Anomalous transport phenomena in Fermi liquids with strong magnetic fluctuations

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Abstract

In this paper, we present recent developments in the theory of transport phenomena based on the Fermi liquid theory. In conventional metals, various transport coefficients are scaled according to the quasiparticles relaxation time, $\tau$, which implies that the relaxation time approximation (RTA) holds well. However, such a simple scaling does not hold in many strongly correlated electron systems. The most famous example would be high-$T_c$ superconductors (HTSCs), where almost all the transport coefficients exhibit a significant deviation from the RTA results. This issue has been one of the most significant unresolved problems in HTSCs for a long time. Similar anomalous transport phenomena have been observed in metals near their antiferromagnetic (AF) quantum critical point (QCP). The main goal of this study is to demonstrate whether the anomalous transport phenomena in HTSC is evidence of a non-Fermi liquid ground state, or just RTA violation in strongly correlated Fermi liquids. Another goal is to establish a unified theory of anomalous transport phenomena in metals with strong magnetic fluctuations. For these purposes, we develop a method for calculating various transport coefficients beyond the RTA by employing field theoretical techniques.

In a Fermi liquid, an excited quasiparticle induces other excited quasiparticles by collision, and current due to these excitations is called a current vertex correction (CVC). Landau noticed the existence of CVC first, which is indispensable for calculating transport coefficients in accord with the conservation laws. Here, we develop a transport theory involving resistivity and the Hall coefficient on the basis of the microscopic Fermi liquid theory, by considering the CVC. In nearly AF Fermi liquids, we find that the strong backward scattering due to AF fluctuations induces the CVC with prominent momentum dependence. This feature of the CVC can account for the significant enhancement in the Hall coefficient, magnetoresistance, thermoelectric power, and Nernst coefficient in nearly AF metals. According to the present numerical study, aspects of anomalous transport phenomena in HTSC are explained in a unified way by considering the CVC, without introducing any fitting parameters; this strongly supports the idea that HTSCs are Fermi liquids with strong AF fluctuations. Further, the present theory also explains very similar anomalous transport phenomena occurring in $\text{CeMIn}_5$ ($M = \text{Co} \text{ or Rh}$), which is a heavy-fermion system near the AF-QCP, and in the organic superconductor $\kappa$-(BEDT-TTF).

In addition, the striking $\omega$-dependence of the ac-Hall coefficient and the remarkable effects of impurities on the transport coefficients in HTSCs appear to fit naturally into the present theory. Many aspects of the present theory are in accord with the anomalous transport phenomena in HTSCs, organic superconductors and heavy-fermion systems near their AF-QCPs. We discuss some of the open questions for future work.

(Some figures in this article are in colour only in the electronic version)

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1. Introduction

1.1. Relaxation time approximation (RTA) and current vertex correction (CVC)

The investigation of transport phenomena in metals is very significant since we can extract a large amount of important information with regard to the electronic states of metals. In conventional metals with weak electron–electron correlation, various transport phenomena are governed by a single parameter, namely, the quasiparticle relaxation time $\tau$. That is, the RTA holds well in such conventional metals [1]. For example, resistivity $\rho$ is proportional to $\tau^{-1}$, which is proportional to $T^2$ in conventional Fermi liquids. The Hall coefficient $R_H$ is independent of $\tau$, and $1/|R_H|$ expresses the approximate carrier density $n$. The magnetoresistance $\Delta \rho/\rho_0$ is proportional to $\tau^{-2}$. The thermoelectric power $S$ is proportional to $T$ for a wide range of temperatures. The signs of $R_H$ and $S$ represent the type of carrier (i.e., electrons or holes). The behaviors of these transport coefficients, which are referred to as Fermi liquid behaviors, are well explained by the RTA.

In strongly correlated electron systems, however, transport coefficients frequently exhibit a prominent deviation from conventional Fermi liquid behaviors. Because of this, it is difficult to obtain information on the electronic states from the transport phenomena. For example, in high-Tc superconductors (HTSCs), both $R_H$ and $S/T$ are very sensitive to temperature; in some compounds, sign changes occur with changes in the temperature [2–12]. Evidently, a simple RTA does not work well for HTSCs. For example, $1/|R_H|$ is considerably greater than the electron density in optimally doped or under-doped systems. Prominent ‘non-Fermi-liquid-like behaviors’ are also observed in other strongly correlated systems such as organic superconductors [13–18] and heavy-fermion systems (f-electron systems) [19–22]. Although these behaviors reveal essential information about the electronic states, only a little was understood about them for a long time. To obtain a significant amount of information from this wealth of treasures, we had to develop the theory of transport phenomena in strongly correlated Fermi liquids.

The main objective of this study is to investigate the non-Fermi-liquid-like transport properties in HTSCs, which have been intensively studied as one of the central issues of HTSC. Thus far, two different kinds of theoretical models have been actively investigated: in the first kind of models, non-Fermi-liquid ground states are assumed. For example, Anderson proposed that the elementary excitations are described by spinons and holons. It is considered that holons, whose density is small in under-doped systems, are responsible for the anomalous transport [5, 23]. In the second kind of models, Fermi-liquid ground states are assumed. For example, transport coefficients were calculated based on the nearly antiferromagnetic (AF) Fermi liquid picture [24]. In previous studies, the RTA has been frequently used [25–27]. However, both these models can explain only a limited number of...
experimental facts involved in HTSC: many other anomalous transport properties remain unsolved. To understand the true ground state of HTSC, it is highly desirable to solve the issue of anomalous transport phenomena in HTSCs in a unified way.

The main aim of this study is to explain that the rich variety of anomalous transport phenomena observed in HTSCs can be understood in a unified way in terms of the Fermi liquid picture. For this purpose, we have to take the CVC into account correctly, which is totally dropped in the RTA. In interacting electron systems, an excited electron induces other particle–hole excitations by collisions. The CVC represents the induced current due to these particle–hole excitations. The CVC is closely related to the momentum conservation law, which is mathematically described using the Ward identity [28–31]. In fact, Landau proved the existence of the CVC, which is called backflow in the phenomenological Fermi liquid theory, as a natural consequence of the conservation law [28]. The CVC can be significant in strongly correlated Fermi liquids owing to strong electron–electron scattering. However, its effect on the transport phenomena has not been studied well until recently.

In this study, we discuss the role of CVC in nearly AF Fermi liquids, such as HTSCs, organic metals and heavy-fermion systems near the magnetic quantum critical point (QCP). For example, the RTA for the highly anisotropic $t_2$ model cannot explain the transport anomaly in HTSC. We find that the RTA is unreliable in the presence of strong AF fluctuations, because the prominently developed CVC entirely modifies the RTA results [32–39]. In a Fermi liquid, the transport coefficients are described by the total current $\vec{j}$, which is expressed as a sum of quasiparticle velocity $\vec{v}_k$ and CVC [40–46]. In the present study, we find that the $\vec{j}_k$ in nearly AF metals shows anomalous $k$-dependence due to the CVC. This is the origin of the transport anomalies in nearly AF metals. Based on the microscopic Fermi liquid theory, we investigate the important role of the CVC in $R_H$, $\Delta \rho/\rho_0$, $S$ and the Nernst coefficient ($\nu$). Furthermore, we confirm this idea by performing a numerical study based on the FLEX + CVC theory. In this approximation, the Coulomb interaction $U$ is the only fitting parameter.

To demonstrate the physical meaning of CVC, we discuss the scattering processes between quasiparticles in an isotropic model [28]. In the RTA, it is assumed that the conductivity due to a quasiparticle at $k$ is proportional to the mean free path $v_k \tau_k$. Since $\tau_k \propto T^{-2}$ in a Fermi liquid [28], the resistivity according to the RTA—$\rho^{\text{RTA}} \propto \tau^{-1}$—is finite. However, this result is not true since the momentum conservation law ensures that $\rho = 0$ in a spherical model in the absence of the Umklapp process [42]. The failure of the RTA originates from the assumption that the velocity of the excited quasiparticle disappears after scattering. Here, we explain that the correct answer (zero resistivity) in the absence of the Umklapp processes is recovered by considering all the relevant normal scattering process as shown in figure 1. When a quasiparticle at $k$ is scattered to $k + q$ after the relaxation time $\tau_k$, a particle–hole pair (at $k' - q$ and $k'$) should be created according to the momentum and energy conservation laws. The CVC represents the current conveyed by the particles at ($k + q$, $k' - q$) and a hole at $k'$, which emerge during the scattering process. Therefore, the momentum and energy conservation laws, which are violated in the RTA, are restored by considering the CVC [42]. The CVC is necessary to reproduce the zero resistivity in the absence of the Umklapp process.

In the RTA, we consider an excited quasiparticle at $k$ (in figure 1) as if it annihilates after the relaxation time. Since the RTA allows such an unphysical process, the conservation law $\partial \rho/\partial t + \vec{V} \cdot \vec{j} = 0$ is violated in the RTA. Since the conservation law is a very important constraint on the transport properties, the RTA frequently yields unphysical results [42, 45]. In later sections, we find that the CVC is significant in nearly AF metals: we explain that the CVC is the origin of a variety of anomalous transport phenomena in such metals.

1.2. Non-Fermi-liquid-like transport phenomena in high-$T_c$ cuprates

In HTSCs, almost all the transport phenomena deviate from the conventional Fermi liquid behaviors, which are referred to as non-Fermi-liquid behaviors [47, 48]. These anomalous transport phenomena have been studied intensively as one of the most important issues in HTSCs, since they offer important clues to reveal the true ground state in HTSCs, which has been unsolved until now. To answer this question, many analytical and numerical studies have been performed to determine the ground state of two-dimensional (2D) systems. For example, Anderson et al. considered that the ground state of a square-lattice Hubbard model is the resonating-valence-bond (RVB) state, where spin and charge degrees of freedom are separated [23]. In the RVB state, the concept of quasiparticle is not valid. According to his philosophy, the possibility of the RVB state in the square-lattice $t$–$J$ model was studied, by using the mean-field theory [49] and the gauge theory [50]. In section 1.4, we will introduce significant works performed in the $t$–$J$ model. Unfortunately, it is difficult to perform quantitative studies of transport coefficients using the $t$–$J$ model.

On the other hand, the Fermi liquid theory has been developed and applied to analyzing the HTSCs, considering that the Fermi liquid ground state is realized [51, 52]. By using the self-consistent renormalization (SCR) theory, Moriya,
Figure 2. Schematic phase diagram of HTSC. AF spin fluctuations start to increase below T0. At the same time, R_H starts to increase. Below T* (in the pseudo-gap region), the AF fluctuations are suppressed since the strong SC fluctuations reduce the density of states at the Fermi level, which is called the pseudo-gap. In this study, the filling area is |1 − n| ≥ 0.1.

Takahashi and Ueda explained that the correct superconducting (SC) order parameter d_x^2−y^2 as well as T-linear resistivity are derived from the strong AF fluctuations [53–55]. Based on the phenomenological spin fluctuation model, Monthoux and Pines performed a quantitative analysis for optimally doped YBCO [56]. Further, Bickers et al. studied the square-lattice Hubbard model according to a self-consistent random-phase-approximation (RPA), which is now called the fluctuation-exchange (FLEX) approximation [57]. These spin fluctuation theories have succeeded in reproducing various non-Fermi-liquid behaviors in the normal state of HTSCs: For example, temperature dependence of nuclear spin-lattice relaxation rate given by NMR/NQR measurements, \(1/T_1 T \propto \sum_\delta \text{Im} \chi_\delta^{ss}(\omega)/\omega \propto T^{-1}\), is reproduced well. Moreover, famous T-linear resistivity in HTSC, \(\rho \propto 1/T \propto T\), is also explained. (The Fermi liquid behavior 1/T \propto T^2 will recover at very low T if we suppress the superconductivity.) The FLEX approximation is also useful for the study of electron-doped systems [33, 58, 59]. These spin fluctuation theories successfully reproduce various non-Fermi-liquid-like behaviors in HTSCs except for the under-doped region \(|1 − n| < 0.1\), where \(n\) is the number of electrons per site. A schematic phase diagram of HTSC is shown in figure 2. The Fermi liquid description of HTSCs is still in progress [60–63]. A d_x^2−y^2-wave symmetry in the SC state was confirmed by phase-sensitive measurements [64–66] and tunneling spectroscopy [67]. A d_x^2−y^2-wave SC state was also derived according to the third-order-perturbation theory with respect to U [68].

Regardless of the remarkable success of the Fermi liquid theory during the 1990s, the non-Fermi-liquid-like transport phenomena observed in HTSCs remained unresolved for a long time. The significant deviation from the RTA results in HTSCs was frequently considered as a hallmark of the breakdown of the Fermi liquid state. This issue has remained one of the most important problems in HTSCs for a long time. In HTSCs, \(|R_H|\) increases below \(T_0 \sim 600\,\text{K}\) as

\[
R_H \propto 1/T
\]

above the pseudo-gap temperature \(T^* \sim 200\,\text{K}\), and \(|R_H| \gg 1/ne at T^* [2, 69]\). The sign of \(R_H\) is positive in hole-doped systems such as La_{2−δ}Sr_{2}CuO_{4} (LSCO) [2, 3], YBa_{2}Cu_{3}O_{7−δ} (YBCO) [4, 5] and Ti_{2}Ba_{2}CuO_{6+δ} (TBCO) [6]; however, it is negative in electron-doped systems such as Nd_{2−δ}Ce_{2}CuO_{4} (NCCO) [7] and Pr_{2−δ}Ce_{2}CuO_{4} (PCCO) [8], even though the angle resolved photoemission (ARPES) measurements resolved hole-like Fermi surfaces [70]. The experimental T-dependences of \(R_H\) for LSCO and NCCO are shown in figure 3. The magnetoresistance of HTSCs also shows strong temperature dependence as

\[
\Delta \rho/\rho_0 \propto T^{-4}
\]

for a wide range of temperatures in LSCO [12, 71, 72], YBCO [71] and TBCO [73]. These results completely contradict Kohler’s rule \((R_H \propto \text{const.} and \Delta \rho/\rho_0 \propto \rho_0^{-2})\) which is derived using the RTA for a single-band model. Interestingly, the following 'modified Kohler’s rule' holds well for optimally doped LSCO [12, 71], 90 and 60 K YBCO [71] and TBCO [73]:

\[
\Delta \rho/\rho_0 \propto \tan^2 \Theta_H,
\]

where \(\Theta_H = \sigma_{xy}/\sigma_{xx}\) is the Hall angle. This experimental fact strongly suggests that the anomalous behaviors of the Hall effect and the magnetoresistance have the same origin. Below \(T^*\), \(R_H\) decreases whereas \(\Delta \rho/\rho_0\) increases further [74]. Therefore, modified Kohler’s rule is not very well applicable for under-doped HTSCs at low temperatures [75–77].

For a long time, anomalous transport phenomena have been considered as one of the strongest objections against the Fermi liquid description of HTSCs. For example, to explain equations (1)–(4), Anderson introduced the Tomonaga–Luttinger model with two types of relaxation times. However, it is not obvious how to describe the crossover from the Tomonaga–Luttinger liquid state in the under-doped region to the Landau–Fermi liquid state with doping. On the other hand, the Fermi liquid description for HTSC, starting from the well-established Fermi liquid state in the over-doped region, appears to account for a wide doping range in HTSCs [52]. Here, we conform to the following principle: before abandoning the Fermi liquid picture in HTSCs, we have to verify whether the RTA is really applicable in HTSCs or not. We stress that the RTA may be unreliable for strongly correlated Fermi liquids since the CVC is not considered, regardless of its importance to satisfy the conservation laws. Due to this inadequacy, the RTA sometimes yields unphysical results in correlated metals [42]. In later sections, we will explain that the non-Fermi liquid-like behaviors in HTSCs can be understood based on the Fermi liquid theory by including the CVC.

Here, we discuss the pseudo-gap phenomena in slightly under-doped systems below \(T^* \sim 200\,\text{K}\), which is also one of the most important issues in HTSC. \(1/T_1 T\) starts to decrease below \(T^*\), which means that the AF fluctuations are
suppressed in the pseudo-gap region [78–82]. According to ARPES measurements, prominent deep pseudo-gap appears in the density of states (DOS) below $T^*$ [83–86]. A simple spin fluctuation theory cannot explain the various anomalous phenomena in the pseudo-gap region. Recent theoretical studies using the $T$-matrix theory [87] and the FLEX+$T$-matrix theory [51, 88–90] have shown that the strong SC amplitude fluctuations, which are induced by the AF fluctuations, are a promising candidate for the origin of pseudo-gap [51, 52, 89].

In section 5, we study transport phenomena below $T^*$ using the FLEX + $T$-matrix theory by including CVCs, and show that the various anomalous transport coefficients are well reproduced in a unified way. This study strongly supports the idea that the pseudo-gap phenomena in under-doped HTSCs are induced by the strong SC amplitude fluctuations with a $d_{x^2−y^2}$-symmetry [51, 52, 89].

We note that wide and shallow pseudo-gap in the DOS (weak pseudo-gap) is observed by ARPES even above $T^*$, which is considered to originate from AF fluctuations that appear below $T_0 \sim 600$ K. We will discuss the weak pseudo-gap phenomena in section 9.2.

1.3. Non-Fermi liquid transport phenomena in Ce$M$In$_5$ ($M =$ Co, Rh, or Ir) and $\kappa$-(BEDT-TTF)

During the last decade, it has been found that in strongly correlated materials, including heavy-fermion systems and organic metals, various transport coefficients exhibit striking deviations from the Fermi liquid behaviors. In particular, anomalous transport properties similar to those in HTSCs (equations (1)–(4)) have been observed in many systems with strong magnetic fluctuations such as $\kappa$-(BEDT-TTF)$_2X$ [$X =$ Cu[N(CN)$_2$]Br [13], $X =$ Cu[N(CN)$_2$]Cl [14, 15], $X =$ Cu(NCS)$_2$ [16, 17]], $\kappa$-(BEDT-TTF)$_2$Hg$_{2.85}$Br$_8$ [18] and Ce$M$In$_5$ ($M =$ Co, Rh) [19, 20]. BEDT-TTF is an abbreviation of bis(ethylenedithio)tetrathiafulvalene. These experimental facts strongly suggest that the transport anomaly given by equations (1)–(4) is not a problem specific to HTSCs, but a universal property in the nearly AF Fermi liquids [20]. The study of these transport phenomena in such systems will serve to resolve the origin of the non-Fermi-liquid-like behaviors in HTSCs.

$\kappa$-(BEDT-TTF) is a layered organic compound made of BEDT-TTF molecules. The d-wave superconductivity can be realized in a wide region of the pressure–temperature ($P$–$T$) phase diagram, adjacent to the AF insulating states [91]. For example, Cu[N(CN)$_2$]Cl salt at ambient pressure is an AF insulator, with its Néel temperature $T_N = 27$ K. With increasing pressure, $T_N$ decreases and superconductivity appears via a weak first-order transition; the maximum $T_c$ is 13 K at 200 bar. An effective theoretical model can be given by the anisotropic triangular lattice Hubbard model at half-filling [92]. According to this model, the phase diagram of $\kappa$-(BEDT-TTF) can be well reproduced by using the FLEX approximation, by assuming that $U_{\text{eff}}/W_{\text{band}}$ decreases by pressure [93–95]. These studies revealed that the $d_{x^2−y^2}$-wave superconductivity occurs due to the strong AF fluctuations $q \sim (\pi, \pi)$. In the metallic phase of $\kappa$-(BEDT-TTF), the relationships $R_H \propto T^{-1}$ and $\tan \Theta_H \propto T^{-2}$ are observed [13–18]. Moreover, the magnitude of $R_H$ decreases with increasing pressure [15, 17, 18]. These behaviors are quantitatively reproduced by the FLEX + CVC theory [96].

Ce$M$In$_5$ is a quasi-two-dimensional (2D) heavy-fermion compound [97–99]. At ambient pressure, CeCoIn$_5$ is a superconductor with $T_c = 2.3$ K. The electronic specific heat coefficient $\gamma$ has been measured to be 300 mJ K$^{-2}$mol at $T \gtrsim T_c$. CeIn$_3$ is also a superconductor at ambient pressure with $T_c = 0.4$ K and $\gamma = 680$ mJ K$^{-2}$mol. CeRhIn$_5$ is an AF metal with Néel temperature $T_N = 3.8$ K at ambient pressure. Under the pressure, CeRhIn$_5$ undergoes a SC
transition at $P_c = 2$ GPa, indicating that the AF quantum critical point (QCP) is located at or in the vicinity of $P_c$. The NMR relaxation rate $T_{1}^{-1}$ measurements indicate the presence of quasi-2D AF spin fluctuations in the normal state [100, 101]. The measurements of the angle resolved thermal conductivity [102, 103] and specific heat [104, 105] revealed that the symmetry of the SC exhibits d-wave symmetry. Furthermore, the Fulde–Ferrell–Larkin–Ovchinnikov (FFLO) SC state has been observed [106].

In CeMIn$_5$, equations (1)–(4) are well satisfied for a wide range of temperatures, and the value of $|R_H/n|e$ reaches $\sim 50$ in CeRhIn$_5$ near the AF-QCP [19, 20]. Similarly, the Nernst coefficient $\nu$ also takes a large positive value at low temperatures [107]. Recently, we studied both $R_H$ and $\nu$ in a quasi-2D Hubbard model, and found that they are prominently enhanced, as large as those in pure 2D systems [108]. In CeMIn$_5$, modified Kohler’s rule (4) is well satisfied for $0 < H \lesssim 3T$, whereas both $\sigma_{xy}/H$ and $(\Delta \rho/\rho_0)/H^2$ are drastically suppressed by a very weak magnetic field ($H > 0.1T$) near the AF-QCP [19, 20]. This surprising fact can be understood in terms of the field dependence of the CVC.

1.4. Fermi liquid or non-Fermi liquid?

In this article, we assume that the Fermi liquid state is realized in a wide range of the phase diagram in HTSCs, and the non-Fermi-liquid-like behaviors are created by strong spin (and SC) fluctuations. According to this idea, we study the anomalous transport phenomena in HTSC, by carefully analyzing the main-body effects that had been overlooked in previous studies. In particular, we intensively study the CVC based on the microscopic Fermi liquid theory. Our final aim is to explain the anomalous transport phenomena in various nearly AF Fermi liquids in a unified way, including HTSC, heavy fermions and organic metals.

However, it is a non-trivial question whether the Fermi liquid state is realized in two-dimensional strongly correlated electron systems near the half-filling. In fact, the Fermi liquid state seems to be broken in heavily under-doped HTSCs. In the infinite dimension Hubbard model, the Fermi liquid state with heavy mass is realized next to the Mott insulating state [109, 110]. In 2D Hubbard models [111], it was rigorously proved that the limit value of the interaction $U$, below which the Fermi liquid state is realized, is finite. However, in strongly correlated 2D systems, an exotic non-Fermi liquid ground state may be realized next to the Mott insulating state due to the strong quantum fluctuations. That is, removal of large tracts of the Hilbert space due to strong correlation effects may lead to a violation of the Fermi liquid state.

To study the strong correlation effect in HTSCs, the $t$–$J$ model has been frequently analyzed. The $t$–$J$ model is derived from the Hubbard model (or $d$–$p$ model) by a canonical transformation, by excluding the double occupancy of holes to represent the strong Coulomb interaction. It is given by

$$H^{t-J} = \sum_{\langle i,j \rangle, \sigma} P_G(t_{i,j} c_{i\sigma} c_{j\sigma} + h.c) P_G + J \sum_{\langle i,j \rangle} S_i \cdot S_j,$$

where $P_G$ represents the exclusion of the doubly occupied state, $t_{i,j}$ is the hopping integral between $(i, j)$ sites and $J$ is the superexchange energy between the neighboring spins. $J = 0.10-0.14$ meV in real HTSCs. In the $t$–$J$ model, the Hilbert space with high-energy state is eliminated by $P_G$, which enables us to perform numerical calculations easier or to invent new approximations. Based on the $t$–$J$ model, various versions of the new fluid have been proposed, with novel kinds of excitations, many involving gauge theories of spin-charge separated spinons and holons [23, 49, 50]. The exact diagonalization technique [112, 113] has been applied to the square-lattice $t$–$J$ model. The ground state phase diagram of the $t$–$J$ model has been studied using the variational Monte Carlo method [114, 115], and it was found that the $d_{x^2-y^2}$-wave SC state is realized in a wide range of the phase diagram. Unfortunately, quantum Monte Carlo (QMC) simulation for the $t$–$J$ model is difficult because of a serious negative sign problem. Instead, QMC simulations for the Hubbard model with a moderate value of $U$ [116, 117] have been performed intensively.

On the other hand, many authors have considered non-Fermi liquid ground states due to novel quantum criticalities, other than a conventional (SCR-type) quantum criticality near the spin density wave (SDW) state [61, 118]. For example, Varma et al proposed that the marginal Fermi liquid state is realized in HTSCs, where the $k$-independent self-energy is given by [118]

$$\Sigma(\omega+i\delta) = \lambda(\omega \ln(\omega_c/x) - ix),$$

where $x = \text{max}(|\omega|, \pi T)$, $\lambda$ is a coupling constant and $\omega_c$ is a cutoff energy. This state is not a Fermi liquid since the quasiparticle renormalization factor $z = (1 - \delta \text{Re} \Sigma/\partial \omega)^{-1} = (1 + \lambda \ln(\omega_c/x))^{-1}$ vanishes logarithmically as $(\omega, T) \to 0$. The self-energy in equation (6) can be derived if electrons couple to the following $k$-independent charge and spin fluctuations that are singular at $T = 0$:

$$\text{Im} P(\omega + i\delta) \propto \min(|\omega|/T, 1).$$

In this model, $\rho \propto -\text{Im} \Sigma(i\delta) = \lambda \pi T$, $1/T_1 T \propto \text{Im} P(\omega)/|\omega| = \propto 1/T$, and the optical conductivity is $\sigma(\omega) \propto (\omega - i\lambda)^{-1}$. They are typical non-Fermi liquid behaviors in HTSCs. Now, a microscopic derivation of the $k$-independent quantum critical fluctuations is an important issue. Varma proposed [119] that the circulating current phase exists in the under-doped regime, and the current fluctuations near the QCP ($\approx$ at the optimum doping) are the origin of the marginal Fermi liquid state.

In this article, we will argue that transport anomaly near the AF-QCP can happen even if the Fermi liquid state ($1 \gtrsim z > 0$) remains intact at the QCP. We will show that the CVC in the Landau–Fermi liquid theory causes various striking quantum critical behaviors.

2. Spin fluctuation theory and model Hamiltonian

2.1. Phenomenological spin fluctuation model

Here, we discuss the functional form of the dynamical spin susceptibility $\chi_d^{(2)}(\omega)$ in nearly AF metals. Hereafter, we use...
the unit $c = \hbar = k_B = 1$. This is the most important physical quantity in such metals since it is the origin of various non-Fermi liquid behaviors in HTSCs. The phenomenological form of $\chi_\sigma^s(\omega)$, which can be obtained by using NMR/NQR spectroscopy and the neutron diffraction measurement, is given by [25, 53, 56, 120]

$$\chi_\sigma^s(\omega) = \sum_{\alpha} 1 + \xi_{\text{AF}}^2(q - Q)^2 - i\omega/\omega_0, \quad (8)$$

where $Q = (\pm \pi, \pm \pi)$ is the antiferromagnetic (AF) wavevector, and $\xi_{\text{AF}}$ is the AF correlation length. This is referred to as the Millis–Monien–Pines model [120]. In HTSCs above the pseudo-gap temperature $T^*$, both $\chi_\sigma$ and $1/\omega_0$ are scaled by $\xi_{\text{AF}}^2$ as follows [121]:

$$\xi_{\text{AF}}^2 \approx \alpha_0/(T + \Theta), \quad \chi_0 \approx \alpha_1 \cdot \xi_{\text{AF}}^2, \quad 1/\omega_0 \approx \alpha_2 \cdot \xi_{\text{AF}}^2, \quad (9)$$

where $\Theta, \alpha_0, \alpha_1$ and $\alpha_2$ are constants. Since $\chi_0 \omega_0 \alpha_0 \propto \xi_{\text{AF}}^0$ in equation (10), the corresponding dynamical exponent $z$ is 2. The coefficient $\alpha_0$ rapidly increases in the under-doped region: $\xi_{\text{AF}}$ reaches $\sim 2a$ in optimally doped YBCO, and it exceeds 100 $a$ in slightly under-doped NCCO just above $T_c$ [122] ($a$ is the unit-cell length; we put $a = 1$ hereafter). The relationship $\omega_0 \geq T$ ($\omega_0 \lesssim T$) is satisfied in the over-doped (under-doped) YBCO. On the basis of the phenomenological $\chi_\sigma^s(\omega)$ model, the SC transition temperature was successfully reproduced by solving the strong-coupling Eliashberg equation [56].

Theoretically, the relationships in equations (9) and (10) can be explained according to the SCR theory, where we consider the renormalization of the dynamical susceptibility due to both self-energy correction and vertex correction. This renormalization effect is referred to as the ‘mode–mode coupling effect’ [123], which represents the destruction of the AF long-range order due to thermal and quantum fluctuations. Relationships (9) and (10) are also reproduced according to the FLEX approximation [57, 124–129]. In this approximation, only the mode–mode coupling correction is taken into account: since the large imaginary part of the self-energy reduces the DOS, the spin susceptibility given by the mean-field approximation (which is equivalent to the RPA) is drastically suppressed in the FLEX approximation. As a result, the magnetic long-range order does not occur in 2D systems, which means that the Mermin–Wagner theorem is satisfied in the FLEX approximation [130].

In nearly AF metals, the FLEX approximation and the SCR theory yield similar results since the mode–mode coupling correction due to the self-energy is dominant [123]. On the other hand, the mode–mode coupling correction due to the second-order vertex correction (Aslamazov–Larkin term) can be important in the nearly ferromagnetic metals.

### 2.2. Model Hamiltonian and FLEX approximation

In the main part of this study, we investigate the transport phenomena in HTSCs based on the Fermi liquid theory, and perform a numerical study using the spin fluctuation theory. In this study, we analyze the following Hubbard model:

$$H = \sum_{k\sigma} e_k^0 c_{k\sigma}^{\dagger} c_{k\sigma} + U \sum_{kk'q} \delta_\sigma \delta_\tau c_{k+q\sigma}^{\dagger} c_{k'q\tau}^{\dagger} c_{k'\tau} c_{k\sigma}, \quad (11)$$

where $U$ is the Coulomb interaction, and $e_k^0$ is the spectrum of the conduction electron. In a square lattice, $e_k^0$ is given by

$$e_k^0 = 2t_0(cos k_x + cos k_y) + 4t_1 cos k_x cos k_y + 2t_2(cos 2k_x + 2k_y). \quad (12)$$

where $c_{k\sigma}^{\dagger}$ is the creation operator of an electron with momentum $k$ and spin $\sigma$, and $U$ is the on-site Coulomb repulsion. We represent the electron filling by $n$, and $n = 1$ corresponds to the half-filling. To fit the band structures given by the local density approximation (LDA) band calculations for YBCO [131], NCCO [132] and LSCO [133, 134] and by the angle resolved photoemission (ARPES) experiments for YBCO [135], NCCO [70] and LSCO [136], we select the following set of parameters [137, 33]. (I) YBCO (hole-doping) and NCCO (electron-doping): $t_0 = -1$, $t_1 = 1/6$ and $t_2 = -1/5$. (II) LSCO (hole-doping): $t_0 = 1$, $t_1 = 1/10$ and $t_2 = -1/10$. Here, $n$ is smaller (larger) than unity in YBCO and LSCO (NCCO). The Fermi surfaces for YBCO and NCCO without interaction are shown in figure 4(i). The deformation of the Fermi surface in the presence of $U$ has been discussed in [33]. Note that the Fermi surface in Bi$_2$Sr$_2$CaCu$_2$O$_8$ (BSCO) is similar to that in YBCO [138]. Since $|\epsilon_0| \sim 400$K in HTSCs, $T > 0.1$ in this study corresponds to 400K.

