1. Introduction

A tiling of the plane is a covering of $\mathbb{R}^2$ by a collection of compact subsets, called tiles, for which two distinct tiles can only meet along their boundaries. The building blocks of a tiling are the prototiles: a finite set of tiles with the property that every tile is the translation of a prototile. A tiling is said to be nonperiodic if it lacks any translational periodicity. One method of producing tilings is by a substitution system; a rule that expands each tile and then breaks each expanded tile into smaller pieces, each of which is an isometric copy of an original tile. A nonperiodic substitution system gives rise to a dynamical system, called the hull, that consists of all tilings whose local patterns appear in a prototile after a finite number of substitutions. The hull becomes a dynamical system with homeomorphism induced by translation. In order to associate a particularly tractable $C^*$-algebra to a nonperiodic tiling, Kellendonk [14] places punctures in each tile that are used to define a discrete subset of the hull, which he called the discrete hull.

In this paper we define spectral triples on Kellendonk’s $C^*$-algebra $A_{punc}$ of a tiling. The fundamental new ingredients for these spectral triples are the recently developed fractal dual substitution tilings [8]. Suppose $T$ is a nonperiodic substitution tiling with finite prototile set $\mathcal{P}$. For each prototile $p$, a fractal dual tiling defines a fractal tree on a self-similar tiling $T_p$ constructed from the substitution system on the prototile $p$. Kellendonk’s construction of the discrete hull of a tiling requires that each tile be endowed with a puncture. Our fractal tree defines a unique fractal path between any two tiles of $T_p$ that respects the hierarchy of the substitution system. The fractal tree is then used to define a length function between any two tiles of $T_p$ using Perron-Frobenius theory. Let $d_{\delta_p}(t, t')$ denote the fractal length between the punctures of two tiles $t$ and $t'$ in $T_p$.

To each substitution tiling with a fractal dual tiling, we construct spectral triples on Kellendonk’s $C^*$-algebra $A_{punc}$. We now outline the construction. For each $p \in \mathcal{P}$, let $H_p := \ell^2(T_p \setminus \{p\})$, with canonical basis $\{\delta_t : t \in T_p \setminus \{p\}\}$, and define an unbounded multiplication operator $D_p \delta_t := \ln(d_{\delta_p}(t, p)) \delta_t$. We show that $(A_{punc}, H_p, D_p)$ is a $\theta$-summable (positive) spectral triple. Now let $H := \bigoplus_{p \in \mathcal{P}} H_p$, and for each function $\sigma : \mathcal{P} \to \{-1, 1\}$ define an unbounded multiplication operator $D_\sigma := \bigoplus_{p \in \mathcal{P}} \sigma(p) D_p$. Then $(A_{punc}, H, D_\sigma)$ is a $\theta$-summable spectral triple. This defines a collection

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of spectral triples on Kellendonk’s algebra $A_{punc}$ that each respect the hierarchy of the substitution system.

Using operator algebras as the basic framework, Alain Connes developed noncommutative geometry [7], and has shown its significance to many fields of mathematics. In particular, one of the overarching themes of noncommutative geometry is to describe a consistent mathematical model for quantum physics. Dynamical systems are particularly well suited to the tools of noncommutative geometry and provide dynamical invariants in a noncommutative framework. Of particular importance to Connes’ program are spectral triples, which typically define a noncommutative Riemannian metric on a $C^*$-algebra. A spectral triple $(A,H,D)$ consists of a $C^*$-algebra $A$ faithfully represented on a separable Hilbert space $H$ and a self-adjoint unbounded operator $D$ on $H$ with compact resolvent, whose commutators with a dense $*$-subalgebra of $A$ are bounded operators.

The noncommutative topology of tilings has a long history. Alain Connes initiated the study of substitution tilings in a noncommutative framework by giving a detailed description of a $C^*$-algebra associated with the Penrose tiling in his seminal book [7]. In 1982, Dan Shechtman discovered quasicrystals [24], a new type of material that is neither crystalline nor amorphous. The mathematical theory explaining Shechtman’s discovery had already been developed in the context of purely mathematical research; nonperiodic tilings provide an excellent model for quasicrystals. In an attempt to understand the physics of quasicrystals, Bellissard defined a crossed product $C^*$-algebra by a family of Schrödinger operators. Years later, Kellendonk defined a discrete version of the continuous hull and constructed a groupoid $C^*$-algebra associated with a tiling [14, 15]. Soon afterwards, Anderson and Putnam [1] showed that the continuous hull $\Omega$ of a tiling is a Smale space, and used this observation to describe the $K$-theory of the crossed product $C(\Omega) \rtimes \mathbb{R}^2$. More recently, Kellendonk’s construction was generalised to tilings with infinite rotational symmetry in [28], and the rotationally equivariant $K$-theory of these algebras was completely worked out in [26].

Only recently has there been a breakthrough in the noncommutative geometry of tilings. The primary interest in spectral triples on tilings is that the hull of a nonperiodic tiling is not only a topological object, it also has rich geometric structure. The groundbreaking spectral triple for tilings appeared in John Pearson’s 2008 thesis [19], and the subsequent joint paper with Bellissard [5]. These spectral triples were defined on the commutative $C^*$-algebra associated with the hull of a tiling. A few years later, the second author constructed spectral triples on the unstable $C^*$-algebra of a Smale space [27, 29], which is strongly Morita equivalent to Kellendonk’s algebra. However, in the special case of tiling algebras, this spectral triple essentially measured the Euclidean distance between two tilings in the groupoid used to define the $C^*$-algebra, and ignored the substitution system. Since Bellissard and Pearson’s seminal result there have been a number of papers on spectral triples of tilings, see for example [12, 13, 16, 18]. The survey article [10] explains these constructions and their relationship to one another.

2. Nonperiodic tilings and their properties

The tilings in this paper are built from prototiles, a finite collection $\mathcal{P} := \{p_1, \ldots, p_n\}$ of labelled compact subsets of $\mathbb{R}^2$ that each contain the origin and are equal to the closure of their interior. We denote the label of $p_i$ by $\ell(p_i)$ and the support of $p_i$ by $\text{supp}(p_i) \subset \mathbb{R}^2$. The labels allow us to have two distinct prototiles with the same support, and we often denote the labels by colours. A tile is defined to be any translation of a prototile. So for any $p \in \mathcal{P}$ and $x \in \mathbb{R}^2$, the labelled subset $t := p + x$ is a tile with support $\text{supp}(t) := \text{supp}(p) + x$ and label $\ell(t) := \ell(p)$.

**Definition 2.1.** Let $\mathcal{P}$ be a set of prototiles. A tiling of the plane is a countable collection of tiles $T = \{t_1, t_2, \ldots\}$, each of which is a translate of a prototile, such that

1. $\cup_{i \in \mathbb{N}} \text{supp}(t_i) = \mathbb{R}^2$; and
Furthermore, let $N_T$ denote the number of tiles in the patch $T$. The scaling factor for this substitution is the golden ratio $\lambda = \frac{1+\sqrt{5}}{2}$. The substitution rule $\omega$ applied to each of the prototiles is also illustrated in Figure 1, and extends to all forty tiles by rotation. A patch of the Penrose tiling appears in Figure 2.

Definition 2.8. For each $p \in \mathcal{P}$ and $n \in \mathbb{N}$ let $N_n(p)$ denote the number of tiles in the patch $\omega^n(p)$. Furthermore, let $N_{\text{max}} := \max\{N_n(p) : p \in \mathcal{P}\}$.
Note that $N_{\text{max}}$ is well defined since prototile sets are finite. Moreover, for each $p \in \mathcal{P}$ and $n \in \mathbb{N}$, it is clear that $N_n(p) \leq N_{\text{max}}$.

**Definition 2.9.** Let $\omega$ be a substitution rule for a set of prototiles $\mathcal{P}$. For each pair of prototiles $p, q \in \mathcal{P}$, define

$$D_{pq} := \{ x \in \mathbb{R}^2 : q + x \in \omega(p) \}.$$  

The matrix $D := (D_{pq})$ is called the **digit matrix** for $\omega$. Further, for each $p, q \in \mathcal{P}$, let $M_{pq} := |D_{pq}|$. That is, $M_{pq}$ is the number of translates of $q$ in the patch $\omega(p)$. The matrix $M := (M_{pq})$ is called the **substitution matrix** for $\omega$.

**Definition 2.10.** Let $\omega$ be a substitution rule on a set of prototiles $\mathcal{P}$ and let $M$ be the substitution matrix for $\omega$. We say $M$ is **primitive** if there exists $N \in \mathbb{N}$ such that $(M^N)_{pq} > 0$ for all $p, q \in \mathcal{P}$. In this case, we say that $\omega$ is a **primitive substitution rule**.

Following Sadun [23], we obtain a tiling from a primitive substitution rule $\omega$ on $\mathcal{P}$ in the following way. Fix a prototile $p \in \mathcal{P}$. Since $\omega$ is primitive, there exists $n \in \mathbb{N}$ sufficiently large such that there exists a translate of $p$, say $t$, in the patch $\omega^n(p)$, and hence there is a fixed point in the tile $t$. Fixing the origin at this fixed point, we obtain the sequence

$$t \subset \omega^n(t) \subset \omega^{2n}(t) \subset \omega^{3n}(t) \subset \cdots .$$

The union $T := \bigcup_{k \in \mathbb{N}} \omega^{kn}(t)$ is a tiling of $\mathbb{R}^2$. We may modify the prototile set and the substitution rule so that $p \subset \omega(p)$ for each $p \in \mathcal{P}$. This implies that the fixed points under the substitution rule previously discussed are at the origin. We make this assumption for the remainder of this paper.

**Definition 2.11.** Let $\omega$ be a primitive substitution rule of a set of prototiles $\mathcal{P}$ whose fixed points are placed on the origin. For each $p \in \mathcal{P}$, the **tiling generated by $p$** is the tiling $T_p := \bigcup_{n \in \mathbb{N}} \omega^n(p)$. 

