Novel accelerating Einstein vacua and smooth inhomogeneous Riemannian manifolds

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Abstract

A novel class of Einstein vacua is presented, which possess non-vanishing cosmological constant and accelerating horizon with the topology of $S^{D-3}$ fibration over $S^1$. After Euclideanization, the solution describes a conformally distorted $S^{D-1}$ fibration over $S^1$, which is smooth, compact and inhomogeneous, and can be regarded as analogue of Don Page’s gravitational instanton.

1 Introduction

Despite its nearly a hundred-year-old age, Einstein gravity continues to be a source of inspirations and surprises. Besides the great success in describing physics in solar system, Einstein gravity also predicts existence of various black holes and even extended objects like black strings [1, 2, 3] and black rings [4, 5, 6, 7] in higher dimensions. Not all aspects of these nontrivial solutions have been fully understood. Even in the pure vacuum sector, Einstein gravity has been shown to possess an unexpected richness – in addition to the well known maximally symmetric vacua (dS, AdS, Minkowski etc), Einstein gravity also admits inhomogeneous vacua such as the anisotropic accelerating vacua [8, 9, 10] and the massless topological black hole vacua [11, 12, 13] and so on. In this article, we shall present a novel class of Einstein vacua which possess accelerating horizons of nontrivial topology. Concretely, the vacua we shall be discussing are accelerating vacua with horizons being conformally distorted sphere bundles over $S^1$. After Euclideanization, the whole vacuum metric

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becomes that of a conformally distorted sphere bundle over $S^1$, which corresponds to a smooth, compact and inhomogeneous Riemannian manifold.

Historically, the first known smooth, compact and inhomogeneous Riemannian manifold of constant scalar curvature was the gravitational instanton discovered by Don Page [14]. Page’s gravitational instanton describes a nontrivial metric of $S^2$ bundle over $S^2$. Such metrics were later generalized to higher dimensions [15], giving rise to constant curvature metrics of nontrivial $S^{D-2}$ bundle over $S^2$. In differential geometry, sphere bundles over $S^1$ are more complicated than sphere bundles over $S^2$, because the latter are simply connected and are known to be Einstein manifolds, while the former are non-simply connected and are not Einstein manifolds. The present work shows that, although $S^{D-1}$ bundles over $S^1$ are in general not Einstein, there exists an Einstein metric in the conformal class of such manifolds.

2 A 5D vacuum solution

We begin our study by presenting a novel Einstein vacuum solution in five dimensions (5D). We start from 5D not because 5D is of any particular importance for the construction, but because we began studying this subject with the aim of finding black rings with cosmological constant. Though we have not yet fulfilled our aim, the result presented here indeed bears some resemblance to black ring solutions, with the exception that instead of black ring horizon of topology $S^2 \times S^1$, we now have a cosmological horizon of the same topology.

The metric of the novel vacuum solution is given as follows,

$$ds^2 = \frac{1}{(1 + \alpha r \cos \theta)^2} \left[ -h(r)dt^2 + \frac{dr^2}{h(r)} + r^2 d\theta^2 + r^2 \sin^2 \theta d\psi^2 + (L + r \cos \theta)^2 d\phi^2 \right],$$

(1)

where

$$h(r) = 1 - \frac{\alpha r^2}{L}. \tag{2}$$

This metric is a generalized version of a 4-dimensional metric found in [16]. It is straightforward to show that this metric is an exact solution to the vacuum Einstein equation with a cosmological constant given by

$$\Lambda = \frac{6\alpha(1 - \alpha L)}{L}.$$ 

For $L > 0$, $\alpha > 0$ and $\alpha L < 1$, this is an inhomogeneous de Sitter spacetime with no singularities. For $\alpha L > 1$, the metric is still free of essential singularities, but the cosmological constant becomes negative, and there exist apparent singularities.
in the conformal factor which indicates non-compactness of the spacetime in this case. In the rest of this article, we assume $\alpha L < 1$, i.e. $\Lambda > 0$.

