NONLINEAR RECONSTRUCTION FOR OPERATOR LEARNING OF PDEs WITH DISCONTINUITIES

A PREPRINT

Samuel Lanthaler  
Computing and Mathematical Science  
California Institute of Technology  
Pasadena, CA, USA  
slanth@caltech.edu

Roberto Molinaro  
Seminar for Applied Mathematics  
ETH Zurich  
Zurich, Switzerland  
roberto.molinaro@math.ethz.ch

Patrik Hadorn  
Seminar for Applied Mathematics  
ETH Zurich  
Zurich, Switzerland

Siddhartha Mishra  
Seminar for Applied Mathematics  
ETH Zurich  
Zurich, Switzerland  
siddhartha.mishra@math.ethz.ch

October 4, 2022

ABSTRACT

A large class of hyperbolic and advection-dominated PDEs can have solutions with discontinuities. This paper investigates, both theoretically and empirically, the operator learning of PDEs with discontinuous solutions. We rigorously prove, in terms of lower approximation bounds, that methods which entail a linear reconstruction step (e.g. DeepONet or PCA-Net) fail to efficiently approximate the solution operator of such PDEs. In contrast, we show that certain methods employing a non-linear reconstruction mechanism can overcome these fundamental lower bounds and approximate the underlying operator efficiently. The latter class includes Fourier Neural Operators and a novel extension of DeepONet termed shift-DeepONet. Our theoretical findings are confirmed by empirical results for advection equation, inviscid Burgers’ equation and compressible Euler equations of aerodynamics.

1 Introduction

Many interesting phenomena in physics and engineering are described by partial differential equations (PDEs) with discontinuous solutions. The most common types of such PDEs are nonlinear hyperbolic systems of conservation laws (Dafermos 2005), such as the Euler equations of aerodynamics, the shallow-water equations of oceanography and MHD equations of plasma physics. It is well-known that solutions of these PDEs develop finite-time discontinuities such as shock waves, even when the initial and boundary data are smooth. Other examples include the propagation of waves with jumps in linear transport and wave equations, crack and fracture propagation in materials (Sun & Jin 2012), moving interfaces in multiphase flows (Drew & Passman 1998) and motion of very sharp gradients as propagating fronts and traveling wave solutions for reaction-diffusion equations (Smoller 2012). Approximating such (propagating) discontinuities in PDEs is considered to be extremely challenging for traditional numerical methods (Hesthaven 2018) as resolving them could require very small grid sizes. Although bespoke numerical methods such as high-resolution finite-volume methods, discontinuous Galerkin finite-element and spectral viscosity methods (Hesthaven 2018) have successfully been used in this context, their very high computational cost prohibits their extensive use, particularly for many-query problems such as UQ, optimal control and (Bayesian) inverse problems (Lye et al. 2020), necessitating the design of fast machine learning-based surrogates.

As the task at hand in this context is to learn the underlying solution operator that maps input functions (initial and boundary data) to output functions (solution at a given time), recently developed operator learning methods can be
employed in this infinite-dimensional setting (Higgins, 2021). These methods include operator networks (Chen & Chen, 1995) and their deep version, DeepONet (Lu et al., 2019, 2021), where two sets of neural networks (branch and trunk nets) are combined in a linear reconstruction procedure to obtain an infinite-dimensional output. DeepONets have been very successfully used for different PDEs (Lu et al., 2021; Mao et al., 2020b; Cai et al., 2021; Lin et al., 2021). An alternative framework is provided by neural operators (Kovachki et al., 2021a), wherein the affine functions within DNN hidden layers are generalized to infinite-dimensions by replacing them with kernel integral operators as in (Li et al., 2020a; Kovachki et al., 2021a; Li et al., 2020b). A computationally efficient form of neural operators is the Fourier Neural Operator (FNO) (Li et al., 2021a), where a translation invariant kernel is evaluated in Fourier space, leading to many successful applications for PDEs (Li et al., 2021a; Pathak et al., 2022).

Currently available theoretical results for operator learning (e.g. Lanthaler et al. (2022); Kovachki et al. (2021a,b); De Ryck & Mishra (2022b); Deng et al. (2022)) leverage the regularity (or smoothness) of solutions of the PDE to prove that frameworks such as DeepONet, FNO and their variants approximate the underlying operator efficiently. Although such regularity holds for many elliptic and parabolic PDEs, it is obviously destroyed when discontinuities appear in the solutions of the PDEs such as in the hyperbolic PDEs mentioned above. Thus, a priori, it is unclear if existing operator learning frameworks can efficiently approximate PDEs with discontinuous solutions. This explains the paucity of theoretical and (to a lesser extent) empirical work on operator learning of PDEs with discontinuous solutions. We rigorously prove that frameworks such as DeepONet, FNO and their variants approximate the underlying operator efficiently.

2 Methods

Setting. Given compact domains $D \subset \mathbb{R}^d$, $U \subset \mathbb{R}^d$, we consider the approximation of operators $\mathcal{G} : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{X} \subset L^2(D)$ and $\mathcal{Y} \subset L^2(U)$ are the input and output function spaces. In the following, we will focus on the case, where $\bar{u} \mapsto \mathcal{G}(\bar{u})$ maps initial data $\bar{u}$ to the solution at some time $t > 0$, of an underlying time-dependent PDE. We assume the input $\bar{u}$ to be sampled from a probability measure $\mu \in \text{Prob}(\mathcal{X})$.

DeepONet. DeepONet (Lu et al., 2021) will be our prototype for operator learning frameworks with linear reconstruction. To define them, let $x := (x_1, \ldots, x_m) \in D$ be a fixed set of sensor points. Given an input function $\bar{u} \in \mathcal{X}$, we encode it by the point values $\mathcal{E}(\bar{u}) = (\bar{u}(x_1), \ldots, \bar{u}(x_m)) \in \mathbb{R}^m$. DeepONet is formulated in terms of two neural networks: The first is the branch-net $\beta$, which maps the point values $\mathcal{E}(\bar{u})$ to coefficients $\beta(\mathcal{E}(\bar{u})) = (\beta_1(\mathcal{E}(\bar{u})), \ldots, \beta_p(\mathcal{E}(\bar{u})))$, resulting in a mapping

$$\beta : \mathbb{R}^m \to \mathbb{R}^p, \quad \mathcal{E}(\bar{u}) \mapsto (\beta_1(\mathcal{E}(\bar{u})), \ldots, \beta_p(\mathcal{E}(\bar{u}))).$$

The second neural network is the so-called trunk-net $\tau(y) = (\tau_1(y), \ldots, \tau_p(y))$, which is used to define a mapping

$$\tau : U \to \mathbb{R}^p, \quad y \mapsto (\tau_1(y), \ldots, \tau_p(y)).$$

While the branch net provides the coefficients, the trunk net provides the “basis” functions in an expansion of the output function of the form

$$\mathcal{N}^{\text{DON}}(\bar{u})(y) = \sum_{k=1}^p \beta_k(\bar{u}) \tau_k(y), \quad \bar{u} \in \mathcal{X}, \ y \in U,$$

with $\beta_k(\bar{u}) = \beta_k(\mathcal{E}(\bar{u}))$. The resulting mapping $\mathcal{N}^{\text{DON}} : \mathcal{X} \to \mathcal{Y}, \ \bar{u} \mapsto \mathcal{N}^{\text{DON}}(\bar{u})$ is a DeepONet.

Although DeepONet were shown to be universal in the class of measurable operators (Lanthaler et al., 2022), the following fundamental lower bound on the approximation error was also established,
Proposition 2.1 (Lanthaler et al. [2022], Thm. 3.4). Let $\mathcal{X}$ be a separable Banach space, $\mathcal{Y}$ a separable Hilbert space, and let $\mu$ be a probability measure on $\mathcal{X}$. Let $\mathcal{G} : \mathcal{X} \to \mathcal{Y}$ be a Borel measurable operator with $\mathbb{E}_{\bar{u} \sim \mu} \| \mathcal{G}(\bar{u}) \|_2^2 < \infty$. Then the following lower approximation bound holds for any DeepONet $\mathcal{N}^{\text{DON}}$ with trunk-/branch-net dimension $p$:

$$e(\mathcal{N}^{\text{DON}}) = \mathbb{E}_{\bar{u} \sim \mu} \left[ \| \mathcal{N}^{\text{DON}}(\bar{u}) - \mathcal{G}(\bar{u}) \|_2^2 \right]^{1/2} \geq e_{\text{opt}} := \sqrt{\sum_{j=p}^{\infty} \lambda_j}, \quad (2.4)$$

where the optimal error $e_{\text{opt}}$ is written in terms of the eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots$ of the covariance operator $\Gamma_{\#\mu} := \mathbb{E}_{\bar{u} \sim \mu} [(u \otimes u)]$ of the push-forward measure $\mathcal{G}_{\#\mu}$.

We refer to SM A for relevant background on the underlying principal component analysis (PCA) and covariance operators. The same lower bound (2.4) in fact holds for any operator approximation of the form $\mathcal{N}(\bar{u}) = \sum_{k=1}^{p} \beta_k(\bar{u}) \tau_k$, where $\beta_k : \mathcal{X} \to \mathbb{R}$ are arbitrary functionals. In particular, this bound continuous to hold for e.g. the PCA-Net architecture of [Hesthaven & Ubbiali 2018]. We will refer to any operator learning architecture of this form as a method with "linear reconstruction", since the output function $\mathcal{N}(\bar{u})$ is restricted to the linear $p$-dimensional space spanned by the $\tau_1, \ldots, \tau_p \in \mathcal{Y}$. In particular, DeepONet are based on linear reconstruction.

**shift-DeepONet.** The lower bound (2.4) shows that there are fundamental barriers to the expressive power of operator learning methods based on linear reconstruction. This is of particular relevance for problems in which the optimal lower bound $e_{\text{opt}}$ in (2.4) exhibits a slow decay in terms of the number of basis functions $p$, due to the slow decay of the eigenvalues $\lambda_j$ of the covariance operator. It is well-known that even linear advection- or transport-dominated problems can suffer from such a slow decay of the eigenvalues [Ohlberger & Rave 2013; Dahmen et al. 2014; Taddei et al. 2015; Petersen et al. 2010], which could hinder the application of linear-reconstruction based operator learning methods to this very important class of problems. In view of these observations, it is thus desirable to develop a non-linear variant of DeepONet which can overcome such a lower bound in the context of transport-dominated problems. We propose such an extension below.

A **shift-DeepONet** $\mathcal{N}^{\text{DON}} : \mathcal{X} \to \mathcal{Y}$ is an operator of the form

$$\mathcal{N}^{\text{DON}}(\bar{u})(y) := \sum_{k=1}^{p} \beta_k(\bar{u}) \tau_k \left( \mathcal{A}_k(\bar{u}) \cdot y + \gamma_k(\bar{u}) \right), \quad (2.5)$$

where the input function $\bar{u}$ is encoded by evaluation at the sensor points $\mathcal{E}(\bar{u}) \in \mathbb{R}^m$. We retain the DeepONet branch- and trunk-nets $\beta, \tau$ defined in (2.1), (2.2), respectively, and we have introduced a **scale-net** $\mathcal{A} = (\mathcal{A}_k)_{k=1}^p$, consisting of matrix-valued functions

$$\mathcal{A}_k : \mathbb{R}^m \to \mathbb{R}^{d_d \times d_d}, \quad \mathcal{E}(\bar{u}) \mapsto \mathcal{A}_k(\bar{u}) := \mathcal{A}_k(\mathcal{E}(\bar{u})), \quad (2.6)$$

and a **shift-net** $\gamma = (\gamma_k)_{k=1}^p$, with

$$\gamma_k : \mathbb{R}^m \to \mathbb{R}^{d_d}, \quad \mathcal{E}(\bar{u}) \mapsto \gamma_k(\bar{u}) := \gamma_k(\mathcal{E}(\bar{u})).$$

All components of a shift-DeepONet are represented by deep neural networks, potentially with different activation functions.

Since shift-DeepONets reduce to DeepONets for the particular choice $\mathcal{A} \equiv 1$ and $\gamma \equiv 0$, the universality of DeepONets (Theorem 3.1 of Lanthaler et al. [2022]) is clearly inherited by shift-DeepONets. However, as shift-DeepONets do not use a linear reconstruction (the trunk nets in (2.5) depend on the input through the scale and shift nets), the lower bound (2.4) does not directly apply, providing possible space for shift-DeepONet to efficiently approximate transport-dominated problems, especially in the presence of discontinuities.

**Fourier neural operators (FNO).** A FNO $\mathcal{N}^{\text{FNO}}$ (Li et al. 2021a) is a composition

$$\mathcal{N}^{\text{FNO}} : \mathcal{X} \to \mathcal{Y} : \mathcal{N}^{\text{FNO}} = Q \circ L_L \circ \cdots \circ L_1 \circ R, \quad (2.6)$$

consisting of a "lifting operator" $\bar{u}(x) \mapsto R(\bar{u}(x), x)$, where $R$ is represented by a (shallow) neural network $R : \mathbb{R}^{d_u} \times \mathbb{R}^{d} \to \mathbb{R}^{d_u}$ with $d_u$ the number of components of the input function, $d$ the dimension of the domain and $d_u$ the "lifting dimension" (a hyperparameter), followed by $L$ hidden layers $L_L : \mathbb{R}^{d_u} \to \mathbb{R}^{d_u}$ of the form

$$v^{l+1}(x) = \sigma \left( W_L \cdot v^l(x) + b_L(x) + \left( K_L v^l \right)(x) \right),$$

where the optimal error $e_{\text{opt}}$ is written in terms of the eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots$ of the covariance operator $\Gamma_{\#\mu} := \mathbb{E}_{\bar{u} \sim \mu} [(u \otimes u)]$ of the push-forward measure $\mathcal{G}_{\#\mu}$.
with $W_\ell \in \mathbb{R}^{d_x \times d_v}$ a weight matrix (residual connection), $x \mapsto b_\ell(x) \in \mathbb{R}^{d_v}$ a bias function and with a convolution operator $K_\ell v^\ell(x) = \int_{\mathbb{R}^d} \kappa_\ell(x-y) v^\ell(y) \, dy$, expressed in terms of a (learnable) integral kernel $x \mapsto \kappa_\ell(x) \in \mathbb{R}^{d_x \times d_v}$. The output function is finally obtained by a linear projection layer $u^{L+1}(x) \mapsto \mathcal{N}^\text{FNO}(\bar{u})(x) = Q \cdot u^{L+1}(x)$.

The convolution operators $K_\ell$ add the indispensable non-local dependence of the output on the input function. Given values on an equidistant Cartesian grid, the evaluation of $K_\ell v^\ell$ can be efficiently carried out in Fourier space based on the discrete Fourier transform (DFT), leading to a representation

$$K_\ell v^\ell = \mathcal{F}_N^{-1} \left( P_\ell(k) \cdot \mathcal{F}_N v^\ell(k) \right),$$

where $\mathcal{F}_N v^\ell(k)$ denotes the Fourier coefficients of the DFT of $v^\ell(x)$, computed based on the given $N$ grid values in each direction, $P_\ell(k) \in \mathbb{C}^{d_x \times d_v}$ is a complex Fourier multiplication matrix indexed by $k \in \mathbb{Z}^d$, and $\mathcal{F}_N^{-1}$ denotes the inverse DFT. In practice, only a finite number of Fourier modes can be computed, and hence we introduce a hyperparameter $k_{\text{max}} \in \mathbb{N}$, such that the Fourier coefficients of $b_\ell(x)$ as well as the Fourier multipliers, $\hat{b}_\ell(k) \equiv 0$ and $P_\ell(k) \equiv 0$, vanish whenever $|k|_\infty > k_{\text{max}}$. In particular, with fixed $k_{\text{max}}$ the DFT and its inverse can be efficiently computed in $O((k_{\text{max}} N)^d)$ operations (i.e. linear in the total number of grid points). The output space of FNO (2.6) is manifestly non-linear as it is not spanned by a fixed number of basis functions. Hence, FNO constitute a nonlinear reconstruction method.

### 3 Theoretical Results.

**Context.** Our aim in this section is to rigorously prove that the nonlinear reconstruction methods (shift-DeepONet, FNO) efficiently approximate operators stemming from discontinuous solutions of PDEs whereas linear reconstruction methods (DeepONet, PCA-Net) fail to do so. To this end, we follow standard practice in numerical analysis of PDEs [Hesthaven 2018] and choose two prototypical PDEs that are widely used to analyze numerical methods for transport-dominated PDEs. These are the linear transport or advection equation and the nonlinear inviscid Burgers’ equation, which is the prototypical example for hyperbolic conservation laws. The exact operators and the corresponding transport-dominated PDEs. These are the linear transport or advection equation and the nonlinear inviscid Burgers’ equation, which is the prototypical example for hyperbolic conservation laws. The exact operators and the corresponding approximation results with both linear and nonlinear reconstruction methods are described below. The computational complexity of the models is expressed in terms of hyperparameters such as the model size, which are described in detail in SM [3].

**Linear Advection Equation.** We consider the one-dimensional linear advection equation

$$\partial_x u + a \partial_x u = 0, \quad u(\cdot, t = 0) = \bar{u}$$

on a $2\pi$-periodic domain $D = \mathbb{T}$, with constant speed $a \in \mathbb{R}$. The underlying operator is $\mathcal{G}_{\text{adv}} : L^1(\mathbb{T}) \cap L^{\infty}(\mathbb{T}) \to L^1(\mathbb{T}) \cap L^{\infty}(\mathbb{T})$, $\bar{u} \mapsto \mathcal{G}_{\text{adv}}(\bar{u}) := u(\cdot, T)$, obtained by solving the PDE (3.1) with initial data $\bar{u}$ up to some final time $t = T$. We note that $\mathcal{X} = L^1(\mathbb{T}) \cap L^{\infty}(\mathbb{T}) \subset L^2(\mathbb{T})$. As input measure $\mu \in \text{Prob}(\mathcal{X})$, we consider random input functions $\bar{u} \sim \mu$ given by the square (box) wave of height $h$, width $w$ and centered at $\xi$,

$$\bar{u}(x) = h1_{[-w/2, +w/2]}(x - \xi),$$

where $h \in [0, h]$, $w \in [w, w]$, $\xi \in [0, 2\pi]$ are independent and uniformly identically distributed. The constants $0 < h \leq \bar{h}$, $0 < w \leq \bar{w}$ are fixed.

**DeepONet fails at approximating $\mathcal{G}_{\text{adv}}$ efficiently.** Our first rigorous result is the following lower bound on the error incurred by DeepONet (2.3) in approximating $\mathcal{G}_{\text{adv}}$.

**Theorem 3.1.** Let $p, m \in \mathbb{N}$. There exists a constant $C > 0$, independent of $m$ and $p$, such that for any DeepONet $\mathcal{N}^{\text{DON}}(\tilde{u})$, with $\sup_{\tilde{u} \sim \mu} \|\mathcal{N}^{\text{DON}}(\tilde{u})\|_{L^\infty} \leq M < \infty$, we have the lower bound

$$\mathcal{E} = \mathbb{E}_{\tilde{u} \sim \mu} \left[ \|\mathcal{G}_{\text{adv}}(\bar{u}) - \mathcal{N}^{\text{DON}}(\bar{u})\|_{L^1} \right] \geq \frac{C}{\min(m, p)}.$$

Consequently, to achieve $\mathcal{E}(\mathcal{N}^{\text{DON}}) \leq \epsilon$ with DeepONet, we need $p, m \gtrsim \epsilon^{-1}$ trunk and branch net basis functions and sensor points, respectively, entailing that size$(\mathcal{N}^{\text{DON}}) \gtrsim pm \gtrsim \epsilon^{-2}$ (cp. SM [3]).

The detailed proof is presented in SM [3]. It relies on two facts. First, following [Lanthaler et al. 2022], one observes that translation invariance of the problem implies that the Fourier basis is optimal for spanning the output space. As the underlying functions are discontinuous, the corresponding eigenvalues of the covariance operator for the push-forward measure decay, at most, quadratically in $p$. Consequently, the lower bound (2.4) leads to a linear decay of error in terms of the number of trunk net basis functions. Second, roughly speaking, the linear decay of error in terms of sensor points is a consequence of the fact that one needs sufficient number of sensor points to resolve the underlying discontinuous inputs.
Shift-DeepONet approximates $G_{adv}$ efficiently. Next and in contrast to the previous result on DeepONet, we have following efficient approximation result for shift-DeepONet (2.5).

