A note on equilibrium Glauber and Kawasaki dynamics for permanental point processes

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Abstract

We construct two types of equilibrium dynamics of an infinite particle system in a locally compact metric space $X$ for which a permanental point process is a symmetrizing, and hence invariant measure. The Glauber dynamics is a birth-and-death process in $X$, while in the Kawasaki dynamics interacting particles randomly hop over $X$. In the case $X = \mathbb{R}^d$, we consider a diffusion approximation for the Kawasaki dynamics at the level of Dirichlet forms. This leads us to an equilibrium dynamics of interacting Brownian particles for which a permanental point process is a symmetrizing measure.

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1 Introduction

Let $X$ be a locally compact Polish space and let $\nu$ be a Radon non-atomic measure on it. Let $\Gamma = \Gamma_X$ denote the space of all locally finite subsets (configurations) in $X$.

A Glauber dynamics (a birth-and-death process of an infinite system of particles in $X$) is a Markov process on $\Gamma$ whose formal (pre-)generator has the form

\[
(L_G F)(\gamma) = \sum_{x \in \gamma} d(x, \gamma \setminus x) (F(\gamma \setminus x) - F(\gamma)) + \int_X \nu(dx) b(x, \gamma) (F(\gamma \cup x) - F(\gamma)), \quad \gamma \in \Gamma.
\] (1.1)

Here and below, for simplicity of notation we write $x$ instead of $\{x\}$. The coefficient $d(x, \gamma \setminus x)$ describes the rate at which particle $x$ of configuration $\gamma$ dies, while $b(x, \gamma)$ describes the rate at which, given configuration $\gamma$, a new particle is born at $x$. 


A Kawasaki dynamics (a dynamics of hopping particles) is a Markov process on $\Gamma$ whose formal (pre-)generator is

$$
(L_K F)(\gamma) = \sum_{x \in \gamma} c(x, y, \gamma \setminus x)(F(\gamma \setminus x \cup y) - F(\gamma)), \quad \gamma \in \Gamma.
$$

(1.2)

The coefficient $c(x, y, \gamma \setminus x)$ describes the rate at which particle $x$ of configuration $\gamma$ hops to $y$, taking the rest of the configuration, $\gamma \setminus x$, into account.

Equilibrium Glauber and Kawasaki dynamics which have a standard Gibbs measure as symmetrizing (and hence invariant) measure were constructed in [19, 20]. In [22], this construction was extended to the case of an equilibrium dynamics which has a determinantal (fermion) point process as invariant measure. For further studies of equilibrium and non-equilibrium Glauber and Kawasaki dynamics, we refer to [3, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 21, 28] and the references therein.

The aim of this note is to show that general criteria of existence of Glauber and Kawasaki dynamics which were developed in [22] are applicable to a wide class of $\alpha$-permanental ($\alpha \in \mathbb{N}$) point processes, proposed by Shirai and Takahashi [30]. This class includes classical permanental (boson) point processes, see e.g. [5, 30]. We will also consider a diffusion approximation for the Kawasaki dynamics at the level of Dirichlet forms (compare with [15]). This will lead us to an equilibrium dynamics of interacting Brownian particles for which an $\alpha$-permanental point process is a symmetrizing measure. As a by-product of our considerations, we will also extend the result of [30] on the existence of $\alpha$-permanental point process.

2 Equilibrium Glauber and Kawasaki dynamics – general results

Let $X$ be a locally compact Polish space. We denote by $\mathcal{B}(X)$ the Borel $\sigma$-algebra on $X$, and by $\mathcal{B}_0(X)$ the collection of all sets from $\mathcal{B}(X)$ which are relatively compact. We fix a Radon, non-atomic measure on $(X, \mathcal{B}(X))$. (For most applications, the reader may think of $X$ as $\mathbb{R}^d$ and $\nu$ as the Lebegue measure.)

The configuration space $\Gamma$ over $X$ is defined as the set of all subsets of $X$ which are locally finite

$$
\Gamma := \{ \gamma \subset X : |\gamma_\Lambda| < \infty \text{ for each } \Lambda \in \mathcal{B}_0(X) \},
$$

where $|\cdot|$ denotes the cardinality of a set and $\gamma_\Lambda := \gamma \cap \Lambda$. One can identify any $\gamma \in \Gamma$ with the positive Radon measure $\sum_{x \in \gamma} \varepsilon_x$, where $\varepsilon_x$ is the Dirac measure with mass at $x$ and $\sum_{x \in \emptyset} \varepsilon_x := \text{zero measure}$. The space $\Gamma$ can be endowed with the vague topology, i.e., the weakest topology on $\Gamma$ with respect to which all maps

$$
\Gamma \ni \gamma \mapsto \langle \varphi, \gamma \rangle := \int_X \varphi(x) \gamma(dx) = \sum_{x \in \gamma} \varphi(x), \quad \varphi \in C_0(X),
$$

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are continuous. Here, $C_0(X)$ is the space of all continuous, real-valued functions on $X$ with compact support. We denote the Borel $\sigma$-algebra on $\Gamma$ by $\mathcal{B}(\Gamma)$. A point process in $X$ is a probability measure on $(\Gamma, \mathcal{B}(\Gamma))$.

We fix a point process $\mu$ which satisfies the so-called condition $(\Sigma_{\nu}')$ [5, 26], i.e., there exist a measurable function $r : X \times \Gamma \to [0, +\infty]$, called the Papangelou intensity of $\mu$, such that

$$
\int \mu(d\gamma) \int_X \gamma(dx) F(x, \gamma) = \int \nu(dx) \int_X r(x, \gamma) F(x, \gamma \cup x) \quad (2.1)
$$

for any measurable function $F : X \times \Gamma \to [0, +\infty]$. The condition $(\Sigma_{\nu}')$ can be thought of as a kind of weak Gibbsianess of $\mu$. Intuitively, we may treat the Papangelou intensity as

$$
r(x, \gamma) = \exp[-E(x, \gamma)], \quad (2.2)
$$

where $E(x, \gamma)$ is the relative energy of interaction between particle $x$ and configuration $\gamma$.

To define an equilibrium Glauber dynamics for which $\mu$ is a symmetrizing measure, we fix a death coefficient as a measurable function $d : X \times \Gamma \to [0, +\infty]$, and then define a birth coefficient $b : X \times \Gamma \to [0, +\infty]$ by

$$
b(x, \gamma) = d(x, \gamma) r(x, \gamma), \quad (x, \gamma) \in X \times \Gamma. \quad (2.3)
$$

To define a Kawasaki dynamics, we fix a measurable function $c : X^2 \times \Gamma^2 \to [0, +\infty]$ which satisfies

$$
r(x, \gamma)c(x, y, \gamma) = r(y, \gamma)c(y, x, \gamma), \quad (x, y, \gamma) \in X^2 \times \Gamma. \quad (2.4)
$$

Formulas (2.3) and (2.4) are called the balance conditions [13, 14]. We will also assume that the function $c(x, y, \gamma)$ vanishes if at least one of the functions $r(x, \gamma)$ and $r(y, \gamma)$ vanishes, i.e.,

$$
c(x, y, \gamma) = c(x, y, \gamma)\chi_{\{r>0\}}(x, \gamma)\chi_{\{r>0\}}(y, \gamma). \quad (2.5)
$$

Here, for a set $A$, $\chi_A$ denotes the indicator function of $A$. We refer to [22, Remark 3.1] for a justification of this assumption, which involves the interpretation of $r(x, \gamma)$ as in (2.2), see also Remark 2.4 below.

We denote by $\mathcal{F}C_b(C_0(X), \Gamma)$ the space of all functions of the form

$$
\Gamma \ni \gamma \mapsto F(\gamma) = g(\langle \phi_1, \gamma \rangle, \ldots, \langle \phi_N, \gamma \rangle), \quad (2.6)
$$

where $N \in \mathbb{N}$, $\phi_1, \ldots, \phi_N \in C_0(X)$ and $g \in C_b(\mathbb{R}^N)$. Here, $C_b(\mathbb{R}^N)$ denotes the space of all continuous bounded functions on $\mathbb{R}^N$. We assume that, for each $\Lambda \in \mathcal{B}_0(X)$,

$$
\int \mu(d\gamma) \int_\Lambda \gamma(dx) d(x, \gamma \setminus x) < \infty, \quad (2.7)
$$

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As easily seen, conditions (2.7) and (2.8) are sufficient in order to define bilinear forms.

\[ \int_{\Gamma} \mu(d\gamma) \int_{X} \gamma(dx) \int_{X} \nu(dy) c(x, y, \gamma \setminus x)(\chi_{\Lambda}(x) + \chi_{\Lambda}(y)) < \infty. \] (2.8)

The following theorem was essentially proved in [22].