The singularity free nature of the metric is best manifested by the calculation of curvature invariants. For instances, we have

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{10\Lambda^2}{9},$$

$$R_{\mu\nu}R^{\mu\nu} = \frac{20\Lambda^2}{9},$$

$$R = \frac{10\Lambda}{3}.$$

Physically, the metric describes an accelerating vacuum of Einstein gravity with the accelerating horizon taking a nontrivial topology. The proper acceleration $a^\mu = u^\nu \nabla_\nu u^\mu$ for the static observers in the spacetime has the norm

$$a^\mu a_\mu = \left( \frac{\alpha L}{L} \right)^2 \left( L^2 + \frac{r^2 - 2rL \cos \theta + \alpha L^2 \cos^2 \theta}{1 - \frac{\alpha}{L} r^2} \right).$$

This quantity has the finite value $\alpha^2$ at $r = 0$ and blows up to infinity at $r = \sqrt{\frac{L}{\alpha}}$. So the $r_H = \sqrt{\frac{L}{\alpha}}$ hyper surface represents an accelerating horizon. Notice that the coordinate $r$ is not the radial variable in polar coordinate system, $r = 0$ corresponds not to the spacial origin but to a circle of radius $L$. Thus, unlike the usual de Sitter spacetime, the acceleration horizon is a topologically nontrivial manifold.

### 2.1 Horizon geometry

To understand the nontrivial topology of the acceleration horizon, we now look at the metric on the horizon surface. We have

$$ds^2_H = \frac{1}{(1 + \alpha r_H \cos \theta)^2} \left\{ r_H^2 (d\theta^2 + \sin^2 \theta d\psi^2) + (L + r_H \cos \theta)^2 d\phi^2 \right\}.$$

Let us temporarily put the conformal factor aside. The 3D hyper surface with the metric

$$ds^2 = r_H^2 (d\theta^2 + \sin^2 \theta d\psi^2) + (L + r_H \cos \theta)^2 d\phi^2$$

has a very nice geometric interpretation. Let us follow the treatment of [17] of this geometry. Consider a global embedding of a 3D hypersurface

$$x^2 + y^2 + (\sqrt{z^2 + w^2} - L)^2 = r_H^2$$

This is the equation of a 3D hyper surface with the metric $ds^2_H$.
into 4D Euclidean space $\mathbb{R}^4$. After introducing the toroidal coordinates \[17\]

\[
\begin{align*}
x &= \frac{\alpha \sin \hat{\theta}}{B} \cos \phi, \\
y &= \frac{\alpha \sin \hat{\theta}}{B} \sin \phi, \\
z &= \frac{\alpha \sinh \eta}{B} \cos \psi, \\
w &= \frac{\alpha \sinh \eta}{B} \sin \psi,
\end{align*}
\]

on $\mathbb{R}^4$, where

\[
B = \cosh \eta - \cos \hat{\theta}, \quad \alpha = \sqrt{L^2 - r_H^2},
\]

the 4D Euclidean line element

\[ds^2 = dx^2 + dy^2 + dz^2 + dw^2\]

becomes

\[ds^2 = \frac{\alpha^2}{B^2} [d\eta^2 + \sinh^2 \eta d\psi^2 + d\hat{\theta}^2 + \sin^2 \hat{\theta} d\phi^2]. \tag{5}\]

The 3D hyper surface (4) corresponds to constant $\eta = \eta_0$, with

\[\cosh \eta_0 = \frac{L}{r_H}.
\]

The line element on this 3-surface is

\[ds^2 = \frac{L^2 - r_H^2}{(L - r_H \cos \hat{\theta})^2} \left[ (L^2 - r_H^2) d\psi^2 + r_H^2 d\hat{\theta}^2 + r_H^2 \sin^2 \hat{\theta} d\phi^2 \right].
\]

After taking the coordinate transformation

\[\sin \theta = \frac{\sqrt{L^2 - r_H^2} \sin \hat{\theta}}{L - r_H \cos \hat{\theta}},\]

the above line element becomes (3).

