ADDITIVITY VIOLATION OF THE REGULARIZED MINIMUM OUTPUT ENTROPY

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ABSTRACT. The problem of additivity of the Minimum Output Entropy is of fundamental importance in Quantum Information Theory (QIT). It was solved by Hastings [Has09] in the one-shot case, by exhibiting a pair of random quantum channels. However, the initial motivation was arguably to understand regularized quantities and there was so far no way to solve additivity questions in the regularized case. The purpose of this paper is to give a solution to this problem. Specifically, we exhibit a pair of quantum channels which unearths additivity violation of the regularized minimum output entropy. Unlike previously known results in the one-shot case, our construction is non-random, infinite dimensional and in the commuting-operator setup. The commuting-operator setup is equivalent to the tensor-product setup in the finite dimensional case for this problem, but their difference in infinite dimensional setting has attracted substantial attention and legitimacy recently in QIT with the celebrated resolutions of Tsirelson’s and Connes embedding problem [JNV+20]. Likewise, it is not clear that our approach works in the finite dimensional setup. Our strategy of proof relies on developing a variant of the Haagerup inequality optimized for a product of free groups.

1. INTRODUCTION

A crucial problem in quantum information theory is the problem of additivity of Minimum Output Entropy (MOE), which asks whether it is possible to find two quantum channels $\Phi_1, \Phi_2$ such that

$$H_{\text{min}}(\Phi_1 \otimes \Phi_2) < H_{\text{min}}(\Phi_1) + H_{\text{min}}(\Phi_2).$$

This problem was stated by King-Ruskai in [KR01] as a natural question in the study of quantum channels. Shor proved in 2004 [Sho04b, Sho04a] that a positive answer to the above question is equivalent to super-additivity of the Holevo capacity, i.e. there exist quantum channels $\Phi_1, \Phi_2$ such that

$$\chi(\Phi_1 \otimes \Phi_2) > \chi(\Phi_1) + \chi(\Phi_2).$$

Heuristically, super-additivity of the Holevo capacity implies that entanglement inputs can be used to increase the transmission rate of classical information. We refer to Section 2.1 for the definitions of the MOE $H_{\text{min}}$ and the Holevo capacity $\chi$. This question attracted lots of attention, and it
was eventually solved by Hastings in 2009 [Has09], with preliminary substantial contributions by Hayden, Winter, Werner, see in particular [HW08]. Subsequently, the mathematical aspects of the proof have been clarified in various directions by [ASW11, FKM10, BaH10, BCN16, Col18, CFZ15].

All previously known examples of additivity violation of MOE rely on subtle random constructions. In particular, to date, no deterministic construction of additivity violation has ever been given. For attempts and partial results in the direction of non-random techniques we refer to [WH02, GHP10, BCLY20], etc.

Note that the above results do not imply anything about the problem of the additivity of the regularized MOE (see Definition 2.3 for details). Indeed, additivity violation is not known to pertain when the MOE is regularized. More precisely, the additivity question for the regularized MOE asks whether it is possible to find two quantum channels \( \Phi_1, \Phi_2 \) such that

\[
\overline{H}_{\text{min}}(\Phi_1 \otimes \Phi_2) < \overline{H}_{\text{min}}(\Phi_1) + \overline{H}_{\text{min}}(\Phi_2),
\]

where \( \overline{H}_{\text{min}} \) stands for the regularized MOE. This question was raised in [Fuk14] and the affirmative answer to this implies superadditivity of classical capacity.

Very few results are known about regularized entropic quantities – see for example [Kin02] or [BCLY20] for partial results. In this paper, we focus on the additivity question of the regularized minimum output entropy, and the tensor product channel will be understood as a composition of two quantum channels whose systems of Kraus operators are commuting (see Section 2.2 for details).

In (quantum) information theory, one key paradigm is to allow repeated uses of a given quantum channel. To do this, we have to analyze a physical system by separated subsystems. In view of quantum strategies for non-local games, there are two natural models to describe separated subsystems. One is the tensor-product model and the other is commuting-operator model. This latter approach is the object of intense research, see for example [PT15, DP16, CLS17, Slo20, CCLP18], culminating with the recent resolution [INV+20] in the negative of the celebrated Connes Embedding problem whose origin dates back to [Con76]. In our case, commuting systems of Kraus operators correspond to a commuting-operator model. We refer to Section 2.2 for details on this.

The main result of this paper is an explicit construction of a pair of quantum channels \( \Phi_1 \) and \( \Phi_2 \) which have commuting systems of Kraus operators and satisfy additivity violation of the regularized MOE. Specifically, our main result can be stated as follows:
Theorem 1.1. There exist systems of operators \( \{E_i\}_{i=1}^m \) and \( \{F_j\}_{j=1}^n \) in \( B(H) \) such that

1. \( E_i F_j = F_j E_i \) for all \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \),
2. \( \sum_{i=1}^m E_i^* E_i = \Id_H = \sum_{j=1}^n F_j^* F_j \),
3. \( \Phi_1, \Phi_2 : \mathcal{T}(H) \to \mathcal{T}(H) \) are quantum channels given by
   \[
   \Phi_1(\rho) = \sum_{i=1}^m E_i \rho E_i^* \quad \text{and} \quad \Phi_2(\rho) = \sum_{j=1}^n F_j \rho F_j^* ,
   \]
4. \( \overline{H}_{\min}(\Phi_1 \circ \Phi_2) < \overline{H}_{\min}(\Phi_1) + \overline{H}_{\min}(\Phi_2) \).

