A Pólya–Vinogradov inequality for short character sums

Matteo Bordignon

School of Science, University of New South Wales Canberra

m.bordignon@student.unsw.edu.au

Abstract

In this paper we obtain a variation of the Pólya–Vinogradov inequality with the sum restricted to a certain height. Assume $\chi$ to be a primitive character modulo $q$, $\epsilon > 0$ and $N \leq q^{1-\gamma}$, with $0 \leq \gamma \leq 1/3$. We prove that

$$\left| \sum_{n=1}^{N} \chi(n) \right| \leq c \left( \frac{1}{3} - \gamma + \epsilon \right) \sqrt{q} \log q$$

with $c = 2/\pi^2$ if $\chi$ is even and $c = 1/\pi$ if $\chi$ is odd. The result is based on the work of Hildebrand [7] and Kerr [4].

1 Introduction

It is of high interest studying the possible upper bounds of the following quantity

$$S(N, \chi) := \left| \sum_{n=1}^{N} \chi(n) \right|,$$

with $N \in \mathbb{N}$ and $\chi$ a primitive Dirichlet character modulo $q$. The interest is principally on primitive characters as a result for these characters can easily be generalized to all non-principal characters. The famous Pólya–Vinogradov inequality tells us that

$$S(N, \chi) \ll \sqrt{q} \log q,$$

and aside for the implied constant, this is the best known result. The best known asymptotic constant is $69/70 + o(1)$, if $\chi$ is even from [6] and $1/3 + o(1)$ if $\chi$ is odd from [8]. For the best completely explicit constant see [2], [3] and [5]. It is interesting to note that it appears that $S(N, \chi)$ assumes its maximum for $N \approx q$, see the work by Bober et al. in [1] and the one by Hildebrand, Corollary 3 of [8], which proves that for even characters we have that $N = o(q)$ implies $S(N, \chi) = o(\sqrt{q} \log q)$. Our work will continue in this direction using tools developed by Hildebrand in [7] that will lead us to prove the following result.
Theorem 1.1. Let $\chi$ be a primitive character modulo $q$, $\epsilon > 0$ and $N \leq q^{1-\gamma}$, with $0 \leq \gamma \leq 1/3$. We have

$$S(N, \chi) \leq c(\frac{1}{3} - \gamma + \epsilon)\sqrt{q}\log q$$

with $c = 2/\pi^2$ if $\chi$ is even and $c = 1/\pi$ if $\chi$ is odd.

It is easy to see that Theorem 1.1, when $N \leq q^{1-\gamma}$, improves on the previous bound in [6] for $\gamma \geq 0.0072$ if $\chi$ is even, although in this range both results are asymptotically superseded by Corollary 3 of [8], and improves on the previous bound in [8] for any $\gamma > 0$ if $\chi$ is odd. Theorem 1.1 can be made completely explicit, though with smaller leading constants, following the approach used in [2]. We note that Theorem 1.1 is also interesting as it could be viewed as a result between the standard Pólya-Vinogradov and the Burgess bound.

In Section 2 we will introduce a lemma, due to Hildebrand, that gives a good upper bound on a certain Gaussian sum and a second lemma related to the sum of trigonometric functions, that is a generalization of a result due to Young and Pomerance. In Section 3 we will then prove Theorem 1.1 drawing inspiration from Hildebrand’s work [7]. Our main improvement will come from dividing a Gauss sum into three parts, instead of into two as done by Hildebrand, and control the first sum using certain bounds on sine and cosine in that range.

2 Two important lemmas

We will need the following result, that is Lemma 3 of [7].

Lemma 2.1. Let $0 < \epsilon < 1/2$ a fixed number. Then we have, for all primitive characters modulo $q$, $q^{1/3+\epsilon} \leq x \leq q$ and real $\alpha$,

$$\left| \sum_{n \leq x} \chi(n)e(\alpha n) \right| \ll_{\epsilon} \frac{x}{\log q}.$$ 

We now prove a lemma that is a variation of two results from [13] and [12].

