ON ZEROES OF RANDOM POLYNOMIALS
AND APPLICATIONS TO UNWINDING

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Abstract. Let \( \mu \) be a probability measure in \( \mathbb{C} \) with a continuous and compactly supported distribution function, let \( z_1, \ldots, z_n \) be independent random variables, \( z_i \sim \mu \), and consider the random polynomial

\[
p_n(z) = \prod_{k=1}^{n} (z - z_k).
\]

We determine the asymptotic distribution of \( \{ z \in \mathbb{C} : p_n(z) = p_n(0) \} \). In particular, if \( \mu \) is radial around the origin, then those solutions are also distributed according to \( \mu \) as \( n \to \infty \). Generally, the distribution of the solutions will reproduce parts of \( \mu \) and condense another part on curves.

We use these insights to study the behavior of the Blaschke unwinding series on random data.

1. Introduction and main results

The purpose of this paper is to discuss an interesting phenomenon of solutions of certain random polynomial equations. In what follows, we will assume that \( \mu \) is an absolutely continuous (with respect to the Lebesgue measure) and compactly supported probability measure on \( \mathbb{C} \) and that \( p_n \) denotes the random polynomial

\[
p_n(z) = \prod_{k=1}^{n} (z - z_k),
\]

where the \( z_k \) are drawn independently from \( \mu \) and \( n \in \mathbb{N} \). Our main result is a reproducing property for radial measures \( \mu \) (see §2 for the motivation that lead us to this result).

Figure 1. Left: roots of 100 polynomials \( p_{30}(z) - p_{30}(0) \) with Gaussian distributed roots are again Gaussian. Right: roots of 100 polynomials \( p_{20}(z) - p_{20}(0) \) with roots uniformly distributed on the boundary of the unit disk.

Theorem 1. Let \( \mu \) be a compactly supported probability measure on \( \mathbb{C} \) with a continuous, radial distribution function. Then the complex numbers \( z_1, \ldots, z_n \) solving \( p_n(z) = p_n(0) \) satisfy

\[
\frac{1}{n} \sum_{k=1}^{n} \delta_{z_k} \to \mu \quad \text{in the sense of distributions as } n \to \infty.
\]

Key words and phrases. Random Polynomials, Zeroes, Potential Theory, Blaschke unwinding.

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Theorem 1 fails for general measures but it is not difficult to construct non-radial measures \( \mu \) that have the same property (see Theorem 2). The assumption on \( \mu \) being compactly supported is clearly not sharp, our proof immediately transfers to probability measures having a certain rate of decay at infinity. The result is similar in spirit to a recent result of Kabluchko \( \cite{8} \) (proving a conjecture of Pemantle \& Rivin \( \cite{10} \)) who showed that the distribution of critical points \( \{ z \in \mathbb{C} : p'_n(z) = 0 \} \) reproduces \( \mu \) for general probability measures \( \mu \). If \( \mu \) is not radial, the situation is not quite as simple. We introduce two sets \( A, B \subset \mathbb{C} \) (and we will keep using \( A, B \) to refer to those sets throughout the rest of the paper)

\[
A = \left\{ z \in \mathbb{C} : \int_{\mathbb{C}} \log |x - z|d\mu(x) > \int_{\mathbb{C}} \log |x|d\mu(x) \right\}
\]

and

\[
B = \left\{ z \in \mathbb{C} : \int_{\mathbb{C}} \log |x - z|d\mu(x) = \int_{\mathbb{C}} \log |x|d\mu(x) \right\}.
\]

A simple description of the result for the general case can be stated as follows.

**Theorem 2** (Main Result). Let \( \mu \) be a probability measure on \( \mathbb{C} \) with a continuous and compactly supported distribution function. Then the distribution of \( \{ z \in \mathbb{C} : p_n(z) = p_n(0) \} \) converges to \( \nu \) in distribution, where \( \nu = \mu \) on \( A \) and \( \nu \) has measure \( 1 - \mu(A) \) supported on \( B \).

