On constructing boundaries for boundary value problems defined by continuous 2-D autonomous systems *

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Abstract: In this paper, we provide a constructive way of specifying initial/boundary data for a given continuous 2-D autonomous system described by a set of linear partial differential equations (PDEs) with real constant coefficients. One of the ways of specifying initial/boundary data is by specifying the values of various derivatives of the solution trajectories at the origin; the derivatives correspond to a standard monomial set obtained using Gröbner basis. However, such an initial/boundary data often lacks physical interpretation. In this paper, we consider subsets of the domain having some algebraic structure (in the form of subspaces and strips of finite width around such subspaces) such that trajectories restricted to these subsets, often called characteristic sets, serve as initial/boundary conditions for the given autonomous system of linear PDEs. We provide a systematic way to construct such characteristic sets with the help of Gröbner bases and Oberst-Riquier algorithm. Thus we bridge the gap between initial/boundary conditions involving standard monomials and more conventional initial/boundary conditions in the form of restrictions on characteristic sets. We also show that every scalar system of PDEs admits such a characteristic set given by a rectangular strip of finite width around a subspace whose dimension equals the Krull dimension of the system’s quotient ring.

Keywords: Distributed parameter systems, boundary control, linear control system.

1. INTRODUCTION

Solving a system of partial differential equations (PDEs) requires initial and/or boundary conditions. In this paper, we consider systems of PDEs with two independent variables having no inputs/free variables. Such systems are also called continuous 2-D autonomous systems in the literature. It is known from the literature that exponential solutions for a given continuous n-D autonomous system can be computed using the Oberst-Riquier algorithm (see (Pal and Pillai, 2014, Algorithm 22)). The Oberst-Riquier algorithm uses Gröbner bases for determining a standard monomial set (Cox et al. (2007)), and then, the initial/boundary data is provided by specifying the values at the origin of various derivatives – corresponding to these standard monomials – of the trajectory to be evaluated. However, such initial/boundary data, based on standard monomials, often lacks a physical interpretation, which might be of crucial importance for problems arising in engineering. This is because the shape of a standard monomial set is often arbitrary, depending on the given system of equations and the term ordering used for computing the requisite Gröbner basis.

A more useful initial/boundary data comes in the form of restrictions of trajectories to proper subsets of the domain having some algebraic structure. For example, such restrictions of trajectories to a subset of the domain play an important role in systems theory, namely, boundary control (Kristic and Smyshlyaev (2008)), causality (Fornasini and Marchesini (1976); Fornasini et al. (1993)), stability analysis (Pillai and Shankar (1998); Valcher (2000); Oberst (2006)), controller design (Shankar (2000)) etc. In this paper, we attempt to bridge this gap by considering subsets of the domain having some algebraic structure (in the form of subspaces and strips of finite width around such subspaces) such that trajectories restricted to these subsets serve as initial/boundary conditions for a given 2-D autonomous system. Such sets are known as characteristic sets in the literature. A characteristic set is a proper subset of the domain (here \( \mathbb{R}^n \)) with the defining property that trajectories restricted to this subset allows to uniquely extend the trajectory over the whole domain. We provide, in this paper, a systematic way to construct characteristic sets (of the form as mentioned above) for continuous scalar 2-D autonomous systems with the help of Gröbner bases and Oberst-Riquier algorithm.

The notion of characteristic sets was originally developed for discrete 2-D systems (that is, for systems of partial difference equations having two independent variables) in Valcher (2000), and was later extended to arbitrary dimensions in Mukherjee and Pal (2016, 2017, 2019). It was shown in Pal (2017) that every discrete autonomous 2-D system admits a characteristic set given by a union of finitely many parallel lines. This rightly generalizes the 1-D case where a characteristic set is always a collection of finitely many points of the domain (Willems (1991)). In this paper, we prove the continuous analogue for 2-D systems.
autonomous systems. It is important to note, here, that
the extension to the continuous case requires significant
developments over the discrete case. This is primarily
because of the difference in action of a discrete and a
continuous operator on a trajectory – in the discrete case
the operator action is a shift action while for a continuous
case the operator acts on a trajectory by differentiation.

