Heat kernels and the range of the trace on completions of twisted group algebras

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Abstract. Heat kernels are used in this paper to express the analytic index of projectively invariant Dirac type operators on $\Gamma$ covering spaces of compact manifolds, as elements in the K-theory of certain unconditional completions of the twisted group algebra of $\Gamma$. This is combined with V. Lafforgue's results in the untwisted case, to compute the range of the trace on the K-theory of these algebras, under the hypothesis that $\Gamma$ is in the class $C'$ (defined by V. Lafforgue).

Introduction

For $\Gamma$ a torsion-free discrete group, one formulation of a standing conjecture of Kaplansky and Kadison states that the range of the canonical trace on the $K$-theory of the reduced $C^*$-algebra of $\Gamma$, is contained in the integers. However, if we twist the convolution by a multiplier (i.e. a normalized $U(1)$-valued 2-cocycle on $\Gamma$) then this is no longer true, as shown by Pimsner-Voiculescu [PiVo] and Rieffel [Ri], who computed the precise range of the canonical trace on the twisted group $C^*$-algebra of $\mathbb{Z}^2$ (which turns out to be the noncommutative torus). The author with Marcolli in [MaMa] settled the case of surface groups by identifying the range of the canonical trace on the twisted group $C^*$-algebra. In the present paper, we study the range of the trace on the $K$-theory of good unconditional completions $\mathcal{A}(\Gamma, \sigma)$ of the twisted group algebra $\mathbb{C}(\Gamma, \sigma)$ (see section 6.1) - an example of such a completion is the $\ell^1$ completion of $\mathbb{C}(\Gamma, \sigma)$. Our approach is to study a twisted analogue of the assembly map, viewed as a homomorphism

$$\mu^\sigma_{\mathcal{A}} : K^*_\Gamma(\mathbb{E}\Gamma) \to K_*(\mathcal{A}(\Gamma, \sigma)),$$

whenever the Dixmier-Douady invariant $\delta(\sigma)$ of the multiplier $\sigma$ on $\Gamma$ is trivial. The map $\mu^\sigma_{\mathcal{A}}$ is a twisted version of the assembly map defined by Lafforgue, and the definition uses Lafforgue’s Banach $KK$-theory. We use the method of heat kernels to study an analytic version of the map [MaMa], called the twisted analytic Baum-Connes map in the paper, and a standard index theorem in section 4.3 establishes that both definitions are equivalent. Using fundamental results in [La], we prove

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in Theorem 4.5 that if \( \Gamma \) is a discrete group in Lafforgue’s class \( C' \), then the twisted assembly map \( \mu_\sigma^A \) is an isomorphism. The class \( C' \) will be described later, but we mention here that it contains all discrete subgroups of connected Lie groups, word hyperbolic groups and amenable groups. Together with a twisted version of an \( L^2 \)-index theorem given in [Ma2] it is then a straightforward corollary to obtain a formula for the range of the trace on the \( K \)-theory of \( A(\Gamma, \sigma) \) in terms of classical characteristic classes on the classifying space \( BT \) for proper actions, as explained in section 5.3. Although this formula is computationally challenging, it can be explicitly computed in low dimensional cases, e.g. in the case when \( \Gamma \) is torsion-free and \( BT \) is a smooth compact oriented manifold of dimension less than or equal to 4, which is done in section 5.4 of the paper. This generalizes earlier results of [CHMM], [MaMa], [PiVo], [Ri], [BaCo].

If in addition \( \Gamma \) has the Rapid Decay property (property RD), then we can choose the good unconditional completion \( A(\Gamma, \sigma) \) to be the Sobolev completion \( H^s(\Gamma, \sigma) \) of \( C(\Gamma, \sigma) \), for \( s \) large and \( H^s(\Gamma, \sigma) \) being a dense \( * \)-subalgebra of the reduced \( C^* \)-algebra \( C^*_r(\Gamma, \sigma) \), stable under the holomorphic functional calculus, the twisted assembly map reduces to the usual twisted assembly map cf. [Ma], (0.2)

\[
\mu_\sigma : K_1^T(E\Gamma) \to K_1(C^*_r(\Gamma, \sigma)).
\]

We prove in Theorem 4.5 that if \( \Gamma \) is a discrete group in the class \( C' \) and \( \Gamma \) has property RD, then the twisted assembly map \( \mu_\sigma \) in (0.2) is an isomorphism. The groups that have property RD include all finitely generated groups of polynomial growth, all finitely generated free groups, all word hyperbolic groups, and certain property (\( T \)) groups such as cocompact lattices in \( \text{SL}(3, \mathbb{F}) \) or the exceptional group \( E_6 \), where \( \mathbb{F} \) is a non-discrete locally compact field or the quaternions. All of these groups are also in the class \( C' \). Using the earlier mentioned procedure, we also obtain a formula for the range of the trace on the \( K \)-theory of \( C^*_r(\Gamma, \sigma) \) in terms of classical characteristic classes on \( BT \).

The last section of the paper is devoted to studying the degree one of the assembly map, and following the construction of Natsume [Na] and Valette et. al. in [BeMaVa], we determine the explicit generators of \( K_1(A(\Gamma, \sigma)) \) and \( K_1(C^*_r(\Gamma, \sigma)) \), whenever \( \Gamma \) is a torsion-free cocompact Fuchsian group.

The appendix, written by Indira Chatterji, establishes useful results on the twisted rapid decay property for \( (\Gamma, \sigma) \) that are used in the text. One interesting result there is that if \( \Gamma \) has property RD, then it has the twisted RD property for any multiplier \( \sigma \) on \( \Gamma \). This means in particular that we do not have to appeal to the Baum-Connes conjecture with coefficients, which is a technical improvement of results in [Ma]. The author thanks Indira Chatterji for helpful discussions.

1. Basics

In this section \( \sigma \) is a multiplier on \( \Gamma \) a discrete group, that is a map \( \sigma : \Gamma \times \Gamma \to U(1) \) satisfying the following identity for all \( \gamma, \mu, \delta \in \Gamma \):

1. \( \sigma(\gamma, \mu)\sigma(\gamma\mu, \delta) = \sigma(\gamma, \mu\delta)\sigma(\mu, \delta) \).
2. \( \sigma(\gamma, 1) = \sigma(1, \gamma) = 1 \).

Recall that the Dixmier-Douady invariant of a multiplier \( \sigma \) is the cohomology class \( \delta(\sigma) \in H^3(\Gamma, \mathbb{Z}) \), the image of \( [\sigma] \) obtained under the map \( \delta \) arising in the long exact sequence in cohomology derived from the short exact sequence of coefficients

\[
0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 0.
\]
We denote by $ET$ the universal cover of $B\Gamma$, the classifying space for $\Gamma$. The following lemma will be used later.

**Lemma 1.1.** Let $\alpha \in Z^2(\mathcal{B}\Gamma, \mathbb{R})$ and $X \subset ET$ a cocompact $\Gamma$-space. Then there is a map $\varphi : \Gamma \to C_0(\mathcal{X})$ such that:

(i) $\varphi_\gamma(x) + \varphi_\mu(x) - \varphi_{\mu\gamma}(x)$ is independent of $x \in X$.

(ii) There is $x_0 \in X$ such that $\varphi_\gamma(x_0) = 0$ for any $\gamma \in \Gamma$.

(iii) $\lambda(\gamma, \mu) = \varphi_\gamma(\mu x_0)$ is an $\mathbb{R}$-valued 2 cocycle that is cohomologous to $\alpha$.

**Proof.** Let $p : ET \to B\Gamma$ be the canonical projection and take a lift $\hat{\alpha} = p^*(\alpha) \in Z^2(\mathcal{E}T, \mathbb{R})$. Since $ET$ is contractible, there is a $\Lambda \in C^1(\mathcal{B}\Gamma, \mathbb{R})$ such that $\hat{\alpha} = d\Lambda$. By definition of $\hat{\alpha}$, we have

$$0 = \gamma^*\hat{\alpha} - \hat{\alpha} = d(\gamma^*\Lambda - \Lambda)$$

for any $\gamma \in \Gamma$.

The element $\eta_\gamma = \gamma^*\Lambda - \Lambda$ hence belongs to $Z^1(\mathcal{B}\Gamma, \mathbb{R})$, so that there exists $c_\gamma \in C^0(\mathcal{B}\Gamma, \mathbb{R})$ with $\eta_\gamma = dc_\gamma$. Let us show that $\mu^*c_\gamma + c_\mu - c_{\gamma\mu} \in C^0(\mathcal{B}\Gamma, \mathbb{R})$ is a constant. To do so, it is enough to see that $d(\mu^*c_\gamma + c_\mu - c_{\gamma\mu}) = 0$. We compute:

$$dc_{\gamma\mu} = \eta_{\gamma\mu} = (\gamma\mu)^*\Lambda - \Lambda = (\mu^*\gamma^*\Lambda - \gamma^*\Lambda + \gamma^*\Lambda - \Lambda)
= \mu^*\eta_\gamma + \eta_\mu = d(\mu^*c_\gamma + c_\mu)$$

Let $x_0 \in X$, we now define

$$\varphi : \Gamma \to C_0(\mathcal{X})$$

$$\gamma \mapsto \varphi_\gamma,$$

where $\varphi_\gamma(x) = c_\gamma(x) - c_\gamma(x_0)$. Then $\varphi$ satisfies (i) and (ii). In particular, from the $\mathbb{R}$-valued closed 2-form $\alpha$ on $B\Gamma$, we have produced an $\mathbb{R}$-valued group 2-cocycle $\lambda(\gamma, \mu) = \varphi_\gamma(\mu x_0)$ on $\Gamma$. Also, the group extension corresponding to $\lambda$ can be described as follows. Let $\Gamma^\lambda = \Gamma \times \mathbb{R}$ with product given by $(\gamma, r)(\gamma', r') = (\gamma\gamma', \lambda(\gamma, \gamma') + r + r')$. If $g_{ij}$ are transition functions for the principal bundle $p : ET \to B\Gamma$, define the lift $\hat{g}_{ij} = (g_{ij}, 0) \in \Gamma^\lambda$. Then $t_{ijk} = \hat{g}_{ij}\hat{g}_{jk}\hat{g}_{ki} = \lambda(g_{ij}, g_{jk}) + \lambda(g_{ik}, g_{ki})$ is the $\mathbb{R}$-valued Cech 2-cocycle on $B\Gamma$ that is associated to the $\mathbb{R}$-valued group 2-cocycle $\lambda$ on $\Gamma$.

If $a_{|U_i} = d\theta_i$, $(\theta_i - \theta_j)|_{U_i \cap U_j} = df_{ij}$, $(f_{ij} + f_{jk} + f_{ki}) = t_{ijk} \in \mathbb{R}$, then the Cech cohomology 2-cocycle corresponding to the de Rham closed 2-form $\alpha$ is $t$, by the well known Cech-de Rham isomorphism.

This shows that $[\alpha] = [t] = [\lambda] \in H^2(B\Gamma, \mathbb{R})$.

In the notation of Lemma 1.1 above, one verifies that $\sigma(\gamma, \mu) = \exp(-i\varphi_\gamma(\mu x_0))$ defines a multiplier on $\Gamma$. The map $\varphi$ is called a phase associated to $\sigma$.

**Definition 1.2.** Let $A$ be a $\Gamma$-$C^*$-algebra, we denote by $\mathbb{C}(\Gamma, A, \sigma)$ the $*$-algebra of finitely supported maps from $\Gamma$ to $A$ endowed with a $\sigma$-twisted convolution given as follows: for all $a, b \in A$ and $\gamma, \mu \in \Gamma$,

$$aT_\gamma \ast_\sigma bT_\mu = a\alpha_\gamma(b)\sigma(\gamma, \mu)T_{\gamma\mu},$$

where $\alpha$ denotes the action of $\Gamma$ on $A$. Here we think of elements of $\mathbb{C}(\Gamma, A, \sigma)$ as finite sums $\sum a_i T_\gamma$, where $a_i \in A$, $T_\gamma$ is a formal letter satisfying $T_\gamma T_\mu = T_\gamma T_{\gamma\mu}$, $T_\gamma aT_\mu = a\gamma\cdot T_\mu$ and $T_\gamma^* = \sigma(\gamma, \gamma^{-1})T_{\gamma^{-1}}$.

