View on N-dimensional spherical harmonics from the quantum mechanical Pöschl-Teller potential well

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Abstract

In this work we propose an approach of obtaining of N-dimensional spherical harmonics based exclusively on the methods of solutions of differential equations and the use of the special functions properties. We deduce the Laplace-Beltrami operator on the N-sphere, indicate some instructive relations for the metric, and demonstrate the procedure of separation of the variables. We show that the ordinary differential equations for every variable, except one, can be reduced to the Schrödinger equation (SE) with the symmetric Pöschl-Teller (SPT) potential well by means of certain substitutions. We also exhibit that the solutions of SE with SPT potential are expressed in terms of the Gegenbauer polynomials. The eigenvalues of the Laplace-Beltrami operator and the characteristic numbers of the spherical harmonics are obtained with the use of the properties of the spectrum of SE with SPT potential. The spherical harmonics are constructed as a product of the eigenfunctions of SE with SPT potential multiplied by a easily computable factor function and expressed in terms of the Gegenbauer polynomials. The work has a pedagogical character to some extent.

1 Introduction

In this paper we propose an approach of obtaining of N-dimensional spherical harmonics based on the method of separation of variables and on the use of the properties of the special functions. We point out the works which are most substantial on the question in our opinion. The first thorough description of obtaining of N-dimensional spherical harmonics was apparently made in the monograph [1] where the harmonic polynomials theory is applied and the generating function of Gegenbauer polynomials is utilized. In [1] the spherical harmonics are obtained in terms of Gegenbauer polynomials. The theorems of orthogonality and completeness of the system of the spherical harmonics are proved, the addition theorem and the other important relations are shown. In the work [2] a generalized

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angular momentum technique is applied. In particular, the eigenvalues of a square of the orbital angular momentum operator in N dimensions are determined. The eigenvectors of this operator can be interpreted as N-dimensional spherical harmonics. In the paper [3] scalar spherical harmonics in N dimensions are obtained as a starting point for discussion of symmetric tensor spherical harmonics in N dimensions. The scalar spherical harmonics are obtained in terms of the associated Legendre functions for the purposes of the work. The necessary properties of the spherical harmonics are demonstrated and eigenvalues of the Laplace-Beltrami operator on N-dimensional sphere are determined. In the paper [4] a construction of N-dimensional spherical harmonics is fulfilled by means of a harmonic projection. This method is proposed as complementary to the methods of the group theory. In addition we mention a recent pedagogical work [5] in which N-dimensional spherical geometry is discussed with a description of the use of Legendre polynomials. In our paper we propose an approach which is distinct from the others described in the accessible literature.

We also note that papers in which the spherical harmonics of various dimensions are utilized in one or another mode for discussions of diverse problems have continued to appear. We mention a few recent papers of this kind. For example, in the paper [6] the scalar harmonics on a four sphere are analyzed using a associated Legendre function. Then, they are used for construction of two types of vector harmonics and three types of tensor harmonics on a four sphere. In [7] the spherical harmonics in N dimensions are used for the construction of summation by parts finite-differencing numerical methods of solution of wave equations in N+2 spacetime dimensions. In [8] the explicit vector spherical harmonics on the 3-sphere are constructed with the use of the technique of p-forms. In [9] the spherical harmonics on 2-sphere are considered from the viewpoint of su(2) Lie algebra symmetry realized in quantization of the magnitude and z-component of angular momentum operator in terms of the azimuthal and magnetic quantum numbers. The azimuthal quantum number is associated with an additional ladder symmetry. In [9] it is shown that simultaneous realization of these two symmetries inherits the positive and negative integer discrete irreducible representations for su(1,1) Lie algebra via the spherical harmonics on the sphere as a compact manifold.

The paper is organized as follows. In Sec. 2 we introduce notations and obtain the Laplace-Beltrami operator on the N-sphere. We reproduce principal steps of computing of the metric and indicate its properties necessary for application of the method of separation of variables. In Sec. 3 we apply the method of separation of variables and obtain a set of the ordinary differential equations of a certain form. In Sec. 4 we describe the solutions of the Schrödinger equation with the symmetric Pöschl-Teller potential and present them in terms of the Gegenbauer polynomials. In Sec. 5 we solve
the set of the ordinary differential equations obtained in Sec. 3 applying the solutions of Sec. 4. In Sec. 6 we construct the N-dimensional spherical harmonics and make some comments. In Sec. 7 we present conclusions of the work.

