PERIODIC FORCING ON DEGENERATE HOPF BIFURCATION

QIGANG YUAN AND JINGLI REN*

Henan Academy of Big Data
Zhengzhou University
Zhengzhou 450001, China

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Abstract. This paper is devoted to the effect of periodic forcing on a system exhibiting a degenerate Hopf bifurcation. Two methods are employed to investigate bifurcations of periodic solution for the periodically forced system. It is obtained by averaging method that the system undergoes fold bifurcation, transcritical bifurcation, and even degenerate Hopf bifurcation of periodic solution. On the other hand, it is also shown by Poincaré map that the system will undergo fold bifurcation, transcritical bifurcation, Neimark-Sacker bifurcation and flip bifurcation. Finally, we make a comparison between these two methods.

1. Introduction. Periodic variation is a widespread phenomenon, observed across a multitude of taxa in both laboratory and natural conditions. It can lead to some unpredictable results in non-intuitive ways, including cyclic behaviour [1], different oscillations [20] and even chaos [4, 25]. Mathematical models are frequently used to study the effect of periodic variation in various fields, such as in chemistry [17], physics [10, 6, 7], material science [27] and even population dynamics [16, 22].

The study of periodically forced differential systems can be traced back to 1980. Rosenblat and Cohen claimed that constant solutions in unforced system are transformed into periodic solutions in periodically forced system [23], and periodic solutions in unforced system are modified into quasiperiodic solutions in periodically forced system [24]. Many studies show that a systems undergoing Hopf bifurcation can appear complex dynamics when it was periodically forced, due to the interaction of Hopf frequency and external forcing frequency [2, 13, 14, 15, 18, 30]. In brief, the ratio of the natural Hopf frequency $\omega_c$ to the forcing frequency $\omega_0$ has a powerful influence on dynamics of the periodically forced system. Gambaudo [13] studied the Poincaré map of a periodically forced system in non-resonant and weakly resonant case between the two frequencies. In [2, 14], authors considered the solutions and their stability in periodically forced system in a harmonic resonance case where $\omega_0 \approx \omega_c$. Namachchiya and Ariaratnam [18] studied bifurcations of solutions in a subharmonic resonance case where $\omega_0 = 2\omega_c$. Zhang and Golubitsky [30] examined the influence of the forcing frequency $\omega_0$ on the number of periodic solutions when the unforced system undergoes supercritical Hopf bifurcation. Moreover, Kuznetsov et al. [15] discovered chaotic attractors in a periodically forced system.
predator-prey system, which were generated through two ways: torus destruction and period doubling. Recently, Li et al. [16] find periodical forcing can also lead to multiple attractors and different bifurcations of periodic solutions.

The previous works all focus on the periodic forcing on system exhibiting generic Hopf bifurcation. Naturally, a new question arise: how does the periodic forcing affect degenerate Hopf bifurcation. The problem is more complex in view of the fact that there will be at least two limit cycle in system undergoing degenerate Hopf Bifurcation. Moreover, the number and stability of the limit cycle of degenerate Hopf Bifurcation are also a significant topic, which is closely related to the Hilbert’s 16th problem [9, 29].

Motivated by this, we investigate the effect of periodic forcing on degenerate Hopf bifurcation by a planar system

\[
\begin{cases}
  \dot{x} = r_1(x - a_1 x^2 - b_1 xy), \\
  \dot{y} = r_2(y - a_2 y^2 - b_2 xy)
\end{cases}
\]  

(1)

where \(r_i, a_i, \) and \(b_i (i = 1, 2)\) are the parameters. This system can be interpreted as a competition model [28] or an information diffusion model of intervention type [21] when the parameters are positive. In this paper, we are concerned with the theoretical analysis on periodic forcing without emphasizing the biological significance of model (1). One effective way to represent periodic forcing is to study the system with periodically varying parameters, i.e.,

\[
a_1(t) = a_1(1 + \epsilon \sin \omega_0 t).
\]  

(2)

where \(\omega_0 (\omega_0 > 0)\) and \(\epsilon (0 < \epsilon < 1)\) are frequency and amplitude of the periodic forcing term, respectively.

In this paper, we verify that the unforced system (1) undergoes degenerate Hopf bifurcation and present some phase diagrams. Then we study bifurcations of the periodically forced system in two ways: theoretical method by using second-order integral averaging operator [5, 11, 19, 26] and numerical method by considering the Poincaré map. Bifurcation diagrams for periodic solutions of period-one and period-two are given by researching the Poincaré map when the unforced system undergoes degenerate Hopf Bifurcation. At last we give some phase portraits of periodic solutions and torus generated in the periodically forced system.

2. Results of the Unforced System. In this section, we consider the equilibria and their local stability of the original unforced system (1). Consider \(a_1 a_2 \neq b_1 b_2\), we obtain that the autonomous system has four equilibria,

\(E_1(0, 0), E_2(0, \frac{1}{a_2}), E_3(\frac{1}{a_1}, 0), E_4(\frac{a_2 - b_1}{a_1 a_2 - b_1 b_2}, \frac{a_1 - b_2}{a_1 a_2 - b_1 b_2})\).

Let \(a_2 \neq b_1, a_1 \neq b_2\), then \(E_4\) is a nontrivial equilibrium and we denote it as \(E_4(x_0, y_0)\), where

\[
x_0 = \frac{a_2 - b_1}{a_1 a_2 - b_1 b_2}, \quad y_0 = \frac{a_1 - b_2}{a_1 a_2 - b_1 b_2}.
\]

The Jacobian matrix of system (1) at any equilibrium \((x, y)\) is

\[
J(x, y) = \begin{pmatrix}
  r_1(1 - 2a_1 x - b_1 y) & -b_1 r_1 x \\
  -b_2 r_2 y & r_2(1 - 2a_2 y - b_2 x)
\end{pmatrix}.
\]

Let \(\lambda_1\) and \(\lambda_2\) be two roots of the characteristic equation of Jacobian matrix \(J(x, y)\) at equilibrium. If \(\lambda_1 < 0\) and \(\lambda_2 < 0\), the corresponding equilibrium is called a sink
Remark 1. If conditions \( \lambda > 0 \) and \( \lambda > 0 \), the corresponding equilibrium is called a source and it is locally asymptotically stable; If \( \lambda \lambda < 0 \), the corresponding equilibrium is called a saddle and it is locally unstable; Both the sink and source are hyperbolic points. If the real part of \( \lambda, \omega \) is equal to zero, the corresponding equilibrium is called a non-hyperbolic point. After a simple calculation, we get following proposition.

**Proposition 1.** The types of equilibria \( E_1, E_2, E_3 \) are as follow

(i) \( E_1 \) is a source;
(ii) \( E_2 \) is a sink for \( a_2 < b_1 \), a saddle for \( a_2 > b_1 \), and a non-hyperbolic point for \( a_2 = b_1 \);
(iii) \( E_3 \) is a sink for \( a_1 < b_2 \), a saddle for \( a_1 > b_2 \), and a non-hyperbolic point for \( a_1 = b_2 \).