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**Figure 1.** The Penrose substitution rule

**Figure 2.** A patch of the Penrose tiling
For each \( p \in \mathcal{P} \), the tiling generated by \( p \) is a self-similar tiling in the sense of [8, Definition 2.5] since \( \omega(T_p) = T_p \).

In the following sections, we will often require the map \( \lambda^{-1}\omega \) which applies the substitution rule on prototiles and scales the resulting patch by \( \lambda^{-1} \). For each \( p, q \in \mathcal{P} \) and \( S \subset \mathbb{R}^2 \), we define the following notation:

\[
S + D_{pq} := \begin{cases} 
\bigcup_{x \in D_{pq}} (S + x) & \text{if } S \neq \emptyset \\
\emptyset & \text{otherwise}
\end{cases}
\]

In this framework, for each \( p \in \mathcal{P} \), we observe that

\[
\omega(p) = \bigcup_{q \in \mathcal{P}} q + D_{pq}.
\]

From the digit matrix, we build a contraction map \( \mathcal{R} \). Consider the set of compact, non-empty subsets of \( \mathbb{R}^2 \) which we denote by \( H(\mathbb{R}^2) \). The domain of \( \mathcal{R} \) is given by \( X := \bigsqcup_{p \in \mathcal{P}} H(\mathbb{R}^2) \). Then, for any \( B \in X \) and \( p \in \mathcal{P} \), we define

\[
\mathcal{R}(B)_p := \bigcup_{q \in \mathcal{P}} \lambda^{-1}(B_q + D_{pq}),
\]

and set \( \mathcal{R}(B) := \bigsqcup_{p \in \mathcal{P}} \mathcal{R}(B)_p \). The map \( \mathcal{R} \) is called the contraction map for \( \omega \). The contraction map \( \mathcal{R} \) may be thought of as the map \( \lambda^{-1}\omega \) applied to an arbitrary set of compact subsets of \( \mathbb{R}^2 \).

As it stands, the contraction map \( \mathcal{R} \) is defined on compact subsets of \( \mathbb{R}^2 \), and not on labeled compact subsets, like tiles. To rectify this, we define the following notation:

\[
\mathcal{R}^n(\mathcal{P})_p := \lambda^{-n}\omega^n(p),
\]

for each \( p \in \mathcal{P} \) and \( n \in \mathbb{N} \). Then, for each \( p \in \mathcal{P} \) and \( n \in \mathbb{N} \), \( \mathcal{R}^n(\mathcal{P})_p \) is a collection of labelled compact subsets of the form \( \lambda^{-n}(q + x) \) for some \( q \in \mathcal{P} \) and \( x \in \mathbb{R}^2 \). Each \( t \in \mathcal{R}^n(\mathcal{P})_p \) is called an \( n \)-subtile. If \( n = 1 \), we call \( t \in \mathcal{R}(\mathcal{P})_p \) a subtile.

### 2.1. The discrete hull

In this section, we will look at a particular subset of the continuous hull of a tiling called the discrete hull. Let \( T \) be a tiling with prototile set \( \mathcal{P} \). Following Kellendonk [14], for each \( p \in \mathcal{P} \), we choose a point in the interior of \( p \). This point is called the puncture of \( p \), denoted \( x(p) \). For any vector \( x \in \mathbb{R}^d \), we choose the puncture of the tile \( t := p + x \) to be \( x(p) + x \). Thus, each tile of every tiling in the continuous hull \( \Omega \) has a puncture.

**Definition 2.12** ([14, Definition 1]). Let \( T \) be a tiling and \( \Omega \) the associated continuous hull. The **discrete hull of \( T \)** is the following subset of \( \Omega \):

\[
\Omega_{\text{punc}} := \{ T' \in \Omega : \text{there exists } t \in T' \text{ with } x(t) = 0 \}.
\]

The tilings in the discrete hull are those that contain a tile whose puncture is at the origin of \( \mathbb{R}^2 \). For each \( T' \in \Omega_{\text{punc}} \), let \( T'(0) \) denote the unique tile \( t \in T' \) whose puncture is the origin. In the relative topology inherited from \( \Omega \), assuming finite local complexity, the discrete hull \( \Omega_{\text{punc}} \) is a Cantor set. In particular, \( \Omega_{\text{punc}} \) is a compact metric space which has a basis of both open and closed sets. Indeed, given a patch \( P \) and a tile \( t \) in \( P \), the set

\[
U(P, t) := \{ T' \in \Omega_{\text{punc}} : P - x(t) \subset T' \}
\]

is both open and closed in \( \Omega_{\text{punc}} \), and all such sets form a basis for \( \Omega_{\text{punc}} \). Let \( T', T'' \in \Omega_{\text{punc}} \). We say that \( T' \) and \( T'' \) are translation equivalent if there exists \( t \in T' \) such that \( T' - x(t) = T'' \). We denote by \([T']\) the translation equivalence class of \( T' \).

Let \( \omega \) be a primitive substitution rule on a set of prototiles \( \mathcal{P} \). We will assume that \( x(p) = 0 \) for each \( p \in \mathcal{P} \) so that each puncture is a fixed point under the substitution rule as in Definition 2.11.
The assumptions we have made concerning the substitution rule $\omega$ and the set of prototiles $P$ force $T(0) \in P$ for all $T \in \Omega_{\text{punc}} \subset \Omega$. Furthermore, for each $p \in P$, $x(p) = 0$ implies $T_p \in \Omega_{\text{punc}}$.

3. Fractal dual tilings and fractal trees

In [8], the authors construct fractal dual substitution tilings from a given substitution tiling. At the core of their construction is the notion of a recurrent pair. In this section we give a basic construction, using dual trees rather than quasi-dual trees, of the fractal dual substitution tilings described in [8]. We note that the constructions in this paper work in full generality of quasi-dual trees, and are stated as such. In each definition we refer the reader to the more general definition for quasi-dual trees found in [8]. We begin with the definition of a consistent dual tree.

**Definition 3.1** (cf. [8, Definition 3.6 and 3.7]). Suppose $T$ is a finite local complexity substitution tiling with prototile set $P$. A dual tree $G_p$ in a prototile $p \in P$ consists of a vertex $v_p$ in the interior of $p$ and a collection of non-overlapping edges connecting $v_p$ to the interior of each edge of $p$. We say that the set of prototiles $P$ has a consistent dual tree $G = \sqcup_{p \in P} G_p$ if for each $p \in P$, $G_p$ is a dual tree on $p$ and if two translated prototile edges meet in the tiling $T$, then the associated boundary vertices of $G$ meet in $T$ as well.

Starting with a consistent dual tree $G$, we consider the graph $R^n(G) \subset P$, and define the edges in $R^n(G)$ to be the images of the edges of $G$ under $R^n$.

**Definition 3.2** (cf. [8, Definition 5.1]). Suppose $T$ is a finite local complexity substitution tiling with substitution $\omega$ and prototile set $P$. A pair of consistent dual trees $(G, S)$ on $P$ is called a recurrent pair if $S \subset R^n(G)$ for some $n \in \mathbb{N}$.

Under certain conditions, which are developed in detail in [8], the attractor of a recurrent pair $(G, S)$ is a consistent dual tree $A$ inscribed in the prototile set $P$. Since the graph $A$ is the attractor of an iterated function system, we call $A$ a fractal graph. Moreover, by construction, $A$ is invariant under the substitution map. It is well known that embedding a consistent dual tree into each prototile of a tiling yields a new tiling, often called the combinatorial dual of $T$. In this case, embedding $A$ into each tile of $T$ we obtain a new tiling that is also a substitution tiling since $A$ is invariant under the substitution of $T$.

**Theorem 3.3** ([8, Theorems 6.6 and 6.9]). Suppose $T$ is a finite local complexity substitution tiling with prototile set $P$ whose tiles meet singly edge-to-edge. Then $T$ has an infinite number of distinct fractal quasi-dual substitution tilings. If the prototiles of $T$ are all convex, then $T$ has an infinite number of distinct fractal dual substitution tilings.

**Example 3.4** (A fractal version of the Penrose tiling). Recall the Penrose tiling constructed in Example 2.7. The trees giving rise to a recurrent pair $(G, S)$ are depicted in Figure 3 on two prototiles, and extend uniquely to all forty prototiles by rotation and reflection. The unique fractal graph $A$ defined by the recurrent pair $(G, S)$ is depicted on the right hand side of Figure 3, and also extends to all forty prototiles.

Placing the graph $A$ in each tile of a Penrose tiling $T$ defines a new tiling whose tiles have fractal borders. Figure 4 shows a patch of the fractal dual tiling of the Penrose tiling overlaid on a patch of the original Penrose tiling. The substitution on the fractal realisation is inherited from the original Penrose tiling. See [8, Appendix A] for several additional examples.

Now that we have defined recurrent pairs and the associated fractal dual, we use the tools developed to define fractal trees. These are defined as decorations on the tiling $T_p$ for each $p \in P$. In short, a fractal tree on a tiling is a network of fractal edges such that the punctures of two distinct tiles are uniquely connected by a fractal path. We use fractal trees to define spectral triples.
Let \((G, S)\) be a recurrent pair of quasi-dual trees such that, for each \(p \in \mathcal{P}\), an interior vertex \(v_p\) of \(G_p\) satisfies \(v_p = x(p)\), where \(x(p)\) is the fixed point from Definition 2.11. The algorithm in the proof of [8, Theorem 6.6] implies that it is always possible to set the corresponding interior vertex in \(S_p\) to be \(x(p)\) as well. This forces an interior vertex of the fractal graph \(A_p\) to be \(x(p)\). We make this additional assumption throughout.

**Theorem 3.5.** Let \(\omega\) be a primitive substitution rule on a set of prototiles \(\mathcal{P}\) with a recurrent pair \((G, S)\) such that, for each \(p \in \mathcal{P}\), an interior vertex of \(G_p\) is the fixed point \(x(p)\) that maps to the corresponding vertex of \(S_p\). Then, for each \(p \in \mathcal{P}\), there exists a geometric graph \(\mathcal{F}_p\) in \(T_p\) such that \(\mathcal{F}_p\) is a connected tree and for each tile \(t \in T_p\), \(x(t) \in \mathcal{F}_p\).

