Complete integrability of derivative nonlinear Schrödinger-type equations

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Abstract. We study matrix generalizations of derivative nonlinear Schrödinger-type equations, which were shown by Olver and Sokolov to possess a higher symmetry. We prove that two of them are ‘C-integrable’ and the rest of them are ‘S-integrable’ in Calogero’s terminology.

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1. Introduction

It has been known that a large part of soliton equations in 1 + 1 dimensions have integrable matrix generalizations, or more generally, integrable multi-field generalizations. Such generalizations have been studied for a wide variety of single (or two)-component integrable systems by virtue of various approaches such as Jordan algebras and Jordan pairs [1, 2, 3], inverse scattering method [4, 5, 6, 7, 8], Hirota’s method [9, 10, 11, 12, 13]. However, as far as the authors know, matrix generalizations of the derivative nonlinear Schrödinger (DNLS)-type systems [14, 15] have not been studied throughly.

In recent papers [16, 17], Olver and Sokolov made a detailed investigation on the DNLS-type systems of the form
\[
P_t = P_{xx} + f(P, S, P_x, S_x)
\]
\[
S_t = -S_{xx} + g(P, S, P_x, S_x).
\]

Here \( P \) and \( S \) take values in an associative algebra. For simplicity, in the following we regard \( P \) and \( S \) as matrix-valued. \( f \) and \( g \) are non-commutative polynomials of weight 5, where the weights of \( \partial_t, \partial_x, P \) and \( S \) are respectively assigned to be 4, 2, 1 and 1. They made a complete list of the DNLS-type systems (1.1) which have one higher symmetry of the following form with weight 9,
\[
P_\tau = P_{xxxx} + \tilde{f}(P, S, P_x, S_x, P_{xx}, S_{xx}, P_{xxx}, S_{xxx})
\]
\[
S_\tau = -S_{xxxx} + \tilde{g}(P, S, P_x, S_x, P_{xx}, S_{xx}, P_{xxx}, S_{xxx}).
\]

Here the commutativity of the two flows, i.e. \( \partial_t \partial_\tau P = \partial_\tau \partial_t P, \partial_t \partial_\tau S = \partial_\tau \partial_t S \), works as a strong constraint on the form of \( f \) and \( g \) in order for the non-commutative polynomials \( \tilde{f} \) and \( \tilde{g} \) to exist. See ref. [18, 19, 20] for a more precise explanation of symmetry approach.

The entries of the list are divided into two kinds. Two systems in the list are interpreted as non-abelian analogues of the following integrable system
\[
p_t = p_{xx} + 2\alpha p^2 s_x + 2\alpha pp_x s - \alpha \beta p^3 s^2
\]
\[
s_t = -s_{xx} + 2\beta s^2 p_x + 2\beta ss_x p + \alpha \beta s^3 p^2
\]

for a particular choice of the constants \( \alpha \) and \( \beta \). The system (1.2) is linearizable by a change of the dependent variables. Thus, we can construct the general solution of the system. We often refer to such linearizable systems as ‘C-integrable’ in Calogero’s terminology [21]. On the other hand, excepting the two entries corresponding to (1.2), the scalar-valued counterparts of the systems in the list by Olver and Sokolov are given by [8]
\[
iq_t + q_{xx} + 4i\delta q^2 r_x + i(4\delta - \alpha)qq_x r + \delta(4\delta + \alpha)q^3 r^2 = 0
\]
\[
ir_t - r_{xx} + 4i\delta r^2 q_x + i(4\delta - \alpha)rr_x q - \delta(4\delta + \alpha)r^3 q^2 = 0
\]

for special choices of the constants \( \alpha \) and \( \delta \). This system was generated via a gauge transformation for the DNLS equation by Kund [22]. We can write down a Lax pair for
the system (1.3) with the help of the gauge transformation. According to Calogero’s terminology, this kind of systems, which are linearizable by the inverse scattering formulation, are called ‘S-integrable’ systems [21].

If a system has one higher symmetry, it has been believed that the system has an infinite series of symmetries and thus is completely integrable. However, a system proposed by Bakirov was recently proved to be a counter-example to this empirical law [23]. Thus, there is no guarantee that the systems in the list by Olver and Sokolov are really integrable, although the counter-example seems very exceptional.

The aim of this paper is to establish the complete integrability of all the matrix-valued systems given in [17]. In a previous paper [8], the authors introduced a Lax pair for the matrix generalization of the Chen-Lee-Liu equation [15],

\[
\begin{align*}
\text{i} Q_t + Q_{xx} - \text{i} QR Q_x &= 0 \\
\text{i} R_t - R_{xx} - \text{i} R_x QR &= 0
\end{align*}
\] (1.4)

which is a member of the list by Olver and Sokolov. In the present paper, we generalize the Lax pair for (1.4) to be applicable for several matrix systems of the DNLS type in [17].

