Adaptive Radar Waveform Design Based on Weighted MI and the Difference of Two Mutual Information Metrics

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This study deals with the problem of radar waveform design based on the weighted mutual information (MI) and the difference of two mutual information metrics (DMI) in signal-dependent interference. Since the target and clutter information are included in the received signal at the beginning of the design, DMI-based waveform is designed according to the following criterion: maximizing the MI between the received signal and target impulse response while minimizing the MI between the received signal and the clutter impulse response. This criterion is equivalent to maximizing the difference between the first MI and the second MI. Then maximizing the difference of two types of MI is used as the objective function, and the optimization model with the transmitted waveform energy constraint is established. In order to solve it, we resort to maximum marginal allocation (MMA) method to find the DMI-based waveform. Since DMI-based waveform does not allocate energy to the frequency band where the clutter power spectral density (PSD) is greater than the target PSD, we propose to weight the MI-based waveform and DMI-based waveform to synthesize the final optimal waveform. It could provide different trade-offs between two types of MI. Simulation results show the proposed algorithm is valid.

1. Introduction

Radar performance is not only dependent on the signal processing at receiving end, but also closely related to the transmitted waveform [1]. The transmitter of conventional radar transmits the same fixed waveforms every time regardless of environment factors. Different from the conventional radar, cognitive radar (CR), which was first proposed by Simon Haykin in 2006, has a closed loop feedback structure from the receiver to the transmitter [2]. It could improve the radar performance by sensing the environment and transmitting adaptive waveforms. Waveform optimization, as one of the important technologies of CR, has attracted attention by lots of scholars in the past years.

There are lots of metrics for adaptive waveform design, such as maximum signal-to-noise ratio (SNR) or maximum signal-to-interference-plus-noise ratio (SINR), mean square error, and information theory [3–24]. Pillai proposed to maximize the output SINR as criterion to optimize the waveform by jointly transmitted waveform and the receiver [4]. In [5], the authors used maximum SINR as the objective function to design waveform to improve target detection performance in the case of unknown target impulse response (TIR). The maximum SINR metric is also applied to the robust waveform design, such as [6, 7]. Some scholars extended maximum SNR/SINR criterion to MIMO radar waveform design with different signal models or different constraints [8–11].

Information theory is one of the main methods. In [3], Bell first proposed to maximize the received signal and target impulse response (TIR) to optimize the transmitted waveform. In [12], the authors used a linear combination of the MI between target signals and the corresponding received beams to optimize transmitted waveform for multiple targets. Goodman et al. conducted in-depth research on the optimization of transmission waveform by using the MI.
method [13–19]. Their studies mainly aimed to design optimal waveform at the target identification and improved the waveform optimization method based on maximizing SNR criterion and maximizing MI criterion. In [13], the authors used sequential probability ratio test to weight each target PSD to optimize the waveform at each transmission. In [17], the authors summarized SNR-based waveform design and MI-based waveform design for known target and stochastic target, respectively; they also derived the relationship of MI and SNR for stochastic signals. In [18], the authors derived an enhanced waveform design algorithm, which derived a strict lower bound of MI based on energy spectral variance and optimized an enhanced waveform by maximizing the lower bound. In [19], the authors proposed a waveform design algorithm based on maximizing MI between the measurements and the target signals for Gaussian mixture target range profiling. Yang and Blum [20] extended the maximizing MI criterion to MIMO radar waveform design for target identification and classification. They also proposed the waveform optimization based on minimum mean square error (MMSE) criterion and pointed out that optimal waveforms based on MI criterion and MMSE criterion have the same solution under the same energy constraint. In [21], the authors aimed to maximize the MI between target echoes and TIR to design waveform for the MIMO radar with STAP to detect target both considering the range cell including target and influence of neighboring range cells with different spatial distributions. Maximizing MI criterion has also been extended to robust waveform design. In the actual radar scene, the target prior information is uncertain, which leads to the waveform designed based on the target prior information and cannot satisfy the requirements of parameter estimation; thus, Wang et al. proposed a robust waveform design method for parameter estimation. They also maximized MI between the radar target echo and the random target spectrum response under the hierarchical game model of radar and jammer [22]. Due to uncorrelation of the successive target echoes in some actual scene, a novel waveform design method is presented, which minimized the MI between successive target echoes as criterion to optimize the transmitted waveform [23]. In [24], the authors minimized the MSE of the target scattering coefficients estimation and then designed the transmitted waveform by using minimization of MI between the radar successive target echoes. Tang et al. proposed an algorithm to design MIMO radar waveforms via relative entropy in the clutter; the proposed algorithm is devised based on minimization-maximization approach, which could reduce complexity. In [25], the authors recommended to maximize Kullback–Leibler divergence locally most powerful (KLD-LMP) metric to optimize the transmitted waveform and indicated the relationship between KLD, MI, and SNR. In [26], the authors developed a close-loop adaptive OFDM waveform design approach based on maximizing MI criterion in the presence of signal-dependent clutter. They updated the complex weights of the transmitted OFDM signal to enhance target detection. In [27], they illustrated the relationship of SINR, MI, and probability of detection criterion. With the same constraints, three criteria could optimize the similar waveform. Numerical results part of [27] indicates the mutual information may not be the largest when the SINR reaches the maximum, which is consistent with the simulation results in our paper.

