On a certain class of locally conformal symplectic structures of the second kind✩

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\textbf{A B S T R A C T}

We study locally conformal symplectic (LCS) structures of the second kind on a Lie algebra. We show a method to construct new examples of Lie algebras admitting LCS structures of the second kind starting with a lower dimensional Lie algebra endowed with a LCS structure and a suitable extension. Moreover, we characterize all LCS Lie algebras obtained with our construction. Finally, we study the existence of lattices in the associated simply connected Lie groups in order to obtain compact examples of manifolds admitting this kind of structure.

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1. Introduction

A \textit{locally conformal symplectic} structure (LCS for short) on the manifold $M$ is a non degenerate 2-form $\omega$ such that there exist an open cover $\{U_i\}$ and smooth functions $f_i$ on $U_i$ such that $\omega_i = \exp(-f_i)\omega$ is a symplectic form on $U_i$. This condition is equivalent to requiring that

\[ d\omega = \theta \wedge \omega \]  

(1)

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for some closed 1-form $\theta$, called the Lee form. The pair $(\omega, \theta)$ will be called a LCS structure on $M$. It is well known that if $(\omega, \theta)$ is a LCS structure on $M$, then $\omega$ is symplectic if and only if $\theta = 0$. Furthermore, $\theta$ is uniquely determined by equation (1), but there is not an explicit formula for the Lee form. If $\omega$ is a non-degenerate 2-form on $M$, with $\dim M \geq 6$, such that (1) holds for some 1-form $\theta$ then $\theta$ is automatically closed and therefore $M$ is LCS.

According to Vaisman (see [19]) there are two different types of LCS structures. If $(\omega, \theta)$ is a LCS structure on $M$, a vector field $X$ is called an infinitesimal automorphism of $(\omega, \theta)$ if $L_X \omega = 0$, where $L$ denotes the Lie derivative. This implies $L_X \theta = 0$ as well and, as a consequence, $\theta(X)$ is a constant function on $M$. We consider $\mathfrak{X}_\omega(M) = \{X \in \mathfrak{X}(M) : L_X \omega = 0\}$ which is a subalgebra of $\mathfrak{X}_\omega(M)$, then the map $\theta|_{\mathfrak{X}_\omega(M)} : \mathfrak{X}_\omega(M) \to \mathbb{R}$ is a well defined Lie algebra morphism called the Lee morphism. If there exists an infinitesimal automorphism $X$ such that $\theta(X) \neq 0$, the LCS structure $(\omega, \theta)$ is said to be of the first kind, and it is of the second kind otherwise. This condition is equivalent to the Lee morphism being either surjective or identically zero. In the literature, there is more information about LCS structures of the first kind, for example, in [19] Vaisman gives relations with contact geometry and he also proves that a manifold with a LCS structure of the first kind admits distinguished foliations. On the other hand, LCS structures of the second kind are less understood.

There is another way to distinguish LCS structures, to do this, we can deform the de Rham differential $d$ to obtain the adapted differential operator

$$d_\theta \alpha = d\alpha - \theta \wedge \alpha,$$

for any differential form $\alpha \in \Omega^*(M)$. Since $\theta$ is $d$-closed, this operator satisfies $d^2_\theta = 0$, thus it defines the Morse-Novikov cohomology $H^*_\theta(M)$ of $M$ relative to the closed 1-form $\theta$ (see [14,15]). Note that if $\theta$ is exact then $H^*_\theta(M) \simeq H^*_\theta(M)$. It is known that if $M$ is a compact oriented $n$-dimensional manifold, then $H^0_\theta(M) = H^n_\theta(M) = 0$ for any non exact closed 1-form $\theta$ (see for instance [7,8]). For any LCS structure $(\omega, \theta)$ on $M$, the 2-form $\omega$ defines a cohomology class $[\omega]_\theta \in H^2_\theta(M)$, since $d_\theta \omega = d\omega - \theta \wedge \omega = 0$. The LCS structure $(\omega, \theta)$ is said to be exact if $\omega$ is $d_\theta$-exact or $[\omega]_\theta = 0$, i.e., $\omega = d\eta - \theta \wedge \eta$ for some 1-form $\eta$, and it is non-exact if $[\omega]_\theta \neq 0$. It was proved in [19] that if the LCS structure $(\omega, \theta)$ is of the first kind on $M$ then $\omega$ is $d_\theta$-exact, i.e., $[\omega]_\theta = 0$. But the converse is not true. Recently in [3] other cohomologies for LCS manifolds were introduced, inspired by the almost Hermitian setting. More precisely, the authors define the LCS-Bott-Chern cohomology and the LCS-Aeppli cohomology on any compact LCS manifold, and compute them for some LCS solvmanifolds in low dimensions.

In the last years, special attention has been devoted to the study of left invariant LCS structures on Lie groups (see for instance [1–4]), with very nice results in the case of LCS structures of the first kind. In this work we focus on LCS structures of the second kind on Lie algebras and solvmanifolds.

We recall that a LCS structure $(\omega, \theta)$ on a Lie group $G$ is called left invariant if $\omega$ is left invariant, and this easily implies that $\theta$ is also left invariant. Accordingly, we say that a Lie algebra $\mathfrak{g}$ admits a locally conformal symplectic (LCS) structure if there exist $\omega \in \bigwedge^2 \mathfrak{g}^*$ and $\theta \in \mathfrak{g}^*$, with $\omega$ non-degenerate and $\theta$ closed, such that (1) is satisfied.

As in the case of manifolds we have that a LCS structure $(\omega, \theta)$ on a Lie algebra $\mathfrak{g}$ can be of the first kind or of the second kind. Indeed, let us denote by $\mathfrak{g}_\omega$ the set of infinitesimal automorphisms of the LCS structure, that is,

$$\mathfrak{g}_\omega = \{X \in \mathfrak{g} : L_X \omega = 0\} = \{X \in \mathfrak{g} : \omega([X,Y],Z) + \omega(Y,[X,Z]) = 0 \text{ for all } Y,Z \in \mathfrak{g}\}$$

where $L$ denotes the Lie derivative. Note that $\mathfrak{g}_\omega \subset \mathfrak{g}$ is a Lie subalgebra, thus the restriction of $\theta$ to $\mathfrak{g}_\omega$ is a Lie algebra morphism called the Lee morphism. The LCS structure $(\omega, \theta)$ is said to be of the first kind if the Lee morphism is surjective, and of the second kind if it is identically zero.
For a Lie algebra $\mathfrak{g}$ and a closed 1-form $\theta \in \mathfrak{g}^*$ we also have the Morse-Novikov cohomology $H^*_\theta(\mathfrak{g})$ defined by the differential operator $d_\theta$ on $\Lambda^* \mathfrak{g}^*$ defined by

$$d_\theta \alpha = d\alpha - \theta \wedge \alpha.$$ 

As in manifolds, we have that a LCS structure $(\omega, \theta)$ on a Lie algebra is said to be exact if $[\omega]_\theta = 0$ or non-exact if $[\omega]_\theta \neq 0$.

