Does gravitational wave propagate in the five dimensional space-time with Kaluza-Klein monopole?

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Abstract

The behavior of small perturbations around the Kaluza-Klein monopole in the five dimensional space-time is investigated. The fact that the odd parity gravitational wave does not propagate in the five dimensional space-time with Kaluza-Klein monopole is found provided that the gravitational wave is constant in the fifth direction.

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Introduction

Kaluza and Klein [1] showed an elegant way of unifying gravity and electromagnetism. In this theory, they are unified as a pure gravity in the five dimensional manifold which is composed of four dimensional Minkowsky space-time $M_4$ and an extra dimensional space $S_1$. Note that the masses of massive modes in the Kaluza-Klein (K-K) theory are tightly related with the radius of $S_1$. The idea has been generalized [2] to include non-abelian gauge theories. Gross and Perry [3] and independently Sorkin [4] found the regular, static and topologically stable magnetic monopoles solution of the five dimensional Einstein equation. Iwazaki [5] discussed the quantum effects of the magnetic monopoles. The discovery of the magnetic monopole strongly indicates the unification of the fundamental interactions. Sundaresan and Tanaka [6] discussed whether the space-time with Kaluza-Klein (K-K) monopole is stable against small perturbations.

In the previous paper [7], we discussed the propagation of the small perturbations around K-K metric without monopole in detail. In this paper, we will investigate the propagation of the small perturbations around the K-K monopole solution.

We consider K-K theory with the Einstein-Hilbert action in five dimensions given by

\[
S = \frac{-1}{16\pi G_5} \int d^5x \sqrt{|g_5|} R_5,
\]

where $G_5$ is the five dimensional gravitational constant, $g_5$ is the determinant of five dimensional metric $g_{AB}$, and $R_5$ denotes the five dimensional curvature scalar of the space-time.

The magnetic monopole solution is the solution of the Einstein equation in the empty space-time

\[
R_{AB} = 0.
\]
We take the metric $g_{AB}$ as

$$g_{AB} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -V^{-1} & 0 & 0 & 0 \\ 0 & 0 & -r^2V^{-1} & 0 & 0 \\ 0 & 0 & 0 & -r^2s^2+A^2V^2 & -AV \\ 0 & 0 & 0 & -AV & -V \end{pmatrix}$$

Here,

$$V^{-1}(r) = 1 + \frac{4M}{r}, \ s = \sin \theta.$$

The gauge field, $A_\mu$, is that of monopole located at the origin

$$A_\mu(x) = (A_t, A_r, A_\theta, A_\varphi = A) = (0, 0, 0, 4M\alpha(\theta)),$$  

where

$$\alpha(\theta) = \begin{cases} 1 - \cos \theta & \text{in } R_a : \{0 \leq r, 0 \leq \theta < \frac{\pi}{2} + \delta, 0 \leq \phi < 2\pi\} \\ -1 - \cos \theta & \text{in } R_b : \{0 \leq r, \frac{\pi}{2} - \delta < \theta \leq \pi, 0 \leq \phi < 2\pi\} \end{cases},$$

and $A_\mu$ satisfies

$$B = \nabla \times A = \frac{4Mr}{r^3}.$$

In order to avoid the so-called NUT singularity, $M$ should satisfy

$$M = \frac{\sqrt{\pi G}}{2e},$$

where $G$ is the four dimensional gravitational constant and $e$ denotes the unit electric charge. Then the magnetic charge $g$ of the monopole is given by

$$g = \frac{4M}{\sqrt{16\pi G}} = \frac{1}{2e}.$$

Thus the monopole has one unit of Dirac charge.

Now, we consider small perturbations around the monopole solution. Then the total metric $g_{AB}^T$ is given by

$$g_{AB}^T = g_{AB} + h_{AB}.$$
According to the approach of Regge and Wheeler[8], we can divide small perturbation into even and odd parity parts:

