Universal spectrum structure at nonequilibrium critical points in the (1+1)-dimensional directed percolation

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Using a tensor renormalization group method with oblique projectors for an anisotropic tensor network, we confirm that the rescaled spectrum of transfer matrices at nonequilibrium critical points in the (1+1)-dimensional directed percolation, a canonical model of nonequilibrium critical phenomena, is scale-invariant and its structure is universal.

Introduction. — The universality of critical phenomena has been found not only in equilibrium systems but also in nonequilibrium ones. The universality of nonequilibrium critical phenomena in the directed percolation (DP) has been extensively studied. There is a preferred direction for the percolation of active objects in the DP. If we regard a preferred direction as a time, the DP is a type of reaction-diffusion process. Since the system cannot escape from a state with no active object, called the absorbing state, the DP process is nonequilibrium. The critical phase transition of the DP process between the absorbing state and the other is universal. Experimental systems of turbulence[1,3] and various theoretical models of reaction-diffusion processes(See a review[4]) belong to the DP universality class.

In the case of equilibrium critical systems, the renormalization group(RG) method[5] is powerful and conceptually important to understand the universality(See textbooks[6]). However, the RG approach for nonequilibrium systems is still challenging. We introduce a tensor network(TN) representation to attack this problem for the DP. In this study, we report a new universal property of DP in a TN representation. Introducing a new tensor RG (TRG) method for the TN of dynamical process, we numerically calculate the renormalized critical tensors of the (1+1)-dimensional equilibrium critical systems[7]. In the following, after we briefly introduce our TRG method, and we will report the spectrum structure of renormalized critical tensors.

Tensor network representation of the (1+1)-dimensional DP.— Domany and Kinzel (DK)[8] proposed a stochastic cellular automaton on a square lattice rotated by 45 degrees in Fig. (1a) as the (1+1)-dimensional DP. We find the universal spectrum structure of renormalized critical tensors, which is similar to the conformal tower in the spectrum of critical tensors of the (1+1)-dimensional equilibrium critical systems[2]. In the following, after we briefly introduce a model of DP and the TN representation, we will explain our TRG method, and we will report the spectrum structure of renormalized critical tensors.

FIG. 1. (a) A lattice of the (1+1)-dimensional DK cellular automaton. The horizontal and vertical directions denote a spatial and a time axis, respectively. Time evolves downward. (b) TN representation of $T_{s_1\cdots s_N}^{\cdots s_N}$ in the (purple) shaded box. (c) Renormalization along a time direction by inserting isometries with a new sign diagonal matrix in a spatial direction. (d) Renormalization along a spatial direction by inserting oblique projectors in a time direction.
In a one-dimensional Markov process with finite local states, the state probability of \( \{ s_i \} \) is written as a rank \( N \) tensor, \( P_{s_1...s_N} \), where \( s_i \) is a local state variable on a site \( i \) and \( N \) is the number of sites. The transfer probability from a state configuration \( \{ s_i \} \) to \( \{ s'_i \} \) is also a tensor as \( T^{s_1...s_N}_{s'_1...s'_N} \). Based on a master equation, the state-probability distribution at the next time is a tensor contraction between \( P \) and \( T \) as 
\[
P_{s_1...s_N}^{\prime} = \sum_{s_1...s_N} T^{s_1...s_N}_{s'_1...s'_N} P_{s_1...s_N}.
\]
Since the interaction in the DK cellular automaton is local, the tensor \( T \) is written as a composite tensor of small local tensors (See Appendix A). Using a diagrammatic notation, we can draw the two time-steps evolution operator as a network of two types of local tensors, \( B \) and \( S \) in Fig. (b). Here, \( S \) is a sign diagonal matrix of which element is 1 or -1. Since a DK cellular automaton has a reflection symmetry in a spatial direction, \( B \) is also invariant under reflection.

**Tensor renormalization group method with oblique projectors.** — The use of a TN representation is expanding not only to equilibrium systems but also to nonequilibrium ones. For example, a one-dimensional TN (Matrix Product States) has been used to calculate a dynamical evolution and a nonequilibrium steady-state (See Appendix A). For the DP, we found that the existence of the absorbing state governs the unique behavior of informational entropy (See Appendix A).

