Characterization of Wave Fronts of Ultradistributions Using Directional Short-Time Fourier Transform

Sanja Atanasova 1,*, Snježana Maksimović 2 and Stevan Pilipović 3

1 Faculty of Electrical Engineering and Information Technologies, Ss. Cyril and Methodius University, Rugjer Boshkovik 18, 1000 Skopje, North Macedonia
2 Faculty of Architecture, Civil Engineering and Geodesy, University of Banja Luka, Stepe Stepanovića 77/3, 78000 Banja Luka, Bosnia and Herzegovina; snjezana.maksimovic@aggf.unibl.org
3 Faculty of Sciences and Mathematics, University of Novi Sad, Trg Obradovića 4, 21000 Novi Sad, Serbia; pilipovic@dmi.uns.ac.rs
* Correspondence: ksanja@feit.ukim.edu.mk

Abstract: In this paper we give a characterization of Sobolev $k$-directional wave front of order $p \in [1, \infty)$ of tempered ultradistributions via the directional short-time Fourier transform.

Keywords: $k$-directional short-time Fourier transform; ultradistributions; wave front sets

MSC: 46F05; 35A18

1. Introduction

The theory of directional sensitive kind of the short-time Fourier transform, in short STFT, was initially introduced and investigated in [1,2] as a blend of Radon transform and time–frequency analysis. It allows to gain information in time and frequency of a function along a certain direction or hyperplane. Following the concept of [1], in [3], the directional STFT was extended to the space of tempered distributions. Moreover, in [4], the $k$-directional short-time Fourier transform, in short $k$-DSTFT, was introduced and the results of [5] were extended to the spaces of tempered ultradistributions of Roumieu class.

Starting from [6], wave fronts have shown to be useful concepts when analyzing the propagation of different type of singularities in the theory of partial differential equations, which led to introducing various wave front sets [7–10].

Following the recent trend on studying integral transforms on the spaces of ultradistribution [11,12], authors in [4] introduce the $k$-directional regular sets to analyze the regularity properties of a tempered ultradistribution of Roumieu class. Furthermore, the wave front set using the $k$-DSTFT ($k$-directional wave front) of a tempered ultradistribution of Roumieu class and the partial wave front in terms of [6] are considered, and it is shown that this partial wave front is equivalent to the $k$-directional wave front.

This paper is a continuation of our work presented in [4] for both Beurling and Roumieu cases. The main result is established in Theorem 2 where we give characterization of the Sobolev wave front of order $p \in [1, \infty)$ via the $k$-DSTFT of tempered ultradistributions. We also consider partial wave fronts in terms of [6,13], and it is shown that these notions are equivalent with the $k$-directional Sobolev wave front.

The main novelty of this work is the proof of Theorem 2 where we follow the idea already proved in [4] but here with another decomposition and estimates of involved integrals. On the basis of this proof we introduce a new kind of wave front and make necessary analysis of it through our main theorem, Theorem 3.
1.1. Notation

For a given multi-index $l = (l_1, \ldots, l_n) \in \mathbb{N}_0^n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we denote $x^l = x_1^{l_1} \cdots x_n^{l_n}$ and $(-i)^{|l|}D^l = \frac{\partial^{|l|}}{\partial x_1^{l_1} \cdots \partial x_n^{l_n}}$, $|l| = l_1 + \cdots + l_n$. Points in $\mathbb{R}^k$ are denoted by $\hat{x} = (x_1, \ldots, x_k)$. The notation $\Omega \subseteq \mathbb{R}^n$ is used for an open set and $K \subset \subset \Omega$ for a compact set $K$ which is contained in $\Omega$. By $\mathcal{F}(f)(x) = \hat{f}(x) = \int_{\mathbb{R}^n} f(t) e^{-2\pi i x t} dt$, $x \in \mathbb{R}^n$, we denote the Fourier transform of a function $f$. The inner product of $f$ and $g$ in $L^2$ is denoted by $\langle f, g \rangle$ and $(f, g)$ means a dual paring. Thus, $\langle f, g \rangle = \langle f, \tilde{g} \rangle$. We also use the notation $\Gamma_\xi$ for a cone neighborhood of $\xi$, $L_\xi(\xi)$ and $B_\xi(\xi)$ for an open and a closed ball with a center $\xi$ and radius $r > 0$, respectively.

1.2. Ultradistribution Spaces

Let $(M_l)_{l \in \mathbb{N}_0}, M_0 = 1$ be a sequence of positive numbers which monotonically increases to infinity and satisfies the following:

(M.1) $M_l^2 \leq M_{l-1}M_{l+1}, l \in \mathbb{N}_0$

(M.2) There exist constants $A, H > 1$ such that

$$M_l \leq AH^l \min_{0 \leq q \leq l} M_qM_{l-q}, l, q \in \mathbb{N}_0;$$

(M.3) There exists a constant $A$ such that $\sum_{l=q+1}^{\infty} M_{l-1}/M_l < Aq M_{q+1}/M_q, q \in \mathbb{N}_0$.

Sometimes we can replace properties (M.2) and (M.3) by the following weaker conditions:

(M.2') There exist constants $A, H > 1$ such that

$$M_{l+1} \leq AH^l M_l, l \in \mathbb{N}_0;$$

(M.3')

$$\sum_{l=1}^{\infty} M_{l-1}/M_l < \infty.$$

We will measure the decay properties of elements of Gelfand–Shilov spaces with respect to the Gevrey sequences $M_l = l^{\alpha}$, $\alpha > 1$.

