ENTROPY OF DIFFEOMORPHISMS OF LINE

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ABSTRACT. For diffeomorphisms of line, we set up the identities between their length growth rate and their entropy. Then, we prove that there is $C^0$-open and $C^r$-dense subset $U$ of $\text{Diff}^r(\mathbb{R})$ with bounded first derivative, $r = 1, 2, \cdots, +\infty$, such that the entropy map with respect to strong $C^r$-topology is continuous on $U$; moreover, for any $f \in U$, if it is uniformly expanding or $h(f) = 0$, then the entropy map is locally constant at $f$.

Also, we construct two examples:
1. there exists open subset $U$ of $\text{Diff}^\infty(\mathbb{R})$ such that for any $f \in U$, the entropy map with respect to strong $C^\infty$-topology, is not locally constant at $f$.
2. there exists $f \in \text{Diff}^\infty(\mathbb{R})$ such that the entropy map with respect to strong $C^\infty$-topology, is neither lower semi-continuous nor upper semi-continuous at $f$.

1. Introduction. Topological entropy and metric entropy are very important invariants. In this paper, we only consider topological entropy. The entropy of diffeomorphisms induces the natural map from diffeomorphisms space to $\mathbb{R}$, which is said to be entropy map. By structural stability of hyperbolic diffeomorphisms of compact manifolds, the entropy map is locally constant. For partially hyperbolic diffeomorphisms of compact manifolds with one dimensional center bundles, the entropy map is continuous at the known examples and is conjectured that \cite{7}:

**Conjecture 1.** The topological entropy is continuous on the space of $C^1$ partially hyperbolic diffeomorphisms of compact manifolds with the dimension of the center equal to one.

During the visiting in University of Paris-Sud, J.Buzzi told us another related question that

**Question 1.** Is the topological entropy locally constant on a dense (necessarily open) subset of the space of $C^1$ partially hyperbolic diffeomorphisms with one-dimensional center (on compact manifolds)?

Our paper on the entropy of diffeomorphisms on line is motivated by these problems.

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For $C^1$ endomorphisms without degenerated critical points on compact interval, the entropy is continuous with $C^1$-topology [4, 5], and locally constant on its' $C^1$-dense and open subset satisfying the structural stability [1]. As we know, the dynamics on non-compact spaces are quite different with dynamics on compact spaces. For example, topological stability preserves entropy of diffeomorphisms of compact manifolds, but it is not valid for the case of non-compact manifolds. Though topological expanding diffeomorphisms on line are conjugated each other, they can have zero entropy, positive entropy, and infinity entropy.

Let $\text{Diff}_r^b(\mathbb{R})$ be the set of $C^r$-diffeomorphisms of line with bounded first derivative, $r = 1, 2, \cdots, +\infty$. Our results in this section are worked under the strong topology of $\text{Diff}_r^b(\mathbb{R})$.

**Theorem 1.1.** There exists open subset $U$ of $\text{Diff}_\infty^b(\mathbb{R})$ such that for any $f \in U$, the entropy map with respect to strong $C^\infty$-topology, is not locally constant at $f$.

**Theorem 1.2.** There exists $f \in \text{Diff}_\infty^b(\mathbb{R})$ such that the entropy map with respect to strong $C^\infty$-topology, is neither lower semi-continuous nor upper semi-continuous at $f$.

The first example shows that Buzzi's corresponding problem is not valid for diffeomorphisms on line. For generic diffeomorphisms on line, we still have the continuity of entropy map.

**Theorem 1.3.** There is $C^0$-open and $C^r$-dense subset $U$ of $\text{Diff}_r^b(\mathbb{R})$, $r = 1, 2, \cdots, +\infty$, such that the entropy map with respect to strong $C^0$-topology is continuous on $U$; moreover, for any $f \in U$ satisfying that it is uniformly expanding or $h(f) = 0$, the entropy map is locally constant at $f$.

By the above theorem and the corollary [2] we give the characteristic of the diffeomorphisms having robustly zero entropy. And by the above theorem, for any uniformly expanding diffeomorphism on line, the entropy map is locally constant at it. By the proof of theorem [4] we can construct non uniformly expanding diffeomorphism on line with positive entropy, where the entropy map is locally constant. So, it is naturally ask the problem:

**Question 2.** How is the union of diffeomorphisms on line where the entropy map is locally constant?

There is an interesting phenomea that “there is $C^1$-dense subset $U$ of diffeomorphisms space on compact manifold such that for any $f \in U$, the entropy map at $f$ is either robustly positive or robustly zero”, hidden in the Palis weak $C^1$-density conjecture [2]. In our case, we show that such phenomea is $C^r$-dense, $r = 1, 2, \cdots, +\infty$. It is a directed consequence of the above theorem on the continuity of entropy map.

**Theorem 1.4.** Let $U^*$ be the union of diffeomorphisms where the entropy map is robustly positive or robustly zero. Then $U^*$ is $C^r$-dense in $\text{Diff}_r^b(\mathbb{R})$, $r = 1, 2, \cdots, +\infty$.

This paper is organized as follows.

In section [2] we introduce length-growth rate of diffeomorphisms and set up the identity between length-growth and entropy.

In section [3] we show that the entropy map with respect to weak topology, is not continuous.

In section [4] we mainly introduce uniformly topologically expanding diffeomorphisms on positive or negative diffeomorphisms, and set up a simpler identity of entropy.
In section 5, we prove the continuity of entropy map and length-growth rate of generic diffeomorphisms on line.

In last two sections, we construction two examples of theorem 1.2 and theorem 1.1.

2. Entropy of diffeomorphisms on line. In this section, we review the definition of entropy of metric space defined in Bowen's way [8], and introduce the definition of the length growth rate of diffeomorphisms on line. Then we show that length-growth rate equals entropy for diffeomorphisms with positive entropy.

