Capacity-Equivocation Region of the Gaussian MIMO Wiretap Channel

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Abstract

We study the Gaussian multiple-input multiple-output (MIMO) wiretap channel, which consists of a transmitter, a legitimate user, and an eavesdropper. In this channel, the transmitter sends a common message to both the legitimate user and the eavesdropper. In addition to this common message, the legitimate user receives a private message, which is desired to be kept hidden as much as possible from the eavesdropper. We obtain the entire capacity-equivocation region of the Gaussian MIMO wiretap channel. This region contains all achievable common message, private message, and private message's equivocation (secrecy) rates. In particular, we show the sufficiency of jointly Gaussian auxiliary random variables and channel input to evaluate the existing single-letter description of the capacity-equivocation region due to Csiszar-Korner.

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1 Introduction

We consider the Gaussian multiple-input multiple-output (MIMO) wiretap channel, which consists of a transmitter, a legitimate user, and an eavesdropper. In this channel, the transmitter sends a common message to both the legitimate user and the eavesdropper in addition to a private message which is directed to only the legitimate user. There is a secrecy concern regarding this private message in the sense that the private message needs to be kept secret as much as possible from the eavesdropper. The secrecy of the private message is measured by its equivocation at the eavesdropper.

Here, we obtain the capacity-equivocation region of the Gaussian MIMO wiretap channel. This region contains all achievable rate triples \((R_0, R_1, R_e)\), where \(R_0\) denotes the common message rate, \(R_1\) denotes the private message rate, and \(R_e\) denotes the private message’s equivocation (secrecy) rate. In fact, this region is known in a single-letter form due to [1]. In this work, we show that jointly Gaussian auxiliary random variables and channel input are sufficient to evaluate this single-letter description for the capacity-equivocation region of the Gaussian MIMO wiretap channel. We prove the sufficiency of the jointly Gaussian auxiliary random variables and channel input by using channel enhancement [2] and an extremal inequality from [3]. In our proof, we also use the equivalence between the Gaussian MIMO wiretap channel and the Gaussian MIMO wiretap channel with public messages [4, Problem 33-c], [5]. In the latter channel model, the transmitter has three messages, a common, a confidential, and a public message. The common message is sent to both the legitimate user and the eavesdropper, while the confidential and public messages are directed to only the legitimate user. Here, the confidential message needs to be transmitted in perfect secrecy, whereas there is no secrecy constraint on the public message. Since the Gaussian MIMO wiretap channel and the Gaussian MIMO wiretap channel with public messages are equivalent, i.e., there is a one-to-one correspondence between the capacity regions of these two models, in our proof, we obtain the capacity region of the Gaussian MIMO wiretap channel with public messages, which, in turn, gives us the capacity-equivocation region of the Gaussian MIMO wiretap channel.

Our result subsumes the following previous findings about the capacity-equivocation region of the Gaussian MIMO wiretap channel: i) The secrecy capacity of this channel, i.e., \(\max R_1\) when \(R_0 = 0, R_e = R_1\), is obtained in [6,7] for the general case, and in [8] for the 2-2-1 case. ii) The common and confidential rate region under perfect secrecy, i.e., \((R_0, R_1)\) region with \(R_e = R_1\), is obtained in [9]. iii) The capacity-equivocation region without a common message, i.e., \((R_1, R_e)\) region with \(R_0 = 0\), is obtained in [5]. iv) The capacity region of the Gaussian MIMO broadcast channel with degraded message sets without a secrecy concern, i.e., \((R_0, R_1)\) region with no consideration on \(R_e\), is obtained in [10]. Here, we obtain the entire \((R_0, R_1, R_e)\) region.
2 Discrete Memoryless Wiretap Channels

The discrete memoryless wiretap channel consists of a transmitter, a legitimate user and an eavesdropper. The channel transition probability is denoted by \( p(y, z|x) \), where \( x \in \mathcal{X} \) is the channel input, \( y \in \mathcal{Y} \) is the legitimate user’s observation, and \( z \in \mathcal{Z} \) is the eavesdropper’s observation. We consider the following scenario for the discrete memoryless wiretap channel:

The transmitter sends a common message to both the legitimate user and the eavesdropper, and a private message to the legitimate user which is desired to be kept hidden as much as possible from the eavesdropper.

An \((n, 2^{nR_0}, 2^{nR_1})\) code for this channel consists of two message sets \( \mathcal{W}_0 = \{1, \ldots, 2^{nR_0}\} \), \( \mathcal{W}_1 = \{1, \ldots, 2^{nR_1}\} \), one encoder at the transmitter \( f : \mathcal{W}_0 \times \mathcal{W}_1 \to X^n \), one decoder at the legitimate user \( g_u : \mathcal{Y}^n \to \mathcal{W}_0 \times \mathcal{W}_1 \), and one decoder at the eavesdropper \( g_e : \mathcal{Z}^n \to \mathcal{W}_0 \).

The probability of error is defined as \( P_n = \max\{P_{n, u}, P_{n, e}\} \), where \( P_{n, u} = \Pr[g_u(Y^n) \neq (W_0, W_1)], P_{n, e} = \Pr[g_e(Z^n) \neq W_0] \), and \( W_j \) is a uniformly distributed random variable in \( \mathcal{W}_j \), \( j = 0, 1 \). The secrecy of the legitimate user’s private message is measured by its equivocation at the eavesdropper \( \mathbb{H} \), i.e.,

\[
\frac{1}{n} H(W_1|W_0, Z^n) \quad (1)
\]

A rate triple \((R_0, R_1, R_e)\) is said to be achievable if there exists an \((n, 2^{nR_0}, 2^{nR_1})\) code such that \( \lim_{n \to \infty} P_n = 0 \), and

\[
R_e = \lim_{n \to \infty} \frac{1}{n} H(W_1|W_0, Z^n) \quad (2)
\]

The capacity-equivocation region of the discrete memoryless wiretap channel is defined as the convex closure of all achievable rate triples \((R_0, R_1, R_e)\), and denoted by \( \mathcal{C} \). The capacity-equivocation region of the discrete memoryless wiretap channel, which is obtained in \( \mathbb{H} \), is stated in the following theorem.

**Theorem 1 (\[\mathbb{H}, \text{Theorem 1}\])** The capacity-equivocation region of the discrete memoryless wiretap channel \( \mathcal{C} \) is given by the union of rate triples \((R_0, R_1, R_e)\) satisfying

\[
0 \leq R_e \leq R_1 \quad (3)
\]

\[
R_e \leq I(V; Y|U) - I(V; Z|U) \quad (4)
\]

\[
R_0 + R_1 \leq I(V; Y|U) + \min\{I(U; Y), I(U; Z)\} \quad (5)
\]

\[
R_0 \leq \min\{I(U; Y), I(U; Z)\} \quad (6)
\]

for some \( U, V, X \) such that

\[
U \to V \to X \to (Y, Z) \quad (7)
\]
We next provide an alternative description for $C$. This alternative description will arise as the capacity region of a different, however related, communication scenario for the discrete memoryless wiretap channel. In this communication scenario, the transmitter has three messages, $W_0, W_p, W_s$, where $W_0$ is sent to both the legitimate user and the eavesdropper, and $W_s, W_p$ are sent only to the legitimate user. In this scenario, $W_s$ needs to be transmitted in perfect secrecy, i.e., it needs to satisfy

$$\lim_{n \to \infty} \frac{1}{n} I(W_s; Z^n, W_0) = 0$$

(8)

and there is no secrecy constraint on the public message $W_p$. To distinguish this communication scenario from the previous one, we call the channel model arising from this scenario the discrete memoryless wiretap channel with public messages. We note that this alternative description for wiretap channels has been previously considered in [4, Problem 33-c], [5].

An $(n, 2^{nR_0}, 2^{nR_p}, 2^{nR_s})$ code for this scenario consists of three message sets $W_0 = \{1, \ldots, 2^{nR_0}\}, W_p = \{1, \ldots, 2^{nR_p}\}, W_s = \{1, \ldots, 2^{nR_s}\}$, one encoder at the transmitter $f : W_0 \times W_p \times W_s \to X^n$, one decoder at the legitimate user $g_u : Y^n \to W_0 \times W_p \times W_s$, and one decoder at the eavesdropper $g_e : Z^n \to W_0$. The probability of error is defined as $P^n_e = \max\{P^n_{e,u}, P^n_{e,e}\}$, where $P^n_{e,u} = \Pr[g_u(Y^n) \neq (W_0, W_p, W_s)]$ and $P^n_{e,e} = \Pr[g_e(Z^n) \neq W_0]$. A rate triple $(R_0, R_p, R_s)$ is said to be achievable if there exists an $(n, 2^{nR_0}, 2^{nR_p}, 2^{nR_s})$ code such that $\lim_{n \to \infty} P^n_e = 0$ and (8) is satisfied. The capacity region $C_p$ of the discrete memoryless wiretap channel with public messages is defined as the convex closure of all achievable rate triples $(R_0, R_p, R_s)$. The following lemma establishes the equivalence between $C$ and $C_p$.

