Mixed-SCORE+ for mixed membership community detection

Huan Qing
Department of Mathematics, China University of Mining and Technology
and
Jingli Wang
School of Statistics and Data Science, Nankai University

December 8, 2020

Abstract

Mixed-SCORE is a recent approach for mixed membership community detection proposed by Jin et al. (2017) which is an extension of SCORE (Jin, 2015). In the note Jin et al. (2018), the authors propose SCORE+ as an improvement of SCORE to handle with weak signal networks. In this paper, we propose a method called Mixed-SCORE+ designed based on the Mixed-SCORE and SCORE+, therefore Mixed-SCORE+ inherits nice properties of both Mixed-SCORE and SCORE+. In the proposed method, we consider $K + 1$ eigenvectors when there are $K$ communities to detect weak signal networks. And we also construct vertices hunting and membership reconstruction steps to solve the problem of mixed membership community detection. Compared with several benchmark methods, numerical results show that Mixed-SCORE+ provides a significant improvement on the Polblogs network and two weak signal networks Simmons and Caltech, with error rates 54/1222, 125/1137 and 94/590, respectively. Furthermore, Mixed-SCORE+ enjoys excellent performances on the SNAP ego-networks.

Keywords: Mixed membership community detection; spectral clustering; Mixed-SCORE; SCORE+; weak signal network
1 Introduction

Mixed membership community detection is a problem that has received substantial attentions, see Airoldi et al. (2008); Goldenberg et al. (2010); Jin et al. (2017); Mao et al. (2017, 2020); Qing and Wang (2020c); Zhang et al. (2020). In a mixed membership network, nodes may share among two or more communities. If a node only belongs to one community, we say this node is pure. For non-mixed membership community detection problem, all nodes are pure. In this paper, we focus on the study of mixed membership community detection.

Consider an undirected, un-weighted, no-loops network $N$ and assume that there are $K$ disjoint blocks $V^{(1)}, V^{(2)}, \ldots, V^{(K)}$ where $K$ is assumed to be known in this paper. Let $A$ be its adjacency matrix such that $A_{ij} = 1$ if there is an edge between node $i$ and $j$, $A_{ij} = 0$ otherwise.

This paper considers the degree-corrected mixed membership (DCMM) model (Jin et al., 2017) which assumes that for each node $i$, there is a Probability Mass Function (PMF) $\pi_i = (\pi_i^{(1)}, \pi_i^{(2)}, \ldots, \pi_i^{(K)})$ such that $\Pr(i \in V^{(k)}) = \pi_i^{(k)}$, $1 \leq k \leq K, 1 \leq i \leq n$. In this sense, DCMM model allows one node belongs to some certain communities with different probabilities. By Jin et al. (2017), under DCMM, we have

$$\Omega = E[A] = \Theta \Pi \Pi' \Theta,$$

where $\Theta$ is an $n \times n$ matrix whose $i$-th diagonal entry is the degree heterogeneity of node $i$, $P$ is a $K \times K$ matrix such that $Pr(A(i, j) = 1 | g_i = k, g_j = l) = \Theta(i, i) \Theta(j, j) P(g_i, g_j)$ (where $g_i$ denotes the community that $i$ belongs to). Therefore, given $(n, P, \Theta, \Pi)$, we can generate $^1$ a random adjacency matrix $A$ under the DCMM model. Let $\theta$ be the $n \times 1$ vector such that $\theta(i) = \Theta(i, i)$. Let $\Pi$ be an $n \times K$ matrix such that its $i$-th row is $\pi_i$ for $1 \leq i \leq n$. For the problem of mixed membership community detection, the chief aim is to estimate $\Pi$ with given $(A, K)$.

The Mixed-SCORE method (Jin et al., 2017) is an extension of the SCORE method (Jin, 2015) to mixed membership community detection problem. As discussed in Jin et al. (2018), traditional spectral clustering methods like SCORE, OCCAM (Zhang et al., 2020), RSC (Qin and Rohe, 2013) can not

