SOME INEQUALITIES FOR TRACE CLASS OPERATORS VIA A KATO’S RESULT

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ABSTRACT. By the use of the celebrated Kato’s inequality we obtain in this paper some new inequalities for trace class operators on a complex Hilbert space $H$. Natural applications for functions defined by power series of normal operators are given as well.

1. INTRODUCTION

We denote by $B(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H; (\cdot, \cdot))$.

If $P$ is a positive selfadjoint operator on $H$, i.e. $\langle Px, x \rangle \geq 0$ for any $x \in H$, then the following inequality is a generalization of the Schwarz inequality in $H$

$$|\langle Px, y \rangle|^2 \leq \langle Px, x \rangle \langle Py, y \rangle,$$

for any $x, y \in H$.

The following inequality is of interest as well, see [18, p. 221].

Let $P$ be a positive selfadjoint operator on $H$. Then

$$\|Px\|^2 \leq \|P\| \langle Px, x \rangle$$

for any $x \in H$.

The "square root" of a positive bounded selfadjoint operator on $H$ can be defined as follows, see for instance [18, p. 240]: If the operator $A \in B(H)$ is selfadjoint and positive, then there exists a unique positive selfadjoint operator $B := \sqrt{A} \in B(H)$ such that $B^2 = A$. If $A$ is invertible, then so is $B$.

If $A \in B(H)$, then the operator $A^*A$ is selfadjoint and positive. Define the "absolute value" operator by $|A| := \sqrt{A^*A}$.

In 1952, Kato [19] proved the following celebrated generalization of Schwarz inequality for any bounded linear operator $T$ on $H$:

$$|\langle Tx, y \rangle|^2 \leq \langle (T^*T)^\alpha x, x \rangle \langle (TT^*)^{1-\alpha} y, y \rangle,$$

for any $x, y \in H, \alpha \in [0, 1]$. Utilizing the modulus notation introduced before, we can write (1.3) as follows

$$|\langle Tx, y \rangle|^2 \leq \langle |T|^{2\alpha} x, x \rangle \langle |T^*|^{2(1-\alpha)} y, y \rangle$$

for any $x, y \in H, \alpha \in [0, 1]$.

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It is useful to observe that, if $T = N$, a normal operator, i.e., we recall that $NN^* = N^*N$, then the inequality (1.4) can be written as

\begin{equation}
|\langle Nx, y \rangle|^2 \leq \langle |N|^{2\alpha} x, x \rangle \langle |N|^{2(1-\alpha)} y, y \rangle,
\end{equation}

and in particular, for selfadjoint operators $A$ we can state it as

\begin{equation}
|\langle Ax, y \rangle| \leq \|A\|^\alpha x \|A^{1-\alpha} y \|
\end{equation}

for any $x, y \in H, \alpha \in [0, 1]$.

If $T = U$, a unitary operator, i.e., we recall that $UU^* = U^*U = 1_H$, then the inequality (1.4) becomes

\begin{equation}
|\langle Ux, y \rangle| \leq \|x\| \|y\|
\end{equation}

for any $x, y \in H$, which provides a natural generalization for the Schwarz inequality in $H$.

The symmetric powers in the inequalities above are natural to be considered, so if we choose in (1.4), (1.5) and in (1.6) $\alpha = 1/2$ then we get for any $x, y \in H$

\begin{align*}
|\langle Tx, y \rangle|^2 &\leq \langle |T| x, x \rangle \langle |T^*| y, y \rangle,
|\langle Nx, y \rangle|^2 &\leq \langle |N| x, x \rangle \langle |N| y, y \rangle,
|\langle Ax, y \rangle| &\leq \|A^{1/2} x \| \|A^{1/2} y \|
\end{align*}

respectively.

It is also worthwhile to observe that, if we take the supremum over $y \in H, \|y\| = 1$ in (1.4) then we get

\begin{equation}
\|Tx\|^2 \leq \|T\|^{2(1-\alpha)} \langle |T|^{2\alpha} x, x \rangle
\end{equation}

for any $x \in H$, or in an equivalent form

\begin{equation}
\|Tx\| \leq \|T\|^{\alpha} \|x\| \|T\|^{1-\alpha}
\end{equation}

for any $x \in H$.

If we take $\alpha = 1/2$ in (1.10), then we get

\begin{equation}
\|Tx\|^2 \leq \|T\| \langle |T| x, x \rangle
\end{equation}

for any $x \in H$, which in the particular case of $T = P$, a positive operator, provides the result from (1.2).

For various interesting generalizations, extension and Kato related results, see the papers [7]-[17], [23]-[29] and [34].

In order to state our results concerning new trace inequalities for operators in Hilbert spaces we need some preliminary facts as follows.

2. Trace of Operators

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\{e_i\}_{i \in I}$ an orthonormal basis of $H$. We say that $A \in \mathcal{B}(H)$ is a Hilbert-Schmidt operator if

\begin{equation}
\sum_{i \in I} \|Ae_i\|^2 < \infty.
\end{equation}
It is well known that, if \( \{e_i\}_{i \in I} \) and \( \{f_j\}_{j \in J} \) are orthonormal bases for \( H \) and \( A \in \mathcal{B}(H) \) then
\[
\sum_{i \in I} \|Ae_i\|^2 = \sum_{j \in J} \|Af_j\|^2 = \sum_{j \in J} \|A^*f_j\|^2
\]
showing that the definition \( (2.1) \) is independent of the orthonormal basis and \( A \) is a Hilbert-Schmidt operator if \( A^* \) is a Hilbert-Schmidt operator.

Let \( \mathcal{B}_2(H) \) the set of Hilbert-Schmidt operators in \( \mathcal{B}(H) \). For \( A \in \mathcal{B}_2(H) \) we define
\[
\|A\|_2 := \left( \sum_{i \in I} \|Ae_i\|^2 \right)^{1/2}
\]
for \( \{e_i\}_{i \in I} \) an orthonormal basis of \( H \). This definition does not depend on the choice of the orthonormal basis.

Using the triangle inequality in \( l^2(I) \), one checks that \( \mathcal{B}_2(H) \) is a vector space and that \( \|\cdot\|_2 \) is a norm on \( \mathcal{B}_2(H) \), which is usually called in the literature as the Hilbert-Schmidt norm.

Denote the modulus of an operator \( A \in \mathcal{B}(H) \) by \( |A| := (A^*A)^{1/2} \).

Because \( \|A\|_2 = \|Ax\| \) for all \( x \in H \), \( A \) is Hilbert-Schmidt iff \( |A| \) is Hilbert-Schmidt and \( \|A\|_2 = \||A||_2 \). From \( (2.2) \) we have that if \( A \in \mathcal{B}_2(H) \), then \( A^* \in \mathcal{B}_2(H) \) and \( \|A\|_2 = \|A^*\|_2 \).

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

**Theorem 1.** We have

(i) \( (\mathcal{B}_2(H), \|\cdot\|_2) \) is a Hilbert space with inner product
\[
\langle A, B \rangle_2 := \sum_{i \in I} \langle Ae_i, Be_i \rangle = \sum_{i \in I} \langle B^*Ae_i, e_i \rangle
\]
and the definition does not depend on the choice of the orthonormal basis \( \{e_i\}_{i \in I} \);

(ii) We have the inequalities
\[
\|A\| \leq \|A\|_2
\]
for any \( A \in \mathcal{B}_2(H) \) and
\[
\|AT\|_2, \|TA\|_2 \leq \|T\| \|A\|_2
\]
for any \( A \in \mathcal{B}_2(H) \) and \( T \in \mathcal{B}(H) \);

(iii) \( \mathcal{B}_2(H) \) is an operator ideal in \( \mathcal{B}(H) \), i.e.
\[
\mathcal{B}(H) \mathcal{B}_2(H) \mathcal{B}(H) \subseteq \mathcal{B}_2(H);
\]

(iv) \( \mathcal{B}_{fin}(H) \), the space of operators of finite rank, is a dense subspace of \( \mathcal{B}_2(H) \);

(v) \( \mathcal{B}_2(H) \subseteq \mathcal{K}(H) \), where \( \mathcal{K}(H) \) denotes the algebra of compact operators on \( H \).

If \( \{e_i\}_{i \in I} \) an orthonormal basis of \( H \), we say that \( A \in \mathcal{B}(H) \) is trace class if
\[
\|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty.
\]
The definition of \( \|A\|_1 \) does not depend on the choice of the orthonormal basis \( \{e_i\}_{i \in I} \). We denote by \( \mathcal{B}_1(H) \) the set of trace class operators in \( \mathcal{B}(H) \).

The following proposition holds:
Proposition 1. If \( A \in \mathcal{B}(H) \), then the following are equivalent:
(i) \( A \in \mathcal{B}_1(H) \);
(ii) \(|A|^\frac{1}{2} \in \mathcal{B}_2(H)\);
(iii) \( A \) (or \(|A|\)) is the product of two elements of \( \mathcal{B}_2(H)\).

The following properties are also well known:
Theorem 2. With the above notations:
(i) We have
\[
\|A\|_1 = \|A^*\|_1 \quad \text{and} \quad \|A\|_2 \leq \|A\|_1
\]
for any \( A \in \mathcal{B}_1(H) \);
(ii) \( \mathcal{B}_1(H) \) is an operator ideal in \( \mathcal{B}(H) \), i.e.
\[
\mathcal{B}(H) \mathcal{B}_1(H) \mathcal{B}(H) \subseteq \mathcal{B}_1(H);
\]
(iii) We have
\[
\mathcal{B}_2(H) \mathcal{B}_2(H) = \mathcal{B}_1(H);
\]
(iv) We have
\[
\|A\|_1 = \sup \{ \langle A, B \rangle_2 : B \in \mathcal{B}_2(H), \|B\| \leq 1 \};
\]
(v) \( (\mathcal{B}_1(H), \|\cdot\|_1) \) is a Banach space.

