Research article

Entire positive $k$-convex solutions to $k$-Hessian type equations and systems

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Abstract: In this paper, we study the existence of entire positive solutions for the $k$-Hessian type equation

$S_k(D^2u + \alpha I) = p(|x|)f^k(u), \ x \in \mathbb{R}^n$

and system

$$\begin{cases}
S_k(D^2u + \alpha I) = p(|x|)f^k(v), \ x \in \mathbb{R}^n, \\
S_k(D^2v + \alpha I) = q(|x|)g^k(u), \ x \in \mathbb{R}^n,
\end{cases}$$

where $D^2u$ is the Hessian of $u$ and $I$ denotes unit matrix. The arguments are based upon a new monotone iteration scheme.

Keywords: $k$-Hessian type equation and system; entire positive $k$-convex solution; monotone iterative; existence

1. Introduction

Consider the existence of entire positive $k$-convex solutions to the following $k$-Hessian type equation

$S_k(D^2u + \alpha I) = p(|x|)f^k(u), \ x \in \mathbb{R}^n,$  \hspace{1cm} (E)

and system

$$\begin{cases}
S_k(D^2u + \alpha I) = p(|x|)f^k(v), \ x \in \mathbb{R}^n, \\
S_k(D^2v + \alpha I) = q(|x|)g^k(u), \ x \in \mathbb{R}^n,
\end{cases}$$  \hspace{1cm} (S)

where $k \in \{1, 2, \ldots, n\}$, $\alpha \geq 0$ is a constant, $I$ is the identity function and $p, q$ are continuous functions on $[0, +\infty)$. Letting $D^2u = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)$ denote the Hessian of $u \in C^2(\mathbb{R}^n)$ and $\lambda_i \ (i \in \{1, 2, \ldots, n\})$ denote the
eigenvalues of $D^2u$, then
\[ S_k(D^2u + \alpha I) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{1}{k!} \delta^{i_1, i_2, \ldots, i_k}_{j_1, j_2, \ldots, j_k} (\lambda_{i_1} + \alpha)(\lambda_{i_2} + \alpha)\cdots(\lambda_{i_k} + \alpha), \]

where $\delta^{i_1, i_2, \ldots, i_k}_{j_1, j_2, \ldots, j_k}$ is the generalized Kronecker symbol, is the $k$-Hessian type operator. When $\alpha = 0$, $S_k(D^2u)$ is the standard $k$-Hessian operator.

Denote
\[ \Gamma_k := \{ \lambda \in \mathbb{R}^n : S_j(\lambda) > 0, 1 \leq j \leq k \}. \]

We call a function $u \in C^2(\mathbb{R}^n)$ $k$-convex in $\mathbb{R}^n$ if $\lambda(D^2u(x) + \alpha I) \in \Gamma_k$ for all $x \in \mathbb{R}^n$.

In particular,
\[ S_1(D^2u + \alpha I) = \sum_{i=1}^n \lambda_i = \Delta u; \]
\[ S_n(D^2u + \alpha I) = \prod_{i=1}^n \lambda_i = \det(D^2u + \alpha I). \]

The $k$-Hessian equation is fully nonlinear PDEs for $k \neq 1$ (see Urbas [1] and Wang [2]), and there are many important applications in fluid mechanics, geometric problems and other applied subjects. Many authors have demonstrated increasing interest in $k$-Hessian equations by different methods, for instance, see ( [3–9]) and the references cited therein ([10–15]). In particular, problem $(E)$ reduces to the problems studied by Keller [16] and Osserman [17] when $k = 1$, $p(|x|) = 1$ on $\mathbb{R}^n$ and $f : [0, \infty) \to [0, \infty)$ is continuous and increasing. The authors studied a necessary and sufficient condition
\[ \int_1^\infty \frac{dt}{\sqrt{2F(t)}} = \infty, \quad F(t) = \int_0^t f(s)ds \]
for the existence of entire large positive radial solutions to $(E)$. When $k = 1$, $f(u) = u^\gamma (\gamma \in (0, 1])$ and $p : [0, \infty) \to [0, \infty)$ is continuous, Lair and Wood [18] showed that $(E)$ admits infinitely many entire large positive radial solutions if and only if
\[ \int_0^\infty rp(r)dr = \infty. \]

