det(\eta + \hat{F})} \left[ (\eta + \hat{F})^{-1} \right]^{\alpha \beta} B_{\beta \alpha} (37)
\]

for the Kalb-Ramond field,

\[
J_{C_{p+1}} = \sqrt{2} c_p V_{p+1} C_{\alpha_0 \alpha_1 \cdots \alpha_p} \varepsilon^{\alpha_0 \alpha_1 \cdots \alpha_p} (38)
\]

for the \((p + 1)\)-form RR potential,

\[
J_{C_{p-1}} = \frac{\sqrt{2} c_p}{2(p - 1)!} V_{p+1} \hat{F}_{\alpha_0 \alpha_1} C_{\alpha_2 \cdots \alpha_p} \varepsilon^{\alpha_0 \alpha_1 \cdots \alpha_p} (39)
\]

for the \((p - 1)\)-form R-R potential and so on. In the above, \( V_{p+1} \) is the world-volume of the brane, \( \varepsilon^{\alpha_0 \cdots \alpha_p} \) is the totally antisymmetric tensor on the D-brane world-volume\(^6\), and we assume that the background field fluctuations depend on only the transverse coordinates to the static brane. In the above, we have used \( c_p = T_p \kappa / g_s = T_p \kappa_{10} \) with the aid of Eqs.(5), (19), (24) and (31). The couplings obtained above are in complete agreement with those obtained in [12] using boundary state approach and will be used in the following section to obtain the long-range forces between two non-threshold bound states.

\(^6\)By conventions, \( \varepsilon^{0,1,\cdots,p} = -\varepsilon_{0,1,\cdots,p} = 1 \).
3 The long-range interactions

In this section, we will calculate the lowest-order contribution to the interaction between two arbitrary \((F, D_p)\) (or \((D_{p-2}, D_p)\)) bound states placed parallel to each other at a given separation \(Y\) due to the exchanges of massless modes, therefore representing the force at large separation. As mentioned in the Introduction, the lower dimensional brane in the bound state can be represented by the corresponding flux on the \(D_p\) brane world-volume. For the present case, the F-strings in \((F, D_p)\) can be represented by an electric flux along a given direction on the p-brane worldvolume while the \(D_{p-2}\) branes in \((D_{p-2}, D_p)\) can be represented by a magnetic flux similarly.

Let us begin with the non-threshold \((F, D_p)\) states. We choose the constant electric flux \(\hat{F}\) the following way

\[
\hat{F} = \begin{pmatrix}
0 & -f \\
-f & 0 \\
& 
\end{pmatrix} \quad \text{(40)}
\]

The couplings derived in the previous section are for a single \(D_p\) brane in the bound state and for multiple \(D_p\) branes, we should replace the \(c_p\) by \(n c_p\) in the couplings with \(n\) an integer. The constant flux is also quantized and is given for an electric flux as [12]

\[
-\frac{nf}{\sqrt{1-f^2}} = mg_s \quad \text{(41)}
\]

with \(m\) an integer. This gives \(f = -m/\Delta_{(m,n)}^{1/2}\) where we have defined

\[
\Delta_{(m,n)} = m^2 + \frac{n^2}{g_s^2} \quad \text{(42)}
\]

Then we have

\[
-\det(\eta + \hat{F}) = 1 - f^2 = \frac{n^2/g_s^2}{\Delta_{(m,n)}} \quad \text{(43)}
\]
and

\[ V \equiv (\eta + \hat{F})^{-1} = \begin{pmatrix} \frac{1}{1-f^2} & -f & -f & \vdots & \frac{1}{1-f^2} \\ \frac{f}{1-f^2} & \frac{1}{1-f^2} & \vdots & \ddots & \frac{f}{1-f^2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \frac{1}{1-f^2} & \frac{f}{1-f^2} & \cdots & \frac{1}{1-f^2} \end{pmatrix}_{(p+1) \times (p+1)} \]

With the above, we have now the couplings using (35)-(39) as

\[ J^i_h = -c_p V_{p+1} \frac{n_i^2}{g_\alpha^\Delta_{(m_i,n_i)}} V_{i}^{\alpha\beta} h_{\beta\alpha}, \quad J^i_\phi = \frac{c_p}{2\sqrt{2}} V_{p+1} \frac{(3-p)n_i^2 - 2m_i^2 g_s^2}{g_\phi^\Delta_{(m_i,n_i)}} \phi \]

\[ J^i_B = -c_p \frac{\sqrt{2}}{V_{p+1}} \frac{n_i^2}{g_\Delta_{(m_i,n_i)}} V_{i}^{\alpha\beta} B_{\beta\alpha}, \]

for the NS-NS fields and

\[ J^i_{C_{p+1}} = \sqrt{2} c_p V_{p+1} n_i C_{01 \cdots p}, \quad J^i_{C_{p-1}} = c_p V_{p+1} \frac{\sqrt{2} n_i m_i}{g_\Delta_{(m_i,n_i)}} C_{23 \cdots p} \]

for the R-R fields. Here \( i \) denotes the respective bound state with \( i = 1, 2 \).

We now calculate the long-range interaction (in momentum space) between two parallel (F, D\( p \)) bound states separated by a transverse distance \( Y \) with each state characterized by a pair of integers \( (m_i, n_i) \), respectively. The gravitational contribution due to the exchange of graviton is

\[ U_h = \frac{1}{V_{p+1}} \left( J^i_h J^i_{(1)} - J^i_h J^i_{(2)} \right) = c_p^2 V_{p+1} \frac{n_1^2 n_2^2}{g_\Delta_{(m_1,n_1)} \Delta_{(m_2,n_2)}} V_{1}^{\alpha\beta} V_{2}^{\gamma\delta} h_{\beta\alpha} h_{\gamma\delta} \]

where the propagator reads

\[ h_{\beta\alpha} h_{\gamma\delta} = \left[ \frac{1}{2} (\eta_{\beta\delta} \eta_{\alpha\gamma} + \eta_{\alpha\delta} \eta_{\beta\gamma}) - \frac{1}{8} \eta_{\alpha\beta} \eta_{\gamma\delta} \right] \frac{1}{k_\perp^2} \]
for the canonically normalized graviton propagating in the transverse directions in the de Donder (harmonic) gauge. The explicit expression for the interaction can be obtained using the matrix $V$ in the second line of (44) as

$$U_h = \frac{c_p^2}{8 g_s^2} \frac{V_{p+1}^{12} 2(7 - p)g_s^2(m_1^2 n_2^2 + m_2^2 n_1^2) + (7 - p)(p + 1)n_1^2 n_2^2}{\Omega} \tag{49}$$

with

$$\Omega \equiv \frac{1}{2} \Delta_{(m_1, n_1)}^{1/2} \Delta_{(m_2, n_2)}^{1/2} = \sqrt{(m_1^2 + n_1^2)(m_2^2 + n_2^2) g_s^2}.$$  \tag{50}$$

The contribution to the interaction due to the exchange of dilaton can be calculated as

$$U_{\phi} = \frac{1}{V_{p+1}^{12}} J_{\phi}^1 J_{\phi}^2 = \frac{c_p^2}{8 g_s^2} \frac{V_{p+1}^{12} 4g_s^4m_1^2 m_2^2 - 2(3 - p)g_s^2(m_1^2 n_2^2 + n_1^2 m_2^2) + (3 - p)^2 n_1^2 n_2^2}{\Omega} \phi_{\phi} \tag{51}$$

where $\Omega$ is given in eq.(50) and the dilaton propagator is

$$\phi_{\phi} = \frac{1}{k^2}.$$ \tag{52}$$

So we have

$$U_{\phi} = \frac{c_p^2}{8 g_s^2} \frac{V_{p+1}^{12} 4m_1^2 m_2^2 - 2(3 - p)g_s^2(m_1^2 n_2^2 + n_1^2 m_2^2) + (3 - p)^2 n_1^2 n_2^2}{\Omega}.$$ \tag{53}$$

