Ball convergence for a fourth order iterative method

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Abstract. Numerical iterative methods are shown to be convergent using hypotheses on higher order derivatives but these derivatives do not appear in the body structure of these methods. Therefore, the usage of them is limited although they may converge. In this article, we demonstrate the convergence but using hypotheses only on the function’s derivative of order one. In this way, we extend the usage of these methods. In addition, we present the computable radii of convergence of the considered scheme and error bounds in accordance with Lipschitz parameters. Moreover, we suggest one counter example where previous studies was not applicable but our results. Finally, we check our results on some other examples and also provide computable radii of convergence.

Keywords: Banach space, nonlinear system of equations, local convergence, Lipschitz constants, Fréchet derivative.

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1. Introduction

Several nonlinear problems [1–29] can be written like

\[ G(u) = 0, \]  

where \( G : \mathbb{D} \subseteq \mathbb{E}_1 \rightarrow \mathbb{E}_2 \) is a differential operator on a convex subset \( \mathbb{D} \) in the sense of Fréchet, \( \mathbb{E}_1 \) and \( \mathbb{E}_2 \) are two Banach spaces.

In 2015, Artidiello et al. [1] were studied a two-step method, which is given by

\[
\begin{align*}
 v_i &= u_i - G'(u_i)^{-1}G(u_i) \\
u_{i+1} &= v_i - H(\mu_i)G'(u_i)^{-1}G(v_i) \quad i = 0, 1, 2, \ldots
\end{align*}
\]  

where \( \mu_i = B_i^{-1}C_i \), \( B_i = (b_1 + b_2)G'(u_i) - b_2[u_i, v_i; G] \), \( C_i = (a_1 + a_2)G'(u_i) - a_2[u_i, v_i; G] \), \( u_0 \) is an initial point, \([\cdot, \cdot; G] : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{L}(\mathbb{E}_1, \mathbb{E}_2)\), \( H : \mathbb{L}(\mathbb{E}_2, \mathbb{E}_2) \rightarrow \mathbb{L}(\mathbb{E}_1, \mathbb{E}_2) \), \( a_1, a_2, b_1, b_2 \in \mathbb{S} \) and \( \mathbb{S} = \mathbb{R} \) or \( \mathbb{C} \). In addition, \( \mathbb{L}(\mathbb{E}_1, \mathbb{E}_2) \) stands for the space of bounded linear operators mapping to \( \mathbb{E}_1 \) in to \( \mathbb{E}_2 \).

Artidiello et al. [1] suggested the convergence analysis of iterative scheme (2) for the particular case when \( \mathbb{E}_1 = \mathbb{E}_2 = \mathbb{R}^m \), using Taylor expansions. In this regards, they assumed hypotheses up to the fourth-order derivative of considered function \( G \), but only the first-order derivative involves in the scheme (2). They also mentioned the benefits of their iterative methods over the

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other existing ones. However, these hypotheses restrict the usage of their scheme (2). Indeed, define a function \( G \) on \( E_1 = [-\frac{1}{2}, \frac{5}{2}] \) as follows:

\[
G(u) = \begin{cases} 
  u^3 \ln u^2 + u^5 - u^4, & u \neq 0 \\
  0, & u = 0
\end{cases}
\]

We obtain for \( u^* = 1 \) that

\[
G'(u) = 3u^2 \ln u^2 + 5u^4 - 4u^3 + 2u^2, \quad G'(1) = 3,
\]

\[
G''(u) = 6u \ln u^2 + 20u^3 - 12u^2 + 10u
\]

\[
G'''(u) = 6 \ln u^2 + 60u^2 - 24u + 22.
\]

Undoubtedly, the third derivative is unbounded in \( E_1 \). We have a plethora of iteration functions for obtaining the zeros of nonlinear functions [1–29]. In these studies, authors mentioned that the starting point \( u_0 \) should be sufficiently near to the required solution \( u^* \), in that case the obtained sequence \( \{u_n\} \) will sure converges to \( u^* \). However, the closeness of the initial is not well defined. Our mean to say that how much closeness of starting guess \( u_0 \) to the required solution should be required for granted convergence. These local outcomes are not giving any details regarding the radius of convergence ball of the corresponding iterative scheme. We suggest the answer for these queries for iterative scheme (2) in section 2. Moreover, we can also use the similar approach for other existing iterative methods.

