Solitaire Chess is NP-complete

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Abstract

"Solitaire Chess" is a logic puzzle published by Thinkfun™ that can be seen as a single person version of traditional chess. Given a chess board with some chess pieces of the same color placed on it, the task is to capture all pieces but one using only moves that are allowed in chess. Moreover, in each move one piece has to be captured. We prove that deciding if a given instance of Solitaire Chess is solvable is NP-complete.

1 Introduction

Solitaire Chess is a one-player puzzle game published by Thinkfun™ [4]. The puzzle is based on chess and can be seen as a single person version of it. As in the original chess game, there are six different types of chess pieces: king, queen, rook, bishop, knight and pawn.

In each instance some of the pieces (where several pieces of the same type are allowed) are placed on a 4 × 4 chess board. Now the task is to capture all but one piece using only moves that are allowed in the original chess game (see e.g. [1]). An additional rule is, that in each move one piece has to be captured. Figure 1 shows an example instance and a sequence of moves solving it.

![Example instance and a sequence of moves solving it.](image-url)

To our knowledge the puzzle has been first published in 2010 and is available as hardware game, iPhone and Android apps and as online puzzle.

In this paper we consider a generalized version of the puzzle where an instance consists of $n$ pieces on a chess board of size $N \times N$, $N \in \mathbb{N}$. Moreover,
every type of piece might appear several times in an instance. We prove that this
generalized version of Solitaire Chess is NP-hard by giving a polynomial reduc-
tion from 3-SAT. Note that the corresponding generalization of the traditional
chess is EXPTIME-complete [3].

2 First Observations

We identify each square of the board by a position \((i, j) \in \mathbb{Z}^2\). If \(i + j \equiv 0 \mod 2\),
we call the square \((i, j)\) white. Otherwise, it is called black. The orientation of
the chess board is only important for pawns, as they are only allowed to move
“forward”. As a moving piece must capture another piece, a pawn on position
\((i, j)\) can only capture a piece at position \((i - 1, j + 1)\) or \((i + 1, j + 1)\).

A move is uniquely defined by a pair of pieces \((p_1, p_2)\) where \(p_1\) captures
\(p_2\). A move \((p_1, p_2)\) is feasible if \(p_1\) can capture \(p_2\) following the rules of chess
(ignoring the color of the pieces).

Lemma 1. Solitaire Chess is in NP.

Proof. The feasibility of a move can be checked in linear time. An instance with
\(n\) chess pieces can be solved, if there exists a sequence of moves such that only
one piece survives. As the number of pieces is reduced by one in each move such
a sequence consists of exactly \(n - 1\) moves. Thus the coding length of such a
sequence is linear in the number of pieces and the feasibility of all moves can be
verified in polynomial time, implying that the problem is in NP. □

3 Reduction from 3-SAT

3-Satisfiability (in short 3-SAT) is a well known problem in complexity theory.
A 3-SAT instance consist of \(n \) variables \(x_1, \ldots, x_n\) and \(m\) clauses \(C_1, \ldots, C_m\),
where each clause contains exactly three literals of \(\{x_1, \ldots, x_n, \overline{x_1}, \ldots, \overline{x_n}\}\).
For a given instance \(I\) the problem is to determine if there exists a truth assignment
\(\tau : \{x_1, \ldots, x_n\} \rightarrow \{\text{true, false}\}\) such that at least one literal of each clause is
satisfied. It is well known that 3-SAT is NP-complete [2].

Before defining the reduction from 3-SAT to Solitaire Chess in detail we give
a short overview of our construction. For every variable we add two variable
columns corresponding to the two possible truth assignments \text{true} and \text{false}
for this variable. A variable rook will move in one of these rows. For every
clause we add three clause rows, one for each literal of the clause. Moreover,
we have two rooks for every clause that can move in two of the three rows. The
third row corresponds to the literal of the clause which has to be satisfied by the
truth assignment we are looking for. In order to interlink the columns and rows
we add for every literal of every clause two literal bishops. They are placed on
the corresponding clause row and variable column, respectively, such that they
can capture each other. If and only if all literal bishops can be captured, the
initial 3-SAT instance is satisfiable. Finally, we require an additional bishop
and several pawns in order to guarantee that all rooks can be captured.

In our construction we have to take care for the rooks not to leave their
clause rows or variable columns. Moreover, the literal bishops should be only
able to capture their partner literal bishop and no other piece.

For our reduction we set \(M = 8m^2\).
3.1 Variables

Let \( x_i, i \in \{1, \ldots, n\} \), be a variable. The columns \( iM \) and \( iM + 2 \) are the variable columns of \( x_i \) that correspond to the truth assignment of \( x_i \), that is, they correspond to \( x_i = \text{true} \) and \( x_i = \text{false} \), respectively. Moreover, we have a variable rook \( r \) for \( x_i \) which is originally placed in one of the two columns.