For the case $k = 1$, system $(S)$ reduces to the following problem
\[ \begin{cases} 
\Delta u = p(|x|)f(v), \ x \in \mathbb{R}^n, \\
\Delta v = q(|x|)g(u), \ x \in \mathbb{R}^n.
\end{cases} \tag{1.1} \]

Lair and Wood [19] analyzed the existence and nonexistence of entire positive radial solutions to Eq (1.1) when $f(v) = v^\beta, \ g(u) = u^\gamma$ ($0 < \beta \leq \gamma$). For the further results, we can see [20–23] and the reference therein.

When $\alpha = 0$, Zhang and Zhou [24] considered the existence of entire positive $k$-convex solutions to problem $(E)$ and system $(S)$. 

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For the case $k = n$, Zhang and Liu [25] studied the existence of entire radial large solutions for a Monge-Ampère type equation

$$\det(D^2 u) - \alpha \Delta u = a(|x|) f(u), \ x \in \mathbb{R}^n$$

(1.2)

and system

$$\begin{cases} 
\det(D^2 u) - \alpha \Delta u = a(|x|) f(v), \ x \in \mathbb{R}^n, \\
\det(D^2 v) - \beta \Delta v = b(|x|) g(u), \ x \in \mathbb{R}^n.
\end{cases}$$

(1.3)

Their results have been improved by Covei [26].

Recently, when $p(|x|) \equiv 1$ on $\mathbb{R}^n$, Dai [27] showed that there exists a subsolution $u \in C^2(\mathbb{R}^n)$ of $(E)$ if and only if

$$\int_0^\infty \left( \int_0^r f(t) dt \right)^{\frac{1}{n}} d\tau = \infty$$

holds.

Motivated by these works mentioned above, in this paper we will obtain some new results on the existence of entire positive $k$-convex radial solutions for equation $(E)$ and system $(S)$. The arguments are based upon a new monotone iteration scheme.

Let $\alpha_0$ denote a positive constant. In the following, we always suppose that

$$\alpha_0 > \frac{n}{k} \left( \frac{nC_{n-1}^k}{k} \right)^\frac{1}{n} a = \frac{n}{k} (C_n^k)^{\frac{1}{n}} a. \quad (1.4)$$

We give the following conditions:

(f1) $f, g : [0, \infty) \to (\alpha_0, \infty)$ are continuous and nondecreasing;

(f2) $p, q : [0, \infty) \to (0, \infty)$ are continuous and nondecreasing.

Define

$$P(\infty) := \lim_{r \to \infty} P(r), \quad P(r) := \int_0^r \left( \frac{I_{k-n}}{C_0} \int_0^s s^{-1} p(s) ds \right)^{\frac{1}{n}} dt, \ r \geq 0;$$

$$Q(\infty) := \lim_{r \to \infty} Q(r), \quad Q(r) := \int_0^r \left( \frac{I_{k-n}}{C_0} \int_0^s s^{-1} q(s) ds \right)^{\frac{1}{n}} dt, \ r \geq 0;$$

(1.5)

(1.6)

where

$$C_0 = \frac{C_{n-1}^k}{k}.$$ 

For an arbitrary $a > 0$, we also define

$$H_{1a}(\infty) := \lim_{r \to \infty} H_{1a}(r), \quad H_{1a}(r) := \int_a^r \frac{d\tau}{f(\tau)}, \ r \geq a; \quad (1.7)$$

$$H_{2a}(\infty) := \lim_{r \to \infty} H_{2a}(r), \quad H_{2a}(r) := \int_a^r \frac{d\tau}{f(\tau) + g(\tau)}, \ r \geq a,$$

(1.8)

and we see that

$$H'_{1a}(r) = \frac{1}{f(r)} > 0, \quad H'_{2a}(r) = \frac{1}{f(r) + g(r)} > 0, \ \forall r > a,$$

and $H_{1a}, H_{2a}$ admit the inverse functions $H'_{1a}$ and $H'_{2a}$ on $[0, H_{1a}(\infty))$ and $[0, H_{2a}(\infty))$ respectively.

