ANALYSIS AND NEW APPLICATIONS OF FRACTAL FRACTIONAL DIFFERENTIAL EQUATIONS WITH POWER LAW KERNEL

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ABSTRACT. We obtain the solutions of fractal fractional differential equations with the power law kernel by reproducing kernel Hilbert space method in this paper. We also apply the Laplace transform to get the exact solutions of the problems. We compare the exact solutions with the approximate solutions. We demonstrate our results by some tables and figures. We prove the efficiency of the proposed technique for fractal fractional differential equations.

1. Introduction. The fractal derivative is a natural extension of Leibniz’s derivative for discontinuous fractal media. It can be categorized as a special local fractional derivative [16]. Implementations of the fractal derivative to fractal media have taken much attention, for example, it can model heat transfer and water permeation in multi-scale fabric and wool fibers [9]. Liouville [11] has investigated an elementary fractional derivative and integral. His main aim was to give the fractional calculus and to apply it to the real-world problems. Thus, the theory and applied part of the fractional calculus cannot be allocated.

Laplace transform is very useful transform to get the exact solutions of the problems. There are many works in the literature related to the Laplace transform. Xiang [14] has investigated the Laplace transforms for approximation of highly oscillatory Volterra integral equations of the first kind. Talbot [12] has worked on the accurate numerical inversion of laplace transforms. Delgado et al. [7] have studied the Laplace homotopy analysis technique for investigating linear partial differential equations. Yan [15] has researched the modified Homotopy Perturbation technique. Jassim et al. [10] have worked the local fractional Laplace variational iteration technique for solving diffusion and wave equations. Fahd et al. [8] have investigated a modified Laplace transform for certain generalized fractional operators.

Fractal fractional differential equations are very remarkable topics nowadays. We investigate the fractal fractional differential equations by the reproducing kernel Hilbert space method and Laplace transform in this paper. The reproducing kernel Hilbert space method, which calculates the numerical solution, is of great attention to many branches of applied sciences. Many works have been dedicated to the

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implementation of the reproducing kernel Hilbert space method to a wide class of
differential equations. The efficiency and power of the reproducing kernel Hilbert
space method were considered by too researchers to investigate many applications.
Reproducing kernel Hilbert space method is very effective method to solve many
problems. This method does not need any discretization to create the effective
results. Akgül et al. [1] have studied the existence of solutions to the Telegraph
equation in binary reproducing kernel Hilbert spaces. Aronszajn [3] has investi-
gated the theory of reproducing kernels. Bouboulis et al. [6] have studied on the
reproducing kernel Hilbert spaces and fractal interpolation. Akgül et al. [2] have
obtained the reproducing kernel functions for difference equations. We organize the
paper as follow. We give some main definitions in Section 2. We construct the main
problems in Section 3. We apply the Laplace transform to the fractal fractional
differential equations in this section. We apply the reproducing kernel method in
Section 4. We demonstrate the numerical results by some tables and figures in this
section. We give the conclusion in the last section.

2. Preliminaries.

Definition 2.1. We assume that \( y(x) \) is continuous in opened interval \((a, b)\), if \( y \)
is fractal differentiable on \((a, b)\) with order \( \beta \) then, the Fractal-Fractional derivative
of \( y \) of order \( \alpha \) in Riemann-Liouville sense with power law is presented as [4]:

\[
_{a}^{F\mathcal{F}P}_{x}D_{x}^{\alpha,\beta}y(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx^{\beta}} \int_{a}^{x} y(t)(x-t)^{-\alpha} dt, \quad 0 < \alpha, \beta \leq 1,
\]

where

\[
\frac{dy(t)}{dt^{\beta}} = \lim_{x \to t} \frac{y(x) - y(t)}{x^{\beta} - t^{\beta}}.
\]

Definition 2.2. Assume that \( y(x) \) is continuous in \((a, b)\). Then the fractal-fractional
integral of \( y \) with order \( \alpha \) is given by [4]:

\[
_{0}^{F\mathcal{F}P}_{x}I_{x}^{\alpha,\beta}y(x) = \frac{\beta}{\Gamma(\alpha)} \int_{0}^{x} t^{\beta-1} y(t)(x-t)^{\alpha-1} dt.
\]

There are many kinds of transforms out there in the world. Laplace transform
is the most important kind of these transforms [5].