We have the following isometric isomorphisms
\[
\mathcal{B}_1(H) \cong K(H)^* \text{ and } \mathcal{B}_1(H)^* \cong \mathcal{B}(H),
\]
where \( K(H)^* \) is the dual space of \( K(H) \) and \( \mathcal{B}_1(H)^* \) is the dual space of \( \mathcal{B}_1(H) \).

We define the trace of a trace class operator \( A \in \mathcal{B}_1(H) \) to be
\[
\text{tr} (A) := \sum_{i \in I} \langle Ae_i, e_i \rangle,
\]
where \( \{e_i\}_{i \in I} \) is an orthonormal basis of \( H \). Note that this coincides with the usual definition of the trace if \( H \) is finite-dimensional. We observe that the series (2.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:
Theorem 3. We have
(i) If \( A \in \mathcal{B}_1(H) \) then \( A^* \in \mathcal{B}_1(H) \) and
\[
\text{tr} (A^*) = \text{tr}(A);
\]
(ii) If \( A \in \mathcal{B}_1(H) \) and \( T \in \mathcal{B}(H) \), then \( AT, TA \in \mathcal{B}_1(H) \) and
\[
\text{tr} (AT) = \text{tr} (TA) \quad \text{and} \quad |\text{tr} (AT)| \leq \|A\|_1 \|T\|;
\]
(iii) \( \text{tr} (\cdot) \) is a bounded linear functional on \( \mathcal{B}_1(H) \) with \( |\text{tr}| = 1 \);
(iv) If \( A, B \in \mathcal{B}_2(H) \) then \( AB, BA \in \mathcal{B}_1(H) \) and \( \text{tr} (AB) = \text{tr} (BA) \);
(v) \( \mathcal{B}_{fin}(H) \) is a dense subspace of \( \mathcal{B}_1(H) \).

Utilising the trace notation we obviously have that
\[
\langle A, B \rangle_2 = \text{tr} (B^* A) = \text{tr} (AB^*) \quad \text{and} \quad \|A\|_2^2 = \text{tr} (A^* A) = \text{tr} (|A|^2)
\]
for any \( A, B \in \mathcal{B}_2(H). \)

For the theory of trace functionals and their applications the reader is referred to [33].
For some classical trace inequalities see [4], [6], [30] and [38], which are continuations of the work of Bellman [2]. For related works the reader can refer to [1], [3], [4], [14], [22], [27], [28], [31] and [35].

3. Trace Inequalities Via Kato’s Result

We start with the following result:

**Theorem 4.** Let $T \in B(H)$.

(i) If for some $\alpha \in (0, 1)$ we have $|T|^{2\alpha}$, $|T^*|^{2(1-\alpha)} \in B_1(H)$, then $T \in B_1(H)$ and we have the inequality

$$|\text{tr} (T)|^2 \leq \text{tr} \left( |T|^{2\alpha} \right) \text{tr} \left( |T^*|^{2(1-\alpha)} \right);$$

(ii) If for some $\alpha \in [0, 1]$ and an orthonormal basis $\{e_i\}_{i \in I}$ the sum

$$\sum_{i \in I} \|T e_i\|^\alpha \|T^* e_i\|^{1-\alpha}$$

is finite, then $T \in B_1(H)$ and we have the inequality

$$|\text{tr} (T)| \leq \sum_{i \in I} \|T e_i\|^\alpha \|T^* e_i\|^{1-\alpha}.$$

Moreover, if the sums $\sum_{i \in I} \|T e_i\|$ and $\sum_{i \in I} \|T^* e_i\|$ are finite for an orthonormal basis $\{e_i\}_{i \in I}$, then $T \in B_1(H)$ and we have

$$|\text{tr} (T)| \leq \inf_{\alpha \in [0, 1]} \left\{ \sum_{i \in I} \|T e_i\|^\alpha \|T^* e_i\|^{1-\alpha} \right\} \leq \min \left\{ \sum_{i \in I} \|T e_i\|, \sum_{i \in I} \|T^* e_i\| \right\}.$$

**Proof.** (i) Assume that $\alpha \in (0, 1)$. Let $\{e_i\}_{i \in I}$ be an orthonormal basis in $H$ and $F$ a finite part of $I$. Then by Kato’s inequality (1.4) we have

$$\sum_{i \in F} \langle T e_i, e_i \rangle \leq \sum_{i \in F} |\langle T e_i, e_i \rangle| \leq \sum_{i \in F} \langle |T|^{2\alpha} e_i, e_i \rangle^{1/2} \langle |T^*|^{2(1-\alpha)} e_i, e_i \rangle^{1/2}.$$

By Cauchy-Buniakowskii-Schwarz inequality for finite sums we have

$$\sum_{i \in F} \langle |T|^{2\alpha} e_i, e_i \rangle^{1/2} \langle |T^*|^{2(1-\alpha)} e_i, e_i \rangle^{1/2} \leq \left( \sum_{i \in F} \langle |T|^{2\alpha} e_i, e_i \rangle^{1/2} \right)^2 \left( \sum_{i \in F} \langle |T^*|^{2(1-\alpha)} e_i, e_i \rangle^{1/2} \right)^2 \frac{1}{2} \left( \sum_{i \in F} \langle |T^*|^{2(1-\alpha)} e_i, e_i \rangle \right) \frac{1}{2} \left( \sum_{i \in F} \langle |T|^{2\alpha} e_i, e_i \rangle \right).$$

Therefore, by (3.4) and (3.5) we have

$$\sum_{i \in F} \langle T e_i, e_i \rangle \leq \left( \sum_{i \in F} \langle |T|^{2\alpha} e_i, e_i \rangle \right)^{1/2} \left( \sum_{i \in F} \langle |T^*|^{2(1-\alpha)} e_i, e_i \rangle \right)^{1/2}$$

for any finite part $F$ of $I$. 
If for some \( \alpha \in (0, 1) \) we have \( |T|^{2\alpha}, |T^*|^{2(1-\alpha)} \in B_1(H) \), then the sums 
\[ \sum_{i \in I} \left| \left\langle T^{2\alpha} e_i, e_i \right\rangle \right| \quad \text{and} \quad \sum_{i \in I} \left| \left\langle T^*^{2(1-\alpha)} e_i, e_i \right\rangle \right| \]
are finite and by (3.6) we have that \( \sum_{i \in I} \left| \langle T e_i, e_i \rangle \right| \) is also finite and we have the inequality (3.1).

(ii) Assume that \( \alpha \in [0, 1] \). Let \( \{e_i\}_{i \in I} \) be an orthonormal basis in \( H \) and \( F \) a finite part of \( I \). Utilising McCarthy’s inequality for the positive operator \( P \), namely
\[ \langle P^\alpha x, x \rangle \leq \langle Px, x \rangle^\beta, \]
that holds for \( \beta \in [0, 1] \) and \( x \in H, \|x\| = 1 \), we have
\[ \left| \left\langle T^{2\alpha} e_i, e_i \right\rangle \right| \leq \left| \left\langle T^2 e_i, e_i \right\rangle \right|^{\alpha} \]
and
\[ \left| \left\langle T^*^{2(1-\alpha)} e_i, e_i \right\rangle \right| \leq \left| \left\langle T^2 e_i, e_i \right\rangle \right|^{1-\alpha} \]
for any \( i \in I \).

Making use of (3.4) we have
\[
(3.7) \quad \left| \sum_{i \in F} \langle T e_i, e_i \rangle \right| \leq \sum_{i \in F} |\langle T e_i, e_i \rangle| \leq \sum_{i \in F} \left| \left\langle T^{2\alpha} e_i, e_i \right\rangle \right|^{1/2} \left| \left\langle T^*^{2(1-\alpha)} e_i, e_i \right\rangle \right|^{1/2} \\
\leq \sum_{i \in F} \left| \langle T^2 e_i, e_i \rangle \right|^{\alpha/2} \left| \langle T^2 e_i, e_i \rangle \right|^{(1-\alpha)/2} \\
= \sum_{i \in F} \langle T^* T e_i, e_i \rangle^{\alpha/2} \langle T T^* e_i, e_i \rangle^{(1-\alpha)/2} \\
= \sum_{i \in F} \|T e_i\|^{\alpha} \|T^* e_i\|^{1-\alpha}.
\]

Utilizing Hölder’s inequality for finite sums and \( p = \frac{1}{\alpha}, q = \frac{1}{1-\alpha} \) we also have
\[
(3.8) \quad \sum_{i \in F} \|T e_i\|^{\alpha} \|T^* e_i\|^{1-\alpha} \\
\leq \left[ \sum_{i \in F} \left( \|T e_i\|^{\alpha} \right) \right]^{1/\alpha} \left[ \sum_{i \in F} \left( \|T^* e_i\|^{1-\alpha} \right) \right]^{1-1/\alpha} \\
= \left[ \sum_{i \in F} \|T e_i\| \right]^{\alpha} \left[ \sum_{i \in F} \|T^* e_i\| \right]^{1-\alpha}.
\]