Let \( f \) be continuous endomorphism on metric space \( X \), take compact subset \( K \subset X \). For any \( \varepsilon > 0 \) and integer \( n > 0 \), a set \( A \) is said to be \((n, \varepsilon)\)-separated if for any \( x, y \in A \), there exists \( i \in [0, n) \) such that \( d(f^i x, f^i y) \geq \varepsilon \). Let \( s(n, \varepsilon, K) \) be the \((n, \varepsilon)\)-separated set contained in \( K \) with maximal number. Define

\[
  h(f, \varepsilon, K) = \limsup_{n \to +\infty} \frac{\ln \# s(n, \varepsilon, K)}{n},
\]

and

\[
  h(f, K) = \lim_{\varepsilon \to 0} h(f, \varepsilon, K).
\]

The topological entropy of \( f \) is defined as

\[
  h(f) = \sup_{K \subset X} h(f, K).
\]

Through the paper, \( I \) denotes some interval of \( \mathbb{R} \), and \( \ell(I) \) denotes the length of \( I \).

For any bounded interval \( I \), the length growth rate of \( f \) on \( I \) is defined as

\[
  l(f, I) = \limsup_{n \to \infty} \frac{\ln \ell(f^n(I))}{n}.
\]

For general interval \( J \), take over bounded interval \( I \subset J \), the length growth rate of \( f \) on \( J \) is defined as

\[
  l(f, J) = \sup_{I \subset J} \limsup_{n \to \infty} \frac{\ln \ell(f^n(I))}{n}.
\]

In particular \( l(f, \mathbb{R}) \) is simplified as \( l(f) \).

**Theorem 2.1.** Let \( f \) be diffeomorphism on \( \mathbb{R} \) and \( J \) be an interval. If \( l(f, J) \geq 0 \), then \( h(f, J) = l(f, J) \). And if \( l(f, J) < 0 \), then \( h(f, J) = 0 \).

**Remark 1.** For diffeomorphisms on compact manifolds, the volume growth rate is study by Yomdin and Newhouse to pursue the continuity of entropy [6, 9]. The above identity has the same flavor with Misiurewicz-Szlenk's identity on the entropy of continuous piecewise monotone maps of the interval [5].

**Proof.** Firstly, we show that for any bounded interval \( I \), any integer \( n > 0 \) and any \( \varepsilon > 0 \), we have the inequality that

\[
  \frac{\ell(f^{n-1}I)}{\varepsilon} \leq \# s(n, \varepsilon, I) \leq \sum_{i=0}^{n-1} \frac{\ell(f^i I)}{\varepsilon} + 1.
\]

Obviously, there exists \( \left[ \frac{\ell(f^{n-1}I)}{\varepsilon} \right] + 1 \) points such that the distance between \( f^{(n-1)} \) iteration of these points each other is not smaller than \( \varepsilon \). Then by the definition of \((n, \varepsilon)\)-separated set, we have the left part of the inequality.
Let \( x_1 < x_2 < \cdots < x_k \) be the \((n, \varepsilon)\)-separated set, and \( I_i = [x_i, x_{i+1}] \), \( i \in [1, k-1] \). By the definition of \((n, \varepsilon)\)-separated set, we have that for any \( i \in [1, k-1] \),
\[
\sum_{j=0}^{n-1} \ell(f^j(I_i)) > \max\{\ell(f^j(I_i)) : j \in [0, n-1]\} \geq \varepsilon.
\]

Then,
\[
\sum_{j=0}^{n-1} \ell(f^j(I)) > \sum_{i=1}^{k-1} \sum_{j=0}^{n-1} \ell(f^j(I_i)) > (k-1)\varepsilon.
\]

So, we have the right part of the inequality.

By the definition of \( l(f, J) \) and \( h(f, J) \), and the left part of the above inequality, we have that \( l(f, J) < h(f, J) \).

Fix any bounded interval \( I \subset J \). By the definitions of \( l(f, J) \), we have that for any \( \delta > 0 \), there exists \( N_\delta \) such that for any \( n \geq N_\delta \), the following inequality satisfies
\[
\frac{\ln(\ell(f^n I))}{n} < l(f, J) + \delta.
\]

Let \( B_\delta(I) = \sum_{i=0}^{N_\delta} \ell(f^i I) \). Then,
\[
\sum_{i=0}^{n-1} \ell(f^i I) < B_\delta(I) + \sum_{i=0}^{n-1} e^{i(l(f, J) + \delta)} = B_\delta(I) + \frac{e^{n(l(f, J) + \delta)} - 1}{e^{l(f, J) + \delta} - 1}.
\]

By the right part of the inequality about \( \#s(n, \varepsilon, I) \), we have that
\[
\#s(n, \varepsilon, I) < 1 + \frac{B_\delta(I) + \frac{e^{n(l(f, J) + \delta)} - 1}{e^{l(f, J) + \delta} - 1}}{\varepsilon}.
\]

So, if \( l(f, J) < 0 \), \( h(f, J) = 0 \). And if \( l(f, J) \geq 0 \), \( h(f, J) \leq l(f, J) + \delta \) for any \( \delta > 0 \). Together with the above inequality \( l(f, J) \leq h(f, J) \), we have that \( h(f, J) = l(f, J) \) if \( l(f, J) \geq 0 \).

\[\square\]

**Remark 2.** In the above proof, we have the more accurate equality that for any \( \varepsilon > 0 \) and any bounded interval \( I \),
\[
h(f, \varepsilon, I) = h(f, I) = \max\{l(f, I), 0\}.
\]

Then, \( f \) can be said to be entropy-expansive.

3. **Entropy map on strong topology and weak topology.** As we know, \( \text{Diff}^r(\mathbb{R}) \) has weak topology and strong topology.

The strong \( C^r \)-topology, \( r = 0, 1, 2, \ldots \), is induced by
\[
d^r_s(f, g) = \sup\{|f^{(i)}(x) - g^{(i)}(x)|, 0 \leq i \leq r, x \in \mathbb{R}\}.
\]

And the strong topology on \( \text{Diff}^\infty(\mathbb{R}) \) is the union of the topologies induced by \( \text{Diff}^\infty(\mathbb{R}) \rightarrow \text{Diff}^r(\mathbb{R}) \) for \( r \) finite.

