Dynamics of trapped Bose gases at finite temperatures

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Abstract

Starting from an approximate microscopic model of a trapped Bose-condensed gas at finite temperatures, we derive an equation of motion for the condensate wavefunction and a quantum kinetic equation for the distribution function for the excited atoms. The kinetic equation is a generalization of our earlier work in that collisions between the condensate and non-condensate \((C_{12})\) are now included, in addition to collisions between the excited atoms as described by the Uehling-Uhlenbeck \((C_{22})\) collision integral. The continuity equation for the local condensate density contains a source term \(\Gamma_{12}\) which is related to the \(C_{12}\) collision term. If we assume that the \(C_{22}\) collision rate is sufficiently rapid to ensure that the non-condensate distribution function can be approximated by a local equilibrium Bose distribution, the kinetic equation can be used to derive hydrodynamic equations for the non-condensate. The \(\Gamma_{12}\) source terms appearing in these equations play a key role in describing the equilibration of the local chemical potentials associated with the condensate and non-condensate components. We give a detailed study of these hydrodynamic equations and show how the Landau two-fluid equations emerge in the frequency domain \(\omega \tau_\mu \ll 1\), where \(\tau_\mu\) is a characteristic relaxation time associated with \(C_{12}\) collisions. More generally, the lack of complete local equilibrium between the condensate and non-condensate is shown to give rise to a new relaxational mode which is associated with the exchange of atoms between the two components. This new mode provides an additional source of
damping in the hydrodynamic regime. Our equations are consistent with the
generalized Kohn theorem for the center of mass motion of the trapped gas
even in the presence of collisions. Finally, we formulate a variational solution
of the equations which provides a very convenient and physical way of esti-
mating normal mode frequencies. In particular, we use relatively simple trial
functions within this approach to work out some of the monopole, dipole and
quadrupole oscillations for an isotropic trap.

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I. INTRODUCTION

At very low temperatures, the dynamics (collective oscillations) of a trapped Bose gas is described by the time-dependent Gross-Pitaevskii (GP) equation \[1\] for the macroscopic wave function \(\Phi(\mathbf{r}, t)\) associated with the Bose condensate. At higher temperatures, when an appreciable fraction of atoms is excited out of the condensate, the dynamics of the trapped gas becomes much more complicated since it now involves the coupled motion of the condensate and the non-condensate degrees of freedom. In recent work \[2\] the finite-temperature excitations of the condensate were calculated within the Hartree-Fock-Popov (HFP) approximation. This treatment of the excitations is equivalent to a description based on a generalized GP equation in which the condensate density \(n_c(\mathbf{r}, t)\) is treated dynamically but the non-condensate density \(\tilde{n}(\mathbf{r}, t)\) is treated as static. That is, one ignores the effect of non-condensate fluctuations on the dynamics of the condensate. Several authors \[3,4\] have recently extended this kind of HFP theory by treating the non-condensate fluctuations \(\delta\tilde{n}(\mathbf{r}, t)\) using a time-dependent Hartree-Fock equation of motion (often referred to as the RPA). This describes the collective modes in the collisionless mean-field regime.

In contrast, in the low frequency collision-dominated regime, one expects a description in terms of hydrodynamic equations. In a recent paper \[5\], we discussed a method for obtaining two-fluid equations starting from a semi-classical kinetic equation for the distribution function \(f(\mathbf{p}, \mathbf{r}, t)\) of the excited atoms. The final result was a closed set of two-fluid hydrodynamic equations in terms of the local densities and velocities of the condensate and non-condensate components. In this paper, we consider a generalized model and present a more complete discussion of the approximations which are involved. As we shall see, this new microscopic model, though approximate, captures all the essential physics of the collision-dominated hydrodynamic behavior involving the condensate and non-condensate components.

Deriving two-fluid equations within the context of a specific microscopic model has several advantages:

1. It allows one to see in a very clear fashion how the condensate and non-condensate degrees of freedom arise and how they are coupled.

2. It gives a closed set of equations with all local thermodynamic properties and relaxation times determined in a self-consistent manner. It thus ensures that the static and time-dependent properties are treated in a unified manner.

3. The equations we obtain can be used to derive the usual Landau equations (originally developed to describe superfluid \(^4\)He \[6,7\]) in a certain limit, and allow a clear identification of the Landau superfluid (normal fluid) density with the condensate (non-condensate) density.

4. The description of the dynamics can be extended beyond the regime in which the Landau phenomenological equations are valid, i.e., when the condensate and non-condensate are not in local diffusive equilibrium.
5. An explicit microscopic formulation provides a starting point for improved descriptions, for example, the inclusion of the effect of the off-diagonal (or anomalous) self-energies which are important at very low temperatures [4].

In the ZGN paper [5], we only included collisions between the excited (non-condensate) atoms. Moreover, we only discussed the solutions of the kinetic equation for $f(p, r, t)$ which correspond to dynamic local equilibrium within the non-condensate. This local equilibrium state is defined by the condition that the collision integral for the excited atoms (denoted by $C_{22}[f]$) vanishes identically. As a result, no explicit dependence on the $C_{22}$ collision integral appeared in the ZGN hydrodynamic equations. In the present paper, we make a major improvement by also considering the effect of collisions between atoms in the condensate with those in the non-condensate. This collision term (denoted by $C_{12}[f]$) allows us to discuss how the condensate is coupled to the non-condensate beyond mean-field effects, and how the condensate comes into dynamic local equilibrium with the non-condensate. In this connection, we should note that the different effects of the $C_{12}$ and $C_{22}$ collisions was first discussed in a qualitative way by Eckern [8]. The generalized set of hydrodynamic equations which arise when the effects of $C_{12}[f]$ are included will be referred to as the ZGN′ equations to distinguish them from the set we derived previously [5].

Even when collisions among non-condensate atoms are sufficiently rapid to enforce the distribution function $f$ to take the form of a local equilibrium Bose distribution with chemical potential $\tilde{\mu}$, it is possible that the non-condensate is not in complete local equilibrium with the condensate. In this situation, the chemical potential $\tilde{\mu}$ of the non-condensate can be different from that of the condensate, $\mu_c$. Thus the variable $\mu_{\text{diff}} \equiv \tilde{\mu} - \mu_c$ emerges as a key new local variable which is needed to represent the equilibration of the condensate and non-condensate components. In the original ZGN analysis, where the effects of $C_{12}$ were neglected, the difference between $\mu_c$ and $\tilde{\mu}$ was implicitly built into the ZGN hydrodynamic equations, but this fact was overlooked in our attempt to reduce these equations to the Landau two-fluid hydrodynamic equations. To arrive at this latter description, we assumed additional local equilibrium thermodynamic relations which implicitly forced $\mu_{\text{diff}} = 0$ (i.e., $\tilde{\mu} = \mu_c$). Thus the ZGN equations are in fact distinct from the Landau two-fluid equations and, in general, will lead to different dynamical predictions. In the present paper, we show how the more general ZGN′ equations reduce to either the ZGN or Landau descriptions, in the appropriate limits. This more complete description involves an additional equation of motion for $\mu_{\text{diff}}$, which is coupled to the condensate and non-condensate velocity fields. The new relaxational mode which emerges involves the transfer of atoms between the two components, and allows one to understand the conditions under which the Landau two-fluid equations are valid.

The possibility of the condensate being out of equilibrium with the non-condensate was studied many years ago, mainly with regard to the behaviour of superfluid helium near the $\lambda$ point. In a well-known paper, Landau and Khalatnikov [9] calculated the absorption of sound near a second-order phase transition due to the coupling to a relaxing order parameter. Pitaevskii [10, 7] subsequently generalized the phenomenological two-fluid equations for superfluid helium to take into account the relaxation of the superfluid density near the $\lambda$ point. The same problem was looked at within the Mori formalism by Miyake and Yamada [11]. All of these approaches are phenomenological in nature since they introduce new dissipation parameters to describe the process of atomic exchange between the condensate
(superfluid) and non-condensate (normal fluid). In contrast, our theory is based on a well-defined microscopic model of a Bose-condensed gas, and as a result, the relaxation times responsible for the dissipation can be calculated explicitly.

We briefly reported on some of our main results in Ref. [12] which discusses the ZGN′ equations for the special case of a uniform Bose gas. Here we give a more complete derivation of the ZGN′ equations and a more detailed discussion of their properties. Our work may be viewed as an extension to the case of a trapped Bose-condensed gas of the pioneering work of Kirkpatrick and Dorfman [13] (referred to in the following as KD). However, these authors concentrated on the derivation of the Landau two-fluid equations with the inclusion of dissipation, and did not address the interesting possibility of the condensate and non-condensate components being out of diffusive local equilibrium ($\mu_c \neq \tilde{\mu}$). In this context, we should also mention Bogoliubov’s rigorous field-theoretic derivation of the Landau two-fluid equations [14], which also implicitly assumed that the superfluid and normal fluid were always in local thermodynamic equilibrium.

In the last few years, several papers dealing with non-equilibrium properties of trapped Bose gases have appeared. We mention specifically the work of Gardiner et al. [15], Proukakis et al. [16], Stoof [17] and Walser et al. [18]. Each of these papers presents a formal derivation of a quantum kinetic equation using quite different theoretical methods. The equations of motion we derive for the condensate and non-condensate are closest in form to those of Stoof [17] and Walser et al. [18], although there are also many formal similarities with the work of Proukakis et al. [16]. However, in contrast to this other work, we focus on deriving hydrodynamic equations, and on establishing the criteria under which such equations are valid. In particular, we identify the crucial role that collisions between condensate and non-condensate atoms have in establishing a state of local equilibrium. The paper of Stoof [17] is also useful for its thorough and insightful discussion of the dynamics of condensate formation and growth, a topic which Gardiner et al. [15] study in considerable detail. In this regard, we note that our kinetic equation can also be used to investigate condensate growth and applications to this problem will be presented elsewhere [19].

A brief summary of our paper follows. In Section II, we derive the equation of motion for the condensate, and in Section III, we obtain the quantum kinetic equation describing the non-condensate. These equations are applicable to the coupled dynamics of the condensate and non-condensate at finite temperatures, and have been derived taking careful account of correlation functions involving three field operators, which describe the transfer of atoms between the two components. Taking moments of our kinetic equation, we derive in Section IV the hydrodynamic equations for the non-condensate, based on the assumption of local equilibrium within the excited-atom component. In Section V, we then analyze the full set of linearized ZGN′ equations and show that a relaxation time $\tau_\mu$ naturally arises which characterizes the equilibration of the condensate and non-condensate chemical potentials. We show (in the simple case of a uniform Bose gas) that the ZGN equations give a good description in the limit $\omega \tau_\mu \gg 1$, while the Landau two-fluid hydrodynamic theory emerges in the opposite limit, $\omega \tau_\mu \ll 1$. In Section VI, we demonstrate that the generalized Gross-Pitaevskii equation for the condensate and the kinetic equation for the non-condensate are consistent with the generalized Kohn theorem for the center of mass motion in a harmonic confining potential. In Section VII, we reformulate the ZGN hydrodynamic equations in terms of a variational functional which is then used to calculate the frequencies (and as-
associated condensate and non-condensate velocities) of the lowest-lying monopole \( (l = 0) \), dipole \( (l = 1) \) and quadrupole \( (l = 2) \) collective modes. The results for the out-of-phase dipole mode were first reported in ZGN \[5\]. Finally, in Section VIII, we assess what we have accomplished and briefly discuss some further extensions of the theory.

**II. HYDRODYNAMIC EQUATIONS FOR THE CONDENSATE ATOMS**

We start with the usual Heisenberg equation of motion for the quantum field operator,

\[
i\hbar \frac{\partial \hat{\psi}(\mathbf{r}, t)}{\partial t} = [\hat{\psi}(\mathbf{r}, t), \hat{H}] = \left( -\frac{\hbar^2 \nabla^2}{2m} + U_{\text{ext}}(\mathbf{r}) \right) \hat{\psi}(\mathbf{r}, t) + g\hat{\psi}^\dagger(\mathbf{r}, t)\hat{\psi}(\mathbf{r}, t)\hat{\psi}(\mathbf{r}, t), \tag{1}\]

where \( U_{\text{ext}}(\mathbf{r}) \) is the confining potential. We have here assumed that the interaction potential can be represented as a short-ranged pseudopotential of strength \( g = 4\pi a_\text{e}^2/m \), where \( a_\text{e} \) is the \( s \)-wave scattering length. The equation for the condensate wave function is obtained by taking an average of (1) with respect to a broken symmetry nonequilibrium ensemble in which the field operator takes a non-zero expectation value (denoted by angular brackets),

\[
\Phi(\mathbf{r}, t) = \langle \hat{\psi}(\mathbf{r}, t) \rangle. \tag{2}\]

Introducing the usual definition of the non-condensate field operator \( \tilde{\psi}(\mathbf{r}, t) \) according to

\[
\hat{\psi}(\mathbf{r}, t) = \Phi(\mathbf{r}, t) + \tilde{\psi}(\mathbf{r}, t), \tag{3}\]

with \( \langle \tilde{\psi}(\mathbf{r}, t) \rangle = 0 \), the expectation value of (1) thus yields

\[
i\hbar \frac{\partial \Phi(\mathbf{r}, t)}{\partial t} = \left( -\frac{\hbar^2 \nabla^2}{2m} + U_{\text{ext}}(\mathbf{r}) + g n_e(\mathbf{r}, t) + g\tilde{n}(\mathbf{r}, t) \right) \Phi(\mathbf{r}, t)
+ g\tilde{n}(\mathbf{r}, t)\Phi^\dagger(\mathbf{r}, t) + g\langle \tilde{\psi}^\dagger(\mathbf{r}, t)\tilde{\psi}(\mathbf{r}, t)\rangle \Phi(\mathbf{r}, t) \hat{\psi}(\mathbf{r}, t), \tag{4}\]

where \( n_e(\mathbf{r}, t) = |\Phi(\mathbf{r}, t)|^2 \) is the non-equilibrium density of atoms in the condensate and \( \tilde{n}(\mathbf{r}, t) = \langle \tilde{\psi}^\dagger(\mathbf{r}, t)\tilde{\psi}(\mathbf{r}, t) \rangle \) is the non-equilibrium non-condensate density. Eq. (4) also involves the off-diagonal non-condensate density \( \tilde{m}(\mathbf{r}, t) = \langle \tilde{\psi}(\mathbf{r}, t)\tilde{\psi}(\mathbf{r}, t) \rangle \) and the three-field correlation function \( \langle \tilde{\psi}^\dagger(\mathbf{r}, t)\tilde{\psi}(\mathbf{r}, t)\tilde{\psi}(\mathbf{r}, t) \rangle \), both of which in principle have non-zero expectation values because of the assumed Bose broken symmetry. We note that (4) is formally exact within the pseudopotential approximation. A more detailed discussion starting from the bare pair interaction is given in Ref. \[1\].

It is useful at this point to discuss some standard approximations to (4). It reduces to the well-known Gross-Pitaevskii (GP) equation \[1\] if all the atoms are in the condensate (i.e., \( \tilde{n} = 0 \)) and the anomalous correlations \( \langle \tilde{n} \rangle \) and \( \langle \tilde{\psi}^\dagger(\mathbf{r}, t)\tilde{\psi}(\mathbf{r}, t) \rangle \) are ignored. This is a very good approximation for \( T \ll T_{\text{BEC}} \); at \( T = 0 \), the non-condensate fraction in trapped atomic gases is estimated to be less than 1% \[20,2\]. As a first step in dealing with finite temperatures, Hutchinson et al. \[4\] (referred to in the following as HZG) calculated the excitations in the HFP approximation \[21\]. The thermal occupation of these quasiparticle excitations provides an expression for the equilibrium non-condensate density \( \tilde{n}_0(\mathbf{r}) \), while the condensate density \( n_{\text{c0}}(\mathbf{r}) \equiv |\Phi_0(\mathbf{r})|^2 \) is obtained from the solution of a generalized GP
equation. The self-consistent calculation of these two densities determines the equilibrium properties of the trapped gas within the Popov approximation and the excitation energies are themselves identified with the collective excitations of the condensate. In the context of Eq. (4), the HZG approximation \cite{2} consists of neglecting $\tilde{m} \mathrel{\text{and}} \langle \tilde{\psi}^\dagger \tilde{\psi} \tilde{\psi} \rangle$ and solving the linearized version of the resulting equation with the additional approximation $\tilde{n}(\mathbf{r}, t) \simeq \tilde{n}_0(\mathbf{r})$.

In physical terms, this latter approximation corresponds to the condensate atoms moving in the static Hartree-Fock field due to the non-condensate atoms. That is, only the dynamics of the Hartree mean field of the condensate atoms is treated self-consistently, while the collective dynamics of the non-condensate atoms is completely ignored.

Returning to the exact equation for the macroscopic wavefunction in (4), it is useful to introduce phase and amplitude variables, $\Phi(\mathbf{r}, t) = |\Phi(\mathbf{r}, t)| e^{i\theta(\mathbf{r}, t)}$. After a little algebra, one finds that (4) is equivalent to

$$
\frac{\partial n_c}{\partial t} + \nabla \cdot (n_c \mathbf{v}_c) = \frac{2g}{\hbar} \text{Im}[ (\Phi^*)^2 \tilde{m} + \Phi^* \langle \tilde{\psi}^\dagger \tilde{\psi} \tilde{\psi} \rangle],
$$

$$(5a)$$

$$
\hbar \frac{\partial \theta}{\partial t} = -\mu_c - \frac{\hbar^2}{2m} (\nabla \theta)^2 = -(\mu_c + \frac{1}{2} m v_c^2).
$$

$$(5b)$$

Here the new local variables are the condensate density

$$n_c(\mathbf{r}, t) \equiv |\Phi(\mathbf{r}, t)|^2,
$$

and the condensate velocity

$$v_c(\mathbf{r}, t) \equiv \frac{\hbar}{m} \nabla \theta(\mathbf{r}, t),
$$

while the local condensate chemical potential is defined by

$$\mu_c(\mathbf{r}, t) \equiv -\frac{\hbar^2}{2m} \sqrt{n_c(\mathbf{r}, t)} + U_{\text{ext}}(\mathbf{r}) + g n_c(\mathbf{r}, t) + 2g \tilde{n}(\mathbf{r}, t) + \frac{g}{\hbar n_c} \text{Re}[ (\Phi^*)^2 \tilde{m} + \Phi^* \langle \tilde{\psi}^\dagger \tilde{\psi} \tilde{\psi} \rangle].
$$

$$(8)$$

The condensate velocity in our theory is clearly identifiable with the superfluid velocity in the usual discussions of superfluid hydrodynamics. It will be seen later when we discuss the kinetic equation for the excited atoms that, in our model, the condensate density $n_c$ can be identified with the superfluid density as defined in the Landau theory \cite{3,4}.

One may give a physical interpretation of the generalized condensate equation in (5b). In static thermal equilibrium, we have $\mathbf{v}_c = 0$ and all variables are time-independent. In this case, the RHS of (5a) must vanish and (5b) has the solution $\theta(t) = -\mu_c t / \hbar$. Thus it is clear that in thermal equilibrium, a condensate atom has an energy $\varepsilon_c$ equal to $\mu_c$, the equilibrium chemical potential. We now suppose that we are in a regime where the variables $\mu_c$ and $\mathbf{v}_c$ are slowly-varying in space and time. In this situation, we have

$$\theta(\mathbf{r}, t) \simeq \theta(\mathbf{r}_0, t_0) + \frac{\partial \theta}{\partial t}(t - t_0) + \nabla \theta \cdot (\mathbf{r} - \mathbf{r}_0) + \ldots$$

$$\equiv \theta(\mathbf{r}_0, t_0) - \frac{1}{\hbar} \varepsilon_c(\mathbf{r}_0, t_0)(t - t_0) + \frac{1}{\hbar} m \mathbf{v}_c(\mathbf{r}_0, t_0) \cdot (\mathbf{r} - \mathbf{r}_0) + \ldots,
$$

$$(9)$$
where \( \varepsilon_c(r_0, t_0) \equiv \mu_c(r_0, t_0) + \frac{1}{2}m v_c^2(r_0, t_0) \). With this phase variation of the condensate wavefunction, it is natural to interpret \( \varepsilon_c \) as the local energy of a condensate atom and \( v_c \) as its local velocity. This interpretation is consistent with the energy and momentum conservation laws we derive later when we consider collisions between condensate and non-condensate atoms (see, in particular, the discussion at the end of Section III and in Appendix A).

The condensate equations summarized by (5-8) are formally exact, but are not a closed set of equations since they depend on the additional non-condensate variables \( \tilde{n}, \tilde{m} \) and \( \langle \tilde{\psi}^\dagger \tilde{\psi} \tilde{\psi} \rangle \). As in Ref. [5], we restrict ourselves in the present paper to the Popov approximation [21] which corresponds to neglecting the anomalous pair correlation \( \tilde{m} \) in (5a) and (8). Some justification for this approximation is given in Appendix A. However a crucial difference from ZGN is that we retain the three-field correlation function \( \langle \tilde{\psi}^\dagger \tilde{\psi} \tilde{\psi} \rangle \) which we evaluate in an approximation consistent with setting \( \tilde{m} \) to zero. We should remark that the importance of \( \langle \tilde{\psi}^\dagger \tilde{\psi} \tilde{\psi} \rangle \) when dealing with collisions involving condensate atoms has also been emphasized in Ref. [16].