The contribution due to the exchange of Kalb-Ramond field can be calculated similarly as

$$U_B = \frac{1}{V_{p+1}^{12}} J_B^1 J_B^2 = \frac{c_p^2}{2 g_s^2} \frac{V_{p+1}^{12} n_1^2 n_2^2 V_1^{\alpha\beta} V_2^{\gamma\delta} B_{\beta\delta} B_{\gamma\alpha}}{\Omega} \tag{54}$$

Using the propagator for the Kalb-Ramond field

$$B_{\beta\delta} B_{\gamma\alpha} = (\eta_{\beta\delta} \eta_{\alpha\gamma} - \eta_{\alpha\delta} \eta_{\beta\gamma}) \frac{1}{k^2} \tag{55}$$

and the explicit expression for the matrices $V_i$, we have

$$U_B = \frac{c_p^2}{8 g_s^2} \frac{V_{p+1}^{12}}{k^2} (-16 m_1 m_2 g_s^4).$$ \tag{56}$$

We now turn to the calculations of the contributions from R-R fields. The contribution from the exchange of R-R potential $C_{01-p}$ is

$$U_{C_{p+1}} = \frac{1}{V_{p+1}^{12}} J_{C_{p+1}}^1 J_{C_{p+1}}^2 = 2c_p^2 V_{p+1}^{12} n_1 n_2 C_{01-p} C_{01-p} \tag{57}$$
Using the propagator for the rank-(p + 1) R-R potential

\[ C_{01 \ldots p} C_{01 \ldots p} = -\frac{1}{k_\perp^2}, \]  

we have

\[ U_{C_{p+1}} = \frac{c_p^2}{8g_s^2} \frac{V_{p+1}}{k_\perp^2} (-16n_1n_2g_s^2). \]  

Similarly we have

\[ U_{C_{p-1}} \equiv \frac{1}{V_{p+1}} J_{C_{p-1}}^1 J_{C_{p-1}}^2 = 2c_p^2 V_{p+1} \frac{m_1m_2n_1n_2}{\Omega} C_{23 \ldots p} C_{23 \ldots p} \]
\[ = \frac{c_p^2}{8g_s^2} \frac{V_{p+1}}{k_\perp^2} 16m_1m_2n_1n_2g_s^2, \]

where we have used the propagator for the rank-(p - 1) R-R potential

\[ C_{23 \ldots p} C_{23 \ldots p} = \frac{1}{k_\perp^2}. \]  

Note that apart from the overall factor \( c_p^2 \frac{V_{p+1}}{k_\perp^2} \), the form field contributions are independent of the dimensionality of the bound state while this is not case for either the graviton or the dilaton contribution.

We would like to point out that each of the components calculated above agrees completely with what has been given in [12] when we set \((m_1, n_1) = (m_2, n_2)\) and \(g_s = 1\), i.e., when the two bound states are identical with string coupling set to one. We here generalize the calculations there for two arbitrary bound states which are characterized by their respective pair of integers \((m_i, n_i)\) with \(i = 1, 2\). The total contribution to the interaction from the NS-NS sector is

\[ U_{\text{NS}} = U_h + U_\phi + U_B \]
\[ = c_p^2 \frac{V_{p+1}}{k_\perp^2} \left[ 2g_s^4m_1^2m_2^2 + g_s^2(m_1^2n_2^2 + m_2^2n_1^2) + 2n_1^2n_2^2 - 2m_1m_2g_s^4\Omega \right] \]
\[ = c_p^2 \frac{V_{p+1}}{k_\perp^2} U_{\text{NS}}(m_1, n_1; m_2, n_2) \]

where in the last line we have made use of the explicit expression for \(\Omega\) given in eq.(50) and

\[ U_{\text{NS}}(m_1, n_1; m_2, n_2) = \frac{g_s^4m_1^2m_2^2 + n_1^2n_2^2 + g_s^4\Omega^2 - 2m_1m_2g_s^4\Omega}{g_s^2\Omega}, \]

while the total R-R contribution is

\[ U_{\text{R-R}} = U_{C_{p+1}} + U_{C_{p-1}} = -c_p^2 \frac{V_{p+1}}{k_\perp^2} U_{\text{R}}(m_1, n_1; m_2, n_2) \]
where
\[ U_R(m_1, n_1; m_2, n_2) = \frac{2n_1n_2(\Omega - m_1m_2)g_s^2}{g_s^2\Omega}. \] (65)

Note that although either the graviton or the dilaton contribution apart from the factor \( c_p \frac{V_{p+1}}{k^2_\perp} \) depends on the dimensionality of the brane, their addition is not. This has to be so since the form field contributions are independent of the dimensionality and “no-force” condition holds once we set the two bound states identical. The total contribution from both sectors is

\[ U = U_{NS-NS} + U_{RR} = c_p^2 \frac{V_{p+1}}{k^2_\perp} \left[ \left( g_s^2 m_1 m_2 + n_1 n_2 \right) - g_s^2 \Omega \right]^2 \frac{g_s^2 \Omega}{g_s^2 \Omega} \geq 0. \] (66)

This clearly shows that the interaction is in general attractive\(^7\) and vanishes only if
\[ g_s^2 m_1 m_2 + n_1 n_2 = g_s^2 \Omega > 0. \] (67)

For non-degenerate case, i.e., \( m_i n_i \neq 0 \) with \( i = 1, 2 \), the above implies \( m_1/n_1 = m_2/n_2 \) and \( n_1 n_2 > 0 \). In showing this, we have made use of the explicit expression for \( \Omega \) given in (50). The vanishing result for the special case of \((m_1, n_1) = (m_2, n_2)\) was previously shown in [12] and we here generalize it to a general case.

We now turn to the case for the non-threshold \((D_{p-2}, D_p)\) bound state. The calculations are similar and we list below only the necessary steps and the main results. The constant magnetic flux \( \hat{F} \) on the world-volume is chosen as

\[ \hat{F} = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & \ddots & \end{pmatrix} + \begin{pmatrix} f & -f & \cdots & 0 \\ -f & f & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & -f & \cdots & f \end{pmatrix}_{(p+1) \times (p+1)}. \] (68)

Here again we need to replace the \( c_p \) for a single \( D_p \) brane in the bound state by \( nc_p \) for multiple branes with \( n \) an integer (also due to charge quantization) in the couplings. The constant magnetic flux is also quantized and in the present case is given by \(-nf = m\) which gives \( f = -m/n\). So we have now

\[ -\det(\eta + \hat{F}) = 1 + f^2 = \frac{n^2 + m^2}{n^2}. \] (69)

\(^7\)We choose conventions here that \( U > 0 \) means attractive which differs from standard one by a sign.
and

\[
V \equiv (\eta + \hat{F})^{-1} = \begin{pmatrix}
-1 & 1 \\
1 & -1 \\
\vdots & \vdots \\
\frac{1}{1+f^2} & \frac{f}{1+f^2} \\
-\frac{f}{1+f^2} & \frac{1}{1+f^2}
\end{pmatrix}_{(p+1) \times (p+1)}
\]

\[
= \begin{pmatrix}
-1 & 1 \\
1 & -1 \\
\vdots & \vdots \\
\frac{n^2}{m^2+n^2} & -\frac{nm}{m^2+n^2} \\
\frac{nm}{m^2+n^2} & \frac{n^2}{m^2+n^2}
\end{pmatrix}_{(p+1) \times (p+1)}
\] (70)