An autonomous scalar system of PDEs having two in-
dependent variables can either be finite dimensional or
be infinite dimensional depending on the Krull dimension
of the system. It is known that in the finite dimensional
case, that is when the system has Krull dimension equal
to zero, a characteristic set for the system is a collection
of finitely many points of the domain of \( \mathbb{R}^2 \) (see Fornasini
et al. (1993)). For the case when the Krull dimension
of the system is equal to one, it is not clear what kind
of subsets of \( \mathbb{R}^2 \) can then qualify as a characteristic set.
We answer this question in this paper. We show that a
rectangular strip of finite width containing a 1-D subspace,
having the special property of being free with respect to
the system, is a characteristic set for the system. We also
show that every scalar 2-D autonomous system admits
a characteristic set given by a rectangular strip of
finite width around a subspace.

The paper is organized as follows. Section 2 discusses
the preliminaries and sets the notation to be used for the
rest of the paper. In Section 3 we characterize initial/boundary
data using the notion of characteristic sets. Section 4 shows
that every scalar system of PDEs admits a characteristic
set given by a rectangular strip of finite width containing
a free subspace.

2. NOTATION AND PRELIMINARIES

2.1 Notation

We use the symbols \( \mathbb{N} \) and \( \mathbb{R} \) to denote the set of
natural numbers and the field of real numbers, respectively.
We use the shorthand \( \partial_i \) to denote differentiation with
respect to the independent variable \( x_i \), that is \( \partial_i := \frac{\partial}{\partial x_i} \), for \( i = 1, 2 \).
The polynomial ring in two indeterminates is denoted by
\( \mathbb{R}[\partial] \), where \( \partial = (\partial_1, \partial_2) \). We use the symbol \( \langle \cdot \rangle \) to denote
the standard inner product in \( \mathbb{R}^2 \).

2.2 System Description

In this paper, we consider linear systems of partial differen-
tial equations (PDEs) with real constant coefficients
having one dependent variable (i.e., a scalar system) and
evolving over two independent variables. Consider the sys-
tem of PDEs given by

\[
\begin{bmatrix}
f_1(\partial) \\
f_2(\partial) \\
\vdots \\
f_r(\partial)
\end{bmatrix} w = 0, \quad (1)
\]

where \( f_i \in \mathbb{R}[\partial] \) for \( i \in \{1, 2, \ldots, r\} \). The solution to this
system of PDEs, \( w \), is called a trajectory. A trajectory \( w \) is
a real valued function from the domain \( \mathbb{R}^2 \) to \( \mathbb{R} \), that is
\( w : \mathbb{R}^2 \to \mathbb{R} \). Such a function can belong to the space of
analytic functions, the space of exponential functions and
so on. It is important to choose a solution space for our
analysis. We consider real entire analytic solutions of the
exponential type as defined below.

**Definition 2.1.** We denote by \( \mathbb{E}(\mathbb{R}^2, \mathbb{R}) \) the set of all
formal power series in two variables

\[
w(x) = \sum_{\nu \in \mathbb{N}^2} \nu! x^{\nu},
\]

where \( \nu := (\nu_1, \nu_2) \) is a 2-tuple, \( x^\nu := x_1^{\nu_1} x_2^{\nu_2} \) and \( \nu! :=
\nu_1! \nu_2! \) with the sequence of real numbers \( \{\nu!\}_{\nu \in \mathbb{N}^2} \) being
such that \( w \) is convergent at all points of the domain, that is
\( w(x) \in \mathbb{R} \) for all \( x \in \mathbb{R}^2 \).

Following Willems (Willems (1991)), the collection of all
trajectories that satisfy a given system of PDEs is called
the behavior, \( \mathcal{B} \), of the system. That is,

\[
\mathcal{B} = \{ w \in \mathbb{E}(\mathbb{R}^2, \mathbb{R}) \mid f_i(\partial)w = 0 \quad \forall i \in \{1, 2, \ldots, r\} \}. \quad (2)
\]

It has been shown in Obst (1990) that the set of ex-
ponential trajectories, \( \mathbb{E}(\mathbb{R}^2, \mathbb{R}) \), is a large injective
cogenerator. Therefore, the one-to-one correspondence between
ideals in \( \mathbb{R}[\partial] \) and scalar autonomous behaviors can be
utilized here. In other words, given a system of PDEs as
in equation (1), let \( a := \{ f_1, f_2, \ldots, f_r \} \subseteq \mathbb{R}[\partial] \) be
the equation ideal generated by the describing PDEs. Then
\( \mathcal{B}(a) := \{ w \in \mathbb{E}(\mathbb{R}^2, \mathbb{R}) \mid f(\partial)w = 0 \quad \forall f \in a \} = \mathcal{B} \). \quad (3)

