Comparison of Ensemble Kalman Filter, Unscented Kalman Filter, and Fractional Kalman Filter for estimating the concentration of CO and NO₂

S E Prakosa 1, D D Lestari 1, R Dianawati 1, E Apriliani 1, D K Arif 1, I Herisman 1, and M S Akbar 2

1 Department of Mathematics, Institut Teknologi Sepuluh Nopember, Surabaya, Indonesia
2 Departement of Statistics, Institut Teknologi Sepuluh Nopember, Surabaya, Indonesia

Email: saneditya@gmail.com

Abstract. Estimation distribution of air pollution is necessary to determine the level of pollution in a location so it can be used to recommend emission minimization. This research use some modification of Kalman Filter method for estimation distribution of air pollution. Three estimation methods namely Fractional Kalman Filter (FKF), Ensemble Kalman Filter (EnKF), Unscented Kalman Filter (UKF). The simulation result of three methods will be compared. Distribution of carbon monoxide and nitrogen dioxide in Surabaya was estimated using three methods. This research showed that Fractional Kalman Filter method is the best estimator among the others because error value is the smallest.

1. Introduction
Air pollution is a major problem in some countries, including Indonesia. If there is no action to solve this problem, it will be damaged for humans, animals, plants, and daily activities. The factors of air pollution is composed by two source, the first is due to natural sources such as volcanic eruptions and natural disaster like tornado, and the second comes from human activities (anthropogenic sources) such as emission from transportation, plant emissions, industrial activities, and etc [1]. To solve this problem, it is necessary to take action to prevent this problem, before it become more dangerous. One way is to estimate the concentration of pollutants to determine the level of air pollution in some locations. Some of the methods that can be used to determine the level of pollutants in the air, such as direct measurements in several places, modeling and simulation, estimation based on measurement results data and so on.

The Ensemble Kalman filter (ENKF) was adopted by Erna Apriliani, DKK in 2011 to estimate groundwater pollution in 100 position based on 15 measurement data and gave the conclusion that the Ensemble Kalman filter (ENKF) method is more accurate than the Kalman filter method [2].

In addition, the method of Unscented Kalman Filter (UKF) has also been applied by Santanu Metia DKK in 2020, to estimate carbonmonoxide (CO) pollutant by comparing the use of the Extended Kalman filter and Unscented Kalman Filter. The results of estimation by using Unscented Kalman Filter greatly correlated with data on monitoring stations or approaching data from monitoring stations and the estimated result with Unscented Kalman filter method is more accurate than using the Extended Kalman
filter. In addition, Unscented Kalman Filter also has high efficiency in estimating the concentration of pollutants [3].

As well, Fractional Kalman Filter (FKF) was adopted by Yessy Vita in 2018, to estimate the pollutant concentration by comparing Kalman Filter and Fractional Kalman Filter, which gave the conclusion that Fractional Kalman Filter is better than Kalman filter, but the computation time is longer than Kalman filter [4].

In this research, several modification methods are used from the Kalman Filter method to estimate the concentration of pollutants in the air. All three methods are used to estimate air pollutants, such as the Ensemble Kalman filter (ENKF), the Unscented Kalman filter (UKF), and the Fractional Kalman Filter (FKF). The measurement data that for the implementation of the three methods is the data on the concentration of carbon monoxide (CO) and Nitrogen dioxide (NO₂) pollutants in the city of Surabaya in 2018. The types of pollutants that can be measured by The Air Quality Monitoring are SO₂, CO, NO₂, O₃ and PM₁₀. In this research, only estimating with carbon monoxide (CO) and nitrogen dioxide (NO₂).

2. The Air Pollution Model with Diffusion and Advection Model (2D)

Based on [2], the pollution for advection-diffusion model can be written by:

\[
\frac{\partial c}{\partial t} = \left[ D_x \frac{\partial^2 c}{\partial x^2} + D_y \frac{\partial^2 c}{\partial y^2} \right] - \left[ V_x \frac{\partial c}{\partial x} + V_y \frac{\partial c}{\partial y} \right]
\]

(1)

With boundary condition:

\[ C((x,y),0) = 0, x,y \geq 0; C((x,y),t) = C_e, t > 0; C((\infty, \infty), t) = 0, t \geq 0 \]

Where \( C \) is concentration of pollutants, \( D_x \) and \( D_y \) is diffusion coefficient in \( x \) and \( y \), and \( V_x \) and \( V_y \) is speed of wind on air in \( x \) and \( y \).

