A note on cycle lengths in graphs of chromatic number five and six

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Abstract

In this note, we prove that every non-complete \((k + 1)\)-critical graph contains cycles of all lengths modulo \(k\), where \(k = 4, 5\). Together with a result in [7], this completely gives an affirmative answer to the question of Moore and West on graphs of given chromatic number.

1 Introduction

The problem of deciding whether a given graph contains cycles of all lengths modulo a positive integer \(k\) shows up in many literatures (see [1–8, 10–13]). Recently, Moore and West [9, Question 2] asked whether every \((k + 1)\)-critical non-complete graph has a cycle of length 2 modulo \(k\). Here, a graph is \(k\)-critical if it has chromatic number \(k\) but deleting any edge will decrease the chromatic number. Very recently, Gao, Huo and Ma [7] partially answered this question by showing the following theorem.

**Theorem 1.1** ( [7] Theorem 1.4). For \(k \geq 6\), every non-complete \((k + 1)\)-critical graph contains cycles of all lengths modulo \(k\).

However, methods in [7] do not work for \(k < 6\). In this note, we give a new method and prove that the conclusion of Theorem 1.1 also holds for \(k = 4, 5\).

**Theorem 1.2.** For \(k = 4, 5\), every non-complete \((k + 1)\)-critical graph contains cycles of all lengths modulo \(k\).

Thus, combined with the Theorems 1.1 and 1.2, we completely give an affirmative answer to the question of Moore and West. See [7, 9] for the history and further references about cycle lengths in graphs of given chromatic number.

The rest of the paper is organized as follows. In Section 2 we introduce the notation. In Section 3, we give a key lemma. In Section 4, we consider graphs of chromatic number five and prove Theorem 1.2 for the case \(k = 4\). In Section 5, we consider graphs of chromatic number six and prove Theorem 1.2 for the case \(k = 5\).

2 Notation

All graphs considered are finite, undirected, and simple. Let \(G\) be a graph and let \(H\) be a subgraph of a graph \(G\). We say that \(H\) and a vertex \(v \in V(G) - V(H)\) are adjacent in \(G\) if \(v\) is adjacent in \(G\) to some vertex in \(V(H)\). Let \(N_G(H) := \bigcup_{v \in V(H)} N_G(v) - V(H)\) and \(N_G[H] := N_G(H) \cup V(H)\). For a subset \(S\) of \(V(G)\), \(G[S]\) denotes the subgraph induced by \(S\) in \(G\), and \(G - S\) denotes the subgraph \(G[V(G) - S]\).

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A vertex is a leaf in \( G \) if it has degree one in \( G \). We say that a path \( P \) is internally disjoint from \( H \) if no vertex of \( P \) other than its endpoints is in \( V(H) \). For two vertex-disjoint subgraphs \( H, H' \) of \( G \), let \( N_H(H') \) be the set of vertices in \( H \) which is adjacent to some vertex in \( H' \).

A cycle or a path is said to be odd (resp. even) if its length is odd (resp. even). Given a cycle \( C \) and an orientation of \( C \), for two vertices \( x \) and \( y \) in \( C \), we denote by \( C[x, y] \) the path on \( C \) from \( x \) to \( y \) in the direction, including \( x \) and \( y \). Let \( C[x, y] := C[x, y] - y, C(x, y) := C[x, y] - x \), and \( C(x, y) := C[x, y] - \{x, y\} \). We use the similar notation to a path \( P \).

Let \( u \) and \( v \) be vertices of a graph. If there are three internally disjoint paths between \( u \) and \( v \), then we call such a graph as \( \theta \)-graph. Note that any \( \theta \)-graph contains an even cycle.

A vertex \( v \) of a graph \( G \) is a cut-vertex of \( G \) if \( G - v \) contains more components than \( G \). A block \( B \) in \( G \) is a maximal connected subgraph of \( G \) such that there exists no cut-vertex of \( B \). So a block is an isolated vertex, an edge or a 2-connected graph. An end-block in \( G \) is a block in \( G \) containing at most one cut-vertex of \( G \). If \( D \) is an end-block of \( G \) and a vertex \( x \) is the only cut-vertex of \( G \) with \( x \in V(D) \), then we say that \( D \) is an end-block with cut-vertex \( x \).

Let \( T \) be a tree, and fix a vertex \( r \) as its root. Let \( v \) be a vertex of \( T \). The parent of \( v \) is the vertex adjacent to \( v \) on the path from \( v \) to \( r \). An ascendant of \( v \) is any vertex which is either the parent of \( v \) or is recursively the ascendant of the parent of \( v \). A child of \( v \) is a vertex of which \( v \) is the parent. A descendant of \( v \) is any vertex which is either the child of \( v \) or is recursively the descendant of any of the children of \( v \). Let \( Y \) be a subset of \( V(T) \). We say a vertex \( x \) is the descendant of \( Y \) if \( x \) is the descendant of some vertex in \( Y \). Let \( a, b \) be two vertices of \( T \). Denote \( T_{a, b} \) the unique path between \( a \) and \( b \) in \( T \).

### 3 Key lemma

Let \( G \) be a 2-connected graph and let \( C \) and \( D \) be two cycles in \( G \). We say that \( (C, D) \) is an opposite pair in \( G \), if \( C \) is odd and \( D \) is even satisfying that \( C \) and \( D \) are edge-disjoint and share at most one common vertex.

**Lemma 3.1.** Let \( G \) be a 2-connected graph of minimum degree at least 4. Let \( (C, D) \) be an opposite pair in \( G \). Then \( G \) contains cycles of all lengths modulo 4.

**Proof.** Suppose to the contrary that \( G \) does not contain cycles of all lengths modulo 4. Since \( G \) is 2-connected and \( |V(C) \cap V(D)| \leq 1 \), there exist two vertex disjoint paths \( P, Q \) between \( C \) and \( D \) satisfying \((V(C) \cap V(D)) - V(Q) = \emptyset \). We take such an opposite pair \( (C, D) \), paths \( P \) and \( Q \) as the following manner:

1. \( |E(P)| \) is as large as possibly,
2. \( |E(Q)| \) is as large as possible subject to (1).