\footnote{For more details about how to generate $A$ under DCMM, please refer to Jin et al. (2017) and Jin (2015).}
deal with weak signal networks (defined in Jin et al. (2018), and we redefined in our Algorithm) such as Simmons and Caltech (Traud et al., 2011; Traud et al., 2012). Therefore, Jin et al. (2018) proposed the SCORE+ as a simple improvement of SCORE to deal with weak signal networks. Some recent spectral clustering community detection methods proposed by Qing and Wang (2020a,b,d) can also successfully detect communities for weak signal networks. In this paper, we find that Mixed-SCORE also fails to detect Simmons and Caltech, which motivates us to design one approach which should successfully deal with mixed membership and weak signal networks. Combining with Mixed-SCORE and SCORE+, we propose Mixed-SCORE+ as a refinement of Mixed-SCORE to weak signal networks, and it also can be deemed as an extension of SCORE+ to mixed membership networks. We list several important differences between Mixed-SCORE+ and Mixed-SCORE as well as SCORE+ as follows:

- SCORE+ is for non-mixing community detection problem and it is designed based on the degree-corrected stochastic block model (DCSBM) (Karrer and Newman, 2011), while Mixed-SCORE and Mixed-SCORE+ are for the mixed membership community detection and designed based on the degree-corrected mixed membership (DCMM) model (Jin et al., 2017).

- Mixed-SCORE+ uses a regularized Laplacian matrix that is slightly different as the one used in SCORE+ and Mixed-SCORE.

- Mixed-SCORE+ has a threshold step while there is no such steps in SCORE+. However, when it turns to mixed membership community detection, there is also a threshold step for mixed-SCORE.

- There are a vertices hunting (VH) step and a membership reconstruction (MR) step in Mixed-SCORE+ and Mixed-SCORE while there is no such steps in SCORE+.

- Mixed-SCORE+ applies the information of the leading \((K + 1)\) eigenvectors and eigenvalues of a regularized Laplacian matrix for estimating \(\Pi\) while Mixed-SCORE applies the leading \(K\) eigenvectors of \(A\). This enables that Mixed-SCORE+ can detect weak signal networks while Mixed-SCORE can not.
2 The algorithm: Mixed-SCORE+

In this paper, for convenience, when we say “leading eigenvalues” or “leading eigenvectors”, we are comparing the magnitudes of the eigenvalues and their respective eigenvectors with unit-norm.

The details of Mixed-SCORE+ are presented in the following Algorithm.

Mixed-SCORE+. Input: A, K, a ridge regularizer $\tau \geq 0$, two thresholds $t > 0$ and $T_n > 0$. Output: $\hat{\Pi}$.

- **SCORE+ step:**
  1. Obtain the regularized graph Laplacian matrix by
     \[ L_\tau = D_\tau^{-1/2}AD_\tau^{-1/2}, \]
     where $D_\tau = D + \tau I$, $D$ is an $n \times n$ diagonal matrix whose $i$-th diagonal entry is $D(i, i) = \sum_{j=1}^{n} A(i, j)$ (a good default $\tau$ is $\tau = 0.1 \frac{d_{\max} + d_{\min}}{2}$, where $d_{\max} = \max_{i} D(i, i)$, $d_{\min} = \min_{i} D(i, i)$).

  2. Assess the aforementioned “signal weakness” by $1 - \left| \frac{\hat{\lambda}_{K+1}}{\hat{\lambda}_{K}} \right|$, and include an additional eigenvector for clustering if and only if
     
     \[ 1 - \left| \frac{\hat{\lambda}_{K+1}}{\hat{\lambda}_{K}} \right| \leq t, \quad \text{conventional choice of } t \text{ is 0.1}, \]
     where $\hat{\lambda}_i$ is the $i$-th leading eigenvalue of $L_\tau$, $1 \leq i \leq (K + 1)$.

  3. Let $M$ be the number of eigenvectors we decide in the last step (so either $M = K$ or $M = K + 1$). Obtain the $n \times (M - 1)$ matrix of entry-wise eigen-ratios by
     
     \[ \hat{R} = \begin{bmatrix} \hat{\eta}_2 & \hat{\eta}_3 & \cdots & \hat{\eta}_M \end{bmatrix}, \quad \text{where } \hat{\eta}_k = \hat{\lambda}_k \hat{\xi}_k, 1 \leq k \leq M, \]
     and $\hat{\xi}_i$ is the $i$-th leading eigenvector with unit-norm of $L_\tau$, $1 \leq i \leq (K + 1)$.

  4. Fixing a threshold $T_n$, define an $n \times (M - 1)$ matrix $\hat{R}^*$ such that for all $1 \leq i \leq n$ and $1 \leq k \leq M$,

     \[
     \hat{R}^*(i, k) = \begin{cases} 
     \hat{R}(i, k), & \text{if } |\hat{R}(i, k)| \leq T_n, \\
     T_n, & \text{if } \hat{R}(i, k) > T_n, \\
     -T_n, & \text{if } \hat{R}(i, k) < -T_n.
     \end{cases}
     \]
where a good default $T_n$ is $\log(n)$.

