Sparse Averages of Partial Sums of Fourier Series

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Abstract

We study convergence properties of sparse averages of partial sums of Fourier series of continuous functions. By sparse averages, we are considering an increasing sequences of integers \( n_0 < n_1 < n_2 < \ldots \) and looking at

\[
\tilde{\sigma}_N(f)(t) = \frac{1}{N+1} \sum_{k=0}^{N} s_{n_k}(f)(t)
\]

to determine the necessary conditions on the sequence \( \{n_k\} \) for uniform convergence. Among our results, we find that convergence is dependent on the sequence: we give a proof of convergence for the linear case, \( n_k = pk \), for \( p \) a positive integer, and present strong experimental evidence for convergence of the quadratic \( n_k = k^2 \) and cubic \( n_k = k^3 \) cases, but divergence for the exponential case, \( n_k = 2^k \). We also present experimental evidence that if we replace the deterministic rules above by random processes with the same asymptotic behavior then almost surely the answer is the same.

1. Introduction

A celebrated result of Fejér asserts the uniform convergence to a continuous function \( f \) on the circle \( \mathbb{R}/2\pi \mathbb{Z} \) of the Cesaro averages

\[
\sigma_N(f)(t) = \frac{1}{N+1} \sum_{n=0}^{N} s_n(f)(t)
\]

of the partial sums \( s_n(f) \) of the Fourier series of \( f \), despite the fact that the partial sums need not converge everywhere (or even remain bounded). But what if we use a different sort of averaging? It is also well-known that heat equation averaging will work as well, but this requires using all the Fourier coefficients at once for each average. In this paper we investigate what happens in the case of what might be called sparse averages. We consider an increasing sequences of integers

\[
n_0 < n_1 < n_2 < \ldots
\]

and look at

\[
\tilde{\sigma}_N(f)(t) = \frac{1}{N+1} \sum_{k=0}^{N} s_{n_k}(f)(t)
\]

We pose the question: under what conditions on the sequence \( \{n_k\} \) do we always have uniform convergence?

Here is a brief summary of our conclusions:

1. It is not always true. The existence of functions with divergent Fourier series yields a nonconstructive proof of this, but we prefer to give an explicit counterexample that is based on another result due to Fejér.

2. In the linear case, \( n_k = pk \), for \( p \) a positive integer, we give a proof of convergence.
In the quadratic $n_k = k^2$ and cubic $n_k = k^3$ cases, we present strong experimental evidence of convergence. This leads us to the conjecture that convergence holds for any polynomial rule.

In the exponential case, $n_k = 2^k$, we present experimental evidence that convergence fails.

We also present experimental evidence that if we replace the deterministic rules above by random processes with the same asymptotic behavior then almost surely the answer is the same.

In Section 2 we describe the problem in detail and reduce it to showing that a certain kernel (the analog of the Fejér kernel) is an approximate identity. In Section 3 we present the counterexample. In Section 4, we prove the result in the linear case. In Section 5 we present experimental evidence in the quadratic and cubic cases. In Section 6 we present experimental evidence in the random cases. In Section 7 we discuss rates of convergence, and outline a different approach that might imply convergence.

The website [1] contains the programs used to generate the experimental evidence, and much more of this evidence. We would like to thank Noam Weinreich, who developed earlier versions of these programs. See [2] for the classical theory of Fourier series.

2. Formulation of the Problem

Let $f$ be a continuous function on the circle $\mathbb{R}/2\pi\mathbb{Z}$. The partial sums of the Fourier series of $f$ may be written as convolutions of $f$ with the Dirichlet Kernel

\begin{equation}
S_n f(x) = \int_{-\pi}^{\pi} D_n(t) f(x - t) \, dt
\end{equation}

for

\begin{equation}
D_n(t) = \frac{1}{2\pi} (1 + 2 \sum_{k=1}^{n} \cos(kt)) = \frac{1}{2\pi} \frac{\sin(n + \frac{1}{2})t}{\sin(\frac{1}{2}t)}
\end{equation}