For any compact subset \( K \subset \mathbb{R} \), define
\[
d^r_K(f, g) = \max\{|f^{(i)}(x) - g^{(i)}(x)|, 0 \leq i \leq r, x \in K\}.
\]

Take compact subsets \( K_n \subset \mathbb{R} \) such that \( \bigcup K_n = \mathbb{R} \). The weak \( C^r \)-topology is induced by the classic metric
\[
d^r_w(f, g) = \sum_{i=1}^{\infty} 2^{-n} \frac{d^r_{K_n}(f, g)}{1 + d^r_{K_n}(f, g)}.
\]
Note that weak topology does not control the infinity of $\mathbb{R}$. While the entropy concerns the infinity of $\mathbb{R}$. So, we have that entropy map on weak topology, is not continuous.

**Proposition 1.** For any $f \in \text{Diff}^r(\mathbb{R})$ with positive entropy, $r \geq 0$, the entropy map is neither upper semi-continuous nor lower semi-continuous at $f$ on the weak $C^r$-topology of $\text{Diff}^r(\mathbb{R})$.

It can easily deduced by the identity between entropy and length growth rate, and the following basic lemma. Also the proof of the lemma is quite basic. So, we omit these proof.

**Lemma 3.1.** Fix any $\lambda > 0$. For any $f \in \text{Diff}^r(\mathbb{R})$, $r \geq 0$ and any $\epsilon > 0$, there exists $g \in \text{Diff}^r(\mathbb{R})$ such that $d_w^r(f, g) < \epsilon$ and $g(x) = \lambda x$ outside some bounded interval.

In the rest part of the paper, we only consider entropy map on the strong topology of $\text{Diff}^r(\mathbb{R})$.

4. **Topologically expanding and topologically contracting maps.** In this section, we introduce the definitions of the uniformly topologically expanding or contracting diffeomorphisms in positive or negative orientation. For these diffeomorphisms, we give the simpler identity of entropy which will be used in the following sections.

**Definition 4.1.** An orientation-preserving diffeomorphism $f$ is said to be uniformly topologically expanding in positive orientation if there exists $K > 0$ and $p$ such that for any $x \geq p$, $f(x) - x > K$.

**Definition 4.2.** An orientation-preserving diffeomorphism $f$ is said to be uniformly topologically expanding in negative orientation if there exists $K > 0$ and $p$ such that for any $x \leq p$, $x - f(x) > K$.

**Proposition 2.** For any uniformly topologically expanding diffeomorphism $f$ in positive orientation, any $p$ that is satisfied in the above definition, and any $a \geq p$, $h(f, [p, +\infty)) = l(f, [p, +\infty)) = \limsup_{n \to \infty} \frac{\ln(|f^n(a)|)}{n}$.

**Proof.** For any bounded interval $I \subset [p, +\infty)$ and any $a \geq p$, there exists positive integers $i, j$ such that $f^i(I) \subset [a, f^j(a)]$. Note that for any $n > 0$, $\ell(f^{n+i})(I) \leq \ell(f^n([a, f^j(a)])) = f^{n+j}(a) - f^n(a)$.

Then, $l(f, [p, +\infty)) \leq \limsup_{n \to \infty} \frac{\ln(|f^n(a)|)}{n}$.

On the other side, by the definition of length growth rate, we have that for any $\epsilon > 0$, there exists $N$ such that for any $n \geq N$, $\ln(f^{n+1}(a) - f^n(a)) < l(f, [p, +\infty)) + \epsilon$.

Note that $l(f, [p, +\infty)) \geq 0$. Then, for any $n > N$, $f^n(a) \leq f^N(a) + e^{(n+1)(l(f, [p, +\infty)) + \epsilon)} / (e^{l(f, [p, +\infty)) + \epsilon} - 1)$.
And then, \( \limsup_{n \to \infty} \frac{\ln(|f^n(a)|)}{n} \leq l(f, [p, +\infty)) + \varepsilon \). So, we have the identity

\[
l(f, [p, +\infty)) = \limsup_{n \to \infty} \frac{\ln(|f^n(a)|)}{n}.
\]

And then by theorem 2.1, we have that \( h(f, [p, +\infty)) = l(f, [p, +\infty)) \). \( \square \)

Let \( \varphi(x) = -x \). For any uniformly topologically expanding diffeomorphism \( f \) in negative orientation, \( \varphi^{-1} f \varphi \) is uniformly topologically expanding in positive orientation, and \( l(f, (-\infty, p)) = l(\varphi^{-1} f \varphi, [-p, +\infty)) \). Therefore, by the above proposition, we can get the similar results for uniformly topologically expanding diffeomorphism in negative orientation.

**Proposition 3.** For any uniformly topologically expanding diffeomorphism \( f \) in negative orientation, any \( p \) that is satisfied in the above definition, and any \( a \leq p \),

\[
h(f, (-\infty, p)) = l(f, (-\infty, p)) = \limsup_{n \to \infty} \frac{\ln(|f^n(a)|)}{n}.
\]

Since both \( l(f, [p, +\infty)) \) and \( l(f, (-\infty, p_-]) \) don’t depend on the choice of \( p_+ \) and \( p_- \) in these two definitions, we let \( l^+(f) = l(f, [p_+, +\infty)) \) and \( l^-(f) = l(f, (-\infty, p_-]) \).

**Remark 3.** If we replace \( K \) by 0 in the above two definitions, these maps are said to be topologically expanding in positive or negative orientation. The above two propositions are also valid for these more general maps.

**Definition 4.3.** An orientation-preserving diffeomorphism \( f \) is said to be positively uniformly contracting if there exists \( K > 0 \) and sequence \( x_n \to +\infty \) such that \( x_n - f(x_n) > K \).

**Definition 4.4.** An orientation-preserving diffeomorphism \( f \) is said to be negatively uniformly contracting if there exists \( K > 0 \) and sequence \( x_n \to -\infty \) such that \( f(x_n) - x_n > K \).

For simplicity, let \( \mathcal{C}^+ \) be the set of uniformly contracting diffeomorphisms in positive orientation contained in \( \text{Diff}^1_b(\mathbb{R}) \), \( \mathcal{E}^+_\mathcal{C} \) be the set of uniformly expanding diffeomorphisms in positive orientation contained in \( \text{Diff}^1_b(\mathbb{R}) \), \( \mathcal{C}^- \) be the set of uniformly contracting diffeomorphisms in negative orientation contained in \( \text{Diff}^1_b(\mathbb{R}) \), and \( \mathcal{E}^- \) be the set of uniformly expanding diffeomorphisms in negative orientation contained in \( \text{Diff}^1_b(\mathbb{R}) \). Obviously, these four sets are open subsets of \( \text{Diff}^1(\mathbb{R}) \).

