GENERALIZED INTELLIGENT STATES AND
$SU(1, 1)$ AND $SU(2)$ SQUEEZING

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Abstract

A sufficient condition for a state $|\psi\rangle$ to minimize the Robertson-Schrödinger uncertainty relation for two observables $A$ and $B$ is obtained which for $A$ with no discrete spectrum is also a necessary one. Such states, called generalized intelligent states (GIS), exhibit arbitrarily strong squeezing (after Eberly) of $A$ and $B$. Systems of GIS for the $SU(1, 1)$ and $SU(2)$ groups are constructed and discussed. It is shown that $SU(1, 1)$ GIS contain all the Perelomov coherent states (CS) and the Barut and Girardello CS while the Bloch CS are subset of $SU(2)$ GIS.

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1 Introduction

The squeezed states of electromagnetic field in which the fluctuations in one of the quadrature components $Q$ and $P$ of the photon annihilation operator $a = (Q+iP)/\sqrt{2}$ are smaller than those in the ground state $|0\rangle$ have attracted due attention in the last decade (see for example the review papers[1, 2] and references there in). In the recent years an interest is devoted to the squeezed states for other observables[3–11]. One looks for non gaussian states which exhibit $Q$-$P$ squeezing[3–7] and/or for states in which the fluctuations of other physical observables are squeezed[3–11].

The aim of the present paper is to construct $SU(1, 1)$ and $SU(2)$ squeezed intelligent states and to consider some general properties of squeezing for an arbitrary pair of quantum observables $A$ and $B$ in states which minimize the Robertson-Schrödinger uncertainty
relation (R-S UR) \[12\]. We call such states generalized intelligent states (GIS) or squeezed intelligent states when the accent is on their squeezing properties. The $Q$-$P$ GIS are well studied and known as squeezed states, two photon coherent states (CS) (see references in \[1, 2\]), correlated states \[13\] or Schrödinger minimum uncertainty states \[14\]. The term intelligent states (IS) \[11\] is referred to states that provide the equality in the Heizenberg UR for $A$ and $B$. The $Q$-$P$ IS are also known as Heizenberg minimum uncertainty states. The spin IS are introduced and studied in \[11\].

## 2 Generalized intelligent states

For any two quantum observables $A$ and $B$ the corresponding second momenta in a given state obey the R-S UR \[12, 13\],

$$\sigma_A^2 \sigma_B^2 \geq \frac{1}{4}(\langle C \rangle^2 + 4\sigma_{AB}^2), \quad C \equiv -i[A, B],$$

(1)

where $\sigma_A$, $\sigma_B$ and $\sigma_{AB}$ are the dispersions and the covariation of $A$ and $B$,

$$\sigma_A^2 = \langle A^2 \rangle - \langle A \rangle^2;$$

$$\sigma_{AB} = \frac{1}{2}(\langle AB \rangle + \langle BA \rangle) - \langle A \rangle \langle B \rangle.$$  

(2)

The states that provide the equality in the R-S UR (1) will be called here generalized intelligent states (GIS). When the covariation $\sigma_{AB} = 0$ then the S-R UR coincides with the Heizenberg one. In paper \[13\] it was proved that if a pure state $|\psi\rangle$ with nonvanishing dispersion of the operator $A$ minimizes the R-S UR then it is an eigenstate of the operator $\lambda A + iB$, where $\lambda$ is a complex number, related to $\langle C \rangle$ and to $\sigma_i(|\psi\rangle), i = A, B, AB$. Here we prove that this is a sufficient condition for any state $|\psi\rangle$.

**Proposition 1** A state $|\psi\rangle$ minimizes the R-S UR (1) if it is an eigenstate of the operator $L(\lambda) = \lambda A + iB$,

$$L(\lambda)|z, \lambda\rangle = z|z, \lambda\rangle,$$

(3)

where the eigenvalue $z$ is a complex number.