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The paper consists of the following. In section 2, we extend a notion of the gauge transformation to the non-abelian case. We give an explicit expression of the Lax pairs for all systems but two in [17] to show their ‘S-integrability’. On the other hand, in section 3, we prove the remaining two systems to be linearizable so that they are ‘C-integrable’. Concluding remarks are given in the last section, section 4. Throughout this paper, capital letters denote matrices while small letters represent scalars. Equation numbers without ‘section-number.’ refer to equations in [17].

2. ‘S-integrable’ systems

We begin with a brief summary of the matrix generalization of the Chen-Lee-Liu-type DNLS equation [8]. A system of linear differential equations for a vector Ψ

\[
\Psi_x = U\Psi \quad \Psi_t = V\Psi
\] (2.1)

is consistent if the following equation is satisfied:

\[
U_t - V_x + UV - VU = O.
\] (2.2)

The matrices U and V are called Lax matrices or a Lax pair. We introduce the following form of the Lax pair,

\[
U = \text{i} \zeta^2 \left[ \begin{array}{c}
-I_1  \\
I_2
\end{array} \right] + \zeta \left[ \begin{array}{c}
Q  \\
R
\end{array} \right] + \text{i} \left[ \begin{array}{c}
O  \\
\frac{1}{2} QR
\end{array} \right],
\]

\[
V = \text{i} \zeta^4 \left[ \begin{array}{c}
-2I_1  \\
2I_2
\end{array} \right] + \zeta^3 \left[ \begin{array}{c}
2Q  \\
2R
\end{array} \right] + \text{i} \zeta^2 \left[ \begin{array}{c}
-QR  \\
R Q
\end{array} \right] + \zeta \left[ \begin{array}{c}
iQ_x + \frac{1}{2} QR Q  \\
-i R_x + \frac{1}{2} R Q R
\end{array} \right] + \text{i} \left[ \begin{array}{c}
O  \\
\frac{1}{2} (R Q_x - R_x Q) + \frac{1}{2} R Q R Q
\end{array} \right].
\] (2.4)
Here, $\zeta$ is the spectral parameter. $I_1$ and $I_2$ are respectively the $n_1 \times n_1$ and the $n_2 \times n_2$ identity matrices. $Q$ is an $n_1 \times n_2$ matrix and $R$ is an $n_2 \times n_1$ matrix. Putting (2.3) and (2.4) into (2.2), we obtain the matrix version of the Chen-Lee-Liu-type DNLS equation
\[
\begin{align*}
\text{i}Q_t + Q_{xx} - \text{i}QRQ_x &= O \\
\text{i}R_t - R_{xx} - \text{i}R_xQR &= O.
\end{align*}
\tag{2.5}
\]

It is to be noted that there is no restriction on the sizes of $Q$ and $R$, that is, on $n_1$ and $n_2$. The system (2.5) was shown to possess at least one higher symmetry \cite{16, 17}. The system is now proved to be completely integrable in the sense that it has a Lax pair and, as a result, an infinite number of conservation laws.

Next, we shall prove the complete integrability of other systems in \cite{17} in the same sense. For this purpose, we introduce a transformation of dependent variables:
\[
Q = F^{-1}QG^{-1} \quad R = GRF
\tag{2.6}
\]
or equivalently
\[
Q = FQG \quad R = G^{-1}RF^{-1}.
\]

Here $F$ and $G$ are invertible matrices, which in general depend on $Q$ and $R$ (or $Q$ and $R$). Then, time evolution equations for $Q$ and $R$, (2.3), are cast into those for $Q$ and $R$:
\[
\begin{align*}
\text{i}Q_t + Q_{xx} - \text{i}QRQ_x - \text{i}F_t F^{-1}Q + \text{i}Q(G^{-1})_t G - (F_t F^{-1})_x Q + Q\{(G^{-1})_x G\}_x \\
-2F_x F^{-1}Q_x + 2Q_x (G^{-1})_x G - 2F_x F^{-1}Q(G^{-1})_x G + (F_x F^{-1})^2 Q \\
+Q\{(G^{-1})_x G\}_x^2 + \text{i}QRF_x F^{-1}Q - \text{i}QRQ(G^{-1})_x G = O
\end{align*}
\tag{2.7a}
\]
\[
\begin{align*}
\text{i}R_t - R_{xx} - \text{i}R_x Q R - \text{i}(G^{-1})_t G R + \text{i}R F^{-1} + \{(G^{-1})_x G\}_x R - R(F_x F^{-1})_x \\
+2(G^{-1})_x G R_x - 2R_x F^{-1} + 2(G^{-1})_x G R F_x F^{-1} - \{(G^{-1})_x G\}_x^2 R \\
-R(F_x F^{-1})^2 + \text{i}(G^{-1})_x G RQR - \text{i}R F^{-1} Q R = O.
\end{align*}
\tag{2.7b}
\]

A sufficient condition for (2.7) to be local and closed equations is that $F_x F^{-1}$, $F_t F^{-1}$, $(G^{-1})_x G$ and $(G^{-1})_x G$ are expressed locally in closed forms in terms of $Q$ and $R$, i.e.
they do not include terms with integrals, infinite sums, etc. We impose this condition on $F$ and $G$ in what follows. A closed expression of the Lax pair for the transformed system is given by performing the gauge transformation
\[
\Psi = g \Phi \quad g = \begin{bmatrix} F^{-1} \\ G \end{bmatrix}.
\]

