Option pricing models without probability

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Abstract. We describe the pricing and hedging practices refraining from the use of probability. We encode volatility in an enhancement of the price trajectory and we give pathwise presentations of the fundamental equations of Mathematical Finance. In particular this allows us to assess model misspecification, generalising the so-called fundamental theorem of derivative trading (see Ellersgaard et al. [EJP17]). Our pathwise integrals and equations exhibit the role of Greeks beyond the leading-order Delta, and makes explicit the role of Gamma sensitivities.

Introduction

In this work we formulate the Black-Scholes technical apparatus for derivatives valuation and hedging without using probability.1 SDE dynamics over-specify the properties of the price trajectories. The minimal information needed is the pair \((S, [S])\), where \(S\) denotes the trajectory and \([S]\) denotes its rough bracket, see [FH14, Chapter 5]. We show moreover how to interpret the rough bracket financially. This also allows us to extend the Black-Scholes technology to a rougher regime, beyond the classical semimartingale setting.

A further aspect that can be assessed in our framework is the consequence of model misspecification. In classical Mathematical Finance, the fundamental theorem of derivative trading provides a formula to compute the profit&loss that a trader incurs into when hedging with the “wrong” volatility. We show the pathwise nature of this formula and we generalise it. The generalisation consists in removing the assumption that the “true” price evolution is governed by an Itô SDE.

To place our result in context, we begin by noticing that the mathematical formulation of Black-Scholes-Merton theory (as formulated for example by Harrison and co-authors, see [HK79], [HP81]) is based on two objects. One object is the equivalent martingale measure, the other object is the Black-Scholes partial differential equation. While the former is only interpretable within a genuinely probabilistic framework, the latter — we argue — has a purely pathwise nature, at a remove from any probabilistic features of the price dynamics. In order to formulate the Black-Scholes partial differential equation we only need to know the quadratic variation of the price path. The quadratic variation is the path property on which H. Föllmer’s work [Foe81] is based. His article initiated the study of those formulas of Stochastic Analysis that hold without reference to a probability space; in the following work, we extend this programme to the classical formulas of Mathematical Finance.

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1 An informal and brief account of the discussion given here, with a historical perspective on how this work developed and on the related literature, is given in [Bri19].
The quadratic variation is not a distributional feature of the price, in the sense that changing to an equivalent probability measure does not affect it. Notwithstanding its non-probabilistic nature, the quadratic variation is usually associated with the diffusion coefficient in a model described using Itô’s theory of stochastic differential equations. In this setting the coefficients of the Black-Scholes PDE are seen as being derived from the characteristics of a certain diffusion process. These coefficients determine the parameters for the variances of diffusion’s marginal laws, which can give rise to confusion between the concept of volatility and of variance which we propose to disentangle.

To the best of our knowledge, C. Bender, T. Sottinen and E. Valkeila were the first to discern this distinction, see [BSV08]. Their work was anticipated by two earlier articles. The first is by Terry J. Lyons, [Lyo95], which focuses on replication arguments and observes how probability is only used to justify lower bounds for option prices. The main theorem in this paper, [Lyo95, Theorem 1], is effectively a precursor to the fundamental theorem of derivative trading. The second work is by D. Brigo and F. Mercurio, [BM00], where the consequences of changing probability measures are addressed. The authors showed, by constructive examples, that the change of measure used in martingale pricing can massively disrupt the distributional features of physical dynamics on arbitrarily fine time grids: close physical evolutions for stock prices can be transformed into pricing counterparts that imply arbitrarily different option prices. We expand on this insight that martingale pricing, although probabilistic in nature, actually entails that only pathwise properties of physical stock evolutions are relevant for option pricing models.

Relying on the result of pathwise stochastic analysis in the book by P. Friz and M. Hairer ([FH14]), we reformulate the Black-Scholes technical apparatus without using probability. Moreover, we use the notion of a rough bracket as an extension of quadratic variation. This is not merely a linguistic exercise: on the one hand, it reveals that any linear transformation of level two of the price path’s signature can take the role of the quadratic variation. On the other hand, it reveals that the Black-Scholes technology extends to the regime of general continuous price trajectories of finite p-variation, where $2 < p < 3$. It continues Bender, Sottinen and Valkeila’s work [BSV08], and it does so in a more radical way, taking probability out of the modelling framework.

As far as option pricing is concerned, specifying a model for stock evolutions can ultimately be reduced to the specification of a rough bracket of the price path. Mathematically this minimal description $(S, [S])$ is a reduced rough path as defined in [FH14, Chapter 5]. We give a financial interpretation to such an object, calling it an enhanced price path. Given a price path $S$, there exist different reduced rough paths that are enhancements of it; a model specification is tantamount to choosing a particular enhancer and diffusion-based models give rise to canonical enhancements. We can also compare different models by comparing their enhancers. We capture the misspecification that arises between two diffusion-enhanced price paths originating from SDE models, yielding a version of the fundamental theorem of derivative trading. We extend this result to compare a diffusion-enhanced price path (used by the trader) and a general enhanced price path.

The article is organised as follows. In Section 1, we expand on D. Brigo and F. Mercurio’s investigation on “alternative continuous-time dynamics for discretely-observed stock prices”, [BM00]. The crucial observation leading to pathwise models for option pricing is contained in Remark 1.17. In Section 2, we introduce the adopted
financial terminology and notation by recalling the main steps in the Black-Scholes
theory. In Section 5, we carry out our project of a pathwise formulation of the Black-
Scholes technology, including the pathwise version of the fundamental theorem of
derivatives trading and the analysis of model misspecification. On the mathematical
side, the technical aspects will be based on pathwise integrals introduced in Section 3; on
the financial side, the guidance for our pathwise description will be provided by
benchmark Markovian models introduced in Section 4. Finally, Section 6 will explore
alternative financial interpretations of pathwise integrals.

1 Disentangling volatility from marginal variances

To understand the difference between the discrete and continuous time settings for
option pricing, D. Brigo and F. Mercurio in [BM00] produced examples of “alterna-
tive continuous-time dynamics for discretely-observed stock prices”, in the following
sense: given two distinct standard Black-Scholes processes \(X_1\) and \(X_2\) (i.e. geometric
Brownian motions), and given a trading grid \(\pi\), they constructively showed the exist-
ence of a continuous price dynamics such that the following hold simultaneously

1. on the grid \(\pi\), all its probabilistic features are those of \(X_1\);
2. it prices contingent claims as \(X_2\) does.

However fine the grid might be, such “alternative dynamics” exist and they span all
the range of no-arbitrage prices. In this respect, the “alternative dynamics” are de-
ceptive, because statistically inferred distributional properties on the grid \(\pi\) would
make a trader prone to use \(X_1\) for pricing and hedging purposes, whereas the “cor-
rect” volatility would be that of \(X_2\). This indicates that Black-Scholes pricing tech-
nology ignores discretely-observed distributional features of the underlying.

A fundamental merit of D. Brigo and F. Mercurio’s construction is to disentangle
the concept of volatility from that of marginal variance. We shall emphasise this in
the following section, devoted to a reformulation of their result. We present simplified
direct proofs that circumvent the original discourse based on Fokker-Planck equa-
tion and evolutions of marginal laws in finite dimensional manifolds of densities (see
[Bri00]). Moreover, the formulation below handles singularities by taking the limit
along sequences of \(\epsilon\)-smoothened “alternative dynamics” around singular points. As
a by-product, we informally observe that a Volterra-type process appears in the mar-
ket price of risk of the “alternative dynamics”. Such a process has features similar
to those empirically found in implied volatilities, and that motivated the introduction
of rough volatility models.

We start by recalling three concepts employed in [BM00]. They concern distribu-
tional properties of stochastic processes.

**Definition 1.1 (“Marginal identity”).** Let \(X\) and \(Y\) be stochastic processes on the time
window \([0, T]\). We say that \(X\) and \(Y\) are marginally identical if their marginal laws are
equal at all times, namely if for all bounded measurable \(f\) and all \(0 \leq t \leq T\) it holds

\[
E f(X_t) = E f(Y_t).
\]

**Remark 1.2.** The condition in equation (1.1) refers to the law of the two processes
\(X\) and \(Y\). Hence, it is not actually necessary to suppose that \(X\) and \(Y\) are defined on
the same probability space. We shall emphasise this in Definition 1.5 below, where
different probability spaces represent different models for stock prices’ evolutions.
However, assuming that \(X\) and \(Y\) are defined on the same probability space does not
affect generality, because a probability space that accommodates both processes can always be constructed.

Let $\pi$ be a partition of $[0, T]$, i.e. a finite ordered collection of points in $[0, T]$ such that the initial time $0$ and the time horizon $T$ are both in $\pi$. Let $t$ be in $[0, T]$. We adopt the following notational convention:

$$t' := \inf \{u \in \pi : u > t\}, \quad [t] := \sup \{u \in \pi : u \leq t\} \quad (1.2)$$

**Definition 1.3** ("$\pi$-Markovianity"). Let $X$ be a stochastic process on $[0, T]$, and let $(\tilde{\mathcal{F}}_t)$ be the minimal filtration generated by $X$. Let $\pi$ be a partition of $[0, T]$. We say that $X$ is $\pi$-Markov if for all $s$ in $\pi$, all $t \geq s$, and all bounded measurable $f$, it holds

$$E[f(X_t)|\tilde{\mathcal{F}}_s] = E[f(X_t)|X_s].$$

**Definition 1.4** ("$\pi$-indistinguishability"). Let $X$ and $Y$ be stochastic processes on $[0, T]$, and let $\pi$ be a partition of $[0, T]$. We say that $X$ and $Y$ are $\pi$-indistinguishable if they are $\pi$-Markov, and if for all $s$ in $\pi$, all $t \geq s$, all $z$ in $\mathbb{R}$ and all bounded measurable $f$, it holds

$$E[f(X_t)|X_s = z] = E[f(Y_t)|Y_s = z]. \quad (1.3)$$

The same observation as in Remark 1.2 applies to the condition in equation (1.3). Moreover we observe that $\pi$-indistinguishability implies that the transition functions of the discrete-time Markov processes $\pi X_t := X_{[t]}$ and $\pi Y_t := Y_{[t]}$ are the same.

The last concept that we introduce is quintessentially financial. The acronym NAP shall stand for no-arbitrage pricing. We fix a deterministic interest rate $r$ so that to include in every market model the riskless asset $S^0_t = S^0_0 \exp(rt)$. Given a price process $X$, the forward price of $X$ at time $t$ is defined to be $e^{-rt}X_t$.

**Definition 1.5** ("NAP-equivalence"). Let $X$ and $Y$ be positive semimartingales defined respectively on $(\Omega, \mathfrak{F}^X, P^X)$ and $(\Omega, \mathfrak{F}^Y, P^Y)$. We say that $X$ and $Y$ induce equivalent pricing kernels / are NAP-equivalent if there exist probability measures $Q^X$ and $Q^Y$, respectively defined on $(\Omega, \mathfrak{F}^X)$ and $(\Omega, \mathfrak{F}^Y)$, and equivalent to $P^X$ and $P^Y$, such that

1. the forward prices of $X$ and $Y$ are respectively $Q^X$ and $Q^Y$-martingales;
2. for all $s < t$, all $z$ in $\mathbb{R}$ and all bounded measurable $f$, it holds

$$E_{Q^X}[f(X_t)|X_s = z] = E_{Q^Y}[f(Y_t)|Y_s = z], \quad (1.4)$$

where $E_{Q^X}$ and $E_{Q^Y}$ denote respectively expectation under $Q^X$ and under $Q^Y$.

**Example 1.6** ("NAP-equivalent geometric Brownian motions and market price of risk"). Notoriously, if $\mu_i$, $i = 1, 2$, are two real numbers and $X^i_t$, $i = 1, 2$, are price processes following the dynamics

$$dX^i = \mu_i X^i dt + \sigma X^i dW^i, \quad (1.5)$$

where $\sigma$ is a fixed volatility coefficient and $W^1$, $W^2$ are standard one-dimensional Brownian motions, then $X^1$ and $X^2$ induce indifferent pricing kernels. The change of measure that brings the physical dynamics (1.5) into their respective pricing dynamics

$$d\tilde{X}^i = r X^i dt + \sigma X^i d\tilde{W}^i$$

is described by

$$\frac{dQ^i}{dP} = \mathcal{E} \left( \frac{\mu_i - r}{\sigma} W^i \right), \quad i = 1, 2,$$

where $\mathcal{E}$ denotes Itô exponential. The coefficient $(\mu_i - r)/\sigma$ is referred to as market price of risk, and it takes the role of the volatility in the dynamics of the Radon-Nykodim derivative of the pricing measure with respect to the physical measure.
If \((X, Q^X)\) and \((Y, Q^Y)\) are time-homogeneous Markov processes, then equation (1.4) is the equivalence of their transition semigroups. This leads to the following

**Proposition 1.7.** Let \((X^1, Y^1)\) and \((X^2, Y^2)\) be two pairs of NAP-equivalent price processes. Assume that under the pricing measure, they are time-homogeneous Markov processes, with \(X^1\) independent from \(X^2\), \(Y^1\) independent from \(Y^2\), and \(X^0_0 = Y^0_0 = 1\). Consider the concatenations

\[
X_t = \begin{cases} 0 \leq t \leq T & \{X^1_t\} \\ X^1_{T - T} & T < t \leq 2T \end{cases}
\]

and

\[
Y_t = \begin{cases} 0 \leq t \leq T & \{Y^1_t\} \\ Y^1_{T - T} & T < t \leq 2T. \end{cases}
\]

Then \(X\) and \(Y\) are NAP-equivalent.

**Proof.** Let \((\Omega, \mathcal{F}, Q)\) be a probability space that accommodates the processes \((X^i, Q^X)\) and \((Y^i, Q^Y)\), \(i = 1, 2\). Firstly, we need to show that for all bounded measurable \(f\) and all \(0 \leq s \leq t \leq 2T\) it holds

\[
E[f(X_t)|X_s = z] = E[f(Y_t)|Y_s = z],
\]

where expectations are computed with respect to \(Q\). This follows from the fact that, under \(Q\), the law of \(\{\log X_t, 0 \leq t \leq 2T\}\) is the same as the law of \(\{\log Y_t, 0 \leq t \leq 2T\}\). To see this, observe that \(\log X\) and \(\log Y\) are Markov and that: 1) the laws of \(\{\log X_t, 0 \leq t \leq T\}\) and of \(\{\log Y_t, 0 \leq t \leq T\}\) are the same by assumption; 2) the laws of \(\{\log X_t, T \leq t \leq 2T\}\) and of \(\{\log Y_t, T \leq t \leq 2T\}\) are the same, since they both coincide with the unique law of the Markov process described by the transition semigroup of \(\log X^2\) and by the initial distribution \(\log X^1_0\).

Secondly, we need to show that the forward prices are \(Q\)-martingales. Again this is clear up to time \(t = T\). If \(s \geq T\), then

\[
E[e^{-rt}X_t|e^{-rs}X_s] = E[E[e^{-rt}X_t|e^{-rt}X^1_t, e^{-r(t-T)}X^2_{T-t}|e^{-rs}X_s] = E[e^{-rt}X^1_T]e^{-r(s-T)}X^2_{T-t}|e^{-rs}X_s] = e^{-rs}X_s.
\]

Finally, if \(s < T < t\) then

\[
E[e^{-rt}X_t|e^{-rs}X_s] = E[e^{-rt}X_T|e^{-rs}X_s]E[e^{-r(t-T)}X^2_{T-t}] = e^{-rs}X_s.
\]

The martingality of \(e^{-rt}Y_t\) is either proved analogously, or deduced from that of \(e^{-rt}X_t\) and the equivalence in law.

A relaxed version of the concept in Definition 1.5 brings to the following

**Definition 1.8 ("Weak NAP-equivalence.")** Let \(X\) and \(Y\) be positive semimartingales, interpreted as price dynamics. We say that \(X\) and \(Y\) are weakly NAP-equivalent if there exist sequences of positive processes \(X^n\) and \(Y^n\), \(n \geq 1\), such that

1. for all \(t\), the log-prices \(\log X^n_t\) and \(\log Y^n_t\) converge respectively to \(\log X_t\) and \(\log Y_t\) in \(L^2(P)\);
2. for every \(n\), the processes \(X^n\) and \(Y^n\) are NAP-equivalent.
The sequences $X^n$ and $Y^n$, $n \geq 1$, in the definition above are referred to as reducing sequences for the weakly NAP-equivalent pair $(X, Y)$.

The possibility to concatenate NAP-equivalent processes extends immediately to weakly NAP-equivalent processes.

**Corollary 1.9.** Let $(X^1, Y^1)$ and $(X^2, Y^2)$ be two pairs of weakly NAP-equivalent processes. Let $(X^{1,n}, Y^{1,n})_n$ and $(X^{2,n}, Y^{2,n})_n$, $n \geq 1$, be their reducing sequences and assume that for every $n$ the NAP-equivalent processes $(X^{1,n}, Y^{1,n})$ and $(X^{2,n}, Y^{2,n})$ satisfy the assumptions of Proposition 1.7. Then the concatenations

$$X_t = \begin{cases} X^1_t & 0 \leq t \leq T \\ X^1_{T-t}X^2_t & T < t \leq 2T \end{cases}$$

and

$$Y_t = \begin{cases} Y^1_t & 0 \leq t \leq T \\ Y^1_{T-t}Y^2_t & T < t \leq 2T. \end{cases}$$

are weakly NAP-equivalent.