**Definition 4.5.** A diffeomorphism \( f \) on \( \mathbb{R} \) is said to be uniformly expanding if there exists positive integer \( n \) and \( \lambda > 1 \) such that for any \( x \in \mathbb{R} \), \( |(f^n)'(x)| > \lambda^n \).

In the end, we give some examples on these definitions.

- Take \( |\lambda| > 1 \). \( f(x) = \lambda x \) is uniformly expanding. If \( \lambda > 1 \), it is contained in \( \mathcal{C}^- \cap \mathcal{E}^+ \).
- Take \( \alpha > 0 \). \( f(x) = x + \alpha \) is contained in \( \mathcal{E}^- \cap \mathcal{C}^+ \).
- Take \( \alpha > 0 \). \( f(x) = x - \alpha \) is contained in \( \mathcal{C}^- \cap \mathcal{E}^+ \).

5. The continuity of length growth rate and entropy map. In this section, we prove the continuity of the length growth rate on \( \mathcal{E}^+ \cup \mathcal{E}^- \). Then by the identity of entropy in above section, we prove the continuity of the entropy map on \( (\mathcal{E}^+ \cup \mathcal{C}^+ ) \cap (\mathcal{E}^- \cup \mathcal{C}^-) \). Moreover, we show that continuity of the entropy map is a generic property on \( \text{Diff}^1_b(\mathbb{R}) \).
Proposition 4. \( l^+() \) with respect to strong \( C^0 \)-topology, is continuous on \( \mathcal{E}^+ \). Moreover, for any \( f \in \mathcal{E}^+ \), if it is uniformly expanding or \( l^+(f) = 0 \), then \( l^+() \) is locally constant around \( f \). It has the same properties on \( \mathcal{E}^- \) and \( l^-() \).

Proof. For any \( f \in \mathcal{E}^+ \), there exists \( p, \lambda > 1, K > 0 \) such that for any \( \varepsilon > 0 \) and any \( g \in \mathcal{E}^+ \) with \( d_0(f, g) = \varepsilon \), we have that

- for any \( x \geq p, \min\{f(x) - x, g(x) - x\} > K \),
- for any interval \( I \) with \( \ell(I) > \varepsilon \), \( \min\{\ell(f(I)), \ell(g(I))\} < \lambda \ell(I) \).

Fix \( f \) and \( g \). By the monotonicity of \( f \), for any positive integer \( n \), there exists \( m_n \) such that \( f^{m_n}(p) \leq g^n(p) < f^{m_n+1}(p) \). Then,

\[
g^{n+1}(p) = f(f^{m_n+1}(p)) + \varepsilon.
\]

Take \( t_\varepsilon = \lfloor n\lambda(\frac{K(\lambda-1)}{\varepsilon} + 1) \rfloor \). By induction, it is not difficult to show that for any \( 0 \leq i \leq t_\varepsilon - 1 \), we have that

\[
g^{n+1+i}(p) < f^{m_n+2+i}(p) + \sum_{j=0}^{i} \lambda^j \varepsilon < f^{m_n+2+i}(p) + K < f^{m_n+3+i}(p).
\]

Then, \( m_{n+1+i} \leq m_n + 2 + i \) for any \( 0 \leq i \leq t_\varepsilon - 1 \). Then, for any \( n > t_\varepsilon \),

\[
\frac{m_n}{n} < \frac{t_\varepsilon + 3}{t_\varepsilon + 1}.
\]

Then,

\[
l^+(g) = \limsup_{n \to +\infty} \frac{\ln(|g^n(p)|)}{n} \leq \limsup_{n \to +\infty} \frac{\ln(|f^{m_n+1}(p)|)}{m_n + 1} \leq l^+(f) \frac{t_\varepsilon + 3}{t_\varepsilon + 1}.
\]

Symmetrically, we have that \( l^-(f) \leq \frac{t_\varepsilon + 3}{t_\varepsilon + 1} l^+(g) \). By \( \lim_{\varepsilon \to 0} t_\varepsilon = +\infty \), we can get the continuity of \( l^+() \) on \( \mathcal{E}^+ \). Particularly when \( l^+(f) = 0 \), it is locally zero at \( f \).

Particularly if \( f \) is uniformly expanding, let \( p' = \frac{p + f(p)}{2} \). For any small perturbation \( g \) of \( f \), we have that \( f^n(p) < g^n(p') < f^{n+1}(p) \) for any \( n > 0 \). By the identity in Proposition 2, we have that \( l^+() \) is locally constant at \( f \).

Let \( \varphi(x) = -x \). For any \( f \in \mathcal{E}^- \), \( \varphi^{-1} f \varphi \in \mathcal{E}^+ \), and \( l^-(f) = l^+(\varphi^{-1} f \varphi) \). Therefore, we can deduce the same result of \( \mathcal{E}^- \) and \( l^-() \), by the result of \( \mathcal{E}^+ \) and \( l^+() \).

Now we give the proof of Theorem 1.3

Proof. Fix \( r \). Let \( \mathcal{U} = \{ f \in \text{Diff}^r(\mathbb{R}) : f^2 \in (\mathcal{E}^+ \cup \mathcal{C}^+ \cap (\mathcal{E}^- \cup \mathcal{C}^-)) \} \). By the \( C^0 \)-openness of \((\mathcal{E}^+ \cup \mathcal{C}^+) \cap (\mathcal{E}^- \cup \mathcal{C}^-)) \), we get the \( C^0 \)-openness of \( \mathcal{U} \).

Let \( \mathcal{U}^+ = \{ f \in \text{Diff}^r(\mathbb{R}) : f^2 \in (\mathcal{E}^+ \cup \mathcal{C}^+) \} \), and \( \mathcal{U}^- = \{ f \in \text{Diff}^r(\mathbb{R}) : f^2 \in (\mathcal{E}^- \cup \mathcal{C}^-) \} \).