Since all the series involved in (3.7) and (3.8) are convergent, then we get
\[
(3.9) \quad \left| \sum_{i \in I} \langle T e_i, e_i \rangle \right| \leq \sum_{i \in I} \|T e_i\|^{\alpha} \|T^* e_i\|^{1-\alpha} \\
\leq \left[ \sum_{i \in I} \|T e_i\| \right]^{\alpha} \left[ \sum_{i \in I} \|T^* e_i\| \right]^{1-\alpha}
\]
for any \( \alpha \in [0, 1] \).
Taking the infimum over $\alpha \in [0, 1]$ in (3.9) produces
\begin{equation}
(3.10) \quad \left| \sum_{i \in I} \langle Te_i, e_i \rangle \right| \leq \inf_{\alpha \in [0, 1]} \left\{ \sum_{i \in F} \|Te_i\|^\alpha \|T^*e_i\|^{1-\alpha} \right\} \\
\leq \inf_{\alpha \in [0, 1]} \left[ \sum_{i \in F} \|Te_i\|^\alpha \right] \left[ \sum_{i \in F} \|T^*e_i\| \right]^{1-\alpha} \\
= \min \left\{ \sum_{i \in F} \|Te_i\|, \sum_{i \in F} \|T^*e_i\| \right\}.
\end{equation}

\[ \square \]

**Corollary 1.** Let $T \in \mathcal{B}(H)$.

(i) If we have $|T|, |T^*| \in \mathcal{B}_1(H)$, then $T \in \mathcal{B}_1(H)$ and we have the inequality
\begin{equation}
(3.11) \quad |\text{tr } (T)|^2 \leq \text{tr } (|T|) \text{ tr } (|T^*|);
\end{equation}

(ii) If for an orthonormal basis $\{e_i\}_{i \in I}$ the sum $\sum_{i \in I} \sqrt{\|Te_i\| \|T^*e_i\|}$ is finite, then $T \in \mathcal{B}_1(H)$ and we have the inequality
\begin{equation}
(3.12) \quad |\text{tr } (T)| \leq \sum_{i \in I} \sqrt{\|Te_i\| \|T^*e_i\|}.
\end{equation}

**Corollary 2.** Let $N \in \mathcal{B}(H)$ be a normal operator. If for some $\alpha \in (0, 1)$ we have $|N|^{2\alpha}, |N|^{2(1-\alpha)} \in \mathcal{B}_1(H)$, then $N \in \mathcal{B}_1(H)$ and we have the inequality
\begin{equation}
(3.13) \quad |\text{tr } (N)|^2 \leq \text{tr } (|N|^{2\alpha}) \text{ tr } (|N|^{2(1-\alpha)}).
\end{equation}

In particular, if $|N| \in \mathcal{B}_1(H)$, then $N \in \mathcal{B}_1(H)$ and
\begin{equation}
(3.14) \quad |\text{tr } (N)| \leq \text{tr } (|N|).
\end{equation}

The following result also holds.

**Theorem 5.** Let $T \in \mathcal{B}(H)$ and $A, B \in \mathcal{B}_2(H)$.

(i) For any $\alpha \in [0, 1]$ we have $|A|^2 |T|^{2\alpha}, |B|^2 |T^*|^{2(1-\alpha)}$ and $B^*TA \in \mathcal{B}_1(H)$ and
\begin{equation}
(3.15) \quad |\text{tr } (AB^*T)|^2 \leq \text{tr } (|A|^2 |T|^{2\alpha}) \text{ tr } (|B|^2 |T^*|^{2(1-\alpha)});
\end{equation}

(ii) We also have
\begin{equation}
(3.16) \quad |\text{tr } (AB^*T)|^2 \leq \min \left\{ \text{tr } (|B|^2) \text{ tr } (|A|^2 |T|^{2\alpha}) \text{ tr } (|B|^2 |T^*|^{2(1-\alpha)}) \right\}.
\end{equation}

**Proof.** (i) Let $\{e_i\}_{i \in I}$ be an orthonormal basis in $H$ and $F$ a finite part of $I$. Then by Kato’s inequality (1.4) we have
\begin{equation}
(3.17) \quad |(TAe_i, Be_i)|^2 \leq \langle |T|^{2\alpha} Ae_i, Ae_i \rangle \langle |T^*|^{2(1-\alpha)} Be_i, Be_i \rangle
\end{equation}
for any $i \in I$. This is equivalent to
\begin{equation}
(3.18) \quad |(B^*TAe_i, e_i)| \leq \langle A^* |T|^{2\alpha} Ae_i, e_i \rangle^{1/2} \langle B^* |T^*|^{2(1-\alpha)} Be_i, e_i \rangle^{1/2}
\end{equation}
for any $i \in I$. 

Using the generalized triangle inequality for the modulus and the Cauchy-Bunyakowsky-Schwarz inequality for finite sums we have from (3.18) that

\[
\sum_{i \in F} |\langle B^* T A e_i, e_i \rangle| \\
\leq \sum_{i \in F} |\langle B^* T A e_i, e_i \rangle| \\
\leq \sum_{i \in F} \left( \left| \langle A^* |T|^{2\alpha} A e_i, e_i \rangle \right|^{1/2} \left| \langle B^* |T^*|^{2(1-\alpha)} B e_i, e_i \rangle \right|^{1/2} \right) \\
\leq \sum_{i \in F} \left( \left( \langle A^* |T|^{2\alpha} A e_i, e_i \rangle \right)^{1/2} \right)^2 \\
\times \left[ \sum_{i \in F} \left( \langle B^* |T^*|^{2(1-\alpha)} B e_i, e_i \rangle \right)^{1/2} \right]^2 \\
= \sum_{i \in F} \left( \langle A^* |T|^{2\alpha} A e_i, e_i \rangle \right)^{1/2} \left[ \sum_{i \in F} \langle B^* |T^*|^{2(1-\alpha)} B e_i, e_i \rangle \right]^{1/2}
\]

for any \( F \) a finite part of \( I \).

Let \( \alpha \in [0, 1] \). Since \( A, B \in \mathcal{B}_2(H) \), then \( A^* |T|^{2\alpha} A, B^* |T^*|^{2(1-\alpha)} B \) and \( B^* T A \in \mathcal{B}_1(H) \) and by (3.19) we have

\[
|\text{tr}(B^* T A)| \leq \left[ \text{tr} \left( A^* |T|^{2\alpha} A \right) \right]^{1/2} \left[ \text{tr} \left( B^* |T^*|^{2(1-\alpha)} B \right) \right]^{1/2}.
\]

Since, by the properties of trace we have

\[
\text{tr}(B^* T A) = \text{tr}(A B^* T), \\
\text{tr}(A^* |T|^{2\alpha} A) = \text{tr}(A A^* |T|^{2\alpha}) = \text{tr}(|A|^2 |T|^{2\alpha})
\]

and

\[
\text{tr}(B^* |T^*|^{2(1-\alpha)} B) = \text{tr}(|B|^2 |T^*|^{2(1-\alpha)}),
\]

then by (3.20) we get (3.15).

(ii) Utilising McCarthy’s inequality [29] for the positive operator \( P \)

\[
\langle P^\beta x, x \rangle \leq \langle P x, x \rangle^\beta
\]

that holds for \( \beta \in (0, 1) \) and \( x \in H, \|x\| = 1 \), we have

\[
\langle P^\beta y, y \rangle \leq \|y\|^{2(1-\beta)} \langle P y, y \rangle^\beta
\]

for any \( y \in H \).

Let \( \{e_i\}_{i \in I} \) be an orthonormal basis in \( H \) and \( F \) a finite part of \( I \). From (3.21) we have

\[
\langle |T|^{2\alpha} A e_i, A e_i \rangle \leq \|A e_i\|^{2(1-\alpha)} \langle |T|^{2\alpha} A e_i, A e_i \rangle^{\alpha}
\]

and

\[
\langle |T^*|^{2(1-\alpha)} B e_i, B e_i \rangle \leq \|B e_i\|^{2\alpha} \langle |T^*|^{2\alpha} B e_i, B e_i \rangle^{1-\alpha}
\]

for any \( i \in I \).
Making use of the inequality (3.17) we get

\[
\langle T A e_i, B e_i \rangle \leq \|A e_i\|^2 (1 - \alpha) \|T^* A e_i, A e_i\|^{\alpha} \|B e_i\|^2 \langle T^* |T^2 B e_i, B e_i\rangle^{1 - \alpha}
\]

and taking the square root we get

\[
(3.22) \quad \langle T A e_i, B e_i \rangle \leq \|B e_i\|^\alpha \langle T^* |T^2 A e_i, A e_i\rangle^{\frac{\alpha}{2}} \|A e_i\|^{1 - \alpha} \langle T^* |T^2 B e_i, B e_i\rangle^{\frac{1 - \alpha}{2}}
\]

for any \( i \in I \).

Using the generalized triangle inequality for the modulus and the Hölder’s inequality for finite sums and \( p = \frac{1}{\alpha}, q = \frac{1}{1 - \alpha} \) we get from (3.22) that

\[
(3.23) \quad \left| \sum_{i \in F} \langle B^* T A e_i, e_i \rangle \right| \\
\leq \sum_{i \in F} \|B e_i\|^\alpha \langle T^* |T^2 A e_i, A e_i\rangle \|A e_i\|^{1 - \alpha} \langle T^* |T^2 B e_i, B e_i\rangle^{\frac{1 - \alpha}{2}}
\]

By Cauchy-Bunyakowsky-Schwarz inequality for finite sums we also have

\[
\sum_{i \in F} \|B e_i\| \langle T |T^2 A e_i, A e_i\rangle^{\frac{1}{2}} \leq \left( \sum_{i \in F} \|B e_i\|^2 \right)^{1/2} \left( \sum_{i \in F} \langle T |T^2 A e_i, A e_i\rangle \right)^{1/2}
\]

\[
= \left( \sum_{i \in F} \langle B^* e_i, e_i \rangle \right)^{1/2} \left( \sum_{i \in F} \langle A^* |T^2 A e_i, e_i\rangle \right)^{1/2}
\]

and

\[
\sum_{i \in F} \|A e_i\| \langle T^* |T^2 B e_i, B e_i\rangle^{\frac{1}{2}} \leq \left( \sum_{i \in F} \|A e_i\|^2 \right)^{1/2} \left( \sum_{i \in F} \langle T^* |T^2 B e_i, B e_i\rangle \right)^{1/2}
\]

\[
= \left( \sum_{i \in F} \langle A^2 e_i, e_i \rangle \right)^{1/2} \left( \sum_{i \in F} \langle B^* |T^2 B e_i, e_i\rangle \right)^{1/2}
\]
and by (3.23) we obtain

\[
\left| \sum_{i \in F} \langle B^*TAe_i, e_i \rangle \right| \\
\leq \left( \sum_{i \in F} |B|^2 e_i, e_i \right)^{\alpha/2} \left( \sum_{i \in F} |A^*|T|^2 A e_i, e_i \right)^{\alpha/2} \\
\times \left( \sum_{i \in F} |A|^2 e_i, e_i \right)^{(1-\alpha)/2} \left( \sum_{i \in F} |B^*|T^*|^2 B e_i, e_i \right)^{(1-\alpha)/2}
\]

for any \( F \) a finite part of \( I \).