Given a Banach norm $\| \cdot \|_B$ on $\mathbb{C}(\Gamma, A, \sigma)$, we denote by $B(\Gamma, A, \sigma)$ the completion of $\mathbb{C}(\Gamma, A, \sigma)$ with respect to the norm $\| \cdot \|_B$. 

In case where $A = \mathbb{C}$ (with a trivial $\Gamma$-action) we simply write $\mathbb{C}(\Gamma, \sigma)$. We often represent it as the $\mathbb{C}$-subalgebra of $B(\ell^2\Gamma)$ generated by $\{T_\gamma | \gamma \in \Gamma\}$, where for $\gamma \in \Gamma$

$$T_\gamma : \ell^2\Gamma \to \ell^2\Gamma,$$

$$\delta_\mu \mapsto \sigma(\gamma, \mu)\delta_\mu,$$

so that an element in $\mathbb{C}(\Gamma, \sigma)$ is a finite $\mathbb{C}$-linear combination of the operators $T_\gamma$, and the convolution reads (for $\gamma, \mu \in \Gamma$)

$$T_\gamma \ast T_\mu = \sigma(\gamma, \mu)T_{\gamma\mu}.$$

We shall consider several completions of $\mathbb{C}(\Gamma, \sigma)$ that we now explain. The $\ell^1$-completion (given by the norm $\|\sum_{\gamma \in \Gamma} a_\gamma T_\gamma\|_1 = \sum_{\gamma \in \Gamma} |a_\gamma|$) yields the $\ell^1$-twisted Banach algebra denoted by $\ell^1(\Gamma, \sigma)$, which is the completion of $\mathbb{C}(\Gamma, \sigma)$ with respect to this $\ell^1$-norm. It is a straightforward computation to show that it is indeed a Banach algebra, contained in $B(\ell^2\Gamma)$. Next we shall consider the operator norm, given by

$$\|f\|_{op} = \sup\{|f(\xi)|_2 : \|\xi\|_2 = 1\},$$

and the completion of $\mathbb{C}(\Gamma, \sigma)$ with respect to this norm is the twisted reduced $C^*$-algebra $C^*_r(\Gamma, \sigma)$. Recall that a length function on $\Gamma$ is a map $\ell : \Gamma \to \mathbb{R}_+$ satisfying:

(a) $\ell(1) = 0$, where 1 denotes the neutral element in $\Gamma$,
(b) $\ell(\gamma) = \ell(\gamma^{-1})$ for any $\gamma \in \Gamma$,
(c) $\ell(\gamma\mu) \leq \ell(\gamma) + \ell(\mu)$ for any $\gamma, \mu \in \Gamma$.

For $\ell$ a length function on $\Gamma$ and $s$ a positive real number, the $s$-weighted $\ell^2$-norm is defined by

$$\|\sum_{\gamma \in \Gamma} a_\gamma T_\gamma\|_s = \sqrt{\sum_{\gamma \in \Gamma} |a_\gamma|^2(1 + \ell(\gamma))^{2s}}$$

and the $s$-Sobolev space is the completion of $\mathbb{C}(\Gamma, \sigma)$ with respect to this norm, denoted by $H^s(\Gamma, \sigma)$. If the length function is chosen to be the word length with respect to a finite set of generators for $\Gamma$, then we just write $H^s(\Gamma, \sigma)$, omitting $\ell$ in the notation. Finally, the space of rapidly decreasing functions (with respect to the length $\ell$) is given by

$$H^\infty(\Gamma, \sigma) = \bigcap_{s \geq 0} H^s(\Gamma, \sigma).$$

$H^\infty(\Gamma, \sigma)$ is not an algebra in general, but it is one if $\Gamma$ has the Rapid Decay property (with respect to the length $\ell$), see Definition 6.3. In fact, if $\Gamma$ has the Rapid Decay property (with respect to the length $\ell$), then $H^s(\Gamma, \sigma)$ is an algebra for $s$ large enough, cf. Corollary 6.8.

**Lemma 1.3 (Sup norm characterization).** If $\Gamma$ has polynomial volume growth (with respect to the length $\ell$), then the space of rapidly decreasing functions (with respect to the length $\ell$) has the following sup norm characterization:

$$H^\infty(\Gamma, \sigma) = \{ f : \Gamma \to \mathbb{C} : \sup_{\gamma \in \Gamma} (|f(\gamma)(1 + \ell(\gamma))|^s < \infty \quad \forall s \in \mathbb{N}\}$$

**Proof.** If $f \in H^\infty(\Gamma, \sigma)$, then for all $s \in \mathbb{N}$, one sees that the function $\gamma \mapsto |f(\gamma)|^2(1 + \ell(\gamma))^{2s}$ is bounded on $\Gamma$, therefore $\gamma \mapsto |f(\gamma)(1 + \ell(\gamma))|^s$ is also a bounded function on $\Gamma$. 


Conversely, suppose that \( f : \Gamma \to C \) is such that
\[
\sup_{\gamma \in \Gamma} (|f(\gamma)|(1 + \ell(\gamma))^{-r}) = C_r < \infty \quad \forall r \in \mathbb{N}.
\]
Then we estimate,
\[
\sum_{\gamma \in \Gamma} |f(\gamma)|^2 (1 + \ell(\gamma))^{2s} \leq C_r^2 \sum_{\gamma \in \Gamma} (1 + \ell(\gamma))^{2(s-r)}.
\]
Since \( \Gamma \) has polynomial growth (with respect to the length \( \ell \)), we see that by choosing \( r \) sufficiently large, the right hand side is finite, proving the lemma. \( \square \)

An important step is to compute the \( K \)-theory of \( C^*_r(\Gamma, \sigma) \). The following lemma shows that we only need to know the cohomology class of the multiplier.

**Lemma 1.4.** Let \( \sigma, \sigma' \in H^2(\Gamma, U(1)) \) be two cohomologous 2-cocycles. Then there exists an isomorphism
\[
\varphi : B(\Gamma, \sigma) \to B(\Gamma, \sigma'),
\]
inducing the identity map on \( K \)-theory. Here \( B(\Gamma, \sigma) \) is any \( * \)-Banach completion of \( \mathbb{C}(\Gamma, \sigma) \).

**Proof.** That the cocycles \( \sigma \) and \( \sigma' \) are cohomologous means that there exists \( f : \Gamma \to U(1) \) such that \( \sigma' = \sigma df \), where for \( \gamma_1, \gamma_2 \in \Gamma \), \( df(\gamma_1, \gamma_2) = f(\gamma_1 \gamma_2) f(\gamma_1)^{-1} f(\gamma_2)^{-1} \). We shall define the map \( \varphi : B(\Gamma, \sigma) \to B(\Gamma, \sigma') \) on the generators \( \{T_\gamma\}_{\gamma \in \Gamma} \) by \( \varphi(T_\gamma) = f(\gamma) T'_\gamma \), extend it by linearity to \( \mathbb{C}(\Gamma, \sigma) \) and by continuity to \( B(\Gamma, \sigma) \). Indeed, it is a \( * \)-homomorphism:
\[
\varphi(T_\gamma T'_\mu) = \sigma(\gamma, \mu) \varphi(T_{\gamma \mu}) = \sigma(\gamma, \mu) f(\gamma \mu) T'_{\gamma \mu}
\]
\[
= \sigma'(\gamma, \mu) f(\gamma) f(\mu) T'_{\gamma \mu} = f(\gamma) f(\mu) T'_{\gamma \mu} = \varphi(T_\gamma) \varphi(T_\mu),
\]
bijection (it is bijective on the generators), hence induces an isomorphism in \( K \)-theory, which is the identity since \( [T'_\gamma] = [f(\gamma) T'_\gamma] \) in \( K_1(B(\Gamma, \sigma')) \) for any \( \gamma \in \Gamma \), the homotopy being given by a path in \( U(1) \) between \( f(\gamma) \) and 1. Similarly at the level of \( K_0 \). \( \square \)

### 2. Good unconditional completions

**Definition 2.1.** Following Lafforgue \( \text{La} \), we say that a norm \( \| \cdot \|_A \) on \( \mathbb{C}(\Gamma, \sigma) \) is **unconditional** if for any two elements \( A = \sum_{\gamma \in \Gamma} a_\gamma T_\gamma \) and \( B = \sum_{\gamma \in \Gamma} b_\gamma T_\gamma \) in \( \mathbb{C}(\Gamma, \sigma) \), \( |a_\gamma| \leq |b_\gamma| \) implies \( \| A \|_A \leq \| B \|_A \). Given an unconditional norm \( \| \cdot \|_A \) on \( \mathbb{C}(\Gamma, \sigma) \), we denote by \( A(\Gamma, \sigma) \) the completion.

For technical reasons, since we use the heat kernel approach in this paper, we introduce the following special case. Assume that an unconditional completion \( \mathcal{A}(\Gamma, \sigma) \) of \( \mathbb{C}(\Gamma, \sigma) \) is such that
\[
\| T_g \|_A \leq C_1 e^{C_2 \ell_g(p)}, \quad \forall g \in \Gamma,
\]
for some positive constants \( C_1, C_2 \) independent of \( g \in \Gamma \) and for some \( p \) such that \( 1 \leq p < 2 \) which is also independent of \( g \in \Gamma \). We shall call such an unconditional completion a **good unconditional completion** of \( \mathbb{C}(\Gamma, \sigma) \).

**Remark 2.2.** Note that \( \ell^1 \) is trivially a good unconditional completion, and that it is straightforward to see that the Sobolev completions are good unconditional completions as well. The operator norm is not unconditional, which means that the reduced (twisted) group \( C^* \)-algebra is **not** an unconditional completion.
Let $\ell^2(\Gamma, H)$ denote the space of $H$-valued square summable functions on the group $\Gamma$, where $H$ is a separable Hilbert space with the trivial action of $\Gamma$. Let $\mathcal{U}_H(\Gamma, \sigma)$ denote the von Neumann algebra of all bounded linear operators on $\ell^2(\Gamma, H)$ that commute with the $(\Gamma, \sigma)$-action. It is a standard observation that any element $A \in \mathcal{U}_H(\Gamma, \sigma)$ can be represented by a strongly convergent series,

$$A = \sum_{\gamma \in \Gamma} T_\gamma \otimes A(\gamma),$$

where $A(\gamma) \in \mathcal{B}(H)$ is a bounded linear operator on $H$, defined by the formula

$$A(\delta_v \otimes v) = \sum_{\gamma \in \Gamma} \delta_\gamma \otimes A(\gamma)v, \quad v \in H.$$

One has the following useful sufficient condition, where $\otimes$ denotes the projective tensor product in the entire paper.

**Lemma 2.3.** Let $\mathcal{A}(\Gamma, \sigma)$ be a good unconditional completion of $\mathbb{C}(\Gamma, \sigma)$ and $\mathcal{K}$ denote the algebra of compact operators on the Hilbert space $H$. If $A \in \mathcal{U}_H(\Gamma, \sigma)$, $A = \sum_{\gamma \in \Gamma} T_\gamma \otimes A(\gamma)$ is such that $A(\gamma) \in \mathcal{K}$ and also satisfies $\|A(\gamma)\| < C_5 e^{-C_4 \ell(\gamma)^2}$ for some positive constants $C_5, C_6$, then $A \in \mathcal{A}(\Gamma, \sigma) \otimes \mathcal{K}$.

**Proof.** Observe that one has the estimate

$$\# \{ \gamma \in \Gamma \mid \ell(\gamma) \leq n \} \leq C_7 e^{C_8 n},$$

for some positive constants $C_7, C_8$, since the growth rate of the volume of balls in $\Gamma$ is at most exponential. We compute,

$$\|A\|_{\mathcal{A} \otimes \mathcal{K}} = \|\sum_{\gamma \in \Gamma} T_\gamma \otimes A(\gamma)\|_{\mathcal{A} \otimes \mathcal{K}} \leq \sum_{\gamma \in \Gamma} \|T_\gamma\| \|A(\gamma)\|$$

$$\leq \sum_{\gamma \in \Gamma} C_1 e^{C_2 \ell(\gamma)^p} C_5 e^{-C_4 \ell(\gamma)^2} = \sum_{n \in \mathbb{N}} \sum_{\ell(\gamma) \leq n} C_1 e^{C_2 \ell(\gamma)^p} C_5 e^{-C_4 \ell(\gamma)^2}$$

$$\leq \sum_{n \in \mathbb{N}} C_7 e^{C_8 n} C_1 e^{C_2 n^p} C_5 e^{-C_4 n^2} < \infty.$$ 

The last sum is convergent since $0 < p < 2$ by the good unconditional hypothesis.