**Theorem 2.1.** (i) Assume that a point process \( \mu \) satisfies (2.1). Assume that conditions (2.3), (2.7), respectively (2.4), (2.5), (2.8), and (2.9) are satisfied. Let \( \sharp = G, K \). Then the bilinear form \((\mathcal{E}_{\sharp}, \mathcal{F}_{\sharp}(C_{0}(x), \Gamma))\) is closable in \( L^{2}(\Gamma, \mu) \) and its closure will be denoted by \((\mathcal{E}_{\sharp}, D(\mathcal{E}_{\sharp}))\). Further there exists a conservative Hunt process (Glauber, respectively Kawasaki dynamics)

\[ M^{\sharp} = \left( \Omega^{\sharp}, \mathcal{F}^{\sharp}, (\mathcal{F}_{t}^{\sharp})_{t \geq 0}, (\Theta_{t}^{\sharp})_{t \geq 0}, (X^{\sharp}(t))_{t \geq 0}, (P_{\gamma}^{\sharp})_{\gamma \in \Gamma} \right) \]

on \( \Gamma \) which is properly associated with \((\mathcal{E}_{\sharp}, D(\mathcal{E}_{\sharp}))\), i.e., for all \((\mu\text{-version of}) \ F \in L^{2}(\Gamma, \mu) \) and \( t > 0 \)

\[ \Gamma \ni \gamma \mapsto p_{\gamma}^{\sharp}F(\gamma) := \int_{\Omega^{\sharp}} F(X^{\sharp}(t)) \, dP_{\gamma}^{\sharp} \]

is an \( \mathcal{E}^{\sharp}\)-quasi continuous version of \( \exp(tL_{\sharp})F \), where \((-L_{\sharp}, D(L_{\sharp}))\) is the generator of \((\mathcal{E}_{\sharp}, D(\mathcal{E}_{\sharp}))\). \( M^{\sharp} \) is up-to \( \mu\)-equivalence unique. In particular, \( M^{\sharp} \) is \( \mu\)-symmetric and has \( \mu \) as invariant measure.

(ii) \( M^{\sharp} \) from (i) is up to \( \mu\)-equivalence unique between all Hunt processes

\[ M' = \left( \Omega', \mathcal{F}', (\mathcal{F}_{t}')_{t \geq 0}, (\Theta_{t}')_{t \geq 0}, (X'(t))_{t \geq 0}, (P_{\gamma}')_{\gamma \in \Gamma} \right) \]

on \( \Gamma \) having \( \mu \) as invariant measure and solving a martingale problem for \((L_{\sharp}, D(L_{\sharp}))\), i.e., for all \( G \in D(H_{\sharp}) \)

\[ \tilde{G}(X'(t)) - \tilde{G}(X'(0)) - \int_{0}^{t} (L_{\sharp}G)(X'(s)) \, ds, \quad t \geq 0, \]
is an \((\mathcal{F}_t')\)-martingale under \(P'_\gamma\) for \(\mathcal{E}_t\)-q.e. \(\gamma \in \Gamma\). Here, \(\tilde{G}\) denotes an \(\mathcal{E}_t\)-quasi-continuous version of \(G\).

(iii) Further assume that, for each \(\Lambda \in \mathcal{B}_0(X)\),
\[
\int_{\Lambda} \gamma(dx) d(x, \gamma \setminus x) \in L^2(\Gamma, \mu), \quad \int_{\Lambda} \nu(dx) b(x, \gamma) \in L^2(\Gamma, \mu),
\]
(2.10)
in the Glauber case, and
\[
\int_{X} \gamma(dx) \int_{X} \nu(dy) c(x, y, \gamma \setminus x)(\chi_\Lambda(x) + \chi_\Lambda(y)) \in L^2(\Gamma, \mu)
\]
(2.11)
in the Kawasaki case. Then \(\mathcal{F}C_b(C_0(X), \Gamma) \subset D(L)\), and for each \(F \in \mathcal{F}C_b(C_0(X), \Gamma)\), \(L\) is given by formulas (1.1) and (1.2), respectively.

Remark 2.1. We refer to [24] for an explanation of notions appearing in Theorem 2.1, see also a brief explanation of them in [22].

Proof of Theorem 2.1. The statement follows from Theorems 3.1 and 3.2 in [22]. Note that, although these theorems are formulated for determinantal point processes only, their proof only uses the \((\Sigma'_\nu)\) property of these point processes. Note also that condition (2.9) is formulated in [22] only for \(v = 1\), however the proof of Lemma 3.2 in [22] admits a straightforward generalization to the case of an arbitrary \(v \in \mathbb{R}\).

Remark 2.2. Part (iii) of Theorem 2.1 states that the operator \((-L^*_\gamma, D(L^*_\gamma))\) is the Friedrichs’ extention of the operator \((-L^*_\gamma, \mathcal{F}C_b(C_0(X), \Gamma))\) defined by formulas (1.1), (1.2), respectively.

Let us fix a parameter \(s \in [0, 1]\) and define
\[
d(x, \gamma) := r(x, \gamma)^{s-1}\chi_{\{r>0\}}(x, \gamma), \quad (x, \gamma) \in X \times \Gamma,
\]
(2.12)
\[
b(x, \gamma) := r(x, \gamma)^s\chi_{\{r>0\}}(x, \gamma), \quad (x, \gamma) \in X \times \Gamma,
\]
(2.13)
\[
c(x, y, \gamma) := a(x, y)r(x, \gamma)^{s-1}r(y, \gamma)^s\chi_{\{r>0\}}(x, \gamma)\chi_{\{r>0\}}(y, \gamma),
\]
(2.14)
\[
(x, y, \gamma) \in X^2 \times \Gamma.
\]
Here the function \(a : X^2 \to [0, +\infty)\) is bounded, measurable, symmetric (i.e., \(a(x, y) = a(y, x)\)), and satisfies
\[
\sup_{x \in X} \int_{X} a(x, y) \nu(dy) < \infty.
\]
(2.15)
Note that the balance conditions (2.3) and (2.4) are satisfied for these coefficients, and so is condition (2.5).

Remark 2.3. Note that, if \(X = \mathbb{R}^d\) and \(a(x, y)\) has the form \(a(x - y)\) for a function \(a : \mathbb{R}^d \to [0, \infty)\), then condition (2.15) means that \(a \in L^1(\mathbb{R}^d, dx)\). (Here and below, in the case \(X = \mathbb{R}^d\), we use an obvious abuse of notation.)
Remark 2.4. Using representation (2.2), we can rewrite formulas (2.12)–(2.14) as follows:

\[
    d(x, \gamma \setminus x) = \exp[(1 - s)E(x, \gamma \setminus x)]\chi_{\{E < +\infty\}}(x, \gamma \setminus x),
\]
\[
    b(x, \gamma \setminus x) = \exp[-sE(x, \gamma \setminus x)]\chi_{\{E < +\infty\}}(x, \gamma \setminus x),
\]
\[
    c(x, y, \gamma \setminus x) = a(x, y) \exp[(1 - s)E(x, \gamma \setminus x) - sE(y, \gamma \setminus x)]
    \times \chi_{\{E < +\infty\}}(x, \gamma \setminus x)\chi_{\{E < +\infty\}}(y, \gamma \setminus x).
\]

So, if the corresponding dynamics exist, one can give the following heuristic description of them: Both dynamics are concentrated on configurations \(\gamma \in \Gamma\) such that, for each \(x \in \gamma\), the relative energy of interaction between \(x\) and the rest of configuration, \(\gamma \setminus x\), is finite; those particles tend to die, respectively hop, which have a high energy of interaction with the rest of the configuration, while it is more probable that a new particle is born at \(y\), respectively \(x\) hops to \(y\), if the energy of interaction between \(y\) and the rest of the configuration is low.

Let us assume that the point process \(\mu\) satisfies:

\[
    \forall \Lambda \in \mathcal{B}_0(X): \int_\Lambda \gamma(dx) \in L^2(\Gamma, \mu).
\]

Then, by choosing \(u = 1 - s\) and \(v = -s\) in (2.9), we conclude that the coefficient \(c\) given by (2.14) satisfies (2.9).

We will construct below a class of point processes \(\mu\) for which the coefficients \(d\), \(b\) and \(c\) given above satisfy the other conditions of Theorem 2.1.

3 Permanental point processes and corresponding equilibrium dynamics

Let \(K\) be a linear, bounded, self-adjoint operator on the real space \(L^2(X, \nu)\). Further assume that \(K \geq 0\) and \(K\) is locally of trace class, i.e., \(\text{Tr}(P_\Lambda KP_\Lambda) < \infty\) for all \(\Lambda \in \mathcal{B}_0(X)\), where \(P_\Lambda\) denotes the operator of multiplication by \(\chi_\Lambda\). Hence, each operator \(P_\Lambda \sqrt{K}\) is of Hilbert–Schmidt class. Following [23] (see also [12, Lemma A.4]), we conclude that \(\sqrt{K}\) is an integral operator whose integral kernel, \(\kappa(x, y)\), satisfies

\[
    \int_\Lambda \int_X \nu(dx)\nu(dy)\kappa(x, y)^2 < \infty \quad \text{for all} \quad \Lambda \in \mathcal{B}_0(X).
\]

In particular,

\[
    \kappa(x, \cdot) \in L^2(X, \nu) \quad \text{for } \nu\text{-a.a.} \quad x \in X.
\]
Hence, $K$ is an integral operator whose integral kernel can be chosen as
\[
k(x, y) = \int_X \kappa(x, z) \kappa(z, y) \nu(dz) = \int_X \kappa(x, z) \kappa(y, z) \nu(dz) = (\kappa(x, \cdot), \kappa(y, \cdot))_{L^2(X, \nu)}. \tag{3.3}\]

We also have, for each $\Lambda \in B_0(X)$,
\[
\text{Tr}(P_{\Lambda}KP_{\Lambda}) = \|\sqrt{KP_{\Lambda}}\|_{\text{HS}}^2 = \int_{\Lambda} \nu(dx) \int_X \nu(dy) \kappa(x, y)^2 = \int_{\Lambda} k(x, x) \nu(dx), \tag{3.4}\]

where $\|\cdot\|_{\text{HS}}$ denotes the Hilbert–Schmidt norm.

