Non-minimal coupling of photons and axions

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Abstract

We establish a new self-consistent system of equations accounting for a non-minimal interaction of gravitational, electromagnetic and axion fields. The procedure is based on a non-minimal extension of the standard Einstein–Maxwell–axion action. The general properties of a ten-parameter family of non-minimal linear models are discussed. We apply this theory to the models with pp-wave symmetry and consider propagation of electromagnetic waves non-minimally coupled to the gravitational and axion fields. We focus on exact solutions of electrodynamic equations, which describe quasi-minimal and non-minimal optical activity induced by the axion field. We also discuss empirical constraints on coupling parameters from astrophysical birefringence and polarization rotation observations.

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1. Introduction

The topic of pseudoscalar-photon interaction and axion theory is a very attractive branch of modern physics, and there are two reasons for this interest. First of all, it is an interest to axions as hypothetical particles, which appear in the context of the strong CP-violation problem and spontaneous breaking of symmetry in the early universe (see, e.g., [1, 2] for review and basic references), and which are considered as one of the dark matter candidates (see, e.g., [3, 4] and references therein). The direct detection of the axions is formulated to be one of the goals of modern experiments in high energy physics (see, e.g., [5]).

The second reason is connected with an interest to the theoretical concept of a pseudoscalar field $\phi$ (axion field) associated with pseudoscalar-photon interaction. This occurs first in the study of electromagnetism and equivalence principles [6–8]. This concept is a base for the so-called Einstein–Maxwell–axion theory, Einstein–Maxwell–dilaton–axion theory and their generalizations (see, e.g., [6–18]).
One of the most significant items in the theory of axion fields is the problem of interaction of electromagnetic and pseudoscalar (axion) fields, the well-known application of the photon–axion coupling being the effect of polarization rotation [6, 19, 20] in the observations of cosmic microwave background (CMB) radiation (see, e.g., [21, 22] and references therein). Several groups are working on the experiments with vacuum birefringence and vacuum dichroism to look for the photon–axion coupling [23–26].

It is well known that the standard effect of optical activity induced by the axion field takes place when the pseudoscalar field $\phi$ has a non-vanishing four-gradient [8, 27], i.e. when $\partial_k \phi \neq 0$. This effect can be described in the framework of classical electrodynamics in terms of linear constitutive equations [8, 27, 28], linking an excitation tensor $H^{ik}$ and the Maxwell tensor $F_{mn}$. When the pseudoscalar field is constant, $\phi = \phi_0$, and thus does not contribute to the vacuum Maxwell equations, new possibilities for the axion–photon interaction exist: one of them is discussed in the work [29], which focuses on reflection and transmission of a wave at an interface between two media. We discuss here another possibility related to non-minimal optical activity induced by photon–axion coupling, which also can occur in the case of a constant pseudoscalar field.

The standard description of the non-minimal coupling of the gravitational field with scalar, electromagnetic and gauge fields is based on the introduction of specific cross-terms into the Lagrangian, which contain the Riemann tensor $R_{ikmn}$, the Ricci tensor, $R_{kn}$, and Ricci scalar, $R$, on the one hand, and the corresponding fields and their derivatives, on the other hand. The theory of non-minimal coupling is elaborated in detail for the scalar field (real and complex fields, as well as Higgs multiplets) (see, e.g., [30] for review and references). Special attention in these investigations is focused on two models: the first of them has the $\xi R/\Phi^2$ coupling, and the second has the so-called non-minimal derivative coupling [31–33]. The study of a non-minimal coupling of gravity with the electromagnetic field started in [34] and this theory has been developed by many authors (see, e.g., [35–48]). The generalization of the idea of non-minimal interactions to the case of torsion coupled to the electromagnetic field has been made in [49, 50]. Exact solutions in the framework of non-minimal Einstein–Yang–Mills and Einstein–Yang–Mills–Higgs theories are discussed in [51–56].

In this paper, we focus on the non-minimal generalization of the Einstein–Maxwell–axion theory. The derivation of master equations is fulfilled in analogy with non-minimal scalar field theory and non-minimal electromagnetic field theory. The items which are specific for the pseudoscalar field theory are discussed in detail. The paper is organized as follows. In section 2, we discuss a formalism of non-minimal Einstein–Maxwell–axion theory; a brief introduction to the minimal theory is given in subsection 2.1; the non-minimal extension of the Lagrangian and the derivation of the corresponding master equations are presented in subsection 2.2; subsection 2.3 contains the description of the model in terms of constitutive equations. In section 3, we apply the obtained master equations to the models with pp-wave symmetry and discuss examples of the exact solutions for the gravitational, pseudoscalar and electromagnetic fields. In section 4, we consider the propagation of test electromagnetic waves coupled to the pseudoscalar field in the pp-wave background and focus on the effect of non-minimal optical activity induced by the axion field. In section 5, we summarize the results.

2. Non-minimal coupling of gravitational, electromagnetic and axion fields

2.1. Minimal model as a starting point

In order to explain the novelty of our approach, let us first introduce the case of gravitational–electromagnetic–axion fields minimally coupled. The simplest minimal action
functional is
\[ S(M) = \int d^4x \sqrt{-g} \left[ \frac{R}{\kappa} + \frac{1}{2} F_{mn} F^{mn} + \frac{1}{2} \phi F^{mn} F_{mn} - g^{mn} \nabla_m \phi \nabla_n \phi + m_A^2 \phi^2 \right], \]  
(1)

where \( g \) is the determinant of the metric tensor \( g_{ik} \), \( \nabla_m \) is a covariant derivative and \( R \) is the Ricci scalar. The Maxwell tensor \( F_{mn} \) is given by
\[ F_{mn} \equiv \nabla_m A_n - \nabla_n A_m, \]  
(2)

where \( A_m \) is an electromagnetic potential four-vector; \( F^{mn} \equiv \frac{1}{2} \epsilon^{mnq} F_{pq} \) is the tensor dual to \( F_{pq} \); \( \epsilon^{mnq} \equiv \frac{1}{\sqrt{-g}} \epsilon^{mnpq} \) is the Levi-Civita tensor and \( \epsilon^{mnpq} \) is the absolutely antisymmetric Levi-Civita symbol with \( \epsilon^{0123} = 1 \). The dual Maxwell tensor satisfies the condition
\[ \nabla_k F^{*ik} = 0. \]  
(3)

The first term in the brackets is the Hilbert–Einstein Lagrangian; the second term is the standard Lagrangian for an electromagnetic field; the third term is the pseudoscalar-photon interaction Lagrangian \([6–8]\) and the fourth and fifth terms constitute the pseudoscalar Lagrangian.

The symbol \( \phi \) stands for a pseudoscalar field; this quantity is dimensionless providing the terms \( \frac{1}{2} F_{mn} F^{mn} \) and \( \frac{1}{2} \phi F^{mn} F_{mn} \) to have the same dimensionality. The axion field itself, \( \phi \), is considered to be proportional to this quantity \( \phi = \phi_0 \) with a constant \( \kappa = \frac{8\pi G}{c^4} \). The term \( m_A \) is proportional to a (hypothetical) mass of an axion,\( m_A = 2\pi c m_{\text{axion}}/h \); \( h \) is the Planck constant. We use the signature \(+−−−\).

The variation of the action functional (1) with respect to the four-vector potential \( A_i \) gives the minimal (vacuum) Maxwell equations:
\[ \nabla_k \left[ F^{*ik} + \phi F^{*ik} \right] = 0, \]  
(4)

which can be transformed into
\[ \nabla_k F^{ik} + F^{*ik} \nabla_k \phi = 0 \]  
(5)
due to equation (3). Equivalently, equation (4) can be written as
\[ \nabla_k H^{ik} = 0, \quad H^{ik} = C^{ikmn} F_{mn}, \]  
(6)

where \( C^{ikmn} \) is the constitutive tensor:
\[ C^{ikmn} = \frac{1}{2} \left( g^{im} g^{kn} - g^{in} g^{km} \right) + \frac{1}{2} \phi \epsilon^{ikmn}, \]  
(7)

and \( H^{ik} \) is the excitation tensor \([8, 27]\). Minimal equations for the axion field can be obtained from action (1) by the variation with respect to the pseudoscalar field \( \phi \), yielding
\[ \left[ \nabla^k \nabla_k + m_A^2 \right] \phi = -\frac{1}{4} F^{*mn} F_{mn}. \]  
(8)

Minimal equations for the gravitational field obtained by the variation of (1) with respect to the metric \( g^{ik} \) are
\[ R_{ik} - \frac{1}{2} R g_{ik} = \kappa \left[ T^{\text{(EM)}}_{ik} + T^{(A)}_{ik} \right]. \]  
(9)

Here \( T^{\text{(EM)}}_{ik} \) is the standard stress–energy tensor of a pure electromagnetic field:
\[ T^{\text{(EM)}}_{ik} \equiv \frac{1}{4} g_{ik} F^{mn} F_{mn} - F_{im} F^{ik}, \]  
(10)

and \( T^{(A)}_{ik} \), given by
\[ T^{(A)}_{ik} \equiv \nabla_i \phi \nabla_k \phi - \frac{1}{2} g_{ik} \left[ \nabla^m \phi \nabla_m \phi - m_A^2 \phi^2 \right], \]  
(11)
is the stress–energy tensor of the pseudoscalar (axion) field \( \phi \).
2.2. Non-minimal extension of the model

2.2.1. Lagrangian and non-minimal susceptibilities. We consider the total action functional as a sum of minimal and non-minimal contributions:

\[ S = S(M) + S(NM), \]