$\square$

In [PR], Packer and Raeburn, inspired by A. Wasserman’s thesis, established a stabilization (or untwisting) trick. We will present a good unconditional version of this, in the simple case of a discrete group $\Gamma$, that we need in this paper. Let $\sigma$ be a multiplier on $\Gamma$ and $\mathcal{K}$ be the algebra of compact operators on $\ell^2(\Gamma)$. Observe that for any $\Gamma$-$C^*$-algebra $A$, one has the following canonical isomorphism,

$$\mathbb{C}(\Gamma, A, \sigma) \otimes \mathcal{K} \cong \mathbb{C}(\Gamma, A \otimes \mathcal{K}),$$

where $\Gamma$ acts diagonally on the tensor product $A \otimes \mathcal{K}$, and is given by the given action of $\Gamma$ on $A$ and the adjoint action, $\gamma \rightarrow \text{Ad}(T_\gamma)$. That is, the twisted convolution on the left hand side of (2.2) becomes an ordinary convolution on the right hand side of (2.2), for all $a, b \in A$, for all $U, V \in \mathcal{K}$ and for all $\gamma, \mu \in \Gamma$,

$$a \otimes \text{Ad}(T_\gamma)U * b \otimes \text{Ad}(T_\mu)V = a a_\gamma(b) \otimes \text{Ad}(T_{\gamma \mu})UV,$$

where $\alpha$ denotes the action of $\Gamma$ on $A$. 


Recall that for any good unconditional completion $\mathcal{A}(\Gamma, A, \sigma)$ of $\mathbb{C}(\Gamma, A, \sigma)$ there are positive constants $C_1, C_2$ independent of $g \in \Gamma$ such that

$$\|T_g\|_A \leq C_1 e^{C_2 \ell(g)^p}, \quad \forall g \in \Gamma,$$

for some $p$ such that $1 \leq p < 2$ which is also independent of $g \in \Gamma$. But this implies that

$$\|\text{Ad}(T_g)\|_A \leq C_2 e^{2C_2 \ell(g)^p}, \quad \forall g \in \Gamma,$$

and conversely. Therefore for any good unconditional completion, one has the canonical isomorphism

$$(2.3) \mathcal{A}(\Gamma, A, \sigma) \otimes \mathcal{K} \cong \mathcal{A}(\Gamma, A \otimes \mathcal{K}).$$

This isomorphism is clearly also true for general unconditional completions.

3. Heat kernels and the analytic twisted Baum-Connes map

3.1. Spin$^C$ manifolds and twisted spin$^C$ Dirac operators. Let $M$ be a smooth $\Gamma$ manifold without boundary. A choice of $\Gamma$-invariant Riemannian metric $g$ on $M$ defines a bundle of Clifford algebras, with fibre at $z \in M$ the complexified Clifford algebra

$$(3.1) \quad \mathcal{C}l_z(M) = \left( \bigoplus_{k=0}^{\infty} (T^*_z M \otimes \mathbb{C})^k \right) / \langle \alpha \otimes \beta + \beta \otimes \alpha - 2(\alpha, \beta)_g, \alpha, \beta \in T^*_z M \rangle.$$

If $\dim M = 2n$, this complexified algebra is isomorphic to the matrix algebra on $\mathbb{C}^{2n}$. In particular the Clifford bundle is an associated bundle to the metric coframe bundle, the principal $\text{SO}(2n)$-bundle $\mathcal{F}$, where the action of $\text{SO}(2n)$ on the Euclidean Clifford algebra $\mathcal{C}l(2n)$ is through the spin group, which may be identified within the Clifford algebra as

$$(3.2) \quad \text{Spin}(2n) = \{ v_1 v_2 \cdots v_{2k} \in \mathcal{C}l(2n); v_i \in \mathbb{R}^{2n}, |v_i| = 1 \}.$$

The non-trivial double covering of $\text{SO}(2n)$ is realized through the mapping of $v$ to the reflection $R(v) \in O(2n)$ in the plane orthogonal to $v$

$$(3.3) \quad p : \text{Spin}(2n) \ni a = v_1 \cdots v_{2k} \mapsto R(v_1) \cdots R(v_{2k}) = R \in \text{SO}(2n).$$

The Spin$^C(2n)$ group, defined as

$$(3.4) \quad \text{Spin}^C(2n) = \{ cv_1 v_2 \cdots v_{2k} \in \mathcal{C}l(2n); v_i \in \mathbb{R}^{2n}, |v_i| = 1, c \in \mathbb{C}, |c| = 1 \},$$

is a central extension of $\text{SO}(2n)$,

$$(3.5) \quad \mathbb{U}(1) \longrightarrow \text{Spin}^C(2n) \longrightarrow \text{SO}(2n),$$

where the quotient map is consistent with the covering of $\text{SO}(2n)$ by Spin($2n$), i.e.

$$(3.6) \quad \text{Spin}^C(2n) = \text{Spin}(2n) \times_{\mathbb{Z}_2} \mathbb{U}(1).$$
The manifold $M$ is said to have a $\Gamma$-equivariant $\text{Spin}^C$ structure, if there is an extension of the coframe bundle to a principal $\text{Spin}^C(2n)$-bundle

$$\begin{align*}
\text{U}(1) & \to \text{U}(1) \\
\text{Spin}^C(2n) & \to \mathcal{F}_L \to M \\
\text{SO}(2n) & \to \mathcal{F} \to M,
\end{align*}$$

where $\mathcal{F}_L$, the $\text{Spin}^C(2n)$ bundle over $M$, may also be viewed as a circle bundle over $\mathcal{F}$, compatible with the $\Gamma$-action. The associated bundles of half spinors on $M$ are defined as

$$S^\pm = \mathcal{F}_L \times_{\text{Spin}^C(2n)} S^\pm,$$

where $S^\pm$ are the fundamental half spin representations of $\text{SO}(2n)$. The $\Gamma$-invariant Levi-Civita connection determines a connection 1-form on $\mathcal{F}$, and together with the choice of a $\Gamma$-invariant connection 1-form on the circle bundle $\mathcal{F}_L$ over $\mathcal{F}$, they determine a connection 1-form on the principal $\text{Spin}^C$ bundle $\mathcal{F}_L$ over $M$, which is $\Gamma$-invariant. That is, one gets a connection

$$\nabla^{S\otimes E} : C^\infty(M, S^+ \otimes E) \to C^\infty(M, T^* M \otimes S^+ \otimes E),$$

defined as $\nabla^{S\otimes E} = \nabla^S \otimes 1 + 1 \otimes \nabla^E$, where $\nabla^E$ is a $\Gamma$-invariant connection on the $\Gamma$-invariant vector bundle $E$ over $M$. Now the contraction given by Clifford multiplication defines a map

$$C : C^\infty(M, T^* M \otimes S^+ \otimes E) \to C^\infty(M, S^- \otimes E).$$

The $\Gamma$-equivariant $\text{Spin}^C$ Dirac operator with coefficients in $E$ is defined as the composition

$$\partial_{E}^{C} = C \circ \nabla^{S\otimes E}.$$

In this section, we will define the analytic index map for an arbitrary torsion-free discrete group $\Gamma$ and for an arbitrary multiplier $\sigma$ on $\Gamma$ with trivial Dixmier-Douady invariant $\delta(\sigma)$. Now let $M$ be a manifold without boundary with a given smooth proper cocompact $\Gamma$ action and a $\Gamma$-equivariant $\text{Spin}^C$ structure, $E \to M$ a $\Gamma$-equivariant complex vector bundle on $M$, and $\phi : M \to \mathbb{E}\Gamma$ a $\Gamma$-equivariant continuous map. We will view the $\Gamma$-equivariant $\text{Spin}^C$ Dirac operator with coefficients in $E$ as an operator on the $L^2$-spaces, $\partial_{E}^{C} : L^2(M, S^+ \otimes E) \to L^2(M, S^- \otimes E)$.

Let $c$ be an $\mathbb{R}$-valued $\Gamma$-equivariant Cech 2-cocycle on $\mathbb{E}\Gamma$ and $\omega$ be a $\Gamma$-equivariant closed 2-form on $M$ such that the $\Gamma$-equivariant cohomology class of $\omega$ is equal to $\phi^*(c)$. Note that $\omega$ is exact, $\omega = d\eta$, since $\mathbb{E}\Gamma$ is contractible. Define $\nabla = d + i\eta$. Then $\nabla$ is a Hermitian connection on the trivial line bundle $L$ over $M$, and the curvature of $\nabla$ is $(\nabla)^2 = i\omega$. Then $\nabla$ defines a projective action of $\Gamma$ on $L^2$ spinors as follows:

For $u \in L^2(M, S \otimes E \otimes L)$, let $S_\gamma u = e^{i\phi_\gamma} u$ (where $\phi$ is the phase for $\sigma$ as explained in Lemma 1.1), $U_\gamma u = \gamma^{-1} u$, and $T_\gamma = U_\gamma S_\gamma$ be the composition, for
all $\gamma \in \Gamma$. Then $T$ defines a projective $(\Gamma, \sigma)$-action on $L^2(M, S \otimes E \otimes \mathcal{L})$, meaning that for any $\gamma, \gamma' \in \Gamma$ one has

$$T_\gamma T_{\gamma'} = \sigma(\gamma, \gamma')T_{\gamma\gamma'}.$$ 

Let $\varphi^+_{E \otimes \mathcal{L}} : L^2(M, S^+ \otimes E \otimes \mathcal{L}) \to L^2(M, S^- \otimes E \otimes \mathcal{L})$ denote the twisted $\Gamma$-equivariant Spin$^c$ Dirac operator.

**Lemma 3.1.** The twisted $\Gamma$-equivariant Spin$^c$ Dirac operator on $M$,

$$\varphi^+_{E \otimes \mathcal{L}} : L^2(M, S^+ \otimes E \otimes \mathcal{L}) \to L^2(M, S^- \otimes E \otimes \mathcal{L}),$$

commutes with the projective $D$-action.

**Proof.** To simplify notation, set $D_\eta = \varphi^+_{E \otimes \mathcal{L}}$ and $D_0 = \varphi^+_{E \otimes \mathcal{L}}$ where we emphasize the dependence on $\eta$. Then $D_\eta = D_0 + i c(\eta)$, where $c(\eta)$ denotes Clifford multiplication by the one-form $\eta$. An easy computation establishes that $U_\gamma D_\eta = D_{\gamma^{-1}} \eta U_\gamma$ and that $S_\gamma D_{\gamma^{-1}} \eta = D_\eta S_\gamma$ for all $\gamma \in \Gamma$. Then $T_\gamma D_\eta = D_\eta T_\gamma$, where $T_\gamma = U_\gamma S_\gamma$ denotes the projective $\gamma$-action. \qed

**3.2. Heat kernels and the analytic index.** Recall the following well-known smoothness properties and Gaussian off-diagonal estimates for the heat kernel, cf. [G], [Ko].

**Lemma 3.2.** The Schwartz kernels $k_\pm(t, x, y)$ of the heat operators $e^{-tD^+D^\mp}$ are smooth for all $t > 0$. Moreover, for any $t > 0$ there are positive constants $C_1, C_2$ such that the following off-diagonal estimate holds

$$|k_\pm(t, x, y)| \leq C_1 e^{-C_2 d(x, y)^2}, \quad x, y \in M,$$

where $d$ denotes the Riemannian distance function on $M$.

For fixed $t > 0$, we will use Lemma 3.2 to show the following.