### III. DERIVATION OF A QUANTUM BOLTZMANN EQUATION

In this section, we turn to the dynamics of the non-condensate. The physical properties of interest are in principle defined by the following equation of motion obtained from (1) and (4),

\[
\begin{align*}
    i\hbar \frac{\partial \tilde{\psi}}{\partial t} & = \left( -\frac{\hbar^2 \nabla^2}{2m} + U_{\text{ext}} + 2gn \right) \tilde{\psi} - 2g\tilde{n}\tilde{\psi} + g\Phi^2 \tilde{\psi}^\dagger \\
    & \quad + g\Phi^* (\tilde{\psi} \tilde{\psi} - \tilde{\psi}^\dagger \tilde{\psi} - \tilde{n}) + g(\tilde{\psi}^\dagger \tilde{\psi} \tilde{\psi} - \langle \tilde{\psi}^\dagger \tilde{\psi} \tilde{\psi} \rangle),
\end{align*}
\] (10)

where \( n = n_c + \tilde{n} \) is the total density. This equation preserves \( \langle \tilde{\psi} \rangle = 0 \) as a function of time. It will allow us to derive a kinetic equation for the excited atoms. Following Kirkpatrick and Dorfman [13], it is convenient to define the time evolution of \( \tilde{\psi} \) by

\[
\tilde{\psi}(r, t) = \hat{S}^\dagger(t, t_0) \tilde{\psi}(r, t_0) \hat{S}(t, t_0),
\] (11)

where the unitary operator \( \hat{S}(t, t_0) \) evolves according to the equation of motion

\[
    i\hbar \frac{d\hat{S}(t, t_0)}{dt} = \hat{H}_{\text{eff}}(t) \hat{S}(t, t_0),
\] (12)

with \( \hat{S}(t_0, t_0) = 1 \). Here, \( t_0 \) is the time at which the initial nonequilibrium density matrix \( \hat{\rho}(t_0) \) is specified. The effective Hamiltonian in (12) is given by

\[
\begin{align*}
    \hat{H}_{\text{eff}}(t) & = \hat{H}_0(t) + \hat{H}'(t) \\
    \hat{H}'(t) & = \hat{H}'_1(t) + \hat{H}'_2(t) + \hat{H}'_3(t) + \hat{H}'_4(t),
\end{align*}
\] (13)

where the various contributions are defined as
\begin{align*}
\hat{H}_0(t) &= \int dr \, \tilde{\psi}^\dagger \left( -\frac{\hbar^2 \nabla^2}{2m} + U(r, t) \right) \tilde{\psi} \\
\hat{H}'(t) &= \int dr \left( L_1(r, t) \tilde{\psi}^\dagger + L_1^*(r, t) \tilde{\psi} \right) \\
\hat{H}_2(t) &= \frac{g}{2} \int dr \left( \Phi^2(r, t) \tilde{\psi}^\dagger \tilde{\psi} + \Phi^* \Phi(r, t) \tilde{\psi} \tilde{\psi} + \Phi(r, t) \tilde{\psi} \tilde{\psi} \right) \\
\hat{H}_3(t) &= g \int dr \left( \Phi^* \Phi(r, t) \tilde{\psi} \tilde{\psi} + \Phi(r, t) \tilde{\psi} \tilde{\psi} + \Phi(r, t) \tilde{\psi} \tilde{\psi} \right) \\
U(r, t) &= U_{\text{ext}}(r) + 2gn(r, t) = U_{\text{ext}}(r) + 2g [\tilde{n}(r, t) + n_c(r, t)] \\
L_1(r, t) &= -g \left( 2\tilde{n}(r, t) \Phi(r, t) + \tilde{n}(r, t) \Phi^*(r, t) + 2 \langle \tilde{\psi}^\dagger \tilde{\psi} \tilde{\psi} \rangle \right).
\end{align*}

It is understood that in all these expressions the arguments of the field operators are \((r, t_0)\); the dependence of the various terms of the Hamiltonian on \(t\) arises through the mean-field expectation values. It can be shown that Eqs. (11)-(14), together with the equal time commutator \([\tilde{\psi}(r, t_0) \tilde{\psi}^\dagger(r', t_0)] = \delta(r - r')\), reproduce the equation of motion for \(\tilde{\psi}\) in (10), as well as that for \(\tilde{\psi}^\dagger\). In writing (13) we consider \(\hat{H}'(t)\) to be a perturbation to the zeroth-order Hamiltonian \(\hat{H}_0(t)\). Noting that \(U(r, t)\) is the total Hartree-Fock (HF) mean field, \(\hat{H}_0(t)\) defines excitations of the system at the level of the time-dependent HF approximation. Other choices of the zeroth-order Hamiltonian are also possible. For example, \(\hat{H}_0(t)\) could be combined with \(\hat{H}_2(t)\) to define a dynamic Bogoliubov Hamiltonian which would then lead to local Bogoliubov quasiparticle excitations. This possible extension (13), which would be appropriate when considering very low temperatures, will not be considered in this paper.

We now consider an arbitrary operator \(\hat{O}(t)\) which is made up of some combination of non-condensate field operators \(\tilde{\psi}(r, t)\) and \(\tilde{\psi}^\dagger(r, t)\), for example, the local density operator \(\tilde{n}(r, t) = \tilde{\psi}^\dagger(r, t) \tilde{\psi}(r, t)\). By making use of (11), the expectation value of \(\hat{O}(t)\) with respect to the initial density matrix \(\hat{\rho}(t_0)\) can be expressed as

\begin{align*}
\langle \hat{O}(t) \rangle &\equiv \langle \hat{O} \rangle_t = \text{Tr} \hat{\rho}(t_0) \hat{O}(t) \\
&= \text{Tr} \hat{\rho}(t_0) \hat{O}(t_0),
\end{align*}

where \(\hat{\rho}(t_0) \equiv \hat{S}(t, t_0) \hat{\rho}(t_0) \hat{S}^\dagger(t, t_0)\) satisfies the following equation

\begin{equation}
\text{i} \hbar \frac{d\hat{\rho}(t_0)}{dt} = [\hat{H}_{\text{eff}}(t), \hat{\rho}(t_0)].
\end{equation}

We refer to Appendix A for further discussion of this equation of motion.

Our ultimate objective is to obtain a quantum kinetic equation for the non-condensate atoms. For this purpose, we define the Wigner operator

\begin{equation}
\hat{f}(\mathbf{p}, r, t_0) \equiv \int d\mathbf{r}' e^{i\mathbf{p} \cdot \mathbf{r}'/\hbar} \tilde{\psi}^\dagger(r + \mathbf{r}'/2, t_0) \tilde{\psi}(r - \mathbf{r}'/2, t_0).
\end{equation}

Its expectation value according to Eq. (13) then yields the Wigner distribution function

\begin{equation}
\langle \hat{f}(\mathbf{p}, r, t) \rangle = \text{Tr} \hat{\rho}(t, t_0) \hat{f}(\mathbf{p}, r, t_0).
\end{equation}
Knowledge of this function allows one to calculate various nonequilibrium expectation values, such as the non-condensate density

\[ \tilde{n}(\mathbf{r}, t) = \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} f(\mathbf{p}, \mathbf{r}, t). \]  

Using (16), the equation of motion for \( f \) is

\[ \frac{\partial f(\mathbf{p}, \mathbf{r}, t)}{\partial t} = \frac{1}{i\hbar} \text{Tr}\hat{\rho}(t, t_0)[\hat{f}(\mathbf{p}, \mathbf{r}, t_0), \hat{H}_{\text{eff}}(t)] \]

\[ = \frac{1}{i\hbar} \text{Tr}\hat{\rho}(t, t_0)[\hat{f}(\mathbf{p}, \mathbf{r}, t_0), \hat{H}_0(t)] + \frac{1}{i\hbar} \text{Tr}\hat{\rho}(t, t_0)[\hat{f}(\mathbf{p}, \mathbf{r}, t_0), \hat{H}'(t)]. \]  

The first term on the right hand side of (20) defines the free-streaming operator in the kinetic equation. With the assumption that \( U(\mathbf{r}, t) \) varies slowly in space, we then have

\[ \frac{\partial f(\mathbf{p}, \mathbf{r}, t)}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla f(\mathbf{p}, \mathbf{r}, t) - \nabla U \cdot \nabla_p f(\mathbf{p}, \mathbf{r}, t) = \frac{1}{i\hbar} \text{Tr}\hat{\rho}(t, t_0)[\hat{f}(\mathbf{p}, \mathbf{r}, t_0), \hat{H}'(t)]. \]  

The right hand side of this equation clearly represents the effect of collisions between the atoms. The reduction of this term to the form of a binary collision integral is a lengthy exercise (some details are given in Appendix A). However, the final result has a physically transparent form. The collision integral is the sum of two contributions:

\[ \frac{\partial f}{\partial t} \bigg|_{\text{coll}} = C_{12}[f] + C_{22}[f], \]

where

\[ C_{22}[f] = \frac{2g^2}{(2\pi)^6\hbar^3} \int d\mathbf{p}_2 \int d\mathbf{p}_3 \int d\mathbf{p}_4 \delta(\mathbf{p} + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \]

\[ \times \delta(\tilde{\varepsilon}_p + \tilde{\varepsilon}_{p_2} - \tilde{\varepsilon}_{p_3} - \tilde{\varepsilon}_{p_4}) \left[(1 + f)(1 + f_2)f_3f_4 - f f_2(1 + f_3)(1 + f_4)\right], \]  

\[ C_{12}[f] = \frac{2g^2n_c}{(2\pi)^3\hbar^3} \int d\mathbf{p}_1 \int d\mathbf{p}_2 \int d\mathbf{p}_3 \delta(m\mathbf{v}_c + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \]

\[ \times \delta(\varepsilon_c + \tilde{\varepsilon}_{p_1} - \tilde{\varepsilon}_{p_2} - \tilde{\varepsilon}_{p_3}) \left[\delta(\mathbf{p} - \mathbf{p}_1) - \delta(\mathbf{p} - \mathbf{p}_2) - \delta(\mathbf{p} - \mathbf{p}_3)\right] \]

\[ \times \left[(1 + f_1)f_2f_3 - f_1(1 + f_2)(1 + f_3)\right], \]

with \( f \equiv f(\mathbf{p}, \mathbf{r}, t), f_i \equiv f(\mathbf{p}_i, \mathbf{r}, t) \). We also note that the variables \( n_c, \mathbf{v}_c, \varepsilon_c \) and \( \tilde{\varepsilon}_p \) are all functions of \( \mathbf{r} \) and \( t \). As a result of Bose statistics, the creation of an atom in a state \( i \) is associated with the statistical factor \((1 + f_i)\). Clearly \( C_{22} \) describes two-body collisions between excited atoms (2 atoms \( \ni 2 \) unexcited atoms) while \( C_{12} \) describes collisions which involve one condensate atom (1 atoms \( \ni 2 \) unexcited atoms). The momentum and energy delta functions in the \( C_{12} \) collision term take into account the fact that a condensate atom locally has energy \( \varepsilon_c(\mathbf{r}, t) \) defined in (17), and momentum \( m\mathbf{v}_c \). On the other hand, the excited atoms locally have the HF energy.
We note that the HF mean-field defined in (14) involves the condensate density \( n_c(r, t) \equiv |\Phi(r, t)|^2 \), which must be determined self-consistently using the condensate equation of motion in (4).

To complete our derivation of a closed set of equations for both the condensate and non-condensate, we must finally deal with the off-diagonal HF self-energy \( \tilde{m} \) and the term \( \langle \tilde{\psi}^\dagger \tilde{\psi} \tilde{\psi} \rangle \) which appear in (5) and (8). We show in Appendix A that \( \tilde{m} \) can be consistently neglected, and that

\[
\langle \tilde{\psi}^\dagger \tilde{\psi} \tilde{\psi} \rangle = -\Gamma_{12}[f] \equiv \frac{2g}{\hbar} \text{Im}[\Phi^* \langle \tilde{\psi}^\dagger \tilde{\psi} \rangle] = -\int \frac{dp}{(2\pi \hbar)^3} C_{12}[f(p, r, t)].
\]

We do not display the real part of \( \langle \tilde{\psi}^\dagger \tilde{\psi} \tilde{\psi} \rangle \) which contributes to the local condensate chemical potential defined in (8). This contribution is of order \( g^2 \) and will be neglected since our approximations do not adequately treat all terms of this order (see Appendix A for further discussion). However, the imaginary part of \( \langle \tilde{\psi}^\dagger \tilde{\psi} \rangle \) shown in (25) contributes an important source term, \(-\Gamma_{12}[f]\), to the right hand side of the continuity equation in (5a).

Comparing (25) with the expression for \( C_{12}[f] \) in (23b), this source term is given by

\[
-\Gamma_{12}[f] = \frac{2g}{\hbar} \text{Im}[\Phi^* \langle \tilde{\psi}^\dagger \tilde{\psi} \rangle] = -\int \frac{dp}{(2\pi \hbar)^3} C_{12}[f(p, r, t)].
\]

The latter form in (26) emphasizes the fact that the \( C_{12} \) collisions can locally change the relative number of condensate and non-condensate atoms and that separate number conservation laws no longer hold for each of these components. As we shall see, the close connection between \( \langle \tilde{\psi}^\dagger \tilde{\psi} \rangle \) in (23) and the \( C_{12} \) collision integral in (23b) is essential for the conservation of the total number of particles. This indicates that the approximations leading to our final equations for the condensate and non-condensate are internally consistent.

To summarize the results of this section, as well as that of Section II, we write out the final form of the equations we have derived for the dynamics of a trapped Bose-condensed gas at finite temperatures. For the condensate, we have the equations

\[
\frac{\partial n_c}{\partial t} + \nabla \cdot (n_c \mathbf{v}_c) = -\Gamma_{12}[f]
\]

\[
m \left( \frac{\partial}{\partial t} + \mathbf{v}_c \cdot \nabla \right) \mathbf{v}_c = -\nabla \mu_c,
\]

where the condensate chemical potential is given by

\[
\mu_c(r, t) = -\frac{\hbar^2 \nabla^2 n_c(r, t)}{2m \sqrt{n_c(r, t)}} + U_{\text{ext}}(r) + gn_c(r, t) + 2g \tilde{n}(r, t).
\]

As we have discussed in detail, these equations are approximations to the exact equations given in (5) and (8). For the non-condensate we have the kinetic equation
\[
\frac{\partial f(p, r, t)}{\partial t} + \frac{p}{m} \cdot \nabla f(p, r, t) - \nabla U \cdot \nabla_p f(p, r, t) = C_{12}[f] + C_{22}[f],
\]
(29)

with the two collision terms given explicitly by the expressions in (23). These equations are coupled through the HF mean fields and must therefore be solved self-consistently. An equation equivalent to (27) was derived by Proukakis et al. [16], while our kinetic equation in (29) is implicitly contained in the work of Stoof [17].

It is apparent that the equations in (27) are already in the form of “hydrodynamic” equations for the condensate variables. In Section IV, we use (29) to derive analogous hydrodynamic equations for the non-condensate atoms. Although these equations have been derived for a trapped gas at finite temperatures, we note that they go over smoothly to the correct \( T = 0 \) theory, since as \( f(p, r, t) \) and hence \( \tilde{n} \) become negligible, the condensate equations (27) and (28) reduce to the standard Gross-Pitaevskii dynamics of a pure condensate at \( T = 0 \) [1,22]. We therefore expect our equations to provide a reasonable description of the dynamics over the full range of temperatures from the strongly degenerate limit well below \( T_{BEC} \) through to the classical high-temperature limit.

Before leaving this Section, we compare these results with some earlier theoretical studies. For temperatures above the Bose-Einstein transition \( (T > T_{BEC}) \) where \( C_{12} \) vanishes, the kinetic equation (29) reduces to the well-known Uehling-Uhlenbeck equation [23], as derived in Ch. 6 of Kadanoff and Baym [24], for example, and studied numerically by several authors [25]. Below \( T_{BEC} \), (29) is analogous to the kinetic equation derived by KD [13] for a homogeneous weakly-interacting Bose-condensed gas at finite temperatures. The main difference is that these authors choose to work with excitations which are defined within the local rest frame of the superfluid (condensate), and thus the excitation energies are measured relative to that of the condensate atoms. In contrast, our kinetic equation involves atoms moving in a self-consistent HF field. It is perhaps useful to make this comparison more precise. Our \( C_{12}[f] \) collision integral involves the energy-momentum conservation factors

\[
\delta(\varepsilon_c + \tilde{\varepsilon}_1 - \tilde{\varepsilon}_2 - \tilde{\varepsilon}_3)\delta(m v_c + p_1 - p_2 - p_3)
\]

where \( \tilde{\varepsilon}_i = \frac{p_i^2}{2m} + 2g(n + n_c) \) (assuming \( U_{ext} = 0 \) for simplicity) and \( \varepsilon_c = \mu_c + \frac{1}{2}m v_c^2 \) is the local energy of an atom in the condensate. On the other hand, in the work of KD, \( C_{12}[f] \) contains the energy-momentum conservation factors

\[
\delta(E_1 - E_2 - E_3)\delta(p'_1 - p'_2 - p'_3),
\]

where \( E_i \) is the quasiparticle (Bogoliubov) excitation energy and \( p'_i = p_i - m v_c \) is the quasiparticle momentum in the local rest frame. In the “high-temperature limit” (large momenta), these excitation energies reduce to \( E_i \approx \mu_i^2 / 2m + gn_c = \tilde{\varepsilon}_i - \mu_c - \mathbf{v} \cdot \mathbf{p}_i + \frac{1}{2}m v_c^2 \) where \( \mu_c = g(n_c + 2\tilde{n}) \) is the local condensate chemical potential as given by (28). One can easily verify that the two energy-momentum conservation factors defined above are completely equivalent in this high-temperature limit, and thus the two seemingly different formulations are essentially the same.

It should be noted that the high-temperature limit in the case of a trapped Bose gas is not as restrictive as it might seem. The work reported in Ref. [26] indicates that the differences arising from using the HF vs. Bogoliubov excitation spectrum in calculating thermodynamic properties are very small down to quite low temperatures, in contrast to
the situation for a uniform Bose gas. A more detailed explanation for the dominance of single-particle excitations in determining the statistical behaviour of trapped Bose gases can be found in Ref. [1].

IV. NON-CONDENSATE HYDRODYNAMIC EQUATIONS

A. General moment equations

Following the standard procedure in kinetic theory, we proceed to take moments of the kinetic equation in (29). It follows immediately from their definitions that

\[ \int d\mathbf{p} \mathbf{p} C_{22} = 0, \quad \int d\mathbf{p} (\mathbf{p} - m \mathbf{v}_c) C_{12} = 0 \]

\[ \int d\mathbf{p} \tilde{\epsilon}_p C_{22} = 0, \quad \int d\mathbf{p} (\tilde{\epsilon}_p - \epsilon_c) C_{12} = 0. \]

(30)

This shows that both the \( C_{12} \) and \( C_{22} \) collision processes conserve energy and momentum. In the case of \( 1 \leftrightarrow 2 \) collisions, the momentum of the non-condensate atoms is conserved in the local rest frame of the condensate; in addition, the energy conservation condition takes into account explicitly the different mean-field energies of condensate and non-condensate atoms. In addition to the exact relations in (30), we have

\[ \int d\mathbf{p} C_{22} = 0, \]

(31)

which implies that the number of excited atoms is conserved in \( C_{22} \) collisions. In contrast, as already mentioned after (29), \( C_{12} \) does not conserve the number of excited atoms and thus we find

\[ \Gamma_{12}[f] \equiv \int \frac{d\mathbf{p}}{(2\pi \hbar)^3} C_{12} \neq 0. \]

(32)

Taking (30)–(32) into account and, after a certain amount of rearrangement, the exact moment equations derived from (29) can be written in the form (\( \mu \) and \( \nu \) denote Cartesian components):

\[ \frac{\partial \tilde{n}}{\partial t} + \nabla \cdot (\tilde{n} \mathbf{v}_n) = \Gamma_{12}[f], \]

(33a)

\[ m\tilde{n} \left( \frac{\partial}{\partial t} + \mathbf{v}_n \cdot \nabla \right) \mathbf{v}_{n\mu} = -\frac{\partial P_{\mu\nu}}{\partial x_\nu} - \tilde{n} \frac{\partial U}{\partial x_\mu} - m(\mathbf{v}_{n\mu} - \mathbf{v}_{c\mu}) \Gamma_{12}[f], \]

(33b)

\[ \frac{\partial \tilde{\epsilon}}{\partial t} + \nabla \cdot (\tilde{\epsilon} \mathbf{v}_n) = -\nabla \cdot \mathbf{Q} - D_{\mu\nu} P_{\mu\nu} + \left[ \frac{1}{2} m(\mathbf{v}_n - \mathbf{v}_c)^2 + \mu_c - U \right] \Gamma_{12}[f]. \]

(33c)

The non-condensate density is defined in (19) while the non-condensate local velocity is defined by
\[ \vec{n}(\mathbf{r}, t) \mathbf{v}_n(\mathbf{r}, t) = \int \frac{d\mathbf{p}}{(2\pi \hbar)^3} \frac{\mathbf{p}}{m} f(\mathbf{p}, \mathbf{r}, t). \]  

In addition, we have

\begin{align}
P_{\mu\nu}(\mathbf{r}, t) &\equiv m \int \frac{d\mathbf{p}}{(2\pi \hbar)^3} \left( \frac{p_\mu}{m} - v_{n\mu} \right) \left( \frac{p_\nu}{m} - v_{n\nu} \right) f(\mathbf{p}, \mathbf{r}, t), \\
Q(\mathbf{r}, t) &\equiv \int \frac{d\mathbf{p}}{(2\pi \hbar)^3} \frac{1}{2m} (\mathbf{p} - m \mathbf{v}_n)^2 \left( \frac{\mathbf{p}}{m} - \mathbf{v}_n \right) f(\mathbf{p}, \mathbf{r}, t), \\
\tilde{\epsilon}(\mathbf{r}, t) &\equiv \int \frac{d\mathbf{p}}{(2\pi \hbar)^3} \frac{1}{2m} (\mathbf{p} - m \mathbf{v}_n)^2 f(\mathbf{p}, \mathbf{r}, t).
\end{align}

The symmetric rate-of-strain tensor appearing in (33c) is defined by

\[ D_{\mu\nu}(\mathbf{r}, t) \equiv \frac{1}{2} \left( \frac{\partial v_{n\mu}}{\partial x_\nu} + \frac{\partial v_{n\nu}}{\partial x_\mu} \right). \]

It is clear from the addition of (27a) and (33a) that the total number of atoms is conserved, as it must be.

