Impurities suspended in incompressible flows are relevant to several physical processes, such as spray combustion, raindrop formation, and transport of pollutants.\cite{1,2,3} These particles, being typically of finite size and heavier than the ambient fluid, cannot be modeled as simple tracers. They indeed possess inertia, which is responsible for the spontaneous generation of strong inhomogeneities in their spatial distribution.\cite{4,5,6} In a turbulent flow, their clustering is more efficient at small scales, below the dissipative scale, where the fluid velocity field is smooth. The intensity of particle clustering there is related to the statistics of the Lyapunov exponents of particle trajectories.\cite{7} The behavior of the Lyapunov exponents of inertial particles and the relation with particle clustering was recently investigated in random short-correlated flows.\cite{8,9}

In this Letter, by means of high-resolution direct numerical simulations, we investigate the Lyapunov spectra of inertial particles, varying their response time $\tau_i$ and the Reynolds number of the carrier turbulent flow. For $\tau_i$ larger than the Kolmogorov-scale turnover time $\tau_{\eta}$, the presence of inertia results in a generic reduction of chaoticity: the leading Lyapunov exponent is smaller than the one of tracers and the Kolmogorov-scale turnover time

$$\tilde{\tau}_{\eta} = \frac{\tau_{\eta}}{H},$$

where $H$ is the fluid density.

The incompressible fluid velocity field $u(x,t)$ evolves according to the Navier-Stokes equations

$\dot{x} = v, \quad \dot{v} = -\frac{1}{\tau_i}[v - u(x(t),t)],$ (1)

where $u$ is the fluid velocity at the location $x$ of the particle that moves with velocity $v$, and $\tau_i = 2\alpha^2/\nu$ is the Stokes time ($\nu$ is the kinematic viscosity of the fluid). Equation (1) holds when the flow surrounding the particle is a Stokes flow; this requires $\alpha \ll \eta$, $\eta$ being the Kolmogorov scale of the flow. The Stokes number is defined as $St = \tau_i/\tau_{\eta}$, where $\tau_{\eta}$ is the eddy turnover time associated with $\eta$. Particles are assumed to behave passively: we neglect their feedback on the flow, which is justified for very diluted suspensions.\cite{10} For comparison, we also study the motion of neutral particles that follow the dynamics $\dot{x} = u(x(t),t)$, which corresponds to the limit $\tau_i \rightarrow 0$ in Eq. (1). We remark that in (1) we also neglect any possible effect of molecular diffusivity. Although this is generally very small in the case of inertial particles, it can affect the evolution of particle density.\cite{12,13}

The incompressible fluid velocity field $u(x,t)$ evolves according to the Navier-Stokes equations

$$\dot{x} = v, \quad \dot{v} = -\frac{1}{\tau_i}[v - u(x(t),t)],$$ (1)
The largest Lyapunov exponent $\lambda_1$ measures the chaotic separation of particle trajectories. To understand how chaoticity is affected by inertia, two mechanisms have to be considered. First, inertial particles have a delay on the fluid motion; this means that their velocity is approximately given by that of tracers with a time filtering over a time window of size $\tau_s$. This effect weakens chaoticity. Second, heavy particles are ejected from persistent vortical structures and concentrate in high-strain regions. Since these portions of the flow are characterized by larger stretching rates, the chaoticity of particle trajectories is increased with respect to tracers that homogeneously visit all the regions. As emphasized in the inset of Fig. 1, the latter effect dominates for $\text{St}<1$, where $\lambda_1$ is larger than the Lyapunov exponent of the fluid tracers ($\text{St}=0$). This is not predicted from analytical and numerical studies done in white-in-time random velocity fields; such flows clearly possess no persistent structures. Conversely, at sufficiently large St, the Lyapunov exponent decreases; preferential concentration is then negligible and the time-filtering approximation becomes relevant. For such a large inertia, the white-in-time models apply and indeed predict a decrease of $\lambda_1$ as a function of St. Note that the competition between filtering and preferential concentration described above also enters in the distribution of particle acceleration.\textsuperscript{14}

As observed in Fig. 1, the time evolution of infinitesimal surfaces is also affected by these two mechanisms. Indeed, at varying St, the second Lyapunov exponent $\lambda_2$ displays a behavior qualitatively similar to that of $\lambda_1$. For the tracers ($\text{St}=0$), incompressibility of the flow implies $\lambda_1+\lambda_2+\lambda_3=0$. Since for a time-reversible dynamics one has $\lambda_3=0$, the ratio $\lambda_2/\lambda_1$ is a measure of the irreversibility of the dynamics. Previous numerical investigations at moderate Reynolds numbers\textsuperscript{16} indicate $\lambda_3/\lambda_1=0.25$; our simulations indicate $\lambda_3/\lambda_1=(0.28\pm0.02)$. This irreversibility stems from the fact that the Navier-Stokes equation itself is not invariant with respect to time reversal.

