D = 11 SUPERMEMBRANE INSTANTONS,  
W∞ STRINGS AND  
THE SUPER TODA MOLECULE 

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Abstract  

Exact instanton solutions to D = 11 spherical supermembranes moving in flat target spacetime backgrounds are constructed. Our starting point is Super Yang-Mills theories, based on the infinite dimensional SU(∞) group, dimensionally reduced to one time dimension. In this fashion the super-Toda molecule equation is recovered preserving only one supersymmetry out of the N = 16 that one would have obtained otherwise. It is conjectured that the expected critical target spacetime dimensions for the (super) membrane, (D = 11) D = 27 is closely related to that of the noncritical (super) W∞ strings. A BRST analysis of these symmetries should yield information about the quantum consistency of the (D = 11) D = 27 dimensional (super) membrane. Comments on the role that Skyrmions might play in the two types of Spinning- Membrane actions constructed so far is presented at the conclusion. Finally, the importance that integrability on light-lines in complex superspaces has in other types of solutions is emphasized.

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1 Introduction

Based on the observation that the theory of extended objects beyond the string could be seen as a gauge theory of p-volume preserving diffeomorphisms [3] (for a review) and that supermembranes in D dimensions (excluding the zero modes), in the lightcone gauge, are essentially D − 1 Super Yang-Mills theories dimensionally reduced to one temporal dimension; we look for solutions to the N = 1 D = 10 SDSYM of the SU(∞) group, i.e; spherical supermembranes moving in flat target space time backgrounds. In II we review the work of [1] in finding special types of solutions to YM equations in R^10n which is crucial in the
construction of the membrane instanton solution presented in III. This solution preserves only one supersymmetry and enables us to arrive at the $SU(\infty)$ $D = 1$ Toda molecule which bears strong resemblance with a higher-conformal-spin-extended version of Liouville-like field theory (dimensionally reduced to one dimension). The geometrical origin of $W_\infty$ symmetries in noncritical string theory is outlined in IV and a plausible connection between the spectra of noncritical $W_\infty$ strings and membranes is raised. A BRST analysis reveals that the target spacetime of noncritical $W_\infty$ strings is closely related to the expected critical dimensions for the membrane. $D = 27$ for the bosonic membrane and $D = 11$ for the supermembrane. Quantization could be implemented through the Quantum Group program based on the $U_q[SU(\infty)]$ Quantum Group. Finally, the relevance that integrability on light-lines in complex superspaces might have in the construction of more general solutions is emphasized. An appendix is included where the $N = \infty$ limit of the Toda molecule is taken and whose solutions can be obtained in terms of solutions to the effective $3D$ $sl(\infty)$ continuous Toda equation. The latter is equivalent to a rotational Killing symmetry reduction of Plebanski’s first heavenly equation in $D = 4$.

2 The Yang-Mills Equations in $R^{4n}$

We are going to follow closely [1]. Solutions of the equations for classical YM theory in Euclidean space $R^{4n}$ for the YM potentials $A_a, a = 1, 2, \ldots, 4n$, and field-strengths, $F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b]$, with values in the semisimple Lie algebra $G$

$$\partial_a F_{ab} + [A_a, F_{ab}] = 0. \quad (1)$$

can be obtained from the ansatz [1]:

$$A_a = -J^\alpha_{ac} T_\alpha(\phi) \partial_c \phi. \quad (2)$$

with the real antisymmetric tensors $J^\alpha_{ab}$ satisfying:

$$J^\alpha_{ac} T^\beta_{bc} = \delta^{\alpha\beta} \delta_{ab} + \epsilon^{\alpha\beta\gamma} J_{ab}^\gamma. \quad (3)$$

where $\epsilon_{\alpha\beta\gamma}$ are the $SU(2)$ structure constants and $\alpha, \beta, \gamma = 1, 2, 3$. With the introduction of these tensors one introduces a quaternionic structure in $R^{4n} = H^n$. Upon substitution of (2) and using the relations (3) one obtains the system of differential equations:

$$\left( \partial T_\alpha / \partial \phi \right) + \frac{1}{2} \epsilon_{\alpha\beta\gamma} [T_\beta, T_\gamma], = 0. \quad (4)$$

These are the Rouhani-Ward equations [1] and

$$2J^\alpha_{ac} \partial_c \partial_b \phi - 2J^\alpha_{bc} \partial_c \partial_a \phi + 2\epsilon_{\beta\gamma} J^\beta_{ac} T^\gamma_{bc} \partial_c \partial_e \phi + J^\alpha_{ab} \Box \phi = 0 \quad (5)$$

One must emphasize that $T_\alpha = T_\alpha^A L_A$ are $G$-valued objects whose Lie algebra bracket is not necessarily that of $SU(2)$. One can also have more general $f_{\alpha\beta\gamma}$ in (4) belonging
to $\mathcal{H}$. In the membrane’s case $T_\alpha$ become a set of three c-number functions depending on the two extra internal spatial variables of the membrane and of the $\phi$ function which depends on the time variable $x_0$ which is in correspondence with the membrane’s clock $\tau = X^+ = 2^{-1/2}(X_0 + X_{10})$. The gauge field of area-preserving diffeomorphisms of the two dimensional spherical membrane $\Sigma$ with two spatial coordinates $\sigma_1, \sigma_2$ is replaced as: $\omega^{ij} = A_0$ which is the extra YM field needed in order to have a $D = 10$ YM theory of the $sdiff \Sigma$ group dimensionally reduced to one time dimension.

Having written down the eqs-(4,5) in $R^{10}$ allows us to relate them to a relative set of equations in $R^4$ as follows: Introduce the tensors $J_{\mu_\nu}(\nu_j) = \delta_{ij} \eta_{\mu\nu}^a$ with the double-index notation : $\mu, \nu,.. = 1, 2, 3, 4; i, j,.... = 1, 2, ...., n$ so that $a, b,.... = 1, 2, ....., 4n$. $\eta_{\mu\nu}^a$ are the ’t Hooft tensors : three real antisymmetric $4 \times 4$ matrices with components $\eta_{\beta\gamma}^a = \epsilon_{\beta\gamma}^a$ and $\eta_{\mu4}^a = -\eta_{4\mu}^a = \delta_{\mu}^a$. Substituting these tensors in (5) one gets the equations :

$$\partial_{\mu i} \partial_{\nu j} \phi = \partial_{\mu j} \partial_{\nu i} \phi. \quad (6a)$$
$$\partial_{\lambda i} \partial_{\lambda j} \phi = 0. \quad (6b)$$

The equation $\Box \phi = \partial_{\lambda j} \partial_{\lambda j} \phi = 0$ follows from (6b) where one sums over the $j = 1, 2, ...., n$ indices only. Having solutions to eqs (4,5,6) will yield solutions of the original YM eqs (1) after using (2). Now that we have presented a brief review of [1] we turn to the membrane’s equations.

