Some remarks on regular integers modulo $n$

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Abstract

An integer $k$ is called regular (mod $n$) if there exists an integer $x$ such that $k^2x \equiv k \pmod{n}$. This holds true if and only if $k$ possesses a weak order (mod $n$), i.e., there is an integer $m \geq 1$ such that $k^{m+1} \equiv k \pmod{n}$. Let $\varrho(n)$ denote the number of regular integers (mod $n$) in the set $\{1, 2, \ldots, n\}$. This is an analogue of Euler’s $\varphi$ function. We introduce the multidimensional generalization of $\varrho$, which is the analogue of Jordan’s function. We establish identities for the power sums of regular integers (mod $n$) and for some other finite sums and products over regular integers (mod $n$), involving the Bernoulli polynomials, the Gamma function and the cyclotomic polynomials, among others. We also deduce an analogue of Menon’s identity and investigate the maximal orders of certain related functions.

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1 Introduction

Throughout the paper we use the notations: $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$, $\mathbb{Z}$ is the set of integers, $[x]$ is the integer part of $x$, $1$ is function given by $1(n) = 1$ ($n \in \mathbb{N}$), id is the function given by $\text{id}(n) = n$ ($n \in \mathbb{N}$), $\phi$ is Euler’s totient function, $\tau(n)$ is the number of divisors of $n$, $\mu$ is the Möbius function, $\omega(n)$ stands for the number of prime factors of $n$, $\Lambda$ is the von Mangoldt function, $\kappa(n) := \prod_{p|n} p$ is the largest squarefree divisor of $n$, $c_n(t)$ are the Ramanujan sums defined by $c_n(t) := \sum_{1 \leq k \leq n, \gcd(k,n)=1} \exp(2\pi ikt/n)$ ($n \in \mathbb{N}, t \in \mathbb{Z}$). Other notations will be fixed inside the paper.

Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Then $k$ is called regular (mod $n$) if there exists $x \in \mathbb{Z}$ such that $k^2x \equiv k \pmod{n}$. This holds true if and only if $k$ possesses a weak order (mod $n$), i.e., there is $m \in \mathbb{N}$ such that $k^{m+1} \equiv k \pmod{n}$. Every $k \in \mathbb{Z}$ is regular (mod 1). If $n > 1$ and its prime power factorization is $n = p_1^{n_1} \cdots p_r^{n_r}$, then $k$ is regular (mod $n$) if and only if for every $i \in \{1, \ldots, r\}$ either $p_i \nmid k$ or $p_i^{n_i} \mid k$. Also, $k$ is regular (mod $n$) if and only if $\gcd(k,n)$ is a unitary divisor of $n$. We recall that $d$ is said to be a unitary divisor of $n$ if $d \mid n$ and $\gcd(d,n/d) = 1$, notation $d \parallel n$.

Note that if $n$ is squarefree, then every $k \in \mathbb{Z}$ is regular (mod $n$). See the papers [11, 12, 13, 14] for further discussion and properties of regular integers (mod $n$), and their connection with the notion of regular elements of a ring in the sense of J. von Neumann.

An integer $k$ is regular (mod $n$) if and only if $k + n$ is regular (mod $n$). Therefore, it is justified to consider the set

$$\text{Reg}_n := \{k \in \mathbb{N} : 1 \leq k \leq n, k \text{ is regular (mod } n)\}$$
and the quantity \( \varrho(n) = \# \text{Reg}_n \). For example, \( \text{Reg}_{12} = \{1, 3, 4, 5, 7, 8, 9, 11, 12\} \) and \( \varrho(12) = 9 \). If \( n \) is squarefree, then \( \text{Reg}_n = \{1, 2, \ldots, n\} \) and \( \varrho(n) = n \). Note that \( 1, n \in \text{Reg}_n \) for every \( n \in \mathbb{N} \). The arithmetic function \( \varrho \) is an analogue of Euler’s \( \phi \) function, it is multiplicative and \( \varrho(p^n) = \phi(p^n) + 1 = p^n - p^{n-1} + 1 \) for every prime power \( p^n \) (\( \nu \in \mathbb{N} \)). Consequently,

\[
\varrho(n) = \sum_{d \mid n} \phi(d) \quad (n \in \mathbb{N}). \tag{1}
\]

See, e.g., [11] for general properties of unitary divisors, in particular the unitary convolution of the arithmetic functions \( f \) and \( g \) defined by \((f \times g)(n) = \sum_{d \mid n} f(d)g(n/d)\). Here \( f \times g \) preserves the multiplicativity of the functions \( f \) and \( g \). We refer to [18] for asymptotic properties of the function \( \varrho \).

The function

\[
\tau_n(t) := \sum_{k \in \text{Reg}_n} \exp(2\pi i kt/n) \quad (n \in \mathbb{N}, t \in \mathbb{Z}),
\]

representing an analogue of the Ramanujan sum \( c_n(t) \) was investigated in the paper [3]. We have

\[
\tau_n(t) = \sum_{d \mid n} c_d(t) \quad (n \in \mathbb{N}, t \in \mathbb{Z}).
\]

It turns out that for every fixed \( t \) the function \( n \mapsto \tau_n(t) \) is multiplicative, \( \tau_n(0) = \varrho(n) \) and \( \tau_n(1) = \varrho(n) \) is the characteristic function of the squarefull integers \( n \).