Having introduced the concepts above, we prepare the construction of “alternative dynamics”.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $W$ be a Brownian motion on it. We consider the probability measure $\mathbb{P}$ as fixed and we refer to it as physical measure. Let $(\mathcal{F}_t)$ be the minimal $\mathbb{P}$-completed right-continuous filtration generated by $W$. We consider processes defined in the time window $[0, T]$. Given $t_0$ in $[0, T]$, the space $L^2(\mathbb{P} \otimes \frac{dt}{T-t_0}) = L^2(\Omega \times [t_0, T], \mathcal{F} \otimes \mathcal{B}[t_0, T], \mathbb{P} \otimes \frac{dt}{T-t_0})$ is the space of square integrable random variables on $\Omega \times [t_0, T]$ with respect to the product measure $\mathbb{P} \otimes \frac{dt}{T-t_0}$, where $dt/(T-t_0)$ is the normalised Lebesgue measure on $[t_0, T]$. We use the symbol $\int dt$ for the integral with respect to such normalised Lebesgue measure. For $\xi$ in $L^2(\mathbb{P} \otimes dt/(T-t_0))$ we set

$$|||\xi||| := \int_{t_0}^T ||\xi(t)||_{L^2(\mathbb{P})} dt$$

(1.7)

and we observe

$$|||\xi||| \leq ||\xi||_{L^2(\mathbb{P} \otimes dt/(T-t_0))}.$$

Let $L^1([0, T]; L^2(\mathbb{P}))$ be the closure of $L^2(\mathbb{P} \otimes dt/(T-t_0))$ with respect to $|||\cdot|||$.

Let $H$ be a strictly positive real number. For $s, t > t_0$ we introduce the functions

$$K_H(t_0, s, t) := \left( \frac{s-t_0}{t-t_0} \right)^{H-\frac{1}{2}}$$

(1.8)

and

$$R_H(t_0, s, t) := \left( \frac{(t-t_0)^{H+\frac{1}{2}}}{(s-t_0)^{H-\frac{1}{2}}} \right).$$

(1.9)

We collect few facts about $K_H$ and $R_H$ in the following two lemmata.

**Lemma 1.10.** Consider the function $K_H$ in equation (1.8). Then,

1. the real-valued function $s \mapsto K_H(t_0, s, t)$ is square integrable over the interval $[t_0, t]$ with

$$\int_{t_0}^t K_H(t_0, u, s)K_H(t_0, u, t) du = R_H(t_0, t, s)/2H,$$

for any $t_0 < s \leq t$;
2. the real-valued function \((s,t) \mapsto K_H(t_0, s, t)\) is square integrable over the simplex \(\{t_0 \leq s \leq t \leq T\}\) with

\[
\int_{t_0}^{t} dt \int_{t_0}^{t} ds K_H^2(t_0, s, t) = R_H^2(t_0, T, T)/4H;
\]

3. for all \(s \leq t\) it holds \(R_H(t_0, t, s) \leq R_H(t_0, t, t).\)

**Lemma 1.11.** Consider the reciprocal \(R_H^{-1}\) of \(R_H\), defined as

\[
R_H^{-1}(t_0, s, t) = \frac{(s - t_0)^{H-\frac{1}{2}}}{(t - t_0)^{H+\frac{1}{2}}}. 
\]

Then,

1. the real-valued function \(s \mapsto R_H^{-1}(t_0, s, t)\) is square integrable over the interval \([t_0, t]\) with

\[
\int_{t_0}^{t} R_H^{-1}(t_0, u, s)R_H^{-1}(t_0, u, t)du = R_H^{-1}(t_0, s, t)/2H,
\]

for all \(t_0 < s \leq t\);

2. the real-valued function \((s, t) \mapsto R_H^{-1}(t_0, s, t)\) is in \(L^q(t < s \leq t \leq T)\) for all \(1 \leq q < 2\), but not square integrable over the simplex \(\{t_0 \leq s \leq t \leq T\}\), with

\[
\int_{t_0}^{t} dt \int_{t_0}^{t} ds R_H^{-q}(t_0, s, t) = R_H^{-q}(t_0, T, T)/(2 - q)(qH + 1 - q/2);
\]

3. for all \(0 < \epsilon_1 \leq \epsilon_2\) it holds

\[
R_H^{-1}(t_0, t + \epsilon_1, t + \epsilon_2) \leq R_H^{-1}(t_0, t, t).
\]

The functions \(K_H\) and \(R_H\) are used to describe the Gaussian processes \(\zeta\) and \(\psi\) introduced in the following two lemmata.

**Lemma 1.12.** The Volterra-type formula

\[
\psi(t, t_0, H) = \int_{t_0}^{t} R_H^{-1}(t_0, s, t)dW_s
\]

defines a centred Gaussian \((\mathfrak{F}_t)\)-adapted process on \([t_0, t]\) with covariance structure

\[
E\psi(s, t_0, H)\psi(t, t_0, H) = R_H^{-1}(t_0, s, t)/2H, \quad s \leq t.
\]

Setting \(\psi(t_0, t_0, H) := 0\), the adapted process \(\{\psi(t, t_0, H) : t_0 \leq t \leq T\}\) is a well defined element of \(L^1([0, T]; L^2(P))\) and it is approximated with respect to \(|||·|||\) by the sequence

\[
\psi_{\epsilon}(t) := \int_{t_0}^{t} R_H^{-1}(t_0, s + \epsilon, t + \epsilon)dW_s \quad (1.10)
\]

of elements of \(L^2(P \otimes dt/(T - t_0))\).

**Remark 1.13.** For every \(\epsilon > 0\) the process \(\psi_{\epsilon}\) of equation (1.10) is a semimartingale adapted to the filtration \((\mathfrak{F}_t)\) of the Brownian motion \(W\).

**Proof of Lemma 1.12.** Consider the function \(g\) in \(C^{1,2}([t_0, T] \times \mathbb{R})\) defined as

\[
g(t, x) := (t - t_0)^{-\frac{1}{2} - H} x.
\]
Let $\epsilon > 0$. Consider the centred Gaussian martingale

$$\xi_\epsilon(t) := \int_{t_0}^t (u + \epsilon - t_0)^{H-\frac{1}{2}} dW_u,$$

and the process

$$\tilde{\psi}_\epsilon(t) := g(t + \epsilon, \xi_\epsilon(t + \epsilon))$$

$$= \int_{t_0}^{t+\epsilon} R^{-1}_H(t_0, s + \epsilon, t + \epsilon) dW_s.$$  \hfill (1.12)

Let $0 < \epsilon_1 \leq \epsilon_2$. We can estimate

$$\int_{t_0}^T \|\tilde{\psi}_{\epsilon_1}(t) - \tilde{\psi}_{\epsilon_2}(t)\|_{L^2(P)} dt$$

$$= \frac{1}{2H} \int_{t_0}^T |R_H^{-1}(t_0, t + \epsilon_1, t + \epsilon_1) - R_H^{-1}(t_0, t + \epsilon_2, t + \epsilon_2)| dt$$

$$- 2R_H^{-1}(t_0, t + \epsilon_1, t + \epsilon_2)|^{1/2} dt$$

$$\leq \frac{1}{2H} \int_{t_0}^T \left| R_H^{-1}(t_0, t + \epsilon_1, t + \epsilon_1) - R_H^{-1}(t_0, t + \epsilon_1, t + \epsilon_2) \right|^{1/2} dt$$

$$+ \frac{1}{2H} \int_{t_0}^T \left| R_H^{-1}(t_0, t + \epsilon_2, t + \epsilon_2) - R_H^{-1}(t_0, t + \epsilon_1, t + \epsilon_2) \right|^{1/2} dt$$

$$\to 0, \quad \text{as } \epsilon_1, \epsilon_2 \to 0.$$

We have used dominated convergence with domination

$$|R_H^{-1}(t_0, t + \epsilon_1, t + \epsilon_1) - R_H^{-1}(t_0, t + \epsilon_1, t + \epsilon_2)|^{1/2}$$

$$\leq 2R_H^*(t_0, t, t).$$

Therefore, $\{\psi(t, t_0, H) : t_0 \leq t \leq T\}$ exists as limit in $L^1([0, T]; L^2(P))$ and defines a Gaussian process on $[t_0, T]$ with the claimed covariance structure. Finally,

$$\tilde{\psi}_\epsilon(t) - \tilde{\psi}_\epsilon(t) = \int_{t}^{t+\epsilon} R^{-1}_H(t_0, s + \epsilon, t + \epsilon) dW_s$$

and

$$E \left( \tilde{\psi}_\epsilon(t) - \tilde{\psi}_\epsilon(t) \right)^2 = (t + \epsilon - t_0)^{-2H-1} \int_{t}^{t+\epsilon} (s + \epsilon - t_0)^{2H-1} ds.$$

In both cases $0 < H < 1/2$ and $H \geq 1/2$, we have

$$E \left( \tilde{\psi}_\epsilon(t) - \tilde{\psi}_\epsilon(t) \right)^2 \lesssim \epsilon(t + \epsilon - t_0)^{-2},$$

so that

$$\int_{t_0}^T \|\tilde{\psi}_\epsilon(t) - \tilde{\psi}_\epsilon(t)\|_{L^2(P)} dt \lesssim \epsilon^{1/2} \int_{t_0}^T (t + \epsilon - t_0)^{-1} dt$$

$$= \epsilon^{1/2} (\log(T + \epsilon - t_0) - \log \epsilon).$$

The right hand side goes to zero as $\epsilon \to 0$. \hfill \Box

**Lemma 1.14.** The Volterra-type formula

$$\zeta(t, t_0, H) = \int_{t_0}^T K_H(t_0, s, t) dW_s$$
defines a centred Gaussian $(\hat{\eta}_t)$-adapted process on $[t_0, T]$ with covariance structure
\[ E\zeta(s, t_0, H)\zeta(t, t_0, H) = R_H(t_0, t, s)/2H, \quad s \leq t. \]

Moreover, $\zeta$ is a semimartingale and for all $t_0 \leq t \leq T$
\[ \zeta(t, t_0, H) = W_t - W_{t_0} + \left( \frac{1}{2} - H \right) \int_{t_0}^{t} \psi(s, t_0, H)ds, \quad (1.13) \]
where equality is meant in $L^2(P)$ and $\psi$ was defined in Lemma 1.12.

**Proof.** Point 2 of Lemma 1.10 yields the first claim. We establish the second claim. Consider the function $f$ in $C^{1,2}([t_0, T] \times \mathbb{R})$ defined as
\[ f(t, x) := (t - t_0)\frac{1}{2} - H x. \]
Let $\epsilon > 0$. Consider the processes
\[ \zeta(t) := f(t + \epsilon, \xi(t)), \]
where $\xi$ was defined in equation (1.11). Since $f$ is twice continuously differentiable on $[t_0 + \epsilon, T] \times \mathbb{R}$, by Itô’s lemma $\zeta$ is a semimartingale in $L^2(P \otimes dt/(T - t_0))$ and
\[ \zeta(t) = f(t_0 + \epsilon, \xi(t_0)) + \int_{t_0}^{t} \partial_x f(s + \epsilon, \xi(s))d\xi(s) \]
\[ + \int_{t_0}^{t} \partial_t f(s + \epsilon, \xi(s))ds \]
\[ = W_t + W_{t_0} + \left( \frac{1}{2} - H \right) \int_{t_0}^{t} \psi(s)ds + o_\epsilon(1), \]
where $\psi$ was defined in equation (1.10) and $o_\epsilon(1)$ is going to 0 in $L^2(P)$ as $\epsilon \downarrow 0$. By Minkowski integral inequality, we have that
\[ \left\| \int_{t_0}^{t} \psi(s) - \psi(s)ds \right\|_{L^2(P)} \leq \int_{t_0}^{t} \left\| \psi(s) - \psi(s) \right\|_{L^2(P)}ds. \]
Therefore, letting $\epsilon \downarrow 0$ yields equation (1.13). \hfill \Box

Consider, for $\epsilon > 0$, the process
\[ \zeta(t, t_0, H) := W_t - W_{t_0} + \left( \frac{1}{2} - H \right) \int_{t_0}^{t} \psi(s)ds, \quad t_0 \leq t \leq T, \quad (1.14) \]
where $\psi$ was defined in equation (1.10). The proof above shows that $\zeta(t, t_0, H)$ converges to $\zeta(t, t_0, H)$ in $L^2(P)$. Since $\text{Var}\psi_\epsilon(t) \leq R_H^{-1}(t_0, t + \epsilon, t + \epsilon)/2H$, we have that
\[ \sup_{t_0 \leq t \leq T} \text{Var}\psi_\epsilon(t) \leq \epsilon^{-1}/2H, \]
and for $0 < \eta < 2H\epsilon$
\[ \sup_{t_0 \leq t \leq T} E \exp \left( \eta\psi_\epsilon^2(t) \right) < \infty. \]
This is a Novikov-type condition, see [RY99, Chapter VIII, (1.40)Exercise]. Therefore, for all $\epsilon > 0$ there exists a probability $P^\epsilon$, equivalent to the physical measure $P$, such that $\zeta(\cdot, t_0, H)$ is a Brownian motion under $P^\epsilon$. More precisely, $P^\epsilon$ is given by the formula
\[ \frac{dP^\epsilon}{dP} |_{\mathcal{G}_t} = \exp \left( (H - \frac{1}{2}) \int_{t_0}^{t} \psi_\epsilon(s)dW_s - \frac{1}{2}(H - \frac{1}{2})^2 \int_{t_0}^{t} \psi_\epsilon^2(s)ds \right), \quad t_0 \leq t \leq T. \]
**Remark 1.15.** Informally\(^3\) passing to the limit as \( \epsilon \downarrow 0 \) in the change of measure above yields the Wick exponential of

\[
(H - \frac{1}{2}) \int_{t_0}^{t} \psi(s, t_0, H) dW_s.
\]

Borrowing the terminology introduced in Example 1.6, we can then refer to \((H - 1/2)\psi(t, t_0, H)\) as the (time-dependent) market price of risk. Its path trajectories are in general rougher than the semimartingales’ ones. Moreover, its behaviour for small times \( t > t_0 \) is proportional to \((t - t_0)^{-\alpha}\), for some \( 0 < \alpha < 1 \). Such behaviour is consistent with the one of the at-the-money skew of the implied volatility, which was empirically observed by Bayer et al. [BFG16]. It is the combination of this small time behaviour with the empirically found roughness of the volatility process that motivated Bayer et al. to introduce the rough Bergomi model. These features are here exhibited at the level of the market price of risk \((H - \frac{1}{2})\psi\).

Let \( \mu \) be a real number, which we fix. With \( \sigma \) in \( \mathbb{R}_+ \), we define the line \( \ell(t, t_0, \mu, \sigma) \) as

\[
\ell(t, t_0, \mu, \sigma) := (\mu - \sigma^2/2)(t - t_0), \quad t_0, t \in [0, T]. \tag{1.15}
\]

The letter \( X \) will refer to geometric Brownian motion, defined for \( t_0 \leq t \leq T \) as

\[
X(t, t_0, \mu, \sigma) := x_0 \exp \left( \ell(t, t_0, \mu, \sigma) + \sigma W_t - \sigma W_{t_0} \right). \tag{1.16}
\]

Example 1.6 has shown that \( \{X(\cdot, t_0, \mu, \sigma) : \mu \in \mathbb{R} \} \) is a family of NAP indifferent processes, whose pricing dynamics is the one of \( X(\cdot, t_0, r, \sigma) \), with \( r \) denoting the fixed interest rate in the market model.

**Proposition 1.16.** Let \( \mu \) be a real coefficient and let \( \sigma_1 \) and \( \sigma_2 \) be two positive real numbers. Then, the process\(^4\)

\[
Y(t, t_0, \sigma_1, \sigma_2) = x_0 \exp \left( \sigma_2 \zeta(t, t_0, \frac{\sigma_1^2}{2\sigma_2^2}) + \ell(t, t_0, \mu, \sigma_1) \right),
\]

\[
t_0 \leq t \leq T,
\]

is simultaneously weakly NAP-equivalent from \( X(\cdot, t_0, \mu, \sigma_2) \) and marginally identical to \( X(\cdot, t_0, \mu, \sigma_1) \).

**Remark 1.17.** The quadratic variation of \( Y \) is

\[
[Y]_{t_0, t} = \sigma_2^2 \int_{t_0}^{t} Y(s) ds.
\]

This is the same as the one of \( X(\cdot, t_0, \mu, \sigma_2) \), since

\[
[X(\cdot, t_0, \mu, \sigma_2)]_{t_0, t} = \sigma_2^2 \int_{t_0}^{t} X(s) ds,
\]

but different from that of \( X(\cdot, t_0, \mu, \sigma_1) \). With this respect, no-arbitrage pricing is sensitive to quadratic variations but blind to marginal variances.

**Proof of Proposition 1.16.** For simplicity we take \( x_0 = 1 \). Consider the process

\[
Y^c(t, t_0, \sigma_1, \sigma_2) = \exp \left( \sigma_2 \zeta(t, t_0, \frac{\sigma_1^2}{2\sigma_2^2}) + \ell(t, t_0, \mu, \sigma_1) \right),
\]

\(^3\)The limiting change of measure is delicate because it entails that some mass is lost; indeed, \( P(\int_{t_0}^{t_0} \psi^2(s, t_0, H) ds = \infty) > 0 \).

\(^4\)The drift \( \mu \) is suppressed from the notation for \( Y \).
where \( \zeta \) was defined in equation (1.14). We know already that \( \log Y_t(t) \rightarrow \log Y(t) \) in \( L^2(P) \). Moreover, there exists a probability measure \( P^\prime \), equivalent to \( P \), such that \((\zeta, P^\prime)\) is a Brownian motion, and thus there exists an equivalent \( Q^\prime \) such that \((\log Y, Q^\prime)\) has the law of \( \log X (\cdot, t_0, r, \sigma_2) \). This shows the asserted weak NAP-equivalence.

