Reproductive strong solutions of Navier-Stokes equations with non
homogeneous boundary conditions

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Abstract

The object of the present paper is to show the existence and the
uniqueness of a reproductive strong solution of the Navier-Stokes equa-
tions, i.e. the solution $u$ belongs to $L^\infty (0,T;V) \cap L^2 (0,T;H^2 (\Omega))$ and
satisfies the property $u(x,T) = u(x,0) = u_0 (x)$. One considers the
case of an incompressible fluid in two dimensions with nonhomogeneous
boundary conditions, and external forces are neglected.

Key Words: Navier-Stokes equations, incompressible fluid, reproductive solu-
tion, nonhomogeneous boundary conditions.

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1 Introduction and notations

Let $\Omega$ be an open and bounded domain of $\mathbb{R}^2$, with a sufficiently smooth bound-
ary $\Gamma$; and let us consider the Navier-Stokes equations:

$$\begin{cases}
\frac{\partial v}{\partial t} - \nu \Delta v + v.\nabla v + \nabla p = 0 & \text{in} \quad Q_T = \Omega \times ]0,T[,
\text{div } v = 0 & \text{in} \quad Q_T,
v = g & \text{on} \quad \Sigma_T = \Gamma \times ]0,T[,
v(0) = v_0 & \text{in} \quad \Omega.
\end{cases}$$

(1)

where $g$, $v_0$ and $T > 0$ are given. We suppose that :

$$\text{div } v_0 = 0 \quad \text{in} \quad \Omega, \quad v_0.n = 0 \quad \text{on} \quad \Gamma,$$

(2)

and

$$g.n = 0 \quad \text{on} \quad \Sigma_T.$$

(3)

One is interested on one hand by the existence of strong solutions of system
(1). On the other hand, one seeks data conditions to establish the existence of a
reproductive solution generalizing the concept of a periodic solution. Kaniel and
Shinbrot [5] showed the existence of these solutions for system (1) in dimensions 2 and 3 with external forces but zero boundary condition i.e. $g = 0$. With another approach using semigroups, one can also point out the work of Takeshita [10] in dimension 2.

We need to introduce the following functional spaces, with $r$ and $s$ positive numbers:

$$H^{r,s}(Q_T) = L^2([0,T];H^r(\Omega)) \cap H^s([0,T];L^2(\Omega))$$

These are Hilbert spaces for the norm

$$\|v\|_{H^{r,s}(Q_T)} = \left( \int_0^T \|v(t)\|^2_{H^r(\Omega)} \, dt + \|v\|^2_{H^s([0,T];L^2(\Omega))} \right)^{1/2}.$$  

Let us recall that for $s = 1$, for example,

$$\|v\|_{H^1([0,T];L^2(\Omega))} = \left( \int_0^T \left( \|v(t)\|^2_{L^2(\Omega)} + \|\partial v/\partial t\|^2_{L^2(\Omega)} \right) \, dt \right)^{1/2}.$$  

In the same manner one defines spaces $H^{r,s}(\Sigma_T)$.

We now introduce the following spaces:

$$V = \{ v \in D(\Omega)^2; \, \text{div} \, v = 0 \text{ in } \Omega \},$$
$$H = \{ v \in L^2(\Omega); \, \text{div} \, v = 0 \text{ in } \Omega, \, v.n = 0 \text{ on } \Gamma \},$$
$$V = \{ v \in H^0_0(\Omega); \, \text{div} \, v = 0 \text{ in } \Omega \},$$

Let us recall that $V$ is dense in $H$ and $V$ for their respective topologies.

Here, $D(\Omega)$ is the class of $C^\infty$ functions with compact support in $\Omega$. The notations $(,.)$ et $(.,.)$ indicate the scalar products in $L^2(\Omega)$ and in $H^0_0(\Omega)$ respectively, and $|.|$ et $||.||$ the associated norms.

In the order to solve problem (1), we will have to remove boundary condition $g$ and consider a new problem with zero boundary condition. We note that if $v \in H^{2,1}(Q_T)$ is solution of (1), then thanks to the Aubin compactness lemma (see J.L. Lions [8], R. Temam [11]) one will have

$$v \in C^0([0,T];H^1(\Omega)) \hookrightarrow C^0([0,T];H^{1/2}(\Gamma))$$

So that a necessary condition for $v$ to exist is that:

$$g(x,0) = v_0(x), \quad x \in \Gamma.$$  

(4)
Combining (2)-(4), one has:

\[ g \cdot n = 0 \quad \text{on} \quad \Gamma \times [0,T]. \]

The following lemma allows us to state hypotheses on \( g \) (voir Lions-Magenes [7]).

**Lemma 1.1.** Suppose that (4) takes place and let

\[ g \in H^{3/2,3/4}(\Sigma_T), \quad v_0 \in H^1(\Omega). \]  \hspace{1cm} (5)

Then there exists a function \( R \in H^{2,1}(Q_T) \) such that

\[ R = g \quad \text{on} \quad \Sigma_T \quad \text{et} \quad R(0) = v_0 \quad \text{in} \quad \Omega, \]  \hspace{1cm} (6)

and satisfying the estimates

\[ \| R \|_{H^{2,1}(Q_T)} \leq C \left( \| g \|_{H^{3/2,3/4}(\Sigma_T)} + \| v_0 \|_{H^1(\Omega)} \right). \]  \hspace{1cm} (7)

We now consider the problem:

For a given \( g \) verifying (5), one seeks \((u, q)\) which satisfies

\[
\begin{cases}
\frac{\partial u}{\partial t} - \nu \Delta u + \nabla q = 0 & \text{in} \quad Q_T, \\
\operatorname{div} u = \operatorname{div} R & \text{in} \quad Q_T, \\
u u = 0 & \text{on} \quad \Sigma_T, \\
 u(0) = 0 & \text{in} \quad \Omega.
\end{cases}
\]  \hspace{1cm} (8)