The main results of this paper can be stated as follows.
Theorem 1.1. Suppose that (f1) and (f2) hold. If \( \alpha = 0 \), then Eq (E) admits an entire positive k-convex radial solution \( u \in C^2(\mathbb{R}^n) \) satisfying

\[
a + \alpha_0 P(r) \leq u \leq H_{1a}^{-1}(P(r)), \quad \forall r \geq 0.
\]

Moreover, if \( P(\infty) = \infty \) and \( H_{1a}(\infty) = \infty \), then \( \lim_{r \to \infty} u(r) = \infty \); if \( P(\infty) < H_{1a}(\infty) < \infty \), then \( u \) is bounded.

If \( \alpha > 0 \) and \( p(|x|) \geq 1 \), then Eq (E) admits an entire positive k-convex radial solution \( u \in C^2(\mathbb{R}^n) \) satisfying

\[
a + \alpha_0 P(r) - \frac{\alpha}{r^2} u \leq H_{1a}^{-1}(P(r)), \quad \forall r \geq 0.
\]

If further suppose \( H_{1a}(\infty) = \infty \), then \( \lim_{r \to \infty} u(r) = \infty \).

Theorem 1.2. Suppose that (f1) and (f2) hold. If \( \alpha = 0 \), then system (S) admits an entire positive k-convex radial solution \( (u, v) \in C^2(\mathbb{R}^n) \times C^2(\mathbb{R}^n) \) satisfying

\[
a \frac{1}{2} + \alpha_0 P(r) \leq u \leq H_{2a}^{-1}(P(r) + Q(r)), \quad \forall r \geq 0;
\]

\[
a \frac{1}{2} + \alpha_0 Q(r) \leq v \leq H_{2a}^{-1}(P(r) + Q(r)), \quad \forall r \geq 0.
\]

Moreover, if \( P(\infty) = \infty = Q(\infty) \) and \( H_{2a}(\infty) = \infty \), then \( \lim_{r \to \infty} u(r) = \lim_{r \to \infty} v(r) = \infty \); if \( P(\infty) + Q(\infty) < H_{2a}(\infty) < \infty \), then \( u \) and \( v \) are bounded.

If \( \alpha > 0 \) and \( p(|x|) \geq 1 \), then system (S) admits an entire positive k-convex radial solution \( (u, v) \in C^2(\mathbb{R}^n) \times C^2(\mathbb{R}^n) \) satisfying

\[
a \frac{1}{2} + \alpha_0 P(r) - \frac{\alpha}{r^2} \leq u \leq H_{2a}^{-1}(P(r) + Q(r)), \quad \forall r \geq 0;
\]

\[
a \frac{1}{2} + \alpha_0 Q(r) - \frac{\alpha}{r^2} \leq v \leq H_{2a}^{-1}(P(r) + Q(r)), \quad \forall r \geq 0.
\]

If further suppose \( H_{2a}(\infty) = \infty \), then \( \lim_{r \to \infty} u(r) = \lim_{r \to \infty} v(r) = \infty \).

2. Preliminary lemmas

For convenience, we give some lemmas for the radial functions before proving the main results.

Let \( r = |x| = \sqrt{x_1^2 + \ldots + x_n^2} \) and \( B_R := \{x \in \mathbb{R}^n : |x| < R\} \) for \( R \in (0, \infty] \).

