On the Nature of the Phase Transition in $SU(N)$, $Sp(2)$ and $E(7)$ Yang-Mills theory

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We study the nature of the confinement phase transition in $d = 3 + 1$ dimensions in various non-abelian gauge theories with the approach put forward in [1]. We compute an order-parameter potential associated with the Polyakov loop from the knowledge of full 2-point correlation functions. For $SU(N)$ with $N = 3, \ldots, 12$ and $Sp(2)$ we find a first-order phase transition in agreement with general expectations. Moreover our study suggests that the phase transition in $E(7)$ Yang-Mills theory also is of first order. We find that it is weaker than for $SU(N)$. We show that this can be understood in terms of the eigenvalue distribution of the order parameter potential close to the phase transition.

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I. INTRODUCTION

Understanding the confinement of gluons is a challenging problem, because characteristic and significant quantities providing both analytical and numerical access are difficult to identify straightforwardly. In recent years, the study of the confinement mechanism based on the IR behavior of gauge-dependent correlation functions has turned out to be both fruitful and inspiring. Most prominently, this has lead to two different confinement scenarios: the Gribov-Zwanziger scenario on one hand and the Kugo-Ojima scenario on the other hand. This scenarios are mutually related and connect signatures of color confinement with the low-momentum behavior of gluon and ghost correlation functions. They have been intensely investigated by a variety of nonperturbative field theoretical tools such as lattice gauge theory as well as functional methods.

Even if color confinement is eventually properly accounted for by a corresponding IR behavior of correlation functions, indicating, e.g., positivity violation and the absence of colored states in the physical state space, the remaining pressing open question is the relation of color confinement to quark confinement. Typical quark-confinement criteria such as those based on the Wilson-loop or Polyakov-loop expectation value in quenched QCD have long remained inaccessible from the pure knowledge of low-order correlation functions of the gauge sector. For instance, a direct computation of the heavy-quark potential requires additional knowledge, e.g., of the quark-gluon vertex or of the static gluon correlator in the Coulomb gauge.

In a more direct relation between color confinement and quark confinement, the relation between quark confinement and chiral symmetry breaking has more recently been studied with the aid of so-called dual observables. These formal and computational advances better our understanding of the mechanisms underlying confinement and chiral symmetry breaking and pave the way towards a first principle access to QCD with functional methods.

In particular, the approach put forward in gives us access not only to an order parameter for confinement, the Polyakov loop, but also to its full effective potential. The latter is a crucial input in Polyakov loop extended effective models such as the PNJL and the PQM models. In these models, the Polyakov loop potential is an external input the parameter of which are fixed to reproduce pure Yang-Mills lattice results. Evidently, these physics constraints do not completely fix the potential, leave aside its extension to full QCD. Different potentials have been studied, for a comparison see e.g. [16], and the physics at finite chemical potential is in fact very sensitive to parameter changes. In this regard the present approach provides the opportunity for a qualitative improvement of the above models since it allows to fix the Yang-Mills potentials completely from first principles, see also [17]. This has motivated further studies such as the development of a framework for the study of QCD with two colors [18]. Moreover the present approach underlies the fully dynamical continuum study of two-flavor QCD at finite temperature and quark chemical potential put forward in [14].

The present work is devoted to a more detailed analysis of the interrelation between color confinement and quark confinement as deduced in [1]. In addition to providing a more comprehensive technical insight into the underlying ideas, we illustrate this interrelation and the generality of the approach by applying it to a variety of non-abelian gauge theories near the deconfinement phase transition. In fact, while the phase transition is of second order in $SU(2)$ Yang-Mills theory, it is well-known from lattice...
simulations that a first-order phase transition occurs in $SU(N)$ gauge theories with $N \geq 3$. This brings up the question on how the nature of the phase transition is related to the properties of the underlying gauge group. It has been conjectured in [13] based on the order-disorder nature of the deconfinement phase transition with respect to center symmetry that the phase transition of $SU(2)$ Yang-Mills theory should fall into the Ising universality class. This observation based on the symmetry properties of the center of the gauge group does not necessarily extend to the other gauge groups with a center symmetry agreeing with a 2nd order universality class. In [20, 21] it has been conjectured that the dynamics near the critical temperature is sensitive to the number of dynamical degrees of freedom in the confined and deconfined phase. In accordance with this conjecture a first-order phase transition has been found for $Sp(2)$ even though the center of the group is $Z(2)$ [20, 21]. In the present paper we study $Sp(2)$ and $E(7)$ gauge theory (the center of both groups is $Z(2)$) and compare the results to our findings for $SU(N)$ gauge theories also in order to shed more light on the conjecture put forward in [21].

Our study of the deconfinement phase transition is based on an order parameter related to the Polyakov loop variable

$$L[A_0] = \frac{1}{N_c} \text{tr} \mathcal{P} \exp \left( ig \int_0^\beta dx_0 A_0(x_0, x) \right),$$

more precisely on the expectation value of $\langle A_0 \rangle$ in Polyakov gauge, see also [17]. Then, $\langle A_0 \rangle$ is sensitive to topological defects related to confinement [22], and also serves as a deconfinement order parameter.

The effective potential of $\langle A_0 \rangle$ is accessible from the knowledge of gauge correlation functions by means of the functional renormalization group (RG), [1]. As an additional characteristic ingredient for a quantitative understanding of the phase transition we introduce and identify eigenvalue distributions of the order parameter which exhibit characteristic traces and facilitate a quantitative understanding of the behavior of the corresponding effective potential.

The paper is organized as follows: In Sect. II and III we discuss general aspects of functional flows for a study of non-abelian gauge theories. In Sect. IV we discuss how background-field RG flows can be constructed from RG flows in Landau-gauge Yang-Mills theories. We discuss a sufficient confinement criterion in Sect. V before we present our study of the nature of the phase transition in $SU(N)$, $Sp(2)$ and $E(7)$ gauge theory in Sect. VI.

II. FUNCTIONAL FLOWS AND OPTIMIZATION

For our study of the phase transition of non-abelian gauge theories we employ the functional RG for the effective action $\Gamma_k$ [23]. This allows us to interpolate between the initial UV action related to the classical action $\Gamma_k = \Lambda \sim S$ and the full quantum effective action $\Gamma = \Gamma_k = 0$, being the 1PI generating functional. The infrared (IR) regulator scale $k$ separates the fluctuations with momenta $p^2 \gtrsim k^2$ which are already included in $\Gamma_k$, from those with smaller momenta which still have to be integrated out. The full RG trajectory is given by the solution to the Wetterich equation ($t = \ln(k/\Lambda)$),

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} \text{STr} \frac{1}{\Gamma_k^{(2)}[\Phi]} \partial_t R_k,$$

where $\Gamma_k^{(2)}$ denotes the second functional derivative with respect to the dynamical field $\Phi$, collectively summarizing gluon and ghost fields in the present context. The super trace STr sums over momenta, internal indices and species of fields and includes a negative sign for the ghost fields. The regulator function $R_k$ specifies the details of the Wilsonian momentum-shell integrations. See [24, 27] for reviews on gauge theories.

