Almost Everywhere Stability of Discrete-Time Dynamical Systems

Özkan Karabacak, Rafael Wisniewski and John-Josef Leth

September 27, 2016

Abstract

It is known that the existence of a Lyapunov-type density function, called Lyapunov densities or Lyapunov measures, implies the convergence of Lebesgue almost all solutions to an equilibrium. Considering the evolution of densities using Perron-Frobenius operator, the Lyapunov density approach can be formulated clearly. In this paper, we consider discrete-time dynamical systems and prove a a Lyapunov density theorem with less assumption than the ones that exist in the current literature.

1 Introduction

Lyapunov density has been proposed by Rantzer [1] and shown to guarantee convergence of almost all solutions to an equilibrium. This approach has been proved to be useful in control theory, in particular in feedback stabilization [2]. In [3], Lyapunov density (called also Lyapunov measure) has been considered in view of Markov processes. In this paper, we improve Theorem 16 in [3] by removing two necessary condition, namely the compactness of the state space and the local almost everywhere stability assumption of the invariant set.

By a measure \( \mu \) on \( X \), we always mean a \( \sigma \)-finite measure defined on the Borel \( \sigma \)-algebra of \( X \). We say that a property is satisfied \( \mu \) almost everywhere of \( \mu \)-a.e. in short, if the set of points (initial conditions for solutions) that do not satisfy the property has \( \mu \)-measure zero. For any set \( V \subset X \), we denote the \( \varepsilon \)-neighborhood of \( V \) by \( B_\varepsilon(V) \), or sometimes just by \( B_\varepsilon \) if the set \( V \) is clear from the context. For \( 0 \in X \), notation \( B_\varepsilon = B_\varepsilon(\{0\}) \) will be used for \( \varepsilon \)-neighborhood of the point set \( \{0\} \). For a set \( A \subset X \), \( A^c := X - A \) denotes the complement of \( A \).

2 Main Results

Consider a discrete-time dynamical system on a metric space \( X \) given by

\[ x(k+1) = T(x(k)) \quad x(0) \in X. \]
Let $m$ be a $(\sigma$-finite) measure defined on $\mathcal{B} = \mathcal{B}(X)$, i.e. the $\sigma$-algebra of Borel subsets of $X$. We assume that the dynamics $T$ is nonsingular, namely $A \in \mathcal{B}$ and $m(A) = 0$ implies that $m(T^{-1}A) = 0$. A pair of measure $\mu_1$ and $\mu_2$ are said to be equivalent if they give rise to the same set of zero measures, i.e. if for any $A \in \mathcal{B}$, $\mu_1(A) = 0 \iff \mu_2(A) = 0$. A measure $\mu$ is said to be properly subinvariant if $\mu(T^{-1}A) < \mu(A)$ whenever $\mu(A) > 0$.

**Theorem 1 (Lyapunov measure)** Let $0 \in X$ be an equilibrium of (1). Assume that there exists a properly subinvariant measure $\mu$ that is equivalent to $m$ and finite on $\mathcal{B}_\varepsilon$ for any $\varepsilon > 0$. Then, solutions of (1) converge to $0$ m.a.e.

### 3 Preliminary Definitions and Tools

A measure $\mu_2$ is said to be weaker than another measure $\mu_1$ if $\mu_1(A) = 0 \implies \mu_2(A) = 0$. In this case, we write $\mu_2 \ll \mu_1$. Clearly, $\mu_1$ and $\mu_2$ are equivalent if and only if $\mu_1 \ll \mu_2$ and $\mu_2 \ll \mu_1$. Radon-Nikodym theorem states that if $\mu$ and $m$ are signed measures with $\mu \ll m$ then there exists a measurable function defined on $X$ such that

$$\mu(A) = \int_A \rho \ d m.$$  \hspace{1cm} (2)

