THE GLASSY PHASE OF THE COMPLEX BRANCHING BROWNIAN MOTION ENERGY MODEL

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ABSTRACT. We identify the fluctuations of the partition function for a class of random energy models, where the energies are given by the positions of the particles of the complex-valued branching Brownian motion (BBM). Specifically, we provide the weak limit theorems for the partition function in the so-called “glassy phase” – the regime of parameters, where the behaviour of the partition function is governed by the extrema of BBM. We allow for arbitrary correlations between the real and imaginary parts of the energies. This extends the recent result of Madaule, Rhodes and Vargas [17], where the uncorrelated case was treated. In particular, our result covers the case of the real-valued BBM energy model at complex temperatures.

1. INTRODUCTION

Phase transitions arise via an analyticity breaking of the logarithm of the partition function. To analyse this phenomenon, the study of partition functions at complex temperatures is of a key interest, as was observed by Lee and Yang [20, 16]. Another motivation to study complex-valued Hamiltonians comes from quantum physics. There, partition functions with complex energies emerge naturally, e.g., from the Schrödinger equation via “imaginary time” Feynman’s path integrals.

It is believed that large classes of models of disordered systems fall in the same universality class and, in particular, share the same shape of the phase diagram. Random energy models were proven to be useful in exploring universality classes in mean-field disordered systems, see the recent books by Bovier [6] and Panchenko [19]. A number of random energy models with complex energies has been considered in the literature. One of the simplest such models (in view of the correlation structure of the energies) is the so called Random Energy Model (REM). For this model, the analyticity of the log-partition function was studied in the seminal work by Derrida [9] and later by Koukiou [14]. The full phase diagram of this model at complex temperatures including the fluctuations and zeros of the partition function were identified by Kabluchko and one of us in [11]. In particular, the case of arbitrary correlations between the imaginary and real parts of the energies was considered in [11]. The same authors answered in [12] similar questions about the Generalized Random Energy model (GREM) – a model with hierarchical correlations – and obtained the full phase diagram. In the complex GREM, the phase diagram turned out to
have a much richer structure than that of the complex REM. This sheds some light on the phase diagrams of the models beyond the complex REM universality class.

Nevertheless, it is known that models with logarithmic correlations between the energies can still belong to the REM universality class. Specifically, this has been shown for a discrete disordered system with complex-valued energies on a tree by Derrida, Evans, and Speer [10], and for a model of complex multiplicative cascades by Barral, Jin and Mandelbrot [5]. Lacoin, Rhodes and Vargas [15] analysed the phase diagram for complex Gaussian multiplicative chaos – a model with logarithmic correlations between the energies on a Euclidean space. There, only the case without correlations between the imaginary and real parts of the energy was treated. It turned out that the phase diagram coincides with the REM one, see Figure 1.

In [15], the analysis of the so-called “glassy” phase $B_2$, see Figure 1, was left open. In this phase, the partition function is dominated by the extreme values of the energies. Phase $B_2$ was analysed by Madaule, Rhodes and Vargas [17] in a continuous model with logarithmic correlations on a tree – the complex BBM energy model, but again only when the imaginary and real parts of the energies are uncorrelated. In this model, a deeper understanding of phase $B_2$ is possible due to recent progress in the analysis of the extremal process, see the contributions by Aïdékon, Berestycki, Brunet, and Shi [1]; and Arguin, Bovier, and Kistler [3]. Madaule, Rhodes, and Vargas [18], have recently analysed the behaviour of the partition function on the boundary between phases $B_1$ and $B_2$ (see Figure 1).

In this article, we extend the result of [17]. Specifically, we prove the weak convergence of the (rescaled) partition function of the complex BBM energy model in phase $B_2$ to a non-trivial distribution. We allow for arbitrary correlations between the real and imaginary

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**Figure 1.** Phase diagram of the REM (and conjecturally of the BBM energy model). The grey curves are the level lines of the limiting log-partition function, cf. (1.15). This paper deals with phase $B_2$ only.
parts of the energy. In particular, this covers the complex temperature case, in which the real and imaginary parts of the random energies have maximal correlation (i.e., they are a.s. equal). This case is especially relevant for the Lee-Yang program.

