Theory of Interaction Effects in N-S Junctions out of Equilibrium

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We consider a normal metal - superconductor (N-S) junction in the regime, when electrons in the normal metal are driven out of equilibrium. We show that the non-equilibrium fluctuations of the electron density in the N-layer cause the fluctuations of the phase of the order parameter in the S-layer. As a result, the density of states in the superconductor deviates from the BCS form, most notably the density of states in the gap becomes finite. This effect can be viewed as a result of the time reversal symmetry breaking due to the non-equilibrium, and can be described in terms of a low energy collective mode of the junction, which couples normal currents in N-layer and supercurrents. This mode is analogous to the Schmid-Schön mode. To interpret their measurements of the tunneling current, Pothier et. al [Phys. Rev. Lett. 79, 3490 (1997)] had to assume that the energy relaxation rate in the normal metal is surprisingly high. The broadening of the BCS singularity of the density of states in the S-layer manifest itself similarly to the broadening of the distribution function. Mechanism suggested here can be a possible explanation of this experimental puzzle. We also propose an independent experiment to test our explanation.

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I. INTRODUCTION

In metals the inelastic scattering rate $1/\tau_{in}$ at low enough energies is determined by electron-electron interactions. In a clean Fermi liquid $\epsilon \tau_e \gtrsim \hbar$ this inelastic rate can be estimated as $\hbar/\tau_{in} \simeq \epsilon^2/\epsilon_F$, where $\epsilon$ is the energy of the quasiparticle, $\tau_e$ is the elastic scattering time, and $\epsilon_F$ is the Fermi energy. This familiar result reflects only the phase volume of the final state for an inelastic process, while the corresponding matrix element is an energy independent constant. In the dirty limit $\epsilon \tau_e < \hbar$ the inelastic rate is significantly enhanced as compared with the clean case due to long range diffusive correlations of single electron wave-functions in the disordered system, see Refs. 3 for more detailed discussion. The inelastic scattering rate is not by itself an observable quantity. However, inelastic collisions of electrons profoundly affect the behavior of the system and, therefore, $1/\tau_{in}$ in many cases can be extracted from experimental data. For example, from magnetoresistance one can evaluate quantitatively the dephasing time $\tau_e$, which often coincides with $1/\tau_{in}$. The dephasing time describes the loss of phase coherence, as electrons move diffusively in the bulk of a metallic sample. This loss of coherence cuts off otherwise divergent weak-localization correction. The dephasing time has been extensively studied, and we believe that the existing theory allows a good understanding of the experimental data.

Another effect of inelastic scattering is the energy relaxation described by the time $\tau_e$. This is the time it takes for a "hot" quasiparticle with energy $\epsilon$ much larger than temperature $T$ to thermalize with all the other electrons. Theoretically, it is given by

$$\frac{\hbar}{\tau(e)} \simeq \frac{\epsilon}{G_m(L_e)}, \quad L_e = \sqrt{\frac{\hbar D}{\epsilon}},$$

(1.1)

where $G_m(L) \propto L^{d-2}$ is the dimensionless (in units of $\epsilon^2/2\pi\hbar$) conductance of a sample of size $L$, $D$ is the diffusion constant and $d$ the dimensionality of the system. This result follows from the Fermi Golden Rule, but now the phase volume is multiplied by the matrix element, which is no longer a constant. Not only this matrix element substantially depends on the energy transferred, but it even diverges at small energies due to the wave function correlation, see Refs. 3.

To determine $\tau_e$ experimentally, one has to apply an external perturbation to drive the system out of equilibrium, and then to measure the distribution function of electrons. Recently, an elegant and important experiment was performed to measure directly the electronic distribution function $f(\epsilon)$. An external voltage $U$ applied to a copper wire caused an electric current $J$, thus driving the wire out of equilibrium. In order to determine the non-equilibrium distribution function the authors of Ref. 4 fabricated an additional electrode connected with the wire by a tunneling contact. The distribution function $f(\epsilon)$ was extracted from the measurements of the tunneling conductance $G_T(V)$ of this contact as a function of the bias voltage $V$ using the procedure as follows. Assuming that the density of electronic states in the wire is energy independent, one can present the tunneling conductance $G_T(V)$ as the convolution of $f(\epsilon)$ with the tunneling density of states in the additional electrode $\rho(\epsilon)$

$$G_T(V) \propto \int d\epsilon \frac{\partial f(\epsilon - eV)}{\partial \epsilon} \rho(\epsilon).$$

(1.2)
As follows from Eq. (1.3), the more pronounced is the energy dependence of the density of states in the additional electrode \( \rho(\epsilon) \), the more precisely one can determine the distribution function \( f(\epsilon) \) measuring \( G_T(V) \). For this reason the authors of Ref. 1 used a superconducting electrode to take advantage of the BCS singularity in the density of states \( \rho(\epsilon) \). The existence of this singularity in the equilibrium was convincingly determined by independent measurements at \( J = 0 \). The data on the tunneling conductance \( G_T(V) \) were fitted by Eq. (1.2) yielding the distribution function \( f(\epsilon) \) and, thus, the energy relaxation time.

This procedure produced quite unexpected results. First of all, the extracted relaxation time turned out to be two orders of magnitude shorter than that of Eq. (1.3). Moreover, no dependence of the relaxation time on the energy \( \epsilon \) was observed. This would mean the failure of the theory lying behind the derivation of Eq. (1.3).

However, the theory is based on regular expansion in inverse powers of the dimensionless conductance \( G_m \) (see Ref. 2 and references therein). The observation of a good metallic conductance \( G_m \gg 1 \) in the experiment justifies the applicability of this theory.

In this paper we attempt to explain the puzzling results of the experiment by lifting the main assumption in the interpretation of the data – independence of the density of states in the superconductor of the electronic distribution in the normal metal. We calculate the tunneling conductance between the superconducting and metallic films explicitly and find that interaction of the tunneling electrons with non-equilibrium fluctuations of the current in the normal layer smears the BCS singularity in the density of states in the superconductor. This effect has nothing to do with energy relaxation. Nevertheless, it effectively broadens the energy dependence of the experimental \( f(\epsilon) \), extracted from Eq. (1.2) with the use of the equilibrium BCS density of states \( \rho(\epsilon) \), whereas the real distribution function remains sharp. As a result, the energy relaxation time appears much shorter, than it really is. We found that this effect is not small as inverse dimensionless conductance of the normal metal \( 1/G_m \). Instead, the magnitude of the effect is proportional to the inverse conductance of the superconductor in the normal state \( 1/G_s \). Under the condition \( G_m \gg G_s \), which we assume in the present paper, this effect dominates the real energy relaxation.

The remainder of the paper is organized as follows. In Sec. II we present a phenomenological derivation of our main results. In the same section we suggest an independent experiment to test our theory. Section III is devoted to the rigorous analysis of the tunneling density of states under non-equilibrium conditions. Our findings are summarized in Conclusions. Some mathematical details are relegated to Appendices.

II. QUALITATIVE DISCUSSION

The purpose of this Section is to describe qualitatively how the the non-equilibrium fluctuations affect the density of states of the superconductor. We need first to classify the collective excitations, which are present in the system, and to understand how non-equilibrium conditions influence them.

We start by recalling the basic physics of phase fluctuations in superconductors, then consider their coupling to currents in the metallic layer. The electric current in the normal metal is accompanied by shot-noise. As we show below, this noise gives rise to the classical phase fluctuations. Finally, we demonstrate that the enhanced fluctuations dramatically affect the BCS density of states. In particular, they lead to a non-zero density inside the BCS gap. This Section is concluded by suggesting an independent experiment to test our theory.

A. Collective modes in N-S sandwich

Consider a superconducting film at zero temperature. It is well known, that all of the excitations with the energy smaller than the superconducting gap \( \Delta \) are associated with the phase \( \theta \) of the order parameter. The time evolution of this phase is governed by hydrodynamic equations, which in the absence of external magnetic fields can be written as

\[
\dot{n}_s + \frac{1}{2e} \nabla \cdot j_s = 0, \quad (2.1a)
\]

\[
j_s = -\pi \hbar D_s \nu_s \Delta \nabla \theta, \quad (2.1b)
\]

\[
\hbar \dot{\theta} = 2 \left( e \varphi + \frac{n_s}{e_s} \right), \quad (2.1c)
\]

where \( n_s \) is the perturbation of the carrier density in the superconductor, \( \nu_s \) is the thermodynamic density of states per unit area in the superconductor, and \( j_s \) is the supercurrent. Equation (2.1a) is the continuity relation. We wrote the London equation (2.1b) for a dirty superconductor and expressed the superfluid density through the diffusion coefficient \( D_s \) in the normal state of the superconductor. Equation (2.1c) is the conventional Josephson relation between the electrochemical potential and the phase of the order parameter \( \theta \). Finally, the electrostatic potential \( \varphi \) is connected to the density variation by the Coulomb law

\[
e \varphi = \int dr' V(r - r')n_s(r'), \quad V(r) = \frac{e^2}{r}. \quad (2.2)
\]

Performing the Fourier transform of Eqs. (2.1) and (2.2), we obtain the dispersion relation for the collective mode

\[
\omega^2 = \frac{\pi}{\hbar} D_s Q^2 [1 + \nu_s V(Q)], \quad V(Q) = \frac{2\pi e^2}{Q} \quad (2.3)
\]
It corresponds to the usual 2D plasmon with dispersion \( \omega \approx \sqrt{Q} \) which has little effect on the behavior of the system because of its small density of states at low frequencies.

