On Generalized Abelian Deformations

Giuseppe Dito

Research Institute for Mathematical Sciences
Kyoto University, Sakyo-ku, Kyoto 606-01, Japan

Département de Mathématiques
Université de Bourgogne, BP 400, F-21011 Dijon Cedex, France

Abstract: We study sun-products on $\mathbb{R}^n$, i.e. generalized Abelian deformations associated with star-products for general Poisson structures on $\mathbb{R}^n$. We show that their cochains are given by differential operators. As a consequence, the weak triviality of sun-products is established and we show that strong equivalence classes are quite small. When the Poisson structure is linear (i.e., on the dual of a Lie algebra), we show that the differentiability of sun-products implies that covariant star-products on the dual of any Lie algebra are equivalent each other.

1 Introduction

A new kind of deformations was introduced in [4] in connection with the quantization of Nambu-Poisson structures (see also [7]). The main feature of these deformations is that they are not of Gerstenhaber’s type [8] in the sense that one does not have a $\mathbb{K}[[\nu]]$-algebra structure on the deformed algebra ($\mathbb{K}$ is the ring over which is defined the original algebra $A$ and $\nu$ denotes the deformation parameter). More precisely, these deformations are not linear with respect to the deformation parameter; the product operation annihilates the deformation parameter so that one has only a $\mathbb{K}$-algebra structure on the deformed algebra $A[[\nu]]$.

The motivation for dealing with these generalized deformations was that they provide non trivial Abelian deformations of the usual product, and this point was essential for the solution proposed in [4] for the quantization of Nambu-Poisson structures. We recall that in Gerstenhaber’s framework, Abelian deformations of the usual product of smooth functions on some manifold are always trivial (it is a consequence of the fact that a symmetric Hochschild 2-cocycle is a coboundary).

Explicit examples of generalized Abelian deformations were constructed in [4, 5]. There are two main classes of generalized Abelian deformations. On the one hand, one has the Zariski products introduced in [4] which involve factorization of polynomials in several variables into irreducible factors. Zariski products are Abelian products on the semi-group algebra generated by irreducible polynomials and can be constructed from any star-product

\footnote{Supported by the Japan Society for the Promotion of Science and the Conseil régional de Bourgogne.}
on $\mathbb{R}^n$. Originally the construction of a Zariski product was performed from a Moyal product and it appeared crucial to go over semi-group algebras with a proper notion of derivatives to fulfill algebraic requirements imposed by the Fundamental Identity of Nambu-Poisson structures. This construction is quite sophisticated and little is known about its properties. Actually the Zariski quantization induced by some Zariski product shares many properties with second quantization (appearance of a Fock space generated by irreducible polynomials, etc.).

On the other hand, sun-products have been studied in [5]. They have much simpler properties than Zariski products and, roughly speaking, they can be as seen as the finite dimensional version of Zariski products. They involve factorization into linear polynomials and can be defined on some algebra of functions over finite dimensional spaces.

Still generalized deformations have to find an appropriate algebraic framework and it is the aim of this paper to study sun-products on $\mathbb{R}^n$ and to clarify their structure. Our main result is that sun-products are differentiable deformations, i.e., their cochains are differential operators vanishing on constants. This fact allows us to find a complete characterization of the cochains of a sun-product: Any sequence of differential operators vanishing on linear polynomials defines a sun-product and vice-versa.

After briefly recalling the most basic facts on star-products and Hochschild cohomology, Sect. 2 provides a study of sun-products associated with star-products on $\mathbb{R}^n$ endowed with a general Poisson structure. We show in Theorem 2 the differentiability of sun-products and deduce some consequences of this property.

We then specialize our discussion to the important case of the dual of a Lie algebra in Sect. 3. Consider a Lie algebra $\mathfrak{g}$. Its dual $\mathfrak{g}^*$ is endowed with a canonical Poisson structure. We show that Gutt’s star-product on $\mathfrak{g}^*$ is the only covariant star-product on $\mathfrak{g}^*$ whose associated sun-product coincides with the usual product on $C^\infty(\mathfrak{g}^*)$. From the differentiability of sun-products one shows that covariant star-products on the dual of any Lie algebra are equivalent each other.

In Sect. 4, as another consequence of the differentiable nature of sun-products we show that sun-products are weakly trivial in the sense of [5]. We said that two sun-products are weakly equivalent if there exists an invertible formal series of differential intertwining these sun-products. The sun-product operation kills all of the non-zero powers of the deformation parameter. Weak triviality of a sun-product means weak equivalence with the usual product (on the undeformed algebra). When one allows the deformation parameter coming from the equivalence operator not to be annihilated by the sun-product, one gets the notion of strong equivalence of sun-products. By a simple argument, we remark that strong equivalence classes are rather small. We think that these results might be helpful or give some hints for the definition of a cohomology adapted to generalized deformations.
2 Sun-products on \(\mathbb{R}^n\)

2.1 Notions on star-products

We summarize here basic facts about star-products that we shall need in the present paper. The general reference on star-products theory are the papers [1, 2].

Let \(M\) be a Poisson manifold with Poisson bracket \(P\). The space of smooth functions \(C^\infty(M)\) carries two natural algebraic structures: It is an Abelian algebra for the pointwise product of functions and also a Lie algebra for the Poisson bracket \(P\). A star-product on \((M, P)\) is a formal associative deformation in the Gerstenhaber’s sense [3] of the Abelian algebra structure of \(C^\infty(M)\). More precisely:

**Definition 1** Let \(C^\infty(M)[[\nu]]\) be the space of formal series in a parameter \(\nu\) with coefficients in \(C^\infty(M)\). A star-product on \((M, P)\) is a bilinear map from \(C^\infty(M) \times C^\infty(M)\) to \(C^\infty(M)[[\nu]]\) denoted by \(f \ast \nu g = \sum_{r \geq 0} \nu^r C_r(f, g)\), \(f, g \in C^\infty(M)\), where (the cochains) \(C_r: C^\infty(M) \times C^\infty(M) \to C^\infty(M)\) are bilinear maps satisfying for any \(f, g, h \in C^\infty(M)\):

i) \(C_0(f, g) = fg\);

ii) \(C_r(c, f) = C_r(f, c) = 0\), for \(r \geq 1\), \(c \in \mathbb{R}\);

iii) \(\sum_{s+t=r} C_s(C_t(f, g), h) \sum_{s+t=r} C_s(f, C_t(g, h))\), for \(r \geq 0\);

iv) \(C_1(f, g) - C_1(g, f) = 2P(f, g)\).