By the monotony of diffeomorphisms, we have that \((\mathcal{E}^+ \cup \mathcal{C}^+) \cap (\mathcal{E}^- \cup \mathcal{C}^-) \subset \mathcal{U} \).

For any \( C^r \) orientation-preserving diffeomorphism \( f \), suppose \( f \not\in \mathcal{E}^+ \). Then

\[
\liminf_{x \to +\infty} f(x) - x \leq 0,
\]

and \( f^2 \not\in \mathcal{E}^+ \). For any \( \varepsilon > 0 \), let \( g = f - \varepsilon \) be the perturbation of \( f \). Then,

\[
\liminf_{x \to +\infty} g(x) - x \leq -\varepsilon.
\]

So, \( \{g, g^2\} \subset \mathcal{C}^+ \).

Take \( \alpha(x) \) be the smooth function on line such that \( \alpha(x) = -1 \) for any \( x < -1 \), \( -1 \leq \alpha(x) \leq 1 \) for any \( -1 \leq x \leq 1 \), and \( \alpha(x) = 1 \) for any \( x > 1 \).
For any $C^r$-diffeomorphism $f$, suppose $f^2 \not\in \mathcal{E}^+$. Then,
\[
\liminf_{x \to +\infty} f^2(x) - x \leq 0.
\]
For any $\varepsilon > 0$, let $g = f + \varepsilon \alpha$ be the perturbation of $f$. Then,
\[
g^2(x) - x = f(f(x) + \varepsilon \alpha(x)) + \varepsilon \alpha(f(x) + \varepsilon \alpha(x)) - x
\]
\[
= f^2(x) - x + \varepsilon f'(\xi)\alpha(x) + \varepsilon \alpha(f(x) + \varepsilon \alpha(x)).
\]
It is not difficult to deduce that $\varepsilon f'(\xi)\alpha(x) + \varepsilon \alpha(f(x) + \varepsilon \alpha(x)) < -\varepsilon$ for big enough $x$. Then, $g^2 \in \mathcal{C}^+$. So, \{\{f^2 : f \in \text{Diff}^r_0(\mathbb{R})\} \subset \overline{\mathcal{C}^+} \cup \mathcal{E}^+\}, and then $\mathcal{U}^+$ is $C^r$-dense subset of $\text{Diff}^r_0(\mathbb{R})$. Similarly, we have that \{\{f^2 : f \in \text{Diff}^r_0(\mathbb{R})\} \subset \overline{\mathcal{C}^-} \cup \mathcal{E}^-\} and then $\mathcal{U}^-$ is $C^r$-dense subset of $\text{Diff}^r_0(\mathbb{R})$. As we know $\text{Diff}^r(\mathbb{R})$ on strong $C^r$-topology, is a Baire space. So, $\mathcal{U} = \mathcal{U}^- \cap \mathcal{U}^+$ is $C^r$-dense subset of $\text{Diff}^r_0(\mathbb{R})$.

For any $f \in (\mathcal{E}^+ \cup \mathcal{C}^+) \cap (\mathcal{E}^- \cup \mathcal{C}^-)$, it has the following four cases:
\begin{itemize}
  \item $f \in \mathcal{E}^- \cap \mathcal{E}^+$, $h(f) = \max\{l^-(f), l^+(f)\}$.
  \item $f \in \mathcal{E}^- \cap \mathcal{C}^+$, $h(f) = l^+(f)$.
  \item $f \in \mathcal{C}^- \cap \mathcal{E}^+$, $h(f) = l^+(f)$.
  \item $f \in \mathcal{C}^- \cap \mathcal{C}^+$, $h(f) = 0$.
\end{itemize}

Then by proposition 4 and $h(f^2) = 2h(f)$, we get the same properties of entropy map on $\mathcal{U}$ with the properties of $l^+(\cdot)$ and $l^-(\cdot)$ on $\mathcal{E}^+ \cup \mathcal{C}^+ \cap \mathcal{E}^- \cup \mathcal{C}^-$.

**Remark 4.** For diffeomorphisms whose first derivatives are away from zero (can have unbounded first derivatives), the corresponding proposition 4 and theorem 1.3 are still valid. But for general diffeomorphisms on line, the proposition is not valid. To show the continuity of their entropy map and length-growth rate, we need additional condition 3.

In the above proof, we show that

**Corollary 1.** $\overline{\mathcal{C}^+} \cup \mathcal{E}^+ = \overline{\mathcal{C}^-} \cup \mathcal{E}^-$ is the set of orientation-preserving diffeomorphisms on line with bounded first derivatives.

**Corollary 2.** The set \{\{f \in \text{Diff}^r_0(\mathbb{R}) : f^2 \in (\mathcal{E}^+ \cup \mathcal{C}^+) \cap (\mathcal{E}^- \cup \mathcal{C}^-) \& h(f) = 0\}\}, is a $C^0$-open subset of $\text{Diff}^r_0(\mathbb{R})$, and $C^r$-dense subset of \{\{f \in \text{Diff}^r_0(\mathbb{R}) : h(f) = 0\}\}, $r = 1, 2, \cdots, \infty$.

**Proof.** The $C^0$-openness of the subset
\[
\{f \in \text{Diff}^r_0(\mathbb{R}) : f^2 \in (\mathcal{E}^+ \cup \mathcal{C}^+) \cap (\mathcal{E}^- \cup \mathcal{C}^-) \& h(f) = 0\}
\]
is the directed property of the above theorem.