2 Laplace-Beltrami operator on the N-sphere

To construct N-dimensional spherical harmonics we solve the Laplace equation in spherical coordinates in $N+1$ dimensions. To this end we use a technique of the Laplace-Beltrami operator. We reproduce some details of computation of the metric and the Laplace-Beltrami operator, mainly for didactical purposes. We use the spherical coordinates in $N+1$ dimensions $(r, \theta_1, \theta_2, \ldots, \theta_N)$ in the following form:

$$x_{N+1} = r \cos \theta_N,$$

$$x_N = r \sin \theta_N \cos \theta_{N-1},$$

$$x_{N-1} = r \sin \theta_N \sin \theta_{N-1} \cos \theta_{N-2},$$

$$\ldots$$

$$x_3 = r \sin \theta_N \sin \theta_{N-1} \ldots \sin \theta_3 \cos \theta_2,$$

$$x_2 = r \sin \theta_N \sin \theta_{N-1} \ldots \sin \theta_3 \sin \theta_2 \cos \theta_1,$$

$$x_1 = r \sin \theta_N \sin \theta_{N-1} \ldots \sin \theta_3 \sin \theta_2 \sin \theta_1$$

with the ranges:

$$0 \leq r < \infty, \ 0 \leq \theta_1 < 2\pi, \ 0 \leq \theta_i \leq \pi, \ i = 2, 3, \ldots, N$$

so that $x_1^2 + x_2^2 + \ldots + x_{N+1}^2 = r^2$. To obtain the Laplacian we use the Laplace-Beltrami operator applied to a scalar function $u$:

$$\nabla^2 u = \frac{1}{\sqrt{|g|}} \partial_i \left( \sqrt{|g|} g^{ij} \partial_j u \right) \quad (2.2)$$

where $g^{ij} = g^{-1}_{ij}$ is the inverse of the metric tensor $g_{ij}$, $g = \det(g_{ij})$ is the determinant of $g_{ij}$, $\partial_i$ is the derivative with respect to the variable $q_i$, $q_{N+1} = r$, $q_i = \theta_i$, $i = 1, 2, \ldots, N$, and summation over $i$, $j$ is implied.

Using the relation of the spherical coordinates with the Cartesian coordinates (2.1) one can determine the metric $g_{ij}$ as follows:

$$g_{ij} = \sum_{k=1}^{N+1} \frac{\partial x_k}{\partial q_i} \frac{\partial x_k}{\partial q_j}. \quad (2.3)$$
Now we compute the Laplace-Beltrami operator:

\[
\frac{\partial x_k}{\partial q_{N+1}} = \frac{x_k}{r}, \quad \frac{\partial x_k}{\partial q_i} = \begin{cases} 
0, & k \geq i + 2, \\
-x_{i+1} \tan \theta_i, & k = i + 1, \\
x_k \cot \theta_i, & k \leq i.
\end{cases} \tag{2.4}
\]

We also note that

\[
\sum_{k=1}^{N+1} x_k^2 = r^2, \quad \sum_{k=1}^{i} x_k^2 = x_{i+1}^2 \tan^2 \theta_i.
\tag{2.5}
\]

Then with the use of Eqs. (2.4), (2.5) one gets the components of the metric:

\[
g_{N+1,N+1} = \sum_{k=1}^{N+1} \left(\frac{\partial x_k}{\partial r}\right)^2 = \sum_{k=1}^{N+1} \frac{x_k^2}{r^2} = 1,
\]

\[
g_{ii} = \sum_{k=1}^{N+1} \left(\frac{\partial x_k}{\partial q_i}\right)^2 = \sum_{k=1}^{i} \left(\frac{\partial x_k}{\partial q_i}\right)^2 + \left(\frac{\partial x_{i+1}}{\partial q_i}\right)^2 + \sum_{k=i+2}^{N+1} \left(\frac{\partial x_k}{\partial q_i}\right)^2
\]

\[
= \sum_{k=1}^{i} x_k^2 \cot^2 \theta_i + x_{i+1}^2 \tan^2 \theta_i = x_{i+1}^2 (1 + \tan^2 \theta_i) = \frac{x_{i+1}^2}{\cos^2 \theta_i}, \quad i \leq N,
\]

