ASYMPTOTIC PROPERTIES OF A STOCHASTIC CHEMOSTAT MODEL WITH TWO DISTRIBUTED DELAYS AND NONLINEAR PERTURBATION

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Abstract. In this paper, a stochastic chemostat model with two distributed delays and nonlinear perturbation is proposed. We first transform the stochastic model into an equivalent high-dimensional system. Then we prove the existence and uniqueness of global positive solution of the model. Based on Khasminskii’s theory, we study the existence of a stationary distribution of the model by constructing a suitable stochastic Lyapunov function. Then we also establish sufficient conditions for the extinction of the plankton. Finally, numerical simulations are carried out to illustrate the theoretical results and to conclude our study, which shows that environmental noise experienced by limiting nutrient completely determines the persistence and extinction of the plankton.

1. Introduction. The importance of nutrient recycling to algal blooms is now widely recognized [1, 15, 25, 28, 45]. The regenerated nutrients from the dead biomass by bacterial decomposition can be absorbed again by plankton and ultimately affect the dynamic structure of the whole food network. It is worth noting that the nutrient recycling process is not instantaneous. A time delay is required for completion of bacterial decomposition from detritus into nutrients and it increases when temperature decreases [32]. In order to simulate the growth of plankton in lakes more accurately, a series of chemostat models with delayed nutrient recycling have been established in recent years. In [1], Beretta et al. first used a distributed delay to describe the effect of nutrient recycling on dynamics of plankton and proposed a single species chemostat model with delayed nutrient recycling. Bischi [3] further studied the model and pointed out that time delay does not destroy the stability of the equilibrium. The authors of [8, 28] generalized the model mentioned above to multi-species competition model with delayed nutrient recycling and found that the competing populations could coexist in the chemostat.

As pointed out in Caperon [6], delayed growth response is almost universal in plankton interactions. One of the reasons is the cell cycle, which characterizes the

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time delay between consumption and growth [11]. Based on the model of [1], Ruan [11, 27] incorporated a discrete delay to characterize the delayed growth response of plankton and investigated its effect on the global dynamics of plankton. Chemostat models involving distributed delay to describe the delayed growth response have also been studied by other authors [2, 33, 34]. Below is a single species chemostat-type model with delayed growth response and delayed nutrient recycling from [2]:

\[
\begin{align*}
N' &= D(N^0 - N) - aU(N)P + b\gamma \int_{-\infty}^{t} K_1(t-s)P(s)ds, \\
P' &= P \left[ -(\gamma + D) + c \int_{-\infty}^{t} K_2(t-s)U(N(s))ds \right],
\end{align*}
\]

(1)

where \( N(t) \) and \( P(t) \) denote the concentrations of limiting nutrient and plankton, respectively. All parameters are assumed to be positive and have the biological meanings listed in Table 1. From the biological meaning, we see that \( c \leq a \) and \( 0 < b < 1 \). The nutrient uptake rate of plankton \( U(N) \) is taken to be a monotonic increasing and differentiable function, that is,

\[
U(0) = 0, \quad U'(N) > 0, \quad \text{for} \quad N > 0 \lim_{N \to \infty} U(N) = 1.
\]

(2)

The delay Kernels \( K_i(s), i = 1, 2 \) are nonnegative bounded functions defined on \([0, \infty)\), where \( K_1(s) \) describes the contribution of the dead planktonic biomass from the past to the nutrient recycled and \( K_2(s) \) describes the delayed growth response of plankton to nutrient uptake. The presence of the distributed time delay must not affect the equilibrium values, so

\[
\int_{0}^{\infty} K_i(s)ds = 1, \quad i = 1, 2.
\]

(3)

The average time lag is defined as

\[
T_{K_i} = \int_{0}^{\infty} sK_i(s)ds < \infty, \quad i = 1, 2.
\]

(4)

Note that model (1) has a stable washout equilibrium \( E_0(N^0, 0) \) when \( U(N^0) < \frac{\gamma + D}{c} \) and a positive equilibrium \( E^*(N^*, P^*) \) when

\[
\gamma + D < c, \quad U(N^0) > \frac{\gamma + D}{c},
\]

(5)

with

\[
N^* = U^{-1}\left(\frac{\gamma + D}{c}\right), \quad P^* = \frac{D(N^0 - N^*)}{aU(N^*) - b\gamma}.
\]
Later, the authors of [12] further studied model (1) in detail and found that when the delays are sufficiently small, the positive equilibrium is globally asymptotically stable.

All the above-mentioned models were developed under the assumption that the biological parameters are constants regardless of time and environmental fluctuations, which is valid at macroscopic scale or for large population sizes due to the law of large numbers, or under homogeneity conditions. Otherwise, the varying environmental effects cannot be neglected [5]. Many researchers have constructed stochastic models to describe the effects of the environmental noises on the dynamics of population [4, 17, 18, 29, 30, 35, 39, 40, 41, 42, 43, 44, 46, 47]. For example, Imhof and Walcher [13] proposed a single microbe species stochastic chemostat model and showed the white noise may make the microorganism extinct. Xu and Yuan [37], and Wang and Jiang [31] further studied the model and obtained respectively the stochastic break-even concentration and the existence of a stationary distribution. With the development of research, the authors of [38] extended the single species stochastic model proposed by [13] to multi-species competition one, and proved that the competitive exclusion principle holds when the noise intensities are relatively small.

Motivated by the above facts, in this paper we consider a stochastic chemostat model with two distributed delays and nonlinear perturbation. Although the dynamics of some similar stochastic models with distributed delay have been studied, for example, a stochastic predator-prey model with weak delay kernel [20], a stochastic SEIR epidemic models with weak delay kernel [19] and a stochastic logistic model with strong delay kernel [21], in general the long time asymptotic behavior for stochastic models involving general delay kernel function is hard to obtain. The reason is that there is no general procedure to follow in constructing a suitable stochastic Liapunov function. In this paper, we try to construct the suitable stochastic Liapunov function step by step in a heuristic way. We believe that our technique can be used to investigate other stochastic models with general delay kernel function.

The paper is organized as follows. In Section 2, we formulate the model and present some preliminary results. The existence of stationary distribution is studied in Section 3 and the extinction is discussed in Section 4. Finally, we conclude our study with a summary and numerical simulations in Section 5.