**Proof.** Let \((G, S)\) be a recurrent pair of quasi-dual trees on \(\mathcal{P}\) with \(A\) the associated fractal graph. We first define a subgraph \(A'\) of \(A\) that does not meet the boundary of any prototile. For each \(p \in \mathcal{P}\), let \(A_p^0\) be the set of vertices of \(A_p\) that intersect the boundary of \(p\). Define

\[
A_p' := A_p \setminus \{ e \in \mathcal{R}(A)_p : e \subset A_p, \ e \cap A_p^0 \neq \emptyset \},
\]

then \(A' := \bigcup_{p \in \mathcal{P}} A_p'\) is a subgraph of \(A\) that does not meet the boundary of any prototile.

We now claim that we can construct a graph \(F_1 \subset \mathcal{R}(A)\) in \(\mathcal{P}\) satisfying
(1) \((F_1)_p\) is a connected tree;
(2) \(\{ t \in \mathcal{R}(\mathcal{P})_p : x(t) \in (F_1)_p \} = \mathcal{R}(\mathcal{P})_p\); and
(3) \((F_1)_p \subseteq \text{int}(\text{supp}(p))\) with \((F_1)_p\) a subgraph of \((F_1)_p\).

Before constructing the graph \(F_1\), we comment on the significance of conditions (1)–(3). Condition (1) ensures that any pair of vertices in \((F_1)_p\) are connected by a unique path for each \(p \in \mathcal{P}\). Condition (2) says that the graph \((F_1)_p\) passes through the puncture of every subtile in the patch \(\mathcal{R}(\mathcal{P})_p\) for each \(p \in \mathcal{P}\). This means that every subtile is connected by the graph \(F_1\) via its puncture. Condition (3) ensures that the graph \((F_1)_p\) is wholly contained within the interior of the support of \(p\) for each \(p \in \mathcal{P}\). This means that the graph contains no edges which have a boundary vertex as one of its endpoints. The final condition also requires that \((F_1)_p\) be a subgraph of \((F_1)_p\).

We now give an algorithm for constructing the graph \(F_1\). Recursively define a sequence of fractal trees \(B^i\) on \(\mathcal{P}\) as follows. Let \(B^0 = A^i\) and if \(x(t) \in B^0\) for all subtiles \(t \in \mathcal{R}(\mathcal{P})\) then \(B^0\) satisfies (1)–(3) and we are done. Otherwise, suppose \(x(t) \notin B^0\) for some subtile \(t \in \mathcal{R}(\mathcal{P})\). Since \(\mathcal{R}(A)\) connects the punctures of the subtiles in \(\mathcal{R}(\mathcal{P})_p\) for each prototile \(p \in \mathcal{P}\), there is a path \(\mu\) in \(\mathcal{R}(A)\) connecting \(x(t)\) to \(B^i\) such that \(x(t)\) is a degree one vertex and \((B^i \cup \mu)_p\) is a connected tree for all \(p \in \mathcal{P}\). Let \(B^{i+1} = B^i \cup \mu\). Since there are only a finite number of prototiles, there exists \(n \in \mathbb{N}\) such that \(x(t) \in B^n\) for all subtiles \(t \in \mathcal{R}(\mathcal{P})\). Then it is routine to check that \(F_1 := B^n\) satisfies (1)–(3), proving the claim.

Now we recursively define a graph \(F_n\), for \(n > 2\), on the prototile set \(\mathcal{P}\) by the formula
\[
F_n := \mathcal{R}(F_{n-1}) \cup F_{n-1}.
\]
We claim that for all \(n \in \mathbb{N}\), \(F_n\) satisfies
(i) \((F_n)_p\) is a connected tree;
(ii) \(\{ t \in \mathcal{R}^n(\mathcal{P})_p : x(t) \in (F_n)_p \} = \mathcal{R}^n(\mathcal{P})_p\); and
(iii) \((F_n)_p \subseteq \text{int}(\text{supp}(p))\).

The proof is by induction on \(n\). By conditions (1)–(3) above, \(F_1\) satisfies (i)–(iii). Now assume \(F_{n-1}\) satisfies (i)–(iii). We prove \(F_n\) does as well.

(i) By our inductive hypothesis, \(\mathcal{R}(F_{n-1})_p\) is a connected tree in each subtile \(t \in \mathcal{R}(\mathcal{P})_p\). Since there is exactly one path connecting each of these subtiles in the connected tree \((F_{n-1})_p\), the union \(\mathcal{R}(F_{n-1})_p \cup (F_{n-1})_p\) is a connected tree in \(\mathcal{R}^n(\mathcal{P})_p\). We remark that the invariance of \(A\) under the map \(\mathcal{R}\) is essential for this step to work, which is why we require fractal trees rather than trees.

(ii) By our inductive hypothesis, the graph \((F_{n-1})_p\) contains the puncture of each \((n-1)\)-subtile in \(\mathcal{R}^{n-1}(\mathcal{P})_p\). Then, \(\mathcal{R}(F_{n-1})_p\) contains the puncture of each \(n\)-subtile in \(\mathcal{R}^n(\mathcal{P})_p\). Since \(\mathcal{R}(F_{n-1})_p \subseteq (F_{n-1})_p\), we have
\[
\mathcal{R}^n(\mathcal{P})_p = \{ t \in \mathcal{R}^n(\mathcal{P})_p : x(t) \in (F_{n-1})_p \}.
\]

(iii) By our inductive hypothesis, \((F_{n-1})_p \subseteq \text{int}(\text{supp}(p))\) and hence, \(\mathcal{R}(F_{n-1})_p\) is contained in the union of the interior of the subtiles in \(\mathcal{R}^n(\mathcal{P})_p\). That is, \(\mathcal{R}(F_{n-1})_p \subseteq \text{int}(\text{supp}(p))\).

Thus by induction, there exists a geometric graph \((F_n)_p\) satisfying conditions (i)–(iii) for each \(n \in \mathbb{N}\).

We will now define a fractal graph \(\mathfrak{F}_n\) on \(\omega^n(\mathcal{P})\) for each \(n \in \mathbb{N}\). For each \(p \in \mathcal{P}\) and \(n \in \mathbb{N}\), define \((\mathfrak{F}_n)_p := \lambda^n(F_n)_p\). By construction, \((\mathfrak{F}_n)_p\) is a geometric graph in \(\omega^n(p)\) containing the puncture of each tile in \(\omega^n(p)\). Furthermore, the construction of \(T_p\) implies that \(\omega^n(p) \subset \omega^{n+1}(p)\) and we have \((\mathfrak{F}_n)_p \subseteq (\mathfrak{F}_{n+1})_p\) for each \(n \in \mathbb{N}\). Thus, we define
\[
\mathfrak{F}_p := \bigcup_{n=1}^{\infty} (\mathfrak{F}_n)_p.
\]
Then \(\mathfrak{F}_p\) is a geometric graph on \(T_p\) connecting the punctures in \(T_p\) by a unique fractal path. \(\square\)
Definition 3.6. For each $p \in \mathcal{P}$, we will refer to a geometric graph $\mathfrak{F}_p$, as in the statement of Theorem 3.5, as a fractal tree on $T_p$.

For fixed $p \in \mathcal{P}$, endow $T_p$ with a fractal tree $\mathfrak{F}_p$. We will define a distance between two tiles in $T_p$ via $\mathfrak{F}_p$. The length of a fractal path is defined using Perron-Frobenius theory. Let $(G, S)$ be the recurrent pair which we used to construct the fractal tree $\mathfrak{F}_p$, as in Theorem 3.5, and let $A$ be the corresponding fractal graph. Since $\mathfrak{F}_p$ consists of edges in $A$, we assign a length to each edge in $A$ and then add the appropriate edge lengths to obtain a length between punctures of tiles in $T_p$.

First, since $(G, S)$ is a recurrent pair, $G$ and $S$ are homeomorphic geometric graphs (planar graphs embedded in $\mathbb{R}^2$). Let $\psi : G \to S$ denote the homeomorphism, and let $E(G) := \sqcup_{p \in \mathcal{P}} E(G_p)$ denote the set of edges in $G$. For each edge $e \in E(G)$, the edge $\psi(e) \in E(S)$ is composed of edges of the form $\lambda^{-1}(f + x)$ where $f \in E(G)$ and $x \in \mathbb{R}^2$. We view $\psi$ as a substitution rule on the set of edges.

Definition 3.7. Let $p, q \in \mathcal{P}$, $e \in G_p$, and $f \in G_q$. Define

$$D^E_{ef} := \{x \in D_{pq} : \lambda^{-1}(f + x) \subseteq \psi(e)\}.$$ 

The matrix $D^E := (D^E_{ef})$ is called the edge digit matrix. Further, let $M^E_{ef} := |D^E_{ef}|$. That is, $M^E_{ef}$ is the number of copies of the edge $f$, on the $\lambda^{-1}$ scale, in $\psi(e)$. The matrix $M^E := (|D^E_{ef}|)$ is called the edge substitution matrix.

When $M^E$ is primitive, the Perron-Frobenius Theorem defines a unique real eigenvalue $\kappa > 1$ with associated eigenvector $v = (v_e)_{e \in E(G)}$ whose entries are all positive real numbers. We note that $M^E$ may not be primitive for all choices of recurrent pairs, however, it is routine to see that a recurrent pair $(G, S)$ with primitive substitution matrix $M^E$ can always be chosen for all tilings satisfying the hypothesis of [8, Theorem 6.6] (see also Theorem 3.3).

Definition 3.8. Let $v = (v_e)_{e \in E(G)}$ be the Perron-Frobenius eigenvector for the primitive edge substitution matrix $M^E$. Then for each $e \in E(A)$, the length of $e$, denoted $l(e)$, is the entry $v_e$ in $v$.

Definition 3.9. Suppose $t, t' \in T_p$ for some $p \in \mathcal{P}$ and $\mathfrak{F}_p$ is a fractal tree on $T_p$. The fractal distance between $t$ and $t'$, denoted $d_{\mathfrak{F}_p}(t, t')$, is the sum of the fractal edges making up the unique fractal path in $\mathfrak{F}_p$ between $x(t)$ and $x(t')$.

