EMBEDDING COPRODUCTS OF PARTITION LATTICES

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Dedicated to Béla Csákány for his 75th birthday

Abstract. We prove that the lattice EqΩ of all equivalence relations on an infinite set Ω contains, as a 0,1-sublattice, the 0-coproduct of two copies of itself, thus answering a question by G. M. Bergman. Hence, by using methods initiated by de Bruijn and further developed by Bergman, we obtain that EqΩ also contains, as a sublattice, the coproduct of 2\textsuperscript{card Ω} copies of itself.

1. Introduction

Whitman’s Theorem [10] states that every lattice L can be embedded into the lattice EqΩ of all equivalence relations on some set Ω. The cardinality of Ω may be taken equal to card L + ℵ\textsubscript{0}. There is not much room for improvement of the cardinality bound, as for example, EqΩ cannot be embedded into its dual lattice. (We believe the first printed occurrence of this result to be Proposition 6.2 in G. M. Bergman’s recent preprint [2], although it may have already been known for some time.) Hence the question of embeddability into EqΩ of lattices of large cardinality (typically, card(EqΩ) = 2\textsuperscript{card Ω}) is nontrivial.

In [2], Bergman also extends results of N. G. de Bruijn [3, 4] by proving various embedding results of large powers or copowers of structures such as symmetric groups, endomorphism rings, and monoids of self-maps of an infinite set Ω, into those same structures. The nature of the underlying general argument is categorical. The problem whether the lattice EqΩ contains a coproduct (sometimes called “free product” by universal algebraists) of two, or more, copies of itself, was stated as an open question in a preprint version of that paper. In the present note, we solve this problem in the affirmative.

The idea of our proof is the following. The lattice EqΩ of all equivalence relations on Ω is naturally isomorphic to the ideal lattice IdK of the lattice K of all finitely generated equivalence relations, that is, those equivalence relations containing only finitely many non-diagonal pairs. Denote by K\Pi\textsuperscript{0}K the coproduct (amalgamation) of two copies of K above the common ideal 0. As K\Pi\textsuperscript{0}K has the same cardinality as Ω, it follows from Jónsson’s proof of Whitman’s Embedding Theorem that the lattice Id(K\Pi\textsuperscript{0}K) embeds into EqΩ. Finally, we prove that the ideal lattice functor preserves the coproduct \Pi\textsuperscript{0} and one-one-ness (Theorem 5.2), in such a way that (Id K)\Pi\textsuperscript{0} (Id K) embeds into Id(K\Pi\textsuperscript{0}K). Then it is easy to extend this result to the usual coproduct (Id K) \Pi (Id K).

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We also present an example (Example 5.3) that shows that the result of Theorem 5.2 does not extend to amalgamation above a common (infinite) ideal. That is, for a common ideal $A$ of lattices $B$ and $C$, the canonical homomorphism from $(\text{Id } B) \sqcup \text{Id } A$ (Id $C$) to Id$(B \sqcup A \sqcup C)$ may not be one-to-one.

2. Basic concepts

We refer to [7] for unexplained lattice-theoretical notions. For any subsets $Q$ and $X$ in a poset (i.e., partially ordered set) $P$, we put

$Q \downarrow X = \{p \in Q \mid (\exists x \in X)(p \leq x)\}$ and $Q \uparrow X = \{p \in Q \mid (\exists x \in X)(p \geq x)\}$.

We also write $Q \downarrow x$, resp. $Q \uparrow x$ in case $X = \{x\}$. A subset $Q$ of $P$ is a lower subset of $P$ if $Q = P \downarrow Q$.

A map $f \colon K \to L$ between lattices is meet-complete if for each $a \in K$ and each $X \subseteq K$, $a = \bigwedge X$ in $K$ implies that $f(a) = \bigwedge f(X)$ in $L$. (Observe that we do not require either $K$ or $L$ to be a complete lattice.) When this is required only for nonempty $X$, we say that $f$ is nonempty-meet-complete. Join-completeness and nonempty-join-completeness of maps are defined dually. We say that $f$ is complete (resp., nonempty-complete) if it is both meet-complete and join-complete (resp., both nonempty-meet-complete and nonempty-join-complete). We say that $f$ is lower bounded if $\{x \in K \mid y \leq f(x)\}$ is either empty or has a least element for each $y \in L$. Upper bounded homomorphisms are defined dually. Lower bounded homomorphisms are nonempty-meet-complete and upper bounded homomorphisms are nonempty-join-complete.

An ideal of a lattice $L$ is a nonempty lower subset of $L$ closed under finite joins. We denote by $\text{Id } L$ the lattice of all ideals of $L$. For a lattice homomorphism $f \colon K \to L$, the map $\text{Id } f \colon \text{Id } K \to \text{Id } L$ defined by

$$(\text{Id } f)(X) = L \downarrow f[X], \quad \text{for each } X \in \text{Id } L,$$

is a nonempty-join-complete lattice embedding. If $L$ is a 0-lattice (i.e., a lattice with least element), the canonical map $L \to \text{Id } L$, $x \mapsto L \downarrow x$ is a 0-lattice embedding. The assignment that to every lattice associates its dual lattice $L^{\text{op}}$ (i.e., the lattice with the same underlying set as $L$ but reverse ordering) is a category equivalence—and even a category isomorphism—from the category of all lattices to itself, that sends 0-lattices to 1-lattices. For every lattice $L$, we denote by $L^{\leq}$ the lattice obtained by adding a new zero element to $L$.

A lattice $L$ is upper continuous if for each $a \in L$ and each upward directed subset $\{x_i \mid i \in I\}$ of $L$ admitting a join, the equality $a \wedge \bigvee_{i \in I} x_i = \bigvee_{i \in I} (a \wedge x_i)$ holds. We shall often use upper continuity in the following form: if $I$ is an upward directed poset and both $(x_i \mid i \in I)$ and $(y_i \mid i \in I)$ are isotope families with respective joins $x$ and $y$, then the family $(x_i \wedge y_i \mid i \in I)$ has join $x \wedge y$.