2. The model and preliminary analysis. According to the biological significance of parameters listed in Table 1, we know that the parameters $DN^0$ represents the supply rate of nutrient to the system and $\gamma + D$ represents the total loss rate of the plankton. Usually they are estimated by an average value plus errors. Following Luo and Mao [22], we assume that the errors are normally distributed, but its standard deviations, known as the noise intensities, are dependent on the state variables $N$ and $P$ in system (1), respectively. Hence, $DN^0$ and $-(\gamma + D)$ may be replaced by

$$DN^0 \rightarrow DN^0 + \sigma_1 N \dot{B}_1(t) \quad \text{and} \quad -(\gamma + D) \rightarrow -(\gamma + D) + \sigma_2 P \dot{B}_2(t),$$

where $B_i(t)$ are mutually independent standard Brownian motions defined the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ and $\sigma_i^2 > 0$ are the intensities of the white noises, $i = 1, 2$. Here, $\Omega$ is the set of events, $\mathcal{F}$ the $\sigma$-algebra, $P$ is the probability measure, and $\{\mathcal{F}_t\}_{t \geq 0}$ is the filtration satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $P$-null sets). Then system (1)
becomes the following form:

\[ dN = [D(N^0 - N) - aU(N)P + b\gamma \int_{-\infty}^{t} K_1(t - s)P(s)ds]dt + \sigma_1 NdB_1(t), \]
\[ dP = P[ - (\gamma + D) + c \int_{-\infty}^{t} K_2(t - s)U(N(s))ds]dt + \sigma_2 P^2dB_2(t). \]

(6)

The complexity of stochastic system (6) makes it difficult to study its dynamic behavior. So, in this paper, we assume that \( K_1(s) \) take the following exponential kernels (see [23]):

\[ K_i(s) = \frac{\alpha_i^r_i}{(r_i - 1)!} s^{r_i - 1} e^{-\alpha_is}, \quad i = 1, 2, \]

(7)

where \( r_i \) are nonnegative integers called the order of delay kernel \( K_i \), and \( \alpha_i \) are positive parameters denoting the rates of decay of effect of past memories which are linked to the mean time delays by

\[ T_{K_i} = \int_{0}^{\infty} \frac{\alpha_i^r_i}{(r_i - 1)!} s^{r_i - 1} e^{-\alpha_is} ds = \frac{r_i}{\alpha_i}, \quad i = 1, 2. \]

In particular, we call (7) the weak kernel when \( r_i = 1 \) and strong kernel when \( r_i = 2 \).

Now we define some new variables:

\[ Y_i(t) = \int_{-\infty}^{t} K_{a_i}^i(t - s)P(s)ds, \quad i = 1, 2, \ldots, r_1, \]
\[ Z_j(t) = \int_{-\infty}^{t} K_{a_2}^j(t - s)U(N(s))ds, \quad j = 1, 2, \ldots, r_2, \]

where \( K_{a_i}^i(s) = \frac{a_i^{k_i} e^{-\alpha_is}}{(k_i - 1)!}, \quad i = 1, 2, \) and \( k_i = 1, 2, \ldots, \max\{r_1, r_2\} \). By the linear chain technique, system (6) can be transformed into an equivalent \( m \) dimensional system (\( m = 2 + r_1 + r_2 \)):

\[
\begin{align*}
  dN & = [D(N^0 - N) - aU(N)P + b\gamma Y_{r_1}]dt + \sigma_1 NdB_1(t), \\
  dP & = P[ - (\gamma + D) + cZ_{r_2}]dt + \sigma_2 P^2dB_2(t), \\
  dY_1 & = -\alpha_1(Y_1 - P)dt, \\
  dY_i & = -\alpha_1(Y_i - Y_{i-1})dt, \quad i = 2, 3, \ldots, r_1, \\
  dZ_1 & = -\alpha_2(Z_1 - U(N))dt, \\
  dZ_j & = -\alpha_2(Z_j - Z_{j-1})dt, \quad j = 2, 3, \ldots, r_2.
\end{align*}
\]

(8)

In this paper, we mainly study the dynamics of high-dimension system (8), including the existence of stationary distribution and the extinction of plankton. Without loss of generality, in the following discussion we suppose

\[ U(N) = \frac{N}{N + L}, \]

which is known as Michaelis-Menten function and \( L > 0 \) is the half-saturation (Michaelis-Menten) constant.

Here are some preliminary results. Let us start with the following stochastic integral equation:

\[ x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds + \sum_{r=1}^{l} \int_{t_0}^{t} g_r(s, x(s))dB_r(s), \quad x \in \mathbb{R}^n, \]

(9)

where \( f(x, t), g_1(x, t), \ldots, g_l(x, t) \) are continuous function of \((x, t)\).
Lemma 2.1. (see [10]) If the coefficients of (9) satisfy the following conditions:

$$|f(s, x) - f(s, y)| + \sum_{r=1}^{l} |g_r(s, x) - g_r(s, y)| \leq \Upsilon |x - y|, \quad (10)$$

in $U_R \subset \mathbb{R}_+^n$ for every $R > 0$, where $\Upsilon$ is some constant, and there exists a nonnegative $C^2$ function $V(x)$ such that $\mathcal{L}V(x) \leq -1$ outside some compact set, where $\mathcal{L}$ denotes the differential operator defined by [24]. Then system (9) exists a solution which is a stationary Markov process.

The following result is about the global existence and uniqueness of the positive solution to system (8), which is the basis for further studying the dynamics of system (8). For simplicity, let $X(t) = (N(t), P(t), Y_1(t), \ldots, Y_r(t), Z_1(t), \ldots, Z_r(t))$.

Theorem 2.2. Given initial value $X(0) \in \mathbb{R}_+^m$, system (8) admits a unique solution $X(t)$ for $t \geq 0$; furthermore, the solution will remain in $\mathbb{R}_+^n$ with probability one.

Proof. The coefficients of (8) satisfy the local Lipschitz condition, but do not satisfy the linear growth condition. Then for given initial value $X(0) \in \mathbb{R}_+^m$, there exists a unique local solution $X(t)$ on $t \in [0, \tau_e)$, where $\tau_e$ is known as the explosion time [24]. Now we show the explosion time $\tau_e = \infty$ almost surely (a.s.), which implies the global existence of the solution. To this end, let $n_0 \geq 1$ be sufficiently large such that $X_i(0)$ all lie within the interval $[\frac{1}{m_0}, n_0], i = 1, 2, \ldots, m$. For each integer $n \geq n_0$, define the stopping as in [9, Theorem 3.1]

$$\tau_n = \inf \left\{ t \in [0, \tau_e) : X_i(t) \notin \left( \frac{1}{n}, n \right), \text{ for } i = 1, 2, \ldots, m \right\}.$$

Clearly, $\tau_n$ is increasing in $n$, and for the empty set $\emptyset$ we have $\inf \emptyset = \infty$. If let $\tau_\infty = \lim_{n \to \infty} \tau_n$, then $\tau_\infty \leq \tau_e$ a.s. Next, we prove that $\tau_\infty = \infty$ by contradiction, implying $\tau_e = \infty$. If this statement is false, then there exist three constants $T > 0, n_1 \geq n_0$ and $\epsilon \in (0, 1)$ such that

$$\mathcal{P}\{\Omega_n\} \geq \epsilon \text{ for all } n \geq n_1, \text{ where } \Omega_n = \{\tau_n \leq T\}.$$

First consider the functions

$$V_{11} = \frac{1}{\theta + 1} N^{\theta + 1} + \frac{1}{\theta} P^\theta + \frac{1}{\alpha_1} \sum_{i=1}^{r_1} \frac{1}{2^{i-1}} Y_i^2 + \frac{1}{\alpha_2} \sum_{j=1}^{r_2} \frac{1}{2^{j-1}} Z_j^2, \quad (11)$$

$$V_{12} = -\ln N - \ln P - \sum_{i=1}^{r_1} \ln Y_i - \sum_{j=1}^{r_2} \ln Z_j, \quad (12)$$

where $0 < \theta < 1$ is a sufficiently small number satisfying

$$\xi := D - \frac{1}{2} \theta \sigma_1^2 - \frac{b \gamma \theta}{\theta + 1} > 0. \quad (13)$$

Obviously, $\lim_{\rho \to \infty, X \in \mathbb{R}_+^m \setminus U_\rho} (V_{11}(X) + V_{12}(X)) = +\infty$, which implies that $V_{11}(X) + V_{12}(X)$ achieves a lower bound at a point $\bar{X} \in \mathbb{R}_+^m$.