**Proposition 3.1.** There exists a random field $(Y(x))_{x \in X}$ on a probability space $(\Omega, \mathcal{A}, P)$ such that the mapping
\[
X \times \Omega \ni (x, \omega) \mapsto Y(x, \omega) \tag{3.5}\]
is measurable, and for $\nu$-a.a. $x \in X$, $Y(x)$ is a Gaussian random variable with mean 0 and such that
\[
\mathbb{E}(Y(x)Y(y)) = k(x, y) \quad \text{for } \nu^{\otimes 2}$-a.a. $(x, y) \in X^2 \text{ and } \nu$-a.a. $x = y \in X. \tag{3.6}\]

**Remark 3.1.** The statement of Proposition [3.1] is well-known if the integral kernel of the operator $K$ admits a continuous version (see e.g. Theorem 1.8 and p. 456 in [30]). In the latter case, $(Y(x))_{x \in X}$ is a Gaussian random field and formula (3.6) holds for all $(x, y) \in X^2$.

**Proof of Proposition 3.1.** Consider a standard triple of real Hilbert spaces
\[
H_+ \subset H_0 = L^2(X, \nu) \subset H_.\]

Here the Hilbert space $H_+$ is densely and continuously embedded into $H_0$, the inclusion operator $H_+ \hookrightarrow H_0$ is of Hilbert–Schmidt class, and the Hilbert space $H_-$ is the dual space of $H_+$ with respect to the center space $H_0$ (see e.g. [2]).

Let $\mathbb{P}$ be the standard Gaussian measure on $H_-$, i.e., the probability measure on the Borel $\sigma$-algebra $\mathcal{B}(H_-)$ which has Fourier transform
\[
\int_{H_-} e^{i\langle \omega, f \rangle} \mathbb{P}(d\omega) = \exp \left[ -\frac{1}{2} \|f\|_{H_0}^2 \right], \quad f \in H_+, \tag{3.7}\]

where $\langle \omega, f \rangle$ denotes the dual pairing between $\omega \in H_-$ and $f \in H_+$. Then the mapping
\[
H_+ \ni f \mapsto \langle \cdot, f \rangle \text{ can be extended by continuity to an isometry}
\]
\[
I : H_0 \rightarrow L^2(H_-, \mathbb{P}). \tag{3.7}\]
For any $f \in H_0$ we denote $\langle \cdot, f \rangle := If$. Thus, for each $f \in H_0$, $\langle \cdot, f \rangle$ is a (complex) Gaussian random variable with mean 0 and for any $f, g \in H_0$
\[
\int_{H_-} \langle \omega, f \rangle \langle \omega, g \rangle \mathbb{P}(d\omega) = \langle f, g \rangle_{L^2(X, \nu)}.
\] (3.8)

Thus, by (3.2), we set for $\nu$-a.a. $x \in X$, $\tilde{Y}(x, \omega) := \langle \omega, k(x, \cdot) \rangle$. Hence $\tilde{Y}(x)$ is a Gaussian random variable and by (3.3) and (3.8), (3.6) holds.

Hence, it remains to prove that there exists a random field $Y = (Y(x))_{x \in X}$ for which the mapping (3.5) is measurable and such that $Y(x, \omega) = \tilde{Y}(x, \omega)$ for $\nu \otimes \mathbb{P}$-a.a. $(x, \omega)$. To this end, we fix any $\Lambda \in \mathcal{B}_0(X)$ and denote by $\mathcal{B}(\Lambda)$ the trace $\sigma$-algebra of $\mathcal{B}(X)$ on $\Lambda$. We define a set $\mathcal{D}_\Lambda$ of the functions $u : \Lambda \times X \to \mathbb{R}$ of the form
\[
u \in \mathbb{R}, \, \nu \otimes \mathbb{P})
\] (3.10)

by setting, for each $u \in \mathcal{D}_\Lambda$ of the form (3.9),
\[
(I_\Lambda u)(x, \omega) = \sum_{i=1}^{n} \chi_{\Delta_i}(x) \langle \omega, f_i \rangle, \quad (x, \omega) \in \Lambda \times H_-. \] Clearly, $I_\Lambda$ can be extended to an isometry
\[
I_\Lambda : L^2(\Lambda \times X, \nu \otimes \mathbb{P}) \to L^2(\Lambda \times H_-, \nu \otimes \mathbb{P}),
\]
and we have $I_\Lambda = I_\Lambda \otimes I$, where $I_\Lambda$ is the identity operator in $L^2(\Lambda, \nu)$ and the operator $I$ is as in (3.7).

Fix any $u \in L^2(\Lambda \times X, \nu \otimes \mathbb{P})$. As easily seen, there exist a sequence $(u_n)_{n=1}^{\infty} \subset \mathcal{D}_\Lambda$ such that $u_n \to u$ in $L^2(\Lambda \times X, \nu \otimes \mathbb{P})$ and for $\nu$-a.a. $x \in \Lambda$, $u_n(x, \cdot) \to u(x, \cdot)$ in $L^2(X, \nu)$

Hence, for $\nu$-a.a. $x \in \Lambda$, $I_\Lambda u_n(x, \cdot) \to I_\Lambda u(x, \cdot)$ in $L^2(H_-, \mathbb{P})$, which implies
\[
(I_\Lambda u)(x, \omega) = \langle \omega, u(x, \cdot) \rangle \quad \text{for $\mathbb{P}$-a.a.} \, \omega \in H_-. \] (3.11)

Now, denote by $x_\Lambda$ the restriction of $x$ to the set $\Lambda \times X$. For $\nu$-a.a. $x \in \Lambda$, we define $Y_\Lambda(x) := (I_\Lambda x_\Lambda)(x, \cdot)$. Hence, by (3.11), for $\nu$-a.a. $x \in \Lambda$, $Y_\Lambda(x) = \tilde{Y}(x)$ $\mathbb{P}$-a.e. Finally, let $(\Lambda_n)_{n=1}^{\infty} \subset \mathcal{B}_0(X)$ be such that $\Lambda_n \cap \Lambda_m = \emptyset$ if $n \neq m$ and $\bigcup_{n=1}^{\infty} \Lambda_n = X$. Setting $Y(x) := Y_{\Lambda_n}(x)$ for $\nu$-a.a. $x \in \Lambda_n, n \in \mathbb{N}$, we conclude the statement. \qed
Let $Y$ be a random field as in Proposition 3.1. For each $\Lambda \in B_0(X)$, we have

$$
\mathbb{E} \left( \int_{\Lambda} Y(x)^2 \nu(dx) \right) = \int_{\Lambda} \mathbb{E}(Y(x)^2) \nu(dx)
= \int_{\Lambda} \nu(dx) \int_X \nu(dy) \kappa(x, y)^2 < \infty.
$$

In particular, the function $Y(x)^2$ is locally $\nu$-integrable $\mathbb{P}$-a.s. Let $l \in \mathbb{N}$ and let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space on which $l$ independent copies $Y_1, Y_2, \ldots, Y_l$ of a random field $Y$ as in Proposition 3.1 are defined. Denote by $\mu^{(l)}$ the Cox point process on $X$ with random intensity $g^{(l)}(x) = \sum_{i=1}^l Y_i(x)^2$, which is locally $\nu$-integrable $\mathbb{P}$-a.s. Thus, $\mu^{(l)}$ is the probability measure on $(\Gamma, \mathcal{B}(\Gamma))$ which satisfies

$$
\int_{\Omega} \mathbb{P}(d\omega) \int_{\Gamma} \pi(g^{(l)}(x, \omega) \nu(dx))(d\gamma) F(\gamma)
= \int_{\Omega} \mathbb{P}(d\omega) \int_{\Gamma} \pi(g^{(l)}(x, \omega) \nu(dx))(d\gamma) \int_X \nu(dx) g(x) F(x, \gamma \cup x)
$$

for each measurable function $F: \Gamma \rightarrow [0, +\infty]$. Here, for a locally $\nu$-integrable function $g: X \rightarrow [0, +\infty)$, we denote by $\pi(g(x) \nu(dx))$ the Poisson point process in $X$ with intensity measure $g(x) \nu(dx)$, see e.g. [5]. This is the unique point process in $X$ which satisfies the Mecke identity

$$
\int_{\Gamma} \pi(g^{(l)}(x, \omega) \nu(dx))(d\gamma) \int_X \gamma(dx) F(x, \gamma) = \int_{\Gamma} \pi(g(x) \nu(dx))(d\gamma) \int_X \nu(dx) g(x) F(x, \gamma \cup x)
$$

for each measurable $F: X \times \Gamma \rightarrow [0, +\infty]$. By (3.12) and (3.13) (compare with e.g. [27]), for each $l \in \mathbb{N}$, the point process $\mu^{(l)}$ satisfies condition $(\Sigma'_l)$ and its Papangelou intensity is given by

$$
\tau^{(l)}(x, \gamma) = \widetilde{\mathbb{E}}(g^{(l)}(x) \mid \mathcal{F})(\gamma) = \widetilde{\mathbb{E}}\left( \sum_{i=1}^l Y_i(x)^2 \mid \mathcal{F} \right)(\gamma).
$$

Here $\widetilde{\mathbb{E}}$ denotes the (conditional) expectation with respect to the probability measure

$$
\widetilde{\mathbb{P}}(d\omega, d\gamma) = \mathbb{P}(d\omega) \pi(g^{(l)}(x, \omega) \nu(dx))(d\gamma)
$$

on $\Omega \times \Gamma$, while $\mathcal{F}$ denotes the $\sigma$-algebra on $\Omega \times \Gamma$ generated by the mappings

$$
\Omega \times \Gamma \ni (\omega, \gamma) \rightarrow F(\gamma) \in \mathbb{R},
$$

where $F: \Gamma \rightarrow \mathbb{R}$ is measurable.