Note that the above discussion for the regularized MOE makes sense since the given channels are generated by finitely many Kraus operators, and given commuting systems will be chosen as an infinite dimensional analogue of i.i.d. Haar distributed unitary matrices, which will be explained in Section 2.2 and Theorem 4.1 in details. One of the biggest benefit from this shift in perspective is that the regularized minimum output entropy becomes computable, whereas for random unitary channels, computing such regularized quantities still seems to remain totally out of reach at this point.

One of the key ingredients is to extend the Haagerup inequality \([Haa79]\) to products of free groups (Proposition 3.2). This result itself is a crucial fact. Indeed, the Haagerup inequality has numerous applications in operator algebras, non-commutative harmonic analysis and geometric group theory \([Boz81, DCH85, Jol89, Laf00, Laf02]\).

This paper is organized as follows. After this introduction, Section 2 gathers some preliminaries about entropic quantities, quantum channels and the infinite dimensional framework. Section 3 contains the proof of a Haagerup-type inequality for products of free groups as well as estimates for the regularized Minimum Output Entropy of our main family of quantum channels. Section 4 explains how we can obtain additivity violation of the regularized MOE in the commuting operator setup, and Section 5 contains concluding remarks.

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2. Preliminaries

2.1. Minimum output entropy in infinite dimensional setting. Let $V : H_A \rightarrow H_B \otimes H_E$ be an isometry. Then partial traces on $H_B$ and $H_E$ define the following completely positive trace preserving maps (aka quantum channels)

\[
\Phi : \mathcal{T}(H_A) \rightarrow \mathcal{T}(H_B), \rho \mapsto (\text{id} \otimes \text{tr})(V \rho V^*) \tag{2.1}
\]

\[
\Phi^c : \mathcal{T}(H_A) \rightarrow \mathcal{T}(H_E), \rho \mapsto (\text{tr} \otimes \text{id})(V \rho V^*) \tag{2.2}
\]

where $\mathcal{T}(H)$ denotes the space of trace class operators on a Hilbert space $H$. The map $\Phi^c$ is called the complementary channel of $\Phi$. The tensor product channels $\Phi^\otimes k : \mathcal{T}((H_A^\otimes k) \rightarrow \mathcal{T}(H_{B}^\otimes k)$ are defined in the obvious way. A (quantum) state in $H$ is a positive element of $\mathcal{T}(H)$ of trace 1, and for a state $\rho$, its Rényi entropy for $p \in (1, \infty)$ is defined as

\[
H_p(\rho) = \frac{1}{1 - p} \log (\text{tr}(\rho^p)).
\]

Its limit as $p \rightarrow 1^+$ is called the von Neumann entropy, and if $\lambda_1(\rho) \geq \lambda_2(\rho) \geq \ldots$ are the eigenvalues of $\rho$ (counted with multiplicity), then the von Neumann entropy is

\[
H(\rho) = - \sum_i \lambda_i(\rho) \log \lambda_i(\rho).
\]

The Holevo capacity of a quantum channel is

\[
\chi(\Phi) = \sup \left\{ H(\Phi(\sum_i \lambda_i \rho_i) - \sum_i \lambda_i H(\Phi(\rho_i)) \right\},
\]

where the supremum is taken over all probability distributions $(p_i)_i$ and all families of states $(\rho_i)_i$. It describes the amount of classical information that can be carried through a single use of a quantum channel. If repeated uses of a given quantum channel is allowed, the ultimate transmission rate of classical information is described by

\[
C(\Phi) = \lim_{k \rightarrow \infty} \frac{1}{k} \chi(\Phi^\otimes k),
\]

which is called the classical capacity.
For a quantum channel $\Phi$, the Minimum Output Entropy (MOE) and the regularized MOE are defined as

$$H_{\text{min}}(\Phi) = \inf_{\xi} H(\Phi(|\xi\rangle\langle\xi|))$$

and

$$\overline{H}_{\text{min}}(\Phi) = \lim_{k \to \infty} \frac{1}{k} H_{\text{min}}(\Phi^{\otimes k}).$$

respectively, where the infimum runs over all unit vectors $\xi$ of $H_A$. If $\Phi$ has finitely many Kraus operators $\{E_1, E_2, \cdots, E_N\}$ satisfying $\Phi(\rho) = \sum_{i=1}^N E_i \rho E_i^*$ (e.g. if $H_E$ is finite dimensional), then

$$H_{\text{min}}(\Phi) + \chi(\Phi) \leq \log(N)$$

and

$$\overline{H}_{\text{min}}(\Phi) + C(\Phi) \leq \log(N).$$

**Remark 2.1.** In Equation (2.3), taking the minimum over all states instead of pure states does not modify the quantity thanks to operator convexity of the function $x \log(x)$, see e.g. [Seg60, NU61].

As in the finite dimensional setting, the following Schmidt decomposition theorem tells us that $\Phi(|\xi\rangle\langle\xi|)$ and $\Phi^c(|\xi\rangle\langle\xi|)$ have the same eigenvalues for each pure state $|\xi\rangle\langle\xi|$ $\in$ $T(H)$.

**Proposition 2.2.** Let $V : H_A \to H_B \otimes H_E$ be an isometry and $\xi \in H_A$ be a unit vector. If we suppose that $\Phi(|\xi\rangle\langle\xi|)$ has the spectral decomposition $\sum_i \lambda_i |e_i\rangle\langle e_i|$ with $\lambda_i > 0$, where $(e_i)_{i \in I}$ is an orthonormal subset of $H_B$, then there exists an orthonormal subset $(f_i)_{i \in I}$ of $H_E$ satisfying

$$V|\xi\rangle = \sum_i \sqrt{\lambda_i} |e_i\rangle \otimes |f_i\rangle$$

and

$$\Phi^c(|\xi\rangle\langle\xi|) = \sum_i \lambda_i |f_i\rangle\langle f_i|.$$  

(2.5)

In particular, $H(\Phi(|\xi\rangle\langle\xi|)) = H(\Phi^c(|\xi\rangle\langle\xi|))$ for each unit vector $\xi \in H_A$.