The gcd-sum function is defined by \( P(n) := \sum_{k=1}^{n} \gcd(k, n) = \sum_{d \mid n} d \varrho(n/d) \), see [20]. The following analogue of the gcd-sum function was introduced in the paper [19]:

\[
\tilde{P}(n) := \sum_{k \in \text{Reg}_n} \gcd(k, n).
\]

One has

\[
\tilde{P}(n) = \sum_{d \mid n} d \varrho(n/d) = n \prod_{p \mid n} \left( 2 - \frac{1}{p} \right) \quad (n \in \mathbb{N}),
\]

the asymptotic properties of \( \tilde{P}(n) \) being investigated in [3, 20, 21, 25].

In the present paper we discuss some further properties of the regular integers (mod \( n \)). We first introduce the multidimensional generalization \( \varrho_r \) (\( r \in \mathbb{N} \)) of the function \( \varrho \), which is the analogue of the Jordan function \( \phi_r \), where \( \phi_r(n) \) is defined as the number of ordered \( r \)-tuples \((k_1, \ldots, k_r) \in \{1, \ldots, n\}^r \) such that \( \gcd(k_1, \ldots, k_r) \) is prime to \( n \) (see, e.g., [11, 16]). Then we consider the sum \( S[\text{reg}_r](n) \) of \( r \)-th powers of the regular integers (mod \( n \)) belonging to \( \text{Reg}_n \). In the case \( r \in \mathbb{N} \) we deduce an exact formula for \( S[\text{reg}_r](n) \) involving the Bernoulli numbers \( B_m \). For a positive real number \( r \) we derive an asymptotic formula for \( S[\text{reg}_r](n) \). We combine the functions \( \tau_n(t) \) and \( \tilde{P}(n) \) defined above and establish identities for sums, respectively products over the integers in \( \text{Reg}_n \) concerning the Bernoulli polynomials \( B_m(x) \), the Gamma function \( \Gamma \), the cyclotomic polynomials \( \Phi_m(x) \) and certain trigonometric functions. We point out that for \( n \) squarefree these identities reduce to the corresponding ones over \( \{1, 2, \ldots, n\} \). We also deduce an analogue of Menon’s identity and investigate the maximal orders of some related functions.
2 A generalization of the function $\varrho$

For $r \in \mathbb{N}$ let $\varrho_r(n)$ be the number of ordered $r$-tuples $(k_1, \ldots, k_r) \in \{1, \ldots, n\}^r$ such that $\gcd(k_1, \ldots, k_r)$ is regular (mod $n$). If $r = 1$, then $\varrho_1 = \varrho$. The arithmetic function $\varrho_r$ is the analogue of the Jordan function $\phi_r$, defined in the Introduction and verifying $\varrho_r(n) = n^r \prod_{p|n} (1 - 1/p^r)$ ($n \in \mathbb{N}$).

**Proposition 1.** i) For every $r, n \in \mathbb{N}$,

$$\varrho_r(n) = \sum_{d|n} \varrho_r(d).$$

ii) The function $\varrho_r$ is multiplicative and for every prime power $p^\nu$ ($\nu \in \mathbb{N}$),

$$\varrho_r(p^\nu) = p^{\nu r} - p^{\nu(r-1)} + 1.$$

**Proof.** i) The integer $\gcd(k_1, \ldots, k_r)$ is regular (mod $n$) if and only if $\gcd(\gcd(k_1, \ldots, k_r), n) \parallel n$, that is $\gcd(k_1, \ldots, k_r, n) = d$ we deduce that

$$\varrho_r(n) = \sum_{(k_1, \ldots, k_r) \in \{1, \ldots, n\}^r, \gcd(k_1, \ldots, k_r) \text{ regular (mod } n)} 1 = \sum_{d|n} \sum_{(k_1, \ldots, k_r) \in \{1, \ldots, n\}^r, \gcd(k_1, \ldots, k_r, n) = d} 1 = \sum_{d|n} \sum_{(\ell_1, \ldots, \ell_r) \in \{1, \ldots, n/d\}^r, \gcd(\ell_1, \ldots, \ell_r, n/d) = 1} 1,$$

where the inner sum is $\varrho_r(n/d)$, according to its definition.

ii) Follows at once by i). \qed

More generally, for a fixed real number $s$ let $\phi_s(n) = \sum_{d|n} d^s \mu(n/d)$ be the generalized Jordan function and define $\varrho_s$ by

$$\varrho_s(n) = \sum_{d|n} \varrho_s(d) \quad (n \in \mathbb{N}). \tag{2}$$

The functions $\phi_s$ and $\varrho_s$ (which will be used in the next results of the paper) are multiplicative and for every prime power $p^\nu$ ($\nu \in \mathbb{N}$) one has $\phi_s(p^\nu) = p^{s \nu} - p^{s(\nu-1)}$ and $\varrho_s(p^\nu) = p^{s \nu} - p^{s(\nu-1)} + 1$. Note that $\phi_{-s}(n) = n^{-s} \prod_{p^\nu|n} (1 - p^s)$ and $\varrho_{-s}(n) = n^{-s} \prod_{p^\nu|n} (p^{s \nu} - p^s + 1)$.