As a result, the magnetic long-range order does not occur in 2D systems, which means that the Mermin–Wagner theorem is satisfied in the FLEX approximation [130]. In nearly AF metals, the FLEX approximation and the SCR theory yield similar results since the mode–mode coupling correction due to the self-energy is dominant [123]. On the other hand, the mode–mode coupling correction due to the second-order vertex correction (Aslamazov–Larkin term) can be important in the nearly ferromagnetic metals.
Figure 4. (i) The Fermi surfaces for YBCO \((n < 1)\) and NCCO \((n > 1)\). The location of the hot spots and the cold spots are shown. According to the FLEX approximation, the hot spot in YBCO shifts to point B, by reflecting the large DOS at \((\pi, 0)\). (ii) Diagrammatic representation of the self-energy in the one-loop (FLEX) approximation.

The FLEX approximation is in good agreement with the results obtained from the quantum Monte Carlo simulations for a moderate value of \(U\) [57]. It has also been applied to the SC ladder compound, \(\text{Sr}_{14-\delta}\text{Ca}_\delta\text{Cu}_{23}\text{O}_{41}\) [127], and the organic SC \(\kappa\)-(BEDT-TTF) compounds [93–95, 141]. Using the FLEX approximation, \(T_c\) was studied in various types of tight-binding models [142]. The FLEX approximation predicts that the AF fluctuations are predominant in a square-lattice Hubbard model near half-filling. This result has been confirmed by renormalization group analyses, which considers both spin and charge fluctuations on the same footing by solving the parquet equation [143–145].

The FLEX approximation cannot reproduce the pseudogap behaviors below \(T^* \sim 200\,\text{K}\) in slightly under-doped systems. However, they are well explained by the FLEX + \(T\)-matrix approximation [51, 88, 89], where the self-energy correction due to strong SC fluctuations, which are induced by AF fluctuations, are considered self-consistently. We will explain about the FLEX+\(T\)-matrix approximation in section 5. In this study, we perform numerical studies from the over-doped region to the slightly under-doped region (i.e. \(n \leq 0.9\) or \(n \geq 1.1\)), where the FLEX+(\(T\)-matrix) approximation yields reasonable results. Note that the FLEX approximation is inappropriate to describe the ‘Mott physics’ in the heavily under-doped region.

Figure 5 shows the spin susceptibility \(\chi^s_\omega(q, 0)\) given by the FLEX approximation, both for YBCO \((n = 0.85; \text{optimum doping})\) at \(T = 0.02\) and for NCCO \((n = 1.20, \text{slightly over-doping})\) at \(T = 0.04\), respectively [33]. Since the nesting of the Fermi surface is not good as shown in figure 4, \(\xi_{\text{AF}} \propto \sqrt{\Delta}\) is moderate even at low temperatures. In optimum YBCO, the Stoner factor \(U\chi^0_\omega(0)\sim 0.98\) at \(T = 0.02\), and the AF correlation length \(\xi_{\text{AF}}\) is approximately 2–3\(\alpha\) (\(\alpha\) denotes the lattice spacing). On the other hand, \(\xi_{\text{AF}}\) in NCCO \((n = 1.20)\) exceeds 10\(\alpha\) at \(T = 0.02\). It should be noted that \(\chi^s_\omega(q, 0)\) given by the RPA shows an incommensurate structure, which is inconsistent with the experimental results. I verified that \(1/T_c\) of the Cu nuclei under \(H \perp \hat{c}\) given by the FLEX approximation for \(n = 0.85\) is \(1–2\,\text{K}^{-1}\,\text{ms}^{-1}\) at 200 K, which is consistent with the experimental results [146].

One of the advantages of the FLEX approximation is that the Mermin–Wagner theorem with respect to the magnetic instability is satisfied: its analytic proof is given in the appendix A of [130]. Hence, the critical region \((U\chi^0_\omega(0) \gtrsim 0.99)\) is stable in 2D systems since the SDW order \((U\chi^0_\omega(0) = 1)\) is...
Figure 6. Bethe–Salpeter equation for two-particle Green function $L(k, k')$. $\Gamma^{\prime}(k, k')$ is the irreducible four-point vertex with respect to the particle–hole channel $G_{k{\rightarrow}q}G_{k{\rightarrow}q}$.

Figure 7. The $k$-dependence of $\gamma_k$ on the Fermi surface at various temperatures. The cold spot in YBCO (NCCO) is point A (B).

2.3. Hot/cold-spot structure and $T$-linear resistivity in nearly AF metals

As we have explained, important advances in HTSCs have been achieved by using the Fermi liquid theory with strong AF fluctuations [24, 25, 52, 54, 56, 57]. One of the most important predictions given by these theories is the “hot/cold-spot structure” of the quasiparticle damping rate, $\gamma_k = \text{Im} \Sigma_k(-i\delta)$ ($\delta_k = 1/2\gamma_k$). That is, $\gamma_k$ becomes anisotropic in the presence of AF fluctuations. This fact is very important to understand the transport phenomena in HTSCs. The portions of the Fermi surface at which $\gamma_k$ takes the maximum and the minimum values are referred to as hot spots and cold spots, respectively [25, 24]. According to the spin fluctuation theory, the hot spots usually exist around the crossing points with the AF Brillouin zone (AFBZ)-boundary, whereas the cold spot is at the points where the distance from the AFBZ-boundary is the largest. Their positions are shown in figure 4 (i). The electronic states around the cold spots play the major role for various transport phenomena. Note that the hot spot in YBCO shifts to the Brillouin zone boundary (point B), by reflecting the large DOS at $(\pi, 0)$.

The critical value of $U$ for a spin density wave (SDW) transition in the RPA (i.e. the mean-field approximation) is $U_{\text{RTA}}^{\text{RPA}} \sim 2.3$ in LSCO ($n = 0.9$). In YBCO and NCCO, $U_{\text{RTA}}^{\text{RPA}}$ takes a much larger value, $U_{\text{RTA}}^{\text{RPA}} \sim 3.5$ for both YBCO ($n = 0.9$) and NCCO ($n = 1.1$), since the nesting is not good due to the large next-nearest and third-nearest hopping integrals ($t_1$ and $t_2$). Here, we consider putting the same $U$ for both YBCO and NCCO. In YBCO, the dimensionless coupling constant $UN(0)$ is large, since the saddle point is close to the Fermi level. Here, $N(0)$ is the DOS at the Fermi level. Therefore, $\text{Im} \Sigma_k(0)$ takes a large value, which significantly reduces the interacting DOS and $\chi_\alpha^2(0)$ at $q = (\pi, \pi)$ [33]. In fact, the obtained $N_{\text{FLEX}}(0)/N_{U<\alpha}(0)$ is considerably smaller than unity in YBCO. On the other hand, the reduction in $\chi_\alpha^2(0)$ prevented by the Mermin–Wagner theorem. That is, $U_{\text{RTA}} = \infty$ in 2D systems. As a result, the $U$ dependence on $R_{\text{H}}$ (and $\chi_\alpha^2$) given by the FLEX $+$ CVC approximation is rather moderate, as shown in [33, 96].

In the next stage, we derive the two-particle Green function $L(k, k')$ which is indispensable for the study of transport phenomena. According to the microscopic Fermi liquid theory [30, 31], $L(k, k')$ can be obtained by the solution of the following Bethe–Salpeter equation:

$$L(k, k; q) = -G_{k{\rightarrow}q}G_{k{\rightarrow}k_1}T - G_{k{\rightarrow}q}G_{k_1{\rightarrow}k_2}T - T \sum_{k''}G_{k{\rightarrow}q}G_{k''{\rightarrow}k_2}\Gamma^{\prime}(k, k''; q)$$

$$\times G_{k''{\rightarrow}q}G_{k_2{\rightarrow}k}L(k'', k'; q),$$  

(18)

where $k = (k, \epsilon_k)$ and $\Gamma^{\prime}(k, k'; q)$ is the full four-point vertex. $\Gamma^{\prime}(k, k'; q)$ is the irreducible four-point vertex, which is given by the Fourier transformation of the Ward identity in real space; $\Gamma^{\prime} = \delta \Sigma/\delta G$. The Bethe–Salpeter equation is expressed by figure 6. In later sections, we show that various linear transport coefficients are described in terms of $L$ (e.g. see equation (51)). In the conserving approximation, transport coefficient obtained by $L(k, k')$ in equation (18) automatically satisfies conservation laws [139, 140]. This is the reason why we refer to it as the conserving approximation. This is a great advantage of the FLEX approximation for the study of transport phenomena. In the FLEX approximation, irreducible four-point vertex will be given in equation (89) in section 4.1.
due to $\text{Im} \Sigma_k(0)$ is small in NCCO since the saddle point is far below the Fermi level. For this reason, $\xi_{\text{AF}}$ for NCCO is much larger than that for YBCO in the FLEX approximation. This result is consistent with experiments.

The real-frequency Green function $G_k(\epsilon)$ is given by the analytic continuation of $G_k(\epsilon_{\sigma})$ in equation (13). In a Fermi liquid, the advanced Green function $G_k^a(\epsilon) = G_k(\epsilon - i\delta)$ in the vicinity of $\epsilon \sim 0$ and $|k| \sim k_F$ can be represented as

$$
G_k^a(\epsilon) = z_k/(\epsilon - E_k^* - i\gamma_k),
$$

(19)

$$
E_k = \epsilon_k^0 + \Sigma_k(\epsilon) - \mu,
$$

(20)

$$
\gamma_k = \text{Im}\,\Sigma_k^+(\epsilon), \quad \gamma_k^* = \bar{z}_k \gamma_k,
$$

(21)

where $z_k$ is the renormalization factor given by $z_k = 1/(1 - \delta\text{Re} \Sigma_k(\epsilon)/\delta\epsilon)_{\mu=0}$, and $E_k^*$ is the renormalized quasiparticle spectrum, which is the solution of $\text{Re} \, G_k^{-1}(E_k^*) = 0$. The quasiparticle weight is given by

$$
\rho_k(\epsilon) = \frac{1}{\pi} \text{Im} \, G_k^a(\epsilon),
$$

(22)

The DOS is expressed as $N(\epsilon) = \sum_k \rho_k(\epsilon)$. In the case of $z_k \gamma_k \ll \mu$,

$$
\rho_k(\epsilon) = z_k \delta(\epsilon - E_k^*),
$$

(23)

for $\epsilon \approx 0$.

In the FLEX approximation, $\gamma_k$ is obtained by the analytic continuation of equation (14):

$$
\gamma_k = \frac{1}{2} \sum_q \int \text{d} \epsilon \left[ \text{cth} \frac{\epsilon}{2T} - \text{th} \frac{\epsilon}{2T} \right] \text{Im} V_q(\epsilon + i\delta) \rho_k q \epsilon(\epsilon),
$$

(24)

where $V_q(\omega + i\delta)$ is given by the analytic continuation of equation (15). In the spin fluctuation model (Millis–Monien–Pines model in equation (8)), $\text{Im} V_q(0)$ in equation (24) is replaced with $(3U^2/2)\text{Im} \chi_q^0(0) = (3U^2/2)\omega \chi_q(\omega^2 + \omega^2)$, where $\omega_k = \omega_0 + \omega_0 \xi_{\text{AF}}^2 q - Q^2$. According to [25, 33], equation (24) is approximately transformed to

$$
\gamma_k \approx \frac{3U^2}{4\pi} \frac{\sum_{q} k_q^d_j}{v} \chi_q^0 \omega \omega_0 \delta \left( \omega_0 - k - \pi T/2 \right).
$$

(25)

According to equation (25), $\gamma_{\text{hot}}(\gamma_k)$ at the hot spot in 2D systems is given by [25]:

$$
\gamma_{\text{hot}} \propto T \xi_{\text{AF}} \quad \text{for} \quad T/2 \omega_{\text{AF}} \gg 1
$$

(26)

$$
\gamma_{\text{hot}} \propto T^2 \xi_{\text{AF}}^2 \quad \text{for} \quad T/2 \omega_{\text{AF}} \ll 1.
$$

(27)

Since $\xi_{\text{AF}}^2 \propto T^{-1}$, $\gamma_{\text{hot}} \propto \sqrt{T}$ for any value of $\omega_{\text{AF}}/T$: this result is recognized in the numerical study in figure 7. Also, $\gamma_{\text{cold}}$ in 2D systems is obtained as [25]

$$
\gamma_{\text{cold}} \propto T \quad \text{for} \quad T/2 \omega_{\text{AF}} \sim (\xi_{\text{AF}} \Delta k_c)^2
$$

(28)

$$
\gamma_{\text{cold}} \propto T^2 \quad \text{for} \quad T/2 \omega_{\text{AF}} \ll (\xi_{\text{AF}} \Delta k_c)^2.
$$

(29)

According to equations (9) and (10), $\pi T/2 \omega_{\text{AF}}$ is constant if $\theta \approx 0$, and it is of the order of $O(1)$ in optimally doped HTSCs. Therefore, $\gamma_{\text{cold}} \propto T$ when $\xi_{\text{AF}} \Delta k_c \sim O(1)$, and $\gamma_{\text{cold}} \propto T^2$ when $\xi_{\text{AF}} \Delta k_c \gg 1$. (Here, $\Delta k_c$ represents the deviation from the nesting condition at the cold spot; see figure 4.) We comment that almost the entire Fermi surface becomes the cold spot in the case of $\xi_{\text{AF}} \Delta k_c \gg 1$.

Now, we discuss the temperature dependence of resistivity according to the SCR theory [54], by dropping the CVC. In the SCR theory, the resistivity is derived from the Born approximation $\rho \propto \langle \gamma_k \rangle_{\text{FS}} = \sum_k \gamma_k \rho_k(0)$. In the case of $\omega_{\text{AF}} \gg T$, equation (25) is simplified as

$$
\gamma_k \approx \sum_q \frac{(\pi T)^2}{2} \text{Im} \frac{\hat{V}_q(0)\rho_k(0) q \epsilon(\epsilon)}{\langle q \rangle_{\text{FS}}}
$$

(30)

where $\hat{V}_q(0) = dV_q(\omega + i\delta)/d\omega|_{\omega=0}$. It can also be derived directly from equation (24) by using the relation $\int d\epsilon\text{cth}(\epsilon/2T) - \text{th}\epsilon(2T) = (\pi T)^2$. Using equations (8) and (30), the temperature dependence of the resistivity is given by [54, 147]

$$
\rho_{\text{SCR}} \propto \langle \gamma_k \rangle_{\text{FS}} \propto T^{2-d'/d} \xi_{\text{AF}}^2
$$

(31)

where $d$ is the dimension of the system. In deriving equation (31), we utilized the fact that the $q$-dependence of $\text{Im} \chi_q^0(0) = (\pi/2) \sum_q \rho_k(0) q \epsilon(\epsilon)\rho_k(0)$ is moderate. Since $\xi_{\text{AF}} \propto T^{-0.5}$ near the AF-QCP [54], $\rho_{\text{SCR}}$ is proportional to $T^{d/2}$ ($d = 2, 3$). We stress that the Fermi liquid behavior $\rho_{\text{SCR}} \propto T^2$ is recovered when $\xi_{\text{AF}}$ constant away from the AF-QCP, like in under-doped systems at low temperatures.

In section 4.2, we will calculate the resistivity using the FLEX approximation (and FLEX + $T$-matrix approximation), based on the linear response theory. In figure 18, ‘FLEX’ represents the resistivity obtained by dropping the CVC, and ‘FLEX + CVC’ represents $\rho$ given by the FLEX + CVC approximation; the latter gives the correct result. Consistently with equation (31), the obtained $\rho$ follows an approximate $T$-linear behavior in under-doped LSCO and NCCO, and it shows a $T^2$-like behavior in the over-doped NCCO. As for the resistivity, the CVC is quantitatively important. In later sections, we explain that the CVC completely changes the $T$-dependence of $R_{\text{H}}, \Delta \rho/\rho_0$ and $\nu$.

However, the Born approximation (equation (31)) gives overestimated values when $\gamma_k$ is highly anisotropic, as pointed out by [24, 148]: According to the linear response theory, the correct resistivity is given by

$$
\rho \propto \frac{1}{\langle \gamma_k \rangle_{\text{FS}}} \sim \gamma_{\text{cold}}.
$$

(32)

According to equations (28) and (29), $\rho$ shows the $T$-linear behavior in the case of $\xi_{\text{AF}} \Delta k_c \sim O(1)$, whereas $\rho \propto T^2$ in the case of $\xi_{\text{AF}} \Delta k_c \gg 1$.

In optimally doped or slightly under-doped YBCO and LSCO, $\rho$ shows an approximate $T$-linear behavior, which means that the relation $\xi_{\text{AF}} \Delta k_c \sim O(1)$ is satisfied in these compounds above $T_c$. In fact, if $\xi_{\text{AF}} \Delta k_c \gg 1$, the dominant part of the Fermi surface should be the cold spot. However, this
result is inconsistent with the ARPES measurements [86]. In electron-doped systems, on the other hand, $\xi_{AF}\Delta k_c \gg 1$ seems to be realized at low temperatures even in optimally doped systems, since $\xi_{AF}$ reaches 100 just above $T_c$ in NCCO [122]. The relationship $\xi_{AF}\Delta k_c \gg 1$ is also satisfied in the present numerical study. In this case, $\gamma_k$ becomes highly anisotropic and $\rho \sim \gamma_{old} \sim T^2$ is realized [25].

According to equation (32), the relationship $\rho \propto T^2$ holds in the close vicinity of the AF-QCP; $\xi_{AF}\Delta k_c \gg 1$. However, Rosch [148] pointed out that the relationship $\rho = a + bT$ ($a, b > 0$) holds even at the AF-QCP when the quasiparticle damping rate due to impurities $\gamma_{imp}$ is finite: in fact, at sufficiently low temperatures where elastic scattering is dominant ($\gamma_{imp} \gg \gamma_k$), equation (32) becomes

$$\rho \propto 1/((\gamma_k + \gamma_{imp})^{-1})_{FS} \propto \gamma_{imp} + (\gamma_k)_{FS},$$

Therefore, according to equation (31), $\rho - \rho_0 \propto T^{d/2}$ ($d = 2, 3$) holds near the AF-QCP in the case of $\gamma_{imp} \gg \gamma_k$.

Here, we explained the $T$-linear resistivity in HTSCs based on the spin fluctuation theory. There are other theories which account for the $T$-linear resistivity. As discussed in section 1.4, the marginal Fermi liquid liquid hypothesis was proposed by Varma et al [118, 119]. Also, the Tomonaga-Luttinger model with two types of relaxation times was proposed by Anderson [23]. Unfortunately, based on these theories, it is difficult to calculate other transport coefficients such as $R_{qK}$ and $\Delta\rho/\rho_0$. In later sections, we will explain various anomalous transport phenomena in HTSCs in a unified way based on the Fermi liquid theory.

3. Anomalous transport phenomena in nearly AF Fermi liquids

3.1. Results by RTA

Before investigating the transport coefficients using the microscopic Fermi liquid theory, we briefly review the results of the relaxation time approximation (RTA) based on the Bloch–Boltzmann theory [1]. The CVC provides correction for the RTA. In later sections, we will see that the CVC becomes crucial near the AF-QCP. The Boltzmann equation in a non-equilibrium steady state is expressed as [1]

$$\frac{\partial f_k}{\partial t}_{\text{scatt}} = -e(E + \mathbf{v}_k \times \mathbf{H}) \cdot \mathbf{V}_k f_k,$$

where we put $c = \hbar = 1$, and $-e$ ($e > 0$) is the charge of an electron. $f_k$ is the distribution function in a non-equilibrium steady state, and $E$ and $H$ are the electric and magnetic fields, respectively. $(\partial f_k/\partial t)_{\text{scatt}}$ represents the rate of change in $f_k$ due to scattering between quasiparticles, which is called the collision integral. Using the scattering amplitude $I(k, k'; q)$ for $(k, k') \leftrightarrow (k + q, k' - q)$, it is given by [1]

$$\frac{\partial f_k}{\partial t}_{\text{scatt}} = -\sum_{k'q} I(k, k'; q)f_k f_{k'}(1 - f_{k+q})(1 - f_{k'-q})$$

$$- f_k(1 - f_k) f_{k+q} f_{k'-q}$$

(35)

where the first (second) term represents the outgoing (incoming) scattering process. To derive the conductivity, we have to linearize the Boltzmann equations (34) and (35) with respect to $E$.

In solving these equations, we frequently apply the RTA: $(\partial f_k/\partial t)_{\text{scatt}} = -g_k / \tau_k$, where $g_k = f_k - f_k^0$ and $f_k^0 = (e^{(g_k - \mu)/k_b T} + 1)^{-1}$ is the equilibrium distribution function. Then, the linearized Boltzmann equation is simplified as

$$\frac{g_k}{\tau_k} = -eE \cdot \mathbf{v} \left( -\frac{\partial f^0}{\partial \epsilon_k} \right) + e(\mathbf{v}_k \times \mathbf{H}) \cdot \mathbf{V}_k g_k.$$

(36)

The solution of equation (36) is given by

$$g_k = -(1 - e\tau_k(\mathbf{v}_k \times \mathbf{H})^{-1})eE \cdot \mathbf{v}_k \left( -\frac{\partial f^0}{\partial \epsilon_k} \right).$$

(37)

Although it is a crude approximation, the RTA can successfully explain the various transport phenomena in metals with weak correlation.

In the RTA, the conductivity is given by $\sigma_{\mu\nu}^{\text{RTA}} = -2e\sum_{\nu} v_{\mu\nu} g_\nu / \epsilon_k$, where the factor 2 is attributed to the spin degeneracy. When the $k$-dependence of $\tau_k$ is moderate, the functional form of $\sigma_{\mu\nu}$ under the magnetic field is $\sigma_{\mu\nu} = \tau F_{\mu\nu}(r, H)$. Here, we assume $H \parallel \hat{z}$. Then, $(\mathbf{v} \times \mathbf{H}) \cdot \mathbf{V}_k = -H_s (\mathbf{v}_k \times \mathbf{V}_k)$. In the RTA, the longitudinal conductivity, Hall conductivity, and magnetococonductivity are given by

$$\sigma_{xx}^{\text{RTA}} = 2e^2 \sum_k \left( -\frac{d f^0}{d \epsilon_k} \right) v_{kx} \cdot \tau_k v_{ky},$$

$$\sigma_{xy}^{\text{RTA}} = -2e^2 H_z \sum_k \left( -\frac{d f^0}{d \epsilon_k} \right) v_{kx} (\tau_k \mathbf{v}_k \times \mathbf{V}_k) \cdot v_{ky},$$

$$\Delta \sigma_{xx}^{\text{RTA}} = 2e^2 H_z^2 \sum_k \left( -\frac{d f^0}{d \epsilon_k} \right) v_{kx} (\tau_k \mathbf{v}_k \times \mathbf{V}_k)^2 v_{ky},$$

where $(\tau_k \mathbf{v}_k \times \mathbf{V}_k) = \tau_k (v_{1x} \delta_{j} - v_{1y} \delta_{y})$. When the $k$-dependence of $\tau_k$ is moderate,

$$\sigma_{xx} \propto \tau, \quad \sigma_{xy} \propto \tau^2 H_z, \quad \Delta \sigma_{xx}^{\text{RTA}} \propto \tau^3 H_z^2,$$

(41)

which is called the Kohler’s rule.

Based on the RTA, Hussey et al studied transport coefficients in heavily over-doped Ti2201 [149] and LSCO [150]. They determined the anisotropy of $\tau_k$ in Ti2201 by measuring the angle-dependent magnetoresistance oscillation (AMRO) along the $c$-axis [151], and calculated transport coefficients using equations (38)–(40). The derived $R_{qK}^{\text{RTA}}$ agrees with the experimental Hall coefficient $R_{qK}^{\text{exp}}$ in heavily over-doped samples ($T_c \approx 10$ K), where the Hall coefficient is small and its temperature dependence is tiny. As the doping decreases, $R_{qK}^{\text{RTA}}$ increases quickly and its temperature dependence becomes prominent, whereas the doping dependence of $R_{qK}^{\text{RTA}}$ is more moderate.

In optimally doped HTSCs, however, the RTA does not work: Stojkovic and Pines [25] attempted to explain the violation of Kohler’s rule in HTSCs based on the RTA. They assumed the highly anisotropic $\tau_k$ model (hot/cold-spot model), where only the quasiparticles near the cold spot...
contributes toward the transport phenomena. In this model, \( R_H \) can take a large value since the 'effective carrier density for the transport phenomena' is reduced. They calculated \( t_k \) based on the Millis–Monien–Pines model; the anisotropy of \( t_k \) reaches 100 in the optimally doped YBCO, which is too large to be consistent with the ARPES measurements. The reason for this overestimation is that self-consistency is not imposed in their calculations. In spite of the large anisotropy of \( t_k \), the obtained enhancement ratio of \( R_H \) is about two according to [27]. Therefore, this scenario cannot account for the large \( R_H \) in under-doped systems. It should be stressed that the magnetoresistance becomes 100 times greater than the experimental value when the anisotropy of \( t_k \) is of the order of 100 [26]. Therefore, the highly anisotropic \( t_k \) model is not applicable for optimally-doped HTSCs. In the next section, we explain the various anomalous transport phenomena in HTSCs 'all together', by considering the CVC.

3.2. Physical meaning of the CVC in nearly AF metals

In the RTA where \( \tilde{g}_k = -g_k/t_k \) is assumed, the deviation from the equilibrium distribution function \( g_k \) dissipates with time since \( g_k \propto e^{-t_k} \). This oversimplification frequently leads to serious unphysical results. For example, in the absence of an electric field, the RTA predicts that \( g_k \) always vanishes when \( t \to \infty \) since \( t_k > 0 \) at finite temperatures. However, \( g_k \) should remain finite when the Umklapp scattering process is absent, because of the momentum conservation laws [1, 42]. Therefore, the RTA violates the conservation laws. To satisfy the conservation laws, the incoming scattering from the other states to \( k \) has to be taken into account by solving equation (35). This scattering process is represented by the CVC in the microscopic Fermi liquid theory. Here, we intuitively discuss the CVC in nearly AF metals, where the quasiparticles are scattered by strong AF fluctuations with \( q \sim \mathbf{Q} = (\pi, \pi) \). Then, the momentum transfer \( q \) in the scattering process is restricted to \( \sim \mathbf{Q} \), as shown in figure 8. According to the momentum conservation law, the quasiparticle at \( k' \) is scattered to \( k' - q \) as shown in figure 8 (i). The current due to the quasiparticle at \( k' - q \) and the hole at \( k' \) is given by \( \tilde{v}_{k' - q} - \tilde{v}_k \). This current almost cancels after performing the \( k' \)-summation: In fact, the current of the particle–hole pair \( (k'' - q, k') \) for \( k'' = -k' \), which is shown in figure 8 (ii), is \( \tilde{v}_{-k' - q} - \tilde{v}_{-k} \approx -\tilde{v}_{k' - q} + \tilde{v}_k \) since \( 2q \approx 2\mathbf{Q} \) is a reciprocal lattice vector. Therefore, the CVC is given only by the quasiparticle at \( k + q \); \( \tilde{v}_{k+q} \). In this case, the conductivity does not diverge because of the existence of the Umklapp processes, e.g. the process in figure 8(ii).

In nearly AF metals, the total current \( \tilde{j}_k \) is approximately parallel to \( \tilde{v}_{k+q} + \tilde{v}_{k+Q} \), which implies that \( \tilde{j}_k \) is not perpendicular to the Fermi surface. This fact is the origin of the enhancement of \( R_H \) [33]. The anomalous \( k \)-dependence of \( \tilde{j}_k \) becomes much more prominent near the AF-QCP due to the multiple backscattering of the quasiparticle between that at \( k \) and that at \( k + Q \). (The schematic behavior of \( \tilde{j}_k \) is shown in figure 14.) Thus far, we have discussed only the two-body scattering process and ignored the higher-order processes. This simplicity is justified in good metals where \( \gamma/E_F \ll 1 \), which is satisfied in HTSCs except for heavily under-doped compounds [28].

3.3. Analysis of the CVC based on the Fermi liquid theory

Here, we calculate the CVC based on the microscopic Fermi liquid theory. In principle, we can also calculate the CVC based on the Bloch–Boltzmann theory, by analyzing equation (35) using the variational principle [1]. However, a systematic calculation of the CVC is very difficult, particularly in the presence of a magnetic field. Therefore, we analyze the CVC based on the linear response theory since we can utilize the powerful field theoretical techniques. In the linear response theory [152–154], the conductivity is given by

\[
\sigma_{\mu\nu}(\omega) = \frac{1}{i\omega} [K_{\mu\nu}^R(\omega) - K_{\mu\nu}^R(0)],
\]

where \( K_{\mu\nu}^R(\omega) \) is the retarded current–current correlation function.

Here, we consider the conductivities in the presence of the uniform magnetic field. Then, the hopping integral between sites \( i \) and \( j \) exhibits the Peierls phase:

\[
t_{m,j} = t^0_{m,j} \exp[-i\epsilon(A_m + A_j) \cdot (r_m - r_j)/2],
\]

which is the expression for the CVC (MT).
where $t_{m,j}$ is the original hopping integral, and $A_m$ is the vector potential at site $m$. Then, the Hamiltonian and the velocity operator under the magnetic field, $H_A$ and $j^A_{\mu}$ respectively, are given by

$$H_A = \sum_{(m,j),\sigma} t_{m,j} c_{m\sigma}^\dagger c_{j\sigma} + \frac{U}{2} \sum_m n_{m\uparrow} n_{m\downarrow},$$

$$j^A_{\mu}(r_m) = \sum_\sigma i[H_A, r_m c_{m\sigma}^\dagger c_{\sigma}].$$

Here, we assume that the vector potential is given by $A_j = Ae^{i\mathbf{q} \cdot \mathbf{r}_j}$, and take the limit $\mathbf{q} \to 0$ at the final stage of the calculation [44]. (The magnetic field is $H = i\mathbf{q} \times \mathbf{A}$. Then, equation (43) can be expanded as $t_{m,j} = t_{m,j}^0, [1-i(e/2)A \cdot (\mathbf{r}_m - \mathbf{r}_j)](e^{\mathbf{q} \cdot \mathbf{r}_j} + e^{\mathbf{q} \cdot \mathbf{r}_j}) + O(A^2)$. After the Fourier transformation, both the Hamiltonian and the velocity operator of the order of $O(A)$ are given by [38]

$$H_A = H - A \cdot j(-q) + O(A^2),$$

$$j^A_{\mu}(q') = j_{\mu}(q') - \sum_{\alpha} A_\alpha j_{\mu\alpha}(q' - q) + O(A^2),$$

where $j_{\mu}(q) = -e \sum_{\alpha} \partial_\alpha e_{\mu \nu}^i (c_{k_{\nu}q_{\alpha}/2} - c_{k_{\nu}q_{\alpha}/2}^\dagger)$ and $j_{\mu\alpha}(q) = e^2 \sum_{\alpha} \partial_{\alpha\mu} e_{\mu \nu}^i (c_{k_{\nu}q_{\alpha}/2} - c_{k_{\nu}q_{\alpha}/2}^\dagger)$. Here, $\partial_\alpha = \partial_j / \partial q_j$, $\partial_{\alpha\beta} = \partial^2 / \partial q_j \partial q_j$, and $-\epsilon (\epsilon > 0)$ is the electron charge. Therefore, $K_{\mu\nu}^R(\omega)$ is given by the analytic continuation of the following function for $\omega_0 \geq 0$ [43,44,38]:

$$K_{\mu\nu}(\omega_0) = \sum_{n=0,1,2} \int_0^{1/T} \mathrm{d}r e^{-i\omega_0r} \langle T_{f} j^A_{\mu}(mq, 0) j^A_{\nu}(0, \tau) \rangle_L$$

$$= \frac{1}{T} \int_0^{1/T} \mathrm{d}r e^{-i\omega_0r} \langle T_{f} j_{\mu}(0, 0) j_{\nu}(0, \tau) \rangle$$

$$+ \sum_\alpha \int_0^{1/T} \mathrm{d}r \mathrm{d}r' e^{-i\omega_0r'} \{-T \cdot \langle T_{f} j_{\mu}(0, 0) \rangle + \langle T_{f} j_{\mu}(0, 0) \rangle_{\text{th}}(\epsilon, \omega)\} + O(A^2),$$

where $\omega_0 \equiv 2\pi T l$ is the Matsubara frequency; here we promise $l$ represents an integer and $n$ is a half-integer, respectively.