According to the above analysis, the waveform design methods based on mutual information are mainly to maximize the target information in the received signal or minimize the uncorrelated parameters as the optimization criterion. In this paper, we proposed a new CR waveform design scheme, which combine DMI-based waveform algorithm and MI-based waveform algorithm to obtain an optimization waveform; the proposed scheme is divided into two steps as follows:

Step 1: The step is to design transmitted waveform based on DMI criterion with the transmitted energy constraint. The main objective is to maximize MI between the received signal and TIR and minimize MI between the received signal and clutter impulse response (CIR).

Step 2: The final optimal waveform is obtained by weighting the DMI-based waveform and MI-based waveform. Let the sum of the weights of DMI-based waveform and MI-based waveform be equal to 1, so as to ensure the transmitted energy constraint.

The main innovations of our work can be described as follows:

1. We first present a waveform design method based on DMI criterion, which ensures that the receive signals could contain more target information and less clutter information compared with conventional radar.

2. Although the DMI-based waveform can improve the radar performance, it has a disadvantage. DMI-based waveform will not allocate energy to the frequency band where the clutter PSD is greater than the target PSD, which results in received signals containing little target information within these bands. In order solve it, we propose to weight the DMI-based waveform and MI-based waveform to obtain the final optimized waveform. By adjusting the weight, the final optimized waveform can allocate main energy to all frequency bands with strong target PSD. Even if the clutter PSD is greater than the target PSD, more target information can be obtained from the received signal while the clutter information can be reduced. Compared with MI-based waveform, the final optimized waveform could reduce more clutter information when the target information is little loss.

The structure of this paper is as follows. The signal model is given in Section 2. In Section 3, we derive the waveform design based on DMI criterion and use MMA to solve the
optimization model for waveform. In Section 4, we present an improved method to solve the problem of DIM-based waveform. In Section 5, we present the simulation results of the proposed method and discuss them. Finally, we conclude the paper in Section 6.

2. Signal Model

The target channel model in clutter environment is shown in Figure 1. The transmitted waveform $x(t)$ has the energy $E_x$ within time interval $[-(T/2), (T/2)]$. $g(t)$ denotes the target impulse response, which is a Gaussian random process with zero mean and variance $\sigma_g^2(f)$. $n(t)$ is additive Gaussian noise with zero mean and one-sided PSD $P_N(f)$. $c(t)$ denotes the clutter impulse response, which is a Gaussian random process with zero mean and variance $\sigma_c^2(f)$. $g(t)$, $c(t)$, and $n(t)$ are statistically independent. The received signal could be written by

$$y(t) = z(t) + d(t) = x(t) \ast g(t) + x(t) \ast c(t) + n(t),$$  

(1)

where $z(t) = x(t) \ast g(t)$, $d(t) = x(t) \ast c(t) + n(t)$, and the symbol $\ast$ denote convolution.

3. DMI-Based Waveform

3.1. Waveform Design Based on DMI Criterion. The classical waveform optimization method based on MI criterion is to maximize the mutual information between the received signal and TIR by optimizing the transmitted waveform, that is, $\max I_T(y(t); g(t)|x(t))$, where $I_T(\cdot)$ denotes mutual information.

In (1), the received signal $y(t)$ contains two components: the target echo $z(t)$ and the interference $d(t)$. The radar system hopes that the received signal contains more target information and less clutter information. Hence, the MI between the received signal and TIR, $I_T(y(t); g(t)|x(t))$, should be maximum. The MI between the received signal and CIR, $I_C(y(t); c(t)|x(t))$, should be minimum. The optimized waveform based on DMI criterion should satisfy the following:

$$\max I_T(y(t); g(t)|x(t)),$$

$$\min I_C(y(t); c(t)|x(t)).$$  

Expression (2) could be equivalent to

$$\max \left[ I_T - I_C \right] = \max T_y \int |X(f)|^2 \ln \left\{ 1 + \frac{2|X(f)|^2}{T_y P_N(f) + 2|X(f)|^2 \sigma_c^2(f)} \right\} df,$$

(3)

where $X(f)$ is the Fourier transform of $x(t)$, $W$ is the transmitted waveform bandwidth, and $T_y$ is the duration of the received signal. The derivation of (3) is shown in Appendix.