**Proposition 2.** If $s > 1$ is a real number, then

$$\sum_{n \leq x} \varrho_s(n) = \frac{x^{s+1}}{s+1} \prod_p \left(1 - \frac{1}{p^{s+1}} + \frac{p - 1}{p(p^{s+1} - 1)}\right) + O(x^s). \tag{3}$$

**Proof.** We need the following asymptotics. Let $s > 0$ be fixed real number. Then uniformly for real $x > 1$ and $t \in \mathbb{N}$,

$$\phi_s(x, t) := \sum_{n \leq x, \gcd(n, t) = 1} \phi_s(n) = \frac{x^{s+1}}{(s+1) \zeta(s+1)} \cdot \frac{t^s \phi(t)}{\phi_{s+1}(t)} + O(x^s 2^{\omega(t)}). \tag{4}$$
To show (4) use the known estimate, valid for every fixed $s > 0$ and $t \in \mathbb{N}$,

$$\sum_{\substack{n \leq x \\ \gcd(n,t) = 1}} n^s = \frac{x^{s+1}}{s+1} \cdot \frac{\phi(t)}{t} + O \left( x^s 2^{\omega(t)} \right).$$

(5)

We obtain

$$\phi_s(x, t) = \sum_{\substack{de=n \leq x \\ \gcd(d,t)=1}} \mu(d) e^s = \sum_{d \leq x} \mu(d) \sum_{\substack{e \leq x/d \\ \gcd(e,t)=1}} e^s = \sum_{d \leq x} \mu(d) \left( \frac{\phi_s(x/d, t)}{s+1} \right) + O \left( \frac{\phi_s(x/d, t)}{s+1} \right),$$

$$= \frac{x^{s+1}}{s+1} \cdot \frac{\phi(t)}{t} \sum_{d \leq x} \frac{\mu(d)}{d^{s+1}} + O \left( x^{s+1} \sum_{d \leq x} \frac{1}{d^{s+1}} \right) + O \left( x^s 2^{\omega(t)} \right),$$

giving (4). Now from (2) and (4),

$$\sum_{n \leq x} \varrho_s(n) = \sum_{\substack{de=n \leq x \\ \gcd(d,e)=1}} \phi_s(e) = \sum_{d \leq x} \phi_s(x/d, d) = \sum_{d \leq x} \phi_s(x/d, d)$$

$$= \frac{x^{s+1}}{(s+1)\zeta(s+1)} \sum_{d=1}^{\infty} \frac{\phi(d)}{d \phi_s(d)} + O \left( x^{s+1} \sum_{d \leq x} \frac{\phi(d)}{d \phi_s(d)} \right) + O \left( x^s \sum_{d \leq x} \frac{2^{\omega(d)}}{d^{s-1}} \right),$$

and for $s > 1$ this leads to (3). \square

Compare (3) to the corresponding formula for the Jordan function $\phi_s$, i.e., to (4) with $t = 1$.

**Remark 1.** For the function $\varrho$ one has

$$\sum_{n \leq x} \varrho(n) = \frac{1}{2} \prod_{p} \left( 1 - \frac{1}{p^2(p+1)} \right) x^2 + R(x),$$

where $R(x) = O(x \log^3 x)$ can be obtained by the elementary arguments given above. This can be improved into $R(x) = O(x \log x)$ using analytic methods. See [18] for references.

### 3 A general scheme

In order to give exact formulas for certain sums and products over the regular integers (mod $n$) we first present a simple result for a general sum over $\text{Reg}_n$, involving a weight function $w$ and an arithmetic function $f$. It would be possible to consider a more general sum, namely over the ordered $r$-tuples $(k_1, \ldots, k_r) \in \{1, \ldots, n\}^r$ such that $\gcd(k_1, \ldots, k_r)$ is regular (mod $n$), but we confine ourselves to the following result. See [22] for another similar scheme concerning weighted gcd-sum functions.
Proposition 3.  
i) Let $w : \mathbb{N}^2 \to \mathbb{C}$ and $f : \mathbb{N} \to \mathbb{C}$ be arbitrary functions and consider the sum

$$R_{w,f}(n) := \sum_{k \in \text{Reg}_n} w(k, n) f(\gcd(k, n)).$$

Then

$$R_{w,f}(n) = \sum_{d \mid n} f(d) \sum_{\substack{j=1 \\gcd(j,n/d)=1}}^{n/d} w(dj, n) \quad (n \in \mathbb{N}). \quad (6)$$

ii) Assume that there is a function $g : (0, 1] \to \mathbb{C}$ such that $w(k, n) = g(k/n)$ ($1 \leq k \leq n$) and let

$$G(n) = \sum_{k=1}^{n} g(k/n) \quad (n \in \mathbb{N}).$$

Then

$$R_{w,f}(n) = \sum_{d \mid n} f(d) \overline{G}(n/d) \quad (n \in \mathbb{N}). \quad (7)$$

Proof.  
i) Using that $k$ is regular (mod $n$) if and only if $\gcd(k,n) || n$ and grouping the terms according to the values of $\gcd(k,n) = d$ and denoting $k = dj$ we have at once

$$R_{w,f}(n) = \sum_{d \mid n} f(d) \sum_{\substack{j=1 \\gcd(j,n/d)=1}}^{n/d} w(dj, n).$$

ii) Now

$$\sum_{j=1}^{n/d} w(dj, n) = \sum_{j=1}^{n/d} g(j/(n/d)) = \overline{G}(n/d).$$

$\square$

Remark 2. For the function $g$ given above let

$$G(n) := \sum_{k=1}^{n} g(k/n).$$

Then we have

$$\overline{G}(n) = \sum_{d \mid n} \mu(d) G(n/d) \quad (n \in \mathbb{N}). \quad (8)$$

Indeed, as it is well known, $\overline{G}(n) = \sum_{k=1}^{n} g(k/n) \sum_{d \mid \gcd(k,n)} \mu(d)$, giving $\square$. 