Hereafter, we ignore the spin indices to simplify expressions. According to equation (49), $K_{\mu\nu}(\omega)$ without the magnetic field is given by [40]

$$K_{\mu\nu}(\omega) = -2e^2 T \sum_{m,m',k,k'} L(kn, k' m; \omega)$$

$$= -2e^2 T \sum_{m,m'} \langle L_{k' n} \rangle_k L(kn, k' m; \omega),$$

where $g^{n,l}_{k' \mu} = G^{n,l}_{k' \mu} G^n_k$, $G^n_k \equiv G_k(\epsilon_0)$ is the Green function where $n$ is a half-integer and $L(kn, k' m; \omega)$ is the two-particle Green function in equation (18). $v_{k\mu}^{0} = \partial E_k / \partial k_{\mu}$ is the velocity of the free electron, and $A_{n,l}^{k,m} = \langle L_{k' n} \rangle_k$ is the three-point vertex.

**Figure 9.** The analytic regions 1–3 as a function of a complex variable $\epsilon$ (or $\epsilon'$). Here, we put $\Im \epsilon > 0$. From each region, $g_k^{(i)}$ and $A^{k,l}_{n,m}$ are analytically continued to become $g_k^{(i)}(\epsilon; \omega)$ and $J_k^{(i)}(\epsilon; \omega)$ ($i = 1–3$), respectively.

In the same way, Kohno and Yamada [43] derived the $H$-linear term of $K_{xy}(i\omega_0)$ from equation (50) as

$$K_{xy}(i\omega_0) = i \cdot e^3 HT \sum_{n,k,\mu,\nu} \langle \partial_{\mu} G_{k}^{\dagger} \cdot G_{k}^{\dagger} - G_{k}^{\dagger} \cdot \partial_{\nu} G_{k}^{\dagger} \rangle$$

$$\times [A^{k,l}_{n,m} \cdot \delta_{\mu\nu} A^{k,l}_{n,m}] \cdot \epsilon_{\mu\nu} + \text{[6 point VC term]},$$

where $\epsilon_{\mu\nu\rho\sigma}$ is an antisymmetric tensor with $\epsilon_{xyz} = 1$. In deriving equation (53), we used the relation $H_S = i(q_k A_y - q_l A_x)$, and took the limit $\mathbf{q} \to 0$ at the final stage of calculation [44]. Several kinds of Ward identities have to be correctly applied to maintain the gauge invariance [43].

In order to derive the conductivity $\sigma_{\mu\nu}(\omega_0)$, we have to perform the analytic continuations of equations (51) and (53); $\omega_0 (\omega_0 > 0) \to \omega_0 + i\delta$. Then, the $\omega_n$ summation is replaced with the integrations along the three cut lines in the complex plane in figure 9, together with the thermal factor $(4\pi T)^{-h}(\omega_0 + i\delta)$ in equation (16). According to equation (17), $K_{\mu\nu}(\omega + i\delta)$ is given by [40]

$$K_{\mu\nu}(\omega + i\delta) = -2e^2 \int_{-\infty}^{\infty} \frac{de}{4\pi T} \left[ \langle \epsilon \rangle + \frac{(\epsilon + \omega_0)}{2T} \right] K^{(i)}_{\mu\nu}(\epsilon; \omega),$$

$$+ \left( \frac{\epsilon + \omega_0}{2T} - \frac{\epsilon}{2T} \right) K^{(2)}_{\mu\nu}(\epsilon; \omega)” \frac{\epsilon + \omega_0}{2T} K^{(3)}_{\mu\nu}(\epsilon; \omega),$$

where $i = 1, 2, 3$. $J_k^{(i)}(\epsilon; \omega)$ is given by the analytic continuation of $A^{k,l}_{n,m}$ from region $i$. It is expressed as

$$J_k^{(i)}(\epsilon, \omega) = v_{k\mu}^{0} + \sum_{k',j=1,2,3} \int_{-\infty}^{\infty} \frac{de'}{4\pi T} T_{k,k'}^{ij}(\epsilon, \epsilon'; \omega) g_k^{(i)}$$

$$\times \langle \epsilon' \rangle, \omega) v_{k'\mu}^{0},$$

where $T_{k,k'}^{ij}(\epsilon, \epsilon'; \omega)$ is given by the analytic continuation of the full four-point vertex $\Gamma$ in equation (18) from region $(i, j)$ for $(\epsilon, \epsilon')$. Thermal factors (such as $(4\pi T)^{\dagger}\text{th}(\chi/2T)$ and $(4\pi T)^{-1}\text{ch}(\chi/2T)$) are included in the definition of $T_{k,k'}^{ij}$. The explicit expression for $T_{k,k'}^{ij}$ is given in equation (12) of [40].
The dc-conductivity $\sigma_{xx}$ is given by $\partial \text{Im} K_{xx}/\partial \omega|_{\omega=0}$. For example, if we take the $\omega$-derivative of the thermal factor (th($(\epsilon + \omega/2T) - \text{th}(\epsilon/2T)$) associated with $g^{(2)}$ in equation (54), we obtain

$$
\left( e^\gamma/\pi \right) \sum_k z_k (-\partial f^0/\partial \epsilon)\epsilon_k v_{k_0}^0 T_{k_0}^{(2)}(0; 0)/\gamma_k,
$$

(57)

where $f^0(\epsilon) = (e^\gamma/\pi + 1)^{-1}$, note that $z_k (-\partial f^0/\partial \epsilon)\epsilon_k = (-\partial f^0/\partial \epsilon)\epsilon_k = \delta(E_k)$ at sufficiently low temperatures. Since $J_{k_0}^{(2)}(\epsilon_0)$ is not singular with respect to $\gamma^{-1}$, equation (57) is proportional to $\gamma^{-1}$, which diverges when $\gamma \to 0$. On the other hand, if we take the $\omega$-derivative of the thermal factor in front of $g^{(2)}$ in equation (54), the obtained term is $(\gamma^0)$. To derive the exact expression for $\sigma_{xx}$ with respect to $O(\gamma^{-1})$, we also have to take the $\omega$-derivative of the thermal factor in front of $g^{(2)}$ in $K_{x_0}^{(1)(3)}(\epsilon; \omega)$. As a result, the exact expression for $\sigma_{xx}$ is given by equation (57) by replacing $v_{k_0}^0$ with $v_{k_0} = v_{k_0}^0 + \partial \text{Re} \Sigma_k(0)/\partial k_k$, [40].

$J_{k_0}^{(2)}$ is called the total current since it contains the CVC discussed in previous sections. Hereafter, we denote $J_{k_0}^{(2)}(\epsilon, 0)$ and $T_{k_0}^{(2)}(\epsilon, \epsilon; 0)$ as $J_k(\epsilon)$ and $T_k(\epsilon, \epsilon')$. The analytic continuation of $K_{x_0}(i \omega_0)$ had been performed in [43, 44]. To summarize, the general expressions for $\sigma_{xx}$ and $\sigma_{xy}$, which are exact within the most divergent term with respect to $O(\gamma^{-1})$, are given by

$$
\sigma_{xx} = e^2 \sum_k z_k \left( -\partial f^0/\partial \epsilon \right)\epsilon_k v_{k_0} J_k(\epsilon)/\gamma_k,
$$

(58)

$$
\sigma_{xy} = -e^2 H \sum_k z_k \left( -\partial f^0/\partial \epsilon \right)\epsilon_k v_{k_0} J_k(\epsilon)(\tilde{v}_k \times \tilde{v}_k_0)/\gamma_k.
$$

(59)

Equation (58) was derived by Eliashberg [40]. Equation (59) was derived by Fukuyama et al. within the Born approximation [44], and it was proved to be correct in Fermi liquids [43]. Note that we dropped the [6 point VC term] in equation (53) since they are less singular with respect to $\gamma^{-1}$ [43]. In particular, its contribution to $\sigma_{xx}$ vanishes in the FLEX approximation. In the same way, the present author has derived the exact expressions for the magnetococonductivity $\Delta \sigma_{xx} \equiv \sigma_{xx}(H) - \sigma_{xx}(0)$ [38] and the Peltier coefficient $\sigma_{xy} = E_y/(-\nabla_x T)$ in the presence of $H$ [39].

Apparently, equations (58) and (59) become equal to the RTA results given in equations (38) and (39) if we replace the total current $\bar{J}_k$ with the quasiparticle velocity $\bar{v}_k$, that is, if we drop the CVC. The RTA had been frequently used in analyzing transport anomaly in high-Tc cuprate. However, we will explain that the neglect of the CVC frequently causes various unphysical results.

According to Eliashberg, the total current $J_k$ can be rewritten as [40]

$$
J_k(0) = v_{k_0} + \int_0^\infty \frac{d\epsilon}{4\pi T} T_{k_0}^{(0)}(0, \epsilon') T_{k_0}^{(2)}(0; 0) J_k(\epsilon),
$$

(60)

where $v_{k_0} = v_{k_0}^0 + \partial \text{Re} \Sigma_k(0)/\partial k_k$, and we put $\epsilon = 0$ in $J_k(\epsilon)$ for simplicity. $T_{k_0}^{(0)}(0, \epsilon')$ is the ‘irreducible’ vertex with respect to $g^{(i)}$, which is given by

$$
T_{k_0}^{(0)}(0, \epsilon') = i \left( \text{cth} \frac{\epsilon'}{2T} - \text{th} \frac{\epsilon'}{2T} \right) 2 \text{Im} \Gamma_{k_0}^{(i)}(0, \epsilon' - i\delta).
$$

(61)

Equation (60) is expressed in figure 10(i). The CVC in the microscopic Fermi liquid theory is also called the backflow in the phenomenological Fermi liquid theory.

Here, we analyze the Bethe–Salpeter equation (60): the solution of equation (60) is real since $T_{k_0}^{(0)}(0, \epsilon')$ in equation (61) is purely imaginary. Since $\text{Im} \Gamma_{k_0}^{(i)}(0, \epsilon') = 0$, we have to extract the $\epsilon'$-linear term of $\text{Im} \Gamma_{k_0}^{(i)}(0, \epsilon')$, which is derived from the cut of the particle–hole pair or that of the particle–particle pair in $\Gamma_{k_0}^{(i)}(0, \epsilon')$ [31]. By this procedure, $\Gamma^{(i)}$ is divided into two $\Gamma$s. Therefore, the $\epsilon'$-linear term of $\text{Im} \Gamma(0, \epsilon')$ is given by [42]

$$
\text{Im} \Gamma_{k_0}^{(i)}(0, \epsilon' - i\delta) = \pi \epsilon' \sum_{q} \Gamma_{k_0}^{(i)}(k, k'; k - q, k + q)
$$

$$
\times \left( \rho_{k_0}^{(0)}(0) \rho_{k_0}^{(-q)}(0) - \frac{1}{\pi} \rho_{k}^{(0)}(0) \rho_{k}^{(-q)}(0) \right),
$$

(62)

where $\Gamma(k, k'; k - q, k + q)$ is the full four-point vertex at the Fermi level, which is a real function of $(k, k', q)$. The factor $\pi$ in front of the second term in the curly bracket in equation (62) is necessary to avoid double counting. Then, $T_{k_0}^{(0)}(0, \epsilon')$ is given by $[\text{cth}(\epsilon'/2T) - \text{th}(\epsilon'/2T)] 2 \text{Im} \Gamma(0, \epsilon' - i\delta)$; it is schematically shown in figure 10(ii).

After changing momentum variables, the Bethe–Salpeter equation (60) is transformed into [42]

$$
\bar{J}_k = \bar{v}_k + \Delta \bar{J}_k,
$$

(63)

$$
\Delta \bar{J}_k = \sum_{k_q} \bar{T}_{k_0}^{(0)(a)}(k, k') \rho_{k}^{(0)}(0) J_{k'} + \sum_{k_q} \bar{T}_{k_0}^{(0)(b)}(k, k') \rho_{k}^{(-q)}(0) J_{k'}
$$

+ \sum_{k_q} \bar{T}_{k_0}^{(0)(c)}(k, k') \rho_{k}^{(-q)}(0) J_{k'},
$$

(64)

where we used the relation $\int_{-\infty}^{\infty} d\epsilon \left[ \text{cth}(\epsilon/2T) - \text{th}(\epsilon/2T) \right] \bar{T}_{k_0}^{(0)(a)}(a = a, b, c)$ are functions of $(k, k', q)$, and they represent the forward scattering amplitude [31, 40]. The expressions for $\bar{T}_{k_0}^{(0)(a)}(a = a, b, c)$ at sufficiently low
Each hatched part represents the real part of the full antisymmetric four point vertex $\Gamma(k,k'; k+q,k-q)$. Each line with arrow represents the imaginary part of the Green function.

Note that (a) becomes different from (b) if we violate the antisymmetric nature of $\Gamma(k,k'; k+q,k-q)$ in the course of approximation. (ii) Diagrammatic representation of equation (68).

temperatures are given by [42]

$$\bar{\rho}_{\mathbf{k},q}(\mathbf{k},\mathbf{k}'; \mathbf{k}'-\mathbf{q}) = \frac{\pi}{2} \Gamma^2(\mathbf{k},\mathbf{k}'; \mathbf{k}'-\mathbf{q}; \mathbf{k}+\mathbf{q},\mathbf{k}) \rho_k(0) \rho_{k+q}(0),$$

(65)

$$\bar{\rho}_{\mathbf{k},q}(\mathbf{k},\mathbf{k}'; \mathbf{k}'-\mathbf{q}) = \frac{\pi}{2} \Gamma^2(\mathbf{k},\mathbf{k}'; \mathbf{k}'-\mathbf{q}; \mathbf{k}+\mathbf{q},\mathbf{k}) \rho_k(0) \rho_{k+q}(0),$$

(66)

$$\bar{\rho}_{\mathbf{k},q}(\mathbf{k},\mathbf{k}'; \mathbf{k}'-\mathbf{q}) = -\frac{\pi}{2} \Gamma^2(\mathbf{k},\mathbf{k}'; \mathbf{k}'-\mathbf{q}; \mathbf{k}+\mathbf{q},\mathbf{k}) \rho_k(0) \rho_{k+q}(0),$$

(67)

Since the CVC due to $\bar{T}^{(0c)}$ represents the hole current, the minus sign appears in equation (67). In equations (65)–(67), we dropped spin indices in $\Gamma$ to simplify equations: if spin indices are taken into account, $\frac{1}{4} \Gamma^2$ in each equation is replaced with $\frac{1}{2} \Gamma^2_{1,1,1,1}$ as explained in [42]. Equations (65)–(67) are expressed in figure 11 (i). Note that equations (65) and (66) are equivalent since $\Gamma(k,k'; k-q,k+q)$ is full antisymmetrized as a consequence of the Pauli principle.

In the same way, $\text{Im} \Sigma_k(-i\delta) = \gamma_k$ at sufficiently low temperatures can be expressed as [31, 42]

$$\gamma_k = \frac{1}{2} \sum_{\mathbf{k}} \int \frac{d\epsilon'}{4\pi i} \bar{T}^{(0c)}(\mathbf{k},\epsilon'; \epsilon') \cdot \rho_k(\epsilon'),$$

(68)

which is proportional to $T^2$ at the zero-temperature limit if the dimension is slightly higher than two. (In pure 2D systems, $\gamma_k \propto -T^2 \ln T$.) Equation (68) is shown in figure 11 (ii). We note that $\gamma_k$ is also given by $\gamma_k = \frac{1}{2} \sum_{\mathbf{k}} \bar{T}^{(0c)}(\mathbf{k},\mathbf{k}'; \mathbf{k}'-\mathbf{q}) \rho_k(\mathbf{k}'-\mathbf{q}) = -\frac{1}{2} \sum_{\mathbf{k}} \bar{T}^{(0c)}(\mathbf{k},\mathbf{k}'; \mathbf{k}'-\mathbf{q}) \rho_k(0).

Here, we calculate the CVC in a free dispersion model in the absence of Umklapp scattering according to Yamada and Yosida [42]. In this case, both $\gamma_k$ and $\Delta J_k$ on the Fermi surface are isotropic, that is, $\gamma_k = \gamma_0$ and $\Delta J_k = J_k / kF$. By noticing the relationship $\Delta J_k + J_k - q = J_k = \Delta J_k$ and using equations (65)–(68), it is easy to verify that the CVC in equation (64) is exactly given by

$$\Delta J_k = \gamma_k,$$

(69)

in a free dispersion model. If we put $\Delta J_k = \epsilon'_k$ (c is a constant), the solution of equation (63) is given by $J_k = \frac{\epsilon_k}{1-c}$, which diverges when $c \rightarrow 1$. Therefore, the conductivity given by equation (58) diverges, which is a natural consequence of the momentum conservation of the system [42]. Yamada and Yosida also showed that $\bar{J}_k$ remains finite in the presence of the Umklapp processes [42].

Next, we discuss the case where $\epsilon'_k$ is anisotropic. Then, the formal solution of equation (60) is given by

$$\bar{J}_k = \sum_{\mathbf{k}} \gamma_k,$$

(70)

where $\gamma_k$ is anisotropic, quasiparticle current is not conserved in the normal scattering process: $\gamma_k + \gamma_k \neq \gamma_k + \gamma_k$. Nonetheless of the fact, one can show that $\text{det} [1 - C] = 0$ as a result of the momentum conservation [46, 155]. Therefore, $\gamma_k$ in equation (70) and $\sigma_{xx}$ diverge in any anisotropic model if the Umklapp processes are absent. (Exactly speaking, $\rho_{xx} = 1/\sigma_{xx}$ is proportional to $T^{2N-2}$ when the $N$-particle Umklapp scattering processes are present [155].)

We stress that the FLEX approximation can reproduce the divergence of the conductivity in the absence of the Umklapp scatterings, if one considers the CVC correctly. This result is assured by the fact that the Ward identity $\Gamma' = \delta \Sigma / \delta G$ is consistently satisfied in the FLEX approximation [139, 140]. Therefore, the FLEX approximation will be appropriate for the study of transport phenomena in correlated electron systems.

Next, we consider the role of the CVC in nearly AF Fermi liquids in the presence of Umklapp scatterings, where the electron-electron scatterings are mainly given by the strong AF fluctuations with $q \sim Q$. In this case, we can approximate the full four-point vertex in equations (65)–(67) as

$$\Gamma_{s_1,s_2,s_3,s_4}(k,k'; k-q,k+q) \approx \frac{U^2}{2} \{ \chi''(0) \bar{\sigma}_{s_3,s_4} \cdot \bar{\sigma}_{s_2,s_3} - \chi''(0) \bar{\sigma}_{s_1,s_3} \cdot \bar{\sigma}_{s_2,s_4} \},$$

(71)

which satisfies the antisymmetric nature of $\Gamma$ that is the consequence of the Pauli principle. The diagrammatic representation of equation (71) is shown in figure 12 (i). Here, $s_j (j = 1,4)$ represents the spin index and $\bar{\sigma}$ is the Pauli matrix.

Figure 11. (i) Diagrammatic representation of equations (65)–(67). Each hatched part represents the real part of the full antisymmetrized four point vertex $\Gamma(k,k'; k+q,k-q)$. Each line with arrow represents the imaginary part of the Green function.

Figure 12. (i) Diagrammatic representation of equation (71). The wavy lines represent the spin fluctuations, $\chi''(0)$. (ii) Diagrammatic representation of $\bar{U}$ in equation (71). $U$ is the bare Coulomb interaction in the Hubbard model and $\bar{U}(k+q,k)$ is the three-point vertex that is irreducible with respect to $U$. 
vector. $\bar{U}$ in equation (71) is the effective interaction between the electrons and spin fluctuations, which is the renormalized Coulomb interaction due to the irreducible three-point vertex shown in figure 12(ii).

In the present section, however, we analyze the CVC using the following more simplified approximation for $\Gamma$:

$$\Gamma_{s_1,s_2,s_3,s_4}(k,k';q,k+q) \approx \frac{\bar{U}_q^2}{2} \gamma_{k+q}^2(0) \tilde{\sigma}_{s_1,s_2} \cdot \tilde{\sigma}_{s_3,s_4}.$$  (72)

Although equation (72) violates the Pauli principle, this approximation produces the dominant CVC (Maki–Thompson term) near AF-QCP. In section 3.4, we will analyze the CVC using equation (71) and show that the analysis based on equation (72) is justified. According to the present approximation, $\bar{T}^{(\alpha \alpha)} (\alpha = a, b, c)$ in equations (65)–(67) are expressed by (A)–(C) in figure 13(i), respectively, where wavy lines represent the spin fluctuations. The CVCs expressed in (A) and (B) correspond to the current due to the quasiparticles at $k+q$ and $k'-q$, respectively, and the CVC in (C) corresponds to the current due to the quasihole at $k'$ in figure 8. In the field theory, (A) is called the Maki–Thompson term, and (B) and (C) are called the Aslamazov–Larkin terms. As explained in figure 8, the Aslamazov–Larkin terms approximately disappear [33] whereas the Maki–Thompson term plays an important role when $\xi_{AF} \gg 1$ and $Q \approx (\pi, \pi)$.

According to equation (72), $\frac{1}{2} \Gamma^2$ in $\bar{T}^{(\alpha \alpha)}$ becomes $(3U^4/4)\gamma_{k+q}^2(0)^2$ after taking the summation of spin indices. However, we replace $\frac{1}{2} \Gamma^2$ with $(3U^4/2)\gamma_{k+q}^2(0)^2$ since the term (A) in figure 13 appears twice if we use equation (71). Then, the CVC in equation (64) and $\gamma_k$ given in equation (68) near the AF-QCP are obtained as

$$\Delta \bar{J}_k = \sum_q (\pi T)^2 \frac{3U^4}{2} \gamma_{k+q}^2(0)^2 \text{Im} \chi_q^0(0) \rho_{k+q} \gamma_k q \bar{J}_{k+q}.$$  (73)

$$\gamma_k = \sum_q (\pi T)^2 \frac{3U^4}{2} \gamma_{k+q}^2(0)^2 \text{Im} \chi_q^0(0) \rho_{k+q} q.$$  (74)

where we used the relation $\text{Im} \chi_q^0(0) = \frac{\pi}{2} \sum_k \rho_k q \rho_{k+q} q$. Since $\text{Im} \chi_q^0(0) = \text{Im} \chi_q^0(0)(\bar{U} \chi_q^0(0))^2$, the Bethe–Salpeter equation (73) and $\gamma_k$ in equation (74) in a nearly AF Fermi liquid are given by

$$\bar{J}_k = \bar{v}_k + \sum_q \frac{3U^2}{4} (\pi T)^2 \text{Im} \chi_q^0(0) \rho_{k+q} \bar{J}_{k+q}. $$  (75)

$$\gamma_k = \sum_q \frac{3U^2}{4} (\pi T)^2 \text{Im} \chi_q^0(0) \rho_{k+q} q.$$  (76)

Note that equation (76) is equivalent to $\gamma_k$ in the spin fluctuation theory in equation (30).

In this section, we have neglected the energy dependence of the four-point vertices (in equations (65)–(67)) and that of the spin susceptibilities (in equations (73)–(76)) to simplify the discussion. For this reason, equations (75) and (76) are appropriate only for $\omega_{sd} \gg T$ where $\omega$-dependence of $\chi_q^0(\omega)$ can be ignored. In the case of $\omega_{sd} \ll T$, $(\pi T)^2$ in equations (75) and (76) should be replaced with $\int d\omega [\text{e}^{|\omega/2T|} - \text{e}^{-|\omega/2T|}] \frac{\rho_{sd}}{2 \pi} = 4T \omega_{sd}$. For any value of $\omega_{sd} / T$, they are expressed by equations (36) and (33) in [33], respectively.

Here, we approximately solve equation (75) in the case of $\xi_{AF} \gg 1$ and $\xi_{AF} \Delta k_c \sim O(1)$. Here, we assume $k$ is close to point A in figure 4(i). Because $\gamma_q^0(0)$ takes a large value only when $|q - Q| \lesssim \xi_{AF}$, the CVC term in equation (75) can be expressed as $\Delta \bar{J}_k \approx \langle \bar{J}_k(q) \rangle \sum_q (3U^4/2)(\pi T)^2 \text{Im} \chi_q^0(0)$, where $(k_x^0, k_y^0) = (-k_y, k_y)$ for $k_x < 0$, and $(k_x^0, k_y^0) = (k_x, k_y)$ for $k_x < 0$ as shown in figure 4(i). Considering equation (76), $\Delta \bar{J}_k \approx \bar{J}_k$ in the case of $\xi_{AF} \gg 1$. The more detailed expression for the CVC term $\Delta \bar{J}_k$ is given by [33]

$$\Delta \bar{J}_k \approx \langle \bar{J}_k(q) \rangle_{\xi_{AF} < 1} \approx \alpha_k \bar{J}_k,$$  (77)

where $\alpha_k \approx (1 - c/\xi_{AF}^2) < 1$ and $c \sim O(1)$ is a constant. The k-dependence of $\alpha_k$ will be moderate if $\xi_{AF} \Delta k_c \gg 1$. By this simplification, equation (75) becomes

$$\bar{J}_k = \bar{v}_k + \alpha_k \bar{J}_k,$$  (78)

and the solution is given by [33]

$$\bar{J}_k = \frac{1}{1 - \alpha_k^2} (\bar{v}_k + \alpha_k \bar{v}_k). $$  (79)

Here, we examine the $k$-dependence of $\bar{J}_k$ given in equation (79): (a) near points A and B, $\bar{J}_k$ is parallel to $\bar{v}_k$ by symmetry. At point A, $\bar{J}_k = \bar{v}_k / (1 + \alpha_k) \sim \frac{1}{2} \bar{v}_k$ since $\bar{v}_k = -\bar{v}_k$. (b) Near point C, $\bar{J}_k \approx (\xi_{AF}^2/2c)(\bar{v}_k + \bar{v}_k)$, which is nearly parallel to the AFBZ. These behaviors of $\bar{J}_k$ together with $\bar{v}_k$ are schematically shown in figure 14(i). In the present discussion, we assumed that $\xi_{AF} \Delta k_c \sim O(1)$, where $\Delta k_c$ represents the deviation from the nesting condition at the cold spot in figure 4(i). This condition seems to be satisfied in slightly under-doped YBCO and LSCO above $T_c$, as discussed in section 2.3. In figure 4(i), we show an ‘effective Fermi surface’ obtained by bending the true Fermi surface such that it is orthogonal to $\bar{J}_k$ around the cold spot [20]. We discuss the role of the CVC in the Hall coefficient based on the concept of the effective Fermi surface in section 4.2.

When $\xi_{AF} \Delta k_c \gg 1$, one may think that the CVC is small around the cold spot, and therefore the increment in $R_{H}$ due to
the effective Fermi surface around the cold spot is the origin of the quasiparticle velocity $\vec{v}_k$ on the Fermi surface. The cold spot is located around the point A (B) in hole-doped (electron-doped) HTSCs. (ii) Effective Fermi surface for located around the point A (B) in hole-doped (electron-doped) the CVC is also small. However, this expectation is not true unless $\chi^d(q)$ is a step function like $\theta(1 - |Q - q|/\xi_{AF})$. In fact, we verified that the CVC around the cold spot is significant even if $\xi_{AF} \Delta k_c$ is much larger than unity, by assuming a phenomenological $\chi^d(q)$ model given in equation (8) [34]. In later sections, we demonstrate that $R_0$ is prominently enhanced by the CVC using the FLEX approximation, which yields a realistic functional form of $\chi^d(q)$. We will show that the CVC is important in NCCO, although $\xi_{AF} \Delta k_c \gg 1$ is realized at low temperatures.

Finally, we comment that the Aslamazov–Larkin terms ((B) and (C) in figure 13) become important when the AF fluctuations are weak. In the FLEX approximation, one can reproduce the divergence of the conductivity in the absence of the Umklapp scatterings, only if all the CVCs ((A)–(C) in figure 13) are taken into consideration.

3.4. CVC beyond one-loop approximation

In the previous section, we analyzed the CVC by assuming equation (72), which corresponds to the one-loop approximation for the self-energy like the FLEX approximation. In this approximation, equations (73) and (74) contain the same Kernel function $[\chi^d(q)(0)]^2 \text{Im} \chi^d(q)(0) \rho_{k+q}(0)$ as a consequence of the Ward identity between the CVC and the imaginary part of the self-energy. For this reason, the factor $\alpha_k$ in equation (79) approaches unity near AF-QCP, which assures the anomalous $k$-dependence of $\tilde{F}_k$ near AF-QCP that is shown in figure 14. However, one may be afraid that this result is an artifact due to the violation of the Pauli principle. In this section, we analyze the self-energy and the CVC beyond the one-loop approximation by assuming equation (71). Here we confirm that the CVC near the AF QCP is important even if the higher-loop diagrams are taken into account.

According to equation (71), $\gamma_k$ in equation (68) is approximately given by

$$\gamma_k = \gamma_k^{(1)} + \gamma_k^{(2)}, \quad (80)$$

where $\gamma_k^{(1)} (> 0)$ and $\gamma_k^{(2)} (< 0)$ are shown in figure 15(ii). The integrand in equation (81) takes a large value when $|q - Q|, |q' - Q| \lesssim \xi_{AF}^{-1}$, under the condition that $\epsilon_{k+q} - \epsilon_{k-q} = \epsilon_{k-q-q'} = \mu$. When $Q = (\pi, \pi)$, $\gamma_k^{(2)}$ can be significant at the hot spot since $k + q \approx k - q' \approx k^*$ and $k + q - q' \approx k$ in modulo $(2\pi, 2\pi)$.

