Welfare Ratios in One-Sided Matching Mechanisms

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We study the Price of Anarchy of mechanisms for the fundamental problem of social welfare maximization in one-sided matching settings, when agents have general cardinal preferences over a finite set of items. We consider both the complete and incomplete information settings and show that the two most well-studied mechanisms in literature, Probabilistic Serial and Random Priority have a Price of Anarchy of \(O(\sqrt{n})\). We complement our results with a lower bound of \(\Omega(\sqrt{n})\) on the Price of Anarchy of all mechanisms. As a result, we conclude that these mechanisms are optimal.

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1. INTRODUCTION

The one-sided matching problem, often referred to as house allocation [Hylland and Zeckhauser 1979] is the fundamental problem of assigning items to agents, such that each agent receives exactly one item. It has numerous applications, such as assigning workers to shifts, students to courses or patients to doctor appointments. In this setting, agents have cardinal valuations which specify the intensity of their preferences over items, i.e. agents assign numerical values to items with the property that higher values are assigned to more preferred items. In game theory, such valuation functions are called von Neumann-Morgenstern utility.
functions [Von Neumann and Morgenstern 2007] and the associated preferences are often referred to as cardinal preferences.

A mechanism is a function that maps agents’ valuations to matchings. However, agents are rational strategic entities that will not always report their valuations truthfully; they may misreport their values if that guarantees a better matching (from their own perspective). The agents report their valuations strategically to maximize their utilities and they naturally end up in some Nash equilibrium of the induced game, i.e. a strategy profile from which no agent wishes to unilaterally deviate.

A natural objective for the designer, is to choose the matching that maximizes the social welfare, i.e. the sum of agents’ valuations for the items they are matched with, which is the prominent measure of aggregate utility in literature. Given the strategic nature of the agents, we are interested in mechanisms that maximize the social welfare in the equilibrium. We use the standard measure of equilibrium inefficiency, the Price of Anarchy [Koutsoupias and Papadimitriou 1999], that compares the maximum social welfare attainable in any matching with the worst-case social welfare that can be achieved in the equilibrium.

We evaluate the efficiency of a mechanism with respect to the Price of Anarchy of the induced game. Our setting involves no monetary transfers. In settings with money, such as combinatorial auctions, a recent body in literature studies the Price of Anarchy of item-bidding auctions with second-price or first-price payment rules (see [Christodoulou et al. 2008; Feldman et al. 2013; Syrgkanis and Tardos 2013; Bhawalkar and Roughgarden 2012; de Keijzer et al. 2013] for an non-exclusive list).

Our work is in a very strong sense analogous to the auction literature but applied to mechanisms without money. We do not impose any restriction on the class of mechanisms we consider. We study both deterministic and randomized mechanisms; in the latter case the output is a probability mixture over matchings, instead of a single matching. We are interested in the general class of cardinal mechanisms, i.e. mechanisms that make full use of the numerical values that they are given as inputs.

In general settings without money like the one we study, one has to fix a canonical representation of the valuations. A common approach in the literature is to consider the unit-sum normalization, i.e. each agent has a total value of 1 for all the items. We will assume that valuations are normalized this way throughout the paper.

1.1. Our results

As our main contribution, we bound the inefficiency of the two most well-known mechanisms in the literature of one-sided matching problems, Probabilistic Serial and Random Priority. We complement this analysis by showing a matching lower bound that applies to all cardinal (randomized) mechanisms. As a result, we conclude that those two ordinal mechanisms are optimal.

We start by proving an $O(\sqrt{n})$ Price of Anarchy guarantee for the two mechanisms mentioned above and then we prove that no mechanism can achieve a Price of Anarchy better than $\Omega(\sqrt{n})$. The fact that those mechanisms are ordinal is quite interesting; our result suggest that even if we allow mechanisms to use the cardinal nature of the reports, we can not achieve a better Price of Anarchy guarantee.

We study both the complete information and the incomplete information settings, where agent valuations are drawn from some known prior distributions. We stress that, in analogy to the literature in auctions, in the first case, our Price of Anarchy bounds extend from the simplest solutions concepts of pure or mixed Nash equilibrium

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1Ordinal mechanisms are mechanisms whose output only depends on preference orderings and not the actual numerical values.
to the very general concepts of correlated and coarse-correlated equilibria. For the incomplete information case, we show how our results extend for Bayes-Nash equilibria.

We also consider deterministic mechanisms and prove that the pure Price of Anarchy of any mechanism (including cardinal mechanisms) is bounded by $\Omega(n^2)$. This result suggests that randomization is essential for non-trivial efficiency guarantees to be achievable.

We provide two extensions to our main results. First, we consider the Price of Stability [Anshelevich et al. 2008], a more optimistic measure of efficiency than the Price of Anarchy. We prove that under a mild “proportionality-like” property, our lower bound of $\Omega(\sqrt{n})$ extends to this case as well. Finally, we consider another common normalization of valuations, the unit-range representation. We prove that our upper bounds extend to this case as well and complement our results with Price of Anarchy lower bounds for deterministic and randomized mechanisms.

1.2. Discussion and related work

The one-sided matching problem was first studied in Hylland and Zeckhauser [1979]. The literature on the problem (as well as more general assignment problems) is deep and extensive and here we will only mention the results that are most relevant to the current paper; the interested reader could read the surveys of [Abdulkadiroğlu and Sönmez 2013] or [Sönmez and Unver 2011]. Over the years, many different mechanisms have been proposed with various desirable properties related to truthfulness, fairness and economic efficiency. Perhaps the two best-studied of all are Probabilistic Serial and Random Priority. Random priority is truthful, but performs worse in terms of economic efficiency and fairness. In fact, the debate on which the best mechanism for one-sided matching problems is is still ongoing. The objective of social welfare provides an alternative way of comparing mechanisms and our paper tackles the debate in that direction.

As we mentioned earlier, in settings without money, one needs to represent the valuations in some canonical way. A common approach is the unit-sum normalization, i.e. each agent has a total value of 1 for all the items. Intuitively, this normalization means that each agent has equal influence within the mechanism and her values can be interpreted as "utility points" that she uses to acquire items. The unit-sum representation is standard for social welfare maximization in many settings without money including fair division and cake-cutting [Brams et al. 2012; Caragiannis et al. 2012; Cohler et al. 2011; Karp et al. 2014; Cole et al. 2013], indivisible and divisible item allocation [Guo and Conitzer 2010; Branzei et al. 2014; Feldman et al. 2009; Filos-Ratsikas et al. 2014] and social choice settings [Boutilier et al. 2012] among others. As an extension, we also consider another common representation, the unit-range normalization [Zhou 1990; Adamczyk et al. 2014; Filos-Ratsikas et al. 2014; Filos-Ratsikas and Miltersen 2014] i.e. each agent values his most preferred item at 1 and his least preferred item at 0.

The objective of social welfare maximization for one-sided matching problems has been studied before in literature but interestingly mainly for truthful mechanisms [Anshelevich and Das 2010; Adamczyk et al. 2014; Filos-Ratsikas et al. 2014]. Note that Random Priority is truthful (i.e. it has a dominant strategy equilibrium) but it does have other equilibria as well; we prove an efficiency guarantee for all equilibria of the mechanism, not just the truth-telling ones. Similar approaches have been adopted for truthful mechanisms like the second price auction in settings with money [Babaioff et al. 2014].