**Proof.** Since $(e_i)_{i}$ is an orthonormal basis of $H_B$, we can write $V|\xi\rangle$ as $\sum_i |e_i\rangle \otimes |\eta_i\rangle$ for a family $(\eta_i)_{i} \subseteq H_E$. Moreover, the given spectral decomposition of $\Phi(|\xi\rangle\langle\xi|)$ tells us that $\langle \eta_j | \eta_i \rangle = \lambda_i \delta_{i,j}$, which is equivalent to that $(f_i)_{i} := \left( \lambda_i^{\frac{1}{2}} |\eta_i\rangle \right)_{i}$ is an orthonormal set. Then we have

$$V|\xi\rangle = \sum_i \sqrt{\lambda_i} |e_i\rangle \otimes |f_i\rangle$$

and

$$\Phi^c(|\xi\rangle\langle\xi|) = \sum_i \lambda_i |f_i\rangle\langle f_i|.$$
2.2. **Commuting systems of Kraus operators.** Let $H$ be a Hilbert space and $(a_{ij})_{(i,j)\in I\times J}$ be a family of bounded operators in $B(H)$ satisfying 
\[
\sum_{i \in I} a_{i,j}^* a_{i,j} = \text{Id}_H \quad \text{for each } j \in J.
\]
We assume that $I$ is finite and $J$ is arbitrary. Let us define a family of quantum channels $(\Phi_j)_{j \in J}$ by
\[
\Phi_j : \mathcal{T}(H) \to \mathcal{T}(H), \quad X \mapsto \sum_{i \in I} a_{i,j} X a_{i,j}^*.
\]
Their complement channels are given by
\[
\Phi_j^c : \mathcal{T}(H) \to M_{|I|}(\mathbb{C}), \quad X \mapsto \sum_{i,j \in I} \text{tr}(a_{i,j} X a_{i,j}^*) |i\rangle \langle i'|.
\]
We say that $(\Phi_j)_{j \in J}$ is in **commuting-operator setup** if the given channels $\Phi_j$ have commuting systems of Kraus operators in the sense that $a_{i,j} a_{i',j'} = a_{i',j'} a_{i,j}$ for any $i, i' \in I$ and $j, j' \in J$ such that $j \neq j'$.

An example of this is the **tensor-product setup** but it is not the only example. A property is that $\Phi_j$ and $\Phi_{j'}$ commute and their products are again quantum channels. If $J$ is finite and $\Phi_1, \ldots, \Phi_{|J|}$ are in commuting-operator setup, then it is natural to ask whether the following additivity property holds when $F$ is one of $H_{\text{min}}, \overline{H}_{\text{min}}, \chi, C$:
\[
F \left( \prod_{j \in J} \Phi_j \right) = \sum_{j \in J} F(\Phi_j).
\]
In particular, in the case $|J| = 2$, the product channel $\Phi_1 \circ \Phi_2$ is called a **local map** in the context of [CKLT19].

Let us construct a non-trivial quantum channel within the commuting-operator setup from the view of abstract harmonic analysis and operator algebra. Let $\mathbb{F}_\infty$ be the free group whose generators are $g_1, g_2, \ldots$ and let us define unitary operators $U_i$ and $V_j$ on $\ell^2(\mathbb{F}_\infty)$ by
\[
(U_i f)(x) = f(g_i^{-1} x) \quad \text{and} \quad (V_j f)(x) = f(x g_j)
\]
for any $f \in \ell^2(\mathbb{F}_\infty)$, $x \in \mathbb{F}_\infty$ and $i, j \in \mathbb{N}$. Since $U_i V_j = V_j U_i$ for all $i, j \in \mathbb{N}$, we have the following quantum channels that have commuting systems of Kraus operators.
\[
\Phi_{N,J} : \mathcal{T}(\ell^2(\mathbb{F}_\infty)) \to \mathcal{T}(\ell^2(\mathbb{F}_\infty)), \quad \rho \mapsto \frac{1}{N} \sum_{i=1}^N U_i \rho U_i^* \quad \text{and}
\]
\[
\Phi_{N,r} : \mathcal{T}(\ell^2(\mathbb{F}_\infty)) \to \mathcal{T}(\ell^2(\mathbb{F}_\infty)), \quad \rho \mapsto \frac{1}{N} \sum_{j=1}^N V_j \rho V_j^*.
\]
Let $J \in B(\ell^2(\mathbb{F}_\infty))$ be a unitary given by

$$(Jf)(x) = f(x^{-1})$$

for any $f \in \ell^2(\mathbb{F}_\infty)$ and $x \in \mathbb{F}_\infty$. Then, since $JU_iJ = V_i$ and $J^2 = \text{Id}$, the above channels $\Phi_{N,i}$ and $\Phi_{N,r}$ are equivalent in the sense that

$$\Phi_{N,r}(\rho) = J\Phi_{N,i}(J\rho J)J$$

for any $\rho \in \mathcal{T}(\ell^2(\mathbb{F}_\infty))$. In particular, $H_{\min}(\Phi_{N,i}^{\otimes k}) = H_{\min}(\Phi_{N,r}^{\otimes k})$ for any $k \in \mathbb{N}$.