Let \( \alpha \in [0,1] \). Since \( A, B \in B_2(H) \), then \( A^*|T|^2 A \) and \( B^*|T^*|^2 B \in B_1(H) \) and by (3.24) we get

\[
|\text{tr}(AB^*T)|^2 \\
\leq \left[ \text{tr} \left( |B|^2 \right) \text{tr} \left( A^*|T|^2 A \right) \right]^{\alpha} \left[ \text{tr} \left( |A|^2 \right) \text{tr} \left( B^*|T^*|^2 B \right) \right]^{1-\alpha} \\
= \left[ \text{tr} \left( |B|^2 \right) \text{tr} \left( |A|^2 |T|^2 \right) \right]^{\alpha} \left[ \text{tr} \left( |A|^2 \right) \text{tr} \left( |B|^2 |T^*|^2 \right) \right]^{1-\alpha}.
\]

Taking the infimum over \( \alpha \in [0,1] \) we get (3.16). \( \square \)

**Corollary 3.** Let \( T \in B(H) \) and \( A, B \in B_2(H) \). We have \( |A^*|^2 |T| \), \( |B^*|^2 |T^*| \) and \( B^*TA \in B_1(H) \) and

\[
|\text{tr}(AB^*T)|^2 \leq \text{tr} \left( |A^*|^2 |T| \right) \text{tr} \left( |B^*|^2 |T^*| \right).
\]

**Corollary 4.** Let \( N \in B(H) \) be a normal operator and \( A, B \in B_2(H) \).

(i) For any \( \alpha \in [0,1] \) we have \( |A^*|^2 |N|^{2\alpha}, |B^*|^2 |N|^{2(1-\alpha)} \) and \( B^*NA \in B_1(H) \) and

\[
|\text{tr}(AB^*N)|^2 \leq \text{tr} \left( |A^*|^2 |N|^{2\alpha} \right) \text{tr} \left( |B^*|^2 |N|^{2(1-\alpha)} \right).
\]

In particular, we have \( |A^*|^2 |N|, |B^*|^2 |N| \) and \( B^*NA \in B_1(H) \) and

\[
|\text{tr}(AB^*N)|^2 \leq \text{tr} \left( |A^*|^2 |N| \right) \text{tr} \left( |B^*|^2 |N| \right).
\]

(ii) We also have

\[
|\text{tr}(AB^*N)|^2 \\
\leq \min \left\{ \text{tr} \left( |B|^2 \right) \text{tr} \left( |A|^2 |N|^2 \right), \text{tr} \left( |A|^2 \right) \text{tr} \left( |B|^2 |N|^2 \right) \right\}.
\]

**Remark 1.** Let \( \alpha \in [0,1] \). By replacing \( A \) with \( A^* \) and \( B \) with \( B^* \) in (3.15) we get

\[
|\text{tr}(A^* BT)|^2 \leq \text{tr} \left( |A|^2 |T|^{2\alpha} \right) \text{tr} \left( |B|^2 |T^*|^{2(1-\alpha)} \right)
\]

for any \( T \in B(H) \) and \( A, B \in B_2(H) \).

If in this inequality we take \( A = B \), then we get

\[
|\text{tr}(B^2 T)|^2 \leq \text{tr} \left( |B|^2 |T|^{2\alpha} \right) \text{tr} \left( |B|^2 |T^*|^{2(1-\alpha)} \right)
\]

for any \( T \in B(H) \) and \( B \in B_2(H) \).
If in (3.30) we take \( A = B^* \) then we get
\[
(3.32) \quad |\text{tr} (B^2 T)|^2 \leq \text{tr} \left( |B^*|^2 |T|^{2\alpha} \right) \text{tr} \left( |B|^2 |T^*|^{2(1-\alpha)} \right)
\]
for any \( T \in \mathcal{B}(H) \) and \( B \in \mathcal{B}_2(H) \).

Also, if \( T = N \), a normal operator, then (3.31) and (3.32) become
\[
(3.33) \quad |\text{tr} \left( |B|^2 N \right)|^2 \leq \text{tr} \left( |B|^2 |N|^{2\alpha} \right) \text{tr} \left( |B|^2 |N|^{2(1-\alpha)} \right)
\]
and
\[
(3.34) \quad |\text{tr} (B^2 N)|^2 \leq \text{tr} \left( |B^*|^2 |N|^{2\alpha} \right) \text{tr} \left( |B|^2 |N|^{2(1-\alpha)} \right),
\]
for any \( B \in \mathcal{B}_2(H) \).

4. Some Functional Properties

Let \( A \in \mathcal{B}_2(H) \) and \( P \in \mathcal{B}(H) \) with \( P \geq 0 \). Then \( Q := A^*PA \in \mathcal{B}_1(H) \) with \( Q \geq 0 \) and writing the inequality (3.31) for \( B = (A^*PA)^{1/2} \in \mathcal{B}_2(H) \) we get
\[
|\text{tr} (A^*PAT)|^2 \leq \text{tr} \left( A^*PA |T|^{2\alpha} \right) \text{tr} \left( A^*PA |T^*|^{2(1-\alpha)} \right),
\]
which, by the properties of trace, is equivalent to
\[
(4.1) \quad |\text{tr} (PA^*AT)|^2 \leq \text{tr} \left( PA |T|^{2\alpha} A^* \right) \text{tr} \left( PA |T^*|^{2(1-\alpha)} A^* \right),
\]
where \( T \in \mathcal{B}(H) \) and \( \alpha \in [0, 1] \).

For a given \( A \in \mathcal{B}_2(H) \), \( T \in \mathcal{B}(H) \) and \( \alpha \in [0, 1] \), we consider the functional \( \sigma_{A,T,\alpha} \) defined on the cone \( \mathcal{B}_+(H) \) of nonnegative operators on \( \mathcal{B}(H) \) by
\[
\sigma_{A,T,\alpha}(P) := \left[ \text{tr} \left( PA |T|^{2\alpha} A^* \right) \right]^{1/2} \left[ \text{tr} \left( PA |T^*|^{2(1-\alpha)} A^* \right) \right]^{1/2} - |\text{tr} (PA^*AT)|.
\]

The following theorem collects some fundamental properties of this functional.

**Theorem 6.** Let \( A \in \mathcal{B}_2(H) \), \( T \in \mathcal{B}(H) \) and \( \alpha \in [0, 1] \).

(i) For any \( P, Q \in \mathcal{B}_+(H) \) we have
\[
(4.2) \quad \sigma_{A,T,\alpha}(P + Q) \geq \sigma_{A,T,\alpha}(P) + \sigma_{A,T,\alpha}(Q) \geq 0,
\]
namely, \( \sigma_{A,T,\alpha} \) is a superadditive functional on \( \mathcal{B}_+(H) \);

(ii) For any \( P, Q \in \mathcal{B}_+(H) \) with \( P \geq Q \) we have
\[
(4.3) \quad \sigma_{A,T,\alpha}(P) \geq \sigma_{A,T,\alpha}(Q) \geq 0,
\]
namely, \( \sigma_{A,T,\alpha} \) is a monotonic nondecreasing functional on \( \mathcal{B}_+(H) \);

(iii) If \( P, Q \in \mathcal{B}_+(H) \) and there exist the constants \( M > m > 0 \) such that \( MQ \geq P \geq mQ \) then
\[
(4.4) \quad M \sigma_{A,T,\alpha}(Q) \geq \sigma_{A,T,\alpha}(P) \geq m \sigma_{A,T,\alpha}(Q) \geq 0.
\]

**Proof.** (i) Let \( P, Q \in \mathcal{B}_+(H) \). On utilizing the elementary inequality
\[
(a^2 + b^2)^{1/2} (c^2 + d^2)^{1/2} \geq ac + bd, \ a, b, c, d \geq 0
\]
and the triangle inequality for the modulus, we have

\[
\sigma_{A,T,\alpha} (P + Q)
= \left[ \text{tr} \left( (P + Q) A |T|^{2\alpha} A^* \right) \right]^{1/2} \left[ \text{tr} \left( (P + Q) A |T|^{2(1-\alpha)} A^* \right) \right]^{1/2}
- \left[ \text{tr} ((P + Q) ATA^*) \right]
= \left[ \text{tr} \left( PA |T|^{2\alpha} A^* + QA |T|^{2\alpha} A^* \right) \right]^{1/2}
\times \left[ \text{tr} \left( PA |T|^{2(1-\alpha)} A^* + QA |T|^{2(1-\alpha)} A^* \right) \right]^{1/2}
- \left[ \text{tr} (PATA^* + QATA^*) \right]
= \left[ \text{tr} \left( PA |T|^{2\alpha} A^* \right) \right]^{1/2}
\times \left[ \text{tr} \left( PA |T|^{2(1-\alpha)} A^* \right) \right]^{1/2}
- \left[ \text{tr} (PATA^*) \right]
\geq \left[ \text{tr} \left( PA |T|^{2\alpha} A^* \right) \right]^{1/2}
\times \left[ \text{tr} \left( PA |T|^{2(1-\alpha)} A^* \right) \right]^{1/2}
- \left[ \text{tr} (PATA^*) \right]
= \sigma_{A,T,\alpha} (P) + \sigma_{A,T,\alpha} (Q)
\]

and the inequality (4.2) is proved.