As for the marginal identity, it suffices to notice that \( \log Y(t, t_0, \sigma_1, \sigma_2) \) is a Gaussian random variable with mean \( \ell((t, t_0, \mu, \sigma_1)) \) and variance

\[
\sigma^2 \text{Var}(\zeta(t, t_0)) = \sigma^2 \left[ R_H(t_0, t, t)/2H \right]_{H=\sigma^2/2\sigma^2}
\]

\[
= \sigma^2(t - t_0).
\]

These are the mean and the variance of the Gaussian random variable \( \log X(t, t_0, \mu, \sigma_1) \).

Let \( \pi \) be a partition of the time window \([0, T]\), and recall the notational convention in equation (1.2). For \( u \) in \( \pi \) consider the process

\[
Z(t, u, \sigma_1, \sigma_2) = \begin{cases} 0 & 0 \leq t \leq u, \\ \ell(t, u, \mu, \sigma_1) + \sigma_2 \zeta(t, u, \sqrt{\frac{\sigma^2}{2\sigma^2}}) & u < t \leq u', \\ \ell(u', u, \mu, \sigma_1) + \sigma_2 \zeta(u', u, \sqrt{\frac{\sigma^2}{2\sigma^2}}) & t > u'. \end{cases}
\]

(1.17)

**Proposition 1.18** ([BM00, Propositions 2.1 and 2.2]). Let \( \sigma_1 \) and \( \sigma_2 \) be two positive real numbers, and correspondingly consider the geometric Brownian motions \( X(\cdot, \sigma_i) = X(\cdot, 0, \mu, \sigma_i), i = 1, 2, \) defined in equation (1.16), for some \( \mu \) in \( \mathbb{R} \). Let \( \pi \) be a time grid in the time window \([0, T]\), and correspondingly define the processes \( Z \) as in equation (1.17). Let \( Y = Y(\cdot, \sigma_1, \sigma_2) \) be the process

\[
Y(t, \sigma_1, \sigma_2) = x_0 \exp \sum_{u \in \pi} Z(t, u, \sigma_1, \sigma_2),
\]

\[
0 \leq t \leq T.
\]

Then, it simultaneously holds

1. \( Y(\cdot, \sigma_1, \sigma_2) \) and \( X(\cdot, \sigma_1) \) are \( \pi \)-indistinguishable;
2. \( Y(\cdot, \sigma_1, \sigma_2) \) and \( X(\cdot, \sigma_2) \) are weakly NAP-equivalent.

**Proof.** We split the proof in two parts, which correspond to the statements.

1. Let \( u \) be a partition point and observe that for \( t > u \) the variable \( \log Y(t, \sigma_1, \sigma_2) \) is independent from \( \zeta_u \). Moreover,

\[
\log Y(t, \sigma_1, \sigma_2) - \log Y(u, \sigma_1, \sigma_2) = \sum_{v \in \pi \atop u \leq v < [t]} \log \frac{Y(v, \sigma_1, \sigma_2)}{Y(v, \sigma_1, \sigma_2)} + \log \frac{Y(t, \sigma_1, \sigma_2)}{Y([t], \sigma_1, \sigma_2)}.
\]

(1.18)

The two summands \( L_1 \) and \( L_2 \) are independent. The second summand, \( L_2 \), is normally distributed with mean \( \ell([t], \mu, \sigma_1) \) and variance \( \sigma^2(t - [t]) \). As for the first summand \( L_1 \), we further notice the independence of the variables \( \log[Y(v', \sigma_1, \sigma_2)/Y(v, \sigma_1, \sigma_2)] \), \( v \in \pi \), which are normally distributed with mean \( \ell(v', \mu, \sigma_1) \) and variance \( \sigma^2(t - v') \). Therefore, \( L_1 \) is normally distributed with mean \( \ell([t], \mu, \sigma_1) \) and variance \( \sigma^2([t] - u) \). Hence, \( \log[Y(t, \sigma_1, \sigma_2)/Y(u, \sigma_1, \sigma_2)] \) is distributed as \( \log[X(t, \sigma_1)/X(u, \sigma_1)] \).
2. On each subinterval \([u, u']\) of \(\pi\), the processes
\[
\begin{align*}
\left\{ \frac{Y(t, \sigma_1, \sigma_2)}{Y(u, \sigma_1, \sigma_2)} : u \leq t \leq u' \right\} \\
\left\{ \frac{X(t, \sigma_2)}{X(u, \sigma_2)} = X(t, u, \mu, \sigma_2) : u \leq t \leq u' \right\}
\end{align*}
\]
are weakly NAP indifferent, as argued in Proposition 1.16. Therefore we conclude by recalling Corollary 1.9.

\[\square\]

2 Notation and preliminaries

We introduce the adopted financial terminology and notation by concisely recalling the main steps in the Black-Scholes theory. This will terminate in the definition of Delta and Gamma sensitivities of a portfolio, which we will later generalise to diffusion models (local volatility models). Such sensitivities will constitute the main ingredients in the hedging strategies.

Suppose that each component of the price vector \(S_t \in \mathbb{R}^d\) of \(d\) non-dividend paying stocks displays the following dynamics
\[
dS^i_t = S^i_t \left( \mu^i dt + \sigma^i dB^i \right), \quad i = 1 \ldots d, \quad S^i_0 = s^i_0 \in \mathbb{R}^d,
\]
on a stochastic base \((\Omega, \mathcal{F}, P, (\mathcal{F}_t)_t, (B_t)_t)\) carrying a standard \(n\)-dimensional Brownian motion \((B_t)_t\). Einstein’s summation convention on double indices is employed and will be throughout all the paper. The vector \(\mu\) in \(\mathbb{R}^d\) and the matrix \(\sigma\) in \(\mathbb{R}^{d \times n}\) are the model parameters. It is assumed that there exists \(\kappa\) in \(\mathbb{R}^n\) such that \(\sigma \kappa = r - \mu\), where \(r = (r, \ldots, r) \in \mathbb{R}^d\) is the vector of constant deterministic interest rate. This guarantees the absence of arbitrage \([Bjo09, \text{Proposition 14.1}]\). The process \((S_t)_t\) is referred to as risky asset. The riskless asset is instead denoted by \(S^0_t\) and follows the one-dimensional deterministic dynamic
\[
dS^0_t = rS^0_t dt, \quad S^0_0 = 1.
\]

Any \((\mathcal{F}_t)_t\)-adapted process \(\phi_t = (H^0_t, H_t) \in \mathbb{R} \times \mathbb{R}^d\) is referred to as strategy, provided that
\[
\int_0^T \left| H^0_u \right| dS^0_u + \sum_{i=1}^d \int_0^T \left| H^i_u S^i_u \right|^2 du < \infty \quad P - \text{a.s.}
\]
The value of its corresponding portfolio is defined as the real-valued process
\[
V_t(\phi) := H^0_t S^0_t + H_t S_t,
\]
and we say that the strategy is self-financing if on a \(P\)-full set
\[
V_t(\phi) = V_0(\phi) + \int_0^t H^0_u dS^0_u + \int_0^t H_u dS_u. \tag{2.1}
\]
Equivalently, \(\phi\) is self-financing if and only if the discounted value \(\tilde{V}_t(\phi) := V_t(\phi)/S^0_t = e^{-rt}V_t(\phi)\) is such that
\[
\tilde{V}_t(\phi) = V_0(\phi) + \int_0^t H_u d\tilde{S}_u, \quad P - \text{a.s.},
\]
where $\tilde{S}_t := S_t/S_0^0 = e^{-rt}S_t$ is the discounted price of the stocks. Notice that $d\tilde{S}_t = \tilde{S}_t((\mu_t - \gamma)dt + \sigma_t^jdB^j_t)$. By Girsanov theorem, there exists a probability measure $Q$ equivalent to $P$ such that under $Q$ the process

$$W_t := B_t - \kappa t$$

is a standard Brownian motion. The measure $Q$ is referred to as pricing measure - as opposed to the “physical” measure $P$ - and is explicitly given by

$$\frac{dQ}{dP}|_{\mathcal{F}_t} = \exp \left\{ \kappa \cdot B_t - \frac{t}{2} |\kappa|^2 \right\}.$$ 

Since $d\tilde{S}_t = \tilde{S}_t \sigma^j_t dW^j_t$, we have that under $Q$ the discounted price vector is a martingale. As a consequence, the discounted portfolio value $\tilde{V}_t$ of any self-financing strategy is a $Q$-martingale too and

$$\tilde{V}_t = E_Q[\tilde{V}_T | \mathcal{F}_t].$$

(2.2)

Let $f(S_T)$ be the payoff of a Vanilla option on the underlying $S$. We assume that $f$ is a continuous and bounded function on $\mathbb{R}^d$. Let

$$h(x) := f(e^{rT}x)$$

(2.3)

and $\tilde{h} := e^{-rT}h$. The payoff is therefore equivalently written as $h(S_T)$, and its discounted value is $\tilde{h}(\tilde{S}_T)$.

Assume that there exists a self-financing $\phi$ such that $\tilde{V}_T(\phi) = \tilde{h}(\tilde{S}_T)$, $P$-almost surely. Then, (2.2) justifies the pricing paradigm according to which the price at time $t < T$ of a contingent claim $f = f(S_T)$ is given by

$$p(t,T) = E_Q[e^{-r(T-t)}f(S_T) | \mathcal{F}_t].$$

We explicitly remark that if $\tilde{V}_T(\phi) = \tilde{h}(\tilde{S}_T)$ then (2.2) reads

$$\tilde{V}_t = \mathbb{P}_{t-\cdot} \tilde{h}(\tilde{S}_t),$$

where $(\mathbb{P}_t)_t$ is the semigroup on $C_b(\mathbb{R}^d)$ generated\(^5\) by the infinitesimal operator

$$A \psi(x) = \frac{1}{2} \sum_k \sigma^k_i \sigma^k_j \partial^2_{i,j} \psi(x), \quad \psi \in C^2(\mathbb{R}^d) \cap C_b(\mathbb{R}^d),$$

of the dynamics of $\tilde{S}$. Hence, the stochastic process $\tilde{V}_t$ is a deterministic function $w = w(t,x)$ of time and space applied after $(t, S_t)$ which solves

\[ \left\{ \begin{array}{l} (\partial_t + A)w = 0 \quad \text{in } [0,T] \times \mathbb{R}^d \\ w(T,x) = h(x) \quad \text{on } \{T\} \times \mathbb{R}^d. \end{array} \right. \]

(2.4)

Equation (2.4) is the discounted version of the celebrated Black-Scholes partial differential equation; solving it amounts to finding the arbitrage-free price of the contingent claim $f(S_T)$. This task does not involve the local mean $\mu$ of the physical dynamic for $S$ which instead is needed for the change of measure above and hence the justification of the pricing paradigm. The choice of focusing on discounted trajectories

\[^5\] By this we mean the semigroup of linear operators on $C_b(\mathbb{R}^d)$ such that for any continuous and bounded $f$, the solution of the Cauchy problem

\[ \left\{ \begin{array}{l} (\partial_t - A)u = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^d \\ u(0,x) = f(x) \quad \text{on } \{0\} \times \mathbb{R}^d. \end{array} \right. \]

is represented as $u(t,x) = \mathbb{P}_t f(x)$, see [LB07, Theorem 2.2.5].
\( \tilde{S}_t \) when elaborating the mathematical discourse reflects this regardlessness of the drift under the physical measure \( P \): for pricing and hedging the dynamic that matters is \( dS^t = S^t (r dt + \sigma^t dW^t) \), with fixed drift determined only by the interest rate and hence constituting not a feature of the price trajectory but one of the environment. Nonetheless, when the mathematical discourse leaves its place to financial considerations, the fundamental object is the (undiscounted) price trajectory \( S_t \), hence constituting not a feature of the price trajectory but one of the environment.

For the undiscounted price of the contingent claim we therefore have \( V_t = v(t, S_t) \), with \( v = v(t, z) \) solution to

\[
\begin{cases}
  \left( \partial_t + L \right)(e^{-rt} v) = 0 & \text{in } [0, T) \times \mathbb{R}^d \\
  v(T, z) = f(z) & \text{on } \{ T \} \times \mathbb{R}^d.
\end{cases}
\]

Notice that \( v(t, z) = e^{rt}w(t, e^{-rt}z) \).

We set the following notation for future reference:

\[
\text{Delta}_t := \nabla_x v(t, S_t) = \nabla_x w(t, \tilde{S}_t),
\]

taking values in \( \mathbb{R}^d \cong \text{Hom}(\mathbb{R}^d, \mathbb{R}) \), and

\[
\text{Gamma}_t := \nabla_{zz} v(t, S_t) = e^{rt} \nabla_{zz} \tilde{w}(t, \tilde{S}_t),
\]

taking values in \( \mathbb{R}^{d \times d} \cong \text{Hom}(\mathbb{R}^d \otimes \mathbb{R}^d, \mathbb{R}) \cong \text{Hom}(\mathbb{R}^d, \text{Hom}(\mathbb{R}^d, \mathbb{R})) \).

**Hedging formulas and heuristics towards pathwise formulation**

We focus on the hedging practice that was justified above, and we extend it to multi-dimensional diffusion models (local volatility models). Exterior to our models are the constant interest rate \( r \geq 0 \) and the bounded continuous function \( f \) that describes the payoff of the contingent claim to hedge. Interior to our models is the specification of the continuous locally \( \alpha \)-Hölder function \( \gamma \) in \( C^0_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^{d \times d}) \), \( 0 < \alpha < 1 \), that describes the volatility of the price. Notice that this generalises the linear case of the previous section.\(^6\) Since we are firstly interested in the mathematics of the models, it is convenient here to work with discounted price trajectories: as recalled above, it is on the discounted diffusion that the economic justification imposes the martingality, and it is the discounted diffusion that the generator is driftless. The volatility \( \sigma \) is thus thought of as a function of the \( x \)-variable (as opposed to the \( z = e^{rt}x \)-variable) which, applied after \( \tilde{S}_t \), gives the diffusion coefficient in the stochastic dynamic of the discounted price trajectory. We set \( a = (a_{i}^{j})_{i,j} = \sigma \sigma^{T} \) and let \( A \) be the second-order differential operator

\[
A \varphi(x) = \frac{a_{i}^{j}(x)}{2} \partial_{i,j} \varphi(x), \quad \varphi \in C^2(\mathbb{R}^d).
\]

\(^6\) We remark that in the non-linear case the change of coordinates that brings the generator \( A = \frac{1}{2}(\sum_k \sigma_k^{i} \sigma_k^{j}(x))\partial_{i,j}^{2} \) of the dynamic of \( S \) into the generator of the dynamic of the undiscounted \( S \) will result in the time-dependent generator

\[
\frac{e^{2rt}}{2} \left( \sum_k \sigma_k^{i} \sigma_k^{j}(e^{-rt}z) \partial_{i,j}^{2} \varphi(z) - rz^{i} \partial_{x,i} \varphi(z) \right).
\]
The operator $A$ is referred to as a volatility operator and it is assumed to be locally uniformly elliptic. We denote by $(X_t)$, the dynamic $dX = \sigma(X) dW$ associated with $A$, where it remains understood that $W_t$ is a standard $n$-dimensional Brownian motion on a stochastic base $(\Omega, \mathcal{F}, Q, (\mathcal{F}_t))$. In this classical probabilistic framework, the discounted trajectory $\hat{S}$ is thought of as a realisation of $X$ under the pricing measure $Q$. Set $P_t := e^{tA}$ for the semigroup of $A$ on $C^0_b(\mathbb{R}^d)$ and define

$$w(t, x) := P_{T-t}\hat{h}(x),$$

where $\hat{h} := e^{-Tt}h$ and $h$ as in equation (2.3). By Itô-Doeblin formula,

$$w(t, X_t) - w(0, X_0) = \int_0^t \nabla_x w(u, X_u) \sigma(X_u) dW_u + \int_0^t (\partial_t + L) w(u, X_u) du$$

$$= \int_0^t \nabla_x w(u, X_u) dX_u. \tag{2.5}$$

Equation (2.5) is the linchpin of hedging. Indeed, under the interpretation of $X$ as the stochastic process that governs the realisation $\hat{S}$, this equation implies that the strategy $\phi_t = (H^0_t, H_t)$ given by

$$H_t := \nabla_x w(t, \hat{S}_t)$$

$$H^0_t := w(t, \hat{S}_t) - H_t \hat{S}_t. \tag{2.6}$$

is such that the undiscounted portfolio process

$$V_t(\phi) = H^0_t e^{rt} + H_t S_t = e^{rt} w(t, \hat{S}_t),$$

is self-financing and replicates the option payoff, i.e. $V_T(\phi) = f(S_T)$.

Consider how equation (2.5) was derived: from the specification of a “feature” of the price – namely, the volatility $\sigma$ –, a diffusion process $X$ on a stochastic base $(\Omega, \mathcal{F}, Q, (\mathcal{F}_t))$ was introduced, with generator $A$ defined in terms of $\sigma$; then, Itô formula applied to $w(t, X_t) = e^{(T-t)A}\hat{h}(X_t)$ yielded equation (2.5). In Section 4 we propose to parsimoniously reconsider such derivation, refraining in particular from the introduction of a probability space. This means analysing the discounted price trajectory in a pathwise sense, and entails the technical difficulty of integration with respect to unbounded variation signals. Section 3 presents how we solve this difficulty by relying on (reduced) rough integration theory, see [FH14]. Once established this, we will directly bridge the specification of the volatility $\sigma$ of a price to an “enhanced” price trajectory that carries such additional information and can be used as pathwise integrator.