It was proved in [4] that:

**Proposition 1.1.** If the Lie algebra $\mathfrak{g}$ is unimodular, a LCS structure on $\mathfrak{g}$ is of the first kind if and only if it is exact.

LCS structures of the first kind on Lie algebras are better understood because they are related with other important geometric structures (see for instance [19] or more recently [4]). On the other hand not much is known about Lie algebras with a LCS structure of the second kind. In [2] the authors study three different types of constructions of LCS Lie algebras. One corresponds to exact LCS Lie algebras. The second one establishes a link between cosymplectic Lie algebras in dimension $2n - 1$ and non-exact LCS structure in dimension $2n$. The third one is related to the existence of Lagrangian ideals (see [2] for more details). In [1] we study LCS structures on almost abelian Lie groups, and we exhibit examples of solvmanifolds with LCS structure of the second kind in any dimension greater than or equal to 6. We do not know many other explicit examples of solvmanifolds with a LCS structure of the second kind. Therefore, we consider that it would be very interesting to find new examples of Lie algebras and solvmanifolds admitting LCS structures of the second kind, hopefully they might be used to understand better this kind of structures.

In this work we deal with this problem and inspired by ideas of [2] we provide another construction of LCS Lie algebras of the second kind which is different from those given in [2]. After we provide this construction in Theorem 3.3 two questions arise naturally:

**Question 1:** Given a LCS structure of the second kind on a Lie algebra, can it be obtained from our construction?

**Question 2:** Do there exist examples of LCS Lie algebras constructed by Theorem 3.3 such that the associated simply connected Lie group admits lattices?

Concerning question 1, we give a nice characterization of Lie algebras constructed with our method. Moreover, we recover most of the known examples in dimension 4. We also answer affirmatively question 2. Indeed, we exhibit lattices for some associated simply connected Lie groups obtaining explicit examples of solvmanifolds admitting LCS structures of the second kind.

The outline of this article is as follows. In Section 2 we prove that a left invariant LCS structure of the second kind on a Lie group induces a LCS structure of the second kind on any compact quotient by a discrete subgroup (see Theorem 2.1). This allows us to study this geometric structure at the Lie algebra level. In Section 3 we give a method to construct new examples of Lie algebras admitting LCS structures of the second kind starting with a Lie algebra endowed with a LCS structure and a compatible extension (see Theorem 3.3). We also have a converse (see Theorem 3.5) obtaining a nice characterization of LCS Lie algebras admitting a non degenerate abelian ideal contained in the kernel of the Lee form. Moreover, we focus on unimodular Lie algebras constructed with our method because we are also interested in compact examples of solvmanifolds admitting this kind of structures. In Section 4 we show the wide range of our construction by reobtaining most of the known examples of LCS Lie algebras on the second kind in dimension 4. We also exhibit examples of Lie algebras in higher dimension starting with a 4-dimensional LCS Lie algebra. Moreover we give a complete list of 4-dimensional LCS Lie algebras which can be used to produce examples.
in higher dimension. Finally, in Section 5 we study the existence of lattices in the associated Lie groups and we give an explicit construction of a family of solvmanifolds admitting a LCS structure of the second kind proving that they are pairwise non homeomorphic.

2. Solvmanifolds with LCS structure of the second kind

Let us consider LCS structures on Lie algebras, or equivalently left invariant LCS structures on Lie groups. If the Lie group is simply connected then any left invariant LCS structure turns out to be globally conformal to a symplectic structure, which is essentially equivalent to having a symplectic structure on the Lie group. Therefore we will study compact quotients of such a Lie group by lattices. Recall that a discrete subgroup $\Gamma$ of a simply connected Lie group $G$ is called a lattice if the quotient $\Gamma \backslash G$ is compact. In this case we have that $\pi_1(\Gamma \backslash G) \cong \Gamma$. The quotient $\Gamma \backslash G$ is called a solvmanifold if $G$ is solvable and it is called a nilmanifold if $G$ is nilpotent.

It is clear that a left invariant LCS structure on a Lie group $G$ induces a LCS structure on any quotient $\Gamma \backslash G$, which will be non simply connected and therefore the inherited LCS structure is “strict”.

It is important to mention that if a Lie group admits a lattice then such Lie group must be unimodular, according to [13]. Besides this necessary condition, there is no general criterion to determine whether a given unimodular solvable Lie group admits a lattice. It is a very difficult problem in itself. However, there is such a criterion for nilpotent Lie groups. Indeed, Malcev proved in [12] that a nilpotent Lie group admits a lattice if and only if its Lie algebra has a rational form, that is, there exists a basis of the Lie algebra such that the corresponding structure constants are all rational. More recently, Bock studied in [5] the existence of lattices in simply connected solvable Lie groups up to dimension 6 and he gave a criterion for the existence of lattices in almost abelian Lie groups.

Concerning LCS structures on a solvmanifold $\Gamma \backslash G$ which arise from a LCS structure on $\mathfrak{g} = \text{Lie}(G)$, it is easy to see that if the LCS structure on $\mathfrak{g}$ is of the first kind, then the induced LCS structure on the quotient $\Gamma \backslash G$ is of the first kind as well. We will prove in the next result that the same happens for LCS structures of the second kind, i.e., a LCS structure of the second kind on $\mathfrak{g}$ induces a LCS structure of the second kind on any compact quotient $\Gamma \backslash G$.

**Theorem 2.1.** Let $\Gamma \backslash G$ be a solvmanifold and $\mathfrak{g} = \text{Lie}(G)$. If $(\omega, \theta)$ is a LCS structure on $\mathfrak{g}$ of the second kind, then the LCS structure induced on the solvmanifold $\Gamma \backslash G$ is of the second kind.

**Proof.** Let $(\omega, \theta)$ be a LCS structure of the second kind on $\mathfrak{g}$. Since $\mathfrak{g}$ is unimodular, it follows from Proposition 1.1 that $(\omega, \theta)$ is not exact, i.e., $0 \neq [\omega] \in H^2_2(\mathfrak{g})$. Let $(\hat{\omega}, \hat{\theta})$ be the induced LCS structure on $\Gamma \backslash G$. According to [11] there exists an injective map $i : H^2_2(\mathfrak{g}) \to H^2_2(\Gamma \backslash G)$. This map arises from the natural inclusion of $\mathfrak{g}$ into $\mathfrak{x}(\Gamma \backslash G)$. Therefore $0 \neq [i(\omega)] = [\hat{\omega}] \in H^2_2(\Gamma \backslash G)$. Then $\hat{\omega}$ is non exact in $\Gamma \backslash G$ and it follows from [19] that the LCS structure $(\hat{\omega}, \hat{\theta})$ in the solvmanifold $\Gamma \backslash G$ is of the second kind. \(\square\)

**Remark 2.2.** Note that the LCS structure induced by Theorem 2.1 is indeed a non-exact LCS structure on $\Gamma \backslash G$.