In the present context, we are interested in the effective potential for an order-parameter quantity which is related to the local part of the full effective action $\Gamma = \Gamma_k = 0$. The latter can formally be obtained from the integrated flow,

$$\Gamma[\Phi] = \Gamma_{\Lambda}[\Phi] - \int_0^\Lambda \frac{dk}{k} \frac{1}{2} \text{STr} \frac{1}{\Gamma_k^{(2)}[\Phi]} \partial_t R_k.$$  (3)

Eq. (3) is an equation for the full quantum effective action $\Gamma$ and has a priori no $\Lambda$ dependence. A partial integration leads to

$$\Gamma = \frac{1}{2} \text{STr} \ln \Gamma^{(2)} + \int_0^\Lambda \frac{dk}{k} \Delta \Gamma_k + \Gamma_{\Lambda} - \frac{1}{2} \text{STr} \ln(\Gamma^{(2)} + R_{\Lambda}).$$  (4)

Note that the first term on the right-hand side does not depend on the regulator function. The initial conditions of the flow at $\Lambda$ including possible subtractions are comprised in the second line of Eq. (4). The second term in the first line of (4) reads

$$\Delta \Gamma_k := -\frac{1}{2} \text{STr} \frac{1}{\Gamma_k^{(2)}} \partial_t \Gamma_k^{(2)}.$$  (5)

For general regulators, $\Delta \Gamma_k$ is only finite upon subtractions contained in the second line of Eq. (4), as $\partial_t \Gamma_k^{(2)}$ does not vanish for large momenta. Whereas the representation (4) thus is of limited practical use in the general case, it is ideally suited for the determination of an order-parameter potential which is UV finite from the beginning, as is the Weiss potential. In fact, our quantitative results for the order-parameter potential are dominated already by the first term of Eq. (4). For the remainder of this section, we will, however, be concerned with an optimized strategy to evaluate the contributions from the $\Delta \Gamma_k$ term.
The evaluation of this ‘RG-improvement term’ \( \sim \partial_t \Gamma_k \) requires two nontrivial ingredients: full gluon and ghost propagators in the presence of an IR regulator, \( G_\kappa = (\Gamma^{(2)}_k + R_k)^{-1} \) and the flow of the inverse propagator \( \partial_t \Gamma^{(2)}_k \). This information has been made available in [29] where the full momentum dependence of Landau-gauge propagators has been computed within optimized RG flows. For earlier RG calculations, acquiring partial knowledge about Yang-Mills propagators and providing evidence for the Kugo-Ojima/Gribov-Zwanziger confinement scenarios, see [8, 29, 30].

Optimization of the RG flow has not only the advantage of a more stable and faster convergent numerical scheme; in a given truncation, it can actually be posed as a stability/convergence problem [31]. It also provides for a link to using propagators obtained from lattice simulations, see below. Full functional optimization can be reformulated as the quest for a minimal flow trajectory for general functional flows [26]. In other words, for a given gap \( 1/k^2_{\text{eff}} \) of the propagator which constitutes the inverse of the physical infrared cut-off, the integrated RG flow is already as close as possible to the full theory, as the remaining flow trajectory is minimal. This results in the following propagator for the corresponding optimal regulator \( R_{\text{opt}} \) [23]:

\[
\frac{1}{\Gamma_k^{(2)} + R_{\text{opt}}} (p^2) = \frac{1}{\Gamma_0^{(2)} (p^2)} \theta(\Gamma_0^{(2)} (p^2) - k^2_{\text{eff}}) + \frac{1}{k^2_{\text{eff}}} \theta(k^2_{\text{eff}} - \Gamma_0^{(2)} (p^2)).
\]

(6)

with \( k^2_{\text{eff}} = \Gamma_0^{(2)} (k^2) \). The propagator \((\Gamma_k^{(2)} + R_{\text{opt}})^{-1}\) is already the full propagator for all eigenvalues of \( \Gamma_0^{(2)} \) belonging to \( \text{spec} \{ \Gamma_0^{(2)} \} \geq k^2_{\text{eff}} \), and is identical to the gap for the remaining eigenvalues, \( \text{spec} \{ \Gamma_0^{(2)} \} < k^2_{\text{eff}} \). The choice [26] requires non-trivial field redefinitions. For the gauge field we have \( A = Z_{\Lambda}^{1/2} \hat{A} \) with \( \partial_t \hat{A} = 0 \) and

\[
\frac{\partial_t Z_A}{Z_A} = \left( \frac{1}{2} \text{STr} \frac{1}{\Gamma_k^{(2)} + R_k} \partial_t R_k \right)^{(2)} (p^2) \times \frac{1}{\Gamma_0^{(2)} (p^2)} \theta(k^2_{\text{eff}} - \Gamma_0^{(2)} (p^2)),
\]

(7)

where the right-hand side is evaluated at \( \Phi = 0 \). Eq. (7) ensures \( \partial_t \Gamma_k^{(2)} \theta(k^2_{\text{eff}} - \Gamma_0^{(2)}) = 0 \), and hence

\[
(\Gamma_k^{(2)} - \Gamma_0^{(2)}) \theta(k^2_{\text{eff}} - \Gamma_0^{(2)}) = 0.
\]

(8)

The conditions (8) and (9) allow us to provide the optimal regulator in an explicit form [22]:

\[
R_{\text{opt}} = (k^2_{\text{eff}} - \Gamma_k^{(2)} (p^2)) \theta(k^2_{\text{eff}} - \Gamma_0^{(2)} (p^2)).
\]

(9)

With the choice (9), the flow of Green’s functions can be computed within an iteration of the integrated flow starting from an initial value for \( \Gamma_0^{(2)} \).

Let us elucidate aspects of the optimized flow in the context of the integrated flow [4]. For the optimized flow the relation (9) follows from a direct integration of the flow: the first term in Eq. (9) relates to integrating the \( \partial_t k_{\text{eff}} \) contributions of the related flow, the second term in Eq. (9) is the \( t \) integral of the contributions \( \sim \partial_t \Gamma_k^{(2)} \). Details of the numerical computation of Eq. (9) can be found in App. [A].

From a general perspective, the effective action [4] together with the optimized regulator can be considered as a DSE within a consistent BPHZ-type non-perturbative renormalization [22], where the \( \Lambda \)-dependent terms provide the classical action and the subtraction terms. The computational benefit in comparison to standard DSE equations is the explicit finiteness of Eq. (9) in any truncation without the need of further additive or multiplicative renormalizations. The second term on the right-hand side of the first line constitutes an RG improvement term.

III. BACKGROUND FIELD FLOWS

In order to arrive at the effective potential for an order-parameter field, we parameterize the fluctuations with respect to a background field which is related to the order parameter. In Yang-Mills theories, this decomposition into fluctuating modes and the background field can be organized such that the resulting background-field action preserves a residual gauge symmetry, e.g. [32]. This approach using the background-field gauge can be understood as a simple extension of Yang-Mills theories within general covariant gauges. The gauge condition \( \partial_\mu A_\mu = 0 \) is generalized to

\[
D_\mu (\bar{A})(A - \bar{A})_\mu = 0,
\]

(10)

for an unspecified background field \( \bar{A} \). Equation (10) implemented on configuration space in a strict sense defines Landau-DeWitt gauge. A less strict Gaussian average over the gauge condition \( D_\mu (\bar{A})(A - \bar{A})_\mu = C \) with a probability distribution \( \sim \exp(-1/\xi \int \mathrm{tr} C^2) \) leads to the background-field equivalent of a general covariant gauge. Such a formulation has the benefit of an auxiliary gauge symmetry for the effective action under a transformation of both, the full gauge field \( A \to A + D\omega \) and the background \( \bar{A} \to \bar{A} + D\omega \). In this manner, the gauge condition (10) is unchanged since \( a = A - \bar{A} \) transforms as a tensor, \( a \to \{ a, \omega \} \).