Let $a > 0$. Following [14], we recall the definitions of some spaces of test functions:

$$\mathcal{E}^a_\alpha(K) := \{ \varphi \in C^\infty(\Omega) : \sup_{l \in K, \varphi \in \mathbb{N}_0^l} \frac{|\partial^{|l|}\varphi(t)|}{|l|} < \infty \};$$

$$\mathcal{D}^a_\alpha(K) := \mathcal{E}^a_\alpha(K) \cap \{ \varphi \in C^\infty(\Omega) : \text{supp } \varphi \subset K \};$$

$$\mathcal{E}^{(a)}(K) := \lim_{a \to \infty} \mathcal{E}^a_\alpha(K); \quad \mathcal{E}^{(a)}(\Omega) := \lim_{K \subset \subset \Omega} \mathcal{E}^{(a)}(K);$$

$$\mathcal{D}^{(a)}(K) := \lim_{a \to \infty} \mathcal{D}^a_\alpha(K); \quad \mathcal{D}^{(a)}(\Omega) := \lim_{K \subset \subset \Omega} \mathcal{D}^{(a)}(K).$$

The elements of the space $\mathcal{D}^{(a)}(\Omega)$ (resp. $\mathcal{D}^{(a)}(\Omega)$) are called ultradifferentiable functions with compact support of Beurling class (resp. Roumieu class). Their strong duals are spaces of ultradistributions $\mathcal{D}'^{(a)}(\Omega)$ (resp. $\mathcal{D}'^{(a)}(\Omega)$), $\mathcal{E}'^{(a)}(\Omega)$ (resp. $\mathcal{E}'^{(a)}(\Omega)$) is a subspace of $\mathcal{D}'^{(a)}(\Omega)$ (resp. $\mathcal{D}'^{(a)}(\Omega)$) that consists of all compactly supported ultradistributions.

Following [11], we introduce the test spaces for spaces of Beurling and Roumieu tempered ultradistributions as a special case of ultradistributional spaces.
Let \( \alpha, \beta > 0 \), and \( \alpha + \beta > 1 \) (resp. \( \alpha + \beta \geq 1 \)). If \( \alpha + \beta = 1 \), then we presume also \( \alpha, \beta \neq 0 \).

Let \( \alpha > 0 \). We denoted by \((S_{a,\beta}^\alpha(\mathbb{R}^n))\) the Banach space of all smooth functions \( \varphi \) on \( \mathbb{R}^n \) for which
\[
\sigma_{a,\beta}^\alpha(\varphi) = \sup_{t \in \mathbb{R}^n, \delta \in \mathbb{N}_0^n} a_{\delta} |t|^\delta |D^\varphi(t)| < \infty.
\]
(1)

The space \( \Sigma_{a,\beta}^\alpha(\mathbb{R}^n) \) (resp. \( \Sigma_{a,\beta}^\beta(\mathbb{R}^n) \)) is defined as a projective (resp. an inductive) limit of the space \((S_{a,\beta}^\alpha(\mathbb{R}^n))\):
\[
\Sigma_{a,\beta}^\alpha(\mathbb{R}^n) = \lim_{\alpha \to 0} (S_{a,\beta}^\alpha(\mathbb{R}^n)) \quad \text{ (resp. } \Sigma_{a,\beta}^\beta(\mathbb{R}^n) = \lim_{\beta \to 0} (S_{a,\beta}^\beta(\mathbb{R}^n))\),
\]
and its strong dual \( \Sigma_{a,\beta}^\alpha(\mathbb{R}^n) \) (resp. \( \Sigma_{a,\beta}^\beta(\mathbb{R}^n) \)) is called the space of ultradistributions of Beurling type (resp. Roumieu type). These spaces \((a + \beta > 1, \text{ resp. } a + \beta \geq 1)\) are closed under translation, dilation, multiplication, differentiation, and under the action of specified infinite order differential operators (see Section 1.2.1). The Roumieu type spaces are the well-known spaces of Gelfand–Shilov.

When \( \alpha = \beta \), we use \( \mathcal{S}^{(a)}(\mathbb{R}^n) \) (resp. \( \mathcal{S}^{(a)}(\mathbb{R}^n) \)) instead of \( \Sigma_{a,\beta}^\alpha(\mathbb{R}^n) \) (resp. \( \Sigma_{a,\beta}^\beta(\mathbb{R}^n) \)).

### 1.2.1. Ultradifferential Operators

It is said that \( P(\xi) = \sum_{l \in \mathbb{N}_0^n} a_l \xi^l \), \( \xi \in \mathbb{R}^n \), is an ultrapolynomial of Beurling class (of Roumieu class), if the coefficients \( a_l \) satisfy:
\[
(\exists a > 0, \exists C_a > 0) \quad (\text{resp. } \forall a > 0, \exists C_a > 0) \quad (\forall l \in \mathbb{N}_0^n) \quad |a_l| \leq C_a |l| / M_l.
\]

The corresponding operator \( P(D) = \sum_{l \in \mathbb{N}_0^n} a_l D^l \) is an ultradifferential operator of Beurling class (resp. Roumieu class). When \( M_l = l! a_l \) it is called ultradifferential operator of class \( (\alpha) \) (resp. class \( \{a\} \)). As \( M_l \) satisfies \((M.2)\), they act continuously on \( \mathcal{E}^{(a)} \) and \( \mathcal{D}^{(a)} \) (resp. \( \mathcal{E}^{(a)} \) and \( \mathcal{D}^{(a)} \)), and the corresponding spaces of ultradistributions.

The following representation theorem holds \([11]\):

For any \( f \in \Sigma_{a,\beta}^\alpha(\mathbb{R}^n) \) (resp. \( f \in \Sigma_{a,\beta}^\beta(\mathbb{R}^n) \)) there exist \( P_1(D) \)-ultradifferential operator of class \( (\alpha) \) (resp. class \( \{a\} \)), an ultrapolynomial \( P_2(\xi) \) of class \( (\beta) \) (resp. class \( \{\beta\} \)) and an \( F \in L^2(\mathbb{R}^n) \) such that
\[
f(\xi) = P_1(D)(P_2(\xi) F(\xi)).
\]
(2)

We will deal only with elliptic operators for which the function \( P(\xi) \) satisfies \([14]\) (Proposition 4.5): there exist \( a > 0 \) and \( C_a > 0 \) (resp. for every \( a > 0 \) there exists \( C_a > 0 \)) such that
\[
C_a^{-\frac{1}{1 - a}} |\xi|^{1 - a} \leq |P(\xi)| \leq C_a |\xi|^{1 - a}, \quad \forall \xi \in \mathbb{R}^n.
\]
(3)

In the quasi-analytic case (when \((M.3)'\) does not hold) we have \([4]\): Let \( r \geq 1 \) there is \( C > 0 \) such that for all \( \xi \in \mathbb{R}^n \), \( l \in \mathbb{N}_0^n \)
\[
|D_\xi^l \frac{1}{P(\xi)}| \leq C |\xi|^l / |P(\xi)|^r.
\]
(4)