When a layer of the normal metal is brought nearby, \( V(Q) \) gets screened, and the dispersion relation Eq. (2.3) becomes linear. To see this, one has to include the normal currents into the set of the hydrodynamic equations (2.3) – (2.2). The equation for the scalar potential (2.2) is modified to

\[
e\varphi = \int dr' V(r - r') \left[ n_s(r') + n_m(r') \right]. \tag{2.4}
\]

Here we neglected for simplicity the thickness of the isolating layer between the normal metal and the superconductor assuming that \( d \ll 1/Q \). All the results are insensitive to this assumption (see Sec. III C). The density of carriers \( n_m \) in the normal metal is governed by the continuity equation and the Ohm’s law:

\[
\dot{n}_m + \frac{1}{e} \nabla \cdot j_m = 0, \tag{2.5a}
\]

\[
j_m = -\sigma_m \nabla \varphi - D_m \nabla n_m. \tag{2.5b}
\]

The charge in the normal metal is redistributed by the electric field according to Eqs. (2.5a) – (2.5b). We evaluate \( n_m \) from these equations, substitute the result into Eq. (2.4) and find, that the potential becomes dynamically screened. At frequencies much smaller, than the plasmon frequency in the normal layer, \( \omega \approx \nu_v V_0(Q) D_m Q^2 \), the screened potential takes the form

\[
V(Q, \omega) = \frac{1}{\nu_m} \frac{-i\omega + D_m Q^2}{D_m Q^2}. \tag{2.6}
\]

We now substitute Eq. (2.6) into the dispersion law of the collective mode Eq. (2.3) and find

\[
\omega_{ph'} = \omega_{ph} - i\omega_{ph''}, \tag{2.7a}
\]

\[
\omega_{ph'} = Q \left( \frac{\pi D_s}{\hbar} \right)^{1/2} \left( 1 + \frac{\nu_s}{\nu_m} \right)^{1/2}, \tag{2.7b}
\]

\[
\omega_{ph''} = \frac{\pi}{2} \left( \frac{\nu_s D_s}{\nu_m D_m} \right) \frac{\Delta}{\hbar}. \tag{2.7c}
\]

Equations (2.7a) – (2.7c) are valid provided that \( \omega_{ph'} > \omega_{ph''} \). This condition is satisfied already at small frequencies \( \hbar \omega \approx \hbar \omega_{ph'} \approx \Delta (G_s/G_m) \ll \Delta \). The lifetime of this mode is finite due to the interaction with the relaxation mode in the normal metal.

Since \( \omega_{ph} > \omega_{ph''} \), Eqs. (2.7a) and (2.7c) describe a well pronounced collective mode. We will call this mode “phason”. According to Eq. (2.7a), the phason has linear dispersion. This is due to the fact that the currents in the normal metal flow along the interface in direction opposite to the currents in the superconducting layer, so the total charge accumulation does not occur. The physics of this mode is essentially similar to the well known Schönhartung mode in the vicinity of the critical temperature or to the second sound in superfluid. The only difference is that the normal excitations are not thermally activated in the superconductor itself but rather exist in the normal metallic layer close to the superconductor. This, however, has little consequence on the charge dynamics.

B. Phase fluctuations due to the current in the normal layer

We have seen that a presence of the metal gives rise to the new collective mode, the phason. This mode corresponds to the oscillating electric currents flowing in the opposite directions in the metallic and the superconducting layers.

Now let us consider what happens, when a current is driven in the normal layer. The average currents in the metal are accompanied by the fluctuations known as the shot noise. Since the currents in the metal are coupled to those in the superconductor, it is natural to expect that in the superconductor the fluctuating currents appear as well, and consequently, the phasons are generated. In other words, a dc-current in the metallic layer should enhance phase fluctuations in the superconductor.

To include these fluctuations in our description of the N-S sandwich we add Langevin sources \( \delta j_\ell \) to the current in the normal metal. Equation (2.5a) takes the form

\[
j_m = -\sigma_m \nabla \varphi - D_m \nabla n_m + \delta j_\ell. \tag{2.8}
\]

The fluctuations \( \delta j_\ell \) are described by their correlator \( \langle \delta j_\ell^\alpha \delta j_\ell^\beta \rangle \). Provided the frequency \( \omega \) is much less than the applied voltage \( \omega \ll eU/\hbar \) and the energy relaxation is negligible, this correlator can be written as

\[
\langle \delta j_\ell^\alpha \delta j_\ell^\beta \rangle \omega_i = \delta_{\alpha\beta} \sigma_m eU \propto \delta \langle j_m \rangle. \tag{2.9}
\]

The superconducting phase \( \theta \) in the presence of the current fluctuations \( \delta \hat{\theta} \) can be determined from the system of equations (2.7a), (2.7b), (2.7c) and (2.8). In addition, we use the Einstein relation \( \sigma_m = e^2\nu_m D_m \). As a result, we can present the phase fluctuation \( \delta \theta \) as

\[
\delta \theta \propto -i \frac{\delta j_\ell \cdot Q}{e \hbar \nu_m D_m Q^2 \omega^2 - \omega_{ph}^2(Q)} \tag{2.10}
\]

where \( \omega_{ph}^2(Q) \) is the phonon dispersion. In Eq. (2.10) and all the subsequent formulas, we have not specified an inessential numerical prefactor, which will be found in the next section. Using the correlator Eq. (2.5a) we obtain
\[ \langle \delta^2 \rangle_{\omega,Q} = \frac{eU}{\hbar^2 v_m D_m Q^2 |\omega^2 - \omega_{ph}^2(Q)|^2}. \]  

(2.11)

Therefore, the correlator of the phase fluctuations \( \langle \delta^2 \rangle_{\omega,Q} \) has a well pronounced phason pole and is proportional to the applied voltage \( U \).

In what follows, we will need the single point correlator of the phase fluctuations, i.e., the integral of Eq. (2.11) over the momentum \( Q \). The main contribution to this integral comes from the pole, which corresponds to the resonant excitation of the phason by the current fluctuations in the normal layer. The logarithmic divergence at \( Q = 0 \) in Eq. (2.11) is not important and, as we show in Sec. III B, disappears due to the gauge invariance. Since the linewidth of the phason decreases with the increase of the diffusion coefficient \( D_m \), see Eq. (2.7c), the large factor \( D_m \) in the denominator of Eq. (2.11) is canceled. As a result, the single point correlator of the phase fluctuations \( \langle \delta^2 \rangle_{\omega} \) does not depend on the parameters of the metallic layer

\[ \langle \delta^2 \rangle_{\omega} = \int d^2Q \langle \delta^2 \rangle_{\omega,Q} = \frac{eU}{\Delta} G_s |\omega|. \]  

(2.12)

Equation (2.12) is valid, provided \( G_s/\Delta G_m \ll \hbar \omega \ll eU \), where \( G_{s,m} \) denote dimensionless conductances of the superconducting (in the normal state) and normal layers respectively: \( G_s = 2\pi h \sigma_{s,m} / e^2 = 2\pi h \sigma_{s,m} D_{s,m} \). These are the conductivities measured in units of \( e^2/2\pi \hbar \) and \( 1/(25.8 K \Omega) \).

C. Effect of the phase fluctuations on the tunneling DOS of the superconductor

We have found that in the presence of the normal layer the phase fluctuations in the superconductor are large due to the resonant excitation of the phasons. Now we are interested in the effect of these fluctuations on a measurable quantity, e.g. on the tunneling conductance, \( G_T = \partial I/\partial V \) of the junction, with \( I \) and \( V \) being the current and the voltage across the junction respectively.

In the lowest order in the tunneling amplitude, the tunneling conductance \( G_T \) is determined by the density of states of the superconductor Eq. (1.2). The density \( \rho(\epsilon) \) depends on single particle excitation energies. In the absence of fluctuations the excitation energy in the superconductor is given by the usual BCS expression \( E = \sqrt{\xi^2 + \Delta^2} \), where \( \xi \) is the energy of the orbital state counted from the Fermi level. The energy \( E \) can not be smaller than \( \Delta \). This prevents electrons (or holes) from the metal with energies smaller than \( \Delta \) from tunneling into the superconductor.

However, in the presence of the phase fluctuations, it becomes possible for an electron with the energy smaller than \( \Delta \) to tunnel and then to absorb a phason to compensate for the energy deficit. As a result, the density of states turns out to be finite even inside the gap \( E < \Delta \).

To describe this effect of the phason assisted tunneling more quantitatively, we first calculate the density of states in the superconductor in the presence of homogeneous phase fluctuations. In this case, due to the orthogonality of the orbital wave functions, all the transitions are confinced to the same orbital (characterized by some orbital energy \( \xi \)). The problem simplifies since within the single orbital we have to consider only four states (one orbital can be occupied by no more than two electrons): \( \psi_0 \) - empty orbital, \( \psi_2 \) filled orbital, \( \psi''_1 \) and \( \psi'_1 \) - singlets.

States \( \psi_0 \) and \( \psi_2 \) are coupled to each other due to the exchange with the condensate, whereas singlets are not. The resulting Schrödinger equations are

\[ i\hbar \dot{\psi}_0 = \Delta e^{i\theta(t)} \psi_2, \]  

(2.13a)

\[ i\hbar \dot{\psi}_2 = 2\xi \psi_2 + \Delta e^{-i\theta(t)} \psi_0, \]  

(2.13b)

\[ i\hbar \dot{\psi}_{\uparrow,\downarrow} = \xi \psi_{\uparrow,\downarrow}. \]  

(2.13c)

At frequencies smaller than \( eU/h \) the occupation number of phasons is large \( \sim eU/h\omega \), and, therefore, \( \theta \) can be treated as a classical variable.

Phase \( \theta \) changes with the characteristic frequency of the order of \( eU/h \ll \Delta/h \). Hence, Eqs. (2.13) can be solved in the adiabatic approximation. The time-dependent ground state of this four-level system is given by

\[ \Psi_{GS}(t) = \left[ u\psi_0 + v\psi_2 e^{i\theta(t)} \right] e^{-\frac{i}{\hbar} \int^t dt' E_{GS}(t')} \]  

(2.14)

where \( u \) and \( v \) are the usual coherence factors, \( u = \cos \alpha \) and \( v = \sin \alpha \) and \( \tan 2\alpha = -\Delta/\xi \). and

\[ E_{GS}(t) = \xi - \sqrt{\xi^2 + \Delta^2}, \]  

(2.15)

\[ \tilde{\xi} = \xi - \hbar \dot{\theta}(t). \]

So far we discussed an isolated superconductor. Let us now consider tunneling of an electron with the energy \( \epsilon \) from the normal metal into the superconductor. Since the initial state has the energy \( E_{GS} + \epsilon \) and the final state is one of the states \( \psi_{\uparrow,\downarrow} \), each having energy \( \xi \), the transition amplitude of this process can be estimated as

\[ \mathcal{A}(t) = \ldots \int_0^t dt' \exp \left( \frac{i}{\hbar} \int_0^{t'} dt''[\epsilon + E_{GS}(t'') - \xi] \right). \]  

(2.16)

The prefactor in Eq. (2.16) denoted by \( \ldots \) includes the tunneling matrix element and the coherence factors. This prefactor is not important for present discussion. The transition rate is given by

\[ \frac{1}{\tau_{\text{tunn}}} \propto \lim_{t \to \infty} \frac{1}{\langle |\mathcal{A}(t)|^2 \rangle} \theta \]  

(2.17)
where \(\langle \ldots \rangle_\theta\) stands for the averaging over the fluctuations of the phase \(\theta\).

