Research Article

Shengjiang Chen* and Aizhu Xu

Uniqueness on entire functions and their $n$th order exact differences with two shared values

https://doi.org/10.1515/math-2020-0022

received April 2, 2019; accepted February 13, 2020

Abstract: Let $f(z)$ be an entire function of hyper order strictly less than 1. We prove that if $f(z)$ and its $n$th exact difference $\Delta^nf(z)$ share $0$ CM and $1$ IM, then $\Delta^nf(z) \equiv f(z)$. Our result improves the related results of Zhang and Liao [Sci. China A, 2014] and Gao et al. [Anal. Math., 2019] by using a simple method.

Keywords: entire function, exact difference, uniqueness, shared values

MSC 2010: 30D35, 39A10

1 Introduction and main results

We assume the reader is familiar with the fundamental results and standard notations of Nevanlinna’s theory, as found in [1,2], such as the characteristic function $T(r,f)$ of a meromorphic function $f(z)$. Notation $S(r,f)$ means any quantity such that $S(r,f) = o(T(r,f))$ as $r \to \infty$ outside of a possible set of finite logarithmic measures. Moreover, the order $\rho(f)$ and hyper order $\rho_2(f)$ of $f(z)$ are defined as usual as follows:

$$\rho(f) = \limsup_{r \to \infty} \frac{\log^* T(r,f)}{\log r} \quad \text{and} \quad \rho_2(f) = \limsup_{r \to \infty} \frac{\log^* \log T(r,f)}{\log r}. $$

For a value $a \in \mathbb{C} \cup \{\infty\}$, we say that two meromorphic functions $f(z)$ and $g(z)$ share $a$ CM (IM) provided that $f$ and $g$ have the same $a$ – points counting multiplicities (ignoring multiplicities).

About 10 years ago, Halburd and Korhonen [3,4] and Chiang and Feng [5] established the difference analogue of Nevanlinna’s theory for finite-order meromorphic functions, independently. Later, Halburd et al. [6] showed in 2014 that it is still valid for meromorphic functions of hyper-order strictly less than 1. So far, it has been a most useful tool to study the uniqueness problems between meromorphic functions $f(z)$ and their shifts $f(z + c)$ or $n$th exact differences $\Delta^nf(z)$ ($n \geq 1$). For some related results in this topic, we refer the reader to [7–12] and so on.

In 2013, Chen and Yi first proved an uniqueness theorem for a meromorphic function $f(z)$ and its first order exact difference $\Delta_c f(z)$ with three distinct shared values CM in [13], which had been improved by Zhang and Liao [14] in 2014 as follows.

**Theorem A.** [14] Let $f$ be a transcendental entire function of finite order, and $a$, $b$ be two distinct constants. If $\Delta_c f(\neq 0)$ and $f$ share $a$, $b$ CM, then $\Delta_c f \equiv f$. Furthermore, $f(z)$ must be of the following form $f(z) = Z^2 h(z)$, where $h(z)$ is a periodic entire function with period 1.

* Corresponding author: Shengjiang Chen, Department of Mathematics, Ningde Normal University, Ningde 352100, Fujian Province, P. R. China, e-mail: chentrent@126.com

Aizhu Xu: Department of Mathematics, Ningde Normal University, Ningde 352100, Fujian Province, P. R. China, e-mail: xuaizhu@126.com

© 2020 Shengjiang Chen and Aizhu Xu, published by De Gruyter. This work is licensed under the Creative Commons Attribution 4.0 Public License.
Theorem A had been improved by Lü and Lü [15] from “entire function” to “meromorphic function” in 2016. More recently, Gao et al. [16] obtained the following uniqueness theorem concerning the $n$th exact difference.

**Theorem B.** [16] Let $f$ be a transcendental meromorphic function of hyper order strictly less than 1 such that $\Delta^n_{c=1} f(z) \neq 0$. If $f(z)$ and $\Delta^n_{c=1} f(z)$ share three distinct periodic functions $a, b, c \in \hat{S}(f)$ with period 1 CM, then $\Delta^n_{c=1} f(z) \equiv f(z)$.

Here, the notation $\hat{S}(f)$ means $S(f) \cup \{\infty\}$, where $S(f)$ is the set of all meromorphic functions $a(z)$ such that $T(r,a) = S(r,f)$. It is obvious that both Theorems A and B require three shared values CM. So, a nature question is: could those conditions of sharing values be weaken?

