Abstract. Consider the product $GX$ of two rectangular complex random matrices coupled by a constant matrix $\Omega$, where $G$ can be thought to be a Gaussian matrix and $X$ is a bi-invariant polynomial ensemble. We prove that the squared singular values form a biorthogonal ensemble in Borodin’s sense, and further that for $X$ being Gaussian the correlation kernel can be expressed as a double contour integral. When all but finitely many eigenvalues of $\Omega^*$ are equal, the corresponding correlation kernel is shown to admit a phase transition phenomenon at the hard edge in four different regimes as the coupling matrix changes. Specifically, the four limiting kernels in turn are the Meijer G-kernel for products of two independent Gaussian matrices, a new critical and interpolating kernel, the perturbed Bessel kernel and the finite coupled product kernel associated with $GX$. In the special case that $X$ is also a Gaussian matrix and $\Omega$ is scalar, such a product has been recently investigated by Akemann and Strahov. We also propose a Jacobi-type product and prove the same transition.

1. Introduction and main results

1.1. Introduction. Given two complex matrices $X_1$ of size $L \times M$ and $X_2$ of size $M \times N$ with $L, M \geq N$, our interest in the present paper is the joint probability density function (PDF for short) which reads

$$P(X_1, X_2) = Z^{-1} \exp\left\{ -\alpha \text{Tr}(X_1 X_1^* + X_2^* X_2) + \text{Tr}(\Omega X_1 X_2 + (\Omega X_1 X_2)^*) \right\}$$

(1.1)

with respect to Lebesgue measure $dX_1 dX_2$ on $\mathbb{R}^{2(L+M)N}$. Here $\alpha > 0$ and $\Omega$ is a non-random $N \times L$ matrix as a coupling of $X_1$ and $X_2$ such that $\Omega^* \prec \alpha^2 I_N$, and the normalization

$$Z = (\pi/\alpha)^{(L+N)M} \det^{-M} \left( I_L - \frac{1}{\alpha^2} \Omega^* \Omega \right),$$

(1.2)

where $I_L$ denotes an identity matrix of size $L \times L$. More precisely, our aim is to study the exact functional form of the joint PDF and correlation kernel for squared singular values of the matrix product $Y_2 = X_1 X_2$, and also to investigate scaling limits at the hard edge. For other local statistical properties such as bulk and soft-edge limits, we leave them to a forthcoming paper.

When all involved matrices are real, the two-matrix model defined in (1.1) is very closely related to testing independence and canonical correlation analysis in Multivariate Statistical Theory; see for example the two excellent monographs [8, Date: September 5, 2017.

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Chapters 9, 12 & 13] and [48, Chapter 11], and [39] for recent developments. To be exact, putting $X_1$ and $X_2$ together we have a sample covariance matrix

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

distributed according to

$$\text{const} \cdot \exp \left\{ -\frac{1}{2} \text{Tr} \left( X_1^\dagger X_2 \right) \left( \Sigma_{11} \Sigma_{12} \Sigma_{21} \Sigma_{22} \right)^{-1} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \right\}.$$  \hfill (1.3)

Then the sample canonical correlations are defined to be the square roots of eigenvalues of the sample canonical correlation matrix

$$(X_2^\dagger X_2)^{-1}(X_1 X_2)^\dagger (X_1 X_2)^{-1} X_1 X_2,$$  \hfill (1.4)

see [8, Sect. 13.4] or [48, Sect. 11.3]. When $X_1$ and $X_2$ are independent and also both have identity covariance matrices, the canonical correlation matrix (1.4) is just the so-called Jacobi/MANOVA ensemble (see e.g. [11] or [26, Chapt. 3.6]) and has been extensively studied, see [38, 54] and references therein. For (1.4) associated with the general PDF (1.3) but restricted to a small rank of the population cross-covariance matrix $\Sigma_{12}$, we refer the reader to [11] and [39] for relevant investigations.

Turning to the complex counterpart of (1.3), the joint PDF (1.1) can be re-expressed as

$$\text{const} \cdot \exp \left\{ - \text{Tr} \left( X_1^\dagger X_2 \right) \Sigma^{-1} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \right\}$$

where

$$\Sigma = \begin{pmatrix} (\alpha I_L - \alpha^{-1} \Omega^* \Omega)^{-1} & \alpha^{-1} \Omega^* (\alpha I_N - \alpha^{-1} \Omega^* \Omega)^{-1} \\ \alpha^{-1} \Omega (\alpha I_L - \alpha^{-1} \Omega^* \Omega)^{-1} & (\alpha I_N - \alpha^{-1} \Omega^* \Omega)^{-1} \end{pmatrix}.$$  \hfill (1.5)

In this case we will come back to study the complex analog of the sample canonical correlations (1.4) and some possible relationships with the complex sample cross-covariance matrix $X_1 X_2$ in the future. Finally, we remark that the global spectral density of $X_1 X_2$ under the joint PDF (1.5) with $L = N$ has been investigated in [53] (this was pointed out to us by Gernot Akemann and Mario Kieburg).

However, our major motivation to consider (1.1) comes from the paper by Akemann and Strahov [3] where $L = N$ and $\Omega$ is set to be a scalar matrix. In this special case, it was applied to Quantum Chromodynamics (QCD) with a baryon chemical potential by Osborn [50] where the complex eigenvalues were determined and a limiting interpolation kernel between the Bessel kernel and the corresponding kernel of complex eigenvalues was derived. This important example inspired Akemann and Strahov to turn to study the singular values for products of two coupled random matrices. Below we just give a brief description of the Osborn-Akemann-Strahov model and refer the reader to [5] for more details. Let $A$ and $B$ be two independent $N \times M$ matrices with i.i.d. standard complex Gaussian entries, Osborn [50] investigated an analogue of the Dirac operator in the context of QCD with a baryon chemical potential and introduced a random matrix ensemble with a coupling parameter $\mu \in [0, 1]$

$$D = \begin{pmatrix} 0 & iA + \mu B \\ iA^* + \mu B^* & 0 \end{pmatrix}.$$  \hfill (1.6)

He further calculated complex eigenvalues of $D$ by reducing them to those of the product $(iA + \mu B)(iA^* + \mu B^*)$. Equivalently, when turning to use the notation in
Let

\[ X_1 = \frac{1}{\sqrt{2}} (A - i\sqrt{\mu}B), \quad X_2 = \frac{1}{\sqrt{2}} (A^* - i\sqrt{\mu}B^*), \]

then \( X_1 \) and \( X_2 \) have a joint PDF as defined in (1.1) but with \( L = N, \alpha = (1 + \mu)/(2\mu) \) and \( \Omega = (1 - \mu)/(2\mu)I_N \); see [5, Sect. 2]. So far, as for singular values of the product matrix \( Y_2 = X_1X_2 \), a very interesting observation from Akemann and Strahov is that as \( \mu \to 0 \) it is equivalent to the classical Laguerre Unitary Ensemble (also called complex sample covariance matrices) while as \( \mu \to 1 \) it corresponds to the product of two independent Gaussian random matrices; see [5, Sect. 3] for detailed discussion.

Another motivation why to study products of coupled random matrices is that they are natural generalizations of products of independent random matrices, as interpreted in [5]. Actually, the topic on products of independent random matrices has attracted tremendous interest in recent years, largely because of the finding of exact solvability for Gaussian matrices [3, 4] and the appearing of some new families of universal patterns [27, 28, 40, 42, 43]. These also afford more examples to support the Wigner-Dyson Universality Conjecture; see [45] for the local statistical properties in the bulk and at the soft edge. For a recent survey, see [2] and references therein. Interestingly, entirely different from the extensively studied products, the singular values for products of two coupled random matrices no longer form a polynomial ensemble (that is, at least one of the two determinants consisting of the joint PDF is the Vandermonde determinant, cf. [42]), but a biorthogonal ensemble with both two sets of “nontrivial” functions [13]; see [5, Sect.3] or Proposition 1.1 below. In this sense, the result derived by Akemann and Strahov affords a very nice example of biorthogonal ensembles, see Borodin and Pečè's paper [14] for another example of the generalized Wishart ensemble distributed proportionally as \( \exp\{-\operatorname{Tr}(S_1XX^* + S_2X^*X)\} \), where \( S_1, S_2 \) are non-random \( N \times N \) positive definite matrices while \( X \) is random with the same size.

Now let’s return to the initial object (1.1). More generally, we can turn to consider the product of two coupled random matrices with matrix entries distributed proportionally as

\[
\exp \left\{ -\alpha \operatorname{Tr}(GG^*) + \operatorname{Tr}(\Omega GX + (\Omega GX)^*) - \operatorname{Tr}(X^*X) \right\} dGdX,
\]

(1.5)

where \( dG = \prod_{j=1}^L \prod_{k=1}^M d\operatorname{Re} G_{j,k} d\operatorname{Im} G_{j,k}, \quad dX = \prod_{j=1}^M \prod_{k=1}^N d\operatorname{Re} X_{j,k} d\operatorname{Im} X_{j,k}, \) and \( V \) is a polynomial with positive leading coefficient. We will show that the squared singular values of \( GX \) have a bi-orthogonal structure; see Corollary 2.2 in Sect. 2 below. When \( L = N \) and \( \Omega \) is a scalar matrix, the joint PDF (1.5) is usually called a coupled chiral two-matrix model and was first introduced by Akemann, Damgaard, Osborn and Splittorff [1] as a chiral analogue of Eynard-Mehta coupled Hermitian matrix model [29]. In this case, Akemann et al. derived the joint PDF of squared singular values of \( G \) and \( X \) and also explicit formulas for all spectral correlation functions, which opens up the possibility of asymptotic analysis for local statistics; see e.g. [1] and [20]. However, when \( L > N \) (at this stage \( \Omega \) must be a rectangular matrix due to the existence of the trace operation in the exponent) or \( \Omega \) is not scalar, to the best of our knowledge, there are no explicit formulas available for the joint PDF of squared singular values of \( G \) and \( X \). But, once we focus on the product \( GX \), its singular values can be exactly expressed as determinantal point processes. At this time, since \( G \) is a Gaussian random matrix given that...
X is fixed, the joint PDF (1.5) can be treated as a coupled multiplication with a Ginibre matrix. Here it is worth emphasizing that the coupled case only preserves biorthogonal ensembles of squared singular values, but not polynomial ensembles; see Theorem 2.1 in Sect. 2 below. This is different from the multiplication with a Ginibre matrix which transforms one polynomial ensemble to another; see [42] or [17, 40, 41] for some nice transformation identities of polynomial ensembles.

The remainder of this article is organized as follows. In the following subsection we summarise the main results on the joint eigenvalue PDF, correlation kernel and scaled kernel for the product of two coupled Gaussian matrices defined in (1.1). In particular, there exists a hard-edge transition phenomenon in four different regimes. Sect. 2 is devoted to the joint PDF of squared singular values for coupled products of a Ginibre (or Jacobi-type) matrix and a bi-unitarily invariant random matrix, which includes (1.1) and (1.5) as special cases. The proofs of Theorems 1.2 and 1.3 in Sect. 1.2 below are respectively given in Sect. 3 and Sect. 4, where the corresponding results are also obtained for a Jacobi-type product. In Sect. 5 further discussions on the four limiting kernels are presented.

1.2. Main results. Let $\nu = M - N \geq 0$, $\kappa = L - N \geq 0$, and let $\delta_1, \ldots, \delta_N$ be singular values of $\Omega$ such that $0 \leq \delta_j < \alpha$ for $j = 1, \ldots, N$. Also let $\Delta(x) = \prod_{1 \leq i < j \leq N} (x_j - x_i)$ denote the Vandermonde determinant. We will frequently use two kinds of modified Bessel functions defined by

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\nu + 1 + k)} \left(\frac{z}{2}\right)^{2k+\nu} \quad (1.6)$$

and

$$K_\nu(z) = \frac{1}{2} \left(\frac{z}{2}\right)^{\nu} \int_0^\infty t^{-\nu-1} e^{-t - \frac{z^2}{4t}} dt, \quad |\arg(z)| < \frac{\pi}{4}, \quad (1.7)$$

(cf. [30, 8.432.6]) and also the hypergeometric function $\,\,_{0}F_{1}$ defined by

$$\,\,_{0}F_{1}(\nu + 1; z) = \sum_{k=0}^{\infty} \frac{1}{(\nu + 1)_k} \frac{z^k}{k!}, \quad (1.8)$$

where the Pochhammer symbol $(a)_k = a(a + 1) \cdots (a + k - 1)$. Viewed from the integral representation (1.7), the argument of $K_\nu(z)$ is restricted to the interval $(-\pi/4, \pi/4)$, however, it can be analytically extended to the domain $\mathbb{C} \setminus (-\infty, 0]$, see e.g. [39] 10.25]. It is worth noting the two relations

$$\,\,_{0}F_{1}(\nu + 1; z) = \Gamma(\nu + 1)(\sqrt{z})^{-\nu} I_{\nu}(2\sqrt{z}) \quad (1.9)$$

and

$$K_{-\nu}(z) = K_\nu(z), \quad (1.10)$$

which respectively show that the RHS of (1.9) is an entire function of $z$ and $K_\nu(z)$ has even parity in its parameter. Here and below the principal square root of a nonzero complex number $z$ is denoted by $\sqrt{z}$ as in the positive real case.

Our first result is an exact formula for the joint PDF of squared singular values of $X_1X_2$ under (1.1), in which modified Bessel functions $I_\nu$ and $K_\nu$ are involved; see Theorem 2.1 of Sect. 2 for the more general results.
Proposition 1.1. With the joint PDF of two matrices $X_1$ and $X_2$ defined in (1.2), let $Y_2 = X_1 X_2$. Then the joint PDF for the squared singular values of $Y_2$ on $[0, \infty)^N$ is given by

$$P_N(x_1, \ldots, x_N) = \frac{1}{Z_N} \det \left[ I_\kappa(2\delta_i \sqrt{x_j}) \right]_{i,j=1}^N \det \left[ \frac{x_i^{\alpha+i}}{x_j^{\alpha}} K_{\nu-\kappa+i-1}(2\alpha \sqrt{x_j}) \right]_{i,j=1}^N,$$

where $0 \leq \delta_j < \alpha$ for $j = 1, \ldots, N$ and the normalization constant

$$Z_N = N! \frac{2^{-N} \alpha^{-N(\kappa+\nu)-\frac{\nu}{2}} N^{N+1}}{\Delta(\delta^2)} \prod_{j=1}^N \left( \Gamma(\nu) \delta_j^\nu \left( 1 - \frac{\delta_j^2}{\alpha^2} \right)^{-\nu-N} \right).$$

We stress that when some of the $\delta_j$’s coincide, L’Hôpital’s rule provides the appropriate eigenvalue density. When $\kappa = 0$ and all $\delta_j$’s are equal, the PDF in Proposition 1.1 has been derived by Akemann and Strahov; see [5, Theorem 3.1].

For the joint eigenvalue PDF (1.11) above as a determinantal point process, we find a double contour integral expression for the correlation kernel, which provides the starting point for further asymptotic analysis. Both our double integral formula and its derivation are different from those given by Akemann and Strahov, see [5] for exact formulae and brilliant derivations. Therein, the authors discussed in details biorthogonal functions, five-term recurrence relations, Christoffel-Darboux formula and relevant contour integral representations.

Theorem 1.2. The correlation kernel for the biorthogonal ensemble (1.11) is given by

$$K_N(x, y) = \frac{2^{\alpha^2}}{(2\pi i)^2} \int_{C_{\text{in}}} \int_{C_{\text{out}}} \frac{du}{u} \frac{dv}{v} K_{-\kappa}(2\alpha \sqrt{(1-u)x}) I_\kappa(2\alpha \sqrt{(1-v)y})$$

$$\times \frac{1}{u-v} \left( \frac{1-u}{1-v} \right)^{\kappa/2} \frac{u-\nu-N}{v-\nu-N} \prod_{l=1}^N \frac{u-(1-\delta_l^2/\alpha^2)}{v-(1-\delta_l^2/\alpha^2)},$$

where $C_{\text{in}}$ is a counterclockwise contour encircling $1-\delta_1^2/\alpha^2, \ldots, 1-\delta_N^2/\alpha^2$, and $C_{\text{out}}$ is a simple contour counterclockwise around the origin with $\text{Re}(z) < 1$ for $z \in C_{\text{out}}$ such that $C_{\text{in}}$ is entirely to the right side of $C_{\text{out}}$. When $0 < \delta_j < \alpha$ for $j = 1, \ldots, N$, we can also choose contours such that $C_{\text{in}}$ is contained entirely in $C_{\text{out}}$.