The following proposition holds (see Dautray-Lions [2], O. A. Ladyzhenskaya [6], V.A. Solonnikov [9]):

**Proposition 1.2.** We suppose that (5) holds,

\[ \operatorname{div} v_0 = 0 \quad \text{on} \quad \Omega, \quad v_0 \cdot n = 0 \quad \text{in} \quad \Gamma, \quad \text{and} \quad g \cdot n = 0 \quad \text{in} \quad \Sigma_T. \]  \hspace{1cm} (9)

Then problem (8) has an unique solution \((u, q)\) such that

\[ u \in H^{2,1}(Q_T), \quad q \in L^2(0,T;H^1(\Omega)^2) \]

with the estimates

\[ \| u \|_{H^{2,1}(Q_T)} + \| q \|_{L^2(0,T;H^1(\Omega)^2)} \leq C \left( \| g \|_{H^{3/2,3/4}(\Sigma_T)} + \| v_0 \|_{H^1(\Omega)} \right). \]  \hspace{1cm} (10)

Thus the function defined by
\[ \mathbf{G} = \mathbf{R} - \mathbf{u} \quad \text{in } Q_T \quad (11) \]

satisfies the estimates (7) and
\[ \text{div } \mathbf{G} = 0 \quad \text{in } Q_T, \quad (12) \]
\[ \mathbf{G} = \mathbf{g} \quad \text{on } \Sigma_T, \quad (13) \]
\[ \mathbf{G}(\mathbf{x},0) = \mathbf{v}(\mathbf{x},0) \quad \mathbf{x} \in \Omega. \quad (14) \]

This yields the following lemma:

**Lemma 1.3.** Let \( \mathbf{g} \) and \( \mathbf{v}_0 \) satisfy (4), (5) and (9). Then there exists \( \mathbf{G} \in \mathbf{H}^{2,1}(Q_T) \) satisfying (12)-(14) and the estimate
\[ \| \mathbf{G} \|_{\mathbf{H}^{2,1}(Q_T)} \leq C \left( \| \mathbf{g} \|_{\mathbf{H}^{3/2,3/4}(\Sigma_T)} + \| \mathbf{v}_0 \|_{\mathbf{H}^1(\Omega)} \right). \]

Moreover, one has the next lemma

**Lemma 1.4.** Let \( \varepsilon > 0 \) and let \( \mathbf{g} \) and \( \mathbf{v}_0 \) satisfy the hypotheses of lemma 1.3. Then there exists \( \mathbf{G}_\varepsilon \in \mathbf{H}^{2,1}(Q_T) \) such that
\[ \text{div } \mathbf{G}_\varepsilon = 0 \quad \text{in } Q_T, \]
\[ \mathbf{G}_\varepsilon = \mathbf{g} \quad \text{on } \Sigma_T, \]
\[ \| \mathbf{G}_\varepsilon(\cdot,0) \|_{\mathbf{H}^1(\Omega)} \leq C_\varepsilon \| \mathbf{G}(\cdot,0) \|_{\mathbf{H}^1(\Omega)} \]
and
\[ \forall \mathbf{v} \in \mathbf{V}, \quad |b(\mathbf{v}, \mathbf{G}_\varepsilon(t), \mathbf{v})| \leq \beta(\varepsilon, t) \| \nabla \mathbf{v} \|_{L^2(\Omega)}^2 \]
with
\[ \sup_{t \in [0,T]} \beta(\varepsilon, t) \to 0 \text{ when } \varepsilon \to 0. \]

Moreover, there exists an increasing function \( L : \mathbb{R}^+ \to \mathbb{R}^+ \), not depending on \( \varepsilon \), such that
\[ \| \mathbf{G}_\varepsilon \|_{\mathbf{H}^{2,1}(Q_T)} \leq L \left( \frac{\varepsilon}{\| \mathbf{g} \|_{\mathbf{H}^{3/2,3/4}(\Sigma_T)} + \| \mathbf{v}_0 \|_{\mathbf{H}^1(\Omega)}} \right) \left( \| \mathbf{g} \|_{\mathbf{H}^{3/2,3/4}(\Sigma_T)} + \| \mathbf{v}_0 \|_{\mathbf{H}^1(\Omega)} \right). \]

**Proof.**

i) **Step 1**: One takes up again the Hopf construction (see Girault & Raviart [4], Temam [11], Lions [8], Galdi [3]).

ii) **Step 2**: The open domain \( \Omega \) being smooth, and since \( \text{div } \mathbf{G}_\varepsilon = 0 \) in \( Q_T \) and
\(G.n = 0\) on \(\Gamma \times [0, T]\), there exists, for all \(t \in [0, T]\), a function \(\psi\) depending on \(x\) and \(t\), such that

\[
G = \text{rot} \ \psi \quad \text{in} \quad \Omega \times [0, T]
\]

with \(\psi = 0\) on \(\Gamma \times [0, T]\), \(\psi \in L^2(0, T; H^3(\Omega))\), \(\frac{\partial \psi}{\partial t} \in L^2(0, T; H^1(\Omega))\) and satisfying the estimate

\[
\|\psi\|_{L^2(0, T; H^3(\Omega))} + \|\psi_t\|_{L^2(0, T; H^1(\Omega))} \leq C\|G\|_{H^{2,1}(Q_T)}.
\]

**iii) Step 3**: Let

\[
G^\varepsilon = \text{rot} (\theta \varepsilon \psi).
\]

One deduces from the properties of \(\theta \varepsilon\), for \(j = 1, 2\):

\[
|G^\varepsilon_j(x, t)| \leq C \left( \frac{\varepsilon}{\rho(x)} |\psi(x, t)| + |\nabla \psi(x, t)| \right) \quad \text{if} \quad \rho(x) \leq 2\delta(\varepsilon)
\]

and \(G^\varepsilon_j = 0\) if \(\rho(x) > 2\delta(\varepsilon)\).

