Proof of Volume Conjecture for twist knots

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Abstract

We prove the volume conjecture for any twist knots by using an equivalence relation, complex analysis, analytic continuation, and function of several complex variables on the basis of colored Jones polynomials.

1 Introduction

H. Murakami-J. Murakami [MM] reformulated Kashaev’s conjecture as the volume conjecture. In this paper, we study the volume conjecture of any oriented twist knots. The volume conjectures has so far proven to be true only for the figure-eight knot by Ekholm (see [M]), $5_2$ by Kashaev-Yokota [KY] and Ohtsuki [O], torus knots by Kashaev-Tirkkonen [KT] and the whitehead doubles of torus knots by Zheng [Z]. These volume conjectures were solved by calculating colored Jones polynomial. We solve the volume conjecture for twist knots by using some equivalence relations, complex analysis, analytic continuations, and function of several complex variables on the basis of colored Jones polynomials. The following is the volume conjecture:

**Conjecture 1.1.** Let $K$ be an oriented hyperbolic knot, $J_N(K, q)$ be the colored Jones polynomial of $K$ and $\text{Vol}(K)$ be the hyperbolic volume of $S^3 \setminus K$. Then we obtain the following equality holds:

$$\lim_{N \to \infty} \frac{2\pi \log |J_N(K, \exp(2\pi \sqrt{-1}/N))|}{N} = \text{Vol}(K). \quad (1)$$

The purpose of this paper is to prove that the volume conjecture is true for any oriented twist knots. Namely, we prove the following theorem.

**Theorem 1.2.** Let $K$ be an oriented twist knot, $J_N(K, q)$ be the colored Jones polynomial of $K$, $\text{Vol}(K)$ be the hyperbolic volume of $S^3 \setminus K$. Then the above equality (1) holds.

Let $K$ be a twist knot as in Figure 1. Note that positive numbers correspond to left-handed twists and negative numbers correspond to right-handed twists respectively.
2 Lemmas necessary for Proof of Volume Conjecture for twist knots

Let $N$ be a positive integer, $n_i$ be positive integers or equal to 0, and $q = \exp(2\pi \sqrt{-1}/N)$. We know the following theorem.

**Theorem 2.1 ([T]).** Let $p$ be a positive integer, $K_{p>0}$ be the usual $p$-th twist knot, $J_N(K; q)$ be the colored Jones polynomial of $K$, and $K^*$ be the mirror image of the knot $K$. We know that $K_{-p}$ is the mirror image of $K_{p>0}$.

$$J_N(K_{p>0}; q) = q^{1-N} \sum_{N-1 \geq n_2p-1 \geq \cdots \geq n_1 \geq 0} (q^{1-N})^{n_2p-1} q^{-Nn_2p-1} \prod_{i=1}^{2p-2} (-1)^{n_i} q^{(-1)^i Nn_i \cdot \binom{n_i+1}{n_i}},$$

where we have used the usual $q$-binomial coefficient

$$\binom{n_i+1}{n_i} = \frac{(q)_{n_i+1}}{(q)_{n_i+1-n_i}(q)_{n_i}}$$

with the standard $q$-hyper geometric notation

$$(q)_n = \prod_{k=0}^{n-1} (1 - q^{k+1}) = (1 - q)^n \Gamma_q(n + 1).$$

**Definition 2.2.** We say that $f(N)$ and $g(N)$ are equivalent if the difference $|f(N) - g(N)|$ tends to zero as $N \to \infty$. Namely, $f(N) \sim g(N)$ if and only if $\lim_{N \to \infty} |f(N) - g(N)| = 0$.

This relation $\sim$ is an equivalence relation. Hence, reflexive relation, symmetric relation, and transitive relation hold.

**Lemma 2.3.** If $i(N) \sim g(N)$, $h(N) \sim i(N)$, and $g(N) \leq f(N) \leq h(N)$, then we have that

$$\lim_{N \to \infty} f(N) = \lim_{N \to \infty} i(N).$$

**Proof.** Because of the assumptions, we have

$$g(N) - i(N) \leq f(N) - i(N) \leq h(N) - i(N),$$

Figure 1: Here $2p$ ($p \in \mathbb{Z} \setminus \{0, 1\}$) denote the numbers of half twists in each box.
and  
\[ \lim_{N \to \infty} |g(N) - i(N)| = \lim_{N \to \infty} |h(N) - i(N)| = 0. \]
Hence,  
\[ \lim_{N \to \infty} |f(N) - i(N)| = 0, \]
and  
\[ \lim_{N \to \infty} f(N) = \lim_{N \to \infty} i(N). \]

We define the following dilogarithm function:
\[ \text{Li}_2(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^2} \quad (|z| < 1). \]

We know that the following equality:
\[ -\log(1 - z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \quad (|z| < 1). \]
Hence, the analytic continuation of the dilogarithm function is given by
\[ \text{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t} dt \quad (z \in \mathbb{C} \setminus [1, \infty)). \]
Here, \( \log(1-t) := \log |1-t| + \sqrt{-1} \text{Arg}(1-t) \) \( (0 \leq \text{Arg}(1-t) < 2\pi) \) is a regular analytic function in \( \mathbb{C} \setminus [1, \infty) \). Hence, \( \text{Li}_2(z) \) is a regular analytic function in \( \mathbb{C} \setminus [1, \infty) \).

**Definition 2.4.** We define the following functions:
\[ f(z_1, \ldots, z_{2p-1}) := -\text{Li}_2\left(\frac{1}{z_1}\right) + \sum_{i=1}^{2p-2} \left( \text{Li}_2(z_i) - \text{Li}_2\left(\frac{z_i}{z_{i+1}}\right) + \text{Li}_2\left(\frac{1}{z_{i+1}}\right) \right) + \text{Li}_2(z_{2p-1}), \]
and
\[ f(z_1^{(i)}, \ldots, z_{2p-1}^{(i)}) := \sum_{i=1}^{2p-2} \left( \text{Li}_2(z_i^{(i)}) - \text{Li}_2\left(\frac{z_i^{(i)}}{z_{i+1}^{(i)}}\right) + \text{Li}_2\left(\frac{1}{z_{i+1}^{(i)}}\right) \right) - \text{Li}_2(z_{2p-1}^{(i)}), \]
where \( z_i^{(i)} := \exp(2\pi \sqrt{-1} n_i / N), 1/z_{i+1}^{(i)} := \exp(2\pi \sqrt{-1} (n_{i+1} - n_i) / N). \)

### 2.1 First reduction

Firstly, we prove that  
\[ \text{Im}[f(z_1^{(1)}, \ldots, z_{2p-1}^{(2p-1)})] \sim \text{Im}[f(\exp(2\pi \sqrt{-1} n_1 / N), \ldots, \exp(2\pi \sqrt{-1} n_{2p-1} / N))], \]
\[ \text{Im}[f(z_1^{(1)}, \ldots, z_{2p-1}^{(2p-1)}) + \text{Li}_2(z_{2p-1}^{(2p-1)})] \sim \text{Im}[f(\exp(2\pi \sqrt{-1} n_1 / N), \ldots, \exp(2\pi \sqrt{-1} n_{2p-1} / N))], \]
\[ \frac{2\pi \log |(q)_n|}{N} \sim \text{Im}[\text{Li}_2(\exp(2\pi \sqrt{-1} / N))]. \]
and the following limit value converges:

\[ \lim_{N \to \infty} \text{Li}_2(2\pi \sqrt{-1}n/N). \]

Hence, the following limit value converges:

\[ \lim_{N \to \infty} f(\text{exp}(2\pi \sqrt{-1}n_1/N), \ldots, \text{exp}(2\pi \sqrt{-1}n_{2p-1}/N)). \]

Moreover, we obtain the following formula:

\[
\frac{2\pi}{N} \log \left( |(q)_{n2p-1}| \prod_{i=1}^{2p-2} \left| \left\lceil \frac{n_{i+1}}{n_i} \right\rceil \right| \right) = \frac{2\pi}{N} \left( \log |(q)_{n2p-1}| + \sum_{i=1}^{2p-2} \log \left| \left\lceil \frac{n_{i+1}}{n_i} \right\rceil \right| \right) \\
\sim \sum_{i=1}^{2p-2} \text{Im}[\text{Li}_2(\text{exp}(2\pi \sqrt{-1}n_i/N)) - \text{Li}_2(\text{exp}(2\pi \sqrt{-1}n_{i+1}/N))] \\
+ \text{Li}_2(\text{exp}(2\pi \sqrt{-1}(n_{i+1} - n_i)/N))] + \text{Im}[-\text{Li}_2(\text{exp}(2\pi \sqrt{-1}n_{2p-1}/N))] \\
= \text{Im}[f(z_1^{(1)}, \ldots, z_{2p-1}^{(2p-1)})] \\
\sim \text{Im}[f(\text{exp}(2\pi \sqrt{-1}n_1/N), \ldots, \text{exp}(2\pi \sqrt{-1}n_{2p-1}/N))].
\]

To show the above formulas, we first prove the property of dilogarithm function \( \text{Li}_2(z) \) of Lemma 2.5, Lemma 2.6 and Lemma 2.7.

**Lemma 2.5.** If \( z \in \mathbb{C} \setminus (-\infty, 0] \), then we obtain the following functional equality:

\[ \text{Li}_2(1 - z) = -\frac{1}{2} \log^2 z - \text{Li}_2(1 - \frac{1}{z}). \]

**Proof.** We obtain the following equality:

\[
\text{Li}_2(1 - z) + \text{Li}_2(1 - \frac{1}{z}) = \text{Li}_2(1 - z) - \int_0^{1 - z} \frac{1 - t}{t} dt \\
= \text{Li}_2(1 - z) - \int_1^{\frac{1}{z}} \log(1 - t) \frac{dt}{t} \\
= \text{Li}_2(1 - z) - \int_1^{\frac{1}{z}} \log t \frac{dt}{t - 1} \\
= \text{Li}_2(1 - z) - \left[ \frac{1}{2} \log^2 t \right]_1^{\frac{1}{z}} + \int_0^{1 - z} \frac{1 - t}{t} dt \\
= \text{Li}_2(1 - z) - \frac{1}{2} \log^2 z - \text{Li}_2(1 - z) = -\frac{1}{2} \log^2 z
\]

\[ \square \]

**Lemma 2.6.** If \( z \in \mathbb{C} \setminus [0, \infty) \), then we obtain the following functional equality:

\[ -\text{Li}_2(\frac{1}{z}) = \text{Li}_2(z) + \frac{\pi^2}{6} + \frac{1}{2} \log^2(-z). \]

**Proof.** We obtain the following equality:

\[
\text{Li}_2(z) + \text{Li}_2(\frac{1}{z}) = \int_z^0 \frac{\log(1 - t)}{t} dt + \int_1/z^{0} \frac{\log(1 - t)}{t} dt
\]
\[
\int_{z}^{-1} \log(1-t) \frac{dt}{t} + \int_{1/z}^{-1} \log(1-t) \frac{dt}{t} + 2 \int_{-1}^{0} \log(1-t) \frac{dt}{t} = 2 \text{Li}_2(-1) + \int_{z}^{-1} \log(1-t) \frac{dt}{t} + \int_{1/z}^{-1} (t-1) \log(t) \frac{dt}{t^2} + 2 \int_{0}^{-1} \log(1-t) \frac{dt}{t} = -\frac{\pi^2}{6} + \int_{z}^{-1} \log(1-t) \frac{dt}{t} + \log(-1) - \log(t) \frac{dt}{t} = -\frac{\pi^2}{6} - \frac{1}{2} \log^2(-z).
\]

\[\square\]

**Lemma 2.7.** If the domain of definition of \(\text{Li}_2(z)\) is \(z \in \mathbb{C} \setminus [1, \infty)\) and that of \(\text{Li}_2(-z)\) is \(z \in \mathbb{C} \setminus (-\infty, -1]\), then we obtain the following functional equality:

\[\text{Li}_2(z) + \text{Li}_2(-z) = \frac{1}{2} \text{Li}_2(z^2).\]

**Proof.** We obtain the following equality:

\[\text{Li}_2(z) + \text{Li}_2(-z) = \sum_{n=1}^{\infty} \left( \frac{z^n}{n^2} + \frac{(-z)^n}{n^2} \right) = \sum_{n=1}^{\infty} \left( \frac{z^{2n-1}}{(2m-1)^2} - \frac{z^{2m-1}}{(2m-1)^2} \right) + \sum_{k=1}^{\infty} \left( \frac{z^{2k}}{(2k)^2} + \frac{z^{2k}}{(2k)^2} \right) = \frac{1}{2} \text{Li}_2(z^2).\]

\[\square\]

**Lemma 2.8.** For any \(n \in \mathbb{R}\), the following limit value converges:

\[\lim_{N \to \infty} \text{Li}_2(\exp(2\pi \sqrt{n})/N).\]