Levin and Nave \cite{13} proposed the first TRG method, a real-space RG method on a two-dimensional tensor network, to calculate the partition function of equilibrium systems. The TRG method assumes an isotropic TN. However, in general, the TN of nonequilibrium systems is not isotropic because a time direction is not equal to a spatial direction.

Xie et al. \cite{14} proposed a simple TRG method based on a higher-order singular value decomposition (HOSVD) called HOTRG. Using HOSVD, we determine an optimal orthogonal projector for each edge of a renormalized tensor. It can be generalized to a higher-dimensional TN and maybe to an anisotropic one. However, there is no reflection symmetry in a time direction. It causes a serious problem by using an orthogonal projector in HOTRG. In fact, since the tensor \( B \) in Fig. (b) has no reflection symmetry in a time direction, an orthogonal projector is not optimal even in the sense of local optimization. Therefore, we extend it to an oblique projector that is optimal for a local TN. We can calculate the optimal oblique projector between two local tensors as in \cite{15, 16} (See Appendix B). A simple TRG method with oblique projectors (OPTRG) \cite{17} consists of the coarse-graining of two neighboring tensors by inserting oblique projectors, as shown in Fig. (c) and (d). We notice that the coarse-graining tensor in Fig. (c) also keeps a reflection symmetry in a spatial direction because the original tensor \( B \) has (See Appendix A).

In general, nonequilibrium critical systems at critical points are strongly anisotropic between a spatial direction and a time direction. The dynamical critical exponent \( z \) is not equal to one. The number of renormalization steps for each direction should not be the same. In practice, we should minimize a truncation error in a renormalization step. Therefore, we always choose a renormalized direction in which the fidelity between an original tensor and a renormalized one is larger than in the other direction.

We can calculate a TN representation of a state-probability distribution at a given time by attaching a TN representation of an initial distribution to that of transfer probability in Fig. (b). The expected value of an observable is an inner product of TN representations of a state-probability distribution and an observable. We renormalize boundary tensors of an initial state-probability and an observable with the same projector of a tensor of transfer probability, \( B \). Then, the OPTRG method drastically improves the accuracy of the TRG calculation of the DK cellular automaton (See Appendix C).

**Renormalized critical tensors of DP.** — The renormalized critical tensor in the TRG calculation for equilibrium critical systems has been well understood \cite{12}. However, we have not studied for nonequilibrium critical systems yet. We will consider the universal property of a renormalized critical tensor of the (1+1)-dimensional DP.

We choose a renormalized direction to reduce a truncation error in the TRG procedure. Figure 2 shows the number of TRG steps in a spatial direction and a time direction of renormalized tensors at three different critical points of the DK cellular automaton. Here, the maximum bond dimension \( D \) is 80 \cite{18}. \( n_x \) and \( n_t \) denote the number of TRG steps in a spatial direction and a time direction, respectively \cite{19}. The number of TRG steps increases so that \( n_t \) is roughly proportional to \( n_x \). The ratio \( n_t/n_x \) at the bond and site DP critical points is close to the dynamical exponent \( z_{\text{DP}} = 1.580745(10) \) \cite{20} of the DP universality class. The ratio at the compact DP critical point is equal to the dynamical exponent \( z_{\text{CDP}} = 2 \) \cite{21} of the compact DP universality class. A renormalized tensor \( B \) represents a \( L_x = 2^n \) time-steps evolution on a region of \( L_x = 2^n \) sites. If \( n_t \) is linearly proportional to \( n_x \) roughly with the coefficient \( z \), then \( L_x \sim L_1^{1/z} \). As in Fig. 2 the exponent \( z \) is roughly consistent with the dynamical exponent for the corresponding universality class for each critical point. The scaling relation corresponds to the scaling relation of spatial and time correlation lengths at DP critical points. To reduce the truncation error of renormalization steps, the aspect ratio of renormalized tensors is automatically proportional to the ratio of spatial and time correlation lengths at DP critical points. Therefore, the approximated scaling relation between the spatial and the time scale of a renormalized tensor is a desired property of a critical tensor.