Equation (3) is the optimization criterion based on DMI. Considering the total energy constraint of transmitter, the optimization model for DMI-based waveform is

$$\max \left[ I_T - I_C \right] = \max T_y \int W \ln \left\{ 1 + \frac{2|X(f)|^2}{T_y P_N(f) + 2|X(f)|^2 \sigma_c^2(f)} \right\} df,$$

(4)

$$\text{s.t.} \quad \int W |X(f)|^2 df = E_x.$$

If the clutter does not exist, the optimization model (4) is consistent with the optimization model in literature [3].

3.2. MMA for DMI-based Waveform Solution. The solution of optimization model (4) is solved by MMA method. First, we discretize model (4):

$$\max \left[ I_T - I_C \right] = \max T_y \sum_{k=0}^{N-1} \ln \left\{ 1 + \frac{2|X(f_k)|^2}{T_y P_N(f_k) + 2|X(f_k)|^2 \sigma_c^2(f_k)} \right\} \Delta f_k,$$

(5)

$$\text{s.t.} \quad \sum_{k=0}^{N} |X(f_k)|^2 \Delta f_k = E_x.$$
Let
\[ u(k) = |X(f_k)|^2, \]
\[ a_k = \frac{\sigma_g^2(f_k)}{\sigma_g^2(f_k) - \sigma_c^2(f_k)}, \]
\[ b_k = \frac{T_y P_N(f_k)}{\sigma_g^2(f_k) - \sigma_c^2(f_k)}. \]
Substituting (6)–(8) into (5), we have
\[
\max_{k} \sum_{k=0}^{N-1} \ln \left[ 1 + \frac{u(k)}{a_k u(k) + b_k} \right] df_k
\]
\[ \text{s.t. } \sum_{k=0}^{N-1} u(k) = \frac{E_x}{\Delta f_k} = u_{\max}. \]

For example, let \( N = 2, P = 4, \Delta = 1 \), and \( u(0) + u(1) + u(2) = u_{\max} = 4 \). We wish to maximize
\[
D = L(u(0), 0) + L(u(1), 1) + L(u(2), 2)
\]
\[
= \ln \left( 1 + \frac{u(0)}{2u(0) + 1} \right) + \ln \left( 1 + \frac{u(1)}{2u(1) + 2} \right) + \ln \left( 1 + \frac{u(2)}{3u(2) + 2} \right)
\] (12)

Either 0, 1, 2, 3, or 4 units could be allocated to \( u(0) \), \( u(1) \), and \( u(2) \) with the energy constraint. The possible values of \( L(u(0), k) \) corresponding to the values of \( L(u(k), k) \) for \( u(k) = 0, 1, 2, 3, 4 \) are shown in Table 1. When the energies are initially allocated \( \Delta = 1 \) units, the values of \( L(u(0), k) \) are 0.2877, 0.2231, and 0.1823, respectively. The maximum value 0.2877 is selected. Thus, 1Δ units are allocated to \( k = 0 \) in the first step.

After 1Δ units are allocated to \( k = 0 \), the new marginal energies are obtained at \( k = 1 \) as shown in Table 2. The marginal energies are 0.0488, 0.2231, and 0.1823. The maximum value 0.2231 is selected. Thus, 1Δ units are allocated to \( k = 1 \) in the second step.

Similarly, the new energies are obtained at \( k = 3 \) in the third step as shown in Table 3. The marginal energies are 0.0488, 0.0646, and 0.1823. The maximum value 0.1823 is selected. Thus, 1Δ units are allocated to \( k = 2 \). Table 4 shows the last energies allocation. The maximum value 0.0646 is selected. Table 5 shows the final energies allocation. 1Δ units are allocated to \( k = 0, 2 \Delta \) units are allocated to \( k = 1 \), and 1Δ units are allocated to \( k = 2 \). More details on MMA method could refer to literature \([28]\).

The waveform optimization problem can be transformed to seek the maximum of \( D = \sum_{k=0}^{N-1} L(u(k), k) \) with the constraint \( \sum_{k=0}^{N-1} u(k) = (E_x/\Delta f_k) = u_{\max} \), where
\[
L(u(k), k) = \ln \left[ 1 + \frac{u(k)}{a_k u(k) + b_k} \right],
\] (10)
and \( u_{\max} = (E/\Delta f_k) \) could be divided equally into \( P \) parts, that is, \( P\Delta = u_{\max} \), and \( u(k) \) take values in the set \( \{0, \Delta, 2\Delta, \ldots, P\Delta\} \) for all \( k \), and \( \Delta \) is defined as the minimum energy unit. The energy unit \( \Delta \) is allocated at each step of MMA until all energy units are allocated. If \( L(u(j), j) > L(u(k), k), k \neq j \), then \( u(j) = \Delta \). Repeat the same procedure, and only in the procedure we need to select the value of \( k \) to make (11) reach the maximum or the \( k \) to make the maximum marginal increase:
\[
\{L(\Delta, 0), L(\Delta, 1), \ldots, L(\Delta, j - 1), L(2\Delta, j) - L(\Delta, j), L(\Delta, j + 1), \ldots, L(\Delta, N)\}. \] (11)

4. Weighted MI- and DMI-Based Waveform

4.1. Problem Formulation. In the previous section, DMI-based waveform could allocate the energy reasonably according to the PSDs of target and clutter. However, DMI-based waveform has two disadvantages.