Note that it is unnecessary to assume $\text{Re}(z) < 1$ for $z \in C_{\text{in}}$ in (1.13), unlike $C_{\text{out}}$, since $z^{-\kappa/2}I_\kappa(2\sqrt{z})$ is an entire function of $z$ (cf. eqn (1.11)). Besides, we will select more specific contours as required in investigating the scaling limits of correlation kernels.

Next, we focus on asymptotic behavior of the correlation kernel under the assumptions of finite-rank perturbation of the matrix $\Omega$ and $\mu$-dependent coupling (see [5] for discussion in details), i.e., for a given nonnegative integer $m$ independent of $N$, $\delta_{m+1} = \cdots = \delta_N = \delta$ and $\alpha = (1+\mu)/(2\mu)$, $\delta = (1-\mu)/(2\mu)$, $0 < \mu \leq 1$, (1.14)
For nonnegative integers \( \nu, \kappa \) and \( m \), we introduce four types of double integrals for correlation kernels as follows. The first kernel is defined to be

\[
K_1(\xi, \eta) = \left( \frac{\eta}{\xi} \right)^{\nu/2} \frac{1}{2\pi i} \int_0^\infty dt \int_{C_0} ds s^{-\nu-1} e^{s-\tau} \times 4 \left( \frac{\xi s}{\eta t} \right)^{\nu/2} K_\nu^{(\text{Bes})} \left( \frac{4\eta}{s}, \frac{4\xi}{t} \right),
\]

where \( C_0 \) is a counterclockwise contour around the origin and the Bessel kernel

\[
K_\nu^{(\text{Bes})}(x, y) = \frac{J_\nu(\sqrt{x})J'_\nu(\sqrt{y}) - J_\nu(\sqrt{y})J'_\nu(\sqrt{x})}{2(x-y)}
\]

with the Bessel function of the first kind \( J_\nu \); cf. \([23, 52]\). Note that this type of convolution representation in \((1.15)\) has been obtained in the product of two independent random matrices for finite matrix size \( N \), see \([17, \text{Theorem 2.8(b)}]\). Actually, in Sect. 5 below this will prove to be the Meijer G-kernel associated with the product of two independent Gaussian matrices which appeared previously in \([12, 43]\).

The second one is a new critical and interpolating kernel between the Meijer G-kernel and the perturbed Bessel kernel, which reads for \( \tau > 0 \) and \( \pi_1, \ldots, \pi_m \in (0, 1) \)

\[
K_{11}(\tau; \xi, \eta) = \frac{2}{(2\pi i)^2} \int_{C_{\text{out}}} du \int_{C_{\text{in}}} dv K_{-\nu}(2\sqrt{(1-u)\xi}) I_\nu(2\sqrt{(1-v)\eta})
\times e^{-\frac{\tau}{2} + \frac{\pi}{4}} \frac{1}{u-v} \left( \frac{1-u}{1-v} \right)^{\nu/2} \left( \frac{u}{v} \right)^{-\nu-m} \prod_{l=1}^m \frac{u-\pi_l}{v-\pi_l}.
\]

The last two kernels are the perturbed Bessel kernel which was first defined in \([21]\) for \( \pi_1, \ldots, \pi_m \in (0, \infty) \)

\[
K_{11}(\xi, \eta) = \frac{2}{(2\pi i)^2} \int_{C_{\text{out}}} du \int_{C_{\text{in}}} dv e^{\sqrt{\xi u - \sqrt{\eta v}}} \left( \frac{1-u}{1-v} \right)^{\nu/2} \left( \frac{u}{v} \right)^{-\nu-m} \prod_{l=1}^m \frac{u-\pi_l}{v-\pi_l},
\]

and the finite coupled product kernel with \( \pi_1, \ldots, \pi_m \in (0, 1) \) and \( m \geq 1 \)

\[
K_{1V}(\xi, \eta) = \frac{2}{(2\pi i)^2} \int_{C_{\text{out}}} du \int_{C_{\text{in}}} dv K_{-\nu}(2\sqrt{(1-u)\xi}) I_\nu(2\sqrt{(1-v)\eta})
\times e^{-\frac{\tau}{2} + \frac{\pi}{4}} \left( \frac{1-u}{1-v} \right)^{\nu/2} \left( \frac{u}{v} \right)^{-\nu-m} \prod_{l=1}^m \frac{u-\pi_l}{v-\pi_l}.
\]

In the definition of last three kernels, \( C_{\text{out}} \) is a simple counterclockwise contour around the origin (with \( \Re(z) < 1, \forall z \in C_{\text{out}} \) for \( K_{11} \) and \( K_{1V} \)) and entirely within it \( C_{\text{in}} \) is a counterclockwise contour encircling \( 0, \pi_1, \ldots, \pi_m \). Note that the last one is actually the correlation kernel \((1.13)\) associated with coupled products of two Gaussian matrices with properly chosen parameters; see Sect. 5 for detailed discussion on the four kernels. Also, it’s worth emphasizing that the kernels defined above may depend on parameters \( \tau > 0, \kappa, \) and \( \pi_1, \ldots, \pi_m \), however, we still use the shorthand notations for simplicity, unless specified.

We are now ready to state the main results which describe a transition of hard edge limits for correlation kernels in four different regimes, by tuning the scale of
1 − δ_j^2/α^2, ..., 1 − δ_m^2/α^2 as μN varies from zero to infinity at different scales. A similar hard edge phase transition occurs in three different regimes for the shifted mean chiral Gaussian ensemble [28]. Recently, some different types of hard-to-soft edge transition have been observed for Gaussian perturbations of hard edge random matrix ensembles by Claey and Doeraene [10]. Also, see [10] for the famous Baik-Ben Arous-Péché phase transition for largest eigenvalues.

Theorem 1.3 (Hard edge limits). Assume that the parameters δ_j satisfy the condition \((1.14)\) and \(0 \leq δ_j < α \text{ for } j = 1, ..., m\). With the correlation kernel \((1.13)\) and with fixed nonnegative integers ν and κ, the following hold uniformly for any ξ and η in a compact set of \((0, ∞)\) as \(N → ∞\).

(i) If \(μN → ∞\), then
\[
\frac{μl}{N}K_N(\frac{μl}{N}ξ, \frac{μl}{N}η) → K_I(ξ, η).
\] (1.20)

(ii) If \(μN → τ/4 \text{ with } τ > 0 \text{ and } 1 − δ_j^2/α^2 → π_l \in (0, 1) \text{ for } l = 1, ..., m\), then
\[
α^−2K_N(α−2ξ, α−2η) → K_{II}(τ; ξ, η).
\] (1.21)

(iii) If \(μN → 0 \text{ and } 1 − δ_j^2/α^2 = 4μNπ_l \text{ with } π_l \in (0, ∞) \text{ for } l = 1, ..., m\), then
\[
\frac{e^{π/√2}}{e^{π/√2} N} K_N\left(\frac{1}{4N^2}ξ, \frac{1}{4N^2}η\right) → K_{III}(ξ, η).
\] (1.22)

(iv) If \(μN → 0 \text{ and } 1 − δ_j^2/α^2 → π_l \in (0, 1) \text{ for } l = 1, ..., m\), then for \(m \geq 1\)
\[
4μ^2K_N(4μ^2ξ, 4μ^2η) → K_{IV}(ξ, η).
\] (1.23)

This theorem says that there are exactly four distinct limiting kernels as the coupling strength μ changes, along with properly chosen scalings of parameters δ_1, ..., δ_m. The same result also appears in a Jacobi-type product ensemble which predicts a universal pattern; see Theorem 4.1 in Sect. 4. Compared with all those known phase transition phenomena mentioned above in Random Matrix Theory (RMT), as far as we know, Theorem 1.3 is the first show of a four-term transition. Usually in RMT the pattern of universality for local eigenvalue statistics depends on some exponent \(c < 1\), with which the limiting density of eigenvalues diverges (hard edge) or vanishes (soft edge) like \(|x − x_0|^{-c}\) as \(x → x_0\) from either side. This leads to a change in fluctuations in powers of matrix size N and thus the scaling of the correlation kernel; see e.g. [10] 28. As to the kernels above, when \(m = 0\) it was argued in [6] Sect. 2] that the exponent \(c = 2/3, 3/4, 3/4\) at the origin corresponding to cases (i), (ii) and (iii) of Theorem 1.3 (if we change variables ξ, η to ξ^2, η^2 in cases (ii) and (iii), then \(c = 1/2\) in both cases, which is consistent with the description given in [6]). Thus, at least, the limit from \(K_{III}\) to \(K_I\) is a candidate for a phase transition. On the other hand, the scaling \(α^−2 ∼ τ^2/(2N)^{−2}\) in case (ii) is the same as in case (i) but different from cases (i) and (iv). This probably indicates a phase transition from \(K_I\) to \(K_{II}\) to \(K_{IV}\).

We remark that although case (iv) can be formally obtained by merely permitting \(τ = 0\) in case (ii), we separate it at least for two reasons: one is, we divide the limits of \(μN\) into three categories: ∞, (0, ∞) and 0, the third of which is again divided into two cases according to the choice of different scalings of parameters δ_1, ..., δ_m; the other is to emphasize that the finite coupled product kernel \(K_{IV}\) will appear as a limiting kernel in RMT like the finite GUE and LUE kernels (cf. [10] 28), and
that it is non-trivial only when the finite rank perturbation \( m \geq 1 \). A few other relevant remarks are as follows.

**Remark 1.4.** We noticed the preprint [6] when it appeared early during the drafting of this article. At that time Theorem 1.2 and Parts (i), (ii) and (iv) of Theorem 1.3 was completed while Part (iii) was later inspired by [6, Theorem 1.5 (a)]. We are grateful to Gernot Akemann for detailed discussions on the main results of [6].

**Remark 1.5.** When \( L = N \) (that is, \( \kappa = 0 \)) and \( \Omega \) is a scalar matrix (equivalently, \( m = 0 \) in (1.14)), we compare Theorem 1.5 with relevant results of Akemann and Strahov as follows. For fixed \( \mu \), Part (i) of Theorem 1.3 was previously obtained by Akemann and Strahov, see [5, Theorem 3.9]. In a subsequent paper [6], with \( \mu = gN^{-\chi} \), they further obtained the hard edge limits in cases \( 0 \leq \chi < 1 \), \( \chi = 1 \) and \( \chi > 1 \), which respectively corresponds to Parts (i), (ii) and (iii), and proved that the limiting kernels in Parts (i) and (iii) agree with the standard integral forms. Although their double integral of correlation kernel at the critical scale is different from ours, these are believed to be the same; see Sect. 3 below for further discussion on the four kernels in Theorem 1.3.

**Remark 1.6.** Note that for biorthogonal ensembles the gap probability that no eigenvalues belong to a given Borel set \( A \subset \mathbb{R} \) has a Fredholm determinant expression (see e.g. [7, Lemma 3.2.4])

\[
\mathbb{P}(x_1 \in A^c, \ldots, x_N \in A^c) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_A \cdots \int_A \det[K_N(t_i, t_j)]^k_{i,j=1} dt_1 \cdots dt_k,
\]

if we strengthen the results in Theorem 1.3 from uniform convergence into the trace norm convergence of the integral operators with respect to the correlation kernels, then as a direct consequence we have the limiting gap probabilities after rescaling, especially including the distribution of smallest squared singular values (cf. [26] Chapters 8 & 9). In the case of Part (iv), we have closed expression for scaling limit of the smallest squared singular values, see equation (1.25) in Sect. 4 below. Since the proof of trace norm convergence is only a technical elaboration that confirms a well-expected result, we do not give the detail.

Finally, we conclude this section with two conjectures. One is the product of two coupled real Gaussian matrices while the other refers to generalizations of matrix entries from Gaussian variables to the more general random variables.

**Conjecture 1.7.** For real counterpart of the joint PDF (1.1), Theorem 1.3 still holds for different limiting kernels with certain Pfaffian structure, but under the same scalings. In particular, the critical scale of \( \mu \) is again expected to be \( 1/N \).

To state the second conjecture, let \( \alpha > 0 \) and \( m \) be a fixed nonnegative integer, assume that \( \delta_1, \ldots, \delta_m \) and \( \delta \) are complex numbers with absolute value less than \( \alpha \). We consider two complex random matrices \( X_1 = [X_1(j, k)]_{1 \leq j \leq N, 1 \leq k \leq M} \) and \( X_2 = [X_2(k, j)]_{1 \leq k \leq M, 1 \leq j \leq N} \) such that the following conditions are satisfied:

(C1) The vector pairs \( \{X_1(j, k), X_2(k, j)\}_{1 \leq j \leq N, 1 \leq k \leq M} \) are independent, and moreover \( \{X_1(j, k), X_2(k, j)\}_{m+1 \leq j \leq N, 1 \leq k \leq M} \) are identically distributed and so are \( \{X_1(j, k), X_2(k, j)\}_{1 \leq k \leq M} \) for any given \( j \in \{1, \ldots, m\} \);

(C2) For any \( j, k \), \( E[X_1(j, k)] = E[X_2(k, j)] = 0 \), \( E[(X_1(j, k))^2] = E[(X_2(k, j))^2] = 0 \), \( E[X_1(j, k)X_2(k, j)] = 0 \);
(C3) When \( j \geq m + 1 \), \( E[|X_1(j, k)|^2] = E[|X_2(k, j)|^2] = \alpha/(\alpha^2 - |\delta|^2) \) and \( E[|X_1(j, k)X_2(k, j)|] = \delta/(\alpha^2 - |\delta|^2) \), while for any given \( j \in \{1, \ldots, m\} \) \( E[|X_1(j, k)|^2] = E[|X_2(k, j)|^2] = \alpha/(\alpha^2 - |\delta_j|^2) \) and \( E[X_1(j, k)X_2(k, j)] = \delta_j/(\alpha^2 - |\delta_j|^2) \);

(C4) For any \( j, k \), \( E[|X_1(j, k)|^4] < \infty \) and \( E[|X_2(k, j)|^4] < \infty \).

Note that the joint PDF (1.1) with \( L = N \) satisfies the above assumptions since \( \Omega \) can be taken to be diagonal according to the invariance of Gaussian random variables.

**Conjecture 1.8.** For \( \alpha = (1 + \mu)/(2\mu) \), \( \delta = (1 - \mu)/(2\mu) \), \( 0 < \mu \leq 1 \), under the above assumptions (C1)-(C4), Theorem 1.3 still holds true but with \( \kappa = 0 \).

## 2. Coupled multiplication with a random matrix

### 2.1. Coupled multiplication with a Ginibre matrix

For complex matrices \( G \) of size \( L \times M \), \( X \) of size \( M \times N \) and \( \Omega \) of size \( N \times L \) with \( L, M \geq N \), suppose that the joint probability distribution of \( G \) and \( X \) is equal to

\[
Z^{-1} \exp \left\{ -\alpha \text{Tr}(GG^*) + \text{Tr}(\Omega GX + (\Omega GX)^*) \right\} h(X)dGdX, \tag{2.1}
\]

where \( dG = \prod_{j=1}^{L} \prod_{k=1}^{M} d\text{Re}G_{j,k}d\text{Im}G_{j,k} \) and \( dX = \prod_{j=1}^{M} \prod_{k=1}^{N} d\text{Re}X_{j,k}d\text{Im}X_{j,k} \), and also suppose that \( h(X) \) is invariant under left and right multiplication with unitary matrices, i.e., \( h(U XV) = h(X) \) for any unitary matrices \( U \in U(M) \) and \( V \in U(N) \). We turn to the product \( Y = GX \) and study the squared singular values of \( Y \).

The main result of this section can be stated as follows.

**Theorem 2.1.** With the joint PDF defined in (2.1), let \( \delta_1, \ldots, \delta_N \) be singular values of \( \Omega \) such that \( 0 \leq \delta_j < \alpha \) for \( j = 1, \ldots, N \). Suppose that \( f_k(t) \) \( (k = 1, \ldots, N) \) are continuous in \((0, \infty)\) such that all \( e^{\alpha t}f_k(t) \) are bounded in \([0, \infty)\), let

\[
h(X) = \frac{1}{\Delta(t)} \det[f_k(t_j)]_{j,k=1}^{N}, \quad 0 < t_1, \ldots, t_N < \infty, \tag{2.2}
\]

where \( t_1, \ldots, t_N \) are eigenvalues of \( X^*X \), then the squared singular values of \( Y = GX \) have a joint PDF on \([0, \infty)^N\)

\[
\mathcal{P}_N(x_1, \ldots, x_N) = \frac{1}{Z_N} \det[\xi_i(x_j)]_{i,j=1}^{N} \det[\eta_i(x_j)]_{i,j=1}^{N}, \tag{2.3}
\]

where \( \xi_i(z) = \frac{1}{\alpha}F_1(L - N + 1; \delta_i^2z) \) and

\[
\eta_i(z) = \int_0^\infty t^{L-N}e^{-\alpha t} \left( \frac{z}{t} \right)^{M-N} f_i(z) \frac{dt}{t}, \tag{2.4}
\]

The normalization constant can be evaluated by

\[
Z_N = N!(L-N)!^\alpha N^{N(L-N+1)} \left\{ \int_0^\infty e^{\delta_j^2/\alpha} x^{M-N} f_k(x)dx \right\}_{j,k=1}^{N}. \tag{2.5}
\]

Theorem 2.1 shows that the coupled product of a complex Ginibre matrix and a bi-invariant polynomial ensemble produces a bi-orthogonal ensemble, with two sets of “nontrivial” functions. This affords us random matrix realizations for a class of determinantal point processes which are bi-orthogonal ensembles but not polynomial ensembles. The following proof is inspired by these of [5, Theorem 3.1] and [42, Theorem 2.1].
Proof. We proceed in three steps.