\[
g_{ij} = \sum_{k=1}^{N+1} \frac{\partial x_k}{\partial q_i} \frac{\partial x_k}{\partial q_j} = \sum_{k=1}^{i} \frac{\partial x_k}{\partial q_i} \frac{\partial x_k}{\partial q_j} + \frac{\partial x_{i+1}}{\partial q_i} \frac{\partial x_{i+1}}{\partial q_j} + \sum_{k=i+2}^{N+1} \frac{\partial x_k}{\partial q_i} \frac{\partial x_k}{\partial q_j}
\]

\[
= \sum_{k=1}^{i} x_k^2 \cot \theta_i \cot \theta_j - \frac{1}{\cos^2 \theta_i} x_{i+1}^2 \tan \theta_i \cot \theta_j = x_{i+1}^2 \tan \theta_i \cot \theta_j - \frac{1}{\cos^2 \theta_i} x_{i+1}^2 \tan \theta_i \cot \theta_j = 0, \quad i < j
\]

therefore

\[
g_{ij} = \text{diag} (\eta_i), \quad \eta_{N+1} = 1, \quad \eta_i = \frac{x_{i+1}^2}{\cos^2 \theta_i} = r^2 \prod_{k=1}^{N} \sin^2 \theta_k, \quad 1 \leq i \leq N. \tag{2.6}
\]

For the determinant we get:

\[
g = \det g_{ij} = \prod_{i=1}^{N+1} g_{ii} = \prod_{i=1}^{N} \frac{x_{i+1}^2}{\cos^2 \theta_i} = \frac{r^{2N} \prod_{k=2}^{N} \sin^{2(k-1)} \theta_k}{N}
\]

and respectively

\[
\sqrt{|g|} = \prod_{i=1}^{N} \frac{x_{i+1}}{\cos \theta_i} = r^N \prod_{k=2}^{N} \sin^{k-1} \theta_k. \tag{2.7}
\]

For the inverse tensor \( g^{ij} \) one has:

\[
g^{ij} = (g_{ij})^{-1} = \text{diag} (h^i), \quad h^{N+1} = 1, \quad h^i = \eta_i^{-1} = r^{-2} \prod_{k=i+1}^{N} \sin^{-2} \theta_k, \quad 1 \leq i \leq N. \tag{2.8}
\]

Now we compute the Laplace-Beltrami operator:

\[
\nabla^2 u = \frac{1}{\sqrt{|g|}} \partial_i \left( \sqrt{|g|} g^{ij} \partial_j u \right) = \frac{1}{\sqrt{|g|}} \partial_i \sqrt{|g|} g^{ij} \partial_j u + (\partial_i g^{ij}) \partial_j u + g^{ij} \partial_i \partial_j u \tag{2.9}
\]
where we imply that the derivatives inside the brackets on the rhs are applied only to the quantities inside the brackets. For the first term of Eq. (2.9) one can see that

$$\frac{1}{\sqrt{|g|}} \left( \partial_{N+1} \sqrt{|g|} \right) = Nr^{-1}, \quad \partial_i \sqrt{|g|} = 0, \quad \frac{1}{\sqrt{|g|}} \left( \partial_i \sqrt{|g|} \right) = (i - 1) \cot \theta_i, \quad 2 \leq i \leq N$$

then

$$\frac{1}{\sqrt{|g|}} \left( \partial_i \sqrt{|g|} \right) g^{ij} \partial_j u = \sum_{i=1}^{N} h^i (i - 1) \cot \theta_i \partial_i u + Nr^{-1} \partial_r u.$$ 

For the second term one can see that $\partial_i g^{ij} = 0, \quad \forall j \quad (\text{no summation})$ then

$$(\partial_i g^{ij}) \partial_j u = 0.$$ 

For the third term one has:

$$g^{ij} \partial_i \partial_j u = \sum_{i=1}^{N+1} h^i \partial_i^2 u = \sum_{i=1}^{N} h^i \partial_i^2 u + \partial_{N+1}^2 u = \sum_{i=1}^{N} h^i \partial_i^2 u + \partial_r^2 u.$$ 

Therefore

$$\nabla^2 u = \sum_{i=1}^{N} h^i \left[ (i - 1) \cot \theta_i \partial_i u + \partial_i^2 u \right] + r^{-N} \partial_r \left( r^N \partial_r u \right).$$ 

It is convenient to introduce the quantities

$$h^i_{(N)} = r^2 h^i, \quad 1 \leq i \leq N, \quad (2.10)$$

which depend only on the angular variables $\theta_i$ and does not depend on the variable $r$. Then we arrive at the final form of the Laplace operator in the spherical coordinates:

$$\nabla^2 u = r^{-2} \nabla^2_{S^N} u + r^{-N} \partial_r \left( r^N \partial_r u \right) \quad (2.11)$$

where

$$\nabla^2_{S^N} u = \sum_{i=1}^{N} h^i_{(N)} \left[ (i - 1) \cot \theta_i \partial_i u + \partial_i^2 u \right]$$

is the Laplace-Beltrami operator on the unit N-sphere $S^N$.