Let \( p \) and \( q \) be the endpoints of \( P \) and \( Q \) in \( D \), respectively.

**Claim 1.** Every even cycle in the block of \( G - (V(C \cup P \cup Q) - \{p, q\}) \) including \( D \) contains both \( p \) and \( q \). In particular, every \( \theta \) graph in the block includes both \( p \) and \( q \).

**Proof of Claim 1.** Let \( H \) be the block of \( G - (V(C \cup P \cup Q) - \{p, q\}) \) including \( D \). Let \( D' \) be an even cycle in \( H \) other than \( D \). Suppose that \( p \notin V(D') \). Since \( H \) is 2-connected, there are two vertex disjoint

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1 We remarked that (i) if \( V(C) \cap V(D) = \emptyset \), then \( P \) and \( Q \) are vertex disjoint, (ii) if \( C \) and \( D \) share one common vertex, then \( V(Q) = V(C) \cap V(D) \).
paths $L_1, L_2$ from $\{p, q\}$ to $D'$ in $H$. We may assume that $L_1$ links $p$ and $D'$. Note that $L_1$ has length at least 1 and $(C, D')$ is an opposite pair. Then $P \cup L_1$ and $Q \cup L_2$ are two internally disjoint paths between $C$ and $D'$ such that $P \cup L_1$ is longer than $P$, a contradiction. Therefore, $p \in V(D')$.

Suppose that $q \notin V(D')$. Since $H$ is 2-connected, there is a path $L_3$ from $q$ to $D'$ internally disjoint from $V(D')$ in $H$. Note that $L_3$ has length at least 1 and $(C, D')$ is an opposite pair. Then $P$ and $Q \cup L_3$ are two internally disjoint paths between $C$ and $D'$ such that $Q \cup L_3$ is longer than $Q$, a contradiction. Therefore, $q \in V(D')$. Since every theta graph contains an even cycle, every theta graph in $H$ includes both $p$ and $q$. This completes the proof of Claim [1].

Since $D$ is an even cycle, we partition $V(D)$ into the sets $A$ and $B$ alternatively along $D$. By symmetry between $A$ and $B$, we may assume that $p \in A$.

**Claim 2.** For any $b \in B - \{q\}$, there is no path from $b$ to $C \cup P \cup Q - \{p, q\}$ internally disjoint from $C \cup D \cup P \cup Q$.

**Proof of Claim 2.** Suppose to the contrary that there is a path $R$ from $b$ to $x \in V(C \cup P \cup Q) - \{p, q\}$ internally disjoint from $C \cup D \cup P \cup Q$. By symmetry, we may assume that $b \in D(p, q)$.

Assume that $|E(D)| \equiv 0$ modulo 4. As $C$ is an odd cycle, there is an even path $X_1$ and an odd path $Y_1$ between $p$ and $q$ in $C \cup P \cup Q$. If $q \in B$, then both $|E(D[p, q])|$ and $|E(D[q, p])|$ are odd, and furthermore, since their sum is 0 modulo 4, they differ by 2 modulo 4. Then $X_1 \cup D[p, q], X_1 \cup D[q, p], Y_1 \cup D[p, q]$ and $Y_1 \cup D[q, p]$ are 4 cycles of different lengths modulo 4, a contradiction. Therefore, we have that $q \in A$.

- Suppose that $x \in V(P) - \{p\}$. Since $C$ is an odd cycle, there is an even path $X_2$ and an odd path $Y_2$ between $b$ and $q$ in $C \cup P \cup Q \cup R$. However, since both $|E(D[b, q])|$ and $|E(D[q, b])|$ are odd and differ by 2 modulo 4, $X_2 \cup D[b, q], X_2 \cup D[q, b], Y_2 \cup D[b, q]$ and $Y_2 \cup D[q, b]$ are 4 cycles of different lengths modulo 4, a contradiction. Thus, $x$ is not contained in $V(P) - \{p\}$.

- Suppose that $x \in V(C \cup Q) - (V(P) \cup \{q\})$. Then there is an even path $X_3$ and an odd path $Y_3$ between $b$ and $p$ in $C \cup P \cup Q \cup R$. However, since both $|E(D[p, b])|$ and $|E(D[p, b])|$ are odd and differ by 2 modulo 4, $X_3 \cup D[p, b], X_3 \cup D[p, b], Y_3 \cup D[p, b]$ and $Y_3 \cup D[p, b]$ are 4 cycles of different lengths modulo 4, a contradiction. Thus, $x$ is not contained in $V(C \cup Q) - (V(P) \cup \{q\})$.$\square$

Therefore, $|E(D)| \equiv 2$ modulo 4. As $C$ is an odd cycle, there is an even path $X_4$ and an odd path $Y_4$ between $p$ and $q$ in $C \cup P \cup Q$. If $q \in A$, then both $|E(D[p, q])|$ and $|E(D[q, p])|$ are even, and furthermore, since their sum is 2 modulo 4, they differ by 2 modulo 4. Then $X_4 \cup D[p, q], X_4 \cup D[q, p], Y_4 \cup D[p, q]$ and $Y_4 \cup D[q, p]$ are 4 cycles of different lengths modulo 4, a contradiction. Therefore, we have that $q \in B$.

- Suppose that $x \in V(C \cup P) - (V(Q) \cup \{p\})$. Since $C$ is an odd cycle, there is an even path $X_5$ between $b$ and $q$ and an odd path $Y_5$ between $b$ and $q$ in $C \cup P \cup Q \cup R$. However, since both $|E(D[b, q])|$ and $|E(D[q, b])|$ are odd and differ by 2 modulo 4, $X_5 \cup D[b, q], X_5 \cup D[q, b], Y_5 \cup D[b, q]$ and $Y_5 \cup D[q, b]$ are 4 cycles of different lengths modulo 4, a contradiction. Thus, $x$ is not contained in $V(C \cup P) - (V(Q) \cup \{p\})$.

