Perturbative and Numerical Analysis of Tilted Cosmological Models of Bianchi type V

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Cosmological models of Bianchi type V and I containing a perfect fluid with a linear equation of state plus cosmological constant are investigated using a tetrad approach where our variables are the Riemann tensor, the Ricci rotation coefficients and a subset of the tetrad vector components. This set, in the following called $S$, describes a spacetime when its elements are constrained by certain integrability conditions and due to a theorem by Cartan this set gives a complete local description of the spacetime. The system obtained by imposing the integrability conditions and Einstein’s equations can be reduced to an integrable system of five coupled first order ordinary differential equations. The general solution is tilted and describes a fluid with expansion, shear and vorticity. With the help of standard bases for Bianchi V and I the full line element is found in terms of the elements in $S$. We then construct the solutions to the linearized equations around the open Friedmann model. The full system is also studied numerically and the perturbative solutions agree well with the numerical ones in the appropriate domains. We also give some examples of numerical solutions in the non-perturbative regime.

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I. INTRODUCTION

In this paper general Bianchi types V and I perfect fluids with linear equation of state and cosmological constant are studied. In general these spacetimes are tilted and in particular there are solutions with rotating matter. It has been difficult to find exact solutions with both expansion and nonzero rotation of the matter flow. To our knowledge the only known exact homogeneous perfect fluid solution with rotation and expansion is the self-similar radiation-filled Bianchi type VI$^{0}$ found by Rosquist [1].

A number of rotating imperfect fluid solutions with heatflow are known, see e.g. [2]. Since for rotating matter the hypersurfaces of homogeneity are tilted with respect to the restframe of matter, local space will not look homogeneous. Hence heatflow is expected and for some solutions the heatflow can be related to a temperature gradient [3], but often with unrealistic coefficient of conductivity. Normally the heat conductivity is negligibly small and a perfect fluid approximation should work well.

For treatments of the properties of homogeneous rotating models in general see [4,5] and for some perturbative calculations see [6–8]. In [9] the qualitative behaviour of locally rotationally symmetric (LRS) Bianchi V solutions is analysed and in particular expressions for different quantities at early and late times are given. There are a number of works on tilted Bianchi cosmologies using a dynamical system approach. The irrotational subcase of type V was studied in [10]. Recently the late time behaviour of tilted Bianchi models including type V was considered in [11]. The stability of non-tilted Bianchi models against tilt was studied in [12].

To find the solutions we use a method for construction of solutions to Einstein’s equations, [13–16], based on the invariant classification scheme by Cartan and Karlhed [17,18]. The method is shortly described in section II. In section III the method is applied to Bianchi V and I models. First choice of frame and the set (called $S$) of quantities needed to specify the spacetime is imposed together with Einstein’s equations. The general system is reduced to an integrable system of five coupled first order ordinary differential equations. With the help of standard bases for Bianchi V and I the full line-element is found up to quadratures in terms of the elements in $S$. The subclass of orthogonal solutions is easily solved, but all these solutions are well known. For LRS dust the equations can also be integrated [19,20].

In section IV we consider first order perturbations around the open Friedmann universe. The general first order solution depends on five constants of integration, the same number as for the general exact solution, and has nonzero expansion, rotation and shear. Finally, in section V the general system is studied numerically and the results agree well with the perturbative calculations in the allowed range. Examples of numerical solutions in the non-perturbative regime are given.

II. CONSTRUCTION OF SOLUTIONS TO EINSTEIN’S EQUATIONS IN TERMS OF CURVATURE INVARIANTS

A brief summary of the method is given here. For more details see [16,21].
According to a theorem by Cartan spacetimes are locally completely determined by a set consisting of the components of the Riemann tensor and a finite number of its covariant derivatives in a frame with constant metric $\eta_{ij}$ [17,18]. This set will henceforth be called $R^{p+1} = \{R_{ijkl}, R_{ijkl;j}, ..., R_{ijkl;m_1} ... m_{p+1}\}$, where $p$ is such that the components in $R^{p+1}$ are functionally dependent on those in $R^p$ (as functions of both the coordinates $x^\alpha$ on the manifold and the parameters of the Lorentz group $\xi^T$). This description can be used to classify geometries and also to construct new solutions to Einstein’s equations in terms of the set $R^{p+1}$.

Assume that we have symmetries such that $R^{p+1}$ only depends on $x^\alpha$, $\alpha = 1, 2, ..., l < n = \text{dimension}$ of spacetime, (in some canonical coordinates) and rotations in the $ab$-planes, $\{a_b\} = 1, ..., m < n(n-1)/2$ (with frames adopted to the rotational symmetries). Here $l = n - \text{dim(orbit)}$ and $m = n(n-1)/2 - \text{dim(isotropy group)}$. The set $R^{p+1}$ is completely determined by the smaller set $S = \{R_{q k l}, \gamma^a_{b kl}, x^a_{ij}\}$ where $x^a_{ij} \equiv X_i(x^\alpha) = X_i^\mu \partial x^\alpha / \partial x^\mu$ are the derivatives with respect to the frame vectors and $\gamma^i_{jk}$ are the Ricci rotation coefficients. Here the numbering is such that $\{P_{q}\} = m+1, ..., n(n-1)/2$ are the complementary rotations to the $\{a_b\}$ ones, i.e., those that keep the set $R^{p+1}$ unchanged.

A set $R^{p+1}$ together with a constant frame metric $\eta_{ij}$ describes a geometry iff certain integrability conditions, being equivalent to the Ricci identities (including the commutators for the essential coordinates) and part of the Bianchi identities, are satisfied. In practice it is often easier to work in a fixed frame than to let the components in $R^{p+1}$ depend explicitly on the parameters $\xi^T$ of the orthogonal group. The Ricci identities then split into the commutators for the essential coordinates $x^\alpha$

$$x^\alpha_{[i,\beta} x^\beta_{j]} + x^\alpha_{m \gamma^m_{ij]} = 0, \quad (1)$$

and the Riemann equations for rotations in the $ab$-planes

$$R^a_{bij} = 2\gamma^a_{[b[i} x^\alpha_{j]} + 2\gamma^k_{[b} \gamma^{(a}_{k]i} + 2\gamma^a_{bkl} \gamma^{(k}_{i)} \quad (2)$$

(the antisymmetrizations are only over $ij$). When using the description in terms of the smaller set $S$ equations (2) merely serve to define the $R^a_{bij}$-components of the Riemann tensor. Note that equations (2) give 36 equations in the case without isotropies and the restriction to 20 independent components come from the cyclic identity below or equations (1). Since not all commutators or Riemann equations are used when the spacetime has symmetries some more integrability conditions are needed. They are parts of the cyclic and Bianchi identities

$$R^i_{[ijk]} = 0, \quad t = l + 1, ..., n, \quad (3)$$

$$R^p_{(ij)k} = 0, \quad \{P_{q}\} = m+1, ..., n(n-1)/2, \quad (4)$$

where $t = l + 1, ..., n$ numbers the symmetry directions (in a suitable numbering of the frame vectors).