1.1. **Branching Brownian motion.** Before stating our results, let us briefly recall the construction of a BBM. Consider a continuous time Galton-Watson (GW) process \[4\] with the fertility distribution \(p_k, k \geq 1\). That is, \(p_k\) is the probability for a particle to give birth to exactly \(k \in \mathbb{Z}_+\) children particles. Clearly, it must hold that \(\sum_{k=1}^{\infty} p_k = 1\). In addition, we assume that \(\sum_{k=1}^{\infty} kp_k = 2\) (i.e., the expected number of children per particle equals two) and \(K = \sum_{k=1}^{\infty} k(k-1)p_k < \infty\) (finite second moment). At time \(t = 0\), the GW process starts with just one particle. At any time \(t\), we label the particles of the process as \(i_1(t), \ldots, i_n(t)\), where \(n(t)\) is the total number of particles at time \(t\). Note that under the above assumptions, we have \(E[n(t)] = e^t\). In addition to the genealogical structure, the particles receive a position in \(\mathbb{R}\). Specifically, the first particle starts at the origin at time zero and performs Brownian motion until the first time the GW process branches. After branching, each new-born particle independently performs Brownian motion (started at the branching location) until their respective next branching times, and so on. We denote the positions of the \(n(t)\) particles at time \(t\) by \(x_1(t), \ldots, x_n(t)\). BBM is defined as the time-evolving collection of the particle positions

\[[X(t) := (x_1(t), \ldots, x_{n(t)}(t)) \in \mathbb{R}^{n(t)}: t \in \mathbb{R}_+\].

Bramson [7, 8] showed that \(m(t) := \sqrt{2t} - \frac{3}{2\sqrt{2}} \log t\) is the order of the maximal position among all BBM particles at large time \(t\), i.e.,

\[\lim_{t \uparrow \infty} \mathbb{P} \left\{ \max_{k \leq n(t)} x_k(t) - m(t) \leq y \right\} = \mathbb{E} \left[ e^{-CZe^{-\sqrt{2}y}} \right], \quad y \in \mathbb{R},\]

where \(C > 0\) is a constant and \(Z\) is the a.s. limit of the so-called derivative martingale:

\[Z := \lim_{t \uparrow \infty} \sum_{k=1}^{n(t)} (\sqrt{2t} - x_k(t))e^{-\sqrt{2}t-x_k(t)}, \quad \text{a.s.}\]

In [1, 3], the point process of the extremal particles in BBM was identified. Specifically, it was shown that the point process,

\[\mathcal{E}_t := \sum_{k=1}^{n(t)} \delta_{x_k(t)-m(t)}, \quad t \in \mathbb{R}_+\]

converges as \(t \uparrow \infty\) in law to the point process

\[\mathcal{E} := \sum_{k,l} \delta_{\eta_k + \Delta_l}\]

where:

(a) \(\{\eta_k\}_{k \in \mathbb{N}} \subset \mathbb{R}\) are the atoms of a Cox process with random intensity measure \(CZe^{-\sqrt{2}y}dy\), where \(C\) and \(Z\) are the same as in (1.3).
1.3. Main results. In this paper, we focus on the glassy phase (cf., Figure 1):

\[ B_2 := \{\sigma + i\tau \in \mathbb{C} : |\sigma| > \sqrt{2}/2, |\sigma| + |\tau| > \sqrt{2}\}. \]  

We start with convergence in the case of the real BBM energy model at complex temperatures. We say that a complex-valued r.v. \( Y \) is isotropic \( \alpha \)-stable if there exists \( c \in \mathbb{R}_+ \) such that

\[ \mathbb{E} [e^{i\Re(izY)}] = e^{-|z|^\alpha}, \quad \text{for all } z \in \mathbb{C}. \]  

Recall the notation in (1.6).

**Theorem 1.1** (Partition function fluctuations for \(|\rho| = 1\)). For \( \beta = \sigma + i\tau \in B_2 \), the rescaled partition function \( \tilde{X}_{\beta,1}(t) := e^{-\beta m(t)}\tilde{X}_{\beta,1}(t) \) converges in law to the r.v.

\[ \mathcal{X}_{\beta,1} := \sum_{k,l \geq 1} e^{\beta (y_k + \Delta_k^{(l)})}, \quad \text{as } t \uparrow \infty. \]  

For \(|\rho| \in (0,1)\), we get the following convergence.