We expand the usage of scheme (2) by adopting hypotheses only on the first order derivative of the considered function \( G \) and the contractions on a Banach space setting. We use Lipschitz parameters instead of Taylor expansions. Moreover, our approach does not require derivatives of high order to illustrate the convergence order of scheme (2). In this way, we extend the usage of iterative scheme (2).

The study is arranged as follows. We study the local convergence in section 2. In addition, we give a radius of convergence ball, uniqueness of the results and computable error, which were not studied in the earlier work. In the concluding section 3, we also discussed some particular cases and numerical experimentation.

### 2. Local convergence

Here, we propose the local convergence study of iterative method (2). Let us consider that \( L > 0, L_0 > 0, L_1 > 0, L_2 > 0, L_3 > 0, M \geq 1, a_1 \in S, a_2 \in S, b_1 \in S - \{0\} \) and \( b_2 \in S - \{0\} \) be some parameters. In addition, we assume that \( H : [0, +\infty) \rightarrow [0, +\infty) \) is a nondecreasing and continuous function. We introduce some functions \( p_1, q, h_q \) on interval \([0, \frac{1}{L_0}]\) in the following way:

\[
p_1(w) = \frac{Lw}{2(1 - L_0w)},
\]

\[
q(w) = \frac{1}{b_1} \left| L_0 \right| b_1 + b_2 \left| L_1 \right| b_2 \left| + \right| b_2 \mid p_1(w) \mid w,
\]

\[
h_q(w) = q(w) - 1
\]

and the value of parameter \( r_1 \) is define as follows:

\[
r_1 = 2 \frac{L}{2L_0 + L}.
\]

Since, the function \( h_q(0) = -1 < 0 \) and \( h_q(w) \rightarrow +\infty \) as \( w \rightarrow \frac{1}{L_0} \). With the help of intermediate value theorem, we can easily claim that function \( h_q \) has at least one zero in the open interval
(0, \frac{1}{L_0}). Let us consider the \( r_q \) is the smallest zero among the other zeros. In addition, we define some more functions \( p_2 \) and \( h_2 \) on interval \([0, r_q)\) in the following way:

\[
p_2(w) = \left[ 1 + \frac{M}{1 - L_0w} H \left( \frac{M \left( |a_1| + |a_2|L_0w \right)}{|b_1| (1 - q(w))} \right) \right] p_1(w)
\]

and

\[
h_2(w) = p_2(w) - 1.
\]

Again, the function \( h_2(0) = -1 < 0 \) and \( h_2(w) \to +\infty \) as \( w \to r_q^- \). We assume that \( r_2 \) is the smallest zero of function \( h_2 \) among other zeros in the open interval \((0, r_q)\). We define

\[
r = \min\{r_1, r_2\}.
\]

Then, we further have

\[
0 < r \leq r_1 < \frac{1}{L_0}
\]

and for each value of \( w \) from \( w \in [0, r) \)

\[
0 \leq p_1(w) < 1,
\]

\[
0 \leq q(w) < 1
\]

and

\[
0 \leq p_2(w) < 1.
\]

Let us consider that \( B(\gamma, \rho) \), \( \overline{B}(\gamma, \rho) \), are the open and closed balls respectively, in the Banach space \( E_1 \) having center \( \gamma \in E_1 \) and radius \( \rho > 0 \).

Now, we propose the local convergence study of iterative scheme (2), by adopting the preceding notations.

**Theorem 2.1** We consider that \( G : \mathbb{D} \subset E_1 \rightarrow E_2 \) is a differential operator in the sense of Fréchet and \([\cdot, \cdot; G] : \mathbb{D} \times \mathbb{D} \rightarrow L(E_1, E_2) \) is a first order divided difference. Let us assume that there exist \( u^* \in \mathbb{D}, L_0 > 0, L > 0, L_1 > 0, L_2 > 0, L_3 > 0, M > 1, a_1 \in S, a_2 \in S, b_1 \in S - \{0\}, b_2 \in S - \{0\}, H : L(E_2, E_2) \rightarrow [0, +\infty) \) nondecreasing and continuous function such that for every \( u \in \mathbb{D}, \ell \in L(E_2, E_2) \)

