THE ERDÖS–KO–RADO THEOREM FOR TWISTED
GRASSMANN GRAPHS

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Abstract. We present a “modern” approach to the Erdős–Ko–Rado theorem for $Q$-polynomial distance-regular graphs and apply it to the twisted Grassmann graphs discovered in 2005 by van Dam and Koolen.

1. Introduction

The 1961 theorem of Erdős, Ko and Rado [8] asserts that the largest possible families $Y$ of $d$-subsets of a $v$-set such that $|x \cap y| \geq t$ for all $x, y \in Y$ where $v > (t+1)(d-t+1)$ are the families of all $d$-subsets containing some fixed $t$-subset. In fact, the exact bound $v > (t+1)(d-t+1)$ was obtained later by Wilson [26] as an application of Delsarte’s linear programming method [6]. It is natural to think of this theorem as a result about (vertex) subsets of the Johnson graphs $J(v, d)$, and analogous theorems are known for several other families of distance-regular graphs, e.g., Hamming graphs $H(d, q)$ ($q \geq t + 2$) [19], Grassmann graphs $J_q(v, d)$ ($v \geq 2d$) [14, 10, 11, 23], bilinear forms graphs $\text{Bil}_q(d, e)$ ($d \leq e$) [15, 11, 23].

In this note, we first distill common algebraic techniques found in some of the proofs of these “Erdős–Ko–Rado theorems” into a unified approach for general $Q$-polynomial distance-regular graphs $\Gamma$. Our approach is also “modern” in the sense that it is based on and motivated by the theory of two parameters, width $w$ and dual width $w^*$, of a subset $Y$ of $\Gamma$ introduced in 2003 by Brouwer et al. [4]. In this setting, the “$t$-intersecting” condition amounts to requiring $w \leq d - t$ where $d$ is the diameter of $\Gamma$, and we shall view the Erdős–Ko–Rado theorem as characterizing those subsets $Y$ with $w = d - t$ and $w^* = t$ by their sizes among all $t$-intersecting families. There are two steps involved: (1) construction of a specific feasible solution to the dual of a linear programming problem; (2) classification of the descendents of $\Gamma$, i.e., those subsets having the property $w + w^* = d$. We demonstrate this approach by deriving the Erdős–Ko–Rado theorem for the twisted Grassmann graphs $\tilde{J}_d(2d + 1, d)$ discovered in 2005 by van Dam and Koolen [5].

2. A “modern” approach to the Erdős–Ko–Rado theorem for $Q$-polynomial distance-regular graphs

Let $\Gamma = (X, R)$ be a finite connected simple graph with diameter $d$ and path-length distance $\partial$, and $\mathbb{R}^{X \times X}$ the set of real matrices with rows and columns indexed by $X$. For each $i$ ($0 \leq i \leq d$), let $A_i \in \mathbb{R}^{X \times X}$ be the adjacency matrix of
the distance-$i$ graph $\Gamma_i$ of $\Gamma$, so $A_0 = I$ and $\sum_{i=0}^d A_i = J$, the all ones matrix. We say $\Gamma$ is distance-regular if $A := \text{span}\{A_0, A_1, \ldots, A_d\}$ is closed under ordinary matrix multiplication; or equivalently, $A$ is a (commutative) algebra. (The reader is referred to [2, 3, 13] for background material on distance-regular graphs.) Throughout this note, suppose $\Gamma$ is distance-regular. We call $A$ the Bose–Mesner algebra of $\Gamma$. It is semisimple (as it is closed under transposition) and therefore has a basis $\{E_i\}_{i=0}^d$ consisting of the primitive idempotents; we always set $E_0 = |X|^{-1} J$. Note that $A$ is also closed under entrywise multiplication, denoted $\circ$. We shall assume $\Gamma$ is $Q$-polynomial with respect to the ordering $\{E_i\}_{i=0}^d$, i.e., $E_1 \circ E_i$ is a linear combination of $E_{i-1}, E_i, E_{i+1}$ with nonzero coefficients for $E_{i-1}, E_{i+1}$ ($0 \leq i \leq d$), where $E_{-1} = E_{d+1} = 0$. Let $Q = (Q_{ij})_{0 \leq i, j \leq d}$ be the second eigenmatrix of $\Gamma$:

$$E_j = \frac{1}{|X|} \sum_{i=0}^d Q_{ij} A_i \quad (0 \leq j \leq d).$$

Let $Y$ be a nonempty subset of $X$ and $\chi \in \mathbb{R}^X$ its (column) characteristic vector. Brouwer et al. [4] defined the width $w$ and dual width $w^*$ of $Y$ as follows:

$$w = \max\{i : \chi^T A_i \chi \neq 0\}, \quad w^* = \max\{i : \chi^T E_i \chi \neq 0\}.$$  

They showed (among other results) that

$$w + w^* \geq d.$$  

We call $Y$ a descendant [24] of $\Gamma$ if $w + w^* = d$. It should be remarked that every descendant is a so-called completely regular code (cf. [17]), and that the induced subgraph is a $Q$-polynomial distance-regular graph provided it is connected; see [4] Theorems 1–3. See also [24] for more information on descendants.

Now fix an integer $t$ ($0 < t < d$) and suppose $w \leq d - t$; in other words, $Y$ is “$t$-intersecting”. We recall the inner distribution $e = (e_0, e_1, \ldots, e_d)$ of $Y$:

$$e_i = \frac{1}{|Y|} \chi^T A_i \chi, \quad (eQ)_i = \frac{|X|}{|Y|} \chi^T E_i \chi \quad (0 \leq i \leq d).$$

It follows that $|Y| = (eQ)_0$ and

$$e_0 = 1, \quad e_1 \geq 0, \ldots, e_{d-t} \geq 0, \quad e_{d-t+1} = \cdots = e_d = 0,$$

$$(eQ)_1 \geq 0, \ldots, (eQ)_d \geq 0.$$  

(Observe that the $E_i$ are positive semidefinite.) Following [6], we view these as a linear programming maximization problem. A vector $f = (f_0, f_1, \ldots, f_d)$ satisfying (2), (3) below gives a feasible solution to its dual problem:

$$f_0 = 1, \quad f_1 = \cdots = f_t = 0, \quad f_{t+1} > 0, \ldots, f_d > 0,$$

$$fQ^T)_1 = \cdots = (fQ^T)_{d-t} = 0.$$  

Indeed, we have

$$|Y| = (eQ)_0 \leq eQ f^T = (fQ^T)_0$$

with equality if and only if $(eQ)_{t+1} = \cdots = (eQ)_d = 0$, i.e., $w^* \leq t$. By virtue of (1), it follows that

**Lemma 1.** Let $Y$ be a nonempty subset of $X$ with $w \leq d - t$. Suppose there is a vector $f = (f_0, f_1, \ldots, f_d)$ satisfying (2), (3). Then $|Y| \leq (fQ^T)_0$, and equality holds if and only if $Y$ is a descendant of $\Gamma$ with $w = d - t$ and $w^* = t$. \vspace{1em}
The vector $f$ above is of independent interest from the point of view of Leonard systems\footnote{Lewis systems provide a linear algebraic framework characterizing the terminating branch of the Askey scheme [10] of (basic) hypergeometric orthogonal polynomials.} and will be discussed in detail in a future paper. Here we mention that $f$ can be found for the following graphs:

| $\Gamma$                                      | $(HQ)^I_0$ |
|-----------------------------------------------|------------|
| $J(v, d)$ ($v > (t + 1)(d - t + 1)$)           | $\binom{v-1}{d-t}$ |
| $H(d, q)$ ($t = d - 1$; or $q \geq d$; or $q = d - 1$, $t < d - 2$) | $q^{d-t}$ |
| $J_q(v, d)$ ($v \geq 2d$)                     | $\binom{v-1}{d-1-1}_q$ |
| $\text{Bil}_q(d, e)$ ($d \leq e$)             | $q^{(d-e)e}$ |