We illustrate the Theorem with a specific example (the measure is a bit more singular than what is covered by the result but it is not difficult to see that the proof carries over to this particular case). We choose \( \mu \) to be the union of the arclength measure of the boundary of two disks of radius 1 in the complex plane (one located in the origin and one centered around 2)

\[
\mu = \frac{1}{4\pi} \left( \mathcal{H}^1 \right|_{|z|=1} \cup \mathcal{H}^1 \right|_{|z-2|=1}.
\]

Theorem 2 implies that the random solutions of \( p_n(z) = p_n(0) \) will partially follow the original measure \( \mu \) and partially concentrate along four new curves. Details behind this example are given after the proof.

![Figure 2](image)

**Figure 2.** Left: the only possible support of limiting measure of solutions of \( p_n(z) = p_n(0) \) as \( n \to \infty \) (bold, the two circles from which roots are drawn are dashed), right: numerical example for \( n = 30 \) with 15 roots chosen randomly from each circle. A repulsion phenomenon leads to a slight visual discrepancy in the left arc (see Theorem 3 for a more precise description).

We observe that Theorem 2 does not make any claim about how the solutions of \( p_n(z) = p_n(0) \) are distributed on \( B \), it only states that their total mass is going to be \( 1 - \mu(A) \). Figure 2 seems to indicate that there might indeed be parts of \( B \) that will not support any part of the new measure \( \nu \), however, there is a somewhat simpler explanation in this case: there is always, by construction, a root in the origin and a root repulsion phenomenon. This has the effect of creating a bubble around 0 in which no roots are found; that bubble shrinks in size as the degree \( n \) increases.
In the generic case, we can give a more precise description of the measure $\nu$ on $B$. Our assumptions will be that $\mu$ is compactly supported and $C \subset B$ is a connected subset of $B$ that is bounded away from the support of $\mu$ and has both

$$\left\| \nabla \int_C \log |x-z| d\mu(z) \right\| \quad \text{and} \quad \left| \frac{\partial}{\partial t} \int_C \arg(\gamma(t)-z) d\mu(z) \right|$$

uniformly bounded away from 0 on $C$, where $\gamma$ is an arclength parametrization of the curve $C$ (the first assumption combined with the definition of $B$ and the implicit function theorem immediately implies that $C$ is a curve). Since the support of $\mu$ is compact and $C$ is bounded away from the support of $\mu$, $\int_C \arg(\gamma(t)-z) d\mu(z)$ is well defined as a continuous single-valued function of $t$.

**Theorem 3 (Structure of $\nu$ on $B$).** Under these assumptions, let $\gamma(t)$ be an arclength parametrization of $C$. The limiting measure of $\{z \in C : p_n(z) = p_n(0)\}$ is absolutely continuous on $C$ and given by

$$\nu|_C = \frac{1}{2\pi} \left| \frac{\partial}{\partial t} \int_C \arg(\gamma(t)-z) d\mu(z) \right|^{-1} d\mathcal{H}^1.$$

Moreover, for degree $n$ sufficiently large, then with high probability the spacing between roots on $B$ becomes uniform in the sense that two consecutive roots on $C$ have distance $\sim n^{-1}$ from each other with the implicit constant determined by the limiting distribution.

It is certainly possible to slightly extend the result to cover other cases as well. For example, here we are not necessarily assuming that $\mu$ is absolutely continuous as long as it is compactly supported and the assumptions hold. However, it certainly already describes the generic situation fairly accurately: in particular, it allows us to deduce that the behavior on $B$ is actually quite regular: the roots decompose into evenly spaced points (with spacing roughly $\sim n^{-1}$ and an implicit constant depending on everything).

![Figure 3](image.png)

**Figure 3.** (Left:) The typical distribution of $p_n(z) = p_n(0)$ for $n = 30$. (Right:) a single instance. Even for this rather small degree, the roots are already well-separated. The fixed root in 0 clears out a uniform area of repulsion.