### 1.3. The \( k \)-DSTFT and the \( k \)-Directional Synthesis Operator

We recall some definitions and assertions from \([4]\), where only the Roumieu case was considered. Here we state also the Beurling case, since the results of \([4]\) also hold for the Beurling-type spaces.
Let \( u^k = (u_1, \ldots, u_k) \), where \( u_i, i = 1, \ldots, k \), are independent vectors of \( S^{n-1} \). Let \( \tilde{g} = (y_1, \ldots, y_k) \in \mathbb{R}^k \) and \( g \in \Sigma^a_B(\mathbb{R}^k) \setminus \{0\} \) (resp. \( g \in S_B^a(\mathbb{R}^k) \setminus \{0\} \)). The \( k \)-directional short-time Fourier transform of \( f \in L^2(\mathbb{R}^n) \) is defined by [4]

\[
DS_{\tilde{g}, u^k} f(\tilde{y}, \xi) = \int_{\mathbb{R}^n} f(t) \tilde{g}(t) g^*(t) dt, \quad \xi \in \mathbb{R}^n,
\]

and the \( k \)-directional synthesis operator of \( F \in L^2(\mathbb{R}^{k+n}) \) is defined by [4]

\[
DS^*_{\tilde{g}, u^k} F(t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} F(\tilde{y}, \xi) g^*(t) d\tilde{y} d\xi, \quad t \in \mathbb{R}^n,
\]

where \( g_{u^k, \tilde{g}^k}(t) = g((u_1 \cdot t, \ldots, u_k \cdot t) - (y_1, \ldots, y_k)) e^{2\pi i \tilde{y}\cdot t}, t \in \mathbb{R}^n \).

It is shown in [4, Proposition 2.4] that for \( f \in S_B^a(\mathbb{R}^n) \) the following reconstruction formula holds,

\[
f(t) = \frac{1}{(g, \varphi)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} DS_{\tilde{g}, u^k} f(\tilde{y}, \xi) \varphi_{u^k, \tilde{g}^k}(t) d\tilde{y} d\xi, \quad t \in \mathbb{R}^n,
\]

where \( \varphi \in S_B^a(\mathbb{R}^k) \) is the synthesis window for \( g \in S_B^a(\mathbb{R}^k) \setminus \{0\} \). The same holds for \( f \in S_B^a(\mathbb{R}^n) \) when \( g, \varphi \in S_B^a(\mathbb{R}^k) \). Thus, the relation (7) takes the form

\[
(DS^*_{\varphi, u^k} \circ DS_{\tilde{g}, u^k}) f = (g, \varphi) f.
\]

For the sake of simplicity we transfer the STFT in direction of \( u^k \) into the STFT in \( e^k \) direction. Recall the procedure (see [4]): Let \( A = [u_{ij}]_{k \times n} \) be a matrix with rows \( u^i, i = 1, \ldots, k \) and \( I \) be the identity matrix of order \( n - k \). Let \( B \) be an \( n \times n \) matrix determined by \( A \) and \( I \) so that \( BT = s \), where \( s_1 = u_{11} t_1 + \cdots + u_{1n} t_n \ldots, s_k = u_{k1} t_1 + \cdots + u_{kn} t_n \), \( s_{k+1} = t_{k+1}, \ldots, s_n = t_n \). The matrix \( B \) is regular, so put \( C = B^{-1} \) and \( e^k = (e_1, \ldots, e_k) \), where \( e_1, \ldots, e_k \) are unit vectors of the coordinate system of \( \mathbb{R}^k \). If we change the variables \( t = Cs \), and \( \eta = C^T \xi \), then for \( f \in L^2(\mathbb{R}^n), g \in \Sigma_B^a(\mathbb{R}^k) \) (resp. \( g \in S_B^a(\mathbb{R}^k) \)), the equality (5) is transformed into:

\[
DS_{\tilde{g}, u^k} f(\tilde{y}, \xi) = (DS_{\tilde{g}, e^k} h(s))(\tilde{y}, \eta) = \int_{\mathbb{R}^n} h(s) \tilde{g}(s - \frac{\eta}{\eta}) e^{-2\pi i \tilde{s} \cdot \eta} ds,
\]

where \( h(s) = \det(C) f(Cs) \) and (6) is transformed, for \( F \in L^2(\mathbb{R}^{k+n}), \ g \in \Sigma_B^a(\mathbb{R}^k) \) (resp. \( g \in S_B^a(\mathbb{R}^k) \)), into:

\[
DS^*_{\tilde{g}, e^k} F(s) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} F(\tilde{y}, \xi) g(s - \frac{\eta}{\eta}) e^{2\pi i \tilde{s} \cdot \eta} d\tilde{y} d\xi, \quad s \in \mathbb{R}^n.
\]

The function \( h(s) = \det(C) f(Cs) \) is in \( \Sigma_B^a(\mathbb{R}^n) \) (resp. \( S_B^a(\mathbb{R}^n) \)) if \( f \in \Sigma_B^a(\mathbb{R}^k) \) (resp. \( f \in S_B^a(\mathbb{R}^k) \)). Additionally, if \( g(s_1, \ldots, s_k) = g_1(s_1) \cdots g_k(s_k) \in (\Sigma_B^a(\mathbb{R}))^k \) (resp. \( g(s_1, \ldots, s_k) = g_1(s_1) \cdots g_k(s_k) \in (S_B^a(\mathbb{R}))^k \)), then

\[
DS_{\tilde{g}, u^k} f(\tilde{y}, \xi) := \int_{\mathbb{R}^n} f(t) \tilde{g}_1(u_1 \cdot t - y_1) \cdots \tilde{g}_k(u_k \cdot t - y_k) e^{-2\pi i \tilde{s} \cdot \tilde{t}} dt = \int_{\mathbb{R}^n} h(s) \tilde{g}_1(s_1 - y_1) \cdots \tilde{g}_k(s_k - y_k) e^{-2\pi i \tilde{s} \cdot \tilde{t}} ds,
\]

and it is referred to as the partial short-time Fourier transform.