**Theorem 3.2.** There exists a constant $C > 0$, such that for any $\epsilon > 0$, there exists a shift-DeepONet $\Lambda^{sDON}_\epsilon$ (2.5) such that

$$\mathcal{E} = \mathbb{E}_{\tilde{u} \sim \mu} \left[ \| G_{adv}(\tilde{u}) - \Lambda^{sDON}_\epsilon(\tilde{u}) \|_{L^1} \right] \leq \epsilon,$$

with uniformly bounded $p \leq C$, and with the number of sensor points $m \leq C\epsilon^{-1}$. Furthermore, we have

$\text{width}(\Lambda^{sDON}_\epsilon) \leq C$, \quad $\text{depth}(\Lambda^{sDON}_\epsilon) \leq C\log(\epsilon^{-1})^2$, \quad $\text{size}(\Lambda^{sDON}_\epsilon) \leq C\epsilon^{-1}$.

The detailed proof, presented in SM [C.3], is based on the fact that for each input, the exact solution can be completely determined in terms of three variables, i.e., the height $h$, width $w$ and shift $\xi$ of the box wave (3.2). Given an input $\tilde{u}$, we explicitly construct neural networks for inferring each of these variables with high accuracy. These neural networks are then combined together to yield a shift-DeepONet that approximates $G_{adv}$, with the desired complexity. The nonlinear dependence of the trunk net in shift-DeepONet (2.5) on the input is the key to encode the shift in the box-wave (3.2) and this demonstrates the necessity of nonlinear reconstruction in this context.

**FNO approximates $G_{adv}$ efficiently.** Finally, we state an efficient approximation result for $G_{adv}$ with FNO (2.6) below.

**Theorem 3.3.** For any $\epsilon > 0$, there exists an FNO $\Lambda^{FNO}_\epsilon$ (2.6), such that

$$\mathbb{E}_{\tilde{u} \sim \mu} \left[ \| G_{adv}(\tilde{u}) - \Lambda^{FNO}_\epsilon(\tilde{u}) \|_{L^1} \right] \leq \epsilon,$$

with grid size $N \leq C\epsilon^{-1}$, and with Fourier cut-off $k_{\text{max}}$, lifting dimension $d_v$, depth and size:

$$k_{\text{max}} = 1, \quad d_v \leq C, \quad \text{depth}(\Lambda^{FNO}_\epsilon) \leq C\log(\epsilon^{-1})^2, \quad \text{size}(\Lambda^{FNO}_\epsilon) \leq C\log(\epsilon^{-1})^2.$$

A priori, one recognizes that $G_{adv}$ can be represented by Fourier multipliers (see SM [C.4]). Consequently, a single linear FNO layer would in principle suffice in approximating $G_{adv}$. However, the size of this FNO would be exponentially larger than the bound in Theorem 3.3. To obtain a more efficient approximation, one needs to leverage the non-linear reconstruction within FNO layers. This is provided in the proof, presented in SM [C.4] where the underlying height, wave and shift of the box-wave inputs (3.2) are approximated with high accuracy by FNO layers. These are then combined together with a novel representation formula for the solution to yield the desired FNO.

**Comparison.** Observing the complexity bounds in Theorems 3.1, 3.2, 3.3, we note that the DeepONet size scales at least quadratically, size $\gtrsim \epsilon^{-2}$, in terms of the error in approximating $G_{adv}$ whereas for shift-DeepONet and FNO, this scaling is only linear and logarithmic, respectively. Thus, we rigorously prove that for this problem, the nonlinear reconstruction methods (FNO and shift-DeepONet) can be more efficient than DeepONet and other methods based on linear reconstruction. Moreover, FNO is shown to have a smaller approximation error than even shift-DeepONet for similar model size.

**Inviscid Burgers’ equation.** Next, we consider the inviscid Burgers’ equation in one-space dimension, which is considered the prototypical example of nonlinear hyperbolic conservation laws [Dafermos 2005]:

$$\partial_t u + \partial_x \left( \frac{1}{2} u^2 \right) = 0, \quad u(\cdot, t = 0) = \bar{u}, \quad (3.4)$$

on the $2\pi$-periodic domain $D = \mathbb{T}$. It is well-known that discontinuities in the form of shock waves can appear in finite-time even for smooth $\bar{u}$. Consequently, solutions of (3.4) are interpreted in the sense of distributions and entropy conditions are imposed to ensure uniqueness [Dafermos 2005]. Thus, the underlying solution operator is $G_{Burg} : L^1(\mathbb{T}) \cap L^\infty(\mathbb{T}) \rightarrow L^1(\mathbb{T}) \cap L^\infty(\mathbb{T}), \bar{u} \mapsto G_{Burg}(\bar{u}) := u(\cdot, T)$, with $u$ being the entropy solution of (3.4) at final time $T$. Given $\xi \sim \text{Unif}([0, 2\pi])$, we define the random field

$$\bar{u}(x) := -\sin(x - \xi), \quad (3.5)$$

and we define the input measure $\mu \in \text{Prob}(L^1(\mathbb{T}) \cap L^\infty(\mathbb{T}))$ as the law of $\bar{u}$. We emphasize that the difficulty in approximating the underlying operator $G_{Burg}$ arises even though the input functions are smooth, in fact analytic. This is in contrast to the linear advection equation.
DeepONet fails at approximating $G_{Burg}$ efficiently. First, we recall the following result, which follows directly from Lanthaler et al. (2022) (Theorem 4.19) and the lower bound (2.4).

**Theorem 3.4.** Assume that $G_{Burg} = u(\cdot, T)$, for $T > \pi$ and $u$ is the entropy solution of (3.4) with initial data $\bar{u} \sim \mu$. Then there exists a constant $C > 0$, such that the $L^2$-error for any DeepONet $\mathcal{N}^{DON}$ with $p$ trunk-/branch-net output functions is lower-bounded by

$$
\mathcal{E}(\mathcal{N}^{DON}) = \mathbb{E}_{\bar{u} \sim \mu} \left[ \| G_{Burg}(\bar{u}) - \mathcal{N}^{DON}(\bar{u}) \|_{L^1} \right] \geq \frac{C}{p}.
$$

(3.6)

Consequently, achieving an error $\mathcal{E}(\mathcal{N}^{DON}) \lesssim \epsilon$ requires at least $\text{size}(\mathcal{N}^{DON}) \geq p \gtrsim \epsilon^{-1}$.

**shift-DeepONet approximate $G_{Burg}$ efficiently.** In contrast to DeepONet, we have the following result for efficient approximation of $G_{Burg}$ with shift-DeepONet,

**Theorem 3.5.** Assume that $T > \pi$. There is a constant $C > 0$, such that for any $\epsilon > 0$, there exists a shift-DeepONet $\mathcal{N}^{sDON}$ such that

$$
\mathcal{E}(\mathcal{N}^{sDON}) = \mathbb{E}_{\bar{u} \sim \mu} \left[ \| G_{Burg}(\bar{u}) - \mathcal{N}^{sDON}(\bar{u}) \|_{L^1} \right] \leq \epsilon,
$$

(3.7)

with a uniformly bounded number $p \leq C$ of trunk/branch net functions, the number of sensor points can be chosen $m = 3$, and we have

$$
\text{width}(\mathcal{N}^{sDON}) \leq C, \quad \text{depth}(\mathcal{N}^{sDON}) \leq C \log(\epsilon^{-1})^2, \quad \text{size}(\mathcal{N}^{sDON}) \leq C \log(\epsilon^{-1})^2.
$$

The proof, presented in SM C.5, relies on an explicit representation formula for $G_{Burg}$, obtained using the method of characteristics (even after shock formation). Then, we leverage the analyticity of the underlying solutions away from the shock and use the nonlinear shift map in (2.3) to encode shock locations.

**FNO approximates $G_{Burg}$ efficiently.** Finally we prove (in SM C.6) the following theorem,

**Theorem 3.6.** Assume that $T > \pi$, then there exists a constant $C$ such that for any $\epsilon > 0$ and grid size $N \geq 3$, there exists an FNO $\mathcal{N}^{FNO}$ (2.6), such that

$$
\mathcal{E}(\mathcal{N}^{FNO}) = \mathbb{E}_{\bar{u} \sim \mu} \left[ \| G_{Burg}(\bar{u}) - \mathcal{N}^{FNO}(\bar{u}) \|_{L^1} \right] \leq \epsilon,
$$

and with Fourier cut-off $k_{\max}$, lifting dimension $d_v$ and depth satisfying,

$$
k_{\max} = 1, \quad d_v \leq C, \quad \text{depth}(\mathcal{N}^{FNO}) \leq C \log(\epsilon^{-1})^2, \quad \text{size}(\mathcal{N}^{FNO}) \leq C \log(\epsilon^{-1})^2.
$$

**Comparison.** A perusal of the bounds in Theorems 3.4, 3.5 and 3.6 reveals that after shock formation, the accuracy $\epsilon$ of the DeepONet approximation of $G_{Burg}$ scales at best as $\epsilon \sim n^{-1}$, in terms of the total number of degrees of freedom $n = \text{size}(\mathcal{N}^{DON})$ of the DeepONet. In contrast, shift-DeepONet and FNO based on a non-linear reconstruction can achieve an exponential convergence rate $\epsilon \lesssim \exp\left(-cn^{1/2}\right)$ in the total number of degrees of freedom $n = \text{size}(\mathcal{N}^{sDON})$, size($\mathcal{N}^{FNO}$), even after the formation of shocks. This again highlights the expressive power of nonlinear reconstruction methods in approximating operators of PDEs with discontinuities.

|                      | ResNet | FCNN  | DeepONet | Shift - DeepONet | FNO   |
|----------------------|--------|-------|----------|------------------|-------|
| Linear Advection     | 14.8%  | 11.6% | 7.95%    | 2.76%            | 0.71% |
| Equation             |        |       |          |                  |       |
| Burgers’ Equation    | 20.16% | 23.23%| 28.5%    | 7.83%            | 1.57% |
| Lax-Sod Shock Tube   | 4.47%  | 8.83% | 4.22%    | 2.76%            | 1.56% |
| 2D Riemann Problem   | 2.6%   | 0.19% | 0.89%    | 0.12%            | 0.12% |

Table 1: Relative median-$L^1$ error computed over 128 testing samples for different benchmarks with different models.
4 Experiments

In this section, we illustrate how different operator learning frameworks can approximate solution operators of PDEs with discontinuities. To this end, we will compare DeepONet \eqref{eq:deeponet} (a prototypical operator learning method with linear reconstruction) with shift-DeepONet \eqref{eq:shiftdeeponet} and FNO \eqref{eq:finitedif} (as nonlinear reconstruction methods). Moreover, two additional baselines (described in detail in SM \cite{D} are also used, namely the well-known ResNet architecture \cite{He2016} and a fully convolutional neural network (FCNN) of \cite{Long2015}. Below, we present results for the best performing hyperparameter configuration, obtained after a grid search, for each model while postponing the description of details for the training procedures, hyperparameter configurations and model parameters to SM \cite{D}.

**Linear Advection.** We start with the linear advection equation \eqref{eq:linearadv} in the domain \([0, 1]\) with wave speed \(a = 0.5\) and periodic boundary conditions. The initial data is given by \eqref{eq:initialdata} corresponding to square waves, with initial heights uniformly distributed between \(h = 0.2\) and \(h = 0.8\), widths between \(w = 0.05\) and \(w = 0.3\) and shifts between 0 and 0.5. We seek to approximate the solution operator \(G_{adv}\) at final time \(T = 0.25\). The training and test samples are generated by sampling the initial data and the underlying exact solution, given by translating the initial data by 0.125, sampled on a very high-resolution grid of 2048 points (to keep the discontinuities sharp), see SM Figure 7 for examples of the input and output of \(G_{adv}\). The relative median test error for all the models is presented in Table 1. We observe from this table that DeepONet performs relatively poorly with a high test error of approximately 8\%, although its outperforms the ResNet and FCNN baselines handily. As suggested by the theoretical results of the previous section, shift-DeepONet is significantly more accurate than DeepONet (and the baselines), with at least a two-fold gain in accuracy. Moreover, as predicted by the theory, FNO significantly outperforms even shift-DeepONet on this problem, with almost a five-fold gain in test accuracy and a thirteen-fold gain vis a vis DeepONet.

**Inviscid Burgers’ Equation.** Next, we consider the inviscid Burgers’ equation \eqref{eq:burgers} in the domain \(D = [0, 1]\) and with periodic boundary conditions. The initial data is sampled from a Gaussian Random field i.e., a Gaussian measure with almost a five-fold gain in test accuracy and a thirteen-fold gain vis a vis DeepONet.

**Inviscid Burgers’ Equation.** Next, we consider the inviscid Burgers’ equation \eqref{eq:burgers} in the domain \(D = [0, 1]\) and with periodic boundary conditions. The initial data is sampled from a Gaussian Random field i.e., a Gaussian measure with almost a five-fold gain in test accuracy and a thirteen-fold gain vis a vis DeepONet.

**Inviscid Burgers’ Equation.** Next, we consider the inviscid Burgers’ equation \eqref{eq:burgers} in the domain \(D = [0, 1]\) and with periodic boundary conditions. The initial data is sampled from a Gaussian Random field i.e., a Gaussian measure with almost a five-fold gain in test accuracy and a thirteen-fold gain vis a vis DeepONet.

**Inviscid Burgers’ Equation.** Next, we consider the inviscid Burgers’ equation \eqref{eq:burgers} in the domain \(D = [0, 1]\) and with periodic boundary conditions. The initial data is sampled from a Gaussian Random field i.e., a Gaussian measure with almost a five-fold gain in test accuracy and a thirteen-fold gain vis a vis DeepONet.

**Inviscid Burgers’ Equation.** Next, we consider the inviscid Burgers’ equation \eqref{eq:burgers} in the domain \(D = [0, 1]\) and with periodic boundary conditions. The initial data is sampled from a Gaussian Random field i.e., a Gaussian measure with almost a five-fold gain in test accuracy and a thirteen-fold gain vis a vis DeepONet.

**Compressible Euler Equations.** The motion of an inviscid gas is described by the Euler equations of aerodynamics. For definiteness, the Euler equations in two space dimensions are,

\[ U_t + F(U)_x + G(U)_y = 0, \quad U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}, \quad F(U) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ (E + p)u \end{pmatrix}, \quad G(U) = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ (E + p)v \end{pmatrix}, \quad (4.1) \]

with \(\rho, u, v\) and \(p\) denoting the fluid density, velocities along \(x\)-and \(y\)-axis and pressure. \(E\) represents the total energy per unit volume

\[ E = \frac{1}{2} \rho (u^2 + v^2) + \frac{p}{\gamma - 1} \]

where \(\gamma = c_p/c_v\) is the gas constant which equals 1.4 for a diatomic gas considered here.

**Shock Tube.** We start by restricting the Euler equations \eqref{eq:euler} to the one-dimensional domain \(D = [-5, 5]\) by setting \(v = 0\) in \eqref{eq:euler}. The initial data corresponds to a shock tube of the form,

\[ \rho_0(x) = \begin{cases} \rho_L & x \leq x_0 \\ \rho_R & x > x_0 \end{cases}, \quad u_0(x) = \begin{cases} u_L & x \leq x_0 \\ u_R & x > x_0 \end{cases}, \quad p_0(x) = \begin{cases} p_L & x \leq x_0 \\ p_R & x > x_0 \end{cases} \quad (4.2) \]
parameterized by the left and right states \((\rho_L, u_L, p_L), (\rho_R, u_R, p_R)\), and the location of the initial discontinuity \(x_0\). As proposed in [Lye et al. (2020)], these parameters are, in turn, drawn from the measure: 
\[
\rho_L = 0.75 + 0.45G(z_1), \rho_R = 0.4 + 0.3G(z_2), u_L = 0.5 + 0.5G(z_3), u_R = 0, p_L = 2.5 + 1.6G(z_4), p_R = 0.375 + 0.325G(z_5), x_0 = 0.5G(z_6),
\]
with \(z = [z_1, z_2, \ldots, z_6] \sim U([0,1])\) and \(G(z) := 2z - 1\). We seek to approximate the operator \(G : [\rho_0, \rho_0 u_0, E_0] \rightarrow E(1.5)\). The training (and test) output are generated with ALSVINN code [Lye (2020)], using a finite volume scheme, with a spatial mesh resolution of 2048 points and examples of input-output pairs, presented in SM Figure 17. We observe that the initial jump discontinuities in density, velocity and pressure evolve into a complex pattern of (continuous) rarefactions, contact discontinuities and shock waves. The (relative) median test errors, presented in Table 1, reveal that shift-DeepONet and FNO significantly outperform DeepONet (and the other two baselines). FNO is also better than shift-DeepONet and approximates this complicated solution operator with a median error of \(\approx 1.5\%\).

Four-Quadrant Riemann Problem. For the final numerical experiment, we consider the two-dimensional Euler equations (4.1) with initial data, corresponding to a well-known four-quadrant Riemann problem [Mishra & Tadmor (2011)] with \(U_0(x, y) = U_{sw}\), if \(x, y < 0\), \(U_0(x, y) = U_{se}\), if \(x < 0, y > 0\), \(U_0(x, y) = U_{nw}\), if \(x > 0, y < 0\) and \(U_0(x, y) = U_{ne}\), if \(x < 0, y > 0\), with states given by \(\rho_{0,nc} = \rho_{0,sw} = \rho_{0,ne} = \rho_{0,se} = 1.1\), \(\rho_{0,nw} = \rho_{0,se} = 0.5065\), \(\rho_{0,sw} = \rho_{0,se} = 0.35\) \([u_{0,nc}, u_{0,sw}, v_{0,nc}, v_{0,sw}] = 0.35[G(z_1), G(z_2), G(z_3), G(z_4)]\) and \([u_{0,se}, u_{0,sw}, v_{0,nc}, v_{0,sw}] = 0.8939 + 0.35(G(z_5), G(z_6), G(z_7), G(z_8))\), with \(z = [z_1, z_2, \ldots, z_8] \sim U([0,1])\) and \(G(z) = 2z - 1\). We seek to approximate the operator \(G : [\rho_0, \rho_0 u_0, \rho_0 v_0, E_0] \rightarrow E(1.5)\). The training (and test) output are generated with the ALSVINN code, on a spatial mesh resolution of 256\(^2\) points and examples of input-output pairs, presented in SM Figure 10. We observe from this table that the initial planar discontinuities in the state variable evolve into a very complex structure of the total energy at final time, with a mixture of curved and planar discontinuities, separated by smooth regions. The (relative) median test errors are presented in Table 1. We observe from this table that the errors with all models are significantly lower in this test case, possibly on account of the lower initial variance and coarser mesh resolution at which the reference solution is sampled. However, the same trend, vis a vis model performance, is observed i.e., DeepONet is significantly (more than seven-fold) worse than both shift-DeepONet and FNO. On the other hand, these two models approximate the underlying solution operator with a very low error of approximately 0.1%.

5 Discussion

Related Work. Although the learning of operators arising from PDEs has attracted great interest in recent literature, there are very few attempts to extend the proposed architectures to PDEs with discontinuous solutions. Empirical results for some examples of discontinuities or sharp gradients were presented in [Mao et al. (2020b)] (compressible Navier-Stokes equations with DeepOnets) and in [Kissas et al. (2022)] (Shallow-Water equations with an attention based framework). However, with the notable exception of [Lanthaler et al. (2022)], where the approximation of scalar conservation laws with DeepOnets is analyzed, theoretical results for the operator approximation of PDEs are not available. Hence, this paper can be considered to be the first where a rigorous analysis of approximating operators arising in PDEs with discontinuous solutions has been presented, particularly for FNOs. On the other hand, there is considerably more work on the neural network approximation of parametric nonlinear hyperbolic PDEs such as the theoretical results of [De Ryck & Mishra (2022a)] and empirical results of [Lye et al. (2020, 2021)]. Also related are results with physics informed neural networks or PINNs for nonlinear hyperbolic conservation laws such as [De Ryck et al. (2022), Jagtap et al. (2022), Mao et al. (2020a)]. However, in this setting, the input measure is assumed to be supported on a finite-dimensional subset of the underlying infinite-dimensional input function space, making them too restrictive for operator learning as described in this paper.

Conclusions. We consider the learning of operators that arise in PDEs with discontinuities such as linear and nonlinear hyperbolic PDEs. A priori, it could be difficult to approximate such operators with existing operator learning architectures as these PDEs are not sufficiently regular. Given this context, we have proved a rigorous lower bound to show that any operator learning architecture, based on linear reconstruction, may fail at approximating the underlying operator efficiently. In particular, this result holds for the popular DeepONet architecture. On the other hand, we rigorously prove that the incorporation of nonlinear reconstruction mechanisms can break this lower bound and pave the way for efficient learning of operators arising from PDEs with discontinuities. We prove this result for an existing widely used architecture i.e., FNO, and a novel variant of DeepONet that we term as shift-DeepONet. For instance, we show that while the approximation error for DeepOnets can decay, at best, linearly in terms of model size, the corresponding approximation errors for shift-DeepOnet and FNO decays exponentially in terms of model size, even in the presence or spontaneous formation of discontinuities.