Lemma 2.1. (Lemma 2.1, [25]) Suppose that \( \phi \in C^2(0, R) \) with \( \phi'(0) = 0 \). Then, for \( u(x) = \phi(r) \), we have \( u(x) \in C^2(B_R) \), and the eigenvalues of \( D^2u + \alpha I \) are

\[
\lambda(D^2u + \alpha I) = \begin{cases} 
(\phi''(r) + \alpha, \frac{\phi'(r)}{r} + \alpha, \ldots, \frac{\phi'(r)}{r} + \alpha), & r \in (0, R), \\
(\phi'(0) + \alpha, \phi''(0) + \alpha, \ldots, \phi''(0) + \alpha), & r = 0,
\end{cases}
\]

and so

\[
S_k(D^2u + \alpha I) = \begin{cases} 
C_{n-1}^k(\phi''(r) + \alpha)^{r^{k-1}} + C_n^k \frac{(\phi'(r)+\alpha)^{r^k}}{r^k}, & r \in (0, R), \\
C_n^k(\phi''(0) + \alpha)^k, & r = 0.
\end{cases}
\]
By Lemma 2.1, we can conclude that \( u(x) = \varphi(r) \) is a \( C^2 \) radial solution of (E) if and only if \( \varphi(r) \) satisfies
\[
C_{n-1}^{k-1}(\varphi''(r) + \alpha \frac{(\varphi'(r) + \alpha r)^{k-1}}{r^{k-1}}) + C_{n-1}^k \left( \frac{(\varphi'(r) + \alpha r)^k}{r^k} \right) = p(r)f^k(\varphi(r)), \ r \in (0, R).
\]

(2.1)

**Lemma 2.2.** Suppose that (f1) and (f2) hold. For any positive number \( a \), let \( \varphi \in C[0, R) \cap C^1(0, R) \) be a solution of the Cauchy problem
\[
\begin{aligned}
\varphi'(r) &= \left( \frac{k-n}{C_0} \int_0^r s^{n-1} p(s)f^k(\varphi(s))ds \right)^{\frac{1}{k}} - \alpha r, \ r > 0, \\
\varphi(0) &= a > 0.
\end{aligned}
\]

(2.2)

Then \( \varphi \in C^2[0, R) \), and it satisfies (2.1) with \( \varphi'(0) = 0 \).

**Proof.** Firstly, we have
\[
\varphi'(0) = \lim_{r \to 0} \frac{\varphi(r) - \varphi(0)}{r - 0} = \lim_{r \to 0} \varphi'(r) = \lim_{r \to 0} \left( \frac{k-n}{C_0} \int_0^r s^{n-1} p(s)f^k(\varphi(s))ds \right)^{\frac{1}{k}} - \alpha r = 0.
\]

Since
\[
\lim_{r \to 0} \varphi'(r) = \lim_{r \to 0} \left( \frac{k-n}{C_0} \int_0^r s^{n-1} p(s)f^k(\varphi(s))ds \right)^{\frac{1}{k}} - \alpha r = 0 = \varphi'(0).
\]

This shows that \( \varphi(r) \in C^1[0, R) \).

Secondly,
\[
\varphi''(0) = \lim_{r \to 0} \frac{\varphi''(r) - \varphi''(0)}{r - 0} = \lim_{r \to 0} \varphi''(r) = \lim_{r \to 0} \left( \frac{k-n}{C_0} \int_0^r s^{n-1} p(s)f^k(\varphi(s))ds \right)^{\frac{1}{k}} - \alpha r
\]
\[
= \lim_{r \to 0} \left( \frac{k-n}{C_0} \int_0^r s^{n-1} p(s)f^k(\varphi(s))ds \right) - \alpha + \lim_{r \to 0} \left( \frac{k-n}{C_0} \int_0^r s^{n-1} p(s)f^k(\varphi(s))ds \right)^{\frac{1}{k}} - \alpha
\]
\[
= \left( \frac{1}{nC_0} \right)^{\frac{1}{k}} f(\varphi(0)) - \alpha.
\]