### 3 Separation of variables

In order to solve the Laplace equation

$$\nabla^2 u = 0$$

we present $u$ as

$$u (r, \theta_1, \theta_2, ..., \theta_N) = R(r) W_N (\theta_1, \theta_2, ..., \theta_N)$$
and we come to
\[
\frac{1}{R} r^{-N} \partial_r \left( r^N \partial_r R \right) = - \frac{1}{W_N} \Delta_{S^N} W_N = \lambda_N = \text{const}
\]

because the lhs of the equality depends only on the radial variable \(r\) and the rhs depends only on the angular variables \(\theta_i\), \(\lambda_N\) is a separation constant.

We concentrate our attention on the equation:
\[
\nabla_{S^N}^2 W_N + \lambda_N W_N = 0 \quad (3.1)
\]

which is the main point of our work and whose solution represents the \(N\)-dimensional spherical harmonics. It is useful to introduce the quantities
\[
h_i^{(k)} = \sin^2 \theta_k h_{(k)}^{(k)}, \quad 1 \leq i \leq k - 1, \quad 2 \leq k \leq N. \quad (3.2)
\]

For a given value \(k\) the set \(\{h_i^{(k)}\}\) depend only on the variables \((\theta_1, ..., \theta_k)\) and \(h_k^{(k)} = 1\). Then we can write Eq. (3.1) as
\[
\sum_{i=1}^{N-1} \sin^{-2} \theta_N h_i^{(N-1)} [(i-1) \cot \theta_i \partial_i W_N + \partial_i^2 W_N] + h_N^{N} ((N-1) \cot \theta_N \partial_N W_N + \partial_N^2 W_N) = -\lambda_N W_N
\]

or in a compact form:
\[
\sin^{-2} \theta_N \nabla_{S^{N-1}}^2 W_N + ((N-1) \cot \theta_N \partial_N W_N + \partial_N^2 W_N) = -\lambda_N W_N \quad (3.3)
\]

where the notation of Eq. (2.12) was utilized. We present \(W_N\) as follows:
\[
W_N (\theta_1, \theta_2, ..., \theta_N) = W_{N-1} (\theta_1, \theta_2, ..., \theta_{N-1}) y_N (\theta_N) \quad (3.4)
\]

then we come to
\[
- \frac{1}{W_{N-1}} \Delta_{S^{N-1}} W_{N-1} = \frac{\sin^2 \theta_N}{y_N} ((N-1) \cot \theta_N \partial_N y_N + \partial_N^2 y_N) + \lambda_N \sin^2 \theta_N = \text{const} = \lambda_{N-1}
\]

because the lhs of the equality depends only on the variables \((\theta_1, ..., \theta_{N-1})\) and the rhs depends only on the variable \(\theta_N\), \(\lambda_{N-1}\) is a new separation constant. Thus we get the equation for \(W_{N-1}\):
\[
\nabla_{S^{N-1}}^2 W_{N-1} = -\lambda_{N-1} W_{N-1}
\]

and the equation for \(y_N (\theta_N)\):
\[
((N-1) \cot \theta_N \partial_N y_N + \partial_N^2 y_N) + \left( \lambda_N - \frac{\lambda_{N-1}}{\sin^2 \theta_N} \right) y_N = 0.
\]

Repeating this process we obtain the set of equations for \(W_k = W_k (\theta_1, \theta_2, ..., \theta_k)\):
\[
\nabla_{S^k}^2 W_k + \lambda_k W_k = 0, \quad 1 \leq k \leq N \quad (3.5)
\]
and for \( y_k = y_k(\theta_k) \):
\[
\partial_k^2 y_k + (k-1) \cot \theta_k \partial_k y_k + \left( \lambda_k - \frac{\lambda_{k-1}}{\sin^2 \theta_k} \right) y_k = 0, \quad 2 \leq k \leq N
\] (3.6)

with the set of \( N \) separation constants \( \lambda_k \). For \( k = 1 \) we get:
\[
\nabla_S^2 y_1 = \partial_1^2 y_1 = -\lambda_1 y_1 .
\] (3.7)