Example 3.10 (A fractal tree for the Penrose tiling). In this example we illustrate the construction of a fractal tree for the recurrent pair defined on the Penrose tiling in Example 3.4. We then use the Perron-Frobenius Theorem to assign lengths to the fractal edges depicted in Figure 5, which extend to the edges in all forty prototiles by rotation. Putting these two steps together defines a fractal distance between the punctures of tiles $t$ and $t'$ in $T_p$, for all $p \in \mathcal{P}$.

![Figure 5. 12 of the fractal edges in the graph A for the Penrose tiling](image-url)
To elucidate the algorithm in the proof of Theorem 3.5, we restrict our attention to prototile $p_1$. Starting with the recurrent pair defined in Example 3.4, the first step in the algorithm is to construct the graph $F_1$ by removing the edges of $\mathcal{R}(A)$ that intersect the boundary. Restricting our attention to prototile $p_1$, the fractal $(F_1)_{p_1}$ appears in Figure 6.

**Figure 6.** $(F_1)_{p_1}$ - prototile $p_1$ after removing the edges of $\mathcal{R}(A)$ that intersect the boundary

The second step in the algorithm is to consider the graph $\mathcal{R}(F_1)$, which will define a non-connected graph in each prototile that hits each puncture in $\mathcal{R}^2(\mathcal{P})$. Figure 7 shows $\mathcal{R}(F_1)_{p_1}$.

**Figure 7.** $\mathcal{R}(F_1)_{p_1}$ - prototile $p_1$ after applying $\mathcal{R}$ to the graph $F_1$

To connect the graphs within each prototile we define $F_2$ to be $\mathcal{R}(F_1) \cup F_1$. We note that it is essential that we begin with a fractal for $F_1$ and $\mathcal{R}(F_1)$, so that the paths intersect along complete edges. Figure 8 shows $F_2$ restricted to prototile $p_1$. Continuing this construction, we obtain fractal graphs $F_n$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, consider the graph $\mathfrak{F}_n = \lambda^n F_n$ which connects the punctures of each patch $\omega^n(p)$ for each $p \in \mathcal{P}$. Finally, the infinite graph $\mathfrak{F}_{p_1}$ on tiling $T_{p_1}$ is defined to be the infinite union $\bigcup_{n=1}^{\infty} (\mathfrak{F}_n)_{p_1}$. Thus we have a fractal tree on the tiling $T_{p_1}$ that jointly extends each of the fractal trees $(\mathfrak{F}_n)_{p_1}$ and the patches $\omega^n(p_1)$. 
We now wish to assign lengths to the edges 1–12 in Figure 5. The edge substitution matrix $M^E$ for the recurrent pair $(G, S)$ defined in Example 3.4 is a primitive $120 \times 120$ matrix. However, since the edges in Figure 5 extend to the 10-fold rotations of the 4 prototiles we need only consider a $12 \times 12$ matrix. The Perron-Frobenius eigenvalue is $\kappa = \frac{1}{2}(3 + \sqrt{21} + 4\sqrt{21}) \approx 4.6357$ with associated eigenvector

$v = [0.6952, 1.3953, 1.2638, 0.6952, 1.2638, 1.3953, 1, 0.5452, 0.5686, 1, 0.5686, 0.5452]^T$.

Each of the entries $v_i$ corresponds with the assigned length $l(i)$ of edge $i$. To obtain the length $d_\mathcal{F}(t, t')$ between two tiles $t, t' \in \mathcal{T}_p$ we add the lengths of the edges between the punctures $x(t)$ and $x(t')$. Thus, the reader can check that the length between the middle pink tile in Figure 8 and each of its three adjacent tiles is $l(t) = l(8) = 1.9405$, $l(t) + l(t) = 2.5275$, and $l(t) + l(t) = 1.3904$ in the northwest, northeast, and south directions, respectively.

Given a fractal distance between tiles in $\mathcal{T}_p$, we explore some of properties associated with a fractal tree and make some new definitions which will aid us in the definition of a spectral triple.

**Definition 3.11.** Suppose $p \in \mathcal{P}$, $n \in \mathbb{N}$ and the patch $\omega^n(p) + x \subset T$ for some $T \in \Omega$ and $x \in \mathbb{R}^2$. The root of $\omega^n(p) + x$ is $p + x$. In particular, $p$ is the root of $\omega^n(p)$ for all $p \in \mathcal{P}$ and $n \in \mathbb{N}$.

**Proposition 3.12.** Fix $p \in \mathcal{P}$ and a recurrent pair $(G, S)$ with fractal tree $\mathcal{F}_p$ on $\mathcal{T}_p$ and Perron-Frobenius eigenvalue $\kappa > 1$. Let $t, t', r, r' \in \mathcal{T}_p$ such that $r$ is the root of $\omega(t)$ and $r'$ is the root of $\omega(t')$. Then, $d_{\mathcal{F}_p}(r, r') = \kappa d_{\mathcal{F}_p}(t, t')$.

**Proof.** Let $t, t' \in \mathcal{T}_p$. Choose $N$ sufficiently large such that the $N$-subtiles $\lambda^{-N}t, \lambda^{-N}t' \in \mathcal{R}^N(\mathcal{P})_p$. By construction, there is a unique fractal path between the punctures of $\lambda^{-N}t$ and $\lambda^{-N}t'$ via the tree $(F_N)_p$, as in the proof of Theorem 3.5. Furthermore, recall that $(F_{N+1})_p = \mathcal{R}(F_N)_p \cup (F_N)_p$. Since $(F_N)_p \subset (F_{N+1})_p$, the fractal path between $\lambda^{-N}t$ and $\lambda^{-N}t'$ in $\mathcal{R}^{N+1}(\mathcal{P})_p$ is identical to the fractal path between $\lambda^{-N}t$ and $\lambda^{-N}t'$ in $\mathcal{R}^N(\mathcal{P})_p$. Scaling appropriately, the fractal path between $r$ and $r'$ is the image under the edge digit matrix of the fractal path between $t$ and $t'$. Since the length of fractal edges were assigned by the eigenvector corresponding to the Perron-Frobenius eigenvalue $\kappa$ of the edge substitution matrix, we conclude that $d_{\mathcal{F}_p}(r, r') = \kappa d_{\mathcal{F}_p}(t, t').$ \hfill $\square$

**Definition 3.13.** For each $p \in \mathcal{P}$ and $n \in \mathbb{N}$, the $n$th major branch radius of $p$ is the number

$$\mathcal{L}_n(p) := \max\{d_{\mathcal{F}_p}(t, p) : t \in \omega^n(p) \setminus \{p\}\}.$$
A tile $t \in \omega^n(p)$ satisfying $d_{\tilde{s}_p}(t, p) = \mathcal{L}_n(p)$ is called an $n$th major leaf of $p$. The major branch radius is $\mathcal{L} := \max\{\mathcal{L}_1(p) : p \in \mathcal{P}\}$. Similarly, for each $p \in \mathcal{P}$ and $m, n \in \mathbb{N}$ such that $m > n$ the $m$th minor branch radius of $p$ over $n$ is the number

$$S_{m,n}(p) = \min\{d_{\tilde{s}_p}(t, p) : t \in \omega^m(p) \setminus \omega^n(p)\}.$$ 

If $n = 0$, we write $S_m(p) := S_{m,n}(p)$ for the $m$th minor branch radius of $p$. A tile $t \in \omega^m(p) \setminus \omega^n(p)$ satisfying $d_{\tilde{s}_p}(t, p) = S_{m,n}(p)$ is called an $m$th minor leaf of $p$ over $n$. Furthermore, a tile $t \in \omega^m(p) \setminus \{p\}$ satisfying $d_{\tilde{s}_p}(t, p) = S_m(p)$ is called an $m$th minor leaf of $p$. The minor branch radius is $S := \min\{S_n(p) : p \in \mathcal{P}, n \in \mathbb{N}\}$.

**Lemma 3.14.** For each $p \in \mathcal{P}$ and $n \in \mathbb{N}$, $\mathcal{L}_n(p) \leq \sum_{k=1}^{n-1} \kappa^k \mathcal{L}$.

*Proof.* We will prove that

$$\mathcal{L}_n(p) \leq \sum_{k=0}^{n-1} \kappa^k \mathcal{L},$$

for all $n \in \mathbb{N}$ by mathematical induction. Consider the case $n = 1$. By definition, we have $\mathcal{L} = \{\mathcal{L}_1(p) : p \in \mathcal{P}\}$, and hence, $\mathcal{L}_1(p) \leq \mathcal{L}$ for all $p \in \mathcal{P}$. Hence, (3.1) holds for $n = 1$. Assume that (3.1) holds for $n = k$ and consider case $n = k + 1$. Fix $p \in \mathcal{P}$. Let $t$ be a first major leaf of $p$ and $r$ be the root of $\omega^k(t)$. Since $p$ is the root of $\omega^k(p)$, then Proposition 3.12 implies

$$d_{\tilde{s}_p}(r, p) = \kappa^k d_{\tilde{s}_p}(t, p) = \kappa^k \mathcal{L}_1(p).$$

This means that if we view the patch $\omega^{k+1}(p)$ as a collection of supertiles of order $k$, the fractal distance from $p$ to the root of one of the supertiles of order $k$ is at most $\kappa^k \mathcal{L}_1(p)$. Let $t'$ be a $(k+1)$th major leaf of $p$. Then $t' \in \omega^k(t'')$ for some $t'' \in \omega^k(p) \setminus \{p\}$. That is, $t'$ must be contained in one of the supertiles of order $k$. Let $r'$ be the root of $\omega^k(t'')$. By our inductive hypothesis, we have

$$d_{\tilde{s}_p}(t', p) \leq d_{\tilde{s}_p}(t', r') + d_{\tilde{s}_p}(r', p) \leq \sum_{i=1}^{k-1} \kappa^i \mathcal{L} + \kappa^k \mathcal{L} = \sum_{i=1}^{k} \kappa^i \mathcal{L}. \quad \square$$