These theoretical results are backed by experimental results where we show that FNO and shift-DeepONet consistently beat DeepONet and other ML baselines by a wide margin, for a variety of PDEs with discontinuities. Together, our
results provide strong evidence for asserting that nonlinear operator learning methods such as FNO and shift-DeepONet, can efficiently learn PDEs with discontinuities and provide further demonstration of the power of these operator learning models.

Moreover, we also find theoretically (compare Theorems 3.2 and 3.3) that FNO is more efficient than even shift-DeepONet. This fact is also empirically confirmed in our experiments. The non-local as well as nonlinear structure of FNO is instrumental in ensuring its excellent performance in this context, see Theorem 3.3 and SM D.2 for further demonstration of the role of nonlinear reconstruction for FNOs.

In conclusion, we would like to mention some avenues for future work. On the theoretical side, we only consider the approximation error and that too for prototypical hyperbolic PDEs such as linear advection and Burgers’ equation. Extending these error bounds to more complicated PDEs such as the compressible Euler equations, with suitable assumptions, would be of great interest. Similarly rigorous bounds on other sources of error, namely generalization and optimization errors [Lanthaler et al., 2022] need to be derived. At the empirical level, considering more realistic three-dimensional data sets is imperative. In particular, it would be very interesting to investigate if FNO (and shift-DeepONet) can handle complex 3-D problems with not just shocks, but also small-scale turbulence. Similarly extending these results to other PDEs with discontinuities such as those describing crack or front (traveling waves) propagation needs to be considered. Finally, using the FNO/shift-DeepONet surrogate for many query problems such as optimal design and Bayesian inversion will be a topic for further investigation.

References

Kaushik Bhattacharya, Bamdad Hosseini, Nikola B Kovachki, and Andrew M Stuart. Model Reduction And Neural Networks For Parametric PDEs. The SMAI journal of computational mathematics, 7:121–157, 2021.

Shengze Cai, Zhicheng Wang, Lu Lu, Tamer A Zaki, and George Em Karniadakis. Deepm&mnet: Inferring the electroconvection multiphysics fields based on operator approximation by neural networks. Journal of Computational Physics, 436:110296, 2021.

Tianping Chen and Hong Chen. Universal approximation to nonlinear operators by neural networks with arbitrary activation functions and its application to dynamical systems. IEEE Transactions on Neural Networks, 6(4):911–917, 1995.

C. M. Dafermos. Hyperbolic Conservation Laws in Continuum Physics (2nd Ed.). Springer Verlag, 2005.

Wolfgang Dahmen, Christian Plesken, and Gerrit Welper. Double greedy algorithms: Reduced basis methods for transport dominated problems. ESAIM: Mathematical Modelling and Numerical Analysis, 48(3):623–663, 2014.

T. De Ryck and S. Mishra. Error analysis for deep neural network approximations of parametric hyperbolic conservation laws. arXiv preprint arXiv:2207.07362, 2022a.

T. De Ryck, S. Mishra, and R. Molinaro. wpinns:weak physics informed neural networks for approximating entropy solutions of hyperbolic conservation laws. arXiv preprint arXiv:2207.08483, 2022.

Tim De Ryck and Siddhartha a Mishra. Generic bounds on the approximation error for physics-informed (and) operator learning. In Advances in Neural Information Processing Systems (NeurIPS), 2022b.

Beichuan Deng, Yeonjong Shin, Lu Lu, Zhongqiang Zhang, and George Em Karniadakis. Approximation rates of deepnets for learning operators arising from advection–diffusion equations. Neural Networks, 153:411–426, 2022. ISSN 0893-6080. doi: https://doi.org/10.1016/j.neunet.2022.06.019. URL https://www.sciencedirect.com/science/article/pii/S0893608022002349.

D. A. Drew and S. L. Passman. Theory of Multicomponent Fluids. Springer Verlag, New York, 1998.

Dennis Elbrächter, Dmytro Perekrestenko, Philipp Grohs, and Helmut Bölcskei. Deep neural network approximation theory. IEEE Transactions on Information Theory, 67(5):2581–2623, 2021.

Ian Goodfellow, Yoshua Bengio, and Aaron Courville. Deep learning. MIT press, 2016.

Kaiming He, Xiangyu Zhang, Shaoqing Ren, and Jian Sun. Deep residual learning for image recognition. In Proceedings of the IEEE conference on computer vision and pattern recognition, pp. 770–778, 2016.

J. S. Hesthaven. Numerical methods for conservation laws: From analysis to algorithms. SIAM, 2018.

Jan S Hesthaven and Stefano Ubbiali. Non-intrusive reduced order modeling of nonlinear problems using neural networks. Journal of Computational Physics, 363:55–78, 2018.

Irina Higgins. Generalizing universal function approximators. Nature Machine Intelligence, 3(3):192–193, 2021.
Ameya D. Jagtap, Zhiping Mao, Nikolaus Adams, and George Em Karniadakis. Physics-informed neural networks for inverse problems in supersonic flows. *Journal of Computational Physics*, 466:111402, October 2022. ISSN 0021-9991. doi: 10.1016/j.jcp.2022.111402. URL https://www.sciencedirect.com/science/article/pii/S0021999122004648.

Georgios Kissas, Jacob H Seidman, Leonardo Ferreira Guilhoto, Victor M Preciado, George J Pappas, and Paris Perdikaris. Learning operators with coupled attention. *Journal of Machine Learning Research*, 23(215):1–63, 2022.

N. Kovachki, Z. Li, B. Liu, K. Azizzadenesheli, K. Bhattacharya, A. Stuart, and A. Anandkumar. Neural operator: Learning maps between function spaces. *arXiv preprint arXiv:2108.08481v3*, 2021a.

Nikola Kovachki, Samuel Lanthaler, and Siddhartha Mishra. On universal approximation and error bounds for fourier neural operators. *Journal of Machine Learning Research*, 22:Art–No, 2021b.

Samuel Lanthaler, Siddhartha Mishra, and George E Karniadakis. Error estimates for DeepONets: A deep learning framework in infinite dimensions. *Transactions of Mathematics and Its Applications*, 6(1):tnc001, 2022. URL https://academic.oup.com/imatrm/article-pdf/6/1/tnc001/42785544/tnc001.pdf.

Zongyi Li, Nikola B Kovachki, Kamyar Azizzadenesheli, Burigede Liu, Kaushik Bhattacharya, Andrew M Stuart, and Anima Anandkumar. Neural operator: Graph kernel network for partial differential equations. *CoRR*, abs/2003.03485, 2020a.

Zongyi Li, Nikola B Kovachki, Kamyar Azizzadenesheli, Burigede Liu, Andrew M Stuart, Kaushik Bhattacharya, and Anima Anandkumar. Multipole graph neural network for parametric partial differential equations. In H. Larochelle, M. Ranzato, R. Hadsell, M. F. Balcan, and H. Lin (eds.), *Advances in Neural Information Processing Systems (NeurIPS)*, volume 33, pp. 6755–6766. Curran Associates, Inc., 2020b.

Zongyi Li, Nikola Borislavov Kovachki, Kamyar Azizzadenesheli, Burigede Liu, Kaushik Bhattacharya, Andrew Stuart, and Anima Anandkumar. Fourier neural operator for parametric partial differential equations. In *International Conference on Learning Representations*, 2021a. URL https://openreview.net/forum?id=c8F9NOVtmmQ.

Zongyi Li, Hongkai Zheng, Nikola Kovachki, David Jin, Haoxuan Chen, Burigede Liu, Kamyar Azizzadenesheli, and Anima Anandkumar. Physics-informed neural operator for learning partial differential equations. *arXiv preprint arXiv:2111.03794*, 2021b.

Chensen Lin, Zhen Li, Lu Lu, Shengze Cai, Martin Maxey, and George Em Karniadakis. Operator learning for predicting multiscale bubble growth dynamics. *The Journal of Chemical Physics*, 154(10):104118, 2021.

Jonathan Long, Evan Shelhamer, and Trevor Darrell. Fully convolutional networks for semantic segmentation. In *Proceedings of the IEEE conference on computer vision and pattern recognition*, pp. 3431–3440, 2015.

Lu Lu, Pengzhan Jin, and George Em Karniadakis. DeepONet: Learning nonlinear operators for identifying differential equations based on the universal approximation theorem of operators. *arXiv preprint arXiv:1910.03193*, 2019.

Lu Lu, Pengzhan Jin, Guoefi Pang, Zhongqiang Zhang, and George Em Karniadakis. Learning nonlinear operators via deeponet based on the universal approximation theorem of operators. *Nature Machine Intelligence*, 3(3):218–229, 2021.

K. O. Lye. *Computation of Statistical Solutions of hyperbolic systems of conservation laws*. ETH Dissertation N. 26728, 2020.

Kjetil O Lye, Siddhartha Mishra, and Deep Ray. Deep learning observables in computational fluid dynamics. *Journal of Computational Physics*, pp. 109339, 2020.

Kjetil O Lye, Siddhartha Mishra, Deep Ray, and Praveen Chandrashekar. Iterative surrogate model optimization (ISMO): An active learning algorithm for pde constrained optimization with deep neural networks. *Computer Methods in Applied Mechanics and Engineering*, 374:113575, 2021.

Z. Mao, A. D. Jagtap, and G. E. Karniadakis. Physics-informed neural networks for high-speed flows. *Computer Methods in Applied Mechanics and Engineering*, 360:112789, 2020a.

Z. Mao, L. Lu, O. Marxen, T. Zaki, and G. E. Karniadakis. DeepMandMnet for hypersonics: Predicting the coupled flow and finite-rate chemistry behind a normal shock using neural-network approximation of operators. Preprint, available from arXiv:2011.03349v1, 2020b.

Siddhartha Mishra and Eitan Tadmor. Constraint preserving schemes using potential based fluxes- ii: Genuinely multi-dimensional schemes for systems of conservation laws,. *SIAM Journal on Numerical Analysis*, 49:1023–1045, 2011.

Mario Ohlberger and Stephan Rave. Nonlinear reduced basis approximation of parameterized evolution equations via the method of freezing. *Comptes Rendus Mathematique*, 351(23):901–906, 2013. ISSN 1631-073X. doi: https:
J. Pathak, S. Subramanian, P. Harrington, S. Raja, A. Chattopadhyay, M. Mardani, T. Kurth, D. Hall, Z. Li, K. Azizzadenesheli, p. Hassanzadeh, K. Kashinath, and A. Anandkumar. Fourcastnet: A global data-driven high-resolution weather model using adaptive fourier neural operators. *arXiv preprint arXiv:2202.11214*, 2022.

Benjamin Peherstorfer. Model reduction for transport-dominated problems via online adaptive bases and adaptive sampling. *SIAM Journal on Scientific Computing*, 42(5):A2803–A2836, 2020.

J. Smoller. *Shock waves and reaction-diffusion equations*. Springer, 2012.

C.T. Sun and Z-H. Jin. *Fracture Mechanics*. Elsevier, 2012.

Tommaso Taddei, Simona Perotto, and ALFIO Quarteroni. Reduced basis techniques for nonlinear conservation laws. *ESAIM: Mathematical Modelling and Numerical Analysis*, 49(3):787–814, 2015.

Qingcan Wang et al. Exponential convergence of the deep neural network approximation for analytic functions. *Science China Mathematics*, 61(10):1733–1740, 2018.

Dmitry Yarotsky. Error bounds for approximations with deep ReLU networks. *Neural Networks*, 94:103–114, 2017. Publisher: Elsevier.
A Principal component analysis

Principal component analysis (PCA) provides a complete answer to the following problem (see e.g. [Bhattacharya et al., 2021; Lanthaler et al., 2022] and references therein for relevant results in the infinite-dimensional context):

Given a probability measure $\nu \in \text{Prob}(\mathcal{Y})$ on a Hilbert space $\mathcal{Y}$ and given $p \in \mathbb{N}$, we would like to characterize the optimal linear subspace $\hat{V}_p \subset \mathcal{Y}$ of dimension $p$, which minimizes the average projection error

$$E_{w \sim \nu} \left[ \| w - \Pi_{\hat{V}_p} w \|_{\mathcal{Y}}^2 \right] = \min_{\text{dim}(V_p) = p} E_{w \sim \nu} \left[ \| w - \Pi_{V_p} w \|_{\mathcal{Y}}^2 \right], \tag{A.1}$$

where $\Pi_{V_p}$ denotes the orthogonal projection onto $V_p$, and the minimum is taken over all $p$-dimensional linear subspaces $V_p \subset \mathcal{X}$.

Remark A.1. A characterization of the minimum in (A.1) is of relevance to the present work, since the outputs of DeepONet, and other operator learning frameworks based on linear reconstruction $\mathcal{N}(u) = \sum_{k=1}^p \beta_k(u) \tau_k$, are restricted to the linear subspace $V_p := \text{span}\{\tau_1, \ldots, \tau_p\}$. From this, it follows that (Lanthaler et al., 2022):

$$E_{w \sim \mu} \left[ \| \mathcal{G}(u) - \mathcal{N}(u) \|_{\mathcal{Y}}^2 \right] \geq E_{\nu \sim \mu} \left[ \| \mathcal{G}(u) - \Pi_{V_p} \mathcal{G}(u) \|_{\mathcal{Y}}^2 \right] = E_{w \sim \mu} \left[ \| w - \Pi_{V_p} w \|_{\mathcal{Y}}^2 \right],$$

is lower bounded by the minimizer in (A.1) with $\nu = \mathcal{G}_{\#} \mu$ the push-forward measure of $\mu$ under $\mathcal{G}$.

To characterize minimizers of (A.1), one introduces the covariance operator $\Gamma_\nu : \mathcal{Y} \to \mathcal{Y}$, by $\Gamma_\nu := E_{w \sim \nu} [w \otimes w]$, where $\otimes$ denotes the tensor product. By definition, $\Gamma_\nu$ satisfies the following relation

$$(v', \Gamma_\nu v)_{\mathcal{Y}} = E_{w \sim \nu} [(v', w)(v, v)_{\mathcal{Y}}], \quad \forall v, v' \in \mathcal{Y}.$$ 

It is well-known that $\Gamma_\nu$ possesses a complete set of orthonormal eigenfunctions $\phi_1, \phi_2, \ldots$, with corresponding eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$. We then have the following result (see e.g. Lanthaler et al., 2022 Thm. 3.8):

Theorem A.2. A subspace $\hat{V}_p \subset \mathcal{Y}$ is a minimizer of (A.1) if, and only if, $\hat{V}_p = \text{span}\{\phi_1, \ldots, \phi_p\}$ can be written as the span of the first $p$ eigenfunctions of an orthonormal eigenbasis $\phi_1, \phi_2, \ldots$ of the covariance operator $\Gamma_\nu$, with decreasing eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots$. Furthermore, the minimum in (A.1) is given by

$$\delta^2_{\text{opt}} = \min_{\text{dim}(V_p) = p} E_{w \sim \mu} \left[ \| w - \Pi_{V_p} w \|_{\mathcal{Y}}^2 \right] = \sum_{j > p} \lambda_j,$$

in terms of the decay of the eigenvalues of $\Gamma_\nu$.

B Measures of complexity for (shift-)DeepONet and FNO

As pointed out in the main text, there are several hyperparameters which determine the complexity of DeepONet/shift-DeepONet and FNO, respectively. Table 1 summarizes quantities of major importance for (shift-)DeepONet and their (rough) analogues for FNO. These quantities are directly relevant to the expressive power and trainability of these operator learning architectures, and are described in further detail below.

| (shift-)DeepONet | FNO |
|------------------|-----|
| spatial resolution | $m$ | $\sim N^d$ |
| “intrinsic” function space dim. | $p$ | $\sim k_{\text{max}}^d \cdot d_v$ |
| # trainable parameters | $\text{size}(\mathcal{N})$ | $\text{size}(\mathcal{N})$ |
| depth | $\text{depth}(\mathcal{N})$ | $\text{depth}(\mathcal{N})$ |
| width | $\text{width}(\mathcal{N})$ | $\sim k_{\text{max}}^d \cdot d_v$ |

Figure 1: Approximate correspondence between measures of the complexity ($d=$dimension of the domain $D \subset \mathbb{R}^d$ of input/output functions).
We also introduce the analogous notions for shift-DeepONet:

\[
\text{size}(\mathcal{N}^{sDON}) := (m + p)\text{width}(\mathcal{N}^{sDON}) + \text{width}(\mathcal{N}^{sDON})^2\text{depth}(\mathcal{N}^{sDON}),
\]

where the second term measures the complexity of the hidden layers, and the first term takes into account the input and output layers. Furthermore, all architectures we consider have a width which scales at least as \(\text{width}(\mathcal{N}^{sDON}) \gtrsim \min(p, m)\), implying the following natural lower size bound,

\[
\text{size}(\mathcal{N}^{sDON}) \gtrsim (m + p) \min(p, m) + \min(p, m)^2\text{depth}(\mathcal{N}^{sDON}). \tag{B.1}
\]

We also introduce the analogous notions for shift-DeepONet:

\[
\begin{align*}
\text{width}(\mathcal{N}^{sDON}) &:= \text{width}(\beta) + \text{width}(\tau) + \text{width}(\mathcal{A}) + \text{width}(\gamma), \\
\text{depth}(\mathcal{N}^{sDON}) &:= \max\{\text{depth}(\beta), \text{depth}(\tau), \text{depth}(\mathcal{A}), \text{depth}(\gamma)\}, \\
\text{size}(\mathcal{N}^{sDON}) &:= (m + p)\text{width}(\mathcal{N}^{sDON}) + \text{width}(\mathcal{N}^{sDON})^2\text{depth}(\mathcal{N}^{sDON}).
\end{align*}
\]

FNO: Quantities of interest for FNO include the number of grid points in each direction \(N\) (for a total of \(O(N^d)\) grid points), the Fourier cut-off \(k_{max}\) (we retain a total of \(O(k_{max}^d)\) Fourier coefficients in the convolution operator and bias), and the lifting dimension \(d_v\). We recall that the lifting dimension determines the number of components of the input/output functions of the hidden layers, and hence the “intrinsic” dimensionality of the corresponding function space in the hidden layers is proportional to \(d_v\). The essential informational content for each of these \(d_v\) components is encoded in their Fourier modes with wave numbers \(|k| \leq k_{max}\) (a total of \(O(k_{max}^d)\) Fourier modes per component), and hence the total intrinsic function space dimension of the hidden layers is arguably of order \(\sim k_{max}^d d_v\). The width of an FNO layer is defined in analogy with conventional neural networks as the maximal width of the weight matrices and Fourier multiplier matrices, which is of order \(\sim k_{max}^d d_v\). The depth is defined as the number of hidden layers \(L\). Finally, the size is by definition the total number of tunable parameters in the architecture. By definition, the Fourier modes of the bias function \(b_v(x)\) are restricted to wavenumbers \(|k| \leq k_{max}\) (giving a total number of \(O(k_{max}^d d_v)\) parameters), and the Fourier multiplier matrix is restricted to wavenumber \(|k| \leq k_{max}\) (giving \(O(k_{max}^d d_v^2)\) parameters). Apriori, it is easily seen that if the lifting dimension \(d_v\) is larger than the number of components of the input/output functions, then \(\text{size}(\mathcal{N}^{FNO}) \lesssim (d_v^2 + d_v k_{max}^d + d_v N^d) \text{depth}(\mathcal{N}^{FNO}),\)

where the first term in parentheses corresponds to \(\text{size}(W_v) = d_v^2\), the second term accounts for \(\text{size}(P_v) = O(d_v k_{max}^d)\) and the third term counts the degrees of freedom of the bias, \(\text{size}(b_v(x_j)) = O(d_v N^d)\). The additional factor \(\text{depth}(\mathcal{N}^{FNO})\) takes into account that there are \(L = \text{depth}(\mathcal{N}^{FNO})\) such layers.