Lemma 1 $(R_0, R_p, R_s) \in C_p$ iff $(R_0, R_s + R_p, R_s) \in C$.

The proof of this lemma is given in Appendix A. Using Lemma 1 and Theorem 1 we can express $C_p$ as stated in the following theorem.

Theorem 2 The capacity region of the discrete memoryless wiretap channel with public messages $C_p$ is given by the union of rate triples $(R_0, R_p, R_s)$ satisfying

$$0 \leq R_s \leq I(V; Y|U) - I(V; Z|U)$$

(9)

$$R_0 + R_p + R_s \leq I(V; Y|U) + \min\{I(U; Y), I(U; Z)\}$$

(10)

$$R_0 \leq \min\{I(U; Y), I(U; Z)\}$$

(11)

for some $(U, V, X)$ such that

$$U \rightarrow V \rightarrow X \rightarrow (Y, Z)$$

(12)
3 Gaussian MIMO Wiretap Channel

The Gaussian MIMO wiretap channel is defined by

\[ Y = H_Y X + N_Y \tag{13} \]
\[ Z = H_Z X + N_Z \tag{14} \]

where the channel input \( X \) is a \( t \times 1 \) vector, \( Y \) is an \( r_Y \times 1 \) column vector denoting the legitimate user’s observation, \( Z \) is an \( r_Z \times 1 \) column vector denoting the eavesdropper’s observation, \( H_Y, H_Z \) are the channel gain matrices of sizes \( r_Y \times t, r_Z \times t \), respectively, and \( N_Y, N_Z \) are Gaussian random vectors with covariance matrices \( \Sigma_Y, \Sigma_Z \), respectively, which are assumed to be strictly positive-definite, i.e., \( \Sigma_Y \succ 0, \Sigma_Z \succ 0 \). We consider a covariance constraint on the channel input as follows

\[ \mathbb{E}[XX^\top] \preceq S \tag{15} \]

We consider a covariance constraint on the channel input as follows

The capacity-equivocation region of the Gaussian MIMO wiretap channel is denoted by \( C(S) \), and is stated in the following theorem.

**Theorem 3** The capacity-equivocation region of the Gaussian MIMO wiretap channel \( C(S) \) is given by the union of rate triples \( (R_0, R_1, R_e) \) satisfying

\[ 0 \leq R_e \leq \frac{1}{2} \log \frac{|H_Y KH_Y^\top + \Sigma_Y|}{|\Sigma_Y|} - \frac{1}{2} \log \frac{|H_Z KH_Z^\top + \Sigma_Z|}{|\Sigma_Z|} \tag{16} \]

\[ R_0 + R_1 \leq \frac{1}{2} \log \frac{|H_Y KH_Y^\top + \Sigma_Y|}{|\Sigma_Y|} \]
\[ + \min \left\{ \frac{1}{2} \log \frac{|H_Y SH_Y^\top + \Sigma_Y|}{|\Sigma_Y|}, \frac{1}{2} \log \frac{|H_Y KH_Y^\top + \Sigma_Z|}{|\Sigma_Z|} \right\} \]

\[ R_0 \leq \min \left\{ \frac{1}{2} \log \frac{|H_Y KH_Y^\top + \Sigma_Y|}{|\Sigma_Y|}, \frac{1}{2} \log \frac{|H_Z SH_Z^\top + \Sigma_Z|}{|\Sigma_Z|} \right\} \tag{17} \]

for some positive semi-definite matrix \( K \) such that \( 0 \preceq K \preceq S \).

Similar to what we did in the previous section, we can establish an alternative statement for Theorem 3 by considering the Gaussian MIMO wiretap channel with **public** messages, where the legitimate user’s private message is divided into two parts such that one part (confidential message) needs to be transmitted in perfect secrecy and there is no secrecy constraint on the other part (public message). The capacity region for this alternative scenario is denoted by \( C_p(S) \), and can be obtained by using Lemma 1 and Theorem 3 as stated in the next theorem.

\footnote{Without loss of generality, we can set \( \Sigma_Y = \Sigma_Z = I \). However, we let \( \Sigma_Y, \Sigma_Z \) be arbitrary for ease of presentation.}
Theorem 4 The capacity region of the Gaussian MIMO wiretap channel with public messages $C_p(S)$ is given by the union of rate triples $(R_0, R_p, R_s)$ satisfying

$$0 \leq R_s \leq \frac{1}{2} \log \frac{|H_YKH_Y^T + \Sigma_Y|}{|\Sigma_Y|} - \frac{1}{2} \log \frac{|H_ZKH_Z^T + \Sigma_Z|}{|\Sigma_Z|}$$

$$R_0 + R_p + R_s \leq \frac{1}{2} \log \frac{|H_YKH_Y^T + \Sigma_Y|}{|\Sigma_Y|} + \min \left\{ \frac{1}{2} \log \frac{|H_YSH_Y^T + \Sigma_Y|}{|H_YKH_Y^T + \Sigma_Y|}, \frac{1}{2} \log \frac{|H_ZSH_Z^T + \Sigma_Z|}{|H_ZKH_Z^T + \Sigma_Z|} \right\}$$

$$R_0 \leq \min \left\{ \frac{1}{2} \log \frac{|H_YSH_Y^T + \Sigma_Y|}{|H_YKH_Y^T + \Sigma_Y|}, \frac{1}{2} \log \frac{|H_ZSH_Z^T + \Sigma_Z|}{|H_ZKH_Z^T + \Sigma_Z|} \right\}$$

for some positive semi-definite matrix $K$ such that $0 \preceq K \preceq S$.

We next define a sub-class of Gaussian MIMO wiretap channels called the aligned Gaussian MIMO wiretap channel, which can be obtained from (13)-(14) by setting $H_Y = H_Z = I$,

$$Y = X + N_Y$$

$$Z = X + N_Z$$

In this work, we first prove Theorems 3 and 4 for the aligned Gaussian MIMO wiretap channel. Then, we establish the capacity region for the general channel model in (13)-(14) by following the analysis in Section V.B of [2] and Section 7.1 of [12] in conjunction with the capacity result we obtain for the aligned channel.

3.1 Capacity Region under a Power Constraint

We note that the covariance constraint on the channel input in (15) is a rather general constraint that subsumes the average power constraint

$$E [X^T X] = \text{tr} (E [XX^T]) \leq P$$

as a special case, see Lemma 1 and Corollary 1 of [2]. Therefore, using Theorem 3 the capacity-equivocation region arising from the average power constraint in (24), $C(P)$, can be found as follows.

Corollary 1 The capacity-equivocation region of the Gaussian MIMO wiretap channel subject to an average power constraint $P$, $C(P)$, is given by the union of rate triples $(R_0, R_1, R_e)$
satisfying
\begin{align}
0 \leq R_e & \leq \frac{1}{2} \log \frac{|H_YK_1H_Y^\top + \Sigma_Y|}{|\Sigma_Y|} - \frac{1}{2} \log \frac{|H_ZK_1H_Z^\top + \Sigma_Z|}{|\Sigma_Z|} \\
R_0 + R_1 & \leq \frac{1}{2} \log \frac{|H_YK_1H_Y^\top + \Sigma_Y|}{|\Sigma_Y|} \\
& \quad + \min \left\{ \frac{1}{2} \log \frac{|H_Y(K_1 + K_2)H_Y^\top + \Sigma_Y|}{|H_YK_1H_Y^\top + \Sigma_Y|}, \frac{1}{2} \log \frac{|H_Z(K_1 + K_2)H_Z^\top + \Sigma_Z|}{|H_ZK_1H_Z^\top + \Sigma_Z|} \right\}
\end{align}

for some positive semi-definite matrices $K_1, K_2$ such that $\text{tr}(K_1 + K_2) \leq P$.