We can make the correspondence of the line element (3) with the 3D hyper surface (4) even more direct. To do this, we simply parametrize the 3-surface as

\[
\begin{align*}
x &= r_H \sin \theta \cos \psi, \\
y &= r_H \sin \theta \sin \psi, \\
z &= (L + r_H \cos \theta) \sin \phi, \\
w &= (L + r_H \cos \theta) \cos \phi.
\end{align*}
\]

In fact, the hypersurface equation (4) describes an $S^2$ fibration over $S^1$, with the circle $S^1$ parametrized by the angle $\phi$. Therefore, the horizon surface is nothing but a conformally distorted $S^2$ bundle over $S^1$. Note that the hypersurface (4) is not of constant scalar curvature.
2.2 Conformal factor

To understand how seriously the conformal factor distorts the geometry of $S^2$ fibration over $S^1$, we need to study the behavior of the conformal factor. For any fixed $r = r_0$ (which is the case when we study the horizon geometry), the square root of the conformal factor, $(1 + \alpha r_0 \cos \theta)^{-1}$, sweeps an ellipse if $\alpha r_0 < 1$, a parabola if $\alpha r_0 = 0$ or a pair of hyperbola if $\alpha r_0 > 1$. The value of $\alpha r_0$ corresponds to the eccentricity of these conics.

Since we shall be mainly interested in compact surfaces, we assume $\alpha r_0 < 1$. Under this condition, if the part of the horizon surface besides the conformal factor were a round sphere, then the effect of the conformal factor will simply be squashing the sphere into an ellipsoid \cite{8}. In the present case, however, the part of the horizon surface besides the conformal factor is an $S^2$ bundle over $S^1$, so the full horizon surface is a conformally squashed $S^2$ bundle over $S^1$.

2.3 Causal structure

To make thorough understanding of the structure of the metric (1), we now study its causal structure. First, we introduce the Eddington-Finkelstein coordinates,

$$u = t - r^*, \quad v = t + r^*, \quad (6)$$

where the tortoise coordinate $r^*$ is defined as

$$r^* = \int \left(1 - \frac{\alpha r^2}{L}\right)^{-1} dr = \frac{r_H}{2} \log \left| \frac{r + r_H}{r - r_H} \right|, \quad (7)$$

and both $u$ and $v$ belong to the range $(-\infty, \infty)$. In this coordinate system, the metric (1) becomes

$$ds^2 = \rho^2 \left[-\left(1 - \frac{\alpha r^2}{L}\right) du dv + d\Omega_3^2\right], \quad (8)$$

where

$$\rho = \left(1 + \alpha r \cos \theta\right)^{-1} \quad (9)$$

and

$$d\Omega_3^2 = r^2 d\theta^2 + r^2 \sin^2 \theta d\psi^2 + (L + r \cos \theta)^2 d\phi^2$$

is the line element on the angular surface, whose geometry is very similar to that of the line element (3) which represents an $S^2$ bundle over $S^1$ as mentioned in Subsection 2.1. The Kruskal coordinates are introduced as

$$\tilde{u} = \pm \exp \left(\frac{u}{r_H}\right), \quad \tilde{v} = \pm \exp \left(-\frac{v}{r_H}\right),$$
where the signs of $\tilde{u}$ and $\tilde{v}$ coincide if $r < r_H$, and are opposite to each other if $r \geq r_H$. So there are totally 4 different combinations, each of which corresponds to a causal patch in the conformal diagrams to be drawn below. In each cases, one finds that

$$\tilde{u} \tilde{v} = \frac{r - r_H}{r + r_H},$$

(10)

and eq.(8) becomes

$$ds^2 = \rho^2 \left[ -(r + r_H)^2 d\tilde{u} d\tilde{v} + d\Omega_3^2 \right],$$

(11)

where $r$ and $\rho$ are to be regarded as functions of $\tilde{u}$ and $\tilde{v}$,

$$r = r_H \frac{1 - \tilde{u} \tilde{v}}{1 + \tilde{u} \tilde{v}},$$

$$\rho = \frac{1 + \tilde{u} \tilde{v}}{(1 + \tilde{u} \tilde{v}) + \alpha r_H (1 - \tilde{u} \tilde{v}) \cos \theta}.$$