It is easy to know that \( \varphi(r) \in C^2(0, R) \) for \( r \in (0, R) \). By calculating,
\[
\lim_{r \to 0} \varphi''(r) = \lim_{r \to 0} \frac{k-n}{k} r^{-\frac{n}{k}} \left( \int_0^r \frac{1}{C_0} p(s)f^k(\varphi(s))s^{n-1}ds \right)^{\frac{1}{k}}
\]
\[
+ \lim_{r \to 0} \frac{1}{kC_0} r^{\frac{n}{k}} \left( \int_0^r \frac{1}{C_0} p(s)f^k(\varphi(s))s^{n-1}ds \right)^{\frac{1}{k}} r^{n-1} p(r)f^k(\varphi(r)) - \alpha
\]
Lemma 2.4. (Lemma 2.2, [25]) Suppose that

\[
\frac{k - n}{k} \left( \frac{1}{nC_0} p(0) \right)^{\frac{1}{k}} f(\varphi(0)) + \frac{n}{k} \left( \frac{1}{nC_0} p(0) \right)^{\frac{1}{k}} f(\varphi(0)) - \alpha
\]

\[= \left( \frac{1}{nC_0} p(0) \right)^{\frac{1}{k}} f(\varphi(0)) - \alpha.
\]

Hence, \( \varphi(r) \in C^2[0, R] \). And by direct calculation, we can prove that \( \varphi(r) \) satisfies (2.1).

Remark 2.3. When \( p(r) \equiv 1 \), Lemma 2.2 is consistent with Lemma 2.3 in [25].

Lemma 2.4. (Lemma 2.2, [25]) Suppose that (f1), (f2) hold and \( \varphi(r) \in C^2[0, R] \) satifies (2.1) with \( \varphi'(0) = 0 \). Then \( \varphi'(r) \geq 0 \) and \( \varphi''(r) + \alpha > 0 \).

Proof. From (2.1), we have

\[C_{n-1}^{\frac{k-1}{k}}(\varphi'(r) + \alpha r)^{k} = k^{p-1} p(r) f^k(\varphi(r)).\]

Noticing that \( \varphi'(0) = 0 \) and integrating from 0 to \( r \), combining with (1.7), we have

\[\varphi'(r) = \left( \frac{1}{C_0} \int_0^r s^{n-1} p(s) f^k(\varphi(s)) \, ds \right)^{\frac{1}{k}} - ar.
\]

If \( \alpha = 0 \), then we can easily prove that \( \varphi'(r) > 0 \); if \( \alpha > 0 \) and \( p(|x|) > 1 \), then we have

\[\varphi'(r) \geq \alpha_0 \left( \frac{1}{C_0} \int_0^r s^{n-1} \, ds \right)^{\frac{1}{k}} - ar
\]

\[> (nC_0)^{\frac{1}{k}} \left( nC_0 \right)^{\frac{1}{k}} - ar - ar
\]

\[= \alpha \left( \frac{n}{k} - 1 \right) r \geq 0.
\]

On the other hand, by calculating, for \( 0 < s < r \), we have

\[\varphi''(r) + \alpha = \frac{k - n}{k} r^{-\frac{k}{2}} \int_0^r \frac{1}{C_0} p(s) f^k(\varphi(s)) s^{n-1} \, ds^{\frac{1}{k}}
\]

\[+ \frac{1}{kC_0} r^{-\frac{n}{2}} \left( \int_0^r \frac{1}{C_0} p(s) f^k(\varphi(s)) s^{n-1} \, ds \right)^{\frac{1}{k}} - r^{n-1} p(r) f^k(\varphi(r))
\]

