Azimuthal angle correlations at large rapidities: revisiting density variation mechanism

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Abstract We discuss the angular correlation present in hadron–hadron collisions at large rapidity difference ($\bar{\alpha}_{S} y_{12} \gg 1$). We find that in the CGC/saturation approach the largest contribution stems from the density variation mechanism. Our principal results are that the odd Fourier harmonics ($v_{2n+1}$) decrease substantially as a function of $y_{12}$, while the even harmonics ($v_{2n}$) increase considerably with the growth of $y_{12}$.

1 Introduction

In this paper we address the problem of the azimuthal angle correlations of two hadrons with transverse momenta $\vec{p}_{T1}$ and $\vec{p}_{T2}$ and rapidities $y_{1}$ and $y_{2}$, at large values of $y_{12} \equiv |y_{1} - y_{1}| \gg 1/\bar{\alpha}_{S}$. Our main theoretical assumption is that these correlations stem from interactions in the initial state. We are aware that, unlike rapidity correlations which at large rapidities originate from the initial state interactions due to causality reasons [1], a substantial part of these correlations could be due to the interactions in the final states [2–4]. On the other hand, it has been demonstrated that at small rapidity difference $\bar{\alpha}_{S} y_{12} < 1$ the interactions in the initial state [5–14] yield the value of the correlations, which describe the major part of the experimentally observed correlations [15–37].

In this paper we concentrate our efforts on calculating the long range rapidity part of angular correlations with large value of the rapidity difference $y_{12}$. All previous calculations assumed that $\bar{\alpha}_{S} y_{12} < 1$ [5–14]. It turns out that in this kinematic region, the main source of the azimuthal angle correlations is the Bose–Einstein correlations of identical gluons, corresponding to the interference diagram in the production of two partonic showers. Intuitively, we expect that the correlations in the process, where two different gluons are produced from two different partonic showers, should not depend on the difference of rapidities ($y_{12}$), nor on the values of $y_{1}$ and $y_{2}$. Using the AGK cutting rules [38] one

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1 In the framework of perturbative QCD for the inclusive cross sections, the AGK cutting rules were discussed and proven in Refs. [44–52]. However, in Ref. [47] it was shown that the AGK cutting rules are violated for double inclusive production. This violation is intimately related to the enhanced diagrams [47–49,52] and to the production of gluon from the triple Pomeron vertex. It reflects the fact that different
The typical QT angle correlations originate from the integration over does not have these features. We will show that the azimuthal correlation function turns out to be proportional to \( G \), which has a non-perturbative origin. Roughly speaking, the gluons with \( \vec{p}_{T1} \) and \( \vec{p}_{T2} \), which have contributions proportional to \( (\vec{p}_{T1} \cdot \vec{Q}_{T})^n (\vec{p}_{T2} \cdot \vec{Q}_{T})^n \). Recall that these kinds of vertices are the only possibilities to obtain angular correlations in the classical Regge analysis [53]. This mechanism for azimuthal angular correlations was suggested in Ref. [54] (see also Refs. [10,55–57]), and in the review of Ref. [57] it was called the density variation mechanism.

The paper is organized as follows. In the next section we discuss the contribution of the diagram of Fig. 2 in the momentum representation. In the remainder of the paper, we will use the mixed representation: the dipole sizes and momentum representation. In the remainder of the paper, we will use the mixed representation: the dipole sizes and momentum representation.

### 2 Correlations in the momentum representation

The double inclusive cross section of Fig. 2 takes the following form:

\[ \text{(expression)} \]

Footnote 1 continued

cuts of the triple BFKL Pomeron vertex with the produced gluon lead to different contributions. We do not consider such diagrams.
\[
\frac{d^2 \sigma}{dy_1 d^2 p_{T1} dy_2 d^2 p_{T2}} \quad (\text{Fig.} \, 2)
= \frac{2 C_F a_s \mu_{\text{soft}}^2}{(2 \pi)^2} \int \frac{d^2 k_T \, d^2 k_{T'} \, d^2 Q_T \, d^2 Q_T'}{(2 \pi)^2} k_T^2 (k_T')^2 (k_T' - Q_T)^2 \left( -\vec{k}_T + \vec{Q}_T; Y - y_1 \right)
\times \phi_H^G (k_T - \bar{Q}_T; -\vec{k}_T + \vec{Q}_T; Y - y_1) \Gamma_{\nu} (-\vec{k}_T + \vec{Q}_T; \vec{p}_{T1}) \times \phi (\vec{Q}_T; y_2) \phi'_{H} (\vec{Q}_T + \vec{Q}_T, \vec{k}_T') \Gamma_{\mu} (\vec{k}_T - \vec{p}_{T2}) \frac{1}{(2 \pi)^2} N (Q_T') \frac{d^2 Q_T'}{(2 \pi)^2} \frac{d^2 Q_T}{(2 \pi)^2} \frac{d^2 k_T}{(2 \pi)^2} \frac{d^2 k_{T'}}{(2 \pi)^2}
\]

where \( \phi_H^G (\vec{k}_T, -\vec{k} + \vec{Q}_T) \), as well as all other functions \( \phi \) of this type, are the correlation functions, which at \( Q_T' = 0 \) give the probability to find a gluon with transverse momentum \( \vec{k}_T \) in the hadron (nucleus) of the projectile (target). \( \phi (\vec{k}_T, -\vec{k} + \vec{Q}_T; \vec{k}_T', -\vec{k}_T' + \vec{Q}_T) \) describes the interaction of two gluons with momenta \( \vec{k}_T \) and \( \vec{k}_T' \), which scatter at momentum transferred \( Q_T' \). \( N (Q_T') \) is a pure phenomenological form factor that describes the probability to find two Pomerons in the projectile or target, with transferred moment \( \vec{Q}_T \) and \( -\vec{Q}_T \). \( C_F = (N_c^2 - 1)/2N_c \) where \( N_c \) is the number of colors. The Lipatov vertex \( \Gamma_{\mu} (k_T, p_{T1}) \) has the following form:

\[
\Gamma_{\mu} (k_T, p_{T1}) = \frac{1}{p_{T1}} \left( k_T^2 p_{T1,\mu} - k_T, p_{T1,\mu}^2 \right). \quad (2.2)
\]

Using Eq. (2.2) we obtain:

\[
2 \Gamma_{\nu} (-\vec{k} + \vec{Q}_T; \vec{p}_{T1}) \Gamma_{\nu} (\vec{k} - \vec{Q}_T - \vec{Q}_T', \vec{p}_{T1}) = \frac{1}{p_{T1}} \left( (-\vec{k} + \vec{Q}_T)^2 (\vec{k}_T - \vec{p}_{T1} - \vec{Q}_T - \vec{Q}_T')^2 + (-\vec{k} + \vec{p}_{T1} - \vec{Q}_T)^2 (\vec{k}_T - \vec{Q}_T - \vec{Q}_T')^2 \right) - Q_T^2;
\]

\[
2 \Gamma_{\mu} (-\vec{k}_T - \vec{p}_{T2} + \vec{Q}_T', \vec{p}_{T2}) \Gamma_{\mu} (\vec{k}_T - \vec{p}_{T2} - \vec{Q}_T, \vec{p}_{T2}) = \frac{1}{p_{T2}} \left( (-\vec{k}_T - \vec{p}_{T2} + \vec{Q}_T')^2 (\vec{k}_T - \vec{Q}_T)^2 + (-\vec{k}_T' + \vec{Q}_T + \vec{Q}_T')^2 (\vec{k} - \vec{p}_{T2} - \vec{Q}_T')^2 \right) - Q_T^2. \quad (2.3)
\]

We can simplify the master equation (see Eq. (2.1)) by observing the dependence on \( Q_T' \) and \( Q_T'' \) is determined by the non-perturbative scale of the projectile (target) structure, which in Eq. (2.1) is absorbed in the phenomenological form factors \( N(Q_T') \) and \( N(Q_T'') \). Therefore, the typical \( Q_T' \) and \( Q_T'' \) turn out to be of the order of the soft scale \( \mu_{\text{soft}} \), which is much smaller that the other typical momenta in Eq. (2.1), assuming that \( p_{T1} \) and \( p_{T2} \) are larger than \( \mu_{\text{soft}} \). Introducing

\[
\mu_{\text{soft}}^2 = \int \frac{d^2 Q_T'}{(2 \pi)^2} N (Q_T') \quad (2.4)
\]

we can neglect \( Q_T' \) and \( Q_T'' \) in the BFKL Pomeron Green functions and rewrite Eq. (2.1) in the form

\[
\frac{d^2 \sigma}{dy_1 d^2 p_{T1} dy_2 d^2 p_{T2}} \quad (\text{Fig.} \, 2) \quad = \frac{2 C_F a_s \mu_{\text{soft}}^2}{(2 \pi)^2} \int \frac{d^2 k_T}{(2 \pi)^2} \frac{d^2 k_{T'}}{(2 \pi)^2} \frac{d^2 Q_T}{(2 \pi)^2} \frac{d^2 Q_T'}{(2 \pi)^2} \frac{d^2 k_T}{(2 \pi)^2} \frac{d^2 k_{T'}}{(2 \pi)^2} \frac{d^2 Q_T}{(2 \pi)^2} \frac{d^2 Q_T'}{(2 \pi)^2} \frac{d^2 k_T}{(2 \pi)^2} \frac{d^2 k_{T'}}{(2 \pi)^2}
\]

with Eq. (2.3), which takes the following form:

\[
2 \Gamma_{\nu} (-\vec{k} + \vec{Q}_T; \vec{p}_{T1}) \Gamma_{\nu} (\vec{k} - \vec{Q}_T - \vec{Q}_T', \vec{p}_{T1}) = \frac{1}{p_{T1}} \left( (-\vec{k} + \vec{Q}_T)^2 (\vec{k}_T - \vec{p}_{T1} - \vec{Q}_T)^2 + (-\vec{k} + \vec{p}_{T1} - \vec{Q}_T)^2 (\vec{k}_T - \vec{Q}_T - \vec{Q}_T')^2 \right) - Q_T^2;
\]

\[
2 \Gamma_{\mu} (-\vec{k}_T - \vec{p}_{T2} + \vec{Q}_T', \vec{p}_{T2}) \Gamma_{\mu} (\vec{k}_T - \vec{p}_{T2} - \vec{Q}_T, \vec{p}_{T2}) = \frac{1}{p_{T2}} \left( (-\vec{k}_T - \vec{p}_{T2} + \vec{Q}_T')^2 (\vec{k}_T - \vec{Q}_T)^2 + (-\vec{k}_T' + \vec{Q}_T + \vec{Q}_T')^2 (\vec{k} - \vec{p}_{T2} - \vec{Q}_T')^2 \right) - Q_T^2. \quad (2.6)
\]

