A Note on the Entropy of Entanglement and Entanglement Swapping Bounds

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Abstract

Using the information content of correlations between multipartite systems, together with the notion of partitioning, we show that some general results about the evolution of correlations in quantum systems can be derived with only elementary methods. In particular, we show that for 2 quantum systems $A$ and $B$, each comprised of a number of sub-systems, in which a partition of $A$ interacts unitarily with a partition of $B$, then the total correlation can only increase (or remain unchanged) and is given simply by the sum of the initial correlation and the correlation that develops as a result of the interaction. We then show that in a 4 qubit entanglement swapping process the transferred degree of entanglement is bounded by the lower of the initial degrees of entanglements of the qubits.

1 Introduction

For 2 quantum sub-systems, which we label as 1 and 2, the degree of entanglement between them is well-characterized by the entropy of entanglement [1]. It is related to the mutual information $I_{12}$, and for pure entangled states the entropy of entanglement is simply just half this quantity. If $\rho$ is the total density operator for the \{12\} joint system and $\rho_k$ is the reduced density operator for sub-system $k$, then the entropy of an individual system is $S_k = \text{Tr}_k \{ \rho_k \ln \rho_k \}$ with $\rho_k = \text{Tr}_{j \neq k} \{ \rho \}$ and the total entropy is $S_{12} = \text{Tr} \{ \rho \ln \rho \}$ then the mutual information is given by [2]

$$I_{12} = S_1 + S_2 - S_{12}$$

In previous work we termed this quantity the ‘index of correlation’. Physically it is a basis-independent measure of the information contained in the correlations between 1 and 2.

If we wish a measure of correlation for quantum systems to be basis-independent and additive then the mutual information, or index of correlation, is the unique
measure satisfying these properties [3]. By additive here it is meant that if 2 systems $A$ and $B$, which could each be comprised of a number of subsystems, are uncorrelated then we require that the total correlation of the $\{AB\}$ system is simply the sum of the correlation within $A$ and $B$ separately.

Previously we have extended this measure to examine the properties of multipartite correlations and entanglement [4]. Characterizing the entanglement of multipartite systems is non-trivial [5], but the information content of the total correlation is a fundamental measure that can be employed to yield useful general properties of the correlation. In this note we explore the use of this parameter to investigate the properties of multipartite correlation where the end-goal is to establish an information-theoretic bound on entanglement swapping.

2 Partitioning

The notion of partitioning of quantum systems is a key idea that we will make use of extensively. In order to illustrate this we consider 4 qubits labelled 1,2,3 and 4. There are various, equivalent, ways we can approach this system. We could, for example, consider qubits 2,3 and 4 to be a single ‘system’. We could then consider the correlation between qubit 1 and qubits 2,3 and 4 taken as a single system. Where several quantum systems are grouped together in this fashion we use the notation $\{234\}$ to emphasize this grouping. In this case we term the correlation between qubit 1 and the system of qubits $\{234\}$ as an ‘external’ correlation denoted by $E_{1\{234\}}$.

The reason for this nomenclature is that the total information content of the correlation between the 4 qubits is simply given by

$$I_{1234} = S_1 + S_2 + S_3 + S_4 - S_{1234}$$

$$= (S_2 + S_3 + S_4 - S_{234}) + (S_1 + S_{234} - S_{1234})$$

$$= I_{234} + E_{1\{234\}}$$

(2)

where $I_{234}$ can be interpreted as the correlation ‘internal’ to the $\{234\}$ system of qubits so that the total correlation is just the sum of the internal correlation plus the external correlation. This is a general property of this entropic measure of correlation. For example, the total correlation for these 4 qubits can also be written as

$$I_{1234} = I_{12} + I_{34} + E_{\{12\}\{34\}}$$

(3)

where here we interpret the total system as comprising 2 sub-systems $\{12\}$ and $\{34\}$. The total correlation is again the sum of the internal correlations $I_{12}$ and $I_{34}$, and the external correlation $E_{\{12\}\{34\}}$.