**Remark 2.3.** Note that in general a LCS structure of the second kind on a Lie algebra induces a LCS structure on the associated simply connected Lie group which is not necessarily of the second kind.

3. A method to construct LCS Lie algebras of the second kind

In this section we give a method to construct new examples of Lie algebras admitting a LCS structures of the second kind starting with a Lie algebra endowed with a LCS structure and a compatible extension (see
Theorem 3.3). Then we characterize all Lie algebras constructed with this method in Theorem 3.5, we also point out when the Lie algebras constructed in Theorem 3.3 are unimodular which is a necessary condition to study lattices in the last section.

Let $\mathfrak{h}$ be a Lie algebra, $(\omega, \theta)$ a LCS structure on $\mathfrak{h}$, and let $(u, \omega_0)$ be a symplectic Lie algebra of dimension $2n$. We consider a Lie algebra homomorphism

$$\pi : \mathfrak{h} \to \text{Der}(u).$$

Let $\mathfrak{g}$ be the Lie algebra given by $\mathfrak{g} = \mathfrak{h} \ltimes_{\pi} u$, equipped with the non degenerate 2-form $\tilde{\omega}$ given by $\tilde{\omega}|_{\mathfrak{h}} = \omega$ and $\tilde{\omega}|_{u} = \omega_0$. In particular $\tilde{\omega}(X, Y) = 0$ for any $X \in \mathfrak{h}$, $Y \in u$. We define the 1-form $\tilde{\theta} \in \mathfrak{g}^*$ by $\tilde{\theta}|_{\mathfrak{h}} = \theta$ and $\tilde{\theta}|_{u} = 0$.

We determine next when the pair $(\tilde{\omega}, \tilde{\theta})$ is a LCS structure on the Lie algebra $\mathfrak{g}$. Computing $d\tilde{\omega} = \tilde{\theta} \wedge \tilde{\omega}$ we can easily see that $(\tilde{\omega}, \tilde{\theta})$ is a LCS structure if and only if the following condition is satisfied:

$$-\omega_0(\pi(X)Y, Z) + \omega_0(\pi(X)Z, Y) = \theta(X)\omega_0(Y, Z),$$

for $X \in \mathfrak{h}$ and $Y, Z \in u$.

We denote by $S$ and $\rho$ the $\omega_0$-symmetric part and $\omega_0$-skew-symmetric part of $\pi$. More precisely, for each $X \in \mathfrak{h}$,

$$\pi(X) = S(X) + \rho(X),$$

where $S(X)$ is $\omega_0$-symmetric and $\rho(X)$ is $\omega_0$-skew-symmetric with respect to the non degenerate 2-form $\omega_0$, that is, $S(X)$ satisfies $\omega_0(S(X)Y, Z) = \omega_0(Y, S(X)Z)$ and $\rho(X)$ satisfies $\omega_0(\rho(X)Y, Z) = -\omega_0(Y, \rho(X)Z)$ for any $X \in \mathfrak{h}$ and $Y, Z \in u$. This condition on $\rho$ is equivalent to saying that $\rho(X) \in \mathfrak{sp}(u, \omega_0)$ for any $X \in \mathfrak{h}$. It is easy to verify that (3) holds if and only if $-2S(X) = \theta(X)\text{Id}$ for any $X \in \mathfrak{h}$.

**Definition 3.1.** With the notation above, if $\pi(X) = -\frac{1}{2}\theta(X)\text{Id} + \rho(X)$ and $\rho(X) \in \mathfrak{sp}(u, \omega_0)$ for all $X \in \mathfrak{h}$, then we say that $(\pi, u)$ is a LCS extension of $\mathfrak{h}$.

**Remark 3.2.** Note that any LCS Lie algebra $\mathfrak{h}$ admits a LCS extension. For example, taking $u = \mathbb{R}^{2n}$ and $\rho = 0$.

Therefore we have the main result, where we provide a method to construct LCS structures of the second kind. In fact, something more general is proved, they are non-exact LSC structures:

**Theorem 3.3.** Let $\mathfrak{h}$ be a Lie algebra with a LCS structure $(\omega, \theta)$, let $(u, \omega_0)$ be a $2n$-dimensional symplectic Lie algebra and $\pi : \mathfrak{h} \to \text{Der}(u)$ a Lie algebra homomorphism. Let $\mathfrak{g} = \mathfrak{h} \ltimes_{\pi} u$ and $(\tilde{\omega}, \tilde{\theta})$ given by $\tilde{\omega}|_{\mathfrak{h}} = \omega$, $\tilde{\omega}|_{u} = \omega_0$, $\tilde{\theta}|_{\mathfrak{h}} = \theta$ and $\tilde{\theta}|_{u} = 0$. We have that

(i) $(\tilde{\omega}, \tilde{\theta})$ is a LCS structure on $\mathfrak{g}$ if and only if $(\pi, u)$ is a LCS extension of $\mathfrak{h}$. Moreover, $\rho : \mathfrak{h} \to \mathfrak{sp}(u, \omega_0)$ is a Lie algebra homomorphism.

(ii) if $u$ is unimodular, then $(\tilde{\omega}, \tilde{\theta})$ is a LCS structure of the second kind. Moreover, it is non-exact LCS structure.

**Proof.** As we mentioned above, $(\tilde{\omega}, \tilde{\theta})$ is a LCS structure on $\mathfrak{g}$ if and only if for any $X \in \mathfrak{h}$ we have $\pi(X) = -\frac{1}{2}\theta(X)\text{Id} + \rho(X)$ with $\rho(X) \in \mathfrak{sp}(u, \omega_0)$. We have to check next that $\rho(X) : \mathfrak{h} \to \mathfrak{sp}(u, \omega_0)$ is a Lie algebra homomorphism. We compute
\[
\pi([X,Y]) = [\pi(X), \pi(Y)] \\
= [S(X) + \rho(X), S(Y) + \rho(Y)] \\
= [\rho(X), \rho(Y)],
\]

since \( S(X) = -\frac{1}{2}\theta(X) \text{Id} \). On the other hand we have that
\[
\pi([X,Y]) = S([X,Y]) + \rho([X,Y]) \\
= -\frac{1}{2}\theta([X,Y]) + \rho([X,Y]) \\
= \rho([X,Y]),
\]
since \( \theta \) is closed and the first part of the result follows.