\[= r^{-\frac{k}{2}} \left( \int_0^r \frac{1}{C_0} p(s) f^k(\varphi(s)) s^{n-1} \, ds \right)^{\frac{1}{k}} - \left[ \frac{k - n}{k} \int_0^r \frac{1}{C_0} p(s) f^k(\varphi(s)) s^{n-1} \, dsight.
\]

\[+ \frac{1}{kC_0} r^{n} p(r) f^k(\varphi(r)) \]

\[\geq r^{-\frac{k}{2}} \left( \int_0^r \frac{1}{C_0} p(s) f^k(\varphi(s)) s^{n-1} \, ds \right)^{\frac{1}{k}} - \left[ \frac{k - n}{k} \frac{1}{C_0} p(r) f^k(\varphi(r)) \right] r^n
\]

\[+ \frac{1}{kC_0} r^{n} p(r) f^k(\varphi(r)) \]

\[= r^{-\frac{k}{2}} \left( \int_0^r \frac{1}{C_0} p(s) f^k(\varphi(s)) s^{n-1} \, ds \right)^{\frac{1}{k}} - \left[ \frac{1}{n} r^n p(r) f^k(\varphi(r)) \right]
\]

\[> 0.
\]

This gives the proof of Lemma 2.4.
Remark 2.5. When \( p(r) \equiv 1 \), Lemma 2.4 is consistent with Lemma 2.2 in [25].

3. Proof of the main results

In this section, we prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Firstly, we consider the equations

\[
C_{n-1}^{k-1}(u''(r) + \alpha) \left( \frac{u'(r) + ar}{r} \right)^{k-1} + C_n^{k-1} \left( \frac{u'(r) + ar}{r} \right)^k = p(r)f^k(u(r)), \quad r > 0, \tag{3.1}
\]

and

\[
u'(r) = \left( \frac{r^{k-n}}{C_0} \int_0^r s^{n-1} p(s)f^k(u(s))ds \right)^{1/2} - ar, \quad r > 0, \quad u(0) = a, \tag{3.2}
\]

Accordingly, solutions in \( C[0, \infty) \) to (3.3) are solutions in \( C[0, \infty) \cap C^1(0, \infty) \) to (3.2).

Let \( \{u_m\}_{m \geq 1} \) be the sequences of positive continuous functions defined on \( [0, \infty) \) by

\[
u_0(r) = a, \quad u_m(r) = a + \int_0^r \left( \frac{r^{k-n}}{C_0} \int_0^s s^{n-1} p(s)f^k(u_{m-1}(s))ds \right)^{1/2} dt - \frac{\alpha}{2}r^2, \quad r \geq 0.
\]

Obviously, for all \( r \geq 0 \) and \( m \in \mathbb{N} \), we have

\[
u_m(r) = a + \int_0^r \left( \frac{r^{k-n}}{C_0} \int_0^s s^{n-1} p(s)f^k(u_{m-1}(s))ds \right)^{1/2} dt - \frac{\alpha}{2}r^2
\]

\[
\geq a + a_0 \int_0^r \left( \frac{r^{k-n}}{C_0} \int_0^s s^{n-1} p(s)ds \right)^{1/2} dt - \frac{\alpha}{2}r^2
\]

\[
\geq a + a_0 P(r) - \frac{\alpha}{2}r^2.
\]

Therefore, \( u_m(r) \geq a, \) and \( u_0(r) < u_1(r). \) Since \( (f1) \) holds, we have \( u_1(r) < u_2(r) \) for \( r \geq 0. \) According to the above reasons, we obtain that the sequences \( \{u_m\} \) is increasing on \( [0, \infty). \) Also, we obtain by \( (f1) \) and \( (f2) \) that for each \( r > 0 \)

\[
u_m'(r) = \left( \frac{r^{k-n}}{C_0} \int_0^r s^{n-1} p(s)f^k(u_{m-1}(s))ds \right)^{1/2} - ar
\]

\[
\leq f(u_m(r)) \left( \frac{r^{k-n}}{C_0} \int_0^r s^{n-1} p(s)ds \right)^{1/2} - ar
\]

\[
\leq f(u_m(r))P'(r).
\]