Due to this transformation, the linear problem and the Lax pair for (2.3) are changed into those for (2.7):
\[
\Phi_x = U' \Phi \quad \Phi_t = V' \Phi
\]
\[
U' = g^{-1} U g - g^{-1} g_x
\]
\[
= \text{i} \zeta^2 \begin{bmatrix} -I_1 \\ I_2 \end{bmatrix} + \zeta \begin{bmatrix} R \\ Q \end{bmatrix} + \text{i} \begin{bmatrix} -F_x F^{-1} \\ \frac{1}{2} R Q - \text{i}(G^{-1})_x G \end{bmatrix}
\tag{2.8}
\]
$V' = g^{-1}Vg - g^{-1}g_t$

$$= \imath \zeta^4 \left[ \begin{array}{c} -2I_1 \\ 2I_2 \end{array} \right] + \zeta^3 \left[ \begin{array}{c} 2Q \\ 2R \end{array} \right] + \imath \zeta^2 \left[ \begin{array}{c} -QR \\ RQ \end{array} \right] + \zeta \left[ \begin{array}{c} V_{12} \\ V_{21} \end{array} \right] + \imath \left[ \begin{array}{c} -iF_tF^{-1} \\ V_{22} \end{array} \right].$$

(2.9)

Here

$$V_{12} = iQ_x + \frac{1}{2}QRQ - iF_x F^{-1}Q + iQ(G^{-1})_x G$$

$$V_{21} = -iR_x + \frac{1}{2}RQR + i(G^{-1})_x GR - iRF_x F^{-1}$$

$$V_{22} = i\frac{1}{2}(RQ_x - R_x Q) + \frac{1}{4}RQRQ - iRF_x F^{-1}Q + i\frac{1}{2}RQ(G^{-1})_x G$$

$$+ i\frac{1}{2}(G^{-1})_x GRQ - i(G^{-1})_t G.$$ The above transformation is a powerful tool; it yields new integrable systems of the DNLS type by appropriate choices of $F$ and $G$. To confirm this, we list six illustrative examples (a)–(f) with the definition of $F$ and $G$, the evolution equations for $Q$ and $R$ and the transformed Lax matrix $U'$:

(a) $F = I_1$

$$(G^{-1})_x = -i\frac{1}{2}G^{-1}RQ = -i\frac{1}{2}RQG^{-1}$$

$$(G^{-1})_t = G^{-1}\left\{ \frac{1}{2}(RQ_x - R_x Q) - \frac{1}{4}RQRQ \right\}$$

$$= \left\{ \frac{1}{2}(RQ_x - R_x Q) - \frac{3}{4}RQRQ \right\}G^{-1}$$

$$iQ_t + Q_{xx} - i(QRQ)_x = 0$$

$$iR_t - R_{xx} - i(RQR)_x = 0$$

(2.10)

$$U' = \imath \zeta^2 \left[ \begin{array}{c} -I_1 \\ I_2 \end{array} \right] + \zeta \left[ \begin{array}{c} Q \\ R \end{array} \right].$$

(2.11)

(b) $G^{-1} = I_2$

$$F_x = -i\frac{1}{2}FQR = -i\frac{1}{2}QRF$$

$$F_t = F\left\{ \frac{1}{2}(Q_x R - QR_x) - \frac{1}{4}QRQR \right\}$$

$$= \left\{ \frac{1}{2}(Q_x R - QR_x) + \frac{1}{4}QRQR \right\}F$$

$$iQ_t + Q_{xx} + iQR_x Q + \frac{1}{2}QRQRQ = 0$$

$$iR_t - R_{xx} + iRQ_x R - \frac{1}{2}RQQR = 0$$

(2.12)
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\[ U' = i\xi^2 \begin{bmatrix} -I_1 & I_2 \end{bmatrix} + \zeta \begin{bmatrix} Q & R \end{bmatrix} + i \begin{bmatrix} -\frac{1}{2}QR & \frac{1}{2}RQ \end{bmatrix} \tag{2.13} \]

(c) \[ F = I \]

\[ (G^{-1})_x = \frac{1}{2}QRG^{-1} = \frac{1}{2}QRG^{-1} \]

\[ (G^{-1})_t = \left\{ \frac{1}{2}(QR_x - Q_xR) + \frac{1}{4}QRQR \right\}G^{-1} \]

\[ = \left\{ \frac{1}{2}(QR_x - Q_xR) + \frac{1}{4}QRQR - \frac{1}{2}Q^2R^2 \right\}G^{-1} \]

\[ iQ_t + Q_{xx} - iQRQ_x + iQ^2R_x + iQ_xQR - \frac{1}{2}Q^2RQR + \frac{1}{2}Q^3R^2 + \frac{1}{2}QRQ^2R = 0 \tag{2.14} \]

\[ iR_t - R_{xx} - iR_xQR + iQ_xQ^2 + iQR^2R_x + \frac{1}{2}Q^2R^2Q^2 - \frac{1}{2}Q^2R^2 - \frac{1}{2}QR^2QR = 0 \]

\[ U' = i\xi^2 \begin{bmatrix} -I & I \end{bmatrix} + \zeta \begin{bmatrix} Q & R \end{bmatrix} + i \begin{bmatrix} O & \frac{1}{2}(RQ + QR) \end{bmatrix} \tag{2.15} \]