We have

\begin{equation}
\int_{-\pi}^{\pi} D_n(t) dt = 1
\end{equation}

but

\begin{equation}
\int_{-\pi}^{\pi} |D_n(t)| dt = O(log n)
\end{equation}

so the Dirichlet Kernel fails to be an approximate identity, and in general the partial sums do not converge to $f$. Fejér observed that the averages

\begin{equation}
\sigma_N(f)(x) = \frac{1}{N+1} \sum_{k=0}^{N} s_k(f)(x)
\end{equation}

are also given by convolution with Kernels $K_N$ given by

\begin{equation}
K_N(t) = \frac{1}{N+1} \sum_{n=0}^{N} D_n(t) = \frac{1}{2\pi(N+1)} \left( \frac{\sin(N+\frac{1}{2})t}{\sin(\frac{1}{2}t)} \right)^2
\end{equation}

But now $K_N(t)$ is nonnegative, so

\begin{equation}
1 = \int_{-\pi}^{\pi} K_N(t) dt = \int_{-\pi}^{\pi} |K_N(t)| dt
\end{equation}

and in fact $K_N$ is an approximate identity. Specifically, we have the estimate

\begin{equation}
\int_{|t| > \epsilon} |K_N(t)| dt \leq \frac{c}{N\epsilon}
\end{equation}

for some $c$ and all $\epsilon > 0$.

So if we define the modulus of continuity of $f$ by
(2.9) \[ m_\epsilon(f) = \sup_{x} \sup_{|t| \leq \epsilon} |f(x-t) - f(x)| \]

then

\[ ||K_N * f - f||_\infty \leq \int_{|t| \leq \epsilon} |f(x-t) - f(x)|K_N(t)dt + \int_{|t| \geq \epsilon} 2||f||_\infty K_N(t)dt \]

(2.10)

\[ \leq m_\epsilon(f) + 2||f||_\infty \frac{1}{N+1} \left( \frac{1}{\sin(\frac{\epsilon}{2})} \right)^2 \]

Thus we obtain Fejér’s theorem that \( K_N * f \) converges uniformly to \( f \) as \( N \to \infty \) in a quantitative form.

Now suppose we are given a sequence

(2.11) \[ n_0 < n_1 < n_2 < n_3 < \ldots \]

and we consider the sparse averages

(2.12) \[ \tilde{\sigma}_N(f) = \frac{1}{N+1} \sum_{k=0}^{N} s_{n_k}(f) \]

analogous to (2.5). Then, analogous to (2.6), we have

(2.13) \[ \tilde{\sigma}_N(f) = Q_N * f \] for

(2.14) \[ Q_N(t) = \frac{1}{N+1} \sum_{k=0}^{N} D_{n_k}(t) = \frac{1}{N+1} \sum_{k=0}^{N} \frac{\sin(n_k + \frac{1}{2})t}{\sin(\frac{t}{2})} \]

In order to show that \( \tilde{\sigma}_N(f) \to f \) uniformly for continuous \( f \) we need to verify that \( Q_N \) is an approximate identity:

(2.15) \[ \int_{-\pi}^{\pi} Q_N(t)dt = 1 \]

(2.16) \[ \int_{-\pi}^{\pi} |Q_N(t)|dt \leq M \text{ for all } N \]

(2.17) \[ \int_{|t| \geq \epsilon} |Q_N(t)|dt \leq \varphi_\epsilon(N) \]

with \( \lim_{N \to \infty} \varphi_\epsilon(N) = 0 \) for all \( \epsilon > 0 \). Indeed, just like (2.10) we obtain

(2.18) \[ ||Q_N * f - f||_\infty \leq Mm_\epsilon(f) + 2||f||_\infty \varphi_\epsilon(N) \]

and hence \( Q_N * f \to f \) uniformly.

Of course, (2.15) is an immediate consequence of (2.3).

Main Question:

Under what conditions on the sequence \( n_j \) do we have (2.16) and (2.17)?

3. A Counterexample

In this section, we show how to modify a construction of Fejér of a continuous function whose Fourier series diverges at a point to exhibit a sequence \( n_j \) and a continuous function such that \( \tilde{\sigma}_N(f)(0) \) is unbounded. The basic building block is the function

(3.1) \[ F_{n,m}(x) = \frac{\cos(m)x}{n} + \frac{\cos(m+1)x}{n-1} + \ldots + \frac{\cos(m+n-1)x}{1} - \frac{\cos(m+n+1)x}{1} - \frac{\cos(m+n+2)x}{2} - \ldots - \frac{\cos(m+2n)x}{n} \]
Note that
\begin{equation}
S_N F_{n,m}(x) = \begin{cases} 0 & \text{if } N < m \\ F_{n,m}(x) & \text{if } N \geq m + 2n \end{cases}
\end{equation}
\begin{equation}
F_{n,m}(0) = 0
\end{equation}
\begin{equation}
S_{n+m} F_{n,m}(0) = O(\log n)
\end{equation}
We also have the uniform boundedness of all $F_{n,m}$ as a consequence of the uniform boundedness of $\sum_{k=1}^{n} \sin(kx)/k$.