We then have the explicit couplings for the respective bound state denoted by index \(i\) with \(i = 1, 2\) from (35)-(39) in the previous section as

\[
J^i_h = -c_p V_{p+1} \sqrt{m_i^2 + n_i^2} V_i^{\alpha \beta} h_{i \beta \alpha}, \quad J^i_\phi = \frac{c_p}{2\sqrt{2}} V_{p+1} \frac{(3-p)(n_i^2 + m_i^2) + 2m_i^2}{\sqrt{m_i^2 + n_i^2}} \phi \]

\[
J^i_B = -\frac{c_p}{\sqrt{2}} V_{p+1} \sqrt{m_i^2 + n_i^2} V_i^{\alpha \beta} B_{i \beta \alpha}
\] (71)

for the NS-NS couplings and

\[
J^i_{C_{p+1}} = \sqrt{2} c_p V_{p+1} n_i C_{01\cdots p}, \quad J^i_{C_{p-1}} = \sqrt{2} c_p V_{p+1} m_i C_{01\cdots p-2}
\] (72)

for the R-R couplings. We then have the long-range interaction due to the exchange of each of the massless fields respectively as

\[
U_\phi = \frac{c_p^2 V_{p+1}}{8 k_\perp^2} \frac{(5-p)^2 m_1^2 m_2^2 + (5-p)(3-p)(m_1^2 n_2^2 + n_1^2 m_2^2) + (3-p)^2 n_1^2 n_2^2}{\Omega},
\]

\[
U_h = \frac{c_p^2 V_{p+1}}{8 k_\perp^2} \frac{(9-p)(p-1)m_1^2 m_2^2 + (7-p)(p-1)(m_1^2 n_2^2 + n_1^2 m_2^2) + (7-p)(p+1)n_1^2 n_2^2}{\Omega},
\]

\[
U_B = \frac{c_p^2 V_{p+1}}{8 k_\perp^2} \frac{16m_1 m_2 n_1 n_2}{\Omega}
\] (73)

for the NS-NS fields and

\[
U_{C_{p+1}} = c_p^2 \frac{V_{p+1}}{k_\perp^2} (-2n_1 n_2), \quad U_{C_{p-1}} = c_p^2 \frac{V_{p+1}}{k_\perp^2} (-2m_1 m_2)
\] (74)
for the R-R fields. In the above, we have defined
\[ \tilde{\Omega} = \sqrt{(m_1^2 + n_1^2)(m_2^2 + n_2^2)}. \]  
(75)

We again have that the interaction contribution due to the exchange of the dilaton or
the graviton in the NS-NS sector apart from the facotr \( c_p V_{p+1} \) still depends on the dimen-
sionality of the world-volume while this is not the case for any form field in either NSNS
sector or the R-R sector. The total contribution to the interac-
tion from the NS-NS sector is

\[ U_{\text{NS-NS}} = U_\phi + U_h + U_B = c_p^2 \frac{V_{p+1}}{k_1^2} U_{\text{NS}}(m_1, n_1; m_2, n_2), \]  
(76)

where

\[ U_{\text{NS}}(m_1, n_1; m_2, n_2) = \frac{2m_1^2 m_2^2 + (m_1^2 n_2^2 + n_1^2 m_2^2) + 2n_1^2 n_2^2 + 2m_1 m_2 n_1 n_2}{\tilde{\Omega}}, \]  
(77)

independent of the dimensionality of the world-volume. The total interaction from the
R-R sector is

\[ U_{\text{R-R}} = U_{C_{p+1}} + U_{C_{p-1}} = -c_p^2 \frac{V_{p+1}}{k_1^2} U_{\text{R}}(m_1, n_1; m_2, n_2), \]  
(78)

where

\[ U_{\text{R}}(m_1, n_1; m_2, n_2) = 2(n_1 n_2 + m_1 m_2). \]  
(79)

The total interaction from both sectors is now

\[ U = U_{\text{NS-NS}} + U_{\text{R-R}} = c_p^2 \frac{V_{p+1} (m_1 m_2 + n_1 n_2 - \tilde{\Omega})^2}{\tilde{\Omega}} \geq 0 \]  
(80)

where in the second line we have used the explicit expression for \( \tilde{\Omega} \) given in eq.(75). This
also clearly shows that the interaction is in general attractive and vanishes only if

\[ m_1 m_2 + n_1 n_2 = \tilde{\Omega} \]  
(81)

which again implies \( m_1/n_1 = m_2/n_2 \) and \( n_1 n_2 > 0 \) for the non-degenerate case, i.e.,
\( m_i n_i \neq 0 \) with \( i = 1, 2 \), the expected supersymmetry preserving result.

We can use Fourier transformation to obtain the corresponding interaction in coordi-
nate space when \( p < 7 \) as

\[ U(Y) = \int \frac{d^p k_\perp}{(2\pi)^p} e^{-i k_\perp \cdot Y} U(k_\perp) = \frac{C(m_1, n_1; m_2, n_2)}{Y^{p-7}} \]  
(82)
where
\[ C(m_1, n_1; m_2, n_2) = \frac{c^2 V_{p+1} U(m_1, n_1; m_2, n_2)}{(7 - p)\Omega_{8-p}} \]  
(83)

with
\[ U(m_1, n_1; m_2, n_2) = \begin{cases} \frac{|(g^2 m_1 m_2 + n_1 n_2)^2 - g^2 \Omega^2_8|}{g^2 \Omega^2} & \text{for the case of } (F, D_p), \\ \frac{|(m_1 m_2 + n_1 n_2) - \tilde{\Omega}|^2}{\Omega} & \text{for the case of } (D_{p-2}, D_p), \end{cases} \]  
(84)

and \( Y^2 = Y_i Y^i \) with the summation index \( i \) along the transverse directions. In the above, we have used the following relation
\[ \int d^\perp k_\perp e^{-i k_\perp \cdot Y} = \frac{1}{(7 - p)Y^{7-p}\Omega_{8-p}}, \]  
(85)

where \( \Omega_q = 2\pi^{\frac{(q+1)}{2}}/\Gamma(\frac{(q+1)}{2}) \) is the volume of unit \( q \)-sphere.

4 The string-level force calculations

We want to go one step further to calculate the forces between two \((F, D_p)\) or \((D_{p-2}, D_p)\) bound states at a separation \( Y \) at the string level as the corresponding interaction vacuum amplitude\(^8\). In addition, we will use the results to discuss certain properties of the underlying systems such as the analytic structure of the amplitude and to calculate the rate of pair production of open strings in the open string channel.

The interaction vacuum amplitude can be calculated via
\[ \Gamma = \langle B(m_1, n_1)|D|B(m_2, n_2) \rangle \]  
(86)

where the bound state with a constant world-volume field in each sector has been given in section 2 and is characterized by a pair of integers \((m_i, n_i)\) with \( i = 1, 2 \) and \( D \) is the closed string propagator defined as
\[ D = \frac{\alpha'}{4\pi} \int_{|z|^2 \leq 1} \frac{d^2z}{|z|^2} z L_0 z \bar{L}_0. \]  
(87)

Here \( L_0 \) and \( \bar{L}_0 \) are the respective left and right mover total zero-mode Virasoro generators of matter fields, ghosts and superghosts. For example, \( L_0 = L_0^X + L_0^\psi + L_0^{gh} + L_0^{sgh} \) where \( L_0^X, L_0^\psi, L_0^{gh} \) and \( L_0^{sgh} \) represent contributions from matter fields \( X^\mu \), matter fields \( \psi^\mu \), ghosts \( b \) and \( c \), and superghosts \( \beta \) and \( \gamma \), respectively, and their explicit expressions can be found in any standard discussion of superstring theories, for example in [18], therefore