In order to discuss the stability of \( E_4 \), we need to calculate its corresponding eigenvalues. It is given as

\[
\lambda_{1,2} = \frac{-a_1 r_1 x_0 + a_2 r_2 y_0}{2} \pm i \sqrt{x_0 y_0 r_1 r_2 (a_1 a_2 - b_1 b_2) - \left( \frac{a_1 r_1 x_0 + a_2 r_2 y_0}{2} \right)^2},
\]

where \( i \) represents an imaginary unit. For simplicity we denote

\[
Re(\lambda) = \frac{-a_1 r_1 x_0 + a_2 r_2 y_0}{2},
\]

\[
Im(\lambda) = \sqrt{x_0 y_0 r_1 r_2 (a_1 a_2 - b_1 b_2) - \left( \frac{a_1 r_1 x_0 + a_2 r_2 y_0}{2} \right)^2}.
\]

Then following proposition can be obtained,

**Proposition 2.** The types of equilibrium \( E_4 \) of system (1) is

(i) a hyperbolic point for \( a_1 r_1 x_0 + a_2 r_2 y_0 \neq 0 \);
(ii) a Hopf bifurcation point for \( a_1 r_1 x_0 + a_2 r_2 y_0 = 0 \) and \( x_0 y_0 (a_1 a_2 - b_1 b_2) > 0 \).

If \( a_1 r_1 x_0 + a_2 r_2 y_0 \neq 0 \), the real part of the corresponding eigenvalues at \( E_4 \) is nonzero values, thus \( E_4 \) is a hyperbolic point. If \( a_1 r_1 x_0 + a_2 r_2 y_0 = 0 \) \( (Re(\lambda) = 0) \) and \( x_0 y_0 (a_1 a_2 - b_1 b_2) > 0 \), the system (1) will undergo Hopf bifurcation, and we need to verify the transversality condition. Assume \( a_1 \) as the bifurcation parameter, the transversality condition can be given as

\[
\frac{dRe(\lambda)}{da_1} = \frac{-r_1 x_0}{2} \neq 0, (r_1 \neq 0, x_0 \neq 0)
\]

In this case, \( E_4 \) is a Hopf bifurcation point and the corresponding Hopf frequency is \( \omega_c = \sqrt{x_0 y_0 r_1 r_2 (a_1 a_2 - b_1 b_2)} \).

**Remark 1.** If conditions \( a_1 r_1 x_0 + a_2 r_2 y_0 = 0 \) and \( x_0 y_0 (a_1 a_2 - b_1 b_2) > 0 \) hold, then \( (a_2 - b_1)(a_1 - b_2) < 0 \). This implies \( x_0 \) and \( y_0 \) have opposite sign, namely, \( E_4 \) will not be an positive equilibrium when it undergoes Hopf bifurcation. As mentioned in Sec.1, this paper focuses on the theoretical analysis on periodic forcing, here we don’t emphasize the biological significance of \( E_4 \).

In order to ascertain stability and direction of bifurcating periodic solutions, we should compute the first Lyapunov coefficient \( l_1 \) of the Hopf Bifurcation. Bring \( E_4 \) to the origin by transformation

\[
\begin{align*}
  x &= \dot{x} + x_0, \\
  y &= \dot{y} + y_0,
\end{align*}
\]

(3)
system (1) becomes
\[
\begin{align*}
\frac{dx}{dt} &= r_1[(1 - 2a_1x_0 - b_1y_0)x - b_1x_0y - b_1\dot{x}\dot{y} - a_1\dot{x}^2], \\
\frac{dy}{dt} &= r_2[(1 - 2a_2y_0 - b_2x_0)y - b_2y_0\dot{x} - b_2\dot{x}\dot{y} - a_2\dot{y}^2].
\end{align*}
\] (4)

For the planar system (4), the first Lyapunov coefficient of the Hopf bifurcation can be calculated by the formula in Theorem 3 of Chapter 3 in [12]. After a massive calculation we can obtain the first Lyapunov coefficient \( l_1 \) as
\[
l_1 = \frac{3\pi}{2r_1b_1}[(a_2b_1b_2^2r_2^2 + a_1b_1^2b_2^2r_2 + a_1^2b_2(\frac{b_2^2r_1^2}{a_1} - a_2b_1r_1(r_1 + 3r_2) + 2a_2^2r_2(r_1 + r_2))]
+ a_1 \left( a_2^2(b_1r_1 - 2b_2^2r_2^2) - a_2b_1(\frac{b_2^2r_1^2}{a_1} + b_1r_1) \right) \sqrt{\frac{a_1a_2 - b_1b_2}{r_1r_2(a_2 - b_1)^3(a_1 - b_2)^3}}.
\] (5)

A degenerate Hopf bifurcation will appear at \( E_4 \) when the parameter values of system (1) satisfy \( l_1 = 0 \). With form (5), it is easy to choose appropriate parameters to make system (1) undergoes degenerate Hopf bifurcation at \( E_4 \). Generally, a degenerate Hopf bifurcation of codimension 2 can be unfolded by two parameters. Two limit cycles with different stability will appear in some subregions of the two-parameter space near a degenerate Hopf bifurcation point. Fig.1(a) and Fig.1(b) are phase portraits of the unforced system (1) near the degenerate Hopf bifurcation under two groups of parameter values, respectively. Take Fig.1(a) as an example,

\[\text{Figure 1.} \ (a) \text{ Phase portrait near a degenerate Hopf bifurcation for } r_1 = 0.3, \ r_2 = 0.6, \ a_1 = 0.42, \ a_2 = 0.6, \ b_1 = 1.0857, \ b_2 = 0.25. \]
\[\text{(b) Phase portrait near a degenerate Hopf bifurcation for } r_1 = 0.3, \ r_2 = 0.6, \ a_1 = 0.447, \ a_2 = 0.6, \ b_1 = 1.13, \ b_2 = 0.25.\]

the parameter values in this case are selected as \( r_1 = 0.3, r_2 = 0.6, a_1 = 0.42, a_2 = 0.6, b_1 = 1.0857, b_2 = 0.25 \). The red points are the four equilibria \( E_1(0,0), E_2(0,1.667), E_3(2.38,0), E_4(25,-8.75) \), they are unstable node, stable node, saddle and stable focus, respectively. One can find that there are two limit cycles around the equilibrium point \( E_4 \). Moreover some of the trajectories outside the circle in the phase portraits look “fat”, the trajectory actually exhibits spiral motion and they are very close together. It also means that the system may have other limit cycles under this group of parameter values, which indicates the system has strong
degeneracy. If we want to know the codimension of Hopf bifurcation, the calculation of the second Lyapunov coefficient is necessary.

In Fig. 1(a), there exits heteroclinic orbits between the trivial equilibria $E_1(0,0)$ and $E_2(0,rac{a_2}{3})$. Generally, heteroclinic orbits are usually studied in Hamiltonian system, it is very difficult to study these orbits for a system that is not Hamiltonian.

We use the scaling logarithm change series (SLS) method [3] to find the heteroclinic orbits in the unforced system (1). This method reduces the connecting orbit as a boundary value problem in an infinite time domain to the initial value problem.

Denote a heteroclinic orbit between $E_1$ and $E_2$ as $S = (x(t), y(t))$. In our case, we have

$$
\lim_{t \to -\infty} S = E_1(0,0), \quad \lim_{t \to +\infty} S = E_2(0, \frac{1}{a_2}).
$$

Introduce a new time $\tau$ by the logarithmic scale $t = -\frac{\ln(\tau)}{T_1}$ for $t > 0$, and $t = \frac{\ln(\tau)}{T_2}$ for $t < 0$, where $T_1$ and $T_2$ are undetermined positive real constant. Obviously, we know from these two time scales that $\tau \to +0$ as $t \to \pm \infty$, therefore $t$ is transformed into $0 < \tau < 1$.