The above description in terms of $R^{p+1}$ can be used to find new solutions to Einstein’s equations. This approach has the advantage of being coordinate invariant. Also, since the components in $R^{p+1}$ correspond to directly measurable quantities, physical requirements are easy to impose. Another advantage is that the set of equations (1), (2), (3) and (4) is a subset of the ones used in different tetrad methods like the ones in, e.g., [22] or the Newman-Penrose method [23]. The procedure is:

1. Choose a set $R^{p+1}$ (or equivalently $S$). Some of the elements in $R^p$ or functions of them are chosen as coordinates. Einstein’s equations are imposed by restricting the Ricci tensor.

2. Impose the integrability conditions (1), (2), (3) and (4), leading to a set of first order differential equations (together with algebraic constraints) for the elements in $R^{p+1}$ ($S$).

Solving this set of equations gives $R^{p+1}$ and hence a complete local description of the geometry. If the geometry does not have any symmetries, one can solve for all the 1-forms from $d\omega^a = x^\alpha \omega^\alpha$ and hence get the full line-element $d^2 s = \eta_{ij} \omega^i \omega^j$.

When there are symmetries one only obtains part of the 1-forms, but one may determine the Lie-algebra of the isometry group, [24], and from this it is often possible to make an ansatz for the remaining 1-forms. Cartan’s equations look the same in $F(M)$, the bundle of frames on $M$, as in $M$

$$d\omega^i = \omega^i \wedge \omega^j \quad (5)$$

$$d\omega^i = -\omega^i_k \wedge \omega^k + \frac{1}{2} R^i_{jkl} \omega^k \wedge \omega^l \quad (6)$$

but now $d = d_x + d_\xi$. The connection forms $\omega^i_j$ will now be linearly independent of the 1-forms $\omega^i$, and can be written as

$$\omega^i_j = \gamma^i_{jk} \omega^k + \tau^i_j \equiv \gamma^i_{jk} \omega^k + \tau^i_{j} \tau \xi^T d\xi^T \quad (7)$$

where $\tau^i_j$ are the generators of the orthogonal group. With the same notation for indices as above Cartan’s equations split into those corresponding to essential coordinates and rotations $(x^\alpha$ and $\tau^a_{b})$ and those corresponding to symmetry coordinates and rotations $(x^t$ and $\tau^p_q)$

$$d\omega^i = \omega^i \wedge \omega^j, \quad d\omega_q = -\omega^p_k \wedge \omega^q_k + \frac{1}{2} R^p_{qkl} \omega^k \wedge \omega^l. \quad (8)$$

The 1-forms that are missing can be found from these equations. Since spacetime locally is completely determined by $R^{p+1}$, it is sufficient to find one linearly independent solution to equations (8). All other solutions will be locally equivalent to this one up to coordinate transformations. When projecting onto the orbits, given by $dx^\alpha = 0$ and $\tau^a_{b} = 0$, equations (8) reduce to the Lie-algebra of the isometry group (since the group acts
simply transitive on the orbits in $F(M)$ [25]) and the structure constants will be given by the elements in $R^{p+1}$ [24]. If one wants to find the full line-element, the procedure can therefore be continued in the following way

3. Solve for the $\omega^\alpha$ (in a suitable numbering) and the $\omega_i^0$ from $R^{p+1}$.

4. Determine the Lie algebra of the isometry group and some standard 1-forms satisfying it.

5. From these make an ansatz $\tilde{\omega}^i$ for the 1-forms over the full spacetime.

6. Calculate the corresponding $\tilde{R}^{p+1}$ and compare with $R^{p+1}$, giving differential or algebraic equations for the coefficients in the metric ansatz. These equations are of course equivalent to (8).

III. BIANCHI V AND I

In this section we consider homogeneous cosmological models of Bianchi type V and I, i.e., those characterized by the symmetric matrix in the Ellis-MacCallum scheme [22] being zero. We assume that matter can be described as a perfect fluid. First the preliminaries, like choice of frame and the elements in the set $S$ ($R^{p+1}$) are given. Einstein's equations with a cosmological constant must also be imposed. The nonzero elements of the Riemann tensor are then given by

$$R_{0101} = C_1 - \frac{1}{2}p - \frac{1}{6}\rho + \frac{1}{3}\Lambda, \quad R_{0102} = R_{1023} = C_2,$$

$$R_{0103} = -R_{1223} = C_3, \quad R_{0112} = R_{0233} = C_4,$$

$$R_{0113} = -R_{0223} = C_5, \quad R_{0123} = C_6,$$

$$R_{0202} = C_7 - \frac{1}{2}p - \frac{1}{6}\rho + \frac{1}{3}\Lambda, \quad R_{0203} = R_{1213} = C_8,$$

$$R_{0212} = -R_{0313} = C_9, \quad R_{0213} = C_{10},$$

$$R_{0303} = -C_1 - C_7 - \frac{1}{2}p - \frac{1}{6}\rho + \frac{1}{3}\Lambda, \quad R_{0312} = R_{1012} = C_{10},$$

$$R_{1212} = C_1 + C_7 - \frac{1}{3}\rho - \frac{1}{3}\Lambda, \quad R_{1313} = -C_7 - \frac{1}{3}\rho - \frac{1}{3}\Lambda,$$

$$R_{2323} = -C_1 - \frac{1}{3}\rho - \frac{1}{3}\Lambda$$

where $C_i$ are the ten independent components of the Weyl tensor.

Some of the rotation coefficients are expressible in terms of the kinematic quantities shear $\sigma_{ij}$, vorticity $\omega_{ij}$, expansion $\theta$ and acceleration $a_i$

$$\gamma_{0ij} = -u_{ij} = -\omega_{ij} - \sigma_{ij} + \frac{1}{3}h_{ij}\theta - a_iu_j,$$

where $\sigma_{ij} = h_i^k h_j^l (u_{(k;lj)} + \frac{1}{2}h_{kl}\theta)$, $\omega_{ij} = h_i^k h_j^l u_{[k;lj]}$, $\theta = u_{i;j}$ and $a_i = u_{ij}u^j$ and $h_{ij} = u_iu_j - \eta_{ij}$ is the projection operator onto the space perpendicular to the 4-velocity. In the next subsection we impose the requirement that the isometry group is of Bianchi type V or I. This will give six restrictions on the rotation coefficients.
B. Symmetry group

The Lie algebra of the isometry group can be determined by projection of Cartan’s equations onto the orbits [24]. Since we do not have any isotropies in this case, the orbits will be the hypersurfaces of homogeneity \( dp = 0 \) in \( M \). From \( dp = 0 \) we get

\[
\frac{dp}{d\rho} = \rho_0 \omega_0| + \rho_0 \omega^0 = 0 \quad \text{or} \quad \omega^0 = -\frac{\rho_0}{\rho_0} \omega^0,
\]

where \( \alpha = 1, 2, 3 \) are the spatial indices and a vertical bar indicates projection onto \( dp = 0 \). From the requirement of no isotropy one has that \( \tau^i_{ij} = 0 \) holds on the orbits and (7) then gives

\[
\omega^i_j = \tau^i_{jk} \omega^k_j.
\]

Hence the orbits are spanned by \( \{\omega^1, \omega^2, \omega^3\} \). By projecting the first pair of Cartan’s equations (5) (the second pair will not give anything new in this case) one obtains

\[
d\omega^\alpha| = \omega^i \wedge \omega_j^\alpha| \equiv \frac{1}{2} C_{\beta\delta}^\alpha \omega^\beta \wedge \omega^\delta,
\]

where the structure constants are given by

\[
\frac{1}{2} C_{\beta\delta}^\alpha = \gamma_{\alpha [\beta\delta]} + \frac{\rho_{\beta\delta}}{\rho_0} \gamma_{\alpha [\beta} \gamma_{\delta]} - \frac{\rho_{\delta}}{\rho_0} \gamma_{\alpha [\beta} \gamma_{\delta]}. \tag{10}
\]

The structure constants are hence given in terms of elements in \( R^{p+1} \), and since they are functions only of \( \rho \) the \( C_{\beta\delta}^\alpha \) are constants on the orbits.

