Unified Theory of Fundamental Interactions

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Abstract

Based on local gauge invariance, four different kinds of fundamental interactions in Nature are unified in a theory which has $SU(3) \otimes SU(2)_L \otimes U(1) \otimes_s$ Gravitational Gauge Group gauge symmetry. In this approach, gravitational field, like electromagnetic field, intermediate gauge field and gluon field, is represented by gauge potential. Four kinds of fundamental interactions are formulated in the similar manner, and therefore can be unified in a direct or semi-direct product group. The model discussed in this paper can be regarded as extension of the standard model to gravitational interactions. The model discussed in this paper is a renormalizable quantum model, so it can be used to study quantum effects of gravitational interactions.

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1 Introduction

It is known that there are four kinds of fundamental interactions in Nature, which are strong interactions, electromagnetic interactions, weak interactions and gravitational interactions. All these fundamental interactions can be described by gauge field theories, which can be regarded as the common nature of all these fundamental interactions. And we can unify different kinds of fundamental interactions in the framework of gauge theory. The first unification of fundamental interactions in human history is the unification of electric interactions and magnetic interactions, which is made by Maxwell in 1864. Now, we know that electromagnetic theory is a $U(1)$ abelian gauge theory. In 1921, H.Weyl tried to unify electromagnetic interactions and gravitational interactions in a theory which has local scale invariance[1, 2]. Weyl’s original attempt is not successful, however in his great attempt, he introduced one of the most important concept in modern physics: gauge transformation and gauge symmetry. After the foundation of quantum mechanics, V.Fock, H.Weyl and W.Pauli found that quantum electrodynamics is a $U(1)$ gauge invariant theory[3, 4, 5].

In 1954, Yang and Mills proposed non-abelian gauge field theory[6]. Soon after, non-abelian gauge field theory is applied to elementary particle theory. In about 1967 and 1968, using the spontaneously symmetry breaking and Higgs mechanism[7, 8, 9, 10, 11, 12, 13, 14], S.Weinberg and A.Salam proposed the unified electroweak theory, which is a $SU(2) \times U(1)$ gauge theory[15, 16, 17]. The unified electroweak theory is now widely called the standard model. The predictions of unified electroweak theory have been confirmed in a large number of experiments, and the intermediate gauge bosons $W^\pm$ and $Z^0$ which are predicted by unified electroweak model are also found in experiments. However, in the traditional standard model, the gravitational interactions are not considered. From nineteen seventies, physicist begin studying Grand Unified theories which try to unify strong, electromagnetic and weak interactions in a simple group. At that time, $SU(5)$ model[18, 19], $SO(10)$ model[20, 21, 22], $E_6$ model[23, 24, 25] and other models[26, 27, 28] are proposed. In all these attempts, gravitational interactions are not considred.

Gauge treatment of gravity was suggested immediately after the gauge theory birth itself[29, 30, 31]. In the traditional gauge treatment of gravity, Lorentz group is localized, and the gravitational field is not represented by gauge potential[32, 33, 34]. It is represented by metric field. The theory has beautiful mathematical forms, but up to now, its renormalizability is not proved. In other words, it is conventionally considered to be non-renormalizable. Recently, Wu proposed a new quantum gauge theory of gravity which is a renormalizable quantum gravity[35, 36]. Based on gauge principle, space-time translation group is selected to be the symmetry of
gravitational interactions, which appears in a new form and is called gravitational
gauge group. After localization of gravitational gauge group, the gravitational field
appears as the corresponding gauge potential. After that work, the unified theory
of electromagnetic interactions and gravitational interactions is discussed[37]. Then
this unified theory is generalized to SU(N) non-abelian case and the the unification
of SU(N) non-abelian gauge interactions and gravitational interactions, which will
be called GSU(N) unification theory, is discussed[38]. GSU(N) unification theory
is also a renormalizable quantum model[39]. Now, using the method prop sed in
literatures [37] and [38], we will try to unify four different kinds of fundamental
interactions in Nature in one theory. This unification is based on the symmetry of
direct or semi-direct product group, which is the direct extension of the standard
model to gravitational interactions. This unification theory is also perturbatively
renormalizable.

2 Gravitational Gauge Field

First, for the sake of integrity, we give a simple introduction to gravitational gauge
theory and introduce some notations which is used in this paper. Details on quantum
gauge theory of gravity can be found in literatures [35] and [36].

In gauge theory of gravity, the most fundamental quantity is gravitational gauge
field $C_{\mu}(x)$, which is the gauge potential corresponding to gravitational gauge
symmetry. Gauge field $C_{\mu}(x)$ is a vector in the corresponding Lie algebra, which, for
the sake of convenience, will be called gravitational Lie algebra in this paper. So it
can expanded as

\begin{equation}
C_{\mu}(x) = C_{\mu}^\alpha(x) \hat{P}_\alpha,
\end{equation}

where $C_{\mu}^\alpha(x)$ is the component field and $\hat{P}_\alpha$ is the generator of gravitational gauge
group. The gravitational gauge covariant derivative is given by

\begin{equation}
D_{\mu} = \partial_{\mu} - igC_{\mu}(x),
\end{equation}

where $g$ is the gravitational coupling constant.

Matrix $G$ is an important quantity in gauge theory of gravity, whose definition is

\begin{equation}
G = (G_{\mu}^\alpha) = (\delta_{\mu}^\alpha - gC_{\mu}^\alpha).
\end{equation}

Its inverse matrix is denoted as $G^{-1}$,

\begin{equation}
G^{-1} = \frac{1}{1 - gC} = (G^{-1}_{\alpha}^\mu).
\end{equation}
They satisfy the following relations,

\[ G_{\mu}^{\alpha}G_{\alpha}^{-1\nu} = \delta_{\mu}^{\nu}, \]  
(2.5)

\[ G_{\beta}^{-1\mu}G_{\mu}^{\alpha} = \delta_{\beta}^{\alpha}. \]  
(2.6)

It can be proved that

\[ D_{\mu} = G_{\mu}^{\alpha}\partial_{\alpha}. \]  
(2.7)

The field strength of gravitational gauge field is defined by

\[ F_{\mu\nu} \triangleq \frac{1}{-ig}[D_{\mu}, D_{\nu}]. \]  
(2.8)

Its explicit expression is

\[ F_{\mu\nu}(x) = \partial_{\mu}C_{\nu}(x) - \partial_{\nu}C_{\mu}(x) - igC_{\mu}(x)C_{\nu}(x) + igC_{\nu}(x)C_{\mu}(x). \]  
(2.9)

\( F_{\mu\nu} \) is also a vector in gravitational Lie algebra,

\[ F_{\mu\nu}(x) = F_{\mu\nu}^{\alpha}(x) \cdot \hat{P}_{\alpha}, \]  
(2.10)

where

\[ F_{\mu\nu}^{\alpha} = \partial_{\nu}C_{\alpha}^{\mu} - \partial_{\mu}C_{\alpha}^{\nu} - gC_{\alpha}^{\beta}\partial_{\beta}C_{\mu}^{\nu} + gC_{\nu}^{\beta}\partial_{\beta}C_{\mu}^{\alpha}. \]  
(2.11)