Besides, it is not difficult to see that without any normalization, non-trivial Price of Anarchy bounds can not be achieved by any mechanism.
Another, somewhat different recent branch of study considers ordinal measures of efficiency instead of social welfare maximization \cite{Bhalgat2011, Krysta2014, Chakrabarty2014}, under the assumption that agents preferences are only expressed through preference orderings over items. While certainly interesting, these measures of efficiency do not accurately encapsulate the socially desired outcome the way that social welfare does, especially since an underlying cardinal utility structure is part of the setting. \cite{Bogomolnaia2001, Hylland1979, Zhou1990, VonNeumann2007}. Our results actually suggest that in order to achieve the optimal welfare guarantees, one does not even need to elicit this utility structure; agents can only be asked to report preference orderings, which is arguably more appealing.

Probabilistic Serial was introduced by Bogomolnaia and Moulin \cite{Bogomolnaia2001} and has been studied extensively ever since \cite{Hashimoto2014, Kesten2006, Katta2006, Kojima2010}. Somewhat surprisingly, the Nash equilibria of the mechanism were only recently studied. Aziz et al. \cite{Aziz2014a} prove that PS has pure Nash equilibria. Ekici and Kesten \cite{Ekici2010} study the ordinal equilibria of the mechanism and prove that the good properties of the mechanism are not necessarily satisfied for those profiles.

Random Priority is a folklore mechanism that has also been the subject of many papers \cite{Bogomolnaia2001, Abdulkadiroglu1998, Sonmez1998, Sönmez2005, Filos-Ratsikas2014, Adamczyk2014, Aziz2013, Krysta2014}. Recently, \cite{Mennle2014} proposed a class of Hybrid mechanisms for achieving tradeoffs between quantified variants of truthfulness and Pareto efficiency. This class contains mechanisms which are convex combinations of Probabilistic Serial and Random Priority.

1.3. Organization
In Section 3, 4 and 5, we prove our main theorems. First in Section 3 and 4 we prove that Probabilistic Serial and Random Priority have Price of Anarchy $O(\sqrt{n})$ for pure Nash, mixed Nash, correlated, coarse-correlated and Bayes-Nash equilibria. Then in Section 5 we prove that these mechanisms are asymptotically optimal among all mechanisms, even among those that use cardinal information. In Section 6 we extend our lower bounds to the Price of Stability of all mechanisms that satisfy a mild “proportionality-like” property. Finally, in Section 7 we discuss how our results extend the other common normalization, unit-range.

2. PRELIMINARIES
Let $N = \{1, \ldots, n\}$ be a finite set of agents and $A = \{1, \ldots, n\}$ be a finite set of indivisible items. An allocation is a matching of agents to items, that is, an assignment of items to agents where each agent gets assigned exactly one item. We can view an allocation $\mu$ as a vector $\left(\mu_1, \mu_2, \ldots, \mu_n\right)$ where $\mu_i$ is the unique item matched with agent $i$. Let $O$ be the set of all allocations. Each agent $i$ has a valuation function $u_i : A \rightarrow \mathbb{R}$ mapping items to real numbers, which is arbitrary. Valuation functions are considered to be well-defined modulo positive affine transformations, that is, for item $j : j \rightarrow \alpha u_i(j) + \beta$ is considered to be an alternative representation of the same valuation function $u_i$. Given this, we fix the canonical representation of $u_i$ to be unit-sum, that is $\sum_j u_i(j) = 1$, with $u_i(j) \geq 0$ for all $i, j$. Equivalently, we can consider valuation functions as valuation vectors $u_i = (u_{i1}, u_{i2}, \ldots, u_{in})$ and let $V$ be the set of all valuation vectors of an agent. Let $u = (u_1, u_2, \ldots, u_n)$ denote a typical valuation profile and let $V^n$ be the set of all valuation profiles with $n$ agents.
We consider strategic agents who might have incentives to lie about their valuations. We define \( s = (s_1, s_2, \ldots, s_n) \) to be a pure strategy profile, where \( s_i \) is the reported valuation vector of agent \( i \). For general mechanisms \( V^n \) is the set of all pure strategy profiles. A direct revelation mechanism without money is a function \( M : V^n \to O \) mapping reported valuation profiles to outcomes. For a randomized mechanism, we define \( M \) to be a random map \( M : V^n \to O \). Let \( M_i(s) \) denote the restriction of the outcome of the mechanism to the \( i \)th coordinate, which is the item assigned to agent \( i \) by the mechanism. For randomized mechanisms, we will let \( p^M_{ij} = \Pr[M_i(s) = j] \) and \( p^M_i = (p^M_{i1}, \ldots, p^M_{in}) \). When it is clear from the context, we will drop one or both of the superscripts from the terms \( p^M_{ij} \) and \( p^M_i \).

The utility of an agent from the outcome of a deterministic mechanism \( M \) on input strategy profile \( s \) is simply \( u_i(M_i(s)) \). For randomized mechanisms, an agent's utility is \( E[u_i(M_i(s))] = \sum p^M_{ijs} u_{ij} \). A subclass of mechanisms that are of particular interest is that of ordinal mechanisms.

**Definition 2.1.** A mechanism \( M \) is ordinal if for any strategy profiles \( s, s' \) such that for all agents \( i \) and for all items \( j, l, s_{ij} < s_{il} \Leftrightarrow s'_{ij} < s'_{il} \), it holds that \( M(s) = M(s') \). A mechanism for which the above does not necessarily hold is cardinal.

Informally, ordinal mechanisms operate solely based on the ordering of items induced by the valuation functions and not the actual numerical values themselves, while cardinal mechanisms take those numerical values into account when outputting an outcome. For ordinal mechanisms, we will define the strategy space to be the set of all permutations of \( n \) items instead of the space of valuation functions \( V^n \). A strategy \( s_i \) of agent \( i \) is a preference ordering of items \( (a_1, a_2, \ldots, a_n) \) where \( a_\ell > a_k \) for \( \ell < k \). We will write \( j >_i j' \) to denote that agent \( i \) prefers item \( j \) to item \( j' \) according to her true valuation function and \( j >_{s_i} j' \) to denote that she prefers item \( j \) to item \( j' \) according to her strategy \( s_i \). When it is clear from the context, we abuse the notation slightly and let \( u_i \) denote the truth-telling strategy of agent \( i \), even when the mechanism is ordinal. Note that agents can be indifferent between items and hence the preference order can be a weak ordering.

Two properties that many mechanisms satisfy are anonymity and neutrality. A mechanism is anonymous if the output is imprecise to renamings of the agents and neutral if the output is imprecise to relabeling of the objects. A subclass is called an equilibrium if a strategy profile in which no agent has an incentive to deviate to different strategy. We consider five standard equilibrium concepts in this paper: pure Nash, mixed Nash, correlated, coarse correlated and Bayes-Nash equilibria. For the first four, the agents have full information. In the Bayesian setting, the valuations are drawn from some distributions and agents know their own valuation and the distributions from which the other valuations are drawn from. We formally define the different equilibrium concepts.