(d) \[ G^{-1} = I \]

\[ F_x = \frac{1}{2}RQF = i\frac{1}{2}RQF \]

\[ F_t = \left\{ \frac{1}{2}(R_xQ - RQ_x) + \frac{1}{4}RQRQ \right\}F \]

\[ = \left\{ \frac{1}{2}(R_xQ - RQ_x) + \frac{1}{4}RQRQ + \frac{1}{2}R^2Q^2 \right\}F \]

\[ iQ_t + Q_{xx} - iQRQ_x - iR_xQ^2 - iRQQ_x + \frac{1}{2}R^2Q^3 - \frac{1}{2}Q^2R^2 = 0 \]

\[ iR_t - R_{xx} - iR_xQR - iR^2Q_x - iR_xRQ - \frac{1}{2}R^2Q^2 - \frac{1}{2}QR^2R = 0 \tag{2.16} \]

\[ U' = i\xi^2 \begin{bmatrix} -I & I \end{bmatrix} + \zeta \begin{bmatrix} Q & R \end{bmatrix} + i \begin{bmatrix} \frac{1}{2}RQ & \frac{1}{2}RQ \end{bmatrix} \tag{2.17} \]

(e) \[ G^{-1} = F \]

\[ F_x = -i\frac{1}{2}FRQ = -i\frac{1}{2}RQF \]

\[ F_t = F\left\{ \frac{1}{2}(RQ_x - R_xQ) - \frac{1}{4}RQRQ \right\} \]

\[ = \left\{ \frac{1}{2}(RQ_x - R_xQ) - \frac{3}{4}RQRQ + \frac{1}{2}R^2Q^2 \right\}F \]

\[ iQ_t + Q_{xx} - iQRQ_x - iR_xQ^2 - iQ_xQR + iR_xQ^2 + iRQQ_x - RQR^2 \]

\[ + \frac{1}{2}R^2Q^3 + \frac{1}{2}QR^2RQ = 0 \]

\[ iR_t - R_{xx} - iR_xQR - iRQQ_x - iR_xRQ - R^2QRQ \]

\[ - \frac{1}{2}R^3Q^2 - \frac{1}{2}QR^2R = 0 \tag{2.18} \]
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\[ U' = \zeta^2 \begin{bmatrix} -1 & 0 \\ I & R \end{bmatrix} + \zeta \begin{bmatrix} 0 & Q \\ R & 0 \end{bmatrix} + i \begin{bmatrix} -\frac{1}{2}Q R & 0 \\ R & 0 \end{bmatrix}. \]  

(2.19)

(f) \[ G^{-1} = F \]

\[ F_x = -i\frac{1}{2}FQR - i\frac{1}{2}QRF \]

\[ F_t = F \left\{ \frac{1}{2}(QxR - QRx) - \frac{1}{4}QRQR \right\} \]

\[ = \left\{ \frac{1}{2}(QxR - QRx) + \frac{1}{4}QRQR - i\frac{1}{2}Q^2 R^2 \right\} F \]

\[ iQ_t + Q_{xx} + iQRxQ - iQ^2 R_x - iQxQR - \frac{1}{2}Q^2 R^2 Q - \frac{1}{2}Q^2 Q R R + \frac{1}{2}Q^3 R^2 + \frac{1}{2}Q R Q R Q = 0 \]  

(2.20)

\[ iR_t - R_{xx} + iRQxR - iQ_x R^2 - iQRRx + \frac{1}{2}R Q^2 R^2 + \frac{1}{2}Q R Q R R - \frac{1}{2}Q Q R^2 R^2 - \frac{1}{2}R Q R Q R = 0 \]

\[ U' = \zeta^2 \begin{bmatrix} -I & 0 \\ I & R \end{bmatrix} + \zeta \begin{bmatrix} 0 & Q \\ R & 0 \end{bmatrix} + i \begin{bmatrix} -\frac{1}{2}Q R & 0 \\ R & 0 \end{bmatrix} + \frac{1}{2}(Q R - Q R) \]  

(2.21)

For all of the six examples (a)–(f), the compatibility conditions for \( F \) and \( G^{-1} \), i.e.

\[(F_x)_t = (F_t)_x \quad \{(G^{-1})_x\}_t = \{(G^{-1})_t\}_x\]

can be checked by a straightforward calculation with the help of (2.5). As is clear from the construction of the gauge transformations, \( Q \) and \( R \) can be rectangular matrices for (a) and (b), while \( Q \) and \( R \) must be square matrices for (c)–(f).