For $t > 0$, using $t = -\frac{\ln(\tau)}{T_1}$, system (1) becomes

$$
\begin{cases}
-T_1 r \frac{dx}{d\tau} = r_1(x - a_1 x^2 - b_1 xy), \\
-T_1 r \frac{dy}{d\tau} = r_2(y - a_2 y^2 - b_2 xy).
\end{cases}
$$

We assume system (6) has a solution $S_1(x(\tau), y(\tau))$ of the form

$$
x(\tau) = 0 + \sum_{i=1}^{\infty} p_i \tau^i, \quad y(\tau) = \frac{1}{a_2} + \sum_{i=1}^{\infty} q_i \tau^i,
$$

where $p_i$ and $q_i$ are undetermined coefficients. Substituting $x(\tau)$ and $y(\tau)$ into system (6) gives

$$
\begin{align*}
-T_1 p_1 \tau - 2T_1 p_2 \tau^2 - 3T_1 p_3 \tau^3 - & \\
& = r_1(p_1 - \frac{b_1 p_1}{a_2}) \tau + r_1[p_2(1 - \frac{b_1}{a_2}) - p_1(a_1 p_1 + b_1 q_1)] \tau^2 + \\
& - T_1 q_1 \tau - 2T_1 q_2 \tau^2 - 3T_1 q_3 \tau^3 - \\
& = -r_2(q_1 + \frac{b_2 p_1}{a_2}) \tau - r_2(q_2 + b_2 p_1 q_1 + q_3 a_2 + \frac{b_2 p_2}{a_2}) \tau^2 + ...
\end{align*}
$$

Comparing the coefficients in (7), we have $T_1 = r_2$ or $r_1 \left( \frac{b_1}{a_2} - 1 \right)$. Take $T_1 = r_2$ as a example, then it implies

$$
p_1 = 0, p_2 = 0, p_3 = 0, ... , p_i = 0, ... \\
q_1 = \theta, p_2 = a_2 \theta^2, q_3 = a_2^3 \theta^3, ..., q_i = a_2^{i-1} \theta^i, ...$$

where $\theta$ is a non-zero parameter to be determined. In this case, the radius of convergence of the series $y(\tau)$ is $R = \frac{1}{a_2 \theta}$. Replacing $\tau$ by $t$, we will obtain a solution as

$$
S_1(0, \frac{1}{a_2} + \frac{\theta e^{-r_2 t}}{1 - a_2 e^{-r_2 t}}).
$$

For $t < 0$, we transform system (1) into a new system by $t = \frac{\ln(\tau)}{T_2}$, and assume the obtained system has a solution $S_2(x(\tau), y(\tau))$ of the form

$$
x(\tau) = 0 + \sum_{i=1}^{\infty} c_i \tau^i, \quad y(\tau) = 0 + \sum_{i=1}^{\infty} d_i \tau^i.$$
Completely similar to the case \( t > 0 \), we can determine \( T_2 \), \( c_i \) and \( d_i \), and obtain a solution as

\[
S_2(0, \frac{\theta e^{r_2 t}}{\theta a_2 e^{r_2 t} - 1}).
\]

The parameter \( \theta \) can be calculated by considering the continuity of the solution at \( t = 0 \) as

\[
\lim_{t \to 0^+} S_1(x(t, \theta), y(t, \theta)) = \lim_{t \to 0^-} S_2(x(t, \theta), y(t, \theta)).
\]

By calculation, \( \theta = -\frac{1}{a_2} \). Substituting \( \theta \) into \( S_1 \) or \( S_2 \), we will obtain an exact heteroclinic orbit as

\[
S(0, \frac{1}{a_2} - \frac{e^{-r_2 t}}{a_2(1 - e^{-r_2 t})}).
\]

Note that for different qualified \( T_1 \) and \( T_2 \), we will obtain different heteroclinic orbits by the SLS method in a similar way. Take \( r_1 = 0.3 \), \( r_2 = 0.6 \), \( a_1 = 0.42 \), \( a_2 = 0.6 \), \( b_1 = 1.0857 \), \( b_2 = 0.25 \) as in Fig. 1(a), we immediately know the system has a heteroclinic orbit

\[
S(0, \frac{5}{3} - \frac{5e^{-0.6 t}}{3(1 - e^{-0.6 t})}).
\]

3. Results by Averaging method. In this section, we use the second-order integral averaging operator to deal with the periodically forced system and study its bifurcations. Here we only consider the periodically forced case \( a_1(t) = a_1(1 + \epsilon \sin \omega_0 t) \), due to the process of averaging on other parameters is analogous. The model can be written as

\[
\begin{aligned}
\dot{x} &= r_1(x - a_1(t)x^2 - b_1xy), \\
\dot{y} &= r_2(y - a_2y^2 - b_2xy),
\end{aligned}
\]

where

\[ a_1(t) = a_1(1 + \epsilon \sin \omega_0 t). \]

The time-periodic function \( a_1(t) \) describes the influence of periodic variability of the parameter \( a_1 \) on the dynamic behaviors. The time is scaled to make a period \( 2\pi/\omega_0 \) in length.

Following we will employ some necessary transformation, and derive the normal form of system (8) at equilibrium \( E_4 \). In order to simplify calculation further, we bring \( E_4 \) to the origin by transformation (3). Rewriting the system and dropping the “hats”, system (8) becomes:

\[
\begin{aligned}
\dot{x} &= r_1[(1 - 2a_1x_0 - b_1y_0)x - b_1x_0y - b_1xy - a_1x^2 - a_1(x + x_0)^2\epsilon \sin \omega_0 t], \\
\dot{y} &= r_2[(1 - 2a_2y_0 - b_2x_0)y - b_2y_0x - b_2xy - a_2y^2].
\end{aligned}
\]

One can find \( \lambda_{1,2} \) are eigenvalues of the linear part of system (9) at the origin. Let \( q_{1,2} \) be the corresponding eigenvectors, we get

\[
q_1 = \left( m + ni, 1 \right),
\]

where

\[
m = \frac{2b_2r_2y_0 - a_1r_1x_0 - a_2r_2y_0}{2r_2(1 - 2a_2y_0 - b_2x_0)},
\]

\[
n = \frac{\sqrt{4x_0y_0r_1r_2(a_1b_2 - a_2b_1) - (a_1r_1x_0 + a_2r_2y_0)^2}}{2r_2(1 - 2a_2y_0 - b_2x_0)}.\]
Denote \( C = \begin{pmatrix} m \\ 1 \end{pmatrix} \) and \( D = \begin{pmatrix} -n \\ 0 \end{pmatrix} \). We shall consider the transformation
\[
\begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} C & D \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}
\] (10)
to simplify the linear part of system (9) to a partially diagonal form. Substituting equation (10) in (9) and rewriting \( \bar{x}, \bar{y} \) as \( x, y \), we obtain following system
\[
\begin{align*}
\dot{x} &= Re(\lambda)x - Im(\lambda)y + A_{11}x^2 + A_{12}xy, \\
\dot{y} &= Im(\lambda)x + Re(\lambda)y + B_{11}x^2 + B_{12}xy + B_{22}y^2 + \epsilon B(x, y) \sin \omega_0 t,
\end{align*}
\]
where
\[
A_{11} = -(2mr_2b_2 + 2r_2a_2), \quad A_{12} = 2nb_2r_2,
\]
\[
B_{11} = \frac{2m}{n} (r_1b_1 + nr_1a_1 - nr_2b_2 - r_2a_2),
\]
\[
B_{12} = 2mr_2b_2 - 2r_1b_1 - 4nr_1a_1, \quad B_{22} = 2nr_1a_1,
\]
\[
B(x, y) = \frac{r_1a_1}{2n} (2mx - 2ny + x_0)^2.
\]

Next we introduce new variables \( \rho \) and \( \phi \) \((\rho \geq 0, 0 \leq \phi \leq 2\pi)\) by means of the transformation
\[
x = \rho \cos \phi, \quad y = \rho \sin \phi.
\]

We have
\[
\begin{align*}
\dot{\rho} &= \dot{x} \cos \phi + \dot{y} \sin \phi, \\
\rho \dot{\phi} &= \dot{y} \cos \phi - \dot{x} \sin \phi.
\end{align*}
\] (12)