Now we choose $n_k, m_k$ so there is no overlap between the exponentials in $F_{n_k,m_k}$. For example, this will hold if $m_k \geq 1 + m_{k-1} + 2n_{k-1}$. Then we choose positive coefficients $a_k$ such that
\begin{equation}
\sum_{k=1}^{\infty} a_k < \infty \text{ and set }
\end{equation}
\begin{equation}
f = \sum_{k=1}^{\infty} a_k F_{n_k,m_k}, \text{ the series converging uniformly.}
\end{equation}
Note that
\begin{equation}
S_{n_k+m_k} f(0) = a_k S_{n_k+m_k} F_{n_k,m_k}(0)
\end{equation}
since all the other terms vanish. Thus
\begin{equation}
S_{n_k+m_k} f(0) = O(a_k \log(n_k))
\end{equation}
and we can make this diverge by the appropriate choice of $n_k$ and $a_k$. For example, $a_k = k^2$ and $n_k = m_k = 2^{(k^2)}$. Thus the sequence $2 \times 2^{(k^2)}$ gives a negative answer to the Main Question in the previous section.

4. The Linear Case

In this section we deal with the case
\begin{equation}
n_k = pk
\end{equation}
where $p$ is a positive integer. The case $p = 1$ gives the Cesaro sums, so $Q_N$ is exactly the Fejér Kernel. We will see that the behavior of $Q_N$ is not as nice as the Fejér Kernel. The statement
\begin{equation}
\lim_{N \to \infty} Q_N(t) = 0
\end{equation}
holds uniformly for any fixed $\epsilon > 0$ for $|t| \geq \epsilon$ for $p = 1$ but it is false for $p \geq 2$, as there are specific choices of $t$ (for example $\frac{2\pi}{p}$) where $Q_N(t)$ is a nonzero constant. Nevertheless, we will prove that $Q_N$ is an approximate identity, using the average decay (2.17) as a substitute for (4.2).

Lemma 4.1. For $n_k = pk$ we have
\begin{equation}
Q_N(t) = \frac{1}{2\pi} \frac{1}{N + 1} \frac{\sin\left(\frac{p}{2}N + \frac{1}{2}t\right) \sin\left(\frac{p}{2}(N + 1)t\right)}{\sin\left(\frac{p}{2}t\right) \sin\left(\frac{p}{2}t\right)}
\end{equation}

Proof. In view of (2.14) it suffices to show
\begin{equation}
\sum_{k=0}^{N} \sin(kp + \frac{1}{2}t) = \frac{\sin\left(\frac{p}{2}N + \frac{1}{2}t\right) \sin\left(\frac{p}{2}(N + 1)t\right)}{\sin\left(\frac{p}{2}t\right)}
\end{equation}
Now the left side of 4.4 is equal to
We need to prove (2.16) and (2.17) since (2.15) is automatic. Note that we
Proof. Let 
\[ Q_N(\pi) = \frac{1}{2\pi N + 1} \frac{\sin(N + \frac{1}{2}\pi)}{\sin(\frac{1}{2}\pi)} \lim_{s \to 0} \frac{\sin(N + 1)(\pi - s)}{\sin(\pi - s)} \]
= \frac{1}{2\pi N + 1} (-1)^N \lim_{s \to 0} \frac{\sin(1) N + 1 + 1) s}{\sin(N + 1) s}
= \frac{1}{2\pi} \]

We have a similar computation for general \( p \).