\(^8\)Actually it is the vacuum free energy.
will not be presented here even though we will need them in our following calculations. The above total vacuum amplitude has contributions from both NS-NS and R-R sectors, respectively, and can be written as \( \Gamma = \Gamma_{\text{NS}} + \Gamma_{\text{R}} \). In calculating either \( \Gamma_{\text{NS}} \) or \( \Gamma_{\text{R}} \), we need to keep in mind that the boundary state used should be the GSO projected one as given in Eq. (1) or Eq. (2). For this purpose, we need to calculate first the following amplitude

\[
\Gamma(\eta', \eta) = \langle B^1, \eta' | D | B^2, \eta \rangle
\]

in each sector with \( \eta' \eta = \pm \) and \( B^i = B(m_i, n_i) \). In doing the calculations, we can set \( \tilde{L}_0 = L_0 \) in the above propagator due to the fact that \( \tilde{L}_0 |B\rangle = L_0 |B\rangle \), which can be used to simplify the calculations. Given the structure of the boundary state in Eq. (3) and Eq. (4), the amplitude \( \Gamma(\eta', \eta) \) can be factorized as

\[
\Gamma(\eta', \eta) = \frac{n_1 n_2 c_\rho^2}{4} \frac{\alpha'}{4\pi} \int_{|z| \leq 1} \frac{d^2 z}{|z|^2} A^X A^{bc} A^\psi(\eta', \eta) A^{\beta\gamma}(\eta', \eta),
\]

where we have replaced the \( c_\rho \) in the boundary state given in section 2 by \( n c_\rho \) with \( n \) an integer to count the multiplicity of the D\( p \) branes in the bound state. In the above,

\[
A^X = \langle B^1_X | z^{2L^X_0} | B^2_X \rangle, \quad A^\psi(\eta', \eta) = \langle B^1_{\psi}, \eta' | z^{2L^\psi_0} | B^2_{\psi}, \eta \rangle,
\]

\[
A^{bc} = \langle B^1_{gh} | z^{2L^gh_0} | B^2_{gh} \rangle, \quad A^{\beta\gamma}(\eta', \eta) = \langle B^1_{sgh}, \eta' | z^{2L^{sgh}_0} | B^2_{sgh}, \eta \rangle.
\]

In order to perform the calculations using the boundary states given in (6)-(8), (12) and (13), we need to specify the worldvolume gauge field and the S-matrix given in (11) for both (F, D\( p \)) and (D\( p-2 \), D\( p \)) bound states, respectively. For the case of (F, D\( p \)), we need to use (40) for \( \hat{F} \) with \( f \) determined by (41), i.e., \( f = -m/\Delta_{(m,n)}^{1/2} \) through the charge quantization. The corresponding longitudinal part of the S matrix as given in (11), is now

\[
S_{\alpha\beta} = \begin{pmatrix}
-\frac{1+f^2}{1-f^2} & \frac{2f}{1-f^2} \\
-\frac{2f}{1-f^2} & \frac{1+f^2}{1-f^2}
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 \\
1
\end{pmatrix} (p+1) \times (p+1)
\]
\[
\begin{pmatrix}
-\frac{g_s^2(\Delta_{(m,n)}+m^2)}{n^2} & -\frac{2mg_s^2\Delta_{(m,n)}^{1/2}}{n^2} \\
\frac{2mg_s^2\Delta_{(m,n)}^{1/2}}{n^2} & \frac{g_s^2(\Delta_{(m,n)}+m^2)}{n^2}
\end{pmatrix}
\begin{pmatrix}
1 \\
1
\end{pmatrix}
\]  
(91)

While for \((D_{p-2}, D_p)\), we need to use (68) for \(\hat{F}\) with the quantized \(f = -m/n\). Now we have the longitudinal part of the S matrix as

\[
S_{\alpha\beta} = \begin{pmatrix}
-1 \\
1 \\
. \\
. \\
. \\
\frac{1-f^2}{1+f^2} & 2f & \frac{1-f^2}{1+f^2} \\
- \frac{2f}{1+f^2} & \frac{1-f^2}{1+f^2} & \frac{1-f^2}{1+f^2} \\
. \\
. \\
\frac{n^2-m^2}{m^2+n^2} & \frac{2nm}{m^2+n^2} & \frac{n^2-m^2}{m^2+n^2}
\end{pmatrix}_{(p+1)\times(p+1)}
\]  
(92)

With the above preparations, we are now ready to perform rather straightforward calculations for the various matrix elements specified in (90) in either NS-NS or R-R sector for either of the bound states under consideration, using (6)-(8), (12) and (13) for the boundary states with \(\hat{F}\) and the matrix \(S\) given in (11) as just described for either of the bound states. We have now

\[
A^X = C_F V_{p+1} e^{-\frac{\psi^2}{2\pi\alpha' L}} \left(2\pi\alpha' t\right)^{-\frac{2}{2}} \prod_{n=1}^{\infty} \left(1 - \lambda |z|^{2n} \right) \left(1 - \lambda^{-1} |z|^{2n} \right) \frac{1}{(1 - |z|^{2n})(1 - |z|^{2n})(1 - |z|^{2n})^8},
\]

\[
A^{bc} = |z|^{-2} \prod_{n=1}^{\infty} (1 - |z|^{2n})^2,
\]

(93)

for both NS-NS and R-R sectors,

\[
A^{\beta\gamma}_{\text{NS}}(\eta', \eta) = |z| \prod_{n=1}^{\infty} \frac{1}{(1 + \eta' \eta |z|^{2n-1})^2},
\]
\[
A_{\text{NS}}^\psi = \prod_{n=1}^{\infty} \left( 1 + \eta \eta' \lambda |z|^{2n-1} \right) \left( 1 + \eta \eta' \lambda^{-1} |z|^{2n-1} \right) \left( 1 + \eta' \eta |z|^{2n-1} \right)^8,
\]
(94)
for NS-NS sector, and
\[
A_{\text{R}}^{\beta' \gamma} (\eta', \eta) A_{\text{R}}^{\psi} (\eta', \eta) = -2^4 |z|^2 D_F \delta_{\eta' \eta} + \prod_{n=1}^{\infty} \left( 1 + \lambda |z|^{2n} \right) \left( 1 + \lambda^{-1} |z|^{2n} \right) (1 + |z|^{2n})^6,
\]
(95)
for the R-R sector. Note that we have \(|z| = e^{-\pi t}\) above and in (95) we have followed the prescription given in [17, 18] not to separate the contributions from matter fields \(\psi^\mu\) and superghosts in the R-R sector in order to avoid the complication due to the respective zero modes. Also in the above, we have
\[
C_F = \begin{cases} 
\sqrt{(1 - f_1^2)(1 - f_2^2)} & \text{for } (F, D_p), \\
\sqrt{(1 + f_1^2)(1 + f_2^2)} & \text{for } (D_{p-2}, D_p),
\end{cases}
\]
(96)
\[
D_F = \begin{cases} 
\sqrt{1 - f_1 f_2} & \text{for } (F, D_p), \\
\sqrt{1 + f_1 f_2} & \text{for } (D_{p-2}, D_p),
\end{cases}
\]
(97)
and
\[
\lambda + \lambda^{-1} = 2(2D_F^2 - 1) = \begin{cases} 
2 \frac{(1 + f_1^2)(1 + f_2^2) - 4f_1 f_2}{(1 - f_1^2)(1 - f_2^2)} & \text{for } (F, D_p), \\
2 \frac{(1 - f_1^2)(1 - f_2^2) + 4f_1 f_2}{(1 + f_1^2)(1 + f_2^2)} & \text{for } (D_{p-2}, D_p),
\end{cases}
\]
(98)
with the previously given
\[
f_i = \begin{cases} 
-\frac{m_i}{\Delta \left( m_i, n_i \right)} & \text{for } (F, D_p), \\
-\frac{m_i}{n_i} & \text{for } (D_{p-2}, D_p),
\end{cases}
\]
(99)
where \(i = 1, 2\) and the explicit expression for \(\Delta \left( m_i, n_i \right)\) is given in (42).

