OPTIMAL DECAY RATE FOR THE WAVE EQUATION ON A SQUARE WITH CONSTANT DAMPING ON A STRIP

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Abstract. We consider the damped wave equation with Dirichlet boundary conditions on the unit square parametrized by Cartesian coordinates $x$ and $y$. We assume the damping $a$ to be strictly positive and constant for $x < \sigma$ and zero for $x > \sigma$. We prove the exact $t^{-2/3}$-decay rate for the energy of classical solutions. Our main result (Theorem 1) answers question (1) of [1, Section 2C].

1. Introduction

1.1. The main result. Let $\Box = (0,1)^2$ be the unit square. We parametrize it by Cartesian coordinates $x$ and $y$. Let $a$ - the damping - be a function on $\Box$ which depends only on $x$ such that $a(x) = a_0 > 0$ for $x < \sigma$ and $a(x) = 0$ for $x > \sigma$ where $\sigma$ is some fixed number from the interval $(0,1)$. We consider the damped wave equation:

\[
\begin{align*}
    u_{tt}(t,x,y) - \Delta u(t,x,y) + 2a(x) u_t(t,x,y) &= 0 \\
    u(t,x,y) &= 0 \\
    u(0,x,y) = u_0(x,y), u_t(0,x,y) = u_1(x,y)
\end{align*}
\]

where $(x,y) \in \Box$. We are interested in the energy

\[
E(t,U_0) = \frac{1}{2} \int \int |\nabla u(t,x,y)|^2 + |u_t(t,x,y)|^2 \, dx \, dy
\]

of a wave at time $t$ with initial data $U_0 = (u_0, u_1)$. Let $D = (H^2 \cap H^1_0) \times H^1_0(\Box)$ denote the set of classical initial data. The purpose of this paper is to prove

**Theorem 1.** Let $\Box$, $a$ and $E(t,U_0)$ be as above. Then $\sup E(t,U_0)^{1/2} \approx t^{-2/3}$ where the supremum is taken over initial data $\|U_0\|_D = 1$.

The exact meaning of ‘$\approx$’ and other symbols is explained in Section 2. In Section 4 we show that this theorem is equivalent to Theorem 3 below. Section 5 is devoted to the proof of Theorem 3.

**Remark 2.** The proof of Theorem 1 shows that a higher dimensional analogue is also true. That is, one can replace $y \in \mathbb{R}$ by $y \in \mathbb{R}^{d-1}$ for any natural number $d \geq 2$. The exact decay rate remains the same for all $d$.

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1.2. The semigroup approach. If we set \( U = (u, u_t) \) and \( U_0 = (u_0, u_1) \) we may formulate the damped wave equation as an abstract Cauchy problem

\[
\dot{U}(t) + AU(t) = 0, \quad U(0) = U_0
\]

where \( A = \begin{pmatrix} 0 & -1 \\ -\Delta & 2a(x) \end{pmatrix} \) on the Hilbert space \( \mathcal{H} = H^1_0 \times L^2(\Box) \). The domain of \( A \) is \( D(A) = (H^2 \cap H^1_0) \times H^1_0(\Box) \). Since \(-A\) is a dissipative (we equip \( H^1_0(\Box) \) with the gradient norm) and invertible operator on a Hilbert space it generates a \( C_0 \)-semigroup of contractions by the Lumer-Phillips theorem. Note that the inclusion \( D(A) \hookrightarrow \mathcal{H} \) is compact by the Rellich-Kondrachov theorem. Thus the spectrum of \( A \) contains only eigenvalues of finite multiplicity.

1.3. Classification of the main result. Our situation is a very particular instance of the so called partially rectangular situation. A bounded domain \( \Omega \) is called partially rectangular if its boundary \( \partial \Omega \) is piecewise \( C^\infty \) and if \( \Omega \) contains an open rectangle \( R \) such that two opposite sides of \( R \) are contained in \( \partial \Omega \). We call these two opposite sides horizontal. One can decompose \( \Omega = R \cup W \), where \( W \) is an open set which is disjoint to \( R \). In our particular situation we can choose \( W \) to be empty. Furthermore it is assumed, that \( a > 0 \) on \( W \) and \( a = 0 \) on \( S \), where \( S \subseteq R \) is an open rectangle with two sides contained in the horizontal sides of \( R \). To avoid the discussion of null-sets we assume for simplicity that either \( a \) is continuous up to the boundary or it is as in subsection 1.1.

