Conditional Aharonov-Bohm Phases with Double Quantum Dots

Roberta Rodriquez\textsuperscript{*1,2} and Jiannis K. Pachos\textsuperscript{‡2}

\textsuperscript{1}Department of Physics, Cavendish Laboratory, University of Cambridge, Cambridge, CB3 0HE, UK,
\textsuperscript{2}Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 0WA, UK.

A quantum dot proposal for the implementation of topological quantum computation is presented. The coupling of the electron charge to an external magnetic field via the Aharonov-Bohm effect, combined with the control dynamics of a double dot, results in a two-qubit control phase gate. The physical mechanisms of the system are analysed in detail and the conditions for performing quantum computation resilient to control errors are outlined and found to be realisable with present technology.

I. INTRODUCTION

Solid state is becoming an increasingly promising arena for future implementations of quantum computation\cite{1,2,3}, as a consequence of the vast progress of classical computation technology in the atomic scale. Quantum manipulation of solid state devices paves the way for many proposals supporting, in principle, the possibility of scalable quantum computation. However, the control of individual quantum systems, such as electrons in quantum dots, demands a much higher degree of control accuracy than currently available. A number of recent proposals address the issue of controllability by employing geometrical and topological effects\cite{4,5}. The key advantage of these methods is that the resulting geometrical and topological gates do not depend on the overall time of the evolution, nor on small deformations in the control parameters. Possible manifestations of geometrical phases are the Berry phases obtained, for example, through a cyclic adiabatic evolution of a system\cite{6}, or through the Aharonov-Bohm effect\cite{7}. Within solid state physics, there have even been proposals to implement Berry phases with Josephson junctions\cite{8} as well as Aharonov-Bohm phases encoded on the different spin states of electrons manipulated in quantum dot structures\cite{9,10}.

In this chapter we present a simple solid state scheme for geometrical quantum computation where the Aharonov-Bohm phase is encoded on the electronic charge\cite{11}. In particular, we consider quantum dots that can either be empty or accommodate one electron. In an array of quantum dots we assume that one is able to lower the potential between any two dots and facilitate the quantum tunneling between them. Intrinsically, the Hamiltonian of this system is governed mainly by three terms, namely the potential wall of height $V_0$ separating two neighbouring dots and the Coulomb interaction $V_C = e^2/r$ between two electrons occupying the same dot or adjacent dots in close proximity.

![Diagram of quantum dots](image)

FIG. 1: Array of quantum dots illustrating the dots within the computational space (1 and 4) and the auxiliary dots (2 and 3). The auxiliary dots get populated only during the operation of the two-qubit gates. Through the triangular configuration (1, 2 and 3) a circular path is obtained. The double dot (3 and 4) provides the conditional dynamics.

Imagine that we have an array of paired dots with one electron in each pair, as shown in Fig. 1. The position of an electron in the pair encodes the states $|0\rangle$ and $|1\rangle$. If the electron is in the left dot the qubit $i$ is in the logical state $|l_i\rangle = |0\rangle$, while if the electron is in the right dot the qubit is in state $|r_i\rangle = |1\rangle$. Preparation of the initial qubit state as well as the final readout are therefore technically easy tasks. Moreover, single-qubit gates are simple to

\textsuperscript{*}r.rodriguez@damtp.cam.ac.uk
\textsuperscript{‡}j.pachos@damtp.cam.ac.uk
achieve in this setting [14]. By lowering the potential barrier $V_0$ between the pair of the dots that compose a qubit, quantum tunnelling between the two dots will create superpositions of the logical states [13] giving, for example, the state $|\psi⟩ = c_0|0⟩ + c_1|1⟩$. Any possible one qubit rotation can thus be performed. For a two-qubit phase-gate we need to perform a controlled operation where the state of one of the qubits is changed conditionally on the state of the second qubit. We shall present a way of performing such a gate in a topological fashion by employing, in addition, a double dot structure. The combination of single-qubit gates and controlled phase-gates can then allow us to carry out any arbitrary quantum computation [15]. Finally, comparison of our model with state of the art experiments is presented together with an outline of its potential advantages.