\[
G(u^*) = 0, \quad G'(u^*)^{-1} \in L(E_2, E_1), \quad (8)
\]

\[
\|H(\ell)\| \leq H(\|\ell\|), \quad (9)
\]

\[
\|G'(u^*)^{-1}(G'(u) - G'(u^*))\| \leq L_0\|u - u^*\|, \quad (10)
\]

and for each \( u, v \in \Omega_0 = \overline{B} \left( u^*, \frac{1}{L_0} \right) \cap \mathbb{D}, \)

\[
\|G'(u^*)^{-1}(G'(u) - G'(v))\| \leq L\|u - v\|, \quad (11)
\]

\[
\|G'(u^*)^{-1}G'(u)\| \leq M, \quad (12)
\]

\[
\|G'(u^*)^{-1}[u, v; G] - G'(u^*)\| \leq L_1\|u - u^*\| + L_2\|v - u^*\|, \quad (13)
\]

\[
\|G'(u^*)^{-1}[u, v; G] - G'(u^*)\| \leq L_3\|u^* - v\|, \quad (14)
\]

and

\[
\overline{B}(u^*, r) \subset \mathbb{D}, \quad (15)
\]
where the radius of convergent ball \( r \) is provided by the expression (3). Then, the sequence \( u_n \in \mathbb{B}(u^*, r) \) and \( \lim u_n = u^* \) so that

\[
\|v_i - u^*\| \leq p_1(\|u_i - u^*\|) \|u_i - u^*\| < \|u_i - u^*\| < r, \tag{16}
\]

and

\[
\|u_{i+1} - u^*\| \leq p_2(\|u_i - u^*\|) \|u_i - u^*\| < \|u_i - u^*\|, \tag{17}
\]

where the functions \( p \) are mentioned above the Theorem 2.1. Further, the limit point \( u^* \) is a unique zero of \( G(u) = 0 \) in \( \Omega_1 = \mathbb{B}(u^*, \frac{2}{L_0}) \cap \mathbb{D} \).

**Proof:** We demonstrate the estimates (16) and (17) by adopting principle of mathematical induction. With the hypothesis \( u_0 \in \mathbb{B}(u^*, r) - \{u^*\} \), expressions (4) and (10), we obtain

\[
\|G'(u^*)^{-1}(G'(u_0) - G'(u^*))\| \leq L_0\|u_0 - u^*\| < L_0 r < 1. \tag{18}
\]

It is clear from expression (18) and Banach Lemma for invertible operators [3, 6, 19, 23, 24, 28], we have \( G'(u_0)^{-1} \in \mathcal{L}(\mathbb{E}_2, \mathbb{E}_1) \) and

\[
\|G'(u_0)^{-1}G'(u^*)\| \leq \frac{1}{1 - L_0\|u_0 - u^*\|}. \tag{19}
\]

Therefore, we can say that \( v_0 \) is a well defined for \( i = 0 \). The expressions (3), (5), (8), (10) and (19), further yields

\[
\|v_i - u^*\| = \|u_i - u^* - G'(u_0)^{-1}G(u_0)\| \\
\leq \|G'(u_0)^{-1}G'(u^*)\| \left| \int_0^1 G'(u^*)^{-1}\left(G'(u^* + \eta(u_0 - u^*)) - G'(u_0)\right)(u_0 - u^*)d\eta \right| \\
\leq \frac{L\|u_0 - u^*\|^2}{2(1 - L_0\|u_0 - u^*\|)} \\
= p_1(\|u_0 - u^*\|)\|u_0 - u^*\| < \|u_0 - u^*\| < r,
\]

which demonstrates (16) for \( i = 0 \) and \( v_0 \in \mathbb{B}(u^*, r) \). Therefore, we can write the expression (8) in the following way:

\[
G(u_0) = G(u_0) - G(u^*) = \int_0^1 G'(u^* + \eta(u_0 - u^*))(u_0 - u^*)d\eta. \tag{21}
\]