In the case of an equilibrium system, based on a TN representation of the partition function, we can write the partition function as a trace of a renormalized tensor and the transfer matrix as a partial trace. Therefore, a renormalized critical tensor itself has a universal property of equilibrium critical systems. In the two-dimensional sys-
the scaling dimensions. The scaling exponents depend on the universality class, and is related to a conformal invariance \[7\]. The rescaled spectrum structure of renormalized critical tensors has been well understood, based on the universality class, the universal structure in a spectrum of a renormalized tensor is also scale-invariant. However, since the aspect ratio of a renormalized tensor changes in the strongly anisotropic case as DP, the raw spectrum is not scale-invariant. Thus, we propose a rescaled spectrum of a transfer matrix along a spatial direction as \(|\lambda_{x,i}| \propto \exp[-c_i L_x/\xi_{\perp}]\).

Since the DP system is strongly anisotropic, we consider two different spectra of a renormalized tensor along a spatial direction and a time direction in Fig. 3 (a) and (b). If a renormalized tensor is critical, it is scale-invariant. Then, the spectrum of a renormalized critical tensor is also scale-invariant. However, since the aspect ratio of a renormalized tensor changes in the strongly anisotropic case as DP, the raw spectrum is not scale-invariant. Thus, we propose a rescaled spectrum of a transfer matrix along a spatial direction in Fig. 3 (a) as

\[
\Delta_{x,i} = -\left(\frac{L_{t}^{1/z}}{L_x}\right) \log \left|\frac{\lambda_{x,i}}{\lambda_{x,0}}\right|,
\]

where \(\lambda_{x,i}\) is the \(i\)-th eigenvalue of a transfer matrix along a spatial direction in descending order and \(\lambda_{x,0}\) is the largest one and \(z\) is a dynamical critical exponent. We consider the absolute value of eigenvalue in (1) because the transfer matrix in Fig. 3 is generally not symmetric. Eq. (1) is a generalization of a universal spectrum of a two-dimensional isotropic critical system proposed in [7] with the exponent \(1/z\). In the isotropic case, the rescaled spectrum corresponds to the conformal tower with a constant factor. Figure 4 shows the rescaled spectrum of renormalized tensors for three different critical DP. The rescaled spectrum is scale-invariant in the wide range from \(L_{t} = 2^5\) to \(2^{14}\) for all critical points. Since the accuracy of a higher spectrum becomes a bit unstable after a TRG procedure along a spatial direction, we only plot the results after the TRG along a time direction. The correlation length along a spatial direction, \(\xi_{\perp}\), is proportional to \(L_{t}^{1/z}\) after the \(L_{t}\) time-steps evolution. Thus, the scale-invariance of \(\Delta_{x,i}\) in (1) is consistent with the expected form of eigenvalues of a transfer matrix along a spatial direction as \(|\lambda_{x,i}| \propto \exp[-c_i L_x/\xi_{\perp}]\).

Some spectra are degenerate in Fig. 4. The degeneracy of the spectrum in the bond DP is equal to that in the site DP as \(1, 2, 4, \ldots\) in the ascending level from \(\Delta_{x,1}\). On the other hand, the degeneracy of the lowest level in the compact DP is 2, not 1. Therefore, the degeneracy depends on the universality class. The degeneracy of the second level in the compact DP is unclear and the degeneracy of the lowest level is slightly violated for renormalized critical tensors larger than \(L_{t} \geq 2^{11}\) because the accuracy of renormalized critical tensors is low. The eigenvalues \(\lambda_{x,0}\) and \(\lambda_{x,1}\) of the bond and the site DPs are real, but the others are complex in Fig. 4. The eigenvalues \(\Delta_{x,0}, \Delta_{x,1}\), and \(\Delta_{x,2}\) of the compact DP are real, but the others are complex. The level interval of the bond DP is proportional to that of the site DP. We summarize the values of rescaled spectra and the ratios in Tab. 1 which is based on the data from \(n_t = 8\) to 11 for the bond and the site DPs and at \(n_t = 8, 10,\) and 12 for the compact DP. The ratios of the second level to the first level for the bond and the site DPs, \(\Delta_{x,2}/\Delta_{x,1}\), are 3.60(1) and 3.59(1), respectively. They agree within error bars. But they do not agree with the compact DP’s ratio. The ratios of the fourth level to the first level for the bond and the site DPs, \(\Delta_{x,4}/\Delta_{x,1}\), are 3.71(1) and 3.69(1), respectively.
DPs, $\Delta_{x,4}/\Delta_{x,1}$, also agree within error bars. Therefore, the rescaled spectrum structure is universal.