It is important to note two things. Firstly, the partitioning is only notional although we could conceive of constructing such partitions physically (for example, we could physically separate qubits 1 and 2 from qubits 3 and 4). Secondly,
the quantities $I$ and $E$ have the same mathematical form being the sum of sub-system entropies minus the total entropy. We distinguish them to emphasize the partitioning into internal and external.

This notion of external and internal correlation is the same as that employed in the Ithaca interpretation of quantum mechanics [6,7] where the notion that correlation between systems is sufficient to describe their properties is developed. Its utility here is that certain partitions have invariant correlation under unitary transformation.

This partitioning extends, in some sense, to the interactions between the sub-systems. For example, if we consider our 4 qubits to be interacting unitarily with one another then we would have interaction terms in the Hamiltonian of the form $H_{jk}$. If we consider the partitioning into qubit 1 and qubits $\{234\}$ then we would have some effective interaction Hamiltonian $H^\text{eff}_{1\{234\}}$ that describes the evolution of the total system. As we have already mentioned, this partitioning is useful in allowing us to construct invariant entropies and correlations in a unitary interaction between quantum systems.

Entanglement swapping is a special case of quantum teleportation [8] and the phenomenon has been experimentally demonstrated [9]. In the usual entanglement swapping scheme we begin with 2 pairs of entangled particles. So we might consider the qubits $\{12\}$ to be initially entangled and the qubits $\{34\}$ to be initially entangled, with no entanglement or correlation between the $\{12\}$ and $\{34\}$ partitions. We let qubits 2 and 3 interact unitarily and perform a measurement (or equivalently we perform a Bell measurement on qubits 2 and 3). This procedure results in qubits 1 and 4 becoming entangled with one another. The central feature here is that qubits 2 and 3 interact. Accordingly in this note we wish to study the evolution of the correlations when the sub-system components are allowed to interact unitarily.

3 Two Interacting Systems

For two quantum systems, which we label as 1 and 2, the information content of the correlation is given as above in (1). This quantity tells us the difference in information between considering the systems 1 and 2 separately and considering them as one entity $\{12\}$. If systems 1 and 2 are themselves comprised of sub-systems then equation (1) gives us the ‘external’ correlation between 1 and 2. This can, of course, be cast more formally in terms of the Hilbert spaces. So system 1 might be described by states in the space $H_1 = H_a \otimes H_b \otimes H_c \ldots$ and system 2 by states in the space $H_2 = H_\alpha \otimes H_\beta \otimes H_\gamma \ldots$ where the subscripts refer to individual quantum systems such as qubits.

The correlation is bounded by [2]

$$I_{12} \leq 2 \inf \{S_1, S_2\}$$

and if $\{12\}$ is in a pure state then $S_1 = S_2$. Let us suppose that these systems are prepared in some initial state (which could be mixed) and we let 1 and 2
interact unitarily then the correlation is time-dependent, but the total entropy remains invariant and we have that
\[ I_{12}(t) = S_1(t) + S_2(t) - S_{12}(0) \] (5)

Noting that the sum of the individual entropies must always be less than or equal to the sum of the individual maximum entropies so that \( S_1(t) + S_2(t) \leq S_{1\text{max}} + S_{2\text{max}} \) we have that
\[ I_{12}(t) \leq S_{1\text{max}} + S_{2\text{max}} - S_{12}(0) \] (6)

If the 2 systems are initially uncorrelated and maximally mixed so that \( S_{12}(0) = S_{1\text{max}} + S_{2\text{max}} \) then it is easy to see that the interaction cannot develop any correlation between the systems. Conversely, if the two systems are initially in a pure state and maximally correlated the interaction can only reduce that correlation.