The set of equations in (33)–(35) are formally exact but they are obviously not closed since the number of local variables (\(n, n_c, \mathbf{v}_n, \mathbf{v}_c\) as well as the stress tensor \(P_{\mu\nu}\), kinetic energy density \(\tilde{\epsilon}\), heat current \(Q\), and \(\Gamma_{12}\)) exceeds the total number (nine) of coupled scalar equations. To proceed we must specify the conditions under which the dynamics of the trapped gas is to be determined. The range of possibilities include: (i) the collisionless regime in which \(C_{12} = C_{22} = 0\). Here, the collisionless kinetic equation for the non-condensate must be solved explicitly in order to determine the various physical quantities of interest; (ii) an intermediate regime in which collisions between excited atoms are sufficiently rapid to establish a state of local equilibrium within the non-condensate component, but in which collisions between the condensate and non-condensate, as described by \(C_{12}\), can be neglected. We shall refer to this as a state of partial local equilibrium which might be expected to arise near \(T_{BEC}\) when the density of condensate atoms is small (note the proportionality of \(C_{12}\) to \(n_c\)); and (iii) a regime in which the collision rates are so high that a condition of complete local equilibrium is established. As will be discussed in detail later, regime (iii) is the one conventionally dealt with using Landau two-fluid hydrodynamics in uniform Bose superfluids [7]. Of course, the most general situation will not fall neatly into any of the categories listed so far and will require a detailed solution of the kinetic equation in (29) together with the quantum hydrodynamic equations for the condensate.

We shall not deal with the collisionless regime (i) in the present paper, but note that some work along these lines has recently appeared [27]. Instead, we shall focus on the other two regimes indicated above, in which the \(C_{12}\) and \(C_{22}\) collisions play a crucial role. Of particular interest is the transition from the partial to complete local equilibrium conditions. An analysis of this regime will clarify the way in which the Landau theory emerges within the context of our explicit microscopic model. This turns out to be a rather subtle problem, which was not fully appreciated in our earlier work [2,9] nor addressed in the analogous work of KD [13].
B. Partial local equilibrium

We now consider the regime in which Eqs. (33)–(35) reduce to a close set of equations for the non-condensate variables. When the collision rate among the excited atoms is high, the collision integral $C_{22}$ drives the distribution function $f(p, r, t)$ towards the “local equilibrium” Bose distribution

$$\tilde{f}(p, r, t) = \frac{1}{e^{\beta \left[ \frac{1}{2} m (p - m v_n)^2 + U - \tilde{\mu} \right]} - 1},$$

where now the temperature parameter $\beta$, local fluid velocity $v_n$, chemical potential $\tilde{\mu}$ and mean field $U$ are all functions of $r$ and $t$. It is important to appreciate that the local chemical potential $\tilde{\mu}$ which appears here is distinct from the local chemical potential $\mu_c$ defined in (28) for the condensate. How these two chemical potentials acquire a common value in the limit of complete local equilibrium requires a careful analysis of the effect of $C_{12}$ (see Section V). For the moment, we shall simply note the consequences that (37) has for the collision integrals.

First, one may immediately verify that the local equilibrium form of $\tilde{f}$ in (37) guarantees that

$$C_{22}[\tilde{f}] = 0.$$

Indeed, it is this condition that defines the “local equilibrium” solution. The result is independent of the value of $\tilde{\mu}$ and only makes use of the energy and momentum conservation factors in the definition of $C_{22}$ in (23a), and the key identity for the Bose distribution

$$1 + f(x) = \frac{e^x}{e^x - 1} = -f(-x).$$

In contrast, one finds from (23b) that $C_{12}[\tilde{f}] \neq 0$, namely

$$C_{12}[\tilde{f}] = \frac{2g^2 n_c}{(2\pi)^2 \hbar^4} \left[ 1 - e^{-\beta (\tilde{\mu} - \frac{1}{2} m (v_n - v_c)^2 - \mu_c)} \right]$$

$$\times \int dp_1 \int dp_2 \int dp_3 \delta (m v_c + p_1 - p_2 - p_3) \delta (\varepsilon_c + \varepsilon_1 - \varepsilon_2 - \varepsilon_3)$$

$$\times \left[ \delta (p - p_1) - \delta (p - p_2) - \delta (p - p_3) \right] \left( 1 + \tilde{f}_1 \right) \tilde{f}_2 \tilde{f}_3.$$

A factor similar to the first square bracket in (40) appears in the growth equation derived by Gardiner et al. [13]. In their work, our time- and space-dependent $\tilde{\mu}$ is replaced by a constant chemical potential characterizing a non-condensate particle reservoir. On the other hand, they use a kinetic description for low-lying non-condensate states which we describe in terms of a local equilibrium distribution with local chemical potential $\tilde{\mu}$. Thus, although different in detail, there is considerable overlap in physical content. The important point is that $C_{12}[\tilde{f}]$ is in general non-vanishing, reflecting the fact that excited atoms in the non-condensate need not be in local diffusive equilibrium with the condensate atoms ($\tilde{\mu} \neq \mu_c$).

Assuming that the collision rate among excited atoms is sufficiently high, the local equilibrium distribution in (37) provides the appropriate zeroth-order solution to the kinetic equation in (29). This is the same reasoning used in classical kinetic theory [23]. Of course,
even in the absence of the $C_{12}$ collision term, $\tilde{f}$ is not an exact solution of (29). A systematic procedure based on the Chapman-Enskog method gives rise to corrections to $\tilde{f}$ which are necessary in order to describe thermal conduction and viscosity. A detailed discussion of these damping processes in a trapped Bose gas above $T_{\text{BEC}}$ is given in Refs. [30] and [31]. Such calculations can be extended to below $T_{\text{BEC}}$ and have been carried out by KD [13] for a uniform Bose-condensed gas. However, we ignore these corrections in the present analysis and proceed with the assumption that $f \simeq \tilde{f}$ is a good approximation (see Section VIII for further remarks.)

Using (37) to evaluate the quantities in (35), we find that the heat current $Q(\mathbf{r}, t) = 0$, and that

$$P_{\mu\nu}(\mathbf{r}, t) = \delta_{\mu\nu} \tilde{P}(\mathbf{r}, t) \equiv \delta_{\mu\nu} \left[ \frac{d\mathbf{p}}{(2\pi\hbar)^3} \frac{p^2}{3m} \tilde{f}(\mathbf{p}, \mathbf{r}, t) \right]_{v_n=0}. \tag{41}$$

In addition, we note that the kinetic energy density (35c) is given by

$$\tilde{\epsilon}(\mathbf{r}, t) = \frac{3}{2} \tilde{P}(\mathbf{r}, t), \tag{42}$$

which is the relation found for a uniform ideal gas. Since the momentum dependence of the integrand in (11) is the same as for a static equilibrium Bose distribution, the integrations can be carried out explicitly to give the expression,

$$\tilde{P}(\mathbf{r}, t) = \frac{1}{\beta \Lambda^3} g_{5/2}(z), \tag{43}$$

where the local thermodynamic variables $\beta$ and $z$ are again functions of $\mathbf{r}$ and $t$. The Bose-Einstein functions are defined as $g_n(z) \equiv \sum_{l=1}^{\infty} z^l / l^n$, the local equilibrium fugacity is

$$z(\mathbf{r}, t) \equiv e^{\beta[r(t)] [\tilde{\mu}(\mathbf{r}, t) - U(\mathbf{r}, t)]}, \tag{44}$$

and the local thermal de Broglie wavelength is

$$\Lambda(\mathbf{r}, t) \equiv \left( \frac{2\pi\hbar^2}{mk_B T(\mathbf{r}, t)} \right)^{1/2}. \tag{45}$$

Finally, we note that the non-condensate density associated with (37) is given by

$$\tilde{n}(\mathbf{r}, t) = \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \tilde{f}(\mathbf{p}, \mathbf{r}, t) \bigg|_{v_n=0} = \frac{1}{\Lambda^3} g_{3/2}(z). \tag{46}$$

To summarize, using the local equilibrium approximation $f \simeq \tilde{f}$, the hydrodynamic equations in (33) simplify to

$$\frac{\partial \tilde{n}}{\partial t} + \nabla \cdot (\tilde{n} \mathbf{v}_n) = \Gamma_{12}[\tilde{f}], \tag{47a}$$

$$m \tilde{n} \left( \frac{\partial}{\partial t} + \mathbf{v}_n \cdot \nabla \right) \mathbf{v}_n = -\nabla \tilde{P} - \tilde{n} \nabla U - m(\mathbf{v}_n - \mathbf{v}_c) \Gamma_{12}[\tilde{f}], \tag{47b}$$
\[
\frac{\partial \tilde{P}}{\partial t} + \nabla \cdot (\tilde{P} \mathbf{v}_n) = -\frac{2}{3} \tilde{P} \nabla \cdot \mathbf{v}_n + \frac{2}{3} \left[ \frac{1}{2} m (\mathbf{v}_n - \mathbf{v}_c)^2 + \mu_c - U \right] \Gamma_{12}[\tilde{f}],
\]
(47c)

where

\[
\Gamma_{12}[\tilde{f}] \equiv \int \frac{d \mathbf{p}}{(2\pi \hbar)^3} C_{12}[\tilde{f}]
= -\frac{2g^2 n_c}{(2\pi)^5 \hbar} \left[ 1 - e^{-\beta(\tilde{\mu} - \mu_c - 4m (\mathbf{v}_n - \mathbf{v}_c)^2)} \right] \int d \mathbf{p}_1 \int d \mathbf{p}_2 \int d \mathbf{p}_3
\times \delta (m \mathbf{v}_c + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \delta (\varepsilon_c + \tilde{\varepsilon}_1 - \tilde{\varepsilon}_2 - \tilde{\varepsilon}_3) (1 + \tilde{f}_1) \tilde{f}_2 \tilde{f}_3.
\]
(48)

It is clear that the \( \Gamma_{12}[\tilde{f}] \) terms in (47) will play a crucial role in the solution of these hydrodynamic equations. In particular, these terms describe the transfer of atoms between the condensate and non-condensate and are responsible for bringing the two components into complete local equilibrium.

In this paper, we concentrate on solving (47) for small amplitude oscillations around static (i.e., absolute) equilibrium. For the non-condensate atoms, absolute equilibrium is specified by a Bose distribution at a uniform temperature \( T_0 \) and chemical potential \( \tilde{\mu}_0 \) (all static equilibrium quantities are indicated by a subscript 0). The equilibrium density and pressure are then given by the expressions \( \tilde{n}_0(\mathbf{r}) = g_{3/2}(z_0)/\Lambda_0^3 \) and \( \tilde{P}_0(\mathbf{r}) = g_{5/2}(z_0)/\beta_0 \Lambda_0^5 \), respectively. These are of the same form as in Eqs. (46) and (43), but with all thermodynamic quantities taking on their equilibrium values. In particular, the equilibrium fugacity is \( z_0(\mathbf{r}) = e^{\beta(\tilde{\mu}_0 - U_0(\mathbf{r}))} \) with \( U_0(\mathbf{r}) = U_{\text{ext}}(\mathbf{r}) + 2g[\tilde{n}_0(\mathbf{r}) + n_{c0}(\mathbf{r})] \). It is clear that the equilibrium condensate density \( n_{c0}(\mathbf{r}) \) is also required to complete the definition of the non-condensate equilibrium variables. This we obtain from the equilibrium condensate wavefunction which is a solution of the static equation

\[
\left[ -\frac{\hbar^2 \nabla^2}{2m} + U_{\text{ext}}(\mathbf{r}) + gn_{c0}(\mathbf{r}) + 2g\tilde{n}_0(\mathbf{r}) \right] \Phi_0(\mathbf{r}) = \mu_{c0} \Phi_0(\mathbf{r}).
\]
(49)

This is the HFP equation for the condensate used previously [2,26], but with the important difference that \( \tilde{n}_0 \) is determined in the HF approximation as described above.

The eigenvalue of (49) can be written as

\[
\mu_{c0} = -\frac{\hbar^2 \nabla^2 \sqrt{n_{c0}(\mathbf{r})}}{2m \sqrt{n_{c0}(\mathbf{r})}} + U_{\text{ext}}(\mathbf{r}) + gn_{c0}(\mathbf{r}) + 2g\tilde{n}_0(\mathbf{r}),
\]
(50)

and is identified with the static equilibrium condensate chemical potential. To see the connection between \( \tilde{\mu}_0 \) and \( \mu_{c0} \), we refer to the expression for \( C_{12}[\tilde{f}] \) in Eq. (40). With \( \mathbf{v}_n = \mathbf{v}_c = 0 \), we note that the factor in the square brackets vanishes if \( \tilde{\mu}_0 = \mu_{c0} \), which is precisely the expected condition for diffusive equilibrium between the condensate and non-condensate. Thus the constraint \( \tilde{\mu}_0 = \mu_{c0} \) must be imposed in order to obtain the self-consistent solution for \( \tilde{n}_0(\mathbf{r}) \) and \( n_{c0}(\mathbf{r}) \). Further discussion regarding the static equilibrium properties of a trapped Bose gas is given in Appendix B.
C. Linearized hydrodynamic equations

To study the collective modes, we linearize the system of equations in (47) around the static equilibrium solutions given above to obtain the non-condensate hydrodynamic equations

\[
\frac{\partial \tilde{\delta} n}{\partial t} = -\nabla \cdot (\tilde{n}_0 \delta \mathbf{v}_n) + \delta \Gamma_{12},
\]

\[
m \tilde{n}_0 \frac{\partial \delta \mathbf{v}_n}{\partial t} = -\nabla \delta \tilde{P} - \delta \tilde{n} \nabla U_0 - 2g\tilde{n}_0 \nabla (\delta \tilde{n} + \delta n_c),
\]

\[
\frac{\partial \delta \tilde{P}}{\partial t} = -\frac{5}{3} \nabla \cdot (\tilde{P}_0 \delta \mathbf{v}_n) + \frac{2}{3} \delta \mathbf{v}_n \cdot \nabla \tilde{P}_0 + \frac{2}{3} (\mu_{c0} - U_0) \delta \Gamma_{12}.
\]

We observe that \( \delta \Gamma_{12} \) does not appear explicitly in (51b) as it contributes a second order correction. These non-condensate equations must be solved in conjunction with the following linearized equations for the condensate [see (27)]

\[
\frac{\partial \delta n_c}{\partial t} = -\nabla \cdot (n_{c0} \delta \mathbf{v}_c) - \delta \Gamma_{12},
\]

\[
m \frac{\partial \delta \mathbf{v}_c}{\partial t} = -\nabla \delta \mu_c,
\]

where

\[
\delta \mu_c(r, t) = \frac{1}{2\Phi_0(r)} \hat{L}_0(r) \left[ \frac{1}{\Phi_0(r)} \delta n_c(r, t) \right] + g\delta n_c(r, t) + 2g\delta \tilde{n}(r, t).
\]

The operator \( \hat{L}_0(r) \) appearing in this equation is defined as

\[
\hat{L}_0(r) \equiv \left[ -\frac{\hbar^2 \nabla^2}{2m} + U_{\text{ext}}(r) + 2g\tilde{n}_0(r) + gn_{c0}(r) - \mu_c \right],
\]

and is essentially the HFP Hamiltonian in Eq. (19). We must finally obtain an expression for \( \delta \Gamma_{12} \) in (51) and (52) which is the linearized form of the expression in (18). We have

\[
\delta \Gamma_{12}[\hat{f}] = -\frac{\beta_0 n_{c0}}{\tau_{12}} \delta \mu_{\text{diff}},
\]

where we have introduced the difference between the local chemical potentials of the condensate and non-condensate,

\[
\mu_{\text{diff}} \equiv \tilde{\mu} - \mu_c,
\]

and the “equilibrium” \( C_{12} \) collision rate

\[
\frac{1}{\tau_{12}} \equiv \frac{2g^2}{(2\pi)^5 \hbar^7} \int d\mathbf{p}_1 \int d\mathbf{p}_2 \int d\mathbf{p}_3 \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \delta(\mu_{c0} + \tilde{\varepsilon}_1 - \tilde{\varepsilon}_2 - \tilde{\varepsilon}_3)(1 + f_1^0 f_2^0 f_3^0).
\]

Here, \( f_i^0 \) is the static equilibrium Bose distribution function.
D. ZGN hydrodynamics

In the remaining part of this Section, we suppose that on the time scale of the collective modes of interest, collisions between condensate and non-condensate atoms can be ignored. This amounts to setting $C_{12}[\hat{f}]$ (and hence $\Gamma_{12}[\hat{f}]$) to zero, and gives the hydrodynamic equations derived in ZGN [5]. The linearized version of these equations is obtained by setting $\delta\Gamma_{12}$ to zero in (51) and (52). They are a closed set of equations for the variables $\delta n_c$, $\delta \tilde{n}$, $v_c$, $v_n$ and $\delta \tilde{P}$ and may be solved to obtain the self-sustained oscillations of the trapped gas at finite temperatures. A powerful variational method for solving equations of this kind is presented in Section VII, where we consider the explicit solution of the ZGN equations for a trapped gas.

An approximation commonly used in the recent literature on trapped Bose gases involves the neglect of the non-local “quantum pressure” terms, i.e., the first terms on the RHS of (50) and (53). This so-called Thomas-Fermi (TF) approximation [22,32] has been found to give quite good results when the number of atoms in the trap is large. Within the TF approximation, the equilibrium condensate chemical potential in (50) simplifies to

$$\mu_{c0} \simeq U_{\text{ext}}(r) + gn_{c0}(r) + 2g\tilde{n}_0(r).$$

At $T = 0$, where $\tilde{n}_0(r) = 0$, (58) gives a condensate density which goes smoothly to zero at points where $U_{\text{ext}}(r) = \mu_{c0}$. However at finite temperatures, the combined solution of (58) and the semiclassical expression for the non-condensate density ($\tilde{n}_0 = g_{3/2}/\Lambda^3_0$) gives rise to unphysical discontinuous behaviour of the densities at the sharp boundary of the condensate [33]. This behaviour is especially problematic when attempting to obtain solutions of Eqs. (51)–(53) for the collective modes, since elaborate boundary conditions must be formulated in order to join the fluctuating variables at the condensate boundary. These problems can be avoided by retaining the quantum pressure terms which ensure that the densities are smooth functions of position in all regions of the trapped Bose-condensed gas. Although these gradient terms might appear to complicate the analysis considerably, it will be seen that they can be handled easily using the variational method of solution developed in Section VII.

For the purpose of comparison with results obtained in Section V, it is useful to consider the ZGN equations in the limit of a uniform Bose gas. In this case, the ZGN equations yield first and second sound modes [28]. Eliminating the variables $\delta n_c$, $\delta \tilde{n}$ and $\delta \tilde{P}$ from Eqs. (51)–(52) (with $\delta\Gamma_{12} \equiv 0$), one obtains two coupled equations [28]:

$$m\frac{\partial^2 \delta v_c}{\partial t^2} = gn_{c0}(\nabla \cdot \delta v_c) + 2g\tilde{n}_0 \nabla(\nabla \cdot \delta v_n),$$

$$(59a)$$

$$m\frac{\partial^2 \delta v_n}{\partial t^2} = 2gn_{c0}(\nabla \cdot \delta v_c) + \left(\frac{5}{3} \frac{\tilde{P}_0}{\tilde{n}_0} + 2g\tilde{n}_0\right) \nabla(\nabla \cdot \delta v_n).$$

$$(59b)$$

Introducing velocity potentials according to $\delta v_c \equiv \nabla \phi_c$ and $\delta v_n \equiv \nabla \phi_n$ and assuming a plane wave solution of the form $\phi_{c,n}(r,t) = \phi_{c,n}(k r - \omega t)$, one obtains

$$m\omega^2 \phi_c = gn_{c0} k^2 \phi_c + 2g\tilde{n}_0 k^2 \phi_n$$

$$(60a)$$

$$m\omega^2 \phi_n = 2gn_{c0} k^2 \phi_c + \left(\frac{5}{3} \frac{\tilde{P}_0}{\tilde{n}_0} + 2g\tilde{n}_0\right) k^2 \phi_n.$$
The nontrivial solution of these equations yields the normal mode frequencies of the uniform gas. These modes are first and second sound waves with the dispersion relation \( \omega = uk \), where the sound velocities \( u \) are the solution of

\[
\frac{5}{3} \frac{\tilde{P}_0}{m} + \frac{2g\tilde{n}_0}{m} + \frac{gn_c\delta}{m} + \frac{5}{3} \frac{\tilde{P}_0}{m} - \frac{2g\tilde{n}_0}{m} = 0.
\]

Writing this equation as \( u^4 - Au^2 + B = 0 \), we see that the coefficients \( A \) and \( B \) are completely determined by equilibrium properties of the gas. These coefficients will be compared in Section V with the corresponding coefficients obtained on the basis of the Landau two-fluid equations. As we show there, the \( A \) and \( B \) coefficients are slightly different in the Landau regime and reflect the different conditions underlying the propagation of first and second sound waves in the two cases.