**Step 1: Reduction.** We claim that the problem can be reduced to the study of case \( M = N \). Let us assume that \( M > N \). Then any matrix \( X \) of size \( M \times N \) can be decomposed as

\[
X = U \begin{pmatrix} X_0 & \end{pmatrix},
\]

where \( U \) is an \( M \times M \) unitary matrix which can be uniquely taken to be in some specific form, \( X_0 \) is an \( N \times N \) complex matrix and \( O \) is a zero matrix of size \( (M - N) \times N \); cf. Lemma 2.1 and Appendix A in [24]. By the results of [24] Sect. 2, we obtain the joint distribution of \( G, X_0 \) and \( U \) proportional to

\[
\det(X_0^* X_0)^{M-N} \exp \left\{ -\alpha \text{Tr}(G G^*) + \text{Tr}(\Omega GU (X_0) + (\Omega G (X_0))^*) \right\} h(X_0) dGdX_0 [dU],
\]

where \([dU]\) denotes the induced measure from the Haar measure of \( M \times M \) unitary group; cf. [24 Eq. (6)].

Make a change of variables \( \tilde{G} = G \) and rewrite \( \tilde{G} = \begin{pmatrix} G_0 & G_1 \end{pmatrix} \) with two blocks \( G_0 \) of size \( L \times N \) and \( G_1 \) of size \( L \times (M - N) \), then \( GX = G_0 X_0 \) and the joint distribution of \( G_0, G_1, X_0 \) and \( U \) can be rewritten to be proportional to

\[
\exp \left\{ -\alpha \text{Tr}(G_0 G_0^* + G_1 G_1^*) + \text{Tr}(\Omega G_0 X_0 + (\Omega G_0 X_0)^*) \right\}
\times \det(X_0^* X_0)^{M-N} h(X_0) dG_0 dX_0 [dU] dG_1.
\]

Noting the invariance of \( h(X) \) given in (2.2) and integrating over \( G_1 \) and \( U \), we immediately see that the joint probability distribution of \( G_0 \) and \( X_0 \) reads

\[
\exp \left\{ -\alpha \text{Tr}(G_0 G_0^*) + \text{Tr}(\Omega G_0 X_0 + (\Omega G_0 X_0)^*) \right\} \det(X_0^* X_0)^{M-N} h(X_0) dG_0 dX_0
\]

up to some constant. Furthermore, both \( GX \) and \( G_0 X_0 \) have the same singular values.

**Step 2: Joint singular value PDF of \( X \) and \( Y \).** To get the squared singular values of the product \( GX \) it suffices to study the distribution defined in (2.3). For simplicity sake, we replace the notation \( G_0 \) and \( X_0 \) with \( G \) and \( X \) respectively.

Since the change of variables of \( G \mapsto Y = GX \) and \( X \mapsto X \) has a Jacobian \( \det(X^* X)^{-L} \) where \( X \) has the full rank \( N \); cf. [17 Theorem 3.2]), \( Y \) and \( X \) have a joint distribution proportional to

\[
\exp \left\{ -\alpha \text{Tr}(Y^* Y (X^* X)^{-1}) + \text{Tr}(\Omega Y + (\Omega Y)^*) \right\} \det(X^* X)^{M-N-L} h(X) dY dX.
\]

Next, let \( \Lambda_x = \text{diag}(x_1, \ldots, x_N) \) and \( \Lambda_t = \text{diag}(t_1, \ldots, t_N) \), according to the singular value decomposition, both \( Y \) and \( X \) can be written as

\[
Y = U \begin{pmatrix} \sqrt{\Lambda_x} \\ O \end{pmatrix} V, \quad X = W \sqrt{\Lambda_t} Q,
\]

where \( U \) is an \( L \times N \) complex matrix with \( U^* U = I_N \), all \( V, W \) and \( Q \) are \( N \times N \) unitary matrices. Then both the Jacobians read

\[
dY \propto \prod_{k=1}^N x_k^{L-N} \Delta(x)^2 dU dV dx_1 \cdots dx_N,
\]

and

\[
dX \propto \Delta(t)^2 dW dQ dt_1 \cdots dt_N,
\]
see e.g. [24] Chapt. 3. Together with (2.12) and (2.13), by the invariance of the Haar measure under the change $Q \mapsto QV$, we know that (2.10) is reduced to the distribution proportional to

$$\exp\left\{-a\text{Tr}(A_xQ^{-1}A_t^{-1}Q) + \text{Tr}(\Omega U \sqrt{A_x}V + (\Omega U \sqrt{A_x}V)^*)\right\} \prod_{k=1}^{N} (x_k^{L-N} t_k^{M-N-L})$$

$$\times \Delta(x)^2 \Delta(t) \det[f_k(t_j)]_{j,k=1}^{N} dU dV dW dQ dx_1 \cdots dx_N dt_1 \cdots dt_N. \quad (2.14)$$

We need to use the Harish-Chandra-Itzykson-Zuber integral formula (cf. [32] and [35])

$$\int_{U(N)} e^{-a\text{Tr}(A_xQ^{-1}A_t^{-1}Q)} dQ = C_N \frac{\det[e^{-a_x/\eta_k}]_{j,k=1}^{N}}{\Delta(x) \Delta(1/t)} \quad (2.15)$$

and its analogue (cf. [31] and [36])

$$\int_{\{U: U^* = I_N\}} \int_{V \in U(N)} e^{-\text{Tr}(\Omega U \sqrt{A_x}V + (\Omega U \sqrt{A_x}V)^*)} dU dV$$

$$= C_{L,N} \frac{\det[aF_1(L - N + 1; x_j \delta^2)]_{j,k=1}^{N}}{\Delta(x) \Delta(\delta^2)} \quad (2.16)$$

where $C_N$ depends only on $N$ and $C_{L,N}$ only on $L$ and $N$. Accordingly, integrate out $U, V, W, Q$ parts and note that $\Delta(1/t) = (-1)^{N(N-1)/2} \prod_{k=1}^{N} t_k^{1-N} \Delta(t)$, we obtain the joint distribution of squared singular values for $Y$ and $X$ which is proportional to

$$\det[e^{-a_x/\eta_k}]_{j,k=1}^{N} \det[aF_1(L - N + 1; x_j \delta^2)]_{j,k=1}^{N} \det[t_k^{M-N} f_k(t_j)]_{j,k=1}^{N}$$

$$\times \frac{1}{\Delta(\delta^2)} \prod_{k=1}^{N} \left(\frac{x_k}{t_k}\right)^{L-N} \frac{dt_1}{t_1} \cdots \frac{dt_N}{t_N} dx_1 \cdots dx_N. \quad (2.17)$$

**Step 3: Singular value PDF of $Y$.** In order to derive the joint PDF for the squared singular values of $Y$, we need to integrate out all variables $t_1, \ldots, t_N$ in (2.17). This can be done with the aid of the Andréief integral identity (see e.g. [19] Sect. 3.1) so that

$$\int_0^\infty \cdots \int_0^\infty \det[e^{-a_x/\eta_k}]_{j,k=1}^{N} \det[t_k^{M-N} f_k(t_j)]_{j,k=1}^{N} \prod_{k=1}^{N} \left(\frac{x_k}{t_k}\right)^{L-N} \frac{dt_1}{t_1} \cdots \frac{dt_N}{t_N}$$

$$= N! \det[\eta_k(x)]_{j,k=1}^{N}, \quad (2.18)$$

where

$$\eta_k(z) = \int_0^\infty e^{-a \frac{t}{t_k}^{M-N}} f_k(t) \left(\frac{z}{t}\right)^{L-N} \frac{dt}{t}$$

$$= \int_0^\infty t^{L-N} e^{-a t} \left(\frac{z}{t}\right)^{M-N} f_k \left(\frac{z}{t}\right) \frac{dt}{t}. \quad (2.19)$$

This gives us the requested joint PDF (2.23).

To evaluate the normalization constant, we make use of the Andréief identity again as follows

$$Z_N = N! \det\left[\int_0^\infty aF_1(L - N + 1; x \delta^2) \eta_k(x) dx\right]_{j,k=1}^{N}. \quad (2.20)$$
Change variables $x \mapsto xt, t \mapsto t$, integrate term by term in the inner integral and we then obtain
\[
\int_0^\infty \int_0^\infty \, f_k(x) \eta_i(x) dx \\
\int_0^\infty \int_0^\infty \, f_k(x) \eta_i(x) dx \\
= \int_0^\infty \int_0^\infty \, f_k(x) \eta_i(x) dx \\
\int_0^\infty \int_0^\infty \, f_k(x) \eta_i(x) dx \\
= \int_0^\infty \int_0^\infty \, f_k(x) \eta_i(x) dx \\
\int_0^\infty \int_0^\infty \, f_k(x) \eta_i(x) dx
\]
from which the normalization constant follows. Here in the second identity above we have applied the Fubini’s theorem, since the assumptions on functions $f_k$ imply $|f_k(x/t)| \leq C e^{-\alpha x/t}$ for some constant $C$.

We can apply Theorem 2.1 to any bi-invariant random matrix ensemble $X$ which can be coupled together with a Ginibre matrix and has a joint singular value PDF as in (2.2). A few examples immediately follow from the above theorem.

**Example 2.2.** For the joint PDF (2.1), suppose that $h(X) = \exp\{-\text{Tr}V(X^*X)\}$ where $V$ is a polynomial with positive leading coefficient and $\delta_1, \ldots, \delta_N$ are singular values of $\Omega$. Then the squared singular values of $Y = GX$ has a joint PDF on $[0, \infty)^N$
\[
\mathcal{P}_N(x_1, \ldots, x_N) = \frac{1}{Z_N} \det[\xi_i(x_j)]_{i,j=1}^N \det[\eta_i(x_j)]_{i,j=1}^N,
\]
where $\xi_i(z) = a F_1(L - N + 1; \delta_i^2 z)$,
\[
\eta_i(z) = \int_0^\infty t^{L-N} e^{-\alpha t - V(z)} (\frac{z}{t})^{M-N+i-1} dt, \quad t \geq 0
\]
and the normalization constant
\[
Z_N = N!((L - N)!)^N \alpha ^{N(L-N+1)} \det \left[ \left( \int_0^\infty x^{M-N+k-1} e^{-V(x)+x \delta_i^2/\alpha} dx \right)^N \right]_{j,k=1}.
\]

Yet another family of random matrix ensembles with singularities of the form
\[
h(X) = \text{const.} \cdot \exp\{-\alpha \text{Tr}(X^*X) - \beta^d \text{Tr}(X^*X)^{-d}\}, \quad \beta > 0 \text{ and } d \in \mathbb{N},
\]
where $X$ is a complex matrix of size $M \times N$ was studied in \cite{9, 15, 57} and a hard edge limiting kernel was obtained in terms of the Painlevé III hierarchy \cite{9, 57}. The singular value PDF for a coupled product with that reads as follows.

**Example 2.3.** With (2.1), let $h(X) = \exp\{-\alpha \text{Tr}(X^*X) - \beta^d \text{Tr}(X^*X)^{-d}\}$ where $\beta > 0$ and $d \in \mathbb{N}$, and let $\delta_1, \ldots, \delta_N$ be singular values of $\Omega$. Then the squared singular values of $Y = GX$ has a joint PDF on $[0, \infty)^N$
\[
\mathcal{P}_N(x_1, \ldots, x_N) = \frac{1}{Z_N} \det[\xi_i(x_j)]_{i,j=1}^N \det[\eta_i(x_j)]_{i,j=1}^N,
\]
where $\xi_i(z) = a F_1(L - N + 1; \delta_i^2 z)$,
\[
\eta_i(z) = \int_0^\infty t^{L-N} e^{-\alpha (z+\beta^d \cdot \cdot \cdot )^d} (\frac{z}{t})^{M-N+i-1} dt, \quad t \geq 0
\]
and the normalization constant
\[ Z_N = N!(L - N)! N^N \alpha^{-N(L-N+1)} \det \left[ \int_0^{\infty} x^{M-N+k-1} e^{-(\alpha - \delta_j^2/\alpha)x - (\beta/\alpha)^2} dx \right]_{j,k=1}^N. \]

We will get back to the random matrix ensembles stated in Examples 2.2 and 2.3 in a forthcoming paper, and expect similar hard edge transition to occur as in Theorem 1.3 but a detailed study would lead us too far.

So far, we see that Proposition 1.1 is just a special case of Example 2.2.

**Proof of Proposition 1.1.** Take \( V(x) = \alpha x \) in Example 2.2 recall (1.1) and we can rewrite
\[ \eta_i(z) = 2(\sqrt{z})^{L+M-2N+i-1} K_{M-L+i-1}(2\alpha \sqrt{z}). \] (2.25)
Accordingly, simple calculation shows that the constant given in (2.24) is reduced to
\[ Z_N = N!(L - N)! N^N \alpha^{-N(L+M)} \Delta(\delta^2) \prod_{j=1}^N \left( \Gamma(M - N + j)(1 - \delta_j^2/\alpha^2)^{-M} \right). \] (2.26)
Combine (2.25) and (2.26), recall (1.1) and we have the desired result. \( \square \)

### 2.2. Coupled multiplication with a Jacobi matrix.
Given two complex matrices \( X_1 \) of size \((\kappa + N) \times (\nu + N)\) and \( X_2 \) of size \((\nu + N) \times N\) with \( \kappa, \nu \geq 0 \), and a positive semidefinite \( N \times N \) matrix \( \Sigma \), we consider the joint PDF which is proportional to
\[ 1F_1(\nu + \nu' + 2N; \kappa + N; X_1 X_2 \Sigma X_2^* X_1^*) \exp \left\{ -\alpha \text{Tr} X_2^* X_2 \right\} \times \det^{\nu'-\kappa}(I_{\nu+N} - X_1 X_1^*) \theta(I_{\nu+N} - X_1 X_1^*) dX_1 dX_2, \] (2.27)
where \( \Sigma < \sqrt{\alpha} I_N \) and \( \nu' \) is a non-negative integer such that \( \nu + \nu' \geq \kappa \). Here \( 1F_1 \) is a hypergeometric function of matrix argument (see [37, 29] for more details) and the Heaviside step function of matrix argument defined on Hermitian matrices \( H \) as
\[ \theta(H) = \begin{cases} 1, & \text{if } H \text{ is positive definite,} \\ 0, & \text{other.} \end{cases} \]

This is expected to be closely related with the non-central distribution in MANOVA (see e.g. [8]), which may be derived from the joint distribution proportional to
\[ \exp \left\{ -\text{Tr}(Z_1 Z_1^* + Z_2 Z_2^*) \Sigma_0^{-1} + \text{Tr}(\Omega Z_1 X_2 + (\Omega Z_1 X_2^*) - \alpha \text{Tr} X_2^* X_2) \right\}, \] (2.28)
where \( Z_1 \) and \( Z_2 \) are rectangular matrices of sizes \((\kappa + N) \times (\nu + N)\) and \((\nu + N) \times (\nu' + N)\), respectively. Set \( X_1 = (Z_1 Z_1^* + Z_2 Z_2^*)^{-1/2} Z_1 \), then \( X_1 \) and \( X_2 \) are expected to be distributed as in (2.27) but with \( \Sigma = \Omega \Sigma_0 \Omega^* \); cf. [37, Sect. 8].