A star-product \(\ast\) is naturally extended to a bilinear map on \(C^\infty(M)[[\nu]]\). The conditions i)–iv) above simply translate, respectively, that a star-product is: i) a deformation of the pointwise product; ii) it preserves the original unit \((1 \ast \nu 1 = f)\); iii) it is an associative product; iv) the associated star-bracket, \([f, g]_{\ast \nu} = (f \ast \nu g - g \ast \nu f)/2\nu\), is a Lie algebra deformation of the Lie-Poisson algebra \((C^\infty(M), P)\).

Usually, one adds one more condition on the cochains \(C_r\) of a star-product by requiring that they should be bidifferential operators (necessarily null on constants by condition ii)). These star-products are called differential star-products. In this paper, star-product will always mean differential star-product. One has a notion of equivalence between star-products given by:

**Definition 2** Two star-products \(\ast\) and \(\ast'\) on \((M, P)\) are said to be equivalent if there exists a formal series \(T = I + \sum_{r \geq 1} \nu^r T_r\), where \(I\) is the identity map on \(C^\infty(M)\) and the \(T_r\)'s are differential operators on \(C^\infty(M)\) vanishing on constants, such that \(T(f \ast \nu g) = T(f) \ast \nu T(g)\), \(f, g \in C^\infty(M)[[\nu]]\).
For a long time, star-products were known to exist on any symplectic manifold (i.e., when the Poisson bracket $P$ is induced by some symplectic form) [3]. Few months ago, as a consequence of his formality conjecture, Kontsevich showed that in fact star-products exist on any Poisson manifold and gave a complete description of their equivalence classes [10].

2.2 Hochschild cohomology

Hochschild cohomology plays a prevailing rôle in the deformation theory of associative algebras. It is well known that the obstructions to equivalence of associative deformations are in second Hochschild cohomology space and the obstructions for extending a deformation, given up to certain order in the deformation parameter, to the next order live in the third Hochschild cohomology space. We shall recall here the definition and basic properties of the Hochschild cohomology in the differentiable (null on constants) case.

Let $A$ be the Abelian algebra $C^\infty(M)$ endowed with the pointwise product. Consider the complex $C^\ast(A, A) = \{C^r(A, A)\}_{r\geq 0}$, where $C^r(A, A)$ is the vector space of $r$-linear differential operators null on constants $\phi: A^r \to A$, with coboundary operator $\delta$, defined on an $r$-cochain $C$ by:

$$\delta C(f_0, \ldots, f_r) = f_0 C(f_1, \ldots, f_r) + \sum_{1 \leq i \leq r} (-1)^i C(f_0, \ldots, f_{i-1} f_i, \ldots, f_r) + (-1)^{r+1} C(f_0, \ldots, f_{r-1}) f_r,$$

for any $f_0, \ldots, f_r$ in $A$. The Hochschild cohomology (with values in $A$) is the cohomology of the cochain complex $(C^\ast(A, A), \delta)$ and shall be denoted by $H^\ast_{\text{diff, nc}}(A)$. A fundamental result is

**Theorem 1 (Vey[16])** The Hochschild cohomology $H^\ast_{\text{diff, nc}}(A)$ is isomorphic to $\Gamma(\wedge^\ast TM)$, the space of skew-symmetric contravariant tensor fields on $M$.

Hence any Hochschild $r$-cocycle $\phi$ can be written as $\phi = \delta \theta + \Lambda$, where $\theta$ is an $(r-1)$-cochain and $\Lambda$ is an $r$-tensor on $M$. In particular, a completely symmetric cocycle is a coboundary.

2.3 Notations and definitions

We start by making precise our notations. The coordinates of $\mathbb{R}^n$ are denoted by $(x_1, \ldots, x_n)$. Let $N$ be the $\mathbb{R}$-algebra of smooth functions on $\mathbb{R}^n$. Let $\text{Pol}$ be the $\mathbb{R}$-subalgebra of $N$ consisting of polynomials in $\mathbb{R}[x_1, \ldots, x_n]$. For a formal parameter $\nu$, we shall denote by $N_{\nu}$ (resp. $\text{Pol}_{\nu}$) the algebra $N[[\nu]]$ (resp. $\text{Pol}[[\nu]]$) of formal series in $\nu$ with coefficients in $N$ (resp. $\text{Pol}$). We distinguish in $N_{\nu}$ a subalgebra $N^0_{\nu}$ consisting of formal series whose zeroth-order coefficient belongs to $\text{Pol}$. $N_{\nu}$, $N^0_{\nu}$ and $\text{Pol}_{\nu}$ are naturally $\mathbb{R}[[\nu]]$-algebras, but we shall often view them as $\mathbb{R}$-algebras.
The natural projection $\pi: N_\nu \to N$ is an $\mathbb{R}$-algebra homomorphism and the same symbol shall be used for the projections of $N_\nu^0$ and $\text{Pol}_\nu$ on $\text{Pol}$.

We now define sun-products. Let $S(\text{Pol})$ denote the symmetric tensor algebra over $\text{Pol}$ with symmetric tensor product $\otimes$, and let $\lambda: \text{Pol} \to S(\text{Pol})$ be the $\mathbb{R}$-algebra homomorphism defined by:

$$\lambda(x_1^{k_1} \cdots x_n^{k_n}) = (x_1^\otimes) \otimes \cdots \otimes (x_n^\otimes), \quad \forall k_1, \ldots, k_n \geq 0. \quad (1)$$

The map $\lambda$ sends a polynomial in $\text{Pol}$ to an element of $S(\text{Pol})$ by replacing the usual product between linear factors by the symmetric tensor product.