- Suppose that $x \in V(Q) - \{q\}$. Since $G$ is 2-connected and $G$ has minimum degree at least 4, there exists a path $T$ from $b$ to $y \in V(C \cup D \cup P \cup Q \cup R) - \{b\}$ internally disjoint from $C \cup D \cup P \cup Q \cup R$. Based on previous analysis, we have that $y \in V(Q \cup D \cup R) - \{b\}$.
– If $y \in V(R \cup Q \cup D(b, p)) - \{b\}$, then $D(b, p) \cup R \cup T \cup Q$ contains a theta graph. It follows that there is an even $D_1$ cycle in $G - (C \cup P - Q)$. Note that $(C, D_1)$ is an opposite pair in $G$. It is easy to see that there are two internally disjoint paths $P'$ and $Q'$ between $C$ and $D'$ satisfying that $P'$ contains $P$ and is longer than $P$ and $Q' \subseteq Q \cup D(b, q)$, a contradiction. Thus, $y$ is not contained in $V(R \cup Q \cup D(b, p)) - \{b\}$.

– Suppose that $y \in V(D[p, b])$. Since $T \cup D[y, b]$ does not contain $q$ and $D[b, q] \cup R \cup Q[x, q]$ does not contain $p$, by the choice of opposite pairs, we have that $T \cup D[y, b]$ and $D[b, q] \cup R \cup Q[x, q]$ are both odd cycles. Since $C$ is an odd cycle, there is an odd path $X'$ and an even path $Y'$ between $y$ and $x$ in $C \cup P \cup Q \cup D[p, y]$. Note that $X'$ and $Y'$ differ by 1 modulo 4, $T$ and $D[y, b]$ differ by 1 modulo 4 and $D[b, q] \cup R \cup Q[x, q]$ and $R$ differ by 1 modulo 4. Then the set \{ $L_1 \cup L_2 \cup L_3 | L_1 \in \{X', Y'\}, L_2 \in \{T, D[y, b]\}, L_3 \in \{D[b, q] \cup Q[x, q], R\}$\} contains cycles of all lengths modulo 4, a contradiction. Thus, $y$ is not contained in $V(D[p, b])$.

This completes the proof of Claim 2.

Let $z$ be a vertex in $B - \{q\}$. By symmetry, we may assume that $z \in V(D(p, q))$. Since $z$ has degree at least 4 in $G$ and $G$ is 2-connected, there is a path $Z$ from $z$ to $C \cup D \cup P \cup Q - \{z\}$ internally disjoint from $C \cup D \cup P \cup Q$. By Claim 2, the endpoint of $Z$ other than $z$ is contained in $D - \{z\}$. Let $r$ be the endpoint of $Z$ other than $z$. Since $z$ has degree at least 4 in $G$ and $G$ is 2-connected, there is a path $S$ from $z$ to $s \in V(C \cup D \cup P \cup Q \cup Z) - \{z\}$ internally disjoint from $C \cup D \cup P \cup Q \cup Z$. By Claim 2, $s$ is contained in $V(D \cup Z) - \{z\}$.

• Suppose that $s \in V(Z) - \{z\}$.

  – If $r \in V(D(z, p))$, then $D[z, r] \cup Z \cup S$ is a theta graph not containing $p$, contradicting Claim 1

  – If $r \in V(D[p, z])$, then $D[r, z] \cup Z \cup S$ is a theta graph not containing $q$, contradicting Claim 1

Thus, $s$ is not contained in $V(Z) - \{z\}$.

• Suppose that $s \in D - \{z, r\}$. By symmetry between $r$ and $s$, we may assume that $s \in V(D(r, z))$.

  – If $r \in V(D(q, z))$, then $D[r, z] \cup Z \cup S$ is a theta graph not containing $q$, contradicting Claim 1

  – If $r \in V(D(z, q))$ and $s \in V(D(r, p))$, then $D[z, s] \cup Z \cup S$ is a theta graph not containing $p$, contradicting Claim 1

  – Therefore $r \in V(D(z, q))$ and $s \in V(D[p, z])$. Since $S \cup D[s, z]$ does not contain $q$ and $D[z, r] \cup Z$ does not contain $p$, by Claim 1 we have that $S \cup D[s, z]$ and $D[z, r] \cup Z$ are both odd cycles. Since $C$ is an odd cycle, there is an odd path $X''$ and an even path $Y''$ between $s$ and $r$ in $C \cup P \cup Q \cup D[s, q] \cup D[r, q]$. Note that $X''$ and $Y''$ differ by 1 modulo 4, $S$ and $D[s, z]$ differ by 1 modulo 4 and $D[z, r]$ and $Z$ differ by 1 modulo 4. Then the set \{ $L_1 \cup L_2 \cup L_3 | L_1 \in \{X'', Y''\}, L_2 \in \{S, D[s, z]\}, L_3 \in \{D[z, r], Z\}$\} contains cycles of all lengths modulo 4, a contradiction.

This completes the proof of Lemma 3.1
4 Graphs of chromatic number five

In this section, we prove the following theorem on 2-connected graphs of minimum degree at least four, from which Theorem 1.2 can be inferred as a corollary for the case \( k = 4 \).

**Theorem 4.1.** Every 2-connected non-bipartite graph of minimum degree at least 4 contains cycles of all lengths modulo 4, except that it is the complete graph of five vertices.