### 4 Proof of Theorem 3 for the Aligned Case

Instead of proving Theorem 3 here we prove Theorem 4, which implies Theorem 3 due to Lemma 1. Achievability of the region given in Theorem 4 can be shown by setting $V = X$ in Theorem 2 and using jointly Gaussian $(U, X = U + T)$, where $U, T$ are independent Gaussian random vectors with covariance matrices $S - K, K$, respectively. In the rest of this section, we provide the converse proof. To this end, we note that since $C_p(S)$ is convex by definition, it can be characterized by solving the following optimization problem

\begin{equation}
f(R_0^*) = \max_{(R_0, R_p, R_s) \in C_p(S)} \mu_0 R_0 + \mu_p R_p + \mu_s R_s
\end{equation}

for all $\mu_p \in [0, \infty), \mu_s \in [0, \infty)$, and all possible common message rates $R_0^*$, which is bounded as follows

\begin{equation}
0 \leq R_0^* \leq \min\{C_Y(S), C_Z(S)\}
\end{equation}

where $C_Y(S), C_Z(S)$ are the single-user capacities for the legitimate user and the eavesdropper channels, respectively, i.e.,

\begin{align}
C_Y(S) &= \frac{1}{2} \log \frac{|S + \Sigma_Y|}{|\Sigma_Y|} \\
C_Z(S) &= \frac{1}{2} \log \frac{|S + \Sigma_Z|}{|\Sigma_Z|}
\end{align}

\footnote{Although characterizing $C_p(S)$ by solving the following optimization problem

\begin{equation}
\max_{(R_0, R_p, R_s) \in C_p(S)} \mu_0 R_0 + \mu_p R_p + \mu_s R_s
\end{equation}

for all $\mu_0, \mu_p, \mu_s$ seems to be more natural, we find working with (29) more convenient. Here, we characterize $C_p(S)$ by solving (29), for all $\mu_p, \mu_s$, for all fixed feasible $R_0^*$.}
We note that the optimization problem in (29) can be expressed in the following more explicit form

\[ f(R^*_{0}) = \max_{U \rightarrow V \rightarrow X \rightarrow (Y, Z)} \mu_p R_p + \mu_s R_s \]  

\[ 0 \leq R_s \leq I(V; Y|U) - I(V; Z|U) \]
\[ R_0^* + R_p + R_s \leq I(V; Y|U) + \min\{I(U; Y), I(U; Z)\} \]
\[ R_0^* \leq \min\{I(U; Y), I(U; Z)\} \]  

We also consider the Gaussian rate region \( R^G(S) \) which is defined as

\[ R^G(S) = \left\{ (R_0, R_p, R_s) : 0 \leq R_s \leq R_s(K) \right. \]
\[ R_0 + R_p + R_s \leq R_s(K) + R_p(K) + \min\{R_{0Y}(K), R_{0Z}(K)\} \]
\[ R_0 \leq \min\{R_{0Y}(K), R_{0Z}(K)\} \]  

for some \( 0 \leq K \leq S \)  

where \( R_s(K), R_p(K), R_{0Y}(K), R_{0Z}(K) \) are given as follows

\[ R_s(K) = \frac{1}{2} \log \frac{|K + \Sigma_Y|}{|\Sigma_Y|} - \frac{1}{2} \log \frac{|K + \Sigma_Z|}{|\Sigma_Z|} \]  

\[ R_p(K) = \frac{1}{2} \log \frac{|K + \Sigma_Z|}{|\Sigma_Z|} \]  

\[ R_{0Y}(K) = \frac{1}{2} \log \frac{|S + \Sigma_Y|}{|K + \Sigma_Y|} \]  

\[ R_{0Z}(K) = \frac{1}{2} \log \frac{|S + \Sigma_Z|}{|K + \Sigma_Z|} \]  

To provide the converse proof, i.e., to prove the optimality of jointly Gaussian \((U, V = X)\) for the optimization problem in (33)-(34), we will show that

\[ f(R_0^*) = g(R_0^*), \quad 0 \leq R_0^* \leq \min\{C_Y(S), C_Z(S)\} \]  

where \( g(R_0^*) \) is defined as

\[ g(R_0^*) = \max_{(R_0^*, R_p, R_s) \in R^G(S)} \mu_p R_p + \mu_s R_s \]  

We show (41) in two parts:

• \( \mu_s \leq \mu_p \)

• \( \mu_p < \mu_s \)
4.1 $\mu_s \leq \mu_p$

In this case, $f(R_0^*)$ can be written as

$$f(R_0^*) = \max_{U \rightarrow V \rightarrow X \rightarrow (Y, Z)} \mu_p(R_p + R_s)$$

(42)

subject to

$$R_0^* + R_p + R_s \leq I(X; Y|U) + \min\{I(U; Y), I(U; Z)\}$$

(43)

where we use the fact that $\mu_s \leq \mu_p$, and the secret message rate $R_s$ can be given up in favor of the private message rate $R_p$. This optimization problem gives us the capacity region of the two-user Gaussian MIMO broadcast channel with degraded message sets, where a common message is sent to both users, and a private message, on which there is no secrecy constraint, is sent to one of the two users [13]. The optimization problem for this case given in (42)-(43) is solved in [10] by showing the optimality of jointly Gaussian $(U, X)$, i.e., $f(R_0^*) = g(R_0^*)$.

This completes the converse proof for the case $\mu_s \leq \mu_p$.

4.2 $\mu_p < \mu_s$

In this case, we first study the optimization problem in (41). We rewrite $g(R_0)$ as follows

$$g(R_0^*) = \max_{0 \leq K \leq S} \mu_p R_p + \mu_s R_s(K)$$

(44)

subject to

$$R_0^* + R_p \leq R_p(K) + \min\{R_{0Y}(K), R_{0Z}(K)\}$$

(45)

Let $(K^*, R_p^*)$ be the maximizer for this optimization problem. The necessary KKT conditions that $(K^*, R_p^*)$ needs to satisfy are given in the following lemma.

Lemma 2 $K^*$ needs to satisfy

$$(\mu_s - \mu_p \lambda - \beta_Y)(K^* + \Sigma_Y)^{-1} + M = (\mu_s - \mu_p \lambda + \beta_Z)(K^* + \Sigma_Z)^{-1} + M_S$$

(46)

for some positive semi-definite matrices $M, M_S$ such that

$$K^* M = M K^* = 0$$

(47)

$$(S - K^*) M_S = M_S(S - K^*) = 0$$

(48)
and for some $\lambda = 1 - \bar{\lambda}$ such that it satisfies $0 \leq \lambda \leq 1$ and

\[
\lambda \begin{cases} 
= 0 & \text{if } R_{0Y}(K^*) > R_{0Z}(K^*) \\
= 1 & \text{if } R_{0Y}(K^*) < R_{0Z}(K^*) \\
\neq 0, 1 & \text{if } R_{0Y}(K^*) = R_{0Z}(K^*)
\end{cases}
\tag{49}
\]

and $(\beta_Y, \beta_Z)$ are given as follows

\[
(\beta_Y, \beta_Z) = \begin{cases} 
(0, 0) & \text{if } R_0^* < \min\{R_{0Y}(K^*), R_{0Z}(K^*)\} \\
(0, >0) & \text{if } R_0^* = R_{0Z}(K^*) < R_{0Y}(K^*) \\
(>0,0) & \text{if } R_0^* = R_{0Y}(K^*) < R_{0Z}(K^*) \\
(>0,>0) & \text{if } R_0^* = R_{0Y}(K^*) = R_{0Z}(K^*)
\end{cases}
\tag{50}
\]

$R_p^*$ needs to satisfy

\[
R_p^* = R_p(K^*) + \min\{R_{0Y}(K^*), R_{0Z}(K^*)\} - R_0^*
\tag{51}
\]

The proof of Lemma 2 is given in Appendix B. We treat three cases separately:

- $R_0^* < \min\{R_{0Y}(K^*), R_{0Z}(K^*)\}$
- $R_0^* = R_{0Y}(K^*) \leq R_{0Z}(K^*)$
- $R_0^* = R_{0Z}(K^*) < R_{0Y}(K^*)$

### 4.2.1 $R_0^* < \min\{R_{0Y}(K^*), R_{0Z}(K^*)\}$

In this case, we have $\beta_Y = \beta_Z = 0$, see (50). Thus, the KKT condition in (46) reduces to

\[
(\mu_s - \mu_p \lambda)(K^* + \Sigma_Y)^{-1} + M = (\mu_s - \mu_p \lambda)(K^* + \Sigma_Z)^{-1} + M_S
\tag{52}
\]

We first note that $K^*$ satisfying (52) achieves the secrecy capacity of this Gaussian MIMO wiretap channel [14], i.e.,

\[
R_s^* = R_s(K^*)
\tag{53}
\]

\[
= C_S(S)
\tag{54}
\]

\[
= \max_{0 \leq K \leq S} \frac{1}{2} \log \frac{|K + \Sigma_Y|}{|\Sigma_Y|} - \frac{1}{2} \log \frac{|K + \Sigma_Z|}{|\Sigma_Z|}
\tag{55}
\]

Next, we define a new covariance matrix $\Sigma_Z$ as follows

\[
(\mu_s - \mu_p \lambda)(K^* + \hat{\Sigma}_Z)^{-1} = (\mu_s - \mu_p \lambda)(K^* + \Sigma_Z)^{-1} + M_S
\tag{56}
\]
This new covariance matrix $\tilde{\Sigma}_Z$ has some useful properties which are listed in the following lemma.

