Existence of good divisors on Mukai manifolds

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Introduction

A normal projective variety $X$ is called Fano if a multiple of the anticanonical Weil divisor, $-K_X$, is an ample Cartier divisor. The importance of Fano varieties is twofold, from one side they give, has predicted by Fano [F a], examples of non rational varieties having plurigenera and irregularity all zero (cfr [Is]); on the other hand they should be the building block of uniruled variety, indeed recently, Minimal Model Theory predicted that every uniruled variety is birational to a fiber space whose general fiber is a Fano variety with terminal singularities.

The index of a Fano variety $X$ is the number

$$i(X) := \sup \{ t \in \mathbb{Q} : -K_X \equiv tH, \text{for some ample Cartier divisor } H \}.$$ 

It is known that $0 < i(X) \leq \dim X + 1$ and if $i(X) \geq \dim X$ then $X$ is either an hyperquadric or a projective space by the Kobayashi–Ochiai criterion, smooth Fano n-folds of index $i(X) = n - 1$, del Pezzo n-folds, have been classified by Fujita [Fu] and terminal Fano n-folds of index $i(X) > n - 2$ have been independently classified by Campana–Flenner [CF] and Sano [Sa].

If $X$ has log terminal singularities, then $Pic(X)$ is torsion free and therefore, the $H$ satisfying $-K_X \equiv i(X)H$ is uniquely determined and is called the fundamental divisor of $X$. Mukai announced, [Mu], the classification of smooth Fano n-folds $X$ of index $i(X) = n - 2$, under the assumption that the linear system $|H|$ contains a smooth divisor.

The main result of this paper is the following

**Theorem 1** Let $X$ be a smooth Fano n-fold of index $i(X) = n - 2$. Then the general element in the fundamental divisor is smooth.

Therefore the result of Mukai [Mu] provide a complete classification of smooth Fano n-folds of index $i(X) = n - 2$, Mukai manifolds.

The ancestors of the theorem, and indeed the lighthouses that directed its proof, are Shokurov’s proof for smooth Fano 3-folds, [Sh] and Reid’s extension to canonical Gorenstein 3-folds using the Kawamata’s base point free technique.
This technique was then applied by Wilson in the case of smooth Fano 4-folds of index 2, \[\text{[Re]}\]. Afterwards Alexeev, \[\text{[Al]}\] did it for log terminal Fano n-folds of index \(i(X) > n - 2\) and recently Prokhorov used it to prove Theorem 1 in dimension 4 and 5, \[\text{[Pr1]} \quad \text{[Pr2]} \quad \text{[Pr3]}\]. As in Reid’s construction we will first prove the existence of a section with canonical singularities. To do this we will use Kawamata’s base point free technique and Kawamata’s notion of centers of log canonical singularities, \[\text{[Ka1]}\] and his subadjunction formula for codimension 2 minimal centers \[\text{[Ka2]}\]. These tools, together with Helmke’s inductive procedure, \[\text{[He]}\], allows to replace difficult non vanishing arguments by a simple Riemann–Roch calculation. Finally the Theorem is proved by an inductive argument that lowers the dimension of \(X\).

A natural extension of this problem, motivated by the Minimal Model Program, should be to ask if for a terminal Fano \(X\) of index \(n - 2\), with fundamental divisor \(H\), it is true that the general element in \(|H|\) has terminal singularities. A first, small, step in this direction is the following.

**Theorem 2** Let \(X\) be a terminal Gorenstein Fano n-fold of index \(n-2\). Then the general element in the fundamental divisor \(H\) has canonical singularities.

While working on this subject I had several discussions with M. Andreatto, who suggested me the direction in which this problem could be tackled, I would like to express him my deep gratitude, I would also like to thank A. Corti for valuable comments.

## 1 Preliminary results

We use the standard notation from algebraic geometry. In particular it is compatible with that of \[\text{[KMM]}\] to which we refer constantly, everything is defined over \(\mathbb{C}\).

A \(\mathbb{Q}\)-divisor \(D\) is an element in \(\mathbb{Z}_{n-1}(X) \times \mathbb{Q}\), that is a finite formal sum of prime divisors with rational coefficients; \(D\) is called \(\mathbb{Q}\)-Cartier if there is an integer \(m\) such that \(mD \in \text{Div}(X)\), where \(\text{Div}(X)\) is the group of Cartier divisors of \(X\). In the following \(\equiv\) (respectively \(\sim\), \(\sim_{\mathbb{Q}}\)) will indicate numerical (respectively linear, \(\mathbb{Q}\)-linear) equivalence of divisors. Let \(\mu : Y \to X\) a birational morphism of normal varieties. If \(D\) is a \(\mathbb{Q}\)-divisor (\(\mathbb{Q}\)-Cartier) then is well defined the strict transform \(\mu^{-1}_* D\) (the pull back \(\mu^* D\)). For a pair \((X, D)\) of a variety \(X\) and a \(\mathbb{Q}\)-divisor \(D\), a log resolution is a proper birational morphism \(\mu : Y \to X\) from a smooth \(Y\) such that the union of the support of \(\mu^{-1}_* D\) and of the exceptional locus is a normal crossing divisor.