Given the equation ideal \( a \subseteq \mathbb{R}[\partial] \), define the quotient ring
\( \mathcal{M} := \mathbb{R}[\partial]/a \), as the set of all equivalence classes defined
by the following equivalence relation: \( f_1, f_2 \in \mathbb{R}[\partial] \) are
related if and only if \( f_1 - f_2 \in a \). For an element \( f \in \mathbb{R}[\partial] \),
the equivalence class of \( f \) is denoted by \( \bar{f} \). This gives the
canonical surjection \( \mathbb{R}[\partial] \to \mathcal{M} \), where every element
is mapped to its equivalence class.

For an \( \mathcal{A} \)-module \( \mathcal{M} \), the annihilator ideal is defined as

\[
\mathfrak{a} \mathcal{M} := \{ f \in \mathcal{A} \mid fm = 0 \quad \forall m \in \mathcal{M} \}. \quad (4)
\]

**Definition 2.2.** An \( \mathcal{A} \)-module \( \mathcal{M} \) is said to be a faithful
module over \( \mathcal{A} \) if \( \mathfrak{a} \mathcal{M} = \{0\} \).

In this paper, we consider autonomous systems. Such
systems have been characterized using various equivalent
conditions in the literature (see Pillai and Shankar (1998);
Rocha and Willems (1989), Pommaret and Quadrat
(1999), Zerz (2000) among others). It follows from the
equivalent characterizations that a scalar 2-D system \( \mathcal{B} \)
is autonomous if and only if the equation ideal \( a \subseteq \mathbb{R}[\partial] \)
is non-zero.

The Krull dimension of the system plays a crucial role in
this paper. By Krull dimension of a system \( \mathcal{B} \), we mean the
Krull dimension of the associated quotient ring \( \mathcal{M} \). The
Krull dimension of a ring \( \mathcal{A} \) is defined to be the supremum
of the lengths of chains of prime ideals in \( \mathcal{A} \), where a chain
of prime ideals of the form \( \mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \ldots \subseteq \mathfrak{p}_\ell \)
is said to be of length \( \ell \).

2.3 Restriction of trajectories to 1-D subspaces

Restriction of trajectories plays an important role in this
paper. In this section, we look at restriction of trajectories
to 1-D subspaces of the domain.

Let \( \mathcal{V} \subseteq \mathbb{R}^2 \) be a 1-D subspace spanned by a non-zero
vector \( v = [v_1, v_2]^T \in \mathbb{R}^2 \). That is,

\[
\mathcal{V} := \{ x \in \mathbb{R}^2 \mid x = vt, t \in \mathbb{R} \}. \quad (5)
\]
Definition 2.3. For a 2-D system \( \mathcal{B} \) and a 1-D subspace \( \mathcal{V} \subseteq \mathbb{R}^2 \) spanned by a non-zero vector \( v \in \mathbb{R}^2 \), the restriction of \( \mathcal{B} \) to \( \mathcal{V} \) is defined as the following set of 1-D trajectories:
\[
\mathcal{B}|_{\mathcal{V}} := \{ w(\tau t) \in \mathfrak{Exp}(\mathbb{R}, \mathbb{R}) \mid w \in \mathcal{B} \}. 
\]  
An immediate question would be whether this subspace is free with respect to the system. We first define free-ness in this context.

Definition 2.4. A subspace \( \mathcal{V} \subseteq \mathbb{R}^2 \) is said to be free with respect to a 2-D system \( \mathcal{B} \) if the restriction of \( \mathcal{B} \) to \( \mathcal{V} \), as defined in equation (6), is equal to the space of 1-D real entire functions of exponential type. That is, \( \mathcal{B}|_{\mathcal{V}} = \mathfrak{Exp}(\mathbb{R}, \mathbb{R}) \).

Later in the paper, we use the notion of vectors being free with respect to a system. By this, we mean the subspace spanned by the vector to be free with respect to the system, according to Definition 2.4.

A necessary and sufficient condition for free-ness of a 1-D subspace with respect to a scalar system was provided in Pal and Pillai (2014). This characterization plays a crucial role in this paper. To state the result some constructions are required which we first note.