**Lemma 4.2.** Let \( j \leq \frac{p}{2} \). Then
\[ Q_N(\frac{2j\pi}{p}) = \frac{1}{2\pi} \]

**Proof.** As before,
\[
\frac{\sin(\frac{p}{2} N + \frac{1}{2}\pi)}{\sin(\frac{1}{2}\pi)} = \frac{\sin(Nj\pi + \frac{j}{p}\pi)}{\sin(\frac{j}{p}\pi)} = (-1)^N j
\]
and
\[
\frac{\sin(\frac{p}{2} (N + 1)t)}{\sin(\frac{1}{2}t)} \bigg|_{t = \frac{2j\pi}{p}} = \lim_{s \to 0} \frac{\sin(\frac{p}{2} (N + 1)(\frac{2}{p}j\pi + s))}{\sin(\frac{2}{p}j\pi + s)}
= \lim_{s \to 0} \frac{\sin(j\pi(N + 1) + \frac{p}{2}(N + 1)s)}{\sin(j\pi + \frac{p}{2}s)}
= (-1)^N \lim_{s \to 0} \frac{\sin(\frac{p}{2} (N + 1)s)}{\sin(\frac{1}{2}s)}
= (-1)^N \]

**Theorem 1.** For \( n_k = pk \), \( Q_n \) is an approximate identity, so \( \sigma_n(f) \to f \) uniformly for continuous \( f \).

**Proof.** We need to prove (2.16) and (2.17) since (2.15) is automatic. Note that we can write (4.3) as
\[ |Q_N(t)| = \frac{1}{2\pi N + 1} h_{pN}(t) h_N(pt) \]
for
\[ h_N(t) = \left| \frac{\sin(N + 1)t}{\sin(t)} \right| \]
Of course when $p = 1$ we obtain the Fejér Kernel which we know satisfies (2.16) and (2.17) by (2.7) and (2.8). By the Cauchy-Schwarz inequality,

$$\int_\epsilon^\pi |Q_N(t)|dt \leq \frac{1}{2\pi} \left( \frac{1}{N+1} \int_\epsilon^\pi h_pN(t)^2 dt \right)^\frac{1}{2} \left( \frac{1}{N+1} \int_\epsilon^\pi h_N(pt)^2 dt \right)^\frac{1}{2}$$

so it suffices to show that $\frac{1}{\pi} \int_\epsilon^\pi h_N(t)^2 dt$ is uniformly bounded. But this is equal to $\frac{1}{\pi} \int_\epsilon^\pi f(t)^2 dt$ by a change of variable and $h_N$ is periodic of period $2\pi$, so this is equal to $\frac{1}{\pi} \int_\epsilon^\pi f(t)^2 dt$, which is uniformly bounded by (2.7).

5. The Quadratic and Cubic Cases

In the quadratic case $n_k = k^2$. We easily compute

$$Q_N(0) = \frac{1}{2\pi} \left( \frac{1}{N+1} \sum_{k=0}^N (2k^2 + 1) \right) = \frac{2N^2 + N + 1}{6\pi}$$

We are led to define

$$q_N(t) = \begin{cases} \frac{6\pi}{2N^2 + N + 1} Q_N(\frac{N^2}{t}) & \text{for } -\pi N^2 \leq t \leq \pi N^2 \\ 0 & \text{for } |t| > \pi N^2 \end{cases}$$

so that

$$q_N(0) = 1$$

and clearly

$$Q_N(t) = \frac{2N^2 + N + 1}{6\pi} q_N(N^2 t) \text{ for } -\pi \leq t \leq \pi.$$  

**Lemma 5.1.** If there exists a positive integrable function $f$ on $\mathbb{R}$ such that

$$\sup_N |q_N(t)| \leq f(t)$$

then $Q_N$ is an approximate identity.

**Proof.** (2.16) holds with $M = \|f\|_1$. For (2.17) we have

$$\int_{|t| \geq \epsilon} |Q_N(t)| dt = 2 \int_{\epsilon N}^{\pi N^2} \frac{2N^2 + N + 1}{6\pi N^2} f(s) ds$$

and this goes to zero as $N \to \infty$ for each $\epsilon > 0$ because $f$ is integrable.  