We emphasize that, in Figure 3, nothing is special about the root $z = 0$. However, combining many different numerical examples has the effect of visually removing the repulsion phenomenon for all roots except the fixed one $z = 0$ which is common to all numerical samples. We consider a simple toy example where $\mu = \delta_1$ is the deterministic measure in 1. Clearly, $p_{2n}(z) = (z-1)^{2n}$ and $p_{2n}(0) = 1$ implying that solutions of $p_{2n}(z) = p_{2n}(0)$ are given by $(z-1)^{2n} = 1$ which are equally spaced points on $|z-1| = 1$. In the framework of Theorem 3, we see that

$$\left\| \nabla \int_C \log |x-z| d\mu(z) \right\| = 1 \quad \text{on} \quad |x-1| = 1$$

as well as

$$\left| \frac{\partial}{\partial t} \int_C \arg(\gamma(t)-z) d\mu(z) \right| = 1$$
and the limiting measure is clearly \((2\pi)^{-1}H^1\) coinciding with what is predicted by Theorem 3.

Generically, one expects the set \(B\) to be a union of bounded lines (though it can be a disk, see the proof of Theorem 1). It might be interesting to understand what happens if the measure \(\mu\) decays at infinity at a certain rate. However, even for compactly supported measures, there are open questions: whenever two of these lines meet at an angle, then clearly the gradient of the logarithmic integral vanishes and Theorem 3 does not apply: is it possible to describe the behavior of solutions of \(p_n(z) = p_n(0)\) in these singular points?

2. Application to the Unwinding Series

**Unwinding.** These results were originally motivated by a study of a nonlinear analogue of Fourier series: given a holomorphic function \(f : \mathbb{C} \rightarrow \mathbb{C}\), its Blaschke factorization is given by

\[
f(z) = \left( \prod_{|\alpha| \leq 1, f(\alpha) = 0} \frac{z - \alpha}{1 - \overline{\alpha}z} \right) g(z),
\]

where the Blaschke product ranges over all roots inside the unit disk and \(g : \mathbb{C} \rightarrow \mathbb{C}\) is holomorphic and has no roots inside the unit disk. Writing \(g(z) = g(0) + (g(z) - g(0))\) produces a new holomorphic function, \(g(z) - g(0)\), which has at least one root inside the unit disk. Iterating the process yields a formal expansion

\[
f(z) = a_0B_0 + a_1B_0B_1 + a_2B_0B_1B_2 + \ldots
\]

This process was introduced by Coifman around 1995, described in a PhD thesis of his student Michel Nahon [7] and followed by several other researchers [11, 12, 13]. It was independently discovered by T. Qian [14] who also studied, jointly with collaborators, different versions of the algorithm [15, 16, 17, 18]. There is a different line of investigation concerned with Blaschke products as a general family of orthogonal functions [4, 5, 6, 9, 20] that we do not discuss here. Convergence of the algorithm in the Hardy spaces \(H^2\) is due to Qian ([14], the proof is also described in [2]), the convergence in a large family of function spaces (including all Sobolev spaces) was given by Coifman and the first author [2]. Ways of computing the expansion for non-analytic signals are due to Coifman and the authors [3]. An extension to Hardy spaces \(H^p\) is due to Coifman & Peyrière [4]. The algorithm seems to have exceptional convergence properties when applied to real signals, a theoretical justification is still open.

**Figure 4.** The signal of a gravity wave (left), the first Blaschke product of the respective signal (middle) and the second Blaschke product (right), from [3].

**Polynomials.** If the function \(f\) is a polynomial of degree \(n\), then the expansion is exact after \(n\) steps (this was already observed by Nahon [7]). For polynomials, the explicit form of the Blaschke products allows for the algorithm to be described in a simpler way: given a polynomial \(f_n\)
(1) define the polynomial $g_{n+1}$ to be the polynomial having the same roots as $f_n$ outside the unit disk and, additionally, the roots $1/\alpha$ for all roots $\alpha \neq 0$ of $f$ inside the unit disk, i.e.

$$g_{n+1} = f_n(z) \left( \prod_{|\alpha| \leq 1, f(\alpha) = 0} \frac{z - \alpha}{1 - \alpha z} \right)^{-1}$$

(2) define $f_{n+1}(z) = g_{n+1}(z) - g_{n+1}(0)$ and, if $f_{n+1} \neq 0$, go to (1)

The main question is with which speed $f_n$ converges to 0 on the boundary of the unit disk. The paper [2] shows that convergence speed in the Dirichlet space can be explicitly connected to how many roots inside the unit disk one would expect $f_n$ to have. Using Theorem 1 of this paper, we can answer the question from [3] and conclude that for typical polynomials (and $n$ large), one cannot expect more than $o(n)$ roots inside the unit disk.