## 3 The Membrane Instantons and the Toda Molecule

A close up study reveals that the $D = 11$ supermembrane admits exact octonionic and quaternionic instanton solutions which are tightly connected to the super Toda molecule equation [4] associated with the minimal embedding of the $SU(2)$ group into the $N \to \infty$ limit of $SU(N)$. A special class of solutions can be found as follows if one chooses the following ansatz: In $D = 10$ take a $10 \times 10$ $J^a_{AB}$ matrix in block form : $J^a_{MN} \otimes T^a$ where $J^a_{MN}$ is a matrix of the type in (3) with $M, N,.. = 0, 1, ...., 7$. $T^a$ is a set of three arbitrary $2 \otimes 2$ matrices for the values of $\alpha, \beta,.. = 1, 2, 3$; and whose matrix indices range over 8,9 only. Due to the fact that the membrane is, in effect, a dimensional reduction to one dimension of a $D = 10$ YM theory, the two YM potentials $A_8, A_9$ are going to be zero after a direct application of the ansatz in (2) and due to the block diagonal form of $J$. i.e; $A_8 \sim J_{98} \partial_{x_0} \phi = 0$. The same applies with $A_9 \Rightarrow 0$. One is not constraining $a priori$ the set two YM potentials $A_8, A_9$ to zero; these are zero a posteriori as a result of the dimensional reduction to one dimension (essentially). The $8 \otimes 8$ $J^a_{MN}$ matrices are of the form :

$$\delta_{ij} \eta_{\mu\nu}^a$$

as explained in the previous section. One could opt to extend the range of indices for $\alpha, \beta,...$ beyond 1, 2, 3 to four indices, instead, and choose for the four $J$ matrices a selected subset of the seven antisymmetric $8 \otimes 8$ matrices involved in the Clifford algebra $Cl(O, 6)$ with generators $\gamma_1, \gamma_2,....., \gamma_6; \gamma^7 = \gamma_1 \gamma_2 ....... \gamma_6$. See [1] for further details where they discuss that the range of indices compatible with the ansatz in (2) can only be those whose $\alpha, ...$ are 4, 5, 6, 7. The sort of equations one gets is an octonionic analog of the Rouhani-Ward-Nahm equations :

$$f_{\alpha\beta\gamma}(\partial T_\gamma / \partial \phi) + [T^a_\alpha G_a, T^b_\alpha G_b] = 0. \quad (7a)$$
where the function $\phi(X_M)$ obeys an equation like (5). There is also another alternative way to solve the effective YM in $d = 8$; although the number of equations increases dramatically, see Popov [6]. These authors projected an arbitrary antisymmetric tensor $F_{AB}$ into the orthogonal 21 and 7 dimensional subspaces of the 28-dim vector space of antisymmetric tensors in $d = 8$ given by the self dual and antiselfdual parts of the tensor $F_{AB} = F^+ + F^-$. The projectors involved completely antisymmetric tensors built entirely from the octonionic structure constants (Cayley numbers) and the standard $\epsilon, \delta$ tensor s. The value of $\phi = 1 + X_M X_M$ is the one which allows to recast the SD equations, for the adequate pertaining ansatz :

$$A^a_M G_a = (-1/6) G_{MNC} X_N W_C(\phi) \quad (7b)$$

to acquire the form :

$$[W_{AB}, W_{CD}] = S_{ABCDMN}(\partial W_{MN}/\partial \phi). \quad (7c)$$

with $S_{ABCDMN}$ the $SO(8)$ structure constants. Similar type of octonionic SDYM equations have been discussed in [5]. We are not going to solve these equations and their octonionic superspace generalizations because there is a simpler way to reach to the super-Toda molecule without having extended supersymmetries in $D = 1$ and without to have to truncate the dimensionally-reduced ten dimensional YM theory. We borrow our results in [4] by first reducing the $d = 8$ YM to $d = 4$ as shown in section II and then concentrating on the $d = 4$ YM theory with YM potentials $A_0, A_1, A_2, A_3$ depending solely on one coordinate which is the $x_o$ coordinate.

Because of YM gauge invariance one has enough freedom to choose the gauge $A_0 = 0$. This is due to the fact that $A_0$ is the relative of the gauge field of $sdiff \Sigma$. [3]. The YM equations in $d = 4$ for this choice of gauge and for the case that the fields depend solely on one coordinate become :

$$A_{i,o} - [A_j, [A_i, A_j]] = 0. \quad [A_{i,o}, A_i] = 0. \quad (7d)$$

where $i, j, k.. = 1, 2, 3$; only . Eqs.(7d) are in fact equivalent to the SDYM in $D = 4$ :

$$\epsilon_{\mu\nu\rho\kappa} F_{\rho\kappa} = 2 F_{\mu\nu}. \quad (8a)$$

Clearly, when the four indices for the $\epsilon$ tensor have only values ranging between 1, 2, 3 one gets zero and when they range over 0, 1, 2, 3 one gets $\epsilon_{ijk}$ so the end result is

$$\epsilon_{ijk} A_{k,o} + [A_i^a G_a, A_j^b G_b] = 0. \quad (8b)$$

For a more general discussion of the equivalence between (7) and (8) see [1]. In the membrane'case one replaces Lie-algebra brackets with Poisson brackets. The semicolons indicate that the derivatives are taken with respect to $x_o$.