We now present experimental evidence that the hypothesis (5.5) is valid. Much more experimental evidence may be found on the website [1]. In Figure 5.1, we show simultaneous plots of $q_N(t)$ for $N$ equal to multiples of 125 from $N = 125$ to $N = 1000$ over the interval $125 \leq t \leq 5000$. In Figures 5.2 and 5.3, we show simultaneous plots of $|q_N(t)|$ for the same choices of $N$ and over the intervals $0 \leq t \leq 125$ and $500 \leq t \leq 2500$ respectively. In Figures 5.4 and 5.5, we show the graph of $\max_N |q_N(t)|$ on the same range of $N$ over the intervals $75 \leq t \leq 1000$ and $500 \leq t \leq 2500$ respectively. In Figures 5.6 and 5.7, we show the graph of $\max_N |q_N(t)|$ on a log-log scale for the same range of $N$ for the intervals $0 \leq t \leq 125$ and $75 \leq t \leq 400$ respectively. All together, this suggests that $f(t) = c(1 + |t|)^{-1.4344}$ will satisfy (5.5).
Figure 5.1. Quadratic evaluation of $q_N(t)$ for $125 \leq t \leq 5000$

Figure 5.2. All Quadratic $|q_N(t)|$ for $0 \leq t \leq 125$

Figure 5.3. All Quadratic $|q_N(t)|$ for $500 \leq t \leq 2500$

Figure 5.4. Maximizer Quadratic $|q_N(t)|$ for $75 \leq t \leq 1000$
Figure 5.5. Maximizer Quadratic $|q_N(t)|$ for $500 \leq t \leq 2500$

Figure 5.6. Log-Log Maximizer Quadratic $|q_N(t)|$, $0 \leq t \leq 125$

A straight line approximation $ax + b$ has

$a = -1.4344$, $b = 0.37321$

Figure 5.7. Log-Log Maximizer Quadratic $|q_N(t)|$, $75 \leq t \leq 400$

A straight line approximation $ax + b$ has

$a = -1.4487$, $b = 0.3975$
Next we consider the cubic case when \( n_k = k^3 \). The analogous formulas are

\[
Q_N(0) = \frac{N^3 + N^2 + 2}{4\pi}
\]

\[
q_N(t) = \begin{cases} 
\frac{4\pi}{N^3 + N^2 + 2} Q_N(\frac{t}{N^3}) & \text{for } -\pi N^3 \leq t \leq \pi N^3 \\
0 & \text{for } |t| > \pi N^3 
\end{cases}
\]

\[
Q_N(t) = \frac{N^3 + N^2 + 2}{4\pi} q_N(N^3 t) & \text{for } -\pi \leq t \leq \pi
\]

With these substitutions, (5.3) and Lemma 5.1 continue to hold.

In Figures 5.8-5.14 we show the same graphs in the cubic case as in Figures 5.1-5.7 in the quadratic case.

**Figure 5.8.** Cubic evaluation of \( q_N(t) \) for \( 125 \leq t \leq 5000 \)
Figure 5.9. All Cubic $|q_N(t)|$ for $0 \leq t \leq 125$

Figure 5.10. All Cubic $|q_N(t)|$ for $500 \leq t \leq 2500$

Figure 5.11. Maximizer Cubic $|q_N(t)|$ for $75 \leq t \leq 500$

Figure 5.12. Maximizer Cubic $|q_N(t)|$ for $500 \leq t \leq 2500$
Figure 5.13. Log-Log Maximizer Cubic $|q_N(t)|$ $0 \leq t \leq 125$
$a = -1.2766$, $b = 0.3533$

Figure 5.14. Log-Log Maximizer Cubic $|q_N(t)|$ $75 \leq t \leq 400$
$a = -1.144$, $b = -0.33313$
See [1] for different zooms of all of the above data.

For the exponential case we choose \( n_k = 2^k \). Again the analogous formulas are

\[(5.1') \quad Q_N(0) = \frac{1}{2\pi} \frac{2^{N+2} + N - 1}{N+1} \]

\[(5.2') \quad q_N(t) = \begin{cases} 
\frac{2\pi(N+1)}{2^{2N+2-N}+1} Q_N\left(\frac{t}{2^{N+1}}\right) & \text{for } -\pi 2^N \leq t \leq \pi 2^N \\
0 & \text{for } |t| > \pi 2^N
\end{cases} \]

\[(5.4') \quad Q_N(t) = \frac{2^{N+2} + N - 1}{2\pi(N+1)} q_N(2^N t) \quad \text{for } -\pi \leq t \leq \pi \]

As we shall see in section 7, it does not appear that (2.16) holds.