The construction will reveal two important findings. Firstly, the role usually assigned to the quadratic variation can more generally be assigned to a linear transformation of level two of the price path signature, which we shall refer to as “enhancer” of the path. This gives rise to the possibility to consider trajectories for which this enhancer is not of bounded variation and, through a P&L formula, to assess the misspecification of the enhancer. On the other hand, the construction reveals that the machinery of Delta hedging can be made work in a rougher regime than the usual semimartingale one. Price trajectories can be assumed only to be continuous of finite $p$-variation for some $p < 3$, rather than for all $p > 2$ as with semimartingales.

We will also address the consequence of time discretisation. In real trading, hedging happens in discrete time and, given a partition $\pi$ of $[0, T]$ of meshsize $|\pi| := \sup\{|u' - u| : u \in \pi\}$, the strategy $H_t$ of equation (2.6) is commonly replaced by

$$\pi H_t := \sum_{u \in \pi} H_u \mathbb{1}\{t \in (u, u']\},$$
where for $u \in \pi$ we denoted by $u' := \inf\{v \in \pi : u < v\}$ its next partition point. The (Itô-) integral of the elementary caglad process $H$ is

$$
(\mathbf{H}\cdot X)_t = \sum_{u \in \pi} H_u X_{u,u'}. 
$$

$H$ being continuous, the discrete-time integral converges in probability to the Itô integral $\int H \, dX$ along any sequence $(\pi^n)_n$ of partitions with meshsize $|\pi^n|$ shrinking to zero (see [RY99, Chapter IV, (2.13)Proposition]). Two issues are to be stressed about this convergence. First, the convergence does not happen pathwise for arbitrary sequences of partitions; if it did, the integrator $X$ would be of bounded variation (see Proposition 2.1 below), which is not the case for semimartingales. Second, the convergence is not uniform with respect to equivalent martingale measures on the same filtered probability space (see Example 2.2 below).

Estimating portfolio increments then necessarily involves higher order expansions than the first order one. To motivate the construction of later sections, we show this heuristically here.

Consider the Itô-enhanced trajectory $X = (X, \mathbb{X})$, where

$$
2\mathbb{X}_{s,t} := (X_t - X_s) \otimes (X_t - X_s) - \langle X, X \rangle_{s,t}. 
$$

The symbol $\langle X, X \rangle_{s,t}$ stands for the quadratic variation of the semimartingale $X$ in the time interval $[s,t]$. Let $H'_t := \nabla_x w(t, X_t)$ be the Gamma sensitivity of the discounted replicating portfolio. By adding and subtracting $(\mathbf{H}\cdot\mathbb{X})_t$ to the difference $w(t, X_t) - w(0, X_0) - (\mathbf{H}\cdot X)_t$ we can estimate

$$
|w(t, X_t) - w(0, X_0) - (\mathbf{H}\cdot X)_t| 
\leq \left| \int_0^t H \, dX - (\mathbf{H}\cdot X)_t - (\mathbf{H}'\cdot\mathbb{X})_t \right| 
+ \sum_{u \in \pi} \left| H'_u \mathbb{X}_{u\land t,u'\land t} \right| 
+ \sum_{u \in \pi} \left| H''_u \mathbb{X}_{u\land t,u'\land t} \right| 
\leq K |\pi|^{-1} t + \sum_{u \in \pi} \left| H''_u \mathbb{X}_{u\land t,u'\land t} \right|. 
$$

(2.7)

In the first step we have used (2.5), in the second we have identified the Itô integral against $X$ with the rough path integral against the Itô-enhancement of $X$, and in the third we have used the “Sewing Lemma” [FH14, Lemma 4.2] (a version of which we give in Section 3). The exponent $\gamma$ is a real number strictly greater than 1, so that the first summand on the upper bound goes to zero as $|\pi| \downarrow 0$. However, in general the second summand does not vanish in the limit $|\pi| \downarrow 0$. This is because the considerations above are those of pathwise analysis. Referring to any $P$-equivalent measure instead, one classically has that the quantity

$$
\sum_{u \in \pi} H'_u \mathbb{X}_{u\land t,u'\land t} = \sum_{u \in \pi} \left[ H'_u X_{u\land t,u'\land t} \otimes X_{u\land t,u'\land t} - H'_u \langle X, X \rangle_{u\land t,u'\land t} \right]
$$

goes to zero in probability along any sequence of partitions with vanishing meshsize (see for example [RY99, Chapter IV, (1.33)Exercise]). This is an instance of the role played, in the classical Black-Scholes model, by the interpretation of the (undiscounted) price trajectory as a realisation $t \mapsto S_t(\omega)$ of a continuous semimartingale on a
stochastic base \((\Omega, \mathcal{F}, P, (\mathbb{F}_t)_t)\). By considering pathwise integrals as limits of compensated Riemann sums we will circumvent this issue. Of course, this raises the question of the interpretation of such integrals. Indeed, because of the compensation, it is not immediately clear that they can be used as representative of portfolio increments. The question is positively answered by N. Perkowski and D. Pr"omel in [PP16] by showing that model-independent integration is still subject of financial interpretation. We will implicitly rely on this when discussing the translation of classical formulas of Mathematical Finance into the pathwise ones.

**Technical motivation for compensated Riemann sums**

There are two technical reasons which motivate the use of compensated Riemann sums. First, if the integrator has semimartingale-type regularity (or worse), then uncompensated Riemann sums cannot converge pathwise (see Proposition 2.1). Second, in the semimartingale setting the rate of convergence of uncompensated Riemann sums to the corresponding Itô integral depends on the underlying probability measure (see Example 2.2).

**Proposition 2.1** ([RY99, Chapter IV, (2.21)Exercise]). Let \(X\) be an \(\mathbb{R}^d\)-valued function on \([0, T]\) such that for every \(H\) in \(C([0, T], \mathbb{R}^d)\) the limit of (uncompensated) Riemann sums

\[
\lim_n(\pi_n H X)_T = \lim_n \sum_{u \in \pi_n} H_n X_{u,u'}
\]

exists in \(\mathbb{R}\) along any sequence \((\pi_n)_n\) of partitions of \([0, T]\) with vanishing meshsize. Then, the limit does not in fact depend on the specific sequence of partitions and \(X\) is of bounded variation.

The statement in Proposition 2.1 is classical. Our reference for it is Exercise 2.21 in Chapter IV of the textbook by D. Revuz and M. Yor [RY99]. Hence, for the reader’s convenience, we recall the proof.

**Proof of Proposition 2.1.** The one-dimensional case \(d = 1\) suffices. Let \((\pi_n)_n\), \((\tilde{\pi}_n)_n\), and \((\tilde{\tilde{\pi}}_n)_n\) be two sequences of partitions of \([0, T]\) with vanishing meshsize. Define \(\tilde{\pi}_{2k+1} := (\pi_k)\) and \(\tilde{\tilde{\pi}}_{2k} := (\tilde{\pi}_k)\) for \(k \in \mathbb{N}\). The assumption guarantees that \(H X)_T, n \geq 1,\) is a Cauchy sequence for every \(H\) in \(C([0, T], \mathbb{R})\). Therefore, the triangulation

\[
|\lim_n(\pi_n H X)_T - \lim_n(\tilde{\pi}_n H X)_T|
\]

yields the first claim. As a consequence, for every \(H\) in \(C([0, T], \mathbb{R})\), we have

\[
\sup \{|(\pi H X)_T| : \pi \text{ partition of } [0, T]\} < \infty.
\]

But the map \(H \mapsto (\pi H X)_T\) is a bounded linear operator on \(C([0, T], \mathbb{R})\) with

\[
|(\pi H X)_T| \leq \|H\|_\infty \sum_{u \in \pi} |X_{u,u'}|.
\]

Furthermore, given \(\pi\) the integrand

\[
S_t = \left\{1 - \frac{t - [t]}{|t| - [t]}\right\} \text{sign}(X_{[t], [t]}) + \left\{\frac{t - [t]}{|t| - [t]}\right\} \text{sign}(X_{[t'], [t']}) \quad 0 \leq t \leq T,
\]
Hence, under $P$

$$\Pi_n = \sum_{\pi \in \Pi} |X_{u, u'}|.$$  

Therefore, an application of the uniform boundedness principle concludes.

**Example 2.2.** Let $\Omega$ be the space $C([0, 1], \mathbb{R})$ of continuous real-valued functions on $[0, 1]$, and let $\mathcal{F}$ be its Borel $\sigma$-algebra. Let $P$ be the Wiener measure on $(\Omega, \mathcal{F})$, so that the coordinate map $X_t(\omega) = \omega(t), \omega \in \Omega$, is a $(\mathcal{F}_t, P)$-Brownian motion, where $\mathcal{F}_t$ is the $P$-completion of $\sigma(X_s : 0 \leq s \leq t)$. Consider the sequence $P^k, k \in \mathbb{N}$, of probability measures on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$ given by

$$dP^k \mid_{\mathcal{F}_t} = \exp \left( k X_t - \frac{k^2}{2} t \right), \quad 0 \leq t \leq 1, \quad k = 1, 2, \ldots.$$  

We observe that for each $k$, the process $X_t - kt$ is a $(\mathcal{F}_t, P^k)$-Brownian motion. For each $k$, for every continuous $(\mathcal{F}_t)_{t \geq 0}$-adapted integrand $H$ and any sequence $(\pi_n)_{n \geq 1}$ of partitions with vanishing meshsize we have that

$$\sup \left\{ \lim_{n,m \to \infty} P^k \left( |(\pi_n H)_1 - (\pi_m H)_1| > \epsilon \right) : \epsilon > 0 \right\} = 0.$$  

Consider the integrand $H_s \equiv s$ and the sequence $\pi_n = \{2^{-n} : l = 0, \ldots, 2^n\}$ of dyadic partitions of $[0, 1]$. We claim that for every $\epsilon > 0$ it simultaneously holds

$$\lim_{n \to \infty} \lim_{k \to \infty} P^k \left( |(\pi_n H)_1 - (\pi_{n+1} H)_1| > \epsilon \right) = 0$$  

and

$$\lim_{n \to \infty} \lim_{k \to \infty} P^k \left( |(\pi_n H)_1 - (\pi_{n+1} H)_1| > \epsilon \right) = 1.$$  

Indeed, since $\pi_{n+1} = \pi_n \cup \{2l + 1 \cdot 2^{-(n+1)} : l = 0, \ldots, 2^n - 1\}$, we have

$$\sum_{v \in \pi_{n+1}} H_v X_{u, u'} - \sum_{u \in \pi_n} \sum_{v \in \pi_n, u < v} H_u X_{u, u'} = \sum_{u \in \pi_n} (H_{u + 2^{-n+1}} - H_u) X_{u + 2^{-(n+1)}, u + 2^{-n}} = 2^{-(n+1)} X_{2^{-(n+1)}, 1}.$$  

Hence, under $P^k$ the difference $(\pi_{n+1} H)_1 - (\pi_n H)_1$ is distributed as

$$2^{-(n+1)} \sigma_n \left( N + k \sigma_n \right),$$  

where $\sigma_n = (1 - 2^{-(n+1)})^{1/2}$ and $N$ is a standard normal random variable. We conclude

$$P^k \left( |(\pi_{n+1} H)_1 - (\pi_n H)_1| > \epsilon \right) = \Phi \left( k \sigma_n \frac{\epsilon}{\sigma_n 2^{n+1}} \right) + \Phi \left( \frac{\epsilon}{\sigma_n 2^{n+1}} - k \sigma_n \right) \xrightarrow{k \to \infty} 1,$$  

where $\Phi$ is the cumulative distribution function of a standard normal random variable.

## 3 Pathwise integrals

Let $B$ be a Banach space and let $X : [0, T] \to B$ be a continuous path with trajectory in $B$. The increments

$$X_{s,t} := X_t - X_s, \quad 0 \leq s, t \leq T,$$  

(3.1)
of such path define a two-parameter function \( X = X_{s,t} \) on the square \([0,T] \times [0,T]\). We employ the notation in (3.1) throughout all our work. Moreover, rather than considering general \( s \) and \( t \) in \([0,T]\), we often restrict to the simplex \( \{(s,t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T \} \subset [0,T]^2 \). A clear property of \( X \) is additivity, in that for all \( 0 \leq s,u,t \leq T \) it holds

\[
X_{s,t} = X_{s,u} + X_{u,t}.
\] (3.2)

Notice that if \( X \) is a priori only defined on the simplex but additive, then it can straightforwardly be extended to an additive function on \([0,T] \times [0,T]\) by setting \( X_{t,s} := -X_{s,t} \).

Additivity characterises those functions \( X \) on \([0,T] \times [0,T]\) that descend from increments of paths, in the following sense.

**Proposition 3.1.** Let \( X : [0,T] \times [0,T] \to B \) be additive. Then, there exists a path \( x \) on \( B \) such that

\[
X_{s,t} = x_t - x_s, \quad \forall 0 \leq s,t \leq T.
\]

Moreover, if \( y \) is another path whose increments coincide with \( X \), then \( y - x \) is constant.

We regard a partition \( \pi \) simultaneously as the finite collection of points and as the finite collection of adjacent subintervals of a given time interval \([s,t]\) that \( \pi \) subdivides. Given a partition \( \pi \) of \([0,T]\) and a time instant \( t \) in \([0,T]\), we adopt the following notational convention:

\[
t' := \inf \{ u \in \pi : u > t \}, \quad [t] := \sup \{ u \in \pi : u \leq t \},
\]

\[
t^- := \sup \{ u \in \pi : u' \leq t \}, \quad t\star := \begin{cases} 
  t^- & \text{if } t \in \pi \\
  [t] & \text{if } t \notin \pi,
\end{cases}
\]

\[
|\pi| := \sup \{|u' - u| : u \in \pi\}, \quad \pi_t := (\pi \cup \{t\}) \cap [0,T].
\] (3.3)

Let \( \pi \) be a partition of the time interval \([s,t]\) under consideration. From the additivity (3.2) it follows that

\[
\sum_{u \in \pi} X_{u,u'} = X_{s,t}.
\] (3.4)

This holds irrespectively of the choice of the partition \( \pi \), so that if \( \pi_n, \ n \geq 1 \), is a sequence of partitions with meshsize shrinking to zero, we can carry formula (3.4) to the limit in \( n \) and write

\[
\lim_{n \to \infty} \sum_{u \in \pi_n} X_{u,u'} = X_{s,t}.
\] (3.5)

Actually, this does not use the fact that the meshsize \( |\pi_n| := \sup \{|u' - u| : u \in \pi_n\} \) goes to zero as \( n \to \infty \). However, restricting to such class of sequences of partitions will become meaningful soon, because we wish to interpret the limit in (3.5) as the integral \( \int_s^t dX \). In order to emphasise that the limit in (3.5) does not depend on the particular sequence of partitions we write

\[
\lim_{|\pi| \to 0} \sum_{u \in \pi} X_{u,u'} = X_{s,t}.
\] (3.6)

In integral notation, this is the trivial, yet fundamental, relation \( \int_s^t dX = X_{s,t} \). Let \( X : \{(s,t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\} \to B \). We say that \( X \) is of finite \( p \)-variation for some \( p \geq 1 \) if

\[
||X||_{p,\text{var},[0,T]} := \sup \left\{ \sum_{u \in \pi} |X_{u,u'}|^p : \pi \text{ partition of } [0,T] \right\} < \infty.
\]

If \( X \) is additive, this notation is the usual \( p \)-variation norm of the underlying path.

For \( s \leq u \leq t \) we introduce the symbol

\[
\delta X_{s,u,t} := X_{s,t} - X_{s,u} - X_{u,t}.
\]
If \( X \) is additive, then \( \delta X \equiv 0 \).

Equation (3.6) is the combination of two statements: a. the limit on the left hand side exists and is the same along every sequence of partitions with vanishing meshsize; b. such limit defines an additive functional on the simplex, hence a path. We have seen that these properties are immediate if we start from an additive \( X \). We will now relax the additivity of \( X \) to obtain the non-trivial statement in Proposition 3.5.

**Definition 3.2 ("Control function").** A control function \( \omega \) is a non-negative continuous function on \( \{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T \} \), null on the diagonal and such that

1. \( \omega(s_1, t_1) \leq \omega(s_2, t_2) \), if the interval \([s_1, t_1]\) is contained in the interval \([s_2, t_2]\);
2. \( \omega(s, u) + \omega(u, t) \leq \omega(s, t) \), for all \( s \leq u \leq t \).

A control function generalizes the concept of the length of an interval. Common controls are \( \omega(s, t) := |t - s| \) and, for a continuous path \( x \) of finite \( p \)-variation, \( \omega(s, t) := \|x\|_{p\text{-var},[s,t]}^p \). From these, new controls can be defined by linear combinations \( c_1 \omega_1 + c_2 \omega_2 \) with non-negative coefficients \( c_1, c_2 \in \mathbb{R}_{\geq 0} \), and by products \( \omega_1 \gamma \omega_2 \gamma_2 \) with exponents \( \gamma_1 \) and \( \gamma_2 \) satisfying \( \gamma_1 + \gamma_2 \geq 1 \), see [FV10, Exercise 1.9].

Given a partition \( \pi \) of \([s, t] \subset [0, T]\) we may use a control function to measure the mesh-size.

