A FUNCTION WITH SUPPORT OF FINITE MEASURE
AND “SMALL” SPECTRUM

FEDOR NAZAROV, ALEXANDER OLEVSKII

Abstract. We construct a function on \( \mathbb{R} \) supported on a set of finite measure whose spectrum has density zero.

1. The result

Let \( F \) be a function in \( L^2(\mathbb{R}) \). We say that it is supported on \( S \) if 
\[
F = 0 \text{ almost everywhere on } \mathbb{R} \setminus S.
\]
Suppose the set \( S \subset \mathbb{R} \) is of finite Lebesgue measure. Then the Fourier transform \( \hat{F} \) of \( F \) is a continuous function, so the spectrum of \( F \) is naturally defined as the closure of the set where \( \hat{F} \) takes non-zero values.

According to the uncertainty principle, the support and the spectrum of a (non-trivial) function \( F \) cannot be both “small sets”. This principle has various versions (see e.g. [HAJ94]).

In particular, the classic uniqueness theorem for analytic functions implies that if \( F \) is supported on an interval and it has a “spectral gap” (that is, \( \hat{F} = 0 \) on an interval ) then \( F = 0 \).

Another important result says that if the support \( S \) and the spectrum \( Q \) of \( F \) are both of finite measure then \( F = 0 \) [Ben74/85], [AB77].

On the other hand, \( F \) may have a support of finite measure and a spectral gap; see [Kr82], where such an example was constructed with \( F = 1_S \).

Answering a question posed by Benedicks, Kargaev and Volberg [KV92] constructed an example of a function \( F \) such that 
\[
|S| < +\infty, |\mathbb{R} \setminus Q| = +\infty
\]
(here and below by \( |A| \) we denote the Lebesgue measure of the set \( A \)).

The goal of this note is to prove the following

Theorem. There is a function \( F \in L^2(\mathbb{R}) \) supported by a set \( S \) of finite measure, such that
\[
|Q \cap (-R, R)| = o(R) \text{ as } R \to \infty.
\]
In addition, \( F \) can be chosen as the indicator function of \( S \).
The proof below is based on a simple construction, completely different from the ones in the cited papers.

2. Proof

2.1. Take a Schwartz function $F_0$ such that

$$0 \leq F_0(t) \leq 1 \quad (t \in \mathbb{R})$$

and its Fourier transform $\hat{F}_0$ is positive on $(-1, 1)$ and vanishes outside that interval. Define a sequence of functions $F_n$ recursively by

$$F_n := F_{n-1} + G_n \quad (n = 1, 2, \ldots),$$

where

$$G_n(t) := F_{n-1}(t)[1 - F_{n-1}(t)] \cos k_n t$$

We are going to prove that if the numbers $k_n$ grow sufficiently fast, then the sequence $F_n$ converges to a function $F$ satisfying the requirements of the theorem.

2.2. Clearly, $F_n$ and $G_n$ are Schwartz functions.

A simple induction shows that for every $t \in \mathbb{R}$, we have

$$|G_n(t)| \leq \max\{F_{n-1}(t), 1 - F_{n-1}(t)\}$$

and

$$0 \leq F_n(t) \leq 1.$$  

The Fourier transforms of $F_{n-1}[1 - F_{n-1}]$, $F_{n-1}^2[1 - F_{n-1}]$, and $F_{n-1}^2[1 - F_{n-1}]^2$ vanish outside a compact interval, so for each $n \geq 1$, we have:

$$\int_\mathbb{R} G_n = \int_\mathbb{R} F_{n-1} G_n = 0$$

and

$$\int_\mathbb{R} G_n^2 = \frac{1}{2} \int_\mathbb{R} F_{n-1}^2[1 - F_{n-1}]^2,$$

provided that $k_n$ is chosen sufficiently large. It follows that

$$\int_\mathbb{R} F_n = \int_\mathbb{R} F_0 =: C$$

and, thereby,

$$I_n := \int_\mathbb{R} F_n (1 - F_n) \leq C$$

(here, as usual, by $C$ we denote a positive constant that may vary from line to line).

Observe also that

$$I_n = \int_\mathbb{R} [F_{n-1} + G_n][1 - F_{n-1} - G_n] = I_{n-1} - \int_\mathbb{R} G_n^2.$$
which implies that
\[ \sum_{n \in [1,N]} \int_{\mathbb{R}} G_n^2 \leq I_0 - I_N \leq C, \]
and so
\[ \sum_n \int_{\mathbb{R}} G_n^2 \leq C \]

2.3. Define the sequence \( Q_n \) of intervals on (another copy of) \( \mathbb{R} \) recursively as follows:

\[ Q_0 := [-1,1], \]
\[ Q_n := \text{conv}(Q_{n-1} \cup [k_n + 2Q_{n-1}] \cup [-k_n + 2Q_{n-1}]) \]

(here \( \text{conv} E \) denotes the convex hull of a set \( E \subset \mathbb{R} \)). Clearly, for every \( n \),

\[ \text{spec } F_{n-1} \subset Q_{n-1}; \]
\[ \text{spec } G_n \subset [k_n + 2Q_{n-1}] \cup [-k_n + 2Q_{n-1}]. \]

Set \( Q := Q_0 \cup \bigcup_n ([k_n + 2Q_{n-1}] \cup [-k_n + 2Q_{n-1}]). \)

Choosing \( k_n \) growing sufficiently fast we can ensure that the spectra of \( G_n \) are pairwise disjoint and
\[ |Q \cap (-R,R)| = o(R) \text{ as } R \to \infty. \]

2.4. Consider the series \( F_0 + G_1 + G_2 + \ldots \). Since the spectra of the terms are pairwise disjoint, this series is orthogonal in \( L^2(\mathbb{R}) \). Then (3) implies that it converges in \( L^2(\mathbb{R}) \) to some non-trivial function \( F \). The partial sums of this series are \( F_n \). Take a subsequence \( F_{n_\ell} \) such that

\[ F_{n_\ell} \to F \text{ almost everywhere on } \mathbb{R} \text{ as } \ell \to \infty. \]

Recall that all \( F_n \) are non-negative functions, so (2) implies that
\[ F \geq 0 \text{ almost everywhere and } \int_{\mathbb{R}} F < \infty. \]

It follows from (1) and (3) that
\[ \sum_n \int_{\mathbb{R}} [F_n(1 - F_n)]^2 = 2 \sum_n \int_{\mathbb{R}} G_n^2 < +\infty, \]
so we must have
\[ F(1 - F) = \lim_{\ell \to \infty} F_{n_\ell}(1 - F_{n_\ell}) = 0 \text{ almost everywhere}, \]
which implies that $F$ is the indicator-function of a set $S$. According to (5), this set has finite measure. Clearly the spectrum of $F$ is a subset of $Q$. Due to (4) it has density zero. This finishes the proof.

**Remark.** Consider the function

$$h(R) := |Q \cap (-R, R)|.$$

In the conditions of the Theorem, it can not be bounded. However the proof above shows that it may increase arbitrarily slowly. It remains an open question, however, if $Q$ can have uniform density 0, i.e., if it is possible that

$$\lim_{R \to \infty} \sup_{x \in \mathbb{R}} \frac{1}{2R} |Q \cap (x - R, x + R)| = 0.$$

**References**

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