**Lemma 3** We have the following facts.

- $\tilde{\Sigma}_Z \preceq \Sigma_Z$
- $\tilde{\Sigma}_Z \preceq \Sigma_Y$
- $(K^* + \tilde{\Sigma}_Z)^{-1}(S + \tilde{\Sigma}_Z) = (K^* + \Sigma_Z)^{-1}(S + \Sigma_Z)$

The proof of Lemma 3 is given in Appendix C. Thus, we have

$$R_{0Z}(K^*) = \frac{1}{2} \log \frac{|S + \Sigma_Z|}{|K^* + \Sigma_Z|}$$

$$= \frac{1}{2} \log \frac{|S + \tilde{\Sigma}_Z|}{|K^* + \tilde{\Sigma}_Z|}$$

$$\geq \frac{1}{2} \log \frac{|S + \Sigma_Y|}{|K^* + \Sigma_Y|}$$

$$= R_{0Y}(K^*)$$

where (58) comes from the third part of Lemma 3, (59) is due to the fact that

$$\frac{|A + B + \Delta|}{|B + \Delta|} \leq \frac{|A + B|}{|B|}$$

for $A \succeq 0$, $\Delta \succeq 0$, $B \succ 0$ by noting the second part of Lemma 3. Therefore, we have

$$R_{0Z}(K^*) \geq R_{0Y}(K^*)$$

where $K^*$ satisfies (52). Using (62) in (51), we find $R_p^*$ as follows

$$R_p^* = R_p(K^*) + R_{0Y}(K^*) - R_0^*$$

We also note the following

$$R_0^* + R_p^* + R_s^* = R_{0Y}(K^*) + R_p(K^*) + R_s(K^*)$$

$$= \frac{1}{2} \log \frac{|S + \Sigma_Y|}{|\Sigma_Y|}$$

$$= C_Y(S)$$

Now, we show that

$$g(R_0^*) = f(R_0^*)$$
To this end, we assume that

\[ g(R_0^*) < f(R_0^*) \]  

which implies that there exists a rate triple \((R_0^*, R_p^0, R_s^0) \in C_p(S)\) such that

\[ \mu_p R_p^s + \mu_s R_s^s < \mu_p R_p^0 + \mu_s R_s^0 \]  

To prove (67), i.e., that (68) is not possible, we note the following bounds

\[ R_s^0 \leq C_S(S) = R_s^* \]  
\[ R_p^0 + R_s^0 \leq C_Y(S) - R_0^* = R_p^* + R_s^* \]

where (70) comes from (55) and the fact that the rate of the confidential message, i.e., \(R_s\), cannot exceed the secrecy capacity, and (71) is due to (66) and the fact that the sum rate \(R_0 + R_p + R_s\) cannot exceed the legitimate user’s single-user capacity. Thus, in view of \(\mu_s > \mu_p\), (70)-(71) imply

\[ \mu_p R_p^0 + \mu_s R_s^0 \leq \mu_p R_p^* + \mu_s R_s^* \]

which contradicts with (69); proving (67). This completes the converse proof for this case.

Before starting the proofs of the other two cases, we now recap our proof for the case \(R_0^* < \min\{R_{0Y}(K^*), R_{0Z}(K^*)\}\). We note that we did not show the optimality of Gaussian signalling directly, instead, we prove it indirectly by showing the following

\[ g(R_0^*) = f(R_0^*) \]  

First, we show that for the given common message rate \(R_0^*\), we can achieve the secrecy capacity, i.e., \(R_s^* = C_S(S)\), see (55)-(55). In other words, we show that \((R_0^*, R_p^*, R_s^*)\) is on the boundary of the capacity region \(C_p(S)\). Secondly, we show that for the given common message rate \(R_0^*\), \((R_p^*, R_s^*)\) achieve the sum capacity of the public and confidential messages, i.e., \(R_s^* + R_p^*\) is sum rate optimal for the given common message rate \(R_0^*\), see (64)-(66) and (71). These two findings lead to the inequalities in (70)-(71). Finally, we use a time-sharing argument for these two inequalities in (70)-(71) to obtain (73), which completes the proof.

4.2.2 \(R_0^* = R_{0Y}(K^*) \leq R_{0Z}(K^*)\)

We first rewrite the KKT condition in (46) as follows

\[ (\mu_s - \mu_p \lambda - \mu_0 \beta)(K^* + \Sigma_Y)^{-1} + M = (\mu_s - \mu_p \lambda + \mu_0 \beta)(K^* + \Sigma_Z)^{-1} + M_S \]  

(74)
by defining $\mu_0 = \beta_Y + \beta_Z$, $\mu_0 \beta = \beta_Y$, and $\mu_0 \bar{\beta} = \beta_Z$. We note that if $R_{0Y}(K^*) < R_{0Z}(K^*)$, we have $\beta = \lambda = 1$, if $R_{0Y}(K^*) = R_{0Z}(K^*)$, we have $0 < \lambda < 1, 0 < \beta < 1$. The proof of these two cases are very similar, and we consider only the case $0 < \lambda < 1, 0 < \beta < 1$, i.e., we assume $R_{0Y}(K^*) = R_{0Z}(K^*)$. The other case can be proved similarly.

Similar to Section 4.2.1 here also, we prove the desired identity

$$g(R_0^*) = f(R_0)$$

(75)

by contradiction. We first assume that

$$g(R_0^*) < f(R_0)$$

(76)

which implies that there exists a rate triple $(R_s^*, R_p^0, R_o^0) \in C_p(S)$ such that

$$\mu_p R_p^* + \mu_s R_s^* < \mu_p R_p^0 + \mu_s R_o^0$$

(77)

where we define $R_s^* = R_s(K^*)$. Since the sum rate $R_0 + R_p + R_s$ needs to be smaller than the legitimate user’s single user capacity, we have

$$R_0^0 + R_p^0 + R_o^0 \leq C_Y(S)$$

(78)

On the other hand, we have the following

$$R_0^* + R_p^* + R_s^* = \min\{R_{0Y}(K^*), R_{0Z}(K^*)\} + R_p(K^*) + R_s(K^*)$$

$$= R_{0Y}(K^*) + R_p(K^*) + R_s(K^*)$$

$$= C_Y(S)$$

(79)

(80)

(81)

where (79) comes from (51), and (80) is due to our assumption that $R_0^* = R_{0Y}(K^*) = R_{0Z}(K^*)$. Equations (78) and (81) imply that

$$R_p^0 + R_o^0 \leq R_p^* + R_s^*$$

(82)

In the rest of this section, we prove that we have $R_s^0 \leq R_s^*$ for the given common message rate $R_0^*$, which, in conjunction with (82), will yield a contradiction with (77); proving (75).

To this end, we first define a new covariance matrix $\tilde{\Sigma}_Y$ as follows

$$(\mu_s - \mu_p \lambda)(K^* + \tilde{\Sigma}_Y)^{-1} = (\mu_s - \mu_p \lambda)(K^* + \Sigma_Y)^{-1} + M$$

(83)

This new covariance matrix $\tilde{\Sigma}_Y$ has some useful properties which are listed in the following lemma.

**Lemma 4** We have the following facts.
• $\Sigma_Y \preceq \Sigma_Y$
• $\Sigma_Y \preceq \Sigma_Z$
• $(K^* + \Sigma_Y)^{-1}\Sigma_Y = (K^* + \Sigma_Y)^{-1}\Sigma_Y$

The proof of this lemma is given in Appendix D. Using this new covariance matrix, we define a random vector $\tilde{Y}$ as

$$\tilde{Y} = X + \tilde{N}_Y$$

(84)

where $\tilde{N}_Y$ is a Gaussian random vector with covariance matrix $\tilde{\Sigma}_Y$. Due to the first and second statements of Lemma 4, we have the following Markov chains

$$U \rightarrow V \rightarrow X \rightarrow \tilde{Y} \rightarrow Y$$

(85)

$$U \rightarrow V \rightarrow X \rightarrow \tilde{Y} \rightarrow Z$$

(86)

We next study the following optimization problem

$$\max_{(R_0, R_p, R_s) \in \mathcal{C}_p(S)} \mu_0 R_0 + (\mu_s - \mu_p \lambda) R_s$$

$$= \max_{\substack{U \rightarrow V \rightarrow X \rightarrow (Y, Z) \rightarrow S \text{ \ s.t.} \ E[XX^\top] \leq S}} \mu_0 \min \{I(U; Y), I(U; Z)\} + (\mu_s - \mu_p \lambda) \left[ I(V; Y|U) - I(V; Z|U) \right]$$

(87)

Since we assume $(R_0^*, R_p^*, R_s^*) \in \mathcal{C}_p(S)$, we have the following lower bound for (87)

$$\mu_0 R_0^* + (\mu_s - \mu_p \lambda) R_s^* \leq \max_{(R_0, R_p, R_s) \in \mathcal{C}_p(S)} \mu_0 R_0 + (\mu_s - \mu_p \lambda) R_s$$