In order to construct a gravitational gauge invariant lagrangian, \( J(C) \) is an important factor. In this paper, it will select to be

\[ J(C) = \sqrt{-\text{det}(g_{\alpha\beta})}, \]  
(2.12)

where

\[ g_{\alpha\beta} \equiv \eta_{\mu\nu}(G^{-1})^{\mu}_{\alpha}(G^{-1})^{\nu}_{\beta}. \]  
(2.13)

### 3 Lagrangian and Action

Now, we know that there are four kinds of fundamental interactions in Nature, which are strong interactions, electromagnetic interactions, weak interactions and gravitational interactions. In the traditional standard model, the first three fundamental
interactions are considered. In this chapter, we will generalize the standard model to gravitational interactions. In other words, the new standard model is a theory that can describe any fundamental physical process which human kind has already known in Nature.

We know that the fundamental particles that we know are fundamental fermions (such as leptons and quarks), gauge bosons (such as photon, gluons, gravitons and intermediate gauge bosons \( W^\pm \) and \( Z^0 \)), and possible Higgs bosons. Our goal is to construct a theory which can describe all possible fundamental interactions among all these elementary particles. We know that the symmetry of the traditional standard model is \( SU(3)_c \times SU(2)_L \times U(1)_Y \). \( SU(3)_c \) is the gauge symmetry for strong interactions, \( SU(2)_L \times U(1)_Y \) is the symmetry for electroweak interactions. The gauge symmetry for gravitational interactions is gravitational gauge group. If we generalize the standard model to gravitational interactions, the symmetry group is

\[
(SU(3)_c \times SU(2)_L \times U(1)_Y) \otimes_s \text{Gravitational Gauge Group. (3.1)}
\]

This is the symmetry of generalized standard model.

Before we construct the lagrangian of the system, let’s first define wave functions of various elementary particles. According to the Standard Model, leptons form left-hand doublets and right-hand singlets. Let’s denote

\[
\psi_L^{(1)} = \left( \begin{array}{c} \nu_e \\ e \end{array} \right)_L, \quad \psi_L^{(2)} = \left( \begin{array}{c} \nu_\mu \\ \mu \end{array} \right)_L, \quad \psi_L^{(3)} = \left( \begin{array}{c} \nu_\tau \\ \tau \end{array} \right)_L,
\]

\[
\psi_R^{(1)} = e_R, \quad \psi_R^{(2)} = \mu_R, \quad \psi_R^{(3)} = \tau_R. \quad (3.2)
\]

Neutrinos have no right-hand singlets. The weak hypercharge for left-hand doublets \( \psi_L^{(i)} \) is \(-1\) and for right-hand singlet \( \psi_R^{(i)} \) is \(-2\). All leptons are \( SU(3)_c \) singlet, so they carry no color charge. The charges of leptons are given by Gell-Mann-Nishijima rule,

\[
Q = T^L_3 + \frac{Y}{2}, \quad (3.4)
\]

where \( Y \) is the hypercharge and \( T^L_3 \) is the weak isospin. In order to define the wave function for quarks, we have to introduce Kabayashi-Maskawa mixing matrix first, whose general form is,

\[
K = \begin{pmatrix}
    c_1 & s_1c_3 & s_1s_3 \\
    -s_1c_2 & c_1c_2c_3 - s_2s_3e^{i\delta} & c_1c_2s_3 + s_2c_3e^{i\delta} \\
    s_1s_2 & -c_1s_2c_3 - c_2s_3e^{i\delta} & -c_1s_2s_3 + c_2c_3e^{i\delta}
\end{pmatrix} \quad (3.5)
\]

where

\[
c_i = \cos \theta_i, \quad s_i = \sin \theta_i \quad (i = 1, 2, 3) \quad (3.6)
\]
and $\theta_i$ are generalized Cabibbo angles. The mixing between three different quarks $d, s$ and $b$ is given by

$$
\begin{pmatrix}
  d \\
  s \\
  b
\end{pmatrix} = K
\begin{pmatrix}
  d \\
  s \\
  b
\end{pmatrix},
$$

(3.7)

Quarks also form left-hand doublets and right-hand singlets,

$$
q_L^{(1)a} = \begin{pmatrix} u^a_L \\ d^a_L \end{pmatrix}, \quad q_L^{(2)a} = \begin{pmatrix} c^a_L \\ s^a_L \end{pmatrix}, \quad q_L^{(3)a} = \begin{pmatrix} t^a_L \\ b^a_L \end{pmatrix},
$$

(3.8)

$$
q_u^{(1)a} = u^a_R, \quad q_s^{(2)a} = c^a_R, \quad q_t^{(3)a} = t^a_R,
$$

(3.9)

where index $a$ is color index. It is known that left-hand doublets have weak isospin $\frac{1}{2}$ and weak hypercharge $\frac{1}{3}$, right-hand singlets have no weak isospin, $q_u^{(j)a}$s have weak hypercharge $\frac{2}{3}$ and $q_s^{(j)a}$s have weak hypercharge $-\frac{2}{3}$. Charges of quarks are also given by Gell-Mann-Nishijima rule.

The symmetry of system is given by eq.(3.1). We can see that there are four different symmetry group. For every kinds of symmetry group, there are corresponding gauge fields which transmit corresponding gauge interactions. For gauge bosons, gravitational gauge field is also denoted by $C^\alpha_{\mu}$. The coupling constant of gravitational interactions is still denoted by $g$. The gluon field is denoted by $A_\mu$,

$$
A_\mu = A^i_\mu \lambda^i \frac{1}{2},
$$

(3.10)

where $\lambda^i$ are Gell-Mann matrix. Color index is denoted by indexes $a, b, c, \cdots$, so

$$
\lambda^i = (\lambda^i_{ab}).
$$

(3.11)

The field strength of gluon field is $A_{\mu\nu}$

$$
A_{\mu\nu} = (D_\mu A_\nu) - (D_\nu A_\mu) - ig_c[A_\mu, A_\nu] = A^i_{\mu\nu} \lambda^i \frac{1}{2},
$$

(3.12)

where $g_c$ is the coupling constant for strong interactions. $A^i_{\mu\nu}$ is the component field strength of gluon field, whose explicit expression is

$$
A^i_{\mu\nu} = (D_\mu A^i_\nu) - (D_\nu A^i_\mu) + g_c f_{ijk} A^j_\mu A^k_\nu,
$$

(3.13)

where $f_{ijk}$ is the structure constant of $SU(3)$ group. $A_{\mu\nu}$ is not $SU(3)$ gauge covariant field strength. $SU(3)$ gauge covariant field strength is defined by

$$
A_{\mu\nu} = A_{\mu\nu} + g G^{-1}_\sigma A_\lambda F^\mu\nu_{\sigma} = A^i_{\mu\nu} \lambda^i \frac{1}{2},
$$