We can substitute the amplitude Eq. (2.17) into the tunneling rate Eq. (2.13) and use the explicit form of \(E_{GS}(t)\) given by Eq. (2.15). In the exponent we have
\[
\epsilon - B - 2\sqrt{\Delta^2 + \hbar^2 \theta}.
\]

The density of states in the superconductor \(\rho(\epsilon)\) is proportional to the tunneling rate \(1/\tau_{\text{tunnel}}\). Evaluating the time integral in the exponent, we obtain
\[
\rho(\epsilon) = \lim_{t \to \infty} \frac{1}{2\pi} \left| \int_0^t dt' e^{i\int_0^t (e^{-\sqrt{\Delta^2 + \hbar^2}} / i\epsilon') dt/\hbar} \right|^2.
\]

(2.18)

The absolute value square of the time integral can be expressed in terms of the double integral over \(t'\) and \(t''\). It is convenient to change variables to the sum and difference of \(t' \pm t''\). The exponent depends only on the time difference. Indeed, the averaging over fluctuations of \(\theta\) can be performed independently and the average functions depend on the time difference only. The integral over \(t' + t''\) then yields \(t\), which cancels with the denominator in the prefactor. In the remaining integral over the difference \(t' - t''\) the upper limit can now be taken to infinity. The last step is to sum over all orbitals, since the electron can tunnel to any of them. This amounts to integration over \(\sum_{\text{orb}} \nu_s \int d\xi\). This integral is independent from the fluctuations of \(\theta\), since they produce shift of \(\xi\) which is irrelevant because of the integration over \(\xi\), and yields the BCS density of states \(\rho_0(\epsilon)\) Fourier transformed to the time domain:
\[
\sum_{\text{orb}} e^{i\int_0^t (e^{-\sqrt{\Delta^2 + \hbar^2}} / \hbar) dt} = \int d\xi \rho_0(\epsilon) e^{i\xi t}.
\]

Thus, we find
\[
\rho(\epsilon) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \frac{d\epsilon'}{2\pi} \rho_0(\epsilon') e^{i\epsilon'(\epsilon - \epsilon') t} |e^{i(\theta(t) - \theta(0))}|_\theta. \quad (2.19)
\]

Therefore, in the presence of the phase fluctuations the density of states in the superconductor becomes modified by the factor, which describes the phason assisted tunneling from the normal layer. To evaluate the average we use the phase correlator Eq. (2.12). It is not correct, since here we considered only the homogeneous fluctuations of the phase, while the correlator Eq. (2.12) includes also inhomogeneous fluctuations. However, the effect of the inhomogeneity can be estimated as \(\hbar D_s Q^2 / \Delta\), while the main contribution comes from the phase mode with \(D_s Q^2 \sim \hbar \omega^2 / \Delta\). Therefore the correction is \(\sim \hbar^2 \omega^2 / \Delta^2\), and should be neglected because the corrections of the same order were already omitted within the adiabatic approximation.

Averaging the phase factor with the help of Eq. (2.12), we find
\[
\rho(\epsilon) \approx \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \frac{d\epsilon'}{2\pi} \rho_0(\epsilon') e^{i\epsilon'(\epsilon - \epsilon') t} \exp \left[ -\frac{\epsilon U}{\Delta G_s} \int_{-\epsilon U}^{\epsilon U} d\omega \frac{e^{i\omega t} - 1}{\omega} \right]. \quad (2.20)
\]

The integration is cut off at \(\omega = \epsilon U\), because at \(\omega > \epsilon U\) the classical description fails, and Eq. (2.12) for the phase fluctuations is not valid.

In the energy interval \(\Delta - \epsilon U \leq \epsilon < \Delta\) one can expand the exponent to the first order, which corresponds to a single phason process. As a result,
\[
\rho(\epsilon) = \left( \frac{\epsilon U}{\Delta G_s} \right) \sqrt{\Delta / \Delta - \epsilon}, \quad (2.21)
\]

where numerical coefficient is the result of rigorous treatment of the following section. Equation (2.21) is the main qualitative result of the paper. It shows that in the presence of the normal layer driven out of equilibrium by the applied voltage \(U\), the superconductor no longer has a hard gap. Instead, there is a dip, and the tunneling density of states is suppressed at \(\epsilon < \Delta\) (see Fig. 1).

![FIG. 1. Sketch of the tunneling density of states \(\rho(\epsilon)\) the superconductor modified by the current fluctuations in the normal metal driven out of equilibrium by the applied voltage \(U\).](image)

At energies smaller than \(\Delta - \epsilon U\) the density of states differs from zero due to multi-phason processes. It means that the exponent in Eq. (2.21) should be expanded to higher orders. However, to describe these multi-phason processes, we need better understanding of the nature of the frequency cut off. Also, we have so far neglected the quantum part of the phase fluctuations, \(\omega > \epsilon U\). These issues are addressed in the technical part of the paper, Sec. III where we perform a rigorous calculation of the tunneling, based on Keldysh diagrammatic technique.
D. Is the effect of phason assisted tunneling independently observable?

We have shown that the phason assisted tunneling manifests itself in the tail in the density of states within superconducting gap. When interpreting the experimental data, this effect could be confused with the broadening of the distribution function $f_{M}(\epsilon)$ in the normal metal [see Eq. (1.2)]. We believe that such a misinterpretation of the experimental data was made in Ref. [1].

We should warn the reader that Eq. (2.2) as well as the subsequent rigorous calculation of the tunneling current [see Eq. (3.69) for the result] were obtained for the simplest model of the N-S junction, namely the 2D sandwich. The spectrum of collective modes is sensitive to the geometry of the system, therefore, our results are not expected to describe the experiment of Ref. [4] in detail. However, we have presented a strong evidence of the existence of the microscopic mechanism responsible for the change of the shape of the tunneling conductance, which is different from the trivial broadening of the distribution function in the normal metal. In this paper we do not intend to evaluate the effect of phason assisted tunneling in various possible experimental realizations of the N-S junction. Instead, we suggest an independent measurement, that can distinguish between the two mechanisms.

![Diagram](image.png)

**FIG. 2.** Scheme of the experiment for observation the effect of non-equilibrium phase fluctuations on the density of states. Switch “S1” corresponds to the experiment of Ref. [2] Switch “S2” corresponds to the measurement of the density of states in the normal metal. In this paper we do not intend to evaluate the effect of phason assisted tunneling in non-equilibrium electrode “M1” itself due to energy relaxation. We suggest to measure the tunneling conductance between the superconductor and the non-equilibrium normal metal.

III. THEORY OF THE INTERACTION EFFECTS IN N-S JUNCTIONS.

In this Section we set $\hbar = 1$ in all intermediate expressions.

A. Tunneling current

In this section we use Keldysh[2] diagrammatic technique to evaluate the tunneling current $I$ between the superconductor and the normal metal. In order to take superconductivity into account properly, we need Green functions $G$ to be matrices in the space $K \otimes N$, which is the direct product of Keldysh space $K$ and Gor'kov-Nambu space $N$. We choose the basis in the Keldysh space, for which electronic Green’s functions are

$$G = \begin{pmatrix} G^R & \tilde{G}^K \\ 0 & G^A \end{pmatrix}_K,$$  \hspace{1cm} (3.1)

Let $[,]_-$ and $[,]_+$ denote a commutator and anticommutator correspondingly. The three components of the Green’s function $G_{\alpha \beta}$, where $\alpha$ and $\beta$ can be either $M$ (metal) or $S$ (superconductor), are defined as

$$G^K_{\alpha \beta}(t_1, t_2) = -i\langle \Psi_{\alpha}(t_1) \otimes \Psi^\dagger_{\beta}(t_2) \rangle_-, \hspace{1cm} (3.2a)$$

$$G^R_{\alpha \beta}(1, 2) = -i\eta(t_1 - t_2)\langle \Psi_{\alpha}(1) \otimes \Psi^\dagger_{\beta}(2) \rangle_+, \hspace{1cm} (3.2b)$$

$$G^A_{\alpha \beta}(1, 2) = i\eta(t_2 - t_1)\langle \Psi_{\alpha}(1) \otimes \Psi^\dagger_{\beta}(2) \rangle_+,$$  \hspace{1cm} (3.2c)

where we used a short-hand notation $n = (\vec{r}_n, t_n)$ for $n = 1, 2$. In Eqs. (3.2) $\eta(x)$ is Heaviside step function, $\Psi$ are Nambu spinors

$$\Psi = \begin{pmatrix} \psi_{\uparrow} \\ \psi^\dagger_{\downarrow} \end{pmatrix}, \hspace{1cm} \Psi^\dagger = \begin{pmatrix} \psi_{\downarrow} & -\psi_{\uparrow} \end{pmatrix},$$  \hspace{1cm} (3.3)

and $\psi_{\uparrow}(n)$, $\psi_{\downarrow}(n)$ are the fermionic operators in the Heisenberg representation. Since we are using the Keldysh technique, $\langle \ldots \rangle$ denotes the average with arbitrary distribution function.

We start our calculation of the tunneling current $I$ between the superconductor and the normal metal with the general expression

$$I = \frac{e}{2} T_0 \text{Tr}[\sigma_2(G_{MS} - G_{SM})],$$  \hspace{1cm} (3.4)
Within our choice of the basis [Eq. 3.1] the current vector in Eq. (3.4) is Pauli matrix $\mathbf{\sigma}_2$ in Keldysh space

$$\mathbf{\sigma}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_K \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_N$$  \hspace{1cm} (3.5)

$\text{Tr}$ in Eq. (3.4) means trace in the $K \otimes N$ space and also assumes that $\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}$, where $\mathbf{r}$ corresponds to the contact point. Below we often omit the spatial coordinates of the Green functions and write explicitly only temporal coordinates. It always assumes that $\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}$.

In the first order in the tunneling amplitude $T_0$ the Green’s functions can be calculated as

$$\mathcal{G}_{MS} = -\pi^2 v_m v_s T_0 \int dt' \mathbf{g}_M(t,t')e^{-i\mathbf{v}_t t'} \mathbf{g}_S(t',t),$$  \hspace{1cm} (3.6)

where $\mathbf{v}$ is the applied voltage and $\mathbf{g}_M$ and $\mathbf{g}_S$ are the Green’s functions in the normal metal and in the superconductor, respectively, defined as

$$\mathbf{g}_\alpha(t_1,t_2) = \frac{i}{\pi v_\alpha} \mathcal{G}_{\alpha\alpha} (\mathbf{r},\mathbf{r};t_1,t_2).$$  \hspace{1cm} (3.7)

In what follows the semiclassical Green’s functions in the $K \otimes N$ space will be denoted by boldfaced symbols, while the $2 \times 2$ matrices in the Nambu space will be denoted by a hat.