Definition 2.3. Let \( y(x) \) be defined for \( x \geq 0 \). The Laplace transform of \( y(x) \)
defined by \( Y(s) \) or \( L\{y(x)\} \), is an integral transform presented by the Laplace
integral:

\[
L\{y(x)\} = Y(s) = \int_{0}^{\infty} \exp(-sx) f(x) dx.
\]

We have some examples of the Laplace transform as [5]:

\[
L\{x^{-\alpha}\} = s^{\alpha-1}\Gamma(1 - \alpha).
\]

\[
L\{x^{\beta}\} = s^{-\beta-1}\Gamma(1 + \beta).
\]

We have the following relation for the Caputo derivative,

\[
_{0}^{C}D_{x}^{\alpha}y(x) = \frac{dy(x)}{dx} * \frac{x^{-\alpha}}{\Gamma(1 - \alpha)}.
\]
We have the Laplace transform of this derivative as [5]:

\[ \mathcal{L}\left\{ C_0 D^\alpha_x y(x) \right\} = (s \mathcal{L}\{y(x)\} - y(0)) s^{\alpha-1}. \] (8)

**Remark 1.** We have the following relation between the Riemann-Liouville and Caputo derivative as [4]:

\[ RL_0^\alpha D_x y(x) = \frac{C}{\alpha} D_x^\alpha y(x) + \frac{y(0)}{\Gamma(1 - \alpha)} x^{-\alpha}. \] (9)

3. **Construction of the problems.** In this section, we consider three problems with Caputo derivative.

3.1. **First problem.** We investigate the following fractal fractional differential equation as [4]:

\[ FFP_0^\alpha,\beta D_x^\alpha y(x) + y(x) = x. \] (10)

We can write this equation as:

\[ \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dx} \int_0^x y(t)(x-t)^{-\alpha} + y(x) = x. \] (11)

Then, we obtain

\[ \frac{1}{\Gamma(1 - \alpha)} \frac{1}{\beta x^{\beta-1}} \frac{d}{dx} \int_0^x y(t)(x-t)^{-\alpha} + y(x) = x. \] (12)

Therefore, we get

\[ RL_0^\alpha D_x^\alpha y(x) + \beta x^{\beta-1} y(x) = \beta x^\beta. \] (13)

We use the relation between Riemann-Liouville and Caputo derivatives and reach

\[ C_0^\alpha D_x^\alpha y(x) + \beta x^{\beta-1} y(x) = \beta x^\beta - \frac{y(0)}{\Gamma(1 - \alpha)} x^{-\alpha}. \] (14)

3.2. **Second problem.** We consider the following fractal fractional differential equation as [4]:

\[ FFP_0^\alpha,\beta D_x^\alpha y(x) = x^2. \] (15)

We can write this equation as:

\[ \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dx} \int_0^x y(t)(x-t)^{-\alpha} = x^2. \] (16)

Then, we obtain

\[ \frac{1}{\Gamma(1 - \alpha)} \frac{1}{\beta x^{\beta-1}} \frac{d}{dx} \int_0^x y(t)(x-t)^{-\alpha} = x^2. \] (17)

Therefore, we get

\[ RL_0^\alpha D_x^\alpha y(x) = \beta x^{\beta+1}. \] (18)

We use the relation between Riemann-Liouville and Caputo derivatives and reach

\[ C_0^\alpha D_x^\alpha y(x) = \beta x^{\beta+1} - \frac{y(0)}{\Gamma(1 - \alpha)} x^{-\alpha}. \] (19)
Then, we apply the Laplace transform to both sides of the above equation:

\[ L \left( C_0 D_x^\alpha y(x) \right) = L \left( \beta x^{\beta+1} \right) - L \left( \frac{y(0)}{\Gamma(1-\alpha)} x^{-\alpha} \right). \]  

(20)

Thus, we acquire

\[ s^\alpha L(y(x)) - y(0)s^{\alpha-1} = \beta \Gamma(2+\beta) s^{-\beta-2} - \frac{y(0)}{\Gamma(1-\alpha)} s^{\alpha-1} \Gamma(1-\alpha). \]  