From the Ellis-MacCallum scheme for the Bianchi classes [25], we can decompose the structure constants into one symmetric \( 3 \times 3 \) matrix \( N_{\beta\delta} \) and a 3-vector \( A^\alpha \) according to

\[
\frac{1}{2} C_{\beta\delta}^\alpha \epsilon^{\beta\delta\gamma} = N^\alpha_\gamma + \epsilon^{\alpha\beta\gamma} A_\beta \quad \text{giving}
\]

\[
N_{\alpha\delta} = \frac{1}{2} \left[ C_{\beta\gamma}^\alpha \epsilon^{\beta\delta\gamma} - C_{\beta\gamma}^\delta \epsilon^{\beta\alpha\gamma} \right]. \tag{11}
\]

Bianchi classes V and I are characterized by \( N_{\alpha\delta} = 0 \). This will give six relations among the \( \gamma_{\alpha [\beta \gamma} \). Note that we cannot simply use the standard form of the structure constants in (10) and obtain nine relations for the Ricci coefficients this way since our choice of frame (giving other relations among the elements in \( S \)) might be inconsistent with that giving the structure constants in standard form. The nonzero Ricci rotation coefficients are then given by \((\alpha, \beta, 1, 2, 3 \) and \( \omega_{12} = 0 \))

\[
\begin{align*}
\gamma_{000} &= -a_0, \\
\gamma_{00\beta} &= -\sigma_{0\beta} - \omega_{0\beta} \quad (\alpha \neq \beta), \\
\gamma_{\alpha0\beta} &= \frac{1}{3} \theta - \sigma_{\alpha\beta}, \\
\gamma_{133} &= \gamma_{122} + \frac{\rho_1}{\rho_0} (\sigma_{11} + 2 \sigma_{22}), \\
\gamma_{223} &= \gamma_{231} = \frac{\rho_1}{\rho_0} (\gamma_{230} - \omega_{23}), \\
\gamma_{223} &= \gamma_{321} + \frac{\rho_1}{\rho_0} (\sigma_{13} + \omega_{13} - \gamma_{130}), \\
\gamma_{233} &= \gamma_{312} + \frac{\rho_1}{\rho_0} (-\sigma_{12} + \gamma_{120}), \\
\gamma_{122} &= \gamma_{121} = \gamma_{120}.
\end{align*}
\]

C. Integrability conditions

The commutator equations (1) give

\[
\frac{d\rho_i}{d\rho} \rho_j - \frac{d\rho_j}{d\rho} \rho_i + \rho_i (\gamma_{kj}^k - \gamma_{kj}^k) = 0 \tag{13}
\]

and the Riemann equations (2)

\[
R^i_{jkli} = 2\gamma^i_{[j\lrfloor [\rho_k]} + 2\gamma^i_{m[l} \gamma_{m}^{[\rho_k]} + 2\gamma^i_{jm} \gamma^m_{[kl]} \tag{14}
\]

(antisymmetrization only over \( kl \)). The cyclic identity is already imposed due to the choice (9) of the Riemann tensor and the Bianchi identities need not be imposed due to the lack of isotropies. (However it is in practice often convenient to use the twice contracted Bianchi identities since they give simple relations, but they may of course be derived from the other equations). Hence the system (13, 14) is the complete system. It is a set of first order ordinary differential equations, where the independent variable is \( \rho \), and algebraic constraints. Some are easily solved and give:

\[
a_2 = a_3 = \gamma_{121} = \omega_{23} = 0, \quad \gamma_{120} = \sigma_{12}, \quad \gamma_{130} = \sigma_{13} + \omega_{13}, \quad \gamma_{230} = -\sigma_{23} \quad \text{(or} \gamma_{131} = 0), \quad \rho_0 = -\gamma \rho, \quad a_1 = (1 - \gamma) \rho \theta, \quad \omega_{13} = \frac{1}{2} \gamma_{131}, \quad \mu_1 \equiv \frac{\rho_1}{\rho_0}, \tag{15}
\]

where we have introduced the tilt

\[
v \equiv \frac{\rho_1}{\rho_0}. \tag{16}
\]

The case \( \gamma_{131} = 0 \) will be treated separately in section III F. We see that the vorticity given by

\[
\omega_{13} = \frac{1}{2} v \gamma_{131} \quad \text{or} \quad \Omega_2 = -\frac{1}{2} v \gamma_{131} \tag{17}
\]

is perpendicular to the acceleration, a result already found in [4]. The ten components, \( C_i \), of the Weyl tensor are also given by (14).

The system then reduces to nine first order differential equations

\[
\dot{f}_i(\rho) = F_i(f_j(\rho), \rho), \tag{18}
\]

where \( \dot{f} \equiv df/d\rho \), for the functions \( f_i(\rho) \)

\[
f_i \in \{v, \theta, \sigma_{11}, \sigma_{22}, \sigma_{12}, \sigma_{13}, \sigma_{23}, \gamma_{131}, B\},
\]

where we use the variable

\[
B \equiv \gamma_{122} - \frac{1}{3} v \theta + v \sigma_{22} \tag{19}
\]

instead of \( \gamma_{122} \), and four algebraic constraints

\[
G_a(f_j(\rho), \rho) = 0. \tag{20}
\]