(3.14)
where
\[ A^i_{\mu\nu} = A^i_{\mu\nu} + g G^{-1\lambda}_\sigma A^i_{\lambda\sigma} F^n_{\mu\nu}. \] (3.15)

The \( U(1)_Y \) gauge field is denoted by \( B_\mu \) and the coupling constant for \( U(1)_Y \) gauge interactions is \( g'_w \). The \( U(1)_Y \) gauge field strength tensor is \( B_{\mu\nu} \)
\[ B_{\mu\nu} = (D_\mu B_\nu) - (D_\nu B_\mu). \] (3.16)

The gauge covariant field strength tensor is
\[ B^\mu_{\mu\nu} = B^\mu_{\mu\nu} + g G^{-1\lambda}_\sigma B^\lambda_{\mu\nu} F^n_{\mu\nu}, \] (3.17)

The \( SU(2)_L \) gauge field is denoted by \( W_\mu \)
\[ W_\mu = W^n_{\mu} \frac{\sigma_n}{2}, \] (3.18)
where \( \sigma_n \) is the Pauli matrix. The \( SU(2)_L \) field strength tensor is \( W^n_{\mu\nu} \),
\[ W^m_{\mu\nu} = (D_\mu W_\nu) - (D_\nu W_\mu) - i_g_w [W_\mu , W_\nu] = W^m_{\mu\nu} \frac{\sigma_n}{2}, \] (3.19)
where \( g_w \) is the weak coupling constant for \( SU(2)_L \) gauge interactions. \( W^m_{\mu\nu} \) is component field strength tensor,
\[ W^m_{\mu\nu} = D_\mu W^m_{\nu} - D_\nu W^m_{\mu} + g_w \epsilon_{lmn} W^l_{\mu} W^m_{\nu}, \] (3.20)
where \( \epsilon_{lmn} \) is the structure constant of \( SU(2) \) group. The \( SU(2)_L \) gauge covariant field strength tensor is \( W^n_{\mu\nu} \),
\[ W^n_{\mu\nu} = W^m_{\mu\nu} + g G^{-1\lambda}_\sigma W^\lambda_{\mu\nu} F^n_{\mu\nu} = W^n_{\mu\nu} \frac{\sigma_n}{2}, \] (3.21)
where
\[ W^n_{\mu\nu} = W^m_{\mu\nu} + g G^{-1\lambda}_\sigma W^m_{\nu} F^n_{\mu\nu}. \] (3.22)

If there exist Higgs particle in Nature, the Higgs field is represented by a complex scalar \( SU(2) \) doublet,
\[ \phi = \left( \begin{array}{c} \phi^+ \\ \phi^0 \end{array} \right). \] (3.23)
The hypercharge of Higgs field \( \phi \) is 1. Higgs field carries no color charge.
The Lagrangian $\mathcal{L}_0$ that describes four kinds of fundamental interactions is given by

$$\mathcal{L}_0 = -\sum_{j=1}^{3} \overline{\psi}_L^{(j)} \gamma^\mu (D_\mu + ig'_w B_\mu - ig_w W_\mu) \psi_L^{(j)} - \sum_{j=1}^{3} \overline{\epsilon}_R^{(j)} \gamma^\mu (D_\mu + ig'_w B_\mu) \epsilon_R^{(j)} - \sum_{j=1}^{3} \overline{\psi}^{(j)} L \gamma^\mu \left( (D_\mu - i\frac{2}{3}g'_w B_\mu) \delta_{ab} - ig_c A^k_\mu (\frac{\lambda^k}{2})_{ab} \right) q_L^{(j)\beta} - \sum_{j=1}^{3} \overline{\epsilon}_u^{(j)} \gamma^\mu \left( (D_\mu - i\frac{1}{3}g'_w B_\mu) \delta_{ab} - ig_c A^k_\mu (\frac{\lambda^k}{2})_{ab} \right) q_u^{(j)\beta} - \sum_{j=1}^{3} \overline{\psi}_{\theta d}^{(j)} \gamma^\mu \left( (D_\mu + i\frac{1}{3}g'_w B_\mu) \delta_{ab} - ig_c A^k_\mu (\frac{\lambda^k}{2})_{ab} \right) q_{\theta d}^{(j)\beta} - \frac{1}{4} \eta^{\mu\nu} \eta^{\rho\sigma} W^\mu_{\mu\nu} W^n_{\rho\sigma} - \frac{1}{4} \eta^{\mu\nu} \eta^{\rho\sigma} B^\mu_{\mu\nu} B^\rho_{\rho\sigma} - \frac{1}{4} \eta^{\mu\nu} \eta^{\rho\sigma} A^i_{\mu\nu} A^i_{\rho\sigma} - \frac{1}{4} \eta^{\mu\nu} \eta^{\rho\sigma} g_{\alpha\beta} F^\alpha_{\mu\nu} F^\beta_{\rho\sigma} - \eta^{\mu\nu} \left[ (D_\mu - i\frac{2}{3}g'_w B_\mu - ig_w W_\mu) \phi \right]^\dagger \cdot \left[ (D_\nu - i\frac{2}{3}g'_w B_\nu - ig_w W_\nu) \phi \right] - \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 - \sum_{j=1}^{3} f^{(j)} \left( \overline{\psi}_R^{(j)} \phi^\dagger L \psi_L^{(j)} + \overline{\psi}_L^{(j)} \phi^\dagger R \right) - \sum_{j=1}^{3} \left( f_u^{(j)} \overline{q}_L^{(j)\alpha} \phi \overline{q}_u^{(j)\alpha} + f_u^{(j)} \overline{q}_u^{(j)\alpha} \phi \overline{q}_L^{(j)\alpha} \right) - \sum_{j,k=1}^{3} \left( f_d^{(j)k} \overline{q}_L^{(j)\alpha} \phi \overline{q}_{\theta d}^{(j)\alpha} + f_d^{(j)k} \overline{q}_{\theta d}^{(j)\alpha} \phi \overline{q}_L^{(j)\alpha} \right),$$

where

$$\overline{\phi} = i\sigma_2 \phi^\dagger = \begin{pmatrix} \phi^{0\dagger} \\ -\phi \end{pmatrix}.$$  \hspace{1cm} (3.25)