Under these constraints one can show that the energy of classical solutions can never decay uniformly faster than \( 1/t^2 \), i.e.

\[
\sup_{U_0 \in D(A)} E(t, U_0)^{\frac{1}{2}} \gtrsim \frac{1}{t}.
\]

This result seems to be well-known. Unfortunately we do not know an original reference to this bound on the energy. A short modern proof using [2, Proposition 1.3] can be found in [1]. But there is also a geometric optics proof using quantified versions of the techniques of [8]. Unfortunately the latter approach seems to be never published anywhere.

On the other hand: If we assume that the damping does not vanish completely in \( R \) (this is an additional assumption only if \( W \) is empty), then

\[
\forall U_0 \in D(A) : E(t, U_0)^{\frac{1}{2}} \lesssim \frac{1}{t^2}.
\]

This is a corollary of one of the main results in [1]. There the authors showed that stability at rate \( t^{-1/2} \) for an abstract damped wave equation is equivalent to an observability condition for a related Schrödinger equation. Earlier contributions towards [2] were given by [3] and [4].

Having the two bounds [1] and [2], at hand a natural question arises: Are these bounds sharp? Concerning the fast decay rates related to [1] this is partly answered by [2] and [3]. Essentially the authors showed that if the damping function is smooth enough than one can get a decay rate as close to \( t^{-1} \) as we wish. Unfortunately they could not characterize the exact decay rate in terms of properties of \( a \). A breakthrough into this direction was achieved in [6] in a slightly different situation (there \( S \) degenerates to a line).

To the best of our knowledge it is completely unknown if the slowest possible rate \( t^{-1/2} \) is attained. To us the only known result towards this direction is due to
Nonnenmacher: If we are in the very particular situation described in subsection 1.1 then
\[
\sup_{U_0 \in D(A)} E(t, U_0)^{\frac{1}{2}} \gtrsim \frac{1}{t^{\frac{2}{3}}}
\]
See [1, Appendix B]. So this situation is a candidate for the slow decay rate. In this paper we show that Nonnenmacher’s bound is actually equal to the exact decay rate.

This of course raises a new question: Is it possible to find a non-vanishing bounded damping in a partially rectangular domain, satisfying the constraints specified above, but discarding the continuity assumptions, such that the exact decay rate for \( E(t, U_0)^{\frac{1}{2}} \) is strictly slower than \( t^{-2/3} \)? We think this is an interesting question for future research.

1.4. From waves to stationary waves. Let \( f \in L^2(\Box) \). Now we consider the stationary damped wave equation with Dirichlet boundary conditions
\[
\left\{ \begin{array}{ll}
P(s)u(x, y) = (-\Delta - s^2 + 2isa(x))u(x, y) = f(x, y) & \text{in } \Box \\
u(x, y) = 0 & \text{on } \partial \Box
\end{array} \right.
\]
As already said above, to prove Theorem 1 is essentially to show

**Theorem 3.** The operator \( P(s) : H^2 \cap H^1_0(\Box) \to L^2(\Box) \) from (3) is invertible for every \( s \in \mathbb{R} \). Moreover

\[
\|P(s)^{-1}\|_{L^2 \to L^2} \approx 1 + |s|^{\frac{1}{2}}.
\]

Actually we only prove a \( \lesssim \)-inequality since the reverse inequality is a consequence of Nonnenmacher’s appendix to [1] together with Proposition 2.4 in the same paper (see Section 4 for more details). Since it is well-known we also do not prove the invertability of \( P(s) \). The (simple) standard proof is based on testing the homogeneous stationary wave equation with \( \Box \). From considering real and imaginary part of the resulting expression one easily checks \( u = 0 \) by a unique continuation principle.