Every algebraic lattice is upper continuous, so, for example, $\text{Id } L \cup \{\emptyset\}$ is upper continuous for any lattice $L$; hence $\text{Id } L$ is also upper continuous. The lattice $\text{Eq } \Omega$ of all equivalence relations on a set $\Omega$, partially ordered by inclusion, is an algebraic lattice, thus it is upper continuous. Other examples of upper continuous lattices that are not necessarily complete are given in [1]. For example, it follows from [1] Corollary 2.2 that every finitely presented lattice is upper continuous.

We denote by $\mathcal{P}(\Omega)$ the powerset of a set $\Omega$, and by $\omega$ the set of all natural numbers.
3. The free lattice on a partial lattice

We recall Dean's description of the free lattice on a partial lattice, see [5] or [6] Section XI.9. A partial lattice is a poset \((P, \leq)\) endowed with partial functions \(\lor\) and \(\land\) from the nonempty finite subsets of \(P\) to \(P\) such that if \(p = \lor X\) (resp., \(p = \land X\)), then \(p\) is the greatest lower bound (resp., least upper bound) of \(X\) in \(P\).

An o-ideal of \(P\) is a lower subset \(A\) of \(P\) such that \(p = \lor X\) and \(X \subseteq A\) implies that \(p \in A\) for each \(p \in P\) and each nonempty finite subset \(X\) of \(P\). The set \(\prod P\) of all o-ideals of \(P\), partially ordered by inclusion, is an algebraic lattice. Observe that \(\prod P = (\text{Id } P) \cup \{ \varnothing \}\) in case \(P\) is a lattice. O-filters are defined dually; again, the lattice \(\prod P\) of all o-filters of \(P\), partially ordered by inclusion, is algebraic. We denote by \(\mathcal{I}(A)\) (resp., \(\mathcal{F}(A)\)) the least o-ideal (resp., o-filter) of \(P\) containing a subset \(A\) of \(P\).

The free lattice \(\mathcal{F}_L(P)\) on \(P\) is generated, as a lattice, by an isomorphic copy of \(P\), that we shall identify with \(P\). (The subscript \(L\) in \(\mathcal{F}_L(P)\) stands for the variety of all lattices, as the “free lattice on \(P\)” construction can be carried out in any variety of lattices.) For each \(x \in \mathcal{F}_L(P)\), the following subsets of \(P\),

\[
\mathcal{I}(x) = P \upharpoonright x = \{ p \in P \mid p \leq x \}
\]

\[
\mathcal{F}(x) = P \downarrow x = \{ p \in P \mid x \leq p \}
\]

are, respectively, an o-ideal and an o-filter of \(P\), which can also be evaluated by the following rules:

\[
\mathcal{I}(x \lor y) = \mathcal{I}(x) \lor \mathcal{I}(y) \text{ in } \prod P, \quad \mathcal{F}(x \lor y) = \mathcal{F}(x) \land \mathcal{F}(y); \quad (3.1)
\]

\[
\mathcal{I}(x \land y) = \mathcal{I}(x) \land \mathcal{I}(y), \quad \mathcal{F}(x \land y) = \mathcal{F}(x) \lor \mathcal{F}(y) \text{ in } \prod P, \quad (3.2)
\]

for all \(x, y \in \mathcal{F}_L(P)\). The natural partial ordering on \(\mathcal{F}_L(P)\) satisfies the following “Whitman-type” condition:

\[
x_0 \land x_1 \leq y_0 \lor y_1 \iff \text{either } (\exists p \in P)(x_0 \land x_1 \leq p \leq y_0 \lor y_1)
\]

or there is \(i < 2\) such that either \(x_i \leq y_0 \lor y_1\) or \(x_0 \land x_1 \leq y_i\), \quad (3.3)

which is also the basis of the inductive definition of that ordering.

4. The 0-coproduct of a family of lattices with zero

Our development of the 0-coproduct of a family of lattices with zero below bears some similarities with the development of coproducts (called there free products) given in [7] Chapter VI. Nevertheless, as we use the known results about the free lattice on a partial lattice (outlined in Section 3), our presentation becomes significantly shorter.

Let \((L_i \mid i \in I)\) be a family of lattices with zero. Modulo the harmless set-theoretical assumption that \(L_i \cap L_j = \{0\}\) for all distinct indices \(i, j \in I\), the coproduct (often called free product by universal algebraists) of \((L_i \mid i \in I)\) can be easily described as \(\mathcal{F}_L(P)\), where \(P\) is the partial lattice whose underlying set is the union \(\bigcup_{i \in I} L_i\), whose underlying partial ordering is the one generated by the partial orders on all the \(L_i\)s, and whose partial lattice structure consists of all existing joins and meets of nonempty finite subsets in each “component” \(L_i\). We denote this lattice by \(L = \coprod_{i \in I} L_i\), the superscript 0 meaning that the coproduct of the \(L_i\)s is evaluated in the category of all 0-lattices and 0-preserving homomorphisms, which we shall often emphasize by saying “0-coproduct” instead of just coproduct. We shall also identify each \(L_i\) with its canonical copy in \(L\). Of course, the coproduct
of any family of lattices \( (L_i \mid i \in I) \) in the variety of all lattices is the sublattice of \( \coprod_{i \in I} (L_i)^{\circ} \) generated by the union of the images of the \( L_i \)s.

Now we shall analyze further the structure of the 0-coproduct \( L \), in a fashion similar to the development in \([7, \text{Chapter VI}]\). We add a new largest element, denoted by \( \infty \), to \( L \), and we set \( L_i = L_i \cup \{ \infty \} \) for each \( i \in I \). The following lemma is an analogue, for 0-coproducts instead of coproducts, of \([7, \text{Theorem VI.1.10}]\).