We note that the quartic equation in (61) yields four solutions while Eqs. (51) and (52) are a total of five equations and must therefore admit five distinct longitudinal modes. The missing solution is, in fact, a zero frequency mode with mode amplitudes \( \delta v_c = \delta v_n = 0 \), \( \delta n_c = -2\delta \tilde{n} \) and \( \delta \tilde{P} = 2g\tilde{n}_0\delta \tilde{n} \). The physical interpretation of such a static mode is not immediately obvious. However, since we require some results from Section V in order to explain its meaning, we shall defer further discussion until then.

V. COMPLETE LOCAL EQUILIBRIUM: LANDAU TWO-FLUID EQUATIONS

In this Section, we show that within the TF approximation, the set of equations in (27) and (47) can be used to derive linearized two-fluid equations which reduce in a certain limit to the standard Landau form [3], as generalized to include an external trapping potential. This derivation is of interest for several reasons. First, it shows that our approximate microscopic model is consistent with the Landau two-fluid equations when conditions of complete local equilibrium exist. Under these conditions, the Landau equations are valid for dense fluids such as superfluid \(^4\)He as well as for dilute Bose gases. These equations have been derived using very general arguments [14], but this very generality tends to obscure the underlying physics which a microscopic model such as ours can reveal in a very clear manner. For example, our model not only leads to the Landau equations, but at the same time gives the static equilibrium thermodynamic functions required in the calculation of the normal modes. An even more significant advantage of our microscopic model is that we can explore the transition from partial to complete local diffusive equilibrium. In this respect, our two-fluid hydrodynamics based on Eqs. (27) and (47) provides a more complete description of the possible dynamical behaviour in trapped Bose gases.

In a previous paper [4], we indicated how the linearized ZGN equations could be combined to give the Landau form of the two-fluid equations for a uniform gas. However, the arguments presented there were incomplete. Here, we present a more careful discussion starting form Eqs. (27) and (47) and show how an explicit consideration of the source term \( \Gamma_1[f] \) associated with the \( C_{12} \) collisions leads to generalized two-fluid equations that reduce to the Landau equations in the appropriate limit. In addition, we shall extend our previous analysis to the case of an inhomogeneous system. In doing so, however, we make use of the Thomas-Fermi approximation which effectively treats the system as locally homogeneous.
If this approximation is not made, one cannot obtain the usual Landau equations since the gradient term of the chemical potential in (28) contributes to the right hand side of (27b). This nonlocal term is a quantum correction which is always ignored in microscopic derivations of the Landau equations for uniform systems [14].

For completeness, we first display the linearized Landau two-fluid hydrodynamic equations [6,7,34], in the form they take in the presence of an external potential [35]:

\[ \frac{\partial \delta n}{\partial t} + \nabla \cdot \delta \mathbf{j} = 0 \]  
\[ m \frac{\partial \delta \mathbf{j}}{\partial t} = -\nabla \delta P - \delta n \nabla U_{\text{ext}} \]  
\[ m \frac{\partial \delta \mathbf{v}_s}{\partial t} = -\nabla \delta \mu \]  
\[ \frac{\partial \delta s}{\partial t} + \nabla \cdot (s_0 \delta \mathbf{v}_n) = 0 , \]

where

\[ m \delta \mathbf{j}(\mathbf{r}, t) = \rho_s(\mathbf{r}) \delta \mathbf{v}_s(\mathbf{r}, t) + \rho_n(\mathbf{r}) \delta \mathbf{v}_n(\mathbf{r}, t) , \]
\[ m \delta n(\mathbf{r}, t) = \delta \rho_s(\mathbf{r}, t) + \delta \rho_n(\mathbf{r}, t) . \]

We have here distinguished the superfluid variables \( \rho_s \) and \( \mathbf{v}_s \) from the corresponding variables for the condensate; a similar distinction applies to the normal fluid variables as opposed to the non-condensate variables. However, the condensate and non-condensate variables can be identified with the more usual two-fluid variables within the context of our model which treats the excitations in the HF approximation. It should be noted that this correspondence will no longer be valid if one were to go beyond this approximation. For example, in the HFP approximation, there is a finite depletion of the condensate even at zero temperature [2] and the superfluid density is no longer precisely equivalent to the condensate density. The other local variables appearing in the above equations are the pressure \( P \), entropy density \( s \) and thermodynamic chemical potential \( \mu \). As we shall show, each of these quantities is given by an explicit expression within our microscopic model.

The linearized ZGN' continuity equations are given in (51a) and (52a). Adding them, we obtain the continuity equation (62a). Combining our two velocity equations in (51b) and (52b), we find

\[ \frac{\partial \delta \mathbf{j}}{\partial t} = -\nabla \delta \tilde{P} - 2g(n_{c0} + \tilde{n}_0) \nabla \delta n - g(n_{c0} + 2\tilde{n}_0) \nabla \delta n_c \\
- 2g \nabla (n_{c0} + \tilde{n}_0) \delta \tilde{n} - \nabla U_{\text{ext}} \delta \tilde{n} . \]

One can show that (64) is equivalent to (62b) if the total pressure is defined as

\[ P = \tilde{P} + \frac{1}{2} g[n^2 + 2n \tilde{n} - \tilde{n}^2] . \]
To prove this, we note that (65) gives
\[ \delta P = \delta \tilde{P} + 2g n_0 \delta \tilde{n} + g(n_{c0} + 2\tilde{n}_0)\delta n_c \] (66)
and hence,
\[ \nabla \delta P = \nabla \delta \tilde{P} + 2g n_0 \nabla \delta \tilde{n} + g(n_{c0} + 2\tilde{n}_0)\nabla \delta n_c \]
\[ + 2g \nabla n_0 \delta \tilde{n} + g(\nabla n_{c0} + 2\nabla \tilde{n}_0)\delta n_c . \] (67)

Making use of the TF result in (58) to rewrite the last term as \(-\delta n_c \nabla U_{\text{ext}}\), and inserting the resulting expression for \(\nabla \delta P\) into (62b), we obtain an equation identical to (64). The results in (64)–(67) were given previously by ZGN [5] for a uniform gas, but the present derivation is more general in that it is based on the ZGN' equations in the context of a nonuniform system.

In order to make contact with the remaining Landau equations (62c) and (62d), we must introduce the appropriate thermodynamic variables corresponding to the chemical potential \(\mu\), local temperature \(T\) and the local entropy \(s\). Within the HF approximation for the non-condensate, the total energy density is given by
\[ \epsilon = \tilde{\epsilon} + \frac{1}{2} g \langle \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi} \rangle + n U_{\text{ext}} \]
\[ = \tilde{\epsilon} + \frac{1}{2} g \left[ n^2 + 2n \tilde{n} - \tilde{n}^2 \right] + n U_{\text{ext}} , \] (68)
where \(\tilde{\epsilon}\) is defined by (41) and (42). For the local entropy we use the definition
\[ s \equiv k_B \int \frac{d\rho}{(2\pi \hbar)^3} \left[ (1 + \tilde{f}) \ln(1 + \tilde{f}) - \tilde{f} \ln \tilde{f} \right] \]
\[ = \frac{1}{T} \left[ \frac{5}{2} \tilde{P} - \tilde{n}(\tilde{\mu} - U) \right] . \] (69)

This result corresponds to the entropy of a uniform ideal gas above \(T_{\text{BEC}}\) with a fugacity \(z = e^{\beta(\tilde{\mu} - U)}\). In static equilibrium, (69) reduces to
\[ s_0 T_0 = \frac{5}{2} \tilde{P}_0 + gn_{c0}\tilde{n}_0 , \] (70)
where we have noted that \(\tilde{\mu}_0 = \mu_{c0} = U_{\text{ext}} + g(2\tilde{n}_0 + n_{c0})\) and \(U_0 = U_{\text{ext}} + 2gn_0\).

From these expressions for \(P\), \(\epsilon\) and \(s\) (all of which depend on \(r\) and \(t\)), we obtain the relation
\[ \epsilon + P - sT = \mu_c n_c + \tilde{\mu} \tilde{n} = \mu_c n + \mu_{\text{diff}} \tilde{n} , \] (71)
where \(\mu_{\text{diff}}\) is the chemical potential difference defined in (56). As we shall see, this variable plays a crucial role in understanding the approach to complete local equilibrium. If \(\mu_{\text{diff}}\) were zero, then (71) would correspond precisely to the usual thermodynamic relation involving these variables, with \(\mu_c\) playing the role of the equilibrium chemical potential.

Having defined these various thermodynamic functions, we can now consider their local variations from absolute equilibrium (recall that the equilibrium properties are still functions
of position. We begin with the kinetic pressure given by (43). Its variation leads to the equation
\[ \delta \tilde{P} = \delta \left( \frac{1}{\beta \Lambda^3} \right) g_{5/2}(z_0) + \frac{1}{\beta_0 \Lambda_0^3} g'_{5/2}(z_0) \delta z \]
\[ = s_0 \delta T + \tilde{n}_0 (\delta \tilde{\mu} - 2g \delta n) \]
(72)
In obtaining this result, we have made use of (70) and the Bose identity
\[ z g_n'(z) = g_n - 1 \]
Combining (66) and (72), we find
\[ \delta P = s_0 \delta T + n_0 \delta \mu_c + \tilde{n}_0 \delta \mu_{\text{diff}} \]
(73)
The appearance of the last term is once again due to the non-condensate having a chemical potential different from that of the condensate. Similarly, from (69) and (72), we find the variation of the entropy to be given by
\[ T_0 \delta s = \frac{3}{2} \delta \tilde{P} + gn_c \delta \tilde{n} \]
(74)
This expression finally allows us to obtain an entropy conservation equation. Taking the time derivative of (74), and making use of the non-condensate density and pressure equations (51a) and (51c), we find
\[ \frac{\partial \delta s}{\partial t} = \frac{3}{2} \frac{\partial \delta \tilde{P}}{\partial t} + \frac{g n_c}{T_0} \frac{\partial \delta \tilde{n}}{\partial t} \]
\[ = - \nabla \cdot (s_0 \delta \mathbf{v}_n) \]
(75)
We note that the source terms on the right hand sides of (71a) and (71c) cancel out, so that a strict conservation law for the nonequilibrium local entropy defined in (69) is obtained.

Finally, a comparison of (52b) and (62c) shows that these two equations are equivalent if \( \delta v_s \) is identified with \( \delta v_c \) and \( \delta \mu \) with \( \delta \mu_c \). With this identification, we arrive at a set of hydrodynamic equations which are precisely of the form of the Landau two-fluid equations. At first sight, it might therefore appear that our starting equations are in fact equivalent to the Landau equations, but this is not the case. The appearance of \( \mu_{\text{diff}} \) in (71) implies that \( \mu_c \) is not related to the other thermodynamic variables in the same way that the equilibrium chemical potential \( \mu \) would be. As a result, \( \delta \mu_c \) and \( \delta s \) are not related directly to the thermodynamic fluctuations \( \delta T \) and \( \delta P \), as assumed in the usual derivations of the first and second sound velocities [34].

To see this more clearly, let us use (70) to define the equilibrium local entropy function, \( T s^{eq} \equiv \frac{5}{2} \tilde{P} + gn_c \tilde{n} \), where the notation \( s^{eq} \) is used to distinguish this quantity from the local entropy given by (69). It is the variation of the equilibrium entropy \( s^{eq} \) which can be expressed in terms of the thermodynamic fluctuations \( \delta T \) and \( \delta P \) via equilibrium thermodynamic derivatives. By considering a quasistatic change in the thermodynamic state of the system, (70) leads to the relation
\[ T_0 \delta s^{eq} = \frac{5}{2} \delta \tilde{P} + gn_c \delta \tilde{n} + g \tilde{n}_0 \delta n_c - s_0 \delta T \]
(76)
and using (72) to eliminate \( \delta T \), we find
\[ T_0 \delta s^{eq} = \frac{3}{2} \delta \tilde{P} + gn_{c0} \delta \tilde{n} + \tilde{n}_0 \delta \mu_{\text{diff}}. \] (77)

This differs from the result for \( \delta s \) given in (74). The last term in (77) indicates that the variation of the equilibrium entropy is not simply given in the present situation by the equilibrium variation. In terms of \( \delta s^{eq} \), the entropy equation analogous to (75) takes the form

\[ \frac{\partial \delta s^{eq}}{\partial t} = - \nabla \cdot (s_0 \delta \nu_{\nu}) + \tilde{n}_0 \frac{\partial \delta \mu_{\text{diff}}}{\partial t}. \] (78)

The last term in (78) can be interpreted as the production of entropy associated with the equilibration of the condensate and non-condensate chemical potentials. Its appearance implies that the thermodynamic fluctuations \( \delta P \) and \( \delta T \) are coupled to fluctuations in \( \delta \mu_{\text{diff}} \), which is directly related to the condensate-non-condensate collision terms \( \delta \Gamma_{12} \) appearing on the right hand side of Eqs. (51a), (51c) and (52a). To this point we have not made any assumptions about \( \delta \Gamma_{12} \), and as a result, the equations as they stand differ from the usual Landau two-fluid theory. To make this final connection we must obtain an equation of motion for \( \delta \mu_{\text{diff}} \).

Noting that \( \tilde{\mu} - 2gn = \mu_{\text{diff}} - gn_c \), (72) can be rewritten in the form

\[ \delta \tilde{P} = s_0 \delta T + \tilde{n}_0 (\delta \mu_{\text{diff}} - g \delta n_c). \] (79)

Similarly, the variation of (46) yields the equation

\[ \delta \tilde{n} = \left( \frac{3}{2} \tilde{n}_0 + \tilde{\gamma}_0 n_{c0} \right) \frac{\delta T}{T_0} + \frac{\tilde{\gamma}_0}{g} (\delta \mu_{\text{diff}} - g \delta n_c), \] (80)

where we have introduced the dimensionless quantity

\[ \tilde{\gamma}_0 = \frac{\beta g}{\Lambda_0^3} g_{1/2}(z_0). \] (81)

This quantity can be related to thermodynamic derivatives by noting that (80) implies the equilibrium variation at constant temperature, \( \delta \tilde{n}|_T = (\tilde{\gamma}_0/g) \delta (\tilde{\mu} - 2gn)|_T = -\tilde{\gamma}_0 (\delta n - \delta \tilde{n})|_T. \)

We thus conclude that

\[ \tilde{\gamma}_0 = \frac{\left( \frac{\partial \tilde{n}}{\partial n} \right)_T}{\left( \frac{\partial \tilde{n}}{\partial T} \right)_n} \right) - 1. \] (82)

In a similar way, one can show that the coefficient of \( \delta T \) in (80) can be expressed as

\[ \left( \frac{3}{2} \tilde{n}_0 + \tilde{\gamma}_0 n_{c0} \right) \frac{1}{T_0} = (1 - \tilde{\gamma}_0) \left( \frac{\partial \tilde{n}}{\partial T} \right)_n. \] (83)

Eqs. (71) and (80) show that the variations in \( \delta T \) and \( \delta \mu_{\text{diff}} \) can each be expressed in terms of \( \delta n_c \), \( \delta \tilde{n} \) and \( \delta \tilde{P} \). Eliminating \( \delta \mu_{\text{diff}} \) from these equations and making use of Eqs. (51a) and (51c), we obtain the equation of motion for \( \delta T \),
where the relaxation time for the chemical potential difference is defined as
\[ \tau \] which shows that any initial difference in chemical potentials decays on a time scale set by
fluctuations which occur on a time scale longer than the relaxation time \( \delta T \).

Alternatively, eliminating \( \delta \) from (86) is

two components. For a uniform gas with spatially homogeneous fluctuations, the solution
of (86) and (87) was quoted in Ref. [12] for the special case of a uniform gas. Here we can see very clearly that
the equilibrium relationships among all the thermodynamic variables we have defined. As a
result, our equations in the previous Section, but take into account the fact that according to (86),
modes in a uniform Bose gas (see also Ref. [12]). We proceed as in the analysis of the ZGN
BEC gas. Here we can see very clearly that
fluctuations in \( \delta \mu \) or its effective is zero (i.e., \( \tilde{\mu} = \mu_c \)). It then follows from (71) that we recover
the equilibrium relationships among all the thermodynamic variables we have defined. As a
result, our equations in the \( \tau \rightarrow 0 \) limit reproduce precisely the hydrodynamic description
as provided by the Landau two-fluid equations.

To demonstrate this reduction explicitly it is again convenient to consider the sound
modes in a uniform Bose gas (see also Ref. [12]). We proceed as in the analysis of the ZGN
equations in the previous Section, but take into account the fact that according to (86),
fluctuations in \( \delta \mu \) are coupled to those of the velocities \( \delta v_c \) and \( \delta v_n \). Introducing velocity
potentials according to \( \delta v_c = \nabla \phi_c \) and \( \delta v_n = \nabla \phi_n \), the equations in (51) and (52) can be
combined to yield

\[ m \frac{\partial^2 \phi_c}{\partial t^2} = gn_{c0} \nabla^2 \phi_c + 2g\tilde{n}_0 \nabla^2 \phi_n + \frac{\sigma}{\tau \mu} \delta \mu \]  
\[ m \frac{\partial^2 \phi_n}{\partial t^2} = \left( \frac{5\tilde{P}_0}{3\tilde{n}_0} + 2g\tilde{n}_0 \right) \nabla^2 \phi_n + 2gn_{c0} \nabla^2 \phi_c - \frac{2\sigma}{3\tau \mu} \delta \mu . \]
These equations in conjunction with (93) constitute a complete set of equations for the normal modes of the system. We look for solutions having a plane wave form \( \phi_{c,n}(r,t) = \phi_{c,n} e^{i(kr - \omega t)} \). In this case, (93) reduces to

\[
\delta \mu_{\text{diff}} = \frac{g_{n_c,0}}{1 - i \omega \tau_{\mu}} \left( \phi_c - \frac{2}{3} \phi_n \right) k^2. \tag{90}
\]

Substituting this result into Eq. (99), we obtain the two coupled equations

\[
m\omega^2 \phi_c = g_{n_c,0} \left( 1 - \frac{\sigma}{1 - i \omega \tau_{\mu}} \right) k^2 \phi_c + 2g\tilde{n}_0 \left( 1 + \frac{\sigma}{3(1 - i \omega \tau_{\mu})} \frac{n_{c,0}}{\tilde{n}_0} \right) k^2 \phi_n \tag{91}
\]

and

\[
m\omega^2 \phi_n = \left( \frac{5\tilde{P}_0}{3\tilde{n}_0} + 2g\tilde{n}_0 \left[ 1 - \frac{2\sigma}{9(1 - i \omega \tau_{\mu})} \frac{n_{c,0}^2}{\tilde{n}_0^2} \right] \right) k^2 \phi_n + 2gn_{c,0} \left( 1 + \frac{\sigma}{3(1 - i \omega \tau_{\mu})} \frac{n_{c,0}}{\tilde{n}_0} \right) k^2 \phi_c, \tag{92}
\]

where the thermodynamic factor \( \sigma \) is defined in (84). In the limit \( \omega \tau_{\mu} \to \infty \), we recover the ZGN equations given in (69). We also see from (90) that \( \delta \mu_{\text{diff}} \to ig_{n_c,0}(\phi_c - 2\phi_n/3)(k^2/\omega) \), which is proportional to \( k \). In the opposite limit \( \omega \tau_{\mu} \to 0 \), we obtain the equations

\[
m\omega^2 \phi_c = g_{n_c,0}(1 - \sigma)k^2 \phi_c + 2g\tilde{n}_0 \left( 1 + \frac{\sigma n_{c,0}}{3\tilde{n}_0} \right) k^2 \phi_n \tag{93a}
\]

and

\[
m\omega^2 \phi_n = \left( \frac{5\tilde{P}_0}{3\tilde{n}_0} + 2g\tilde{n}_0 \left[ 1 - \frac{2\sigma n_{c,0}^2}{9\tilde{n}_0^2} \right] \right) k^2 \phi_n + 2gn_{c,0} \left( 1 + \frac{\sigma n_{c,0}}{3\tilde{n}_0} \right) k^2 \phi_c, \tag{93b}
\]

with \( \delta \mu_{\text{diff}} \to g_{n_c,0}\tau_{\mu}(\phi_c - 2\phi_n/3)k^2 \). We see from this that \( \delta \mu_{\text{diff}} \) vanishes in the \( \omega \tau_{\mu} \to 0 \) limit while \( \delta \Gamma_{12} \), which is proportional to \( \delta \mu_{\text{diff}}/\tau_{\mu} \), remains finite even when \( \tau_{\mu} \to 0 \). Thus, the \( \delta \Gamma_{12} \) terms in equations (11) and (12) are still present in the Landau limit, reflecting the fact that the establishment of complete local equilibrium requires the continued local readjustment of the number of atoms in the condensate and non-condensate. In other words, the strong collisional coupling between the two components, which ensures \( \delta \mu_{\text{diff}} = 0 \), implies that the condensate and non-condensate densities are not separately conserved as they are in the ZGN (\( \omega \tau_{\mu} \to \infty \)) limit.