Finally, the Carter-Penrose coordinates can be introduced by the usual arctangent mappings of $\tilde{u}$ and $\tilde{v}$

$$U = \arctan \tilde{u}, \quad V = \arctan \tilde{v},$$

$$T \equiv U + V, \quad R \equiv U - V,$$

in terms of which the metric becomes

$$ds^2 = \frac{\rho^2 r_H^2}{\cos^2 R} \left[ -dT^2 + dR^2 + \frac{1}{r_H^2} \cos^2 R \, d\Omega_3^2 \right].$$

(12)

The values of the product $\tilde{u} \tilde{v}$ at $r = 0$ ($\rho = 1$), $r = r_H$, $r = \infty$ ($\rho = 0$) and $\rho = \pm \infty$ are respectively

$$\lim_{r \to 0} \tilde{u} \tilde{v} = 1, \quad \Rightarrow U + V = \pm \frac{\pi}{2},$$

$$\lim_{r \to r_H} \tilde{u} \tilde{v} = 0, \quad \Rightarrow UV = 0,$$

$$\lim_{r \to \infty} \tilde{u} \tilde{v} = -1, \quad \Rightarrow U - V = \pm \frac{\pi}{2},$$

$$\lim_{\rho \to \pm \infty} \tilde{u} \tilde{v} = \frac{1 + \alpha r_H \cos \theta}{1 - \alpha r_H \cos \theta}.$$

The curves corresponding to $r = 0$, $r = r_H$ and $r = \infty$ can be easily depicted on the $(T, R)$ plane and they form various boundaries in the Carter-Penrose diagrams. However, from (9), it follows that $\rho = \pm \infty$ will never be reachable if $\cos \theta \geq 0$. So, only for $\cos \theta < 0$, the $\rho = \pm \infty$ curves on the $(T, R)$ plane can possibly play as boundaries in the Carter-Penrose diagram. When this the case, these curves will separate our spacetime into two patches: the $(+)$ patch with $\rho(r, \theta) > 0$ and the $(-)$ patch with $\rho(r, \theta) < 0$. Below we shall consider 3 different sub cases in detail.
$r = 0$

Figure 1: Carter-Penrose diagrams: the shaded areas correspond to the ($-$) patch, while the unshaded areas correspond to the ($+$) patch.

- $\cos \theta > 0$ ($\theta \in [0, \pi/2]$). In this case, $\rho \in [0, 1)$ and there is no conformal infinity in the causal diagram. The Carter-Penrose diagram is consisted purely of the first 3 kinds of boundaries and is depicted in Fig.1 (a).

- $\cos \theta = 0$ ($\theta = \pi/2$). In this case, $\rho$ takes the fixed value 1, so is also no conformal infinities in the causal diagram. The Carter-Penrose diagram is depicted in Fig.1 (b).

- $\cos \theta < 0$ ($\theta \in (\pi/2, \pi]$). In this case, $\rho \in (-\infty, 0] \cup (1, \infty)$. Moreover, the $\rho = +\infty$ and the $\rho = -\infty$ curves coincide on the $(T, R)$ plane, which separate the spacetime into ($+$) and ($-$) patches. It can be easily inferred from above that $-1 < \lim_{\rho \to \pm\infty} \tilde{u} \tilde{v} < 0$ and its value is varying with $\theta$. The corresponding Carter-Penrose diagram is depicted in Fig.1 (c), in which the shaded areas correspond to the ($-$) patch, while the unshaded area corresponds to the ($+$) patch. The variable $\rho$ is discontinuous in this case: it takes values in $(0, \infty)$ in the ($+$) patch and in $(-\infty, 0]$ in the ($-$) patch.

## 3 Limiting cases

The metric depends on two parameters $\alpha$ and $L$, which can be chosen to take various limits.

The first limit we can consider is taking $\alpha \to 0$ while keeping $L$ finite. In this limit, both the conformal factor and the function $f(r)$ goes to unity and the metric becomes

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\psi^2 + (L + r \cos \theta)^2 d\phi^2,$$

which describes a Ricci flat spacetime. The spacial slices of the spacetime are nothing but a trivial fibration of $\mathbb{R}^3$ over $S^1$. The corresponding spacetime has
vanishing proper accelerations for any static observer and there is no acceleration horizon. If, at this point, we require further \( L \rightarrow 0 \), then the line element becomes that of the 5D Minkowski spacetime,

\[
ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2_3,
\]

where

\[
d\Omega^2_3 = d\theta^2 + \sin^2 \theta d\psi^2 + \cos^2 \theta d\phi^2
\]
is the metric of \( S^3 \) written in the standard Hopf coordinates. It is amazing that the presence of the acceleration parameter produces both the horizon and its nontrivial topology.