**Proof.** For sufficiently large \(N\) and \(N'\) such that \(N > N'\), we obtain the following inequality:

\[
|\text{Li}_2(\exp(2\pi \sqrt{n})/N) - \text{Li}_2(\exp(2\pi \sqrt{N'}))| = \left| \int_{\exp(2\pi \sqrt{n})/N}^{\exp(2\pi \sqrt{N'})} \log(1-t) \frac{dt}{t} \right| 
\leq \int_{\exp(2\pi \sqrt{n})/N}^{\exp(2\pi \sqrt{N'})} \left| \log(1-\exp(\sqrt{-\theta})) \right| |\exp(\sqrt{-\theta})| d\theta = \int_{\exp(2\pi \sqrt{n})/N}^{\exp(2\pi \sqrt{N'})} \left| \log(1-\exp(\sqrt{-\theta})) \right| d\theta 
\leq \int_{\exp(2\pi \sqrt{n})/N}^{\exp(2\pi \sqrt{N'})} (1-\log|\theta|) d\theta = \left[ 2\theta - \theta \log|\theta| \right]_{\exp(2\pi \sqrt{n})/N}^{\exp(2\pi \sqrt{N'})} 
= 2\frac{\pi n}{N'} \left( 2 - \log \left( \frac{2\pi n}{N'} \right) \right) - \frac{2\pi n}{N} \left( 2 - \log \left( \frac{2\pi n}{N} \right) \right) \to 0 \ (N > N' \to \infty).
\]

\[\square\]

**Lemma 2.9.** With the same notation as in Theorem 2.1 we have that

\[\frac{2\pi \log(\sqrt{2\pi n})}{N} \sim \text{Im}[\text{Li}_2(\exp(2\pi \sqrt{n})/N)].\]
Proof. Because of Lemma 2.5 and Lemma 2.6, we obtain the following equality:

$$\sqrt{-1} \text{Li}_2(\exp(2\pi \sqrt{-1} \frac{n}{N})) - \frac{\pi^2 \sqrt{-1}}{6} - \text{Im}[\text{Li}_2(\exp(2\pi \sqrt{-1} \frac{n}{N}))]$$

$$= \sqrt{-1}(\text{Re}[\text{Li}_2(\exp(2\pi \sqrt{-1} \frac{n}{N}))] + \sqrt{-1}\text{Im}[\text{Li}_2(\exp(2\pi \sqrt{-1} \frac{n}{N}))]) - \frac{\pi^2 \sqrt{-1}}{6} + \text{Im}[\text{Li}_2(\exp(2\pi \sqrt{-1} \frac{n}{N}))]$$

$$= \sqrt{-1}(\text{Re}[\text{Li}_2(\exp(2\pi \sqrt{-1} \frac{n}{N}))] - \frac{\pi^2}{6})$$

$$= \sqrt{-1}(\text{Re}[-\frac{1}{2} \log^2(1 - \exp(2\pi \sqrt{-1} \frac{n}{N})) - \text{Li}_2(1 - \frac{1}{1 - \exp(2\pi \sqrt{-1} \frac{n}{N})})] - \frac{\pi^2}{6})$$

$$= \sqrt{-1}(\text{Re}[-\frac{1}{2} \log^2(1 - \exp(2\pi \sqrt{-1} \frac{n}{N})) - \text{Li}_2\left(\frac{\exp(2\pi \sqrt{-1} \frac{n}{N})}{\exp(2\pi \sqrt{-1} \frac{n}{N}) - 1}\right) - \frac{\pi^2}{6})$$

$$= \sqrt{-1}(\text{Re}[\text{Li}_2\left(\frac{\exp(2\pi \sqrt{-1} \frac{n}{N}) - 1}{\exp(2\pi \sqrt{-1} \frac{n}{N})}\right) - \log(\exp(2\pi \sqrt{-1} \frac{n}{N}))\log(1 - \exp(2\pi \sqrt{-1} \frac{n}{N}))$$

$$+ \frac{1}{2} \log^2(\exp(2\pi \sqrt{-1} \frac{n}{N}))] \to 0 \ (N \to \infty).$$

Hence, we obtain the following equivalence relation:

$$\frac{2\pi \log |(q)_n|}{N} = 2\pi \log \prod_{k=1}^{n} (1 - q^{k+1}) = 2\pi \sum_{k=1}^{n} \log |1 - q^k|$$

$$\sim 2\pi \int_{0}^{n/N} \log |1 - \exp(2\pi \sqrt{-1} x)| \ dx$$

$$= \sqrt{-1} \text{Li}_2(\exp(2\pi \sqrt{-1} \frac{n}{N})) + \frac{\pi^2 \sqrt{-1} \frac{n}{N}}{N^2} - \frac{2\pi n \log(1 - \exp(2\pi \sqrt{-1} \frac{n}{N}))}{N} + \frac{\pi n \log 4}{N}$$

$$+ \frac{\pi n \log \sin(\frac{\pi}{2})}{N} \sim \frac{\pi^2 \sqrt{-1}}{6}$$

$$\sim \text{Im}[\text{Li}_2(\exp(2\pi \sqrt{-1} \frac{n}{N}))].$$

\[\square\]

Corollary 2.10. With the same notation as in Theorem 2.1 we have that

$$2\pi \log \left[\frac{n_{i+1}}{n_i}\right] \sim \text{Im}[\text{Li}_2(\exp(2\pi \sqrt{-1} n_i / N)) - \text{Li}_2(\exp(2\pi \sqrt{-1} n_{i+1} / N))] + \text{Li}_2(\exp(2\pi \sqrt{-1} (n_{i+1} - n_i) / N))].$$

Lemma 2.11. \(f(z_{1}^{(1)}, \ldots, z_{2p-1}^{(2p-1)})\) and \(f(z_{1}, \ldots, z_{2p-1})\) in Definition 2.4 above. We obtain the following equivalence relations:

$$\text{Im}[f(z_{1}^{(1)}, \ldots, z_{2p-1}^{(2p-1)})] \sim \text{Im}[f(\exp(2\pi \sqrt{-1} n_{1}/N), \ldots, \exp(2\pi \sqrt{-1} n_{2p-1}/N))],$$

and

$$\text{Im}[f(z_{1}^{(1)}, \ldots, z_{2p-1}^{(2p-1)}) + \text{Li}_2(z_{2p-1}^{(2p-1)})] \sim \text{Im}[f(\exp(2\pi \sqrt{-1} n_{1}/N), \ldots, \exp(2\pi \sqrt{-1} n_{2p-1}/N))].$$
Proof. Firstly, for a sufficiently large \( N \),
\[
\left| -\text{Li}_2\left( \frac{z^{(i)}}{z_i^{(i)}} \right) + \text{Li}_2\left( \frac{\exp(2\pi\sqrt{-1}n_i/N)}{\exp(2\pi\sqrt{-1}n_{i+1}/N)} \right) \right|
= \left| -\text{Li}_2(2\pi\sqrt{-1}n_{i+1}/N) + \text{Li}_2(2\pi\sqrt{-1}(n_i - n_{i+1})/N) \right|
\leq \int_{2\pi(n_i - n_{i+1})}^{2\pi(n_{i+1})} \left| \log(1 - \exp(-1\theta)) \right| d\theta
\leq \int_{2\pi(n_i - n_{i+1})}^{2\pi(n_{i+1})} (1 - \log |\theta|) d\theta = \left[ 2\theta - \theta \log |\theta| \right]_{2\pi(n_i - n_{i+1})}^{2\pi(n_{i+1})}
= \frac{2\pi n_{i+1}}{N} (2 - \log \frac{2\pi n_{i+1}}{N}) + \frac{2\pi n_{i+1}}{N} (2 - \log \frac{2\pi(n_{i+1} - n_i)}{N}) \to 0 \quad (N \to \infty).
\]

Secondly, for a sufficiently large \( N \),
\[
\left| \text{Li}_2\left( \frac{1}{z_i^{(i)}} \right) - \text{Li}_2\left( \frac{1}{\exp(2\pi\sqrt{-1}n_i/N)} \right) \right|
= |\text{Li}_2(2\pi\sqrt{-1}n_{i+1}/N) - \text{Li}_2(2\pi\sqrt{-1}n_i/N) |
\leq \int_{2\pi(n_i - n_{i+1})}^{2\pi(n_{i+1})} \left| \log(1 - \exp(-1\theta)) \right| d\theta
\leq \int_{2\pi(n_i - n_{i+1})}^{2\pi(n_{i+1})} (1 - \log |\theta|) d\theta = \left[ 2\theta - \theta \log |\theta| \right]_{2\pi(n_i - n_{i+1})}^{2\pi(n_{i+1})}
= \frac{2\pi n_{i+1}}{N} (2 - \log \frac{2\pi n_{i+1}}{N}) + \frac{2\pi n_{i+1}}{N} (2 - \log \frac{2\pi(n_{i+1} - n_i)}{N}) \to 0 \quad (N \to \infty).
\]

Thirdly, for a sufficiently large \( N \), because of Lemma 2.7,
\[
\left| -\text{Li}_2\left( \frac{2(2p-1)}{z^{(2p-1)}} \right) + \frac{1}{\exp(2\pi\sqrt{-1}n_1/N)} + \pi^2 \right|
= |\text{Li}_2(2\pi\sqrt{-1}n_{2p-1}/N) - \text{Li}_2(2\pi\sqrt{-1}n_1/N) | - \frac{\pi^2}{6}
\leq \int_{2\pi(n_1)}^{2\pi(n_{2p-1})} \left| \log(1 - \exp(-1\theta)) \right| d\theta + | - 2\text{Li}_2(-\exp(2\pi\sqrt{-1}n_{2p-1}/N)) - \frac{\pi^2}{6}|
\leq \int_{2\pi n_1}^{2\pi n_{2p-1}} (1 - \log |\theta|) d\theta + | - 2\text{Li}_2(-\exp(2\pi\sqrt{-1}n_{2p-1}/N)) - \frac{\pi^2}{6}|
= \frac{2\pi n_{2p-1}}{N} (2 - \log \frac{2\pi n_{2p-1}}{N}) + \frac{2\pi n_1}{N} (2 - \log \frac{2\pi n_1}{N}) + | - 2\text{Li}_2(-\exp(2\pi\sqrt{-1}n_{2p-1}/N)) - \frac{\pi^2}{6}|
\to 0 \quad (N \to \infty).
\]
Fourthly, for a sufficiently large \( N \),
\[
\left| -\text{Li}_2(\exp(2\pi \sqrt{-1}n_{2p-1}/N)) + \text{Li}_2(\frac{1}{\exp(2\pi \sqrt{-1}n_1/N)}) \right|
\leq \int_{-\frac{2\pi n_1}{N}}^{\frac{2\pi n_2-1}{N}} \log(1 - \exp(\sqrt{-1}\theta))d\theta
\leq \int_{-\frac{2\pi n_1}{N}}^{\frac{2\pi n_2-1}{N}} \log(1 - \exp(\sqrt{-1}\theta))d\theta
\leq \frac{2\pi n_2-1}{N}(2 - \log \frac{2\pi n_2-1}{N}) + \frac{2\pi n_1}{N}(2 - \log \frac{2\pi n_1}{N})
\rightarrow 0 \ (N \rightarrow \infty).
\]
Hence, we obtain the following formulas:
\[
\left| \text{Im}[f(z^{(1)}_i, \ldots, z^{(2p-1)}_{2p-1})] - \text{Im}[f(\exp(2\pi \sqrt{-1}n_1/N), \ldots, \exp(2\pi \sqrt{-1}n_{2p-1}/N))] \right|
\leq \sum_{i=1}^{2p-2} \left( \left| \text{Li}_2(z^{(i)}_i) - \text{Li}_2(\exp(2\pi \sqrt{-1}n_i/N)) \right| + \left| -\text{Li}_2\left(\frac{z^{(i)}_i}{\exp(2\pi \sqrt{-1}n_{i+1}/N)}\right) + \text{Li}_2\left(\frac{1}{\exp(2\pi \sqrt{-1}n_{i+1}/N)}\right) \right| \right)
\rightarrow 0 \ (N \rightarrow \infty),
\]
and
\[
\left| \text{Im}[f(z^{(1)}_i, \ldots, z^{(2p-1)}_{2p-1})] + \text{Li}_2(\exp(2\pi \sqrt{-1}n_{2p-1}/N)) \right|
\leq \sum_{i=1}^{2p-2} \left( \left| \text{Li}_2(z^{(i)}_i) - \text{Li}_2(\exp(2\pi \sqrt{-1}n_i/N)) \right| + \left| -\text{Li}_2\left(\frac{z^{(i)}_i}{\exp(2\pi \sqrt{-1}n_{i+1}/N)}\right) + \text{Li}_2\left(\frac{1}{\exp(2\pi \sqrt{-1}n_{i+1}/N)}\right) \right| \right)
\rightarrow 0 \ (N \rightarrow \infty).
\]
2.2 Second reduction

Secondly, we prove that the following limit value is compactly uniformly and absolutely-convergent for all \( N \in \mathbb{N} \).

\[
f(\exp(2\pi \sqrt{-1} n_1/N), \ldots, \exp(2\pi \sqrt{-1} n_{2p-1}/N))
= \sum_{\nu_1, \ldots, \nu_{2p-1} = 0}^{\infty} \frac{1}{\nu_1! \cdots (\nu_{2p-1})!} \frac{\partial^{\mid \nu \mid} f(a_1, \ldots, a_{2p-1})}{\partial z_1^{\nu_1} \cdots \partial z_{2p-1}^{\nu_{2p-1}}} \times (\exp(2\pi \sqrt{-1} n_1/N) - a_1)^{\nu_1} \cdots (\exp(2\pi \sqrt{-1} n_{2p-1}/N) - a_{2p-1})^{\nu_{2p-1}}
\]

Here, \( \mid \nu \mid : = \nu_1 + \cdots + \nu_{2p-1} \).