**Definition 1.1** Let \(X\) be a normal variety and \(D = \sum d_i D_i\) an effective \(\mathbb{Q}\)-divisor such that \(K_X + D\) is \(\mathbb{Q}\)-Cartier. If \(\mu : Y \to X\) is a log resolution of the
pair \((X, D)\), then we can write

\[ K_Y + \mu^{-1}_* D = \mu^*(K_X + D) + F \]

with \(F = \sum_j e_j E_j\) for the exceptional divisors \(E_j\). We call \(e_j \in \mathbb{Q}\) the discrepancy coefficient for \(E_j\), and regard \(-d_i\) as the discrepancy coefficient for \(D_i\).

The pair \((X, D)\) is said to have log canonical (LC) (respectively purely log terminal (pLT), Kawamata log terminal (KLT)) singularities if \(d_i \leq 1\) (resp. \(d_i \leq 1, d_i < 1\)) and \(e_j \geq -1\) (resp. \(e_j > -1, e_j > -1\)) for any \(i, j\) of a log resolution \(\mu : Y \to X\). In particular if \(X\) is smooth at the generic point of \(Z\), with \(\text{cod}_X Z = a\) and \(D\) is a Weil divisor with \(\text{mult}_Z D = r\), then \((X, \gamma D)\) is LC for some \(\gamma \leq a/r\).

**Definition 1.2** A log-Fano variety is a pair \((X, \Delta)\) with KLT singularities and such that for some positive integer \(m\), \(m(K_X + \Delta)\) is an ample Cartier divisor.

The index of a log-Fano variety \(i(X, \Delta) := \sup \{ t \in \mathbb{Q} : -(K_X + \Delta) \equiv tH \text{ for some ample Cartier divisor } H \}\) and the \(H\) satisfying \(-(K_X + \Delta) \equiv i(X, \Delta)H\) is called fundamental divisor. In case \(\Delta = 0\) we have log terminal Fano variety.

We will start recalling some results on log Fano varieties, essentially due to the Kawamata–Viehweg vanishing theorem.

**Lemma 1.3** ([Al]) Let \((X, \Delta)\) be a log-Fano \(n\)-fold of index \(r\), \(H\) the fundamental divisor in \(X\) and \(H^n = d\). Then

- If \(r > n - 2\) then \(\dim |H| = n - 2 + d(r - n + 3)/2 > 0\)
- If \(r = n - 2\) and \(X\) has canonical Gorenstein singularities, then \(\dim |H| = g + n - 2 \geq n\), where \(2g - 2 = d, g \in \mathbb{Z}, g \geq 2\).

Let us recall the notion and properties of minimal center of log canonical singularities as introduced in [Ka1]

**Definition 1.4** ([Ka1]) Let \(X\) be a normal variety and \(D = \sum d_i D_i\) an effective \(\mathbb{Q}\)-divisor such that \(K_X + D\) is \(\mathbb{Q}\)-Cartier. A subvariety \(W\) of \(X\) is said to be a center of log canonical singularities for the pair \((X, D)\), if there is a birational morphism from a normal variety \(\mu : Y \to X\) and a prime divisor \(E\) on \(Y\) with the discrepancy coefficient \(e \leq -1\) such that \(\mu(E) = W\). The set of all the centers of log canonical singularities is denoted by \(\text{CLC}(X, D)\), for a point \(x \in X\), define \(\text{CLC}(X, x, D) := \{ W \in \text{CLC}(X, D) : x \in W \}\). The union of all the subvarieties in \(\text{CLC}(X, D)\) is denoted by \(\text{LLC}(X, D)\).

**Theorem 1.5** ([Ka1]) Let \(X\) be a normal variety and \(D\) an effective \(\mathbb{Q}\)-Cartier divisor such that \(K_X + D\) is \(\mathbb{Q}\)-Cartier. Assume that \(X\) is KLT and \((X, D)\) is LC.
i) If $W_1, W_2 \in \text{CLC}(X, D)$ and $W$ is an irreducible component of $W_1 \cap W_2$, then $W \in \text{CLC}(X, D)$. In particular, if $(X, D)$ is not KLT at a point $x \in X$ then there exists a unique minimal element of $\text{CLC}(X, x, D)$.

ii) If $W \in \text{CLC}(X, D)$ is a minimal center then $W$ is normal.

iii) Assume that $D \equiv cL$, with $c < 1$, for some ample Cartier divisor $L$. If $\{x\} \in \text{CLC}(X, D)$ is a minimal center then there is a section of $K_X + L$ not vanishing at $x$.