In this study, we shall prove a uniqueness theorem for entire functions that share two finite values “$1 CM + 1 IM$” with their $n$th exact differences, by using a simple method which is very differential to the proof of Theorems A and B. In fact, we obtain the following result.

**Theorem 1.1.** Let $f$ be a transcendental entire function of hyper order $\rho(f) < 1$, and let $c \in \mathbb{C}\setminus\{0\}$ such that $\Delta^c_n f(z) \neq 0$. If $f$ and $\Delta^c_n f$ share 0 CM and 1 IM, then $\Delta^c_n f(z) \equiv f(z)$.

**Remark 1.** There exist many entire functions satisfying Theorem 1.1, which are arranged in Section 4. Here, we shall only give an example to illustrate it as follows.

**Example 1.** Let $f(z) = e^{az}e^{iz}$, where $a = \frac{\log(2)}{\pi}$ and $c = \pi$. Then, the $n$th exact difference of $f(z)$ is as follows:

$$
\Delta^n f(z) \equiv \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(z + kc) \equiv \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} e^{az} e^{ikz} \equiv f(z).
$$

**Remark 2.** It is obvious that Theorem 1.1 is invalid for polynomials $f(z)$. Actually, if $f$ and $\Delta^c_n f$ share 0 CM, then we know that the degrees of $f$ and $\Delta^c_n f$ are the same. But, on the other hand, the degree of $\Delta^c_n f$ is strictly less than the degree of $f$ in this case for $n \geq 1$.

**Remark 3.** As per Theorems A and B, we all hope that the restriction on the growth of $f$ can be dropped. But it seems not to be easy. However, we can also find out many entire functions satisfying the difference equation $\Delta^n f(z) = f(z)$ with hyper order greater than 1. Those discussions are arranged in Section 4.

## 2 Some lemmas

To prove our result, we need the following auxiliary results.

**Lemma 2.1.** [3,6] Let $f(z)$ be a nonconstant meromorphic function of hyper order $\rho(f) < 1$ and $c \in \mathbb{C}\setminus\{0\}$, $n$ is a positive integer. Then, for any $a \in \mathbb{C}$, we have

$$
m\left(r, \frac{\Delta^n f(z)}{f(z) - a}\right) = S(r, f).
$$
Lemma 2.2. [2, Theorem 1.38] Suppose that \( f(z) \) is a meromorphic function in the complex plane, and \( a_1, a_2, a_3 \) are three distinct small functions of \( f(z) \). Then,

\[
T(r, f) \leq \sum_{j=1}^{3} N\left(r, \frac{1}{f - a_j}\right) + S(r, f).
\]

To estimate \( N(r, f(z + c)) \) and \( T(r, f(z + c)) \), we need the next result.

Lemma 2.3. [6] Let \( T : [0, +\infty) \to [0, +\infty) \) be a non-decreasing continuous function and let \( s \in (0, \infty) \). If the hyper order of \( T \) is strictly less than one, i.e.,

\[
\limsup_{r \to \infty} \frac{\log \log T(r)}{\log r} = \rho_2 < 1,
\]

and \( \delta \in (0, 1 - \rho_2) \), then

\[
T(r + s) = T(r) + o\left(\frac{T(r)}{r^\delta}\right),
\]

where \( r \) runs to infinity outside of a set of finite logarithmic measures.

Lemma 2.4. [2, Theorem 1.45] Suppose \( h(z) \) is a nonconstant entire function and \( f(z) = e^{h(z)} \), then \( \rho_2(f) = \rho(h) \).

3 Proof of Theorem 1.1

As

\[
\Delta^n f(z) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(z + kc),
\]

we get from Lemma 2.3 that

\[
T(r, \Delta^n f) \leq \sum_{k=0}^{n} T(r, f(z + kc) + S(r, f) \leq (n + 1)T(r, f) + S(r, f). \tag{3.1}
\]

On the other hand, by the assumptions of Theorem 1.1, we know that

\[
T(r, f) \leq N(r, f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f - \frac{1}{T}}\right) + S(r, f) = N\left(r, \frac{1}{\Delta^n f}\right) + N\left(r, \frac{1}{\Delta^n f - 1}\right) + S(r, f) \leq 2T(r, \Delta^n f) + S(r, f). \tag{3.2}
\]