Since, \( \|u^* - u^* + \eta(u_0 - u^*)\| = \|u_0 - u^*\| < r \), so \( u^* + \eta(u_0 - u^*) \in \mathbb{B}(u^*, r) \). By using the expressions (12) and (21), we obtain

\[
\|G'(u^*)^{-1}G(u_0)\| = \left| \int_0^1 G'(u^*)^{-1}G'(u^* + \eta(u_0 - u^*))(u_0 - u^*)d\eta \right| \\
\leq M\|u_0 - u^*\|. \tag{22}
\]

In the similar fashion, we have

\[
\|G'(u^*)^{-1}G(v_0)\| \leq M\|v_0 - u^*\| \\
\leq Mp_1(\|u_0 - u^*\|)\|u_0 - u^*\|, \tag{23}
\]
by (20) and since \( v_0 \in \mathbb{B}(u^*, r) \).

Next, we show that \( B_0^{-1} \in L(\mathbb{E}_2, \mathbb{E}_1) \). By using the expressions (3), (4),(6), (8), (10), (13) and (20), we further yields

\[
\|(b_1 G'(u^*))^{-1}(B_0 - b_1 G'(u^*))\| \leq \frac{1}{|b_1|}( |b_1 + b_2| \|G'(u^*)^{-1}(G'(u_0) - G'(u^*))\|
\]

\[
+ |b_2| \|G'(u^*)^{-1}([u_0, v_0; G] - G'(u^*))\|
\]

\[
\leq \frac{1}{|b_1|}( |b_1 + b_2| |L_0\|u_0 - u^*|)
\]

\[
+ |b_2| (L_1\|u_0 - u^*\| + L_2\|v_0 - u^*\|)
\]

\[
\leq \frac{1}{|b_1|}|L_0| |b_1 + b_2| + |L_1| |b_2|
\]

\[
+ |b_2| p_1(\|u_0 - u^*\|)\|u_0 - u^*\|
\]

\[
= q(\|u_0 - u^*\|) < q(r) < 1. \tag{24}
\]

Hence, \( u_1 \) is well defined and we obtain

\[
\|B_0^{-1} G'(u^*)\| \leq \frac{1}{|b_1| (1 - q(\|u_0 - u^*\|))}. \tag{25}
\]

From the expressions (14),(19) and (22), we get the following estimate

\[
\|G'(u^*)^{-1}C_0\| \leq |a_1| \|G'(u^*)^{-1}G'(u_0)\| + |a_2| \|G'(u^*)^{-1}(G'(u_0) - [u_0, v_0; G])\|
\]

\[
\leq |a_1| |M + |a_2| L_3\|u_0 - v_0\|
\]

\[
\leq |a_1| |M + |a_2| L_3\|G'(u_0)^{-1}G'(u^*)\|\|G'(u^*)^{-1}G'(u_0)\|
\]

\[
\leq |a_1| |M + \frac{|a_2| L_3 M\|u_0 - u^*\|}{1 - L_0\|u_0 - u^*\|}. \tag{26}
\]

Then, using (3),(4), (7), (9), (19), (20), (23), (25) and (26), we have

\[
\|u_1 - u^*\| \leq \|v_0 - u^*\| + H\left(\frac{M(|a_1| + |a_2| L_3\|u_0 - u^*\|)}{|b_1| (1 - q(\|u_0 - u^*\|))}\right)\|G'(u_0)^{-1}G'(u^*)\|\|G'(u^*)^{-1}G'(u_0)\|
\]

\[
\leq p_1(\|u_0 - u^*\|) H\left(\frac{M(|a_1| + |a_2| L_3\|u_0 - u^*\|)}{|b_1| (1 - q(\|u_0 - u^*\|))}\right) \frac{M}{1 - L_0\|u_0 - u^*\| + 1}\|u_0 - u^*\|
\]

\[
= p_2(\|u_0 - u^*\|)\|u_0 - u^*\| < \|u_0 - u^*\| < r,
\tag{27}
\]

which demonstrates (17) for \( i = 0 \) and \( u_1 \in \mathbb{B}(u^*, r) \).