Recall that a point process $\mu$ in $X$ is said to have correlation functions if, for each $n \in \mathbb{N}$, there exist a non-negative, measurable, symmetric function $k^{(n)}_{\mu}$ on $X^n$ such
that, for any measurable, symmetric function \( f^n : X^n \to [0, +\infty] \),

\[
\int_{\Gamma} \sum_{\{x_1, \ldots, x_n\} \subset \gamma} f(n)(x_1, \ldots, x_n) \mu(d\gamma) = \frac{1}{n!} \int_{X^n} f(n)(x_1, \ldots, x_n) k^{(n)}_{\mu}(x_1, \ldots, x_n) \nu(dx_1) \cdots \nu(dx_n). \tag{3.16}
\]

As well known (e.g. [5]), for a locally \( \nu \)-integrable function \( g : X \to [0, +\infty] \), the Poisson point process \( \pi_g(x) \nu(dx) \) has correlation functions

\[
k^{(n)}_{\mu}(x_1, \ldots, x_n) = g(x_1) \cdots g(x_n). \tag{3.17}
\]

Let us recall the notion of \( \alpha \)-permanent [31], called \( \alpha \)-determinant in [30]. For a square matrix \( A = (a_{ij})_{i,j=1}^n \) and \( \alpha \in \mathbb{R} \), we set

\[
\text{per}_\alpha A := \sum_{\sigma \in S_n} \alpha^{n-m(\sigma)} \prod_{i=1}^n a_{i\sigma(i)},
\]

where \( S_n \) is the group of all permutations of \( \{1, \ldots, n\} \) and \( m(\sigma) \) denotes the number of cycles in \( \sigma \). In particular, \( \text{per}_1 A \) is the usual permanent of \( A \), while \( \text{per}_{-1} A \) is the usual determinant of \( A \). Analogously to [30] subsec. 6.4], we conclude from (3.12), (3.16) and (3.17) that the point process \( \mu^{(l)} \) has correlation functions

\[
k_{\mu^{(l)}}^{(n)}(x_1, \ldots, x_n) = \text{per}_l (lk(x_i, x_j))_{i,j=1}^n \text{ for } \nu^{\otimes n}\text{-a.a. } (x_1, \ldots, x_n) \in X^n. \tag{3.18}
\]

For \( l = 2 \), the point process \( \mu^{(2)} \) is often called a boson point process, see e.g. [5, 23]. Thus, we have proved the following

**Proposition 3.2.** For each \( l \in \mathbb{N} \), there exists a point process \( \mu^{(l)} \) in \( X \) whose correlation functions are given by (3.18). The \( \mu^{(l)} \) satisfies condition (\( \Sigma'_\nu \)) and its Papangelou intensity is given by (3.14).

**Remark 3.2.** Recall that in [30], under the same assumptions on the operator \( K \), the existence of a point process with correlation functions (3.18) was proved for even \( l \in \mathbb{N} \), and for odd \( l \in \mathbb{N} \) the statement of Proposition 3.2 was proved under the additional assumption of continuity of the integral kernel \( k(\cdot, \cdot) \).

We will now prove that, for a point process \( \mu^{(l)} \) as in Proposition 3.2, Glauber and Kawasaki dynamics with coefficients (2.12), (2.13) and (2.14), respectively exist.

**Theorem 3.1.** (i) For each point process \( \mu^{(l)} \) as in Proposition 3.2, the coefficients \( d(x, \gamma) \) and \( b(x, \gamma) \) defined by (2.12) and (2.13), satisfy conditions (2.3) and (2.7) and so statements (i) and (ii) of Theorem 2.7 hold, in particular, a corresponding Glauber dynamics exists.
Assume additionally that \( k(x, x) \) is bounded outside a set \( \Delta \in \mathcal{B}_0(X) \). Then for a point process \( \mu(l) \) as in Proposition 3.2, the coefficient \( c(x, y, \gamma) \) defined by (2.14), satisfies (2.4), (2.5), (2.8) and (2.9), and so statements (i) and (ii) of Theorem 2.1 hold, in particular, a corresponding Kawasaki dynamics exists.

**Proof.** We start with the following

**Lemma 3.1.** For each \( n \in \mathbb{N} \) and for \( \nu \)-a.a. \( x \in X \)

\[
\int_\Gamma r(x, \gamma)^n \mu(d\gamma) \leq \frac{(2n)!}{2^n n!} k(x, x)^n. \tag{3.19}
\]

**Proof.** Using Jensen’s inequality for conditional expectation and the formula for moments of a Gaussian measure (see e.g. [2, Chapter 2, Section 2, Lemma 2.1]), we have

\[
\int_\Gamma r(x, \gamma)^n \mu(d\gamma) = \mathbb{E}(\mathbb{E}(Y(x)^2 | \mathcal{F})^n) \leq \mathbb{E}(\mathbb{E}(Y(x)^{2n} | \mathcal{F})) = \mathbb{E}(Y(x)^{2n}) \leq \frac{(2n)!}{2^n n!} \|x(x, \cdot)\|_{L^2(X, \nu)}^2 = \frac{(2n)!}{2^n n!} k(x, x)^n
\]

for \( \nu \)-a.a. \( x \in X \).

We will only prove statement (ii) of Theorem 3.1 as the proof of statement (i) is similar and simper. Also, for simplicity of notation, we will only consider the case \( l = 1 \) (for \( l > 1 \) the proof being similar). We will also omit the upper index (1) from our notation. By (2.1) we have, for each \( \Lambda \in \mathcal{B}_0(X) \),

\[
\int_\Gamma \mu(d\gamma) \int_X \gamma(dx) \int_X \nu(dy) c(x, y, \gamma \setminus x)(\chi_\Lambda(x) + \chi_\Lambda(y))
\]

\[
= \int_\Gamma \mu(d\gamma) \int_X \nu(dx) \int_X \nu(dy) r(x, \gamma)c(x, y, \gamma)(\chi_\Lambda(x) + \chi_\Lambda(y))
\]

\[
= \int_\Gamma \mu(d\gamma) \int_X \nu(dx) \int_X \nu(dy) a(x, y)r(x, \gamma)^*r(y, \gamma)^*\chi_{\{r>0\}}(x, \gamma)
\]

\[
\times \chi_{\{r>0\}}(y, \gamma)(\chi_\Lambda(x) + \chi_\Lambda(y)) \tag{3.20}
\]

\[
\leq \int_\Gamma \mu(d\gamma) \int_X \nu(dx) \int_X \nu(dy) a(x, y)r(x, \gamma)^*r(y, \gamma)^*(\chi_\Lambda(x) + \chi_\Lambda(y))
\]

\[
= 2 \int_\Gamma \mu(d\gamma) \int_\Lambda \nu(dx) \int_X \nu(dy) a(x, y)r(x, \gamma)^*r(y, \gamma)^*
\]

\[
\leq 2 \int_\Gamma \mu(d\gamma) \int_\Lambda \nu(dx) \int_X \nu(dy) a(x, y)(1 + r(x, \gamma))(1 + r(y, \gamma)).
\]

By (2.15)

\[
\int_\Gamma \mu(d\gamma) \int_\Lambda \nu(dx) \int_X \nu(dy) a(x, y) < \infty. \tag{3.21}
\]
Below, $C_i, i = 1, 2, 3, \ldots$, will denote positive constants whose explicit values are not important for us. We have, by (2.15)

$$
\int_{\Gamma} \mu(d\gamma) \int_{\Lambda} \nu(dx) \int_{X} \nu(dy) a(x, y) r(x, \gamma) \\
= \int_{\Gamma} \mu(d\gamma) \int_{\Lambda} \nu(dx) r(x, \gamma) \left( \int_{X} \nu(dy) a(x, y) \right) \\
\leq C_1 \int_{\Gamma} \mu(d\gamma) \int_{\Lambda} \nu(dx) r(x, \gamma) \\
= C_1 \int_{\Gamma} \mu(d\gamma) \int_{\Lambda} \gamma(dx) = C_1 \int_{\Lambda} k(x, x) \nu(dx) < \infty.
$$

(3.22)

Next, by (3.14)

$$
\int_{\Gamma} \mu(d\gamma) \int_{\Lambda} \nu(dx) \int_{X} \nu(dy) a(x, y) r(y, \gamma) \\
= \int_{\Lambda} \nu(dx) \int_{X} \nu(dy) a(x, y) \int_{\Gamma} \mu(d\gamma) r(y, \gamma) \\
= \int_{\Lambda} \nu(dx) \int_{X} \nu(dy) a(x, y) k(y, y) \\
= \int_{\Lambda} \nu(dx) \int_{\Delta} \nu(dy) a(x, y) k(y, y) + \int_{\Lambda} \nu(dx) \int_{\Delta^c} \nu(dy) a(x, y) k(y, y) \\
\leq C_2 \int_{\Lambda} \nu(dx) \int_{\Delta} \nu(dy) k(y, y) + C_3 \int_{\Lambda} \nu(dx) \int_{\Delta^c} \nu(dy) a(x, y) < \infty,
$$