**Proof.** Let \( G \) be a 2-connected non-bipartite graph of minimum degree at least 4. Assume that \( G \) is not a \( K_5 \) and does not contain cycles of all lengths modulo 4. Let \( C := v_0v_1\ldots v_2t v_0 \) be an odd cycle in \( G \) such that \( |V(C)| \) is minimum, where the indices are taken under the additive group \( \mathbb{Z}_{2t+1} \). Note that \( C \) is induced. Let \( H := G - V(C) \). By Lemma 3.1 there is no opposite pairs in \( G \), hence \( H \) does not contain an even cycle. It follows that every block of \( H \) is either an odd cycle, an edge or an isolated vertex.

**Claim.** \( G \) does not contain a triangle.

**Proof of Claim.** Suppose that \( G \) contains a triangle. Then \( C \) is a triangle. Let \( H_1 \) be a component of \( H \). Since \( G \) has minimum degree at least 4, \( H_1 \) has at least two vertices. Suppose that \( H_1 \) contains an odd cycle \( C_1 \).

- If \( H_1 \) is not 2-connected, then there exists an end-block \( B_1 \) of \( H_1 \) with cut-vertex \( b_1 \) such that \( (V(B_1) - \{b_1\}) \cap C_1 = \emptyset \). As \( B_1 \) is either an odd cycle or an edge, there exists \( w \in V(B_1) - \{b_1\} \) such that \( w \) is at least two neighbors on \( C \). Since \( C \) is an odd cycle, \( G[C \cup \{w\}] \) contains an even cycle \( D_1 \). Then \( C_1 \) and \( D_1 \) form an opposite pair in \( G \), a contradiction.

- Therefore, \( H_1 \) is 2-connected, that is \( H_1 \) is an induced odd cycle, we denote \( H_1 := u_0u_1\ldots u_2h u_0 \), where the indices are taken under the additive group \( \mathbb{Z}_{2h+1} \). Since \( G \) has minimum degree at least 4, \( u_0 \) and \( u_2 \) have at least two neighbors on \( C \). Without loss of generality, we may assume that \( u_0 \) is adjacent to \( v_0 \) and \( v_1 \) and \( u_2 \) is adjacent to \( v_0 \). Then \( C, u_0v_0v_2v_1u_0, u_0u_1u_2v_0v_1u_0 \) and \( u_0u_1u_2v_0v_2v_1u_0 \) are cycles of lengths 3, 4, 5 and 6, respectively, a contradiction.

Therefore every component of \( H \) does not contain an odd cycle, that is, every component of \( H \) is a tree.

- If \( |V(H_1)| = 2 \), then \( G[C \cup H_1] \) is a \( K_5 \). Suppose that there is another component \( H_2 \neq H_1 \) of \( H \). Since \( G \) is 2-connected, there are two disjoint path \( L_1 \) and \( L_2 \) from \( H_2 \) to \( C \) internally disjoint from \( C \) in \( G[H_2 \cup C] \). Without loss of generality, we may assume that \( V(L_i) \cap V(C) = \{v_i\} \) for \( i = 1, 2 \). Concatenating \( L_1 \), \( L_2 \) and a path in \( H_2 \), there exists a path \( L \) from \( v_1 \) to \( v_2 \) internally disjoint from \( C \) in \( G[H_2 \cup C] \). As there are paths of lengths 1, 2, 3 and 4 from \( v_1 \) to \( v_2 \) in \( G[H_1 \cup C] \), we could easily obtain 4 cycles of consecutive lengths, a contradiction. Therefore, \( H = H_1 \). It follows that \( G = G[C \cup H_1] \), a contradiction.

- Therefore \( |V(H_1)| \geq 3 \). For any two leaves \( x, y \) of \( H_1 \), let \( T \) be the fixed path between \( x \) and \( y \) in \( H_1 \). Since \( G \) has minimum degree at least 4, \( x \) and \( y \) have at least two neighbors on \( C \). Without loss of generality, we may assume that \( x \) is adjacent to \( v_0 \) and \( y \) is adjacent to \( v_0 \). If \( T \) is even, then \( C \) and \( v_0g T x v_0 \) form an opposite pair, a contradiction. Therefore \( T \) is odd. Suppose that there exist three leaves \( x, y \) and \( z \) in \( H_1 \). Let \( T_{x,y}, T_{y,z} \) and \( T_{z,x} \) be the fixed paths between \( x \) and \( y \), \( y \) and \( z \) and \( z \) and \( x \) in \( H_1 \), respectively. Note that all of them are odd. However, there sum is even, a contradiction. Therefore, \( H_1 \) is a path. Let \( H_1 := z_0z_1z_2\ldots z_n \) for some \( n \geq 2 \). Since \( G \) has minimum degree at least 4, \( z_0 \) is adjacent to all vertices of \( C \) and \( z_2 \) is adjacent to at least 2
vertices of $C$. Without loss of generality, we may assume that $z_2$ is adjacent to $v_0$ and $v_1$. Then $C, z_0v_0v_2v_1z_0, z_0z_1z_2v_0v_1z_0$ and $z_0z_1z_2v_0v_2v_1z_0$ are cycles of lengths 3, 4, 5 and 6, respectively, a contradiction.

This completes the proof of Claim.

By Claim, $G$ does not contain a triangle. Suppose that there is a vertex $u$ of degree at most one in $H$. Since $G$ has minimum degree at least 4, $u$ has at least three neighbors on $C$. Since $C$ is odd, there exist two distinct neighbors $v_i, v_j$ of $u$ on $C$ such that the odd path between $v_i$ and $v_j$ on $H$ has no internal vertices which are the neighbors of $u$ in $G$. Let $Q_o, Q_e$ be the odd and even paths between $v_i$ and $v_j$ in $C$ respectively. Let $C' := uv_i \cup Q_o \cup v_ju$. Note that $C'$ is an odd cycle. By the choice of $C$, we have that $|E(C')| \geq |E(C)|$. This fores that $|E(Q_e)| = 2$ and $u$ is adjacent to all vertices of $V(Q_e)$. It follows that there is a triangle in $G$, a contradiction. Therefore, $H$ has minimum degree at least 2.