With the gauge fixing (10), the effective action now depends also on the auxiliary field, \( \Gamma = \Gamma[\Phi, \bar{A}] \) with \( \Phi = \{ a, C, \bar{C} \} \). We emphasize that the background-field gauge transformation is an auxiliary symmetry. The effective action \( \Gamma = \Gamma[\Phi, \bar{A}] \) still carries non-trivial symmetry constraints, namely the Slavnov-Taylor identities (STI). These follow from a gauge or BRST transformation of the field \( \Phi \) at fixed background. Indeed, the underlying STI are that of a standard covariant gauge. Even though the gauge invariance is an auxiliary symme-
try, it facilitates the construction of a (physically) gauge-invariant effective action $\Gamma[\Phi] = \Gamma[0, A = A]$. The flow equation for $\Gamma[\Phi, \bar{A}]$ in such a setting reads
\[
\partial_t \Gamma_k[\Phi, \bar{A}] = \frac{1}{2} \text{Tr} \frac{1}{\Gamma_k^{(2,0)}[\Phi, \bar{A}]} R_k + R_k, \tag{11}
\]
where
\[
\Gamma_k^{(n,m)} = \frac{\delta^n}{\delta \Phi^n} \frac{\delta^m}{\delta A^m} \Gamma_k. \tag{12}
\]
The action $\Gamma_k[\Phi, \bar{A}]$ is still gauge invariant under background gauge transformations provided the regulator is still gauge invariant under background-field flow (14), we have to compute it separately. From now on, we restrict ourselves to the Landau-DeWitt gauge \cite{23} with the gauge parameter set to $\xi = 0$.

This gauge has several benefits: first, it projects on (covariantly) transversal degrees of freedom, and second, the longitudinal components of Green functions decouple from the dynamics of the transversal ones, and thirdly it is a fixed point of the flow \cite{30}. As the longitudinal components are subject to modified Slavnov-Taylor identities, this minimizes the truncation error. This is also related to a second issue, namely the gauge dependence of the background-field effective action $\Gamma_k[\bar{A}]$. For Landau-DeWitt gauge, this action is identical to the geometrical effective action, see e.g. \cite{24, 30, 11}, which is gauge invariant also with respect to gauge transformations. There, the fluctuation field agrees with $a = A - \bar{A}$ only in leading order. The background-field approach in Landau-DeWitt gauge can indeed be understood as the leading order of a manifestly gauge-invariant approach to functional RG flows \cite{23, 41}.

We proceed with constructing the key input $\Gamma_k^{(2,0)}[0, A]$. First we remark that $\Gamma_k^{(2,0)}[0, 0](p^2)$ is simply the propagator in Landau gauge which has been computed on the lattice \cite{23} as well as by functional methods \cite{30, 24}. The full RG trajectory $\Gamma_k^{(2,0)}[0, 0](p^2)$ has been computed in \cite{23} \cite{24} for the optimized regulator $R_{\text{opt}}$. Now the auxiliary background gauge symmetry comes to our aid. It constrains the extension of the Landau-gauge two-point function $\Gamma_k^{(2,0)}[0, 0]$ to $\Gamma_k^{(2,0)}[0, A]$ as the latter has to transform as a tensor under gauge transformations. We conclude that
\[
(\Gamma_k^{(2,0)}[0, A])_{\mu \nu}^a b = (\Gamma_k^{(2,0)}[0, 0](-D^2))_{\mu \nu}^a b + F_{\rho \sigma \mu \nu}^a b c d (D), \tag{16}
\]
where $F_{\rho \sigma \mu \nu}^a b c d$ denotes the field strength tensor in the adjoint algebra, and the function $f(x)$ is non-singular at $x = 0$. Note that covariantly longitudinal correction terms in Eq. (16) are irrelevant as we are in the Landau-DeWitt gauge. The $f$ terms cannot be obtained from the Landau gauge propagator. They are indeed related to higher Landau-gauge Green functions.

Next we briefly recall the results for the Landau gauge propagators \cite{23, 24, 28, 30}: the ghost and gluon propagators can be parameterized as
\[
\Gamma_{k, A}^{(2,0)}[0, 0](p^2) = p^2 Z_A(p^2) \Pi_T(p) \mathbb{I} + p^2 \frac{Z_L(p^2)}{\xi} \Pi_L(p) \mathbb{I}, \tag{17}
\]
where
\[
(\Pi_T)_{\mu \nu}(p) = \delta_{\mu \nu} - \frac{p_\mu p_\nu}{p^2}, \quad (\Pi_L)_{\mu \nu}(p) = \frac{p_\mu p_\nu}{p^2}, \quad \mathbb{I}_{ab} = \delta_{ab}, \tag{18}
\]
for the gluon and
\[
\Gamma_{k, C}^{(2,0)}[0, 0](p^2) = p^2 Z_C(p^2) \mathbb{I}, \tag{19}
\]
for the ghost. For the longitudinal dressing function, we have $Z_L = 1 + O(\xi)$. Hence, it drops out of all diagrams.

IV. BACKGROUND-FIELD FLOWS AND LANDAU-GAUGE YANG-MILLS THEORY

The crucial ingredient for our studies of the confinement phase transition is the two-point function $\Gamma_k^{(2,0)}[0, A]$. As it is not the output of the pure background-field flow \cite{14}, we have to compute it separately. From now on, we restrict ourselves to the Landau-DeWitt gauge \cite{10} with the gauge parameter set to $\xi = 0$. However, within the present context of the order-parameter potential, it is crucial to go beyond the approximation \cite{15} used in \cite{32}, as the confinement-deconfinement phase transition is rather sensitive to the correct mid-momentum and infrared behavior of the fluctuation-field propagator $\Gamma_k^{(2,0)}[0, A]$. Therefore, a proper distinction between the fluctuation-field and background-field dependence of the action is mandatory.
Beyond one loop. In the deep infrared, the dressing functions $Z_{A,C}$ exhibit a leading momentum behavior

$$Z_A(p^2 \to 0) \simeq (p^2)^{\kappa_A}, \quad Z_C(p^2 \to 0) \simeq (p^2)^{\kappa_C}. \quad (20)$$

Landau-gauge Yang-Mills theory admits a one-parameter family of infrared solutions consistent with RG invariance [9], the underlying structure still being subject to current research. This family of solutions can be parameterized by an infrared boundary condition for the ghost propagator, specifying a value for $Z_C(p^2 = 0)$. This fact is reflected in recent lattice solutions on relatively small lattices, [42], and in the strong-coupling limit [43], for an alternative point of view see [44].

