The large-\( j \) limit for certain 9-\( j \) symbols—power law behaviour

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In a previous work, certain unitary 9-\( j \) symbols were shown to go asymptotically to zero in the large-\( j \) limit. In this work, we examine this in more detail and find an approximate power law for some unitary 9-\( j \)'s in the large-\( j \) limit and exponential decrease for others in this same limit.

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I. INTRODUCTION

A unitary 9-\( j \) (U9-\( j \)) coefficient is related to a 9-\( j \)-symbol via

\[
\langle (j_1 j_2)^{J_{12}} (j_3 j_4)^{J_{34}} (j_1 j_3)^{J_{13}} (j_2 j_4)^{J_{24}} \rangle = ([2 J_{12} + 1](2 J_{34} + 1)(2 J_{13} + 1)(2 J_{24} + 1))^{1/2} \times \left\{ \begin{array}{ccc} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{array} \right\} 
\]

(1)

We first note the well-known normalization relation

\[
\sum_{J_{12},J_{24}} \left| \langle (j_1 j_2)^{J_{12}} (j_3 j_4)^{J_{34}} (j_1 j_3)^{J_{13}} (j_2 j_4)^{J_{24}} \rangle \right|^2 = 1.
\]

Another useful relation is:

\[
\sum_{J_{12},J_{24}} (-1)^{(J_{13}+J_{24})} \times \left| \langle (j_1 j_2)^{J_{12}} (j_3 j_4)^{J_{34}} (j_1 j_3)^{J_{13}} (j_2 j_4)^{J_{24}} \rangle \right|^2 = 0
\]

(3)

if \( J_{12} \) does not equal \( J_{34} \). When \( J_{12} \) is equal to \( J_{34} \), one gets \((-1)^J\).

In a previous work, we noted that the “coupling” U9-\( j \) \( \langle (j j)^{2j}(jj)^{2j}(jj)^{2j-2}\rangle^{I=2} \) decreased rapidly with increasing \( j \) and went asymptotically to zero. Indeed the decrease is roughly exponential in \( j \). This will be discussed later.

The motivation for considering this class of U9-\( j \)'s was that they enter into the overlap of the approximate wave functions for two \( I = 2 \) states. The components of these wave functions are the following U9-\( j \)'s:

\[
\langle (jj)^{2j}(jj)^{2j}(jj)^{I=2} \rangle
\]

(4a)

\[
\langle (jj)^{2j}(jj)^{2j-2}(jj)^{I=2} \rangle
\]

(4b)

It was found that the overlap was very small and so (4a) and (4b) are good approximations to the lowest two \( I = 2^+ \) states when the E(9) interaction is used. In this interaction for the \( g_{9/2} \) shell, all two-body matrix elements are set equal to zero except the one for \( J = J_{\text{max}} = 9 \).

If there were no restriction on the integers \( J_p \) and \( J_n \), (4a) and (4b) would be orthogonal. However, we were considering a system of two protons and two neutrons in the single \( j \) shell. To satisfy the Pauli principle, \( J_p \) and \( J_n \) had to be even. As shown in Ref. [1], the overlap, with this restriction, is:

\[
-1/2 \langle (jj)^{2j}(jj)^{2j}(jj)^{I=2} \rangle
\]

The relation, generalized to any even total angular momentum \( I \), is

\[
\sum_{J_p,J_n} \langle (jj)^{2j}(jj)^{2j}(jj)^{I=2} \rangle \times \langle (jj)^{2j}(jj)^{2j-2}(jj)^{I=2} \rangle = -1/2 \langle (jj)^{2j}(jj)^{2j-2}(jj)^{I=2}(jj)^{2j-2} \rangle.
\]

(5)

For odd total angular momentum \( I \), the appropriate relation is

\[
\sum_{J_p,J_n} \langle (jj)^{2j}(jj)^{I=2}(jj)^{I=2} \rangle \times \langle (jj)^{2j}(jj)^{I=2}(jj)^{I=2} \rangle = -1/2 \langle (jj)^{2j}(jj)^{I=2}(jj)^{I=2}(jj)^{I=2} \rangle.
\]