If the bias \(b_v\) is constrained to have Fourier coefficients \(\hat{b}_v(k) = 0\) for \(|k| > k_{max}\) (as we assumed in the main text), then the representation of \(b_v\) only requires \(\text{size}(\hat{b}_v(k)) = O(d_v k_{max}^d)\) degrees of freedom. This is of relevance in the regime \(k_{max} \ll N\), reducing the total FNO size from \(O(d_v^2 N^d L)\) to

\[
\text{size}(\mathcal{N}^{FNO}) \lesssim d_v^2 k_{max}^d \text{depth}(\mathcal{N}^{FNO}). \tag{B.2}
\]

Practically, this amounts to adding the bias in the hidden layers in Fourier \(k\)-space, rather than physical \(x\)-space.

C Mathematical Details

In this section, we provide detailed proofs of the Theorems in Section 3. We start with some preliminary results below,
C.1 ReLU DNN building blocks

In the present section we collect several basic constructions for ReLU neural networks, which will be used as building blocks in the following analysis. For the first result, we note that for any $\delta > 0$, the following approximate step-function

\[ \zeta_\delta(x) := \begin{cases} 0, & x < 0, \\ \frac{x}{\delta}, & 0 \leq x \leq \delta, \\ 1, & x > \delta, \end{cases} \]

can be represented by a neural network:

\[ \zeta_\delta(x) = \sigma\left(\frac{x}{\delta}\right) - \sigma\left(\frac{x - \delta}{\delta}\right), \]

where $\sigma(x) = \max(x, 0)$ denotes the ReLU activation function. Introducing an additional shift $\xi$, multiplying the output by $h$, and choosing $\delta > 0$ sufficiently small, we obtain the following result:

**Proposition C.1** (Step function). Fix an interval $[a, b] \subset \mathbb{R}$, $\xi \in [a, b]$, $h \in \mathbb{R}$. Let $h 1_{[x > \xi]}$ be a step function of height $h$. For any $\epsilon > 0$ and $p \in [1, \infty)$, there exist a ReLU neural network $\Phi_\epsilon : \mathbb{R} \rightarrow \mathbb{R}$, such that

\[ \text{depth}(\Phi_\epsilon) = 1, \quad \text{width}(\Phi_\epsilon) = 2, \]

and

\[ \| \Phi_\epsilon - h 1_{[x > \xi]} \|_{L^p([a, b])} \leq \epsilon. \]

The following proposition is an immediate consequence of the previous one, by considering the linear combination $\Phi_\delta(x - a) - \Phi_\delta(x - b)$ with a suitable choice of $\delta > 0$.

**Proposition C.2** (Indicator function). Fix an interval $[a, b] \subset \mathbb{R}$. Let $1_{[a, b]}(x)$ be the indicator function of $[a, b]$. For any $\epsilon > 0$ and $p \in [1, \infty)$, there exist a ReLU neural network $\Phi_\epsilon : \mathbb{R} \rightarrow \mathbb{R}$, such that

\[ \text{depth}(\Phi_\epsilon) = 1, \quad \text{width}(\Phi_\epsilon) = 4, \]

and

\[ \| \Phi_\epsilon - 1_{[a, b]} \|_{L^p([a, b])} \leq \epsilon. \]

A useful mathematical technique to glue together local approximations of a given function rests on the use of a “partition of unity”. In the following proposition we recall that partitions of unity can be constructed with ReLU neural networks (this construction has previously been used by Yarotsky (2017); cp. Figure 2):

**Proposition C.3** (Partition of unity). Fix an interval $[a, b] \subset \mathbb{R}$. For $J \in \mathbb{N}$, let $\Delta x := (b - a)/J$, and let $x_j := a + j \Delta x$, $j = 0, \ldots, J$ be an equidistant grid on $[a, b]$. Then for any $\epsilon \in (0, \Delta x/2]$, there exists a ReLU neural network $\Lambda : \mathbb{R} \rightarrow \mathbb{R}^J, x \mapsto (\Lambda_1(x), \ldots, \Lambda_J(x))$, such that

\[ \text{width}(\Lambda) = 4J, \quad \text{depth}(\Lambda) = 1, \]

Figure 2: Illustration of partition of unity network for $J = 5$, $[a, b] = [0, 1]$. 


each \( \Lambda_j \) is piecewise linear, satisfies

\[
\Lambda_j(x) = \begin{cases} 
0, & (x \leq x_j - \epsilon), \\
1, & (x_j - \epsilon \leq x \leq x_j + \epsilon), \\
0, & (x \geq x_j + \epsilon), 
\end{cases}
\]

and interpolates linearly between the values 0 and 1 on the intervals \([x_j - \epsilon, x_j - \epsilon]\) and \([x_j - \epsilon, x_j + \epsilon]\). In particular, this implies that

- \( \text{supp}(\Lambda_j) \subset [x_j - \epsilon, x_j + \epsilon] \), for all \( j = 1, \ldots, J \),
- \( \Lambda_j(x) \geq 0 \) for all \( x \in \mathbb{R} \),
- The \( \{\Lambda_j\}_{j=1}^{\ldots,J} \) form a partition of unity, i.e.

\[
\sum_{j=1}^{J} \Lambda_j(x) = 1, \quad \forall x \in [a, b].
\]

We also recall the well-known fact that the multiplication operator \( (x, y) \mapsto xy \) can be efficiently approximated by ReLU neural networks (cp. Yarotsky [2017]):

**Proposition C.4** (Multiplication, Yarotsky [2017, Prop. 3]). There exists a constant \( C > 0 \), such that for any \( \epsilon \in (0, \frac{1}{2}] \), \( M \geq 2 \), there exists a neural network \( \times_{\epsilon,M} : [-M, M] \times [-M, M] \to \mathbb{R} \), such that

\[
\text{width}(\tilde{x}_{\epsilon,M}) \leq C, \quad \text{depth}(\tilde{x}_{\epsilon,M}) \leq C \log(M \epsilon^{-1}), \quad \text{size}(\tilde{x}_{\epsilon,M}) \leq C \log(M \epsilon^{-1}),
\]

and

\[
\sup_{x,y \in [-M, M]} |\tilde{x}_{\epsilon,M}(x, y) - xy| \leq \epsilon.
\]

We next state a general approximation result for the approximation of analytic functions by ReLU neural networks. To this end, we first recall

**Definition C.5** (Analytic function and extension). A function \( F : (\alpha, \beta) \to \mathbb{R} \) is analytic, if for any \( x_0 \in (\alpha, \beta) \) there exists a radius \( r > 0 \), and a sequence \( (a_k)_{k \in \mathbb{N}_0} \) such that \( \sum_{k=0}^{\infty} |a_k| r^k < \infty \), and

\[
F(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k, \quad \forall |x - x_0| < r.
\]

If \( f : [a, b] \to \mathbb{R} \) is a function, then we will say that \( f \) has an analytic extension, if there exists \( F : (\alpha, \beta) \to \mathbb{R} \), with \( [a, b] \subset (\alpha, \beta) \), with \( F(x) = f(x) \) for all \( x \in [a, b] \) and such that \( F \) is analytic.

We then have the following approximation bound, which extends the main result of Wang et al. [2018]. In contrast to Wang et al. [2018], the following theorem applies to analytic functions without a globally convergent series expansion.

**Theorem C.6**. Assume that \( f : [a, b] \to \mathbb{R} \) has an analytic extension. Then there exist constants \( C, \gamma > 0 \), depending only on \( f \), such that for any \( L \in \mathbb{N} \), there exists a ReLU neural network \( \Phi_L : \mathbb{R} \to \mathbb{R} \), with

\[
\sup_{x \in [a, b]} |f(x) - \Phi_L(x)| \leq C \exp(-\gamma L^{1/2}),
\]

and such that

\[
\text{depth}(\Phi_L) \leq CL, \quad \text{width}(\Phi_L) \leq C.
\]

**Proof.** Since \( f \) has an analytic extension, for any \( x \in [a, b] \), there exists a radius \( r_x > 0 \), and an analytic function \( F_x : [x - r_x, x + r_x] \to \mathbb{R} \), which extends \( f \) locally. By the main result of Wang et al. [2018] Thm. 6), there are constants \( C_x, \gamma_x > 0 \) depending only on \( x \in [a, b] \), such that for any \( L \in \mathbb{N} \), there exists a ReLU neural network \( \Phi_{x,L} : \mathbb{R} \to \mathbb{R} \), such that

\[
\sup_{|\xi - x| \leq r_x} |F_x(x) - \Phi_{x,L}(\xi)| \leq C_x \exp(-\gamma_x L^{1/2}),
\]
and \( \text{depth}(\Phi_x) \leq C x L, \text{width}(\Phi_x) \leq C x \). For \( J \in \mathbb{N} \), set \( \Delta x = (b - a)/J \) and consider the equidistant partition \( x_j := a + j \Delta x \) of \([a, b]\). Since the compact interval \([a, b]\) can be covered by finitely many of the intervals \((x - r_x, x + r_x)\), then by choosing \( \Delta x \) sufficiently small, we can find \( x^{(j)} \in [a, b] \), such that \([x_{j-1}, x_j] \subset (x^{(j)} - r_x^{(j)}, x^{(j)} + r_x^{(j)})\) for each \( j = 1, \ldots, J \).

By construction, this implies that for \( \bar{C} := \max_{j=1, \ldots, J} C x_{j,i} \), and \( \gamma := \min_{j=1, \ldots, J} \gamma_{x^{(j)}} \), we have that for any \( L \in \mathbb{N} \), there exist neural networks \( \Phi_{j,L} := \Phi_{x^{(j)}, L} : \mathbb{R} \to \mathbb{R} \), such that

\[
\sup_{x \in [x_{j-1}, x_j]} |f(x) - \Phi_{j,L}(x)| \leq \bar{C} \exp(-\gamma L^{1/2}),
\]

and such that \( \text{depth}(\Phi_{j,L}) \leq \bar{C} L, \text{width}(\Phi_{j,L}) \leq \bar{C} \).

Let now \( \Lambda : \mathbb{R} \to \mathbb{R}^J \) be the partition of unity network from Proposition C.3 and define

\[
\Phi_L(x) := \sum_{j=1}^J \tilde{x}_{M,\epsilon}(\Lambda_j(x), \Phi_{j,L}(x)),
\]

where \( \tilde{x}_{M,\epsilon} \) denotes the multiplication network from Proposition C.4, with \( M := 1 + \bar{C} \exp(-\gamma) + \sup_{x \in [a, b]} |f(x)| \), and \( \epsilon := J^{-1} \bar{C} \exp(-\gamma L^{1/2}) \). Then we have

\[
\text{depth}(\Phi_L) \leq \text{depth}(\tilde{x}_{M,\epsilon}) + \text{depth}(\Lambda) + \max_{j=1, \ldots, J} \text{depth}(\Phi_{j,L})
\]

\[
\leq C' \log(M \epsilon^{-1}) + 1 + \bar{C} L
\]

\[
\leq C(1 + L),
\]

where the constant \( C > 0 \) on the last line depends on \( \sup_{x \in [a, b]} |f(x)| \), \( \gamma \) and on \( \bar{C} \), but is independent of \( L \). Similarly, we find that

\[
\text{width}(\Phi_L) \leq \text{width}(\Lambda) + \max_{j=1, \ldots, J} \text{width}(\Phi_{j,L}) \leq 4J + \bar{C},
\]

is bounded independently of \( L \). After potentially enlarging the constant \( C > 0 \), we can thus ensure that

\[
\text{depth}(\Phi_L) \leq CL, \quad \text{width}(\Phi_L) \leq C,
\]

with a constant \( C > 0 \) that depends only on \( f \), but is independent of \( L \). Finally, we note that

\[
|\Phi_L(x) - f(x)| \leq \sum_{j=1}^J |\tilde{x}_{M,\epsilon}(\Lambda_j(x), \Phi_{j,L}(x)) - \Lambda_j(x) f(x)|
\]

\[
\leq \sum_{j=1}^J \left| \tilde{x}_{M,\epsilon}(\Lambda_j(x), \Phi_{j,L}(x)) - \Lambda_j(x) \Phi_{j,L}(x) \right|
\]

\[
+ \sum_{j=1}^J \Lambda_j(x) |\Phi_{j,L}(x) - f(x)|.
\]

By construction of \( \tilde{x}_{M,\epsilon} \), and since \( |\Phi_{j,L}(x)| \leq M, \Lambda_j \leq M \), the first sum can be bounded by \( J \epsilon = \bar{C} \exp(-\gamma L^{1/2}) \). Furthermore, each term in the second sum is bounded by \( \Lambda_j(x) C \exp(-\gamma L^{1/2}) \), and hence

\[
\sum_{j=1}^J \Lambda_j(x) |\Phi_{j,L}(x) - f(x)| \leq \left( \sum_{j=1}^J \Lambda_j(x) \right) \bar{C} \exp(-\gamma L^{1/2}) = \bar{C} \exp(-\gamma L^{1/2}),
\]

for all \( x \in [a, b] \). We conclude that \( \sup_{x \in [a, b]} |\Phi_L(x) - f(x)| \leq 2 \bar{C} \exp(-\gamma L^{1/2}) \), with constants \( \bar{C}, \gamma > 0 \) independent of \( L \). Setting \( \gamma := \bar{\gamma} \) and after potentially enlarging the constant \( C > 0 \) further, we thus conclude: there exist \( C, \gamma > 0 \), such that for any \( L \in \mathbb{N} \), there exists a neural network \( \Phi_L : \mathbb{R} \to \mathbb{R} \) with \( \text{depth}(\Phi_L) \leq CL, \text{width}(\Phi_L) \leq C \), such that

\[
\sup_{x \in [a, b]} |\Phi_L(x) - f(x)| \leq C \exp(-\gamma L^{1/2}).
\]

This conclude the proof of Theorem C.6.
By combining a suitable ReLU neural network approximation of division \( a \mapsto 1/a \) based on Theorem C.6 (division is an analytic function away from 0), and the approximation of multiplication by \cite{yarotsky} (cp. Proposition C.4 above), we can also state the following result:

**Proposition C.7** (Division). Let \( 0 < a \leq b \) be given. Then there exists \( C = C(a, b) > 0 \), such that for any \( \epsilon \in (0, \frac{1}{2}] \), there exists a ReLU network \( \tilde{\tau}_{a,b,\epsilon} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), with

\[
\text{depth}(\tilde{\tau}_{a,b,\epsilon}) \leq C \log \left( \epsilon^{-1} \right)^2, \quad \text{width}(\tilde{\tau}_{a,b,\epsilon}) \leq C, \quad \text{size}(\tilde{\tau}_{a,b,\epsilon}) \leq C \log \left( \epsilon^{-1} \right)^2,
\]

satisfying

\[
\sup_{x,y \in [a,b]} \left| \frac{x}{y} \right| \leq \epsilon.
\]

We end this section with the following result.

**Lemma C.8.** There exists a constant \( C > 0 \), such that for any \( \epsilon > 0 \), there exists a neural network \( \Xi_\epsilon : \mathbb{R} \to \mathbb{R} \), such that

\[
\sup_{\xi \in [0,2\pi - \epsilon]} |\xi - \Xi_\epsilon(\cos(\xi), \sin(\xi))| \leq \epsilon,
\]

with \( \Xi_\epsilon(\cos(\xi), \sin(\xi)) \in [0,2\pi] \) for all \( \xi \in [0,2\pi] \), and such that

\[
\text{depth}(\Xi_\epsilon) \leq C \log(\epsilon^{-1})^2, \quad \text{width}(\Xi_\epsilon) \leq C.
\]

**Sketch of proof.** We can divide up the unit circle \( \{ (\cos(\xi), \sin(\xi)) \mid \xi \in [0,2\pi] \} \) into 5 subsets, where

\[
\begin{align*}
\xi \in [0, \pi/4], \quad &\iff x \geq 1/\sqrt{2}, \quad y \geq 0, \\
\xi \in [\pi/4, 3\pi/4], \quad &\iff y > 1/\sqrt{2}, \\
\xi \in [3\pi/4, 5\pi/4], \quad &\iff x \leq -1/\sqrt{2}, \\
\xi \in [5\pi/4, 7\pi/4], \quad &\iff y < -1/\sqrt{2}, \\
\xi \in [7\pi/4, 2\pi], \quad &\iff x \leq -1/\sqrt{2}, \quad y < 0,
\end{align*}
\]

On each of these subsets, one of the mappings

\[
x \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right] \to \mathbb{R}, \quad x = \cos(\xi) \mapsto \xi,
\]

or

\[
y \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right] \to \mathbb{R}, \quad y = \sin(\xi) \mapsto \xi,
\]

is well-defined and possesses an analytic, invertible extension to the open interval \((-1, 1)\) (with analytic inverse). By Theorem C.6 it follows that for any \( \epsilon > 0 \), we can find neural networks \( \Phi_1, \ldots, \Phi_5 \), such that \(|\Phi_j(\cos(\xi), \sin(\xi)) - \xi| \leq \epsilon\) on an open set containing the corresponding domain, and

\[
\text{depth}(\Phi_j) \leq C \log(\epsilon^{-1})^2, \quad \text{width}(\Phi_j) \leq C.
\]

By a straight-forward partition of unity argument based on Proposition C.3, we can combine these mappings to a global map\(^1\) which is represented by a neural network \( \Xi_\epsilon : \mathbb{R}^2 \to [0,2\pi] \), such that

\[
\sup_{\xi \in [0,2\pi - \epsilon]} |\Xi_\epsilon(\cos(\xi), \sin(\xi)) - \xi| \leq \epsilon,
\]

and such that

\[
\text{depth}(\Xi_\epsilon) \leq C \log(\epsilon^{-1})^2, \quad \text{width}(\Xi_\epsilon) \leq C.
\]

\(\square\)

\(1\)this step is where the \( \epsilon \)-gap at the right boundary \( 2\pi - \epsilon \) is needed, as the points at angles \( \xi = 0 \) and \( \xi = 2\pi \) are identical on the circle.
C.2 Proof of Theorem 3.1

The proof of this theorem follows from the following two propositions,

**Proposition C.9 (Lower bound in \( p \)).** Consider the solution operator \( G_{\text{adv}} : L^1 \cap L^\infty(\mathbb{T}) \to L^1 \cap L^\infty(\mathbb{T}) \) of the linear advection equation, with input measure \( \mu \) given as the random law of box functions of height \( h \in [h, \tilde{h}] \), width \( w \in [w, \overline{w}] \) and shift \( \xi \in [0, 2\pi] \). Let \( M > 0 \). There exists a constant \( C = C(M, \mu) > 0 \), depending only on \( \mu \) and \( M \), with the following property: If \( \mathcal{N}(\tilde{u}) = \sum_{k=1}^{l} \beta_k (\tilde{u}) \xi_k \) is any operator approximation with linear reconstruction dimension \( p \), such that \( \sup_{\tilde{u} \sim \mu} \| \mathcal{N}(\tilde{u}) \|_{L^\infty} \leq M < \infty \), then

\[
\mathbb{E}_{\tilde{u} \sim \mu} [\| \mathcal{N}(\tilde{u}) - G_{\text{adv}}(u) \|_{L^1}] \geq \frac{C}{p}.
\]

**Sketch of proof.** The argument is an almost exact repetition of the lower bound derived in Lanthaler et al. (2022), and therefore we will only outline the main steps of the argument, here: Since the measure \( \mu \) is translation-invariant, it can be shown that the optimal PCA eigenbasis with respect to the \( L^2(\mathbb{T}) \)-norm (cp. SM[A]) is the Fourier basis. Consider the (complex) Fourier basis \( \{ e^{ikx} \}_{k \in \mathbb{Z}} \), and denote the corresponding eigenvalues \( \{ \lambda_k \}_{k \in \mathbb{Z}} \). The \( k \)-th eigenvalue \( \lambda_k \) of the covariance-operator \( \Gamma_\mu = \mathbb{E}_{u \sim \mu}[(u \otimes u)] \) satisfies

\[
\lambda_k = \int_{-\pi}^{\pi} \int_{w}^{\pi} h^2 |\hat{\psi}_w(k)|^2 \frac{dw}{\Delta w} \frac{dh}{\Delta h} \geq h^2 \int_{w}^{\pi} |\hat{\psi}_w(k)|^2 \frac{dw}{\Delta w},
\]

where \( \hat{\psi}_w(k) \) denotes the \( k \)-th Fourier coefficient of \( \psi_w(x) := 1_{[-w/2,w/2]}(x) \). Since \( \psi_w(x) \) has a jump discontinuity of size 1 for any \( w > 0 \), it follows from basic Fourier analysis, that the asymptotic decay of \( |\hat{\psi}_w(k)| \sim C/|k| \), and hence, there exists a constant \( C = C(h, \tilde{h}, \overline{w}) > 0 \), such that

\[
\lambda_k \geq C|k|^{-2}.
\]