For $Z_C(p^2 \to 0) \to 0$, there is a unique scaling solution, [15, 16]. The two exponents $\kappa_A$ and $\kappa_C$ are then related by the sum rule arising from a non-renormalization theorem for the ghost-gluon vertex [47].

$$0 = \kappa_A + 2\kappa_C + \frac{4 - d}{2}, \quad (21)$$

in $d$ dimensional spacetime [4, 12, 48]. Admissible solutions are bound to lie in the range $\kappa_C \in [1/2, 1]$. In pure non-abelian gauge theories, $\kappa_C$ has been computed by a variety of methods [3, 6]. The precise value depends on the IR behavior of the ghost-gluon vertex [48]. In most DSE and FRG computations we are led to ($d = 4$)

$$\kappa_C = 0.59535... \quad \text{and} \quad \kappa_A = -2\kappa_C = -1.1907..., \quad (22)$$

being the value for the optimized regulator [8]. The regulator dependence in FRG computations leads to a $\kappa_C$ range of $\kappa_C \in [0.539, 0.595]$, see [8]; for a specific flow, see [30]. These results entail the Kugo-Ojima/Gribov-Zwanziger confinement scenario. The gluon is infrared screened, the propagator even tends to zero, see [21], [22], whereas the ghost is infrared enhanced. Due to the non-renormalization property of the ghost-gluon vertex [17, 48], a running coupling can be defined in terms of

$$\alpha_s(p^2) = \frac{g^2}{4\pi Z_A(p^2)Z_C(p^2)}, \quad (23)$$

which runs towards an IR fixed point, see Eq. (20). In Fig. 1 we show the momentum dependence of the ghost- and gluon propagator as obtained from a functional RG study [3] in comparison to lattice results [6].

A different type of decoupling solution is found for non-vanishing $Z_C(0)$: here, the gluon propagator tends to a constant in the infrared, $p^2 Z_A(p^2) \to m^2$, for related work see e.g. [8, 49]. It should be stressed that the gluon propagator then does not correspond to the propagator of a massive physical particle, but clear indications for positivity violation related to gluon confinement are observed [8, 50]. Still, the gluon decouples from the dynamics as a massive particle. The qualitative infrared behavior is then characterized by the exponents

$$\kappa_A = -1, \quad \text{and} \quad \kappa_C = 0. \quad (24)$$

Though the infrared exponents for the scaling [22] and decoupling solutions [24] deviate from another, the propagators do only differ in the deep infrared. It has been suggested in [9] that the infrared boundary condition is directly related to the global part of the gauge fixing, hence reflecting different resolutions of the Gribov problem. Indeed, the infrared boundary condition has been used as a global completion of the gauge fixing in [42].

For most parts of the present work, the difference between the scaling and the decoupling solution is of minor importance. For concrete numerical computations, we combine information about the propagators as obtained from the lattice as well as functional methods, completing these propagators in the deep IR with the scaling solution. The latter is actually singled out by the requirement of global BRST for Landau gauge Yang-Mills with standard local BRST invariance.

The discussion so far applies to Yang-Mills theory at zero temperature. Several modifications arise in the presence of a thermal bath, such as the immediate replacement of continuous loop energies $p_0$ by Matsubara frequencies $\omega_n = 2\pi nT$ in the imaginary-time formalism. Moreover, the gluon propagator acquires an additional component, as the contributions longitudinal and transversal to the heat bath become independent.

In this work, we neglect the finite-temperature modifications of the propagators, but work with zero-temperature propagators evaluated at the Matsubara frequencies. In scalar theories it has been shown that this approximation already provides a quantitative insight into the finite-temperature phase structure [52].

First results for finite-temperature gluon- and ghost-propagators indeed indicate that the propagators are little modified [52, 53] for Matsubara frequencies $2\pi T n$ with $|n| \gtrsim 2, 3$. Significant changes have been found for the gluon propagator longitudinal to the heat bath which is increased compared to the transversal counterpart. We stress that the inclusion of the full temperature dependence of the propagators as well as the order-parameter fluctuations is inevitable for an accurate determination of, e.g., the critical exponents or the thermodynamic properties of the theory (see, e.g., [54]).

The presence of finite temperature also takes influence on the form of the propagator at finite background field, as another field invariant, the Polyakov loop $L$, exists. This adds further terms to the right-hand side of Eq. (10). Specializing to constant background fields $\bar{A} = (\langle A_0 \rangle)$, we find schematically

$$\langle \Gamma_k^{(2,0)}[0, (\langle A_0 \rangle)] \rangle_{\mu\nu} = \langle \Gamma_k^{(2,0)}[0, 0](-D^2) \rangle_{\mu\nu} + L \text{ terms}, \quad (25)$$

since the $f$ term in (10) vanishes, $F(\langle A_0 \rangle) = 0$, for $\langle A_0 \rangle =$const. In this work we drop the $L$ contribution in (25) which is related to the second derivative of the order-parameter potential via Nielsen identities [23, 33]. We expect that this term affects our results only on a quantitative level, e.g., the quality of critical exponents. For example in $SU(2)$ gauge theory, we expect a second-order
phase transition. Here, the critical dynamics encoded in the critical exponents is sensitive to order-parameter fluctuations as is well known from studies of scalar $O(N)$ theories. The $L$ terms take a direct influence on the spectrum of order-parameter fluctuations, so that we expect these terms to be relevant at criticality. Indeed, the role of order-parameter fluctuations has been studied in Ref. [17] for $SU(2)$ where the correct $Z_2$ critical exponents have been found. However, the phase transition in $SU(N)$ gauge theories ($N \geq 3$) is of first order and therefore less affected by our approximation of dropping the $L$ terms.

Let us finally stress that our approximations at finite temperature do not take any influence on our conclusions about confinement in the zero-temperature limit, discussed below. In particular, the background covariantization of the transverse propagator in Eq. (10) becomes exact in this limit, representing a first important result of our present work. This paves the way for a fully consistent low-energy RG analysis of QCD in the background-field formalism, see Refs. [55, 56] for a study of 1-flavor QCD.

From Eq. (10) and the above results for the Landau gauge propagators we already conclude that the truncation (15) is not working well in the (deep) infrared ($p \ll Q_{\text{CD}}$). There we expect a fixed point for the coupling, (23), which entails a constant dressing for the propagator of the background field, $\Gamma^{(0,2)}$: background gauge invariance leads to RG invariance of $gA$, and hence to

$$Z_\tilde{A} \sim Z_g^{-1},$$

which results in a constant dressing. Therefore the background field propagator $1/\Gamma^{(0,2)}$ diverges in the infrared whereas the propagator of the dynamical fluctuation field $a$, $1/\Gamma^{(2,0)}$, is suppressed in the infrared for both scaling and decoupling solution. Moreover, the ghost propagator is infrared enhanced for the scaling solution in contradistinction to the one-loop truncation used so far in most background field flows, that is $Z_C = 1$ which would only be compatible with the decoupling solution. We conclude that important aspects of the infrared physics can easily be missed by truncations based on [15].