(6)

Regardless of the motivation, we here consider the behavior of the U9-\( j \)'s \( \langle (jj)^{2j}(jj)^{2j}(jj)^{2j-2}\rangle^{I=2} \) for even \( I \) and \( \langle (jj)^{2j}(jj)^{2j-1}(jj)^{2j-3}\rangle^{I=2} \) for odd \( I \). For the sake of convenience, we will use the notation \( M_{\text{even}}^j(I) \) and \( M_{\text{odd}}^j(I) \) for the latter even-\( I \) and odd-\( I \) U9-\( j \)'s, respectively.

II. RESULTS

A. Even \( I \)

We will here adopt a very simple approach. We just calculate a number of U9-\( j \)'s and make reasonable guesses at the extrapolations. We start by giving in Table II the values of \( M_{\text{even}}^j(I) \) for the \( g_{9/2} \) shell, i.e., we consider

\[
M_{\text{even}}^j(I) = \langle (jj)^{2j}(jj)^{2j}(jj)^{2j-2}\rangle^{I=2}.
\]

A striking result is that all the U9-\( j \)'s are small except for the one with the maximum value of \( I \), namely \( I = 16 \). A reasonable speculation is that \( M_{\text{even}}^j(I) \) will vanish in the limit of large \( j \) for all \( I \) except \( I_{\text{max}} = 4j - 2 \). We can even dare to speculate that the last one approaches a value of 1/2 in the large-\( j \) limit.
Table I: Values of $M_j^{\text{even}}(I)$ (see text) for all even total angular momenta $I$ in the $g_{9/2}$ shell.

| $I$  | $M_j^{\text{even}}(I)$ |
|------|------------------------|
| 2    | -0.000182              |
| 4    | 0.000173               |
| 6    | -0.000260              |
| 8    | 0.000536               |
| 10   | -0.001513              |
| 12   | 0.006055               |
| 14   | -0.037896              |
| 16   | 0.491530               |

To test this in our simple approach, we go to a much higher $j$ shell: $j = 21/2$. The values of $M_j^{\text{even}}(I)$ for selected angular momenta that we find are shown in Table II.

Table II: Selected values of $M_j^{\text{even}}(I)$ (see text) for the $j = 21/2$ shell.

| $I$  | $M_j^{\text{even}}(I)$ |
|------|------------------------|
| 2    | $-3.57861 \times 10^{-11}$ |
| 34   | $-8.35524 \times 10^{-5}$ |
| 36   | $9.27451 \times 10^{-4}$  |
| 38   | $-1.52261 \times 10^{-2}$ |
| 40   | 0.496870                |

These results strongly support our speculations. The $I = 2$ value is now extremely small (of the order of $10^{-11}$) and all the others are small except for $I = 40$. The value for $I = 40$ in the $21/2$ shell is closer to 1/2 than is the $I = 16$ result in the $9/2$ shell: 0.496870 vs. 0.491530.

The $U9$-$j$'s that go to a finite value in the large-$j$ limit are said to exhibit classical behaviour and those that go to zero, non-classical behaviour. Thus, we have only one $U9$-$j$ exhibiting classical behaviour, the one with $I = 2j + (2j - 2) = 4j - 2$. We can gain some insight into this behavior by noting that the relation of Eq. (5) for the case $I = I_{\text{max}} = 4j - 2$ involves only one term:

$$\langle (jj)^{2j}(jj)^{2j}(jj)^{(2j-2)} \rangle_{\text{max}} =$$

$$= -2\langle (jj)^{2j}(jj)^{2j}(jj)^{(2j-1)}(jj)^{(2j-1)} \rangle_{\text{max}} \times$$

$$\times \langle (jj)^{(2j)}(jj)^{(2j-2)}(jj)^{(2j-1)}(jj)^{(2j-1)} \rangle_{\text{max}}. \quad (7)$$

The first $U9$-$j$ on the right hand side approaches $1/\sqrt{2}$ in the large-$j$ limit. The second one approaches $-1/(2\sqrt{2})$ in the same limit.