In the same way, the CVC in equation (64) is given by the six diagrams shown in figure 15. As discussed in section 3.3, terms (B) and (C) are negligible in nearly AF Fermi liquids. Other terms are given by

$$\rho \gamma_k = \rho \gamma_k \delta_k, \quad (82)$$

Here, $\chi^d(q)(0) \chi^d(q')(0)$ takes a large value only when $|q - Q|, |q' - Q| \lesssim \xi_{AF}^{-1}$. In (D)–(F), the set of momenta $(k, k', q)$ is changed to $(k, q, q')$ due to the transformation of variables. Applying the same approximation that was used in equation (77), the total CVC is given by

$$\Delta \tilde{F}_k = \alpha_k \rho_k \tilde{F}_k^{(1)} - \alpha_k \rho_k \tilde{F}_k^{(2)}, \quad (84)$$

where $(1 - \alpha_k)$, $(1 - \alpha_k')$ are positive, and they approach zero in proportion to $\xi_{AF}^{-2}$ near the AF-QCP. Then, the solution of the Bethe–Salpeter equation $\tilde{F}_k = \tilde{v}_k + \Delta \tilde{F}_k$ is
Figure 16. Diagrammatic representation of $\tilde{\Gamma}(q)$ in equation (87).

Figure 15. (i) According to the approximation in equation (71), $\tilde{T}^{(0a)}_{q,k,k'q'}$, $\tilde{T}^{(0b)}_{q,k,k'q}$ and $\tilde{T}^{(0c)}_{q,k,k'}$ in equations (65)–(67) are approximated as (A)–(F), respectively. (A), (B), (D) and (E) come from $\tilde{T}^{(0a)}$ and $\tilde{T}^{(0b)}$. (C) and (F) come from $\tilde{T}^{(0c)}$. (ii) The expression for $\gamma_k$ according to the approximation in equation (71).

The third term of equation (88) is expected to be smaller than the second term since the integrand takes large values only when $k - k' \sim q$. Since the q-dependence of $\chi_q(0)$ is moderate, the present analysis suggests that the approximate relation $\chi_q(0) \propto \Gamma(q)$ holds for $q \sim Q$. For this reason, the solution of $\tilde{J}_k$ obtained in this section, equation (85) or (86), will be valid even in higher-loop approximations.

As a result, the anomalous $k$-dependence of $\tilde{J}_k$ in figure 14, which is derived in the previous section within the one-loop approximation, is expected to be realized even in the close vicinity of the AF-QCP where the higher-loop diagrams become important. In later sections, we present a numerical study of the transport phenomena based on the FLEX + CVC approximation. Although it is a one-loop approximation with respect to the spin fluctuation, the numerical results obtained will be qualitatively reliable even in the under-doped region, since the CVC due to the higher-loop diagrams do not alter the one-loop approximation results qualitatively.

4. Transport phenomena in HTSCs above $T^*$

4.1. Total current $\tilde{J}_k$

Here, we discuss the transport phenomena in HTSCs above the pseudo-gap temperature $T^*$ where the SC fluctuations are negligibly small. In the FLEX + CVC approximation, the self-energy is given by equation (14), and the vertex function in the Bethe–Salpeter equation, $\tilde{T}^{(0)}_{k,k'}$, is given by equation (61). The total current $\tilde{J}_k$ is obtained by solving the Bethe–Salpeter equation [33]. According to the Ward identity, the irreducible four-point vertex is given by $\Gamma^I = g \Sigma / \partial G$. In the FLEX approximation, the self-energy is given by the convolution of $G$ and $V$ as in equation (14). Then, $\Gamma^I$ contains one Maki–Thompson term that is given by taking the derivative of $G$, and two Aslamazov–Larkin terms that are given by taking the derivative of $V$, which is composed of infinite $G$’s. These three terms correspond to (A)–(C) in figure 13 [33, 57].
Considering the relation \( \delta \chi^0 \delta G^0_k = -(G_{k+q} + G_{k-q}) \), we can derive \( \Gamma_{k,k'}^{(2)}(\epsilon_n, \epsilon_{n'}) = V_{k-k'}(\epsilon_n - \epsilon_{n'}) \)
\[
-\bar{T} \sum_{q,l} U^2 \left[ \frac{3}{2} (U \chi_q^0(\omega_l) + 1)^2 + \frac{1}{2} (U \chi_q^0(\omega_l) - 1)^2 - 1 \right] \times G_{k-q}(\epsilon_n - \omega_l) G_{k-q}(\epsilon_n - \omega_l) \\
-\bar{T} \sum_{q,l} U^2 \left[ \frac{3}{2} (U \chi_q^0(\omega_l) + 1)^2 + \frac{1}{2} (U \chi_q^0(\omega_l) - 1)^2 - 1 \right] \times G_{k+q}(\epsilon_n + \omega_l) G_{k-q}(\epsilon_n - \omega_l).
\] (89)

The first term \( V q(\omega_l) \) corresponds to the Maki–Thompson vertex correction. The last two terms in equation (89), both of which contain \( |\chi_q^0|^2 \), are the Aslamazov–Larkin vertex corrections. Hereafter, we drop the Aslamazov–Larkin terms since they are negligible in nearly AF Fermi liquids \( |\chi_q^0| \gg |\chi_q^0|^0 \), as we have discussed in previous sections.

After the analytic continuation of the Maki–Thompson term \( V \to \omega + i\delta \) from the region \( \epsilon_n, \epsilon_{n'} < 0 \) and \( \epsilon_n + \omega_l, \epsilon_{n'} + \omega_l > 0 \), the Bethe–Salpeter equation in the FLEX+CVC theory for \( \omega = 0 \) is given by [33],
\[
\bar{J}_k(\epsilon) = \bar{v}_k(\epsilon) + \sum_q \int \frac{d\epsilon'}{2\pi} \left[ \frac{c\tan(\epsilon'/2T) - \tan(\epsilon'/2T)}{2T} \right] \times \text{Im} \chi_g(\epsilon + \epsilon') \chi_q(\epsilon + \epsilon' + i\delta) \cdot \bar{J}_{k+q}(\epsilon + \epsilon') \\
\approx \bar{v}_k + \sum_q \frac{(\pi T)^2}{2} \text{Im} \chi_g(\epsilon) \chi_q(\epsilon) \bar{J}_{k+q},
\] (90)

where \( \bar{v}_k(\epsilon) = \bar{v}_k(\epsilon^0 + \text{Re} \Sigma_k(\epsilon)) \). In deriving equation (90), we used the relation \( \int [\text{c} \tan(\epsilon'/2T) - \tan(\epsilon'/2T)]d\epsilon = (\pi T)^2 \) and assumed the relation \( \omega_{ML} \gg T \). If we approximate \( V \approx 3U^2 \chi_q^0/2 \), equation (90) is equivalent to equation (75).

(Not that \( U = U \) in the FLEX approximation.) The numerical solution of equation (90) is shown in figure 17, whose schematic behavior is shown in figure 14(i). This singular \( k \)-dependence of \( J_k \) is realized because \( \epsilon_k \) in equation (77) approaches unity for \( \xi_{AF} \to \infty \). As discussed in section 3.4, since the same vertex function \( T_{k+q}^{[0]} \) appears in both (60) and (68) in the microscopic Fermi liquid theory, which is one of the Ward identities. For this reason, singular \( k \)-dependence of \( J_k \) in the case of \( \xi_{AF} \gg 1 \), which is shown in figure 17, is not specific to the FLEX approximation, but is a universal behavior in Fermi liquids near the AF-QCP.

In hole-doped systems, \( \bar{J}_k \) attains its minimum (maximum) at point A (point B) as shown in figure 7 [24, 25]. At low \( T \), the local minimum of \( |\bar{J}_k| \) is located at points A and B, and its maximum is located at point C; this is schematically shown in figure 14(i). Further, figure 17(ii) shows that the mean free path with CVC, \( \bar{L}_k = |\bar{J}_k|/\bar{J}_k \), attains its local minimum at point A for \( T \leq 0.02 \), since \( |\bar{J}_k| \) rapidly increases as \( k \) deviates from point A. In fact, equation (79) suggests that \( |\bar{J}_k| \propto |\xi_{AF}|^{\alpha_k} \bar{v}_k + \bar{v}_{k+1} \). Therefore, \( |\bar{J}_k| \) takes a large value near the AF-QCP except for points A and B. In later sections, we explain that the local minimum structure of \( \bar{L}_k \) at point A becomes prominent below \( T^* \), which causes significant increases in the Nernst coefficient and magnetoresistance.

In [33], we solved the Bethe–Salpeter equation by considering both the Maki–Thompson term and the Aslamazov–Larkin terms, and calculated both \( \rho \) and \( R_{II} \). As a result, we verified numerically that the Aslamazov–Larkin terms are negligible in nearly AF metals.

4.2 Resitivity and Hall coefficient

First, we analyze the conductivity \( \sigma_{xx} \) using equation (58). According to the approximate expression for \( \bar{J}_k \) given in equation (79),
\[
|\bar{v}_k, J_k|_{\text{cold}} = \left[ \frac{v^2_{k+1}}{1 + \alpha_k} \right]_{\text{cold}}
\] (91)
at the cold spot of YBCO [point A]. We have used the relationship \( k = -k^* \) at the cold spot. Since \( 1/(1 + \alpha_k) \sim 1/2 \) when \( \xi_{AF} \gg 1 \), then \( \sigma_{xx} \sim \sigma_{xx}^{\text{RTA}}/2 \). Therefore, the resistivity \( \rho \) is slightly increased by the CVC [33]. Since the effect of the CVC on \( \rho \) is not large, the RTA result for \( \rho \) can be qualitatively justified.
Figure 18. (i) $\rho$ for LSCO obtained by the FLEX (without CVC) and FLEX + CVC approximations. $\rho$ is slightly enhanced because of the CVC. We also plot $\rho$ obtained by the FLEX + $T$-matrix (without CVC) and [FLEX + $T$-matrix]+CVC approximations. The FLEX + $T$-matrix approximation coincides with the FLEX approximation for $T > T^*$, where SC fluctuations disappear. $T = 0.2$ corresponds to 800 K. (ii) $\rho$ for NCCO obtained by the FLEX (without CVC) and FLEX + CVC approximations.

Figure 18(i) shows the numerical results of $\rho$ for LSCO ($n = 0.92$ and $U = 4.5$) obtained from the FLEX approximation and the FLEX+$T$-matrix approximation. In the FLEX approximation without CVC, $\rho$ shows an approximate $T$-linear behavior for $T > 0.02$ ($\sim 80$ K). Because of the CVC, $\rho$ is slightly enhanced for a wide range of temperatures. Moreover, $\rho$ obtained by the FLEX + CVC approximation shows a tiny ‘kink’ structure at $\sim T_B$, below which AF fluctuations grows prominently. This result is consistent with the experimental results [167]. The kink becomes more prominent in the FLEX + $T$-matrix approximation, which will be discussed in section 5.2. In optimally doped YBCO and LSCO, the resistivity at 300 K is 200–300 $\mu\Omega$ cm, and it increases to 400–600 $\mu\Omega$ cm in slightly under-doped compounds ($n \sim 0.1$) [156]. In figure 18(i), $\rho = 1 \sim 250$ $\mu\Omega$ cm at $T = 0.08 \sim 320$ K; the resistivity increases with $U$, and $\rho \sim 450$ $\mu\Omega$ cm for YBCO with $U = 8$ and $n = 0.9$ [33], which is consistent with experimental values. However, the FLEX + CVC method cannot reproduce the huge resistivity in heavily under-doped compounds, since the self-consistency condition in the FLEX approximation tends to suppress $\gamma_k$ too strongly; see section 9.2. In under-doped compounds, one has to take account of the effect of residual disorder since the residual resistivity grows prominently. We will explain the reason in section 7.2 based on the Fermi liquid theory with strong AF fluctuations.

Figure 18(ii) shows the $\rho$ for NCCO ($U = 5.5$) obtained from the FLEX approximation (within the RTA) and the FLEX + CVC approximation. In the case of $n = 1.20$ (over-doped), $\rho$ shows a $T^2$-like behavior below $T = 0.1 \sim 400$ K, which is consistent with experiments. In the case of $n = 1.10$ (under-doped), $\rho$ shows an approximate $T$-linear behavior when the CVC is included in the calculation. Interestingly, $\rho_{\text{FLEX+CVC}} < \rho_{\text{FLEX+RTA}}$ for $n = 1.10$ below $T = 0.06$, since $|J_k| \gg |\tilde{v}_k|$ due to the CVC in NCCO at low temperatures.

Next, we discuss the Hall coefficient. Using the Onsager’s relation $\sigma_{xy} = -\sigma_{yx}$, the general expression for $\sigma_{xy}$ in equation (59) and $\sigma_{xy}^{\text{RTA}}$ in equation (39) in 2D systems can be rewritten as [33]

$$\sigma_{xy}/H_c = \frac{e^3}{4} \int_{\text{FS}} \frac{dk_1}{(2\pi)^2} \left( L_k \times \frac{\partial L_k}{\partial k_1} \right) \frac{1}{\gamma_k},$$

where $L_k = \tilde{J}_k/\gamma_k$, $\tilde{v}_k = \frac{1}{\gamma_k} \partial \tilde{v}_k/\partial k_1$, $\tilde{v}_k$ is the momentum tangent to the Fermi surface, which is depicted in figure 14(ii). Note that $|L_k| = |\tilde{J}_k/\gamma_k|$ represents the mean free path by considering the CVC. In deriving the above equations, we used the relation $|\tilde{u}_k/\gamma_k|$ exhibiting strong $\gamma_k$ dependence in 2D systems. Using the relation $|v_k^x + v_k^y| \frac{\partial}{\partial k_1} = (\tilde{v}_k \times \tilde{v}_k)_z$, $\sigma_{xy}$ in 3D systems is given by

$$\sigma_{xy}/H_c = \frac{e^3}{4} \int_{\text{FS}} \frac{dS_k}{(2\pi)^3} \sqrt{|v_k^x + v_k^y|^2} \tilde{J}_k^2 \frac{1}{|\tilde{v}_k|} \frac{1}{\gamma_k^2},$$

where $dS_k$ represents the Fermi surface element, and $dk_1$ is the momentum tangent to the Fermi surface and parallel to the $xy$-plane. We stress that the $k$-derivative of $|\tilde{u}_k|$ does not enter into the expression of $\sigma_{xy}$, while it exists in expressions of $\Delta \sigma_{xy}$ and $\alpha_{xy}$.

In the expression of $\sigma_{xy}^{\text{RTA}}$, $(-d\tilde{u}_k^x/dk_1)$ represents the curvature of the Fermi surface [168, 169]; in both hole-doped and electron-doped HTSCs, $(-d\tilde{u}_k^x/dk_1) \sim 1/k_F$ ($> 0$) on the Fermi surface. On the other hand, $(-d\tilde{u}_k^y/dk_1)$ exhibits strong $k$-dependence in nearly AF metals, as shown in figure 14(i); $(-d\tilde{u}_k^y/dk_1)$ is positive around point $\Lambda$, whereas it is negative around point $B$. Since the cold spot is located around point $\Lambda$ in hole-doped systems [33], this present study predicts that $R_{HI} > 0$ in hole-doped systems when the AF fluctuations are strong. (Note that the charge of electron is $-e$ ($e > 0$) in this study.) On the other hand, $R_{HI} < 0$ in electron-doped systems since the cold spot in electron-doped systems is $B$ [33].

For an intuitive understanding of the CVC, we introduce an ‘effective Fermi surface’ obtained by bending the true
Fermi surface such that it is orthogonal to \( \vec{J}_k \) around the cold spot [20]. The effective Fermi surface in HTSCs is shown in figure 14(ii). It can be seen that the curvature of the effective Fermi surface, which is equal to \( (\partial \theta J / \partial k) \) by definition, takes a large positive (negative) value around the cold spot in hole-doped (electron-doped) systems. Therefore, \( R_{\text{H}} \) in hole-doped (electron-doped) systems exhibits a large positive (negative) value at low temperatures [33].

Figures 19(i), (iii) and (iv) show the numerical results for \( R_{\text{H}} \) obtained from the FLEX approximation by including the CVC. In hole-doped systems \( (n < 1) \), \( R_{\text{H}} \) increases as the doping \( \delta = |1 - n| \) decreases. On the other hand, \( R_{\text{H}} \) for electron-doped systems \( (n > 1) \) becomes negative below \( T = 0.09 - 400 \text{ K} \). These results are consistent with the experimental results. Previously, negative \( R_{\text{H}} \) for electron-doped systems was considered to be strong evidence of the deformation of the Fermi surface at low temperatures [33]: the deformation of the Fermi surface is caused by the strong \( k \)-dependence of the self-energy, which becomes prominent when AF fluctuations are strong. A tiny increment in \( R_{\text{H}}^{\text{RTA}} \) in YBCO below \( T = 0.02 \) is caused by the strong anisotropy of \( \gamma_k \). In the present calculation, \( \gamma_{\text{hot}} / \gamma_{\text{cold}} \) is approximately 3 at \( T = 0.02 \).

Figure 19(ii) shows the \( k_z \)-dependence of \( \Delta \sigma_{xx}(k_i) = \tilde{\sigma}_{xx}(\vec{L}_k)/|\tilde{\sigma}| \) and \( \Delta \sigma_{xy}(k_i) = |\tilde{\sigma}_{xy}(\vec{L}_k)|/|\tilde{\sigma}| \), where \( \sigma_{xx} = \int_{B\text{FS}} |\partial \theta J / \partial k_i| \Delta \sigma_{xx}(k_i) \) and \( \sigma_{xy} = \int_{B\text{FS}} |\partial \theta J / \partial k_i| \Delta \sigma_{xy}(k_i) \). In both \( \sigma_{xx} \) and \( \sigma_{xy} \), the quasiparticles around the cold spot give the dominant contributions. We can see that \( \sigma_{xx} \) is slightly reduced by including the CVC. On the other hand, \( \sigma_{xy} \) obtained from the FLEX + CVC theory is considerably larger than \( \sigma_{xx}^{\text{RTA}} \), since \( |\partial \theta J / \partial k_i| \gg |\partial \theta J / \partial k_i| \) at the cold spot. \( \sigma_{xy}(k_i) \) is large only around the cold spot, and it becomes negative around the hot spot.

Here, we discuss the temperature dependence of \( R_{\text{H}} \) in detail using the approximate expression for \( J_k \) in equation (79). Since \( \partial \sigma / \partial k_{||} = 0 \) at the cold spot (point A) because of the symmetry,

\[
\frac{1}{1 - \sigma_k^{\text{RTA}}} \left( \frac{d\tilde{\sigma}_k}{dk_{||}} + \alpha_k \frac{d\tilde{\sigma}_{xy}}{dk_{||}} \right)
\]
at the cold spot. By noticing the relationships \((\vec{v}_k \times d\vec{v}_k/dk_1)_z = - (\vec{v}_k \times d\vec{v}_k/dk_2)_z\) and \((\vec{v}_k \times d\vec{v}_k/dk_3)_z = - (\vec{v}_k \times d\vec{v}_k/dk_3)_z\), we obtain that

\[
\left( \vec{J}_k \times \frac{d\vec{J}_k}{dk} \right)_z = |\vec{J}_k|^2 \frac{d\vec{J}_k}{dk} \frac{1}{\alpha k}^2 \left( \begin{array}{c} \frac{d\vec{J}_k}{dk_1} \\ \frac{d\vec{J}_k}{dk_2} \\ \frac{d\vec{J}_k}{dk_3} \end{array} \right)_{\text{cold}}
\]

\[
= \frac{1}{1 - \alpha k} |\vec{J}_k|^2 \frac{d\vec{J}_k}{dk} \frac{1}{\alpha k}^2 ,
\]

which is proportional to \((1 - \alpha k)^{-1} \propto \xi_{AF}^2\) [33]. In fact, figure 17 (i) shows that equation (96) increases as the temperature decreases. As a result, \(R_H\) behaves as [33]

\[
|R_H| \propto \xi_{AF}^2 .
\]

Therefore, both the sign and the \(T\)-dependence of \(R_H\) in hole-doped HTSCs are successfully reproduced in the present approach.

Finally, we discuss electron-doped systems. At point B in figure 4(i), \(k_B = k_B - Q\) is equal to \(k_B = k_B - Q\) for \(Q = (\pi, -\pi)\). Then, the CVC at \(k_B\), which is given by the second term of equation (75), is approximately proportional to \(\vec{J}_B\). Therefore, \(\alpha k\) is given by the simplified Bethe–Salpeter equation (78) and \(\vec{J}_B \approx \vec{v}\) at point B. Since \(\alpha k\) rapidly increases if \(k\) deviates from point B, \(- d\theta^2/dk\) attains a large negative value around the cold spot of NCCO. Therefore, the negative sign of \(R_H\) is realized by considering the CVC.

Finally, we stress that we have also studied the CVC by using a widely used phenomenological AF fluctuation model given in equation (8). Assuming a reasonable set of parameters, we find that \(R_H\) is prominently enhanced due to the CVC [34]. Thus, the enhancement of \(R_H\) in nearly AF metals is not an artifact in the FLEX approximation, but a universal phenomenon near the AF-QCP.

### 4.3. Magnetoresistance

Next, we study the orbital magnetoresistance \(\Delta \rho/\rho_0\) in HTSCs by considering the CVC. According to the linear response theory [1], the magnetoresistance is given by

\[
\Delta \rho/\rho_0 \equiv - \Delta \sigma_{xx}/\sigma_{xx}^0 - (\sigma_{xy}/\sigma_{xx}^0)^2 \quad \text{(98)}
\]

where \(\sigma_{xx}^0\) denotes the conductivity without a magnetic field, and \(\Delta \sigma_{xx} \equiv \sigma_{xx}(H) - \sigma_{xx}^0\) is the magnetoconductivity, which is always negative.

To derive the magnetoresistance, we have to calculate the magnetoresistance, which is given in equation (40) in the RTA. By performing the partial integration, equation (40) for 2D systems is rewritten as

\[
\Delta \sigma_{xx}^{\text{RTA}} = -H_c^2 \frac{e^2}{8} \int_{FS} \frac{dk}{(2\pi)^2} |\vec{v}| \left[ \left| \vec{v}_k \right|^2 \left( \frac{d\theta^2}{dk_1} \right)^2 + \left( \frac{d\vec{v}_k}{dk_1} \right)^2 \right] \frac{1}{\gamma k} \quad \text{(99)}
\]

where \(\vec{v}_k = \vec{v}_k/\gamma k\). Here, we used the relation \(|v_k| \partial/\partial k_1 = (\hat{\xi} \times \vec{v}_k) \cdot \vec{v}_k = (\hat{\xi} \times \vec{v}_k \times \vec{v}_k)_z\) in 2D systems. When the Fermi surface is spherical, the second term in brackets in equation (99) vanishes identically. In this case, \(\sigma_{xx}^0 = e^2 k_F v_F/4\pi\gamma\), \(\sigma_{xx} = (e v_F/2\gamma)\sigma_{xx}^0\), and \(\Delta \sigma_{xx} = -(e v_F/2\gamma)^2 \sigma_{xx}^0\), according to the RTA, where \(k_F\) and \(v_F\) are the Fermi momentum and the Fermi velocity, respectively. Therefore, \(\Delta \rho/\rho_0\) given in equation (98) becomes zero. Except for this special case, the orbital magnetoresistance is always positive [1].

The general expression for the magnetoresistance in Fermi liquids was derived in [38], which is exact of order \(O(\gamma^{-3})\). It is derived by performing the analytic continuation of equation (48) for \(m = 2\). This work had enabled us to calculate the magnetoresistance along with satisfying the conservation laws. At low temperatures, the magnetoresistance can be expressed by a simple form: \(\Delta \sigma_{xx} = \Delta \sigma_{xx}^0 + \Delta \sigma_{xx}^0\),

\[
\Delta \sigma_{xx}^a = -H_c^2 \frac{e^2}{4} \sum_k \left( \frac{\partial \theta^2}{\partial \chi_s} \right)_{\text{cold}} \frac{1}{\gamma k},
\]

\[
\Delta \sigma_{xx}^b = -H_c^2 \frac{e^4}{4} \sum_k \left( \frac{\partial \theta^2}{\partial \chi_s} \right)_{\text{cold}} d_k, \frac{1}{\gamma k},
\]

\[
D_{k} = \int_{k} \frac{1}{\gamma k} \frac{\partial L}{\partial \chi_s} \frac{1}{\gamma k},
\]

where \(\theta = \vec{J}_B/\gamma k \) and \(\theta(k,0)\) is the irreducible four-point vertex in the particle–hole channel introduced in equation (60) [40]. \(\Delta \sigma_{xx}^{\text{RTA}}\) in equation (40) is given by equations (100) and (102), by replacing \(\vec{J}_B\) with \(\vec{v}\). In the FLEX approximation, \(J_{k}^{(0)}(0,\epsilon) \approx (3\epsilon^2/2)(\pi T)^2 \Im \partial \chi_s^4(\epsilon)/\partial \epsilon_{\text{cold}}^0\), as shown in equation (78).

Here, we can rewrite \(\Delta \sigma_{xx}^a\) as

\[
\Delta \sigma_{xx}^a = -H_c^2 \frac{e^2}{8} \int_{FS} \frac{dk}{(2\pi)^2} |\vec{v}| \left[ \left| \vec{v}_k \right|^2 \left( \frac{d\theta^2}{dk_1} \right)^2 + \left( \frac{d\vec{v}_k}{dk_1} \right)^2 \right] \frac{1}{\gamma k},
\]

where \(\vec{L}_k = \vec{J}_B//\gamma k\). According to equation (96), the first term in the brackets in equation (104) is proportional to \(d\theta^2/dk1^2 \gamma \propto \xi_{AF}^2 \gamma \propto \xi_{AF}^2 \gamma \) [35]. Further, remaining terms (the second term in equation (104) and \(\Delta \sigma_{xx}^b\)) are also proportional to \(\xi_{AF}^2 \gamma \) for a wide range of temperatures, since the \(k_1\)-derivative of \(\xi_{AF}^2 \gamma \) yields a factor proportional to \(\xi_{AF}^2 \gamma \) [33,35]. In section 5.2, we will explain that \(\vec{L}_k\) becomes highly anisotropic below \(T^*\) due to AF+SC fluctuations. For this reason, due to the second term in equation (104), \(\Delta \rho/\rho_0\) is prominently enhanced below \(T^*\) in under-doped HTSCs.

Therefore, the magnetoresistance in nearly AF metals behaves as

\[
\Delta \rho/\rho_0 \propto \xi_{AF}^2 \rho_0^{-2} \propto T^{-4}
\]

above \(T^*\) due to the CVC since \(\rho_0 \propto T\) and \(\xi_{AF}^2 \propto T^{-1}\). Equation (105) is consistent with experimental results in LSCO.
exhibits a strong \(\zeta\) in YBCO and BSCCO [12, 71, 72], and it increases (decreases) as the doping increases (decreases) [12]. These experimental results are reproduced by the present study and they are shown in figure 20. In LSCO, the Fermi surface is very close to the van-Hove singularity point \((\pi, 0)\) since both \(|t'|/t\) and \(|t''/t|\) in equation (12) are smaller than those in other systems. Since \(\nu_k = 0\) at \((\pi, 0)\), the anisotropy of the velocity \(|\nu_k|\) on the Fermi surface is larger in LSCO. Hence, the second term in equation (104) (or equation (99)) takes a large value in LSCO. In heavily over-doped LSCO at \(\delta = 0.225\), where CVC is expected to be unimportant, \(\zeta\) exceeds 100 [12]. This result is consistent with recent ARPES measurement [136], which shows that the Fermi surface in LSCO passes through \((\pi, 0)\) in the over-doped region \(\delta = 0.2 \sim 0.22\).

Therefore, the experimental value of \(\zeta\) gives us a useful measure of the anisotropy of \(L_\beta = j_k/\gamma_k (\kappa_k = \nu_k/\gamma_k)\) in weakly correlated systems. In CeMIn5 (\(M = \text{Rh, Co})\), \(\zeta\) is of order \(O(100)\) [20], which indicates that the anisotropy of \(L_\beta\) is large in the main Fermi surface.

4.4. Thermoelectric power

We also discuss the thermoelectric power, \(S\). In HTSCs, \(S\) takes a large value in under-doped systems, and it increases as \(T\) decreases from room temperature. Except for overdoped compounds, \(S > 0\) in hole-doped systems [2,157–162] whereas \(S < 0\) in electron-doped systems [7,163] below room temperature. Interestingly, the peak temperature of \(S\) is nearly equal to the pseudo-gap temperature \(T^\ast\) in many hole-doped compounds; in \(\text{HgBa}_2\text{CuO}_4\) [159,160], LSCO, YBCO [161] and \(\text{Bi}_2\text{Sr}_2\text{Ru}_x\text{O}_{6+y}\) \((R = \text{Ca, Y, Pr, Dy or Er})\) [162]. By neglecting the CVC, Hildebrand et al calculated the thermoelectric power for YBCO using the FLEX approximation [164]. Here, we study the thermoelectric power both for hole-doped and electron-doped systems using the FLEX + CVC method.

According to the linear response theory [39,165,166], the thermoelectric power in a Fermi liquid system is given by

\[
S = \alpha_{xx}/\sigma_{xx},
\]

where \(\alpha_{xx}\) is the diagonal Peltier conductivity; \(j_x = \alpha_{xx}(-\nabla_x T)\). It is given by [39,165,166],

\[
\alpha_{xx} = K_{xx}^a (0\omega_0) / i\omega_0,\]

where \(K_{xx}^a (0\omega_0)\) is given by the analytic continuation of [39]

\[
K_{xx}^a (i\omega_0) = \frac{1}{T} \int_0^\beta d\tau e^{-\omega_0 T} (T; j_x^0 (0), j_x (\tau)),
\]
pseudo-gap behavior in AF fluctuations, $k$ is the prominent NCCO (room temperature, and it increases in the under-doped case. In sufficiently low temperatures, where $\vec{\nabla} \cdot \vec{j}_Q(\epsilon) \approx 0$. Then, $\vec{j}_Q$ contains complex two-body terms in the presence of electron–electron correlation. Fortunately, the Ward identity allows us to rewrite the heat current as $\vec{j}_Q^0(\epsilon) = e\vec{v}_k$, where $\vec{v}_k = \nabla_k (\epsilon_k^0 + \text{Re} \Sigma_k)$ [39]. By performing the analytic continuation of equation (110), we can derive the exact expression of $\alpha_{xx}$ of order $O(E_F/\gamma)$ as [39]

$$\alpha_{xx} = -\frac{e}{T} \sum_k \int \frac{d\epsilon}{\pi} \left( -\frac{\partial f^0}{\partial \epsilon} \right) \epsilon v_{kk}(\epsilon) \times \left( J_{kr}(\epsilon)|G_k(\epsilon)|^2 - 2v_{kr}(\epsilon)\text{Re}G_k(\epsilon)^2 \right),$$

(111)

where $J_{kr}(\epsilon)$ is the total current given in equation (60). At sufficiently low temperatures, $\alpha_{xx}$ is approximately simplified as

$$\alpha_{xx} = -\frac{e^2 T}{3} \int_{FS} \frac{dk_{||}}{(2\pi)^2} \frac{1}{v_k(E_k^0)} \int_{-\pi T}^{\pi T} \frac{d\epsilon}{\pi} \frac{\partial}{\partial k_{\perp}} \left( v_{kk}(E_k^0) J_k(E_k^0) \right),$$

(112)

where $-e (e > 0)$ is the electron charge. $dk_{\perp}$ represents the momentum vertical to the Fermi surface. When $\gamma$ is energy-independent, equation (112) yields $S = -e\pi T/6\gamma \sigma_{xx} = -\pi T m^*/3\hbar$ ($m^* = m/z$ is the effective mass) in the 2D isotropic system with $E_k = k^2/2m - \mu$.