The full Lagrangian is given by

$$\mathcal{L} = J(C) \mathcal{L}_0,$$ \hspace{1cm} (3.26)

and the action of the system is

$$S = \int d^4x \mathcal{L}.$$ \hspace{1cm} (3.27)
4 Symmetry of the Model

Now, let’s discuss symmetry of the system. First, let’s study $SU(3)_c$ symmetry. Denote the $SU(3)_c$ transformation matrix as $U_3$. Under $SU(3)_c$ transformation, transformations of various fields and operators are:

\begin{align*}
\psi_L^{(j)} \rightarrow \psi'_L^{(j)} &= \psi_L^{(j)}, \\
e_R^{(j)} \rightarrow e'_R^{(j)} &= e_R^{(j)}, \\
q_L^{(j)a} \rightarrow q'_L^{(j)a} &= U_{3ab} q_L^{(j)b}, \\
q_u^{(j)a} \rightarrow q'_u^{(j)a} &= U_{3ab} q_u^{(j)b}, \\
q_{\theta d}^{(j)a} \rightarrow q'_{\theta d}^{(j)a} &= U_{3ab} q_{\theta d}^{(j)b}, \\
C_\mu \rightarrow C'_\mu &= C_\mu \\
D_\mu \rightarrow D'_\mu &= D_\mu,
\end{align*}

\begin{align*}
A_\mu &\rightarrow A'_\mu = U_3 A_\mu U_3^{-1} - \frac{1}{ig_c} U_3 (D_\mu U_3^{-1}), \\
B_\mu &\rightarrow B'_\mu = B_\mu, \\
W_\mu &\rightarrow W'_\mu = W_\mu, \\
\phi &\rightarrow \phi' = \phi, \\
J(C) &\rightarrow J'(C') = J(C).
\end{align*}

According to above transformation rules, gauge field strength tensors transform as

\begin{align*}
W_{\mu\nu} &\rightarrow W'_{\mu\nu} = W_{\mu\nu}, \\
B_{\mu\nu} &\rightarrow B'_{\mu\nu} = B_{\mu\nu}, \\
F^{\sigma}_{\mu\nu} &\rightarrow F'^{\sigma}_{\mu\nu} = F^{\sigma}_{\mu\nu},
\end{align*}

\begin{align*}
A_{\mu\nu} &\rightarrow A'_{\mu\nu} = U_3 A_{\mu\nu} U_3^{-1} - \frac{ig}{g_c} F^{\sigma}_{\mu\nu} U_3 (\partial_\sigma U_3^{-1}), \\
A_{\mu\nu} &\rightarrow A'_{\mu\nu} = U_3 A_{\mu\nu} U_3^{-1}.
\end{align*}

Using all these transformation rules, we can prove that the lagrangian density $L$ is invariant. Therefore, the system has strict local $SU(3)_c$ symmetry.

Denote the transformations matrix of $SU(2)_L$ gauge transformation as $U_2$. Under $SU(2)_L$ transformation, transformations of various fields are:

\begin{align*}
\psi_L^{(j)} &\rightarrow \psi'_L^{(j)} = U_2 \psi_L^{(j)},
\end{align*}
\[
\begin{align*}
\epsilon_R^{(j)} & \rightarrow \epsilon_R'^{(j)} = \epsilon_R^{(j)}, \\
q_L^{(j)a} & \rightarrow q_L'^{(j)a} = U_2 q_L^{(j)a}, \\
q_u^{(j)a} & \rightarrow q_u'^{(j)a} = q_u^{(j)a}, \\
q_{\theta d}^{(j)a} & \rightarrow q_{\theta d}'^{(j)a} = q_{\theta d}^{(j)a}, \\
C_\mu^\alpha & \rightarrow C_{\mu}'^{\alpha} = C_\mu^\alpha, \\
D_\mu & \rightarrow D_{\mu}' = D_\mu, \\
A_\mu & \rightarrow A_{\mu}' = A_\mu, \\
B_\mu & \rightarrow B_{\mu}' = B_\mu, \\
W_\mu & \rightarrow W_\mu' = U_2 W_\mu U_2^{-1} - \frac{1}{ig_W} U_2 (D_\mu U_2^{-1}), \\
\phi & \rightarrow \phi' = U_2 \phi, \\
J(C) & \rightarrow J'(C') = J(C).
\end{align*}
\]

According to above transformation rules, gauge field strength tensors transform as
\[
\begin{align*}
A_{\mu\nu} & \rightarrow A_{\mu\nu}' = A_{\mu\nu}, \\
B_{\mu\nu} & \rightarrow B_{\mu\nu}' = B_{\mu\nu}, \\
F_{\mu\nu}^\sigma & \rightarrow F_{\mu\nu}'^\sigma = F_{\mu\nu}^\sigma, \\
W_{\mu\nu} & \rightarrow W_{\mu\nu}' = U_2 W_{\mu\nu} U_2^{-1} - \frac{ig}{g_W} F_{\mu\nu}' U_2 (\partial_\mu U_2^{-1}), \\
W_{\mu\nu} & \rightarrow W_{\mu\nu}' = U_2 W_{\mu\nu} U_2^{-1}.
\end{align*}
\]

We can prove that the action of the system is invariant under the above \(SU(2)_L\) gauge transformations.

Under \(U(1)_Y\) gauge transformation, transformations of various fields are:
\[
\begin{align*}
\psi_L^{(j)} & \rightarrow \psi_L'^{(j)} = e^{i\alpha(x)/2} \psi_L^{(j)}, \\
\epsilon_R^{(j)} & \rightarrow \epsilon_R'^{(j)} = e^{i\alpha(x)} \epsilon_R^{(j)}, \\
q_L^{(j)a} & \rightarrow q_L'^{(j)a} = e^{-i\alpha(x)/6} q_L^{(j)a}, \\
q_u^{(j)a} & \rightarrow q_u'^{(j)a} = e^{-2i\alpha(x)/3} q_u^{(j)a}, \\
q_{\theta d}^{(j)a} & \rightarrow q_{\theta d}'^{(j)a} = e^{i\alpha(x)/3} q_{\theta d}^{(j)a}, \\
C_\mu^\alpha & \rightarrow C_{\mu}'^{\alpha} = C_\mu^\alpha.
\end{align*}
\]
$$D_\mu \rightarrow D'_\mu = D_\mu, \quad (4.41)$$
$$A_\mu \rightarrow A'_\mu = A_\mu, \quad (4.42)$$
$$B_\mu \rightarrow B'_\mu = B_\mu - \frac{1}{g_\mu W} (D_\mu \alpha (x)), \quad (4.43)$$
$$W_\mu \rightarrow W'_\mu = W_\mu, \quad (4.44)$$
$$\phi \rightarrow \phi' = e^{-i\alpha(x)/2} \phi, \quad (4.45)$$
$$\bar{\phi} \rightarrow \bar{\phi}' = e^{i\alpha(x)/2} \bar{\phi}, \quad (4.46)$$
$$J(C) \rightarrow J'(C') = J(C). \quad (4.47)$$

According to above transformation rules, gauge field strength tensors transform as

$$A_{\mu\nu} \rightarrow A'_{\mu\nu} = A_{\mu\nu}, \quad (4.48)$$
$$W_{\mu\nu} \rightarrow W'_{\mu\nu} = W_{\mu\nu}, \quad (4.49)$$
$$F^\sigma_{\mu\nu} \rightarrow F'^\sigma_{\mu\nu} = F^\sigma_{\mu\nu}, \quad (4.50)$$
$$B_{\mu\nu} \rightarrow B'_{\mu\nu} = B_{\mu\nu} + \frac{g}{g_\mu W} F^\sigma_{\mu\nu} (\partial_\sigma \alpha (x)), \quad (4.51)$$
$$B_{\mu\nu} \rightarrow B'_{\mu\nu} = B_{\mu\nu} \quad (4.52)$$

We can also prove that the action of the system is invariant under the above $U(1)_Y$ gauge transformations.