B. Odd $I$

We can also consider odd total angular momentum. In that, we use Eq. (6). For $I = 4j - 3$ (the largest odd $I$), we again have that there is only one term in the sum:

$$\langle (jj)^{2j}(jj)^{(2j-1)}(jj)^{2j}(jj)^{(2j-3)}(4j-3) \rangle =$$

$$= -2\langle (jj)^{2j}(jj)^{(2j-1)}(jj)^{(2j-1)}(jj)^{(2j-1)}(4j-3) \rangle \times$$

$$\times \langle (jj)^{2j}(jj)^{(2j-3)}(jj)^{(2j-1)}(jj)^{(2j-1)}(4j-3) \rangle. \quad (8)$$

For $j = 9/2$, the largest odd angular momentum is $I = 15$, and the left hand side of Eq. (8) is equal to 0.42564827. We expect that this will approach 1/2 in the large-$j$ limit.

Just as in the even-$I$ case, for odd $I$ less than 15, the values are small and go to zero in the large-$j$ limit, as we can see in Table III.

Table III: Values of $M_j^{\text{odd}}(I)$ (see text) for all odd total angular momenta $I$ in the $g_{9/2}$ shell.

| $I$  | $M_j^{\text{odd}}(I)$ |
|------|------------------------|
| 3    | 1.410648 $\times 10^{-3}$ |
| 5    | -1.002940 $\times 10^{-3}$ |
| 7    | 1.325594 $\times 10^{-3}$ |
| 9    | -2.673475 $\times 10^{-3}$ |
| 11   | 8.145222 $\times 10^{-3}$ |
| 13   | -4.025313 $\times 10^{-2}$ |
| 15   | 0.425648               |

We briefly compare with $j = 21/2$. The $U9$-$j$ is now $M_{21/2}^{\text{odd}}(I) = \langle (\frac{21}{2} \frac{21}{2} \frac{21}{2} \frac{21}{2} \frac{21}{2} \frac{21}{2} \frac{21}{2} \frac{21}{2} \frac{21}{2} \frac{21}{2}) \rangle_{\text{max}}$. We show in Table IV some relevant selected values.

Table IV: Selected values of $M_j^{\text{odd}}(I)$ (see text) for the $j = 21/2$ shell.

| $I$  | $M_j^{\text{odd}}(I)$ |
|------|------------------------|
| 15   | 9.084676 $\times 10^{-10}$ |
| 35   | 9.407980 $\times 10^{-4}$ |
| 37   | -1.424514 $\times 10^{-2}$ |
| 39   | 0.430223               |

Thus, we see that the odd-$I$ case is similar to the even-$I$ case in the sense that the $U9$-$j$ approaches zero in the large-$j$ limit for all $I$ except for the largest possible value $I = 4j - 3$.

### III. POWER LAW BEHAVIOUR NEAR $I = I_{\text{max}}$

We again consider even $I$. We re-examine the $U9$-$j$ $\langle (jj)^{2j}(jj)^{2j}(jj)^{2j}(jj)^{(2j-2)} \rangle$ close to $I = I_{\text{max}}$. We already noted that for $I = I_{\text{max}}$ this $U9$-$j$ increases asymptotically to 1/2 in the large-$j$ limit. We next consider $I = I_{\text{max}} - 2$, $I_{\text{max}} - 4$, $I_{\text{max}} - 6$, $I_{\text{max}} - 8$, and $I_{\text{max}} - 10$. We find numerically that the first one goes slowly to zero approximately as $1/j$, the second one as $1/j^2$, and so
on. All these intriguing and varying behaviors deserve further study. We here give the details.