In figure 21(i), we show the numerical results for $S$ of YBCO and NCCO given by the FLEX + CVC method. In LSCO, $S > 0$ in the optimally doped case ($n = 0.85$) below room temperature, and it increases in the under-doped case. In NCCO ($n = 1.10$), $S < 0$ and has a peak around 200 K. These results are consistent with the experimental results [2, 157–161]. The origin of the non-Fermi-liquid-like behavior is the prominent $k$- and $e$-dependences of $\gamma_k(\epsilon)$ due to strong AF fluctuations. According to the spin fluctuation theory, $\partial \gamma_k(E_k^0)/\partial k_{\perp} > 0$ at point A in figure 4(i) because $\gamma_k$ becomes large near the AFBZ. Since point A is the cold spot of YBCO, equation (112) becomes positive in YBCO. Therefore, $S > 0$ in under-doped YBCO and LSCO. On the other hand, $S < 0$ in NCCO because $\partial \gamma_k(E_k^0)/\partial k_{\perp} < 0$ at point B. In YBCO, $S$ slightly decreases if the CVC is included, since $|\vec{j}_Q| < |\vec{v}_k|$ around the cold spot. In NCCO, in contrast, the absolute value of $S$ is enhanced by the CVC. In both cases, the effect of the CVC on $S$ is smaller than that on $R_{HH}$. Figure 21(ii) shows the $S$ for YBCO ($n = 0.90$), together with $S_{in}$ ($S_{out}$) which denotes the contribution from the inside (outside) of the 2D isotropic system with $E_k = k^2/2m - \mu$.

4.5. Summary of this section and comments on other transport coefficients

In HTSCs, anomalous transport phenomena (such as the violation of Kohler’s rule) had been frequently considered strong evidence for the breakdown of the quasiparticle picture. For example, the RTA for the highly anisotropic $t_2$ model and cannot reproduce relationships (1)–(4) for hole-doped systems at the same time for a wide range of temperatures. Furthermore, the RTA cannot explain the negative Hall coefficient in electron-doped systems, since the curvature of the true Fermi surface is positive everywhere. The explanation for the nearly symmetric behavior of $R_{HH}$ and $S$ with respect to the type of carrier doping (shown in figure 3) was desirable for a long time. These highly non-trivial transport phenomena had been one of the central issues in HTSCs, which should serve to elucidate the actual electric ground state of HTSCs.

To resolve this long-standing problem, we developed a method to calculate various transport coefficients based on the microscopic Fermi liquid theory. In the RTA, the momentum...
and energy transfers between the quasiparticles by scattering are not treated correctly. Therefore, the RTA results frequently yield erroneous results. To overcome this defect, we study the role of the CVC in nearly AF Fermi liquids. We find that the total current $J_k$ shows an anomalous $k$-dependence because of the CVC in nearly AF metals, which gives rise to the non-Fermi liquid behaviors such as $R_H \propto \varepsilon_{AF}^2 \propto T^{-1}$ and $\Delta \rho / \rho_0 \propto \rho_0^{-2} \varepsilon_{AF}^4 \propto T^{-4}$. Consequently, the modified Kohler’s rule $\Delta \rho / \rho_0 \propto (R_H / \rho)^2 \propto T^{-4}$ is realized in HTSCs. We also studied the $S$ for HTSCs, which increases as the temperature decreases above $T^*$. This experimental fact is reproduced by considering the strong $k$- and $\varepsilon$-dependences of $\tau_k(\varepsilon)$ [36]. Therefore, various anomalous transport phenomena in HTSCs above $T^*$ can be well explained by the FLEX + CVC method.

Recently, Tsukada et al. measured the $R_H$ in clean heavily over-doped LSCO samples [170] and found that $R_H$ becomes almost temperature-independent for $\delta \gtrsim 0.24$; further, its sign smoothly changes from positive to negative between $\delta = 0.28$ and 0.32, corresponding to the change in the curvature of the Fermi surface. Hussey et al. also showed that the RTA analysis is successful in heavily over-doped Ti2201 [149] and LSCO [150]. These experimental results are consistent with the present theory: according to the present analysis, the RTA works well in heavily over-doped systems since the CVC is unimportant when the AF fluctuations are very weak. This fact is well explained in [167].

Finally, we comment on other important theoretical studies involving the CVC in strongly correlated systems. In the interacting electron-gas model, the electron cyclotron frequency $\omega_c$ is given by $eH/mc$, where $m$ is the bare electron mass. In 1961, Kohn proved that $\omega_c$ is unchanged by electron–electron interaction due to the consequence of the angular momentum conservation law, which is called the ‘Kohn’s theorem’ [171]. The Kohn’s theorem is also proved by the phenomenological Fermi liquid theory, by correctly considering the CVC [172]. Kanaki and Yamada studied this problem based on the microscopic Fermi liquid theory, and found that $\omega_c$ is influenced by the electron–electron correlation if the Umklapp processes are present [173]. The Umklapp processes also reduces the Drude weight in $\sigma(\omega)$ [174] and the penetration depth in the SC state [175]. Moreover, the effect of the vertex correction on the Raman spectroscopy in HTSC was studied [176].

5. Transport phenomena in HTSCs below $T^*$

5.1. Mechanism of pseudo-gap phenomena

In under-doped HTSCs, a deep pseudo-gap in the DOS emerges at the chemical potential below $T^* \sim 200$ K. The origin of the pseudo-gap has been a central issue with regard to HTSCs. The pseudo-gap does not originate from the spin fluctuations since the strength of the spin fluctuations decreases below $T^*$ [78–82]. According to recent ARPES measurements [83–86], the $k$-dependence of the pseudo-gap coincides with that of the $d_{x^2-y^2}$-wave SC gap function. This pseudo-gap structure in the quasiparticle spectrum $\sigma(k, \omega) = \text{Im}G(k, \omega - i\delta)/\pi$ starts to appear around $\omega = 0$ below $T^*$. Below $T_c$, the quasiparticle spectrum shows sharp peaks at the edge of the SC gap ($\omega \sim \Delta$) since the inelastic scattering is reduced by the SC gap. At the same time, a dip-hump structure is induced around $\omega_{\text{D}} \gtrsim \Delta$ by the strong resonance peak in Im$\chi^*(\omega)$ at $\omega_L \sim 40$ meV in YBCO and Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ (BiSCCO). We note that $\omega_{\text{D}} \approx \Delta + \omega_L$ and $\omega_L < \Delta$. However, the overall gap-like structure in the quasiparticle spectrum remains unchanged around $T_c$.

Motivated by these experimental facts, various strong-coupling theories of SC fluctuations have been studied to reproduce the pseudo-gap phenomena [51, 87–89]. Several groups developed the ‘FLEX + $T$-matrix theory’, where the normal self-energy correction induced by the strong SC fluctuations, $\Sigma_k^{\text{SCF}}(\varepsilon)$, has been self-consistently included into the FLEX approximation. The SC ‘amplitude’ fluctuations are given by the $T$-matrix (particle-particle scattering amplitude) induced by the AF fluctuations, which prominently increase below $T^*$ [37, 51, 52, 88, 89]. The $T$-matrix with respect to the AF fluctuations, which is responsible for the Thouless instability for the $d_{x^2-y^2}$-channel, is given by

$$T_{k,k}(q, \varepsilon_n, \varepsilon_m; \omega_l) = V_{k,k}^{\text{AFF}}(\varepsilon_n - \varepsilon_m) + T \sum_{m,p} V_{k,k}^{\text{AFF}}(\varepsilon_n - \varepsilon_m) \times G_{k}(\varepsilon_m)G_{-p,q}(\varepsilon_m + \omega_l) + \text{AFF}(\varepsilon_n),$$

which is shown in figure 22(i). $V_{k,k}^{\text{AFF}}$ is given in equation (15). In the FLEX + $T$-matrix approximation, the Green function and the self-energy are given by

$$G_k(\varepsilon_n) = [\varepsilon_n + \mu - \varepsilon_0^0 - \Sigma_k^{\text{AFF}}(\varepsilon_n) - \Sigma_k^{\text{SCF}}(\varepsilon_n)]^{-1},$$

$$\Sigma_k^{\text{AFF}}(\varepsilon_n) = T \sum_q \Delta_{k,q}(\omega)T_{k,k}(q, \varepsilon_n, \varepsilon_m; \omega_l),$$

where $\Sigma_k^{\text{AFF}}$ and $\Sigma_k^{\text{SCF}}$ are shown in figure 22(ii). In the self-consistent FLEX + $T$-matrix approximation, we have to solve equations (14), (15) and (113)–(115) self-consistently. Unfortunately, it is very difficult to calculate the $T$-matrix in equation (113) since a lot of computer memory is required. Fortunately, equation (113) can be simplified by considering only the $d_{x^2-y^2}$-channel and dropping other pairing channels as $T_{k,k,q}(\varepsilon_n, \varepsilon_m; \omega_l) \approx \psi_k(\varepsilon_n)\psi^*_k(\varepsilon_m)\delta(q_{\omega_l})$, where $\psi_k \propto \cos k_x - \cos k_y$ [37, 51, 52, 88, 89]. This approximation is expected to be reasonable when the temperature is close to $T_c$ in HTSCs. Then, the $k$-dependence of $\Sigma_k^{\text{SCF}}$ becomes $\Sigma_k^{\text{SCF}} \propto T_{k,k,q} \propto \psi_k^2$.

In the present approximation, $T^*$ is defined as the temperature below which $\Sigma_k^{\text{SCF}}$ takes considerable values. $T^*$ is slightly greater than the $\xi_{\text{FLEX}}$ in the FLEX approximation. In a pure 2D system, $T_c = 0$ since the FLEX + $T$-matrix theory satisfies the Mermin–Wagner–Hohenberg theorem. $T_c$ becomes finite when weak three-dimensionality is assumed. Sufficiently above $T^*$, where $\Sigma_k^{\text{SCF}} \approx 0$, the FLEX + $T$-matrix theory is equivalent to the FLEX approximation. The obtained DOS and the self-energy are shown in figures 22(iii) and (iv), respectively. Below $T^* \sim 0.04$, a large pseudo-gap structure appears in the FLEX + $T$-matrix approximation, due to the cooperation of the real part and the imaginary part of $\Sigma_k^{\text{SCF}}$ [37].
The AF fluctuations are suppressed below \( T^* \) in accordance with the pseudo-gap in the DOS [51, 52, 88, 89]. For this reason, \( \text{Im} \Sigma_{k}(\omega) \) in the FLEX + \( T \)-matrix approximation is much smaller than \( \text{Im} \Sigma_{k}(0) \) in the FLEX approximation, as recognized in figure 22(iv). Below \( T_{c} \), the pseudo-gap in the FLEX + \( T \)-matrix approximation smoothly changes to the SC gap [87].

Here, we discuss transport phenomena in the pseudo-gap region. In YBCO [9, 74], Bi2201 and Bi2212 [177], \( R_{H} \) shows a maximum around the pseudo-gap temperature \( T^* \), at which \( 1/T_{c} \) shows the maximum value. Also, the peak temperature of \( S \) is nearly equal to \( T^* \) in HgBa2CuO4+\( \delta \) [159, 160], LSCO, YBCO [161] and Bi2Sr2RCu2O8 (\( R = \text{Ca, Y, Pr, Dy and Er} \) [162]. As discussed in [33], these behaviors are naturally understood since both of them are strikingly enhanced by the AF fluctuations above \( T^* \). Therefore, both \( R_{H} \) and \( S \) should decrease below \( T^* \) in accordance with the reduction in the AF fluctuation. This behavior is easily reproduced by the FLEX + \( T \)-matrix method [33, 51]. The obtained \( \rho, R_{H} \) and \( S \) for hole-doped systems by using the [FLEX + \( T \)-matrix] + CVC method are shown in figures 18(i), 19(i) and 21(i), respectively. In this approximation, \( T_{c}^{(0)}(\epsilon, \epsilon') \) in equation (60) is given by \( j^t \sum_{\omega}^{\Sigma} \text{Im} \{ k_{\omega}^{-} - k_{\omega}^{+} - T_{k,k,q}^{(0)}(\epsilon, \epsilon' - \epsilon') \} \), where \( T_{k,k,q}^{(0)} \) represents the \( T \)-matrix in equation (113). The detailed method of calculation is explained in [37].

As shown in figures 19(i) and 21(i), both \( R_{H} \) and \( S \) start to decrease below \( T^* \sim 0.04 \sim 0.160 \) \( K \) in the FLEX + \( T \)-matrix approximation. These results are consistent with experiments [159–161]. Figure 18(i) shows the resistivity given by the FLEX approximation and the FLEX + \( T \)-matrix approximation, both of which are given by including CVCs. As pointed out in [167], \( \rho \) given by the FLEX + CVC method shows a tiny ‘kink’ structure at \( T_{0} \sim 0.12 \) is experimentally observed in LSCO [185] and YBCO [186]. The kink becomes more prominent in the [FLEX + \( T \)-matrix] + CVC method since the inelastic scattering is reduced by the formation of the pseudo-gap. As shown in figure 22(iv), \( \text{Im} \Sigma_{k}(\omega) \) decreases prominently below \( T^* \), in accordance with the emergence of \( \text{Im} \Sigma_{k}^{\text{SCF}}(\omega) \). As a result, \( \rho \) is approximately proportional to \( T^{2} \) below \( T^* \). To summarize, \( \rho, R_{H} \) and \( S \) are suppressed in the pseudo-gap region in the [FLEX + \( T \)-matrix] + CVC method.

According to the FLEX + \( T \)-matrix theory, the SC ‘amplitude’ fluctuations are the origin of the pseudo-gap formation. Other than this scenario, various mechanisms of pseudo-gap formation have been proposed. For example, Kivelson and Emery proposed the SC ‘phase’ fluctuation scenario [178]: they considered that the amplitude of the SC order-parameter develops in the pseudo-gap region, whereas the global phase coherence is absent. Moreover, several hidden-order scenarios, such as d-density wave formation [179], have been proposed. To elucidate the correct scenario, anomalous transport phenomena in the pseudo-gap region are significant and they severely constrain the theories. In the following, we show that the scenario involving SC ‘amplitude’ fluctuations enables us to understand the anomalous transport phenomena below \( T^* \) in a unified way.

5.2. Enhancement of Nernst coefficient and magnetoresistance

In contrast to \( R_{H} \) and \( S \), the Nernst coefficient \( v \) (\( v = S_{xy} / H = -E_{y} / H \nabla_{x} T \); off-diagonal thermoelectric power under a magnetic field) prominently increases in the pseudo-gap region by approximately 100 times that in conventional metals [180–182]. Moreover, in the pseudo-gap region, the magnetoresistance \( \Delta \rho / \rho_{0} \) increases faster than \( T^{4} \) [75, 76]. Therefore, modified Kohler’s rule in equation (106) is violated below \( T^* \). Such non-trivial behaviors of \( v \) and \( \Delta \rho / \rho_{0} \) have attracted considerable attention as a key phenomenon closely related to the origin of the pseudo-gap.
Here, we study the Nernst coefficient. According to the linear response theory \([39, 165, 166]\), \(\nu\) is given by
\[
\nu = \frac{a_{\alpha\gamma}/a_{\alpha\alpha} - S_{\alpha\gamma}/a_{\alpha\alpha}}{H_z},
\]
(116)
where \(a_{\alpha\alpha} = j_\alpha/(\nabla \cdot \mathbf{T})\), and \(a_{\alpha\gamma} = j_\alpha/(\nabla \cdot \mathbf{T})\) is the off-diagonal Peltier conductivity under the magnetic field \(H_z\). In conventional metals with simple Fermi surfaces, \(\nu\) is small because of an approximate cancellation between the first and the second terms in equation (116), which is known as the Sondheimer cancellation. (Sondheimer cancellation is exact only when \(\epsilon^0_\mathbf{k} = k^2/2m_0\).) In HTSCs, however, Sondheimer cancellation is totally violated since \(\alpha_{\alpha\gamma}\) is considerably enhanced. That is, \(\alpha_{\alpha\gamma}\) is the origin of giant Nernst coefficient in LSCO below \(T^*\) and in NCCO below \(T_\Omega[180, 181]\).

In this study, we investigate \(\alpha_{\alpha\gamma}\) due to the quasiparticle transport, and find that \(\alpha_{\alpha\gamma}\) is prominently enhanced below \(T^*\) if we include the CVC caused by SC and AF fluctuations. As a result, \(\nu \approx a_{\alpha\gamma}/a_{\alpha\alpha}\) takes a large value below \(T^*\). Moreover, we show that all the transport anomalies below \(T^*\) are understood as the quasiparticle transport phenomena in a unified way. Note that Yip showed that the Maki–Thompson-type CVC due to SC fluctuations disappears in d-wave superconductors when the inelastic scatterings are negligible \([184]\). However, this is inappropriate for HTSC since inelastic scatterings are much larger than elastic scatterings. We find that the Maki–Thompson type CVC is significant in HTSC.

We note that Ussishkin et al studied \(\nu\) induced by short-lived Cooper pairs, which is independent of the quasiparticle lifetime and order \(O(\gamma^0)\) \([183]\). It will be observed in the close vicinity of \(T_c\). However, quasiparticle transport is dominant in good metals where \(\kappa_{\theta\theta} \sim E_F/\gamma \gg 1\). (Here, \(l = \hbar v_F/\gamma\) is the mean free path.) In HTSCs, \(\kappa_{\theta\theta} = 1\) corresponds to 1700 \(\mu\Omega\) cm if we omit the \(k\)-dependence of \(\kappa_{\theta\theta}\). Therefore, quasiparticle transport will be dominant in slightly underdoped HTSC (\(\rho \lesssim 200 \Omega\) cm above \(T_c\)) for a wide range of temperatures. Hereafter, we show that \(\alpha_{\alpha\gamma}\) due to quasiparticle transport is strongly enhanced by the CVC.

According to the microscopic Fermi liquid theory, \(\alpha_{\alpha\gamma}\) is given by the correlation function between the heat current and the charge current in the presence of \(H_z \neq 0\). Therefore, \(\alpha_{\alpha\gamma}\) is given by equation (50) by replacing \(j_\mu\) with \(j^{0}_\mu\). After the analytic continuation, the exact expression for \(\alpha_{\alpha\gamma}\) of the order \(\gamma^{-1}\) is given by \([39]\)
\[
\alpha_{\alpha\gamma} = \frac{H_z \cdot q^2}{T} \sum_k \int \frac{d\epsilon}{2\pi} \left( \frac{-\partial f_0'}{\partial \epsilon} \right) |\text{Im} G_{\mathbf{k}}(\epsilon)| |G_{\mathbf{k}}^R(\epsilon)|^2 \times |\tilde{q}_\mathbf{k}(\epsilon)|^2 \chi_\mathbf{k}(\epsilon) A_{\mathbf{k}}(\epsilon),
\]
(117a)
where
\[
A_{\mathbf{k}}(\epsilon) = \left( \tilde{Q}_{\mathbf{k}}(\epsilon) \times \frac{\partial \tilde{L}_{\mathbf{k}}(\epsilon)}{\partial k_1} \right)_z,
\]
(117b)
\[
\tilde{Q}_{\mathbf{k}}(\epsilon) = \tilde{q}_\mathbf{k}(\epsilon) + \sum_k \int \frac{d\epsilon'}{4\pi i T_{\mathbf{k}}^0(\epsilon, \epsilon')} |G_{\mathbf{R}}^0(\epsilon')|^2 \tilde{Q}_{\mathbf{k}}(\epsilon'),
\]
(117c)
where \(\tilde{L}_{\mathbf{k}}(\epsilon) = \tilde{J}_\mathbf{k}(\epsilon)/|\chi_\mathbf{k}(\epsilon)|\), \(\tilde{q}_\mathbf{k}(\epsilon) = \epsilon \cdot \tilde{v}_\mathbf{k}\) is the quasiparticle heat velocity, and \(\tilde{Q}_\mathbf{k}(\epsilon)\) is the total heat current \([37, 39]\). We stress that the CVC term in equation (117) vanishes if we omit the energy dependence of \(T_{\mathbf{k}}^0(\epsilon, \epsilon')\).

This fact means that the heat CVC is small and thus \(\tilde{Q}_\mathbf{k}(\epsilon) \sim \tilde{q}_\mathbf{k}(\epsilon)\) in general cases \([37, 39]\). In the FLEX + \(T\)-matrix approximation, \(T_{\mathbf{k}}^0(\epsilon, \epsilon')\) is given by \(|i\text{th} \epsilon^{\mathbf{k} \cdot \mathbf{q}}_{\mathbf{k}, \mathbf{k}} + \text{th} \epsilon^{\mathbf{k} \cdot \mathbf{q}}_{\mathbf{k}, \mathbf{k}}| |\tilde{V}_{\mathbf{k}-\mathbf{k}}(\epsilon - \epsilon; \epsilon' - \epsilon')|\).

In contrast to \(R_H\) and \(S\), the Nernst coefficient \(\nu[180, 181]\) and \(\Delta/\rho_0[12, 75, 76]\) in LSCO rapidly increase below \(T^*\). These experimental facts are also reproduced by the [FLEX+\(T\)-matrix] + CVC method as shown in figure 23(ii) and (ii). Since \(R_H\) decreases whereas \(\Delta/\rho_0\) increases drastically below \(T^*\), the plot of \(\Delta/\rho_0/\rho_0\) as a function of \(T^2\) for \(\gamma_0\) forms an ‘inverse S-shape’. Therefore, modified Kohler’s rule is completely violated below \(T^*\). This result is very similar to the experimental results provided in \([12]\). Next, we discuss \(\nu\) in NCCO using the FLEX (not the FLEX + \(T\)-matrix) approximation since SC fluctuations is considered to be absent in NCCO. The obtained \(\nu\) is shown in figure 23(iii); it takes a large value only when the CVC due to the AF fluctuations is considered. In NCCO, \(\nu\) increases gradually as \(T\) decreases, and it starts to decrease below the maximum temperature of 120 K. The obtained behaviors of \(\nu\) are semiquantitatively consistent with the experimental results \([7]\). Therefore, the present numerical study can explain the experimental behaviors of \(\nu\) and \(\Delta/\rho_0\) in HTSCs both above and below \(T^*\).

Here, we explain a theoretical reason why \(\nu\) is enhanced in both LSCO and NCCO: \(A_k\) in equation (118) can be rewritten as
\[
A_k = Q_k L_k(\partial^2 f'/\partial k_1^2) \cos(\theta_{Q_k} - \theta_{J_k}) + Q_k (\partial^2 f'/\partial k_1^2) \sin(\theta_{Q_k} - \theta_{J_k}),
\]
(120)
where \(\theta_{Q_k} = \tan^{-1}(J_k/J_k/\kappa_0, \theta_{Q_k} = \tan^{-1}(Q_k/\kappa_0)\), and \(L_k = J_k/\kappa_k\) is the mean free path with CVC. In the RTA, the heat current is given by \(\tilde{q}_\mathbf{k} = \epsilon \cdot \tilde{v}_\mathbf{k}\). Since \(\theta_{Q_k} = \theta_{J_k}\), the second term is absent in the RTA. If one includes the CVC, \(\tilde{Q}_\mathbf{k}\) is not parallel to \(\tilde{J}_\mathbf{k}\) because the CVC for \(\tilde{q}_\mathbf{k}\) is usually small (i.e. \(\tilde{Q}_\mathbf{k} \sim \tilde{q}_\mathbf{k}\) as we will discuss below \([37, 39]\). Therefore, we obtain \(\theta_{Q_k} = \theta_{J_k}\). In this case, the second term gives rise to an enhancement of \(\nu\) if \(L_k \approx 0\) is highly anisotropic around the cold spot. Figure 17(ii) shows \(L_k\) in LSCO given by the FLEX + CVC approximation. We can see that \(\partial L_k/\partial \kappa_0\) takes a large value near the cold spot below \(T = 0.02\). (Note that \(\partial L_k/\partial \kappa_0 = 0\) just at point A.) This is the origin of increment in \(v_{\text{FLEX}}\) in figure 23(ii). As shown in figure 23(iii), \(v_{\text{FLEX}}\) for NCCO is much larger than \(v_{\text{FLEX}}\) for LSCO; one reason is that the anisotropy of \(T^{\text{FLEX}}_k\) in NCCO is greater than \(\xi_A \gg 1\) in NCCO. Another reason is that \(\gamma^{-1}_k = \gamma^{-1}_N\) in NCCO is larger than that in LSCO. For these reasons, the FLEX+CVC method can explain the huge \(\nu\) observed in NCCO.

Next, we consider the increment of \(\nu\) in the pseudo-gap region in LSCO using the FLEX + \(T\)-matrix method, which is shown as \(v_{\text{FLEX+T}}\) in figure 23(ii). The CVC due to SC fluctuations (Maki–Thompson term) represents the acceleration of quasiparticles caused by the short-lived Cooper pairs. Since HTSCs are \(d_{xy}\) wave superconductors, the CVC due to the SC fluctuations magnifies \(J_k\) except at the nodal point (point A in figure 14). In fact, when the Ginzburg–Landau correlation length is longer than \(\xi_A\), the total current...
The anisotropy of $L_k = |\tilde{J}_k/\chi_k|$ becomes considerably prominent below $T^*$ since $\text{Im} \Sigma_k^{\text{SCF}}(0) \propto \psi_k^2$ increases whereas $\text{Im} \Sigma_k^{\text{AFF}}(0)$ decreases below $T^*$ as shown in figure 22(iv). For this reason, the second term in equation (120) takes a large value in the neighborhood of point A, as explained in [37]. Thus, the [FLEX + $T$-matrix] + CVC method can explain the rapid increment in $\nu$ in LSCO below $T^*$.

If the AF fluctuations are absent, $\theta_k^d = \theta_k^Q = \theta_k^e$ even if the CVC due to SC fluctuations are taken into account. Therefore, to explain the enhancement in $\nu$ in the pseudo-gap region, we have to include CVCs due to both the AF fluctuations and SC fluctuations. In contrast, $\tilde{R}_0$ decreases below $T^*$ in proportion to $\xi_{\text{AF}}^2$ since the second term in equation (120) is absent if we replace $\tilde{Q}_k$ with $\tilde{J}_k$.

Finally, we explain why the heat CVC is usually small. As we have explained, the CVC represents the current of the Fermi sea that is transferred from the excited quasiparticles by the electron-electron scattering. The charge CVC can be large because of the momentum conservation law. However, the heat current $\tilde{q}_k = (\epsilon_k^0 - \mu)\tilde{Q}_k$ is not conserved even in the free dispersion model, that is, $\tilde{q}_k + \tilde{q}_k' \neq \tilde{q}_{k+q} + \tilde{q}_{k-q}$ even if $\epsilon_k^0 + \epsilon_k^0 = \epsilon_{k+q}^0 + \epsilon_{k-q}^0$. For this reason, the heat CVC is small in general, and therefore the thermal conductivity $\kappa$ is finite even if the Unklapp process is absent: in a 3D free dispersion model, $\kappa = (9/8)\kappa_{\text{RTA}}$ within the second-order perturbation theory with respect to $U$ [39]. The heat CVC is also small in nearly AF metals: since the direction of the heat CVC due to the quasiparticle at $\tilde{k} + \tilde{q}$ in figure 8 (the MT term) depends on the sign of $\epsilon_{k+q}^0 - \mu$, the heat CVC becomes small due to cancellation after the $q$-summation for $|\tilde{q} - \tilde{Q}| \lesssim \xi_{\text{AF}}^{-1}$.

5.3. Summary of this section

In conclusion, we studied $\rho$, $R_0$, $\Delta\rho/\rho_0$, $S$ and $\nu$ in HTSCs using the FLEX + $T$-matrix approximation. The results are shown in figures 18(i), 19(i), 21(i) and 23(i) and (ii). We can explain that (i) $R_0$ and $S$ start to decrease below $T^*$, (ii) $\rho$ starts to deviate from the $T$-linear behavior at $T_0$, and it is approximately proportional to $T^2$ below $T^*$. Moreover, (iii) $\nu$ and $\Delta\rho/\rho$ increase further below $T^*$. Therefore, this study gives us a unified understanding of the various anomalous transport phenomena both below and above $T^*$.

This study provides strong evidence that the SC fluctuations are predominant in the pseudo-gap region. The striking increment in $\nu$ below $T^*$ is a cooperative phenomenon between the d-wave SC fluctuations and AF fluctuations. We stress that the RTA cannot explain the enhancement of $\nu$ since the second term in equation (120) vanishes identically in the RTA. Further, a very large Nernst signal appears also in electron-doped systems, which starts to increase in proportion to $T^{-1}$ below $T_0 \sim 600$ K. This experimental fact is reproduced by the CVC due to the AF fluctuations, even in the absence of SC fluctuations. In section 5.2, we found that the origin of rapid increment in $\nu$ is the second term of equation (120), which emerges only when the CVC is included. In NCCO (and CeMIn$_3$), $\nu$ starts to increase below $T_0$ because of the

![Diagram](image-url)
CVC due to strong AF fluctuations; $\xi_{\text{AF}} \gg 1$. In LSCO, on the other hand, increment in $v$ below $T_0$ is small since $\xi_{\text{AF}}$ is of order 1; $v$ in LSCO starts to increase rapidly below $T^* \ll T_0$, with the aid of the strong SC fluctuations.

Here, we comment on the transport phenomena in Nd-doped LSCO, La$_{1-x}$Nd$_x$Sr$_{2}$CuO$_4$, which shows the static stripe ordered phase at $T_{	ext{st}} \sim 80$ K [187]. In this compound, $\rho$ increases whereas $R_H$ decreases monotonically below $T_{\text{st}}$, which indicates that the system becomes one-dimensional due to the stripe order. This experimental fact might tempt us to consider that the stripe order is the origin of the reduction in $R_H$ in other HTSCs in the pseudo-gap region. However, the stripe order scenario cannot explain the reduction in $\rho$ nor increment in $v$ below $T^*$. On the other hand, the overall transport phenomena in HTSCs can be explained by considering the CVCs due to AF+SC fluctuations.

In previous sections, we discussed the dc transport phenomena in $\rho$ and $R_H$ originate from the reduction of AF fluctuations, and the pseudo-gap behavior of $v$ reflects the increment of SC fluctuations. The pseudo-gap temperature determined by $\rho$, $T^*_\rho$, is larger than $T_{\text{st}}^\text{R}$ and $T^*_\rho$ determined by $R_H$ and $v$, respectively. Recently, impurity (Zn) effect on the pseudo-gap behavior in YBCO was studied [188], and it was found that $T^*_\rho$ and $T_{\text{st}}^\text{R}$ are independent of impurities, whereas $T^*_v$ decreases with Zn-doping in parallel to $T_c$. It is an important future problem to show whether these experimental facts can be explained by the FLEX + $T$-matrix approximation or not. In section 7, we will show that the CVC due to fluctuations is usually sensitive to impurities. Therefore, to reconcile this problem, we have to study the impurity effect on the CVC seriously.

It should be noted that $v$ takes a very large value in the vortex-liquid state above $H_{c2}$ in a clean 2D sample, reflecting the high mobility of the vortices. Therefore, $v$ is frequently used as a sensitive probe for the mixed state. Based on his observation, Ong et al proposed that spontaneous vortex–antivortex pairs emerge in under-doped systems below $T^*$, and they govern the transport phenomena in the pseudo-gap region [180]. However, this assumption seems to contradict the other transport coefficients; for example, the flux-flow resistance does not appear below $T^*$.

6. AC transport phenomena in HTSCs

In previous sections, we discussed the dc transport phenomena in nearly AF metals and found that the CVC produces various non-Fermi liquid-like behaviors. In principle, the ac transport phenomena can yield further useful and decisive information about the electronic status. Unfortunately, the measurements of the ac transport coefficients are not common because of the difficulty in their observations, except for the optical conductivity $\sigma_{xx}(\omega)$ measurements.