**Acknowledgments.** This paper was inspired and motivated by [1, Appendix B] (by S. Nonnenmacher) and [3]. I am grateful to Ralph Chill for reading and correcting the very first version of this paper.

2. Notations and conventions

**Convention.** Because of the symmetry of (3) we have \( \|P(-s)^{-1}\|_{L^2 \to L^2} = \|P(s)^{-1}\|_{L^2 \to L^2} \). Therefore in the following we always assume \( s \) to be positive.

**Constants.** We use two special constants \( c > 0 \) and \( C > 0 \). Special means, that they may change their value from line to line. The difference between these two constants is, that their usage implicitly means that we could always replace \( c \) by a smaller constant and \( C \) by a larger constant - if this is necessary. So one should keep in mind that \( c \) is a small number and \( C \) a large number.

**Landau notation.** For this subsection let us denote by \( \phi, \phi_1, \phi_2 \) and \( \psi \) complex valued functions defined on \( \mathbb{R} \setminus K \), where \( K \) is a compact interval. Furthermore we
always assume $\phi, \phi_1$ and $\phi_2$ to be real valued and (not necessary strictly) positive. We define

$$\phi_1(s) \lesssim \phi_2(s) :\iff \exists s_0 > 0, C > 0 \forall |s| \geq s_0 : \phi_1(s) \leq C \phi_2(s),$$

$$\phi_1(s) \approx \phi_2(s) :\iff \phi_1(s) \lesssim \phi_2(s) \text{ and } \phi_2(s) \lesssim \phi_1(s).$$

Furthermore we define the following classes (sets) of functions:

$$O(\phi(s)) := \{ \psi; |\psi(s)| \lesssim \phi(s) \},$$

$$o(\phi(s)) := \{ \psi; \forall \varepsilon > 0 \exists s_\varepsilon > 0 \forall |s| \geq s_\varepsilon : |\psi(s)| \leq \varepsilon \phi(s) \}.$$

By abuse of notation we write for example $\psi(s) = O(\phi(s))$ instead of $\psi \in O(\phi(s))$ or $\phi(s) = \phi_1(s) + O(\phi_2(s))$ instead of $|\phi(s) - \phi_1(s)| \lesssim \phi_2(s)$. By $O(s^{-\infty})$ we denote the intersection of all $O(s^{-N})$ for $N \in \mathbb{N}$.

**Function spaces.** As usual, by $L^2(\Omega)$ we mean the space of square-integrable functions on some open subset $\Omega$ of $\mathbb{R}^n$ for some $n \in \mathbb{N}$. For $k$ a natural number $H^k(\Omega)$ denotes the space of functions from $L^2(\Omega)$ whose distributional derivatives up to order $k$ are square integrable, too. Finally the space $H^1_0(\Omega)$ denotes the closure of the set of compactly supported smooth functions in $H^1(\Omega)$. We equip $H^1_0(\Omega)$ with the norm $(\int_\Omega |\nabla u|^2 \, dx)^{1/2}$ which is equivalent to the usual norm.

### 3. Proof of Theorem 3

Here is the plan for the proof: First we separate the $y$-dependence of the stationary wave equation from the problem. As a result we are dealing with a family of one dimensional problems which are parametrized by the vertical wave number $n \in \mathbb{N}$. Then we derive explicit solution formulas for the separated problems. These formulas allow us to estimate the solutions of the separated problems by their right-hand side with a constant essentially depending explicitly on $s$ and $n$. In the final step we introduce appropriate regimes for $s$ relative to $n$ which allow us to drop the $n$-dependence of the constant by a (short) case study.