Let \( (a_1, \ldots, a_{2p-1}) \in (\mathbb{C} \setminus [0, \infty))^{2p-1} \) be a solution of \( \partial f/\partial z_1 = \cdots = \partial f/\partial z_{2p-1} = 0 \). The solution of these equations which induces the complete hyperbolic structure of the knot complement exists uniquely by virtue of the Lemma 2.3 of [Y].

\( \text{Im}[f(a_1, \ldots, a_{2p-1})] \) is the hyperbolic volume of a twist knot [CMY].

A uniformly convergent sequence is always pointwise convergence to the same limit. Moreover, rearrangements do not change the value of the sum.

**Remark 2.12.** \( (a_1, \ldots, a_{2p-1}) \in (\mathbb{C} \setminus [0, \infty))^{2p-1} \) satisfies the following three equalities:

\[
\begin{cases}
1 - \frac{z_1}{z_2} = (1 - z_1)(1 - \frac{1}{z_1}) \\
(1 - \frac{z_i}{z_{i+1}})(1 - \frac{1}{z_i}) = (1 - z_i)(1 - \frac{z_i - 1}{z_i}) & (i = 2, \ldots, 2p - 2) \\
1 - \frac{1}{z_{2p-1}} = (1 - z_{2p-1})(1 - \frac{z_{2p-2}}{z_{2p-1}})
\end{cases}
\]

Lemma 2.13 and Lemma 2.17 prove \( \text{Im}[a_i] \neq 0 \) \( (1 \leq i \leq 2p - 1) \). Lemma 2.14, Lemma 2.15, Lemma 2.16, and Lemma 2.18 prove \( \text{Re}[a_i] \neq 1 \) \( (1 \leq i \leq 2p - 1) \). These are used to denote the following limit value is compactly uniformly and absolutely-convergent for all \( N \in \mathbb{N} \).

\[
\sum_{\nu_1, \ldots, \nu_{2p-1} = 0}^{\infty} \frac{1}{\nu_1! \cdots (\nu_{2p-1})!} \frac{\partial^{\mid \nu \mid} f(a_1, \ldots, a_{2p-1})}{\partial z_1^{\nu_1} \cdots \partial z_{2p-1}^{\nu_{2p-1}}} (\exp(2\pi \sqrt{-1} n_1/N) - a_1)^{\nu_1} \cdots (\exp(2\pi \sqrt{-1} n_{2p-1}/N) - a_{2p-1})^{\nu_{2p-1}}
\]

Here, \( \mid \nu \mid : = \nu_1 + \cdots + \nu_{2p-1} \)

**Lemma 2.13.** \( \text{Im}[a_i] \neq 0 \).

**Proof.** Let \( a \) be real number, \( z_1 \) be \( a \). Since

\[
1 - \frac{z_1}{z_2} = (1 - z_1)(1 - \frac{1}{z_1})
\]

holds, we obtain the following equalities:

\[
z_2 = \frac{a^2}{a^2 - a + 1}
\]
\( b = \frac{a^2}{a^2 - a + 1}, \)

and

\( c = 0. \)

Since

\[ a^2 - a + 1 = (a - \frac{1}{2})^2 + \frac{3}{4} > 0 \]

holds, we obtain the following inequality:

\[ z_2 = \frac{a^2}{a^2 - a + 1} > 0. \]

Because of \( z_2 \in \mathbb{C} \setminus [0, \infty) \), this is a contradiction. Hence, \( \text{Im}[z_1] \neq 0. \)

**Lemma 2.14.** \( \text{Re}[a_1] \neq 1 \) or \( \text{Re}[a_2] \neq 1. \)

**Proof.** Let \( a \) be real number, \( z_1 \) be \( 1 + \sqrt{-1}a \). Since

\[ 1 - \frac{z_1}{z_2} = (1 - z_1)(1 - \frac{1}{z_1}), \]

holds, we obtain the following equality:

\[ z_2 = \frac{a^4 + 1}{a^4 - a^2 + 1} + \sqrt{-1} \frac{a - a^3}{a^4 - a^2 + 1}. \]

Since \( \text{Re}[z_2] = 1 \) holds, we obtain the following equality:

\[ a = 0. \]

Hence, \( z_1 = z_2 = 1 \). This is a contradiction.

**Lemma 2.15.** If \( \text{Re}[z_4] \neq 1 \), then \( \text{Re}[z_1] \neq 1. \)

**Proof.** Let \( a \) be real number, \( z_1 \) be \( 1 + \sqrt{-1}a \). Since

\[ 1 - \frac{z_1}{z_2} = (1 - z_1)(1 - \frac{1}{z_1}), \]

and

\[ z_{i+2} = \frac{z_{i+1}}{z_{i+1} - z_i + 1} \]

hold, we obtain the following equality:

\[ z_4 = \frac{2a^8 - 2a^6 - a^2 + 1}{4a^8 - 3a^6 + a^4 - 2a^2 + 1} + \sqrt{-1} \left( \frac{-3a^7 + a^5 - a^3 + a}{4a^8 - 3a^6 + a^4 - 2a^2 + 1} \right). \]

Hence,

\[ 1 = \frac{2a^8 - 2a^6 - a^2 + 1}{4a^8 - 3a^6 + a^4 - 2a^2 + 1}. \]

There exist

\[ a = \pm \sqrt{\frac{1 - \frac{5}{\sqrt{46 + 3\sqrt{249}}} + \frac{1}{\sqrt{46 + 3\sqrt{249}}}}{6}} \in \mathbb{R} \]

such that \( \text{Re}[z_4] = 1 \). We consider the contrapositive of the above statement. If, for any \( a \in \mathbb{R}, \text{Re}[z_4] \neq 1 \), then \( \text{Re}[z_1] \neq 1. \)
Lemma 2.16. If $\text{Re}[z_3] \neq 1$, then $\text{Re}[z_2] \neq 1$.

Proof. Let $a$, $b$, and $c$ be real numbers, $z_1$ be $a + \sqrt{-1}b$, and $z_2$ be $1 + \sqrt{-1}c$. Since

$$1 - \frac{z_1}{z_2} = (1 - z_1)(1 - \frac{1}{z_1}),$$

and

$$z_3 = \frac{z_2}{z_2 - z_1 + 1}$$

hold, we obtain the following equalities:

$$z_2 = \frac{a^4 - a^3 + 2a^2b^2 + a^2 - ab^2 + b^4 - b^2}{a^4 - 2a^3 + a^2(2b^2 + 3) - 2a(b^2 + 1) + b^4 - b^2 + 1} - \sqrt{-1}\left(\frac{b(a^2 - 2a + b^2)}{a^4 - 2a^3 + a^2(2b^2 + 3) - 2a(b^2 + 1) + b^4 - b^2 + 1}\right),$$

and

$$z_3 = \frac{-a - bc + c^2 + 2}{a^2 - 4a + b^2 - 2bc + c^4 + 4} + \sqrt{-1}\frac{-ac + b + c}{a^2 - 4a + b^2 - 2bc + c^4 + 4}.$$

Hence,

$$1 = \frac{a^4 - a^3 + 2a^2b^2 + a^2 - ab^2 + b^4 - b^2}{a^4 - 2a^3 + a^2(2b^2 + 3) - 2a(b^2 + 1) + b^4 - b^2 + 1},$$

$$c = \frac{b(a^2 - 2a + b^2)}{a^4 - 2a^3 + a^2(2b^2 + 3) - 2a(b^2 + 1) + b^4 - b^2 + 1},$$

and

$$1 = \frac{-a - bc + c^2 + 2}{a^2 - 4a + b^2 - 2bc + c^4 + 4}.$$

There exist

$$(a, b, c) = \left(\frac{1}{\sqrt{2}} \pm \sqrt{-5 + 4\sqrt{2}}, \pm \frac{1}{7} (5\sqrt{-5 + 4\sqrt{2}} + 4\sqrt{-10 + 8\sqrt{2}})\right) \in \mathbb{R}^3$$

such that $\text{Re}[z_3] = 1$. We consider the contrapositive of the above statement. If, for any $a \in \mathbb{R}$, $b \in \mathbb{R}$, and $c \in \mathbb{R}$, $\text{Re}[z_3] \neq 1$, then $\text{Re}[z_2] \neq 1$. □

Lemma 2.17. $\text{Im}[a_i] \neq 0$ (1 ≤ i ≤ 2p - 1).

Proof. We prove that if $\text{Im}[z_{i+1}] = 0$, then we obtain the following formula:

$$\text{Im}[z_i] = 0.$$

Since

$$\text{Im}[z_{i+1}] = \text{Im}[\frac{z_i}{z_i - z_{i-1} + 1}] = 0$$

holds, we obtain the following formulas:

$$\begin{align*}
zh_{i-1} &= 1, & (\text{Im}[z_i] \neq 0), \\
zh_{i-1} &= 1, & (\text{Re}[z_i] \neq 0), \\
\text{Re}[z_{i-1}] &= 1, & (\text{Re}[z_i] = 0), \\
\text{Re}[z_{i-1}] &\neq 1, & (\text{Im}[z_i] = \frac{\text{Im}[z_{i-1}]\text{Re}[z_i]}{\text{Re}[z_{i-1}] - 1}).
\end{align*}$$

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Because of $z_{i-1} \in \mathbb{C} \setminus [0, \infty)$, we only consider the following formulas:

$$\text{Re}[z_{i-1}] = 1 \text{ and } \text{Re}[z_{i}] = 0,$$
or

$$\text{Re}[z_{i-1}] \neq 1 \text{ and } \text{Im}[z_{i}] = \frac{\text{Im}[z_{i-1}] \text{Re}[z_{i}]}{\text{Re}[z_{i-1}] - 1}.$$  

(i) In case of $\text{Re}[z_{i-1}] = 1$ and $\text{Re}[z_{i}] = 0$, let $a$ and $b$ be real numbers, $z_{i-1}$ be $1 + \sqrt{-1}a$, and $z_{i}$ be $\sqrt{-1}b$. Since $z_{i+1} = \frac{z_{i}}{z_{i} - z_{i-1} + 1} \in \mathbb{C} \setminus [0, \infty)$ holds, we obtain the following equality:

$$z_{i+1} = \frac{b}{b-a} < 0.$$

We prove that this case dose not hold by using mathematical induction on $i \ (2 \leq i \leq 2p - 2)$.

(i)-1. When $i = 2p - 2$, we obtain the following equalities:

$$z_{2p-1} = \frac{b}{b-a}, \quad z_{2p-2} = \sqrt{-1}b, \quad \text{and} \quad z_{2p-3} = 1 + \sqrt{-1}a.$$

Since

$$1 - \frac{1}{z_{2p-1}} = (1 - z_{2p-1})(1 - \frac{z_{2p-2}}{z_{2p-1}})$$

holds, we obtain the following equality:

$$a = 0 \text{ and } b \neq 0.$$

Hence, $z_{2p-1} = z_{2p-3} = 1$. Since

$$z_{i} = \frac{(z_{i-1} - 1)z_{i+1}}{z_{i+1} - 1}$$

holds, we obtain the following equality:

$$z_{2p-2} = \frac{(z_{2p-3} - 1)z_{2p-1}}{z_{2p-1} - 1} = \frac{(z_{2p-1} - 1)z_{2p-1}}{z_{2p-1} - 1} = z_{2p-1} = 1$$

(i)-2. When $i = 2p - k$ and $i = 2p - (k + 1)$, we assume the following equalities hold:

$$z_{2p-k} = z_{2p-(k+1)} = 1.$$

Since

$$z_{i} = \frac{z_{i+1}}{z_{i+2}} + z_{i+1} + 1$$

holds, we obtain the following equality:

$$z_{2p-(k+2)} = 1.$$

Hence, $z_{1} = z_{2} = \cdots = z_{2p-2} = z_{2p-1} = 1$. This is a contradiction.