These two special instances are well-known and obvious properties of correlations for these initial states but they illustrate the general approach we shall take here. We consider unitarily interacting sub-systems and examine the entropy and correlation invariants of that interaction in order to yield general properties for the evolution of the correlations.

4 Three Interacting Systems

As a precursor to the situation relevant to entanglement swapping we now consider 3 quantum systems labelled 1, 2 and 3. We shall assume, for convenience, these systems are not comprised of internal sub-systems so that we can set their ‘internal’ correlation to zero. We shall further assume that system 3 is initially uncorrelated with the system described by the \{12\} partition. System 3 interacts unitarily with system 2 for a time \( t \). The total \( \{123\} \) system thus evolves unitarily according to \( \hat{U} = \hat{I}_1 \otimes \hat{U}_{23} \) where \( \hat{I}_1 \) is the identity operator for the subspace of system 1 and \( \hat{U}_{23} \) is the unitary interaction between 2 and 3. We now ask the following questions. How does the interaction affect the total degree of correlation \( I_{123} \)? How does the interaction between 2 and 3 affect the correlation \( I_{12} \) between 1 and 2?

4.1 Entropy and Correlation Invariants

Since systems 2 and 3 interact unitarily certain entropies, and hence correlations, are invariant. For example, considering the entropy of system 1 we note that no local unitary operation on \{23\} will change this and so \( S_1 \) is time-invariant. Considerations of this sort allow us to write down the following invariant entropies
\[
\begin{align*}
S_1(t) &= S_1(0) \\
S_{23}(t) &= S_{23}(0) \\
S_{123}(t) &= S_{123}(0)
\end{align*}
\] (7)
All other entropies being time-dependent. Considering the correlation between system 1 and the partition \{23\} which is given by the external correlation \( E_{1\{23\}} = S_1 + S_{23} - S_{123} \) then it is clear that this external correlation is also time-invariant so that

\[
E_{1\{23\}} (t) = E_{1\{23\}} (0)
\]

where this latter condition merely expresses the fact that for 2 quantum systems \( A \) and \( B \) no local unitary operation on \( B \) will affect the degree of entanglement between \( A \) and \( B \). The entropies \( S_2 \) and \( S_3 \) are clearly time-dependent since \( \rho_2 \) and \( \rho_3 \) undergo non-unitary evolutions.

### 4.2 The Total Correlation

Using the notion of partitioning the total correlation can be decomposed into external and internal as follows

\[
I_{123} (t) = I_{23} (t) + E_{1\{23\}} (t)
\]

where here we recall that we have assumed no internal correlation for the individual sub-systems so that \( I_1 = 0 \). The external correlation is invariant and therefore simply equal to the initial correlation between 1 and 2 (since we have assumed system 3 is initially uncorrelated). The total correlation is therefore

\[
I_{123} (t) = I_{123} (0) + I_{23} (t)
\]

which gives the appealing and intuitive result that the total correlation is simply the sum of the initial correlation and the correlation that develops between 2 and 3 as a result of their interaction. Since we have \( I_{23} (t) \geq 0 \) we can also see that the interaction between 2 and 3 can only increase the total correlation (or leave it unchanged).

### 4.3 The Correlation Between 1 and 2

The total correlation for the 3 systems can be written in two equivalent ways by considering the partition into 1 and \{23\} and the partition into \{12\} and 3 so that

\[
I_{123} (t) = I_{12} (t) + E_{3\{12\}} (t)
\]

A simple rearrangement gives us that

\[
E_{1\{23\}} (t) - I_{12} (t) = E_{3\{12\}} (t) - I_{23} (t)
\]

However, \( E_{1\{23\}} (t) \) is an invariant so that \( E_{1\{23\}} (t) = E_{1\{23\}} (0) = I_{12} (0) \) which gives

\[
I_{12} (0) - I_{12} (t) = E_{3\{12\}} (t) - I_{23} (t)
\]
Strong subadditivity [9] gives us the condition that $E_{3\{12\}}(t) \geq I_{23}(t)$ and so we obtain the result that $I_{12}(0) - I_{12}(t) \geq 0$. This establishes the following theorem

If we have 3 quantum systems 1, 2 and 3 such that 3 is initially uncorrelated with either 1 or 2 and we let 2 interact unitarily with 3, then the interaction always reduces the correlation between 1 and 2, or leaves it unchanged.