Also, in order to express $k$-fold tensor product quantum channels $\Phi_{N,i}^{\otimes k}$, we will use the following notation

$$U_m = U_{m_1} \otimes U_{m_2} \otimes \cdots \otimes U_{m_k} \in B(\ell^2(\mathbb{F}_\infty))$$

for any $m = (m_1, \cdots, m_k) \in I^k$, where $I = \{1, 2, \cdots, N\}$.

3. Generalized Haagerup inequality and regularized MOE

In this section, we prove that the Haagerup inequality extends naturally to $r$-products of free groups $\mathbb{F}_\infty^r$. Then we explain how this generalization allows to prove lower bounds of the regularized minimum output entropies (MOE) for $\Phi_{N,i}$. Let us simply write $\Phi_{N,i}$ as $\Phi_N$ in this section.

3.1. A generalized Haagerup inequality. For $x$ in the free group $\mathbb{F}_\infty$, we call $|x|$ its reduced word length with respect to the canonical generators and their inverses. We consider products of free groups $\mathbb{F}_\infty^r$ for any $r \in \mathbb{N}$. Let us use the following notations $E_j = \{x \in \mathbb{F}_\infty : |x| = j\}$ for any $j \in \mathbb{N}_0$ and $E_m = E_{m_1} \times E_{m_2} \times \cdots \times E_{m_r} \subseteq \mathbb{F}_\infty^r$ for any $m = (m_1, \cdots, m_r) \in \mathbb{N}_0^r$.

We view $\mathbb{F}_\infty^r$ as an orthonormal basis that generates the Hilbert space $\ell^2(\mathbb{F}_\infty^r)$. As an algebraic vector space, $\mathbb{F}_\infty^r$ spans $\mathbb{C}[\mathbb{F}_\infty^r]$, on which we may define the convolution $f \star g$ and the pointwise product $f \cdot g$. For $A \subseteq \mathbb{F}_\infty^r$, $\chi_A$ denotes the indicator function of $A$. First of all, we can generalize Lemma 1.3 of [Haa79] as follows:

**Lemma 3.1.** Let $l, m, k \in \mathbb{N}_0$ and let $f, g$ be supported in $E_k$ and $E_l$ respectively. Then

$$\|(f \star g) \cdot \chi_{E_m}\|_{\ell^2(\mathbb{F}_\infty^r)} \leq \|f\|_{\ell^2(\mathbb{F}_\infty^r)} \cdot \|g\|_{\ell^2(\mathbb{F}_\infty^r)}$$

(3.1)

if $|k_j - l_j| \leq m_j \leq k_j + l_j$ and $k_j + l_j - m_j$ is even for all $1 \leq j \leq r$. Otherwise, we have $\|(f \star g) \cdot \chi_{E_m}\|_{\ell^2(\mathbb{F}_\infty^r)} = 0$.

**Proof.** Let us suppose that $|k_j - l_j| \leq m_j \leq k_j + l_j$ and $k_j + l_j - m_j$ is even for all $1 \leq j \leq r$. If not, it is not difficult to see that $(f \star g) \chi_{E_m} = 0$. Also, it is enough to suppose that $f, g$ are finitely supported since $(f, g) \mapsto (f \star g) \cdot \chi_{E_m}$ is bilinear.
Let us use the induction argument with respect to $r \in \mathbb{N}$. The first case $r = 1$ follows from \cite{Haa79} Lemma 1.3 and let us suppose that (3.1) holds true for $\mathbb{F}_r^\infty$. Under the notation $m = (m_0, m') \in \mathbb{N}_0^r+1$, we have

\begin{equation}
\left\| (f * g)\chi_{E_m} \right\|_{L^2(\mathbb{F}_r^\infty)}^2 = \sum_{s \in E_m} \sum_{t, u \in \mathbb{F}_r^\infty : t : u = s} f(t)g(u)^2 \tag{3.2}
\end{equation}

\begin{equation}
= \sum_{s_0 \in E_{m_0}} \sum_{s' \in E_{m'}} \sum_{t_0, u_0 \in \mathbb{F}_r^\infty : t_0 u_0 = s_0} \sum_{t', u' \in \mathbb{F}_r^\infty : t' u' = s'} f(t_0, t')g(u_0, u')^2 \cdot \tag{3.3}
\end{equation}

First of all, if we suppose that $m_0 = k_0 + l_0$, then we have

\begin{equation}
= \sum_{s_0 \in E_{m_0}} \sum_{s' \in E_{m'}} \sum_{t_0, u_0 \in \mathbb{F}_r^\infty : t_0 u_0 = s_0} \sum_{t', u' \in \mathbb{F}_r^\infty : t' u' = s'} f(t_0, t')g(u_0, u')^2
\end{equation}

since there is a unique choice of $t_0 \in E_{k_0}$ and $u_0 \in E_{l_0}$ satisfying $t_0 u_0 = s_0$. Let us define functions $f_{t_0}(t') = f(t_0, t')$ and $g_{u_0}(u') = g(u_0, u')$ on $\mathbb{F}_r^\infty$. Then the above is written as

\begin{equation}
\left\| (f * g)\chi_{E_m} \right\|_{L^2(\mathbb{F}_r^\infty)}^2 = \sum_{s_0 \in E_{m_0}} \sum_{t_0, u_0 \in \mathbb{F}_r^\infty : t_0 u_0 = s_0} \left\| (f_{t_0} * g_{u_0})\chi_{E_{m'}} \right\|_{L^2(\mathbb{F}_r^\infty)}^2 , \tag{3.4}
\end{equation}

which is dominated by

\begin{equation}
\leq \sum_{s_0 \in E_{m_0}} \sum_{t_0, u_0 \in \mathbb{F}_r^\infty : t_0 u_0 = s_0} \left\| f_{t_0} \right\|_{L^2(\mathbb{F}_r^\infty)}^2 \left\| g_{u_0} \right\|_{L^2(\mathbb{F}_r^\infty)}^2 \tag{3.5}
\end{equation}

\begin{equation}
\leq \sum_{t_0 \in E_{k_0}} \sum_{u_0 \in E_{l_0}} \left\| f_{t_0} \right\|_{L^2(\mathbb{F}_r^\infty)}^2 \left\| g_{u_0} \right\|_{L^2(\mathbb{F}_r^\infty)}^2 = \left\| f \right\|_{L^2(\mathbb{F}_r^{r+1})}^2 \left\| g \right\|_{L^2(\mathbb{F}_r^{r+1})}^2 . \tag{3.6}
\end{equation}