Re-ordering these eigenvalues \( \lambda_k \) in descending order (and renaming), \( \lambda_1 \geq \lambda_2 \geq \ldots \), it follows that for some constant \( C = C(h, \tilde{h}, \overline{w}) > 0 \), we have

\[
\sum_{j > p} \lambda_j \geq C \sum_{j > p} j^{-2} \geq C p^{-1}.
\]

By Theorem 2.1 this implies that (independently of the choice of the functionals \( \beta_k(\tilde{u}) \)), in the Hilbert space \( L^2(\mathbb{T}) \), we have

\[
\mathbb{E}_{\tilde{u} \sim \mu} [\| \mathcal{N}(\tilde{u}) - G(\tilde{u}) \|_{L^2}^2] \geq \sum_{j > p} \lambda_j \geq \frac{C}{p}.
\]

for a constant \( C > 0 \) that depends only on \( \mu \), but is independent of \( p \). To obtain a corresponding estimate with respect to the \( L^1 \)-norm, we simply observe that the above lower bound on the \( L^2 \)-norm together with the a priori bound \( \sup_{\tilde{u} \sim \mu} \| G_{\text{adv}}(\tilde{u}) \|_{L^\infty} \leq \tilde{h} \) on the underlying operator and the assumed \( L^\infty \)-bound \( \sup_{\tilde{u} \sim \mu} \| \mathcal{N}(\tilde{u}) \|_{L^\infty} \leq M \), imply

\[
\frac{C}{p} \leq \mathbb{E}_{\tilde{u} \sim \mu} [\| \mathcal{N}(\tilde{u}) - G(\tilde{u}) \|_{L^1}^2]
\]

\[
\leq \mathbb{E}_{\tilde{u} \sim \mu} [\| \mathcal{N}(\tilde{u}) - G(\tilde{u}) \|_{L^\infty} \| \mathcal{N}(\tilde{u}) - G(\tilde{u}) \|_{L^1}]
\]

\[
\leq (M + \tilde{h}) \mathbb{E}_{\tilde{u} \sim \mu} [\| \mathcal{N}(\tilde{u}) - G(\tilde{u}) \|_{L^1}].
\]

This immediately implies the claimed lower bound. \( \square \)

**Proposition C.10 (Lower bound in \( m \)).** Consider the solution operator \( G_{\text{adv}} : L^1(\mathbb{T}) \cap L^\infty(\mathbb{T}) \to L^1(\mathbb{T}) \cap L^\infty(\mathbb{T}) \) of the linear advection equation, with input measure \( \mu \) given as the law of random box functions of height \( h \in [h, \tilde{h}] \), width \( w \in [w, \overline{w}] \) and shift \( \xi \in [0, 2\pi] \). There exists an absolute constant \( C > 0 \) with the following property: If \( \mathcal{N}_{\text{DON}} \) is a DeepONet approximation with \( m \) sensor points, then

\[
\mathbb{E}_{\tilde{u} \sim \mu} [\| \mathcal{N}(\tilde{u}) - G_{\text{adv}}(\tilde{u}) \|_{L^1}] \geq \frac{C}{m}.
\]
Proof. We recall that the initial data $\bar{u}$ is a randomly shifted box function, of the form

$$\bar{u}(x) = h_{1_{[-w/2, +w/2]}(x - \xi)},$$

where $\xi \in [0, 2\pi]$, $h \in [-h, h]$ and $w \in [w, \bar{w}]$ are independent, uniformly distributed random variables.

Let $x_1, \ldots, x_m \in (0, 2\pi]$ be an arbitrary choice of $m$ sensor points. In the following, we denote $\bar{u}(X) : = (\bar{u}(x_1), \ldots, \bar{u}(x_m)) \in \mathbb{R}^m$. Let now $(x, \bar{u}(X)) \mapsto \Phi(x; \bar{u}(X))$ be any mapping, such that $x \mapsto \Phi(x; \bar{u}(X)) \in L^1(\mathbb{T})$ for all possible random choices of $\bar{u}$ (i.e. for all $\bar{u} \sim \mu$). Then we claim that

$$\mathbb{E}_{\bar{u} \sim \mu}[\| \mathcal{G}_{\text{adv}}(\bar{u}) - \Phi(\cdot; \bar{u}(X)) \|_{L^1}] \geq \frac{C}{m},$$

for a constant $C = C(h, \bar{w}) > 0$, holds for all $m \in \mathbb{N}$. Clearly, the lower bound [C.1] immediately implies the statement of Proposition[C.10] upon making the particular choice

$$\Phi(x; \bar{u}(X)) = \mathcal{N}(\bar{u})(x) \equiv \sum_{k=1}^{p} \beta_k(\bar{u}(x_1), \ldots, \bar{u}(x_m)) \tau_k(x).$$

To prove [C.1], we first recall that $\bar{u}(x) = \bar{u}(x; h, w, \xi)$ depends on three parameters $h, w$ and $\xi$, and the expectation over $\bar{u} \sim \mu$ in [C.1] amounts to averaging over $h \in [-h, h]$, $w \in [w, \bar{w}]$ and $\xi \in [0, 2\pi]$. In the following, we fix $w$ and $h$, and only consider the average over $\xi$. Suppressing the dependence on the fixed parameters, we will prove that

$$\frac{1}{2\pi} \int_0^{2\pi} \| \bar{u}(x; \xi) - \Phi(x; \bar{u}(X; \xi)) \|_{L^1} d\xi \geq \frac{C}{m}$$

with a constant that only depends on $h, \bar{w}$. This clearly implies [C.1].

To prove [C.2], we first introduce two mappings $\xi \mapsto I(\xi)$ and $\xi \mapsto J(\xi)$, by

$$I(\xi) = i \Leftrightarrow \xi - \frac{w}{2} \in [x_i, x_{i+1}], \quad J(\xi) = j \Leftrightarrow \xi + \frac{w}{2} \in [x_j, x_{j+1}],$$

where we make the natural identifications on the periodic torus (e.g. $x_{m+1}$ is identified with $x_1$ and $\xi \pm w/2$ is evaluated modulo $2\pi$). We observe that both mappings $\xi \mapsto I(\xi), J(\xi)$ cycle exactly once through the entire index set $\{1, \ldots, m\}$ as $\xi$ varies from $0$ to $2\pi$. Next, we introduce

$$A_{ij} := \{ \xi \in [0, 2\pi) \mid I(\xi) = i, J(\xi) = j \}, \quad \forall i, j \in \{1, \ldots, m\}.$$ 

Clearly, each $\xi \in [0, 2\pi)$ belongs to only one of these sets $A_{ij}$. Since $\xi \mapsto I(\xi)$ and $\xi \mapsto J(\xi)$ have $m$ jumps on $[0, 2\pi)$, it follows that the mapping $\xi \mapsto (I(\xi), J(\xi))$ can have at most $2m$ jumps. In particular, this implies that there are at most $2m$ non-empty sets $A_{ij} \neq \emptyset$ (these are all sets of the form $A_{i,j}(\xi), J(\xi), \xi \in [0, 2\pi)$), i.e.

$$\# \{ A_{ij} \neq \emptyset \mid i, j \in \{1, \ldots, m\} \} \leq 2m.$$ 

(C.3)

Since $\bar{u}(x; \xi) = h_{1_{[-w/2, +w/2]}(x - \xi)}$, one readily sees that when $\xi$ varies in the interior of $A_{ij}$, then all sensor point values $\bar{u}(X; \xi)$ remain constant, i.e. the mapping

$$\text{interior}(A_{ij}) \mapsto \mathbb{R}^m, \quad \xi \mapsto \bar{u}(X; \xi) = \text{const.}$$

We also note that $A_{ij} = [x_i + w/2, x_{i+1} + w/2] \cap [x_j - w/2, x_{j+1} - w/2]$ is in fact an interval. Fix $i, j$ such that $A_{ij} \neq \emptyset$ for the moment. We can write $A_{ij} = [a - \Delta, a + \Delta]$ for some $a, \Delta \in \mathbb{T}$, and there exists a constant $\bar{U} \in \mathbb{R}^m$ such that $\bar{U} \equiv \bar{u}(X; \xi)$ for $\xi \in [a - \Delta, a + \Delta)$. It follows from the triangle inequality that

$$\int_{A_{ij}} \| \bar{u}(x; \xi) - \Phi(x; \bar{u}(X; \xi)) \|_{L^1} d\xi = \int_{a - \Delta}^{a + \Delta} \| \bar{u}(x; \xi) - \Phi(x; \bar{U}) \|_{L^1} d\xi$$

$$= \int_0^\Delta \| \bar{u}(x; a - \xi') - \Phi(x; \bar{U}) \|_{L^1} d\xi'$$

$$+ \int_0^\Delta \| \bar{u}(x; a + \xi') - \Phi(x; \bar{U}) \|_{L^1} d\xi'$$

$$\geq \int_0^\Delta \| \bar{u}(x; a + \xi') - \bar{u}(x; a - \xi') \|_{L^1} d\xi'.$$
We observe that
\[ u(x; \xi) = h 1_{[-w/2,w/2]}(x) \],
we have, by a simple change of variables
\[ \|u(x; a + \xi') - u(x; a - \xi')\|_{L^1} = h \int_{\mathbb{T}} |1_{[-w/2,w/2]}(x) - 1_{[-w/2,w/2]}(x + 2\xi')| \, dx. \]

The last expression is of order \( \xi' \), provided that \( \xi' \) is small enough to avoid overlap with a periodic shift (recall that we are on working on the torus, and \( 1_{[-w/2,w/2]}(x) \) is identified with its periodic extension). To avoid such issues related to periodicity, we first note that \( \xi' \leq \Delta \leq \pi \), and then we choose a (large) constant \( C := C_0(\overline{w}) \), such that for any \( \xi' \leq \pi/C \) and \( w \leq \overline{w} \), we have
\[ \int_{\mathbb{T}} |1_{[-w/2,w/2]}(x) - 1_{[-w/2,w/2]}(x + 2\xi')| \, dx = \int_{-w/2}^{w/2} 1 \, dx + \int_{w/2}^{w/2} 1 \, dx = 4\xi'. \]

From the above, we can now estimate
\[ \int_{A_{ij}} \|\tilde{u}(x; \xi) - \Phi(x; \tilde{u}(X; \xi))\|_{L^1} \, d\xi \geq \int_{0}^{\Delta/C} \|\tilde{u}(x; a + \xi') - \tilde{u}(x; a - \xi')\|_{L^1} \, d\xi' \geq h \int_{0}^{\Delta/C} 4\xi' \, d\xi' \geq 2h \frac{\Delta^2}{C^2} \geq C |A_{ij}|^2, \]

where \( C = C(h, \overline{w}) \) is a constant only depending on the fixed parameters \( h, \overline{w} \).

Summing over all \( A_{ij} \neq \emptyset \), we obtain the lower bound
\[ \int_{0}^{2\pi} \|\tilde{u}(x; \xi) - \Phi(x; \tilde{u}(X; \xi))\|_{L^1} \, d\xi = \sum_{A_{ij} \neq \emptyset} \int_{A_{ij}} \|\tilde{u}(x; \xi) - \Phi(x; \tilde{u}(X; \xi))\|_{L^1} \, d\xi \geq C \sum_{A_{ij} \neq \emptyset} |A_{ij}|^2. \]

We observe that \( [0, 2\pi) = \bigcup A_{ij} \) is a disjoint union, and hence \( \sum_{A_{ij} \neq \emptyset} |A_{ij}| = 2\pi \). Furthermore, as observed above, there are at most \( 2m \) non-zero summands \( |A_{ij}| \neq 0 \). To finish the proof, we claim that the functional \( \sum_{k=1}^{2m} |\alpha_k|^2 \) is minimized among all \( \alpha_1, \ldots, \alpha_{2m} \) satisfying the constraint \( \sum_{k=1}^{2m} |\alpha_k| = 2\pi \) if, and only if, \( |\alpha_1| = \cdots = |\alpha_{2m}| = \pi/m \). Given this fact, it then immediately follows from the above estimate that
\[ \frac{1}{2\pi} \int_{0}^{2\pi} \|\tilde{u}(x; \xi) - \Phi(x; \tilde{u}(X; \xi))\|_{L^1} \, d\xi \geq C \sum_{A_{ij} \neq \emptyset} |A_{ij}|^2 \geq \frac{2C\pi^2}{m}. \]

where \( C = C(h, \overline{w}) > 0 \) is independent of the values of \( w \in [\overline{w}, \overline{w}] \) and \( h \in [h, \overline{h}] \). This suffices to conclude the claim of Proposition \( \text{[C.10]} \).

It remains to prove the claim: We argue by contradiction. Let \( \alpha_1, \ldots, \alpha_{2m} \) be a minimizer of \( \sum_{k=1}^{2m} |\alpha_k|^2 \) under the constraint \( \sum_{k=1}^{2m} |\alpha_k| = 2\pi \). Clearly, we can wlog assume that \( 0 \leq \alpha_1 \leq \cdots \leq \alpha_{2m} \) are non-negative numbers. If the claim does not hold, then there exists a minimizer, such that \( \alpha_1 < \alpha_{2m} \). Given \( \delta > 0 \) to be determined below, we define \( \beta_k \) by
\[ \beta_1 = \alpha_1 + \delta, \quad \beta_{2m} = \alpha_{2m} - \delta, \]
and \( \beta_k = \alpha_k \), for all other indices. Then, by a simple computation, we observe that
\[ \sum_{k} \alpha_k^2 - \sum_{k} \beta_k^2 = 2\delta(\alpha_{2m} - \alpha_1 - \delta). \]

Choosing \( \delta > 0 \) sufficiently small, we can ensure that the last quantity is \( > 0 \), while keeping \( \beta_k \geq 0 \) for all \( k \). In particular, it follows that \( \sum_{k} |\beta_k| = \sum_{k} |\alpha_k| = 2\pi \), but
\[ \sum_{k} \alpha_k^2 > \sum_{k} \beta_k^2, \]
in contradiction to the assumption that \( \alpha_1, \ldots, \alpha_{2m} \) minimize the last expression. Hence, any minimizer must satisfy \( |\alpha_1| = \cdots = |\alpha_{2m}| = \pi/m \).
C.3 Proof of Theorem 3.2

Proof. We choose equidistant grid points \(x_1, \ldots, x_m\) for the construction of a shift-DeepONet approximation to \(G_{\text{adv}}\). We may wlog assume that the grid distance \(\Delta x = x_2 - x_1 < \frac{1}{m}\), as the statement is asymptotic in \(m \to \infty\). We note the following points:

**Step 1:** We show that \(h\) can be efficiently determined by max-pooling. First, we observe that for any two numbers \(a, b\), the mapping
\[
\left( \begin{array}{c}
a \\
b
\end{array} \right) \mapsto \left( \begin{array}{c}
\max(0, a - b) \\
\max(0, b)
\end{array} \right) \mapsto \max(0, a - b) + \max(0, b) - \max(0, -b) = \max(a, b),
\]

is exactly represented by a ReLU neural network \(\overline{\max}(a, b)\) of width 3, with a single hidden layer. Given \(k\) inputs \(a_1, \ldots, a_k\), we can parallelize \(O(k/2)\) copies of \(\overline{\max}\), to obtain a ReLU network of width \(\leq 3k\) and with a single hidden layer, which maps
\[
\left( \begin{array}{c}
a_1 \\
a_2 \\
\vdots \\
a_{k-1} \\
a_k
\end{array} \right) \mapsto \left( \begin{array}{c}
\max(a_1, a_2) \\
\vdots \\
\max(a_{k-1}, a_k)
\end{array} \right).
\]

Concatenation of \(O(\log_2(k))\) such ReLU layers with decreasing input sizes \(k, \lceil k/2 \rceil, \lceil k/4 \rceil, \ldots, 1\), provides a ReLU representation of max-pooling
\[
\left( \begin{array}{c}
a_1 \\
a_2 \\
\vdots \\
a_k
\end{array} \right) \mapsto \left( \begin{array}{c}
\max(a_1, a_2) \\
\vdots \\
\max(a_{k-1}, a_k)
\end{array} \right) \mapsto \left( \begin{array}{c}
\max(a_1, a_2, a_3) \\
\vdots \\
\max(a_{k-3}, a_{k-2}, a_{k-1}, a_k)
\end{array} \right) \mapsto \cdots \mapsto \max(a_1, \ldots, a_k).
\]

This concatenated ReLU network \(\text{maxpool} : \mathbb{R}^k \to \mathbb{R}\) has width \(\leq 3k\), depth \(O(\log(k))\), and size \(O(k \log(k))\).

Our goal is to apply the above network maxpool to the shift-DeepONet input \(u(x_1), \ldots, u(x_m)\) to determine the height \(h\). To this end, we first choose \(\ell_1, \ldots, \ell_k \in \{1, \ldots, m\}\), such that \(x_{\ell_{j+1}} - x_{\ell_j} \leq \bar{w}\), with \(k \in \mathbb{N}\) minimal. Note that \(k\) is uniformly bounded, with a bound that only depends on \(\bar{w}\) (not on \(m\)). Applying the maxpool construction above the \(u(x_{\ell_1}), \ldots, u(x_{\ell_k})\), we obtain a mapping
\[
\left( \begin{array}{c}
u(x_1) \\
\vdots \\
u(x_m)
\end{array} \right) \mapsto \left( \begin{array}{c}
u(x_{\ell_1}) \\
\vdots \\
u(x_{\ell_k})
\end{array} \right) \mapsto \text{maxpool}(u(x_{\ell_1}), \ldots, u(x_{\ell_k})) = h.
\]

This mapping can be represented by \(O(\log(k))\) ReLU layers, with width \(\leq 3k\) and total (fully connected) size \(O(k^2 \log(k))\). In particular, since \(k\) only depends on \(\bar{w}\), we conclude that there exists \(C = C(\bar{w}) > 0\) and a neural network \(\overline{h} : \mathbb{R}^m \to \mathbb{R}\) with
\[
\text{depth}(\overline{h}) \leq C, \quad \text{width}(\overline{h}) \leq C, \quad \text{size}(\overline{h}) \leq C, \quad (C.4)
\]
such that
\[
\overline{h}(\tilde{u}(X)) = h, \quad (C.5)
\]
for any initial data of the form \(\tilde{u}(x) = h_1[x_{-w/2}, x_{w/2}](x - \xi)\), where \(h \in [\underline{h}, \overline{h}], w \in [\underline{w}, \overline{w}]\), and \(\xi \in [0, 2\pi]\).

**Step 2:** To determine the width \(\underline{w}\), we can consider a linear layer (of size \(m\)), followed by an approximation of division, \(\overline{\div}(a; b) \approx a/b\) (cp. Proposition C.7).
\[
\tilde{u}(X) \mapsto \Delta x \sum_{j=1}^{m} \tilde{u}(x_j) \mapsto \overline{\div} \left( \Delta x \sum_{j=1}^{m} \tilde{u}(x_j) ; \overline{h}(\tilde{u}(X)) \right)
\]
Denote this by \( \hat{w}(\hat{u}(X)) \). Then
\[
|w - \hat{w}| = \left| \frac{1}{h} \int_0^{2\pi} \hat{u}(x) \, dx - \frac{\Delta x}{h} \sum_{j=1}^{m} \hat{u}(x_j) \right| \\
+ \left| \frac{\Delta x}{h} \sum_{j=1}^{m} \hat{u}(x_j) - \frac{\pi}{h} \int_{\mathbb{R}} \left( \Delta x \sum_{j=1}^{m} \hat{u}(x_j); h \right) \right| \\
\leq \frac{2\pi}{m} + \epsilon.
\]
And we have depth(\( \hat{w} \)) \( \leq C \log(\epsilon^{-1})^2 \), width(\( \hat{w} \)) \( \leq C \), size(\( \hat{w} \)) \( \leq C \left( m + \log(\epsilon^{-1})^2 \right) \), by the complexity estimate of Proposition C.7.