It turns out that the differentiated constraints, when using (18), all are satisfied, i.e.,

\[
G_a = \frac{\partial G_a}{\partial f_j} \dot{f}_j + \frac{\partial G_a}{\partial \rho} \frac{dG_a}{d\rho} = \frac{\partial G_a}{\partial f_j} F_j + \frac{\partial G_a}{d\rho} = 0.
\]
Hence the system can be reduced to five differential equations and 4 constraints [15]. The constraints are given by

\[ \gamma_{131} [-2v^2 (3\sigma_{11} + (3\gamma - 4)\theta) - 3vB - 18 (\sigma_{11} + \sigma_{22}) (1 - v^2)] - 18B\sigma_{13} = 0, \]

\[ B\sigma_{12} - \gamma_{131}\sigma_{23} (1 - v^2) = 0, \]

\[ 9\gamma_{131} (\gamma_{131} v + 2\sigma_{13}) (1 - v^2) + 6v (2B^2 + \gamma\rho) - 2v^2 B (2(3\gamma - 4)\theta - 3\sigma_{11}) - 18B\sigma_{13} = 0, \]

\[ 9\gamma_{131} [\gamma_{131} (12 - 5v^2) - 8v\sigma_{13}] + 48vB\theta + 36 [\sigma_{11}^2 + \sigma_{11}\sigma_{22} + \sigma_{23}^2 + \sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2 + \Lambda + 3B^2 + \rho - v^2 (\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{23}) + 4\theta^2 [(6\gamma - 5)v^2 - 3] + 12\sigma_{11} [6vB + (3\gamma - 2)v^2\theta] = 0 \]

that are easily solved for \( \sigma_{13}, \sigma_{23} \) and \( \sigma_{11} \) in which the three first constraints are linear. Finally the last constraint gives a second order polynomial in \( \sigma_{22} \). The differential equations are

\[ \dot{v} = v \left( \frac{-\sigma_{11}}{\gamma \rho \theta} + \frac{4}{3\gamma} - 1 \right) \frac{1}{\rho}, \]

\[ \dot{\gamma}_{131} = \gamma_{131} \left( \frac{\sigma_{11} + \sigma_{22}}{\gamma \rho \theta} + 1 \right) \frac{1}{3\gamma \rho} \]

\[ \dot{\sigma}_{12} = \sigma_{12} \left( \frac{\sigma_{11} - \sigma_{22}}{\gamma \rho \theta} + 1 \right) \frac{2v\gamma_{131}\sigma_{23}}{\gamma \rho \theta} \]

\[ \dot{B} = B \left( \frac{-\sigma_{11}}{\gamma \rho \theta} + 1 \right) \frac{1}{3\gamma \rho} \]

\[ (\theta^2) = \left[ 3\gamma_{131} v^2 + 12 (\gamma - 1)\rho \theta (B + v\sigma_{11}) - 2\theta^2 + 2(\gamma - 1)(6\gamma - 5)v^2\theta^2 - 3 ((3\gamma - 2)\rho - 2\Lambda) - 12 (\sigma_{11}^2 + \sigma_{11}\sigma_{22} + \sigma_{12}^2 + \sigma_{13}^2 + \sigma_{22}^2 + \sigma_{23}^2) \right] / [3\gamma \rho ((\gamma - 1)v^2 - 1)] \]

If one instead wants to use proper time for a comoving observer as independent coordinate the equation

\[ \frac{d\rho}{d\tau} = \rho_0 = -\gamma \rho \theta, \]

obtained by putting \( dx = dy = dz = 0 \) in the line-element (see equation (26) below), should be added to the system (22). The derivatives in (22) are then expressed as

\[ \dot{f}_i = \frac{df_i}{d\rho} \frac{d\rho}{d\tau} \frac{d\tau}{d\rho} = -\frac{df_i}{d\tau} \frac{1}{\gamma \rho \theta}. \]

The system has a unique solution for given initial conditions provided \( F_i \) for the reduced system (22) satisfy a Lipschitz condition (for example if \( F_i \) are \( C^1 \) in a compact convex domain). The general solution hence depends on five constants of integration and in general its matter flow has both expansion, shear and vorticity.

One could of course choose to solve for four other quantities than \( \sigma_{13}, \sigma_{23}, \sigma_{11} \) and \( \sigma_{22} \) from (21). When doing a first order perturbative calculation around an isotropic universe, as in section IV, both \( \sigma_{11} \) and \( \sigma_{22} \) cannot be solved for from (21) since the last constraint will be quadratic in \( \sigma_{22} \) or \( \sigma_{11} \). Hence we solve for \( \theta \) from the last constraint and use the differential equation for \( \sigma_{22} \)

\[ \dot{\sigma}_{22} = \frac{\dot{\theta}}{3} + [(\gamma - 2)\rho - 4 (\sigma_{12}^2 - (v^2 - 1)\sigma_{23}^2) - 2\Lambda + 2 (\frac{\sigma_{22}}{\gamma}) (\frac{v^2}{2\sigma_{11} - (6\gamma - 5)\frac{\theta}{3}} - \theta) - 4B^2 + 2vB (2(\sigma_{22} - \sigma_{11}) - \gamma \theta) + 4\gamma_{131} (\nu\sigma_{13} - \gamma_{131})] / [2\gamma \rho \theta (v^2 - 1)] \]

instead of the one for \( \theta \).

This system is not in a suitable form for a dynamical system analysis. The system should then be made autonomous and compact dimensionless variables should be introduced. In [10] irrotational tilted Bianchi type V cosmologies were studied with this method. The field equations are derived in terms of expansion-normalized variables making the state space compact. The existence of a monotonic function shows that the dynamics to a large extent is determined by the invariant subset of locally rotationally symmetric models. A complete analysis of the orbits with non-extreme tilt was obtained. In [11] tilted Bianchi models of solvable type, including Bianchi type V, were considered with emphasis on the late-time behaviour. The equilibrium points were given in [10], but here the stability analysis was performed in the full state space. It was found that for \( 2/3 < \gamma < 2 \) Bianchi type V models approach the Milne universe in the asymptotic future. To complete the analysis a study of the behaviour for early times (high densities) would be of interest.

### D. The metric

From equations (10) for the structure constants we find the Lie-algebra of the isometry group to be

\[ d\omega^1 = \gamma_{131}\omega^1 \wedge \omega^3 \]
\[ d\omega^2 = B\omega^2 \wedge \omega^3 + \gamma_{131}\omega^2 \wedge \omega^3 \]
\[ d\omega^3 = B\omega^1 \wedge \omega^3 \]

where \( B \) is given by (19). As expected it is not in the standard form for Bianchi V, unless \( B = 0 \). Guided by the above algebra and standard 1-forms for Bianchi V, we make the following simplified, but sufficient, ansatz for a basis of 1-forms for the full spacetime
\[\omega^0 = -\frac{\ddens}{\gamma^2} - v\omega^1, \quad \omega^1 = f_1 e^{-\xi} dy\]
\[\omega^2 = g_1 e^{-\xi} dy + g_2 e^{-\xi} dz + g_3 dx\]
\[\omega^3 = \frac{1}{\gamma_{131}} dx + h_1 B e^{-\xi} dy\]  \hspace{1cm} \text{(26)}

where \(f_1, g_1, g_2, g_3\) and \(h_1\) are functions of \(\rho\) to be determined. (The case \(\gamma_{131} = 0\) is treated separately in section III F.) From (26) we calculate the set \(\tilde{S} = \{\tilde{\rho}_{ij}, \tilde{\gamma}_{ijk}\}\). A comparison with the set \(S\) gives

\[f_1 = h_1 = C_1 \rho^{\frac{1}{2}} - v, \quad g_1 = \gamma_{131} \rho^{\frac{1}{2}} - \frac{2}{3} v I_1\]
\[g_2 = C_2 \gamma_{131} \rho^{\frac{1}{2}} - \frac{2}{3} v I_2\]  \hspace{1cm} \text{(27)}

where \(C_1\) and \(C_2\) are (nonzero) constants of integration that can be absorbed in the definitions of the coordinates \(y\) and \(z\) and the integrals \(I_1\) and \(I_2\) are given by

\[I_1 = \frac{2C_1}{\gamma} \int \frac{\gamma^{\frac{1}{2}} \gamma_{131} + \sigma_{23} B}{v \gamma^{\frac{1}{2}} \gamma_{131}} \sigma_{12} \gamma_{131} + \sigma_{23} B \]  \hspace{1cm} \text{(28)}

respectively. From the 1-forms the metric is given by \(ds^2 = \eta_{ij} \omega^i \omega^j\). The full metric is hence given in terms of quadratures once the set \(S\) has been constructed. The solution depends on some arbitrary constants of integration and an even more general ansatz than (26) could have been made, but all these metrics give the same set \(S\) and hence they are locally equivalent.