(12)

where \( S(M) \) is given by (1). The non-minimal contribution \( S(NM) \) is, generally, a nonlinear function of all independent invariants containing the Riemann tensor \( R_{iklm} \) and the dual ones \( \ast R_{iklm}, R^*_{iklm}, \) Ricci tensor \( R_{mn} \) and Ricci scalar \( R \) in convolutions with the Maxwell tensor and its dual, as well as with derivatives of a pseudoscalar field. The set of such invariants for the pure electromagnetic field was discussed in [47]; then it was supplemented by the invariants containing scalar Higgs fields in [53, 54, 56]. For the axion field, this set of invariants can be constructed analogously, and here we focus only on the following Lagrangian linear in the curvature:

\[ S(NM) = \int \! d^4 x \sqrt{-g} \left\{ \frac{1}{2} R^{ikmn} F_{ik} F_{mn} + \frac{1}{2} \chi^{ikmn} \phi F_{ik} F^*_{mn} - \eta^{mn} \nabla_m \phi \nabla_n \phi + \eta(A) \phi \right\}. \]

(13)

The quantity \( R^{ikmn} \) is a non-minimal three-parameter susceptibility tensor [47], which has the form

\[ R^{ikmn} = q_1 R g^{ikmn} + q_2 \eta^{ikmn} + q_3 R^{ikmn}, \]

(14)

where

\[ g^{ikmn} = \frac{1}{2} (g^{im} g^{kn} - g^{in} g^{km}), \]

(15)

\[ \eta^{ikmn} = \frac{1}{2} (R^{im} g^{kn} - R^{in} g^{km} + R^{km} g^{in} - R^{km} g^{in}). \]

(16)

The constants \( q_1, q_2 \) and \( q_3 \) are non-minimal parameters describing the linear coupling of the Maxwell tensor \( F_{mn} \) with curvature [47]. The quantity \( \chi^{ikmn} \) given by

\[ \chi^{ikmn} = Q_1 R g^{ikmn} + Q_2 \eta^{ikmn} + Q_3 R^{ikmn}, \]

(17)

where \( Q_1, Q_2 \) and \( Q_3 \) are also constants, is the non-minimal susceptibility tensor describing the linear coupling of the dual tensor \( F^*_{mn} \) with curvature. As in the previous case, the combination \( \phi F^*_{mn} \) gives the tensor quantity. The tensor

\[ \eta^{mn} \]

(18)

describes a non-minimal susceptibility for the pseudoscalar field in analogy with [53], but in this case it contains a term linear in the Maxwell tensor. This term describes effects analogous to the so-called derivative coupling in the non-minimal scalar field theory [31–33]. As for the tenth coupling constant \( \eta(A) \), it is a direct analog of the well-known coupling constant \( \xi \) in the non-minimal scalar field theory (see, e.g., [30] for review and references).

The tensors \( R^{ikmn} \) and \( \chi^{ikmn} \), defined by (14) and (17), are skew-symmetric with respect to transposition of the indices \( i \) and \( k \), as well as \( m \) and \( n \). In addition, the following relations take place:

\[ R^{ikmn} = R^{mnik}, \quad \chi^{ikmn} = \chi^{mnik}, \]

(19)

which guarantee that the model under consideration does not contain the solutions of the skewon type [27]. The tensor \( \eta^{mn} \) is explicitly symmetric, i.e. \( \eta^{mn} = \eta^{nm} \).
2.2.2. Non-minimal electrodynamic equations. Electrodynamic equations, which correspond to the Lagrangian (12) with (1) and (13), are linear and have the standard form
\[ \nabla_i H^{ik} = I^i. \]  
(20)

The excitation tensor \( H^k \) and the Maxwell tensor \( F_{mn} \) are linked by the linear constitutive law:
\[ H^{ik} \equiv F^{ik} + \mathcal{R}^{ikmn} F_{mn} + \left[ \phi \left( F^{ik} + \chi_{(A)}^{ikmn} F_{mn} \right) \right]. \]  
(21)

The second term on the right-hand side is the curvature-induced polarization–magnetization, appearing in the non-minimally extended pure Einstein–Maxwell model [47]. The contribution detailed in square brackets is the one of the axion type, the terms \( \phi \) and \( F^{ik} \) enter the equations in the multiplicative form only (see, e.g., [8, 35] for more details). The non-minimal axion contribution \( \phi \chi_{(A)}^{ikmn} F_{mn} \) is a new term proposed here.

The four-vector of an effective electric current
\[ I^i = \frac{1}{2} \eta \nabla_i \left[ \left( R^{km} \nabla^i \phi - R^{im} \nabla^k \phi \right) \nabla_m \phi \right] \]  
(22)
is proportional to the coupling parameter \( \eta_1 \) and is due to the first term in (18). This four-vector does not contain the Maxwell tensor and satisfies the conservation law \( \nabla_i I^i = 0 \). It contains both the Ricci tensor and the pseudoscalar field; thus, it describes the electric current induced by curvature and axion field derivatives.

2.2.3. Non-minimal equation for the pseudoscalar field. The non-minimally extended master equation for the pseudoscalar \( \phi \) takes the form
\[ \nabla_m \left[ \left( g^{mn} + \gamma^{(A)}_{(A)} \right) \nabla_n \phi \right] + \left[ m_1^2 + \eta_{(A)} R \right] \phi = -\frac{1}{4} F^{mn} F_{mn} - \frac{1}{4} \chi_{(A)}^{ikmn} F_{ik} F_{mn}, \]  
(23)
where \( \gamma^{(A)}_{(A)} \) and \( \chi_{(A)}^{ikmn} \) are given by (18) and (17), respectively. This equation is a non-minimal generalization of (8).

2.2.4. Non-minimal equations for the gravitational field. Variation of the action functional (12) with (1) and (13) with respect to \( g^{ik} \) gives the non-minimally extended equations for the gravitational field:
\[ (R_k - \frac{1}{2} R g_{ik} (1 + \kappa \Theta)) = \kappa \left[ T_{ik}^{(EM)} + T_{ik}^{(A)} + T_{ik}^{(NMEM)} + T_{ik}^{(NMA)} \right]. \]  
(24)

Here the scalar \( \Theta \) stands for the quantity
\[ \Theta = \eta_{(A)} \phi^2 + \frac{1}{2} q_1 F_{mn} F^{mn} + \frac{1}{2} Q_1 \phi F_{mn} F^{mn} + \left( \frac{1}{2} \eta_3 - \eta_2 \right) \nabla_m \phi \nabla^n \phi, \]  
(25)
with the tensors \( T_{ik}^{(EM)} \) and \( T_{ik}^{(A)} \) given by (10) and (11), respectively. The non-minimal extension of the stress–energy tensor contains two contributions: first, \( T_{ik}^{(NMEM)} \) describing pure non-minimal electromagnetic part (see, e.g., [47] for details); second, the non-minimal axion part \( T_{ik}^{(NMA)} \). These tensors can be specified as follows:
\[ T_{ik}^{(NMEM)} = q_1 T_{ik}^{(1)} + q_2 T_{ik}^{(2)} + q_3 T_{ik}^{(3)}, \]  
(26)
\[ T_{ik}^{(NMA)} = Q_1 T_{ik}^{(1)} + Q_2 T_{ik}^{(2)} + Q_3 T_{ik}^{(3)} + \eta_1 T_{ik}^{(4)} + \eta_2 T_{ik}^{(5)} + \eta_3 T_{ik}^{(6)} + \eta_{(A)} T_{ik}^{(7)}, \]  
(27)
where
\[ T_{ik}^{(1)} = -RF_{im} F^m_k + \frac{1}{2} \left[ \nabla_i F_m^{\ell} - g_{ik} \nabla^j F_{mj} [F_{mn} F^{mn}] \right], \]  
(28)
\[ T_{ik}^{(2)} = -\frac{1}{2} g_{ik} \left[ \nabla_m \nabla_i \left( F^{mn} F^{\ell} - R^{ij} F^{mn} F_{mn} \right) \right] - F^{mn} \left[ R_{ij} F_{kn} + R_{k1} F_{in} \right] - R_{mn} F_{im} F_{kn} - \frac{1}{2} \left( \nabla_m \nabla_i \left( F_{kn} F^{\ell} \right) - \frac{1}{2} \nabla \left( \nabla_i \left( F_{kn} F^\ell \right) + \nabla_k \left( F_{in} F^{\ell} \right) \right) \right], \]  
(29)
\[ T^{(3)}_{ijk} = \frac{1}{2} g_{ik} R^{mnls}_{j} F^{m}_{n} F^{ls}_{j} - \frac{1}{4} g_{ik} \left( F^{n}_{j} R^{mnls}_{k} + F^{l}_{j} R^{mlns}_{k} \right) - \frac{1}{2} \nabla_{m} \nabla_{n} \left[ F^{n}_{j} F^{m}_{k} + F^{l}_{j} F^{m}_{k} \right]. \quad (30) \]

\[ T^{(1)}_{ik} = \frac{1}{2} \left[ \nabla_{i} \nabla_{k} - g_{ik} \nabla^{i} \nabla_{j} \right] \left[ \phi F^{mn}_{km} \right] - \frac{1}{4} g_{ik} R \phi F^{mn}_{km} F^{mn}_{km}. \quad (31) \]

\[ T^{(2)}_{ik} = -\frac{1}{4} \phi F^{mn}_{km} R_{in} F_{kn} + \frac{1}{4} \nabla_{i} \left[ \nabla_{k} \left[ \phi \left( F^{n}_{j} F^{m}_{k} + F^{l}_{j} F^{m}_{k} \right) \right] \right] + \frac{1}{2} \nabla_{i} \nabla_{n} \left[ \phi \left( F^{m}_{j} F^{n}_{k} + F^{l}_{j} F^{n}_{k} \right) \right] \quad \text{with} \quad \phi F^{mn}_{km} F^{mn}_{km}. \quad (32) \]