**Lemma 3.15.** For each $p \in \mathcal{P}$ and $m, n \in \mathbb{N}$ with $m > n$, $S_{m,n}(p) \geq \kappa^{n-1} S$.

*Proof.* Let $p \in \mathcal{P}$ and $m \in \mathbb{N}$. Suppose that $n = 1$. Then

$$S_{m,1}(p) = \min\{d_{\tilde{s}_p}(t, p) : t \in \omega^m(p) \setminus \omega(p)\} \geq \min\{d_{\tilde{s}_p}(t, p) : t \in \omega^m(p) \setminus \{p\}\} = S_m(p) \geq S.$$ 

Suppose that $n > 1$. Let $t$ be a first minor leaf of $p$. Let $r$ be the root of $\omega^{n-1}(t)$, then $r \in \omega^m(p)$ and satisfies $d_{\tilde{s}_p}(r, p) = \kappa^{n-1} S_1(p) \geq \kappa^{n-1} S$. Let $t'$ be an $m$th minor leaf of $p$ over $n$, then the fractal path between $p$ and $t'$ must pass through the root of $\omega^{n-1}(t'')$ for some $t'' \in \omega(p) \setminus \{p\}$. Thus,

$$S_{m,n}(p) = d_{\tilde{s}_p}(t', p) \geq d_{\tilde{s}_p}(r, p) \geq \kappa^{n-1} S. \quad \square$$

We now have a means of estimating the various major and minor branch radii for prototiles. We will use these estimates later in the proofs of our results in Section 5.

## 4. $C^*$-Algebras Associated to Nonperiodic Tilings

In this section, we will investigate the $C^*$-algebra associated to a nonperiodic substitution tiling given by Kellendonk in [14], on which we will define a spectral triple. Let $\omega$ be a primitive substitution rule on a set of prototiles $\mathcal{P}$ and $\Omega_{\text{punc}}$ be the associated discrete hull. Recall the translational equivalence relation on the discrete hull, given in Section 2.1, which may be written as:

$$R_{\text{punc}} := \{(T, T') \in \Omega_{\text{punc}} \times \Omega_{\text{punc}} : T' = T - x(t) \text{ for some } t \in T\}.$$
We endow $R_{\text{punc}}$ with a metric given by
\[ d_R((T, T - x(t)), (T', T' - x(t'))) := d(T, T') + |x(t) - x(t')|, \]
where $T, T' \in \Omega_{\text{punc}}$, $t \in T$ and $t' \in T'$. In this topology, $R_{\text{punc}}$ is an étale equivalence relation. Note that the topology here is not the same as the topology inherited from the product topology on $\Omega_{\text{punc}} \times \Omega_{\text{punc}}$. In the latter, $R_{\text{punc}}$ is not étale.

Following the construction of groupoid $C^*$-algebras given by Renault in [22], Kellendonk constructed a $C^*$-algebra in [14], which we will denote by $A_{\text{punc}}$. We now outline this construction. Consider the vector space $C_c(R_{\text{punc}})$; the continuous functions of compact support on $R_{\text{punc}}$. Given $f, g \in C_c(R_{\text{punc}})$ and $(T, T') \in R_{\text{punc}}$, endow $C_c(R_{\text{punc}})$ with multiplication given by the convolution
\[ (f \cdot g)(T, T') := \sum_{T'' \in [T]} f(T, T'')g(T'', T'), \]
and involution
\[ f^*(T, T') := \overline{f(T', T)}. \]
Under these operations, $C_c(R_{\text{punc}})$ is a $*$-algebra. For each $T \in \Omega_{\text{punc}}$, consider the induced representation from the unit space $\pi_T : C_c(R_{\text{punc}}) \to B(\ell^2(T))$ given in [28]. Explicitly,
\[ (\pi_T(f)\xi)(t) := \sum_{t' \in T} f(T - x(t), T - x(t'))\xi(t'), \]
where $f \in C_c(R_{\text{punc}})$, $\xi \in \ell^2(T)$, and $t \in T$. It is well known that each induced representation $\pi_T$ is non-degenerate [22]. Completing $C_c(R_{\text{punc}})$ in the reduced $C^*$-algebra norm given by
\[ \|f\|_{\text{red}} := \sup\{\|\pi_T(f)\| : T \in \Omega_{\text{punc}}\}, \]
where $f \in C_c(R_{\text{punc}})$, we obtain Kellendonk’s $C^*$-algebra $A_{\text{punc}}$.

Given Kellendonk’s $C^*$-algebra $A_{\text{punc}}$, we now describe a dense spanning set. Let $P$ be a patch and let $t, t'$ be tiles in $P$. Define a function $e(P, t, t') : R_{\text{punc}} \to \mathbb{C}$ by
\[ e(P, t, t')(T, T') := \begin{cases} 
1 & \text{if } T \in U(P, t) \text{ and } T' = T - x(t') \\
0 & \text{otherwise} \end{cases}. \]
Then, $e(P, t, t')$ is a partial isometry in $C_c(R_{\text{punc}})$. The linear span of all such functions $e(P, t, t')$ forms a dense spanning set for $A_{\text{punc}}$. We note some of the relations satisfied by these functions. Let $P_1$ and $P_2$ be patches, $t_1, t'_1$ be tiles in $P_1$ and $t_2, t'_2$ be tiles in $P_2$. It is easy to check that
\[ e(P_1, t_1, t'_1)^* = e(P_1, t'_1, t_1). \]
Furthermore, if there exists a patch $P$ such that $P_1 \subset P$ and $P_2 \subset P$, then
\[ e(P_1, t_1, t'_1) \cdot e(P_2, t_2, t'_2) = e(P_1 \cup P_2, t_1, t'_2). \]
We note that $A_{\text{punc}}$ is a unital $C^*$-algebra, with unit
\[ 1_{A_{\text{punc}}} := \sum_{p \in \mathcal{P}} e(\{p\}, p, p). \]

We now define another representation that we require in the next section. Note that in the following, we are making an identification between $[T]$ and $T$ via $T - x(t) \longleftrightarrow t$ for each $t \in T$. For each $p \in \mathcal{P}$, consider the map $\pi_p : C_c(R_{\text{punc}}) \to B(\ell^2(T_p \setminus \{p\}))$ defined by
\[ (\pi_p(f)\xi)(t) := \sum_{t' \in T_p \setminus \{p\}} f(T_p - x(t), T_p - x(t'))\xi(t'), \]
for each $p \in \mathcal{P}$.
where \( f \in C_c(R_{punc}), \xi \in \ell^2(T_p \setminus \{p\}) \) and \( t \in T_p \setminus \{p\} \). We note that the map \( \pi_p \) is similar to an induced representation of the unit space. In fact, the only difference is that we are removing the tile \( p \) from the tiling \( T_p \). The reason we have removed this particular tile will become apparent in the next section. That \( \pi_p \) is a non-degenerate representation on \( C_c(R_{punc}) \) is similar to the proof for the induced representation found in [22]. Since the substitution on \( \Omega_{punc} \) is assumed to be primitive, every finite patch appearing in any tiling in \( \Omega_{punc} \) also appears in \( T_p \setminus \{p\} \). Using this fact, it is a routine argument to show that \( A_{punc} \) is isometrically isomorphic to the closure of \( C_c(R_{punc}) \) is the operator norm of \( \mathcal{B}(\ell^2(T_p \setminus \{p\})) \). Thus, \( \pi_p \) extends to a faithful representation of \( A_{punc} \).

5. Spectral triples on nonperiodic tilings from fractal trees

In this section, we define a spectral triple on Kellendonk’s \( C^* \)-algebra \( A_{punc} \), introduced in the previous section. We begin with the definition of a spectral triple on a \( C^* \)-algebra.

**Definition 5.1.** Let \( A \) be a unital \( C^* \)-algebra. A spectral triple over \( A \), \((A,H,D)\), consists of a separable Hilbert space \( H \), a faithful representation \( \pi : A \to \mathcal{B}(H) \) and an unbounded, self-adjoint operator \( D \) on \( H \) satisfying the following properties:

1. \( \{a \in A : \pi(a) \text{ Dom}(D) \subset \text{Dom}(D) \text{ and } [D, \pi(a)] \in \mathcal{B}(H)\} \) is dense in \( A \); and
2. the operator \((1 + D^2)^{-1}\) is compact on \( H \).

A spectral triple \((A,H,D)\) is \( \theta \)-summable if the operator \( \exp(-\alpha D^2) \) is trace class for all \( \alpha > 0 \).

We are now able to state the main result of the paper, which makes use of the fractal trees and distance function defined in Section 3.

**Theorem 5.2.** Suppose \( \omega \) is a nonperiodic primitive substitution on a prototile set \( \mathcal{P} \) and, for each \( p \in \mathcal{P} \), let \( T_p \) be the self-similar tiling from Definition 2.11. Let \((G,S)\) be a recurrent pair on \( \mathcal{P} \) satisfying the hypotheses of Theorem 3.5 so that there is a fractal tree \( \mathfrak{F}_p \) on \( T_p \) for all \( p \in \mathcal{P} \).

1. For each \( p \in \mathcal{P} \), there is a \( \theta \)-summable spectral triple \((A_{punc}, H_p, D_p)\) where \( H_p := \ell^2(T_p \setminus \{p\}) \), \( \pi_p : A_{punc} \to \mathcal{B}(H_p) \) is defined in (4.1), and \( D_p \) is an unbounded self-adjoint operator on \( H_p \) defined on the canonical basis \( \{\delta_t : t \in T_p \setminus \{p\}\} \) by \( D_p \delta_t := \ln(d_{\mathfrak{F}_p}(t,p)) \delta_t \).

