Three Field Dynamics in (1+1)-dimensions

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Abstract: In a model of nonlinear system of three scalar fields the problem on dynamics of a massive particle moving in effective potential provided by two relativistic fields is solving. The potentials for these fields are chosen in the form of anti-Higgs and Higgs potentials. It is shown that the effective potential has the shape of two-hump barrier localized in spacetime. It tends to constant attractive potential at spacetime infinity. The magnitude of this constant constituent is determined by the Higgs condensate. It is shown that nonlinear equation of motion of a particle has the solutions which describe the capture of a particle by the barrier and the scattering on the barrier.

1. Introduction. Study of the dynamics of few fields interacting between themselves is of interest in the connexion with the problems of physics of nonlinear systems which arise in quantum field theory \[1\], in plasma, in hydrodynamics, in optics and so on \[2\]. In QCD, for example, if one takes into account spin, colour and flavour degrees of freedom in quark-gluon system the requirement of gauge invariance of the theory leads to clear structure of the Lagrangian \[3\]. However the equations of motion which follow from this Lagrangian prove to be too complicated for analysis and solution. The modelling of peculiar features of the Lagrangians of few-field systems with nonlinear self-interaction and study on this basis of toy models allowing exact or approximate solutions of the nonlinear equations of motion seem to be important.

In the present article nonlinear system of three scalar fields in (1+1)-dimensions is considered. One field is massive while the potentials of two other fields are described by the anti-Higgs and Higgs potentials. In the case when one can separate the slow and rapid motion in spacetime the effective potential of interaction of massive particle with “force field” provided by two relativistic fields is obtained. The mechanism of the formation of this potential is cleared up. It is shown that in general case it has the shape of two-hump barrier localized in spacetime. The potential tends to constant negative value at spacetime infinity. This constant attractive interaction represents the energy determined by the Higgs condensate. It arises in the system as a result of reorganization of vacuum of the field with self-interaction described by the Higgs potential. The equation of motion of massive particle takes into account self-interaction of its field and it is nonlinear. It is shown that the solutions of this equation can describe both capture of a particle by the barrier and the scattering on the barrier. The character of solutions depends on coupling constants of relativistic fields.

2. Effective potential. We shall consider the nonlinear system of three interacting scalar fields $\phi(x^\mu)$, $\varphi_1(x^\mu)$ and $\varphi_2(x^\mu)$ in two-dimensional spacetime $x^\mu = \{t, z\}$. Let us suppose that the field $\phi$ is massive and complex and the fields $\varphi_i$ are real. We define the

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Lorentz-invariant Lagrangian in the form

\[ \mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - \phi^* m^2 \phi + \sum_{i=1}^{2} \left[ \frac{1}{2} \partial_\mu \varphi_i \partial^\mu \varphi_i - \varphi^* U_i(\varphi_i) \phi \right], \]  

(1)

where \( m \) is the mass of the field \( \phi \). The functions \( U_i(\varphi_i) \) are chosen in the form of anti-Higgs (\( i = 1 \)) and Higgs (\( i = 2 \)) potentials respectively

\[ U_i(\varphi_i) = (-1)^i \left[ -\frac{\mu_i^2}{2} \varphi_i^2 + \frac{\lambda_i}{4} \varphi_i^4 \right], \]  

(2)

with coupling constants \( \mu_i^2 > 0 \) and \( \lambda_i > 0 \). Variational principle being applied to the Lagrangian \((1)\) leads to the set of three nonlinear field equations:

\[ \partial^2 \phi = - \left[ \sum_i U_i + m^2 \right] \phi, \]  

(3)

\[ \partial^2 \varphi_i = - |\phi|^2 \frac{\partial U_i}{\partial \varphi_i}. \]  

(4)

We shall limit ourselves to the consideration of the excitation spectrum of the field \( \phi \) with energies \( E' \ll 2m \). The unitary transformation

\[ \phi = \psi \exp(-imt) \]  

(5)

in mentioned nonrelativistic limit leads to the Schrödinger equation

\[ i \partial_t \psi = \frac{1}{2m} \left[ -\partial_z^2 + V \right] \psi, \]  

(6)

which describes the motion of a particle with the mass \( m \) in the effective potential

\[ V = \sum_i U_i(\varphi_i) \]  

(7)

provided by the relativistic fields \( \varphi_i \). The equations \((3) - (7)\) describe the system in which one can separate two types of motion: slow variations of the Schrödinger field \( \psi \) in the spacetime and rapid variations of the fields \( \varphi_i \). Neglecting the dependence of the amplitude \( |\phi| = |\psi| \) on variables \((t, z)\) we shall find the solution of equations \((4)\) in the form of solitary waves \([2, 3]\)

\[ \varphi_1 = \pm \sqrt{\frac{2\mu_1^2}{\lambda_1}} \sech \left( \mu_1 |\psi| s_1 \right), \]  

(8)

\[ \varphi_2 = \pm \sqrt{\frac{\mu_2}{\lambda_2}} \tanh \left( \frac{\mu_2}{\sqrt{2}} |\psi| s_2 \right), \]  

(9)
where \( s_i = (z - u_i t) / \sqrt{1 - u_i^2} \), and \( u_i \) are the velocities (free parameters), \(-1 < u_i < 1\).