Instead of (2.27), with the same notations as in (2.1) we now turn to a more general PDF proportional to
\[ 1F_1(\nu + \nu' + 2N; \kappa + N; GX \Sigma X^* G^*) h(X) \times \det^{\nu'-\kappa}(I_{\nu+N} - GG^*) \theta(I_{\nu+N} - GG^*) dGdX, \] (2.29)
where \( \Sigma < \sqrt{\alpha} I_N \) and \( \nu, \kappa, \nu' \) are non-negative integers such that \( \nu + \nu' \geq \kappa \). Likewise, the squared singular values of \( GX \) also forms a determinantal point process.
Theorem 2.4. With the joint PDF defined in (2.29), let $\delta_j^2/\alpha, \ldots, \delta_N^2/\alpha$ be eigenvalues $\Sigma$ with $\alpha > 0$ and all $0 \leq \delta_j < \alpha$. Suppose that $f_k(t)$ ($k = 1, \ldots, N$) are continuous in $(0, \infty)$ such that all $e^{zt}f_k(t)$ are bounded in $[0, \infty)$, let

$$h(X) = \frac{1}{\Delta(t)} \det[f_k(t)]_{j,k=1}^N, \quad 0 < t_1, \ldots, t_N < \infty,$$

where $t_1, \ldots, t_N$ are eigenvalues of $X^*X$, then the squared singular values of $Y = G X$ have a joint PDF on $[0, \infty)^N$

$$P_N(x_1, \ldots, x_N) = \frac{1}{Z_N} \det[\xi_i(x_j)]_{i,j=1}^N \det[\eta_i(x_j)]_{i,j=1}^N,$$

where $\xi_i(z) = z^{\kappa_i} F_i(\nu + \nu' + N + 1; \kappa + 1; \delta_i^2 z/\alpha)$ and

$$\eta_i(z) = \int_0^1 (1 - t)^{\nu' + \nu - \kappa + N - 1} \left(\frac{z}{t}\right)^{\nu - \kappa} f_i(z) t \, dt.$$

The normalization constant can be evaluated by

$$Z_N = N! \left(\frac{\Gamma(\kappa + 1) \Gamma(\nu + \nu' - \kappa + N)}{\Gamma(\nu + \nu' + N + 1)}\right)^N \det \left[\int_0^\infty x^{\nu + \nu' - \kappa + N - 1} (\frac{z}{x})^{\nu - \kappa} e^{\frac{dt}{\alpha}}\right]_{j,k=1}^N.$$

A corollary immediately follows from the above theorem.

Corollary 2.5. For the joint PDF (2.29), let $h(X) = \exp\{-\alpha \text{Tr}(X^*X)\}$. With the same notations as in Theorem 2.4, then the squared singular values of $Y = G X$ have a joint PDF on $[0, \infty)^N$

$$P_N(x_1, \ldots, x_N) = \frac{1}{Z_N} \det[\xi_i(x_j)]_{i,j=1}^N \det[\eta_i(x_j)]_{i,j=1}^N, \quad (2.30)$$

where $\xi_i(z) = z^{\kappa_i} F_i(\nu + \nu' + N + 1; \kappa + 1; \delta_i^2 z/\alpha)$,

$$\eta_i(z) = \int_0^1 (1 - t)^{\nu' + \nu - \kappa + N - 1} \left(\frac{z}{t}\right)^{\nu - \kappa} e^{-\frac{dt}{\alpha}}.$$

with $0 \leq \delta_j < \alpha$ for $j = 1, \ldots, N$ and the normalization constant

$$Z_N = N! \alpha^{-\frac{1}{2}N(N-1)} \Delta(\delta^2) \times \left(\frac{\Gamma(\kappa + 1) \Gamma(\nu + \nu' - \kappa + N)}{\Gamma(\nu + \nu' + N + 1)}\right)^N \prod_{j=1}^N \left(\Gamma(\nu + j) \left(\alpha - \frac{\delta_j^2}{\alpha}\right)^{-\nu - N}\right). \quad (2.31)$$

Proof of Theorem 2.4 Since we can proceed almost in the same steps as in Theorem 2.1, we just point out some different places and leave the details to the reader.

Step 1: Reduction. Let us assume that $\nu > 0$. Since any matrix $X$ of size $M \times N$ can be decomposed as

$$X = U \begin{pmatrix} X_0 \\ O \end{pmatrix},$$

where $U$ is a $(\nu + N) \times (\nu + N)$ unitary matrix and $X_0$ is an $N \times N$ complex matrix, setting $G U = (G_0 G_1)$ we arrive at the joint distribution of $G_0, G_1, X_0$ and $U$ proportional to

$$1 \tilde{F}_1(\nu + \nu' + 2N; \kappa + N; G_0 X_0 \Sigma X_0^* G_0^*) h(X_0) \det'(X_0^* X_0) \times \det^{\nu' - \kappa} \left(I - G_0 G_0^* - G_1 G_1^*\right) \theta \left(I - G_0 G_0^* - G_1 G_1^*\right) dG_0 dG_1 dX_0 [dU].$$
Make a change of variables $G_1 \mapsto \left( I - G_0 G_0^* \right)^{1/2} G_1$ and integrate over $G_1$ and $U$, we immediately see that the joint probability distribution of $G_0$ and $X_0$ reads

$$1 \tilde{F}_1 (\nu + \nu' + 2N; \kappa + N; G_0 X_0 \Sigma X_0^* G_0^*) h(X_0) \det' (X_0^* X_0)$$

$$\times \det^\nu + \nu - \kappa \left( I - G_0 G_0^* \right) \theta \left( I - G_0 G_0^* \right) dG_0 dX_0. \quad (2.33)$$

up to some constant. Furthermore, both $G X$ and $G_0 X_0$ have the same singular values, which shows that we only need to focus on the joint distribution (2.33).

**Step 2: Joint singular value PDF of $X$ and $Y$.** For convenience, we next replace the notation $G_0$ and $X_0$ with $G$ and $X$ respectively in (2.33). Since the change of variables of $G \mapsto Y = G X$ and $X \mapsto X$ has a Jacobian $\det(X^* X)^{-L}$ (cf. 17 Theorem 3.2], $Y$ and $X$ have a joint distribution proportional to

$$1 \tilde{F}_1 (\nu + \nu' + 2N; \kappa + N; Y \Sigma Y^*) h(X) \det' (X^* X)$$

$$\times \det^\nu + \nu - \kappa \left( I - (X^* X)^{-1} Y^* Y \right) \theta \left( X^* X - Y^* Y \right) dY dX. \quad (2.34)$$

Next, let $\Lambda_x = \text{diag} (x_1, \ldots, x_N)$, according to the singular value decomposition, write

$$Y = U \left( \sqrt{\Lambda_x} \right) V, \quad X = W \sqrt{\Lambda_t} Q,$$

note the fact $1 \tilde{F}_1 (\cdot; Y \Sigma Y^*) \rightarrow 1 \tilde{F}_1 (\cdot; \Sigma Y^* Y)$ and change $Q \mapsto Q V$, due to the invariance of the Haar measure we know that (2.34) is reduced to the distribution proportional to

$$1 \tilde{F}_1 (\cdot; \Sigma V^{-1} \Lambda_x V) \det' \nu + \nu - \kappa \left( I - \Lambda_t^{-1} Q \Lambda_x Q^{-1} \right) \theta \left( \Lambda_t - Q \Lambda_x Q^{-1} \right) \prod_{k=1}^N \left( x_k^{\nu} t_k^{\nu - N} \right)$$

$$\times \Delta(x)^2 \Delta(t) \det \left[ f_k(t_j) \right]_{j,k=1}^N dU dV dW dQ dx_1 \cdots dx_N dt_1 \cdots dt_N.$$

We need to use the following two integral formulas over the unitary group

$$\int_{U(N)} 1 \tilde{F}_1 (\nu + \nu' + 2N; \kappa + N; \Sigma V^{-1} \Lambda_x V) dV \propto$$

$$\frac{1}{\Delta(x) \Delta(\delta^2)} \det \left[ F_1 (\nu + \nu' + N + 1; \kappa + 1; x j \delta^2 / \alpha) \right]_{j,k=1}^N, \quad (2.35)$$

where the ordinary hypergeometric function appears inside the determinant, and

$$\int_{U(N)} \det' \nu + \nu - \kappa \left( I - \Lambda_t^{-1} Q \Lambda_x Q^{-1} \right) \theta \left( \Lambda_t - Q \Lambda_x Q^{-1} \right) dQ \propto$$

$$\frac{1}{\Delta(x) \Delta(1/t)} \det \left[ \left( 1 - \frac{x_j}{t_k} \right)^{\nu + \nu' + N - 1} \right]_{j,k=1}^N, \quad (2.36)$$

where $x_+ = \max\{0, x\}$; see e.g. [29 Sect. 4] or [44 Sect. 2] for the first formula, and [30] Theorem 2.3 for the later. Accordingly, integrate out $U, V, W, Q$ parts and note that $\Delta(1/t) = (-1)^{N(N-1)/2} \prod_{k=1}^N \left[ t_k^{1/2} \Delta(t) \right]$, we thus arrive at the joint
distribution of squared singular values for \( Y \) and \( X \) proportional to
\[
\det \left[ x_j^k F_1 (\nu + \nu' + N + 1; \kappa + 1; x_j \delta^2_k / \alpha) \right]_{j,k=1}^{N} \det \left[ \tau_j^{\nu - \kappa - 1} f_k (t_j) \right]_{j,k=1}^{N} \times \det \left[ \left( 1 - \frac{x_j}{t_k} \right)^{\nu' + \nu - \kappa + N - 1} \right]_{j,k=1}^{N}.
\]

(2.37)

**Step 3: Singular value PDF of \( Y \).** Integrating out all variables \( t_1, \ldots, t_N \) in (2.37) and using the Andréief integral identity, we have the requested joint PDF after some simple manipulations. To evaluate the normalization constant, we make use of the Andréief identity again to obtain
\[
Z_N = N! \det \left[ \int_0^\infty x_k^1 F_1 (\nu + \nu' + N + 1; \kappa + 1; \delta^2 x / \alpha) \eta_k (x) dx \right]_{j,k=1}^{N}.
\]

Change variables \( x \to xt, t \to t \), integrate term by term in the inner integral and we then get for \( a := \nu + \nu' + N \)
\[
\int_0^\infty x^a F_1 (\nu + \nu' + N + 1; \kappa + 1; \delta^2 x / \alpha) \eta_k (x) dx
\]
\[
= \int_0^\infty \int_0^1 x^a F_1 (a + 1; \kappa + 1; \delta^2 x / \alpha) (1 - t)^{\nu - \kappa - 1} \left( \frac{x}{t} \right)^{\nu' - \kappa} f_k \left( \frac{x}{t} \right) \frac{dt}{t} dx
\]
\[
= \int_0^\infty \left( \int_0^1 t^n (1 - t)^{\nu - \kappa - 1} F_1 (a + 1; \kappa + 1; \delta^2 xt / \alpha) dt \right)^N f_k (x) dx
\]
\[
= \frac{\Gamma (\kappa + 1) \Gamma (\nu + \nu' - \kappa + N)}{\Gamma (\nu + \nu' + N + 1)} \int_0^\infty e^{\delta^2 x / \alpha} x^\nu f_k (x) dx,
\]
from which the desired normalization constant follows. Here in the second identity above we have applied the Fubini’s theorem, since the assumptions on functions \( f_k \) imply \( |f_k (x/t)| \leq Ce^{-\alpha x/t} \) for some constant \( C \).

3. Double integrals for correlation kernels

As mentioned earlier, the joint eigenvalue density (2.3) is an example of biorthogonal ensembles in Borodin’s sense [13]
\[
Q_N (x_1, \ldots, x_N) = \frac{1}{Z_N} \det [\eta_i (x_j)]_{i,j=1}^{N} \det [\xi_i (x_j)]_{i,j=1}^{N}, \quad x_1, \ldots, x_N \in I,
\]
(3.1)
where \( I \) is a union of finite intervals of \( \mathbb{R} \). The significance of the structure (3.1) is that there exists a systematic way to compute the corresponding \( k \)-point correlation functions defined by
\[
\rho_k (x_1, \ldots, x_k) = \frac{N!}{(N-k)!} \int \cdots \int Q_N (x_1, \ldots, x_N) dx_{k+1} \cdots dx_N,
\]
see e.g. [20 Eq.(5.1)]. The following proposition due to Borodin provides a solution to derive the correlation kernel which is of vital importance in the study of determinantal point processes.

**Proposition 3.1** ([13 Proposition 2.2]). Let \( g_{i,j} := \int \eta_i (x) \xi_j (x) dx \), suppose that \( [g_{i,j}]_{i,j=1}^{N} \) be invertible for each \( n = 1, 2, \ldots \) Defining \( c_{i,j} \) by
\[
([c_{i,j}]_{i,j=1}^{N})^t = \left( [g_{i,j}]_{i,j=1}^{N} \right)^{-1},
\]
(3.2)
and setting
\[ K_N(x, y) = \sum_{i,j=1}^{N} c_{i,j} \eta_i(x) \xi_j(y), \] (3.3)
we then have
\[ \rho_k(x_1, \ldots, x_k) = \det[K_N(x_j, x_l)]_{j,l=1}^{k}. \]

Next, we first use Proposition 3.1 to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Starting with the eigenvalue PDF (1.11), with (3.1) in mind we set
\[ \eta_i(x) = \frac{\mu}{\alpha^\nu} K_{\nu-\kappa+i-1}(2\alpha \sqrt{x}), \quad \xi_i(x) = I_{\kappa}(2\delta_i \sqrt{x}). \] (3.4)
In order to calculate the integral \( g_{i,j} \) as presented in Proposition 3.1 we make use of the integral formula involving Bessel functions as in [13],
\[ \int_0^\infty \mu^{\nu+1} \kappa \mu(a t) I_{\mu}(b t) dt = \frac{2\mu^{\nu+\mu} \Gamma(\mu + 1)(1 - \frac{b^2}{a^2})^{\nu - 1}}{a^{\nu + 2}}, \] (3.5)
which can be derived by applying Euler’s transformation for Gaussian hypergeometric functions \( _2F_1 \) in [20, 6.576.5] and then by taking a special case (noting that we have in essence given a direct derivation for the formula (3.6) in the proof of Proposition 1.1 cf. (2.26)). Here \( a > b > 0, \) and \( \mu + \nu + 1 > 0. \) Then by (3.5) we have
\[ g_{i,j} = \int_0^\infty \frac{\mu^{\nu+\mu+1} K_{\nu-\kappa+i-1}(2\alpha \sqrt{x})}{I_{\kappa}(2\delta_j \sqrt{x})} \Gamma(\nu + i) \delta_j N \frac{2\mu^{\nu+\nu+1}}{a^{\nu + 2}} \left(1 - \frac{\delta_j^2}{\alpha^2}\right)^{N-i} \] (3.6)
According to Proposition 3.1 with \( G = [g_{i,j}]_{i,j=1}^{N}, \) let \( C = (G^{-1})^t, \) the entries \( c_{i,j} \) of \( C \) then satisfy
\[ \sum_{i=1}^{N} \left(1 - \frac{\delta_j^2}{\alpha^2}\right)^{N-i} c_{i,j} = \delta_{i,k}, \] (3.7)
that is,
\[ \sum_{i=1}^{N} \left(1 - \frac{\delta_j^2}{\alpha^2}\right)^{N-i} c_{i,j} = \delta_{i,k} \left(1 - \frac{\delta_j^2}{\alpha^2}\right)^{N-i}. \]
Without loss of generality, we assume that \( \delta_1, \ldots, \delta_N \) are pairwise distinct; otherwise, the requested result follows from taking proper limit and using of L’Hospital’s rule. In this case, the above equations imply
\[ \sum_{i=1}^{N} \left(1 - \frac{\delta_j^2}{\alpha^2}\right)^{N-i} c_{i,j} = \delta_{i,k} \left(1 - \frac{\delta_j^2}{\alpha^2}\right)^{N-i} \prod_{l=1, l \neq j}^{N} \frac{u - \left(1 - \frac{\delta_l^2}{\alpha^2}\right)}{\left(1 - \frac{\delta_j^2}{\alpha^2}\right) - \left(1 - \frac{\delta_l^2}{\alpha^2}\right)}, \] (3.8)
as can be verified by noting that both sides are polynomials of degree \( N - 1 \) in \( u \) and take the same values at \( N \) different points since (3.7) holds true.