5
4 Power sums of regular integers (mod $n$)

In this section we investigate the sum of $r$-th powers ($r \in \mathbb{N}$) of the regular integers (mod $n$). Let $B_m$ ($m \in \mathbb{N}_0$) be the Bernoulli numbers defined by the exponential generating function

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}.$$ 

Here $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $B_m = 0$ for every $m \geq 3$, $m$ odd and one has the recurrence relation

$$B_m = \sum_{j=0}^{m} \binom{m}{j} B_j \quad (m \geq 2).$$ (9)

It is well known that for every $n,r \in \mathbb{N}$,

$$S_r(n) := \sum_{k=1}^{n} k^r = \frac{1}{r+1} \sum_{m=0}^{r} (-1)^m \binom{r+1}{m} B_m n^{r+1-m}$$

$$= \frac{n^r}{2} + \frac{1}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \binom{r+1}{2m} B_{2m} n^{r+1-2m}. \quad (10)$$

From here one obtains, using the same device as that given in Remark 2 that for every $n,r \in \mathbb{N}$ with $n \geq 2$,

$$S[\text{relpr}]_r(n) := \sum_{\gcd(k,n)=1}^{n} k^r = \frac{n^r}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \binom{r+1}{2m} B_{2m} \phi_{1-2m}(n), \quad (11)$$

where $\phi_{1-2m}(n) = n^{1-2m} \prod_{p|n} (1 - p^{2m-1})$. Formula (11) was given in [17]. Here we prove the following result.

**Proposition 4.** For every $n,r \in \mathbb{N}$,

$$S[\text{reg}]_r(n) := \sum_{k \in \text{Reg}_n} k^r = \frac{n^r}{2} + \frac{n^r}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \binom{r+1}{2m} B_{2m} \varrho_{1-2m}(n), \quad (12)$$

where

$$\varrho_{1-2m}(n) = n^{1-2m} \prod_{p^{\nu} || n} \left( p^{(2m-1)\nu} - p^{2m-1} + 1 \right)$$

is the generalized $\varrho$ function, discussed in Section 2.

**Proof.** Applying (10) for $w(k,n) = k^r$ and $f = 1$ we have

$$S[\text{reg}]_r(n) = \sum_{d||n} \sum_{\gcd(j,n/d)=1}^{n/d} (dj)^r = \sum_{d||n} d^r S[\text{relpr}]_r(n/d).$$
Now by (11) we deduce

\[
S_{\text{reg}}(n) = n^r + \sum_{d \mid n, d < n} d^r \left( \frac{(n/d)^r}{r+1} \right) \sum_{m=0}^{[r/2]} \left( \frac{r+1}{2m} \right) B_{2m} \phi_{1-2m}(n/d)
\]

\[
= n^r + \frac{n^r}{r+1} \sum_{m=0}^{[r/2]} \left( \frac{r+1}{2m} \right) B_{2m} \phi_{1-2m}(n/d)
\]

\[
= n^r + \frac{n^r}{r+1} \sum_{m=0}^{[r/2]} \left( \frac{r+1}{2m} \right) B_{2m} \phi_{1-2m}(d)
\]

\[
= n^r - \frac{n^r}{r+1} \sum_{m=0}^{[r/2]} \left( \frac{r+1}{2m} \right) B_{2m} + \frac{n^r}{r+1} \sum_{m=0}^{[r/2]} \left( \frac{r+1}{2m} \right) B_{2m} \phi_{1-2m}(d).
\]

Here \( \sum_{d \mid n} \phi_{1-2m}(n) = q_{1-2m}(d) \) by (2). Also, by (9),

\[
\sum_{m=0}^{[r/2]} \left( \frac{r+1}{2m} \right) B_{2m} = \frac{r+1}{2}
\]

and this completes the proof.

For example, in the cases \( r = 1, 2, 3, 4 \) we deduce that for every \( n \in \mathbb{N} \),

\[
S_{\text{reg}}(n) = n \left( \frac{\varrho(n) + 1}{2} \right),
\]

(13)

\[
S_{\text{reg}}(n) = \frac{n^2}{2} + \frac{n^2 \varrho(n)}{3} + \frac{n}{6} \prod_{p^\nu \mid n} (p^\nu - p + 1),
\]

(14)

\[
S_{\text{reg}}(n) = \frac{n^3}{2} + \frac{n^3 \varrho(n)}{4} + \frac{n^2}{4} \prod_{p^\nu \mid n} (p^\nu - p + 1),
\]

\[
S_{\text{reg}}(n) = \frac{n^4}{2} + \frac{n^4 \varrho(n)}{5} + \frac{n^3}{3} \prod_{p^\nu \mid n} (p^\nu - p + 1) - \frac{n}{30} \prod_{p^\nu \mid n} (p^{3\nu} - p^3 + 1).
\]

The formula (13) was obtained in [18, Th. 3] and [3, Sec. 2], while (14) was given in a different form in [3, Prop. 1]. Note that if \( n \) is squarefree, then (12) reduces to (10).