At high energies the parton densities \( \phi_{H}^G (-\vec{l}_T, \vec{l}_T; Y - y_2) \) in Eqs. (2.1) and (2.5) are proportional to \( \exp (\Delta_{\text{BFKL}} Y - y_2) \) for the BFKL Pomeron, where \( \Delta_{\text{BFKL}} = 2.8 a_s \) is the intercept of the BFKL Pomeron. Bearing this in mind, one can see that the interference diagram for the double inclusive cross section does not depend on \( y_1, y_2 \) or on \( y_{12} \). As \( \phi_{H}^G (-\vec{l}_T, \vec{l}_T; Y - y_2) \propto \exp (\Delta_{\text{BFKL}} Y - y_2) \) the main diagram of Fig. 1a also does not depend on the rapidities.

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3.1 BFKL Pomeron: the simplest approach for a one parton shower

The single inclusive cross section resulting from the one BFKL Pomeron is well known, and it is equal to

\[
\frac{d^2\sigma}{dy d^2p_T} = \frac{2C_F\alpha_s}{(2\pi)^2} \int d^2k_T \phi_H^G(\vec{k}_T; Q_T = 0; y) \times \phi_H^G(\vec{k}_T - \vec{p}_T; Q_T = 0; y) \times \Gamma_\nu(\vec{k}_T, \vec{p}_T) \Gamma_\nu(-\vec{k}_T + \vec{Q}_T, \vec{p}_T).
\]

(3.1)

The relation between the parton densities \(\phi\) and the Green function of the BFKL Pomeron has been given in Ref. [46]:

\[
N_{BFKL}^{\nu}(r, r_1; y, Q_T = 0) = \frac{\alpha_s}{2} \int d^2k_T \left(1 - e^{i\vec{k}_T \cdot \vec{r}}\right) \phi_H^G(\vec{k}_T, Q_T = 0; y)
\]

(3.2)

where \(N_{BFKL}^{\nu}(r, r_1; Y)\) is given by Eq. (A.1) or by Eq. (A.9), in the high energy limit. Equation (3.2) can be rewritten as follows:

\[
\phi_H^G(\vec{k}_T, Q_T = 0; y) = \frac{2}{\alpha_s} \int d^2r e^{i\vec{k}_T \cdot \vec{r}} \nabla_r^2 N_{BFKL}^{\nu}(r, r_1; y, Q_T = 0).
\]

(3.3)

We have

\[
\Gamma_\nu(\vec{k}_T, \vec{p}_T) \Gamma_\nu(-\vec{k}_T + \vec{Q}_T, \vec{p}_T) = \frac{k_T^2 (\vec{k}_T - \vec{p}_T)^2}{p_T^2}.
\]

(3.4)

Substituting Eq. (3.2) and also Eq. (3.4) into Eq. (3.1) we obtain [46]

\[
\frac{d^2\sigma}{dy d^2p_T} = \frac{8C_F}{\alpha_s (2\pi)^2} \int d^2r e^{i\vec{r} \cdot \vec{r}_T} \nabla_r^2 N_{BFKL}^{\nu}(r, r_1; Y, y, Q_T = 0) \times (r, r_2; y, Q_T = 0)
\]

(3.5)

where \(N_{pr}\) and \(N_t\) denote the probability to find a dipole in the projectile and target, respectively. \(r_1\) and \(r_2\) are the typical dipole sizes in the projectile and target.

As can be seen from Eq. (2.1) we need to generalize Eq. (3.5) for the case \(Q_T \neq 0\). Equation (3.1) has to be replaced by

\[
\frac{d^2\sigma}{dy d^2p_T} (Q_T \neq 0) = \frac{2C_F\alpha_s}{(2\pi)^2} \int d^2k_T \phi_H^G(\vec{k}_T, Q_T = 0; y) \times \phi_H^G(\vec{k}_T - \vec{p}_T, Q_T = 0; y) \times \Gamma_\nu(\vec{k}_T, \vec{p}_T) \Gamma_\nu(-\vec{k}_T + \vec{Q}_T, \vec{p}_T). 
\]

(3.6)

Taking into account Eq. (3.2) for \(Q_T \neq 0\) and

\[
\Gamma_\nu(\vec{k}_T, \vec{p}_T) \Gamma_\nu(-\vec{k}_T + \vec{Q}_T, \vec{p}_T) = \frac{1}{2} \left[ \left(\vec{k}_T - \vec{Q}_T\right)^2 \left(\vec{k}_T - \vec{p}_T\right)^2 + \left(\vec{k}_T\right)^2 \left(\vec{k}_T - \vec{p}_T - \vec{Q}_T\right)^2 - Q_T^2 \right]
\]

(3.7)

we rewrite Eq. (3.5) in the form

\[
\frac{d^2\sigma}{dy d^2p_T} (Q_T \neq 0) = \frac{4C_F}{\alpha_s (2\pi)^2} \frac{1}{p_T^2} \int d^2r e^{i\vec{r} \cdot \vec{r}_T} \times \left\{ - \nabla_r^2 N_{BFKL}^{\nu}(r, r_1; Y, y, Q_T) \times (i\nabla_r - \vec{Q}_T)^2 N_{BFKL}^{\nu}(r, r_2; y, Q_T) + (i\nabla_r - \vec{Q}_T)^2 (\nabla_r^2) N_{BFKL}^{\nu}(r, r_2; y, Q_T) \times (r, r_1; Y, y, Q_T) N_{BFKL}^{\nu}(r, r_2; y, Q_T) \right\}
\]
3.2 General estimates

It should be stressed that the single inclusive production has the form of Eqs. (3.5) and (3.8) for the general structure of the single parton shower, as was shown in Ref. [46]. For example, for the process shown in Fig. 1c. We need only to substitute \( N_u^G (r, r_2; y, Q_T) \) for \( 2N_u^{BFKL} (r, r_2; y, Q_T) \) where

\[
2N_u^{BFKL} (r, r_2; y, Q_T) \rightarrow N_u^G (r, r_2; y, Q_T)
\]

\[
= 2N_u (r, r_2; y, Q_T) - \int d^2 Q_T' N_u (r, r_2; y, Q_T') \nonumber
\]

\[
- \tilde{Q}_T) N_u \left( r, r_2; y, \tilde{Q}_T \right) ;
\]

(3.9)

\( N_u (r, r_2; y, Q_T) \) is a solution to the non-linear evolution equation. For the case of inclusive production, we can considerably simplify the estimates noting that

\[
\int d^2 Q_T N_u (r, r_2; y, Q_T) \rightarrow N_u^{BFKL} (r, r_2; y, Q_T) \ll 1;
\]

\[
\int d^2 Q_T N_u (r, r_2; y, Q_T) \rightarrow N_u^{BFKL} (r, r_2; y, Q_T) \ll 1;
\]

(3.10)

where \( Q_s(y) \) denotes the saturation momentum.

In other words, the main contribution to inclusive production comes from the vicinity of the saturation scale, where \( r^2 Q_s^2 \approx 1 \). Fortunately, the behavior of \( N \) in this kinematic region is determined by the linear BFKL evolution equation [62–67] and has the following form [68]:

\[
N_u (r, r_2; y, Q_T = 0) \propto \left( r^2 Q_s^2 (y) \right)^{1 - \gamma_{cr}} \text{ with } Q_s^2
\]

\[
= (1/r_2^2) \exp \left( \frac{\omega (y = 1/2 + i \nu = \gamma_{cr})}{1 - \gamma_{cr}} \right)
\]

(3.11)

where \( \gamma_{cr} = 0.37 \).

From Eq. (A.8) we see that, for \( Q_T \neq 0 \), the scattering amplitude decreases at \( Q_s^2(r^2/r_2^2) \gg 1 \). Therefore, we need to consider rather small values of \( Q_T \): \( Q_s^2(r^2/r_2^2) \ll 1 \). The product of vertices that determines the amplitude has two terms (see Eq. (A.5)) which are proportional to \( (r^2/r_2^2)^{4ν} \) and to \( (Q_s^2(r^2/r_2^2))^{4ν} \). Therefore, the maximum of \( \nabla_s^2 \) can be reached if \( r^2/r_2^2 e^{κ_y} \sim 1 \) and \( Q_s^2(r^2/r_2^2 e^{κ_y}) \sim 1 \) and the amplitude then has the following form:

\[
N_u (r, r_2; y, Q_T) \propto c_1 \left( r^2/r_2^2 e^{κ_y} \right)^{1 - \gamma_{cr}}
\]

\[
+ c_2 \left( Q_s^2(r^2/r_2^2 e^{κ_y}) \right)^{1 - \gamma_{cr}} .
\]

(3.12)

The first term does not depend on \( Q_T \) and, therefore, the upper limit of the integral over \( Q_T \), goes up to \( (Q_s^2)^2) \approx 1/(r r_2) \). The second term, both for \( Q_s^2(r^2/r_2^2 < e^{-κ_y} \) and for \( Q_s^2(r^2/r_2^2 > e^{-κ_y} \), turns out to be small. Indeed, in the first region the amplitude is small, while in the second region we are deep in the saturation domain where \( \nabla_s^2 \) is small. Hence, we expect that in the integral over \( Q_T \), the first term gives a larger contribution than the second term, and we will only keep this contribution in our estimates.