Denote by \( \langle v, \partial \rangle \) the linear polynomial \( v_1 \partial_1 + v_2 \partial_2 \). Note that, the polynomial \( \langle v, \partial \rangle \) is transcendental over \( \mathbb{R} \). Therefore, the \( \mathbb{R} \)-algebra \( \mathbb{R}[\langle v, \partial \rangle] \) is isomorphic to the polynomial ring in one indeterminate. Further note that \( \mathbb{R}[\langle v, \partial \rangle] \) is a sub-algebra of \( \mathbb{R}[\partial] \). Consider the \( \mathbb{R} \)-algebra homomorphism \( \Phi : \mathbb{R}[\langle v, \partial \rangle] \to \mathcal{M} \) defined as follows:
\[
\Phi : \mathbb{R}[\langle v, \partial \rangle] \to \mathcal{M}, \quad p \mapsto p \mapsto \bar{p}. 
\]  
Observe that \( \ker \Phi = a \cap \mathbb{R}[\langle v, \partial \rangle] \). Since \( \Phi \) is an \( \mathbb{R} \)-algebra homomorphism, \( \ker \Phi \) is an ideal in \( \mathbb{R}[\langle v, \partial \rangle] \). We call this the intersection ideal of \( a \) and denote it by \( a_{\mathcal{V}} \), that is \( a_{\mathcal{V}} \) := \( a \cap \mathbb{R}[\langle v, \partial \rangle] \). Corresponding to the intersection ideal \( a_{\mathcal{V}} \), we have the following 1-D behavior
\[
\mathcal{B}_{\mathcal{V}} := \{ \bar{w} \in \mathfrak{Exp}(\mathbb{R}, \mathbb{R}) \mid f(\frac{d}{dt})\bar{w} = 0 \forall f(\langle v, \partial \rangle) \in a_{\mathcal{V}} \}. 
\]  
Proposition 2.5 relates the 1-D behavior, as defined in equation (8), to the behavior restricted to a 1-D subspace as defined in equation (6). The proof of the result can be found in Pal and Pillai (2014).

Proposition 2.5. Consider a scalar autonomous 2-D system \( \mathcal{B} \) with equation ideal \( a \subseteq \mathbb{R}[\partial] \). Let \( \mathcal{V} \subseteq \mathbb{R}^2 \) be a 1-D subspace spanned by a non-zero vector \( v \in \mathbb{R}^2 \). Let \( \mathcal{B} \) restricted to \( \mathcal{V} \) be as defined in Definition 2.3. Consider the 1-D behavior as defined in equation (8). Then \( \mathcal{B}|_{\mathcal{V}} \subseteq \mathcal{B}_{\mathcal{V}} \).

We now return to the question of free-ness of a subspace with respect to a given behavior. Proposition 2.6 gives an algebraic characterization of free-ness. The proof can be found in Pal and Pillai (2014).

Proposition 2.6. Consider a scalar autonomous 2-D system \( \mathcal{B} \) with equation ideal \( a \subseteq \mathbb{R}[\partial] \). Let \( \mathcal{V} \subseteq \mathbb{R}^2 \) be a 1-D subspace spanned by a non-zero \( v \in \mathbb{R}^2 \). Then the following are equivalent:

1. \( \mathcal{V} \) is free with respect to \( \mathcal{B} \).
2. The intersection ideal \( a_{\mathcal{V}} = a \cap \mathbb{R}[\langle v, \partial \rangle] \) is the zero ideal.
3. The \( \mathbb{R} \)-algebra homomorphism \( \Phi : \mathbb{R}[\langle v, \partial \rangle] \to \mathcal{M} \), as defined in equation (7), is injective.

It also follows from the characterization in Proposition 2.6 that free subspaces are abundant in \( \mathbb{R}^2 \). In other words, a 1-D subspace \( \mathcal{V} \subseteq \mathbb{R}^2 \) chosen at random will almost always be free with respect to a given 2-D behavior \( \mathcal{B} \). For details and proof please see Pal and Pillai (2014).

3. Characterization of Initial Data

In this section, we provide a constructive way of specifying initial data for a given system of PDEs using the notion of characteristic sets. Characteristic sets were initially defined for discrete 2-D systems (Valcher (2000)) and later extended to \( n \)-D system in Mukherjee and Pal (2016, 2017, 2019). Here, we define a characteristic set for the continuous case, that is for autonomous systems described by linear PDEs with real constant coefficients.