\[ T^{(3)}_{ik} = -\frac{1}{4} \nabla_{m} \nabla_{n} \left[ \phi \left( F^{mn}_{km} F^{mn}_{kn} + F^{sn}_{k} F^{mn}_{km} \right) \right] + \frac{1}{2} \phi F^{mn}_{km} \left( R_{i} F^{mn}_{km} + F^{sl}_{k} F^{mn}_{im} \right). \quad (33) \]

\[ T^{(4)}_{ik} = \frac{1}{2} g_{ik} \left( R^{n}_{m} \nabla^{m}_{n} \phi \left( F^{mn}_{km} \right) + \frac{1}{2} R_{n} \nabla_{n} \phi \left( F^{m}_{j} \nabla^{m}_{k} \phi + F^{l}_{j} \nabla^{m}_{k} \phi \right) \right) + \frac{1}{2} \nabla_{i} \left[ \nabla_{m} \phi \left( F^{mn}_{km} \right) \right] + \frac{1}{2} \nabla_{i} \left[ \nabla_{m} \phi \left( F^{mn}_{km} \right) \right] \quad \text{with} \quad \phi F^{mn}_{km} F^{mn}_{km}. \quad (34) \]

\[ T^{(5)}_{ik} = \frac{1}{4} g_{ik} \left( R^{n}_{m} \nabla^{m}_{n} \phi + R^{l}_{m} \nabla^{m}_{l} \phi \right) - \frac{1}{4} R_{ik} \nabla_{m} \phi \left( F^{mn}_{km} \right) + \frac{1}{2} \nabla_{i} \nabla_{m} \left[ \nabla^{m} \phi \right] - \frac{1}{2} \nabla_{m} \phi \left( \nabla_{i} \nabla_{k} \right) \quad \text{with} \quad \phi F^{mn}_{km} F^{mn}_{km}. \quad (35) \]

\[ T^{(6)}_{ik} = \nabla_{i} \nabla_{k} - g_{ik} \nabla^{i} \nabla^{m} \phi \quad \text{with} \quad \phi F^{mn}_{km} F^{mn}_{km}. \quad (36) \]

\[ T^{(7)}_{ik} = \left( \nabla_{i} \nabla_{k} - g_{ik} \nabla^{i} \nabla^{m} \phi \right) \phi. \quad (37) \]

Straightforward calculations show that the following identity takes place:

\[
\nabla^{i} \left\{ (1 + \kappa) \Theta^{-1} \left[ T^{(EM)}_{ik} + T^{(NME)}_{ik} + T^{(NMA)}_{ik} \right] \right\} = 0, \quad \text{i.e. the total effective stress–energy tensor written in the braces is a conserved quantity.} \quad (38)
\]

2.3. Non-minimal constitutive equations for the electromagnetic field

Relation (21) is a linear constitutive equation [8, 27, 57] of the following type:

\[ H^{ik} = C^{ikmn} F_{mn} \]

\[ \text{where} \]

\[ C^{ikmn} = g^{ikmn} + \frac{1}{2} \phi \epsilon^{ikmn} + \mathcal{R}^{ikmn} + \frac{1}{2} \phi \left[ \chi^{ikmn} + \chi^{ikmn} \phi \right]. \quad (40) \]

Here we use the standard definitions for the right and left dualization:

\[ \chi^{ikmn} \equiv \frac{1}{2} \epsilon^{ikpq} \epsilon^{mn} \rho q, \quad \chi^{ikmn} \phi \equiv \frac{1}{2} \epsilon^{ikpq} \phi \epsilon^{mn} \rho q. \quad (41) \]

The tensor \( C^{ikmn} \) describes the linear response of the material to the electromagnetic field action and contains the information about non-minimal dielectric permittivity and magnetic permeability, as well as about the non-minimal magneto-electric coefficients in analogy with the standard continuum electrodynamics [57–59].

Let us mention that in many works (e.g. in [27], [60–62]), a constitutive tensor density \( \tilde{\chi}^{ikmn} = \sqrt{-g} C^{ikmn} \) is used for the description of linear response instead of a true tensor \( C^{ikmn} \). In terms of the quantity \( \tilde{\chi}^{ikmn} \), the condition for no birefringence (no splitting, no retardation) for electromagnetic wave propagation in all directions in the weak field limit
gives ten constraint equations on the components of constitutive tensor density (see, e.g., [60–63]). With these ten constraints \( \hat{\chi}^{ikmn} \) can be rewritten in the following form:

\[
\hat{\chi}^{ikmn} = \frac{1}{2} \sqrt{-H} \psi \left( H^{im} H^{kn} - H^{in} H^{km} \right) + \phi E^{ikmn},
\]

(42)

where \( H = \det(H_{ik}) \) is a determinant of an effective metric \( H_{ik} \), which generates the light cone for electromagnetic wave propagation; \( \psi \) is some dilation factor; \( \phi \) differs from our \( \phi \) by the coefficient 2, \( \phi = 2\phi \). In the case when (42) is not satisfied, birefringence will occur. In the case when (42) is satisfied, a pseudoscalar field \( \phi \) will give a polarization rotation (an optical activity induced by \( \phi \)). The effects of birefringence and polarization rotation described in terms of constitutive tensor density are studied in the 1970s and 1980s. In particular, constraints on birefringence from pulsar signal observations give the following estimates for the birefringence part \( \delta \hat{\chi}^{ikmn} \) of \( \hat{\chi}^{ikmn} \) in a weak field:

\[
|\delta \hat{\chi}^{ikmn}| < 10^{-14} - 10^{-16}.
\]

(43)

Using the medium velocity four-vector \( U^i \), normalized by \( U^i U_i = 1 \), one can decompose \( C^{ikmn} \) uniquely as

\[
C^{ikmn} = \frac{1}{2} \left( \epsilon^{ik} U^k U^n - \epsilon^{in} U^k U^m + \epsilon^{kn} U^i U^m - \epsilon^{km} U^i U^n \right)
+ \frac{1}{2} \left[ -\eta^{ik} (\mu^{-1})_{ik} \eta^{mn} + \eta^{kn} (U^m U^n - U^n U^m) + \eta^{lm} \left( U^i U_k^j - U^k U_i^j \right) \right].
\]

(44)

Here \( \epsilon^{ik} \) is the dielectric permittivity tensor, \((\mu^{-1})_{ik}\) is the magnetic impermeability tensor and \( \nu^{pm}_m \) is the tensor of magneto-electric coefficients. These quantities are defined as follows:

\[
\epsilon^{im} = 2 \epsilon^{ikmn} U_k U_n, \quad (\mu^{-1})_{pq} = -\frac{1}{2} \eta_{ik} C^{ikmn} \eta_{mnp},
\]

(45)

\[

v^{pm}_m = \eta_{ip} C^{ikmn} U_n = U_k C^{kinn} \eta_{mnp}.
\]

We use the symbols \( \eta_{mnl} \) and \( \Delta^i_k \) for the tensors

\[
\eta_{mnl} = \epsilon_{mnl} U^i, \quad \Delta^i_k = g^{ik} - U^i U^k,
\]

(46)

orthogonal to \( U^i \). The tensors \( \epsilon^i_k \) and \((\mu^{-1})_{ik}\) are symmetric, but \( v_{ik} \) is in general non-symmetric. These three tensors are orthogonal to \( U^i \),

\[
\epsilon^i_k U_k^j = 0, \quad (\mu^{-1})_{ik} U^j = 0, \quad v^k U^j = 0 = \nu^k U_k.
\]

(47)

Using expression (40), one can calculate the tensors \( \epsilon^{im} \), \((\mu^{-1})_{im}\) and \( v^{pm} \) explicitly. The symmetric dielectric permittivity tensor

\[
\epsilon^{im} = \Delta^{im} \left[ 1 + q_1 R + \frac{q_2}{2} R^{pq} U_p U_q \right] + q_2 R_{pq} \Delta^{ip} \Delta^{mq} + 2 q_3 R^{pmpq} U_p U_q + Q_3 \phi U_p U_q, \quad R^{pmpq} = R^{ipmpq} + R^{ipqp},
\]

(48)

as well as the symmetric magnetic impermeability tensor

\[
(\mu^{-1})^{im} = \Delta^{im} \left[ 1 + q_1 R + \frac{q_2}{2} R^{pq} U_p U_q \right] - q_2 R_{pq} \Delta^{ip} \Delta^{mq} - 2 q_3 R^{pmpq} U_p U_q + Q_3 \phi U_p U_q, \quad R^{pmpq} = R^{ipmpq} + R^{ipqp}
\]

(49)

do not contain the coupling parameters \( Q_1 \) and \( Q_2 \). The cross-tensor

\[
v^{pm} = q_2 \eta^{pm} R_{ij} U^i U^j + 2 q_3 R^{pmlmn} U^i U^j - \Delta^{pm} \left[ 1 + \left( Q_1 + \frac{1}{2} Q_2 \right) R \right] + Q_3 \phi U_p U_q, \quad R^{pmlmn} = R^{pmlmn} - R^{plmn},
\]

(50)

which gives the optical activity effects (see, e.g., [59]), contains two explicitly distinguished parts. The first part does not contain a pseudoscalar field \( \phi \), is linear in the parameters \( q_2 \) and \( q_3 \) and describes optical activity induced by direct non-minimal interaction between electromagnetic and gravitational fields (such a model was analyzed in [54, 65]). It is important to mention that this part of the cross-tensor \( v^{pm} \) contains both skew-symmetric and symmetric
Table 1. Ten non-minimal (NM) coupling parameters are divided into four subgroups: the first $q_1, q_2, q_3$; second $Q_1, Q_2, Q_3$; third $\eta_1, \eta_2, \eta_3$; and fourth $\eta(A)$. In the second column the terms in the Lagrangian are given in front of which the corresponding coupling parameters are introduced; the parameters of the first subgroup introduce the terms without the pseudoscalar field $\phi$; the parameters of the second subgroup relate to the terms linear in $\phi$; the terms indicated by $\eta_1, \eta_2, \eta_3$ are quadratic in the four-gradient of $\phi$; and finally, $\eta(A)$ introduces the term quadratic in $\phi$. In the last column, we point out the physical meaning of these non-minimal terms, based on decompositions of the constitutive tensors for the electromagnetic (EM) and pseudoscalar fields.