Here, the first inequality comes from the induction hypothesis. Furthermore, the same idea applies whenever $m_j = k_j + l_j$ for some $0 \leq j \leq r$. Now, let us suppose that $m_j < k_j + l_j$ and put $q_j = \frac{k_j + l_j - m_j}{2}$ for all $0 \leq j \leq r$. Also, denote by $q = (q_0, \cdots, q_r)$ and define two functions $F$ and $G$ on $\mathbb{F}_r^{r+1}$ as follows:

\begin{equation}
F(x) = \begin{cases} \left( \sum_{v \in E_q} \left| f(xv) \right|^2 \right)^{\frac{1}{2}} & \text{for any } x \in E_{k-q} \\ 0 & \text{otherwise} \end{cases} \tag{3.7}
\end{equation}

\begin{equation}
G(y) = \begin{cases} \left( \sum_{v \in E_q} \left| g(v^{-1}y) \right|^2 \right)^{\frac{1}{2}} & \text{for any } y \in E_{l-q} \\ 0 & \text{otherwise} \end{cases} \tag{3.8}
\end{equation}
Note that $F$ and $G$ are supported in $E_{k-q}$ and $E_{l-q}$ respectively with

$$\|F\|_{l^2(\ell^n_{\infty}+1)}^2 = \sum_{x \in E_{k-q}} \sum_{v \in E_q} |f(xv)|^2 = \|F\|_{l^2(\ell^n_{\infty}+1)}$$
and

$$\|G\|_{l^2(\ell^n_{\infty}+1)}^2 = \sum_{v \in E_q} \sum_{y \in E_{l-q}} |g(v^{-1}y)|^2 = \|G\|_{l^2(\ell^n_{\infty}+1)}^2.$$  (3.10)

Then we can show that the convolution $F * G$ dominates $|f * g|$ on $E_m$. Indeed, for any $s \in E_m$, there exists a unique $(x, y) \in E_{k-q} \times E_{l-q}$ such that $s = xy$ and we have

$$|(f * g)(s)| = \left| \sum_{t \in E_{k-q}, u \in E_{l-q}} f(t)g(u) \right| = \left| \sum_{v \in E_q} f(xv)g(v^{-1}y) \right|$$
$$\leq \left( \sum_{v \in E_q} |f(xv)|^2 \right)^{\frac{1}{2}} \left( \sum_{v \in E_q} |g(v^{-1}y)|^2 \right)^{\frac{1}{2}} = F(x)G(y) = (F * G)(s)$$

Finally, since $F$ and $G$ are supported in $E_{k-q}$ and $E_{l-q}$ respectively with $m = (k-q) + (l-q)$, we can conclude that

$$\|(f * g)\chi_{E_m}\|_{l^2(\ell^n_{\infty}+1)} \leq \|(F * G)\chi_{E_m}\|_{l^2(\ell^n_{\infty}+1)}$$
$$\leq \|F\|_{l^2(\ell^n_{\infty}+1)} \|G\|_{l^2(\ell^n_{\infty}+1)} = \|f\|_{l^2(\ell^n_{\infty}+1)} \|g\|_{l^2(\ell^n_{\infty}+1)}.$$  □

Then we can generalize the Haagerup inequality to products of free groups $\mathbb{F}^r_{\infty}$ as follows:

**Proposition 3.2.** Let $n = (n_1, \cdots, n_r) \in \mathbb{N}_0^r$ and $f$ be supported on $E_n \subseteq \mathbb{F}^r_{\infty}$. Then

$$\|L_f\| \leq (n_1 + 1) \cdots (n_r + 1) \|f\|_{l^2(\mathbb{F}^r_{\infty})},$$

where $L_f$ is the convolution operator on $l^2(\mathbb{F}^r_{\infty})$ given by $g \mapsto f \ast g$.

**Proof.** By density arguments, we may assume that $f$ is finitely supported and it is enough to consider finitely supported functions to evaluate the norm of the associated convolution operator $L_f$. Let $g \in l^2(\mathbb{F}^r_{\infty})$ be finitely supported and define $g_k = g \cdot \chi_{E_k}$ for each $k \in \mathbb{N}_0^r$. Then $g = \sum_{k \in \mathbb{N}_0^r} g \cdot \chi_{E_k}$ and we have

$$h := f \ast g = \sum_{k \in \mathbb{N}_0^r} f \ast g_k.$$
Then, by Lemma 3.1, we have the following estimate for $h_m = h \cdot \chi_{E_m}$ with $m = (m_1, \ldots, m_r) \in \mathbb{N}_0^r$ as follows:

$$\|h_m\|_{\ell^2(\mathcal{F}^\infty)} = \left\| \sum_{k \in \mathbb{N}_0^r} (f * g_k) \cdot \chi_{E_m} \right\|_{\ell^2(\mathcal{F}^\infty)} \leq \sum_{k \in \mathbb{N}_0^r} \|f \cdot \chi_{E_m}\|_{\ell^2(\mathcal{F}^\infty)} \leq \left\|f\right\|_{\ell^2(\mathcal{F}^\infty)} \|g_k\|_{\ell^2(\mathcal{F}^\infty)} =: A$$