2. For each \( \sigma : \mathcal{P} \to \{-1,1\} \) there is a \( \theta \)-summable spectral triple \((A_{punc}, H, D_\sigma)\) where

\[
H := \bigoplus_{p \in \mathcal{P}} H_p, \quad \pi := \bigoplus_{p \in \mathcal{P}} \pi_p, \quad \text{and} \quad D_\sigma := \bigoplus_{p \in \mathcal{P}} \sigma(p) D_p.
\]

The remainder of the paper will be dedicated to proving Theorem 5.2. We note that the spectral triples constructed in (1) are bounded below, and therefore \( D_p \) may be taken to be a positive operator (by suitably scaling the Perron-Frobenious eigenvector so that its minimal entry is greater than 0.5). This is the primary reason for the more sophisticated spectral triple appearing in (2).

We first concentrate on part (1) of Theorem 5.2. Fix \( p \in \mathcal{P} \) and consider the self-similar tiling \( T_p \). Endow \( T_p \) with a fractal tree \( \mathfrak{F}_p \) with associated scaling factor \( \kappa > 1 \). In the previous section we looked at the \( C^* \)-algebra \( A_{punc} \), along with the faithful representation \( \pi_p \) on the Hilbert space \( H_p := \ell^2(T_p \setminus \{p\}) \) as in (4.1). For \( t, t' \in T_p \), recall the fractal distance \( d_{\mathfrak{F}_p}(t,t') \) defined in Definition 3.9. We define an operator \( D_p \) on \( H_p \) by

\[
D_p \delta_t := \ln(d_{\mathfrak{F}_p}(t,p)) \delta_t,
\]

where \( t \in T_p \setminus \{p\} \), and extend linearly to span \( \{\delta_t : t \in T_p \setminus \{p\}\} \). Since \( p \not\in T_p \setminus \{p\} \), \( d_{\mathfrak{F}_p}(t,p) \neq 0 \) for all \( t \in T_p \setminus \{p\} \), and hence \( \ln(d_{\mathfrak{F}_p}(t,p)) \) is well-defined. Since \( D_p \delta_t \in H_p \) for each \( t \in T_p \setminus \{p\} \), \( D_p \) is a densely defined operator on \( H_p \). Furthermore, \( D_p \) is an unbounded operator. To see this,
fix a tile \( t \in T_p \setminus \{ p \} \) and \( n \in \mathbb{N} \). Choose \( N \in \mathbb{N} \) such that \( d_{\delta_p}(t, p)\kappa^N > e^n \) and let \( t' \) be the root of \( \omega^N(t) \). Since \( p \) is the root of \( \omega^N(p) \), Proposition 3.12 implies
\[
\ln(d_{\delta_p}(t', p)) = \ln(\kappa^N d_{\delta_p}(t, p)) > \ln(e^n) = n.
\]
Thus,
\[
\| D_p \| = \sup\{\| D_p \| : \xi \in \text{Dom}(D_p), \| \xi \| \leq 1 \} \geq \| D_p \| = \| \ln(d_{\delta_p}(t', p)) \| = \ln(d_{\delta_p}(t', p)) > n,
\]
and \( D_p \) is unbounded. Finally, we show that \( D_p \) is self-adjoint. That \( D_p \) is symmetric follows immediately from the definition of \( D_p \) on \( H_p \). To see that \( D_p \) is self-adjoint, consider
\[
\text{Dom}(D^*_p) = \{ \eta \in H_p : \xi \mapsto \langle D_p \xi, \eta \rangle \text{ from } \text{Dom}(D_p) \to \mathbb{C} \text{ is bounded} \}.
\]
Since \( D_p \) is symmetric, we have \( \text{Dom}(D_p) \subset \text{Dom}(D^*_p) \). We show that \( \text{Dom}(D^*_p) \subset \text{Dom}(D_p) \).

Let \( \eta \in \text{Dom}(D^*_p) \). Then the map \( \xi \mapsto \langle D_p \xi, \eta \rangle \) from \( \text{Dom}(D_p) \) to \( \mathbb{C} \) is bounded. By the Riesz Representation Theorem, there exists \( \varphi \in H_p \) such that \( \langle D_p \xi, \eta \rangle = \langle \xi, \varphi \rangle \) for all \( \xi \in \text{Dom}(D_p) \). Now let \( \xi = \delta_t \) for some \( t \in T_p \setminus \{ p \} \). Then,
\[
\langle D_p \delta_t, \eta \rangle = \langle \ln(d_{\delta_p}(t, p)) \delta_t, \eta \rangle = \ln(d_{\delta_p}(t, p)) \sum_{t' \in T_p \setminus \{ p \}} \delta_t(t')\eta(t') = \ln(d_{\delta_p}(t, p))\eta(t).
\]
Furthermore,
\[
\langle \delta_t, \varphi \rangle = \sum_{t' \in T_p \setminus \{ p \}} \delta_t(t')\varphi(t') = \varphi(t).
\]
Thus, \( \varphi(t) = \ln(d_{\delta_p}(t, p))\eta(t) \) and
\[
\sum_{t \in T_p \setminus \{ p \}} \ln(d_{\delta_p}(t, p))^2|\eta(t)|^2 = \sum_{t \in T_p \setminus \{ p \}} |\ln(d_{\delta_p}(t, p))\eta(t)|^2 = \sum_{t \in T_p \setminus \{ p \}} |\varphi(t)|^2 < \infty.
\]
Hence, \( \eta \in \text{Dom}(D_p) \), and \( \text{Dom}(D^*_p) \subset \text{Dom}(D_p) \). Thus, \( D_p \) is self-adjoint. Given our unbounded, self-adjoint operator \( D_p \) on \( H_p \) we aim to show that \( (A_{\text{punc}}, H_p, D_p) \) is a spectral triple. We will show that conditions (1) and (2) in Definition 5.1 are satisfied.

**Proposition 5.3.** Let \( P \) be a patch in a tiling \( T \in \Omega_{\text{punc}} \). Then, for all \( t, t' \in P \),
\[
[D_p, \pi_p(e(P, t, t'))] \in \mathcal{B}(H_p).
\]

Proposition 5.3 implies condition (1) in Definition 5.1 since the linear span of functions \( e(P, t, t') \) are dense in \( A_{\text{punc}} \). We require a sequence of lemmas before arriving at the proof of this proposition. Given a patch \( P \) in a tiling \( T \in \Omega_{\text{punc}} \), we want to partition \( T_p \) in such a way that we confine where \( P \) can appear in \( T_p \). The following lemma will help us to do this, and the corollary which follows describes what sort of control we have gained. First, we will need some definitions.

For the next definition, we will denote the boundary of a compact subset \( X \subset \mathbb{R}^2 \) by \( \partial X \).

**Definition 5.4.** For each \( p \in \mathcal{P} \) and \( n \in \mathbb{N} \), the \( n \)th coronal radius of \( p \) is the number:
\[
\text{corad}_n(p) := \inf\{|x - y| : x \in \text{supp}(\omega^n(p)) \text{ and } y \in \partial \text{supp}(\omega^{n+1}(p))\}.
\]
We note that for each \( p \in \mathcal{P} \) and \( n \in \mathbb{N} \), \( \text{corad}_n(p) = \lambda^n \text{corad}_0(p) \).

**Lemma 5.5.** Fix \( p \in \mathcal{P} \) and \( r > 0 \). There exists \( N \in \mathbb{N} \) such that for all \( x \in \mathbb{R}^2 \), either \( B(x, r) \subset \text{supp}(\omega^{N+1}(p)) \) or \( B(x, r) \subset \text{supp}(\omega^{k+2}(p)) \setminus \text{supp}(\omega^k(p)) \) for some \( k \geq N \).

**Proof.** Let \( x \in \mathbb{R}^2 \). Since \( \lambda > 1 \), fix \( N \in \mathbb{N} \) with \( \lambda^N \text{corad}_0(p) > 2r \). If \( B(x, r) \subset \text{supp}(\omega^{N+1}(p)) \), then there is nothing to check. Suppose then, that \( B(x, r) \not\subset \text{supp}(\omega^{N+1}(p)) \). Then, either
\[
(1) \ B(x, r) \cap \text{supp}(\omega^{N+1}(p)) \neq \emptyset; \text{ or}
\]
Let \( \kappa > 1 \), so that the limit is \( \{F\} \). Otherwise, \( \kappa = 1 \), so that the limit is \( \{G\} \).

Furthermore, since \( \text{corad}_N(p) = \text{corad}_{N+1}(p) > 2r \), we have \( B(x, r) \cap \text{supp}(\omega^{N+1}(p)) = \emptyset \).

If (2) occurs, then choose \( k > N \) minimal such that \( B(x, r) \cap \text{supp}(\omega^{k+1}(p)) = \emptyset \). By choice of \( k \), \( B(x, r) \cap \text{supp}(\omega^k(p)) = \emptyset \). Furthermore, \( \text{corad}_{k+1}(p) = \lambda^{k-N} \text{corad}_{N+1}(p) > 2r \) implies that \( B(x, r) \subset \text{supp}(\omega^{k+2}(p)) \).

In either case, there exists \( k \geq N \) such that \( B(x, r) \subset \text{supp}(\omega^{k+2}(p)) \setminus \text{supp}(\omega^k(p)) \). □

**Corollary 5.6.** Let \( P \) be a patch in \( T_p \) for some \( p \in \mathcal{P} \). There exists \( N \in \mathbb{N} \) such that either \( P \subset \omega^{N+1}(p) \) or \( P \subset \omega^{k+2}(p) \setminus \omega^k(p) \) for some \( k \geq N \).

For the following lemma, recall Definition 3.13.