For (ii), it is enough to prove that the LCS structure is non-exact. If we assume that the LCS structure \((\tilde{\omega}, \tilde{\theta})\) is exact, that is, there exists a 1-form \( \eta \in g^* \) such that \( \tilde{\omega} = d\eta - \tilde{\theta} \wedge \eta \). Then \( \omega_0 = \tilde{\omega}|_{u \times u} = d\eta|_{u \times u} \) since \( \tilde{\theta}|_u = 0 \). Therefore, \( \omega_0 \) is an exact symplectic form on \( u \), which cannot be possible since \( u \) is unimodular (see [6]). Therefore, the LCS structure is non-exact. \( \square \)

Theorem 3.3 provides us with a method to construct new examples of Lie algebras equipped with a non-exact LCS structure (in particular LCS structures of the second kind), starting with a LCS Lie algebra and a suitable extension. Note that the LCS structure on the initial Lie algebra can be of the first or of the second kind. We believe that this method is interesting because, as we mentioned before, there are not many general results about LCS structures of the second kind.

**Remark 3.4.** This method can also be used to construct Lie algebras admitting other kind of structures. For example, if we start with a symplectic Lie algebra \((\mathfrak{h}, \omega)\), that is \( \theta = 0 \), then it is clear that we obtain a new symplectic Lie algebra \((\mathfrak{g}, \tilde{\omega})\).

The LCS Lie algebra \((\mathfrak{g}, \tilde{\omega}, \tilde{\theta})\) constructed in Theorem 3.3 has an ideal which is non degenerated with respect to the restriction of the fundamental form \( \tilde{\omega} \), namely, \((u, \omega_0)\). Moreover, it is contained in \( \ker \tilde{\theta} \). Then, \( u \) is a symplectic ideal of \((\mathfrak{g}, \tilde{\omega})\). We show next a sort of converse of Theorem 3.3. More precisely, we prove that any LCS Lie algebra with a symplectic ideal can be constructed as in Theorem 3.3.

Let \( \mathfrak{g} \) be a Lie algebra endowed with a LCS structure \((\omega', \theta')\). Let \( u \) be a non degenerate ideal, and we consider the complement \( u^\perp \) given by \( u^\perp = \{ X \in \mathfrak{g} : \omega'(X, U) = 0, \forall U \in u \} \). Since \( u \) is non degenerate we have that \( \mathfrak{g} = u^\perp \oplus u \) as vector spaces.

**Theorem 3.5.** Let \((\mathfrak{g}, \omega', \theta')\) be a LCS Lie algebra admitting a non degenerate ideal \( u \). If \( u \subset \ker \theta' \), then \( u^\perp \) is a subalgebra. Moreover, \((\operatorname{ad} u) \) with \( \operatorname{ad} : u^\perp \to \operatorname{Der}(u) \) is a LCS extension of \( u^\perp \) and \((\mathfrak{g}, \omega', \theta')\) is isomorphic to \((u^\perp \ltimes u, \tilde{\omega}, \tilde{\theta})\) with the LCS structure of Theorem 3.3 (i). Therefore, if \( u \) is unimodular the LCS structure \((\omega', \theta')\) is non-exact, hence of the second kind.

**Proof.** We show \( u^\perp \) is a subalgebra, that is \( \omega'([X,Y], U) = 0 \) for all \( U \in u \) and \( X, Y \in u^\perp \). We compute
\[
\omega'([X,Y], U) = -d\omega'(X,Y,U) - \omega'([Y,U],X) - \omega'(U,[X,Y]) \\
= -\theta'(X)\omega'(Y,U) - \theta'(Y)\omega'(U,X) - \theta'(U)\omega'(X,Y) \\
= 0,
\]
where we used the LCS condition and the fact that \( u \subset \ker \theta' \).
Let \((\omega, \theta)\) be the restriction of \((\omega', \theta')\) to \(u^\perp\). It is clear that \(\omega\) is non-degenerate on \(u^\perp\) and \(\theta \neq 0\) since \(u \subset \ker \theta'\) and \(\theta' \neq 0\). Therefore, \((\omega, \theta)\) satisfies the LCS condition on the subalgebra \(u^\perp\). On the other hand, the restriction of \(\omega'\) to the non-degenerate ideal \(u\), denoted by \(\omega_0\), is clearly a symplectic form. We can decompose \(g\) as a semidirect product \(g = u^\perp \ltimes u\) and the LCS structure \((\omega', \theta')\) is exactly the LCS structure \((\tilde{\omega}, \tilde{\theta})\) constructed in Theorem 3.3 with initial data \((u^\perp, \omega, \theta)\) and \((u, \omega_0)\). Therefore, it follows from Theorem 3.3 that \(\text{ad} : u^\perp \to \text{Der}(u)\) is a LCS extension of \(u^\perp\) and \((g, \omega', \theta')\) is isomorphic to \((u^\perp \ltimes u, \tilde{\omega}, \tilde{\theta})\).

Finally, it follows from Theorem 3.3 that \((\omega', \theta')\) is a non-exact LCS structure on \(g\). □

**Remark 3.6.** Using Theorem 3.5 it is easy now to determine whether a given Lie algebra endowed with a LCS structure can be constructed by Theorem 3.3. This gives us a nice characterization of LCS Lie algebras admitting a symplectic ideal contained in the kernel of the Lee form. One can see Theorem 3.5 as a kind of reduction of the LCS condition.

**Remark 3.7.** This construction has some similar ideas to the ones in [2, Proposition 1.17] since both are related with special ideals contained in the kernel of the Lee form. In our work we look for a symplectic ideal instead of a Lagrangian ideal.

As a conclusion, we should mention that Theorem 3.3 and Theorem 3.5 establish a correspondence between LCS Lie algebras \(g\) admitting a non degenerate ideal \(u\) contained in the kernel of the Lie form and pairs \((\mathfrak{h}, u)\) with a suitable LCS extension \(\pi\), where \(\mathfrak{h}\) is a LCS Lie algebra and \(u\) is a symplectic Lie algebra. Moreover, if \(u\) is unimodular the LCS structure on \(g\) is non-exact.

### 3.1. Unimodular LCS Lie algebras of the second kind

Since we are interested in finding examples of solvmanifolds equipped with a LCS structure of the second kind, we determine next when a Lie algebra \(g\) constructed in Theorem 3.3 is unimodular. Note that, according to Proposition 1.1, if \(g\) is unimodular then a LCS structure is of the second kind if and only if it is non-exact.