Fix $r$. By the corollary [4] we have that
\[
\{f^2 : f \in \text{Diff}^r_0(\mathbb{R})\} \subset (\overline{\mathcal{C}^+} \cup \mathcal{E}^+) \cap (\overline{\mathcal{C}^-} \cup \mathcal{E}^-).
\]
Then for any diffeomorphism $f \in \text{Diff}^r_0(\mathbb{R})$, $f^2$ is contained in $\overline{\mathcal{C}^+} \cap \mathcal{C}^- \text{ or } \overline{\mathcal{C}^+} \cap \mathcal{E}^-$ or $\overline{\mathcal{E}^+} \cap \mathcal{E}^-$ or $\overline{\mathcal{C}^+} \cap \mathcal{C}^-$. Note that for any $g \in \mathcal{C}^+ \cup \mathcal{C}^-$, the corresponding length growth rate is zero. Then by proposition 4 we have that if $h(f) = 0$, there exists $g$ arbitrarily $C^r$-closed to $f$ such that $g$ has robustly zero entropy and $g^2 \in (\mathcal{E}^+ \cup \mathcal{C}^+) \cap (\mathcal{E}^- \cup \mathcal{C}^-)$.
\[\square\]
6. **Proof of Theorem 1.2** In this section, we construct diffeomorphism where the entropy map is neither lower semi-continuous nor upper semi-continuous. At first, we give a sensitive diffeomorphism on interval, which has a non-hyperbolic fixed point.

**Lemma 6.1.** Let \( \alpha' \) be \( C^\infty \) increasing function on \([0, 10]\) such that \( 0.1 \leq \alpha'(x) \leq 10 \) and
\[
\begin{align*}
\alpha'(x) = 0.1 & \quad 0 \leq x \leq 1.2 \\
\alpha'(x) = 10 & \quad 1.9 \leq x \leq 10
\end{align*}
\]
And let
\[
\alpha_{w_0}(x) = \int_0^x \alpha'(s)ds + w.
\]
Let \( w_0 = \inf \{w : \sup_{n>0}\{\alpha_{w_n}^n(0)\} > 1.9\} \). Then, \( w_0 > 1 \) and \( \min\{\alpha_{w_n}(x) - x : x \in [0, 10]\} = 0 \).

And for any \( 0.1 > \varepsilon > 0 \), \( \alpha_{w_n+\varepsilon}(0) > 1.9 \).

**Proof.** Note that the curves \( \alpha_{w_n} \) are convex. Then, \( w_0 \) is the value such that the curve \( \alpha_{w_n} \) is tangent with the line \( y = x \). The other properties are quite trivial. \( \square \)

Before giving the example of Theorem 1.2 we talk about the main idea of the construction. It is constructed by induction. In its n-th “periodic”, it is uniformly expanding, then uniformly contracting, and behaves the same dynamics with some construction. It is constructed by induction. In its n-th “periodic”, it is uniformly expanding, then uniformly contracting, and behaves the same dynamics with some \( \alpha_{w_n} \) (\( w_n > w_0 \)) in the end, besides very few connecting iterations. By the above lemma, the iteration number of the dynamics \( \alpha_{w_n} \) can become suddenly smaller by small perturbation. This makes the entropy suddenly become bigger.

**Proof.** By lemma 6.1 we can take a sequence \( 1 < w_n < 10 \) such that for any \( n \geq 1 \), we have that
\[
\alpha_{w_n}^n(0) = 1.9, \alpha_{w_n}^{n+1}(0) < 10
\]
Take \( \beta(x) \) to be the smooth decreasing function on \([0, 1]\) such that
\[
\begin{align*}
\beta(x) = 10 & \quad x \in [0, 0.1] \\
0.1 < \beta(x) < 10 & \quad x \in (0.1, 0.9) \\
\beta(x) = 0.1 & \quad x \in [0.9, 1].
\end{align*}
\]

The example \( f \) satisfying \( 0.1 \leq f' \leq 10 \), is topologically contracting in negative orientation, and topologically expanding on \([0, \infty)\), which is constructed by induction. Let \( f^{-1}(0) = -10 \) and \( f'(x) = 10, x \in [f^{-1}(0), f^{2^{-1}}(0)] \). Then
\[
f^{2^2}(0) - f^{2^{-1}}(0) = 10f^0(0) - f^{-1}(0).
\]
Note that \( f^{2^2}(0) - f^{2^{-1}}(0) > 11 \). Then there exists \( f^{2^{-1}}(0) < a_1 < f^{2^2}(0) - 1 \) such that
\[
\begin{align*}
f'(x) = 10 & \quad f^{2^{-1}}(0) \leq x \leq a_1 \\
f'(x) = \beta(x - a_1) & \quad a_1 \leq x \leq a_1 + 1 \\
f'(x) = 0.1 & \quad a_1 + 1 \leq x \leq f^{2^2}(0)
\end{align*}
\]
and
\[
f^{2^2+1}(0) - f^{2^2}(0) = 10^{k_1}w_1,
\]
here \( k_1 \) is positive integer.