Because of model at Eq.(1) is continued model, we must doing discretize model into discrete form before doing Estimation with Kalman Filter. Here we use the forward difference method that respect to time, and we use the central difference method that respect to space or position [2]. The discrete form of equation (1) is:

\[
C_{i,j}^{k+1} = a C_{i+1,j}^k + b C_{i,j}^k + c C_{i-1,j}^k + d C_{i,j+1}^k + e C_{i,j-1}^k
\]

(2)

Where:

\[
a = \left[ \frac{D_x}{(\Delta x)^2} - \frac{V_x}{2\Delta x} \right] \Delta t
\]

\[
b = \left[ \frac{1}{\Delta t} - \frac{2D_x}{(\Delta x)^2} - \frac{2D_y}{(\Delta y)^2} - \frac{V_x}{2\Delta x} + \frac{V_y}{2\Delta y} \right] \Delta t
\]

\[
c = \left[ \frac{D_y}{(\Delta y)^2} + \frac{V_x}{2\Delta x} \right] \Delta t
\]

\[
d = \left[ \frac{D_x}{(\Delta x)^2} - \frac{V_y}{2\Delta y} \right] \Delta t
\]

\[
e = \left[ \frac{D_y}{(\Delta y)^2} + \frac{V_y}{2\Delta y} \right] \Delta t
\]

With \( i = 1, 2, 3, ..., 10 \) and \( j = 1, 2, 3, ..., 10 \) also \( C_{i,0}^k = C_{0,j}^k = 0 \).

We arrange and add the system noise such that we have the stochastic dynamical sistem:

\[
C_{k+1} = AC_k + Gw_k
\]

(3)
With $w_k$ is system noise with zero-mean and $w_k \sim N(0, Q_k)$.

3. The Kalman Filter and its modification

There are many ways to estimate an unknown quantity from available data. This chapter basically covers mean-square estimation and the Wiener filter. A discussion of recursive estimation is included, since it provides a good background for the Kalman Filter [7]. Based on continuous research about estimation with Kalman Filter, that method was developed with any modification, such as Ensemble Kalman Filter, Unscented Kalman Filter, and Fractional Kalman Filter. In this part, we will discussed about that methods.

3.1 Kalman Filter

Kalman filter is an algorithm that provides estimates of some unknown variables given the measurements observed over time [8]. On 1960’s, Rudolph E. Kalman discovered Kalman Filter with his article that explain of recursive solution for linear filtering problem for discrete data. This method can estimate for some state based on measurement data that could get from measurements result from measurement properties, because this KF combines models and measurement so can generate estimation result close to real data (measurement data). Kalman Filter are used to estimated state based on linear dynamical system in state space format. The process model defines the evolution of the state from time $k$ to time $k + 1$ with dynamical stochastic system:

$$x_{k+1} = Ax_k + Bu_k + Gw_k$$  \hspace{1cm} (4)

The measurement equation is

$$z_k = Hx_k + v_k$$  \hspace{1cm} (5)

Where $x_{k+1}$ is a state variable in time $k + 1$, $u_k$ is input vector, $A$ is state matrix, $B$ is input matrix, and $H$ is matrix which connects between measurements data $z_k$ and the state variables $x_k$. And $w_k$, $v_k$ are system white noise and measurement white noise with zero-mean and covariance $Q, R$ respectively.