This is consistent with the monogamy property of quantum mechanics in which maximal pairwise entanglement can not be established for more than 1 pair of a 3 component system [10]. Although in the above we have only considered systems with no degree of internal correlation for convenience, this result is easily extended to the case where 1, 2, and 3 are each comprised of a number of sub-systems.

Of course the content of this theorem is intuitive and obvious; if we have systems 1 and 2 with some initial degree of correlation then we would not expect some local process on 2 to increase the degree of correlation. However, as the next example shows we must sometimes be careful in relying on our intuition where correlation is concerned.

### 4.4 Non-Transitivity of Correlation

If we consider 3 systems $A, B$ and $C$ then if $A$ is correlated with $B$ and $B$ is correlated with $C$ then it would seem intuitive to suppose that $A$ has to be correlated to $C$. This, however, is not always true as the following counter-example shows.

Let us suppose that $A$ is a single qubit, qubit 1, system $B$ is comprised of 2 qubits, qubits 2 and 3 and $C$ is a single qubit, qubit 4. If we prepare these qubits in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle_1 |0\rangle_2 + |1\rangle_1 |1\rangle_2) \otimes \frac{1}{\sqrt{2}} (|0\rangle_3 |0\rangle_4 + |1\rangle_3 |1\rangle_4)$$  \quad (14)

then it is easy to see that $I_{1B} \neq 0$, $I_{4B} \neq 0$ but $I_{14} = 0$. Qubit 1 is correlated to qubit 2, and qubit 3 is correlated to qubit 4, but system $B$ is comprised of qubits 2 and 3 which are not correlated with one another. At this 'system' level, therefore, it is not possible to demonstrate a transitivity property for correlation because internally the chain of correlation can be broken within a given system.

The state given by (14) is, of course, that considered in typical entanglement-swapping schemes where the qubit pairs are initially maximally entangled. This state possesses the maximum possible total correlation $I_{1234}$ for 4 qubits, even though the overall $\{1234\}$ system is not maximally entangled. In order to have maximal entanglement we have to have a state that simultaneously optimizes the pairwise correlations [4]. The state (14) clearly does not simultaneously optimize the pairwise correlation between the qubits since qubits 2 and 3 are uncorrelated whereas the qubit pairs $\{12\}$ and $\{34\}$ are maximally correlated.
4.5 Entanglement Exchange in an Atom-Field Interaction

In the usual entanglement-swapping scheme there are 4 qubits such that \{12\} are entangled and \{34\} are entangled with no entanglement between these partitions so that \( E_{\{12\}\{34\}}(0) = 0 \). The initial entanglement is ‘swapped’ by ineracting qubits 2 and 3 followed by a subsequent measurement on these qubits. The result is that the initial entanglement can be transferred to qubits 1 and 4, which have never previously interacted. Entanglement swapping can be viewed as an instance of quantum teleportation [11].

The key feature here is that the measurement can be viewed as projecting the \{14\} qubits into an entangled state. The measurement process is, of course, non-unitary. It is, however, possible to exchange entanglement to 2 qubits that have never directly interacted using only unitary processes. To illustrate this we consider an idealized example consisting of 2 two-level atoms interacting with a single field mode in a lossless cavity [12]. The appropriate interaction Hamiltonian is given by the Jaynes-Cummings Hamiltonian in the rotating-wave approximation

\[
\hat{H}_{\text{int}} = \hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-
\]  

(15)

where \( \hat{a}, \hat{a}^\dagger \) are the field annihilation and creation operators, respectively, and \( \hat{\sigma}_-, \hat{\sigma}_+ \) are the atomic lowering and raising operators, respectively.