**Proposition 3.3.** Let $A(\Gamma, \sigma)$ be a good unconditional completion of $C(\Gamma, \sigma)$. Then for fixed $t > 0$, the heat operators $e^{-tD^-D^+}$ and $e^{-tD^+D^-}$ belong to $A(\Gamma, \sigma) \otimes K_+ \otimes A(\Gamma, \sigma) \otimes K_-$ respectively, where $K_\pm$ denotes the algebra of compact operators on the Hilbert space $\mathcal{H}_\pm = L^2(\mathcal{F}, S^\pm \otimes E | \mathcal{F})$, and $\mathcal{F}$ denotes a connected fundamental domain of the action of $\Gamma$ on $M$.

**Proof.** We have $e^{-tD^+D^\mp} \in \mathcal{U}_\mathcal{H}_\pm(\Gamma, \sigma)$, so that

$$e^{-tD^+D^\mp} = \sum_{\gamma \in \Gamma} T_\gamma \otimes h_\pm^\gamma(\gamma),$$

where $h_\pm^\gamma(\gamma) \in \mathcal{B}(\mathcal{H}_\pm)$ has Schwartz kernel $k_\pm(t, x, \gamma y)$ for $x, y \in \mathcal{F}$. By Lemma 3.2 we have

$$\|h_\pm^\gamma(\gamma)\| \leq \|k_\pm(t, x, \gamma y)\| \leq C_1 e^{-C_2 d(\gamma)^2},$$

where $d(\gamma) = \inf\{d(x, \gamma y) : x, y \in \mathcal{F}\}$.

It is well known that

$$\ell(\gamma) \leq C_4(d(\gamma) + 1),$$

for some positive constant $C_4$. From (3.12) and Lemma 3.2 we get

$$\|h_\pm^\gamma(\gamma)\| \leq C_5 e^{-C_4 \ell(\gamma)^2},$$

for some positive constants $C_5, C_6$. We conclude using Lemma 2.3. \qed
We shall now give a brief description of the Baum-Connes-Douglas \([\text{BaDo}]\) version of the twisted analytic index map.\footnote{\textcite{BaCo}} Since \(\mathcal{A}(\Gamma, \sigma)\) is a Banach algebra, one has the invariance property of \(K\)-theory under stable isomorphism, \(K_*(\mathcal{A}(\Gamma, \sigma)) \cong K_*(\mathcal{A}(\Gamma) \otimes \mathcal{K})\). Using this isomorphism, the \(\mathcal{A}\)-twisted analytic index is defined as

\[
e - \text{index}_A(D^+) = [e_t(D)] - [E_0] \in K_0(\mathcal{A}(\Gamma, \sigma)),
\]

where \(\mathcal{A}(\Gamma, \sigma) \otimes \tilde{\mathcal{K}}\) denotes the unital algebra associated with \(\mathcal{A}(\Gamma, \sigma) \otimes \mathcal{K}\). It is the analogue of the Wasserman idempotent, see e.g. Connes and Moscovici \(\text{CoMo}\). Since \(\mathcal{A}(\Gamma, \sigma)\) is a Banach algebra, one has the invariance property of \(K\)-theory under stable isomorphism, \(K_*(\mathcal{A}(\Gamma, \sigma)) \cong K_*(\mathcal{A}(\Gamma) \otimes \mathcal{K})\). Using this isomorphism, the \(\mathcal{A}\)-twisted analytic index is defined as

\[
\text{index}_A(D^+) = [e_t(D)] - [E_0] \in K_0(\mathcal{A}(\Gamma, \sigma)),
\]

where \(t > 0\) and \(E_0\) is the idempotent

\[
E_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathcal{A}(\Gamma, \sigma) \otimes \tilde{\mathcal{K}}).
\]

Since the difference \(e_t(D^+) - E_0\) is in \(M_2(\mathcal{A}(\Gamma, \sigma) \otimes \mathcal{K})\), we see that the right hand side of equation (3.14) is in \(K_0(\mathcal{A}(\Gamma, \sigma))\) as asserted.

### 3.3. Topological \(K\)-homology and the analytic twisted Baum-Connes map

We shall now give a brief description of the Baum-Connes-Douglas \(\text{BaDo}\) version of the \(K\)-homology groups \(K_j^\Gamma(E\Gamma)\) \((j = 0, 1)\).

The basic objects are \(\Gamma\)-equivariant \(K\)-cycles. A \(\Gamma\)-equivariant \(K\)-cycle on \(E\Gamma\) is a triple \((M, E, \phi)\), where:

- (i) \(M\) is a manifold without boundary with a smooth proper cocompact \(\Gamma\)-action and a \(\Gamma\)-equivariant \(\text{Spin}^C\) structure.
- (ii) \(E \to M\) is a \(\Gamma\)-equivariant complex vector bundle on \(M\).
- (iii) \(\phi : M \to E\Gamma\) is a \(\Gamma\)-equivariant continuous map.

Two \(\Gamma\)-equivariant \(K\)-cycles \((M, E, \phi)\) and \((M', E', \phi')\) are said to be isomorphic if there is a \(\Gamma\)-equivariant diffeomorphism \(h : M \to M'\) preserving the \(\Gamma\)-equivariant \(\text{Spin}^C\) structures on \(M, M'\) such that \(h^*(E') \cong E\) and \(h^*\phi' = \phi\). Let \(\Pi^F(E\Gamma)\) denote the collection of all \(\Gamma\)-equivariant \(K\)-cycles on \(E\Gamma\). The following operations on \(\Gamma\)-equivariant \(K\)-cycles will enable us to define an equivalence relation on \(\Pi^F(E\Gamma)\).

**Bordism:** Two \(\Gamma\)-equivariant \(K\)-cycles \((M_i, E_i, \phi_i) \in \Pi^F(E\Gamma)\) \((i = 0, 1)\) are said to be bordant if there is a triple \((W, E, \phi)\), where \(W\) is a manifold with boundary \(\partial W\), with a smooth proper cocompact \(\Gamma\)-action and a \(\Gamma\)-equivariant \(\text{Spin}^C\) structure; \(E \to W\) is a \(\Gamma\)-equivariant complex vector bundle on \(W\) and \(\phi : W \to X\) is a \(\Gamma\)-equivariant continuous map such that \((\partial W, E|_{\partial W}, \phi|_{\partial W})\) is isomorphic to the disjoint union \((M_0, E_0, \phi_0) \cup (-M_1, E_1, \phi_1)\). Here \(-M_1\) denotes \(M_1\) with the reversed \(\Gamma\)-equivariant \(\text{Spin}^C\) structure.

**Direct sum:** Suppose that \((M, E, \phi) \in \Pi^F(E\Gamma)\) and that \(E = E_0 \oplus E_1\). Then \((M, E, \phi)\) is isomorphic to \((M, E_0, \phi) \cup (M, E_1, \phi)\).

**Vector bundle modification:** Let \((M, E, \phi) \in \Pi^F(E\Gamma)\) and \(H\) be an even dimensional \(\Gamma\)-equivariant \(\text{Spin}^C\) vector bundle over \(M\). Let \(\widetilde{M} = S(H \oplus 1)\) denote the sphere bundle of \(H \oplus 1\). Then \(\widetilde{M}\) is canonically a \(\Gamma\)-equivariant \(\text{Spin}^C\) manifold. Let \(S\)
denote the \( \Gamma \)-equivariant bundle of spinors on \( H \). Since \( H \) is even dimensional, \( S \) is \( \mathbb{Z}_2 \)-graded,
\[
S = S^+ \oplus S^-,
\]
into \( \Gamma \)-equivariant bundles of 1/2-spinors on \( M \). Define \( \hat{E} = \pi^*(S^{**} \otimes E) \), where \( \pi : \hat{M} \to M \) is the projection. Finally, set \( \hat{\phi} = \pi^* \phi \). Then \((\hat{M}, \hat{E}, \hat{\phi}) \in \Pi^\Gamma(E\Gamma) \) is said to be obtained from \((M, E, \phi) \) and \( H \) by vector bundle modification.

Let \( \sim \) denote the equivalence relation on \( \Pi^\Gamma(E\Gamma) \) generated by the operations of bordism, direct sum and vector bundle modification. Notice that \( \sim \) preserves the parity of the dimension of the \( K \)-cycle. Let
\[
K^\Gamma_0(E\Gamma) = \Pi^\Gamma_{\text{even}}(E\Gamma)/\sim,
\]
where \( \Pi^\Gamma_{\text{even}}(E\Gamma) \) denotes the set of all even dimensional \( \Gamma \)-equivariant \( K \)-cycles in \( \Pi^\Gamma(E\Gamma) \), and let
\[
K^\Gamma_1(E\Gamma) = \Pi^\Gamma_{\text{odd}}(E\Gamma)/\sim,
\]
where \( \Pi^\Gamma_{\text{odd}}(E\Gamma) \) denotes the set of all odd dimensional \( \Gamma \)-equivariant \( K \)-cycles in \( \Pi^\Gamma(E\Gamma) \).

The analytic twisted Baum-Connes map is defined as
\[
a - \mu^\Lambda_\sigma : K^\Gamma_0(E\Gamma) \to K_0(\mathcal{A}(\Gamma, \sigma))
\]
(3.15)
\[
a - \mu^\Lambda_\sigma ([M, E, \phi]) = a - \text{index}^\Lambda_\sigma (D^+),
\]
(3.16)
where \( D^+ \) is the twisted \( \Gamma \)-equivariant \( \text{Spin}^\mathbb{C} \) Dirac operator defined as in section 3.4.

4. Twisted Baum-Connes conjecture in Lafforgue’s settings

4.1. On Lafforgue’s Banach \( KK \)-theory. We here recall Lafforgue’s definitions of Banach \( KK \)-theory in \[La\], and its compatibility with Kasparov \( KK \)-theory. A Banach algebra \( A \) is called a \( \Gamma \)-Banach algebra if \( \Gamma \) acts on \( A \) by isometric automorphisms. We shall briefly sketch how Lafforgue associates to a pair of \( \Gamma \)-Banach algebras \((A, B)\) an abelian group
\[
KK^\text{ban}_1(A, B).
\]

An important concept in this setting is the notion of \( \Gamma - (A, B) \)-Banach bimodule \( E \): to start with, \( E \) is a \( B \)-pair, that is a pair of Banach spaces \( E = (E^<, E^>) \) each of which is endowed with a \( B \)-action (left and right respectively), and with a \( B \)-valued and \( \mathbb{C} \)-linear bracket satisfying
\[
\langle bx, y \rangle = b \langle x, y \rangle, \quad \langle x, y \rangle b = \langle x, y \rangle b, \quad \| \langle x, y \rangle \| B \leq \| x\| \| y\|,
\]
(where the norms of \( x \) and \( y \) are taken in \( E^< \) and \( E^> \) respectively). A \( B \)-pair \( E \) is called an \((A, B)\)-bimodule if it is endowed with a Banach algebra morphism from \( A \) into \( L(E) \). If \( A \) and \( B \) are \( \Gamma \)-Banach algebras, then a \( B \)-pair \( E \) endowed with an isometric \( \Gamma \)-action is called a \( \Gamma \)-\( B \)-pair, and a \( \Gamma \) – \((A, B)\)-Banach bimodule if \( E \) is both an \((A, B)\)-bimodule and a \( \Gamma \)-\( B \)-pair such that the morphism \( A \to L(E) \) is \( \Gamma \)-equivariant. Denote by \( E^\text{ban}(A, B) \) the isomorphism classes of pairs \( \alpha = (E, T) \), where \( E \) is a \( \mathbb{Z}_2 \)-graded \( \Gamma - (A, B) \)-Banach bimodule and \( T \in L(E) \) an operator reversing the graduation and such that for any \( a \in A \), \([a, T]\) and \( a(\text{Id}_E - T^2) \) are compact operator on \( E \). Two cycles \( \alpha \) and \( \beta \) in \( E^\text{ban}(A, B) \) are said homotopic if they are the image of the evaluation in 0 and 1 respectively of a single element...
in $E^{ban}(A, B[0, 1])$, where $B[0, 1]$ denotes the Banach algebra of continuous maps from the interval $[0, 1]$ into $B$. $KK^n_{ban}(A, B)$ is the quotient of $E^{ban}(A, B)$ by the equivalence relation induced by homotopy. This defines an abelian group, and Lafforgue’s Banach $KK^{ban}$ theory is compatible with Kasparov’s $KK$-theory in the sense that the forgetful morphism $\iota : E_\Gamma(A, B) \to E^f_\Gamma(A, B)$ induces a well-defined morphism $\iota : KK_\Gamma(A, B) \to KK^{ban}_\Gamma(A, B)$ which is functorial in $A$ and $B$ in the case where those are $\Gamma$-$C^*$-algebras. It is well-known that $K^n_\Gamma(X) \cong KK_\Gamma(C_0(X), \mathbb{C})$, where $X$ is any $\Gamma$-CW-complex.