Now, we can simply replace \( u_0, v_0, u_1 \) by \( u_i, v_i, u_{i+1} \) in the preceding estimates. Then, we reach at the estimates (16) and (17). In addition, the estimate \( \|u_{i+1} - u^*\| < \|u_i - u^*\| < r \), deduce that \( \lim u_i = u^* \) and \( u_{i+1} \in \mathbb{B}(u^*, r) \). Finally, we want to demonstrate the uniqueness part, let

\[
F = \int_0^1 G'(u^* + \eta(u^* - v^*)\,d\eta \text{ for some } v^* \in \bar{\mathbb{B}}(u^*, T) \text{ with } G(v^*) = 0. \tag{28}
\]

Using (10) we get that

\[
\|G'(u^*)^{-1}(F - G'(u^*))\| \leq \int_0^1 L_0\|v^* - u^* + \eta(u^* - v^*)\|\,d\eta
\]

\[
\leq \frac{1}{2} L_0(1 - \eta)\|u^* - v^*\|\,d\eta \leq \frac{L_0}{2} T < 1.
\]

It is clear from the expression (28) and the Banach Lemma for invertible operators that \( F \) is also invertible. Finally, the identity \( 0 = G(v^*) - G(u^*) = F(u^* - v^*) \), conclude that \( v^* = u^* \).
3. Numerical Examples

Here, we confirm our theoretical conclusions which we have presented in the earlier section 2. First of all, we provide the numerical outcomes obtained by using the presented methods on a scalar equation which is display in the test example 3.1. In addition, we assume \([u, v; G] = \frac{1}{2} \int_0^1 G' v + \eta(u - v) \, \text{d}t\), \(H(w) = 1 + 2w\), \(a_1 = 0\) and \(a_2 = b_1 = b_2 = 1\). Moreover, we also display the starting point, radius of convergence and minimum iterations are needed to reach the desired accuracy for corresponding solution of the considered problem. Further, we want to cross verify the theoretical convergence order. In this regard, we determine the computational convergence order by adopting the following formula

\[
\rho = \frac{\ln \frac{\|u_{i+2} - u^*\|}{\|u_{i+1} - u^*\|}}{\ln \frac{\|u_{i+1} - u_{i+2}\|}{\|u_{i} - u_{i+1}\|}}, \quad i = 0, 1, 2, 3, \ldots \tag{29}
\]

or in the case where \(u^*\) is not available, then we use the following approximate computational convergence order (ACOC) [16]

\[
\rho' = \frac{\ln \frac{\|u_{i+2} - u_{i+1}\|}{\|u_{i+1} - u_{i}\|}}{\ln \frac{\|u_{i+1} - u_{i}\|}{\|u_{i} - u_{i-1}\|}}, \quad i = 1, 2, 3, 4, \ldots \tag{30}
\]

We adopt the following stopping criteria computer programming in order to solve nonlinear equations:

(i) \(|u_{i+1} - u_i| < \epsilon\) and (ii) \(|G(u_{i+1})| < \epsilon\),

where we consider the tolerance error as \(\epsilon = 10^{-550}\).

In the case of nonlinear systems, we assume two standard systems of nonlinear equations (in examples 3.2 and 3.3) for checking our theoretical results. We consider a \(H(w) = 1 + 2w\), \(a_1 = 0\) and \(a_2 = b_1 = b_2 = 1\). That is we consider King’s family [1, 23, 29]. We display the starting point, radius of convergence and minimum iterations are needed to reach the required accuracy for corresponding solution of the considered problem. In addition, we calculate the computational convergence order by adopting the multi-dimensional version of above mentioned formulas namely, (29) or (30) to verify the theoretical convergence order. We performed all the calculations/computations Mathematica 9 (programming package) with multiple precision arithmetic for nonlinear equations and system. We adopt the following stopping criteria:

(i) \(\|u^{(i+1)} - u^{(i)}\| < \epsilon\) and (ii) \(\|G(u^{(i+1)})\| < \epsilon\), where we assume the tolerance error for nonlinear system as \(\epsilon = 10^{-50}\).

Example 3.1 Looking back at the motivational example, we have \(L_0 = L = 96.662907\), \(L_1 = L_2 = L_3 = \frac{L}{2}\) and \(M = 2\). Then, we obtain

\[
r_1 = 0.006897, \quad r_2 = 0.001128,
\]

and as a consequence by (3), we deduce that

\[
r = 0.001128.
\]

Finally, we obtain \(\rho = 4.000000\) with the initial guess \((1.0009)\) and \(n = 4\) is the minimum iterations needed to attain this.