3.1 Characteristic sets

Definition 3.1. Given a system \( \mathcal{B} \), a subset \( S \) of the domain is called a characteristic set for \( \mathcal{B} \) if for every trajectory \( w \in \mathcal{B} \), the restriction of \( w \) to \( S \) allows to uniquely determine the remaining portion of \( w \), that is \( w|_{\mathbb{R}^2 \setminus S} \) can be uniquely determined if \( w|_{S} \) is known.

It is often infeasible to ascertain if a given arbitrary subset of the domain is a characteristic set. In this paper, we provide a way of constructing a characteristic set for a given scalar 2-D autonomous system \( \mathcal{B} \).

Consider a 2-D autonomous system \( \mathcal{B} \) with equation ideal \( a \subseteq \mathbb{R}[\partial] \) and corresponding quotient ring \( \mathcal{M} \). The Krull dimension of \( \mathcal{M} \) as an \( \mathbb{R}[\partial] \)-module can either be zero or one. For the case when the Krull dimension of \( \mathcal{M} \) equals zero, \( \mathcal{M} \) becomes a finite dimensional vector space over \( \mathbb{R} \). In that case, \( \mathcal{B} \) is a strongly autonomous system and a characteristic set for \( \mathcal{B} \) is a set of finitely many points of the domain (see Fornasini et al. (1993)). The other possibility for an autonomous 2-D system is to have Krull dimension of \( \mathcal{M} \) to be equal to one. It is not yet known what kind of subsets of \( \mathbb{R}^2 \) can then qualify as a characteristic set in this case. We address this in the following section.

3.2 Rectangular strips of finite width as characteristic sets

In this section, we show that rectangular strips of finite width containing a 1-D subspace, which is free with respect to the system, is a characteristic set for the system. This result is proved by first showing that for a system \( \mathcal{B} \) having Krull dimension equal to one and a 1-D subspace \( \mathcal{V} \) which is free with respect to \( \mathcal{B} \), following Definition 2.4, the quotient ring \( \mathcal{M} \) is a finitely generated faithful module over \( \mathbb{R}[\langle v, \partial \rangle] \). This result uses the idea of integral ring extension. We briefly discuss this here; for details please refer to (Atiyah and MacDonald, 1969, Chapter 5).

Let \( A_1 \) and \( A_2 \) be rings such that \( A_1 \subseteq A_2 \) as a subring. Then an element \( \alpha \in A_2 \) is said to be integral over \( A_1 \) if
α satisfies a monic polynomial equation with coefficients from A₁. When A₁ ⊆ A₂ as a subring, A₂ is said to be an integral extension of A₁ if every element of A₂ is integral over A₁. Proposition 3.2 summarizes the results on integral ring extension required for this paper. For proofs, please see (Atiyah and MacDonald, 1969, Chapter 5).

Proposition 3.2. Let A₁ and A₂ be rings. Further, let A₂ be a finitely generated algebra over A₁. Then the following are equivalent:

1. A₂ is integral over A₁,
2. A₂ is a finitely generated module over A₁.

Theorem 3.3. Consider a 2-D autonomous system B with equation ideal a ⊆ R[∂] and corresponding quotient ring M having Krull dimension equal to one. Let V ⊆ R² be a 1-D subspace spanned by a non-zero v ∈ R² such that V is free with respect to B. Recall the sub-algebra R[(v, ∂)] of R[∂]. Then M is a finitely generated faithful module over R[(v, ∂)].

Proof: Recall the R-algebra homomorphism Φ : R[(v, ∂)] → M as defined in equation (7). Since V is free with respect to B, Φ is injective (Statement 3 of Proposition 2.6). This shows that M is a faithful module over R[(v, ∂)].

Note that, M is naturally a finitely generated algebra over R. Since R ⊆ R[(v, ∂)], M is a finitely generated algebra over R[(v, ∂)] as well. To show that M is a finitely generated module over R[(v, ∂)], it suffices to show that Φ : R[(v, ∂)] → M is integral (Proposition 3.2). We prove this by contradiction. Suppose Φ is not integral. Then there exists an element x ∈ M transcendental over R[(v, ∂)] such that x ∉ R[(v, ∂)]. Let A := R[∂] be a sub-algebra of R[(v, ∂)] such that A ∩ R = R[∂]. Define the linear polynomial (u, ∂) := u₁∂₁ + u₂∂₂ as before, (u, ∂) is transcendental over R.

Lemma 3.4. Let V ⊆ R² be a 1-D subspace spanned by a non-zero v ∈ R². Let U ⊆ R² be a complementary subspace of V spanned by u ∈ R² such that R² = V ⊕ U. Then R[∂] = R[(v, ∂), (u, ∂)] as R-algebras.