(1) DMI-based waveform only allocates the energy to the frequency bands where target PSD is greater than that of clutter (see Figure 2 in Section 4).

(2) There may be no solution to DMI-based waveform in the frequency domain.

Proof. The optimization model for DMI-based waveform is (4). If using Lagrange multiplier to model (4), we have
\[
L(|X(f)|^2, \lambda) = T_y \int_w \ln \frac{|X(f)|^2 |\sigma_g^2(f) - \sigma_c^2(f)| |X(f)|^2 \sigma_c^2(f)}{T_y P_N(f) + |X(f)|^2 \sigma_c^2(f)} df
\]
\[
- \lambda \left[ E_x - \int_w |X(f)|^2 df \right].
\] (13)

Equation (13) can be equivalently maximized as
\[
\bar{L}(|X(f)|^2, \lambda) = T_y \int \ln \left[ 1 + \frac{|X(f)|^2 |\sigma_g^2(f) - \sigma_c^2(f)|}{T_y P_N(f) + |X(f)|^2 \sigma_c^2(f)} \right] - \lambda |X(f)|^2.
\] (14)

The first derivatives of \( \bar{L}(|X(f)|^2, \lambda) \) with respect to \(|X(f)|^2\) are
The second derivatives of $\tilde{L}(|X(f)|^2, \lambda)$ with respect to $|X(f)|^2$ are

$$\frac{d^2 \tilde{L}(|X(f)|^2, \lambda)}{d(|X(f)|^2)^2} = \frac{-A(\sigma_p^2(f) - \sigma_c^2(f))}{(T_y P_N(f) + |X(f)|^2 \sigma_p^2(f)(T_y P_N(f) + |X(f)|^2 \sigma_c^2(f)))^2}.$$(16)
where
\[ A = \left[T_s \cdot P_N(f)\left(\sigma_y^2(f) + \sigma_c^2(f)\right) + 2|X(f)|^2\sigma_y^2(f)\sigma_c^2(f)\right]T_s^* P_N(f). \]  

(17)

The denominator of (16) is greater than 0, and \( A > 0 \). If DMI-based waveform has solution for optimization model (4), (16) should be less than 0. Thus, \( (\sigma_y^2(f) - \sigma_c^2(f)) \) must be greater than 0. That is, the reason for DMI-based waveform allocates the energy to the frequency band where target PSD is greater than that of the clutter PSD; see Figure 2 in Section 5. If \( (\sigma_y^2(f) - \sigma_c^2(f)) < 0 \), the DMI-based waveform has no solution in all frequency bands. □

4.2. Final Scheme. In order to solve the above disadvantages of DMI-based waveform, the final scheme for waveform design based on weighted MI and DMI criterion is proposed.

The optimization model for waveform based on MI criterion with energy constraint is [29].

\[
\max_{|X(f)|^2} \int_{W} \left[ \ln \frac{2|X(f)|^2\sigma_y^2(f)}{T_s^* P_N(f) + 2|X(f)|^2\sigma_c^2(f)} \right] df 
\]

s.t. \( \int_{W} |X(f)|^2 df = E_x. \)  

(18)

Assuming that \( |\tilde{X}(f)|^2 \) is the optimal solution to model (18) (it can be solved by Lagrange multiplier, MMA, or other methods).

Assuming that \( |\tilde{X}(f)|^2 \) is the optimal solution to model (4) (if \( (\sigma_y^2(f) - \sigma_c^2(f)) > 0 \)), then, the final optimized waveform is obtained by weighting MI-based waveform \( |\tilde{X}(f)|^2 \) and DMI-based waveform \( |\tilde{X}(f)|^2 \); that is, 

\[
|\tilde{X}(f)|^2 = c_1|\tilde{X}(f)|^2 + c_2|\tilde{X}(f)|^2, 
\]

where \( c_1 \) and \( c_2 \) are weights and 

\[ c_1 + c_2 = 1. \]  

(20)

Equation (20) could ensure that final optimal waveform \( |\tilde{X}(f)|^2 \) still has the total energy \( E_x \).

When \( (\sigma_y^2(f) - \sigma_c^2(f)) < 0 \), the optimal waveform \( |\tilde{X}(f)|^2 \) has no solution. In this situation, let \( c_1 = 1 \) and \( c_2 = 0 \). The final optimal waveform \( |\tilde{X}(f)|^2 \) is MI-based waveform \( |\tilde{X}(f)|^2 \).