(21)

\[ L(y(x)) = \beta \Gamma(2+\beta) s^{\alpha+\beta+1}. \]  

(22)

After taking the inverse Laplace transform of the above equation, we get

\[ y(x) = \frac{\beta \Gamma(2+\beta) x^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)}. \]  

(23)

3.3. Third problem. We consider the following fractal fractional differential equation as [4]:

\[ {}^c_0 FDP_x^{\alpha,\beta} y(x) + \sin(y(x) + 1) = \frac{AB(\alpha) \Gamma(\alpha+4)}{1-\alpha} \left[ 30 x^{\alpha+5} E_{\alpha,\alpha+6} \left( -\frac{\alpha}{1-\alpha} x^{\alpha} \right) 
- 15 x^{\alpha+4} E_{\alpha,\alpha+5} \left( -\frac{\alpha}{1-\alpha} x^{\alpha} \right) 
+ 2 x^{\alpha+3} E_{\alpha,\alpha+4} \left( -\frac{\alpha}{1-\alpha} x^{\alpha} \right) 
+ \sin(x^{\alpha+5} - 3 x^{\alpha+4} + x^{\alpha+3} + 1) \right] \]

We can write this equation as:

\[ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x y(t) (x-t)^{-\alpha} = \frac{AB(\alpha) \Gamma(\alpha+4)}{1-\alpha} \left[ 30 x^{\alpha+5} E_{\alpha,\alpha+6} \left( -\frac{\alpha}{1-\alpha} x^{\alpha} \right) 
- 15 x^{\alpha+4} E_{\alpha,\alpha+5} \left( -\frac{\alpha}{1-\alpha} x^{\alpha} \right) 
+ 2 x^{\alpha+3} E_{\alpha,\alpha+4} \left( -\frac{\alpha}{1-\alpha} x^{\alpha} \right) 
+ \sin(x^{\alpha+5} - 3 x^{\alpha+4} + x^{\alpha+3} + 1) - \sin(y(x) + 1) \right] \]

Therefore, we get
\[ \frac{RL}{0} D_\alpha^\alpha y(x) = \frac{\beta x^{\beta - 1} AB(\alpha) \Gamma(\alpha + 4)}{1 - \alpha} \left[ 30 x^{\alpha+5} E_{\alpha,\alpha+6} \left( -\frac{\alpha}{1 - \alpha} x^\alpha \right) 
right. 

- 15 x^{\alpha+4} E_{\alpha,\alpha+5} \left( -\frac{\alpha}{1 - \alpha} x^\alpha \right) + 2 x^{\alpha+3} E_{\alpha,\alpha+4} \left( -\frac{\alpha}{1 - \alpha} x^\alpha \right) 

\left. \right] + \beta x^{\beta - 1} \left( + \sin(x^{\alpha+5} - 3 x^{\alpha+4} + x^{\alpha+3} + 1) - \sin(y(x) + 1) \right) \]

We use the relation between Riemann-Liouville and Caputo derivatives and reach

\[ C_0 D_\alpha^\alpha y(x) = \frac{\beta x^{\beta - 1} AB(\alpha) \Gamma(\alpha + 4)}{1 - \alpha} \left[ 30 x^{\alpha+5} E_{\alpha,\alpha+6} \left( -\frac{\alpha}{1 - \alpha} x^\alpha \right) 
right. 

- 15 x^{\alpha+4} E_{\alpha,\alpha+5} \left( -\frac{\alpha}{1 - \alpha} x^\alpha \right) + 2 x^{\alpha+3} E_{\alpha,\alpha+4} \left( -\frac{\alpha}{1 - \alpha} x^\alpha \right) 

\left. \right] + \beta x^{\beta - 1} \left( + \sin(x^{\alpha+5} - 3 x^{\alpha+4} + x^{\alpha+3} + 1) - \sin(y(x) + 1) \right) 

\frac{y(0)}{\Gamma(1 - \alpha)} x^{-\alpha}. \]

4. Reproducing kernel Hilbert space method. We investigate our problems by reproducing kernel Hilbert space method in this section. We construct some useful reproducing kernel Hilbert spaces and obtain very important reproducing kernel functions in these spaces. We obtain numerical results of the problems and demonstrate them by the following tables. We prove the accuracy of the reproducing kernel method for fractal fractional differential equations with the power law kernel.