Proof: Note that, R[(v, ∂), (u, ∂)] ⊆ R[∂]. To show R[∂] ⊆ R[(v, ∂), (u, ∂)], we need to show that R[∂] can be expressed as R-linear combinations of (v, ∂) and (u, ∂) for i = 1, 2. In other words, we need to show that there exists a matrix A ∈ R²×² such that

\[
\begin{bmatrix}
\alpha \\
\beta
\end{bmatrix} = A \begin{bmatrix}
(v, ∂) \\
(u, ∂)
\end{bmatrix}.
\]

Define T := [v u] ∈ R²×². Note that, T is non-singular. It can be easily verified that A = (TT⁻¹)⁻¹, where TT⁻¹ is the transpose of T.

Using Corollary 3.5, we prove Lemma 3.6 which gives an explicit list of generators for M as a module over R[(v, ∂)]. Lemma 3.6. Consider a 2-D autonomous system B with equation ideal a ⊆ R[∂] and corresponding quotient ring M having Krull dimension equal to one. Let V ⊆ R² be a 1-D subspace spanned by a non-zero v ∈ R² such that V is free with respect to B. Then there exist u ∈ R² and d ∈ N such that

\[
\begin{bmatrix}
\partial_1^d \\
\partial_2^d
\end{bmatrix} \forall j \leq d - 1
\]

is a generating set for R[∂]/a as a module over R[(v, ∂)].

Proof: Since Krull dimension of M is equal to one and V ⊆ R² is a 1-D subspace it follows from Theorem 3.3 that M is a finitely generated faithful module over R[(v, ∂)]. It follows from Lemma 3.4 that, there exists a complementary subspace U ⊆ R² of V spanned by u ∈ R² such that R[∂] = R[(v, ∂), (u, ∂)]. Define \( \partial_e := \langle v, ∂ \rangle \) and \( \partial_a := \langle u, ∂ \rangle \). Then every polynomial in R[∂] can be rewritten as a finite R-linear combination of monomials of the form \( \partial_a\partial_b^n \), where a, b ∈ N. Since M is a finitely generated faithful module over R[(v, ∂)], according to Corollary 3.5, there exists d ∈ N such that

\[
p(\partial_e, \partial_a) := a_d\partial_a^{-1} + \ldots + a_1\partial_a + a_0\partial_e \in a,
\]

where a_i ∈ R[∂] for i ∈ \{0, 1, \ldots, d - 1\}. Now p(\partial_e, \partial_a) being a monic polynomial in \( \partial_a\), \( \partial_a\partial_b^n \) can be divided by \( \partial_a\partial_b^n \) using the Euclidean division algorithm. Thus we have

\[
\partial_a\partial_b^n p(\partial_e, \partial_a) = r(\partial_e, \partial_a),
\]

where p(\partial_e, \partial_a)q(\partial_e, \partial_a) ∈ a and the remainder r(\partial_e, \partial_a) is an R[∂]-linear combination of monomials \( \{1, \partial_a, \ldots, \partial_a^{-1}\} \). Under the canonical surjection we have \( \partial_a\partial_b^n = r(\partial_e, \partial_a) \). Thus every monomial \( \partial_a\partial_b^n \), where a, b ∈ N, is equal to an R[∂]-linear combination of monomials \( \{T, \overline{\partial_a}, \ldots, \overline{\partial_a^{-1}}\} \).
Proof: To show that a characteristic set for $\mathfrak{B}$ is given by a rectangular strip of finite width containing $\mathcal{V}$ we show that to evaluate a trajectory at an arbitrary point $x = (x_1, x_2) \in \mathbb{R}^2$, the information of the trajectory on this rectangular strip is sufficient. Recall, from Definition 2.1, that, $w(x)$ can be computed if $w_\nu$ is known for all $\nu \in \mathbb{N}^2$. Note that, $w_\nu = (\partial^\nu ) w|_{x=0}$. Applying Lemma 3.4, the solution trajectory in the transformed domain can be written as:

$$w(x) = \sum_{\nu \in \mathbb{N}^2} \tilde{w}_\nu \xi^\nu,$$

where $x = T\xi$ with $T = [v \ w] \in \mathbb{R}^{2 \times 2}$ and $\tilde{w}_\nu = (\partial_1^\nu \partial_2^\nu ) w|_{x=0}$. Using Lemma 3.6, $\partial_1^\nu \partial_2^\nu$ is equivalent modulo the equation ideal to a finite $\mathbb{R}[\partial_\nu]$-linear combination of monomials $1, \partial_\nu, \ldots, \partial_\nu^{d-1}$ for some $d \in \mathbb{N}$. Thus to calculate $\tilde{w}_\nu$, we must know the action of these monomials on a trajectory $w \in \mathfrak{B}$ evaluated at $x = 0$. In other words,

$$\{(\partial_1^\nu \partial_2^\nu ) w|_{x=0} | 0 \leq i \leq d - 1, j \in \mathbb{N}\}$$