**Proposition 3.8.** Let \(\mathfrak{h}\) be a Lie algebra with a LCS structure \((\omega, \theta)\), \((\pi, u)\) a \(2n\)-dimensional LCS extension of \(\mathfrak{h}\) and \(g = \mathfrak{h} \ltimes_{\pi} u\) the Lie algebra with LCS structure \((\tilde{\omega}, \tilde{\theta})\) constructed as above. Then \(g\) is unimodular if and only if \(\text{tr}(\text{ad}^h_X) = n\theta(X)\) for any \(X \in \mathfrak{h}\) and \(u\) is unimodular.

**Proof.** Given \(X \in \mathfrak{h}\), the operator \(\text{ad}^g_X : g \to g\) can be written as

\[
\text{ad}^g_X = \begin{pmatrix} \text{ad}^h_X & 0 \\ \pi(X) & \pi(X) \end{pmatrix},
\]

for some bases of \(\mathfrak{h}\) and \(u\). In the same way, we have

\[
\text{ad}^u_X = \begin{pmatrix} 0 & \text{ad}^u_X \\ \pi(X) & 0 \end{pmatrix},
\]

for any \(X \in u\). Therefore, \(g\) is unimodular if and only if \(u\) is unimodular and \(\text{tr}(\pi(X)) = -\text{tr}(\text{ad}^h_X)\) for all \(X \in \mathfrak{h}\), and the latter happens if and only if

\[
\text{tr}(\text{ad}^h_X) = -\text{tr} \left( -\frac{1}{2} \theta(X) \text{Id} + \rho(X) \right) = n\theta(X),
\]

for any \(X \in \mathfrak{h}\), since \(\pi\) is a LCS extension. □
In particular we have the following corollary, which will be used later:

**Corollary 3.9.** Let $\mathfrak{h}$ be a Lie algebra with a LCS structure $(\omega, \theta)$. If there exists $n \in \mathbb{N}$ such that

$$\operatorname{tr}(\operatorname{ad}_X^\mathfrak{h}) = n\theta(X)$$

for all $X \in \mathfrak{h}$, then for any LCS extension $(\pi, \mathfrak{u})$ with $\mathfrak{u}$ unimodular and $\dim \mathfrak{u} = 2n$, the LCS Lie algebra $\mathfrak{g} = \mathfrak{h} \ltimes \pi \mathfrak{u}$ is unimodular.

Note that, in particular, one can consider $\mathfrak{u} = \mathbb{R}^{2n}$ with its canonical symplectic form. Therefore the method of Theorem 3.3 together with condition (4) allow us to construct unimodular Lie algebras admitting a LCS structure of the second kind starting from a non unimodular Lie algebra with a LCS structure and a suitable extension.

### 3.2. Center of LCS Lie algebras of the second kind

The center of a Lie algebra with a LCS structure was studied in [2], in particular the authors characterized the center of a nilpotent LCS Lie algebra and they proved that the dimension of the center is at most 2. Note that the nilpotency condition implies that the LCS structure is of the first kind. On the other hand, as a consequence of Theorem 3.3 it is easy to verify that there is no restriction for the dimension of the center of a Lie algebra admitting a LCS structure of the second kind, as we show in the following example.

**Example 3.10.** Consider the 4-dimensional Lie algebra $\mathfrak{rr}_{3, \lambda}$ with structure constants $(0, -12, -\lambda 13, 0)$. It means that we fix a coframe $\{e^1, e^2, e^3, e^4\}$ for $(\mathfrak{rr}_{3, \lambda})^*$ such that $de^1 = 0$, $de^2 = -e^1 \wedge e^2$, $de^3 = -\lambda e^1 \wedge e^3$ and $de^4 = 0$. According to [2] this Lie algebra admits a LCS structure given by $\omega = e^{12} + e^{34}$ with Lee form $\theta = -\lambda e^1$. Consider now the $(2n + 4)$-dimensional Lie algebra $\mathfrak{g} = \mathfrak{rr}_{3, \lambda} \ltimes \mathbb{R}^{2n}$ where $\pi(e_i) = 0$ for $i = 2, 3, 4$ and $\pi(e_1) \in M(2n, \mathbb{R})$ is given by

$$\pi(e_1) = \begin{pmatrix} \frac{1}{2} \text{Id}_{n \times n} & \lambda \text{Id}_{n \times n} \\ \lambda \text{Id}_{n \times n} & -\frac{1}{2} \text{Id}_{n \times n} \end{pmatrix} = \begin{pmatrix} \lambda \text{Id}_{n \times n} \\ 0_{n \times n} \end{pmatrix},$$

in a basis $\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ of $\mathbb{R}^{2n}$. More precisely, the Lie brackets on $\mathfrak{g}$ are:

$$[e_1, e_2] = e_2, \quad [e_1, e_3] = \lambda e_3, \quad [e_1, u_k] = \lambda u_k,$$

for $k = 1, \ldots, n$. Then $e_4 \in \mathfrak{z}(\mathfrak{g})$ and $v_i \in \mathfrak{z}(\mathfrak{g})$ for $i = 1, \ldots, n$, therefore the dimension of $\mathfrak{z}(\mathfrak{g})$ is $n + 1$. Moreover, it can be seen that $(\pi, \mathbb{R}^{2n})$ is a LCS extension of $\mathfrak{rr}_{3, \lambda}$, hence according to Theorem 3.3 it determines a LCS structure on the Lie algebra $\mathfrak{g}$ given by $\tilde{\omega} = e^{12} + e^{34} + \sum_{i=1}^n u^i \wedge v^i$ with Lee form $\tilde{\theta} = -\lambda e^1$.

### 4. Examples of Lie algebras with LCS structures of the second kind

In this section we show first that our construction is quite general. Indeed, we can reobtain with this construction most of the known examples of Lie algebras admitting a LCS structure of the second kind. More precisely, we see that every unimodular 4-dimensional Lie algebra admitting a LCS structure of the second kind has a non degenerate abelian ideal contained in the kernel of the Lee form, and then by Theorem 3.5 they can be obtained with our construction. We also use Theorem 3.3 to construct new examples of unimodular Lie algebras admitting LCS structures of the second kind in dimension higher than 4.
### 4.1. Dimension 4

We start by recalling in Table 1 the unimodular 4-dimensional Lie algebras admitting a LCS structure of the second kind (see [2]).