#### 3.1. Separation of variables.** First recall that the functions $s_n(y) = \sqrt{2} \sin(n \pi y)$ for $n \in \{1, 2, \ldots \}$ form a complete orthonormal system of $L^2(0, 1)$. Thus considering $u$ and $f$ satisfying \ref{eq:wave_equation} we may write

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x)s_n(y) \quad \text{and} \quad f(x, y) = \sum_{n=1}^{\infty} f_n(x)s_n(y).$$

In terms of this separation of variables the stationary wave equation is equivalent to the one dimensional problem $P_n(s)u_n = f_n$ where

$$P_n(s) = -\partial_x^2 - k_n^2 + 2isa(x), \quad \text{and} \quad k_n^2 = s^2 - (n \pi)^2.$$

Note that $k_n$ might be an imaginary number. In a few lines we see that only the real case is important. In that case we choose $k_n \geq 0$. But first we prove the following simple

**Lemma 4.** Let $\phi : \mathbb{R} \to (0, \infty)$. Then the estimate $\|P_n(s)^{-1}\|_{L^2 \to L^2} \lesssim \phi(s)$ uniformly in $n$ is equivalent to the estimate $\|P(s)^{-1}\|_{L^2 \to L^2} \lesssim \phi(s)$. 

Proof. Let \( P(s)u = f \) and expand \( u \) and \( f \) as in (4). Then the implication from the left to the right is a consequence of the following chain of equations and inequalities:

\[
\|u\|_{L^2}^2 = \sum_{n=1}^{\infty} \|u_n\|_{L^2}^2 \lesssim s(\phi(s))^2 \sum_{n=1}^{\infty} \|f_n\|_{L^2}^2 = \phi(s)^2 \|f\|_{L^2}^2.
\]

The reverse implication follows from looking at \( \|f\|_{L^2}^2 = \|f_n\|_{L^2}^2 \) for some universal constant \( c > 0 \).

So below we are concerned with the separated stationary wave equation

\[
(\phi(s))^2 \sum_{n=1}^{\infty} \|u_n\|_{L^2}^2 \lesssim \phi(s)^2 \sum_{n=1}^{\infty} \|f_n\|_{L^2}^2 = \phi(s)^2 \|f\|_{L^2}^2.
\]

Let \( u_n(x) = s_n(x) \) and \( u(x,y) = u_n(x)s_n(y) \).

So below we are concerned with the separated stationary wave equation

\[
\begin{cases}
P_n(s)u_n(x) = f_n(x) & \text{for } x \in (0,1) \\
u_n(0) = u_n(1) = 0
\end{cases}
\]

where \( P_n(s) \) is defined in (5). In view of Lemma 4 we are left to show \( \|u_n\|_{L^2} \lesssim s^{1/2} \|f_n\|_{L^2} \) uniformly in \( n \) in order to prove Theorem 3. It turns out that such an estimate is easy to prove if \( k_n \) is imaginary. More precisely:

**Lemma 5.** There exists a constant \( c > 0 \) such that \( \|P_n(s)^{-1}\|_{L^2 \to H^1_0} \lesssim 1 \) holds uniformly in \( n \) whenever \( s^2 \leq (\pi n)^2 + c \).

Note that \( P_n(s)^{-1} \) is considered as an operator mapping to \( H^1_0(0,1) \). But it does not really matter since we will only use this estimate after replacing \( H^1_0 \) by \( L^2 \).

**Proof.** Testing equation (4) by \( \overline{u_n} \) and taking the real part leads to

\[
\int_0^1 |u_n'|^2 - c \int_0^1 |u_n|^2 \leq \int_0^1 |f_n u_n|.
\]

Recall that \( \|v\|_{L^2}^2 \geq \pi^2 \|v\|_{L^2}^2 \) for all \( v \in H^1_0(0,1) \) since \( \pi^2 \) is the lowest eigenvalue of the Dirichlet-Laplacian on the unit interval. Thus the conclusion of the Lemma holds for all \( c < \pi^2 \).

This lemma allows us to assume

\[
k_n = \sqrt{s^2 - (\pi n)^2} > c
\]

for some universal constant \( c > 0 \) not depending on neither \( s \) nor \( n \).