We consider the first atom to be prepared in its excited state and the field to be in its vacuum state. The first atom is sent through the cavity with a cavity transit time such that there is a probability of 1/2 of the atom-field interaction resulting in the atom leaving the cavity in its ground state. The state of the total system after this interaction is therefore given by a state of the form

\[
|\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle_{A_1} |1\rangle_F + |1\rangle_{A_1} |0\rangle_F) \otimes |\varphi\rangle_{A_2}
\]  

(16)

where the subscripts \( A \) and \( F \) refer to the atoms and field, respectively. We consider this first interaction to be a state preparation phase that generates an initial correlation between atom 1 and the field. If we now consider the second atom to be prepared in its ground state and sent through the cavity with a transit time such that there would be a unit probability of the atom absorbing the photon if the field were in the state \( |1\rangle_F \) then the total state after this interaction is given by

\[
|\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle_{A_1} |1\rangle_{A_2} + |1\rangle_{A_1} |0\rangle_{A_2}) \otimes |0\rangle_F
\]  

(17)

The field state has been completely ‘decoupled’ by the second interaction and the atom-field entanglement after the first interaction has been transferred to an entanglement between the 2 atoms. In our general notation and terminology above, atom 1 would be system 1, the field system 2, and atom 2 would be system 3. Thus the unitary interaction of 2 and 3 has ‘decoupled’ system 2 and the initial entanglement between 1 and 2 transferred to an entanglement between 1 and 3.
The entropy of atom 1 during this second interaction remains unchanged; the
degree of mixing of the state of atom 1 (after the first preparation interaction)
is invariant when the field interacts with atom 2. The initial correlation that
exists between atom 1 and the field is reduced and the correlation between
atom 1 and atom 2 increases. We have chosen interaction times to generate
maximally entangled states here, but it is straightforward to generalize this
to arbitrary cavity transit times for the atoms. The correlations that develop
between the atoms and field are consistent with the general properties (10) and
(13) above.

5 Four Quantum Systems

We now consider the situation most pertinent to entanglement swapping where
we have 4 quantum systems, labelled 1, 2, 3 and 4. Here, however, we are going
to consider a more general scenario in which these 4 quantum systems can each
be comprised of a number of sub-systems. The usual entanglement swapping
scheme is just a special case of this more general situation.

As before we shall let systems 2 and 3 interact unitarily and we shall also
assume that there is no initial external correlation between the \{12\} and \{34\}
partitions.\footnote{Since systems 1, 2, 3 and 4 are themselves possibly comprised of sub-systems then these can
also be viewed as partitions of the total system. The partition \{12\} is really then a partition of partitions although we shall simply refer to it as the \{12\} partition.} This can be represented schematically as

\[
\begin{array}{c}
1 \Diamond \\
\uparrow \\
\downarrow \\
2 \Diamond
\end{array}
\quad
\begin{array}{c}
\Diamond 4 \\
\uparrow \\
\downarrow \\
\Diamond 3
\end{array}
\quad
\hat{U}_{23}
\]

This is a quite general model for interacting systems. If we wish to consider
the interaction of 2 systems prepared in mixed states, for example, then 1 and
4 might be taken to be the supplementary quantum systems required for the
purification of the \{12\} and \{34\} partitions. Our goal here, as in the previous
section, is to examine the general properties of the correlations that develop as
a result of the interaction.

It is clear from the previous section that the unitary interaction between 2
and 3 will reduce any initial external correlation between 1 and 2 (or at best
leave it unchanged) so that \(E_{12}(t) \leq E_{12}(0)\). It is also clear, from symmetry,
that the initial external correlation between 3 and 4 will also be reduced by the
interaction (or at best unchanged) so that \(E_{34}(t) \leq E_{34}(0)\).