Suppose that $H$ has more than one component. Let $W_1$ and $W_2$ be two components of $H$. Since every vertex in $W_1$ has degree at least 2, we have that $W_1$ contains an odd cycle $C_2$. Since $G$ is 2-connected and $C$ is an odd cycle, there is an even cycle $D_2$ in $G[V(C) \cup W_2]$. Thus, $C_2$ and $D_2$ form an opposite pair, a contradiction. Therefore, $H$ is connected.

Note that $H$ has minimum degree at least 2 and every block of $H$ is either an odd cycle, an edge or an isolated vertex. There is a vertex $t$ of $H$ which has at least two neighbors on $C$. Since $C$ is odd, there exist two distinct neighbors $v_i, v_j$ of $t$ on $C$ such that the odd path between $v_i$ and $v_j$ on $C$ has no internal vertices which are the neighbors of $t$ in $G$. Let $Q'_o, Q'_e$ be the odd and even paths between $v_i$ and $v_j$ in $C$ respectively. Let $C'' := tv_i \cup Q'_o \cup v_jt$. Note that $C''$ is an odd cycle. By the choice of $C$, we have that $|E(C'')| \geq |E(C)|$. This fores that $|E(Q'_e)| = 2$. Without loss of generality, we may assume that $i = j + 2$. Let $s$ be the neighbor of $v_j + \ell + 1$ in $H$. Since $H$ is connected, there is a path $L$ between $t$ and $s$ in $H$. Then $C[v_j + 2, v_j + \ell + 1] \cup v_j + \ell + 1s \cup L \cup tv_{j + 2}, C[v_j + \ell + 1, v_j] \cup v_jt \cup L \cup sv_{j + \ell + 1}, C[v_j, v_j + \ell + 1] \cup v_j + \ell + 1s \cup L \cup tv_j, C[v_j + \ell + 1, v_{j + 2}] \cup v_{j + 2}t \cup L \cup sv_{j + \ell + 1}$ are 4 cycles of consecutive lengths, a contradiction. This completes the proof of Theorem 4.1.

We remark that Theorem 4.1 is best possible by the following examples. For any positive integer $t$, let $P_t := v_0v_1 \ldots v_{2t + 1}$ and $Q_t := u_0u_1 \ldots u_{2t + 1}$ be two vertex disjoint paths. Let $H_t$ be the graph obtained from $P_t \cup Q_t$ by adding edges in $\{v_2u_{2t + 1}, u_2v_{2t + 1}, u_0v_0, u_{2t + 1}v_{2t + 1}\}$. We see that $H_t$ is a 2-connected non-bipartite graph of minimum degree 3 without cycles of length 1 modulo 4.

![Diagram](image)

Figure 1: Graphs without cycles of length 1 modulo 4

5 Graphs of chromatic number six

In this section, we consider graphs of chromatic number six and prove Theorem 1.2 for the case $k = 5$. We need the following theorems in [7].
Theorem 5.1 ([7] Theorem 3.2). Let $G$ be a connected graph of minimum degree at least three and $(A, B)$ be a non-trivial partition of $V(G)$. For any cycle $C$ in $G$, there exist $A$-$B$ paths of every length less than $|V(C)|$ in $G$, unless $G$ is bipartite with the bipartition $(A, B)$.

Theorem 5.2 ([7] Theorem 4.1). Let $k \geq 3$ be an integer. Let $G$ be a 2-connected graph of minimum degree at least $k$. If $G$ is $K_3$-free, then $G$ contains a cycle of length at least $2k + 2$, except that $G = K_{k,n}$ for some $n \geq k$.

Theorem 5.3 ([7] Theorem 5.2). Let $k \geq 2$ be an integer. Every 2-connected graph $G$ of minimum degree at least $k$ containing a triangle $K_3$ contains $k$ cycles of consecutive lengths, except that $G = K_{k+1}$.

Theorem 5.4. Every graph of chromatic number six contains all lengths modulo five.

Proof. It suffices to consider 6-critical graphs $G$. Suppose that $G$ does not contain five cycles of all lengths modulo five. It is well-known that $G$ is a 2-connected graph of minimum degree at least five. By Theorem 5.2 we may assume that $G$ is $K_3$-free. Fix a vertex $r$ and let $T$ be the breadth first search tree in $G$ with root $r$. Let $L_0 = \{r\}$ and $L_i$ be the set of vertices of $T$ at distance $i$ from its root $r$.

Lemma 5.5. Every component of $G[L_i]$ has chromatic number at most 3, for all $i \geq 0$.

Proof. Suppose to the contrary that there exists a component $D$ of $G[L_t]$ which has chromatic number at least 4 for some $t$. Let $H$ be a 4-critical subgraph of $D$. It is clear that $H$ is a 2-connected non-bipartite graph of minimum degree at least 3. By Theorem 5.2 $H$ contains a cycle of length at least 8. Let $T'$ be the minimal subtree of $T$ whose set of leaves is precisely $V(H)$, and let $r'$ be the root of $T'$. Let $h$ denote the distance between $r'$ and vertices in $H$ in $T'$. Since $G$ is $K_3$-free, $h \geq 2$. By the minimality of $T'$, $r'$ has at least two children in $T'$. Let $x$ be one of its children. Let $A$ be the set of vertices in $H$ which are the descendants of $x$ in $T'$ and let $B = V(H) - A$. Then both $A, B$ are nonempty and for any $a \in A$ and $b \in B$, $T_{a,b}$ has the same length $2h$. By Theorem 5.1 there are 7 subpaths of $H$ from a vertex of $A$ to a vertex of $B$ of length $1, 2, \ldots, 7$, respectively. It follows that $G$ contains 7 cycles of consecutive lengths, a contradiction. This completes the proof of Lemma 5.5. \quad \blacksquare

For a connected graph $D$, a vertex in $D$ is called good if it is not contained in the minimal connected subgraph of $D$ which contains all 2-connected blocks of $D$, and bad otherwise.