**Theorem 1.6 ([Ka2])** Let $X$ be a normal variety which has only KLT singularities, $D$ and effective $\mathbb{Q}$-Cartier divisor such that $(X, D)$ is LC, and $W$ a minimal element of $\text{CLC}(X, D)$. Assume that $\text{cod}W = 2$. Then there exist canonically determined effective $\mathbb{Q}$-divisors $M_W$ and $D_W$ on $W$ such that $(K_X + D)_|W \sim Q K_W + M_W + D_W$. If $X$ is affine then there exists an effective $\mathbb{Q}$-divisor $M'_W$ such that $M'_W \sim D_W \sim O_W$ and the pair $(W, M'_W + D_W)$ is KLT.

**Remark** Note that in particular a cod2 minimal center has rational singularities and if $M_W + D_W \equiv 0$, then $W$ is KLT. In fact it is enough to choose an open affine covering $\{U_i\}$ of $X$, then $V_i = W \cap U_i$ we have $(V_i, M'_i + D_{V_i})$ is KLT and therefore $V_i$ has rational singularities, [KMM]. Furthermore if $M_W + D_W \equiv 0$ then $M_W \sim D_W \sim O_W$ and therefore $M_{V_i} \sim O_{V_i}$ and these glue together to give that globally $W$ is KLT.

**Definition 1.7 ([He])** Let $X$ be smooth at $x$ and $(X, D)$ be log canonical at $x$, let $\pi : \tilde{X} \to X$ the blow up of $x$. Following Helmke, the local discrepancy of $(X, D)$ is the rational number

$$b_x(X, D) = \inf \left\{ t \left| \begin{array}{l} \text{There is a center of log canonical singularity} \\ \text{of } (\tilde{X}, \pi^*D - (n-1)E + tE) \text{ contained in } E \end{array} \right. \right\}$$

**Claim** Let $(X, D)$ be LC and $Z \in \text{CLC}(X, D)$ with $x \in Z$ and $Z$, $X$ smooth at $x$, then $b = b_x(X, D) \leq \dim Z$.

**proof of the claim** Let $\pi : Y \to X$ the blow up of $x$, with exceptional divisor $E$ and $Z' = \pi^{-1}Z \cap E$ since $Z$ is a center of log canonical singularities for $(X, D)$ then $\pi^*D$ has multiplicity at least $2\text{cod}Z$ along $Z'$. Therefore by definition

$$b \leq -(2\text{cod}Z - n + 1) + \text{cod}Z' = \dim Z$$

\[ \diamond \]

The following inductive procedure due to Helmke (this is a particular case of his more general Theorem) allows us to decrease the dimension of a minimal center.
Proposition 1.8 (He) Let $L$ an ample divisor on $X$ and $D$ an effective $\mathbb{Q}$-divisor with $D \equiv \gamma L$ for some rational number $0 \leq \gamma < 1$. Assume that $X$ is smooth at $x$ and $(X, D)$ is log canonical with local discrepancy $b = b_x(X, D)$ at $x$. Let $Z$ be the minimal center of $\text{CLC}(X, x, D)$ assume that $d = \dim Z > 0$ and $Z$, $X$ smooth at $x$. If $L^d \cdot Z > p^d$, where $p = \frac{b}{1-\gamma}$, (1.8.1)
then there is a $\mathbb{Q}$-divisor $D_1 \equiv \gamma_1 L$, with $\gamma < \gamma_1 < 1$ such that $(X, D_1)$ is log canonical at $x$ with minimal center $Z_1$ properly contained in $Z$ and $p_1 = \frac{b_1}{1-\gamma_1} < p$ where $b_1 = b_x(X, D_1)$.

We will sketch the proof for reader’s convenience.

Step 1. Produce a section $D_0 \in |kL|_Z|$, for $k \gg 0$, with $\text{mult}_x D_0 > pk$. This is accomplished by R–R theorem using inequality (1.8.1).

Step 2. Using Serre’s vanishing and Bertini Theorem extend $D_0$ to a section $D' \in |kL|$ which is smooth away from $Z$. Let $\gamma' = \sup \{t|(X, D+tD')is \text{ LC at } x\}$, then $D_1 = D + \gamma' D' \equiv \gamma_1 L$.

Step 3. Use the definition of $p$ and the minimality of $Z$ to prove that $\gamma_1 < 1$, and then a straightforward computation gives the assert.

2 Existence of a canonical section

For this step we will use Kawamata’s base point free technique, as explained in Reid [Re]. Let us start with some lemmas.