**Proof.** Let first restrict the parameter $\lambda$ in the eigenvalue eqn. (3), $\text{Re} \lambda \neq 0$. Then we express $A$ and $B$ in terms of $L(\lambda)$ and $L^\dagger(\lambda)$ and obtain

$$\sigma_A^2(z, \lambda) = \frac{\langle C \rangle}{2\text{Re} \lambda}, \quad \sigma_B^2(z, \lambda) = |\lambda|^2 \frac{\langle C \rangle}{2\text{Re} \lambda},$$

$$\sigma_{AB}(z, \lambda) = -\frac{\langle C \rangle}{2\text{Re} \lambda},$$

(4)

where $\langle C \rangle = \langle \lambda, z|C|z, \lambda \rangle$. The obtained second momenta (4) obey the equality in R-S UR (1).

Let now the eigenvalue equation (3) holds for $\text{Re} \lambda = 0$. This means that the state $|z, \lambda\rangle$ is an eigenstate of the Hermitean operator $rA + B$ where $r = \text{Im} \lambda$. We consider now the
mean value of the non negative operator $F^*(r)F(r)$, where $F(r) = rA + B - (r\langle A \rangle + \langle B \rangle)$ and $r$ is any real number. Herefrom we get the uncertainty relation

$$\sigma_A^2 \sigma_B^2 \geq \sigma_{AB}^2, \quad (5)$$

the equality holding in the eigenstates of $F(r)$ only. One can consider the equality in (5) as the desired equality in the Robertson-Schrödinger UR if in these states the mean value of the operator $C$ vanishes. And this is the case. Indeed, consider in $|z, ir\rangle$ the mean values of the operators $A(rA + B)$ and $(rA + B)A$. We easily get the coincidence of the two mean values, wherefrom we obtain $\langle ir, z|C|z, ir\rangle = 0$.

Thus all eigenstates $|z, \lambda\rangle$ are GIS. One can prove that when the operator $A$ has no discrete spectrum then for any $|\psi\rangle$ $\sigma_A(\psi) \neq 0$, thereby the condition (3) is also necessary and all $A$-$B$ GIS (for any $B$) are of the form $|z, \lambda\rangle$. Such are for example the cases of canonical $Q$-$P$ GIS\cite{14} and the $SU(1,1)$ GIS, considered below. The above result stems from the following property of the dispersion of quantum observables:

$$\sigma_A^2(\psi) = 0 \iff A|\psi\rangle = a|\psi\rangle. \quad (6)$$

As a consequence of the second part of the proof of the Proposition 1 we have the following

**Proposition 2** If the commutator $C = -i[A, B]$ is a positive operator then the operator $rA + B$ with real $r$ has no eigenstates in the Hilbert space.

In terms of GIS $|z, \lambda\rangle$ the above Proposition\cite{2} gives the restriction on $\lambda$: Re$\lambda \neq 0$ in cases of positive $C$.

Before going to examples let us point out that the $A$-$B$ IS $|z, \lambda = 1\rangle \equiv |z\rangle$ are noncorrelated and with equal variances,

$$L|z\rangle = z|z\rangle, \quad L = L(\lambda = 1) = A + iB, \quad (7)$$
$$\sigma_A^2(z) = \frac{1}{2}\langle z|C|z\rangle = \sigma_B^2(z). \quad (8)$$

We shall call such states equal variances IS or non squeezed IS, adopting the Eberly and Wodkiewicz\cite{7} definition of $A$-$B$ squeezed states. It is convenient to describe this squeezing by means of the dimensionless parameter $q_A$\cite{8}

$$q_A = \frac{\langle C\rangle/2 - \sigma_A^2}{\langle C\rangle/2}, \quad (9)$$

in terms of which the 100% squeezing corresponds to $q_A = 1$. In the equal variances IS $|z\rangle$ $q_A = 0 = q_B$.