**Definition 3.3.** The modulus of continuity of \( \omega \) on a scale smaller or equal than the meshsize \(|\pi|\) is given by

\[
\text{osc}(\omega, |\pi|) := \sup \{ \omega(s, t) : |t - s| \leq |\pi| \}.
\]

**Definition 3.4 ("Approximate additivity").** A function \( \Xi : \{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\} \rightarrow B \) is said approximately additive if

1. it is null and right-continuous on the diagonal, i.e. \( \Xi_{s,s} = \lim_{t \downarrow s} \Xi_{s,t} = 0 \) for all \( s \) in \([0, T]\);
2. there exist \( \gamma > 1 \) and a control function \( \omega \) such that

\[
|\Xi_{s,t} - \Xi_{s,u} - \Xi_{u,t}| \leq \omega_{\gamma}(s, t),
\]

for all \( s \leq u \leq t \).

Notice that equation (3.7) implies that for all \( 1 < \gamma' < \gamma \)

\[
\|\delta \Xi\|_{\omega, \gamma'} := \sup_{s \leq u \leq t} \frac{|\delta \Xi_{s,u,t}|}{\omega_{\gamma'}(s, t)} \leq \omega_{\gamma-\gamma'}(0, T).
\]

Therefore, condition 2 above is equivalent to the existence of a control \( \omega \) and some \( \gamma > 1 \) such that \( \|\delta \Xi\|_{\omega, \gamma} < \infty \).

**Proposition 3.5 ("Sewing Lemma").** Let \( \Xi : \{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\} \rightarrow B \) be approximately additive and let the control \( \omega \) and the exponent \( \gamma > 1 \) be such that \( \|\delta \Xi\|_{\omega, \gamma} < \infty \). Then, there exists a unique continuous path

\[
\int \Xi : [0, T] \rightarrow B,
\]

whose increments we denote by \( f^t_s \Xi \), such that for all \( 0 \leq s \leq t \leq T \)

1. \( f^t_s \Xi = \lim_{|\pi| \downarrow 0} \sum_{u \in \pi} \Xi_{u,u'} \) with limit in \( B \):
2.
\[
\left| \int_s^t \Xi - \Xi_{s,t} \right| \leq \|\delta \Xi\|_{\omega,1} \sum_{n \geq 1} \frac{1}{n^\gamma} \omega(s,t).
\]  
(3.8)

**Remark 3.6.** With respect to the formulation in [FH14, Lemma 4.2], Proposition 3.5 extends the so-called Sewing Lemma to the case of general control \( \omega \). Hence, it allows to handle the case of \( p \)-variation regularity, which is more general than the case of \( 1/p \)-Hölder regularity.

**Proof of Proposition 3.5.** Given a partition \( \pi \) of \( [s,t] \subset [0,T] \), let us set
\[
\int_\pi \Xi := \sum_{u \in \pi} \Xi_{u,u'}.
\]

We start by showing that, for any pair \( \pi, \tilde{\pi} \) of partitions of \( [s,t] \), it holds
\[
\left| \int_\pi \Xi - \int_{\tilde{\pi}} \Xi \right| \leq 2^2 \zeta(\gamma) \omega(s,t) \|\delta \Xi\|_{\omega,1} \left( \text{osc}(\omega, |\pi|)^{\gamma - 1} - \text{osc}(\omega, |\tilde{\pi}|)^{\gamma - 1} \right),
\]  
(3.9)

where \( \zeta(\gamma) := \sum_{n \geq 1} n^{-\gamma} \) is the zeta function, and \( \text{osc}(\omega, |\pi|) := \sup \{ \omega(s,t) : |t-s| \leq |\pi| \} \) is the modulus of continuity of \( \omega \) on a scale smaller or equal than the meshsize \( |\pi| \).

Let \( \pi \) be a partition of \( [s,t] \subset [0,T] \) with at least two subintervals and let
\[
m := \# \{ [u,u'] \in \pi \} \geq 2
\]
denote the number of subintervals of \( \pi \). It is easily seen by contradiction that there must exists some internal point \( u \) of \( \pi \) such that \( [u-, u], [u, u'] \in \pi \) and
\[
\omega(u-, u') \leq \frac{2}{m-1} \omega(s,t).
\]

We estimate
\[
\left| \int_{\pi \setminus \{u\}} \Xi - \int_\pi \Xi \right| = |\Xi_{u-, u'} - \Xi_{u-, u} - \Xi_{u, u'}| 
\leq \|\delta \Xi\|_{\omega,1} \omega(s, u-u') 
\leq \|\delta \Xi\|_{\omega,1} \frac{2^\gamma}{(m-1)^\gamma} \omega(s, t).
\]

By iteration we see
\[
\left| \Xi_{s,t} - \int_\pi \Xi \right| \leq \|\delta \Xi\|_{\gamma,1} \left[2\omega(s,t)\right]^{\gamma} \sum_{n \geq 1} \frac{1}{n^\gamma}.
\]  
(3.10)

Now, if \( \tilde{\pi} \) is a partition that refines \( \pi \) we have
\[
\int_\pi \Xi - \int_{\tilde{\pi}} \Xi = \sum_{u \in \pi} \left\{ \Xi_{u,u'} - \int_{\tilde{\pi} \cap [u,u']} \Xi \right\}
\]
and equation (3.10) yields
\[
\left| \int_\pi \Xi - \int_{\tilde{\pi}} \Xi \right| \leq \sum_{u \in \pi} \|\delta \Xi\|_{\gamma,1} \left[2\omega(u, u')\right]^{\gamma} \zeta(\gamma) 
\leq 2^2 \zeta(\gamma) \|\delta \Xi\|_{\omega,1} \text{osc}(\omega, |\pi|)^{\gamma - 1} \omega(s,t).
\]

The general case for \( \pi, \tilde{\pi} \) can be reduced to the case where \( \tilde{\pi} \) refines \( \pi \). This proves (3.9) and says that \( \int \Xi \) is well-defined and consistent as pointwise limit of \( t \mapsto \int_{\pi_t} \Xi \).
along any sequence \((\pi^n)_n\) of partitions of \([0, T]\) with meshsizes \(|\pi^n|\) shrinking to zero. Here we have used the notation \(\pi^n_n := (\pi^n \cup \{t\}) \cap [0, t].\)

In order to get the bound in (3.8), we consider the dyadic sequence of partitions of \([s, t]\), i.e. \(\pi_0 = \{[s, t]\}\) and

\[
\pi_{n+1} = \bigcup_{u \in \pi_n} \{[u, \hat{u}]; [\hat{u}, u']\}, \quad n \geq 0,
\]

where \(\hat{u} := \inf\{v > u : \omega(u, v) \geq 2^{-(n+1)}\omega(s, t)\}\). We are assuming, without loss of generality, that \((\pi_n)_n\) has vanishing meshsize, i.e. that \(\omega\) is strictly increasing, in the sense that \(\omega(s, t) > 0\) if \(s < t\). Notice that by continuity of \(\omega\) it holds \(\omega(u, \hat{u}) = 2^{-(n+1)}\omega(s, t)\) and by subadditivity \(\omega(\hat{u}, u') \leq 2^{-(n+1)}\omega(s, t)\). Thus,

\[
\int_{\pi_{n+1}} \Xi = \int_{\pi_n} \Xi - \sum_{u \in \pi_n} \delta \Xi_{u, \hat{u}, u'}
\]

and

\[
\left|\int_{\pi_{n+1}} \Xi - \int_{\pi_n} \Xi\right| \leq \sum_{u \in \pi_n} \|\delta \Xi\|_\gamma \omega^\gamma(u, u') \leq \|\delta \Xi\|_\gamma \omega^\gamma(s, t) \sum_{u \in \pi_n} 2^{-n\gamma} = \|\delta \Xi\|_\gamma \omega^\gamma(s, t) 2^n(1-\gamma).
\]

The right hand side is summable in \(n\). Hence,

\[
\left|\int_{s}^{t} \Xi - \Xi_{s,t}\right| \leq \sum_{n \geq 0} \left|\left(\int_{\pi_{n+1}} - \int_{\pi_n}\right) (\Xi)\right| \leq \|\delta \Xi\|_\gamma \frac{\omega^\gamma(s, t)}{1 - 2^{1-\gamma}}.
\]

Having obtained (3.8), the continuity of the path \(\int \Xi\) follows from the assumption \(\lim_{t \downarrow s} \Xi_{s,t} = 0\).

\[\square\]

Owing to Proposition 3.5, we can regard the integral as the map

\[
\int : \{\text{approximatively additive functionals}\} \rightarrow \{\text{additive functionals}\}.
\]

Owing to Proposition 3.1, we can unambiguously replace the range \(\{\text{additive functionals}\}\) with the space of continuous paths on \(B\) starting at 0 \(\in B\). Let \(AA_p,\var{0, T}; B\) be the family of approximately additive functions \(\Xi : \{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\} \rightarrow B\) that are of finite \(p\)-variation, \(p \geq 1\). Then, we can state the following

**Corollary 3.7.** The restriction of the integral map \(\int\) to \(AA_p,\var{0, T}; B\) takes value in the space \(C^p_0,\var{0, T}; B\) of continuous paths on \(B\) that start at the origin \(0 \in B\) and are of finite \(p\)-variation. Moreover,

\[
\int : AA_p,\var{0, T}; B \rightarrow C^p_0,\var{0, T}; B
\]

\[
\Xi \mapsto \lim_{|\tau| \downarrow 0} \sum_{u \in \tau} \Xi_{u, u'}
\]

is continuous in \(p\)-variation norm.
Proof. Immediate from equation (3.8).

Let $X : [0, T] \to B$ be continuous and of finite $p$-variation. Let $H : [0, T] \to W$ be continuous and of finite $q$-variation, where $W = \text{Hom}(B; V)$ and $V$ is a Banach space. We say that $p$ and $q$ are Young complementary if $1/p + 1/q > 1$.

**Proposition 3.8** (Young integral\footnote{The original article by L. C. Young is [You36], where the extension of Stieltjes integral was introduced. Our reference is the rough path-oriented presentation of the Young integral contained in [FV10, Chapter 6].}). Let $p$ and $q$ be Young complementary and set

$$
Ξ_{s,t} := H_s X_{s,t}
$$

for all $0 \leq s \leq t \leq T$, or $Ξ_{s,t} := H_t X_{s,t}$ for all $0 \leq s \leq t \leq T$. Then, $Ξ$ is approximately additive and of finite $p$-variation. As a consequence, the integral

$$
H.X := \lim_{|π| \downarrow 0} \sum_{u \in π} Ξ_{u,u'}
$$

(3.11)
defines a continuous path in $V$ of finite $p$-variation. The integral in (3.11) does not depend on whether $Ξ$ is defined according to $Ξ_{s,t} := H_s X_{s,t}$ or to $Ξ_{s,t} := H_t X_{s,t}$.

**Remark 3.9.** The Young integral is a classical and well-known construction (see for example [FV10, Chapter 6]). In Proposition 3.8, this construction is connected to the general version of Sewing Lemma given above.

**Proof of Proposition 3.8.** Let $ω_X$ and $ω_H$ be respectively the $p$-variation and the $q$-variation controls of $X$ and of $H$. Using additivity of $X$ we see that

$$
H_s X_{s,t} - H_s X_{s,u} - H_u X_{u,t} = -H_{s,u} X_{u,t}
$$

and

$$
H_t X_{s,t} - H_u X_{s,u} - H_t X_{u,t} = H_{u,t} X_{u,t}.
$$

Therefore in both cases

$$
|δ Ξ_{s,u,t}| \leq ω_H^{1/q} ω_X^{1/p} Ξ(s,t).
$$

This shows the claimed approximate additivity. Moreover with $1/p' = 1 - 1/p$ we can estimate

$$
\sum_{u \in π} |H_u X_{u,u'} - H_{u'} X_{u,u'}| \leq \left( \sum_{u \in π} |H_u u'|^{p'} \right)^{1/p'} \left( \sum_{u \in π} |X_{u,u'}|^p \right)^{1/p}
$$

$$
\leq \text{osc}^p \frac{p'}{p} (H, |π|) \omega_H^{1/p'} \omega_X^{1/p} (0, T)
$$

$$
\to 0 \quad \text{as} \quad |π| \downarrow 0.
$$

The continuity of $H$ was only used to show that the choice to evaluate $H$ at the beginning or at the end of the partition subintervals does not affect the integral. The two choices are respectively referred to as adapted evaluation and terminal evaluation. If $H$ is not continuous but of bounded variation, the Young integral is defined (because $q = 1$), but depends on the evaluation choice. If $π$ is a partition of $[0, T]$, we set

$$
π_H_t := \sum_{u \in π} H_u \mathbb{1} \{ t \in (u, u'] \},
$$

which denotes the piecewise constant caglad approximation of $H$ on the grid $π$. We let \footnote{The original article by L. C. Young is [You36], where the extension of Stieltjes integral was introduced. Our reference is the rough path-oriented presentation of the Young integral contained in [FV10, Chapter 6].} $H.X$ be the Young integral of $H$ against $X$ with terminal evaluation, namely

$$
(π_H.X)_{0,t} := \sum_{u \in π_t} H_u X_{u,u'}.
$$
In this way, for $H$ continuous and of finite $q$-variation, $1/p + 1/q > 1$, we can write

$$H.X = \lim_{|\tau| \to 0} \tau^H.X.$$  \hfill (3.12)

When the complementary regularities of integrand $H$ and integrator $X$ are not sufficient for Young integration, we resort to compensated Riemann sums. In particular this is the case if $H$ and $X$ have the same $p$-variation regularity for some $p$ greater than 2.

As above, let $X$ be a continuous path of finite $p$-variation with trajectory in the Banach space $B$. Recall that $W$ denotes $\text{Hom}(B; \mathcal{V})$. We use the identification $\text{Hom}(\mathcal{B}, W) \cong \text{Hom}(\mathcal{B} \otimes \mathcal{B}; \mathcal{V})$, and we write $\text{Hom}_\text{sym}(\mathcal{B} \otimes \mathcal{B}; \mathcal{V})$ for the subset of those $\ell$ in $\text{Hom}(\mathcal{B} \otimes \mathcal{B}; \mathcal{V})$ such that $\ell(\alpha \otimes b) = \ell(b \otimes a)$ for all $a, b \in \mathcal{B}$. Also, the symbol $\mathcal{B} \circ \mathcal{B}$ will denote the symmetric tensor product of the Banach space $\mathcal{B}$, so that we can identify $\text{Hom}_\text{sym}(\mathcal{B} \otimes \mathcal{B}; \mathcal{V}) \cong \text{Hom}(\mathcal{B} \circ \mathcal{B}; \mathcal{V})$. We say that a continuous path $H : [0, T] \to W$ admits a symmetric Gubinelli derivative $H'$ with respect to $X$ if there exists a continuous path $H' : [0, T] \to \text{Hom}_\text{sym}(\mathcal{B} \circ \mathcal{B}; \mathcal{V})$ of finite $q$-variation such that

1. $q$ and $p/2$ are Young complementary;
2. $R^H_{s,t} := H_{s,t} - H'_sX_{s,t}$ is of finite $pq/(p + q)$-variation.

In this case we say that the pair $(H, H')$ is $X$-controlled of $(p, q)$-variation regularity. Notice that the regularities of $R^H$ and of $X$ imply that $H$ is of finite $p$-variation.

**Definition 3.10** ("Enhancement of a path"). Let $X$ be in $C^{p\text{-var}}([0, T]; \mathcal{B})$ and let $A$ be in $C^{p/2\text{-var}}([0, T]; \mathcal{B} \circ \mathcal{B})$. The $A$-enhancement of $X$ is the pair $X = (X, \Xi)$, where

$$2\Xi_{s,t} = X_{s,t} \otimes X_{s,t} - A_{s,t}.$$  

Similarly, we speak of an enhanced path $^8$ $X = (X, \Xi)$ of $p$-variation regularity if $X$ is in $C^{p\text{-var}}([0, T]; \mathcal{B})$ and $(s, t) \mapsto X_{s,t} \otimes X_{s,t} - 2\Xi_{s,t}$ defines an additive $\mathcal{B} \circ \mathcal{B}$-valued function of finite $p/2$-variation. The path $A_{s,t} := X_{s,t} \otimes X_{s,t} - 2\Xi_{s,t}$ is called the enhancer of $X$ and we often denote such enhancer with the symbol $[X]_{s,t} := X_{s,t} \otimes X_{s,t} - 2\Xi_{s,t}$.

The symbol $[X]$ will be referred to as volatility enhancer when the financial meaning of it is to be stressed. We say that $X = (X, \Xi)$ is a bounded-variation enhancement of $X$ if

$$\sup \left\{ \sum_{u \in \pi} |[X]_{u,u}| : \pi \text{ partition of } [0, T] \right\} < \infty.$$  

Notice that $\delta \Xi$ does not depend on the enhancer because $[X]$ is additive; moreover, for all $s \leq u \leq t$ the following reduced Chen identity holds

$$\delta \Xi_{s,u,t} = X_{s,u} \otimes X_{u,t}.$$  

**Lemma 3.11.** Let $X = (X, \Xi)$ be an enhanced path and let $(H, H')$ be $X$-controlled of $(p, q)$-variation regularity, with $H'$ being symmetric. Then,

$$\Xi_{s,t} := H_{s,t}X_{s,t} + H'_s\Xi_{s,t}$$  

is approximately additive.

