A SHARP EFFECTIVENESS RESULT OF DEMAILLY’S STRONG
OPENNESS CONJECTURE

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Abstract. In this article, we establish a sharp effectiveness result of De-
mailly’s strong openness conjecture. We also establish a sharp effectiveness
result related to a conjecture posed by Demailly and Kollár.

1. Introduction

The multiplier ideal sheaf related to a plurisubharmonic function plays an impor-
tant role in complex geometry and algebraic geometry, which was widely discussed
(see e.g. [34, 24, 29, 6, 7, 3, 8, 22, 31, 32, 4]). We recall the definition as follows.

Let \( \varphi \) be a plurisubharmonic function (see [5, 27, 28]) on a complex manifold. It
is known that the multiplier ideal sheaf \( \mathcal{I}(\varphi) \) was defined as the sheaf of germs of
holomorphic functions \( f \) such that \( |f|^2 e^{-\varphi} \) is locally integrable (see [4]).

In [4] (see also [3]), Demailly posed the strong openness conjecture for multiplier
ideal sheaves (SOC for short), i.e.

\[
\mathcal{I}(\varphi) = \mathcal{I}_+(\varphi) := \cup_{p>1} \mathcal{I}(p\varphi).
\]

The two-dimensional case of SOC was proved by Jonsson-Mustață [19]. An
important case of SOC so-called the openness conjecture (OC for short) was proved
by Berndtsson [2] and the two-dimensional case of OC was proved by Favre-Jonsson
[10, 9].

Recently, SOC was proved in [15] (see also [23, 18]). After that, stimulated
by the effectiveness result in Berndtsson’s solution of the openness conjecture, an
effectiveness result of SOC was established in [10] as continuous work of the solution
of SOC. Note that the effectiveness result of SOC is not sharp, then it is natural
to ask:

Can one establish a sharp effectiveness result of SOC?

In the following section, we give an affirmative answer to the above question.

One of the innovations in the present article is that, instead of the single minimal
\( L^2 \) integral on the whole domain considered in previous articles (e.g. [10, 13, 14,
12]), we consider the minimal \( L^2 \) integrals on all sublevel sets \( \{ \varphi < -t \} \), e.g. the
function \( G(t) \) (details see Section 2.1).

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1.1. A sharp effectiveness result of Demailly’s strong openness conjecture.

Let $F$ be a holomorphic function on pseudoconvex domain $D \subset \mathbb{C}^n$ (see [5]) containing the origin $o \in \mathbb{C}^n$, and let $\varphi$ be a negative plurisubharmonic function on $D$.

Recall that $c^F_\varphi(\varphi) := \sup\{ c \geq 0 : |F|^2 e^{-2c\varphi} \text{ is } L^1 \text{ on a neighborhood of } o \}$ is the jumping number (see [20]). Especially, when $F \equiv 1$, $c^F_\varphi(\varphi)$ will degenerate to the complex singularity exponent $c_\varphi(\varphi)$ (or log canonical threshold) (see [34, 26, 21, 7], etc.).

If $c^F_\varphi(\varphi) \neq +\infty$, $C_{F,I,+(2c^F_\varphi(\varphi))c,o}(D) := \inf\{ \int_D |\tilde{F}|^2 |(\tilde{F} - F, o) \in I_+(2c^F_\varphi(\varphi)) \odot & \tilde{F} \in \mathcal{O}(D) \}$. If $c^F_\varphi(\varphi) = +\infty$, $C_{F,I,+(2c^F_\varphi(\varphi))c,o}(D) := \int_D |F|^2$.

In this section, we establish a sharp effectiveness result of SOC.

**Theorem 1.1.** Assume that $\int_D |F|^2 e^{-\varphi} < +\infty$. Then for any $p > 1$ satisfying

$$\theta(p) > \frac{\int_D |F|^2 e^{-\varphi}}{C_{F,I,+(2c^F_\varphi(\varphi))c,o}(D)},$$

we have

$$(F,o) \in \mathcal{I}(p\varphi)_o,$$

i.e. $|F|^2 e^{-\varphi}$ is locally integrable near $o$, where $\theta(p) = \frac{p}{p-1}$, which is sharp.

When $D$ is the unit disc $\Delta \subseteq \mathbb{C}$, $F \equiv 1$ and $\varphi = \frac{2}{p} \log |z|$, note that $\int_D e^{-\varphi} = \int_{\Delta} \frac{1}{|z|^{2/p}} = \frac{\pi}{p-1}$ and $C_{F,I,+(2c^F_\varphi(\varphi))c,o}(D) = \pi$, then it is clear that $\frac{\int_p |F|^2 e^{-\varphi}}{C_{F,I,+(2c^F_\varphi(\varphi))c,o}(D)} = \frac{p}{p-1}$, which implies that $\theta(p) = \frac{p}{p-1}$ is sharp.

**Remark 1.1.** For the case $\theta(p) = \left( \frac{1}{(p-1)^{\frac{1}{p}} - 1} \right)^p$, the effectiveness result of SOC was established in [15], which implies (a more precise but non-sharp version of) Berndtsson’s effectiveness result of OC ([2], see also [16]).

It follows from inequality [2.1] that Theorem [1.1] degenerates to the following sharp version of Berndtsson’s effectiveness result of OC.

**Corollary 1.1.** Assume that $\int_D e^{-\varphi} < +\infty$, where $\varphi$ is a negative plurisubharmonic function on pseudoconvex domain $D$. Then for any $p > 1$ satisfying

$$\frac{p}{p-1} > K_D(o) \int_D e^{-\varphi},$$

we have $e^{-p\varphi}$ is locally integrable near $o$, where $K_D$ is the Bergman kernel on $D$.

Let $D$ be the unit disc $\Delta \subseteq \mathbb{C}$, and $\varphi = \frac{2}{p} \log |z|$. Note that $\int_D e^{-\varphi} = \int_{\Delta} \frac{1}{|z|^{2/p}} = \frac{\pi}{p-1}$, and $K_D(o) = \frac{1}{\pi}$, then it is clear that $K_D(o) \int_D e^{-\varphi} = \frac{p}{p-1}$. Then Corollary [1.1] is sharp.

1.2. A sharp effectiveness result related to a conjecture posed by Demailly and Kollár.

In this section, we present the following sharp effectiveness result related to a conjecture posed by Demailly and Kollár.
Theorem 1.2. Let $F$ be a holomorphic function on pseudoconvex domain $D \subset \mathbb{C}^n$, and let $\varphi$ be a negative plurisubharmonic function on $D$. If $c_0^F(\varphi) < +\infty$, then
\[ \frac{1}{r^{2c_0^F(\varphi)}} \int_{\{\varphi < \log r\}} |F|^2 \geq C_{F,\mathcal{I}_+}(2c_0^F(\varphi)\varphi)_+(D) > 0 \]
holds for any $r \in (0,1)$.

Especially, if $C_{F,\mathcal{I}_+}(2c_0^F(\varphi)\varphi)_+(D) = +\infty$, then $\int_{\{\varphi < \log r\}} |F|^2 = +\infty$ for any $r \in (0,1)$.

Let $D = \Delta \subset \mathbb{C}$ be the unit disc, and let $\varphi = \log |z|$ and $F \equiv 1$. It is clear that $c_0^F(\varphi) = 1$, $\int_{\{\varphi < \log r\}} |F|^2 = \pi r^2$, $C_{F,\mathcal{I}_+}(2c_0^F(\varphi)\varphi)_+(D) = \pi$, which imply the sharpness of Theorem 1.2.

When $F \equiv 1$, Theorem 1.2 degenerates to

Corollary 1.2. Let $\varphi$ be a negative plurisubharmonic function on pseudoconvex domain $D \subset \mathbb{C}^n$. If $c_0^F(\varphi) < +\infty$, then
\[ \frac{1}{r^{2c_0(\varphi)}} \int_{\{\varphi < \log r\}} 1 \geq K_D^{-1}(o) \]
holds for any $r \in (0,1)$.

Let $D = \Delta \subset \mathbb{C}$ be the unit disc, and let $\varphi = \log |z|$. It is clear that $c_0(\varphi) = 1$, $\int_{\{\varphi < \log r\}} 1 = \pi r^2$, $K_D^{-1}(o) = \pi$, which imply the sharpness of Corollary 1.2.

In [7] (see also [20]), Demailly and Kollár conjectured that
\[ \lim_{r \to 0} \inf \frac{1}{r^{2c_0(\varphi)}} \int_{\{\varphi < \log r\}} 1 > 0. \]

Depending on the truth of OC, the above conjecture was proved in [10] (the two-dimensional case was proved by Favre-Jonsson [10]). Note that the proof of Theorem 1.2 doesn’t depend on the truth of OC, then we obtain a new approach to the above conjecture with sharp effectiveness (inequality 1.2).

2. Preparations

In this section, we will do some preparations.

2.1. Some properties of $C_{f,\varphi}(D)$.

Let $D \subset \mathbb{C}^n$ be a pseudoconvex domain. Let $f$ be a holomorphic function near $o$, and let $I \subset \mathcal{O}_o$ be an ideal. $C_{f,I}(D)$ denotes $\inf \{ \int_D |\hat{f} - f| \} (\hat{f}, o) \in I$ and $\hat{f} \in \mathcal{O}(D)$ as a generalized version of $C_{F,\mathcal{I}_+}(2c_0^F(\varphi)\varphi)_+(D)$.

If there is no holomorphic function $\hat{f}$ satisfying both $(\hat{f} - f, o) \in I$ and $\hat{f} \in \mathcal{O}(D)$, then we set $C_{f,I}(D) = -\infty$. Especially, if $I = \mathcal{I}(\varphi)_o$, then $C_{f,\varphi}(D)$ denotes $C_{f,I}(D)$.

In this section, we will recall and present some properties related to $C_{f,\varphi}(D)$.

Lemma 2.1. $(f, o) \notin \mathcal{I}(\varphi)_o \Leftrightarrow C_{f,\varphi}(D) \neq 0$ (maybe $-\infty$ or $+\infty$). Especially, if $f \equiv 1$ and $\mathcal{I}(\varphi)_o \neq \mathcal{O}_o$, then $C_{1,\varphi}(D) \geq K_D^{-1}(o)$.

Note that $C_{1,\mathcal{I}_+}(2c_0(\varphi)\varphi)_+(D) = C_{1,\mathcal{I}(p_0\varphi)_o}(D)$ for some $p_0 > 2c_0(\varphi)$ (Noetherian of $\mathcal{O}_o$), then it follows from $C_{1,\mathcal{I}(p_0\varphi)_o}(D) \geq K_D(o)$ (Lemma 2.1) that
\[ C_{1,\mathcal{I}_+}(2c_0(\varphi)\varphi)_+(D) \geq K_D(o). \]
Let Lemma 2.2. holomorphic function on $f_o$ topology of compact convergence (see [11]) that $(f, o) \notin \mathcal{I}(\varphi)$, which implies $(f, o) \notin \mathcal{I}(\varphi)$. Then we obtain $(f, o) \notin \mathcal{I}(\varphi)$ or $(f, o) \notin \mathcal{I}(\varphi)$ if $(f, o) \notin \mathcal{I}(\varphi)$. Secondly, we prove $C_{1,\varphi}(D) \geq K_{\varphi}^{-1}(O_o)$. Then we obtain $(\hat{f} - f)(o) = 0$ i.e. $\hat{f}(o) = 1$, then we have $\int_{|f|}^2 \geq K_{\varphi}^{-1}(O_o)$, which implies $C_{1,\varphi}(D) \geq K_{\varphi}^{-1}(O_o)$. Lemma 2.1 has thus been proved.