The next limit we can consider is a double scaling limit \( \alpha \rightarrow 0, L \rightarrow 0 \) whilst \( \alpha/L \rightarrow 1/l^2 \) is kept finite. In this limit, the spacetime becomes the standard spherically symmetric de Sitter spacetime \( dS_5 \) with radius \( l \). Notice that we cannot take the limit \( L \rightarrow 0 \) while keeping \( \alpha \) finite, because in that case the function \( h(r) \) becomes undefined.

The third limit we can consider can be achieved by an overall rescaling of the metric. Let us make a rescaling \( t \rightarrow \sqrt{\frac{L}{\alpha}} \tilde{t}, r \rightarrow \sqrt{\frac{L}{\alpha}} \tilde{r} \), so that the resulting metric becomes

\[
ds^2 = \frac{A}{(1 + \ell \tilde{r} \cos \theta)^2} \left\{ - h(\tilde{r}) d\tilde{t}^2 + \frac{d\tilde{r}^2}{h(\tilde{r})} + \tilde{r}^2 d\theta^2 + \tilde{r}^2 \sin^2 \theta d\psi^2 + (\ell + \tilde{r} \cos \theta)^2 d\phi^2 \right\},
\]

where

\[
A = \frac{L}{\alpha}, \quad \ell = \sqrt{\alpha L},
\]

and

\[
h(\tilde{r}) = 1 - \tilde{r}^2.
\]

Notice that the cosmological constant can be expressed in terms of \( A \) and \( \ell \) as

\[
\Lambda = 6A^{-1}(1 - \ell^2).
\]

So we have \( 0 \leq \ell < 1 \) for de Sitter case, \( \ell = 1 \) for flat case and \( \ell > 1 \) for AdS case.

We can drop the overall constant factor \( A \) in the line element, which just results in a rescaling of the cosmological constant. The resulting metric will then depend only on a single parameter \( \ell \). We can further take the limit \( \ell \rightarrow 0 \), in which case the metric becomes that of the standard 5D de Sitter of unit radius. Note that this limit cannot be achieved if we had not made the rescaling of the metric before.
4 Euclideanization

Euclideanization of Einstein vacua is an important subject of study from both physics and mathematical perspectives. The physics motivation for studying the Euclideanized vacua is to study the vacuum transitions in Euclidean quantum gravity. The known examples of [14] and [15] were found for this purpose. Mathematically, Euclideanized Einstein metrics often gives explicit examples of smooth compact Riemannian manifolds, which is an important class of manifold in differential geometry. In the following, we shall see that there exist a Riemannian metric of constant scalar curvature in the conformal class of $S^4$ fibration over $S^1$. The corresponding metric is nothing but the Euclideanized version of our Einstein vacuum solution.

Let us start from the line element (13). Making a Wick rotation $\tilde{t} \to i\chi$ and letting $\tilde{r} = \sin \sigma$, we get\(^1\)

$$
\begin{aligned}
\bar{ds}_5^2 &= \frac{A}{(1 + \ell \sin \sigma \cos \theta)^2} \left\{ d\sigma^2 + \cos^2 \sigma \, d\chi^2 \\
&\quad + \sin^2 \sigma \,(d\theta^2 + \sin^2 \theta \, d\psi^2) + (\ell + \sin \sigma \cos \theta)^2 \, d\phi^2 \right\}.
\end{aligned}
$$

(14)

In the absence of the conformal factor, this metric is just an $S^4$ fibration over $S^1$ which is not a constant curvature manifold\(^2\). The coordinate $(\sigma, \chi, \theta, \psi)$ parametrizes the $S^4$ fiber, which can be regarded as a generalization of the standard Hopf coordinates for $S^3$. The ranges of these coordinates are given in (18) in the appendix. The conformal factor squashes the round $S^4$ fiber and makes the Ricci scalar of the full space a constant. In other words, there is a constant scalar curvature manifold in the conformal class of $S^4$ fibration over $S^1$. The study of conformal class of a given Riemannian manifold is an important subject of study in differential geometry, because this subject is intimately related to the analysis of geometric flows.