From [1] : $x_0 = p_1 x_{01} + p_2 x_{02}$ so that $(\partial/\partial x_{01}) = p_1(\partial/\partial x_0)....$ and the potentials are obtained for $\phi(x_o) = x_o$ :

$$A_6 = p_2 n^o_{20} A_o(x_o; \sigma^1, \sigma^2) = p_2 A_2(x_o, \sigma^1, \sigma^2). \quad A_2 = p_1 A_2.$$  

$$A_1 = p_1 A_1. \quad A_5 = p_2 A_1. \quad A_3 = p_1 A_3. \quad A_7 = p_2 A_3.$$
The last equation is consistent with the gauge choice \( A_0 = 0 \).

Equations (7a) in the \( R^4 \) case, and for the special choice of the \( J \) matrices discussed in section II, have a Lax-type representation in terms of a spectral parameter \( \lambda \). Applying the inverse scattering transform method allows to construct solutions of Lax’s equation in terms of theta functions for any semisimple Lie algebra \( \mathcal{G} \) and can be reduced to Toda lattice equations [6]. On the other hand, eqs. (7a,7b) in \( d = 8 \) for the special case that the Lie algebra \( \mathcal{G} \equiv \mathcal{H} \) may be obtained from classical Yang-Baxter equations. For simple Lie algebras these admit three classes of solutions: elliptic, trigonometric and rational solutions [6]. However, this is not necessary for the \( SU(\infty) \) group when one uses Poisson brackets instead. Initially, the authors [7] expressed solutions of the self dual membrane in \( D = 4 + 1 \) in terms of solutions of the Toda molecule for \( SU(2) \) after an appropriate truncation of the YM potentials in the expansion of spherical harmonics. In [4] we found solutions of the self dual supermembrane in \( 4 + 1 \) without a truncation. The ansatz which allows to recast the \( SU(\infty) \) Nahm’s equations as a super-Toda molecule is:

\[
\{ A_y, A_{\bar{y}} \} = -i \sum_{l=1}^{\infty} \exp(K_{ll'} \Phi_{l'}) Y_{l0}(\sigma_1, \sigma_2). \tag{9a}
\]

and:

\[
A_3 = -\sum_{l=1}^{\infty} (\partial \Phi_l / \partial \tau) Y_{l0}. \tag{9b}
\]

where \( A_y \sim A_1 + iA_2 \), \( A_{\bar{y}} \sim A_1 - iA_2 \), and \( A_y = \sum A_{yl}(x^0_L, \theta^+) Y_{l+1}(\sigma_1, \sigma_2) \). \( A_{\bar{y}} \) is expanded in terms of \( Y_{l-1} \) and \( A_3 \) in terms of \( Y_{l0} \).

Taking \( \partial A_3 / \partial \tau = i\{ A_y, A_{\bar{y}} \} \) gives:

\[
-\frac{\partial^2 \Phi_l}{\partial (x^0_L)^2} = \exp(K_{ll'} \Phi_{l'}). \tag{10}
\]

where \( \tau = x^0_L \). This is the \( SU(N) \) super-Toda molecule equation in Minkowski form for the \( \Phi_1, \Phi_2, \Phi_3, \ldots \) superfields where the \( SU(2) \) has been minimally embedded into \( SU(N) \). This explains the presence of the spherical harmonics in (9). \( x^0_L \) is a left-handed coordinate and \( \theta^+ \) is the light-cone chiral Grassmanian variable [8]. The anti self dual YM equations require the use of anti-chiral light-cone superfields \( \Phi(x^0_L, \theta^+) \).

\( K_{ll'} \) is the \( SU(N) \) Cartan matrix. \( \Phi_1, \Phi_2, \ldots \) are light-cone chiral superfields in Lorentzian superspace encoding the true physical propagating degrees of freedom of YM in \( 3 + 1 \) dimensions [8]. Gilson et al by integrating out the inappropriate light-cone projections of the superspace Grassmanian variables obtained a SDSYM formulation with only physical propagating modes. Auxiliary fields and nonpropagating modes were completely eliminated. The end results are not manifestly Lorentz covariant and are supersymmetric under the light-cone supersymmetry \( Q_{+\alpha} \). This light-cone chiral superfield formulation is compatible with the initial light-cone supermembrane action insofar as the light-like directions of the \( D = 11 \) supermembrane is concerned. There is a correspondence (not an strict identification ! ) between the membrane’s coordinates and the YM potentials. Also, the membrane’s clock
is $X_+ = X_0 + X_{10}$ which corresponds to the $x_o$ coordinate of the \( D = 10 \) YM theory. If one identifies $X_+$ with $x_0$ which, in turn, is seen as $X_0$, one would have to constrain the membrane’s $X_0, X_{10}$ coordinates! The value of the left handed coordinate is therefore: $x_0^L \equiv x^0 + i\sigma^0 \theta$. The fact that only half of the supersymmetries are linearly realized in the $D = 11$ light-cone supermembrane [3] is also consistent with the fact that the light-cone chiral superfield formulation has only manifest supersymmetry under $Q_{+\alpha}$.

Furthermore, a naive dimensional reduction of the $N = 1 \ D = 3 + 1$ SDSYM theory to $D = 1$ would have generated $N = 4$ supersymmetries. And, as it is known, a naive dimensional reduction of $N = 1 \ D = 10 \ SYM$ yields $N = 16 \ D = 1 \ SYM$. Therefore, the light-cone chiral-superspace formulation generates only one supersymmetry out of the $N = 16$ . Octonionic superstring soliton solutions to the low-energy heterotic-field-theory equations of motion in $D = 10$ preserving only one supersymmetry out of the sixteen were found in [9]. The soliton described a ten dimensional cosmic (not fundamental) string in Minkowski space acting as a source for massless spacetime fields transverse to the two dimensional world of the string. Since the transverse space of the $d = 2 + 1$ supermembrane in $D = 10 + 1$ is the same as that of the $d = 1 + 1$ superstring in $D = 9 + 1$ it is not surprising that similarities occur. Also the role of octonions in the construction of the exceptional group $E_8$ and the triality properties of the transverse compact group $SO(8)$ hint to important connections between the two. Dualities between strings and five-branes seem to point out that these two are dual description of the same physics [10]. It is warranted to explore all these connections deeper and see what new physical results are encountered. It was emphasized [4] that although the Lorenztian superspace solutions of the SDSYM equations are complex valued (which would also force the membrane’s coordinates to be as well)- after the dimensional reduction has taken place from $4 \rightarrow 1$ a reality condition on the complex superfields can be met [8]. This is consistent with the fact that $N = 1 \ D = 4$ Euclidean supersymmetries can only be analytically continued to Minkowski space iff the number of Minkowski supersymmetries is $N = 2$. There are no Majorana spinors in four Euclidean dimensions but there are in ten and eight Euclidean [11], which was our starting point to construct instanton solutions. At the end of the road in Euclidean $D = 1$ a Majorana spinor condition can be imposed. Whereas in Minkowski $D = 1$ space a pseudo-Majorana spinor exists. In the appendix we show that the $N \rightarrow \infty$ limit of (10) (the bosonic piece) is nothing but a dimensional reduction of the $3D$ continuous Toda equation [37] given by Leznov and Saveliev:

$$\partial_z \partial_{\bar{z}} u = -\partial_t^2 (e^u).$$

(11)

where $z, \bar{z}$ are complex coordinates and $t$ is a continuous parameter. A dimensional reduction from $3D$ to $2D$ yields:

$$\partial_r^2 u = -\partial_t^2 (e^u). \quad r \equiv z + \bar{z}.$$  

(12)

Solutions to (12) have been given by Saveliev [37] in terms of an exact series expansion of the quantity $\mu = \frac{d(t) e^{\phi(t)}}{(\phi(t))^2}$ where $d(t)$ and $\phi(t)$ are arbitrary functions of $t$:

$$e^x = e^{x_0} [1 - \mu + 1/2 \mu^2 + \ldots].$$

(13a)

where $x$ is a solution of

$$\partial_r^2 x = e^{\partial_t^2 x}.$$ 

(13b)
\[ \partial_t^2 x_o = r \phi + \ln d. \] (13c)

where one has set the coupling constant to one and \( x_o \) is an asymptotic solution at \( r = \infty \). For more details see [37]. An exact quantization of the Toda lattice has been given by Leznov and Saveliev in chapter seven of their book [38]. In principle a quantization of the continuous Toda theory could be achieved by taking the continuum limit in all of their equations. After quantization the solutions (13) will receive \( \hbar \) corrections. There is no assurance that these limits would behave smoothly and probably renormalization would be required. A Quantum Group approach certainly is very useful in order to evaluate the operator products of Liouville-type fields exponentials however as far as we know the Quantum Group associated with the \( SL(\infty) \) algebra has not been constructed. Saveliev gave an explicit realization of the \( W_\infty \) algebra in terms of the 3D continuous Toda field since the continuous Toda exhibits a \( W_\infty \) symmetry. Quasi finite Unitary Highest Weight representations of the \( W_\infty \) algebra have been given in [39]. Thus the quantum spectrum of the continuous Toda theory can in principle be computed. We won’t go into these details but instead we suggest the connection to noncritical \( W_\infty \) strings in the next section.

4 Anomalies, Spectrum and Light-like Integrability

Using the techniques and results [6] of constructing solutions of the Toda lattice and Yang-Baxter equations would facilitate the construction of solutions to (10) and a subsequent quantization process will provide for us the spectrum of the supermembrane. The authors in [12] gave a rigorous proof that the Hamiltonians of Supersymmetric Gauge Quantum Mechanical Models (SGQMM) for finite matrices have a continuous spectrum starting at zero and, in particular, with no mass gap. For this reason they concluded that the supermembrane was unstable against deformations into long stringlike configurations of zero area (zero energy). In the bosonic case quantum effects saved the day and prevented these instabilities to occur [13]. Nevertheless, [12] emphasized that matters could change in the \( N = \infty \) limit. Mainly, this limit is not unique: it is basis dependent. Therefore discontinuities do in fact occur. For this reason one cannot be absolutely certain that results for the finite \( N \) truncation of the supermembrane apply as well for the full fledged supermembrane theory. It is for this reason that solutions to the super-Toda molecule equation and their subsequent quantization will in principle provide for us the long-sought spectrum of the supermembrane. Supersymmetry is present in (10) without invoking to Witten index evaluations nor worrying if it makes sense to evaluate it in the continuous spectrum case and whether is infinite or not.

The underlying origins of \( W_\infty \) symmetries of the continuous Toda equation is closely related to the \( W_\infty \) symmetry in \( W_\infty \) strings as we shall see below. These stem from \( D = 4 \ SU(\infty) \) SDYM as follows:

In [15] a geometrical meaning to \( W_\infty \) gravity as a gauge theory of volume-preserving diffs in the space of dimensionally-reduced \( SU^*(\infty) \) instantons was presented. The origins of a universal linear and nonlinear \( W_\infty \) algebras was provided in terms of the geometry of the \( SU^*(\infty) \) SDYM in \( D = 4 \) (An infinite dimensional moduli space). A dimensional reduction gives SD gravity in \( D = 4 \) (Plebanski’s second heavenly equation). A Darboux change of variables converts it into the first heavenly equation and a further Killing symmetry
reduction furnishes the \( sl(\infty) \) continuous Toda equation.

The latter is an effective \( D = 3 \) equation whose symmetry algebra contains the linear classical \( w_\infty \) algebra [16,37].