**Definition 2.2.** Given a mechanism \( M \), let \( q \) be a distribution over strategies. Also, for any distribution \( \Delta \) let \( \Delta_{-i} \) denote the marginal distribution without the \( i \)th index. Then a strategy profile \( q \) is called a

1. pure Nash equilibrium if \( q = s \) and \( u_i(M_i(s)) \geq u_i(M_i(s'_i, s_{-i})) \),
2. mixed Nash equilibrium if \( q = \times q_i, E_{s \sim q}[u_i(M_i(s))] \geq E_{s \sim q_{-i}}[u_i(M_i((s'_i, s_{-i})))] \)
3. correlated Nash equilibrium if \( E_{s \sim q}[u_i(M_i(s))|s_i] \geq E_{s \sim q}[u_i(M_i((s'_i, s_{-i})))|s_i] \),

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(4) coarse correlated Nash equilibrium if \( \mathbb{E}_{\mathbf{s} \sim \mathbf{q}}[u_i(M_i(s))] \geq \mathbb{E}_{\mathbf{s} \sim \mathbf{q}}[u_i(M_i((s', s_{-i})))] \),

(5) Bayes-Nash equilibrium for a distribution \( \Delta_u \) where each \( (\Delta_u)_i \) is independent, if when \( u \sim \Delta_u \) then \( q(u) = \times_i q_i(u_i) \) and for all \( u \) in the support of \( (\Delta_u)_i \)
\[
\mathbb{E}_{u_{-i}, s \sim \mathbf{q}(u)}[u_i(M_i(s))] \geq \mathbb{E}_{u_{-i}, s_{-i} \sim \mathbf{q}_{-i}}[u_i(M_i(s', s_{-i}))],
\]
where the given inequalities hold for all agents \( i \), and \( (\text{pure}) \) deviating strategies \( s'_i \).

Also notice that for randomized mechanisms definitions are with respect to an expectation over the random choices of the mechanism.

It is well known that for the first four classes each is contained in the next class, i.e., pure \( \subset \) mixed \( \subset \) correlated \( \subset \) coarse correlated. If we regard the full information setting as a special case of Bayesian setting, we also have pure \( \subset \) mixed \( \subset \) Bayesian. This means that for the complete information setting, when proving efficiency guarantees, it suffices to consider the coarse correlated equilibria of a mechanism and in the incomplete information setting, we only need to consider Bayes-Nash equilibria.

Let \( S^M_0 \) denote the set of all pure Nash equilibria of mechanism \( M \) under truthful valuation profile \( u \). The measure of efficiency that we will use is the pure Price of Anarchy,

\[
\text{PoA}(M) = \sup_{u \in \mathcal{V}^n} \frac{SW_{OPT}(u)}{\min_{s \in S^M_0} SW_M(u, s)}
\]

where \( SW_M(u, s) = \sum_{i=1}^{n} \mathbb{E}[u_i(M_i(s))] \) is the expected social welfare of mechanism \( M \) on strategy profile \( s \) under true valuation profile \( u \), and \( SW_{OPT}(u) = \max_{u \in \mathcal{O}} \sum_{i=1}^{n} u_i(\mu_i) \) is the social welfare of the optimal matching. Let \( OPT \) be the mechanism that outputs that matching. Similarly, we can define the mixed, correlated, coarse correlated and Bayesian Price of Anarchy.

3. PROBABILISTIC SERIAL

First, we consider Probabilistic Serial, which we will refer to as \( PS \) for short. Informally, the mechanism is the following. Each item can be viewed as an infinitely divisible good that all agents consume at the same speed over the unit interval \([0, 1]\). In the beginning, each agent consumes his most preferred item (or one of his most preferred items in case of ties) until the item is entirely consumed. Then, the agent would be if she were consuming the item she is matched with in the optimal allocation of item \( s_{ij} \) consumed by agent \( i \) is then interpreted as the probability that agent \( i \) will be matched with item \( j \) under the mechanism.

We prove that the Price of Anarchy of \( PS \) for all solution concepts is \( O(\sqrt{n}) \). Recall that Probabilistic Serial has pure Nash equilibria [Aziz et al. 2014] and hence we will first prove our upper bound for pure Price of Anarchy and then generalize them.

We start by proving the following lemma, which states that in a pure Nash equilibrium of the mechanism an agent’s utility cannot be much worse than what her utility would be if she were consuming the item she is matched with in the optimal allocation from the beginning of the mechanism until the item is entirely consumed. Let \( t_j(s) \) be the time when item \( j \) is entirely consumed on profile \( s \) under \( PS(s) \).

**Lemma 3.1.** Let \( u \) be the profile of true agent valuations and let \( s \) be a pure Nash equilibrium. Let \( i \) be any agent and let \( j = OPT_i(u) \). Then, \( \sum_{l=1}^{n} p^*_{il} u_{il} \geq \frac{1}{4} \cdot t_j(s) \cdot u_{ij} \).

**Proof.** Let \( s' = (s'_i, s_{-i}) \) be the strategy profile obtained from \( s \) when agent \( i \) deviates to the strategy \( s'_i \) where \( s'_i \) is some strategy such that \( j \succ s_l \) for all items \( l \neq j \). If \( s'_i = s_i \), i.e., agent \( i \) is already consuming item \( j \) from the beginning, her utility
\[ u_i(PS_i(s)) = \sum_{t=1}^{n} p_{it}^s u_{it} \] is at least \( t_j(s) \cdot u_{ij} \) and we are done. Hence assume that \( s_i \neq s' \). Obviously, agent i’s utility \( u_i(PS_i(s')) = \sum_{t=1}^{n} p_{it}^s u_{it} \) is at least \( t_j(s') \cdot u_{ij} \) so since \( s \) is a pure Nash equilibrium, it suffices to prove that \( t_j(s') \geq \frac{1}{\alpha} \cdot t_j(s) \).

First, note that if agent i is the only one consuming item \( j \) for the duration of the mechanism; then \( t_j(s') = 1 \) and we are done. So assume at least one other agent consumes item \( j \) at some point, and let \( \tau \) be the time when the first agent besides agent \( i \) starts consuming item \( j \) in \( s' \). Obviously, \( t_j(s') \geq \tau \), therefore if \( \tau \geq \frac{1}{\alpha} \cdot t_j(s) \) then \( t_j(s') \geq \frac{1}{\alpha} \cdot t_j(s) \) and we are done. So assume that \( \tau < \frac{1}{\alpha} \cdot t_j(s) \).

Next observe that in the interval \([\tau, t_j(s')]\), agent i can consume at most half of what remains of item i because there exists at least one other agent consuming the item for the same duration. Overall, agent i’s consumption is at most \( \frac{1}{\alpha} + \frac{1}{\beta} \cdot t_j(s) \) so at least \( \frac{1}{\alpha} - \frac{1}{\beta} \cdot t_j(s) \) of the item will be consumed by the rest of the agents.