**Lemma 4.1.** For each \( x \in L \) and each \( i \in I \), there are a largest element of \( L_i \) below \( x \) and a least element of \( L_i \) above \( x \) with respect to the ordering of \( L \cup \{ \infty \} \). Furthermore, if we denote these elements by \( x(i) \) and \( x^{(i)} \), respectively, then the following formulas hold:

\[
p(i) = p^{(i)} = p, \text{ if } p \in L_i;
\]

\[
p(i) = 0 \text{ and } p^{(i)} = \infty, \text{ if } p \in P \setminus L_i;
\]

\[
(x \lor y)(i) = x(i) \lor y(i) \text{ and } (x \land y)(i) = x(i) \land y(i);
\]

\[
(x \lor y)^{(i)} = x^{(i)} \lor y^{(i)};
\]

\[
(x \land y)^{(i)} = \begin{cases} 0, & \text{if } x(j) \land y(j) = 0 \text{ for some } j \in I, \\ x^{(i)} \land y^{(i)}, & \text{otherwise}, \end{cases}
\]

for each \( x, y \in L \) and each \( i \in I \).

**Proof.** For an element \( x \) of \( L \), abbreviate by "\( x(i) \) exists" (resp., "\( x^{(i)} \) exists") the statement that \( L_i \downarrow x \) is a principal ideal in \( L_i \) (resp., \( L_i \uparrow x \) is a principal filter in \( L_i \)), and then denote by \( x(i) \) (resp., \( x^{(i)} \)) the largest element of \( L_i \downarrow x \) (resp., the least element of \( L_i \uparrow x \)). Denote by \( K \) the set of all \( x \in L \) such that both \( x(i) \) and \( x^{(i)} \) exist for each \( i \in I \). It is clear that \( K \) contains \( P \) and that both \( p(i) \) and \( p^{(i)} \) are given by the first two formulas of (4.1), for any \( p \in P \). Furthermore, it follows immediately from the definition of \( K \) that

\[
\exists(z) = \bigcup_{i \in I} (L_i \downarrow z(i)),
\]

(4.2)

\[
\forall(z) = \bigcup_{i \in I} (L_i \uparrow z^{(i)}),
\]

(4.3)

for each \( z \in K \). We shall establish that \( K \) is a sublattice of \( L \). So let \( x, y \in K \), put \( u = x \land y \) and \( v = x \lor y \). It is straightforward that for each \( i \in I \), both \( u(i) \) and \( v^{(i)} \) exist, and

\[
u(i) = x(i) \land y(i), \quad v^{(i)} = x^{(i)} \lor y^{(i)},
\]

(4.4)

Now we shall prove that \( v(i) \) exists and is equal to \( x(i) \lor y(i) \). By the induction hypothesis, (4.2) holds at both \( x \) and \( y \). So, as \( \exists(v) = \exists(x) \lor \exists(y) \), in order to get the asserted existence and description of the elements \( v(i) \), it suffices to prove that

\[
\bigcup_{i \in I} (L_i \downarrow x(i)) \lor \bigcup_{i \in I} (L_i \downarrow y(i)) = \bigcup_{i \in I} (L_i \downarrow (x(i) \lor y(i))).
\]

(4.5)

The containment from left to right is obvious, and each \( x(i) \lor y(i) \) is contained in any \( \omega \)-ideal of \( P \) containing \( \{x(i), y(i)\} \), so it suffices to prove that the right hand side of (4.5) is an \( \omega \)-ideal of \( P \). As the join operation in \( P \) is internal to each \( L_i \), this set is closed under joins. As each \( L_i \) is a lower subset of \( P \), this set is also a lower subset of \( P \). This establishes the desired result for the \( v(i) \)s.
It remains to prove that \( u^{(i)} \) exists and is equal to \( z_i \), where \( z_i = x^{(i)} \land y^{(i)} \) if \( x^{(j)} \land y^{(j)} \neq 0 \) for all \( j \), and \( z_i = 0 \) otherwise. By the induction hypothesis, (4.3) holds at both \( x \) and \( y \). So, as \( \mathcal{F}(u) = \mathcal{F}(x) \lor \mathcal{F}(y) \), in order to get the asserted existence and description of the elements \( u^{(i)} \), it suffices to prove that

\[
\bigcup_{i \in I} (L_i \uparrow x^{(i)}) \lor \bigcup_{i \in I} (L_i \uparrow y^{(i)}) = \bigcup_{i \in I} (L_i \uparrow z_i).
\]

(4.6)

The containment from left to right is obvious. If an \( \alpha \)-filter \( U \) of \( P \) contains \( \{x^{(i)}, y^{(i)}\} \) for all \( i \in I \), then it also contains all elements \( x^{(i)} \land y^{(i)} \); in particular, it is equal to \( P \) in case \( x^{(i)} \land y^{(i)} = 0 \) for some \( i \). In any case, \( z_i \in U \) for all \( i \in I \). So it suffices to prove that the right hand side of (4.6) is an \( \alpha \)-filter of \( P \). This is trivial in case \( z_i = 0 \) for some \( i \), so suppose that \( z_i \neq 0 \) for all \( i \). As the meet operation in \( P \) is internal to each \( L_i \), the right hand side of (4.6) is closed under meets. As each \( L_i \setminus \{0\} \) is an upper subset of \( P \), this set is also an upper subset of \( P \). This establishes the desired result for the \( u^{(i)} \)s.

Lemma 4.2. Let \( K_i \) be a 0-sublattice of a lattice \( L_i \), for each \( i \in I \). Then the canonical 0-lattice homomorphism \( f: \prod_{i \in I}^0 K_i \to \prod_{i \in I}^0 L_i \) is an embedding.