A Moyal deformation of the Poisson bracket yields the centerless \( W_\infty \) algebra [17]. Central extensions are obtained by means of the cocycle formula in [17]. Central terms of this sort were also considered in terms of the symmetries of the \( \tau \) function associated with the \( sdiff \Sigma \) group by [18]. A nonlinear bracket based on nonlinear gauge theories of the Kyoto school was developed in [15] which made contact with the nonlinear \( W_\infty \) algebras of [19]. Also, the mapping of solutions (after a dimensional reduction) from Plebanski equations to those of the KP equation was made in [15] and the role of many different types of \( W_\infty \) algebras as subalgebras of the KP equation was found. Since the KP hierarchy contains many of the known integrable hierarchies of nonlinear differential equations in lower dimensions it is not surprising that the Toda molecule (a generalization of the dimensionally reduced Liouville-like theory) appears within this context because we had started from a supermembrane which is a gauge theory of the \( sdiff \Sigma \sim SU(\infty) \) group. In view of this fact it is very plausible that the spectrum of the super-Toda molecule is related to the infinite tower of massless higher-superconformal-spin analogs of the super-Liouville modes (after a dimensional reduction) present in noncritical linear and nonlinear \( W_\infty \) strings. Therefore, the membrane's spectrum (from the point of view of the target spacetime) might be related to that of the noncritical \( W_\infty \) strings moving in flat target spacetimes. This is worth looking into. The presence of the Liouville-like modes signals strings in non-critical dimensions. The regularized central charge for critical linear \( W_\infty \) strings was \( c = -2 \) [28]. Clearly one cannot have a membrane moving in negative spacetime dimensions. Since the Liouville modes are present then \( c_m + c_L + c_g = 0 \Rightarrow c_m \neq -2 \). Furthermore, the vast literature on Quantum Groups should provide insights as to how quantize the membrane: the Quantum Liouville theory has been extensively studied from many points of view; mainly in the continuum limit, matrix models ...

[29]. The group to be studied is \( U_q[SU^*((\infty))] \sim U_q[SL((\infty),H)] \).

This is what we are going to do now: Establish the correspondence between the target spacetime of noncritical \( W_\infty \) strings and the membrane’s embedding coordinates and find that \( D = 27; 11 \)-the expected number of spacetime dimensions for the critical bosonic membrane and supermembrane-can be accommodated within the context of a \( W_\infty \) conformal field theory.

\( W \) symmetries of Toda models have been known for sometime. The Toda theory can be obtained from a constrained \( WZNW \) model [31]. Also, the induced action of \( W_n \) gravity in the conformal gauge takes the form of a Toda action for the scalar fields and the \( W \) currents take the familiar free field form [32]. Each of the Toda actions possesses a \( W_n \) symmetry. Furthermore, \( W_\infty \) gravity in the light-cone gauge possesses an underlying \( sl(\infty) \) Kac-Moody symmetry [33] which is to be contrasted with the fact that \( sl(\infty) \) Toda theory has \( w_\infty \) for underlying symmetry. A Moyal deformation and the use of the nonlinear bracket [15] yields the linear and nonlinear \( W_\infty \) algebras.

Our aim is to show that the spectrum of (10) could be classified in terms of representations of \( W_\infty \) algebras [38] and the instanton sector of the membrane (excluding the zero modes) hereby discussed is closely related to the spectrum noncritical \( W_\infty \) strings.

A BRST quantization of the continuous Toda action will encounter anomalies in the quantum \( W_\infty \) algebra as a result of normal ordering ambiguities, per example, as it occurs in
the string. This would destroy unitarity in the spectrum present in the physical Hilbert space of states and hence full Lorentz invariance of the target spacetime will not hold. Pope et al [34] have shown that an anomaly-free quantum theory can be constructed if the ”matter” realization of the $W_\infty$ algebra has for central charge a regularized value of $c_m = -c_{gh} = -2$ in order to have a nilpotent BRST charge operator. If this is so the quantum theory will be devoid of universal gauge anomalies. The $W_\infty$ algebra involved here is the $W$ algebra associated with the Lie algebra of $SU(\infty)$ and this is the $A_\infty$.

$W_\infty$ string is a generalization of ordinary string theory in the sense that instead of gauging the two-dimensional Virasoro algebra one gauges the higher conformal-spin algebra generalization :the $W_\infty$ algebra. The spectrum can be computed exactly [35] and is equivalent to a set of spectra of Virasoro strings with unusual central charges and intercepts. In particular, the critical $W_N$ string ( linked to the $A_{N-1}$ algebra ) has for central charge the value :

$$c = 26 - (1 - \frac{6}{N(N+1)}),$$  \hspace{1cm} (14)

for which unitarity is achieved if the conformal-spin two-sector intercept is

$$\omega_2 = 1 - \frac{k^2 - 1}{4N(N+1)}. \hspace{1cm} (15)$$

$k$ is an integer ranging between 1 and $N - 1$.

The role of the unitary Virasoro minimal models in the expression for the central charge given by (14) was explained and clarified by [29] when they constructed in general the BRST operator corresponding to a $W_N$ algebra as a nested sum of nilpotent and mutually anticommuting BRST operators. This required a new basis in the Hilbert space. In this fashion the exact cohomology can be calculated by an iterative procedure.

In the noncritical $W_N$ string case [29] the matter and Liouville sector of the $W_N$ algebra can be realized in terms of $N - 1$ scalars, $\phi_k; \sigma_k$, respectively. These realizations in general have background charges which are fixed by the Miura transformations [29,36]. The non-critical $W_N$ string is characterized by the central charges of the matter and Liouville sectors $c_m, c_L$ respectively. To achieve a nilpotent BRST operator these central charges must satisfy :

$$c_m + c_L = -c_{gh} = 2 \sum_{s=2}^{N} (6s^2 - 6s + 1) = 2(N-1)(2N^2 + 2N + 1). \hspace{1cm} (16)$$

Where one has summed over all the conformal spins $s = 2, 3.....N$. In the $N \to \infty$ limit one has to perform a zeta function regularization [34] scheme which yields the value of $-2$ for the right hand side of (16).

The authors in [29] have shown that the nested basis can be chosen either for the Liouville sector or the matter sector but not both. [29] chose the nested basis for the Liouville sector and found that :

$$c_L = (N-1)[1 - 2x^2N(N+1)]. \hspace{1cm} (17)$$

where $x$ is an arbitrary parameter which makes it possible to avoid the relation with the minimal models. By choosing $x$ appropriately one can of course get the $q$ th unitary minimal models by having

$$x_o^2 = -2 - \frac{1}{2q(q+1)}. \hspace{1cm} (18)$$
where \( q \) is an integer. In this case from (16,17) one gets for the central charge of the matter sector:

\[
c_m = (N - 1)(1 - \frac{N(N + 1)}{q(q + 1)}).
\]