Substituting equation (12) in (11), we obtain following
\[
\begin{align*}
\dot{\rho} &= Re(\lambda)\rho + A_{11}\rho^2 \cos^3 \phi + (A_{12} + B_{11})\rho^2 \cos^2 \phi \sin \phi \\
&\quad + B_{12}\rho^2 \sin^3 \phi \cos \phi + B_{22}\rho^2 \sin^3 \phi + \epsilon B(\rho \cos \phi, \rho \sin \phi) \sin \phi \sin \omega_0 t, \\
\rho \dot{\phi} &= Im(\lambda)\rho + B_{11}\rho^2 \cos^3 \phi + (B_{12} - A_{11})\rho^2 \sin \phi \cos^2 \phi \\
&\quad + (B_{22} - A_{12})\rho^2 \sin^2 \phi \cos \phi + \epsilon B(\rho \cos \phi, \rho \sin \phi) \cos \phi \sin \omega_0 t.
\end{align*}
\]

From proposition 2 we know if \( E_4 \) is a Hopf bifurcation point, then \( Re(\lambda) = 0 \) and \( Im(\lambda) = \omega_c \). We obtain the standard form of system (8) at the Hopf bifurcation point,
\[
\begin{align*}
\dot{\rho} &= A_{11}\rho^2 \cos^3 \phi + (A_{12} + B_{11})\rho^2 \cos^2 \phi \sin \phi + B_{22}\rho^2 \sin^3 \phi \\
&\quad + B_{12}\rho^2 \sin^2 \phi \cos \phi + \epsilon \frac{r_1a_1x_0^2}{2n} \sin \phi \sin \omega_0 t, \\
\rho \dot{\phi} &= \omega_c \rho + B_{11}\rho^2 \cos^3 \phi + (B_{12} - A_{11})\rho^2 \sin \phi \cos^2 \phi \\
&\quad + (B_{22} - A_{12})\rho^2 \sin^2 \phi \cos \phi + \epsilon \frac{r_1a_1x_0^2}{2n} \cos \phi \sin \omega_0 t.
\end{align*}
\] (13)

Here we discuss stability and bifurcation of system (13) by using the averaging procedure. Before this, we shall employ the necessary transformation
\[
\phi = \theta + \Omega t
\]
then we get
\[
\begin{align*}
\dot{\theta} &= \omega_c - \Omega + B_{11}\rho \cos^3 \phi + (B_{12} - A_{11})\rho \sin \phi \cos^2 \phi \\
&\quad + (B_{22} - A_{12})\rho \sin^2 \phi \cos \phi + \epsilon \frac{r_1a_1x_0^2}{2n} \cos \phi \sin \omega_0 t.
\end{align*}
\] (14)

To insure the dynamics of system (13) contained in the averaged system, we should choose suitable rescaling. Here we use the rescaling given in [11]. Let
\[
\rho = \sigma^{-\frac{1}{2}} \tilde{\rho},
\] (15)
where $\sigma$ is small parameter. Now $\epsilon$ are to be considered small, so we put

$$\epsilon = \sigma \tilde{\epsilon}. \quad (16)$$

Next we have to choose $\Omega$ in such a way that $\dot{\theta}$ becomes small whenever $\sigma$ does. Based on the assumption in [14], we choose

$$\Omega = k \omega_0 + \sigma \tilde{\alpha}, \quad (17)$$

where $k > 0$ and $\alpha \neq 0$.

With the scalings of (15), (16) and (17), we obtain the system

$$\begin{cases}
\dot{\rho} = \sigma \frac{4}{3} \left( A_{11} \bar{\rho}^2 \cos^3 \phi + (A_{12} + B_{11}) \bar{\rho}^2 \cos^2 \phi \sin \phi + B_{22} \bar{\rho}^2 \sin^3 \phi \\
+ B_{12} \bar{\rho}^2 \sin^2 \phi \cos \phi \right) + \sigma \frac{4}{3} \frac{r_1 a_1 x_0^2}{2n} \sin \phi \sin \omega_0 t, \\
\dot{\theta} = \omega_c - k \omega_0 + \sigma \frac{4}{3} \left( B_{11} \bar{\rho} \cos^3 \phi + (B_{12} - A_{11}) \bar{\rho} \sin \phi \cos^2 \phi \\
+ (B_{22} - A_{12}) \bar{\rho} \sin^2 \phi \cos \phi \right) + \sigma \frac{4}{3} \left( -\alpha + \frac{r_1 a_1 x_0}{2n} \cos \phi \sin \omega_0 t \right),
\end{cases} \quad (18)$$

where the coefficients of (18) can be compute directly.

Assume the nature frequency $\omega_c$ and the forcing frequency satisfy $\omega_c = k \omega_0, k > 0$, then we can apply the averaging to system (18). In addition, it is harmonic resonance case for $k = N$ ($N \in \mathbb{N}^+$), and subharmonic resonance case for $k = 1/N$ ($N \in \mathbb{N}^+$). It is easily seen that the first average of (18) vanish, such that averaging to second order is necessary [11]. After a long calculation and rescaling the time we obtain the averaged equations. Rewrite $\bar{\rho}$, $\tilde{\epsilon}$ as $\rho$, $\epsilon$ respectively, we have

$$\begin{cases}
\dot{\rho} = \mu \rho^3 + \beta \cos \theta, \\
\dot{\theta} = \nu \rho^2 - \alpha - \frac{\beta}{\rho} \sin \theta,
\end{cases} \quad (19)$$

where

$$\mu = \frac{A_{11}(A_{12} - B_{22}) - B_{12}(B_{11} + B_{22})}{16}, \quad \beta = \frac{r_1 a_1 x_0^2}{4n},$$

$$\nu = \frac{5B_{11}^2 - 2(A_{11}^2 + A_{12}^2 + B_{12}^2 + B_{22}^2) + 5(B_{11}B_{22} - A_{11}B_{12} - A_{12}B_{22}) - A_{12}B_{11}}{48}.$$

Notice that system (8) has no nontrivial equilibrium due to it is nonautonomous system. Accurately, the equilibria of the averaged system correspond to the periodic solutions in system (8), and they have same stability, because we employ the translations $x = \rho \cos \phi$ and $\phi = \theta + \Omega t$. Therefore, bifurcations of systems (19) correspond to bifurcations of periodic solutions in system (8).

The equilibria of the averaged system (19) are given by

$$\begin{cases}
\mu \rho^3 + \beta \cos \theta = 0, \\
\nu \rho^2 - \alpha - \frac{\beta}{\rho} \sin \theta = 0.
\end{cases} \quad (20)$$

Then eliminate $\theta$ by squaring and addition, we have

$$(\mu^2 + \nu^2) \rho^6 - 2\alpha \nu \rho^4 + \alpha^2 \rho^2 - \beta^2 = 0. \quad (21)$$

Clearly, the number of equilibria is determined by the number of real roots of (21). For simplicity, assume $\rho^2 = \eta$, we obtained

$$F(\eta) := (\mu^2 + \nu^2) \eta^3 - 2\alpha \nu \eta^2 + \alpha^2 \eta - \beta^2 = 0. \quad (22)$$

In view of $\rho^2 \geq 0$ and $\eta \geq 0$, the number of positive roots of equation (21) and (22) is equal. Derivative

$$F'(\eta) = 3(\mu^2 + \nu^2) \eta^2 - 4\alpha \nu \eta + \alpha^2.$$

If $\nu^2 - 3\mu^2 \geq 0$, we assume $\eta_\pm = \frac{2\alpha \nu \mp \sqrt{\alpha^2(\nu^2 - 3\mu^2)}}{3(\mu^2 + \nu^2)}$. Then we have following
**Lemma 1.** The averaged system (19) has at least one positive equilibrium for $\beta^2 > 0$ in equation (22), and at most three positive equilibria in region $\{(\rho, \phi) \mid \rho \geq 0, 0 \leq \phi \leq 2\pi\}$.