(ii) Let \( P, Q \in \mathcal{B}_+ (H) \) with \( P \geq Q \). Utilising the superadditivity property we have

\[
\sigma_{A,T,\alpha} (P)
= \sigma_{A,T,\alpha} ((P - Q) + Q) \geq \sigma_{A,T,\alpha} (P - Q) + \sigma_{A,T,\alpha} (Q)
\geq \sigma_{A,T,\alpha} (Q)
\]

and the inequality (4.3) is obtained.

(iii) From the monotonicity property we have

\[
\sigma_{A,T,\alpha} (P) \geq \sigma_{A,T,\alpha} (mQ) = m \sigma_{A,T,\alpha} (Q)
\]

and a similar inequality for \( M \), which prove the desired result (4.4).

\( \square \)

**Corollary 5.** Let \( A \in \mathcal{B}_2 (H) \), \( T \in \mathcal{B} (H) \) and \( \alpha \in [0, 1] \). If \( P \in \mathcal{B} (H) \) is such that there exist the constants \( M > m > 0 \) with \( M1_H \geq P \geq m1_H \), then we have

\[
(4.5) \quad M \left( \left[ \text{tr} \left( A |T|^{2\alpha} A^* \right) \right]^{1/2} \left[ \text{tr} \left( A |T|^{2(1-\alpha)} A^* \right) \right]^{1/2} - \left| \text{tr} (ATA^*) \right| \right)
\geq \left[ \text{tr} \left( PA |T|^{2\alpha} A^* \right) \right]^{1/2} \left[ \text{tr} \left( PA |T|^{2(1-\alpha)} A^* \right) \right]^{1/2} - \left| \text{tr} (PATA^*) \right|
\geq m \left( \left[ \text{tr} \left( A |T|^{2\alpha} A^* \right) \right]^{1/2} \left[ \text{tr} \left( A |T|^{2(1-\alpha)} A^* \right) \right]^{1/2} - \left| \text{tr} (ATA^*) \right| \right).
\]
For a given $A \in \mathcal{B}_2(H)$, $T \in \mathcal{B}(H)$ and $\alpha \in [0,1]$, if we take $P = |V|^2$ with $V \in \mathcal{B}(H)$, we have

$$
\sigma_{A,T,\alpha}(|V|^2) = \left[ \text{tr} \left( |V|^2 A |T|^{2\alpha} A^* \right) \right]^{1/2} \left[ \text{tr} \left( |V|^2 A |T^*|^{2(1-\alpha)} A^* \right) \right]^{1/2} \\
- \left[ \text{tr} \left( |V|^2 ATA^* \right) \right] \\
= \left[ \text{tr} \left( V^* VA |T|^{2\alpha} A^* \right) \right]^{1/2} \left[ \text{tr} \left( V^* VA |T^*|^{2(1-\alpha)} A^* \right) \right]^{1/2} \\
- \left[ \text{tr} \left( V^* VAT A^* \right) \right] \\
= \left[ \text{tr} \left( A^* V^* VA |T|^{2\alpha} \right) \right]^{1/2} \left[ \text{tr} \left( A^* V^* VA |T^*|^{2(1-\alpha)} \right) \right]^{1/2} \\
- \left[ \text{tr} \left( A^* V^* VAT \right) \right] \\
= \left[ \text{tr} \left( |VA|^2 |T|^{2\alpha} \right) \right]^{1/2} \left[ \text{tr} \left( |VA|^2 |T^*|^{2(1-\alpha)} \right) \right]^{1/2} - \left[ \text{tr} \left( |VA|^2 T \right) \right].
$$

Assume that $A \in \mathcal{B}_2(H)$, $T \in \mathcal{B}(H)$ and $\alpha \in [0,1]$. If we use the superadditivity property of the functional $\sigma_{A,T,\alpha}$ we have for any $V, U \in \mathcal{B}(H)$ that

$$
\text{(4.6)} \quad \left[ \text{tr} \left( (|VA|^2 + |UA|^2) |T|^{2\alpha} \right) \right]^{1/2} \left[ \text{tr} \left( (|VA|^2 + |UA|^2) |T^*|^{2(1-\alpha)} \right) \right]^{1/2} \\
- \left[ \text{tr} \left( (|VA|^2 + |UA|^2) T \right) \right] \\
\geq \left[ \text{tr} \left( |VA|^2 |T|^{2\alpha} \right) \right]^{1/2} \left[ \text{tr} \left( |VA|^2 |T^*|^{2(1-\alpha)} \right) \right]^{1/2} - \left[ \text{tr} \left( |VA|^2 T \right) \right] \\
+ \left[ \text{tr} \left( |UA|^2 |T|^{2\alpha} \right) \right]^{1/2} \left[ \text{tr} \left( |UA|^2 |T^*|^{2(1-\alpha)} \right) \right]^{1/2} - \left[ \text{tr} \left( |UA|^2 T \right) \right] \geq 0.
$$

Also, if $|V|^2 \geq |U|^2$ with $V, U \in \mathcal{B}(H)$, then

$$
\text{(4.7)} \quad \left[ \text{tr} \left( |VA|^2 |T|^{2\alpha} \right) \right]^{1/2} \left[ \text{tr} \left( |VA|^2 |T^*|^{2(1-\alpha)} \right) \right]^{1/2} - \left[ \text{tr} \left( |VA|^2 T \right) \right] \\
\geq \left[ \text{tr} \left( |UA|^2 |T|^{2\alpha} \right) \right]^{1/2} \left[ \text{tr} \left( |UA|^2 |T^*|^{2(1-\alpha)} \right) \right]^{1/2} - \left[ \text{tr} \left( |UA|^2 T \right) \right] \geq 0.
$$

If $U \in \mathcal{B}(H)$ is invertible then

$$
\frac{1}{\|U^{-1}\|} \|x\| \leq \|Ux\| \leq \|U\| \|x\| \text{ for any } x \in H,
$$

which implies that

$$
\frac{1}{\|U^{-1}\|^2} 1_H \leq |U|^2 \leq \|U\|^2 1_H.
$$
Utilising (4.5) we get
\[(4.8) \quad ||U||^2 \left( \left[ \text{tr} \left( |A|^2 |T|^{2\alpha} \right) \right]^{1/2} - \left| \text{tr} \left( |A|^2 |T|^{2(1-\alpha)} \right) \right| \right) \]
\[\geq \left[ \text{tr} \left( |U|A^2 |T|^{2\alpha} \right) \right]^{1/2} - \left[ \text{tr} \left( |U|A^2 |T|^{2(1-\alpha)} \right) \right]^{1/2}
\[\geq \frac{1}{||U^{-1}||^2} \left( \left[ \text{tr} \left( |A|^2 |T|^{2\alpha} \right) \right]^{1/2} - \left| \text{tr} \left( |A|^2 |T|^{2(1-\alpha)} \right) \right| \right).\]

5. Inequalities for Sequences of Operators

For \( n \geq 2 \), define the Cartesian products \( B^{(n)}(H) := B(H) \times ... \times B(H) \), \( B_2^{(n)}(H) := B_2(H) \times ... \times B_2(H) \) and \( B_+^{(n)}(H) := B_+(H) \times ... \times B_+(H) \) where \( B_+(H) \) denotes the convex cone of nonnegative selfadjoint operators on \( H \), i.e. \( P \in B_+(H) \) if \( \langle Px, x \rangle \geq 0 \) for any \( x \in H \).

**Proposition 2.** Let \( P = (P_1, ..., P_n) \in B_+^{(n)}(H) \), \( T = (T_1, ..., T_n) \in B^{(n)}(H) \), \( A = (A_1, ..., A_n) \in B_2^{(n)}(H) \) and \( z = (z_1, ..., z_n) \in \mathbb{C}^n \) with \( n \geq 2 \). Then
\[(5.1) \quad \left| \text{tr} \left( \sum_{k=1}^{n} z_k P_k A_k T_k A_k^* \right) \right|^2 \]
\[\leq \text{tr} \left( \sum_{k=1}^{n} |z_k|^2 P_k A_k |T_k|^{2\alpha} A_k^* \right) \text{tr} \left( \sum_{k=1}^{n} |z_k|^2 P_k A_k |T_k|^{2(1-\alpha)} A_k^* \right)
\]
for any \( \alpha \in [0, 1] \).