Lemma 2.2. Let $\varphi$ be a negative plurisubharmonic function on $D$, and let $F$ be a holomorphic function on $\{ \varphi < -t \}$. Assume that $C_{F,\varphi}(\{ \varphi < -t \}) < +\infty$. Then there exists a unique holomorphic function $F_t^0$ on $\{ \varphi < -t \}$ satisfying $(F_t^0 - F, o) \notin \mathcal{I}(\varphi)$, and $f_t^0 \equiv f$. Furthermore, for any holomorphic function $\hat{F}$ on $\{ \varphi < -t \}$ satisfying $(\hat{F} - F, o) \notin \mathcal{I}(\varphi)$ and $\int_{|\varphi|}^2 \hat{F}^2 < +\infty$, we have the following equality

$$
\int_{\{ \varphi < -t \}} |F_t^0|^2 + \int_{\{ \varphi < -t \}} |\hat{F} - F_t^0|^2 = \int_{\{ \varphi < -t \}} |\hat{F}|^2. \quad (2.2)
$$

Proof. Firstly, we prove the existence of $F_t^0$. As $C_{F,\varphi}(\{ \varphi < -t \}) < +\infty$ then there exists holomorphic functions $\{ f_j \}_{j \in \mathbb{N}^+}$ on $\{ \varphi < -t \}$ such that $\int_{|f|}^2 \rightarrow C_{F,\varphi}(\{ \varphi < -t \})$, and $(f_j - F, o) \notin \mathcal{I}(\varphi)$. Then there exists a subsequence of $\{ f_j \}$ compactly convergent to a holomorphic function $f$ on $\{ \varphi < -t \}$ satisfying $\int_{|f|}^2 \leq C_{F,\varphi}(\{ \varphi < -t \})$ for any compact set $K \subset \{ \varphi < -t \}$, which implies $\int_{|f|}^2 \leq C_{F,\varphi}(\{ \varphi < -t \})$ by Levi’s Theorem. Note that the closedness of the sections of coherent analytic sheaves under the topology of compact convergence (see [11]) implies that $(f - F, o) \notin \mathcal{I}(\varphi)$, then we obtain the existence of $F_t^0(= f)$.

Secondly, we prove the uniqueness of $F_t^0$ by contradiction: if not, there exist two different holomorphic functions $f_1$ and $f_2$ on $\{ \varphi < -t \}$ satisfying $\int_{|f|}^2 = \int_{|f_1 - f_2|^2} = C_{F,\varphi}(\{ \varphi < -t \})$, $(f_1 - F, o) \notin \mathcal{I}(\varphi)$, and $(f_2 - F, o) \notin \mathcal{I}(\varphi)$, and $(f_1 + f_2 - F, o) \notin \mathcal{I}(\varphi)$. Note that $\int_{|f|}^2 = \int_{|f_1 - f_2|^2} = \int_{|f_1 + f_2 - f|^2} = C_{F,\varphi}(\{ \varphi < -t \})$, then we obtain $\int_{|f_1 - f_2|^2} < C_{F,\varphi}(\{ \varphi < -t \})$, and $(\frac{f_1 + f_2}{2} - F, o) \notin \mathcal{I}(\varphi)$, which contradicts the definition of $C_{F,\varphi}(\{ \varphi < -t \})$.

Finally, we prove equality (2.2). For any holomorphic function $f$ on $\{ \varphi < -t \}$ satisfying $\int_{|f|}^2 < +\infty$ and $(f, o) \notin \mathcal{I}(\varphi)$, it is clear that for any complex number $\alpha$, $F_t + \alpha f$ satisfying $((F_t^0 + \alpha f - F, o) \notin \mathcal{I}(\varphi)$, and $\int_{|F_t - f|^2} < \int_{|f-\alpha|^2} < +\infty$. Note that $\int_{|f|}^2 |F_t + \alpha f|^2 - \int_{|f-\alpha|^2} |F_t|^2 > 0$ implies $\Re \int_{|f|}^2 |F_t f| = 0$ by considering $\alpha \rightarrow 0$, then we obtain $\int_{|F_t|^2}^2 + \int_{|f|^2}^2$. Choosing $f = \hat{F} - F_t$, we obtain equality (2.2).
Let $F$ be a holomorphic function on $D$. $G(t)$ denotes $C_{F,T}(\{\varphi < -t\})$. In the following part of this section, we will consider the properties of $G(t)$. The following Lemma will be used to prove Proposition 2.3.

**Lemma 2.3.** Assume that $G(0) < +\infty$. Then $G(t)$ is decreasing with respect to $t \in [0, +\infty)$, such that $\lim_{t \to t_0^+} G(t) = G(t_0)$ ($t_0 \in [0, +\infty)$), $\lim_{t \to t_0^-} G(t) \geq G(t_0)$ ($t_0 \in (0, +\infty)$), and $\lim_{t \to +\infty} G(t) = 0$, where $t_0 \in [0, +\infty)$. Especially $G(t)$ is lower semi-continuous on $[0, +\infty)$.

**Proof.** By the definition of $G(t)$, it is clear that $G(t)$ is decreasing on $[0, +\infty)$ and $\lim_{t \to t_0^-} G(t) \geq G(t_0)$. It suffices to prove $\lim_{t \to t_0^+} G(t) = G(t_0)$. We prove it by contradiction: if not, then $\lim_{t \to t_0^+} G(t) < G(t_0)$.

By Lemma 2.2 there exists a unique holomorphic function $F_t$ on $\{\varphi < -t\}$ satisfying $(F_t - F, \varphi)$ and $\int_{\{\varphi < -t\}} |F_t|^2 = G(t)$. Note that $G(t)$ is decreasing implies that $\int_{\varphi < -t} |F_t|^2 \leq \lim_{t \to t_0^+} G(t)$ for any $t < t_0$, then for any compact subset $K$ of $\{\varphi < -t_0\}$, there exists $\{F_{t_j}\}$ ($t_j \to t_0 - 0$, as $j \to +\infty$) uniformly convergent on $K$, which implies that there exists a subsequence of $\{F_{t_j}\}$ (also denoted by $\{F_{t_j}\}$) convergent on any compact subset of $\{\varphi < -t_0\}$.

Let $\hat{F}_{t_0} := \lim_{j \to +\infty} F_{t_j}$, which is a holomorphic function on $\{\varphi < -t_0\}$. Then it follows from the decreasing property of $G(t)$ that $\int_K |\hat{F}_{t_0}|^2 \leq \lim_{j \to +\infty} \int_K |F_{t_j}|^2 \leq \lim_{j \to +\infty} G(t_j) \leq \lim_{t \to t_0^+} G(t)$ for any compact set $K \subset \{\varphi < -t_0\}$. It follows from Levi’s theorem that $\int_D |\hat{F}_{t_0}|^2 \leq \lim_{t \to t_0^+} G(t)$. Then we obtain that $G(t_0) \leq \int_D |\hat{F}_{t_0}|^2 \leq \lim_{t \to t_0^+} G(t)$, which contradicts $\lim_{t \to t_0^+} G(t) < G(t_0)$.

We prove Lemma 2.3 by the following Lemma, whose various forms already appear in [13, 14] etc.:

**Lemma 2.4.** (see [10], see also [13, 14]) Let $B \in (0, 1]$ be arbitrarily given. Let $D$ be a pseudoconvex domain in $\mathbb{C}^n$ containing $\partial D$. Let $\varphi$ be a negative plurisubharmonic function on $D$, such that $\varphi(o) = -\infty$. Let $F$ be an $L^2$ integrable holomorphic function on $\{\varphi < -t_0\}$. Then there exists a holomorphic function $\tilde{F}$ on $D$, such that

$$\tilde{F} - F, o \in \mathcal{I}(\varphi)_o$$

and

$$\int_D |\tilde{F} - (1 - b_{t_0,B}(\varphi))F|^2 \leq (1 - e^{-(t_0+B)}) \int_D \frac{1}{B} d_{\{(t_0-B < t < -t_0) \circ \varphi\}} |F|^2 e^{-\varphi},$$

(2.3)

where $d_{\{-t_0-B < t < -t_0\}}$ is the character function of set $\{-t_0 - B < t < -t_0\}$,

$$b_{t_0,B}(t) = \int_{-\infty}^t \frac{1}{B} d_{\{-t_0-B < s < -t_0\}} ds,$$

and $t_0 \geq 0$.

Although the $F$ in Lemma 2.1 in [10] is holomorphic on $D$, in fact the condition that $F$ is an $L^2$ integrable holomorphic function on $\{\varphi < -t_0\}$ is enough for the proof in [10] (details see Section 4.2).

Using Lemma 2.4, we present the following Lemma, which will be used to prove Proposition 2.4.

**Lemma 2.5.** Assume that $G(0) < +\infty$. Then for any $t_0 \in [0, +\infty)$, we have

$$G(0) - G(t_0) \leq (e^{t_0} - 1) \liminf_{B \to 0^+} \left( \frac{G(t_0 + B) - G(t_0)}{B} \right).$$
Proof. By Lemma 2.2, there exists a holomorphic function $F_{t_0}$ on $\{\phi < t_0\}$, such that, $(F_{t_0} - F, o) \in \mathcal{I}(\phi)_o$ and $\int_{\{\phi < -t_0\}} |F_{t_0}|^2 = G(t_0)$.

It suffices to consider that $\liminf_{B \to 0+0} \frac{G(t_0 + B) - G(t_0)}{B} \in (-\infty, 0]$ because of the decreasing property of $G(t)$. Then there exists $B_j \to 0+0$ ($j \to +\infty$) such that

$$\lim_{j \to +\infty} \frac{G(t_0 + B_j) - G(t_0)}{B_j} = \liminf_{B \to 0+0} \frac{G(t_0 + B) - G(t_0)}{B}$$

and $\{\frac{G(t_0 + B_j) - G(t_0)}{B_j}\}_{j \in \mathbb{N}^+}$ is bounded.

By Lemma 2.3, it follows that for any $B_j$, there exists holomorphic function $\tilde{F}_j$ on $D$, such that,

$$\left(\tilde{F}_j - F_{t_0}, o\right) \in \mathcal{I}(\phi)_o \quad (\Rightarrow (\tilde{F}_j - F, o) \in \mathcal{I}(\phi)_o)$$

and

$$\int_D |\tilde{F}_j - (1 - b_{t_0, B_j}(\phi))F_{t_0}|^2 \leq (1 - e^{-(t_0 + B_j)}) \int_D \frac{1}{B_j} |\mathbb{I}_{\{t_0 - B_j < \phi < -t_0\}} \circ \phi|F_{t_0}|^2 e^{-\phi} \tag{2.4}$$

Firstly, we will prove that $\int_D |\tilde{F}_j|^2$ is bounded with respect to $j$.