4.2. Twisted Assembly map - the idempotent method. For any separable $\Gamma$-$C^*$-algebra $C$, there is a dilation homomorphism

$$\tau_{C, \Gamma} : KK^{ban}_\Gamma(A, B) \to KK^{ban}_\Gamma(C \otimes A, C \otimes B),$$

where as in the entire paper, $\otimes$ denotes the projective tensor product, cf. [La]. The following stability property is also proved in [La]:

$$KK^{ban}_\Gamma(A, B) \cong KK^{ban}_\Gamma(\mathcal{K} \otimes A, \mathcal{K} \otimes B),$$

where $\mathcal{K}$ denotes the Banach algebra of compact operators on a Hilbert space, such as $\ell^2(\Gamma)$.

**Proposition 4.1 (Twisted descent map).** For any two $\Gamma$-$C^*$-algebras $A$ and $B$ there is a twisted descent map

$$j_{\Gamma, A, \sigma} : KK^{ban}_\Gamma(A, B) \to KK^{ban}_\Gamma(A(\Gamma, A, \sigma), A(\Gamma, B, \sigma)),$$

which is compatible with the canonical homomorphism $j_{\Gamma, \sigma}$ of Proposition 2.1 in [Ma].

**Proof.** The twisted descent map is defined as the composition of the following three homomorphisms. The first is the dilation homomorphism,

$$\tau_{\mathcal{K}, \Gamma} : KK^{ban}_\Gamma(A, B) \to KK^{ban}_\Gamma(\mathcal{K} \otimes A, \mathcal{K} \otimes B),$$

where the action of $\Gamma$ on $\mathcal{K}$ is determined by $\sigma$ and is given as in the unconditional version of the Packer-Raeburn stabilization theorem, [23]. The second is Lafforgue’s descent homomorphism [La],

$$j_{\Gamma, A} : KK^{ban}_\Gamma(\mathcal{K} \otimes A, \mathcal{K} \otimes B) \to KK^{ban}_\Gamma(A(\Gamma, \mathcal{K} \otimes A), A(\Gamma, \mathcal{K} \otimes B)),$$

where $\Gamma$ acts diagonally on $\mathcal{K} \otimes A$ and on $\mathcal{K} \otimes B$. The third isomorphism is obtained as a result of the unconditional version of the Packer-Raeburn stabilization theorem [24], together with stability of $KK^{ban}$ as above [4,1],

$$KK^{ban}_\Gamma(A(\Gamma, \mathcal{K} \otimes A), A(\Gamma, \mathcal{K} \otimes B)) \cong KK^{ban}_\Gamma(A(\Gamma, A, \sigma), A(\Gamma, B, \sigma)) .$$

The composition of the homomorphisms [1,3], [4,3], and [4,5] yields the twisted descent map in equation (4.2). \hfill \Box

To follow Lafforgue’s construction of the assembly map we shall now define a canonical element in $KK(\mathbb{C}, A(\Gamma, C_0(X), \sigma)) \simeq K_0(A(\Gamma, C_0(X), \sigma))$, where $X \subset ET$ is a free cocompact $\Gamma$-CW-complex.

**Lemma 4.2 (Canonical idempotent).** Take $h \in C_0(X)$ such that $\sum_{\gamma \in \Gamma} h(\gamma x)^2 = 1$ and let $\varphi$ be the phase associated to the cocycle $\sigma$. The element

$$e(\gamma, x) = h(x)h(\gamma^{-1}x)e^{-i\varphi, (\gamma^{-1}x)} \in A(\Gamma, C_0(X), \sigma),$$

(4.6)
is an idempotent, which defines a class \([e] \in K_0(\mathcal{A}(\Gamma, C_0(X), \sigma))\) that is independent of the choice of \(h\).

**Proof.** That \(e\) belongs to \(\mathcal{A}(\Gamma, C_0(X), \sigma)\) is clear since it is finitely supported. We now compute

\[
(e \ast e)(\gamma, x) = \sum_{g \in \Gamma} h(x)h(g^{-1}x)e^{-i\varphi_\sigma(g^{-1})}h(g^{-1}x)h(\gamma^{-1}x)e^{-i\varphi_\sigma(\gamma^{-1})}\sigma(g, g^{-1}\gamma)
\]

\[
= h(x)h(\gamma^{-1}x)\sum_{g \in \Gamma} h(g^{-1}x)^2 e^{-i(\varphi_\sigma(g^{-1}) + \varphi_\sigma(\gamma^{-1}))}\sigma(g, g^{-1}\gamma)
\]

\[
= h(x)h(\gamma^{-1}x)\sum_{g \in \Gamma} h(g^{-1}x)^2 e^{i\varphi_\sigma(\gamma^{-1})}
\]

\[
= h(x)h(\gamma^{-1}x)e^{i\varphi_\sigma(\gamma^{-1})} = e(\gamma, x),
\]

where the last equality follows from the relations described under Lemma 1.1. Since the set of all \(h\) as in the lemma is convex, one sees that the class \([e] \in K_0(\mathcal{A}(\Gamma, C_0(X), \sigma))\) is independent of the choice of \(h\). \(\square\)

We denote by

\[p : KK^{ban}(\mathcal{A}(\Gamma, C_0(X), \sigma), \mathcal{A}(\Gamma, \sigma)) \rightarrow K_0(\mathcal{A}(\Gamma, \sigma)),\]

the map determined by the idempotent \(e\), i.e. \(p(\xi) = [e] \otimes A(\Gamma, C_0(X), \sigma)\xi \in K_0(\mathcal{A}(\Gamma, \sigma))\) for all \(\xi \in KK^{ban}(\mathcal{A}(\Gamma, C_0(X), \sigma), \mathcal{A}(\Gamma, \sigma))\), cf. Lemma 1.6, as done by Lafforgue in [La] page 42 in the untwisted case.

The twisted assembly map,

\[(4.7)\]

\[t - \mu^A_\sigma : K^\Gamma_0(ET \Gamma) \simeq K K^\Gamma_0(C_0(ET \Gamma), \mathbb{C}) \rightarrow K_0(\mathcal{A}(\Gamma, \sigma))\]

is then defined as the inductive limit over cocompact \(\Gamma\)-CW-complexes \(X\) of the following maps:

\[t - \mu^A_{X^\Gamma} : K^\Gamma_0(X) \simeq K K^\Gamma_0(C_0(X), \mathbb{C}) \rightarrow K_0(\mathcal{A}(\Gamma, \sigma))\]

where each map \(t - \mu^A_{X^\Gamma}\) is given as the composition \(p \circ j_{\Gamma^\Gamma} \circ \iota\), that is,

\[KK^\Gamma_0(C_0(X), \mathbb{C}) \overset{\mu^A}{\rightarrow} KK^{ban}(\mathcal{A}(\Gamma, C_0(X), \sigma), \mathcal{A}(\Gamma, \sigma)) \overset{p}{\rightarrow} K_0(\mathcal{A}(\Gamma, \sigma)).\]

**4.3. On the equivalence of the analytic twisted Baum-Connes and twisted assembly maps.** We sketch the equivalence of the twisted assembly maps given by equations (1.7) and (3.10).

As in section 3.3, let \((M, E, \phi)\) denote a \(\Gamma\) equivariant \(K\)-cycle. Then the analytic twisted Baum-Connes map is defined as in 3.14 in terms of the analytic index,

\[(4.8)\]

\[a - \mu^A_\sigma : K^\Gamma_0(ET \Gamma) \rightarrow K_0(\mathcal{A}(\Gamma, \sigma))\]

\[(4.9)\]

\[a - \mu^A_\sigma([M, E, \phi]) = a - \text{index}^A(D^+),\]

where \(D\) is the twisted Spin\(^C\) Dirac operator defined as in section 3.1.

On the other hand, section 3.2 defines a twisted assembly map in terms of the class of an idempotent, \([e] \in K_0(\mathcal{A}(\Gamma, C_0(X), \sigma))\), as

\[(4.10)\]

\[t - \mu^A_\sigma : K^\Gamma_0(ET \Gamma) \rightarrow K_0(\mathcal{A}(\Gamma, \sigma))\]
\[ t - \mu^A_\sigma([M, E, \phi]) = t - \text{index}^A_\sigma(D^+), \]
where \( t - \text{index}^A_\sigma(D^+) = [e] \otimes_{A(\Gamma, c_0(X), \sigma)} \text{index} A_\sigma(D^+), \)

A direct application of the scheme of section 4, [Ga2], establishes the following index theorem,

\[ a - \text{index}^A_\sigma(D^+) = t - \text{index}^A_\sigma(D^+) \in K_0(A(\Gamma, \sigma)). \]

Therefore the analytic twisted Baum-Connes map \( a - \mu^A_\sigma \) and the twisted assembly map \( t - \mu^A_\sigma \) are equal, so we will henceforth denote either of these by \( \mu^A_\sigma \).

4.4. Unconditional analog of the twisted Baum-Connes conjecture.

The following conjecture is natural in view of the above computations combined with Lafforgue’s work, and it amounts to a twisted Bost conjecture in case where we choose the unconditional completion to be \( \ell^1 \).

**Conjecture 1.** Let \( \Gamma \) be a countable group and \( \sigma \) a multiplier on \( \Gamma \) with trivial Dixmier-Douady invariant. Then for any unconditional completion \( A(\Gamma, \sigma) \) of \( C(\Gamma, \sigma) \), the twisted assembly map

\[ \mu^A_\sigma : K^*_j(E\Gamma) \to K_j(A(\Gamma, \sigma)), \quad j = 0, 1, \]

is an isomorphism.

This conjecture is strongly related to a twisted version of the Baum-Connes conjecture (see [BaCo]).

**Conjecture 2.** Let \( \Gamma \) be a countable group and \( \sigma \) a multiplier on \( \Gamma \) with trivial Dixmier-Douady invariant. Then the twisted assembly map

\[ \mu_\sigma : K^*_j(E\Gamma) \to K_j(C^*_r(\Gamma, \sigma)), \quad j = 0, 1, \]

is an isomorphism.

To prove Conjecture 2 in some cases, we first prove Conjecture 1 and deduce Conjecture 2 from Conjecture 1 when the groups in addition have property RD, using Proposition 6.11. To prove Conjecture 1 in case where the group \( \Gamma \) is in Lafforgue’s class \( C' \) we need to first recall some facts and definitions. Let \( A \) be a proper \( \Gamma \)-C*-algebra. Then a Dirac element \( \alpha \in KK^\Gamma_0(A, C) \) and a dual Dirac element \( \beta \in KK^\Gamma_0(C, A) \) satisfy the following conditions,

\[ \alpha \otimes_C \beta = 1 \in KK^\Gamma_0(A, A) \]

\[ \beta \otimes_A \alpha = \gamma \in KK^\Gamma_0(C, C), \]

where \( \gamma \) is the idempotent as defined by Kasparov in [Ka], Lafforgue in [La] or Valette in [Va]. The Dirac element \( \alpha \) gets its name as it is constructed using a Spin\(^C\) Dirac operator.

**Definition 4.3.** We say that a group \( \Gamma \) has the **Banach Dirac-Dual Dirac property** if the element \( \gamma \in KK^\Gamma_1(C, C) \) is trivial in \( KK^\Gamma_{ban}(C, C) \).

Recall that Lafforgue’s class \( C' \) defined in [La] contains all countable discrete groups acting properly and by isometries either on a Hilbert space (those are said to have the Haagerup property, see [a1], which include amenable groups, free groups and the property is closed under free and direct products), on a strongly bolic space (e.g. CAT(0) groups, hyperbolic groups due to Mineyev and Yu [MiYu], or on some non-positively curved Riemannian manifolds (as linear groups).
Theorem 4.4 (Lafforgue \[L]\)). Any group $\Gamma$ in the class $C'$ has the Banach Dirac-Dual Dirac property.