\begin{equation}
\tag{11}
h_{AB}^{\text{even}}(t, r, \theta, \varphi, x^5) = \sum_{n^5=-\infty}^{\infty} \left( \begin{array}{cccc}
H_0 & H_1 & H_3 \nabla_2 & H_3 \nabla_3 \\
\text{Sym} & H_2 & H_4 \nabla_2 & H_4 \nabla_3 \\
\text{Sym} & \text{Sym} & r^2 \gamma_{22} K + L \nabla_2 \nabla_2 & L \nabla_2 \nabla_3 \\
\text{Sym} & \text{Sym} & \text{Sym} & r^2 \gamma_{33} K + L \nabla_3 \nabla_3 \\
\text{Sym} & \text{Sym} & \text{Sym} & \text{Sym}
\end{array} \right)_{n^5} \times Y_{q\ell m}(\theta, \varphi)e^{-i\omega t}e^{in^5x^5/R}\end{equation}

and

\begin{equation}
\tag{12}
h_{AB}^{\text{odd}}(t, r, \theta, \varphi, x^5) = \sum_{n^5=-\infty}^{\infty} \left( \begin{array}{cccc}
0 & 0 & \epsilon_2^2 h_0 \nabla_3 & \epsilon_3^2 h_0 \nabla_2 \\
\text{Sym} & 0 & \epsilon_2^3 h_1 \nabla_3 & \epsilon_3^2 h_1 \nabla_2 \\
\text{Sym} & \text{Sym} & 2\epsilon_2^3 h_2 \nabla_3 \nabla_2 & h_2[\epsilon_2^3 \nabla_3 \nabla_3 + \epsilon_3^2 \nabla_2 \nabla_2] \\
\text{Sym} & \text{Sym} & \text{Sym} & 2\epsilon_3^2 h_2 \nabla_3 \nabla_2 \\
\text{Sym} & \text{Sym} & \text{Sym} & \text{Sym}
\end{array} \right)_{n^5} \times Y_{q\ell m}(\theta, \varphi)e^{-i\omega t}e^{in^5x^5/R}\end{equation}

Here, \( R \) is the radius of the circle in the fifth dimension and we have used

\begin{equation}
\tag{13}
\gamma_{AB} = \frac{g_{AB}}{r^2},
\end{equation}

\begin{equation}
\tag{14}
\epsilon_2^2 \equiv g^{2A} \sqrt{|g_5|} \epsilon_{012A5} = -\frac{1}{s},
\end{equation}

and

\begin{equation}
\tag{15}
\epsilon_3^2 \equiv g^{2A} \sqrt{|g_5|} \epsilon_{013A5} = s,
\end{equation}

where \( \epsilon_{ABCDE} \)’s are totally antisymmetric Levi-Chivita symbols in five dimensions. In Eqs.(11) and (12), \( Y_{q\ell m} \) denotes the monopole harmonics[9], which is introduced
so as to avoid the singularity at $\theta = \pi$ and which satisfies following eigen value equations.

\begin{equation}
L^2 Y_{q\ell m} \equiv \left\{ -\frac{1}{s} \partial_\theta s \partial_\theta - \frac{1}{s^2} \partial_\varphi^2 - \frac{2q \alpha(\theta)L_z}{s^2} \right\} Y_{q\ell m} = \ell(\ell + 1) Y_{q\ell m},
\end{equation}

\begin{equation}
L_z Y_{q\ell m} \equiv \left\{ -i \partial_\varphi - q \alpha(\theta) - q \cos \theta \right\} Y_{q\ell m} = m Y_{q\ell m}.
\end{equation}

The monopole harmonics $Y_{q\ell m}$ coincides with the usual spherical harmonics $Y_{\ell m}$ when $q = 0$. The number $q$ is defined by $q = eg$ and which equals to $\frac{1}{2}$ in the case of K-K monopole.

We will consider small perturbations with $n^5 = 0$ that is we consider only perturbations which are constant in the fifth direction or equivalently we consider massless mode only. We impose the gauge conditions on the small perturbations as follows

\begin{equation}
\nabla_B h^{AB} = 0,
\end{equation}

where $h^{AB} = -g^{AC} g^{BD} h_{CD}$. We derive equations for small perturbation $h$ as follows. The Ricci tensor will be $R_{AB}$ if it is calculated from $g_{AB}$ and $R_{AB} + \delta R_{AB}$ if it is calculated from $g_{AB} + h_{AB}$. We can get second order equations on the perturbation $h$ if we impose the condition $\delta R_{AB} = 0$. This condition implies that the perturbed space is also empty space. The explicit form of the equations are

\begin{equation}
\delta R_{AB} = R_A^C \gamma^D_B h_{CD} + \frac{\Box}{2} h_{AB} = 0.
\end{equation}
Propagation of odd perturbations

We have studied the gravitational wave propagation in the K-K vacuum without magnetic monopole\[7\]. In this paper, we investigate whether the odd perturbations can propagate in the five dimensional space-time with K-K monopole provided that the perturbations are constant in the fifth direction.

