ON CHEREDNIK-MACDONALD–MEHTA IDENTITIES

Pavel Etingof and Alexander Kirillov, Jr

INTRODUCTION

In this note we give a proof of Cherednik’s generalization of Macdonald–Mehta identities for the root system $A_{n-1}$ using the representation theory of quantum groups. These identities, suggested and proved in [Ch2], give an explicit formula for the integral of a product of Macdonald polynomials with respect to a “difference analogue of the Gaussian measure”. They can be written for any reduced root system, or, equivalently, for any semisimple complex Lie algebra $g$. Assuming for simplicity that $g$ is simple and simply-laced, these identities are given by the following formula:

$$\frac{1}{|W|} \int \delta_k \, P_\lambda P_\mu \gamma \, dx = q^{\lambda^2 + (\mu, \mu + 2k \rho)} P_\mu (q^{-2(\lambda + k \rho)})$$

$$\times q^{-2k(k-1)|R_+|} \prod_{\alpha \in R_+} \prod_{i=0}^{k-1} \left(1 - q^{2(\alpha, \lambda + k \rho) + 2i}\right)$$

where $\lambda, \mu$ are dominant integral weights, $k$ is a positive integer, $P_\lambda$ are Macdonald polynomials associated with the corresponding root system, with parameters $q^2, t = q^{2k}$ (see [M1, M2] or a review in [K2]), $\delta_k$ is the $q$-analogue of $k$-th power of the Weyl denominator $\delta = \delta_1$:

$$\delta_k = \prod_{\alpha \in R_+} \prod_{i=0}^{k-1} \left(e^{\alpha/2} - q^{-2i} e^{-\alpha/2}\right),$$

and $\gamma$ is the Gaussian, which we define by

$$\gamma = \sum_{\lambda \in P} e^{\lambda} q^{\lambda^2},$$

where $P$ is the weight lattice. We consider $\gamma$ as a formal series in $q$ with coefficients from the group algebra of the weight lattice. In a more standard terminology $\gamma$ is called the theta-function of the lattice $P$. All other notations, which are more or less standard, will be explained below.

These identities were formulated in the form we use in a paper of Cherednik [Ch2], who also proved them using double affine Hecke algebras (note: our notations are somewhat different from Cherednik’s ones). We refer the reader to [Ch2] for the discussion of the history of these identities and their role in difference harmonic analysis.
As an important intermediate step, we also prove the following identity for the Gaussian:

\[
\gamma = \left( \prod_{\alpha \in R_+} \left( 1 - q^{2(\alpha, \rho)} \right) \right) \sum_{\nu \in P_+} q^{(\nu, \nu + 2\rho)} (\dim_{q} L_{\nu}) \chi_{\nu}.
\]

Here \( \chi_{\nu} \) is the character of the irreducible finite-dimensional module \( L_{\nu} \) over \( g \), and \( \dim_{q} L_{\nu} := \chi_{\nu}(q^{2\rho}) \) is the quantum dimension of \( L_{\nu} \). This identity was known to experts and is not difficult to prove; however, we were unable to locate a proof in the literature.

**Notations.** We use the same notations as in \([E1, E2]\) with the following exceptions: we replace \( q \) by \( q^{-1} \) (note that this does not change the Macdonald’s polynomials) and we use the notation \( \phi_{\lambda} \) for “generalized characters” (see below), reserving the notation \( \chi_{\lambda} \) for usual (Weyl) characters. In particular, we define \( e_{\lambda} = e_{-\lambda}, \bar{q} = q \), and for \( f \in \mathbb{C}_{q}[P] \), we define \( f(q^{\lambda}), \lambda \in P \) by \( e^{\mu}(q^{\lambda}) = q^{(\mu, \lambda)}. \) For brevity, we also write \( \lambda^{2} \) for \( (\lambda, \lambda) \). Finally, we denote by \( \int dx : \mathbb{C}_{q}[P] \to \mathbb{C}_{q} \) the functional of taking the constant term: \( \int e^{\lambda} dx = \delta_{\lambda, 0}. \)

### 1. Rewriting the Gaussian

In this Section, we prove formula (4) for an arbitrary simple Lie algebra \( g \).