**Definition 4.1.** We construct the first reproducing kernel Hilbert space \( A_2[0, 1] \) as [1]:

\[ A_2[0, 1] = \{ u \text{ is absolutely continuous function in the } [0, 1] : u' \in L^2[0, 1] \}. \]

We need this reproducing kernel Hilbert space for range. The inner product and the norm of this reproducing kernel Hilbert space are given as:

\[ \langle a, b \rangle_{A_2} = a(0)b(0) + \int_0^1 a'(x)b'(x)dx. \]

and

\[ \| a \|_{A_2[0, 1]} = \sqrt{\langle a, a \rangle_{A_2[0, 1]}}, \quad a \in A_2[0, 1]. \]

The reproducing kernel function has been obtained as:

\[ C_y(x) = \begin{cases} 
1 + x, & 0 \leq x \leq y \leq 1, \\
1 + y, & 0 \leq y < x \leq 1.
\end{cases} \quad (24) \]

**Definition 4.2.** We construct the second reproducing kernel Hilbert space \( A_2^2[0, 1] \) as [1]:

\[ A_2^2[0, 1] = \{ u, \ u' \text{ are absolutely continuous functions in the } [0, 1], \]

\[ u'' \in L^2[0, 1], \ u(0) = 0 \}. \]

We need this reproducing kernel Hilbert space for domain. The inner product and the norm of this reproducing kernel Hilbert space are given as:
\[ \langle a, b \rangle_{A^2_2[0,1]} = a(0)b(0) + a'(0)b'(0) + \int_0^1 a''(x)b''(x)dx, \quad a, b \in A^2_2[0,1], \]

and
\[ ||a||_{A^2_2[0,1]} = \sqrt{\langle a, a \rangle_{A^2_2[0,1]}}, \quad A \in A^2_2[0,1]. \]

The reproducing kernel function has been obtained as:
\[ E_y(x) = \begin{cases} 
 xy + \frac{1}{2}x^2y - \frac{x^3}{6}, & 0 \leq x \leq y \leq 0, \\
 yx + \frac{1}{2}y^2x - \frac{y^3}{6}, & 0 \leq y < x \leq 1.
\end{cases} \tag{25} \]

We obtain the solutions of the problems in the reproducing kernel Hilbert space \( A^2_2[0,1] \).

\[ H : A^2_2[0,1] \to A^1_2[0,1] \]

as:
\[ Hy(x) = \int_0^x D^\alpha \phi(x,y) + M(x)y(x). \tag{26} \]

Thus, we obtain
\[ \begin{cases} 
 Hy(x) = O(x,y(x)), \quad x \in [0,1], \\
 y(0) = 0.
\end{cases} \tag{27} \]

We construct \( \varphi_i(x) = C_x(x) \) and \( \psi_i(x) = H^* \varphi_i(x) \). The operator \( H^* \) is the adjoint operator of \( H \). The orthonormal system \( \{\hat{\psi}_i(x)\}_{i=1}^\infty \subseteq A^2_2[0,1] \) can be constructed by Gram-Schmidt orthogonalization process of \( \{\psi_i(x)\}_{i=1}^\infty \):
\[ \hat{\psi}_i(x) = \sum_{k=1}^i \beta_{ik} \psi_k(x), \quad (\beta_{ii} > 0, i = 1, 2, \ldots). \tag{28} \]

**Theorem 4.3.** Assume that \( y(x) \) is the exact solution of (27). Then, we will obtain
\[ y(x) = \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} O(x_k,y_k) \hat{\psi}_i(x). \tag{29} \]

**Proof.** We will use the properties of the adjoint operator and reproducing property to obtain the proof of the theorem. We use the feature of the complete system and get
\[ y(x) = \sum_{i=1}^\infty \left< y(x), \hat{\psi}_i(x) \right>_{A^2_2} \hat{\psi}_i(x). \tag{30} \]

Now, we use the Eq. (28) and obtain
\[ y(x) = \sum_{i=1}^\infty \sum_{k=1}^i \beta_{ik} \left< y(x), \psi_k(x) \right>_{A^2_2} \hat{\psi}_i(x) \]