Lemma 5.6. Let $H_1$ be a non-bipartite component of $G[L_i]$ and $H_2$ be a non-bipartite component of $G[L_{i+1}]$ for some $i \geq 1$. If $N_{H_1}(H_2) \neq \emptyset$, then every vertex in $N_{H_1}(H_2)$ is a good vertex of $H_1$.

Proof. Suppose that there exists a bad vertex $v$ of $H_1$ which has a neighbor in $H_2$. Let $T'$ be the minimal subtree of $T$ whose set of leaves is precisely $V(H_1)$, and let $r'$ be the root of $T'$. Let $h$ denote the distance between $r'$ and vertices in $H_1$ in $T'$. Since $G$ is $K_3$-free, $h \geq 2$. By the minimality of $T'$, $r'$ has at least two children in $T'$. Let $(X, Y)$ be a non-trivial partition of all children of $r'$ in $T'$. Let $A$ be the set of vertices in $H_1$ which are the descendants of $X$ in $T'$ and let $B$ be the set of vertices in $H_1$ which are the descendants of $Y$ in $T'$. Note that $(A, B)$ is a non-trivial partition of $V(H_1)$. Note that every vertex in $B$ is the descendants of $Y$ in $T'$. Let $A'$ be the set of vertices in $L_i - A$ which are the descendants of $X$ in $T$. Let $B'$ be the set of vertices in $L_i - B$ which are the descendants of $Y$ in $T$. Let $M := L_i - (A \cup A' \cup B \cup B')$. Note that $A, A', B, B'$ and $M$ form a partition of $L_i$. Note that every vertex of $H_2$ has a neighbor in $L_i$.

Suppose that there exists a vertex $m \in V(H_2)$ which has a neighbor $m'$ in $M$. Recall that $H_1$ is non-bipartite and $K_3$-free. There exists a path $z_1z_2z_3z_4z_5$ of length 4 in $H_1$ with $z_1 = v$. It is easy to see that $T_{z_i,m}$ contains $r'$ for $i \in [5]$, so they have the same length. Let $P$ be a fixed path from $u$ to $m$ in $H_2$. Then $P \cup uz_1z_2 \ldots z_i \cup T_{z_i,m'} \cup m'm$, for $i \in [5]$ are 5 cycles of consecutive lengths in $G$, a
contradiction. Therefore $N_M(H_2) = \emptyset$, that is every vertex in $H_2$ has a neighbor in $A \cup A' \cup B \cup B'$.

For a vertex in $V(H_2)$, we call it type-A if it has a neighbor in $A \cup A'$ and it type-B if it has a neighbor in $B \cup B'$. 

Let $C = v_0v_1 \ldots v_n$ be an odd cycle of $H_1$, where $n \geq 4$. Suppose that $V(C) \subseteq A$. Since $B$ is non-empty, we choose an arbitrary vertex $b$ in $B$. Let $b$ be a vertex in $B$. Since $H_1$ is connected, there exists a path $P$ from $b$ to $V(C)$ internal disjoint from $V(C)$. Without loss of generality, we assume that $V(P) \cap V(C) = \{v_0\}$. Then $P \cup C[v_0, v_i] \cup T_{b, v_i}$ for $i = 0, 1, \ldots, 4$ give 5 cycles of consecutive lengths, a contradiction. Therefore, $B \cap V(C) \neq \emptyset$, and similarly, $A \cap V(C) \neq \emptyset$. Then there must be an $A$-$B$ path of length 4 in $C$ (otherwise, since 4 and $|C|$ is co-prime and $|C| \geq 5$, one can deduce that all vertices of $C$ are contained in one of the two parts $A$ and $B$, a contradiction).

Without loss of generality, we may assume that $v_0, v_1 \in A$ and $v_2 \in B$. Then $T_{v_1, v_2} \cup v_2v_1$ and $T_{v_0, v_2} \cup v_2v_1v_0$ are two cycles of lengths $2h + 1$ and $2h + 2$, respectively. We have showed that there exists some $A$-$B$ path of length 4 in $C$ which gives a cycle of length $2h + 4$, so we may assume that there is no $A$-$B$ path of length 3 or 5 in $C$. This would force that one of the following holds.

### 5.1 There is no $A$-$B$ path of length 3 in $H_1$.

This would force that for any path $P' = u_0u_1 \ldots u_k$ in $H_1$ with $u_1 = v_0, u_2 = v_1, u_3 = v_2$, we can derive that $u_j \in B$ if $j \equiv 0$ modulo 3 and $u_j \in A$ if $j \equiv 1$ or 2 modulo 3. Moreover, we have that $v_3j, v_{3j+1} \in A$ and $v_{3j+2} \in B$ for each possible $j \geq 0$. So $|C| \geq 9$ and $G$ contains a cycle of length $\ell \in \{2h + 1, 2h + 2, 2h + 4, 2h + 5, 2h + 7, 2h + 8\}$. In particular, since $H_1$ is connected, for any vertex $b \in B$, there exists a path of length 2 in $H_1$ from $b$ to some vertex in $A$. And for any bad vertex $a \in A$, there exists a path $b_1a_1b_2$ satisfying $b_1, b_2 \in B$ and $a, a_1 \in A$.

- Suppose that $N_{A \cup A'}(H_2) \neq \emptyset$ and $N_{B \cup B'}(H_2) \neq \emptyset$. Since $H_2$ is connected and every vertex of $H_2$ has a neighbor in $A \cup A' \cup B \cup B'$, there exist two adjacent vertices $p, q$ of $H_2$ such that $p$ has a neighbor $p'$ in $A \cup A'$ and $q$ has a neighbor $q'$ in $B \cup B'$. Then $p'pq' \cup T_{p', q'}$ is a cycle of length $2h + 3$. It follows that $G$ contains 5 cycles of lengths $2h + 1, 2h + 2, 2h + 3, 2h + 4$ and $2h + 5$, respectively, a contradiction.