Theorem 4.5. Suppose that $\Gamma$ is a discrete group that has the Banach Dirac-Dual Dirac property, and that $\sigma$ is a multiplier on $\Gamma$ with trivial Dixmier-Douady invariant. Then Conjecture \[\Box\] is true.

If in addition, $\Gamma$ has property RD, then Conjecture \[\Box\] is true.

Sketch. It follows from Lafforgue’s work that we can find a proper $\Gamma$-C*-algebra $A$ and a Dirac element $\alpha \in KK^1_*(A, C)$ and a dual Dirac element $\beta \in KK_{\text{ban}}^1(C, A)$ such that

$$\gamma = \beta \otimes_A \alpha = 1 \text{ in } KK_{\text{ban}}^1(C, C).$$

Then consider the following commutative diagram:

Then consider the following commutative diagram:

$$
\begin{array}{ccc}
K^1_*(E \Gamma) & \rightarrow & KK^1_*(E \Gamma, A) \\
\otimes_{\alpha} & & \otimes_{\alpha} \\
\mu^\Delta & & \mu^\Delta \\
K_*(A(\Gamma, \sigma)) & \rightarrow & K_*(A(\Gamma, A, \sigma)) \\
\otimes_{\alpha} & & \otimes_{\alpha} \\
\mu^\Delta & & \mu^\Delta \\
K_*(A(\Gamma, \sigma)) & \rightarrow & K_*(A(\Gamma, \sigma)),
\end{array}
$$

with the fact that, using Proposition 4.1, composites on the top and the bottom lines are identity. \[\Box\]

5. On the range of the trace, conjectures and applications

5.1. Characteristic classes. We recall some basic facts about some well-known characteristic classes that will be used in this paper, cf. \[Hir\].

Let $E \rightarrow M$ be a Hermitian vector bundle over the compact manifold $M$ that has dimension $n = 2m$. The Chern classes of $E$, $c_j(E)$, are by definition integral cohomology classes. The Chern character of $E$, $\text{Ch}(E)$, is a rational cohomology class

$$\text{Ch}(E) = \sum_{r=0}^{m} \text{Ch}_r(E),$$

where $\text{Ch}_r(E)$ denotes the component of $\text{Ch}(E)$ of degree $2r$. Then $\text{Ch}_0(E) = \text{rank}(E)$, $\text{Ch}_1(E) = c_1(E)$ and in general

$$\text{Ch}_r(E) = \frac{1}{r!} P_r(E) \in H^{2r}(M, \mathbb{Q}),$$

where $P_r(E) \in H^{2r}(M, \mathbb{Z})$ is a polynomial in the Chern classes of degree less than or equal to $r$ with integral coefficients, that is determined inductively by the Newton formula

$$P_r(E) - c_1(E) P_{r-1}(E) + \ldots + (-1)^{r-1} c_{r-1}(E) P_1(E) + (-1)^r r c_r(E) = 0,$$

and by $P_0(E) = \text{rank}(E)$. The next two terms are $P_1(E) = c_1(E)$, $P_2(E) = c_1(E)^2 - 2c_2(E)$.

The Todd-genus characteristic class of the Hermitian vector bundle $E$ is a rational cohomology class in $H^{2\bullet}(M, \mathbb{Q})$,

$$\text{Todd}(E) = \sum_{r=0}^{m} \text{Todd}_r(E),$$
where \( \text{Todd}_r(E) \) denotes the component of \( \text{Todd}(E) \) of degree \( 2r \). Then \( \text{Todd}_r(E) = B_r Q_r(E) \), where \( Q_r(E) \) is a polynomial in the Chern classes of degree less than or equal to \( r \), with integral coefficients, and \( B_r \neq 0 \), \( B_r \in \mathbb{Q} \) are the Bernoulli numbers. In particular, \( \text{Todd}_0(E) = B_0 Q_0 = 1 \).

For the rest of this section, we use the notation of Section 5.1.

5.2. An \( L^2 \) index theorem. Let \( \tau \) be the canonical trace on \( \mathcal{A}(\Gamma, \sigma) \) defined by evaluation at the identity element of \( \Gamma \). It induces a linear map \([\tau] : K_0(\mathcal{A}(\Gamma, \sigma)) \to \mathbb{R}\), which is called the trace map in \( K \)-theory. Explicitly, first \( \tau \) extends to matrices with entries in \( \mathcal{A}(\Gamma, \sigma) \) as (with \( \text{Trace} \) denoting matrix trace):

\[
\tau(f \otimes r) = \text{Trace}(r) \tau(f).
\]

Then the extension of \( \tau \) to \( K_0 \) is given by \([\tau](e - f) = \tau(e) - \tau(f)\), where \( e \) and \( f \) are idempotent matrices with entries in \( \mathcal{A}(\Gamma, \sigma) \).

We will compute \([\tau] \circ \mu_\sigma^A(K_0^\Gamma(E_\Gamma))\) as follows.

\[
[\tau] \circ \mu_\sigma^A([M, E, \phi]) = [\tau][e_t(D)] - [E_0] = \tau(e^{-tD^-}D^+) - \tau(e^{-tD^+}D^-) = c_0 \int_{M/\Gamma} \text{Todd}(M) \wedge e^\omega \wedge \text{Ch}(E),
\]

where \( D^+ = \partial^+_{E \otimes L} \) denotes the twisted \( \Gamma \)-equivariant Dirac operator and \( D^- \) denotes its adjoint. Here the local index theorem is used to deduce the last line, cf. the Appendix in [Ma2]. Here \( c_0 = 1/(2\pi)^{n/2} \) is the universal constant determined by the Atiyah-Singer index theorem, see [AtSi], \( n = \dim M \), Todd and Ch denote the Todd-genus and the Chern character respectively, \( \omega \) is the curvature of the connection on the trivial line bundle \( L \) that is described in Section 3.1. This theorem is also a consequence of section 4.3.

5.3. Range of the canonical trace. Here we will present some consequences of the twisted Bost conjecture above and the twisted \( L^2 \) index theorem described in subsection 5.2 above.

The following result is an easy modification of a result in [Ma].

**Theorem 5.1 (Range of the trace theorem).** Suppose that \( (\Gamma, \sigma) \) satisfies Conjecture 4.1. Then the range of the canonical trace on \( K_0(\mathcal{A}(\Gamma, \sigma)) \) is given by

\[
\left\{ c_0 \int_{M/\Gamma} \text{Todd}(M) \wedge e^\omega \wedge \text{Ch}(E) ; \text{ for all } (M, E, \phi) \in \Pi^\Gamma_{\text{even}}(E_\Gamma) \right\}.
\]

**Remarks 5.2.** The set

\[
\left\{ c_0 \int_{M/\Gamma} \text{Todd}(M) \wedge e^\omega \wedge \text{Ch}(E) ; \text{ for all } (M, E, \phi) \in \Pi^\Gamma_{\text{even}}(E_\Gamma) \right\},
\]

is a countable discrete subgroup of \( \mathbb{R} \), but it is not in general a subgroup of \( \mathbb{Z} \).
REMARKS 5.3. When \( \Gamma \) is the fundamental group of a compact Riemann surface of positive genus, it follows from \( R \) in the genus one case, and \( CHMM \) in the general case, that the set
\[
\left\{ c_0 \int_{M/\Gamma} \text{Todd}(M) \wedge e^\omega \wedge \text{Ch}(E); \text{ for all } (M, E, \phi) \in \Pi^\Gamma_{\text{even}}(E \Gamma) \right\},
\]
reduces to the countable discrete group \( \mathbb{Z} + \theta \mathbb{Z} \), where \( \theta \in [0, 1) \) corresponds to the multiplier \( \sigma \) under the isomorphism \( H^2(\Gamma; \mathbb{U}(1)) \cong \mathbb{R}/\mathbb{Z} \).

PROOF OF THEOREM 5.1. By hypothesis, the twisted assembly map \( \mu^\Lambda_\sigma \) is an isomorphism. Therefore to compute the range of the trace map on \( K_0(A(\Gamma, \sigma)) \), it suffices to compute the range of the trace map on elements of the form
\[
\mu^\Lambda_\sigma([M, E, \phi]), \quad [M, E, \phi] \in K_0^\Gamma(E \Gamma).
\]
Here \( (M, E, \phi) \in \Pi^\Gamma_{\text{even}}(E \Gamma) \). By the \( L^2 \) index theorem described in section 5.2 above, one has
\[
[\tau](\mu^\Lambda_\sigma([M, E, \phi])) = c_0 \int_{M/\Gamma} \text{Todd}(M) \wedge e^\omega \wedge \text{Ch}(E),
\]
as desired. \( \square \)

Therefore we deduce the following.

COROLLARY 5.4. Suppose that \( (\Gamma, \sigma) \) satisfies Conjecture \( 4 \), then the range of the trace map on \( K_0(C^*_r(\Gamma, \sigma)) \) is
\[
\left\{ c_0 \int_{M/\Gamma} \text{Todd}(M) \wedge e^\omega \wedge \text{Ch}(E); \text{ for all } (M, E, \phi) \in \Pi^\Gamma_{\text{even}}(E \Gamma) \right\}.
\]

5.4. The 3 and 4 dimensional cases. We explicitly determine the range of the trace in the special case when \( \Gamma \) is torsion-free and \( B \Gamma \) is either a three or a four dimensional smooth compact manifold. In the three dimensional case we get the following.

THEOREM 5.5. Let \( \Gamma \) be a torsion-free group such that \( (\Gamma, \sigma) \) satisfies Conjecture \( 4 \) and such that \( B \Gamma \) is a smooth, compact oriented three dimensional manifold. Then the range of the trace map is
\[
(5.1) \quad [\text{tr}](K_0(A(\Gamma, \sigma))) = \mathbb{Z} + \sum_{i=1}^{b_1} \mathbb{Z} \theta_i,
\]
where for \( i = 1, \ldots, b_1 \), \( \theta_i = c_0 \langle \eta_i \cup \omega, [B \Gamma] \rangle \) and the \( \eta_i \)'s are generators for \( H^1(B \Gamma, \mathbb{Z}) \cap H^1(B \Gamma, \mathbb{R}) \), \( b_1 = \dim H^1(B \Gamma, \mathbb{R}) \) and \( \sigma = e^\omega \).

PROOF. Since any smooth, compact oriented three dimensional manifold is a spin manifold, it satisfies Poincaré duality,
\[
K_0(B \Gamma) \cong K^1(B \Gamma).
\]
The range of the trace \( [\text{tr}](K_0(A(\Gamma, \sigma))) \) simplifies to
\[
(5.2) \quad \left\{ c_0 \int_{B \Gamma} \text{Todd}(B \Gamma) \wedge e^\omega \wedge \text{Ch}^{\text{odd}}(E); \text{ for all } E \in K^1(B \Gamma) \right\}.
\]
For dimension reasons, $\text{Todd}(BG) = 1$, $e^\omega = 1 + \omega$ and $\text{Ch}^{\text{odd}}(E) = c_1^{\text{odd}}(E) + \text{Ch}^3_3^{\text{odd}}(E)$. Therefore equation (5.2) reduces to

$$\left\{ c_0 \int_{BG} c_1^{\text{odd}}(E) \wedge \omega + c_0 \int_{BG} \text{Ch}^3_3^{\text{odd}}(E); \text{ for all } E \in K^1(BG) \right\}.$$  

By the Atiyah-Singer index theorem [AtSi], one knows that

$$c_0 \int_{BG} \text{Ch}^3_3^{\text{odd}}(E) \in \mathbb{Z} \quad \text{for all } E \in K^1(BG).$$

The proof is concluded from the fact that $c_1^{\text{odd}}(E) = c_1^{\text{odd}}(\det E) \in H^1(BG, \mathbb{Z}) \cap H^1(BG, \mathbb{R})$. \hfill \Box

We now turn to the four dimensional case. Let $Q(a, b) = \langle a \cup b, [BG] \rangle$, for $a, b \in H^2(\Gamma, \mathbb{R})$, be the intersection form on $BG$. Define the linear functional $T_\omega : H^2(\Gamma, \mathbb{Z}) \to \mathbb{R}$ as $T_\omega(a) = Q(\omega, a)$. Then the following is a consequence of Theorem 5.1 and the proof of Theorem 2.5 in [MaMa].