(10)

lets us uniquely determine $w$ at an arbitrary point $x \in \mathbb{R}^2$.

Let $\mathcal{S} \subseteq \mathbb{R}^2$ be a strip containing the subspace $\mathcal{V}$ spanned by a non-zero $v \in \mathbb{R}^2$. That is,

$$\mathcal{S} := \{x \in \mathbb{R}^2 | \exists \xi_1, \xi_2 \in \mathbb{R} \text{ such that } x = \xi_1 v + \xi_2 u, \text{ where } -\epsilon \leq \xi_1, \xi_2 \leq \epsilon\}.$$

(11)

Then for all $x \in \mathcal{S}$, the action of $\partial_1^\nu \partial_2^\nu$ on $w$ evaluated at $x$ is given by

$$(\partial_1^\nu \partial_2^\nu ) w|_x = \left((\partial_1^\nu \partial_2^\nu ) w\right)(\xi_1 v + \xi_2 u) = \frac{\partial}{\partial \xi_2} \left(\frac{\partial^{\nu} w_T}{\partial \xi_1^\nu} \right)$$

(12)

where $w_T := w \circ T$. Thus $w_T$ must be known for all $\xi_1 \in \mathbb{R}$ and $-\epsilon \leq \xi_1, \xi_2 \leq \epsilon$. Therefore the initial data, as specified in equation (10), can be calculated from $w|_\mathcal{S}$ which in turn helps in uniquely calculating $w$. □

Remark 3.8. Note that, a characteristic set only refers to a subset of the domain having some special property. The property of uniquely extending the trajectory to the whole domain requires an algorithm for computing trajectories at various points in the domain. One such algorithm is the Oberst-Riquier algorithm as stated in (Pal and Pillai, 2014, Algorithm 22).

Remark 3.9. The Oberst-Riquier algorithm crucially uses Gröbner basis for computing a standard monomial set. A characteristic set is analogous to the standard monomial set having some nice structure and useful system theoretic properties.

4. EXISTENCE OF A RECTANGULAR STRIP OF FINITE WIDTH AS A CHARACTERISTIC SET

In Section 3.2 we have shown that for a given scalar autonomous 2-D system $\mathfrak{B}$ and a 1-D subspace $\mathcal{V} \subseteq \mathbb{R}^2$, which is free with respect to $\mathfrak{B}$, a characteristic set for $\mathfrak{B}$ is given by a rectangular strip of finite width containing this 1-D subspace. In this section, we provide a systematic way of finding a 1-D subspace, with the desired specification of being free with respect to the given 2-D system, such that a characteristic set can be constructed using it. This follows from the application of a well-known result in commutative algebra – the Noether’s Normalization Lemma (see Atiyah and MacDonald (1969)).

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an invertible linear map represented by a non-singular matrix

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \in \mathbb{R}^{2 \times 2}.$$

Let $x$ and $y$ denote the co-ordinate functions of the domain and co-domain, respectively. Then $y = Tx$. Define the $T$-induced linear map between the tangent spaces, $T^* : T^*_y \mathbb{R}^2 \rightarrow T^*_x \mathbb{R}^2$, in the following way. Let $y \mapsto w(y)$ be in $\mathfrak{E}_y^p(\mathbb{R}^2, \mathbb{R})$. Then for $i = 1, 2$,

$$\left(T^* \frac{\partial}{\partial x_i}\right)(w(y)) := \frac{\partial}{\partial y_j} w(Tx).$$

(13)

Applying the definition of $T^*$ (equation (13)) to the co-ordinate function $y_1, y_2$, we have

$$\left(T^* \frac{\partial}{\partial x_i}\right) y_j = \frac{\partial}{\partial y_j} \sum_{k=1}^2 t_{jk} x_k = t_{ji}.$$