Therefore, this case dose not hold.
(ii) In case of \(\text{Re}[z_{i-1}] \neq 1\) and \(\text{Im}[z_i] = \frac{\text{Im}[z_{i-1}]\text{Re}[z_i]}{\text{Re}[z_{i-1}] - 1}\), let \(a, b, c, d,\) and \(e\) be real numbers, \(z_{i-1}\) be \(a + \sqrt{-1}b\) \((a \neq 1)\), \(z_i\) be \(c + \sqrt{-1}d\), and \(z_{i+1}\) be \(e\). Since

\[
\text{Re}[z_i] = \frac{\text{Im}[z_{i-1}]\text{Re}[z_i]}{\text{Re}[z_{i-1}] - 1},
\]

and

\[
z_i = \frac{(z_{i-1} - 1)z_{i+1}}{z_{i+1} - 1}
\]

hold, we obtain the following equalities:

\[
e = \frac{(a - 1)e}{e - 1},
\]

and

\[
d = \frac{be}{e - 1} = \frac{bc}{a - 1}.
\]

When \(b = 0\), we obtain the following equality:

\[
d = \frac{be}{e - 1} = \frac{bc}{a - 1} = 0.
\]

Namely,

\[
\text{Im}[z_i] = 0.
\]

When \(b \neq 0\), we prove that \(\text{Im}[z_i] = 0\) by using mathematical induction on \(i\) \((1 \leq i \leq 2p - 2)\).

(ii)-1. When \(i = 2p - 2\), since

\[
z_{2p-2} - 1 = z_{2p-1}
\]

holds, we obtain the following equalities:

\[
c - 1 = e \quad \text{and} \quad d = 0.
\]

Namely,

\[
\text{Im}[z_{2p-2}] = 0.
\]

(ii)-2. When \(i = 2p - 3\), since

\[
z_{2p-2} - 1 = z_{2p-1},
\]

and

\[
z_{i+2} = \frac{z_{i+1}}{z_{i+1} - z_i + 1}
\]

hold, we obtain the following equalities:

\[
z_{2p-1} = z_{2p-2} - 1 = e - 1,
\]

and

\[
z_{2p-1} = \frac{z_{2p-2}}{z_{2p-2} - z_{2p-3} + 1} = \frac{e}{e - (c + \sqrt{-1}d) + 1} = \frac{e(e - c + 1) + \sqrt{-1}ed}{(e - c + 1)^2 + d^2}.
\]

Because of \(e \neq 0\), we obtain \(d = 0\). Namely,

\[
\text{Im}[z_{2p-3}] = 0.
\]

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When \( i = 2p - k \) and \( i = 2p - (k + 1) \), we assume the following equalities hold:

\[
\mathrm{Im}[z_{2p-k}] = \mathrm{Im}[z_{2p-(k+1)}] = 0.
\]

Then, when \( i = 2p - (k + 2) \), since \( z_i = \frac{z_{i+1}}{z_{i+2}} + z_{i+1} + 1 \) holds, we obtain the following equality:

\[
\mathrm{Im}[z_{2p-(k+2)}] = 0.
\]

Hence, \( d = 0 \). Namely,

\[
\mathrm{Im}[z_i] = 0.
\]

We consider the contrapositive of the above statement. If \( \mathrm{Im}[z_i] \neq 0 \), then we obtain the following formula:

\[
\mathrm{Im}[z_{i+1}] \neq 0.
\]

Because of Lemma 2.13, we obtain \( \mathrm{Im}[a_i] \neq 0 \) (1 \( \leq i \leq 2p - 1 \)).

**Lemma 2.18.** \( \mathrm{Re}[a_i] \neq 1 \) (1 \( \leq i \leq 2p - 1 \)).

**Proof.** We prove that if \( \mathrm{Re}[z_{i+2}] = 1 \), then we obtain the following equalities:

\[
\mathrm{Re}[z_i] = 1 \quad \text{and} \quad \mathrm{Re}[z_{i+1}] = 1.
\]

Since

\[
\mathrm{Re}[z_{i+2}] = \mathrm{Re}\left[\frac{z_{i+1}}{z_{i+1} - z_i + 1}\right] = 1
\]

holds, we obtain the following formulas:

\[
\begin{align*}
\mathrm{Re}[z_i] & = 1 \quad (\mathrm{Im}[z_{i-1}] = \mathrm{Im}[z_i]), \\
\mathrm{Re}[z_i] & \neq 1 \quad (\mathrm{Im}[z_{i+1}] = \frac{\mathrm{Im}[z_i]^2 - \mathrm{Re}[z_i] \mathrm{Re}[z_{i+1}] + \mathrm{Re}[z_i]^2 - 2 \mathrm{Re}[z_i] + 1 + \mathrm{Re}[z_{i+1}]}{\mathrm{Im}[z_i]}), \\
\mathrm{Im}[z_i] & = 0 \quad (\mathrm{Re}[z_{i+1}] = \mathrm{Re}[z_i] - 1), \\
z_{i-1} & = 1 \quad \text{(otherwise)}.
\end{align*}
\]

Because of \( z_i \in \mathbb{C} \setminus [0, \infty) \) and Lemma 2.17, we only consider the following equality:

\[
\begin{align*}
\mathrm{Re}[z_i] & \neq 1 \quad \text{and} \quad \mathrm{Im}[z_{i+1}] = \frac{\mathrm{Im}[z_i]^2 - \mathrm{Re}[z_i] \mathrm{Re}[z_{i+1}] + \mathrm{Re}[z_i]^2 - 2 \mathrm{Re}[z_i] + 1 + \mathrm{Re}[z_{i+1}]}{\mathrm{Im}[z_i]}. \\
\end{align*}
\]

Let \( a, b, c, d, \) and \( e \) be real numbers, \( z_i = a + \sqrt{-1}b \), \( z_{i+1} = c + \sqrt{-1}d \), and \( z_{i+2} = 1 + \sqrt{-1}e \). Since

\[
z_{i+1} = \frac{(z_i - 1)z_{i+2}}{z_{i+2} - 1}
\]

hold, we obtain the following equalities:

\[
c = a + \frac{b}{e} - 1,
\]
\[ d = -a + \frac{be + 1}{c}, \]

and

\[ bd = b^2 - ac + a^2 - 2a + 1 + c. \]

We prove that this case does not hold by using mathematical induction on \( i \) (1 \( \leq \) \( i \) \( \leq \) 2\( p \) − 3, \( i \) \( \neq \) 2\( p \) − 4).

(i)-1. When \( i = 1 \), since

\[ z_2 = \frac{z_1^2}{z_1^2 - z_1 + 1} \]

holds, we obtain the following equalities:

\[ c = \frac{c^2 - c^3 + c^4 - d^2 - c d^2 + 2 c^2 d^2 + d^4}{1 - 2 c^3 + c^4 - d^2 + d^4 - 2 c (1 + d^2) + c^2 (3 + 2 d^2)} \]

and

\[ d = \frac{-d (-2 c + c^2 + d^2)}{1 - 2 c^3 + c^4 - d^2 + d^4 - 2 c (1 + d^2) + c^2 (3 + 2 d^2)}. \]

Hence \( a = 1 \), \( b = 0 \), \( d = 0 \), and \( e = 0 \). Namely,

\[ z_1 = z_3 = 1. \]

Since

\[ z_4 = \frac{(z_3 - 1) z_4}{z_4 - 1} \]

holds, we obtain the following equality:

\[ z_2 = \frac{(z_1 - 1) z_3}{z_3 - 1} = \frac{(z_3 - 1) z_3}{z_3 - 1} = z_3 = 1. \]

(i)-2. When \( i = k \) and \( i = k + 1 \), we assume the following equalities hold:

\[ z_k = z_{k+1} = 1. \]

Then, when \( i = k + 2 \), since

\[ z_i = \frac{z_{i-1}}{z_{i-1} - z_{i-2} + 1} \]

holds, we obtain the following equality:

\[ z_{k+2} = 1 \]

Hence, \( z_1 = z_2 = \cdots = z_{2p-2} = z_{2p-1} = 1 \). This is a contradiction. Therefore, this case does not hold.

(ii) When \( i = 2p - 4 \), since

\[ z_{2p-1} = z_{2p-2} - 1 \]

holds, we obtain the following equality:

\[ z_{2p-1} = 1 + \sqrt{-1} e - 1 = \sqrt{-1} e. \]
Hence,
\[ z_{2p-3} = c + \sqrt{-1}d = -\frac{1 + \sqrt{-1}e}{\sqrt{-1}e} + 1 + \sqrt{-1} = 1 + \sqrt{-1}\left(e + \frac{1}{e}\right). \]
Thus, \( \text{Re}[z_{2p-4}] = e^2/(e^2 + 1) \neq 1 \) and \( \text{Re}[z_{2p-3}] = 1 \).

Hence, \( \text{Re}[z_i] = 1 \) or \( \text{Re}[z_{2p-3}] = 1 \). Let \( a \) and \( b \) be real numbers, \( z_i = 1 + \sqrt{-1}a \), \( z_{i+2} = 1 + \sqrt{-1}b \).

Since
\[(1 - \frac{z_i}{z_{i+1}})(1 - \frac{1}{z_i}) = (1 - z_i)(1 - \frac{z_i - 1}{z_i}) \quad (3 \leq i \leq 2p - 5)\]
holds, we obtain the following equalities:
\[
\text{Re}[z_{i-2}] = \frac{1 + a^2(2 + b^2)}{(1 + a^2)(1 + b^2)} = 1,
\]
\[
\text{Re}[z_{i-1}] = \frac{ab^2 + 2a - b}{ab^2 + a},
\]
\[
z_{i+1} = \frac{a}{b} + \sqrt{-1}a,
\]
\[
\text{Re}[z_{i+3}] = \frac{b(b(2^2 + 2) - a(b^2 + 1))}{a^2(b^2 + 1) - 2a(b^2 + 2)b + b^2(b^2 + 4)}
\]
and
\[
\text{Re}[z_{i+4}] = \frac{1}{a^2(b^2 + 1) - 2a(b^2 + 2)b + b^4 + 3b^2 + 1} = 1.
\]
Hence, \( a = b \). Thus, \( \text{Re}[z_{i-1}] = \text{Re}[z_{i+1}] = \text{Re}[z_{i+3}] = 1 \). Therefore, \( \text{Re}[z_i] = 1 \) and \( \text{Re}[z_{i+1}] = 1 \) \((1 \leq i \leq 2p - 3)\).

We consider the contrapositive of the above statement. If \( \text{Re}[z_i] \neq 1 \) or \( \text{Re}[z_{i+1}] \neq 1 \), then we obtain the following formula:
\[ \text{Re}[z_{i+2}] \neq 1. \]

Because of Lemma 2.14, Lemma 2.15, and Lemma 2.16, \( \text{Re}[a_i] \neq 1 \) \((1 \leq i \leq 2p - 1)\).

**Lemma 2.19.** If the following limit value converges:
\[
\sum_{v_1, \ldots, v_{2p-1}=0}^{\infty} \frac{1}{v_1! \cdots (v_{2p-1})!} \frac{\partial^{v_1}f(a_1, \ldots, a_{2p-1})}{\partial z_1^{v_1} \cdots \partial z_{2p-1}^{v_{2p-1}}} (\exp(2\pi\sqrt{-1}n_1/N) - a_1)^{v_1} \cdots (\exp(2\pi\sqrt{-1}n_{2p-1}/N) - a_{2p-1})^{v_{2p-1}},
\]
then the above limit value is compactly uniformly and absolutely-convergent for all \( N \in \mathbb{N} \). Here, \(|v| := v_1 + \cdots + v_{2p-1}\).

