Error mitigation in quantum metrology via zero noise extrapolation

Zhuo, Zhao and Kok Chuan, Tan

School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore 637371, Republic of Singapore

We consider Zero Noise Extrapolation (ZNE) as an error mitigation strategy in quantum metrology. It is shown that noise expansion can be systematically performed over sufficiently short time scales for general Markovian noise models described by the time homogeneous Lindblad master equation. This suggests that ZNE can be an effective, resource efficient error mitigation alternative when strategies employing full quantum error correcting codes are unavailable. The ZNE method is then applied quantum phase estimation in a Mach-Zehnder interferometer subject to photon losses. Numerical simulations show a significant recovery of measurement sensitivity by employing first order ZNE corrections, which can be further improved upon using higher order corrections at the cost of additional measurements.

I. INTRODUCTION

Quantum metrology seeks to exploit nonclassical properties of quantum systems to enhance measurement precision [1–3]. Examples of such nonclassical properties include entanglement[4, 5], quantum coherence[6, 7], and negative quasiprobabilities[8, 9]. However, unavoidable coupling between the system and environment can cause the quantum system to decohere[10, 11], which diminishes the actual quantum enhancement that is accessible to the experimenter[12–14]. Quantum systems may be partially shielded from detrimental environmental effects via methods such as cooling[15, 16], cancellation of field noise[17] or mitigation of mechanical vibrations[18, 19]. The applicability as well as the cost of such implementations typically depend on the system being considered.

Quantum error correction has also emerged as a powerful method of suppressing environmental noise in quantum metrology[20–25]. It was recently shown that given full quantum control of the system, access to noiseless ancillary qubits, and in the limit of infinitely many quantum gates and measurements, Heisenberg scaling can be retrieved for large classes of environmental noise[24]. However, full quantum error correcting codes are currently prohibitively expensive and unfeasible to execute for many systems, which limits the applicability of this approach in the near term. In particular, it is challenging to achieve full quantum control of individual photons in quantum optical systems due to the weakly interacting nature of light. Such quantum light sources are widely used in interferometry[26–29].

In order to keep the cost of noise suppression low, we explore the use of alternative error mitigating approaches which avoids the assumption of full quantum control. In this paper, we propose to use Zero Noise Extrapolation (ZNE) as a strategy to mitigate noise in quantum metrological devices. ZNE has emerged as a promising new technique to mitigate quantum noise[30–33] in near term quantum computers, otherwise called Noisy Intermediate Scale Quantum (NISQ) devices[35]. Instead of suppressing environmental noise, ZNE achieves error mitigation by expanding the noise at the level of quantum gates in a controlled manner. Measurement statistics are then extrapolated to the zero noise level thus mitigating the effects of a noisy environment at the cost of performing additional measurements. In this paper, we will exploit this in quantum metrology, in particular demonstrating its potential use in optical phase estimation. We show that ZNE results in a significant enhancement of the measurement sensitivity as compared to a non-error mitigated state, and that higher-order ZNE corrections is able to retrieve nearly noiseless measurement sensitivity.

II. ZNE FOR MARKOVIAN NOISE MODELS

In line with the assumptions of previous error correcting approaches[23–24], we will assume that the noise is Markovian and can be described by the time homogeneous Lindblad master equation[36, 37]. In its diagonal form, the Lindblad master equation can be written as:

$$\frac{\partial \rho}{\partial t} = -i[H, \rho] + \sum_k \left( L_k \rho L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho \right).$$

The operators $L_k$ can take any form and are called the Lindblad jump operators. From the master equation, we see that the dynamics can be split into two distinct parts: the Hamiltonian contribution from $H$, which generates the unitary “noiseless” dynamics, and contribution due to jump operators $L_k$, which describe the nonunitary, dissipative portion of the dynamics. Without any loss in generality, we can assume that $L_k$ is traceless.