Let now consider the cases when the commutator $C = -i[A, B]$ is a positive operator: $\langle \psi|C|\psi\rangle > 0$. In such cases Re$\lambda \neq 0$ and we can safely devide by $\langle \psi|C|\psi\rangle$. Then from eqns (3) we get the quite general result for squeezing in GIS $|z, \lambda\rangle$ with positive $C$,

$$q_A(z, \lambda) = 1 - \frac{1}{2\text{Re} \lambda}, \quad q_B(z, \lambda) = 1 - \frac{|\lambda|^2}{2\text{Re} \lambda}. \quad (10)$$
We see that the squeezing parameter $q$ depends on $\lambda$ only and 100% squeezing of $A$ is obtained at $\text{Re}\, \lambda \to \infty$ (and of $B$ at $\lambda = 0$).

In many cases the IS $|z\rangle$ are constructed. Except of the canonical $Q$-$P$ case we point out also the cases of lowering and raising operators of some semisimple Lie groups (the $SU(2)$ and the $SU(1,1)$[15] for example) and for the quantum group $SU(1,1)_q$, constructed recently[10]. The GIS $|z,\lambda\rangle$ are eigenstates of the linearly transformed operator

$$L \rightarrow L(\lambda) = uL + vL^\dagger,$$

where $u = (\lambda + 1)/2$, $v = (\lambda - 1)/2$, $L^\dagger = A - iB$. If this is a similarity transformation then GIS can be obtained by acting on $|z\rangle$ with the transforming operator $S(\lambda)$ (the generalized squeezing operator) as it was done by Stoler (see the reference in[1, 2]) in the canonical case. In the examples below we construct GIS by solving the eigenvalue equations of $L(\lambda)$.

### 3 SU(1,1) squeezed intelligent states

In this section we construct and discuss $K_1$-$K_2$ GIS, where $K_1$ and $K_2$ are the generators of the discrete series $D^+(k)$ of representations of $SU(1,1)$ with Casimir operator $C_2 := k(k - 1)$. From the commutation relation $[K_1, K_2] = -iK_3$ we see that one can apply the corresponding formulas of the previous section with $A = K_1, B = -K_2$ and $C = K_3$. The operator $K_3$ is positive with eigenvalues $k + m$ where $m = 0, 1, 2, \ldots, \ldots$. Then as a consequence of the Proposition 2 the GIS $|z,\lambda;k\rangle$ exist only if $\text{Re}\, \lambda \neq 0$ and one can safely use formulas (4) for the second momenta of $K_{1,2}$ in the $SU(1,1)$ GIS $|z,\lambda;k\rangle$. Since the operator $K_1$ has no discrete spectrum the condition (3) is also necessary for GIS.

The $SU(1,1)$ equal variances IS $|z;k\rangle$ (the eigenstates of $K_1 - iK_2 \equiv K_-$) have been constructed and studied by Barut and Girardello as ‘new “coherent” states associated with noncompact groups’[15]. These states form an overcomplete family of states and provide a representation of any state $|\psi\rangle$ in terms of entire annalytic function $\langle \psi|z;k\rangle$ of $z$ of order 1 and type 1 (exponential type). In the Hilbert space of such entire analytic functions the generators of $SU(1,1)$ act as the following differential operators[15] (we shall call this BG-representation)

$$K_3 = k + z\frac{d}{dz}, \quad K_+ = K_- = z,$$

$$K_- = 2k\frac{d}{dz} + z\frac{d^2}{dz^2}.$$  

(12)

We use the BG-representation to construct the $SU(1,1)$ GIS $|z',\lambda;k\rangle$ (we denote for a while the eigenvalue by $z'$). The eigenvalue equation (3) now reads

$$\left[u(2k\frac{d}{dz} + z\frac{d^2}{dz^2}) + vz\right] \Phi_{z'}(z) = z'\Phi_{z'}(z),$$