**TABLE I. Values and ratios of rescaled spectra along a spatial direction.**

|      | $\Delta_{t,1}$ | $\Delta_{t,2}/\Delta_{t,1}$ | $\Delta_{t,3-7}/\Delta_{t,1}$ |
|------|----------------|-------------------------------|-------------------------------|
| bond DP | 1.948(1) | 3.60(1) | 3.91(2) |
| site DP | 1.660(2) | 3.59(1) | 3.91(1) |
| compact DP | 2.69(2) | 2.65(4) |          |

For the time direction, we propose a rescaled spectrum of a transfer matrix in Fig. 5 (b) as

$$\Delta_{t,i} = -\left(\frac{L_z^x}{L_t}\right) \log \left| \frac{\lambda_{t,i}}{\lambda_{t,0}} \right|,$$

(2)

where $\lambda_{t,i}$ is the $i$-th eigenvalue of a transfer matrix along a time direction in descending order and $\lambda_{t,0}$ is the largest one and $z$ is a dynamical critical exponent. As in Figure 5, $\Delta_{t,i}$ is scale-invariant in the wide range for all critical points. All eigenvalues in $\Delta_{t,i}$ of Fig. 5 are real. The scale-invariance of $\Delta_{t,i}$ in (2) is consistent with the expected form of eigenvalues of a transfer matrix along a time direction because $\xi \parallel L_z^x$. We summarize the values of rescaled spectra and the ratios in Tab. II which is based on the data from $n_t = 8$ to $10$ for the bond and the site DPs and at $n_t = 8, 10$, and $12$ for the compact DP. $\Delta_{t,3}$ and $\Delta_{t,4}$ are degenerate in the bond and the site DPs. The first spectrum $\Delta_{t,1}$ of the bond and the site DPs are small, and their accuracy are low. Thus, we compare the ratios $\Delta_{t,3-7}/\Delta_{t,2}$ for both DPs. As in Tab. II, they agree within error bars. However, the spectrum structure of the compact DP is different from that of the bond and the site DPs. The first eigenvalue $\lambda_{t,1}$ is degenerate to the largest eigenvalue, $\lambda_{t,0}$. It is consistent with the existence of two absorbing states in the compact DP. The other degeneracy is also different from that of the bond and the site DPs. Interestingly, the ratio of each level to the second level of the compact DP is an integer within error bars. As in Tab. II, the rescaled spectrum structures depend on the universality class. Therefore, the rescaled spectrum structure of a transfer matrix along a time direction is universal.

**TABLE II. Values and ratios of rescaled spectra along a time direction.**

|      | $\Delta_{t,1}$ | $\Delta_{t,2}/\Delta_{t,1}$ | $\Delta_{t,3-7}/\Delta_{t,2}$ |
|------|----------------|-------------------------------|-------------------------------|
| bond DP | 0.053(2) | 1.334(2) | 1.110(1) |
| site DP | 0.063(5) | 1.716(8) | 1.111(1) |
| compact DP | 0.6172(4) | 2.000(2) | 4.000(4) |

**Conclusion.**—To understand the universality of nonequilibrium critical systems, we tried the numerical

RG approach for the one-dimensional DK stochastic cellular automaton that is a dynamical model of the (1+1)-dimensional DP. Using a TRG method with oblique projectors for the TN representation of the time evolution operator, we numerically calculated renormalized critical tensors for the bond, the site, and the compact DPs. We proposed the rescaled spectrum of a renormalized critical tensor with a strongly anisotropic criticality. We numerically confirmed that the rescaled spectra of renormalized critical tensors at nonequilibrium critical points of the (1+1)-dimensional DP are scale-invariant and the rescaled spectrum structure is universal. Future works need to understand the universal spectrum structure. We calculated the RG fixed points in the tensor space for the (1+1)-dimensional DP. Our approach is applicable to the other nonequilibrium criticalities or the higher dimensional cases. To improve the accuracy of renormalization steps, the generalization of techniques in [22, 23] for the strongly anisotropic criticality is promising.