(3.23)

where we used that the function $a$ is bounded and $k(y, y)$ is bounded on $\Delta^c$. Analogously, using Lemma 3.1, we have

$$
\int_{\Gamma} \mu(d\gamma) \int_{\Lambda} \nu(dx) \int_{X} \nu(dy) a(x, y) r(x, \gamma) r(y, \gamma) \\
\leq \int_{\Lambda} \nu(dx) \int_{X} \nu(dy) a(x, y) \| r(x, \cdot) \|_{L^2(\mu)} \| r(y, \cdot) \|_{L^2(\mu)} \\
\leq C_4 \int_{\Lambda} \nu(dx) \int_{X} \nu(dy) a(x, y) k(x, x) k(y, y) \\
\leq C_5 \int_{\Lambda} \nu(dx) k(x, x) \int_{\Delta} \nu(dy) k(y, y) \\
+ C_6 \int_{\Lambda} \nu(dx) k(x, x) \int_{\Delta^c} \nu(dy) a(x, y) < \infty.
$$

(3.24)

Thus, by (3.20)–(3.24), the theorem is proven.
Theorem 3.2. (i) Let \( s \in \left[ \frac{1}{2}, 1 \right] \), and let the conditions of Theorem 3.1 (i) be satisfied. Then the coefficients \( d(x, \gamma) \) and \( b(x, \gamma) \) defined by (2.12) and (2.13), satisfy condition (2.10). Thus, \( \mathcal{F}C_b(C_0(X), \Gamma) \subset D(L_G) \), and for each \( F \in \mathcal{F}C_b(C_0(X), \Gamma) \), \( L_G F \) is given by formula (1.1).

(ii) Let \( s \in \left[ \frac{1}{2}, 1 \right] \), and let the conditions of Theorem 3.1 (ii) be satisfied. Further assume that either

\[
\forall \Lambda \in B_0(X) \ \exists \Lambda' \in B_0(X) \ \forall x \in \Lambda \ \forall y \in (\Lambda')^c : \ a(x, y) = 0, \tag{3.25}
\]

or

\[
\int_{\Delta} k(x, x)^2 \nu(dx) < \infty, \tag{3.26}
\]

where \( \Delta \) is as in Theorem 3.1 (ii). Then the coefficient \( c(x, y, \gamma) \) defined by (2.14), satisfies condition (2.11). Thus, \( \mathcal{F}C_b(C_0(X), \Gamma) \subset D(L_K) \), and for each \( F \in \mathcal{F}C_b(C_0(X), \Gamma) \), \( L_K F \) is given by formula (1.2).

Remark 3.3. If \( X = \mathbb{R}^d \) and the function \( a \) is as in Remark 2.3, then condition (3.25) means that the function \( \tilde{a} \) has a compact support.

Proof of Theorem 3.2. We again prove only the part related to Kawasaki dynamics and only in the case \( l = 1 \), omitting the upper index (1) from our notation. We first assume that (3.25) is satisfied. Since the function \( a \) is bounded and satisfies (3.25), it suffices to show that, for each \( \Lambda \in B_0(X) \),

\[
\int_{\Lambda} \gamma(dx) \int_{\Lambda} \nu(dy) r(x, \gamma \setminus x)^{s-1} r(y, \gamma \setminus x)^s \chi_{\{r>0\}}(x, \gamma \setminus x) \chi_{\{r>0\}}(y, \gamma \setminus x) \in L^2(\mu). \tag{3.27}
\]

We note that, for \( s \in \left[ \frac{1}{2}, 1 \right], \ 2s - 1 \in [0, 1] \). Therefore, by the Cauchy inequality, we have

\[
\begin{align*}
\int_{\Gamma} \mu(d\gamma) \left( \int_{\Lambda} \gamma(dx) r(x, \gamma \setminus x)^{s-1} \chi_{\{r>0\}}(x, \gamma \setminus x) \\
\times \int_{\Lambda} \nu(dy) r(y, \gamma \setminus x)^s \chi_{\{r>0\}}(y, \gamma \setminus x) \right)^2 \\
\leq \int_{\Gamma} \mu(d\gamma) \int_{\Lambda} \gamma(dx) r(x, \gamma \setminus x)^{2(s-1)} \chi_{\{r>0\}}(x, \gamma \setminus x) \\
\times \left( \int_{\Lambda} \nu(dy) r(y, \gamma \setminus x)^s \chi_{\{r>0\}}(y, \gamma \setminus x) \right)^2 \gamma(\Lambda) \\
= \int_{\Gamma} \mu(d\gamma) \int_{\Lambda} \nu(dx) r(x, \gamma)^{2s-1} \chi_{\{r>0\}}(x, \gamma) \\
\times \left( \int_{\Lambda} \nu(dy) r(y, \gamma)^s \chi_{\{r>0\}}(y, \gamma) \right)^2 \gamma(\Lambda) + 1
\end{align*}
\]
Then, by the Cauchy inequality,

\[
\leq \int_{\Gamma} \mu(d\gamma) \left( \int_{\Lambda} \nu(dx)(1 + r(x, \gamma)) \right)^3 (\gamma(\Lambda) + 1)
\]

\[
\leq \left( \int_{\Gamma} \mu(d\gamma) \left( \int_{\Lambda} \nu(dx)(1 + r(x, \gamma)) \right)^6 \right)^{1/2} \left( \int_{\Gamma} \mu(d\gamma)(\gamma(\Lambda) + 1)^2 \right)^{1/2}.
\]

(3.28)

By Lemma 3.1, we have, for each \( n \in \mathbb{N} \),

\[
\int_{\Gamma} \mu(d\gamma) \left( \int_{\Lambda} \nu(dx) r(x, \gamma) \right)^n
\]

\[
= \int_{\Lambda} \nu(dx_1) \cdots \int_{\Lambda} \nu(dx_n) \int_{\Gamma} \mu(d\gamma) r(x_1, \gamma) \cdots r(x_n, \gamma)
\]

\[
\leq \int_{\Lambda} \nu(dx_1) \cdots \int_{\Lambda} \nu(dx_n) \|r(x_1, \cdot)\|_{L^\nu(\mu)} \cdots \|r(x_n, \cdot)\|_{L^\nu(\mu)}
\]

\[
\leq \frac{(2n)!}{2^n n!} \left( \int_{\Lambda} \nu(dx) k(x, x) \right)^n < \infty.
\]

(3.29)

Now, (3.27) follows from (3.28) and (3.29).

Next, we assume that (3.26) is satisfied. We fix \( \Lambda \in \mathcal{B}_0(X) \) and denote

\[u(x, y) := a(x, y)(\chi_\Lambda(x) + \chi_\Lambda(x)).\]

Then, by the Cauchy inequality,

\[
\int_{\Gamma} \mu(d\gamma) \left( \int_{X} \gamma(dx) \int_{X} \nu(dy) u(x, y) r(x, \gamma \setminus x)^{s-1} \chi_{\{r > 0\}}(x, \gamma \setminus x) \times r(y, \gamma \setminus x)^{s-1} \chi_{\{r > 0\}}(y, \gamma \setminus x) \right)^2
\]

\[
\leq \int_{\Gamma} \mu(d\gamma) \int_{X} \gamma(dx) \int_{X} \nu(dy) u(x, y) r(x, \gamma \setminus x)^{2(s-1)} \chi_{\{r > 0\}}(x, \gamma \setminus x)
\]

\[
\times r(y, \gamma \setminus x)^{2s} \chi_{\{r > 0\}}(y, \gamma \setminus x) \int_{X} \gamma(dx') \int_{X} \nu(dy') u(x', y')
\]

\[
= \int_{\Gamma} \mu(d\gamma) \int_{X} \nu(dx) \int_{X} \nu(dy) u(x, y) r(x, \gamma)^{2s-1} \chi_{\{r > 0\}}(x, \gamma)
\]

\[
\times r(y, \gamma)^{2s} \chi_{\{r > 0\}}(y, \gamma) \int_{X} (\gamma + \varepsilon_x)(dx') \int_{X} \nu(dy') u(x', y')
\]

\[
\leq \int_{\Gamma} \mu(d\gamma) \int_{X} \nu(dx) \int_{X} \nu(dy) u(x, y)(1 + r(x, \gamma))(1 + r(y, \gamma))^2
\]

\[
\times \left( \int_{X} \gamma(dx') \int_{X} \nu(dy') u(x', y') + \int_{X} \nu(dy') u(x, y') \right).
\]

By (2.15), it suffices to prove that

\[
\int_{\Gamma} \mu(d\gamma) \left( \int_{X} \nu(dx) \int_{X} \nu(dy) u(x, y)(1 + r(x, \gamma))(1 + r(y, \gamma))^2 \right)^2 < \infty.
\]

(3.30)
We first prove (3.31). We have, by Proposition \(3.2\)
\[
\int_{\Gamma} \mu(d\gamma) \left( \int_X \gamma(dx) \int_X \nu(dy) \ u(x,y) \right)^2 < \infty.
\] (3.31)

Next, we prove (3.30). By Lemma \(3.1\) and (3.26), we have
\[
\int_{\Gamma} \mu(d\gamma) \left( \int_X \gamma(dx) \int_X \nu(dy) \ u(x,y) \right)^2 < \infty.
\]
$\times (1 + \|r(x', \cdot)\|_{L^4(\mu)}) \left( 1 + \|r(y, \cdot)\|_{L^4(\mu)} \right) \left( 1 + \|r(y', \cdot)\|_{L^4(\mu)} \right) \\
\leq C_\delta \left( \int_X \nu(dx) \int_X \nu(dy) u(x, y) (1 + k(x, x))(1 + k(y, y))^2 \right)^2 < \infty.$

Thus, the theorem is proven. \hfill \Box

4 Diffusion approximation

From now on, we set $X = \mathbb{R}^d$, $d \in \mathbb{N}$, and $\nu$ to be Lebesgue measure. We will show that, under an appropriate scaling, the Dirichlet form of the Kawasaki dynamics converges to a Dirichlet form which identifies a diffusion process on $\Gamma$ having a permenental point process $\mu^{(l)}$ as a symmetrizing measure. The way we scale the Kawasaki dynamics will be similar to the ansatz of [15].