Finally, in the linear case, we have

\[(5.1'') \quad Q_N(0) = \frac{pN + 1}{2\pi} \]

\[(5.2'') \quad q_N(t) = \begin{cases} 
\frac{2\pi}{pN+1} Q_N\left(\frac{t}{pN}\right) & \text{for } -\pi pN \leq t \leq \pi pN \\
0 & \text{for } |t| > \pi pN
\end{cases} \]

and so

\[(5.4'') \quad Q_N(t) = \frac{pN + 1}{2\pi} q_N(pN t) \quad \text{for } -\pi \leq t \leq \pi \]

We do not show data in the linear case since we have already proved convergence, but the analogous formulas are used in the random linear case in the next section.

6. Random Cases

For each of the cases we have considered so far, we can also look at random choices of \( n_k \) that mimic the qualitative features of the growth of \( \{n_k\} \).

1. Linear Case \( p \geq 3 \)
   Take \( n_k = p_k - 1, p_k, \) or \( p_k + 1 \) with equal probability, chosen independently.

2. Quadratic Case
   Choose \( n_k \) independently with equal probability in the interval \( k^2 \leq n_k \leq (k + 1)^2 - 1 \).

3. Cubic Case
   Choose \( n_k \) independently with equal probability in the interval \( k^3 \leq n_k \leq (k + 1)^3 - 1 \).

In the Figures, we show comparisons among the same graphs in the random and deterministic cases (\( N = 125 \)). In Figures 6.1-6.3 we look at \( q_N(t) \) for \(-25 \leq t \leq 25\) for the linear (\( p = 5 \)), quadratic, and cubic cases and find the random case is similar. In Figure 6.4-6.6, we shift to look at the same graphs but for \(25 \leq t \leq 300\) and see that the random case is significantly noisier. Finally in 6.7-6.9 we see that further along the \( t \) axis, the random case fails to follow the regularity of the deterministic case, though it does approximate the converging behavior. We then look at the maximizer for the quadratic and cubic cases in Figures 6.10 and 6.11, and examine the log-log graph to examine the rate of convergence in Figures 6.12 and 6.13. We see that while all cases converge (\(|a| > 1\)), the deterministic cases have a larger slope (converge faster).
Figure 6.1. Linear $q_N(t)$ comparison, $N = 125$, $n_k = 5k$

Figure 6.2. Quadratic $q_N(t)$ comparison, $N = 125$

Figure 6.3. Cubic $q_N(t)$ comparison, $N = 125$
Figure 6.4. Linear $q_N(t)$, $25 \leq t \leq 300$, $N = 125$

Figure 6.5. Quadratic $q_N(t)$, $25 \leq t \leq 300$, $N = 125$

Figure 6.6. Cubic $q_N(t)$, $25 \leq t \leq 300$, $N = 125$
Figure 6.7. Linear $q_N(t)$, $300 \leq t \leq 600$, $N = 125$

Figure 6.8. Quadratic $q_N(t)$, $300 \leq t \leq 600$, $N = 125$

Figure 6.9. Cubic $q_N(t)$, $300 \leq t \leq 800$, $N = 125$
Figure 6.10. Quadratic $|q_N(t)|$ Random v Deterministic

Figure 6.11. Cubic $|q_N(t)|$ Random v Deterministic

Figure 6.12. Quad $|q_N(t)|$ Rand. v Determ. Log-Log
A straight line approximation $ax + b$ has
Determin: $a = -1.45$, $b = 0.4047$
Rand: $a = -1.005$, $b = -1.0913$

Figure 6.13. Cubic $|q_N(t)|$ Rand. v Determ. Log-Log
A straight line approximation $ax + b$ has
Determin: $a = -1.2758$, $b = 0.31202$
Rand: $a = -1.0669$, $b = -0.20448$
7. Rates of Convergence

As discussed in section 2, the estimates (2.16) for a specific constant M and (2.17) for a specific function \( \varphi(N) \) yield a quantitative rate of convergence given in (2.18). Thus it is interesting to try to control these quantities in each of the cases we have considered. There is not much to be said about the constant M, other than that it should not be too large, but the rate of vanishing of the limit of \( \varphi(N) \) as \( N \to \infty \) allows many possibilities. One of these is an estimate of the form

\[
\varphi(N) \leq \frac{c}{N^{\alpha \epsilon \beta}}
\]

for fixed positive \( c, \alpha, \beta \).