**Proof.** We put the following equalities:
\[
\begin{align*}
x_i &:= \lim_{N \to \infty} |\text{Arg}(\exp(2\pi\sqrt{-1}n_i/N) - a_i)| \\
csc \frac{x}{2} &:= \max\{\csc\left(\frac{x_i}{2}\right) \mid 1 \leq i \leq 2p - 1\} \\
y_i &:= \lim_{N \to \infty} |\text{Arg}(\exp(2\pi\sqrt{-1}n_i/N) - a_i) - \frac{\pi}{2}| \\
csc \frac{y}{2} &:= \max\{\csc\left(\frac{y_i}{2}\right) \mid 1 \leq i \leq 2p - 1\}
\end{align*}
\]
Because of Lemma 2.17, \(x_i \neq 0\), \(x_i \neq \pi\), and \(\cos(x/2) \neq \infty\). Because of Lemma 2.18, \(y_i \neq 0\), \(y_i \neq \pi\), and \(\csc(y/2) \neq \infty\). There exists a real number \(C > 0\) such that

\[
\left| \frac{1}{(\nu_1)! \cdots (\nu_{2p-1})!} \frac{\partial^{[\nu]} f(a_1, \ldots, a_{2p-1})}{\partial z_1^{\nu_1} \cdots \partial z_{2p-1}^{\nu_{2p-1}}} \left( \exp(2\pi\sqrt{-1}n_1/N) - a_1 \right)^{\nu_1} \cdots \left( \exp(2\pi\sqrt{-1}n_{2p-1}/N) - a_{2p-1} \right)^{\nu_{2p-1}} - C \right| < C.
\]

We put \(r_i \exp(\sqrt{-1}\theta_i) := \exp(2\pi\sqrt{-1}n_i/N) - a_i\). Because of \(|\exp(\sqrt{-1}\theta_i)| = 1\), we obtain the following inequality:

\[
\left| \frac{1}{(\nu_1)! \cdots (\nu_{2p-1})!} \frac{\partial^{[\nu]} f(a_1, \ldots, a_{2p-1})}{\partial z_1^{\nu_1} \cdots \partial z_{2p-1}^{\nu_{2p-1}}} \left( \exp(2\pi\sqrt{-1}n_1/N) - a_1 \right)^{\nu_1} \cdots \left( \exp(2\pi\sqrt{-1}n_{2p-1}/N) - a_{2p-1} \right)^{\nu_{2p-1}} \right| < C|\exp(\sqrt{-1}\theta_1 \nu_1) \cdots \exp(\sqrt{-1}\theta_{2p-1} \nu_{2p-1})| \leq C|\Re\{\exp(\sqrt{-1}\theta_1 \nu_1) \cdots \exp(\sqrt{-1}\theta_{2p-1} \nu_{2p-1})\}| + C|\Im\{\exp(\sqrt{-1}\theta_1 \nu_1) \cdots \exp(\sqrt{-1}\theta_{2p-1} \nu_{2p-1})\}|.
\]

Moreover,

\[
\exp(\sqrt{-1}\theta_1 \nu_1) \cdots \exp(\sqrt{-1}\theta_{2p-1} \nu_{2p-1}) = \exp(\sqrt{-1}\sum_{i=1}^{2p-1} \theta_i \nu_i) = \cos\left(\sum_{i=1}^{2p-1} \theta_i \nu_i\right) + \sqrt{-1}\sin\left(\sum_{i=1}^{2p-1} \theta_i \nu_i\right).
\]

Hence,

\[
\left| \sum_{\nu_1, \ldots, \nu_{2p-1} = 0}^{\infty} \frac{1}{(\nu_1)! \cdots (\nu_{2p-1})!} \frac{\partial^{[\nu]} f(a_1, \ldots, a_{2p-1})}{\partial z_1^{\nu_1} \cdots \partial z_{2p-1}^{\nu_{2p-1}}} \left( \exp(2\pi\sqrt{-1}n_1/N) - a_1 \right)^{\nu_1} \cdots \left( \exp(2\pi\sqrt{-1}n_{2p-1}/N) - a_{2p-1} \right)^{\nu_{2p-1}} \right|
\]

\[
< \left| \sum_{\nu_1, \ldots, \nu_{2p-1} = 0}^{\infty} C(\cos\left(\sum_{i=1}^{2p-1} \theta_i \nu_i\right) + \sqrt{-1}\sin\left(\sum_{i=1}^{2p-1} \theta_i \nu_i\right)) \right|
\]

\[
\leq \left| C \sum_{\nu_1, \ldots, \nu_{2p-1} = 0}^{\infty} \cos\left(\sum_{i=1}^{2p-1} \theta_i \nu_i\right) \right| + \left| C \sum_{\nu_1, \ldots, \nu_{2p-1} = 0}^{\infty} \sin\left(\sum_{i=1}^{2p-1} \theta_i \nu_i\right) \right|
\]

\[
\leq \left( \sum_{\alpha_1, \ldots, \alpha_{2p-1} = 0}^{\infty} \cos\left(\sum_{i=1}^{2p-1} x_i \nu_i\right) \right) + \left( \sum_{\beta_1, \ldots, \beta_{2p-1} = 0}^{\infty} \sin\left(\sum_{i=1}^{2p-1} y_i \nu_i\right) \right.
\]

\[
(2)
\]

We prove converges the series (2) by using mathematical induction on \(i\) of \(\alpha_i\) and \(\beta_i\).

(i) When \(i = 1\), we obtain the following formulas:

\[
\lim_{\alpha_1 \to \infty} \left| \sum_{\nu_1 = 1}^{\alpha_1} \cos(x_1 \nu_1) \right| = \lim_{\alpha_1 \to \infty} \left| \csc\left(\frac{x_1}{2}\right) \cdot \sin\left(\frac{\alpha_1 x_1}{2}\right) \cdot \cos\left(\frac{x_1 (\alpha_1 + 1)}{2}\right) \right| \leq \left| \csc\left(\frac{x_1}{2}\right) \right| \leq \left| \csc\frac{x_1}{2} \right|,
\]

\[
\lim_{\alpha_1 \to \infty} \left| \sum_{\nu_1 = 1}^{\alpha_1} \sin(x_1 \nu_1) \right| = \lim_{\alpha_1 \to \infty} \left| \csc\left(\frac{x_1}{2}\right) \cdot \sin\left(\frac{\alpha_1 x_1}{2}\right) \cdot \sin\left(\frac{x_1 (\alpha_1 + 1)}{2}\right) \right| \leq \left| \csc\left(\frac{x_1}{2}\right) \right| \leq \left| \csc\frac{x_1}{2} \right|,
\]

\[
\lim_{\beta_1 \to \infty} \left| \sum_{\nu_1 = 1}^{\beta_1} \cos(y_1 \nu_1) \right| = \lim_{\beta_1 \to \infty} \left| \csc\left(\frac{y_1}{2}\right) \cdot \sin\left(\frac{\beta_1 y_1}{2}\right) \cdot \cos\left(\frac{y_1 (\beta_1 + 1)}{2}\right) \right| \leq \left| \csc\left(\frac{y_1}{2}\right) \right| \leq \left| \csc\frac{y_1}{2} \right|,
\]

\[
\lim_{\beta_1 \to \infty} \left| \sum_{\nu_1 = 1}^{\beta_1} \sin(y_1 \nu_1) \right| = \lim_{\beta_1 \to \infty} \left| \csc\left(\frac{y_1}{2}\right) \cdot \sin\left(\frac{\beta_1 y_1}{2}\right) \cdot \sin\left(\frac{y_1 (\beta_1 + 1)}{2}\right) \right| \leq \left| \csc\left(\frac{y_1}{2}\right) \right| \leq \left| \csc\frac{y_1}{2} \right|.
\]
Then, when \( \alpha = \lim_{\beta_1 \to \infty} \sum_{\nu_1 = 1}^{\beta_1} \sin(y_1 \nu_1) \), and
\[
\lim_{\beta_1 \to \infty} \left| \sum_{\nu_1 = 1}^{\beta_1} \sin(y_1 \nu_1) \right| = \lim_{\beta_1 \to \infty} \left| \csc \left( \frac{y_1}{2} \right) \cdot \sin \left( \frac{\beta_1 y_1}{2} \right) \cdot \sin \left( \frac{y_1(\beta_1 + 1)}{2} \right) \right| \leq \left| \csc \left( \frac{y_1}{2} \right) \right| \leq \left| \csc \frac{y}{2} \right|.
\]

(ii) When \( i = \ell \), we assume the following formulas hold:
\[
\left| \sum_{\nu_1, \ldots, \nu_\ell = 0}^{\alpha_1, \ldots, \alpha_\ell} \cos \left( \sum_{i=1}^{\ell} x_i \nu_i \right) \right| \leq 2^{\ell-1} \left| \csc \frac{x}{2} \right|, \\
\left| \sum_{\nu_1, \ldots, \nu_\ell = 0}^{\alpha_1, \ldots, \alpha_\ell} \sin \left( \sum_{i=1}^{\ell} x_i \nu_i \right) \right| \leq 2^{\ell-1} \left| \csc \frac{x}{2} \right|, \\
\left| \sum_{\nu_1, \ldots, \nu_\ell = 0}^{\alpha_1, \ldots, \alpha_\ell} \sin \left( \sum_{i=1}^{\ell} y_i \nu_i \right) \right| \leq 2^{\ell-1} \left| \csc \frac{y}{2} \right|,
\]
and
\[
\left| \sum_{\nu_1, \ldots, \nu_\ell = 0}^{\alpha_1, \ldots, \alpha_\ell} \cos \left( \sum_{i=1}^{\ell} y_i \nu_i \right) \right| \leq 2^{\ell-1} \left| \csc \frac{y}{2} \right|.
\]

Then, when \( i = \ell + 1 \), we obtain the following formulas:
\[
\lim_{(\alpha_1, \ldots, \alpha_{\ell+1}) \to \infty} \left| \sum_{\nu_1, \ldots, \nu_{\ell+1} = 0}^{\alpha_1, \ldots, \alpha_{\ell+1}} \cos \left( \sum_{i=1}^{\ell+1} x_i \nu_i \right) \right|
\]
\[
= \lim_{(\alpha_1, \ldots, \alpha_{\ell+1}) \to \infty} \left| \sum_{\nu_1, \ldots, \nu_{\ell+1} = 0}^{\alpha_1, \ldots, \alpha_{\ell+1}} \left( \cos \left( \sum_{i=1}^{\ell} x_i \nu_i \right) \cos \left( x_{\ell+1} \nu_{\ell+1} \right) - \sin \left( \sum_{i=1}^{\ell} x_i \nu_i \right) \sin \left( x_{\ell+1} \nu_{\ell+1} \right) \right) \right|
\]
\[
= \lim_{(\alpha_1, \ldots, \alpha_{\ell+1}) \to \infty} \left| \sum_{\nu_1, \ldots, \nu_{\ell+1} = 0}^{\alpha_1, \ldots, \alpha_{\ell+1}} \left( \cos \left( \sum_{i=1}^{\ell} x_i \nu_i \right) \cos \left( x_{\ell+1} \nu_{\ell+1} \right) - \sin \left( \sum_{i=1}^{\ell} x_i \nu_i \right) \sin \left( x_{\ell+1} \nu_{\ell+1} \right) \right) \right|
\]
\[
\leq \lim_{(\alpha_1, \ldots, \alpha_{\ell+1}) \to \infty} \left| \sum_{\nu_1, \ldots, \nu_{\ell+1} = 0}^{\alpha_1, \ldots, \alpha_{\ell+1}} \left( \sum_{i=1}^{\ell} x_i \nu_i \right) \sum_{\nu_{\ell+1} = 0}^{\alpha_{\ell+1}} \cos \left( x_{\ell+1} \nu_{\ell+1} \right) \right|
\]
\[
+ \left| \sum_{\nu_1, \ldots, \nu_{\ell+1} = 0}^{\alpha_1, \ldots, \alpha_{\ell+1}} \left( \sum_{i=1}^{\ell} \sin \left( x_i \nu_i \right) \right) \sum_{\nu_{\ell+1} = 0}^{\alpha_{\ell+1}} \sin \left( x_{\ell+1} \nu_{\ell+1} \right) \right|
\]
\[
\leq \lim_{(\alpha_1, \ldots, \alpha_{\ell+1}) \to \infty} \left( 2^{\ell-1} \left| \csc \frac{x}{2} \right| \left| \csc \frac{x_{\ell+1}}{2} \right| + 2^{\ell-1} \left| \csc \frac{x}{2} \right| \left| \csc \frac{x_{\ell+1}}{2} \right| \right)
\]
\[
\leq 2^\ell \left| \csc \frac{x}{2} \right| \left| \csc \frac{x_{\ell+1}}{2} \right| = 2^\ell \left| \csc \frac{x}{2} \right| \left| \csc \frac{x}{2} \right|,
\]
and
\[
\lim_{(\alpha_1, \ldots, \alpha_{\ell+1}) \to \infty} \left| \sum_{\nu_1, \ldots, \nu_{\ell+1} = 0}^{\alpha_1, \ldots, \alpha_{\ell+1}} \sin \left( \sum_{i=1}^{\ell+1} x_i \nu_i \right) \right|
\]
\[
= \lim_{(\alpha_1, \ldots, \alpha_{\ell+1}) \to \infty} \left| \sum_{\nu_1, \ldots, \nu_{\ell+1} = 0}^{\alpha_1, \ldots, \alpha_{\ell+1}} \left( \sin \left( \sum_{i=1}^{\ell} x_i \nu_i \right) \cos \left( x_{\ell+1} \nu_{\ell+1} \right) + \cos \left( \sum_{i=1}^{\ell} x_i \nu_i \right) \sin \left( x_{\ell+1} \nu_{\ell+1} \right) \right) \right|
\]
\[
= \lim_{(\alpha_1, \ldots, \alpha_{\ell+1}) \to \infty} \left| \sum_{\nu_1, \ldots, \nu_{\ell+1} = 0}^{\alpha_1, \ldots, \alpha_{\ell+1}} \left( \sin \left( \sum_{i=1}^{\ell} x_i \nu_i \right) \cos \left( x_{\ell+1} \nu_{\ell+1} \right) + \cos \left( \sum_{i=1}^{\ell} x_i \nu_i \right) \sin \left( x_{\ell+1} \nu_{\ell+1} \right) \right) \right|
\]
Thirdly, we prove that the following equality:

\[
\lim_{\alpha_1, \ldots, \alpha_{\ell+1} \to \infty} \left| \sum_{\nu_1, \ldots, \nu_{\ell+1}=0}^{\alpha_{\ell+1}} \left( \sum_{i=1}^{\ell} x_i \mu_i \cos(x_{\ell+1} \nu_{\ell+1}) + \cos(\sum_{i=1}^{\ell} x_i \mu_i) \sin(x_{\ell+1} \nu_{\ell+1}) \right) \right|
\]

\[
\leq \lim_{\alpha_1, \ldots, \alpha_{\ell+1} \to \infty} \left| \sum_{\nu_1, \ldots, \nu_{\ell+1}=0}^{\alpha_{\ell+1}} \sin(\sum_{i=1}^{\ell} x_i \mu_i) \cos(x_{\ell+1} \nu_{\ell+1}) \right|
\]

\[
+ \left| \sum_{\nu_1, \ldots, \nu_{\ell+1}=0}^{\alpha_{\ell+1}} \cos(\sum_{i=1}^{\ell} x_i \mu_i) \sin(x_{\ell+1} \nu_{\ell+1}) \right|
\]

\[
\leq \lim_{\alpha_1, \ldots, \alpha_{\ell+1} \to \infty} \left( 2^{\ell-1} \left| \csc \frac{x}{2} \right| \cos \left( \frac{x_{\ell+1}}{2} \right) + 2^{\ell-1} \left| \csc \frac{x}{2} \right| \cos \left( \frac{x_{\ell+1}}{2} \right) \right)
\]

\[
\leq 2^\ell \left| \csc \frac{x}{2} \right| \left| \csc \frac{x}{2} \right|^{\ell+1}
\]

Similarly, we obtain the following formulas:

\[
\lim_{\beta_1, \ldots, \beta_{\ell+1} \to \infty} \left| \sum_{\nu_1, \ldots, \nu_{\ell+1}=0}^{\beta_{\ell+1}} \cos(\sum_{i=1}^{\ell+1} y_i \beta_i) \right| \leq 2^\ell \left| \csc \frac{y}{2} \right|^{\ell+1},
\]

and

\[
\lim_{\beta_1, \ldots, \beta_{\ell+1} \to \infty} \left| \sum_{\nu_1, \ldots, \nu_{\ell+1}=0}^{\beta_{\ell+1}} \sin(\sum_{i=1}^{2p-1} y_i \beta_i) \right| \leq 2^\ell \left| \csc \frac{y}{2} \right|^{\ell+1}.
\]

Because of Lemma 2.8 and Lemma 2.19, rearrangements do not change the value of the sum. We obtain the following equality:

\[
\lim_{N \to \infty} f(\exp(2\pi \sqrt{-1} n_1/N), \ldots, \exp(2\pi \sqrt{-1} n_{2p-1}/N)) - f(a_1, \ldots, a_{2p-1})
\]

\[
= \lim_{N \to \infty} \sum_{i=1}^{2p-1} \left| \frac{\partial f}{\partial z_i} \right| \left( \exp(2\pi \sqrt{-1} n_i/N) - a_i \right) v_i
\]

\[
+ \lim_{N \to \infty} \sum_{\nu_1, \ldots, \nu_{2p-1}=0}^{\infty} \frac{1}{(v_1)^{\nu_1} \cdots (v_{2p-1})^{\nu_{2p-1}} \partial z_1^{\nu_1} \cdots \partial z_{2p-1}^{\nu_{2p-1}}} \times \left( \exp(2\pi \sqrt{-1} n_1/N) - a_1 \right)^{\nu_1} \cdots \left( \exp(2\pi \sqrt{-1} n_{2p-1}/N) - a_{2p-1} \right)^{\nu_{2p-1}}.
\]

Here, \(|\nu| = \nu_1 + \cdots + \nu_{2p-1} \geq 2\).

### 2.3 Third reduction

Thirdly, we prove the following equality:

\[
\lim_{N \to \infty} f(\exp(2\pi \sqrt{-1} n_1/N), \ldots, \exp(2\pi \sqrt{-1} n_{2p-1}/N)) = f(a_1, \ldots, a_{2p-1}).
\]

We set the simply connected domain of holomorphy \(D_i\) including the two points: \(a_i\) and \(\exp(2\pi \sqrt{-1} n_i/N)\) and let \(\overline{D}_i\) be \(\partial D_i \cup D_i\). For any \(1 \leq \ell \leq 2p-1\), \([0, \infty) \cap \overline{D}_i = \emptyset\). \(D := D_1 \times \cdots \times D_{2p-1}\) be the open polydisc, and \(\partial D' := \partial D_1 \times \cdots \times \partial D_{2p-1}\) be the special boundary. Then \(f(z_1, \ldots, z_{2p-1})\) satisfies the following two conditions.
• \( f(z_1, \ldots, z_{2p-1}) \) is continuous on \( D \).

• \( f(z_1, \ldots, z_{2p-1}) \) is holomorphic in each variable separately.

We obtain the following Lemma by using Hartogs’s Lemma, Rouché’s Theorem and Dominated Convergence Theorem.

**Lemma 2.20.** We obtain the following equality (3) holds:

\[
\lim_{N \to \infty} f(\exp(2\pi \sqrt{-1}n_1/N), \ldots, \exp(2\pi \sqrt{-1}n_{2p-1}/N)) = f(a_1, \ldots, a_{2p-1}) \in \mathbb{C}.
\]  

(3)

**Proof.** For any \( (z_1, \ldots, z_{2p-1}) \in D \),

\[
f(z_1, \ldots, z_{2p-1}) = \sum_{\nu_1, \ldots, \nu_{2p-1}=0}^{\infty} \frac{1}{(\nu_1)!(\nu_{2p-1})!} \frac{\partial^{\nu_1} f(a_1, \ldots, a_{2p-1})}{\partial z_1^{\nu_1}} \frac{\partial^{\nu_{2p-1}} f(a_1, \ldots, a_{2p-1})}{\partial z_{2p-1}^{\nu_{2p-1}}} (z_1 - a_1)^{\nu_1} \cdots (z_{2p-1} - a_{2p-1})^{\nu_{2p-1}}
\]

holds. Here \( |\nu| := \nu_1 + \cdots + \nu_{2p-1} \geq 2 \). We obtain the following equality:

\[
f(\exp(2\pi \sqrt{-1}n_1/N), \ldots, \exp(2\pi \sqrt{-1}n_{2p-1}/N))
= \frac{1}{(2\pi \sqrt{-1})^{2p-1}} \int_{\partial D'} (z_1 - \exp(2\pi \sqrt{-1}n_1/N)) \cdots (z_{2p-1} - \exp(2\pi \sqrt{-1}n_{2p-1}/N)) dz_1 \cdots dz_{2p-1}
= \frac{1}{(2\pi \sqrt{-1})^{2p-1}} \int_{\partial D'} (z_1 - \exp(2\pi \sqrt{-1}n_1/N)) \cdots (z_{2p-1} - \exp(2\pi \sqrt{-1}n_{2p-1}/N)) dz_1 \cdots dz_{2p-1}
+ \frac{1}{(2\pi \sqrt{-1})^{2p-1}} \int_{\partial D'} \frac{f(z_1, \ldots, z_{2p-1}) - f(a_1, \ldots, a_{2p-1})}{(z_1 - \exp(2\pi \sqrt{-1}n_1/N)) \cdots (z_{2p-1} - \exp(2\pi \sqrt{-1}n_{2p-1}/N))} dz_1 \cdots dz_{2p-1}
= f(a_1, \ldots, a_{2p-1})
+ \frac{1}{(2\pi \sqrt{-1})^{2p-1}} \int_{\partial D'} \frac{f(z_1, \ldots, z_{2p-1}) - f(a_1, \ldots, a_{2p-1})}{(z_1 - \exp(2\pi \sqrt{-1}n_1/N)) \cdots (z_{2p-1} - \exp(2\pi \sqrt{-1}n_{2p-1}/N))} dz_1 \cdots dz_{2p-1}
\]

We define the following functions:

\[
g(z_1, \ldots, z_{2p-1}) := \sum_{\nu_1, \ldots, \nu_{2p-1}=0}^{\infty} \frac{1}{(\nu_1)!(\nu_{2p-1})!} \frac{\partial^{\nu_1} f(a_1, \ldots, a_{2p-1})}{\partial z_1^{\nu_1}} \frac{\partial^{\nu_{2p-1}} f(a_1, \ldots, a_{2p-1})}{\partial z_{2p-1}^{\nu_{2p-1}}} \times (z_1 - a_1)^{\nu_1} \cdots (z_{2p-1} - a_{2p-1})^{\nu_{2p-1}},
\]

\[
h(z_1, \ldots, z_{2p-1}) := \frac{g(z_1, \ldots, z_{2p-1})}{(z_1 - \exp(2\pi \sqrt{-1}n_1/N)) \cdots (z_{2p-1} - \exp(2\pi \sqrt{-1}n_{2p-1}/N))}.
\]

Here, \( |\nu| \geq 2 \). There exist \( N_i \in \mathbb{N} \) such that for any \( N > N_i \),

\[
\exp(2\pi \sqrt{-1}n_i/N) \neq a_i \ (1 \leq i \leq 2p - 1),
\]

\[
h(a_1, \ldots, a_{2p-1}) = 0,
\]

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and an isolated zero \((a_1, \ldots, a_{2p-1}) \in \mathbb{C}^{2p-1}\) is one that has no other zeros close to it. Because of Hartogs’s Lemma, every isolated zero is removable. Hence, \(1/h(z_1, \ldots, z_{2p-1})\) is a holomorphic function on \(D\). Thus,

\[
\int_{\partial D'} \frac{1}{h(z_1, \ldots, z_{2p-1})} dz_1 \cdots dz_{2p-1} = 0.
\]

Namely,

\[
\int_{\partial D'} \frac{(z_1 - \exp(2\pi \sqrt{-1} n_1/N)) \cdots (z_{2p-1} - \exp(2\pi \sqrt{-1} n_{2p-1}/N))}{g(z_1, \ldots, z_{2p-1})} dz_1 \cdots dz_{2p-1} = 0.
\]

We define the following function:

\[
i(z_1, \ldots, z_{2p-1}) = \frac{1}{h(z_1, \ldots, z_{2p-1})} + \sum_{i=1}^{2p-1} \frac{z_i - \exp(2\pi \sqrt{-1} n_i/N)}{N \cdot g(z_1, \ldots, z_{2p-1})}\]

\[
= \frac{(z_1 - \exp(2\pi \sqrt{-1} n_1/N)) \cdots (z_{2p-1} - \exp(2\pi \sqrt{-1} n_{2p-1}/N))}{g(z_1, \ldots, z_{2p-1})} + \sum_{i=1}^{2p-1} \frac{z_i - \exp(2\pi \sqrt{-1} n_i/N)}{N \cdot g(z_1, \ldots, z_{2p-1})}.
\]

There exist \(N_j \in \mathbb{N}\) such that for any \(z_j \in \partial D_j\),

\[
\left| \frac{z_j - \exp(2\pi \sqrt{-1} n_j/N_j)}{N_j} \right| < \left| \frac{g(z_1, \ldots, z_j, \ldots, z_{2p-1})}{h(z_1, \ldots, z_j, \ldots, z_{2p-1})} \right| \quad (1 \leq j \leq 2p - 1).
\]

Because of Rouche’s Theorem, \(g(z_1, \ldots, z_{2p-1})/h(z_1, \ldots, z_{2p-1})\) and \(g(z_1, \ldots, z_{2p-1}) \cdot i(z_1, \ldots, z_{2p-1})\) have the same number of zeros inside \(D_j\). It is only \(z_j = \exp(2\pi \sqrt{-1} n_j/N_j)\). Namely,

\[
i(\exp(2\pi \sqrt{-1} n_1/N_1), \ldots, \exp(2\pi \sqrt{-1} n_{2p-1}/N_{2p-1})) = 0.
\]

Since \(\partial D_k\) is a compact space and \([0, \infty) \cap D_k = \emptyset\), there exists a real number \(C_k > 0\) such that for any \(z_k \in \partial D_k\),

\[
0 < C_k \leq |z_k - 1|.
\]