It follows from (3.1) and (3.2) that

\[
S(r, \Delta^n f) = S(r, f) = S(r)
\]

and

\[
\rho_2(\Delta^n f) = \rho_2(f) < 1. \tag{3.3}
\]

Since \( f \) and \( \Delta^n f \) share 0 CM, we have

\[
\Delta^n f = e^h, \tag{3.4}
\]
where $h$ is some entire function. In addition, by using Lemma 2.1, we have

$$T(r, e^h) = S(r).$$  \hspace{1cm} (3.5)

Now, we suppose on the contrary that the assertion of Theorem 1.1 is not true, i.e., $\Delta^nf \neq f$. Hence,

$$e^h \neq 1.$$  \hspace{1cm} (3.6)

Next, by the assumption that $f$ and $\Delta^nf$ share 1 IM, we can deduce from (3.4) and (3.5) that

$$N\left(r, \frac{1}{f - e^h}\right) = N\left(r, \frac{1}{\Delta^nf - 1}\right) \leq S(r).$$  \hspace{1cm} (3.7)

Rewrite formula (3.4) as

$$\Delta^nf - 1 = e^h(f - e^h).$$  \hspace{1cm} (3.8)

Together (3.8) with (3.7), we have

$$N\left(r, \frac{1}{f - e^h}\right) = S(r).$$  \hspace{1cm} (3.9)

Finally, by using the second main theorem for three small functions (Lemma 2.2), we deduce from (3.6), (3.7) and (3.9) that

$$T(r, f) \leq N(r, f) + N\left(r, \frac{1}{f - 1}\right) + N\left(r, \frac{1}{f - e^h}\right) + S(r) = S(r),$$

which is impossible. And this completes the proof of Theorem 1.1.

### 4 Examples and discussions

To construct the proper examples for Theorem 1.1, we recall a result obtained by Ozawa [17]. That is, for an arbitrary number $\sigma \in [1, \infty)$, there exists a periodic entire function $D(z)$ with period $c \neq 0$ such that $\rho(D) = \sigma$. Throughout this section, the notation $D(z)$ always means such an entire function.

**Example 2.** Let $g(z) = e^{a\pi z}D(z)$, where $D(z + c) \equiv D(z)$. Then, we have $g(z + kc) = -g(z)$ if $k$ is odd, and $f(z + kc) = f(z)$ if $k$ is even. And let $f(z) = e^{a\pi}g(z)$ where $a = \frac{\log(2)}{\pi}$. Thus,

$$\Delta^nf(z) \equiv \sum_{k=0}^{n} (-1)^{n-k} \begin{pmatrix} n \\ k \end{pmatrix} f(z + kc) \equiv \sum_{k=0}^{n} (-1)^{n-k} \begin{pmatrix} n \\ k \end{pmatrix} e^{a\pi} e^{akc}g(z + kc)$$

$$\equiv \sum_{k=0}^{n} (-1)^{n-k} \begin{pmatrix} n \\ k \end{pmatrix} (-2)^k e^{a\pi} (-1)^k g(z) \equiv \left[ \sum_{k=0}^{n} (-1)^{n-k} \begin{pmatrix} n \\ k \end{pmatrix} 2^k \right] e^{a\pi} g(z) \equiv f(z).$$

It is clear that there exist many entire functions satisfying Theorem 1.1 from Example 2.

Next, we shall show that there also exist many entire functions satisfying the difference equation $\Delta^nf(z) \equiv f(z)$ with hyper order greater than or equal to 1.
Example 3. Let \( g(z) = e^{\sin z} - e^{\sin z} \) and \( c = \pi \). Then, we also have \( g(z + kc) = -g(z) \) if \( k \) is odd, and \( g(z + kc) = g(z) \) if \( k \) is even. And let \( f(z) = e^{azg(z)} \) where \( a = \frac{\log(-2)}{\pi} \). Thus, the \( n \)th exact difference of \( f(z) \) is as follows:

\[
\Delta^n f(z) \equiv \sum_{k=0}^{n} (-1)^{n-k} \left( \frac{n}{k} \right) e^{az+akc}g(z + kc) = \sum_{k=0}^{n} (-1)^{n-k} \left( \frac{n}{k} \right)^2 e^{az}g(z) \equiv f(z).
\]

In general, we have the following example.