Using this implicit formula for \( \{c_{i,j}\} \) we are ready to show that (3.3) implies the double contour integral formula (1.13). Keep (1.7) in mind and also note that for a positive integer \( l \) (cf. Hankel’s formula for the reciprocal gamma function)
\[ z^{l-1} = \frac{\Gamma(l)}{2\pi i} \int_{c_{\text{out}}} u^{-l} e^{zu} du, \] (3.9)
we have from \(3.3\) that
\[
K_N(x, y) = \sum_{j=1}^{N} \xi_j(y) \sum_{i=1}^{N} \frac{1}{2} x^\frac{2}{\alpha} \int_0^\infty \frac{dt}{t} \left( \frac{\alpha x}{t} \right)^{i+\nu-1-\kappa} e^{-t-\frac{\alpha^2}{t}} c_{i,j} \]
\[
= \sum_{j=1}^{N} \xi_j(y) \frac{1}{2} x^\frac{2}{\alpha} \int_0^\infty \frac{dt}{t} \left( \frac{x}{t} \right)^{-\kappa} e^{-t-\frac{\alpha^2}{t}} \]
\[
\times \frac{2\alpha^2}{2\pi i} \sum_{i=1}^{N} \int_{C_{\text{out}}} du e^{\frac{\alpha^2}{2} u} \frac{\Gamma(\nu+i)}{2\alpha^{\nu+\kappa+1}+1} u^{-\nu-i} c_{i,j},
\]
(3.10)

where the simple closed contour \(C_{\text{out}}\) is chosen such that it doesn’t depend on any parameters \(\alpha, \delta_1, \ldots, \delta_N\) and \(\text{Re}(z) < 1\) for \(z \in C_{\text{out}}\). Then for \(x > 0\) the integrand with variables \(u, t\) on the RHS of the second identity of (3.10) permits us to exchange the order of integration. Combine the identity (3.8) and we thus get
\[
K_N(x, y) = \frac{2\alpha^2}{2\pi i} \sum_{j=1}^{N} \xi_j(y) \delta_j^{-\kappa} \left( 1 - \frac{\delta_j^2}{\alpha^2} \right)^{\nu+N} \int_{C_{\text{out}}} du u^{-\nu-N} \]
\[
\times \prod_{l=1, l \neq j}^{N} \frac{u - (1 - \frac{\delta_l^2}{\alpha^2})}{(1 - \frac{\delta_j^2}{\alpha^2}) - (1 - \frac{\delta_l^2}{\alpha^2})} \frac{1}{2} x^\frac{2}{\alpha} \int_0^\infty \frac{dt}{t} \left( \frac{\alpha x}{t} \right)^{-\kappa} e^{-t-\frac{\alpha^2}{t}(1-u)}. \quad (3.11)
\]

Finally, recall (1.7) and (3.3), we rewrite the summation in (3.11) as
\[
K_N(x, y) = \frac{2\alpha^2}{2\pi i} \sum_{j=1}^{N} I_{\kappa}(2\delta_j \sqrt{y}) \delta_j^{-\kappa} \left( 1 - \frac{\delta_j^2}{\alpha^2} \right)^{\nu+N} \int_{C_{\text{out}}} du u^{-\nu-N} (\alpha \sqrt{1-u})^{-\kappa} \]
\[
\times K_{-\kappa}(2\alpha \sqrt{(1-u)x}) \prod_{l=1, l \neq j}^{N} \frac{u - (1 - \frac{\delta_l^2}{\alpha^2})}{(1 - \frac{\delta_j^2}{\alpha^2}) - (1 - \frac{\delta_l^2}{\alpha^2})}. \quad (3.12)
\]

We recognise the above summation over \(j\) as the summation of the residues at \(\{1 - \frac{\delta_j^2}{\alpha^2}\}\) of the \(v\)-function
\[
v^{\nu+N} (\alpha \sqrt{1-v})^{-\kappa} I_{\kappa}(2\alpha \sqrt{(1-v)y}) \prod_{l=1}^{N} \frac{u - (1 - \frac{\delta_l^2}{\alpha^2})}{v - (1 - \frac{\delta_l^2}{\alpha^2})}. \quad (3.13)
\]

application of the residue theorem then gives the required result. Here \(C_{\text{in}}\) is a counterclockwise contour encircling \(1 - \frac{\delta_j^2}{\alpha^2}, \ldots, 1 - \frac{\delta_N^2}{\alpha^2}\) but not any \(u \in C_{\text{out}}\). In particular, we can choose the two contours as described in the theorem. \(\Box\)

Note that in order to derive the double contour integral in Theorem 1.2 we have made the best of nice formulas for integrals of Bessel functions, a question arises naturally: Are there double contour integrals for correlation kernels of the bi-orthogonal ensembles \(2.3\)? And even more specifically, is there a relationship between the correlation kernels associated with singular values of \(GX\) and \(X\)? When \(G\) and \(X\) are independent, for \(G\) being a Ginibre or truncated unitary matrix, Claeys, Kuijlaars and Wang found a nice relation; see [17], Lemma 2.14. It is really a challenge for us to extend their result to the coupled product case.

Secondly, we have a Jacobi-type analogue of Theorem 1.2. For this purpose, we need to define two functions which can be treated as being of mutual duality for
an integral representation of correlation kernel. One is, as an entire function of \( z \), for \( \nu > \kappa > -1 \),

\[
f_1(\nu, \kappa; z) = \frac{\Gamma(\nu + 1)}{\Gamma(\kappa + 1)\Gamma(\nu - \kappa)} F_1(\nu + 1; \kappa + 1; z) \tag{3.14}
\]

\[
= \frac{\Gamma(\nu + 1)}{\Gamma(\nu - \kappa)\Gamma(\nu - \kappa)} \frac{1}{2\pi i} \int_{C_0} s^{-\kappa - 1} e^s \left( 1 - \frac{z}{s} \right)^{-\nu - 1} ds, \tag{3.15}
\]

where \( C_0 \) is a counterclockwise contour around the origin. The other is, for \( \nu > 0 \) and \( \kappa \in \mathbb{R} \),

\[
f_2(\nu, \kappa; z) = \int_0^1 (1 - t)^{\nu - 1} t^{\kappa - 1} e^{-\frac{z}{t}} dt, \quad |\text{arg}(z)| < \frac{\pi}{2} \tag{3.16}
\]

**Theorem 3.2.** The correlation kernel for the biorthogonal ensemble (2.30) is given by

\[
K_N(x, y) = \frac{\alpha}{(2\pi i)^2} \left( \frac{y}{x} \right)^\kappa \int_{C_{\text{in}}} \, du \int_{C_{\text{out}}} \, dv \, f_2(\nu + \nu' - \kappa + N, \kappa; \alpha(1 - u)x) \times f_1(\nu + \nu' + N, \kappa; \alpha(1 - v)y) \frac{1}{u - v} \left( \frac{u}{v} \right)^{-\nu - N} \prod_{l=1}^N \frac{u - \left( 1 - \frac{\delta_l^2}{\alpha^2} \right)}{v - \left( 1 - \frac{\delta_l^2}{\alpha^2} \right)}, \tag{3.17}
\]

where \( C_{\text{in}} \) is a counterclockwise contour encircling \( 1 - \delta_1^2/\alpha^2, \ldots, 1 - \delta_N^2/\alpha^2 \), and \( C_{\text{out}} \) is a simple counterclockwise contour around the origin with \( \text{Re}(z) < 1 \) for \( z \in C_{\text{out}} \) such that \( C_{\text{in}} \) is entirely to the right side of \( C_{\text{out}} \). When \( 0 < \delta_j < \alpha \) for \( j = 1, \ldots, N \), we can also choose contours such that \( C_{\text{in}} \) is contained entirely in \( C_{\text{out}} \).

**Proof.** We proceed in a similar way as in Theorem 1.2 and just give a brief derivation as follows. With Corollary 2.5 in mind, simple calculation in the same way as in (2.30) shows us

\[
g_{i,j} := \int_0^\infty \eta_i(x)\xi_j(x)dx = \frac{\Gamma(\kappa + 1)\Gamma(\nu + \nu' - \kappa + N)}{\Gamma(\nu + \nu' + N + 1)} \frac{\Gamma(\nu + i)}{\alpha^{\nu+i}} \left( 1 - \frac{\delta_j^2}{\alpha^2} \right)^{-\nu-i}. \tag{3.18}
\]

According to Proposition 3.1 with \( G = [g_{i,j}]_{i,j=1}^N \), let \( C = (G^{-1})^t \), the entries \( c_{i,j} \) of \( C \) then satisfy identical equations

\[
\sum_{i=1}^N \frac{\Gamma(\nu + i)}{2\alpha^{\nu+i+1}} u^{N-i} c_{i,j} = \frac{\Gamma(\nu + \nu' + N + 1)}{\Gamma(\kappa + 1)\Gamma(\nu + \nu' - \kappa + N)} \times \left( 1 - \frac{\delta_j^2}{\alpha^2} \right)^{\nu+N} \prod_{l=1, l \neq j}^N \frac{u - \left( 1 - \frac{\delta_l^2}{\alpha^2} \right)}{1 - \left( 1 - \frac{\delta_l^2}{\alpha^2} \right)} \tag{3.19}
\]

By Hankel’s formula (3.10), we have from (3.33) that

\[
K_N(x, y) = \frac{\alpha}{2\pi i} \sum_j \xi_j(y) \int_0^1 \frac{dt}{t} \left( \frac{x}{t} \right)^{\kappa} (1 - t)^{\nu + \nu' - \kappa - N - 1} e^{-\frac{z}{t}} \times \sum_i \frac{\Gamma(\nu + i)}{\alpha^{\nu+i}} \int_{C_{\text{out}}} \, du \, e^{\frac{\alpha}{\nu+i}u} u^{-\nu-i} c_{i,j} \tag{3.20}
\]
Thus, using the identity (3.19), exchanging the order of integration and then
applying residue theorem imply the required result. □

Finally, let’s extract some key ideas behind the proofs of both Theorems 1.2 and
3.2 and draw a general procedure in giving double contour integrals for correlation
kernels in a class of bi-orthogonal ensembles; see [10, 17, 22, 28] for relevant
examples.

Remark 3.3. For the bi-orthogonal ensemble (3.1), suppose that the following con-
ditions hold true:

(i) There exist two functions $g(t, x), \Phi(t, x)$ and $N$ generic parameters $a_1, \ldots, a_N$
such that

$$\eta_i(x) = \int (xt)^{i-1} g(t, x) dt, \quad \xi_j(x) = \Phi(a_j, x), \quad x \in I. \quad (3.21)$$

(ii) There exist $h(x), q(x)$ and polynomials $L_k(x)$ of degree $k$ such that

$$\int \eta_i(x) \xi_j(x) dx = \frac{1}{b_i h(a_j)} L_{i-1}(a_j) \quad (3.22)$$

where

$$b_i = \int z^{i-1} q(z) dz. \quad (3.23)$$

(iii) There exist $\tilde{g}(t, x)$ and a contour $C_1$ not containing $\{a_j\}$ such that

$$z^{i-1} = \int_{C_1} \frac{1}{h(u)} L_{i-1}(u) \tilde{g}(u, z) du. \quad (3.24)$$

Setting

$$\Psi(u, x) = \int \tilde{g}(u, xzt) q(z) g(t, x) dz dt, \quad (3.25)$$

if both $h(z)$ and $\Phi(z, x)$ are analytic functions of $z$ in some proper domain contain-
ing all $a_j$, then with certain conditions such as integrability on $g, \tilde{g}, \Psi$ and $q$ the
correlation kernel should be given by

$$K_N(x, y) = \frac{1}{2\pi i} \int_{C_1} du \int_{C_2} dv \Psi(u, x) \Phi(v, y) \frac{h(v)}{h(u)} \frac{1}{u - v} \prod_{l=1}^{N} \frac{u - a_l}{v - a_l}, \quad (3.26)$$

where $C_2$ is a counterclockwise contour encircling $a_1, \ldots, a_N$, but does not intersect
with $C_1$. The integral transform (3.24) connects polynomials $L_i(z)$ and $z^i$, and in a
practical application of Proposition 3.3 the most difficult part usually lies in finding
of a suitable kernel function $\tilde{g}(t, x)$ as stated in Condition (iii).

4. Hard edge limits

In this section we are devoted to the proof of Theorem 1.3 and hard edge limits of
the kernel (3.17), for which the same hard edge transition phenomenon is observed.
Proof of Theorem [1.3] First, under the assumptions [1.13], we can rewrite the correlation kernel [1.13] as

\[ K_N(x, y) = \frac{2\alpha^2}{(2\pi i)^2} \int_{\mathcal{C}_{\text{out}}} du \int_{\mathcal{C}_{\text{in}}} dv K_{\nu}(2\alpha \sqrt{(1-u)x}) I_\nu(2\alpha \sqrt{(1-v)y}) \]

\[ \times \frac{1}{u-v \left(1 - \frac{1}{N} \right)^{\nu}} \left( \frac{1}{u} \right)^{\nu+m} \left( \frac{1}{1 - \frac{1}{N} \left(1 + \mu \right)^2} \right)^{N-m} \prod_{l=1}^{m} \frac{u - (1 - \frac{\delta^2}{N})}{v - (1 - \frac{\delta^2}{N})}, \]  

(4.1)

where \( \mathcal{C}_{\text{in}} \) encircles \( 1 - \frac{\delta^2}{\alpha^2}, \ldots, 1 - \frac{\delta^2}{\alpha^2} \) and \( 4\mu/(1+\mu)^2 \). Next, we prove Parts (i)–(iv) under the corresponding conditions respectively.

For Part (i) where \( \mu N \to \infty \) and \( 0 \leq \delta_j < \alpha \) for \( j = 1, \ldots, m \), we choose the contours such that \( \mathcal{C}_{\text{out}} \) goes around the origin with \( \text{Re}(z) < 1 \) for \( z \in \mathcal{C}_{\text{out}} \) and \( \mathcal{C}_{\text{in}} \) is entirely to the right side of \( \mathcal{C}_{\text{out}} \). In order to take limits smoothly, we need to substitute \( K_{\nu} \) and \( I_\nu \) into (4.1) with their integral representations respectively given by [1.17] and

\[ I_\nu(z) = \left( \frac{\pi}{2} \right) \frac{1}{2\pi i} \int_{\mathcal{C}_0} ds s^{-\nu-1} e^{zs} \]  

(4.2)

which can be obtained by applying the integral representation of the reciprocal gamma function (cf. [1.19]) to the RHS of (4.4). Changing \( u \) to \( 4\mu/(1+\mu)^2 N \) and \( v \) to \( 4\mu/(1+\mu)^2 Nv \), we then use Fubini’s theorem to get

\[ \frac{\mu}{N} K_N\left( \frac{\mu}{N}, \frac{\mu}{N} \right) = \left( \frac{\eta}{\xi} \right)^{\nu/2} \frac{1}{2\pi i} \int_{\mathcal{C}_0} dt \int_{\mathcal{C}_{\text{out}}} ds t^{\nu-1} s^{-\nu-1} e^{s t} \tilde{K}_N\left( \frac{\eta}{s} \frac{\xi}{t} \right) \]  

(4.3)

where

\[ \tilde{K}_N\left( \frac{\eta}{s} \frac{\xi}{t} \right) = \exp \left\{ \frac{\eta (1+\mu)^2}{4\mu N} - \frac{\xi (1+\mu)^2}{4\mu N} \right\} \frac{1}{(2\pi i)^2} \int_{\mathcal{C}_{\text{out}}} du \int_{\mathcal{C}_{\text{in}}} dv e^{\frac{u}{v} - \frac{\mu}{N}} \]

\[ \times \frac{1}{u-v \left(1 - \frac{1}{N} \right)^{\nu}} \left( \frac{1}{u} \right)^{\nu+m} \left( \frac{1}{1 - \frac{1}{N} \left(1 + \mu \right)^2} \right)^{N-m} \prod_{l=1}^{m} \frac{u - (1 - \frac{\delta^2}{N})}{v - (1 - \frac{\delta^2}{N})}. \]  

(4.4)

Here \( \mathcal{C}_{\text{out}} \) is a counterclockwise contour around the origin and entirely to its right side \( \mathcal{C}_{\text{in}} \) encircles \( 1/N \) and \( a_l := (4\mu N)^{-1}(1+\mu)^2(1 - \frac{\delta^2}{\alpha^2}) \) for \( l = 1, \ldots, m \).