Let $P$ be a Poisson bracket on $\mathbb{R}^n$. Given a star-product $\ast_\nu$ on $(\mathbb{R}^n, P)$, we define an $\mathbb{R}$-linear map $T_{\ast_\nu}: S(\text{Pol}) \to N_\nu^0$ by:

$$T_{\ast_\nu}(f_1 \otimes \cdots \otimes f_k) = \frac{1}{k!} \sum_{\sigma \in S_k} f_{\sigma(1)} \ast_\nu \cdots \ast_\nu f_{\sigma(k)}, \quad \forall k \geq 1, \quad (2)$$

where $f_i \in \text{Pol}$, $1 \leq i \leq k$, and $S_k$ is the permutation group on $k$ elements. By convention, we set $T_{\ast_\nu}(I) = 1$, where $I$ is the identity of $S(\text{Pol})$. Notice that the zeroth-order coefficient on the right-hand side of (2) is the product of polynomials $f_1 \cdots f_k \in \text{Pol}$, but in general the coefficient of $\nu^r$ for $r \geq 1$ is in $N$.

**Definition 3** To a star-product $\ast_\nu$ on $(\mathbb{R}^n, P)$, we associate a new product on $N_\nu^0$ by the following formula:

$$f \circ_\nu g = T_{\ast_\nu}(\lambda(\pi(f)) \otimes \lambda(\pi(g))), \quad f, g \in N_\nu^0. \quad (3)$$

This product is called the $\circ_\nu$-product (or sun-product) associated to $\ast_\nu$.

In words, a sun-product on $\mathbb{R}^n$ associates to two polynomials $f, g \in \text{Pol}$ the element $f \circ_\nu g \in N_\nu^0$ obtained by replacing the usual product between linear factors (in some given order) in $fg$ by a star-product $\ast_\nu$ and then by completely symmetrizing the expression found.

The extension of the product to $f, g \in N_\nu^0$ is obtained by applying the previous procedure to the zeroth-order coefficient of $fg$. Hence a sun-product annihilates any non-zero powers of the deformation parameter.

Basic properties of sun-products are collected in the following lemma:

**Lemma 1** A sun-product $\circ_\nu$ on $\mathbb{R}^n$ is an Abelian, associative product on $N_\nu^0$. It fails to be $\mathbb{R}[[\nu]]$-bilinear, but it is $\mathbb{R}$-bilinear. $N_\nu^0$ endowed with a product $\circ_\nu$ is an Abelian $\mathbb{R}$-algebra.

**Proof.** That the product $\circ_\nu$ is Abelian is clear from (3). Associativity follows from $\pi(f \circ_\nu g) = \pi(f) \pi(g)$ for $f, g \in N_\nu^0$, and from the fact that both $\lambda$ and $\pi$ are $\mathbb{R}$-algebra homomorphisms: $f \circ_\nu (g \circ_\nu h) = T_{\ast_\nu}(\lambda(\pi(f)) \otimes \lambda(\pi(g \circ_\nu h))) = T_{\ast_\nu}(\lambda(\pi(fh))) = (f \circ_\nu g) \circ_\nu h$, for $f, g, h \in N_\nu^0$.

Clearly a sun-product does not have a unit on $N_\nu^0$, nevertheless one has $1 \circ_\nu f = f$ when $f$ is a linear polynomial in $\text{Pol}$.
From the preceding proof, we see that to every sun-product \( \circ_{\nu} \), we can associate a formal series of linear maps \( \rho = \sum_{0 \leq r} \nu^r \rho_r \), where \( \rho_0 = \text{Id} \) is the identity map on \( \text{Pol} \), and \( \rho_r: \text{Pol} \to \mathbb{N} \) for \( r \geq 1 \), such that \( f \circ_{\nu} g = \rho(\pi(fg)) \) for \( f, g \in \mathbb{N}^0_{\nu} \). We shall (abusively) call the maps \( \rho_r \) the cochains of the sun-product \( \circ_{\nu} \).

### 2.4 Differentiability

An example of sun-product has been explicitly computed in [5] for some star-product on the dual of the Lie algebra \( \mathfrak{su}(2) \) seen as Poisson manifold when endowed with its natural Lie-Poisson bracket. A remarkable feature of this sun-product is that its cochains are differential operators. In the following, we shall show that this fact corresponds to the general situation. As a consequence, any sun-product admits a natural extension from \( \mathbb{N}^0_{\nu} \) to \( \mathbb{N}_{\nu} \).

**Theorem 2** The cochains \( \rho_r \) of a sun-product \( \circ_{\nu} \) associated to some star-product \( *_{\nu} \) on \((\mathbb{R}^n, P)\) are given by the restriction to \( \text{Pol} \) of differential operators on \( \mathbb{N} \).

Before proving this theorem, we shall derive few lemmas. We consider a sun-product \( \circ_{\nu} \) associated with some star-product \( *_{\nu} \) on \((\mathbb{R}^n, P)\). The cochains of the sun-product (resp. star-product) are denoted by \( \rho_r \) (resp. \( C_r \)).

For any map \( \phi: \mathbb{R}^k \to E \), where \( E \) is a vector space, \( \sum_{(i_1, \ldots, i_k)} \phi(x_{i_1}, \ldots, x_{i_k}) \) denotes the sum over cyclic permutations of \((x_{i_1}, \ldots, x_{i_k})\).

**Lemma 2** Let \( \psi: \text{Pol} \to \mathbb{N} \) be a linear map such that \( \psi(1) = \psi(x_i) = 0 \), for \( 1 \leq i \leq n \). Let \( \phi: \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) be a bidifferential operator null on constants. If the Hochschild coboundary \( \delta \psi \) satisfies for any \( k \geq 2 \) and indices \((i_1, \ldots, i_k)\):

\[
\sum_{(i_1, \ldots, i_k)} \delta \psi(x_{i_1}, x_{i_2} \cdots x_{i_k}) = \sum_{(i_1, \ldots, i_k)} \phi(x_{i_1}, x_{i_2} \cdots x_{i_k}),
\]

then \( \psi \) is the restriction to \( \text{Pol} \) of a differential operator null on constants.