Also, since

\[
\lim_{N \to \infty} \exp(2\pi \sqrt{-1} n_k/N) = 1
\]

holds, for any \(0 < \varepsilon_k < C_k\), there exist \(N_k \in \mathbb{N}\) such that for any \(N > N_k\),

\[
|\exp(2\pi \sqrt{-1} n_k/N) - 1| < C_k - \varepsilon_k.
\]

Hence, for any \(N > \max\{N_i, N_j, N_k \mid 1 \leq i, j, k \leq 2p - 1\},\)

\[
1 \leq \frac{\prod_{k=1}^{2p-1} |z_k - 1| - |\exp(2\pi \sqrt{-1} n_k/N_k)) - 1|}{\prod_{k=1}^{2p-1} |z_k - 1| - (C_k - \varepsilon_k)}.
\]
We put $N' = \max\{N_i, N_j, N_k \mid 1 \leq i, j, k \leq 2p - 1\}$ and $\varepsilon = \min\{\varepsilon_k \mid 1 \leq k \leq 2p - 1\}$. Since $\partial D'$ is a compact space, there exists a real number $C > 0$ such that for any $(z_1, \ldots, z_{2p-1}) \in \partial D'$ and for any $N > N'$,

$$\frac{1}{|i(z_1, \ldots, z_{2p-1})|} \leq \frac{|g(z_1, \ldots, z_{2p-1})|}{\prod_{k=1}^{2p-1} \varepsilon_k} < \frac{|g(z_1, \ldots, z_{2p-1})|}{\varepsilon^{2p-1}} \leq C.$$ 

Hence, $1/i(z_1, \ldots, z_{2p-1})$ is uniformly bounded on $\partial D'$. Because of Dominated Convergence Theorem (DCT), we obtain the following equality:

$$\lim_{N \to \infty} \int_{\partial D'} \frac{1}{i(z_1, \ldots, z_{2p-1})} dz_1 \cdots dz_{2p-1} = \int_{\partial D'} \lim_{N \to \infty} \frac{1}{i(z_1, \ldots, z_{2p-1})} dz_1 \cdots dz_{2p-1}$$

$$= \int_{\partial D'} \lim_{N \to \infty} \frac{g(z_1, \ldots, z_2)}{(z_1 - \text{exp}(2\pi \sqrt{-1} n_1/N)) \cdots (z_{2p-1} - \text{exp}(2\pi \sqrt{-1} n_{2p-1}/N)) + \sum_{i=1}^{2p-1} (z_i - \text{exp}(2\pi \sqrt{-1} n_i/N))/N} dz_1 \cdots dz_{2p-1}$$

$$= \int_{\partial D'} \frac{g(z_1, \ldots, z_2)}{(z_1 - 1) \cdots (z_{2p-1} - 1)} dz_1 \cdots dz_{2p-1}$$

$$= 0.$$

Moreover,

$$\int_{\partial D'} h(z_1, \ldots, z_{2p-1}) dz_1 \cdots dz_{2p-1} \sim \int_{\partial D'} \frac{1}{i(z_1, \ldots, z_{2p-1})} dz_1 \cdots dz_{2p-1}.$$

Because of Lemma 2.8 and Lemma 2.19, we obtain the following formula:

$$\left| \lim_{N \to \infty} f(\text{exp}(2\pi \sqrt{-1} n_1/N), \ldots, \text{exp}(2\pi \sqrt{-1} n_{2p-1}/N)) - f(a_1, \ldots, a_{2p-1}) \right|$$

$$= \lim_{N \to \infty} \sum_{\nu_1, \ldots, \nu_{2p-1} = 0}^{\infty} \sum_{(\nu_1, \ldots, \nu_{2p-1}) \neq (0, \ldots, 0)} \frac{1}{(\nu_1)! \cdots (\nu_{2p-1})!} \frac{\partial^{\nu_1} f(a_1, \ldots, a_{2p-1})}{\partial z_1^{\nu_1} \cdots \partial z_{2p-1}^{\nu_{2p-1}}}$$

$$\times \left( \text{exp}(2\pi \sqrt{-1} n_1/N - a_1)^{\nu_1} \cdots (\text{exp}(2\pi \sqrt{-1} n_{2p-1}/N - a_{2p-1})^{\nu_{2p-1}} \right)$$

$$\times \left| \nu \right| \geq 2$$

$$= \lim_{N \to \infty} \int_{\partial D'} \frac{1}{(2\pi \sqrt{-1})^{2p-1}} h(z_1, \ldots, z_{2p-1}) dz_1 \cdots dz_{2p-1}$$

$$= \lim_{N \to \infty} \int_{\partial D'} \frac{1}{(2\pi \sqrt{-1})^{2p-1}} h(z_1, \ldots, z_{2p-1}) dz_1 \cdots dz_{2p-1}$$

$$= \lim_{N \to \infty} \int_{\partial D'} \frac{1}{(2\pi \sqrt{-1})^{2p-1}} i(z_1, \ldots, z_{2p-1}) dz_1 \cdots dz_{2p-1}$$

$$= \lim_{N \to \infty} \int_{\partial D'} \frac{1}{(2\pi \sqrt{-1})^{2p-1}} i(z_1, \ldots, z_{2p-1}) dz_1 \cdots dz_{2p-1}$$

$$= 0.$$

Therefore, we obtain Equation (3). □

3 Proof of Volume Conjecture for twist knots

In this section, we prove Volume Conjecture for twist knots by using Lemmas of the previous Section.
Lemma 3.1. With the same notation as in Theorem 2.1. If
\[
\frac{2\pi \log |\psi(N)|}{N} \to 0 \quad (N \to \infty),
\]
then we have that
\[
\frac{2\pi}{N} \log \left( |\psi(N)| |(q)_{n_{2p-1}} \prod_{i=1}^{2p-2} \left[ \frac{n_{i+1}}{n_i} \right] \right) \sim \text{Im} \left[ f(2\pi \sqrt{-1} m_{n_{2p-1}} / N) \right].
\]

Proof. We obtain the following formula:
\[
\frac{2\pi}{N} \log \left( |\psi(N)| |(q)_{n_{2p-1}} \prod_{i=1}^{2p-2} \left[ \frac{n_{i+1}}{n_i} \right] \right) = \frac{2\pi}{N} \left( \log |\psi(N)| + |(q)_{n_{2p-1}}| + \sum_{i=1}^{2p-2} \log \left[ \frac{n_{i+1}}{n_i} \right] \right)
\]
\[
\sim \frac{2p-2}{N} \text{Im} \left[ \text{Li}_2(2\pi \sqrt{-1} (n_{i+1} - n_i) / N) \right] - \text{Li}_2(2\pi \sqrt{-1} n_{2p-1} / N)
\]
\[
+ \text{Li}_2(2\pi \sqrt{-1} (n_{i+1} - n_i) / N)) + \text{Im} \left[ -\text{Li}_2(2\pi \sqrt{-1} n_{2p-1} / N) \right]
\]
\[
= \text{Im} \left[ f(z_1^{(1)}, \ldots, z_{2p-1}^{(2p-1)}) \right]
\]
\[
\sim \text{Im} \left[ f(2\pi \sqrt{-1} n_{2p-1} / N) \right].
\]

Here, Equivalent relation (4) is shown by using Lemma 2.9 and Corollary 2.10. Equation (5) is shown by
using Lemma 2.11. \[\square\]

Lemma 3.2. With the same notation as in Theorem 2.1 we have that
\[
\lim_{N \to \infty} \frac{2\pi \log |J_n(K_{p>0}:q)|}{N} \leq \text{Im} \left[ f(a_1, \ldots, a_{2p-1}) \right].
\]

Proof. We obtain the following inequality:
\[
\frac{2\pi \log |J_n(K_{p>0}:q)|}{N} \leq \frac{2\pi}{N} \log \sum_{N-1 \geq m_{2p-1} \geq \cdots \geq m_1 \geq 0} |(q)_{m_{2p-1}} | \prod_{i=1}^{2p-2} \left[ \frac{m_{i+1}}{m_i} \right].
\]

When \(N-1 \geq m_{2p-1} \geq \cdots \geq m_1 \geq 0\), because of [CK],
\[
\Gamma_q(z+1) = \frac{1-q^z}{1-q} \Gamma_q(z),
\]

and
\[
\left[ \begin{array}{c} m_{2p-1} + 1 \\ s \end{array} \right] = \left[ \begin{array}{c} m_{2p-1} \\ s - 1 \end{array} \right] + q^s \left[ \begin{array}{c} m_{2p-1} \\ s \end{array} \right]
\]

hold. By using the above equalities, we obtain the following formulas:
In case of \(m_1\),
\[
\left| \left[ \begin{array}{c} m_2 \\ m_1 + 1 \\ m_1 \end{array} \right] \right| = \left| \frac{(q)_{m_2-m_1}(q)_{m_1}}{\Gamma_q(m_2-(m_1+1))} \right| \leq 1.
\]
Hence,
\[ m_1 = \frac{m_2 \log(1 - q) - \log(1 - q) - \sqrt{-1} \pi}{2 \log(1 - q)} \rightarrow \frac{m_2 - 1}{2} \quad (N \rightarrow \infty). \]

In fact, when we substitute \( m_1 = (m_2 - 1)/2 \) for \(|(1 - q)^{m_2 - 2m_1 - 1}|\), we obtain the following equality:
\[ |(1 - q)^{m_2 - (m_2 - 1) - 1}| = 1. \]

In case of \( m_2, \ldots, m_{2p-2} \),
\[
\begin{bmatrix}
  m_{i+1} \\
  m_i + 1 \\
  m_i \end{bmatrix}
\begin{bmatrix}
  m_i + 1 \\
  m_{i-1} \\
  m_i \\
  m_{i+1} \\
  m_i \\
  m_{i-1} \\
\end{bmatrix}
= \left| (q)_{m_{i+1}-m_i} (q)_{m_i-m_{i-1}} \right| = |(1 - q)^{m_{i+1}-2m_i+m_{i-1}-1}| \leq 1.
\]

Hence,
\[
m_i = \frac{2m_{i+1} \log 2 + 2m_{i-1} \log 2 + m_{i+1} \log(\sin^2 \frac{x}{2}) + m_{i-1} \log(\sin^2 \frac{x}{2}) - 2 \log(2 \sqrt{\sin^2 \frac{x}{2}})}{4 \log 2 + 2 \log(\sin^2 \frac{x}{2})} \rightarrow \frac{m_{i+1} + m_{i-1} - 1}{2} \quad (N \rightarrow \infty).
\]

In fact, when we substitute \( m_i = (m_{i+1} + m_{i-1} - 1)/2 \) for \(|(1 - q)^{m_{i+1}-2m_i+m_{i-1}-1}|\), we obtain the following equality:
\[ |(1 - q)^{m_{i+1}-(m_{i+1}+m_{i-1}+1)+m_{i-1}-1}| = 1. \]

In case of \( m_{2p-1} \),
\[
[q]_x = \frac{(-2\pi \sqrt{-1})^x \Gamma(x+1)}{N^x} + O\left(\frac{1}{N^{x+1}}\right) \quad (0 \leq x).
\]

Hence, \(|(q)_x|\) is a monotonically decreasing function of \( x \). Hence, \( x = 0 \) is a maximal value of \(|(q)_0| = 1\).