We have \( DS_{\tilde{g}, e^k} : \Sigma_B^a(\mathbb{R}^n) \times \Sigma_B^a(\mathbb{R}^k) \rightarrow \Sigma_B^a(\mathbb{R}^{k+n}) \) (resp. \( DS_{\tilde{g}, e^k} : \Sigma_B^a(\mathbb{R}^n) \times \Sigma_B^a(\mathbb{R}^k) \rightarrow \Sigma_B^a(\mathbb{R}^{k+n}) \)) is a continuous bilinear mapping. This is proved in Theorem 2.3 in [4] in the Roumieu case. With Theorem 2.5 and Corollary 2.7 in [4] in Roumieu case, and similarly in the Beurling case, it follows that \( DS^*_{\tilde{g}, e^k} : \Sigma_B^a(\mathbb{R}^{k+n}) \times \Sigma_B^a(\mathbb{R}^k) \rightarrow \Sigma_B^a(\mathbb{R}^n) \)
Theorem 1. Let $u^k = (u_1, \ldots, u_k)$, where $u_i, i = 1, \ldots, k$ are independent vectors of $\mathbb{R}^{n-1}$. Let $\varphi_\gamma \in S(\alpha)(\mathbb{R}^k)$ (resp. $\varphi_\gamma, \gamma_1 \in S(\alpha)(\mathbb{R}^k)$) where $\gamma_1$ is the synthesis window for $g$ and $\gamma_0 \in S(\alpha)(\mathbb{R}^{n-k})$ (resp. $\gamma_0 \in S(\alpha)(\mathbb{R}^{n-k})$) so that $\int_{\mathbb{R}^{n-k}} \gamma_0(t_{n-k+1}, \ldots, t_n)dt_{n-k+1} \cdots dt_n \neq 0$. Put
\[
\gamma(t_1, \ldots, t_n) = \gamma_1(t_1, \ldots, t_k)\gamma_0(t_{n-k+1}, \ldots, t_n).
\]
Let $f \in \Sigma(\mathbb{R}^n)$ (resp. $f \in \Sigma^h(\mathbb{R}^n)$), then
\[
DS_{\varphi, u^k}f(\xi, \eta) = (DS_{\varphi, e^k}f(\tilde{s}, \zeta)) \ast (DS_{\varphi, u^k}e^{\gamma(\tilde{s}, \zeta)})(\xi, \eta),
\]
\[
\tilde{s}, \zeta \in \mathbb{R}^k, \eta, \zeta \in \mathbb{R}^n.
\]

2. The Main Results

The STFT in the direction of $u^k$ can be used in the detection of singularities determined by the hyperplanes orthogonal to vectors $u_1, \ldots, u_k$. For this purpose, we introduce $k$-directional regular sets and wave front sets for the Beurling (resp. Roumieu)-tempered ultradistributions using the STFT in the direction of $u^k$. To simplify our exposition we transfer the STFT in direction of $u^k$ into the STFT in $e^k$ direction by the use of (8).

As in [5], let $\Pi_0 := \{0\}$ and let $\Pi_0(\varphi)\nu = \Pi_0(\varphi)$ for each $\nu \geq 0$, we consider direction $e^k = (e_1, \ldots, e_k)$. Let $k = 1$ and $y_0 = y_{0,1} \in \mathbb{R}$, and let $\Pi_1(\varphi) \nu = \Pi_1(\varphi) := \{t \in \mathbb{R}^n : |t_1 - y_0| < e\}$. It is a part of $\mathbb{R}^n$ between two hyperplanes orthogonal to $e_1$, that is,
\[
\Pi_{0,e} = \bigcup_{y \in (y_0 - e, y_0 + e)} P_y, \quad (y_0 = (y_0, 0, \ldots, 0), y = (y, 0, \ldots, 0)),
\]
and $P_y$ denotes the hyperplane orthogonal to $e_1$ passing through $y$. Let
\[
\Pi_1(\varphi) \nu = \Pi_{1,y_1} \cap \ldots \cap \Pi_{k,y_k} \quad \Pi_1(\varphi) = \Pi_{1,y_1} \cap \ldots \cap \Pi_{k,y_k}.
\]
The set $\Pi_1(\varphi) \nu$ is a parallelepiped in $\mathbb{R}^k$. In $\mathbb{R}^n$ this parallelepiped is determined by $2k$ finite edges while the other edges are infinite. The set $\Pi_1(\varphi) \nu$ equals $\mathbb{R}^{n-k}$ translated by vectors $y_1, \ldots, y_k$. We call it $n - k$-dimensional element of $\mathbb{R}^n$ and it is denoted by $P_1(\varphi) \nu \in \mathbb{R}^{n-k}$. When $k = n$, we have the point $y = (y_1, \ldots, y_n)$.

Definition 1. Let $f \in S(\alpha)(\mathbb{R}^n)$ (resp. $f \in S(\alpha)(\mathbb{R}^n)$), $\alpha > 1$ and $\nu \in [1, \infty]$. It is said that $f$ is $(\alpha, k)$-directionally microlocally regular (in short, $(\alpha, \nu, k)$-d.m.r.) (resp. $(\alpha, \nu, k)$-directionally microlocally regular (in short, $(\alpha, k, \nu, k, \nu, k)$-d.m.r.)) at $P_{\varphi, e^k, y_0} = \mathbb{R}^k \times (\mathbb{R}^n \setminus \{0\})$, that is, at every point of the form $(y_0, \zeta_0)$ if there exist $g \in D(\alpha)(\mathbb{R}^k)$ (resp. $g \in D(\alpha)(\mathbb{R}^k)$), $g(\tilde{0}) \neq 0$, a product of open balls $L_{\nu}(\tilde{y}_0 \nu) = L_{\nu}(y_{0,1}) \times \ldots \times L_{\nu}(y_{0,k}) \in \mathbb{R}^k$, a cone $\Gamma_{\nu}$ and for each $N \in \mathbb{N}$ (resp. for some $N \in \mathbb{N}$) there exists $C_N > 0$ such that
\[
\sup_{\tilde{y} \in L_{\nu}(y_{0,\nu})} \left| DS_{\varphi, u^k}f(\tilde{y}, \zeta)(\tilde{y}, \zeta)||\nu ||^{1/\nu} \right|_{L^p(\Gamma_{\nu})} \leq C_N.
\]
If $k = n$, Definition 1 gives the classical Hörmander’s regularity [6].