**Theorem 1.2** (Partition function fluctuations for \(|\rho| \in (0,1)\)). For \( \beta = \sigma + i\tau \in B_2 \), and \(|\rho| \in (0,1)\), the rescaled partition function \( \tilde{X}_{\beta,\rho}(t) := e^{-\sigma m(t)}\tilde{X}_{\beta,\rho}(t) \) converges in law to the r.v. \( \mathcal{X}_{\beta,\rho} \) as \( t \uparrow \infty \). Conditionally on \( Z \), \( \mathcal{X}_{\beta,\rho} \) is a complex isotropic \( \sqrt{2}/\sigma \)-stable r.v.

For \( \rho = 0 \), Theorem 1.2 was proven in [17].
Remark. Note that the fluctuations of the partition function in the complex BBM energy model (cf., Theorems 1.1, 1.2) are governed by the extremal process \( \mathcal{E} \). Thus, the fluctuations are different from the ones in the complex REM [11, Theorems 2.8, 2.20] which are governed by a Poisson point process. Despite the differences in fluctuations, we conjecture that in the limit as \( t \uparrow \infty \) the log-partition function

\[
 p_t(\beta) := \frac{1}{t} \log |\tilde{X}_{\beta,\rho}(t)|, \quad t \in \mathbb{R}_+, \quad \beta \in \mathbb{C}
\]  

(1.14)

of the complex BBM energy model is the same as in the complex REM.

**Conjecture 1.3 (Phase diagram).** For any \( \rho \in [-1, 1] \), the complex BBM energy model has the same free energy and the phase diagram (cf., Figure 1) as the complex REM, i.e.,

\[
 \lim_{t \uparrow \infty} p_t(\beta) =: p(\beta) =
\begin{cases}
 1 + \frac{1}{2}(\sigma^2 - \tau^2), & \beta \in B_1, \\
 \sqrt{2}|\sigma|, & \beta \in B_2, \\
 \frac{1}{2} + \sigma^2, & \beta \in B_3,
\end{cases}
\]

and the convergence in (1.15) holds in probability and in \( L^1 \).

Remark. Convergence in probability for \( \beta \in B_2 \) in (1.15), follows from Theorems 1.1 and 1.2 by [11, Lemma 3.9 (1)].

**Organization of the rest of the paper.** The proofs of Theorems 1.1 and 1.2 consist of two main steps. First, we show that only the extremal particles can contribute to the partition function in the limit as \( t \uparrow \infty \) (cf., Proposition 2.1 and its proof in Section 3). Second, we use the continuous mapping theorem to deduce Theorems 1.1 and 1.2 from the behaviour of the extremal process. This is done in Section 2.

2. Convergence of the Partition Function

First, we state that in the glassy phase \( B_2 \) only the extremal particles can contribute to the limit of the partition function as \( t \) tends to infinity.

**Proposition 2.1.** If \( |\rho| \in (0, 1] \) and \( \beta \in B_2 \), then, for all \( \delta, \epsilon > 0 \), there exists \( A_0 > 0 \) such that, for all \( A > A_0 \) and all \( t \) sufficiently large,

\[
 \mathbb{P} \left\{ \left| \sum_{k=1}^{n(t)} e^{\sigma(x_k(t)-m(t)) + i r y_k(t)} 1_{\{x_k(t) - m(t) < -A\}} \right| > \delta \right\} < \epsilon. \tag{2.1}
\]

The proof of Proposition 2.1 is postponed until Section 3. Using Proposition 2.1 together with the continuous mapping theorem, we now prove Theorem 1.1

**Proof of Theorem 1.1** Consider for \( A \in \mathbb{R}_+ \) the functional \( \Phi_{\beta, A} \) that maps a locally finite counting measure \( \zeta = \sum_{i \in I} \delta_{x_i} \) to \( \Phi_{\beta, A}(\zeta) = \sum_{i \in I} e^{\beta x_i} 1_{\{x_i > -A\}} \), where \( I \) is a countable index set. This functional is continuous for all \( \zeta \) that do not charge \(-A\) and \(+\infty\), which is a set of full measure with respect to the law of \( \mathcal{E} \). Hence, by the continuous mapping theorem, it follows that \( \Phi_{\beta, A}(\mathcal{E}_t) \) converges in law to \( \Phi_{\beta, A}(\mathcal{E}) \), which is equal to

\[
 \sum_{k,l \geq 1} e^{\beta (\eta_k + \Delta_t^{(k)} + \xi_l)} 1_{\{\eta_k + \Delta_t^{(k)} + \xi_l \geq -A\}}. \tag{2.2}
\]