V. CONFINEMENT CRITERION

Our study of the deconfinement phase transition is based on an order parameter related to the Polyakov loop variable, see Eq. (1). The negative logarithm of the Polyakov loop expectation value $\langle L \rangle$ can be interpreted as the free energy of a single static color source in the fundamental representation of the gauge group [57]. In this sense, an infinite free energy associated with confinement is indicated as $\langle L \rangle \to 0$, whereas $\langle L \rangle \neq 0$ signals deconfinement. For gauge groups with a nontrivial center, $\langle L \rangle$ measures whether center symmetry is realized by the ensemble under consideration [57]. As $\langle L \rangle$ transforms nontrivially under center transformations a center-symmetric (disordered) ground state automatically ensures $\langle L \rangle = 0$, whereas deconfinement $\langle L \rangle \neq 0$ is related to the breaking of this symmetry, pointing to an ordered phase.

The background-field formalism used in this work allows us to fix the fluctuations with respect to Landau-DeWitt gauge and simultaneously maintain gauge invariance for the background field $\tilde{A}$ which we relate to the Polyakov loop in the following manner: we use the Polyakov gauge [22] by gauge-rotating the background field into the Cartan subalgebra and imposing $\partial_0 A_0 = 0$. From the knowledge of $A_0$, the value of the corresponding Polyakov loop $L[A_0]$ can immediately be inferred. Once the effective action $\Gamma[A = \tilde{A}]$ is constructed within the background-field formalism, the minimum of the action represents the expectation value of the fluctuating quantum field, $A = \tilde{A} = \langle A \rangle$. The corresponding Polyakov-loop value then is $L[A_0] = L[\langle A_0 \rangle]$.

In addition to $\langle L \rangle$, also $L[\langle A_0 \rangle]$ serves as an order parameter for confinement: first, $L[\langle A_0 \rangle]$ is an upper bound for the Polyakov loop expectation value due to the Jensen inequality, $L[\langle A_0 \rangle] \geq \langle L[A_0] \rangle$, and therefore is nonzero in the deconfined phase. Second, it has been shown in [17]
that \( L([A_0]) \) vanishes identically in the center-symmetric phase where also \( L([A_0]) = 0 \). We conclude that together with \( L([A_0]) \) also \( A_0 = \overline{A_0} = \langle A_0 \rangle = 0 \) is an order parameter for center symmetry and confinement in the Polyakov gauge. In the following, we indeed concentrate on the effective potential \( V(\langle A_0 \rangle) \) for this order parameter.

Let us now recapitulate the confinement criterion put forward in Ref. \( ^1 \). This criterion relates the IR behavior of gluon and ghost 2-point functions to the effective potential for the order parameter \( \langle A_0 \rangle \), starting from the full flow as displayed in Eq. \( \text{(4)} \).

The following simplified analytical discussion is based on the assumption that the second term in Eq. \( \text{(4)} \) is proportional to \( \sim \partial_t \Gamma_k^{(2)} \) (cf. Eq. \( \text{(9)} \)) is subleading. This term with the explicit \( k \) integral resembles the full integrated flow except for the substitution \( \partial_t R_k \rightarrow \partial_t \Gamma_k^{(2)} \).

In the UV, its subleading role is obvious, since \( \partial_t \Gamma_k^{(2)} \) is of order \( \alpha_s \) whereas \( \partial_t R_k \) is of order one. In the deep infrared such an ordering cannot be found. Nevertheless, our full numerical study shows that the term depending on \( \partial_t \Gamma_k^{(2)} \) is subleading for a study of the Polyakov loop on all scales studied in this work.

Anticipating the subdominance of the \( \partial_t \Gamma_k^{(2)} \) term, we study the influence of the first term of Eq. \( \text{(4)} \) on the Polyakov-loop potential in the UV and IR regime. (The remaining terms are irrelevant for this discussion, as the potential is finite and does not require counterterms.) In the UV regime \( (p^2 \gg T^2) \), perturbation theory holds and the inverse propagators of the longitudinal and transversal gluons and the ghosts are given by \( \Gamma_k^{(2), \text{pert}} (p^2) = p^2 \).

In the presence of a constant background field \( \langle A_0 \rangle \), the momentum is replaced by the background covariant derivative, i.e. \( p_0 \rightarrow -i D_0 \). With the parameterization

\[
\beta g \langle A_0 \rangle = 2 \pi \sum_{T = \text{Cartan}} T^a \phi^a = 2 \pi \sum_{T = \text{Cartan}} T^a v^a |\phi|, \quad v^2 = 1, \quad (27)
\]

the spectrum of the background covariant Laplacian becomes

\[
p^2 \rightarrow \text{spec} \{- D^2 |\langle A_0 \rangle|\} = \hat{p}^2 + (2 \pi T)^2 (n - |\phi| v^2), \quad (28)
\]

where \( n \in \mathbb{Z} \), and \( (T^a)^{bc} = -i f^{abc} \) denotes the generators of the adjoint representation of the gauge group under consideration. \( \nu_k \) denotes the eigenvalues of the hermitian color matrix occurring in Eq. \( \text{(27)} \),

\[
\nu_k = \text{spec} \{(T^a v^a)^{bc} |v^2 = 1\}, \quad (29)
\]

and therefore depends on the direction of the unit vector \( v^a \). The index \( \ell \) labels these eigenvalues, the number of which is equal to the dimension \( d_{\text{adj}} \) of the adjoint representation of the gauge group, \( \ell = 1, \ldots, d_{\text{adj}} \), e.g., \( d_{\text{adj}} = N^2 - 1 \) for \( SU(N) \). For each non-vanishing eigenvalue \( \nu_k \) there exists an eigenvalue \(-\nu_k \). For \( SU(2) \), we have \( \nu_k = \pm 1, 0 \). Equation \( \text{(25)} \) reveals that \( \phi^a = |\phi| v^a \) denotes a set of compact variables, as an arbitrary shift of \( \phi^a \) can be mapped back onto a compact domain for \( \phi^a \) by a corresponding shift of \( n \), i.e., the Matsubara frequency.

With these prerequisites, the perturbative limit of the effective order-parameter potential \( V \) in \( d > 2 \) dimensions is given by

\[
\frac{V^{\text{UV}}(\phi)}{T^d} = \frac{(2 - d) \Gamma(\frac{d}{2})}{\pi^{d/2}} \sum_{\ell=1}^{d_{\text{adj}}} \sum_{n=1}^{\infty} \cos \frac{2 \pi n |\phi| \nu_k}{n^2} \quad (30)
\]

Here, we have dropped a temperature- and field-independent constant. The dimensionality of the potential is determined by the dimension of the Cartan (sub)algebra. This perturbative \( V^{\text{UV}} \) corresponds to the well-known Weiss potential \( ^{18} \), generalized to \( d \) dimensions \( ^{29} \). It exhibits maxima at the center-symmetric points where \( L([A_0]) = 0 \) (and thus also \( \langle L \rangle = 0 \)), implying that the perturbative ground state is not confining, i.e. \( \langle L \rangle \neq 0 \). Since the eigenvalues \( \nu_k \) are pairwise identical with respect to their absolute values, the Weiss potential for a given gauge group can be considered as a superposition of \( SU(2) \) potentials with different periodicities determined by the eigenvalues \( \nu_k \). The eigenvalues can be viewed as Fourier frequencies of the order-parameter potential. We stress that this also holds in non-perturbative studies of the Weiss potential. Hence we have

\[
V(\phi) = \frac{1}{2} \sum_{l} V_{SU(2)}(\nu_l |\phi|) \cdot (31)
\]