Gravitational gauge transformations of various fields are

$$\psi^{(j)}_L \rightarrow \psi'^{(j)}_L = (\hat{U}_\epsilon \psi^{(j)}_L), \quad (4.53)$$
$$e^{(j)}_R \rightarrow e'^{(j)}_R = (\hat{U}_\epsilon e^{(j)}_R), \quad (4.54)$$
$$q^{(j)a}_L \rightarrow q'^{(j)a}_L = (\hat{U}_\epsilon q^{(j)a}_L), \quad (4.55)$$
$$q^{(j)a}_u \rightarrow q'^{(j)a}_u = (\hat{U}_\epsilon q^{(j)a}_u), \quad (4.56)$$
$$q^{(j)a}_{\theta d} \rightarrow q'^{(j)a}_{\theta d} = (\hat{U}_\epsilon q^{(j)a}_{\theta d}), \quad (4.57)$$

$$C_\mu \rightarrow C'_\mu = \hat{U}_\epsilon C_\mu \hat{U}^{-1}_\epsilon - \frac{1}{ig} \hat{U}_\epsilon (\partial_\mu \hat{U}^{-1}_\epsilon), \quad (4.58)$$
$$D_\mu \rightarrow D'_\mu = \hat{U}_\epsilon D_\mu \hat{U}^{-1}_\epsilon, \quad (4.59)$$
$$A_\mu \rightarrow A'_\mu = (\hat{U}_\epsilon A_\mu), \quad (4.60)$$
$$B_\mu \rightarrow B'_\mu = (\hat{U}_\epsilon B_\mu), \quad (4.61)$$
According to the above transformation rules, gauge field strength tensors transform as

\[ W_\mu \rightarrow W'_\mu = (\hat{U}_e W_\mu), \]  

\[ \phi \rightarrow \phi' = (\hat{U}_e \phi), \]  

\[ \bar{\phi} \rightarrow \bar{\phi}' = (\hat{U}_e \bar{\phi}), \]  

\[ J(C) \rightarrow J'(C') = J \cdot (\hat{U}_e J(C)), \]  

\[ \eta_{1\alpha}^\mu \rightarrow \eta'_{1\alpha}^\mu = \Lambda_\alpha^{\alpha_1} (\hat{U}_e \eta_{1\alpha_1}^\mu), \]  

\[ \eta_{2\alpha\beta} \rightarrow \eta'_{2\alpha\beta} = \Lambda_\alpha^{\alpha_1} \Lambda_\beta^{\beta_1} (\hat{U}_e \eta_{2\alpha_1\beta_1}). \]  

According to these transformations, the lagrangian density \( L_0 \) transforms covariantly,

\[ \hat{L}_0 \rightarrow \hat{L}'_0 = (\hat{U}_e L_0). \]  

So,

\[ L \rightarrow L' = J \cdot (\hat{U}_e L). \]  

Using eq.(2.54), we can prove that action \( S \) is invariant under gravitational gauge transformations,

\[ S \rightarrow S' = S. \]  

Therefore, the system has gravitational gauge symmetry.

Now, as a whole, we discuss

\[ (SU(3)_c \times SU(2)_L \times U(1)_Y) \otimes_s \text{Gravitational Gauge Group} \]
gauge symmetry. In order to do this, we need define generator operators. The generator operators of \( SU(3)_c \) group are denoted by \( \hat{T}_3^j \). The \( SU(3)_c \) transformation operator \( \hat{U}_3 \) is defined by

\[ \hat{U}_3 = e^{-i\alpha/2 \hat{T}_3^j}. \]  

Matrix \( U_3 \) is defined by

\[ U_3 = e^{-i\alpha/2 \lambda_j}. \]  

When \( \hat{T}_3^j \) acts on fields, it will becomes the corresponding representation matrix of generators. So,

\[ \hat{T}_3^j \psi_L^{(j)} = 0, \]
\[
\hat{T}_{3i} e_R^{(j)} = 0, \\
\hat{T}_{3i} q_L^{(j)a} = \left( \frac{\lambda^i}{2} \right)_{ab} q_{L}^{(j)b}, \\
\hat{T}_{3i} q_u^{(j)a} = \left( \frac{\lambda^i}{2} \right)_{ab} q_{u}^{(j)b}, \\
\hat{T}_{3i} q_{\theta d}^{(j)a} = \left( \frac{\lambda^i}{2} \right)_{ab} q_{\theta d}^{(j)b}, \\
\hat{T}_{3i} \phi = 0, \\
[\hat{T}_{3i}, A_\mu] = 0, \\
[\hat{T}_{3i}, B_\mu] = 0, \\
[\hat{T}_{3i}, W_\mu] = 0, \\
[\hat{T}_{3i}, F_{\mu\nu}] = 0, \\
[\hat{T}_{3i}, A_{\mu\nu}] = \left[ \frac{\lambda^i}{2}, A_{\mu\nu} \right], \\
[\hat{T}_{3i}, \mathbb{B}_{\mu\nu}] = 0, \\
[\hat{T}_{3i}, \mathbb{W}_{\mu\nu}] = 0.
\]

The generator operators of \( SU(2)_L \) group are denoted by \( \hat{T}_{2i} \). The \( SU(2)_L \) transformation operator \( \hat{U}_2 \) is defined by
\[
\hat{U}_2 = e^{-i \alpha^i \hat{T}_{2i}}.
\]

Matrix \( U_2 \) is defined by
\[
U_2 = e^{-i \alpha^i \sigma_i / 2}.
\]

When \( \hat{T}_{2i} \) acts on fields, it will becomes the corresponding representation matrix of generators. So,
\[
\hat{T}_{2i} \psi_L^{(j)} = \frac{\sigma^i}{2} \psi_L^{(j)} \\
\hat{T}_{2i} e_R^{(j)} = 0, \\
\hat{T}_{2i} q_L^{(j)a} = \frac{\sigma^i}{2} q_L^{(j)a}, \\
\hat{T}_{2i} q_u^{(j)a} = 0, \\
\hat{T}_{2i} q_{\theta d}^{(j)a} = 0.
\]
\[
\hat{T}_{2l}q_{bd}^{(j)a} = 0, 
\]
\[
\hat{T}_{2l}\phi = \frac{\sigma_1}{2}\phi 
\]
\[
[\hat{T}_{2l}, C_\mu^a] = 0, 
\]
\[
[\hat{T}_{2l}, A_\mu] = 0, 
\]
\[
[\hat{T}_{2l}, B_\mu] = 0, 
\]
\[
[\hat{T}_{2l}, W_\mu] = \left[ \frac{\sigma_1}{2}, W_\mu \right], 
\]
\[
[\hat{T}_{2l}, F_\sigma^\mu] = 0, 
\]
\[
[\hat{T}_{2l}, A_\mu^\sigma] = 0, 
\]
\[
[\hat{T}_{2l}, B_\mu^\sigma] = 0, 
\]
\[
[\hat{T}_{2l}, W_{\mu^\sigma}] = \left[ \frac{\sigma_1}{2}, W_{\mu^\sigma} \right]. 
\]