Lemma 2.1 Let $X$ be a log terminal Fano $n$-fold, with $n \geq 3$ and $H$ an ample Cartier divisor with $-K_X \equiv (n-2)H$ and $G$ a $\mathbb{Q}$-Cartier divisor with $(X, G)$ LC. Assume that $Z \in \text{CLC}(X, G)$ is a minimal center and $G \equiv \gamma H$, with $\gamma < \text{cod}Z-1$ then there is a section of $H$ not vanishing identically on $Z$.

Proof. We proceed as in [Ka1, Prop 2.3]. Let $M \in |mH|$, for $m \gg 0$, be a general member among Cartier divisors containing $Z$, let $G_1 = (1-\epsilon_1)(G+\epsilon_2 M)$, for $\epsilon_i \ll 1/m$, then $G_1 \equiv \gamma_1 H$, with $\gamma_1 < \text{cod}Z-1$. Furthermore we may assume that $(X, G_1)$ is LC and $Z$ is an isolated element of $\text{LLC}(X, G_1)$. Let $\mu : Y \to X$ a log resolution of $(X, G_1)$, then $K_Y + E - A + F = \mu^*(K_X + G_1)$, where $\mu(E) = Z$, $A$ is an integral $\mu$-exceptional divisor and $|F| = 0$. Let $N(t) := -K_Y - E - F + A + \mu^*(tH)$,
then \( N(t) \equiv \mu^*(t + (n-2) - \gamma_1)H \) and \( N(t) \) is nef and big whenever \( t + (n-2) - \gamma_1 > 0 \), hence by hypothesis this is true whenever \( t \geq -n + 1 + \text{cod}Z \). Thus K–V vanishing yields

\[
H^i(Y, \mu^*(tH) - E + A) = 0 \quad H^i(E, (\mu^*(tH) + A)_{|E_0}) = 0 \tag{2.1.2}
\]

for \( i > 0 \) and \( t \geq -n + 1 + \text{cod}Z \), and consequently

\[
H^0(Y, \mu^*H + A) \to H^0(E, \mu^*H + A) \to 0;
\]

since \( A \) is effective and \( \mu \)-exceptional, then any section in \( H^0(Y, \mu^*H + A) \), not vanishing on \( E \), pushes forward to give a section of \( H \) not vanishing on \( Z \). To conclude the proof it is, therefore, enough to prove that \( h^0(E, \mu^*H + A) = 0 \).

Let \( p(t) = \chi(E, N(t)) \), then by equation (2.1.3), \( p(0) \geq 0 \) and \( p(t) = 0 \) for \( 0 > t \geq -n + 1 + \text{cod}Z = -\text{dim}Z + 1 \). Since \( \deg p(t) = \text{dim}Z \) and \( p(t) > 0 \) for \( t \geq 0 \) then \( h^0(E, N(1)) = p(1) > 0 \).

The above lemma allows us, essentially, to treat minimal centers of codimension \( \geq 3 \). In the next couple of lemmas we will treat codimension 2 minimal centers.

**Lemma 2.2** Let \( X \) be a log terminal Fano \( n \)-fold, with \( n \geq 3 \), and \( H \) an ample Cartier divisor with \( -K_X \equiv (n-2)H \), let \( L \sim (n-1)H \) and \( D \) a \( \mathbb{Q} \)-Cartier divisor with \( (X, D) \) LC. Assume that \( D \equiv H \) and \( Z \in \text{CLC}(X, D) \) a cod 2 minimal center, then for \( k \gg 0 \) and \( \delta \geq 0 \)

\[
h^0(Z, KL_{|Z}) \geq \frac{(n-1)^{n-2}}{(n-2)!}k^{n-2} + \frac{(n-3+\delta)(n-1)^{n-3}}{2(n-3)!}k^{n-3} + \text{lower terms in } k.
\]

Furthermore, keeping the notation of Theorem 1.6, if \( M_Z + D_Z \not\equiv 0 \) then \( \delta > 0 \).

**Proof.** By Theorem 1.6 there are effective \( \mathbb{Q} \)-divisors \( M_Z \) and \( D_Z \) such that

\[-(n-3)H_{|Z} \equiv (K_X + D)_{|Z} \sim_{\mathbb{Q}} K_Z + M_Z + D_Z.
\]

Let \( f : Y \to Z \) a log resolution of \((Z, M_Z + D_Z)\) then

\[K_Y + \Delta = f^*(K_Z + M_Z + D_Z) + \sum e_iE_i,
\]

where \( \Delta = f_*^{-1}(M_Z + D_Z) \) is effective and the \( E_i \) are \( f \)-exceptional. In particular

\[-K_Y \cdot (f^*H_{|Z})^{n-3} = -(K_X + D)_{|Z} \cdot H_{|Z}^{n-3} + \Delta \cdot f^*H_{|Z}^{n-3} \geq n - 3 + \delta
\]