---

8 An enhanced path is what in [FH14, Chapter 5] is called reduced rough path. An enhanced path satisfies the following two properties, which are taken as the defining properties of reduced rough paths:

1. the symmetric second order process $X = \text{sym}X$ is of finite $p/2$-variation;
2. the reduced Chen’s identity holds, i.e. for all $s \leq u \leq t$

$$X_{s,t} - X_{s,u} - X_{u,t} = X_{s,u} \otimes X_{u,t}.$$  

In [FH14, Lemma 5.4] these two properties are shown to necessarily imply the more explicit formulation that we adopted.
where \( \omega \) is every (appropriate exponents. Since the space of \( R \)-entiable in space with all the \( n \)-th time derivative of local \( \beta \)-Hölder regularity . Notice that nothing is assumed about the cross derivatives in time and space-gradient integrands associated with \( q \)-moderate pairs

If \( J \) is a time interval, \( n \) and \( m \) are non-negative integers and \( \alpha, \beta \) are in \( [0, 1) \), consider the space

\[
C^{m+\beta, n+\alpha}_{\text{loc}}(J \times \mathbb{R}^d; \mathbb{R}^r)
\]

of \( \mathbb{R} \)-valued functions that are \( m \) times continuously differentiable in time with the \( m \)-th time derivative of local \( \beta \)-Hölder regularity, and \( n \) times continuously differentiable in space with all the \( n \)-th order space derivatives of local \( \alpha \)-Hölder regularity. Notice that nothing is assumed about the cross derivatives in time and space of functions in \( C^{m+\beta, n+\alpha}_{\text{loc}} \). Let \( C^{m+\beta, n+\alpha}_{\text{cross}}([0, T] \times \mathbb{R}^d; \mathbb{R}^r) \) be the subspace of \( C^{m+\beta, n+\alpha}_{\text{loc}}([0, T] \times \mathbb{R}^d; \mathbb{R}^r) \) consisting of functions \( f \) such that

1. for every multiindex \( I \) with \( |I| = n \) and every compact \( K \subset \mathbb{R}^d \),
   \[
   \sup \left\{ \| \partial^I f(t, \cdot) \|_{\alpha \text{-Hö}} : 0 \leq t \leq T \right\} < \infty;
   
2. for every compact \( K \subset \mathbb{R}^d \),
   \[
   \sup \left\{ \| \partial^m f(\cdot, x) \|_{\beta \text{-Hö}} : x \in K \right\} < \infty.
   
Let \( C^\alpha \) be the space

\[
C^\alpha := C^{1+\alpha/2, 2+\alpha}_{\text{loc}}([0, T] \times \mathbb{R}^d) \cap C([0, T] \times \mathbb{R}^d).
\]  

**Definition 3.12 ("\( q \)-Moderation").** Let \( w \) be in \( C^\alpha \) and let \( X \) be a continuous path on \( \mathbb{R}^d \) of finite \( p \)-variation, with \( p - 2 < \alpha < 1 \). We say that the pair \((w, X)\) is \( q \)-moderate if

1. the paths
   \[
   H : \ t \mapsto \nabla_x w(t, X_t)
   
   H' : \ t \mapsto \nabla^2_{xx} w(t, X_t),
   \]
   \( 0 \leq t < T \),

   can be continuously extended up to \([0, T]\), and \( H' \) is of finite \( q \)-variation for some \( 1 - 2/p < 1/q < \alpha/p \);
2. there exists a control function \( \omega \) such that for all \( x \) in the trace \( X[0,T] \) and all \( 0 \leq s \leq t \leq T \)

\[
|\nabla_x w(t, x) - \nabla_x w(s, x)|^{p*} \leq \omega(s, t),
\]

where \( p* = pq/(p + q) \):

3.

\[
\sup_{0 \leq s \leq T} \| \nabla^2_{xx} w(s, \cdot) \|_{\alpha\text{-Hölder, } \text{Conv } X[0,T]} < \infty,
\]

where \( \text{Conv } X[0,T] \) is the convex hull of the trace of \( X \).

**Remark 3.13.** Let \( 0 < \alpha < 1 \) and \( p, q \geq 1 \) be such that \( 1 - 2/p < 1/q < \alpha/p \). Assume that \( w \in C^{1+\alpha/2, 1+\alpha}_{\text{loc}}([0,T] \times \mathbb{R}^d; \mathbb{R}) \) is such that \( \nabla_x w \) is in \( C^{1/p+1/q, 1+\alpha}_{\text{cross}}([0,T] \times \mathbb{R}^d; \mathbb{R}^d) \) and \( \nabla^2_{xx} w \) is in \( C^{1/\alpha}_{\text{cross}}([0,T] \times \mathbb{R}^{d} \times \mathbb{R}^{d}) \). Then for all \( X \in C^{p\text{-var}}([0,T]; \mathbb{R}^d) \) the pair \((w, X)\) is \( q\)-moderate. In particular this holds if \( w \) is twice continuously differentiable in the combined time-space variable \((t, x)\) with second derivatives of \( \alpha\)-Hölder regularity.

**Lemma 3.14.** Let \( w \) be in \( C^\alpha \) and let \( X \) be a continuous \( \mathbb{R}^d \)-valued path of finite \( p \)-variation, with \( p - 2 < \alpha < 1 \). Assume that the pair \((w, X)\) is \( q\)-moderate, \( 1 - 2/p < 1/q < \alpha/p \). Then,

\[
(H, H') := \{ \nabla_x w(t, X_t), \nabla^2_{xx} w(t, X_t) \}
\]

is a Gubinelli \( X \)-controlled path of \((p, q)\)-variation regularity.

**Proof.** Let \( p* \) and \( \omega \) be as in the definition of \( q\)-moderation. Then,

\[
|\nabla_x w(t, X_t) - \nabla_x w(s, X_s) - \nabla^2_{xx} w(s, X_s)X_{s,t}| \\
\leq |\nabla_x w(t, X_t) - \nabla_x w(s, X_t)| \\
+ |\nabla_x w(s, X_t) - \nabla_x w(s, X_s) - \nabla^2_{xx} w(s, X_s)X_{s,t}| \\
\leq \omega^{1/p*}(s, t) \\
+ \left| \int_0^1 \left[ \nabla^2_{xx} w(s, (1 + y)X_s + yX_t) - \nabla^2_{xx} w(s, X_s) \right] X_{s,t} \, dy \right| \\
\leq \omega^{1/p*}(s, t) \\
+ \left\| \nabla^2_{xx} w(s, \cdot) \right\|_{\alpha\text{-Hölder, } \text{Conv } X[0,T]} \frac{\omega_X}{1+\alpha}(s, t)
\]

The symbol \( \omega_X \) denotes the \( p\)-variation control of the path \( X \). By assumption \( 1/q < \alpha/p \) and so \( 1+\alpha/p > 1 \). This says that the \( p*\)-power of the right hand side is of bounded variation.

### 4 Benchmark Markovian models and enhanced paths of diffusion type

In [BSV08], the theory of hedging and no-arbitrage is extended to processes other than semimartingales by introducing the concept of model classes dependent on the quadratic variation. A model class consists of filtered probability spaces \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}_0)\) that accommodate stochastic processes (i.e. models) with prespecified and fixed quadratic variation. The quadratic variation is representative of the class, in the sense that it is the common feature that all the models within a class share. The paper shows that in fact the models in a class all yield the same pricing and hedging formulas. Our discussion at the beginning of the paragraph devoted to hedging in Section 2 mirrored this; indeed, we started from \( \sigma \) and went directly to the generator \( A \) of the discounted diffusion under the pricing measure, overlooking the details of the stochastic process that models the physical stock price evolution.

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Relying on the pathwise integrals introduced in Section 3, we now describe a framework to assess the pathwise feature of price trajectories that actually affects the hedging practice, disregarding the overabundant probabilistic specifications of the stochastic models. We will consider enhanced price paths (as defined in Definition 3.10) that embed the essential feature of the stochastic models. Among these the Markovian one remains the benchmark, because within such model the martingale justification of the PDE pricing and hedging technology can be fully argued. However, beyond the semimartingale setting, such a justification is not possible. In [BSV08] prices are justified by showing that, in a sufficiently large class of strategies, there are no arbitrage opportunities and option payoffs can be replicated: the option prices will then be the costs of such replicating strategies. We will defer these considerations about pricing to a later work.

A benchmark Markovian model consists of the pair $((Ω, F, Q,(F_t)), A)$, where $(Ω, F, Q,(F_t))$ is a filtered probability space and $A$ is a diffusion generator. Until further notice, we adopt the perspective of discounted prices, so that only the second order part of $A$ is considered, with coefficients thought of as functions of the discounted stock price.

**Definition 4.1 (“α-Hölder volatility operator”).** Let $α$ be in the open interval $(0, 1)$. An $α$-Hölder volatility operator is a second order elliptic differential operator of the form

$$A = \text{trace } \left( a \nabla^2 \right) / 2 = a^{i,j} \partial_{i,j}^2 / 2,$$

where $a = (a^{i,j})_{1 \leq i,j \leq d}$ is symmetric and such that all coefficients $a^{i,j} : \mathbb{R}^d \to \mathbb{R}$, $1 \leq i, j \leq d$, are $α$-Hölder regular.

Notice that, for the definition of the classical delta hedging of equation (2.6), only the diffusion generator of the market model is relevant, whereas the stochastic base is not. With this respect, we sometimes write “$A$-delta hedging”, in order to emphasize that it is defined in terms of the semigroup $e^{tA}$ of $A$ as explained in Section 2.

Given an $α$-Hölder volatility operator $A = \text{trace } \left( a \nabla^2 \right) / 2$ and a continuous path $X : [0, T] \to \mathbb{R}^d$ of finite $p$-variation, we can consider the $A$-enhancement of $X = (X, X)$ of $X$ given be

$$X_{s,t} = \frac{1}{2}(X_{s,t} \otimes X_{s,t}) - \int_s^t a(X_u) du.$$

Notice that such construction yields a bounded variation enhancement. The converse construction, which starts from a bounded variation enhancement and defines a differential operator, is formalised in the following

**Definition 4.2 (“Enhanced path of $α$-diffusion type”).** Let $X = (X, X)$ be an enhanced path of $p$-variation regularity. We say that $X$ is of $α$-diffusion type, $p - 2 < α < 1$, if by setting

$$m^{i,j}(s, t) := \frac{\partial}{\partial t} \left[ X_{s,t} \right]_{i,j}, \quad 0 \leq t \leq T, \quad 1 \leq i, j \leq d,$$

(4.1)

absolute continuous measures are defined on the interval $[0, T]$, and if their densities with respect to the Lebesgue measure are given by

$$\frac{dm^{i,j}}{dt} = a^{i,j}(X_t),$$

for some $a = (a^{i,j})_{1 \leq i,j \leq d}$ in $C^{α-Hölder}_{loc}(\mathbb{R}^d, \mathbb{R}^{d \times d})$ satisfying the ellipticity condition

$$a^{i,j}(x) |ξ|^2 \geq c(x) |ξ|^2, \quad \forall x, ξ \in \mathbb{R}^d,$$

---

9 Here we are abusing notation: Definition 3.10 prescribed to put the rough bracket $A = A_{n,t}$ in front of the word “enhancement”, so that actually we should have written $\int_0^t a(X_u) du$-enhancement. However, the employed notation distortion does not cause confusion and rather stresses the nature of an enhancement of diffusion-type.
with some continuous strictly positive $c : \mathbb{R}^d \to \mathbb{R}_+$. The operator $A[X] := a^{i,j}(x)\partial_{i,j}/2$ is called $[X]$-volatility operator, and we say that a diffusive price with Markov generator $L$ is $[X]$-compatible if the second order part of $\mathbf{L}$ is equal to $A[X]$.

An enhanced path of $\alpha$-diffusion type is the minimal information that the PDE pricing technology requires from a probabilistic model. Indeed, assume that we wish to use the PDE pricing technology to price a contingent claim $h(X_T)$, where $h$ is in $C_b(\mathbb{R}^d)$ and $X_T$ is the terminal value of a continuous price path $X$ of finite $\rho$-variation. Let $X = (X, X)$ be an enhancement of $X$ of $\alpha$-diffusion type and consider the equation

$$
\begin{cases}
(\partial_t + A[X])w = 0 & \text{in } [0, T) \times \mathbb{R}^d \\
w(T, \cdot) = h(\cdot) & \text{on } \{T \} \times \mathbb{R}^d.
\end{cases}
$$

(4.2)

Then, the Cauchy problem (4.2) admits a solution $w \in C^\alpha$ and, on any $[X]$-compatible market model, the value $w(t, X_t)$ is the discounted price at time $t < T$ of the option maturing at $T$ and yielding $h(X_T)$.

We are in the position to give the pathwise counterpart to equation (2.5), which – as argued in Section 2 – is the linchpin of Delta hedging.

**Proposition 4.3.** Let $X = (X, X)$ be an enhanced path of $\alpha$-diffusion type. Let $w$ be the solution to (4.2) and assume that the pair $(w, X)$ is $\rho$-moderate, for some $1 - 2/\rho < 1/\rho'<1/\rho$. Then,

$$(H_t, H'_t) := (\nabla_x w(t, X_t), \nabla^2_{xx} w(t, X_t))$$

is a Gubinelli $X$-controlled path of $(\rho, \rho')$-variation regularity, and it is such that

$$(H, H')(X, X)_{s,t} = w(t, X_t) - w(s, X_s)$$

(4.3)

for all $0 \leq s \leq t \leq T$.

**Proof.** The fact that $(H, H')$ is $X$-controlled follows from Lemma 3.14. We can expand the increments of $w_t := w(t, X_t)$ as

$$
w(t, X_t) - w(s, X_s) = (t - s) \int_0^1 \left[ \partial_t w(s + y(t - s), X_t) - \partial_t w(s, X_t) \right] dy$$

$$- \frac{t - s}{2} \left[ a^{i,j}(X_t)\partial_{i,j}^2 w(s, X_t) - a^{i,j}(X_s)\partial_{i,j}^2 w(s, X_s) + \partial_t w(s, X_s)(t - s) \right]$$

$$+ \left( \int_0^1 \int_0^1 \left\{ \nabla^2_{xx} w(s, X_s + y_1 y_2 X_{s,t}) \right. \right.$$

$$\left. \left. - \nabla^2_{xx} w(s, X_s) \right)y_1 \right] dy_2 dy_1 \right) (X_{s,t} \otimes X_{s,t})$$

$$+ \nabla_x w(s, X_s) X_{s,t} + \frac{1}{2} \nabla^2_{xx} w(s, X_s)(X_{s,t} \otimes X_{s,t}).$$

We have used (4.2) on the second line to re-express time derivatives as spatial ones. The assumed $\rho$-moderation allows to control the three increment-type summands in the expansion. Let $\text{Conv}X[0, T]$ be the convex hull of the trace of $X$ and let $K := \sup_{0 \leq s \leq T} \|\nabla^2_{xx} w(s, \cdot)\|_{1, \text{Hölder}} \text{Conv}X[0, T]$. Then,

$$\left| \int_0^1 \left[ \partial_t w(s + y(t - s), X_t) - \partial_t w(s, X_t) \right] dy \right| \leq \|a\|_{1, \infty, X[0, T]} \left[ K\omega^\alpha_X + \omega^{1/\rho'}_{H'} \right](s, t);$$

---

$^{10}$ The existence and regularity of a solution to (4.2) is proved for example in [LB07, Theorem 2.2.1]. Recall that the function space $C^\alpha$ was defined in equation (3.13).
and
\[
\left| a^{i,j}(X_t) \partial^2_{i,j} w(s, X_s) - a^{i,j}(X_s) \partial^2_{i,j} w(s, X_s) \right|
\]
\[
\leq ||a||_{\infty, X[0,T]} K \omega^{\alpha/p}_X(s, t)
\]
\[
+ ||H'||_{\infty, [0,T]} ||a||_{\alpha-\text{Hölder, Conv}, X[0,T]} \omega^{\alpha/p}_X(s, t);
\]
and
\[
\left| \left( \int_0^1 \int_0^1 \left[ \nabla^2_{x,x} w(s, X_s + y g_2 X_s, t) - \nabla^2_{x,x} w(s, X_s) y_1 g_2 y_1 \right] dy_2 dy_1 \right) \left( X_{s,t} \otimes X_{s,t} \right) \right|
\]
\[
\leq \frac{K}{(1 + \alpha)(2 + \alpha)} \omega^{(2 + \alpha)/p}_X(s, t).
\]

Recall that, in particular, \( \frac{2 + \alpha}{p} > 1 \) by the choice of \( \alpha \) in the definition of enhanced path of \( \alpha \)-diffusion type. Then, the three estimations above say that, for the expansion of the increments \( w_{s,t} \), the following holds: there exists a control \( \omega \) and an exponent \( \gamma > 1 \) such that
\[
\left| w_{s,t} - \nabla_x w(s, X_s) X_{s,t} - \nabla^2_{x,x} w(s, X_s) X_{s,t}
\]
\[
- \partial_t w(s, X_s) - \frac{1}{2} \nabla^2_{x,x} w(s, X_s) [X]_{s,t}
\]
\[
= \left| w_{s,t} - \partial_t w(s, X_s)
\]
\[
- \nabla_x w(s, X_s) X_{s,t} - \nabla^2_{x,x} w(s, X_s) (X_{s,t} \otimes X_{s,t}) \right|
\]
\[
\leq \omega^{\gamma}(s, t),
\]

Hence,
\[
w_{s,t} = \lim_{|\pi| \to 0} \sum_{u \in \pi \cap [s,t]} \left[ \partial_t w(u, X_u)(u' - u) + \frac{1}{2} \partial^2_{i,j} w(u, X_u) [X]_{u,u'} \right]
\]
\[
+ \lim_{|\pi| \to 0} \sum_{u \in \pi \cap [s,t]} \left[ H_u X_{u\wedge t, u'} + H'_u X_{u\wedge t, u'} \right].
\]
\[
=: \left( \langle a H, a H' \rangle (X_X) \right)_{s,t}
\]
\[
=: \langle (H, H') (X_X) \rangle_{s,t}.
\]

The possibility to split the limit descends from the already-known convergence of \( \langle a H, a H' \rangle (X_X) \) as \(|\pi| \to 0\). For any \( i, j \) the discrete sum \( \sum_{u \in \pi} \partial^2_{i,j} w(u, X_u) [X]_{u,u'} \) approximates the Stieltjes integral of the continuous function \( u \mapsto \partial^2_{i,j} w(u, X_u) \) against the measure \( m^{i,j} \) of (4.1). Hence, in the limit as \(|\pi| \to 0\) it converges to \( \int_0^1 \partial^2_{i,j} w(u, X_u) a^{i,j}(X_u) du \). The cancellation guaranteed by (4.2) then implies (4.3).