The generator operators of \(U(1)_Y\) group are denoted by \(\hat{T}_1\). \(2\hat{T}_1\) is the hypercharge operator. The \(U(1)_Y\) transformation operator \(\hat{U}_1\) is defined by
\[
\hat{U}_1 = e^{-i\hat{T}_1}. 
\]

Matrix \(U_1\) is defined by
\[
U_1 = e^{-i\alpha}. 
\]

When \(2\hat{T}_1\) acts on fields, it will becomes the hypercharge of the corresponding fields. So,
\[
\hat{T}_1\psi_L^{(j)} = -\frac{1}{2}\psi_L^{(j)} 
\]
\[
\hat{T}_1e_R^{(j)} = -e_R^{(j)} 
\]
\[
\hat{T}_1q_{L}^{(j)a} = \frac{1}{6}q_{L}^{(j)a}, 
\]
\[
\hat{T}_1q_{u}^{(j)a} = \frac{2}{3}q_{u}^{(j)a}, 
\]
\[
\hat{T}_1q_{bd}^{(j)a} = -\frac{1}{3}q_{bd}^{(j)a} 
\]
\[
\hat{T}_1\phi = \frac{1}{2}\phi 
\]
\[
[\hat{T}_1, C_\mu^a] = 0, 
\]
\[
[\hat{T}_1, A_\mu] = 0, 
\]
\[ [\hat{T}_1, B_\mu] = B_\mu, \quad (4.117) \]
\[ [\hat{T}_1, W_\mu] = 0, \quad (4.118) \]
\[ [\hat{T}_1, F^\sigma_{\mu\nu}] = 0, \quad (4.119) \]
\[ [\hat{T}_1, A_{\mu\nu}] = 0, \quad (4.120) \]
\[ [\hat{T}_1, B_{\mu\nu}] = B_{\mu\nu}, \quad (4.121) \]
\[ [\hat{T}_1, \mathbb{W}_{\mu\nu}] = 0. \quad (4.122) \]

Different generator operators act on different spaces, so they commute each other,
\[ [\hat{T}_1, \hat{T}_{2l}] = 0, \quad (4.123) \]
\[ [\hat{T}_1, \hat{T}_{3i}] = 0, \quad (4.124) \]
\[ [\hat{T}_1, \hat{P}_\alpha] = 0, \quad (4.125) \]
\[ [\hat{T}_{2l}, \hat{T}_{3i}] = 0, \quad (4.126) \]
\[ [\hat{T}_{2l}, \hat{P}_\alpha] = 0, \quad (4.127) \]
\[ [\hat{T}_{3i}, \hat{P}_\alpha] = 0. \quad (4.128) \]

As we have mention before, what generators commute each other does not means that group elements commute each other.

A general element of semi-direct product group
\[ (SU(3)_c \times SU(2)_L \times U(1)_Y) \otimes_s \text{Gravitational Gauge Group} \]
is denoted by \( g(x) \). It can be proved that the \( g(x) \) can be written into the following form
\[ g(x) = \hat{U}_1 \hat{U}_2 \hat{U}_3. \quad (4.129) \]

Define quark color triplet states,
\[ q_L^{(j)} = \begin{pmatrix} q_L^{(j)1} \\ q_L^{(j)2} \\ q_L^{(j)3} \end{pmatrix}, \quad (4.130) \]
\[ q_u^{(j)} = \begin{pmatrix} q_u^{(j)1} \\ q_u^{(j)2} \\ q_u^{(j)3} \end{pmatrix}, \quad (4.131) \]
\[ q^{(j)}_{\bar{a}d} = \begin{pmatrix} q^{(j)1}_{\bar{d}d} \\ q^{(j)2}_{\bar{d}d} \\ q^{(j)3}_{\bar{d}d} \\ q^{(j)4}_{\bar{d}d} \end{pmatrix}. \]  

(4.132)

Then, we have the following relations

\[ \hat{T}_{3j}q^{(j)}_{L} = \frac{\lambda_j}{2} q^{(j)}_{L}, \]

(4.133)

\[ \hat{T}_{3j}q^{(j)}_{u} = \frac{\lambda_j}{2} q^{(j)}_{u}, \]

(4.134)

\[ \hat{T}_{3j}q^{(j)}_{\bar{a}d} = \frac{\lambda_j}{2} q^{(j)}_{\bar{a}d}. \]

(4.135)

Under gauge transformations of semi-direct product group, various fields and operators transform as

\[ \psi^{(j)}_{L} \rightarrow \psi'^{(j)}_{L} = (g(x) \psi^{(j)}_{L}), \]

(4.136)

\[ e^{(j)}_{R} \rightarrow e'^{(j)}_{R} = (g(x) e^{(j)}_{R}), \]

(4.137)

\[ q^{(j)\alpha}_{L} \rightarrow q'^{(j)\alpha}_{L} = (g(x) q^{(j)\alpha}_{L}), \]

(4.138)

\[ q^{(j)\alpha}_{u} \rightarrow q'^{(j)\alpha}_{u} = (g(x) q^{(j)\alpha}_{u}), \]

(4.139)

\[ q^{(j)\alpha}_{\bar{a}d} \rightarrow q'^{(j)\alpha}_{\bar{a}d} = (g(x) q^{(j)\alpha}_{\bar{a}d}), \]

(4.140)

\[ C_{\mu} \rightarrow C'_{\mu} = \hat{U}_{\epsilon} C_{\mu} \hat{U}_{\epsilon}^{-1} - \frac{1}{i g} \hat{U}_{\epsilon} (\partial_{\mu} \hat{U}_{\epsilon}^{-1}), \]

(4.141)

\[ D_{\mu} \rightarrow D'_{\mu} = \hat{U}_{\epsilon} D_{\mu} \hat{U}_{\epsilon}^{-1}, \]

(4.142)

\[ F'_{\mu\nu} = \hat{U}_{\epsilon} F_{\mu\nu} \hat{U}_{\epsilon}^{-1}, \]

(4.143)

\[ A_{\mu} \rightarrow A'_{\mu} = g(x) \left[ A_{\mu} - \frac{1}{i g} (D_{\mu} U^{-1}_3) U_3 \right] g^{-1}(x), \]

(4.144)

\[ W_{\mu} \rightarrow W'_{\mu} = g(x) \left[ W_{\mu} - \frac{1}{i g W} (D_{\mu} U^{-1}_2) U_2 \right] g^{-1}(x), \]