The spherical harmonics \( W_N \) are finally expressed in the form of a product of \( y_k(\theta_k) \):
\[
W_N(\theta_1, \theta_2, ..., \theta_N) = y_1(\theta_1) y_2(\theta_2) ... y_{N-1}(\theta_{N-1}) y_N(\theta_N) .
\] (3.8)

4 Symmetric Pöschl-Teller potential

Now we describe the solutions of the Schrödinger equation with the symmetric Pöschl-Teller potential which will be used later in searching for solutions of the equations (3.6). We follow the exposition of Ref. [10] (problem 38). In accordance with [10] the solutions of the Schrödinger equation with the Pöschl-Teller potential of a general form:
\[
- \frac{d^2 \psi}{dx^2} + V_{PT}(x) \psi = q^2 \psi, \quad V_{PT}(x) = c^2 \left( \frac{\mu (\mu - 1)}{\sin^2 (\mu x)} + \frac{\kappa (\kappa - 1)}{\cos^2 (\mu x)} \right), \quad 0 \leq x \leq \frac{\pi}{2c}; \quad \mu > 1, \quad \kappa > 1 \quad (4.1)
\]
are given by the functions:
\[
\psi_n(x) = \sin^\mu (\mu x) \cos^\kappa (\mu x) \text{ } _2F_1 \left( -n, n + \mu + \kappa, \mu + \frac{1}{2}; \sin^2 (\mu x) \right)
\] (4.2)

with the spectrum:
\[
q_n^2 = c^2 (2n + \mu + \kappa)^2, \quad n = 0, 1, 2, ...
\] (4.3)

where \( _2F_1(a, b; c; x) \) is the hypergeometric function. In the case of the symmetric Pöschl-Teller potential we have \( \mu = \kappa \) and we can write the potential in the form:
\[
V_{SPT}(x) = 4c^2 \frac{\mu (\mu - 1)}{\sin^2 (2\mu x)} .
\]

Introducing the variable \( \theta = 2\mu x \), we come to the equation
\[
- \frac{d^2 \psi}{d\theta^2} + \frac{\mu (\mu - 1)}{\sin^2 \theta} \psi = \sigma^2 \psi, \quad \sigma^2 = \frac{q^2}{4c^2}, \quad 0 \leq \theta \leq \pi, \quad \mu > 1
\] (4.4)

whose solutions are given by the functions
\[
\psi_n(\theta) = \sin^\mu \theta \text{ } _2F_1 \left( -n, n + 2\mu, \mu + \frac{1}{2}; \frac{1}{2} (1 - \cos \theta) \right)
\] (4.5)
and the spectrum is

\[ \sigma_n^2 = (n + \mu)^2, \ n = 0, 1, 2, \ldots \] (4.6)

We note that the hypergeometric function in Eq. (4.5) represents the Gegenbauer polynomials \( C_n^\mu(z) \) up to a numeric factor, [11], Sec. 3.15, Eq. (3) (or in [12], 8.932.1):

\[ C_n^\mu(z) = \frac{\Gamma(n + 2\mu)}{\Gamma(n + 1) \Gamma(2\mu)} \, {}_2 F_1 \left(-n, n + 2\mu, \mu + \frac{1}{2}, \frac{1}{2} (1 - z)\right). \] (4.7)

Then we can write the solutions of Eq. (4.5) as

\[ \psi_n(\theta) = \sin^\mu \theta C_n^\mu(\cos \theta). \] (4.8)

For further use we recall that the Gegenbauer polynomials are solutions of the Gegenbauer equation, [11], Sec. 3.15, Eq. (21) (or in [12], 8.928):

\[ (1 - z^2)y'' - (2\mu + 1)zy' + n(n + 2\mu)y = 0 \] (4.9)

where \( n \) are non-negative integers, \( y(z) = C_n^\mu(z) \). The Gegenbauer polynomials are orthogonal on the interval \( z \in [-1, +1] \) with the weight \( (1 - z^2)^{\mu-1/2} \). If Eq. (4.7) is used for the definition of the Gegenbauer polynomials they are normalized by

\[ \int_{-1}^{1} C_n^\mu(z) C_m^\mu(z) (1 - z^2)^{\mu-1/2} \, dz = \int_{0}^{\pi} C_n^\mu(\cos \theta) C_m^\mu(\cos \theta) \sin^{2\mu} \theta d\theta = \frac{2^{1-2\mu} \pi \, \Gamma(n + 2\mu)}{n! (\mu + 1) [\Gamma(\mu)]^2} \delta_{nm} \] (4.10)

where \( \delta_{nm} \) stands for the Kronecker delta, [11], Sec. 3.15, Eqs. (16), (17).