We note that

\[
\psi \in C \left([0, T]; H^2(\Omega)\right) \hookrightarrow C \left([0, T]; L^\infty(\Omega)\right).
\]

Therefore,

\[
|G^\varepsilon_j(x, t)| \leq C \left( \frac{\varepsilon}{\rho(x)} + |\nabla \psi(x, t)| \right) \quad \text{if} \quad \rho(x) \leq 2\delta(\varepsilon).
\]

Thus, for all \(v \in H^1_0(\Omega),

\[
\|v_i G^\varepsilon_j\|_{L^2(\Omega)} \leq C \left[ \varepsilon \left\|\frac{v_i}{\rho}\right\|_{L^2(\Omega)} + \left( \int_{\rho(x) \leq 2\delta(\varepsilon)} v_i^2 |\nabla \psi|^2 \, dx \right)^{1/2} \right]
\]

\[
\|v_i G^\varepsilon_j\|_{L^2(\Omega)} \leq C \varepsilon \|\nabla v_i\|_{L^2(\Omega)} + C \|\nabla v_i\|_{L^2(\Omega)} \times \left( \int_{\rho(x) \leq 2\delta(\varepsilon)} |\nabla \psi|^3 \, dx \right)^{1/3}
\]

Setting

\[
\beta(\varepsilon, t) = \left( \int_{\rho(x) \leq 2\delta(\varepsilon)} |\nabla \psi|^3 \, dx \right)^{1/3},
\]
it’s clear that
\[ \lim_{\varepsilon \to 0} \beta(\varepsilon, t) = 0 \text{ uniformly on } [0, T]. \]

The second inequality of lemma 1.4 is a consequence of Hölder inequality. The first inequality follows from Hardy inequality for \( H^1(\Omega) \)-functions and properties of \( \theta_\varepsilon \).

\[ \square \]

2 Existence of strong solutions

Let us make a change of the unknown function in problem (1), by setting
\[ u = v - G_\varepsilon, \quad u_0 = v_0 - G_\varepsilon (\cdot, 0), \]
where \( G_\varepsilon \) is the function given by lemma 1.4. Problem (1) then becomes:

\[
\begin{cases}
\frac{\partial u}{\partial t} - \nu \Delta u + u \cdot \nabla u + u \cdot \nabla G_\varepsilon + G_\varepsilon \cdot \nabla u + \nabla p = f_\varepsilon & \text{in } Q_T \\
\text{div } u = 0 & \text{in } Q_T \\
uu u = 0 & \text{on } \Sigma_T \\
uu u(0) = u_0^\varepsilon & \text{in } \Omega
\end{cases}
\tag{16}
\]

with
\[
f_\varepsilon = -\frac{\partial G_\varepsilon}{\partial t} + \nu \Delta G_\varepsilon - G_\varepsilon \cdot \nabla G_\varepsilon \quad \text{and} \quad u_0^\varepsilon = v_0 - G_\varepsilon (\cdot, 0).
\tag{17}
\]

We note that \( u_0^\varepsilon \in V \) and
\[
\| u_0^\varepsilon \|_{H^1(\Omega)} \leq C_\varepsilon \left( \| g \|_{H^{3/2,3/4}(\Sigma_T)} + \| v_0 \|_{H^1(\Omega)} \right).
\tag{18}
\]

Moreover, \( f_\varepsilon \in L^2(0, T; L^2(\Omega)) \) and
\[
\| f_\varepsilon \|_{L^2(0, T; L^2(\Omega))} \leq C_\varepsilon \left( \| g \|_{H^{3/2,3/4}(\Sigma_T)} + \| v_0 \|_{H^1(\Omega)} \right).
\tag{19}
\]

Now we are able to announce and to establish the following theorem:

**Theorem 2.1.** Let \( v_0 \) and \( g \) satisfy the hypotheses of lemma 1.3. Then problem (16) has a unique solution \((u, p)\) such that

\[ u \in L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; V), \quad \frac{\partial u}{\partial t} \in L^2(0, T; H), \quad p \in L^2(0, T; H^1(\Omega)), \]

\[ p \text{ being unique up to an } L^2(0, T)-\text{function of the single variable } t. \]

**Proof.**
2.1 Approximate solutions

We use the Galerkin method. Let \( m \in \mathbb{N}^* \) and \( u_{0m} \in \langle w_1, w_2, \ldots, w_m \rangle \) such that

\[
u m \to u_0^m \text{ in } V, \text{ if } m \to \infty,
\]

where \( w_j \) are the Stokes operator eigenfunctions. For each \( m \), one defines an approximate solution of (16) by :

\[
\begin{align*}
\mathbf{u}_m(t) &= \sum_{j=1}^{m} g_{jm}(t)w_j \\
\mathbf{u}_m(t) &= \mathbf{u}_m(0) + \int_0^t \mathbf{b}(\mathbf{u}_m(s), \mathbf{G}_\varepsilon(s), \mathbf{u}_m(s)) \, ds + \int_0^t \mathbf{f}_\varepsilon(s) \, ds \\
&= \mathbf{u}_m(0) + \int_0^t \mathbf{b}(\mathbf{u}_m(s), \mathbf{G}_\varepsilon(s), \mathbf{u}_m(s)) \, ds + \int_0^t \mathbf{f}_\varepsilon(s) \, ds
\end{align*}
\]