A typical scenario in quantum parameter estimation is to consider a signal Hamiltonian of the form $H := \theta G$, where $G$ is a known Hermitian operator describing the interaction, and $\theta$ is some real but unknown parameter describing the strength of the interaction. The goal of parameter estimation is to estimate the value of $\theta$. This is achieved by allowing the system to evolve under this Hamiltonian for a fixed time $\Delta t$, and then extracting an estimate of $\theta$ by performing a measurement on the time evolved state. In this case, the parameter $\theta$ is encoded only by the unitary portion of the dynamics. The goal in noisy quantum parameter estimation is to mitigate or eliminate the contribution due to the dissipative dynamics, i.e. the contributions due to $L_k$.

Suppose that the jump operators $L_k = L_k(\phi)$ depends on some nonnegative parameter $\phi$ that $L_k(\phi = 0) = \mathbb{1}$. Note the
The dissipative contribution then becomes

\[ L_k(\phi)\rho L_k^\dagger(\phi) - \frac{1}{2}\left[L_k^\dagger(\phi)L_k(\phi),\rho\right] \]

for any real number \( R \).

At \( \phi = 0 \) the dynamic is completely unitary and noiseless. We therefore interpret \( \phi \) as some parameter quantifying the strength of the dissipation. Performing a Taylor series expansion about \( \phi = 0 \) for each \( k \), we get

\[ L_k(\phi) = 1 + L_k'(0)\phi + L_k''(0)\frac{\phi^2}{2} + O(\phi^3). \]

The dissipative contribution then becomes

\[ L_k(\phi)\rho L_k^\dagger(\phi) - \frac{1}{2}\left[L_k^\dagger(\phi)L_k(\phi),\rho\right] \]

\[ = \sum_{j=1}^{\infty} R^{(j)}_k(\phi) + O(\phi^2) \]

where \( R^{(j)}_k \) is the jth order coefficient in the expansion. For instance, the first two orders are given by the expressions:

\[ R^{(1)}_k = L'_k(0)\rho + \rho L'_{k^\dagger}(0) - \frac{1}{2}\left\{L''_k(0)\rho,\frac{\phi^2}{2}\right\} \]

\[ R^{(2)}_k = L'_k(0)\rho L'_{k^\dagger}(0) + \left[L''_k(0)\rho,\frac{\phi^2}{2}\right]. \]

We note that if \( L_k(\phi) \) is Hermitian, then first order expansion is identically equals to 0, since

\[ [L'_k(0),\rho] + [\rho,L'_{k^\dagger}(0)] = 0. \]

In this case, the first nonzero coefficient is the second order \( R^{(2)}_k \). In general, the lowest nonzero coefficient determines how ZNE is will be performed. Otherwise, as we shall see, the exact expression for \( R^{(j)}_k \) is not necessary for ZNE.

If the system evolves over some finite time \( \Delta t \), to the first order in time, the evolution is described by

\[ \rho(t = \Delta t | H, \phi) = \rho + \frac{\partial \rho}{\partial t} \Delta t \]

\[ = \rho - i[H,\rho]\Delta t + \left(\sum_k R^{(1)}_k\Delta t\right)\phi \]

\[ + \left(\sum_k R^{(2)}_k\Delta t\right)\phi^2 + O(\phi^3). \]

Let \( O \) be some measurement observable. Observe that

\[ \text{Tr}[\rho(t = \Delta t | H, \phi)O] \]

\[ = A + B\phi + C\phi^2 + O(\phi^3), \]

where \( A := \text{Tr}([\rho - i[H,\rho]\Delta t]O) \), \( B := \text{Tr}([\sum_k R^{(1)}_k\Delta t]O) \), and \( C := \text{Tr}([\sum_k R^{(2)}_k\Delta t]O) \). To the first order, we see that the relationship with \( \phi \) is linear, and that the y-intercept \( A \) is the desired noiseless quantity.

Suppose we are able to "expand" the noise such that \( \phi_0 \rightarrow \phi_0 + \Delta \phi \). The y-intercept can be obtained by evaluating \( \text{Tr}[\rho(t = \Delta t | H, \phi)O] \) at the points \( \phi = \phi_0 \) and \( \phi = \phi_0 + \Delta \phi \) and performing a linear regression. Higher order ZNE corrections are also possible by performing more measurements and then finding a polynomial fit.