Now consider all agents other than \( i \) in profile \( s \) and let \( \alpha \) be the the amount of item \( j \) that they have consumed by time \( t_j(s) \). Notice that the total consumption speed of an item is non-decreasing in time which means in particular that for any \( 0 \leq \beta \leq 1 \), agents other than \( i \) need at least \( \beta \cdot t_j(s) \) time to consume \( \alpha \cdot \beta \) in profile \( s \). Next, notice that since agent i starts consuming item \( j \) at time \( 0 \) in \( s' \) and all other agents use the same strategies in \( s \) and \( s' \), it holds that every agent \( k \neq i \) starts consuming item \( j \) in \( s' \) no sooner than she does in \( s \). This means that in profile \( s' \), agents other than \( i \) will need more time to consume \( \beta \cdot \alpha \); in particular they will need at least \( \beta \cdot t_j(s) \) time, so \( t_j(s') \geq \beta t_j(s) \). However, from the previous paragraph we know that they will consume at least \( \frac{1}{\alpha} - \frac{1}{\beta} \cdot t_j(s) \), so letting \( \beta = \frac{1}{\alpha} \left( \frac{1}{\beta} - \frac{1}{4} \cdot t_j(s) \right) \) we get

\[ t_j(s') \geq \beta t_j(s) \geq t_j(s) \left( \frac{1}{\alpha} - \frac{1}{\beta} \cdot t_j(s) \right) \frac{1}{\alpha} \geq t_j(s) \left( \frac{1}{\alpha} - \frac{1}{4} \cdot t_j(s) \right) \geq \frac{1}{4} \cdot t_j(s) \]

\( \square \)

We can now prove the pure Price of Anarchy guarantee of the mechanism.

**Theorem 3.2.** The pure Price of Anarchy of the Probabilistic Serial mechanism is \( O(\sqrt{n}) \).

**Proof.** Let \( u \) be any profile of true agent valuations and let \( s \) be any pure Nash equilibrium. First, note that by reporting truthfully, every agent \( i \) can get an allocation that is at least as good as \( (\frac{1}{n}, \ldots, \frac{1}{n}) \), regardless of other agents’ strategies. To see this, first consider time \( t = 1/n \) and observe that during the interval \([0,1/n]\), agent \( i \) is consuming her favorite item (say \( a_1 \)) and hence \( \text{SW}_{u(a_1)} \geq \frac{1}{n} \). Next, consider time \( \tau = 2/n \) and observe that during the interval \([0,2/n]\), agent \( i \) is consuming one or both of her two favorite items (\( a_1 \) and \( a_2 \)) and hence \( \text{SW}_{u(a_1)} + \text{SW}_{u(a_2)} \geq \frac{2}{n} \). By a similar argument, for any \( k \), it holds that \( \sum_{j=1}^{n} p_{ia_j} \geq \frac{k}{n} \). This implies that regardless of other agents’ strategies, agent \( i \) can achieve a utility of at least \( \frac{1}{n} \sum_{j=1}^{n} u_{ij} \). Since \( s \) is a pure Nash equilibrium, it holds that \( u_i(PS_i(s)) \geq \frac{1}{n} \sum_{j=1}^{n} u_{ij} \) as well. Summing over all agents, we get that \( \text{SW}_{PS(u,s)} \geq \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{n} u_{ij} = 1 \). If \( \text{SW}_{OPT(u)} \leq \sqrt{n} \) then we are done, so assume \( \text{SW}_{OPT(u)} > \sqrt{n} \).

Because \( PS \) is neutral we can assume \( t_j(s) \leq t_j(s') \) for \( j < j' \) without loss of generality. Observe that for all \( j = 1, \ldots, n \), it holds that \( t_j(s) \geq \frac{1}{\alpha} \). This is true because for any \( t \in [0,1] \), by time \( t \), exactly \( t \cdot n \) mass of items must have been consumed by the agents. Since \( j \) is the \( j \)th item that is entirely consumed, by time \( t_j(s) \), the mass of items that must have been consumed is at least \( j \). By this, we get that \( t_j(s) \geq \frac{j}{n} \).
For each \( j \) let \( i_j \) be the agent that gets item \( j \) in the optimal allocation and for ease of notation, let \( w_i_j \) be her valuation for the item. Now by Lemma 3.1 it holds that \( u_{i_j}(PS(s)) \geq \frac{1}{4} \cdot \frac{1}{\sqrt{n}} \cdot w_{i_j} \) and \( SW_{PS}(u, s) \geq \frac{1}{4} \sum_{j=1}^{n} w_{i_j} \). The Price of Anarchy is then at most \( \frac{4 \sum_{j=1}^{n} w_{i_j}}{\sum_{j=1}^{n} \frac{1}{\sqrt{j}} w_{i_j}/n} \).

Consider the case when the above ratio is maximized and let \( k \) be an integer such that \( k \leq \sum_{j=1}^{n} w_{i_j} \leq k + 1 \). Then it must be that \( w_{i_j} = 1 \) for \( j = 1, \ldots, k \) and \( w_{i_j} = 0 \), for \( k + 2 \leq i_j \leq n \). Hence the maximum ratio is \( \frac{k + w_{i_{k+1}}}{aw_{i_{k+1}} + b} \), for some \( a, b > 0 \), which is monotone for \( w_{i_{k+1}} \) in \([0, 1]\). Therefore, the maximum value of \( \frac{k + w_{i_{k+1}}}{aw_{i_{k+1}} + b} \) is achieved when either \( w_{i_{k+1}} = 0 \) or \( w_{i_{k+1}} = 1 \). As a result, the maximum value of the ratio is obtained when \( \sum_{i=1}^{n} w_{i_{k+1}} = k \) for some \( k \). By simple calculations, the Price of Anarchy should be at most:

\[
\frac{4k}{\sum_{j=1}^{k} \frac{1}{\sqrt{j}} w_{i_j}/n} \leq \frac{4k}{\frac{8n}{2n}} = \frac{8n}{k - 1},
\]

so the Price of Anarchy is maximized when \( k \) is minimized. By the argument earlier, \( k > \sqrt{n} \) and hence the ratio is \( O(\sqrt{n}) \). \( \square \)

In the following, we extend Theorem 3.2 to the case when the solution concept is the coarse correlated Nash equilibrium. Recall that the class of coarse correlated equilibria includes the classes of mixed Nash and correlated equilibria so since we are proving a Price of Anarchy bound, the result covers those solution concepts as well.

**Theorem 3.3.** The coarse correlated Price of Anarchy of Probabilistic Serial is \( O(\sqrt{n}) \).

**Proof.** Let \( u \) be any valuation profile. Let \( i \) be any agent and let \( j = OPT_i(u) \). The intuition here is that in the proof of Lemma 3.1 the inequality \( t_j(s') \geq \frac{1}{4} t_j(s) \) holds for every strategy profile. In particular, it holds for any pure strategy profile \( s \) where \( s_i \) is in the support of the distribution of the mixed strategy \( q_i \) of agent \( i \), for any coarse correlated equilibrium \( q \). Now let \( s'_i \) be the pure strategy that places item \( j \) on top of agent \( i \)’s preference list. This implies that

\[
\mathbb{E}_{s \sim q} [u_i(PS_i(s))] \geq \mathbb{E}_{s \sim q} [u_i(PS_i(s'_i, s_{-i}))] \geq \mathbb{E}_{s \sim q} [u_{ij}t_j(s'_i, s_{-i})] \geq \frac{1}{4} u_{ij}t_j(s).
\]

where the last inequality holds by the discussion above on Lemma 3.1 Using this, we can use very similar arguments to the arguments of the proof of Theorem 3.2 and obtain the bound. \( \square \)

For the incomplete information setting, when valuations are drawn from some publicly known distributions, we can prove the same upper bound on the Bayesian Price of Anarchy of the mechanism.