**Corollary 4** (Invariance of certain random polynomials under Blaschke factorization.) Let $p_n$ be a random polynomial with $n$ roots that are independently and identically distributed following a probability measure that can be written as $\mu = \phi(\sqrt{x^2 + y^2})dx dy$ for some $\phi \in C_c^\infty((1, \infty))$. For every such polynomial $p_n$, we may determine the Blaschke (or inner-outer) Factorization

$$p_n(z) = p_n(0) + B \cdot G.$$ 

Then, $G$ is a random polynomial whose roots are also distributed according to $\mu$ as $n \to \infty$.

More precisely, let $p_n$ be a random polynomial created in the way described at above for some radial probability measure $\mu$ that is compactly supported outside a neighborhood of the unit disk. Then Theorem 1 implies that the roots of $p_n(z) - p_n(0)$ are again distributed according to the measure $\mu$ as $n \to \infty$. The proof of Theorem 1 also implies that with high probability all solutions of $p_n(z) = p_n(0)$ except the trivial root in the origin are outside the unit disk since they are exponentially close to the $n$ roots with high likelihood. We observe that in this case, when $n$ is sufficiently large, the Blaschke unwinding series reduces to a simple power series expansion. A similar phenomenon was already observed to occur for functions whose power series expansion has exponentially decaying coefficients in [3]. It seems likely that polynomials with roots outside the unit disk exhibit exponentially decaying coefficients at least in the generic case – simple power series expansion then naturally leads to exponentially convergence in the unit disk.

### 3. Proofs

We start by first proving Theorem 2. Theorem 1 will follow from a small modification of the same argument. Theorem 3 follows from a different line of reasoning.

The whole argument is based on establishing the fact that if $p_n(z) = 0$ for some $z \in A$, then with high probability there is a solution of $p_n(z) = p_n(0)$ that is exponentially close (in the degree) to $z$. We start from giving a heuristic argument, which will be made rigorous in the following proof. First, for any fixed $z_0 \in \mathbb{C}$ the distance to the nearest root is at most exponentially distributed and at scale $\sim n^{-1/2}$ in the sense that it is not going to be closer, but it might be further away. On the other hand, by a direct expansion we expect

$$\log |p_n(z)| \sim n \int_{\mathbb{C}} \log |z - x| d\mu(x).$$

Since the entire theorem is invariant under scaling all the roots by (the same) scalar $\lambda \in \mathbb{R}$, we may assume without loss of generality that this integral is positive. Based on the above two facts, due to the root separation at scale $\sim n^{-1/2}$, since $\log |p_n(z)| = \sum_{k=1}^n \log |z - z_k|$, in order for a single root $z^*$ to substantially contribute to $\log |p_n(z)|$, we would require that $\log |z - z^*| \sim n$ which requires that $z^*$ is exponentially close to $z$. 

3.1. Proof of Theorem 2.

Proof. The proof is based on understanding the expected size of $\mathbb{E} n^{-1} \log |p_n|$. For any fixed $z \in \mathbb{C}$,

$$
\mathbb{E} \frac{\log |p_n(z)|}{n} = \mathbb{E} \frac{1}{n} \sum_{k=1}^{n} \log |z - z_k| = \int_{\mathbb{C}} \log |z - x| d\mu(x).
$$

Our assumptions on $\mu$ imply that this integral is well defined, continuous in $z$, and finite everywhere (and, as can easily be seen, this would also hold for non-compactly supported measures that decay at a certain speed). Let us now assume that $z \in A$. Since $\mu$ has an absolutely continuous probability measure associated with the Lebesgue measure $dx$ with a continuous distribution function, if we subdivide the support of the measure into finitely many boxes of equal size, we know that for $n$ sufficiently large, each box contains a number of roots proportional to the measure assigned to that box by $\mu$. This shows, by approximating the integral with a Riemann sum over these boxes, that we have a large deviation principle: for any $c > 0$,

$$
\mathbb{P} \left( \left| \frac{\log |p_n(z)|}{n} - \int_{\mathbb{C}} \log |z - x| d\mu(x) \right| \geq c \right) \text{ is exponentially decaying in } n.
$$