Now define a nonnegative $C^2$ function $V_1 : \mathbb{R}_+^m \to \mathbb{R}_+$ as

$$V_1(X) = V_{11}(X) + V_{12}(X) - V_{11}(\bar{X}) - V_{12}(\bar{X}).$$
Applying Itô’s formula and Young’s inequality, we have
\[
\mathcal{L} \left( \frac{1}{\theta + 1} N^{\theta+1} \right) = N^\theta \left[ D(N^0 - N) - aU(N)P + b\gamma Y_{r_1} + \frac{1}{2} \theta \sigma_1^2 N^{\theta+1} \right]
\leq DN^0 N^\theta - (D - \frac{1}{2} \theta \sigma_1^2) N^{\theta+1} + b\gamma \left( \frac{1}{\theta + 1} Y_{r_1} + \frac{\theta}{\theta + 1} N^{\theta+1} \right)
= DN^0 N^\theta - \xi N^{\theta+1} + \frac{b\gamma}{\theta + 1} Y_{r_1}.
\]

\[
\mathcal{L} \left( \frac{1}{\theta} P^\theta \right) = P^\theta \left[ - (\gamma + D) + c Z_{r_2} \right] + \frac{1}{2} (\theta - 1) \sigma_2^2 P^\theta + 1 \leq c Z_{r_2} P^\theta - \frac{1}{2} (1 - \theta) \sigma_2^2 P^\theta + 1,
\]

\[
\mathcal{L} \left( \frac{1}{\alpha_1} \sum_{i=1}^{r_1} \frac{1}{2i-1} Y_i^2 \right) = 2(Y_i P - Y_i^2) + \sum_{i=2}^{r_1} \frac{1}{2i-2} (Y_{i-1} Y_i - Y_i^2)
\leq P^2 - Y_i^2 + \sum_{i=2}^{r_1} \frac{1}{2i-2} (Y_{i-1}^2 - Y_i^2) \leq P^2 - \sum_{i=1}^{r_1} \frac{Y_i^2}{2i},
\]

\[
\mathcal{L} \left( \frac{1}{\alpha_2} \sum_{j=1}^{r_2} \frac{1}{2j-1} Z_j^2 \right) = 2(Z_i U(N) - Z_i^2) + \sum_{j=2}^{r_2} \frac{1}{2j-2} (Z_{j-1} Z_j - Z_j^2)
\leq U^2(N) - Z_i^2 + \sum_{j=2}^{r_2} \frac{1}{2j-2} (Z_{j-1}^2 - Z_j^2) \leq 1 - \sum_{j=1}^{r_2} \frac{Z_j^2}{2j}.
\]

Then,
\[
\mathcal{L} V_{11} \leq DN^0 N^\theta - \xi N^{\theta+1} + \frac{c\theta}{\theta + 1} P^\theta + P^2 - \frac{1}{2} (1 - \theta) \sigma_2^2 P^\theta + 1 + \sum_{i=1}^{r_1} \frac{Y_i^2}{2i} + \frac{b\gamma}{\theta + 1} Y_{r_1} + \frac{1}{2} \frac{Z_{r_2}^2}{2j} + \frac{c}{\theta + 1} Z_{r_2}^\theta + 1. \tag{14}
\]

Similarly,
\[
\mathcal{L} V_{12} = - \frac{DN^0}{N} + D + \frac{1}{2} \sigma_1^2 + \frac{aU(N)}{N} P - \frac{b\gamma Y_{r_1}}{N} + \frac{\gamma + D - c Z_{r_2}}{N} + \frac{1}{2} \sigma_2^2 P^2
+ \alpha_1 r_1 - \alpha_1 \frac{P}{Y_1} - \alpha_1 \sum_{i=2}^{r_1} \frac{Y_{i-1}}{Y_i} + \alpha_2 r_2 - \alpha_2 \frac{U(N)}{Z_1} - \alpha_2 \sum_{j=2}^{r_2} \frac{Z_{j-1}}{Z_j}
\leq \gamma + \alpha_1 r_1 + \alpha_2 r_2 + 2D + \frac{1}{2} \sigma_1^2 + \frac{a}{L} P + \frac{1}{2} \sigma_2^2 P^2. \tag{15}
\]

Then it follows from (14) and (15) that
\[
\mathcal{L} V_i \leq \gamma + \alpha_1 r_1 + \alpha_2 r_2 + 2D + \frac{1}{2} \sigma_1^2 + \Theta := C,
\]

where
\[
\Theta = \max_{X \in \mathbb{R}_+} \left\{ DN^0 N^\theta - \xi N^{\theta+1} + \frac{c\theta}{\theta + 1} P^\theta + P^2 - \frac{1}{2} (1 - \theta) \sigma_2^2 P^\theta + 1 + \sum_{i=1}^{r_1} \frac{Y_i^2}{2i} + \frac{b\gamma}{\theta + 1} Y_{r_1} + \frac{1}{2} \frac{Z_{r_2}^2}{2j} + \frac{c}{\theta + 1} Z_{r_2}^\theta + \frac{a}{L} P + \frac{1}{2} \sigma_2^2 P^2 \right\}. \tag{16}
\]
Consequently,

\[ EV_1(X(\tau_n \wedge T)) \leq V_1(X(0)) + C\mathbb{E}(\tau_n \wedge T) \leq V_1(X(0)) + CT. \]  \hfill (17)

Moreover for all \( \omega \in \Omega_n \), \( X_1(\tau_n, \omega) \) or \( X_2(\tau_n, \omega), \ldots, \) or \( X_n(\tau_n, \omega) \) equals either \( n \) or \( \frac{1}{n} \). So \( V_1(X(\tau_n, \omega)) \) is no less than either

\[ \frac{1}{\theta + 1} n^{\theta + 1} - \ln n \text{ or } \frac{1}{\theta + 1} (\frac{1}{n})^{\theta + 1} + \ln n \text{ or } \frac{1}{\theta} n^{\theta} - \ln n \text{ or } \frac{1}{\theta} (\frac{1}{n})^{\theta} + \ln n \]

\[ \frac{1}{2^{i-1} \alpha_1} n^2 - \ln n \text{ or } \frac{1}{2^{i-1} \alpha_1} (\frac{1}{n})^2 + \ln n \]

Then

\[ V_1(X(\tau_n, \omega)) \geq \left( \frac{1}{\theta + 1} n^{\theta + 1} - \ln n \right) \land \left( \frac{1}{\theta + 1} (\frac{1}{n})^{\theta + 1} + \ln n \right) \land \left( \frac{1}{\theta} n^{\theta} - \ln n \right) \land \left( \frac{1}{\theta} (\frac{1}{n})^{\theta} + \ln n \right) \]

\[ \land \left( \frac{1}{2^{i-1} \alpha_1} n^2 - \ln n \right) \land \left( \frac{1}{2^{i-1} \alpha_2} (\frac{1}{n})^2 + \ln n \right). \]