| Term in the Lagrangian | Physical meaning |
|------------------------|------------------|
| $q_1 \frac{1}{2} R^{mn} F_{mn}$ | NMEM susceptibility linear in the Ricci scalar |
| $q_2 R^{mn} F_{nk} F^{nk}$ | NMEM susceptibility linear in the Ricci tensor |
| $q_3 \frac{1}{2} \phi R^{mn} F_{mn}$ | NMEM susceptibility induced by the axion with the Ricci scalar |
| $Q_1 \frac{1}{2} \phi R^{mn} F_{mn}^{*}$ | NMEM susceptibility induced by the axion with the Ricci tensor |
| $Q_2 \frac{1}{2} \phi R^{mn} F_{nk} F^{nk}$ | NMEM susceptibility induced by the axion with the Riemann tensor |
| $Q_3 \frac{1}{2} \phi R^{mn} F_{nk} F^{nk}$ | NMEM susceptibility induced by the axion with the Riemann tensor |
| $\eta_1 \phi R^{mn} \nabla_{m} \phi \nabla_{n} \phi$ | NMEM current induced by the axion field gradient |
| $\eta_2 \phi R^{mn} \nabla_{m} \phi \nabla_{n} \phi$ | NM axion–graviton derivative coupling with the Ricci scalar |
| $\eta_3 \phi R^{mn} \nabla_{m} \phi \nabla_{n} \phi$ | NM axion–graviton derivative coupling with the Ricci tensor |
| $\eta(A) R \phi^{2}$ | NM correction to the mass square of the axion |

terms (see the terms with $\eta_{pmk}$ and $^* R^{plmn} U_{i} U_{k}$). The second contribution to $\nu^{pm}$, which is proportional to the pseudoscalar field $\phi$, is symmetric with respect to transposition of indices (see the terms, containing the projector $\Delta^{pm}$ and the tensor $^* R^{plmn} = R^{plmn}$). It describes an optical activity induced by the pseudoscalar field $\phi$, is symmetric with respect to transposition of indices and the terms proportional to $Q_1, Q_2, Q_3$ relate to non-minimal effects. It is well known that the non-vanishing cross-tensor $\nu^{ik}$ indicates that the medium is optically active, and the rotation of the Faraday type takes place in the course of electromagnetic wave propagation. Thus, one can see directly from (50) that the non-minimal interaction between electromagnetic and axion fields produces curvature-induced optical activity of a new type. Below we consider this effect explicitly by the example of a model with pp-wave symmetry.

In table 1, we summarize briefly the information about non-minimal coupling constants.

3. Non-minimal models with pp-wave symmetry

3.1. Reduced non-minimal field equations

We consider the line element for the spacetime with pp-wave symmetry in the form [66]

$$ ds^2 = 2du dv - L^2 \{ \cosh 2\gamma [e^{2\beta} (dx^2)^2 + e^{-2\beta} (dx^3)^2] + 2 \sinh 2\gamma dx^2 \, dx^3 \}, $$

(51)

where $u = \frac{ct - x^1}{\sqrt{2}}$ and $v = \frac{ct + x^1}{\sqrt{2}}$ are the retarded and the advanced times, respectively; $\beta(u)$ and $\gamma(u)$ are functions of retarded time only; they determine the polarization of the gravitational wave field. The case when the gravitational pp-wave field is characterized by one polarization only ($\gamma = 0$) is investigated in detail in [64]. Various properties of pp-waves are discussed, e.g., in [67–69]. For the metric (51), the Riemann tensor has only three non-vanishing components $R_{2u2u}, R_{3u3u}$ and $R_{2u3u}$. The Ricci tensor has only one (generally) non-vanishing component $
\( R_{\mu\nu} \) and the Ricci scalar is equal to zero identically \( R = 0 \). For this metric the three Killing vectors, which form an Abelian subgroup of the group \( G_5 \) [69], are

\[
\xi^{(1)}_{(v)} = \delta^1_v, \quad \xi^{(2)}_{(v)} = \delta^2_v, \quad \xi^{(3)}_{(v)} = \delta^3_v.
\]  

(52)

The Killing vector \( \xi^{(v)} \) is a covariant constant null vector orthogonal to \( \xi^{(1)}_{(v)} \) and \( \xi^{(3)}_{(v)} \). We suggest here that the axion and electromagnetic field potentials inherit the spacetime symmetry, i.e. the Lie derivatives satisfy the following relations:

\[
\xi^{(5)}_v \phi = 0, \quad \xi^{(5)}_v A_\mu = 0,
\]  

(53)

where \( (u) \) take the values \( (v), (2) \) and \( (3) \). As a consequence of (53), the pseudoscalar field and electromagnetic field potential depend on the retarded time only, \( \phi = \phi(u), A_\mu = A_\mu(u) \).

Taking into account the Lorentz gauge \( \nabla \xi A^\mu = 0 \), we can choose the potential in the form

\[
A_\mu = A_k = 0, \quad A_2 = A_2(u), \quad A_3 = A_3(u).
\]  

(54)

The next step is to verify that pp-wave symmetry is admitted by the total system of non-minimal master equations. Direct calculations show that for such an ansatz about symmetry the non-minimal equations for the pseudoscalar field \( \phi(u) \), i.e. equations (23) with (18), are satisfied identically for arbitrary \( Q_1, \ldots, Q_3 \) and \( \eta_{(A)} \), when the axion field is massless, i.e. \( m_{(A)} = 0 \). Also, the non-minimal dyadic equations (20) with (21) and (22) are satisfied identically for arbitrary coupling constants, when the potential takes the form (54). Finally, the non-minimal equations for the gravitational field (24) with (25)–(37) can be reduced to one equation only, namely

\[
-\frac{2}{\kappa} \left[ \frac{L''}{L} + (\beta')^2 \cosh^2 2\gamma + (\gamma')^2 \right] [1 + \kappa \eta_{(A)} \phi^2] = (\phi')^2 (1 + 2\eta_{(A)}) + 2\eta_{(A)} \phi \phi''
\]

\[
+ \frac{1}{L^2} \{- \cosh 2\gamma [\{A'_2 \, e^{-\beta}\}^2 + (A'_3 \, e^{\beta})^2] - 2 \sinh 2\gamma \, A'_2 \, A'_3 \}.
\]  

(55)

The prime denotes here and below the derivative with respect to the retarded time \( u \). The right-hand side of this equation includes, as usual, the minimal contributions from the tensors \( T_{ik}^{(A)} \) and \( T_{ik}^{(EM)} \). Only two non-minimal contributions came from the term \( T_{ik}^{(7)} \) (37) and from the term \( \kappa \eta_{(A)} \phi^2 \) in the expression for \( \Theta \) (25). The non-minimal coupling constants \( q_1, q_2, q_3, Q_1, Q_2, Q_3, \eta_1, \eta_2, \eta_3 \) are non-vanishing, but they happen to be hidden in this pp-wave model. Thus, in the presented model, six functions \( \phi(u), A_2(u), A_3(u), \beta(u), \gamma(u) \) and \( L(u) \) are linked by one equation only (55), and this model admits very wide possibilities in searching for exact solutions.

3.2. Exact solutions of the pp-wave type: quasi-minimal models with \( \eta_{(A)} = 0 \)

When the non-minimal parameter \( \eta_{(A)} \) is vanishing, we deal with a model which looks like minimal, i.e. the key equation for the pp-wave gravitational field

\[
-\frac{L''}{L} = (\beta')^2 \cosh^2 2\gamma + (\gamma')^2 + \frac{\kappa}{2} (\phi')^2
\]

\[
+ \frac{\kappa}{2L^2} \{- \cosh 2\gamma [\{A'_2 \, e^{-\beta}\}^2 + (A'_3 \, e^{\beta})^2] - 2 \sinh 2\gamma \, A'_2 \, A'_3 \}
\]  

(56)

do not contain non-minimal parameters at all. That is why we indicate this case with arbitrary but hidden parameters \( q_1, q_2, q_3, Q_1, Q_2, Q_3, \eta_1, \eta_2, \eta_3 \) and vanishing \( \eta_{(A)} \) as quasi-minimal.

Clearly, the quantity \( \left( -\frac{L''}{L} \right) \) on the left-hand side of (56) has to be non-negative. This property of \( L(u) \), which has initial value \( L(0) = 1 \), provides \( L(u) \) to vanish at some moment \( u = u^* \) (\( L(u^*) = 0 \)) (see, e.g., [68]). This moment of the retarded time relates to the well-known singularity, the physical sense of which is discussed in detail in many books. In [69], a number
of papers are quoted, in which exact solutions of (56) are obtained for the pure electromagnetic source (\(\phi = 0\)). In addition, exact solutions of the model of this type are known for vanishing \(A_k\), when the (pure) scalar field \(\Phi\) is the source of the pp-wave gravity field (see, e.g., \([70]\)). Finally, when \(A_k = 0\) and \(\phi = 0\), we deal with the so-called pure gravitational pp-wave (see, e.g., \([64, 67–69]\)). We would like to list here only three examples from the wide collection of minimal solutions of equation (56), which relates to the models with symmetric spacetime of the pp-wave type. The models with symmetric spacetime are characterized by the condition \([67, 69]\)

\[\nabla_k R^i_{\text{met}} = 0. \tag{57}\]

In particular, when \(\gamma(u) = 0\), one obtains from (57) that

\[
\begin{align*}
R^2_{\text{a2u}} &= - \left[ \frac{L''}{L} + (\beta')^2 \right] - \left[ 2 \frac{L'}{L} \beta' + \beta'' \right] = \lambda_1, \\
R^3_{\text{a3u}} &= - \left[ \frac{L''}{L} + (\beta')^2 \right] + \left[ 2 \frac{L'}{L} \beta' + \beta'' \right] = \lambda_2,
\end{align*}
\]

where \(\lambda_1\) and \(\lambda_2\) are some constants. The summation of (58) and (59) yields

\[
R_{uu} = \left[ \frac{L''}{L} + (\beta')^2 \right] = \lambda_1 + \lambda_2. \tag{60}
\]

Thus, for the symmetric spacetime, the key equation for the (quasi) minimal model can be reduced to

\[
\frac{2}{\kappa} (\lambda_1 + \lambda_2) = (\phi')^2 + \frac{1}{L^2} [(A_k' e^{-\beta})^2 + (A_k' e^{\beta})^2]. \tag{61}
\]

In more appropriate terms, when

\[
F''(u) + \lambda_1 F = 0, \quad G''(u) + \lambda_2 G = 0, \tag{63}
\]

which can be easily solved if the signs of the constants \(\lambda_1\) and \(\lambda_2\) are fixed.