Note that

$$\int_D \frac{1}{B_j} |\mathbb{I}_{\{t_0 - B_j < \phi < -t_0\}} \circ \phi|F_{t_0}|^2 \leq \int_{\{\phi < -t_0\}} |F_{t_0}|^2 - \int_{\{\phi < -t_0 - B_j\}} |F_{t_0}|^2 \leq \frac{G(t_0) - G(t_0 + B_j)}{B_j}, \tag{2.5}$$

and

$$\left(\int_D |\tilde{F}_j - (1 - b_{t_0, B_j}(\phi))F_{t_0}|^2\right)^{1/2} \geq \left(\int_D |\tilde{F}_j|^2\right)^{1/2} - \left(\int_D |(1 - b_{t_0, B_j}(\phi))F_{t_0}|^2\right)^{1/2} \tag{2.6}$$

then it follows from inequality 2.4 that

$$\left(\int_D |\tilde{F}_j|^2\right)^{1/2} \leq e^{(t_0 + B_j)} - 1) \left(\frac{G(t_0) - G(t_0 + B_j)}{B_j}\right)^{1/2} + \left(\int_D |(1 - b_{t_0, B_j}(\phi))F_{t_0}|^2\right)^{1/2}. \tag{2.7}$$

Since $\{\frac{G(t_0 + B_j) - G(t_0)}{B_j}\}_{j \in \mathbb{N}^+}$ is bounded and $0 \leq b_{t_0, B_j}(\phi) \leq 1$, then $\int_D |\tilde{F}_j|^2$ is bounded with respect to $j$.

Secondly, we will prove the main result.

Note that $b_{t_0}(\phi) = 1$ on $\{\phi \geq -t_0\}$, the it follows that

$$\int_D |\tilde{F}_j - (1 - b_{t_0, B_j}(\phi))F_{t_0}|^2 = \int_{\{\phi \geq -t_0\}} |\tilde{F}_j|^2 + \int_{\{\phi < -t_0\}} |\tilde{F}_j - (1 - b_{t_0, B_j}(\phi))F_{t_0}|^2 \tag{2.8}$$
It is clear that
\[
\int_{\{\varphi < -t_0\}} |\hat{F}_j - (1 - b_{t_0}(\varphi))F_{t_0}|^2
\geq (\int_{\{\varphi < -t_0\}} |\hat{F}_j - F_{t_0}|^2)^{1/2} - (\int_{\{\varphi < -t_0\}} |b_{t_0,B_j}(\varphi)F_{t_0}|^2)^{1/2})^2
\geq \int_{\{\varphi < -t_0\}} |\hat{F}_j - F_{t_0}|^2 - 2\int_{\{\varphi < -t_0\}} |\hat{F}_j - F_{t_0}|^2)^{1/2}(\int_{\{\varphi < -t_0\}} |b_{t_0,B_j}(\varphi)F_{t_0}|^2)^{1/2}
\geq \int_{\{\varphi < -t_0\}} |\hat{F}_j - F_{t_0}|^2 - 2\int_{\{\varphi < -t_0\}} |\hat{F}_j - F_{t_0}|^2)^{1/2}(\int_{\{-t_0-B_j<\varphi<-t_0\}} |F_{t_0}|^2)^{1/2},
\]
(2.9)
where the last inequality follows from 0 \leq b_{t_0,B_j}(\varphi) \leq 1 and b_{t_0,B_j}(\varphi) = 0 on \{\varphi \leq -t_0 - B_0\}.

Combining equality 2.8, inequality 2.9 and equality 2.2, we obtain that
\[
\int_D |\hat{F}_j - (1 - b_{t_0,B_j}(\varphi))F_{t_0}|^2
= \int_{\{\varphi \geq -t_0\}} |\hat{F}_j|^2 + \int_{\{\varphi < -t_0\}} |\hat{F}_j - (1 - b_{t_0,B_j}(\varphi))F_{t_0}|^2
\geq \int_{\{\varphi \geq -t_0\}} |\hat{F}_j|^2 + \int_{\{\varphi < -t_0\}} |\hat{F}_j - F_{t_0}|^2
- 2\int_{\{\varphi < -t_0\}} |\hat{F}_j - F_{t_0}|^2)^{1/2}(\int_{\{-t_0-B_j<\varphi<-t_0\}} |F_{t_0}|^2)^{1/2},
\]
(2.10)

It follows from equality 2.2 that
\[
(\int_{\{\varphi < -t_0\}} |\hat{F}_j - F_{t_0}|^2)^{1/2} \leq (\int_{\{\varphi < -t_0\}} |\hat{F}_j|^2)^{1/2} \leq (\int_D |\hat{F}_j|^2)^{1/2}.
\]
(2.11)
Since \int_D |\hat{F}_j|^2 is bounded with respect to j, then it follows from inequality 2.11 that (\int_{\{\varphi < -t_0\}} |\hat{F}_j - F_{t_0}|^2)^{1/2} is bounded with respect to j. Using the dominated convergence theorem and \int_{\{\varphi < -t_0\}} |F_{t_0}|^2 = G(t_0) \leq G(0) < +\infty, we obtain that
\[
\lim_{j \to +\infty} \int_{\{-t_0-B_j<\varphi<-t_0\}} |F_{t_0}|^2d\lambda_n = 0,
\]
Then it follows that
\[
\lim_{j \to +\infty} (\int_{\{\varphi < -t_0\}} |\hat{F}_j - F_{t_0}|^2)^{1/2}(\int_{\{-t_0-B_j<\varphi<-t_0\}} |F_{t_0}|^2)^{1/2} = 0.
\]
Combining with equality 2.10, we obtain
\[
\liminf_{j \to +\infty} \int_D |\hat{F}_j - (1 - b_{t_0,B_j}(\varphi))F_{t_0}|^2 \geq \int_D |\hat{F}_j|^2 - \int_{\{\varphi < -t_0\}} |F_{t_0}|^2.
\]
(2.12)
Proof. For any \( \varphi \) be a negative plurisubharmonic function on \( \Phi \), Lemma 2.6.

Let \( \varphi \geq (4th \ " \geq \") \), we obtain

\[
\begin{align*}
(e^{t_0} - 1) \lim_{j \to +\infty} & \left( -\frac{G(t_0 + B_j) - G(t_0)}{B_j} \right) \\
& = \lim_{j \to +\infty} (e^{(t_0 + B_j)} - 1)\left( -\frac{G(t_0 + B_j) - G(t_0)}{B_j} \right) \\
& \geq \lim_{j \to +\infty} \inf (e^{(t_0 + B_j)} - 1) \int_D \frac{1}{B_j}(\tilde{I}_{\{t_0 - B < t < t_0\} \circ \varphi})|F_t|^2 \\
& \geq \lim_{j \to +\infty} \inf \int_D |\tilde{F}_j - (1 - b_{t_0, B_j}(\varphi))F_{t_0}|^2 \\
& \geq \int_D |\tilde{F}_j|^2 - \int_{\{\varphi < t_0\}} |F_{t_0}|^2 \geq G(0) - G(t_0).
\end{align*}
\]

Then Lemma 2.6 has thus been proved. \( \square \)

The following Lemma will be used to prove Theorem 1.1.

**Lemma 2.6.** Let \( F \) be a holomorphic function on pseudoconvex domain \( D \), and let \( \varphi \) be a negative plurisubharmonic function on \( D \). Assume that \( \int_D |F|^2 e^{-\varphi} < +\infty \). Then

\[
\int_D |F|^2 e^{-\varphi} = \int_{-\infty}^{+\infty} \left( \int_{\varphi < -t} |F|^2 \right) e^t dt.
\]

**Proof.** For any \( M \in \mathbb{N}^+ \), note that \( \frac{1}{2m} \sum_{i=1}^{2mM} \mathbb{I}_{\{\varphi < \frac{1}{2m} + i\}} \) is increasing with respect to \( m \) and convergent to \( e^{-\max\{\varphi, -\log M\}} \geq 0 \) \((m \to +\infty)\), then it follows from Levi’s Theorem that

\[
\begin{align*}
\int_D |F|^2 e^{-\varphi} &= \lim_{m \to +\infty} \sum_{i=1}^{2mM} \int_D |F|^2 \left( \frac{1}{2m} \sum_{i=1}^{2mM} \mathbb{I}_{\{\varphi < \frac{1}{2m} + i\}} \right) \\
& = \lim_{m \to +\infty} \frac{1}{2m} \sum_{i=1}^{2mM} \left( \int_D |F|^2 \mathbb{I}_{\{\varphi < \frac{1}{2m} + i\}} \right) \\
& = \lim_{m \to +\infty} \frac{1}{2m} \sum_{i=1}^{2mM} \left( \int_{\{\varphi < \frac{1}{2m} + i\}} |F|^2 \right),
\end{align*}
\]

where \( \mathbb{I}_A \) is the character function of set \( A \).

As \( \int_D |F|^2 \leq \int_D |F|^2 e^{-\varphi} < +\infty \), then \( \int_{\{\varphi > s\}} |F|^2 \) is finite and non-negative for any \( s \) and decreasing with respect to \( s \), which implies \( \int_{\{\varphi > s\}} |F|^2 \) is Riemann integrable and

\[
\int_0^M \left( \int_{\{\varphi > s\}} |F|^2 \right) ds = \lim_{m \to +\infty} \frac{1}{2m} \sum_{i=1}^{2mM} \left( \int_{\{\varphi > \frac{1}{2m} + i\}} |F|^2 \right), \quad (2.15)
\]

holds for any \( M \in \mathbb{N}^+ \). Note that \( \int_D |F|^2 e^{-\varphi} < +\infty \), then it follows from equality \( 2.13 \) and \( 2.15 \) that

\[
\int_D |F|^2 e^{-\max\{\varphi, -\log M\}} = \int_0^M \left( \int_{\{\varphi > s\}} |F|^2 \right) ds. \quad (2.16)
\]
As $\int_D |F|^2 e^{-\varphi} < +\infty$, then it follows from the Levi’s theorem that

$$\lim_{M \to +\infty} \int_D |F|^2 e^{-\max\{\varphi, -\log M\}} = \int_D |F|^2 e^{-\varphi} < +\infty. \quad (2.17)$$

Combining equality 2.16 and equality 2.17, we obtain that

$$\lim_{M \to +\infty} \int_0^M \left( \int_{\{e^{-\varphi} > s\}} |F|^2 ds \right) = \lim_{M \to +\infty} \int_D |F|^2 e^{-\max\{\varphi, -\log M\}}$$

$$= \int_D |F|^2 e^{-\varphi} < +\infty. \quad (2.18)$$

Note that $\int_0^{+\infty} (\int_{\{e^{-\varphi} > s\}} |F|^2 ds) ds = \lim_{M \to +\infty} \int_0^M (\int_{\{e^{-\varphi} > s\}} |F|^2 ds) ds < +\infty$, then it follows from equality 2.18 that

$$\int_0^{+\infty} \left( \int_{\{e^{-\varphi} > s\}} |F|^2 ds \right) ds = \int_D |F|^2 e^{-\varphi}. \quad (2.19)$$

Let $s = e^t$, then

$$\int_0^{+\infty} \left( \int_{\{e^{-\varphi} > s\}} |F|^2 ds \right) ds = \int_{-\infty}^{+\infty} \left( \int_{\{e^{-\varphi} > e^t\}} |F|^2 de^t \right)$$

$$= \int_{-\infty}^{+\infty} \left( \int_{\{\varphi < -t\}} |F|^2 \right) e^t dt. \quad (2.20)$$

Combining equality 2.19 and equality 2.20, we obtain Lemma 2.6.

2.2. **A sharp lower bound of the volume of the sublevel sets of plurisubharmonic functions with a multiplier.**

In order to prove Theorem 1.1 and Theorem 1.2, we present the following sharp lower bound of the volume of the sublevel sets of plurisubharmonic functions with a multiplier.

