Evolution equation for soft physics at high energy

P. Brogueira
Departamento de Física, IST, Av. Rovisco Pais, 1049-001 Lisboa, Portugal
E-mail: pedro@fisica.ist.utl.pt

J. Dias de Deus
CENTRA, Departamento de Física, IST, Av. Rovisco Pais, 1049-001 Lisboa, Portugal
E-mail: jdd@fisica.ist.utl.pt

Abstract. Based on the non-linear logistic equation we study, in a qualitative and semi-quantitative way, the evolution with energy and saturation of the elastic differential cross-section in $pp(\bar{p}p)$ collisions at high energy. Geometrical scaling occurs at the black disk limit, and scaling develops first for small values of the scaling variable $|t|/\sigma_{\text{tot}}$. Our prediction for $d\sigma/dt$ at LHC, with two zeros and a minimum at large $|t|$ differs, as far as we know, from all existing ones.

PACS numbers: 13.85.Dz,13.85.Lg
Saturation phenomena are expected to dominate QCD physics at high energy and high matter density \([1, 2]\), as it may happen at LHC and cosmic rays at ultra high energies. Non linear differential equations include, in a natural way, saturation effects. This happens with the well known logistic equation\([3]\), which can be seen as a simplified version of the B-K equation\([4]\). See\([5]\) and \([6]\) for discussions on evolution and saturation.

We shall concentrate here in the evolution of the imaginary part of the impact parameter elastic amplitude, or the profile function \(\Gamma(b, R) \equiv \text{Im} B(b, R)\), where \(b\) is the impact parameter, related to angular momentum \(\ell\) by

\[
b \simeq \frac{2}{\sqrt{s}} \ell,
\]

where \(\sqrt{s}\) is the centre of mass energy, and \(R\) is an increasing with energy radial scale parameter. Partial wave unitarity constrains \(\Gamma(b, R)\):

\[
0 \leq \Gamma(b, R) \leq 1.
\]

We now write two logistic equations, in \(R\) and \(b\), respectively:

\[
\frac{\partial \Gamma}{\partial R} = \frac{1}{\gamma} (\Gamma - \Gamma^2),
\]

and

\[
\frac{\partial \Gamma}{\partial b} = -\frac{1}{\gamma} (\Gamma - \Gamma^2),
\]

where \(\gamma > 0\) is practically a constant. From \([3]\) one sees that \(\partial \Gamma/\partial R > 0\) and that, for fixed \(b\) and \(\Gamma > 0\), \(\Gamma\) reaches the black disk limit: \(\Gamma = \Gamma^2 = 1\). From \([3]\) one sees that in general \(\Gamma\) is a decreasing function of \(b\) and that, for large \(b\), \(\Gamma\) decreases, as expected, exponentially (\(\Gamma \sim \exp -b/\gamma\)), saturation occurring first at small \(b\).

A solution of \([3]\) and \([4]\), not the most general one, is:

\[
\Gamma(b, R) = \frac{1}{\exp \frac{b - R}{\gamma} + 1}.
\]

The total and elastic cross-section are written as

\[
\sigma_{\text{tot}}(s) = 2\pi \int \Gamma(b, R) db^2
\]

and, neglecting real part contributions,

\[
\sigma_{\text{el}}(s) = \pi \int |\Gamma(b, R)|^2 db^2,
\]

respectively. The imaginary part of the elastic amplitude \(\text{Im} F(t, R)\) is the Fourier-Bessel transform of \(\Gamma(b, R)\) and the differential elastic cross-section is written as

\[
\frac{d\sigma}{dt} = \frac{\sigma_{\text{tot}}^2}{16\pi} \frac{\text{Im} F(t, R)^2}{\text{Im} F(0, R)^2}.
\]

It should be noticed that in regions of energy where \(\gamma/R \simeq \text{const.}\), as it happens at ISR energies \((20 \lesssim \sqrt{s} \lesssim 60 \text{ GeV})\), \(\Gamma(b, R)\) satisfies geometrical scaling \([7]\),

\[
\Gamma(b, R) \xrightarrow{\gamma/R \approx \text{const.}} \Gamma(\beta),
\]
Evolution equation for soft physics at high energy

with

\[ \beta \equiv \frac{b}{R}, \]  

and,

\[ \frac{d\sigma}{dt} \sim R^2 \left( f(tR^2) \right)^2. \]  