4 Double inclusive cross section for two parton shower production

4.1 The simplest diagram

In this section we calculate the simplest diagram of Fig. 2.

We need to integrate the product of two BFKL Pomerons over \( Q_T \) (see Eq. (2.5)):

\[
I = \int d^2 Q_T V_{i_1} (\vec{r}_1, \vec{Q}_T) V_{i_2} (\vec{r}_2, \vec{Q}_T) V_{i_1} (\vec{r}_1, \vec{Q}_T) V_{i_2} (\vec{r}_2, \vec{Q}_T).
\]

(4.1)

From Eq. (2.5) in the momentum representation, we see that \( r_1 \neq r_1' (r_2 \neq r_2') \), however, they are close to each other, being determined by the same momentum \( k_T \). We assume that \( p_T = k_T \) and \( Q_s (Y - y_1) \approx μ_{soft} \). Considering \( r_1 \approx r_1' \approx r_2 \approx r_2' \) we will show that in the integral over \( Q_T \), the typical \( Q_T \approx 1/r_2 \). In other words, the dependence of \( Q_T \) is determined by the largest of interacting dipoles.

From Eq. (A.8) we see that, for large \( Q_T \), when \( r_1^2 r_2^2 Q_T^2 \gg 1 \) and \( r_1^2 r_2^2 Q_T^2 \gg 1 \), the integrand is proportional to \( 1/Q_T^4 \) and converges. The main region of interest is \( r_1^2 Q_T^2 \gg 1 \) and \( r_1^2 Q_T^2 \ll 1 \). In this kinematic region for the vertices

\[
V_{i_1} (\vec{r}_1, \vec{Q}_T) \text{ and } V_{i_2} (\vec{r}_2, \vec{Q}_T) ,
\]

we can use Eq. (A.6), where the conjugated vertices are still in the regime of Eq. (A.8). Eq. (4.1) then takes the form:

\[
I = 2^{6i(\nu_1 + \nu_2)} (16\nu_1 \nu_2) \pi
\]

\[
\times \int_{1/r_2^2} d Q_T^2 \left\{ \left( Q_1^2 r_1^2 \right)^{-i\nu_1} - \left( Q_1^2 r_1^2 \right)^{-i\nu_1} \right\}
\]

\[
\times \left\{ \left( Q_2^2 r_2^2 \right)^{i\nu_2} - \left( Q_2^2 r_2^2 \right)^{i\nu_2} \right\}
\]

\[
\times \frac{\cos^2 \left( \frac{1}{2} \vec{Q}_T \cdot \vec{r}_1 \right)}{Q_T^2 r_2^2}.
\]

(4.2)

Assuming that both \( \nu_1 \) and \( \nu_2 \) are small, we see that all four terms are equal to each other, and the integral can be written as follows:
I = 2^{6i(v_1+v_2)} \left(-2^{6} v_1 v_2\right) \frac{1}{i (v_1 + v_2)} \left(\frac{r_1^2}{r_2^2}\right)^{i (v_1+v_2)} \frac{1}{r_2^2}.

(4.3)

The appearance of the pole \(v_1 = -v_2\) indicates that the contribution from this kinematic region is large.

Closing the contour of integration on \(v_2\) over the pole, we obtain

\[ I = 2^6 \pi v_1^2 \frac{1}{r_1^2}. \tag{4.4} \]

Actually, the double inclusive cross section depends on \(V^2N\) as we argued in the previous section. Repeating the procedure for

\[ \mathcal{I} = \int d^2 \zeta_T \mathcal{V}_j^2 \left( r_1, v_1 \left( \tilde{r}_1, \tilde{Q}_T \right) \right) \mathcal{V}_j^2 \left( r_1', v_1' \left( \tilde{r}_2, \tilde{Q}_T \right) \right) \]

\[ \times \mathcal{V}_{r_1} \left( r_2, v_2 \left( \tilde{r}_1', \tilde{Q}_T \right) \right) \mathcal{V}_{r_2} \left( r_2', v_2' \left( \tilde{r}_2', \tilde{Q}_T \right) \right). \tag{4.5} \]

we obtain for small \(v_1\) and \(v_2\)

\[ \mathcal{I} = 2^6 \pi v_1^2 \frac{1}{r_1 r_1' r_2^2 r_2'}. \tag{4.6} \]

Using the method of steepest descent, to integrate over \(v_1\), we obtain the following contribution:

\[ \mathcal{I} = 2^5 \frac{1}{r_1 r_1' r_2^2 r_2'} \sqrt{\frac{\pi}{2 D y_{12}}} e^{2 \Delta_{\text{BFKL}} y_{12}} \tag{4.7} \]

where \(\Delta_{\text{BFKL}}\) and \(D\) are defined in Eq. (A.2).

Rewriting Eq. (2.5) in the coordinate representation we obtain

\[
\frac{d^2 \sigma}{dy_1 d^2 p_{T1} d^2 y_2 d^2 p_{T2}} \quad (\text{Fig. 2})
\]

\[
= \left(\frac{2C_F \mu_0^2}{\alpha_s(2\pi)^2}\right)^2 \frac{1}{p_{T1} p_{T2}} \int \frac{d^2 Q_T}{(2\pi)^2}
\]

\[
\times \int d^2 r_1 d^2 r_2 d^2 \tilde{r}_1 d^2 \tilde{r}_2 e^{-i \tilde{\rho}_r \tilde{\rho}_{r'} \delta(2)} \left( \tilde{r}_1 + \tilde{r}_2 - \tilde{r}_1 - \tilde{r}_2 \right)
\]

\[
\times \mathcal{V}_{r_1} N_{tr} \left(r_1; Y - y_1\right) e^{i \tilde{Q}_T \tilde{\rho}_r} \mathcal{V}_{r_1} N_{tr} \left(r_1'; Y - y_1\right)
\]

\[
\times \mathcal{V}_{r_2} N_{tr} \left(r_2; Q_T; y_{12}\right) e^{i \tilde{Q}_T \tilde{\rho}_{r'} \delta(2)} \left( \tilde{r}_2 + \tilde{r}_2' - \tilde{r}_1 - \tilde{r}_2\right)
\]

\[
\times \mathcal{V}_{r_2} N_{tr} \left(r_2; y_2\right) e^{i \tilde{Q}_T \tilde{\rho}_{r'} \delta(2)} \mathcal{V}_{r_2} N_{tr} \left(r_2'; y_2\right). \tag{4.8} \]

In Eq. (4.8) we have neglected the terms which are proportional to \(Q_T^2\) (see Eq. (2.5)), since, as we have argued, the typical \(Q_T\) are small, and because these terms do not lead to additional correlations in the azimuthal angles. In Appendix B we calculate this integral and obtain the final expression for the double inclusive cross section:

\[
\frac{d^2 \sigma}{dy_1 d^2 p_{T1} d^2 y_2 d^2 p_{T2}} \quad (\text{Fig. 2})
\]

\[
= 4\pi \left(\frac{2C_F}{\alpha_s(2\pi)^2}\right)^2 \frac{1}{p_{T1} p_{T2}} \sqrt{\frac{1}{2 D \left(Y - y_1\right)}} \sqrt{\frac{1}{2 D y_{12}}} e^{2 \Delta_{\text{BFKL}} y_{12}}. \tag{4.9} \]

\[ \Delta \]

4.2 The CGC/saturation approach

The integral over \(k_T^2\) in Eq. (B.6) has an infrared singularity with a cutoff at \(p_{T2}\), since we assume that \(p_{T2}\) is the smallest momentum. This reflects the principal feature of the BFKL Pomeron parton cascade, which has diffusion, both in the region of small and large transverse momenta. On the other hand, we know that the CGC/saturation approach suppressed the diffusion in the small momenta [44], providing the natural cutoff for the infrared divergency. We expect that such a cutoff will be the value of the smallest saturation momenta: \(Q_s(Y - y_1)\) or \(Q_s(y_2)\), which will replace one of the \(p_{T2}\) in the denominator of Eq. (4.9). Therefore, we anticipate that for a realistic structure of the one parton shower cascade, (see Fig. 1c for example), the contribution for the double inclusive cross section will be different.