**Proposition 1.** Let \( \gamma \) be defined by (3). Then

\[
\gamma = \left( \prod_{\alpha \in R_+} \left( 1 - q^{2(\alpha, \rho)} \right) \right) \sum_{\nu \in P_+} q^{(\nu, \nu + 2\rho)} (\dim_{q} L_{\nu}) \chi_{\nu}.
\]

**Proof.** The proof is straightforward and uses Weyl character formula along with the following result: if we extend the definition of \( \chi_{\nu} \) to all \( \nu \in P \) by letting \( \chi_{\nu} = (\sum (-1)^{|w|} e^{w(\nu + \rho)}) / \delta \) (recall that \( \delta = \delta_{1} \) is the Weyl denominator) then \( \chi_{w.\nu} = (-1)^{|w|} \chi_{\nu} \), where \( w.\nu = w(\nu + \rho) - \rho \). In particular, \( \chi_{\nu} = 0 \) if \( \nu \) lies on one of the walls, i.e. if \( s_\alpha.\nu = \nu \) for some root \( \alpha \). The same applies to \( \dim_{q} L_{\nu} = \chi_{\nu}(q^{2\rho}). \) Using this, we rewrite the right-hand side of (4) as follows:

\[
\sum_{\nu \in P_+} q^{(\nu, \nu + 2\rho)} (\dim_{q} L_{\nu}) \chi_{\nu} = q^{-\rho^{2}} \sum_{\nu \in P_+} q^{(\nu, \nu + \rho)^2} (\dim_{q} L_{\nu}) \chi_{\nu} = q^{-\rho^{2}} \frac{1}{|W|} \sum_{\nu \in P} q^{\nu^{2}} (\dim_{q} L_{\nu - \rho}) \chi_{\nu - \rho}.
\]

By Weyl character formula, we can write

\[
(\dim_{q} L_{\nu - \rho}) \chi_{\nu - \rho} = \frac{1}{\delta(q^{2\rho})} \frac{1}{\delta} \sum (\chi_{\nu - \rho}(q^{s_{1}w_{1}2}(\nu, w_{1}(\rho))) e^{w_{2}(\nu)}).
\]
Substituting this in the previous identity, we get

\[
\sum_{\nu \in P_+} q^{(\nu,\nu+2\rho)} (\dim_q L_\nu) \chi_\nu = \frac{q^{-2\rho^2}}{|W|}\frac{1}{\delta(q^{2\rho})} \delta \sum_{\nu \in \rho} q^{(\nu+w_1(\rho))^2} (-1)^{|w_1 w_2|} e^{w_2(\nu)}
\]

\[
= \frac{q^{-2\rho^2}}{|W|}\frac{1}{\delta(q^{2\rho})} \delta \sum_{w_1, w_2 \in W} q^{\lambda^2} (-1)^{|w_1 w_2|} e^{w_2(\lambda-w_1(\rho))}
\]

\[
= \frac{q^{-2\rho^2}(-1)^{|R_+|}}{|W|\delta(q^{2\rho})} \sum_{w_2 \in W} q^{\lambda^2} e^{w_2(\lambda)} = \frac{q^{-2\rho^2}(-1)^{|R_+|}}{\delta(q^{2\rho})} \sum_{\lambda \in P} q^{\lambda^2} e^{\lambda}
\]

Simplifying this, we get the statement of the Proposition. \(\square\)

**Example.** Let \(\mathfrak{g} = \mathfrak{sl}_2\). Then Proposition 1 gives the following identity, which can be verified directly:

\[
\sum_{n \geq 0} q^{n(n+2)/2}[n+1](x^n + x^{n-2} + \cdots + x^{-n}) = \frac{1}{1-q^2} \sum_{l \in \mathbb{Z}} x^l q^{l^2/2},
\]

where \([n] = \frac{q^n-q^{-n}}{q-q^{-1}}\).

Finally, we note that the Gaussian can be defined in any semisimple ribbon category \(\mathcal{C}\), i.e., a tensor category, with possibly non-trivial commutativity isomorphism, and a “Casimir element” (also called “ribbon element”) satisfying certain compatibility properties (see, e.g., [Kas] or [K1]). Namely, we define the Gaussian to be the following element of the suitable completion of the Grothendieck ring \(K(\mathcal{C})\):

\[
\gamma_\mathcal{C} = \sum_i C_i \dim X_i \langle X_i \rangle,
\]

where \(X_i\) are simple objects in \(\mathcal{C}\), \(C_i\) is the value of the Casimir element in \(X_i\) (in [K1], these numbers are denoted by \(\theta_i\)), \(\dim X_i\) is the \(q\)-dimension of \(X_i\), and \(\langle X_i \rangle\) is the class of \(X_i\) in the Grothendieck ring. In particular, if we take the category of representations of the quantum group \(U_q \mathfrak{h}\) corresponding to the Cartan subalgebra \(\mathfrak{h} \subset \mathfrak{g}\) considered as a commutative Lie algebra, then its irreducible representations are parametrized by \(\lambda \in P\), and they are all one-dimensional. One can check that defining the universal \(R\)-matrix by \(R|_{X_\lambda \otimes X_\mu} = q^{(\lambda,\mu)}\), and the Casimir element by \(C|_{X_\lambda} = q^{\lambda^2}\) endows this category with a structure of ribbon category. Thus, the Gaussian \(\gamma_{U_q \mathfrak{h}} = \gamma\) for this category is exactly given by the formula (3). On the other hand, if we consider the category of representations of the quantum group \(U_q \mathfrak{g}\), then the Casimir element \(C\) in this category is defined by \(C = q^{2\rho} u^{-1}\), where \(u\) is the Drinfeld’s element (see details in [Kas, Chapter XIV.6], where the element \(\theta = C^{-1}\) is discussed), and \(C|_{L_\lambda} = q^{(\lambda,\lambda+2\rho)}\). Thus, Gaussian for this category is given by \(\gamma_{U_q \mathfrak{g}} = \sum_{\nu \in P_+} q^{(\nu,\nu+2\rho)} (\dim_q L_\nu) \chi_\nu\). Therefore, Proposition 1 can be rewritten as