### III. Noise Expansion

From the previous section, we see that ZNE is contingent on the ability to perform controlled noise expansion. We consider how this may be done systematically while keeping in mind the metrological setting.

Recall the master equation from Eq. 2. The dynamics consist of two parts: the unitary dynamics generated by the Hamiltonian \( H \), and the dissipative dynamics generated by the jump operators \( L_k \).

Suppose we prepare a probe state \( \rho_0 \). As part of the state preparation, we can also subject the probe to environmental noise sans the unitary interaction for a fixed time \( \Delta t \). In this case, the master equation reads

\[ \frac{\partial \rho}{\partial t} = \sum_k \left(L_k\rho L_k^\dagger - \frac{1}{2}\left[L_k^\dagger L_k,\rho\right]\right) \]

\[ := \mathcal{L}(\rho). \]

In general, the Lindbladian \( \mathcal{L} \) generates a completely positive, trace preserving (CPTP) map \( \Phi_{\Delta t}(\rho) \). Furthermore, it is known that due to the Markovian nature of the dissipation, this CPTP map satisfies the semigroup property \( \Phi_{\Delta t} \circ \Phi_{\Delta t}(\rho) = \Phi_{2\Delta t}(\rho) \). We will exploit the semigroup property in order to perform noise expansion.

We recall the first order time expansion in Eq. 9 which includes the interaction with the noiseless unitary. Using the above definition, we can rewrite it as

\[ \rho(t = \Delta t) \]

\[ \approx \rho + \mathcal{L}(\rho)\Delta t - i[H,\rho]\Delta t \]

\[ \approx \Phi_{\Delta t}(\rho) - i[H,\rho]\Delta t. \]

Using \( \Phi_{\Delta t}(\rho) \) as the input state in the expression above, we perform the substitution \( \rho(t = 0) = \Phi_{\Delta t}(\rho_0) \) to get

\[ \rho(t = \Delta t) \]

\[ \approx \Phi_{\Delta t} \circ \Phi_{\Delta t}(\rho_0) - i[H,\Phi_{\Delta t}(\rho_0)]\Delta t \]

\[ = \Phi_{\Delta t}(\rho_0) - i[H,\Phi_{\Delta t}(\rho_0)]\Delta t \]

\[ = \rho_0 - i[H,\rho_0]\Delta t + \sum_{j=1}^{\infty} R^{(j)}_k(\Delta t + \Delta t)^j \]

\[ + \Delta t O(\Delta t) + O((\Delta t + \Delta t)^{n+1}). \]
Note the similarity between Eq.\[22\] in terms of powers of \((\Delta t + \Delta \tau)\) and the corresponding expression in Eq.\[13\] in terms of powers of \(\phi\). For sufficiently small \(\Delta t\) and \(\Delta \tau\), the higher order contributions are negligible, so noise expansion can be performed by changing \(\Delta \tau\) during the state preparation phase. ZNE can then be performed by extrapolating the data to \(\Delta t + \Delta \tau = 0\).

IV. EXAMPLE: MITIGATING PHOTON LOSS ERROR IN OPTICAL INTERFEROMETRY

To illustrate the ZNE procedure, we consider the Mach-Zehnder interferometer (MZI) and estimate the relative phase shift \(\theta\) when the system is subject to photon loss (see Fig. 1). In order to model the photon loss, the Lindblad jump operator is chosen to be the photon annihilation operator \(a_k\), where \(k = 1, 2\) denotes the upper and the lower paths respectively.

\[
\frac{d\rho}{dt} = -i\theta[G, \rho] + \mathcal{L}(\rho)
\]

\(\mathcal{L}(\rho)\) describes photon loss. Light emerging from the two output ports is measured to estimate \(\theta\).

The Hermitian operator \(G\) which generates the noiseless signal and the dissipation \(\mathcal{L}\) in the MZI are described as follows \[26\]:

\[
G = \frac{a_1^2 a_2 - a_2^* a_1^*}{2}
\]

\[
\mathcal{L}(\rho) = \sum_{k=1,2} \gamma(a_k^* a_k - \frac{1}{2} \{a_k^* a_k, \rho\}).
\]

The operator \(G\) generates the noiseless unitary evolution \(U_\theta := \exp(-i\theta G)\), and the goal is to estimate the value of \(\theta\) by performing some measurement \(M\) on the output ports of the MZI, which gives the expectation value \(\langle M \rangle = \text{Tr}(M \rho_{out})\).