The behavior of $\lambda_3$ as a function of St markedly differs from that of the first two exponents. Due to the dissipative nature of the inertial particle dynamics, volumes in physical space are not conserved. Indeed, the volume growth rate, defined as $\Lambda=\lambda_1+\lambda_2+\lambda_3$, which identically vanishes for fluid tracers, is negative for all Stokes numbers in the range $0<\text{St}\lesssim1.72$ (see Fig. 2). This means that all volumes from physical space contract to zero at large times. Such clustering, which happens at scales much smaller than $\eta_s$, is a consequence of the convergence of particle trajectories toward a dynamically evolving (multi)fractal set.\textsuperscript{7} The fractal dimension of this attractor can be estimated by means of the Kaplan-Yorke (or Lyapunov) dimension\textsuperscript{17} as $d_f=J+\sum_{j=1}^{J-1}1/|\lambda_{j+1}|$, where $J$ is the largest integer such that $\sum_{j=1}^{J}1/\lambda_j>0$. The fractal dimension of inertial particles is represented in Fig. 2 as a function of St. The minimum at $\text{St}\approx0.5$ corresponds to maximal clustering. For $\text{St}\geq1.72$, the fractal dimension becomes greater than $d=3$, indicating that the spatial distribution of particles is not fractal anymore.

\[ \frac{\partial u + u \cdot \nabla u}{\nabla} = -(1/\rho_0) \nabla p + \nu \Delta u + f, \quad \nabla \cdot u = 0, \quad (2) \]

where the forcing provides an external energy input at a rate $\varepsilon=\langle f \cdot u \rangle$. These equations are integrated on a $d=3$ dimensional grid of size $N=128,256,512$ with periodic boundary conditions, by means of a fully de-aliased pseudospectral parallel code with second-order Adams-Bashforth time-stepping. Energy is injected by keeping constant the spectral content of the two smallest wave-number shells. Viscosity is stepping. Energy is injected by keeping constant the spectral injection rates. They asymptotically converge to the Lyapunov exponents as a function of St for the largest value chosen to resolve well the small-scale velocity content of the two smallest wave-number shells. Viscosity is stepping. Energy is injected by keeping constant the spectral content of the two smallest wave-number shells. Viscosity is stepping. Energy is injected by keeping constant the spectral content of the two smallest wave-number shells. Viscosity is stepping. The Reynolds numbers based on Taylor’s microscale $R_T=65:185$ (see Ref. 14 for further details).

Once the fluid flow has reached a statistically stationary state, particles are homogeneously seeded with initial velocities equal to the fluid velocity at their locations. We followed 33 sets of 2000 particles with Stokes numbers ranging from 0 to 2.2 for a time $\approx200\tau_n$ after relaxation of transients.

In order to compute the Lyapunov spectrum, we follow along each particle trajectory the time evolution of $2 \times d$ infinitesimal displacements in the position-velocity space obtained by linearizing the particle dynamics (1). The infinitesimal volume $V^i$, defined by $j$ linear independent tangent vectors, grows in time with an exponential rate $\Sigma_{i=1}^j \gamma_i(T) = (1/\tau_s) \ln |V^i(T)/V^i(0)|$, which defines the finite-time Lyapunov exponents $\gamma_i(T)$ (FTLE), also called stretching rates. They asymptotically converge to the Lyapunov exponents $\lambda_i = \lim_{T \to \infty} \gamma_i(T)$, which by definition are labeled in decreasing order $\lambda_1 \geq \cdots \geq \lambda_{2d}$. To compute these exponents numerically, we make use at each time lag $\tau_n$ of a standard technique based on the orthonormalization of the infinitesimal displacements by a Gram-Schmidt procedure (see, e.g., Ref. 15).