This estimate says that the size of $\log |p_n(z)|$ is linear in $n$ with high probability (we will argue in a localized fashion and can without loss of generality, after possibly rescaling the roots by a real scalar $\lambda > 0$, assume the coefficient of front of the linear growth to be positive; this is not important for any of the subsequent arguments but may be helpful in visualizing the argument). Let $r$ be one of the roots of $p_n(z)$ located in $A$. We write $p_n(z) = (z - r)q_n(z)$. When $|z - r| < 1$, we have

$$
\log |q_n(z)| - \log |p_n(z)| = \log \frac{1}{|z - r|} > 1
$$

We thus conclude that, with likelihood tending to 1 at an exponential rate in the degree, that $|q_n(z)|$ is exponentially larger than $|p_n(0)|$ when $z$ is close to $r$. We now use Rouche’s theorem to conclude the existence of a solution of $p_n(z) = (z - r)q_n(z) = p_n(0)$ for some $z$ very close to $r$. We will find an exponentially small (in the degree $n$) $\delta > 0$ such that

$$
|p_n(z)| = |(z - r)q_n(z)| > |p_n(0)| \quad \text{for all } |z - r| = \delta
$$

That is, we need to control the size of $|q_n(z)|$ for all $|z - r| = \delta$ so that $|q_n(z)| > |p_n(0)|/\delta$ for all $|z - r| = \delta$. Our argument is as follows:

- With likelihood exponentially close to 1, the size of $|q_n(r)|$ is an exponential factor in $n$ larger than $|p_n(0)|$: since roots are chosen in an i.i.d. fashion, $q_n$ is merely $p_n$ with a root removed. It thus behaves exactly like $p_{n-1}$ would in that area and the usual large deviation principle, combined with $r \in A$, implies that $|p_{n-1}(r)|$ is an exponential factor larger than $|p_n(0)|$ (this follows from the large deviation principle and the fact that $r \in A$).

- We expect $q_n$ to oscillate rather wildly when $n$ is large. However, the oscillation is actually well controlled. We have

$$
\frac{q'_n(z)}{q_n(z)} = \sum_{k=1}^{n-1} \frac{1}{z - z_k},
$$

where the sum ranges over the $n - 1$ remaining roots. Using the triangle inequality to bound the sum from above, we see that we typically expect

$$
\left| \sum_{k=1}^{n-1} \frac{1}{z - z_k} \right| \lesssim n \log n
$$

by standard logarithmic blowup. Naturally, since the sum is again a random object, it could be much bigger if there happen to be other roots close to $r$. This part of the argument actually has a lot of wiggle room and we only require that the sum grows less than $\sim (1 + \varepsilon)^n$ with high likelihood for any fixed $\varepsilon > 0$. We know that $|q'_n|$ is at most a polynomial factor larger than $|q_n|$ with likelihood exponentially close to 1 when $z$ is not extremely close to any of the roots (and, with high likelihood, $r$ is not very close to any other root). This means that $|q_n(z)|$ is still rather large in a small neighborhood of $r$. 

Choose a small neighborhood of \( r \) of size \( \delta \) being exponentially small, that is, \( \delta \sim c^n \) with \( c < 1 \) depending on the size of the logarithmic integral at 0 and at \( r \). As a result, on the neighborhood of \( r \) of size \( \delta \), \((z - r)\) yields an exponential contribution that can be played off against the exponential gain coming from the ratio \(|q_n(r)|/|p_n(0)|\).