We need to specify the behavior of the scattering amplitude in the vicinity of the saturation scale. We have discussed the basic formulas [68] of Eq. (3.11), but for integration over the dipole sizes we need to know the size of this region. The scattering amplitude can be written in the form

\[
N \left(r_1, r_2; Y\right) = \int_{\epsilon - i \infty}^{\epsilon + i \infty} \frac{dy}{2\pi} n_{in} \left(\gamma\right) e^{i \omega(\gamma, 0) Y - (1 - \gamma) \xi} \tag{4.10} \]

where \(\omega(\gamma, 0)\) is given by Eq. (A.2), replacing \(\frac{1}{2} + i v \equiv \gamma\) and \(\xi = \ln \left(r_1^2 / r_2^2\right)\). The saturation scale is determined by the line on which the amplitude is a constant \(C\), of the order one. This leads to the following equation for the saturation scale [62,68]:

\[
\omega \left(\gamma_{cr}, 0\right) Y - (1 - \gamma_{cr}) \xi_\gamma = 0; \quad \omega' \left(\gamma, 0\right) Y - \xi_\gamma = 0, \tag{4.11} \]

which results in the value of \(\gamma_{cr}\) given by the equation

\[
\frac{\omega \left(\gamma_{cr}, 0\right)}{1 - \gamma_{cr}} = \omega' \left(\gamma, 0\right), \tag{4.12} \]

and gives \(\gamma_{cr} = 0.37\), with the equation for the saturation momentum:

\[
\xi_\gamma = \ln \left(Q_s^2 / r_2^2\right) = \kappa Y = \frac{\omega \left(\gamma_{cr}\right)}{1 - \gamma_{cr}} Y. \tag{4.13} \]
Expanding the phase \( \omega(y',0) \) of \( 1-(1-y)\xi \) in the vicinity \( \Delta \xi = \xi - \xi_s \) and \( \Delta \gamma = \gamma - \gamma_C \) we obtain

\[
N(r_1, r_2; Y) = C \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dy}{2\pi} \left( \frac{r_1^2 Q_s^2}{Q_{sT}} \right)^{1-\gamma_C} \int \frac{d\Delta y}{2\pi} e^{\gamma_{\gamma}^{\Delta} (y,0) Y (\Delta y^2 + \Delta y \Delta \xi)} \]

\[
= \left( \frac{r_1^2 Q_s^2}{Q_{sT}} \right)^{1-\gamma_C} C \sqrt{\frac{\pi}{D Y}} e^{-\frac{(\Delta \xi)^2}{4 \Delta \gamma}}. \tag{4.14}
\]

At first sight, Eq. (4.14) shows that the amplitude has a maximum at \( \tau = r_1^2 Q_s^2 = 1 \). However, this is not correct. Equation (4.14) gives the correct behavior for \( \tau < 1 \), while for \( \tau > 1 \) we need to take into account the interaction of the BFKL Pomerons and the non-linear evolution, generated by these interactions. The general result of this evolution is the fact that the amplitude depends on one variable \([69-72] \tau \), i.e. \( \tau (\tau) \) (as it shows geometric scaling behavior). The peak at \( \tau = 1 \) appears in

\[
\nabla^2 N(r_1, r_2; Y) = 4 Q_s^2(Y) \frac{1}{\tau} \frac{d}{d\tau} N(\tau). \tag{4.15}
\]

From Eq. (4.15) we can conclude that the width of the distribution in \( r_1^2 \) is of the order of \( Q_s^2 \), but it depends crucially on the model for the Pomeron interaction. In Fig. 3a we plot this value for the behavior of the scattering amplitude deep in the saturation domain (see Ref. [72]).

This approach is not correct for \( \tau \rightarrow 1 \) and \( -\nabla^2 N = 1.58 \) at \( \tau = 1 \), but it starts to be small at \( \tau > 2 \), which could be large enough to trust the formulas of Ref. [72]. At least such a conclusion can be justified considering the fit of the DIS data in the saturation model of Refs. [73,74], which is based on the idea of Ref. [75], and which has the correct behavior of the scattering amplitude, both deep in the saturation domain, and near \( \tau = 1 \). Hence, we expect that \( \nabla^2 N \) decreases faster than we can see from Eq. (4.14). Bearing these conclusions in mind, we will calculate the contribution of Fig. 2, keeping all \( N \) in Eq. (4.8) in the vicinities of the saturation scales, by replacing \( \int_0^\infty d\tau (-\nabla^2 N) = -\int_1^1 d\tau (-\nabla^2 N) \).

We will show in the following that we cannot integrate over the dipole sizes, so that all six Pomerons will be in the vicinity of the saturation scale. At least two of the Pomerons occur either deep in the saturation domain, or in the perturbative QCD region. We believe that the largest contribution stems from the exchange of two Pomerons between rapidities \( y_1 \) and \( y_2 \) (see Fig. 3b), which are in the perturbative QCD region. Unfortunately, we cannot use the AGK cutting rules [38], which state that these Pomerons will not be affected by the Pomeron interaction, and the contributions of these interactions (see the red Pomeron in Fig. 3b) are canceled. Indeed, it has been proven that for the double inclusive production [47] they are not applicable in perturbative QCD. On the other hand, these Pomerons carry transverse momentum \( Q_T \), unlike the others in the diagram, which is larger than the saturation scale \( Q_s(y_2) \); hence, their contributions are suppressed in comparison with the other Pomerons in Fig. 2.

In addition our choice leads to the natural matching with the region \( \bar{a}s \gamma_{12} < 1 \).

The integration over \( Q_T \) will produce the same result as Eq. (4.7), as in the previous section. In Appendix C we discuss making estimates for the integrals over the dipole sizes which lead for \( p_T \ll Q_s^2(y_1) \) to the following cross section:

\[
\frac{d^2\sigma}{dy_1 d^2 p_{T1}} (Q_T = 0; \text{Eq. (3.5)}) = \frac{12\pi^2 C_F}{\alpha_s(2\pi)^2} \int d^2 r e^{i\vec{r} \cdot \vec{R}} \frac{1}{p_T^2} \frac{1}{Q_s^2} N_BFKL(r, r_1; y, Q_T = 0) \times N_BFKL(r, r_2; y, Q_T = 0) \times C^2 \exp \left( \frac{-\ln^2(Q_s^2(y,0)/Q_s^2(y))}{4D y} \right).
\tag{4.16}
\]

In Eq. (4.16) we used backward evolution from the saturation boundary where \( N = C \).

The ratio of the two contributions takes the following form:

\[
R = \frac{d^2\sigma}{dy_1 d^2 p_{T1} dy_2 d^2 p_{T2}} (\text{Fig. 3b}) \frac{d^2\sigma}{dy_1 d^2 p_{T1} dy_2 d^2 p_{T2}} (\text{Fig. 1a})
\]
in the leading order of perturbative QCD with $\bar{\alpha}_S = 0.25$. In e we take $\Delta_{\text{BFKL}} = 0.25$ and $Q_2^2(Y) \propto \exp(\lambda Y)$ with $\lambda = 0.25$. These numbers correspond to the BFKL phenomenology.

5 Azimuthal angle correlations

The azimuthal angle correlations stem from terms $(\vec{Q}_T \cdot \vec{r}_i)^n$ in the vertices (see Eqs. (A.5), (A.6)). Indeed, after integrating over $r_i$ these terms transform to expressions of the following type [54]: $(\vec{Q}_T \cdot \vec{r}_1)^{m_1} (\vec{Q}_T \cdot \vec{r}_2)^{m_2}$, which lead to the term $(\vec{p}_{T1} \cdot \vec{p}_{T2})^m$. We have illustrated in Eqs. (A.5) and (A.6) how these originate from the general form of the BFKL Pomeron vertices in the coordinate representation. From Eqs. (A.5) and (A.6) only terms proportional to $(\vec{Q}_T \cdot \vec{r}_i)^n$ with even $n$ appear in the expansion. Therefore, the azimuthal angle ($\phi$) correlation function contains only terms $\cos^{2n}(\phi)$, and it is invariant with respect to $\phi \rightarrow \pi - \phi$.

In other words, $v_n$ with odd $n$ turn out to be zero. Hence, we have the first prediction: the value $v_n$ with odd $n$ should decrease with $y_{12}$, and their dependence should follow the dotted lines in Fig. 4a.

We return to Eq. (4.1) and integrate over $Q_T$, collecting terms that depend on the angles between $\vec{Q}_T$ and $\vec{r}_1$, which we have neglected in the previous section. As we have learned, the typical values of $Q_T \propto 1/r_1 \sim 1/r_2'$ where $r_2$ and $r_2'$ are larger than $r_1$ and $r_1'$. In other words, we showed that the main contributions stem from the kinematic regions $r_1^2 Q_2^2(y - y_1) \sim 1$ ($r_1^2 Q_2^2(y - y_1) \sim 1$) and $r_2^2 Q_2^2(y_2) \sim 1$ ($r_2^2 Q_2^2(y_2) \sim 1$). Assuming that $Q_2(y - y_1) \gg Q_2(y_2)$ we conclude that $r_1(r_1') \ll r_2(r_2')$. The typical $Q_T$ is determined by the largest dipoles and, therefore, we expect $Q_T(1/r_1^2(1/r_2^2) \sim 1)$, as has been demonstrated above. Bearing these estimates in mind, we can replace vertices $V_{i1} (\vec{r}_1, \vec{Q}_T)$ and $V_{i2} (\vec{r}_1, \vec{Q}_T)$ in Eq. (4.1) by Eq. (A.6) in which we put $Q_T = 1/r_2$ and $Q_T = 1/r_2'$, respectively. Taking into account that $r_1/r_2 \ll 1(r_1'/r_2' \ll 1)$, we obtain

$$\frac{1}{(2 D y_{12})^3} e^{2\Delta_{\text{BFKL}} y_{12}} < C.$$ (4.18)
At first sight Eq. (5.1) should enter two angles between \( \vec{Q}_T \) and \( \vec{r}_1 \) and \( \vec{r}_1' \), respectively. However, in the integrand for integration over \( r_1 \) (see Eq. (B.1)) it depends only on one vector \( \vec{p}T_1 \). Therefore, after integration over all angles, we find that the angle \( \phi \) in Eq. (5.1) is the angle between \( \vec{Q}_T \) and \( \vec{p}T_1 \).