For $\Gamma = J(v, d)$ or $J_q(v, d)$ (with $v, d$ as in the table), Wilson and Frankl [29] constructed a matrix $B \in A$ such that (i) $B_{xy} = 0$ if $\partial(x, y) \leq d - t$; (ii) $B + I - \binom{v-1}{d-t}J$ is positive semidefinite and its $i$th eigenvalue $\lambda_i$ is positive precisely when $t + 1 \leq i \leq d$, where we interpret $\binom{m}{n}$ as $(\binom{m}{n})_{\Gamma}$ for $J(v, d)$ and $(\binom{m}{n})_{J_q}$ for $J_q(v, d)$. We define $f$ by $f_0 = 1$, $f_1 = \cdots = f_t = 0$, and $f_{t+1} = \binom{v-1}{d-t}_q^{-1}\lambda_i$ for $t + 1 \leq i \leq d$. For $\Gamma = \text{Bil}_q(d, e)$ ($d \leq e$), Delsarte [7] constructed a Singleton system, i.e., a subset whose inner distribution $e' = (e_0', e_1', \ldots, e_q')$ satisfies $e_1' = \cdots = e_t' = 0$ and $(e'Q)_{t+1} = \cdots = (e'Q)_{d-1} = 0$. It follows that $e_{t+1}', \ldots, e_q'$ are positive; see [23, §4]. We define $f = e' \cdot \text{diag}(k_0, k_1, \ldots, k_d)^{-1}$ where $k_i$ is the valency of $\Gamma_i$ ($0 \leq i \leq d$). For $\Gamma = H(d, q)$, a subset having the above properties is known as an MDS code [13, Chapter 11]. MDS codes may not exist for some families; see [4, 23, 24]. Moon [19] showed that the upper bound $q^{d-t}$ for $H(d, q)$ and the characterization of its descendents as optimal intersecting families are valid under the (in general) weaker assumption $q \geq t + 2$. Dual polar graphs discussed in [23] do not always possess $f$ even for the case $t = 1$ [22]; see [21], however, for a description of optimal 1-intersecting families.

3. The Erdős–Ko–Rado theorem for twisted Grassmann graphs

Let $\Gamma$ be a prime power and fix a hyperplane $H$ of $F_q^{2d+1}$. Let $X_1$ be the set of $(d + 1)$-dimensional subspaces of $F_q^{2d+1}$ not contained in $H$, and $X_2$ the set of $(d - 1)$-dimensional subspaces of $H$. The twisted Grassmann graph $\Gamma = \text{J}_q(2d + 1, d)$ [5] has vertex set $X = X_1 \cup X_2$, and two vertices $x, y \in X$ are adjacent if $\dim x + \dim y - 2 \dim x \cap y = 2$. It has the same parameters (i.e., the structure constants of $A$) as $\text{J}_q(2d + 1, d)$. The twisted Grassmann graphs provide the first known family of non-vertex-transitive distance-regular graphs with unbounded diameter. See [12, 21, 29] for more information.

The Erdős–Ko–Rado theorem for $\text{J}_q(2d + 1, d)$ can now be rapidly obtained. Note that $J_q(2d + 1, d)$ and $\text{J}_q(2d + 1, d)$ share the same $Q$. Hence we may use the vector $f$ for $\text{J}_q(2d + 1, d)$ constructed in [2] and Lemma I applies. The descendents
of $\tilde{J}_q(2d + 1, d)$ have recently been classified by the author [24, Theorem 8.20]. To summarize:

**Theorem 2.** Let $Y$ be a nonempty subset of $\tilde{J}_q(2d + 1, d)$ with width $w \leq d - t$, where $0 < t < d$. Then $|Y| \leq \left\lceil \frac{2d + 1 - t}{d - t} \right\rceil$, and equality holds if and only if $Y = \{ x \in X_2 : u \subseteq x \}$ for some subspace $u$ of $H$ with $\dim u = t - 1$.

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