**Step 3:** To determine the shift \( \xi \in [0, 2\pi) \), we note that
\[
\Delta x \sum_{j=1}^{m} \hat{u}(x_j) e^{-ixj} = \int_0^{2\pi} \hat{u}(x) e^{-ix} \, dx + O \left( \frac{1}{m} \right) \\
= 2 \sin(w/2) e^{-i\xi} + O \left( \frac{1}{m} \right).
\]
Using the result of Lemma C.8 combined with the approximation of division of Proposition C.7 and the observation that \( w \in (w, \pi) \) implies that \( \sin(w/2) \geq \sin(w/2) \) \( \geq 0 \) is uniformly bounded from below for all \( w \in [w, \bar{w}] \), it follows that for all \( \epsilon \in (0, \frac{1}{2}) \), there exists a neural network \( \hat{\xi} : \mathbb{R}^m \to [0, 2\pi) \), of the form
\[
\hat{\xi}(\hat{u}(X)) = \Xi_{\epsilon} \left[ \frac{1}{h} \left( \Delta x \sum_{j=1}^{m} \hat{u}(x_j) e^{-ixj}; \hat{M}, \epsilon \right) \left( \hat{h}, 2\sin(\hat{w}/2) \right) \right]
\]
such that
\[
\left| \xi - \hat{\xi}(\hat{u}(X)) \right| \leq \epsilon,
\]
for all \( \xi \in [0, 2\pi - \epsilon) \), and
\[
\text{depth}(\hat{\xi}) \leq C \log(\epsilon^{-1})^2, \quad \text{width}(\hat{\xi}) \leq C, \quad \text{size}(\hat{\xi}) \leq C \left( m + \log(\epsilon^{-1})^2 \right).
\]

**Step 4:** Combining the above three ingredients (Steps 1–3), and given the fixed advection velocity \( a \in \mathbb{R} \) and fixed time \( t \), we define a shift-DeepONet with \( p = 6 \), scale-net \( A_k \equiv 1 \), and shift-net \( \gamma \) with output \( \gamma_k(\hat{u}) \equiv \xi + at \), as follows:
\[
\mathcal{N}^{\text{DON}}(\hat{u}) = \sum_{k=1}^{p} \beta_k(\hat{u}) \tau_k(x - \gamma_k(\hat{u})) \\
:= \sum_{j=1}^{1} \widehat{h_{[0, \infty)}}(x - \xi - at + \hat{w}/2 + 2\pi j) \\
- \sum_{j=1}^{1} \widehat{h_{[0, \infty)}}(x - \xi - at - \hat{w}/2 + 2\pi j),
\]
where \( \widehat{h} = \widehat{h}(\hat{u}(X)) \), \( \hat{w} = \hat{w}(\hat{u}(X)) \), and \( \hat{\xi} = \hat{\xi}(\hat{u}(X)) \), and where \( \widehat{1}_{[0,\infty)} \) is a sufficiently accurate \( L^1 \)-approximation of the indicator function \( 1_{[0,\infty)}(x) \) (cp. Proposition C.2). To estimate the approximation error, we denote \( \overline{1}_{[-\hat{w}/2,\hat{w}/2]}(x) := \overline{1}_{[0,\infty)}(x + \hat{w}/2) - \overline{1}_{[0,\infty)}(x - \hat{w}/2) \) and identify it with its periodic extension to \( \mathbb{T} \), so that we can more simply write
\[
\mathcal{N}^{\text{DON}}(\hat{u})(x) = \widehat{h}_{\overline{1}_{[-\hat{w}/2,\hat{w}/2]}(x - \xi - at)}.
\]

We also recall that the solution \( u(x, t) \) of the linear advection equation \( \partial_t u + a \partial_x u = 0 \), with initial data \( u(x, 0) = \hat{u}(x) \) is given by \( u(x, t) = \hat{u}(x - at) \), where \( at \) is a fixed constant, independent of the input \( \hat{u} \). Thus, we have
\[
G_{\text{adv}}(\hat{u})(x) = \hat{u}(x - at) = h \overline{1}_{[-\hat{w}/2,\hat{w}/2]}(x - \xi - at).
\]
We can now write
\[ |G_{\text{adv}}(\tilde{u})(x) - N^{\text{DON}}(\tilde{u})(x)| = |h 1_{[-w/2,w/2]}(x - \xi - at) - \tilde{h} \tilde{1}_{[-\tilde{w}/2,\tilde{w}/2]}(x - \tilde{\xi} - at)|. \]

We next recall that by the construction of Step 1, we have \( \tilde{h}(\tilde{u}) \equiv h \) for all inputs \( \tilde{u} \). Furthermore, upon integration over \( x \), we can clearly get rid of the constant shift \( at \) by a change of variables. Hence, we can estimate
\[ \|G_{\text{adv}}(\tilde{u}) - N^{\text{DON}}(\tilde{u})\|_{L^1} \leq \tilde{h} \int_{\xi} \left| 1_{[-w/2,w/2]}(x - \xi) - \tilde{1}_{[-\tilde{w}/2,\tilde{w}/2]}(x - \tilde{\xi}) \right| \, dx. \]  
(C.6)

Using the straight-forward bound
\[ \left| 1_{[-w/2,w/2]}(x - \xi) - \tilde{1}_{[-\tilde{w}/2,\tilde{w}/2]}(x - \tilde{\xi}) \right| \leq \left| 1_{[-w/2,w/2]}(x - \xi) - 1_{[-w/2,w/2]}(x - \tilde{\xi}) \right| + \left| 1_{[-w/2,w/2]}(x - \tilde{\xi}) - 1_{[-\tilde{w}/2,\tilde{w}/2]}(x - \tilde{\xi}) \right| + \left| 1_{[-\tilde{w}/2,\tilde{w}/2]}(x - \tilde{\xi}) - \tilde{1}_{[-\tilde{w}/2,\tilde{w}/2]}(x - \tilde{\xi}) \right|, \]

one readily checks that, by Step 3, the integral over the first term is bounded by
\[ \|(I)\|_{L^1} \leq C \int_{0}^{2\pi-\epsilon} |\xi - \tilde{\xi}| \, d\xi + \int_{2\pi-\epsilon}^{2\pi} 2 \, d\xi \leq (C + 2)\epsilon. \]
where \( C = C(w, \bar{w}) > 0 \). By Step 2, the integral over the second term can be bounded by
\[ \|(II)\|_{L^1} \leq C |w - \tilde{w}| \leq C \left( \frac{1}{m} + \epsilon \right). \]

Finally, by Proposition C.1 by choosing \( \epsilon \) sufficiently small (recall also that the size of \( \tilde{1}^\epsilon \) is independent of \( \epsilon \), we can ensure that
\[ \|(III)\|_{L^1} = \|\tilde{1}_{[-\tilde{w}/2,\tilde{w}/2]} - 1_{[-w/2,w/2]}\|_{L^1} \leq \epsilon, \]
holds uniformly for any \( \tilde{w} \). Hence, the right-hand side of (C.6) obeys an upper bound of the form
\[ \mathbb{E}_{\tilde{u} \sim \mu} \left[ \|G_{\text{adv}}(\tilde{u}) - N^{\text{DON}}(\tilde{u})\|_{L^1} \right] \leq \tilde{h} \frac{\pi}{2} \int_{\bar{w}}^{w} \int_{\tilde{w}}^{\tilde{w}} \int_{\xi} \int_{\tilde{\xi}} \, d\xi \, d\tilde{\xi} \|G_{\text{adv}}(\tilde{u}) - N^{\text{DON}}(\tilde{u})\|_{L^1} \]
\[ \leq \tilde{h} \frac{\pi}{2} \int_{\bar{w}}^{w} \int_{\tilde{w}}^{\tilde{w}} \int_{\xi} \int_{\tilde{\xi}} \, d\xi \, d\tilde{\xi} \left\{ \|I\|_{L^1} + \|II\|_{L^1} + \|III\|_{L^1} \right\} \]
\[ \leq C \left( \epsilon + \frac{1}{m} \right), \]

for a constant \( C = C(w, \bar{w}, \tilde{h}) > 0 \). We also recall that by our construction,
\[ \text{depth}(N^{\text{DON}}) \leq C \log(\epsilon^{-1})^2, \quad \text{width}(N^{\text{DON}}) \leq C, \quad \text{size}(N^{\text{DON}}) \leq C \left( m + \log(\epsilon^{-1})^2 \right). \]

Replacing \( \epsilon \) by \( \epsilon/2C \) and choosing \( m \sim \epsilon^{-1} \), we obtain
\[ \mathbb{E}_{\tilde{u} \sim \mu} \left[ \|G_{\text{adv}}(\tilde{u}) - N^{\text{DON}}(\tilde{u})\|_{L^1} \right] \leq \epsilon, \]
with
\[ \text{depth}(N^{\text{DON}}) \leq C \log(\epsilon^{-1})^2, \quad \text{width}(N^{\text{DON}}) \leq C, \quad \text{size}(N^{\text{DON}}) \leq C \epsilon^{-1}, \]
where \( C \) depends only on \( \mu \), and is independent of \( \epsilon \). This implies the claim of Theorem 3.2 \( \square \)

C.4 Proof of Theorem 3.3

For the proof of Theorem 3.3 we will need a few intermediate results:

Lemma C.11. Let \( \tilde{u} = h 1_{[-w/2,w/2]}(x - \xi) \) and fix a constant \( at \in \mathbb{R} \). There exists a constant \( C > 0 \), such that given \( N \) grid points, there exists an FNO with
\[ k_{\text{max}} = 1, \quad d_v \leq C, \quad \text{depth} \leq C, \quad \text{size} \leq C, \]
such that
\[ \sup_{\tilde{h},w,\xi} \left| N^{\text{FNO}}(\tilde{u})(x) - \sin(w/2) \cos(x - \xi - at) \right| \leq \frac{C}{N}. \]
Lemma C.12. Fix $0 < \frac{h}{\bar{h}} < 1$ and $0 < \frac{w}{\bar{w}} < 1$. There exists a constant $C = C(\frac{h}{\bar{h}}, \frac{w}{\bar{w}}) > 0$ with the following property: For any input function $\bar{u}(x) = \bar{h} 1_{[-w/2,w/2]}(x - \xi)$ with $h \notin [\bar{h}, \bar{w}]$ and $w \notin [\bar{w}, \bar{w}]$, and given $N$ grid points, there exists an FNO with constant output function, such that

$$\sup_{h,w,:} |\mathcal{N}^{\text{FNO}}(\bar{u}) - w| \leq \frac{C}{N},$$

and with uniformly bounded size,

$$k_{\text{max}} = 0, \quad d_v \leq C, \quad \text{depth} \leq C, \quad \text{size} \leq C.$$

Proof. We can define a FNO mapping

$$\bar{u} \mapsto 1_{[-w/2,w/2]}(x) \mapsto \frac{2\pi}{N} \sum_{j=1}^{N} 1_{[-w/2,w/2]}(x_j) = w + O(N^{-1}),$$

where we observe that the first mapping is just $\bar{u} \mapsto \max(h^{-1}\bar{u}(x), 1)$, which is easily represented by an ordinary ReLU NN of bounded size. The second mapping above is just projection onto the 0-th Fourier mode under the discrete Fourier transform. In particular, both of these mappings can be represented exactly by a FNO with $k_{\text{max}} = 0$ and uniformly bounded $d_v$, depth and size. To conclude the argument, we observe that the error $O(N^{-1})$ depends only on the grid size and is independent of $w \in [\bar{w}, \bar{w}]$. \qed

Lemma C.13. Fix $0 < \frac{w}{\bar{w}} < 1$. There exists a constant $C = C(\frac{w}{\bar{w}}) > 0$, such that for any $\epsilon > 0$, there exists a FNO such that for any constant input function $\bar{u}(x) \equiv w \in [\bar{w}, \bar{w}]$, we have

$$\left|\mathcal{N}^{\text{FNO}}(\bar{u})(x) - \frac{1}{2} \sin(w)\right| \leq \epsilon, \quad \forall x \in [0, 2\pi],$$

and

$$k_{\text{max}} = 0, \quad d_v \leq C, \quad \text{depth} \leq C \log(\epsilon^{-1})^2, \quad \text{size} \leq C \log(\epsilon^{-1})^2.$$
Proof. It follows e.g. from [Elbracht et al., 2021, Thm. III.9] (or also Theorem C.6 above) that there exists a constant $C = C(w, w) > 0$, such that for any $\epsilon > 0$, there exists a ReLU neural network $S_\epsilon$ with size($S_\epsilon$) $\leq C \log(\epsilon^{-1})^2$, depth($S_\epsilon$) $\leq C \log(\epsilon^{-1})^2$ and width($S_\epsilon$) $\leq C$, such that

$$\sup_{w \in [w, \bar{w}]} \left| S_\epsilon(w) - \frac{1}{2} \sin(w) \right| \leq \epsilon.$$ 

To finish the proof, we simply note that this ReLU neural network $S_\epsilon$ can be easily represented by a FNO $S_\epsilon$ with $k_{\text{max}} = 0$, $d_v \leq C$, depth($S_\epsilon$) $\leq C \log(\epsilon^{-1})^2$ and size($S_\epsilon$) $\leq C \log(\epsilon^{-1})^2$; it suffices to copy the weight matrices $W_\ell$ of $S_\epsilon$, set the entries of the Fourier multiplier matrices $P_\ell(k) \equiv 0$, and choose constant bias functions $b_\ell(x) = \text{const.}$ (with values given by the corresponding biases in the hidden layers of $S_\epsilon$).

Lemma C.14. Let $\bar{u} = h 1_{[-w/2, w/2]}(x - \xi)$. Assume that $2\pi/N \leq w$. For any $\epsilon > 0$, there exists an FNO with constant output function, such that

$$\sup_{h, w, \xi} |\mathcal{A}^\text{FNO}(\bar{u})(x) - h| \leq \epsilon,$$

and

$$k_{\text{max}} = 0, \quad d_v \leq C, \quad \text{depth} \leq C \log(\epsilon^{-1})^2.$$ 

Proof. The proof follows along similar lines as the proofs of the previous lemmas. In this case, we can define a FNO mapping

$$\bar{u} \mapsto \left[ h 1_{[-w/2, w/2]}(x) \right] \mapsto \left[ \sum_{j=1}^{N} 1_{[-w/2, w/2]}(x_j) \right] \mapsto \frac{\bar{u}}{h} \left( \sum_{j=1}^{N} 1_{[-w/2, w/2]}(x_j) \right).$$

The estimate on $k_{\text{max}}$, $d_v$, depth follow from the construction of $\frac{\bar{u}}{h}$ in Proposition C.7

Proof of Theorem 3.3. We first note that (the $2\pi$-periodization of) $1_{[-w/2, w/2]}(x - \xi - at)$ is 1 if, and only if

$$\cos(x - \xi - at) \geq \cos(w/2) \iff \sin(w/2) \cos(x - \xi - at) \geq \frac{1}{2} \sin(w). \quad (C.7)$$

The strategy of proof is as follows: Given the input function $\bar{u}(x) = h 1_{[-w/2, w/2]}(x - \xi)$ with unknown $w \in [w, \bar{w}]$, $\xi \in [0, 2\pi]$ and $h \in [h, \bar{h}]$, and for given $a, t \in \mathbb{R}$ (these are fixed for this problem), we first construct an FNO which approximates the sequence of mappings

$$\bar{u} \mapsto \left[ \frac{h}{\sin(w/2) \cos(x - \xi)} \right] \mapsto \left[ \frac{h}{\frac{1}{2} \sin(w)} \right] \mapsto \left[ \frac{h}{\sin(w/2) \cos(x - \xi - at)} \right].$$

Then, according to (C.7), we can approximately reconstruct $1_{[-w/2, w/2]}(x - \xi - at)$ by approximating the identity

$$1_{[-w/2, w/2]}(x - \xi - at) = 1_{[0, \infty)} \left( \sin(w/2) \cos(x - \xi - at) - \frac{1}{2} \sin(w) \right),$$

where $1_{[0, \infty)}$ is the indicator function of $[0, \infty)$. Finally, we obtain $G_{\text{adv}}(\bar{u}) = h 1_{[-w/2, w/2]}(x - \xi - at)$ by approximately multiplying this output by $h$. We fill in the details of this construction below.

Step 1. The first step is to construct approximations of the mappings above. We note that we can choose a (common) constant $C_0 = C_0(h, \bar{h}, w, w) > 0$, depending only on the parameters $h, \bar{h}, w$ and $\bar{w}$, such that for any grid size $N \in \mathbb{N}$ all of the following hold:

1. There exists a FNO $\mathcal{H}_N$ with constant output (cp. Lemma C.14), such that for $\bar{u}(x) = h 1_{[-w/2, w/2]}(x - \xi)$,

$$\sup_{w, h} |\mathcal{H}_N(\bar{u}) - h| \leq \frac{1}{N}. \quad (C.8)$$

and with

$$k_{\text{max}} \leq 1, \quad d_v \leq C_0, \quad \text{depth} \leq C_0 \log(N)^2, \quad \text{size} \leq C_0 \log(N)^2.$$
2. Combining Lemma C.12 and C.13, we conclude that there exists a FNO $\mathcal{S}_N$ with constant output, such that for $\bar{u}(x) = h \mathbb{1}_{[-w/2,w/2]}(x - \xi)$, we have

$$\sup_{x \in [w-1,w+1]} \left| \mathcal{S}_N(\bar{u}) - \frac{1}{2} \sin(w) \right| \leq \frac{C_0}{N}. \quad (C.9)$$

and with

$$k_{\max} = 0, \quad d_v \leq C_0, \quad \text{depth} \leq C_0 \log(N)^2, \quad \text{size} \leq C_0 \log(N)^2.$$

3. There exists a FNO $\mathcal{C}_N$ (cp. Lemma C.11), such that for $\bar{u} = h \mathbb{1}_{[-w/2,w/2]}$,

$$\sup_{x,\xi,w} |\mathcal{C}_N(\bar{u})(x) - \sin(w/2) \cos(x - \xi - at)| \leq \frac{C_0}{N}, \quad (C.10)$$

where the supremum is over $x, \xi \in [0,2\pi]$ and $w \in [w,\bar{w}]$, and such that

$$k_{\max} = 1, \quad d_v \leq C_0, \quad \text{depth} \leq C_0, \quad \text{size} \leq C_0.$$

4. There exists a ReLU neural network $\tilde{1}_{[0,\infty)}^N$ (cp. Proposition C.1), such that

$$\|\tilde{1}_{[0,\infty)}^N\|_{L^\infty} \leq 1, \quad \tilde{1}_{[0,\infty)}^N(z) = \begin{cases} 0, & (x < 0), \\ 1, & (x \geq \frac{1}{N}). \end{cases} \quad (C.11)$$

with

$$\text{width}(\tilde{1}_{[0,\infty)}^N) \leq C_0, \quad \text{depth}(\tilde{1}_{[0,\infty)}^N) \leq C_0.$$

5. There exists a ReLU neural network $\tilde{x}_N$ (cp. Proposition C.4), such that

$$\sup_{a,b} |\tilde{x}_N(a,b) - ab| \leq \frac{1}{N}, \quad (C.12)$$

where the supremum is over all $|a|,|b| \leq \frac{1}{2N}$, and

$$\text{width}(\tilde{x}_N) \leq C_0, \quad \text{depth}(\tilde{x}_N) \leq C_0 \log(N).$$

Based on the above FNO constructions, we define

$$\mathcal{N}^{\text{FNO}}(\bar{u}) := \tilde{x}_N \left( H_N(\bar{u}), \tilde{1}_{[0,\infty)}^N \left( \mathcal{C}_N(\bar{u}) - \mathcal{S}_N(\bar{u}) \right) \right). \quad (C.13)$$

Taking into account the size estimates from points 1–5 above, as well as the general FNO size estimate B.2, it follows that $\mathcal{N}^{\text{FNO}}$ can be represented by a FNO with

$$k_{\max} = 1, \quad d_v \leq C, \quad \text{depth} \leq C \log(N)^2, \quad \text{size} \leq C \log(N)^2. \quad (C.14)$$

To finish the proof of Theorem 3.3, it suffices to show that $\mathcal{N}^{\text{FNO}}$ satisfies an estimate

$$\sup_{\bar{u} \sim \mu} \|\mathcal{N}^{\text{FNO}}(\bar{u}) - G_{\text{adv}}(\bar{u})\|_{L^2} \leq \frac{C}{N},$$

with $C > 0$ independent of $N$.

**Step 2:** We claim that if $x \in [0,2\pi]$ is such that

$$|\sin(w/2) \cos(x - \xi) - \frac{1}{2} \sin(w)| \geq \frac{2C_0 + 1}{N},$$

with $C_0$ the constant of Step 1, then

$$\tilde{1}_{[0,\infty)}^N(\mathcal{C}_N(\bar{u})(x) - \mathcal{S}_N(\bar{u})) = \mathbb{1}_{[-w/2,w/2]}(x - \xi).$$

To see this, we first assume that

$$\sin(w/2) \cos(x - \xi) - \frac{1}{2} \sin(w) \geq \frac{2C_0 + 1}{N} > 0.$$
Then
\[ C_N(\bar{u})(x) - S_N(\bar{u}) \geq \sin(w/2) \cos(x - \xi) - \frac{1}{2} \sin(w) \]
\[ - |C_N(\bar{u})(x) - \sin(w/2) \cos(x - \xi)| - |S_N(\bar{u}) - \frac{1}{2} \sin(w)| \]
\[ \geq \frac{2C_0 + 1}{N} - \frac{C_0}{N} - \frac{C_0}{N} = \frac{1}{N} > 0. \]

Hence, it follows from (C.11) that
\[ \bar{1}_N(0, \infty)(C_N(\bar{u})(x) - S_N(\bar{u})) \]
\[ = 1_{[0, \infty)}(\sin(w/2) \cos(x - \xi) - \frac{1}{2} \sin(w)) \]
\[ = 1_{[-w/2, w/2]}(x). \]

The other case,
\[ \sin(w/2) \cos(x - \xi) - \frac{1}{2} \sin(w) \leq -\frac{2C_0 + 1}{N}, \]
is shown similarly.