(i) System (19) has two positive equilibria if one of the following conditions holds,

1. $F(\eta_-) = 0$, $F(\eta_+) < 0$;  
2. $F(\eta_+) = 0$, $F(\eta_-) > 0$;

(ii) System (19) has three positive equilibria if and only if $\nu^2 - 3\mu^2 \geq 0$, $F(\eta_-) > 0$, $F(\eta_+) < 0$.

Let $\rho^2 = \eta_*$, we can get the characteristic equation of averaged system (19)

$$\lambda^2 - 4\mu\eta_* \lambda + 3(\mu^2 + \nu^2)\eta_*^2 - 4\alpha\nu\eta_* + \alpha^2 = 0$$

(23)

Then we have following theorem

**Theorem 1.** The types of $E_*$ is

(i) a hyperbolic equilibrium if $\mu \neq 0$ and $F'(\eta_*) \neq 0$;

(ii) a bifurcation point of codimension 1 if $\mu \neq 0$ and $F'(\eta_*) = 0$, or a fold bifurcation point if any of the two conditions in (i) of Lemma 1 holds;

(iii) a Hopf bifurcation point if $\mu = 0$ and $F'(\eta_*) > 0$.

Here we consider the averaged system (19) directly, and see $\mu$, $\nu$, $\alpha$, $\beta$ as the coefficients of (19). The transversality condition of the Hopf bifurcation can be easily verified when we choose $\mu$ as the bifurcation parameter, we do not present it. In fact, the bifurcation point of codimension 1 mentioned in Theorem 1 is transcritical bifurcation point which will be analysed in Section 4.

Figure 2 is the phase portraits of the averaged system under two group of parameter values. Figure 2(a) shows the system has two unstable focus when $\mu = 0.06$,
We will analyse the periodically forced system (1) with four different periodically driven cases obtained by using software package AUTO [8].

4. Results by Poincaré map. We investigate the bifurcations of the periodically forced system by Poincaré map in this section. Some bifurcation diagrams for different periodically driven cases are obtained by using software package AUTO [8]. We will analyse the periodically forced system (1) with four different periodically driven mechanisms respectively. They are

- Case 1: \( a_1(t) = a_1(1 + \epsilon \sin \omega_0 t) \),
- Case 2: \( b_1(t) = b_1(1 + \epsilon \sin \omega_0 t) \),
- Case 3: \( a_2(t) = a_2(1 + \epsilon \sin \omega_0 t) \),
- Case 4: \( b_2(t) = b_2(1 + \epsilon \sin \omega_0 t) \).

The periodically forcing can be done by adding a nonlinear oscillator with the desired periodic as one of the solution components. Here we take such an oscillator

\[
\begin{align*}
\dot{v} &= v + \omega_0 w - v(v^2 + w^2), \\
\dot{w} &= w - \omega_0 v - w(v^2 + w^2),
\end{align*}
\]

which has the asymptotically stable solution \( v = \sin \omega_0 t, w = \cos \omega_0 t \). To simplify matter we take \( \omega_0 = 2\pi \) in system (8). Then for

- Case 1: \( a_1(t) = a_1(1 + \epsilon \sin 2\pi t) \),

the forced system (8) can be transformed into a autonomous four dimensional system

\[
\begin{align*}
\dot{x} &= r_1(x - a_1(1 + \epsilon v)x^2 - b_1 xy), \\
\dot{y} &= r_2(y - a_2 y^2 - b_2 xy), \\
\dot{v} &= v + 2\pi w - v(v^2 + w^2), \\
\dot{w} &= w - 2\pi v - w(v^2 + w^2).
\end{align*}
\]  

Clearly, if the original unforced system (1) has equilibrium \((x^*, y^*)\), then system (25) has periodic solution \((x^*, y^*, \sin 2\pi t, \cos 2\pi t)\). For \( \epsilon \neq 0 \), we like to investigate the bifurcations of the transformed system (25) by Poincaré map. The first return map of the continuous four dimensional system can be defined as

\[\mathcal{P} : (x(0), y(0), v(0), w(0)) \mapsto (x(1), y(1), v(1), w(1)).\]

The stable (unstable) fixed points of the \( k \)th iterate of the map correspond to the stable (unstable) periodic solutions with period \( k \) of the forced system. For simplicity, we use the notations \( h^{(k)}, f^{(k)}, t^{(k)} \) to denote the Hopf (Neimark-Sacker) bifurcation curve, flip (Period doubling) bifurcation curve and fold (Tangent) bifurcation curve respectively, which are generated by the fixed point with period \( k \). Some other notations, \( A, B, C, D \) are used to denote strong \( 1 : 1 \) resonance, strong \( 1 : 2 \) resonance, strong \( 1 : 3 \) resonance and strong \( 1 : 4 \) resonance, respectively. Detailed descriptions can be found in [20].

We have to mention that the description of the following diagrams is about the Poincaré map \( \mathcal{P} \). The periodically forced system is done by adding a nonlinear oscillator (24), hence the corresponding Poincaré map \( \mathcal{P} \) has four multipliers. One can find that although the parameters of the periodically forced system are varying, two of the four multipliers are unchangeable in the parameter space, due to the asymptotically stable periodic solution generated by the nonlinear oscillator (24). We denote the unchangeable multipliers as \( |\mu_3| = 1, |\mu_4| = c \), where \( c \) is a constant.
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Figure 3. (b) Bifurcation diagram of the forced system (8) in \((a_1, \epsilon)-\text{plane for } r_1 = 0.3, b_1 = 1.13, r_2 = 0.6, a_2 = 0.6, b_2 = 0.25\). (c) and (d) Partial enlargements of (b). The solutions of system (8) are as follows, region 1-unstable period-one solution and stable quasiperiodic solution; region 2-unstable period-one solution and unstable period-two solution; region 3-unstable period-one solution, unstable period-two solutions, and period-four solution; region 4-stable and unstable period-one solutions; region 5-stable and unstable period-one solutions.

In order to describe dynamical behaviours of the Poincaré map \(P\) clearly, we just focus on the variation of multipliers \(\mu_1\) and \(\mu_2\). And we have following illustrations in this paper. If \(|\mu_1| > 1\) and \(|\mu_2| > 1\), the corresponding fixed point is locally unstable and it is called a repelling point. If \(|\mu_1| < 1\) and \(|\mu_2| < 1\), the corresponding fixed point is locally asymptotically stable and it is called a attracting point. If \(|\mu_1| > 1\) and \(|\mu_2| < 1\) (or \(|\mu_1| < 1, |\mu_2| > 1\)), the corresponding fixed point is locally unstable and it is called a saddle. To make the diagrams more readable, some necessary explanations for solutions of the four dimensional system in different regions are given in figure captions.

Fig.3(b) is the bifurcation diagram of the forced system (8) in \((a_1, \epsilon)-\text{plane for the case } r_1 = 0.3, b_1 = 1.13, r_2 = 0.6, a_2 = 0.6, b_2 = 0.25\). In order to make the diagram more readable some partial enlargement of Fig.3(b) are given, see Fig.3(c).
and Fig. 3(d). On the \(a_1\)-axis in Fig. 3(b), the point \(H\) corresponds to a degenerate Hopf bifurcation point in the unforced system (1), and it is the root of curve \(h^{(1)}\). The point \(T\) corresponds to a transcritical bifurcation point in the unforced system and it is the root of transcritical bifurcation curve (the black curve). These mean if \(\epsilon = 0\), the corresponding unforced system undergoes a transcritical bifurcation and a degenerate Hopf bifurcation.