And let
\[
f'(x) = 0.1, x \in [f^{2^2}(0), f^{2^2+k_1}(0)],
\]
Then
\[ f^{2^i+k_i+1}(0) - f^{2^i+k_i}(0) = w_1. \]
Let
\[ f'(x) = \alpha'(x - f^{2^i+k_i}(0)), x \in [f^{2^i+k_i}(0), f^{2^i+k_i}(0) + 10]. \]
Then,
\[ f(x) - f^{2^i+k_i}(0) = \alpha w_1(x - f^{2^i+k_i}(0)), x \in [f^{2^i+k_i}(0), f^{2^i+k_i}(0) + 10]. \]
Let \( t_0 = T_0 = 0, t_1 = 2^2 + k_1 + 1, \) and \( T_1 = T_0 + t_1. \) Let \( n = 2. \) Then for any \( 1 \leq i < n, \) we have that
1. \( f'(x) = 10, \) for any \( x \in [f^{T_{i-1}}(0), f^{T_{i-1}+2^i}(0)] \)
2. \( f'(x) = 0.1, \) for any \( x \in [f^{T_{i-1}+2^i}(0), f^{T_{i-1}+2^i+k_1}(0)], \)
3. for any \( x \in [f^{T_{i-1}+2^i+k_1}(0), f^{T_{i-1}+2^i+k_1}(0) + 10], \)
\[ f(x) - f^{T_{i-1}+2^i+k_1}(0) = \alpha w_1(x - f^{T_{i-1}+2^i+k_1}(0)). \]
Suppose for any \( i < n, \) \( f(x) \) satisfies the above properties. Then we construct \( f'(x), x \in [f^{T_{i-1}}(0), f^{T_{i}}(0)], \) which is almost the copy of \( f'(x), x \in [0, f^{T_{i}}(0)]. \) (We replace \( 2^1 \) by \( 2^n, 2^2 \) by \( 2^n+1 \), \( \alpha_1 \) by \( a_n, w_1 \) by \( w_n \), and \( k_1 \) by \( k_n \) respectively. If it is the number of iterations of \( f, \) it should plus the number \( T_{n-1}. \))
Let \( f'(x) = 10, x \in [f^{T_{n-1}}(0), f^{T_{n-1}+2^n}(0)]. \) Then
\[ f^{T_{n-1}+2^n}(0) - f^{T_{n-1}+2^n-k_1}(0) = 10^{2^n} (f^{T_{n-1}}(0) - f^{T_{n-1}-1}(0)). \]
Note that
\[ f^{T_{n-1}+2^n}(0) - f^{T_{n-1}+2^n-k_1}(0) > 10^{2^n} (f^{T_{n-1}}(0) - f^{T_{n-1}-1}(0)), \]
Then there exists \( f^{T_{n-1}+2^n}(0) - f^{T_{n-1}+2^n-k_1}(0) = 10^{2^n} (f^{T_{n-1}}(0) - f^{T_{n-1}-1}(0)). \)
Then for any \( i < n, \) \( f(x) \) satisfies the above three properties. Then
\[ f'(x) = 0.1, x \in [f^{T_{n-1}+2^n}(0), f^{T_{n-1}+2^n+k_n}(0)], \]
Then
\[ f^{T_{n-1}+2^n+k_n+1}(0) - f^{T_{n-1}+2^n+k_n}(0) = w_n. \]
And let
\[ f'(x) = \alpha'(x - f^{T_{n-1}+2^n+k_n}(0)), x \in [f^{T_{n-1}+2^n+k_n}(0), f^{T_{n-1}+2^n+k_n}(0) + 10]. \]
Then,
\[ f(x) - f^{T_{n-1}+2^n+k_n}(0) = \alpha w_n(x - f^{T_{n-1}+2^n+k_n}(0)). \]
Let \( t_n = 2^{n+1} + k_n + 1, \) and \( T_n = t_n - T_{n-1}. \)
Then for any \( i > 0, f(x) \) satisfies the above three properties.
Here is the graph of the first derivative of \( f \) in its n-th “periodic”:
Now we show that \( f \) has positive entropy. For any \( n \geq 0, \) by the third property of \( f, \) and the choice of \( \alpha w_n, \) we have that \( f^{T_{n-1}+2^n+k_n+1}(0) - f^{T_{n-1}+2^n+k_n}(0) = w_n > 1, \) and then
\[ 10^{-2^n} < f^{T_{n}}(0) - f^{T_{n}-1}(0) < 10. \]
Then, \( f^{T_n+2^{n+1}}(0) - f^{T_n+2^{n+1}-1}(0) > 10^{2^n} \), and \( f^{T_n+2^{n+1}+1}(0) - f^{T_n+2^{n+1}}(0) = 10^{k_{n+1}} w_{n+1} < 10^{2^{n+1}+2} \).

By \( w_{n+1} > 1 \), we have that \( k_{n+1} < 2^{n+1} + 2, t_{n+1} < 2^{n+3} \) and \( T_{n+1} < 2^{n+4} \). Then,

\[
h(f) = l^+(f) \geq \limsup_{n \to \infty} \frac{\ln f^{T_n+2^{n+1}}(0)}{T_n + 2^{n+1}} > \frac{\ln 10}{18} > 0.
\]

Note that \( f \not\in \mathcal{L}^+ \). Then by the corollary 1, we have that there is arbitrarily small perturbation \( g \) of \( f \) such that \( h(g) = 0 \). Then, the entropy map at \( f \) is not lower semi-continuous.

In the end, we check that the entropy map at \( f \) is not upper semi-continuous. For any \( 0.1 > \varepsilon > 0 \), let \( g(x) = f(x) + \varepsilon \) be the small perturbation of \( f \). For any \( n > 0 \), by the third property of \( f \), the choice of \( \alpha_{w_n} \), and lemma 6.1 we have that,

\[
g^{[\frac{3}{2}] + 1}(f^{T_{n-1}+2^n+k_n}(0)) > f(f^{T_{n-1}+2^n+k_n}(0) + 1.9) = f^{T_n}(f^{T_{n-1}}(0)).
\]

Let \( t_n(g) = 2^n + k_n + [\frac{3}{2}] + 1, T_0(g) = 0 \) and \( T_{n+1}(g) = T_n(g) + t_n(g) \) for any \( n \geq 0 \). Note that \( g \geq f \). Then for any \( n > 0 \),

\[
g^{T_n}(f^{T_n-1}(0)) > f^{T_n}(f^{T_{n-1}}(0)).
\]

Then by induction, we can deduce that for any \( n > 0 \),

\[
g^{T_n}(0) > f^{T_n}(0).
\]

Then for any \( m > 0 \), let \( m = T_n + i, 0 \leq i < t_{n+1} \), we have that

\[
h(f) = l^+(f) = \limsup_{m \to \infty} \frac{\ln f^m(0)}{m} \leq \limsup_{m \to \infty} \frac{\ln g^{T_n+1}(0)}{T_n + i}
\]

\[
= \limsup_{m \to \infty} \frac{\ln g^{T_n+1}(0)}{T_n + i} \frac{T_n(g) + i}{T_n + i} \leq h(g) \limsup_{n \to \infty} \frac{T_n(g) + i}{T_n + i} \leq \frac{13h(g)}{15}
\]

Remark 5. Similarly, we can construct \( f \) with zero-entropy such that the entropy map at \( f \) is not upper semi-continuous. The construction is very similar with the above example. By lemma 6.1 we can take a sequence \( 1 < w_n < 10 \) such that for any \( n \geq 1 \), we have that \( \alpha_{w_n}^{1n}(0) = 1.9 \). Here is the graph of the first-order derivative of the example \( f \) in its n-th “periodic”:
7. Proof of Theorem 1.1. In this section, we will give open subset $\mathcal{U}$ of $\text{Diff}_0^r(\mathbb{R})$ such that for any $f \in \mathcal{U}$, the entropy map is not locally constant at $f$. The entropy map is continuous at these diffeomorphisms. It is also constructed by induction. In its $n$-th “periodic”, it is uniformly expanding, then uniformly contracting, and is the translation in the end, besides very few connecting iterations. It is not difficult to deduce that, the iteration number of uniformly expanding and uniformly contracting is almost not changed by small perturbation. But the iteration number of the translation is changed by small perturbation. Fortunately, we can accurately estimate the number.