**Proof.** Using the properties of modulus and the inequality (4.1) we have
\[\left| \text{tr} \left( \sum_{k=1}^{n} z_k P_k A_k T_k A_k^* \right) \right| \]
\[= \sum_{k=1}^{n} z_k \text{tr} \left( P_k A_k T_k A_k^* \right) \leq \sum_{k=1}^{n} |z_k| \left| \text{tr} \left( P_k A_k T_k A_k^* \right) \right|
\[\leq \sum_{k=1}^{n} |z_k| \left[ \text{tr} \left( P_k A_k |T_k|^{2\alpha} A_k^* \right) \right]^{1/2} \left[ \text{tr} \left( P_k A_k |T_k|^{2(1-\alpha)} A_k^* \right) \right]^{1/2}.
\]
Utilizing the weighted discrete Cauchy-Bunyakovsky-Schwarz inequality we also have
\[\sum_{k=1}^{n} |z_k| \left[ \text{tr} \left( P_k A_k |T_k|^{2\alpha} A_k^* \right) \right]^{1/2} \left[ \text{tr} \left( P_k A_k |T_k|^{2(1-\alpha)} A_k^* \right) \right]^{1/2}
\[\leq \left( \sum_{k=1}^{n} |z_k|^2 \left[ \text{tr} \left( P_k A_k |T_k|^{2\alpha} A_k^* \right) \right]^{1/2} \right)^{1/2}
\[\times \left( \sum_{k=1}^{n} |z_k|^2 \left[ \text{tr} \left( P_k A_k |T_k|^{2(1-\alpha)} A_k^* \right) \right]^{1/2} \right)^{1/2}
\[= \left( \sum_{k=1}^{n} |z_k| \text{tr} \left( P_k A_k |T_k|^{2\alpha} A_k^* \right) \right) \left( \sum_{k=1}^{n} |z_k| \text{tr} \left( P_k A_k |T_k|^{2(1-\alpha)} A_k^* \right) \right)^{1/2},\]
where

\[ \text{Proposition 3.} \]

\[ \text{provided that } T = (T_1, \ldots, T_n) \in \mathcal{B}^n(H), \ A = (A_1, \ldots, A_n) \in \mathcal{B}_2^n(H), \ \alpha \in [0, 1] \]

and \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \).

We consider the functional for \( n \)-tuples of nonnegative operators \( P = (P_1, \ldots, P_n) \in \mathcal{B}_+^n(H) \) as follows:

\[ \text{Remark 2.} \]

\[ \text{If we take } P_k = 1_H \text{ for any } k \in \{1, \ldots, n\} \text{ in (5.1), then we have the simpler inequality} \]

\[ \frac{\sum_{k=1}^{n} z_k |A_k|^2 T_k}{\left( \sum_{k=1}^{n} |z_k|^2 |T_k|^{2\alpha} \right)^{1/2}} \]

\[ \leq \frac{\sum_{k=1}^{n} |z_k|^2 |T_k|^{2\alpha}}{\left( \sum_{k=1}^{n} |z_k|^2 |T_k|^{2\alpha} \right)^{1/2}} \]

\[ \text{the desired result (5.1).} \square \]

\[ \text{We have} \]

\[ \text{which imply the desired result (5.1).} \]

\[ \text{Proposition 3.} \]

\[ \text{provided that } T = (T_1, \ldots, T_n) \in \mathcal{B}^n(H), \ A = (A_1, \ldots, A_n) \in \mathcal{B}_2^n(H), \ \alpha \in [0, 1] \]

\[ \text{and } z = (z_1, \ldots, z_n) \in \mathbb{C}^n. \]

\[ \text{We consider the functional for } n \text{-tuples of nonnegative operators } P = (P_1, \ldots, P_n) \in \mathcal{B}_+^n(H) \text{ as follows:} \]

\[ \sigma_{A,T,\alpha}(P) := \left[ \frac{\sum_{k=1}^{n} P_k A_k |T_k|^{2\alpha} A_k^*}{\left( \sum_{k=1}^{n} P_k A_k |T_k|^{2\alpha} A_k^* \right)^{1/2}} \right] \]

\[ \text{where } T = (T_1, \ldots, T_n) \in \mathcal{B}^n(H), \ A = (A_1, \ldots, A_n) \in \mathcal{B}_2^n(H) \text{ and } \alpha \in [0, 1]. \]

\[ \text{Utilising a similar argument to the one in Theorem 6 we can state:} \]

\[ \text{Proposition 3.} \]

\[ \text{Let } T = (T_1, \ldots, T_n) \in \mathcal{B}^n(H), \ A = (A_1, \ldots, A_n) \in \mathcal{B}_2^n(H) \text{ and } \alpha \in [0, 1]. \]

\[ \text{(i) For any } P, Q \in \mathcal{B}_+^n(H) \text{ we have} \]

\[ \sigma_{A,T,\alpha}(P + Q) \geq \sigma_{A,T,\alpha}(P) + \sigma_{A,T,\alpha}(Q) \geq 0, \]

\[ \text{naming, } \sigma_{A,T,\alpha} \text{ is a superadditive functional on } \mathcal{B}_+^n(H); \]

\[ \text{(ii) For any } P, Q \in \mathcal{B}_+^n(H) \text{ with } P \geq Q \text{, namely } P_k \geq Q_k \text{ for all } k \in \{1, \ldots, n\} \text{ we have} \]

\[ \sigma_{A,T,\alpha}(P) \geq \sigma_{A,T,\alpha}(Q) \geq 0, \]

\[ \text{naming, } \sigma_{A,B} \text{ is a monotonic nondecreasing functional on } \mathcal{B}_+^n(H); \]

\[ \text{(iii) If } P, Q \in \mathcal{B}_+^n(H) \text{ and there exist the constants } M > m > 0 \text{ such that } M Q \geq P \geq m Q \text{ then} \]

\[ M \sigma_{A,T,\alpha}(Q) \geq \sigma_{A,T,\alpha}(P) \geq m \sigma_{A,T,\alpha}(Q) \geq 0. \]

\[ \text{If } P = (p_1 1_H, \ldots, p_n 1_H) \text{ with } p_k \geq 0, k \in \{1, \ldots, n\} \text{ then the functional of real nonnegative weights } p = (p_1, \ldots, p_n) \text{ defined by} \]

\[ \sigma_{A,T,\alpha}(p) := \left[ \frac{\sum_{k=1}^{n} p_k |A_k|^2 |T_k|^{2\alpha}}{\left( \sum_{k=1}^{n} p_k |A_k|^2 |T_k|^{2\alpha} \right)^{1/2}} \right] \]

\[ \times \left[ \frac{\sum_{k=1}^{n} p_k |A_k|^2 |T_k|^{2(1-\alpha)}}{\left( \sum_{k=1}^{n} p_k |A_k|^2 |T_k|^{2(1-\alpha)} \right)^{1/2}} \right] \]

\[ \text{has the same properties as in Theorem 6.} \]
Moreover, we have the simple bounds

\[
\max_{k \in \{1, \ldots, n\}} \{p_k\} \left( \frac{\max_{\alpha} \sum_{k=1}^{n} |A_k|^2 |T_k|^{2\alpha}}{\left( \sum_{k=1}^{n} |A_k|^2 |T_k|^{2(1-\alpha)} \right)^{1/2}} \right)^{1/2} \\
\times \left[ \frac{\sum_{k=1}^{n} p_k |A_k|^2 |T_k|^{2(1-\alpha)}}{\left( \sum_{k=1}^{n} p_k |A_k|^2 |T_k|^{2(1-\alpha)} \right)^{1/2}} \right]^{1/2} \\
\geq \left[ \frac{\sum_{k=1}^{n} p_k |A_k|^2 |T_k|^{2\alpha}}{\left( \sum_{k=1}^{n} p_k |A_k|^2 |T_k|^{2\alpha} \right)^{1/2}} \right]^{1/2} \\
- \left[ \frac{\sum_{k=1}^{n} p_k |A_k|^2 |T_k|^{2\alpha}}{\left( \sum_{k=1}^{n} p_k |A_k|^2 |T_k|^{2\alpha} \right)^{1/2}} \right]^{1/2} \\
\geq \min_{k \in \{1, \ldots, n\}} \{p_k\} \left( \frac{\max_{\alpha} \sum_{k=1}^{n} |A_k|^2 |T_k|^{2\alpha}}{\left( \sum_{k=1}^{n} |A_k|^2 |T_k|^{2(1-\alpha)} \right)^{1/2}} \right)^{1/2} \\
\times \left[ \frac{\sum_{k=1}^{n} p_k |A_k|^2 |T_k|^{2(1-\alpha)}}{\left( \sum_{k=1}^{n} p_k |A_k|^2 |T_k|^{2(1-\alpha)} \right)^{1/2}} \right]^{1/2} \\
- \left[ \frac{\sum_{k=1}^{n} p_k |A_k|^2 |T_k|^{2(1-\alpha)}}{\left( \sum_{k=1}^{n} p_k |A_k|^2 |T_k|^{2(1-\alpha)} \right)^{1/2}} \right]^{1/2}.
\]

6. **Inequalities for Power Series of Operators**

Denote by:

\[
D(0, R) = \begin{cases} 
\{z \in \mathbb{C} : |z| < R\}, & \text{if } R < \infty \\
\mathbb{C}, & \text{if } R = \infty,
\end{cases}
\]

and consider the functions:

\[
\lambda \mapsto f(\lambda) : D(0, R) \to \mathbb{C}, \quad f(\lambda) := \sum_{n=0}^{\infty} \alpha_n \lambda^n
\]

and

\[
\lambda \mapsto f_\alpha(\lambda) : D(0, R) \to \mathbb{C}, \quad f_\alpha(\lambda) := \sum_{n=0}^{\infty} |\alpha_n| \lambda^n.
\]

As some natural examples that are useful for applications, we can point out that, if

\[
\begin{align*}
\lambda &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \lambda^n = \ln \frac{1}{1+\lambda}, \quad \lambda \in D(0, 1); \\
g(\lambda) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \lambda^{2n} = \cos \lambda, \quad \lambda \in \mathbb{C}; \\
h(\lambda) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \lambda^{2n+1} = \sin \lambda, \quad \lambda \in \mathbb{C}; \\
l(\lambda) &= \sum_{n=0}^{\infty} (-1)^n \lambda^n = \frac{1}{1+\lambda}, \quad \lambda \in D(0, 1);
\end{align*}
\]
then the corresponding functions constructed by the use of the absolute values of the coefficients are

\[ f_a(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n} \lambda^n = \ln \frac{1}{1-\lambda}, \quad \lambda \in D(0, 1); \]

\[ g_a(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n} = \cosh \lambda, \quad \lambda \in \mathbb{C}; \]

\[ h_a(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1} = \sinh \lambda, \quad \lambda \in \mathbb{C}; \]

\[ l_a(\lambda) = \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}, \quad \lambda \in D(0, 1). \]

Other important examples of functions as power series representations with non-negative coefficients are:

\[ \exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n \quad \lambda \in \mathbb{C}, \]

\[ \frac{1}{2} \ln \left( \frac{1 + \lambda}{1 - \lambda} \right) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \quad \lambda \in D(0, 1); \]

\[ \sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}{\sqrt{\pi} (2n+1) n!} \lambda^{2n+1}, \quad \lambda \in D(0, 1); \]

\[ \tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \quad \lambda \in D(0, 1); \]

\[ _2F_1(\alpha, \beta, \gamma, \lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha) \Gamma(n+\beta) \Gamma(\gamma)}{n! \Gamma(\alpha) \Gamma(\beta) \Gamma(n+\gamma)} \lambda^n, \alpha, \beta, \gamma > 0, \]

\[ \lambda \in D(0, 1); \]

where \( \Gamma \) is Gamma function.