The transversality conditions become

\[
\nabla A h^{0A}_{\text{odd}} = 0 \text{ (automatically satisfied)},
\]

\[
\nabla A h^{1A}_{\text{odd}} = \frac{e^{-i\omega t}AV^2(1 + V)Y_{\ell m}^{(1, 0)}}{s r^3} h_3 = 0,
\]

\[
\nabla A h^{2A}_{\text{odd}} = \frac{e^{-i\omega t}V}{s^3 r^4} \left[ h_3 \left\{ 2AcsV Y_{\ell m}^{(1, 0)} + AVY_{\ell m}^{(0, 2)} + s^2 r(V - 1)Y_{\ell m}^{(1, 0)} \right\} \right. \\
+ h_2 \left\{ -V s^2 \triangle_{\theta \phi} Y_{\ell m}^{(0, 1)} + s^2 V(V^2 - 2V - 1)Y_{\ell m}^{(0, 1)} \right\} \\
- s^2 VY_{\ell m}^{(0, 1)} (r^2 h_1)' - i\omega h_0 s^2 r^3 Y_{\ell m}^{(0, 1)} \right] = 0,
\]

\[
\nabla A h^{3A}_{\text{odd}} = \frac{e^{-i\omega t}V}{s^3 r^5} \left[ h_2 V \left\{ AV(V - 1)s^2 \triangle_{\theta \phi} Y_{\ell m} + r \left( -c^2 s Y_{\ell m}^{(1, 0)} - 2cY_{\ell m}^{(0, 2)} + cs^2 Y_{\ell m}^{(2, 0)} + sY_{\ell m}^{(1, 2)} + s^3 Y_{\ell m}^{(3, 0)} - s^3 VY_{\ell m}^{(1, 0)} + 2s^3 VY_{\ell m}^{(1, 0)} \right) \right\} \\
- AsrV h_3 Y_{\ell m}^{(1, 1)} + is^3 \omega r^3 h_0 Y_{\ell m}^{(1, 0)} + s^3 r VY_{\ell m}^{(1, 0)} (r^2 h_1)' \right] = 0
\]

and

\[
\nabla A h^{5A}_{\text{odd}} = -\frac{e^{-i\omega t}AV}{s^3 r^6} \left[ h_2 V \left\{ AV(V - 1)s^2 \triangle_{\theta \phi} Y_{\ell m} + r \left( -c^2 s Y_{\ell m}^{(1, 0)} - 2cY_{\ell m}^{(0, 2)} + cs^2 Y_{\ell m}^{(2, 0)} + sY_{\ell m}^{(1, 2)} + s^3 Y_{\ell m}^{(3, 0)} - s^3 VY_{\ell m}^{(1, 0)} + 2s^3 VY_{\ell m}^{(1, 0)} \right) \right\} \\
- AsrV h_3 Y_{\ell m}^{(1, 1)} + is^3 \omega r^3 h_0 Y_{\ell m}^{(1, 0)} + s^3 r VY_{\ell m}^{(1, 0)} (r^2 h_1)' \right] = 0
\]
Here, we have used

\[ \Delta_{\theta\varphi} = s^{-1} \partial_{\theta} s \partial_{\theta} + s^{-2} \partial_{\varphi}^2, \]

and

\[ Y_{q\ell m}^{(j,k)} = \partial_{\theta}^{j} \partial_{\varphi}^{k} Y_{q\ell m}. \]