As \( \gamma \), contrary to \( R \), does not show in general a consistent dependence on energy, in the limit \( R \to \infty \), \( \gamma/R \to 0 \) and one obtains again scaling [8],

\[ \Gamma(b, R) \xrightarrow{R \to \infty} \Gamma(\beta) \equiv \begin{cases} 1, & 0 \leq \beta < 1 \\ 0, & \beta > 1 \end{cases}, \]  

with \( \sigma_{\text{tot}} \sim 10 \) and \( \sigma_{\text{el}}/\sigma_{\text{tot}} \to \text{const.} = 1/2. \)

It should be also noticed that the parameter \( R \) in (5) separates the region of negative curvature, \( b < R \), from the region of positive curvature, \( b > R \). In fact, \( R \) plays the role of the angular momentum \( L \), used in the proof of Froissart bound [9] by Martin and collaborators [10], that separates the region that contributes in a significant way to the total cross-section, \( \ell < L \), from the region that is negligible, \( \ell > L \):

\[ R = \frac{2}{\sqrt{s}} L = \frac{1}{\sqrt{t_0}} \ln \left( \frac{s}{s_0} \right), \]  

with \( \sqrt{t_0} = 2m_\pi. \)

When comparing our model to experimental \( d\sigma/dt \) one finds that \( \gamma \) takes values always of the order \( \gamma \simeq 1 \) mb\(^{1/2} \), while \( R \) is a monotonically increasing function of energy.

In fact \( \gamma \), the parameter that controls the low density region, \( b > R \), can be seen as a measure of the range of the interaction, with \( \gamma \sim (2m_\pi)^{-1} \) in the Yukawa picture. The evolution of the cross-sections with the energy is controlled by the single parameter \( R(s) \), the effective impact parameter radius.

When studying the evolution of the amplitude (5) with energy one finds three regimes (see Fig. 1):

i) \( \sqrt{s} \lesssim 20 \) GeV, Fig. 1a).

This is the region corresponding to linear evolution, with \( \Gamma \) small and with exponential behavior, \( d\sigma/dt \) being a monotonically decreasing function of \(-t\).

ii) \( 20 \lesssim \sqrt{s} \lesssim 63 \) GeV (ISR energies), Figs. 1b) and c).

In this region a dip, which is a minimum, not a zero, appears at \( -t \approx 1.4 \) GeV\(^2 \) and slowly moves to the left as energy increases. Conventional wisdom says that the dip results from a zero: interference between one-Pomeron and two-Pomeron exchanges [12].

iii) \( 500 \lesssim \sqrt{s} \lesssim 1.8 \) TeV, Figs. 1d), e) and f).

In this region the minimum becomes negative, originating a pair of zeros. Instead of the clean second maximum of region ii) one has now a kind of shoulder, but with a cross-section higher by an order of magnitude.
Evolution equation for soft physics at high energy

Figure 1. \(d\sigma/dt\) as function of \(-t\) at different energies, showing the sequence: no structure in a), one minimum in b) and c), two zeros in d) and e) and two zeros and one minimum in f) at LHC. Values of \(\gamma\) and \(R\): a) \(\gamma = 1.020, R = 1.972\); b) \(\gamma = 1.026, R = 2.171\); c) \(\gamma = 1.090, R = 2.259\); d) \(\gamma = 1.000, R = 2.575\); e) \(\gamma = 1.016, R = 2.496\); f) \(\gamma = 1.078, R = 2.683\); g) \(\gamma = 1, R = 3.770\). Data from [11]. Dashed line: only imaginary part contribution. Full line: the real part of the amplitude is included.

In Fig. 1g) we have also included our expectation for LHC (assuming \(\sigma_{\text{tot.}} \approx 110\) mb - see [13] for expected range of values - and \(\gamma = 1\) mb\(^{1/2}\)). Our LHC curve clearly shows
Figure 2. $d(\sigma/dt) / (d\sigma/dt(0))$ as a function of the scaling variable $|t|\sigma_{\text{tot.}}$ showing the approach to black disk geometrical scaling from small to larger values of the scaling variable. Scaling only applies to the imaginary part of the amplitude. The parameter $\gamma$ was put equal 1.

how the evolution towards the black disk continues: from a monotonically decreasing curve at large $-t$ a minimum starts developing which at some stage generates a pair of zeros to join the previous pair. And so on! In the black disk limit we just have a sequence of pairs of zeros.