**Proposition 2.1.** Let $F$ be a holomorphic function on $D$, and let $\varphi$ be a negative plurisubharmonic function $\varphi$ on $D$. Then the inequality

$$\int_{\{\varphi < -t\}} |F|^2 \geq e^{-t} C_{F,\varphi}(D) \quad (2.21)$$

holds for any $t \geq 0$, which is sharp.

Especially, if $C_{F,\varphi}(D) = +\infty$, then $\int_{\{\varphi < -t\}} |F|^2 = +\infty$ for any $t \geq 0$.

Let $D = \Delta \subset \mathbb{C}$ be the unit disc, and let $\varphi = 2 \log |z|$ and $F \equiv 1$. It is clear that $C_{F,\varphi}(D) = \pi$, and $\int_{\{\varphi < -t\}} |F|^2 = e^{-t} \pi$, which gives the sharpness of Proposition 2.1.

**Proof.** We prove Proposition 2.1 in two steps, i.e. the case $C_{F,\varphi}(D) < +\infty$ and the case $C_{F,\varphi}(D) = +\infty$.

Step 1. We prove the case $C_{F,\varphi}(D) < +\infty$ As $\int_{\{\varphi < -t\}} |F|^2 \geq G(t)$ for any $t \in [0, +\infty)$, then it suffices to prove that $G(t) \geq e^{-t} G(0)$ for any $t \in [0, +\infty)$.

Let $H(t) := G(t) - e^{-t} G(0)$. We prove $H(t) \geq 0$ by contradiction: if not, then there exists $t$ such that $H(t) < 0$.

Note that $G(t) \in [0, G(0)]$ is bounded on $[0, \infty)$, then $H(t)$ is also bounded on $[0, \infty)$, which implies that $\inf_{[0, +\infty)} H(t)$ is finite.
By Lemma 2.3 it is clear that \( \lim_{t \to 0+0} H(t) = H(0) = 0 \) and \( \lim_{t \to +\infty} H(t) = \lim_{t \to +\infty} G(t) - \lim_{t \to +\infty} e^{-t} G(0) = 0 - 0 = 0 \). Then it follows from \( \inf_{[0, +\infty)} H(t) < 0 \) that there exists a closed interval \([a, b] \subseteq (0, \infty)\) such that \( \inf_{[a, b]} H(t) = \inf_{[0, +\infty)} H(t) \). Since \( G(t) \) is lower semi-continuous (Lemma 2.3) and \( e^{-t} G(0) \) is continuous, then it follows that \( H(t) \) is lower semi-continuous, which implies that there exists \( t_0 \in [a, b] \) such that \( H(t_0) = \inf_{[0, +\infty)} H(t) < 0 \).

In the following part of Step 1, we will consider the negativeness of \( (e^{t_0} - 1) \liminf_{t \to t_0+0} \left( -\frac{H(t) - H(t_0)}{t - t_0} \right) + H(t_0) \) and get a contradiction.

As \( H(t_0) = \inf_{[0, +\infty)} H(t) \), then it follows that \( \liminf_{t \to t_0+0} \left( -\frac{H(t) - H(t_0)}{t - t_0} \right) = -\limsup_{t \to t_0+0} \left( \frac{H(t) - H(t_0)}{t - t_0} \right) \leq 0 \). Combining with \( H(t_0) < 0 \), then we obtain that

\[
(e^{t_0} - 1) \liminf_{t \to t_0+0} \left( -\frac{H(t) - H(t_0)}{t - t_0} \right) + H(t_0) < 0. \tag{2.22}
\]

Note that

\[
(e^{t_0} - 1) \liminf_{t \to t_0+0} \left( -\frac{H(t) - H(t_0)}{t - t_0} \right) + H(t_0) \\
= (e^{t_0} - 1) \liminf_{t \to t_0+0} \left( -\frac{(G(t) - e^{-t} G(0)) \left( G(t_0) - e^{-t_0} G(0) \right)}{t - t_0} \right) \\
+ (G(t_0) - e^{-t_0} G(0)) \\
= (e^{t_0} - 1) \liminf_{t \to t_0+0} \left( -\frac{G(t) - G(t_0)}{t - t_0} \right) \\
+ (e^{t_0} - 1) \liminf_{t \to t_0+0} \left( \frac{e^{-t} G(0) - e^{-t_0} G(0)}{t - t_0} \right) + (G(t_0) - e^{-t_0} G(0)) \\
= (e^{t_0} - 1) \liminf_{t \to t_0+0} \left( -\frac{G(t) - G(t_0)}{t - t_0} \right) \\
- (e^{t_0} - 1)e^{-t_0} G(0) + (G(t_0) - e^{-t_0} G(0)) \\
= (e^{t_0} - 1) \liminf_{t \to t_0+0} \left( -\frac{G(t) - G(t_0)}{t - t_0} \right) + G(t_0) - G(0), \tag{2.23}
\]

then it follows from Lemma 2.4 that

\[
(e^{t_0} - 1) \liminf_{t \to t_0+0} \left( -\frac{H(t) - H(t_0)}{t - t_0} \right) + H(t_0) \geq 0, \tag{2.24}
\]

which contradicts inequality 2.22. The Case \( C_{F, \varphi}(D) < +\infty \) has thus been proved.

Step 2. We prove the case \( C_{F, \varphi}(D) = +\infty \) by contradiction: if not, then integral \( \int_{\{\varphi < -t_0\}} |F|^2 \) is finite for some \( t_0 \geq 0 \). It follows from Lemma 2.4 that for \( B = 1 \), there exists holomorphic function \( F \) on \( D \) satisfying

\[
(F - F, o) \in \mathcal{I}(\varphi)_{o}
\]

and

\[
\int_D |F - (1 - b_{t_0, B}(\varphi)) F|^2 \leq (e^{(t_0 + B)} - 1) \int_D \frac{1}{B} |(1 - B_{t_0-B < t < -t_0} o \varphi)|^2 |F|^2. \tag{2.25}
\]
Note that
\[ \int_D |\tilde{F} - (1 - b_{t_0,B}(\varphi))F|^2 \geq (\int_D |\tilde{F}|^2)^{1/2} - (\int_D |(1 - b_{t_0,B}(\varphi))F|^2)^{1/2}, \]  
(2.26)
then it follows from inequality (2.23) that
\[ (e^{(t_0+B)} - 1) \int_D \frac{1}{B} |(\tilde{F} - (1 - b_{t_0,B}(\varphi))F)|^2 \geq (\int_D |\tilde{F}|^2)^{1/2} - (\int_D |(1 - b_{t_0,B}(\varphi))F|^2)^{1/2}, \]  
(2.27)
which implies
\[ ((e^{(t_0+B)} - 1) \int_D \frac{1}{B} |(\tilde{F} - (1 - b_{t_0,B}(\varphi))F)|^2)^{1/2} \geq (\int_D |\tilde{F}|^2)^{1/2}. \]  
(2.28)
Since \( b_{t_0,B}(\varphi) = 1 \) on \( \{ \varphi \geq t_0 \} \), \( 0 \leq b_{t_0,B}(\varphi) \leq 1 \) and \( \int_{\{ \varphi < -t_0 \}} |F|^2 < +\infty \), then we obtain that \( \int_D |(1 - b_{t_0,B}(\varphi))F|^2 = \int_{\{ \varphi < -t_0 \}} |(1 - b_{t_0,B}(\varphi))F|^2 \leq \int_{\{ \varphi < -t_0 \}} |F|^2 < +\infty \) and \( \int_D \frac{1}{B} |(\tilde{F} - (1 - b_{t_0,B}(\varphi))F)|^2 < +\infty \). It is clear that the LHS of inequality (2.28) is finite, which implies that the RHS of inequality (2.28) is also finite. Then we obtain that \( C_{F,\varphi}(D) \leq \int_D |\tilde{F}|^2 < +\infty \), which contradicts \( C_{F,\varphi}(D) = +\infty \). The case \( C_{F,\varphi}(D) = +\infty \) has thus been proved.

\[ \square \]

3. Proofs of Theorem 1.1 and Theorem 1.2

In this section, we prove Theorem 1.1 and Theorem 1.2.

3.1. Proof of Theorem 1.1

It suffices to consider the case \( e^F(\varphi) \neq +\infty \).

Firstly, we prove the following proposition.

Proposition 3.1. Let \( F \) be a holomorphic function on pseudoconvex domain \( D \), and let \( \psi \) be a negative plurisubharmonic function on \( D \). Assume that \( C_{F,\psi}(D) \in (0, +\infty] \). Then for any \( p > 1 \), we have
\[ \int_D |F|^p e^{-\frac{\psi}{p}} \geq \frac{p}{p-1} C_{F,\psi}(D). \]  
(3.1)

Proof. If \( C_{F,\psi}(D) = +\infty \), then it is clear that \( \int_D |F|^p e^{-\frac{\psi}{p}} \geq C_{F,\psi}(D) = +\infty \).

It suffices to consider the situation that \( C_{F,\psi}(D) \in (0, +\infty) \) and \( \int_D |F|^p e^{-\frac{\psi}{p}} < +\infty \) both hold. Then Lemma 2.6 \((\varphi \sim \frac{\psi}{p})\) implies that
\[ \int_D |F|^p e^{-\frac{\psi}{p}} = \int_{-\infty}^{+\infty} (\int_{\{ \psi < t \}} |F|^2 e^{\varphi}) dt. \]  
(3.2)

It follows from \( C_{F,\psi}(D) \in (0, +\infty) \) and Proposition 2.4 that for \( t \geq 0 \),
\[ \int_{\{ \psi < -pt \}} |F|^2 = \int_{\{ \psi < -pt \}} |F|^2 \geq e^{-pt} C_{F,\psi}(D). \]
Then we obtain
\[
\int_0^{+\infty} \left( \int_{\{ p < -t \}} |F| e^t \right) dt \geq \int_0^{+\infty} e^{-pt} e^t C_{F,\psi}(D) = \frac{1}{p-1} C_{F,\psi}(D). \quad (3.3)
\]

As \( \int_{\{ p < -t \}} |F|^2 \geq C_{F,\psi}(D) \) holds for any \( t < 0 \), then it is clear that
\[
\int_{-\infty}^{0} \left( \int_{\{ p < -t \}} |F|^2 e^t \right) dt \geq C_{F,\psi}(D) \int_{-\infty}^{0} e^t dt = C_{F,\psi}(D) \quad (3.4)
\]

Combining equality \( 3.2 \), inequality \( 3.3 \) and inequality \( 3.4 \) we obtain Proposition \( 3.1 \). □

In the following part, we prove Theorem \( 1.2 \) by using Proposition \( 3.1 \).

Let \( \psi = p \varphi \). If \( p > 2c_o^F(\varphi) \), then it follows from \( C_{F,p\varphi}(D) \geq C_{F,I_+(2c_o^F(\varphi)\varphi)\varphi}(D) > 0 \) (maybe \( +\infty \)) and Proposition \( 3.1 \) that
\[
\int_D |F|^2 e^{-\varphi} \geq \frac{p}{p-1} C_{F,I_+(2c_o^F(\varphi)\varphi)\varphi}(D). \quad (3.5)
\]
Taking \( p \to 2c_o^F(\varphi) + 0 \), it is clear that equality \( 3.5 \) also holds for \( p \geq 2c_o^F(\varphi) \). Then we obtain that if
\[
\int_D |F|^2 e^{-\varphi} < \frac{p}{p-1} C_{F,I_+(2c_o^F(\varphi)\varphi)\varphi}(D),
\]
then \( p < 2c_o^F(\varphi) \), i.e. \( (F,o) \in \mathcal{I}(p\varphi,o) \). Theorem \( 1.1 \) has thus been proved.