In the normal metal the semiclassical Green’s function $\mathbf{g}_M$ satisfies the Usadel equation

$$-D_m \nabla_R^2 (\mathbf{g}_M \circ \nabla_R \mathbf{g}_M) + \frac{\tau^3}{3} \frac{\partial \mathbf{g}_M}{\partial t_1} + \frac{\partial \mathbf{g}_M}{\partial t_2} \tau^3 - i[\Phi, \mathbf{g}_M] = 0,$$  \hspace{1cm} (3.8)

where the matrix $\tau^3$ is

$$\tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_K \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_N,$$  \hspace{1cm} (3.9)

the commutator designates

$$[\Phi, \mathbf{g}_M] = \Phi(t_1)\mathbf{g}_M(t_1,t_2) - \mathbf{g}_M(t_1,t_2)\Phi(t_2),$$  \hspace{1cm} (3.10)

and

$$\mathbf{g}_M \circ \mathbf{g}_M(t_1,t_2) = \int dt_3 \mathbf{g}_M(t_1,t_3) \mathbf{g}_M(t_3,t_2).$$  \hspace{1cm} (3.11)

The scalar potential is also a matrix

$$\Phi = \begin{pmatrix} \varphi_+ & \varphi_- \\ \varphi_- & \varphi_+ \end{pmatrix}_K \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_N.$$  \hspace{1cm} (3.12)

While the uniform scalar potential can be gauged out and does not affect the behaviour of the system, it can be shown [2] that the slow spatial variation of $\varphi$ can be taken into account by means of linear functional $K[\varphi]$ so that the resulting Green’s function can be written as

$$\mathbf{g}_M(t_1,t_2) = e^{iK[\varphi](t_1)t_3} (\mathbf{g}_{M0} + \mathbf{g}_{M1}) e^{-iK[\varphi](t_2)t_3},$$  \hspace{1cm} (3.13)

where $\mathbf{g}_{M0}(t_1 - t_2)$ is the noninteracting Green’s function, while $\mathbf{g}_{M1}(t_1,t_2)$ contains all of the corrections not captured by the transformation Eq. (3.13).

The functional $K[\varphi]$ can be specifically chosen in such a way, that in equilibrium the correction $\mathbf{g}_{M1}(t_1,t_2)$ is proportional to the square of the gradient $\mathbf{g}_M \sim (\nabla K[\varphi])^2$. When the system is out of equilibrium, $\mathbf{g}_{M1}(t_1,t_2)$ also acquires a term proportional to the deviation of the distribution function $f_M(\epsilon)$ from equilibrium one $f_{M,eq}(\epsilon)$, i.e. $\mathbf{g}_{M1} \sim \Delta K[\varphi](f_M(\epsilon) - f_{M,eq}(\epsilon))$. Such functional is given by [3] (see also Appendix A)

$$\left( \begin{array}{c} K[\varphi_+] \\ K[\varphi_-] \end{array} \right) = \mathcal{K} \left( \begin{array}{c} \varphi_+ \\ \varphi_- \end{array} \right),$$  \hspace{1cm} (3.14a)

$$\mathcal{K} = \begin{pmatrix} (-i\omega + D_m Q^2)^{-1} & -2\text{Re} \frac{N(\omega)}{-i\omega + D_m Q^2} \\ 0 & -(i\omega + D_m Q^2)^{-1} \end{pmatrix},$$  \hspace{1cm} (3.14b)

where $N(\omega)$, which can be named “bosonic distribution function” is defined as

$$N(\omega) = \int \frac{d\epsilon}{\omega} \left[ 2f_M(\epsilon)f_M(\epsilon + \omega) - f_M(\epsilon) - f_M(\epsilon + \omega) \right].$$  \hspace{1cm} (3.15)

In this case the corrections $\mathbf{g}_{M1}(t_1,t_2)$ in Eq. (3.13) come from the effects of energy relaxation and interaction localization corrections (the effects of weak localization were neglected from the very beginning, as discussed above). These corrections are inversely proportional to the metal conductance $G_m$ (in 2D; in 1D the corrections are $\sim 1/G_m(\ell_{ph})$ where $\ell_{ph}$ is the dephasing length; for details see Ref. 4). Therefore, these corrections can also be neglected.

The retarded and advanced Green functions $g_{M0}^{RA}$ in the time domain are delta functions

$$g_{M0}^R(t) = -g_{M0}^A(t) = \delta(t).$$  \hspace{1cm} (3.16)

The Keldysh function in the energy domain is related to the distribution function by

$$g_{M0}^K(\epsilon) = 2(1 - 2f_M(\epsilon)).$$  \hspace{1cm} (3.17)

Fluctuating electric fields give rise to the fluctuations of the phase $\theta$ of the order parameter $\Delta$ in the superconductor. The fluctuations of the absolute value of the order parameter $\Delta$ do not couple to the fluctuations of the electric field, and their spectrum has a gap $\Delta$. Therefore they can be ignored. The superconducting Green’s function $\mathbf{g}_S$ can be evaluated by solving the Usadel equation in the superconductor.
where \(g_{S0}(t_1 - t_2)e^{-i\theta_K(t_1-t_2)^3},\) (3.21)

where \(g_{S0}\) is the usual BCS Green’s function.

We now substitute the Green’s functions in the normal and superconducting layers Eqs. (3.13) and (3.21) into Eq. (3.6) and substitute the result into the expression for the tunneling current Eq. (3.23). The next step is to average over the phase and Coulomb fluctuations. The current is then given by the integral

\[ I = \frac{G_0}{2} \int dt e^{-i eV't} \text{Tr} \{ \sigma_2 [g_{MO}(t-t')g_S(t'-t)] \}, \]

where \(G_0 = 2\pi \nu_m \nu_s e^2 T_0.\) We used the cyclic property of the trace and an obvious fact that the matrices \(\sigma_2\) and \(\tau^1\) to obtain the averaged function

\[ g_S(t_1 - t_2) = (ei\phi(t_1)^3) g_{S0}(t_1 - t_2)e^{-i\phi(t_2)^3},\] (3.23)

where the fluctuation field equals to

\[ \phi = \theta - K[\varphi].\] (3.24)

So far our analysis was quite general and independent of the geometry of the sample. Even though we were considering tunneling through point contacts, Eqs. (3.22) and (3.23) can be straightforwardly generalized for the wide contacts with large number of channels.

We now calculate the trace in Keldysh space and use the explicit form of the metallic Green’s functions Eq. (3.16) and Eq. (3.17) to evaluate the time integral. The differential tunneling conductance \(G_T = \partial I/\partial V\) can be written as

\[ G_T = \frac{G_0}{2} \int \frac{d\epsilon}{2\pi} \frac{\partial f_M}{\partial \epsilon} (\epsilon - eV) \text{Re} \text{Tr} \{ g_S^R(\epsilon)^3 \}. \] (3.25)

The last step is to perform the average in Eq. (3.23). In the leading approximation in \(1/G_m\) the fluctuations are Gaussian, and we obtain (see Appendix C for details) for the function \(h_{\phi\phi}^R(\epsilon) = \text{Tr} [g_S^R(\epsilon)^3]\)

\[ h_{\phi\phi}^R(\epsilon) = \int_0^\infty dt e^{i\epsilon t} \left[ h_0^-(t)e^{2i\Delta} - h_0^+(t)e^{-2i\Delta} \right], \]

where \(h_0^+(t > 0) = -h_0^-(t > 0) = \frac{\Delta}{\pi} K_1(i\Delta t)\) and \(K_1\) is the modified Bessel function. The fluctuation propagators are defined as

\[ \langle \phi_+ \phi_+ \rangle = iD_{\phi\phi}^K(t_1 - t_2), \]

\[ \langle \phi_+ \phi_- \rangle = iD_{\phi\phi}^R(t_1 - t_2), \]

\[ \langle \phi_- \phi_- \rangle = iD_{\phi\phi}^A(t_1 - t_2), \]

\[ \langle \phi_- \phi_- \rangle = 0, \]

and

\[ D_{\phi\phi}^{+/-}(\epsilon) = (D_{\phi\phi}^K \pm D_{\phi\phi}^R)/2. \] (3.29)

The conductance can finally be expressed in terms of the function \(h_{\phi\phi}^R(\epsilon)\) Eq. (3.26) as

\[ G_T = \frac{G_0}{2} \int \frac{d\epsilon}{2\pi} \frac{\partial f_M}{\partial \epsilon} (\epsilon - eV) \text{Re} h_{\phi\phi}^R(\epsilon). \] (3.30)

So far our results are quite general. They describe the effect of the low frequency \(\omega < \Delta\) collective mode on the tunneling conductance. In the following sections we evaluate the tunneling conductance Eq. (3.30) for our particular setup. To do this we need the fluctuation propagators \(D\), which is evaluated in the following subsection.

**B. Propagators for low energy excitations.**

Throughout this paper we are using the small parameter \(1/G_m\). When the fluctuation propagators \(D_{\phi\phi}\) is calculated, this parameter allows us to restrict ourselves by ladder diagrams, i.e., to use RPA. These diagrams can be summed up by means of solving the Dyson-type equation

\[ D = D_0 + D_0 \Pi D. \] (3.31)

In this equation the propagators \(D\) and the polarization operator \(\Pi\) are \(8 \times 8\) matrices defined in the \(K \otimes MS \otimes \varphi\theta\) space. The \(2 \times 2\) space \(\varphi\theta\) describes the two fluctuation fields we have in the problem. Since these fluctuations
take place in normal metal as well as in the superconductor, another $2 \times 2$ space $MS$ is needed.

In the metal sector only the $\varphi \varphi$ polarization operator is present, which yields the usual diffusion pole

$$\left( \Pi^{\varphi \varphi}_{MM} \right)^{R} = \frac{u_{m} D_{m} Q^{2}}{D_{m} Q^{2} - i \omega}, \quad (3.32)$$

where $D_{m}$ is the diffusion constant in the normal metal.