Since the metric (14) represents a conformally distorted $S^4$ fibration over $S^1$, we shall refer to the corresponding geometry as a “ring geometry”, with the ring (i.e. the $S^1$ factor) parametrized by the angle $\phi$. Fibers of the ring geometry at a given $\phi$ is a conformal 4-sphere. The overall constant $A$ signifies the size of the ring surface, and the parameter $\ell$ represents the relative radius of the $S^1$ with respect to that of the $S^4$. We have mentioned previously that when $0 < \ell < 1$, the corresponding geometry is smooth, compact and inhomogeneous, with a positive constant scalar curvature. Perhaps it is the first known example of such metrics

\(^1\)By convention, we denote the Euclideanized line elements with a bar and the dimension of the Euclideanized space is specified by a suffix.

\(^2\)That the metric (14) without the conformal factor describes an $S^4$ fibration over $S^1$ can also be understood from an extrinsic geometric point of view, just like the way we understood the horizon geometry in Section 2. We put this extrinsic geometric description in the appendix.
on $S^4$ fibration over $S^1$. The case $\ell = 0$ corresponds to a degenerate case, i.e. a conformal $S^5$.

It is interesting to evaluate the volume of this compact space. Without loss of generality, we set $A = 1$ in the following calculations. We have, by direct calculation,

$$\sqrt{g} = \frac{\cos \sigma \sin^2 \sigma \sin \theta (\ell + \sin \sigma \cos \theta)}{(1 + \ell \sin \sigma \cos \theta)^5}.$$ 

Using the coordinate range (18), it is easy to evaluate the 5-volume of the space. The result is

$$V_5 = \int d^5x \sqrt{g} = 0.$$ 

Since the space is of constant scalar curvature, the corresponding Einstein-Hilbert action is proportional to the 5-volume, so its value is also zero. Compact Einstein metrics of zero volume are not rare. See [18] for other examples.

## 5 General dimensions

So far we have been restricting ourselves in five dimensions. As mentioned earlier, 5D is not of any particular importance in the construction. There is a higher dimensional cousin for the metric we studied in the previous sections. The dimensional metric can be written as

$$ds^2_D = \frac{1}{(1 + \alpha r \cos \theta)^2} \left\{ -h(r) dt^2 + \frac{dr^2}{h(r)} + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\Omega^2_{D-4} + (L + r \cos \theta)^2 d\phi^2 \right\},$$

where $h(r)$ is as before, and $d\Omega^2_{D-4}$ is the line element of a round $(D - 4)$-sphere. The associated cosmological constant is given by

$$\Lambda_D = \frac{(D - 1)(D - 2) \alpha}{2} \frac{\alpha}{L}(1 - \alpha L).$$

The physical and mathematical properties of this metric is extremely similar to the 5D case. For instance, the metric interpreted as Einstein vacuum represents an accelerating vacuum in which the acceleration horizon has the topology of $S^{D-3}$ fibration over $S^1$. The Euclideanized version of the metric represents a conformally distorted metric on the full space of an $S^{D-1}$ fibration over $S^1$ and this metric is only compact for small value of $\ell$, etc.

The Euclideanized version of the higher dimensional metric is given as follows,

$$ds^2_D = \frac{A}{(1 + \ell \sin \sigma \cos \theta)^2} \left\{ d\sigma^2 + \cos^2 \sigma d\chi^2 + \sin^2 \sigma (d\theta^2 + \sin^2 \theta d\Omega^2_{D-4}) + (\ell + \sin \sigma \cos \theta)^2 d\phi^2 \right\}. \quad (15)$$
This metric describes a conformally squashed $S^{D-1}$ fibration over $S^1$. The cosmological constant can be written in terms of the parameters $A$ and $\ell$ as

$$\Lambda_D = \frac{(D-1)(D-2)}{2} \left( \frac{1 - \ell^2}{A} \right).$$  \hspace{1cm} (16)

**Remarks:**

The metric (15) makes sense for all dimensions $D \geq 3$. We have already analyzed in detail the $D = 5$ case. For $D = 4$, we just remove the $d\Omega_{D-4}^2$ term from the metric and recovers the metric given in [16]. For $D = 3$, we remove the $d\theta^2$ term altogether, and set $\theta = 0$ everywhere else.