Step 3: We note that there exists \( C = C(\omega, \pi) > 0 \), such that for any \( \delta > 0 \), the Lebesgue measure
\[ \text{meas} \{ x \in [0, 2\pi] \left| \sin(w/2) \cos(x - \xi) - \frac{1}{2} \sin(w) < \delta \right. \} \leq C\delta. \]

Step 4: Given the previous steps, we now write
\[ G_{\text{adv}}(\bar{u}) - N_{\text{FNO}}(\bar{u}) = h \bar{1}_{[-w/2, w/2]}(x) - \bar{\times}_N \left( H_N, \bar{1}_N(0, \infty)(C_N - S_N(\bar{u})) \right) \]
\[ = \left[ h \bar{1}_{[-w/2, w/2]}(x) - h \bar{1}_N(0, \infty)(C_N - S_N) \right] \]
\[ + (h - H_N) \bar{1}_N(0, \infty)(C_N - S_N) \]
\[ + H_N \bar{1}_N(0, \infty)(C_N - S_N(\bar{u})) - \bar{\times}_N \left( H_N, \bar{1}_N(0, \infty)(C_N - S_N(\bar{u})) \right) \]
\[ =: (I) + (II) + (III). \]

The second \((II)\) and third \((III)\) terms are uniformly bounded by \( N^{-1} \), by the construction of \( \bar{\times}_N \) and \( H_N \). By Steps 2 and 3 (with \( \delta = (2C_0 + 1)/N \)), we can estimate the \( L^1 \)-norm of the first term as
\[ \| (I) \|_{L^1} \leq 2\pi \text{meas} \{ \sin(w/2) \cos(x - \xi) - 2^{-1} \sin(w) < \delta \} \leq C/N, \]
where the constant \( C \) is independent of \( N \), and only depends on the parameters \( h, \bar{h}, \bar{w} \) and \( \bar{\pi} \). Hence, \( N_{\text{FNO}} \) satisfies
\[ \sup_{\bar{u} \sim \mu} \| N_{\text{FNO}}(\bar{u}) - G_{\text{adv}}(\bar{u}) \|_{L^1} \leq C/N, \]
for a constant \( C > 0 \) independent of \( N \), and where we recall (cp. (C.14) above):
\[ k_{\text{max}} = 1, \quad d_0 \leq C, \quad \text{depth}(N_{\text{FNO}}) \leq C \log(N)^2, \quad \text{size}(N_{\text{FNO}}) \leq C \log(N)^2. \]
The claimed error and complexity bounds of Theorem 3.3 are now immediate upon choosing \( N \sim \epsilon^{-1} \). \( \square \)

C.5 Proof of Theorem 3.3

To motivate the proof, we first consider the Burgers’ equation with the particular initial data \( \bar{u}(x) = -\sin(x) \), with periodic boundary conditions on the interval \( x \in [0, 2\pi] \). The solution for this initial datum can be constructed via the well-known method of characteristics; we observe that the solution \( u(x, t) \) with initial data \( \bar{u}(x) \) is smooth for time \( t \in [0, 1) \), develops a shock discontinuity at \( x = 0 \) (and \( x = 2\pi \)) for \( t \geq 1 \), but remains otherwise smooth on the interval \( x \in (0, 2\pi) \) for all times. In fact, fixing a time \( t \geq 0 \), the solution \( u(x, t) \) can be written down explicitly in terms of the bijective mapping (cp. Figure 3)
\[ \Psi_t : [x_0, 2\pi - x_0] \rightarrow [0, 2\pi], \quad \Psi_t(x_0) = x_0 - t \sin(x_0), \]

\[ w \]
We are ultimately interested in solutions for more general periodic initial data of the form $\bar{x}$.

We note that for given $\bar{x}$, there exist constants $C, \gamma > 0$ and $\epsilon > 0$ (domain (blue limits) allowing for bijective analytic continuation, before shock formation, (c) $\Psi$).

Theorem C.16 is then given in terms of $\Psi_t$, by

$$ u(x, t) = -\sin\left(\Psi_t^{-1}(x)\right), \quad \text{for } x \in [0, 2\pi]. $$

We are ultimately interested in solutions for more general periodic initial data of the form $\bar{u}(x) = -\sin(x - \xi)$; these can easily be obtained from the particular solution (C.16) via a shift. We summarize this observation in the following lemma:

**Lemma C.15.** Let $\xi \in [0, 2\pi)$ be given, fix a time $t \geq 0$. Consider the initial data $\bar{u}(x) = -\sin(x - \xi)$. Then the entropy solution $u(x, t)$ of the Burgers’ equations with initial data $\bar{u}$ is given by

$$ u(x, t) = \begin{cases} 
-\sin(\Psi_t^{-1}(x - \xi + 2\pi)), & (x < \xi), \\
-\sin(\Psi_t^{-1}(x - \xi)), & (x \geq \xi), 
\end{cases} $$

for $x \in [0, 2\pi], t \geq 0$.

**Lemma C.16.** Let $t > 1$, and define $U : [0, 2\pi] \to \mathbb{R}$ by $U(x) := -\sin(\Psi_t^{-1}(x))$. There exists $\Delta_t > 0$ (depending on $t$), such that $x \mapsto U(x)$ can be extended to an analytic function $\tilde{U} : (-\Delta_t, 2\pi + \Delta_t) \to \mathbb{R}$, $x \mapsto \tilde{U}(x)$; i.e., such that $\tilde{U}(x) = U(x)$ for all $x \in [0, 2\pi]$.

**Corollary C.17.** Let $U : [0, 2\pi] \to \mathbb{R}$ be defined as in Lemma C.16. There exists a constant $C > 0$, such that for any $\epsilon > 0$, there exists a ReLU neural network $\Phi_\epsilon : \mathbb{R} \to \mathbb{R}$, such that

$$ \sup_{x \in [0, 2\pi]} |\Phi_\epsilon(x) - U(x)| \leq \epsilon, $$

and

$$ \text{depth}(\Phi_\epsilon) \leq C \log(\epsilon^{-1}), \quad \text{width}(\Phi_\epsilon) \leq C, $$

**Proof.** By Lemma C.16 $U$ can be extended to an analytic function $\tilde{U} : (-\Delta_t, 2\pi + \Delta_t) \to \mathbb{R}$. Thus, by Theorem C.6 there exist constants $C, \gamma > 0$, such that for any $L \in \mathbb{N}$, there exists a neural network $\Phi_L : \mathbb{R} \to \mathbb{R}$, such that

$$ \sup_{x \in [0, 2\pi]} |U(x) - \Phi_L(x)| \leq C \exp(-\gamma L^{1/2}), $$

$$ \text{depth}(\Phi_L) \leq CL, \quad \text{width}(\Phi_L) \leq C. $$
Given $\epsilon > 0$, we can thus choose $L \geq \gamma^{-2} \log(\epsilon^{-1})^2$, to obtain a neural network $\Phi_\epsilon := \tilde{\Phi}_L$, such that

$$\sup_{x \in [0,2\pi]} |U(x) - \Phi_\epsilon(x)| \leq \epsilon,$$

and

$$\text{depth}(\Phi_\epsilon) \leq C\log(\epsilon^{-1})^2, \quad \text{width}(\Phi_\epsilon) \leq C,$$

for a constant $C > 0$, independent of $\epsilon$.

\[\square\]

**Lemma C.18.** Let $t > 1$, and let $U : [-2\pi, 2\pi] \to \mathbb{R}$ be given by

$$U(x) := \begin{cases} -\sin(\Psi^{-1}(x + 2\pi)), & (x < 0), \\ -\sin(\Psi^{-1}(x)), & (x \geq 0). \end{cases}$$

There exists a constant $C = C(t) > 0$, depending only on $t$, such that for any $\epsilon \in (0, \frac{1}{2}]$, there exists a neural network $\Phi_\epsilon : \mathbb{R} \to \mathbb{R}$, such that

$$\text{depth}(\Phi_\epsilon) \leq C\log(\epsilon^{-1})^2, \quad \text{width}(\Phi_\epsilon) \leq C,$$

and such that

$$\|\Phi_\epsilon - U\|_{L^1([-2\pi, 2\pi])} \leq \epsilon.$$

**Proof.** By Corollary C.17, there exists a constant $C > 0$, such that for any $\epsilon > 0$, there exist neural networks $\Phi^-, \Phi^+ : \mathbb{R} \to \mathbb{R}$, such that

$$\text{depth}(\Phi^+) \leq C\log(\epsilon^{-1})^2, \quad \text{width}(\Phi^+) \leq C,$$

and

$$\|\Phi^-(x) - U(x)\|_{L^\infty([-2\pi, 0])} \leq \epsilon, \quad \|\Phi^+(x) - U(x)\|_{L^\infty([0, 2\pi])} \leq \epsilon.$$

This implies that

$$\|\Phi^-(x) - U(x)\|_{L^\infty([-2\pi, 0])} \leq \epsilon, \quad \|\Phi^+(x) - U(x)\|_{L^\infty([0, 2\pi])} \leq \epsilon.$$

By Proposition C.2 (approximation of indicator functions), there exist neural networks $\chi^\pm : \mathbb{R} \to \mathbb{R}$ with uniformly bounded width and depth, such that

$$\|\chi^- - 1_{[-2\pi, 0]}\|_{L^1} \leq \epsilon, \quad \|\chi^+ - 1_{[0, 2\pi]}\|_{L^1} \leq \epsilon,$$

and $\|\chi^\pm(x)\|_{L^\infty} \leq 1$. Combining this with Proposition C.4 (approximation of multiplication), it follows that there exists a neural network $\Phi_\epsilon : \mathbb{R} \to \mathbb{R}$, $\Phi_\epsilon(x) = \tilde{\chi}_\epsilon(\chi^+_\epsilon, \Phi^+) + \tilde{\chi}_\epsilon(\chi^-_\epsilon, \Phi^-)$, such that

$$\|\Phi_\epsilon - [1_{[-2\pi, 0]}\Phi^- + 1_{[0, 2\pi]}\Phi^+]\|_{L^1([-2\pi, 2\pi])} \leq \|\tilde{\chi}_{M, \epsilon}(\chi^+, \Phi^+) - 1_{[0, 2\pi]}\Phi^+\|_{L^1([-2\pi, 2\pi])} + \|\tilde{\chi}_{M, \epsilon}(\chi^-, \Phi^-) - 1_{[-2\pi, 0]}\Phi^-\|_{L^1([-2\pi, 2\pi])}.$$

By construction of $\tilde{\chi}_{M, \epsilon}$, we have

$$\|\tilde{\chi}_{M, \epsilon}(\chi^+, \Phi^+) - 1_{[0, 2\pi]}\Phi^+\|_{L^1} \leq \|\tilde{\chi}_{M, \epsilon}(\chi^+, \Phi^+) - \chi^+\Phi^+\|_{L^1} + \|\chi^+ - 1_{[0, 2\pi]}\|_{L^1}\|\Phi^+\|_{L^\infty} \leq (4\pi + 2)\epsilon.$$

And similarly for the other term. Thus, it follows that

$$\|\Phi_\epsilon - [1_{[-2\pi, 0]}\Phi^- + 1_{[0, 2\pi]}\Phi^+]\|_{L^1([-2\pi, 2\pi])} \leq 2(4\pi + 2)\epsilon,$$

and finally,

$$\|U - \Phi_\epsilon\|_{L^1([-2\pi, 2\pi])} \leq 4\pi \|U - [1_{[-2\pi, 0]}\Phi^- + 1_{[0, 2\pi]}\Phi^+]\|_{L^\infty([-2\pi, 2\pi])} + \|1_{[-2\pi, 0]}\Phi^- + 1_{[0, 2\pi]}\Phi^+\|_{L^1([-2\pi, 2\pi])} - \Phi_\epsilon\|_{L^1([-2\pi, 2\pi])} \leq 4\pi\epsilon + 2(4\pi + 2)\epsilon = (12\pi + 4)\epsilon.$$

for a neural network $\Phi_\epsilon : \mathbb{R} \to \mathbb{R}$ of size:

$$\text{depth}(\Phi_\epsilon) \leq C\log(\epsilon^{-1})^2, \quad \text{width}(\Phi_\epsilon) \leq C.$$

Replacing $\epsilon$ with $\epsilon = \epsilon/(12\pi + 4)$ yields the claimed estimate for $\Phi_\epsilon$. \[\square\]
Based on the above results, we can now prove the claimed error and complexity estimate for the shift-DeepONet approximation of the Burgers’ equation, Theorem 3.3.

**Proof of Theorem 3.3** Fix $t > 1$. Let $U : [-2\pi, 2\pi] \to \mathbb{R}$ be the function from Lemma C.18. By Lemma C.15 the exact solution of the Burgers’ equation with initial data $\bar{u}(x) = -\sin(x - \xi)$ at time $t$, is given by

$$u(x, t) = U(x - \xi), \quad \forall x \in [0, 2\pi].$$

From Lemma C.18 (note that $x - \xi \in [-2\pi, 2\pi]$), it follows that there exists a constant $C > 0$, such that for any $\epsilon > 0$, there exists a neural network $\Phi_{\epsilon} : \mathbb{R} \to \mathbb{R}$, such that

$$\|u(\cdot, t) - \Phi_{\epsilon}(\cdot - \xi)\|_{L^1([0, 2\pi])} \leq \epsilon,$$

and

$$\text{depth}(\Phi_{\epsilon}) \leq C \log(\epsilon)^{-2}, \quad \text{width}(\Phi_{\epsilon}) \leq C.$$  

We finally observe that for equidistant sensor points $x_0, x_1, x_2 \in [0, 2\pi], x_j = 2\pi j/3$, there exists a matrix $A \in \mathbb{R}^{2 \times 3}$, which for any function of the form $g(x) = \alpha_0 + \alpha_1 \sin(x) + \alpha_2 \cos(x)$ maps

$$[g(x_0), g(x_1), g(x_2)]^T \mapsto A \cdot [g(x_0), g(x_1), g(x_2)]^T = [-\alpha_1, \alpha_2]^T.$$  

Clearly, the considered initial data $\bar{u}(x) = -\sin(x - \xi)$ is of this form, for any $\xi \in [0, 2\pi]$, or more precisely, we have

$$\bar{u}(x) = -\sin(x - \xi) = -\cos(\xi) \sin(x) - \sin(\xi) \cos(x),$$

so that

$$A \cdot [\bar{u}(x_0), \bar{u}(x_1), \bar{u}(x_2)]^T = [\cos(\xi), \sin(\xi)].$$

As a next step, we recall that there exists $C > 0$, such that for any $\epsilon > 0$, there exists a neural network $\Xi_{\epsilon} : \mathbb{R}^2 \to [0, 2\pi]$ (cp. Lemma C.8), such that

$$\sup_{\xi \in [0, 2\pi - \epsilon]} |\xi - \Xi_{\epsilon}(\cos(\xi), \sin(\xi))| < \epsilon,$$

such that $\Xi_{\epsilon}(\cos(\xi), \sin(\xi)) \in [0, 2\pi]$ for all $\xi$, and

$$\text{depth}(\Xi_{\epsilon}) \leq C \log(\epsilon)^{-2}, \quad \text{width}(\Xi_{\epsilon}) \leq C.$$  

Based on this network $\Xi_{\epsilon}$, we can now define a shift-DeepONet approximation of $\mathcal{N}^{\text{DON}}(\bar{u}) \approx g_{\text{Burg}}(\bar{u})$ of size:

$$\text{depth}(\mathcal{N}^{\text{DON}}) \leq C \log(\epsilon)^{-2}, \quad \text{width}(\mathcal{N}^{\text{DON}}) \leq C,$$

by the composition

$$\mathcal{N}^{\text{DON}}(\bar{u})(x) := \Phi_{\epsilon}(x - \Xi_{\epsilon}(A \cdot \bar{u}(X))), \quad \text{(C.18)}$$

where $\bar{u}(X) := [\bar{u}(x_0), \bar{u}(x_1), \bar{u}(x_2)]^T$, and we note that (denoting $\Xi_{\epsilon} := \Xi_{\epsilon}(A \cdot \bar{u}(X))$, we have for $\xi \in [0, 2\pi - \epsilon]$:

$$\|g_{\text{Burg}}(\bar{u}) - \mathcal{N}^{\text{DON}}(\bar{u})\|_{L^1} = \|U(\cdot - \xi) - \Phi_{\epsilon}(\cdot - \Xi_{\epsilon})\|_{L^1}$$

$$\leq \|U(\cdot - \xi)\|_{L^1} + \|\Phi_{\epsilon}(\cdot - \Xi_{\epsilon})\|_{L^1}$$

$$\leq C|\xi - \Xi_{\epsilon}| + \epsilon$$

$$\leq (C + 1)\epsilon,$$

where $C$ only depends on $U$, and is independent of $\epsilon > 0$. On the other hand, for $\xi > 2\pi - \epsilon$, we have

$$\|g_{\text{Burg}}(\bar{u}) - \mathcal{N}^{\text{DON}}(\bar{u})\|_{L^1} = \|U(\cdot - \xi) - \Phi_{\epsilon}(\cdot - \Xi_{\epsilon})\|_{L^1}$$

$$\leq \|U(\cdot - \xi)\|_{L^1} + \|\Phi_{\epsilon}(\cdot - \Xi_{\epsilon})\|_{L^1}$$

$$\leq 2\pi \|U(\cdot - \xi)\|_{L^\infty} + \|\Phi_{\epsilon}(\cdot - \Xi_{\epsilon})\|_{L^\infty}$$

$$\leq 6\pi,$$

is uniformly bounded. It follows that

$$\mathbb{E}_{\bar{u} \sim \mu} \left[ \|g_{\text{Burg}}(\bar{u}) - \mathcal{N}^{\text{DON}}(\bar{u})\|_{L^1} \right] = \int_0^{2\pi - \epsilon} \int_0^{2\pi - \epsilon} \|g_{\text{Burg}}(\bar{u}) - \mathcal{N}^{\text{DON}}(\bar{u})\|_{L^1} d\xi$$

$$\leq 2\pi(C + 1)\epsilon + 6\pi \epsilon,$$

with a constant $C > 0$, independent of $\epsilon > 0$. Replacing $\epsilon$ by $\tilde{\epsilon} = \epsilon/C'$ for a sufficiently large constant $C' > 0$ (depending only on the constants in the last estimate above), one readily sees that there exists a shift-DeepONet $\mathcal{N}^{\text{DON}}$, such that

$$\mathbb{E}(\mathcal{N}^{\text{DON}}) = \mathbb{E}_{\bar{u} \sim \mu} \left[ \|g_{\text{Burg}}(\bar{u}) - \mathcal{N}^{\text{DON}}(\bar{u})\|_{L^1} \right] \leq \epsilon,$$

and such that

$$\text{width}(\mathcal{N}^{\text{DON}}) \leq C, \quad \text{depth}(\mathcal{N}^{\text{DON}}) \leq C \log(\epsilon)^{-2},$$

and $\text{size}(\mathcal{N}^{\text{DON}}) \leq C \log(\epsilon)^{-2}$, for a constant $C > 0$, independent of $\epsilon > 0$. This concludes our proof. □
C.6 Proof of Theorem 3.6

Proof. **Step 1:** Assume that the grid size is $N \geq 3$. Then there exists an FNO $\mathcal{N}_1$, such that

$$\mathcal{N}_1(\bar{u}) = [\cos(\xi), \sin(\xi)],$$

and $\text{depth}(\mathcal{N}_1) \leq C$, $d_v \leq C$, $k_{\max} = 1$.