3.2. **Proof of Theorem 1.2**

We prove Theorem \( 1.2 \) for the cases \( C_{F,I_+(2c_o^F(\varphi)\varphi)\varphi}(D) < +\infty \) and \( = +\infty \) respectively.

Step 1. We prove the case \( C_{F,I_+(2c_o^F(\varphi)\varphi)\varphi}(D) < +\infty \).

Proposition \( 2.1 \) shows that
\[
\int_{\{ p \varphi < -t \}} |F|^2 \geq e^{-t} C_{F,p\varphi}(D) \quad (3.6)
\]
holds for any \( t \geq 0 \) and \( p > 2c_o^F(\varphi) \).

By the Noetherian property of \( \mathcal{O}_o \), it follows that there exists \( p_0 > 2c_o^F(\varphi) \), such that \( \mathcal{I}_+(2c_o^F(\varphi)\varphi)\varphi = \mathcal{I}(p_0\varphi,o) \), which implies
\[
\lim_{p \to 2c_o^F(\varphi)+0} C_{F,p\varphi}(D) = C_{F,p_0\varphi}(D) = C_{F,I_+(2c_o^F(\varphi)\varphi)\varphi}(D). \quad (3.7)
\]

It follows from Lemma \( 2.1 \) that \( C_{F,p_0\varphi}(D) > 0 \) (maybe \( +\infty \)). Combining with equality \( 3.7 \) we obtain
\[
\lim_{p \to 2c_o^F(\varphi)+0} C_{F,p\varphi}(D) = C_{F,I_+(2c_o^F(\varphi)\varphi)\varphi}(D) > 0 \) (maybe \( +\infty \)) \quad (3.8)
\]

It follows from Levi’s theorem that
\[
\lim_{p \to 2c_o^F(\varphi)+0} \int_{\{ p \varphi < -t \}} |F|^2 = \int_{\{ 2c_o^F(\varphi)\varphi \leq -t \}} |F|^2 \quad (3.9)
\]
holds for any \( t \geq 0 \).
Combining equality 3.9 equality 3.6 and equality 3.8 we obtain

\[
\int \{2c_F^p(\varphi)\varphi \leq -t\} |F|^2 = \lim_{p \to 2c_F^p(\varphi)+0} \int \{p\varphi < -t\} |F|^2 = \lim_{p \to 2c_F^p(\varphi)+0} e^{-t} C_{F,p\varphi}(D) \geq e^{-t} C_{F,I+}(2c_F^p(\varphi)\varphi)_a(D) > 0, (\text{maybe } +\infty).
\]

Note that \(\{2c_F^p(\varphi)\varphi < -t\} = \bigcup_{t' > t} \{2c_F^p(\varphi)\varphi \leq -t'\}\), then it follows from inequality 3.10 that \(\int \{2c_F^p(\varphi)\varphi < -t\} |F|^2 \geq \sup_{(t' > t)} e^{-t'} C_{F,I+}(2c_F^p(\varphi)\varphi)_a(D) > 0, (\text{maybe } +\infty)\).

Let \(r = e^{-\frac{c_F^p(\varphi)}{t}}\), then the case \(C_{F,I+}(2c_F^p(\varphi)\varphi)_a(D) < +\infty\) has thus been proved.

Step 2. We prove the case \(C_{F,I+}(2c_F^p(\varphi)\varphi)_a(D) = +\infty\). By the Noetherian property of \(O_o\), it follows that there exists \(p_0 > 2c_F^p(\varphi)\), such that \(I_+(2c_F^p(\varphi)\varphi)_o = I(p_0\varphi)_o\). Then it is clear that \(C_{F,p\varphi}(D) = C_{F,I+}(2c_F^p(\varphi)\varphi)_a(D) = +\infty\). It follows from Proposition 2.1 that for any \(t \in [0, +\infty)\), \(\int \{p\varphi < -t\} |F|^2 = +\infty\), which implies that \(\int \{\varphi < -t\} |F|^2 = +\infty\) for any \(t \in [0, +\infty)\). Then the case \(C_{F,I+}(2c_F^p(\varphi)\varphi)_a(D) = +\infty\) has thus been proved.

4. APPENDIX: CONCAVITY OF MINIMAL L^2 INTEGRALS ON SUBLEVEL SETS OF PLURISUBHARMONIC FUNCTIONS

In section 2 we consider the minimal \(L^2\) integrals \(G(t)\) on the sublevel sets of the weights related to multiplier ideals, and obtain that \(G(t) \geq e^{-t} G(0)\).

In the present section, we consider a generalized version of \(G(t)\) (details see 4.2), and reveal the concavity of \(G(-\log r)\), which was contained in section 2.

**Proposition 4.1.** If \(G(0) < +\infty\), then \(G(-\log r)\) is concave with respect to \(r \in (0, 1]\).

Especially, the above result is a generalization of \(G(t) \geq e^{-t} G(0)\).

Choosing \(\psi\) as the polar function \(\psi + \log |z_n|^2\) in 25 (see also 12), \(G(t)\) is the minimal \(L^2\) extension with negligible weight on \(\{\psi + \log |z_n|^2 < -t\}\) in 25 (see also 12).

4.1. SOME RESULTS CONTAINED IN SECTION 2 In the following part, we recall and reveal some results contained section 2.

Let \(D \subset \mathbb{C}^n\) be a pseudoconvex domain containing \(o \in \mathbb{C}^n\), and let \(\varphi\) be a locally upper bounded Lebesgue measurable function on \(D\). Let \(f\) be a holomorphic function near \(o\), and let \(I \subset O_o\) be an ideal. \(C_{f,I}(D, \varphi)\) denotes \(\inf \left\{ \int_D |\hat{f}|^2 e^{-\varphi} |(\hat{f} - f, o) \in I \& \hat{f} \in O(D)\right\}\).

If there is no holomorphic function \(\hat{f}\) satisfying both \((\hat{f} - f, o) \in I\) and \(\hat{f} \in O(D)\), then we set \(C_{f,I}(D, \varphi) = -\infty\). Especially, if \(I = I(\psi)_o\), then \(C_{f,\psi}(D, \varphi)\) denotes \(C_{f,I}(D, \varphi)\).

The proof of Lemma 2.1 contains

**Lemma 4.1.** \((f, o) \notin I(\psi)_o \Leftrightarrow C_{f,\psi}(D, \varphi) \neq 0\) (maybe \(-\infty\) or \(+\infty\)).

The proof of Lemma 2.2 contains

**Lemma 4.2.** Let \(\varphi + \psi < 0\) be plurisubharmonic functions on \(D\), and let \(F\) be a holomorphic function on \(\{\psi < -t\}\). Assume that \(C_{F,\varphi+\psi}(\{\psi < -t\}, \varphi) < \)
Then there exists a unique holomorphic function $F_t$ on $\{\psi < -t\}$ satisfying $(F_t - F, o) \in \mathcal{I}(\varphi + \psi)$, and $\int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi} = C_{F, \varphi + \psi}(\{\psi < -t\}, \varphi)$. Furthermore, for any holomorphic function $F$ on $\{\psi < -t\}$ satisfying $(F - F, o) \in \mathcal{I}(\varphi + \psi)$, and $\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} < +\infty$, we have the following equality

$$\int_{\{\psi < -t\}} |F_t|^2 e^{-\varphi} + \int_{\{\psi < -t\}} |\hat{F} - F_t|^2 e^{-\varphi} = \int_{\{\psi < -t\}} |\hat{F}|^2 e^{-\varphi}. \quad (4.1)$$

Let $F$ be a holomorphic function on $D$, and let $\varphi + \psi$ and $\psi < 0$ be plurisubharmonic functions on $D$. Denote that

$$G(t) := C_{F, \varphi + \psi}(\{\psi < -t\}, \varphi). \quad (4.2)$$

The proof of Lemma 2.3 contains

**Lemma 4.3.** Assume that $G(0) < +\infty$. Then $G(t)$ is decreasing with respect to $t \in [0, +\infty)$, such that $\lim_{t \to t_0 + 0} G(t) = G(t_0)$ ($t_0 \in [0, +\infty)$), $\lim_{t \to t_0 - 0} G(t) = G(t_0)$ ($t_0 \in (0, +\infty)$), and $\lim_{t \to +\infty} G(t) = 0$, where $t_0 \in [0, +\infty)$. Especially $G(t)$ is lower semi-continuous on $[0, +\infty)$.

The proof of Lemma 2.1 in [10], implies the following Lemma (details see section 4.2), whose various forms already appear in [13 [14].

**Lemma 4.4.** (see [13 [14]) Let $B \subset (0, 1]$ be arbitrarily given. Let $D$ be a pseudo-convex domain in $\mathbb{C}^n$ containing $o$. Let $\psi$ be a negative plurisubharmonic function on $D$, such that $\psi(o) = -\infty$. Let $\varphi$ be a locally upper bounded function on $D$, such that $\varphi + \psi$ is plurisubharmonic on $D$. Let $F$ be a holomorphic function on $\{\psi < -t_0\}$ such that $\int_{\{\psi < -t_0\}} |F|^2 e^{-\varphi} < +\infty$. Then there exists a holomorphic function $\hat{F}$ on $D$, such that,

$$(\hat{F} - F, o) \in \mathcal{I}(\varphi + \psi).$$

and

$$\int_D |\hat{F} - (1 - b_{t_0, B}(\psi))F|^2 e^{-\varphi} \leq (1 - e^{-(t_0 + B)}) \int_D \frac{1}{B} (\mathbb{I}_{\{t_0 - B < t < -t_0\}} \circ \psi)|F|^2 e^{-\varphi - \psi}, \quad (4.3)$$

where $\mathbb{I}_{\{t_0 - B < t < -t_0\}}$ is the character function of set $\{t_0 - B < t < -t_0\}$, $b_{t_0, B}(t) = \int_{t_0 - B}^t \frac{1}{B} \mathbb{I}_{\{t_0 - B < s < -t_0\}} ds$, and $t_0 \geq 0$.

It follows from Lemma 4.4 that the proof of Lemma 2.5 contains

**Lemma 4.5.** Assume that $G(0) < +\infty$. Then for any $t_0 \in [0, +\infty)$, we have

$$G(0) - G(t_0) \leq (e^{t_0} - 1) \liminf_{B \to 0+0} \left(-\frac{G(t_0 + B) - G(t_0)}{B}\right).$$

By replacing $D$ with the component of $\{\psi < -t_1\}$ containing $o$ and replacing $\psi$ with $\psi + t_1$, it follows from Lemma 4.4 that the proof of Lemma 2.5 contains

**Lemma 4.6.** Assume that $G(0) < +\infty$. Then for any $t_0, t_1 \in [0, +\infty)$, we have

$$G(t_1) - G(t_1 + t_0) \leq (e^{t_0} - 1) \liminf_{B \to 0+0} \left(-\frac{G(t_0 + t_1 + B) - G(t_0 + t_1)}{B}\right),$$

i.e.

$$\frac{G(t_1) - G(t_1 + t_0)}{e^{-t_1} - e^{-t_0+t_1}} \leq \liminf_{B \to 0+0} \frac{G(t_0 + t_1 + B) - G(t_0 + t_1)}{e^{-t_0+t_1+B} - e^{-t_0+t_1}} \quad (4.4)$$
As $G(-\log r)$ is lower semicontinuous (Lemma 4.3), then it follows from the following well-known property of concave functions (Lemma 4.7) that Lemma 4.6 is equivalent to Proposition 4.1.