II. TWO QUBIT PHASE-GATES

We would like to implement a control phase gate by adopting an intrinsically geometrical evolution. Consider a homogeneous magnetic field $B$ in the neighbourhood of the dots that comprise the logical array of qubits. An electron which spans a closed trajectory (loop) inside the magnetic field will acquire a phase factor proportional to the flux of the magnetic field encircled by the loop [7]. If this trajectory is spanned in a conditional way, depending on the presence of an electron in a neighbouring dot, the resulting effect can be used as the basis for a two-qubit phase-gate.

We assume that the magnetic field we employ, $\vec{B}$, is large enough so that the electronic spins in the dots are aligned along the direction of $\vec{B}$, and henceforth play no role in the subsequent evolution. However, errors will be introduced in general if there is a small deformation in the geometry of the loop due to fluctuations in the control parameters, giving eventually different phases. We can remedy this by discretising the trajectory of the electron by allowing tunnelling between three different dots, as shown in Fig. 2. This structure, which eliminates statistical deformations in the electron trajectory, is then repeated as shown in Fig. 1 and is the basis of our proposal.

By using external control we allow the activation and deactivation of tunneling between neighbouring sites. We assume that all tunnel couplings $t_{ij}$ between adjacent sites $i$ and $j$, $\{i,j\} = \{1,2,3\}$, can take the same value, $J$. In addition, we require a double dot structure in order to implement the conditional phase gate; the purpose of this structure is to suppress tunnelling between $1 \rightarrow 3$ and $2 \rightarrow 3$ when dot 4 is occupied. This is achieved through the Coulomb blockade, where the Coulomb repulsion between two electrons in each dot of the double dot is strong enough to prevent any tunnelling towards dot 3. In this regime, by turning on the couplings $t_{12}, t_{23}, t_{12}$ and $t_{31}$ in succession for a sufficient time to allow complete transition from one dot to another, it is impossible to circulate an electron initially in dot 1 around dots 1, 2, 3 and back to 1. If, however, dot 4 is empty, an electron can go around the closed path acquiring a phase due to the Aharonov-Bohm effect. This phase is given by $\phi = ABe/\hbar$ where $A$ is the area of the triangle spanned by the dots 1, 2 and 3. In both cases, at the end of this evolution, the system returns back to the computational space where dots 2 and 3 are unoccupied. We can describe this evolution, in the computational basis, by

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\phi} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (1)

The resulting two-qubit gate is therefore a control-phase gate.

FIG. 2: Quantum dot implementation of the traversing loop by tunnelling transitions in the presence of a magnetic field. The possible occupancy of dot 4 changes the tunneling coupling towards dot 3 from $J$ to $I \ll J$ eventually forbidding the circulation of the electron in dot 1.
We now discuss the physical conditions required in order to implement the control procedures outlined in the previous section. The main element that provides the controlled evolution is the double dot. If dot 4 is occupied we would like to prohibit tunneling towards dot 3 from dots 1 or 2. This, in effect, can be achieved by Coulomb blockade due to strong Coulomb interaction between electrons in dots 3 and 4. At the same time we would like to exclude any unwanted tunneling transitions between the different sites of the double dot in order to protect the computational space. How these conditions are met in a realistic setup is presented in the following.

A. Coulomb blockade

Assume that dot 4 is occupied. We would like to show that for sufficiently strong Coulomb repulsion due to the presence of an electron in dot 4 the Coulomb blockade effect takes place. Hence, tunneling from, e.g. dot 1, to dot 3 will be suppressed. The Hamiltonian for the relevant quantum dot system is given by

\[ H = -J(a_4^\dagger a_3 + a_1 a_3^\dagger) + U a_4^\dagger a_4 \]  

(2)

where \( a, a^\dagger \) are annihilation and creation operators, \( U \) is the Coulomb repulsion between the two adjacent sites 3 and 4, and \( J \) is the tunneling of an electron between two empty dots. In this description we do not allow the possibility of double occupation, as we assume that the barrier between sites 3 and 4 is very large so that no tunneling can occur. In matrix representation the Hamiltonian is given by

\[ H = \begin{pmatrix} 0 & -J \\ -J & U \end{pmatrix} \]  

(3)

in the basis \( |01\rangle \equiv a_4^\dagger |0\rangle, |10\rangle \equiv a_3^\dagger |0\rangle \) where \( |0\rangle \) represents the vacuum state. In the regime of \( U \gg J \), the Coulomb interaction is much larger than the tunnelling, and we can calculate an ‘effective tunnelling’ parameter, \( I \), felt by an electron in dot 1. This is related to the tunnelling rate of an electron going from dot 1 to dot 3 when 4 is occupied. Diagonalisation of the Hamiltonian and expansion with respect to large \( U \) gives to first order \( I = J^2/U \). Hence, for large enough \( U \) the tunneling from dot 1 to dot 3 is eliminated, as \( I \ll J \).