We evaluate the selected $U_9$-j’s for $j = 25/2$ and $j = 41/2$. Then we use $j = 25/2$ to predict what happens for $j = 41/2$ via the formula

$$ U_{\text{predicted}}(j = 41/2) = U(j = 25/2) \left( \frac{25}{41} \right)^n, \quad (9) $$

such that $I = I_{\text{max}} - 2n$. We see in Table V that there is close but not perfect agreement with the power law behaviour $1/j^n$. The ratios predicted/actual for $j = 41/2$ for $n = 1, 2, 3, 4, 5$ are shown in the last column. The agreement is best for small $n$, i.e. as $I$ gets closer to $I_{\text{max}}$.

### IV. EXPONENTIAL BEHAVIOUR FOR $I = 2$

We now come back to the $I = 2$ case. We again consider the U9-j symbol $U(j) = \langle (jj) | (jj) \rangle | (jj) \rangle (jj)^{(2j-2)} > I = 2$. We speculate that the asymptic form is $C e^{(-aj)}$. To put this to the test we consider 3 values - $j$, $j+1$ and $j+2$. If it were striickly exponential then the ratio $R_1[U(j)]/U(j+1)$ would be the same as the ratio $R_2[U(j+1)]/U(j+2)$. We present results for 3 values of $j$, $40.5, 60.5,$ and $100.5$.

| $I$ | $n$ | $U(j = 25/2)$ | $U(j = 41/2)$ | $U_{\text{predicted}}(j = 41/2)$ | Ratio |
|-----|-----|---------------|---------------|-------------------------------|-------|
| 80  | 0   | 0.497390      | 0.498435      | 0.497390                      |       |
| 78  | 1   | $-0.126985 \times 10^{-1}$ | $-0.763220 \times 10^{-2}$ | $-0.774296 \times 10^{-2}$ | 1.0145 |
| 76  | 2   | $0.641513 \times 10^{-3}$ | $0.22983 \times 10^{-3}$ | $0.238516 \times 10^{-3}$ | 1.0405 |
| 74  | 3   | $-0.47053 \times 10^{-4}$ | $-0.100101 \times 10^{-4}$ | $-0.108152 \times 10^{-4}$ | 1.0804 |
| 72  | 4   | $0.471819 \times 10^{-5}$ | $0.589299 \times 10^{-6}$ | $0.652293 \times 10^{-6}$ | 1.1069 |
| 70  | 5   | $-0.579665 \times 10^{-6}$ | $-0.427933 \times 10^{-7}$ | $-0.488858 \times 10^{-7}$ | 1.1423 |

For $j = 100.5$, $U(j) = -0.426386 \times 10^{-117}$, $R_1 = 0.157652 \times 10^2$, $R_2 = 0.157675 \times 10^2$, $R_2-R_1 = 0.231934 \times 10^{-2}$.

We see that the difference $R_2-R_1$ gets smaller and smaller with increasing $j$. The agreement is remarkable. For $j = 100.5$ it is better than 1 part in 6000.

We now try a more elaborate form $C j^m e^{(-aj)}$ and look for $m$. Consider the ratio $R_2/R_1$. For the functional form just given it is equal to

$$ (j+1)^{2m}/(j(j+2))^m $$

For $m=0$ this has a value of one. By comparing with the exact value of $R_2/R_1$ obtained from explicit values of $U_9$-j’s we conclude that $m=1.5$. For example for $j=40.5$ the values of $R_2/R_1$ is $1.00089299679$ for $m=1.5$ whereas the exact value is $1.0008477742$. The corresponding values for $j=80.5$ are $1.00022864342$ and $1.00022150836$.

Early works on 3-j, 6-j and 9-j symbols were performed by atomic and nuclear physicists, especially Wigner [2] and Racah [3]. More recently, there have been extensive works by researchers in chemistry and quantum gravity [4, 5]. Very recently Van Isacker and Macchiavelli [6] have considered the large $j$ limit of shell model matrix elements in the context of the problem of shears mechanisms in nuclei. This involved 12-j symbols. In this work we have shown that one can get a variety of behaviors of unitary 9-j’s in the large $j$ limit. Some go to a finite value, some go to zero exponentially and others go to zero via power laws i.e. $1/j^n$ with varying $n$. We have here found the results numerically. It will be a challenge to get a better and perhaps more analytic understanding of these behaviors.

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