In Fig. 1, we show the behavior of the first three Lyapunov exponents as a function of St for the largest value of the Reynolds number. These three exponents rule the time evolution of infinitesimal elements in the physical space. For the range of Stokes numbers investigated here, we observe $\lambda_4 - \lambda_3 \approx \lambda_0 \approx -1/\tau_n$, signaling the relaxation of particle velocities to the fluid.

$\lambda_4 - \lambda_3 \approx \lambda_0 \approx -1/\tau_n$.
091702-3 Lyapunov exponents of heavy particles

Phys. Fluids 18, 091702 (2006)

St = 1.72, the volume growth rate $\Gamma$ vanishes, meaning that the dynamics of such particles preserve volumes on average. However, the finite-time volume growth rate $\Gamma = \gamma_1 + \gamma_2 + \gamma_3$ experiences large fluctuations, as shown in the inset of Fig. 2. As a result, strong local inhomogeneities are present in the particle concentration also at large values of St.

We now turn to the study of the Lyapunov exponent dependence on the Reynolds number of the flow. The inset of Fig. 3 shows the first Lyapunov exponent for neutral particles (i.e., St=0) as a function of $R_\lambda$. Since $\lambda_1$ is a small-scale turbulent quantity with the dimension of an inverse time, one expects $\lambda_1 \tau_\eta \approx \text{const}$ and thus $\lambda_1 \propto R_\lambda$. However, as a consequence of the intermittent fluctuations of the velocity gradients in turbulent flows, one can predict an anomalous dependence on Reynolds number $\lambda_1 \sim R_\lambda^\alpha$ with $\alpha < 1$.18 This implies that $\lambda_1 \tau_\eta$ decreases with Reynolds, as indeed confirmed by our simulations.

Intermittency is actually expected to affect the whole probability distribution function (PDF) of the largest finite-time Lyapunov exponents $\gamma_1(T)$. For $T$ sufficiently large, the distribution of FTLE is expected to obey a large-deviation principle, i.e., $p_T(\gamma_1) \propto \exp[-\eta \gamma(T)]$. The Cramér (or rate) function $S(\gamma)$ is a non-negative concave function that vanishes at its minimum, attained for $\gamma_1 = \lambda_1$. Small fluctuations occurring when $|\gamma_1 - \lambda_1| \ll T^{-1/2}$ are described by the central-limit theorem, which amounts to approximating $S(\gamma)$ by a parabola in the vicinity of its minimum. The effect of intermittency on such small fluctuations can be measured from the variance $\sigma^2 = \langle (\gamma_1 - \lambda_1)^2 \rangle$ of the FTLE, or more particularly from the reduced variance $\mu = T \sigma^2$, which measures the width of the Cramér function. As predicted in Ref. 18, intermittency is responsible for an anomalous dependence of $\mu$ on the Reynolds number. More particularly, $\mu \tau_\eta$ is expected to grow with $R_\lambda$. This tendency is qualitatively confirmed by our simulations as shown in the inset of Fig. 3. The signature of intermittency on the higher-order statistics of $\gamma_1$ can hardly be measured in a reliable way. Indeed, as shown in Fig. 3, the PDFs of $\gamma_1$ for the three $R_\lambda$ considered, once centered and normalized, almost collapse for fluctuations as large as $3 \sigma$. However, it is still possible to observe a systematic deviation from the Gaussian distribution. Because of incompressibility, the left tail of the PDF is bounded by the constraint that $\gamma_1 > 0$. It thus has to decrease faster than a Gaussian, as indeed observed. The right part of the PDF is related to strong velocity gradients. Such events apparently lead to a tail that is faster than Gaussian. Turbulent intermittency should therefore mainly affect the right tail.

For inertial particles, intermittent corrections act in the same direction as for fluid tracers. Figure 4 shows the behavior of $\lambda_1$ and of the reduced variance $\mu$ as a function of the Stokes number for various Reynolds numbers. For tracers, for any given St, $\lambda_1 \tau_\eta$ decreases while $\mu \tau_\eta$ increases with $R_\lambda$. The inset of Fig. 4 shows, for $R_\lambda = 185$, the Cramér function of the FTLE $\gamma_1(T)$ for both neutral particles (St=0) and
inertial particles with two different St. For St<1, the whole distribution shifts to higher values, signaling the increased chaoticity. For St>1, the distribution of FTLE shifts to lower values and fluctuations become less probable. The asymmetry observed in the PDF of tracer stretching rates decreases with inertia. At the largest values of St that are considered, the Cramér function γ₁ becomes indistinguishable from a parabola. Finally, it is worth noticing that the dependence on the Reynolds number is less significant for the volume growth rate 091702-4 Bec et al. Phys. Fluids 18, 091702 (2006).