Note that by Proposition 2.1, for all \( \epsilon > 0 \) and \( \delta > 0 \), there exists \( A_0 \) such that, for all \( A > A_0 \) and all \( t \) sufficiently large,

\[
 \mathbb{P} \{ |X_{\beta,\rho}(t) - \Phi_{\beta, A}(\mathcal{E}_t)| > \delta \} < \epsilon. \tag{2.3}
\]
Hence, by Slutsky’s Theorem (see, e.g., [13 Theorem 13.18]), \( \mathcal{X}_{\beta,1}(t) \) converges in law to
\[
\lim_{\delta \to 0} \sum_{k,l \geq 1} e^{\beta \left( \eta_k + \Delta^k(l) \right)} 1_{\{\eta_k + \Delta^k(l) \geq -\delta\}}
\]
which is equal to \( \mathcal{X}_{\beta,1} \).

We turn to the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Using Representation (1.9), we have that \( \mathcal{X}_{\beta,\rho}(t) \) is in distribution equal to
\[
\sum_{k=1}^{n(t)} e^{(\sigma + i \rho \tau)(z_k - m(t)) + i \sqrt{1 - \rho^2} \tau z_k(t) - i \rho \tau m(t)},
\]
where \( (z_k(t), k \leq n(t)) \) are the particles from a BBM that is independent from \( X(t) \) (but with respect to the same underlying GW process). If \( |\rho| \neq 1 \), then by [17, see Lemma 3.2 and the subsequent discussion before Eq (3.7) therein] we get that
\[
G(t) := \sum_{k=1}^{n(t)} \delta_{(x_k(t) - m(t), \exp(i \sqrt{1 - \rho^2} \tau z_k(t) - i \rho \tau m(t)))}
\]
converges weakly as \( t \to \infty \) to
\[
\mathcal{G} := \sum_{k,l \geq 1} \delta_{(\eta_k + \Delta^k(l), U(l)W_l^k)},
\]
where \( (U(k))_{k \geq 1} \) are i.i.d. uniformly distributed on the unit circle and \( W_l^k \) are the atoms of a point process on the unit circle. The description of \( W_l^k \) could be made more explicit using the description of the cluster process \( \Delta \) obtained in [1 Theorem 2.3] that encodes the genealogical structure of \( \Delta \).

For \( A \in \mathbb{R}_+ \), consider the functional \( \tilde{\Phi}_{\beta,A} \) that maps a locally finite counting measure \( \tilde{\zeta} = \sum_{k \in I} \delta_{(x_k,z_k)} \) to \( \tilde{\Phi}_{\beta,A}(\tilde{\zeta}) = \sum_{k \in I} e^{\beta z_k} z_k 1_{\{x_k > -A\}} \), where \( I \) is a countable index set. This functional is continuous w.r.t. \( \tilde{\zeta} \) that do not change \((A, \cdot)\) and \((+\infty, \cdot)\). Hence, by the continuous mapping theorem, it follows that \( \tilde{\Phi}_{\sigma+i\rho \tau,A}(\mathcal{G}_t) \) converges in law to \( \tilde{\Phi}_{\sigma+i\rho \tau,A}(\mathcal{G}) \), which is equal to
\[
\sum_{k,l \geq 1} e^{(\sigma + i \rho \tau)(\eta_k + \Delta^k(l))} U(l)W_l^k 1_{\{\eta_k + \Delta^k(l) \geq -A\}}.
\]

Observe that (2.8) is by the definition of the uniform distribution on the circle equal in distribution to
\[
\sum_{k,l \geq 1} e^{\sigma (\eta_k + \Delta^k(l))} U(l)W_l^k 1_{\{\eta_k + \Delta^k(l) \geq -A\}}.
\]