For vertices \( V_{v_1} (\vec{r}_2, \vec{Q}_T) \) and \( V_{v_2} (\vec{r}_2', \vec{Q}_T) \) in Eq. (4.1) we use Eq. (A.8). Finally, we need to evaluate the integral

\[
I_Q = -16v_1v_2 \int Q_TdQ_T \{ V_{v_1} (\vec{r}_1, \vec{Q}_T) V_{v_2} (\vec{r}_1', \vec{Q}_T) \}_{\text{Eq. (5.1)}} \times (2\pi)^{-i(v_1+v_2)} \left( \frac{\vec{Q}_T \cdot \vec{r}_2 - r_2}{Q_T r_2^2} \right) \cos \left( \theta - \theta' \right) \nu \nu' \right)
\]

with better accuracy that we did in Sect. 5.1, keeping the dependence on the angle between \( \vec{Q}_T \) and \( \vec{r}_2 \). Note that the factor \( \cos \left( \theta - \theta' \right) \) comes from \( \exp(i \vec{Q}_T \cdot \vec{r}_2) \) in Eq. (4.8). Taking this integral we substitute for the terms in parentheses in Eq. (5.1), \( |Q_T^2| = 1/r_2^2 \langle 1/r_2^2 \rangle \) to

\[
I_Q = 2^{6i(v_1+v_2)} \left( -2^7 v_1 v_2 \right) \left( \frac{r_1^2}{r_2^2} \right)^{i(v_1+v_2)} \left( \frac{1}{r_2^2} \right) \times \left( 1 - \frac{2}{2^4} \frac{(n \cdot \vec{r}_1)^2}{r_2^2} + \frac{1}{2^8} \frac{(n \cdot \vec{r}_1')^4}{r_2^4} \right) \times \left( 1 - \frac{2}{2^4} \frac{(n \cdot \vec{r}_1)^2}{r_2^2} + 1 \frac{1}{2^8} \frac{(n \cdot \vec{r}_1')^4}{r_2^4} \right) \times \left( \frac{1}{i(v_1+v_2)} - \frac{9}{32} \cos(2\phi_2) + \frac{3}{16} \cos(4\phi_2) \right) \nu \nu' \right)
\]

where \( \vec{n} = \vec{Q}_T/Q_T \), and \( \phi_2 \) is the angle between \( \vec{n} \) and \( \vec{n}_2 = \vec{r}_2/r_2 \). In Eq. (5.3) the terms in \( (\ldots) (\ldots) \) stem from the expansion with respect to \( r_1^2/r_2^2 \ll 1 \). However, for the terms in \( (\ldots) \) there are no such small parameters, and we expand the function of \( \phi_2 \) in a Fourier series.

Integrating over \( \vec{n} \) one obtains

\[
(...)(...) = \left[ \frac{1}{i(v_1+v_2)} + \frac{9 r_1^2}{2^{10} r_2^2} \left( (\vec{n}_1 \cdot \vec{n}_2)^2 + (\vec{n}_1' \cdot \vec{n}_2)^2 \right) + \frac{3}{2^{12} r_2^4} \left( (\vec{n}_1 \cdot \vec{n}_2)^4 + (\vec{n}_1' \cdot \vec{n}_2)^4 \right) \right] \nu \nu' \right)
\]

where \( \vec{n}_1 = \vec{r}_1/r_1, \vec{n}'_1 = \vec{r}_1'/r_1 \) and \( \vec{n}_2 = \vec{r}_2/r_2 \). Deriving Eq. (5.4) we neglected the extra powers of \( r_1^2/r_2^2 \), which are small. Finally

\[
I_Q (\vec{r}_1, \vec{r}_2, \vec{r}_2'; v_1, v_2) = 2^{6i(v_1+v_2)} \left( -2^7 v_1 v_2 \right) \left( \frac{1}{r_2^2} \right)^{i(v_1+v_2)} \left( \frac{1}{r_2^2} \right) \times \left( \frac{1}{i(v_1+v_2)} + \frac{9 r_1^2}{2^{10} r_2^2} \left( (\vec{n}_1 \cdot \vec{n}_2)^2 + (\vec{n}_1' \cdot \vec{n}_2)^2 \right) + \frac{3}{2^{12} r_2^4} \left( (\vec{n}_1 \cdot \vec{n}_2)^4 + (\vec{n}_1' \cdot \vec{n}_2)^4 \right) \right) \nu \nu' \right)
\]

From Eq. (4.8) we can see that the integration over \( r_1 \) can be written in the form

\[
\frac{d^2\sigma}{dy_1 dy_2 dp_{T1} dp_{T2}} = \left[ \frac{2CF \mu_{\text{soft}}^2}{\alpha_s (2\pi)^2} \right]^2 \left( \frac{1}{p_{T1} p_{T2}} \right) \times \left( \frac{1}{i(v_1+v_2)} + \frac{9 r_1^2}{2^{10} r_2^2} \left( (\vec{n}_1 \cdot \vec{n}_2)^2 + (\vec{n}_1' \cdot \vec{n}_2)^2 \right) + \frac{3}{2^{12} r_2^4} \left( (\vec{n}_1 \cdot \vec{n}_2)^4 + (\vec{n}_1' \cdot \vec{n}_2)^4 \right) \right) \nu \nu' \right)
\]

The tedious calculations of the integral over the dipole sizes in Eq. (5.6) are collected in Appendix D.

The values of \( v_2 \) and \( u_4 \) can be determined from the following representation of the double inclusive cross section:

\[
\frac{d^3\sigma}{dy_1 dy_2 dp_{T1} dp_{T2}} \propto 1 + 2 \sum_n v_{n,n} (p_{T1}, p_{T2}) \cos(n \varphi) \nu \nu' \right)
\]
where $\varphi$ is the angle between $\vec{p}_{T1}$ and $\vec{p}_{T2}$, $v_n$ is calculated from $v_{n,n}(p_{T1}, p_{T2})$.

1. $v_n(p_T) = \sqrt{v_{n,n}(p_T, p_T^\text{Ref})}$;
2. $v_n(p_T) = \frac{v_{n,n}(p_T, p_T^\text{Ref})}{\sqrt{v_{n,n}(p_T^\text{Ref}, p_T^\text{Ref})}}$. (5.8)

Equations (5.8)-1 and (5.8)-2 depict two methods of how the values of $v_n$ have been extracted from the experimentally measured $v_{n,n}(p_{T1}, p_{T2})$, where $p_T^\text{Ref}$ denotes the momentum of the reference trigger. These two definitions are equivalent if $v_{n,n}(p_{T1}, p_{T2})$ can be factorized as $v_{n,n}(p_{T1}) v_n(p_{T2})$. In this paper we use the definition in Eq. (5.8)-1.

Introducing the angular correlation function as

$$C(p_T, \phi) \equiv \frac{d^2\sigma}{dy_1 dy_2 d^2p_{T11}d^2p_{T22}} (\text{Fig. 3b})$$
$$\frac{d^2\sigma}{dy_1 dy_2 d^2p_{T1}d^2p_{T2}} (\text{Fig. 1a})$$

we obtain

$$v_{n,n} = \int_0^{2\pi} d\phi \sqrt{C(p_T, \phi)} \cos(n \phi)$$

$$v_n = \sqrt{v_{n,n}};$$

(5.10)

In Eq. (4.17) we have calculated the part of $C(p_T, \phi)$ which does not depend on $\phi$, which coincides with $C(p_T, \phi = 0) = R$ of Eq. (4.17) for $Q_s(Y - y_1) \gg Q_s(y_2)$. To calculate the contribution to $C$, which depends on $\phi$, we need to take the separate integrals over $v_1$ and $v_2$, since the terms, which are proportional to $\cos^n(\phi)$ and $\cos^2(\phi)$ do not have a pole at $v_1 = -v_2$ (see Eq. (5.5)). These integrations lead to the following extra factor in $C(p_T, \phi) - C(p_T, \phi = 0)$:

$$C(p_T, \phi) - C(p_T, \phi = 0) \propto R \frac{p_T^2}{Q_s^2(Y - y_1) Q_s^2(Y - y_1)} C(p_T, \phi = 0);$$

$$R = 2\xi^2 \left[\frac{1}{(2D y_1 y_2)^2} \exp(-2\xi^2/(4D y_1 y_2))\right]$$

(5.11)

where $\xi = \ln(Q_s^2(Y - y_1)/Q_s^2(y_2))$. We took factors proportional to $p_T$ from the expression for $A(k_T, p_{T1})$ and $A(k_T, p_{T1})$ putting $p_{T1} = p_{T2} = p_T$. To find the final correlation function and $v_{2,2}$ and $v_{4,4}$, we need to collect all numerical factors that come from $A(k_T, p_{T1})$, $A(k_T, p_{T1})$ and Eq. (5.6), and to integrate over $\phi$, as given in Eq. (5.9).

Note that in the case of symmetric kinematics, where $Y - y_1 = y_2 = \frac{1}{2}(Y - y_1)$, $\xi = 0$ and Eq. (D.7) vanishes. In this case, we have to use Eq. (A.5) instead of Eq. (A.6), keeping track of the corrections, which are proportional to $v_1$. As a result, we can consider $\xi = 0$ in Eq. (D.7), but we need to replace the factor $\xi^2$ by 1.

Equations (4.17) and (D.7) contain numerical uncertainties, which stem both from the values of the soft parameters $\bar{\mu}_\text{soft}$ and $\mu_\text{soft}$, as well as the values of the saturation scale at low energies, and from the integration in Eqs. (C.3) and (C.5), which were taken neglecting contribution from the region $r' < 1$. On the other hand, the contribution to the double inclusive cross sections of the diagram of Fig. 2 at $\tilde{a}_S y_{12} \ll 1$ coincides with the contribution of Fig. 1b,

$$\frac{d^2\sigma}{dy_1 dy_2 d^2p_{T1}d^2p_{T2}} (\text{Fig. 2}) \propto \frac{d^2\sigma}{dy_1 dy_2 d^2p_{T1}d^2p_{T2}} (\text{Fig. 1b}).$$

(5.12)

Therefore, to obtain the realistic estimate we use the following procedure of matching:

$$v_2(p_T = 5 \text{ GeV}, y_1 = 2) |_{\text{Fig. 2}} = v_2(p_T = 5 \text{ GeV}) |_{\text{Fig. 1b}};$$

$$v_4(p_T = 5 \text{ GeV}, y_1 = 2) |_{\text{Fig. 2}} = v_4(p_T = 5 \text{ GeV}) |_{\text{Fig. 1b}};$$

(5.13)

where $v_2(p_T = 5 \text{ GeV}) |_{\text{Fig. 1b}}$ and $v_4(p_T = 5 \text{ GeV}) |_{\text{Fig. 1b}}$ are taken from Ref. [14] where the estimates were performed based on the model for soft interaction which describes all features of the soft interaction at high energy and provides an interface with the hard processes.