Comparing the above results with those by Olver and Sokolov, we find that (2.3) and (10) in [17], (2.10) and (7) in [17], (2.12) and (12) in [17], (2.14) and (14) in [17], (2.16) and (16) in [17], (2.18) and (15) in [17], (2.20) and (17) in [17] are respectively identical up to scalings of variables. The system (2.5) was surveyed in a previous paper from the viewpoint of the inverse scattering method [8]. The system (2.10) is a well-known matrix generalization of the DNLS equation of Kaup-Newell type [5]. For systems (2.12), (2.14), (2.16) and (2.20), or (12), (14)–(17) in [17], we have obtained the Lax representations by virtue of a non-commutative version of gauge transformations for the first time in this paper.

The systems (2.3), (2.18) and (2.20) are interpreted as matrix generalizations of (1.3) with \( \alpha = 1, \delta = 0 \). The systems (2.10) and (2.16) reduce to (1.3) with \( \alpha = 1, \delta = -1/4 \) in the commutative case. The systems (2.12) and (2.14) correspond to (1.3) with \( \alpha = 1, \delta = 1/4 \) when \( Q \) and \( R \) are scalar-valued. The symmetry approach shows that matrix generalizations of (1.3) are essentially exhausted by the above statement up to scalings and the transposition [17]. It is remarkable that the system (1.3) has matrix generalizations only for some choices of \( \alpha \) and \( \delta \). This feature of matrix generalizations
may be explained from our approach in the following way. In the case of scalar variables, equation (1.3) is generated by the gauge transformation

\[ q = \frac{1}{\alpha} q \exp\left\{ -2i \frac{\delta}{\alpha} \int r q d{x}' \right\} \]

\[ r = \frac{1}{\alpha} r \exp\left\{ 2i \frac{\delta}{\alpha} \int q r d{x}' \right\} \]

for the DNLS equation by Chen, Lee and Liu [15]

\[ i q_t + q_{xx} - i q r q_x = 0 \]

\[ i r_t - r_{xx} - i r q r_x = 0 . \]

Why cannot we generalize this transformation to the matrix case for the continuous parameters \( \alpha \) and \( \delta \)? As an illustrative example, we choose \( F \) and \( G^{-1} \) to satisfy

\[ F_x = i \gamma R Q F \quad \lim_{x \to x_0} F = I \quad G^{-1} = I \]

where \( \gamma \) is a scalar parameter. The explicit form of \( F \) is given by

\[ F = \mathcal{E} \exp\left\{ i \gamma \int_{x_0}^{x} R Q d{x}' \right\} \]

\[ = I + \sum_{n=1}^{\infty} (i \gamma)^n A_n . \]

Here \( \mathcal{E} \) is the path ordering operator and

\[ A_n = \int_{x_0}^{x} dx_1 \int_{x_0}^{x_1} dx_2 \cdots \int_{x_0}^{x_{n-1}} dx_n R(x_1)Q(x_1) \cdots R(x_n)Q(x_n) . \]

We can calculate the time derivative of \( F \), \( F_t \), with the help of (2.5). However, it is observed that \( F_t F^{-1} \) cannot be expressed in a closed form in general, i.e. it includes infinitely multiple integrals. An exception is the case of \( \gamma = 1/2 \), which gives

\[ F_t = \left\{ \frac{1}{2} (R_x Q - R Q_x) + i \frac{1}{4} R Q R Q \right\} F \]

and, as a result, example (d). The example explains why the transformation (2.6) is effective for finite choices of \( F \) and \( G \) in the matrix case.

It is a well-known fact that the spatial part of the Lax formulation for the Chen-Lee-Liu-type DNLS equation is common with that for the massive Thirring model, i.e. they belong to the same hierarchy. Thus, employing an appropriate time part in the Lax formulations in correspondence with the Lax matrices \( U \) (or \( U' \)) in this section, we can obtain new matrix generalizations of the massive Thirring model (see appendix).