Substitution of these solutions into (8) gives the effective potential

\[
V(t, z) = \frac{\mu_1^4}{\lambda_1} \text{sech}^2 (\mu_1 |\psi| s_1) \tanh^2 (\mu_1 |\psi| s_1) + \frac{\mu_2^4}{4 \lambda_2} \left[ \text{sech}^4 \left( \frac{\mu_2}{\sqrt{2}} |\psi| s_2 \right) - 1 \right].
\] (10)

Its explicit form depends on the relations between the parameters \( \mu_i, \lambda_i \) and \( u_i \). In Fig. 1 the typical variants of the potential \( V(t, z) \) in the stationary case \( (u_i = 0) \) as functions of \( z \) are shown. The interactions which have the shape of two-hump barrier are the most interesting. As it is well known \([4]\) the scattering of a particle with positive energy on such potential shows the resonant states. That corresponds to the formation of quasistationary state and subsequent quantum tunneling of a particle through the barrier into the region of large values of \( |z| \). The wave function of such process out of range of barrier action has the form

\[
\psi(z, t) = A(q) \left[ e^{-iqz} - S(q) e^{iqz} \right] e^{-iEt}.
\] (11)

where \( S(q) \) is the S-matrix, \( q \) is the momentum of particle, \( E \) is the energy, and the factor \( A(q) \) is determined by the density of the incident flux. Since the amplitude \( |\psi| \) is the function of \( z \) which does not decrease on infinities the potential takes the constant negative value in the limit \( |\psi| s_2 \to \pm \infty \),

\[
V_c = -\frac{\mu_2^4}{4 \lambda_2}.
\] (12)

Hence the equation (8) being nonlinear with respect to unknown \( \psi \) reduces to the ordinary Schrödinger equation with constant attractive potential in the shape of square step in the limit of large values of \( |z| \). If the energy \( E = k^2 / 2m > 0 \) counted from the value \( V(0, 0) = 0 \) then the momentum \( q \) in (11) equals to \( q = \sqrt{k^2 - V_c} \). By means of simple renormalization of energy, \( \tilde{E} = E - V_c / 2m \), the problem can be reduced to the problem of scattering on the potential \( \tilde{V} = V - V_c \) which is short-range. We note that such renormalization may be introduced already in the initial Lagrangian (1) by adding the constant term \(-V_c\) to the potential \( U_2 \) which does not change the solitary wave (9). New potential \( \tilde{U}_2 \) takes the form of the Higgs potential

\[
U_2 \to \tilde{U}_2 = \frac{\lambda_2}{4} \left[ \varphi_2^2 - \frac{\mu_2^2}{\lambda_2} \right]^2,
\] (13)

which is widely used in different applications of the field theory. Such normalization of \( U_2 \) in particular is convenient in cosmology during the study of the early Universe \([3, 4]\). For example, it can give the equation of state between the pressure \( p \) and the density \( \epsilon \) in the form \( p = -\epsilon \) which leads to the exponential expansion of the Universe in inflationary models \([4]\).

In order to clarify the physical contents of the terms in (11) we rewrite it as

\[
V(t, z) = \frac{1}{2 |\psi|^2} \sum_{i=1}^{2} (1 - u_i^2) \varepsilon_i(s_i) - \frac{\mu_2^4}{4 \lambda_2},
\] (14)

where \( \varepsilon_i \) are the energy densities of the fields \( \varphi_i \) and \( \varepsilon_2 \) is renormalized according to (13). The functions \( \varepsilon_i \) are localized in spacetime and corresponding total energies \( E_i = \int_{-\infty}^{\infty} \varepsilon_i \, dz \)
Figure 1: The effective potential (10) in the stationary case ($u_i = 0$) for different relations between the parameters $\mu_i, \lambda_i$. 
are finite. It means that the fields (8) and (9) form the local spacetime particle-like formations which are the carriers of the effective interaction in the nonlinear three-field system. The field $\varphi_1$ (8) generates the part of interaction in two-hump barrier shape and the field $\varphi_2$ (9) smooths it so that potential $V$ takes the value (12) in the limit of large $|z|$. This constant can be explained as a result of spontaneous symmetry breaking [8] of the Lagrangian (1) when the field $\varphi_2$ comes from unstable vacuum state with $\varphi_2 = 0$ to the state with stable vacuum $\varphi_2 = v$, where

$$v = \pm \frac{m_H}{2\sqrt{\lambda}} = \pm \frac{m_H}{2\sqrt{\lambda}}, \quad (15)$$

is the constant field (the Higgs condensate), $m_H = 2\mu_2 m$ is the Higgs mass, $\lambda = \lambda_2 m^2$ is Higgs self-constant corresponding to the free field $\varphi_2$, $m_H = m_H |\psi|$ and $\lambda = |\psi|^2$ are the effective values of these quantities in the presence of the field $\psi$. Taking (13) into account the constant constituent (12) can be expressed through the Higgs parameters,