At high energy, $\sqrt{s} \gtrsim 60$ GeV, when $\sigma_{pp} \simeq \sigma_{\bar{p}p}$ is not difficult to use the derivative dispersion relations [14] to estimate the real part contribution to the differential cross section. In Figs. 1(c) to 1(g) we show, in full line, $d\sigma/dt$ including the real part correction. The real part contribution is important at the zeros of $\Gamma$.

In Fig. 2 we show the geometrical scaling plot (see (11)) of $d\sigma/dt / d\sigma/dt(t=0)$ as function of the variable $|t|\sigma_{\text{tot.}}$ for different values of $R$ and for $\gamma = 1$ mb$^{1/2}$, and the black disk limit. The way the approach to the scaling curve is achieved seems clear: as the energy, or $R$, increases scaling is satisfied for larger $|t|\sigma_{\text{tot.}}$ values. The LHC curve corresponds to $\sigma_{\text{tot.}} = 110$ mb, $\sigma_{\text{el.}}/\sigma_{\text{tot.}} \simeq 0.28$.

In Fig. 3 we show the correlation between $\sigma_{\text{el.}}/\sigma_{\text{tot.}}$ and $\sigma_{\text{tot.}}$ in comparison with data. Note the transient geometrical scaling at ISR energies, $\sigma_{\text{el.}}/\sigma_{\text{tot.}} \simeq \text{const.}$.

It should be mentioned that the curves shown in Fig. 1 are not fits to data, but represent qualitative descriptions. This means that $\sigma_{\text{el.}}$ and $\sigma_{\text{tot.}}$, which are controlled by the first points in the low $|t|$ region, are not constrained. That is the reason why the values of $R(s)$ in Fig. 1 do not coincide with the values used in Fig. 3. For instance, at ISR energies, in Fig. 1 $\sigma_{\text{el.}}$ and $\sigma_{\text{tot.}}$ are larger by a factor of the order of 15% relative to the true values (in Fig. 3).

Recently, in several papers [15, 16, 17] the questions of soft physics were addressed from different points of view. However, contrary to the present paper, no attempts
Figure 3. $\sigma_{el.}/\sigma_{tot.}$ as a function of $\sigma_{tot.}$, making use of (5). Data points, from left to right: $\sqrt{s} = 23.5$ GeV, $\gamma = 1.065$, $R = 1.710$; $\sqrt{s} = 30.6$ GeV, $\gamma = 1.090$, $R = 1.731$; $\sqrt{s} = 44.9$ GeV, $\gamma = 1.095$, $R = 1.805$; $\sqrt{s} = 52.8$ GeV, $\gamma = 1.107$, $R = 1.810$; $\sqrt{s} = 200$ GeV, $\gamma = 1.123$, $R = 2.100$; $\sqrt{s} = 540$ GeV, $\gamma = 1.080$, $R = 2.498$; $\sqrt{s} = 900$ GeV, $\gamma = 1.026$, $R = 2.662$; $\sqrt{s} = 1800$ GeV, $\gamma = 1.046$, $R = 3.050$. We expect for LHC $\sigma_{el.}/\sigma_{tot.} = 0.28$ and $\sigma_{tot.} = 110$ mb. Data from [11].

were made to explain the qualitative aspects of $d\sigma/dt$ evolution with energy. Instead, interesting problems related to inelastic diffraction, as Higgs production, were addressed. The black disk perspective varied significantly from [15] to [16]: the black disk limit being already present at LHC energies in [15], but occurring at extremely high energies in [16]. For a discussion on the black disk limit see also [18].

The relevance or not of the Froissart bound (see [19] and [20]) and a dynamical interpretation of it (see [21] and [22]) are still matters open to discussion.

Finally, we summarize our work. Starting from the non-linear logistic equation we obtained a solution for the high-energy imaginary part of the amplitude, and we were able to describe in a qualitative and semi-quantitative way the essential features of the evolution of the differential elastic cross-section with energy, namely the sequence: no structure in $|t|$, one minimum, two zeros and so on. Our prediction for $d\sigma/dt$ at LHC energies is different from all the ones we are aware of (see, for instance, [23]).