In calculating \(A^X\) and \(A^\psi (\eta', \eta)\) as given explicitly above, we have made use of an important property for the S matrix
\[
S_{\mu \nu}^{T} = \delta_{\mu \nu},
\]
(100)
with \(T\) denoting the transpose. We can check this using, for example, the explicit expression (11) for \(S_{\mu \nu}\) with the indices raised or lowered using the corresponding metric. This property enables us to perform unitary transformations of the respective operators in the boundary states (6)-(8) such that the S matrix appearing in one of the boundary
states, for example, in the boundary state originally denoting as ‘1’ above, completely disappears, while leaving the other one (originally denoting as ‘2’) with a new S matrix as $S = S_2S_1^T$, in the course of evaluating the respective $A^X$ or $A^\psi$. This new S matrix shares the same property (100) as the original $S_1$ and $S_2$ do but its determinant is always equal to one. Therefore this S matrix under consideration can always be diagonalized to give two eigenvalues $\lambda$ and $\lambda^{-1}$ with their sum as given in (98) above and the other eight eigenvalues all equal to one. This is the basis for the structure appearing in the contributions due to the respective oscillators to the $A^X$ and $A^\psi(\eta, \eta')$ as given in (93)-(95) above.

We can now have the vacuum amplitude in the NS-NS sector as

$$\Gamma_{\text{NS}} = \langle B^1 | D | B^2 \rangle_{\text{NS}}$$

$$= \frac{n_1 n_2 c_2^2 V_{p+1} C_F}{32\pi(2\pi^2\alpha')^{7/2}} \int_0^\infty \frac{dt}{t} e^{-\frac{\lambda^2}{2\pi\alpha'}} t^{-\frac{7+\nu}{2}} x |z|^{-1} \left[ \prod_{n=1}^\infty \frac{(1 + \lambda |z|^{2n})(1 + \lambda^{-1} |z|^{2n})(1 + |z|^{2n})}{(1 - \lambda |z|^{2n})(1 - \lambda^{-1} |z|^{2n})(1 - |z|^{2n})^6} \right. - \left. \prod_{n=1}^\infty \frac{(1 - \lambda |z|^{2n})(1 - \lambda^{-1} |z|^{2n})(1 - |z|^{2n})}{(1 - \lambda |z|^{2n})(1 - \lambda^{-1} |z|^{2n})(1 - |z|^{2n})^6} \right], \quad (101)$$

where we have used the GSO projected boundary state in (1) for $|B^i\rangle_{\text{NS}}$ ($i = 1, 2$) with $B^i$ as defined previously and have made use of the matrix elements in (93) and (94). Also we have used in the above

$$\int_{|z|\leq 1} \frac{d^2z}{|z|^2} = 2\pi \int_0^\infty dt, \quad (102)$$

with $|z| = e^{-\pi t}$. The corresponding vacuum amplitude in the R-R sector is now

$$\Gamma_{\text{R}} = \langle B^1 | D | B^2 \rangle_{\text{R}}$$

$$= -\frac{n_1 n_2 c_2^2 V_{p+1} C_F}{2\pi(2\pi^2\alpha')^{7/2}} \int_0^\infty \frac{dt}{t} e^{-\frac{\lambda^2}{2\pi\alpha'}} t^{-\frac{7+\nu}{2}} x \prod_{n=1}^\infty \frac{(1 + \lambda |z|^{2n})(1 + \lambda^{-1} |z|^{2n})(1 + |z|^{2n})}{(1 - \lambda |z|^{2n})(1 - \lambda^{-1} |z|^{2n})(1 - |z|^{2n})^6}, \quad (103)$$

where we have used the GSO projected boundary state in (2) for $|B^i\rangle_{\text{R}}$ ($i = 1, 2$) again with $B^i$ as defined previously and made use of the matrix elements in (93) and (95) as well as the equation (102). In the above, we always assume both $n_1$ and $n_2$ are positive integers and the p-branes in the non-threshold bound states are both D$p$ branes (or both anti D$p$ branes). In the case when the p-branes in either of the non-threshold bound states (but not both) are anti D$p$ branes, the corresponding $\Gamma_{\text{R}}$ will switch sign from the one above but the $\Gamma_{\text{NS}}$ will remain the same. In what follows, we will focus on that the p-branes
in both non-threshold bound states are D\(_p\)-branes, i.e., (101) and (103) are valid. The case when the p-branes in either of the bound states are anti D\(_p\)-branes can be similarly analyzed.

We would like to pause here to make a few checks of the above results (101) and (103) against known ones. When we set \( n_1 = n_2 = 1 \) and switch off the worldvolume gauge fields, i.e., setting \( f_1 = f_2 = 0 \) (therefore \( C_F = D_F = 1 \) and \( \lambda = \lambda^{-1} = 1 \)), our above \( \Gamma_{NS} \) and \( \Gamma_R \) agree with the well-known results between two identical Dp-branes placed parallel to each other and separated by a distance \( Y \). For example, our results completely agree with the calculations given in Eq. (9.285) and Eq. (9.289) in [18] when we set \( p = p' \), i.e., \( \nu = 0 \), in their case if we notice that

\[
\frac{c_p^2}{32\pi(2\pi^2\alpha')^{\frac{3}{2}}} = \frac{1}{(8\pi^2\alpha')^{\frac{3}{2}}} \times \frac{1}{2},
\]

(104)

where we have used the explicit expression (5) for \( c_p \). For the case of (F, D\(_p\)) bound state, when two such bound states are identical, i.e., \( f_1 = f_2 = -m/\Delta_{(m,n)} \), the results for \( \Gamma_{NS} \) and \( \Gamma_R \) with the string coupling set to unit were given in [12] as mentioned earlier. Applying the same conditions to our calculations for the (F, D\(_p\)) case, we again find perfect agreements if we make use of (104) and notice the following: 1) \( S_1 = S_2 \), therefore the matrix \( S = S_1 S_2^T \) is now a unit matrix and so \( \lambda = \lambda^{-1} = 1 \); 2)

\[
D_F = 1, \quad C_F = 1 - f^2 = \frac{n^2}{g_s^2 \Delta_{(m,n)}},
\]

(105)

with \( \Delta_{(m,n)} \) given in (42) and \( g_s \) set equal to unit; 3) Their integration variable \( t \) is \( \pi \) times ours.