**Theorem 7.** Let \( f(\lambda) := \sum_{n=1}^{\infty} \alpha_n \lambda^n \) be a power series with complex coefficients and convergent on the open disk \( D(0, R), \ R > 0 \). Let \( N \in \mathcal{B}(H) \) be a normal operator. If for some \( \alpha \in (0, 1) \) we have \( |N|^{2\alpha}, |N|^{2(1-\alpha)} \in \mathcal{B}_1(H) \) with \( \text{tr} \left( |N|^{2(1-\alpha)} \right) \), \( \text{tr} \left( |N|^{2(1-\alpha)} \right) < R \), then we have the inequality

\[ \text{tr} \left( f(\lambda) \lambda \right) \leq \text{tr} \left( f_a(\lambda) \lambda \right) \text{tr} \left( f_a(\lambda) \lambda \right). \]

**Proof.** Since \( N \) is a normal operator, then for any natural number \( k \geq 1 \) we have \( |N|^{2\alpha} = |N|^{2ak} \) and \( |N|^{2(1-\alpha)} = |N|^{2(1-\alpha)} \).

By the generalized triangle inequality for the modulus we have for \( n \geq 2 \)

\[ \text{tr} \left( \sum_{k=1}^{n} \alpha_k N^k \right) = \left| \sum_{k=1}^{n} \alpha_k \text{tr} (N^k) \right| \leq \sum_{k=1}^{n} |\alpha_k| \left| \text{tr} (N^k) \right|. \]

If for some \( \alpha \in (0, 1) \) we have \( |N|^{2\alpha}, |N|^{2(1-\alpha)} \in \mathcal{B}_1(H) \), then by Corollary 2 we have \( N \in \mathcal{B}_1(H) \). Now, since \( N, |N|^{2\alpha}, |N|^{2(1-\alpha)} \in \mathcal{B}_1(H) \) then any natural
power of these operators belong to \( \mathcal{B}_1(H) \) and by (3.13) we have
\[
|\text{tr} (N^k)|^2 \leq \text{tr} \left( |N|^{2\alpha k} \right) \text{tr} \left( |N|^{2(1-\alpha)k} \right),
\]
for any natural number \( k \geq 1 \).

Making use of (6.6) we have
\[
\sum_{k=1}^{n} |\alpha_k| |\text{tr} (N^k)| \leq \sum_{k=1}^{n} |\alpha_k| \left( \text{tr} \left( |N|^{2\alpha k} \right) \right)^{1/2} \left( \text{tr} \left( |N|^{2(1-\alpha)k} \right) \right)^{1/2}. \tag{6.7}
\]

Utilising the weighted Cauchy-Bunyakovsky-Schwarz inequality for sums we also have
\[
\sum_{k=1}^{n} |\alpha_k| \left( \text{tr} \left( |N|^{2\alpha k} \right) \right)^{1/2} \left( \text{tr} \left( |N|^{2(1-\alpha)k} \right) \right)^{1/2} \leq \left[ \sum_{k=1}^{n} |\alpha_k| \left( \text{tr} \left( |N|^{2\alpha k} \right) \right)^{1/2} \right]^2 \leq \left[ \sum_{k=1}^{n} |\alpha_k| \left( \text{tr} \left( |N|^{2(1-\alpha)k} \right) \right)^{1/2} \right]^2 = \sum_{k=1}^{n} |\alpha_k| \text{tr} \left( |N|^{2\alpha k} \right) \sum_{k=1}^{n} |\alpha_k| \text{tr} \left( |N|^{2(1-\alpha)k} \right)^{1/2}. \tag{6.8}
\]

Making use of (6.5), (6.7) and (6.8) we get the inequality
\[
\left| \text{tr} \left( \sum_{k=1}^{n} \alpha_k N^k \right) \right|^2 \leq \text{tr} \left( \sum_{k=1}^{n} |\alpha_k| |N|^{2\alpha k} \right) \text{tr} \left( \sum_{k=1}^{n} |\alpha_k| |N|^{2(1-\alpha)k} \right) \tag{6.9}
\]
for any \( n \geq 2 \).

Due to the fact that \( \text{tr} \left( |N|^{2\alpha} \right), \text{tr} \left( |N|^{2(1-\alpha)} \right) < R \) it follows by (3.13) that \( \text{tr}(|N|) < R \) and the operator series
\[
\sum_{k=1}^{\infty} \alpha_k N^k, \quad \sum_{k=1}^{\infty} |\alpha_k| |N|^{2\alpha k} \quad \text{and} \quad \sum_{k=1}^{\infty} |\alpha_k| |N|^{2(1-\alpha)k}
\]
are convergent in the Banach space \( \mathcal{B}_1(H) \).

Taking the limit over \( n \to \infty \) in (6.9) and using the continuity of the \( \text{tr} (\cdot) \) on \( \mathcal{B}_1(H) \) we deduce the desired result (6.4).

**Example 1.** a) If we take in \( f(\lambda) = (1 \pm \lambda)^{-1} - 1 = \mp \lambda \left( (1 \pm \lambda)^{-1} \right), |\lambda| < 1 \) then we get from (6.4) the inequality
\[
\left| \text{tr} \left( N \left( (1 \pm N)^{-1} \right) \right) \right|^2 \leq \text{tr} \left( |N|^{2\alpha \left( 1 - |N|^{2\alpha} \right)^{-1}} \right) \text{tr} \left( |N|^{2(1-\alpha) \left( 1 - |N|^{2(1-\alpha)} \right)^{-1}} \right), \tag{6.10}
\]
provided that \( N \in \mathcal{B}(H) \) is a normal operator and for \( \alpha \in (0, 1) \) we have \( |N|^{2\alpha}, |N|^{2(1-\alpha)} \in \mathcal{B}_1(H) \) with \( \text{tr} \left( |N|^{2\alpha} \right), \text{tr} \left( |N|^{2(1-\alpha)} \right) < 1 \).
b) If we take in (6.4) \( f(\lambda) = \exp(\lambda) - 1, \lambda \in \mathbb{C} \) then we get the inequality

\[
(6.11) \quad |\text{tr}(\exp(N) - 1_H)|^2 \leq \text{tr}\left(\exp\left(|N|^{2\alpha}\right) - 1_H\right) \text{tr}\left(\exp\left(|N|^{2(1-\alpha)}\right) - 1_H\right),
\]

provided that \( N \in \mathcal{B}(H) \) is a normal operator and for \( \alpha \in (0, 1) \) we have \( |N|^{2\alpha}, |N|^{2(1-\alpha)} \in \mathcal{B}_1(H) \).

The following result also holds:

**Theorem 8.** Let \( f(\lambda) := \sum_{n=0}^{\infty} \alpha_n \lambda^n \) be a power series with complex coefficients and convergent on the open disk \( D(0,R), R > 0 \). If \( T \in \mathcal{B}(H), A \in \mathcal{B}_2(H) \) are normal operators that double commute, i.e. \( TA = AT \) and \( TA^* = A*T \) and \( \text{tr}\left(|A|^2 |T|^{2\alpha}\right), \text{tr}\left(|A|^2 |T|^{2(1-\alpha)}\right) < R \) for some \( \alpha \in [0,1] \), then

\[
(6.12) \quad \left|\text{tr}\left(f\left(|A|^2 T\right)\right)\right|^2 \leq \text{tr}\left(f_a\left(|A|^2 |T|^{2\alpha}\right)\right) \text{tr}\left(f_a\left(|A|^2 |T|^{2(1-\alpha)}\right)\right).
\]

**Proof.** From the inequality (5.2) we have

\[
(6.13) \quad \left|\text{tr}\left(\sum_{k=0}^{n} \alpha_k |A|^k |T|^k\right)\right|^2 \leq \text{tr}\left(\sum_{k=0}^{n} |\alpha_k| |A|^k |T|^k\right)^2 \text{tr}\left(\sum_{k=0}^{n} |\alpha_k| |A|^k |T|^k\right)^2.
\]

Since \( A \) and \( T \) are normal operators, then \( |A|^k = |A|^{2k}, |T|^k = |T|^{2\alpha} \) and \( |T|^{2(1-\alpha)} = |T|^{2(1-\alpha)k} \) for any natural number \( k \geq 0 \) and \( \alpha \in [0,1] \).

Since \( T \) and \( A \) double commute, then it is easy to see that

\[
|A|^{2k} T^k = \left(|A|^2 T\right)^k, \quad |A|^{2k} |T|^{2\alpha k} = \left(|A|^2 |T|^{2\alpha}\right)^k
\]

and

\[
|A|^{2k} |T|^{2(1-\alpha)k} = \left(|A|^2 |T|^{2(1-\alpha)}\right)^k
\]

for any natural number \( k \geq 0 \) and \( \alpha \in [0,1] \).