If the transcritical bifurcation curve (the black curve in Fig. 3(b)) is crossed from region 4 to the below, two period-one fixed points (stable and unstable) exchange the stability. Curve \(h^{(1)}\) is formed by continuation of a Neimark-Sacker bifurcation of Poincaré map \(P\). It contains three codimension two bifurcation points, \(1:2\) resonance \(B\), \(1:3\) resonance \(C\) and \(1:4\) resonance \(D\). Continuing curve \(h^{(1)}\) to the right, the multiplier \(\mu_{1,2}^{(1)}\) varies smoothly. One can find from Fig. 3(c) that the strong \(1:2\) resonance \(B\) is a crossover point of curve \(h^{(1)}\) and curve \(f^{(1)}\), this is due to the strong resonance \(B\) has multipliers \(\mu_{1,2}^{(1)} = -1\). As the parameter value of \(a_1\) crosses curve \(h^{(1)}\) from region 4 to region 1, a stable fixed point of \(P\) becomes repelling point and a stable closed invariant curve appears, see Fig. 3(b). In other words, the stable cycle of period-one of system (8) bifurcates into a stable torus. Curve \(h^{(1)}\) (green line) is the fold bifurcation curve of period-one fixed point, it has two branches which generates a small cone-shaped region 5 resembling region 3, see Fig. 3(d)). Another pair of period-one fixed points, stable and unstable fixed point, exist in region 5. When curve \(h^{(1)}\) is crossed from region 5 to the outside, the pair of period-one points will collide on \(h^{(1)}\) then disappear due to the fold bifurcation. From the partial enlargement one can find there is no crossover point between the flip bifurcation curve \(f^{(1)}\) and \(a_1\)-axis, so region 1 has two subregions, see Fig. 3(c). If \(f^{(1)}\) is crossed from region 1 to region 2, a repelling period-one fixed point changes as a saddle, and a pair of unstable period-two fixed points appear. If \(f^{(1)}\) is crossed from region 4 to region 2, a stable period-one fixed point becomes a saddle and a pair of unstable period-two fixed points appear. If \(f^{(2)}\) is crossed from region 2 to region 3, the pair of repelling period-two fixed points becomes saddles, and a pair of period-four fixed points appear. Besides, the flip bifurcation curve \(f^{(4)}\), \(f^{(8)}\), ..., also exist in region 3, here we do not present them again.

Fig. 4 is the bifurcation diagram of the forced system (8) in the \((a_1, \epsilon)\)-plane for the case \(r_1 = 0.3, b_1 = 0.85, r_2 = 0.4, a_2 = 0.6, b_2 = 0.4\). It is easy to find the bifurcation curves have no crossover point with \(a_0\) axis from the partial enlargement Fig. 4(b). This indicates that the unforced system does not undergo fold bifurcation, Hopf bifurcation or transcritical bifurcation at this group of parameter values, and the bifurcation results of system (8) are all attribute to periodically forcing. The fold bifurcation curve \(t_1^{(1)}\), flip bifurcation curve \(f^{(1)}\) and \(f^{(2)}\) all have two branches which form cone-shaped region in \((a_1, \epsilon)\)-plane like region 1, see Fig. 4(b). A flip bifurcation curve \(f^{(1)}\) passes through \(B\), which is obtained by continuation from a flip bifurcation point of \(P\) in two directions. Starting from \(B\), an \(1:2\) resonance, there is a Hopf bifurcation curve \(h^{(1)}\) connecting \(B\) and \(A\), where \(A\) is an \(1:1\) resonance on curve \(t_1^{(1)}\). Curve \(h^{(1)}\) also contains another two resonances, they are \(1:3\) resonance \(C\) and \(1:4\) resonance \(D\). When the curve \(h^{(1)}\) is crossed from region 2 to region 3, a attracting period-one fixed point becomes repelling point and a stable closed invariant curve appear. Two period-one fixed points collide on curve \(t_1^{(1)}\) then disappear if \(t_1^{(1)}\) is crossed from region 2 or region 3 to region 4, see 4(b). In other words, there is no periodic solution in region 4. If the curve \(f^{(1)}\)
Figure 4. (a) Bifurcation diagram of the forced system (8) in $(\epsilon, a_1)$-plane for $r_1 = 0.3, b_1 = 0.85, r_2 = 0.4, a_2 = 0.6, b_2 = 0.4$. (b), (c) Partial enlargements of (a). The solutions of system (8) are as follows, region 1-unstable period-one solutions, unstable period-two solutions, and period-four solution; region 2-stable and unstable period-one solutions; region 3-unstable period-one solutions and stable quasiperiodic solution; region 4-no periodic solution; region 5-unstable period-one solutions and unstable period-two solutions; region 6-stable and unstable period-one solutions.

is crossed from region 3 (region 2) to region 5, a repelling (attracting) period-one fixed point becomes a saddle and a pair of period-two fixed points appear. If the curve $f^{(2)}$ is crossed from region 5 to region 1, a pair of period-two saddles become two repelling fixed points and a pair of period-four fixed points appear. In Fig.4(a) there is another fold bifurcation curve $t_2^{(1)}$ which forms a small cone-shaped region 6, see the partial enlargement Fig.4(c). Another pair of period-one fixed points, one is stable and the other is unstable, exist in region 6, and they will disappear if the curve $t_2^{(1)}$ is crossed from region 6 to the outside.

In the following we will study the dynamics of system

\[
\begin{align*}
\dot{x} &= r_1(x - a_1 x^2 - b_1(t)xy), \\
\dot{y} &= r_2(y - a_2 y^2 - b_2 xy),
\end{align*}
\] (26)
Case 2 : \( b_1(t) = b_1(1 + \epsilon \sin 2\pi t) \)

is periodically forcing term. The time is scaled to make a period one in length. Here we assume \( 0 < \epsilon < 1 \), and the periodically forced system (26) is also done by adding a nonlinear oscillator (24). We use the Poincaré map to investigate the dynamics of system (24), and some bifurcation diagrams are given by using software package AUTO.

Fig. 5 is the bifurcation diagram of the forced system (26) in \((b_1, \epsilon)\)-plane for \( r_1 = 0.3, a_1 = 0.2, r_2 = 0.4, a_2 = 0.4, b_2 = 0.162 \). (b) Partial enlargements of (a). The solutions of system (26) are as follows, region 1-stable and unstable period-one solutions, unstable quasiperiodic solution; region 2- unstable period-one solutions; region 3-unstable period-one solutions, unstable period-two solutions, and period-four solution; region 4-unstable period-one solutions, stable period-two solutions; region 5-stable and unstable period-one solutions.