Remark 1. (a) If $f$ is $(\alpha, k)$-d.m.r. (resp. $(\alpha, k)$-d.m.r.) at $P_{\varphi, e^k, y_0}$, then there exist an open ball $L_{\nu}(\tilde{y}_0)$ and an open cone $\Gamma \subset \Gamma_{\nu}$ so that $f$ is $(\alpha, \nu, k)$-d.m.r. (resp. $(\alpha, k)$-d.m.r.) at
This implies that the union of all \( (\alpha)-p\text{-}k\text{-}d.m.r. \) points \((P_{\varepsilon,\theta_0},\theta_0)\) for any \( \varepsilon_0 \in L_r(\tilde{g}_0) \) and \( \theta_0 \in \Gamma \) is an open set of \( \mathbb{R}^k \times (\mathbb{R}^n \setminus \{0\}) \).

(b) Denote by \( \text{Pr}_k \) the projection of \( \mathbb{R}^n \) onto \( \mathbb{R}^k \). Then, the \( (\alpha)-p\text{-}k\text{-}d.m.r. \) point \( (P_{\varepsilon,\theta_0},\theta_0) \), considered in \( \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \) with respect to the first \( k \) variables, equals \((\text{Pr}_k^{-1} \times I_2)(P_{\varepsilon,\theta_0},\theta_0)(I_2 \text{ is the identity matrix on } \mathbb{R}^n)\).

A \( p \)-Sobolev \( k \)-directional wave front of Beurling (resp. Roumieu) type is defined as the complement in \( \mathbb{R}^k \times (\mathbb{R}^n \setminus \{0\}) \) of all \((\alpha)-p\text{-}k\text{-}d.m.r. \) points \((P_{\varepsilon,\theta_0},\theta_0)\), and we denoted as \( \Sigma WF_f(\varepsilon) \) (resp. \( SWF_f(\varepsilon) \)).

2.1. Independence with Respect to a Window Function

One of the main results in [4] (Theorem 3.4) shows that the wave front set does not depend on the used window. Here, we prove the same assertion for the \( p \)-Sobolev \( k \)-directional wave fronts. The idea is similar to the one in [6,13] (see [4]) but here the decomposition of the involved integrals and the use of ultradifferential operators make the proof more complex.

**Theorem 2.** If (12) holds for some \( g \in D(\alpha) (\mathbb{R}^k) \) (resp. \( g \in D(\alpha) (\mathbb{R}^k) \)), \( \tilde{g}(\theta_0) \neq 0 \), then it holds for every \( h \in D(\alpha) (\mathbb{R}^k) \) (resp. \( h \in D(\alpha) (\mathbb{R}^k) \)), \( h(\theta) \neq 0 \) supported by a ball \( B_p(\theta) \), where \( \rho \leq \rho_0 \) and \( \rho_0 \) depends on \( r \) in (12).

**Proof.** We will focus only on the Roumieu-type spaces. The proof in the Beurling case will follow similarly. We assume that \( \varphi, g, \gamma_1 \), belong to \( S_0^0(\mathbb{R}^k) \) where \( \gamma_1 \) defined by (10) is the synthesis window for \( g \) and \( \gamma_0 \in S^{(\alpha)}(\mathbb{R}^{n-k}) \). Additionally, suppose that \( f \) is a continuous function which satisfies (3), since we can use the methods of oscillatory integral and transfer the differentiation from \( f \) on other factors in integral expressions. Using [11] (Theorem 3.2.2), we obtain that \( f = P_0(D)F \), where \( F \) is a continuous function which satisfies

\[
\forall \alpha > 0, \ \exists C_\alpha > 0, \ \forall \xi \in \mathbb{R}^n \ |F(\xi)| \leq C_\alpha e^{\alpha |\xi|^{1/\alpha}}
\]

and \( P_0(D) \) is a differential operator.

We use Theorem 1, that is, the form (11). Assume that (12) holds. The constructions of balls we repeat from [5]. The window function \( \gamma \) is chosen so that \( \text{supp } \gamma \subset B_p(0) \) and \( \rho_1 < r - r_0 \). Let \( h \in D(\alpha) (\mathbb{R}^k) \) and \( \text{supp } h \subset B_p(0) \). The aim is to find \( \rho_0 \) such that (12) holds for \( DS_{h,\varepsilon}f(\tilde{x}, \eta) \), with \( \tilde{x} \in B(\tilde{y}_0), \eta \in \Gamma_1 \subset \subset \Gamma_{\varepsilon_0}, \) for \( \rho \leq \rho_0 \) (\( \Gamma_1 \subset \subset \Gamma_{\varepsilon_0} \) implies that \( \Gamma_1 \cap S^{n-1} \) is a compact subset of \( \Gamma_{\varepsilon_0} \cap S^{n-1} \)).

We choose \( \rho_0 \) such that \( \rho_0 + \rho_1 < r - r_0 \) and

\[
\rho + \rho_1 + r_0 < r \text{ holds for } \rho \leq \rho_0.
\]

This implies that

\[
|\tilde{y} - \tilde{y}_{\varepsilon}| < r
\]

as a consequence of

\[
|\tilde{y}| \leq \rho_1, \quad |\tilde{x} - \tilde{y}_0| \leq r_0 \quad \text{and} \quad |\tilde{y} - ((\tilde{x} - \tilde{y}_0) - (\tilde{y} - \tilde{y}_0))| \leq \rho.
\]

Let \( \Gamma_1 \subset \subset \Gamma_{\varepsilon_0} \). Then, there exists \( c \in (0,1) \) such that \( \eta \in \Gamma_1, |\eta| > 1 \) and

\[
|\eta - \xi| \leq c |\eta| \Rightarrow \eta \in \Gamma_{c\xi_0}, |\eta - \xi| \leq c |\eta| \Rightarrow |\eta| \leq (1 - c)^{-1}|\xi|.
\]

(15)