**Theorem 3.4.** The Bayesian Price of Anarchy of Probabilistic Serial is \( O(\sqrt{n}) \).

**Proof.** The proof is again similar to the proof of Theorem 3.2. Let \( u \) be a valuation profile drawn from some distribution satisfying the unit-sum constraint. Let \( i \) be any agent and let \( j_u = OPT_i(u), i \in [n] \). Note that by a similar argument as the one used in the proof of Theorem 3.2, the expected social welfare of \( PS \) is at least 1 and hence we can assume that \( \mathbb{E}_u[SW_{OPT}(u)] \geq 2\sqrt{2n} + 1 \). Observe that in any Bayes-Nash equilibrium...
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4. RANDOM PRIORITY

We now turn our attention to another mechanism, Random Priority, or RP for short, often referred to as Random Serial Dictatorship. The mechanism first fixes an ordering of the agents uniformly at random and then according to that ordering, sequentially matches them with their most preferred item that is still available. Random Priority is truthful, but it does have other equilibria as well. From [Filos-Ratsikas et al. 2014], we know that in the truth-telling equilibrium, the welfare of the mechanism and the maximum social welfare differ by a multiple of at most $O(\sqrt{n})$. We prove here that this ratio is guaranteed in all equilibria of the mechanism, for any of the equilibrium notions. We start with an interesting lemma when agents’ valuation for items are all distinct.

**Lemma 4.1.** If valuations are distinct, the social welfare is the same in all mixed Nash equilibria of Random Priority.

**Proof.** Let $i$ be an agent, and let $B$ be a subset of the items. Let $s$ be a mixed Nash equilibrium with the property that with positive probability, $i$ will be chosen to select an item at a point when $B$ is the set of remaining items. In that case (by distinctness of $i$’s values), $i$’s strategy should place agent $i$’s favourite item in $B$ on the
top of the preference list among items in \( B \). Suppose that for items \( j \) and \( j' \), there is no set of items \( B \) that may be offered to \( i \) with positive probability, in which either \( j \) or \( j' \) is optimal. Then \( i \) may rank them either way, i.e. can announce \( j \succ_i j' \) or \( j' \succ_i j \). However, that choice has no effect on the other agents, in particular it cannot affect their social welfare.

Given the main theorem in [Filos-Ratsikas et al. 2014], Lemma 4.1 implies the following.

**Corollary 4.2.** If valuations are distinct, the Price of Anarchy of Random Priority is \( \Theta(\sqrt{n}) \).

The same guarantee on the Price of Anarchy holds even when the true valuations of agents are not necessarily distinct.

**Theorem 4.3.** The Price of Anarchy of Random Priority is \( O(\sqrt{n}) \), even if valuations are not distinct.

**Proof.** We know from [Filos-Ratsikas et al. 2014] that the social welfare of Random Priority given truthful reports, is within \( O(\sqrt{n}) \) of the social optimum. The social welfare of a (mixed) Nash equilibrium \( q \) cannot be worse than the worst pure profile from \( q \) that occurs with positive probability, so let \( s \) be such a pure profile. We will say that agent \( i \) misranks items \( j \) and \( j' \) if \( j \succ_i j' \), but \( j' \succ_s j \).

If an agent misranks two items for which she has distinct values, it is because she has 0 probability in \( s \) to receive either item. So we can change \( s \) so that no items are misranked, without affecting the social welfare or the allocation. For items that the agent values equally (which are then not misranked) we can apply arbitrarily small perturbations to make them distinct. Profile \( s \) is thus consistent with rankings of items according to perturbed values and is truthful with respect to these values, which, being arbitrarily close to the true ones, have optimum social welfare arbitrarily close to the true optimal social welfare.

Theorem 4.3 can be extended to solution concepts more general than the mixed Nash equilibrium. The next theorem extends the result to the large class of coarse correlated equilibria.

**Theorem 4.4.** The coarse correlated Price of Anarchy of Random Priority is \( O(\sqrt{n}) \).

**Proof.** The argument is very similar to the one used in the proof of Theorem 4.3. Again, if any strategy in the support of a correlated equilibrium \( q \) misranks two items \( j \) and \( j' \) for any agent \( i \), it can only be because agent \( i \) has 0 probability of receiving those items, otherwise agent \( i \) would deviate to truth-telling, violating the equilibrium condition. The remaining steps are exactly the same as in the proof of Theorem 4.3.

Again, for the incomplete information case, we prove the same Price of Anarchy guarantee in the Bayes-Nash equilibria of the mechanism.

**Theorem 4.5.** The Bayesian Price of Anarchy of Random Priority is \( O(\sqrt{n}) \).

**Proof.** Consider any Bayes-Nash equilibrium \( q(u) \) and let \( u \) be a any sampled valuation profile. The expected social welfare of the Random Priority can be written as \( E_u [E_{s \sim q(u)} [u_i(s)]] \). Using the same argument as the one in the proof of Theorem 4.3, we can lower bound the quantity \( E_{s \sim q(u)} [u_i(s)] \) by \( \Omega \left( \frac{SW_{OPT}(u)}{\sqrt{n}} \right) \) and the bound follows.
5. LOWER BOUNDS

Here, we prove our main lower bound. Note that the result holds for any mechanism, including randomized and cardinal mechanisms. Since we are interested in mechanisms with good properties, it is natural to consider those mechanisms that have pure Nash equilibria.

**Theorem 5.1.** The pure Price of Anarchy of any mechanism is $\Omega(\sqrt{n})$.

**Proof.** Assume $n = k^2$ for some $k \in \mathbb{N}$. Let $M$ be a mechanism and consider the following valuation profile $u$. There are $\sqrt{n}$ sets of agents and let $G_j$ denote the $j$-th set. For every $j \in \{1, \ldots, \sqrt{n}\}$ and every agent $i \in G_j$, it holds that $u_{ij} = \frac{1}{n} + \alpha$ and $u_{ik} = \frac{1}{n} - \frac{\alpha}{2n}$, for $k \neq j$, where $\alpha$ is sufficiently small. Let $s$ be a pure Nash equilibrium and for every set $G_j$, let $i_j = \arg\min_{i \in G_j} u_{ij}^{M,s}$ (break ties arbitrarily). Observe that for all $j = 1, \ldots, \sqrt{n}$, it holds that $p^{M,s}_{ij} \leq \frac{1}{\sqrt{n}}$ and let $I = \{i_1, i_2, \ldots, i_{\sqrt{n}}\}$. Now consider the valuation profile $u'$ where:

- For every agent $i \in I$, $u_i' = u_i$.
- For every agent $i_j \in I$, let $u_{i_j,j} = 1$ and $u_{i_j,k} = 0$ for all $k \neq j$.