Fig. 6 is the bifurcation diagram of the forced system (26) in \((b_1, \epsilon)\)-plane for the case \( r_1 = 0.3, b_1 = 0.6, r_2 = 0.4, a_2 = 0.6, b_2 = 0.4 \). In Fig. 6(a), the point \( T \) corresponds to a transcritical bifurcation point in the unforced system (1) and it is the root of transcritical bifurcation curve (the black curve). In fact, there is no
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Figure 6. (a) Bifurcation diagram of the forced system (26) in $(b_1, \epsilon)$-plane for $r_1 = 0.3, b_1 = 0.6, r_2 = 0.4, a_2 = 0.6, b_2 = 0.4$. (b) bifurcation curves of a period-one saddle. (c) and (d) Partial enlargements of (a). The solutions of system (26) are as follows, region 1-stable and unstable period-one solutions; region 2- unstable period-one solutions and stable quasiperiodic solution; region 3-unstable period-one solutions, unstable period-two solutions, and period-four solution; region 4-stable period-one solutions; region 5- unstable period-one solutions, unstable period-two solutions; region 6-stable and unstable period-one solutions.

intersection point among transcritical bifurcation curve, fold bifurcation curve $t_1^{(1)}$ and flip bifurcation curve, because these bifurcation curves belong to different fixed points of Poincaré map $P$. Therefore, we show some bifurcation curves separately in Fig.6(b), which are generated by a period-one saddle. A stable period-one point and a period-one saddle in region 1 will exchange their stability, when the transcritical bifurcation curve is crossed from region 1 to the below. They will collide on $t_2^{(1)}$ then disappear, when the curve $t_2^{(1)}$ is crossed from region 1 to the above. On the other hand, the fold bifurcation curve $t_2^{(1)}$ passes through $A$, a strong 1 : 1 resonance, which is codimension two bifurcation point. Starting from $A$, there is a Hopf bifurcation curve $h^{(1)}$, which is generated by a stable period-one fixed point.
When the curve $h^{(1)}$ is crossed from region 4 to region 2, a attracting period-one fixed point becomes repelling point and a stable closed invariant curve appear. If the curve $f^{(1)}$ is crossed from region 2 to region 5, a repelling period-one fixed point becomes a saddle, and a pair of repelling period-two fixed points appear, see Fig. 6(c). If the curve $f^{(2)}$ is crossed from region 5 to region 3, the pair of repelling period-two fixed points become a pair of period-two saddles, and a pair of period-four fixed points appear. Furthermore, flip bifurcation curve $f^{(4)}, f^{(8)}, \ldots$, also exist in region 3. In Fig.6(a) there is another fold bifurcation curve $t^{(1)}_1$ which forms a small cone-shaped region 6, see the partial enlargement Fig.6(d). Another pair of period-one fixed points, one is stable and the other is unstable, exist in region 6, and they will disappear if the curve $t^{(1)}_1$ is crossed from region 6 to the outside.

Following we study the periodically forced system

$$\begin{cases}
\dot{x} = r_1(x - a_1x^2 - b_1xy), \\
\dot{y} = r_2(y - a_2(t)y^2 - b_2xy),
\end{cases}$$

(27)

where

Case 3: $a_2(t) = a_2(1 + \epsilon \sin 2\pi t)$.

Assume $0 < \epsilon < 1$. The periodically forced system (27) is also done by adding a nonlinear oscillator (24), and some bifurcation diagrams are given in Fig.7.

Fig.7(a) is the bifurcation diagram of the forced system (27) in the $(a_2, \epsilon)$-plane for the case $r_1 = 0.3, a_1 = 0.2, b_1 = 0.5, r_2 = 0.4, b_2 = 0.162$. The point $H$ corresponds to a degenerate Hopf bifurcation point in the unforced system (1) and it is the root of curve $h^{(1)}$. Bifurcation in Fig.7(a) are similar with the case in Fig.3, except there is no transcritical bifurcation curve. Fig.7(b) is the bifurcation diagram of the forced system (27) in the $(a_2, \epsilon)$-plane for the case $r_1 = 0.3, a_1 = 0.6, b_1 = 0.83, r_2 = 0.4, b_2 = 0.4$. The bifurcation results in Fig.7(b) are similar with the case in Fig.4, here we do not repeat it.

Next we study the dynamics of system

$$\begin{cases}
\dot{x} = r_1(x - a_1x^2 - b_1xy), \\
\dot{y} = r_2(y - a_2y^2 - b_2(t)xy),
\end{cases}$$

(28)
where

\[ \text{Case 4 : } b_2(t) = b_2(1 + \epsilon \sin 2\pi t) \]

is periodically forcing. Here we assume \( 0 < \epsilon < 1 \), and the periodically forced system (28) is also done by adding a nonlinear oscillator (24).

The bifurcation results of the forced system (28) in \((b_2, \epsilon)\)-plane for the case \( r_1 = 0.3, a_1 = 0.2, b_1 = 0.83, r_2 = 0.4, a_2 = 0.4 \) are shown in Fig. 8, which are equivalent to the cases displayed in Fig. 5. So we do not give the details again.

Fig. 9 is the bifurcation diagram of the forced system (28) in the \((b_2, \epsilon)\)-plane for the case \( r_1 = 0.3, a_1 = 0.6, b_1 = 0.85, r_2 = 0.4, a_2 = 0.6 \). At this group of parameter values, the unforced system (1) dose not undergo fold bifurcation, Hopf bifurcation or transcritical bifurcation. The fold bifurcation curve \( t_{(1)}^{(2)} \), flip bifurcation curve \( f_{(1)}^{(1)} \), \( f_{(2)}^{(1)} \), \( f_{(2)}^{(2)} \) all have two branches which form some cone-shaped region in \((b_2, \epsilon)\)-plane like region 1, see Fig. 9(a). A flip bifurcation curve \( f_{(1)}^{(1)} \) passes through \( B \), which is obtained by continuation from a flip bifurcation point of \( \mathcal{P} \) in two directions. Starting from \( B \), a 1 : 2 resonance, there is a Hopf bifurcation curve \( h_{(1)}^{(1)} \) connecting \( B \) and \( A \), where \( A \) is an 1 : 1 resonance on curve \( t_{(1)}^{(1)} \). Curve \( h_{(1)}^{(1)} \) also contains another resonances, 1 : 4 resonance \( D \), these resonances are codimension two bifurcation points. When the curve \( h_{(1)}^{(1)} \) is crossed from region 2 to region 3, an attracting period-one fixed point becomes repelling point, and a stable closed invariant curve appear. Two period-one fixed points collide on curve \( t_{(1)}^{(1)} \) then disappear if \( t_{(1)}^{(1)} \) is crossed from region 2 or region 3 to region 4, see Fig. 9(b). If the curve \( f_{(1)}^{(1)} \) is crossed from region 3 (region 2) to region 4, a repelling (attracting) period-one fixed point becomes a saddle and a pair of period-two fixed points appear. If the curve \( f_{(2)}^{(1)} \) is crossed from region 4 to region 1, a pair of repelling period-two fixed become period-two saddles and a pair of period-four fixed points appear, see Fig. 9(b) or Fig. 9(c). In the partial enlargement Fig. 9(d) there are fold bifurcation curve \( t_{(1)}^{(1)} \) and flip bifurcation curve \( f_{(2)}^{(2)} \), which forms two small cone-shaped region 5 and region 6, respectively. Another pair of period-one fixed points, one is stable and the other is unstable, exist in region 5, and they will disappear if the curve \( t_{(1)}^{(1)} \) is crossed from region 5 to the outside. If the curve \( f_{(2)}^{(2)} \) is crossed to region 6, another pair of period-two saddles and a pair of period-four fixed will appear. Moreover, the flip bifurcation curve \( f^{(4)}, f^{(8)}, \ldots \), exist in region 1.
Figure 9. (a) Bifurcation diagram of the forced system (28) in $(b_2, \epsilon)$-plane for $r_1 = 0.3, a_1 = 0.6, b_1 = 0.85, r_2 = 0.4, a_2 = 0.6$. (b), (c) and (d) Partial enlargements of (a). The solutions of system (28) are as follows, region 1-unstable period-one solution, unstable period-two solutions, and period-four solution or chaos in some subregion; region 2-stable and unstable period-one solution; region 3-unstable period-one solution and stable quasiperiodic solution; region 4-unstable period-one solutions and unstable period-two solutions; region 5-stable and unstable period-one solutions; region 6-unstable period-one solution, unstable period-two solutions, and period-four solution.