- **Vertices Hunting (VH) step:**
  5. Perform K-means clustering on the rows of $\hat{R}^*$ and obtain $K$ estimated cluster centers $\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_K \in \mathcal{R}^{1 \times (M-1)}$, i.e.,

  \[
  \{\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_K\} = \arg \min_{\hat{v}_1, \ldots, \hat{v}_K} \frac{1}{n} \sum_{i=1}^{n} \min_{\hat{v} \in \{\hat{v}_1, \ldots, \hat{v}_K\}} \|\hat{R}^*_i - \hat{v}\|_2.
  \]

  Form the $K \times (M-1)$ matrix $\hat{V}$ such that the $i$-th row of $\hat{V}$ is $\hat{v}_i, 1 \leq i \leq K$.

- **Membership Reconstruction (MR) step:**
  6. Obtain the $K \times M$ matrix $\hat{V}^*$ by $\hat{V}^* = [1, \hat{V}]$, where 1 is a $K \times 1$ vector with all entries being 1. Meanwhile, obtain an $n \times M$ matrix $\hat{R}^*$ by $\hat{R}^* = [1, \hat{R}^*]$, where 1 is an $n \times 1$ vector with all entries being 1.

  7. Project the rows of $\hat{R}^*$ onto the spans of $K$ rows of $\hat{V}^*$, i.e., compute the $n \times K$ matrix $\hat{Y}$ such that $\hat{Y} = \hat{R}^* \hat{V}^* (\hat{V} \hat{V}^*)^{-1}$.

  8. If there exists any node $i$ such that all entries of the $i$-th row of $\hat{Y}$ are negative, we set $\hat{Y}_i = -\hat{Y}_i$ (i.e., let all negative entries of $\hat{Y}_i$ be positive).

  9. For $1 \leq i \leq n, 1 \leq k \leq K$, let $\hat{Y}(i, k) = \max(0, \hat{Y}(i, k))$.

  10. Estimate $\pi_i$ by $\hat{\pi}_i = \hat{Y}_i / \|\hat{Y}_i\|_1, 1 \leq i \leq n$. Obtain the estimated membership matrix $\hat{\Pi}$ such that its $i$-th row is $\hat{\pi}_i, 1 \leq i \leq n$.

Several remarks about Mixed-SCORE+ method are listed in order.

- The regularized Laplacian matrix in Mixed-SCORE+ is slightly different from that of SCORE+, where we set $\tau = 0.1 \frac{d_{\max} + d_{\min}}{2}$ instead of the $0.1d_{\max}$ in SCORE+ since such setting provides us with slightly better numerical results.

- In step 2, we measure the “signal weakness” slightly different as that in Jin et al. (2018), where we use $1 - |\hat{\lambda}_{K+1}|$ instead of the $1 - \frac{\hat{\lambda}_{K+1}}{\hat{\lambda}_K}$ in Jin et al. (2018) since we find that the leading eigenvalues are measured by magnitude, which means that $\hat{\lambda}_{K+1}$ may have different sign as that of $\hat{\lambda}_K$.

- Similar as Mixed-SCORE, in step 4, we need the threshold $T_n$ to guarantee the performances of Mixed-SCORE+. The default of $T_n$ is set as $\log(n)$. Meanwhile, if one ignores step 4, then respective method can not deal with some of the empirical networks (such as SNAP ego-networks) in Section 4.2.
In the VH step, unlike applying K-medians in OCCAM or vertex hunting algorithm in Mixed-SCORE for hunting the $K$ centers (also known as vertices) of $\hat{R}^*$, we state that it is enough for our Mixed-SCORE+ to apply K-means in the VH step, and it performs satisfactory both numerically and empirically. Actually, one can also apply the VH algorithm in Mixed-SCORE or the K-medians technique in OCCAM to find the $K$ centers in Mixed-SCORE+, in this paper we use K-means.

In step 6, we need to construct $\hat{V}$ and $\hat{R}$ by adding one columns with entries 1 to $\hat{V}$ and $\hat{R}$, respectively. Actually, there is a similar procedure in the MR step of Mixed-SCORE, and such procedure is related with the convex linear combination stated in Jin et al. (2017).