Proof. By the amalgamation property for lattices [7, Section V.4], the \( i \)-th co-projection from \( K_i \) to \( K \) is an embedding, for each \( i \in I \). Put \( L'_i = K \sqcup K \sqcup L_i \) for each \( i \in I \). Comparing the universal properties, it is immediate that the 0-coproduct \( L = (L_i \mid i \in I) \) is also the coproduct of \( (L'_i \mid i \in I) \) over \( K \). Again by using the amalgamation property for lattices, all canonical maps from the \( L'_i \)s to \( L \) are embeddings. So, in particular, the canonical map from their common sublattice \( K \) to \( L \) is an embedding.

We shall call the adjoint maps \( \alpha_i: x \mapsto x^{(i)} \) and \( \beta_i: x \mapsto x^{(i)} \) the canonical lower, resp. upper adjoint of \( x \) on \( L \), resp. \( L_i \). Observe that these maps may not be defined in the case of amalgamation of two lattices over a common sublattice, as Example 5.3 will show. (In that example, there is no largest element of \( B \) below \( b_0 \lor c_0 \).

The following result is an immediate consequence of well-known general properties of adjoint maps.

Corollary 4.3. The canonical embedding from \( L_i \) into \( L \) is both lower bounded and upper bounded, for each \( i \in I \). In particular, it is a nonempty-complete lattice homomorphism. Furthermore, the lower adjoint \( \alpha_i \) is meet-complete while the upper adjoint \( \beta_i \) is nonempty-join-complete.

In the following lemma, we shall represent the elements of \( L = \bigsqcup_{i \in I} L_i \) in the form \( p(\bar{a}) \), where \( p \) is a lattice term with variables from \( I \times \omega \) and the “vector” \( \bar{a} = (a_{i,n} \mid (i,n) \in I \times \omega) \) is an element of the cartesian product \( \Pi = \prod_{(i,n) \in I \times \omega} L_i \). Define a support of \( p \) as a subset \( J \) of \( I \) such that \( p \) involves only variables from \( J \times \omega \). Obviously, \( p \) has a finite support. It is straightforward from (4.4) that \( p(\bar{a}) = 0 \) and either \( p(\bar{a}) = 0 \) or \( p(\bar{a})^{(i)} = \infty \), for each \( i \) outside a support of \( p \).

Lemma 4.4. Let \( \Lambda \) be an upward directed poset, let \( (\bar{a}^\lambda \mid \lambda \in \Lambda) \) be an isotone family of elements of \( \Pi \) with supremum \( \bar{a} \) in \( \Pi \), and let \( p \) be a lattice term. If all the lattices \( L_i \) are upper continuous, then \( p(\bar{a}) = \bigvee_{\lambda \in \Lambda} p(\bar{a}^\lambda) \) in \( L \).
Again, Example 5.3 will show that Lemma 4.4 fails to extend to the amalgam of two lattices over a common ideal.

Proof. As \( p(\vec{a}) \) is clearly an upper bound for all elements \( p(\vec{a}^\lambda) \), it suffices to prove that for each lattice term \( \mathbf{q} \) on \( I \times \omega \) and each \( \vec{b} \in \Pi \) such that \( p(\vec{a}^\lambda) \leq \mathbf{q}(\vec{b}) \) for all \( \lambda \in \Lambda \), the inequality \( p(\vec{a}) \leq \mathbf{q}(\vec{b}) \) holds. We argue by induction on the sums of the lengths of \( p \) and \( \mathbf{q} \). The case where \( p \) is a projection follows immediately from the second sentence of Corollary 4.3. The case where either \( p \) is a join or \( \mathbf{q} \) is a meet is straightforward.

Now suppose that \( p = p_0 \land p_1 \) and \( \mathbf{q} = q_0 \lor q_1 \). We shall make repeated uses of the following easily established principle, which uses only the assumption that \( \Lambda \) is upward directed:

For every positive integer \( n \) and every \( X_0, \ldots, X_{n-1} \subseteq \Lambda \), if \( \bigcup_{i<n} X_i \) is cofinal in \( \Lambda \), then one of the \( X_i \)'s is cofinal in \( \Lambda \).

Now we use (3.3). If there exists a cofinal subset \( \Lambda' \) of \( \Lambda \) such that

\[
(\forall \lambda \in \Lambda')(\exists i < 2) (\text{either } p_i(\vec{a}^\lambda) \leq \mathbf{q}_i(\vec{b}) \text{ or } p(\vec{a}^\lambda) \leq q_i(\vec{b})),
\]

then there are \( i < 2 \) and a smaller cofinal subset \( \Lambda'' \) of \( \Lambda' \) such that

either \( (\forall \lambda \in \Lambda'')(p_i(\vec{a}^\lambda) \leq \mathbf{q}_i(\vec{b})) \)

or \( (\forall \lambda \in \Lambda'')(p(\vec{a}^\lambda) \leq q_i(\vec{b})) \).

In the first case, it follows from the induction hypothesis that \( p_i(\vec{a}) \leq \mathbf{q}(\vec{b}) \). In the second case, it follows from the induction hypothesis that \( p(\vec{a}) \leq q_i(\vec{b}) \). In both cases, \( p(\vec{a}) \leq \mathbf{q}(\vec{b}) \). It remains to consider the case where there exists a cofinal subset \( \Lambda' \) of \( \Lambda \) such that

\[
(\forall \lambda \in \Lambda')(\exists c_\lambda \in P)(p(\vec{a}^\lambda) \leq c_\lambda \leq \mathbf{q}(\vec{b})).
\]

It follows from the induction hypothesis that

\[
p(\vec{a}) = \bigvee_{\lambda \in \Lambda'} p(\vec{a}^\lambda), \quad \text{for all } \ell < 2. \tag{4.7}
\]