(i) First example. The first model of such type relates to the pure gravitational wave, i.e. to the case when \(R_{uu} \equiv 0\). Suggesting that \(\lambda_1 = -\lambda_2 \equiv \nu^2\), we obtain immediately from (63) the well-known Petrov’s solution \([67]\)

\[
ds^2 = 2du dv - \cos^2 \nu u (dx^2)^2 - \cosh^2 \nu u (dx^3)^2, \tag{64}
\]

with

\[
F(u) = \cos \nu u, \quad G(u) = \cosh \nu u. \tag{65}
\]

This solution is admissible when \(\phi = \phi_0 = \text{const}, \ A_k = 0\). The quantity \(\sqrt{-g} \equiv \cos \nu u \cosh \nu u\) vanishes at \(u^* = \pi / 2\nu\).

(ii) Second example. When \(A_2 = A_3 = 0, \ \beta = \gamma = 0\) and function \(\phi(u)\) is linear in the retarded time, i.e. \(\phi(u) = \phi_0 + \omega u\), then (56) gives

\[
L(u) = \cos \sqrt{\frac{\kappa}{2}} \omega u, \quad L(0) = 1, \quad L'(0) = 0. \tag{66}
\]

For such a metric one obtains that

\[
R^2_{\text{a2u}} = R^3_{\text{a3u}} = - \frac{L''}{L} = \frac{\kappa \omega^2}{2} = \lambda_1 = \lambda_2. \tag{67}
\]

The first zero of the function \(L\) is \(u = u^* = \frac{\pi}{\omega \sqrt{2\kappa}}\).
(iii) Third example. When \( \lambda_1 = \nu_2^2, \lambda_1 = -\mu_2^2 \) and \( \phi = 0 \), there is an exact solution

\[
F(u) = \cos \nu u, \quad G(u) = \cosh \mu u,
\]

with

\[
\nu_2^2 = \mu_2^2 + \frac{\kappa E_0^2}{2}.
\]

The electromagnetic field in this model is presented by the following potentials:

\[
A_2(u) = \frac{E_0}{(\omega^2 - \nu_2^2)} [\omega \sin \omega u \cos \nu u - \nu \sin \nu u \cos \omega u],
\]

\[
A_3(u) = \frac{E_0}{(\omega^2 + \mu_2^2)} [\mu \sin \omega u \sinh \mu u - \omega \cos \omega u \cosh \mu u].
\]

This electromagnetic wave can be considered as a circularly polarized wave, since the physical components of the electric field satisfy the following conditions:

\[
E_2^2 \equiv -(E_2 E_2^2 + E_3 E_3^2) \equiv -g^{\alpha \beta} A_\alpha' A_\beta' = E_0^2, \quad (\alpha, \beta = 2, 3).
\]

When \( \eta_{(A)} \neq 0 \), the quantity \( \left(-\frac{L''}{T}\right) \) can change the sign, since the term \( \eta_{(A)} (\phi^2)'' \) on the right-hand side of (55) is not positively defined in the general case. This means that the singularity \( L = 0 \) can, in principle, be avoided.

3.3. Exact solutions of the pp-wave type: non-minimal models with \( \eta_{(A)} \neq 0 \)

Consider now exactly solvable models with the non-vanishing coupling constant \( \eta_{(A)} \).

3.3.1. Regular model. Let us consider a model with \( L(u) \equiv 1 \) and \( \gamma(u) \equiv 0 \). It can be indicated as the regular one, since \( \det(g_{ik}) = -L^4 \equiv -1 \) and cannot vanish. The equation for \( \beta \) reduces to

\[
-\frac{\kappa}{2} (\beta')^2 \left[ 1 + \kappa \eta_{(A)} \phi^2 \right] = (\phi')^2 + \eta_{(A)} (\phi^2)'' + (A_2 e^{-\beta})^2 + (A_3 e^{\beta})^2.
\]

When \( \eta_{(A)} = 0 \), there is no real solutions of this equation, but such a possibility appears in the non-minimal case. We consider only one example of the exact regular models; it is characterized by

\[
\phi = \phi_0, \quad A_2(u) = A_2(0) e^{\beta(u)}, \quad A_3(u) = A_3(0) e^{-\beta(u)},
\]

and is possible when \( \eta_{(A)} < 0 \) and

\[
\phi_0^2 |\eta_{(A)}| = 1 + \frac{\kappa}{2} [A_2^2(0) + A_3^2(0)].
\]

The function \( \beta(u) \) is arbitrary; we prefer to use the following periodic finite function:

\[
\beta(u) = \frac{1}{2} \beta_{(\text{max})} (1 - \cos 2\nu u), \quad \beta(0) = 0, \quad \beta'(0) = 0.
\]

The metric for this non-minimal model is regular:

\[
ds^2 = 2 \, du \, dv - |\exp[2\beta_{(\text{max})}] \sin^2 \lambda u| (dx^2)^2 + \exp[-2\beta_{(\text{max})}] \sin^2 \lambda u |(dx^3)^2|;
\]

the potentials of the electromagnetic field and their derivatives are also periodic and regular.
3.3.2. Non-minimal models with symmetric spacetime of the pp-wave type. The key equation for the metric coefficients can be reduced in this case to the relation

$$\frac{2}{\kappa} (\lambda_1 + \lambda_2) [1 + \kappa \eta(A) \phi^2] = (\phi')^2 (1 + 2\eta(A)) + 2\eta(A) \phi \phi'' + \frac{1}{L^2} [(A') e^{-\beta})^2 + (A') e^\beta]^2. \ (78)$$

Let us consider three exact solutions of this equation, which can be indicated as the non-minimal ones.

(1) First exact solution. Let us suppose that $\lambda_1 = -\lambda_2 \equiv \nu^2$, and we again deal with the Petrov solution (64). This solution is admissible when

$$\{(\phi')^2 (1 + 2\eta(A)) + 2\eta(A) \phi \phi'' - [A'' e^\nu + A' \cos \nu u]^2 \} \cos^2 \nu u \cosh^2 \nu u + \left[ A'' e^\nu - A' \cos \nu u \right]^2 = 0. \ (79)$$

One of the solutions of this equation, which relates to the case $\eta(A) = -\frac{1}{4}$, is given by

$$\phi(u) = \phi_0 \cos \sigma u, \quad \phi_0 = \pm \sqrt{\frac{2}{\sigma}}, \quad \sigma = \text{const}, \quad \eta(A) < 0, \quad \kappa E_0^2 = 4\mu^2 (|\eta(A)| \phi_0^2 - 1) > 0. \ (80)$$

(2) Second exact solution. In the case $\lambda_1 = \lambda_2 = -\mu^2$, an appropriate example of the model is characterized by

$$\beta(u) = 0, \quad L(u) = F(u) = G(u) = \cosh \mu u, \quad L(0) = 1, \quad L'(0) = 0, \quad \mu = \sqrt{-\kappa E_0^2 (1 + 2\eta(A) \phi_0^2)} \ (81)$$

$$\beta''(u) = \frac{E_0}{(\sigma^2 + \mu^2)} \left[ \sigma \sin ou \cos \nu u - \nu \sin \nu u \cos ou \right], \quad \beta'(0) = 0 \ (82)$$

$$\beta'(u) = \frac{E_0}{(\sigma^2 + \mu^2)} \left[ \nu \sin ou \sinh \nu u - \omega \cos \nu u \cosh \nu u \right]. \ (83)$$

Here $E_0$, $\sigma$, $\omega$ and $\nu$ are arbitrary constants. When $\omega^2 = \nu^2$, the expression for $A_3(u)$ should be replaced by $A_3(u) = \frac{E_0}{\sigma} (2\nu u + \sin 2\nu u)$. Let us mention that in the paper [70], the case $\xi = 1/4$ is also indicated as the special one ($\eta(A)$ in our notations corresponds to $\xi$ in [70]).

(3) Third exact solution. When $\lambda_1 = \nu^2$, $\lambda_2 = -\mu^2$, there is a simple non-minimal modification of the solution (68), (69) with the constant non-vanishing pseudoscalar field

$$F(u) = \cos \nu u, \quad G(u) = \cosh \mu u, \quad \phi(u) = \phi_0, \quad \nu^2 = \mu^2 + \frac{\kappa E_0^2}{2(1 + \kappa \eta(A) \phi_0^2)} \ (84)$$

with electromagnetic potentials given by (70), (71).
3.3.3. Special model. When a pseudoscalar field is constant, i.e. \( \phi(u) = \phi_0 \), and the electromagnetic field is absent, the equation for the gravitational field takes the form

\[
\left[ \frac{L''}{L} + (\beta')^2 \cosh^2 2\gamma + (\gamma')^2 \right] [1 + \kappa \eta(A) \phi_0^2] = 0.
\] (90)

When \( \phi_0^2 = -1/\kappa \eta(A) \), the equations for the gravity field are satisfied for arbitrary \( L(u), \beta(u) \) and \( \gamma(u) \), and we deal with a non-trivial special case. It corresponds to the case when the Ricci scalar disappears from the Lagrangian.