Therefore (6.13) is equivalent to

\[
(6.14) \quad \left|\text{tr}\left(\sum_{k=0}^{n} \alpha_k \left(|A|^2 T\right)^k\right)\right|^2 \leq \text{tr}\left(\sum_{k=0}^{n} |\alpha_k| \left(|A|^2 |T|^{2\alpha}\right)^k\right) \text{tr}\left(\sum_{k=0}^{n} |\alpha_k| \left(|A|^2 |T|^{2(1-\alpha)}\right)^k\right),
\]

for any natural number \( n \geq 1 \) and \( \alpha \in [0,1] \).

Due to the fact that \( \text{tr}\left(|A|^2 |T|^{2\alpha}\right), \text{tr}\left(|A|^2 |T|^{2(1-\alpha)}\right) < R \) it follows by (5.2) for \( n = 1 \) that \( \text{tr}\left(|A|^2 T\right) < R \) and the operator series

\[
\sum_{k=1}^{\infty} \alpha_k N^k, \sum_{k=1}^{\infty} |\alpha_k| |N|^{2\alpha k} \text{ and } \sum_{k=1}^{\infty} |\alpha_k| |N|^{2(1-\alpha)k}
\]

are convergent in the Banach space \( \mathcal{B}_1(H) \).

Taking the limit over \( n \to \infty \) in (6.14) and using the continuity of the \( \text{tr}(\cdot) \) on \( \mathcal{B}_1(H) \) we deduce the desired result (6.12). \( \square \)
Example 2. a) If we take \( f(\lambda) = (1 \pm \lambda)^{-1}, |\lambda| < 1 \) then we get from (6.12) the inequality

\[
\begin{align*}
(6.15) & \quad | \text{tr} \left( \left( 1 + |A|^2 |T|^{-1} \right) \right) |^2 \\
& \leq \text{tr} \left( \left( 1 - |A|^2 |T|^{2\alpha} \right) \right) \text{tr} \left( \left( 1 - |A|^2 |T|^{2(1-\alpha)} \right) \right),
\end{align*}
\]

provided that \( T \in \mathcal{B}(H), A \in \mathcal{B}_2(H) \) are normal operators that double commute and

\[
\text{tr} \left( |A|^2 |T|^{2\alpha} \right), \text{tr} \left( |A|^2 |T|^{2(1-\alpha)} \right) < 1 \text{ for } \alpha \in [0, 1].
\]

b) If we take in (6.12) \( f(\lambda) = \exp(\lambda), \lambda \in \mathbb{C} \) then we get the inequality

\[
(6.16) \quad | \text{tr} \left( \exp \left( |A|^2 |T|^{2\alpha} \right) \right) |^2 \leq \text{tr} \left( \exp \left( |A|^2 |T|^{2(1-\alpha)} \right) \right) \text{tr} \left( \exp \left( |A|^2 |T|^{2(1-\alpha)} \right) \right),
\]

provided that \( T \in \mathcal{B}(H) \) and \( A \in \mathcal{B}_2(H) \) are normal operators that double commute and \( \alpha \in [0, 1] \).

Theorem 9. Let \( f(z) := \sum_{j=0}^{\infty} p_j z^j \) and \( g(z) := \sum_{j=0}^{\infty} q_j z^j \) be two power series with nonnegative coefficients and convergent on the open disk \( D(0, R), R > 0. \) If \( T \in \mathcal{B}(H), A \in \mathcal{B}_2(H) \) are normal operators that double commute and \( \text{tr} \left( |A|^2 |T|^{2\alpha} \right), \text{tr} \left( |A|^2 |T|^{2(1-\alpha)} \right) < R \) for \( \alpha \in [0, 1], \) then

\[
(6.17) \quad \left[ \text{tr} \left( f \left( |A|^2 |T|^{2\alpha} \right) + g \left( |A|^2 |T|^{2\alpha} \right) \right) \right]^{1/2} \\
\times \left[ \text{tr} \left( f \left( |A|^2 |T|^{2(1-\alpha)} \right) + g \left( |A|^2 |T|^{2(1-\alpha)} \right) \right) \right]^{1/2} \\
- \left[ \text{tr} \left( f \left( |A|^2 |T|^{2\alpha} \right) + g \left( |A|^2 |T|^{2\alpha} \right) \right) \right] \\
\geq \left[ \text{tr} \left( f \left( |A|^2 |T|^{2\alpha} \right) \right) \right]^{1/2} \left[ \text{tr} \left( f \left( |A|^2 |T|^{2(1-\alpha)} \right) \right) \right]^{1/2} \\
- \left[ \text{tr} \left( f \left( |A|^2 |T|^{2\alpha} \right) \right) \right] \\
+ \left[ \text{tr} \left( g \left( |A|^2 |T|^{2\alpha} \right) \right) \right]^{1/2} \left[ \text{tr} \left( g \left( |A|^2 |T|^{2(1-\alpha)} \right) \right) \right]^{1/2} \\
- \left[ \text{tr} \left( g \left( |A|^2 |T|^{2\alpha} \right) \right) \right] \quad (\geq 0).
\]

Moreover, if \( p_j \geq q_j \) for any \( j \in \mathbb{N} \), then, with the above assumptions on \( T \) and \( A \), we have

\[
(6.18) \quad \left[ \text{tr} \left( f \left( |A|^2 |T|^{2\alpha} \right) \right) \right]^{1/2} \left[ \text{tr} \left( f \left( |A|^2 |T|^{2(1-\alpha)} \right) \right) \right]^{1/2} \\
- \left[ \text{tr} \left( f \left( |A|^2 |T|^{2\alpha} \right) \right) \right] \\
\geq \left[ \text{tr} \left( g \left( |A|^2 |T|^{2\alpha} \right) \right) \right]^{1/2} \left[ \text{tr} \left( g \left( |A|^2 |T|^{2(1-\alpha)} \right) \right) \right]^{1/2} \\
- \left[ \text{tr} \left( g \left( |A|^2 |T|^{2\alpha} \right) \right) \right] \quad (\geq 0).
\]

The proof follows in a similar way to the proof of Theorem 8 by making use of the superadditivity and monotonicity properties of the functional \( \sigma_{A,T,\alpha}(\cdot) \). We omit the details.
Example 3. Now, observe that if we take

\[ f(\lambda) = \sinh \lambda = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1} \]

and

\[ g(\lambda) = \cosh \lambda = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n} \]

then

\[ f(\lambda) + g(\lambda) = \exp \lambda = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n \]

for any \( \lambda \in \mathbb{C} \).

If \( T \in \mathcal{B}(H) \), \( A \in \mathcal{B}_2(H) \) are normal operators that double commute and \( \alpha \in [0, 1] \), then by (6.17) we have

\[
(6.19) \quad \left[ \text{tr} \left( \exp \left( |A|^2 |T|^{2\alpha} \right) \right) \right]^{1/2} \left[ \text{tr} \left( \exp \left( |A|^2 |T|^{2(1-\alpha)} \right) \right) \right]^{1/2} \\
- \left[ \text{tr} \left( \exp \left( |A|^2 T \right) \right) \right] \\
\geq \left[ \text{tr} \left( \sinh \left( |A|^2 |T|^{2\alpha} \right) \right) \right]^{1/2} \left[ \text{tr} \left( \sinh \left( |A|^2 |T|^{2(1-\alpha)} \right) \right) \right]^{1/2} \\
- \left[ \text{tr} \left( \sinh \left( |A|^2 T \right) \right) \right] \\
+ \left[ \text{tr} \left( \cosh \left( |A|^2 |T|^{2\alpha} \right) \right) \right]^{1/2} \left[ \text{tr} \left( \cosh \left( |A|^2 |T|^{2(1-\alpha)} \right) \right) \right]^{1/2} \\
- \left[ \text{tr} \left( \cosh \left( |A|^2 T \right) \right) \right] (\geq 0). 
\]

Now, consider the series \( \frac{1}{1-\lambda} = \sum_{n=0}^{\infty} \lambda^n \), \( \lambda \in D(0, 1) \) and \( \ln \frac{1}{1-\lambda} = \sum_{n=1}^{\infty} \frac{1}{n} \lambda^n \), \( \lambda \in D(0, 1) \) and define \( p_n = 1 \), \( n \geq 0 \), \( q_0 = 0 \), \( q_n = \frac{1}{n} \), \( n \geq 1 \), then we observe that for any \( n \geq 0 \) we have \( p_n \geq q_n \).

If \( T \in \mathcal{B}(H) \), \( A \in \mathcal{B}_2(H) \) are normal operators that double commute, \( \alpha \in [0, 1] \) and \( \text{tr} \left( |A|^2 |T|^{2\alpha} \right) \), \( \text{tr} \left( |A|^2 |T|^{2(1-\alpha)} \right) \) < 1, then by (6.18) we have

\[
(6.20) \quad \left[ \text{tr} \left( \left( 1 - |A|^2 |T|^{2\alpha} \right)^{-1} \right) \right]^{1/2} \left[ \text{tr} \left( \left( 1 - |A|^2 |T|^{2(1-\alpha)} \right)^{-1} \right) \right]^{1/2} \\
- \left[ \text{tr} \left( \left( 1 - |A|^2 T \right)^{-1} \right) \right] \\
\geq \left[ \text{tr} \left( \ln \left( 1 - |A|^2 |T|^{2\alpha} \right)^{-1} \right) \right]^{1/2} \left[ \text{tr} \left( \ln \left( 1 - |A|^2 |T|^{2(1-\alpha)} \right)^{-1} \right) \right]^{1/2} \\
- \left[ \text{tr} \left( \ln \left( 1 - |A|^2 T \right)^{-1} \right) \right] (\geq 0). 
\]

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