This is a nonlinear differential system of \( m \) equations in \( m \) unknowns \( g_{jm} \), \( j = 1, \ldots, m \) :

\[
\sum_{i=1}^{m} (w_i, w_j) g_{im}(t) + \nu \sum_{i=1}^{m} (w_i, w_j) g_{im}(t) + \sum_{i,t=1}^{m} b(w_i, w_j) g_{im}(t) + \sum_{j,t=1}^{m} (f_{\varepsilon}(t), w_j) g_{im}(t) = (f_{\varepsilon}(t), w_j), \quad j = 1, \ldots, m
\]

2.2 Estimates I

Let us multiply (20) by \( g_{jm}(t) \) and sum over \( j \) :

\[
\frac{1}{2} \frac{d}{dt} |\mathbf{u}_m(t)|^2 + \nu \|\mathbf{u}_m(t)\|^2 = - \mathbf{b}(\mathbf{u}_m(t), \mathbf{G}_\varepsilon(t), \mathbf{u}_m(t)) + (f_{\varepsilon}(t), \mathbf{u}_m(t)) \\
\leq |f_{\varepsilon}(t)| \|\mathbf{u}_m(t)\| + |\mathbf{b}(\mathbf{u}_m(t), \mathbf{G}_\varepsilon(t), \mathbf{u}_m(t))|
\]

One deduces from lemma 1.4 that :

\[
\frac{1}{2} \frac{d}{dt} |\mathbf{u}_m(t)|^2 + \nu \frac{1}{2} \|\mathbf{u}_m(t)\|^2 \leq \frac{1}{2\nu C^2(\Omega)} \|f_{\varepsilon}(t)\|^2 + \beta(\varepsilon, t) \|\mathbf{u}_m(t)\|^2.
\]

As \( \sup_{t \in [0,T]} \beta(\varepsilon, t) \to 0 \) when \( \varepsilon \to 0 \), for a fixed and small \( \varepsilon > 0 \), one has:

\[
\frac{d}{dt} |\mathbf{u}_m(t)|^2 + \frac{\nu}{2} \|\mathbf{u}_m(t)\|^2 \leq \frac{1}{\nu C^2(\Omega)} \|f_{\varepsilon}(t)\|^2.
\]

Integrating (21) from 0 to \( s \), one deduces that:

\[
|\mathbf{u}_m(s)|^2 \leq |\mathbf{u}_0|^2 + \frac{1}{\nu C^2(\Omega)} \int_0^s \|f_{\varepsilon}(t)\|^2 \, dt \\
\leq |\mathbf{u}_0|^2 + \frac{1}{\nu C^2(\Omega)} \|f_{\varepsilon}(t)\|^2_{L^2[0,T;L^2(\Omega)]} \\
\leq C_{\varepsilon} \left( \|g\|^2_{H^{3/2.3}(\Sigma_T)} + \|\mathbf{v}_0\|^2_{H^1(\Omega)} \right)
\]

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Let us multiply (20) by $2.3$. Estimates II

$$u_m \in L^\infty(0, T; H),$$

and $\{u_m\}$ is an equibounded sequence in $L^\infty(0, T; H)$.

Next, thanks to (21), one has:

$$u_m \in L^2(0, T; V),$$

and the sequence $\{u_m\}$ is equibounded in $L^2(0, T; V)$.

### 2.3 Estimates II

Let us multiply (20) by $\lambda_j g_{jm}(t)$ and sum over $j$:

$$\frac{1}{2} \frac{d}{dt} \|u_m(t)\|^2 + \nu |Au_m(t)|^2 + b(u_m(t), u_m(t), Au_m(t)) +$$

$$b(G_\varepsilon(t), u_m(t), Au_m(t)) + b(u_m(t), G_\varepsilon(t), Au_m(t)) = (f_\varepsilon, Au_m(t))$$

where $A$ is the Stokes operator. Let us begin by considering the nonlinear terms.

For the first term, thanks to the Gagliardo-Nirenberg inequality one has

$$|b(u_m(t), u_m(t), Au_m(t))| \leq \|u_m(t)\|_{L^4(\Omega)} \|\nabla u_m(t)\|_{L^4(\Omega)} |Au_m(t)|$$

$$\leq C |u_m(t)|^{1/2} \|u_m(t)\| |Au_m(t)|^{3/2}$$

$$\leq C \|u_m(t)\|^4 + \frac{\nu}{8} |Au_m(t)|^2.$$

In the same way,

$$|b(G_\varepsilon(t), u_m(t), Au_m(t))| \leq \|G_\varepsilon(t)\|_{L^4(\Omega)} \|\nabla u_m(t)\|_{L^4(\Omega)} |Au_m(t)|$$

$$\leq C \|G_\varepsilon(t)\|_{H^1(\Omega)} \|u_m(t)\|^{1/2} |Au_m(t)|^{3/2}$$

$$\leq C \|G_\varepsilon(t)\|_{H^1(\Omega)}^4 \|u_m(t)\|^2 + \frac{\nu}{8} |Au_m(t)|^2.$$

We remark that, according to lemma 1.4, one has:

$$\|G_\varepsilon\|_{L^\infty(0, T; H^1(\Omega))} \leq C \left(\|g\|_{H^{3/2, 3/4}(\Sigma_T)} + \|v_0\|_{H^1(\Omega)}\right).$$

So that

$$|b(G_\varepsilon(t), u_m(t), Au_m(t))| \leq C \|u_m(t)\|^2 + \frac{\nu}{8} |Au_m(t)|^2.$$

Finally,

$$|b(u_m(t), G_\varepsilon(t), Au_m(t))| \leq \|u_m(t)\|_{L^4(\Omega)} \|\nabla G_\varepsilon(t)\|_{L^4(\Omega)} |Au_m(t)|$$

$$\leq C \|u_m(t)\|^2 \|G_\varepsilon(t)\|_{H^2(\Omega)}^2 + \frac{\nu}{8} |Au_m(t)|^2.$$