In summary, for a typical root \( r \) (in the sense of the exceptional set being \( o(n) \)) contained in the set \( A \), by Rouché’s theorem, there is a solution to \( p_n(z) = p_n(0) \) exponentially close to \( r \). This gives a complete description of the process in the set \( A \) and explains why the measure \( \mu \) is being reproduced as \( n \to \infty \). It remains to show the second part of the statement saying that a total measure of \( 1 - \mu(A) \) is concentrated on \( B \). This is rather easy to see: if \( z_0 \notin A \cup B \), then this implies that

\[
\mathbb{E} \frac{\log |p_n(z_0)|}{n} = \int_C \log |z_0 - x| d\mu(x) < \int_C \log |x| d\mu(x) = \mathbb{E} \frac{\log |p_n(0)|}{n}.
\]

The associated large deviation principle implies that \(|p_n(z)| \ll |p_n(0)|\) in that region which shows that the number of roots there has to be \( o(n) \).

We remark that the argument shows a somewhat stronger statement: roots in the set \( A \) have a solution of \( p_n(z) = p_n(0) \) exponentially close nearby if they are somewhat separated from the other roots (or, put differently, the jump from \( p_n(z) = 0 \) to \( p_n(z) = p_n(0) \) is in the perturbative regime). The precise exponential factor depends on the value of the logarithmic integral in 0 and in the point in which the root occurs.

### 3.2. Proof of Theorem 1.

**Proof.** We use Theorem 2 and compute the sets \( A \) and \( B \). We start by showing that for radial measures \( \mu \), the function

\[
\int_C \log |z - x| \, d\mu(x)
\]

has a global minimum in the origin.

This can be seen rather easily from the elementary observation that

\[
\frac{1}{2\pi} \int_0^{2\pi} \log |z - re^{it}| \, dt = \begin{cases} \log |z| & \text{if } |z| > r \\ \log |r| & \text{if } |z| < r. \end{cases}
\]

Using \( \phi(r) \) to denote the Radon-Nikodym derivative of \( \mu \) with respect to the Lebesgue measure, can write

\[
\int_C \log |z - x| \, d\mu(x) = \int_0^\infty \phi(r) r \int_0^{2\pi} \log |z - re^{it}| \, dt \, dr
\]

\[
= 2\pi \int_0^{\log |z|} \phi(r) r \log |z| \, dr + 2\pi \int_0^z \phi(r) r \log r \, dr
\]

\[
= \int_C \log |x| \, d\mu(x) + 2\pi \int_0^{\log |z|} \phi(r) r \log \frac{|z|}{r} \, dr.
\]

The second integral is always nonnegative, this shows that there is a global minimum in \( z = 0 \). It also allows us to determine

\[
B = \begin{cases} B(0, R) & \text{if } \phi \equiv 0 \text{ on } (0, R) \\ \{0\} & \text{otherwise} \end{cases} \quad \text{and} \quad A = \mathbb{C} \setminus B.
\]

This implies that \( \mu(B) = 0 \) and Theorem 2 then implies that the distribution accurately reproduces \( \mu \) on \( A \). This implies the result. \( \square \)
3.3. Proof of Theorem 3.

Proof. Let $p_n(z)$ be a random polynomial and let $C \subset B$ be a set in the complex plane bounded away from the support of $\mu$ for which

$$\left| \nabla \int_C \log |x-z|d\mu(z) \right| \quad \text{and} \quad \left| \frac{\partial}{\partial t} \int_C \arg(\gamma(t) - z)d\mu(z) \right|$$

are both bounded away from 0 uniformly on $C$. We note that compactness of the support of $\mu$ implies compactness of $B$ and thus $C$ is necessarily bounded. We recall that $B$ is defined as a level set of the logarithmic integral, the first condition thus implies that this level set is non-degenerate and thus $C$ is necessarily a curve. $C$ will be a natural limit set for the set $C_n$ associated to a random polynomial $p_n$ and defined by

$$C_n = \left\{ z \in \mathbb{C} : \sum_{k=1}^n \log |z-z_k| \right\}.$$ 

$C_n$ is the level set of a superposition of random functions and does a priori look quite complicated. However, since we will only be studying it away from the support of $\mu$ in a neighborhood of $C$ and recall the large deviation principle from the proof of Theorem 2, we see that these objects are actually rather rigid. On parts of $C_n$ that are uniformly bounded away from the support of $\mu$, it is easy to see that

$$\lim_{n \to \infty} n^{-1} \sum_{k=1}^n \nabla \log |z-z_k| = \nabla \int_C \log |x-z|d\mu(z) \quad \text{in probability}.$$ 