The final optimization scheme for waveform design is as follows:

Step 1: The radar system estimates the PSDs of the target and clutter.

Step 2: MI-based waveform \( e \) is derived according to model (18).

Step 3: Determine whether \( (\sigma_y^2(f) - \sigma_c^2(f)) \) is greater than 0, if \( (\sigma_y^2(f) - \sigma_c^2(f)) > 0 \), DMI-based waveform \( |\tilde{X}(f)|^2 \) is derived according to model (28), and Step 4 is executed. Otherwise, step 5 is executed.

Step 4: According to (19) and (20), the final optimal waveform is derived by weighting \( |\tilde{X}(f)|^2 \) and \( |X(f)|^2 \).

Step 5: If \( (\sigma_y^2(f) - \sigma_c^2(f)) < 0 \), the final optimal waveform is MI-based waveform \( |X(f)|^2 \).

The algorithm flow is shown in Figure 3.

5. Simulation and Results

In this section, we demonstrate the effectiveness of the proposed method. The signal energy is \( E_s = 10 \) (energy unit). The noise PSD is \( P_N = 0.5 \). The clutter-to-noise ratio (CNR) is CNR = −5.7 dB and the target-to-noise ratio (TNR) is TNR = −12.38 dB. \( P \) is set to 1000. Figure 4 shows the spectrum signatures of the target and clutter.

Figure 2 shows the energy spectral densities (ESD) of MI-based waveform and DMI-based waveform, respectively. The main characteristics of DMI-based waveform are as follows:

(1) DMI-based waveform could allocate the most energy to the frequency bands with the stronger target PSD. Although the target PSD is symmetric about the 0 frequency point, the clutter PSD is stronger in the frequency band \((-5.0, 0)\) than that in the frequency band \((-0.5, 0)\); the DMI-based waveform allocates more energy to the sub two frequency bands with the stronger target PSD in the range of frequency band \((-0.5, 0)\).

(2) Since DMI-based waveform takes into account the minimum mutual information between the received signal and clutter, the energy of DMI-based waveform is only allocated to the frequency bands whose target PSD is stronger than the clutter PSD. When the clutter PSD is greater than target PSD, DMI-based waveform will not allocate energy even if target PSD is still strong.

The performance of DMI-based waveform is verified as shown in Figure 5, and the linear frequency modulated (LFM) waveform is used as a benchmark. Figure 5 shows detection probability curves corresponding to the DMI-based waveform, MI-based waveform, and LFM waveform, respectively. The performance of DMI-based waveform and MI-based waveform
are better than that of LFM waveform. However, the detection probability of DMI-based waveform is slightly lower than that of MI-based waveform. This is due to the fact that, in order to minimize the mutual information between the received signal and CIR, DMI-based waveform does not allocate energy to the frequency band, wherein the clutter PSD is greater than that of target PSD. To solve this problem, weighted MI- and DMI-based waveforms are proposed.

MMA algorithm is applied to solve DMI-based waveform. Since $P\Delta = u_{\text{max}}$, the larger $P$ is, the smaller $\Delta$ is. Figure 6 shows the DMI-based waveform based on MMA algorithm with different $P$ values. If $P$ is growing larger, $\Delta$ will become smaller, and the optimal waveform will be more accurate. Figure 7 shows the performance of different DMI-based waveforms corresponding to different $P$ values. The performance of DMI-based waveform with $P = 1000$ is the worst. The DMI-based waveform with $P = 100$ performs the best. It demonstrates that $P$ is great significance in MMA algorithm; however, when $P$ increases to a certain value, the performance improvement is not obvious.

Figure 8 shows weighted MI- and DMI-based waveforms corresponding to $(c_1 = c_2 = 0.5)$, $(c_1 = 0.8, c_2 = 0.2)$, and $(c_1 = 0.1, c_2 = 0.9)$, respectively. The main characteristics of weighted MI- and DMI-based waveforms are as follows:

1. The ESD of weighted MI- and DMI-based waveforms is between ESDs of MI-based waveform and DMI-based waveform at each frequency point. Weighted MI- and DMI-based waveforms allocate the energy to all frequency bands with strong PSD of target.

2. When both target PSD and clutter PSD are strong and clutter PSD is greater than target PSD, compared with MI-based waveform and DMI-based waveform, weighted MI- and DMI-based waveforms will allocate less energy in these frequency bands rather than no energy, which can not only ensure the acquisition of target information but also reduce clutter information.

The performance of the weighted MI- and DMI-based waveforms is shown in Figure 9; we plot the curves of detection probabilities of several weighted MI- and DMI-based waveforms corresponding to $(c_1 = c_2 = 0.5)$, $(c_1 = 0.8, c_2 = 0.2)$, and $(c_1 = 0.1, c_2 = 0.9)$.
Figure 6: DMI-based waveforms with different $P$ values.