Hence,
\[ \frac{d}{dt} \| u_m(t) \|^2 + \nu |A u_m(t)|^2 \leq \frac{C}{\nu} |f_\varepsilon(t)|^2 + C \left[ \| u_m(t) \|^4 + \| u_m(t) \|^2 \left( 1 + \| G_\varepsilon(t) \|^2_{H^2(\Omega)} \right) \right]. \]

Let 
\[ \sigma_m(t) = C \left[ \| u_m(t) \|^2 + (1 + \| G_\varepsilon(t) \|^2_{H^2(\Omega)}) \right]. \]

One knows that 
\[ \sigma_m(t) \in L^1(0,T); \]
so that, according to the Gronwall lemma and (24), one has:
\[ u_m \in L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)), \tag{25} \]
and \( \{ u_m \} \) is an equibounded sequence in \( L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)). \)

### 2.4 Estimates III

Let us multiply (20) by \( g_j'(t) \) and sum over \( j \) from 1 to \( m \). Then
\[ |u'_m(t)|^2 = \nu (Au_m(t),u'_m(t)) - b(u_m(t),u_m(t),u'_m(t)) \]
\[ - b(G_\varepsilon(t),u_m(t),u'_m(t)) - b(u_m(t),G_\varepsilon(t),u'_m(t)) + (f_\varepsilon,u'_m(t)). \]

From this, one deduces that
\[ |u'_m(t)|^2 \leq \nu |Au_m(t)| |u'_m(t)| + C \| u_m(t) \|_{L^4(\Omega)} \| \nabla u_m(t) \|_{L^4(\Omega)} |u'_m(t)| \]
\[ + C \| G_\varepsilon(t) \|_{L^4(\Omega)} \| \nabla G_\varepsilon(t) \|_{L^4(\Omega)} |u'_m(t)| + |f_\varepsilon(t)| |u'_m(t)| \]

Using the Gagliardo-Nirenberg inequality, estimates (25) and (19), and lemma 1.4 giving the estimate of \( G_\varepsilon \), one deduces that
\[ u'_m \in L^2(0,T;H), \tag{26} \]
and \( \{ u'_m \} \) is an equibounded sequence in \( L^2(0,T;H). \)

### 2.5 Taking the limit.

It is a consequence of the above estimates that the sequence \( u_m \) has a subsequence \( u_m \), the same notation being used to avoid unnecessary notation overload:
\[ u_m \rightharpoonup u \text{ weakly* in } L^\infty(0,T;V), \tag{27} \]
\[ u_m \rightharpoonup u \text{ weakly in } L^2(0,T;H^2(\Omega)), \tag{28} \]
\[ u'_m \rightharpoonup u' \text{ weakly in } L^2(0,T;H). \tag{29} \]

But we have a compact embedding
\{ v \in L^2 (0, T; H^2 (\Omega) \cap V) , \ V' \in L^2 (0, T; H) \} \quad \rightarrow \quad L^2 (0, T; V)

So that
\[ u_m \rightarrow u \text{ strongly in } L^2 (0, T; V) \text{ and a.e. in } QT \]

Let \( m_0 \) be fixed and \( v \in \langle w_1, w_2, ..., w_{m_0} \rangle \). Let \( m \) tend towards +\( \infty \) in (20).

Then
\[
(u' (t), v) + \nu ((u (t), v)) + b (u (t), u (t), v) + b (u (t), G_\varepsilon (t), v) + b (G_\varepsilon (t), u (t), v) = (f_\varepsilon (t), v),
\]

This last relation being valid for all \( m_0 \), it remains true for all \( v \in \langle w_1, w_2, ..., w_m \rangle \), \( \forall m \in \mathbb{N}^* \).

Finally let \( v \in V \). There exists \( v_m \in \langle w_1, w_2, ..., w_m \rangle \) such that \( v_m \rightarrow v \) in \( V \) and

\[
(u' (t), v) + \nu ((u (t), v)) + b (u (t), u (t), v) + b (u (t), G_\varepsilon (t), v) + b (G_\varepsilon (t), u (t), v) = (f_\varepsilon (t), v)
\]

Now let us note that for all \( t \in [0, T] \),
\[ u_m (t) \rightarrow u (t) \text{ weakly in } V, \]

and thus
\[ u_m (0) = u_{0m} \rightarrow u (0) \text{ weakly in } V. \]

Since
\[ u_{0m} \rightarrow u_0^\varepsilon \text{ in } V, \]

we have:
\[ u (0) = u_0^\varepsilon. \]

### 2.6 Existence of pressure.

From (31), one has, for all \( v \in V \),
\[
\langle u' - \nu \Delta u + B (u, u) + B (u, G_\varepsilon) + B (G_\varepsilon, u) - f_\varepsilon , v \rangle_{H^{-1}(\Omega) \times H_0^1 (\Omega)} = 0.
\]

Consequently, there exists a unique function \( p \) of \( L^2 (0, T) \) satisfying (16) and such that :
\[ p \in L^2 (0, T; H^1 (\Omega)). \]

This ends the proof of theorem 2.1. \( \square \)
3 Uniqueness Theorem

Theorem 3.1 Problem (16) has a unique solution.