In the superconductor there exist the phase as well as the Coulomb fluctuations. Corresponding polarization operators can be obtained from the general relation (omitting the SS superscript)

$$\frac{1}{\nu_{s}} \Pi_{\mu \nu}^{ik} = \pi \delta_{\varphi k}^{\nu} \delta_{\varphi k}^{\mu} \text{Tr} \left[ \Gamma_{i}^{\nu} g_{S} \right] + \delta_{i \varphi} \delta_{k \varphi}, \quad (3.33)$$

where the indices $\mu, \nu$ can be $+ \text{ or } -$. The polarization operator $\Pi_{\mu \nu}^{ik}$ is the matrix in Keldysh space:

$$\Pi_{ik} = \begin{pmatrix} \Pi_{ik}^{+} & \Pi_{ik}^{-} \\ 0 & \Pi_{ik}^{+} \end{pmatrix}_{K} \quad (3.34)$$

Indices $i, k$ (which can be $\varphi$ or $\theta$) label the fluctuation parameter space. The second term in Eq. (3.33) is the “anomalous” contribution of the large $\epsilon$ region which is not captured by the Usadel equation (for details see, for example, Ref. 5). The vertex $\Gamma_{i}$ is a vector in the $\varphi \theta$ space (as indicated by index $i$) and a matrix in $K \otimes N$ space given by

$$\Gamma_{\varphi}^{\nu} = i \Delta (\tau^{\nu})_{K} \otimes \begin{pmatrix} 0 & e^{2i\theta_{K}} \\ e^{-2i\theta_{K}} & 0 \end{pmatrix}_{N}, \quad (3.35a)$$

$$\Gamma_{\varphi}^{\nu} = i (\tau^{\nu})_{K} \otimes 1_{N}, \quad (3.35b)$$

where $(\tau^{+})_{K} = 1_{K}$ and $(\tau^{-})_{K} = (\sigma_{2})_{K}$.

To proceed with this program and evaluate the polarization operators Eq. (3.33) we need an explicit form of the Green’s function $g_{S}$. The approximate solution Eq. (3.23) which was obtained by the gauge transformation is not sufficient here, since the polarization operator is a gauge invariant quantity. For this reason substitution of Eq. (3.23) into Eq. (3.33) immediately gives zero. Therefore we have to take into account the gradient terms, which were neglected in Eq. (3.23). To do that we expand the Usadel equation Eq. (3.18) to the first order in fluctuation fields

$$D_{s} Q^{2} g_{S0}(\epsilon_{1}) g_{S1}(\epsilon_{1}, \epsilon_{2}) - i \left[ H_{0}, g_{S1} \right] = i [\delta H, g_{S0}], \quad (3.36)$$

where $H_{0} = 1_{K} \otimes \hat{H}_{0}, \delta H = \Phi + 2i \Delta \Theta + \delta \Delta$. The last term is irrelevant, since the fluctuations of the absolute value of the order parameter are gapped. Nambu space matrix $H_{0}$ is given by

$$\hat{H}_{0} = \begin{pmatrix} \epsilon & \Delta \\ -\Delta & -\epsilon \end{pmatrix}_{N}. \quad (3.37)$$

The gapless phase fluctuations have the following matrix structure (see also Appendix B)

$$\Theta = \begin{pmatrix} \theta_{+} & \delta_{+} \\ \theta_{-} & \delta_{-} \end{pmatrix}_{K} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{N}. \quad (3.38)$$

The solution of the first order equation Eq. (3.36) is the following Keldysh matrix (omitting the subscript $S$ herefrom)

$$g_{1}(\epsilon_{1}, \epsilon_{2}) = \begin{pmatrix} \hat{g}^{R}_{1} & \hat{g}^{K}_{1} \\ \hat{g}^{K}_{1} & \hat{g}^{A}_{1} \end{pmatrix}_{K}. \quad (3.39)$$

The anomalous function $\hat{g}^{X}$ is only present before averaging over the fluctuations.

$$\hat{g}^{X}_{1}(\epsilon_{1}, \epsilon_{2}) = \frac{i}{B_{++}} \left[ g_{0}^{A}(\epsilon_{1}) \delta \hat{H}_{-}(\epsilon_{1} - \epsilon_{2}) \hat{g}_{0}^{R}(\epsilon_{2}) - \delta \hat{H}_{-}(\epsilon_{1} - \epsilon_{2}) \right], \quad (3.40)$$

where

$$B_{++} = D_{s} Q^{2} - i S_{-}(\epsilon_{1}) - i S_{+}(\epsilon_{2}), \quad (3.41a)$$

$$S_{\pm}(\omega) = \sqrt{\epsilon \pm i0}^{2} - \Delta^{2}. \quad (3.41b)$$

The rest of the functions are given by

$$\hat{g}_{1}^{R} = \frac{i}{B_{++}} \left[ \hat{g}_{0}^{A} \delta \hat{H}_{+} \hat{g}_{0}^{R} - \delta \hat{H}_{+} + \hat{g}_{0}^{K} \delta \hat{H}_{-} \hat{g}_{0}^{R} + \hat{g}_{0}^{A} \delta \hat{H}_{+} \hat{g}_{0}^{A} \right] + \hat{g}^{X}_{1} \operatorname{tanh} \frac{\epsilon_{1}}{2T} (S_{+}(\epsilon_{1}) - S_{-}(\epsilon_{1})), \quad (3.42)$$

$$\hat{g}_{1}^{A} = \frac{i}{B_{++}} \left[ \hat{g}_{0}^{A} \delta \hat{H}_{+} \hat{g}_{0}^{A} - \delta \hat{H}_{+} + \hat{g}_{0}^{A} \delta \hat{H}_{-} \hat{g}_{0}^{K} + \hat{g}_{0}^{A} \delta \hat{H}_{+} \hat{g}_{0}^{K} \right] + \hat{g}^{X}_{1} \operatorname{tanh} \frac{\epsilon_{2}}{2T} (S_{+}(\epsilon_{2}) - S_{-}(\epsilon_{2})), \quad (3.43)$$

and

$$\hat{g}_{1}^{K} = \frac{i}{B_{++}} \left[ \hat{g}_{0}^{R} \delta \hat{H}_{+} \hat{g}_{0}^{K} + \hat{g}_{0}^{K} \delta \hat{H}_{-} \hat{g}_{0}^{R} + \hat{g}_{0}^{R} \delta \hat{H}_{+} \hat{g}_{0}^{A} + \hat{g}_{0}^{K} \delta \hat{H}_{-} \hat{g}_{0}^{A} \right] + \hat{g}^{A} \operatorname{tanh} \frac{\epsilon_{1}}{2T} (S_{+}(\epsilon_{1}) - S_{-}(\epsilon_{1}))$$

$$\quad + \hat{g}^{K} \operatorname{tanh} \frac{\epsilon_{2}}{2T} (S_{+}(\epsilon_{2}) - S_{-}(\epsilon_{2})), \quad (3.44)$$

in the obvious notation (compare with Eqs. (3.40) and (3.41)).
Using the first order solution we can calculate the polarization operators Eq. (3.33) for the phase and the Coulomb fluctuations in the superconductor. For the retarded component of the polarization operator we obtain

\[(\Pi^{SS})_R = \nu_S \left( \frac{1}{\nu \omega^2 - \pi \Delta D \phi Q^2} \right) \phi \theta. \quad (3.45)\]

The solution of Dyson equation Eq. (3.34) is

\[D = [D^{-1} - \Pi]^{-1}, \quad (3.46)\]

where II is the polarization operator of the form

\[\Pi = \left( \begin{array}{cc} \Pi^{SS} & 0 \\ 0 & \Pi^{MM} \end{array} \right)_{MS}. \quad (3.47)\]

The unperturbed propagator \(D_0\) has non-zero elements only in the Coulomb interaction channel

\[D_0 = -V(Q) \left( \begin{array}{c} 1 \\ e^{-Qd} \\ e^{-Qd} \end{array} \right)_{SM} \otimes 1_K \otimes \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right)_{\phi \theta}, \quad (3.48)\]

where the Coulomb interaction in two dimensions \(V(Q) = 2\pi e^2/Q\) and \(d\) is the thickness of the sample. This form of the unperturbed propagator model depends. It corresponds to the particular setup of the S-N sandwich, which we are discussing in this paper.

To obtain the retarded propagators we have to invert the retarded block of the 8 \(\times\) 8 matrix. Since we need only \(\phi \phi\) propagator in Eq. (3.26), there are three elements of the remaining 4 \(\times\) 4 matrix which are relevant:

\[D_0^{\phi \phi} = \frac{1}{M} \left[ \left( \frac{1}{V} + \Pi_{\phi \phi} \right) \Pi_{\theta \theta} - \Pi_{\phi \theta} \Pi_{\theta \phi} \right], \quad (3.49a)\]

\[D_0^{\phi \theta} = -\frac{1}{M} \left[ \frac{1 - Qd}{V} \Pi_{\phi \phi} \right], \quad (3.49b)\]

\[D_0^{\theta \theta} = \frac{1}{M} \left[ \left( \frac{1}{V} + \Pi_{\phi \phi} \right) \left( \frac{1}{V} + \Pi_{MM} \right) - \frac{(1 - Qd)^2}{V^2} \right], \quad (3.49c)\]

where

\[M = \left( \frac{1}{V} + \Pi_{MM} \right) \left[ \Pi_{\phi \theta} \Pi_{\phi \phi} - \Pi_{\phi \phi} \Pi_{\theta \phi} \right], \quad (3.50a)\]

\[\Pi_{\phi \phi} = \Pi_{\phi \phi} + \frac{\Pi}{1 + VI_{\Pi M}}, \quad \tilde{V} = 4\pi e^2d. \quad (3.50b)\]

Equations (3.49) are written for the case \(Qd \ll 1\).

Substituting the polarization operators Eqs. (3.45) and (3.32) into Eqs. (3.49) we obtain

\[D_0^{\phi \phi} = -\frac{\nu^2}{M} \left[ \frac{\omega^2}{V\nu_s - \pi \Delta D \phi Q^2} \left( 1 + \frac{1}{V\nu_s} \right) \right], \quad (3.51a)\]

\[D_0^{\phi \theta} = -\frac{1}{M} \frac{i\nu_s}{V}, \quad (3.51b)\]

\[D_0^{\theta \theta} = -\frac{1}{M} \frac{\zeta D_m Q^2 - i\nu_s}{\nu_s} - i\omega + D_m Q^2, \quad (3.51c)\]

where \(\zeta = \nu_m + \nu_s (1 + \nu_m \tilde{V})\) and Eq. (3.50a) takes the form

\[M = -\pi \Delta \nu_s D_0 \zeta \left( \frac{D_m Q^2}{V} - \frac{i\nu_s}{\nu_s} \right) P(Q^2), \quad (3.52a)\]

\[P(Q^2) = Q^2 - \frac{i\omega \nu_s}{\zeta D_m} - \frac{\omega^2 \nu_m}{\pi \Delta \zeta D_s}. \quad (3.52b)\]

The pole \(P(Q^2)\) in the fluctuation propagators means that there is a well-defined collective mode with linear dispersion relation. This mode is the phason, discussed in Sec II A. The imaginary term in Eq. (3.52b) determines the finite lifetime of the phason. However, this term is small, since \(D_m/D_s \gg 1\), and the phason lifetime is large by this parameter.