Fix a common finite support \( J \) of \( p_0, p_1, q_0, q_1 \). Each \( c_\lambda \) belongs to \( L_i \), for some \( i \) in the given support \( J \). By using the finiteness of \( J \) and by extracting a further cofinal subset of \( \Lambda' \), we may assume that all those \( i \) are equal to the same index \( j \in J \). Hence we have reduced the problem to the case where

\[
(\forall \lambda \in \Lambda')(p(\vec{a}^\lambda) \leq c_\lambda \leq \mathbf{q}(\vec{b})), \quad \text{where } c_\lambda = p(\vec{a}^\lambda)^{(j)} \in L_j. \tag{4.8}
\]

If \( p_0(\vec{a})^{(j)} \land p_1(\vec{a})^{(j)} = 0 \) for some \( i \in I \), then \( p(\vec{a}) \leq \mathbf{q}(\vec{b}) \) and we are done. Now suppose that \( p_0(\vec{a})^{(j)} \land p_1(\vec{a})^{(j)} \neq 0 \) for all \( i \in I \). By using (4.7), the finiteness of \( J \), and the upper continuity of \( L_i \), we obtain that there exists a cofinal subset \( \Lambda'' \) of \( \Lambda' \) such that

\[
(\forall \lambda \in \Lambda'')(\forall i \in J)(p_0(\vec{a}^\lambda)^{(i)} \land p_1(\vec{a}^\lambda)^{(i)} \neq 0).
\]

In particular, both \( p_0(\vec{a}^\lambda) \) and \( p_1(\vec{a}^\lambda) \) are nonzero for each \( \lambda \in \Lambda'' \). As \( J \) is a common support of \( p_0 \) and \( p_1 \), the equality \( p_0(\vec{a}^\lambda)^{(i)} \land p_1(\vec{a}^\lambda)^{(i)} = \infty \) holds for all \( \lambda \in \Lambda'' \) and all \( i \in I \setminus J \), hence

\[
(\forall \lambda \in \Lambda'')(\forall i \in I)(p_0(\vec{a}^\lambda)^{(i)} \land p_1(\vec{a}^\lambda)^{(i)} \neq 0).
\]
Thus it follows from Lemma 4.8 that \( c_\lambda = p(\bar{a}_\lambda)^{(j)} = p_0(\bar{a}_\lambda)^{(j)} \wedge p_1(\bar{a}_\lambda)^{(j)} \) for each \( \lambda \in \Lambda' \).

Hence, by the upper continuity of \( L_j \) (and thus of \( \mathcal{L}_j \)), (4.7), and the previously observed fact that the upper adjoint \( \beta_j \) is nonempty-join-complete, \( \{c_\lambda \mid \lambda \in \Lambda'\} \) has a join in \( L_j \), which is equal to \( p_0(\bar{a})^{(j)} \wedge p_1(\bar{a})^{(j)} = p(\bar{a})^{(j)} \).

Therefore, it follows from Lemma 4.8 that \( p(\bar{d}) \leq p(\bar{a})^{(j)} \leq q(\bar{b}) \).

5. Ideal lattices and 0-coproducts

In this section we fix again a family \( (L_i \mid i \in I) \) of lattices with zero, pairwise intersecting in \( \{0\} \), and we form \( L = \prod_{i \in I} L_i \). We denote by \( \varepsilon_i : \text{Id } L_i \hookrightarrow \text{Id } L \) the 0-lattice homomorphism induced by the canonical embedding \( L_i \hookrightarrow L \), for each \( i \in I \). By the universal property of the coproduct, there exists a unique 0-lattice homomorphism \( \varepsilon : \prod_{i \in I} \text{Id } L_i \rightarrow \text{Id } L \) such that \( \varepsilon_i = \varepsilon|\text{Id } L_i \) for each \( i \in I \).

Observe that in case \( p \) is finite, the lattice \( \prod_{i \in I} \text{Id } L_i \) has \( \bigvee_{i \in I} L_i \) as a largest element, and this element is sent by \( \varepsilon \) to \( L \) (because every element of \( L \) lies below some join of elements of the \( L_i \)s). Hence, if the index set \( I \) is finite, then the map \( \varepsilon \) preserves the unit as well.

**Lemma 5.1.** Let \( p \) be a lattice term on \( I \times \omega \) and let \( \bar{X} = (X_{i,n} \mid (i,n) \in I \times \omega) \) be an element of \( \prod_{i,n \in I} \text{Id } L_i \). We put \( \varepsilon \bar{X} = (\varepsilon_i(X_{i,n}) \mid (i,n) \in I \times \omega) \in (\text{Id } L)^{I \times \omega} \).

Then the following equality holds:

\[
p(\varepsilon \bar{X}) = L \downarrow \{ p(\bar{x}) \mid \bar{x} \not\in \bar{X} \},
\]

where \( \bar{x} \not\in \bar{X} \) stands for \( \forall (i,n) \in I \times \omega)(x_{i,n} \in X_{i,n}) \).

**Proof.** We argue by induction on the length of the term \( p \). If \( p \) is a projection, then the result follows immediately from the definition of the maps \( \varepsilon_i \). If \( p \) is either a join or a meet, then the result follows immediately from the expressions for the join and the meet in the ideal lattice of \( L_i \), in a fashion similar to the end of the proof of [7, Lemma I.4.8].

**Theorem 5.2.** The canonical map \( \varepsilon : \prod_{i \in I} \text{Id } L_i \rightarrow \text{Id } \left( \prod_{i \in I} L_i \right) \) is a 0-lattice embedding.

**Proof.** We put again \( L = \prod_{i \in I} L_i \). Let \( p, q \) be lattice terms in \( I \times \omega \) and let \( \bar{X} \in \prod_{i,n \in I} \text{Id } L_i \) such that \( p(\varepsilon \bar{X}) \leq q(\varepsilon \bar{X}) \) in \( \text{Id } L \).