Lemma 2.2. Uniformly for $0 \leq \gamma \leq 1/3$ and real $\alpha$ we have

$$\sum_{q^{\gamma} \leq n \leq q^{1/3+\epsilon}} \frac{1 - \cos(\alpha n)}{n} \leq (\frac{1}{3} - \gamma + \epsilon)\log q + O(1)$$  \hspace{1cm} (1)$$

and

$$\sum_{q^{\gamma} \leq n \leq q^{1/3+\epsilon}} \frac{\left| \sin(\alpha n) \right|}{n} \leq \frac{2}{\pi} (\frac{1}{3} - \gamma + \epsilon)\log q + O(1).$$  \hspace{1cm} (2)$$

Proof. It is easy to show that we can assume $q^{1/3+\epsilon} \geq 5q^{\gamma} + 6$. Define

$$\sigma(a, b)(\alpha) := \sum_{a \leq n \leq b} \cos(\alpha n).$$
We start proving (1) noting that, assuming $C \geq 0$, the result would follow from
\[
\sigma(q^\gamma, q^{1/3+\epsilon})(\alpha) \geq -C. \tag{3}
\]
Taking $v = p(2m + 1)$, with $p$ an odd integer and $m$ any integer, equation (10) from [13] states that
\[
\sigma(m + 1, v)(\alpha) \geq \sigma(m + 1, v)(\pi) - \frac{1}{2q}.
\]
and noting that
\[
\sigma(m + 1, v)(\pi) \geq -1,
\]
we have
\[
\sigma(m + 1, v)(\alpha) \geq -5 - \frac{1}{2q}. \tag{4}
\]
Taking $m = \lceil q^\gamma \rceil$, $p = \begin{cases} 
\lfloor q^{1/3+\epsilon}/2m + 1 \rfloor - 1 & \text{if } \lfloor q^{1/3+\epsilon}/2m + 1 \rfloor \text{ is even,} \\
\lfloor q^{1/3+\epsilon}/2m + 1 \rfloor & \text{if } \lfloor q^{1/3+\epsilon}/2m + 1 \rfloor \text{ is odd},
\end{cases}
\]
and $v = p(2m + 1)$, we can rewrite the sum in (3), as we assumed $q^{1/3+\epsilon} \geq 5q^\gamma + 6$, as follows
\[
\sigma(q^\gamma, q^{1/3+\epsilon})(\alpha) = \sigma(q^\gamma, v)(\alpha) + \sigma(v, q^{1/3+\epsilon})(\alpha) - \frac{\cos(\alpha v)}{v}. \tag{5}
\]
Observing that $q^{1/3+\epsilon} - v \leq 4m + 2$ and using $q^{1/3+\epsilon} \geq 5q^\gamma + 6$, we obtain
\[
\left| \sum_{q^\gamma \leq n \leq q^{1/3+\epsilon}} \frac{\cos(\alpha n)}{n} \right| \leq \frac{q^{1/3+\epsilon} - v}{v} \leq 2. \tag{6}
\]
Thus (3) follows applying (4) and (6) to (5).
We are thus left with proving (2). We need the Fourier expansion for $|\sin(\theta)|$ that can be found in [11] Problem 34 in Part VI,
\[
|\sin(\theta)| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\cos 2m\theta}{4m^2 - 1}.
\]
Thus,
\[
\sum_{q^\gamma \leq n \leq q^{1/3+\epsilon}} \frac{|\sin(\alpha n)|}{n} = \frac{2}{\pi} \sum_{q^\gamma \leq n \leq q^{1/3+\epsilon}} \frac{1}{n} - \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{1}{4m^2 - 1} \sum_{q^\gamma \leq n \leq q^{1/3+\epsilon}} \frac{\cos 2mn\alpha}{n},
\]
and (2) now follows from (3). \qed

We can note that, in the above result, the $O(1)$ could be easily made explicit, but for the following applications this is not necessary.
3 Proof of Theorem 1.1

We take $\chi$ primitive and start with

$$\chi(n) = \frac{1}{\tau(\chi)} \sum_{a=1}^{q} \overline{\chi}(a) e\left(\frac{a n}{q}\right) = \frac{1}{\tau(\chi)} \sum_{0 < |a| < q/2} \overline{\chi}(a) e\left(\frac{a n}{q}\right),$$

where $\tau(\chi)$ is the Gaussian sum. Summing over $1 \leq n \leq N$, we obtain

$$\sum_{n=1}^{N} \chi(n) = \frac{1}{\tau(\chi)} \sum_{0 < |a| < q/2} \overline{\chi}(a) \sum_{n=1}^{N} e\left(\frac{a n}{q}\right) = \frac{1}{\tau(\chi)} \sum_{0 < |a| < q/2} \overline{\chi}(a) \frac{e\left(\frac{a N}{q}\right) - 1}{1 - e\left(\frac{-a}{q}\right)}.$$