### 6 Conclusion and discussion

We presented a novel class of accelerating Einstein vacua with accelerating horizon bearing a nontrivial topology of $S^{D-3}$ bundle over $S^1$. Such solutions contain two parameters, one corresponds to the acceleration, the other corresponds to the relative radius of the $S^1$ base with respect to the $S^{D-3}$ fiber. There are various limiting cases for these parameters. Among these, the zero acceleration limit corresponds to a Ricci flat vacuum with no horizon. The double scaling limit gives rise to the standard de Sitter vacua. Upon Euclideanization, the full space becomes a smooth compact inhomogeneous Riemannian manifold with a positive constant scalar curvature. Such Euclidean manifolds can be regarded as analogues of Page’s gravitational instanton or its generalizations, but the topologies are now conformally distorted $S^{D-1}$ bundle over $S^1$, rather than sphere bundles over $S^2$.

Since for generic values of the parameters, the vacua we obtained possess a positive cosmological constant and a ring-like acceleration horizon, we expect that such vacua should be the starting point to construct black rings with cosmological constant. Asymptotically flat black ring solutions in Einstein gravity were found over ten years ago, but so far no black ring solutions with cosmological constant were found. One of the major obstacle for constructing such solutions is the unmatching topologies: black ring solutions have horizons with $S^2 \times S^1$ topology, whilst the usual 5D de Sitter spacetime has only an accelerating horizon of $S^3$ topology. The new vacua we found in this article do have the matching topology with black rings. To actually construct black rings which asymptote to our vacuum solutions, some complicated mathematical constructions are yet to be carried out. Presumably the Kerr-Schild method which led to the discoveries of rotating black holes in higher dimensions with cosmological constants [19, 20] is a good starting point. We shall continue our explorations in this direction.
Appendix: extrinsic geometric description of $S^4$ fibration over $S^1$

Consider the following surface

\[ x_1^2 + x_2^2 + x_3^2 + x_4^2 + (\sqrt{x_5^2 + x_6^2} - \ell)^2 = 1 \quad (17) \]

in the 6D Euclidean space with metric $ds^2 = \sum_{i=1}^{6} dx_i^2$. It is straightforward to parametrize this surface as

\[
\begin{align*}
    x_1 &= \cos \sigma \sin \chi, \\
    x_2 &= \cos \sigma \cos \chi, \\
    x_3 &= \sin \sigma \sin \theta \cos \psi, \\
    x_4 &= \sin \sigma \sin \theta \sin \psi, \\
    x_5 &= (\ell + \sin \sigma \cos \theta) \sin \phi, \\
    x_6 &= (\ell + \sin \sigma \cos \theta) \cos \phi,
\end{align*}
\]

where the angular coordinates $(\sigma, \chi, \theta, \psi, \phi)$ must be chosen such that these parametrization equations cover the surface (17) exactly once. The ranges for these angular coordinates are given as follows:

\[
\begin{align*}
    \sigma &\in [0, \pi/2], \quad \theta \in [0, \pi], \\
    \chi &\in [0, 2\pi], \quad \psi \in [0, 2\pi], \quad \phi \in [0, 2\pi].
\end{align*}
\quad (18)
\]

Inserting the parametrization equations into the 6D Euclidean line element we get the desired result

\[ ds^2 = d\sigma^2 + \cos^2 \sigma d\chi^2 + \sin^2 \sigma (d\theta^2 + \sin^2 \theta d\psi^2) + (\ell + \sin \sigma \cos \theta)^2 d\phi^2, \]

which is identical to the line element (14) without the conformal factor. The Ricci scalar associated with the above metric reads

\[ R = \frac{4(3\ell + 5 \sin \sigma \cos \theta)}{\ell + \sin \sigma \cos \theta}, \]

which is not a constant. The surface (17) clearly describes an $S^4$ fibration over $S^1$.

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