For a real number \( s \) consider now the slightly more general sum

\[
S_{\text{reg}}(n, x) := \sum_{k \leq x} k^s.
\]

\[
k \text{ regular (mod } n)\]
Proposition 5. Let \( s \geq 0 \) be a fixed real number. Then uniformly for real \( x > 1 \) and \( n \in \mathbb{N} \),
\[
S[\text{reg}]_s(n, x) = \frac{x^{s+1}}{s+1} \cdot \frac{\varphi(n)}{n} + O\left(\frac{x^s 3^{s \omega(n)}}{n}\right).
\]

Proof. Similar to the proof of Proposition 3,
\[
S[\text{reg}]_s(n, x) = \sum_{k \leq x, \gcd(k, n) || n} k^s = \sum_d d^s \sum_{j \leq x/d, \gcd(j/d, n) = 1} j^s.
\]

Now using the estimate (5) we deduce
\[
T_r(n, x) = \sum_{d || n} d^s \left(\frac{(x/d)^{s+1} \phi(n/d)}{(s+1)(n/d)} + O\left(\frac{x^s 3^{s \omega(n/d)}}{n}\right)\right)
\]
\[
= \frac{x^{s+1}}{(s+1)n} \sum_{d || n} \phi(n/d) + O \left(\frac{x^s \sum_{d || n} 2^{\omega(n/d)}}{n}\right),
\]
and using (1) the proof is complete. \(\square\)

5 Identities for other sums and products over regular integers \( \text{(mod } n) \)

5.1 Sums involving Bernoulli polynomials

Let \( B_m(x) \ (m \in \mathbb{N}_0) \) be the Bernoulli polynomials defined by
\[
\frac{te^{xt}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!}.
\]

Here \( B_0(x) = 1, B_1 = x - 1/2, B_2 = x^2 - x + 1/6, B_3 = x^3 - 3x/2 + x/2, B_m(0) = B_m \) \((m \in \mathbb{N}_0)\) are the Bernoulli numbers already defined in Section 4 and one has the recurrence relation
\[
B_m(x) = \sum_{j=0}^{m} \binom{m}{j} B_j x^{m-j} \quad (m \in \mathbb{N}_0).
\]

It is well known (see, e.g., [5, Sect. 9.1]) that for every \( n, m \in \mathbb{N}, m \geq 2 \),
\[
T_m(n) := \sum_{k=1}^{n} B_m(k/n) = \frac{B_m}{n^{m-1}}. \tag{15}
\]

Furthermore, applying (8) one obtains from (15) that for every \( n, m \in \mathbb{N}, m \geq 2 \),
\[
T[\text{relpr}]_m(n) := \sum_{k=1, \gcd(k, n) = 1}^{n} B_m(k/n) = B_m \phi_{1-m}(n), \tag{16}
\]
where \( \phi_{m-1}(n) = n^{1-m} \prod_{p | n} (1 - p^{m-1}) \). See [5, Sect. 9.9, Ex. 7]. We now show the validity of the next formula:
Proposition 6. For every \( n, m \in \mathbb{N}, m \geq 2 \),
\[
T_{\text{reg}}^m(n) := \sum_{k \in \text{Reg}_n} B_m(k/n) = B_m \varrho_{1-m}(n), \quad (17)
\]
where \( \varrho_{1-m}(n) = n^{1-m} \prod_{\nu \mid n} (p^{(m-1)\nu} - p^{m-1} + 1) \).

Proof. Choosing \( g(x) = B_m(x) \) and \( f = 1 \) we deduce from (7) by using (16) that
\[
T_{\text{reg}}^m(n) = \sum_{d \mid n} T_{\text{relpr}}^m(d)
= B_m \sum_{d \mid n} \varphi_{1-m}(d) = B_m \varrho_{1-m}(n),
\]
according to (2). \( \square \)

Remark 3. In the case \( m = 1 \) a direct computation and (13) show that \( T_{\text{reg}}^1(n) = 1/2 \). Also, (17) can be put in the form
\[
\sum_{k=0}^{n-1} B_m(k/n) = B_m \varrho_{1-m}(n),
\]
which holds true for every \( n, m \in \mathbb{N} \), also for \( m = 1 \).

5.2 Sums involving \( \gcd \)'s and the \( \exp \) function

Consider in what follows the function
\[
P_{\text{reg}}^{f,t}(n) := \sum_{k \in \text{Reg}_n} f(\gcd(k,n)) \exp(2\pi ikt/n) \quad (n \in \mathbb{N}, t \in \mathbb{Z}),
\]
where \( f \) is an arbitrary arithmetic function. For \( t = 0 \) and \( f(n) = n \ (n \in \mathbb{N}) \) we reobtain the function \( \overline{P}(n) \) and for \( f = 1 \) we have \( \overline{\tau}_n(t) \), the analogue of the Ramanujan sums, both given in the Introduction. We have

Proposition 7. For every \( f \) and every \( n \in \mathbb{N} \) and \( t \in \mathbb{Z} \),
\[
P_{\text{reg}}^{f,t}(n) = \sum_{d \mid n} f(d)c_{n/d}(t).
\]

If \( f \) is integer valued and multiplicative (in particular, if \( f = \text{id} \)), then \( n \mapsto P_{\text{reg}}^{f,t}(n) \) also has these properties.

Proof. Choosing \( g(x) = \exp(2\pi i tx/n) \) from (7) we deduce at once that
\[
P_{\text{reg}}^{f,t}(n) = \sum_{d \mid n} f(d) \sum_{j=1}^{n/d} \exp(2\pi ijt/(n/d)) = \sum_{d \mid n} f(d)c_{n/d}(t).
\] \( \square \)
For \( t = 1 \) and \( f = \text{id} \) this gives the multiplicative function

\[
P_{\text{id,1}}(n) = \sum_{d \mid n} d\mu(n/d),
\]

not investigated in the literature, as far as we know. Here \( P_{\text{id,1}}(p^\nu) = p - 1 \) for every prime \( p \) and \( P_{\text{id,1}}(p^\nu) = p^\nu \) for every prime power \( p^\nu \) with \( \nu \geq 2 \).