**ACKNOWLEDGMENTS**

K.H appreciates fruitful comments from N. Kawashima. This work was supported by JSPS KAKENHI Grant Nos. 17K05576 and No. 20K03766. The computation in this work has been done using the facilities of the Supercomputer Center, the Institute for Solid State Physics, the University of Tokyo and the facilities of the Supercomputer Center at Kyoto University.
Appendix A: Tensor network representation of the time evolution operator of the one-dimensional Domany-Kinzel cellular automaton

Since the interaction in the one-dimensional Domany-Kinzel (DK) cellular automaton [8] consists is local, the transfer probability tensor $T$ is written as a composite tensor of small local tensors. Using a diagrammatic notation, we can draw it as a network of two types of local tensors, $w$ and $\delta$, in Fig. 6(a), which is a tensor network (TN) representation of $T$. Here, a local tensor is a graphical object with edges. An edge denotes the index of a local tensor, and a connected edge means a tensor contraction between the corresponding indexes of two local tensors. The definition of local tensors, $w$ and $\delta$, is as follows:

$$w_{lm}^{n} = (1 - n) + 2n - 1) \Lambda[l + m],$$
$$\delta_{op}^{n} = \begin{cases} 1 & \text{if } n = o = p, \\ 0 & \text{otherwise}. \end{cases}$$

Here, $P[0], P[1]$, and $P[2]$ are free parameters which define the DK cellular automaton. We introduce a new local tensor $A$ as $A_{lm}^{op} = \sum_n \delta_{op}^n w_{lm}^{n}$ in Fig. 6(a). Since it can be transformed into a symmetric matrix $A_{(l)},(m)p$ along a horizontal direction, it can be diagonalized as $A = VS^t$. Here, $S$ is a sign diagonal matrix of which diagonal element is 1 or -1. Combining four local tensors $\delta, w, V$, and $V^t$, we introduce a new tensor $B$ in Fig. 6(b). Then, the two time-steps evolution operator can be transformed into the TN of $B$ and $S$.

Appendix B: Optimal oblique projector

We consider an approximation of a matrix product of Fig. 7(a) into Fig. 7(b) with a reduced bond dimension. To minimize the Frobenius norm between them under a fixed bond dimension $\chi'$, we keep the $\chi'$ largest singular values of $ASB$ by inserting an oblique projector $PQ'$ between $A$ and $B$. Such $P, Q$, and $S'$ can be calculated as the following procedures [15, 16].

1. Singular value decomposition: $M = ASB = U\Lambda V^t$, where $U$ and $V$ are unitary, $\Lambda$ is a diagonal matrix of singular values.

2. Extract the $\chi'$ largest components of singular values: $\Lambda \rightarrow \tilde{\Lambda}, U \rightarrow \tilde{U}, V \rightarrow \tilde{V}$.

3. $P = SB\tilde{V}|\tilde{\Lambda}|^{-\frac{1}{2}}, Q = |\tilde{\Lambda}|^{-\frac{1}{2}}\tilde{U}^t AS, S' = \text{sign}(\tilde{\Lambda})$.

If a matrix $ASB$ is Hermitian, $B = A^t$, then $M = UAU^t$ and $P = Q^t$.

![FIG. 6. (a) TN representation of a transfer probability tensor $T$ in the (yellow) shaded box. The new local tensor $A$ has a reflection symmetry along a horizontal direction (b) Decomposition of $A$ and a new local tensor $B$ and $S$.](image)

![FIG. 7. (a) Matrix product with a sign diagonal matrix. (b) Reducing a bond dimension from $\chi$ to $\chi'$ with an oblique projector with a new sign diagonal matrix.](image)
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