The total vacuum amplitude is now

\[
\Gamma = \Gamma_{NS} + \Gamma_R = \frac{n_1 n_2 V_{p+1} C_F}{2(2\pi^2\alpha')^{\frac{3}{2}}} \int_0^\infty \frac{dt}{t} e^{-\frac{3}{2\pi\alpha'} t} \left( \frac{t}{2} \right)^{-\frac{3}{2}}
\times \left\{ |z|^{-1} \left[ \prod_{n=1}^{\infty} \frac{(1 + \lambda |z|^{2n-1})(1 + \lambda^{-1} |z|^{2n-1})(1 + |z|^{2n-1})^6}{(1 - \lambda |z|^{2n})(1 - \lambda^{-1} |z|^{2n})(1 - |z|^{2n})^6} \right] - \prod_{n=1}^{\infty} \frac{(1 - \lambda |z|^{2n})(1 - \lambda^{-1} |z|^{2n})(1 - |z|^{2n})^6}{(1 - \lambda |z|^{2n})(1 - \lambda^{-1} |z|^{2n})(1 - |z|^{2n})^6} \right] \right. 
-2^4 D_F \prod_{n=1}^{\infty} \frac{(1 + \lambda |z|^{2n})(1 + \lambda^{-1} |z|^{2n})(1 + |z|^{2n})^6}{(1 - \lambda |z|^{2n})(1 - \lambda^{-1} |z|^{2n})(1 - |z|^{2n})^6 \left( \prod_{n=1}^{\infty} \frac{(1 + \lambda |z|^{2n})(1 + \lambda^{-1} |z|^{2n})(1 + |z|^{2n})^6}{(1 - \lambda |z|^{2n})(1 - \lambda^{-1} |z|^{2n})(1 - |z|^{2n})^6} \right)^6}, \quad (106)
\]

where we have used the explicit expression (5) for \( c_p \) and Eq. (104). This is our basic result of this paper in addition to the long-distance one given in the previous section. At
first look, this is completely different from the calculation given in [25] for \( p = 1 \), i.e. the D-string case in the Wick rotated version using the light-cone boundary state. In what follows, we will show that our result above is indeed the same as theirs for \( p = 1 \) using various \( \theta \)-function relations. For this purpose, let us express our amplitude (106) in terms of \( \theta \)-functions and the Dedekind \( \eta \)-function with their standard definitions as given, for example, in [26]. We then have

\[
\Gamma = \frac{n_1 n_2 V_{p+1} C_F \sin \pi \nu}{(8 \pi^2 \alpha')^{1/2}} \int_0^\infty \frac{dt}{t} e^{-\frac{\nu^2}{8\pi \alpha'} t^{-\frac{\pi}{4}}} \times \frac{1}{\eta^9(it)} \left[ \frac{\theta_3(\nu|it) \theta_3^3(0|it)}{\theta_1(\nu|it)} - \frac{\theta_4(\nu|it) \theta_4^3(0|it)}{\theta_1(\nu|it)} - \frac{\theta_2(\nu|it) \theta_2^3(0|it)}{\theta_1(\nu|it)} \right],
\]

(107)

where we have defined \( \lambda = e^{2\pi i \nu} \) and used the fact \( \cos \pi \nu = D_F \) which can be obtained from \( \lambda + \lambda^{-1} = 2(2D_F^2 - 1) \) as given in (98). Note that \( \nu = i \nu_0 \) with \( 0 \leq \nu_0 < \infty \) for the case of \((F, D_p)\) while \( \nu = \nu_0 \) with \( 0 \leq \nu_0 < 1 \) for \((D_{p-2}, D_p)\). Further \( \nu_0 \to \infty \) when \( f_1 \neq f_2 \) and either of \( |f_i| \to 1 \) (or both \(|f_i| \to 1 \) when \( f_1 = -f_2 \)) in the former case while \( \nu_0 \to 1 \) when \( f_1 = -f_2 \) with \( |f_i| \to \infty \) in the latter case but \( \nu_0 = 0 \) when \( f_1 = f_2 \) in both cases. Now we use the following identity for \( \theta \)-functions

\[
2 \theta_1^4(\nu|\tau) = \theta_3(2\nu|\tau) \theta_3^3(0|\tau) - \theta_4(2\nu|\tau) \theta_4^3(0|\tau) - \theta_2(2\nu|\tau) \theta_2^3(0|\tau),
\]

(108)

which is obtained from (iv) on page 468 in [27]. With the identity (108), the amplitude (107) is greatly simplified to

\[
\Gamma = \frac{2 n_1 n_2 V_{p+1} C_F \sin \pi \nu}{(8 \pi^2 \alpha')^{1/2}} \int_0^\infty \frac{dt}{t} e^{-\frac{\nu^2}{8\pi \alpha'} t^{-\frac{\pi}{4}}} \times \frac{1}{\eta^9(it)} \left[ \frac{\theta_3(\nu|it) \theta_3^3(0|it) - \theta_4(\nu|it) \theta_4^3(0|it) - \theta_2(\nu|it) \theta_2^3(0|it)}{\theta_1(\nu|it)} \right],
\]

(109)

where in the second equality, we have made use of

\[
\sin^4 \frac{\pi \nu}{2} = \frac{1}{4} \cos^2 \frac{\pi \nu}{2} = \frac{1}{4} (D_F - 1)^2, \quad n_1 n_2 C_F (D_F - 1)^2 = U(m_1, m_1; m_2, n_2),
\]

(110)

\footnote{In obtaining the above identity from the more general one (iv) there, we have made choices of variables \( x' = y' = z' = 0 \) and \( w' = 2z \) which give \( w = -z \) and \( x = y = z \) in their notation. Note also that their notation for \( \theta \)-functions is \( \theta_r(z) = \theta_r(z|\tau) \) with \( r = 1, 2, 3, 4 \). We also use the facts that \( \theta_1(0|\tau) = 0 \) and \( \theta_1(-z|\tau) = -\theta_1(z|\tau) \).}
with \( U(m_1, n_1; m_2, n_2) = U_{\text{NS}}(m_1, n_1; m_2, n_2) - U_{\text{R}}(m_1, n_1; m_2, n_2) \) as given by (84) for either case under consideration and with the respective quantization for \( f_i \) as given previously, and in the third equality we have made use of explicit expressions for the Dedekind \( \eta \)-function and the theta-function \( \theta_1 \).

One can check now that our above amplitude in the present various forms does agree with the calculations given in [25] for the \( p = 1 \) case in the light-cone approach up to an overall constant factor\(^{10}\) of \( 1/(8\pi^6) \). In making the comparison, we need also to consider that in their calculations they chose \( \alpha' = 2 \) and their parameter \( \alpha \) is related to our \( \nu \) as \( \alpha = 2\pi\nu \).

We now consider the large \( Y \) limit of the amplitude (109). This amounts to accounting for the massless-mode contribution of closed string and therefore the result should agree with our low-energy effective field theory calculations performed in the previous section. We will find that this is indeed true\(^{11}\). For large \( Y \), the separation dependent exponential suppression factor in (109) implies that the contribution to the amplitude comes from the large \( t \) integration. Note that for large \( t \), \(|z| = e^{-\pi t} \to 0\) and

\[
\theta_1(\nu | it) \to 2e^{-\frac{\pi t}{\nu}} \sin \pi \nu, \quad \theta_1\left(\frac{\nu}{2} | it\right) \to 2e^{-\frac{\pi t}{\nu}} \sin \frac{\pi \nu}{2}, \quad \eta(it) \to e^{-\frac{\pi t}{\nu}}. \tag{111}
\]

So

\[
\Gamma \to \frac{U(m_1, n_1; m_2, n_2) V_{p+1}}{2(8\pi^2\alpha')^{\frac{1}{4}p}} \sin \frac{\pi \nu}{2} \int_0^\infty \frac{dt}{t} e^{-\frac{\nu^2}{2\pi\alpha' t}} t^{-\frac{7+p}{2}} \frac{1}{e^{-\frac{\pi t}{4}}} 2^4 e^{-\pi t} \sin^4 \frac{\pi \nu}{2},
\]

\[
= \frac{4 U(m_1, n_1; m_2, n_2) V_{p+1}}{(8\pi^2\alpha')^{\frac{1}{4}p}} \int_0^\infty \frac{dt}{t} e^{-\frac{\nu^2}{2\pi\alpha' t}} t^{-\frac{7+p}{2}},
\]

\[
= \frac{4 U(m_1, n_1; m_2, n_2) V_{p+1}}{(8\pi^2\alpha')^{\frac{1}{4}p}} \left( \frac{2\pi\alpha'}{Y^2} \right)^{\frac{7+p}{2}} \Gamma\left( \frac{7+p}{2} \right),
\]

\[
= \frac{C(m_1, n_1; m_2, n_2)}{Y^{7-p}}, \tag{112}
\]

where \( C(m_1, n_1; m_2, n_2) \) is given by (83). So this is in complete agreement with our low-energy result (82), as expected, which in turn shows that even our normalization constant is also correct. In reaching the last equality, we have made use of (104) and \((7-p)\Omega_{8-p} = 4\pi\pi^{(7-p)/2}/\Gamma((7-p)/2)\) with \( \Omega_q \) the volume of unit \( q \)-sphere.