9K. Duncan, B. Mehlig, S. Ostlund, and M. Wilkinson, “Clustering by mixing flows,” Phys. Rev. Lett. 95, 240602 (2005).

We have seen that two mechanisms enter the dynamics of inertial particles: they concentrate in high-strain regions and they lag behind the fluid flow. In the limit of either small or large inertia, one of the two effects dominates the other. At present, tackling analytically the behavior of the largest Lyapunov exponent as a function of the Stokes number could only be done in these asymptotics. For the range of Stokes numbers considered here, both effects compete and influence the Lyapunov exponents, preventing a complete analytical description. A numerical confirmation of the present theoretical predictions would require greater computational resources.

We acknowledge useful discussions with A. Celani and A. Lanotte. This work has been partially supported by the EU under Contract No. HPRN-CT-2002-00300, and the Galileo program on Lagrangian Turbulence. Numerical simulations have been performed at CINECA (Italy) and IDRIS (France) under the HPC-Europa program, Contract No. RI3-CT-2003-506079.

1S. Post and J. Abraham, “Modeling the outcome of drop-drop collisions in diesel sprays,” Int. J. Multiphase Flow 28, 997 (2002).

2G. Falkovich, A. Fouxon, and M. Stepanov, “Acceleration of rain initiation by cloud turbulence,” Nature (London) 419, 151 (2002).

3J. Seinfeld, Atmospheric Chemistry and Physics of Air Pollution (Wiley, New York, 1986).

4J. K. Eaton and J. R. Fessler, “Preferential concentrations of particles by turbulence,” Int. J. Multiphase Flow 20, 169 (1994).

5T. Elperin, N. Kleeorin, and I. Rogachevskii, “Self-excitation of fluctuations of inertial particle concentration in turbulent fluid flow,” Phys. Rev. Lett. 77, 5373 (1996).

6A. Pumir and G. Falkovich, “Intermittent distribution of heavy particles in a turbulent flow,” Phys. Fluids 16, L47 (2004).

7J. Bec, “Fractal clustering of inertial particles in random flows,” Phys. Fluids 15, L81 (2003).

8M. Wilkinson and B. Mehlig, “Caustics in turbulent aerosols,” Europhys. Lett. 71, 186 (2005).

9K. Duncan, B. Mehlig, S. Ostlund, and M. Wilkinson, “Clustering by mixing flows,” Phys. Rev. Lett. 95, 240602 (2005).

10M. R. Maxey and J. Riley, “Equation of motion of a small rigid sphere in a nonuniform flow,” Phys. Fluids 26, 883 (1983).

11J. R. Fessler, J. D. Kulick, and J. K. Eaton, “Preferential concentration of heavy particles in a turbulent channel flow,” Phys. Fluids 6, 3742 (1994).

12E. Balkovsky, G. Falkovich, and A. Fouxon, “Intermittent distribution of inertial particles in turbulent flows,” Phys. Rev. Lett. 86, 2790 (2001).

13T. Elperin, N. Kleeorin, V. S. L’vov, I. Rogachevskii, and D. Sokoloff, “Clustering instability of the spatial distribution of inertial particles in turbulent flows,” Phys. Rev. E 66, 036302 (2002).

14J. Bec, L. Biferale, G. Boffetta, A. Celani, M. Cencini, A. Lanotte, S. Musacchio, and F. Toschi, “Acceleration statistics of heavy particles in turbulence,” J. Fluid Mech. 550, 349 (2006).

15A. Crisanti, G. Paladin, and A. Vulpiani, Product of Random Matrices (Springer Verlag, Berlin, 1993).

16S. Girimaji and S. Pope, “Material element deformation in isotropic turbulence,” J. Fluid Mech. 220, 427 (1990).

17J.-P. Eckmann and D. Ruelle, “Ergodic theory of chaos and strange attractors,” Rev. Mod. Phys. 57, 617 (1985).

18A. Crisanti, M. H. Jensen, G. Paladin, and A. Vulpiani, “Intermittency and predictability in turbulence,” Phys. Rev. Lett. 70, 166 (1993).