Since \( Z \) has rational singularities and \( L \) is ample then, for \( k \gg 0 \),

\[
h^0(Z, kL_{|Z}) = \chi(Z, kL_{|Z}) = \chi(Y, kf^*L_{|Z}).
\]
Therefore by R–R formula
\[
h^0(Z, kL|_Z) \geq \frac{(n-1)^{n-2} - 2(n-2)!}{2(n-3)!} k^{n-3} + \text{lower terms in } k.
\]

Lemma 2.3 Let \( X \) be a log terminal Gorenstein Fano n-fold, with \( n \geq 3 \), and \( H \) an ample Cartier divisor with \( -K_X \sim (n-2)H \), let \( L \sim (n-1)H \) and \( D \) a \( \mathbb{Q} \)-Cartier divisor with \( (X, D) \) LC. Assume that \( D \equiv H \) and \( Z \in \text{CLC}(X, D) \) a cod 2 minimal center with \( Z \not\subset \text{Sing}(X) \). Then there exists a section of \( H \sim K + L \) not vanishing identically on \( Z \).

Proof. By Theorem 1.6 there are effective \( \mathbb{Q} \)-divisors \( M_Z \) and \( D_Z \) such that
\[
-(n-3)H|_Z \equiv (K_X + D)|_Z \sim Q K_Z + M_Z + D_Z.
\]

If \( n = 3 \) then \( Z \) is a smooth curve, by Theorem 1.5 and \( g(Z) \leq 0 \), thus \( h^0(Z, H) > 0 \); if \( n > 3 \) and \( (Z, M_Z + D_Z) \) is KLT then \( (Z, M_Z + D_Z) \) is a log-Fano variety of index \( i(Z, M_Z + D_Z) = \dim Z - 1 \), therefore by Lemma 1.3, \( h^0(Z, H) > 0 \). As in the proof of Lemma 2.1 let us replace \( D \) with \( D_1 \) such that \( (X, D_1) \) is LC, \( Z \) is isolated in \( \text{LLC}(X, D_1) \) and \( D_1 \equiv \gamma H \), for \( \gamma < 1 + \epsilon \), with \( \epsilon \ll 1 \). Let \( \mu : Y \to X \) a log resolution of \( (X, D_1) \) with \( K_Y + E - A + F = \mu^*(K_X + D_1) \), where \( f(E) = Z \), \( A \) is an integral \( \mu \)-exceptional divisor and \( |F| = 0 \). Let \( N(t) := -K_Y - E - F + A + \mu^*(tH) \), then \( N(1) \equiv \mu^*(1 + (n-2) - \gamma_1)H \) is nef and big and consequently
\[
H^0(Y, \mu^*H + A) \to H^0(E, \mu^*H + A) \to 0.
\]

Therefore the sections in \( H^0(Z, H) \) extends to sections of \( H^0(X, H) \) not vanishing identically on \( Z \).

By the remark after Theorem 1.7 we can, therefore assume that \( M_Z + D_Z \not\equiv 0 \). Fix a smooth point \( x \in Z \) outside of \( \text{Sing}(X) \), such that \( Z \) is the minimal element of \( \text{CLC}(X, x, D) \). Let us mimic Helmke’s arguments; in the notation of Proposition 1.8, \( \gamma = 1/(n-1) \) and
\[
p = \frac{b}{1-\gamma} \leq \frac{(n-1)(n-2)}{n-2} \leq n - 1.
\]

The first step is accomplished using Lemma 2.2; in fact
\[
h^0(Z, \mathcal{O}_Z/I_{Z,x}^k) = \frac{1}{(n-2)!} k^{n-2} + \frac{(n-3)}{2(n-3)!} k^{n-3} + \text{lower terms in } k,
\]
therefore by Lemma 2.2 there exists a section \( D' \in |kL| \), for \( k \gg 0 \), such that
\[
\text{mult}_x D' > pk.
\]
It is now enough to follow word by word Helmke’s arguments.
to conclude that there is a \( \mathbb{Q} \)-divisor \( D_1 \equiv \gamma_1 L \), with \( \gamma < \gamma_1 < 1 \) such that \( (X, D_1) \) is log canonical at \( x \), with minimal center \( Z_1 \ni x \) properly contained in \( Z \). Since \( x \in Z_1 \) and \( p_1 < p \leq n - 1 \) then \( Z_1 \not\subseteq \text{Sing}(X) \) and we can choose a smooth point \( x_1 \in Z_1 \) and apply directly Proposition 1.8 to \( (X, D_1) \) and \( x_1 \).