(13)

where the parameters $u, v$ have been defined in formula (11). By means of a simple substitutions the above equation is reduced to the Kummer equation for the confluent
hypergeometric function $\Phi(z) = \exp(cz) \mp_1 F_1(a, b; -2cz)$, so that we have the following solution of eqn. (13)

$$\Phi(z) = \exp(cz) \mp_1 F_1(a, b; -2cz),$$

(14)

$$a = k - \frac{z'}{2uc}, \quad b = 2k; \quad c^2 = \frac{v}{u}.$$  

(15)

This solution obey the requirements of the BG representation iff

$$|c| = \sqrt{|v/u|} < 1 \Leftrightarrow \Re \lambda > 0,$$

(16)

which is exactly the restriction on $\lambda$ imposed by the positivity of the commutator $C \equiv K_3$, according to the Proposition 2. No other constrains on $z'$ and $\lambda$ are needed. Thus we obtain the $SU(1, 1)$ GIS $|z', \lambda; k\rangle$ in the BG-representation in the form

$$\langle k; \lambda, z'|z; k\rangle = \exp(c'z) \mp_1 F_1(a'^*, b; -2c'^*z),$$

(17)

where the parameters $a, b$ and $c$ are given by formulas (3.4). Using the power series of $\mp_1 F_1(a, b; z)$\,[16] we get the coincidence of our solution (17) at $\lambda = 1 (u = 1, v = 0)$ with the solution of Barut and Girardello\,[15],

$$\langle k; \lambda = 1, z'|z; k\rangle = 0 F_1(2k; zz'^*) = \langle k; z'|z; k\rangle.$$  

(18)

We note the twofold degeneracy of the eigenvalues of the operator $L(\lambda \neq 1)$ as it is seen from eqn. (3.4). We denote the two solutions as $\langle \pm; k; \lambda, z'|z; k\rangle$. The degeneracy is removed at $\lambda = 1$ as it is known from the BG-solution. Thus this point is a branching point for the operator $L(\lambda)$. It worth noting that the degeneracy is also removed by the following constrain on the two complex parameters $z'$ and $\lambda$ in eqn. (3.6)

$$z' = 2k\sqrt{-uv} = k\sqrt{1 - \lambda^2}.$$  

(19)

Using the properties of the function $\mp_1 F_1(a, b; z)$\,[16] we get from (17) in both ($\pm$) cases the same expression $\exp(z\sqrt{-v*/u^*})$ which can be seen to be nothing but the BG-representation of the Perelomov $SU(1, 1)$ CS $|\zeta; k\rangle$\,[17] with $\zeta = \sqrt{-v/u},$

$$|\zeta; k\rangle = (1 - |\zeta|^2)^k \exp(\zeta K_+)|k; k\rangle.$$  

(20)

If we impose $z' = -2k\sqrt{-uv}$ we get CS $| - \zeta; k\rangle$. One can directly check (using the $SU(1, 1)$ commutation relations only) that CS (20) are indeed eigenstates of $L(\lambda)$, eqn. (11), with eigenvalue (19) provided $\zeta^2 = -v/u$. We calculate explicitly the first and second momenta of the generators $K_i$ in CS $|\zeta; k\rangle$ (for $\sigma_{K_i}$ see also\,[8])

$$\sigma_{K_1K_2} = -2k \frac{\Re \zeta \Im \zeta}{(1 - |\zeta|^2)^2},$$

$$\sigma_{K_1}^2 = \frac{k}{2} \frac{|1 + \zeta|^2}{(1 - |\zeta|^2)^2}; \quad \sigma_{K_2}^2 = \frac{k}{2} \frac{|1 - \zeta|^2}{(1 - |\zeta|^2)^2}$$

(21)

and convince that the equality in the R-S UR (1) is satisfied.
Thus all the Perelomov $SU(1,1)$ CS are GIS. They are represented by the points of the two dimensional surface [13] in the four dimensional space of points $(z, \lambda)$. The BG CS [13] form another subset of $SU(1,1)$ GIS isomorphic to the plane $\lambda = 1$.