(88)
Now we solve the optimization problem in (87) as follows

\[
\max_{(R_0, R_p, R_s) \in \mathcal{C}_p(S)} \mu_0 R_0 + (\mu_s - \mu_p \lambda) R_s
\]

\[
= \max_{U \rightarrow V \rightarrow X \rightarrow (Y, Z)} \min_{E[XX^\top] \geq S} \mu_0 \min \{I(U; Y), I(U; Z)\} + (\mu_s - \mu_p \lambda) [I(V; Y|U) - I(V; Z|U)]
\]

\[(89)\]

\[
\leq \max_{U \rightarrow V \rightarrow X \rightarrow (Y, Z)} \min_{E[XX^\top] \geq S} \mu_0 \beta I(U; Z) + \mu_0 \beta I(U; Y) + (\mu_s - \mu_p \lambda) [I(V; Y|U) - I(V; Z|U)]
\]

\[(90)\]

\[
\leq \max_{U \rightarrow V \rightarrow X \rightarrow (Y, Z)} \min_{E[XX^\top] \geq S} \mu_0 \beta I(U; Z) + \mu_0 \beta I(U; Y) + (\mu_s - \mu_p \lambda) [I(X; Y|U) - I(X; Z|U)]
\]

\[(91)\]

\[
\leq \max_{U \rightarrow X \rightarrow (Y, Z)} \min_{E[XX^\top] \geq S} \mu_0 \beta I(U; Z) + \mu_0 \beta I(U; Y) + (\mu_s - \mu_p \lambda) [I(X; Y|U) - I(X; Z|U)]
\]

\[(92)\]

\[
\leq \frac{\mu_0 \bar{\beta}}{2} \log \frac{|S + \Sigma_Z|}{|K^* + \Sigma_Z|} + \frac{\mu_0 \beta}{2} \log \frac{|S + \Sigma_Y|}{|K^* + \Sigma_Y|} + \frac{\mu_s - \mu_p \lambda}{2} \left[ \log \frac{|K^* + \Sigma_Y|}{|\Sigma_Y|} - \log \frac{|K^* + \Sigma_Z|}{|\Sigma_Z|} \right]
\]

\[(93)\]

\[
= \mu_0 \beta R_{0Z}(K^*) + \mu_0 \beta R_{0Y}(K^*) + \frac{\mu_s - \mu_p \lambda}{2} \left[ \log \frac{|K^* + \Sigma_Y|}{|\Sigma_Y|} - \log \frac{|K^* + \Sigma_Z|}{|\Sigma_Z|} \right]
\]

\[(94)\]

\[
= \mu_0 \beta R_{0Z}(K^*) + \mu_0 \beta R_{0Y}(K^*) + \frac{\mu_s - \mu_p \lambda}{2} \left[ \log \frac{|K^* + \Sigma_Y|}{|\Sigma_Y|} - \log \frac{|K^* + \Sigma_Z|}{|\Sigma_Z|} \right]
\]

\[(95)\]

\[
= \mu_0 \beta R_{0Z}(K^*) + \mu_0 \beta R_{0Y}(K^*) + (\mu_s - \mu_p \lambda) R_s(K^*)
\]

\[(96)\]

\[
= \mu_0 R_0^* + (\mu_s - \mu_p \lambda) R_s^*
\]

\[(97)\]

where (90) comes from the fact that $0 < \beta = 1 - \bar{\beta} < 1$, (91)-(92) are due to the Markov chains in (85)-(86), respectively, (93) can be obtained by using the analysis in [9 eqns (30)-(32)], (95) comes from the third part of Lemma 4 and (97) is due to our assumption that $R_0^* = R_{0Y}(K^*) = R_{0Z}(K^*)$. Thus, (97) implies

\[
\max_{(R_0, R_p, R_s) \in \mathcal{C}_p(S)} \mu_0 R_0 + (\mu_s - \mu_p \lambda) R_s \leq \mu_0 R_0^* + (\mu_s - \mu_p \lambda) R_s^*
\]

\[(98)\]

Comparing (88) and (98) yields

\[
R_0^* \leq R_s^*
\]

\[(99)\]

Using (82) and (99) and noting $\mu_s > \mu_p$, we can get

\[
\mu_p R_p^0 + \mu_s R_s^0 \leq \mu_p R_p^0 + \mu_s R_s^*
\]

\[(100)\]

which contradicts with (177); proving (72). This completes the converse proof for this case.

Before providing the proof for the last case, we recap our proof for the case $R_0^* =
\[ R_{0Y}(K^*) \leq R_{0Z}(K^*) \]. Similar to Section 4.2.1 here also, we prove the optimality of Gaussian signalling indirectly, i.e., we show the desired identity

\[ g(R^*_0) = f(R^*_0) \]  

(101)

indirectly. First, we show that for the given common message rate \( R^*_0 \), \( R^*_p + R^*_s \) is sum rate optimal, i.e., \((R^*_p, R^*_s)\) achieve the sum capacity of the public and confidential messages, by obtaining (82). Secondly, we show that \((R^*_0, 0, R^*_s)\) is also on the boundary of the capacity region \( C_p(S) \) by obtaining (98). These two findings give us the inequalities in (82) and (99). Finally, we use a time-sharing argument for these two inequalities in (82) and (99) to establish (101), which completes the proof.

4.2.3 \( R^*_0 = R_{0Z}(K^*) < R_{0Y}(K^*) \)

In this case, we have \( \lambda = \beta_Y = 0 \), see (49)-(50). Hence, the KKT condition in (46) reduces to

\[ \mu_s(K^* + \Sigma_Y)^{-1} + M = (\mu_s + \beta_Z)(K^* + \Sigma_Z)^{-1} + M_S \]  

(102)

We again prove the desired identity

\[ g(R^*_0) = f(R^*_0) \]  

(103)

by contradiction. We first assume that

\[ g(R^*_0) < f(R^*_0) \]  

(104)

which implies that there exists a rate triple \((R^*_0, R^*_p, R^*_s)\) in \( C_p(S) \) such that

\[ \mu_p R^*_p + \mu_s R^*_s < \mu_p R^*_p + \mu_s R^*_s \]  

(105)

In the rest of the section, we show that

\[ \mu_p R^*_p + \mu_s R^*_s \geq \mu_p R^*_p + \mu_s R^*_s \]  

(106)

to reach a contradiction, and hence, prove (103). To this end, we define a new covariance matrix \( \tilde{\Sigma}_Y \) as follows

\[ \mu_s(K^* + \tilde{\Sigma}_Y)^{-1} = \mu_s(K^* + \Sigma_Y)^{-1} + M \]  

(107)

This new covariance matrix \( \tilde{\Sigma}_Y \) has some useful properties listed in the following lemma.

**Lemma 5** We have the following facts.
\[ \tilde{\Sigma}_Y \preceq \Sigma_Y \]
\[ \tilde{\Sigma}_Y \preceq \Sigma_Z \]
\[ (K^* + \tilde{\Sigma}_Y)^{-1}\tilde{\Sigma}_Y = (K^* + \Sigma_Y)^{-1}\Sigma_Y \]

The proof of this lemma is very similar to the proof Lemma 4, and hence is omitted. Using this new covariance matrix \( \tilde{\Sigma}_Y \), we define a random vector \( \tilde{Y} \) as

\[ \tilde{Y} = X + \tilde{N}_Y \quad (108) \]

where \( \tilde{N}_Y \) is a Gaussian random vector with covariance matrix \( \tilde{\Sigma}_Y \). Due to the first and second statements of Lemma 5, we have the following Markov chains

\[ U \to V \to X \to \tilde{Y} \to Y \quad (109) \]
\[ U \to V \to X \to \tilde{Y} \to Z \quad (110) \]

Next, we study the following optimization problem

\[ \max_{(R_0, R_p, R_s) \in C_p(S)} \left( \mu_p + \beta Z \right) R_0 + \mu_p R_p + \mu_s R_s \quad (111) \]

We note that since \((R_0^o, R_p^o, R_s^o) \in C_p(S)\), we have the following lower bound for the optimization problem in (111)

\[ (\mu_p + \beta Z) R_0^o + \mu_p R_p^o + \mu_s R_s^o \leq \max_{(R_0, R_p, R_s) \in C_p(S)} \left( \mu_p + \beta Z \right) R_0 + \mu_p R_p + \mu_s R_s \quad (112) \]

We next obtain the maximum for (111). To this end, we introduce the following lemma which provides an explicit form for this optimization problem.

**Lemma 6** For \( \mu_s > \mu_p \), we have

\[ \max_{(R_0, R_p, R_s) \in C_p(S)} \left( \mu_p + \beta Z \right) R_0 + \mu_p R_p + \mu_s R_s = \max_{\mathclap{U \to V \to X \to (Y, Z)}} \left( \mu_p + \beta Z \right) \min \{ I(U; Y), I(U; Z) \} + \mu_p I(V; Z|U) \]
\[ + \mu_s \left[ I(V; Y|U) - I(V; Z|U) \right] \quad (113) \]

The proof of this lemma is given in Appendix E.

Next we introduce the following extremal inequality from [3], which will be used subsequently in the solution of (113).