Figure 5 shows the $p_T$ and $y$ dependence of the $v_2$ and $v_4$ using Eq. (D.8) for normalization. In addition we take $\Delta_{\text{BFKL}} = 0.25$ and $Q^2_s(y) \propto \exp(\lambda y)$ with $\lambda = 0.25$. These values correspond to the BFKL Pomeron phenomenology. We believe that this figure illustrates the scale of rapidity dependence and will be instructive for future experimental observations.

6 Conclusions

In this paper we generalize the interference diagram that described the Bose–Einstein correlation for small rapidity difference $\tilde{a}_S y_{12} \ll 1$, to include the emission of the gluons with rapidities ($y_1$) between $y_1$ and $y_2$ ($y_1, y_2 < y_2$). We calculate the resulting diagram in the CGC/saturation approach and make two observations, which we consider as the main result of this paper. The first one is a substantial decrease of the odd Fourier harmonics $v_{2n+1}$ as a function of the rapidity difference $y_{12}$ (see Fig. 4c). The second result is that even Fourier harmonics $v_{2n}$ have a rather strong dependence on $y_{12}$, showing a considerable increase in the region of large $y_{12}$ (see Fig. 5). We believe that our calculations, which have been performed both for the simplest diagrams
and for the CGC/saturation approach, will be instructive for further development of the approach especially in the part that is related to the integration of the momenta transferred by the BFKL Pomerons.

We demonstrated in this paper the general origin of the density variation mechanism, whose nature does not depend on the technique that has been used. This mechanism has to be taken into account, since it leads to the values of the Fourier harmonics that are large and of the order of $v_n$, which have been observed experimentally.

We hope that the paper will be useful in the clarification of the origin of the angular correlation, especially for hadron–hadron scattering at high energy. We firmly believe that the experimental observation of both phenomena, the sharp decrease of $v_n$ with odd $n$ and the substantial increase of $v_n$ with even $n$ as a function of $y_{12}$, will be a strong argument for the CGC/saturation nature of the angular correlations.

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Appendix A: BFKL Pomeron in the mixed representation

In this appendix we discuss the BFKL Pomeron Green function $G(\vec{r}, \vec{R}, \vec{Q}_T; Y)$ in the mixed representation, where $r$ and $R$ are the sizes of two interacting dipoles, $Q_T$ denotes the momentum transferred by the Pomeron, and $Y$ the rapidity between the two dipoles. This Green function has the following form [42,43,58–60]:

$$G(\vec{r}, \vec{R}, \vec{Q}_T; Y) = \frac{r R}{16} \sum_{n=\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dv}{(v^2 + \frac{1}{2}(n-1)^2)} \left( v^2 + \frac{1}{2}(n+1) \right) \times V_{r,n}(\vec{r}, \vec{Q}_T) V_{r,n}^*(\vec{R}, \vec{Q}_T) e^{i(v,n)Y}$$

(A.1)
where

\[
\omega(v, n) = 2 \bar{\alpha}_s \text{Re} \left( \psi \left( \frac{1}{2} + \frac{1}{2} |n| + v \right) - \psi(1) \right);
\]

\[
\omega(v, 0) = 2 \bar{\alpha}_s \text{Re} \left( \psi \left( \frac{1}{2} + v \right) - \psi(1) \right) \xrightarrow{v \ll 1} \Delta_{\text{BFKL}} - D v^2, \tag{A.2}
\]

where \(\psi(z)\) is the Euler \(\psi\)-function (see Ref. [61] formulas 8.36) and \(\Delta_{\text{BFKL}} = \bar{\alpha}_s 4 \ln 2, D = \bar{\alpha}_s 14 \xi(3), \xi = \ln \left( \frac{r_1^2}{r_2^2} \right)\).

Each term in Eq. (A.1) has a very simple structure, being the typical contribution of the Regge-pole exchange: the product of two vertices and Regge-pole propagator. From Eq. (A.2) one can see that at large \(Y\) the main contribution comes from the term with \(n = 0\), and in what follows we will concentrate on this particular term. The vertices with \(n = 0\) have been determined in Refs. [42, 43, 58–60], and they have an elegant form in the complex number representation for the point on the two dimensional plane, viz.,

For \(\tilde{r}(x, y)\): \(\rho = x + iy; \rho^* = x - iy;\)

For \(\tilde{Q}_T(Q_x, Q_y)\): \(q = Q_x + i Q_y; q^* = Q_x - i Q_y.\)

(A.3)

Using this notation the vertices have the following structure:

\[
V_v \left( \tilde{r}, \tilde{Q}_T \right) = \left( \frac{\bar{Q}_T}{Q_T^2} \right)^{i v} \Gamma^2 \left( 1 - i v \right) \times \left\{ J_{-i v} \left( \frac{1}{4} q^* \rho \right) J_{-i v} \left( \frac{1}{4} q \rho^* \right) - J_{i v} \left( \frac{1}{4} q^* \rho \right) J_{i v} \left( \frac{1}{4} q \rho^* \right) \right\}. \tag{A.4}
\]

At \(Q_T \rightarrow 0\) this vertex takes the form

\[
2^{6i v} V_v \left( \tilde{r}, \tilde{Q}_T \right) \xrightarrow{Q_T \ll 1} \left( \frac{\bar{Q}_T}{Q_T^2} \right)^{-i v} \times \left( (v + i) \left( 8 (\tilde{Q}_T \cdot \tilde{r})^4 - 8 (\tilde{Q}_T \cdot \tilde{r}) Q_T^2 r^2 + 5 \xi Q_T^2 r^2 \right) + (2i + v) Q_T^2 r^2 \right) \right. \\
\left. \quad \quad \quad \times \frac{64^2 (v + 2i) (1 + iv)^2 + 2 + (v + i) (2 (\tilde{Q}_T \cdot \tilde{r})^2 - Q_T^2 r^2)}{32 (1 - iv)^2} \right) \times \left( (v + 2i) \left( 8 (\tilde{Q}_T \cdot \tilde{r})^4 - 8 (\tilde{Q}_T \cdot \tilde{r}) Q_T^2 r^2 + 5 \xi Q_T^2 r^2 \right) + (2i + v) Q_T^2 r^2 \right) \\
\left. \quad \quad \quad \times \frac{2^2 (2 + 2i) (1 + iv)^2}{2^6 (1 + iv)^2} \right) \times \frac{-1}{1 + iv}.
\tag{A.5}
\]

For small values of \(v\) (which are related to the region of large \(\bar{\alpha}_s Y \gg 1\)), Eq. (A.5) can be simplified and reduced to the form

\[
\left\{ \begin{array}{l}
2^{6i v} V_v \left( \tilde{r}, \tilde{Q}_T \right) \\
\quad \quad \times \frac{r_T^2}{\sqrt{2 \pi}} \left( (\tilde{Q}_T \cdot \tilde{r})^4 - (\tilde{Q}_T \cdot \tilde{r})^2 Q_T^2 r^2 + \frac{9}{16} Q_T^2 r^4 \right) \\
\times \frac{2 (\tilde{Q}_T \cdot \tilde{r})^2 - Q_T^2 r^2}{2^5 + 1} \end{array} \right) \tag{A.6}
\]

Using

\[
J_{-i v} (z) \xrightarrow{z \gg 1} \sin \left( \frac{1}{4} \pi z + \frac{1}{2} i \pi v \right) \sqrt{\frac{2}{\pi}} \sqrt{\frac{1}{z}} \tag{A.7}
\]

at \(v \ll 1\) we obtain for \(Q_T^2 r^2 \gg 1\)

\[
V_v \left( \tilde{r}, \tilde{Q}_T \right) \xrightarrow{Q_T \gg 1} \left( \frac{Q_T^2}{Q_T r} \right)^{i v} \Gamma^2 \left( 1 - i v \right) \cos \left( \frac{1}{2} \tilde{Q}_T \cdot \tilde{r} \right) \frac{4 i v}{Q_T r}. \tag{A.8}
\]

The contribution of the first term in Eq. (A.1) can be reduced to the following form for the scattering amplitude of two dipoles with sizes \(r_1\) and \(r_2\):

\[
N(r_1, r_2; Y) = \frac{r_1 r_2}{16} \int \frac{d v}{(v^2 + 1/4)^2} V_v \left( r_1, Q_t \right) \rightarrow 0. \tag{A.9}
\]

\[
V_v^* \left( r_2, Q_T \rightarrow 0 \right) \xrightarrow{v \rightarrow 0} \frac{r_1 r_2}{16} \int \frac{d v}{(v^2 + 1/4)^2} e^{i v \nu(0,0) Y} \times \left\{ \text{Re} \{ Q_T^2 r_2^2 \} - \text{Im} \{ Q_T^2 r_2^2 \} \right\} \times \left( \frac{r_T^2}{2^4} \right)^{i v} \left( \frac{Q_T^2 r_2^2}{2^4} \right)^{-i v} \tag{A.9}
\]

\[
= \frac{r_1 r_2}{16} \int \frac{d v}{(v^2 + 1/4)^2} e^{i v \nu(0,0) Y} \frac{r_2^2}{r_1^2} \tag{A.9}
\]

\[
\xrightarrow{r_1 \gg 1; v \ll 1} 2 r_1 r_2 \int d v \exp \left( i \bar{\alpha}_s 4 \ln 2 \frac{r_3^2}{r_1^2} \right) \tag{A.9}
\]

\[
\frac{1}{2} \bar{\alpha}_s 14 \xi(3) v^2 Y \left( \frac{r_3^2}{r_1^2} \right)^{i v} \tag{A.9}
\]

\[
= r_1 r_2 \sqrt{\frac{2 \pi}{D Y}} \exp \left( \Delta_{\text{BFKL}} - \frac{\bar{\alpha}_s}{4 D Y} \right) \tag{A.9}
\]

where \(\Delta_{\text{BFKL}}\) and \(D\) are defined in Eq. (A.2).
In the derivation of Eq. (A.9) we neglected the contributions that are proportional to \( \left( \frac{Q^2 T 2^2 \xi^2}{2 T^2} \right)^{-i \nu} \), since this contribution will be the same as in Eq. (A.9), but with \( \xi = \ln \left( \frac{Q^2 T 2^2 \xi^2}{2 T^2} \right) \gg 1 \). To integrate over \( \nu \), we use the method of steepest descent, and the expansion of \( \omega(v, 0) \) at small \( v \) (diffusion approximation; see the second equation in Eq. (A.2)).