**Proof.** On the right-hand side of Eq. (4), it is clear that is sufficient to consider bidifferential operators of the form (only these are contributing to Eq. (4)):

\[
\phi(f, g) = \sum_{1 \leq i \leq n} \sum_{|J| \geq 1} \phi^{i, J} \partial_i f \partial_J g,
\]

where \( J = (j_1, \ldots, j_n) \) is a multi-index, \( |J| = \sum_{1 \leq s \leq n} j_s \), \( \partial_i = \partial / \partial x_i \), \( \partial_J = \partial^{|J|} / \partial x_1^{j_1} \cdots \partial x_n^{j_n} \) and, for fixed \( i \) and \( J \), \( \phi^{i, J} \) is a smooth function on \( \mathbb{R}^n \) vanishing if \( |J| \) is greater than some integer. Consider the differential operator

\[
\tilde{\psi}(f) = - \sum_{1 \leq i \leq n} \sum_{|J| \geq 1} \frac{1}{|J| + 1} \phi^{i, J} \partial_i f.
\]
where $\partial_1 f$ means $\partial^{|J|+1} f / \partial x_i^{j_1} \cdots \partial x_i^{j_{|J|+1}}$ for $J = (j_1, \ldots, j_n)$. Notice that $\tilde{\psi}(1) = \psi(x_i) = 0$, for $1 \leq i \leq n$. The following property of $\tilde{\psi}$ is established by a straightforward computation:

$$\sum_{(i_1, \ldots, i_k)} \delta \tilde{\psi}(x_{i_1}, x_{i_2} \cdots x_{i_k}) = \sum_{(i_1, \ldots, i_k)} \phi(x_{i_1}, x_{i_2} \cdots x_{i_k}), \tag{5}$$

for any $k \geq 2$ and indices $(i_1, \ldots, i_k)$. Then, for $\psi : \text{Pol} \to \mathbb{N}$ satisfying the hypothesis of the lemma, we have:

$$\sum_{(i_1, \ldots, i_k)} \delta(\psi - \tilde{\psi})(x_{i_1}, x_{i_2} \cdots x_{i_k}) = 0 \tag{6}$$

for any $k \geq 2$ and indices $(i_1, \ldots, i_k)$. Let $\eta = \psi - \tilde{\psi}$. Since $\eta(x_i) = 0$, $1 \leq i \leq n$, we have $\delta \eta(x_i, f) = x_i \eta(f) - \eta(x_i f)$ for $1 \leq i \leq n$ and $f \in \text{Pol}$. Then Eq. (8) implies that

$$\eta(x_{i_1} \cdots x_{i_k}) = \frac{1}{k} \sum_{(i_1, \ldots, i_k)} x_{i_1} \eta(x_{i_2} \cdots x_{i_k}),$$

and by induction on $k$, we find that $\eta = 0$ on $\text{Pol}$, i.e. $\psi = \tilde{\psi}|_{\text{Pol}}$. □

**Lemma 3** Let $\circ_\nu$ be the sun-product associated with some star-product $*_{\nu}$ on $(\mathbb{R}^n, P)$. The first cochain $\rho_1$ of $\circ_\nu$ is a differential operator null on constants whose Hochschild coboundary satisfies $\delta \rho_1 = P - C_1$, where $C_1$ is the first cochain of the star-product $*_{\nu}$.

**Proof.** From Def. 3, we have for $k \geq 2$ and indices $(i_1, \ldots, i_k)$:

$$\rho(x_{i_1} \cdots x_{i_k}) = x_{i_1} \circ_\nu \cdots \circ_\nu x_{i_k}$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} x_{i_{\sigma(1)}} *_{\nu} \cdots *_{\nu} x_{i_{\sigma(k)}},$$

$$= \frac{1}{k} \sum_{(i_1, \ldots, i_k)} x_{i_1} *_{\nu} \rho(x_{i_2} \cdots x_{i_k}). \tag{7}$$

The first-order term in $\nu$ in the last equation is:

$$\rho_1(x_{i_1} \cdots x_{i_k}) = \frac{1}{k} \sum_{(i_1, \ldots, i_k)} C_1(x_{i_1}, x_{i_2} \cdots x_{i_k}) + \frac{1}{k} \sum_{(i_1, \ldots, i_k)} x_{i_1} \rho_1(x_{i_2} \cdots x_{i_k}),$$

which can be written as

$$\sum_{(i_1, \ldots, i_k)} (\delta \rho_1 + C_1)(x_{i_1}, x_{i_2} \cdots x_{i_k}) = 0, \tag{8}$$

7
since $\rho_1(x_i) = 0$. The associativity condition for a star-product implies that $C_1$ is a Hochschild 2-cocycle and Theorem [1] and condition iv) in Def. [1] tell us that $C_1 = P + \delta \theta$ where $\theta$ is a differential operator null on constants. We can always take $\theta$ such that $\theta(x_i) = 0$, $1 \leq i \leq n$, by adding a suitable 1-cocycle to it (e.g., $\theta(x_i) = \theta(x_i) - \sum_i \theta(x_i) \partial_i$). The Poisson bracket $P$ is a 2-tensor and does not contribute to the left-hand side of Eq. (8). The same argument used in the proof of Lemma 2 (cf. Eq. (6)) leads us to the conclusion that $\rho_1 = -\theta$ and, consequently, $\delta \rho_1 = P - C_1$. □

Proof of Theorem 2. Using that the cochains of a sun-product satisfy $\rho_r(x_i) = 0$, $1 \leq i \leq r$, we can write the equation of the term of order $r$ in Eq. (7) as:

$$\sum_{(i_1, \ldots, i_k)} \delta \rho_r(x_{i_1}, x_{i_2}, \ldots x_{i_k}) =$$

$$- \sum_{(i_1, \ldots, i_k)} C_r(x_{i_1}, x_{i_2}, \ldots x_{i_k}) - \sum_{(i_1, \ldots, i_k)} \sum_{a,b=0}^{r} C_a(x_{i_1}, \rho_b(x_{i_2}, \ldots x_{i_k})),$$

(9)

for $k \geq 2$ and $r \geq 1$ (for $r = 1$, the right-hand side has only one sum). Notice that in the right-hand side of Eq. (9) only the first $r - 1$ cochains of the sun-product $\circ_\nu$ appear. We already know that $\rho_1$ is a differential operator null on constants from Lemma 3 and with the help of Lemma 2 a simple induction on $r$ proves the theorem. □

Remark 1 A direct consequence of Theorem 2 is that we can extend sun-products, originally defined on $N_0^0$, to $N_\nu$ by the formula $f \circ_\nu g = \pi(fg) + \sum_{r \geq 1} \nu^r \rho_r(\pi(fg))$ for $f, g \in N_\nu$.

Theorem 2 has very simple consequences. We shall end this section by deriving some results about the cochains of a sun-product. In Sect. 3 we shall see that differentiability of sun-products allows one to deduce interesting properties for star-products on the dual of a Lie algebra. The cochains of a sun-product can be used to construct equivalence operators and this turns out to be a quite powerful tool to establish equivalence relation between certain type of star-products without any cohomological computations.