We denote by $\mathcal{FC}_b^\infty(C_0^\infty(\mathbb{R}^d), \Gamma)$ the space of all functions of the form \[2.6\] where $N \in \mathbb{N}$, $\varphi_1, \ldots, \varphi_N \in C_0^\infty(\mathbb{R}^d)$ and $g \in C_0^\infty(\mathbb{R}^N)$. Here, $C_0^\infty(\mathbb{R}^d)$ denotes the space of smooth functions on $\mathbb{R}^d$ with compact support, and $C_0^\infty(\mathbb{R}^N)$ denotes the space of all smooth bounded functions on $\mathbb{R}^N$ whose all derivatives are bounded. Clearly,

$$\mathcal{FC}_b^\infty(C_0^\infty(\mathbb{R}^d), \Gamma) \subset \mathcal{FC}_b(C_0(\mathbb{R}^d), \Gamma),$$

and the set $\mathcal{FC}_b^\infty(C_0^\infty(\mathbb{R}^d), \Gamma)$ is a core for the Dirichlet form $(\mathcal{E}_K, D(\mathcal{E}_K))$.

We fix $s = 1/2$. Let us assume that the function $a(x, y)$ is as in Remark \[2.3\]. Thus, the coefficient $c(x, y, \gamma)$ has the form

$$c(x, y, \gamma) = a(x - y)r(x, \gamma)^{-1/2}r(y, \gamma)^{1/2} \chi_{\{r > 0\}}(x, \gamma) \chi_{\{r > 0\}}(y, \gamma). \quad (4.1)$$

Note that $y - x$ describes the change of the position of a particle which hops from $x$ to $y$. We now scale the function $a$ as follows: for each $\varepsilon > 0$, we denote

$$a_\varepsilon(x) := \varepsilon^{-d-2}a(x/\varepsilon), \quad x \in \mathbb{R}^d. \quad (4.2)$$

The Dirichlet form $(\mathcal{E}_K, D(\mathcal{E}_K))$ which corresponds to the choice of function $a$ as in \[1.2\] will be denoted by $(\mathcal{E}_\varepsilon, D(\mathcal{E}_\varepsilon))$.

**Theorem 4.1.** Assume that the function $a$ has compact support, and the value $a(x)$ only depends on $|x|$, i.e., $a(x) = \tilde{a}(|x|)$ for some function $\tilde{a} : [0, \infty) \to \mathbb{R}$. Further assume that the function $\mathcal{K}(x, y)$ has the form $\mathcal{K}(x - y)$ for some $\mathcal{K} : \mathbb{R}^d \to \mathbb{C}$, and

$$\lim_{y \to 0} \int_{\mathbb{R}^d} (\mathcal{K}(x) - \mathcal{K}(x + y))^2 dx = 0. \quad (4.3)$$

For each $l \in \mathbb{N}$, define a bilinear form $(\mathcal{E}_0, \mathcal{FC}_b^\infty(C_0^\infty(\mathbb{R}^d), \Gamma))$ by

$$\mathcal{E}_0(F, G) := c \int_\Gamma \mu^{(l)}(d\gamma) \int_{\mathbb{R}^d} dx r(x, \gamma)(\nabla_x F(\gamma \cup x), \nabla_x G(\gamma \cup x)). \quad (4.4)$$
Here
\[ c := \frac{1}{2} \int_{\mathbb{R}^d} a(x)x_1^2 \, dx \]
(x_1 denoting the first coordinate of \( x \in \mathbb{R}^d \), \( \nabla_x \) denotes the gradient in the \( x \) variable, and \( \langle \cdot, \cdot \rangle \) stands for the scalar product in \( \mathbb{R}^d \). Then, for any \( F, G \in FC_0^\infty(\mathbb{R}^d), \Gamma \),
\[ \mathcal{E}_\varepsilon(F, G) \to \mathcal{E}_0(F, G) \quad \text{as} \; \varepsilon \to 0. \]

Remark 4.1. Assume that the function \( \kappa \) is differentiable on \( \mathbb{R}^d \). Denote
\[ K(x, \delta) := \sup_{y \in B(x, \delta)} |\nabla \kappa(y)|, \quad x \in \mathbb{R}^d, \; \delta > 0. \]
Here \( B(x, \delta) \) denotes the closed ball in \( \mathbb{R}^d \) centered at \( x \) and of radius \( \delta \). Assume that, for some \( \delta > 0 \), \( K(\cdot, \delta) \in L^2(\mathbb{R}^d, dx) \).

Then condition (4.3) is clearly satisfied. Note that condition (4.5) is slightly stronger than the condition \( |\nabla \kappa| \in L^2(\mathbb{R}^d, dx) \).

Proof of Theorem 4.1. Again we will only present the proof in the case \( l = 1 \), omitting the upper index (1). We start with the following

Lemma 4.1. Fix any \( \Lambda \in B_0(\mathbb{R}^d) \) and \( \alpha \in (0,1] \). Then, under the conditions of Theorem 4.1,
\[ r(x + \varepsilon y, \gamma) \to r(x, \gamma) \quad \text{in} \quad L^2(\Omega \times \Lambda \times \mathbb{R}^d, d\omega \, d\gamma \, dy \, a(y)) \quad \text{as} \; \varepsilon \to 0. \]

Proof. We first prove the statement for \( \alpha = 1 \). Thus, equivalently we have to prove that
\[ r(x + \varepsilon y, \gamma) \to r(x, \gamma) \quad \text{in} \quad L^2(\Omega \times \Gamma \times \Lambda \times \mathbb{R}^d, \tilde{P}(d\omega, d\gamma) \, dy \, a(y)) \quad \text{as} \; \varepsilon \to 0. \]

We have, using Jensen’s inequality for conditional expectation,
\[ \int_{\Lambda} dx \int_{\mathbb{R}^d} dy \, a(y) \int_{\Omega \times \Gamma} \tilde{P}(d\omega, d\gamma) \left( r(x + \varepsilon y) - r(x, \gamma) \right)^2 \]
\[ = \int_{\Lambda} dx \int_{\mathbb{R}^d} dy \, a(y) \int_{\Omega \times \Gamma} \tilde{P}(d\omega, d\gamma) \tilde{E}(Y(x + \varepsilon y)^2 - Y(x)^2 \mid \mathcal{F})^2 \]
\[ \leq \int_{\Lambda} dx \int_{\mathbb{R}^d} dy \, a(y) \int_{\Omega \times \Gamma} \tilde{P}(d\omega, d\gamma)(Y(x + \varepsilon y)^2 - Y(x)^2)^2 \]
\[ = \int_{\Lambda} dx \int_{\mathbb{R}^d} dy \, a(y) \int_{\Omega} d\tilde{P}(Y(x + \varepsilon y)^4 + Y(x)^4 - 2Y(x + \varepsilon y)^2 Y(x)^2). \]
Using the formula for moments of a Gaussian measure, we have

\[
\int_{\Omega} Y(x + \varepsilon y)^4 dP
\]
\[
= 3 \left( \int_{\mathbb{R}^d} \varphi(x + \varepsilon y - u)^2 du \right)^2
\]
\[
= 3 \left( \int_{\mathbb{R}^d} \varphi(x - u)^2 du \right)^2
\]
\[
= \int_{\Omega} Y(x)^4 dP.
\]  
(4.8)

Analogously, using condition (4.3) and the dominated convergence theorem, we get

\[
\int_{\Lambda} dx \int_{\mathbb{R}^d} dy a(y) \int_{\Omega} dP Y(x + \varepsilon y)^2 Y(x)^2
\]
\[
= \int_{\Lambda} dx \int_{\mathbb{R}^d} dy a(y) \left[ \int_{\mathbb{R}^d} \varphi(x + \varepsilon y - u)^2 du \cdot \int_{\mathbb{R}^d} \varphi(x - u')^2 du' + 2 \left( \int_{\mathbb{R}^d} \varphi(x + \varepsilon y - u) \varphi(x - u) du \right)^2 \right]
\]
\[
\to \int_{\Lambda} dx \int_{\mathbb{R}^d} dy a(y) \int_{\Omega} dP Y(x)^4 \quad \text{as} \quad \varepsilon \to 0.
\]  
(4.9)

By (4.7)–(4.9), statement (4.6) follows.

To prove the result for \( \alpha \in (0, 1) \), it is now sufficient to show the following

Claim. Let \( (\Lambda, \mathcal{A}, m) \) be a measure space and let \( m(A) < \infty \). Let \( f_\varepsilon \in L^2(m) \), \( f_\varepsilon \geq 0, \varepsilon \in [-1, 1] \), and let \( f_\varepsilon \to f_0 \) in \( L^2(m) \) as \( \varepsilon \to 0 \). Then, for each \( \alpha \in (0, 1) \), \( f_\varepsilon^\alpha \to f_0^\alpha \) in \( L^2(m) \) as \( \varepsilon \to 0 \).