Figure 7: Performance of different DMI-based waveforms corresponding to different $P$ values.

Figure 8: Weighted MI- and DMI-based waveforms with different weights.

Figure 9: Performance of different optimal waveforms.
and $(c_1 = 0.9, c_2 = 0.1)$, respectively. As can be seen from Figure 9, all weighted MI- and DMI-based waveforms with different weights perform better than MI-based waveform. The performance of the weighted MI- and DMI-based waveforms with different weights is similar to that of the MI-based waveform in our simulation scheme. Depending on different weights selected, some weighted MI- and DMI-based waveforms have better performance than MI-based waveform, while some are slightly inferior to MI-based waveform such that when $c_1 = 0.8$ and $c_2 = 0.2$, the performance of the weighted MI- and DMI-based waveforms is very approximate to that of MI-based waveform, and when $c_1 = 0.9$ and $c_2 = 0.1$, the performance of the weighted MI- and DMI-based waveforms is better than that of MI-based waveform, which indicates that different weights affect the performance of optimal waveforms. How to select optimal weights to achieve the best performance of optimal waveform is our future research.

Table 6 shows the SINR, MI between received signal and TIR and MI corresponding to different optimal waveforms in our simulation scheme. DMI-based waveform could obtain the minimum value of MI between the clutter and received signal, while the SINR is also the worst, since, at the same time of reducing the clutter information, the target information also lost the most. Thus, the DMI-based waveform has the worst performance. Weighted waveform $(c_1 = 0.9, c_2 = 0.1)$ has the best SINR.

### 6. Conclusions

In this paper, we have developed radar waveform design based on weighted MI and DMI criterion in the presence of clutter, which takes into account the fact that the received signal contains more target information and as little clutter information as possible. Firstly, the DMI criterion, the maximizing MI between the received signal and TIR and the minimizing MI between the received signal and CIR, is used to establish optimization model with transmitted energy constraint, which is solved by MMA. Since DMI-based waveform is solved only when the target PSD is stronger than the clutter PSD in the same frequency band, we propose weighted MI and DMI-based waveform; it is a trade-off between maximum target information and minimum clutter information. If the clutter PSD is stronger than the target PSD in the entire frequency band, the weighted MI and DMI-based waveform is equal to the MI-based waveform. The performance of weighted MI and DMI-based waveform is dependent on selection of weights. How to optimize the weights adaptively is our future research.

### Appendix

#### Proof for Equation (3):

Considering a random variable $x$ with probability density function (PDF) $p(x)$, the differential entropy is defined by

$$h(x) = -\int_{-\infty}^{\infty} p(x) \log p(x) \, dx. \quad (A.1)$$

Assume there exists another random variable $y$, $x$ and $y$ are jointly distributed. The conditional differential entropy is defined by

$$h(x|y) = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y) \log p(x|y) \, dx \, dy, \quad (A.2)$$

where $p(x|y)$ is the PDF of $x$ conditioned on $y$. The mutual information between two jointly distributed random variables $x$ and $y$ is defined by [3].

$$I(x; y) = h(x) - h(x|y). \quad (A.3)$$

Assuming the random variable $x$ is Gaussian variable with zero mean and variance $\sigma_x^2$, its PDF is

$$p(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\left(\frac{x^2}{2\sigma_x^2}\right)}. \quad (A.4)$$

Substituting (A.4) into (A.1), the differential entropy of $x$ is
\[
 h(x) = -\int_{-\infty}^{\infty} f(x) \log \frac{1}{\sqrt{2\pi\sigma^2_x}} e^{-\frac{x^2}{2\sigma^2_x}} \, dx \\
 = -\int_{-\infty}^{\infty} f(x) \left[ \log \sqrt{2\pi\sigma^2_x} + \frac{x^2}{2\sigma^2_x} \right] \, dx \\
 = \frac{1}{2} \log 2 \pi \sigma^2_x + \frac{1}{2} \\
 = \frac{1}{2} \log 2 \pi \sigma^2_x.
\]

Equation (A.5) demonstrates the entropy of the Gaussian variable with zero mean is dependent on the variance of the Gaussian random variable.

Figure 10 shows the channel of additive Gaussian noise and signal-dependent clutter. \( Z \) is the Gaussian random variable with zero mean and variance \( \sigma^2_Z \) and \( D \) is the signal-dependent clutter, which is Gaussian random process with zero mean and variance \( \sigma^2_D \). \( N \) is the additive Gaussian noise with zero mean and variance \( \sigma^2_N \). The output \( Y \) is the sum of \( Z \), \( D \), and \( N \).