Algorithm of Kalman Filter working with this step:

First Step: Initialization

Initial value : $\hat{x}_0 = \bar{x}_0$

Covariance : $P_0 = E[(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^T]$  

Second Step: Prediction

We get prediction value from $x_k$ from the previous step by adding noise system $w_k$

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k$$

With error covariance : $P_{k+1} = AP_kA^T + Q$

Third Step: Correction

Kalman Gain : $K_k = P_k^{-1}H^T(HP_k^{-1}H^T + R)^{-1}$

Estimation : $\hat{x}_k = \hat{x}_k + K(z_k - H\hat{x}_k)$
Error Covariance: \( P_k = (1 - K_k^*H)P_k^- \)

3.2 Ensemble Kalman Filter

The Kalman Filter is an optimal estimation for linear dynamical. For non-linear dynamical stochastic system, we cannot apply Kalman Filter, but we can use Extended Kalman Filter (EKF), Unscented Kalman Filter (UKF), or Ensemble Kalman Filter (EnKF). The EnKF is proposed by Evensen (1994) [2]. In EnKF, we generate the ensemble of scalar/vector for initial estimation, and to estimate error covariance on time update. System and measurement noise, \( w_k \) and \( v_k \) also generated with ensemble form. Suppose we have a dynamical stochastic system:

\[
x_{k+1} = f(x_k, u_k) + w_k
\]

With measurement equation is

\[
z_k = Hx_k + v_k
\]

Where \( x_{k+1} \) is a state variable in time \( k + 1 \), \( u_k \) is input vector, \( A \) is state matric, \( B \) is input matric, and \( H \) is matric which connects between measurements data \( z_k \) and the state variables \( x_k \). And \( w_k \) , \( v_k \) are system white noise and measurement white noise with zero-mean and covariance \( Q, R \) respectively [2].

Algorithm of EnKF working with this step:

First Step: Initialization
Generate the \( N \)-ensemble of initial condition:

\[
X_{0,i} = [X_{0,1}, X_{0,2}, X_{0,3}, \cdots, X_{0,N}]
\]

where \( x_{0,i} \sim N(\bar{x}_0, P_0) \)

Second Step: Time update
Generate the \( N \)-ensemble of time update estimation:

\[
\hat{X}_{k,i}^* = AX_{k-1,i} + w_{k,i}
\]

where \( x_{k,j} \sim N(0, Q) \) are the ensemble of system noises.

Mean of time update estimation is:

\[
\hat{X}_k^- = \frac{1}{N} \sum_{i=1}^{N} \hat{X}_{k,i}^*
\]

Covariance of error time update is:

\[
\hat{X}_k^* = \frac{1}{N} \sum_{i=1}^{N} \hat{X}_{k,i}^*
\]

Where \( i = 1, 2, 3, \ldots, N \)

Third Step: Measurement Update
Generate \( N \)-ensemble of measurement data

\[
z_{k,j} = z_k + v_{k,j}
\]

Where \( v_{k,j} \sim N(0, R) \) are the ensemble of measurement noises

Kalman Gain is defined as:

\[
K_k = P_k^- H^T [HP_k^- H^T + R_k]^{-1}
\]

Measurement update estimation are:

\[
\hat{X}_{k,j}^- = \hat{X}_{k,j}^* + K_k (z_{k,j} - H\hat{X}_{k,j}^*)
\]

Mean of measurement update estimation are:
\[ \hat{x}_k = \frac{1}{N} \sum_{i=1}^{N} \hat{x}_{k,i} \]

With covariance of error measurement update are:
\[ P_k = [I - K_k H] P_k^- \]

3.3 Unscented Kalman Filter

Other methods of estimation that used nonlinear system are Unscented Kalman Filter (UKF). This method estimates non-linear system model with Unscented transformation. The unscented transformation (UT) is a method for calculating the statistics of a random variable which undergoes a nonlinear transformation [9].

For example given a density function discrete opportunity has a \( y_k = f(x_k, \kappa) \) random variable \( x \) of a nonlinear model with the dimensions of \( L \), mean \( \hat{x} \) and covariant. Function approached with \( p_{x,y_k} = f(x_k, \kappa) \) unscented transformation. Then mean the mean and the covariant is used to determine the spread of \( 2L + 1 \) the points sigma around \( \hat{x} \). The points sigma in vector form Sigma \( X_i \), can be obtained using the following equation:
\[
X_0 = \hat{x} \\
X_i = \hat{x} + \left( \sqrt{(L + \lambda)P_x^k} \right)_i, \quad i = 1, ..., L \\
X_i = \hat{x} - \left( \sqrt{(L + \lambda)P_x^k} \right)_{i-L}, \quad i = (L + 1), ..., 2L
\]

With
\[ \lambda = \alpha^2 (n + \kappa) - L \]