Proof.
Let \( u \) and \( v \) be two solutions satisfying the hypotheses of theorem 2.1 and let \( w = u - v \). Then one has

\[
\frac{\partial w}{\partial t} - \nu \Delta w + w.\nabla u + v.\nabla w + w.\nabla G_\varepsilon + G_\varepsilon.\nabla w = 0
\]

Multiplying by \( w \), we obtain

\[
\frac{1}{2} \frac{d}{dt} |w(t)|^2 + \nu \|w(t)\|^2 = -(w.\nabla u, w) - (v.\nabla w, w) - (w.\nabla G_\varepsilon, w) - (G_\varepsilon.\nabla w, w)
\]

But \( b(v, w, w) = 0 \) and \( b(G_\varepsilon, w, w) = 0 \). This yields

\[
\frac{1}{2} \frac{d}{dt} |w(t)|^2 + \nu \|w(t)\|^2 = -b(w, u, w) - b(w, G_\varepsilon, w).
\]

One then integrates with respect to \( t \) and we get

\[
\frac{1}{2} |w(t)|^2 + \nu \int_0^t \|w(s)\|^2 ds = - \int_0^t b(w, u, w) ds - \int_0^t b(w, G_\varepsilon, w) ds.
\]

Since

\[
\left| \int_0^t b(w, u, w) ds \right| \leq C_1 \int_0^t \|w(s)\|_{L^2(\Omega)} \|u(s)\|_{L^2(\Omega)} ds
\]

\[
\leq C_2 \int_0^t \|w(s)\| \|u(s)\| ds
\]

\[
\leq \frac{\nu}{2} \int_0^t \|w(s)\|^2 ds + C_3 \int_0^t \|w(s)\|^2 \|u(s)\|^2 ds.
\]

and, by the same way,

\[
\int_0^t b(w, G_\varepsilon, w) ds \leq \frac{\nu}{2} \int_0^t \|w(s)\|^2 ds + C_4 \int_0^t \|w(s)\|^2 \|\nabla G_\varepsilon(s)\|^2 ds.
\]

it follows that

\[
|w(t)|^2 \leq C_5 \int_0^t |w(s)|^2 \left( |\nabla G_\varepsilon(s)|^2 + \|u(s)\|^2 \right) ds
\]

Thanks to the Gronwall lemma, one deduces \( w = 0 \).

4 Existence of strong reproductive solution

We first recall results obtained by Kaniel et Shinbrot [5] in the study of the following problem :

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \nu \Delta u + u.\nabla u + \nabla p &= f & \text{in} & & QT \\
\text{div} u &= 0 & \text{in} & & QT \\
u &= 0 & \text{on} & & \Sigma_T \\
u(0) &= u_0 & \text{in} & & \Omega
\end{aligned}
\]
where $\Omega$ is an open and bounded domain of $\mathbb{R}^3$, with a smooth boundary $\Gamma$.

The following result establishes the property of a reproductive solution

**Theorem 4.1.** Let $T > 0$, and $f \in B_{R,T}$ with $f$ small enough. Then, there exists an unique function $u_0$, independent of $t$, with $\nabla u_0 \in B_{R,T}$ and such that the solution of (32) reproduces its initial value at $t = T$:

$$u(x,T) = u(x,0) = u_0(x),$$

where

$$B_{R,T} = \left\{ u \in L^\infty(0,T;L^2(\Omega)) : \|u\|_{L^\infty(0,T;L^2(\Omega))} \leq R \right\}.$$

We begin by recalling the following lemma.

**Lemma 4.2.** If

$$u \in L^2(0,T;H^2(\Omega) \cap V) \quad \text{and} \quad u' \in L^2(0,T;H)$$

then

$$u \in C([0,T];V)$$

and

$$\frac{d}{dt}\|u(t)\|^2 = -2(u'(t),\nabla u(t)).$$

Now, let

$$v_0 \in H^1(\Omega) \cap H, \quad w_0 \in H^1(\Omega) \cap H, \quad g \in H^{3/2,3/4}(\Sigma_T) \quad (33)$$

with

$$g.n = 0 \quad \text{on} \quad \Sigma_T \quad \text{and} \quad v_0(x) = w_0(x) = g(x,0) \quad x \in \Gamma. \quad (34)$$

With these assumptions, it follows from theorem 2.1 that system (1), with data $(v_0, g)$, (respectively $(w_0, g)$), has an unique solution

$$v \in L^2(0,T;H^2(\Omega) \cap H) \cap L^\infty(0,T;H^1(\Omega)) \quad \text{and} \quad v' \in L^2(0,T;H),$$

(respectively

$$w \in L^2(0,T;H^2(\Omega) \cap H) \cap L^\infty(0,T;H^1(\Omega)) \quad \text{and} \quad w' \in L^2(0,T;H).$$

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Let us now set \( z = v - w \). Then

\[
\begin{aligned}
\partial_t z - \nu \Delta z + w \cdot \nabla z + z \cdot \nabla v + \nabla r &= 0 \\
\text{div } z &= 0 \\
z &= 0 \\
z(0) &= v_0 - w_0
\end{aligned}
\]

in \( Q_T \), (35)

where \( r = p - q \) (\( q \) being the pressure corresponding to \( w \)).

**Lemma 4.3.** If

\[
\max \left( \|v\|_{L^\infty(0,T;H^1(\Omega))}, \|w\|_{L^\infty(0,T;H^1(\Omega))} \right) \leq M
\]

(36)

under the assumptions (33) and (34) with \( 0 < M < 1 \), then

\[
\frac{d}{dt} \|z(t)\|^2 + \nu \|z(t)\|^2 \leq 0
\]

(37)

and thus, for all \( t \in [0,T] \),

\[
\|v(t) - w(t)\| \leq \|v_0 - w_0\| \exp(-\nu t).
\]

(38)

**Proof.**

Let \( P: \mathbf{L}^2(\Omega) \rightarrow \mathbf{H} \) be the orthogonal projection operator. Then

\[
\forall \varphi \in \mathbf{H}, (\nabla r, \varphi) = 0.
\]

In particular, let us multiply (35) by \( P\nabla z = Az \):

\[
\frac{1}{2} \frac{d}{dt} \|z(t)\|^2 + \nu \|Az\|^2 = -(w \cdot \nabla z, Az) - (z \cdot \nabla v, Az)
\]