This with (17) infers

\[ V_1(X(0)) + CT \geq \mathbb{E}(I_{\Omega_n} V_1(X(\tau_n, \omega))) \]

\[ \geq \epsilon \left( \frac{1}{\theta + 1} n^{\theta + 1} - \ln n \right) \land \left( \frac{1}{\theta + 1} (\frac{1}{n})^{\theta + 1} + \ln n \right) \land \left( \frac{1}{\theta} n^{\theta} - \ln n \right) \]

\[ \land \left( \frac{1}{\theta} (\frac{1}{n})^{\theta} + \ln n \right) \land \left( \frac{1}{2^{i-1} \alpha_1} n^2 - \ln n \right) \land \left( \frac{1}{2^{i-1} \alpha_2} (\frac{1}{n})^2 + \ln n \right), \]

where \( I_{\Omega_n} \) denotes the indicator function of \( \Omega_n \). This leads to a contradiction as we let \( n \to \infty \),

\[ \infty > V_1(X(0)) + CT = \infty. \]

So we have \( \tau_e = \infty \) a.s. This completes the proof. \( \square \)

3. Existence of stationary distribution. In this section, we investigate the existence of stationary distribution.

**Theorem 3.1.** Let \( X(t) \) be the solution of system (8) with given initial value \( X(0) \in \mathbb{R}_+^m \). If

\[ U(X^0) > \frac{\gamma + D}{C} + \frac{\eta N^0 \sigma^2}{2}, \]

where \( \eta > \frac{L}{\text{var}(X^0)} \), then the solution \( X(t) \) of system (8) is a stationary Markov process.

**Proof.** According to Remark 5 of Xu et al. [36], the global existence of the solutions of system (8) guarantees that the condition (10) is true. Therefore, to complete the proof of Theorem 3.1, we only need to construct a \( C^2 \) function \( V_2 : \mathbb{R}_+^m \to \mathbb{R}_+ \) and a closed set \( E_c \subset \mathbb{R}_+^m \) such that \( \mathcal{L}V_2 \leq -1 \) for \( X(t) \in \mathbb{R}_+^m \setminus E_c \). To this end, let
\[ V_{21} = \eta \left( N - N^0 - N^0 \ln \frac{N}{N^0} \right) - \frac{U'(N^0)}{D} N + \frac{\eta b \gamma}{\alpha_1} \sum_{i=1}^{r_1} Y_i - \frac{1}{\alpha_2} \sum_{j=1}^{r_2} Z_j - \frac{1}{c} \ln P, \]
\[ V_{22} = -\ln N - \sum_{i=1}^{r_1} \ln Y_i - \sum_{j=1}^{r_2} \ln Z_j, \]

Then define \( \tilde{V}_2 = V_{11} + M V_{21} + V_{22}, \) where \( V_{11} \) is defined by (11), \( M \) is a sufficiently large number satisfying
\[ -M \lambda + \Psi \leq -2. \] (18)

Here,
\[ \lambda = U(N^0) - \frac{\gamma + D}{c} - \frac{\eta N^0 \sigma_1^2}{2} > 0, \]
\[ \Psi = \max_{X \in \mathbb{R}^n_+} \left\{ DN^0 N^\theta - \frac{1}{2} \xi N^{\theta+1} + \frac{\alpha}{L} P + \frac{c \theta}{\theta + 1} \rho^{\theta+1} + P^2 - \frac{1}{4} (1 - \theta) \sigma_2^2 \rho^{\theta+2} \right\}, \]
\[ -\sum_{i=1}^{r_1} \frac{Y_i^2}{2^{r_1 + 1}} + \frac{b \gamma}{\theta + 1} Y_i^{\theta+1} + 1 - \sum_{j=1}^{r_2} \frac{Z_j^2}{2^{r_2 + 1}} + \frac{c}{\theta + 1} Z_j^{\theta+1} + D + \alpha_1 r_1 + \alpha_2 r_2 + \frac{1}{2} \sigma_1^2 \right\}. \] (19)

Similar to Theorem 2.2, let \( \tilde{X} \in \mathbb{R}^n_+ \) be the minimum point of \( \tilde{V}_2 \). Then the nonnegative \( C^2 \) function \( V_2 \) can be defined by
\[ V_2 = \tilde{V}_2(X) - \tilde{V}_2(\tilde{X}). \]

Note that from Itô’s formula we have
\[ \mathcal{L} \left( N - N^0 - N^0 \ln \frac{N}{N^0} \right) = \frac{N - N^0}{N} \left[ D(N^0 - N) - a U(N) P + b \gamma Y_{r_1} \right] + 2 \frac{N^0 \sigma_1^2}{2} \]
\[ \leq - \frac{D(N - N^0)^2}{N} + \frac{a N^0 U(N) P}{N} + b \gamma Y_{r_1} + \frac{1}{2} N^0 \sigma_1^2, \]
\[ \mathcal{L}(-N) = D(N - N^0) + a U(N) P - b \gamma Y_{r_1} \leq D(N - N^0) + a U(N) P, \]
\[ \mathcal{L} \left( \sum_{i=1}^{r_1} Y_i \right) = \alpha_1 P - \alpha_1 Y_{r_1}, \]
\[ \mathcal{L} \left( - \sum_{j=1}^{r_2} Z_j \right) = \alpha_2 Z_{r_2} - \alpha_2 U(N), \]
\[ \mathcal{L}(-\ln P) = \gamma + D - c Z_{r_2} + \frac{1}{2} \sigma_2^2 P^2. \]

Then,
\[ \mathcal{L} V_{21} \leq \frac{\gamma + D}{c} + \frac{\eta N^0 \sigma_1^2}{2} + \left[ \frac{a U'(N^0)}{D} U(N) + \eta \alpha N^0 U(N) \right] \frac{1}{N} + \eta b \gamma \right] P \]
\[ + \frac{\sigma_2^2}{2c} P^2 - U(N) + U'(N^0)(N - N^0) - \frac{\eta D(N - N^0)^2}{N} \]
\[ = -\lambda + G(N) P + \frac{\sigma_2^2}{2c} P^2 + F(N), \]

where
\[ F(N) = U(N^0) - U(N) + U'(N^0)(N - N^0) - \frac{\eta D(N - N^0)^2}{N}, \]
\[ G(N) = \frac{a U'(N^0)}{D} U(N) + \eta \alpha N^0 U(N) \frac{1}{N} + \eta b \gamma. \]
By simple calculations, we get
\[ F'(N)|_{N=N^0} = 0, \quad F''(N) = -U''(N) - \frac{2\eta D(N^0)^2}{N^3}. \]
Notice that \( U(N) = \frac{N}{L+N}, \) thus
\[ F''(N) = \frac{2L}{(L+N)^2} - \frac{2\eta D(N^0)^2}{N^3} < \frac{2L - 2\eta D(N^0)^2}{N^3} < 0, \]
which implies that
\[ F(N) \leq F(N^0) = 0. \]
And we also have
\[ G(N) \leq \frac{aU'(N^0)}{D} + \frac{\eta aN^0}{L} + \eta \gamma := d. \]
Therefore,
\[ -\lambda + dP + \frac{2\alpha}{2c} P^2. \tag{20} \]
Moreover,
\[ LV_{22} = -\frac{DN^0}{N} + D + \frac{1}{2} \sigma_1^2 + \frac{aU(N)}{N} - \frac{b\gamma Y_i}{N} - \alpha_2 \sum_{j=2}^{r_2} \frac{Z_j - 1}{Z_j} \]
\[ + \alpha_1 r_1 - \frac{\alpha_1 \sigma_2}{Y_1} - \alpha_1 \sum_{i=2}^{r_1} Y_i - \alpha_2 r_2 - \alpha_2 \sum_{j=2}^{r_2} \frac{Z^j}{Z_j}, \tag{21} \]
It follows from (14), (20) and (21) that
\[ LV_2 \leq M \left[ -\lambda + dP + \frac{2\alpha}{2c} P^2 \right] - \frac{DN^0}{N} - \alpha_1 \frac{P}{Y_1} - \alpha_1 \sum_{i=2}^{r_1} \frac{Y_i - 1}{Y_i} - \alpha_2 \frac{U(N)}{Z_1} \]
\[ - \alpha_2 \sum_{j=2}^{r_2} \frac{Z_j - 1}{Z_j} - \frac{1}{2} \xi N^{\theta + 1} - \frac{1}{4}(1-\theta)\alpha_2 P^{\theta + 2} - \sum_{i=1}^{r_1} \frac{Y_i^2}{2\xi + 1} - \sum_{j=1}^{r_2} \frac{Z_j^2}{2\xi + 1} + \Psi, \tag{22} \]
where \( \Psi \) is defined by (19).