\(^{10}\)In making the comparison, we have considered both zero-mode contribution (77) and the oscillator contribution (82) in [25] for the magnetic flux. For the case of electric flux, one should send \( f_1 \to if_1 \) and \( f_2 \to if_2 \) as well as \( \alpha \to i\alpha \) as mentioned there. In their calculation, the volume factor was not considered and the overall constant factor difference mentioned in the text should not be concerned here since it is well-known that the light-cone calculations alone cannot fix the overall constant.

\(^{11}\)One can also show that \( \Gamma_{\text{NS}} \) (101) and \( \Gamma_{\text{R}} \) (103) give also their corresponding low energy limits as discussed in the previous section in a similar fashion.
The interaction amplitude (109) vanishes when $U(m_1, n_1; m_2, n_2) = 0$ which gives $m_1/n_1 = m_2/n_2$ (note $n_1n_2 > 0$) as shown in the previous section (now $\nu = 0$ since $f_1 = f_2$), reflecting the BPS property of the system. If we take one pair of integers, say the pair $(m_2, n_2)$, as co-prime, then the vanishing amplitude would need $(m_1, n_1) = k(m_2, n_2)$ with $k$ a positive integer. Note that unlike the single brane case, the non-threshold bound states have infinite many stable fundamental states with each characterized by a different pair of co-prime integers $(m, n)$. When placing a brane with a pair of integers $k(m, n)$ parallel to one with its pair of integers $k'(m, n)$, we have the system breaking no supersymmetry and being BPS if $kk' > 0$, i.e., integer $k$ and integer $k'$ have the same sign. When $(m_1, n_1)$ and $(m_2, n_2)$ are both co-prime, the interaction vanishes only if $(m_1, n_1) = (m_2, n_2)$. Further when none of the above is satisfied, we have $U(m_1, n_1; m_2, n_2) > 0$. Note that each numerator in the infinite product in the integrand of (109)

$$
1 - e^{i\pi \nu |z|^{2n}} 1 - e^{-i\pi \nu |z|^{2n}} = 1 - 2\cos\pi \nu |z|^{2n} + |z|^{4n})^4 > 0,
$$

so the sign of the interaction amplitude will depend on that of the factor in each denominator in the infinite product in the integrand

$$
(1 - e^{2i\pi \nu |z|^{2n}})(1 - e^{-2i\pi \nu |z|^{2n}}) = (1 - 2\cos 2\pi \nu |z|^{2n} + |z|^{4n})^4 > 0,
$$

which is always positive for the case of $(D_{p-2}, D_p)$ (now $\nu$ is real) while it is positive for large $t$ but it can be negative for small $t$ for the case of $(F, D_p)$ for which $\nu$ is purely imaginary. So for the case of $(D_{p-2}, D_p)$, the interaction amplitude is now greater than zero and is solely determined by the positiveness of $U(m_1, n_1; m_2, n_2)$. In this aspect it shares the same feature as its long distance interaction shown in the previous section, reflecting the attractive nature of the interaction. For the case of $(F, D_p)$, while the long distance interaction amplitude is again now greater than zero (implying attractive interaction) and is also solely determined by the positiveness of the corresponding $U(m_1, n_1; m_2, n_2)$ as shown in the previous section, the sign of the small separation amplitude (corresponding to small $t$ contribution) is uncertain in the present representation of integration variable $t$ since even with the factor in (114) less than zero, the sign of the product of infinite such factors in the integrand remains indefinite. So one would expect some interesting physics to appear in this case for small $t$.

The small $t$ contribution to the amplitude mainly concerns about the physics for small separation $Y$. The appropriate frame for describing the underlying physics as well as the analytic structure as a function of the separation in the short cylinder limit $t \to 0$ is in terms of an annulus, which can be achieved by the Jacobi transformation $t \to t' = 1/t$. This is also stressed in [28] that the lightest open string modes now contribute most and
the open string description is most relevant. So in terms of the annulus variable $t'$, noting

$$\eta(\tau) = \frac{1}{(-i\tau)^{1/2}} \eta\left(\frac{-1}{\tau}\right),$$

$$\theta_1(\nu|\tau) = i \frac{e^{-i\nu^2/\tau}}{(-i\tau)^{1/2}} \theta_1\left(\frac{\nu}{\tau}\right),$$

(115)

the second equality in (109) now becomes

$$\Gamma = -i \frac{U(m_1, n_1; m_2, n_2) V_{p+1}}{2(8\pi^2\alpha')^{\frac{1}{2}} \sin \frac{\pi \nu}{2}} \int_0^\infty \frac{dt'}{t'} e^{-\frac{\nu^2 t'}{2\pi \alpha'}} t'^{\frac{1}{2}} \frac{1}{\eta(\nu t') \theta_1\left(\frac{-\nu t'}{2}\right)} \theta_1^{4}\left(\frac{-i\nu t'}{2}\right) |it'|,$$

$$\times \prod_{n=1}^{\infty} \frac{\left(1 - e^{\nu t'}|z|^{2n}\right)^4 \left(1 - e^{-\nu t'}|z|^{2n}\right)^4}{\left(1 - e^{2\nu t'}|z|^{2n}\right) \left(1 - e^{-2\nu t'}|z|^{2n}\right)},$$

(116)

with now $|z| = e^{-\nu t'}$. We follow [25] to discuss the underlying analytic structure and the possible associated physics of the amplitude of (116). For the case of $(D_{p-2}, D_p)$, we limit ourselves to the interesting non-BPS amplitude, i.e., $\nu = \nu_0$ with $0 < \nu_0 < 1$, and for this the above amplitude is real and has no singularities unless $Y \leq 2\pi \sqrt{\nu \alpha'}$, i.e. on the order of string scale, for which the integrand is dominated by, in the short cylinder limit $t' \to \infty$,

$$\lim_{t' \to -\infty} e^{-\frac{\nu^2 t'}{2\pi \alpha'}} \theta_1^{4}\left(\frac{-i\nu t'}{2}\right) \eta(\nu t') \theta_1\left(\frac{-\nu t'}{2}\right) \sim \lim_{t' \to -\infty} e^{-\frac{\nu^2 t'}{2\pi \alpha'}} \sin^{4}\left(\frac{-i\nu t'}{2}\right) \sim \lim_{t' \to \infty} e^{-\frac{t'}{2\pi \alpha'}}(\nu^2 - 2\nu \alpha'').$$

The contribution of the annulus to the vacuum amplitude (free energy) should be real if the integrand in (116) have no simple poles on the positive $t'$-axis since the imaginary part of the amplitude is given by the sum of residues at the poles times $\pi$ due to the integration contour passing to the right of all poles as dictated by the proper definition of the Feynman propagator[29]. In the present case, the amplitude appears purely real but there are no simple poles on the positive $t'$-axis, therefore giving zero imaginary amplitude, i.e., zero pair-production (absorptive) rate, which is consistent with the conclusion reached in [30] in quantum field theory context and also pointed out in a similar context in [31]. When $Y \leq \pi \sqrt{2\nu_0 \alpha'}$, i.e., on the order of string scale, the integration in (116) diverges and this therefore gives a divergent amplitude which indicates the breakdown of the calculations and behaves similarly to the situation of brane/antibrane systems as studied in [32, 33], signalling the possible onset of tachyonic instability now caused instead by the magnetic...
fluxes\textsuperscript{12} and the relaxation of the system to form new non-threshold bound state. However, the detail of this requires further dynamical understanding.