But

\[
|(w \cdot \nabla z, Az)| \leq \|w\|_{L^1(\Omega)} \|\nabla z\|_{L^2(\Omega)} |Az|
\]

\[
\leq C \|w\| \|Az\|^2
\]

and

\[
|(z \cdot \nabla v, Az)| \leq \|z\|_{L^\infty(\Omega)} \|v\| |Az|
\]

\[
\leq C \|v\| \|Az\|^2.
\]

So that if

\[
C \left( \|v\|_{L^\infty(0,T;H^1(\Omega))} + \|w\|_{L^\infty(0,T;H^1(\Omega))} \right) \leq \frac{\nu}{2}
\]

then

\[
\frac{d}{dt} \|z(t)\|^2 + \nu \|z(t)\|^2 \leq 0
\]

and one deduces (38). \( \square \)
4.1 The main result

Lemma 4.4. Suppose that \( g \) and \( v_0 \) satisfy hypotheses (4)-(5) and (9). Let us suppose moreover that \( f_\varepsilon \in L^\infty (0, T; L^2(\Omega)) \) and that

\[
\|g\|_{H^{3/2, 3/4}(\Sigma_T)} + \|v_0\|_{H^1(\Omega)} \leq \alpha \quad (39)
\]

\[
\|f_\varepsilon\|_{L^\infty(0, T; L^2(\Omega))} \leq K \quad (40)
\]

with \( \alpha > 0 \) and \( 0 < K < 1 \). Then, if \( u \) is the solution given by theorem 2.1, one has:

\[
\sup_{t \in [0, T]} \|\nabla u(t)\|_{L^2(\Omega)} \leq M \quad (41)
\]

Remark 4.5. Let us recall that

\[
u_0 = v_0 - G_\varepsilon (., 0)
\]

Consequently, if hypothesis (39) takes place, one has from lemma 1.4 :

\[
\|u_0\| \leq \|u_0\|_{H^1(\Omega)} \leq \|v_0\|_{H^1(\Omega)} + \|G_\varepsilon (., 0)\|_{H^1(\Omega)}
\]

\[
\leq \|v_0\|_{H^1(\Omega)} + L \left( \|g\|_{H^{3/2, 3/4}(\Sigma_T)} + \|v_0\|_{H^1(\Omega)} \right)
\]

\[
\leq \alpha (L + 1) = M. \square
\]

Proof of lemma 4.4. (see Batchi [5])

Let us multiply (16) by \( Au \) and integrate on \( \Omega \) :

\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 + \nu \|Au\|^2 \leq \int_{\Omega} f_\varepsilon . Au dx - \int_{\Omega} (u.Au) . Au dx
\]

\[
- \int_{\Omega} (u.\nabla G_\varepsilon) . Au dx - \int_{\Omega} (G_\varepsilon . Au) . Au dx
\]

But

\[
\left| \int_{\Omega} (u.\nabla u) . Au \ dx \right| \leq \|u\|_{L^\infty(\Omega)} \|u\| \|Au\|
\]

\[
\leq C_1 \|u\| \|Au\|^2,
\]

where \( C_1 \) is such that \( \|u\|_{L^\infty(\Omega)} \leq C_1 \|Au\| \).

In the same way, one also has

\[
\left| \int_{\Omega} (u.\nabla G_\varepsilon) . Au \ dx \right| \leq C_1 \|\nabla G_\varepsilon\|_{L^2(\Omega)} \|Au\|^2
\]

But thanks to the lemma 1.4, one knows that

\( G_\varepsilon \in L^\infty (0, T; H^1(\Omega)) \)

and
Let \( \phi \) which implies that
\[
\|G\|_{H^{1/2}(Q_T)} + \|v_0\|_{H^1(\Omega)}
\]
\[
\leq C_3 a.
\]

It then follows that
\[
\int_\Omega (G \cdot \nabla u) \cdot Au \, dx \leq \|G\|_{L^1(\Omega)} \|\nabla u\|_{L^1(\Omega)} |Au| + C_4 \|G\|_{H^{1/2}(\Omega)} |Au| \|\nabla u\|^{1/2}_{L^2(\Omega)} \|\nabla^2 u\|^{1/2}_{L^2(\Omega)}
\]
\[
\leq C_5 \alpha \|u\|^{1/2} |Au|^{3/2}
\]
\[
\leq C_5 \alpha \sqrt{c_6} |Au|^2,
\]
with \( \|u\| \leq C_6 |Au| \).

Thus,
\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 + \nu |Au|^2 \leq |f_e| |Au| + C_1 \|u\| |Au|^2 + C_1 C_3 \alpha |Au|^2 + C_5 \alpha \sqrt{c_6} |Au|^2.
\]

Let \( \varphi(t) = \|u(t)\| \)

i) Let us first suppose that \( \|u_0\| < M \).

Let \( t_0 > 0 \) be the smallest \( t > 0 \) such that \( \varphi(t_0) = M \). According to (41), one then has
\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|^2_{t=t_0} + \nu |Au(t_0)|^2 \leq K |Au(t_0)| + C_1 M |Au(t_0)|^2
\]
\[
+ C_1 C_3 \alpha |Au(t_0)|^2 + C_5 \alpha \sqrt{c_6} |Au(t_0)|^2.
\]

Let us choose \( \alpha \) sufficiently small and \( K \) such that
\[
K = \frac{\nu}{8} \frac{1}{c_6} M, \quad (C_1 M + C_1 C_3 \alpha + C_5 \alpha \sqrt{c_6}) \leq \frac{3\nu}{8}
\]

Then
\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|^2_{t=t_0} + \nu |Au(t_0)|^2 \leq \frac{\nu}{8} \frac{1}{c_6} M |Au(t_0)| + \frac{3\nu}{8} |Au(t_0)|^2
\]
\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|^2_{t=t_0} + \nu |Au(t_0)|^2 \leq \frac{\nu}{8} \frac{1}{c_6} \|u(t_0)\| |Au(t_0)| + \frac{3\nu}{8} |Au(t_0)|^2
\]
\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|^2_{t=t_0} + \nu |Au(t_0)|^2 \leq \frac{\nu}{2} |Au(t_0)|^2.
\]