Moreover,
\[
\begin{bmatrix}
  m_{2p-1} + 1 \\
  m_{2p-2} \\
  m_{2p-1} \end{bmatrix}
\begin{bmatrix}
  m_{2p-2} \\
  m_{2p-1} - 1 \\
  m_{2p-2} \\
\end{bmatrix}
= \left| q^{m_{2p-2}} + \frac{m_{2p-2} - 1}{m_{2p-1}} \right| = \left| q^{m_{2p-2}} + \frac{1-q^{m_{2p-2}}}{1-q^{m_{2p-1}-m_{2p-2}+1}} \right| \leq 1.
\]

Hence,
\[
m_{2p-1} = \frac{2(\pi m_{2p-2} - \pi) + \sqrt{-1} \log \frac{1+q}{2}}{2\pi} \rightarrow \frac{m_{2p-2} - 1}{2} \quad (N \rightarrow \infty).
\]

In fact, when we substitute \( m_{2p-1} = m_{2p-2}/2 - 1 \) for \(|q^{m_{2p-2}} + (1-q^{m_{2p-2}})/(1-q^{m_{2p-1}-m_{2p-2}+1})|\), we obtain the following equality:
\[
\left| q^{m_{2p-2}} + \frac{1-q^{m_{2p-2}}}{1-q^{m_{2p-1}-m_{2p-2}+1}} \right| = \left| \frac{1-q^{m_{2p-2}}}{1-q^{m_{2p-1}-m_{2p-2}+1}} \right| = 1.
\]

Therefore, there exist the following positive real sequence \( n_i \in \mathbb{N} \quad (1 \leq i \leq 2p - 1):\)
\[ n_2 = 2n_1 + 1, \quad n_{i+1} = 2n_i - n_{i-1} + 1 \quad (i = 2, \ldots, 2p - 3), \quad \text{and} \quad n_{2p-1} = \frac{n_{2p-2}}{2} - 1 \quad (6) \]
such that for any \((m_1, \ldots, m_{2p-1}) \in \mathbb{N}^{2p-1}\),
\[
|\langle q \rangle_{m_{2p-1}} \prod_{i=1}^{2p-2} \left[ m_{i+1} \right] |  
\leq |\langle q \rangle_0 \prod_{i=1}^{2p-2} \left[ n_{i+1} \right] |.
\]
Hence, we obtain the following formula:
\[
\frac{2\pi \log |J_n(K_p^*; q)|}{N} \leq \frac{2\pi}{N} \log \left( \sum_{N-1 \geq m_{2p-1} \geq \cdots \geq m_1 \geq 0} |\langle q \rangle_{m_{2p-1}} \prod_{i=1}^{2p-2} \left[ m_{i+1} \right] | \right)
\]
\[
\leq \frac{2\pi}{N} \log \left( \log N^{2p-1} + \log |\langle q \rangle_0 | + \sum_{i=1}^{2p-2} \log |\prod_{i=1}^{n_{i+1}} | \right)
\]
\[
\sim \text{Im} \left[ f(z_1^{(1)}, \ldots, z_{2p-1}^{(2p-1)}) + \text{Li}_2(z_{2p-1}^{(2p-1)}) \right] \quad (7)
\]
\[
\sim \text{Im} \left[ f(\exp(2\pi\sqrt{-1}m_1/N), \ldots, \exp(2\pi\sqrt{-1}m_{2p-1}/N)) \right] \quad (8)
\]
\[
\rightarrow \text{Im} \left[ f(a_1, \ldots, a_{2p-1}) \right] \quad (N \to \infty). \quad (9)
\]
Here, Equivalent relation (8) is shown by using Lemma 2.11 and Limit value (9) is shown by using Equation (3).

**Lemma 3.3.** Let \(\{a_s\}\) be a complex sequence. For any \(s\) and \(t\),
\[
2|a_s| - |a_t| \leq \sum_{u=0}^{N-1} a_u
\]
holds.

**Proof.** Because of triangle inequality, we obtain the following inequality:
\[
|a_s| = |a_s + \sum_{u=0, u \neq s}^{N-1} a_u - \sum_{u=0, u \neq s}^{N-1} a_u| \leq |a_s + \sum_{u=0, u \neq s}^{N-1} a_u| + | - \sum_{u=0, u \neq s}^{N-1} a_u| \leq | \sum_{u=0}^{N-1} a_u | + \sum_{u=0}^{N-1} | a_u |.
\]
Hence,
\[
|a_s| - \sum_{u=0, u \neq s}^{N-1} | a_u | \leq | \sum_{u=0}^{N-1} a_u |.
\]
Similarly,
\[
-|a_t| + \sum_{u=0, u \neq t}^{N-1} a_u \geq - | \sum_{u=0}^{N-1} a_u |.
\]
Hence,
\[
- | \sum_{u=0}^{N-1} a_u | \leq 2(|a_s| - |a_t|) \leq | \sum_{u=0}^{N-1} a_u |.
\]
Hence, we obtain the claim. \(\square\)
Lemma 3.4. With the same notation as in Theorem 2.1 we have that

\[ |J_N(K^*_p > 0; q)| \geq 2^{2p-1} |(q)_0| \prod_{i=1}^{2p-2} \left\lfloor \frac{n_{i+1}}{n_i} \right\rfloor - 2N^{2p-2}. \]

Proof. Since, for any \( m_i, m_{i+1} \in \mathbb{N} \),

\[ \left\lfloor \frac{m_{i+1}}{m_i} \right\rfloor = 0 \quad (m_{i+1} < m_i) \]

holds, we obtain the following equality:

\[ q^{1-N} \sum_{N-1 > m_1 > \cdots > m_{2p-1} > 0} (q^{1-N})_{m_{2p-1}} q^{-N m_{2p-1}} \prod_{i=1}^{2p-2} (-1)^{m_i} q^{-N m_i + \binom{m_i}{2} - m_{i+1}} \left\lfloor \frac{m_{i+1}}{m_i} \right\rfloor = 0. \]

Hence,

\[ J_N(K^*_p > 0; q) \]

\[ = q^{1-N} \sum_{N-1 > m_1 > \cdots > m_{2p-1} > 0} (q^{1-N})_{m_{2p-1}} q^{-N m_{2p-1}} \prod_{i=1}^{2p-2} (-1)^{m_i} q^{-N m_i + \binom{m_i}{2} - m_{i+1}} \left\lfloor \frac{m_{i+1}}{m_i} \right\rfloor \]

\[ + q^{1-N} \sum_{N-1 > m_1 > \cdots > m_{2p-1} > 0} (q^{1-N})_{m_{2p-1}} q^{-N m_{2p-1}} \prod_{i=1}^{2p-2} (-1)^{m_i} q^{-N m_i + \binom{m_i}{2} - m_{i+1}} \left\lfloor \frac{m_{i+1}}{m_i} \right\rfloor \]

\[ = q^{1-N} \sum_{m_{2p-1}, \ldots, m_1 = 0}^{N-1} (q^{1-N})_{m_{2p-1}} q^{-N m_{2p-1}} \prod_{i=1}^{2p-2} (-1)^{m_i} q^{-N m_i + \binom{m_i}{2} - m_{i+1}} \left\lfloor \frac{m_{i+1}}{m_i} \right\rfloor. \]

We define the following sequence:

\[ j(m_{2p-1}, \ldots, m_1) := \sum_{m_{2p-1}, \ldots, m_1 = 0}^{N-1} (q^{1-N})_{m_{2p-1}} q^{-N m_{2p-1}} \prod_{i=1}^{2p-2} (-1)^{m_i} q^{-N m_i + \binom{m_i}{2} - m_{i+1}} \left\lfloor \frac{m_{i+1}}{m_i} \right\rfloor. \]

Because of Lemma 3.3 and Sequence (6), there exist the following positive real sequence \( n_i \in \mathbb{N} \) (1 \( \leq \) \( i \) \( \leq \) \( 2p-1 \)):

\[ n_2 = 2n_1 + 1, \quad n_{i+1} = 2n_i - n_{i-1} + 1 \quad (i = 2, \ldots, 2p-3), \quad \text{and} \quad n_{2p-1} = \frac{n_{2p-2}}{2} - 1 \]

such that for any \( (m_1, \ldots, m_{2p-1}) \in \mathbb{N}^{2p-1} \),

\[ |(q)_{m_{2p-1}} \prod_{i=1}^{2p-2} \left\lfloor \frac{m_{i+1}}{m_i} \right\rfloor| \leq |(q)_0| \prod_{i=1}^{2p-2} \left\lfloor \frac{n_{i+1}}{n_i} \right\rfloor. \]

Moreover, since for any \( m_i, m_{i+1} \in \mathbb{N} \),

\[ \left\lfloor \frac{m_{i+1}}{m_i} \right\rfloor = 0 \quad (m_{i+1} < m_i) \]

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holds, we obtain the following equality: for any \( 1 \leq i \leq 2p - 2 \),
\[
j(m_{2p-1}, \ldots, m_{i+1}, m_i, \ldots, m_1) = 0 \quad (m_{i+1} < m_i).
\]
Hence, we obtain the following formula:
\[
\left| \sum_{m_{2p-1}, \ldots, m_1 = 0}^{N-1} j(m_{2p-1}, \ldots, m_1) \right| \geq 2 \left| \sum_{m_{2p-2}, m_1 = 0}^{N-1} j(n_{2p-1}, m_{2p-2}, \ldots, m_1) \right| - \left| \sum_{m_{2p-2}, m_1 = 0}^{N-1} j(0, m_{2p-2}, \ldots, m_1) \right| \\
\geq 2 \left( \left| \sum_{m_{2p-2}, m_1 = 0}^{N-1} j(n_{2p-1}, m_{2p-2}, \ldots, m_1) \right| - \sum_{m_{2p-2}, m_1 = 0}^{N-1} \right) \\
= 2 \left( \left| \sum_{m_{2p-2}, m_1 = 0}^{N-1} j(n_{2p-1}, m_{2p-2}, \ldots, m_1) \right| - N^{2p-2} \right) \\
\geq 2 \left( 2 \left| \sum_{m_{2p-3}, m_1 = 0}^{N-1} j(n_{2p-1}, n_{2p-2}, \ldots, m_1) \right| \\
\quad - \left| \sum_{m_{2p-3}, m_1 = 0}^{N-1} j(n_{2p-1}, n_{2p-2} + 1, \ldots, m_1) \right| - N^{2p-2} \right) \\
= 2 \left( 2 \left| \sum_{m_{2p-3}, m_1 = 0}^{N-1} j(n_{2p-1}, n_{2p-2}, \ldots, m_1) \right| - N^{2p-2} \right) \\
= 4 \left| \sum_{m_{2p-3}, m_1 = 0}^{N-1} j(n_{2p-1}, n_{2p-2}, \ldots, m_1) \right| - 2N^{2p-2}.
\]
We repeat the above calculation \( 2p - 3 \) times. Then, we obtain the following inequality:
\[
\left| \sum_{m_{2p-1}, m_{2p-2}, m_1 = 0}^{N-1} j(m_{2p-1}, m_{2p-2}, \ldots, m_1) \right| \geq 2^{2p-1} |j(n_{2p-1}, n_{2p-2}, n_{2p-3}, \ldots, m_1)| - 2N^{2p-2}.
\]
Hence,
\[
|J_N(K^+_p; q)| \geq 2^{2p-1} |(q)_0| \prod_{i=1}^{2p-2} \left| \binom{n_{i+1}}{n_i} \right| - 2N^{2p-2}.
\]

\[ \square \]

**Lemma 3.5.** With the same notation as in Theorem 2.1 we have that

\[
\text{Im}[f(a_1, \ldots, a_{2p-1})] \leq \lim_{N \to \infty} \frac{2\pi \log |J_N(K^+_p; q)|}{N}
\]

**Proof.** Because of Lemma 3.4, for a sufficiently large \( N \), we obtain the following formula:
\[
2^{2p-1} |(q)_0| \prod_{i=1}^{2p-2} \left| \binom{n_{i+1}}{n_i} \right| \leq |J_N(K^+_p; q)| + 2N^{2p-2}.
\]

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Hence, we obtain the following formula:

\[
\frac{2\pi \log(|J_N(K_{p>0}^*; q)| + 2N^{2p-2})}{N} \geq \frac{2\pi}{N} \log \left( 2^{2p-1} |(q)_0| \prod_{i=1}^{2p-2} \left\lfloor \frac{n_{i+1}}{n_i} \right\rfloor \right) \quad (10)
\]

\[
= \frac{2\pi}{N} \left( \log 2^{2p-1} + \log |(q)_0| + \sum_{i=1}^{2p-2} \left\lfloor \frac{n_{i+1}}{n_i} \right\rfloor \right) \geq \frac{2\pi}{N} \left( \log 2^{2p-1} + \log |(q)_0| + \log |(q)_{2p-1}| + \sum_{i=1}^{2p-2} \log \left\lfloor \frac{n_{i+1}}{n_i} \right\rfloor \right)
\]

\[
\sim \text{Im}[f(z_1^{(1)}, \ldots, z_{2p-1}^{(2p-1)})] 
\sim \text{Im}[f(\exp(2\pi \sqrt{-1}n_1/N), \ldots, \exp(2\pi \sqrt{-1}n_{2p-1}/N))] 
\rightarrow \text{Im}[f(a_1, \ldots, a_{2p-1})] \quad (N \to \infty). 
\quad (11)
\]

Here, Inequality (10) is shown by using Lemma 3.4. Equivalent relation (11) is shown by using Lemma 3.1. Limit value (12) is shown by using Equation (3). Moreover, we obtain the following relation:

\[
\frac{2\pi \log(|J_N(K_{p>0}^*; q)| + 2N^{2p-2})}{N} \sim \frac{2\pi}{N} \log |J_N(K_{p>0}^*; q)|.
\]

Therefore,

\[
\text{Im}[f(a_1, \ldots, a_{2p-1})] \leq \lim_{N \to \infty} \frac{2\pi}{N} \log |J_N(K_{p>0}^*; q)|.
\]

**Theorem 3.6.** Theorem 1.2 is true.

*Proof.* Because of Lemma 2.3, Lemma 3.2, and Lemma 3.5, we obtain Theorem 1.2. \qed

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