Note that again by Proposition 2.1, for all \( \epsilon > 0 \) and \( \delta > 0 \), there exists \( A_0 \) such that, for all \( A > A_0 \) and all \( t \) sufficiently large,
\[
\mathbb{P} \left\{ \left| \mathcal{X}_{\beta,\rho}(t) - \tilde{\Phi}_{\sigma+i\rho \tau,A}(\mathcal{G}_t) \right| > \delta \right\} < \epsilon.
\]

Hence, by Slutsky’s theorem (see, e.g., [13 Theorem 13.18]), \( \mathcal{X}_{\beta,\rho}(t) \) converges in law to
\[
\lim_{\delta \to 0} \sum_{k,l \geq 1} e^{\sigma (\eta_k + \Delta^k(l))} U(l)W_l^k 1_{\{\eta_k + \Delta^k(l) \geq -A\}} = \sum_{k,l \geq 1} e^{\sigma (\eta_k + \Delta^k(l))} U(l)W_l^k.
\]
We rewrite (2.11) as

$$\sum_{k \geq 1} e^{\sigma m} U^{(k)} W^{(k)}, \tag{2.12}$$

where $W^{(k)} := \sum_i e^{\sigma \Delta^{(k)}_i} \tilde{W}^{(k)}_i$, $k \geq 1$ are i.i.d. r.v.'s. From (2.12), it follows that conditionally on $Z$, the distribution of $X_{\beta,\rho}$ is complex isotropic $\sqrt{2/\sigma}$-stable.

$$\square$$

3. PROOF OF PROPOSITION 2.1

Due to symmetry, we only prove Proposition 2.1 for $\sigma, \tau > 0$. In the proof of Proposition 2.1, we distinguish two cases:

(a) $\sigma > \sqrt{2}$;

(b) $\sqrt{2}/2 < \sigma \leq \sqrt{2}$ but $\sigma + \tau > \sqrt{2}$.

In case (a), the proof works as in the independent case treated in [17, Lemma 3.5]. For completeness, we also give the proof in this case. We use a first moment computation together with the upper bound on the maximal position of all particles obtained in [2, Theorem 2.2].

Proof of Proposition 2.1 in case (a). By [2, Theorem 2.2], for $0 < \gamma < \frac{1}{2}$, there exists $r_u > 0$ such that for all $r > r_u$ and $t > 3r$

$$P \{ \exists k \leq n(t): x_k(s) > U_{t,\gamma} \text{ for some } s \in [r, t-r] \} < \frac{\epsilon}{2}, \tag{3.1}$$

where $U_{t,\gamma}(s) := \frac{2}{t} m(t) + \left( s \wedge (t-s) \right) \gamma$. Define the following set on the space of paths

$$U_{t,r,\gamma} := \{ x(\cdot) \in C(\mathbb{R}_+, \mathbb{R}): x(s) \leq \frac{s}{t} m(t) + \left( s \wedge (t-s) \right) \gamma, \forall s \in [r, t-r] \}. \tag{3.2}$$

By (3.1), to show (2.1), it suffices to check that, for sufficiently large $A > 0$,

$$P \left\{ \sum_{k=1}^{n(t)} e^{\sigma(x_k(t)-m(t)) + i\tau y_k(t)} 1_{\{x_k(t)-m(t)<A\} \cap \{x_k \in U_{t,\gamma}\}} > \delta \right\} < \epsilon/2. \tag{3.3}$$

By Markov’s inequality, the probability in (3.3) is bounded from above by

$$\frac{1}{\delta} \mathbb{E} \left[ \sum_{k=1}^{n(t)} e^{\sigma(x_k(t)-m(t)) + i\tau y_k(t)} 1_{\{x_k(t)-m(t)<A\} \cap \{x_k \in U_{t,\gamma}\}} \right] \leq \frac{1}{\delta} \mathbb{E} \left[ \sum_{k=1}^{n(t)} e^{\sigma(x_k(t)-m(t))} 1_{\{x_k(t)-m(t)<A\} \cap \{x_k \in U_{t,\gamma}\}} \right]. \tag{3.4}$$