This is characterized by the error propagation formula

\[
(\Delta \theta)^2 = (\Delta M)^2 \left| \frac{\partial \langle M \rangle}{\partial \theta} \right|^2.
\]

The uncertainty of the estimate of \(\theta\), and therefore the sensitivity of the measurement, is inversely proportional to the slope of \(\langle M \rangle\).

Suppose the input state is a classical light source, i.e. the coherent state \(|\psi_{in}\rangle = |\alpha\rangle |0\rangle\) at the input ports. It is well known that the maximum measurement sensitivity is achieved by the visibility measurement \(M_{\mu} := a_1^* a_1 - a_2^* a_2\) at the output ports of the MZI. This essentially achieves the shot noise limit, which the maximum sensitivity for classical light sources.

This can be improved upon if the input states are non-classical. It is well known that the N00N state, \(|\psi_{in}\rangle = (|N\rangle |0\rangle + |0\rangle |N\rangle)/\sqrt{2}\) achieves this. The optimal measurement for the N00N state is \[38\]:

\[
M_{\text{N00N}} = |N\rangle \langle 0| + |0\rangle \langle N|.
\]

This choice of measurement saturates the quantum Fisher information \[39\] \[40\]. By using nonclassical input states one is able to beat the classical shot noise limit \[27\] \[41\] \[42\]. Under ideal conditions, the N00N state is \(N\) times more sensitive than a classical state with the same expected photon number.

The N00N state is especially sensitive to photon losses since a single lost photon will destroy the entanglement of the state, which diminishes the sensitivity back to below the shot noise limit (see Fig. 2). We will apply ZNE to the N00N state under lossy conditions in order to extract better measurement sensitivity.

Fig. 2 illustrates the effects of photon loss on the N00N states. Expectation values \(\langle M \rangle\) for both the coherent as well as the N00N states are shown. The input states are the N00N states \((|N\rangle + e^{i\mu} |0\rangle)/\sqrt{2}\), as well as the rotated coherent states \(\exp(-i\mu G) |\alpha\rangle |0\rangle\) where \(\mu \in [0, 2\pi]\). We see that for the N00N state, photon losses significantly diminishes amplitude of the signal and the slope of \(\langle M \rangle\). This has the effect of reducing the measurement sensitivity. Applying a first order ZNE correction is able to partially recovers some of this loss in sensitivity.

![FIG. 1: Schematic of a Mach-Zehnder Interferometer. The relative phase shift between the upper and lower paths is \(\theta\). \(\mathcal{L}(\rho)\) describes photon loss. Light emerging from the two output ports is measured to estimate \(\theta\).](image)

![FIG. 2: Comparison of normalized expectation values \(\langle M \rangle\) for rotated coherent states, \(\exp(-i\mu G) |\alpha\rangle |0\rangle\), and N00N states, \((|N\rangle + e^{i\mu} |0\rangle)/\sqrt{2}\), where \(\mu \in [0, 2\pi]\). The amplitude of the lossless signals are normalized to 1.](image)
correction, we perform a polynomial fit with \( n + 1 \) measurement points to perform extrapolation to zero noise. The noise expansion is performed in steps of \( \Delta \tau \) and the states are measured at the points \( \Delta t + m\Delta \tau \) where \( m = 1, 2, \ldots, n \). Fig. 3 illustrates the enhancement due to higher order ZNE corrections. The relevant parameters are in this case \( N = 3, \theta = 0.2, \gamma = 0.5, \Delta t = 1, \) and \( \Delta \tau = 0.01 \).

FIG. 3: The normalized expectation values \( \langle M \rangle \) for various orders of ZNE corrections. Values are normalized such that the lossless signal has amplitude 1.