We claim that $s$ is a pure Nash equilibrium under $u'$ as well. For agents not in $I$, the valuations have not changed and hence they have no incentive to deviate. Assume now for contradiction that some agent $i \in I$ whose most preferred item is item $j$ could deviate to some beneficial strategy $s'_i$. Since agent $i$ only values item $j$, this would imply that $p^{M,(s_i',s_{-i})}_{ij} > p^{M,s}_{ij}$. However, since agent $i$ values all items other than $j$ equally under $u_i$ and her most preferred item is item $j$, such a deviation would also be beneficial under profile $u$, contradicting the fact that $s$ is a pure Nash equilibrium.

Now consider the expected social welfare of $M$ under valuation profile $u'$ at the pure Nash equilibrium $s$. For agents not in $I$ and taking $\alpha$ to be less than $\frac{1}{\sqrt{n}}$, the contribution to the social welfare is at most $1$. For agents in $I$, the contribution to the welfare is then at most $\frac{1}{\sqrt{n}} \cdot \sqrt{n} + 1$ and hence the expected social welfare of $M$ is at most $3$. As the optimal social welfare is at least $\sqrt{n}$, the bound follows. \(\square\)

Interestingly, if we restrict our attention to deterministic mechanisms only, then we can prove that only trivial pure Price of Anarchy guarantees are achievable.

**Theorem 5.2.** The pure Price of Anarchy of any deterministic mechanism is $\Omega(n^2)$.

**Proof.** Let $M$ be a deterministic mechanism that always has a pure Nash equilibrium. Now let $u$ be a valuation profile such that for for all agents $i$ and $i'$, it holds that $u_i = u_{i'}$, $u_{i,1} = \frac{1}{n} + \frac{1}{\sqrt{n}}$ and $u_{i,j} > u_{i,k}$ for $j < k$. Let $s$ be a pure Nash equilibrium for this profile and assume without loss of generality that $M_i(s) = i$.

Now fix another true valuation profile $u'$ such that $u_i' = u_i$ and for agents $i = 2, \ldots, n$, $u_{i,i-1}' = 1 - c_{i,i-1}'$ and $u_{i,j}' = c_{i,j}'$ for $j \neq i - 1$, where $0 \leq c_{i,j}' \leq \frac{1}{n}$, $\sum_{j \neq i-1} c_{i,j}' = c_{i,i-1}'$ and $c_{i,j}' > c_{i,k}'$ if $j < k$ when $j, k \neq i - 1$. Intuitively, in profile $u'$, each agent $i \in \{2, \ldots, n\}$ has valuation close to 1 for item $i - 1$ and small valuations for all other items. Furthermore, she prefers items with smaller indices, except for item $i - 1$.

We claim that $s$ is a pure Nash equilibrium under true valuation profile $u$ as well. Assume for contradiction that some agent $i$ has a benefiting deviation, which matches her with an item that she prefers more than $i$. But then, since the set of items that she prefers more than $i$ in both $u$ and $u'$ is $\{1, \ldots, i\}$, the same deviation would match her with a more preferred item under $u$ as well, contradicting the fact that $s$ is a pure...
Nash equilibrium. It holds that $SW_{OPT}(u) \geq n - 2$ whereas the social welfare of $M$ is at most $\frac{2}{n}$ and the theorem follows. \hfill \square

The optimal mechanism, i.e. the mechanism that naively tries to maximize the sum of the reported valuations with no regard to incentives, when equipped with a lexicographic tie-breaking rule has pure Nash equilibria and also achieves the above ratio in the worst-case, which means that the bounds are tight.

6. EXTENSION 1: PRICE OF STABILITY

Theorem [5.1] bounds the Price of Anarchy of all mechanisms. A more optimistic measure of efficiency is the Price of Stability, i.e. the worst-case ratio over all valuation profiles of optimal social welfare over the welfare attained at the best equilibrium, formally defined as

$$PoS(M) = \sup_{u \in V^n} \frac{SW_{OPT}(u)}{\max_{s \in S_M} SW_M(u, s)}.$$  

In this section we extend the Theorem [5.1] to the Price of Stability of all mechanisms that satisfy a "proportionality-like" property. This class is quite large and contains most well-known mechanisms, including Probabilistic Serial and Random Priority. We start with a few definitions.

Definition 6.1 (Stochastic Dominance). Let $a_1 \succ_i a_2 \succ_i \cdots \succ_i a_n$ be the (possibly weak) preference ordering of agent $i$. A random assignment vector $p_i$ for agent $i$ stochastically dominates another random assignment vector $q_i$ if $\sum_{j=1}^k p_{ia_j} \geq \sum_{j=1}^k q_{ia_j}$, for all $k = 1, 2, \cdots, n$. The notation that we will use for this relation is $p_i \succ_i^{sd} q_i$.

Definition 6.2 (Safe strategy). Let $M$ be a mechanism. A strategy $s_i$ is a safe strategy if for any strategy profile $s_{-i}$ of the other players, it holds that $M_i(s_i, s_{-i}) \succ_i^{sd} (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$.

We will say that a mechanism $M$ has a safe strategy if every agent $i$ has a safe strategy $s_i$ in $M$. We now state our lower bound.

Theorem 6.3. The pure Price of Stability of any mechanism that has a safe strategy is $\Omega(\sqrt{n})$.

Proof. Let $M$ be a mechanism and let $I = \{k+1, \ldots, n\}$ be a subset of agents. Let $u$ be the following valuation profile.

— For all agents $i \in I$, let $u_{ij} = \frac{1}{k}$ for $j = 1, \ldots, k$ and $u_{ij} = 0$ otherwise.

— For all agents $i \notin I$, let $u_{ii} = 1$ and $u_{ij} = 0$, $j \neq i$.

Now let $s$ be a pure Nash equilibrium on profile $u$ and let $s'_i$ be a safe strategy of agent $i$. The expected utility of each agent $i \in I$ in the pure Nash equilibrium $s$ is $E[u_i(s)] = \sum_{j \in [n]} p_{ij}(s_i, s_{-i}) v_{ij} \geq \sum_{j \in [n]} p_{ij}(s'_i, s_{-i}) v_{ij} \geq \frac{1}{n} \sum_{j \in [n]} v_{ij} = \frac{1}{n}$, due to the fact that $s$ is pure Nash equilibrium and $s'_i$ is a safe strategy of agent $i$.

On the other hand, the utility of agent $i \in I$ can be calculated by $E[u_i(s)] = \sum_{j \in [n]} p_{ij}(s_i, s_{-i}) v_{ij} = \frac{1}{k} \sum_{j=1}^k p_{ij}$. Because $s$ is a pure Nash equilibrium, it holds that $E[u_i] \geq \frac{1}{n}$, so we get that $\sum_{j=1}^k p_{ij} \geq \frac{k}{n}, i \in I$. As for the rest of the agents, $\sum_{i \notin I} \sum_{j=1}^k p_{ij} = k - \sum_{i \in I} \sum_{j=1}^k p_{ij} \leq k - (n-k) \frac{k}{n} = \frac{k^2}{n}$.