(19)

which corresponds to the \( q \) th minimal model of the \( W_N \) string. In the present case one has the freedom of selecting the minimal model since the value of \( q \) is arbitrary. If one chooses \( q = N \) then \( c_m = 0 \) and the theory effectively reduces to that of the “critical” \( W_N \) string. We are now approaching the main point of this work. In order to find a spacetime interpretation, the coordinates \( X^\mu \) must be related to the single scalar field of the Liouville sector \( \sigma_1 \) (since one decided to choose the nested basis in the Liouville sector) appearing in the Miura basis as well as in the nested basis. It happens that the stress energy momentum tensor is not modified when one performs a field redefinition from the Miura to the nested basis [29]. The other higher spin currents are clearly modified [29]. The field \( \sigma_1 \) is on special footing because it always appears through its energy momentum tensor so it can be replaced with an effective \( T_{\text{eff}} \) of any conformal field theory as long as it has the same value of the central charge. If one switches-off all the matter fields as if one had effectively a ”critical” string comprised only on the Liouville sector, one could replace:

\[
T(\sigma_1) = -1/2 \left( \partial_z \sigma_1 \right)^2 + \partial_\alpha \partial_\beta \sigma_1 \sigma_1.
\]

(20)

with a central value

\[
c = 1 + 12(\alpha_0)^2.
\]

(21)

by an effective stress energy tensor associated with \( D \) worldsheet scalars, \( X^\mu \), with a background charge vector \( \alpha_\mu \):

\[
T_{\text{eff}} = -1/2 \partial X_\mu \partial X^\mu - \partial_\alpha \partial_\beta X^\mu.
\]

(22)

so that a \( c_{\text{eff}} = D + 12\alpha_\mu \alpha^\mu \) equal to (21) will do the job. This will be sufficient to ensure closure of the \( W_N \) algebra (after switching-off the matter sector) once one includes the extra fields \( \sigma_2, \sigma_3, \ldots \). In [29] it was emphasized that noncritical strings involve two copies of the \( W_N \) algebra. One for the matter sector and one for the Liouville sector. Since \( W_N \) is nonlinear one cannot naively add two realizations of it and obtain a third realization. Nevertheless there are is a way in which this is possible [29]. In any case in order to get a nested sum of nilpotent BRST operators, \( Q^N_\text{BRST} \), one needs to have all the matter fields, \( \phi_k \); the scalars \( \sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_{N-2} \), of the Liouville sector and the ghost and antighost fields of the spin-\( n \), \( n+1 \), \ldots, \( N \) symmetries where \( n \) ranges between 2 and \( N \). Central charges were computed for each set of the nested set of stress energy tensors, \( T^N_{\text{BRST}} \) depending on the above fields which appear in the construction of the \( Q^N_\text{BRST} \) BRST charges. The important point is that the value of \( x \) can be chosen at will. Later we shall see the connection that this choice has with Toda field theory and constrained \( SL(N,R) \) WZW models at level \( k \). Therefore, the effective central charge in the noncritical \( W_N \) string case is now \( c_{m_0} + 1 + 12(\alpha_0)^2 \) in contradistinction to the critical case \( 1 + 12(\alpha)^2 \) where we define:

\[
c_{\text{eff}} \equiv 1 - 12x^2 = (1 - 12x^2) + c_{m_0} - 12x^2 = -12x^2 + c_{m_0}.
\]

(23)

with \( c_{m_0} \) given by (19). Thus in this way one has coupled \( W_N \) matter to \( W_N \) gravity consistently where the Liouville modes have now been switched-on and a unitary noncritical \( W_N \) string spectrum can be constructed when the central charge equals : \( c_{m_0} + 1 + 12(\alpha_0)^2 \).
As stated earlier the induced covariant $W_N$ action- which is tantamount of coupling $W_N$ matter to $W_N$ gravity- in the conformal gauge reduces to a Toda action \[32\]. The constraints following after the gauge fixing of the $W_N$ generalizations of the Beltrami differentials form a closed $W_N$ classical algebra. The $W_N$ currents are comprised of a "matter" sector and a Toda sector. Quantization can be achieved by constructing a nilpotent BRST operator in the lines of \[29\]. This is the connection between the membrane solutions (8c) in terms of (10) via (9a,9b) and the solutions to the continuous Toda equation.

If one chooses the value for $x_0^2$ given in (18) with the particular value of $q = N + 1$, one gets that $c = 1 + 12(\alpha_o)^2 = 1 - 12x_0^2$ becomes in the $N \to \infty$ limit \[29\]: $1 - 12x_0^2 \to 25$. (we are setting $q = N+1$). Since we do not wish to break the target spacetime Lorentz invariance one cannot have background charges for the $D X^\mu$ coordinates. Therefore if one chooses for the effective stress energy tensor the $D X^\mu$ coordinates without background charges one has in this case that an effective central charge given by (23): $c_m + 1 - 12x_0^2 = c_m + 25$ can be achieved by having:

$$D = c_m + 25 = 1 - 12x^2.$$  (24)

$c_m$ is explicitly given by eq-(19) which in the case that $q = N + 1$ it becomes:

$$c_m = 2 \frac{N - 1}{N - 2} \to 2.$$  (25)

in the $N \to \infty$ limit. Therefore, in the $N \to \infty$ limit, (24,25) become:

$$D = 2 + 25 = 27 \Rightarrow x^2 = -13/6$$  (26)

and the $D = 27 \ X^\mu$ scalars without background charges could be seen as the membrane’s embedding coordinates.

The value for the total central charge of the matter sector is $c_m = c_m + \frac{1}{24}$ after a zeta function regularization. The central charge for the Liouville sector is $c_L = -4 - \frac{1}{24}$. The value of $c_m$ (after a regularisation) corresponds to the central charge of the first unitary minimal model of $WA_{n-1}$ after $n$ is analytically continued to a negative value of $n = -146 \Rightarrow 2/(n + 2) = 2 + 1/24$. The value of $c_L$ does not correspond to a minimal model but nevertheless corresponds to a very special value of $c$ where the $WA_{n-1}$ algebra truncates to that of the $W$ algebra associated with noncompact coset models \[40\]:

$$WA_{n-1} \Rightarrow W(2, 3, 4, 5) \sim \frac{sl(2, R)_n}{U(1)}.$$  (26b)

This occurs at the value $c(n) = 2/(n - 2) = -4 - 1/24$ for $n = 146$. Two other values for $c(n)$ are possible, in particular the first unitary minimal model. One should not confuse $c_{eff}$ with $c_m$ and $x^2$ with $x_0^2$.