\( N(r_1, r_2; Y) \) denotes the imaginary part of the dipole-dipole scattering amplitude at \( Q_T = 0 \), which is related to the cross section. One can check that Eq. (A.9) has the correct dimension.

**Appendix B: Calculation of the integrals for the contribution of the simplest diagram**

In this appendix we discuss the integrations in Eq. (2.5). The integral over \( Q_T \) has been considered in Sect. 4.1 and it has the form of Eq. (4.7). The extra \( e^{i p_{r1} \cdot \vec{r}_2} \) give an additional numerical factor, replacing \( 2^L \) by \( 2^L \) in Eq. (4.7). To integrate over \( k_T \) and \( k_F \) we replace

\[
\int \prod d \phi_i e^{-i \vec{p}_{r1} \cdot \vec{r}_1} \delta^{(2)}(\vec{r}_1 + \vec{p}_T \vec{r}_1 - \vec{r}_1') \rightarrow (2\pi)^4 \int k_T dk_T J_0(k_T r_1) J_0(k_T r_1') \frac{i}{(k_T)} J_0(k_T r_1) J_0(k_T r_1') .
\]  

(B.1)

Now we can take the integrals over \( r_1 \) bearing in mind Eq. (4.7) and

\[
N_{pr}(r_1, Y - y_1) = \int \frac{d \nu}{2\pi} \left( \mu_2^{soft} r_1^2 \right)^{1+i \nu} e^{\omega(v_0, 0)(Y - y_1)} .
\]  

(B.2)

The integrals over \( \vec{r}_1 \) and \( \vec{r}_1' \) have the following form (see Ref. [61], Eq. 6.511(6)):

\[
\int \vec{r}_2 \ J_0(k_T \vec{r}_1) \frac{d \vec{r}_1}{2\pi} = \vec{r}_2 \ J_0(k_T \vec{r}_1) \frac{1}{\pi} \ 
\]

if \( k_T \vec{r}_2 \ll 1 \),

\[
\int \vec{r}_2 \ J_0(k_T \vec{r}_1) \frac{d \vec{r}_1}{2\pi} = \frac{1}{\pi} \ 
\]

if \( k_T \vec{r}_2 \gg 1 \).

Using Eq. (B.2) we obtain

\[
\int_0^\infty \ r_1 \ d r_1 \ J_0(k_T r_1) \cdot \nabla^2 r_1 \ N_{pr}(r_1, Y - y_1)
\]

\[
= \frac{1}{k} \left( \frac{4 \mu_2^{soft}}{k^2} \right)^{1+i \nu} e^{\omega(v_0, 0)(Y - y_1)}.
\]  

(B.4)

Collecting Eqs. (B.2), (B.3) and (B.4) we see that the main contribution stems from the region \( k_T \vec{r}_2 \ll 1 \) and the integral over \( k_T \) has the form

\[
\vec{r}_2 \frac{1}{r_1 r_1'} \int \frac{d \vec{r}_2^2}{2\pi} \frac{d k_T^2}{k_T^2} \left( \frac{4 \mu_2^{soft}}{k_T^2} \right)^{1+i \nu} e^{\omega(v_0, 0)+(0, v_1'))(Y - y_1)}
\]

\[
= \frac{1}{2} 1 + \frac{1}{(r_1 + v_1') \ 4 \mu_2^{soft}} \frac{1}{(r_2)^2} \ e^{\omega(v_0, 0)+(0, v_1'))(Y - y_1)}
\]

after integration over \( v_1, v_2 \) \( \frac{1}{2} \ \pi \ \frac{1}{r_2} \)

(B.5)

The integral over \( k_T \) has the same structure while the integration in Eq. (B.3) goes to infinity. As a result we can reduce the integral to the form

\[
\int \frac{d k_T^2}{k_T^2} \left( \frac{4 \mu_2^{soft}}{k_T^2} \right)^{1+i \nu} e^{\omega(v_0, 0)+(0, v_1'))(Y - y_1)} \rightarrow \int \frac{d k_T^2}{k_T^2} \left( \frac{4 \mu_2^{soft}}{k_T^2} \right)^{1+i \nu} e^{\omega(v_0, 0)+(0, v_1'))(Y - y_1)}
\]

(B.6)

Finally, we obtain Eq. (4.9).

**Appendix C: Integration over dipole sizes in the CGC/saturation approach**

The integration over \( Q_T \) will produce the same result as Eq. (4.7), as in the previous section. We rewrite the integral over \( \phi_i \) (see Eq. (B.1)) in the following way:

\[
\int d \phi e^{-i \vec{p}_{r1} \cdot \vec{r}_1} \delta^{(2)}(\vec{r}_1 + \vec{p}_T \vec{r}_1 - \vec{r}_1') \rightarrow (2\pi)^4 \int d \phi e^{i \vec{p}_{r1} \cdot \vec{r}_1} \int k_T dk_T J_0(k_T r_1) J_0(k_T r_1') J_0(k_T r_1) J_0(k_T r_1').
\]  

(C.1)

We see that the integrals over \( r_1' \) and \( r_1' \) lead to \( r_1 \sim 1/Q_s(Y - y_1) \) and \( r_1' \sim 1/Q_s(Y - y_1) \). The same holds for the integrals over \( r_1' \) and \( r_1' \), leading to \( r_2 \sim 1/Q_s(y_2) \) and \( r_2' \sim 1/Q_s(y_2) \). Assuming that \( Q_s(Y - y_1) > Q_s(y_2) \) we conclude that \( r_1 \) and \( r_1' \) are much smaller than \( r_2 \) and \( r_2' \). Replacing

\[
\nabla^2_{r_1} N_{pr}(r_1; Y - y_1) e^{i \vec{p}_{r1} \cdot \vec{r}_1} \nabla^2_{r_1} N_{pr}(r_1' ; Y - y_1)
\]

\[
\rightarrow 2^8 \frac{\vec{r}_1^2}{r_1 r_1'} (r_1^2 Q_s^2(Y - y_1)) \frac{\vec{r}_1^2}{r_1^2} Q_s^2(Y - y_1) \vec{r}_1
\]  

(C.2)
where \( \tilde{Y} = 1 - \gamma_{cr} \), we see from Eq. (C.1) that integration over \( r \) takes the form

\[
\frac{1}{Q_s} \frac{1}{1 + 2\tilde{Y}} \int_0^1 d\tau J_0 \left( k_T \sqrt{\tau} \right) \tau \tilde{Y} \frac{d\tau}{2\sqrt{\tau}} = \frac{1}{Q_s} \frac{1}{1 + 2\tilde{Y}} F_2 \left( \left\{ \frac{1}{2} + \tilde{Y} \right\}, \left\{ 1, \frac{3}{2} + \tilde{Y} \right\}, -\frac{k_T^2}{4 Q_s^2} \right).
\]

(C.3)

Recall that we consider \( Q_s = Q_s(Y - Y_1) \) in Eq. (C.3). For \( pT_1 \ll Q_s(Y - y_1) \) we can replace \( e^{-i\vec{r}_1 \cdot \vec{r}_1} = 1 \) in Eq. (C.1). In this case the integral has the form

\[
\frac{1}{Q_s^2} \frac{1}{(1 + 2\tilde{Y}^2)^2} \int_0^1 d\tau' \left( \frac{1}{Q_s^2}, \frac{1}{1 + 2\tilde{Y}^2}^{-1} \times F_2 \left( \left\{ \frac{1}{2} + \tilde{Y} \right\}, \left\{ 1, \frac{3}{2} + \tilde{Y} \right\}, -\frac{1}{4 \tau'} \right) \right)^2 d\tau' \approx 0.18/Q_s^2
\]

(C.4)

where \( \tau' = k_T^2/Q_s^2 \).

The integral over \( r \) in the lower part of the diagram takes the form

\[
\int \frac{d^2r'}{r'^2} J_0 (k_T \cdot r) = \pi \ln \left( \frac{k_T^2}{(4\mu_{\text{soft}}^2)} \right).
\]

(C.5)

Using Eq. (C.5) for \( pT_2 \ll Q_s(y_2) \) the integral over \( k_T' \) can be reduced to

\[
\frac{1}{(1 + 2\tilde{Y}^2)^2} \int_0^1 dr' \left( \frac{1}{Q_s^2}, \frac{1}{1 + 2\tilde{Y}^2}^{-1} \times F_2 \left( \left\{ \frac{1}{2} + \tilde{Y} \right\}, \left\{ 1, \frac{3}{2} + \tilde{Y} \right\}, -\frac{1}{4 \tau'} \right) \right)^2 \times \left( \frac{\ln (r')}{r'} \right)^2 = 3.50.
\]

(C.6)

Finally, collecting all numerical coefficients, we obtain

\[
\frac{d^2\sigma}{dy_1 d^2 pT_1 d^2 pT_2} = \frac{C^4}{2^3 3^2 \pi^4} \left( \frac{2C_F}{\alpha_s(2\pi)^2} \right)^2 \frac{1}{pT_1 pT_2} \times 0.183.5 \frac{Q_s^2}{Q_s^2(Y - y_1)} \frac{1}{(2 Y_{12})^3} e^{2\Delta_{\text{BFKL},12}}
\]

(C.7)

where the constant \( C \) is the value of the amplitude at \( \tau = 1 \).