B. The double dot

As a next step we are interested in studying the regime of parameters that allows the double dot to behave as a control device. In particular, we demand that in the case that one side of the quantum dot is occupied, there is no significant effective tunneling coupling, \( J \), due to the Coulomb blockade. This is achieved by having a large Coulomb repulsion, \( U \), between two electrons occupying each site of the double dot. In order to achieve this, the positions of the trapping minima for these electrons are required to be close enough to each other. At the same time these minima cannot be too close as we demand that no significant tunnelling occurs between the two sides of the double dot. This can also be prohibited by maintaining a large potential barrier between them. The regime where these conditions are fulfilled is derived analytically in the following.

The Hamiltonian governing the double dot system is given by

\[ H = \int dr \psi^\dagger(r) \left( -\frac{\nabla^2}{2m} + V(r) \right) \psi(r) + \frac{1}{2} \int dr dr' \psi^\dagger(r) V(r-r') \psi(r') \]  

(4)

where \( V(r_i) \) is the external trapping potential, which is taken here to be quartic in the \( x \) direction with its minima separated by distance \( d \), and harmonic in the \( y \) and \( z \) directions

\[ V(r_i) = \frac{m}{2} \left[ \frac{\omega^2}{d^2} (x^2 - \frac{d^2}{4})^2 + \omega_y^2 y^2 + \omega_z^2 z^2 \right] \]  

(5)

and \( V_C \) is the Coulomb potential. For electrons well localised in the single particle wells (harmonic wells) it is convenient to expand the theory in terms of localised Wannier functions. The transformed second quantised operators in the Wannier basis are given by \( \psi(r_i) = \sum_i a_i \phi_i(r-r_i) \) and hence Eq. (4) becomes

\[ H = -\sum_{ij} t_{ij} a_i^\dagger a_j + \sum_{ii'jj'} U_{ii'jj'} a_i^\dagger a_j a_{i'} a_{j'}^\dagger \]  

(6)
where

\[ t_{ij} = \int dr w^*(r - r_i) \left( -\frac{\nabla^2}{2m} + V(r) \right) w(r - r_j) \quad (7) \]

\[ U_{ii'jj'} = \frac{1}{2} \int dr \int dr' V(r - r')w^*(r - r_i)w(r' - r_j')w(r - r_j)w^*(r' - r_j') \quad (8) \]

For tight enough traps one can assume that the separation between the ground state and the first excited state is large so that the particles are well confined to the ground state in which the potential is parabolic to good approximation. Hence the wave functions have the form \( w(r) = w(x)w(y)w(z) \), each \( w \) being the ground state of the harmonic potential. It is therefore possible to calculate the tunneling and collisional couplings. In particular for the tunneling coupling we have

\[ t_{ij} = \frac{\hbar}{2} (\omega_x + \omega_y + \omega_z)e^{-\alpha^2 d^2/4} \quad (9) \]

where \( \alpha^2 = m\omega/\hbar \). For well localised electrons in the quantum wells we can consider the interaction component of the Hamiltonian taking the form of the Coulomb repulsion, \( U_{ij}n_in_j \), where \( d \) is the separation between the dots, and \( U_{ij} \) is given by

\[ U_{ij} = \frac{1}{2} \frac{q^2}{4\pi\epsilon_0kd} \int dr dr' |w(r - r_j)|^2|w(r - r_i)|^2 = \frac{q^2}{8\pi\epsilon_0kd} e^{-\alpha^2 d^2/2} \quad (10) \]

The conditions on the nearest-neighbour Coulomb repulsion are stringent enough not to allow electrons to occupy adjacent dots. We can thus safely ignore the probability of double occupation within the same dot.