**Lemma 7** ([3, Corollary 4]) Let \((U, X)\) be an arbitrarily correlated random vector, where \(X\) has a covariance constraint \( E[XX^T] \preceq S \) and \( S \succ 0 \). Let \( N_1, N_2 \) be Gaussian random...
vectors with covariance matrices $\Sigma_1, \Sigma_2$, respectively. They are independent of $(U, X)$. Furthermore, $\Sigma_1, \Sigma_2$ satisfy $\Sigma_1 \preceq \Sigma_2$. Assume that there exists a covariance matrix $K^*$ such that $K^* \preceq S$ and

$$\nu(K^* + \Sigma_1)^{-1} = \gamma(K^* + \Sigma_2)^{-1} + M_S$$

(114)

where $\nu \geq 0, \gamma \geq 0$ and $M_S$ is positive semi-definite matrix such that $(S - K^*)M_S = 0$. Then, for any $(U, X)$, we have

$$\nu h(X + N_1|U) - \gamma h(X + N_2|U) \leq \frac{\nu}{2} \log |(2\pi e)(K^* + \Sigma_1)| - \frac{\gamma}{2} \log |(2\pi e)(K^* + \Sigma_2)|$$

(115)

Now we use Lemma 7. To this end, we note that using (107) in (102), we get

$$\mu_s(K^* + \tilde{\Sigma}_Y)^{-1} = (\mu_s + \beta_Z)(K^* + \Sigma_Z)^{-1} + M_S$$

(116)

In view of (116) and the fact that $\tilde{\Sigma}_Y \preceq \Sigma_Z$, Lemma 7 implies

$$\mu_s h(\tilde{Y}|U) - (\mu_s + \beta_Z)h(Z|U) \leq \frac{\mu_s}{2} \log |(2\pi e)(K^* + \tilde{\Sigma}_Y)| - \frac{\mu_s + \beta Z}{2} \log |(2\pi e)(K^* + \Sigma_Z)|$$

(117)
We now consider the maximization in (113) as follows

\[
\begin{align*}
& \max_{(R_0, R_p, R_s) \in \mathcal{C}(\mathbf{S})} (\mu_p + \beta Z) R_0 + \mu_p R_p + \mu_s R_s \\
= & \max_{U \rightarrow V \rightarrow X \rightarrow (Y, Z)} \frac{1}{E[XX^\top] \preceq \mathbf{S}} (\mu_p + \beta Z) \min \{I(U; Y), I(U; Z)\} + \mu_p I(V; Z | U) \\
& \quad + \mu_s [I(V; Y | U) - I(V; Z | U)] \\
\leq & \max_{U \rightarrow V \rightarrow X \rightarrow (Y, Z)} \frac{1}{E[XX^\top] \preceq \mathbf{S}} (\mu_p + \beta Z) I(U; Z) + \mu_p I(V; Z | U) + \mu_s [I(V; Y | U) - I(V; Z | U)] \\
\leq & \max_{U \rightarrow V \rightarrow X \rightarrow (Y, Z)} \frac{1}{E[XX^\top] \preceq \mathbf{S}} (\mu_p + \beta Z) I(U; Z) + \mu_p I(X; Z | U) + \mu_s [I(V; Y | U) - I(V; Z | U)] \\
\leq & \max_{U \rightarrow X \rightarrow (Y, Z)} \frac{1}{E[XX^\top] \preceq \mathbf{S}} (\mu_p + \beta Z) I(U; Z) + \mu_p I(X; Z | U) + \mu_s [I(X; \tilde{Y} | U) - I(X; Z | U)] \\
= & \max_{U \rightarrow X \rightarrow (Y, Z)} \frac{1}{E[XX^\top] \preceq \mathbf{S}} (\mu_p + \beta Z) h(Z) + \mu_p h(\tilde{Y} | U) - (\mu_s + \beta Z) h(Z | U) \\
& \quad - \frac{\mu_s}{2} \log |(2\pi e)\Sigma_Y| + \frac{\mu_s - \mu_p}{2} \log |(2\pi e)\Sigma_Z| \\
\leq & \frac{\mu_p + \beta Z}{2} \log |(2\pi e)(\mathbf{S} + \Sigma_Z)| + \max_{U \rightarrow X \rightarrow (Y, Z)} \frac{1}{E[XX^\top] \preceq \mathbf{S}} \mu_s h(\tilde{Y} | U) - (\mu_s + \beta Z) h(Z | U) \\
& \quad - \frac{\mu_s}{2} \log |(2\pi e)\Sigma_Y| + \frac{\mu_s - \mu_p}{2} \log |(2\pi e)\Sigma_Z| \\
\leq & \frac{\mu_p + \beta Z}{2} \log |(2\pi e)(\mathbf{S} + \Sigma_Z)| + \frac{\mu_s}{2} \log |(2\pi e)(\mathbf{K}^* + \Sigma_Y)| - \frac{\mu_s + \beta Z}{2} \log |(2\pi e)(\mathbf{K}^* + \Sigma_Z)| \\
& \quad - \frac{\mu_s}{2} \log |(2\pi e)\Sigma_Y| + \frac{\mu_s - \mu_p}{2} \log |(2\pi e)\Sigma_Z| \\
= & \frac{\mu_p + \beta Z}{2} \log \frac{|\mathbf{S} + \Sigma_Z|}{|\mathbf{K}^* + \Sigma_Z|} + \frac{\mu_p}{2} \log \frac{|\mathbf{K}^* + \Sigma_Z|}{|\Sigma_Z|} + \mu_s \left[ \log \frac{|\mathbf{K}^* + \Sigma_Y|}{|\Sigma_Y|} - \log \frac{|\mathbf{K}^* + \Sigma_Z|}{|\Sigma_Z|} \right] \\
= & (\mu_p + \beta Z) R_{0Z}(\mathbf{K}^*) + \mu_p R_p(\mathbf{K}^*) + \frac{\mu_s}{2} \left[ \log \frac{|\mathbf{K}^* + \Sigma_Y|}{|\Sigma_Y|} - \log \frac{|\mathbf{K}^* + \Sigma_Z|}{|\Sigma_Z|} \right] \\
= & (\mu_p + \beta Z) R_{0Z}(\mathbf{K}^*) + \mu_p R_p(\mathbf{K}^*) + \mu_s R_s(\mathbf{K}^*) \\
= & (\mu_p + \beta Z) R_0^* + \mu_p R_p^* + \mu_s R_s^* 
\end{align*}
\]
where (119) is due to \( \min\{a, b\} \leq a \), (120) is due to the Markov chain in (110), (121)-(122) come from the Markov chains in (109)-(110), respectively, (124) is due to the maximum entropy theorem [15], (125) comes from (117), and (128) is due to the third part of Lemma 5. Comparing (130) and (112) yields

\[
\mu_p R_p^o + \mu_s R_s^o \leq \mu_p R_p^* + \mu_s R_s^*
\]  

which contradicts with our assumption in (105); implying (103). This completes the converse proof for this case.

We note that contrary to Sections 4.2.1 and 4.2.2, here we prove the optimality of Gaussian signalling, i.e.,

\[
g(R_0^*) = f(R_0^*)
\]  

directly. In other words, to show (132), we did not find any other points on the boundary of the capacity region \( \mathcal{C}_p(S) \) and did not have to use a time-sharing argument between these points to reach (132). (This was our strategy in Sections 4.2.1 and 4.2.2.) Instead, we define a new optimization problem given in (113) whose solution yields (132).

## 5 Proof of Theorem 3 for the General Case

The achievability of the region given in Theorem 3 can be shown by computing the region in Theorem 1 with the following selection of \( (U, V, X) \): \( V = X, X = U + T \) where \( T, U \) are independent Gaussian random vectors with covariance matrices \( K, S - K \), respectively, \( U = U \). In the rest of this section, we consider the converse proof. We first note that following the approaches in Section V.B of [2] and Section 7.1 of [12], it can be shown that a new Gaussian MIMO wiretap channel can be constructed from any Gaussian MIMO wiretap channel described by (13)-(14) such that the new channel has the same capacity-equivocation region with the original one and in the new channel, both the legitimate user and the eavesdropper have the same number of antennas as the transmitter, i.e., \( r_Y = r_Z = t \). Thus, without loss of generality, we assume that \( r_Y = r_Z = t \). We next apply singular-value decomposition to the channel gain matrices \( H_Y, H_Z \) as follows

\[
H_Y = U_Y \Lambda_Y V_Y^T
\]
\[
H_Z = U_Z \Lambda_Z V_Z^T
\]  

20
where $U_Y, U_Z, V_Y, V_Z$ are $t \times t$ orthogonal matrices, and $\Lambda_Y, \Lambda_Z$ are diagonal matrices. We now define a new Gaussian MIMO wiretap channel as follows

\begin{align}
\mathbf{Y} &= \mathbf{H}_Y \mathbf{X} + \mathbf{N}_Y \\
\mathbf{Z} &= \mathbf{H}_Z \mathbf{X} + \mathbf{N}_Z
\end{align}

(135)

where $\mathbf{H}_Y, \mathbf{H}_Z$ are defined as

\begin{align}
\mathbf{H}_Y &= U_Y (\Lambda_Y + \alpha \mathbf{I}) V_Y^\top \\
\mathbf{H}_Z &= U_Z (\Lambda_Z + \alpha \mathbf{I}) V_Z^\top
\end{align}