Although not immediately apparent, the equations in (93) yield first and second sound velocities which are in precise agreement with those determined by the Landau two-fluid equations. The latter are usually expressed in terms of pressure and temperature fluctuations and yield velocities, \( u \), as solutions of an equation of the form \( u^4 - Au^2 + B = 0 \), where \( A \) and \( B \) are expressed in terms of equilibrium thermodynamic quantities [28]. If these thermodynamic quantities are evaluated for our model of the equilibrium properties, one can show that the Landau \( A \) and \( B \) coefficients correspond precisely to those following from (93). These differ from the ZGN \( A \) and \( B \) coefficients given in (61), and as a result, the first and second sound speeds found in these two limits also differ. However, as shown in Fig. 1, these differences are very small in the case of a weakly interacting Bose gas. That is, the transfer of atoms required to equilibrate the condensate and non-condensate is playing
a relatively minor role when it comes to determining the magnitude of the first and second sound speeds.

The damping of these sound modes is another matter. The modes are undamped in either the ZGN ($\omega \tau_\mu \to \infty$) or Landau ($\omega \tau_\mu \to 0$) limits, but at intermediate values of $\omega \tau_\mu$, the damping arising from the equilibration of the condensate and non-condensate can be quite significant. To obtain the complex mode frequencies, $\omega = \Omega - i\Gamma$, we can solve either the hydrodynamic equations in (51) and (52) [from (55), we recall that $\delta\Gamma_{12}$ is proportional to $\delta\mu_{\text{diff}}$], or Eqs. (89) together with the relaxation equation (86) for the chemical potential difference. These equations yield five mode frequencies: four modes corresponding to damped first and second sound ($\Omega = \pm u_1 k$, $u_2 k$) and a purely imaginary relaxational mode. The latter is associated with the zero-frequency mode mentioned at the end of Section IV and will be discussed further at the end of this Section.

In Fig. 2 we show the damping of the second sound mode for a uniform gas as a function of $\Omega \tau_\mu$, for $T/T_{BEC} = 0.9$ and $gn/k_B T_{BEC} = 0.2$. The relative damping ($\Gamma/\Omega$) of the first sound mode is barely visible on the scale of Fig. 2 and therefore is not displayed. It can be seen that the relative damping peaks at $\Omega \tau_\mu \simeq 1$, which is the transition between the ZGN and Landau regimes. The damping of the second sound mode is especially large since it involves an out-of-phase oscillation of the condensate and non-condensate density fluctuations. Such an out-of-phase oscillation would be expected to have the largest imbalance of the local chemical potentials, and thus the largest rate of transfer of atoms between the condensate and non-condensate. This damping mechanism will arise whenever the equilibration of the condensate and non-condensate is incomplete.

The results found here for the uniform gas may be relevant to the MIT studies of collective excitations in highly anisotropic traps [36], which are believed (on the basis of $C_{22}$ collisions) to be in the transition regime between collisionless dynamics and hydrodynamics. The $m = 0$ quadrupolar oscillations of both the condensate and thermal cloud were observed, as well as an out-of-phase dipole mode. Both the condensate quadrupolar mode and the dipole mode involve the condensate and non-condensate oscillating out-of-phase (see Section VII). According to the above discussion for a uniform gas, this will entail a large chemical potential difference between the two components and we would therefore expect to see a large damping rate associated with the equilibration of chemical potentials when the number of non-condensate atoms is appreciable. Although such large damping rates were indeed observed [36], they cannot be attributed entirely to this source since there will also be contributions to the damping from viscosity and thermal conduction which our present analysis does not include (see the discussion at the end of this Section and in Section VIII). It is nevertheless clear that the relaxation of the local chemical potentials of the two components must be taken into account if one wants to provide a quantitative description of the damping processes in condensed Bose gases at finite temperatures.

In addition to the first and second sound modes discussed above, the ZGN' equations also yield a nonpropagating relaxational mode which, in the $\tau_\mu \to 0$ limit, has a frequency $\omega \simeq -i/\tau_\mu$, consistent with (88). In the opposite limit, $\tau_\mu \to \infty$, collisions between condensate and non-condensate atoms no longer play a role, and the mode becomes the zero-frequency mode of the ZGN description. As discussed at the end of Section IV, this mode is characterized by the fluctuating variables $\delta v_c = \delta v_n = 0$, $\delta n_c = -2\delta \tilde{n}$ and $\delta P = 2g_0\delta \tilde{n}$. This mode persists even into the normal phase, in which case the variables $n_c$ and $v_c$ do
not exist, and the mode is then characterized by amplitudes satisfying \( \delta \tilde{P} = -2g \tilde{n}_0 \delta \tilde{n} \) and \( \delta v_n = 0 \).

In either case, one can show from (66) that the zero-frequency mode is characterized by \( \delta P = 0 \). In other words, even though there is a local change in the total density, this mode has a zero pressure fluctuation, and as a result, the fluid remains in mechanical equilibrium. This is possible since the change in kinetic pressure, \( \delta \tilde{P} \), is exactly balanced by a corresponding change in the mean-field potential. It is important to point out that the same situation also arises in a conventional fluid having some arbitrary equation of state \( P(n,T) \). Variations in the pressure are then given by \( \delta P = (\partial P/\partial n)_T \delta n + (\partial P/\partial T)_n \delta T \), and we see that \( \delta P \) can be zero if \( \delta n/\delta T = -(\partial P/\partial T)_n/(\partial P/\partial n)_T \). Our present model is a particular realization of this possibility, and we find from (72) that the fluctuations in the local temperature and non-condensate chemical potential are related by \( s_0 \delta T = -\tilde{n}_0 \delta \tilde{\mu} \). The zero-frequency mode in the ZGN description therefore has associated with it a static temperature fluctuation. This is consistent from a thermodynamic point of view, but of course is not expected in a more complete description of the fluid dynamics.

Extending the ZGN hydrodynamic equations to include heat conduction will allow the temperature fluctuation to relax in time, and as a result, the zero-frequency ZGN mode will become the usual thermal diffusion mode with a purely imaginary frequency. However, even in the absence of heat conduction, we have found that the zero-frequency mode is converted into a relaxational mode within the ZGN' description. It is therefore clear that the further inclusion of heat conduction within the ZGN' description will simply combine with the collisional relaxation effects due to \( C_{12} \) to produce a hybrid relaxational mode. These two effects will also contribute to the damping of the sound modes. In conventional discussions based on the Landau two-fluid equations, the thermal diffusion mode found above the transition is replaced by damped second sound modes below the transition [4,37]. Since there is one additional dynamical equation in both the ZGN and ZGN' descriptions as compared to the Landau theory, there must necessarily be an additional mode. This is the relaxational mode we have been discussing. However, this additional mode is strongly damped in the \( \omega \tau_\mu \rightarrow 0 \) limit and we then recover the usual results based on the Landau two-fluid equations. A more complete discussion of the combined effects of collisional relaxation and thermal conduction will be given elsewhere.

### VI. CENTER OF MASS (KOHN) MODE

In this section, we discuss the center of mass motion of the coupled condensate and non-condensate systems. For confinement in an arbitrary harmonic trap potential of the form

\[
U_{\text{ext}}(r) = \frac{1}{2} m \left( \omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2 \right),
\]

the center of mass degrees of freedom separate from all other internal degrees of freedom, with the general consequence that there exists a special normal mode in which the equilibrium density profile oscillates rigidly without distortion. In the electron gas literature, this is referred to as the generalized Kohn theorem [38], but the result is independent of statistics.
and applies to any harmonically confined system at any temperature. The experimental observation of this mode provides a convenient way of determining the trap frequencies.

In the context of a Bose-condensed trapped gas, the theorem implies that the mode densities have the form

\[ n_c(r, t) = n_{c0}(r - \eta(t)), \quad (95a) \]

\[ \tilde{n}(r, t) = \tilde{n}_0(r - \eta(t)), \quad (95b) \]

where the center of mass displacement \( \eta(t) \) has a time-dependence given by

\[ \frac{\partial^2 \eta_\alpha(t)}{\partial t^2} = -\omega^2_\alpha \eta_\alpha(t). \quad (96) \]

An approximate theory of the dynamics should, at the very least, be consistent with a mode of this type. This is indeed the case for the zero-temperature dynamics of a Bose condensate based on the nonlinear GP equation [39], including the TF limit [22]. However, the simplest extension to finite temperatures as considered by HZG [2] fails to meet this criterion. In fact, it was the small deviation of the center of mass frequency from the trap frequency as a function of temperature that emphasized the limitation of the static HFP theory. As discussed in Section II, this approximation does not include the collective dynamics of the non-condensate.

We now prove that our set of coupled equations (27)–(29) are consistent with the generalized Kohn theorem. We showed previously [5] that this theorem was satisfied for the linearized ZGN equations in (51) and (52), which are a special case of the more general equations being considered here. However, it is important to demonstrate that this conclusion remains valid even when collisions between the atoms are included. By doing so, we shall be able to clearly see how the effects of interactions, both in the form of mean fields and collisions, cancel out in the determination of the center of mass mode frequency.

We start by considering the condensate density. From (95a), we obtain

\[ \frac{\partial n_c(r, t)}{\partial t} = -\dot{\eta}(t) \cdot \nabla n_{c0}(r - \eta(t)) = -\nabla \cdot (\dot{\eta}(t)n_c(r, t)). \quad (97) \]

Comparing this with (27a), we see that (95a) is a solution of the continuity equation if the velocity field is given by

\[ \mathbf{v}_c(r, t) = \frac{\partial \eta(t)}{\partial t} \equiv \dot{\eta}(t), \quad (98) \]

and provided the source term \( \Gamma_{12}[f] \) is zero. This latter result will be demonstrated shortly, once the form of the distribution function \( f \) corresponding to (95a) is specified. Using (98) in (27b), we have

\[ m\frac{\partial^2 \eta_\alpha(t)}{\partial t^2} = -\frac{\partial \mu_c(r, t)}{\partial x_\alpha}. \quad (99) \]

From the dependence of \( \mu_c(r, t) \) on the densities given in (28), we see that (95) implies
\[
\mu_c(r, t) = \mu_{c0}(r) + \frac{1}{2} \sum_{\alpha} m \omega_{\alpha}^2 x_\alpha \eta_\alpha - \frac{1}{2} \sum_{\alpha} m \omega_{\alpha}^2 \eta_\alpha^2,
\]
(100)
since as indicated by (50), \( \mu_{c0} \) is a constant, independent of position. Inserting (100) into (99) leads immediately to (96). The condensate equations are thus seen to admit a solution consistent with the assumed form of the densities in (95).

A similar analysis of the non-condensate continuity equation (33a) shows that (95b) implies a non-condensate velocity
\[
v_n(r, t) = \frac{\partial \eta(t)}{\partial t},
\]
(101)
again assuming that \( \Gamma_{12}[f] = 0 \). This velocity field and the non-condensate density in (95b) are clearly generated by the distribution function
\[
f_\eta(p, r, t) = \frac{1}{e^{\beta_0 \left[ \frac{1}{2} m (p - \mathbf{v}_c(r, t))^2 + U_0(r, t) - \tilde{\mu}_0 \right]} - 1},
\]
(102)
where \( \beta_0 \) and \( \tilde{\mu}_0 \) are equilibrium parameters. By noting that (95) implies
\[
U_0(r - \eta(t)) = U(r, t) + U_{ext}(r - \eta(t)) - U_{ext}(r),
\]
(103)
we see that (102) is of the local equilibrium form in (37), with the non-condensate chemical potential given by
\[
\tilde{\mu}(r, t) = \tilde{\mu}_0 + U_{ext}(r) - U_{ext}(r - \eta(t)).
\]
(104)
Since \( \tilde{\mu}_0 = \mu_{c0} \) in static equilibrium [see the discussion after (50)], this local chemical potential of the non-condensate is equal to the condensate chemical potential in (100).

We now show that (102) is indeed a solution of our kinetic equation in (29). The \( C_{22} \) collision integral on the RHS of (29) is seen to vanish as an immediate consequence of (102), namely, that \( f_\eta \) is a local equilibrium Bose distribution. In addition, from (40) we see that the \( C_{12} \) collision integral vanishes since \( v_n = v_c \) and \( \tilde{\mu}(r, t) = \mu_c(r, t) \). This, of course, implies that \( \Gamma_{12} = 0 \), as we assumed in the argument leading to (98) and (101). A straightforward calculation of the LHS of the kinetic equation in (29) using \( f_\eta \) in (102) shows that it vanishes if
\[
m \frac{\partial^2 \eta}{\partial t^2} = - \nabla [U(r, t) - U_0(r, t)].
\]
(105)
Using the expression in (103), we see that (105) reduces to the equation of motion in (96).

In summary, we have proven in full generality that the set of dynamical equations in (27)-(29) are consistent with the generalized Kohn theorem for a harmonic trap described by (94). In other words, these equations admit a solution consisting of a rigid in-phase oscillation of the equilibrium condensate and non-condensate density profiles, the center of mass displacement \( \eta(t) \) having the trap frequency \( \omega_\alpha \) for a displacement in the \( \alpha \)-th direction. This result is an important check of the internal consistency of our treatment of the dynamics. We also find that this particular solution is, in fact, independent of collisions and is therefore valid in the collisionless [27] as well as in the collision-dominated regime. Finally, we note that no assumptions have been made regarding the magnitude of the center of mass displacement. These results are therefore valid for arbitrary amplitude of the center of mass motion.
Our purpose in this Section is to develop a practical means of solving hydrodynamic equations of the kind we have derived in this paper. For simplicity, we shall restrict ourselves to the ZGN equations given in (51) and (52), although a similar approach could be developed for the more general ZGN′ equations. From a practical point of view, the two sets of equations give fairly similar results for the frequencies of the first and second sound modes in a uniform system, even though they describe different regimes. We would expect a similar correspondence in the case of a trapped gas. However, it should be noted that the ZGN equations neglect the damping associated with the relaxation of the chemical potentials, which is included in the more general ZGN′ equations of Section V.

A direct solution of the ZGN equations is exceedingly difficult, partly for the number of equations involved and partly for the fact that one is dealing with a strongly inhomogeneous system. For this reason it is advantageous to reformulate the problem in terms of a variational principle. In doing so, accurate estimates of mode frequencies can be obtained by using simple trial functions for the velocity fields whose choice is guided by physical considerations. Our immediate objective is therefore to transcribe the ZGN hydrodynamic equations into a variational form.

Although not essential, it is convenient to replace the velocities in terms of displacement fields. Such a description is particularly useful for small amplitude oscillations about equilibrium, since in this case each fluid element makes small excursions from its equilibrium position and the motion of the fluid is analogous to that of an elastic medium. The displacement field for the non-condensate, \( u(r, t) \), and the condensate, \( w(r, t) \), are defined by the relations

\[
\begin{align*}
v_n(r, t) & = \frac{\partial u(r, t)}{\partial t}, \quad v_c(r, t) = \frac{\partial w(r, t)}{\partial t}.
\end{align*}
\] (106)

Assuming a harmonic time-dependence with frequency \( \omega \), these definitions are equivalent to

\[
\begin{align*}v_n(r) & = -i\omega u(r), \quad v_c(r) = -i\omega w(r),
\end{align*}
\] (107)

and in terms of these variables, the continuity equations take the form

\[
\begin{align*}
\delta n_c(r) & = -\nabla \cdot (n_{c0} w)
\delta \tilde{n}(r) & = -\nabla \cdot (\tilde{n}_0 u).
\end{align*}
\] (108)

Introducing these definitions into (51) and eliminating the variables \( \delta n_c, \delta \tilde{n} \) and \( \delta \tilde{P} \), we obtain

\[
- m\tilde{n}_0 \omega^2 u_i = \frac{\partial}{\partial x_i} \left( u_j \frac{\partial \tilde{P}_0}{\partial x_j} + \frac{5}{3} \tilde{P}_0 \frac{\partial u_j}{\partial x_j} \right) + \frac{\partial U_0}{\partial x_i} \frac{\partial (\tilde{n}_0 u_j)}{\partial x_j} \\
+ 2g\tilde{n}_0 \frac{\partial^2}{\partial x_i \partial x_j} (\tilde{n}_0 u_j + n_{c0} w_j).
\] (109)

This equation can be simplified considerably by collecting together terms involving similar derivatives of \( u_i \), and by making use of the equilibrium condition \( \nabla \tilde{P}_0 + \tilde{n}_0 \nabla U_0 = 0 \). We find
\[ m\tilde{n}_0\omega^2 u_i = -\frac{\partial}{\partial x_i} \left( \lambda \frac{\partial u_j}{\partial x_j} \right) - \frac{\partial}{\partial x_j} \left( \mu \frac{\partial u_j}{\partial x_i} \right) + \tilde{n}_0 u_j \frac{\partial^2}{\partial x_i \partial x_j} (U_{\text{ext}} + 2gn_{c0}) \]
\[ - 2g\tilde{n}_0 \frac{\partial^2}{\partial x_i \partial x_j} (n_{c0} w_j), \] (110)

where
\[ \lambda = \frac{2}{3} \tilde{P}_0 + g\tilde{n}_0^2, \quad \mu = \tilde{P}_0 + g\tilde{n}_0^2. \] (111)

The first two terms on the right hand side of (110) are reminiscent of the divergence of the stress tensor of an anisotropic elastic medium. To make this correspondence more apparent, we introduce the symmetric and antisymmetric strain tensors
\[ u_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \]
\[ \bar{u}_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right). \] (112)

The first two terms on the right hand side of (110) thus become
\[ -\frac{\partial}{\partial x_i} (\lambda u_{jj}) - \frac{\partial}{\partial x_j} (\mu u_{ij}) + \frac{\partial}{\partial x_j} (\mu \bar{u}_{ij}) \]

The part of this expression depending on the symmetric strain tensor will be recognized as representing an elastic medium, with \( \lambda \) and \( \mu \) playing the role of Lamé constants [40].

We are now in a position to define energy functionals for the non-condensate. First, we introduce the kinetic energy functional
\[ K_n[u] = \frac{1}{2} \int d^3r m\tilde{n}_0 u^2. \] (113)

Its variation, \( \delta K_n/\delta u_i = m\tilde{n}_0 u_i \), gives the coefficient of \( \omega^2 \) on the left hand side of (110). Similarly, one can easily check that the variation of the potential energy functional
\[ U_n[u] = \frac{1}{2} \int d^3r \left\{ \lambda u_{ii}^2 + \mu u_{ij}^2 + u_i u_j \tilde{n}_0 \frac{\partial^2}{\partial x_i \partial x_j} (U_{\text{ext}} + 2gn_{c0}) \right\} \] (114)
yields the desired \( u_i \)-dependent terms on the right hand side of (110). The two terms proportional to \( \mu \) can be written in several different ways:
\[ \mu(u_{ij}^2 - \bar{u}_{ij}^2) = \mu u_{ij}^2 - \frac{1}{2} \mu (\nabla \times u) \cdot (\nabla \times u) \]
\[ = \mu \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \]
\[ = 2\mu u_{ij}^2 - \mu \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}. \] (115)

The first equality in particular shows that the antisymmetric strain tensor is associated with the solenoidal part of \( u_i \).
The last term in (110) accounts for the force exerted on the non-condensate by fluctuations of the condensate. It can be represented by the interaction energy functional

\[ U_{cn}[u, w] = 2g \int d^3r \frac{\partial(\bar{n}_0u_i)}{\partial x_i} \frac{\partial(n_0w_j)}{\partial x_j}. \]  

(116)

We have here indicated explicitly that \( U_{cn} \) depends on both of the displacement fields.

We next consider the condensate. With the elimination of \( \delta n_c \) and \( \delta \bar{n} \) from Eqs. (52), we arrive at the equation

\[ mn_0\omega^2 w_i = -n_c \frac{\partial}{\partial x_i} \hat{K} \frac{\partial}{\partial x_j} (n_0w_j) - gn_c \frac{\partial^2}{\partial x_i \partial x_j} (n_0w_j) - 2gn_c \frac{\partial^2}{\partial x_i \partial x_j} (\bar{n}_0u_j). \]  

(117)

Here we have defined the operator

\[ \hat{K} \equiv \frac{1}{2\Phi_0(r)} \hat{L}_0(r) \frac{1}{\Phi_0(r)}, \]  

(118)

with \( \hat{L}_0(r) \) as given in (54). We emphasize that (117) is equivalent to the linearized form of the time-dependent GP equation in (4) (of course with the last two terms excluded). In particular, we do not make the TF approximation which neglects the quantum mechanical kinetic energy.

The energy functionals for the condensate analogous to those of the non-condensate are

\[ K_c[w] = \frac{1}{2} \int d^3r mn_0w^2 \]  

(119)

and

\[ U_c[w] = \frac{1}{2} \int d^3r \frac{\partial(n_0w_i)}{\partial x_i} [\hat{K} + g] \frac{\partial(n_0w_j)}{\partial x_j}. \]  

(120)

The last term in (117) follows from the variation of (116) with respect to \( w_i \).