Inductively the dimension of the minimal center is lowered and we find a divisor \( D_l \equiv \gamma_l L \), with \( c_l < 1 \), which has zero dimensional minimal center. Conclude by Theorem 1.3 (iii) that there exists a section of \( H \sim K_X + L \) not vanishing on \( Z \).

\[\diamond\]

We will need the forthcoming lemma only in the next section, to be able to apply an inductive procedure on the Fano variety, but we place it here since the flavor and the proof are close to the previous one.

**Lemma 2.4** Let \( X \) be a log terminal Gorenstein Fano \( n \)-fold, with \( n > 3 \), and \( H \) an ample Cartier divisor with \( -K_X \sim (n-2)H \), let \( L \sim (n-1)H \) and \( D \) a \( \mathbb{Q} \)-Cartier divisor with \( (X, D) \) LC. Assume that \( D \equiv 2H \), \( Z \in \text{CLC}(X, D) \) is a codimension 3 minimal center not contained in \( \text{Sing}(X) \). Furthermore assume that there exist \( S \in |H| \) and an effective \( \mathbb{Q} \)-divisor \( D_S \) such that \( (S, D_S) \) satisfy the hypothesis of Theorem 1.4 and Lemma 2.3. Then there is a section of \( H \sim K_X + L \) not vanishing identically on \( Z \).

**Proof.** Let us simply sketch the proof since it is similar to that of Lemma 2.3. By Theorem 1.4 there exist effective \( \mathbb{Q} \)-divisors \( M_Z \) and \( D_Z \) such that

\[-(n-4)H|_Z \equiv K_Z + M_Z + D_Z.\]

Let us, again, replace \( D \) with \( D_1 \) such that \( (X, D_1) \) is LC, \( Z \) is isolated in \( \text{LLC}(X, D_1) \) and \( D_1 \equiv \gamma_1 H \), for \( \gamma_1 < 2 + \epsilon \), with \( \epsilon \ll 1 \). Let \( \mu : Y \to X \) a log resolution of \( (X, D_1) \) with \( K_Y + E - A + F = \mu^*(K_X + D_1) \), where \( f(E) = Z \), \( A \) is an integral \( \mu \)-exceptional divisor and \( |F| = 0 \).

Let \( N(t) := -K_Y - E - F + A + \mu^*(tH) \), then \( N(1) \equiv \mu^* (1+(n-2) - \gamma_1)H \) is nef and big.

If \( n = 4 \) then \( Z \) is a smooth curve of non positive genus, therefore \( h^0(Z, H) > 0 \); if \( n \geq 5 \) and \( (Z, M_Z + D_Z) \) is KLT then it is a log-Fano variety of index \( i(Z, M_Z + D_Z) = \dim Z - 1 \), therefore as above the sections in \( H^0(Z, H) \) extends to sections of \( H^0(X, H) \) not vanishing identically on \( Z \).

Again we can assume that \( M_Z + D_Z \neq 0 \), and choose a smooth point in \( Z \) with \( x \not\in \text{Sing}(X) \), in Helmke’s notations, \( \gamma = 2/(n-1) < 1 \) and

\[p \leq \frac{(n-1)(n-3)}{n-1-2} \leq n-1\]

and by Lemma 2.2 for \( k \gg 0 \) there is a section \( D' \in |kL|_Z \) with \( \text{mult}_x D' > pk \); then conclude as in Lemma 2.3.

\[\diamond\]
Proposition 2.5 Let $X$ be a log terminal Gorenstein Fano $n$-fold and $H$ an ample Cartier divisor with $-K_X \sim (n-2)H$. Assume that $\text{codSing}(X) > 2$ and $n \geq 3$. Then the general element in $|H|$ has at worst canonical singularities.

Proof. By Lemma 1.3 we know that $\dim |H| \geq 1$. Let $S \in |H|$ a generic element and assume that $S$ has worse than canonical singularities. Since both $H$ and $K_X$ are Cartier divisors then $(X, S)$ is not pLT, that is there exists $\gamma \leq 1$ such that $(X, \gamma S)$, is LC with $Z$ a minimal center in $CLC(X, \gamma S)$, and by Bertini theorem $Z \subset Bsl|H|$. We will derive a contradiction, producing a section of $|H|$ not vanishing identically on $Z$.

If either $\dim Z \leq n-3$ or $\dim Z = n-2$ and $\gamma < 1$ then apply Lemma 2.1. If $\dim Z = n-2$ and $\gamma = 1$, by hypothesis $Z \not\subset \text{Sing}(X)$ hence apply Lemma 2.3.

To conclude we have to exclude the case $\dim Z = n-1$. Assume that $|H|$ has a fixed component $F$, by [Al, Prop 3.2] $F$ must have multiplicity 1, that is $\gamma = 1$. Since $H$ is connected and movable then $S$ must be singular along a codimension 2 set $Z \subset F$, therefore $F$ is not minimal in $CLC(X, S)$, see Definition 1.1.