**Proposition 8.** We have

\[
\sum_{n \leq x} P_{\text{id,1}}(n) = \frac{x^2}{2} \prod_{p} \left( 1 - \frac{1}{p^2} + \frac{1}{p^3} \right) + O(x \log^2 x).
\]

**Proof.** Using (5) for \( s = 1 \) we deduce

\[
\sum_{n \leq x} P_{\text{id,1}}(n) = \sum_{d \leq x} \mu(d) \sum_{\delta \leq x/d} \frac{\phi(d)}{\delta^2} + O((x/d)2^{\omega(d)})
\]

\[
= \frac{x^2}{2} \sum_{d=1}^{\infty} \frac{\mu(d)\phi(d)}{d^3} + O \left( x^2 \sum_{d>x} \frac{1}{d^2} \right) + O \left( x \sum_{d \leq x} \frac{2^{\omega(d)}}{d} \right),
\]

giving the result. \( \square \)

### 5.3 An analogue of Menon’s identity

Our next result is the analogue of Menon’s identity ([10], see also [21])

\[
\sum_{k=1 \atop \gcd(k,n)=1}^{n} \gcd(k-1,n) = \phi(n)\tau(n) \quad (n \in \mathbb{N}).
\]

**Proposition 9.** For every \( n \in \mathbb{N} \),

\[
\sum_{k \in \text{Reg}_n} \gcd(k-1,n) = \sum_{d \mid n} \phi(d)\tau(d) = \prod_{p^\nu \mid n} (p^{\nu-1}(p-1)(\nu+1)+1).
\]

**Proof.** Applying (6) for \( w(k,n) = \gcd(k-1,n) \) and \( f = 1 \) we deduce

\[
S_n := \sum_{k \in \text{Reg}_n} \gcd(k-1,n) = \sum_{d \mid n} \sum_{j=1 \atop \gcd(j,n/d)=1}^{n/d} \gcd(dj-1,n)
\]

\[
= \sum_{d \mid n} \sum_{j=1 \atop \gcd(j,n/d)=1}^{n/d} \gcd(dj-1,n/d),
\]
since $\gcd(dj - 1, d) = 1$ for every $d$ and $j$. Now we use the identity
\[
\sum_{k=1}^{\frac{n}{\gcd(k,n)}} \gcd(ak - 1, n) = \phi(n)\tau(n) \quad (n \in \mathbb{N}),
\]
valid for every fixed $a \in \mathbb{N}$ with $\gcd(a, n) = 1$, see [21, Cor. 14] (for $a = 1$ this reduces to [18]). Choose $a = d$. Since $d \mid n$ we have $\gcd(d, n/d) = 1$ and obtain
\[
S_n = \sum_{d \mid n} \phi(n/d)\tau(n/d) = \sum_{d \mid n} \phi(d)\tau(d).
\]

5.4 Trigonometric sums

Further identities for sums over $\text{Reg}_n$ can be derived. As examples, consider the following known trigonometric identities. For every $n \in \mathbb{N}$, $n \geq 2$,
\[
\sum_{k=1}^{n} \cos^2 \left( \frac{k\pi}{n} \right) = \frac{n}{2},
\]
furthermore, for every $n \in \mathbb{N}$ odd number,
\[
\sum_{k=1}^{n} \tan^2 \left( \frac{k\pi}{n} \right) = n^2 - n,
\]
and also for every $n \in \mathbb{N}$ odd,
\[
\sum_{k=1}^{n} \tan^4 \left( \frac{k\pi}{n} \right) = \frac{1}{3}(n^4 - 4n^2 + 3n).
\]

See, for example, [4] for a discussion and proofs of these identities. See [14, Appendix 3] for other similar identities. By the approach given in Remark 2 we deduce that for every $n \in \mathbb{N}$,
\[
\sum_{k=1}^{n} \frac{\cos^2 \left( \frac{k\pi}{n} \right)}{\gcd(k,n)=1} = \frac{\phi(n) + \mu(n)}{2},
\]
for every $n \in \mathbb{N}$ odd number,
\[
\sum_{k=1}^{n} \frac{\tan^2 \left( \frac{k\pi}{n} \right)}{\gcd(k,n)=1} = \phi_2(n) - \phi(n),
\]
and for every $n \in \mathbb{N}$ odd,
\[
\sum_{k=1}^{n} \frac{\tan^4 \left( \frac{k\pi}{n} \right)}{\gcd(k,n)=1} = \frac{1}{3}(\phi_4(n) - 4\phi_2(n) + 3\phi(n)).
\]

This gives the next results. The proof is similar to the proofs given above.
Proposition 10. For every \( n \in \mathbb{N} \),
\[
\sum_{k \in \text{Reg}_n} \cos^2 \left( \frac{k\pi}{n} \right) = \frac{\phi(n) + \mu(n)}{2},
\]
where \( \mu(n) = \sum_{d|n} \mu(d) \) is the characteristic function of the squarefull integers \( n \), given in the Introduction.