Example given state variables:
\[ x = [x_1 \ x_2 \ ... \ x_L]^T \]

Point Sigma on equation (8) can be expressed in the form of a matrix as follows:
\[ X = [X_1 \ X_2 \ ... \ X_L \ X_{L+1} \ X_{L+2} \ ... \ X_{2L}]^T \]

Because \( y_k = f(x_k) \), then the vector deployment of Sigma is:
\[ y_i = f(X_i), \quad i = 0, ..., 2L \]

Point Sigma for are:
\[ y_i = \begin{bmatrix} f(X_0), \\ f(X_1), \\ . \\ . \\ f(X_L), \\ f(X_{L+1}), \\ . \\ . \\ f(X_{L+2}), \\ . \\ f(X_{2L}) \end{bmatrix} \]

A weighted mean and covariant as follows:
\[ W_0^{(m)} = \frac{\lambda}{L + \lambda} \]
\[
W_0^{(c)} = \frac{\lambda}{L + \lambda} + 1 - \alpha^2 + \beta
\]
\[
W_i^{(m)} = W_i^{(c)} = \frac{1}{2(L + \lambda)}, \quad i = 1, \ldots, 2L
\]  

(12)

By using the point Sigma on the equation (8), the mean and weighted covariants of the equation (12) are obtained mean as follows:
\[
\hat{y} = \sum_{i=0}^{2L} [W_i^{(m)} f(X_i)]
\]

(13)

Or
\[
\hat{y} = \left( \frac{\lambda}{L + \lambda} \right) f(X_0) = \frac{1}{2(L + \lambda)} \sum_{i=1}^{2L} [f(X_i)]
\]

(14)

As for counting the covariants of used the following equation:
\[
P_y = \sum_{i=0}^{2L} [W_i^{(c)} (f(X_i) - \hat{y})(f(X_i) - \hat{y})^T]
\]

(15)

Or
\[
P_y = \left( \frac{\lambda}{L + \lambda} + 1 - \alpha^2 + \beta \right) (f(X_i) - \hat{y})(f(X_i) - \hat{y})^T + \frac{1}{2(L + \lambda)} \sum_{i=1}^{2L} [W_i^{(c)} (f(X_i) - \hat{y})(f(X_i) - \hat{y})^T]
\]

(16)

Estimation and covariant error on unscented kalman Filter equations are obtained using unscented transformation.

The algorithm of Unscented Kalman Filter working with this step:

**First step: Initialization for \( k=0 \)**

Initial value : \( \hat{x}_0 = E[x_0] \)
\( \hat{x}_0^a = E[x^a] = [\hat{x}_0^T \ 0]^T \)

Covariance : \( P_0 = E[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T] \)
\[
P_0^a = E[(x_0^a - \hat{x}_0^a)(x_0^a - \hat{x}_0^a)^T] = \begin{bmatrix} P_x & 0 & 0 \\ 0 & P_w & 0 \\ 0 & 0 & P_v \end{bmatrix}
\]

**Second step: Initialization for \( k \in \{1, \ldots, \infty\} \)**

Calculate sigma point : \( X_{k-1}^a = [\hat{x}_{k-1}^a \ \hat{x}_{k-1}^a + \gamma \sqrt{P_{k-1}} \ \hat{x}_{k-1}^a - \gamma \sqrt{P_{k-1}}]^T \)
where \( \gamma = \sqrt{L + \lambda} \); \( \lambda = \alpha^2 (n + k) - L \)

**Third step: Time update**

Calculate sigma point : \( X_{k|k-1} = f[X_{k-1}^x, X_{k-1}^w] \)

Estimation : \( \hat{x}_k = \sum_{i=0}^{2L} W_i^{(m)} X_{i,k|k-1} \)

Covariance : \( P_k = \sum_{i=0}^{2L} W_i^{(c)} [X_{i,k|k-1} - \hat{x}_k] [X_{i,k|k-1} - \hat{x}_k]^T \)

Sigma point update : \( z_{k|k-1} = H[X_{k|k-1}^x, X_{k|k-1}^y] \)

Measurement estimation : \( \hat{z}_k = \sum_{i=0}^{2L} W_i^{(m)} z_{i,k|k-1} \)
Fourth step: Measurement update