Further, wave equations for small perturbation with \( B = 5 \) are given by

\[ 2 \delta R_{05}^{\text{odd}} = -\frac{e^{-i\omega t} V^2 (V - 1) h_0 \Delta_{\theta\varphi} Y_{q\ell m}}{r^3} = 0, \]

\[ 2 \delta R_{15}^{\text{odd}} = -\frac{e^{-i\omega t} V^2 (V - 1) h_1 \Delta_{\theta\varphi} Y_{q\ell m}}{r^3} = 0, \]

\[ 2 \delta R_{25}^{\text{odd}} = -\frac{e^{-i\omega t} V^2 (V - 1) h_2 \left\{ A s^2 V (V - 1)^2 \Delta_{\theta\varphi} Y_{q\ell m} + s^3 r (V - 1) \Delta_{\theta\varphi} Y_{q\ell m}^{(1,0)} \right\}}{s^3 r^4} \]

\[ -c r (V - 1) \left( c s Y_{q\ell m}^{(1,0)} + 2 Y_{q\ell m}^{(0,2)} \right) - s^3 r V (V - 1) (V - 2) Y_{q\ell m}^{(1,0)} \]

\[ + s r h_3 \left\{ A V^2 (V - 1) Y_{q\ell m}^{(1,1)} - s r \left( V \Delta_{\theta\varphi} - 2 V^2 (V - 1) + r^2 \omega^2 \right) Y_{q\ell m}^{(0,1)} \right\} \]

\[ - s^2 r^4 V Y_{q\ell m}^{(0,1)} h_3'' + s^3 r^2 V^2 (2 V + 1) (V - 1) Y_{q\ell m}^{(1,0)} h_1 = 0, \]

\[ 2 \delta R_{35}^{\text{odd}} = \frac{e^{-i\omega t}}{s^2 r^4} \left[ h_3 \left\{ s \left( A^2 V^3 (3 V - 1) (V - 1) - 2 A c r V^2 (V - 1) + c^2 r^2 V \right. \right. \right. \]

\[ + s^2 r^2 V^3 - s^2 r^4 \omega^2 \right) Y_{q\ell m}^{(1,0)} + s^3 r^2 V \Delta_{\theta\varphi} Y_{q\ell m}^{(1,0)} - r V \left( A V (V - 1) - 2 c r \right) Y_{q\ell m}^{(0,2)} \right\} \]

\[ - s^3 r^4 V^3 Y_{q\ell m}^{(1,0)} h_3'' + s^3 r^2 V^2 h_2 \left\{ - (V - 1) \Delta_{\theta\varphi} Y_{q\ell m}^{(0,1)} + (V - 1) (V^2 - 2 V - 1) Y_{q\ell m}^{(0,1)} \right\} \]

\[ - s^2 r^2 V^2 (2 V + 1) (V - 1) Y_{q\ell m}^{(0,1)} h_1 = 0 \]
and

\[ 2\delta R^{\text{odd}}_{55} = \frac{e^{-i\omega t}V^2(V - 1)h_3}{s^2r^4} \left[ AsV(3V - 1)Y^{(1,0)}_{q\ell m} - 2s^2r\Delta_{\theta \phi}Y_{q\ell m} \right] = 0 \]

Eq.(20) implies

\[ h_3 = 0, \]

because \( Y^{(1,0)}_{q\ell m} \) with \( q \neq 0 \) does not vanish identically. Eq.(26) implies

\[ h_0 = 0. \]

From Eq.(27), we find

\[ h_1 = 0. \]

Eqs.(29), (31) and (33) tell us

\[ h_2 = 0. \]

Thus, we find that small perturbations with odd parity are forbidden, provided that the perturbations are constant in the fifth direction.

**Summary and conclusion**

In this paper, we have studied the properties of the small perturbations around the five dimensional Kaluza-Klein metric with magnetic monopole. We found that in the Kaluza-Klein space-time with monopole, there was no odd parity solution provided that the solution is independent of the fifth coordinate. The non-existence of the time dependent solution implies the classical stability of the Kaluza-Klein monopole. One may expect that far away from the magnetic monopole, for example, the space-time is asymptotically approximated by the usual vacuum and all the vacuum
perturbation must be allowed. But this expectation is not true, because K-K monopole metric, which is given in Eq.(3), does not approaches to the vacuum metric even in the space far away from the origin. The existence of the K-K monopole gives non-local intervention.

Our arguments were restricted to the five dimensions, but our arguments would be applicable to more realistic theories.

We have discussed only the odd parity perturbations when the K-K monopole have been taken into account. It would be of interest to study the even parity perturbations. We hope to return to this problem in the future.

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