Writing $k_j = m_j + n_j - 2l_j$ for all $1 \leq j \leq r$, we obtain

$$A = \|f\|_{\ell^2(\mathcal{F}^\infty)} \sum_{l_1, \ldots, l_r} \|g_{m+n-2l}\|_{\ell^2(\mathcal{F}^\infty)} \leq \|f\|_{\ell^2(\mathcal{F}^\infty)} \left( \sum_{l_1, \ldots, l_r} \|g_{m+n-2l}\|_{\ell^2(\mathcal{F}^\infty)}^2 \right)^{\frac{1}{2}}.$$

Therefore,

$$\|h\|_{\ell^2(\mathcal{F}^\infty)}^2 = \sum_{m \in \mathbb{N}_0^r} \|h_m\|_{\ell^2(\mathcal{F}^\infty)}^2 \leq (1 + n_1) \cdots (1 + n_r) \|f\|_{\ell^2(\mathcal{F}^\infty)}^2 \sum_{m \in \mathbb{N}_0^r} \sum_{l_1, \ldots, l_r} \|g_{m+n-2l}\|_{\ell^2(\mathcal{F}^\infty)}^2$$

$$= (1 + n_1) \cdots (1 + n_r) \|f\|_{\ell^2(\mathcal{F}^\infty)}^2 \sum_{l_1, \ldots, l_r} \sum_{m_1, \ldots, m_r} \|g_{m+n-2l}\|_{\ell^2(\mathcal{F}^\infty)}^2$$

$$= (1 + n_1) \cdots (1 + n_r) \|f\|_{\ell^2(\mathcal{F}^\infty)}^2 \sum_{l_1, \ldots, l_r} \sum_{k_1, \ldots, k_r} \|g_k\|_{\ell^2(\mathcal{F}^\infty)}^2$$

$$\leq (1 + n_1) \cdots (1 + n_r) \|f\|_{\ell^2(\mathcal{F}^\infty)}^2 \sum_{l_1, \ldots, l_r} \sum_{k \in \mathbb{N}_0^r} \|g_k\|_{\ell^2(\mathcal{F}^\infty)}^2$$

$$= (1 + n_1)^2 \cdots (1 + n_r)^2 \|f\|_{\ell^2(\mathcal{F}^\infty)}^2 \|g\|_{\ell^2(\mathcal{F}^\infty)}^2,$$

which gives us

$$\|L_f\| \leq (1 + n_1) \cdots (1 + n_r) \|f\|_{\ell^2(\mathcal{F}^\infty)}.$$

□
3.2. **A norm estimate.** In this subsection, we investigate the operator norm of the following elements

\[ \sum_{v, w \in E^k} a_{vw}(U_v)^* U_w \text{ with } a_{vw} \in \mathbb{C}, \]

where \( E = \{ g_1, g_2, \cdots \} \) is the set of generators of \( \mathbb{F}_\infty \) and \( E^k = E \times \cdots \times E \subseteq E_{(1, \cdots, 1)} \). Indeed, in Section 3.3 this estimate will be needed to evaluate the regularized MOE. Our estimate is as follows:

**Theorem 3.3.** For any \( a = (a_{vw})_{v, w \in E^k} \in M_N(\mathbb{C}) \) such that \( \text{tr}(a) = 0 \), we have

\[ \left\| \sum_{v, w \in E^k} a_{vw}(U_v)^* U_w \right\| \leq N^k \sqrt{1 + 9N^{-1}} - 1 \left\| a \right\|_2. \]

**Proof.** Now we are summing over \((N^k)^2\) elements and we have to split the sum according to whether there are simplification or not. To do this, for any subset \( K \) of \( \{ 1, 2, \cdots, k \} \), we define

\[ E_K = \{(v, w) \in E^k \times E^k : v_i = w_i \text{ for all } i \in K \}. \]

Note that \( E^2 = \bigcup_{K \subseteq \{ 1, \cdots, k \}} E_K \). Then, by the triangle inequality, we have

\[
\left\| \sum_{v, w \in E^2} a_{vw}(U_v)^* U_w \right\| \leq \sum_{K \subseteq \{ 1, \cdots, k \}} \left\| \sum_{(v, w) \in E_K} a_{vw}(U_v)^* U_w \right\| \\
= \sum_{s=0}^k \sum_{|K|=s} \left\| \sum_{(v, w) \in E_K} a_{vw}(U_v)^* U_w \right\|.
\]

Note that \( |K| = k \) implies \( \sum_{(v, w) \in E_K} a_{vw}(U_v)^* U_w = \sum_{v \in E^k} a_{vv} = 0 \). From now on, let us suppose that \( |K| = s < k \). Then \( E_K \) can be identified with the set \( \{(z, x, y) \in E^s \times E^{k-s} \times E^{k-s} : x_j \neq y_j \forall 1 \leq j \leq k - s \} \) for each \( K \). Under this notation, \( (a_{vw}) \) can be written as \( (a_{x, y}^{K, z}) \) and we have

\[
\sum_{K \subseteq \{ 1, \cdots, k \}} \left\| \sum_{(v, w) \in E_K} a_{vw}(U_v)^* U_w \right\| \leq \sum_{|K|=s} \sum_{(x, y) \in E^s} \sum_{x_j \neq y_j} a_{x, z}^{K, z}(U_x)^* U_y.
\]
Moreover, Proposition 3.2 gives us