3.3.4. Cheshire smile. When the spacetime is the Minkowski one, and \( L = 1, \beta = 0, \gamma = 0 \), equation (56) yields

\[
0 = (\phi')^2 + \eta(A) (\phi^2)'' + (A_2')^2 + (A_3')^2,
\] (91)

and this equation admits real solutions when the non-minimal parameter \( \eta(A) \) is non-vanishing.

The exotic situation when the (pure) scalar field does not curve the spacetime is called in [71] the gravitational Cheshire smile. Here we deal with the pseudoscalar field instead of the pure scalar one; nevertheless, the physical sense of such a solution remains the same as in [71].

When the electromagnetic field is described by the circularly polarized wave with

\[
A_2(u) = \frac{E}{\lambda} \cos \lambda u, \quad A_3(u) = \frac{E}{\lambda} \sin \lambda u,
\] (92)

then the pseudoscalar field satisfies the equation

\[
(\phi')^2 + \eta(A) (\phi^2)'' + E^2 = 0,
\] (93)

whose solution can be represented in quadratures as

\[
\pm (u + C_2) = \int \frac{d\phi}{\sqrt{C_1 \phi^{12\eta(A)} - \frac{E^2}{\Omega^{2\eta(A)}}}}.
\] (94)

One of the explicit particular solutions to (94) is

\[
\phi = \phi_0 \cosh ku, \quad \eta(A) = -\frac{1}{4}, \quad \phi_0 = \sqrt{2} \frac{E}{k}.
\] (95)

Again we used the special value of the non-minimal parameter \( \eta(A) = -\frac{1}{4} \) as in [70]. This solution is infinite at \( u \to \infty \).

In table 2, we summarize and classify exact solutions obtained above for the model with pp-wave symmetry.

4. Propagation of electromagnetic waves coupled to the background axion and gravitational pp-wave fields

4.1. Evolutionary equations for the potentials

Consider now test electromagnetic waves propagating in an arbitrary direction in the pure pp-wave background with \( \gamma(u) = 0 \). The test electromagnetic field satisfies the non-minimal evolutionary equations (20), (21) with a vanishing current on the right-hand side, and the solutions, describing the axion field \( \phi(u) \) and the pp-wave gravitational field, are assumed to be unperturbed. We deal below with pure gravitational waves; thus, the Ricci scalar and the Ricci tensor vanish \( (R = 0, R_{\text{mun}} = 0) \), providing that the nonvanishing components of
Table 2. All the obtained solutions are classified with respect to the pseudoscalar field type: \( \phi \) is a non-vanishing constant \( \phi_0 \); \( \phi \) depends on retarded time and \( \phi'(u) \neq 0 \); \( \phi \) vanishes (see horizontal lines); and with respect to values of the guiding parameter: \( \eta(A) = 0 \) or \( \eta(A) \neq 0 \). We distinguish solutions of pure gravitational wave type (GW), co-moving gravito-electromagnetic waves (GEMW), as well as axion–GEMW and axion–EMW. The cases indicated as the solutions for symmetric spacetime relate to the model with the covariantly constant Riemann tensor, the solutions of the Petrov type being the particular subcase of such spacetime with vanishing Ricci tensor. In the parentheses we indicate the initial formula in the text related to the corresponding solution. The solutions with \( \phi(u) \equiv 0 \) are not new; they are introduced to complete the table.

| \( \eta(A) = 0 \) | \( \eta(A) \neq 0 \) |
|-----------------|------------------|
| \( \phi(u) = \phi_0 \) | GW of Petrov’s type (64) | Regular periodic GEMW (74) |
| \( \phi_0 \neq 0 \) | Regular aperiodic GEMW (83) | GW with periodic EMW (88) |
| | Symmetric spacetime of GW type with the axion field linear in time (66) | Special solution (90) |
| \( \phi'(u) \neq 0 \) | Axion–GEMW of Petrov’s type (80) | Axion–EMW Cheshire smile (91) |
| \( \phi(u) \equiv 0 \) | Co-moving GEMW (68) | Co-moving GEMW (68) |



the Riemann tensor are of opposite signs \( (R^2_{u2u} = -R^3_{u3u}) \). Also, we require the potential four-vector to satisfy the Lorentz gauge condition

\[
\nabla_k A^k = 0 \quad \Rightarrow \quad \partial_v A_u + \partial_u A_v + g^{\alpha\beta} \partial_{\alpha} A_{\beta} + \frac{2L'}{L} A_v = 0. \quad (96)
\]

Let us mention that the problem of propagation of electromagnetic waves coupled to axions in the geometric optics approximation is studied in the paper [72] on the basis of investigation of the Fresnel equation. Here we do not restrict ourselves to this approximation and consider exact solutions of the equations of axion electrodynamics in analogy with solutions of the model of non-minimal optical activity discussed in [73].

Due to condition (96) the first equation from (20) (with \( i = u \)) accounting for (21) gives the following simple equation for the longitudinal potential \( A_v = \xi_{(v)} A_i \):

\[
\hat{D} A_v = \frac{-2L'}{L} \partial_v A_v, \quad (97)
\]

where \( \hat{D} \) is the differential operator:

\[
\hat{D} \equiv g^{\alpha\beta} \partial_\alpha \partial_\beta = 2\partial_u \partial_v - L^{-2} (e^{-2\beta} \partial_v^2 + e^{2\beta} \partial_u^2). \quad (98)
\]

Clearly, (97) is an equation for \( A_v \) only; it does not include other components of the potential four-vector. The exact solution of this equation takes the form (see [74])

\[
A_v = L^{-1}(u) B_v(W), \quad (99)
\]

where \( B_v(W) \) is an arbitrary function of the phase scalar \( W \), given by

\[
W \equiv W_0 + \frac{1}{2k_v} \left[ k_2^2 \int du L^{-2} e^{-2\beta} + k_3^2 \int du L^{-2} e^{2\beta} \right] + k_v v + k_2 x^2 + k_3 x^3, \quad (100)
\]

with arbitrary constant values of \( W_0, k_2, k_3 \) and \( k_v \). The wave four-vector \( K_i \),

\[
K_i \equiv \nabla_i W = -\delta_i^v \frac{1}{2k_v} (g^{22} k_2^2 + g^{33} k_3^2) + \delta_i^v k_v + \delta_i^2 k_2 + \delta_i^3 k_3, \quad (101)
\]
is the null-four vector, i.e. $g^{\mu\nu} K_\mu K_\nu = 0$ and the longitudinal component of the electromagnetic potential $A_\mu$ propagates with the speed of light in vacuum. A special solution $B_i(W)$ has been motivated in [74]. This solution relates to the Landau gauge condition $\xi^{(\nu)}_{\nu\alpha} A_\alpha = 0$, and we will use it below. Equations for the transversal components $A_2$ and $A_3$ form a coupled system

$$\begin{align*}
[\hat{D} + 2q_3 R^2_{a2a} \partial_v^2 - 2\beta' \partial_v] A_2 + e^{2\beta} [\phi' \partial_v + 2Q_3 \phi R^2_{a2a} \partial_v^2] A_3 &= 0, \\
[\hat{D} + 2q_3 R^3_{a3a} \partial_v^2 + 2\beta' \partial_v] A_3 - e^{-2\beta} [\phi' \partial_v + 2Q_3 \phi R^3_{a3a} \partial_v^2] A_2 &= 0.
\end{align*}$$

(102)

(103)

The replacement

$$A_2 = e^\beta B_2, \quad A_3 = e^{-\beta} B_3$$

(104)

simplifies these equations yielding

$$\begin{align*}
[\hat{D} + 2q_3 R^2_{a2a} \partial_v^2] B_2 + [\phi' \partial_v + 2Q_3 \phi R^2_{a2a} \partial_v^2] B_3 &= 0, \\
[\hat{D} + 2q_3 R^3_{a3a} \partial_v^2] B_3 - [\phi' \partial_v + 2Q_3 \phi R^3_{a3a} \partial_v^2] B_2 &= 0.
\end{align*}$$

(105)

(106)

Below we obtain and discuss exact solutions of these equations, suggesting that the functions $\phi(u)$, $L(u)$ and $\beta(u)$ are presented by one of the exact solutions listed in the previous section.

4.2. Birefringence induced by curvature in the case of a vanishing pseudoscalar field ($\phi = 0$)

In order to clarify a physical meaning of the parameter $q_3$, let us consider the simplest model with $\phi = 0$, i.e. the model without axion–photon interaction. As was shown in [74], the solutions of (102) and (103) at $\phi = 0$ are

$$A_2 = e^\beta B_{(2)}(W_{(2)}), \quad A_3 = e^{-\beta} B_{(3)}(W_{(3)}),$$

(107)

$$A_v = 0, \quad A_u = \frac{1}{k_v L^3} [k_2 e^{-\beta} B_{(2)}(W_{(2)}) + k_3 e^{\beta} B_{(3)}(W_{(3)})].$$

(108)

Here $B_{(2)}$ and $B_{(3)}$ are arbitrary functions of their arguments, $W_{(2)}$ and $W_{(3)}$, given by

$$W_{(2)} = W - q_3 k_v \int_0^u d\tilde{\upsilon} R^2_{a2a}(\tilde{\upsilon}), \quad W_{(3)} = W - q_3 k_v \int_0^u d\tilde{\upsilon} R^3_{a3a}(\tilde{\upsilon}),$$

(109)

with the phase $W$ found in (100). The numbers (2) and (3) indicate here the electromagnetic waves with polarization vectors directed along the $O\xi^2$ and $O\xi^3$ axes, respectively. For estimations it is convenient to consider a particular case with harmonic functions:

$$B_{(2)} = B_{(2)}^0 \cos W_{(2)}, \quad B_{(3)} = B_{(3)}^0 \cos W_{(3)},$$

(110)

and the parameter $k_v$,

$$k_v = \frac{\omega_{(EM)}(1 - \cos \theta)}{c \sqrt{2}}.$$  

(111)

explicitly expressed in terms of the electromagnetic wave frequency $\omega_{(EM)}$ and angle $\theta$ between the directions of propagation of the gravitational wave ($O\xi^1$ in our case) and of the electromagnetic one.