In the MR step, setting $\hat{Y} = \hat{R}v\hat{v}'(\hat{v}v')^{-1}$ in our Mixed-SCORE+ guarantees that it can deal with weak signal networks since $\hat{v}v'$ is a $K \times K$ nonsingular matrix when $K$ is much smaller than $n$. Meanwhile, if simply setting $\hat{Y}$ as $\hat{R}v(\hat{v})^{-1}$, then method designed based on such setting performs poor and can not successfully detect empirical networks used in this paper.

In the MR step, steps 8 and 9 guarantee that $||\hat{Y}_i||$ is nonzero and all entries of $\hat{Y}_i$ are nonnegative (and at least one entry is strictly positive) for any $i \in \{1, 2, \ldots, n\}$. This two steps make sure that $\hat{\pi}_i$ is well defined and nonnegative (since weights should be nonnegative for any node).

However, it is challenging to provide the respective theoretical guarantees of Mixed-SCORE+ under the degree-corrected mixed membership (DCMM) model, and we leave it for our future work.

3 Simulations

We investigate the performance of our Mixed-SCORE+ by comparing it with Mixed-SCORE (Jin et al., 2017), GeoNMF (Mao et al., 2017), SPACL (Mao et al., 2020) and OCCAM (Zhang et al., 2020) on various simulations in this section. Note that in this paper, we only compare our Mixed-SCORE+ with methods designed for mixed membership community detection problem. It is not our intention to compare Mixed-SCORE+ with community detection methods such as those applied in Jin et al. (2018).
For each method, we measure the performance of mixed membership community detection method by the mixed-Hamming error rate which is defined as

$$\min_{O \in \{K \times K \text{ permutation matrix}\}} \frac{1}{n} \| \hat{\Pi} O - \Pi \|_1,$$

where $\Pi$ and $\hat{\Pi}$ are the true and estimated mixed membership matrices respectively. For simplicity, we write the mixed-Hamming error rate as

$$\sum_{i=1}^n \| \hat{\pi}_i - \pi_i \|_1 / n.$$ 

Unless specified, for all experiments, we set $n = 500$ and $K = 3$. For $0 \leq n_0 \leq 160$, let each block own $n_0$ number of pure nodes. For the top $3n_0$ nodes $\{1, 2, \ldots, 3n_0\}$, we let these nodes be pure and let nodes $\{3n_0+1, 3n_0+2, \ldots, 500\}$ be mixed. Fixing $x \in [0, \frac{1}{2})$, let all the mixed nodes have four different memberships $(x, x, 1-2x), (x, 1-2x, x), (1-2x, x, x)$ and $(1/3, 1/3, 1/3)$, each with $\frac{500-3n_0}{4}$ number of nodes. Fixing $\rho \in (0, 1)$, the mixing matrix $P$ has diagonals 0.8 and off-diagonals $\rho$. There are two settings about $\theta$, one is $\theta(i) = 0.2 + 0.8(i/n)^2$; the other is: fix $z \geq 1$, generate the degree parameters such that $1/\theta(i) \sim U(1, z)$, where $U(1, z)$ denotes the uniform distribution on $[1, z]$. For each parameter setting, we report the mixed-Hamming error rate $\sum_{i=1}^n \| \hat{\pi}_i - \pi_i \|_1 / n$ averaged over 50 repetitions. Based on these settings we designed four experiments to illustrate the proposed method from different aspects.

**Experiment 1: Fraction of pure nodes.** Fix $(x, \rho) = (0.4, 0.3)$ and let $n_0$ range in $\{40, 60, 80, 100, 120, 140, 160\}$. A larger $n_0$ indicates a case with higher fraction of pure nodes. In Experiment 1(a), set $\theta(i) = 0.2 + 0.8(i/n)^2$. In Experiment 1(b), set $z = 4$. The numerical results are shown in panels (a) and (b) of Figure 1, from which we can find that all methods perform poor when the fraction of pure nodes is small. Under the setting of Experiment 1(a), our Mixed-SCORE+ significantly outperforms its competitors, and it is interesting to find that Mixed-SCORE, OCCAM, GeoNMF and SACL always perform unsatisfactory under this setting even when $n_0$ is quite large. For Experiment 1(b), Mixed-SCORE+ performs similar as Mixed-SCORE and both two algorithms outperform OCCAM, GeoNMF and SACL.
Figure 1: Estimation errors of Experiments 1 (y-axis: $\sum_{i=1}^{n} n^{-1}\|\hat{\pi}_i - \pi_i\|_1$).