Later on we will consider perturbations around the open Friedmann model, and will be interested in the limit \(v \to 0\) as \(\epsilon \to 0\). As seen some of the components in the metric will then diverge. This coordinate singularity can be avoided by changing coordinates according to

\[y = \epsilon \bar{y}, \quad z = \frac{\bar{z}}{\epsilon}\].

E. Orthogonal solutions

For the orthogonal case \(v = 0\), when also the vorticity is zero, it is better to use a frame where the shear tensor is diagonal, \(\sigma_{12} = \sigma_{13} = \sigma_{23} = 0\). The obtained system is easily solved and all solutions are well known. Essentially only two types of solutions appear. These are the Bianchi V solutions

\[\sigma_{11} = -\sigma_{22} = -k_1 \rho^{\frac{1}{2}}, \quad \gamma_{131} = \gamma_{232} = k_2 \rho^{\frac{1}{2}}, \]
\[\theta = \pm \sqrt{3} \sqrt{3k_2^{\frac{1}{2}} + k_2^2 \rho^{\frac{1}{2}} + \rho + \Lambda}\]  \hspace{1cm} \text{(29)}

and the Bianchi class I solutions

\[\sigma_{11} = k_1 \rho^{\frac{1}{2}}, \quad \sigma_{22} = k_2 \rho^{\frac{1}{2}}, \]
\[\theta = \pm \sqrt{3} \sqrt{(k_1^2 + k_2^2 + k_1 k_2) \rho^{\frac{1}{2}} + \rho + \Lambda}\].  \hspace{1cm} \text{(30)}

All orthogonal solutions can be found from these interchanging the 1-, 2- and 3-directions. The corresponding metrics are

\[\omega^0 = -\frac{\ddens}{\gamma^2}, \quad \omega^1 = c_1 e^{-\int \frac{\gamma_{131}}{\gamma^2} \rho^{-\frac{1}{2}}} dy,\]
\[\omega^2 = c_2 e^{-\int \frac{\gamma_{131}}{\gamma^2} \rho^{-\frac{1}{2}}} dz, \quad \omega^3 = \frac{1}{k_2} \rho^{-\frac{1}{2}} dx\]

and

\[\omega^0 = -\frac{\ddens}{\gamma^2}, \quad \omega^1 = c_1 e^{-\int \frac{\gamma_{131}}{\gamma^2} \rho^{-\frac{1}{2}}} dx,\]
\[\omega^2 = c_2 e^{-\int \frac{\gamma_{131}}{\gamma^2} \rho^{-\frac{1}{2}}} dy, \quad \omega^3 = c_3 e^{-\int \frac{\gamma_{131}}{\gamma^2} \rho^{-\frac{1}{2}}} dz\]

respectively. The integrals can be performed for specific values of \(\gamma\).

F. Irrotational tilted solutions

In [10] this class for non-extreme tilt \((v < 1)\) was studied using a dynamical systems approach. It was found that the dynamics to a large extent is determined by the invariant subset of locally rotationally symmetric (LRS) models. This subset was studied in detail in [9] for different values of \(\gamma\). There are solutions for which the tilt always is less than one and hence the hypersurfaces of homogeneity remain spacelike when going backwards in time. However, it may approach one for large times for certain values of \(\gamma\) including the radiation case \(\gamma = 4/3\). There are also solutions for which the tilt gets larger than one for small times and hence the hypersurfaces change from being spacelike to being timelike. For this class singularities may occur for finite densities. For example, in the \(\gamma = 4/3\) case some of the kinematic quantities and the Weyl tensor diverge for \(v = \sqrt{3}\).

The irrotational solutions are given by \(\gamma_{131} = 0\), which implies that the vorticity vanishes and the frame cannot be fixed by demanding \(\Omega^3 = 0\). Instead the frame is fixed by putting \(\sigma_{23} = 0\). Some care should be taken in using equations (21) and (22) directly since an extra constraint is introduced and they were derived assuming \(\gamma_{131} \neq 0\). Yet the only nontrivial cases are those obtained from this system with \(\gamma_{131} = \sigma_{23} = 0\) and \(B \neq 0\). From the two first constraints \(\sigma_{12} = \sigma_{13} = 0\) is obtained. \(\sigma_{11}\) and \(\sigma_{22}\) can be solved for from the two others. The system of differential equations is reduced to a system for \(\dot{v}, \dot{B}\), and \(\dot{\theta}\). This can be reduced to a system of two differential equations since the equations for \(\dot{v}\) and \(\dot{B}\) in this case can be combined to

\[\frac{\dot{v}}{v} = \frac{\dot{B}}{B} + \left(\frac{1}{\gamma} - 1\right) \frac{1}{\rho}\]

with solution

\[B = C_1 v^{\frac{2\gamma - 1}{\gamma(\gamma - 2)}}\]  \hspace{1cm} \text{(31)}
where $C_1$ is a constant of integration. The metric is in this case given by:

\[
\begin{align*}
\omega^0 &= -\frac{d\rho}{\gamma\rho^2} - \nu\omega^1, \quad \omega^1 = -\frac{1}{B} dx \\
\omega^2 &= k_2\rho^{-\frac{1}{2}}e^{\frac{3}{2}k_2\rho}e^{-\nu}dy, \\
\omega^3 &= k_3\rho^{-\frac{1}{2}}e^{\nu + \frac{3}{2}k_2\rho}e^{-\nu}dz.
\end{align*}
\]

1. LRS solutions

LRS solutions are obtained by putting $\sigma_{11} = -2\sigma_{22}$ in the above equations (the LRS symmetry lies in the 23-plane). If one solves the constraints for $\sigma_{22}$ and $\theta$ the system is reduced to one differential equation for $v$.