3.2. **Explicit formula for** \( P_n(s)^{-1} \). From now on we consider (4) under the constraint (7). To avoid cumbersome notation we drop the subscript \( n \) from \( k_n \), i.e. we write \( k \) instead from now on. Next let \( v = u_n|_{(0,\sigma)}, g = f_n|_{(0,\sigma)} \) and \( w = u_n|_{(\sigma,1)}, h = f_n|_{(\sigma,1)} \). We may write (4) as a coupled system consisting of a wave equation with constant damping and an undamped wave equation:

\[
\begin{cases}
(\partial_x^2 - k^2 + 2isa_0)v(x) = g(x) & \text{for } x \in (0,\sigma), \\
(\partial_x^2 - k^2)w(x) = h(x) & \text{for } x \in (\sigma,1), \\
v(0) = w(1) = 0, \\
v(\sigma) = w(\sigma), v'(\sigma) = w'(\sigma).
\end{cases}
\]

3.2.1. **Solution of the homogeneous equation.** The following ansatz satisfies the first three lines of (5) with \( g, h = 0 \):

\[
v_0(x) = \frac{1}{k'} \sin(k'x), \quad w_0(x) = \frac{1}{k} \sin(k(1-x)),
\]

where \( k' \) is the solution of \( k'^2 = k^2 - 2isa_0 \) which has negative imaginary part.
3.2.2. Solution of the inhomogeneous equation. The following ansatz satisfies the first three lines of (8):

\[ v_g(x) = -\frac{1}{k'} \int_0^x \sin(k'(x-y))g(y)dy, \quad w_h(x) = -\frac{1}{k} \int_x^1 \sin(k(y-x))h(y)dy. \]

This is simply the variation of constants (or Duhamel’s) formula. It is useful to know the derivatives of these particular solutions:

\[ v'_g(x) = -\int_0^x \cos(k'(x-y))g(y)dy, \quad w'_h(x) = +\int_1^x \cos(k(y-x))h(y)dy. \]

3.2.3. General solution. The general solution of the first three lines of (6) has the form

\[ v = av_0 + v_g, \quad w = bw_0 + w_h. \]

Our task is to find the coefficients \( a = a(s,n) \) and \( b = b(s,n) \). Therefore we have to analyze the coupling condition in line four of (8). A short calculation shows that it is equivalent to

\[ \begin{vmatrix} \left(\frac{v_0}{v'_g} - \frac{w_0}{w'_h}\right) \bigg|_{x=\sigma} \end{vmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{vmatrix} \left(\frac{w_h - v_g}{w'_h - v'_g}\right) \bigg|_{x=\sigma} \end{vmatrix}. \]

From the preceding equation we easily deduce

\[ a = \frac{1}{\det M} \left[ w'_0(v_g - w_h) - w_0(v'_g - w_h) \right]_{x=\sigma}, \]

\[ b = \frac{1}{\det M} \left[ v'_0(v_g - w_h) - v_0(v'_g - w_h) \right]_{x=\sigma}. \]

Moreover

\[ \det M = \frac{1}{k'} \sin(k'\sigma) \cos(k(1-\sigma)) + \frac{1}{k} \cos(k'\sigma) \sin(k(1-\sigma)). \]

3.3. Proving a general estimate \( \|u_n\|_{L^2} \leq C(k,k',M) \|f_n\|_{L^2} \). For this inequality we will derive an explicit formula for \( C \) in terms of \( k, k' \) and \( M \). In the next subsection we identify the qualitatively different regimes in which \( s \) can live. By regime we mean a relation which says how big \( s \) - the full momentum - is compared to \( n\pi \) - the momentum in \( y \)-direction. For each of these regimes we then easily translate the explicit \( k,k',M \) dependence of \( C \) to a an explicit dependence on \( s \).