(136)

for some $\alpha > 0$. We denote the capacity-equivocation region of the Gaussian MIMO wiretap channel defined in (135)-(136) by $C_\alpha(S)$. Since $\mathbf{H}_Y, \mathbf{H}_Z$ are invertible, the capacity-equivocation region of the channel in (135)-(136) is equal to the capacity-equivocation region of the following aligned channel

\begin{align}
\mathbf{Y} &= \mathbf{X} + \mathbf{N}_Y \\
\mathbf{Z} &= \mathbf{X} + \mathbf{N}_Z
\end{align}

(137)

Thus, using the capacity result for the aligned case, which was proved in the previous section, we obtain $C_\alpha(S)$ as the union of rate triples $(R_0, R_1, R_e)$ satisfying

\begin{align}
0 \leq R_e &\leq \frac{1}{2} \log \frac{|\mathbf{H}_Y \mathbf{K} \mathbf{H}_Y^\top + \Sigma_Y|}{|\Sigma_Y|} - \frac{1}{2} \log \frac{|\mathbf{H}_Z \mathbf{K} \mathbf{H}_Z^\top + \Sigma_Z|}{|\Sigma_Z|} \\
R_0 + R_1 &\leq \frac{1}{2} \log \frac{|\mathbf{H}_Y \mathbf{K} \mathbf{H}_Y^\top + \Sigma_Y|}{|\Sigma_Y|} \\
&\quad + \min \left\{ \frac{1}{2} \log \frac{|\mathbf{H}_Y \mathbf{K} \mathbf{H}_Y^\top + \Sigma_Y|}{|\Sigma_Y|}, \frac{1}{2} \log \frac{|\mathbf{H}_Z \mathbf{K} \mathbf{H}_Z^\top + \Sigma_Z|}{|\Sigma_Z|} \right\} \\
R_0 &\leq \min \left\{ \frac{1}{2} \log \frac{|\mathbf{H}_Y \mathbf{K} \mathbf{H}_Y^\top + \Sigma_Y|}{|\Sigma_Y|}, \frac{1}{2} \log \frac{|\mathbf{H}_Z \mathbf{K} \mathbf{H}_Z^\top + \Sigma_Z|}{|\Sigma_Z|} \right\}
\end{align}

(141)

(142)

(143)

for some positive semi-definite matrix $\mathbf{K}$ such that $0 \preceq \mathbf{K} \preceq \mathbf{S}$.

We next obtain an outer bound for the capacity-equivocation region of the original Gaussian MIMO wiretap channel in (13)-(14) in terms of $C_\alpha(S)$. To this end, we first note the following Markov chains

\begin{align}
\mathbf{X} &\rightarrow \mathbf{Y} \\
\mathbf{X} &\rightarrow \mathbf{Z} \\
\mathbf{X} &\rightarrow \mathbf{Y} \rightarrow \mathbf{Y}
\end{align}

(144)

(145)
which imply that if the messages \((W_0, W_1)\) with rates \((R_0, R_1)\) are transmitted with a vanishingly small probability of error in the original Gaussian MIMO wiretap channel given by (13)-(14), they will be transmitted with a vanishingly small probability of error in the new Gaussian MIMO wiretap channel given by (135)-(136) as well. However, as opposed to the rates \(R_0, R_1\), we cannot immediately conclude that if an equivocation rate \(R_e\) is achievable in the original Gaussian MIMO wiretap channel given in (13)-(14), it is also achievable in the new Gaussian MIMO wiretap channel in (135)-(136). The reason for this is that both the legitimate user’s and the eavesdropper’s channel gain matrices are enhanced in the new channel given by (135)-(136), see (137)-(138) and/or (144)-(145), and consequently, it is not clear what the overall effect of these two enhancements on the equivocation rate will be. However, in the sequel, we show that if \((R_0, R_1, R_e) \in C(S)\), then we have \((R_0, R_1, R_e - \gamma) \in C_\alpha(S)\).

This will let us write down an outer bound for \(C(S)\) in terms of \(C_\alpha(S)\). To this end, we note that if \((R_0, R_1, R_e) \in C(S)\), we need to have a random vector \((U, V, X)\) such that the inequalities given in Theorem 1 hold. Assume that we use the same random vector \((U, V, X)\) for the new Gaussian MIMO wiretap channel in (135)-(136), and achieve the rate triple \((R_1, R_0, R_e)\). Due to the Markov chains in (144)-(145), we already have \(R_1 \leq R_1, R_0 \leq R_0\).

Furthermore, following the analysis in Section 4 of [9], we can bound the gap between \(R_e\) and \(R_e\), i.e., \(\gamma\), as follows

\[
\gamma = R_e - R_e \leq \frac{1}{2} \log \left| \frac{H_Z S \bar{H}_Z^\top + \Sigma_Z}{|\Sigma_Z|} \right| - \frac{1}{2} \log \left| \frac{H_Z S \bar{H}_Z^\top + \Sigma_Z}{|\Sigma_Z|} \right| \quad \text{(146)}
\]

Thus, we have

\[
C(S) \subseteq C_\alpha(S) + G(S) \quad \text{(147)}
\]

where \(G(S)\) is

\[
G(S) = \left\{ (0, 0, R_e) : 0 \leq R_e \leq \frac{1}{2} \log \left| \frac{H_Z S \bar{H}_Z^\top + \Sigma_Z}{|\Sigma_Z|} \right| - \frac{1}{2} \log \left| \frac{H_Z S \bar{H}_Z^\top + \Sigma_Z}{|\Sigma_Z|} \right| \right\} \quad \text{(148)}
\]

Taking \(\alpha \to 0\) in (147), we get

\[
C(S) \subseteq \lim_{\alpha \to 0} C_\alpha(S) \quad \text{(149)}
\]

where we use the fact that

\[
\lim_{\alpha \to 0} \frac{1}{2} \log \left| \frac{H_Z S \bar{H}_Z^\top + \Sigma_Z}{|\Sigma_Z|} \right| - \frac{1}{2} \log \left| \frac{H_Z S \bar{H}_Z^\top + \Sigma_Z}{|\Sigma_Z|} \right| = 0 \quad \text{(150)}
\]

which follows from the continuity of \(\log |\cdot|\) in positive semi-definite matrices, and the fact
that \( \lim_{\alpha \to 0} \overline{H}_Z = H_Z \). Finally, we note that
\[
\lim_{\alpha \to 0} C_{\alpha}(S) = \lim_{\alpha \to 0} \overline{H}_Y = H_Y, \lim_{\alpha \to 0} \overline{H}_Z = H_Z; \quad \text{completing the proof.}
\]

6 Conclusions

We study the Gaussian MIMO wiretap channel in which a common message is sent to both the legitimate user and the eavesdropper in addition to the private message sent only to the legitimate user. We first establish an equivalence between this original definition of the wiretap channel and the wiretap channel with public messages, in which the private message is divided into two parts as the confidential message, which needs to be transmitted in perfect secrecy, and public message, on which there is no secrecy constraint. We next obtain capacity regions for both cases. We show that it is sufficient to consider jointly Gaussian auxiliary random variables and channel input to evaluate the single-letter description of the capacity-equivocation region due to [1]. We prove this by using channel enhancement [2] and an extremal inequality from [3].

A Proof of Lemma 1

The proof of this lemma for \( R_0 = 0 \) is outlined in [4, Problem 33-c], [5]. We extend their proof to the general case of interest here. We first note the inclusion \( C_p \subseteq C \), which follows from the fact that if \((R_0, R_p, R_s) \in C_p\), we can attain the rate triple \((R_0, R_1 = R_s + R_p, R_e = R_s)\), i.e., \((R_0, R_s + R_p, R_s) \in C\). To show the reverse inclusion, we use the achievability proof for Theorem 1 given in [4]. According to this achievable scheme, \( W_1 \) can be divided into two parts as \( W_1 = (W_p, W_s) \) with rates \((R_1 - R_e, R_e)\), respectively, and we have
\[
H(W_1|W_0, Z^n) = H(W_p, W_s|Z^n, W_0) \geq H(W_s|Z^n, W_0) \geq H(W_s) - n\gamma_n
\]
for some \( \gamma_n \) which satisfies \( \lim_{n \to \infty} \gamma_n = 0 \). Hence, using this capacity achieving scheme for \( C \), we can attain the rate triple \((R_0, R_p = R_1 - R_e, R_s = R_e) \in C_p\). This implies \( C \subseteq C_p \); completing the proof of the lemma.
Since the program in (44)-(45) is not necessarily convex, the KKT conditions are necessary but not sufficient. The Lagrangian for this optimization problem is given by

\[
L = \mu_s R_s(K) + \mu_p R_p + \lambda_Y [R_p(K) + R_{0Y}(K) - R_p - R_0^*] + \lambda_Z [R_p(K) + R_{0Z}(K) - R_p - R_0^*] \\
+ \beta_Y [R_{0Y}(K) - R_0^*] + \beta_Z [R_{0Z}(K) - R_0^*] + \text{tr}(KM) + \text{tr}((S - K)M_S)
\]

(155)

where \(M, M_S\) are positive semi-definite matrices, and \(\lambda_Y \geq 0, \lambda_Z \geq 0, \beta_Y \geq 0, \beta_Z \geq 0\).