We know that \( \psi_i(x) = H^* \varphi_i(x) \). Therefore, we have
\[
y(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle y(x), H^* \varphi_k(x) \rangle A_{i2} \hat{\psi}_i(x)
\]

Then, we obtain

\[
y(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle Hy(x), \varphi_k(x) \rangle A_{i2} \hat{\psi}_i(x).
\]

by the properties of the adjoint operator \(H^*\). We use the (27) and obtain

\[
y(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle O(x, y), C_{x_k} A_{i2} \hat{\psi}_i(x) \rangle.
\]

Finally, we obtain the following desired result by reproducing property as:

\[
y(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} O(x_k, y_k) \hat{\psi}_i(x).
\]

This completes the proof. \(\Box\)

We construct the approximate solution of the problem as:

\[
y_n(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} O(x_k, y_k) \hat{\psi}_i(x).
\]

We investigate three problems by reproducing kernel Hilbert space method. We demonstrate the numerical results of first problem by Table 1. In this table, we find the approximate solutions of the problem for different values \(\alpha\) and \(\beta\). We present the numerical results of the second problem by Tables 2-4. In the Table 2, we show the absolute errors of the problem for different values \(\alpha\) and \(\beta\). In the Table 3, we demonstrate the relative errors of the problem for different values \(\alpha\) and \(\beta\). In the Table 4, we show the approximate solutions, exact solutions, absolute errors and relative errors of the problem for \(\alpha = 0.5\) and \(\beta = 1\). We present the approximate solutions of the third problem for different values \(\alpha\) and \(\beta\) in Table 5. We choose 50 dense points for our all problems. We use MAPLE 18 to obtain the numerical results for all problems. We demonstrate the exact solution of the second problem by Figures 1–3. In the tables and figures \(\alpha\) shows the dimension of fractional derivative and \(\beta\) describes the dimension of the fractal. We present our numerical results for different values of \(\alpha\) and \(\beta\). Therefore, we can see the effects of the dimension of the fractals.

5. Application of the fractal fractional Malkus waterwheel model. We consider the following problem:
\[ x_t' = y(t) - ax(t) \]
\[ y_t' = bx(t)z(t) - \frac{y(t)}{2} \]
\[ z_t' = cx(t)y(t) - \frac{1 - z(t)}{2} \]

We use the fractal fractional derivatives and obtain:

\[ \frac{D^{\gamma, \beta}}{0} x(t) = y(t) - ax(t) \]
\[ \frac{D^{\gamma, \beta}}{0} y(t) = bx(t)z(t) - \frac{y(t)}{2} \]
\[ \frac{D^{\gamma, \beta}}{0} z(t) = cx(t)y(t) - \frac{1 - z(t)}{2} \]

Then, we get

\[
\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t x(\tau)(t-\tau)^{-\alpha} d\tau = y(t) - ax(t) \\
\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t y(\tau)(t-\tau)^{-\alpha} d\tau = bx(t)z(t) - \frac{y(t)}{2} \\
\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t z(\tau)(t-\tau)^{-\alpha} d\tau = cx(t)y(t) - \frac{1 - z(t)}{2} 
\]

We use the relation between the classical derivative and the fractal derivative. Then, we obtain

\[
\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t x(\tau)(t-\tau)^{-\alpha} d\tau = \beta t^{\beta-1} (y(t) - ax(t)) \\
\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t y(\tau)(t-\tau)^{-\alpha} d\tau = \beta t^{\beta-1} \left( bx(t)z(t) - \frac{y(t)}{2} \right) \\
\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t z(\tau)(t-\tau)^{-\alpha} d\tau = \beta t^{\beta-1} \left( cx(t)y(t) - \frac{1 - z(t)}{2} \right) 
\]

For simplicity, we define

\[
A(t, x, y, z) = \beta t^{\beta-1} (y(t) - ax(t)) \\
B(t, x, y, z) = \beta t^{\beta-1} \left( bx(t)z(t) - \frac{y(t)}{2} \right) \\
C(t, x, y, z) = \beta t^{\beta-1} \left( cx(t)y(t) - \frac{1 - z(t)}{2} \right) 
\]