Let
\[ E_c = \left\{ X \in \mathbb{R}^m_+ : \epsilon \leq N \leq \frac{1}{\epsilon}, \epsilon \leq P \leq \frac{1}{\epsilon}, \epsilon^{i+1} \leq Y_i \leq \frac{1}{\epsilon^{i+1}}, \epsilon^{j+1} \leq Z_j \leq \frac{1}{\epsilon^{j+1}} \right\}, \]
i = 1, 2, \ldots, r_1, j = 1, 2, \ldots, r_2, and \( \epsilon \) meets the following conditions:
\[ \min \left\{ \frac{DN^0}{\epsilon}, \frac{\alpha_1}{\epsilon}, \frac{\alpha_2}{\epsilon}, \frac{1}{2} \xi \left( \frac{1}{\epsilon^{i+1}} \right), \frac{1}{2\xi + 1} \left( \frac{1}{\epsilon^{i+1}} \right)^2, \frac{1}{2\xi + 1} \left( \frac{1}{\epsilon^{i+1}} \right)^2 \right\} \geq 1 + \Phi_1, \tag{23} \]
\[ M \left[ -\lambda + d\epsilon + \frac{2\alpha}{2c} \epsilon^2 \right] + \Psi \leq -1, \tag{24} \]
\[ \Phi_2 - \frac{1}{8}(1-\theta)\alpha_2^2 \left( \frac{1}{\epsilon} \right)^{\theta + 2} \leq -1. \tag{25} \]
Similarly, for any \( L \in \mathbb{R}_+^m \setminus E_\epsilon \), we divide \( \mathbb{R}_+^m \setminus E_\epsilon \) into \( \bigcup_{k=1}^{2m} E_{\epsilon}^k \), where

\[
E_{\epsilon}^1 = \{ X \in \mathbb{R}_+^m : 0 < N < \epsilon \}, \quad E_{\epsilon}^2 = \{ X \in \mathbb{R}_+^m : 0 < P < \epsilon \},
\]
\[
E_{\epsilon}^{2+i} = \{ X \in \mathbb{R}_+^m : 0 < Y_i < \epsilon^{i+1}, Y_{i-1} > \epsilon^i, i = 2, \ldots, r_1 \},
\]
\[
E_{\epsilon}^{i+1} = \{ X \in \mathbb{R}_+^m : 0 < Z_i < \epsilon^2, N > \epsilon \},
\]
\[
E_{\epsilon}^{2+r_1+j} = \{ X \in \mathbb{R}_+^m : 0 < Z_j < \epsilon^{j+1}, Z_{j-1} > \epsilon^j, j = 2, \ldots, r_2 \},
\]
\[
E_{\epsilon}^{1+m} = \{ X \in \mathbb{R}_+^m : N > 1/\epsilon \}, \quad E_{\epsilon}^{2+m} = \{ X \in \mathbb{R}_+^m : P > 1/\epsilon \},
\]
\[
E_{\epsilon}^{2+m+i} = \{ X \in \mathbb{R}_+^m : Y_i > 1/\epsilon^{i+1}, i = 1, 2, \ldots, r_1 \},
\]
\[
E_{\epsilon}^{2+m+r_1+j} = \{ X \in \mathbb{R}_+^m : Z_j > 1/\epsilon^{j+1}, j = 1, 2, \ldots, r_2 \}.
\]

And then, we will show \( \mathcal{L}V_2 \leq -1 \) in each \( E_{\epsilon}^k, \ k = 1, \ldots, 2m \) implying the conclusion on \( \mathbb{R}_+^m \setminus E_\epsilon \). We consider six cases:

**Case 1:** For any \( X \in E_{\epsilon}^1 \), by (22) and (23) we have

\[
\mathcal{L}V_2 \leq M \left[ dP + \frac{\sigma_2^2}{2c} P^2 \right] - \frac{DN_0}{N} - \frac{1}{4}(1-\theta)\sigma_2^2 P^{\theta+2} + \Psi \leq \Phi_1 - \frac{D_{N_0}}{\epsilon} \leq -1,
\]

where

\[
\Phi_1 = \max_{X \in \mathbb{R}_+^m} \left\{ -\frac{1}{4}(1-\theta)\sigma_2^2 P^{\theta+2} + M \left[ dP + \frac{\sigma_2^2}{2c} P^2 \right] + \Psi \right\}.
\]  

Similarly, for any \( X \in E_{\epsilon}^3 \) or \( E_{\epsilon}^{2+i}, i = 2, \ldots, r_1 \), we can prove

\[
\mathcal{L}V_2 \leq \Phi_1 - \frac{\alpha_1}{\epsilon} \leq -1.
\]

**Case 2:** For any \( X \in E_{\epsilon}^2 \), (22) and (24) imply that

\[
\mathcal{L}V_2 \leq M \left[ -\lambda + dP + \frac{\sigma_2^2}{2c} P^2 \right] + \Psi \leq -1.
\]

**Case 3:** For any \( X \in E_{\epsilon}^{2+r_1} \), by (22) and (23), then

\[
\mathcal{L}V_2 \leq M \left[ dP + \frac{\sigma_2^2}{2c} P^2 \right] - \frac{U(N)}{Z_1} - \frac{1}{4}(1-\theta)\sigma_2^2 P^{\theta+2} + \Psi \leq \max_{X \in \mathbb{R}_+^m} \left\{ \frac{1}{4}(1-\theta)\sigma_2^2 P^{\theta+2} + M \left[ dP + \frac{\sigma_2^2}{2c} P^2 \right] + \Psi \right\} - \frac{U(N)}{Z_1} \leq \Phi_1 - \frac{\alpha_2}{\epsilon(L+\epsilon)} \leq -1.
\]

Similarly, for any \( X \in E_{\epsilon}^{2+r_1+j}, j = 2, \ldots, r_2 \), we can see that

\[
\mathcal{L}V_2 \leq \Phi_1 - \frac{\alpha_2}{\epsilon} \leq -1.
\]
Case 4: For any \( X \in E_1^{1+m} \), it follows that from (22) that
\[
\mathcal{L}V \leq M \left[ dP + \frac{\sigma^2}{2c} P^2 \right] - \frac{1}{2} \xi N^\theta + \frac{1}{4} (1 - \theta) \sigma^2 P^\theta + \Psi
\]
\[
\leq \max_{X \in \mathbb{R}^m} \left\{ \frac{1}{4} (1 - \theta) \sigma^2 P^\theta + M \left[ dP + \frac{\sigma^2}{2c} P^2 \right] + \Psi \right\} - \frac{1}{2} \xi N^\theta
\]
\[
\leq \Phi_1 - \frac{1}{2} \left( \frac{1}{\epsilon} \right)^{\theta+1} \leq -1.
\]
Then, by (23), we have \( \mathcal{L}V \leq -1 \) for any \( X \in E_1^{1+m} \).