To take limit as \( N \to \infty \) in (4.4), we need to deform the two contours. For this, denote by \( \mathcal{L}_{c_1, c_2, r} \) with \( c_1 < c_2 \) and \( r > 0 \) a rectangular contour connecting four points \( (c_1, \pm r), (c_2, \pm r) \) in a counterclockwise direction. Take \( b_1, b_2 \) such that \( 0 < b_1 < \min \{1/N, a_1, \ldots, a_m\} \) and \( b_2 > \max \{1/N, a_1, \ldots, a_m\} \), so we can specify \( \mathcal{C}_{\text{out}} \) and \( \mathcal{C}_{\text{in}} \) with rectangular contours \( \mathcal{L}_{-b_1/2, b_1/2, 2} \) and \( \mathcal{L}_{b_1, b_2; 2} \), respectively. For convenience, let’s use an abbreviated notation for the RHS of (4.4). We thus arrive at

\[ \tilde{K}_N\left( \frac{\eta}{s} \frac{\xi}{t} \right) = \int_{\mathcal{L}_{-b_1/2, b_1/2, 2}} du \int_{\mathcal{L}_{b_1, b_2; 2}} dv \left( \cdot \right) \]

\[ = \int_{\mathcal{L}_{-b_1/2, b_1/2, 2}} du \int_{\mathcal{L}_{b_1, b_2; 2}} dv \left( \cdot \right) - \int_{\mathcal{L}_{b_1, b_2; 2}} du \int_{\mathcal{L}_{-b_1/2, b_1/2, 2}} dv \left( \cdot \right) \]  

(4.5)

\[ = \int_{\mathcal{L}_{-b_1/2, b_1/2, 2}} du \int_{\mathcal{L}_{b_1, b_2; 2}} dv \left( \cdot \right), \]  

(4.6)
where the second integral in (1.2.5) is actually equal to zero because the integrand has no pole with respect to $u$. Moreover, we can deform the resulting integral \((1.6)\) again and get
\[
\tilde{K}_N(\frac{n}{s}, \frac{\xi}{t}) = \int_{L_{2b_2, 2b_2, 2}} du \int_{L_{-b_2, b_2, 1}} dv \left( \cdot \right), \quad (4.7)
\]

Since $\mu N \to \infty$ as $N \to \infty$, noting that $0 < \mu \leq 1$ and $0 \leq \delta_l < \alpha$ ($l = 1, \ldots, m$), for $N$ large sufficient (for instance, $N > 1$ and $\mu N > 1$), set $b_2 = 1$ in \((1.7)\), application of Lebesgue’s dominated convergence theorem provides us
\[
\tilde{K}_N(\frac{n}{s}, \frac{\xi}{t}) \to \tilde{K}_\infty(\frac{n}{s}, \frac{\xi}{t}) := \frac{1}{(2\pi i)^2} \int_{L_{2, 2, 2}} du \int_{L_{-1, 1, 1}} dv e^{\frac{\xi u}{2} - \frac{u^2}{2\nu} + \frac{1}{2} \frac{1}{u - v}} \frac{v^\nu}{u}. \quad (4.8)
\]

This limit has been identified as the Bessel kernel by Desrosiers and Forrester (cf. \cite{21} eqns (1.20) and (6.20)]) with a specific relation
\[
\tilde{K}_\infty(s, t) = 4 \left( \frac{\xi s}{\eta t} \right)^{\nu/2} K_{\nu}^{(\text{Bes})} \left( \frac{4\eta}{s}, \frac{4\xi}{t} \right), \quad (4.9)
\]
from which the requested conclusion follows.

In the case of Part (ii) where $\mu N \to \tau/4$ with $\tau > 0$ and $1 - \delta_l^2/\alpha^2 \to \pi_l \in (0, 1)$ for $l = 1, \ldots, m$, for large $N$ sufficient we see all $1 - \delta_l^2/\alpha^2 \in (0, r_1)$ with a given positive number $r_1$ satisfying $1 > r_1 > \max\{\pi_1, \ldots, \pi_m\}$. Given $1 > r_2 > r_1$, let $C_i$ and $C_{out}$ be circles with radius $r_1$ and $r_2$ and center at the origin. Note that the involved function is continuous in the given bounded contours and as $N \to \infty$
\[\left( 1 - \frac{1}{z} \right)^{(1 - \frac{\delta_l^2}{\alpha^2})} \to e^{-\frac{z}{2}},\]
take limit in the integrand of the right-hand side of \((1.1)\) and we have the required conclusion \((1.21)\).

For Part (iii) where $\mu N \to 0$ and $1 - \delta_l^2/\alpha^2 = 4\mu N \pi_l$ with $\pi_l \in (0, \infty)$ for $l = 1, \ldots, m$, change $u$ to $4\mu N u$ and $v$ to $4\mu N v$ in the integrand of the right-hand side of \((1.1)\), recalling \((1.14)\) and the assumptions on $\delta_1, \ldots, \delta_m$, we have
\[
\frac{1}{4N^2} K_N \left( \frac{1}{4N^2}, \frac{1}{4N^2} \right) = \frac{\mu}{N} \frac{2\alpha^2}{(2\pi i)^2} \int_{C_{out}} du \int_{C_{in}} dv
\]
\[K_{\nu} \left( \frac{\alpha N}{\sqrt{1 - 4\mu N u}} \frac{\xi}{\sqrt{1 - 4\mu N v}} \right) I_{\nu} \left( \frac{\alpha N}{\sqrt{1 - 4\mu N v}} \eta \right) \frac{1}{u - v} \frac{(v)^{\nu + m}}{\nu + m} \]
\[\times \left( 1 - \frac{1}{4\mu N u} \right)^{\kappa/2} \left( \frac{1 - \frac{1}{(1 + \mu)^2 N u}}{1 - \frac{1}{(1 + \mu)^2 N v}} \right)^{N - m} \prod_{l=1}^{m} \frac{u - \pi_l}{v - \pi_l},\]
where the two contours are chosen such that $\text{Re}(z) < (4\mu N)^{-1}$ for any $z$ in $C_{out}$ and $C_{in}$, and $C_{in}$ encircles $0, (1 + \mu)^{-2} N^{-1}, \pi_1, \ldots, \pi_m$ and is wholly within $C_{out}$ (here the same notations of the contours are used, for simplicity). Moreover, since $\mu N \to 0$ as $N \to \infty$, for $N$ large enough we can always assume that both the contours are selected and are independent of $N$. For instance, let $c_0 = \max\{1, \pi_1, \ldots, \pi_m\}$, when $4\mu N \leq 1/(2c_0 + 8)$, we can take $C_{out}$ and $C_{in}$ as two circles centered at zero with radius $c_0 + 2$ and $c_0 + 1$ respectively.

We need to make use of asymptotic formulas of modified Bessel functions, see e.g. \cite{49} 10.40 (i). For any given constant $\delta$ such that $0 < \delta < \pi/2$, then as
\[ z \to \infty, \text{ with } \nu \text{ fixed, the following hold uniformly with respect to } \arg(z) \text{ in the corresponding sectors} \]
\[ I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}} \left( 1 + \mathcal{O}\left(\frac{1}{z}\right) \right), \quad |\arg(z)| \leq \frac{1}{2} \pi - \delta, \quad (4.10) \]
and
\[ K_\nu(z) = \frac{\pi}{2} \frac{1}{z} e^{-z} \left( 1 + \mathcal{O}\left(\frac{1}{z}\right) \right), \quad |\arg(z)| \leq \frac{3}{2} \pi - \delta. \quad (4.11) \]

Notice the assumption that \( N \to 0 \) as \( N \to \infty \), for any given two closed contours \( C_{\text{out}} \) and \( C_{\text{in}} \) (say, the two circles with radius \( c_0 + 2 \) and \( c_0 + 1 \) described previously), independent of \( N \), we can choose sufficiently large \( N \) such that \(|4\mu N u| \leq 1/2 \) and \(|4\mu N v| \leq 1/2 \) uniformly for \( u \in C_{\text{out}} \) and \( v \in C_{\text{in}} \). These show that \(|\arg(1 - 4\mu N u)| \leq \pi/8 \) and \(|\arg(1 - 4\mu N v)| \leq \pi/8 \) for any \( u \in C_{\text{out}}, v \in C_{\text{in}} \). Noting \( \alpha/N \to \infty \) as \( N \to \infty \) and the Taylor expansion
\[ \alpha/N \sqrt{1 - 4\mu N v} = \frac{\alpha}{N} - (1 + \mu)v + \mathcal{O}(\mu N), \quad \mu N \to 0, \]
applying (4.10) thus gives rise to
\[ I_\kappa' \left( \frac{\alpha}{N} \sqrt{(1 - 4\mu N v)\eta} \right) \sim \frac{1}{\sqrt{2\pi}} \left\{ \frac{N}{\alpha} \right\}^{-\frac{1}{4}} e^{\sqrt{\eta \xi}} e^{-\sqrt{\eta v}} \]
uniformly for any \( v \in C_{\text{in}} \). Likewise, we see from (4.11) that
\[ K_{-\kappa} \left( \frac{\alpha}{N} \sqrt{(1 - 4\mu N u)\xi} \right) \sim \frac{\pi}{2} \left\{ \frac{N}{\alpha} \right\}^{-\frac{1}{4}} e^{-\sqrt{\xi \eta}} e^{-\sqrt{\xi u}} \]
uniformly for any \( u \in C_{\text{out}} \). Taken together, the desired result immediately follows from application of Lebesgue’s dominated convergence theorem.

In the case of Part (iv) where \( \mu N \to 0 \) and \( 1 - \delta_l^2/\alpha^2 \to \pi_l \in (0, 1) \) for \( l = 1, \ldots, m \), as in Part (ii) for large \( N \) sufficient we can let \( C_{\text{in}} \) and \( C_{\text{out}} \) be two circles with center and radius independent of \( N \). Note that \( \mu N \to 0 \) implies
\[ \left( 1 - \frac{1}{\frac{1}{2} \alpha^2} \right)^{N-m} \to 1, \]
we get the required conclusion (4.23) by taking limit in the integrand of the right-hand side of (4.11).

Obviously, the results hold uniformly for \( \xi, \eta \) in a given compact set of \((0, \infty)\). Therefore, we have completed the proof of the given statement. \( \square \)

We now state a similar result associated with correlation kernel (3.17).

**Theorem 4.1.** With the kernel (3.17) and the assumptions given in (1.14), and with fixed nonnegative integers \( \nu, \nu' \) and \( \kappa \) such that \( \nu + \nu' \geq \kappa \), the following hold uniformly for any \( \xi \) and \( \eta \) in a compact set of \((0, \infty)\) as \( N \to \infty \).

(i) If \( \mu N \to \infty \), then
\[ \frac{1 + \mu}{2N^2} K_N \left( \frac{1 + \mu}{2N^2} \xi, \frac{1 + \mu}{2N^2} \eta \right) \to \left( \frac{\eta}{\xi} \right)^{\kappa/2} K_1(\xi, \eta). \]

(ii) If \( \mu N \to \tau/4 \) with \( \tau > 0 \) and \( 1 - \delta_l^2/\alpha^2 \to \pi_l \in (0, 1) \) for \( l = 1, \ldots, m \), then
\[ \frac{1}{\alpha N} K_N \left( \frac{\xi}{\alpha N}, \frac{\eta}{\alpha N} \right) \to \left( \frac{\eta}{\xi} \right)^{\kappa/2} K_{II}(\tau; \xi, \eta). \]
(iii) If $\mu N \to 0$ but $\mu N^2 \to \infty$, and $1 - \delta_l^2 / \alpha^2 = 4\mu N \pi_l$ with $\pi_l \in (0, \infty)$ for $l = 1, \ldots, m$, then

$$
eq \frac{e^{\pi\sqrt{\xi}}}{e^{\pi\sqrt{\eta}} / 16\alpha \mu^2 N^3} K_N \left( \frac{\xi}{16\alpha \mu^2 N^3}, \frac{\eta}{16\alpha \mu^2 N^3} \right) \to \left( \frac{\eta}{\xi} \right)^{\kappa/2} K_{III}(\xi, \eta).$$

(iv) If $\mu N \to 0$ and $1 - \delta_l^2 / \alpha^2 \to \pi_l \in (0, 1)$ for $l = 1, \ldots, m$, then for $m \geq 1$

$$
\frac{1}{\alpha N} K_N \left( \frac{\xi}{\alpha N}, \frac{\eta}{\alpha N} \right) \to \left( \frac{\eta}{\xi} \right)^{\kappa/2} K_{IV}(\xi, \eta).
$$

Proof. We proceed in a similar way as in the proof of Theorem [13]. First, recall the assumptions [11.1] and rewrite the kernel (3.17) as

$$K_N(x, y) = \frac{\alpha}{(2\pi i)^2} \left( \frac{y}{x} \right)^{\kappa} \int_{\mathcal{C}_{\text{out}}} du \int_{\mathcal{C}_{\text{in}}} dv f_2(\cdot, \kappa; \alpha(1-u)x) f_1(\cdot, \kappa; \alpha(1-v)y)$$

$$\times \frac{1}{u-v} \left( \frac{v}{u} \right)^{\nu+m} \left( 1 - \frac{1}{u} \left( 1 - \frac{\delta^2}{\alpha^2} \right) \right)^{N-m} \prod_{l=1}^{m} \left( u - \left( 1 - \frac{\delta^2}{\alpha^2} \right) \right), \quad (4.12)$$

For Part (i), without loss of generality we assume that $\mu < 1$ and $1 - \delta_l^2 / \alpha^2 \to \pi_l < 1$ for $l = 1, \ldots, m$ (otherwise, see the proof of Part (i) of Theorem [13]), then we can choose the contours such that $\mathcal{C}_{\text{in}}$ is wholly within $\mathcal{C}_{\text{out}}$. In order to take limits smoothly, substituting (3.15) and (3.13), changing $t$ to $t/N$, $u$ to $(1 - \delta_l^2 / \alpha^2)Nu$ and $v$ to $(1 - \delta_l^2 / \alpha^2)Nv$, we then use Fubini’s theorem to get

$$
\frac{1 + \mu}{2N^2} K_N \left( \frac{1 + \mu}{2N^2} \xi, \frac{1 + \mu}{2N^2} \eta \right) = \left( \frac{\eta}{\xi} \right)^{\kappa} \frac{1}{2\pi i} \int_{0}^{\infty} \frac{dt}{t} \int_{\mathcal{C}_{\text{in}}} \frac{du}{s} t^{s-\kappa} e^{-t} \hat{K}_N(s, t)
$$

where

$$
\hat{K}_N(s, t) = N^{-\kappa-1} \Gamma(\mu + \nu' + N + 1) \Gamma(\mu + \nu' - \kappa + N) \left( 1 - \frac{t}{N} \right)^{\nu + \nu' + N - \kappa - 1} e^{t^2 [N]}(t) \int du \int dv
$$

$$\frac{1}{(2\pi i)^2} \exp \left\{ \left( u - \frac{(1 + \mu)^2}{4\mu N} \right) \xi \right\} \left( 1 - \left( \frac{(1 + \mu)^2}{4\mu N} - v \right) \frac{\eta v}{s} \right)^{-\nu - \nu' - N - 1}
$$

$$\times \frac{1}{u-v} \left( \frac{v}{u} \right)^{\nu+m} \left( 1 - \frac{1}{Nv} \right)^{N-m} \prod_{l=1}^{m} \left( u - \left( \frac{(1 + \mu)^2}{4\mu N} \right) (1 - \frac{\delta_l^2}{\alpha^2}) \right). \quad (4.13)$$

Since $\mu N \to \infty$ as $N \to \infty$, noting that $0 < \mu \leq 1$ and $0 \leq \delta_l < \alpha$ ($l = 1, \ldots, m$), application of Lebesgue’s dominated convergence theorem provides us

$$\hat{K}_N(s, t) \to \hat{K}_\infty(s, t) := \frac{1}{(2\pi i)^2} \int_{\mathcal{C}_{\text{out}}} du \int_{\mathcal{C}_{\text{in}}} dv e^{\frac{u}{t} - \frac{1}{t} - \frac{\nu}{t} + \frac{1}{u-v} \left( \frac{v}{u} \right)^{\nu}},$$

from which the requested conclusion follows (cf. eq. (1.9)).

We easily verify Parts (ii) and (iv) as in the proof of Theorem [13]. For Part (iii), with (3.15) and (3.16) in mind, change of variables $t \mapsto t/(\mu N^2), t \mapsto t/(\mu N^2), \ldots$. 