$\rho$ is called Radon-Nikodym derivative of $\mu$ with respect to $m$. If $\mu$ is a positive measure equivalent to a positive measure $m$ then $\rho$ must be strictly positive (m.a.e.). The Radon-Nikodym derivative of a measure is unique up to a set of $m$-measure zero. On the other hand, if $m$ is a positive measure and $\rho \geq 0$ (resp. $\rho > 0$) is an $m$-measurable function on $X$, then $\mu(A) := \int_A \rho \ d m$ defines a measure weaker than (resp. equivalent to) $m$. Therefore, there is a one-to-one correspondence between the set of positive measures (resp. signed measures) that are weaker than $m$ and the set of equivalence classes of nonnegative measurable functions (resp. all measurable functions), where equivalence classes are defined as sets of functions that differs only on $m$-measure zero points.

### 3.1 Perron-Frobenious Operator

Let $\mathcal{M}$ denote the linear vector space of signed measures on $X$. The evolution of distributions under the dynamics of (1) can be captured by a linear operator $P : \mathcal{M} \to \mathcal{M}$ defined as

$$(P\mu)(A) := \mu(T^{-1}A).$$  \hspace{1cm} (3)

Assume that $\mu$ is weaker than $m$. Since $T$ is nonsingular $m(A) = 0 \implies m(T^{-1}A) = 0 \implies \mu(T^{-1}A) = 0 \implies (P\mu)(A) = 0$. Then $P\mu$ is also weaker than $m$. Therefore, $P$ maps weaker (than $m$) measures to weaker (than $m$) measures. Equivalently, $P$ can be seen to act over the space of measurable functions (Radon-Nikodym derivatives of measures). In other words, $P\rho$ is defined as the Radon-Nikodym derivative of $P\mu$ with respect
Therefore, 
\[ \int_{A} P \rho dm = \int_{T^{-1}A} \rho dm. \] (4)

\( P \) maps integrable functions to integrable functions. Therefore, the restriction of \( P \) as \( P : L_1 \rightarrow L_1 \) is a Markov operator:

- \( \rho \geq 0 \Rightarrow P\rho \geq 0 \)
- \( \|P\rho\| \leq \|\rho\| \).

### 3.2 Koopman Operator

A dual method to capture the statistical behaviour of the deterministic system (1) is via the Koopman operator \( U \), which describes the evolution of the values of observables under the dynamics of (1). Let \( \mathcal{O} = \mathcal{O}(X) \) denote the set of \( \nu \) equivalence classes of measurable functions on \( X \).

Define \( U : \mathcal{O} \rightarrow \mathcal{O} \) as 
\[ (Uf)(x) := f(Tx). \] (5)

Clearly, \( U \) is linear and maps positive functions to positive functions. It also maps bounded functions to bounded functions. Hence \( U \) can be restricted to \( L_\infty \), the normed vector space of equivalence classes of bounded measurable functions.

### 3.3 Duality between Perron-Frobenious and Koopman Operators

If \( \rho \in L_1 \) and \( f \in L_\infty \), then 
\[ < \rho, f > := \int \rho f dm \] (6)

is finite. To see the duality between \( P : L_1 \rightarrow L_1 \) and \( U : L_\infty \rightarrow L_\infty \), observe that 
\[ < P\rho, 1_A > = \int P\rho 1_A dm = \int_A P\rho dm = \int_{T^{-1}A} \rho dm = \int \rho 1_{T^{-1}A} dm = < \rho, 1_{T^{-1}A} > = < \rho, U1_A >. \]
Since any function in \( L_\infty \) can be approximated by characteristic functions we have the following duality 
\[ < P\rho, f > = < \rho, Uf > \quad \rho \in L_1, \ f \in L_\infty. \] (7)

This duality persists even for general measurable functions whenever the integral is finite.