Moreover, by the same argument this extends to higher derivatives on $C$ since all higher derivatives are uniformly bounded. This shows that for $n$ sufficiently large, $C_n$ is a curve (a segment of which converges uniformly (together with its derivatives) to $C$ as $n \to \infty$). Let us assume that $\gamma_n$ is an arclength parametrization of a segment of $C_n$ on which the assumptions of Theorem 3 apply. $\gamma_n$ then parametrizes a curve on which $|p_n(z)| = |p_n(0)|$. It remains to see whether the arguments of the complex numbers can be matched to produce a solution of the equation. We note that

$$\frac{\partial}{\partial t} \arg \prod_{k=1}^n (\gamma_n(t) - z_k) = \frac{n}{\partial t} \frac{1}{n} \sum_{k=1}^n \arg(\gamma_n(t) - z_k).$$

For $n$ sufficiently large, this quantity converges to

$$\frac{\partial}{\partial t} \frac{1}{n} \sum_{k=1}^n \arg(\gamma_n(t) - z_k) \to \frac{\partial}{\partial t} \int_C \arg(\gamma(t) - z)d\mu(z) \quad \text{in probability},$$

where $\gamma(t)$ is some curve satisfying $\gamma'(t) = \lim_{n \to \infty} \gamma_n'(t)$ (this, of course, leads exactly to an arclength parametrization of $C$). This shows that the argument is asymptotically moving linearly in $n$. Therefore, when $n$ is sufficiently large, with high probability, the argument of $p_n(\gamma_n(t))$ hits the argument $p_n(0)$ at a linear rate. As a result, we have a regular distribution of solutions of the equation along the level set: the argument needs to complete a total revolution of $2\pi$ which accounts for the arising pre-factor. Since the linear rate is $\left| \frac{\partial}{\partial t} \int_C \arg(\gamma(t) - z)d\mu(z) \right|$, the associated measure on $C$ is thus described in $[\square]$. 

It is not difficult to see that the argument can be extended to the setting where $C$ and the measure of $\mu$ are not disjoint (but $\mu$ is still assumed to be absolutely continuous with respect to the Lebesgue measure): the random curve $C_n$ is only minorly impacted by roots nearby (which would need to be exponentially close to have an impact which becomes increasingly unlikely), we leave the details to the interested reader.
3.4. An explicit example. This section is devoted to an explicit computation for what to expect in the example

$$\mu = \frac{1}{4\pi} \left( H_1^{\dagger} \big|_{|z|=1} \cup H_1^{\dagger} \big|_{|z-2|=1} \right)$$

(see Fig. 2). Summarizing the proof, we can fix a point $z \in \mathbb{C}$ and compute

$$E_n^{-1} \log |p_n(z)| = \frac{1}{n} \sum_{k=1}^{n} \log |z - z_k| \to \int_{\mathbb{C}} \log |z - x| d\mu(x)$$

because the likelihood of having singularities nearby is small. Moreover, we have

$$\frac{1}{2\pi} \int_{0}^{2\pi} \log |z - e^{it}| dt = \begin{cases} \log |z| & \text{if } |z| > 1 \\ 0 & \text{if } |z| < 1. \end{cases}$$

Thus,

$$\int_{\mathbb{C}} \log |z - x| d\mu(x) = \begin{cases} \frac{1}{2} \log |z - 2| & \text{if } |z| < 1 \\ \frac{1}{2} \log |z| & \text{if } |z - 2| < 1 \\ \frac{1}{2} \log |z| + \frac{1}{2} \log |z - 2| & \text{otherwise.} \end{cases}$$

This also shows that we expect exponential growth in the origin

$$E_n^{-1} \log |p_n(0)| = \log \frac{2}{2}.$$ 

It remains to find all points in the complex plane for which the logarithmic integral equals that quantity and those are displayed in Figure 2.

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