**Theorem 5.6.** Let $\Gamma$ be a torsion-free group such that $(\Gamma, \sigma)$ satisfies Conjecture 1 and such that $BG$ is a smooth, compact oriented four dimensional manifold. Then the range of the trace map is

$$\text{tr}(K_0(A(\Gamma, \sigma))) = \mathbb{Z}\theta + \mathbb{Z} + B,$$

where $2(2\pi)^2\theta = \langle [\omega \cup \omega], [BG] \rangle$, and $B = \text{range}(T_\omega)$.

**Remarks 5.7.** Here $\omega$ is as in subsection 5.2. If $a_1, \ldots, a_r$ are generators of $H^2(BG, \mathbb{Z}) \cap H^2(BG, \mathbb{R})$, where $r = \dim H^2(BG, \mathbb{R})$, then we can express equation (5.3) as,

$$\text{tr}(K_0(A(\Gamma, \sigma))) = \mathbb{Z}\theta + \mathbb{Z} + \sum_{j=1}^r \mathbb{Z}\theta_j,$$

where $\theta_j = \langle [\omega \cup a_j], [BG] \rangle$ for $j = 1, \ldots, r$.

**5.5. The trace conjecture for unconditional twisted group completions.** The calculations done earlier in the section validate the following bold conjecture.

**Conjecture 3.** Let $\Gamma$ be a torsion-free group such that $(\Gamma, \sigma)$ satisfies Conjecture 1 and such that the classifying space $BG$ is a smooth, compact, oriented manifold.

(1) (Even dimensional case) Suppose that $BG$ is of dimension $2n$. If $a_1(j), \ldots, a_{b_2}(j)$ are generators of $H^{2j}(BG, \mathbb{Z}) \cap H^{2j}(BG, \mathbb{R})$, where $b_{2j} = \dim H^{2j}(BG, \mathbb{R})$, then the range of the trace map is

$$\text{tr}(K_0(A(\Gamma, \sigma))) = \mathbb{Z} + \mathbb{Z}\theta + \sum_{j=1}^{n-1} \sum_{k=1}^{b_{2j}} \mathbb{Z} r_{k, j, n} \theta_k(j),$$

where $\theta_k(j) = \langle [\omega^{n-j} \cup a_{k}(j)], [BG] \rangle$ for $k = 1, \ldots, b_{2j}$, $2(2\pi)^n\theta = \langle [\omega^n], [BG] \rangle$ and $r_{k, j, n}$ are universal constants.
(2) (Odd dimensional case) Suppose that $BG$ is of dimension $2n-1$. If $a_{1}(j), \ldots, a_{b_{2j-1}}(j)$ are generators of $H^{2j-1}(BG, \mathbb{Z}) \cap H^{2j-1}(BG, \mathbb{R})$, where $b_{2j-1} = \dim H^{2j-1}(BG, \mathbb{R})$, then the range of the trace map is

$$
\text{tr}(K_{0}(A(\Gamma, \sigma))) = \mathbb{Z} + \sum_{j=1}^{n-1} \sum_{k=1}^{b_{2j-1}} \mathbb{Z}r'_{k,j,n} \theta_{k}(j),
$$

where $\theta_{k}(j) = ([\omega^{n-j} \cup a_{k}(j)], [BG])$ for $k = 1, \ldots, b_{2j-1}$ and $r'_{k,j,n}$ are universal constants.

Remark 5.8. In section we have a more explicit form of Conjecture whenever the dimension of $BG$ is less than or equal to 4.

6. Generators for $K_{1}$ of twisted group algebra completions of surface groups

In this section we shall focus on the degree one part of conjecture and more precisely we shall identify the generators for $K_{1}$ of a fundamental group of a Riemann surface. We assume that all the groups are torsion-free. We shall more generally obtain partial results in low-dimensional homology ($BG$ of dimension less than or equal to 2). In this case, there exist (as shown by Natsume in in the untwisted case) natural homomorphisms

$$
\beta_{1} : H_{1}(\Gamma, \mathbb{Z}) = \Gamma_{ab} \to K_{1}(BG),
$$

$$
\beta_{ab} : H_{1}(\Gamma, \mathbb{Z}) = \Gamma_{ab} \to K_{1}(A(\Gamma, \sigma)),
$$

such that $\beta_{ab} = \mu \beta_{1}$. Here $A(\Gamma, \sigma)$ is any good unconditional completion of the twisted group algebra $C(\Gamma, \sigma)$. Defining $\beta_{1} : \Gamma_{ab} \to K_{1}(BG)$ has been done by Valette in and we recall here the construction. Since $\pi_{1}(BG) = \Gamma$, an element $\gamma \in \Gamma$ can be viewed as a pointed continuous map $\gamma : S^{1} \to BG$, inducing a map in $K$-homology,

$$
\gamma_{*} : K_{1}(S^{1}) \to K_{1}(BG).
$$

The generator of $K_{1}(S^{1}) \cong \mathbb{Z}$ can be described by the class of the cycle $(\pi, D)$ where $\pi$ is the representation of $C(S^{1})$ on $L^{2}(S^{1})$ by pointwise multiplication and

$$
D = -i \frac{t}{dt}.
$$

An element $\gamma \in \Gamma$ gets then mapped to the class of the cycle $\gamma_{*}(\pi, D) = (\gamma_{*}\pi, D)$, where for $X$ a compact subset of $BG$ containing $\gamma(S^{1})$ and $f \in C(X)$, $\gamma_{*}f = \pi(f \circ \gamma)$ is the pointwise multiplication by $f \circ \gamma$ on $L^{2}(S^{1})$. In other terms one defines

$$
\tilde{\beta}_{1} : \Gamma \to K_{1}(BG),
$$

$$
\gamma \mapsto [(\gamma_{*}\pi, D)].
$$

According to Valette in , the map $\tilde{\beta}_{1} : \Gamma \to K_{1}(BG)$ is a group homomorphism and hence factors through

$$
\beta_{1} : \Gamma_{ab} \to K_{1}(BG).
$$

To define $\beta_{ab} : \Gamma_{ab} \to K_{1}(A(\Gamma, \sigma))$, simply map a representative $[\gamma]$ of $\Gamma_{ab}$ to the class $[T_{[\gamma]}]$ of the invertible operator $T_{[\gamma]}$ in $A(\Gamma, \sigma)$. The following is then an adaptation of a result due to Natsume and the proof we give here is an easy
Consider then the diagram

\[ \begin{array}{ccc}
\mathbb{Z} & \xrightarrow{\beta_a} & K_1(A(\mathbb{Z}, \sigma)) \\
\beta_t & \downarrow{\gamma_a} & \xrightarrow{\beta_t} K_1(A(\Gamma, \sigma)) \\
K_1(S^1) & \xrightarrow{\gamma_*} & K_1(B\Gamma)
\end{array} \]

where by abuse of notation \( \sigma \) denotes also the multiplier \( \sigma \) restricted to \( \mathbb{Z} \). That \( \beta_t \circ \gamma = \gamma_a \circ \beta_a \) is a simple computation, \( \beta_t \circ \gamma = \gamma_a \circ \beta_t \) by definition of \( \beta_t \), and \( \gamma_* \circ \mu_* = \mu_a \circ \gamma_* \) by naturality of the twisted assembly map (see [Ma]). That \( \beta_a = \mu^A_a \circ \beta_t \) follows from the proof of the isomorphism [2] for \( \mathbb{Z} \) (see [Ma]) and we conclude the proof by a diagram chase.

**Corollary 6.2.** Let \( \Gamma_g \) be the fundamental group of a compact Riemann surface of genus \( g \geq 1 \). Then the map \( \beta_a \) is an isomorphism.

**Proof.** It is well known that \( \beta_t \) is an isomorphism in this case. Also \( \Gamma_g \) is in class \( C' \) and has property RD, so that \( \mu^A_a \) is an isomorphism. The result now follows from Proposition [6.1]. \( \square \)

**Remark 6.3.** This corollary shows that we have obtained in particular the explicit generators for \( K_1(C^*_r(\Gamma_g, \sigma)) \), since \( \Gamma_g \) has property RD. More explicitly, consider the standard presentation of \( \Gamma_g \) in terms of generators and relations, namely,

\[ \Gamma_g = \left\{ a_j, b_j : \prod_{j=1}^g [a_j, b_j] = 1 \right\}. \]

Then by Corollary [6.2] the unitary operators \( \{T_{a_j}, T_{b_j} \in U(\ell^2(\Gamma)) : j = 1, \ldots, g\} \) form a natural set of generators for \( K_1(C^*_r(\Gamma_g, \sigma)) \) over \( \mathbb{Z} \). The corollary at the same time gives explicit generators for \( K_1(A(\Gamma_g, \sigma)) \) for any good unconditional completion.

### 6.1. K-homology

Thanks to [CHMM], \( K_1(C^*_r(\Gamma, \sigma)) \cong \mathbb{Z}^{2g} \) and \( K_2(C^*_r(\Gamma, \sigma)) \cong \mathbb{Z}^2 \) whenever \( \Gamma = \Gamma_g \) as above. Moreover, we have determined a natural set of generators for \( K_1(C^*_r(\Gamma_g, \sigma)) \). It was also shown in [CHMM] that \( C^*_r(\Gamma, \sigma) \) is a \( K \)-amenable \( C^* \)-algebra. These two facts together with the universal coefficient theorem [RS] enable us to also compute the \( K \)-homology groups as

\[ K^1(C^*_r(\Gamma, \sigma)) \cong \mathbb{Z}^{2g}, \]
and also
\[ K^0(C^*_r(\Gamma, \sigma)) \cong \mathbb{Z}^2, \]
where the $K$-homology groups $K^i(C^*_r(\Gamma, \sigma))$ are defined as usual as the Kasparov groups $KK^i(C^*_r(\Gamma, \sigma), \mathbb{C})$ for $i = 0, 1$.

**Appendix: Twisted Rapid Decay**

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Throughout this appendix, $\Gamma$ is a finitely generated group, endowed with a length function $\ell$, and $\sigma$ is a multiplier on $\Gamma$. We adopt the notations used in the first paragraph of the paper.

**Definition 6.4.** We will say that the group $\Gamma$ has $\sigma$-twisted Rapid Decay property (with respect to the length $\ell$) if
\[ H^\infty_\ell(\Gamma, \sigma) \subseteq C^*_r(\Gamma, \sigma). \]
We just say that the group $\Gamma$ has the Rapid Decay property (with respect to the length $\ell$), if it has the $\sigma$-twisted Rapid Decay property (with respect to the length $\ell$) for the constant multiplier $1$. For short, we shall say that a group $\Gamma$ has property $\sigma$-RD if there exists a length function $\ell$ with respect to which $\Gamma$ has the $\sigma$-twisted Rapid Decay property.

**Remark 6.5.** In the context of noncommutative geometry, the reduced $C^*$-algebra $C^*_r(\Gamma, \sigma)$ represents the space of continuous functions on a noncommutative manifold, and $H^\infty_\ell(\Gamma, \sigma)$ the space of smooth functions on the same noncommutative manifold. This comes from the abelian case, where using Fourier transforms, one easily sees that $C^*_r(\mathbb{Z}^n) \cong C(\mathbb{T}^n)$ and that $H^\infty_\ell(\mathbb{Z}^n) \cong C^\infty(\mathbb{T}^n)$ (for the word length associated to the generating set $S = \{ \pm 1, 0, \ldots \}$ of $\mathbb{Z}^n$). The ($\sigma$-twisted) Rapid Decay property can be rephrased as the desirable property that every smooth function on the noncommutative manifold is also a continuous function.