Case 5: For any \( X \in E_2^{1+m} \), according to (22) and (25), then
\[
\mathcal{L}V \leq M \left[ dP + \frac{\sigma^2}{2c} P^2 \right] - \frac{1}{4} (1 - \theta) \sigma^2 P^\theta + \Psi
\]
\[
\leq \Phi_2 - \frac{1}{8} (1 - \theta) \sigma^2 P^\theta + \Psi
\]
\[
\leq \Phi_2 - \frac{1}{8} (1 - \theta) \sigma^2 \left( \frac{1}{\epsilon} \right)^{\theta+2} \leq -1,
\]
where
\[
\Phi_2 = \max_{X \in \mathbb{R}^m} \left\{ -\frac{1}{8} (1 - \theta) \sigma^2 P^\theta + M \left[ dP + \frac{\sigma^2}{2c} P^2 \right] + \Psi \right\}.
\]

Case 6: For any \( X \in E_2^{1+m+i}, i = 1, 2, \ldots, r_1 \), by (22) and (23),
\[
\mathcal{L}V \leq M \left[ dP + \frac{\sigma^2}{2c} P^2 \right] - \frac{Y_i^2}{2^{i+1}} - \frac{1}{4} (1 - \theta) \sigma^2 P^\theta + \Psi
\]
\[
\leq \max_{X \in \mathbb{R}^m} \left\{ \frac{1}{4} (1 - \theta) \sigma^2 P^\theta + M \left[ dP + \frac{\sigma^2}{2c} P^2 \right] + \Psi \right\} - \frac{Y_i^2}{2^{i+1}}
\]
\[
\leq \Phi_1 - \frac{1}{2^{i+1}} \left( \frac{1}{\epsilon^{i+1}} \right) \leq -1.
\]

Similarly, for any \( X \in E_2^{1+m+r_1+j}, j = 1, 2, \ldots, r_2 \), we can show that
\[
\mathcal{L}V \leq \Phi_1 - \frac{1}{2^{j+1}} \left( \frac{1}{\epsilon^{j+1}} \right) \leq -1.
\]

In short, we have \( \mathcal{L}V \leq -1 \) on \( \mathbb{R}^m \setminus E_\epsilon \). By Lemma 2.1, we conclude that system (8) has a solution which is a stationary Markov process. The proof is completed. \( \square \)

Remark 1. (i) Theorem 3.1 gives a sufficient condition for the existence of stationary distribution, which to some extent reflects the weak stability and persistence of system (8). In other words, when the death rate of the plankton and the dilution rate are small enough respect to the input concentration of nutrient, the plankton may persist in a random environment.

(ii) It is noteworthy that the main idea of proving Theorem 3.1 relies on constructing a suitable stochastic Liapunov function. Generally, for any function that satisfies the condition
\[
U(N) \in C^2([0, +\infty), [0, +\infty)), \quad \frac{U(N)}{N} \leq K, \quad N \in (0, +\infty),
\]
where $K > 0$, we can discuss the existence of a stationary distribution of model (8) by use of the same idea. Common functions of this kind include generalized Holling type III or sigmoidal \cite{7, 31} which is given by

$$U(N) = \frac{N^2}{N^2 + AN + L},$$

and Ivlev functional response \cite{14, 26} which is given by

$$U(N) = 1 - e^{-\mu N},$$

where $L$ and $\mu$ are positive constants and $A$ is constant.

4. Extinction of the plankton. In this section, we investigate the conditions for the extinction of plankton.

**Theorem 4.1.** Let $X(t)$ be the solution of system (8) with given initial value $X(0) \in \mathbb{R}^m_+$. If $c < \gamma + D$, then

$$\limsup_{t \to \infty} \frac{\ln P(t)}{t} \leq - (\gamma + D) + c < 0 \text{ a.s.},$$

i.e., the plankton is extinct exponentially with probability one.

**Proof.** By Itô’s formula, we have

$$d \ln P(t) = [- (\gamma + D) + c Z_{r_2}(t) - \frac{1}{2} \sigma_2^2 P^2(t)] dt + \sigma_2 P(t) dB_2(t),$$

which further implies that

$$\frac{1}{t} \ln \frac{P(t)}{P(0)} = - (\gamma + D) + \frac{c}{t} \int_0^t Z_{r_2}(s) ds - \frac{\sigma_2^2}{2t} \int_0^t P^2(s) ds + \frac{M(t)}{t},$$

where $M(t) = \sigma_2 \int_0^t P(s) dB_2(s)$. Note that $\langle M, M \rangle_t = \sigma_2^2 \int_0^t P^2(s) ds$. It then follows from the exponential martingale inequality that

$$\mathcal{P} \left\{ \sup_{0 \leq t \leq k} \left[ M(t) - \frac{1}{2} \langle M, M \rangle_t \right] > 2 \ln k \right\} \leq \frac{1}{k^2},$$

where $k \in \mathbb{N}_+$. This, together with Borel-Cantelli Lemma, yields that for almost all $\omega \in \Omega$, there is a random integer $k_0(\omega)$ such that for $k \geq k_0(\omega)$,

$$\sup_{0 \leq t \leq k} \left[ M(t) - \frac{1}{2} \langle M, M \rangle_t \right] \leq 2 \ln k.$$

This implies that

$$M(t) \leq 2 \ln k + \frac{1}{2} \sigma_2^2 \int_0^t P^2(s) ds, \quad 0 \leq t \leq k, \quad k \geq k_0(\omega).$$

Then, from (29) and (30), we obtain

$$\frac{1}{t} \ln \frac{P(t)}{P(0)} \leq - (\gamma + D) + \frac{c}{t} \int_0^t Z_{r_2}(s) ds + \frac{2 \ln k}{k},$$

for $0 \leq t \leq k, \quad k \geq k_0(\omega)$. Consequently, for $0 \leq k - 1 \leq t \leq k, \quad k \geq k_0(\omega)$, one have

$$\frac{1}{t} \ln \frac{P(t)}{P(0)} \leq - (\gamma + D) + \frac{c}{t} \int_0^t Z_{r_2}(s) ds + \frac{2 \ln k}{k - 1}.$$
On the other hand, for
\[ V_3(t) = \sum_{j=1}^{r_2} Z_j(t), \]
we find from the positivity of \( Z_j(t) \) that
\[ \liminf_{t \to \infty} \frac{V_3(t)}{t} \geq 0 \text{ a.s.} \quad (32) \]
Note that from system (8) we have
\[ dV_3(t) = \alpha_2 (U(N) - Z_{r_2}) dt, \]
which implies that
\[ \frac{1}{t} \int_0^t \alpha_2 Z_{r_2}(s) ds = \frac{1}{t} \int_0^t \alpha_2 U(N(s)) ds - \frac{V_3(t) - V_3(0)}{t}. \quad (33) \]
Hence, by (31) and (33), we have
\[ \frac{1}{t} \ln \frac{P(t)}{P(0)} \leq -(\gamma + D) + \frac{c}{t} \int_0^t U(N(s)) ds + \frac{2 \ln k}{k - 1} - \frac{c}{\alpha_2} \frac{V_3(t) - V_3(0)}{t}, \]
which, together with (32), further implies that
\[ \limsup_{t \to \infty} \frac{1}{t} \ln \frac{P(t)}{P(0)} \leq - (\gamma + D) + \limsup_{t \to \infty} \frac{c}{t} \int_0^t U(N(s)) ds \]
\[ \leq - (\gamma + D) + c < 0 \text{ a.s.} \]
This completes the proof. \( \Box \)