We must prove that \( p(\bar{X}) \leq q(\bar{X}) \) in \( \prod_{i \in I} \text{Id } L_i \). For each \( \bar{x} \not\in \bar{X} \), the inequalities \( L \downarrow p(\bar{x}) \leq p(\varepsilon \bar{X}) \leq q(\varepsilon \bar{X}) \) hold in \( \text{Id } L \), thus, by Lemma 5.1, there exists \( \bar{y} \not\in \bar{X} \) such that \( L \downarrow p(\bar{x}) \leq L \downarrow q(\bar{y}) \) in \( \text{Id } L \), that is, \( p(\bar{x}) \leq q(\bar{y}) \) in \( L \).

Therefore, by applying the canonical map from \( L = \prod_{i \in I} L_i \) to \( \prod_{i \in I} \text{Id } L_i \) and putting \( \bar{L} \downarrow \bar{x} = (L_i \downarrow x_{i,n} \mid (i,n) \in I \times \omega) \in I \times \omega \), we obtain

\[
p(\bar{L} \downarrow \bar{x}) \leq (p(\bar{L} \downarrow \bar{x}) \mid \bar{x} \not\in \bar{X} \} = \prod_{i \in I} \text{Id } L_i \).
\]

As \( \bar{X} \) is equal to the directed join \( \bigvee_{\bar{x} \not\in \bar{X}}(L_i \downarrow \bar{x}) \) in \( \prod_{i,n \in I} \text{Id } L_i \) and each \( \text{Id } L_i \) is upper continuous, it follows from Lemma 4.8 that

\[
p(\bar{X}) = \bigvee \left( \{ p(\bar{L} \downarrow \bar{x}) \mid \bar{x} \not\in \bar{X} \} \right) \in \prod_{i \in I} \text{Id } L_i .
\]

Therefore, it follows from Lemma 5.1 that

\[
p(\bar{X}) \leq q(\bar{X}) \in \prod_{i \in I} \text{Id } L_i .
\]
The following example shows that Theorem 5.2 does not extend to the amalgam $B \amalg_A C$ of two lattices $B$ and $C$ above a common ideal $A$. The underlying idea can be traced back to Grätzer and Schmidt in [8, Section 5].

**Example 5.3.** Lattices $B$ and $C$ with a common ideal $A$ such that the canonical lattice homomorphism $f : (\text{Id}_B) \amalg \text{Id}_A (\text{Id}_C) \to \text{Id}(B \amalg_A C)$ is not one-to-one.

**Proof.** Denote by $K$ the poset represented in Figure 1. We claim that the subsets $A, B,$ and $C$ of $K$ defined by

$$A = \{a_n \mid n < \omega\} \cup \{p_n \mid n < \omega\} \cup \{q_n \mid n < \omega\},$$

$$B = A \cup \{b_n \mid n < \omega\},$$

$$C = A \cup \{c_n \mid n < \omega\},$$

are as required. Observe that $B$ and $C$ are isomorphic lattices and that $A$ is an ideal of both $B$ and $C$.

![Figure 1. The poset $K$.](image)

The map $f$ is the unique lattice homomorphism that makes the diagram of Figure 2 commute. Unlabeled arrows are the corresponding canonical maps.

![Figure 2. The commutative diagram defining the homomorphism $f$.](image)

Put $D = B \amalg_A C$ and identify $B$ and $C$ with their images in $D$. Further, we endow $\text{Id} B \cup \text{Id} C$ with its natural structure of partial lattice, that is, the ordering is
Similarly, the union of the orderings of \( \text{Id } B \) and \( \text{Id } C \) (remember that \( A = B \cap C \) is an ideal of both \( B \) and \( C \)) and the joins and meets are those taking place in either \( \text{Id } B \) or \( \text{Id } C \).

Observe that \( \text{Id } A = \text{Id } B \cap \text{Id } C \) and \( \text{Id } (B) \sqcup_{\text{Id } A} (\text{Id } C) \) is the free lattice on the partial lattice \((\text{Id } B) \cup (\text{Id } C)\). As the latter is identified with its canonical image in \((\text{Id } B) \sqcup_{\text{Id } A} (\text{Id } C)\), the elements \( A, B \downarrow b_0 \), and \( C \downarrow c_0 \) belong to \((\text{Id } B) \sqcup_{\text{Id } A} (\text{Id } C)\).

We prove by induction that \( a_n \leq b_0 \lor c_0 \) in \( D \) for all \( n < \omega \). This is trivial for \( n = 0 \). Suppose that \( a_n \leq b_0 \lor c_0 \). Then \( a_n \lor b_0 \leq b_0 \lor c_0 \), but \( B \) is a sublattice of \( D \) containing the subset \( \{a_n, b_0\} \) with join \( b_n \), thus \( b_n \leq b_0 \lor c_0 \), and thus \( p_n \leq b_0 \lor c_0 \).

Similarly, \( q_n \leq b_0 \lor c_0 \), but \( A \) is a sublattice of \( D \) containing the subset \( \{p_n, q_n\} \) with join \( a_{n+1} \), and thus \( a_{n+1} \leq b_0 \lor c_0 \), which completes the induction step.

So we have established the inequality

\[
\forall (\text{Id } A \sqcup_{\text{Id } B} \text{Id } C) = \text{Id } D, \quad (5.2)
\]

Now observe that \( \{B \downarrow x \mid x \in B\} \cup \{C \downarrow y \mid y \in C\} \) is an \( n \)-ideal of the partial lattice \((\text{Id } B) \cup (\text{Id } C)\), containing \( \{B \downarrow b_0, C \downarrow c_0\} \) and to which \( A \) does not belong. Hence, \( A \not\subseteq \exists \{B \downarrow b_0, C \downarrow c_0\} \), which means that \( A \not\subseteq (B \downarrow b_0) \lor (C \downarrow c_0) \) in \((\text{Id } B) \sqcup_{\text{Id } A} (\text{Id } C)\).