We note that the aboved formulas for the first and second momenta of $K_i$ in CS $|\zeta; k\rangle$ hold also for the (non square integrable) Lipkin-Cohen representation with Bargman index $k = 1/4$ (but not for $k = 3/4$).

$$K_1 = \frac{1}{4}(Q^2 - P^2), \quad K_2 = -\frac{1}{4}(QP + PQ), \quad K_3 = \frac{1}{4}(Q^2 + P^2).$$

Due to the expressions of $K_i$ in terms of the canonical pair $Q, P$ the CS $|\zeta; k = 1/2, 1/4, 3/4\rangle$ ($|\zeta; k = 1/4, 3/4\rangle$ are eigenstates of the squared boson operator $a^2$) are of interest for $Q$-$P$ squeezing [3, 14, 18]. One can also calculate the fluctuations of $Q$ and $P$ [18] and show that CS $|\zeta; k = 1/4\rangle$ exhibit about 56% ordinary squeezing (Bužek [4]). The squeezing of $K_{1,2}$ in CS $|\zeta; k\rangle$ has been studied in [8]: the 100% squeezing (in the sense of the parameter $q$, eqn. (9) for $K_1$ is obtained at $\zeta = i$. We note however that

$$\sigma_i^2(\zeta; k) \geq \frac{k}{2} = \sigma_i^2(0; k), \quad i = K_1, K_2,$$

i.e. no squeezing of $\sigma_i$ in $|\zeta; k\rangle$ in comparison with the ground state $|0; k\rangle$.

In conclusion to this section we note that for $SU(1,1)$ GIS the squeezing operator $S(\lambda)$ exists and can be defined by means of the relation $|z, \lambda; k\rangle = S(\lambda)|z; k\rangle$ since the spectra of $L$ and $L(\lambda)$ coincide. It belongs again to the $SU(1,1)$ (but not to the series $D^+(k)$ since one can show that it is not unitary) and its matrix elements $\langle k; z|S|z; k\rangle$ are explicitly given by the functions [17] with $z' = z$. These diagonal matrix elements determine $S$ uniquely due to the analyticity property of the BG-representation [13]. We recall that the same property of the diagonal matrix elements holds in the canonical (Glauber) CS representation (see for example [2] and references therein).

4 $SU(2)$ squeezed intelligent states

Let now $A, B$ and $C$ be the generators $J_1, -J_2$ and $-J_3$ of $SU(2)$ group, i.e. the spin operators of spin $j = 1/2, 1, \ldots$. In this example the commutator $C = -J_3$ is not positive (the limit $\text{Re} \lambda = 0$ can be taken) and the operator $A = J_1$ has a discrete spectrum (some of its eigenstates are examples of exceptional GIS which are not eigenstates of $L(\lambda)$). In paper [11] there were constructed the eigenstates (in their notations) $|w_N(\tau)\rangle$ of the operator $J(\alpha) = J_1 - i\alpha J_2$, where $N = 0, 1, 2 \ldots, 2j$, $\tau^2 = (1 - \alpha)/(1 + \alpha)$, $\alpha$ being arbitrary complex number. These states are eigenstates also of $L(\lambda) = \lambda J_1 - iJ_2$, thereby they all are $J_1$-$J_2$ GIS, minimizing the R-S UR [1]. They can be represented in the general form $|z_N, \lambda; j\rangle$ with the eigenvalues $z_N = (j - N)\sqrt{\lambda^2 - 1}$. Among them (for $N = 0$ and $N = 2j$) are the Bloch (the spin or the $SU(2)$) CS $|\tau; -j\rangle$ and $| -\tau; -j\rangle$ ($\tau$ is any complex number)

$$|\tau; -j\rangle = (1 + |\tau|^2)^{-j} \exp (\tau J_+)| -j\rangle.$$  (23)
The mean values of \( J_i, i = 1, 2, 3 \) and \( J_i^2 \) (and the dispersions \( \sigma_{J_1} \) and \( \sigma_{J_2} \)) in Bloch CS are known\cite{11, 19}. Calculating also the covariation,

\[
\sigma_{J_1,J_2}(\tau) = 2j \frac{\text{Re} \tau \text{Im} \tau}{(1 + |\tau|^2)^2}
\]

we can directly check that in CS \(|\tau\rangle\) the equality in the R-S UR \((1)\) holds for the spin operators \( J_{1,2} \). Thus the Bloch CS are a subset of the \( SU(2) \) GIS.