3.3.1. Elementary estimates for \( w_0 \) and \( w_h \). Directly from the definition of \( w_0 \) (see (9)) we deduce

\[ \|w_0\|_\infty \leq \frac{1}{k}, \|w'_0\|_\infty \leq 1 \text{ and } \|w_0\|_2 \leq \frac{\sqrt{1-\sigma}}{k}. \]

In the same manner for \( w_h \) from (10) and (11) we deduce:

\[ \|w_h\|_\infty \leq \frac{\sqrt{1-\sigma}}{k} \|h\|_2, \|w'_h\|_\infty \leq \sqrt{1-\sigma} \|h\|_2 \text{ and } \|w_h\|_2 \leq \frac{1-\sigma}{k} \|h\|_2. \]
3.3.2. Estimating $w$. Recall from (12) that $w = bw_0 + w_h$. Recall the formula (14) for $b$. Note that
\[
(v_0'v_g - v_0'v_g') = \frac{1}{k'} \int_0^\sigma \sin(k'y)g(y)dy.
\]
Thus it seems to be natural to decompose
\[
b = \frac{1}{\det M} \left[ (v_0w_0' - v_0'w_0h) + (v_0'v_g - v_0v_g') \right]_{x=\sigma} =: b_1 + b_2.
\]
This leads to the decomposition of $w = b_1w_0 + b_2w_0 + w_h$ into three parts. With the help of (15) and (17) each part can easily be estimated as follows:
\[
\|b_1w_0\|_2 \lesssim e^{\|xk'|\sigma} \left( \frac{1}{k} + \frac{|k'|}{k^2} \right) \|h\|_2,
\]
\[
\|b_2w_0\|_2 \lesssim e^{\|xk'|\sigma} \frac{\|g\|_2}{k} \|w_h\|_2 \lesssim \frac{1}{k} \|h\|_2.
\]
We could now add all three single estimates to get the desired estimate on $w$ but we wait until we have done the same thing for $v$.

3.3.3. Estimating $v$. Recall from (12) that $v = av_0 + v_h$. Recall the formula (13) for $a$. Note that
\[
(v_0w_0' - v_0'w_0h)(\sigma) = \frac{1}{k} \int_0^1 \sin(k(1-y))h(y)dy \quad \text{and} \quad v_g = \frac{(-w_0v_0 + w_0v_0')(\sigma)}{\det M} v_g =: v_{g,2} + v_{g,3}.
\]
Thus it seems to be natural to decompose
\[
a = \frac{1}{\det M} \left[ (v_0w_0' - v_0'w_0h) + w_0'v_g - w_0v_g' \right]_{x=\sigma} =: a_1 + a_2 + a_3.
\]
This in turn leads to a decomposition of $v = a_1v_0 + (a_2v_0 + v_{g,2}) + (a_3v_0 + v_{g,3})$ into three parts. Essentially it leaves to find a good representation of the second and the third part of $v$. First let us write
\[
a_{2}v_0 + v_{g,2} = \frac{w_0'(\sigma)}{k'\det M} \left( v_g(\sigma) \sin(k'x) - k'v_0(\sigma)v_g(x) \right), =: I(x)
\]
\[
a_3v_0 + v_{g,3} = \frac{w_0(\sigma)}{k'\det M} \left( -v_g'(\sigma) \sin(k'x) + k'v_0'(\sigma)v_g(x) \right). =: II(x)
\]
Simple calculations yield
\[
-2I(x) = \int_0^\sigma \cos(k'(\sigma - x - y))g(y)dy - \int_0^x \cos(k'(\sigma - x + y))g(y)dy
- \int_x^\sigma \cos(k'(\sigma + x - y))g(y)dy.
\]
and
\[ 2II(x) = \int_x^\sigma \sin(k'(\sigma + x - y))g(y)dy - \int_0^x \sin(k'(\sigma - x + y))g(y)dy - \int_0^\sigma \sin(k'(\sigma - x + y))g(y)dy. \]

Using this and again the elementary estimates (16) and (17) for \( w_0 \) and \( w_h \) we deduce
\[ \| a_3 v_0 + v_{g,3} \|_2 \lesssim \frac{e^{|3k'|\sigma}}{|k'\det M| k} \| g \|_2, \]
(19)
\[ \| a_2 v_0 + v_{g,2} \|_2 \lesssim \frac{e^{|3k'|\sigma}}{|k'\det M| \| g \|_2}, \| a_1 v_0 \|_2 \lesssim \frac{e^{|3k'|\sigma}}{|k'\det M| k} \| h \|_2. \]