Let \( \tilde{x} \in B_0(\tilde{y}_0), \eta \in \Gamma_1 \) and \( K = e^{N|\eta|^{1/\alpha} - \epsilon|\eta|^{1/\alpha}} |DS_{h,\varepsilon}f(\tilde{x}, \eta)| \). Then, by (8) and (11)

\[
\sup_{\tilde{x} \in L_0(\tilde{y}_0)} \int_{\Gamma_1} |K|^p d\eta = \sup_{\tilde{x} \in L_0(\tilde{y}_0)} \int_{\Gamma_1} e^{-\epsilon|\eta|^{1/\alpha}} e^{N|\eta|^{1/\alpha}} | \int_{\mathbb{R}^k} d\tilde{y} |
\]
We continue to estimate $I_1$

\[
I_1 \leq C \sup_{\mathcal{S} \in \mathcal{L}_n(g_0)} \int_{\Gamma_1} e^{-\epsilon p|\eta|^{1/\alpha}} \int_{\mathbb{R}^k} d \tilde{g}(\int_{\Gamma_n} e^{pN|\eta - \zeta|^{1/\alpha}} (\int_{\mathbb{R}^q} |f(t)\tilde{g}(l - \tilde{y})| dt)^p
\]

By the assumptions that $g, h, \gamma$ are with compact support, integrals over $\mathbb{R}^k$ and $\mathbb{R}^q$ are finite while the integral over $\Gamma_{\tilde{y}, \zeta}$ is finite because of the assumption (1).

Now we consider $I_2$:

\[
I_2 = \sup_{\mathcal{S} \in \mathcal{L}_n(g_0)} \int_{\Gamma_1} e^{-\epsilon p|\eta|^{1/\alpha}} e^{pN|\eta - \zeta|^{1/\alpha}} \int_{\mathbb{R}^k} d \tilde{g}(\int_{|\eta - \zeta| \leq c|\eta|} DS_{\hat{g}, \hat{e}} f(\tilde{y}, \eta - \zeta)
\]

\[
DS_{\hat{h}, \hat{e}} \gamma(\tilde{x} - \tilde{y}, \zeta) d\zeta | d\eta.
\]

Let $\Omega = \{ \xi : |\eta - \zeta| \geq c|\eta| \}$. By $k_{\alpha, 0}^0 = 0 < d < 1$, we denote the characteristic function of $\Omega_d = \bigcup_{\xi \in \Omega} L_d(\xi)$, where $\Omega_d$ is an open $d$-neighborhood of $\Omega$. Then, put

\[
k_d = k_{\alpha, 0}^0 * \rho_d,
\]

where $\rho_d = \frac{1}{d^2} \rho(\cdot / d)$, $\rho \in \mathcal{D}(\mathbb{R}^n)$ is non-negative, supported in the ball $B_1(0)$ and equals 1 on $B_1/2(0)$. By the construction we have that $k_d$ equals one on $\Omega$, it is supported in $\Omega_{2d}$, and all the derivatives of $k_d$ are bounded. We note that

\[
|\int_{\Omega} \ldots d\xi| \leq |\int_{\Omega_d} k_d(\xi) \ldots d\xi| + |\int_{\Omega_{2d}} \{ |\eta - \zeta| \leq c|\eta| \} k_d(\xi) \ldots d\xi|.
\]

Then,

\[
I_2 \leq \sup_{\mathcal{S} \in \mathcal{L}_n(g_0)} \int_{\Gamma_1} e^{-\epsilon p|\eta|^{1/\alpha}} (\int_{\mathbb{R}^k} k_d e^{N|\eta - \zeta|^{1/\alpha}} DS_{\hat{g}, \hat{e}} f(\tilde{y}, \eta - \zeta)
\]

\[
+ \sup_{\mathcal{S} \in \mathcal{L}_n(g_0)} \int_{\Gamma_1} e^{-\epsilon p|\eta|^{1/\alpha}} (\int_{\mathbb{R}^k} \int_{\Omega_{2d}} \{ |\eta - \zeta| \leq c|\eta| \} k_d e^{N|\eta - \zeta|^{1/\alpha}} DS_{\hat{g}, \hat{e}} f(\tilde{y}, \eta - \zeta)
\]

\[
DS_{\hat{h}, \hat{e}} \gamma(\tilde{x} - \tilde{y}, \zeta) d\zeta | d\eta = I_{2,1} + I_{2,2}.
\]

We first estimate $I_{2,1}$.

\[
I_{2,1} \leq C \sup_{\mathcal{S} \in \mathcal{L}_n(g_0)} \int_{\Gamma_1} e^{-\epsilon p|\eta|^{1/\alpha}} | \int_{\mathbb{R}^k} d \tilde{g}(\int_{\mathbb{R}^k} e^{N|\eta - \zeta|^{1/\alpha}} (\int_{\mathbb{R}^q} |f(t)\tilde{g}(l - \tilde{y})| dt)^p.
\]

\[
e^{pN|\xi|^{1/\alpha}} \int_{\mathbb{R}^n} P(D_q)(\gamma(q)\tilde{h}(\tilde{y} - (\tilde{x} - \tilde{y}))|.}
\]
\[
\frac{P(D_\xi)(\kappa_d(\eta - \xi)e^{-2\pi\imath q \cdot (\eta - \xi)})}{P(2\pi \cdot (\eta - \xi))} \, dq \, d\xi \, |d\eta|
\leq C \sup_{x \in \text{Loc}(\eta_0)} \int_{\Gamma_1} e^{-\varepsilon \eta \cdot \eta^1/s} \left| \int_{\mathbb{R}^d} d\gamma \left( \int_{\mathbb{R}^d} e^{P(\eta \cdot \eta^1/s)} \left( \int_{\mathbb{R}^t} \frac{f(t) \gamma(t - \gamma)}{P(2\pi t)} \, dt \right)^p \, d\xi \right) \right| \, d\eta
\]\n
is finite.

Since integration goes through a subset of \( \{ \xi : |\eta - \xi| \leq c(|\eta|) \} \), we can conclude that \( I_{2,2} \) can be estimated similarly as \( I_1 \). Additional term \( \kappa_d \) does not cause any problems since it belongs to \( C^{(\alpha)} \). \[\square\]

2.2. Equivalent Definition

In this section we characterize the wave front sets given Definition 1 with the ones formulated in the next definition. We will use the Fourier transform as well as the cut-off function, and since we will show that these definitions are equivalent but we need to distinguish them, we add the prefix “locally” in front of this notation in the Definition 2.

We follow our ideas outlined in [10] and prove that both definitions determine the same sets. For the sake of completeness, we give all the details of the proof although it is the repetition of our proof of the theorem in [10] where we have considered distributions instead of ultradistributions.