Example 4. Let \( g(z) = e^{D(z)\sin(z \pi)} - e^{-D(z)\sin(z \pi)} \), where \( D(z + c) = D(z) \). It is clear that the hyper order of \( g \) is greater than or equal to 1 and that \( g(z + c) = -g(z) \). Set \( f(z) = e^{azg(z)} \) where \( a = \frac{\log(-2)}{\pi} \). Then,

\[
\Delta^n f(z) \equiv \sum_{k=0}^{n} (-1)^{n-k} \left( \frac{n}{k} \right) e^{az+akc}g(z + kc) \equiv f(z).
\]

Inspired by the above example, we raise the following open problem.

Problem. If \( f(z) \) is a transcendental entire function solution of the difference equation \( \Delta^n f(z) = f(z) \), then \( f(z) \) must be of the form \( f(z) = e^{azg(z)} \), where \( a = \frac{\log(-2)}{\pi} \) and \( g \) satisfies \( g(z + c) = -g(z) \)?

Acknowledgments: This project was supported by the National Natural Science Foundation of China (Grant No. 11801291), the Natural Science Foundation of Fujian Province (Grant No. 2018J01424) and the Training Program of Outstanding Youth Research Talents in Fujian (2018).

References

[1] W. K. Hayman, *Meromorphic Function*, Clarendon Press, Oxford, 1964.
[2] C. Yang and H. Yi, *Uniqueness Theory of Meromorphic Function*, Kluwer Academic Publishers, Dordrecht, 2003.
[3] R. Halburd and R. Korhonen, *Nevanlinna theory for the difference operator*, Ann. Acad. Sci. Fenn. Math. 31 (2006), no. 2, 463–487.
[4] R. Halburd and R. Korhonen, *Difference analogue of the lemma on the logarithmic derivative with applications to difference equations*, J. Math. Anal. Appl. 314 (2006), no. 2, 477–487.
[5] Y. Chiang and S. Feng, *On the Nevanlinna characteristic of f(z + n) and difference equations in the complex plane*, Ramanujan J. 16 (2008), no. 1, 105–129.
[6] R. Halburd, R. J. Korhonen and K. Tohge, *Holomorphic curves with shift-invariant hyperplane preimages*, Trans. Amer. Math. Soc. 366 (2014), no. 8, 4267–4298.
[7] X. Li, H. Yi and C. Kang, *Results on meromorphic functions sharing three values with their difference operators*, Bull. Korean Math. Soc. 52 (2015), no. 5, 1601–1422.
[8] S. Chen, *On uniqueness of meromorphic functions and their difference operators with partially shared values*, Comput. Methods Funct. Theory 18 (2018), no. 3, 529–536.
[9] Z. Chen, *Complex Differences and Difference Equations*, Science Press, Beijing, 2014.
[10] J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo and J. Zhang, *Value sharing results for shifts of meromorphic functions, and sufficient conditions for periodicity*, J. Math. Anal. Appl. 355 (2009), no. 1, 352–363.
[11] J. Heittokangas, R. Korhonen, I. Laine and J. Rieppo, *Uniqueness of meromorphic functions sharing values with their shifts*, Complex Var. Elliptic Equ. 56 (2011), no. 1–4, 81–92.
[12] S. Li and Z. Gao, *Entire functions sharing one or two finite values CM with their shifts or difference operators*, Arch. Math. 97 (2011), no. 5, 475–483.
[13] Z. Chen and H. Yi, *On sharing values of meromorphic functions and their differences*, Results Math. 63 (2013), no. 1–2, 557–565.
[14] J. Zhang and L. Liao, *Entire functions sharing some values with their difference operators*, Sci. China A. 57 (2014), no. 10, 2143–2152.
[15] F. Lü and W. Lü, *Meromorphic functions sharing three values with their difference operators*, Comput. Methods Funct. Theory 17 (2017), no. 3, 395–403.
[16] Z. Gao, R. Kornonen, J. Zhang and Y. Zhang, *Uniqueness of meromorphic functions sharing values with their nth order exact differences*, Anal. Math. 45 (2019), no. 2, 321–334.
[17] M. Ozawa, *On the existence of prime periodic entire functions*, Kodai Math. Sem. Rep. 29 (1978), no. 3, 308–321.