Thus
\[
\frac{d}{dt} \|u(t)\|^2_{t=t_0} + \nu |Au(t_0)|^2 \leq 0
\]

which implies that
\[
\frac{d}{dt} \|u(t)\|^2_{t=t_0} \leq 0
\]

Consequently, there exists \( t^* \in [0, t_0] \) such that
\[
\varphi(t^*) > \varphi(t_0), \text{ in contradiction with the definition of } t_0.
\]
Therefore
\[ \forall t \in [0, T], \varphi(t) < M. \]

ii) Suppose now that \( \|u_0\| = M. \)

According to the above calculations, one verifies that \( \varphi'(0) < 0 \) and thus there exists \( t^* > 0 \) such that
\[ \forall t \in [0, t^*], \varphi(t) < M. \]

Repeating the reasoning made in i), one shows that on \([t^*, T], \varphi(t) < M, \) and this ends the proof. □

**Remark 4.6.** From now on, we assume that \( g \) does not depend on time. More precisely, it is supposed that
\[ g \in H^{3/2}(\Gamma), \quad g \cdot n = 0 \text{ on } \Gamma. \] (43)

One recalls that \( v_0 \in H^1(\Omega) \) satisfies
\[ \text{div} \, v_0 = 0 \text{ in } \Omega, \quad v_0 \cdot n = 0 \text{ on } \Gamma \] (44)
and that
\[ v_0 = g \quad \text{on } \Gamma. \] (45)

One knows that there exists \( G \in H^2(\Omega) \) such that
\[ \begin{cases} \text{div} \, G = 0 & \text{in } \Omega, \\ G = g & \text{on } \Gamma, \end{cases} \] (46)
with
\[ \|G\|_{H^2(\Omega)} \leq C \|g\|_{H^{3/2}(\Gamma)}. \] (47)

Processing as in lemma 1.4, one shows the existence, for all \( \varepsilon > 0, \) of \( G_\varepsilon \in H^2(\Omega) \) satisfying (44)-(47) and the estimates:
\[ \forall v \in V, \quad |b(v, G_\varepsilon, v)| \leq \varepsilon \|g\|^2 \] (48)

The right side \( f_\varepsilon \) in system (16) then becomes independent of time and satisfies
\[ f_\varepsilon \in L^\infty\left(0, T; L^2(\Omega)^2\right) \] (49)

In the same way, \( u_0^\varepsilon \) becomes
\[ u_0^\varepsilon = v_0 - G_\varepsilon \] (50)
with \( G_\varepsilon \) depends only on \( g. \Box \)
4.2 Reproductive solution result

With these assumptions on \( g \) and \( v_0 \), lemma 4.2 remains naturally valid and one is able to establish the theorem which follows:

**Theorem 4.7.** Let \( g \in H^{3/2}(\Gamma) \) such that \( g \cdot n = 0 \) on \( \Gamma \) and

\[
\|g\|_{H^{3/2}(\Gamma)} \leq \alpha
\]  

with \( 0 < \alpha << 1 \). Then, there exists \( v_0 \in H^1(\Omega) \) such that \( \text{div} v_0 = 0 \) in \( \Omega \) and \( v_0 = g \) on \( \Gamma \), and such that the solution \( v = u + G_\varepsilon \) where \( u \) is given by theorem 2.1, is reproductive:

\[
v(T) = v(0) = v_0.
\]

**Proof.** Let \( G_\varepsilon \in H^2(\Omega) \) be the extension of \( g \) satisfying (45)-(47) and

\[
f_\varepsilon = \nu \Delta G_\varepsilon - G_\varepsilon \cdot \nabla G_\varepsilon
\]

Let \( u^0_\varepsilon = v_0 - G_\varepsilon \in V \) and \( u \in L^2 \left( 0, T; H^2(\Omega) \right) \cap L^\infty \left( 0, T; V \right) \) be the unique solution of (16). We note that the function \( v = u + G_\varepsilon \) is the unique solution of the initial problem (1). As in the proof of lemma 4.3, it is clear that if \( \|u^0_\varepsilon\| < M \), then

\[
\sup_{t \in [0,T]} \|u(t)\| \leq M
\]

provided that \( \|f_\varepsilon\|_{L^2(\Omega)} \) is sufficiently small, which follows from (49).

Let us define the application

\[
L : u^0_\varepsilon \rightarrow u(., T)
\]

\[
B_M \rightarrow B_M
\]

where \( B_M = \{ z \in V, \|z\| \leq M \} \);

\( u(., T) \) being the unique solution of (16) at \( t = T \).

Moreover, as in remark 4.5, it is clear that if \( \|v_0\| \leq \alpha \) and \( \|w_0\| \leq \alpha \) then

\[
\|u^0_\varepsilon\| \leq M \quad \text{and} \quad \|w^0_\varepsilon\| \leq M,
\]

with \( y^0_\varepsilon = w_0 - G_\varepsilon \).

So that

\[
L u^0_\varepsilon(t) - Ly^0_\varepsilon(t) = u(t) - y(t) = u(t) - G_\varepsilon - (y(t) - G_\varepsilon) = v(t) - w(t),
\]

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and, according to lemma 4.2

$$\|L u_0^\varepsilon (t) - L y_0^\varepsilon (t)\| = \|v (T) - w (T)\|$$

$$\leq \| v_0 - w_0 \| \exp (-\nu T)$$

$$\leq \| u_0^\varepsilon - y_0^\varepsilon \| \exp (-\nu T)$$

Thus $L$ is a contraction and has a fixed point. $\square$

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