We now define the functional

\[ J[u, w] \equiv \frac{U[u, w]}{K[u, w]}, \]  

(121)

where

\[ U[u, w] = U_n[u] + U_c[w] + U_{cn}[u, w] \]

\[ K[u, w] = K_n[u] + K_c[w]. \]  

(122)

The requirement that the functional \( J \) be stationary with respect to variations of \( u_i \) and \( w_i \) leads to the equations

\[ \frac{\delta U}{\delta u_i} - \omega^2 \frac{\delta K}{\delta u_i} = 0 \]

\[ \frac{\delta U}{\delta w_i} - \omega^2 \frac{\delta K}{\delta w_i} = 0, \]  

(123)
where we have identified $\omega^2$ with the stationary value of the functional $J$. These equations are identical to (110) and (117), which confirms that variation of the functional $J$ leads to the desired hydrodynamic equations. More importantly, the functional $J$ offers a convenient variational method of estimating the normal mode frequencies of a trapped Bose gas.

One useful approach is the Rayleigh-Ritz method, in which the velocity components are expanded in some complete set of functions. This method leads to expressions for the kinetic and potential energy functionals which are quadratic forms in the expansion coefficients. Variation of $J$ with respect to these coefficients results in a set of linear equations which schematically have the form

$$\sum_\beta \left[ U_{\alpha\beta} - \omega^2 K_{\alpha\beta} \right] c_\beta = 0,$$

where $U_{\alpha\beta}$ and $K_{\alpha\beta}$ are matrix elements of the energy functionals. A simplified version of this method will be used to determined several of the normal modes of interest.

We now establish some general properties of the solutions to Eqs. (110) and (117). For this purpose, it is convenient to define the six-component displacement vector

$$\eta = \begin{pmatrix} u \\ w \end{pmatrix}.$$  

Eqs. (110) and (117) can then be combined into one matrix equation

$$L\eta = \omega^2 D\eta,$$

where the matrix $L$ has the block structure

$$L = \begin{pmatrix} \hat{L}_{11} & \hat{L}_{12} \\ \hat{L}_{21} & \hat{L}_{22} \end{pmatrix}.$$  

The elements of this matrix are three-by-three matrix operators defined as

$$(\hat{L}_{11} u)_i = -\frac{\partial}{\partial x_i} \left( \lambda \frac{\partial u_j}{\partial x_j} \right) - \frac{\partial}{\partial x_j} \left( \mu \frac{\partial u_i}{\partial x_i} \right) + \tilde{n}_0 u_j \frac{\partial^2}{\partial x_i \partial x_j} (U_{\text{ext}} + 2 gn_{c0})$$

$$(\hat{L}_{12} w)_i = -2gn\tilde{n}_0 \frac{\partial^2}{\partial x_i \partial x_j} (n_{c0} w_j)$$

$$(\hat{L}_{21} u)_i = -2gn_{c0} \frac{\partial^2}{\partial x_i \partial x_j} (\tilde{n}_0 u_j)$$

$$(\hat{L}_{22} w)_i = -n_{c0} \frac{\partial}{\partial x_i} (\hat{K} + g) \frac{\partial}{\partial x_j} (n_{c0} w_j).$$

Similarly, the matrix $D$ is block-diagonal ($\hat{D}_{12} = \hat{D}_{21} = 0$) with elements

$$\hat{D}_{11} = m\tilde{n}_0 \hat{1}$$

$$\hat{D}_{22} = mn_{c0} \hat{1}.$$  

Here, $\hat{1}$ is a three-by-three unit matrix. From the definition of $L$, it can easily be checked that
\[ \int d^3r \, \zeta^* \cdot (L\eta) = \int d^3r \, (L\zeta)^* \cdot \eta. \] (130)

which shows that \( L \) is a Hermitian operator. This property can be used in (126) to show that the eigenvalues \( \omega^2 \) of \( L \) are real and that the corresponding eigenvectors \( \eta_\alpha \) satisfy the orthonormality relation

\[ \int d^3r \, \eta_\alpha^* \cdot (D\eta_\beta) = \delta_{\alpha\beta}. \] (131)

**A. In-phase and out-of-phase dipole oscillations and the center of mass mode**

As our first application of the above formalism, we consider the center of mass motion of a gas trapped in a parabolic potential given by (94). According to our earlier discussion in Section VI, we expect a solution whereby the total density oscillates harmonically without distortion. One can easily show that a uniform displacement \( u_i = w_i = A \) of both components along one of the Cartesian directions is a solution of (110) and (117), with frequency \( \omega = \omega_i, \ i = x, y, z \). According to (108), this implies a density fluctuation \( \delta n_c(r) = -A \partial n_0 \partial x_i \) for the condensate and \( \delta \tilde{n}(r) = -A \partial \tilde{n}_0 \partial x_i \) for the non-condensate. These forms follow from (95) on expanding in the displacement \( \eta \).

This center of mass mode has the condensate and non-condensate oscillating in phase with the same amplitude. One would also expect a dipole mode in which the two components oscillate out of phase. In this case, there is no reason to expect the mode densities to be given by the equilibrium profiles as for the center of mass mode. However, in obtaining a variational estimate of the mode frequencies which is accurate to second order in the error of the mode density, it is reasonable to assume constant displacements for the two components, but in general with different amplitudes. Thus we take \( u_i = A_n \) and \( w_i = A_c \) for a particular direction \( i \). For these displacements, the equations in (123) become

\[
\begin{align*}
\frac{\partial U}{\partial A_c} - \omega^2 \frac{\partial K}{\partial A_c} &= 0, \\
\frac{\partial U}{\partial A_n} - \omega^2 \frac{\partial K}{\partial A_n} &= 0.
\end{align*}
\] (132)

Using the constant displacements to evaluate the energy functionals, we obtain

\[
K = \frac{1}{2} M_c A_c^2 + \frac{1}{2} M_n A_n^2,
\]

\[
U = \frac{1}{2} M_c \omega_i^2 A_c^2 + \frac{1}{2} M_n \omega_i^2 A_n^2 - \frac{1}{2} k_i (A_c - A_n)^2,
\] (133)

where \( M_c = mN_c, \ M_n = m\tilde{N} \) and

\[
k_i \equiv -2g \int d^3r \left( \frac{\partial n_0}{\partial x_i} \right) \left( \frac{\partial \tilde{n}_0}{\partial x_i} \right).
\] (134)
(Here, the repeated index is not summed.) This energy corresponds to two one-dimensional oscillators confined in an external parabolic potential of frequency $\omega_i$ and coupled together by a spring of force constant $k_i$.

Substituting (133) into (132), we obtain the pair of coupled equations

$$\begin{pmatrix}
\omega_i^2 + \omega_1^2 - \omega^2 \\
-\omega_2^2 \\
-\omega_2^2 + \omega_1^2 - \omega^2
\end{pmatrix}
\begin{pmatrix}
A_c \\
A_n
\end{pmatrix}
= 0 \tag{135}$$

where $\omega_1^2 = k_i/M_c$ and $\omega_2^2 = k_i/M_n$. One solution of (135) is $\omega = \omega_i$ with $A_c = A_n$, which is the center of mass mode discussed more generally in the previous section. However, the present analysis gives another solution

$$\omega^2 = \omega_i^2 + \omega_1^2 + \omega_2^2 = \omega_i^2 - 2gM_c M_n/M_c M_n \int d^3r \left( \frac{\partial n_0}{\partial x_i} \right) \left( \frac{\partial \tilde{n}_0}{\partial x_i} \right), \tag{136}$$

with the corresponding amplitudes satisfying $M_c A_c + M_n A_n = 0$. This implies that this mode is essentially an \textit{out-of-phase} oscillation of the condensate and non-condensate and that the center of mass remains stationary. To the extent that the displacement fields for this mode are indeed uniform, one finds that $N_c/\tilde{N} = -A_n/A_c$ and hence a measurement of the relative amplitudes would provide a direct estimate of the relative number of atoms in the condensate and non-condensate.

These results for the in-phase and out-of-phase oscillations of the condensate and non-condensate were previously reported in ZGN [5]. The first observation of such an out-of-phase dipole oscillation was made by the MIT group [36] for an anisotropic cigar-shaped trap. They find that this mode has a frequency slightly below the axial frequency of the trap, and is strongly damped at higher temperatures. Our prediction would give a frequency slightly \textit{above} the trap frequency since the slopes of the condensate and non-condensate tend to have opposite signs (see Fig. 3) in the region of space contributing to the integral in (136). It is possible that the damping of the mode not included in the ZGN equations would have to be taken into account in order to resolve this discrepancy.

### B. Monopole and Quadrupole Modes in Isotropic Traps

We next consider some of the other low-lying modes of the trapped gas. For simplicity, we shall work with an isotropic trap with a common oscillator frequency $\omega_0$ in all directions. The particular modes we shall consider here are the monopole and quadrupole modes, which respectively have $l = 0$ and $l = 2$ angular character. The lowest monopole mode is expected to be a breathing mode whose mode density would have the approximate form $\lambda^{-3}n_0(\mathbf{r}/\lambda)$ where $\lambda$ is a time-dependent, but spatially independent, scaling parameter. The equilibrium situation corresponds to $\lambda = 1$ and small deviations from this value imply a displacement field of the form $\mathbf{u} \propto \mathbf{r}$. The quadrupole mode can be viewed as an anisotropic scaling solution with $l = 2$ symmetry. This requires a displacement field with components $(x,y, -2z)$. Both the $l = 0$ and $l = 2$ modes can be represented by displacement fields of the kind

$$\mathbf{u} = (ax, ay, bz) \tag{137a}$$

$$\mathbf{w} = (cx, cy, dz), \tag{137b}$$
where \(a, b, c\) and \(d\) are constant variational parameters. For the monopole mode, we have \(a = b\) and \(c = d\), while for the quadrupole mode, \(b = -2a\) and \(d = -2c\). The various terms required in the evaluation of the energy functional \(J\) for these modes are given in Appendix C, where the expressions are reduced to the form of radial integrals over equilibrium quantities.

One advantage of a variational approach is that the calculated mode frequencies can be improved systematically by introducing trial displacements which are more general than those given, for example, by (137). In the case of the monopole modes, displacement fields of the form \(u = r \sum_{\nu=0}^{n} a_{\nu} r^{2\nu}\) were examined [41], where the number of terms \(n\) in the expansion was varied from 1 to 5. The expansion coefficients \(a_{\nu}\) (together with a similar set for the condensate) play the role of the variational parameters, and the functional \(J\) can be minimized with respect to this set. The total number of collective modes generated in this way is \(2n\), with the lowest pair corresponding to the modes found using (137). One can examine the convergence of the mode frequencies with \(n\) and one finds that the results for the lowest monopole modes are already quite good with \(n = 1\). We therefore have confidence in the accuracy of the present results obtained using the simplest variational form given in (137).

In Fig. 4, we present our results for the lowest \(l = 0\) and \(l = 2\) modes in an isotropic trap with radial frequency \(\nu_0 = 200\) Hz. As for the case of the dipole modes discussed earlier, the restricted form of the variational displacement fields gives rise to two modes for each symmetry. The calculations were performed for 5000 \(^{87}\text{Rb}\) atoms having an s-wave scattering length of \(a = 110\ a_B\). In the lower part of Fig. 4 we show the condensate fraction as a function of temperature. The transition temperature is found to be \(T_{\text{BEC}} \approx 149\) nK. In Fig. 3, we show the condensate and non-condensate radial densities at \(T = 100\) nK, where the number of condensate and non-condensate atoms is approximately equal. The depletion in the non-condensate density at the centre of the trap due to the repulsive interaction with the condensate in the overlap region is clearly evident. These results are very similar to those obtained within the static HFP approximation [4], and confirmed by other calculations [26].

In the upper part of Fig. 4 we show the mode frequencies as a function of temperature. The results below \(T_{\text{BEC}}\) can be viewed in a first approximation as a superposition of the condensate modes given by the HFP scheme [4] and the non-condensate modes found above \(T_{\text{BEC}}\) [12], extended to lower temperatures. At \(T = 0\), the condensate modes start at \(\omega \approx 2.25\omega_0\) for \(l = 0\) and \(\omega \approx 1.5\omega_0\) for \(l = 2\), which are close to the TF limits (\(\sqrt{5\omega_0}\) and \(\sqrt{2\omega_0}\), respectively [22]). Apart from some hybridization effects to be discussed, these two modes basically follow the HFP behaviour with increasing temperature, merging together as \(T\) approaches \(T_{\text{BEC}}\). The two non-condensate modes are analogous to the \(l = 0\) mode at \(\omega = 2\omega_0\) and the \(l = 2\) mode at \(\omega = \sqrt{2}\omega_0\), as found by Griffin et al. [12] above \(T_{\text{BEC}}\). However, unlike this earlier work, the present results include the effect of interactions, and even above \(T_{\text{BEC}}\), the mode frequencies are shifted slightly from their non-interacting values, although it turns out that the \(l = 2\) mode frequency (\(^2\) remains at \(\sqrt{2}\omega_0\) within the present variational approximation. For temperatures below \(T_{\text{BEC}}\), the non-condensate density fluctuations couple with the condensate fluctuations and these modes show an increasing frequency shift with decreasing \(T\). In addition, hybridization of the modes is observed at points were the uncoupled modes cross. For example, the \(\omega = 2\omega_0\) mode is seen in Fig. 4 to hybridize with the condensate mode just below \(T_{\text{BEC}}\), and once again at lower temperatures.

This hybridization is most clearly revealed by looking at the amplitudes of the condensate
and non-condensate fluctuations associated with each of the modes. In Fig. 5 we show the amplitudes for the two $l = 0$ modes, where the upper $l = 0$ mode is labelled $0^+$ and the lower, $0^-$. Starting at low temperatures, the $0^+$ mode is essentially a non-condensate oscillation ($|A_{n+}| \approx 1$) with a very small condensate component. After passing through the hybridization point near $T \approx 35$ nK, the upper $l = 0$ mode changes over to a condensate oscillation, with a small out-of-phase non-condensate amplitude. At the second hybridization point near 136 nK, the amplitude of the non-condensate grows dramatically as the mode switches over to the $2\omega_0$ non-condensate mode. In the temperature range between this hybridization point and $T_{BEC} \approx 149$ nK, the condensate amplitude is still appreciable; however, the condensate has a minimal effect on the non-condensate oscillation since the condensate fraction is quite small here. One can simply view the condensate oscillation as being driven by the much more massive non-condensate component. The amplitudes for the lower frequency $l = 0$ mode ($0^-$) are shown in the lower part of Fig. 5. Above the $T \approx 136$ nK hybridization point, this mode is seen to be essentially a condensate oscillation ($|A_{c-}| \approx 1$), with a very small non-condensate amplitude. A mode of this type is to be expected since the condensate is relatively massive and the condensate simply oscillates in the presence of a static non-condensate density. In this respect, this mode is clearly equivalent to the mode found in the HZG approximation \[2\], which ignored the collective dynamics of the non-condensate completely.

The behaviour of the $l = 2$ amplitudes shown in Fig. 6 follows a similar pattern. The high frequency $l = 2$ mode ($2^+$) starts off as a non-condensate oscillation at low $T$ and then switches over to a condensate oscillation all the way up to $T_{BEC}$, since in this case there is no second hybridization point. The lower $l = 2$ mode ($2^-$) is mainly a non-condensate oscillation for all temperatures, although a small condensate amplitude is evident as well. Interestingly, the condensate amplitude goes to zero at $T \approx 125$ nK, which implies that at this temperature, the condensate is effectively stationary in the presence of the oscillating non-condensate. A similar zero-crossing can be seen in Fig. 5 in the case of the lower $l = 0$ mode. This behaviour can perhaps be understood most easily in the case of the breathing mode, which corresponds to a dilation of the density. By referring to the non-condensate density in Fig. 3, one can see that the corresponding density fluctuation, $\delta n \propto 3\tilde{n}_0 + r(\partial \tilde{n}_0 / \partial r)$, will be both positive and negative in different regions of space. Thus the zero-amplitude condensate oscillation arises when the radial force on the condensate due to the breathing non-condensate averages to zero. Although the situation for the $l = 2$ mode is not as easy to interpret, presumably the force on the condensate is being averaged to zero in a similar way. We conclude that a simultaneous measurement of the condensate and non-condensate amplitudes as a function of temperature should reveal a particular temperature at which the condensate amplitude passes through a minimum. This would be an especially interesting experimental signature of the modes we have been discussing.

We have checked that the results shown here for 5000 atoms are in fact qualitatively similar to those for other values of $N$, from a lower limit of 2000 to an upper limit of 20,000 atoms. There is no reason why this qualitative behaviour should not persist to arbitrarily large $N$, where a hydrodynamic description would be expected to be valid. Thus the numerical calculations we have performed can be viewed as representative of the modes in the hydrodynamic regime.

We are not claiming that a trapped gas containing 5000 atoms, approximately the number
of atoms in the experiments of Jin et al. [43], is actually in the hydrodynamic regime. The crossover into this regime occurs at $\omega \tau_c \approx 1$, where $\tau_c$ is the mean time between collisions. If one uses the classical gas expression $\tau_c^{-1} = \sigma \bar{n} \bar{v}$ (where $\sigma = 8 \pi a^2$ is the s-wave cross-section, $\bar{n}$ is the maximum non-condensate density, and $\bar{v}$ is a characteristic thermal velocity), one obtains for the JILA experiment [43] the estimate $\omega \tau_c \approx 4$, which places this experiment in the collisionless regime. However, this estimate based on the classical collision time is not entirely secure. Our recent work [12] has shown that collision times defined by the $C_{12}$ and $C_{22}$ collision integrals can in fact be much smaller than $\tau_c$ in the region around $T_{\text{BEC}}$. Thus there is the possibility that the JILA data near $T_{\text{BEC}}$ is actually closer to the hydrodynamic regime than would be expected on the basis of the classical gas expression for $\tau_c$. In any case, it would be very useful to have further experimental studies of collective modes at finite temperatures in traps containing a much larger number of atoms. Performing such experiments in isotropic traps would further facilitate a comparison with our theoretical predictions, not only with regard to mode frequencies, but also with respect to the relative amplitude of the condensate and non-condensate oscillations.

Finally, we comment on the results of Shenoy and Ho [35], which were obtained using the Landau two-fluid hydrodynamic equations. Above $T_{\text{BEC}}$, they find hydrodynamic modes in agreement with our results, namely, an $l = 0$ mode at $2\omega_0$, the center of mass mode at $\omega_0$ and an $l = 2$ mode at $\sqrt{2}\omega_0$. Below $T_{\text{BEC}}$, these modes continue as non-condensate-like modes (i.e., the non-condensate has the larger amplitude as indicated in Figs. 5 and 6) and their results are again in agreement with ours. However, there are large differences in the condensate-like oscillations which, in both approaches, are essentially out-of-phase oscillations of the two components. Referring to Fig. 4, we see that our $l = 0$ and $l = 2$ condensate modes are essentially the modes obtained [4] from the $T = 0$ time-dependent GP equation, but with the condensate number given by the finite-temperature value $N_c(T)$. These modes then become degenerate as $T$ approaches $T_{\text{BEC}}$, with a common value close to $2\omega_0$. In the work of Shenoy and Ho, however, the lowest $l = 0$ condensate mode frequency falls steadily from $2\omega_0$ at $T_{\text{BEC}}$ with decreasing temperature and gives no indication of approaching the expected GP frequency of about $2.25\omega_0$ at low temperatures. Their lowest $l = 2$ condensate mode starts out at $\sqrt{2}\omega_0$ at $T_{\text{BEC}}$ and stays close to this value with decreasing temperature.

As a partial explanation of these differences, we recall that our results are based on the ZGN equations which assume only partial local equilibrium and therefore are not in the Landau regime studied by Shenoy and Ho. However, as shown in Fig. 1, the results for the first and second sound mode frequencies of a uniform gas are in fact very similar in the two regimes, which would lead one to expect the differences also to be small in the case of a trapped Bose gas. Perhaps the major difference can be accounted for by the different equilibrium properties assumed in the two calculations. As discussed in Appendix B, Shenoy and Ho based their calculations on the Lee-Yang equation of state [44], which is only correct to first order in the interactions and, moreover, leads to a uniform non-condensate density in the region of overlap with the condensate. In contrast, our calculations make use of condensate and non-condensate densities determined self-consistently as shown in Fig. 3. It would be useful to recalculate the hydrodynamic modes in the Landau limit with these equilibrium densities.
VIII. CONCLUSIONS

In this paper, we have derived a set of equations which describe the dynamics of a trapped Bose-condensed gas at finite temperatures. These equations consist of a generalized Gross-Pitaevskii equation for the condensate order parameter $\Phi(r,t)$ and a semiclassical kinetic equation for the excited atom (non-condensate) distribution function $f(p,r,t)$. By limiting ourselves to higher temperatures, we arrive at a simple and intuitive picture in which the excited atoms behave as particles moving in a dynamic Hartree-Fock field. Collisions between all the atoms are included and in particular, allow for the transfer of atoms between the two components.