In this section, we present experimental evidence concerning such estimates. Because \( Q_N(t) \) is even we just deal with integrals for \( t \geq 0 \). Let

\[
I_N(\epsilon) = \int_\epsilon^\pi |Q_N(t)|dt
\]

To estimate the constant M in (2.16) we just double the surpemum of \( I_N(0) \) over \( N \). Thus, we compute \( I_N(0) \) for as large values of \( N \) as we can with confidence of moderate accuracy. To guess an estimate of the form (7.1) with \( \varphi(N) \) in place of \( \varphi(N) \) we need a more accurate computation of \( I_N(\epsilon) \). Thus we look at a sequence of \( \epsilon \) values decreasing toward zero, and for each \( \epsilon \) we try to fit a curve \( b(\epsilon)N^a \) to the data \( I_N(\epsilon) \). This amounts to finding a straight line approximation to \( \log I_N(\epsilon) \approx \log(N) + \log(b(\epsilon)) \) as a function of \( \log(N) \). The value of \( a \) should be negative, of course, but \( b(\epsilon) \) must be allowed to grow as \( \epsilon \to 0 \). If we can find a value for \( a \) that is consistent across the range of \( \epsilon \) values, we can choose \( a \) in (7.1) to be the negative of this value. Then we can go back and try to fit the values of \( N^aI_N(\epsilon) \) to the curve \( c\epsilon^{-\beta} \), or \( \log(N) + \log(I_N(\epsilon)) \approx -\beta \log(\epsilon) + \log(\epsilon) \). This allows us to estimate the value \( \beta \) in (7.1).

In Figures 7.1 - 7.4, we show the graphs of \( I_N(0) \) as a function of \( N \) in the different cases. Figure 7.1 shows the full set of cases (linear, quadratic, cubic, exponentials), Figure 7.2 shows the linear case with varying value of \( p \), Figure 7.3 shows the random linear case, and Figure 7.4 shows the deterministic versus random quadratic and cubic cases. Figure 7.1 shows that already for \( n_k = 2^k \) the value of \( I_N(0) \) grows with \( N \), but for the linear, quadratic, and cubic cases, the value of \( I_N(0) \) gives very strong evidence that it is bounded. We split the next sets of figures into linear (Figures 7.5-7.8), quadratic (7.9 - 7.12), and cubic (7.13 - 7.16) cases. In each case, the first two figures are deterministic and then random graphs of \( I_N(\epsilon) \) for various values of \( \epsilon \) on a log-log scale, the slope of which is considered our \( \alpha \) value.

We see that the \( \alpha \) values for different choices of \( \epsilon \) are relatively the same, so we make the following choices for \( \alpha \) in order to estimate \( \beta \):

- Linear \( \alpha = 0.93 \)
- Quadratic \( \alpha = 0.5 \)
- Cubic \( \alpha = 0.5 \)

- Random Linear \( \alpha = 0.6 \)
- Random Quadratic \( \alpha = 0.5 \)
- Random Cubic \( \alpha = 0.5 \)

For each case, the second set of graphs show the approximate lines for fitting to \( \beta \) values. These do not closely resemble straight lines over the whole range of \( \epsilon \) values, so we used values of \( \epsilon \) closer to zero in estimating \( \beta \).

It is striking that the \( \alpha \) value in the linear case is close to 1 while in all the other cases it is close to \( \frac{1}{2} \). In other words, randomization severely perturbs \( \alpha \) in the linear case, but has almost no effect in the other cases. We see the specific \( \alpha \) values in the final figures (Table 1-3). We consider this to be even stronger evidence of the approximate identity property in these cases than that presented in sections 5 and 6.
Figure 7.1. $I_N(0)$ for different cases
From top down: $2^{1+n^3}$, $2^{n^2}$, $2^n$ (unbounded), $n^3$, $n^2$, $3n$

Figure 7.2. Linear $I_N(0)$ for different values of $p$

Figure 7.3. Random Linear $I_N(0)$

Figure 7.4. Quadratic and Cubic Random $I_N(0)$ Comparison
Figure 7.5. Linear $I_N(\epsilon)$ for $n_k = 5k$