**Definition 2.** Let \( f \in S^{(\alpha)}(\mathbb{R}^n) \) (resp. \( f \in S^{(\alpha)}(\mathbb{R}^n), p \in [1, \infty) \) and \( s \in \mathbb{R} \). The point \( (\tilde{y}_0, \tilde{\xi}_0) \in \mathbb{R}^k \times (\mathbb{R}^n \setminus \{0\}) \) is locally \((\alpha)\)-\(p\)-\(k\)-analytically regular, in short locally \((\alpha)\)-\(p\)-\(k\)-m.r., (resp. locally \(\{\alpha\}\)-\(p\)-\(k\)-m.r.) for \( f \) if there exists \( \chi \in D^{(\alpha)}(\mathbb{R}^k) \) (resp. \( \chi \in D^{(\alpha)}(\mathbb{R}^k) \)) so that \( \chi(\tilde{y}_0) \neq 0 \) and a cone \( \Gamma_{\tilde{\xi}_0} \) such that

\[
\|e^{i|\xi|^{1/2}}F(\chi(\tilde{y})f(y))(\tilde{\xi})\|_{L^p(\Gamma_{\tilde{\xi}_0})} < \infty,
\]

\[
y = (\tilde{y}, y_{k+1}, \ldots, y_n) \in \mathbb{R}^k \times \mathbb{R}^{n-k}.
\]

For the complements we use the notation \( L - \Sigma \text{WF}_{\psi}(f) \) (resp. \( L - \text{SWF}_{\psi}(f) \)).

**Theorem 3.** Let \( f \in S^{(\alpha)}(\mathbb{R}^n) \) (\( f \in S^{(\alpha)}(\mathbb{R}^n) \) and \( p \in [1, \infty) \). The following conditions are equivalent.

(i) \((\tilde{y}_0, \tilde{\xi}_0) \notin L - \Sigma \text{WF}_{\psi}(f) \) (resp. \( (\tilde{y}_0, \tilde{\xi}_0) \notin L - \text{SWF}_{\psi}(f) \)).

(ii) There exist a compact neighborhood \( \tilde{K} \) of \( \tilde{y}_0 \) and a cone \( \Gamma_{\tilde{\xi}_0} \) such that for every \( s > 0 \) (resp. for some \( s > 0 \)) the mapping \( \chi \rightarrow e^{i|\xi|^{1/2}}F(\chi(\tilde{y})f)(\tilde{\xi}) \) \( D^{(\alpha)}(\tilde{K}) \rightarrow L^p(\Gamma_{\tilde{\xi}_0}) \) (resp. \( D^{(\alpha)}(\tilde{K}) \rightarrow L^p(\Gamma_{\tilde{\xi}_0}) \)) is well-defined and continuous.

(iii) There exist a compact neighborhood \( \tilde{K} \) of \( \tilde{y}_0 \) a cone \( \Gamma_{\tilde{\xi}_0} \) and \( C, a > 0 \) such that for all \( \chi \in D^{(\alpha)}(\tilde{K} - \{\tilde{y}_0\}) \) and \( s > 0 \) (resp. for all \( \chi \in D^{(\alpha)}(\tilde{K} - \{\tilde{y}_0\}) \)) and some \( s > 0 \) there holds

\[
\sup_{\tilde{y} \in \tilde{K}} \|e^{i|\xi|^{1/2}}DS_{\chi,\psi}(\tilde{y}, \xi)\|_{L^p(\Gamma_{\tilde{\xi}_0})} \leq C \sup_{t \in [0, \infty)} \|D^t\chi\|_{L^\infty(\mathbb{R}^k)}.
\]

Here, the set \( \tilde{K} - \{\tilde{y}_0\} = \{\tilde{y} \in \mathbb{R}^k : |\tilde{y} + \tilde{y}_0| \in \tilde{K}\} \).

(iv) There exist a compact neighborhood \( \tilde{K} \) of \( \tilde{y}_0 \) a cone \( \Gamma_{\tilde{\xi}_0} \), such that for all \( s > 0 \) and corresponding \( \chi \in D^{(\alpha)}(\mathbb{R}^k) \) (resp. for some \( s > 0 \) and corresponding \( \chi \in D^{(\alpha)}(\mathbb{R}^k) \) with \( \chi(0) \neq 0 \) there holds \( \sup_{\tilde{y} \in \tilde{K}} \|e^{i|\xi|^{1/2}}DS_{\chi,\psi}(\tilde{y}, \xi)\|_{L^p(\Gamma_{\tilde{\xi}_0})} < \infty \).
We take where

Now let \( \Gamma \) be a cone such that \( \Gamma_{\xi_0} \subseteq \Gamma_{\xi_0}' \cup \{0\} \). One can find \( 0 < c < 1 \) such that

We take \( \psi \in D^{|\alpha|}(K) \), then \( \mathcal{F}(\psi(\tilde{y})\chi(\tilde{y})f(y)) = \mathcal{F}(\psi(\tilde{y})) \ast \mathcal{F}(\chi(\tilde{y})f(y)) \). By the Minkowski integral inequality we have

where

Using the change of variable \( \xi - \eta \) in the inner integral in \( I_1 \) we have

For the inequality above, we have used \( \{ \xi \in \Gamma_{\xi_0} - \{ \eta \} | |\xi + \eta| \geq |\eta|/c \} \subseteq \Gamma_{\xi_0}' \), which follows from (16). For \( I_2 \) we have

\[
I_2 \leq C_1 \int_{\mathbb{R}^n_{\eta}} |\mathcal{F}(\psi(\tilde{y})\chi(\tilde{y})f(y))| \left( \int_{|c\xi - \eta| < |\xi + \eta|} e^{ps|\xi + \eta|^{1/\alpha} + pr|\xi - \eta|^{1/\alpha}} d\xi \right)^{1/p} d\eta
\leq C_1 (1 + c^{-1}) e^{-s - n - 1} \|e^{((n-1)|1/\alpha}) \|_{L^p(\mathbb{R}^n)},
\]
From the estimates of $I_1$ and $I_2$, we conclude that there exists $C_\chi > 0$ such that
\[
\|e^{i|\cdot|^{1/\alpha}}\mathcal{F}(\psi(y)\chi(y)f(y))\|_{L^p(\Gamma_{\delta_0})} \leq C_\chi \|e^{i(s+r+n+1)|\cdot|^{1/\alpha}}\mathcal{F}(\psi(y))\|_{L^1(\mathbb{R}^n)},
\]
$\forall \psi \in D^{(a)}(\mathcal{K})$. Now, the claim in (ii) can be deduced since for $\psi \in D^{(a)}(\mathcal{K})$, we have $\psi f = (\psi/\chi)f$ with $\psi/\chi \in D^{(a)}(\mathcal{K})$.