This implies that the contribution to the social welfare from agents not in $I$ is at most $\frac{k^2}{n}$ and the expected social welfare of $M$ will be at most $1 + \frac{k^2}{n}$. It holds that $SW_{OPT}(u) \geq k$ and the bound follows by choosing $k = \sqrt{n}$. \hfill \square
Due to Theorem 6.3, in order to obtain an $\Omega(\sqrt{n})$ bound for a mechanism $M$, it suffices to prove that $M$ has a safe strategy. We observe that this is indeed the case for a large class of mechanisms in the literature, including for example the well-known class of ordinal, envy-free mechanisms:

**Definition 6.4 (Envy-freeness).** A mechanism $M$ is (ex-ante) envy-free if for all agents $i$ and $r$ and all profiles $s$, it holds that $\sum_{j=1}^{n} p_{ij} s_{ij} \geq \sum_{j=1}^{n} p_{rj} s_{rj}$. Furthermore, if $M$ is ordinal, then this implies $p_{ij}^{M,s} \geq s_{ij} p_{r}^{M,s}$.

**Lemma 6.5.** Let $M$ be an ordinal, envy-free mechanism. Then for any agent $i$, the truthtelling strategy $u_i$ is a safe strategy.

**Proof.** Let $s = (u_i, s_{-i})$ be the strategy profile in which agent $i$ is truthtelling and the rest of the agent are playing some strategies $s_{-i}$. Since $M$ is envy-free and ordinal, it holds that $\sum_{j=1}^{n} p_{ij}^{s} \geq \sum_{j=1}^{n} p_{rj}^{s}$ for all agents $r \in \{1, \ldots, n\}$ and all $l \in \{1, \ldots, n\}$. Summing up these inequalities for agents $r = 1, 2, \ldots, n$ we obtain

$$n \sum_{j=1}^{l} p_{ij}^{s} \geq \sum_{j=1}^{l} \sum_{r=1}^{n} p_{rj}^{s} = l,$$

which implies that $\sum_{j=1}^{l} p_{ij}^{s} \geq \frac{l}{n}$, for all $i \in \{1, \ldots, n\}$, for all $l \in \{1, \ldots, n\}$. $\square$

**Lemma 6.6.** Random Priority has truthtelling as a safe strategy.

**Proof.** Since Random Priority first fixes an ordering of agents uniformly at random, every agent $i$ has a probability of $\frac{1}{n}$ to be selected first or choose an item, a probability of $\frac{2}{n}$ to be selected first or second and so on. If the agent ranks her items truthfully, then for every $l = 1, \ldots, n$, it holds that $\sum_{i=1}^{l} p_{ij}^{s} \geq \frac{l}{n}$. $\square$

Recently, [Mennle and Seuken 2014] defined the class of Hybrid Mechanisms to obtain tradeoffs between quantified versions of Pareto efficiency and truthfulness. Hybrid mechanisms are convex combinations of Probabilistic Serial and Random Priority, which as we will show, also have safe strategies.

**Lemma 6.7.** Any mechanism $M$ in the class of Hybrid Mechanisms has truth tellling as a safe strategy.

**Proof.** Mechanism $M$ can be written as a convex combination of $PS$ and $RP$ i.e. for every agent $i$ and strategy profile $s = (u_i, s_{-i})$, it holds that $p_{ij}^{M,s} = \alpha p_{ij}^{PS,s} + (1 - \alpha) p_{ij}^{RP,s}$. Since truthtelling is a safe strategy for both $PS$ and $RP$, it also holds that $\sum_{j=1}^{l} p_{ij}^{PS,s} \geq \frac{l}{n}$ and $\sum_{j=1}^{l} p_{ij}^{RP,s} \geq \frac{l}{n}$, for all $l = 1, \ldots, n$. This means that for every $l = 1, \ldots, n$, it holds that $\sum_{j=1}^{l} p_{ij}^{M,s} \geq \alpha \frac{l}{n} + (1 - \alpha) \frac{l}{n} = \frac{l}{n}$ and hence $M_i(s) > s_i$ \((\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})\). $\square$

Probabilistic Serial is ordinal and envy-free [Bogomolnaia and Moulin 2001] and hence from Lemma 6.5 and Lemma 6.7 we obtain the following corollaries.

**Corollary 6.8.** The Price of Stability of any ordinal, envy-free mechanism (including Probabilistic Serial) is $\Omega(\sqrt{n})$.

**Corollary 6.9.** The Price of Stability of any Hybrid Mechanism (including Random Priority) is $\Omega(\sqrt{n})$. 

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7. EXTENSION 2: UNIT-RANGE REPRESENTATION

In this section, we discuss how our results extend to the another normalization that is also common in literature, the unit-range representation, i.e., \( \max_j u_i(j) = 1 \) and \( \min_j u_i(j) = 0 \). In short, the Price of Anarchy guarantees from Section 3 and 4 extend directly to the unit-range case. The lower bound from Theorem 5.2 is replaced by an analogous theorem with a different (but still tight) bound whereas the lower bound from Theorem 5.1 is replaced by an \( \Omega(n^{1/4}) \) lower bound on the Price of Anarchy of any mechanism with respect to \( \epsilon \)-approximate pure Nash equilibria, for all \( \epsilon > 0 \).

7.1. Price of Anarchy guarantees

We extend the Price of Anarchy guarantees of Probabilistic Serial and Random Priority first.

**Theorem 7.1.** The Price of Anarchy of Probabilistic Serial is \( O(\sqrt{n}) \) for the unit-range representation.

**Proof.** First, observe that Lemma 3.1 holds independently of the representation. Secondly, in the proof of Theorem 3.2 it now holds that \( SW_{PS}(u,s) \geq \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} u_{ij} \geq 1 \), which is sufficient for bounding the Price of Anarchy when \( SW_{OPT}(u) \leq \sqrt{n} \). Finally, the arguments for the case when \( SW_{OPT}(u) \leq \sqrt{n} \) hold for both representations. It is easy to see that the extension applies to all the other equilibrium notions as well. \( \square \)

**Theorem 7.2.** The Price of Anarchy of Random Priority is \( O(\sqrt{n}) \) for the unit-range representation.

**Proof.** First observe that Theorem 4.1 is independent of representation. Secondly, since the main result in [Filos-Ratsikas et al. 2014] also holds for the unit-range representation, the proof of Theorem 4.3 extends to unit-range as well. Again, the result holds for all the other solution concepts as well. \( \square \)

7.2. Lower bounds for unit-range

Next, we present a Price of Anarchy lower bound for deterministic mechanisms. First, we prove the following lemma about the structure of equilibria of deterministic mechanisms. Note that the lemma holds independently of the choice of representation.

**Lemma 7.3.** The set of pure Nash equilibria of any deterministic mechanism is the same for all valuation profiles that induce the same preference orderings of valuations.

**Proof.** Let \( u \) and \( u' \) be two different valuation profiles that induce the same preference ordering. Let \( s \) be a pure Nash equilibrium under true valuation profile \( u \) and assume for contradiction that it is not a pure Nash equilibrium under \( u' \). Then, there exists an agent \( i \) who by deviating from \( s \) is matched to a more preferred item according to \( u'_i \). But that item would also be more preferred according to \( u_i \) and hence she would have an incentive to deviate from \( s \) under true valuation profile \( u \), contradicting the fact that \( s \) is a pure Nash equilibrium. \( \square \)

Using Lemma 7.3 we can then prove the following theorem.

**Theorem 7.4.** The Price of Anarchy of any deterministic mechanism that always has pure Nash equilibria is \( \Omega(n) \) for the unit-range representation.