3.3.4. Conclusion. Putting (18) and (19) together we get the desired inequality
\[ \| u_n \|_{L^2} \lesssim \left[ \frac{e^{|3k'|\sigma}}{|k'\det M|} \left( 1 + \frac{|k'|}{K^2} \right) + \frac{1}{k} \right] \| f_n \|_{L^2}. \]
(20)

3.4. Regimes where \( s \) can live. Keeping (20) in mind, our task is now to find asymptotic dependencies of \( k \) and \( k' \) on \( s \) and a lower bound for \( |k'\det M| \). A priori there is no unique asymptotic behavior of \( k = \sqrt{s^2 - (n\pi)^2} \) as \( s \) tends to infinity because of \( k' \)'s dependence on \( n \). To overcome this difficulty we introduce the following four regimes:

(i) \( c \leq k \leq cs^\frac{1}{2}, \) (ii) \( cs^\frac{1}{2} \leq k \leq Cs^\frac{1}{2}, \) (iii) \( Cs^\frac{1}{2} \leq k \leq cs, \) (iv) \( cs \leq k < s. \)

Recall from Section 2 that \( c \) (resp. \( C \)) means a small (resp. big) number. Both constants may be different in each regime. But by the convention made in section 2 we may assume that consecutive regimes overlap.

Since we want to investigate the asymptotics \( s \to \infty \) we always may assume \( s > s_0 \) for some sufficiently large number \( s_0 > 0. \)

3.4.1. Regime (i): \( c \leq k \leq cs^\frac{1}{2}. \) For sufficiently small \( c \) the first order Taylor expansion of the square root at 1 gives a good approximation of
\[ k' = \sqrt{2a_0 s^\frac{1}{2} e^{1+\frac{1}{2}}} \left( 1 + \frac{ik^2}{a_0 s} + O(k^4 s^{-2}) \right). \]

In particular \( \Im k' = -\sqrt{a_0 s^\frac{1}{2}}(1 + O(k^2 s^{-1})) \) tends with a polynomial rate to minus infinity as \( s \) tends to infinity. Therefore \( \cot(k'\sigma) = i + O(s^{-\infty}) \). Together with (15) this gives us the following useful formula for
\[ \det M = \frac{\sin(k'\sigma)}{k'} \left[ \cos(k(1 - \sigma)) + \frac{k'}{k}(i + O(s^{-\infty})) \sin(k(1 - \sigma)) \right]. \]
(21)

It is not difficult to see that the term within the brackets is bounded away from zero. Thus \( |k'\det M| \gtrsim \exp(|\Im k'| |\sigma|) \). From (20) now follows (recall also (7))
\[ \| u_n \|_{L^2} \lesssim \left( 1 + \frac{|k'|}{k^2} \right) \| f_n \|_{L^2} \lesssim s^\frac{1}{2} \| f_n \|_{L^2} \] uniformly in \( n \).
3.4.2. **Regime (ii):** \(cs^{\frac{1}{2}} \leq k \leq Cs^{\frac{1}{2}}\). Because of \(k'^2 = k^2 - 2ia_0\) we see that both \(\Re k'\) and \(-\Im k'\) are of order \(s^{\frac{1}{2}}\). Therefore (21) is valid also in this regime. Again the term within the brackets is bounded away from zero. Thus \(|k' \det M| \gtrsim \exp(|3k'| \sigma)\) and (20) imply
\[
\|u_n\|_{L^2} \lesssim \|f_n\|_{L^2} \quad \text{uniformly in } n.
\]