Fortunately, Drew’s group has performed intensive measurements of the ac-Hall coefficient $R_H(\omega) = \sigma_{xy}(\omega)/\sigma_{xx}(\omega)$ in YBCO [189–192], BSCCO [193], LSCO [194] and PCCO [195]. They found that the $\omega$-dependence of $R_H(\omega)$ in HTSC shows amazing non-Fermi liquid-like behaviors, which have been a big challenge for researchers for a long time. Here, we show that this crucial experimental constraint is well satisfied by the numerical study using the FLEX + CVC method.

In the RTA, both $\sigma_{xx}(\omega)$ and $\sigma_{xy}(\omega)$ in a single-band model will follow the following ‘extended Drude forms’:

\[
\sigma_{xx}^{\text{RTA}}(\omega) = \Omega_{1xx}(2\eta(\omega) - iz^{-1}\omega)^{-1},
\]

\[
\sigma_{xy}^{\text{RTA}}(\omega) = \Omega_{1xy}(2\eta(\omega) - iz^{-1}\omega)^{-2},
\]

where $z^{-1}$ is the mass-enhancement factor and $\eta(\omega)$ is the $\omega$-dependent damping rate in the optical conductivity, which is approximately given by $\gamma(\omega) = (\gamma_{\text{cold}}(\omega/2) + \gamma_{\text{cold}}(-\omega/2))/2$ for small $\omega$. According to the spin fluctuation theory [56], $\gamma(\omega) \propto \text{max}(|\omega/2, \pi T|)$, which is observed by the optical conductivity measurements. The $\omega$-dependence of $z$ is important in heavy-fermion systems ($1/z \gg 1$ at $\omega = 0$), whereas it will not be so important in HTSC since $1/z$ is rather small. Expressions (123) and (124) are called the ‘extended Drude form’.

Within the RTA, the ac-Hall coefficient is independent of $\omega$ even if the $\omega$-dependence of $z$ is considered:

\[
R_H^{\text{RTA}}(\omega) = \Omega_{1xx}/\Omega_{1xy}^2 \sim 1/\eta n e.
\]

Very interestingly, Drew’s group has revealed that $R_H(\omega)$ in HTSC decreases drastically with $\omega$: as shown in figure 24, $\text{Im}R_H(\omega)$ shows a peak at $\omega_0 \sim 50$ cm$^{-1}$ in optimally doped YBCO [192]. Moreover, $\text{Im}R_H(\omega)$ is as large as $\text{Re}R_H(\omega)$ for $\omega \gtrsim \omega_0$, as a consequence of the Kramers–Kronig relation between $\text{Re}R_H(\omega)$ and $\text{Im}R_H(\omega)$. Such a large $\omega$-dependence of $R_H$ cannot be explained by the RTA, even if one assumes an arbitrary $(k,\omega)$-dependence of the quasiparticle damping rate $\gamma(\omega)$. Therefore, the ac-Hall effect severely constrains the theories involving the normal state of HTSCs.

Recently, we studied both $\sigma_{xx}(\omega)$ and $\sigma_{xy}(\omega)$ in HTSC using the FLEX + CVC method, by performing the analytic continuation of equations (51) and (53) using the Padé approximation [196, 197]. Since the $\omega$-dependence of the CVC is correctly considered, the obtained $\sigma_{xx}(\omega)$ and $\sigma_{xy}(\omega)$ satisfy the $f$-sum...
Bethe–Salpeter equation for \( \omega \neq 0 \) is given by

\[
\tilde{J}_k(\epsilon; \omega) = \tilde{v}_k + \int \frac{d\epsilon'}{4\pi^2} \sum_{k'q} \tilde{T}^{(0)}_{k'kq}(\epsilon, \epsilon'; \omega) \times G^R_{k'kq}(\epsilon' + \omega/2) G^A_{kq}(\epsilon' - \omega/2) \tilde{J}_k(\epsilon'; \omega),
\]

which is equivalent to equation (60) when \( \omega = 0 \). A simplified Bethe–Salpeter equation for \( \omega = 0 \) is given in equation (78). Here, we extend equation (78) to the case of \( \omega \neq 0 \). By noticing the relationship \( G^R_{k'kq}(\epsilon' + \omega/2) G^A_{kq}(\epsilon' - \omega/2) \approx \pi \rho_k(\epsilon') \cdot 2/(2\gamma_k - i\omega^{-1}) \), equation (126) is simplified for \( |\omega| \ll \gamma \) as

\[
\tilde{J}_k(\omega) = \tilde{v}_k + \frac{\alpha_k \cdot 2\gamma_k}{2\gamma_k - i\omega^{-1}} \tilde{v}_k,
\]

where \((1 - \alpha_k)^{-1} \propto \xi_{AF}^{-2}\). In deriving equation (127), we assumed that the \( \epsilon \)-dependence of \( J_k(\epsilon + i\delta) \) is small for \( |\epsilon| \ll \gamma \). The solution of equation (127) is given by

\[
\tilde{J}_k = \frac{(2\gamma - i\omega^{-1})^2}{(2\gamma - i\omega^{-1})^2 - (\alpha_k \cdot 2\gamma)^2} \left[ \tilde{v}_k + \frac{\alpha_k \cdot 2\gamma_k}{2\gamma_k - i\omega^{-1}} \tilde{v}_k \right].
\]

As a result, a \( \sigma_{xy}(\omega) \) for \( |\omega| \ll \gamma_0 \) is approximately given by

\[
\sigma_{xy}(\omega)/|\omega| \approx \frac{-\alpha_k \cdot 2\gamma_k}{2\gamma_k - i\omega^{-1}} \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \Omega_{xy} \]

\[
\approx \frac{(1 - \alpha_k)\gamma_0 - i\omega^{-1})((1 + \alpha_k)\gamma_0 - i\omega^{-1})}{(1 + \alpha_k)^2 \gamma_0 - i\omega^{-1}}.
\]

If we omit the CVC (i.e., \( \alpha_k = 0 \)), equation (130) is equivalent to the extended Drude form in equation (124). According to equation (130), the solution of \( \sigma_{xy}(\omega) \) is given by

\[
\sigma_{xy}(\omega) = \frac{\alpha_k \cdot 2\gamma_k}{2\gamma_k - i\omega^{-1}} \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \frac{1}{(2\gamma - i\omega^{-1})^2} \left( \tilde{J}_k \times \frac{\partial \tilde{J}_k}{\partial k_{||}} \right)_z
\]

\[
\approx \frac{(1 - \alpha_k)\gamma_0 - i\omega^{-1})((1 + \alpha_k)\gamma_0 - i\omega^{-1})}{(1 + \alpha_k)^2 \gamma_0 - i\omega^{-1}}.
\]

Furthermore, the effect of CVC is still large even for the far-infrared region \( \omega \geq 0.3 \ (<1000\text{cm}^{-1}) \), which is consistent with the experimental results. This fact requires the satisfaction of the \( f \)-sum rule, \( \int_{-\infty}^{\infty} d\omega \sigma_{xy}(\omega) = 0 \), since the dc \( \sigma_{xy} \) is significantly enhanced. Schmid et al [13] observed \( \sigma_{xy}(\omega) \) in optimally doped BSCCO for a wide range of \( \omega \leq 0.3 \ (<1000\text{cm}^{-1}) \), and found that \( \sigma_{xy}(\omega) \) follows a 'simple Drude form': \( \sigma_{xy} \propto (2\gamma_0 - i\omega)^{-2} \), where \( \omega \)-dependence of \( \gamma_0 \) is much smaller than \( \gamma_0 \) and \( \gamma_0 \ll \gamma_0 \). The experimental \( \omega \)-dependence of \( \sqrt{B/\sigma_{xy}} \) and the theoretical result are shown in figure 26(i). Moreover, figure 26(ii) shows that \( \alpha_k \gamma_0 \) depends on \( T \) sensitively. Therefore, \( \gamma_0 \) is independent of \( \omega \) in the infrared region. In terms of the CVC, this experimental result highly contradicts the fact that \( \gamma_0 \) is independent of \( T \); \( \gamma_0 \propto \max(|\omega|, \pi T) \). This
mysterious behavior is well reproduced by the FLEX + CVC method, as indicated by the solid dots in figure 26(ii). This numerical result suggests that the effect of the ω-dependence of γ_{xy}(ω) on the ac-Hall conductivity, by which \(\sigma_{xy}^{RTD}(\omega)\) deviates from the simple Drude form, approximately cancels that of the CVC.

Interestingly, the Hall angle \(\theta_H(\omega) = \sigma_{xy}(\omega)/\sigma_{xx}(\omega)\) follows an approximate simple Drude form in both YBCO [190] and LSCO [194]—\(\theta_H(\omega) \propto (2\gamma_H - i\omega)^{-1}\) with a ω-independent constant \(\gamma_H \propto T^{-\alpha} (n = 1.5 \sim 2)\). (Exactly speaking, both \(\sigma_{xy}\) and \(\theta_H\) cannot follow the simple Drude forms at the same time when \(\sigma_{xx}\) exhibits an extended Drude form.) This experimental fact has also been theoretically reproduced [196]. Therefore, the anomalous \((\omega, T)\)-dependences of \(\sigma_{xy}(\omega)\) and \(R_H(\omega)\) in HTSCs can be semiquantitatively explained by the FLEX + CVC method—a microscopic theory without any fitting parameters except for \(U\). At the present stage, the FLEX + CVC method is the only theory with the capability to explain the anomalous ac and dc transport coefficients in a unified way. We briefly comment on the carrier-doping dependence of \(\Omega_H \equiv -\omega/1{\mathrm{Im[\theta_H^{-1}(\omega)]}}\): the experimental value of \(\Omega_H / H\) for optimally doped YBCO \((n \sim 0.85)\) is 0.15 (cm\(^{-1}\)T\(^{-1}\)) and it increases to 0.3 (cm\(^{-1}\)T\(^{-1}\)) in slightly under-doped YBCO and BSCCO \((n \sim 0.9)\). This experimental doping dependence can be quantitatively reproduced by using the FLEX + CVC method for \(n \leq 0.9\) [196]. However, this method cannot explain the large experimental value of \(\Omega_H / H\) \((0.5 \mathrm{ (cm}^{-1}\mathrm{T}^{-1})\) in heavily under-doped YBCO \((T_c = 0)\) [189].

7. Impurity effects in nearly AF metals

In earlier sections, we studied the transport phenomena in nearly AF metals without randomness. By including CVCs, we succeeded in explaining the anomalous dc and ac transport phenomena in a unified way, in both hole- and the electron-doped HTSCs. However, we have neglected the impurity or disorder effects on the transport phenomena, although they are prominent in real under-doped HTSCs. In fact, the residual resistivity due to disorder increases drastically as the system approaches the half-filling [156]. Moreover, STM/STS measurements revealed that the electronic states in under-doped HTSCs are highly inhomogeneous at the nanoscale, reflecting the random potential induced by the disordered atoms outside of the CuO\(_2\) plane [199].

These experimental facts are reproduced by assuming that the SC pairing potential is strongly influenced by the impurity potential [200, 201].

![Figure 26](image-url)

**Figure 26.** (i) ω-dependence of \(\sqrt{B/\sigma_{xy}}\) in BSCCO at 300 K. Open squares and diamonds represent the real and imaginary parts of the theoretical result given by the FLEX + CVC method. (ii) \(\sqrt{B/\sigma_{xy}}\) in optimally doped BSCCO at 950 cm\(^{-1}\). The solid dots represent the result of the FLEX + CVC method. [193].

Further, the anomalous transport phenomena near the AF QCP are sensitive to randomness. For example, within the Born approximation, the CVC due to electron–electron interaction vanishes at zero temperature since only elastic scattering exists at \(T = 0\). In fact, the CVC term in equation (75) vanishes at \(T = 0\) if we replace \(\gamma_{kq}\) with \(\gamma_{imp} \neq 0\). However, the Born approximation is applicable only when the impurity potential is weak. In fact, we will show that a ‘strong’ local impurity potential in under-doped HTSCs induces drastic and widespread changes in the electron–electron correlation around the impurity site. Thus, the impurity effect in HTSCs depends on the strength of the impurity potential. Hereafter, we study the non-trivial impurity effects in nearly AF Fermi liquids.

7.1. Hall coefficient in the presence of weak local impurities

First, we study the effect of ‘weak local impurities’ on the transport phenomena within the Born approximation, where the quasiparticle damping rate due to impurity scattering is given by

\[
\gamma_{imp} = n_{imp} I^2 \sum_k \text{Im}G_k (-i\delta) = \pi n_{imp} I^2 N(0),
\]

where \(n_{imp}\) is the impurity density, \(I\) is the impurity potential and \(N(0)\) is the DOS at the chemical potential. Then, the total
weak local impurities', when the elastic scattering is dependent, and the kernel of the Bethe–Salpeter equation is not allowed for a large $I$, as we will show in section 7.2.)

Also, the CVC due to local impurities vanishes identically within the Born approximation. Therefore, in the case of $\gamma_{\text{imp}} \neq 0$, the Bethe–Salpeter equation (75) is changed to become

$$\tilde{J}_k = \tilde{v}_k + \sum_q \frac{3U^2}{2} (\pi T)^2 \text{Im} \chi'_q(0) \rho_{kq}(0) 2\tilde{v}_{kq} \tilde{J}_{kq} \approx \tilde{v}_k + \tilde{a}_k \tilde{J}_k,$$

(133)

where $\tilde{a}_k \approx \alpha_k \approx \gamma_k / \tilde{v}_k$. Thus, an approximate solution of equation (133) is

$$\tilde{J}_k = \frac{1}{1 - \tilde{a}_k} (\tilde{v}_k + \tilde{a}_k \tilde{J}_k).$$

(134)

In the absence of impurities, $\tilde{J}_k$ exhibits singular $k$-dependence because $\alpha_k \approx (1 - c/\xi^2_{\text{AF}})$ approaches one. In the case of $\gamma_k \sim \gamma_{\text{imp}}$, on the other hand, $\tilde{a}_k \ll \alpha_k \lesssim 1$ and therefore $\tilde{J}_k \sim \tilde{v}_k$. As a result, the CVC is strongly suppressed by high density ‘weak local impurities’, when the elastic scattering is comparable to the inelastic scattering due to AF fluctuations.

In [34], we have pointed out that the enhancement of $R_H$ due to the CVC is easily suppressed by the weak impurities. Figure 27 shows the impurity effects on $\rho$ and $R_H$ obtained by the FLEX + CVC method for electron-doped systems. As expected, $R_H$ is suppressed by a small amount of impurities, although the induced residual resistivity is small. (Note that $\rho = 1$ corresponds to $250 \mu \Omega$ cm.) As a result, $R_H$ becomes positive at low temperatures in the presence of impurities, which is consistent with the experimental behavior of $R_H$ in PCCO for $\delta = 0.16-0.18$ [8]. Finally, we consider the case where the impurity potential is widespread (nonlocal).

In this case, the impurity potential $I(q)$ is momentum-dependent, and the kernel of the Bethe–Salpeter equation in equation (133), $(3U^2/2)(\pi T)^2 \text{Im} \chi'_q(0)$, is replaced with $n_{\text{imp}} I^2(q) + (3U^2/2)(\pi T)^2 \text{Im} \chi'_q(0)$. When $I(q)$ is large only for $q \sim 0$ (forward scattering), both the residual resistivity and the reduction in $R_H$ due to the impurities will be small. In HTSCs, it is considered that impurities outside of the CuO$_2$ plane causes the forward impurity scattering [202].

Figure 28 shows the impurity effects on $\cot \theta_H = \rho / R_H$ obtained by the FLEX + CVC method for LSCO. The relationship $\cot \theta_H \propto T^{1.6} + c$ holds well, and $c \approx \gamma_{\text{imp}}$. Such a parallel shift of $\gamma_{\text{imp}}$ by impurity doping is observed in various hole-doped HTSCs [169].

7.2. Effect of strong local impurities near AF-QCP

Now, we discuss the effect of ‘strong local impurities’ in HTSCs. According to a recent LDA study [203], a Zn atom introduced in the CuO$_2$ plane of HTSCs induces a large positive potential (less than 10 eV), and the potential radius is only ~1 Å. In HTSCs, however, Zn doping causes a non-trivial widespread change of the electronic states. In Zn-doped YBCO compounds, site-selective $^{89}$Y NMR
measurements revealed that both local spin susceptibility [204, 205] and staggered susceptibility [206] are prominently enhanced around the Zn site, within a radius of the AF correlation length $\xi_{AF}$. The same result was obtained by the $^7$Li Knight shift measurement in Li-doped YBCO compounds [207], and by the $^{63}$Cu NMR measurement in Zn-doped YBCO compounds [208]. Moreover, a small concentration of Zn induces a huge residual resistivity, which is significantly greater than the s-wave unitary scattering limit [156]. These non-trivial impurity effects were frequently considered as evidence for the breakdown of the Fermi liquid state in under-doped HTSCs.

Up to now, many theorists have studied this important issue. The single-impurity problem in a cluster $t$–$J$ model has been studied by using the exact diagonalization method [209, 210]. When the number of holes is two, both AF correlations and electron density increase near the impurity site. This problem was also studied by using the extended Gutzwiller approximation [211]. Although these methods of calculation are founded, the upper limit of the cluster size is rather small. Moreover, these studies are restricted to $T = 0$.

In this section, we study a single-impurity problem in a large size square-lattice (say $64 \times 64$) Hubbard model based on the nearly AF Fermi liquid theory [130]:

$$H = \sum_{i,j,\sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow} + I (n_{i\uparrow} + n_{i\downarrow}),$$

where $I$ is the local impurity potential at site $i = 0$. It is a difficult problem since we have to consider two different types of strong interactions ($U$ and $I$) on the same footing. Moreover, the absence of translational symmetry severely complicates the numerical analysis. To overcome these difficulties, we developed the $GV^I$ method—a powerful method for calculating the electronic states in real space in the presence of impurities [130]. The $GV^I$ method is applicable for finite temperatures since the thermal fluctuation effect is taken into account appropriately. Based on the $GV^I$ method, we successfully explain the non-trivial impurity effects in HTSCs in a unified way without assuming any exotic mechanisms.

In the $GV^I$ method, the real-space spin susceptibility $\hat{\chi}^{I \sigma}(\omega)$ is given by

$$\hat{\chi}^{I \sigma}(\omega) = \tilde{\Pi}^I \left( 1 - U \tilde{\Pi}^I \right)^{-1},$$

where $\tilde{\Pi}^I(r_i, r_j; \omega) = -T \sum_{e} G^I(r_i, r_j; e + \omega) G^I(r_j, r_i; e)$.

Here, $G^I$ is solved by the following Dyson equation:

$$\tilde{G}^I(e_a) = \hat{G}^0(e_a) \tilde{G}^I(e_a),$$

where $(\tilde{G}^I e_a) = I_{\delta_{i,0}} \delta_{i,0}$ and $\hat{G}^0$ is the real-space Green function in the FLEX approximation without the impurity potential $I$. The solution of equation (138) is

$$G^I_{i,j} = G^0_{i,j} + I \sum_{\delta} G^0_{i,0} G^0_{\delta,j} / (1 - I G^0_{0,0}),$$

where the position of the impurity potential is $i = 0$.

In figure 29, we show the numerical results for spin susceptibilities, $\hat{\chi}^{I \sigma}$, for $I = \infty$. Surprisingly, a non-magnetic impurity induces a huge Curie-like component in the uniform susceptibility; $\Delta \chi \approx n_{imp} \cdot \mu_{eff}^2 / 3T$. This result explains the long-standing experimental problem [212–214]. The obtained value of $\mu_{eff}$ is $0.74 \mu_B$, which is close to the experimental value $\mu_{eff} \sim 1 \mu_B$ in YBa$_2$Cu$_3$O$_{6+6}$ ($T_c \approx 60$ K) [212] and in LSCO ($\delta = 0.1$) [214]. We also emphasize that both the local and staggered susceptibilities get enhanced around the impurity site within a radius of about 3 ($\sim \xi_{AF}$) at $T = 0.02$. Here, we discuss the physical reason for this drastic impurity effect. In the FLEX approximation, the AF order (in the mean-field level) is suppressed by the self-energy correction (see section 2), which represents the destruction of the long-range order due to thermal and quantum fluctuations. In the FLEX approximation, the reduction in the DOS due to the large quasiparticle damping rate $\gamma_k = 1 \Sigma_k$ renormalizes the spin susceptibility. Around the impurity site, the self-energy effect due to electron–electron correlation is expected to be smaller. In fact, quantum fluctuation is reduced near the vacant site in the $s = 1/2$ Heisenberg model [215]. For this reason, in the $GV^I$ method, the AF correlations are prominently enhanced around the impurity site due to
the reduction of thermal and quantum fluctuations. The examples of the cross terms between $I$ and $U$ for susceptibility are shown in figure 30(i). The same physics occurs in quantum spin systems with vacancies; AF spin correlations are realized even when $V$ is zero, and a large residual resistivity will be induced by the non-s-wave scattering channels. In the $GV^I$ method, the self-energy effect is reduced by introducing an impurity. Therefore, we have to use the $GV^I$ method for investigating the impurity problem in HTSCs.

Next, we discuss the transport phenomena in the presence of impurities according to the $GV^I$ method [130]. The enhanced susceptibilities due to the impurity induce the additional self-energy correction $\delta \Sigma \equiv \Sigma^I - \Sigma^{I=0}$, around the impurity site. If the area of $\delta \Sigma \neq 0$ is large, a large residual resistivity will be induced by the non-s-wave scattering channels. In the $GV^I$ method, $\delta \Sigma$ is given by

$$\delta \Sigma(r_i, r_j; \epsilon_n) = T \sum_i V_i(r_i, r_j; \omega_l + \epsilon_n) V_i^I(r_i, r_j; \omega_l)$$

(140)

$$\chi_{Ic}^I(r, \omega_l) = U^2 \left( 2^{-1} \chi^{Ic}(r, \omega_l) - \Pi^I(r, \omega_l; \omega_l) \right),$$

(141)

where $\Sigma^I$ is the self-energy given by the FLEX approximation without the impurity potential ($I = 0$). The Green function $G(r_i, r_j; \epsilon_n)$ is obtained by solving the following Dyson equation in real space:

$$\hat{G}(\epsilon_n) = \hat{G}_i(\epsilon_n) + \hat{G}_i(\epsilon_n) \delta \hat{\Sigma}(\epsilon_n) \hat{G}(\epsilon_n).$$

(142)

In the $GV^I$ method, we solve equations (140) and (142) self-consistently. The local DOS given by the $GV^I$ method in real space — $\rho(r, \omega) = \text{Im} G(r, \omega - i\delta)/\pi$ — is shown in figure 30(ii). The local DOS decreases around the impurity site within the radius of approximately $3a$ (a is the lattice spacing), since the quasiparticle lifetime is very short due to the large Im$\delta \Sigma$. We verified that the radius of Im$\delta \Sigma$ increases as the filling number $n$ approaches unity, in proportion to $\xi_{AF}$. As explained in [130], we should not solve $V^I$ in equation (141) self-consistently, since the feedback effect on $V^I$ introduced by iteration is cancelled by the vertex correction for $V^I$ that is absent in the FLEX approximation.

To derive the resistivity in the presence of impurities, we have to obtain the $t$-matrix, $T(\epsilon)$, which is defined as $\hat{G} = \hat{G}^0 + \hat{G}^0 \delta \hat{\Sigma} \hat{G}^0$. The expression of $T(\epsilon)$ in the case of $n_{imp} \ll 1$ is derived in [130]. Using the $t$-matrix, the quasiparticle damping rate due to the impurity is given by [40,41]

$$\gamma_k^{imp}(\epsilon) = \frac{n_{imp}}{N^2} \sum_l \text{Im} T_l(\epsilon - i\delta) e^{i a k r_l},$$

(143)

where $T_l(\epsilon) = \sum_m t_m \sum_{m' \neq m} t_m' \sum_{l \neq m} t_l$ and $n_{imp}$ is the density of the impurities. In the case of $n_{imp} \neq 0$, the total quasiparticle damping rate is $\gamma_k(\epsilon) = \gamma_k^{0}(\epsilon) + \gamma_k^{imp}(\epsilon)$, where $\gamma_k^{0}(\epsilon) = \text{Im} \Sigma_k^{0}(\epsilon - i\delta)$. The resistivity for $n_{imp} \neq 0$ is approximately given by $\rho \propto \gamma_{cold}$.

Figure 31(i) shows $k$-dependences of $\gamma_k^{imp}$ ($n_{imp} = 0.05$) and $\gamma_k^{0}$. We emphasize that $\gamma_k^{imp}$ exhibits strong $k$-dependence that is similar to the $k$-dependence of $\gamma_k^{0}$. This result suggests that the effective impurity potential $\delta \Sigma$ is non-local. As a result, the structure of the ‘hot spot’ and the ‘cold spot’ is not smeared out by the strong non-magnetic impurities, although it will be smeared out by the weak local impurities. This finding strongly suggests that the enhancement of the Hall coefficient
near the AF-QCP, which is induced by the strong CVC around the cold spot \[32, 33\], does not decrease due to the strong impurities. Moreover, the enhancement of the cold spot \[32, 33\], does not decrease due to the strong near the AF-QCP, which is induced by the strong CVC around \(T_x\), reflecting an extremely short quasiparticle lifetime near the impurities. In other words, the considerable residual resistivity is caused by the large scattering cross section of the ‘effective impurity potential’, \(\delta \Sigma\), as shown in figure 30(ii). Note that \(T_x\) increases with \(n_{\text{imp}}\). This amazing result strongly suggests that the insulating behaviour of \(\rho\) observed in under-doped LSCO \[223, 224\] and NCCO \[225\] is caused by the residual disorder in the CuO\(_2\) plane. As shown in figure 31(iii), the parallel shift of the resistivity at finite temperatures due to impurities (\(\Delta \rho\)) grows drastically in the under-doped region; \(\Delta \rho\) exceeds the \(s\)-wave unitary scattering limit in the 2D electron-gas model, \(\Delta \rho = (\hbar/e^2)(4n_{\text{imp}}/n)\). This result effectively explains the experimental carrier-doping dependence of \(\Delta \rho\) \[156\].

The enhancement of the residual resistivity is also observed near the AF-QCP in heavy-fermion systems such as CeAl\(_3\) \[226\] and CeCu\(_3\)Au \[227\] and in the organic superconductor \(\kappa\)-(BEDT-TTF)\(_2\)Hg\(_2\)\(_{89}\)Br\(_8\) \[239\]. In these compounds, \(\Delta \rho\) quickly decreases with pressure, as the distance from the AF-QCP increases. In \(\kappa\)-(BEDT-TTF)\(_2\)Hg\(_2\)\(_{89}\)Br\(_8\), \(\Delta \rho\) under 2 GPa is six times smaller than the value at 0.5 GPa. Such a large change in \(\Delta \rho\) is difficult to be explained by the pressure dependence of the DOS. In fact, according to the \(t\)-matrix approximation, \(\gamma_{\text{imp}}^{\text{imp}} = (\pi I^2 N(0)/2)/(1 + \pi IN(0)/2)^2\). The residual resistivity in 2D free dispersion model \((\epsilon^2_k = k^2/2m)\) is \(\Delta \rho = 2\pi N(0)\gamma_{\text{imp}}/e^2n\). Therefore, \(\Delta \rho\) is given by

\[
\Delta \rho = \frac{\hbar}{e^2} \frac{4\pi I^2 N(0)/2}{1 + \pi IN(0)/2} \frac{n_{\text{imp}}}{n},
\]

in the 2D free dispersion model. Note that the renormalization factor \(\zeta\) does not appear in the expression of \(\Delta \rho\). In the case of \(IN(0) \gg 1\), equation (144) gives the \(s\)-wave unitary scattering value; \(\Delta \rho = (\hbar/e^2)(4n_{\text{imp}}/n)\). In the case of weak impurity scattering where \(IN(0) \ll 1\) (Born limit), we obtain the relation \(\Delta \rho \propto I^2 N^2(0)\), which will decrease under pressure since \(N(0) \propto 1/W_{\text{band}}\). However, pressure dependence of \(\Delta \rho\) in \(\kappa\)-(BEDT-TTF)\(_2\)Hg\(_2\)\(_{89}\)Br\(_8\) seems too strong to be explained by the Born approximation. We comment that the increment in \(\Delta \rho\) due to charge fluctuations is discussed in some heavy-fermion systems \[228, 229\].

In summary, this study revealed that a single impurity strongly influences the electronic states in a wide area around the impurity site near the AF-QCP. Using the \(GV^I\) method, the characteristic impurity effects in under-doped HTSCs are well explained in a unified way, without assuming any exotic non-Fermi-liquid ground states. We successfully explain the non-trivial impurity effects in HTSCs in a unified way in terms of a spin fluctuation theory, which strongly suggests that the ground state of HTSCs is a Fermi liquid. We expect that the novel impurity effects in other metals near the AF-QCP, such as heavy-fermion systems and organic metals, will be explained.
Moreover, as shown in figure 33, modified Kohler’s rule given in equation (106) is well satisfied in CeRhIn$_5$ [20] for over four orders of magnitude. Furthermore, this is well satisfied in CeCoIn$_5$ (M = Co or Rh) and κ-(BEDT-TTF).

8. Anomalous transport behaviors in CeMIn$_5$ (M = Co or Rh) and κ-(BEDT-TTF)

Thus far, we have shown that the CVC due to strong AF+SC fluctuations induces various anomalous transport phenomena in HTSCs. To validate this idea, we have to study the various nearly AF systems other than HTSCs. In general, the electronic structure of heavy-fermion systems and organic metals are very sensitive to the pressure. Therefore, the distance from the AF-QCP can be easily changed by applying the pressure, without introducing disorders in the compounds. This is a great advantage with respect to investigating the intrinsic electronic states near the AF-QCP, free from the disorder effects. A useful theoretical review for heavy-fermion systems near the AF-QCP is given in [233]. Recently, detailed measurements of the transport phenomena under pressure have been performed in the heavy-fermion superconductor CeMIn$_5$ (M = Co or Rh) and in the organic superconductor κ-(BEDT-TTF). They exhibit striking non-Fermi-liquid-like behaviors as observed in HTSCs—equations (1)–(4). Hereafter, we explain the experimental and theoretical studies on the transport phenomena in these systems.

8.1. CeMIn$_5$ (M = Co or Rh)

CeMIn$_5$ is a quasi-2D heavy-fermion superconductor with $T_c = 2.3$ K. According to the angle resolved measurements of thermal conductivity [102] and specific heat [105], the symmetry of the SC gap is d-wave. Figure 32 shows the temperature dependence of $R_{H1}$ in CeCoIn$_5$ and CeRhIn$_5$ in the limit of $H_c = 0$. In CeRhIn$_5$, the critical pressure under which the AF-QCP is realized is $P_c \approx 2.01$ GPa. At $P = P_c$, $R_{H1}(2.3$ K)/$R_{H1}(300$ K) reaches 50, whereas the magnitude of $R_{H1}$ rapidly decreases as the pressure is increased. A similar pressure dependence of $R_{H1}$ is observed in CeCoIn$_5$, where $P_c$ is slightly below the ambient pressure. In both these compounds, $R_{H1}$ is inversely proportional to $T$ at higher temperatures. Moreover, as shown in figure 33, modified Kohler’s rule given in equation (106) is well satisfied in CeRhIn$_5$ [20] for over four orders of magnitude. Furthermore, this is well satisfied in CeCoIn$_5$ [19, 20]. Therefore, both $R_{H1}$ and $\Delta \rho / \rho_0$ in CeMIn$_5$ show anomalous behaviors which are similar to HTSCs. This experimental fact strongly suggests that the CVC due to the strong AF fluctuations is the origin of the anomalous transport phenomena.