If the BRST quantization of the continuous Toda action is devoid of $W_\infty$ anomalies the net central charge of the matter plus Toda sector must equal to $-2$ which is the regularized value obtained by \[34\]. Saveliev \[37\] gave a highly nontrivial realization of the $W_\infty$ algebra in terms of the $3D$ continuous Toda field $\Phi(z, \bar{z}, t)$ as we discussed in the previous section. One has to check that indeed such a realization of the $W_\infty$ algebra, after a BRST quantization or
another quantization scheme, does have the correct central charge in order to have \( c_m + c_L \) equal to \(-2\). We recall once more the fact that 32] have shown that the Toda action is the action one gets after one couples \( W_N \) matter to \( W_N \) gravity and the conformal gauge is chosen. It is well known to the experts by now that [41] Quantum Toda theories are conformally invariant and the conformally improved stress energy tensor obeys a Virasoro algebra with an adjustable central charge which depends on the value of the coupling constant \( \beta \) appearing in the exponential potential. This value for the central charge \( c(\beta) \) coincides precisely with the one obtain from a Quantum Drinfeld-Sokolov reduction of the \( SL(N,R) \) Kac-Moody algebra at the level \( k \). The value of the coupling is:

\[
\beta = \frac{1}{\sqrt{k + N}}, \quad c(\beta) = (N - 1) - 12|\beta \rho - 1/\beta \rho^v|^2. \tag{26c}
\]

where \( \rho, \rho^v \) are the Weyl vectors of the (dual) \( A_N \) Lie algebra. [41] . We see that indeed (26c) has the same form as (17) where the value of \( x^2 \) is related to the coupling \( \beta \):

\[
|x^2| = 13/6 \sim (1/\sqrt{k + N} - \sqrt{k + N})^2. \quad k = -\infty, \quad k + N = constant. \tag{26d}
\]

So we learn that in the \( N \to \infty \) limit one must have \( k = -\infty \) such as \( k + N = constant \) in order to obey (26d). The check now is to evaluate the central charge directly from Saveliev's realization of the \( W_\infty \) algebra in terms of the \( 3D \) continuous Toda theory. From eqs-(9a,9b) one has that the continuous Toda field \( \Phi \) is a function of the YM potentials, \( A_\mu \), which play the role of coordinates :\( A_\mu \to X_\mu \) so that \( \Phi = \Phi[X_\mu] \). Upon computing the operator products of the \( W_\infty \) currents in terms of \( \Phi \) one can reexpress them in terms of OPE involving the \( X_\mu \) coordinates which from the point of view of a three-dimensional observer are just \( 3D \) scalar fields. Therefore, in order to be selfconsistent the central charge obtained from such a procedure must match the central charge obtained from the value of \( c(\beta) \) above in eqs-(26). This selfconsistency in the matching of these central charges would be an indication that \( x^2 \) is in fact equal to \(-13/6\) and that the zeta function regularisation was indeed consistent with the Saveliev's realization. The value of \( x^2 = -13/6 \) is the required value so that \( c_{\text{eff}} = 27 \). Fixing \( x^2 \) yields the coupling constant \( \beta \) in (26c, 26d). Notice that:

\[
D - 2 = 25 = 26 - [1 - \frac{6}{q(q + 1)}]. \tag{27}
\]

in the \( N \to \infty \) limit. So a connection with the unitary Virasoro minimal models is also established for those values of \( q = N + 1 \).

In the supersymmetric case one could again establish the connection with the unitary Virasoro superconformal minimal models. In the \( N \to \infty \) limit the value of the central charge becomes \( c_{\text{superconformal}} \to 3/2 \). Since 10 is the critical dimension of the superstring the value of the central charge when one has ten worldshet scalars and ten fermions is \( 10 + 10/2 = 30/2 \). Therefore in order to have an effective ”critical ” super \( W_\infty \) string one has for the supersymmetric analog of the right hand side of (27) the value of :

\[
10(1 + 1/2) - c_{\text{superconformal}} = 30/2 - 3/2 = 27/2. \tag{28}
\]

The super Liouville sector is comprised of the infinite number of super Toda fields present in the continous super \( SL(\infty) \) Toda action and must have a total central charge \( c_L \) so that
\(c_m + c_L\) is equal to \(-3\) which is the required value to have an anomaly-free "matter" realization of the quantum super \(W_\infty\) algebra \([34]\). Going through the same reasoning as in the bosonic case \([29]\) and choosing the appropriate value for the arbitrary parameter \(x^2\) in order to make contact with the bosonic sector of the unitary minimal models of the super \(W_N\) algebra ones gets for the central charge of the matter sector of the super \(W_N\) algebra the value of \(c_{m_0} = 3 = 2(1 + 1/2)\).

Having \(D \ X^\mu\) and \(D \ \psi^{\mu}\) (anticommuting spacetime vectors and worldsheet spinors) without background charges yields a central charge equal to \(D + D/2 = 3D/2\). Equating now:

\[
3D/2 = c_{m_0} + [30/2 - 3/2] = 3 + 27/2 \Rightarrow D = 11.
\]

This is the expected critical dimension for the supermembrane. So one obtains the expected critical dimensions for the (super) membrane if one adjoins a \(q = N + 1\) unitary minimal model of the \(W_N\) algebra in the \(N \to \infty\) limit to a critical \(W_\infty\) string spectrum. The same goes for the super \(W_N\) algebra.

The issue of global anomalies remains to be settled. Global anomalies in the \(E_8 \times E_8\) and \(SO(32)\) superstring theories in arbitrary target spacetimes were discussed in \([20]\). Cancellation of global anomalies in the \(E_8\) theory results in constraints on the topology of \(M\). For the \(SO(32)\) it seems unlikely that any nontrivial manifold leads to an anomaly-free theory. Since the exceptional algebra \(E_8\) is intimately tied up with the octonion division algebra which is so essential to formulate a SDYM in \(R^8\) \([1, 9]\) referred above, prior to the reduction to \(R^4\), it seems that an anomaly-free supermembrane can only propagate in those special manifolds (after one spatial dimensional reduction from \(D = 11 \to D = 10\)) discussed in \([20]\).