By e.g. [1, Theorems 21.2 and 21.4], \( f_\varepsilon \to f_0 \) in \( L^2(m) \) implies that

(i) \( f_\varepsilon \to f_0 \) in measure;

(ii) \( \sup_{\varepsilon \in [-1, 1]} \int f_\varepsilon^2 dm < \infty \);

(iii) For each \( \theta > 0 \) there exist \( h \in L^1(m) \) and \( \delta > 0 \) such that, for all \( 0 < |\varepsilon| \leq 1 \) and for each \( A \in \mathcal{A} \)

\[
\int_A h dm \leq \delta \Rightarrow \int_A f_\varepsilon^2 dm \leq \theta.
\]

Hence, for \( \alpha \in (0, 1) \), we get

a) \( f_\varepsilon^\alpha \to f_0^\alpha \) in measure;
b) \( \sup_{\varepsilon \in [-1, 1]} \int f_{\varepsilon}^{2\alpha} dm \leq \sup_{\varepsilon \in [-1, 1]} (1 + f_{\varepsilon}^2) dm < \infty; \)

c) Let \( \theta, h, \) and \( \delta \) be as in (iii). Set \( h' := h + \frac{\delta}{\theta}. \) Clearly, \( h \in L^1(m). \) Assume that, for some \( A \in \mathcal{A}, \int_A h' dm \leq \delta. \) Hence \( \int_A h dm \leq \delta, \) and therefore \( \int_A f_{\varepsilon}^2 dm \leq \delta \) for all \( 0 < |\varepsilon| \leq 1. \) Furthermore, we get \( \int_A \frac{\delta}{\theta} dm \leq \delta, \) and therefore \( m(A) \leq \theta. \)

Now
\[
\int_A f_{\varepsilon}^2 dm \leq \int_A (1 + f_{\varepsilon}^2) dm \leq 2\theta.
\]

Applying again [\( \Pi \) Theorems 21.2 and 21.4], we conclude the claim. \( \square \)

Fix any \( F \in \mathcal{F}C^\infty_0(C^\infty_0(\mathbb{R}^d), \Gamma). \) We have

\[
\mathcal{E}_\varepsilon(F, F) = \frac{1}{2} \int_{\Gamma} \mu(d\gamma) \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \varepsilon^{-d-2} a((x - y)/\varepsilon) r(x, \gamma)^{1/2} r(y, \gamma)^{1/2} (F(\gamma \cup x) - F(\gamma \cup y))^2
\]

\[
= \frac{1}{2} \int_{\Gamma} \mu(d\gamma) \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy a(y) r(x + \varepsilon y, \gamma)^{1/2} r(x, \gamma)^{1/2} \left( \frac{F(\gamma \cup \{x + \varepsilon y\}) - F(\gamma \cup x)}{\varepsilon} \right)^2.
\]

Assume that \( 0 < |\varepsilon| \leq 1. \) Noting that the function \( F \) is local (i.e., there exists \( \Delta \in \mathcal{B}_0(\mathbb{R}^d) \) such that \( F(\gamma) = F(\gamma_\Delta) \) for all \( \gamma \in \Gamma \)) and that the function \( a \) has a compact support, we conclude that there exists \( \Lambda \in \mathcal{B}_0(\mathbb{R}^d) \) such that

\[
\mathcal{E}_\varepsilon(F, F) = \int_{\Gamma} \mu(d\gamma) \int_{\Lambda} dx \int_{\mathbb{R}^d} dy a(y) r(x + \varepsilon y, \gamma)^{1/2} r(x, \gamma)^{1/2}
\]

\[
\times \left( \frac{F(\gamma \cup \{x + \varepsilon y\}) - F(\gamma \cup x)}{\varepsilon} \right)^2.
\] (4.10)

By the dominated convergence theorem

\[
r(x, \gamma)^{1/2} \left( \frac{F(\gamma \cup \{x + \varepsilon y\}) - F(\gamma \cup x)}{\varepsilon} \right)^2 \to r(x, \gamma)^{1/2} \langle \nabla_x F(\gamma \cup x), y \rangle^2 \quad (4.11)
\]

in \( L^2(\Gamma \times \Lambda \times \mathbb{R}^d, \mu(d\gamma) dx dy a(y)) \) as \( \varepsilon \to 0. \) By Lemma [\( \Pi \) 4.1] with \( \alpha = 1/2, \) (4.10) and (4.11)

\[
\mathcal{E}_\varepsilon(F, F) \to \frac{1}{2} \int_{\Gamma} \mu(d\gamma) \int_{\Lambda} dx \int_{\mathbb{R}^d} dy a(y) r(x, \gamma) \langle \nabla_x F(\gamma \cup x), y \rangle^2.
\] (4.12)

Since \( a(y) = \tilde{a}(|y|), \) for any \( i, j \in \{1, \ldots, d\}, \ i \neq j, \) we have

\[
\int_{\mathbb{R}^d} a(y) y_i y_j dy = 0
\]
and
\[ c = \frac{1}{2} \int_{\mathbb{R}^d} a(y)y_i^2 \, dy, \quad i = 1, \ldots, d. \]

Therefore, by (4.12),
\[ \mathcal{E}_\varepsilon (F, F) \to c \int_{\Gamma} \mu(d\gamma) \int_{\mathbb{R}^d} dx r(x, \gamma) |\nabla_x F(\gamma \cup x)|^2. \]

From here the theorem follows by the polarization identity. \qed

We will now show that the limiting form \((\mathcal{E}_0, \mathcal{F}C_0^\infty(\mathbb{R}^d, \Gamma))\) is closable and its closure identifies a diffusion process.

In what follows, we will assume that the conditions of Theorem 4.1 are satisfied. We have
\[ k(x, y) = \int_{\mathbb{R}^d} \kappa(x - u) \kappa(y - u) \, du \]
\[ = \int_{\mathbb{R}^d} \kappa(u - y) \kappa(u - x) \, du = \int_{\mathbb{R}^d} \kappa(u) \kappa(u + y - x) \, du. \]

Hence, by (4.3), the function \(k(x, y)\) is continuous on \((\mathbb{R}^d)^2\). Thus, by Remark 3.1, \((Y(x))_{x \in X}\) is a Gaussian random field and formula (3.6) holds for all \((x, y) \in (\mathbb{R}^d)^2\).

Consider the semimetric
\[ D(x, y) := \frac{1}{2} \left( \int_{\Omega} (Y(x) - Y(y))^2 \, d\mathbb{P} \right)^{1/2} \]
\[ = \frac{1}{2} \left( k(x, x) + k(y, y) - 2k(x, y) \right)^{1/2} \]
\[ = \left( \int_{\mathbb{R}^d} \kappa(u) \left( \kappa(u) - \kappa(u + y - x) \right) \, du \right)^{1/2}, \quad x, y \in \mathbb{R}^d. \]

The associated metric entropy \(H(D, \delta)\) is defined as \(H(D, \delta) := \log N(D, \delta)\), where \(N(D, \delta)\) is the minimal number of points in a \(\delta\)-net in \(B(0, 1) = \{x \in \mathbb{R}^d \mid |x| \leq 1\}\) with respect to the semimetric \(D\), i.e., points \(x_i\) such that the open balls centered at \(x_i\) and of radius \(\delta\) (with respect to \(D\)) cover \(B(0, 1)\). The expression
\[ J(D) := \int_0^1 \sqrt{H(D, \delta)} \, d\delta \]
is called the Dudley integral. The following result holds, see e.g. [4, Corollary 7.1.4] and the references therein.

**Theorem 4.2.** Assume that \(J(D) < \infty\). Then the Gaussian random field \((Y(x))_{x \in \mathbb{R}^d}\) has a continuous modification.
Remark 4.2. Let \( \varkappa \) be as in Remark 4.1. Then, by (4.13), for any \( x, y \in B(0, 1) \)

\[
D(x, y)^2 \leq \| \varkappa \cdot \|_{L^2(\mathbb{R}^d, dx)} \left( \int_{\mathbb{R}^d} (\varkappa(u) - \varkappa(u + y - x))^2 du \right)^{1/2}
\]

\[
\leq \| \varkappa \cdot \|_{L^2(\mathbb{R}^d, dx)} \| K(\cdot, 2) \|_{L^2(\mathbb{R}^d, dx)} |y - x|,
\]

where we assumed that \( K(\cdot, 2) \in L^2(\mathbb{R}^d, dx) \). Then \( J(D) < \infty \), see e.g. [1, Example 7.1.5].

Denote by \( \tilde{\Gamma} \) the space of all multiple configurations in \( \mathbb{R}^d \). Thus, \( \tilde{\Gamma} \) is the set of all Radon \( Z_+ \cup \{ +\infty \} \)-valued measures on \( \mathbb{R}^d \). In particular, \( \Gamma \subset \tilde{\Gamma} \).

Theorem 4.3. Let \( \varkappa(x, y) \) be of the form \( \varkappa(x - y) \) for some \( \varkappa \in L^2(\mathbb{R}^d, dx) \). Let \( J(D) < \infty \). Let \( l \in \mathbb{N} \) and \( c > 0 \). Then

(i) The bilinear form \((\mathcal{E}_0, \mathcal{F}C_0^\infty(\mathcal{C}_0^\infty(\mathbb{R}^d), \Gamma))\) defined by (4.4) is closable on \( L^2(\Gamma, \mu^{(l)}) \) and its closure will be denoted by \((\mathcal{E}_0, D(\mathcal{E}_0))\).