**Lemma 4.7.** Let $a(r)$ be a lower semicontinuous function on $(0, 1]$. Then $a(r)$ is concave if and only if

\[
\frac{a(r_1) - a(r_2)}{r_1 - r_2} \leq \liminf_{r_3 \to r_2^{-}} \frac{a(r_3) - a(r_2)}{r_3 - r_2},
\]

holds for any $0 < r_2 < r_1 \leq 1$.

4.2. A unified approach to Lemma 2.4 and Lemma 4.4. We prove the following Lemma, which is a unified approach to Lemma 2.4 and Lemma 4.4 whose various forms already appear in [13, 14] etc.:

**Lemma 4.8.** Let $B \in (0, +\infty)$ and $t_0 \geq 0$ be arbitrarily given. Let $D$ be a pseudoconvex domain in $\mathbb{C}^n$. Let $\psi$ be a negative plurisubharmonic function on $D$. Let $\varphi$ be a plurisubharmonic function on $D$. Let $F$ be a holomorphic function on \( \{ \psi < -t_0 \} \), such that

\[
\int_{K \cap \{ \psi < -t_0 \}} |F|^2 < +\infty
\]

for any compact subset $K$ of $D$, and

\[
\int_D \frac{1}{B} \mathbb{I}_{\{ -t_0 - B < \psi < -t_0 \}} |F|^2 e^{-\varphi} d\lambda_n \leq C < +\infty.
\]

Then there exists a holomorphic function $\tilde{F}$ on $D$, such that

\[
\int_D |\tilde{F} - (1 - b(\psi))F|^2 e^{-\varphi + \psi} d\lambda_n \leq (1 - e^{-(t_0 + B)})C
\]

where $b(t) = \int_{-t_0}^{t} \frac{1}{B} \mathbb{I}_{\{ -t_0 - B < s < -t_0 \}} ds$, and $\psi(t) = \int_{t_0}^{t} b(s) ds$.

It is clear that $\mathbb{I}_{\{ -t_0, +\infty \}} \leq b(t) \leq \mathbb{I}_{\{ -t_0 - B, +\infty \}}$ and $\max\{ t, -t_0 - B \} \leq \psi(t) \leq \max\{ t, -t_0 \}$.

It suffices to consider the case of Lemma 4.8 that $D$ is a strongly pseudoconvex domain and $\varphi$ and $\psi$ are plurisubharmonic functions on an open set $U$ containing $D$, and $F$ is a holomorphic function on $U \cap \{ \psi < -t_0 \}$ such that $\int_{D \cap \{ \psi < -t_0 \}} |F|^2 < +\infty$. In the following remark, we recall some standard steps (see e.g. [29, 13, 14]) to illustrate it.

**Remark 4.1.** It is well-known that there exist strongly pseudoconvex domains $D_1 \subset \cdots \subset D_j \subset \cdots \subset D_{j+1} \subset \cdots$ such that $\bigcup_{j=1}^{\infty} D_j = D$.

If inequality (4.7) holds on any $D_j$ and inequality (4.6) holds on $D$, then we obtain a sequence of holomorphic functions $\tilde{F}_j$ on $D_j$ such that

\[
\int_{D_j} |\tilde{F}_j - (1 - b(\psi))F|^2 e^{-\varphi + \psi} d\lambda_n \\
\leq (1 - e^{-(t_0 + B)}) \int_{D_j} \frac{1}{B} \mathbb{I}_{\{ -t_0 - B < \psi < -t_0 \}} |F|^2 e^{-\varphi} d\lambda_n \leq (1 - e^{-(t_0 + B)})C
\]

is bounded with respect to $j$. Note that for any given $j$, $e^{-\varphi + \psi}$ has a positive lower bound, then it follows that for any given $j$, $\int_{D_j} |\tilde{F}_j' - (1 - b(\psi))F|^2$ is
bounded with respect to \( j' \geq j \). Combining with
\[
\int_{D_j} \left| (1 - b(\psi))F \right|^2 \leq \int_{D_j \cap \{ \psi < -t_0 \}} |F|^2 < +\infty \tag{4.9}
\]
and inequality [4.7] one can obtain that \( \int_{D_j} |\tilde{F}_{j'}|^2 \) is bounded with respect to \( j' \geq j \).

By diagonal method, there exists a subsequence \( F'_{j'} \) uniformly convergent on any \( D_j \) to a holomorphic function on \( D \) denoted by \( \tilde{F} \). Then it follows from inequality [4.9] and the dominated convergence theorem that
\[
\int_{D_j} |\tilde{F} - (1 - b(\psi))F|^2 e^{-\max\{\varphi - v(\psi), -M\}} d\lambda_n \leq (1 - e^{-(t_0 + B)})C
\]
for any \( M > 0 \), which implies
\[
\int_{D_j} |\tilde{F} - (1 - b(\psi))F|^2 e^{-(\varphi - v(\psi))} d\lambda_n \leq (1 - e^{-(t_0 + B)})C,
\]
then one can obtain Lemma [4.5] when \( j \) goes to \( +\infty \).

For the sake of completeness, we recall some lemmas on \( L^2 \) estimates for some \( \bar{\partial} \) equations, and \( \bar{\partial}^* \) means the Hilbert adjoint operator of \( \bar{\partial} \).

**Lemma 4.9. (see [29], see also [1])** Let \( \Omega \subset \subset \mathbb{C}^n \) be a domain with \( C^\infty \) boundary \( b\Omega \), \( \Phi \in C^\infty(\overline{\Omega}) \). Let \( \rho \) be a \( C^\infty \) defining function for \( \Omega \) such that \( |d\rho| = 1 \) on \( b\Omega \). Let \( \eta \) be a smooth function on \( \overline{\Omega} \). For any \((0,1)\)-form \( \alpha = \sum_{j=1}^n \alpha_j d\bar{z}^j \in \text{Dom}_\Omega(\bar{\partial}^*) \cap C^\infty_{(0,1)}(\overline{\Omega}) \),

\[
\int_{\Omega} \eta|\bar{\partial}_\Phi \alpha|^2 e^{-\Phi} d\lambda_n + \int_{\Omega} \eta|\bar{\partial}_\alpha \alpha|^2 e^{-\Phi} d\lambda_n = \sum_{i,j=1}^n \int_{\Omega} \eta|\bar{\partial}_{ij} \alpha_j|^2 d\lambda_n
\
+ \sum_{i,j=1}^n \int_{\Omega} \eta(\bar{\partial}_\alpha \bar{\partial}_\rho \alpha)\alpha_i \bar{\alpha}_j e^{-\Phi} dS + \sum_{i,j=1}^n \int_{\Omega} \eta(\bar{\partial}_\rho \bar{\partial}_\alpha \Phi)\alpha_i \bar{\alpha}_j e^{-\Phi} d\lambda_n \tag{4.10}
\
+ \sum_{i,j=1}^n \int_{\Omega} -\bar{\partial}_\alpha \eta \alpha_i \bar{\alpha}_j e^{-\Phi} d\lambda_n + 2\text{Re}(\bar{\partial}_\Phi \alpha, \alpha \cdot (\bar{\partial}_\eta)^\sharp)_{\Omega,\Phi},
\]

where \( d\lambda_n \) is the Lebesgue measure on \( \mathbb{C}^n \), and \( \alpha \cdot (\bar{\partial}_\eta)^\sharp = \sum_j \alpha_j \bar{\partial}_j \eta \).

The symbols and notations can be referred to [14]. See also [29], [30], or [33].

**Lemma 4.10. (see [1], see also [14])** Let \( \Omega \subset \subset \mathbb{C}^n \) be a strictly pseudoconvex domain with \( C^\infty \) boundary \( b\Omega \) and \( \Phi \in C^\infty(\overline{\Omega}) \). Let \( \lambda \) be a \( \bar{\partial} \) closed smooth form of bidegree \((n,1)\) on \( \overline{\Omega} \). Assume the inequality
\[
|\langle \lambda, \alpha \rangle_{\Omega,\Phi}|^2 \leq C \int_{\Omega} |\bar{\partial}_\Phi \alpha|^2 \frac{e^{-\Phi}}{\mu} d\lambda_n < \infty,
\]
where \( \frac{1}{\mu} \) is an integrable positive function on \( \Omega \) and \( C \) is a constant, holds for all \((n,1)\)-form \( \alpha \in \text{Dom}_\Omega(\bar{\partial}^*) \cap \text{Ker}(\bar{\partial}) \cap C^\infty_{(n,1)}(\overline{\Omega}) \). Then there is a solution \( u \) to the equation \( \bar{\partial}u = \lambda \) such that
\[
\int_{\Omega} |u|^2 \mu e^{-\Phi} d\lambda_n \leq C.
\]
Proof. (Proof of Lemma 4.8)

For the sake of completeness, let’s recall some steps in our proof in [13] (see also [14, 16]) with some slight modifications in order to prove Lemma 4.8.

By Remark 4.1, one can assume that $D$ is strongly pseudoconvex (with smooth boundary), and $\psi$ and $\varphi$ are plurisubharmonic on an open set $U$ containing $D$, and $F$ is holomorphic on $U \cap \{ \psi < -t_0 \}$ and

$$\int_{\{ \psi < -t_0 \} \cap D} |F|^2 < +\infty.$$  \hspace{1cm} (4.11)

Then it follows from method of convolution (see e.g. [5]) that there exist smooth plurisubharmonic functions $\psi_m$ and $\varphi_m$ on an open set $U \subset D$ decreasing convergent to $\psi$ and $\varphi$ respectively, such that $\sup_m \sup_D \psi_m < 0$ and $\sup_m \sup_D \varphi_m < +\infty$.

**Step 1: recall some Notations**

Let $\varepsilon \in (0, \frac{1}{8} B)$. Let $\{ v_\varepsilon \}_{\varepsilon \in (0, \frac{1}{8} B)}$ be a family of smooth increasing convex functions on $\mathbb{R}$, which are continuous functions on $\mathbb{R} \cup \{ -\infty \}$, such that:

1. $v_\varepsilon(t) = t$ for $t \geq -t_0 - \varepsilon$, $v_\varepsilon(t) = \text{constant}$ for $t < -t_0 - B + \varepsilon$;
2. $v''_\varepsilon(t)$ are pointwise convergent to $\frac{1}{B} \mathbb{I}_{(-t_0 - B, -t_0)}$, when $\varepsilon \to 0$, and $0 \leq v''_\varepsilon(t) \leq \frac{2}{B} \mathbb{I}_{(-t_0 - B + \varepsilon, -t_0 - \varepsilon)}$ for any $t \in \mathbb{R}$;
3. $v'_\varepsilon(t)$ are pointwise convergent to $b(t)$ which is a continuous function on $\mathbb{R} \cup \{ -\infty \}$, when $\varepsilon \to 0$, and $0 \leq v'_\varepsilon(t) \leq 1$ for any $t \in \mathbb{R}$.