**Proposition 6.6.** Let $\sigma$ be a multiplier on $\Gamma$ and $\ell$ be a length function on $\Gamma$. The following are equivalent:

1. $\Gamma$ has $\sigma$-twisted Rapid Decay (with respect to the length $\ell$).
2. There exist constants $C, s > 0$ such that for any $f \in C(\Gamma, \sigma)$
   \[ \| f \|_{op} \leq C \| f \|_s. \]
3. There exists a polynomial $P$ such that for any $f \in C(\Gamma, \sigma)$ and $f$ supported in a ball of radius $r$
   \[ \| f \|_{op} \leq P(r) \| f \|_{\ell^2 \Gamma}. \]
4. There exists a polynomial $P$ such that for any $f, g \in C(\Gamma, \sigma)$ and $f$ supported in a ball of radius $r$
   \[ \| f *_{\sigma} g \|_{\ell^2 \Gamma} \leq P(r) \| f \|_{\ell^2 \Gamma} \| g \|_{\ell^2 \Gamma}. \]
PROOF. (1) $\Leftrightarrow$ (2) As in the case of untwisted Rapid Decay, the inclusion $H_{L}^\infty(\Gamma, \sigma) \subseteq C^r(\Gamma, \sigma)$ is continuous since both inclusions $H_{L}^\infty(\Gamma, \sigma) \subseteq \ell^2\Gamma$ and $C^r(\Gamma, \sigma) \subseteq \ell^2\Gamma$ are continuous. Since $H_{L}^\infty(\Gamma, \sigma)$ is a Fréchet space, the continuity of the inclusion $H_{L}^\infty(\Gamma, \sigma) \subseteq C^r(\Gamma, \sigma)$ rephrases as the statement of (2). The converse is obvious since $H_{L}^{s+1}(\Gamma) \subseteq H_{L}^s(\Gamma)$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) Take $f \in C(\Gamma, \sigma)$ supported in a ball of radius $r$, then

$$
\|f\|_{op} \leq C\|f\|_{s} = C\sqrt{\sum_{\gamma \in \Gamma} |f(\gamma)|^2(1 + \ell(\gamma))^{2s}} \leq C(1 + r)^s\|f\|_{\ell\Gamma}.
$$

Hence (3) follows. Since $\|f\|_{op} = \sup\{\|f \ast g\|_{\ell\Gamma} : 0 \neq g \in \ell^2\Gamma\}$ we deduce (4) as well. That (4) implies (3) is by definition of the operator norm.

(3) $\Rightarrow$ (2) For $n \in \mathbb{N}$, denote by $S_n = \{\gamma \in \Gamma | n \leq \ell(\gamma) < n + 1\}$ the sphere of radius $n$. For $f \in C(\Gamma, \sigma)$ we have:

$$
\|f\|_{op} = \| \sum_{n=0}^{\infty} \lambda_\sigma(f|S_n)\|_{op} \leq \sum_{n=0}^{\infty} \|f|S_n\|_{op},
$$

so that using (3) we get the following bound

$$
\|f\|_{op} \leq \sum_{n=0}^{\infty} P(n+1)\|f|S_n\|_{\ell\Gamma} \leq \sum_{n=0}^{\infty} C(n+1)^k \|f|S_n\|_{\ell\Gamma}
$$

$$
\leq C\sum_{n=0}^{\infty} (n+1)^{-2} \sqrt{\sum_{n=0}^{\infty} (n+1)^{2k+2}\|f|S_n\|^2_{\ell\Gamma}} \leq C'\|f\|_{k+1}
$$

where $C'$ is some constant bigger than $C\pi/6$.

The following proposition was known by Ji and Schweitzer [JiSc], but the proof we give here might be shorter.

**Lemma 6.7.** Let $\ell$ be a length function on $\Gamma$. If $\Gamma$ has Rapid Decay (with respect to the length $\ell$), then $\Gamma$ has $\sigma$-twisted Rapid Decay (with respect to the length $\ell$) for any multiplier $\sigma$.

**Proof.** Take $\gamma \in \Gamma$, then:

$$
|f \ast_\sigma g(\gamma)| = |\sum_{\mu \in \Gamma} f(\gamma^{-1}\mu)g(\mu)\sigma(\gamma^{-1}\mu, \mu)| \leq \sum_{\mu \in \Gamma} |f(\gamma^{-1}\mu)||g(\mu)| = |f|\ast |g|(\gamma)
$$

so that summing and squaring over $\gamma \in \Gamma$ yields

$$
\|f \ast_\sigma g\|_{\ell\Gamma} \leq \| |f| \ast |g|\|_{\ell\Gamma} \leq P(r)\|f\|_{\ell\Gamma}\|g\|_{\ell\Gamma}
$$

and we conclude that $\Gamma$ has $\sigma$-twisted Rapid Decay using the previous proposition.

The following corollary is the first part of Proposition 2.1 in [La2] with an obvious modification.

**Corollary 6.8** (Noncommutative Sobolev Embedding Theorem). Let $\ell$ be a length function on $\Gamma$. If $\Gamma$ has Rapid Decay (with respect to the length $\ell$), then there is a constant $S$ sufficiently large such that for any multiplier $\sigma$ on $\Gamma$ and any $s \geq S$, $H_{L}^s(\Gamma, \sigma)$ is a Banach algebra such that $H_{L}^s(\Gamma, \sigma) \subseteq C^r(\Gamma, \sigma)$.
Proof. Let $s$ be bigger than the degree of the polynomial of point (3) in Proposition 6.9. We first have to show that there is a constant $K = K(s)$ such that for any $f, g \in C(\Gamma, \sigma)$, $\|f * g\|_s \leq K \|f\|_s \|g\|_s$. But this is true since $\|f * g\|_s \leq \|f\|_s \|g\|_s$ and $\|f\|_s \leq K \|f\|_s \|g\|_s$ by Proposition 2.1 part (a) in [La2] (see also Proposition 8.15 in [Va]), since we have assumed that $\Gamma$ has Rapid Decay (with respect to the length $\ell$). Therefore $H^s_\ell(\Gamma, \sigma)$ is a Banach algebra. By Lemma 6.7, we know that since $\Gamma$ has property RD, $\Gamma$ has property $\sigma$-twisted RD for any multiplier $\sigma$ on $\Gamma$, and hence $H^s_\ell(\Gamma, \sigma) \subseteq C^*_\sigma(\Gamma, \sigma)$ follows from Proposition 6.6 part (2). □

Remark 6.9. In the context of noncommutative geometry, Corollary 6.8 can be viewed as the analog of the Sobolev Embedding Theorem for a compact manifold $M$, a simplified version of which says that any function in the Sobolev space $W^{s, 2}(M)$ for $s > \dim M/2$ is actually continuous. Indeed, using Fourier transforms, one can see that $W^{s, 2}(\mathbb{T}^n) \simeq H^s_\ell(\mathbb{T}^n)$ for the word length associated to the generating set $S = \{ \pm 1, 0, \ldots \}$, and that $C^*_\sigma(\mathbb{T}^n) \simeq C(\mathbb{T}^n)$.

Example 6.10. Groups having Rapid Decay notably include: Polynomial growth groups (Jolissaint [Jo]), free groups (Haagerup [Ha]) and more generally Gromov hyperbolic groups (Jolissaint-de la Harpe [JdH]), cocompact lattices in $SL_2(F)$ where $F$ is the $p$-adic field $\mathbb{Q}_p$, $\mathbb{R}$, $\mathbb{C}$ or $E_{6(-26)}$, as well as finite products of rank one Lie groups (see Rammage-Robertson-Steger [RaRoSt], Lafforgue [La2] and [Ch]) and all lattices in a rank one Lie group, see [ChR].

Question: Is it possible to find a group $\Gamma$ which doesn’t have Rapid Decay, but which has $\sigma$-twisted Rapid Decay for some multiplier $\sigma$ on $\Gamma$ (or does the converse of Lemma 6.7 hold)?

The following is the second part of Proposition 1.2 of [La2] with a trivial change. But we still recall Lafforgue’s proof below for the sake of completeness.

Proposition 6.11. Let $\ell$ be a length function on $\Gamma$. If $\Gamma$ has Rapid Decay (with respect to the length $\ell$), then for any multiplier $\sigma$ on $\Gamma$ and for $s$ sufficiently large (and also for $s = \infty$), the inclusion $H^s_\ell(\Gamma, \sigma) \rightarrow C^*_\sigma(\Gamma, \sigma)$ induces an isomorphism in $K$-theory.

Proof. The idea of the proof is as follows. By Corollary 6.8 there exists $S > 0$ and finite such that for any $s \geq S$, $H^s_\ell(\Gamma, \sigma) \subseteq C^*_\sigma(\Gamma, \sigma)$, and since $\mathbb{C}(\Gamma, \sigma) \subseteq H^s_\ell(\Gamma, \sigma)$, it follows that $H^s_\ell(\Gamma, \sigma)$ is a dense $*$-subalgebra of $C^*_\sigma(\Gamma, \sigma)$. All we have to show is that the inclusion $H^s_\ell(\Gamma, \sigma) \subseteq C^*_\sigma(\Gamma, \sigma)$ is spectral, then it follows (see e.g. Proposition 8.14 of [Va]) that the inclusion $H^s_\ell(\Gamma, \sigma) \rightarrow C^*_\sigma(\Gamma, \sigma)$ induces an isomorphism in $K$-theory.

Now, for two number $s, t$ such that $S < t < s$ the first step is to show that $H^s_\ell(\Gamma, \sigma)$ is stable by holomorphic functional calculus in $H^t_\ell(\Gamma, \sigma)$. To do so, and since $H^s_\ell(\Gamma, \sigma)$ is dense in $H^t_\ell(\Gamma, \sigma)$, it is enough (see Remark 8.13 in [Va]) to prove that the spectral radius $\rho_\ell(f)$ of $f \in H^s_\ell(\Gamma, \sigma)$ is the same as $\rho_\ell(f)$, the one of $f \in H^t_\ell(\Gamma, \sigma)$, namely that

$$\lim_{n \to \infty} \|f^{\ast^n}\|_t^{1/n} = \lim_{n \to \infty} \|f^{\ast^n}\|_t^{1/n},$$

where for $n \in \mathbb{N}$ we set $f^{\ast^n} = f *_{\sigma} f *_{\sigma} \cdots *_{\sigma} f$. Notice that since $t < s$, then

$$\|f\|_t \leq \|f\|_s$$

and hence $\rho_\ell(f) \leq \rho_s(f)$, so we only need to prove the other inequality.
For $\gamma \in \Gamma$, using the triangle inequality one sees that
\[
|f^{s_n}(\gamma)| \leq \sum_{\gamma_1, \ldots, \gamma_{n-1} \in \Gamma} |f(\gamma_1^{-1})||f(\gamma_1\gamma_2^{-1})| \cdots |f(\gamma_{n-2}\gamma_{n-1}^{-1})||f(\gamma_{n-1})|
\]
\[
= \sum_{\gamma_1, \ldots, \gamma_n = \gamma} |f(\gamma_1)| \cdots |f(\gamma_n)|
\]
Therefore, using that $(1 + \ell(\gamma))^{s-t} \leq n^{s-t} \sum_{i=1}^{n} (1 + \ell(\gamma_i))^{s-t}$ if $\gamma_1 \ldots \gamma_n = \gamma$ (which follows easily from Lemma 1.1.4 (3) in [Jo]) we deduce that
\[
\|f^{s_n}\|_s = \|(1 + \ell)^{s-t} f^{s_n}\|_t \leq n^{s-t+1} K^{n-1} \|f\|_s \|f\|^{n-1}_t,
\]
where $K = K(t)$ is the constant in the proof of Corollary 6.3. Taking the $n$-th root and the limit shows that $\lim_{n \to \infty} \|f^{s_n}\|_s^{1/n} \leq K\|f\|_t$. Replacing $f$ by $f^{s_m}$ in the previous inequality, taking the $m$-th root and the limit shows $\rho_s(f) \leq \rho_t(f)$. We can now show that $H^*_f(\Gamma, \sigma) \subseteq C^*_r(\Gamma, \sigma)$ is spectral, namely that for $f \in H^*_f(\Gamma, \sigma)$, its spectral radius $\rho_s(f)$ equals $\rho_t(f)$, its spectral radius as an element of $C^*_r(\Gamma, \sigma)$. If $\rho_s(f) = 0$ it is clear because $\rho_s(f) \leq \rho_t(f)$. Otherwise, Hölder’s inequality shows that
\[
\|f\|_t \leq \|f\|_s^{\frac{t}{s}} \|f\|_s^{\frac{t-s}{s}}
\]
and hence
\[
|f^{s_n}| \geq \|f^{s_n}\|_t \|f^{s_n}\|_s^{\frac{t-s}{s}},
\]
so that we conclude using equality (6.1). \qed

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