### 3. ‘C-integrable’ systems

In the previous section, we have verified that all but two systems proposed by Olver and Sokolov are ‘S-integrable’, i.e. they have Lax representations and can be linearized by the inverse scattering method. In this section, we show that two systems left for further analysis are ‘C-integrable’, i.e. they can be linearized by a certain type of transformation of dependent variables which resembles that in the previous section.
The two systems, (11) and (13) in [17], are matrix generalizations of (1.2) with \( \alpha = 0, \beta = 1 \). We briefly summarize a solution of (1.2) for investigating (11) and (13). The pair of equations (1.2) is rewritten in a linearized form as
\[
\begin{align*}
(p e^\alpha \int_{x_0}^{x} s p dx')_t - (p e^\alpha \int_{x_0}^{x} s p dx')_{xx} &= 0 \\
(s e^{-\beta} \int_{x_0}^{x} p s dx')_t + (s e^{-\beta} \int_{x_0}^{x} p s dx')_{xx} &= 0
\end{align*}
\]
under the boundary conditions: \( \lim_{x \to x_0} p = \lim_{x \to x_0} s = 0 \). In terms of functions \( y(x, t) \) and \( z(x, t) \), which satisfy a pair of heat equations
\[
\begin{align*}
y_t - y_{xx} &= 0 \\
z_t + z_{xx} &= 0
\end{align*}
\]
and the boundary conditions, \( \lim_{x \to x_0} y = \lim_{x \to x_0} z = 0 \), the general solution of (1.2) is given by
\[
\begin{align*}
p &= ye^{-\alpha} \int_{x_0}^{x} s p dx' = y\left\{1 + (\alpha - \beta) \int_{x_0}^{x} y z dx'\right\}^{-\frac{\alpha}{\alpha - \beta}} \\
s &= ze^{\beta} \int_{x_0}^{x} p s dx' = z\left\{1 + (\alpha - \beta) \int_{x_0}^{x} y z dx'\right\}^{-\frac{\beta}{\alpha - \beta}}
\end{align*}
\]
for \( \alpha \neq \beta \) (cf. [24] for \( \alpha = -\beta \)) and
\[
\begin{align*}
p &= ye^{-\alpha} \int_{x_0}^{x} s p dx' = ye^{-\alpha} \int_{x_0}^{x} y z dx' \\
s &= ze^{\beta} \int_{x_0}^{x} p s dx' = ze^{\beta} \int_{x_0}^{x} y z dx'
\end{align*}
\]
for \( \alpha = \beta \).

We proceed to solve (11) and (13) in [17] by generalizing the above method to the matrix case. We write (11) in [17]:
\[
\begin{align*}
P_t &= P_{xx} \\
S_t &= -S_{xx} + 2SP_x S + 2SPSP_x.
\end{align*}
\]
(3.1)
Here \( P \) is an \( n_1 \times n_2 \) matrix and \( S \) is an \( n_2 \times n_1 \) matrix. The boundary conditions
\[
\lim_{x \to x_0} P = O \quad \lim_{x \to x_0} S = O
\]
are assumed for convenience. In terms of \( A \) defined by
\[
A_x = A(SP) \quad A_t = A(SP_x - SP - SPSP)
\]
the time evolution equation for \( S \) in (3.1) is rewritten in a linearized form as
\[
(AS)_t + (AS)_{xx} = 0.
\]
Here the consistency condition, \( A_{xt} = A_{tx} \), is checked by a direct calculation by use of (3.1). We introduce an \( n_1 \times n_2 \) matrix \( Y(x, t) \) and an \( n_2 \times n_1 \) matrix \( Z(x, t) \) which satisfy a pair of matrix heat equations
\[
\begin{align*}
Y_t - Y_{xx} &= 0 \\
Z_t + Z_{xx} &= 0
\end{align*}
\]
(3.3)
and the boundary conditions
\[
\lim_{x \to x_0} Y = O \quad \lim_{x \to x_0} Z = O.
\]
The general solution of (3.3) is obtained by means of the Fourier transformation. Thus, if we set

\[ P = Y, \quad S = A^{-1}Z \]

this gives the general solution of (3.1). Due to the relation \( A_x = -ZY \), we obtain

\[ A = I_2 - \int_{x_0}^{x} Z(x', t) Y(x', t) dx'. \]

Here we have assumed the boundary condition, \( \lim_{x \to x_0} A = I_2 \), with \( I_2 \) being the \( n_2 \times n_2 \) identity matrix. In conclusion, an explicit form of the general solution of (3.1) is given by

\[ P(x, t) = Y(x, t) \quad S(x, t) = \{ I_2 - \int_{x_0}^{x} Z(x', t) Y(x', t) dx' \}^{-1} Z(x, t). \] (3.4)

Finally, we shall derive the general solution of the only system left to solve, (13) in [17]. For this purpose, we set \( n_1 = n_2 \) and perform a change of the dependent variables:

\[ P = AP, \quad S = SA^{-1}. \]

Noting the fact that (3.2) is rewritten in terms of the new variables \( P \) and \( S \) as

\[ A_x = A(-SP) \quad A_t = A(S_x P - SP_x - SPS^2P + 2S^2P^2) \]

we obtain the evolution equations for \( P \) and \( S \):

\[
\begin{align*}
P_t &= P_{xx} - 2S_xP^2 - 2SPP_x + 2SPSP^2 - 2S^3P^3 \\
S_t &= -S_{xx} - 2S^2P_x - 2S_xSP + 2SPS_x + 2SP_xS \\
&\quad + 2SPS^2P + 2S^3P^2 - 2S^2PSP - 2S^2P^2S.
\end{align*}
\] (3.5)

This is nothing but the system (13) in [17] up to scalings of variables. Thus, by virtue of the derivation in the above, the general solution of (3.5), which is an alternative expression of (13), is obtained straightforwardly:

\[
\begin{align*}
P(x, t) &= A^{-1}P = \{ I - \int_{x_0}^{x} Z(x', t) Y(x', t) dx' \}^{-1} Y(x, t) \\
S(x, t) &= SA = \{ I - \int_{x_0}^{x} Z(x', t) Y(x', t) dx' \}^{-1} Z(x, t) \{ I - \int_{x_0}^{x} Z(x', t) Y(x', t) dx' \}. \quad (3.6)
\end{align*}
\]

It should be noted that all of the matrices in the above expression are square matrices.