For $\gamma = 1$ and $\Lambda = 0$ the equations can be completely integrated [19,20]. The solution in our variables is given by:

\[
\begin{align*}
\sigma_{11} &= -2\sigma_{22} = \pm \frac{\kappa^2 (3C\kappa + 2)}{3 (D - \kappa\sqrt{1 + C\kappa})}, \\
\theta &= \mp \frac{3D\kappa\sqrt{1 + C\kappa} - \kappa^2 (2 + \frac{3}{2}C\kappa)}{D - \kappa\sqrt{1 + C\kappa}}, \\
v &= \mp \frac{\kappa}{D - \kappa\sqrt{1 + C\kappa}}, \quad \gamma_{122} = -\kappa
\end{align*}
\]

where $C$ and $D$ are constants of integration and $\kappa$ and $\rho$ are related as

\[
\rho = \frac{3CD\kappa^3}{D - \kappa\sqrt{1 + C\kappa}}.
\]

The basis 1-forms are

\[
\begin{align*}
\omega^0 &= \pm \frac{d\kappa}{\kappa^2\sqrt{1 + C\kappa}} \mp dx, \quad \omega^1 = -\frac{D - \kappa\sqrt{1 + C\kappa}}{\kappa} dx, \\
\omega^2 &= \frac{e^{Dx}}{\kappa}dy, \quad \omega^3 = \frac{e^{Dx}}{\kappa} dz.
\end{align*}
\]

From (33) one finds essentially two types of solutions. If both $C$ and $D$ are positive the density rises from zero to infinity when $\kappa$ goes from zero to $\k_1$, where $\k_1$ is given by

\[
D - \k_1\sqrt{1 + C\k_1} = 0.
\]

The tilt $v$ then also increases from zero to infinity, and hence the hypersurfaces of homogeneity remain spacelike in this interval. The maximum value is obtained for $\kappa_2$

\[
C\kappa_2^2 + 2D\sqrt{1 + C\kappa_2} = 0.
\]

giving

\[
|v_{\text{max}}| = \frac{\sqrt{1 + C\kappa_2}}{1 + \frac{3}{2}C\kappa_2}
\]

that is less then one and hence the hypersurfaces of homogeneity remain spacelike for all times.

A perturbative calculation (see IV for the perturbative method) with a small $\gamma_{131} = \epsilon\gamma_{131}^{(1)}$ around the exact solution gives

\[
\gamma_{131}^{(1)} = F \frac{\kappa}{D - \kappa\sqrt{1 + C\kappa}}
\]

where $F$ is a constant of integration. Hence the perturbation grows as the other quantities for the first case ($C, D > 0$) as $\kappa \to \kappa_1$. However, for the second case ($D < 0$) it goes to zero when $\kappa \to \infty$.

G. Solutions with a timelike homothetic motion

A spacetime has a homothetic Killing vector $\xi$ if

\[
\xi_{\mu,\nu} + \xi_{\nu,\mu} = 2c\epsilon_{\mu\nu}
\]

is satisfied for some constant $c$. This implies that quantities of the same dimension scale in the same way in the direction of $\xi$. For a timelike homothetic Killing vector it is hence easy to show that for perfect fluids all components of the Riemann tensor, $R_{ijkl}$, are proportional to $\rho$, the Ricci rotation coefficients to $\rho^{1/2}$ and $\rho_i$ to $\rho^{3/2}$, see e.g. [26]. The field equations are then reduced to algebraic equations.

Unfortunately this limitation is quite restrictive and the only solution of this type is the flat Friedmann universe (of Bianchi type I).

IV. PERTURBATIVE SOLUTIONS

In this section we consider perturbative solutions to the system (21) and (22). In general perturbative solutions to systems of non-linear equations with constraints need not correspond to exact solutions, but in the present case the system can be reduced to (22), that under reasonable assumptions is known to be integrable with five constants of integration. Hence a perturbative solution, obtained by solving the Taylor expanded equations, should agree with the Taylor expansion of some exact solution. This is also favoured by comparing the perturbative solutions with a numerical solving of the full system, see section V.

We here first briefly recall some basic results of perturbation theory. Calling the four functions solved for from the algebraic constraints (20) $g_a, a, b,.. = 1,.., 4$, the reduced system can be written as
\[ f_i = F_i(f_j, g_a, \rho), \quad i, j, \ldots = 1, \ldots, 5, \]
\[ 0 = G_a(f_j, g_b, \rho). \]  
(36)

The functions \( f_i \) and \( g_a \) are expanded around the solution \( f_i^{(0)} \) and \( g_a^{(0)} \) of (36) in the small parameter \( \epsilon \) as
\[ \Delta f_i = f_i - f_i^{(0)} = \epsilon f_i^{(1)} + \epsilon^2 f_i^{(2)} + \ldots \]
\[ \Delta g_a = g_a - g_a^{(0)} = \epsilon g_a^{(1)} + \epsilon^2 g_a^{(2)} + \ldots \]  
(37)

giving, with \( H_p \) equal to \( F_i \) or \( G_a \),
\[ H_p = H_p(f_j^{(0)}, g_a^{(0)}, \rho) + \frac{\partial H_p}{\partial f_j} \Delta f_j + \frac{\partial H_p}{\partial g_a} \Delta g_a + \frac{1}{2} \frac{\partial^2 H_p}{\partial f_j \partial g_a} \Delta f_j \Delta g_a + \epsilon \left( \frac{\partial H_p}{\partial f_j} f_j^{(1)} + \frac{\partial H_p}{\partial g_a} g_a^{(1)} \right) + \epsilon^2 \left( \frac{\partial^2 H_p}{\partial f_j \partial f_j} f_j^{(1)} f_j^{(1)} + \frac{\partial^2 H_p}{\partial f_j \partial g_a} f_j^{(1)} g_a^{(1)} \right) \ldots \]  
(38)

where the partial derivatives are evaluated at \( f_i = f_i^{(0)} \) and \( g_a = g_a^{(0)} \). Identifying equal powers of \( \epsilon \) in (38), using (37) and (38) with \( H_p = F_i \) or \( G_a \), one can solve for \( g_a^{(n)} \) in terms of \( f_i^{(n)} \) and lower order quantities as
\[ g_a^{(n)} = - (G^{-1})_a \frac{\partial G_b}{\partial f_j} f_j^{(n)} + G_a^{(n-1)} \]  
(39)

where \( (G^{-1})_a \) is the inverse of \( \frac{\partial G_b}{\partial g_a} \) (assuming that it is invertible) and \( G_a^{(n-1)} \) only depends on \( f_j^{(m)} \) and \( g_a^{(m)} \) up to order \( n-1 \), i.e. \( m \leq n-1 \). Substitution of (39), (37) and (38) with \( H_p = F_i \) into (36) and identification of equal powers of \( \epsilon \) now gives
\[ f_i^{(n)} = - \left( \frac{\partial F_i}{\partial f_j} - \frac{\partial F_j}{\partial f_i} (G^{-1})_a \frac{\partial G_b}{\partial f_j} \right) f_j^{(n)} = F_i^{(n-1)} \]  
(40)

where \( F_i^{(n-1)} \) only depends on \( f_j^{(m)} \) and \( g_a^{(m)} \) up to order \( n-1 \) and for \( n = 1 \) reduces to \( F_i^{(0)} = F_j^{(0)} = 0 \). From this equation we recall the result that the homogeneous parts of the differential equations are the same to each order in \( \epsilon \), and hence the integration constants add up as \( k_i = k_i^{(1)} + \epsilon^2 k_i^{(2)} + \ldots \), so that the correct number of constants is maintained to any order.

**A. First order perturbations**

We here consider perturbations to first order in the small parameter \( \epsilon \) around the open Friedmann universe.