We now combine these propagators into a single retarded propagator of the field \(\phi\) using the definition of \(\phi\)

\[\langle \phi \phi \rangle = \langle K \phi \rangle [K \phi] - 2 \langle K \phi \phi \rangle + \langle \theta \theta \rangle. \quad (3.53)\]

Substituting the explicit form Eq. (3.14) of the linear functional of the Coulomb fluctuations in the normal metal and the propagators Eq. (3.51) we find

\[D_0^{\phi \phi}(\omega; Q) = \frac{\tilde{V}}{\nu_s \pi \Delta D \phi Q^2} \left[ \frac{\zeta}{V} + \frac{\nu_s}{V} \frac{i\omega}{-i\omega + D_m Q^2} \right] P(Q^2) + \frac{\nu_s^2}{D_m} \left( 1 - \frac{1}{\nu_s} \right) \frac{\pi \Delta D_s}{\zeta} \left( \frac{1}{V} + \frac{1}{\nu_s} \right) \frac{D_m}{\nu_s} \left( -i\omega + D_m Q^2 \right). \quad (3.54)\]

Note, that although each of the propagators Eq. (3.51) contains the \(1/Q^2\) factor, the combined propagator Eq. (3.54) does not. This is a manifestation of the gauge invariance: the zero mode here corresponds to a uniform perturbation with the constant scalar potential, which can be gauged away and can not affect the physics.

Finally, to obtain the time-dependent propagator, which is needed to calculate the average Eq. (3.26) we integrate Eq. (3.54) over the momentum. The last two terms in brackets do not contribute by their analytic properties, while the first term gives for the retarded propagator

\[D_0^{\phi \phi}(\omega) = \frac{i\nu_s \omega}{2\Delta G_s}. \quad (3.55)\]

The corresponding Keldysh propagator is

\[D_0^{\phi \phi}(\omega) = \frac{i\nu_s \omega}{2\Delta G_s} N(\omega), \quad (3.56)\]
where the dimensionless conductance of the superconductor $G_s = 2\pi\nu_s D_s$. The distribution function $N(\omega)$ can be obtained from Eq. (3.15) using the particular form of the metallic distribution function $f_M(\epsilon)$ corresponding to the case when the system is driven out of equilibrium by the applied voltage $U$

$$f_M(\epsilon) = \frac{1}{2} \left[ \eta \left( \epsilon + \frac{eU}{2} \right) + \eta \left( \epsilon - \frac{eU}{2} \right) \right]. \quad (3.57)$$

This results in the bosonic distribution function

$$\omega N(\omega) = \left\{ \begin{array}{ll}
\frac{1}{2} |\omega|, & \text{if } |\omega| > eU \\
\frac{1}{2} (eU + |\omega|), & \text{if } |\omega| < eU
\end{array} \right.,$$

$$\quad (3.58)$$

C. Results for tunneling conductance.

Using the propagators Eqs. (3.53) and (3.56) we now calculate the average Eq. (3.26). First, let us separate the equilibrium ($V = 0$) and non-equilibrium contributions

$$D^{+\pm}_{\phi\phi}(t) - D^{+\pm}_{\phi\phi}(0) = D_{ne} + D_{eq}. \quad (3.59)$$

The non-equilibrium part comes entirely from the Keldysh propagator Eq. (3.56):

$$D_{ne}(t) = \frac{1}{2} \left[ D^K_{\phi\phi}(t) - D^K_{\phi\phi}(0) \right] eU$$

$$- \frac{1}{2} \left[ D^K_{\phi\phi}(t) - D^K_{\phi\phi}(0) \right] eU = 0. \quad (3.60)$$

Taking the Fourier transform we get

$$D_{ne}(t) = \frac{i eU}{2\pi G_s} \left[ S_1(eUt) + \frac{\sin eUt}{eUt} - 1 \right], \quad (3.61)$$

where $S_1(z) = \ln (z + \gamma - C_i(z))$, $C_i(z)$ is the integral cosine function and $\gamma = 0.577\ldots$ is the Euler constant.

The remaining equilibrium part is determined by the Fourier transform of

$$D_{eq}(t) = \frac{1}{2} \left[ D^K_{\phi\phi}(t) - D^K_{\phi\phi}(0) \right] eU + \frac{1}{2} D^K_{\phi\phi}(t), \quad (3.62)$$

which is given by the integral

$$D_{eq}(t) = \frac{i}{\Delta G_s} \int_0^\infty \frac{d\omega}{2\pi} \left( 1 - e^{-i\omega t} \right)$$

$$\approx \frac{i}{2\pi G_s} - \frac{1}{G_s \Delta t}; \quad t \gg \frac{1}{\Delta}. \quad (3.63)$$

Here we have used the fact that the advanced propagator vanishes at $t > 0$.

It is easy to see that when substituted into Eq. (3.26), only the non-equilibrium propagator Eq. (3.61) gives a noticeable deviation from the non-interacting (Golden-rule-type) behaviour. Consider Eq. (3.26) in the absence of interactions. Then it is the BSC Green’s function given by Eq. (C1), which at frequencies near the gap diverges as $\sim 1/\sqrt{\Delta - |\epsilon|}$. The equilibrium correction Eq. (3.64) at long times behaves like $1/(\Delta t)$ (since the upper integration limit in Eq. (3.63) is cut off by the gap). Expanding the exponential will only give rise to extra factors of $|\Delta - \epsilon|$ in the numerator, thus being small. Therefore we can neglect the equilibrium correction.

The non-equilibrium contribution in Eq. (3.26) is thus dominant and the average Green’s function Eq. (3.24) at $\epsilon > 0$ is

$$\text{Re} \tilde{D}^R_S = \sqrt{\frac{\Delta}{|\Delta - \epsilon|}} F \left( \frac{eU}{\epsilon - \Delta}; \frac{eU}{\pi \Delta G_s} \right), \quad (3.64)$$

where $F$ is a function of two dimensionless variables $y = eU/(\epsilon - \Delta)$ and $z = eU/\pi \Delta G_s$. It is given by

$$F(y; z) = \text{Re} \int_0^\infty \frac{dx}{\sqrt{4\pi x}} e^{i\pi x y} \frac{1}{|y|^z} \left[ S_1(|y|\pi z) + \frac{\sin y \pi z}{y \pi z} - 1 \right]. \quad (3.65)$$

The tunneling conductance Eq. (3.30) depends on the real part of the Green’s function Eq. (3.64). The external voltage in the experimental setting is always less than the superconducting gap, so that $z \ll 1$. To probe the vicinity of the gap we need to look at $\epsilon \sim \Delta$ corresponding to the limit $y \gg 1$. In that limit Eq. (3.65) gives

$$F(y; z) = |y|^{-z} \left\{ \begin{array}{ll}
\frac{3}{16}, & \text{if } y < 0 \\
\frac{3}{16}, & \text{if } y > 0, \quad |y| \gg 1.
\end{array} \right. \quad (3.66)$$

That describes the behaviour of the density of states in the superconductor in the vicinity of the gap edge, in particular the appearance of non-zero density of states inside the gap ($y < 0$ result).

For completeness we include the solution in the opposite limit $y \ll 1$ which describes the tail of the density of states as we go deeper into the gap. When $\epsilon > \Delta$ we can expand the exponential to obtain

$$F(y; z) = 1 + \frac{3}{16} |z| y^2, \quad y > 0; \quad y \ll 1. \quad (3.67)$$

However, deep in the gap the perturbation theory gives zero (due to energy conservation: no single particle process is possible at $\epsilon < \Delta - eU$). Therefore, we have to evaluate the full integral. This yields the non-perturbative result, which describes the multi-phason processes

$$F(y; z) \approx \frac{1}{\sqrt{2}} \left( \frac{|y| eU}{2} \right)^{1/4} \left( \ln 2 \frac{|z| |y|}{|y|} \right)^{3/4}, \quad y < 0, \quad |y| \ll 1. \quad (3.68)$$

Here $e = 2.718\ldots$

Finally, we express the result for the tunneling conductance Eq. (3.30) in terms of the function $F$
\[ G_T = \frac{1}{2} G_0 \left| \frac{\Delta}{\Delta - |\epsilon|} \right|^\frac{1}{2} \sum_{\epsilon = eV \pm eU/2} F \left( \frac{eU}{|\epsilon| - \Delta} \cdot \frac{eU}{\pi |\Delta - \epsilon|} \right) , \]

(3.69)

where \( F \) is given by Eq. (3.67) and its asymptotic behavior is described by Eqs. (3.66)-(3.68).

IV. CONCLUSIONS

In this paper we have studied the effect of electron-electron interactions on the tunneling conductance of N-S junction with the metallic layer driven out of equilibrium. Contrary to the common belief, the tunneling current can not be presented as a convolution of bare BCS density of states and the derivative of the non-equilibrium distribution function in metal. We have demonstrated, that the interaction between the tunneling electrons and those in the normal metal drastically affect this current modifying the superconducting density of states. This modification is manifested in particular in appearance of finite density of states at energies even smaller than \( \Delta \), see Sec. III C. This effect can complicate the clear determination of the distribution function by tunneling experiments similar to Ref. 1.

The reason for the strong effect of the non-equilibrium state is due to the appearance of low-energy collective mode on the N-S junction, analogous to the Schmidt-Schön mode or the second sound, which we called “phason”, Sec. II A. The tunneling can be accompanied by emission or absorption of phasons. The shot noise in the normal layer generates these phasons, thus, affecting the tunneling density of states.

The particular form of the phason spectrum Eq. (2.7) and consequently the result for the tunneling conductance Eq. (3.69) depend on the geometry of the junction. In this paper we considered the simplest model of the junction, namely the 2D sandwich. Calculations of the effect of the phason assisted tunneling on various experimental realizations of the N-S junction are outside of the scope of this paper. Therefore our results can not be expected to be described in the experiment.1 However, we have demonstrated an existence of a microscopic mechanism changing the tunneling conductance, which is different from the trivial broadening of the distribution function in the normal metal. To test which mechanism dominates in reality we have suggested the independent experiment, see Sec. III D.

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APPENDIX A:

Here we demonstrate the choice Eq. (3.14) of the functional \( K[\phi] \) in Eq. (3.13). We simply substitute the Green’s function Eq. (3.13) into the Usadel equation Eq. (3.8) and require that in equilibrium the correction \( g_{M1}(t_1, t_2) \) is proportional to the square of the gradient \( g_{M1} \sim (\nabla K[\phi])^2 \) and out of equilibrium to the variation of the distribution function \( g_{M1} \sim \Delta K[\phi](f_M(\epsilon) - f_{M, eq}(\epsilon)) \).