**Proof.** The example $f$ is topologically contracting in negative orientation. Let $T_n = 3(2^{n+1} - 2)$ for any $n \geq 0$. By induction, we construct the diffeomorphism $f$ satisfying the following properties that for any $n \geq 1$,

- $f'(x) = 10$ for any $x \in [fT_{n-1}(0), fT_{n-1}+2^n(0)]$,
- $f'(x) = 0.1$ for any $x \in [fT_{n-1}+2^n(0), fT_{n-1}+2^{n+1}(0)]$,
- $f(x) = x + 1$ for any $x \in [fT_{n-1}+2^n(0), fT_{n-1}+2^{n+1}(0)]$ and $fT_{n-1}+2^{n+1}(0) - fT_{n-1}+2^n(0) = 1$,
- for any $x > 0$, $f(x) - x > 0.5$.

Note that $fT_{n-1}+2^n(0) - fT_{n-1}+2^{n+1}(0) > 9(fT_{n-1}+2^n(0) - fT_{n-1}(0))$, and $fT_{n-1}+2^{n+1}(0) - fT_{n-1}+2^n(0) > 9(fT_{n-1}+2^n(0) - fT_{n-1}+2^{n+1}(0))$. Then, there is enough space for our construction such that $f$ satisfies the former part of the third property above.

The induction construction is basic and something similar with the above example. So, we omit details of the construction. Here is the graph of the first-order derivative of the example $f$ in its $n$-th “periodic”:

Let $\mathcal{U}$ be the small $C^\infty$-neighborhood of $f$ such that $d_{C^0}(f, g) < 0.01$ for any $g \in \mathcal{U}$. Fix $g \in \mathcal{U}$. Note that $g(x) - x > f(x) - x - 0.01 > 0.4$ for any $x > 0$. Then
by the uniformly expanding of $f$ on the interval $[f^{T_{n-1}}(0), f^{T_{n-1}+2^n-1}(0)]$, we have that for any $n > 0$,
\[ g^{2^n+1}(f^{T_{n-1}}(0)) > f^{2^n}(f^{T_{n-1}}(0)) > g^{2^n-1}(f^{T_{n-1}}(0)). \]
By the uniformly contracting of $f$ on the interval $[f^{T_{n-1}+2^n}(0), f^{T_{n-1}+2^{n+1}-1}(0)]$, we have that for any $n > 0$,
\[ g^{2^n}(f^{T_{n-1}+2^n}(0)) > f^{2^n-1}(f^{T_{n-1}+2^n}(0)) > g^{2^n-2}(f^{T_{n-1}+2^n}(0)). \]
Then
\[ g^{2^n+1+3}(f^{T_{n-1}}(0)) > f^{T_{n-1}+2^{n+1}}(0) > g^{2^n+1-3}(f^{T_{n-1}}(0)). \]
For any $x \in [f^{T_{n-1}+2^{n+1}}(0), f^{T_{n-1}+3*2^n-2}(0)] = [f^{T_{n-1}+2^{n+1}}(0), f^{T_{n-1}+2^{n+1}}(0) + 2^n - 2]$, $f(x) - x = 1$, and then $0.99 < g(x) - x < 1.01$. For any $n \geq 1$, let $a_n(g)$ be the number of iterations of $g$ in the interval $[f^{T_{n-1}+2^{n+1}}(0), f^{T_{n-1}+3*2^n-2}(0)]$
\[ a_n(g) = \max\{i : g^i(f^{T_{n-1}+2^{n+1}}(0)) \leq f^{T_{n-1}+3*2^n-2}(0)\}. \]
Then,
\[ \left[\frac{2^n - 2}{0.99}\right] \geq a_n(g) \geq \left[\frac{2^n - 2}{1.01}\right]. \]
For any $n \geq 1$, let
\[ t_n(g) = \max\{i : g^i(f^{T_{n-1}}(0)) \leq f^T(0)\}. \]
be the number of iterations of $g$ in the interval $[f^{T_{n-1}}(0), f^T(0)]$.
Then for any $n > 100$,
\[ 2^{n+1} < t_n(g) < (2^{n+1} + 3) + \left[\frac{2^n - 2}{0.99}\right] + 3 < 2^{n+2}. \]
Then,
\[ h(g) \geq \limsup_{n \to \infty} \frac{\ln 10^{2^n}}{\sum_{i=1}^{t_n(g)} t_i(g)} \geq \frac{\ln 10}{8} > 0. \]
For any $g \in U$, and let $g_\varepsilon(x) = g(x) + \varepsilon \in U$, $\varepsilon > 0$. Then
\[ a_n(g) - a_n (g_\varepsilon) \geq \left[\frac{\varepsilon}{1.01}\right] a_n (g_\varepsilon) \geq 2^{n-1} \varepsilon - 3\varepsilon - 1. \]
Then for big enough $n$, we have that
\[ \frac{t_n(g_\varepsilon)}{t_n(g)} < 1 - 10^{-1} \varepsilon. \]
Fix $g$ and $g_\varepsilon$. For any $n > 1$, there exists unique $m_n$ such that $g^n(0) \leq g_\varepsilon^m(0) < g^{n+1}(0)$. Note that for any $n > 100$ and any $g \in U$, $2^{n+1} < t_n(g) < 2^{n+2}$. Then for big enough $n$, by induction, it is not difficult to deduce that
\[ \frac{m_n}{n} < 1 - 10^{-2} \varepsilon. \]
Then by the identity of entropy, we have that
\[ h(g) = h^+(g) \leq \limsup_{n \to \infty} \frac{\ln g^n(0)}{n} \leq \limsup_{m \to \infty} \frac{\ln g_\varepsilon^m(0)}{m} \frac{m_n}{n} < (1 - 10^{-3}) h(g_\varepsilon). \]

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