(4.145)

\[ B_{\mu} \rightarrow B'_{\mu} = g(x) \left[ B_{\mu} - \frac{1}{i g W} (D_{\mu} U^{-1}_1) U_1 \right] g^{-1}(x), \]

(4.146)

\[ \phi \rightarrow \phi' = (g(x) \phi), \]

(4.147)

\[ J(C) \rightarrow J' (C') = J \cdot (\hat{U}_{\epsilon} J(C)), \]

(4.148)

\[ \eta^{\mu}_{1\alpha} \rightarrow \eta'^{\mu}_{1\alpha} = \Lambda^{\alpha \alpha_{1}} g(x) \eta^{\mu}_{1\alpha} g^{-1}(x), \]

(4.149)

\[ \eta_{2\alpha\beta} \rightarrow \eta'^{2}_{2\alpha\beta} = \Lambda^{\alpha \alpha_{1}} \Lambda^{\beta \beta_{1}} g(x) \eta_{2\alpha\beta} g^{-1}(x), \]

(4.150)
\[ A_{\mu\nu} \rightarrow A'_{\mu\nu} = g(x)A_{\mu\nu}g^{-1}(x), \quad (4.151) \]
\[ \mathbb{W}_{\mu\nu} \rightarrow \mathbb{W}'_{\mu\nu} = g(x)\mathbb{W}_{\mu\nu}g^{-1}(x), \quad (4.152) \]
\[ B_{\mu\nu} \rightarrow B'_{\mu\nu} = g(x)B_{\mu\nu}g^{-1}(x). \quad (4.153) \]

It can be proved that the action of the system has strict local gauge symmetry of semi-direct product group.

## 5 Spontaneously Symmetry Breaking

It is known that, SU\((3)\)\(_c\) color symmetry and gravitational gauge symmetry are strict symmetry. SU\((2)\)_L \(\times\) U\((1)\)_Y symmetry are not strict symmetry, which is broken to U\((1)\)_Q symmetry. Now, let’s discuss spontaneously symmetry breaking of the system. The potential of Higgs field is

\[ -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2. \quad (5.1) \]

If,

\[ \mu^2 > 0, \quad \lambda > 0, \quad (5.2) \]

the symmetry of vacuum will be spontaneously broken. Suppose that the vacuum expectation value of neutral Higgs field is non-zero, that is

\[ \langle \phi \rangle_0 = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}, \quad (5.3) \]

where,

\[ v = \sqrt{\frac{\mu^2}{\lambda}}. \quad (5.4) \]

After a local SU\((2)\)_L gauge transformation, we can select the Higgs field \(\phi(x)\) as,

\[ \langle \phi \rangle_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + \varphi(x) \end{pmatrix}. \quad (5.5) \]

After symmetry breaking, the Higgs potential becomes,

\[ V(\varphi) = \mu^2\varphi^2 + \lambda v\varphi^3 + \frac{\lambda}{4}\varphi^4 - \frac{\mu^4}{4\lambda}, \quad (5.6) \]

from which we know that the mass of Higgs field is \(2\mu^2\).
Define
\[ W^\pm \equiv \frac{1}{\sqrt{2}} (W^1_\mu \mp iW^2_\mu) \quad (5.7) \]
\[ A^e_\mu \equiv \cos \theta_W B_\mu + \sin \theta_W W^3_\mu, \quad (5.8) \]
\[ Z_\mu \equiv \sin \theta_W B_\mu - \cos \theta_W W^3_\mu, \quad (5.9) \]

where
\[ \tan \theta_W = \frac{g'_W}{g_W}. \quad (5.10) \]

\( A^e_\mu \) in eq. (5.8) is the electromagnetic field. Define two mass parameters,
\[ m_W = \frac{1}{2} g_W v, \quad (5.11) \]
\[ m_Z = \frac{1}{2} \sqrt{g'^2_W + g^2_W} v. \quad (5.12) \]

It is known that, in the standard model, \( m_W \) is the mass of \( W^\pm \) bosons and \( m_Z \) is the mass of \( Z \) bosons. The coupling constant of electromagnetic interactions is denoted by \( e \),
\[ e \equiv g'_W \cos \theta_W = \frac{g_W g'_W}{\sqrt{g'^2_W + g^2_W}}. \quad (5.13) \]

The current of strong interactions are denoted by \( J^\mu_{ci} \),
\[ J^\mu_{ci} = i \left( \bar{u} \gamma^\mu \frac{\lambda^i}{2} u + \bar{c} \gamma^\mu \frac{\lambda^i}{2} c + \bar{t} \gamma^\mu \frac{\lambda^i}{2} t + \bar{d} \gamma^\mu \frac{\lambda^i}{2} d + \bar{s} \gamma^\mu \frac{\lambda^i}{2} s + \bar{b} \gamma^\mu \frac{\lambda^i}{2} b \right). \quad (5.14) \]

The current of electromagnetic interactions is
\[ J^\mu_{em} = i \left( \bar{e} \gamma^\mu e - \bar{\mu} \gamma^\mu \mu - \bar{\tau} \gamma^\mu \tau + \frac{2}{3} \bar{\nu} \gamma^\mu \nu + \frac{2}{3} \bar{\chi} \gamma^\mu \chi \right. \]
\[ + \frac{2}{3} \bar{d} \gamma^\mu t - \frac{1}{2} \bar{d} \gamma^\mu d - \frac{1}{3} \bar{s} \gamma^\mu s - \frac{1}{3} \bar{b} \gamma^\mu b \right). \quad (5.15) \]

The currents for weak interactions are
\[ J^\mu_{W} = \frac{i}{2\sqrt{2}} \left[ \bar{e} \gamma^\mu (1 + \gamma_5) e + \bar{\mu} \gamma^\mu (1 + \gamma_5) \mu + \bar{\tau} \gamma^\mu (1 + \gamma_5) \tau \right. \]
\[ + \bar{\nu} \gamma^\mu (1 + \gamma_5) \nu + \bar{\mu} \gamma^\mu (1 + \gamma_5) s + \bar{\tau} \gamma^\mu (1 + \gamma_5) b \theta \right], \quad (5.16) \]
\[ J^\mu_{W} = \frac{i}{2\sqrt{2}} \left[ \bar{e} \gamma^\mu (1 + \gamma_5) \nu + \bar{\mu} \gamma^\mu (1 + \gamma_5) \nu + \bar{\tau} \gamma^\mu (1 + \gamma_5) \nu \right. \]
\[ + \bar{\nu} \gamma^\mu (1 + \gamma_5) \nu + s \gamma^\mu (1 + \gamma_5) c + \bar{b} \gamma^\mu (1 + \gamma_5) t \right), \quad (5.17) \]
\[ J^\mu_z = J^\mu_3 - \sin^2 \theta_W J^\mu_{em}, \quad (5.18) \]
where
\[ J_3^\mu = \frac{i}{2} (+\bar{\nu}_\mu \gamma^\mu \nu_{\bar{e}L} + \bar{\nu}_{\mu L} \gamma^\mu \rho_{\nu L} + \bar{\nu}_{\tau L} \gamma^\mu \nu_{\tau L} - \bar{c}_L \gamma^\mu e_L - \bar{\mu}_L \gamma^\mu \mu_L - \bar{\tau}_L \gamma^\mu \tau_L \\
+ \bar{u}_L \gamma^\mu u_L + \bar{c}_L \gamma^\mu c_L + \bar{\ell}_L \gamma^\mu t_L - \bar{d}_L \gamma^\mu d_L - \bar{s}_L \gamma^\mu s_L - \bar{b}_L \gamma^\mu b_L). \] (5.19)