Then, we get
\[ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t x(\tau)(t-\tau)^{-\alpha} \, d\tau = A(t, x, y, z) \]
\[ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t y(\tau)(t-\tau)^{-\alpha} \, d\tau = B(t, x, y, z) \]
\[ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t z(\tau)(t-\tau)^{-\alpha} \, d\tau = C(t, x, y, z) \]

If we apply the fractal fractional integral we will obtain

\[ x(t) - x(0) = \frac{1}{\Gamma(\alpha)} \int_0^t A(\tau, x, y, z)(t-\tau)^{\alpha-1} \, d\tau \]
\[ y(t) - y(0) = \frac{1}{\Gamma(\alpha)} \int_0^t B(\tau, x, y, z)(t-\tau)^{\alpha-1} \, d\tau \]
\[ z(t) - z(0) = \frac{1}{\Gamma(\alpha)} \int_0^t C(\tau, x, y, z)(t-\tau)^{\alpha-1} \, d\tau \]

We discretize the above system as:

\[ x(t_{n+1}) = x(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{n+1}} A(\tau, x, y, z)(t_{n+1} - \tau)^{\alpha-1} \, d\tau \]
\[ y(t_{n+1}) = y(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{n+1}} B(\tau, x, y, z)(t_{n+1} - \tau)^{\alpha-1} \, d\tau \]
\[ z(t_{n+1}) = z(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{n+1}} C(\tau, x, y, z)(t_{n+1} - \tau)^{\alpha-1} \, d\tau \]

Then, we get

\[ x(t_{n+1}) = x(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{n+1}} A(\tau, x, y, z)(t_{n+1} - \tau)^{\alpha-1} \, d\tau \]
\[ y(t_{n+1}) = y(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{n+1}} B(\tau, x, y, z)(t_{n+1} - \tau)^{\alpha-1} \, d\tau \]
\[ z(t_{n+1}) = z(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{n+1}} C(\tau, x, y, z)(t_{n+1} - \tau)^{\alpha-1} \, d\tau \]

We use the two-step Lagrange polynomial as [13]:

\[ p_s(\tau) = \frac{\tau - t_{s-1}}{t_s - t_{s-1}} A(t_s, x, y, z) - \frac{\tau - t_s}{t_s - t_{s-1}} A(t_{s-1}, x, y, z). \]  
\[ q_s(\tau) = \frac{\tau - t_{s-1}}{t_s - t_{s-1}} B(t_s, x, y, z) - \frac{\tau - t_s}{t_s - t_{s-1}} B(t_{s-1}, x, y, z). \]  
\[ r_s(\tau) = \frac{\tau - t_{s-1}}{t_s - t_{s-1}} C(t_s, x, y, z) - \frac{\tau - t_s}{t_s - t_{s-1}} C(t_{s-1}, x, y, z). \]

Thus, we will get
\[ x(t_{n+1}) = x(0) + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{n} \int_{t_s}^{t_{s+1}} p_s(\tau)(t_{n+1} - \tau)^{\alpha-1} d\tau \]

\[ = x(0) + \sum_{s=0}^{n} \left[ h^\alpha A(t_s, x, y, z) \frac{(n+1-s)^\alpha (n-s+2+\alpha)}{\Gamma(\alpha+2)} \right] - (n-s)^\alpha (n-s+2+2\alpha)] \]

\[ - \sum_{s=0}^{n} \left[ h^\alpha A(t_{s-1}, x, y, z) \frac{(n+1-s)^\alpha (n-s) (n-s+1+\alpha)}{\Gamma(\alpha+2)} \right] \]

\[ y(t_{n+1}) = y(0) + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{n} \int_{t_s}^{t_{s+1}} q_s(\tau)(t_{n+1} - \tau)^{\alpha-1} d\tau \]

\[ = y(0) + \sum_{s=0}^{n} \left[ h^\alpha B(t_s, x, y, z) \frac{(n+1-s)^\alpha (n-s+2+\alpha)}{\Gamma(\alpha+2)} \right] - (n-s)^\alpha (n-s+2+2\alpha)] \]

\[ - \sum_{s=0}^{n} \left[ h^\alpha B(t_{s-1}, x, y, z) \frac{(n+1-s)^\alpha (n-s) (n-s+1+\alpha)}{\Gamma(\alpha+2)} \right] \]