Table 1: Unimodular 4-dimensional Lie algebras admitting LCS structure of second kind.

| Lie algebra | Structure equations | LCS structure of the second kind |
|-------------|---------------------|--------------------------------|
| \( \mathfrak{r}_3, -1 \) | \((0, -12, 13, 0)\) | \( \theta = e^1 \) |
| \( \mathfrak{r}_4, \alpha, -(1+\alpha) \) | \((14, \alpha 24, -(1 + \alpha)34, 0)\) | \( \theta = \alpha e^4, \alpha \neq -\frac{1}{2} \) |
| \( -1 < \alpha \leq -(1 + \alpha) \leq 1 \) | \( \alpha \neq 0 \) | \( \omega = e^{12} + e^{34} \) |
| \( \mathfrak{d}_4 \) | \((14, -24, -12, 0)\) | \( \omega = e^{12} - e^{34} + e^{24} \) |

We explain in details how to obtain the first case in Table 1 using our construction. Let \( \mathfrak{g} = \mathfrak{r}_3, -1 \) with the LCS structure \( \omega = e^{12} + e^{34}, \theta = e^1 \). It is clear that \( \mathfrak{u} = \langle e_3, e_4 \rangle \) is a non-degenerate abelian ideal of \( \mathfrak{g} \). It follows from Theorem 3.5 that \( \mathfrak{g} = \mathfrak{u} \ltimes \mathfrak{u} \) where \( \mathfrak{u}^\perp = \langle e_1, e_2 \rangle \) is isomorphic to the 2-dimensional non-abelian Lie algebra \( \text{aff}(\mathbb{R}) \). The LCS structure on \( \mathfrak{u}^\perp \) is given by \( \omega = e^{12}, \theta = e^1 \), note that since \( \dim \mathfrak{u}^\perp = 2 \), this structure is in fact a symplectic structure. Therefore \( \mathfrak{r}_3, -1 \simeq \text{aff}(\mathbb{R}) \ltimes \mathfrak{r}_1 \mathbb{R}^2 \) with \( \pi_1(e_1) = \text{diag}(-1, 0) = \text{diag}(-\frac{1}{2}, -\frac{1}{2}) + \text{diag}(-\frac{1}{2}, \frac{1}{2}) \) and \( \pi_1(e_2) = 0 \). Clearly, \( (\pi_1, \mathbb{R}^2) \) is an LCS extension of \( \text{aff}(\mathbb{R}) \).

It can be easily seen that for any of the Lie algebras and any of the LCS structures of Table 1 we can proceed in the same way with the exception of \( \mathfrak{d}_4 \) with the LCS structure \( \omega = e^{12} - e^{34} + e^{24} \) and \( \theta = e^1 \). This is the only LCS structure of the second kind on a 4-dimensional unimodular Lie algebra which does not satisfy the condition of Theorem 3.5, that is, it does not have a non-degenerate ideal contained in \( \ker \theta \). Indeed, a 2-dimensional ideal is generated by \( \{e_1, e_3\} \) or \( \{e_2, e_3\} \) which are not symplectic ideals.

To summarize we have the following result:

**Proposition 4.1.** Any unimodular 4-dimensional LCS Lie algebra of the second kind, with the only exception of \((\mathfrak{d}_4, \omega = e^{12} - e^{34} + e^{24}, \theta = e^1)\), can be reobtained by Theorem 3.3 for a suitable LCS extension.

According to [2] the simply connected Lie groups associated with the Lie algebras of Table 1 (for a countable set of parameters \( \alpha \) and \( \delta \)) admit lattices (see also [5]). Then we have that:

**Corollary 4.2.** Any unimodular 4-dimensional LCS Lie algebra of the second kind (except for \((\mathfrak{d}_4, e^{12} - e^{34} + e^{24}, e^4)\)) associated with a compact solvmanifold can be obtained by Theorem 3.3 for a suitable LCS extension.

Moreover, it follows from Theorem 2.1 that the induced LCS structures on any quotient are LCS structures of the second kind.
4.2. Higher dimension

We focus now on providing examples in higher dimension. In order to construct examples of unimodular Lie algebras admitting LCS structures we need to start with a non unimodular Lie algebra with a LCS structure in lower dimension. In [2] the authors classify the 4-dimensional solvable Lie algebras admitting LCS structures up to automorphism of the Lie algebra. Using their classification we show in Table 2 all the 4-dimensional solvable Lie algebras with their associated LCS structure satisfying condition (4), which means that they can be extended to a higher dimensional unimodular Lie algebra admitting a LCS structure given by Theorem 3.3.

We explain in details how to extend one example.

**Example 4.3.** Let $t_2'$ be the Lie algebra with structure constants $(0, 0, -13 + 24, -14 - 23)$. According to [2] this Lie algebra admits 4 non equivalent LCS structures up to Lie algebra automorphism.
\[
\begin{aligned}
\theta &= \sigma e^1 + \tau e^2 \\
\omega &= e^{13} - \tau e^{14} - \frac{1+\tau^2}{1+\sigma} e^{24} \\
&\text{with } \sigma \neq -1, 0, \quad \tau > 0
\end{aligned}
\]
\[
\begin{aligned}
\theta &= -2e^1 \\
\omega &= \sigma e^{12} + e^{34} \\
&\text{with } \sigma \neq 0
\end{aligned}
\]
\[
\begin{aligned}
\theta &= \tau e^2 \\
\omega &= e^{13} - \tau e^{14} - (1+\tau^2) e^{24} \\
&\text{with } \tau > 0
\end{aligned}
\]
\[
\begin{aligned}
\theta &= -2e^1 \\
\omega &= \sigma e^{12} + e^{34} \\
&\text{with } \sigma \neq 0
\end{aligned}
\]
We verify next if each LCS structure satisfies condition (4). If we consider the cases \(\theta = \sigma e^1 + \tau e^2, \theta = \tau e^2\) or \(\theta = -2e^1\) the condition (4) does not hold, then the possible LCS extension will not be unimodular. Finally we consider the LCS structure with Lee form \(\theta = \sigma e^1\) and \(\sigma \neq -1, 0\). In this case (4) is satisfied for \(2 = n\sigma\). Therefore for any \(n \in \mathbb{N}\) and \(\sigma = \frac{2}{n}\), the LCS structure on \(\mathfrak{r}_2^*\) given by
\[
\begin{aligned}
\theta &= \sigma e^1 \\
\omega &= e^{13} - \frac{1}{1+\sigma} e^{24} \\
&\text{with } \sigma \neq -1, 0
\end{aligned}
\]
can be extended to the \((2n + 4)\)-dimensional unimodular Lie algebra
\[
\mathfrak{g} = \mathfrak{r}_2^* \ltimes_{\pi} \mathbb{R}^{2n},
\]
where \((\pi, \mathbb{R}^{2n})\) is a suitable LCS extension of \(\mathfrak{r}_2^*\). It follows from Theorem 3.3 that \(\mathfrak{g}\) admits a LCS structure \((\tilde{\omega}, \tilde{\theta})\) given by \(\tilde{\omega}\mid_{\mathfrak{h}} = \omega, \tilde{\omega}\mid_{\mathbb{R}^{2n}} = \omega_0, \tilde{\theta}\mid_{\mathfrak{h}} = \theta\) and \(\tilde{\theta}\mid_{\mathbb{R}^{2n}} = 0\), where \(\omega_0\) is any symplectic form on \(\mathbb{R}^{2n}\). To be more specific we consider \(n = 2\), then \(\sigma = 1\). Let \(\pi : \mathfrak{r}_2^* \to \mathfrak{gl}(4, \mathbb{R})\) be a Lie algebra homomorphism given by
\[
\pi(e_1) = \begin{pmatrix} 0 & -1 & -1 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -\frac{1}{2} \text{Id} + \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix},
\]
and \(\pi(e_2) = \pi(e_3) = \pi(e_4) = 0\), in a basis \(\{e_5, e_6, e_7, e_8\}\) of \(\mathbb{R}^4\) with \(\omega_0 = e^{56} + e^{78}\). It is easy to see that \((\pi, \mathbb{R}^4)\) is a LCS extension of \(\mathfrak{r}_2^*\). Then the 8-dimensional Lie algebra \(\mathfrak{g} = \mathfrak{r}_2^* \ltimes_{\pi} \mathbb{R}^4\) has the following Lie brackets
\[
\begin{aligned}
[e_1, e_3] &= e_3, & [e_2, e_3] &= e_4, & [e_1, e_6] &= -e_6, \\
[e_1, e_4] &= e_4, & [e_2, e_4] &= -e_3, & [e_1, e_7] &= -e_7,
\end{aligned}
\]
and the LCS structure is given by
\[
\begin{aligned}
\tilde{\omega} &= e^{13} - \frac{1}{2} e^{24} + e^{56} + e^{78} \\
\tilde{\theta} &= e^1.
\end{aligned}
\]