Therefore,

\[
\int_a^{u_m(r)} \frac{1}{f(r)} dr \leq P(r), \quad r > 0.
\]

This shows that

\[
H_1u_m(r) \leq P(r), \quad \forall r \geq 0, \tag{3.4}
\]
and
\[ u_m(r) \leq H_{1d}^{-1}(P(r)), \forall r \geq 0. \] (3.5)

It follows that the sequences \( \{u_m\}, \{u'_m\} \) are bounded on \([0, R_0]\) for an arbitrary \(R_0 > 0\). By Arzelà-Ascoli theorem, \( \{u_m\} \) has subsequences converging uniformly to \(u\) on \([0, R_0]\). Since \( \{u_m\} \) is increasing on \([0, \infty)\), we see that \( \{u_m\} \) itself converges uniformly to \(u\) on \([0, R_0]\). By arbitrariness of \(R_0\) and Lemma 2.2, we get that \(u\) is an entire positive \(k\)-convex radial solution to \((E)\), and \(u\) satisfies
\[ a + \alpha_0 P(r) - \frac{\alpha}{2} r^2 < u(r) \leq H_{1d}^{-1}(P(r)), \forall r \geq 0. \] (3.6)

If \(\alpha = 0\), by (3.6), it is easy to obtain that if \(P(\infty) = \infty\) and \(H_{1d}(\infty) = \infty\), then \(\lim_{r \to \infty} u(r) = \infty\); if \(P(\infty) < H_{1d}(\infty) < \infty\), then \(u\) is bounded. If \(\alpha > 0\), combining the fact that \(\rho(|x|) \geq 1\), \(\alpha_0 > \frac{\alpha}{2}(C^k_n)^2\alpha\) and \(H_{1d}(\infty) = \infty\), it is obvious that \(\lim_{r \to \infty} u(r) = \infty\). This finishes the proof of Theorem 1.1.

Remark 3.1. Theorem 1.1 generalizes Theorem 1.1 with \(\alpha > 0\) in [24]. In the case \(\alpha > 0\), since \(P(\infty) = \infty\) for the positivity of \(u\), it is difficult to ensure if there is bounded positive entire solution of \((E)\).

Proof of Theorem 1.2. Consider the following systems
\[
\begin{align*}
  C_{n-1}^{k-1}(u''(r) + \alpha) \frac{(s''(r) + \alpha r^k)}{r^{k-1}} + C_{n-1}^{k} \frac{(s''(r) + \alpha r^k)}{r^{k-1}} &= p(r)f^k(v(r)), \ r > 0, \\
  C_{n-1}^{k-1}(v''(r) + \alpha) \frac{(s''(r) + \alpha r^k)}{r^{k-1}} + C_{n-1}^{k} \frac{(s''(r) + \alpha r^k)}{r^{k-1}} &= q(r)g^k(u(r)), \ r > 0,
\end{align*}
\]

and
\[
\begin{align*}
  u(r) &= \frac{\alpha}{2} + \int_0^r \left( \frac{\frac{n}{k}}{\frac{k}{c}} \int_0^t s^{n-1} p(s) f^k(v(s))ds \right)^{\frac{1}{2}} dt - \frac{\alpha}{2} r^2, \ r \geq 0, \\
  v(r) &= \frac{\alpha}{2} + \int_0^r \left( \frac{\frac{n}{k}}{\frac{k}{c}} \int_0^t s^{n-1} q(s) g^k(u(s))ds \right)^{\frac{1}{2}} dt - \frac{\alpha}{2} r^2, \ r \geq 0.
\end{align*}
\]