Acknowledgments

We would like to thank Carlos Pajares and Jaime Alvarez-Muniz for discussions on cosmic rays at very high energy. Special thanks are due to Teresa Peña for information on zeros and minima in nuclear reactions.
References

[1] L.V. Gribov, E. Levin and M. Ruskin, Phys. Rep. 49 (1994).
[2] L. McLerran and R. Venugopalan, Phys. Rev. D49 (1994) 2233, arXiv:9309289 [hep-ph]; Phys. Rev. D49 (1994) 3352, arXiv:9311205 [hep-ph]; Phys. Rev. D50 (1994) 2225, arXiv:9402335 [hep-ph].
[3] P.-F. Verhulst, Nouv. Mm. De lAcademie Royale des Sci. et Belles-Lettres de Bruxelles 18 (1845) 1; Nouv. Mm. De lAcademie Royale des Sci. et Belles-Lettres de Bruxelles 20 (1847) 1.
[4] I. Balitsky, Nucl. Phys. B463 (1996) 99, arXiv:9509348 [hep-ph]; Y.V. Kovchegov, Phys. Rev. D60 (1999) 034008, arXiv:9901281 [hep-ph].
[5] O. V. Selyugin and J.-R. Cudell, Diffraction 2006, PoS (DIFF 2006) 057; J. R. Cudell and O. V. Selyugin, arXiv:0612046 [hep-ph].
[6] J. Dias de Deus and J.G. Milhano, Nucl. Phys. A795 (2007) 98, arXiv:0701215 [hep-ph].
[7] J. Dias de Deus, Nucl. Phys. B59 (1973) 231; A.J. Buras and J. Dias de Deus, Nucl. Phys. B71 (1974) 481.
[8] G. Auberson, T. Kinoshita and A. Martin, Phys. Rev. D3 (1970) 3185.
[9] M. Froissart, Phys. Rev. 123 (1961) 1053.
[10] A. Martin, Nuovo Cim. A42 (1965) 930; Y.S. Jin and A. Martin, Phys. Rev. 135B (1964) 1375; L. Lukaszuk and A. Martin, Nuovo Cim. A52 (1967) 122.
[11] N. Amos et al., Phys. Lett. B120 (1983) 460; Phys. Lett. B128 (1983) 343; U. Amaldi and K.R. Schubert, Nucl. Phys. B166 (1980) 301; M. Bozzo et al., Phys. Lett. B147 (1984) 392; N. Amos et al., Nucl. Phys. B262 (1985) 689; F. Abe et al., Phys. Rev. D50 (1994) 5550; A. Bohm, Czechoslovak Journal of Physics B26 (1976) 363.
[12] P.V. Landshoff, Soft Collisions of Hadrons, in 6th Topical Workshop on proton-antiproton collider physics, pg. 561, World Scientific (1987).
[13] T.Wibig, arXiv:0810.5281 [hep-ph], 28 October 2008.
[14] J. Bronzan, G. Kane, P.Sukhatme, Phys. Lett. B 49 (1974) 272.
[15] L. Frankfurt, C.E. Hyde, M. Strikman and C. Weiss, Rapidity gap survival, 12th International Conference on Elastic and Diffractive Scattering, Hamburg (2007), arXiv:0710.2942 [hep-ph]; L. Frankfurt, M. Strikman and C. Weiss, Phys. Rev. D69 (2004) 114010.
[16] E. Gotsman, E. Levin and U. Maor, arXiv:0805.0418 [hep-ph]; arXiv:0708.1596 [hep-ph].
[17] A.D. Martin, M.G. Ryskin and V.A. Khoze, arXiv:0903.2980 [hep-ph]; V.A. Khoze, A.D. Martin and M.G. Ryskin, Eur. Phys. J. C18 (2000) 167.
[18] M. M. Block and R. N. Cahn, Phys. Lett. B149 (1984) 245.
[19] A. Martin, arXiv:0812.0680 [hep-ph];
[20] S. Nussinov, arXiv:0805.1540 [hep-ph].
[21] E. Ferreiro, E. Iancu, K. Itakura and L. McLerran, Nucl. Phys. A710 (2002) 373.
[22] A. Kovner and U. Wiedemann, Phys. Lett. B551 (2003) 311.
[23] C. Bourrely, arXiv:0810.0565 [hep-ph], C. Bourrely, J. Soffer and T.T. We, Phys. Rev. D19 (1979) 3249.