Figure 7.6. Random Linear $I_N(\epsilon)$

Figure 7.7. Linear $\beta$ Fitting

Figure 7.8. Random Linear $\beta$ Fitting
Figure 7.9. Quadratic $I_N(\epsilon)$

Figure 7.10. Random Quadratic $I_N(\epsilon)$

Figure 7.11. Quadratic $\beta$ Fitting

Figure 7.12. Random Quadratic $\beta$ Fitting
Figure 7.13. Cubic $I_N(\epsilon)$

Figure 7.14. Random Cubic $I_N(\epsilon)$

Figure 7.15. Cubic $\beta$ Fitting

Figure 7.16. Random Cubic $\beta$ Fitting
### Table 1. Linear $\alpha$ Comparison

| Epsilon | Determ: alpha | Rand: alpha |
|---------|---------------|-------------|
| 0.0500  | 0.9753        | 0.6275      |
| 0.0600  | 0.9248        | 0.5967      |
| 0.0700  | 0.9286        | 0.5922      |
| 0.0750  | 0.9392        | 0.5899      |
| 0.0800  | 0.9355        | 0.5846      |
| 0.0850  | 0.9227        | 0.5775      |
| 0.0900  | 0.9151        | 0.5734      |
| 0.0950  | 0.9027        | 0.5712      |
| 0.1000  | 0.9182        | 0.5690      |
| 0.2000  | 0.8925        | 0.5306      |
| 0.3000  | 0.8831        | 0.5241      |
| 0.4000  | 0.8780        | 0.5343      |
| 0.5000  | 0.8748        | 0.5347      |
| 0.6000  | 0.8724        | 0.5391      |
| 0.7000  | 0.8706        | 0.5430      |
| 0.8000  | 0.8689        | 0.5470      |
| 0.9000  | 0.8671        | 0.5412      |
| 1       | 0.8649        | 0.5491      |

### Table 2. Quad $\alpha$ Comparison

| Epsilon | Determ: alpha | Rand: alpha |
|---------|---------------|-------------|
| 0.0500  | 0.4849        | 0.4979      |
| 0.0600  | 0.4900        | 0.4982      |
| 0.0650  | 0.4943        | 0.4969      |
| 0.0700  | 0.4938        | 0.5000      |
| 0.0750  | 0.4920        | 0.4970      |
| 0.0800  | 0.4907        | 0.4945      |
| 0.0850  | 0.4914        | 0.4844      |
| 0.0900  | 0.4919        | 0.4826      |
| 0.0950  | 0.4935        | 0.4810      |
| 0.1000  | 0.4936        | 0.4820      |
| 0.2000  | 0.4960        | 0.4848      |
| 0.3000  | 0.4961        | 0.4815      |
| 0.4000  | 0.4962        | 0.4864      |
| 0.5000  | 0.4980        | 0.4954      |
| 0.6000  | 0.4949        | 0.4945      |
| 0.7000  | 0.4963        | 0.5009      |
| 0.8000  | 0.4952        | 0.5140      |
| 0.9000  | 0.4938        | 0.5017      |
| 1       | 0.4950        | 0.5009      |

### Table 3. Cubic $\alpha$ Comparison

| Epsilon | Determ: alpha | Rand: alpha |
|---------|---------------|-------------|
| 0.0500  | 0.4851        | 0.4883      |
| 0.0550  | 0.4863        | 0.4957      |
| 0.0600  | 0.4890        | 0.4964      |
| 0.0650  | 0.4830        | 0.4918      |
| 0.0700  | 0.4836        | 0.4876      |
| 0.0750  | 0.4880        | 0.4818      |
| 0.0800  | 0.4899        | 0.4819      |
| 0.0850  | 0.4910        | 0.4810      |
| 0.0900  | 0.4925        | 0.4808      |
| 0.0950  | 0.4950        | 0.4835      |
| 0.1000  | 0.4925        | 0.4833      |
| 0.2000  | 0.4954        | 0.4817      |
| 0.3000  | 0.4916        | 0.4867      |
| 0.4000  | 0.4940        | 0.4925      |
| 0.5000  | 0.4995        | 0.4983      |
| 0.6000  | 0.5017        | 0.4966      |
| 0.7000  | 0.5064        | 0.4951      |
| 0.8000  | 0.5033        | 0.5004      |
| 0.9000  | 0.5023        | 0.4989      |
| 1       | 0.4907        | 0.4960      |
References

1. http://www.math.cornell.edu/~egoolish/sparse.html
2. A. Zygmund, *Trigonometric Series*, Cambridge University Press, Cambridge, 2003.

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