Remark In particular the above argument shows that $H$ is smooth in codimension 1 and there are not fixed component.

Proof. (of Theorem 2) By Lemma 1.3 $h^0(X, H) \geq 2$; furthermore terminal singularities are smooth in codimension 2. It is, therefore enough to apply Proposition 2.5.

Remark It is not true, in general, that terminal Fano $X$ of index $i(X) = n-2$ are Gorenstein; consider a terminal Fano 3-fold with an Enriques surface as section of the fundamental divisor, this varieties are studied by Conte–Murre [CM]. In this case $X$ has 8 singular points, which are cones over the Veronese surface, and $X$ is 2-Gorenstein; nevertheless $H$ has a smooth (terminal) section.

3 Proof of the main Theorem

By a direct calculation, for instance Lemma 1.3, $h^0(X, H) \geq n$ therefore by Proposition 2.5 there exists a section $S \in |H|$ with canonical singularities. Our aim is to apply inductively Proposition 2.5, to do this we have to prove that $S$ is smooth in codimension 2. Assume the contrary, in particular $S$ is not terminal and there is a center $Z \subset Bsl|H|$ of canonical singularities in $S$ with $\dim Z = n-3$.

Case 1 Assume that all sections of $|H|$ are singular at $Z$, let $H_i \in |H|$ generic elements and $D = 1/2(H_1 + H_2)$.

Claim $(X, \gamma D)$ is log canonical for some $\gamma \leq 3/2$ with a minimal center $W \subseteq Z$ of codimension $\geq 3$. 

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Observe that by the claim we can apply Lemma 2.1 to produce a section of $|H|$ not vanishing on $Z$ and derive in this way a contradiction.

**Proof.** (of the claim) Let $f : Y \to X$ the blow up of $Z$ let $f^*S = S' + rE$, since $X$ is smooth at the generic point of $Z$ then $K_Y = f^*K_X + 2E$. By adjunction formula

$$K_{S'} = (K_Y + S')|_{S'} = f^*K_S + (2 - r)E|_{S'},$$

since $S$ is canonical and is singular at $Z$ then $r = 2$. $|H|$ has not fixed components and its general element is smooth in codimension 1 therefore for some $\gamma \leq 3/2$, $(X, \gamma D)$ is log canonical with a minimal center $W \subseteq Z$ of codimension $\geq 3$.

\hfill \Diamond

**Case 2** Assume that there are infinitely many such codimension 3 components $Z_i \subset \text{Bl}_1|H|$ centers of canonical singularities for $H_i \in |H|$. Let $H_1$ a generic element in $|H|$, we can assume that $H_1$ is singular along $Z_1$, with $Z_1 \subset \text{Bl}_1|H|$ and $Z_1 \not\subset \text{Sing}(X)$. Let $D = 1/2(H_1 + H_2)$, with $H_2 \in |H|$, a general element; by construction $(X, \gamma D)$ is log canonical for some $\gamma \leq 2$ with a minimal center $Z$ of codimension $\geq 3$. If either $\gamma < 2$ or $\text{cod} Z > 3$ then conclude by Lemma 2.1. Assume that $\gamma = 2$ and $\text{cod} Z = 3$, we can assume without loss of generality that $Z = Z_1$, let $S \in |H|$ a generic element smooth at the generic point of $Z$ and $D_S = H_{1|S}$, then $(S, D_S)$ and $Z$ satisfy the hypothesis of Theorem 1.4 and Lemma 2.2, thus, we derive a contradiction by Lemma 2.4 if $n \geq 4$.

At each inductive step we loose only one section of $|H|$, therefore $|H_{1|S}|$ is always movable; furthermore by K–V vanishing theorem

$$H^0(X, H) \to H^0(S, H_{1|S}) \to 0,$$

hence it is possible to study the singularities of $S$ through the linear system $H_{1|S}$.

To carry on induction in Case 1 we need the following

**Claim** Let $S_0 = X$ and $S_j = X \cap H_1 \cap \ldots \cap H_j$, for $H_i \in |H|$ general elements. Assume that $S_j$ has canonical singularities and is singular at $Z_j$, with $\text{cod} X Z_j = j + 2$. If $X$ is smooth at $Z_j$ then $S_{j-1}$ is smooth at $Z_j$.