**Remark 2.** According to Theorem 4.1, the condition \( c < \gamma + D \) ensures the extinction of plankton. Biologically speaking, it is reasonable for species to be extinct due to large death rate and large dilution rate, for instance, if the dilution rate is large, the nutrient is removed very fast and the species cannot get it and species is also removed. And also, we have \( U(N^0) < \frac{\gamma + D}{c} \) when \( c < \gamma + D \), which, together with Theorem 3.1, implies that parameter \( U(N^0) - \frac{\gamma + D}{c} \) is the critical value of plankton between persistence and extinction when \( \sigma_1 = 0 \).

5. **Numerical simulations and discussions.** Environmental noises are ubiquitous in nature and prominently affect population dynamics. This may be especially true for plankton population due to unpredictability of weather, temperature and many other physical factors embedded in aquatic ecosystems. Based on [2, 12], in this paper, we proposed a stochastic chemostat model with two distributed delays and nonlinear perturbation. The complexity of the system makes it difficult to study its dynamic behavior. So, by use of linear chain technique, we first transformed the system into an equivalent high-dimensional system (8). Then we showed the global existence and uniqueness of positive solution to system (8). And then, based on Khasminskii’s theory, we proved the existence of a stationary distribution of the model by constructing a suitable stochastic Lyapunov function. In addition, we also studied the extinction of the plankton.

From Theorem 3.1, we find that when environmental noise experienced by nutrients is not too large, \( i.e., \)
\[ \sigma_1^2 < \frac{2}{\eta N^0} \left[ U(N^0) - \frac{\gamma + D}{c} \right], \]
system (8) exists a stationary distribution, see Fig. 1. More interestingly, we find from Fig. 2 that environmental noise experienced by plankton only affects the fluctuation amplitude of sample trajectories, but it does not change the persistence
of plankton. This finding is very interesting and different from the existing literatures [13, 16, 31, 37, 48, 49]. In brief, stochastic system (8) still preserves some stability and exhibits oscillation around the interior equilibrium of its corresponding deterministic model when the intensity of noise $\sigma_1$ is relatively small.

At the same time, Theorem 4.1 implies that the extinction of plankton has nothing to do with environmental noise. It is worth noting that when $\sigma_1 = 0$, the survival of plankton only depends on $U(N^0) - \gamma + Dc$. That is, parameter $U(N^0) - \gamma + Dc$ is the critical value determining persistence and extinction of plankton, see Fig. 3.

To examine the effect of the delays in nutrient recycling and growth response, first we fix $T_{K_1} = 10$ and decrease $T_{K_2}$ from 100, 10 to 1. In this case, the plankton is extinct, see Fig. 4(a). And then we fix $T_{K_2} = 10$ and decrease $T_{K_1}$ from 1000, 100

Figure 1. The left is the stochastic trajectories of system (8); the right is the density functions of $N(t)$ and $P(t)$, respectively. Here, $r_1 = r_2 = 1$, $D = 0.3$, $N^0 = 2$, $a = 0.7$, $L = 0.3$, $b = 0.3$, $\gamma = 0.2$, $\alpha_1 = 0.1$, $\alpha_2 = 0.15$, $c = 0.58$, $\sigma_1 = 0.01$, $\sigma_2 = 0.08$.

Figure 2. The solutions of stochastic system (8) and its corresponding deterministic system. Here, all parameter values are taken as in Fig. 1 except $\sigma_2$.
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$\text{Figure 3.}$ The solutions of stochastic system (8) when $\sigma_1 = 0$. Here, (a) $c = 0.5, \sigma_2 = 2.8$; (b) $c = 0.59, \sigma_2 = 2.8$. Other parameter values are taken as in Fig. 1.

$\text{Figure 4.}$ The solutions of stochastic system (8). Here, (a) $c = 0.48, \sigma_1 = 1.5, \sigma_2 = 2.8, \alpha_1 = 0.1, T_{K_1} = 10, T_{K_2} = 100$ (blue), 10 (red), 1 (green); (b) $c = 0.58, \sigma_1 = 0.01, \sigma_2 = 0.15, \alpha_2 = 0.1, T_{K_2} = 10, T_{K_1} = 1000$ (blue), 100 (red), 1 (green). Other parameters take the same values as in Fig. 1.

To sum up, this paper is concerned with asymptotic properties of a stochastic chemostat model with two distributed delays and nonlinear perturbation. One of the most interesting findings is that plankton’s survival only depends on environmental noise experienced by nutrients, but has nothing to do with that experienced by plankton. The obtained results may enrich the research of dynamics in chemostat. In addition, we only study the effect of Gaussian white noise on asymptotic
properties of the chemostat model with two distributed delays. It is also interesting to study the effect of other environmental noise, such as Lévy noise and telephone noise, on delayed chemostat models. We leave these for future investigations.