Next we perform the same analysis in the IR. With the parameterizations \( \text{(17)} \) and \( \text{(19)} \), the dressing functions \( Z_A(p^2), Z_C(p^2) \) are characterized by the power-law behavior \( ^{20} \) in the deep IR, \( p^2 \ll \Lambda_{\text{QCD}}^2 \). Quantitatively, the effective potential \( V(\phi) \) is dominantly induced by fluctuations with momenta near the temperature scale \( p^2 \sim (2 \pi T)^2 \). At low temperatures \( (2 \pi T) \ll \Lambda_{\text{QCD}} \), the first term in Eq. \( \text{(4)} \) thus induces an effective potential which arises dominantly from fluctuations in the deep IR, characterized by the exponents \( \kappa_{A,C} \). By coupling the fluctuations to the background field, \( p^2 \rightarrow -D^2 |\langle A_0 \rangle| \), we obtain the following low-temperature effective potential from the power-law behavior of the two-point Green functions in the deep IR:

\[
V^{\text{IR}}(\phi) = \left\{1 + \frac{(d - 1) \kappa_A - 2 \kappa_C}{d - 2}\right\} V^{\text{UV}}(\phi).
\]

Compared to the perturbative Weiss potential \( ^{30} \) we observe that the effective potential is reversed if

\[
2 \kappa_C - (d - 1) \kappa_A > d - 2. \quad (32)
\]

In this case, the confining center-symmetric points of the Weiss potential turn from maxima to minima: the order parameter acquires a center-symmetric value, such that \( L([A_0]) = \langle L \rangle = 0 \). We conclude that Eq. \( \text{(32)} \) serves as a criterion for quark confinement. Provided that the term \( \sim \Gamma_k^{(2)} \) that we dropped for this discussion does not modify this result, this criterion is sufficient for the occurrence of a center-symmetric confining phase at low temperatures.
Let us discuss this criterion in the light of the IR solutions for the propagators available in the Landau gauge. For the scaling solution, the IR exponents are related by the sum rule \( \frac{\kappa_1}{\kappa_C} = 1 \), simplifying the confinement criterion to
\[
\kappa \equiv \kappa_C > \frac{d - 3}{4}. \tag{33}
\]

It is instructive to compare this simple criterion for quark confinement with related criteria in the \( d = 4 \) case: \( \kappa_{d=4} > 1/4 \). This criterion includes the Kugo-Ojima criterion for color confinement \( \kappa > 0 \) as well as the Zwanziger horizon condition for the ghost \( \kappa > 0 \), which are both necessary but not sufficient criteria. The corresponding horizon condition for the gluon, \( \kappa > 1/2 \), is stronger, since the latter is a sufficient but not necessary condition for the transversal gluons to exhibit positivity violation.

For the decoupling solution with \( \kappa_C = 0 \) and \( \kappa_A = -1 \) in \( d = 4 \), the criterion \( \frac{\kappa}{d - 1} \) is satisfied as well and, hence, the whole one-parameter family of Landau-gauge IR solutions is confining.

To summarize: in color-confined gauge theories, the suppressed gluon and the enhanced (or constant) ghost fluctuations in the IR induce an effective potential for the Polyakov loop which corresponds to a center-ordered confining ground state; this implies an infinite free energy for a single quark and thus relates color confinement to quark confinement.

\section{VI. NUMERICAL RESULTS}

In the following, we present our results for the order parameter \( L[A_0] \) as a function of temperature for \( SU(N) \), \( Sp(2) \) and \( E(7) \) Yang-Mills theory. Recall that \( L[A_0] \geq \langle L[A_0] \rangle \) in the deconfined phase. Our computation of the effective potential \( V(A_0) \) for \( SU(N) \), \( Sp(2) \) and \( E(7) \) Yang-Mills theory involves several approximations which can easily be improved on, once more precise propagator data from the lattice or from functional methods is available: first, we employ the same solution for the ghost and gluon propagators as obtained from a functional RG study \cite{9, 28} for all gauge groups, see also Fig. 1. In a first approximation, this can be justified, since the propagators are identical in leading order in a \( 1/N \) expansion where \( N \) is the number of colors. However, even for a small number of colors, it has indeed been found on the lattice that \( SU(2) \) and \( SU(3) \) propagators agree within errors \cite{60}.

As a second approximation, we do not take a possible modification of the functional form of zero-temperature and finite-temperature propagators into account. We also neglect that the transversal gluon propagator splits into independent components longitudinal and transversal to the heat bath at finite \( T \). Our approximation to the finite-temperature propagators corresponds to inserting Matsubara frequencies into the momentum argument of the zero-temperature propagator functions. From an RG point of view, this represents the zeroth-order approximation to the full temperature-dependent propagators. Nevertheless, we expect that this already provides a quantitative insight into the finite-temperature phase structure for the following reasons: finite-temperature modifications of the propagators are expected to occur for momentum scales below the temperature scale and, more prominently, for temperatures below \( T_c \). By contrast, the effective order-parameter potential is dominantly built up from momentum modes near the scale \( 2\pi T \). Therefore, detecting \( T_c \) from the above, the IR properties of the propagators are hardly probed and only the decisive mid-momentum region together with the perturbative high-momentum tail of the propagators effectively enters in the computation of the potential. Neglecting a potentially strong explicit temperature dependence of the propagators for small temperatures \( T \ll T_c \) and/or momentum modes below the temperature scale is thus an acceptable approximation for detecting the phase boundary. The validity of this approximation has been verified explicitly for scalar theories in Ref. \cite{81}.

In addition to our analytical discussion of the confinement criterion, we have now used the full functional flow equation including the term depending on \( \partial_k \Gamma_k^{(2)} \) in Eq. (3) in our numerical study. We observe that the order of the phase transition for a given gauge theory remains unchanged upon the inclusion of this term. Moreover the phase transition temperature increases only by \( \lesssim 7\% \) when this term is added. For a qualitative understanding of the order-parameter potential as discussed in the preceding section, the omission of this term is hence justified which confirms the picture arising from the our confinement criterion. For details on the numerical computation of the order-parameter potential, we refer to App. A.

In order to convert our results into physical units, we fix our propagators relative to the lattice scales. In turn, the propagators on the lattice can be converted into physical units by measuring lattice momenta in units of, e.g., the string tension. In this manner, we can determine \( T_c \) in physical units corresponding to a string tension of \( \sigma = 440 \text{ MeV} \). In our studies we keep the position of the peak of the gluon propagator fixed for all gauge groups. This provides a prescription for a comparison of lattice results for \( T_c \) and our results.