Since $|\tau(\chi)| = \sqrt{q}$ for primitive characters and

$$\frac{1}{1 - e\left(\frac{-a}{q}\right)} = \frac{q}{2\pi i a} + O(1),$$

for $0 < |a| < q/2$, it follows that

$$S(N,\chi) \leq \frac{\sqrt{q}}{2\pi} \left| \sum_{0 < |a| < q/2} \overline{\chi}(a) \left( \frac{e\left(\frac{a N}{q}\right) - 1}{a} \right) \right| + O(\sqrt{q}).$$

Now we split the inner sum in three parts: $\Sigma_1$ with $0 < |a| < q^1/(2\pi)$, $\Sigma_2$ with $q^1/(2\pi) \leq |a| \leq q^{1/3+\epsilon}$ and $\Sigma_3$ with $q^{1/3+\epsilon} < |a| < q/2$. Therefore

$$S(N,\chi) \leq \frac{\sqrt{q}}{2\pi} \left( \left| \Sigma_1 \right| + \left| \Sigma_2 \right| + \left| \Sigma_3 \right| \right) + O(\sqrt{q}). \quad (7)$$

We have

$$\Sigma_1 = \begin{cases} 2i \sum_{1 \leq a < q^1/(2\pi)} \frac{\overline{\chi}(a) \sin\left(\frac{2\pi a N}{q}\right)}{a} & \text{if } \chi \text{ is even,} \\ -2 \sum_{1 \leq a < q^1/(2\pi)} \frac{\overline{\chi}(a) \left(1 - \cos\left(\frac{2\pi a N}{q}\right)\right)}{a} & \text{if } \chi \text{ is odd.} \end{cases}$$

Now observing that for $0 < |a| < q^1/(2\pi)$ we have that $\left|\frac{2\pi a N}{q}\right| < 1$ and thus

$$\left|\sin\left(\frac{2\pi a N}{q}\right)\right| \ll \frac{2\pi a N}{q} \quad \text{and} \quad \left|1 - \cos\left(\frac{2\pi a N}{q}\right)\right| \ll \left(\frac{2\pi a N}{q}\right)^2,$$
where, remembering that $N \leq q^{1-\gamma}$, easily give $\Sigma_1 \ll 1$.

We also have

$$\Sigma_2 = \begin{cases} 
2i \sum_{q^{\gamma}/(2\pi) \leq a \leq q^{1/3+\epsilon}} \frac{\chi(a) \sin(\frac{2\pi a N}{q})}{a} & \text{if } \chi \text{ is even,} \\
-2 \sum_{q^{\gamma}/(2\pi) \leq a \leq q^{1/3+\epsilon}} \frac{\chi(a) \left(1 - \cos(\frac{2\pi a N}{q})\right)}{a} & \text{if } \chi \text{ is odd,}
\end{cases}$$

and applying Lemma 2.2, observing that the difference between having the lower bound of the sum starting from $q^{\gamma}/(2\pi)$ and not $q^{\gamma}$ is bounded by a constant, we obtain

$$\Sigma_2 = \begin{cases} 
\frac{4}{\pi} \left(\frac{1}{3} - \gamma + \epsilon\right) \log q + O(1) & \text{if } \chi \text{ is even,} \\
2\left(\frac{1}{3} - \gamma + \epsilon\right) \log q + O(1) & \text{if } \chi \text{ is odd.}
\end{cases}$$

By partial summation and Lemma 2.1 we have

$$|\Sigma_3| \ll (\log q) \max_{q^{1/3+\epsilon} \leq x \leq q} \left| \frac{1}{x} \sum_{a \leq x} \overline{\chi}(a) \left(e\left(\frac{aN}{q}\right) - 1\right) \right| \ll \epsilon 1.$$

Thus, by the above upper bounds on $|\sum_i|$, for $i = 1, 2, 3$, and (7), we obtain the desired result.

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