Now, we discuss the magnetic field dependence of the transport coefficients. Interestingly, $R_{H1} \equiv \frac{\partial n_H(H)}{\partial H}$ in CeMIn$_5$ near $P_c$ is easily suppressed only by a small magnetic field, as shown in the inset of figure 32. At the same time, $(\Delta \rho / \rho_0)H_{c2}^{-2}$ is also significantly suppressed. Therefore, the relationships $\sigma_{xy} \propto H_c$ and $\Delta \sigma_{xx} \propto H_{c2}^2$ are satisfied only below $\sim 0.1$ T near the QCP [19, 20]. These behaviors cannot be attributed to the orbital effect (i.e. the cyclotron motion of conduction electrons) since $\omega_c^* \tau^* = (eH_c/m^*c)\tau^* \gtrsim 1$ is satisfied only when $H_c \gg H_{c2} = 5$ T and $T \ll 1$ K [234]. The condition $\omega_c^* \tau^* \ll 1$ is also recognized from the relationship $\Delta \rho / \rho_0 \lesssim 0.1$ for $T > 2$ K and $H < 3$ T, as shown in figure 33.

When $\omega_c^* \tau^* \ll 1$ is satisfied, we can safely expand the Nakano–Kubo formula given in equation (42) with respect to the vector potential, as we did in sections 4–7. Then the following relationships derived in section 4

$$\rho_{H1} \propto H_{c2}^2 \xi_{AF}^2,$$  
(145)

$$\Delta \rho / \rho_0 \propto H_{c2}^2 \xi_{AF}^4 / \rho_0^2,$$  
(146)

are expected to be valid for CeMIn$_5$. Here, the factors $\xi_{AF}^2$ and $\xi_{AF}^4$ in equations (145) and (146), respectively, come from the...
CVCs. Therefore, experimental striking non-linear behaviors of \( R_H \) and \( \Delta \rho/\rho_0 \) with respect to \( H \) should originate from the field-dependence of \( \chi_0(0) \propto \xi_{AF}^2 \); the suppression of \( \xi_{AF} \) due to the magnetic field results in reducing both equations (145) and (146). Since the correlation length is sensitive to the outer parameters in the vicinity of the QCP [235], the anomalous sensitivity of \( \rho_H \) and \( \Delta \rho/\rho_0 \) to the magnetic field in CeMIn\(_3\) originates from the field dependence of the CVC. Recently, \( \rho_H \) and \( \Delta \rho/\rho_0 \) in PCCO were measured in a magnetic field up to 60 T, and it is found that both of them shows striking non-linear behaviors with respect to \( H \). [236]. Their behaviors will be explained by the field dependence of the CVC.

Even if the field dependence of CVC is prominent, modified Kohler’s rule in equation (106) should be satisfied for a wide range of the magnetic field strength, since both \( \Delta \rho/\rho_0 \) and \( \cot^2 \theta_H \) are proportional to \( \xi_{AF}^2 \rho^2 \) if the theory of CVC is correct. In fact, the modified Kohler’s rule is well satisfied for \( T = 2.5-30 \, \text{K} \), and \( \cot \theta_H \lesssim 3 \, \text{T} \) in CeRhIn\(_5\) as shown in figure 33, regardless of the fact that conventional Kohler’s rule is violated only for 0.1 T. This fact strongly suggests that both \( \sigma_{xy} \) and \( \Delta \sigma_{xx} \) are enhanced by the same origin, namely, the CVC due to the AF fluctuations. Therefore, the anomalous transport phenomena in CeCoIn\(_3\) are consistently described by the theory of CVC in nearly AF Fermi liquids.

The Nernst signal \( v \) in CeCoIn\(_3\) is also very anomalous [107]. Below 20 K, \( v \) starts to increase approximately in proportion to \( T^{-1} \), exhibiting anomalously large values (\( v \sim 1 \, \text{mV K}^{-1} \, \text{T}^{-1} \)) below 4 K. This behavior is very similar to that of \( v \) in electron-doped HTSC, whose transport phenomena in quasi 2D Hubbard model are described by the FLEX approximation in the Hubbard model. (Even in the FLEX approximation, relatively large values \( v \sim 1 \, \text{mV K}^{-1} \, \text{T}^{-1} \)) below 4 K, which corresponds to \( T = 0.8 \), is obtained in the periodic Anderson model, \( 1/\nu = m^*/m_{\text{hand}} \sim 50 \) in CeCoIn\(_3\). Since the mass-enhancement factor in the present FLEX approximation is \( 1/\nu_{\text{FLEX}} \sim 3 \), we can experimentally estimate the effective mass-enhancement factor as \( 1/\nu = 1/(1/\nu_{\text{FLEX}}) \), where \( 1/\nu \) is 50/3 is the mass-enhancement factor which cannot be described by the FLEX approximation in the Hubbard model. (Even in the FLEX approximation, relatively large \( 1/\nu_{\text{FLEX}} \sim 10 \) is obtained in the periodic Anderson model, which is an effective model for heavy-fermion systems [237]. We will comment on this fact in section 9.2.) We present \( v = \nu_{\text{FLEX}}/z^* \) in figure 34(iii), where \( \nu_{\text{FLEX}} \) is given by the FLEX approximation, using the relation \( k_B a_{2D}^2/\nu = 28 \, \text{nV K}^{-1} \, \text{T}^{-1} \). Recently, we extended the FLEX approximation to reproduce appropriate results under a finite magnetic field and calculated the field dependence of \( R_H \) and \( v \). It was suggested that both these quantities were rapidly suppressed by the magnetic field near AF-QCP, reflecting the reduction of AF fluctuations. The result obtained is in good agreement with the experimental results.

In CeCoIn\(_3\), \( R_H \) exhibits a peak at \( T^*_a \sim 4 \, \text{K} \) at ambient pressure, and \( T^*_a \) increases with pressure (see figure 32). This is different from the pseudo-gap behavior in under-doped HTSCs, since \( T^*_a \) in CeCoIn\(_3\) increases as the system goes
away from the AF-QCP. In particular, $R_{\text{HI}}$ of CeRhIn$_5$ at $P = P_c$ maintains its increasing trend just above $T_c$. These behaviors can be understood as the effect of the weak (local) residual disorders, as we have discussed in section 7.1: as shown in figure 27(ii), $|R_{\text{HI}} - R_{\text{HI}}^{\text{RTA}}|$ starts to decrease at lower temperatures when $\gamma_{\text{imp}} > 0$. The peak temperature $T_{\text{HI}}^{\gamma}$ increases with $\gamma_{\text{imp}}$. In CeCoIn$_5$, the resistivity at $T_{\text{HI}}$ is $\rho(T = T_{\text{HI}}) \sim 6 \mu\Omega \cdot \text{cm}$ [20] for $P = 0$–2.5 GPa. On the other hand, in CeRhIn$_5$ at $P = 2$ GPa, $\rho(T \gtrsim T_c) \approx 10 \mu\Omega \cdot \text{cm}$ because of the large inelastic scattering. Since $\gamma_{\text{imp}}$ in CeCoIn$_5$ and that in CeRhIn$_5$ are expected to be similar, $\tilde{\alpha}_k = \alpha_k \cdot (\gamma_k/\gamma_k + \gamma_{\text{imp}})$ in equation (134) will be smaller in CeCoIn$_5$. That is, reduction of CVC due to impurities is more prominent in CeCoIn$_5$. Therefore, we can explain the different behaviors of $R_{\text{HI}}$ in CeRhIn$_5$ and CeCoIn$_5$ at low temperatures as the effect of residual disorders.

One may ascribe the temperature dependence of $R_{\text{HI}}$ in CeMnIn$_5$ to the multiband effect. For example, a sign change in $R_{\text{HI}}$ can occur if a hole-like Fermi surface and an electron-like Fermi surface coexist and their mean free paths have different $T$-dependences. However, it is very difficult to explain the relation $|R_{\text{HI}}| \gg 1/n_e$ in a multiband model based on the RTA. In fact, in order to explain the pressure dependence of $R_{\text{HI}}$ in CeCoIn$_5$, one has to assume that a small Fermi surface governs the transport phenomena at 0 GPa (near AF-QCP), whereas a large Fermi surface should consequently govern the transport phenomena under 2.5 GPa. The same drastic change in the electronic states should occur by applying a magnetic field $H \sim 1$ T. This unnatural assumption is not true since the other transport coefficients cannot be explained at all, as discussed in [20] in detail. In particular, the elegant modified Kohler’s rule plot for CeRhIn$_5$ shown in figure 33 for over four orders of magnitude is strong evidence that the CVC is the origin of the anomalous transport phenomena, and the anomalous transport phenomena are mainly caused by a single large Fermi surface with heavy quasiparticles. Since the enhancements of both $R_{\text{HI}}$ and $p/(\rho_0)$ originate from a small portion of the Fermi surface (i.e., the cold spot) as shown in figure 19(ii), the multiband effect is unimportant in CeMnIn$_5$. Therefore, modified Kohler’s rule will be realized near the AF-QCP even in multiband systems like CeMnIn$_5$.

In usual heavy-fermion compounds, $R_{\text{HI}}$ exhibits a Curie-like behavior above the coherent temperature $T_0$ due to the large anomalous Hall effect, which is caused by the angular momenta of the f-orbitals as we will discuss in section 9.4. For $T \ll T_0$, $R_{\text{HI}}$ due to the AHE is proportional to $\rho^2$. In heavy-fermion compounds near the AF-QCP, on the other hand, both the ordinary Hall coefficient $R_{\text{HI}}^{\text{n}}$ and the anomalous Hall coefficient $R_{\text{HI}}^{\text{AHE}}$ can show large temperature dependences. Pasc hen et al extracted $R_{\text{HI}}^{\text{n}}$ from the experimental Hall coefficient in YbRh$_2$Si$_2$ and discussed the critical behavior of $R_{\text{HI}}^{\text{n}}$ near AF-QCP, which is realized under the magnetic field [22]. In contrast, $R_{\text{HI}}$ in CeCoIn$_5$ $(M = \text{Co}, \text{Rh})$ is almost constant above 50 K, as shown in figure 32. Therefore, the anomalous Hall effect in CeMnIn$_5$ is very small, and therefore $R_{\text{HI}}^{\text{AHE}} \approx R_{\text{HI}}^{\text{n}}$, as explained in detail in [19, 20]. This fact is a
the ordinary Hall effect in CeIn5 is independent of pressure at room temperature. Therefore, the pressure is increased, the AF fluctuations get reduced and strongest at the lowest experimental pressure (0.19 GPa). As under homogeneous pressure. Here, the AF correlations are the great advantage for studying the anomalous T-dependence of the ordinary Hall effect in CeMIn5.

In CeCoIn5, novel kinds of critical behaviors are observed near $H_{c2} \sim 5$ T for $T \ll 1$ K, where a prominent increment in the effective mass was observed. This phenomenon is referred to as a ‘field-induced QCP’. One possible origin will be the field-induced SDW state that is hidden in the SC state. We will discuss this important future problem in section 9.2 in more detail.

8.2. $\kappa$-(BEDT-TTF)

The measurements of $R_{H}$ under pressure have been intensively performed in $\kappa$-(BEDT-TTF)$_2$X. Figure 35 shows the temperature dependence of $R_{H}$ and $\cot \theta_{H}$ for $X = \text{Cu}(NCS)_2$ under homogeneous pressure. Here, the AF correlations are the strongest at the lowest experimental pressure (0.19 GPa). As the pressure is increased, the AF fluctuations get reduced and a conventional Fermi liquid state is realized. At the same time, $R_{H}$ is reduced and exhibits a constant value. The observed $R_{H}$ is independent of pressure at room temperature. Therefore, the origin of the increment in $R_{H}$ below 100 K cannot be related to deformation of the Fermi surface under pressure. That a similar increment in $R_{H}$ is observed in $X = \text{Cu}[\text{N(CN)}_2]\text{Cl}$, which has a single elliptical Fermi surface [92]. Because the observed $T$ dependences of $R_{H}$ and $\cot \theta_{H}$ are very similar to the observations involving HTSCs and in CeMIn5, the CVC is expected to play a significant role in $\kappa$-(BEDT-TTF)$_2$X.

As discussed in section 2.3, the resistivity in 2D systems in the presence of AF fluctuations is [147],

$$\rho \propto T^2 \xi_{AF}^2.$$  \hspace{1cm} (147)

This relationship is reliable when $\omega_{AF} \gtrsim T$, which is satisfied in optimally or over-doped HTSCs. The $T$-dependence of $\omega_{AF}$ is given in equation (10). We consider equation (147) is realized in $X = \text{Cu}(NCS)_2$ since the AF fluctuations are not so prominent. Since $R_{H}$ is proportional to $\xi_{AF}^2$ as shown in equation (97), we obtain

$$\cot \theta_{H} = \rho / R_{H} \propto T^2.$$ \hspace{1cm} (148)

Figure 35(ii) shows that equation (148) is well satisfied below 80 K, except at $P = 0.19$ GPa. (The thermal contraction of the sample might modify this relationship at the lowest pressure.) The success of the scaling relationships (97), (147) and (148) is strong evidence that the enhancement of $R_{H}$ in $\kappa$-(BEDT-TTF)$_2$X is caused by the CVC due to AF fluctuations.

In many $\kappa$-(BEDT-TTF)$_2$X compounds, the phase transition between the Mott insulating phase and the metallic (SC) phase induced by pressure is weak first order. Therefore, the AF fluctuations are not so strong even in the vicinity of the AF insulating phase. As a result, the enhancement of $R_{H}$ in $\kappa$-(BEDT-TTF)$_2$X is much smaller than that in CeMIn5. On the contrary, carrier doped (11% doping) $\kappa$-type superconductor $\kappa$-(BEDT-TTF)$_2$Hg$_{2.89}$Br$_3$ ($T_c = 4$ K at ambient pressure) exhibits very strong AF fluctuations, and therefore the critical pressures are expected to be slightly below 0 kbar. According to NMR measurement, $1/T_1 T \propto \xi_{AF}^2$ increases with decreasing temperature above $\sim 10$ K, in proportion to $(T + \Theta)^{-1}$ with $\Theta = 13$ K [238]. The P–T phase diagram and transport properties of $\kappa$-(BEDT-TTF)$_2$Hg$_{2.89}$Br$_3$ are carefully measured by Taniguchi et al [239]: it was found that $T_c$ shows a two-peak structure under pressure. Also, the power $n$ in $\rho = \rho_0 + AT^n$ increased from 1 to 2 with increasing pressure, which indicates that the AF fluctuations are suppressed under pressure.

Taniguchi et al also measured the $R_{H}$ in $\kappa$-(BEDT-TTF)$_2$Hg$_{2.89}$Br$_3$ under pressure, and found that $R_{H}$ exhibits a Curie–Weiss temperature dependence; $R_{H} \propto (T + \Theta_{RH})^{-1}$ [18]. At 0.19 GPa where AF fluctuations are strong, $R_{H}(10 \text{ K}) / R_{H}(300 \text{ K})$ reaches 10, whereas the enhancement of $R_{H}$ is totally suppressed by 1 GPa. The behavior of $R_{H}$ in $\kappa$-(BEDT-TTF)$_2$Hg$_{2.89}$Br$_3$ is very similar to the observation in CeMIn5. $\Theta_{RH}$ increases with pressure, and its extrapolated value to 0 GPa ($\sim 13$ K) coincides with the Weiss temperature of $1/T_1 T$ at ambient pressure. This is strong evidence that the Hall coefficient in $\kappa$-(BEDT-TTF)$_2$Hg$_{2.89}$Br$_3$ is proportional to $\xi_{AF}^2$, due to the CVC induced by AF fluctuations. It is noteworthy that the residual resistivity $\Delta \rho$ in $\kappa$-(BEDT-TTF)$_2$Hg$_{2.89}$Br$_3$ drastically decreases with pressure [239]. As we have discussed in section 7.2, the large residual resistivity near 0 GPa is expected to be given by the enlarged effective impurity potential due to the many-body effect.

Note that $R_{H}$ in $\kappa$-(BEDT-TTF)$_2$Cu[N(CN)$_2$]Br changes to be negative under high pressures and low temperatures, when the electron-electron correlation becomes very weak. The
lattice Hubbard model, since the increment approximation. The large (small) dots represent the d-wave model for compound [13].

The geometric frustration (\( t_1/t_0 \approx 0.7 \)). Therefore, a consistent understanding of the \( P-T \) phase diagram as well as the \( P \)-dependence of \( R_H \) are achieved by using the FLEX + CVC method.

9. Discussions

9.1. Summary of the present study

In this study, we have investigated the mechanism of non-Fermi-liquid-like transport phenomena in strongly correlated Fermi liquids in the presence of strong magnetic fluctuations. This problem was first realized in the study of HTSC, which has been one of the central issues in HTSC. For example, the relaxation time approximation (RTA) for the highly anisotropic \( \tau_q \) model cannot reproduce relationships (1)–(4) at the same time, even for a narrow range of temperatures. Recent experiments have been revealed that these anomalous transport properties are not a special phenomena in HTSC, but a universal phenomena in Fermi liquids near AF-QCP. We discussed two typical examples—CeMIn5 (\( M = \text{Co or Rh} \)) and \( \kappa \)-(BEDT-TTF)\(_2\)X. The ratio \( R_H(2.5 \text{ K})/R_H(300 \text{ K}) \) in CeRhIn\(_5\) reaches \( \sim 50 \), which is much larger than the ratio in HTSCs.

Here, we have studied this long-standing problem by developing the microscopic Fermi liquid theory. In the RTA, the momentum and energy conservation laws in quasiparticles scattering are violated, as explained in sections 1 and 3. To overcome this defect, we take account of the CVC to satisfy the conservation laws. The total current \( \vec{J}_k \), which is responsible for the transport coefficients, is given by the summation of the quasiparticle velocity \( \vec{v}_k \) and the CVC \( \Delta \vec{J}_k \). In interacting electron systems, an excited quasiparticle induces other particle–hole excitations by collisions. The CVC represents the current due to these particle–hole excitations. Since the CVC is caused by quasiparticle interactions, it can be significant in strongly correlated Fermi liquids. However, the effects of CVC in HTSCs have not been studied in detail until recently. Since the CVC is totally ignored in the RTA, the RTA frequently yields unphysical results in strongly correlated systems.

In the presence of strong AF fluctuations, we find that (a) the quasiparticle damping rate \( \gamma_k = 1/2\tau_k \) becomes anisotropic, and the portion of the Fermi surface with small \( \gamma_k \) (cold spot) governs the transport phenomena [24, 25]. At the same time, (b) the total current \( \vec{J}_k \) becomes highly anisotropic due to the prominent CVC [33]. (a) and (b) should occur simultaneously since both of them are caused by the same origin; highly anisotropic incoherent scattering due to AF fluctuations. That is, both (a) and (b) are induced by the strong backward scattering between \( k \) and \( k' \approx \frac{Q}{k} \approx Q \), as shown in figure 4(i). Mathematically, \( \gamma_k = \text{Im} \Sigma_k (-i\delta) \) and the CVC are closely connected by the Ward identity near AF-QCP, that is, the same vertex function \( T_k^{(0)} \) appears in equations (68) and (60). The facts (a) and (b) naturally explain the enhancement of \( R_H, \Delta \rho/\rho_0 \) and \( v \) in nearly AF metals in a unified way. As shown in figure 14(ii), the large curvature of the effective Fermi...
surface around the cold spot gives rise to the enhancement of $R_H$. In section 3.4, we proved the significance of the CVC beyond the one-loop (such as the FLEX) approximation.

It is also very important to demonstrate to what extent the CVC can reproduce the aspects of anomalous transport properties based on a standard spin fluctuations theory. For this purpose, we study $R_H$, $\Delta \rho / \rho_0$, $S$ and $v$ based on the FLEX (or FLEX + $T$-matrix) approximation by including CVCs. The results obtained semiquantitatively reproduce the various experimental facts in slightly under-doped HTSCs. This study strongly suggests that the striking deviation from the Fermi liquid behaviors (such as equations (1)–(4)) are ubiquitous in strongly correlated metals near the AF-QCP, not specific to HTSCs. Note that the increment in $R_H$ is also observed in non-SC metals near the AF phase, such as $\text{V}_{2-x}\text{O}_3$ [240] and $\text{R}_{2-\delta}\text{Bi}_2\text{Ru}_2\text{O}_7$ ($R = \text{Sm or Eu}$) [241].

The main results in this study are as follows.

(i) Hall coefficient: due to the CVC, $R_H \propto \xi^2_{AF} \propto T^{-1}$. The large curvature of the effective Fermi surface around the cold spot is the origin of the enhancement of $R_H$. In particular, we can explain that the $R_H$ for NCCO and PCCO are negative even above $T_N \sim 150\,\text{K}$, since the curvature of the ‘effective Fermi surface’ at point B in figure 14 is negative, although the curvature of the true Fermi surface is positive.

(ii) Magnetoresistance: $\Delta \rho / \rho_0 \propto \xi^2_{AF} / \rho_0^2 \propto T^{-4}$ due to the CVC, which means that conventional Kohler’s rule $\Delta \rho / \rho_0 \propto \rho_0^{-2}$ is violated. On the other hand, the modified Kohler’s rule $\Delta \rho / \rho_0 \propto (R_H / \rho_0)^2$ is realized since the same factor $\xi^2_{AF}$ appears in both sides.

(iii) Thermolectric power: for $T > T^*$, $|S|$ increases as $T$ decreases since the $\epsilon$-dependence of $\tau_0(\epsilon)$ becomes prominent when the AF fluctuations are strong. Further, $S > 0$ ($S < 0$) in many hole-doped (electron-doped) systems.

(iv) Nernst coefficient: due to the CVC, $v$ gradually increases in NCCO below room temperature. With regard to LSCO, a rapid increase in $v$ in the pseudo-gap region is well explained by the FLEX + $T$-matrix approximation, due to the CVC induced by the strong AF + SC fluctuations.

(v) ac-Hall effect: the strong $\omega$-dependence of $R_H(\omega)$ appears due to the frequency dependence of the CVC. Both $T$- and $\omega$-dependences of $R_H(\omega)$ are well reproduced by the FLEX+CVC method.

(vi) Impurity effect: the strong CVC near the AF QCP is suppressed by the weak local impurities. For strong local impurities, the residual resistivity exceeds the s-wave unitary scattering value since the local and staggered susceptibilities are strongly enhanced around the impurity sites.

**Results** (i)–(vi) are given by the same mechanism—the hot/cold-spot structure and the singular $k$-dependence of the total current $\hat{J}_k$ in nearly AF metals, which is shown in figure 14. We emphasize that the above-mentioned results are not derived from a special defect of the FLEX approximation. They are also reproduced by another spin fluctuation theory. In fact, Kanki and Kontani [34] calculated $\hat{J}_k$ based on the Millis–Monien–Pines model in equation (8), by assuming a realistic set of parameters for optimally doped YBCO. The $\hat{J}_k$ obtained represents the characteristic $k$-dependence in nearly AF metals as shown in figure 14. Moreover, we proved the significance of the CVC near AF-QCP beyond the one-loop (such as the FLEX) approximation in section 3.4. As a result, it is general that $R_H$ is prominently enhanced by the CVC when the AF fluctuations are strong, independent of the types of the spin fluctuation theories.

## 9.2. Applicable scope of the present study and future problems

In this study, we explained that the CVC plays a significant role in nearly AF metals, based on the microscopic Fermi liquid theory. To illustrate this theoretical idea, we performed numerical studies using the FLEX+CVC method in sections 4–7. Since the FLEX is a conserving approximation, we can perform a reliable study of the transport phenomena by including CVCs. As we discussed in section 2.2, the FLEX approximation can explain characteristic electronic properties in optimally doped HTSC. For example, the appropriate spin fluctuation $\chi^s(\omega)$ and the hot/cold-spot structure of quasiparticle damping rate $\gamma_s$ are reproduced satisfactorily. Although the FLEX approximation cannot explain the ‘strong pseudo-gap behavior’ below $T^* \sim 200\,\text{K}$, the FLEX + $T$-matrix approximation can reproduce various transport anomalies below $T^*$ in slightly under-doped systems as discussed in section 5. In the FLEX+$T$-matrix approximation, we take account of the strong SC fluctuations that are induced by AF fluctuations. The success of the FLEX + $T$-matrix method is strong evidence that SC fluctuations are the origin of strong pseudo-gap.

Here, we discuss the applicable scope of the FLEX approximation. First, we discuss the ‘weak pseudo-gap behavior’ below $T_0 \sim 600\,\text{K}$, where the DOS [242–245], the Knight shift [80,246,247] and the uniform susceptibility [248,249] decrease inversely with the growth of AF fluctuations. The weak pseudo-gap in the DOS is shallow and wide in energy. Although weak pseudo-gap behaviors are reproduced by the FLEX approximation, the gap behaviors obtained are too moderate. (Since the weak pseudo-gap affects transport phenomena only slightly, FLEX + CVC approximation can explain abnormal transport phenomena in HTSC.) This failure of the FLEX will originate from the fully self-consistent determination of the self-energy and the Green function. In fact, although a large weak pseudo-gap in the DOS is reproduced at the first iteration stage, it vanishes in the course of iteration. Mathematically, additional self-energy correction introduced by iteration (i.e. the feedback effect) should be canceled by the vertex correction in the self-energy that is introduced by iteration (i.e. the feedback effect). The same result is reported in the GW approximation: this is a first principle calculation for the self-energy, which is given by the convolution of the Green function $G$ and the screened interaction $W$ within the RPA. Although the descriptions of the bandwidth reduction and satellite structure in the quasiparticle spectrum are satisfactory in a partially self-consistent GW approximation, their applicability is restricted to the weak pseudo-gap limit.
method, they are smeared out in the fully self-consistent GW due to the feedback effect by iteration [250–252].

To produce a weak pseudo-gap successfully, it is better to derive the self-energy by performing (i) fully self-consistent calculation with vertex corrections or (ii) partially (or no) self-consistent calculation. Along the lines of method (i), Schmalian and Pines calculated the self-energy with all the vertex corrections by applying a high-temperature approximation [253]. Using a similar technique, Fujimoto calculated the pseudo-gap since the vertex corrections for the self-energy are considered [255]. This method also reproduces the difference in the fourth-order-perturbation theory with respect to fluctuations [254]. It is noteworthy that the fourth-order-perturbation theory by iteration [250–252].

GW energy correction due to spin fluctuations [259, 260]. These theories can explain the weak pseudo-gap in the DOS. However, they are not suitable for the study of the transport phenomena since they are not conserving approximations. For this reason, we use the FLEX approximation in the numerical study for the transport phenomena. Since the FLEX approximation tends to underestimate the anisotropy of the effective Coulomb interaction due to spin fluctuations [259, 260]. These theories can explain the weak pseudo-gap in the DOS.

Finally, we present important future problems. Since the FLEX (or FLEX + T-matrix) approximation is a one-loop approximation, it does not work satisfactorily in under-doped systems. However, the analysis in section 3.4 has shown that the enhancements of $R_H$ and $\Delta \rho_{\text{mag}}$ given by the FLEX + CVC method might be underestimated quantitatively.

In under-doped systems, a small amount of impurities drastically changes the electronic states of the system as discussed in section 7.2. Therefore, to understand transport phenomena in under-doped systems, we have to develop the theory of transport phenomena in the presence of disorder.

According to the ARPES measurements for under-doped Bi2212 [85, 264–266] and LSCO [267], the intensity of the spectrum $\rho_k(\omega) = \operatorname{ImG}(\omega - i\delta)/\pi$ on the Fermi surface at $\omega = 0$ takes a very small value around the hot spot below the pseudo-gap temperature $T^*$. As a result, $\rho_k(0)$  

 heavy-fermion systems, the Fermi surfaces are much more complicated. As discussed in section 8.1, the huge $R_H (\gg 1/\eta_e)$ in CeMIn$_5$ cannot be explained simply by the RTA in the multiband system. However, we have to study the multiband system for a more realistic study. In fact, we recently studied the 2D and 3D periodic Anderson model, which is a two-band model for a heavy-fermion system, and found that a relatively large mass-enhancement factor $z^{-1} \approx 10$ is obtained in the FLEX approximation [237]. Since $\rho$, $R_H$, and $\Delta \rho/\rho_0$ are independent of $z^{-1}$, we could discuss their critical behaviors using the Hubbard model as we have explained in section 8.1. Since $S$ and $v$ are proportional to $z^{-1}$, on the other hand, we have to take account of the experimental $z^{-1}$ if one compares the theoretical results with the experimental values.

Finally, we comment on the ‘field-induced QCP’ in CeCoIn$_5$ near $H_2 \sim 5$ T, where a prominent increment in the effective mass $m^*$ was inferred experimentally. Paglione et al. [234] measured the in-plane resistivity near $H_2$, and derived the relation $A \propto (H - H_2)^{1/3}$, where $A$ is the coefficient of the $T^2$-term in the resistivity. According to the Kadowaki–Woods relation $\sqrt{A} \propto m^*$ (see equation (164)), $m^*$ diverges at $H = H_2$. This fact strongly suggests that the field-induced QCP occurs at $H = H_2 \approx 5$ T in CeCoIn$_5$. The prominent increment of $m^*$ around the field-induced QCP is also confirmed by thermoelectric transport measurements $S$ and $\nu$ [261]. Interestingly, a similar field-induced QCP is also realized in heavily over-doped Bi2212 ($T_c = 10$ K) [262]. Up to now, the origin of the critical behavior is unknown. One possible origin will be the (field-induced) SDW state that is hidden in the SC state. In CeCoIn$_5$, Tanatar et al. [263] measured the anisotropy of the resistivity near $H_2$, and found that the in-plane resistivity ($\rho_\parallel$) is linear-in-$T$ above $4$ K, whereas it decreases more quickly below $4$ K. The observed $T$-dependence of $\rho_\parallel$ looks similar to the theoretical result in figure 34(i). In contrast, inter-plane resistivity ($\rho_\perp$) shows a complete $T$-linear dependence from $25$ mK to $16$ K, which might suggest that the CVC is unimportant for $\rho_\perp$. Moreover, the Wiedemann–Franz law $(\kappa/\sigma T = (\pi^2/3)(k_B/e)^2; \kappa$ being the thermal conductivity) is strongly violated only for the inter-plane direction. This strong anisotropy in transport may be difficult to explain by the current theory. Since the transport phenomena under high magnetic field are outside the scope of this study, this is an important future problem of transport phenomena.

Up to now, electronic properties in strongly correlated metals under high magnetic field have not been studied sufficiently. This issue is an important future challenge. We should note that the $H_2$-line in CeCoIn$_5$ is a weak first-order line for $T \ll T_\text{wo}$ [106]. Therefore, strictly speaking, the field-induced QCP in CeCoIn$_5$ cannot be a true QCP.

9.3. Fermi arc picture and transport phenomena

According to the ARPES measurements for under-doped Bi2212 [85, 264–266] and LSCO [267], the intensity of the spectrum $\rho_k(\omega) = \operatorname{ImG}(\omega - i\delta)/\pi$ on the Fermi surface at $\omega = 0$ takes a very small value around the hot spot below the pseudo-gap temperature $T^*$. As a result, $\rho_k(0)$
Figure 37. (i) Quasiparticle spectrum $\rho_k(0) = \text{Im} G_k(-i\delta)/\pi$ for LSCO at $T = 0.02$ given by the FLEX approximation. $\rho_k(0)$ is shown only when $\rho_k(0) > 0.1$. The contour is shown on the basal plane. (ii) $\rho_k(0)$ given by the FLEX + $T$-matrix approximation. The contour shows a ‘Fermi arc structure’.