Experiment 2: Connectivity across communities. Fix $(x, n_0) = (0.4, 100)$ and let $\rho$ range in $\{0, 0.05, 0.1, \ldots, 0.35\}$. A larger $\rho$ generates more edges across different communities (hence a dense network). In Experiment 2(a), set $\theta(i) = 0.2 + 0.8(i/n)^2$. In Experiment 2(b), set $z = 4$. The results are displayed in Figure 2. We can find that all methods perform poorer as $\rho$ increases, this phenomenon occurs due to the fact that more edges across different communities lead to a case that different communities tend to be into a giant community and hence a case that is more challenging to detect for any algorithms. Under the setting of Experiment 2(a), our Mixed-SCORE+ outperforms its competitors obviously, and the 4 competitors always perform poorly even for a small $\rho$. Meanwhile, in Experiment 2(b), Mixed-SCORE+ performs slightly better than Mixed-SCORE while both two approaches outperform OCCAM, GeoNMF and SPACL.

Experiment 3: Purity of mixed nodes. Fix $(n_0, \rho) = (100, 0.3)$, and let $x$ range in $\{0, 0.05, \ldots, 0.5\}$. As $x$ increases to $1/3$, these mixed nodes become less pure and they become more pure as $x$ increases further. In Experiment 3(a), set $\theta(i) = 0.2 + 0.8(i/n)^2$. In Experiment 3(b), set $z = 4$. Figure 3 records the numerical results of this experiment. It is obvious to find that Mixed-SCORE+ outperforms the other four methods in Experiment 3(a), and it performs similar as Mixed-SCORE+ while both two perform better than OCCAM, GeoNMF and SPACL.

Experiment 4: Degree heterogeneity. Fix $(n_0, \rho, x) = (100, 0.3, 0.4)$, and let $z$ range in $\{1, 2, \ldots, 8\}$. In Experiment 4(a), set $\theta(i) = z/10 + 0.8*(i/n)^2$. In Experiment 4(b), set $1/\theta(i) \sim U(1, z), i = 1, 2, \ldots, n$. From the
results in Figure 4 we can conclude that this experiment shares similar conclusions with the above experiments.

4 Application to empirical datasets

In this section, we apply two kinds of empirical datasets to investigate the performance of our Mixed-SCORE+. For the community detection problem, we use the eight real-world networks with known label information; for the mixed membership community detection problem, we use the SNAP ego-networks with known membership information where the SNAP ego-networks
are applied in Zhang et al. (2020) and Qing and Wang (2020c).

4.1 Eight empirical networks with known label information for community detection

The details of the eight real-world networks can be found in Appendix A. To measure the performances of these methods on the eight networks, we first introduce the Hamming error rate.

When all nodes are pure, the community information can be expressed by an $n \times 1$ nodes labels vector $\ell$ where $\ell_i$ takes values in set $\{1, \cdots, K\}$ and denotes the node $i$ belongs to the $\ell_i$-th community. Let $\hat{\ell}$ be an estimation of $\ell_i$. For community detection, since each node belongs to exactly one community, $\hat{\ell}_i$ and $\ell_i$ take one value from $\{1, 2, \ldots, K\}$ for $1 \leq i \leq n$. $\hat{\ell}_i$ for Mixed-SCORE+ can be computed as below

$$\hat{\ell}_i = \arg \max_{1 \leq k \leq K} \hat{\Pi}_{ik}.$$ 

Then the clustering error rate is measured by the Hamming error rate (Jin, 2015) which is defined as

$$\min_{\{o: \text{permutation over } \{1,2,\ldots,K\}\}} \frac{1}{n} \sum_{i=1}^{n} 1\{o(\hat{\ell}_i) \neq \ell_i\},$$

where $\ell_i$ and $\hat{\ell}_i$ are the true and estimated labels of node $i$. 