The nonzero elements in the set \( S \) for the Friedmann universe are given by
\[ \theta^{(0)} = \pm \sqrt{3} \sqrt{3 k_1^{(0)} \rho^{\mathcal{H}} + \rho + \Lambda}, \]
\[ \gamma_{133}^{(0)} = \gamma_{122}^{(0)}, \]
\[ \gamma_{131}^{(0)} = \gamma_{232}^{(0)} = \pm \sqrt{3 k_1^{(0)} \rho^{\mathcal{H}} - (\gamma_{122}^{(0)})^2} \]  
(41)

where \( k_1^{(0)} \) is a constant of integration. In the following we choose \( \gamma_{122}^{(0)} = 0 \), so that \( \gamma_{131}^{(0)} = \gamma_{232}^{(0)} = k_1^{(0)} \rho^{\mathcal{H}} \). The freedom in one of the Ricci rotation coefficients is due to that only the \( \gamma_{0ij} \) appear in \( S \) for the Friedmann universe due to the isotropy. Note, however, that the resulting perturbed solutions are depending on this choice. For example, if we instead had chosen \( \gamma_{131}^{(0)} = 0 \), the first order perturbations would have had zero vorticity.

Instead of solving for \( \sigma_{22} \) from the last of equations (21) we solve this equation for \( \theta \), and hence the differential equation for \( \theta \) in (22) is replaced by the corresponding for \( \sigma_{22}, (24) \).

To first order we write the elements in \( S \) as
\[ v = e \nu \left( 1, \gamma_{ijk} = \gamma_{ijk}^{(0)} + \epsilon \gamma_{ijk}^{(1)} \right) \]

Expanding the system (21) and (22) (with the last equation replaced by (24)) to first order then gives
\[ a_1 = e k_3 \rho^{\mathcal{H} - 1} (1 - \gamma), \]
\[ \sigma_{11} = - \sigma_{22} = - e k_3 \rho^{\mathcal{H}}, \quad \sigma_{12} = e k_4 \rho^{\mathcal{H}}, \]
\[ \sigma_{13} = - e \frac{k_4 k_3}{k_1} \rho^{\mathcal{H} - 1} - \frac{k_1 k_3}{k_2} \rho^{\mathcal{H} - 1}, \]
\[ \omega_{13} = e \frac{k_4 k_3}{k_2} \rho^{\mathcal{H} - 1}, \quad \gamma_{131} = k_1 \rho^{\mathcal{H}}, \]
\[ \gamma_{122} = \epsilon \gamma_{122}^{(1)} \]  
where \( \gamma_{122}^{(1)} = k_5 \rho^{\mathcal{H} - 1} - \frac{k_3}{6} \rho^{\mathcal{H}} \int \frac{2}{\gamma} (\gamma - 1) \rho^{\mathcal{H} - 2} + \frac{\rho^{\mathcal{H} - 1}}{\theta} \]  
dP \rho \]
\[ \theta = \pm \sqrt{3} \sqrt{3 k^2 \rho^{\mathcal{H}} + \rho + \Lambda}, \]
\[ v = e k_3 \rho^{\mathcal{H} - 1} \]  
(42)

where \( k_1 = k_1^{(0)} + e k_1^{(1)}, \quad k_2 = k_2^{(1)} \), \( k_3 = k_3^{(1)} \), \( k_4 = k_4^{(1)} \) and \( k_5 = k_5^{(1)} \) are the five constants of integration.

The integral in \( \gamma_{122} \) can be evaluated for certain values of \( \gamma \) if \( \Lambda = 0 \). For \( \gamma = 1 \) (dust) one gets
\[ \gamma_{122}^{(1)} = k_5 \rho^{\mathcal{H}} \pm \frac{k_4}{9} (\rho^{\mathcal{H}} - 6 k_1^2) \],
for \( \gamma = \frac{4}{3} \) (radiation)
\[ \gamma_{122}^{(1)} = k_5 \rho^{\mathcal{H}} - \frac{k_4}{3} \rho^{\mathcal{H}} \pm k_1 \ln \left( \frac{\rho^{\mathcal{H}} \pm \frac{1}{\sqrt{3}} \rho^{\mathcal{H}} - \sqrt{3 k_1}}{\rho^{\mathcal{H}} \pm \frac{1}{\sqrt{3}} \rho^{\mathcal{H}} + \sqrt{3 k_1}} \right) \]
and for $\gamma = 2$ (stiff matter), with $\dot{\gamma}_5 = \ddot{\gamma}_5 \pm k_3/2$,
\[
\gamma_{122}^{(1)} = \ddot{\gamma}_5 \rho^{\frac{1}{3}} \pm \frac{\sqrt{3} k_3 \rho^{\frac{1}{3}}}{6} \left[ 4 \ln \left( \frac{\rho^{\frac{1}{3}}}{\sqrt{3} k_1} \pm \frac{\rho^{\frac{1}{3}} - \frac{1}{3} k_1 \rho^{\frac{1}{3}}}{\sqrt{3} k_1} \right) - 9 \sqrt{3} k_1^2 \rho^{\frac{1}{3}} \pm \theta \right].
\]
The integral can also be evaluated with nonzero $\Lambda$ for $\gamma = 1$ if $k_1 = 0$, corresponding to that the background metric is of Bianchi type I (but the perturbed one is of type V), giving
\[
\gamma_{122}^{(1)} = \ddot{\gamma}_5 \rho^{\frac{1}{3}} \pm \frac{k_3}{3 \sqrt{3}} \rho^{\frac{1}{3}} \sqrt{\rho + \Lambda}. \tag{43}
\]

B. The metric to first order

The form of the metric (26-28) is not suitable for a perturbative calculation since some of the coefficients diverge when $\epsilon \to 0$. This can be avoided by introducing new coordinates according to
\[
y = \epsilon \tilde{y}, \quad z = \frac{\tilde{z}}{\epsilon}. \tag{44}
\]
When expanding the 1-forms to first order in $\epsilon$ the tilt $v$ to second order, given in IV.C, will be needed. To first order in $\epsilon$ one then obtains the following 1-forms (with $C_1 = C_2 = 1$)
\[
\omega^0 = - \frac{d\rho}{\gamma \rho \theta} - \epsilon \rho^{\frac{1}{3}} e^{-\epsilon x} d\tilde{y}
\]
\[
\omega^1 = \frac{1}{k_1} \rho^{\frac{1}{3}} e^{-\epsilon x} (1 + \epsilon A_1) d\tilde{y} + \frac{k_3 \rho^{\frac{1}{3}}}{k_2 k_3} e^{-\epsilon x} d\tilde{z}
\]
\[
\omega^2 = - \epsilon \frac{2 k_2}{k_3} e^{-\epsilon x} (1 + \epsilon A_1) d\tilde{y}
\]
\[
\omega^3 = \frac{1}{k_1} \rho^{\frac{1}{3}} e^{-\epsilon x} d\tilde{z} + \frac{\epsilon}{k_1} e^{-\epsilon x} \left( \frac{1}{k_3} \rho^{\frac{1}{3}} \gamma^{(1)}_{122} - \frac{1}{3} \rho^{\frac{1}{3}} \theta^{\frac{1}{3}} \right) d\tilde{y}
\]
where $\theta$ and $\gamma_{122}^{(1)}$ are given by (42) and $A_1$ by
\[
A_1(\rho) = - k_2 \int \frac{\rho^{\frac{1}{3}}}{\gamma \theta^{-1}} d\rho. \tag{45}
\]
For dust ($\gamma = 1$) and $\Lambda = 0$ $A_1$ becomes
\[
A_1 = - \frac{2 k_2 \theta}{3} \left( 1 - 6 k_2^2 \rho^{\frac{1}{3}} \right)
\]
and with $\Lambda \neq 0$ and $k_1 = 0$
\[
A_1 = - \frac{2 k_2 \theta}{3}.
\]

C. Second order perturbations

From (39) we see that the second order perturbations can be obtained up to quadratures. A full second order calculation will be done in a forthcoming paper. Here we focus on the tilt, $v$, whose $n$:th order equations decouple from other $n$:th order quantities, and also since it will be needed to second order to obtain the metric to first order.