As anticipated in Section 2, the deployment of higher order sensitivities and pathwise integration allows to estimate errors arising from time discretisation of integral quantities. Instances of time discretisation of integral quantities appear in the costs associated with hedging. Indeed, consider the cost of financing of a hedging strategy, defined as
\[
C_t(\phi) := \phi^0_t S^0_t + \phi^1_t S_t - (\phi^0_0.S^0)_t - (\phi^1_0.S)_t,
\]
where \( (\phi^0, \phi^1) \in \mathbb{R} \times \mathbb{R}^d \) is the strategy and \( S^0 \) the riskless asset and the risky asset. The symbols \( (\phi^0_0.S^0)_t \) and \( (\phi^1_0.S)_t \) denote the time-\( t \) marginals of the integral processes of \( \phi^0 \) and \( \phi^1 \) respectively against \( S^0 \) and \( S \). Thus, the cost of financing in equation (4.4) is the difference between the value of the portfolio at time
and the cost of rebalancing the portfolio during the time window $[0, t]$ in order to follow the hedging strategy. If continuous hedging were possible and one were able to take $(\phi^0, \phi^1) = (H^0, H)$ as defined in (2.6), then this cost would match $V_0 = w(0, X_0)$, the price at time $t = 0$ of the option, on a $P$-full set. We remark that the probability $P$ is the measure of the stochastic base on which in the continuous-time case the Itô integral $(\phi^1, S)_t$ would be defined. In practice, the cost of financing has two components: theorem the price $V_0$ and the cost arising from time discretisation, which is $C_T(\phi) - V_0$. For the latter, with $\phi$ replaced by the discretisation $(\bar{\pi} H^0, \bar{\pi} H)$ of (2.6), we now provide a pathwise estimate that relies on integration bounds. Recall that $X$ in Proposition 4.3 plays the role of the discounted trajectory $\tilde{S}_t = e^{-rt} S_t$.

**Corollary 4.4.** Assume the setting of Proposition 4.3. Let $\omega$ be the control function whose $(2/p + 1/q)$-th power asserts the approximate additivity of $H_s X_{s,t} + H_s' X_{s,t}$. Along any partition $\pi$ of $[0, T]$, the discretised strategy $(\bar{\pi} H^0, \bar{\pi} H)$ stemming from (2.6) with $\tilde{S} = X$ has a cost of financing $C(\bar{\pi} H^0, \bar{\pi} H)$ that is bounded as follows:

$$C_T(\bar{\pi} H^0, \bar{\pi} H) \leq |V_0| + e^{T}\left(K\omega(0,T)\text{osc}(\omega, |\pi|)^{2/p+1/q-1} + |w|_{T-0,T}\right) + \left|\sum_{u \in \pi} e^{\tilde{\pi} H'_u X_{u,u'}} \right|,$$

where $\text{osc}(\omega, |\pi|)$ is the modulus of continuity of $\omega$ on a scale smaller or equal than the meshsize of the partition, and $w_{T-0,T}$ is the difference between $w(T, X_T) = h(X_T)$ and the discounted value $w(T-, X_{T-})$ of the option at the second last node of the partition. The path-dependent constant $K$ appearing in the bound is not greater than

$$\frac{1}{1 - 2^{-(2/p + 1/q)}} \left(\omega\bar{\pi}^{1/p+1/q}(0,T) ||X||_{p,\text{var},[0,T]} + ||H'||_{q,\text{var},[0,T]} ||X||_{p/2,\text{var},[0,T]} \right),$$

where $\omega \bar{\pi}$ is the $pq/(p + q)$-variation control of $H_{s,t} - H_{s,t}' X_{s,t}$.

**Proof.** Let $w_t$ be the path $t \mapsto w(t, X_t)$. Fix a partition $\pi$ of $[0, T]$ and recall the notation in (3.3). We preliminarily observe that

$$(\bar{\pi} H^0, S^0)_t + (\bar{\pi} H, S)_t = \sum_{u \in \pi} \left[ w_u S^0_{u \wedge t, u' \wedge t} + H_u S^0_{u' \wedge t, u' \wedge t} \tilde{S}_{u \wedge t, u \wedge t} \right]$$

$$= w_t S^0_t - w_0 + \sum_{u \in \pi} S^0_{u' \wedge t} \left[ - w_u S^0_{u \wedge t, u' \wedge t} + H_u \tilde{S}_{u \wedge t, u \wedge t} \right],$$

where in the second line we have used summation by parts. Then,

$$C_T(\bar{\pi} H^0, \bar{\pi} H) = w_t S^0_t - \bar{\pi} H_t \tilde{S}_t S^0_t + \bar{\pi} H_t S_t - w_t S^0_t$$

$$+ w_0 + \sum_{u \in \pi} S^0_{u \wedge t} \left[ w_u S^0_{u \wedge t, u' \wedge t} - H_u \tilde{S}_{u \wedge t, u \wedge t} \right]$$

$$= S^0_t H_t (\tilde{S}_t - \tilde{S}_t) + V_0 + \sum_{u \in \pi} S^0_{u' \wedge t} \left[ w_u S^0_{u \wedge t, u' \wedge t} - H_u \tilde{S}_{u \wedge t, u \wedge t} \right]$$

$$= V_0 + S^0_t w_{t,t} + \sum_{u \in \pi} S^0_{u'} \left[ w_u - H_u \tilde{S}_{u, u'} \right].$$

By adding and subtracting the compensation, we can apply the Sewing Lemma (Proposition 3.5) and conclude.

---

11 In the continuous-time abstraction, the term $(\phi^1, S)_t$ is to be read as the Itô integral of the continuous adapted process $\phi^1$ against the continuous semimartingale $S$; the term $(\phi^0, S^0)_t$ would instead refer to the Lebesgue integral $\int_t^0 \phi^0 e^{\pi} du$.
Until now, we have worked with the identification $X = \tilde{S}$, i.e., the enhanced path at hand has represented the actual enhanced path of the discounted stock price. In other words, the market models have been $[\tilde{S}]$-compatible. This amounts to considering the square $a = \sigma \sigma^T$ of co-volatilities a true parameter. In Corollary 4.5 below, we no longer do so and we distinguish the modelled enhancer of $X$ from the actual enhancer of $\tilde{S}$. The only assumption on $\tilde{S}$ is that it is an enhanced path, i.e., its trace $\tilde{S}$ is a continuous path of finite $p$-variation, $2 < p < 3$, and its second order process $\tilde{S} = (\tilde{S} \circ \tilde{S} - [\tilde{S}])/2$ is a continuous two-parameter function of finite $p/2$-variation with values in $\mathbb{R}^d$; the enhancer $[\tilde{S}]$ is not required to be of bounded variation and the integrals against it will be interpreted as Young integrals.

**Corollary 4.5.** Let $\tilde{S} = (\tilde{S}, \tilde{S})$ be an enhanced path above the $\mathbb{R}^d$-valued discounted price trajectory $\tilde{S}$ of $p$-variation regularity. Let $A$ be an $\alpha$-Hölder volatility operator, with $\alpha > p - 2$. Consider the $A$-enhancement $X = (\tilde{S}, \tilde{S})$ of $\tilde{S}$. If $h$ and $w$ are as in Proposition 4.3, then $(H, H) := (\nabla_x w(t, \tilde{S}), \nabla^2_{xx} w(t, \tilde{S}))$ is a Gubinelli $\tilde{S}$-controlled path of $(p, q)$-variation regularity and

$$\tilde{h}(\tilde{S}_T) - V_0 = ((H, H'), (\tilde{S}, \tilde{S}))_{0,T} + \frac{1}{2} \langle H', [\tilde{S}] - [X] \rangle_{0,T}, \tag{4.7}$$

where the second summand on the right-hand side is a well-defined Young integral. As a consequence, if $\tilde{h}$ denotes the strategy obtained by discretising along $\pi$ the $A$-delta hedging, then its cost of financing $C_T(\tilde{h}^0, \tilde{h})$ is bounded by

$$\left| V_0 \right| + \sum_{u \in \pi} e^{\tau_u} H_{u} \tilde{S}_{u,u'} + e^{\tau_T} \left( K \omega(0, T) \text{osc}(\omega, |\pi|)^{2/p+1/q-1} + |w_{T-}| + K_H \left\| [\tilde{S}] - [X] \right\|_{p/2, \text{var}, [0,T]} \right), \tag{4.8}$$

where $\omega$, $K$ and $|w_{T-}|$ are as in Corollary 4.4 and

$$K_H = \frac{2^{-1} (2/p)^2}{1 - 2^{-1} (4/p+1/q)} \left\| H' \right\|_{q, \text{var}, [0,T]} + 2^{-1 - 1/2/p} \left\| H' \right\|_{\infty, [0,T]}.$$

**Proof.** The fact that $(\nabla_x w(t, \tilde{S}), \nabla^2_{xx} w(t, \tilde{S}))$ is $\tilde{S}$-controlled of $(p, q)$-variation regularity is already contained in Proposition 4.3, because it does not involve the second-order component of $\tilde{S}$. Also, the Taylor expansion of Proposition 4.3 yields a control function $\omega$ and an exponent $\gamma > 1$ such that however chosen a subinterval $[s, t]$ of $[0, T]$, it holds

$$w(t, \tilde{S}_t) - w(s, \tilde{S}_s) = \nabla_x w(s, \tilde{S}_s) \tilde{S}_{s,t} + \partial_t w(s, \tilde{S}_s)(t - s) + \frac{1}{2} \nabla^2_{xx} w(s, \tilde{S}_s) \tilde{S}_{s,t} \otimes \tilde{S}_{s,t} + O(\omega^\gamma(s, t))$$

$$= \nabla_x w(s, \tilde{S}_s) \tilde{S}_{s,t} + \nabla^2_{xx} w(s, \tilde{S}_s) \tilde{S}_{s,t} + \partial_t w(s, \tilde{S}_s)(t - s) + \frac{1}{2} \nabla^2_{xx} w(s, \tilde{S}_s) [X]_{s,t}$$

$$+ \frac{1}{2} \nabla^2_{xx} w(s, \tilde{S}_s) \left( [\tilde{S}]_{s,t} - [X]_{s,t} \right) + O(\omega^\gamma(s, t)).$$

Therefore, by considering the subintervals $[u, u']$ of a partition $\pi$ of $[s, t]$, summing over these, and letting $|\pi| \to 0$, we obtain

$$w(t, \tilde{S}_t) - w(s, \tilde{S}_s) = ((H, H'), (\tilde{S}, \tilde{S}))_{s,t} + \frac{1}{2} \langle H', [\tilde{S}] - [X] \rangle_{s,t}, \tag{4.9}$$

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and in particular (4.7). The second summand on the right hand side is a well-defined Young integral because \( t \mapsto \nabla S_{S_t} w(t, \tilde{S}_t) \) is of bounded \( q \)-variation, \( q < p/\alpha \), and \( \alpha > p - 2 \) by assumption.

Write \( w_{s,t} \) for the increments \( w(t, \tilde{S}_t) - w(s, \tilde{S}_s), 0 \leq s \leq t \leq T \). Owing to (4.9), for every subinterval \([u, u']\) of a partition \( \pi \) we can write

\[
\begin{align*}
    w_{u,u'} - H_u \tilde{S}_{u,u'} &= ((H, H'), (\tilde{S}, \tilde{S}))_{u,u'} - H_u \tilde{S}_{u,u'} - H' u \tilde{S}_{u,u'} \\
    &\quad + H' u \tilde{S}_{u,u'} + \frac{1}{2} (H' \left( [\tilde{S}] - [X] \right))_{u,u'}.
\end{align*}
\]

Therefore,

\[
\sum_{u \in \pi} \sum_{u' < t} \left[ w_{u,u'} - H_u \tilde{S}_{u,u'} \right] \leq e^{rT} K \omega(0, T) \text{osc}(\omega, |\pi|)^{2/p+1/q-1} + \sum_{u \in \pi} e^{r u'} H' u \tilde{S}_{u,u'} \]

\[
+ \frac{1}{2} e^{rT} \left\| H' \left( [\tilde{S}] - [X] \right) \right\|_{p/2\text{-var},[0,T]}.
\]

where, by applying the bounds in [FV10, Theorem 6.8] we see

\[
\left\| H' \left( [\tilde{S}] - [X] \right) \right\|_{p/2\text{-var},[0,T]} \leq 2^{(1-\frac{1}{2p})\frac{2}{q}} \left\| [\tilde{S}] - [X] \right\|_{p/2\text{-var},[0,T]} \]

\[
\left( \frac{1}{1 - 2^{1-(4/p+1/q)}} \right) \left\| H' \right\|_{p\text{-var},[0,T]} + \left\| H' \right\|_{\text{sc},[0,T]}.
\]

Therefore, by plugging in (4.6), we conclude.

\( \square \)

5 Pathwise formulation of fundamental equations of hedging

By adopting the perspective of undiscounted price paths, we recover the classical formulas of Mathematical Finance within our pathwise setting. Given a price path \( S \), we say that a model for \( S \) has been specified when a choice for the enhancement \( \mathbf{S} = (S, \mathbf{S}) \) is made. This means choosing the enhancer \( [S] \), see Section 3. We speak of an \( \alpha \)-diffusive model specification if the enhancer is given by

\[
[S]_{i,j}^{u,v} = \int_u^v e^{2rt} a^{i,j} (e^{-rt} S_t) dt, \quad 0 \leq u \leq v \leq T, \quad 1 \leq i, j \leq d,
\]

where \( a^{i,j} \), \( 1 \leq i, j \leq d \) are the coefficients of an \( \alpha \)-Hölder volatility operator and \( r \) is the constant interest rate. In other words, an \( \alpha \)-diffusive model specification is the undiscounted counterpart to an \( A \)-enhancement of some discounted price path, where \( A \) is an \( \alpha \)-Hölder volatility operator as defined in Definition 4.1.

**Theorem 5.1.** Let \( f(S_T) \) be a contingent claim, where \( f \) is in \( C_b(\mathbb{R}^d) \) and \( S_T \) is the terminal value of a continuous \( d \)-dimensional price path \( S \) of finite \( p \)-variation. Let \( \mathbf{S} = (S, \mathbf{S}) \) be an \( \alpha \)-diffusive model specification, with \( \alpha > p - 2 \), and let \( A = a^{i,j} \partial_{i,j}^2 / 2 \) be the corresponding volatility operator. Then, the Black-Scholes partial differential equation

\[
\begin{align*}
    e^{2rt} a^{i,j} (e^{-rt} z) \partial_{z,z}^2 v + rz_i \partial_j v + \partial_t v &= rv \quad \text{in } [0, T) \times \mathbb{R}^d \\
    v(T, z) &= f(z) \quad \text{on } \{T\} \times \mathbb{R}^d
\end{align*}
\]

(5.1)
admits a solution $v$ in $\mathcal{C}^\alpha$ and such solution is unique. Moreover, for every $0 \leq t \leq T$, the quantity $V_t := v(t, S_t)$ is the fair value at time $t$ of the contingent claim $f(S_T)$ in the benchmark Markovian model $((\Omega, \mathcal{F}, Q, (\mathcal{F}_t)_t), A)$.

**Remark 5.2.** In line with what we argued in the introduction to our work, the statement of Theorem 5.1 shows that the pricing technology only relies on pathwise features; it is the economic justification of its fairness that requires probability.

**Proof of Theorem 5.1.** The change of variable $x := e^{-rt}z$ allows to rewrite equation (5.1) as

$$
\left\{ \begin{array}{ll}
(\partial_t + A)w &= 0 &\text{in } [0, [0, T]] \times \mathbb{R}^d \\
w(T, x) &= e^{-rT}f(e^{rT}x) &\text{on } \{T\} \times \mathbb{R}^d,
\end{array} \right.
$$

where $w(t, x) = e^{-rt}v(t, z)$. Therefore, existence, uniqueness and regularity of the solution follow from those of equation (4.2).

Let $p(t, T)$ be the fair value of $f(S_T)$ in the benchmark Markovian model $((\Omega, \mathcal{F}, Q, (\mathcal{F}_t)_t), A)$. This means that the discounted price path $\bar{S}$ is thought of as a realisation of a Markov diffusion process on $(\Omega, \mathcal{F}, Q)$ with generator $A$, and such diffusion process is a $Q$-martingale. On the one hand, by the pricing paradigm

$$
p(t, T) = \mathbb{E}_Q[e^{-(r-T)t}h(\bar{S}_T)|\bar{S}_t],
$$

where $h(x) := f(e^{rT}x)$ and $e^{tA}$ is the semigroup associated with $A$. On the other hand, the Itô integral $V_t := \int_0^t \nabla_z v(u, S_u) d\bar{S}_u$ is such that $\dot{V}_t = e^{-rt}h(\bar{S}_T)$, and thus

$$
p(t, T) = e^{rt}\mathbb{E}_Q[\dot{V}_T|\bar{S}_t] = e^{rt}\bar{V}_t,
$$

because $\bar{V}$ is a martingale. Combining (5.2) and (5.3) we obtain the second claim. 