Measurement covariance error: $P_{z_k} = \sum_{i=0}^{2L} W_i^{(c)} \left[ z_{i,k} - \hat{z}_k \right] \left[ z_{i,k} - \hat{z}_k \right]^T$

Cross covariance: $P_{x_k} = \sum_{i=0}^{2L} W_i^{(c)} \left[ x_{i,k} - \hat{x}_k \right] \left[ z_{i,k} - \hat{z}_k \right]^T$

Kalman Gain: $K_k = P_{x_k} P_{z_k}^{-1}$

Estimation: $\hat{x}_k = \hat{x}_k + K_k (z_k - \hat{z}_k)$

Covariance: $P_k = P_{x_k} - K_k P_{x_k} K_k^T$

3.4 Fractional Kalman Filter

One of other modification of Kalman Filter that estimate the variable state is Fractional Kalman Filter (FKF). Fractional Kalman Filter was modified from Kalman Filter with Calculus Fractional. In FKF, we use a definition of fractional discrete derivative, Grunwald-Letnikov, definition to get fractional order difference [4].

Based on Grunwald-Letnikov definition, order difference is given by the following equation [10]:

$$\Delta^n x_k = \frac{1}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} x_{k-j}$$  \hspace{1cm} (17)

When $n$ is a fractional order, $h$ is a sampling interval that equal to 1, $k$ is a number of sample that will be calculated. The factor $\binom{n}{j}$ can be obtained from [10]:

$$\binom{n}{j} = \frac{n(n-1)(n-j+1)}{j!}$$  \hspace{1cm} for $j > 0$

In this research, we will use second order for Fractional Kalman Filter method. But before we going to construct algorithm of second order FKF, we must construct from first order, because every fractional order was dependend by others. From Eq. (17), we will get the discrete state model for fractional first order as follow:

$$\Delta^1 x_{k+1} = x_{k+1} - x_k$$  \hspace{1cm} (18)

Using formula of stochastic system in Eq.(4), Eq.(5) and Eq.(18), we can get linier fractional first order stochastic discrete state system as follow:

$$\Delta^1 x_{k+1} = A_d x_k + Bu_k + w_k$$

$$\Delta^2 x_{k+1} = x_{k+1} - x_k$$

$$z_k = H x_k + v_k$$  \hspace{1cm} (19)

Where $\Delta^1 x_{k+1}$ is the first order fractional for state $x_{k+1}$, $w_k$ is a system input, $w_k$ and $v_k$ are system noise and measurement noise, $A_d = A - I$ and $I$ is an identity matrix. Variables $w_k \sim N(0, Q)$ and $v_k \sim N(0, R)$ are assumed white noise with zero-mean.

With Eq.(17), we can construct for the second order, that we will get the discrete state model for second order fractional as follow:

$$\Delta^2 x_{k+1} = x_{k+1} - 2x_k + x_{k-1}$$  \hspace{1cm} (20)

Using formula of stochastic system in Eq.(5), Eq.(19), and Eq.(20), we can get linier second order fractional stochastic discrete state system as follow:

$$\Delta^2 x_{k+1} = (A^2 - 2A + I)x_{k-1} + (A - 2I)Bu_{k-1} + (A - 2I)w_{k-1} + Bu_k + w_k$$

$$\Delta^2 x_{k+1} = x_{k+1} - 2x_k + x_{k-1}$$

$$z_k = H x_k + v_k$$  \hspace{1cm} (21)

The algorithm of the Fractional Kalman Filter working with this step:

First Step: Initialization

Initial value: $\hat{x}_0 = \bar{x}_0$
Covariance : \( P_0 = E[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T] \)

Second Step: Prediction (time update)

We get prediction value from \( x_k \) from the previous step

\[
\hat{x}_{k+1} = (A^2 - 2A + I) \hat{x}_{k-1} + (A - 2I)Bu_{k-1} + Bu_k
\]

With error covariance:

\[
P_{k+1} = (A^2 - 2A + I)(A^2 - 2A + I)^T P_{k-1} + (A - 2I)(A - 2I)^T Q_{k-1} + Q_k
\]

Third step: Correction

Kalman Gain : \( K_k = P_k^{-1} H^T (HP_k^{-1} H^T + R)^{-1} \)