One can construct the family $\{ v_\varepsilon \}_{\varepsilon \in (0, \frac{1}{8} B)}$ by the setting

$$v_\varepsilon(t) := \int_{-\infty}^{t} \left( \int_{-\infty}^{t_1} \left( \frac{1}{B - 4 \varepsilon} \mathbb{I}_{(-t_0 - B + 2 \varepsilon, -t_0 - 2 \varepsilon)} \ast \rho_{\frac{1}{4} \varepsilon}(s) \right) ds \right) dt_1 \hspace{1cm} (4.12)$$

where $\rho_{\frac{1}{4} \varepsilon}$ is the kernel of convolution satisfying $\text{supp}(\rho_{\frac{1}{4} \varepsilon}) \subset (-\frac{1}{4} \varepsilon, \frac{1}{4} \varepsilon)$. Then it follows that

$$v''_\varepsilon(t) = \frac{1}{B - 4 \varepsilon} \mathbb{I}_{(-t_0 - B + 2 \varepsilon, -t_0 - 2 \varepsilon)} \ast \rho_{\frac{1}{4} \varepsilon}(t),$$

and

$$v'_\varepsilon(t) = \int_{-\infty}^{t} \left( \frac{1}{B - 4 \varepsilon} \mathbb{I}_{(-t_0 - B + 2 \varepsilon, -t_0 - 2 \varepsilon)} \ast \rho_{\frac{1}{4} \varepsilon}(s) \right) ds.$$  

It suffices to consider the case that

$$\int_{D} \frac{1}{B} \mathbb{I}_{\{ \psi < -t_0 \}} |F|^2 e^{-\psi - \varphi} d\lambda_n < +\infty.  \hspace{1cm} (4.13)$$

Let $\eta = s(-v_\varepsilon(\psi_m))$ and $\phi = u(-v_\varepsilon(\psi_m))$, where $s \in C^\infty((0, +\infty))$ satisfies $s \geq 0$, and $u \in C^\infty((0, +\infty))$, satisfies $\lim_{t \to +\infty} u(t) = 0$, such that $u'' - s'' > 0$, and $s' - s'' = 1$. It follows from $\sup_m \sum_D \psi_m < 0$ that $\phi = u(-v_\varepsilon(\psi_m))$ are uniformly bounded on $D$ with respect to $m$ and $\varepsilon$, and $u(-v_\varepsilon(\psi_m))$ are uniformly bounded on $D$ with respect to $\varepsilon$. Let $\Phi = \phi + \varphi_m$.

**Step 2: Solving $\bar{\partial}$—equation with smooth polar function and smooth weight**
Now let \( \alpha = \sum_{j=1}^{n} \alpha_j d\bar{z}^j \in \text{Dom}_{D} (\partial^*) \cap \text{Ker}(\partial) \cap C_{(0,1)}^\infty (\overline{D}) \). By Cauchy-Schwarz inequality, it follows that

\[
2\text{Re}(\partial^*_\Phi \alpha, \alpha L(\partial \eta))^2_{D, \Phi} \geq - \int_{D} g^{-1} |\partial^*_\Phi \alpha|^2 e^{-\Phi} d\lambda_n + \sum_{j,k=1}^{n} \int_{D} (-g(\partial_j \eta) \partial_k \eta) \alpha_j \bar{\alpha}_k e^{-\Phi} d\lambda_n.
\] (4.14)

Using Lemma 4.9 and inequality 4.14, since \( s \geq 0 \) and \( \psi_m \) is a plurisubharmonic function on \( \overline{D} \), we get

\[
\int_{D} (\eta + g^{-1}) |\partial^*_\Phi \alpha|^2 e^{-\Phi} d\lambda_n \geq \sum_{j,k=1}^{n} \int_{D} (-\partial_j \partial_k \eta + \eta \partial_j \partial_k \Phi - g(\partial_j \eta) \partial_k \eta) \alpha_j \bar{\alpha}_k e^{-\Phi} d\lambda_n
\]

\[
\geq \sum_{j,k=1}^{n} \int_{D} (-\partial_j \partial_k \eta + \eta \partial_j \partial_k \phi - g(\partial_j \eta) \partial_k \eta) \alpha_j \bar{\alpha}_k e^{-\Phi} d\lambda_n,
\] (4.15)

where \( g \) is a positive continuous function on \( D \). We need some calculations to determine \( g \).

We have

\[
\partial_j \partial_k \eta = -s'(v_{\epsilon}(\psi_m)) \partial_j \partial_k (v_{\epsilon}(\psi_m)) + s''(-v_{\epsilon}(\psi_m)) \partial_j v_{\epsilon}(\psi_m) \partial_k v_{\epsilon}(\psi_m), \quad (4.16)
\]

and

\[
\partial_j \partial_k \phi = -u'(-v_{\epsilon}(\psi_m)) \partial_j \partial_k v_{\epsilon}(\psi_m) + u''(-v_{\epsilon}(\psi_m)) \partial_j v_{\epsilon}(\psi_m) \partial_k v_{\epsilon}(\psi_m) \quad (4.17)
\]

for any \( j, k \) (\( 1 \leq j, k \leq n \)).

We have

\[
\sum_{1 \leq j, k \leq n} (-\partial_j \partial_k \eta + \eta \partial_j \partial_k \phi - g(\partial_j \eta) \partial_k \eta) \alpha_j \bar{\alpha}_k
\]

\[
=(s' - su') \sum_{1 \leq j, k \leq n} \partial_j \partial_k v_{\epsilon}(\psi_m) \alpha_j \bar{\alpha}_k
\]

\[
+((u''s - s'') - gs'^2) \sum_{1 \leq j, k \leq n} \partial_j (-v_{\epsilon}(\psi_m)) \partial_k (v_{\epsilon}(\psi_m)) \alpha_j \bar{\alpha}_k
\]

\[
=(s' - su') \sum_{1 \leq j, k \leq n} (v_{\epsilon}'(\psi_m) \partial_j \partial_k \psi_m + v_{\epsilon}'(\psi_m) \partial_j (\partial_k \psi_m)) \alpha_j \bar{\alpha}_k
\]

\[
+((u''s - s'') - gs'^2) \sum_{1 \leq j, k \leq n} \partial_j (-v_{\epsilon}(\psi_m)) \partial_k (v_{\epsilon}(\psi_m)) \alpha_j \bar{\alpha}_k.
\] (4.18)

We omit composite item \(-v_{\epsilon}(\psi_m)\) after \( s' - su' \) and \((u''s - s'') - gs'^2\) in the above equalities.

Let \( g = \frac{u''s - s''}{su'^2}(-v_{\epsilon}(\psi_m)) \). It follows that \( \eta + g^{-1} = (s + \frac{u''s - s''}{su'^2})(-v_{\epsilon}(\psi_m)) \).

Because of \( v_{\epsilon}' \geq 0 \) and \( s' - su' = 1 \), using inequalities 4.15 we have

\[
\int_{D} (\eta + g^{-1}) |\partial^*_\Phi \alpha|^2 e^{-\Phi} d\lambda_n \geq \int_{D} (v_{\epsilon}' \circ \psi_m) |\alpha L(\partial \psi_m)|^2 e^{-\Phi} d\lambda_n.
\] (4.19)
As $F$ is holomorphic on $\{\psi < -t_0\} \supset \text{Supp}(v'_\varepsilon(\psi_m))$, then $\lambda := \overline{\partial}(1 - v'_\varepsilon(\psi_m)F)$ is well-defined and smooth on $D$. By the definition of contraction, Cauchy-Schwarz inequality and inequality \[4.19\] it follows that
\[
|\langle \lambda, \alpha \rangle_{D, \Phi}|^2 = |\langle \nu'_\varepsilon(\psi_m), \overline{\partial}\psi_mF, \alpha \rangle_{D, \Phi}|^2 = |\langle \nu''_\varepsilon(\psi_m), \alpha \rangle_{D, \Phi}|^2 \leq \int_D (\nu''_\varepsilon(\psi_m)|F|^2 e^{-\Phi} d\lambda) \int_D (\nu''_\varepsilon(\psi_m)|\alpha \overline{\partial}\psi_m|^2 e^{-\Phi} d\lambda) \lesssim \int_D \nu''_\varepsilon(\psi_m)|F|^2 e^{-\Phi} d\lambda_n(\int_D (\eta + g^{-1})|\overline{\partial}\alpha| e^{-\Phi} d\lambda_n).
\]

Let $\mu := (\eta + g^{-1})^{-1}$. Using Lemma \[4.10\] we have locally $L^1$ function $u_{m,m',\varepsilon}$ on $D$ such that $\partial u_{m,m',\varepsilon} = \lambda$, and
\[
\int_D |u_{m,m',\varepsilon}|^2 (\eta + g^{-1})^{-1} e^{-\Phi} d\lambda_n \leq \int_D (\nu''_\varepsilon(\psi_m))|F|^2 e^{-\Phi} d\lambda_n. \tag{4.21}
\]
Assume that we can choose $\eta$ and $\phi$ such that $e^{\nu_\varepsilon \varphi_m} e^{\nu_\varepsilon \varphi_m} = (\eta + g^{-1})^{-1}$. Then inequality \[4.21\] becomes
\[
\int_D |u_{m,m',\varepsilon}|^2 e^{\nu_\varepsilon \varphi_m - \nu_\varepsilon \varphi_{m'}} d\lambda_n \leq \int_D (\nu''_\varepsilon(\psi_m))|F|^2 e^{-\Phi} d\lambda_n. \tag{4.22}
\]
Let $F_{m,m',\varepsilon} := -u_{m,m',\varepsilon} + (1 - v'_\varepsilon(\psi_m))F$. Then inequality \[4.22\] becomes
\[
\int_D |F_{m,m',\varepsilon} - (1 - v'_\varepsilon(\psi_m))F|^2 e^{\nu_\varepsilon \varphi_m - \nu_\varepsilon \varphi_{m'}} d\lambda_n \leq \int_D (\nu''_\varepsilon(\psi_m))|F|^2 e^{-\Phi} d\lambda_n. \tag{4.23}
\]

**Step 3: Singular polar function and smooth weight**

As $\sup_{m,\varepsilon}|\phi| = \sup_{m,\varepsilon}|u(-v'_\varepsilon(\psi_m))| < +\infty$ and $\varphi_{m'}$ is continuous on $\overline{D}$, then $\sup_{m,\varepsilon} e^{-\nu_\varepsilon \varphi_{m'}} < +\infty$. Note that
\[
\nu''_\varepsilon(\psi_m)|F|^2 e^{-\nu_\varepsilon \varphi_{m'}} \lesssim \frac{2}{B} \int_{\{\psi < -t_0\}} |F|^2 \sup_{m,\varepsilon} e^{-\nu_\varepsilon \varphi_{m'}}
\]
on $D$, then it follows from inequality \[4.11\] and the dominated convergence theorem that
\[
\lim_{m \to +\infty} \int_D \nu''_\varepsilon(\psi_m)|F|^2 e^{-\nu_\varepsilon \varphi_{m'}} d\lambda_n = \int_D \nu''_\varepsilon(\psi)|F|^2 e^{-u(-v'_\varepsilon(\psi)) - \varphi_{m'}} d\lambda_n \tag{4.24}
\]
Note that $\inf_m \inf_D e^{v'_\varepsilon(\psi_m) - \varphi_{m'}} > 0$, then it follows from inequality \[4.23\] and \[4.24\] that $\sup_m \int_D |F_{m,m',\varepsilon} - (1 - v'_\varepsilon(\psi_m))F|^2 < +\infty$. Note that
\[
|(1 - v'_\varepsilon(\psi_m))F| \leq |\mathbb{I}_{\{\psi < -t_0\}} F|, \tag{4.25}
\]
then it follows from inequality \[4.11\] that $\sup_m \int_D |F_{m,m',\varepsilon}|^2 < +\infty$, which implies that there exists a subsequence of $\{F_{m,m',\varepsilon}\}_m$ (also denoted by $F_{m,m',\varepsilon}$) compactly convergent to a holomorphic $F_{m',\varepsilon}$ on $D$.