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\( u \mapsto 4\mu Nu \) and \( v \mapsto 4\mu Nv \) in [1.12], as \( N \to \infty \) we have

\[
\frac{1}{16\alpha^2 N^3} K_N \left( \frac{\xi}{16\alpha^2 N^3}, \frac{\eta}{16\alpha^2 N^3} \right)
\]

\[
\sim \left( \frac{\eta}{\xi} \right)^{\kappa} \frac{1}{4\mu N^2} \frac{1}{\Gamma(\kappa + 1)} \frac{1}{(2\pi)^2} \int_{\Gamma} du \int_{\Gamma} dv g_1,N(u)g_2,N(v)
\]

\[
\times \left( 1 - \frac{1}{(1 + \frac{\eta}{\xi} u)^N} \right)^{N-m} \frac{1}{u-v} \left( \frac{v}{u} \right)^{\nu+m} \prod_{l=1}^{m} \frac{u - \pi_l}{v - \pi_l},
\]

where

\[
g_1,N(u) = \int_0^{\mu N^2} dt t^{\kappa-1} \left( 1 - \frac{t}{\mu N^2} \right)^{\nu+\nu' + N - \kappa - 1} \exp \left\{ \frac{\eta \xi}{4t} - \frac{\eta \xi}{16\mu N t} \right\}
\]

and

\[
g_2,N(v) = \frac{1}{2\pi i} \int_{C_0} ds s^{-\kappa-1} e^{\pi i s} \left( 1 + \frac{vn s}{4Ns} - \frac{\eta}{16\mu N^2 s} \right)^{-\nu-\nu' - \kappa - 1}.
\]

It suffices to find the leading coefficients for both functions. For this purpose we use the method of steepest decent; see e.g. [56]. For \( g_1,N(u) \), we use the inequality \( 1 - x \leq e^x \) \((0 \leq x \leq 1)\) to get

\[
|g_1,N(u)| \leq \int_0^{\mu N^2} dt |e^{\frac{\xi}{2t}}| t^{\kappa-1} \exp \left\{ - \frac{\mu N}{t} \left( t + \frac{\xi}{16t} \right) - (\nu + \nu' - \kappa - 1) \frac{t}{\mu N^2} \right\}.
\]

Note that the function \( t + \frac{\xi}{16t} \) attains its unique minimum at \( t_0 = \sqrt{\xi}/4 \) over \((0, \infty)\) and both \( 1/(\mu N) \) and \( \mu N^2 \) go to infinity, the leading contribution must come from the neighbourhood of \( t_0 \). By Taylor expansion, we easily see that

\[
g_1,N(u) \sim 2^{1-2\kappa} \sqrt{\pi \mu N \xi} \frac{\xi}{2} e^{\sqrt{\xi} u - \frac{\xi}{2\sqrt{\xi} u}}.
\]

For \( g_2,N(v) \), noting

\[
g_2,N(v) \sim \frac{1}{2\pi i} \int_{C_0} ds s^{-\kappa-1} \exp \left\{ \frac{1}{\mu N} \left( s + \frac{\eta}{16s} \right) - \frac{vn s}{4s} \right\},
\]

let \( C_0 \) be a circle of radius \( \sqrt{\eta}/4 \), it is easy to verify that \( \text{Re} \{ s + \frac{\eta}{16s} \} \) attains a unique maximum at \( s_0 = \sqrt{\eta}/4 \). Thus the steepest decent argument leads us to

\[
g_2,N(v) \sim 2^{2\kappa} \sqrt{\mu N/\pi \xi} \frac{\xi}{2} e^{-\sqrt{\xi} v - \frac{\xi}{2\sqrt{\xi} v}}.
\]

Substitution of (4.15) and (4.16) in (4.14) completes Part (iii).

Obviously, the results hold uniformly for \( \xi, \eta \) in a given compact set of \((0, \infty)\). \( \square \)

Compare Part (iii) in Theorems 1.3 and 1.4 there is a technical restriction on the rate of \( \mu N \) in the latter. We believe this can be removed such that the same result holds true as in the former.

5. ON THE FOUR LIMITING KERNELS

5.1. Comparison. We first introduce a few families of contour integrals and rewrite the kernels defined as before. Setting

\[
\tilde{K}^{(k)}(x) = \frac{1}{\pi i} \int_{C_0} du K_{-\kappa}(2\sqrt{1-u}x)(1-u)^{\kappa/2} u^{-\nu-m} e^{-x} \prod_{l=1}^{k-1} (u - \pi_l),
\]
\[ \Lambda_{II}^{(k)}(x) = \frac{1}{2\pi i} \int_{C_\nu} dv I_\kappa(2\sqrt{(1-v)x})(1-v)^{-\kappa/2} u^{\nu+m} e^{\frac{1}{u-v}} \prod_{l=1}^{k} \frac{1}{v-\pi_l} \] (5.2)

where \( C_\nu \) denotes a contour enclosing the origin and \( C_\tau \) encloses \( \pi_1, \ldots, \pi_k \), and

\[ K_{II}^{(0)}(\tau; \xi, \eta) = \frac{2}{(2\pi)^2} \int_{C_\nu} du \int_{C_\tau} dv K_\kappa(2\sqrt{(1-u)\xi}) I_\kappa(2\sqrt{(1-v)\eta}) \times e^{-\frac{\tau}{\xi+\frac{1}{\nu}}} \frac{1}{u-v} \left( 1 - u \right)^{\kappa/2} \left( \frac{u}{v} \right)^{-\nu-m}, \] (5.3)

then the use of the identity (see e.g. [21, Eq.(5.12)])

\[ \frac{1}{u-v} \prod_{l=1}^{m} \frac{u-\pi_l}{v-\pi_l} = \frac{1}{u-v} + \sum_{k=1}^{m} \prod_{l=1}^{k-1} (u-\pi_l) \prod_{l=k}^{m} \frac{1}{v-\pi_l} \] (5.4)

immediately gives us

\[ K_{II}(\tau; \xi, \eta) = K_{II}^{(0)}(\tau; \xi, \eta) + \sum_{k=1}^{m} \Lambda_{II}^{(k)}(\xi) \Lambda_{II}^{(k)}(\eta). \] (5.5)

Likewise, setting

\[ \tilde{\Lambda}_{III}^{(k)}(x) = \frac{1}{2\pi i} \frac{1}{2\xi} \int_{C_\nu} du e^{\sqrt{\xi} u - \frac{\tau}{\xi} u^{-\nu-m} \prod_{l=1}^{k-1} (u-\pi_l)}, \] (5.6)

\[ \Lambda_{III}^{(k)}(x) = \frac{1}{2\pi i} \frac{1}{2\eta} \int_{C_\nu} du e^{\sqrt{\eta} v + \frac{\tau}{\eta} v^{\nu+m} \prod_{l=1}^{k} \frac{1}{v-\pi_l}}, \] (5.7)

and

\[ K_{III}^{(0)}(\xi, \eta) = \frac{2}{(2\pi)^2} \frac{1}{4(\xi)\frac{\tau}{\eta}} \int_{C_\nu} du \int_{C_\tau} dv e^{\sqrt{\xi} u - \sqrt{\eta} v - \frac{\tau}{\xi} u^{-\nu-m} \prod_{l=1}^{k-1} (u-\pi_l)}, \] (5.8)

then

\[ K_{III}(\xi, \eta) = K_{III}^{(0)}(\xi, \eta) + 2 \sum_{k=1}^{m} \tilde{\Lambda}_{III}^{(k)}(\xi) \Lambda_{III}^{(k)}(\eta). \] (5.9)

Again, by defining

\[ \tilde{\Lambda}_{IV}^{(k)}(x) = \frac{1}{\pi i} \int_{C_\nu} dv K_\kappa(2\sqrt{(1-u)x})(1-u)^{\kappa/2} u^{-\nu-m} \prod_{l=1}^{k-1} (u-\pi_l), \] (5.10)

\[ \Lambda_{IV}^{(k)}(x) = \frac{1}{2\pi i} \int_{C_\nu} dv I_\kappa(2\sqrt{(1-v)x})(1-v)^{-\kappa/2} v^{\nu+m} \prod_{l=1}^{k} \frac{1}{v-\pi_l} \] (5.11)

we have

\[ K_{IV}(\xi, \eta) = \sum_{k=1}^{m} \tilde{\Lambda}_{IV}^{(k)}(\xi) \Lambda_{IV}^{(k)}(\eta). \] (5.12)

Next, we compare the four limiting kernels with the known limiting kernels in random matrix theory one after another. When \( \kappa = 0 \) and \( m = 0 \), according to the result of Akemann and Strahov (cf. [5, Theorem 3.9]), the kernel \( K_I(x, y) \) is
expected to be the Meijer G-kernel $K_{\nu,\kappa}(x, y)$ up to a transformation like $f(x)/f(y)$ for some function $f(x)$ where

$$
K_{\nu,\kappa}(x, y) = \frac{1}{(2\pi i)^2} \int_{-1/2-i\infty}^{-1/2+i\infty} du \int_{\Sigma} dv \frac{\sin \pi u}{\sin \pi v} \times \frac{\Gamma(u + 1)\Gamma(\nu + u + 1)\Gamma(\kappa + u + 1) x^u y^{-u-1}}{\Gamma(\nu + v + 1)\Gamma(\kappa + v + 1)}
$$

with $\Sigma$ a contour enclosing the positive real axis but not $u$. Actually, since the case of $\mu = 1$ (cf. (1.1) and (1.14) in Sect. 1) reduces to the product of two independent Gaussian rectangular matrices, $K_I(x, y)$ is strongly believed to be $K_{\nu,\kappa}(x, y)$, which was first found by Kuijlaars and Zhang [43] in this context, up to a factor $f(x)/f(y)$. They are indeed equal according to the following proposition. Actually, this type of convolution representation has been obtained in the product of two independent random matrices for finite matrix size $N$, see [17, Theorem 2.8(b)]. Thus the limiting case is also expected.

**Proposition 5.1.** For the correlation kernels (1.15) and (5.13), we have

$$
K_I(\xi, \eta) = (\eta/\xi)^{\kappa/2} K_{\nu,\kappa}(\eta, \xi).
$$

**Proof.** Start from the representation of the Bessel kernel (see e.g. [13, Example 3.1] and [43, Sect. 5.3])

$$
4K_{\nu}^{(\text{Bessel})}(x, y) = \int_0^1 J_{\nu}(2\sqrt{wx})J_{\nu}(2\sqrt{yw})dw,
$$

we have

$$
(\eta/\xi)^{-\kappa/2} K_I(\xi, \eta) = \int_0^1 dw \frac{1}{2\pi i} \int_{\mathcal{C}_0} ds s^{-\kappa-1} e^{s\eta w/s} J_{\nu}(2\sqrt{\eta w})
\times \int_0^\infty dt t^{\kappa-1} e^{-t} \left(\frac{\xi w}{t}\right)^{\nu/2} J_{\nu}(2\sqrt{\xi w/t}).
$$

Integrate term by term and then use the relation between hypergeometric functions and Meijer G-functions (cf. [40, Sect. 5.2]), we get

$$
\frac{1}{2\pi i} \int_{\mathcal{C}_0} ds s^{-\kappa-1} e^{s\eta w/s} J_{\nu}(2\sqrt{\eta w}) = \frac{1}{\Gamma(\kappa + 1)} \frac{1}{\Gamma(\nu + 1)} {}_2F_1(\kappa + 1, \nu + 1; -\eta w)
= G_{0,3}^{1,0}(0, -\nu, -\kappa | \eta w).
$$

On the other hand, noting

$$
\left(\frac{\xi w}{t}\right)^{\nu/2} J_{\nu}(2\sqrt{\xi w/t}) = G_{0,2}^{1,0}(\nu, 0 | \xi w/t),
$$

the Mellin convolution formula (see e.g. [42, Appendix eqn(A.3)]) gives us

$$
\int_0^\infty dt t^{\kappa-1} e^{-t} G_{0,2}^{1,0}(\nu, 0 | \xi w/t) = G_{0,3}^{2,0}(\nu, \kappa, 0 | \xi w).
$$
Therefore,
\[(\eta/\xi)^{-\kappa/2} K_1(\xi, \eta) = \int_0^1 dw \, G_{0,3}^{1,0}(0, -\nu, -\kappa | \eta w) G_{0,3}^{0,2}(\nu, \kappa, 0 | \xi w). \] (5.20)

By Theorem 5.3 of [43] (noting \(\nu_3 = 0\) therein), the RHS of (5.20) is indeed another integral representation of the kernel \(K_{\nu,\kappa}(\eta, \xi)\), from which the desired result immediately follows. \(\square\)

Here it’s worth stressing that the Meijer G-kernels \(K_{\nu,\kappa}(x, y)\) already appeared in the works of Bertola, Gekhtman and Szmigielski on the Cauchy-Laguerre two matrix model [12], Kuijlaars and Zhang [41] on products of two independent Gaussian rectangular matrices, Forrester on the product with the inverse [27]. This shows that the kernel is universal. It’s probably worth pointing out that although the Borodin’s kernel from [13] can be written in terms of Meijer G-functions, it does not agree with kernels stemming from products of random matrices as the indices obtained are different. See [42] for the inter-relation between Borodin’s kernel and Meijer G-kernels.

Under the same conditions of \(\kappa = 0\) and \(m = 0\), at the critical scale of \(\mu = g/N\) with \(g \in (0, \infty)\), for the rescaled kernel \((1/(4N^2))K_N(x^2/(4N^2), y^2/(4N^2))\)

Akemann and Strahov obtained the hard edge limiting kernel defined by
\[S(x, y; g) = \left(\frac{4}{(x^2 - y^2)^2}g \right) \int_{\Sigma} du \, \int_{\Sigma} dv \, \frac{\Gamma(-u)\Gamma(-v)}{(u + \nu + 1)\Gamma(v + \nu + 1)} x^u y^v
\times \left(A(u, v, \nu) - g(u^2 + v^2 - uv + \nu u)\right) I_v \left(\frac{x}{2g}\right) K_{a+u} \left(\frac{y}{2g}\right) \] (5.21)

where \(\Sigma\) is a contour enclosing the positive real axis and
\[A(u, v, \nu) = \frac{1}{4}(v - u)(u^2 + v^2 + (\nu - 1)(u + v) - \nu), \] (5.22)
see [6, Theorem 1.5 (b)]. In this case it remains as a challenge for us to directly verify the equivalence of both the critical kernels \(S(x, y; g)\) and \(K_{11}^{(0)}(\tau; x, y)\). However, using integral representations of Bessel functions and noting (4.8) and (4.9), it is easy to rewrite the kernel defined by (5.3) in terms of the Bessel kernel as
\[K_{11}^{(0)}(\tau; \xi, \eta) = \left(\frac{\xi}{\eta}\right)^{\kappa/2} \frac{1}{2\pi i} \int_0^\infty dt \int_{c_0}^\infty ds t^{\kappa-1} s^{-\kappa-1} e^{\eta s - \xi t + \frac{4}{4}} \times 4\tau \left(\frac{s}{t}\right)^{(v+m)/2} K_{v+m}^{(B)} \left(\left.4\tau \frac{\xi}{\eta} \right. \right). \] (5.23)

With change of variables, the kernel \(K^{(0)}_{11}\) has been identified by Desrosiers and Forrester [21] as the hard edge limiting kernel for the spiked complex sample covariance matrices. In particular, we have the following relation (cf. [21] eqns (1.20) and (6.20))
\[K^{(0)}_{11}(\xi, \eta) = (\xi/\eta)^{(\nu+m)/4} 2(\xi\eta)^{-1/4} K_{\nu+m}^{(B)} \left(4\sqrt{\eta}, 4\sqrt{\xi}\right). \] (5.24)

The fourth kernel \(K_{IV}(x, y)\) is essentially the kernel (1.13) for the product of two coupled Gaussian random matrices but with \(N \rightarrow m, \alpha \rightarrow 1\) and \(\delta_i^2 \rightarrow 1 - \pi_l, l = 1, \ldots, m\). Moreover, as the correlation kernel of a determinantal point process it corresponds to the joint PDF given in (1.11). So in that sense, it appears as one of limiting kernels for the smallest singular values in random matrix theory, like the Gaussian Unitary Ensemble with source for the largest eigenvalues or the noncentral
Wishart matrices (also being called as shifted mean chiral Gaussian matrices) for
the smallest singular values; see [10] and [28]. Particularly for the case of \( m = 1 \),
note (1.11) and Remark 1.6 under the same assumptions as in Theorem 1.3 (iv) we have

\[
P(x_1 \leq 4\mu^2, \ldots, x_N \leq 4\mu^2) \rightarrow \frac{2(1 - \pi_1)}{\Gamma(\nu + 1)\pi_1^{\nu/2}} \int_0^y t^{\nu/2} K_{\nu-k}(2\sqrt{t}) I_k(2\sqrt{\pi_1 t}) dt
\]

as \( N \to \infty \).

Finally, we conclude this subsection with a transition from the critical kernel
\( K_{II}(x, y) \) to the other three kernels, which shows that \( K_{II}(x, y) \) is an interpolation
between them. This is to be expected, as then the parameter effectively \( \mu \sim \tau/(4N) \)
and the coupled product tends to the classical Laguerre Unitary Ensemble as \( \mu \to 0 \) while it corresponds to the product of two independent Gaussian random matrices
as \( \mu \to 1 \). For \( \kappa = 0 \) and \( m = 0 \), similar results been obtained by Akemann and
Strahov [6].

**Theorem 5.2.** With the kernels defined in (1.15)–(1.19), the following hold uni-
formly for any \( x \) and \( y \) in a compact set of \((0, \infty)\).