We also will need the non-singular solutions of the equation

\[ \partial^2_\theta y + (k - 1) \cot \theta \partial_\theta y + \lambda y = 0 \] (4.11)

which in terms of the variable \( z = \cos \theta \) takes the form:

\[ (1 - z^2) \partial^2_z y - kz \partial_z y + \lambda y = 0. \]

Then due to Eq. (4.9), we can write the solutions of Eq. (4.11) as

\[ y(\theta) = C_n^{(k-1)/2}(\cos \theta), \ \lambda = n(n + k - 1), \ n \geq 0. \] (4.12)

5 Solution of the ordinary differential equations

In this section we obtain solutions of the equations (3.6), (3.7). For Eq. (3.7) we have

\[ y_1 = e^{\pm im_1 \theta_1}, \ \lambda_1 = n_1^2, \ n_1 \in \mathbb{Z} \] (5.1)
since \( y_1 \) must obey the condition of periodicity \( y_1 (\theta_1 + 2\pi) = y_1 (\theta_1) \).

For the set of the ordinary differential equations equations (3.6) first we obtain solutions for \( \lambda_{k-1} = 0 \). In this case the equations (3.6) take the form:

\[
\partial_k^2 y_k + (k - 1) \cot \theta_k \partial_k y_k + \lambda_k y_k = 0.
\]

(5.2)

Due to Eqs. (4.11), (4.12) their solutions are given by

\[
y_k = C_{n_k}^{(k-1)/2} (\cos \theta_k), \quad \lambda_k = n_k (n_k + k - 1), \quad n_k \geq 0.
\]

(5.3)

To solve the equations (3.6) for \( \lambda_{k-1} > 0 \) we present \( y_k \) as follows:

\[
y_k (\theta_k) = f_k (\theta_k) v_k (\theta_k).
\]

Then from Eq. (3.6) we come to the equation for \( v_k \):

\[
v_k'' + \left( 2 \frac{f_k'}{f_k} + (k - 1) \cot \theta_k \right) v_k' + \left( \frac{f_k''}{f_k} + (k - 1) \cot \theta_k \frac{f_k'}{f_k} + \lambda_k - \frac{\lambda_{k-1}}{\sin^2 \theta_k} \right) v_k = 0.
\]

(5.4)

To make vanish the coefficient of \( v_k' \) in Eq. (5.4) we set

\[
2 \frac{f_k'}{f_k} + (k - 1) \cot \theta_k = 0
\]

that gives

\[
f_k = (\sin \theta_k)^{-\frac{1}{2}(k-1)}.
\]

(5.5)

Then for the two first terms in the coefficient of \( v_k \) in Eq. (5.4) we get:

\[
\frac{f_k''}{f_k} + (k - 1) \cot \theta_k \frac{f_k'}{f_k} = \frac{1}{2} (k - 1) \left( 1 - \frac{1}{2} (k - 1) \right) \frac{1}{\sin^2 \theta_k} + \frac{1}{4} (k - 1)^2.
\]

So that we come to the equation for \( v_k \):

\[
v_k'' + \left[ \alpha_k^2 - \frac{\mu_k (\mu_k - 1)}{\sin^2 \theta_k} \right] v_k = 0,
\]

(5.6)

\[
\mu_k (\mu_k - 1) = \lambda_{k-1} - \frac{1}{2} (k - 1) \left( 1 - \frac{1}{2} (k - 1) \right),
\]

\[
\alpha_k^2 = \frac{1}{4} (k - 1)^2 + \lambda_k.
\]