Most of our discussion is devoted to the hydrodynamic regime and the derivation of a closed set of generalized hydrodynamic equations for the two components. These equations are based on the assumption that collisions between excited atoms are sufficiently rapid to drive the distribution function $f(p,r,t)$ toward the local equilibrium Bose-Einstein distribution $\tilde{f}(p,r,t)$, with the consequence that the $C_{22}[f]$ collision integral for the excited atoms vanishes. However, the $C_{12}[\tilde{f}]$ collision integral describing collisions between the condensate and non-condensate atoms remains finite, and as a result, our equations can be used to describe the situation in which the condensate is not in diffusive local equilibrium with the non-condensate. As noted in Section I, this situation was discussed many years ago in the context of superfluid helium near the $\lambda$ point, but using a more phenomenological approach [10,11]. Previous microscopic studies [14,13] of the hydrodynamics of Bose superfluids have generally ignored this possibility and simply assumed that the superfluid and normal fluid were in local equilibrium with each other.

We find that the equilibration process associated with the transfer of atoms between the two components leads to the existence of a new relaxational mode characterized by the relaxation time $\tau_\mu$. If we consider the dynamics of the system on time scales set by the frequency $\omega$, we find that the linearized form of our theory is equivalent to the Landau two-fluid equations in the limit $\omega \tau_\mu \ll 1$. In the opposite limit, $\omega \tau_\mu \gg 1$, the $C_{12}[f]$ collisions can be neglected and we recover the ZGN limit recently considered by the authors [5]. The transition between these two limits can be analyzed straightforwardly for the case of a uniform Bose gas. As shown in Section V, we find that the first and second sound modes are coupled to the new relaxational mode and that this coupling gives rise to anomalous sound absorption near $\omega \tau_\mu \approx 1$. Of course, a complete theory of sound absorption must also take into account other dissipative mechanisms such as viscosity and thermal conduction [30]. We briefly discussed the qualitative effect of a finite thermal conductivity at the end of Section V.

One outcome of our study has been the realization that the various collision and relaxation times which appear are strongly temperature dependent. In Ref. [12], these were evaluated for a uniform Bose gas, where it was found that $\tau_{12}$ and $\tau_{22}$ become very small as $T \to T_{BEC}$, but that $\tau_\mu$ becomes very long. It is therefore possible to have a situation in which $\omega \tau_{22} \ll 1$, as required by our local equilibrium assumption, but with $\omega \tau_\mu \gg 1$. These conditions will always occur close enough to $T_{BEC}$ and define the regime in which the ZGN equations can be used. The Landau regime will only be valid at intermediate temperatures where $\tau_\mu \simeq \tau_{22}$ and $\omega \tau_\mu \ll 1$. In a separate paper, we shall give a detailed discussion of how these relaxation times depend on position within a trapped Bose gas. Since the the edge
of the condensate is effectively in the critical regime, we anticipate anomalous collision and relaxation times in this region.

In Section VII, we developed a powerful variational approach to solve our generalized hydrodynamic equations. Although we restricted our discussion to the collective modes in isotropic traps, the method can also be applied to anisotropic traps, where a direct solution of the hydrodynamic equations (as in Ref. [35]) would be extremely difficult. Using this variational approach to obtain solutions of the ZGN equations, we found significant differences in some of the low-lying hydrodynamic oscillations from the results of Shenoy and Ho [35] based on the Landau two-fluid equations. At present, we believe these differences arise largely from the use in Ref. [35] of the TF approximation in conjunction with the Yang-Lee [44] free energy approximation for the equilibrium thermodynamic properties (see Appendix B). To confirm this, however, will require detailed calculations based on the full ZGN equations. Such calculations will have the added benefit of allowing us to explore the full range of behaviour from the ZGN limit to the Landau two-fluid regime.

There are other possible generalizations and extensions of the work presented in this paper. We briefly discuss some of these, partly for the purpose of pointing out some of the limitations of the present work. Perhaps the most obvious limitation of our simple model is the use of the Hartree-Fock approximation, as given in (24), to describe the dynamics of the thermally excited atoms. As noted at the end of Section III, this corresponds to thermal excitations having an energy $E_p(r,t) = p^2/2m + gn_c(r,t)$, which is the correct high momentum limit of the Popov approximation for the excitation spectrum [21]. This emphasizes the fact that our kinetic equation (29) can only be used to describe low energy collective modes at finite temperatures, where (24) gives a good description of the dominant elementary excitations determining the thermodynamic properties.

In the region of very low temperatures, the HF spectrum (24) we have used in this paper is clearly inadequate. A first step in improving our treatment would be to use the full Popov approximation for the excitation spectrum. In this approximation, the excitations are identical to the Bogoliubov spectrum, with the exception that the condensate density is now temperature dependent. Indeed, KD [13] have used this Popov spectrum in the case of a uniform gas to derive a kinetic equation valid at very low temperatures. The main difference this leads to in the $C_{12}$ collision integral is the appearance of Bose coherence factors which result from the transformation from a description in terms of excited atoms to one in terms of Popov-Bogoliubov quasiparticles.

In this connection, we note that generalized kinetic equations can be derived using the well-known Kadanoff-Baym method [24], which starts from a specific self-energy approximation for non-equilibrium single-particle Green’s functions. The full, self-consistent Hartree-Fock-Bogoliubov (HFB) approximation for the thermal excitations includes the off-diagonal self-energy of the non-condensate ($\tilde{m}$) neglected in the Popov approximation [21] used here. (See Section 6 of Ref. [46] for a recent attempt to classify the various approximations used in the treatment of static and dynamic properties of Bose-condensed gases.) As is well-known in the Bose gas literature [15,21], the full first order HFB does not give a consistent “gapless” approximation of the thermal excitations, since the off-diagonal self-energies are second order in the interaction $g$. Nevertheless, the kinetic equation based on the HFB spectrum does allow one to discuss collective modes which are consistent with conservation laws. The HFB approximation was recently used to derive [17] coupled kinetic equations
for a Bose-condensed gas in the collisionless regime. Work to include collisions within this scheme is currently in progress.

Turning to another basic limitation of our present analysis, we recall that the derivation of the ZGN' hydrodynamic equations depended on the assumption that the excited atoms were in local equilibrium with each other, i.e., \( f(p, r, t) \approx \tilde{f}(p, r, t) \), where by definition \( C_{22}[\tilde{f}] = 0 \). The next step would be to look at the effect of small deviations from this local Bose distribution, \( f \approx \tilde{f} + \delta f \). This procedure parallels the Chapman-Enskog treatment of transport processes in classical gases. It was implemented recently for trapped Bose gases [30] to generate corrections to the ZGN hydrodynamic equations due to shear viscosity and thermal conductivity. This gives explicit expressions for hydrodynamic damping associated with the Uehling-Uhlenbeck collision integral \( C_{22} \) in (23a). Similar corrections to the ZGN' equations will be discussed elsewhere.

Finally, it should be noted that the situation in a trapped Bose gas is quite different from that in a uniform gas. Due to the decreasing density in the tail of a trapped thermal cloud, the assumption of local equilibrium enforced by collisions must always break down. This leads to the interesting result that the damping of certain collective modes above \( T_{BEC} \) is governed by the low-density tail of the thermal cloud. It was shown in recent work [31,30] that a good estimate of this damping could be obtained by introducing a spatial cutoff in the hydrodynamic equations. Although this type of calculation can also be extended to below \( T_{BEC} \), it is clear that a more rigorous solution of the kinetic equation in (29) is needed in order to deal with the kinetics in the low-density tail of a trapped Bose gas.

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APPENDIX A: DERIVATION OF COLLISION INTEGRALS

In this Appendix, we give a more detailed derivation of the expressions for the collision integrals given by Eqs. (23a) and (23b), as well as the three-field correlation function given by (25). We closely follow the approach of Kirkpatrick and Dorfman [13] in their calculation of $C_{12}$ and $C_{22}$, apart from the fact that we work in the lab frame as opposed to the local rest frame. In order to obtain a closed kinetic equation for the distribution function $f(p, r, t)$, we must express higher order correlation functions in terms of $f(p, r, t)$. For this purpose, we treat $H'(t)$ as a perturbation to the zeroth-order Hamiltonian $\hat{H}_0(t)$, as defined in (14). We shall effectively calculate collision integrals to second order in $g$, while explicitly keeping interaction effects in excitation energies and chemical potentials only to first order in $g$.

The formal solution to (10) can be written as (with $\hbar = 1$)

$$\tilde{\rho}(t, t_0) = \hat{S}_0(t, t_0)\tilde{\rho}(t_0)\hat{S}_0^\dagger(t_0, t_0) - i \int_{t_0}^{t} dt' \hat{S}_0(t, t')[\hat{H}'(t'), \tilde{\rho}(t', t_0)]\hat{S}_0^\dagger(t, t'),$$

(A1)

where the unperturbed evolution operator $S_0(t, t_0)$ satisfies the equation of motion

$$i\frac{dS_0(t, t_0)}{dt} = \hat{H}_0(t)\hat{S}_0(t, t_0).$$

(A2)

The solution of this equation is

$$\hat{S}_0(t, t_0) \equiv T \exp \left[ -i \int_{t_0}^{t} dt' \hat{H}_0(t') \right]$$

(A3)

where $T$ is the time-ordering operator. Iterating (A1) to first order in $\hat{H}'$, one has

$$\tilde{\rho}(t, t_0) \simeq \hat{S}_0(t, t_0)\tilde{\rho}(t_0)\hat{S}_0^\dagger(t_0, t_0) - i \int_{t_0}^{t} dt' \hat{S}_0(t, t')[\hat{H}'(t'), \hat{S}_0(t', t_0)\tilde{\rho}(t_0)\hat{S}_0^\dagger(t', t_0)]\hat{S}_0^\dagger(t', t').$$

(A4)

Using this in (15), the expectation value of an arbitrary operator $\hat{O}(t)$ can be expressed as

$$\langle \hat{O} \rangle_t = \langle \hat{O}(t) \rangle = \text{Tr}\tilde{\rho}(t_0) \left\{ \hat{S}_0^\dagger(t, t_0)\hat{O}(t_0)\hat{S}_0(t, t_0) - i \int_{t_0}^{t} dt' \hat{S}_0^\dagger(t', t_0)[\hat{S}_0^\dagger(t', t')\hat{O}(t_0)\hat{S}_0(t, t'), \hat{H}'(t')\hat{S}_0^\dagger(t', t_0)]\hat{S}_0(t', t_0) \right\}.$$  

(A5)

We first consider the three-field correlation function $\langle \tilde{\psi}^\dagger \tilde{\psi} \tilde{\psi} \rangle$, which appears in (23a). Referring to (A3), we can see that there are two ways in which such an anomalous average can arise. The first term on the RHS of (A5) gives the initial correlation, while the term proportional to $\hat{H}'$ defines those correlations (to lowest order in $g$) which build up in time as a result of the Bose broken symmetry. It is the latter which we identify with collision processes. Thus, for the purpose of evaluating higher order correlation functions, we assume that the initial density matrix is such that the initial correlations vanish. This is the basic assumption we make in the following (13). Since the evolution according to $\hat{S}_0(t, t_0)$ is number conserving, this assumption implies that in evaluating various correlation functions we need retain only those terms which preserve the total number of particles. Thus, to lowest order in the interaction $g$, the three-field correlation function is given by
and using (A7) and (A8), one contribution to the commutator in (A6) is

\[
\langle \tilde{\psi}^\dagger(r, t) \tilde{\psi}(r, t) \tilde{\psi}(r, t) \rangle
\]

\[
= -i \text{Tr} \hat{\rho}(t_0) \int_{t_0}^{t} dt' \hat{S}_0^\dagger(t', t_0) \left[ \hat{S}_0^\dagger(t, t') \tilde{\psi}^\dagger(r, t_0) \tilde{\psi}(r, t_0) \hat{S}_0(t, t') + \hat{H}'_1(t') + \hat{H}'_3(t') \right] \times \hat{S}_0(t', t_0), \tag{A6}
\]

where it is understood that only those parts of \( \hat{H}'_1 \) and \( \hat{H}'_3 \) are kept which yield an equal number of creation and annihilation operators.

Let us first consider the contribution coming from \( \hat{H}'_1(t') \), which is defined in (14). The evaluation of (A3) and similar correlation functions is facilitated by two key assumptions (i) that the effect of \( \hat{H}'_1(t') \) in the interval \( t_0 \leq t' \leq t \) is essentially a collision process, which occurs on a time scale much shorter than all other time scales in the problem, and (ii) that the hydrodynamic variables vary slowly in space and time. One therefore expects the dominant contribution from the commutator of field operators \( \tilde{\psi}(r, t) \) and terms contained in the effective Hamiltonian to come from values of \( r' \) and \( t' \) close to \( r \) and \( t \). In this situation, we can expand the quantities \( n_c, \tilde{n}, \theta \) and \( U \) that appear in \( \hat{H}_{\text{eff}} \) about this point. It is sufficient to use

\[
n_c(r', t') \simeq n_c(r, t), \quad \tilde{n}(r', t') \simeq \tilde{n}(r, t), \quad U(r', t') \simeq U(r, t),
\]

\[
\theta(r', t') \simeq \theta(r, t) + \frac{\partial \theta}{\partial t}(t' - t) + \nabla \theta \cdot (r' - r)
\]

\[
= \theta(r, t) - \varepsilon_c(r, t)(t' - t) + m \mathbf{v}_c(r, t) \cdot (r' - r), \tag{A7}
\]

and

\[
\hat{S}_0(t, t') \simeq e^{-i \hat{H}_0(t)(t-t')}.
\]

Introducing the Fourier transform of the non-condensate field operators according to \( V \) is the volume of the system

\[
\tilde{\psi}(r, t_0) = \frac{1}{\sqrt{V}} \sum_p a_p e^{i p \cdot r}, \quad \tilde{\psi}^\dagger(r, t_0) = \frac{1}{\sqrt{V}} \sum_p a_p^\dagger e^{-i p \cdot r}, \tag{A9}
\]

and using (A7) and (A8), one contribution to the commutator in (A6) is

\[
[\hat{S}_0^\dagger(t, t') \tilde{\psi}(r, t_0) \tilde{\psi}(r, t_0) \hat{S}_0(t, t'), \hat{H}'_1(t')]
\]

\[
= -\frac{2g}{V^2} \sum_{p_1, p_2, p_3, p_4} e^{-i(p_{c}+p_1-p_2-p_3) \cdot r} e^{i(\varepsilon_{c}+\varepsilon_1-\varepsilon_2-\varepsilon_3)(t-t')} \int dr' \tilde{n}(r', t') \Phi(r', t') e^{-i p_1 \cdot r'} [a_{p_1}^\dagger a_{p_2} a_{p_3} a_{p_4}^\dagger],
\]

\[
\simeq -2g \tilde{n}(r, t) \sqrt{n_c(r, t)} e^{i \theta(r, t)} \frac{1}{V} \sum_{p_1, p_2, p_3, p_4} e^{-i(p_{c}+p_1-p_2-p_3) \cdot r} e^{i(\varepsilon_{c}+\varepsilon_1-\varepsilon_2-\varepsilon_3)(t-t')}
\]

\[
\times \delta_{p_4, p_c} \left[ a_{p_1}^\dagger a_{p_2} a_{p_3} a_{p_4}^\dagger + a_{p_1}^\dagger a_{p_3} a_{p_2} a_{p_4}^\dagger \right], \tag{A10}
\]

where we have defined the condensate momentum \( p_c \equiv m \mathbf{v}_c \). Substituting this expression into (A6), we obtain

\[
\langle \tilde{\psi}^\dagger(r, t) \tilde{\psi}(r, t) \tilde{\psi}(r, t) \rangle \tag{A11}
\]

\[
= 2i g \tilde{n}(r, t) \sqrt{n_c(r, t)} e^{i \theta(r, t)} \frac{1}{V} \sum_{p_1, p_2, p_3, p_4} e^{-i(p_{c}+p_1-p_2-p_3) \cdot r} \delta_{p_4, p_c}
\]

\[
\times \int_{t_0}^{t} dt' e^{i(\varepsilon_{c}+\varepsilon_1-\varepsilon_2-\varepsilon_3)(t-t')} \left[ \delta_{p_3, p_4} (a_{p_1}^\dagger a_{p_2}) \delta_{p_{2}, p_4} (a_{p_1}^\dagger a_{p_3}) \right].
\]
where
\[ \langle a^+_p a^+_q \rangle_{t'} = \text{Tr} \hat{\rho}(t_0) \hat{S}^+_p(t', t_0) a^+_p a^+_q \hat{S}_0(t', t_0) \approx e^{i(\xi_1 - \xi_2)(t'-t_0)} \langle a^+_p a^+_q \rangle_{t_0}. \] (A12)

Following KD [13], we assume that the initial statistical density matrix \( \hat{\rho}(t_0) \) has the form appropriate for the HF Hamiltonian \( \hat{H}_0(t) \), in which case
\[ \langle a^+_p a^+_q \rangle_{t_0} \approx \delta_{p_1, p_2} f(p_1, r, t). \] (A13)

This identification of the expectation value with the distribution function at \( r \) and \( t \) is based on the picture that the collision processes are short-ranged in both space and time. Finally, the remaining time integral in (A11) is performed by setting \( t_0 \to -\infty \) and introducing the convergence factor \( e^{-\delta(t-t')} \), with the result
\[ \int_{-\infty}^{t} dt' e^{i(\xi_1 + \xi_2 - \xi_3)(t'-t')} \approx \pi \delta(\xi_1 + \xi_2 - \xi_3) + i P \left( \frac{1}{\xi_1 + \xi_2 - \xi_3} \right). \] (A14)

It is useful to note that the approximations made in the above calculations can be reproduced simply by replacing \( \hat{H}_1'(t') \) by
\[ H_1'(t') \approx -2g V^{1/2} \tilde{n} n_c^{1/2} \sum_p \delta_{p, p_r} [e^{-i\theta} e^{i\xi_c(t'-t)} e^{ip_r r} a_p + h.c.], \] (A15)

where the variables \( \tilde{n} \), \( n_c \), \( v_c \), \( \varepsilon_c \) and \( \theta \) are all evaluated at \( r \) and \( t \). In the subsequent calculations it is convenient to make similar replacements for the other parts of the interaction Hamiltonian. In particular, we have
\[ H_2'(t') \approx \frac{gn_c}{2} \sum_{p_1, p_2} \delta_{p_1 + p_2, p_r} [e^{-i\theta} e^{i2\varepsilon_c(t'-t)} e^{ip_r r} a_{p_1} a_{p_2} + h.c.], \] (A16a)
\[ H_3'(t') \approx \frac{gn_c^{1/2}}{\sqrt{V^{1/2}}} \sum_{p_1, p_2, p_3} \delta_{p_1 + p_2 + p_3} [e^{-i\theta} e^{i\varepsilon_c(t'-t)} e^{ip_r r} a_{p_1} a_{p_2} a_{p_3} + h.c.], \] (A16b)
\[ H_4'(t') \approx \frac{g}{2V} \sum_{p_1, p_2, p_3, p_4} \delta_{p_1 + p_2 + p_3 + p_4} a^+_p a^+_q a^+_r a^+_s - 2g\tilde{n} \sum_p a^+_p a_p. \] (A16c)

One thus ends up with expressions for \( \hat{H}_1'(t') \) which now depend explicitly on \( (r, t) \). These forms are of course restricted to the evaluation of correlation functions such as (A10) involving the field operators \( \hat{\psi}(r, t) \).