Let \(\{u_m\}_{m \geq 1}\) and \(\{v_m\}_{m \geq 1}\) be the sequences of positive continuous functions defined on \([0, \infty)\) by
\[
\begin{align*}
  v_0 &= \frac{\alpha}{2}, \\
  u_m(r) &= \frac{\alpha}{2} + \int_0^r \left( \frac{\frac{n}{k}}{\frac{k}{c}} \int_0^t s^{n-1} p(s) f^k(v_m(s))ds \right)^{\frac{1}{2}} dt - \frac{\alpha}{2} r^2, \ r \geq 0, \\
  v_m(r) &= \frac{\alpha}{2} + \int_0^r \left( \frac{\frac{n}{k}}{\frac{k}{c}} \int_0^t s^{n-1} q(s) g^k(u_m(s))ds \right)^{\frac{1}{2}} dt - \frac{\alpha}{2} r^2, \ r \geq 0.
\end{align*}
\]

Similarly, for all \(r \geq 0\) and \(m \in \mathbb{N}\), when \(m \geq 1\), we have
\[
\begin{align*}
  u_m(r) > &\ \frac{\alpha}{2} + \alpha_0 P(r) - \frac{\alpha}{2} r^2, \\
  v_m(r) > &\ \frac{\alpha}{2} + \alpha_0 Q(r) - \frac{\alpha}{2} r^2.
\end{align*}
\]

Therefore, \(u_m(r) \geq \frac{\alpha}{2}, v_m(r) \geq \frac{\alpha}{2}\) and \(v_0(r) < v_1(r)\). Since \(f, g\) are continuous and nondecreasing, we have \(u_1(r) < u_2(r), \forall r \geq 0\), and \(v_1(r) < v_2(r), \forall r \geq 0\). According to the above reasons, we obtain that the sequences \(\{u_m\}\) and \(\{v_m\}\) are increasing on \([0, \infty)\).

Moreover, for \(r > 0\), by (f1) and (f2), one can prove that
\[
\begin{align*}
  u_m'(r) \leq (f(v_m(r) + u_m(r)) + g(v_m(r) + u_m(r)))P'(r);
\end{align*}
\]
\[
v_m'(r) \leq (f(v_m(r) + u_m(r)) + g(v_m(r) + u_m(r)))Q'(r),
\]
and
\[
u_m'(r) + v_m'(r) \leq [f(v_m(r) + u_m(r)) + g(v_m(r) + u_m(r))](P'(r) + Q'(r)).
\]
Therefore,
\[
\int_a^{u_m(r)+v_m(r)} \frac{1}{f(\tau) + g(\tau)} d\tau \leq P(r) + Q(r), \quad r > 0,
\]
which shows that
\[
H_{2a}(u_m(r) + v_m(r)) \leq P(r) + Q(r), \quad \forall r \geq 0,
\]
and
\[
u_m(r) + v_m(r) \leq H_{2a}^{-1}(P(r) + Q(r)), \quad \forall r \geq 0.
\]

It so follows that the sequences \{u_m\}, \{u_m'\} and \{v_m\}, \{v_m'\} are bounded on \([0, R_0]\) for an arbitrary \(R_0 > 0\). By Arzelà-Ascoli theorem, \{u_m\} and \{v_m\} have subsequences converging uniformly to \(u\) and \(v\) respectively on \([0, R_0]\). Since \{u_m\}, \{v_m\} are increasing on \([0, \infty)\), we see that \{u_m\} itself converges uniformly to \(u\) on \([0, R_0]\), so is \{v_m\}. By arbitrariness of \(R_0\) and Lemma 2.2, we get that \((u, v)\) is an entire positive \(k\)-convex radial solution to \((S)\).

The rest proof is similar to that of Theorem 1.1. So we omit it here.

4. Conclusions

In this paper, we use a new monotone iteration scheme to obtain some new existence results of entire positive solutions for a \(k\)-Hessian type equation and system.

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Conflict of interest

The authors declare there is no conflict of interest.

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