As far as Lorentz anomalies is concerned we have seen how the cancellation of \(W_\infty\) anomalies for the noncritical \(W_\infty\) strings (the net value of \(c = 0\)) via a BRST analysis might yield information of the supermembrane’s critical dimension. Conformally-invariant Spinning membrane actions have been given in \([21]\) in a nonpolynomial form, and in polynomial form \([22]\). The polynomial form \([22]\) was supersymmetric solely under the \(Q\) super symmetry of the superconformal algebra in \(D = 3\) and was not invariant under \(S\) supersymmetry nor \(K\) symmetry (conformal boosts). It was invariant under translations, Lorentz and dilations and the subalgebra comprised of these invariance-generators does in fact close. The action \([22]\), in effect, was a 3\(D\)-conformally invariant analog of the spinning string with the main difference that one needs the explicit presence of the 3\(D\)-gauge field of dilations, \(b_\mu\) which depends on the matter fields (the membrane’s coordinates) in a highly complicated manner due to the presence of the quartic derivative terms. It is unfortunate that we have encountered so many difficulties in having many readers accept this point.

It is essential to study the plausible existence of conformal anomalies and see if constraints on the dimension of the target spacetime can be obtained as they do for the ordinary string. These latter spinning membrane actions are the supersymmetric extension of the generalization of the \(SU(2)\) chiral models in three spacetime dimensions discussed in \([23]\). Finite, stable energy and static kink solutions carrying a topologically conserved charge were found without the usual scaling instability problems present in \(D = 2\) (Derrick’s theorem). This required the choice of the nonpolynomial action \([21]\) whereas the polynomial action \([22]\) bears a strong resemblance with Skyrme’s model. A simple count of the transverse dimensions of
the supermembrane in 4, 5, 7, 11 dimensions suggests that because \(11 - 3 = 8 = (3)^2 - 1\) one can incorporate these membrane’s degrees of freedom into a \(SU(3)\) generalization of Skyrme’s model. The other values of \(D\) do not fit.

Finally we wish to mention the importance of these (instanton) solutions of the \(D = 11\) supermembrane in connection to the the geometry of SYM. Firstly, a complexification yields the \(SU^*(\infty)\) group which is isomorphic to \(SL(\infty, H)\) and hence the role of quaternions in these theories. All this seems to point at the fact that it is quaternionic field theory in \(D = 4\) [24] the one which may contain many of the hidden symmetries of string theory and in particular might give us the clues how to build a background-independent string-field theory in terms of the spaces of all quaternionic field theories. Secondly, the abundant research into twistor-like formulations of null super-\(p\)-branes [25] by extending the configuration space through the addition of auxiliary spinor coordinates of the Newman-Penrose dyades for \(D = 4\), will give us important clues as to how implement quaternionic analyticity into the picture. (in terms of quaternionic twistors).

And, finally, the notion of integrability on light-like lines [26] is tightly connected with supertwistors (a parameter space of light-like lines in complex superspace). It was such integrability in \(D = 10\), \(N = 1\) and \(D = 4\) for \(N\) greater than 2 super YM theory that led to the on-shell equations of motion. Per example: \(N = 1\) \(D = 6\) SYM in complexified-superspace can be written down in terms of \(SL(2, H) \sim SU^*(4)\) spinors. A dim reduction to \(D = 4\) implies a fourth-folding in terms of the number of real supersymmetries and one has then a \(D = 4\) \(N = 4\) SYM which many believe it is finite theory at all loops [27]. Since in \(D = 10\) SYM the number of transverse degrees of freedom is eight which is the same as those of the supermembrane it would be interesting if extending the notion of light-like integrability in loop superspaces or their generalization to higher-dimensional loops, yields exact on-shell solutions to the supermembrane’s equations of motion. i.e; Instead of light-like lines one has null-\(p\)-branes directions in higher dimensional loop spaces. After all it has been known for some time that SDYM amounts to an integrability condition. The construction of higher dimensional loop algebras with central extensions and their relevance to extended objects has been given by Goteborg group [42]. We hope that the super-Toda molecule equation is an alternative to the SGQMM. This is the main conclusion of this work.

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6 Appendix

We shall give in this appendix the continuum limit of (10). For simplicity we shall only work with the bosonic sector. It is given by

\[
\frac{-\partial^2 \Psi}{\partial t^2} = \int K(y, y') \exp[\Psi(t, y')] dy'.
\] (A-1)
The continuum field $\Psi(t, y = y_0 + l\epsilon) \equiv c\Phi_l(t)$. Where one partitions the interval $[y_0, y_f]$ in $N$ intervals and performs the linear interpolation such as:

$$\frac{\Psi(t, y_f) - \Psi(t, y_0)}{\Phi_N - \Phi_1} \sim \frac{y_f - y_0}{N} = \epsilon. \quad (A - 2)$$

In this fashion, when $\epsilon \to 0$ the continuum field $\Psi(t, y_0) = \Psi(t, y_f) \to 0$ so that $\Phi_1, \Phi_\infty$ are well defined. The continuum Cartan matrix is: $K(y, y') = cK_{ll'}$ where $y_0 + l\epsilon \leq y \leq y_0 + (l + 1)\epsilon$ and $y_0 + l'\epsilon \leq y' \leq y_0 + (l' + 1)\epsilon$. Another way of recasting the continuum Cartan matrix is: $\delta^\epsilon(y - y')$ (we thank I. Bakas for pointing this out to us) so that (A-1) becomes:

$$-\partial^2_{t^2}\Psi = \partial^2_{y^2} \exp \Psi. \quad (A - 3)$$

In solving equation (A-3) one could recurr to a dimensional reduction from $3 \to 2$ of the $sl(\infty)$ Toda equation: given by Leznov and Saveliev:

$$\partial_z \partial_{\bar{z}} u = -\partial^2_t (e^u). \quad (A - 4)$$

where $u(z, \bar{z}, t)$ and $z = x + iy, \bar{z} = x - iy$. Therefore, the solutions to the Toda molecule equation are tied-up to Killing symmetry reductions of $D = 4$ SDG.

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