One can see that Eq. (5.6) presents the Schrödinger equation with the symmetric Pöschl-Teller potential for \( \mu_k > 1 \), Eq. (4.4), whose solutions are given by Eq. (4.8). We will formally use these solutions and prove the condition \( \mu_k > 1 \) during the process of computations. For \( k = 2 \) we have from Eq. (5.6):

\[
\mu_2 (\mu_2 - 1) = \left( \mu_2 - \frac{1}{2} \right)^2 - \frac{1}{4} = \lambda_1 - \frac{1}{4} = n_1^2 - \frac{1}{4}.
\]
Then choosing a positive sign of the root we get:

\[ \mu_2 = |n_1| + \frac{1}{2}. \]  \hfill (5.7)

Here \( |n_1| \geq 1 \), because we consider the case \( \lambda_1 > 0 \), so that the condition \( \mu_2 > 1 \) is obeyed, then

\[ \mu_2 = l_1 + \frac{1}{2}, \quad l_1 = |n_1|. \]  \hfill (5.8)

Starting from \( k = 2 \) we consequently obtain the solutions for \( k \geq 3 \). For \( \mu_k > 1 \) the solutions of Eq. (5.6) are given by

\[ v_k = (\sin \theta_k)^{\mu_k} C_{n_k}^{\mu_k} (\cos \theta_k), \quad \alpha_k^2 = (n_k + \mu_k)^2, \quad n_k \geq 0 \]  \hfill (5.9)

with

\[ \lambda_k = (n_k + \mu_k)^2 - \frac{1}{4} (k - 1)^2 \]

\[ = \left( n_k + \mu_k - \frac{1}{2} (k - 1) \right) \left( n_k + \mu_k + \frac{1}{2} (k - 1) \right). \]  \hfill (5.10)

With the use of Eqs. (5.6), (5.10) one can write:

\[ \mu_k (\mu_k - 1) = (n_{k-1} + \mu_{k-1})^2 - \frac{1}{4} \]

then for \( \mu_k \) with \( k \geq 3 \) one gets a recurrence relation:

\[ \mu_k = n_{k-1} + \mu_{k-1} + \frac{1}{2}, \quad k \geq 3 \]  \hfill (5.11)

where we choose a positive sign of the root. Repeating it \( (k - 2) \) times we get:

\[ \mu_k = \sum_{j=2}^{k-1} n_j + \frac{1}{2} (k - 2) + \mu_2 = \sum_{j=2}^{k-1} n_j + l_1 + \frac{1}{2} (k - 1). \]

Introducing the quantity

\[ l_k = l_{k-1} + n_k \]  \hfill (5.12)

we can write

\[ \mu_k = l_{k-1} + \frac{1}{2} (k - 1) \]  \hfill (5.13)

then for \( \lambda_k \) we get:

\[ \lambda_k = l_k (l_k + k - 1). \]  \hfill (5.14)

We note that since for \( k \geq 2 \) all \( n_k \geq 0 \) then we have the conditions for \( l_k \) from Eq. (5.12):

\[ l_k \geq l_{k-1}, \quad 2 \leq k \leq N. \]  \hfill (5.15)
We also note that from Eq. (5.14) one gets \( \lambda_k = 0 \) if \( l_k = 0 \). The case \( \lambda_k > 0 \) is realized for \( l_k \geq 1 \) that gives the condition for \( \mu_k \):

\[
\mu_k \geq \frac{1}{2} (k + 1) > 1 \text{ if } k \geq 2.
\] (5.16)

Eq. (5.16) confirms appropriateness of the use of the solutions of the quantum mechanical Pöschl-Teller potential well problem.

With the use of Eqs. (5.12), (5.13) we can write \( v_k \) in the form:

\[
v_k = (\sin \theta_k)^{l_k-1} \frac{1}{2} C_{l_k-l_{k-1}}^{l_k-1} (\cos \theta_k)
\]

therefore

\[
y_k = f_k v_k = (\sin \theta_k)^{l_k-1} C_{l_k-l_{k-1}}^{l_k-1} (\cos \theta_k), \quad \lambda_k = l_k (l_k + k - 1), \quad l_k \neq 0, \quad 2 \leq k \leq N.
\] (5.17)

We note that the solutions for \( \lambda_{k-1} = 0 (l_{k-1} = 0) \) given in Eq. (5.3) have the same structure as the solutions of Eq. (5.17), then it is possible to unite the solutions for all values of \( l_k \) within a unique formula:

\[
y_k = (\sin \theta_k)^{l_k-1} C_{l_k-l_{k-1}}^{l_k-1} (\cos \theta_k), \quad \lambda_k = l_k (l_k + k - 1), \quad l_k \geq l_{k-1} \geq 0, \quad 2 \leq k \leq N.
\] (5.18)