5.1 Entropy and Correlation Invariants

Since the interaction between 2 and 3 is assumed to be unitary then there are
various entropies, and hence correlations, that remain invariant under the inter-
action. There are 7 invariant entropies and these are; \(S_{1234}, S_1, S_4, S_{23}, S_{14}, S_{123}\),
and $S_{234}$. All other entropies are time-dependent. There are 30 possible correlations we can consider; the 4 internal correlations $I_k$, the correlations internal to a given partition, and the various external correlations between the partitions. There are 6 invariant correlations from the 30 possibilities. For our purposes we shall consider only the following 4 invariants

\[
I_1 (t) = I_1 (0) \\
I_4 (t) = I_4 (0) \\
E_{\{234\}} (t) = E_{\{234\}} (0) \\
E_{\{23\}4} (t) = E_{\{23\}4} (0) \tag{18}
\]

### 5.2 The Total Correlation

The total correlation can be partitioned as

\[
I_{1234} (t) = I_1 (t) + I_{\{234\}} (t) + E_{\{234\}} (t) = const + I_{\{234\}} (t) \tag{19}
\]

where we have used the invariants (18). The internal correlation $I_{\{234\}} (t)$ can also be partitioned as $I_{\{234\}} (t) = I_4 (t) + E_{\{23\}4} (t) + I_{23} (t)$. The quantity $I_{23} (t)$ is just the correlation that develops between 2 and 3 as a result of their interaction. Since $I_4$ and $E_{\{23\}4}$ are invariant, the total correlation can be written as

\[
I_{1234} (t) = const + I_{23} (t) = I_{1234} (0) + I_{23} (t) \tag{20}
\]

where for the latter identity we have assumed that 2 and 3 are initially uncorrelated. We therefore arrive at the following general theorem

If $A$ and $B$ are 2 initially uncorrelated quantum systems each comprised of a number of sub-systems and a partition of $A$ interacts unitarily with a partition of $B$ then the total correlation that develops is greater than or equal to the total initial correlation and is simply the sum of the initial correlation and the correlation that develops between the partitions.

This, again, is an intuitive and appealing general result. It is interesting that the interaction reduces certain correlations between systems (the initial external correlation between 1 and 2 reduces, for example) but in such a way that the total correlation increases if the initial correlation is not maximal.

In an optimal entanglement swapping scheme qubits $\{12\}$ are maximally entangled, as are qubits $\{34\}$. In this case it is clear that $I_{23} (t) = 0$ so that no correlation develops between 2 and 3 as a result of their (unitary) interaction. It is only when the initial internal correlations for the partitions $\{12\}$ and $\{34\}$ are not maximal will there be any correlation developed between qubits 2 and
3. This gives us the seemingly paradoxical property that it is only when no
correlation develops between 2 and 3 (which implies that qubits 2 and 3 are in
maximally mixed states) can we transfer maximal entanglement to qubits 1 and
4.

6 Entanglement Swapping

We now consider the general problem of entanglement swapping for qubits. If
the qubit pairs are not initially maximally entangled, then what is the maximum
correlation, or entanglement, that can be swapped? A general and elegant
approach to this in terms of concurrence has been developed [13], but here we
show how a simple information-theoretic bound can be established.

In the previous sections we have considered a unitary evolution of the state;
in entanglement swapping a measurement is necessary to transfer the entan-
glement to the 2 qubits that have not directly interacted. Consider 4 qubits,
(labelled 1,2,3,4 from left to right, as necessary), prepared in the state

\[
|\psi\rangle = (a |00\rangle + b |11\rangle) \otimes (c |00\rangle + d |11\rangle)
\]  

which we can represent diagrammatically as

\[
\begin{array}{c}
\diamond \Leftrightarrow \diamond \\
\otimes \end{array} \begin{array}{c}
\diamond \Leftrightarrow \diamond
\end{array}
\]