This contribution is proportional to

\[
e^{-2\Delta_{\text{BFKL},12}} / Q_s^2(Y - y_1)
\]

for \( pT_1 \ll Q_s(Y - y_1) \) and \( pT_2 \ll Q_s(y_2) \). Note that \( Q_s^2(Y - y_1) > Q_s^2(y_2) \).

We need to estimate the diagram of Fig. 1a (see Eq. (2.7)). This diagram can be rewritten as

\[
\frac{d^2\sigma}{dy_1 d^2 pT_1 d^2 pT_2} \quad (\text{Fig. 1a-})
\]

\[
= \frac{\bar{\nu}^2_{\text{soft}} d^2\sigma}{dy_1 d^2 pT_1} \quad (Q_T = 0; \text{Eq. (3.5)}) \times \frac{d^2\sigma}{dy_2 d^2 pT_2} \quad (Q_T = 0; \text{Eq. (3.5)})
\]

where \( \bar{\nu}^2_{\text{soft}} = \int d^2 Q T N^2(Q_T) \).

(C.8)

Examining Eq. (3.5), one can see that in the general case when \( Y - y_1 \neq y_1 \) and \( Y - y_2 \neq y_2 \) all four Pomeron cannot be in the vicinity of the saturation scale. Actually we have two kinematic regions which give the maximal contributions (assuming \( Q_s(Y - y_1) > Q_s^2(y_1) \)):

1. \( r^2 Q_s^2(Y - y_1) \ll 1 \) but \( r^2 Q_s^2(y_1) \rightarrow Q_s^2(Y - y_1) / Q_s^2(y_1) \ll 1 \);
2. \( r^2 Q_s^2(y_1) \ll 1 \) but \( r^2 Q_s^2(y_1) \rightarrow Q_s^2(y_1) / Q_s^2(Y - y_1) \gg 1 \).

In region 1 the upper Pomeron is in the vicinity of the saturation scale, while the lower Pomeron is in the perturbative QCD region. In region 2 the lower Pomeron is in the vicinity of the saturation scale, and the upper Pomeron is deep inside the saturation domain. As we have discussed (see Fig. 3a) \( \nabla^2 N \) decreases in the saturation region much faster than in the perturbation QCD region and, therefore, we assume that the kinematic region 1 gives the largest contribution. Hence, for \( pT_1 \ll Q_s^2(y_1) \) we obtain Eq. (4.16).

**Appendix D: Integration over dipole sizes in the angle correlation function**

In this appendix we collect tedious integration over the dipole sizes in Eq. (5.6).

Each term in this equation can be factorized as a product of two functions which depend on \( r_1^2 \) and on \( r_2^2 \). Bearing this feature in mind we calculate each term going to the momentum representation using Eq. (C.1). We obtain a product of functions of \( k_T \). Each of these functions has the following general form:

\[
\int d^2 r e^{i k_T \cdot r} \prod_{i=1}^{j} r_{\mu_i} F (r) = \left( -i \vec{\nabla}_{k_T} \right)^j \int d^2 r e^{i k_T \cdot r} F (r) = 2\pi \left( -i \vec{\nabla}_{k_T} \right)^j \int d^2 r J_0 (k_T r) F (r) .
\]

(D.1)
As we have seen the dependence on \( \vec{r}_j \) stems from the integration over \( Q_T \) or, in other words, from \( (I_Q) \). In \( I_Q \) the dependence on \( r_1 \) and \( r'_1 \) can be extracted explicitly, leading to \( F (r) \propto 1/r \). Hence the momentum image for Eq. (D.1) has a simple form:

\[
\int d^2 \vec{r} e^{i \vec{k}_T \cdot \vec{r}} \prod_{i=1}^j r_{\mu_i} F (r) = 2 \pi (-i \sqrt{r} \vec{\nabla}_{\vec{k}_T})^j \frac{1}{k_T}.
\]  

(D.2)

For \( j = 2 \) and \( j = 4 \), which we need to calculate considering Eq. (5.6), we have

\[
\begin{align*}
(-i \vec{\nabla}_{\vec{k}_T})^2 \frac{1}{k_T} &= \frac{3}{k_T} k_{T,i} k_{T,i'} - \frac{1}{k_T^2} \delta_{i,i'}; \\
(-i \vec{\nabla}_{\vec{k}_T})^4 \frac{1}{k_T} &= \frac{105}{k_T^2} k_{T,i} k_{T,i'} k_{T,j} k_{T,j'} - 15 \delta_{i,j} k_{T,i} k_{T,j} + \delta_{i,j} k_{T,i} k_{T,j} + \delta_{i,j'} k_{T,i} k_{T,j'} + \delta_{i,j'} k_{T,j} k_{T,i'} + \delta_{i,j'} k_{T,j} k_{T,i'} + \delta_{i,j'} k_{T,i} k_{T,i'} + \frac{3}{k_T} \left( \delta_{i,j} \delta_{j,i'} + \delta_{i,j'} \delta_{j,i} + \delta_{i,j} \delta_{j,i'} \right).
\end{align*}
\]

Note that, for integration over \( \vec{r}_1 \), Eq. (D.2) takes the form

\[
\int d^2 \vec{r}_1 e^{i (\vec{k}_T + \vec{p}_{T1}) \cdot \vec{r}_1} \prod_{i=1}^j r_{1,\mu_i} F (r_1) = 2 \pi (-i \sqrt{\vec{k}_T + \vec{p}_{T1}}) \frac{1}{(\vec{k}_T + \vec{p}_{T1})^2}.
\]  

(D.3)

The term \( r_1^2 (\vec{n}_1 \cdot \vec{n}_2)^2 + r_1' (\vec{n}_1' \cdot \vec{n}_2')^2 \) can be rewritten as \( (r_{1,\mu} r_{1,v} + r_{1,\mu} r_{1,v}') r_{2,\mu} r_{2,v} \) and in the momentum representation it is

\[
\int d\phi \left\{ \frac{3}{k_T^2} k_{T,i} k_{T,i'} - \frac{\delta_{i,i'}}{k_T^2} \right\} \frac{1}{\sqrt{k_T^2 + p_{T1}^2 + 2 \cos(\phi) k_T p_{T1}}} \\
+ \left( \frac{3}{\sqrt{k_T^2 + p_{T1}^2 + 2 \cos(\phi) k_T p_{T1}}} \right)^2 (\vec{k}_T + \vec{p}_{T1}) (\vec{k}_T + \vec{p}_{T1})' \frac{1}{k_T} \\
+ \frac{\delta_{i,i'}}{\sqrt{k_T^2 + p_{T1}^2 + 2 \cos(\phi) k_T p_{T1}}}' \right\} \frac{1}{k_T} \\
= \frac{A}{p_{T1,i} p_{T1,i'}} + B \delta_{i,i'}.
\]  

(D.4)

The expressions for \( A \) and \( B \) can be written in a general form. Assuming that both \( p_{T1} \) and \( p_{T2} \) are smaller than \( Q_s (y_2) \), we can expand the answer, taking into account only terms that are proportional to \( p_{T1}^2 / k_T^2 \) and \( p_{T2}^2 / k_T^2 \). We obtain

\[
A (k_T, p_{T1}) = \frac{3 p_{T1}^2}{4 k_T^3} \left( -13 k_T^2 + 50 p_{T1}^2 \right); \\
B (k_T, p_{T1}) = \frac{1}{8 k_T^4} \left( 8 k_T^4 + 65 k_T^2 p_{T1}^2 - 150 p_{T1}^4 \right).
\]  

(D.5)

The integrations over \( r'_2 \) and \( r_2 \) differ from the integrations over \( r_1 \) and \( r'_1 \), due to the extra factor \( 1/r_2^2 \), which comes from the integration over \( Q_T \) in Eqs. (4.2) and (4.3). Since \( r_2^2 \approx 1/Q_s^2 (y_2) \) we replace it by \( 1/r_2^2 = Q_s^2 (y_2) \). In the case the integral over \( k'_T \) takes the same form as the integral over \( k_T \), leading to the following expression, which is proportional to \( \cos^2 (\phi) \), where \( \phi \) is the angle between \( \vec{p}_{T1} \) and \( \vec{p}_{T2} \):

\[
\frac{d^2 \sigma}{dy_{1} d^2 p_{T1} dy_{2} d^2 p_{T2}} \left( \text{Fig. 2} \right) \propto Q_s^2 (y_2) A (k_T, p_{T1}) A (k'_T, p_{T2}) \cos^2 (\phi),
\]  

(D.6)

which is responsible for the appearance of \( v_{2.2} \) and \( v_2 \).

Using the second expression in Eq. (D.5) we can calculate the term which is proportional to \( \cos^4 (\phi) \) and has the form

\[
\frac{d^2 \sigma}{dy_{1} d^2 p_{T1} dy_{2} d^2 p_{T2}} \left( \text{Fig. 2} \right) \propto Q_s^2 (y_2) A^{(4)} (k_T, p_{T1}) A^{(4)} (k'_T, p_{T2}) \cos^4 (\phi)
\]  

(D.7)

with

\[
A^{(4)} (k_T, p_{T1}) = 15 \frac{573}{8} \frac{1}{k_T^2} \frac{p_{T1}^2}{k_T^2}.
\]  

(D.8)

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