4. Concluding remarks

In this paper, we have studied matrix-valued systems of the derivative nonlinear Schrödinger (DNLS) type. Applying a kind of gauge transformation to a matrix version of the DNLS equation of the Chen-Lee-Liu type with a Lax pair [8], we have derived Lax representations for all but two systems proposed in [17]. Hence, these systems can be linearized through the inverse scattering formulation and proved to be ‘S-integrable’ in the terminology of Calogero. As has been clarified in section 2, these systems each are connected with the others through transformations of the dependent variables. However, it is noteworthy that the transformations cannot be written in a closed form in terms of
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the matrix-valued dependent variables. More explicitly, if $F$ (or $G^{-1}$) is not the identity, we may not express $F$ (or $G^{-1}$) for the examples (a)–(f) in section 2 without using the path ordering operator, infinitely multiple integrals, etc.

For the two systems in [17] left to prove their complete integrability, we have shown that both of them are linearizable and connected with each other by a change of dependent variables. The transformations which linearize the two systems can be explicitly written in a closed form in terms of the variables $Y$ and $Z$. This enables us to obtain the general solutions of the two systems, which directly proves their ‘C-integrability’.

To conclude, we have proved for the first time that all the matrix-valued systems proposed in [17] can be integrated by the inverse scattering method or the transformations of the dependent variables. The dependent variables of the systems take values in either square matrices or, more generally, rectangular matrices of arbitrary size. However, it is noteworthy that not all of integrable multi-component systems can be expressed in terms of matrix variables of arbitrary size. For instance, we consider the following Lax pair,

$$
U = i \xi^2 \begin{bmatrix} -I_1 & \cdots & 0 \\ 0 & \cdots & -I_2 \end{bmatrix} + \xi \begin{bmatrix} Q \\ R \end{bmatrix} + i \begin{bmatrix} \frac{1}{2}QR \\ \frac{1}{2}RQ \end{bmatrix} + \frac{1}{2}(QRx - R_xQ) + \frac{3}{4}QRQR
$$

$$
V = i \xi^4 \begin{bmatrix} -2I_1 & \cdots & 0 \\ 0 & \cdots & 2I_2 \end{bmatrix} + \xi^3 \begin{bmatrix} 2Q \\ 2R \end{bmatrix} + i \xi^2 \begin{bmatrix} -QR \\ RQ \end{bmatrix} + \frac{1}{2}(QxR - QRx) + \frac{3}{4}QRQR
$$

Substituting (4.1) and (4.2) into (2.2), we obtain the evolution equations for $Q$ and $R$,

$$
iQ_t + Q_{xx} - iQR_xQ - 2iQQR_x = 0
$$

$$
iR_t - R_{xx} - iRQ_xR - 2iR_xQR = 0
$$

and the commutation relation,

$$[QR, Q_xR - QR_x] = O.
$$

Setting

$$Q = (q_1, q_2, \cdots, q_m) = q$$

$$R = (r_1, r_2, \cdots, r_m) = r$$

which automatically satisfies the constraint (4.4), we obtain a system of the Kaup-Newell-type DNLS equations [8],

$$i\langle q_t + q_{xx} - i < q, r_x > q - 2i < q, r > q_x = 0
$$

$$i\langle r_t - r_{xx} - i < r, q_x > r - 2i < r, q > r_x = 0.
$$

Here $<,>$ denotes the inner product between vectors. The vector Kaup-Newell system (4.5) is ‘S-integrable’ because it possesses the Lax pair (4.1) and (4.2) with changing
$Q \rightarrow q, \ R \rightarrow r$. However, the matrix-valued extension of (1.3) with $\alpha = 1, \delta = -1/4,$ (4.3), is not ‘$S$-integrable’ in general for $Q$ and $R$ of arbitrary size.

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Appendix. Massive Thirring model

In this appendix, we show a list of matrix generalizations of the massive Thirring model, which are respectively a member of the DNLS-type hierarchies studied in section 2. For this purpose, we consider Lax pairs with the following dependence on the spectral parameter $\zeta$,

\begin{align}
U & = i \zeta^2 \left[ \begin{array}{cc} -I_1 \\
I_2 \end{array} \right] + \zeta \left[ \begin{array}{cc} Q \\
R \end{array} \right] + i \left[ \begin{array}{cc}
U_{11} \\
U_{22} \end{array} \right] \\
V & = i \frac{m^2}{4\zeta^2} \left[ \begin{array}{cc} -I_1 \\
I_2 \end{array} \right] + m \frac{2i}{2\zeta} \left[ \begin{array}{cc} P \\
S \end{array} \right] + i \left[ \begin{array}{cc}
V_{11} \\
V_{22} \end{array} \right].
\end{align}