\subsection{SU(N)}

Let us first consider the gauge groups \( SU(N) \). In Fig. 2, we show our results for the order parameter \( L[A_0] \) for \( N = 3, 5, 7, 9, 11 \) as a function of \( T/T_c \); the corresponding results for \( SU(2) \) and \( SU(12) \) can be found in Figs. 3 and 4 respectively. We find a second order phase transition for \( SU(2) \) and a first-order phase transition for \( SU(N) \) \((N = 3, 4, \ldots, 12)\). For \( SU(2) \) the phase transition occurs at \( T_c \approx 265 \text{ MeV} \). For \( SU(3) \) we find...
$T_c \approx 291$ MeV. With increasing rank of the gauge group, the phase transition temperature increases slightly and approaches $T_c \approx 295$ MeV for $SU(5)$. For $N \geq 5$ our results for the order-parameter are essentially independent of $N$. In other words, our results for the phase transition temperature for $SU(3)$ is already close to the large-$N$ value. This independence of the Polyakov loop on $N$ for $N \geq 5$ is in accordance with recent lattice studies of $SU(N)$ Yang-Mills theories [61]. Since we employ the same propagators for all $SU(N)$, the increase of $T_c$ is only due to the increase in the rank of the gauge group. In accordance with the weak dependence of $T_c$ on $N > 2$, the order-parameter as a function of $T/T_c$ depends only slightly on the rank of the gauge group. Note that our result for the Polyakov loop $L[(A_0)]$ for $T/T_c > 1$ is higher than the corresponding expectation value $\langle L[A_0] \rangle$ of the Polyakov loop as obtained from lattice simulations, being in perfect agreement with the Jensen inequality $L[(A_0)] \geq \langle L[A_0] \rangle$.

At this point we would like to emphasize that our studies are of course not bound to $N \leq 12$. Our approach can be straightforwardly generalized to $N > 12$ with the aid of Eq. (31). Our numerical study of a given gauge group involves three simple steps: computing $V_{SU(2)}$, finding the eigenvalues of the generators of the Cartan subalgebra in the adjoint representation for the gauge group under consideration, and finally minimizing Eq. (31). Therefore the computation of the order parameter for very large gauge groups is not considerably more involved than for smaller ones.

In view of the approximations listed above, we expect corrections to our results from modifications of the propagators due to finite temperature and due to order-parameter fluctuations. Whereas finite-temperature corrections of the propagators affect the results for the order parameter of all gauge groups, order-parameter fluctuations play a particularly important role in $SU(2)$ since it has a second-order phase transition.

### B. $Sp(2)$

The fact that confinement and center symmetry are related naively suggests that gauge groups with the same center may show similar phase transition properties. This is, however, not the case as the prime counter-example of $SU(2)$ vs. the symplectic group $Sp(2)$ demonstrates: both gauge groups have the same center $Z(2)$, but exhibit qualitatively different phase-transition properties. Our results for the order parameter $L[(A_0)]$ as a function of $T/T_c$ for $SU(2)$ and $Sp(2)$ are depicted in Fig. 3. We find a second-order phase transition for $SU(2)$ and a first-order phase transition for $Sp(2)$. Therefore, $SU(2)$ falls into the Ising universality class [57] but $Sp(2)$ does not. Moreover the phase transition temperature for $Sp(2)$ gauge theory is close to the value of $SU(3)$ gauge theory; we find $T_c \approx 286$ MeV. Since we use the same propagators for both gauge groups and the center of both groups is $Z(2)$, it is natural to relate the different nature of the phase transition to the different dimensionality of the two groups [21]. In fact, the number of degrees of freedom in the deconfined phase is much larger in $Sp(2)$ than in $SU(2)$. This strong mismatch in the number of dynamical degrees of freedom in the confined and deconfined phase appears to enforce a first-order phase transition in $Sp(2)$. In this respect these findings resemble the situation in the case of $SU(3)$ in $2+1$ and $3+1$ space-time dimensions. While the phase transition in $SU(3)$ is of

1 In our conventions $Sp(1)$ is isomorphic to $SU(2)$. 

---

FIG. 2: Polyakov loop $L[(A_0)]$ as a function of temperature for $SU(3)$, $SU(5)$, $SU(7)$, $SU(9)$, $SU(11)$. We observe that the phase transition is of first order. Hardly any difference for the order parameter is visible for $N > 5$, suggesting a close proximity of these gauge groups to the large-$N$ limit.

FIG. 3: Polyakov loop $L[(A_0)]$ for $SU(2)$ (blue/dashed line) and $Sp(2)$ (black/solid line). The phase transition is of second order for $SU(2)$ and of first order for $Sp(2)$.
first order in $3 + 1$ dimensions, it is of second order in $2 + 1$ dimensions \cite{62}. Again, this might be traced back to the fact that the mismatch in the number of dynamical degrees of freedom in the deconfined phase is smaller in $d = 2 + 1$ than it is in $d = 3 + 1$.

Our findings for the nature of the $Sp(2)$ phase transition are in accordance with lattice simulations \cite{20, 21}. Quantitatively, our approximation of neglecting terms $\propto V''(\langle A_0 \rangle)$ on the right-hand of Eq. \eqref{eq:4}, which account for order-parameter fluctuations, might be more severe in $Sp(2)$ due to its similarity to $SU(2)$. The inclusion of these fluctuations may only lead to a weaker first-order jump and an increase of the critical temperature.

C. $E(7)$

Another interesting test of the proposal that the order of the phase transition is related to the size of the gauge group \cite{21} is the following comparison between $SU(12)$ and $E(7)$ gauge theory. The dimension of the adjoint representation of these two gauge groups is about the same: We have $d_{\text{adj}} = 133$ for $E(7)$ and $d_{\text{adj}} = 143$ for $SU(12)$. The gauge groups differ with respect to their center, being $Z(2)$ for $E(7)$ and $Z(12)$ for $SU(12)$.

In Fig. \ref{fig:4} our result for the Polyakov loop $L(\langle A_0 \rangle)$ for both theories is depicted as a function of $T/T_c$. As discussed above, the phase transition in $SU(12)$ is of first order and occurs at $T_c \approx 295$ MeV. For $E(7)$, our RG approach predicts a first-order phase transition at $T_c \approx 295$ MeV as well.

Our study is thus compatible with the suggestive relation of the order of the phase transition and the mismatch of the number of degrees of freedom above and below the phase transition – provided the glueball spectrum below the phase transition in $E(7)$ is similar to that of $SU(N)$.

However, we also observe that the height of the jump of the order parameter is smaller in $E(7)$ than in $SU(N)$ for all values of $N$ studied in the present paper. Even though the Polyakov loop $L(\langle A_0 \rangle)$ is not an RG invariant quantity, our approach of studying the associated eigenvalue distribution allows us to give the height of the jump a physical meaning. This suggests that the mismatch in the number of degrees of freedom is not the only mechanism that determines the nature of the phase transition.

In order to gain a better understanding of the nature of the phase transition, we study the eigenvalue distribution $N(\langle \nu_\ell \rangle)$ of the spectrum of the color matrix in the Cartan subgroup as defined in Eq. \eqref{eq:29}. In Fig. \ref{fig:5} we show $N(\langle \nu_\ell \rangle)$ as a function of the normalized eigenvalues $\nu_\ell/\nu_{\ell \text{max}}$ at the ground state of the order-parameter potential for $T \to T_+^\pm$, approaching the critical temperature from above. Here, the eigenvalues have been binned with a bin size of $\Delta \nu_\ell = 0.005$.