\[ z(t_{n+1}) = z(0) + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{n} \int_{t_s}^{t_{s+1}} r_s(\tau)(t_{n+1} - \tau)^{\alpha-1} d\tau \]

\[ = z(0) + \sum_{s=0}^{n} \left[ h^\alpha C(t_s, x, y, z) \frac{(n+1-s)^\alpha (n-s+2+\alpha)}{\Gamma(\alpha+2)} \right] - (n-s)^\alpha (n-s+2+2\alpha)] \]

\[ - \sum_{s=0}^{n} \left[ h^\alpha C(t_{s-1}, x, y, z) \frac{(n+1-s)^\alpha (n-s) (n-s+1+\alpha)}{\Gamma(\alpha+2)} \right] \]

We demonstrate the numerical simulations by Figures 4-7. In these figures we use the initial conditions as \(x(0) = 0.51\), \(y(0) = 0.60407\) and \(z(0) = 0.07\). We choose \(a = 1 = c\) and \(b = 5\). In Figures 4-5, we demonstrate the numerical simulations for \(x-y\), \(x-z\), \(y-z\) and \(x-y-z\). In Figure 4, we choose \(\alpha = 1 = \beta\). In this case we get the classical derivative. In Figure 5, we choose \(\alpha = 0.98\) and \(\beta = 0.99\). In this case we check the effect of the fractional order \(\alpha\) and the effect of the fractal dimension \(\beta\). In Figure 6, we would like to see the effect of the fractal dimension. Therefore, we choose \(\alpha = 1\) and \(\beta = 0.5, 0.6, 0.7, 0.8\). We show the numerical simulations for \(x-y\) in the Figure 6. In Figure 7, we would like to show the effect of the fractional order \(\alpha\). Therefore, we demonstrate the numerical simulation \((y-z)\) for \(\beta = 1\) and \(\alpha = 0.8, 0.85, 0.9, 0.95\).

6. **Conclusion.** We investigated the fractal fractional differential equations with power law kernel by Laplace transform and reproducing kernel Hilbert space method. We researched three problems in the reproducing kernel Hilbert space. We found the numerical results of the problems and showed them by tables. We proved the efficiency of the reproducing kernel Hilbert space method for the fractal fractional differential equations with the power law kernel.
\begin{tabular}{cccc}
\hline
 & $\alpha = \beta = 0.1$ & $\alpha = \beta = 0.5$ & $\alpha = \beta = 0.9$ \\
0.1 & 0.03854888333 & 0.0307043774 & 0.0078792973 \\
0.2 & 0.05333186893 & 0.0614101561 & 0.0259058607 \\
0.3 & 0.06270054818 & 0.0921156879 & 0.0516560594 \\
0.4 & 0.06958768905 & 0.1228211511 & 0.0837457401 \\
0.5 & 0.07504844747 & 0.1535265840 & 0.121235367 \\
0.6 & 0.07958273648 & 0.1842320012 & 0.1633740132 \\
0.7 & 0.08346678875 & 0.2149374096 & 0.2096297969 \\
0.8 & 0.08686918110 & 0.2456428118 & 0.2595266634 \\
0.9 & 0.08990026700 & 0.2763482087 & 0.3126795251 \\
1.0 & 0.09263619896 & 0.3070536026 & 0.3687589104 \\
\hline
\end{tabular}

Table 1. Approximate solutions of the first problem.

\begin{tabular}{cccc}
\hline
 & $\alpha = \beta = 0.1$ & $\alpha = \beta = 0.55$ & $\alpha = \beta = 0.95$ \\
0.1 & 0.000000062152 & 0.000000676818 & 0.000000856466 \\
0.2 & 0.00000032204 & 0.000000490756 & 0.00001243897 \\
0.3 & 0.00000022407 & 0.000000407914 & 0.0000217489 \\
0.4 & 0.00000017321 & 0.000000358476 & 0.00003698863 \\
0.5 & 0.00000014600 & 0.000000324696 & 0.00005597480 \\
0.6 & 0.00000012155 & 0.000000299770 & 0.00008452815 \\
0.7 & 0.00000010765 & 0.000000282100 & 0.00009461450 \\
0.8 & 0.00000009145 & 0.000000269290 & 0.00012236140 \\
0.9 & 0.000000018475 & 0.000001399970 & 0.00021363610 \\
1.0 & 0.000000018475 & 0.000001399970 & 0.00021363610 \\
\hline
\end{tabular}

Table 2. Absolute Errors for the second problem.