\textbf{Remark 4.4.} Note that this example is not covered by the construction given in [2, Proposition 1.8].

We construct now other example of a LCS Lie algebra starting with a pair of Lie algebras \((\mathfrak{h}, \mathfrak{u})\) where \(\mathfrak{u}\) is not abelian as in the previous example.

\textbf{Example 4.5.} Let \(\mathfrak{h} = \mathfrak{r}_{3, \lambda}\) be the Lie algebra with structure constants \((0, -12, -\lambda 13, 0)\) with LCS structure \(\omega = e^{12} + e^{34}\) and Lee form \(\theta = -\lambda e^1\), and let \(\mathfrak{u} = \mathfrak{r}_{3, \lambda}\) be the Lie algebra with structure constants \((0, 0, -12, 0)\) and symplectic form \(\omega_0 = f^{13} + f^{24}\).
We consider now the Lie algebra \( g = h \ltimes u \) where \( \pi : h \to \text{Der}(u) \) is defined by \( \pi(e_2) = \pi(e_3) = \pi(e_4) = 0 \) and \( \pi(e_1) = \text{diag}(\frac{1}{2}, 0, \frac{1}{2}, \lambda) \). It is easy to check that \((\pi, u)\) is a LCS extension of \( h \) and therefore it follows from Theorem 3.3 that \( \tilde{\omega} = e^{12} + e^{34} + f^{13} + f^{24} \) is a LCS form on \( g \) with Lee form \( \tilde{\theta} = -\lambda e^1 \).

5. Examples of solvmanifolds with LCS structures of the second kind

In the section we use Theorem 3.3 and Theorem 2.1 to construct a family of solvmanifolds \( \Gamma_m \backslash G \) admitting a LCS structure of the second kind, where \( G \) is the simply connected Lie group associated to the Lie algebra considered in the Example 4.3. This Lie algebra can be decomposed as \( g = \mathbb{R}e_2 \ltimes \mathbb{R}e_1 \ltimes \mathbb{R}^6 \) where the adjoint actions of \( e_1 \) and \( e_2 \) are given by

\[
\text{ad}_{e_1} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \quad \text{ad}_{e_2} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

in the reordered bases \( \{e_3, e_6, e_5, e_4, e_7, e_8\} \) and \( \{e_1, e_3, e_6, e_5, e_4, e_7, e_8\} \) respectively. The simply connected Lie group associated to \( g \) is

\[
G = \mathbb{R}e_2 \ltimes \psi(\mathbb{R}e_1 \ltimes \varphi \mathbb{R}^6),
\]

where \( \varphi(t) = \exp(t \text{ad}_{e_1}) \) and \( \psi(t) = \exp(t \text{ad}_{e_2}) \). Next we construct a lattice in \( G \), to do this we start by considering the Lie subgroup \( H := \mathbb{R}e_1 \ltimes \varphi \mathbb{R}^6 \). Note that \( H \) is an almost abelian Lie group. According to [5] the Lie group \( H \) admits a lattice if and only if there exists \( t_0 \in \mathbb{R}, t_0 \neq 0 \), such that \( \varphi(t_0) \) is conjugated to an integer matrix.

We can write the matrix \( \varphi(t) \) in the basis \( \{e_3, e_6, e_5, e_4, e_7, e_8\} \) as

\[
\varphi(t) = \begin{pmatrix} e^t & e^{-t} & 1 & e^t & e^{-t} \\ e^{-t} & 1 & e^{-t} & e^t \\ 1 & e^t & e^{-t} & e^t \end{pmatrix}.
\]

We consider only the block

\[
M = \begin{pmatrix} e^t & e^{-t} \\ e^{-t} & 1 \end{pmatrix}.
\]

The characteristic polynomial of the matrix \( M \) is

\[
p(x) = (x - 1)(x - e^t)(x - e^{-t}).
\]

Fixing \( m \in \mathbb{N}, m > 2 \), we define \( t_m = \text{arccosh}(\frac{m}{2}) \), \( t_m > 0 \), and we have that \( (x - e^{tm})(x - e^{-tm}) = x^2 - mx + 1 \in \mathbb{Z}[x] \). Then the characteristic polynomial of \( M \) for \( t = t_m \) can be written as \( p(x) = x^3 - (m + 1)x^2 + (m + 1)x - 1 \). Therefore, it is easy to see that \( M \) is conjugated to the companion matrix \( C_m \) of the polynomial \( p \), that is, \( M = Q_mC_mQ_m^{-1} \) where
\[ C_m = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1-m \\ 0 & 1 & 1+m \end{pmatrix} \quad \text{and} \quad Q_m = \begin{pmatrix} 1 & e^{t_m} & e^{2t_m} \\ 1 & e^{-t_m} & e^{-2t_m} \\ 1 & 1 & 1 \end{pmatrix}. \]

We can copy this process in the second block of \( \phi(t) \) and we easily can check that

\[ \phi(t_m) = P_mD_mP_m^{-1}, \]

where

\[ D_m = \begin{pmatrix} C_m & 0 \\ 0 & C_m \end{pmatrix} \quad \text{and} \quad P_m = \begin{pmatrix} Q_m & 0 \\ 0 & Q_m \end{pmatrix}. \]