(ii) There exists a conservative diffusion process

\[
M^0 = (\Omega^0, \mathcal{F}^0, (\mathcal{F}^0_t)_{t \geq 0}, (\mathcal{G}^0_t)_{t \geq 0}, (X^0(t))_{t \geq 0}, (P^0_{\gamma})_{\gamma \in \tilde{\Gamma}})
\]

on \( \tilde{\Gamma} \) which is properly associated with \((\mathcal{E}_0, D(\mathcal{E}_0))\). In particular, \( M^0 \) is \( \mu^{(l)} \)-symmetric and has \( \mu^{(l)} \) as invariant measure. In the case \( d \geq 2 \), the set \( \tilde{\Gamma} \setminus \Gamma \) is \( \mathcal{E}^0 \)-exceptional, so that \( \tilde{\Gamma} \) may be replaced by with \( \Gamma \) in the above statement.

Proof. We again discuss only the case \( l = 1 \), omitting the upper index (1). By (4.4), for any \( F, G \in \mathcal{F}C_0^\infty(\mathcal{C}_0^\infty(\mathbb{R}^d), \Gamma) \),

\[
\mathcal{E}_0(F, G) = c \int_{\Omega \times \Gamma} \bar{P}(d\omega, d\gamma) \int_{\mathbb{R}^d} dx \bar{E}(Y(x, \omega)^2 | \mathcal{F}) \langle \nabla_x F(\gamma \cup x), \nabla_x G(\gamma \cup x) \rangle
\]

\[
= \int_{\Omega \times \Gamma} \bar{P}(d\omega, d\gamma) \int_{\mathbb{R}^d} dx Y(x, \omega)^2
\]

\[
\times \langle \nabla_x (F(\gamma \cup x) - F(\gamma)), \nabla_x (G(\gamma \cup x) - G(\gamma)) \rangle. \tag{4.14}
\]

Fix \( (\omega, \gamma) \in \Omega \times \Gamma \). Denote

\[
f(x) := F(\gamma \cup x) - F(\gamma), \quad g(x) := G(\gamma \cup x) - G(\gamma).
\]

Clearly, \( f, g \in C_0^\infty(\mathbb{R}^d) \). In view of Theorem 4.2, \( Y(x, \omega)^2 \) is a continuous function of \( x \in \mathbb{R}^d \). Hence, by [6, Theorem 6.2], the bilinear form

\[
\mathcal{E}(f, g) := \int_{\mathbb{R}^d} \langle \nabla f(x), \nabla g(x) \rangle Y(x, \omega)^2 dx, \quad f, g \in C_0^\infty(\mathbb{R}^d),
\]

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is closable on $L^2(\mathbb{R}^d, |Y(x, \omega)|^2 dx)$. Now the closability of $(\mathcal{E}_0, FC^\infty_0(C^\infty_0(\mathbb{R}^d), \Gamma))$ on $L^2(\Gamma, \mu^{(0)})$ follows by a straightforward generalization of the proof of [6, Theorem 6.3]. Part (ii) of the theorem can be shown completely analogously to [25, 29], see also [20].

**Remark 4.3.** Heuristically, the generator of $(\mathcal{E}_0, D(\mathcal{E}_0))$ has the form

$$(LF)(\gamma) = \sum_{x \in \gamma} \left( \Delta_x F(\gamma) + \left\langle \nabla_x r(x, \gamma \setminus x), \nabla_x F(\gamma) \right\rangle \right).$$

Here, for $x \in \gamma$, $\nabla_x F(\gamma) := \nabla_y F(\gamma \setminus x \cup y)|_{y=x}$ and analogously $\Delta_x$ is defined. However, we should not expect that $r(x, \gamma)$ is differentiable in $x$.

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**References**

[1] H. Bauer, Measure and Integration Theory, Walter de Gruyter & Co., Berlin, 2001.

[2] Y. M. Berezansky, Y. G. Kondratiev, Spectral Methods in Infinite Dimensional Analysis, Kluwer Acad. publ., Dordrecht/Boston/London, 1994.

[3] L. Bertini, N. Cancrini, F. Cesi, The spectral gap for a Glauber-type dynamics in a continuous gas, Ann. Inst. H. Poincaré Probab. Statist. 38 (2002) 91–108.

[4] V. I. Bogachev, Gaussian Measures, Providence, RI, 1998.

[5] D. J. Daley, D. Vere-Jones, An introduction to the theory of point processes. Vol. I. Elementary theory and methods. Second edition, Springer-Verlag, New York, 2003.

[6] J. da Silva, Y. Kondratiev, M. Röckner, On a relation between intrinsic and extrinsic Dirichlet forms for interacting particle systems, Math. Nachr. 222 (2001) 141–157.

[7] D. Finkelshtein, Y. Kondratiev, Regulation mechanisms in spatial stochastic development models, J. Stat. Phys. 136 (2009) 103–115.
[8] D. Finkelshtein, Y. Kondratiev, O. Kutoviy, Individual based model with competition in spatial ecology, SIAM J. Math. Anal. 41 (2009) 297–317.

[9] D. L. Finkelshtein, Y. G. Kondratiev, E. W. Lytvynov, Equilibrium Glauber dynamics of continuous particle systems as a scaling limit of Kawasaki dynamics, Random Oper. Stoch. Equ. 15 (2007) 105–126.

[10] D. L. Finkelshtein, Y. G. Kondratiev, M. J. Oliveira, Markov evolutions and hierarchical equations in the continuum. I. One-component systems, J. Evol. Equ. 9 (2009) 197–233.

[11] R. A. Holley, D. W. Stroock, Nearest neighbor birth and death processes on the real line, Acta Math. 140 (1978) 103–154.

[12] H.-O. Georgii, H. J. Yoo, Conditional intensity and Gibbsianness of determinantal point processes, J. Stat. Phys. 118 (2005) 55–84.

[13] E. Glötzl, Time reversible and Gibbsian point processes. I. Markovian spatial birth and death processes on a general phase space, Math. Nachr. 102 (1981) 217–222.

[14] E. Glötzl, Time reversible and Gibbsian point processes. II. Markovian particle jump processes on a general phase space, Math. Nachr. 106 (1982) 63–71.

[15] Y. G. Kondratiev, O. V. Kutoviy, E. W. Lytvynov, Diffusion approximation for equilibrium Kawasaki dynamics in continuum, Stochastic Process. Appl. 118 (2008) 1278–1299.

[16] Y. Kondratiev, O. Kutoviy, R. Minlos, On non-equilibrium stochastic dynamics for interacting particle systems in continuum, J. Funct. Anal. 255 (2008) 200–227.

[17] Y. Kondratiev, O. Kutoviy, S. Pirogov, Correlation functions and invariant measures in continuous contact model, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 11 (2008) 231–258.

[18] Y. G. Kondratiev, O. V. Kutoviy, E. Zhizhina, Nonequilibrium Glauber-type dynamics in continuum, J. Math. Phys. 47 (2006), 113501, 17 pp.

[19] Y. G. Kondratiev, E. Lytvynov, Glauber dynamics of continuous particle systems, Ann. Inst. H. Poincaré Probab. Statist. 41 (2005) 685–702.

[20] Y. G. Kondratiev, E. Lytvynov, M. Röckner, Equilibrium Kawasaki dynamics of continuous particles systems, Infin. Dimen. Anal. Quant. Prob. Rel. Top. 10 (2007) 185–210.

[21] Y. Kondratiev, R. Minlos, E. Zhizhina, Self-organizing birth-and-death stochastic systems in continuum, Rev. Math. Phys. 20 (2008) 451–492.
[22] E. Lytvynov, N. Ohlerich, A note on equilibrium Glauber and Kawasaki dynamics for fermion point processes, Methods Funct. Anal. Topology 14 (2008) 67–80.

[23] E. Lytvynov, L. Mei, On the correlation measure of a family of commuting Hermitian operators with applications to particle densities of the quasi-free representations of the CAR and CCR, J. Funct. Anal. 245 (2007) 62–88.

[24] Z.-M. Ma, M. Röckner, An Introduction to the Theory of (Non-Symmetric) Dirichlet Forms, Springer-Verlag, Berlin, 1992.

[25] Z.-M. Ma, M. Röckner, Construction of diffusions on configuration spaces, Osaka J. Math. 37 (2000) 273–314.

[26] K. Matthes, W. Warmuth, J. Mecke, Bemerkungen zu einer Arbeit von Nguyen Xuan Xahn und Hans Zessin, Math. Nachr. 88 (1978) 117–127.

[27] J. Møller, R. P. Waagepetersen, Modern statistics for spatial point processes, Scand. J. Statist. 34 (2007) 643–684.

[28] C. Preston, Spatial birth-and-death processes, in Proceedings of the 40th Session of the International Statistical Institute (Warsaw, 1975), Vol. 2, Bull. Inst. Internat. Statist., Vol. 46, 1975, pp. 371–391.

[29] M. Röckner, B. Schmuland, A support property for infinite-dimensional interacting diffusion processes, C. R. Acad. Sci. Paris 326 (1998), Série I, 359–364.

[30] T. Shirai, Y. Takahashi, Random point fields associated with certain Fredholm determinants I: fermion, Poisson and boson point processes, J. Funct. Anal. 205 (2003) 414–463.

[31] D. Vere-Jones, A generalization of permanents and determinants, Linear Algebra Appl. 111 (1988) 119–124.