\[
\leq \sum_{K \subset \{1, \ldots, k\}} \sum_{z \in E^s} 3^{k-s} \left( \sum_{(x,y) \in E^{k-s} \times E^{k-s}} |a_{K,z}^{x,y}|^2 \right)^{\frac{1}{2}}
\]

and, since we are summing \(N^s \binom{k}{s}\) elements, the Cauchy-Schwarz inequality tells us that

\[
\leq 3^{k-s} N^s \binom{k}{s} \left( \sum_{K \subset \{1, \ldots, k\}} \sum_{z \in E^s} \sum_{(x,y) \in E^{k-s} \times E^{k-s}} |a_{K,z}^{x,y}|^2 \right)^{\frac{1}{2}}
\]

\[
= 3^{k-s} N^s \binom{k}{s} \left( \sum_{K \subset \{1, \ldots, k\}} \sum_{(u,v) \in E_K} |a_{vw}|^2 \right)^{\frac{1}{2}}.
\]

Here, the Haagerup constant \(3^{k-s}\) appears due to \(s\) cancellations. Applying the Cauchy-Schwartz inequality once more, we obtain that

\[
\left\| \sum_{v, w \in E^s} a_{vw} (U_v)^* U_w \right\| = \sum_{s=0}^{k-1} \sum_{K \subset \{1, \ldots, k\}} \left\| \sum_{(v,w) \in E_K} a_{vw} (U_v)^* U_w \right\|
\]

\[
\leq \sum_{s=0}^{k-1} 3^{k-s} N^s \binom{k}{s} \left( \sum_{K \subset \{1, \ldots, k\}} \sum_{(u,v) \in E_K} |a_{vw}|^2 \right)^{\frac{1}{2}}
\]

\[
\leq \left( \sum_{s=0}^{k-1} 9^{k-s} N^s \binom{k}{s} \right)^{\frac{1}{2}} \left( \sum_{K \subset \{1, \ldots, k\}} \sum_{(u,v) \in E_K} |a_{vw}|^2 \right)^{\frac{1}{2}}
\]

\[
= \sqrt{(N + 9)^k - N^k} \|a\|_2
\]

\[
= N^\frac{k}{2} \sqrt{(1 + 9N^{-1})^k - 1} \|a\|_2.
\]
3.3. The regularized minimum output entropy of $\Phi_N$. Theorem 3.3 enables us to show that for any density matrix $S$

$$\left\| (\Phi_N^c)^{\otimes k}(S) - \frac{1}{N^k} \text{Id}_N^{\otimes k} \right\|_2$$

is sufficiently small with respect to the Hilbert-Schmidt norm. This generalizes [Col18, Theorem 3.1]. Specifically, we prove the following theorem:

**Theorem 3.4.** For each $k \in \mathbb{N}$, we have

$$\sup_S \left\| (\Phi_N^c)^{\otimes k}(S) - \frac{1}{N^k} \text{Id}_N^{\otimes k} \right\|_2 \leq \frac{\sqrt{(1 + 9N^{-1})^k - 1}}{\sqrt{N}}$$

where $S$ runs over all density matrices in $T(\ell^2(\mathbb{F}_\infty^k))$.

**Proof.** Let $X = (x_{i,j})_{i,j \in I} = (\Phi_N^c)^{\otimes k}(S) - \frac{1}{N^k} \text{Id}_N^{\otimes k}$. Since $\text{tr}(X) = 0$, we have

$$\text{tr}(X^2) = \text{tr}((\Phi_N^c)^{\otimes k}(S)X) = \text{tr}(S ((\Phi_N^c)^{\otimes k})^*(X))$$

where $((\Phi_N^c)^{\otimes k})^*$ denotes the adjoint map of $(\Phi_N^c)^{\otimes k}$. Moreover,

$$((\Phi_N^c)^{\otimes k})^*(X) = \frac{1}{N^k} \sum_{i,j \in I^k \atop i \neq j} x_{i,j} U_i^* U_j$$

where $I = \{1, 2, \cdots, N\}$. Hence, we have

$$\text{tr}(S((\Phi_N^c)^{\otimes k})^*(X)) \leq \frac{1}{N^k} \left\| \sum_{i,j \in I^k \atop i \neq j} x_{i,j} U_i^* U_j \right\|.$$ 

According to Theorem 3.3, we get

$$\|X\|_2^2 = \text{tr}(X^2) \leq \frac{\sqrt{1 + 9N^{-1}} - 1}{N} \|X\|_2,$$

as claimed. 

This allows us to estimate the regularized minimum output entropies of $\Phi_N$ as follows:

**Theorem 3.5.** For any $k \in \mathbb{N}$ we have

$$H_{\min}(\Phi_N^{\otimes k}) \geq k \log(N) - 2 \log(1 + \sqrt{(1 + 9N^{-1})^k - 1}).$$