Definition 4 $\mathcal{E}(P)$ is the set of star-products on $(\mathbb{R}^n, P)$ such that their associated sun-product $\circ_\nu$ coincide with the usual product on $\text{Pol}$, i.e. the cochains $\rho_r = 0$ for $r \geq 1$.

Corollary 1 Any star-product on $(\mathbb{R}^n, P)$ is equivalent to a star-product belonging to $\mathcal{E}(P)$.

Proof. Let $*_\nu$ be a star-product and let $\{\rho_r\}_{r \geq 1}$ be the cochains of its associated sun-product. The maps $\rho_r$ are defined on $\mathbb{N}$ and we shall denoted by the same symbol their $\mathbb{R}[\nu]$-linear extension to $N_\nu$. Let us define another star-product $*_\nu'$ by equivalence from $*_\nu$ with equivalence operator $T = I + \sum_{r \geq 1} \nu^r \rho_r$, that is to say: $T(f*_\nu' g) = T(f)*_\nu T(g)$, $f, g \in N_\nu$. 

8
Since $T(x_i) = x_i$, $1 \leq i \leq n$, we have for $k \geq 2$: $T(x_{i_1} \ast_{\nu'} \cdots \ast_{\nu'} x_{i_k}) = x_{i_1} \ast_{\nu} \cdots \ast_{\nu} x_{i_k}$, and complete symmetrization gives:

$$T(x_{i_1} \odot_{\nu'} \cdots \odot_{\nu'} x_{i_k}) = x_{i_1} \odot_{\nu} \cdots \odot_{\nu} x_{i_k}.$$ 

By definition $T$ is invertible and notice that $x_{i_1} \odot_{\nu'} \cdots \odot_{\nu'} x_{i_k} = T(x_{i_1} \cdots x_{i_k})$, for any $k \geq 2$, i.e. the cochains of $\odot_{\nu'}$ satisfy $\rho'_{r} = 0$ for $r \geq 1$. Hence $\ast_{\nu'}$ belongs to $E(P)$.

In view of the preceding corollary, the problem of classification of equivalence classes of star-products on $(\mathbb{R}^n, P)$ reduces to classifying equivalence classes in $E(P)$. An order-by-order analysis in $\nu$ of star-products in $E(P)$ makes the second Lichnerowicz-Poisson cohomology space appear explicitly here. It plays the same rôle in the Poisson case as the one played by the second de Rham cohomology space for the classification of equivalences classes in the symplectic case.

**Corollary 2** Let $\{\eta_i\}_{i \geq 1}$ be a sequence of differential operators on $\mathbb{N}$ such that $\eta_i(1) = \eta_i(x_k) = 0$, $1 \leq i, 1 \leq k \leq n$, and let $\ast_{\nu}$ be some star-product on $(\mathbb{R}^n, P)$. There exists a star-product $\ast_{\nu'}$, equivalent to $\ast_{\nu}$, such that the cochains of the sun-product $\odot_{\nu'}$ associated with $\ast_{\nu'}$ are precisely the $\eta_i$'s.

**Proof.** Any star-product $\ast_{\nu}$ is equivalent to a star-product $\ast_{\nu''}$ in $E(P)$. For $\{\eta_i\}_{i \geq 1}$ satisfying the hypothesis of the corollary, we consider a third star-product $\ast_{\nu'}$ defined by equivalence: $T(f \ast_{\nu''} g) = T(f) \ast_{\nu'} T(g)$ where $T = I + \sum_{i \geq 1} \nu^i \eta_i$. It is easily verified that the sun-product associated with $\ast_{\nu'}$ admits the $\eta_i$'s as cochains.

This shows that the set of possible cochains for a sun-product on $\mathbb{R}^n$ coincides with the set of differential operators on $\mathbb{R}^n$ vanishing on polynomial of degree less or equal to one. Also it is sufficient to consider only one equivalence class of star-products to generate all of the sun-products on $\mathbb{R}^n$. As one could have guessed, there is almost no constraints imposed by the associativity condition on the possible cochains of a sun-product. This fact in our opinion makes the cohomology problem for generalized deformations quite difficult (see the discussion in Sect. 4).

### 3 Sun-products on $\mathfrak{g}^*$

We shall specialize our discussion to the case of the dual of a Lie algebra. Let $\mathfrak{g}$ be a real Lie algebra of dimension $n$. The dual $\mathfrak{g}^*$ of $\mathfrak{g}$ carries a canonical Poisson structure and, by choosing a basis of $\mathfrak{g}$, we can identify $\mathfrak{g}^*$ as Poisson manifold with $\mathbb{R}^n$ endowed with the following Poisson bracket:

$$P_C(F, G) = \sum_{i,j,k=1}^{n} C^k_{ij} x_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}, \quad \forall f, g \in \mathbb{N},$$

(10)
where the $C^k_{ij}$’s are the structure constants of the Lie algebra $\mathfrak{g}$ expressed in some basis.

A particular class of star-products which are important for physical applications and in star-representation theory are the covariant star-products:

**Definition 5** Let $\mathfrak{g}$ be a Lie algebra of dimension $n$. A star-product $\ast_\nu$ on $\mathbb{R}^n$ is said to be $\mathfrak{g}$-covariant if

$$\frac{1}{2\nu} (x_i \ast_\nu x_j - x_j \ast_\nu x_i) = PC(x_i, x_j) = \sum_{k=1}^n C^k_{ij} x_k, \quad \forall 1 \leq i, j \leq n,$$

(11)

where the $C^k_{ij}$’s are the structure constants of the Lie algebra $\mathfrak{g}$ in a given basis.

Star-products on the dual of a Lie algebra were known from the very beginning of the theory of star-products. The well known Moyal product is such an example, another for $\mathfrak{so}(n)^*$ appears in [2] in relation with the quantization of angular momentum. The general case was treated by S. Gutt [9] who defined a star-product on the cotangent bundle of any Lie group $T^*G$. Gutt’s star-product admits a restriction to $\mathfrak{g}^*$ that we shall call Gutt’s star-product on $\mathfrak{g}^*$.