Figure 4: Estimation errors of Experiments 4 (y-axis: $\sum_{i=1}^{n} n^{-1}\|\hat{\pi}_i - \pi_i\|_1$).
The error rates of the eight empirical networks are summarized in Table 1, where we use default parameters for Mixed-SCORE+. The results show that Mixed-SCORE+ outperforms its competitors on the three large networks: Polblogs, Simmons and Caltech, with error rates 54/1222, 125/1137, and 94/590 respectively. As discussed in Jin et al. (2018), Simmons and Caltech are two weak signal \(^2\) networks whose \((K + 1)\)-th leading eigenvalue is close to the \(K\)-th leading eigenvalue of the adjacency matrix \(A\) or its variants, suggesting that the leading \((K + 1)\) eigenvector may contain information about nodes labels. While, for the five small strong signal networks, we see that all methods enjoy similar performances.

\begin{table}[h]
\centering
\begin{tabular}{lcccccccc}
\hline
Methods & Karate & Dolphins & Football & Polbooks & UKfaculty & Polblogs & Simmons & Caltech \\
\hline
Mixed-SCORE & 0/34 & 2/62 & 4/110 & 3/92 & 6/79 & 60/1222 & 261/1137 & 174/590 \\
OCCAM & 0/34 & 1/62 & 4/110 & 3/92 & 5/79 & 60/1222 & 268/1137 & 192/590 \\
GeoNMF & 0/34 & 1/62 & 5/110 & 3/92 & 4/79 & 64/1222 & 383/1137 & 229/590 \\
SPACL & 0/34 & 1/62 & 5/110 & 3/92 & 4/79 & 61/1222 & 413/1137 & 185/590 \\
Mixed-SCORE+ & 1/34 & 1/62 & 6/110 & 2/92 & 2/79 & 54/1222 & 125/1137 & 94/590 \\
\hline
\end{tabular}
\caption{Error rates on the eight empirical data sets.}
\end{table}

4.2 SNAP ego-networks with known mixed membership information for mixed membership community detection

SNAP ego-networks contains substantial ego-networks from three platforms Facebook, GooglePlus, and Twitter. There are 7 communities with total 1656 nodes in Facebook, 58 communities with total 25127 nodes in GooglePlus, and 255 communities with total 15463 nodes in Twitter. For more details please refer to Zhang et al. (2020) and Qing and Wang (2020c). Here we use the newest version of SNAP ego-networks (those used in Qing and Wang (2020c)) to investigate the performances of Mixed-SCORE+ and its competitors.

Since the ground truth communities of mixed membership (i.e., \(\Pi\)) of SNAP ego-networks are known in advance, we can use the mixed-Hamming error rate to measure these methods’ performances directly. To compare the

\(^2\)Readers interested in the details of weak signal networks and strong signal networks please refer to Jin et al. (2018).
performances of these methods, similar as that in Zhang et al. (2020), we report the average performances over each of the social platforms and the corresponding standard deviation in Table 2. Meanwhile, recall that in the VH step of Mixed-SCORE+, we argue that we apply K-means method for vertices hunting instead of K-medians. Here, we use Mixed-SCORE+(Kmedians) to denote the Mixed-SCORE+ method designed based on K-medians clustering technique. We also report the numerical results of Mixed-SCORE+(Kmedians) on the SNAP ego-networks in Table 2, which tells us that Mixed-SCORE+ shares similar performances as that of Mixed-SCORE+(Kmedians). Since K-means is faster than K-medians, the default vertices hunting technique for Mixed-SCORE+ is K-means in this paper. From Table 2, we can find that, for Facebook networks, SPACL has smallest error rate, while GeoNMF and Mixed-SCORE+ have similar results. OCCAM performs poorest on Facebook networks. For GooglePlus and Twitter networks, our proposed methods Mixed-SCORE+ and Mixed-SCORE+(Kmedians) perform best and share similar error rates. At the same time, we see that Mixed-SCORE performs poorest on GooglePlus and Twitter, suggesting that our Mixed-SCORE+ provides a significant improvement of Mixed-SCORE.

Table 2: Mean (SD) of mixed-Hamming error rates for ego-networks.

|                | Facebook          | GooglePlus        | Twitter           |
|----------------|-------------------|-------------------|-------------------|
| Mixed-SCORE    | 0.2496(0.1322)    | 0.3766(0.1053)    | 0.3088(0.1296)    |
| OCCAM          | 0.2610(0.1367)    | 0.3564(0.1210)    | 0.2864(0.1406)    |
| GeoNMF         | 0.2537(0.1266)    | 0.3520(0.1078)    | 0.2858(0.1292)    |
| SPACL          | **0.2371**(0.1233)| 0.3616(0.1077)    | 0.3068(0.1268)    |
| Mixed-SCORE+   | 0.2536(0.1289)    | 0.3341(0.1157)    | **0.2659**(0.1411)|
| Mixed-SCORE+(Kmedians) | 0.2561(0.1292) | **0.3332**(0.1168)| 0.2665(0.1422) |