We denote the index of correlation before the entanglement swapping process
with \(I\) and afterwards by \(I^M\) where the superscript reminds us that we are
considering the situation before and after the Bell measurement. Writing the
Bell basis in the usual fashion as

\[
|\Psi^\pm\rangle = \frac{1}{\sqrt{2}} (|00\rangle \pm |11\rangle)
\]

\[
|\Phi^\pm\rangle = \frac{1}{\sqrt{2}} (|01\rangle \pm |10\rangle)
\]

our initial state of the 4 qubits can be written as

\[
|\psi\rangle_{1234} = \frac{1}{\sqrt{2}} (ac |00\rangle_{14} + bd |11\rangle_{14}) \otimes |\Psi^+_23\rangle + \frac{1}{\sqrt{2}} (ac |00\rangle_{14} - bd |11\rangle_{14}) \otimes |\Psi^-_{23}\rangle
\]

\[
+ \frac{1}{\sqrt{2}} (ad |01\rangle_{14} + bc |10\rangle_{14}) \otimes |\Phi^+_23\rangle + \frac{1}{\sqrt{2}} (ad |01\rangle_{14} - bc |10\rangle_{14}) \otimes |\Phi^-_{23}\rangle
\]  

(23)
Defining the normalized states

\[ |\psi_{\pm}\rangle = n^{-1/2}_\psi (ac|00\rangle \pm bd|11\rangle) \]

\[ |\varphi_{\pm}\rangle = n^{-1/2}_\varphi (ad|01\rangle \pm bc|10\rangle) \]  

(24)

with \( n_\psi = a^2c^2 + b^2d^2 \) and \( n_\varphi = a^2d^2 + b^2c^2 \) (where the modulus has been dropped for convenience) we can write the initial state as

\[ |\psi\rangle_{1234} = \frac{1}{\sqrt{2}} n^{1/2}_\psi (|\psi_+\rangle |\Psi_+\rangle + |\psi_-\rangle |\Psi_-\rangle) + \frac{1}{\sqrt{2}} n^{1/2}_\varphi (|\varphi_+\rangle |\Phi_+\rangle + |\varphi_-\rangle |\Phi_-\rangle) \]  

(25)

where the states appearing to the left side of the tensor products describe the \{1, 4\} qubits and those to the right the \{2, 3\} qubits. So that a Bell measurement on the \{2, 3\} qubits projects the \{1, 4\} qubits into the states

\[ |\psi_{\pm}\rangle \text{ with probability } \frac{1}{2} n_\psi \]

\[ |\varphi_{\pm}\rangle \text{ with probability } \frac{1}{2} n_\varphi \]

6.1 Upper Bound on Correlation

Let us assume that the Bell measurement on the \{2, 3\} qubits has been performed with the result \(|\Psi_+\rangle\) obtained. This result is communicated to the holders of the \{1, 4\} qubits. The \{1, 4\} qubits can then be assigned the pure state

\[ |\psi_+\rangle = n^{-1/2}_\psi (ac|00\rangle + bd|11\rangle) \]  

(26)

The density operator for qubit 1 is therefore

\[ \hat{\rho}_1^M = \frac{a^2c^2}{n_\psi} |0\rangle \langle 0| + \frac{b^2d^2}{n_\psi} |1\rangle \langle 1| \]

\[ = \left( \frac{1}{2} + \varepsilon_1^M \right) |0\rangle \langle 0| + \left( \frac{1}{2} - \varepsilon_1^M \right) |1\rangle \langle 1| \]  

(27)

where \( |\varepsilon_1^M| \) is the bias. A qubit state of higher entropy has a lower bias and vice versa.