Clearly, $W_{(2)} \neq W_{(3)}$, when the corresponding components of the curvature tensor do not coincide, $R^2_{a2a} \neq R^3_{a3a}$, and thus we deal with the birefringence effect induced by the curvature. The wave four-vectors $K_{(2)} = \nabla_i W_{(2)}$ and $K_{(3)} = \nabla_i W_{(3)}$ can be represented as follows:

$$K_{(2)} = K_i - \delta_i^\alpha q_3 k_v R^2_{a2a}, \quad K_{(3)} = K_i - \delta_i^\alpha q_3 k_v R^3_{a3a},$$

(112)
where $K_i$ is given by (101). The wave four-vectors $K_i^{(2)}$ and $K_i^{(3)}$ are not longer null four-vectors, since
\[g^{ij}K_i^{(2)}K_j^{(2)} = -2q_3k_3^2R_{u_{2n}u_{2n}}, \quad g^{ij}K_i^{(3)}K_j^{(3)} = -2q_3k_3^2R_{u_{3n}u_{3n}}.\] (113)

The frequencies of the waves with polarization along $Ox^2$ and $Ox^3$, respectively (see, e.g., [75] for details), are
\[\omega_{(2)} = \frac{ck_v}{\sqrt{2}} \left[ 1 - \frac{1}{2k_v^2} (g^{22}k_v^2 + g^{33}k_v^2) - q_3R_{u_{2n}}^2 \right],\]
\[\omega_{(3)} = \frac{ck_v}{\sqrt{2}} \left[ 1 - \frac{1}{2k_v^2} (g^{22}k_v^2 + g^{33}k_v^2) - q_3R_{u_{3n}}^2 \right].\] (114)

These frequencies differ when $R_{u_{2n}}^2 \neq R_{u_{3n}}^2$. The refraction indices $n_{(2)}$ and $n_{(3)}$ can be calculated as follows (see, e.g., [75] for details):
\[n_{(2)}^2 = 1 - \left( \frac{c}{\omega_{(2)}} \right)^2 g^{ij}K_i^{(2)}K_j^{(2)} = 1 + 2q_3 \left( \frac{ck_v}{\omega_{(2)}} \right)^2 R_{u_{2n}}^2,\]
\[n_{(3)}^2 = 1 - \left( \frac{c}{\omega_{(3)}} \right)^2 g^{ij}K_i^{(3)}K_j^{(3)} = 1 + 2q_3 \left( \frac{ck_v}{\omega_{(3)}} \right)^2 R_{u_{3n}}^2.\] (115)

The difference $\Delta n = n_{(2)} - n_{(3)}$ can be calculated for all models mentioned in the previous section, but we focus now on one example only. Let the test electromagnetic wave propagate in the direction $-\mathbf{x}_1$, and the solution for the gravitational field be of the Petrov type, i.e. $R_{u_{2n}}^2 = -R_{u_{3n}}^2 = v^2$. Then one obtains the explicit expressions for the phase velocities:
\[\frac{v_{ph}^{(2)}}{c} = \frac{\omega_{(2)}}{ck_v} = 1 - q_3v^2, \quad \frac{v_{ph}^{(3)}}{c} = \frac{\omega_{(3)}}{ck_v} = 1 + q_3v^2, \quad k = \frac{k_v}{\sqrt{2}}.\] (116)

and for the refraction indices:
\[n_{(2)} = \frac{1 + q_3v^2}{1 - q_3v^2}, \quad n_{(3)} = \frac{1 - q_3v^2}{1 + q_3v^2}, \quad \Delta n = \frac{4q_3v^2}{1 - q_3^2v^2}.\] (117)

This means, first, that the phase velocities differ from each other, one of them being less than the speed of light in vacuum $c$, while the second being bigger than $c$. Second, the non-minimal (curvature induced) difference between the refraction indices for the waves with orthogonal polarizations depends on the curvature $\nu^2$ and the non-minimal coupling parameter $q_3$. Thus, in this illustration, we deal with birefringence in the absence of polarization rotation.

The constraints on the non-minimal coupling parameter from solar system observations and from pulsar data analysis were studied in [76] (the parameter $\lambda$ in this paper relates to $q_3/4$ in our case). From the timing observations of the binary pulsar PSR B1534+12 signals, Prasanna and Mohanty gave a constraint on $q_3$ to be $|q_3| < 2.4 \times 10^{11}$ cm$^3$. From more precise observations of the double pulsar system PSR J0737-3039 A/B, the Shapiro time delay (passing PSR J0737-3039 B of mass 1.25 $M_{\odot}$ and radius 10 km) is measured to be $6.21 \pm 0.33 \mu s$ [77]. The theoretical estimate using general relativity is $6.153 \pm 0.26 \mu s$. If we take an upper limit of 0.1 $\mu s$ for the maximum contribution of the polarization-dependent correction, we have an upper bound $|q_3| < 2.4 \times 10^{10}$ cm$^3$.

4.3. Quasi-minimal model of optical activity ($q_3 = 0$, $Q_3 = 0$) induced by varying pseudoscalar field ($\phi \neq \phi_0$)

Now we consider the model in which the birefringence is absent, but the rotation of polarization is present. Let two non-minimal coefficients $q_3$ and $Q_3$ be vanishing; then the exact solutions of (96), (97), (102), (103) are
\[ A_2 \equiv e^{\beta(u)} \left\{ E(W) \cos \frac{1}{2} [\phi(u) - \phi(0)] + F(W) \sin \frac{1}{2} [\phi(u) - \phi(0)] \right\}, \tag{118} \]

\[ A_3 \equiv e^{-\beta(u)} \left\{ E(W) \sin \frac{1}{2} [\phi(u) - \phi(0)] - F(W) \cos \frac{1}{2} [\phi(u) - \phi(0)] \right\}, \tag{119} \]

\[ A_v = 0, \quad A_u = \frac{1}{k_v L} [k_2 e^{-2\beta(u)} A_2 + k_3 e^{2\beta(u)} A_3]. \tag{120} \]

Here \( E(W) \) and \( F(W) \) are arbitrary functions of one phase scalar \( W \) \((100)\); thus, there is no birefringence. But now we deal with the effect of the Faraday rotation, induced by the photon–axion interactions. This is a kind of optical activity induced by the axion field. The effect of the Faraday rotation is described by the rotation phase \( \phi(u) - \phi(0) \), so this effect disappears when the pseudoscalar field is constant. When the pseudoscalar field is a linear function of retarded time (see \((66), (67)\)), the Faraday rotation is uniform. Let us mention again that we indicate this model as quasi-minimal since we suppose that only two non-minimal parameters are vanishing, but other ones are non-vanishing.

4.4. Non-minimal optical activity induced by photon–axion interaction in the case of a constant pseudoscalar field \( \phi = \phi_0 \)

When non-minimal coupling constants are non-vanishing, i.e. \( q_3 \neq 0 \) and \( Q_3 \neq 0 \), we search for exact solutions to the system \((105), (106)\) as follows:

\[ B_2 = h_1(u) \cos W + h_2(u) \sin W, \quad B_3 = -h_3(u) \cos W + h_4(u) \sin W. \tag{121} \]

Again \( W \) is given by \((100)\), and four functions of the retarded time \( h_1, h_2, h_3 \) and \( h_4 \) satisfy the following system of linear ordinary first-order differential equations with a coefficient depending on retarded time \( u \):

\[
\begin{align*}
    h_1'(u) & = -\frac{1}{2} \phi' h_4 - q_3 k_v R^2 u_{2u} h_2 + Q_3 \phi k_v R^2 u_{2u} h_3, \\
    h_2'(u) & = +\frac{1}{2} \phi' h_3 + q_3 k_v R^2 u_{2u} h_1 + Q_3 \phi k_v R^2 u_{2u} h_4, \\
    h_3'(u) & = -\frac{1}{2} \phi' h_2 - q_3 k_v R^2 u_{3u} h_4 + Q_3 \phi k_v R^2 u_{3u} h_1, \\
    h_4'(u) & = +\frac{1}{2} \phi' h_1 + q_3 k_v R^2 u_{3u} h_3 + Q_3 \phi k_v R^2 u_{3u} h_2. 
\end{align*}
\tag{122}
\]

Let the function \( \phi(u) \) be presented by a constant value \( \phi(u) = \phi_0 \). Introducing a new convenient variable \( \tau \):

\[ \tau \equiv k_v \int_0^u R^2 u_{2u}(\tilde{u}) \, d\tilde{u}, \tag{123} \]

we reduce the system \((122)\) to the following system with constant coefficients:

\[
\begin{align*}
    \tilde{h}_1(\tau) & = -q_3 h_2 + Q_3 \phi_0 h_3, \\
    \tilde{h}_2(\tau) & = q_3 h_1 + Q_3 \phi_0 h_4, \\
    \tilde{h}_3(\tau) & = -Q_3 \phi_0 h_1 + q_3 h_4, \\
    \tilde{h}_4(\tau) & = -Q_3 \phi_0 h_2 - q_3 h_3. 
\end{align*}
\tag{124}
\]

where the dot denotes the derivative with respect to \( \tau \). The characteristic equation for this system of equations with constant coefficients takes the form

\[
(\lambda^2 + Q_3^2 \phi_0^2 + q_3^2)^2 = 0, \tag{125}
\]

and its solutions \( \lambda = \pm i \Omega \) are the double ones, where

\[ \Omega \equiv \sqrt{q_3^2 + Q_3^2 \phi_0^2}. \tag{126} \]