(ii) $\Rightarrow$ (iii) Let $\tilde{K}_1$ be a compact neighborhood of $\tilde{y}_0$, and we choose a cone $\Gamma_{\delta_0}$ such that the mapping
\[
\chi \mapsto e^{i|\cdot|^{1/\alpha}}\mathcal{F}(\chi(y)f(y)), \quad D^{(a)}(\tilde{K}_1) \to L^p(\Gamma_{\delta_0}),
\]
is well-defined and continuous. Under the assumption that $\tilde{K}_1 = B_{r}(\tilde{y}_0)$, for some $r > 0$, there exist $C > 0$ and $a > 0$ such that
\[
\|e^{i|y|^{1/\alpha}}\mathcal{F}(\chi(y)f(y))(\xi)\|_{L^p(\Gamma_{\delta_0})} \leq C \sup_{l \in \mathbb{N}_0} \left\| D^l \chi \right\|_{L^\infty(\tilde{K}_1)},
\]
$\forall \chi \in D^{(a)}(\tilde{K}_1)$.

Let $\tilde{K} = B_{r/2}(\tilde{y}_0)$. For $\chi \in D^{(a)}(\tilde{K} - \{\tilde{y}_0\})$ and $\tilde{y} \in \tilde{K}$, the function $\tilde{i} \mapsto \chi(\tilde{i} - \tilde{y})$ belongs to $D^{(a)}(\tilde{K}_1)$ and, as $\mathcal{F}(\chi(\cdot - \tilde{y})f(y))(\xi) = DS_{\chi} e^{i\phi(y,\xi)}$, we have
\[
\sup_{\tilde{y} \in \tilde{K}} \|e^{i|\cdot|^{1/\alpha}}DS_{\chi} e^{i\phi(y,\xi)}\|_{L^p(\Gamma_{\delta_0})} \leq C \sup_{\tilde{y} \in \tilde{K}} \sup_{l \in \mathbb{N}_0} \left\| D^l \chi(\cdot - \tilde{y}) \right\|_{L^\infty(\tilde{K}_1)} = C \sup_{l \in \mathbb{N}_0} \left\| D^l \chi \right\|_{L^\infty(\mathbb{R}^n)}.
\]
The implication (iii) $\Rightarrow$ (iv) is trivial. If we let $\tilde{y} = \tilde{y}_0$, the implication (iv) $\Rightarrow$ (i) follows immediately. □

**Author Contributions:** Individual contributions of the authors were equally distributed in writing the original draft of the manuscript, editing or revising it. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** The data is available from the authors, upon reasonable request.

**Acknowledgments:** This paper was supported by the project “Time-frequency methods”, No. 174024 financed by the Ministry of Science, Republic of Serbia, by the project “Localization in the phase space: theoretical, numerical and practical aspects”, No. 19.032/961-103/19. of the Republic of Srpska Ministry for Scientific and Technological Development, Higher Education and Information Society, and by the bilateral project &eMicrolocal analysis and applications& between the Macedonian and Serbian academies of sciences and arts.

**Conflicts of Interest:** The authors declare that there is no conflict of interest between them regarding the publishing in this journal, and no support from any organisation for the submitted work is obtained.

**References**
1. Giv, H.H. Directional short-time Fourier transform. *J. Math. Anal. Appl.* 2013, 399, 100–107. [CrossRef]
2. Grafakos, L.; Sansing, C. Gabor frames and directional time-frequency analysis. *Appl. Comput. Harmon. Anal.* 2008, 25, 47–67. [CrossRef]
3. Saneva, K.; Atanasova, S. Directional short-time Fourier transform of distributions. *J. Inequal. Appl.* 2016, 124, 1–10.
4. Atanasova, S.; Maksimović, S.; Pilipović, S. Directional Short-Time Fourier Transform of Ultradistributions. *Bull. Malays. Math. Sci. Soc.* **2021**, *44*, 3069–3087. [CrossRef]

5. Atanasova, S.; Pilipović, S.; Saneva, K. Directional Time–Frequency Analysis and Directional Regularity. *Bull. Malays. Math. Sci. Soc.* **2019**, *42*, 2075–2090. [CrossRef]

6. Hörmander, L. *The Analysis of Linear Partial Differential Operators I*, 2nd ed.; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 1990.

7. Coriasco, S.; Johansson, K.; Toft, J. Local wave-front sets of Banach and Fréchet types, and pseudo-differential operators. *Monatsh. Math.* **2013**, *169*, 285–316. [CrossRef]

8. Coriasco, S.; Johansson, K.; Toft, J. Global Wave-front Sets of Banach, Fréchet and Modulation Space Types, and Pseudo-differential Operators. *J. Differ. Equ.* **2013**, *254*, 3228–3258. [CrossRef]

9. Dimovski, P.; Prangoski, B. Wave front sets with respect to Banach spaces of ultradistributions. Characterisation via the short-time fourier transform. *Filomat* **2019**, *33*, 5829–5836. [CrossRef]

10. Pilipović, S.; Prangoski, B. On the characterization of wave front sets via the short-time Fourier transform. *Math. Notes* **2019**, *105*, 153–157. [CrossRef]

11. Carmichael, R.; Kamiński, A.; Pilipović, S. *Boundary Values and Convolution in Ultradistribution Spaces*; World Scientific: Singapore, 2007.

12. Teofanov, N. Gelfand-Shilov spaces and localization operators. *Funct. Anal. Approx. Comput.* **2015**, *7*, 135–158.

13. Hörmander, L. *Lectures on Nonlinear Hyperbolic Differential Equations*; Mathématiques et Applications 26; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 1997.

14. Komatsu, H. Ultradistributions, I: Structure theorems and a characterization. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **1973**, *20*, 25–105.