In particular, we have the following estimate for the regularized MOE

$$\overline{H}_{\min}(\Phi_N) \geq \log(N) - \log(1 + \frac{9}{N}) \geq \log(N) - \frac{9}{N}.$$
Proof. Thanks to the fact that the von Neumann entropy is bigger than Rényi entropies of order $\alpha = 2$ [MLDS+13], we have
\[
H((\Phi_N^c \otimes S))^k(\rho) \geq \alpha \frac{1}{1 - \alpha} \log \left( \left\| (\Phi_N^c)^{\otimes k}(S) \right\|_\alpha \right)
\geq -2 \log \left( N^{-\frac{k}{\alpha}} (1 + \sqrt{(1 + 9N^{-1})^k - 1}) \right)
= k \log(N) - 2 \log \left( 1 + \sqrt{(1 + 9N^{-1})^k - 1} \right)
\]
for any density matrix $S \in \mathcal{T}(\ell^2(F^k))$ by Theorem 3.4. In particular, we have
\[
H_{\text{min}}(\Phi_N^c)^k \geq k \log(N) - 2 \log(1 + \sqrt{(1 + 9N^{-1})^k - 1})
\]
and the last conclusion follow from the following computation with L'Hôpital's rule:
\[
\lim_{k \to \infty} \frac{\log(1 + \sqrt{(1 + 9N^{-1})^k - 1})}{k} = \lim_{k \to \infty} \frac{\frac{1}{2} \log(1 + \frac{9}{N})}{1 + \sqrt{(1 + \frac{9}{N})^k - 1}} = \frac{1}{2} \log(1 + \frac{9}{N}).
\]
\[\square\]

4. Additivity Violation of the Regularized MOE

In this section, we choose two copies of $\Phi_N$ as $\Phi_{N,l}$ and $\Phi_{N,r}$. Indeed, these two quantum channels $\Phi_{N,l}$ and $\Phi_{N,r}$ are equivalent as explained in Section 2.2 and are in commuting-operator setup.

Then we can obtain the following additivity violation of the regularized MOE by generalizing Winter-Holevo-Hayden-Werner trick for $\Phi_{N,l} \circ \Phi_{N,r}$:

**Theorem 4.1.** The regularized MOE is not additive: For any $N > e^{18}$, we have
\[
\overline{H}_{\text{min}}(\Phi_{N,l} \circ \Phi_{N,r}) < \overline{H}_{\text{min}}(\Phi_{N,l}) + \overline{H}_{\text{min}}(\Phi_{N,r}).
\]

**Proof.** Note that under notations from Subsection 2.2,
\[
(\Phi_{N,l} \circ \Phi_{N,r})(\rho) = \frac{1}{N^2} \sum_{i,j=1}^N U_i V_j \rho V_j^* U_i^*.
\]
Since $|e\rangle\langle e|$ is an invariant for $U_i V_i$, we have
\[
(\Phi_{N,l} \circ \Phi_{N,r})(|e\rangle\langle e|) = \frac{1}{N} |e\rangle\langle e| + \frac{1}{N^2} \sum_{i,j:i \neq j} |g_i g_j^{-1}\rangle\langle g_i g_j^{-1}|,
\]
which implies
\[
\overline{H}_{\text{min}}(\Phi_{N,l} \circ \Phi_{N,r}) \leq H_{\text{min}}((\Phi_{N,l} \circ \Phi_{N,r})(|e\rangle\langle e|)) \\
= \frac{\log(N)}{N} + (N^2 - N) \cdot \frac{\log(N^2)}{N^2} = 2 \log(N) - \frac{\log(N)}{N}.
\]

Moreover, \(\Phi_{N,l}\) and \(\Phi_{N,r}\) are copies of \(\Phi_N\), so that we have
\[
2 \log(N) - \frac{\log(N)}{N} < 2 \log(N) - \frac{18}{N} \leq \overline{H}_{\text{min}}(\Phi_{N,l}) + \overline{H}_{\text{min}}(\Phi_{N,r})
\]
by Theorem 3.5 if \(N > e^{18}\).

5. CONCLUDING REMARKS

(1) Various versions of \(C^*\)-tensor products can be used to obtain commuting systems of operators. For example, let \(A, B\) be unital \(C^*\)-algebras and take families of operators \((E_i)_{i=1}^m \subseteq A\) and \((F_j)_{j=1}^n \subseteq B\). Also, suppose that \(A \otimes_{\text{max}} B \subseteq B(K)\). Then

\[
\{E_i \otimes 1_B\}_{i=1}^m \text{ and } \{1_A \otimes F_j\}_{j=1}^n
\]
give us commuting systems of operators in \(B(K)\). Moreover, if we suppose that \(C_r^*(\mathbb{F}_\infty) \otimes_{\text{max}} C_r^*(\mathbb{F}_\infty) \subseteq B(K)\) where \(C_r^*(\mathbb{F}_\infty)\) is the reduced group \(C^*\)-algebra of the free group \(\mathbb{F}_\infty\), then commuting systems \(\{U_i \otimes \text{Id}\}_i\) and \(\{\text{Id} \otimes U_j\}_j\) give another example of additivity violation in the commuting-operator setup.

(2) Since Haagerup type inequalities exist for other groups (e.g. hyperbolic groups \([dhH88]\)) or certain reduced free products of \(C^*\)-algebras \([Boz91]\), it is natural to expect that similar results should hold and will yield other examples of additivity violation phenomena.

(3) It is worthwhile to compare the main results of this paper and the cases of random unitary channels. On the side of random unitary channels, the regularized MOE is unknown, whereas our Theorem 3.5 gives us a strong estimate for the regularized MOE of \(\Phi_N\).

(4) One might wonder if we can evaluate the classical capacity of \(\Phi_{c,N}\) whose output space is finite dimensional. Thanks to Theorem 3.5 and a standard argument, the classical capacity of \(\Phi_{c,N}\) is upper bounded by
\[
C(\Phi_{c,N}) \leq \log(N) - \overline{H}_{\text{min}}(\Phi_{c,N}) \leq \frac{9}{N}.
\]

However, unlike in the tensor-product setup \([Fuk14, \text{Theorem 6.1}]\), it is not clear whether additivity violation of the regularized MOE implies superadditivity of the classical capacity within commuting-operator framework, so the question of the additivity of the classical capacity remains open.
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