The current for gravitational interactions is
\[ J_{\mu g} = \bar{\nu}_\mu \gamma^\mu \rho_{\nu L} - \bar{\nu}_{\mu L} \gamma^\mu \rho_{\nu L} - \bar{\tau}_L \gamma^\mu \tau_L - \bar{\rho}_L \gamma^\mu \rho_L + \bar{\nu}_L \gamma^\mu \nu_L + \bar{\mu}_L \gamma^\mu \mu_L + \bar{\tau}_L \gamma^\mu \tau_L + \bar{u}_L \gamma^\mu u_L + \bar{c}_L \gamma^\mu c_L \\
+ \bar{\ell}_L \gamma^\mu t_L + \bar{d}_L \gamma^\mu d_L + \bar{s}_L \gamma^\mu s_L + \bar{b}_L \gamma^\mu b_L. \] (5.20)

Denote
\[ J_e = \frac{1}{v}(m_e \bar{e}e + m_\mu \bar{\mu} \mu + m_\tau \bar{\tau} \tau + m_u \bar{u} u + m_c \bar{c} c + m_t \bar{t} t + m_d \bar{d} d + m_s \bar{s} s + m_b \bar{b} b). \] (5.21)

Then lagrangian density \( \mathcal{L}_0 \) can be written into the following form,
\[ \mathcal{L}_0 = -\bar{\nu}_\mu \gamma^\mu \rho_{\nu L} - \bar{\nu}_{\mu L} \gamma^\mu \rho_{\nu L} - \bar{\tau}_L \gamma^\mu \tau_L - \bar{\rho}_L \gamma^\mu \rho_L + \bar{\nu}_L \gamma^\mu \nu_L + \bar{\mu}_L \gamma^\mu \mu_L + \bar{\tau}_L \gamma^\mu \tau_L + \bar{u}_L \gamma^\mu u_L + \bar{c}_L \gamma^\mu c_L \\
+ \bar{\ell}_L \gamma^\mu t_L + \bar{d}_L \gamma^\mu d_L + \bar{s}_L \gamma^\mu s_L + \bar{b}_L \gamma^\mu b_L. \] (5.22)

According to above discussions, before spontaneously symmetry breaking, the system has
\[ (SU(3)_c \times SU(2)_L \times U(1)_Y) \otimes \text{Gravitational Gauge Group} \]
gauge symmetry. After spontaneously symmetry breaking, the system has

\[(SU(3)_c \times U(1)_Q) \otimes_s Gravitational\ Gauge\ Group\]

gauge symmetry. Four different kinds of fundamental interactions are unified in the same lagrangian. So, the lagrangian density \(\mathcal{L}\) which is given by eq.(3.24) and eq.(3.26) can be used to calculate any fundamental interaction process in Nature.

6 Summary and Discussions

In this paper, we have studied unifications of four kinds of fundamental interactions in Nature, which is unified in the direct or semi-direct product group. This model is the direct extension of the standard model to gravitational interactions.

It is know that the real spirit of the traditional unification is to try to unify different kinds of fundamental interactions in a single simple Lie-group, where only one coupling constant is used in the unification. But now, we find that it is impossible to unify gravitational interactions with other fundamental interactions in a theory which has only one independent parameter for coupling constant. The reason is that, the generators of gravitational gauge group have mass dimension, while the generators of ordinary \(SU(N)\) group are dimensionless. If we use only one coupling constant in a unified theory, we will need another independent parameters to balance the difference of the dimensions of generators. It means that we really use two independent parameters for coupling constant, which is equivalent to the theory with two independent coupling constants. Therefore, any unification theory of gravity must need at least two independent parameters for coupling constant. In other words, it is impossible to unify four kinds of fundamental interactions in a simple group in which only one independent parameter for coupling constant is used.

Quantization of the unified theory of fundamental interactions can be performed in the path integral method. In order to keep the unitarity of the S-matrix, ghost fields for gravitational gauge symmetry, \(SU(2)_L\) symmetry and \(SU(3)_c\) color symmetry are needed[40]. We can obtain Feynmann rules for various interaction vertices and propagaters for matter fields, gauge fields and ghost fields. After obtain Feynmann rules, we can calculate Feynmann dagrams for various interaction processes and determine finite quantum modificaitons in the classical theory of gravity.

It is known that quantum gauge theory of gravity is a perturbatively renormalizable quantum theory. Detailed proof on its renormalizability can be found in literature [35, 36]. The foundations of the unified theory of fundamental interactions
which is discussed in this paper is the $GSU(N)$ unification theory which is discussed in literature [37, 38]. Because the theory has strict $GSU(N)$ symmetry, it is believed that the theory is also renormalizable. Detailed study shows that the strict local $GSU(N)$ symmetry gives out two sets of generalized BRST transformations which will give out two sets of generalized Ward-Takahashi identities. Using these two sets of generalized Ward-Takahashi identities, we can determine the general form of the divergent part of the generating functional of regular vertex. After the divergent part of the generating functional is cancelled by counterterm, the renormalized generating functional is just a scale transformation of the non-renormalized generating functional. Comparing the renormalized theory with the non-renormalized theory, we will find that the only effect of the renormalization is to redefine the fields, coupling constants and some other parameters of the original theory. Only a few new parameters are introduced in the renormalization, so the $GSU(N)$ unification theory is indeed a renormalizable quantum theory[39]. Detailed proof on its renormalizability is quite complicated mathematically, which will be discussed in a separate paper. Unified theory of fundamental interactions is just an application of the $GSU(N)$ unification theory to fundamental interactions in Nature. Because the $GSU(N)$ unification theory is renormalizable and from the standard model, we know that the spontaneously symmetry breaking does not affect the renormalizability of the theory, the unified theory of fundamental interactions which is discussed in this paper is a renormalizable quantum theory. Detailed and strict proof on its renormalizability will be found in our future work[40]. This is the first renormalizable quantum model which contains four different kinds of fundamental interactions in Nature. In the future, we will use this model to study quantum effects of gravitational interactions and new effects which are originated from the non-Abelian nature of $GSU(N)$ group and lead to direct coupling between gravitational gauge field and ordinary gauge fields(such as electromagnetic field, intermediate gauge field and gluon field).

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