In the periodically forced system, the equilibrium of system (1) becomes the periodic solution of period $T = 1$ due to the adding nonlinear oscillator with frequency $\omega = 2\pi$. Since there are no equilibrium solutions in periodically forced system, the bifurcation types are those of periodic orbits. Periodic forcing is a key feature in the model, and it can induce different bifurcations, which can lead to the appearance of various solutions, see Fig.10.

5. Conclusion. The dynamics of periodically forced degenerate Hopf bifurcation is investigated in two ways: the averaging method and Poincaré map. These two
methods all reveal the periodically forced system undergoes Hopf bifurcation, transcritical bifurcation and fold bifurcation, where the bifurcations are all for periodic solution.

The averaging method reveals that the forced system undergoes degenerate Hopf bifurcation of periodic solution if the unforced system exhibits a degenerate Hopf bifurcation of equilibrium point. After we perform the averaging transform, a non-autonomous system becomes a autonomous one and we can use the qualitative theory to study the bifurcations. By investigating Poincaré map, we obtain from bifurcation diagrams that a degenerate Hopf bifurcation in unforced system can be extend to the periodically forced case as bifurcations of periodic solutions and seen as a root of a Hopf bifurcation curve in bifurcation diagrams. On the other hand, the averaging method transforms the periodically forced system to a continuous system, while the Poincaré map is a discrete system (return map) of the forced system. For the averaging method, it is not available to study the period doubling bifurcation of periodic solution, because there is no essential distinction between period-one solution and period-two solution after we perform the averaging transform. It is shown by the Poincaré map that the periodically forced system can experience
period doubling bifurcation of periodic solution for all of the four periodical forcing cases, and some more complex dynamics such as 1:1 resonance, 1:2 resonance, 1:3 resonance and 1:4 resonance. Not only that, under some groups of parameter values, though the unforced system (1) has no bifurcations the periodically forced system can also generate different types of bifurcations including fold bifurcation, Neimark-Sacker bifurcation and period doubling bifurcation, see Fig.4, Fig.7(b) and Fig.9.

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REFERENCES

[1] F. Barraquand, S. Louca and K. C. Abbott, et al., Moving forward in circles: Challenges and opportunities in modelling population cycles, Ecol. Lett., 20 (2017), 1074–1092.
[2] A. K. Bajaj, Resonant parametric perturbations of the Hopf bifurcation, J. Math. Anal. Appl., 115 (1986), 214–224.
[3] J. H. Bao and Q. G. Yang, A new method to find homoclinic and heteroclinic orbits, Appl. Math. Comput., 217 (2011), 6526–6540.
[4] E. Benincà, B. Ballantine and S. P. Ellner, et al., Species fluctuations sustained by a cyclic succession at the edge of chaos, P. Natl. Acad. Sci. USA, 112 (2015), 6389–6394.
[5] S. N. Chow and M. P. John, Integral averaging and bifurcation, J. Differ. Equations, 26 (1977), 112–159.
[6] Z. B. Cheng and F. F. Li, Positive periodic solutions for a kind of second–order neutral differential equations with variable coefficient and delay, Mediterr. J. Math., 15 (2018), Paper No. 134, 19 pp.
[7] Z. B. Cheng and Q. G. Yuan, Damped superlinear duffing equation with strong singularity of repulsive type, J. Fix. Point Theory A, 22 (2020), Paper No. 37, 18 pp.
[8] E. J. Doedel and B. E. Oldeman, AUTO–07P: continuation and bifurcation software for ordinary differential equations, http://cmvl.cs.concordia.ca/auto., 2012.
[9] W. W. Farr, C. Z. Li, I. S. Labouriau and W. F. Langford, Degenerate Hopf bifurcation formulas and Hilbert’s 16th problem, SIAM J. Math. Anal., 20 (1989), 13–30.
[10] J. M. González–Miranda, On the effect of circadian oscillations on biochemical cell signaling by NF–B, J. Theor. Biol., 335 (2013), 283–294.
[11] P. Gross, On harmonic resonance in forced nonlinear oscillators exhibiting a Hopf bifurcation, IMA J. Appl. Math., 50 (1993), 1–12.
[12] L. Perko, Differential Equations and Dynamical Systems, Springer-Verlag, New York, 1991.
[13] J. M. Gambaudo, Perturbation of a Hopf bifurcation by an external time–periodic forcing, J. Differ. Equations, 57 (1985), 172–199.
[14] W. L. Kath, Resonance in periodically perturbed Hopf bifurcation, Stud. Appl. Math., 65 (1981), 95–112.
[15] Y. A. Kuznetsov, S. Muratori and S. Rinaldi, Bifurcations and chaos in a periodic predator–prey model, Int. J. Bifurcat. Chaos, 2 (1992), 117–128.
[16] X. P. Li, J. L. Ren and S. A. Campbell, et al., How seasonal forcing influences the complexity of a predator–prey system, Discrete Cont. Dyn.–B, 23 (2018), 785–807.
[17] M. A. McKarnin, L. D. Schmidt and R. Aris, Response of nonlinear oscillators to forced oscillations: Three chemical reaction case studies, Chem. Eng. Sci., 43 (1988), 2833–2844.
[18] N. S. Namachchivaya and S. T. Ariaratnam, Periodically Perturbed Hopf Bifurcation, SIAM J. Appl. Math., 47 (1987), 15–39.
[19] L. Perko, Higher order averaging and related methods for perturbed periodic and quasi–periodic systems, SIAM J. Appl. Math., 17 (1969), 698–724.
[20] J. L. Ren and Q. G. Yuan, Bifurcations of a periodically forced microbial continuous culture model with restrained growth rate, Chaos, 27 (2017), 083124, 15pp.
[21] J. L. Ren and L. P. Yu, Codimension–two bifurcation, chaos control in a discrete–time information diffusion model, J. Nonlinear Sci., 26 (2016), 1895–1931.
[22] J. L. Ren and X. P. Li, Bifurcations in a seasonally forced predator–prey model with generalized Holling type IV functional response, Int. J. Bifurcat. Chaos, 26 (2016), 1650203, 19pp.
[23] S. Rosenblat and D. S. Cohen, Periodically perturbed bifurcation—1. Simple bifurcation, *Stud. Appl. Math.*, 63 (1980), 1–23.
[24] S. Rosenblat and D. S. Cohen, Periodically perturbed bifurcation. II. Hopf bifurcation, *Stud. Appl. Math.*, 64 (1981), 143–175.
[25] A. Rego–Costa, F. Debarre and L. M. Chevin, Chaos and the (un)predictability of evolution in a changing environment, *Evolution*, 72 (2018), 375–385.
[26] J. A. Sanders, F. Verhulst and J. Murdock, *Averaging Methods in Nonlinear Dynamical Systems* (2nd edition) (Springer, New York, NY), 2007.
[27] Y. W. Tao, X. P. Li and J. L. Ren, A repeated yielding model under periodic perturbation. *Nonlinear Dynam.*, 94 (2018), 2511–2525.
[28] Y. Takeuchi, *Global Dynamical Properties of Lotka–Volterra Systems*, (World Scientific), 1996.
[29] D. M. Xiao and H. P. Zhu, Multiple focus and Hopf bifurcations in a predator–prey system with nonmonotonic functional response, *SIAM J. Appl. Math.*, 66 (2006), 802–819.
[30] Y. Y. Zhang and M. Golubitsky, Periodically forced Hopf bifurcation, *SIAM J. Appl. Dyn. Syst.*, 10 (2011), 1272–1306.

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*E-mail address*: yqg@gs.zzu.edu.cn
*E-mail address*: renjl@zzu.edu.cn