The contribution of \( H_2'(t') \) to the three-field correlation function in (A10) can now be obtained in a similar way using (A16a). In the course of the calculation we encounter higher order correlation functions of the type \( \langle a^+_p a^+_q a^+_r \rangle \), which we express in terms of the normal products \( \langle a^+_p a \rangle \) by means of Wick’s theorem. We thus find
\[ \langle \hat{\psi}^+(r, t) \hat{\psi}(r, t) \hat{\psi}(r, t) \rangle_{(3)} = -i2gn_c^{1/2} e^{-i\theta} \frac{1}{V^{1/2}} \sum_{p_1, p_2, p_3} e^{i(p_2 + p_3 - p_1 - p_r) r} \]
\[ \times \int_{t_0}^{t} dt' e^{i(\varepsilon_1 + \varepsilon_2 - \varepsilon_3)(t'-t)} \sum_{p_1', p_2', p_3'} \delta_{p_1 + p_2', p_2 + p_3'} \]
\[ \times [\delta_{p_2, p_2'} \delta_{p_3, p_3'} \langle a^+_p a^+_q \rangle_{t'} + \delta_{p_2, p_3'} \langle a^+_p a^+_q \rangle_{t'} \langle a^+_p a^+_q \rangle_{t'} - \delta_{p_1, p_1'} \langle a^+_p a^+_q \rangle_{t'} \langle a^+_p a^+_q \rangle_{t'} - \delta_{p_2, p_2'} \langle a^+_p a^+_q \rangle_{t'} \langle a^+_p a^+_q \rangle_{t'} + \delta_{p_2, p_2'} \langle a^+_p a^+_q \rangle_{t'} \langle a^+_p a^+_q \rangle_{t'} + \delta_{p_3, p_3'} \langle a^+_p a^+_q \rangle_{t'} \langle a^+_p a^+_q \rangle_{t'}], \] (A17)
where \(\langle a_{p_1}^\dagger a_{p_2}\rangle_t\) is given by (A12). The last two terms in (A17) have a special form. Isolating this pair of terms, we have the sum

\[
\sum_{p_1, p_2, p_3'} \delta_{p_1 + p_1', p_2 + p_2'} \left[ \delta_{p_2, p_2'} \langle a_{p_1}^\dagger a_{p_3}\rangle_t \langle a_{p_4}^\dagger a_{p_4'}\rangle_t + \delta_{p_3, p_3'} \langle a_{p_1}^\dagger a_{p_2}\rangle_t \langle a_{p_4}^\dagger a_{p_4'}\rangle_t \right]
\]

\[
= \sum_{p_1, p_2, p_3'} \delta_{p_1 + p_1', p_2 + p_2'} \left[ \delta_{p_2, p_2'} \langle a_{p_1}^\dagger a_{p_3}\rangle_t + \delta_{p_3, p_3'} \langle a_{p_1}^\dagger a_{p_2}\rangle_t \right] \langle a_{p_4}^\dagger a_{p_4'}\rangle_t
\]

\[
= nV \sum_{p_2'} \delta_{p_2, p_2'} \left[ \delta a_{p_1}^\dagger a_{p_3}\rangle_t + \delta a_{p_1}^\dagger a_{p_2}\rangle_t \right] \langle a_{p_4}^\dagger a_{p_4'}\rangle_t,
\]

where in going from the first to the second line we have interchanged the \(p_2\) and \(p_3\) summation variables, and have used (A13) to obtain the last line. When this expression is inserted back into (A17), we see that the last two terms in this equation exactly cancel the contribution from \(H_1'\) in (A11). We thus obtain the following explicit expression for the three-field correlation function

\[
\langle \tilde{\psi} \dagger \tilde{\psi} \rangle (r, t) \tilde{\psi} (r, t) = -i2\pi gn_c^{1/2}e^{i\theta} \frac{1}{V^2} \sum_{p_1, p_2, p_3} \delta (\epsilon_c + \epsilon_1 - \epsilon_2 - \epsilon_3) \left[ \frac{P}{\pi} \frac{1}{\epsilon_c + \epsilon_1 - \epsilon_2 - \epsilon_3} \right]
\]

\[
\times \delta_{p_1 + p_1', p_2 + p_2'} [f_1(1 + f_2)(1 + f_3) - (1 + f_1)f_2f_3],
\]

(A18)

with \(f_i \equiv f(p_i, r, t)\). We emphasize that in deriving (A18), we have assumed that initial values of anomalous correlations, such as \(\langle a_{p_1} a_{p_2}\rangle_{t_0}\), vanish. In addition, we have treated the system as locally homogeneous, with the consequence that the local HF single-particle energies, \(\tilde{\epsilon}_p (r, t) = p^2/2m + U(r, t)\), appear. We finally note that the three-field correlation function in (A18) is explicitly proportional to \(\sqrt{n_c}\) and the interaction strength \(g\). Thus it vanishes above \(T_{BEC}\) as well as in a non-interacting Bose gas.

Following the same procedure, one can also calculate the anomalous pair correlation function

\[
\tilde{m}(r, t) \equiv \langle \tilde{\psi} \dagger \tilde{\psi} \rangle (r, t)
\]

\[
= \frac{1}{V} \sum_{p_1, p_2} \langle a_{p_1}^\dagger a_{p_2}\rangle_t e^{i(p_1 + p_2) \cdot r}.
\]

(A19)

The final result analogous to (A18) is

\[
\tilde{m}(r, t) = -i\pi g \Phi \frac{1}{V^2} \sum_{p_1, p_2} \delta_{p_1 + p_2, 2p_0} [1 + f_1 + f_2] \left[ \delta (\tilde{\epsilon}_1 + \tilde{\epsilon}_2 - 2\epsilon_c) \frac{P}{\tilde{\epsilon}_1 + \tilde{\epsilon}_2 - 2\epsilon_c} \right].
\]

We note that if (A20) is evaluated for a static equilibrium situation (\(v_c = 0, \epsilon_c = \mu_c\) and \(f_i\) equilibrium Bose distributions), the imaginary part vanishes and the real part simplifies to

\[
\tilde{m}_0 (r) = -gn_c (r) \int \frac{dp}{(2\pi)^3} \frac{1 + 2f_0^0}{2E_p (r)}.
\]

(A21)
where $E_p(r) = \tilde{\varepsilon}_p(r) - \mu_{c0}$ is the local HF excitation energy. This result for the anomalous pair correlation is consistent with the well-known HFB expression \cite{18} for a uniform Bose gas in the HF limit.

The real parts of (A18) and (A20) contribute to the local condensate chemical potential in (8). As is well known (see, for example, Section 5 of Ref. [40] and Ref. [49]), the anomalous pair correlation in (A21) is ultraviolet divergent. This divergence is removed in the more complete Beliaev approximation \cite{18}, which includes all self-energy contributions to second order in $g$. Since we do not include renormalization effects consistently to order $g^2$ in the present treatment, we will drop these contributions to the chemical potential. The imaginary parts of (A18) and (A20) contribute to the source term on the right hand side of (5a). However, the contribution from $\tilde{m}$ involves the energy-conserving delta function $\delta(\tilde{\varepsilon}_1 + \tilde{\varepsilon}_2 - 2\varepsilon_c)$. Taking the momentum conservation condition in (A20) into account, one finds that the argument of the delta function in the TF limit is equal to $(p_1 - p_2)^2/4m + 2gn_c$. Since this expression is positive definite, the imaginary part of $\tilde{m}$ makes no contribution to the source term in (5a).

Using similar techniques and approximations, we next evaluate the collision terms given in the right hand side of (21). The $C_{12}$ collision integral is defined as the contribution from the $\hat{H}_3'$ perturbation in (A16f),

$$C_{12}[f] \equiv -i\text{Tr}\hat{\rho}(t, t_0)[\hat{f}(p, r, t_0), \hat{H}_3'(t)]$$

$$\simeq -ign_c^{1/2}V^{-1/2} \sum_p \sum_{p_1, p_2, p_3} e^{ip_r} \delta_{p_1 + p_2 + p_3} e^{-i\theta} e^{ip_c - p_{c'}}$$

$$\times \left[ \delta_{p_1, p + q/2} \langle a_{p_1}^\dagger a_{p + q/2}^\dagger a_{p_3} \rangle_t - \delta_{p_2, p - q/2} \langle a_{p_2}^\dagger a_{p - q/2}^\dagger a_{p_3} \rangle_t - \delta_{p_3, p - q/2} \langle a_{p_3}^\dagger a_{p - q/2}^\dagger a_{p_3} \rangle_t - \text{h.c.} \right]. \tag{A22}$$

Eq. (A18) can be used to identify the three-field correlation function $\langle a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} \rangle_t$. Inserting the result in (A22), we find

$$C_{12}[f] = 4\pi g^2 n_c V^{-1} \sum_{p_1, p_2, p_3} \delta(\varepsilon_c + \tilde{\varepsilon}_1 - \tilde{\varepsilon}_2 - \tilde{\varepsilon}_3) \delta_{p_1 + p_2 + p_3}$$

$$\times [\delta_{p_1, p} - \delta_{p_2, p} - \delta_{p_3, p}] [(1 + f_1)f_2f_3 - f_1(1 + f_2)(1 + f_3)]. \tag{A23}$$

The $C_{22}$ collision term is the contribution from the $\hat{H}_3'$ perturbation defined in (A16c). We note that the second term in (A16c) is proportional to the number operator which commutes with any number-conserving operator. We thus have

$$C_{22}[f] \equiv -i\text{Tr}\hat{\rho}(t, t_0)[\hat{f}(p, r, t_0), \hat{H}_3'(t)]$$

$$\simeq -ig \frac{2V}{2} \sum_p \sum_{p_1, p_2, p_3, p_4} e^{ip_r} \delta_{p_1 + p_2 + p_3 + p_4}$$

$$\times \left[ \delta_{p_1, p + q/2} \langle a_{p_1}^\dagger a_{p + q/2}^\dagger a_{p_3} \rangle_t + \delta_{p_2, p - q/2} \langle a_{p_2}^\dagger a_{p - q/2}^\dagger a_{p_3} \rangle_t + \delta_{p_3, p - q/2} \langle a_{p_3}^\dagger a_{p - q/2}^\dagger a_{p_3} \rangle_t - \delta_{p_4, p - q/2} \langle a_{p_4}^\dagger a_{p - q/2}^\dagger a_{p_3} \rangle_t \right]. \tag{A24}$$

According to (X5),

$$\langle a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} a_{p_4} \rangle_t = \text{Tr}\hat{\rho}(t_0) \left\{ \hat{S}^+(t, t_0) a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} a_{p_4} \hat{S}(t, t_0) - i \int_{t_0}^t dt' e^{i(\tilde{\varepsilon}_1 + \tilde{\varepsilon}_2 - \tilde{\varepsilon}_3) (t - t')} \hat{S}^+(t', t_0) a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} a_{p_4} \hat{S}(t', t_0) \right\}. \tag{A25}$$
We find that the first term of order \( g^0 \) in (A25), while finite, makes no contribution to (A24). Thus the \( C_{22}[f] \) collision integral is explicitly of second order in \( g \). Using Wick’s theorem to evaluate the second term in (A25), we obtain the relevant contribution

\[
\langle \hat{a}_{p_1}^\dagger \hat{a}_{p_2}^\dagger \hat{a}_{p_3} \hat{a}_{p_4} \rangle_t \simeq -\frac{2\pi ig}{V} \delta(\bar{\epsilon}_1 + \bar{\epsilon}_2 - \bar{\epsilon}_3 - \bar{\epsilon}_4) \delta_{p_1+p_2,p_3+p_4} \times \left[ f_1 f_2 (1 + f_3)(1 + f_4) - (1 + f_1)(1 + f_2)f_3 f_4 \right].
\]

(A26)

Inserting (A26) into (A24), we finally obtain the expression

\[
C_{22}[f] = \frac{\pi g^2}{V^2} \sum_{p_{1},p_{2},p_{3},p_{4}} \delta(\bar{\epsilon}_1 + \bar{\epsilon}_2 - \bar{\epsilon}_3 - \bar{\epsilon}_4) \delta_{p_1+p_2,p_3+p_4} \\
\times \left[ \delta_{p_1,p_2} + \delta_{p_2,p_3} - \delta_{p_1,p_3} - \delta_{p_2,p_4} \right] \times \left[ f_1 f_2 (1 + f_3)(1 + f_4) - (1 + f_1)(1 + f_2)f_3 f_4 \right].
\]

(A27)

Replacing the momentum sum \( (1/V) \sum_p \) by the integral \( \int d\mathbf{p}/(2\pi)^3 \), and the Kronecker delta function \( V\delta_{p,0} \) by the Dirac delta function \( (2\pi)^3\delta(p) \) in (A18), (A23) and (A27), we recover the expressions written down in Eqs. (25), (23b) and (23a), respectively. As discussed in the body of the paper, there is a close relation between the expressions in (A18) and (A23). This ensures that the separate continuity equations for the condensate and the non-condensate [see (27a) and (47a)] are consistent with the requirement that the total number of atoms is conserved. (We note that KD [13] did not obtain the equivalent of these equations since they did not explicitly evaluate \( \langle \bar{\psi}^\dagger \bar{\psi} \rangle \).)

APPENDIX B: EQUILIBRIUM PROPERTIES

In this Appendix, we briefly discuss the difference between the static equilibrium functions we use for a weakly-interacting Bose gas and the well-known results obtained by Lee and Yang [44] at finite temperatures \( (T \ll T_{BEC}) \). The Lee-Yang expressions for the thermodynamic functions were derived as \textit{first-order} corrections in the interaction \( g \) to the ideal Bose-condensed gas results. We list them here for convenience:

\[
P = \bar{P}_{cr} + \frac{1}{2}g(n^2 + \bar{n}_{cr}^2),
\]

\[
\epsilon = \frac{3}{2} \bar{P}_{cr} + \frac{1}{2}g(n^2 - n\bar{n}_{cr} + 2\bar{n}_{cr}^2),
\]

\[
T_s = \frac{5}{2} \bar{P}_{cr} + \frac{3}{2}g(\bar{n}_{cr}^2 - n\bar{n}_{cr}).
\]

(B1)

The associated free energy density is

\[
f \equiv \epsilon - T_s = -\bar{P}_{cr} + \frac{1}{2}g(n^2 + 2n\bar{n}_{cr} - \bar{n}_{cr}^2).
\]

(B2)

Here

\[
\bar{P}_{cr} \equiv \frac{1}{\beta \Lambda^2} g_{5/2}(z = 1)
\]
\[ \tilde{n}_{cr} \equiv \frac{1}{\beta \Lambda^3} g_{3/2}(z = 1), \]  

(B3)

are the ideal-gas values of the pressure and non-condensate density at \( T_{BEC} \). One can see that (B1) and (B2) explicitly involve only first order corrections in \( g \). To this order, these expressions are entirely consistent with our thermodynamic expressions given by (65), (68) and (69). For example, we have

\[
\tilde{P} = \frac{1}{\beta \Lambda^3} g_{5/2}(z = e^{-\beta g n_c}) \\
\simeq \frac{1}{\beta \Lambda^3} g_{5/2}(z = 1) - g n_c \frac{1}{\Lambda^3} g_{3/2}(1) + \cdots \\
= \tilde{P}_{cr} - g \tilde{n}_{cr} (n - \tilde{n}_{cr}).
\]  

(B4)

Substituting this into (63) with \( \tilde{n} = \tilde{n}_{cr} \), we obtain the expression for the pressure in (B1). However, it is clear that our description of the equilibrium properties (based on the HF approximation) goes beyond the Lee-Yang description, which is only correct to first order in \( g \).

The implications of this become apparent when these results are considered in the context of a trapped Bose-condensed gas. Within our semiclassical HF description, the non-condensate density is given by

\[ \tilde{n} = \frac{1}{\Lambda^3} g_{3/2}(z), \]  

(B5)

with the fugacity given by

\[ z = e^{\beta (\mu_c - U)}. \]  

(B6)

Here, \( \mu_c \) is the condensate eigenvalue as obtained from (49) and \( U(r) = U_{ext}(r) + 2gn(r) \) is the effective potential as seen by the non-condensate atoms. For (B3) to be well-defined, \( \mu_c \) must lie below the minimum of the effective potential \( U(r) \). Because of the localized condensate, the non-condensate atoms experience a localized repulsive interaction \( 2gn_c(r) \) which has the effect of depleting the non-condensate density at the center of the trap where the non-condensate density overlaps with the condensate. This is evident in Fig. 3 (see also Refs. [2] and [26]), and gives rise to corresponding spatial variations in the other thermodynamic functions.

In contrast, in the Lee-Yang description, the non-condensate takes on the critical value \( \tilde{n}_{cr} \) wherever the condensate density is finite. This is the approximation used by Shenoy and Ho [35] in their calculation of the normal modes of a trapped Bose gas. Working within the Thomas-Fermi approximation, their condensate density is given by

\[ n_c(r) = \frac{1}{g} (\mu - U_{ext}(r) - 2g \tilde{n}_{cr}), \]  

(B7)

which goes to zero at a radius \( R_{TF} \) defined by \( U_{ext}(R_{TF}) = \mu - 2g \tilde{n}_{cr} \). For \( r < R_{TF} \), all the thermodynamic coefficients required in the Landau equations were evaluated using the Lee-Yang free energy density in (B2).
Above $T_{BEC}$, Lee and Yang \cite{44} obtain the non-perturbative result for the free energy density (generalized to include a trap)

$$f = -\frac{1}{\beta\Lambda^3}g_{3/2}(z_0) - gn^2(r) + n\mu_0 - nU_{ext}(r), \quad (B8)$$

where the density $n(r)$ is obtained from (B5) with the fugacity given by

$$z_0 = e^{\beta(\mu_0 - U_{ext}(r) - 2gn(r))}. \quad (B9)$$

The chemical potential $\mu_0$ is chosen to give the desired total number of atoms, $N$. The other thermodynamic functions are given in this temperature range by

$$P = \frac{1}{\beta\Lambda^3}g_{5/2}(z_0) + gn^2,$$
$$\epsilon = \frac{3}{2}\frac{1}{\beta\Lambda^3}g_{3/2}(z_0) + gn^2 + nU_{ext}(r),$$
$$sT = \frac{5}{2}\frac{1}{\beta\Lambda^3}g_{5/2}(z_0) - n\kappa_B T \ln z_0. \quad (B10)$$

These are the results used by Shenoy and Ho \cite{35} for the thermal cloud in those regions where $\tilde{n}_0 < \tilde{n}_{cr}$, that is, for $r > R_{TF}$ when $T < T_{BEC}$. We note that these expressions take interactions into account in a self-consistent fashion, unlike the situation for $r < R_{TF}$ where the non-condensate density takes on the ideal gas value of $\tilde{n}_{cr}$.

**APPENDIX C:**

We list here the various integrals required for the evaluation of the energy functional $J$ in Section VII. The form of the non-condensate displacement is $u(r) = (ax, ay, bz)$ and that of the condensate is $w(r) = (cx, cy, dz)$. For the kinetic energy terms, we have

$$K_n[u] = \frac{1}{6}(2a^2 + b^2)m \int d^3r r^2\tilde{n}_0(r)$$
$$K_c[w] = \frac{1}{6}(2c^2 + d^2)m \int d^3r r^2n_{c0}(r).$$

In obtaining these results, we have made use of the spherical symmetry of the equilibrium densities. The various potential energy contributions are the terms appearing in (114), (116) and (120):

$$\frac{1}{2} \int d^3r u_{ii}^2 = \frac{1}{2}(2a + b)^2 \int d^3r \lambda(r)$$
$$\frac{1}{2} \int d^3r \mu(u_{ij}^2 - \bar{u}_{ij}^2) = \frac{1}{2}(2a^2 + b^2) \int d^3r \mu(r)$$
$$\frac{1}{2} \int d^3r u_{ij} \tilde{n}_0 \frac{\partial^2}{\partial x_i x_j}U_{ext} = \frac{1}{6}(2a^2 + b^2)m\omega_0^2 \int d^3r r^2\tilde{n}_0$$
$$\frac{1}{2} \int d^3r u_{ij} \tilde{n}_0 \frac{\partial^2}{\partial x_i x_j}2gn_{c0} = \frac{2}{15}(a - b)^2 g \int d^3r r\tilde{n}_0 \frac{\partial n_{c0}}{\partial r}.$$
\[2g \int d^3r \frac{\partial (\tilde{n}_0 u_i)}{\partial x_i} \frac{\partial (n_{c0} w_j)}{\partial x_j} = \frac{1}{15} (8a^2 + 4ab + 3b^2) g \int d^3r r^2 \tilde{n}_0 \frac{\partial^2 n_{c0}}{\partial r^2} \]
\[+ \frac{2}{15} (8ac + 2ad + 2bc + 3bd) g \int d^3r r^2 \partial \tilde{n}_0 \frac{\partial n_{c0}}{\partial r} \frac{\partial n_{c0}}{\partial r} \]
\[= \frac{1}{30} (8c^2 + 4cd + 3d^2) g \int d^3r r^2 \left( \frac{\partial n_{c0}}{\partial r} \right)^2 \]
\[= \frac{1}{15} (8c^2 + 4cd + 3d^2) \int d^3r \left( r \frac{\partial \Phi_0}{\partial r} \right) \hat{\mathcal{L}}_0 \left( r \frac{\partial \Phi_0}{\partial r} \right) \]
\[+ \frac{4}{15} (c - d)^2 \frac{\hbar^2}{m} \int d^3r \left( \frac{\partial \Phi_0}{\partial r} \right)^2 .\]
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Figure Captions

Fig. 1: Squares of the first and second sound velocities (normalized by the first sound velocity of the ideal gas at $T = T_{BEC}$) vs. $T/T_{BEC}$. The solid lines are for the Landau two-fluid theory while the dashed lines are for the ZGN hydrodynamics. The curves were generated for $gn/k_B T_{BEC} = 0.2$.

Fig. 2: Damping of the second sound mode in a uniform gas due to the relaxation of the condensate and non-condensate chemical potentials at a temperature $T/T_{BEC} = 0.9$. The damping, $\Gamma_2$, relative to the mode frequency, $\Omega_2$, is plotted as a function of $\Omega_2 \tau_\mu$.

Fig. 3: Density of atoms as a function of position in an isotropic parabolic trap containing 5000 $^{87}$Rb atoms. Parameters defining the trap and atomic interactions are given in the body of the paper. The solid line is the total density and the dashed line is the non-condensate density. The parameter $d$ is the oscillator length, $d = \sqrt{\hbar/m\omega_0}$.

Fig. 4: Monopole ($l = 0$) and quadrupole ($l = 2$) mode frequencies, in units of the trap frequency $\omega_0$, vs. temperature for 5000 $^{87}$Rb atoms in an isotropic parabolic trap. The ‘+’ and ‘−’ signs merely distinguish the two modes of a given angular character. The lower part of the figure shows the condensate fraction as a function of $T$.

Fig. 5: Variational condensate ($A_c$) and non-condensate ($A_n$) amplitudes for the monopole modes as a function of temperature. Panel (a) is for the $0^+$ mode and panel (b) is for the $0^-$ mode. The vertical line at $T \simeq 149$ nK denotes the BEC transition temperature.

Fig. 6: As in Fig. 5, but for the quadrupole $2^+$ and $2^-$ modes.
$n(r) \text{ (d}^{-3})$

$r \text{ (d)}$

$T = 100 \ nK$