First the rook must pass a column changing gate consisting of three pawns \( p_1, p_2, p_3 \) as shown in Figure 2. Depending in which direction the rook passes the gate, it will end in the left or the right column. If the rook executes the moves \((r, p_1), (r, p_2), (r, p_3)\), it ends in the right column. Otherwise the rook ends in the left column after executing the moves \((r, p_3), (r, p_2)\) and \((r, p_1)\). Note, that the three pawns cannot capture other pieces and have to be captured by \( r \). The remaining pieces are placed in such a way that a variable rook can leave its column neither after passing the column changing gate nor after capturing a piece that does not belong to the gate.

Initially, the variable rook for \( x_i \) is placed at position \((iM + 2, 5m^2 + 8i)\) and the pawns of the columns changing gate at \((iM, 5m^2 + 8i), (iM, 5m^2 + 8i - 2)\) and \((iM + 2, 5m^2 + 8i - 2)\).

![Figure 2: A variable rook, a column changing gate and two variable columns.](image)

3.2 Clause

Let \( C_j = (l^1_j \lor l^2_j \lor l^3_j), j \in \{1, \ldots, m\} \), be a clause. The three rows \( 6mj + 2, 6mj + 4 \) and \( 6mj + 6 \) correspond to the literals \( l^1_j, l^2_j \) and \( l^3_j \), respectively. Moreover we have two clause rooks \( r_1 \) and \( r_2 \) for clause \( C_j \) originally placed in the rows \( 6mj + 6 \) and \( 6mj + 4 \), respectively. They will represent the two literals that might be unsatisfied by a feasible truth assignment. To this end, they must be able to change the row where they are initially placed. We introduce a row changing gate consisting of three pawns \( p_1, p_2 \) and \( p_3 \) that works as the column changing gate for variable rooks: If the rook \( r_1 \) executes the moves \((r_1, p_1), (r_1, p_2)\) and \((r_1, p_3)\) it remains in its original row. Otherwise it can change the row by executing the moves \((r_1, p_3), (r_1, p_2)\) and \((r_1, p_1)\). Note that pawns of a row changing gate have to be captured by clause rooks. A second row changing gate is required for the second rook. Figure 3 shows the initial placement of the clause rooks and pawns of the row changing gate.

![Figure 3: A clause rook and a row changing gate.](image)
By using the row changing gates we can assure that the two rooks are in any two of the three clause rows. Note that it is also possible, that one of the rooks captures the other one. In this case the instance can be solved only if there exists a truth assignment such that at least two of the literals of the clause are satisfied.

Initially the positions of the clause rooks for clause \( C_j \) are \((Mn+10j, 6mj+4)\) and \((Mn + 10j - 4, 6mj + 6)\).

\[ \text{Figure 3: Clause rows of a clause } C_j \text{ and the initial placement of the two clause rooks and two row changing gates.} \]

### 3.3 Literal Bishops

Up to now we have defined the pieces, rows and columns required to represent clauses and variables. Now we interlink the clauses and the corresponding literals. Let \( C_j, j \in \{1, \ldots, m\} \), be a clause and \( l^k_j \), \( k \in \{1, 2, 3\} \), be one of its literals. Assume \( l^1_j \) is a literal of the variable \( x_i \), \( i \in \{1, \ldots, n\} \). We add a pair of literal bishops \( b_1 \) and \( b_2 \) which interlink the \( k \)'th clause row of clause \( C_j \) and the left or right variable column of \( x_i \) (depending on \( l^1_j \) being positive or negative). We place \( b_1 \) and \( b_2 \) such that \( b_1 \) can only be captured by \( b_2 \) or one of the clause rooks of \( C_j \) and \( b_2 \) can be captured by \( b_1 \) or the literal rook of \( x_i \).

We place \( b_1 \) at position

\[ (iM - 7 - 2(i + j \mod m) + 2k, 6mj + 2k) \] (1)

and \( b_2 \) at position

\[ (iM, 6mj + 7 + 2(i + j \mod m)) \quad \text{if literal } l^1_j \text{ is positive and} \]
\[ (iM + 2, 6mj + 9 + 2(i + j \mod m)) \quad \text{if literal } l^1_j \text{ is negative.} \] (2)

**Lemma 2.** The only feasible move for a literal bishop is to capture a piece at the position of its opponent.

**Proof.** First note that the literal bishops are on black squares and all other pieces are on white squares. Thus a literal bishop can only capture pieces that are currently placed on the original positions of other literal bishops.

Let \( b \) be a literal bishop for a literal of clause \( C_j \) and of variable \( x_i \) and \( b' \) be a literal bishop for a literal of clause \( C_j' \) and variable \( x_{i'} \) with \((i, j) \neq (i', j')\).
Let \((x, y)\) and \((x', y')\) be the positions of \(b\) and \(b'\), respectively. We want to prove, that \(b\) cannot capture \(b'\). It suffices to show that \(x - y \neq x' - y'\) and \(x + y \neq x' + y'\). To prove the first inequality observe that 

\[(i - 1)M < x - y < iM.\]

Thus if \(i \neq i'\) then \(x - y \neq x' - y'\). If \(i = i'\) then

\[x - y = iM - 6mj - 7 - 2(i + j \mod m) \equiv -7 - 2(i + j \mod m) \mod 6m\]

and therefore \(x - y \neq x' - y'\) if \(j \neq j'\).