To proceed with this program we start with the transformation

\[ g_M(t_1, t_2) = e^{iK[\phi](t_1)t_2}h(t_1, t_2)e^{-iK[\phi](t_2)t_1} . \]  

(A1)

After the substitution into the Usadel equation Eq. (3.8) we find the equation for the function \( h(t_1, t_2) \)

\[ -D_m\nabla^2_R(h) + \tau^3 \frac{\partial h}{\partial t_1} + \frac{\partial h}{\partial t_2} e^{3} + i[\frac{\partial K}{\partial t} - \Phi, h]_t = 0 , \]

(A2)

where the covariant derivative now includes the commutator of the gradient of \( K[\phi] \) and the function \( h \)

\[ \nabla^R_k h = \nabla_R h + i[\nabla_R K[\phi], h] . \]  

(A3)

Here we assumed that the functional \( K[\phi] \) (which is a matrix in the Keldysh space) commutes with the scalar potential matrix \( \Phi \) which we will verify later using the explicit form of \( K[\phi] \).

We now treat Eq. (A2) in perturbation theory \( h = g_{M0} + g_{M1} \), where \( g_{M0} \) is the free Green’s function determined by Eq. (3.13). The equation becomes

\[ -D_m\int dt g_{M0}(t_1 - t_3)\left[ \nabla^2 K[\phi](t_3)\tau^3 g_{M0}(t_3 - t_2) - g_{M0}(t_3 - t_2)\tau^3 \nabla^2 K[\phi](t_2) \right] + \left( \frac{\partial K[\phi]}{\partial t_1} - \Phi(t_1) \right) g_{M0}(t_1 - t_2) \]

\[ - g_{M0}(t_1 - t_2) \left( \frac{\partial K[\phi]}{\partial t_2} - \Phi(t_2) \right) \]

\[ = \mathcal{F} \left[ (\nabla K)^2; g_{M1}; \nabla K \nabla g_{M1} \right] . \]  

(A4)

The functional \( \mathcal{F} \) in the right hand side of Eq. (A4) (which explicit form is not needed for what follows) contains higher powers of \( K[\phi] \). We now choose such a form
of $K[\varphi]$ that all linear terms (the left hand side of Eq. (A4)) cancel with the external potential $\Phi$. To do that we treat the left hand side of Eq. (A4) as the equation for $K[\varphi]$ (with zero in the right hand side).

To simplify the solution we notice that the non-interacting Green’s function has the matrix structure

$$g_{M0} = \hat{g}_{M0} \otimes \tau^3$$

(A5)

where $\hat{g}_{M0}$ is the $2 \times 2$ matrix in Keldysh space and $\tau^3$ acts in Nambu space. Therefore each term in the equation is proportional to $\tau^3$ which therefore can be omitted. Then, using the normalization $g_{M0} \circ g_{M0} = 1$ and performing the Fourier transform in space variable (only $K[\varphi]$ depends on $R$), we obtain the equation for the functional $K[\varphi]$

$$D_m Q^2 \left[ \int dt_3 \hat{g}_{M0}(t_1 - t_3) K[\varphi](t_3) \hat{g}_{M0}(t_3 - t_2) 
- \frac{\partial K[\varphi]}{\partial t_1} \delta(t_1 - t_2) \right] + \left( \frac{\partial K[\varphi]}{\partial t_2} - \hat{\varphi}(t_1) \right) \hat{g}_{M0}(t_1 - t_2) 
- \hat{g}_{M0}(t_1 - t_2) \left( \frac{\partial K[\varphi]}{\partial t_2} - \hat{\varphi}(t_2) \right) = 0, \quad (A6)$$

where $\hat{\varphi}$ is the matrix

$$\hat{\varphi} = \begin{pmatrix} \varphi_+ & \varphi_- \\ \varphi_- & \varphi_+ \end{pmatrix}_K,$$

(A7)

and the functional $K[\varphi]$ has the same structure

$$K[\varphi] = \begin{pmatrix} K_+ & K_- \\ K_- & K_+ \end{pmatrix}_K.$$

(A8)

The Green’s function $\hat{g}_{M0}$ is given by

$$\hat{g}_{M0}(t_1 - t_2) = \begin{pmatrix} \delta(t_1 - t_2) & 2f(t_1 - t_2) \\ 0 & -\delta(t_1 - t_2) \end{pmatrix}_K,$$

(A9)

where the Keldysh component is related to the distribution function (see Eq. (3.17)).

We substitute Eqs. (A7) - (A9) into the matrix equation Eq. (A6) and investigate each component separately. The diagonal elements (“11” and “22”) of Eq. (A4) differ only in sign and give the equation for $K_-$

$$D_m Q^2 K_-(t) - \left( \frac{\partial K_-}{\partial t} - \varphi_-(t) \right) = 0, \quad (A10)$$

which gives the solution in frequency domain (see Eq. (3.14))

$$K_- = \frac{-\varphi_-}{i\omega + D_m Q^2}.$$  \(A11\)

This solution also satisfies the lower non-diagonal (“21”) element of Eq. (A4). The upper non-diagonal (Keldysh) element gives the equation for $K_+$ which we write in Fourier space

$$(D_m Q^2 - i\omega) K_+(\omega)[f(\epsilon) - f(\epsilon - \omega)]
+ 2D_m Q^2 K_-(\omega)\varphi_-\left[f(\epsilon)f(\epsilon - \omega) - 1\right] = \varphi_+\left[f(\epsilon) - f(\epsilon - \omega)\right].$$ (A12)

The last trick is to introduce the bosonic “distribution function” $N(\omega)$ Eq. (3.15). In the second term of Eq. (A.13) we rewrite

$$f(\epsilon)f(\epsilon - \omega) - 1 = -N(\omega)[f(\epsilon) - f(\epsilon - \omega)]
+ \left[N(\omega)[f(\epsilon) - f(\epsilon - \omega)] + f(\epsilon)f(\epsilon - \omega) - 1\right].$$ (A13)

In equilibrium the expression in square brackets is equal to zero, which is just a reflection of the detailed balance principle. Then, using the solution Eq. (A11) for $K_-$, we obtain the solution

$$K_+ = \frac{-\varphi_+}{-i\omega + D_m Q^2} - 2\varphi_- Re\frac{N(\omega)}{i\omega + D_m Q^2}.$$ (A14)

which was previously given in the matrix form in Eq. (3.14).

Out of equilibrium the solution Eq. (A14) is not exact, since the detailed balance principle is no longer valid and the square bracket in Eq. (A13) gives rise to corrections, which are proportional to the difference between equilibrium and non-equilibrium distribution functions. These corrections should then be included in the next order Green’s function $g_{M1}$.

The calculation of the correction $g_{M1}$ introduces the Altshuler-Aroonov corrections to the conductivity as well as the energy relaxation. We will not dwell on this issue in the present paper.

APPENDIX B:

Here we show that the gauge transformation Eq. (3.21) captures the dominant effect of phase and Coulomb fluctuations in the superconductor. Since we integrate the fluctuation propagators over momentum before using them to calculate the tunneling conductance, we can here set the momentum to zero from the very beginning taking the integrated Keldysh propagator Eq. (3.56) as known. We only need the Keldysh propagator since we are interested in non-equilibrium effects (see the paragraph following Eq. (3.63)). In the straightforward perturbation theory described in Sec. III the terms describing the phase fluctuations contained the large factor of $\Delta$. These contributions can be taken into account by means of the gauge transformation (similar to Eq. (3.21))
\[ \mathbf{g}_S(t_1, t_2) = e^{i\theta_K(t_1)\tau^3} \mathbf{g}(t_1, t_2)e^{-i\theta_K(t_2)\tau^3}, \]  
(B1)

where now the function \( \mathbf{g}(t_1, t_2) \) is not the BCS Green’s function but is to be determined from the Usadel equation, which after the transformation becomes

\[- D_s \nabla^c(g \circ \nabla^c g) + [\mathbf{H}_0, \mathbf{g}]_\epsilon + [\delta \mathbf{H}, \mathbf{g}]_\epsilon = 0, \]  
(B2)

where

\[ [\delta \mathbf{H}, \mathbf{g}]_\epsilon = \int \frac{d\epsilon_3}{2\pi} \left[ \delta \mathbf{H}(\epsilon_1 - \epsilon_3) \mathbf{g}(\epsilon_3, \epsilon_2) - \mathbf{g}(\epsilon_1, \epsilon_3) \delta \mathbf{H}(\epsilon_3 - \epsilon_2) \right], \]  
(B3)

the covariant derivative after the gauge transformation includes a commutator with the gradient of \( \theta \)

\[ \nabla^c \mathbf{g} = \nabla \mathbf{g} + i[\nabla \theta_K \tau^3, \mathbf{g}], \]  
(B4)

and \( \delta \mathbf{H} = \Phi + \Theta \).

In addition to the equation Eq. (B2) the function \( \mathbf{g}(t_1, t_2) \) has to satisfy the constraint

\[ g \circ g = \mathbf{1} \delta(t_1 - t_2), \]  
(B5)

which fixes the normalization.

Now instead of \( \Delta \) the phase fluctuation term has a factor of \( \omega \). In what follows we show that all the corrections calculated by perturbation theory will be small at least as a power of \( \omega / \Delta \) and thus the gauge transformation indeed captures the dominant contribution. Since we already have the factor of \( \omega \) in the numerator, the only possibility to obtain a strong correction is to excite a soft mode. Therefore to simplify the discussion we first separate the gapped modes which cannot give rise to strong corrections (since we are interested in energies much smaller than \( \Delta \)) and then treat the remaining soft modes in perturbation theory.

The separation of the modes can already be seen in the absence of fluctuations. The free (BCS) solution is given by Eq. (C3) and can be written in the following way

\[ \mathbf{g}_S(\epsilon) = \begin{pmatrix} \epsilon \sigma_3 + \Delta \sigma_2 \\ \frac{\epsilon \sigma_3 + \Delta \sigma_2}{\sqrt{(\epsilon \pm i0)^2 - \Delta^2}} \end{pmatrix} \approx \frac{\Delta}{\epsilon \pm i0 - \Delta} \left[ (\sigma_3 + \sigma_2) + \frac{\epsilon - \Delta}{\Delta} (\sigma_3 - \sigma_2) \right], \]  
(B6)

where only in this section we denote by \( \sigma_2 \) and \( \sigma_3 \) the following matrices in the Nambu space

\[ \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  
(B7)

Here the large and small parts of the solution (in the limit \( \epsilon - \Delta \ll \Delta \)) are turned out to have different matrix structure.