Gutt’s star-product on $\mathfrak{g}^*$ has a simple expression that we briefly recall here (see [9] for further details). Polynomials on $\mathfrak{g}^*$ can be considered as elements of the symmetric algebra over $\mathfrak{g}$, $S(\mathfrak{g})$. Let $S_r$ be the set of homogeneous polynomials of degree $r$ and let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. The symmetrization map $\phi: S(\mathfrak{g}) \to U(\mathfrak{g})$ defined by:

$$\phi(X_{i_1} \cdots X_{i_r}) = \frac{1}{r!} \sum_{\sigma \in S_r} X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(k)},$$

(where $\otimes$ is the product in $U(\mathfrak{g})$) is a bijection. Let $\mathcal{U}_r = \phi(S_r)$, one has $U(\mathfrak{g}) = \bigoplus_{r \geq 0} \mathcal{U}_r$ and each $u \in U(\mathfrak{g})$ can be decomposed as $u = \bigoplus_{r \geq 0} u_r$, where $u_r \in \mathcal{U}_r$. Now define a product between $P \in S_p$ and $Q \in S_q$, by:

$$P \times_\nu Q = \sum_{r \geq 0} (2\nu)^k \phi^{-1}((\phi(P) \otimes \phi(Q))_{p+q-r}),$$

and extend it by linearity to all of $S(\mathfrak{g})$. It can be shown that the product $\times_\nu$ is associative and is defined by differential operators. Hence one gets a star-product on $S(\mathfrak{g})$ which is naturally extended to $C^\infty(\mathfrak{g}^*)$. This star-product is $\mathfrak{g}$-covariant. We shall see that Gutt’s star-product plays a special rôle in relation with sun-products on $\mathfrak{g}^*$.

**Lemma 4** Let $\mathfrak{g}$ be a fixed Lie algebra of dimension $n$. The set of $\mathfrak{g}$-covariant star-products belonging to $\mathcal{E}(PC)$ has only one element. In words, there is only one $\mathfrak{g}$-covariant star-product on $\mathfrak{g}^*$ whose sun-product coincides with the usual product on $\text{Pol}$. 

10
Proof. Let \( *_\nu \) be a \( g \)-covariant star-product on \( (\mathbb{R}^n, P_C) \) with associated sun-product \( \odot_\nu \) which coincides with usual product on \( \text{Pol} \). Let \( \text{Lin} \subseteq \text{Pol} \) be the subspace of linear homogeneous polynomials on \( \mathbb{R}^n \). It is easy to verify that the \( *_\nu \)-powers, the \( \odot_\nu \)-powers and the usual powers of any \( X \in \text{Lin} \) are identical:

\[ X^m = X^m \odot_\nu = X^m, \quad \forall X \in \text{Lin}, m \geq 0. \quad (12) \]

Obviously, we also have that \( X^m \ast_\nu = X^m \), for any \( X \in \text{Lin}[\nu], m \geq 0 \). (As usual, \( \text{Lin}[\nu] \) denotes the set of formal series in \( \nu \) with coefficients in \( \text{Lin} \).

For \( X \in \text{Lin}[\nu] \), consider its \( \ast_\nu \)-exponential defined by:

\[ \exp_{\ast_\nu}(X) = \sum_{r \geq 0} \frac{1}{r!} X^r \ast_\nu, \quad (13) \]

it is an element of \( \mathbb{N}_\nu \) and here \( \exp_{\ast_\nu}(X) \) is identical to the usual exponential \( \exp(X) \) for any \( X \in \text{Lin}[\nu] \). The fact that \( \ast_\nu \) is a \( g \)-covariant star-product allows us to make usage of the Campbell-Hausdorff formula in the following form (in the sense of formal series):

\[ \exp_{\ast_\nu}(sX) \ast_\nu \exp_{\ast_\nu}(tY) = \exp_{\ast_\nu}(Z(sX,tY)), \quad X,Y \in \text{Lin}, s,t \in \mathbb{R}, \quad (14) \]

where \( Z(X,Y) = \sum_{r \geq 0} \nu^r Z_r(X,Y) \in \text{Lin}[\nu] \), and the \( Z_r \)'s are related to the Campbell-Hausdorff coefficients by \( Z_r(X,Y) = 2^r c_{r+1}(X,Y) \) (where \( c_1(X,Y) = X + Y \), \( c_2(X,Y) = P_C(X,Y)/2 \), etc.). As the \( \ast_\nu \)-exponential of \( X \in \text{Lin}[\nu] \) is simply the usual exponential, Eq. (14) yields

\[ \exp(sX) \ast_\nu \exp(tY) = \exp(Z(sX,tY)), \quad X,Y \in \text{Lin}, s,t \in \mathbb{R}. \quad (15) \]

Hence a \( g \)-covariant star-product for which the associated \( \odot_\nu \)-product is the usual product must satisfy the preceding relation.

Actually Eq. (15) determines the star-product \( \ast_\nu \) completely. Notice that a bidifferential operator \( B: \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) is completely characterized by the functions \( B(X^a,Y^b), a,b \in \mathbb{N}, X,Y \in \text{Lin} \). The functions \( C_r(X^a,Y^b), 0 \leq a, b, r, X, Y \in \text{Lin} \), which completely determine the cochains \( \ast_\nu \) can be easily computed by differentiation with respect to \( s \) and \( t \) on both sides of Eq. (13) of the coefficient of \( \nu^r \) and by evaluation at \( s = t = 0 \). Therefore there is at most one star-product whose associated \( \odot_\nu \)-product is the usual product on \( \text{Pol} \).

It is easy to show that the star-product defined by Eq. (15) has the usual product as associated \( \odot_\nu \)-product. By setting \( Y = X \) in Eq. (13), we find that \( X^a \ast_\nu X^b = X^{a+b} \), \( \forall X \in \text{Lin}, 0 \leq a, b \), which implies by induction that \( X^{a_\nu} = X^a, \forall X, a, b \). By Eq. (12) we have \( X^{\odot_\nu} = X^a, \forall X, a \), and since \( \odot_\nu \) is Abelian, it implies that \( \odot_\nu \) is the usual product on \( \text{Pol} \) and this proves the lemma. \( \square \)
Lemma 5 Let $\mathfrak{g}$ be a Lie algebra. The $\mathfrak{g}$-covariant star-product characterized in Lemma 4 is Gutt’s star-product on $\mathfrak{g}^*$.