Estimation : \( \hat{x}_k = \hat{x}_k^- + K(z_k - H\hat{x}_k^-) \)

Error Covariance : \( P_k = (1 - K_k H)P_k^- \)

4. Simulation and Result

In this research, we applied EnKF, UKF, and FKF to estimate the concentration of air pollution. We choose \( CO \) and \( NO_2 \) as particles of air pollution. We assume that air pollution that researched was in Surabaya city, and based on maps of Surabaya City, we divided into 10 × 10 sub area. Measurement data or real data have been got from Dinas Lingkungan Hidup Kota Surabaya, and also we take 100 data of concentration of \( CO \) and \( NO_2 \) that measured every 30 minutes along 2 days past 2 hours. Based on measurement data, Air Pollution Monitor Engineering measures on 3 locations, there are Kebonsari, Wonorejo, and Ketabang Kali. In this simulation, for EnKF, number of ensemble is 1000, and for FKF, we use second order fractional. Computational time and the accuracy of estimation are compared with all of these methods. For doing simulation, we must give the value from these parameter on Table 1

| Parameter | Value |
|-----------|-------|
| \( V_x \) | 2 km/h |
| \( V_y \) | 2 km/h |
| \( D_x \) | 0.25 km²/h |
| \( D_y \) | 0.25 km²/h |
| \( \alpha \) | 0.001 |
| \( \beta \) | 2 |
| \( dt \) | 0.1 |
| \( Q \) | 0.01 |
| \( R \) | 0.0001 |
| \( P_0 \) | 0.01 |
| \( \kappa \) | 0 |
4.1 Simulation using 100 data.

Figure 1. Estimation of CO at Kebonsari

Figure 2. Estimation of CO at 56th time in Kebonsari

Figure 3. Estimation of CO at Wonorejo

Figure 4. Estimation of CO at 9th time in Wonorejo

Figure 5. Estimation of CO at Ketabang Kali

Figure 6. Estimation of CO at 87th time in Ketabang Kali
Figure 1 until Figure 6 shows the estimation of concentration of $CO$, and Figure 7 until Figure 12 shows the estimation of concentration of $NO_2$. All of plots were using measurement data, EnKF, UKF, and FKF that shows on every plots.
4.2 3D Plot Estimation

In this part, we will doing 3d plot estimation for knowing the estimation on one time for the sample time. We choose 75\textsuperscript{th} time, on 2\textsuperscript{nd} of January 2018 at 13.30 WIB. Here is the 3D plot estimation:

**Figure 13.** 3D plot estimation of CO by UKF

**Figure 14.** 3D plot estimation of CO by EnKF

**Figure 15.** 3D plot estimation of CO by FKF

**Figure 16.** 3D plot estimation of NO\textsubscript{2} by UKF

**Figure 17.** 3D plot estimation of NO\textsubscript{2} by EnKF

**Figure 18.** 3D plot estimation of NO\textsubscript{2} by FKF
Figure 13 until 15 shows the 3D estimation of \( CO \), and Figure 16 until Figure 18 shows the 3D estimation of \( NO_2 \). Based on comparison by the methods, Fraksional Kalman Filter gives the evenly distribution of concentration of air pollution. Based on comparison by the type of air pollution, \( CO \) is more evenly distribution of its concentration than \( NO_2 \). It shows that many areas on plot are flat.

4.3 Contour Plot

![Figure 19. Contour plot estimation of CO by UKF](image1)

![Figure 20. Contour plot estimation of CO by EnKF](image2)

![Figure 21. Contour plot estimation of CO by FKF](image3)

![Figure 22. Contour plot estimation of NO\(_2\) by UKF](image4)

![Figure 23. Contour plot estimation of NO\(_2\) by EnKF](image5)

![Figure 24. Contour plot estimation of NO\(_2\) by FKF](image6)
Figure 19 until Figure 21 shows the contour plot of $CO$, and Figure 22 until Figure 24 shows the contour plot of $NO_2$ that used by EnKF, UKF, and FKF. Based on its distribution from its colour, Fractional Kalman Filter gives the evenly distribution of concentration of air pollutant, and $CO$ gives the evenly distribution of its concentration than $NO_2$.