Example 3.2 Let us consider that \(E_1 = E_2 = \mathbb{R}^3, D = \bar{B}(0, 1), u^* = (0, 0, 0)^T\) (in examples 3.2 and 3.3) for checking our theoretical results. We consider a \(H(w) = 1 + 2w\), \(a_1 = 0\) and \(a_2 = b_1 = b_2 = 1\). Moreover, we also display the starting point, radius of convergence and minimum iterations are needed to reach the required accuracy for corresponding solution of the considered problem. Further, we want to cross verify the theoretical convergence order. In this regard, we determine the computational convergence order by adopting the following formula

\[
\rho = \frac{\ln \frac{|u_{i+2} - u^*|}{|u_{i+1} - u^*|}}{\ln \frac{|u_{i+1} - u_{i+2}|}{|u_{i} - u_{i+1}|}}, \quad i = 0, 1, 2, 3, \ldots \tag{29}
\]

or in the case where \(u^*\) is not available, then we use the following approximate computational convergence order (ACOC) [16]

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\[
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and as a consequence by (3), we deduce that

\[
r = 0.001128.
\]

Finally, we obtain \(\rho = 4.000000\) with the initial guess \((1.0009)\) and \(n = 4\) is the minimum iterations needed to attain this.

Example 3.2 Let us consider that \(E_1 = E_2 = \mathbb{R}^3, D = \bar{B}(0, 1), u^* = (0, 0, 0)^T\). We consider the following function \(G\) on \(D\) for \(u = (u_1, u_2, u_3)^T\)

\[
G(u) = (e^{u_1} - 1, \frac{e - 1}{2}u_2^2 + u_2, u_3)^T.
\]
The derivative of above function in the sense of Fréchet is defined as follows

$$G'(u) = \begin{bmatrix} e^{u_1} & 0 & 0 \\ 0 & (e - 1)u_2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ 

Notice, we obtain $L_0 = e - 1$, $L = 1.789572397$, $M = 2$, $L_1 = L_2 = \frac{e-1}{2}$ and $L_3 = \frac{1.789572397}{2}$. Then, by the definition of the $r_1$ and $r_2$, we obtain

$$r_1 = 0.382692, \quad r_2 = 0.126408,$$

and as a consequence by (3), we conclude that

$$r = 0.126408.$$ 

Further, we get $\rho = 3.93786$ by considering the initial guess $(0.1, 0.09, 0.1)^T$ and $n = 4$ is the minimum iterations needed to attain the required accuracy.

**Example 3.3** Here, we assume that $E_1 = E_2 = \mathbb{R}^2$, $D = \bar{B}(0,1)$, $u^* = (3.470, -2.470)^T$. We choose the following function $F$ on $D$ for $u = (u_1, u_2)^T$

$$G(u) = (u_1 \cos u_2 + e^{u_1+u_2}, u_1 - 1 + u_2)^T.$$ 

The Fréchet-derivative of above function is given as follows

$$G'(u) = \begin{bmatrix} \cos u_2 + e^{u_1+u_2} & e^{u_2+u_2} - u_1 \sin u_2 \\ 1 & 1 \end{bmatrix}.$$ 

Therefore, we have $L_0 = L = 2(1 + e)$, $L_1 = L_2 = L_3 = 1 + e$ and $M = 2$. By adopting the definition of $r_1$ and $r_2$, we have

$$r_1 = 0.896471, \quad r_2 = 0.030022,$$

and as a consequence by (3), we further yield

$$r = 0.030022.$$ 

Further, we obtain $\rho = 4.00000$ by assuming the initial guess $(3.4, -2.4)^T$ and $n = 4$ is the least number of iterations needed to get the needed accuracy.
References

[1] Artidiello S, Cordero A, Torregrosa J R and Vassileva M P 2015 Multidimensional generalization of iterative methods for solving nonlinear problems. Appl. Math. Comput. 268 pp.1064–1071.

[2] Amat S, Hernández M A and Romero N 2012 Semilocal convergence of a sixth order iterative method for quadratic equations. Appl. Numer. Math. 62 pp.833–841.

[3] Argyros I K 2017 Computational theory of iterative methods. Series: Studies in Computational Mathematics. Elsevier Publ. Co. New York, U.S.A.