**Proposition 5.3.** Let $f$ and $S$ be as in Theorem 5.1. Let $S = (S, \mathcal{F})$ be an $\alpha$-diffusive model specification, with $\alpha > p - 2$, and let $v = v(t, z)$ solve equation (5.1). If $(v, S)$ is $q$-moderate, for some $1 - 2/p < 1/q < \alpha/p$, then

$$(\Delta_t, \Gamma_t) := (\nabla_z v(t, S_t), \nabla^2_{zz} v(t, S_t))$$

is a Gubinelli $S$-controlled path of $(p, q)$-variation regularity, and

$$
V_t - V_0 = (\Delta_t, \Gamma_t, (S, \mathcal{F}))_{0, t} + \int_0^t (V_u - \Delta_u S_u) dS^0_u,
$$

where $V_t = v(t, S_t)$ and $S^0_t = \exp(\gamma t)$.

**Proof.** The proof is analogous to the one of Proposition 4.3. Indeed, the same Taylor expansion shows that for some $\gamma > 1$ and some control function $\omega$, on the subintervals $[u, u']$ of any partition $\pi$, it holds

$$
v(u', S_{u'}) - v(u, S_u) = \nabla_z v(u, S_u) S_{u, u'} + \nabla^2_{zz} v(u, S_u) S_{u, u'} \gamma + \partial_t v(u, S_u)(u' - u) + \frac{1}{2} \nabla^2_{zz} v(u, S_u) [S]_{u, u'} \\
+ O(\omega^\gamma(u, u')).
$$

By applying the operator $\lim_{|\pi| \to 0} \sum_{u \in \pi}$ to both sides of this expansion, we obtain (5.4) since $v$ solves the Black-Scholes partial differential equation (5.1). 

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The pathwise differential equation in (5.4) syntactically coincides with the classical Stochastic Differential Equation for the portfolio process in the Delta Hedging. In addition, the definition of the pathwise integral \((\text{Delta, Gamma}) \cdot (S, Z)\) explicitly expresses the dependence on the Gamma sensitivity, which is not captured by the classical stochastic integral. This provides a theoretical underpinning to the usage of Greeks beyond the leading first-order Delta.

As is apparent, the formulas for pricing and hedging heavily depend on the diffusive model specification. In classical terms of Mathematical Finance, such specification amounts to specifying the diffusion coefficient (volatility) in Itô’s price dynamics. Volatility is not directly observable and consequently a trader is liable to misspecify volatility and to use coefficients that do not faithfully represent the true price dynamics. The Fundamental Theorem of Derivative Trading addresses such misspecification. It provides a formula that computes the profit & loss that a trader incurs when hedging with the wrong volatility – a reference for this classical formula is [EJP17]. Proposition 5.4 contributes to the assessment of model misspecification in two ways: on the one hand, it shows the pathwise nature of the P&L formula (this aligns with the unifying theme of the section); on the other hand, it provides a generalisation of the classical P&L formula. The generalisation consists in removing the assumption that the “true” price evolution is governed by an Itô SDE: it captures the misspecification that arises not just between two diffusive enhancements but between a diffusive enhancement (used by the trader) and a general enhanced path (the “true” dynamics).

**Proposition 5.4 (“Fundamental Theorem Of Derivative Trading”).** Let \(f(S_T)\) be a contingent claim, where \(f\) is in \(C_1([0, T])\) and \(S_T\) is the terminal value of a continuous \(d\)-dimensional price path \(S\) of finite \(p\)-variation. Let \(S^{\text{true}} = (S, S^\gamma)\) be the true enhanced path above the trace \(S\). Let \(S = (S, Z)\) be an \(\alpha\)-diffusive model specification, \(\alpha > p - 2\), and let \(A, V, \Delta\) and \(\Gamma\) be as in Proposition 5.3. Then,

\[
P\&L = V_T - f(S_T) = \frac{1}{2} \Gamma \cdot \partial_{uu} \bigg( \frac{(S_t - [S^\text{true}])_t}{\gamma} \bigg)_{0,T},
\]

where the integral on the right hand side is a well-defined Young integral, and \(V_t\) is the value at time \(0 \leq t \leq T\) of the A-hedging portfolio applied to the true enhancement \(S^{\text{true}}\), namely

\[
V_t = v(0, S_0) + ((\text{Delta, Gamma}) \cdot (S, S^{\text{true}}))_{0,t} + \int_0^t \left( v(u, S_u) - \Delta u S_u \right) dS_u.
\]

**Remark 5.5.** In order to recognise the extension of the classical Fundamental Theorem of Derivative Trading, we rewrite the Young integral in equation (5.5) as

\[
\frac{1}{2} \int_0^T \Delta v \partial_{uu} v(t, S_t) dt \bigg( [S]_t - [S^{\text{true}}]_t \bigg).
\]

In the case where \(S^{\text{true}}\) is a diffusive enhancement, we have that \([S^{\text{true}}]_t = \int_0^t e^{2ru} a^{ij}_{\text{true}}(e^{-ru} S_u) du\), so that the integral is turned in the familiar form

\[
\frac{1}{2} \int_0^T e^{2ru} \partial_{u,t}^2 v(t, S_t) \left( a^{ij}(e^{-rt} S_t) - a^{ij}_{\text{true}}(e^{-rt} S_t) \right) dt.
\]

**Proof of Proposition 5.4.** We manipulate the Taylor expansion in the proof of Proposition 5.3 and, for \(0 \leq u \leq t \leq T\), we write

\[
v(t, S_t) - v(u, S_u) = \Delta v \cdot [v(u, S_u)]_{u,t} + \Delta^2 v \cdot [v(u, S_u)]_{u,t} + \partial_t [v(u, S_u)]_{u,t} + \frac{1}{2} \partial_{u,t}^2 [v(u, S_u)]_{u,t} + \frac{1}{2} \partial_{u,t}^2 [v(u, S_u)]_{u,t} \bigg( [S^{\text{true}}]_{u,t} - [S]_{u,t} \bigg) + O(\omega^2(u, t)),
\]

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where \( v \) is the solution to the \( d \)-dimensional Black-Scholes partial differential equation (5.1), \( \omega \) is a control function and \( \gamma > 1 \). We sum over the nodes of a partition and then we let the mesh size shrink to zero, obtaining (5.5). The good definition of the Young integral of \( \Gamma \) against \([S_{\text{true}}]\) and \([S]\) holds as in Corollary 4.5.

### 6 Enlarged hedging strategies

Given an enhanced price path \( S = (S, \mathcal{F}) \), we interpreted the pathwise integral \((H, H')\) \((S, \mathcal{F})\) as the portfolio trajectory arising from the position \( H \) on the risky asset \( S \). As mentioned in Section 2, this interpretation relies upon the possibility to recover the pathwise integral from limits of uncompensated Riemann sums along carefully chosen sequences of partitions. This result was established in [PP16]. Alternatively, in the continuous martingale setting, one sees that Itô enhanced paths are in fact such that the compensation of the Riemann sums vanishes almost surely.

In this section, we explore the possibility to modify the interpretation of \((H, H')(S, \mathcal{F})\). We will not only consider it as representing the values of the position \( H \) on \( S \), but we will give a financial interpretation to the compensation within the rebalancing mechanics, we can refrain from resorting to probability the axiomatic condition \((\pi(S), \pi(H))\),\(^{12}\) the cost of rebalancing the portfolio from \((u^-, u] \) to \((u, u'] \) is

\[
\text{rebal}^\pi(u) = \pi H^0_0 S^0_0 + \pi H'_0 S_u - \pi H^0_0 S^0_u - \pi H_u S_u.
\]

Such discretised strategy is self-financing on the grid \( \pi \) if and only if for all \( u > 0 \) in \( \pi \) it holds \( \text{rebal}^\pi(u) = 0 \), or equivalently if and only if

\[
H^0_0 S^0_u + H'_0 S_u - H^0_0 S^0_u - H_u S_u = H^0_0 S^0_{u,u'} + H_u S_{u,u'}.
\]

Given \( t \) in \((0, T]\), set \( \pi_t := (\pi \cup \{t\}) \cap [0, t) \). By summing over \( u \in \pi_t, u < t \), we have

\[
H^0_0 S^0_t + H_t S_t - H^0_0 S^0_t - H_0 S_0 = \sum_{u \in \pi_t} H^0_0 S^0_{u,u'} + \sum_{u \in \pi_t} H_u S_{u,u'}.
\]

If \( S \) is a semimartingale on \((\Omega, \mathcal{F}, P, (\mathcal{F}_t)_t)\), then taking the \( P \)-limit as \(|\pi| \to 0 \) justifies the axiomatic condition (2.1), owing in particular to

\[
\sup \left\{ \limsup_{|\pi| \to 0} \mathbb{P} \left( \left| (\pi S)_t - \int_0^t H \omega \right| > \epsilon \right) : \epsilon > 0 \right\} = 0.
\]

Here the probabilistic model comes into play to guarantee the convergence of the Riemann sums to the Itô integral \( \int_0^t H dS \) of \( H \) against the semimartingale \( S = S_t(\omega) \), of which the actual price trajectory is thought of as a realisation.

Considering an enhancement \( S \) of \( S \) and incorporating the appropriate compensation within the rebalancing mechanics, we can refrain from resorting to probability when assessing continuously rebalanced hedging strategies.

Given a symmetric \( G_i \) in \( \mathbb{R}^{d \times d} \cong \text{Hom}(\mathbb{R}^d \otimes \mathbb{R}^d, \mathbb{R}) \) and a subinterval \([s, t] \subset [0, T]\) we interpret the real quantity \( G_i S_{s,t} \) as the sum of the payoffs at time \( t \) of the \( d(d-1)/2 \) positions \( 2G^{ij}_{s,t} = 2G^{ji}_{s,t}, 1 \leq i < j \leq d \), on the swap contracts

\[
[S]_{s,t}^{i,j} - [S]_{s,t}^{i,j}, \quad 1 \leq i < j \leq d.
\]

\(^{12}\)Recall that given a (continuous) path \( \varphi \) in \( \mathbb{R}^m \) and a partition \( \pi \) we denote by \( \tau \) the following piecewise constant càglàd approximation:

\[
\tau \varphi_t = \sum_{u \in \pi} \varphi_u 1_{\{t \in (u, u']\}}.
\]
and of the $d$ positions $C^i_{s,t}$, $1 \leq i \leq d$, on the swap contracts
\[(S^i_{s,t})^2 - [S^i_{s,t}]^1, \quad 1 \leq i \leq d.\]
Hence, for every continuous $\phi = (\phi^0_1, \phi^1_1, \phi^2_1) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d \times d}$ we can interpret
\[\tilde{\gamma}_u^0 S^{0}_u + \tilde{\gamma}_u^1 S^{1}_u + \tilde{\gamma}_u^2 S^{2}_u,\]
as the value of our portfolio at time $u$ if on the subinterval $(u, u)$ we have held $\tilde{\gamma}_u^0 = \phi^0_0$ positions in cash, $\tilde{\gamma}_u^1 = \phi^1_1$ positions in stocks and $\tilde{\gamma}_u^2 = \phi^2_1$ positions in swaps. Strategies that adopt positions in cash, stocks and swaps shall be referred to as enlarged strategies. For an enlarged strategy, the rebalancing cost from $(u, u)$ to $(u, u)$ is
\[\text{rebal}^r(u) = \phi^0_0 S^{0}_u + \phi^1_1 S^{1}_u + \phi^2_1 p(u, u') - \{ \phi^0_0 S^{0}_u + \phi^1_1 S^{1}_u + \phi^2_1 S^{2}_u \},\]
where, for $0 \leq s < t \leq T$ and $1 \leq i, j \leq d$, the amount $p^{i,j}(s, t) = p^{i,j}(s, t)$ denotes the (exogenously-given) price at time $s$ of the swap $S^{i,j}_t$ with maturity $t$. Notice that, since swap contracts are not primitive financial instruments, in the equation above the payoff $S^{i,j}_u$ at time $u$ is disentangled from the price $p(u, u')$ required at time $u$ to take a unit position on the next swap $S^{i,j}_u$.

We assume that the price $p(s, t)$ of the swap contracts $S_{s,t}$ defines a $\mathbb{R}^d \circ \mathbb{R}^d$-valued function on $\{(s, t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$, null and right-continuous on the diagonal, and such that $p(s, t)$ is of finite $p/2$-variation. Let $\phi^2$ be a continuous path of finite $q$-variation on $\text{Hom}(\mathbb{R}^d \circ \mathbb{R}^d; \mathbb{R})$, where $q$ and $p/2$ are Young complementary. Then, the integral path
\[Y_t := (\phi^2 p)_{0,t}\]
e exists and represents the accumulated cost in the time interval $[0, t]$ consumed by a continuously rebalanced enlarged strategy in order to adopt the positions $\phi^2$ on the swap contracts.

**Definition 6.1.** Let $f(S_T)$ be a contingent claim, where $f$ is in $C_b(\mathbb{R}^d)$ and $S_T$ is the terminal value of a continuous $d$-dimensional price path $S$ of finite $p$-variation. Let $S = (S, S)$ be an $\alpha$-diffusive model specification, $\alpha > p - 2$, and let $A$, $v$, Delta and Gamma be as in Proposition 5.3. Let $C$ be a continuous real valued function on $[0, T]$. Then, the $C$-enlarged Delta hedging is the enlarged strategy defined as
\[\phi^0_C = C t e^{-rt} - \text{Delta}_t S_t e^{-rt} - Y_t e^{-rt}, \quad \phi^1_C = \text{Delta}_t, \quad \phi^2_C = \text{Gamma}_t, \tag{6.1}\]
where $Y_t := (\phi^2 p)_{0,t}$.

A desirable property of a hedging strategy is the self-financing condition, i.e. the fact that the strategy does not require money to readjust its positions during the hedging period. The following Proposition 6.2 gives the explicit formula for $C$ in (6.1) that guarantees a null rebalancing cost of the $C$-enlarged Delta hedging.

**Proposition 6.2.** The continuous real valued function
\[C_t = v(t, S_t) - r \int_0^t e^{r(t-u)} Y_u du, \tag{6.2}\]
where $Y_t := (\text{Gamma} p)_{0,t}$, is such that the $C$-enlarged Delta hedging has zero cost of continuous rebalancing.

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13By this we mean: $p(s, s) = \lim_{t \downarrow s} p(s, t) = 0$ for all $0 \leq s \leq T$. 

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Proof. We adopt the notation in Definition 6.1. Furthermore, we set
\[ y_t := -r \int_0^t e^{r(t-u)}Y_u du. \]
We can write
\[ y_{0,t} - r \int_0^t (y_u - Y_u) du = 0. \] (6.3)
The cost of rebalancing along a partition \( \pi \) is
\[ \text{rebal}^\pi(u) = \bar{v}_u^0.S_0^u + \Delta\phi_u^\pi.S_0^u + \Gamma, \gamma_u p(u, u') \]
\[ = C_{u-u} + \Delta\phi_u p(u, u') - Y_{u-u} \]
\[ - \{ \phi_{u-u}^0.S_0^{u-u} + \Delta\phi_{u-u}^0.S_0^{u-u} + \Gamma, \gamma_{u-u} \}. \]
Hence, summing over \( u \in \pi_t, u > 0 \), we have
\[ \sum_{u \in \pi_t, u > 0} \text{rebal}^\pi(u) = V_{0,t} + y_{0,t} - Y_t + \sum_{u \in \pi_t} \Gamma, \gamma_u p(u, u') - \Gamma, \gamma_0 p(0, 0') \]
\[ - (\bar{v}_0^0.S_0^t) - (\gamma, \gamma, \gamma). (S, S)_t. \]
In the limit as \( |\pi| \to 0 \) we conclude
\[ \lim_{|\pi| \to 0} \sum_{u \in \pi_t, u > 0} \text{rebal}^\pi(u) = V_{0,t} + y_{0,t} - r \int_0^t V_u du \]
\[ - r \int_0^t (y_u - Y_u) du + r \int_0^t \Delta\phi_u S_u du \]
\[ - (\gamma, \gamma, \gamma). (S, S)_0^0 \]
\[ = 0, \]
owing to (5.4) and (6.3).

The classical Delta hedging is such that the initial endowment \( V_0 = v(0, S_0) \) is precisely what the replicating strategy requires in order to yield the amount \( f(S_T) \) at maturity \( T \). Therefore, the writer of an option invests \( V_0 \) in the Delta hedging strategy, and such strategy will yield exactly the amount of money that the buyer of the option will demand at maturity. Since Delta hedging has no additional costs of financing (i.e., rebalancing the portfolio does not consume money) the writer’s profit&loss is null. For the \( C \)-enlarged Delta hedging in Proposition 6.2, the self-financing condition holds. Therefore, the option writer’s P&L is exclusively given by the cost of replication, namely by the difference between the due payment \( f(S_T) \) and the final value \( \phi_T^0 S_T^0 + \phi_T^1 S_T \) of the portfolio. Notice that the latter does not comprise the payoff of the swaps, because such endowments are consumed in the rebalancing process.

**Proposition 6.3.** The profit&loss of the \( C \)-enlarged Delta hedging with \( C \) given as in (6.2) is
\[ P&L = Y_T + r \int_0^T e^{r(T-t)}Y_t dt, \]
where \( Y_t = (\Gamma, \gamma, S)_0^t. \)

Proof. Immediate from the definition. \( \square \)
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