**Lemma 5.7.** Let \( P \subset T_p \setminus \{p\} \) be a patch in \( T_p \) and \( t, t' \in P \). There exists \( N \in \mathbb{N} \) such that

\[
0 < \frac{d_{\bar{\gamma}}(t, p)}{d_{\bar{\gamma}}(t', p)} \leq \max \left\{ \frac{\mathcal{L}(r)}{S}, \frac{\kappa^3 \mathcal{L}}{S(\kappa - 1)} \right\}.
\]

**Proof.** By Corollary 5.6, there exists an \( N \in \mathbb{N} \) such that \( P \subset \omega^{N+1}(p) \) or \( P \subset \omega^{k+2}(p) \setminus \omega^k(p) \) for some \( k \geq N \). First, suppose that \( P \subset \omega^{N+1}(p) \). By Definition 3.13, we have

\[
0 < \frac{d_{\bar{\gamma}}(t, p)}{d_{\bar{\gamma}}(t', p)} \leq \frac{\mathcal{L}(r)}{S} \leq \frac{\mathcal{L}(r)}{S} \leq \frac{\mathcal{L}(r)}{S} \leq \frac{\kappa^3 \mathcal{L}}{S(\kappa - 1)}.
\]

Otherwise, \( P \subset \omega^{k+2}(p) \setminus \omega^k(p) \) for some \( k \geq N \). By definition of the \((k+2)\)th minor branch radius of \( p \) over \( k \), the fact that \( t \in \omega^{k+2}(p) \) and Lemmas 3.14 and 3.15, we have

\[
0 < \frac{d_{\bar{\gamma}}(t, p)}{d_{\bar{\gamma}}(t', p)} \leq \frac{\mathcal{L}(r)}{S} \leq \frac{\mathcal{L}(r)}{S} \leq \frac{\mathcal{L}(r)}{S} \leq \frac{\kappa^3 \mathcal{L}}{S(\kappa - 1)}.
\]

where the final inequality holds since \( \{\sum_{i=0}^{k-1} \kappa^{-i}\} \) is an increasing sequence of partial sums with \( \kappa > 1 \), so that the limit is \( \frac{1}{\kappa^{-1}} \). Thus, combining (5.2) and (5.3), for any \( t, t' \in P \), we have

\[
0 < \frac{d_{\bar{\gamma}}(t, p)}{d_{\bar{\gamma}}(t', p)} \leq \max \left\{ \frac{\mathcal{L}(r)}{S}, \frac{\kappa^3 \mathcal{L}}{S(\kappa - 1)} \right\},
\]

as required. □

Let \( P \) be a patch in a tiling \( T \in \Omega_{\text{punc}}, t, t' \in P \) and \( t'' \in T_p \setminus \{p\} \). Making the necessary modifications to the equation immediately preceding [28, Proposition 3.3], we have the following formula for the representation of \( e(P, t, t') \) in \( B(H_p) \):

\[
\pi_p(e(P, t, t'))\delta_v = \begin{cases} \delta_{v'' - (x(t'') - x(t))} & \text{if } T_p - x(t'') \in U(P, t') \\ 0 & \text{otherwise} \end{cases}
\]

It is important to note that if \( t'' - (x(t'') - x(t)) = p \), then \( \pi_p(e(P, t, t'))\delta_v \equiv 0 \).

**Proof of Proposition 5.3.** Let \( P \) be a patch in a tiling \( T \in \Omega_{\text{punc}}, t, t' \in P \) and \( t'' \in T_p \setminus \{p\} \). If \( T_p - x(t'') \in U(P, t) \), let \( s \) denote the tile \( t'' - (x(t'') - x(t)) \in T_p \). We begin by calculating

\[
[D_p, \pi_p(e(P, t, t'))]_{\delta_v} = (D_p \pi_p(e(P, t, t')) - \pi_p(e(P, t, t'))D_p)_{\delta_v} = (D_p \pi_p(e(P, t, t')) - \pi_p(e(P, t, t'))D_p)_{\delta_v} = D_p \delta_s - \ln(d_{\bar{\gamma}}(t'', p))\pi_p(e(P, t, t'))_{\delta_v}
\]

...
= \ln(d_{\delta_p}(s,p))\delta_s - \ln(d_{\delta_p}(t'',p))\delta_s
\]
= \ln \left[ \frac{d_{\delta_p}(s,p)}{d_{\delta_p}(t'',p)} \right] \delta_s.

By equation (5.4), if \( T_p - x(t'') \not\in U(P,t') \) then \([D_p,\pi_p(e(P,t,t'))]|_{\delta''} = 0 \). Suppose then, that \( T_p - x(t'') \in U(P,t') \). Thus, to prove that \([D_p,\pi_p(e(P,t,t'))] \in B(H_p)\), we show that
\[
\left| \ln \left[ \frac{d_{\delta_p}(s,p)}{d_{\delta_p}(t'',p)} \right] \right| \leq M,
\]
for some fixed \( M \geq 0 \). Using Lemma 5.7, fix \( N \in \mathbb{N} \) such that (5.1) holds. We claim that the number:
\[
M := \ln \left[ \max \left\{ \frac{L_{N+1}(p)}{S}, \frac{\kappa^2 L}{S(\lambda - 1)} \right\} \right],
\]
does the job. Suppose that \( d_{\delta_p}(s,p) \geq d_{\delta_p}(t'',p) \). Then, by Lemma 5.7, we have
\[
1 \leq \frac{d_{\delta_p}(s,p)}{d_{\delta_p}(t'',p)} \leq \max \left\{ \frac{L_{N+1}(p)}{S}, \frac{\kappa^2 L}{S(\lambda - 1)} \right\} \implies 0 \leq \ln \left[ \frac{d_{\delta_p}(s,p)}{d_{\delta_p}(t'',p)} \right] \leq M,
\]
and hence
\[
0 \leq \left| \ln \left[ \frac{d_{\delta_p}(s,p)}{d_{\delta_p}(t'',p)} \right] \right| \leq M.
\]
Otherwise, \( d_{\delta_p}(s,p) \leq d_{\delta_p}(t'',p) \) and again by Lemma 5.7, we have
\[
1 \leq \frac{d_{\delta_p}(t'',p)}{d_{\delta_p}(s,p)} \leq \max \left\{ \frac{L_{N+1}(p)}{S}, \frac{\lambda^3 L}{S(\lambda - 1)} \right\} \implies 0 \leq \ln \left[ \frac{d_{\delta_p}(t'',p)}{d_{\delta_p}(s,p)} \right] \leq M.
\]
Noting that
\[
\left| \ln \left[ \frac{d_{\delta_p}(t'',p)}{d_{\delta_p}(s,p)} \right] \right| = \left| \ln \left[ \frac{d_{\delta_p}(s,p)}{d_{\delta_p}(t'',p)} \right] \right|,
\]
we have
\[
0 \leq \ln \left[ \frac{d_{\delta_p}(t'',p)}{d_{\delta_p}(s,p)} \right] \leq M \implies 0 \leq \ln \left[ \frac{d_{\delta_p}(s,p)}{d_{\delta_p}(t'',p)} \right] = \ln \left[ \frac{d_{\delta_p}(t'',p)}{d_{\delta_p}(s,p)} \right] \leq M,
\]
as required. \( \square \)

So we have shown that \((A_{\text{panc}}, H_p, D_p)\) satisfies condition (1) of Definition 5.1. For condition (2), we need to show that the operator \((1 + D_p^2)^{-1}\) is compact. We will show that \(\exp(-\alpha D_p^2)\) is trace class for all \(\alpha > 0\) and invoke the following result.

**Lemma 5.8** ([6]). Suppose \( D : \text{Dom}(D) \to H \) is an unbounded operator on a separable Hilbert space \( H \). If \(\exp(-\alpha D^2)\) is trace class for all \(\alpha > 0\), then \((1 + D^2)^{-1} \in K(H)\).

This will imply that \((A_{\text{panc}}, H_p, D_p)\) is a \(\theta\)-summable spectral triple hence proving part (1) of Theorem 5.2. We will need the following technical lemma.

**Lemma 5.9.** For each \( n \in \mathbb{N} \), there exists a constant \( c > 0 \) such that \( x^{\ln x} \geq cx^n \) for all \( x > 0 \).

**Proof.** First, we claim that \( x^{\ln x} \geq 1 \) for all \( x > 0 \). Let \( x > 0 \). Suppose that \( x \geq 1 \), then \( \ln x \geq 0 \). Hence, \( x^{\ln x} \geq 1 \). Otherwise \( 0 < x < 1 \), and \( x^{-1} > 1 \). Then
\[
(x^{-1})^{\ln(x^{-1})} \geq 1 \implies (x^{-1})^{-\ln x} \geq 1 \implies x^{\ln x} \geq 1.
\]
Thus, \( x^{\ln x} \geq 1 \) for all \( x > 0 \), as claimed. Fix \( n \in \mathbb{N} \) and let \( x > 0 \). Suppose \( 0 < x \leq e^n \), then \( x^n \leq (e^n)^n = e^{n^2} \). By our previous claim, \( e^{-n^2} x^n \leq 1 \leq x^{\ln x} \). On the other hand, suppose \( x > e^n \).
Then, \( \ln x > \ln(e^n) = n \). Thus, \( x^{\ln x} > x^n \). Since \( 0 < e^{-n^2} < 1 \), we have \( x^{\ln x} > e^{-n^2} x^n \). Hence, \( x^{\ln x} \geq e^{-n^2} x^n \) for all \( x > 0 \), so \( c := e^{-n^2} \) can be taken as the desired constant.

\[ \Box \]

**Proposition 5.10.** The operator \( \exp(-\alpha D_p^2) \), where \( D_p \) is defined above, is trace class for all \( \alpha > 0 \).