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\[
\begin{array}{cccc}
0.1 & 0.0001037114955 & 0.00247111550600 & 0.019860952690 \\
0.2 & 0.0000233907536 & 0.00041794915520 & 0.003864445499 \\
0.3 & 0.000010048043 & 0.00014826368870 & 0.002084659184 \\
0.4 & 0.0000054760999 & 0.00007121228610 & 0.001539514613 \\
0.5 & 0.0000035331934 & 0.00004037018634 & 0.001219745100 \\
0.6 & 0.0000023623478 & 0.00002541507333 & 0.000777074981 \\
0.7 & 0.0000017388686 & 0.00001730286530 & 0.000682296305 \\
0.8 & 0.0000012705894 & 0.000012705894 & 0.000682296305 \\
0.9 & 0.0000009316650 & 0.000009316650 & 0.000682296305 \\
1.0 & 0.0000001945157 & 0.000001945157 & 0.000682296305 \\
\end{array}
\]

Table 3. Relative Errors for the second problem.

| \(x\) | AS | ES | AE | RE |
|-------|----|----|----|----|
| 0.1   | 0.019195134 | 0.0190306572 | 0.0019195134 | 0.008642736713 |
| 0.2   | 0.010795265 | 0.01076536543 | 0.00002416107 | 0.002244333475 |
| 0.3   | 0.0296830910 | 0.02966585872 | 0.00001723228 | 0.000580879190 |
| 0.4   | 0.0610499710 | 0.06089810315 | 0.00001518674 | 0.0002493802633 |
| 0.5   | 0.1065047070 | 0.10638460810 | 0.00012009890 | 0.001128912369 |
| 0.6   | 0.1680971830 | 0.16781543890 | 0.00002817441 | 0.0001678892609 |
| 0.7   | 0.2469466021 | 0.24671689310 | 0.00002297090 | 0.000931063119 |
| 0.8   | 0.3444606528 | 0.34449169370 | 0.00003104290 | 0.00090106381 |
| 0.9   | 0.4653131081 | 0.4624497090 | 0.00339997550 | 0.006202115669 |
| 1.0   | 0.6052021980 | 0.6018022250 | 0.00007121228 | 0.005649655938 |

Table 4. Approximate Solution (AS), Exact Solution (ES), Absolute Error (AE) and Relative Error (RE) for the second problem for \(\alpha = 0.5\) and \(\beta = 1\).

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| $x$  | $AS$       |
|------|------------|
| 0.1  | 0.00048062670 |
| 0.2  | 0.00395841402 |
| 0.3  | 0.01287525177 |
| 0.4  | 0.02728111814 |
| 0.5  | 0.04281856069 |
| 0.6  | 0.05461350578 |
| 0.7  | 0.05592662854 |
| 0.8  | 0.03911508097 |
| 0.9  | $-0.0027967424$ |
| 1.0  | $-0.0752782621$ |

Table 5. Approximate Solution (AS) for the third problem for $\alpha = \beta = 0.5$.

Figure 1. Exact Solutions (ES) of the second problem for $\alpha = \beta = 0.1$ and $\alpha = \beta = 0.9$.

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Figure 2. Exact Solutions (ES) of the second problem for $\alpha = \beta = 0.5$ and $\alpha = \beta = 0.9$.

Figure 3. Exact Solutions (ES) of the second problem for $\alpha = \beta = 1.0$ and $\alpha = \beta = 0.9$.

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Figure 4. The dynamical behavior of the chaotic attractor for $\alpha = 1 = \beta$.

Figure 5. The dynamical behavior of the chaotic attractor for $\alpha = 0.98$ and $\beta = 0.99$.

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Figure 6. The dynamical behavior of the chaotic attractor for $\alpha = 0.1$ and different values of $\beta$.

Figure 7. The dynamical behavior of the chaotic attractor for $\beta = 1$ and different values of $\alpha$.

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