- Suppose that $N_{L_1}(H_2) \subseteq B \cup B'$. Since $N_{A \cup B}(H_2) \neq \emptyset$, we have that $v \in B$. Let $u$ be any vertex in $N_{H_2}(v)$. Choose $v_1 \in V(H_2)$ such that there exists a path $Q$ of length 2 from $u$ to $v_1$ in $H_2$. Let $w_2$ be a neighbor of $v_1$ in $B \cup B'$. Suppose that $w_2 \neq v$. Note that there is a path $R := vv''v'$ such that $v'' \in B$ and $v' \in A$. Then $R \cup vu \cup Q \cup w_1w_2 \cup T_{w_1, v'}$ is a cycle of length $2h + 6$. So $G$ contains cycles of lengths $2h + 4, 2h + 5, 2h + 6, 2h + 7$ and $2h + 8$, a contradiction. Therefore $w_2 = v$ and $w_1 \in N_{H_2}(v)$. That says, every vertex in $H_2$ of distance 2 from a neighbor of $v$ is a neighbor of $v$. Continuing to apply this along with a path from $u$ to an odd cycle $C_0$ in $H_2$, we could obtain that $v$ is adjacent to all vertices of $C_0$, which contradicts that $G$ is $K_3$-free. Therefore, $N_{B \cup B'}(H_2) = \emptyset$.

- Now we see that $N_{L_1}(H_2) \subseteq A \cup A'$. This forces that $v \in A$. For any neighbor $u'$ of $v$ in $H_2$, let $w_3 \in V(H_2)$ satisfies that there exists a path $Q'$ of length 2 from $u'$ to $w_3$ in $H_2$. Note that $v \in A$ is bad in $H_1$, we can infer that there exists a path $b_2a_1b_1$ in $H_1$ such that $a_1 \in A$ and $b_1, b_2 \in B$. Note that $v$ and $a_1$ are symmetric. Let $w_4$ be a neighbor of $w_3$ in $A \cup A'$. Suppose that $w_4 \notin \{v, a_1\}$. Then $vu' \cup Q' \cup w_2w_4 \cup T_{w_4, b_1} \cup b_1a_1v$ is a cycle of length $2h + 6$. So again, $G$ contains cycles of lengths $2h + 4, 2h + 5, 2h + 6, 2h + 7$ and $2h + 8$, a contradiction. Therefore, $w_4 \in \{v, a_1\}$. That is, every vertex in $H_2$ of distance 2 from a neighbor of $v$ or $a_1$ is adjacent to

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2We remark that a vertex can be both type-A and type-B.
one of $v, a_1$. Continuing to apply this along with a path from $u'$ to an odd cycle $C_1$ in $H_2$, we could obtain that every vertex of $C_1$ is adjacent to one of $v, a_1$. But this would force a $K_3$ in $G$. This final contradiction completes the proof of this subsection.

5.2 There is an $A$-$B$ path of length 3 in $H_1$.

Therefore, we may assume that there is no $A$-$B$ paths of length 5 in $H_1$.

**Claim.** Let $t_1t_2t_3$ be a path in $H_1$ satisfying that $t_1$ and $t_3$ are in different parts. Then $t_2$ does not have a neighbor in $V(H_2)$.

**Proof of Claim.** Without loss of generality, we may assume that $t_1, t_2 \in A$ and $t_3 \in B$. Suppose that $t_2$ has a neighbor in $H_2$. Let $s$ be any vertex in $N_{H_2}(t_2)$. Choose $s' \in V(H_2)$ such that there exists a path $Q$ of length 2 from $s$ to $s'$ in $H_2$. Let $t$ be a neighbor of $s'$ in $L_i - M$. Suppose that $t \neq t_2$. If $t \in A \cup A'$, then $t_3t_2s \cup Q \cup s't \cup T_{t, t_3}$ is a cycle of length $2h + 5$. So $G$ contains cycles of lengths $2h + 1, 2h + 2, 2h + 3, 2h + 4$ and $2h + 5$, a contradiction. Therefore $t \in B \cup B'$, then $t_1t_2s \cup Q \cup s't \cup T_{t, t_1}$ is a cycle of length $2h + 5$. So $G$ contains cycles of lengths $2h + 1, 2h + 2, 2h + 3, 2h + 4$ and $2h + 5$, a contradiction. Therefore $t = t_2$ and $w_1$ is the neighbor of $t_2$. That says, every vertex in $H_2$ of distance 2 from a neighbor of $t_2$ is a neighbor of $t_2$. Continuing to apply this along with a path from $s$ to an odd cycle $C_2$ in $H_2$, we could obtain that $t_2$ is adjacent to all vertices of $C_2$, which contradicts that $G$ is $K_3$-free. This completes the proof of Claim.

- Suppose that $N_{A \cup A'}(H_2) \neq \emptyset$ and $N_{B \cup B'}(H_2) \neq \emptyset$. Suppose that there exists a path $p_0p_1p_2p_3$ in $H_2$ such that $p_0$ is type-$A$ and $p_3$ is type-$B$. Let $q$ be the neighbor of $p_0$ in $A \cup A'$ and $q'$ be the neighbor of $p_3$ in $B \cup B'$. Then $q_{p_0p_1p_2p_3q'} \cup T_{q', q}$ is a cycle of length $2h + 5$. So $G$ contains cycles of lengths $2h + 1, 2h + 2, 2h + 3, 2h + 4$ and $2h + 5$, a contradiction. This forces that every two vertices which are linked by a path of length 3 in $H_2$ have the same type. Note that $N_{A \cup A'}(H_2) \neq \emptyset$ and $N_{B \cup B'}(H_2) \neq \emptyset$. By symmetry between $A \cup A'$ and $B \cup B'$, there exists a path $z_0z_1z_2$ in $H_2$ such that $z_0$ and $z_1$ are type-$A$ and $z_2$ is type-$B$. Moreover, for any path $P' := u_0u_1 \ldots u_s$ in $H_2$ with $u_0 = z_0, u_1 = z_1, u_2 = z_2$, we can derive that $u_j$ is type-$A$ if $j \equiv 0$ or 1 modulo 3 and $u_j$ is type-$B$ if $j \equiv 2$ modulo 3. Moreover, for any path $P'' := u_0u_1 \ldots u_s$ in $H_2$ with $u_0 = z_2, u_1 = z_1, u_2 = z_0$, we can derive that $u_j$ is type-$A$ if $j \equiv 2$ modulo 3 and $u_j$ is type-$B$ if $j \equiv 1$ or 2 modulo 3. This forces that every cycle in $H_2$ has length 0 modulo 3. Since $H_2$ is non-bipartite and $K_3$-free, there is an odd cycle $C_3 := w_0w_1 \ldots w_mw_0$ of length at least 9. Note that $w_0$ and $w_8$ have different types. If follows that there is a cycle of length $2h + 10$. So $G$ contains cycles of lengths $2h + 1, 2h + 2, 2h + 3, 2h + 4$ and $2h + 10$, a contradiction.