Proposition 11. For every \( n \in \mathbb{N} \) odd number,
\[
\sum_{k \in \text{Reg}_n} \tan^2 \left( \frac{k\pi}{n} \right) = \rho_2(n) - \phi(n),
\]
\[
\sum_{k \in \text{Reg}_n} \tan^4 \left( \frac{k\pi}{n} \right) = \frac{1}{3}(\rho_4(n) - 4\rho_2(n) + 3\phi(n)).
\]

5.5 The product of numbers in \( \text{Reg}_n \)

It is known (see, e.g., [14, p. 197, Ex. 24]) that for every \( n \in \mathbb{N} \),
\[
\mathcal{Q}[\text{relpr}](n) := \prod_{\substack{k = 1 \\ \gcd(k,n) = 1}}^{n} k = n^{\phi(n)} A(n),
\]
where
\[
A(n) = \prod_{d|n} (d!/d^n)^{\mu(n/d)}.
\]

We show that

Proposition 12. For every \( n \in \mathbb{N} \),
\[
\mathcal{Q}[\text{reg}](n) := \prod_{k \in \text{Reg}_n} k = n^\phi(n) \prod_{d|n} A(d).
\]

Proof. Choosing \( w(k, n) = \log k \) and \( f = 1 \) in Proposition 3, we have
\[
\log \mathcal{Q}[\text{reg}](n) = \sum_{k \in \text{Reg}_n} \log k = \sum_{d|n} \sum_{\substack{j=1 \\ \gcd(j,n/d) = 1}}^{n/d} \log(dj)
\]
\[
= \sum_{d|n} (\phi(n/d) \log d + \log \mathcal{Q}[\text{relpr}](n/d))
\]
\[
= \sum_{d|n} (\phi(d) \log(n/d) + \log \mathcal{Q}[\text{relpr}](d))
\]
\[
= (\log n) \sum_{d|n} \phi(d) - \sum_{d|n} \phi(d) \log d + \sum_{d|n} \log \mathcal{Q}[\text{relpr}](d).
\]

Hence,
\[
\mathcal{Q}[\text{reg}](n) = n^\phi(n) \prod_{d|n} \frac{\mathcal{Q}[\text{relpr}](d)}{d^\phi(d)}.
\]

Now the result follows from the identity [19].
5.6 Products involving the Gamma function

Let $\Gamma$ be the Gamma function defined for $x > 0$ by
$$\Gamma(x) = \int_0^\infty e^{-t^x} \, dt.$$  

It is well known that for every $n \in \mathbb{N},$
$$R(n) := \prod_{k=1}^n \Gamma(k/n) = \frac{(2\pi)^{n-1/2}}{\sqrt{n}}, \quad (20)$$
which is a consequence of Gauss’ multiplication formula. Furthermore, for every $n \in \mathbb{N}, n \geq 2,$
$$R[relpr](n) := \prod_{\gcd(k,n)=1} \Gamma(k/n) = \frac{(2\pi)^{\phi(n)/2}}{\exp(\Lambda(n)/2)}, \quad (21)$$
see [15, 9].

**Proposition 13.** For every $n \in \mathbb{N},$
$$R[reg](n) := \prod_{k \in \text{Reg}_n} \Gamma(k/n) = \frac{(2\pi)^{\phi(n)/2}}{\sqrt{\kappa(n)}}. \quad (22)$$

**Proof.** Choosing $g = \log \Gamma$ and $f = 1$ in (7) and using (21) we deduce
$$\log R[reg](n) = \sum_{k \in \text{Reg}_n} \log \Gamma(k/n) = \sum_{d \mid n} \log R[relpr](d)$$
$$= \sum_{d \mid n} \left( \frac{\log 2\pi}{2} \phi(d) - \frac{1}{2} \Lambda(d) \right)$$
$$= \sum_{d \mid n} \left( \frac{\log 2\pi}{2} \phi(d) - \frac{1}{2} \Lambda(d) \right) - \frac{\log 2\pi}{2} = \frac{\log 2\pi}{2} (\phi(n) - 1) - \frac{1}{2} \sum_{d \mid n} \Lambda(d),$$
where the last sum is $\log \kappa(n).$ \hfill \Box

For squarefree $n$ (22) reduces to (20).

5.7 Identities involving cyclotomic polynomials

Let $\Phi_n(x) \ (n \in \mathbb{N})$ stand for the cyclotomic polynomials (see, e.g., [7, Ch. 13]) defined by
$$\Phi_n(x) = \prod_{k=1}^n (x - \exp(2\pi ik/n)).$$

Consider now the following analogue of the cyclotomic polynomials $\Phi_n(x):$
$$\Phi[reg]_n(x) = \prod_{k \in \text{Reg}_n} (x - \exp(2\pi ik/n)).$$

The application of Proposition 3 gives the following result.
Proposition 14. For every \( n \in \mathbb{N} \),

\[
\Phi_{\text{reg}}(x) = \prod_{d \mid n} \Phi_d(x).
\]

Here the degree of \( \Phi_{\text{reg}}(x) \) is \( \rho(n) \). If \( n \) is squarefree, then \( \Phi_{\text{reg}}(x) = x^n - 1 \) and for example, \( \Phi_{\text{reg}}_{12}(x) = \Phi_1(x)\Phi_3(x)\Phi_4(x)\Phi_{12}(x) = x^9 - x^6 + x^3 - 1 \).

It is well known that for every \( n \in \mathbb{N}, n \geq 2 \),

\[
U(n) := \prod_{\substack{k=1 \atop \gcd(k,n)=1}}^{n} \sin \left( \frac{k\pi}{n} \right) = \frac{\Phi_n(1)}{2^{\phi(n)}}, \tag{23}
\]

where

\[
\Phi_n(1) = \begin{cases} 
  p, & n = p^\nu, \nu \geq 1 \\
  1, & \text{otherwise, i.e., if } \omega(n) \geq 2,
\end{cases}
\]

and for \( n \geq 3 \),

\[
V(n) := \prod_{\substack{k=1 \atop \gcd(k,n)=1}}^{n} \cos \left( \frac{k\pi}{n} \right) = \frac{\Phi_n(-1)}{(-4)^{\phi(n)/2}}. \tag{24}
\]

where

\[
\Phi_n(-1) = \begin{cases} 
  2, & n = 2^\nu, \\
  p, & n = 2p^\nu, p > 2 \text{ prime}, \nu \geq 1 \\
  1, & \text{otherwise}.
\end{cases}
\]