5 Discussion

In this paper, Mixed-SCORE+ focus on detecting network memberships for the problem of mixed membership community detection, and it can also detect two weak signal networks Simmons and Caltech. Such advantage of

\(^3\)Actually, Mixed-SCORE+ also shares almost the same error rates as that of Mixed-SCORE+(Kmedians) on the eight real-world networks in Table 1.
Mixed-SCORE+ mainly comes from the fact we apply the information of the leading \((K+1)\) eigenvector and eigenvalue of the regularized Laplacian matrix when dealing with weak signal networks. Although Mixed-SCORE+ is an extension of Mixed-SCORE, Mixed-SCORE can not utilize such information (for the details, please refer to those remarks after our Mixed-SCORE+ algorithm). Numerical studies of substantial simulations and empirical datasets show that Mixed-SCORE+ enjoys satisfactory performances and it performs better than most of the benchmark methods both numerically and empirically.

There remain several problems unsolved: (a) Jin et al. (2017) provided full theoretical analysis for Mixed-SCORE while there is no such studies for Mixed-SCORE+ in this paper due to the fact that it is challenge and difficult to study the theoretical guarantee of Mixed-SCORE+. Hence, it is meaningful to build theoretical frameworks for Mixed-SCORE+. (b) Whether there exist optimal parameters \(\tau\) and \(T_n\) both theoretically and numerically is an interesting topic for further study. (c) In Ali and Couillet (2018), the authors studied the existence of an optimal value \(\alpha_{opt}\) of the parameter \(\alpha\) for community detection methods based on \(D^{-\alpha}AD^{-\alpha}\) for community detection problem. Recall that our Mixed-SCORE+ is designed based on \(D^{-\alpha}AD^{-\alpha}\tau\), we argue that whether there exist optimal \(\alpha_0\) and \(\beta_0\) such that mixed membership community detection method (say Mixed-SCORE+) designed based on \(D^{\alpha_0}A^{\alpha_0}\tau\) outperforms methods designed based on \(D^{\alpha_0}A^{\beta_0}\tau\) for any choices of \(\alpha\) and \(\beta\). For reasons of space, we leave studies of these problems to the future.

A Description of eight real-word data

- **Karate**: this network consists of 34 nodes where each node denotes a member in the karate club (Zachary, 1977). As there is a conflict in the club, the network divides into two communities: Mr. Hi’s group and John’s group. Zachary (1977) records all labels for each member and we use them as the true labels.

- **Dolphins**: this network consists of frequent associations between 62 dolphins in a community living off Doubtful Sound. In Dolphins network, node denotes a dolphin, and edge stands for companionship (Lusseau et al., 2003; Lusseau, 2003, 2007). The network splits natu-
rally into two large groups females and males (Lusseau, 2003; Newman and Girvan, 2004), which are seen as the ground truth in our analysis.

- **Football**: this network is for American football games between Division I-A college teams during the regular football season of Fall (Girvan and Newman, 2002). Nodes in Football denote teams and edges represent regular-season games between any two teams (Girvan and Newman, 2002). The original network contains 115 nodes in total, since 5 of them are called “Independent” and the remaining 110 nodes are manually divided into 11 conferences for administration purpose, for community detection, we remove the 5 independent teams in this paper.

- **Polbooks**: this network is about US politics published around the 2004 presidential election and sold by the online bookseller Amazon.com. In Polbooks, nodes represent books, edges represent frequent co-purchasing of books by the same buyers. Full information about edges and labels can be downloaded from [http://www-personal.umich.edu/~mejn/netdata/](http://www-personal.umich.edu/~mejn/netdata/). The original network contains 105 nodes labeled as either “Conservative”, “Liberal”, or “Neutral”. Nodes labeled “Neutral” are removed for community detection in this paper.

- **UKfaculty**: this network reflects the friendship among academic staffs of a given Faculty in a UK university consisting of three separate schools (Nepusz et al., 2008). The original network contains 81 nodes, in which the smallest group only has 2 nodes. The smallest group is removed for community detection in this paper.

- **Polblogs**: this network consists of political blogs during the 2004 US presidential election (Adamic and Glance, 2005). Each blog belongs to one of the two parties liberal or conservative. As suggested by Karrer and Newman (2011), we only consider the largest connected component with 1222 nodes and ignore the edge direction for community detection.