The mutual information between \( Y \) and \( Z \) is

\[
 I(Y; Z) = h(Y) - h(Y|Z).
\]  

Assuming that \( Z \), \( D \), and \( N \) are statistically independent, hence \( Y \) is still Gaussian random variable and its variance is

\[
 \sigma^2_Y = \sigma^2_Z + \sigma^2_D + \sigma^2_N.
\]

According to (A.5), the entropy \( h(Y) \) is

\[
 h(Y) = \frac{1}{2} \ln 2 \pi e \sigma^2_Y = \frac{1}{2} \ln 2 \pi e \left( \sigma^2_Z + \sigma^2_D + \sigma^2_N \right).
\]

According to literature [3], the conditional differential entropy \( h(Y|Z) \) is

\[
 h(Y|Z) = \frac{1}{2} \ln 2 \pi e \left( \sigma^2_Z + \sigma^2_N \right).
\]

Substituting (A.8) and (A.9) into (6.6), the authors have

\[
 I(Y; Z) = h(Y) - h(Y|Z) \\
 = \frac{1}{2} \ln 2 \pi e \left( \sigma^2_Z + \sigma^2_D + \sigma^2_N \right) - \frac{1}{2} \ln 2 \pi e \left( \sigma^2_D + \sigma^2_N \right) \\
 = \frac{1}{2} \ln \left( \frac{\sigma^2_Z + \sigma^2_D + \sigma^2_N}{\sigma^2_D + \sigma^2_N} \right) \\
 = \frac{1}{2} \ln \left( 1 + \frac{\sigma^2_Z}{\sigma^2_D + \sigma^2_N} \right).
\]

Discussing Figure 1 again, the signals \( \tilde{y}_k(t), \tilde{z}_k(t), \tilde{d}_k(t), \) and \( \tilde{\eta}_k(t) \) with frequency components confined to the frequency interval \( F_k = [f_k, f_k + \Delta f] \), according to sampling theorem; each signal can be replaced by a series of samples obtained from uniform sampling rate of \( 2\Delta f \). Assuming that the spectra \( X(f), Z(f), D(f), \) and \( Y(f) \) are smooth and have a constant value for all \( f \in F_k \), the samples of the Gaussian process sampled are statistically independent.

The samples \( \tilde{z}_k(t) \) are independent and identically distributed random variables with zero mean and variance \( \sigma^2_Z \). Over the time interval \( T \), the energy \( E_{\tilde{z}} \) in \( \tilde{z}_k(t) \) is

\[
 E_{\tilde{z}} = 2\Delta f |X(f)|^2 \sigma^2_Z(f_k).
\]  

The energy \( E_{\tilde{z}} \) is evenly spread among \( 2\Delta f T \) statistically independent samples. Hence, the variance of each sample, \( \sigma^2_{\tilde{z}} \), is

\[
 \sigma^2_{\tilde{z}} = \frac{E_{\tilde{z}}}{2\Delta f T} = \frac{2\Delta f |X(f)|^2 \sigma^2_Z(f_k)}{2\Delta f T} = \frac{|X(f)|^2 \sigma^2_Z(f_k)}{T}.
\]

Similarly, the energy \( E_{\tilde{d}} \) in \( \tilde{d}_k(t) \) is

\[
 E_{\tilde{d}} = 2\Delta f |X(f)|^2 \sigma^2_D(f_k).
\]

The energy \( E_{\tilde{d}} \) is evenly spread among \( 2\Delta f T \) statistically independent samples. Hence, the variance of each sample, \( \sigma^2_{\tilde{d}} \), is

\[
 \sigma^2_{\tilde{d}} = \frac{E_{\tilde{d}}}{2\Delta f T} = \frac{2\Delta f |X(f)|^2 \sigma^2_D(f_k)}{2\Delta f T} = \frac{|X(f)|^2 \sigma^2_D(f_k)}{T}.
\]

The energy \( E_N \) of noise \( \tilde{\eta}_k(t) \) is

\[
 E_N = \Delta f P_N(f_k) T.
\]

The energy \( E_N \) is evenly spread among \( 2\Delta f T \) statistically independent samples. Hence, the variance of each sample, \( \sigma^2_{\tilde{\eta}} \), is

\[
 \sigma^2_{\tilde{\eta}} = \frac{E_N}{2\Delta f T} = \frac{P_N(f_k) T}{2}.
\]

Substituting (A.13), (A.14), and (A.16) into (A.10), for each sample \( Z_m \) of \( \tilde{z}_k(t) \) and corresponding sample \( Y_m \) of \( \tilde{y}_k(t) \), the mutual information between \( Z_m \) and \( Y_m \) is

\[
 I(Y_m; Z_m) = \frac{1}{2} \ln \left( 1 + \frac{\sigma^2_Z}{\sigma^2_D + \sigma^2_N} \right) \\
 = \frac{1}{2} \ln \left( 1 + \frac{2|X(f_k)|^2 \sigma^2_Z(f_k)/T + 2|X(f_k)|^2 \sigma^2_D(f_k)/T}{P_N(f_k) T + 2|X(f_k)|^2 \sigma^2_D(f_k)/T} \right) \\
 = \frac{1}{2} \ln \left( 1 + \frac{2|X(f_k)|^2 \sigma^2_Z(f_k)}{P_N(f_k) T + 2|X(f_k)|^2 \sigma^2_D(f_k)/T} \right).
\]