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REFERENCES

[1] E. Beretta, G. I. Bischi and F. Solimano, Stability in chemostat equations with delayed nutrient recycling, J. Math. Biol., 28 (1990), 99–111.
[2] E. Beretta and Y. Takeuchi, Qualitative properties of chemostat equations with time delays: boundedness, local and global asymptotic stability, Differential Equations and Dynamical Systems, 2 (1994), 19–40.
[3] G. I. Bischi, Effects of time lags on transient characteristics of a nutrient cycling model, Math. Biosci., 109 (1992), 151–175.
[4] Y. L. Cai, Y. Kang, M. Banerjee and W. M. Wang, A stochastic epidemic model incorporating media coverage, Commun. Math. Sci., 14 (2016), 893–910.
[5] F. Campillo, M. Joannides and I. Larramendy-Valverde, Stochastic modeling of the chemostat, Ecol. Model., 222 (2011), 2676–2689.
[6] J. Caperon, Time lag in population growth response of isochrysis galbana to a variable nitrate environment, Ecology, 50 (1969), 188–192.
[7] R. M. Etoua and C. Rousseau, Bifurcation analysis of a generalized Gause model with prey harvesting and a generalized Holling response function of type III, J. Differ. Equ., 249 (2010), 2316–2356.
[8] H. I. Freedman and Y. T. Xu, Models of competition in the chemostat with instantaneous and delayed nutrient recycling, J. Math. Biol., 31 (1993), 513–527.
[9] A. Gray, D. Greenhalgh, L. Hu, X. Mao and J. Pan, A stochastic differential equation SIS epidemic model, SIAM J. Appl. Math., 71 (2011), 876–902.
[10] R. Z. Has’minskii, Stochastic Stability of Differential Equations, Sijthoff and Noordhoff, Alphen aan den Rijn, The Netherlands, 1980.
[11] X.-Z. He and S. G. Ruan, Global stability in chemostat-type plankton models with delayed nutrient recycling, J. Math. Biol., 37 (1998), 253–271.
[12] X.-Z. He, S. G. Ruan and H. X. Xia, Global stability in chemostat-type equations with distributed delays, SIAM J. Math. Anal., 29 (1998), 681–696.
[13] L. Imhof and S. Walcher, Exclusion and persistence in deterministic and stochastic chemostat models, J. Differ. Equ., 217 (2005), 26–53.
[14] V. S. Ivlev, Experimental Ecology of the Feeding of Fishes, Yale University Press, New Haven, 1961.
[15] D. X. Jia, T. H. Zhang and S. L. Yuan, Pattern dynamics of a diffusive toxin producing phytoplankton-zooplankton model with three-dimensional patch, Int. J. Bifurcat. Chaos, 29 (2019), 1930011.
[16] J. Jiang, A. L. Shen, H. Wang and S. L. Yuan, Regulation of phosphate uptake kinetics in the bloom-forming dinoflagellates Procentrum donghaiense with emphasis on two-stage dynamic process, J. Theor. Biol., 463 (2019), 12–21.
[17] D. Li, S. Q. Liu and J. A. Cui, Threshold dynamics and ergodicity of an SIRS epidemic model with Markovian switching, J. Differ. Equ., 263 (2017), 8873–8915.
[18] M. Liu and M. L. Deng, Permanence and extinction of a stochastic hybrid model for tumor growth, Appl. Math. Lett., 94 (2019), 66–72.
[19] Q. Liu and D. Q. Jiang, Stationarity and periodicity of positive solutions to stochastic SEIR epidemic models with distributed delay, Discrete Contin. Dyn. Syst., 22 (2017), 2479–2500.
[20] Q. Liu and D. Q. Jiang, Stationary distribution and extinction of a stochastic predator-prey model with distributed delay, Appl. Math. Lett., 78 (2018), 79–87.
[21] Q. Liu, D. Q. Jiang, T. Hayat and A. Alsaedi, Long-time behavior of a stochastic logistic equation with distributed delay and nonlinear perturbation, Physica A, 508 (2018), 289–304.
S. G. Ruan, Persistence and coexistence in zooplankton-phytoplankton-nutrient models with instantaneous nutrient recycling, *J. Math. Biol.*, 31 (1993), 633–654.

S. G. Ruan, The effect of delays on stability and persistence in plankton models, *Nonlinear Analysis*, 24 (1995), 575–585.

S. G. Ruan and X.-Z. He, Global stability in chemostat-type competition models with nutrient recycling, *SIAM J. Appl. Math.*, 58 (1998), 170–192.

K. Y. Song, W. B. Ma, S. B. Guo and H. Yan, A class of dynamic models describing microbial flocculant with nutrient competition and metabolic products in wastewater treatment, *Adv. Differ. Equ.*, 2018 (2018), Paper No. 33, 14 pp.

W. M. Wang, Y. L. Cai, J. L. Li and Z. J. Gui, Periodic behavior in a FIV model with seasonality as well as environment fluctuations, *J. Frankl. Inst.*, 354 (2017), 7410–7428.

L. Wang and D. Q. Jiang, A note on the stationary distribution of the stochastic chemostat model with general response functions, *Appl. Math. Lett.*, 73 (2017), 22–28.

R. H. Whittaker, *Communities and Ecosystems*, Macmillan, New York, 1975.

G. S. K. Wolkowicz, H. X. Xia and S. G. Ruan, Competition in the chemostat: A distributed delay model and its global asymptotic behavior, *SIAM J. Appl. Math.*, 57 (1997), 1281–1310.

G. S. K. Wolkowicz, H. X. Xia and J. H. Wu, Global dynamics of a chemostat competition model with distributed delay, *J. Math. Biol.*, 38 (1999), 285–316.

D. M. Wu, H. Wang and S. L. Yuan, Stochastic sensitivity analysis of noise-induced transitions in a predator-prey model with environmental toxins, *Math. Biosci. Eng.*, 16 (2019), 2141–2153.

D. Y. Xu, Y. M. Huang and Z. G. Yang, Existence theorems for periodic Markov process and stochastic functional differential equations, *Discrete Contin. Dyn. Syst.*, 24 (2009), 1005–1023.

C. Q. Xu and S. L. Yuan, An analogue of break-even concentration in a simple stochastic chemostat model, *Appl. Math. Lett.*, 48 (2015), 62–68.

C. Q. Xu and S. L. Yuan, Competition in the chemostat: A stochastic multi-species model and its asymptotic behavior, *Math. Biosci.*, 280 (2016), 1–9.

C. Q. Xu, S. L. Yuan and T. H. Zhang, Average break-even concentration in a simple chemostat model with telegraph noise, *Nonlinear Anal. Hybrid. Syst.*, 29 (2018), 373–382.

X. W. Yu, S. L. Yuan and T. H. Zhang, The effects of toxin-producing phytoplankton and environmental fluctuations on the planktonic blooms, *Nonlinear Dyn.*, 91 (2018), 1653–1668.

X. W. Yu, S. L. Yuan and T. H. Zhang, Persistence and ergodicity of a stochastic single species model with Allee effect under regime switching, *Commun. Nonlinear Sci. Numer. Simul.*, 59 (2018), 359–374.

X. W. Yu, S. L. Yuan and T. H. Zhang, Asymptotic properties of stochastic nutrient-phytoplankton food chain models with nutrient recycling, *Nonlinear Anal. Hybrid. Syst.*, 34 (2019), 209–225.

X. W. Yu, S. L. Yuan and T. H. Zhang, Survival and ergodicity of a stochastic phytoplankton-zooplankton model with toxin-producing phytoplankton in an impulsive polluted environment, *Appl. Math. Comput.*, 347 (2019), 249–264.

S. Q. Zhang, X. Z. Meng, T. Feng and T. H. Zhang, Dynamics analysis and numerical simulations of a stochastic non-autonomous predator-prey system with impulsive effects, *Nonlinear Anal. Hybrid. Syst.*, 26 (2017), 19–37.

S. N. Zhao, S. L. Yuan and H. Wang, Threshold behavior in a stochastic algal growth model with stoichiometric constraints and seasonal variation, *J. Differ. Equ.*

Y. Zhao, L. You, D. Burkow and S. L. Yuan, Optimal harvesting strategy of a stochastic inshore-offshore hairtail fishery model driven by Lévy jumps in a polluted environment, *Nonlinear Dyn.*, 95 (2019), 1529–1548.

Y. Zhao, S. L. Yuan and J. Ma, Survival and stationary distribution analysis of a stochastic competitive model of three species in a polluted environment, *B. Math. Biol.*, 77 (2015), 1286–1326.
[48] Y. Zhao, S. L. Yuan and T. H. Zhang, Stochastic periodic solution of a non-autonomous toxic-producing phytoplankton allelopathy model with environmental fluctuation, Commun. Nonlinear Sci. Numer. Simul., 44 (2017), 266–276.

[49] Y. Zhao, S. L. Yuan and T. H. Zhang, The stationary distribution and ergodicity of a stochastic phytoplankton allelopathy model under regime switching, Commun. Nonlinear Sci. Numer. Simul., 37 (2016), 131–142.

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