Proof. We shall use the notations introduced in the proof of Lemma 4. Let $\mathfrak{g}$ be of dimension $n$ and let $*_{\nu}$ be the star-product characterized in Lemma 4 by Eq. (15). The identification of the coefficients of $\nu^r$ in (15) gives:

$$C_r(\exp(sX), \exp(tY)) = F_r(sX, tY) \exp(sX + tY), \quad \forall X, Y \in \text{Lin}, s, t \in \mathbb{R},$$

where the $F_r$'s are polynomial functions of the (normalized) Campbell-Hausdorff coefficients $Z_{r}(sX, tY)$ and are defined by the following recursive relation with $F_0 = 1$:

$$F_r = \frac{1}{r} \sum_{k=0}^{k=r-1} (r - k)Z_{r-k}F_k, \quad r \geq 1. \quad (17)$$

By induction, one finds the explicit expression for $F_r$ for $r \geq 1$ to be:

$$F_r = \sum_{k=1}^{k=r} \sum_{\substack{m_1, \ldots, m_k \geq 1 \\text{ such that } n_1 \ldots n_k = r}} \frac{1}{n_1! \ldots n_k!} (Z_{m_1})^{n_1} \cdots (Z_{m_k})^{n_k}. \quad (18)$$

Now we shall derive an explicit expression for $X *_{\nu} \exp(Y)$, $X, Y \in \text{Lin}$. Notice that this relation also characterizes $*_{\nu}$ as any polynomial can be expressed as a $*_{\nu}$-polynomial (it is a simple consequence of Eq. (12)). In general, the Campbell-Hausdorff coefficients $\{c_i\}_{i \geq 1}$ ($c_1(X, Y) = X + Y$, $c_2(X, Y) = \frac{1}{2}[X, Y]$, etc.) have the following properties:

$$c_i(0, X) = c_i(X, 0) = 0, \quad i \geq 2;$$

$$\frac{\partial}{\partial s} c_i(sX, Y)|_{s=0} = \frac{B_{i-1}}{(i-1)!} (\text{ad}_Y)^{i-1}(X), \quad i \geq 2; \quad (19)$$

where $\text{ad}_Y : X \mapsto [Y, X]$, and $B_n$ are the Bernoulli numbers. These can be easily derived from the standard recursive formula for the $c_i$’s, see e.g. [13].

Also, using Eqs. (18) and (19), along with the definition of $Z_r$ ($Z_r = 2^r c_{r+1}$), one finds that

$$F_r(0, Y) = 0, \quad r \geq 1;$$

$$\frac{\partial}{\partial s} F_r(sX, Y)|_{s=0} = \frac{\partial}{\partial s} Z_r(sX, Y)|_{s=0} = \frac{2^r B_r}{r!} (\text{ad}_Y)^r(X), \quad r \geq 1.$$
Therefore we can write

\[ C_r(X, \exp(tY)) = \frac{\partial}{\partial s}(F_r(sX, tY) \exp(sX + tY))|_{s=0} = 2^r B_r(ad_{tY})(X) \exp(tY), \quad r \geq 1. \] (20)

For \( r = 0 \), we simply have: \( C_r(X, \exp(tY)) = X \exp(tY) \). Equation (20) is also characterizing Gutt’s star-product on \( g^* \) (compare with Eq. (3.2) in [9]). □

As a simple consequence of Lemmas 4 and 5 we have the following corollary which tells us that any two covariant star-products on the dual of a Lie algebra are equivalent.

**Corollary 3** Any covariant star-product on the dual of a Lie algebra \( g \) is equivalent to Gutt’s star-product on \( g^* \).

**Proof.** Let \( *_\nu \) be a \( g \)-covariant star-product on \( g^* \), the dual of a Lie algebra of dimension \( n \). From Corollary 1, \( *_\nu \) is equivalent to a star-product \( *'_\nu \) belonging to \( \mathfrak{E}(P_C) \), where \( P_C \) is the Lie-Poisson structure on \( g^* \). The equivalence operator is constructed out from the cochains of the sun-product associated with \( *_\nu \) and it leaves invariant linear polynomials, i.e. \( T(x_i) = x_i, \quad 1 \leq i \leq n \). Consequently, \( *'_\nu \) is also a \( g \)-covariant star-product. According to Lemmas 4 and 5, \( *'_\nu \) must be Gutt’s star-product on \( g^* \). □

**Remark 2** Though the de Rham cohomology of \( g^* \) is trivial, not all star-products on \( g^* \) are equivalent. Indeed, in the symplectic case, the second de Rham cohomology space classifies equivalence classes of star-products. In the Poisson case, one has to consider the Lichnerowicz-Poisson cohomology \([14]\) instead, and this cohomology is not in general trivial for the Lie-Poisson structure on \( g^* \). See \([14]\), for explicit computations of some of the (Chevalley-Eilenberg) cohomology spaces for the dual of a Lie algebra.

### 4 Weak and strong equivalences

In the deformation theory of some algebraic structure one has the notion of equivalent deformations. The equivalence of star-products given Def. 4 is adapted to the associative (differential) case and one has similar notions of equivalence for other algebraic structures (e.g., Lie algebras, Abelian algebras, etc.). Moreover, as mentioned in Sect. 2.2, it is a general result of Gerstenhaber \([5]\) that obstructions for equivalence of deformations reside in the second cohomology space of an appropriate cohomology. For associative, Lie, Abelian deformations the associated cohomologies are, respectively, Hochschild, Chevalley-Eilenberg, Harrison cohomologies. One may wonder what is the corresponding cohomology for generalized Abelian deformations. Before discussing on that matter, it is important to bear in mind
that in Gerstenhaber’s theory of deformations a deformed algebraic structure has a structure of $\mathbb{K}[[\nu]]$-algebra, where $\mathbb{K}$ is the ground ring of the original structure. This feature, which is crucial to determine the appropriate cohomology, does not hold anymore in the case of generalized deformations.

The answer to the cohomology issue raised by generalized deformations might be, as advocated by M. Flato [6], that one has to give a noncommutative ring structure on the space of formal parameters in such a way that $\mathbb{R}[[\nu]]$-bilinearity would be restored. This should lead to a noncommutative deformation theory and the first steps toward this program were taken by Pinczon [15] who considered the case where the deformation parameter is acting by different left and right endomorphisms on the algebra (hence the deformation parameter is not required to commute with the undeformed algebra). This point of view produced very interesting results (e.g., deformation of the Weyl algebra yields supersymmetric algebras), but still generalized deformations do not fit in the particular framework considered in [15].