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Similarly, for each sample $D_m$ of $\bar{\Delta}_k(t)$ and corresponding sample $Y_m$ of $\bar{y}_k(t)$, the mutual information between $D_m$ and $Y_m$ is

$$I(Y_m; D_m) = \frac{1}{2} \ln \left(1 + \frac{\sigma_D^2}{\sigma_Z^2 + \sigma_N^2}\right)$$

$$= \frac{1}{2} \ln \left(1 + \frac{2|X(f_k)|^2\sigma_g^2(f_k)}{P_N(f_k)T + 2|X(f_k)|^2\sigma_g^2(f_k)}\right)$$

$$= \frac{1}{2} \ln \left(1 + \frac{2|X(f_k)|^2\sigma_g^2(f_k)}{P_N(f_k)T + 2|X(f_k)|^2\sigma_g^2(f_k)}\right).$$

(A.18)

In the time interval $T$, $Z_m$ and $Y_m$ contain $2\Delta f T$ statistically independent samples. Thus,

$$I(\bar{y}_k(t); \bar{z}_k(t)) = 2\Delta f TI(Y_m; Z_m) = \Delta f T \ln \left(1 + \frac{2|X(f_k)|^2\sigma_g^2(f_k)}{P_N(f_k)T + 2|X(f_k)|^2\sigma_g^2(f_k)}\right),$$

(A.19)

$$I(\bar{y}_k(t); \bar{d}_k(t)) = 2\Delta f TI(Y_m; D_m) = \Delta f T \ln \left(1 + \frac{2|X(f_k)|^2\sigma_g^2(f_k)}{P_N(f_k)T + 2|X(f_k)|^2\sigma_g^2(f_k)}\right).$$

The bandwidth $W$ is divided into many continuous disjoint intervals, and the interval bandwidth is $\Delta f$. According to literature [3], when $\Delta f \rightarrow 0$, the mutual information between $y(t)$ and $z(t)$ can be obtained as

$$I_y(y(t); g(t)|x(t)) = I_y(y(t); z(t)|x(t))$$

$$= T \int \ln \left[1 + \frac{2|X(f)|^2\sigma_g^2(f)}{T_yP_N(f) + 2|X(f)|^2\sigma_g^2(f)}\right] df.$$  

(A.20)

The mutual information between $y(t)$ and $d(t)$ can be obtained as

$$I_C(y(t); c(t)|x(t)) = I_C(y(t); d(t)|x(t))$$

$$= T \int \ln \left[1 + \frac{2|X(f)|^2\sigma_g^2(f)}{T_yP_N(f) + 2|X(f)|^2\sigma_g^2(f)}\right] df.$$  

(A.21)

Substituting (A.20) and (A.21) into $\max_{|X(f)|^2} [I_T - I_C]$, the authors have

$$\max_{|X(f)|^2} [I_T - I_C] = \max_{|X(f)|^2} \left\{ \int \ln \left[1 + \frac{2|X(f)|^2\sigma_g^2(f)}{T_yP_N(f) + 2|X(f)|^2\sigma_g^2(f)}\right] df - \int \ln \left[1 + \frac{2|X(f)|^2\sigma_g^2(f)}{T_yP_N(f) + 2|X(f)|^2\sigma_g^2(f)}\right] df \right\}$$

$$= \max_{|X(f)|^2} \left\{ \int \ln \left(1 + \frac{2|X(f)|^2\sigma_g^2(f)}{T_yP_N(f) + 2|X(f)|^2\sigma_g^2(f)}\right) df \right\}$$

$$= \max_{|X(f)|^2} \left\{ \int \ln \left(1 + \frac{2|X(f)|^2\sigma_g^2(f)}{T_yP_N(f) + 2|X(f)|^2\sigma_g^2(f)}\right) df \right\}$$

$$= \max_{|X(f)|^2} \left\{ \int \ln \left(1 + \frac{2|X(f)|^2\sigma_g^2(f)}{T_yP_N(f) + 2|X(f)|^2\sigma_g^2(f)}\right) df \right\}$$

$$= \max_{|X(f)|^2} \left\{ \int \ln \left[1 + \frac{2|X(f)|^2\sigma_g^2(f)}{T_yP_N(f) + 2|X(f)|^2\sigma_g^2(f)}\right] df \right\}.$$  

(A.22)
Data Availability

All data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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