(i) \[
\lim_{\tau \to \infty} \frac{1}{\tau} K_{II}(\tau; \frac{x}{\tau}, \frac{y}{\tau}) = K_1(x, y).
\]

(ii) Given \( q \leq m \), suppose that \( \pi_l = \tau \hat{\pi}_l \) for \( l = 1, \ldots, q \) and \( \pi_{q+1}, \ldots, \pi_m \) are
fixed, then

\[
\lim_{\tau \to 0} e^{\hat{\pi}^\tau(x-y)} \frac{1}{\tau} K_{II}(\tau; \frac{x}{\tau^2}, \frac{y}{\tau^2}) = K_{III}(x, y) \big|_{m \to q, \nu \mapsto \nu + m - q, \pi \mapsto \hat{\pi}}.
\]

(iii) \[
\lim_{\tau \to 0} K_{II}(\tau; x, y) = K_{IV}(x, y).
\]

**Proof.** For Part (i), change \( u \) to \( \tau u \) and \( v \) to \( \tau v \) in the integrand of (1.17), we have

\[
\frac{1}{\tau} K_{II}(\tau; \frac{x}{\tau}, \frac{y}{\tau}) = \frac{2}{(2\pi i)^2} \int_{C_{out}} du \int_{C_{in}} dv K_{-k}(2\sqrt{1 - \tau u}) I_k(2\sqrt{1 - \tau v})
\times e^{-\frac{1}{\tau} + \frac{1}{\tau} (\frac{u}{\tau})^{\nu - m} \left( \frac{1 - \tau u}{1 - \tau v} \right)^{\nu/2} \prod_{j=1}^m \frac{\tau u - \pi_j}{\tau v - \pi_j}}.
\]

where \( C_{out} \) is a simple counterclockwise contour around the origin with \( \text{Re}(z) < 1/\tau, \forall z \in C_{out} \) and entirely within it \( C_{in} \) is a counterclockwise contour encircling
\( 0, \pi_1/\tau, \ldots, \pi_m/\tau \). Next, we will use the similar argument as in the proof of Part
(i) of Theorem 1.3 to complete it.

Substitute \( K_u \) and \( I_u \) into (5.26) with their integral representations respectively
given by (1.7) and (4.2). Use Fubini’s theorem and we rewrite the integral appearing
in (5.26) as

\[
\frac{1}{\tau} K_{II}(\tau; \frac{x}{\tau}, \frac{y}{\tau}) = \frac{1}{2\pi i} \int_0^\infty dt \int_{C_0} ds t^{\nu - 1} e^{-\tau s} K_{\frac{y}{s}}(\tau; \frac{x}{s})
\]

(5.27)
where
\[ K(\tau; \frac{y}{s}, \frac{x}{t}) = \frac{1}{(2\pi i)^2} \int_{C_{\text{out}}} du \int_{C_{\text{in}}} dv \, e^{\frac{y}{s} u - \frac{x}{t} v} \left( e^{\frac{y}{s} u - \frac{x}{t} v} - \frac{1}{u - v} \right) \]
\[ \times \frac{1}{u - v} \left( \frac{u}{v} \right)^{\nu + m} \prod_{l=1}^{m} \frac{u - \pi_l}{v - \pi_l} \, . \] (5.28)

Note that it is unnecessary to assume \( \text{Re}(z) < 1/\tau \) for \( z \in C_{\text{out}} \) in (5.28). In particular, it is seen from \( 0 < \pi_l < 1 \) (\( l = 1, \ldots, m \)) that when \( \tau > 2 \) we can choose \( C_{\text{in}} \) and \( C_{\text{out}} \) as two circles with radius 1 and 2 and both with center at the origin. Note that the involved function is continuous in the given contours and as \( \tau \to \infty \) application of Lebesgue’s dominated convergence theorem provides us
\[ K(\tau; \frac{y}{s}, \frac{x}{t}) \to \frac{1}{(2\pi i)^2} \int_{C_{\text{out}}} du \int_{C_{\text{in}}} dv \, e^{\frac{y}{s} u - \frac{x}{t} v} \left( e^{\frac{y}{s} u - \frac{x}{t} v} - \frac{1}{u - v} \right) \] \[ \times \frac{1}{u - v} \left( \frac{u}{v} \right)^{\nu + m} \prod_{l=1}^{m} \frac{u - \pi_l}{v - \pi_l} \, . \] This limit has been identified as the Bessel kernel by Desrosiers and Forrester [21] and the requested conclusion then follows (cf. (1.8) and (1.9) in Sect. I).

For Part (ii), change \( u \) to \( \tau u \) and \( v \) to \( \tau v \) in the integrand of (1.17), we have
\[ \frac{1}{\tau^2} K_{\Pi}(\tau; \frac{x}{\tau}, \frac{y}{\tau^2}) = \frac{2}{(2\pi i)^2} \int_{C_{\text{out}}} du \int_{C_{\text{in}}} dv \, K_{-\kappa}\left( \frac{2}{\tau} \sqrt{1 - \tau u}, \frac{2}{\tau} \sqrt{1 - \tau v} \right) \]
\[ \times e^{-\frac{1}{2} \left( \frac{u}{v} \right)^{\nu + m} \prod_{l=1}^{m} \frac{u - \pi_l}{v - \pi_l}} \, . \] As \( \tau \to 0 \), by (4.10) and (4.11) simple calculation gives us
\[ K_{-\kappa}\left( \frac{2}{\tau} \sqrt{1 - \tau u}, \frac{2}{\tau} \sqrt{1 - \tau v} \right) \sim \frac{\tau}{4(xy)^{\frac{\nu}{2}}} e^{-\frac{1}{4} \left( \sqrt{u} + \sqrt{v} \right)^2} \, . \] Obviously, the function of variables \( u \) and \( v \) in the limit above is continuous in the bounded contours \( C_{\text{out}} \) and \( C_{\text{in}} \), application of Lebesgue’s dominated convergence theorem thus provides us Part (ii) as \( \tau \to 0 \).

Lastly, taking limit in the definition of (1.17), we have Part (iii). \( \Box \)

5.2. Integrable form of the critical kernel. Recall that a correlation kernel \( K(x, y) \) is called integrable in the sense of Its, Isergin, Korepin and Slavnov [34] if it can be represented as
\[ K(x, y) = \sum_{i=1}^{k} f_i(x) g_i(y), \quad \text{with} \quad \sum_{i=1}^{k} f_i(x) g_i(x) = 0 \] (5.29)
for some integer \( k \geq 2 \) and certain functions \( f_i \) and \( g_i \). The kernels of standard universality classes in Random Matrix Theory, for instance, sines, Airy and Bessel kernels, all belong to the class of integrable kernels. Recently, the Meijer G-kernel \( K_I \) (cf. Eqns (5.13) and (5.14)) has turned out to be integrable, see [12, 43] or [51, 53] for relevant Hamiltonian differential equations. On the other hand, noting the fact that
\[ \sum_{i=1}^{k} f_i(x) g_i(y) = \frac{1}{x - y} \sum_{i=1}^{k} (xf_i(x)g_i(y) - f_i(x)yg_i(y)) \] (5.30)
and \( K^{(0)}_{III} \) is integrable, it is easy to verify from (5.9) and (5.12) that \( K_{III} \) and \( K_{IV} \) are also integrable. As for the critical kernel \( K_{II} \), we argue that the new limiting
kernel $K_{11}^{(0)}$ can be represented in an integrable form in terms of two functions and their derivatives up to third order, and so does the kernel $K_{11}$ because of (5.5). However, in the case of $\kappa = \nu = 0$ Akemann and Strahov gave an integrable form of their limiting kernel (5.21) (cf. [6, Sect. 1.5]).

In order to state the integrable form of the critical kernel, we need two functions defined by integrals involving Bessel functions for non-negative integers $\alpha, \kappa$

$$f(x) = \int_0^\infty dt \, t^{-(\xi/2)-3} e^{-xt- \xi/2} J_\alpha \left( \sqrt{\frac{4\tau}{t}} \right), \quad x > 0,$$

and

$$g(x) = \frac{1}{2\pi i} \int_{C_0} ds \, s^{(\eta/2)-3} e^{xs+ \eta/2} J_\alpha \left( \sqrt{\frac{4\tau}{s}} \right), \quad -\infty < x < \infty.$$  

**Proposition 5.3.** Let $f(x)$ and $g(x)$ be defined by (5.31) and (5.32), and let $\alpha = \nu + m$. Then

$$K_{11}^{(0)}(\tau; \xi, \eta) = \left( \frac{\xi}{\eta} \right)^{\kappa/2} \frac{1}{\eta - \xi} \left( \xi \eta f''''(\xi) g'''(\eta) + f'''(\eta) (g(\eta) - (\alpha - 2\kappa - \tau - 1)g'(\eta)) + g''(\eta) (f(\xi) + (\alpha - 2\kappa - \tau + 1)f'(\xi)) - (\xi + \eta + \alpha \kappa - \kappa^2) f''(\xi) g''(\eta) - f'(\xi) g'(\eta) \right),$$

and moreover, $f(x)$ and $g(x)$ are respectively particular solutions of the fourth order ODEs

$$x^2 f^{(4)} - (\alpha - 2\kappa - 1)x f''' - (2x + \alpha \kappa - \kappa^2) f'' + (\alpha - 2\kappa - \tau + 1)f' = 0$$

and

$$x^2 g^{(4)} + (\alpha - 2\kappa + 1)x g''' - (2x + \alpha \kappa - \kappa^2) g'' - (\alpha - 2\kappa - \tau - 1)g' + g = 0.$$  

**Proof.** For convenience, we use the shorthand notation $J(s) = J_\alpha(2\sqrt{\tau}/s)$. First, we see from the Bessel differential equation

$$z^2 J''_\alpha(z) + z J'_\alpha(z) + (z^2 - \alpha^2) J_\alpha(z) = 0$$

that

$$\frac{d}{ds} \left( \sqrt{\frac{4\tau}{s}} J'_\alpha \left( \sqrt{\frac{4\tau}{s}} \right) \right) = \left( \frac{4\tau}{2s^2} - \frac{\alpha^2}{2s} \right) J(s).$$  

Together with the formula

$$\sqrt{\frac{4\tau}{s}} J'_\alpha \left( \sqrt{\frac{4\tau}{s}} \right) = -2s \frac{d}{ds} J(s),$$

simple calculations give us

$$\left( \frac{\partial}{\partial s} + \frac{\partial}{\partial t} \right) \left( J_\alpha \left( \sqrt{\frac{4\tau}{s}} \right) \sqrt{\frac{4\tau}{t}} J'_\alpha \left( \sqrt{\frac{4\tau}{t}} \right) - J_\alpha \left( \sqrt{\frac{4\tau}{t}} \right) \sqrt{\frac{4\tau}{s}} J'_\alpha \left( \sqrt{\frac{4\tau}{s}} \right) \right)$$

$$= 2(s - t) \frac{\partial}{\partial s} J(s) \frac{\partial}{\partial t} J(t) + \left( \frac{2\tau}{t^2} - \frac{\alpha^2}{2t} - \frac{2\tau}{s^2} + \frac{\alpha^2}{2s} \right) J(s) J(t).$$

Noting the simple fact

$$(\eta - \xi)e^{\eta s - \xi t} = \left( \frac{\partial}{\partial s} + \frac{\partial}{\partial t} \right) e^{\eta s - \xi t},$$
combine (5.38) and (5.39), integrate by parts and we thus get from (5.23) that

\[
(\xi/\eta)^{-\kappa/2}(\eta - \xi)K_{11}^{(1)}(\tau, \xi, \eta) = \frac{1}{2\pi i} \int_0^\infty dt \int_{c_0} ds e^{\eta s - \xi t} \left( \frac{\partial}{\partial s} + \frac{\partial}{\partial t} \right) \left( \frac{s}{t} \right)^{\frac{\alpha}{2} - k}
\]

\[
\times \left( \frac{4\tau}{s} \right) \sqrt{\frac{4\tau}{s} t} J_\alpha \left( \sqrt{-\frac{4\tau}{s} t} \right) - J_\alpha \left( \sqrt{-\frac{4\tau}{s} t} \right) \right) \left( \frac{\tau(s + t)}{s^2t^2} - \frac{\alpha^2}{4st} \right) J(s)J(t)
\]

\[
+ \frac{\partial}{\partial s} J(s) \frac{\partial}{\partial t} J(t) + \left( \frac{s + t}{s^2t^2} - \frac{\tau}{st} \right) \left( J(t)s \frac{\partial}{\partial s} J(s) - J(s)t \frac{\partial}{\partial t} J(t) \right)
\].

(5.40)

Now integrate by parts a second time, with (5.31) and (5.32) in mind, we arrive at

\[
\int_0^\infty dt t^{-\frac{\alpha}{2} - k} e^{-\xi t} \frac{\partial}{\partial t} J(t) = f'(\xi) + \left( \frac{\alpha}{2} - \kappa \right)f''(\xi) - \xi f'''(\xi),
\]

\[
\int_0^\infty dt t^{-\frac{\alpha}{2} - k} \frac{\partial}{\partial t} J(t) = -f(\xi) - \left( \frac{\alpha}{2} - \kappa + 1 \right)f'(\xi) + \xi f''(\xi),
\]

and

\[
\frac{1}{2\pi i} \int_{c_0} ds s^{\frac{\alpha}{2} - k} e^{\eta s + \frac{1}{2}} \frac{\partial}{\partial s} J(s) = g'(\eta) - \left( \frac{\alpha}{2} - \kappa \right)g''(\eta) - \eta g'''(\eta),
\]

\[
\frac{1}{2\pi i} \int_{c_0} ds s^{\frac{\alpha}{2} - k} e^{\eta s + \frac{1}{2}} \frac{1}{s} \frac{\partial}{\partial s} J(s) = g(\eta) - \left( \frac{\alpha}{2} - \kappa - 1 \right)g'(\eta) - \eta g''(\eta).
\]

Substitution of the above formulas into (5.40), careful calculations result in the desired formula (5.33).

Next, we turn to the proof of (5.33) while that of (5.34) is similar. Recalling (5.38), integrate by parts two times and we get

\[
\frac{1}{2\pi i} \int_{c_0} ds s^{\frac{\alpha}{2} - k} e^{\eta s + \frac{1}{2}} \frac{d}{ds} \left( \sqrt{\frac{4\tau}{s}} J_\alpha \left( \sqrt{\frac{4\tau}{s}} s \right) \right)
\]

\[
= -\frac{1}{2\pi i} \int_{c_0} ds s^{\frac{\alpha}{2} - k} e^{\eta s + \frac{1}{2}} \left( x - \frac{1}{s^2} + \frac{\alpha}{2} - \kappa \right) (-2s) \frac{d}{ds} J(s)
\]

\[
= -\frac{2}{2\pi i} \int_{c_0} ds s^{\frac{\alpha}{2} - k} e^{\eta s + \frac{1}{2}} J(s) \left( x^2 + \frac{1}{s^2} + s \left( x - \frac{1}{s^2} + \frac{\alpha}{2} - \kappa \right)^2 \right)
\]

\[
= -2 \left( x^2 g(4) + (\alpha - 2\kappa + 1) x g''' + \left( \frac{\alpha}{2} - \kappa \right)^2 - 2x \right) g'' - (\alpha - 2\kappa - 1) g' + g.
\]

By using (5.38), the RHS of the above equation is equal to \(2\tau g' - \frac{\alpha^2}{2} g''\). Then (5.33) follows.

Finally, we conclude this subsection with a remark on the two functions \(f(x)\) and \(g(x)\) given in (5.31) and (5.32). Define the pairing

\[
[f(x), g(y)] = xyf'''(x)g'''(y) + (g(y) - (\alpha - 2\kappa - \tau - 1)g'(y) - yg''(y))f''(x)
\]

\[
+ (f(x) + (\alpha - 2\kappa - \tau + 1)f'(x) - xf''(x))g''(y)
\]

\[
- (\alpha - \kappa)^2 f'(x)g''(y) - f'(x)g'(y)
\]

(5.41)

and denote \([f, g](x) = [f(x), g(x)]\) which is the bilinear concomitant. Then
\[
\frac{d}{dx}[f, g](x) = \left(g(x) - (\alpha - 2\kappa - \tau - 1)g'(x) - (2x + \alpha\kappa - \kappa^2)g''(x)\right)
+ (\alpha - 2\kappa + 1)xg'''(x) + x^2g^{(4)}(x)\right)f'''(x)
+ \left(f(x) + (\alpha - 2\kappa - \tau + 1)f'(x) - (2x + \alpha\kappa - \kappa^2)f''(x)\right)
- (\alpha - 2\kappa - 1)xf'''(x) + x^2f^{(4)}(x)\right)g'''(x).
\] (5.42)

This shows that the bilinear concomitant \([f, g](x)\) is constant whenever \(f\) and \(g\) satisfy the respective differential equations.

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