The eigenvalues correspond to Fourier frequencies of $SU(2)$ Weiss potentials, cf. Eq. \eqref{eq:30} and Eq. \eqref{eq:31}. For instance, the higher the dominating Fourier frequency in the high-temperature phase, the closer the $\phi$ minimum is to $\phi = 0$, implying that $L(\langle A_0 \rangle)$ is closer to $L(\langle A_0 \rangle) = 1$. By contrast, if lower eigenvalues dominate, $L(\langle A_0 \rangle)$ can approach the center ordered state in a smoother fashion. This is precisely what we observe for $E(7)$ in contradistinction to the $SU(N > 2)$ gauge groups, where the eigenvalues cluster around $\nu_\ell/\nu_{\ell \text{max}} \approx 0.25, 0.5, 0.75$, leading to an almost constructive interference of $SU(2)$ potentials with almost identical periodicity. We stress that the eigenvalue distribution $N(\langle \nu_\ell \rangle)$ depends on the actual position $\langle A_0 \rangle_{\text{min}}$ of the ground-state of the potential. Therefore the eigenvalue distribution and hence the strength of the first-order phase transition depends on the actual trajectory $\langle A_0 \rangle_{\text{min}}(T)$ of the physical ground-state close to $T_c$ in the space spanned by the generators of the Cartan subalgebra. Of course, since $L(\langle A_0 \rangle)$ provides only an upper bound for $\langle L \rangle$ it is not immediately clear whether the difference in $L(\langle A_0 \rangle)$ which we observe for $E(7)$ and $SU(N)$ also translates into a similar difference in $\langle L \rangle$. If so, we expect the phase transition for $E(7)$ to be smoother than for $SU(N)$. Taking into account that order parameter fluctuations dropped so far can smoothen the phase transition even further, our results may not even be taken as a strict excluding evidence for a second order phase transition in $E(7)$.

With respect to our study of $Sp(2)$ we indeed find that the eigenvalue distribution $N(\langle \nu_\ell \rangle)$ exhibits a pattern very similar to the one of $SU(3)$ resulting in a jump of the order parameter at the phase transition with a height comparable to the one of $SU(3)$ Yang-Mills theory, see Figs. \ref{fig:2} and \ref{fig:3}.

VII. CONCLUSIONS

In the present paper we have discussed the nature of the phase transition in various gauge groups and based on a simple confinement criterion put forward in Ref. \cite{1}.
The order-parameter potential can be considered as a destructive interference/superposition of SU(2) potentials favoring a first-order phase transition. Moreover, we have a stronger mismatch in the number of the dynamical degrees of freedom in Sp(2) in the confined and deconfined phase compared to SU(2) Yang-Mills theory [20, 21].

For E(7) gauge theory we find that the phase transition is of first order as well. Here, the mismatch in the number of dynamical degrees of freedom in the confined and deconfined phase is even stronger than it is in Sp(2), suggesting that the first-order phase transition is even stronger. However, our RG study suggests that the first-order phase transition can be traced back to the eigenvalue distribution at the phase transition. In contrast to Sp(2) and SU(N) we have found that the distribution exhibits distinct equidistant maxima resulting in an almost constructive interference of SU(2) potentials. In this respect E(7) is closer to SU(2) than to SU(N) with $N \geq 3$. However, further studies are needed to establish this picture. For example, a RG study of SU(3) and Sp(2) Yang-Mills theory in 2+1 dimensions may help to shed more light on the underlying mechanisms of the deconfinement phase transition since it is known from lattice simulations that the nature of the phase transition in both gauge groups changes from first to second order when the number of dimensions is reduced [20, 21, 62].

Another interesting case is the gauge group G(2) with non-trivial center. In this case it has been found [20, 63] that the Polyakov loop exhibits a jump but is non-vanishing for all temperatures. A verification of our quark confinement criterion with the aid of G(2) Yang-Mills theory is under way and will help us to establish our findings and to improve our understanding of confinement in gauge theories.

Acknowledgments

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Appendix A: Details on the computation of the order-parameter potential

In this addendum we discuss some details on the computation of the order-parameter potential. The order-parameter potential can be obtained directly from an evaluation of Eq. (1). While the first term in Eq. (1) is independent of our choice of the regulator function, the second term is not. Since we employed the optimized regulator \( \mathcal{R} \), we encounter expressions involving unit-step functions in the second term on the right-hand side of Eq. (1). These unit-step functions depend on the background field \( \langle A_0 \rangle \). This dependence on \( \langle A_0 \rangle \) generates divergent RG flows for \( \Lambda \to \infty \). In principle one can deal with these divergences by computing appropriate counter-terms at the initial UV scale. In the present paper we sought for a different approach to circumvent this problem and introduced ‘smeared’ unit-step functions:

\[
f_{\theta}(x[\langle A_0 \rangle], \epsilon) = e^{-(x[\langle A_0 \rangle])^\epsilon},
\]

\[
\lim_{\epsilon \to \infty} f_{\theta}(x[\langle A_0 \rangle], \epsilon) = \theta(1 - x[\langle A_0 \rangle]), \tag{A1}
\]

where \( x[\langle A_0 \rangle] \) is an arbitrary function depending on the background field \( \langle A_0 \rangle \). Using \( f_{\theta}(x[\langle A_0 \rangle], \epsilon) \) instead of \( \theta(1 - x[\langle A_0 \rangle]) \) yields an order-parameter potential periodic in \( \langle A_0 \rangle \) for any finite value of \( \epsilon \) and allows to get conveniently rid of the unphysical divergent parts of the flow. For our numerical study of the deconfinement phase transition we have used \( \epsilon = 7 \). In Fig. 6 we illustrate the dependence of the position of the minimum \( \phi^{\text{fit}} = \beta \langle A_0 \rangle_{\text{min}}/(2\pi) \) of the potential on the ‘smearing’ parameter \( \epsilon \) for \( T = 300 \text{ MeV} \) for \( SU(2) \) Yang-Mills theory. From an extrapolation of our results to \( \epsilon \to \infty \) using the two functions

\[
\phi_{\text{min}}^{\text{fit}}(\epsilon) = \begin{cases} 
\text{const.} + a e^{-b \epsilon} \\
\text{const.} + \frac{\epsilon}{\epsilon}
\end{cases}, \tag{A2}
\]

where \( a, b \) and \( c \) are fit parameters, we estimate that the theoretical error is less than 1\% when \( \epsilon = 7 \) is used. Note that the fit function in the second line of Eq. (A2) can be formally deduced from a Taylor expansion around \( \epsilon \) of the integral of a general polynomial in \( x[\langle A_0 \rangle] \) weighted by \( f_{\theta}(x[\langle A_0 \rangle], \epsilon) \).

Appendix B: Generators of \( Sp(2) \)

Our definition of the generators of \( Sp(2) \) in the fundamental representation is as follows:

\[
C_1 = \begin{pmatrix}
0 & i & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
C_2 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
C_3 = \begin{pmatrix}
i & 0 & 0 & 0 \\
0 & -i & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
C_4 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & i \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
C_5 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
C_6 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & i & 0 & 0 \\
0 & 0 & 0 & -i
\end{pmatrix},
\]

\[
C_7 = \begin{pmatrix}
0 & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & \frac{-i}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & \frac{i}{\sqrt{2}} & 0 & 0
\end{pmatrix},
\]
\[ C_8 = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}, \]

\[ C_9 = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{i}{\sqrt{2}} & 0 \\ 0 & \frac{i}{\sqrt{2}} & 0 & 0 \\ \frac{i}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix}, \]

\[ C_{10} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix}. \]
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