- **Simmons**: this network contains one largest connected component with 1137 nodes. It is observed in Traud et al. (2011); Traud et al. (2012) that the community structure of the Simmons College network exhibits a strong correlation with the graduation year-students since students in the same year are more likely to be friends.
- **Caltech**: this network has one largest connected component with 590 nodes. The community structure is highly correlated with which of the 8 dorms a user is from, as observed in Traud et al. (2011); Traud et al. (2012).

**References**

Adamic, L. A. and N. Glance (2005). The political blogosphere and the 2004 us election: divided they blog. pp. 36–43.

Airoldi, E. M., D. M. Blei, S. E. Fienberg, and E. P. Xing (2008). Mixed membership stochastic blockmodels. *Journal of Machine Learning Research* 9, 1981–2014.

Ali, H. T. and R. Couillet (2018). Improved spectral community detection in large heterogeneous networks. *Journal of Machine Learning Research* 18(225), 1–49.

Girvan, M. and M. E. Newman (2002). Community structure in social and biological networks. *Proceedings of the national academy of sciences* 99(12), 7821–7826.

Goldenberg, A., A. X. Zheng, S. E. Fienberg, and E. M. Airoldi (2010). A survey of statistical network models. *Foundations and Trends® in Machine Learning archive* 2(2), 129–233.

Jin, J. (2015). Fast community detection by SCORE. *Annals of Statistics* 43(1), 57–89.

Jin, J., Z. T. Ke, and S. Luo (2017). Estimating network memberships by simplex vertex hunting. *arXiv preprint arXiv:1708.07852*.

Jin, J., Z. T. Ke, and S. Luo (2018). Score+ for network community detection. *arXiv preprint arXiv:1811.05927*.

Karrer, B. and M. E. J. Newman (2011). Stochastic blockmodels and community structure in networks. *Physical Review E* 83(1), 16107.

Lusseau, D. (2003). The emergent properties of a dolphin social network. *Proceedings of the Royal Society of London. Series B: Biological Sciences* 270(suppl_2), S186–S188.
Lusseau, D. (2007). Evidence for social role in a dolphin social network. *Evolutionary ecology* 21(3), 357–366.

Lusseau, D., K. Schneider, O. J. Boisseau, P. Haase, E. Slooten, and S. M. Dawson (2003). The bottlenose dolphin community of Doubtful Sound features a large proportion of long-lasting associations. *Behavioral Ecology and Sociobiology* 54(4), 396–405.

Mao, X., P. Sarkar, and D. Chakrabarti (2017). On mixed memberships and symmetric nonnegative matrix factorizations. In *International Conference on Machine Learning*, pp. 2324–2333.

Mao, X., P. Sarkar, and D. Chakrabarti (2020). Estimating mixed memberships with sharp eigenvector deviations. *Journal of the American Statistical Association*, 1–13.

Nepusz, T., A. Petróczi, L. Négyessy, and F. Bazsó (2008). Fuzzy communities and the concept of bridgeness in complex networks. *Physical Review E* 77(1), 016107.

Newman, M. E. and M. Girvan (2004). Finding and evaluating community structure in networks. *Physical review E* 69(2), 026113.

Qin, T. and K. Rohe (2013). Regularized spectral clustering under the degree-corrected stochastic blockmodel. In *Advances in Neural Information Processing Systems* 26, pp. 3120–3128.

Qing, H. and J. Wang (2020a). Community detection by principal components clustering methods. *arXiv preprint arXiv:2011.04377*.

Qing, H. and J. Wang (2020b). Dual regularized laplacian spectral clustering methods on community detection. *arXiv preprint arXiv:2011.04392*.

Qing, H. and J. Wang (2020c). Estimating network memberships by mixed regularized spectral clustering. *arXiv preprint arXiv:2011.12239*.

Qing, H. and J. Wang (2020d). An improved spectral clustering method for community detection under the degree-corrected stochastic blockmodel. *arXiv preprint arXiv:2011.06374*.
Traud, A. L., E. D. Kelsic, P. J. Mucha, and M. A. Porter (2011). Comparing community structure to characteristics in online collegiate social network. *Siam Review* 53(3), 526–543.

Traud, A. L., P. J. Mucha, and M. A. Porter (2012). Social structure of facebook networks. *Physica A-statistical Mechanics and Its Applications* 391(16), 4165–4180.

Zachary, W. W. (1977). An information flow model for conflict and fission in small groups. *Journal of anthropological research* 33(4), 452–473.

Zhang, Y., E. Levina, and J. Zhu (2020). Detecting overlapping communities in networks using spectral methods. *SIAM Journal on Mathematics of Data Science* 2(2), 265–283.