By varying $j$, we have

$$\left(T^* \frac{\partial}{\partial x_i}\right) \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix} w(Tx) = T^* \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix} w(y) = T^* \begin{pmatrix} \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \end{pmatrix} w(y),$$

(14)

where $T^T$ denotes the transpose of $T$. Define $\partial_k := \left\{\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\right\}$ and $\partial'_k := \left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right\}$. Define the $T$-induced $\mathbb{R}$-algebra homomorphism $\psi : \mathbb{R}[\partial_k] \rightarrow \mathbb{R}[\partial'_k]$ defined in the following way

$$\psi : \mathbb{R}[\partial_k] \rightarrow \mathbb{R}[\partial'_k], \partial_k \mapsto T^T \partial_k.$$

(15)

Since $T$ is non-singular, $T^T$ is also non-singular and thus $\psi$ is an isomorphism of $\mathbb{R}$-algebras. Lemma 4.1 below relates the behaviors corresponding to equations written using the partial differential operators $\partial_k$ and $\partial'_k$ owing to the co-ordinate transformation of the domain. The proof can be found in Pal and Pillai (2014).

Lemma 4.1. Let $T \in \mathbb{R}^{2 \times 2}$ be a linear co-ordinate transformation on the domain $\mathbb{R}^2$ defined by $x \mapsto Tx := y$. Consider the $\mathbb{R}$-algebra isomorphism $\psi : \mathbb{R}[\partial_k] \rightarrow \mathbb{R}[\partial'_k]$, induced by $T$, as defined in equation (15). Let $a \subseteq \mathbb{R}[\partial'_k]$ be an ideal. Then $\psi(a) \subseteq \mathbb{R}[\partial_k]$ is also an ideal. Consider the behaviors

$$\mathfrak{B}_x := \{w(x) \in \mathfrak{E}_x(\mathbb{R}^2, \mathbb{R}) | m(\partial_k) w = 0 \forall m \in a\},$$

$$\mathfrak{B}_y := \{w(y) \in \mathfrak{E}_y(\mathbb{R}^2, \mathbb{R}) | m(\partial'_k) w = 0 \forall m \in \psi(a)\}.$$
We now prove Theorem 4.3.

**Theorem 4.3.** Consider a 2-D autonomous system $B$ with equation ideal $a \subseteq R[\partial]$ having Krull dimension equal to one. Then there exists a 1-D subspace $V \subseteq R^2$ such that $V$ is free with respect to $B$.

**Proof:** Let $T \in R^{2 \times 2}$ be the non-singular matrix representing the co-ordinate transformation on the domain $R^2$ as defined in Proposition 4.2. It further follows from Proposition 4.2 that the $R$-linear map $R[\partial y_{i}] \rightarrow R[\partial y]/\psi(a)$ is injective and integral. Injectivity of the map implies that $e_1 = [1 0]^T$ is free with respect to $B_p$. Then $v := T^{-1}e_1$ is free with respect to $B$. Thus the 1-D subspace $V \subseteq R^2$ that is free with respect to $B$ is given by the span of $v \in R^2$.

Therefore, given a 2-D autonomous system $B$ with equation ideal $a \subseteq R[\partial]$, there exists a non-singular matrix $T \in R^{2 \times 2}$ defined by $x \mapsto Tx =: y$ and the corresponding $T$-induced map $\psi: R[\partial y_{i}] \rightarrow R[\partial y]/\psi(a)$ is a finitely generated faithful module over $R[\partial y_{i}]$. Using Theorem 3.7, a characteristic set for $B_p$ is given by a rectangular strip of finite width containing $e_1$ in the transformed domain. Applying the inverse transformation we obtain a characteristic set for $B$. Thus, every scalar 2-D autonomous system, described by a set of linear PDEs having exponential solution, admits a characteristic set given by a rectangular strip of finite width containing a free subspace.

**Remark 4.4.** The theory of Gröbner bases can be applied for implementation and verification of the results presented in this paper.

### 5. CONCLUDING REMARKS

In this paper, we provided a method of constructing a region (in the form of subspaces and strips of finite width around such subspaces) in the domain such that trajectories restricted to these regions serve as initial/boundary data for the given scalar 2-D autonomous system of PDEs. Note that, we have considered only exponential trajectories; on bigger function spaces – like infinitely often differentiable functions or distributions – the main results of this paper often fail to hold. Formulating characteristic sets for these function spaces would be a matter of future research.

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