\[
\gamma_{U_q \mathfrak{g}} = \left( \prod_{\alpha \in R_+} \frac{1}{1-q^{2(\alpha,\rho)}} \right) \gamma_{U_q \mathfrak{h}},
\]

which is closely connected with the Weyl formula for a compact group \(G\), which relates the measure on \(G/\text{Ad } G = T/W\) coming from the Haar measure on \(G\) with the Haar measure on \(T\).
2. Proof of Cherednik–Macdonald–Mehta identities

In this section, we give a proof of the Cherednik–Macdonald–Mehta identities (1) for $\mathfrak{g} = \mathfrak{sl}_n$. The proof is based on the realization of Macdonald’s polynomials as “vector-valued characters” for the quantum group $U_q\mathfrak{sl}_n$, which was given in [EK1]. For the sake of completeness, we briefly outline these results here, referring the reader to the original paper for a detailed exposition.

Let us fix $k \in \mathbb{Z}_+$ and denote by $U$ the finite-dimensional representation of $U_q\mathfrak{sl}_n$ with highest weight $n(k-1)\omega_1$, where $\omega_1$ is the first fundamental weight. We identify the zero weight subspace $U[0]$, which is one-dimensional, with $\mathbb{C}_q$.

For $\lambda \in P_+$, we denote by $\Phi_\lambda$ the unique intertwiner

$$\Phi_\lambda : L_{\lambda+(k-1)\rho} \to L_{\lambda+(k-1)\rho} \otimes U$$

and define the generalized character $\varphi_\lambda \in \mathbb{C}_q[P] \otimes U[0] \simeq \mathbb{C}_q[P]$ by $\varphi_\lambda(q^x) = \text{Tr}_{L_{\lambda+(k-1)\rho}}(\Phi_\lambda g^x)$.

We can now summarize the results of [EK1] as follows:

$$\varphi_0 = \prod_{\alpha \in R_+} \prod_{i=1}^{k-1} (e^{\alpha/2} - q^{2i}e^{-\alpha/2}) = \delta_k/\delta$$

$$\varphi_\lambda/\varphi_0 = P_\lambda$$

where $P_\lambda$ is the Macdonald polynomial with parameters $q^2, t = q^{2k}$.

We can also rewrite Macdonald’s inner product in terms of the generalized characters as follows. Recall that Macdonald’s inner product on $\mathbb{C}_q[P]$ is defined by

$$\langle f, g \rangle_k = \frac{1}{|W|} \int \delta_k \delta_k fg \, dx$$

(this differs by a certain power of $q$ from the original definition of Macdonald). Obviously, one has

$$\langle P_\lambda, P_\mu \rangle_k = \langle \varphi_\lambda, \varphi_\mu \rangle_1.$$
where \( \nu^* = -w_0(\nu) \) is the highest weight of the module \((L_\nu)^*\) (here \(w_0\) is the longest element of the Weyl group).

Of course, the coefficients \(C_{\lambda\mu}^\nu\) are very difficult to calculate. However, the formula above is still useful. For example, it immediately shows that \( \langle \varphi_\lambda, \varphi_\mu \rangle_1 = 0 \) unless \( \lambda = \mu \), which was the major part of the proof of the formula \( \varphi_\lambda / \varphi_0 = P_\lambda \) in [EK1]. It turns out that this formula also allows us to prove the Cherednik-Macdonald–Mehta identities.

**Theorem 2.** Let \( \varphi_\lambda \) be the renormalized Macdonald polynomials for the root system \( A_{n-1} \) given by (6), and let \( \gamma \) be the Gaussian (3). Then

\[
\frac{1}{|W|} \int \delta \varphi_\lambda \varphi_\mu \gamma \, dx = q^{(\lambda+\kappa\rho)^2} q^{(\mu+\kappa\rho)^2} \varphi_\mu (q^{-2(\lambda+\kappa\rho)}) \times \left( \prod_{\alpha \in R_+} \left( 1 - q^{2(\alpha,\rho)} \right) \right) q^{-2\rho^2} \|P_\lambda\|^2 \dim_q L_{\lambda+(k-1)\rho},
\]  

where \( \|P_\lambda\|^2 = \langle P_\lambda, P_\lambda \rangle_k \).