### 4 Proofs

We will use the following characterization for almost sure attractivity of the equilibrium:

**Lemma 1** \( \lim_{n \to \infty} x(n) = 0 \ m \text{-a.e. if and only if the series} \sum_{k=0}^{\infty} U^k 1_{B_\varepsilon} \) is finite \( m \text{-a.e. for all} \ \varepsilon > 0. \)

\(^1\)We allow here the integral to be infinite.
Proof. Consider a trajectory \( x(n) \) with initial condition \( x_0 \in X \). Then, \( \sum_{k=0}^{\infty} U^k 1_{B_{\varepsilon}}(x_0) = \sum_{k=0}^{\infty} 1_{B_{\varepsilon}}(T^n x_0) \) is equal to the number of visits of the trajectory \( x(n) \) to the closed set \( B_{\varepsilon} \). Hence, it is finite if and only if the set of limit points of \( x(n) \) is nonempty and contained in \( B_{\varepsilon} \). Therefore, \( \sum_{k=0}^{\infty} U^k 1_{B_{\varepsilon}}(x_0) \) is finite if and only if the set of limit points of \( x(n) \) is nonempty and contained in \( B_{\varepsilon} \) for \( m \)-a.e. initial points \( x_0 \). Invoking this statement for a sequence \( \varepsilon_i \to 0 \) and considering the fact that \( \bigcap_i B_{\varepsilon_i} = \{0\} \) and that any intersection of countably many full measure sets is a full measure set give the result.

Proof of Theorem \( \blacksquare \) We assume that there exists a properly subinvariant equivalent measure \( \mu \) that is finite on \( B_{\varepsilon} \) for every \( \varepsilon > 0 \). Let \( \rho > 0 \) be the Radon-Nikodym derivative of \( \mu \) with respect to \( m \). Then, \( m \)-a.e. \( \mathbb{P} \rho < \rho \). Define \( \rho_0 := \rho - \mathbb{P} \rho \). Clearly \( \rho_0 \) is positive \( m \)-a.e.

Note that

\[
\rho_0 := \sum_{k=0}^{\infty} \mathbb{P}^k \rho_0
\]

\[
= \rho_0 + \mathbb{P} \rho_0 + \mathbb{P}^2 \rho_0 + \cdots
\]

\[
= \rho - \mathbb{P} \rho + \mathbb{P} \rho + \mathbb{P}^2 \rho + \cdots
\]

\[
= \rho - \lim_{n \to \infty} \mathbb{P}^n \rho.
\]

The last limit exists \( m \)-a.e. since \( \mathbb{P}^n \rho \) is a decreasing sequence bounded from below. Since \( \rho \) and therefore all \( \mathbb{P}^n \rho \) has finite integral on sets \( B_{\varepsilon} \), we conclude that \( \rho_0 \) has finite integral on sets \( B_{\varepsilon} \). Therefore,

\[
\text{finite} = \langle \rho_0, 1_{B_{\varepsilon}} \rangle
\]

\[
= \sum_{k=0}^{\infty} \langle \rho_0, 1_{B_{\varepsilon}} \rangle
\]

\[
= \sum_{k=0}^{\infty} \rho_0 \mathbb{P}^k 1_{B_{\varepsilon}}
\]

\[
= \langle \rho_0, \sum_{k=0}^{\infty} \mathbb{P}^k 1_{B_{\varepsilon}} \rangle
\]

Since \( \rho_0 \) is positive \( m \)-a.e., \( \sum_{k=0}^{\infty} \mathbb{P}^k 1_{B_{\varepsilon}} \) is finite \( m \)-a.e. and from Lemma \( \blacksquare \) 0 is attracting \( m \)-a.e.. \( \blacksquare \)

References

[1] Anders Rantzer. A dual to Lyapunov’s stability theorem. *Systems and Control Letters*, 42:1–17, 2001.

[2] Stephen Prajna, Pablo a. Parrilo, and a. Rantzer. Nonlinear control synthesis by convex optimization. *IEEE Transactions on Automatic Control*, 49(2):310–314, 2004.

[3] Umesh Vaidya and Prashant G. Mehta. Lyapunov measure for almost everywhere stability. *IEEE Transactions on Automatic Control*, 53(1):307–323, 2008.