6 Construction of the spherical harmonics

With the use of Eqs. (5.11), (5.18), and (5.8) we can write down the N-dimensional spherical harmonics, which are the solutions of Eq. (3.1), in the form:

\[
Y_{l_1,l_2,...,l_N} (\theta_1, \theta_2, ..., \theta_N) = W_N (\theta_1, \theta_2, ..., \theta_N) = \prod_{k=1}^{N} y_k (\theta_k)
\] (6.1)

\[
= \prod_{k=2}^{N} (\sin \theta_k)^{l_k-1} C_{l_k-l_{k-1}}^{l_k-1} (\cos \theta_k) e^{\pm il_1 \theta_1},
\]

\[
\lambda_N = l_N (l_N + N - 1), \quad l_N \geq l_{N-1} \geq ... \geq l_2 \geq l_1 = |n_1| \geq 0
\]

where \( l_N, l_{N-1}, ..., l_2, l_1 = |n_1| \) are the characteristic numbers which obey the specified conditions.

The form of the spherical harmonics of Eq. (6.1) coincides with the corresponding expressions of Refs. [1], [3], [4] up to the used notations. For the sake of completeness we also compute a normalization factor for the spherical harmonics of Eq. (6.1). To this end we compute the integral over N-sphere \( S^N \):

\[
[N_{l_1,l_2,...,l_N}]^{-1/2} = \int_{S^N} Y_{l_1,l_2,...,l_N}^* Y_{l_1,l_2,...,l_N} d\Omega
\]
\[
= \int_{\theta_N=0}^{\pi} \ldots \int_{\theta_2=0}^{\pi} \int_{\theta_1=0}^{2\pi} Y^*_1, l_2, \ldots, l_N Y_1, l_2, \ldots, l_N \prod_{k=1}^N \sin^{k-1} \theta_k d\theta_k
\]
\[
= \prod_{k=2}^N \int_{\theta_k=0}^{\pi} (\sin \theta_k)^{2l_k-1} \left[ C_{l_k-l_k-1}^{l_k-1} \left( \cos \theta_k \right) \right]^2 \sin^{k-1} \theta_k d\theta_k \int_{\theta_1=0}^{2\pi} d\theta_1
\]
\[
= 2\pi \prod_{k=2}^N \int_{\theta_k=0}^{\pi} \left[ C_{l_k}^{n_k} \left( \cos \theta_k \right) \right]^2 (\sin \theta_k)^{2n_k} d\theta_k = 2\pi \prod_{k=2}^N \frac{2^{1-2n_k} \pi \Gamma (n_k + 2n_k)}{n_k! (n_k + n_k) \Gamma (n_k) \Gamma (n_k)}
\]

where Eq. (4.10) was used. We also write it in terms of the characteristic numbers \( l_1, l_2, \ldots, l_N \):
\[
[N_1, l_2, \ldots, l_N]^{-1/2} = \frac{1}{2} (4\pi)^N \prod_{k=2}^N \frac{2^{-2l_k-1} \Gamma (l_k + l_k-1 + k - 1)}{(l_k - l_k-1)! (l_k + (k - 1)/2) \Gamma (l_k-1 + (k - 1)/2)}
\]

Finally, we remark that orthogonality of the system of the spherical harmonics of Eq. (6.1) can be deduced from the orthogonal property of the Gegenbauer polynomials, Eq. (4.10).

7 Conclusions

In this paper we present another view on the problem of construction of the N-dimensional spherical harmonics. In our approach we effectively use the solutions of the quantum mechanical Pöschl-Teller potential well problem. We use the eigenfunctions of the symmetric Pöschl-Teller potential well problem to construct the functional expression and the eigenvalues for obtaining of the characteristic numbers of the spherical harmonics. The conditions for the characteristic numbers arise naturally in the process of solution. We expect that our approach to the problem may be usefully applicable in consideration of problems which involve spherical geometry of higher dimensions. For example, it can be incorporated in schemes of construction of vector, tensor or spin spherical harmonics. Besides that, we believe that the approach is comprehensible for students of final years of undergraduate courses of universities. It could be utilized in teaching of the theme of the usual two-dimensional spherical harmonics, leastwise as complementary to the usual methods, and it allows in a simple way to pass to higher dimensional generalizations.

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