**Proof.** Fix \( \alpha > 0 \). For any \( t \in T_p \setminus \{p\} \), we have

\[
\exp(-\alpha D_p^2) \delta_t = \exp(-\alpha (\ln(d_{\delta_p}(t,p)))^2) \delta_t = d_{\delta_p}(t,p)^{-\alpha \ln(d_{\delta_p}(t,p))} \delta_t.
\]

From this, we calculate

\[
\text{Tr}(\exp(-\alpha D_p^2)) = \sum_{t \in T_p \setminus \{p\}} \langle \exp(-\alpha D_p^2) \delta_t, \delta_t \rangle = \sum_{t \in T_p \setminus \{p\}} (d_{\delta_p}(t,p)^{-\ln(d_{\delta_p}(t,p))})^{-\alpha}.
\]

Since \( \kappa > 1 \) and \( \alpha > 0 \), we have \( \kappa^\alpha > 1 \). Fix \( N \in \mathbb{N} \) such that \( \mathcal{N}_{\text{max}} < \kappa^N \). By Lemma 5.9, there exists \( c > 0 \) such that \( x^{\ln x} > c x^N \) for all \( x > 0 \). Since \( d_{\delta_p}(t,p) > 0 \) for all \( t \in T_p \setminus \{p\} \) we have

\[
\text{Tr}(\exp(-\alpha D_p^2)) = \sum_{t \in T_p \setminus \{p\}} (d_{\delta_p}(t,p)^{-\ln(d_{\delta_p}(t,p))})^{-\alpha} \leq c^{-\alpha} \sum_{t \in T_p \setminus \{p\}} d_{\delta_p}(t,p)^{-\alpha N}.
\]

From here, let us restrict our attention to the final sum in equation (5.5). We will rewrite this sum as follows. Let \( t \in T_p \setminus \{p\} \). Then \( t \in \omega(p) \setminus \{p\} \) or \( t \in \omega^{2m+1}(p) \setminus \omega^{2m-1}(p) \) for some \( m \in \mathbb{N} \). To simplify notation, define \( \omega^{2m \pm 1}(p) := \omega^{2m+1}(p) \setminus \omega^{2m-1}(p) \) for each \( m \in \mathbb{N} \). Thus,

\[
\sum_{t \in \omega^{2m \pm 1}(p)} d_{\delta_p}(t,p)^{-\alpha N} = \sum_{t \in \omega(p) \setminus \{p\}} d_{\delta_p}(t,p)^{-\alpha N} + \sum_{m=1}^{\infty} \sum_{t \in \omega^{2m \pm 1}(p)} d_{\delta_p}(t,p)^{-\alpha N}.
\]

The first sum on the right hand side of (5.6) is finite since \( \omega(p) \setminus \{p\} \) is a patch. As a result, we will only concern ourselves with the second sum on the right hand side of (5.6). Fix \( m \in \mathbb{N} \) and consider the patch \( \omega^{2m \pm 1}(p) \). By Lemma 3.15 (with \( n = 2m - 1 \)), we may deduce that for any \( t \in \omega^{2m \pm 1}(p) \), we have \( d_{\delta_p}(t,p) \geq \kappa^{2m-2} \mathcal{S} \). Thus,

\[
\sum_{m=1}^{\infty} \sum_{t \in \omega^{2m \pm 1}(p)} d_{\delta_p}(t,p)^{-\alpha N} \leq \sum_{m=1}^{\infty} \sum_{t \in \omega^{2m \pm 1}(p)} (\kappa^{2m \pm 2} \mathcal{S})^{-\alpha N} = (\kappa^{-2} \mathcal{S})^{-\alpha N} \sum_{m=1}^{\infty} \sum_{t \in \omega^{2m \pm 1}(p)} \kappa^{-2mN}.
\]

For each \( m \in \mathbb{N} \), the sum over \( t \in \omega^{2m \pm 1}(p) \) in the final sum above no longer depends on \( t \) and so we simply multiply \( \kappa^{-2mN} \) by the number of tiles in \( \omega^{2m \pm 1}(p) \). The number of tiles in \( \omega^{2m \pm 1}(p) \) is

\[
\mathcal{N}_{2m+1}(p) - \mathcal{N}_{2m-1}(p) \leq \mathcal{N}_{2m+1}(p) \leq \mathcal{N}^{2m+1}_{\text{max}}.
\]

Thus, we have

\[
\sum_{m=1}^{\infty} \sum_{t \in \omega^{2m \pm 1}(p)} d_{\delta_p}(t,p)^{-\alpha N} \leq (\kappa^{-2} \mathcal{S})^{-\alpha N} \sum_{m=1}^{\infty} \mathcal{N}^{2m+1}_{\text{max}} \kappa^{-2mN} = \mathcal{N}^{max}_{\text{max}} (\kappa^{-2} \mathcal{S})^{-\alpha N} \sum_{m=1}^{\infty} \left( \mathcal{N}^{2}_{\text{max}} \kappa^{-2N} \right)^m.
\]

By choice of \( N \), we have \( \mathcal{N}^{2}_{\text{max}} < \kappa^{2N} \) and so the sum on the right is a convergent geometric series. Following this back to (5.5), we obtain \( \lim_{\alpha \to 0} \text{Tr}(\exp(-\alpha D_p^2)) < \infty \). Since \( \alpha > 0 \) was arbitrary, \( \exp(-\alpha D_p^2) \) is trace class for every \( \alpha > 0 \).

\[ \Box \]

We are now able to prove part (1) of Theorem 5.2.

**Proof of Theorem 5.2 (1).** The operator \( D_p \) is a self-adjoint operator on \( H_p \) which satisfies condition (1) of Definition 5.1 by Proposition 5.3. Furthermore, since the operator \( \exp(-\alpha D_p^2) \) is trace class for all \( \alpha > 0 \) by Proposition 5.10, Lemma 5.8 implies that \( (1 + D_p^2) \) is compact on \( H_p \) which is condition (2) of Definition 5.1. Thus, \( (A_{\text{punc}}, H_p, D_p) \) is a \( \theta \)-summable spectral triple.

\[ \Box \]
We now turn our attention to part (2) of Theorem 5.2. In proving part (1) of Theorem 5.2, for each prototile \( p \in \mathcal{P} \), we considered a fractal tree \( \mathfrak{F}_p \) on the self-similar tiling \( T_p \) and defined a spectral triple \((A_{punc}, H_p, D_p)\) over Kellendonk’s \( C^*\)-algebra \( A_{punc} \). We now consider a collection of fractal trees \( \mathfrak{F}_p \) on \( T_p \) for each prototile \( p \in \mathcal{P} \) and a map \( \sigma : \mathcal{P} \rightarrow \{-1, 1\} \). We then have a collection of spectral triples \((A_{punc}, H_p, D_p)\) for each \( p \in \mathcal{P} \). Define

\[
H := \bigoplus_{p \in \mathcal{P}} H_p, \quad \pi := \bigoplus_{p \in \mathcal{P}} \pi_p, \quad \text{and} \quad D_\sigma := \bigoplus_{p \in \mathcal{P}} \sigma(p)D_p.
\]

Consider the map \( \pi : A_{punc} \rightarrow \mathcal{B}(H) \) defined by \( \pi := \bigoplus_{p \in \mathcal{P}} \pi_p \). Then \( \pi \) is a non-degenerate faithful representation of \( A_{punc} \) on \( H \) since \( \pi_p \) is a non-degenerate faithful representation of \( A_{punc} \) on \( H_p \) for each \( p \in \mathcal{P} \). Thus, to show that \((A_{punc}, H, D_\sigma)\) is a spectral triple, we need check that \( D_\sigma \) is an unbounded, self-adjoint operator satisfying conditions (1) and (2) of Definition 5.1.

**Lemma 5.11.** The operator \( D_\sigma \) is an unbounded, self-adjoint operator on \( H \) with

\[
\text{Dom}(D_\sigma) = \bigoplus_{p \in \mathcal{P}} \text{Dom}(D_p).
\]

**Proof.** Since \( \|D_\sigma\| = \max\{\|D_p\| : p \in \mathcal{P}\} \) and each \( D_p \) is unbounded on \( H_p \), \( D_\sigma \) is an unbounded operator on \( H \). Lemma 5.3.7 in [20] implies that \( D_\sigma \) is self-adjoint and (5.7) holds. \( \square \)

**Lemma 5.12.** Let \( P \) be a patch in a tiling \( T \in \Omega_{punc} \). Then, for all \( t, t' \in P \),

\[
[D_\sigma, \pi(e(P, t, t'))] \in \mathcal{B}(H).
\]

**Proof.** We have,

\[
[D_\sigma, \pi(e(P, t, t'))] = D_\sigma \pi(e(P, t, t')) - \pi(e(P, t, t'))D_\sigma
\]

\[
= \bigoplus_{p \in \mathcal{P}} \sigma(p)D_p \pi_p(e(P, t, t')) - \sigma(p) \pi_p(e(P, t, t'))D_p
\]

\[
= \bigoplus_{p \in \mathcal{P}} \sigma(p) [D_p, \pi_p(e(P, t, t'))] \in \mathcal{B}(H),
\]

since Proposition 5.3 implies \([D_p, \pi_p(e(P, t, t'))] \in \mathcal{B}(H_p)\) for each \( p \in \mathcal{P} \). \( \square \)

**Lemma 5.13.** The operator \( \exp(-\alpha D_\sigma^2) \), where \( D_\sigma \) is defined above, is trace class for all \( \alpha > 0 \).

**Proof.** Proposition 5.10 implies that \( \text{Tr}(\exp(-\alpha D_\sigma^2)) < \infty \) for each \( \alpha > 0 \) and \( p \in \mathcal{P} \). Fix \( \alpha > 0 \). Then, by the holomorphic functional calculus we obtain

\[
\text{Tr}(\exp(-\alpha D_\sigma^2)) = \text{Tr}(\bigoplus_{p \in \mathcal{P}} \exp(-\alpha D_p^2)) = \sum_{p \in \mathcal{P}} \text{Tr}(\exp(-\alpha D_p^2)) < \infty.
\]

Since \( \alpha > 0 \) was arbitrary, the operator \( \exp(-\alpha D_\sigma^2) \) is trace class for all \( \alpha > 0 \). \( \square \)

**Proof of Theorem 5.2 (2).** The operator \( D_\sigma \) is a self-adjoint operator on \( H \) which satisfies condition (1) of Definition 5.1 by Lemma 5.12. Furthermore, since the operator \( \exp(-\alpha D_\sigma^2) \) is trace class for all \( \alpha > 0 \) by Lemma 5.13, Lemma 5.8 implies that \((1 + D_\sigma^2)\) is compact on \( H \) which is condition (2) of Definition 5.1. Thus, \((A_{punc}, H, D_\sigma)\) is a \( \theta \)-summable spectral triple. \( \square \)

**Example 5.14.** Recall the fractal trees constructed for the Penrose tiling in Example 3.10. Theorem 5.2 implies that we have a spectral triple \((A_{punc}, H, D_\sigma)\) on \( A_{punc} \) for each function \( \sigma : \mathcal{P} \rightarrow \{-1, 1\} \).
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