Therefore, by (5.2), \( f \) is not an embedding. \( \square \)

As observed before, this example shows that Lemma 4.4 fails to extend to the amalgam of two lattices over a common ideal. Indeed, while \( A = \bigvee_{n}(A \downarrow a_n) \) in \( \text{Id } B \), the same equality fails in \((\text{Id } B) \sqcup_{\text{Id } A} (\text{Id } C)\). The reason for this is that \( A \downarrow a_n \leq (B \downarrow b_0) \lor (C \downarrow c_0) \) for each \( n \), while \( A \not\subseteq (B \downarrow b_0) \lor (C \downarrow c_0) \).

6. Embedding coproducts of infinite partition lattices

Whitman’s Embedding Theorem states that every lattice embeds into \( \text{Eq } \Omega \), for some set \( \Omega \). We shall use a proof of Whitman’s Theorem due to B. Jónsson [9], see also [7] Section IV.4. The following result is proved there.

**Lemma 6.1.** For every lattice \( L \) with zero, there are an infinite set \( \Omega \) and a map \( \delta: \Omega \times \Omega \to L \) satisfying the following properties:

1. \( \delta(x, y) = 0 \) iff \( x = y \), for all \( x, y \in \Omega \).
2. \( \delta(x, y) = \delta(y, x) \), for all \( x, y \in \Omega \).
3. \( \delta(x, z) \leq \delta(x, y) \lor \delta(y, z) \), for all \( x, y, z \in L \).
4. For all \( x, y \in \Omega \) and all \( a, b \in L \) such that \( \delta(x, y) \leq a \lor b \), there are \( z_1, z_2, z_3 \in \Omega \) such that \( \delta(x, z_1) = a, \delta(z_1, z_2) = b, \delta(z_2, z_3) = a, \) and \( \delta(z_3, y) = b \).

Observe, in particular, that the map \( \delta \) is surjective. Furthermore, a straightforward Löwenheim-Skolem type argument (“keeping only the necessary elements in \( \Omega’ \)) shows that one may take \( \text{card } \Omega = \text{card } L + \aleph_0 \).

The following is the basis for Jónsson’s proof of Whitman’s Embedding Theorem.

**Corollary 6.2.** For every lattice \( L \) with zero and every set \( \Omega \) such that \( \text{card } \Omega = \text{card } L + \aleph_0 \), there exists a complete lattice embedding from \( \text{Id } L \) into \( \text{Eq } \Omega \).

**Proof.** Any map \( \delta \) as in Lemma 6.1 gives rise to a map \( \varphi: \text{Id } L \to \text{Eq } \Omega \) defined by the rule

\[
\varphi(A) = \{(x, y) \in \Omega \times \Omega \mid \delta(x, y) \in A\}, \quad \text{for each } A \in \text{Id } L, \quad (6.1)
\]

and conditions (1)–(4) above imply that \( \varphi \) is a complete lattice embedding. \( \square \)
Theorem 6.3. Let $\Omega$ be an infinite set. Then there exists a 0,1-lattice embedding from $(\text{Eq}\,\Omega)\Pi^0 (\text{Eq}\,\Omega)$ into $\text{Eq}\,\Omega$.

Proof. Denote by $K$ the sublattice of $\text{Eq}\,\Omega$ consisting of all compact equivalence relations of $\Omega$. Thus the elements of $K$ are exactly the equivalence relations containing only finitely many non-diagonal pairs. In particular, $\text{Eq}\,\Omega$ is canonically isomorphic to $\text{Id}\,K$.

Now we apply Corollary 5.2 to $L = K \Pi^0 K$. As $\text{card}\,L = \text{card}\,\Omega$, we obtain a complete lattice embedding $\varphi: \text{Id}\,L \hookrightarrow \text{Eq}\,\Omega$. However, $\text{Id}\,L = \text{Id}(K \Pi^0 K)$ contains, by Theorem 5.2 and the last sentence of the first paragraph of Section 5, a complete lattice embedding $\psi$ containing only finitely many non-diagonal pairs. In particular, $\text{Eq}\,\Omega$ is canonically isomorphic to $\text{Id}\,K$.

For any nonempty set $\Omega$, form $\Omega = \Omega \cup \{\infty\}$ for an outside point $\infty$. As there exists a retraction $\rho: \Omega \twoheadrightarrow \Omega$ (pick $p \in \Omega$ and send $\infty$ to $p$), we can form a meet-complete, nonempty-join-complete lattice embedding $\eta: \text{Eq}\,\Omega \hookrightarrow \text{Eq}\,\Omega$ by setting $\eta(\theta) = \{(x, y) \in \Omega \times \Omega \mid (\rho(x), \rho(y)) \in \theta\}$, for each $\theta \in \text{Eq}\,\Omega$, and $\eta$ sends the zero element of $\text{Eq}\,\Omega$ to a nonzero element of $\text{Eq}\,\Omega$. Hence, in case $\Omega$ is infinite, $(\text{Eq}\,\Omega)^\varnothing$ completely embeds into $\text{Eq}\,\Omega$. As $(\text{Eq}\,\Omega)\Pi (\text{Eq}\,\Omega)$ is the sublattice of $(\text{Eq}\,\Omega)^\varnothing \Pi^0 (\text{Eq}\,\Omega)^\varnothing$ generated by the union of the images of $\text{Eq}\,\Omega$ under the two canonical coprojections, it follows from Theorem 6.3 and Lemma 4.2 that $(\text{Eq}\,\Omega)\Pi (\text{Eq}\,\Omega)$ has a 1-lattice embedding into $\text{Eq}\,\Omega$. If we denote by $\theta$ the image of zero under this embedding, then $(\text{Eq}\,\Omega)\Pi (\text{Eq}\,\Omega)$ has a 0,1-lattice embedding into $\text{Eq}(\Omega/\theta)$, and thus, as $\text{card}(\Omega/\theta) \leq \text{card}\,\Omega$, into $\text{Eq}\,\Omega$. Hence we obtain

Theorem 6.4. Let $\Omega$ be an infinite set. Then there exists a 0,1-lattice embedding from $(\text{Eq}\,\Omega)\Pi (\text{Eq}\,\Omega)$ into $\text{Eq}\,\Omega$.