Figure 38. Schematic picture of the ‘Fermi arc structure’ in a hole-doped system. The ‘effective Fermi surface (FS)’ above $T^*$, which was explained in figure 14(ii), is also described. The smallness of the effective Fermi surface is the origin of the enhancement of $R_H$ above $T^*$. Below $T^*$, the effective Fermi surface approaches the true Fermi surface since the CVC due to AF fluctuations are reduced. Therefore, $R_H$ decreases below $T^*$.

SDW order. Below $T_N$, $R_H$ is negative by reflecting the small electron-like Fermi surface around $(\pi, 0)$, which is caused by the reconstruction of the Fermi surface due to the SDW order. Surprisingly, $R_H$ remains negative even above $T_N$, although SDW-induced Fermi surface reconstruction is absent. Moreover, no anomaly in $R_H$ is observed at $T_N$. This highly non-trivial fact is explained by the Fermi liquid theory by considering the CVC, as shown in figure 19(iv). According to the ARPES measurements [269], a pseudo-gap state due to AF fluctuations is observed even above $T_N$. Therefore, a Fermi arc state appears to be realized in electron-doped systems.

When a distinct Fermi arc structure exists, the ‘effective carrier density at the Fermi level ($n_{\text{eff}}$)’ reduces in proportion to the length of the Fermi arc. Since $R_H \propto 1/n_{\text{eff}}$ in the RTA, it is sometimes claimed that the enhancement of $R_H$ in HTSCs can be explained by the reduction in $n_{\text{eff}}$ due to the emergence of the Fermi arc. This idea seems to be valid in heavily under-doped compounds [270]. However, this idea is not true for slightly under-doped systems since $R_H$ starts to decrease below $T^*$, whereas the length of the Fermi arc shrinks due to SC fluctuations. The reduction in $R_H$ in the pseudo-gap region is explained by the reduction in the AF fluctuation, as explained in section 5.1.

Here, we discuss the Hall coefficient in the presence of the Fermi arc structure due to strong AF fluctuations, by taking the CVC into consideration. The cases we consider correspond to the hole-doped systems above $T^*$ and the electron-doped systems above $T_N$. Figure 38 shows the Fermi arc structure due to the weak pseudo-gap formation and the ‘effective Fermi surface’ that is defined to be perpendicular to the total current $\vec{J}_c$, as introduced in figure 14(ii). (Note that the ‘effective Fermi surface’ is not visible in the ARPES measurements.) Within the RTA, $\sigma_{xx}^{\text{RTA}} \propto l_c/l_{\text{cold}}$ and $\sigma_{xy}^{\text{RTA}} \propto l_c/l_{\text{cold}}^2$, where $l_c$ is the length of the cold spot (Fermi arc). Therefore,

$$R_H^{\text{RTA}} \propto 1/l_c.$$  (149)
As we have discussed, $\sigma_{xy}$ is proportional to the curvature of the effective Fermi surface at the cold spot. In the presence of strong AF fluctuations above $T^*$, $\theta_k^*$ rotates approximately by $\pi$ around the cold spot, as shown in figure 38. Therefore, the curvature of the effective Fermi surface at the cold spot in figure 38 is proportional to $1/l_c$. As a result, due to the CVC above $T^*$,

$$R_H \propto 1/l_c^2 \tag{150}$$

approximately. The large curvature of the effective Fermi surface around the cold spot is the origin of the enhancement of $R_H$. In conclusion, when the Fermi arc emerges, the magnitude of $R_H$ is much larger than the value obtained by the RTA that is given in equation (149).

We note that if the distinct Fermi arc structure due to the highly anisotropic $\tau_k$ is realized, it should lead to a huge magnetoresistance within the RTA: in fact, $\Delta \sigma_{xx}$ diverges when the $k$-dependence of the mean free path $l_k = v_k \tau_k$ is not continuous on the Fermi surface; in this case, the second term in equation (99) diverges, as pointed out in [26]. Therefore, the modified Kohler’s rule given by equation (4) cannot be satisfied. In summary, a unified understanding of anomalous transport phenomena in HTSC cannot be achieved by the RTA even if the Fermi arc structure is taken into account.

9.4. Unconventional transport phenomena in multiorbital systems: anomalous Hall effect and grand Kadowaki–Woods relation

In previous sections, we studied single-band models and showed that the CVC induces various striking non-Fermi-liquid-like behaviors in the presence of AF fluctuations. For example, $R_H$ shows strong temperature dependence due to the CVC. In multiband systems, however, $R_H$ can exhibit temperature dependence or sign change within the RTA, if a hole-like Fermi surface and an electron-like Fermi surface coexist and their relaxation times have different $T$-dependences. There are many such examples even in conventional metals. In this study, we did not discuss the multiband effect, since the magnitude $|R_H|$ will remain $\sim 1/\rho e$ in this mechanism. Thus, the multiband mechanism is impossible to explain the huge $|R_H|$ observed in various systems near the AF-QCP.

However, a kind of multiorbital effect causes a huge Hall coefficient in $d$- and $f$-electron systems, which is known as the ‘anomalous Hall effect (AHE)’. In the presence of the AHE, the Hall resistivity is given by

$$\rho_{xy} = R_H B + R_H^\text{AH}, \tag{151}$$

where $B$ is the magnetic field and $R_H$ is the ordinary Hall coefficient. $R_H^\text{AH}$ is the anomalous Hall coefficient, which is (generally) proportional to the magnetization $M$. In general, $R_H^\text{AH}$ and $R_H^\text{M}$ are even functions of $B$ and $M$. In ferromagnets, the second term $R_H^\text{AH} M$ takes a finite value even if $B = 0$. Study of the AHE due to the multiband effect was initiated by Kaplans and Luttinger [271], and by Luttinger [272]. They found that the anomalous Hall conductivity (AHC) $\sigma_{xy}^\text{AHC} = (R_H^\text{AH} M/\mu^2)$ is independent of the resistivity $\rho$. This Kaplans-Luttinger term is called the ‘intrinsic AHE’ because it exists even in systems without impurities. Later, Smit presented a mechanism of ‘extrinsic AHE’ [273]: he found that the spin-polarized electrons are scattered asymmetrically around an impurity site in the presence of spin–orbit interaction. The AHC due to this skew-scattering mechanism is linearly proportional to $\rho$.

In paramagnetic heavy-fermion systems, the observed Hall coefficient is given by

$$R_H^\text{HF} = \mu_B/B = R_H + R_H^\text{AH}, \tag{152}$$

where $R_H^\text{AH} = R_H M/\chi$ and $\chi = M/B$ is the uniform magnetic susceptibility. In heavy-fermion systems, the AHE due to the second term of equation (152) takes a large value since the uniform susceptibility $M/B = \chi$ is widely enhanced due to the strong Coulomb interaction. Due to the AHE, $R_H^\text{HF}$ starts to increase with increasing temperature from 0 K, and it begins to decrease after exhibiting its maximum value around the coherent temperature $T_\text{coh}$. The maximum value of $|R_H^\text{HF}|$ is more than one order of magnitude greater than $1/|\rho e| \sim 10^{-9} \text{m}^2\text{C}^{-1}$. The AHE cannot be derived from the RTA since the interband hopping of electrons is important. (In the AHE, the effect of CVC is not expected to be crucial.) Experimentally, $R_H^\text{HF}$ is positive in the usual Ce- and U-based heavy-fermion systems [274, 275].

Here we discuss the AHE in heavy-fermion systems. The bandstructure of Ce- and Yb-based heavy-fermion systems is described by the following orbitally degenerate periodic Anderson model [276, 277]:

$$H = \sum_{k \sigma} \epsilon_k c_k^{\dag \sigma} c_k \sigma + \sum_{kM} E_f f^\dagger_{kM} f_{kM} + \sum_{kM\sigma} (V_{kM\sigma} f^\dagger_{kM} c_{k\sigma} + \text{h.c.}) + \frac{U}{2} \sum_{kkM\sigma} f^\dagger_{k+qM\sigma} f^\dagger_{k-qM\sigma} f_{k'\sigma} f_{k'M\sigma}, \tag{153}$$

where $c_{k\sigma}^\dagger$ is the creation operator of the conduction electron ($f$-electron) with $\sigma = \pm 1$ ($M = J_1, J_2, \ldots, J_3$). f-orbital degeneracy is $N_f = 2J + 1$. $\epsilon_k$ ($E_f$) is the spectrum for conduction electrons ($f$-electrons). In the case of Ce-compound ($J = 5/2$), the complex $c$-f mixing potential is given by

$$V_{kM\sigma} = \sigma \sqrt{\frac{4\pi}{3}} \sqrt{\frac{7/2 - 2M\sigma}{\pi}} Y_{J+1/2}^{M-\sigma} (\theta_k, \varphi_k),$$

where $Y_{J+1/2}^{M-\sigma} (\theta_k, \varphi_k)$ is the spherical harmonic function. In the case of Yb-compound ($J = 7/2$), $V_{kM\sigma} = \sqrt{\frac{4\pi}{3}} \sqrt{\frac{7/2 + 2M\sigma}{\pi}} Y_{J+1/2}^{M-\sigma} (\theta_k, \varphi_k)$.

Note that the relation

$$\sum_{M=-J}^{M=J} |V_{kM\sigma}|^2 = V_{0\sigma}^2 \tag{154}$$

holds.

The Green function for the $c$ electron is given by [276, 277]

$$G^c(\epsilon) = \left( \frac{\epsilon + \mu - \epsilon_k - (V_0^c)^2}{\omega + \mu + E_c - \Sigma(\epsilon)} \right)^{-1},$$

$$\approx \left( \frac{\epsilon + \mu - \epsilon_k - (V_0^c)^2}{\omega - E_c - i\gamma^*} \right)^{-1}, \tag{154}$$

where $V_0^c = \sqrt{2} V_0$, $E_c = z(\epsilon_c + \Re \Sigma(0) - \mu)$, $

\gamma^* = z \Im \Sigma(0)$ and $z = (1 - \partial \Re \Sigma(\omega)/\partial \omega)|_{\omega=0}$ is the...
renormalization factor. Since the quasiparticle energy $E_k^*$ satisfies $\text{Re} [1/G_k (E_k^*)] = 0$, the equation for $E_k^*$ is given by

$$E_k^* + \mu - \epsilon_k - \frac{(V_\gamma^n)^2 (E_k^* - \tilde{E}_f)}{(E_k^* - \tilde{E}_f)^2 + (\gamma^n)^2} = 0.$$  (155)

Here we derive the quasiparticle energy by analysing equation (155): at zero temperature where $\gamma^n = 0$, a heavy quasiparticles band is formed due to c–f hybridization. Below $T_{coh}$ the quasiparticles band is

$$E_k^* = \frac{1}{2} (\epsilon_k - \mu + \tilde{E}_f \pm \sqrt{(\epsilon_k - \mu - \tilde{E}_f)^2 + 4 (V_\gamma^n)^2}).$$  (156)

where $E_k^*$ ($E_k^{* -}$) represents the lower (upper) quasiparticles band. $\gamma^n$ grows monotonically as temperature increases, and $|\tilde{E}_f| \sim |\gamma^n|$ at the coherent temperature $T_{coh}$. When $T \gg T_{coh}$, c–f hybridization is prohibited by $\gamma^n \gg |\tilde{E}_f|$.

We summarize the electronic states in heavy-fermion systems: (i) Below $T_{coh}$, Fermi liquid state with heavy quasiparticles is realized due to c–f hybridization. Mass-enhancement factor $z^{-1}$ and uniform susceptibility $\chi \propto z^{-1}$ are constant. The heavy quasiparticle bandwidth $W_{HF}$ is approximately given by $W_{HF} \sim \min [E_{k^*} - E_{k^*}] \sim (V_\gamma^n)^2 / W_c \sim |\tilde{E}_f|$, where $W_c$ is the c-electron bandwidth. (ii) Above $T_{coh}$, c–f hybridization ceases, and localized f-electrons causes Curie–Weiss susceptibility. Experimentally, the temperature of maximum resistivity $T_p^*$ is larger than $T_{coh}$.

In paramagnetic heavy-fermion systems, the Hall coefficients take huge values due to the AHE [274, 275, 278]. In the early stage, Coleman et al [279] and Fert and Levy [280] developed theories of extrinsic AHE: they studied the extrinsic mechanism based on the f-electron impurity Anderson models with d-orbital channels, and predicted the relation $R_{H}^{AHE} \propto \chi \rho$ above $T_{coh}$ when the d-orbital phase-shift is finite.

On the other hand, Kontani and Yamada discussed the intrinsic AHE due to the multifield effect. They derived the AHC in the $J = 5/2$ periodic Anderson model, which a model in the strong limit of spin–orbit interaction [277, 281]. In this model, the anomalous velocity originates from the angular momentum of the localized f-electrons. That is, the AHE originates from the transfer of angular momentum of the f-electron to the conduction electron. Based on the linear response theory, they derived a large intrinsic AHC: $\sigma_{xy}^f / H [1/\Omega \text{cm G}] = 5.8 \times 10^{-8} (n_0 [\text{cm}]^{-3})^{1/3} / \tilde{E}_f [\text{K}]$ where $\tilde{E}_f$ is the renormalized f-level measured from the Fermi energy. $|\tilde{E}_f|$ gives approximate quasiparticle bandwidth. This study predicts that $R_{H}^{AHE} > 0$ (<0) in Ce (Yb) based heavy-fermion systems since $\tilde{E}_f > 0$ (<0). This prediction is consistent with the experimental results [274, 275]. If we substitute $\tilde{E}_f = 10 \text{ K}$ and $n_0 = 10^{22} \text{ cm}^{-3}$, $\sigma_{xy}^f / H = 1.36 \times 10^{10} [\Omega \text{cm} \text{G}]$, which is a typical experimental value. The predicted temperature dependence of $R_{H}^{AHE} = (\sigma_{xy}^f / H) \rho^2$ is $R_{H}^{AHE} \propto (\gamma^{2/3} / (\tilde{E}_f + \gamma^{2/3}) \chi$ [277]. Therefore, $R_{H}^{AHE}$ shows the following crossover behavior:

$$R_{H}^{AHE} \propto \chi \rho^2 [\chi \rho^2] : \text{ below } T_{coh} (\gamma^n < |\tilde{E}_f|),$$  (157)

$$R_{H}^{AHE} \propto \chi [\propto 1 / T ] : \text{ above } T_{coh} (\gamma^n > |\tilde{E}_f|).$$  (158)

In typical Ce-based heavy-fermion systems, $1/z \sim 100$ and $|\tilde{E}_f| \sim 10 \text{ K}$. Below $T_{coh}$, equation (157) is proportional to $\rho^2$ since $\chi$ is constant for $T < T_{coh}$. Figure 39 shows the $R_{H}$ in U$_3$Ni$_5$Sn$_4$ ($\gamma = 380 \text{ mJ K}^{-2}$, where the relation $R_{H}^{AHE} = R_{H}^{HF} - R_{H}^{CE} \propto \rho^2$ holds well below $\sim 0.3 T_{coh} \sim 25 \text{ K}$. Note that the extrinsic type AHE [279, 280] and the intrinsic type AHE [277] can coexist. However, the relation $R_{H}^{AHE} \propto \rho^2$ suggests that the intrinsic-type AHE is dominant at least below $T_{coh}$. Above $T_{coh}$, the AHE due to interband transition is suppressed when c–f mixing is prohibited. Therefore, equation (158) is independent of $\rho$. This coherent–incoherent crossover of intrinsic AHE was first theoretically derived in [277]. This crossover behavior is not restricted to heavy-fermions systems but also observed in various transition ferromagnets [282].

In heavy-fermion systems near the AF-QCP, $R_{H}$ shows strong temperature dependence due to the CVC. To extract $R_{H}^{AHE}$ from the observed Hall coefficient, we have to seriously consider the $T$-dependence of $R_{H}^{AHE}$. Very fortunately, AHE is vanishingly small in CeM$_3$In$_5$ [19, 20]. Therefore, we could perform a reliable analysis of the ordinary Hall effect in CeM$_3$In$_5$ as discussed in section 8.1, without the necessity of subtracting the AHE. Note that the AHE vanishes when the crystal-field splitting of the f-levels is much larger than $T_{coh}$. This may be the reason for the small AHE in CeM$_3$In$_5$. Recently, the theory of the intrinsic AHE in (ferromagnetic) d-electron systems has been developed based on realistic multiorbital tight-binding models [283, 284]. It was revealed that a large anomalous velocity emerges in general multiorbital
given by the linear combination of $|m|$-component of the spin–orbit interaction $\sum_i (\hat{l}_z \cdot \mathbf{s}_i)$. The effective magnetic flux for up-electrons is negative. Both $|x\rangle$ and $|y\rangle$ are given by the linear combination of $|l_z = \pm 1\rangle$. Since $\tilde{l}_z^2 = 1$ in the present basis, the rotation operator $R(\theta) = e^{-2i\theta\hat{l}_z} = \cos \theta - \hat{i} \sin \theta$ is equal to $-\hat{i} \sin \theta$. Therefore, $\hat{l}_z |x\rangle = i R(\pi/2) |x\rangle = i |y\rangle$ and $\hat{l}_z |y\rangle = i R(\pi/2) |y\rangle = -i|x\rangle$.

d-electron systems because of the inter-orbital hopping. This is the origin of the large AHE in d-electron systems, which is very similar to the origin of the AHE in f-electron systems [277].

The intrinsic AHEs for Fe [286] and in Sr2RuO4 [287] were also calculated based on the LDA band calculations.

Here we present an intuitive explanation of the AHE due to the multiorbital effect [284, 288]. In a multiorbital system, a conduction electron acquires the ‘effective Aharonov–Bohm phase factor’ due to d-atomic angular momentum with the aid of the spin–orbit interaction and the inter-orbital hoppings, which is responsible for the Hall effect. The intuitive explanation is based on the tight-binding model, which is given in figure 40, which represents a square-lattice $(d_{xz}, d_{yz})$-orbital tight-binding model with $z$-component of the spin–orbit interaction $\sum_i (\hat{l}_z \cdot \mathbf{s}_i)$ [284], which is a simplified model for (Ca,Sr)$_2$RuO$_4$; this compound shows large AHE under the magnetic field [285]. Its magnitude is comparable with the large AHE in f-electron systems (such as UPt$_3$). In figure 40, $\pm t'$ represents the interorbital hopping integral between nearest neighbors. Now, let us consider the motion of a down-spin electron along a triangle of half unit cell: an electron in the $d_{xz}$-orbital can transfer to $d_{yz}$-orbital and vice versa using the spin–orbit interaction $\pm \hbar \lambda l_z$, where $\langle yz | l_z | xz \rangle = -(xz | l_z | yz) = i$. By a combination of angular dependence of inter-orbital hopping and the spin–orbit interaction, any clockwise (anti-clockwise) motion along any triangle path with spin–orbit interaction causes the factor $+i$ ($-i$). This factor can be interpreted as the ‘Aharonov–Bohm phase factor’ $\exp(2\pi i \phi/\phi_0) [\phi_0 = \hbar c/|e|]$, where $\phi = \phi_0/4$ represents the ‘effective magnetic flux’ in the half unit cell. We revealed the fact that the effective magnetic flux, which is inherent in multiorbital d-electron systems, causes huge AHE in various transition metals.

In many d-electron ferromagnets, intrinsic AHE ($R_H^{\text{I}} \propto \rho^2$) seems to be observed experimentally. In particular, Asamitsu et al. measured AHE in various transition-metal ferromagnets, and found a universal crossover behavior from $R_H^{\text{I}} \propto \rho$ to $R_H^{\text{I}} \propto \rho^n (n = 0 \sim 0.5)$ as the resistivity increases, around $\rho \sim 100 \mu \Omega \text{cm}$. A recent theoretical study in [284] has revealed that the experimental result is explained by the following coherent–incoherent crossover

$$R_H^{\text{I}} \propto \rho^2 \quad : \gamma < \Delta, \quad (159)$$

$$R_H^{\text{I}} \propto \text{const} \quad : \gamma > \Delta, \quad (160)$$

where $\Delta$ represents the minimum bandsplitting near the Fermi level. $\Delta \geq 0.1 \text{eV}$ in usual transition-metal ferromagnets. It is apparent that equations (159) and (160) correspond to equations (157) and (158) for heavy-fermion systems. [Note that $R_H^{\text{IHE}} = R_H^{\text{I}} \chi$.] Therefore, a conventional behavior $R_H^{\text{I}} \propto \rho^2$ is violated in high-resistivity metals since the interband particle–hole excitation, which is the origin of the AHE, is suppressed when $\gamma$ is larger than bandsplitting energy $\Delta$.

There is a simple intuitive explanation for this coherent–incoherent crossover of intrinsic AHE: when $\gamma$ is sufficiently small, the intrinsic Hall conductivity is proportional to the lifetime of the interband particle–hole excitation: $\hbar/\Delta$ [271, 272, 277]. In the high-resistivity regime where $\gamma \gg \Delta$, the SHC decreases drastically with $\gamma$ since the interband excitation is suppressed when the quasiparticle lifetime $\hbar/\gamma$ is shorter than $\hbar/\Delta$. According to the above discussion, coherent–incoherent crossover of intrinsic AHE should be universal and is widely observed in various multiorbital p-, d- and f-electron systems.

It is noteworthy that the large anomalous velocity also induces a sizable spin Hall effect (SHE) in paramagnetic multiorbital systems [284, 288, 289]. The SHE is the phenomenon that an applied electric field induces a spin current $j^s \equiv j_1 - j_\perp$ in a transverse direction. Recently, SHE has attracted great attention due to its fundamental interest and its potential application in spintronics. Karpus and Luttinger [271] showed that an applied electric field induces a spin-dependent transverse current in the presence of spin–orbit interaction. This mechanism causes the AHE in ferromagnetic metals and the SHE in paramagnetic metals. Recently, the spin Hall conductivity (SHC) in Pt was observed [290], and it was found that the SHC in Pt is $10^4$ times larger than that observed in n-type semiconductors. This has attracted considerable attention to the study of the SHE in transition metals.

Recently, we proposed a new mechanism for the giant SHE originating from the d-orbital degrees of freedom, which is absent in semiconductors [289, 290, 291]. In a multiorbital system, a conduction electron acquires the ‘effective Aharonov–Bohm phase factor’ due to d-atomic angular momentum, as explained in figure 40. We studied the SHEs in Sr2RuO4 [288], which is the first theoretical study of the SHE in d-electron systems, in Pt [289] and in various 4d and 5d transition metals [291]. It is found that the explanation in figure 40 seems to capture the characteristics of SHE in d-electron systems. In contrast, the ‘Dirac monopole mechanism’ [292] is appropriate when massless Dirac cone dispersion exists close to the Fermi level as in semiconductors. The present mechanism of SHE is also different from that in the Rashba-type 2D electron-gas model due to momentum-dependent SOI [293]. Thus, giant SHE due to atomic orbital degrees of freedom is ubiquitous in various p-, d- and f-electron systems.
Finally, we discuss the multiorbital eﬀect on the Kadowaki–Woods (KW) ratio $A \gamma^{-2}$, where $A$ is the coeﬃcient of the $T^2$ term in the resistivity, and $\gamma$ is the coeﬃcient of the $T$-linear term of the electric speciﬁc heat. Experimentally, the KW ratio in Ce- and U-based heavy-fermion compounds shows an approximate universal value $A \gamma^{-2} \approx 5 \times 10^{-5}$ [µΩ cm (mol K mJ$^{-1}$)], which is known as the KW relation [294]. Although it was believed to be universal in heavy-fermion systems for a long time, recent experimental activities have revealed that the KW relation is strongly violated in many Yb-based compounds. Recently, the author derived a generalized KW relation that is applicable for systems with general f-orbital degeneracy $N_f$ for Ce- and Yb-based compounds [295] and for Sm- and Er-based compounds [296]. By considering the material dependence of $N_f$, the mystery of the failure of the KW relation was resolved.

Here, we analyze the KW ratio in terms of the DMFT ($d = \infty$-limit) [109, 110], which is believed to be useful in heavy-fermion systems that are not close to the AF-QCP. By using the DMFT, we can utilize the strong-coupling Fermi liquid theory for the impurity Anderson model developed by Yamada and Yoshida [297]; using the Ward identity, they derived the exact Wilson ratio $R \equiv \langle \chi / \chi^0 \rangle / (\gamma / \gamma^0) = 2$ for $N_f = 2$ in the strong-coupling limit, that is, $\gamma^{-1} \gg 1$ and the f-electron charge susceptibility is zero (Kondo model limit). Here $\chi^0$ and $\gamma^0$ represent the non-interacting values. Using the DMFT, $\gamma$ and $\text{Im} \Sigma(0)$ in the PAM (in equation (153)) are given by [295]$^2$

$$\gamma = N_A k_B \pi^2 \frac{2}{3} N_f (N_f - 1) \Gamma_{\text{loc}} (0, 0) \rho_f (0)^2,$$

$$\text{Im} \Sigma(0) = \frac{\pi^2 (k_B T)^2}{2} (N_f - 1) (\Gamma_{\text{loc}} (0, 0) \rho_f (0))^3,$$

in the strong-coupling limit. Here, $N_A = 6.02 \times 10^{23}$ is the Avogadro constant and $\Gamma_{\text{loc}} (0, 0)$ is the local four-point vertex. $\rho_f (0)$ is the DOS for f-electron per channel; $N_f \rho_f (0)$ is the total DOS at the Fermi level. When $\epsilon_F$, in equation (153) is a free dispersion, the conductivity is given by

$$\sigma = e^2 / (3\pi^2) \frac{N_f}{N_A} \rho_f (0) \cdot \text{Im} \Sigma(0),$$

(163)

where $n (= k_B^2 / 3\pi^2)$ is the density of quasiparticles that form the conduction band and $a$ is the unit cell length. According to equations (161)–(163), we find the scaling properties $\gamma \propto N_f (N_f - 1) \Gamma_{\text{loc}} \rho_f^2$ and $\Lambda \propto N_f (N_f - 1) \Gamma_{\text{loc}}^2 \rho_f^{1/2}$. Therefore, we obtain the ‘grand KW relation’ that is valid for any $N_f$ ($\geq 2$) [295, 296, 298]$^2$: $A \gamma^{-2} \approx \frac{9 (3\pi)^{1/3}}{2} \frac{1}{e^2 k_B^2 4 n^{1/3} \Lambda N_A^2 1/2 N_f (N_f - 1)} \approx \frac{1}{2} \times 10^{-5}$ [µΩ cm (mol K mJ$^{-1}$)],

$$\text{Im} \Sigma(0) = \frac{\pi^2 (k_B T)^2}{2} (N_f - 1) (\Gamma_{\text{loc}} (0, 0) \rho_f (0))^3,$$

where we put $h/e^2 = 2.6 \times 10^4 \Omega$, $k_B = 1.38 \times 10^{-23}$ J K$^{-1}$ and we assumed $1/4 n^{1/3} \Lambda^3 \approx 4 \times 10^{-8}$ cm.

$^2$ In [295, 296], we put $\gamma = (\pi T)^3 / 6 \cdot N_f \rho_f (0)$ by mistake; the correct relation is $\gamma = (\pi T)^3 / (3 \cdot N_f \rho_f (0))$. For this reason, equation (161) is two times equation (8) in [295]. Since the author of [295] assumed $1/4 n^{1/3} \Lambda^3 \approx 1 \times 10^{-4}$ cm, the grand-KW relation in [295] is equal to equation (164) in this article.

Figure 41 (a) Grand Kadowaki–Woods relation in [298], where $A = 2A / N_f (N_f - 1)$ and $\gamma = 2\gamma / N_f (N_f - 1)$. (b) A scaling relation between $E_{\text{mag}}$ and $\sqrt{\gamma / \gamma_{\text{loc}}}$ [304]. $E_{\text{mag}}$ corresponds to $2\gamma$ and $\sqrt{\gamma / \gamma_{\text{loc}}} \propto \sqrt{\gamma}$ (Kondo ratio) is a constant of order $O(1)$ [305].

Without crystal-field splitting of the f-level, $N_f = 2J + 1 = 6$ for Ce$^{3+}$ and Sm$^{3+}$ ions, and $N_f = 8$ for Yb$^{3+}$ and Er$^{3+}$ ions. Therefore, the previous KW relation turned out to be valid only when $N_f = 2$ (Kramers doublet case due to strong crystal-field splitting). A similar universal relation limit $T \to 0$ $eS / T \gamma \approx \pm 1$ ($S$ is the Seebeck coeﬃcient) was recently found [299, 300]. The characteristics of the electronic state in heavy-fermion systems are (i) large mass enhancement and (ii) small charge susceptibility since the f-electron is almost localized. The grand-KW relation (164) is derived only by imposing these constraints on the microscopic Fermi liquid theory. This fact illustrates a remarkable advantage of the Fermi liquid theory for the analysis of strongly correlated systems.

Figure 41 (i) shows the experimental veriﬁcation of equation (164) for various heavy-fermion compounds, where $N_f$ in each compound was determined by the temperature dependence of $\chi$ and the inelastic neutron scattering [298]. Tsuji’s study conﬁrmed that $N_f \sim 2$ in many Ce-based compounds, where crystal-field splitting is larger than the renormalized Fermi energy $W_{\text{HF}}$. On the other hand, $N_f \sim 8$ in many Yb-based ones, where crystal-field splitting is smaller than $W_{\text{HF}}$. Torikachvili et al also found that other Yb-based
heavy-fermion systems YbT2Zn20 (T = Fe, Co, Ru, Rh, Os, or Ir) follow the grand-KW relation shown in equation (164) [301]. When Nf is larger than the number of conduction band Nc, almost unhybridized f-levels exist near the Fermi level, except when the crystal-field splitting is very large. This situation induces a huge anomalous Hall coefficient [277] as well as the large Van Vleck susceptibilities [302, 303] in Kondo insulators and in singlet SC heavy-fermion systems.

We briefly discuss the absence of the multiband effect on the KW relation: in the usual heavy-fermion compound, there are one or two (relatively) large Fermi surfaces composed of heavy quasiparticles, and several (relatively) small Fermi surfaces composed of light quasiparticles. In the presence of impurities, AT2 ≈ ⟨γk⟩1FS, as explained in equation (33). Therefore, heavy quasiparticles on the large Fermi surfaces give the dominant contribution to both the specific heat and the A-term. As a result, the grand-KW relation is universally realized even in multiband systems. Moreover, in many heavy-fermions, there is evidence that the heavy quasiparticles on the large Fermi surfaces give the dominant contribution to the conductivity. For example, |R2|| ≈ 1/ne seems to be realized in many heavy-fermions away from AF-QCPs, although |R2|| ≫ 1/ne should be realized when small Fermi surfaces are the most conductive.

Figure 41(ii) shows the mid-infrared peak energies Epeak of the optical conductivity σ(ω) in various heavy-fermion systems [304]. According to the Fermi liquid theory, Epeak ≈ √2eV is satisfied in the PAM, where V is the c–f mixing potential and 1/2 = m* /m is the mass-enhancement factor. Okamura confirmed that the relation Epeak ≈ γ−1/2 is universally satisfied in various heavy-fermion compounds, as shown in figure 41(ii). This scaling relation is well satisfied regardless of Nf. Their study established the validity of the PAM and the Fermi liquid theory in various heavy-fermion compounds.

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