(A.1)

(A.2)

Here $U_{11}, V_{11}$ and $U_{22}, V_{22}$ are $\zeta$-independent square matrices whose size are respectively $n_1 \times n_1$ and $n_2 \times n_2$. $Q$ and $P$ are $n_1 \times n_2$ matrices. $R$ and $S$ are $n_2 \times n_1$ matrices. $m$ is a nonzero constant. We have derived new Lax pairs for several matrix generalizations of the DNLS-type equation (1.3) in section 2. To obtain matrix versions of the massive Thirring model, we have only to change the time part of the Lax pairs as above. In this formulation, the new pair of potentials $P$ and $S$ appears. In correspondence with the choices of $U_{jj}$ ($j = 1, 2$), we can determine $V_{jj}$ ($j = 1, 2$) so that the compatibility condition (2.2) yields a consistent set of evolution equations.

We can obtain four matrix generalizations of the massive Thirring model by the above-mentioned method. The result of the choices of $U_{jj}, V_{jj}$ ($j = 1, 2$) and the corresponding evolution equations is listed as follows.

(a) \quad U_{11} = O \quad U_{22} = \frac{1}{2} RQ \quad V_{11} = -\frac{1}{2} PS \quad V_{22} = O

\begin{align}
Q_t - imP + \frac{1}{2} PSQ &= O \\
R_t + imS - \frac{1}{2} RPS &= O \\
\partial_t P &= -imQ + \frac{1}{2} PRQ = O \\
\partial_t S &= +imR - \frac{1}{2} RQS = O.
\end{align}

(A.3)

(b) \quad U_{11} = O \quad U_{22} = O \quad V_{11} = -\frac{1}{2} PS \quad V_{22} = \frac{1}{2} S P

\begin{align}
Q_t - imP + \frac{1}{2} PSQ &= O \\
R_t + imS - \frac{1}{2} RPS &= O \\
\partial_t P &= -imQ + \frac{1}{2} PRQ = O \\
\partial_t S &= +imR - \frac{1}{2} RQS = O.
\end{align}
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\[
Q_t - \text{im}\mathcal{P} + \frac{1}{2}(Q\mathcal{S}\mathcal{P} + \mathcal{P}\mathcal{S}Q) = O \\
R_t + \text{im}\mathcal{S} - \frac{1}{2}(\mathcal{S}\mathcal{P}R + R\mathcal{P}\mathcal{S}) = O \\
\mathcal{P}_x - \text{im}Q = O \\
\mathcal{S}_x + \text{im}R = O. 
\]

(A.4)

\[
U_{11} = O \quad U_{22} = \frac{1}{2}(RQ + QR) \quad V_{11} = -\frac{1}{2}\mathcal{P}\mathcal{S} \quad V_{22} = -\frac{1}{2}\mathcal{P}\mathcal{S} \\
Q_t - \text{im}\mathcal{P} + \frac{1}{2}(\mathcal{P}\mathcal{S}Q - Q\mathcal{P}\mathcal{S}) = O \\
R_t + \text{im}\mathcal{S} - \frac{1}{2}(R\mathcal{P}\mathcal{S} - \mathcal{P}SR) = O \\
\mathcal{P}_x - \text{im}Q + \frac{1}{2}\mathcal{P}(RQ + QR) = O \\
\mathcal{S}_x + \text{im}R - \frac{1}{2}(RQ + QR)\mathcal{S} = O. 
\]

(A.5)

\[
U_{11} = -\frac{1}{2}RQ \quad U_{22} = O \quad V_{11} = \frac{1}{2}(\mathcal{S}\mathcal{P} - \mathcal{P}\mathcal{S}) \quad V_{22} = \frac{1}{2}\mathcal{S}\mathcal{P} \\
Q_t - \text{im}\mathcal{P} + \frac{1}{2}(Q\mathcal{S}\mathcal{P} + \mathcal{P}\mathcal{S}Q - \mathcal{S}\mathcal{P}\mathcal{Q}) = O \\
R_t + \text{im}\mathcal{S} - \frac{1}{2}(\mathcal{S}\mathcal{P}R + R\mathcal{P}\mathcal{S} - R\mathcal{S}\mathcal{P}) = O \\
\mathcal{P}_x - \text{im}Q + \frac{1}{2}RQ\mathcal{P} = O \\
\mathcal{S}_x + \text{im}R - \frac{1}{2}\mathcal{S}RQ = O. 
\]

(A.6)

In view of the Lax matrix $U$ the cases (a)–(d) respectively correspond to (2.5), (a), (c) and (e) in section 2. Other choices of $U$ obtained in section 2 lead to systems which coincide with one of (a)–(d) up to the exchange of $t$ and $x$, etc. All of the systems (a)–(d) are shown to be connected with the others through the gauge transformations utilized in section 2.

The system (A.3) has been obtained by the authors in [25]. The others seem to be new integrable systems. For the existence of products of the matrices in the evolution equations, $Q$, $R$, $P$ and $S$ must be square matrices of the same size in (c) and (d).

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