3.4.3. **Regime (iii):** \(Cs^{\frac{1}{2}} \leq k \leq cs\). Using first order Taylor expansion for the square root at 1 gives
\[
k' = k \left(1 - ia_0sk^{-2} + O(s^2k^{-4})\right).
\]
In particular: If we choose \(C\) big enough we can assume the ratio \(k'/k\) to be as close to 1 as we wish. Similarly: If we choose \(c\) small enough we may assume \(-\Im k'\) to be as large as we want. Therefore we may assume \(\cot(k'\sigma)\) to be as close to \(i\) as we wish. This means that the following variant of (21) is true for this regime
\[
det M = \frac{\sin(k'\sigma)}{k'} \left[\cos(k(1 - \sigma)) + (i + \varepsilon)\sin(k(1 - \sigma))\right],
\]
where \(\varepsilon \in \mathbb{C}\) is some error term with a magnitude as small as we wish. If we choose \(c\) and \(C\) such that \(|\varepsilon| \leq 1/2\) we see that the term within the brackets is bounded away from zero. Thus \(|k' \det M| \gtrsim \exp(|3k'| \sigma)\) and (20) imply
\[
\|u_n\|_{L^2} \lesssim \|f_n\|_{L^2} \quad \text{uniformly in } n.
\]

3.4.4. **Regime (iv):** \(cs \leq k < s\). As in the previous regime
\[
k' = k \left(1 - ia_0sk^{-2} + O(s^{-2})\right).
\]
In particular \(k'/k = 1 + O(s^{-1}) \to 1\) and \(\Im k' = -a_0sk^{-1} + O(s^{-1})\) is bounded away from 0, \(+\infty\) and \(-\infty\). Thus
\[
det M = \frac{1}{k'} \left[\sin(k'\sigma) \cos(k(1 - \sigma)) + \cos(k'\sigma) \sin(k(1 - \sigma))\right] + O(s^{-2})
\]
\[
= \frac{\sin(k + (k' - k)\sigma)}{k'} + O(s^{-2}).
\]
This implies that \(|k' \det M| \approx 1\). Thus from (20) we deduce
\[
\|u_n\|_{L^2} \lesssim \|f_n\|_{L^2} \quad \text{uniformly in } n.
\]

3.5. **Conclusion.** Let \(u_n\) solve \(P_n(s)u_n(x) = f_n(x)\), where \(P_n(s)\) is defined in (19). Section 3.4 together with Lemma 4 shows that the estimate \(\|u_n\|_{L^2} \lesssim s^{1/2}\|f_n\|_{L^2}\) holds uniformly for any \(n\). Therefore, Lemma 4 implies Theorem 1.

4. **Exact decay rate for the damped wave equation**

Now we want to prove Theorem 1. Therefore recall the definition of the energy \(E\) and the damped wave operator \(A\) from Section 1. Then [4, Theorem 2.4] together with [2, Proposition 1.3] restricted to our situation says in particular that for any \(\alpha > 0\)
\[
(22) \quad \sup_{\|U_0\|_{\mathcal{D}(A)} = 1} E(t, U_0)^{\frac{1}{2}} \approx t^{-\frac{1}{2}} \iff \|(is + A)^{-1}\| \approx s^\alpha.
\]
In [1, Proposition 2.4] it was shown in particular that
\[
(23) \quad \|(is + A)^{-1}\| \approx s^\alpha \iff \|P(s)^{-1}\|_{L^2 \to L^2} \approx s^{\alpha - 1}.
\]
Actually this equivalence is stated there with ‘≈’ replaced by ‘≲’. But the ‘≳’-version is included in [1, Lemma 4.6]. In the appendix of [1] Stéphane Nonnenmacher proved

**Proposition 6** (Nonnenmacher, 2014). *The spectrum of* $A$ *contains an infinite sequence* $(z_j)$ *with* $\Im z_j \to \infty$ *such that* $0 < \Re z_j \lesssim (\Im z_j)^{-3/2}$.

Actually he proved this theorem under periodic boundary conditions, but the proof applies also to Dirichlet or Neumann boundary conditions. Note that Proposition 6 together with (23) establishes the ‘≳’-inequality of Theorem 3.

Using (22) and (23) together with Theorem 3 yields Theorem 1.

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