The cohomology problem is still open and in a previous work [5] we have nevertheless considered two notions of equivalence for sun-products. They are mimicking the usual notion of equivalence and take into account that sun-products are not $\mathbb{R}[[\nu]]$-bilinear operations, but only $\mathbb{R}$-bilinear. Let us recall their definitions.

**Definition 6** Two sun-products $\circ_\nu$ and $\circ_\nu'$ on $(\mathbb{R}^n, P)$ are said to be (a) weakly ((b) strongly) equivalent, if there exists an $\mathbb{R}[[\nu]]$-linear map $S_\nu: N_\nu \mapsto N_\nu$, where $S_\nu = \sum_{r \geq 0} \nu^r S_r$, with $S_r: \mathbb{N} \rightarrow \mathbb{N}$, $r \geq 1$, being differential operators and $S_0 = I$, such that for $f, g \in N$ the following holds:

(a) $S_\nu(f \circ_\nu g) = S_\nu(f) \circ_\nu' S_\nu(g),$

(b) $S_\nu(f \circ_\nu g) = S_\mu(f) \circ_\nu' S_\mu(g)|_{\mu = \nu}.$

For weak equivalence, condition (a) above can be equivalently replaced by $S_\nu(f \circ_\nu g) = f \circ_\nu' g$, as sun-products annihilate the deformation parameter $\nu$. In the case of strong equivalence, condition (b), when written in terms of the cochains of the sun-products, simply states that:

$$\sum_{r,s \geq 0} S_r(\rho_s(fg)) = \sum_{r,a+b=t} \rho_s'(S_a(f)S_b(g)), \quad f,g \in N, t \geq 0, \quad (21)$$

where the $\rho_s$’s (resp. $\rho_s'$’s) are the cochains of $\circ_\nu$ (resp. $\circ_\nu'$). It can be easily checked that Def. [6] indeed defines equivalence relations on the set of sun-products. Weak or strong triviality has to be understood as weak or strong equivalence with the pointwise product on $N$.

We shall now draw some conclusions for weak and strong equivalences of sun-products from Theorem 2. It was shown in [5] that a sun-product is weakly trivial if its cochains are differential operators. Hence as a corollary of Theorem 2, we simply have:
Corollary 4 Let $\circ_\nu$ be a sun-product on $(\mathbb{R}^n, P)$, then $\circ_\nu$ is weakly trivial.

Proof. Let $\rho_i$ be the cochains of $\circ_\nu$. They are differential operators null on constants by Theorem 2. Then define $S_\nu$ to be the formal inverse of $\sum_{r \geq 0} \nu^r \rho^r$. The map $S_\nu$ satisfies $S_\nu(f \circ_\nu g) = f \cdot g$ for $f, g \in \mathbb{N}$, where $\cdot$ denotes the pointwise product, hence $\circ_\nu$ is weakly equivalent to the pointwise product. □

On the other hand, we shall see that strong equivalence puts severe conditions on the equivalence operator $S_\nu$. By setting $g = 1$ in Eq. (21), we get with shortened notations that $S_\nu \rho = \rho' S_\nu$ and by substituting this relation in Eq. (21), we find that the equivalence operator should satisfy $S_\nu(fg) = S_\nu(f)S_\nu(g)$. Hence $S_\nu$ can be nothing else than the exponential of a formal series of derivations of the pointwise product. Actually there are still some supplementary constraints on $S_\nu$, but we do not need to be concerned with them. We conclude that strong equivalence classes are very small and can even reduce to a single point in some situations (e.g., the equivalence class of strongly trivial sun-products).

Although, we do not know whether weak and strong equivalences are induced by the cohomology of some complexes, these notions provide limiting cases between which a proper notion of equivalence for generalized deformations should lie.

Acknowledgements. The author would like to thank Moshé Flato and Daniel Sternheimer for very useful discussions, and Izumi Ojima for great hospitality at RIMS where this work was finalized.

References

[1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer. Deformation theory and quantization. I. Deformations of symplectic structures. *Ann. Physics* **111**, 61–110 (1978).

[2] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer. Deformation theory and quantization. II. Physical applications. *Ann. Physics* **111**, 111–151 (1978).

[3] M. De Wilde and P. B. A. Lecomte. Existence of star-products on and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds. *Lett. Math. Phys.* **7**, 487–496 (1983).

[4] G. Dito, M. Flato, D. Sternheimer, and L. Takhtajan. Deformation quantization and Nambu mechanics. *Comm. Math. Phys.* **183**, 1–22 (1997).

[5] G. Dito and M. Flato. Generalized abelian deformations: application to Nambu mechanics. *Lett. Math. Phys.* **39**, 107–125 (1997).

[6] M. Flato. Private communication (1996).
[7] M. Flato, G. Dito, and D. Sternheimer. Nambu mechanics, n-ary operations and their quantization. In Deformation theory and symplectic geometry (Ascona, 1996), volume 20 of Math. Phys. Stud., pages 43–66. Kluwer Acad. Publ., Dordrecht, 1997.

[8] M. Gerstenhaber. On the deformation of rings and algebras. Ann. of Math. 79, 59–103 (1964).

[9] S. Gutt. An explicit \( \ast \)-product on the cotangent bundle of a Lie group. Lett. Math. Phys. 7, 249–258 (1983).

[10] M. Kontsevich. Deformation quantization of Poisson manifolds I. Preprint I.H.E.S. q-alg/9709040 (1997).

[11] A. Lichnerowicz. Les variétés de Poisson et leurs algèbres de Lie associées. J. Differential Geometry 12, 253–300 (1977).

[12] D. Mélotte. Cohomologie de Chevalley associée aux variétés de Poisson. Bull. Soc. Roy. Sci. Liège 58, 319–413 (1989).

[13] M. A. Naïmark and A. I. Štern. Theory of group representations, volume 246 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science]. Springer-Verlag, New York, 1982.

[14] R. Nest and B. Tsygan. Algebraic index theorem. Comm. Math. Phys. 172, 223–262 (1995).

[15] G. Pinczon. Noncommutative deformation theory. Lett. Math. Phys. 41, 101–117 (1997).

[16] J. Vey. Déformation du crochet de Poisson sur une variété symplectique. Comment. Math. Helv. 50, 421–454 (1975).