**Proof.** (of the claim) We will prove it by induction on $j$. If $j = 1$ then it follows by hypothesis. Let $f : Y \to X$ the blow up of $Z_j$, with $f^*H_i = H'_i + rE$, since $X$ is smooth at the generic point of $Z_j$ then $K_Y = f^*K_X + (j + 1)E$. By adjunction formula

$$K_{S'_j} = (K_Y + \sum_i H'_i)|_{S'_j} = f^*K_{S_j} + (j + 1 - jr)E|_{S'_j},$$

where $S'_j = Y \cap H'_1 \cap \ldots \cap H'_j$. Since $S_j$ has canonical singularities then $j + 1 - jr \geq 0$, and consequently $r = 1$, that is the generic element in $|H|$ is smooth at $Z_j$. On the other hand $Z_j = S_1 \cap H_2 \cap \ldots \cap H_j$, where $S_1 \in |H|$ is
a general element smooth at $Z_j$, and $\text{cod}_{S_j}Z_j = j - 1$, therefore by induction hypothesis $S_{j-1}$ is smooth at $Z_j$.

By the inductive process we are reduced to a canonical Gorenstein 3-fold smooth in codimension 2, $S_3 = X \cap (\bigcap_{i=1}^{n-3} H_i)$ with a line bundle $H_3 = H_{|S_3}$ satisfying the following conditions:

- $h^0(S_3, H_3) \geq 3$
- $\dim Bsl[H_3] = 1$

Let $H_1 \in |H_3|$ a general element and $B$ a curve contained in the base locus of $|H_3|$. Assume, without loss of generalities that $x_1 \in B \cap H_1$ is such that $x_1 \notin \text{Sing}(S_3)$ and $H_1$ singular at $x_1$. Let $A = \{M \in |H| \mid M \text{ is singular at } x_1\}$ since $h^0(S_3, H_3) \geq 3$ and $\dim Bsl[H_3] = 1$ then $\dim A \geq 1$. Let $H_i \in A$, for $i = 1, 2$ be general elements and $D = 1/2(D_1 + D_2)$. Then $(X, \gamma D)$ is log canonical for some $\gamma \leq 3/2$ with zero dimensional minimal center thus Lemma 2.1 apply to derive a contradiction and the theorem is proved.

\begin{itemize}
  \item References
\end{itemize}

[A] Alexeev, V. Theorem about good divisors on log Fano varieties Lect. Notes in Math 1479 (1991) 1-9.

[CF] Campana, F. Flenner, H. Projective threefolds containing a smooth rational surface with ample normal bundle, J. reine angew. Math 440 (1993), 77-98.

[CM] Conte, A. Murre, J.P. Algebraic varieties of dimension three whose hyperplane sections are Enriques surfaces, Ann. Sc. Norm. Sup. Pisa, Cl. Sci. IV Ser. 12 (1985), 43-80.

[Fa] Fano, G. Sopra alcune varietà algebriche a tre dimensioni aventi tutti i generi nulli, Atti Acc. Torino, 43 (1908).

[Fu] Fujita, T., On the structure of polarized manifolds with total deficiency one I, II and III, J. Math. Soc. Japan 32 (1980) 709-725, 33 (1981) 415-434 and 36 (1984) 75-89.

[He] Helmke, S. On Fujita’s conjecture, preprint 1996

[Is] V.A. Iskovskikh, Birational automorphisms of three dimensional algebraic varieties, J. Soviet Math. 13 (1980) 815-868.

[Ka1] Kawamata, J. On Fujita’s freeness conjecture for 3-folds and 4-folds, preprint 1996

[Ka2] Kawamata, J. Subadjunction of log canonical divisors for a subvariety of codimension 2, preprint 1996

[KMM] Kawamata, Y., Matsuda, K., Matsuki, K., Introduction to the Minimal Model Program in Algebraic Geometry, Sendai, Adv. Studies in Pure Math. 10, Kinokuniya–North-Holland (1987), 283-360.
[Mu] Mukai, S. Biregular classification of Fano threefolds and Fano manifolds of coindex 3, Proc. Nat. Acad. Sci. USA 86 (1989) 3000-3002.

[Pr1] Prokhorov, Y.G. The existence of smooth divisors on Fano fourfolds of index 2, Russian Acad. Sci. Sb. Math. 83 (1995) 119-131.

[Pr2] Prokhorov, Y.G. On the existence of good divisors on Fano varieties of coindex 3, preprint.

[Pr3] Prokhorov, Y.G. On the existence of good divisors on Fano varieties of coindex 3, II Contemporary Math. and its App. Plenum 24 (1995)

[Re] Reid, M. Projective morphism according to Kawamata, Warwick Preprint 1983.

[Sa] Sano, T. Classification of $\mathbb{Q}$-Fano d-folds of index greater than $d-2$, Nagoya Math. J. 142 (1996) 133-143.

[Sh] Shokurov V.V. Smoothness of the general anticanonical divisor of a Fano 3-fold, Math. USSR Izv. 14 (1980) 395-405.

[Wi] Wilson, P.M.H. Fano fourfold of index greater than one, J. Reine Angew. Math. 389 (1987) 172-184.

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