Therefore \( \phi(t_m) \) is conjugate to the integer matrix \( D_m \). It follows from [5] that \( H \) admits a lattice

\[ \Gamma_m = t_m\mathbb{Z} \ltimes P_m\mathbb{Z}^6. \]

We now write the matrix \( \psi(t) \) in the basis \( \{e_1, e_3, e_6, e_5, e_4, e_7, e_8\} \) as

\[ \psi(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos(t) & 0 & 0 & -\sin(t) & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \sin(t) & 0 & 0 & \cos(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \]

Clearly, \( \psi(2\pi) \) preserves the lattice \( \Gamma_m \). Thus

\[ \Lambda_m = 2\pi\mathbb{Z} \ltimes \Gamma_m = 2\pi\mathbb{Z} \ltimes (t_m\mathbb{Z} \ltimes P_m\mathbb{Z}^6) \]

is a lattice in \( G \) for any \( m > 2 \). Therefore we obtain an explicit construction of examples of solvmanifolds \( \Lambda_m \backslash G \) admitting a LCS structure of the second kind.

Since \( \psi(2\pi) = \text{Id} \), we have that \( \Lambda_m \simeq 2\pi\mathbb{Z} \ltimes \Gamma_m \), and therefore it can be considered as a lattice in \( G' = \mathbb{R} \times H \). Then, according to [16, Theorem 3.6] the corresponding solvmanifolds \( M_m := \Lambda_m \backslash G \) and \( M'_m := \Lambda_m \backslash G' \) are diffeomorphic. Note that \( M' \) is diffeomorphic to the product of \( S^1 \) and the solvmanifold \( \Gamma_m \backslash H \). We also note that \( G' \) can be seen as an almost abelian Lie group, more explicitly, \( G' = \mathbb{R} \ltimes \rho \mathbb{R}^7 \), where

\[ \rho(t) = \begin{pmatrix} 1 & e^t & e^{-t} \\ e^t & 1 & e^{-t} \\ e^{-t} & 1 \end{pmatrix}. \]

The Lie algebra \( g' \) of \( G' \) is given by \( g' = \mathbb{R}e_1 \ltimes \mathbb{R}^7 \) with

\[ \text{ad}_{e_1} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}. \]
in the basis \( \{e_2, e_3, e_6, e_5, e_4, e_7, e_8\} \) of \( \mathbb{R}^7 \). Note that the Lie group \( G' \), unlike \( G \), is completely solvable, that is, \( \text{ad}_X \) has real eigenvalues for all \( X \in \mathfrak{g}' \). It is easy to see that \( \rho(t_m) \) is conjugated to the integer matrix

\[
R_m = \begin{pmatrix} 1 & 0 \\ 0 & P_m \end{pmatrix}.
\]

Using this identification between \( M_m \) and \( M'_m \) we can prove that:

**Proposition 5.1.** The solvmanifolds \( \Lambda_m \backslash G \) are pairwise non homeomorphic.

**Proof.** We assume that \( M_m \) and \( M_n \) are homeomorphic, then \( M'_m \) and \( M'_n \) are homeomorphic as well, and therefore their fundamental groups \( \pi_1(M'_m) \) and \( \pi_1(M'_n) \) are isomorphic. Since \( G' \) is simply connected, we have that these fundamental groups are isomorphic to the lattices, and therefore \( \Lambda_m \cong \Lambda_n \). Since \( G' \) is completely solvable, we can use the Saito’s rigidity theorem [17] to extend this isomorphism to an automorphism of \( G' \). Since the lattices differ by an automorphism of \( G' \), it follows from [10, Theorem 3.6] that the integer matrix \( R_m \) is conjugated either to \( R_n \) or to \( R_n^{-1} \). Finally comparing the eigenvalues of \( R_m \) and \( R_n \) we obtain that this happens if and only if \( m = n \). □

We study now the de Rham and the Morse-Novikov cohomology of the solvmanifolds \( \Lambda_m \backslash G \cong \Lambda_m \backslash G' \). Since \( G' \) is completely solvable these cohomologies can be computed in terms of the Lie algebra cohomology. Indeed, it follows from [9] (see also [1]) that \( H^*_{dR}(\Lambda_m \backslash G') \cong H^*(\mathfrak{g}') \) and \( H^*_\theta(\Lambda_m \backslash G') \cong H^*_\theta(\mathfrak{g}') \) for any \( m \in \mathbb{N} \).

According to [18], the \( k \)th Betti number of \( \mathfrak{g}' \), \( \beta_k = \dim H^k(\mathfrak{g}') \), can be computed in terms of the dimension of \( Z^j(\mathfrak{g}') = \{ \alpha \in \wedge^j \mathfrak{g}^* : d\alpha = 0 \} \) as follows:

\[
\beta_k = \dim H^k(\mathfrak{g}') = \dim Z^k(\mathfrak{g}') + \dim Z^{k-1}(\mathfrak{g}') - \binom{8}{k-1},
\]

for \( k > 2 \). Note that \( \beta_0 = 1 \) and \( \beta_1 = \dim(\mathfrak{g}'/[\mathfrak{g}', \mathfrak{g}']) = 4 \). Using equation (5) it can be seen that \( \beta_2 = 10 \), \( \beta_3 = 20 \) and \( \beta_4 = 26 \). Finally, due to Poincaré duality, we obtain that \( \beta_5 = 20 \), \( \beta_6 = 10 \), \( \beta_7 = 4 \) and \( \beta_8 = 1 \).

For the Morse-Novikov cohomology, the corresponding Betti numbers \( \beta_k^\theta = \dim H^k_\theta(\mathfrak{g}') \) satisfy a similar equation. Indeed, setting \( Z^j_\theta(\mathfrak{g}) = \{ \alpha \in \wedge^j \mathfrak{g}^* : d_\theta \alpha = 0 \} \) we have

\[
\beta_k^\theta = \dim H^k_\theta(\mathfrak{g}') = \dim Z^k_\theta(\mathfrak{g}') + \dim Z^{k-1}_\theta(\mathfrak{g}') - \binom{8}{k-1},
\]

for \( k > 2 \). It is easy to see that \( \beta_0^\theta = 0 \) and \( \beta_1^\theta = 2 \). Then, after some computations and using (6), it can be seen that \( \beta_2^\theta = 8 \), \( \beta_3^\theta = 14 \), \( \beta_4^\theta = 16 \), \( \beta_5^\theta = 14 \), \( \beta_6^\theta = 8 \), \( \beta_7^\theta = 2 \) and \( \beta_8^\theta = 0 \).

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