**Proof.** From (7) and (4), we get

\[
\int \delta \varphi_\lambda \varphi_\mu \gamma \, dx = \left( \prod_{\alpha \in R_+} \left( 1 - q^{2(\alpha,\rho)} \right) \right) \sum_{\nu \in P^+} q^{(\nu,\nu+2\rho)} (\dim_q L_\nu) C_{\lambda\mu}^\nu.
\]  

On the other hand, let \( C \) be the Casimir element for \( U_q g \) discussed above. Consider the intertwiner \((\Phi_\lambda \circ \Phi_\mu) \Delta(C) : V \to V\), where, as before, \( V = L_{\lambda+(k-1)\rho} \otimes L_{\mu+(k-1)\rho} \). Then it follows from \( \Delta(C) = q^{(\lambda+\rho,\lambda+2\rho)} \) that

\[
\text{Tr}_V \left( (\Phi_\lambda \circ \Phi_\mu) \Delta(C) \Delta(q^{2\rho}) \right) = \sum_{\nu \in P^+} C_{\lambda\mu}^\nu q^{(\nu,\nu+2\rho)} \dim_q L_\nu,
\]

which is exactly the sum in the right hand side of (9). On the other hand, using \( \Delta(C) = (C \otimes C)(R^{21}R) \), we can write

\[
\text{Tr}_V \left( (\Phi_\lambda \circ \Phi_\mu) \Delta(C) \Delta(q^{2\rho}) \right) = q^{-2\kappa^2} q^{(\lambda+\kappa\rho)^2} q^{(\mu+\kappa\rho)^2} \text{Tr}_V \left( (\Phi_\lambda \circ \Phi_\mu)(R^{21}R) \Delta(q^{2\rho}) \right)
\]

This last trace can be calculated, which was done in [EK2, Corollary 4.2], and the answer is given by

\[
\text{Tr}_V \left( (\Phi_\lambda \circ \Phi_\mu)(R^{21}R) \Delta(q^{2\rho}) \right) = \varphi_\mu (q^{-2(\lambda+\kappa\rho)}) \|P_\lambda\|^2 \dim_q L_{\lambda+(k-1)\rho}.
\]

Combining these results, we get the statement of the theorem. \( \square \)

The norms \( \|P_\lambda\|^2 \) appearing in the right-hand side of (8) are given by Macdonald’s inner product identities

\[
\|P_\lambda\|^2 = \prod_{\alpha \in R_+} \prod_{i=1}^{k-1} \frac{1 - q^{-2(\alpha,\lambda+\kappa\rho)-2i}}{1 - q^{-2(\alpha,\lambda+\kappa\rho)+2i}},
\]

which were conjectured in [M1, M2] and proved for root system \( A_{n-1} \) by Macdonald himself [M3]; see also [EK2] for the proof based on representation theory of \( U_q sl_2 \).
and [Ch1] or a review in [K2] for a proof for arbitrary root systems. Using this formula and rewriting the statement of Theorem 2 in terms of Macdonald polynomials \( P_\lambda \) rather than \( \varphi_\lambda \), we get the Cherednik–Macdonald–Mehta identities (1).

**Remarks.** 1. Note that the left-hand side of (8) is symmetric in \( \lambda, \mu \). Thus, the same is true for the right-hand side, which is exactly the statement of Macdonald’s symmetry identity (compare with the proof in [EK2]).

2. The proof of Cherednik–Macdonald–Mehta identities given above easily generalizes to the case when \( q \) is a root of unity (see [K1] for the discussion of the appropriate representation-theoretic setup). In this case, we need to replace the set \( P_+ \) of all integral dominant weights by an appropriate (finite) Weyl alcove \( C \) (see [K1]), and the integral \( \int \delta \delta f \, dx \) should be replaced by \( \text{const} \sum_{\lambda \in C} f(q^{2(\lambda+\rho)}) \dim_q L_\lambda \).

Using the following obvious property of the Gaussian:

\[
\gamma(q^{2(\lambda+\rho)}) = q^{-(\lambda,\lambda)+2\rho}\gamma(q^{2\rho})
\]

(which in this case coincides with formula (1.7) in [K1]), it is easy to see that in this case the Cherednik–Macdonald–Mehta identities are equivalent to

\[
S^{-1}T^{-1}S = TST
\]

where the matrices \( S, T \) are defined in [K1, Theorem 5.4]. This identity is a part of a more general result, namely, that these matrices \( S, T \) give a projective representation of the modular group \( SL_2(\mathbb{Z}) \) on the space of generalized characters (see [K1, Theorem 1.10] and references therein).

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