To second order one obtains
\[
v = \epsilon k_3 \rho^{\frac{1}{3}} \left( 1 - \frac{e k_2}{\gamma} \int \frac{\rho^{\frac{1}{3}}}{\gamma \theta^{-1}} d\rho \right) \tag{46}
\]
where now $k_3 = k_3^{(1)} + k_3^{(2)}$. For $\Lambda = 0$ and $\gamma = 1$ we have
\[
v = \epsilon k_3 \rho^{\frac{1}{3}} \left( 1 - \frac{2 e k_2}{\gamma} \theta (1 - 6 k_2^2 \rho^{\frac{1}{3}}) \right) = \epsilon k_3 \rho^{\frac{1}{3}} \left( 1 \mp \frac{2 e k_2}{\gamma} \sqrt{\rho^{\frac{1}{3}} + 3 k_1^2} \rho^{\frac{1}{3}} - 6 k_1^2 \right)
\]
and for $\Lambda \neq 0$, $\gamma = 1$ and $k_1 = 0$
\[
v = \epsilon k_3 \rho^{\frac{1}{3}} \left( 1 \mp \frac{2 e k_2}{\gamma} \sqrt{\rho^{\frac{1}{3}} + \Lambda} \right).
\]
For $\Lambda = 0$ and $\gamma = 4/3$ the tilt is given by
\[
v = \epsilon k_3 \left( 1 - e k_2 \rho^{\frac{1}{3}} \right) = \epsilon k_3 \left( 1 \mp e k_2 \sqrt{3/\rho^{\frac{1}{3}} + 3 k_1^2} \right).
\]
These results are in accordance with the exact equation (22) for $v$ that shows that the behaviour is determined by the sign of $\sigma_{11}/\theta$ ($\sigma_{11} = \epsilon k_2 \rho^{1/\gamma}$).
V. NUMERICAL SOLUTIONS

In this section we solve the system (21-22) numerically using the Runge-Kutta method with a truncation error of order $h^4$ in the step length $h$. A comparison between the code and the exact solutions (29) gives agreement to high accuracy.

A. Comparison with perturbative calculations

To check the perturbative calculation in section IV (or conversely the numerical method) we have solved the system numerically for small deviations (at start) from the Friedmann models and compared with the perturbative solutions. The constants of integration are chosen so that the numerical and perturbative solutions agree for the starting value of the independent variable $\rho$. In the two following figures first and second order perturbations together with the numerical solutions for $v$ with $\gamma = 1$ (fig. 1) and $\gamma = 4/3$ (fig. 2) are depicted. The agreement is similar for the other quantities in $S$ and other values of $\gamma$.

B. Asymptotic behaviour

In [11] a dynamical system analysis was used to find that for $2/3 < \gamma < 2$ Bianchi type V models approach the Milne universe in the asymptotic future (for low densities). Figure 3 shows the components of the shear normalized with expansion $(\sigma_{ij}/\theta)$ for $\gamma = 1$, and they all vanish in the limit $\rho \to 0$. The tilt $v$ also approaches zero as shown in figure 4.
Figure 5 show the corresponding behaviour with a non-zero cosmological constant.

As found for the LRS case in [9] anisotropies may remain for late times with $\gamma \neq 1$. This is not in conflict with the result of [11]. An anisotropy remains in the matter fields, but since matter is getting infinitely diluted space-time still approaches isotropy. In figure 6 $\omega_{13}/\theta$ and $v$ are plotted for $\gamma = 4/3$. As seen they do not approach zero for small $\rho$.

For the LRS case two different behaviours close to the initial singularity was found in [9]. One where the tilt $v$ goes to zero and one where it passes the extreme value of one and grows towards infinity. In the second case, however, often singularities in some of the other quantities occur while $v$ and the density still are finite. The numerical runs seem to show that generically the tilt grows towards one, even if it initially may be decreasing for a long density interval. In figures 7 and 8 $v$, $\sigma_{11}$ and $\sigma_{22}$ for $\gamma = 1$ and $\gamma = 4/3$ are shown respectively. For these particular cases $v$ was decreasing for as far the numerical calculations could be carried out (about 20 times as long as shown in the picture). Note that asymptotically $\sigma_{22} = -\sigma_{11}$ and hence these spacetimes do not approach LRS.
Small changes of the initial values are sufficient to make \( v \) eventually turn and start growing as shown in figure 9.

Due to an apparent singularity in the equations for \( v = 1 \) we have not been able to follow the evolution beyond this value. For the LRS case it is possible to rewrite the only remaining differential equation to avoid this problem, but in the general case the expressions become quite complex. The equation for \( \gamma = 4/3 \) in the LRS case is given by

\[
\frac{\dot{v}}{v^2} = \frac{(3B^2 + 2\rho) \left[ 2v (9B^2 + 2v^2\rho) \pm \right]}{2\rho (3B^2(9 - v^2) + 4v^2\rho) (9B^2 + (v^2 + 3)v)} \pm (v^2 - 3) \sqrt{81B^4 + 9B^2v^2\rho + 27B^2\rho + 4v^2\rho^2},
\]

(47)

where \( B \) is given by (31). For the negative root the tilt may grow larger than one as seen in figure 10:

For this case, however, the kinematic quantities as well as the Weyl tensor diverge for \( v = \sqrt{3} \), as also found in [9].

VI. CONCLUSIONS

In this paper it was shown that the general Bianchi V cosmology with linear equation of state and cosmological constant can be reduced to an integrable system of five ordinary first order differential equations for quantities that give a complete local description of the geometry. The full line-element was found in terms of quadratures of these quantities. In general the solutions have expansion, shear and vorticity. The system was cast in a form suitable for perturbative calculations and the first order perturbations around the open Friedmann model with vorticity, being approximations to exact solutions, were constructed. Perturbative calculations to higher orders would be straightforward up to quadratures.

A numerical study was done and the results agree well with the perturbative ones in the appropriate domains. For large times (small densities) the results agree well with previous works [9–11]. Numerically we found that the tilt probably falls off towards zero when the density grows for special initial values, but generically it seems as if the tilt eventually always grows unlimited for large densities. In [10,11] the late time behaviour of (among others) Bianchi V solutions were studied using dynamical systems analysis. It would be of great interest to extend this analysis to early times.
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