To see this structure in the equation we first explicitly separate the fast oscillating part of the Green’s functions writing the solution as

\[ \mathbf{g}(t_1, t_2) = e^{i\Delta(t_1 - t_2)} \mathbf{g}_1(t_1, t_2). \]  
(B8)

Here we are focusing on positive energies.

Now the functions \( \mathbf{g}_1 \) varies slowly in time (since the external potential \( \delta \mathbf{H} \) is a slow function) and we get the equation

\[- D_s \nabla^c(\mathbf{g}_1 \circ \nabla^c \mathbf{g}_1) + [\mathbf{H}, \mathbf{g}_1]_\epsilon + [\delta \mathbf{H}, \mathbf{g}_1]_\epsilon = 0, \]  
(B9)

where the Hamiltonian \( \mathbf{H} \) has an additional term

\[ \mathbf{H} = \mathbf{H}_0 + i\Delta \mathbf{1}_K \otimes \sigma_3. \]  
(B10)

We now see that the large term in the BCS Green’s function Eq. (B6) commutes with the Hamiltonian Eq. (B10) (neglecting the \( \epsilon \) term for the moment), while the small term anticommutates. In the same manner we will look for a solution of Eq. (B9) which contains matrices commuting and anticommuting with \( \mathbf{H} \)

\[ \mathbf{g}_1 = \mathbf{\hat{G}}_1 \otimes \mathbf{1}_N + \mathbf{\hat{G}}_2 \otimes (\sigma_3 + \sigma_2) \]  
(B11)

\[ + i\mathbf{\hat{H}}_1 \otimes \sigma_1 + \mathbf{\hat{H}}_2 \otimes (\sigma_3 - \sigma_2), \]  
(B11)

where hat denotes matrices in Keldysh space, while the \( \sigma_j \) act in Nambu space. The anticommuting terms \( \mathbf{\hat{H}}_j \) are small and the commuting terms \( \mathbf{\hat{G}}_j \) are large. In the absence of fluctuations \( \mathbf{\hat{G}}_1 = \mathbf{\hat{H}}_1 = 0 \) and \( \mathbf{\hat{G}}_2 \) and \( \mathbf{\hat{H}}_2 \) form the BCS Green’s function Eq. (B6).

When we substitute the solution Eq. (B11) into the transformed Usadel equation Eq. (B8), in the first order we only use the non-commuting terms \( \mathbf{\hat{H}}_j \) in the commutator with the Hamiltonian, neglecting the smaller contribution from the other terms in the equation. As a result we obtain

\[ (\epsilon_1 - \epsilon_2) [\mathbf{\hat{G}}_1 \otimes \sigma_3 + \mathbf{\hat{G}}_2 \otimes \mathbf{1}_N] + (\epsilon_1 + \epsilon_2) \mathbf{\hat{G}}_2 \otimes \sigma_1 \]  
\[ + [\delta \mathbf{H}, \mathbf{\hat{G}}_1]_\epsilon \otimes \mathbf{1}_N + [\delta \mathbf{H}, \mathbf{\hat{G}}_2]_\epsilon \otimes (\sigma_3 + \sigma_2) \]  
\[ + 2\Delta \mathbf{\hat{H}}_1 \otimes (\sigma_3 + \sigma_2) - 4\Delta \mathbf{\hat{H}}_2 \otimes \sigma_1 + \cdots = 0, \]  
(B12)

where \( \cdots \) denote gradient terms which we do not include here due to their complexity but will restore in the next section.

Collecting terms with the matrix structure corresponding to four matrices in Eq. (B11) we obtain the final equations for \( \mathbf{\hat{H}}_j \) and \( \mathbf{\hat{G}}_j \)
\((\epsilon_1 - \epsilon_2) \hat{G}_1 - D_s \nabla A_4 \)

\[- \frac{i}{2} D_s [\nabla \theta_K, (A_1 + A_2)] = 0, \quad \text{(B13a)}\]

\((\epsilon_1 - \epsilon_2) \hat{G}_2 + [\delta \mathbf{H}, \hat{G}_1]_\epsilon - D_s \nabla A_1 - i D_s [\nabla \theta_K, (A_4 + A_3)] + 2 D_s [\nabla \theta_K, \hat{G}_2 \circ \nabla \theta_K \hat{G}_2] = 0, \quad \text{(B13b)}\]

\[\hat{H}_1 = - \frac{1}{2 \Delta} [\delta \mathbf{H}, \hat{G}_2]_\epsilon + 2 i D_s \hat{G}_2 \circ \nabla^2 \theta_K \hat{G}_2, \quad \text{(B16d)}\]

\[H_{2}^{R, A} = \frac{1}{2} \epsilon \sqrt{\frac{\Delta}{(\epsilon + i 0)^2 - \Delta^2}}, \quad \text{(B16c)}\]

where \(\epsilon\) is counted from \(\Delta\) after the transformation Eq. (B8).

Thus we see that \(\delta \mathbf{H}\) does not couple soft modes to each other and so it can introduce only perturbative corrections of the order \((\delta \mathbf{H}/\Delta)^2\). Therefore in the leading order in \(1/\Delta\) the dominant effect of the phase fluctuations is indeed captured by the gauge transformation Eq. (B.21).

**APPENDIX C:**

Here we perform the averaging over the fluctuating field in Eq. (3.23). The BCS Green’s function \(g_0\) has the following structure in frequency domain

\[g_0^{R(A)}(\epsilon) = \frac{\hat{H}_0}{\sqrt{(\epsilon + i 0)^2 - \Delta^2}}, \quad \text{C1a}\]

\[g_0^K(\epsilon) = \tanh \frac{\epsilon}{2T} (g_0^R - g_0^A), \quad \text{C1b}\]

where the matrix in Nambu space is

\[\hat{H}_0 = \begin{pmatrix} \epsilon & \Delta \\ -\Delta & -\epsilon \end{pmatrix}_{SC}. \quad \text{C2}\]

In time domain the matrix structure is the same and we can represent \(g_0\) as a sum of two parts

\[g_0 = h_0 \tau^3 + l_0 \tau^2. \quad \text{C3}\]

To calculate the tunneling current we only need the normal component of \(g_0\), therefore we can focus on averaging \(h_0\). Then the matrix \(\tau^3\) can be carried through, so that

\[\bar{h}_0 = \langle e^{i\phi(t_1)} h_0(t_1 - t_2) e^{-i\phi(t_2)} \rangle. \quad \text{C4}\]

Since only the retarded function enters the expression for the tunneling current Eq. (3.30), we now multiply the Keldysh space matrices to get \(\bar{h}_0\). The Green’s function has it’s usual (rotated) form

\[h_0 = \begin{pmatrix} h_0^R & h_0^K \\ h_0^{-} & h_0^{+} \end{pmatrix}_K, \quad \text{C5}\]

while the fluctuating field is

\[\phi = \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}_K. \quad \text{C6}\]
After matrix multiplication we get for the retarded function
\[
\tilde{h}_0^R(t_1 - t_2) = e^{i(\phi_+(t_1) - \phi_+(t_2))} \left[ h_0^R \cos \phi_-(t_1) \cos \phi_-(t_2) - h_0^K \cos \phi_-(t_1)i \sin \phi_-(t_2) + h_0^A \sin \phi_-(t_1) \sin \phi_-(t_2) \right]
\] (C7)

By definition a retarded Green’s function is only non-zero when its time argument is positive. In that region an advanced Green’s function is zero. Therefore we can simplify Eq. (C7) as
\[
\tilde{h}_0^R(t_1 > t_2) = \frac{1}{2} (e^{i(\phi_+(t_1) - \phi_+(t_2))} \cos \phi_-(t_1)
\]

\[
\left[ (h_0^R + h_0^K) e^{-i\phi_-(t_2)} + (h_0^R - h_0^K) e^{i\phi_-(t_2)} \right].
\] (C8)

We are now ready to perform the average:
\[
\langle e^{i(\phi_+(t_1) - \phi_+(t_2)) + i(\phi_-(t_1) - \phi_-(t_2))} \rangle = e^{(\phi_+ + \phi_+ - (\phi_+^2)(0) + (\phi_- + \phi_-) + (\phi_+ + \phi_-)}
\]
\[
(\phi_+(t_1) - \phi_+(t_2)) - i(\phi_-(t_1) + \phi_-(t_2))
\] (C9a)
\[
= e^{(\phi_+ + \phi_+ - (\phi_+^2)(0) - (\phi_- + \phi_-) + (\phi_+ + \phi_-)}
\] (C9b)

where all the correlators \(\langle \phi \phi \rangle\) depend on the difference \(t_1 - t_2\), except where indicated.

We now introduce propagators \(D\) of the fluctuating fields
\[
\langle \phi_+ \phi_+ \rangle = iD^K_{\phi \phi}(t_1 - t_2),
\] (C10a)
\[
\langle \phi_+ \phi_- \rangle = iD^R_{\phi \phi}(t_1 - t_2),
\] (C10b)
\[
\langle \phi_- \phi_+ \rangle = iD^A_{\phi \phi}(t_1 - t_2).
\] (C10c)

At \(t_1 > t_2\) the advanced propagator is zero. Therefore only sums and differences of the retarded and the Keldysh functions are present in Eq. (C8). These can be expressed in terms of functions of the original Keldysh basis as \(h_0^K + h_0^R = 2h_0^+(\tau^+)\) and similarly with the fluctuation propagators to obtain the final expression in time domain
\[
\tilde{h}_0^R(t_1 > t_2) = h_0^+(t_1 - t_2)e^{2i(D^+_{\phi \phi}(t_1 - t_2) - D^{+\phi}_{\phi \phi}(0))}
\]
\[
- h_0^-+(t_1 - t_2)e^{2i(D^-_{\phi \phi}(t_1 - t_2) - D^{-\phi}_{\phi \phi}(0))}.
\] (C11)

When Fourier transforming to the frequency domain, we notice that the first term in Eq. (C11) contributes at positive frequencies, while the second term contributes only at negative frequencies. Therefore
\[
\tilde{h}_0^R(\epsilon > 0) = \int_0^{\infty} dt e^{i\epsilon t} h_0^R(\epsilon > 0) e^{2i(D^+_{\phi \phi}(t) - D^{+\phi}_{\phi \phi}(0))}.
\] (C12)

For reference purposes we list here the explicit form of the BCS function \(h_0^+\) in the time domain
\[
h_0^+(t > 0) = \frac{\Delta}{\pi} K_1(i\Delta t),
\] (C13)
where \(K_1\) is the modified Bessel function.

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