- Therefore, all vertices in $H_2$ have the same type. Without loss of generality, we may assume that $N_{L_i}(H_2) \subseteq A \cup A'$. Therefore $v \in A$ and let $f_0$ be a neighbor of $v$ in $H_2$. Since $H_2$ is $K_3$-free and non-bipartite, there is a path $f_0f_1f_2$ in $H_2$. Since $H_1$ is a $K_3$-free non-bipartite graph and $v$ is a bad vertex in $H_1$, there is a path $a_0a_1a_2a_3$ in $H_1$. Since there is no $A$-$B$ path of length 5 in $H_1$, we have that for any path $Q' := u_0u_1 \ldots u_s$ in $H_1$ with $u_0 = a_0, u_1 = a_1, u_2 = v, u_3 = a_2, u_4 = a_3$, we can derive that $u_j$ and $v_k$ are in the same part if $j \equiv k$ modulo 5. Also, we have that for any path $Q' := u_0u_1 \ldots u_s$ in $H_1$ with $u_0 = a_3, u_1 = a_2, u_2 = v, u_3 = a_1, u_4 = a_0$, we can derive that $u_j$ and $v_k$ are in the same part if $j \equiv k$ modulo 5. Based on previous analysis, We have that $a_1$ and $a_2$ have the same type.

- Suppose that $a_1, a_2 \in A$. Since $V(H_1) \cap B \neq \emptyset$, we have that one of $a_0$ and $a_3$ is in $B$. Without loss of generality, we may assume that $a_0 \in B$. Let $w$ be a neighbor of $f_1$ in $H_1$. We
have that \( w \in A \cup A' \). Since \( G \) is \( K_3\)-free, \( w \neq u \). Note that \( a_0a_1v \) satisfying that \( a_0 \) and \( v \) are in different parts of \( H_1 \). By Claim, we have that \( w \neq a_1 \). Therefore, \( w f_1 f_0 v a_1 a_0 \cup T_{a_0,w} \) is a cycle of length \( 2h + 5 \). So \( G \) contains cycles of lengths \( 2h + 1, 2h + 2, 2h + 3, 2h + 4 \) and \( 2h + 5 \), a contradiction.

- Therefore, \( a_1, a_2 \in B \). Let \( w' \) be a neighbor of \( v_1 \) in \( H_1 \). We have that \( w' \in A \cup A' \). Suppose that \( w' \neq v \). Then \( w' f_2 f_1 f_0 v a_1 \cup T_{a_1,w'} \) is a cycle of length \( 2h + 5 \). So \( G \) contains cycles of lengths \( 2h + 1, 2h + 2, 2h + 3, 2h + 4 \) and \( 2h + 5 \), a contradiction. Therefore \( w' = v \). That says, every vertex in \( H_2 \) of distance 2 from a neighbor of \( v \) is a neighbor of \( v \). Continuing to apply this along with a path from \( f_0 \) to an odd cycle \( C_4 \) in \( H_2 \), we could obtain that \( v \) is adjacent to all vertices of \( C_4 \), which contradicts that \( G \) is \( K_3\)-free.

This completes the proof of Lemma \ref{lemma5.6}.

Now, we define a coloring \( c : V(G) \rightarrow \{1, 2, 3, 4, 5\} \) as following. Let \( D \) be any bipartite component of \( G[L_i] \) for some \( i \). If \( i \) is even, we color one part of \( D \) with color 1 and the other part with color 2, and if \( i \) is odd, we color one part of \( D \) with color 4 and the other part with color 5. Let \( F \) be any non-bipartite component of \( G[L_j] \) for some \( j \). If \( j \) is even, by using the block structure of \( F \), we can properly color \( V(F) \) with colors 1, 2 and 3 by coloring bad vertices with colors 1, 2 and 3 and coloring good vertices with colors 1 and 2. If \( j \) is odd, then we also can properly color \( V(F) \) with colors 3, 4 and 5 by coloring bad vertices with colors 3, 4 and 5 and coloring good vertices with colors 4 and 5.

Next, we argue that \( c \) is a proper coloring on \( G \). Let \( H_1 \) be a component of \( G[L_i] \) and \( H_2 \) be a component of \( G[L_{i+1}] \) for \( i \geq 0 \) such that there exists an edge between \( H_1 \) and \( H_2 \). If one of them is bipartite, then \( c \) is proper on \( V(H_1) \cup V(H_2) \). Therefore, both \( H_1 \) and \( H_2 \) are non-bipartite. By the above claim, all vertices of \( H_2 \) are not adjacent to vertices of color 3 in \( H_1 \). It follows that \( c \) is proper on \( V(H_1) \cup V(H_2) \). Therefore, \( c \) is a proper 5-coloring of \( G \), which contradicts that \( G \) is 6-critical. This completes the proof of Theorem \ref{thm5.3}.

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