### IV. QUANTITATIVE ESTIMATES AND PARAMETER REGIMES

In the previous sections we established the requirements for the double dot to provide the controlled circulation of the electron around the dots 1, 2 and 3. This, as we have seen, produces the control phase gate because of the Aharonov-Bohm effect, conditional on the presence of an electron in dot 4. In this section we want to establish the feasibility of the above procedure by estimating its physical requirements on the system. In particular, we want to ensure that the time taken for an electron to make a complete loop, i.e. the total time for the control gate, is shorter than the decoherence time.

![FIG. 3: Plots showing the variation of the total gate time T and of the tunneling strength t_{44} within the double dot as a function of the barrier height V_0 and the distance d between the two dots. The physical setup we consider here is quantum dots in GaAs, where the effective electron mass is m = 0.067m_e and the dielectric constant is \( \kappa = 13.1 \).](image-url)

The time taken for an electron to tunnel to an adjacent site is given by \( t = h\pi/J \). Recent experiments [14] place a lower bound on the decoherence time of charge qubits in GaAs/AlGaAs heterostructure devices of the order of \( ns \), which gives us an indication of the maximum total gate time, \( T = 4t \), that can be tolerated. Since \( T \) is a function of the interdot tunnelling, it is also a function of the confinement potential, characterised by \( V_0 \), the height of the barrier, and \( d \), the distance between the two dots. The distance between the dots also determines the strength of the Coulomb repulsion, \( U \), according to Eq. [10], which we needed to be much greater than \( J \) for the Coulomb blockade regime to hold. This places an upper bound on \( J \), and, in turn, a lower bound on \( T \).
A plot of $T$ vs $d$ and $V_0$ is shown in Fig. 3. We also show a plot of the intradot tunnelling in the double dot, $t_{34}$, as a function of $d$ and $V_0$. We can see from the plots that for the selected range of $d$ and $V_0$, achievable with present technology, it is possible to obtain short gate times well below ns. In reality, however, the gate time will also be limited by the finite pulse length which controls the switching of the tunnel couplings. The plots also show that the intradot tunnelling $t_{34}$ can also be made small enough to be safely neglected.

V. DISCUSSION AND CONCLUSIONS

It is possible to manipulate the tunnel couplings in a unified fashion to relax the stringent experimental requirement of their individual control. We can consider for example the case where $t_{12}$ and $t_{13}$ have a common control procedure and $t_{23}$ is controlled independently. Indeed in this case we can set the time for the different processes in order to achieve the conditional phase presented in the previous section. In particular, the interaction time has to satisfy both $Jt/\hbar = m\pi$ and $\sqrt{2}Jt/\hbar = (2n + 1)\pi$, for integers $m$ and $n$ in order to ensure full population transfer between the different dots in the case of dot 4 being occupied or empty. This can be done by taking the time intervals for the duration of the tunnel couplings in the case of the double dot being occupied or empty to be equal, up to an integer multiple, given in the lowest case by $m = 5$ and $n = 3$ respectively. This choice of integers gives an error in the phase-gate due to timing mismatch of the order of 1%.

However, our analysis does not take into account other possible sources of errors, such as decoherence due to the coupling of the electron to the environment. The environment can, for example, act as a projective measurement determining the position of the electron and thereby destroying any superposition of the electron occupancy states of different dots. Alternatively, the environment can destroy the phase coherence between different elements of the superposition. For the particular analysis of the behaviour of geometric phases under classical and quantum noise see, for example, [17]. To successfully compensate for this kind of error, in parallel to the methods presented here, we will, most likely, have to resort to other existing methods such as quantum error correcting codes [18] or error avoiding methods such as decoherence-free subspaces [19]. Furthermore, another possible realisation of this scheme which may yield a better accuracy in the manipulation of the electron around the loop is to use a solenoid inside the loop rather than a magnetic field [3]. Mathematically, however, this is completely equivalent to using a uniform magnetic field, and may be harder in practice to realise. Note finally that we can interpret our structure as generating a single electron circulating current (vortex) by the electron in dot $r_1$ conditional on the presence of an electron in $l_2$. This is a very interesting physical system in its own right that could be potentially used for other applications of quantum circuits, for example in measuring the flux of the magnetic field by the resulting relative phase between the two distinct possible evolutions. A similar model with three potential wells has also been used to describe generation of vortices in trapped Bose-condensates [20]. An elaborated analysis of our proposal including various decoherence mechanisms is therefore very much worthwhile and will be presented elsewhere.

Acknowledgments

This work was supported by a Royal Society URF and the Schiff Foundation.

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