Let us briefly discuss the properties of the \( SU(2) \) GIS. First of all for a given parameter \( \lambda \) there are \( 2j+1 \) independent GIS \(|z_N, \lambda; j\rangle\). There is only one equal variances IS, namely \(|-j\rangle\), the point \( \lambda = 1 \) being again the branching point of the \( L(\lambda) \). From this fact it follows that squeezing operator does not exist. Since the commutator \( C = i[J_1, J_2] = -J_3 \) the limit \( \text{Re} \lambda = 0 \) in GIS is allowed and in the fluctuations formulas \((1)\) as well since at this limit \( \langle C \rangle = \langle J_3 \rangle = 0 \). The operator \( A = J_1 \) has a discrete spectrum, therefore \( \sigma_A \geq 0 \). From the explicit formula

\[
\sigma^2_{J_1}(\tau) = \frac{j}{2} \frac{1 - |\tau|^2}{(1 + |\tau|^2)^2}
\]

we see that this fluctuation vanishes at \( \tau^2 = 1 \). Therefore in virtue of the property \((1)\) the Bloch CS \(|\tau = \pm 1; -j\rangle\) are eigenstates of \( J_1 \) which can be checked also directly, the eigenvalues being \( \pm j \). The other eigenstates of \( J_1 \) are exactly those exceptional states which minimize the R-S UR \((1)\) but are not of the form \(|z, \lambda\rangle\) (i.e. dont obey eqn.\((3)\)).

The final note we make about \( SU(2) \) GIS is that except for the eigenvalue \( z_N = 0 \) (when \( N = j \)) all the others are not degenerate (unlike the \( SU(1,1) \) case).

5 Concluding remarks

We have presented a method for construction of squeezed intelligent states (called here generalized intelligent states (GIS)) for any two quantum observables \( A \) and \( B \) in which 100\% squeezing (after Eberly) can be obtained. GIS minimize the Robertson-Schrödinger uncertainty relation and can be considered as a generalization of the canonical \( Q-P \) squeezed states\cite{13}. When the operators \( A \) and/or \( B \) are expressed in terms of the canonical pair \( Q, P \) one can look in the \( A-B \) GIS for the squeezing of \( Q \) and/or \( P \) as well. Such are for example the cases of \( SU(1,1) \) GIS for the representations with Bargman indexes \( k = 1/4, 1/2, 3/4 \). The \( SU(1,1) \) GIS form a larger set of states which contains as two different subsets the Perelomov CS and the Barut and Girardello CS.

The method is based on the minimization of the Robertson-Schrödinger UR \((1)\) for which the eigenvalue equation \((3)\) for the operator \( L(\lambda) = \lambda A + i B \) is a sufficient condition. In case of \( A \) with continuous spectrum this is also a necessary condition independently on \( B \). In view of this the method provides the possibility (when one is interested in squeezing of the fluctuations of \( A \)) to look for the best squeezing partner of \( A \). Thus for example if \( A = P \) then one can show that the eigenstates of \( L(\lambda) \) exist for a series \( B = Q^n, n = 1, 5, 9, \ldots \).

When the \( A-B \) GIS can be obtained from the equal variances IS \(|z\rangle\) by means of the invertable squeezing operator \( S(\lambda) \) the latter belongs to \( SU(1,1) \) as it can be derived
from (11). This fact shows that $SU(1, 1)$ plays important role in a wide class of squeezing phenomena (not only in Q-P case).

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