For every \( n \in \mathbb{N}, \prod_{k \in \text{Reg}_n} \sin(k\pi/n) = 0 \), since \( n \in \text{Reg}_n \). This suggests to consider also the modified products

\[
U[\text{regmod}](n) := \prod_{k=1 \atop \text{k regular (mod } n)}^{n-1} \sin \left( \frac{k\pi}{n} \right),
\]

\[
V[\text{regmod}](n) := \prod_{k=1 \atop \text{k regular (mod } n)}^{n-1} \cos \left( \frac{k\pi}{n} \right).
\]

We show that \( U[\text{regmod}](n) \) is nonzero for every \( n \geq 2 \). More precisely, define the modified polynomials

\[
\Phi[\text{regmod}](x) = (x - 1)^{-1} \Phi[\text{reg}](x) = \prod_{d \mid n, d > 1} \Phi_d(x).
\]

Here, for example, \( \Phi[\text{regmod}]_{12}(x) = \Phi_3(x)\Phi_4(x)\Phi_{12}(x) = x^8 + x^7 + x^6 + x^2 + x + 1 \). All of the polynomials \( \Phi[\text{regmod}](x) \) have symmetric coefficients. By arguments similar to those leading to the formulas (23) and (24) we obtain the following identities.

Proposition 15. For every \( n \in \mathbb{N}, n \geq 2 \),

\[
U[\text{regmod}](n) = \frac{\Phi[\text{regmod}](n)}{2^{\phi(n)-1}} = \frac{\kappa(n)}{2^{\phi(n)-1}},
\]

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and for every $n \in \mathbb{N}$, $n \geq 3$ odd,

$$V[\text{regmod}](n) = \frac{\Phi[\text{regmod}](n)(-1)}{(-4)^{(\gamma(n)-1)/2}} = (-1/4)^{(\gamma(n)-1)/2}.$$  

Note that $\rho(n)$ is odd for every $n \in \mathbb{N}$ odd.

6 Maximal orders of certain functions

Let $\sigma(n)$ be the sum of divisors of $n$ and let $\psi(n) = n \prod_{p|n} (1 + 1/p)$ be the Dedekind function. The following open problems were formulated in [2]: What are the maximal orders of the functions $\rho(n)\sigma(n)$ and $\rho(n)\psi(n)$?

The answer is the following:

Proposition 16.

$$\limsup_{n \to \infty} \frac{\rho(n)\sigma(n)}{n^2 \log \log n} = \limsup_{n \to \infty} \frac{\rho(n)\psi(n)}{n^2 \log \log n} = \frac{6}{\pi^2} e^\gamma,$$

where $\gamma$ is the Euler-Mascheroni constant.

Proof. Apply the following general result, see [23 Cor. 1]: If $f$ is a nonnegative real-valued multiplicative arithmetic function such that for each prime $p$,

i) $\rho(p) := \sup_{\nu \geq 0} f(p^\nu) \leq (1 - 1/p)^{-1}$, and

ii) there is an exponent $e_p = p^{e(1)} \in \mathbb{N}$ satisfying $f(p^{e_p}) \geq 1 + 1/p$,

then

$$\limsup_{n \to \infty} \frac{f(n)}{\log \log n} = e^\gamma \prod_p \left(1 - \frac{1}{p^2}\right) \rho(p).$$

Take $f(n) = \rho(n)\sigma(n)/n^2$. Here $f(p) = 1 + 1/p$ and $f(p^\nu) = 1 + 1/p + 1/p^2 + 1/p^{\nu+3} + \ldots + 1/p^{2\nu} < 1 + 1/p$ for every prime $p$ and every $\nu \geq 2$. This shows that $\rho(p) = 1 + 1/p$ and obtain that

$$\limsup_{n \to \infty} \frac{f(n)}{\log \log n} = e^\gamma \prod_p \left(1 - \frac{1}{p^2}\right) = \frac{6}{\pi^2} e^\gamma.$$

The proof is similar for the function $g(n) = \rho(n)\psi(n)/n^2$. In fact, $g(p) = f(p) = 1 + 1/p$ and $g(p^\nu) \leq f(p^\nu)$ for every prime $p$ and every $\nu \geq 2$, therefore the result for $g(n)$ follows from the previous one. \(\square\)

Remark 4. Let $\sigma_s(n) = \sum_{d|n} d^s$. Then for every real $s > 1$,

$$\limsup_{n \to \infty} \frac{\sigma_s(n)\sigma(n)}{n^{2s}} = \frac{\zeta(s)}{\zeta(2s)},$$

where $\zeta$ is the Riemann zeta function, as usual. This follows by observing that for $f_s(n) = \sigma_s(n)\sigma(n)/n^{2s}$, $f_s(p) = 1 + 1/p^s$ and $f_s(p^\nu) = 1 + 1/p + 1/p^{s+1} + 1/p^{s+2} + \ldots + 1/p^{s\nu} < 1 + 1/p^s$ for every prime $p$ and every $\nu \geq 2$. Hence, for every $n \in \mathbb{N}$,

$$f_s(n) \leq \prod_{p|n} \left(1 + \frac{1}{p^s}\right) < \prod_{p} \left(1 + \frac{1}{p^s}\right) = \frac{\zeta(s)}{\zeta(2s)},$$

and the lim sup is attained for $n = n_k = \prod_{1 \leq j \leq k} p_j$ with $k \to \infty$, where $p_j$ is the $j$-th prime.
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