**Proof.** Let \( M \) be a deterministic mechanism that always has a pure Nash equilibrium and let \( u \) be a valuation profile such that for all agents \( i \) and \( i' \), it holds that \( u_i = u_{i'} \) and \( u_{ij} > u_{ik} \), for all items \( i < k \). Let \( s \) be a pure Nash equilibrium for this profile and assume without loss of generality that \( M_i(s) = i \). By Lemma 7.3 \( s \) is a pure
Nash equilibrium for any profile $u$ that induces the above ordering of valuations. In particular, it is a pure Nash equilibrium for a valuation profile satisfying

— For agents $i = 1, \ldots, \frac{n}{2}$, $u_{i1} = 1$ and $u_{ij} < \frac{1}{n^3}$, for $j > 1$.

— For agents $i = \frac{n}{2} + 1, \ldots, n$, $u_{ij} > 1 - \frac{1}{n^3}$ for $j = 1, \ldots, n/2$ and $u_{ij} < \frac{1}{n^3}$ for $j = \frac{n}{2} + 1, \ldots, n$.

It holds that $OPT(u) \geq \frac{2}{n}$, whereas the social welfare of $M$ is at most 2 and the theorem follows. □

Note that the optimal mechanism, that maximizes the sum of the reported valuations achieves the above bound and hence the bound is tight.

We now prove a general lower bound for the class of all mechanisms when the solution concept is the $\epsilon$-approximate pure Nash equilibrium. A strategy profile is an $\epsilon$-approximate pure Nash equilibrium if no agent can deviate to a different strategy and improve her utility by more than $\epsilon$. For $\epsilon$-approximate pure Nash equilibria the measure of efficiency is the $\epsilon$-approximate pure Price of Anarchy.

**Theorem 7.5.** Let $M$ be a mechanism and let $\epsilon \in (0, 1)$. The $\epsilon$-approximate Price of Anarchy of $M$ is $\Omega(n^{1/4})$ for the unit-range representation.

**Proof.** Assume $n = k^2$, where $k \in \mathbb{N}$ will be the size of a subset $I$ of “important” agents. We consider valuation profiles where, for some parameter $\delta \in (0, 1),$

— all agents have value 1 for item 1,
— there is a subset $I$ of agents with $|I| = k$ for which any agent $i \in I$ has value $\delta^2$ for any item $j \in \{2, \ldots, k+1\}$ and 0 for all other items,
— for agent $i \not\in I$, $i$ has value $\delta^3$ for items $j \in \{2, \ldots, k+1\}$ and 0 for all other items.

Let $u$ be such a valuation profile and let $s$ be a Nash equilibrium. In the optimal allocation members of $I$ receive items $\{2, \ldots, k+1\}$ and such an allocation has social welfare $k\delta^2 + 1$.

First, we claim that there are $k(1 - 2\delta)$ members of $I$ whose payoffs in $s$ are at most $\delta$; call this set $X$. If that were false, then there would be more than $2k\delta$ members of $I$ whose payoffs in $s$ were more than $\delta$. That would imply that the social welfare of $s$ was more than $2k\delta^2$, which would contradict the optimal social welfare attainable, for large enough $n$ (specifically, $n > 1/\delta^4$).

Next, we claim that there are at least $k(1 - 2\delta)$ non-members of $I$ whose probability (in $s$) to receive any item in $\{1, \ldots, k+1\}$ is at most 4$(k+1)/n$; call this set $Y$. To see this, observe that there are at least $\frac{k}{4}n$ agents who all have probability $\leq 4/n$ to receive item 1. Furthermore, there are at least $\frac{k}{4}n$ agents who all have probability $\leq 4k/n$ to receive an item from the set 2, \ldots, $k+1$. Hence there are at least $\frac{k}{4}n$ agents whose probabilities to obtain these items satisfy both properties.

We now consider the operation of swapping the valuations of the agents in sets $X$ and $Y$ so that the members of $I$ from $X$ become non-members, and vice versa. We will argue that given that they were best-responding beforehand, they are $\delta$-best-responding afterwards. Consequently $s$ is an $\delta$-NE of the modified set of agents. The optimal social welfare is unchanged by this operation since it only involves exchanging the payoff functions of pairs of agents. We show that the social welfare of $s$ is some fraction of the optimal social welfare, that goes to 0 as $n$ increases and $\delta$ decreases.

Let $I'$ be the set of agents who, after the swap, have the higher utility of $\delta^2$ for getting items from $\{2, \ldots, k+1\}$. That is, $I'$ is the set of agents in $Y$, together with $I$ minus the agents in $X$.
Following the above valuation swap, the agents in $X$ are $\delta$-best responding. To see this, note that these agents have had a reduction to their utilities for the outcome of receiving items from $\{2, \ldots, k+1\}$. This means that a profitable deviation for such agents should result in them being more likely to obtain item 1, in return for them being less likely to obtain an item from $\{2, \ldots, k+1\}$. However they cannot have probability more than $\delta$ to receive item 1, since that would contradict the property that their expected payoff was at most $\delta$.

After the swap, the agents in $Y$ are also $\delta$-best responding. Again, these agents have had their utilities increased from $\delta^3$ to $\delta^2$ for the outcome of receiving an item from $\{2, \ldots, k+1\}$. Hence any profitable deviation for such an agent would involve a reduction in the probability to get item 1 in return for an increased probability to get an item from $\{2, \ldots, k+1\}$. However, since the payoff for any item from $\{2, \ldots, k+1\}$ is only $\delta^2$, such a deviation pays less than $\delta^2$.

Finally, observe that the social welfare of $s$ under the new profile (after the swap) is at most $1 + 3k\delta^3$. To see this, note that (by an earlier argument and the definition of $I'$) $k(1 - 2\delta)$ members of $I'$ have probability at most $4(k + 1)/n$ to receive any item from $\{1, \ldots, k+1\}$. To upper bound the expected social welfare, note that item 1 contributes 1 to the social welfare. Items in $\{2, \ldots, k+1\}$ contribute in total, $\delta^2$ times the expected number of members of $I'$ who get them, plus $\delta^3$ times the expected number of non-members of $I'$ who get them, which is at most $\delta^2 k 2\delta + \delta^3 k (1 - 2\delta)$ which is less than $3k\delta^3$.

Overall, the price of anarchy is at least $(k\delta^2 + 1)/3k\delta^3$, which is more than $1/\delta$. The statement of the theorem is obtained by choosing $\delta$ to be less than $\epsilon$, $n$ large enough for the arguments to hold for the chosen $\delta$, i.e. $n > 1/\delta^4$. \hfill $\square$

8. CONCLUSION AND FUTURE WORK

We studied the Price of Anarchy of mechanisms for one-sided matching problems and proved that two very well-known mechanisms, Probabilistic Serial and Random Priority are asymptotically best among all mechanisms. Our work provides a different perspective to the debate on which the best mechanisms for one-sided matching problems are. Our main lower bound also extends to the Price of Stability for a large class of mechanisms, those that have a safe strategy, containing most well-known mechanisms in literature. It would be very interesting to generalize our Price of Stability lower bound to the class of all mechanisms. As for the unit-range setting, proving Price of Anarchy or Price of Stability bounds that match our upper bounds is worth looking into.

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