By applying the category equivalence $L \mapsto L^{\text{op}}$ to Theorems 6.3 and 6.4, and denoting by $\Pi^1$ the coproduct of 1-lattices, we obtain the following result.

Theorem 6.5. Let $\Omega$ be an infinite set. Then there are 0,1-lattice embeddings from $(\text{Eq}\,\Omega)^{\text{op}}\Pi^1 (\text{Eq}\,\Omega)^{\text{op}}$ into $(\text{Eq}\,\Omega)^{\text{op}}\Pi (\text{Eq}\,\Omega)^{\text{op}}$ and from $(\text{Eq}\,\Omega)^{\text{op}}\Pi (\text{Eq}\,\Omega)^{\text{op}}$ into $(\text{Eq}\,\Omega)^{\text{op}}$.

By using the results of [2], we can now fit the copower of the optimal number of copies of $L = \text{Eq}\,\Omega$ into itself. The variety $V$ to which we apply those results is, of course, the variety of all lattices with zero. The functor to be considered sends every set $I$ to $F(I) = \prod_{i \in I} L$, the 0-coproduct of $I$ copies of $L$. If we denote by $e_i^I: L \rightarrow F(I)$ the $i$-th coprojection, then, for any map $f: I \rightarrow J$, $F(f)$ is the unique 0-lattice homomorphism from $F(I)$ to $F(J)$ such that $F(f) \circ e_i^I = e_j^{F(I)}$ for all $i \in I$. Observe that even in case both $I$ and $J$ are finite, $F(f)$ does not preserve the unit unless $f$ is surjective. The condition labeled (9) in [2, Section 3], stating that every element of $F(I)$ belongs to the range of $F(a)$ for some $a: n \mapsto I$, for some positive integer $n$, is obviously satisfied. Hence, by [2, Theorem 3.1], $F(\mathfrak{P}(\Omega))$ has a 0-lattice embedding into $F(\omega)^\varnothing$. Furthermore, it follows from [2, Lemma 3.3] that $F(\omega)$ has a 0-lattice embedding into $\prod_{i \leq \omega} F(n)$. By Lemma 4.2 and Theorem 6.3, each $F(n)$ has a 0,1-lattice embedding into $L$. As, by the final paragraph of [2, Section 2], $L^\varnothing$ has a 0-lattice embedding into $L$, we obtain the following theorem.

Theorem 6.6. Let $\Omega$ be an infinite set. Then the following statements hold:

1. $\prod_{i \in I} \text{Eq}\,\Omega$ has a 0-lattice embedding into $\text{Eq}\,\Omega$. 


EMBEDDING COPRODUCTS OF PARTITION LATTICES

(2) \( \prod_{\Omega} \text{Eq} \Omega \) has a lattice embedding into \( \text{Eq} \Omega \).

(3) \( \prod_{\Omega} (\text{Eq} \Omega)^{\text{op}} \) has a \( 1 \)-lattice embedding into \( (\text{Eq} \Omega)^{\text{op}} \).

(4) \( \prod_{\Omega} (\text{Eq} \Omega)^{\text{op}} \) has a lattice embedding into \( (\text{Eq} \Omega)^{\text{op}} \).

This raises the question whether \( (\text{Eq} \Omega) \prod^1 (\text{Eq} \Omega) \) embeds into \( \text{Eq} \Omega \), which the methods of the present paper do not seem to settle in any obvious way. More generally, we do not know whether, for a sublattice \( A \) of \( \text{Eq} \Omega \), the amalgam \( (\text{Eq} \Omega) \prod_A (\text{Eq} \Omega) \) of two copies of \( \text{Eq} \Omega \) over \( A \) embeds into \( \text{Eq} \Omega \).

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REFERENCES

[1] K. V. Adaricheva, V. A. Gorbunov, and M. V. Semenova, *On continuous noncomplete lattices*, The Viktor Aleksandrovich Gorbunov memorial issue, Algebra Universalis 46 (2001), no. 1-2, 215–230.

[2] G. M. Bergman, *Some results on embeddings of algebras, after de Bruijn and McKenzie*, Indag. Math. (N.S.), to appear. Available online at http://math.berkeley.edu/~gbergman/papers/ and arXiv:math.RA/0606407.

[3] N. G. de Bruijn, *Embedding theorems for infinite groups*, Nederl. Akad. Wetensch. Proc. Ser. A. 60=Indag. Math. 19 (1957), 560–569.

[4] N. G. de Bruijn, *Addendum to “Embedding theorems for infinite groups”*, Nederl. Akad. Wetensch. Proc. Ser. A 67–Indag. Math. 26 (1964), 594–595.

[5] R. A. Dean, *Free lattices generated by partially ordered sets and preserving bounds*, Canad. J. Math. 16 (1964), 136–148.

[6] R. Freese, J. Ježek, and J.B. Nation, “Free Lattices”. Mathematical Surveys and Monographs 42. American Mathematical Society, Providence, RI, 1995. viii+293 p.

[7] G. Grätzer, “General Lattice Theory. Second edition”, new appendices by the author with B. A. Davey, R. Freese, B. Ganter, M. Greferath, P. Jipsen, H. A. Priestley, H. Rose, E. T. Schmidt, S. E. Schmidt, F. Wehrung, and R. Wille. Birkhäuser Verlag, Basel, 1998. xx+663 p.

[8] G. Grätzer and E. T. Schmidt, *A lattice construction and congruence-preserving extensions*, Acta Math. Hungar. 66, no. 4 (1995), 275–288.

[9] B. Jónsson, *On the representation of lattices*, Math. Scand. 1 (1953), 193–206.

[10] P. M. Whitman, *Lattices, equivalence relations, and subgroups*, Bull. Amer. Math. Soc. 52 (1946), 507–522.

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