To prove the second inequality observe that

\[iM + 6mj - 3m < x + y < iM + 6mj + 3m.\]

Thus \([x + y + 3m)/M] = i\) and \([((x + y \mod M) + 3m)/6m] = j\) implying \(x + y \neq x' + y'\) if \((i, j) \neq (i', j')\).

Now let \(b_1\) and \(b_2\) be the two literal bishops of the same literal at position \((x_1, y_1)\) and \((x_2, y_2)\), respectively. By definition \(x_1 - y_1 = x_2 - y_2\), thus \(b_1\) can capture \(b_2\) and vice versa.

![Figure 4: Possible positions for the two literal bishops of a literal. The black bishops indicate feasible positions for \(b_1\) and the white bishops positions for \(b_2\).](image)

### 3.4 Cleaning Pieces

Finally, we require some more pieces whose purpose is to capture all remaining pieces in a feasible instance.

To this end we add a **cleaning bishop** at position \((0, 0)\). Moreover, we add for every clause row \(y\) a pawn at position \((-y, y)\) and for every variable row \(x\) a pawn at position \((x, -x)\). Thus we add in total \(3m + 2n\) **cleaning pawns**. We have already seen, that the clause and variable rooks can capture all pawns that
belong to row and column changing gates. Moreover, each rook can capture one of the cleaning pawns. After this, the cleaning bishop can capture all pawns and rooks at positions \((i, -i), i \in \mathbb{Z}\). Thus if there are no literal bishops, we can capture all but one piece and the instance is solvable.

So the challenge of such an instance is to capture all literal bishops.

### 3.5 NP-completeness

Before finally proving that the transformed Solitaire Chess instance is solvable if and only if the original 3-SAT instance is satisfiable, we need some more observations, which pieces can be captured by other ones. First note, that literal bishops cannot capture cleaning pawns or the cleaning bishop: Initially all literal bishops are on black squares, while all pawns and the cleaning bishop are on white squares.

**Proposition 3.** A clause rook can change its row only in row changing gates and variable rooks can change their column only in their column changing gate.

*Proof.* Let \(r\) be a clause rook positioned in row \(y\). In order to change its row there must be two pieces at positions \((x, y)\) and \((x, y')\) for \(x, y, y' \in \mathbb{Z}, y \neq y'\). But by construction for any piece in row \(y\) at position \((x, y)\) that does not belong to a row changing gate there is no other piece with the same x-coordinate. The same arguments apply to variable rooks by exchanging rows and columns. 

**Proposition 4.** If the Solitaire Chess instance is solvable, then in an optimal sequence of moves at least one of the two literal bishops is captured by a variable or a clause rook.

*Proof.* By Lemma 2 a literal bishop can capture only a piece at the position of the opponent bishop. But then this bishop has to be captured by a rook in the corresponding row or column.

**Theorem 5.** Solitaire Chess is NP-complete.

*Proof.* The transformation is polynomial: For a 3-SAT instance with \(n\) variables and \(m\) clauses we require \(17m + 1 + 4n\) chess pieces and all pieces are placed on a board of polynomial size.

Assume \(I\) is a 3-SAT instance that is satisfied by a truth assignment \(\tau\). We give a sequence of moves that solves the corresponding Solitaire Chess instance. For every clause \(C_j, j \in \{1, \ldots, m\}\), there exists at least one literal that is satisfied by \(\tau\). We set \(z_j \in \{1, 2, 3\}\) such that \(l_j^{z_j}\) is satisfied by \(\tau\).

For every variable \(x_i, i \in \{1, \ldots, n\}\), the variable rook uses the column selecting gate in order to get into the row corresponding to \(\tau(x_i)\). The two rooks of every clause \(C_j, j \in \{1, \ldots, m\}\), use the row selecting gates to get into the two rows corresponding to the two literals in \(\{l_j^1, l_j^2, l_j^3\} \setminus \{l_j^{z_j}\}\). Consider a literal \(l_j^z, z \in \{1, 2, 3\}\). If \(z = z_j\), then the literal bishop \(b_1\) captures \(b_2\) and is captured by the variable rook. Otherwise, \(b_2\) captures \(b_1\) and is itself captured by the clause rook in the corresponding row. By this all literal rooks are captures. Finally, the rooks capture their cleaning pawns and the cleaning bishops capture all remaining pieces. We conclude that the Solitaire Chess instance is solvable.

Now assume that the Solitaire Chess instance is solvable. We define a truth assignment \(\tau\) by setting \(\tau(x_i) = \text{true}\) if the variable rook corresponding to \(x_i\),
uses the right column and $\tau(x_i) = \text{false}$ otherwise. Now consider a clause $C_j$, $j \in \{1, \ldots, m\}$. We have to show that at least one of the literals $l^1_j, l^2_j$ or $l^3_j$ is satisfied by $\tau$. At most two of the corresponding literal bishop pairs have been captured by their clause rooks. Thus the third bishop pair must be captured by the corresponding variable rook. But then this literal is satisfied by $\tau$.

We finish the proof by observing that by Lemma 1 the problem is in NP.

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