Let us move to the case of \((F, D_\alpha)\). We have now \(\nu = i\nu_0\) with \(0 < \nu_0 < \infty\) \((\nu_0 = 0\) corresponds to BPS case and is not considered here). The amplitude \((116)\) is now

\[
\Gamma = \frac{4U(m_1, n_1; m_2, n_2)V_{p+1}}{(8\pi^2\alpha')^{1+p}} \frac{\sinh \pi\nu_0}{\sinh^4 \frac{\pi\nu_0}{2}} \int_0^\infty \frac{dt'}{t'} e^{-\frac{\nu_0^2 t'}{2\pi\alpha'}} t'^{\frac{1-p}{2}} \frac{\sin^4 \left(\frac{\pi\nu_0 t'}{2}\right)}{\sin \left(\pi\nu_0 t'\right)} \times \prod_{n=1}^\infty \frac{(1 - e^{i\pi\nu_0 t'}|z|^{2n})^4 (1 - e^{-i\pi\nu_0 t'}|z|^{2n})^4}{(1 - |z|^{2n})^6 (1 - e^{2i\pi\nu_0 t'}|z|^{2n}) (1 - e^{-2i\pi\nu_0 t'}|z|^{2n})}. \quad (118)
\]

Exactly the same as the \(p = 1\) case given in \([25]\), the above integrand has also an infinite number of simple poles on the positive real \(t'\)-axis at \(t' = (2k+1)/\nu_0\) with \(k = 0, 1, 2, \cdots\). This leads to an imaginary part of the amplitude, which is given as the sum over the residues of the poles as described in \([29, 34]\). Therefore the rate of pair production of open strings per unit worldvolume in a constant electric flux in the present context is

\[
\mathcal{W} \equiv -\frac{2\text{Im} \Gamma}{V_{p+1}},
\]

\[
= \frac{8U(m_1, n_1; m_2, n_2)}{\nu_0(8\pi^2\alpha')^{1+p}} \frac{\sinh \pi\nu_0}{\sinh^4 \frac{\pi\nu_0}{2}} \sum_{k=0}^\infty \left(\frac{\nu_0}{2k+1}\right)^{\frac{1+p}{2}} e^{-\frac{(2k+1)\nu_0^2}{2\pi\alpha'}} \prod_{n=1}^\infty \left(\frac{1 + e^{-2n\pi(2k+1)/\nu_0}}{1 - e^{-2n(2k+1)/\nu_0}}\right)^8,
\]

\[
= \frac{32n_1n_2}{\nu_0(8\pi^2\alpha')^{1+p}} \frac{\triangle_{(m_1, n_1)} - \triangle_{(m_2, n_2)}}{\sum_{k=0}^\infty \left(\frac{\nu_0}{2k+1}\right)^{\frac{1+p}{2}} e^{-\frac{(2k+1)\nu_0^2}{2\pi\alpha'}} \prod_{n=1}^\infty \left(\frac{1 + e^{-2n(2k+1)/\nu_0}}{1 - e^{-2n(2k+1)/\nu_0}}\right)^8},
\]

(119)

where \(\triangle_{(m,n)}\) is defined in \((42)\) and \(\nu_0\) can be determined from

\[
\cosh \pi \nu_0 = \frac{g_s^2 (\Omega - m_1m_2)}{n_1n_2}\quad (120)
\]

with \(\Omega\) defined in \((50)\). This rate has been calculated in different context before \([29, 34, 35, 36]\) but as stressed in \([25]\) for the \(p = 1\) case, the rather complicated sum over spin structures obtained in those papers reduces to our simple expression of \((116)\) or \((118)\) or \((119)\). Note that the above rate is suppressed by the brane separation and the integer \(k\) but increases with the value of \(\nu_0\) which is expected. Let us consider \(\nu_0 \to 0\) and \(\nu_0 \to \infty\)

\textsuperscript{12}Without the presence of the magnetic flux, the system is a BPS one and the amplitude vanishes. With the presence of the magnetic flux, in addition to the evidence given in the text, that the open string tachyon mode appears to arise is also indicated from the leading term \(e^{\pi \nu_0 t'}\), which diverges in the short cylinder limit \(t' \to \infty\), in the expansion of the \(\theta\)-functions and \(\eta\)-function in \((116)\) in the open string channel.
limits for the above rate, respectively. The former limit corresponds to the near extremal limit for which we can set $f_1 = f_2 + \epsilon$ with $|\epsilon| \ll 1$ while the latter corresponds to the critical field limit for which one can set either $|f_i| \to 1$ while keeping the other less than unit (but fixed) or set $f_1 = -f_2$ with both $|f_i| \to 1$ as mentioned earlier. The definition for $f_i$ with $i = 1, 2$ is given in (99). For the near extremal limit, we have, to leading order,

$$\nu_0 \approx \frac{|\epsilon|}{\pi(1 - f_2^2)},$$

(121)

the rate (119) is now well approximated by the $k = 0$ term as

$$W \approx \frac{32n_1n_2 |\epsilon|}{(8\pi^2\alpha')^{1/4}} \left( \frac{|\epsilon|}{\pi(1 - f_2^2)} \right)^{p+1} e^{-\frac{\sqrt{2}(1-f_2^2)}{2\alpha'|\epsilon|}},$$

(122)

very tiny as expected. For the critical field limit mentioned above, now $\nu_0 \to \infty$ and it is easy to see that each term in the summation of (119) diverges and so does the rate, signalling also an instability as mentioned in a similar context in [37].

5 Summary

In this paper, we calculate explicitly the interaction amplitude between two ($F, D_p$) or ($D_{(p-2)}, D_p$) non-threshold bound states with a separation. In doing so, we make use of their respective boundary state representation with a quantized world-volume electric (or magnetic) flux. Each such non-threshold bound state is therefore characterized by a pair of integers $(m_i, n_i)$ with $i = 1, 2$. When the two bound states are ($D_{p-2}, D_p$), the interaction is in general attractive but this remains so and can be certain only at large brane separation when the two states are ($F, D_p$). In both cases, the interaction vanishes only if $m_1/n_1 = m_2/n_2$ and $n_1n_2 > 0$. We also calculate the respective long-distance interaction independently from the low energy field theory approach and each agrees with the long-distance part of the corresponding general string amplitude. We also study the analytic structure of the amplitude and in particular we calculate the rate of pair production of open strings for the case of ($F, D_p$). In general, one expects that the interacting system is unstable and will relax itself by releasing the exceed energy via so-called tachyonic condensation[38] to form eventually a BPS non-threshold bound state, characterized by a pair of integers $(m_1 + m_2, n_1 + n_2)$. If $m_1 + m_2$ and $n_1 + n_2$ are co-prime, this state will be stable otherwise it will be marginally unstable. Similar to the brane/antibrane systems studied in [32, 33], the open string tachyonic condensation manifests itself for the case of ($D_{p-2}, D_p$) by showing a divergent amplitude but now caused by the presence of magnetic fluxes when the brane separation is on the order of
string scale. However, for the case of \((F, D_p)\), this manifests itself by the pair production of open strings which takes the exceed energy away so that the system can lower its energy and relax itself to form the final BPS bound state. By all means, what has been said here is just an indication being responsible for forming the final BPS states of the systems under consideration. To determine whether this actually leads to the formation of final BPS states requires a more detailed dynamical understanding which is beyond the scope of this paper.

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