The necessary KKT conditions that they need to satisfy are given as follows

\[
\frac{\partial L}{\partial R_p} \bigg|_{R_p = R_p^*} = 0 \quad (156)
\]

\[
\nabla_K L \bigg|_{K = K^*} = 0 \quad (157)
\]

\[
\text{tr}(K^* M) = 0 \quad (158)
\]

\[
\text{tr}((S - K^*) M_S) = 0 \quad (159)
\]

\[
\lambda_Y [R_p(K^*) + R_{0Y}(K^*) - R_p^* - R_0^*] = 0 \quad (160)
\]

\[
\lambda_Z [R_p(K^*) + R_{0Z}(K^*) - R_p^* - R_0^*] = 0 \quad (161)
\]

\[
\beta_Y (R_{0Y}(K^*) - R_0^*) = 0 \quad (162)
\]

\[
\beta_Z (R_{0Z}(K^*) - R_0^*) = 0 \quad (163)
\]

The first KKT condition in (156) implies \(\lambda_Y + \lambda_Z = \mu_p\). We define \(\lambda_Y = \mu_p \bar{\lambda}, \lambda_Z = \mu_p \bar{\lambda}\) and consequently, we have \(0 \leq \bar{\lambda} = 1 - \lambda \leq 1\). The second KKT condition in (157) implies (46).

Since \(\text{tr}(AB) = \text{tr}(BA)\) and \(\text{tr}(AB) \geq 0\) for \(A \succeq 0, B \succeq 0\), (158)-(159) imply (47)-(48). The KKT conditions in (160)-(161) imply (51). Furthermore, the KKT conditions in (160)-(161) state the conditions that if \(R_{0Y}(K^*) > R_{0Z}(K^*)\), \(\lambda = 0\), if \(R_{0Y}(K^*) < R_{0Z}(K^*)\), \(\lambda = 1\), and if \(R_{0Y}(K^*) = R_{0Z}(K^*)\), \(\lambda\) is arbitrary, i.e., \(0 < \lambda < 1\). Similarly, the KKT conditions in (162)-(163) imply (50).

We note the following identities

\[
(\mu_s - \mu_p \lambda)(K^* + \tilde{\Sigma}_Z)^{-1} = (\mu_s - \mu_p \lambda)(K^* + \Sigma_Z)^{-1} + M_S \quad (164)
\]

\[
(\mu_s - \mu_p \lambda)(K^* + \tilde{\Sigma}_Z)^{-1} = (\mu_s - \mu_p \lambda)(K^* + \Sigma_Y)^{-1} + M \quad (165)
\]
where (164) is due to (56), and (165) is obtained by plugging (164) into (52). Since \( M \succeq 0 \), \( M_S \succeq 0 \), (164)-(165) implies

\[
(\mu_s - \mu_p \lambda)(K^* + \tilde{\Sigma}_Z)^{-1} \succeq (\mu_s - \mu_p \lambda)(K^* + \Sigma_Z)^{-1} \tag{166}
\]

\[
(\mu_s - \mu_p \lambda)(K^* + \tilde{\Sigma}_Z)^{-1} \succeq (\mu_s - \mu_p \lambda)(K^* + \Sigma_Y)^{-1} \tag{167}
\]

Using the fact that for \( A \succ 0 \), \( B \succ 0 \), if \( A \preceq B \), then \( A^{-1} \succeq B^{-1} \) in (166)-(167), we can get the first and second parts of Lemma 3. We next show the third part of Lemma 3 as follows

\[
(K^* + \tilde{\Sigma}_Z)^{-1}(S + \tilde{\Sigma}_Z) = I + (K^* + \tilde{\Sigma}_Z)^{-1}(S - K^*) \tag{168}
\]

\[
= I + \left[(K^* + \Sigma_Z)^{-1} + \frac{1}{\mu_s - \mu_p \lambda} M_S\right](S - K^*) \tag{169}
\]

\[
= I + (K^* + \Sigma_Z)^{-1}(S - K^*) \tag{170}
\]

\[
= (K^* + \Sigma_Z)^{-1}(S + \Sigma_Z) \tag{171}
\]

where (169) is due to (164), and (170) comes from (48). The proof is complete.

### D Proof of Lemma 4

We note the following

\[
(\mu_s - \mu_p \lambda)(K^* + \tilde{\Sigma}_Y)^{-1} = (\mu_s - \mu_p \lambda)(K^* + \Sigma_Y)^{-1} + M \tag{172}
\]

\[
(\mu_s - \mu_p \lambda)(K^* + \tilde{\Sigma}_Y)^{-1} = (\mu_s - \mu_p \lambda + \mu_0 \beta)(K^* + \Sigma_Z)^{-1} + \mu_0 \beta(K^* + \Sigma_Y)^{-1} + M_S \tag{173}
\]

where (172) is (83), and (173) comes from plugging (172) into (74). Since \( M \succeq 0 \), (172) implies

\[
(\mu_s - \mu_p \lambda)(K^* + \tilde{\Sigma}_Y)^{-1} \succeq (\mu_s - \mu_p \lambda)(K^* + \Sigma_Y)^{-1} \tag{174}
\]

Using the fact that for \( A \succ 0 \), \( B \succ 0 \), if \( A \preceq B \), then \( A^{-1} \succeq B^{-1} \) in (174) yields the first statement of the lemma. Since \( 0 \leq \beta = 1 - \bar{\beta} \leq 1 \) and \( M_S \succeq 0 \), (173) implies

\[
(\mu_s - \mu_p \lambda)(K^* + \tilde{\Sigma}_Y)^{-1} \succeq (\mu_s - \mu_p \lambda)(K^* + \Sigma_Z)^{-1} \tag{175}
\]
Using the fact that for $A \succ 0$, $B \succ 0$, if $A \preceq B$, then $A^{-1} \succeq B^{-1}$ in (175) yields the second statement of the lemma. We next consider the third statement of this lemma as follows

$$(K^* + \hat{\Sigma}_Y)^{-1} \hat{\Sigma}_Y = I - (K^* + \hat{\Sigma}_Y)^{-1} K^*$$  \hfill (176)

$$= I - \left[ (K^* + \Sigma_Y)^{-1} + \frac{1}{\mu_s - \mu_p \lambda} M \right] K^*$$  \hfill (177)

$$= I - (K^* + \Sigma_Y)^{-1} K^*$$  \hfill (178)

$$= (K^* + \Sigma_Y)^{-1} \Sigma_Y$$  \hfill (179)

where (177) is due to (172) and (178) comes from (17).  

### E Proof of Lemma 6

The optimization problem in (113) can be written as

$$\max_{U \rightarrow V \rightarrow X \rightarrow (Y, Z)} \mu_s R_s + \mu_p R_p + (\mu_p + \beta Z) R_0$$  \hfill (180)

subject to

$$0 \leq R_s \leq I(V; Y|U) - I(V; Z|U)$$

$$R_s + R_p + R_0 \leq I(V; Y|U) + \min \{ I(U; Y), I(U; Z) \}$$

$$R_0 \leq \min \{ I(U; Y), I(U; Z) \}$$  \hfill (181)

For a given $(U, V, X)$, we can rewrite the cost function in (180) as follows

$$\mu_s R_s + \mu_p R_p + (\mu_p + \beta Z) R_0$$

$$\leq \mu_s R_s + \mu_p [ I(V; Y|U) + \min \{ I(U; Y), I(U; Z) \} - R_s - R_0 ] + (\mu_p + \beta Z) R_0$$  \hfill (182)

$$= (\mu_s - \mu_p) R_s + \mu_p [ I(V; Y|U) + \min \{ I(U; Y), I(U; Z) \} ] + \beta Z R_0$$  \hfill (183)

$$\leq (\mu_s - \mu_p) [ I(V; Y|U) - I(V; Z|U) ] + \mu_p [ I(V; Y|U) + \min \{ I(U; Y), I(U; Z) \} ] + \beta Z R_0$$  \hfill (184)

$$= \mu_s [ I(V; Y|U) - I(V; Z|U) ] + \mu_p [ I(V; Z|U) + \min \{ I(U; Y), I(U; Z) \} ] + \beta Z R_0$$  \hfill (185)

$$\leq \mu_s [ I(V; Y|U) - I(V; Z|U) ] + \mu_p [ I(V; Z|U) + \min \{ I(U; Y), I(U; Z) \} ] + \beta Z \min \{ I(U; Y), I(U; Z) \}$$  \hfill (186)

$$= \mu_s [ I(V; Y|U) - I(V; Z|U) ] + \mu_p I(V; Z|U) + (\mu_p + \beta Z) \min \{ I(U; Y), I(U; Z) \}$$  \hfill (187)

where (182) comes from the second constraint in (181), (184) is due to the first constraint in (181) and the assumption $\mu_s > \mu_p$, and (186) comes from the third constraint in (181). The proof can be concluded by noting that the upper bound on the cost function given in (187) is attainable.
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