4.4 Accuracy and Computational Time

Beside simulate the estimation, we also simulate error plot for knowing accuracy of every method and knowing which method is the best for estimate, and computational time for knowing which method is the fastest for compute. Here are the error plot and computational time for every methods:

![Error plot of CO](image1)

![Error plot of NO$_2$](image2)

### Table 2. RMSE of every pollutants and methods

| Pollutants | RMSE value |
|------------|------------|
|            | Unscented KF | Ensemble KF | Fractional KF |
| $CO$       | 0.0162      | 0.0093      | 0.0030        |
| $NO_2$     | 0.1512      | 0.1449      | 0.0587        |

Figure 25 and Figure 26 shows the error of estimation of air pollutants, that explain more on Table 2. Based on Table 2, Fractional Kalman Filter is most accurate method than others, but Ensemble Kalman Filter is more accurate than Unscented Kalman Filter.

### Table 3. Comparison of time computation of every methods

| Methods      | Computation Time (s) |
|--------------|----------------------|
| Unscented KF | 5.734                |
| Ensemble KF  | 14.4064              |
| Fractional KF| 0.3817               |

Based on Table 3, time computation of Fractional Kalman Filter is the fastest than others, but Unscented Kalman Filter is more fast than Unscented Kalman Filter.

5. Conclusion

Based on simulation result and analysis, we can conclude that:

1. Ensemble Kalman Filter, Unscented Kalman Filter, and Fractional Kalman Filter can be used to estimate air pollution on Surabaya on 100 position based on 100 measurement data with 3 point of measurement point.
2. Based on RMSE, Fractional Kalman Filter is the most accurate method, and Ensemble Kalman Filter is more accurate than Unscented Kalman Filter.
3. Based on computation time, Fractional Kalman Filter is the fastest method from all, and Unscented Kalman Filter more fast than Ensemble Kalman Filter.
4. Based on 3D Estimation and Contour plot, Fractional Kalman Filter gives the distribution of estimation more evenly than the other methods, and CO gives the distribution of concentration more evenly than NO₂.

Acknowledgments
This research is supported by Dinas Lingkungan Hidup Kota Surabaya for their data of concentration of air pollution that supporting for doing the simulation on this research, also we thankfull to the Modelling and Simulation Laboratoty, Departement of Mathematics, Institut Teknologi Sepuluh Nopember for support throughout this research.

References
[1] Simanjuntak A G 2007 Pencemaran udara. Bulletin Limbah, Vol 11 No 1.
[2] Apriliani E, Sanjoyo A B, and Adzkiya D 2011 The Groundwater Pollution Estimation by The Ensemble Kalman Filter. Canadian Journal on Science and Engineering Mathematics, Vol 2 No. 2, Pages 60-63.
[3] Metia S, Odoro S D, Sinha A P 2020 Pollutant Profile Estimation Using Unscented Kalman Filter. In: Basu T., Goswami S., Sanyal N. Advances in Control, Signal Processing and Energy Systems. Lecture Notes in Electrical Engineering, Vol 591, Pages 17-28.
[4] Oktaviana Y V 2018 Fractional Kalman Filter to Estimate the Concentration of Air Pollution. Journal of Physics: Conference Series 1008-012008.
[5] Apriliani E, Hanafi, and Wahyinghinsih 2011 Metode Estimasi Penyebaran Polutan di Udara. Jurnal Purifikasi Vol 12 No.2, Pages 53-62.
[6] Suyono 2014 Pencemaran Kesehatan Lingkungan. Jakarta: Penerbit Buku Kedokteran EGC.
[7] Lewis F L, Xie L, and Popa 2008 Optimal and Robust Estimation. London, New York : CRC Press
[8] Kim Y, and Bang H Introduction to Kalman Filter and Its Applications IntechOpen.
[9] Wan E A, Merwe R V D, The Unscented Kalman Filter for Nonlinear Estimation. Conference: Adaptive Systems for Signal Processing, Communications, and Control Symposium 2000, Vol 153-158, Pages 153-158.
[10] Ortigueira M D, Truillo, and Juan J 2011 Generalized Grunwald-Letnikov Fractional Derivative and Its Laplace and Fourier Transforms. Journal of Computational and Nonlinear Dynamics Vol 6.