From Fig. 3, it is observed that higher order corrections lead to higher measurement sensitivity, and the 5th order correction essentially recovers the noiseless signal to within 1%. The enhancement in sensitivity for various orders of correction is summarized in Table I. The first order ZNE correction, which requires two measurement points to perform the extrapolation, improves the sensitivity by a factor of 2.49 while the fifth order ZNE correction improves it by a factor 4.46. In comparison, repeating the same measurements \( \nu \) times will increase the sensitivity by a factor \( \sqrt{\nu} \) at best. This can be seen from the Cramér-Rao bound, where the number of measurements performed and \( I_F (\rho) \) is the quantum Fisher information quantity. We observe that the improvement from both first and fifth order ZNE corrections exceed the thresholds expected from repeating the same measurements two and six times respectively.

Finally, the effects of ZNE over longer interaction times \( \Delta t \) is simulated in Fig. 4. The parameters chosen are in this case \( N = 2, \mu = 0, \theta = 0.2, \gamma = 0.5, \) and \( \Delta t = 0.01 \Delta \tau \). We see that over sufficiently short interaction times \( \Delta \tau \), all orders of ZNE corrections essentially retrieve the noiseless signal, as was argued in Eq. 22. Over longer interaction times, the noiseless signal is only partially recovered, with higher order ZNE corrections recovering greater amounts of the noiseless signal over longer time scales.

| Order, \( n \) | Relative Sensitivity | Enhancement Ratio |
|---------------|---------------------|------------------|
| 0             | 0.223               | 1                |
| 1             | 0.554               | 2.49             |
| 2             | 0.803               | 3.61             |
| 3             | 0.929               | 4.17             |
| 4             | 0.978               | 4.39             |
| 5             | 0.993               | 4.46             |

TABLE I: Table of measurement sensitivities relative to the noiseless signal for various orders of correction, \( n \). The enhancement ratio is the ratio between the sensitivities of the \( n \)th order and the zeroth order ZNE corrections, where \( n = 0 \) implies no ZNE corrections are applied.

FIG. 4: Comparison of normalized expectation values \( \langle M \rangle \) for the N00N state where \( N = 2 \) and \( \mu = 0 \) over a range of interaction times \( \Delta \tau \in [0, \pi] \). Higher order corrections preserves the signal over longer periods of time.

V. CONCLUSION

We studied an implementation of ZNE to mitigate the effects of environmental noise and to improve the sensitivity of quantum measurement devices. In order to implement ZNE, there needs to be a systematic method of performing noise expansion. For Markovian noise models, we show that this is possible when the interaction time with the environment is sufficiently short. Interestingly, the types of noise where ZNE applies is not restrictive, with the only requirement being that the noise is time homogeneous. In comparison, other techniques that employ full quantum error correction typically require that the signal is not parallel to the noise. For instance in Ref. [24], the basic underlying assumption is that the generator \( G \) is not an element of the Lindblad span, i.e. \( G \) cannot be written as a linear combination of the Lindblad jump operators \( L_k \), \( L^\dagger_k \), and \( L_i \) for all \( i, j \). In the case of ZNE, we observe that because the noise expansion only impacts the dissipative part of the evolution, errors can be mitigated regardless of the nature of the generator \( G \) or jump operators \( L_k \). In addition, ZNE does not require interaction with ancillas, which makes
it suitable for weakly interacting systems such as optical systems, nor does it require other strong assumptions such as full quantum control, access of noiseless ancillae or infinitely fast quantum gates, which are typical of methods employing quantum error correcting codes. This suggests that ZNE can be a cost efficient alternative in situations where previous proposals are not feasible.

As an example, we demonstrated the use of ZNE in noisy interferometry. As the probe is in this case a nonclassical light source, it is challenging to implement error correcting codes within the interferometer. We numerically simulated a N00N state in a MZI subject to photon losses. It is observed that within the interferometer, we numerically simulated a N00N source, it is challenging to implement error correcting codes.

Quantum gates, which are typical of methods employing quantum control, access of noiseless ancillae or infinitely fast systems, nor does it require other strong assumptions such as full quantum control, access of noiseless ancillae or infinitely fast quantum gates, which are typical of methods employing quantum error correcting codes. This suggests that ZNE can be a cost efficient alternative in situations where previous proposals are not feasible.

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