To see this, we note that for any $\xi \in [0, 2\pi]$, the input function $\bar{u}(x) = -\sin(x - \xi) = -\cos(\xi)\sin(x) - \sin(\xi)\cos(x)$ can be written in terms of a sine/cosine expansion with coefficients $\cos(\xi)$ and $\sin(\xi)$. For $N \geq 3$ grid points, these coefficients can be retrieved exactly by a discrete Fourier transform. Therefore, combining a suitable lifting to $d_v = 2$ with a Fourier multiplier matrix $P$, we can exactly represent the mapping

$$\bar{u} \mapsto \mathcal{F}_N^{-1}(P \cdot \mathcal{F}_N(R \cdot \bar{u})) = \begin{bmatrix} \cos(\xi)\sin(x) \\ \sin(\xi)\cos(x) \end{bmatrix},$$

by a linear Fourier layer. Adding a suitable bias function $b(x) = [\sin(x), \cos(x)]^T$, and composing with an additional non-linear layer, it is then straightforward to check that there exists a (ReLU-)FNO, such that

$$\bar{u} \mapsto \begin{bmatrix} \cos(\xi)\sin(x) + \sin(x) \\ \sin(\xi)\cos(x) + \cos(x) \end{bmatrix} = \begin{bmatrix} (\cos(\xi) + 1)\sin(x) \\ (\sin(\xi) + 1)\cos(x) \end{bmatrix}$$

$$\mapsto \begin{bmatrix} |\cos(\xi) + 1| |\sin(x)| \\ |\sin(\xi) + 1| |\cos(x)| \end{bmatrix}$$

$$\mapsto \begin{bmatrix} |\cos(\xi) + 1|\sum_{j=1}^N |\sin(x_j)| \\ |\sin(\xi) + 1|\sum_{j=1}^N |\cos(x_j)| \end{bmatrix}$$

$$\mapsto \begin{bmatrix} |\cos(\xi) + 1| - 1 \\ |\sin(\xi) + 1| - 1 \end{bmatrix} = \begin{bmatrix} \cos(\xi) \\ \sin(\xi) \end{bmatrix}.$$

**Step 2:** Given this construction of $\mathcal{N}_1$, the remainder of the proof follows essentially the same argument as in the proof of Theorem 3.5. We again observe that the solution $u(x, t)$ with initial data $\bar{u}(x) = -\sin(x - \xi)$ is well approximated by the composition

$$\mathcal{N}^{\text{FNO}}(\bar{u})(x) := \Phi_e(x - \Xi_e(\mathcal{N}_1(\bar{u}))),$$

such that (by verbatim repetition of the calculations after (C.18) for shift-DeepONets)

$$\mathcal{E}(\mathcal{N}^{\text{FNO}}) = \mathbb{E}_{\bar{u} \sim \mu} \| \mathcal{G}_{\text{Burg}}(\bar{u}) - \mathcal{N}^{\text{FNO}}(\bar{u}) \|_{L^1} \leq \epsilon,$$

and where $\Phi_e : \mathbb{R} \to \mathbb{R}$ is a ReLU neural network of width

$$\text{depth}(\Phi_e) \leq C \log(\epsilon^{-1})^2, \quad \text{width}(\Phi_e) \leq C,$$

and $\Xi_e : \mathbb{R}^2 \to [0, 2\pi]$ is an ReLU network with

$$\text{depth}(\Xi_e) \leq C \log(\epsilon^{-1})^2, \quad \text{width}(\Xi_e) \leq C.$$

Being the composition of a FNO $\mathcal{N}_1$ satisfying $k_{\max} = 1$, $d_v \leq C$, $\text{depth}(\mathcal{N}_1) \leq C$ with the two ordinary neural networks $\Phi_e$ and $\Xi_e$, it follows that $\mathcal{N}^{\text{FNO}}$ can itself be represented by a FNO with $k_{\max} = 1$, $d_v \leq C$ and $\text{depth}(\mathcal{N}^{\text{FNO}}) \leq C \log(\epsilon^{-1})^2$. By the general complexity estimate (B.2),

$$\text{size}(\mathcal{N}^{\text{FNO}}) \lesssim d_v^2 k_{\max} \text{depth}(\mathcal{N}^{\text{FNO}}),$$

we also obtain the claimed an upper complexity bound $\text{size}(\mathcal{N}^{\text{FNO}}) \leq C \log(\epsilon^{-1})^2$. \[ \square \]

D Details of Numerical Experiments and Further Experimental Results.

D.1 Training and Architecture Details

Below, details concerning the model architectures and training are discussed.
D.1.1 Feed Forward Dense Neural Networks

Given an input \( y \in \mathbb{R}^n \), a feedforward neural network (also termed as a multi-layer perceptron), transforms it to an output, through a layer of units (neurons) which compose of either affine-linear maps between units (in successive layers) or scalar non-linear activation functions within units [Goodfellow et al. (2016)], resulting in the representation,

\[
u_\theta(y) = C_L \circ \sigma \circ C_{L-1} \circ \ldots \circ \sigma \circ C_2 \circ \sigma \circ C_1(y).
\]

(D.1)

Here, \( \circ \) refers to the composition of functions and \( \sigma \) is a scalar (non-linear) activation function. For any \( 1 \leq \ell \leq L \), we define

\[
C_\ell z_\ell = W_\ell z_\ell + b_\ell, \quad \text{for } W_\ell \in \mathbb{R}^{d_\ell \times d_\ell}, \quad z_\ell \in \mathbb{R}^{d_\ell}, \quad b_\ell \in \mathbb{R}^{d_\ell+1},
\]

and denote,

\[
\theta = \{W_\ell, b_\ell\}_{\ell=1}^L,
\]

(D.3)

to be the concatenated set of (tunable) weights for the network. Thus in the terminology of machine learning, a feed forward neural network (D.1) consists of an input layer, an output layer, and \( L \) hidden layers with \( d_\ell \) neurons, \( 1 < \ell < L \). In all numerical experiments, we consider a uniform number of neurons across all the layer of the network \( d_\ell = d_{\ell-1} = d \), \( 1 < \ell < L \). The number of layers \( L \), neurons \( d \) and the activation function \( \sigma \) are chosen though cross-validation.

D.1.2 ResNet

A residual neural network consists of residual blocks which use skip or shortcut connections to facilitate the training procedure of deep networks [He et al. (2016)]. A residual block spanning \( k \) layers is defined as follows,

\[
r(z_\ell, z_{\ell-k}) = \sigma(W_\ell z_\ell + b_\ell) + z_{\ell-k}.
\]

(D.4)

In all numerical experiments we set \( k = 2 \).

The residual network takes as input a sample function \( \overline{u} \in \mathcal{X} \), encoded at the Cartesian grid points \((x_1, \ldots, x_m)\), \( \mathcal{E}(\overline{u}) = (\overline{u}(x_1), \ldots, \overline{u}(x_m)) \in \mathbb{R}^m \), and outputs the output sample \( \mathcal{G}(\overline{u}) \in \mathcal{Y} \) encoded at the same set of points, \( \mathcal{E}(\mathcal{G}(\overline{u})) = (\mathcal{G}(\overline{u})(x_1), \ldots, \mathcal{G}(\overline{u})(x_m)) \in \mathbb{R}^m \). For the compressible Euler equations the encoded input is defined as

\[
\mathcal{E}(\overline{u}) = \left( \rho_0(x_1), \ldots, \rho_0(x_m), \rho_0(x_1)u_0(x_1), \ldots, \rho_0(x_m)u_0(x_m), E_0(x_1), \ldots, E_0(x_m) \right) \in \mathbb{R}^{3m}
\]

\[
\mathcal{E}(\overline{u}) = \left( \rho_0(x_1), \ldots, \rho_0(x_m), \rho_0(x_1)v_0(x_1), \ldots, \rho_0(x_m)v_0(x_m), \rho_0(x_1)\rho_0(x_1)u_0(x_1), \ldots, \rho_0(x_m)\rho_0(x_m)u_0(x_m) \right) \in \mathbb{R}^{4m^2}
\]

for the 1d and 2d problem, respectively.

D.1.3 Fully Convolutional Neural Network

Fully convolutional neural networks are a special class of convolutional networks which are independent of the input resolution. The networks consist of an encoder and decoder, both defined by a composition of linear and non-linear transformations:

\[
E_{\theta_e}(y) = C^e_L \circ \sigma \circ C^e_{L-1} \circ \ldots \circ \sigma \circ C^e_2 \circ \sigma \circ C^e_1(y), \quad D_{\theta_d}(z) = C^d_L \circ \sigma \circ C^d_{L-1} \circ \ldots \circ \sigma \circ C^d_2 \circ \sigma \circ C^d_1(z),
\]

\[
u_\theta(y) = D_{\theta_d} \circ E_{\theta_e}(y).
\]

(D.6)

The affine transformation \( C_\ell \) commonly corresponds to a convolution operation in the encoder, and transposed convolution (also know as deconvolution), in the decoder. The latter can also be performed with a simple linear (or bilinear) upsampling and a convolution operation, similar to the encoder.

The (de)convolution is performed with a kernel \( W_\ell \in \mathbb{R}^{k_\ell} \) (for 1d-problems, and \( W_\ell \in \mathbb{R}^{k_\ell \times k_\ell} \) for 2d-problems), stride \( s \) and padding \( p \). It takes as input a tensor \( z_\ell \in \mathbb{R}^{w_\ell \times h_\ell} \) (for 1d-problems, and \( z_\ell \in \mathbb{R}^{w_\ell \times h_\ell \times c_\ell} \) for 2d-problems), with \( c_\ell \) being the number of input channels, and computes \( z_{\ell+1} \in \mathbb{R}^{w_{\ell+1} \times h_{\ell+1}} \) (for 1d-problems, and \( z_{\ell+1} \in \mathbb{R}^{w_{\ell+1} \times h_{\ell+1} \times c_{\ell+1}} \) for 2d-problems). Therefore, a (de)convolutional affine transformation can be uniquely identified with the tuple \((k_\ell, s, p, c_\ell, c_{\ell+1})\).

The main difference between the encoder’s and decoder’s transformation is that, for the encoder \( h_{\ell+1} < h_\ell, w_{\ell+1} < w_\ell, c_{\ell+1} > c_\ell \) and for the decoder \( h_{\ell+1} > h_\ell, w_{\ell+1} > w_\ell, c_{\ell+1} < c_\ell \).

For the linear advection equation and the Burgers’ equation we employ the same variable encoding of the input and output samples as ResNet. On the other hand, for the compressible Euler equations, each input variable is embedded in
where blocks, each defined as:

On the other hand, for the compressible Euler equation we use a convolutional network obtained as a composition of

\[ C \circ BN \circ \sigma \circ C \circ B \circ \sigma \circ C \circ B \circ \sigma \circ C \circ B(z) \]

In the experiments, the number of channel

an individual channel. More precisely, \( E(\bar{u}) \in \mathbb{R}^{m \times 3} \) for the shock-tube problem, and \( E(\bar{u}) \in \mathbb{R}^{m \times 4} \) for the 2d Riemann problem. The architectures used in the benchmarks examples are shown in figures 4, 5, 6.

In the experiments, the number of channel \( c \) (see figures 4, 5, 6 for an explanation of its meaning) and the activation function \( \sigma \) are selected with cross-validation.

**D.1.4 DeepONet and shift-DeepONet**

The architectures of branch and trunk are chosen according to the benchmark addressed. In particular, for the first two numerical experiments, we employ standard feed-forward neural networks for both branch and trunk-net, with a skip connection between the first and the last hidden layer in the branch.

On the other hand, for the compressible Euler equation we use a convolutional network obtained as a composition of \( L \) blocks, each defined as:

\[ B(z_\ell) = BN \circ \sigma \circ C_\ell(z_\ell), \quad 1 < \ell < L, \]

\[ (D) \]
We use the implementation of the FNO model provided by the authors of Li et al. (2021a). Specifically, the lifting $R$ with $BN$ denotes a batch normalization. The convolution operation is instead defined by $k_{12} = 3, s = 2, p = 1$, $c_{12} = c_{12} = d_{12}$, for all $1 \leq \ell \leq L - 1$. The output is then flattened and forwarded through a multi layer perceptron with 2 layers and 256 neurons and activation function $\sigma$.

For the $shift$ and $scale$-nets of shift-DeepONet, we use the same architecture as the branch.

Differently from the rest of the models, the training samples for DeepONet and shift-DeepONet are encoded at $m$ and $n$ uniformly distributed random points, respectively. Specifically, the encoding points represent a randomly picked subset of the grid points used for the other models. The number of encoding points $m$ and $n$, together with the number of layers $L$, units $d$ and activation function of trunk and branch-nets, are chosen through cross-validation.

### D.1.5 Fourier Neural Operator

We use the implementation of the FNO model provided by the authors of Li et al. (2021a). Specifically, the lifting $R$ is defined by a linear transformation from $\mathbb{R}^{d_v \times d_m}$ to $\mathbb{R}^{d_v \times d_m}$, where $d_m$ is the number of inputs, and the projection $Q$ to the target space is performed by a neural network with a single hidden layer with 128 neurons and GeLU activation function. The same activation function is used for all the Fourier layers, as well. Moreover, the weight matrix $W_{\ell}$ used in the residual connection derives from a convolutional layer defined by $(k_{12} = 1, s = 1, p = 0, c_{12} = d_v, c_{12} = d_v)$, for all $1 \leq \ell \leq L - 1$. We use the same samples encoding employed for the fully convolutional models. The lifting dimension $d_v$, the number of Fourier layers $L$ and $k_{\text{max}}$, defined in $\mathbb{E}$ are the only objectives of cross-validation.

### D.1.6 Training Details

For all the benchmarks, a training set with 1024 samples, and a validation and test set each with 128 samples, are used. The training is performed with the ADAM optimizer, with learning rate $5 \cdot 10^{-4}$ for 10000 epochs and minimizing the $L^1$-loss function. We use the learning rate schedulers defined in Table 2. We train the models in mini-batches of size 10. A weight decay of $10^{-6}$ is used for ResNet (all numerical experiments), DON and sDON (linear advection equation, Burgers’ equation, and shock-tube problem). On the other hand, no weight decay is employed for remaining experiments and models. At every epoch the relative $L^1$-error is computed on the validation set, and the set of trainable parameters resulting in the lowest error during the entire process saved for testing. Therefore, no early stopping is used. The models hyperparameters are selected by running grid searches over a range of hyperparameter values and selecting the configuration realizing the lowest relative $L^1$-error on the validation set. For instance, the model size (minimum and maximum number of trainable parameters) that are covered in this grid search are reported in Table 3.

The results of the grid search i.e., the best performing hyperparameter configurations for each model and each benchmark, are reported in tables 4, 5, 6, 7, and 8.

### D.2 Further Experimental Results.

In this section, we present some further experimental results which supplement the results presented in Table 1 of the main text. We start by presenting more statistical information about the median errors shown in Table 1. To this end, in Table 9 we show the errors, for each model on each benchmark, corresponding to the 0.25 and 0.75 quantiles, within the test set. We observe from this table that the same trend, as seen in Table 1 also holds for the statistical spread. In

| Benchmark | Learning Rate Sched. | Model | Architecture | Hyperparameters |
|-----------|----------------------|-------|--------------|----------------|
| Advection Equation | Step-wise decay 100 steps, $\gamma = 0.999$ | ResNet | Step-wise loss 50 steps, $\gamma = 0.999$ | DeepONet | Exponential decay $\gamma = 0.999$ |
| Burgers’ Equation | Step-wise decay 100 steps, $\gamma = 0.999$ | BN | Step-wise decay 50 steps, $\gamma = 0.999$ | Shift - DeepONet | None |
| Shock Tube | Step-wise decay 100 steps, $\gamma = 0.999$ | L | Step-wise decay 50 steps, $\gamma = 0.999$ | FNO | None |

Table 2: Learning rate scheduler for different benchmarks and different models: $\gamma$ denotes the learning rate decay factor.
Table 3: Minimum (Top sub-row) and maximum (Bottom sub-row) number of trainable parameters among the grid-search hyperparameters configurations.

| Problem        | Advection Equation | Burgers' Equation | Shocktube Problem | 2D Riemann Problem |
|----------------|--------------------|-------------------|-------------------|--------------------|
| ResNet Params  | Trainable          | Trainable         | Trainable         | Trainable          |
|                | 576,768            | 1,156,449         | 519,781           | 1,018,825          | 22,945             |
|                | 1,515,008          | 1,8240,545        | 892,561           | 1,835,297          | 352,961             |

Table 4: ResNet best performing hyperparameters configuration for different benchmark problems.

| Problem        | Advection Equation | Burgers' Equation | Shocktube Problem | 2D Riemann Problem |
|----------------|--------------------|-------------------|-------------------|--------------------|
|                | Trainable          | Trainable         | Trainable         | Trainable          |
|                | Leaky ReLU         | Leaky ReLU        | Leaky ReLU        | Leaky ReLU         |
| σ              | 4                  | 8                 | 8                 | 8                  |
| L              | 128                | 256               | 256               | 256                |

Table 5: Fully convolutional neural network best performing hyperparameters configuration for different benchmark problems.

| Problem        | Advection Equation | Burgers' Equation | Shocktube Problem | 2D Riemann Problem |
|----------------|--------------------|-------------------|-------------------|--------------------|
|                | Trainable          | Trainable         | Trainable         | Trainable          |
|                | Leaky ReLU         | Leaky ReLU        | Leaky ReLU        | ReLU               |
| σ              | 8                  | 16                | 16                | 16                 |
| L              | 256                | 512               | 512               | 64^2               |
| n              | 512                | 256               | 256               | 256^2              |
| p              | 200                | 50                | 200               | 100                |
| L_{branch}     | 3                  | 3                 | 3                 | 3                  |
| L_{trunk}      | 6                  | 4                 | 6                 | 6                  |
| d_{branch}     | 256                | 256               | 256               | 32                 |
| d_{trunk}      | 256                | 256               | 256               | 256                |
| σ_{branch}     | Leaky ReLU         | Leaky ReLU        | Leaky ReLU        | SoftSign           |
| σ_{trunk}      | Leaky ReLU         | Leaky ReLU        | Leaky ReLU        | Leaky ReLU         |
| Trainable Params | 761,233            | 618,085           | 3,190,673         | 607,433            |

Table 6: DeepONet best performing hyperparameters configuration for different benchmark problems.
particular, FNO and shift-DeepONet outperform DeepONet and the other baselines on every experiment. Similarly, FNO outperforms shift-DeepONet handily on each experiment, except the four-quadrant Riemann problem associated with the Euler equations of gas dynamics.

|              | ResNet | FCNN  | DeepONet | Shift - DeepONet | FNO    |
|--------------|--------|-------|----------|------------------|--------|
| Linear Advection Equation | 10.1% − 22.8% | 8.2% − 17.3% | 4.9% − 13.8% | 1.4% − 5.4% | 0.35% − 1.25% |
| Burgers’ Equation      | 18.8% − 22%   | 20.3% − 25.7% | 25.4% − 32.4% | 6.7% − 9.6% | 1.3% − 1.9%  |
| Lax-Sod Shock Tube     | 3.6% − 5.6%   | 7.0% − 10.25% | 3.4% − 5.4%  | 2.0% − 3.75% | 1.2% − 2.1%  |
| 2D Riemann Problem     | 2.4% − 2.9%   | 0.17% − 0.21% | 0.77% − 1.1%  | 0.10% − 0.15% | 0.10% − 0.14% |

Table 9: 0.25 and 0.75 quantile of the relative $L^1$ error computed over 128 testing samples for different benchmarks with different models.
Nonlinear Reconstruction for Operator Learning of PDEs with Discontinuities

Table 10: Testing error obtained by training FNO with different number of modes and corresponding error obtained with linear Fourier projection.

| $k_{\text{max}}$ | $L$ | $d_v$ | Trainable Params | FNO - Median $L^1$-error | FFT - Median $L^1$-error |
|------------------|-----|-------|------------------|--------------------------|-------------------------|
| 0                | 3   | 192   | 247,169          | 2.30%                    | 164.2%                  |
| 1                | 3   | 160   | 252,097          | 1.21%                    | 137.9%                  |
| 3                | 3   | 128   | 263,169          | 0.48%                    | 63.3%                   |
| 7                | 3   | 92    | 241,113          | 0.54%                    | 38.9%                   |

This experiment clearly brings out the role of the nonlinearities in FNO in enhancing its expressive power on advection-dominated problems.

Figure 7: Illustration of two input (blue) and output (orange) samples for the advection equation.

a) Sample 1

a) Sample 2
Figure 8: Illustration of two input (blue) and output (orange) samples for the Burgers’ equation.
Figure 9: Illustration of two input (blue) and output (orange) samples for the shock-tube problem.

(a) Sample 1

(b) Sample 2
Figure 10: Illustration of two input (left) and output (right) samples for the 2-dimensional Riemann problem.

(a) Sample 1

(b) Sample 2