[4] Argyros I K 2011 A semilocal convergence analysis for directional Newton methods. Math. Comput. 80 pp.327–343.

[5] Argyros I K and Hilout S 2012 Weaker conditions for the convergence of Newton’s method. J. Comput. Appl. Math. 252 pp.364–387.

[6] Argyros I K and Hilout S 2013 Computational methods in nonlinear analysis. Efficient algorithms, fixed point theory and applications. World Scientific.

[7] Argyros I K and Ren H 2012 Improved local analysis for certain class of iterative methods with cubic convergence. Numer. Algor. 59 pp.1665–1675.

[8] Candela V and Marquina A 1990 Recurrence relations for rational cubic methods I: The Halley method. Computing 44 pp.169–184.

[9] Candela V and Marquina A 1990 Recurrence relations for rational cubic methods II: The Chebyshev method. Computing 45(4) pp.355–367.

[10] Chun C, Stănică P and Neta B 2011 Third-order family of methods in Banach spaces. Comput. Math. Appl. 61 pp.1665–1675.

[11] Cordero A, Martínez E and Torregrosa J R 2009 Iterative methods of order four and five for systems of nonlinear equations. J. Comput. Appl. Math. 231 pp.541–551.

[12] Cordero A, Hueso J, Martínez E and Torregrosa J R 2010 A modified Newton-Jarratt’s composition. Numer. Algor. 55 pp.87–98.

[13] Cordero A, Torregrosa J R and Vasileva M P 2013 Increasing the order of convergence of iterative schemes for solving nonlinear systems. J. Comput. Appl. Math. 252 pp.86–94.

[14] Ezzati R and Azandegan E 2009 A simple iterative method with third-order convergence by using Potra and Pták’s method. Math. Sci. 8 pp.191–200.

[15] Gutiérrez J M, Magreñán A A and Romero N 2013 On the semi-local convergence of Newton-Kantorovich method under center-Lipschitz conditions. Appl. Math. Comput. 221 pp.79–88.

[16] Hasanov V I, Ivanov I G and Nebzhibov F 2002 A new modification of Newton’s method. Appl. Math. Eng. 27 pp.278–286.

[17] Hernández M A and Salanova M A 2000 Modification of the Kantorovich assumptions for semi-local convergence of the Chebyshev method. J. Comput. Appl. Math. 126 pp.131–143.

[18] Jaiswal J P 2016 Semilocal convergence of an eighth-order method in Banach spaces and its computational efficiency. Numer. Algor. 71(4) pp.933–951.

[19] Kantorovich L V and Akilov G P 1982 Functional Analysis. Pergamon Press, Oxford.

[20] Kon J S, Li Y T and Wang X H 2006 A modification of Newton method with third-order convergence. Appl. Math. Comput. 181 pp.1106–1111.

[21] Magreñán À A 2014 Different anomalies in a Jarratt family of iterative root finding methods. Appl. Math. Comput. 233 pp.29–38.

[22] Magreñán À A 2014 A new tool to study real dynamics: The convergence plane. Appl. Math. Comput. 248 pp.29–38.

[23] Petkovic M S, Neta B, Petkovic L and Džunić J. 2013 Multipoint methods for solving nonlinear equations. Elsevier.

[24] Potra F A and Pták V 1984 Nondiscrete Induction and Iterative Processes, in: Research Notes in Mathematics. 103 Pitman, Boston.

[25] Ren H and Wu Q 2009 Convergence ball and error analysis of a family of iterative methods with cubic convergence. Appl. Math. Comput. 209 pp.369–378.

[26] Rheinboldt WC 1978 An adaptive continuation process for solving systems of nonlinear equations. Banach Center Publications 3(1) 129–142.

[27] Sharma J R, Guha P K and Sharma R 2013 An efficient fourth order weighted-Newton method for systems of nonlinear equations. Numer. Algor. 62(2) pp.307–323.

[28] Soleymani F, Lotfi T and Babhliari P 2014 A multi-step class of iterative methods for nonlinear systems. Optim. Lett. 8 pp.1001–1015.

[29] Traub J F 1964 Iterative methods for the solution of equations. Prentice-Hall, Englewood Cliffs.