Let us assume without loss of generality that the pre-measurement indices of correlation satisfy \( I_{12} \geq I_{34} \) which implies that \( c^2 \geq a^2 \) and that both \( a^2, b^2 > d^2 \). We further assume, again without any essential loss of generality, that \( a^2 > b^2 \). The pre-measurement biases for qubits 1 and 3 are therefore

\[ \varepsilon_1 = a^2 - \frac{1}{2} \]

\[ \varepsilon_3 = c^2 - \frac{1}{2} \]  

(28)
with the post-measurement bias for qubit 1 being given by

\[ \varepsilon_1^M = \frac{a^2c^2}{a^2c^2 + b^2d^2} - \frac{1}{2} \]

\[ = c^2 \left( \frac{1}{c^2 + d^2 b^2 a^2} \right) - \frac{1}{2} \] (29)

but

\[ \left( \frac{1}{c^2 + d^2 b^2 a^2} \right) > 1 \] (30)

Hence \( \varepsilon_1^M > \varepsilon_3 \). This implies that the entropy of qubit 1 after measurement is lower than the entropy of qubit 3 before measurement and so we have \( I_{14}^M \leq I_{34} \).

Similar arguments apply to all possible output states after the measurement and so we have the result that

\[ I_{14}^M \leq \inf \{ I_{12}, I_{34} \} \] (31)

The degree of entanglement that can be transferred is therefore limited by the lower of the initial existing degrees of entanglement.

6.2 Example

With the choices \( a^2 = 3/4 \) and \( c^2 = 7/8 \) and assuming the result of the Bell measurement is \(|\Psi_+\rangle\) then the density operator for qubit 1 post-measurement is

\[ \hat{\rho}_1^M = \frac{21}{22} |0\rangle \langle 0| + \frac{1}{22} |1\rangle \langle 1| \] (32)

which is a good deal less mixed than the pre-measurement density operators for the qubits. Note that any iteration of the entanglement swapping process in which we begin with less than perfect entanglement will rapidly drive the \{1, 4\} qubits into an uncorrelated state.

7 Conclusions

Entanglement remains one of the most intriguing features of quantum mechanics. Indeed, many have argued that it is the central feature of quantum mechanics that distinguishes it from a classical perspective. Characterizing the entanglement of multipartite systems is a difficult, and still largely unresolved, problem. In this note we have emphasized a measure of correlation based on the information content of the correlation. For bi-partite systems in a pure state this is just proportional to the entropy of entanglement. The generalization of this to multipartite systems that we have used here does not provide a similarly
straightforward measure of entanglement. It does, however, give a useful measure of the overall degree of correlation within any given partition of a quantum system and between those partitions. It is an observable-independent characterization that provides the unique measure satisfying the additivity property that if two quantum systems, each comprised of sub-systems, are uncorrelated then the total correlation is simply the sum of the correlations within those two multi-component systems.

In this note we have used this measure, together with the notion of partitioning, to derive some general properties of the correlation of interacting quantum systems. Partitioning is only notional unless we take steps to physically create the partitions, but it allows us to identify the various entropy and correlation invariants of the interaction. Once these invariants have been identified it only requires very elementary techniques to establish these general properties. In particular, it is easy to demonstrate using this approach the intuitive result that if we have 2 initially uncorrelated quantum systems $A$ and $B$, each comprised of a number of sub-systems, and a partition of $A$ interacts unitarily with a partition of $B$, then the total correlation is simply the sum of the initial correlation within $A$ and within $B$ and the time-dependent correlation due to the interaction. It would certainly be surprising if it were otherwise, but the index of correlation applied to multipartite systems allows us to quantify this precisely in information-theoretic terms.

These general results apply only to unitary interactions between the partitions. In entanglement swapping a non-unitary process (i.e. measurement) is employed to transfer the entanglement to 2 systems that have never previously directly interacted. Viewing entanglement as a resource, two remote parties can make use of this entanglement provided they are given the supplementary information about the result of the measurement. The index of correlation allows us to place an upper bound to the amount of entanglement that can be transferred if the initial systems are not prepared in perfectly entangled states. The maximum amount of entanglement that can be transferred in entanglement swapping is the lower of the 2 initial entanglements.

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