$$V_c = -\frac{1}{64 \lambda} \frac{m_H^4}{m^2}. \quad (16)$$

This constant addition to the energy of interaction is proved to be important during the calculation of the time delay caused by the particle being located within the region inside the barrier [9]. This addition permits us introduce the natural cut off of the confinement potential at distances of the order of the range of barrier and provides suppression of the probability of tunneling of a particle through the barrier of constant height and width to the observed (or expected) value. The expression (16) itself can be considered as a relation between the Higgs mass and the Higgs self-constant with the weight multiplier $V_c$ being fixed independently [9].

3. Localization of massive field. According to (10) and (13) the renormalization potential $\tilde{V}$ is local in spacetime. Its form is determined by the sech-function and the dependence of the $|\psi|$ on the variables $(t, z)$ does not change the general behaviour of the effective potential. Therefore the equation (6) approximately can be considered as the ordinary Schrödinger equation. This approximation will be good enough everywhere except the cases when

$$\mu_1^2 |\psi|^2 s_1^2 \ll \frac{3}{5} \quad \text{and/or} \quad \mu_2^2 |\psi|^2 s_2^2 \ll \frac{8}{3}, \quad (17)$$

In this region of values of $s_2^2$ the form of the potential is determined by the wave function $\psi$ itself and the equation (6) takes the form of the nonlinear Schrödinger equation

$$i \partial_t \psi + \frac{1}{2m} \left[ \partial_z^2 + \gamma |\psi|^2 \right] \psi = 0 \quad (18)$$

with the effective coupling constant

$$\gamma = \frac{\mu_2^6}{4A_2} s_2^2 - \frac{\mu_1^6}{\lambda_1} s_1^2. \quad (19)$$

If $\gamma > 0$ then the motion of a particle is determined mainly by the part of the effective interaction formed by the kink/antikink (10) and the interaction is attractive. If $\gamma < 0$ then the main contribution to $V$ is made by the energy density of sech-wave (8) which forms the barrier. Since $s_2^2$ suppose to be small then their dependence on spacetime variables in
the equation (18) can be neglected. In such approximation it has single-soliton solution

$$\psi(z,t) = P \sqrt{\frac{2}{\gamma}} \sech \left[ P \left( z - \frac{Q}{m} t \right) \right] \exp \left[ iQz + i \frac{1}{2m} (P^2 - Q^2) t \right]$$

(20)

at $\gamma > 0$ and the boundary condition $\psi(z,t) \to 0$ at $|z| \to \infty$. Here $P$ and $Q$ are the free parameters (momenta). This solution describes the traveling wave in the form of the local formation with the finite total energy. The case $\gamma > 0$ is realized in the potentials of the type shown in Fig. 1 (third plot). They have an attractive part in the region inside the barrier. The function (20) describes the particle which is captured by the potential well. This phenomenon can be interpreted as capture of a particle by the barrier with formation of long-live (quasistationary) state. The particle can tunnel outside the barrier with small but finite probability caused by long tails of the wave function (20) coming from the sech-function.

This example shows that nonlinear character of motion of a particle (i.e. a motion with self-interaction originated from the interaction with the scalar fields $\varphi_i$) can lead to the localization of its wave function in finite region of spacetime.

From the wave function (20) one can restore the form of the wave $\varphi_2$,

$$\varphi_2 \sim \tanh \left[ P \left( z - \frac{Q}{m} t \right) \right].$$

(21)

Comparison with (11) shows that the approximation (17) describes the case of kink/antikink moving with the velocity $u_2 = (Q/m) \ll 1$ and the amplitude $|\psi|$ equals to the effective value $\sqrt{2P}/\mu_2$.

At $\gamma < 0$ the particle gets to the repulsion region and corresponding exact particular solution of the equation (18) can be written in the form

$$\psi(z,t) = P \sqrt{\frac{2}{|\gamma|}} \sec \left[ P \left( z - \frac{Q}{m} t \right) \right] \exp \left[ iQz - i \frac{1}{2m} (P^2 + Q^2) t \right].$$

(22)

It describes the wave propagating in whole space with the energy $E = (P^2 + Q^2)/2m$ and amplitude which has the singularities in the points where the function of secant turns into infinity. These singularities apparently do not have the physical meaning and are connected with the approximate character of the calculations. The solution (22) describes the scattering of a particle on the barrier at small values of the parameters $s_i^2$.

At $u_1 = (Q/m) \ll 1$ the argument of secant is proportional to $Ps_1 \sim |\psi|s_1 \ll 1$ and we can write

$$\psi(z,t) \approx P \sqrt{\frac{2}{|\gamma|}} \exp [iQz - i Et],$$

where the plane wave is distorted by the influence of the barrier and the coefficient $P \sqrt{2/|\gamma|}$ is the effective scattering amplitude.

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