Let us mention that \((124)\) in this case can be rewritten in terms of second-order equations:

\[ \tilde{h}_0 + \Omega^2 \tilde{h}_0 = 0, \tag{127} \]
where $a = 1, 2, 3, 4$. That is why the exact solution of the system (124),

\[
\begin{align*}
    h_1(\tau) &= h_1(0) \cos \Omega \tau - \frac{1}{\Omega} \sin \Omega \tau [q_3 h_2(0) - Q_3 \phi_0 h_3(0)], \\
    h_2(\tau) &= h_2(0) \cos \Omega \tau + \frac{1}{\Omega} \sin \Omega \tau [q_3 h_1(0) + Q_3 \phi_0 h_4(0)], \\
    h_3(\tau) &= h_3(0) \cos \Omega \tau + \frac{1}{\Omega} \sin \Omega \tau [q_3 h_4(0) - Q_3 \phi_0 h_1(0)], \\
    h_4(\tau) &= h_4(0) \cos \Omega \tau - \frac{1}{\Omega} \sin \Omega \tau [q_3 h_3(0) + Q_3 \phi_0 h_2(0)],
\end{align*}
\]  

(128)

contains the trigonometric functions $\cos \Omega \tau$ and $\sin \Omega \tau$ only. Thus, the quantity $\Omega$ plays a role of frequency of the curvature-induced rotation of the Faraday type. Let us stress that in this model the non-minimal coupling parameters $Q_3$ and $q_3$ play equivalent roles, predetermining the pure rotation of the polarization vector without damping. When the pp-wave spacetime is symmetric, i.e. the component $R^2_{\mu\nu\rho\sigma}$ is constant, one obtains immediately from (123) that $\tau$ is linear in the retarded time, e.g. for solution (64) one obtains $\tau = v^2 k_c u$.

4.4.1. Constraints on the non-minimal coupling parameters from CMB polarization observations. The parameter $Q_3$ enters the key formulas in the product $Q_3 \phi_0$ and thus relates to the non-minimal axion contribution to the effect of polarization rotation. Let us assume that $q_3 = 0$ and the following simple initial conditions take place:

\[
\begin{align*}
    h_1(0) &= h_3(0) = h_4(0) = 0, \\
    h_2(0) &= B_0 \to B_2 = B_0 \sin W, \\
    B_3 &= 0,
\end{align*}
\]  

(129)

i.e. initially the electromagnetic wave was polarized along the $0x^2$ axis only. Then we obtain from (128)

\[
A_2 = B_0 e^\beta \cos (Q_3 \phi_0 \tau) \sin W, \\
A_3 = -B_0 e^{-\beta} \sin (Q_3 \phi_0 \tau) \cos W.
\]  

(130)

Clearly, when $Q_3 \phi_0 \neq 0$, the orthogonal component of the electromagnetic potential $A_3$ appears; thus, the polarization rotation can exist when the parameter $Q_3 \phi_0$ is non-vanishing. The polarization ellipticity can be described in this case by the formula

\[
\frac{(A_2 e^{-\beta} / B_0)^2}{\cos^2 (Q_3 \phi_0 \tau)} + \frac{(A_3 e^{\beta} / B_0)^2}{\sin^2 (Q_3 \phi_0 \tau)} = 1.
\]  

(131)

Thus, the polarization rotation rate is predetermined by the function $Q_3 \phi_0 \tau$, and the quantity

\[
[\nabla \phi]_{\text{effective}} = Q_3 \phi_0 \sqrt{2} \left[ \frac{H_{(GW)}}{\lambda_{(GW)}} \right] \frac{\nu_1(EM)}{c} (1 - \cos \theta)
\]  

(132)

can be considered as an equivalent of the gradient of the pseudoscalar field for the estimates of the effect. The frequency dependence is different from the Faraday rotation in an ionized medium. From (132), the order of magnitude of $[\nabla \phi]_{\text{effective}}$ from a microwave of frequency 100 GHz is

\[
[\nabla \phi]_{\text{effective}} \simeq 4.7 Q_3 \phi_0 \left[ \frac{H_{(GW)}}{\lambda_{(GW)}} \right] \text{cm}^{-1};
\]  

(133)

hence,

\[
|\Delta \phi|_{\text{effective}} \simeq 4.7 \text{ cm}^{-1} Q_3 \phi_0 \left[ \frac{H_{(GW)}}{\lambda_{(GW)}} \right].
\]  

(134)

Let $H_{(GW)} = 10^{-15} \xi$, where $\xi \leq 1$; then for

\[
\lambda_{(GW)} = \text{Hubble distance} = 1.3 \times 10^{28} \text{ cm},
\]  

(135)
we have
\[ |\Delta \phi_{\text{effective}}| \simeq 3.6 \times 10^{-43} \xi \text{cm}^{-2} Q_3 \phi_0. \]  
(136)

Constraints on the cosmic polarization rotation angle \( \varphi \) come from three kinds of observations: (i) polarization observations of radio galaxies; (ii) optical/UV polarization observations of radio galaxies; (iii) CMB polarization observations. Radio observations put a limit of \( \Delta \varphi \lesssim 0.17\text{–}1.0 \) rad over the cosmological distance from various types of analysis (see, e.g., [21] for a review). Optical/UV observations put a limit of \( \Delta \varphi \lesssim 0.17 \) rad [78, 79]. The QUaD observations [80] on CMB polarization give the most stringent constraint on \( \Delta \varphi \) [81]:
\[ \Delta \varphi = 0.0096 \pm 0.0143 \pm 0.008 \text{ rad}. \]  
(137)

For gravitational waves with a wavelength of the order of the Hubble distance, this is the constraint, \( \Delta \varphi_{\text{HD}} \); for shorter wavelength, \( \lambda \), we have to use the estimate
\[ \Delta \varphi_{\lambda} \simeq \Delta \varphi_{\text{HD}} \left\{ \frac{\lambda}{\text{Hubble distance}} \right\}. \]  
(138)

In the general case, \( \Delta \varphi \) depends on directions [22]. Constraints on polarization rotation could also come from solar system observations: the analysis of these data are under study.

Combining the estimate (136) with (138) and using the relation \( \phi = 2 \varphi \), we have the constraint
\[ Q_3 \phi_0 < 2 \times 10^{41} \xi^{-1} \text{cm}^2. \]

5. Conclusions

(1) We formulated a new \textit{non-minimal ten-parameter} Einstein–Maxwell–axion model, i.e., on the basis of the Lagrangian approach, we derived a non-minimally extended self-consistent (coupled) system of equations for electromagnetic (see equations (20), (21), (22)), pseudoscalar (see equation (23)) and gravitational (see (24)–(37)) fields. The Lagrangian is linear in the curvature tensor and its contractions, and is quadratic in the Maxwell tensor, i.e., we deal with one of the versions of \textit{non-minimal linear axion electrodynamics}.

(2) The constitutive tensor (see (40) with (14)–(18)), associated with this model, is manifestly symmetric with respect to transposition of pairs of indices, i.e., this model excludes skewons [27] and describes non-minimal interaction of photons and axions only. An extension of the model accounting for skewon-type interactions will be considered separately.

(3) Decomposition of the constitutive tensor demonstrates explicitly that the dielectric permittivity, magnetic permeability tensors and the tensor of cross-effects (or the tensor of magneto-electric coefficients, in other words) acquire non-minimal contributions, including terms proportional to the pseudoscalar (axion) field. Since the tensor of non-minimal cross-effects is non-vanishing, one may conclude that, generally, the curvature-induced effect of \textit{optical activity} is expected, even if the pseudoscalar (axion) field is constant.

(4) We applied the non-minimal Einstein–Maxwell–axion model to the case when the spacetime has the \textit{pp-wave} symmetry and all the physical fields inherit this symmetry. The reduced system of equations contains in this case only one key equation (55). We presented nine examples of exact solutions for the non-minimally coupled electromagnetic, axion and gravitational fields with pp-wave symmetry. One can emphasize three new solutions among them. First, there is a new \textit{regular solution} (see section 3.3.1), which can appear in the non-minimal model only; the term regularity means that gravitational,
electromagnetic and pseudoscalar fields have no singularities and are described by the functions finite everywhere. Second, there exist exact solutions describing circularly polarized electromagnetic waves coupled non-minimally to the axion field and plane gravitational waves (see (80)–(82)). Third, the solution which is known as the Cheshire smile is also admissible in this model (see section 3.3.4).

(5) Discussing the optical activity induced by photon–axion interaction, we solved exactly the problem of propagation of test electromagnetic waves coupled non-minimally to the pseudoscalar and gravitational fields in the pp-wave background. According to the classification of the models of photon–axion interaction given in the paper [72], we deal here with the case when the four-gradient of the pseudoscalar field is a null four-vector, i.e. \( \nabla_k \phi \nabla^k \phi = 0 \). We show explicitly that the non-minimal coupling of the photon and axions with gravitational field generally leads to the birefringence effect. We discussed explicit exact solutions which describe the effect of optical activity.

(6) The non-minimal Einstein–Maxwell–axion model can give the cosmological solutions of the FLRW and Bianchi-I type, as well as static spherically symmetric solutions of the Reissner–Nordström type. We intend to consider the corresponding exact solutions in the near future.

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