Note that $v'_\varepsilon(\psi_m) - \varphi_{m'}$ are uniformly bounded on $D$ with respect to $m$, then it follows from $|F_{m,m',\varepsilon} - (1 - v'_\varepsilon(\psi_m))F|^2 \leq 2(|F_{m,m',\varepsilon}|^2 + |(1 - v'_\varepsilon(\psi_m))F|^2 \leq$
\[ 2(|F_{m',\varepsilon}|^2 + \|\psi_{t_{0}}\| F^2) \] and the dominated convergence theorem that
\[
\lim_{\varepsilon \to +\infty} \int_{K} |F_{m,\varepsilon}|^2 e^{u_{\varepsilon}(\psi_{m})} - \varphi_{m'} d\lambda_n
\]
\[
= \int_{K} |F_{m',\varepsilon}|^2 e^{u_{\varepsilon}(\psi_{m})} - \varphi_{m'} d\lambda_n
\] (4.26)
holds for any compact subset \( K \) on \( D \). Combining with inequality (4.23) and (4.24), one can obtain that
\[
\int_{K} |F_{m',\varepsilon}|^2 e^{u_{\varepsilon}(\psi_{m})} - \varphi_{m'} d\lambda_n \leq \int_{D} v''_{\varepsilon}(\psi)|F|^2 e^{-u(\psi)} - \varphi_{m'} d\lambda_n,
\] (4.27)
which implies
\[
\int_{D} |F_{m',\varepsilon}|^2 e^{u(\psi)} - \varphi_{m'} d\lambda_n \leq \int_{D} v''_{\varepsilon}(\psi)|F|^2 e^{-u(\psi)} - \varphi_{m'} d\lambda_n,
\] (4.28)

**Step 4: Nonsmooth cut-off function**

Note that \( \sup_{\varepsilon} \inf_{D} e^{-u_{\varepsilon}(\psi)} - \varphi_{m'} > 0 \), then it follows from inequality (4.11) and the dominated convergence theorem that
\[
\lim_{\varepsilon \to 0} \int_{D} v''_{\varepsilon}(\psi)|F|^2 e^{-u_{\varepsilon}(\psi)} - \varphi_{m'} d\lambda_n
\]
\[
= \int_{D} \frac{1}{B} \|_{\{t_{0} - B < \psi < -t_{0}\}}|F|^2 e^{-u(\psi)} - \varphi_{m'} d\lambda_n
\] (4.29)
\[
\leq (\sup_{\varepsilon} e^{-u(\psi)}) \int_{D} \frac{1}{B} \|_{\{t_{0} - B < \psi < -t_{0}\}}|F|^2 e^{-\varphi_{m'}} < +\infty.
\]

Note that \( \inf_{\varepsilon} \inf_{D} e^{u_{\varepsilon}(\psi)} - \varphi_{m'} > 0 \), then it follows from inequality (4.28) and (4.29) that \( \sup_{\varepsilon} \int_{D} |F_{m',\varepsilon}|^2 < +\infty \). Combining with
\[
\sup_{\varepsilon} \int_{D} |(1 - v'_{\varepsilon}(\psi))|F|^2 \leq \int_{\{\psi < -t_{0}\}} |F|^2 < +\infty,
\] (4.30)
one can obtain that \( \sup_{\varepsilon} \int_{D} |F_{m',\varepsilon}|^2 < +\infty \), which implies that there exists a subsequence of \( \{F_{m',\varepsilon}\}_{\varepsilon \to 0} \) (also denoted by \( \{F_{m',\varepsilon}\}_{\varepsilon \to 0} \)) compactly convergent to a holomorphic function on \( D \) denoted by \( F_{m'} \).

Note that \( \sup_{\varepsilon} \sup_{D} e^{u_{\varepsilon}(\psi)} - \varphi_{m'} < +\infty \) and \( |F_{m',\varepsilon} - (1 - v'_{\varepsilon}(\psi))|F|^2 \leq 2(|F_{m',\varepsilon}|^2 + \|_{\{\psi < -t_{0}\}} F^2) \), then it follows from inequality (4.30) and the dominated convergence theorem on any given \( K \subset D \) (with dominant function \( 2(\sup_{\varepsilon} \sup_{K} (|F_{m',\varepsilon}|^2) + \|_{\{\psi < -t_{0}\}} F^2) \sup_{\varepsilon} \inf_{D} e^{u_{\varepsilon}(\psi)} - \varphi_{m'} \) that
\[
\lim_{\varepsilon \to 0} \int_{K} |F_{m',\varepsilon} - (1 - v'_{\varepsilon}(\psi))|F|^2 e^{u_{\varepsilon}(\psi)} - \varphi_{m'} d\lambda_n
\]
\[
= \int_{K} |F_{m'} - (1 - b(\psi))|F|^2 e^{u(\psi)} - \varphi_{m'} d\lambda_n.
\] (4.31)
Combining with inequality \[4.29\] and \[4.28\] one can obtain that
\[
\int_K |F_{m'} - (1 - b(\psi))F|^2 e^{v(\psi)-\varphi_{m'}} d\lambda_n
\leq \left( \sup_D e^{-u(-v(\psi))} \right) \int_D \frac{1}{B} \mathbb{I}_{\{-n - B < \psi < -t_0\}} |F|^2 e^{-\varphi_{m'}} \tag{4.32}
\]
which implies
\[
\int_D |F_{m'} - (1 - b(\psi))F|^2 e^{v(\psi)-\varphi_{m'}} d\lambda_n
\leq \left( \sup_D e^{-u(-v(\psi))} \right) \int_D \frac{1}{B} \mathbb{I}_{\{-n - B < \psi < -t_0\}} |F|^2 e^{-\varphi_{m'}}. \tag{4.33}
\]

**Step 5: Singular weight**

Note that
\[
\int_D \frac{1}{B} \mathbb{I}_{\{-n - B < \psi < -t_0\}} |F|^2 e^{-\varphi_{m'}} \leq \int_D \frac{1}{B} \mathbb{I}_{\{-n - B < \psi < -t_0\}} |F|^2 e^{-\varphi} < +\infty, \tag{4.34}
\]
and \(\sup_D e^{-u(-v(\psi))} < +\infty\), then it from \[4.33\] that
\[
\sup_{m'} \int_D |F_{m'} - (1 - b(\psi))F|^2 e^{v(\psi)-\varphi_{m'}} d\lambda_n < +\infty.
\]
Combining with \(\inf_{m'} \inf_D e^{v(\psi)-\varphi_{m'}} > 0\), one can obtain that \(\sup_{m'} \int_D |F_{m'} - (1 - b(\psi))F|^2 d\lambda_n < +\infty\). Note that
\[
\int_D |(1 - b(\psi))F|^2 d\lambda_n \leq \int_D \mathbb{I}_{\{\psi < -t_0\}} |F|^2 d\lambda_n < +\infty. \tag{4.35}
\]
Then \(\sup_{m'} \int_D |F_{m'}|^2 d\lambda_n < +\infty\), which implies that there exists a compactly convergent subsequence of \(\{F_{m'}\}\) denoted by \(\{F_{m''}\}\), which is convergent a holomorphic function \(\bar{F}\) on \(D\).

Note that \(\sup_{m''} \sup_D e^{v(\psi)-\varphi_{m''}} < +\infty\), then it follows from inequality \[4.35\] and the dominated convergence theorem on any given compact subset \(K\) of \(D\) (with dominant function \(2[\sup_{m''} \sup_K (|F_{m''}|^2) + \mathbb{I}_{\{\psi < -t_0\}} |F|^2] \sup_D e^{v(\psi)-\varphi_{m''}}\)) that
\[
\lim_{m'' \to +\infty} \int_K |F_{m''} - (1 - b(\psi))F|^2 e^{v(\psi)-\varphi_{m''}} d\lambda_n = \int_K |\bar{F} - (1 - b(\psi))F|^2 e^{v(\psi)-\varphi_{m''}} d\lambda_n. \tag{4.36}
\]
Note that for any \(m'' \geq m', \varphi_{m''} \leq \varphi_{m'}\) holds, then it follows from inequality \[4.33\] and \[4.34\] that
\[
\lim_{m'' \to +\infty} \int_K |F_{m''} - (1 - b(\psi))F|^2 e^{v(\psi)-\varphi_{m''}} d\lambda_n
\leq \limsup_{m'' \to +\infty} \int_K |F_{m''} - (1 - b(\psi))F|^2 e^{v(\psi)-\varphi_{m''}} d\lambda_n
\leq \limsup_{m'' \to +\infty} \sup_D e^{-u(-v(\psi))} \int_D \frac{1}{B} \mathbb{I}_{\{-n - B < \psi < -t_0\}} |F|^2 e^{-\varphi_{m''}}
\leq (\sup_D e^{-u(-v(\psi)))} C < +\infty. \tag{4.37}
\]
Combining with equality $[4.36]$ one can obtain that
\[
\int_K |\tilde{F} - (1 - b(\psi))F|^2 e^{v(\psi) - \varphi} \, d\lambda_n \leq \left( \sup_D e^{-u(\psi)} \right) C,
\]
for any compact subset of $D$, which implies
\[
\int_D |\tilde{F} - (1 - b(\psi))F|^2 e^{v(\psi) - \varphi} \, d\lambda_n \leq \left( \sup_D e^{-u(\psi)} \right) C.
\]
When $m' \to +\infty$, it follows from Levi’s Theorem that
\[
\int_D |\tilde{F} - (1 - b(\psi))F|^2 e^{v(\psi) - \varphi} \, d\lambda_n \leq \left( \sup_D e^{-u(\psi)} \right) C. \tag{4.38}
\]

**Step 6: ODE system**

It suffices to find $\eta$ and $\phi$ such that $(\eta + \psi - 1) = e^{-\psi} e^{-\phi}$ on $D$. As $\eta = s(-v_1(\psi_m))$ and $\phi = u(-v_1(\psi_m))$, we have $(\eta + \psi - 1) e^{v_1(\psi_m)} e^\phi = \left( s + \frac{s^2}{u'' - s''} \right) e^{-t} e^u \circ (-v_1(\psi_m))$.

Summarizing the above discussion about $s$ and $u$, we are naturally led to a system of ODEs (see [12, 13, 14, 16]):
\[
\begin{align*}
1). & \quad \left( s + \frac{s^2}{u'' - s''} \right) e^{u'} = 1, \\
2). & \quad s' - su' = 1, \tag{4.39}
\end{align*}
\]

where $t \in [0, +\infty)$.

It is not hard to solve the ODE system \[4.39\] and get $u = -\log(1 - e^{-t})$ and $s = \frac{t}{1 - e^{-t}} - 1$. It follows that $s \in C^\infty([0, +\infty))$ satisfies $s \geq 0$, $\lim_{t \to +\infty} u(t) = 0$ and $u \in C^\infty((0, +\infty))$ satisfies $u'' - s'' > 0$.

As $u = -\log(1 - e^{-t})$ is decreasing with respect to $t$, then it follows from $0 \geq v(t) \geq \max\{t, -t_0 - B_0\} \geq -t_0 - B_0$ for any $t \leq 0$ that
\[
\sup_D e^{-u(\psi)} \leq \sup_{t \in (0, t_0 + B]} e^{-u(t)} = \sup_{t \in (0, t_0 + B]} (1 - e^{-t}) = 1 - e^{-(t_0 + B)}, \tag{4.40}
\]
therefore we are done. Thus we prove Lemma \[4.38\]

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