We rewrite the expectation in the r.h.s. of (3.4) as $\sum_{B>A} S(B, t)$, where

$$S(B, t) := \mathbb{E} \left[ \sum_{k=1}^{n(t)} e^{\sigma(x_k(t)-m(t))} 1_{\{x_k(t)-m(t)\in(-B+1,-B]\} \cap \{x_k \in U_{t,\gamma}\}} \right]. \tag{3.5}$$

Next, we manipulate the event

$$\{x_k(t) - m(t) \in (-B + 1, -B]\} \cap \{x_k \in U_{t,\gamma}\} \cap \{x_k(t) - m(t) \epsilon (-B + 1, -B]\} \cap \{\xi(s) \leq \frac{s}{t} B + (s \wedge (t-s)) \gamma, \forall s \in [r, t-r]\}, \tag{3.6}$$
where \(\xi_k(s) := x_k(s) - \frac{s}{t} x_k(t)\) is a Brownian bridge from 0 to 0 in time \(t\) that is independent from \(x_k(t)\). Hence, we can bound \(S(B, t)\) from above by

\[
\mathbb{E} \left[ \sum_{k=1}^{n(t)} e^{\sigma(x_k(t)-m(t))} \mathbb{1}_{\{x_k(t)-m(t)\in (-B+1,-B] \cap \xi_k(s) \leq \frac{s}{t} B + (s \wedge (t-s))\gamma, \forall s \in [r, t-r] \}} \right] \tag{3.7}
\]

where \(x(t)\) is normal distributed with mean 0 and variance \(t\) and \(\xi(\cdot)\) is a Brownian bridge from 0 to 0 in time \(t\) independent from \(x(t)\). The expectation in the second line of (3.7) is equal to

\[
\int_{m(t)-B}^{m(t)-1} e^{\sigma(x-m(t))} e^{-x^2/2t} \sqrt{2\pi t} \, dx = e^{-\sigma m(t)+x_0^2/2} \int_{m(t)-B-\sigma t}^{m(t)-B+1-\sigma t} e^{-w^2/2t} \, dw, \tag{3.8}
\]

where we changed variables \(x = w + \sigma t\). Since \(\sigma > \sqrt{2}\), by the definition of \(m(t)\) it holds that \(m(t) - B - \sigma t < (\sqrt{2} - \sigma) t < 0\), for all \(t > 1\). Therefore, using the standard Gaussian tail bound,

\[
\int_{-\infty}^{-x} e^{-w^2/2} \, dw \leq \frac{1}{\sqrt{2\pi x}} e^{-x^2/2}, \quad x > 0, \tag{3.9}
\]

we can bound (3.8) using \(m^2(t) = 2t - 3t \log t + \left((3 \log t)/(2 \sqrt{2})\right)^2\) from above by

\[
\frac{\sqrt{2}}{\sqrt{2\pi}(B-1+\sigma t-m(t))} e^{-\sigma m(t)+x_0^2/2} \int_{m(t)-B-\sigma t}^{m(t)-B+1-\sigma t} e^{-w^2/2t} \, dw \sim \frac{t}{t+\infty} e^{-t+(\sqrt{2}-\sigma)(B-1)} \tag{3.10}
\]

Next, we analyse the probability in the r.h.s. of (3.7). We bound it, for \(B < t/3\), from above by

\[
\mathbb{P} \left\{ \xi(s) \leq 2(s \wedge (t-s))\gamma, \forall s \in [r \vee B^{1/\gamma}, (t-B^{1/\gamma}) \wedge (t-r)] \right\}. \tag{3.11}
\]

By the proof of [2] Theorem 2.3, see (5.55), for all \(r\) large enough, probability (3.11) is bounded from above by

\[
\mathbb{P} \left\{ \xi(s) \leq 0, \forall s \in [r \vee B^{1/\gamma}, (t-B^{1/\gamma}) \wedge (t-r)] \right\} \leq \frac{2(B^{1/\gamma} \wedge r)}{t-2(B^{1/\gamma} \wedge r)} (1 + \epsilon), \tag{3.12}
\]

where in the last step we used [2] Lemma 3.4. Plugging the estimates from (3.10) and (3.12) into (3.7), we get

\[
S(B, t) \leq \left( \frac{2(B^{1/\gamma} \vee r)}{t-2(B^{1/\gamma} \vee r)} (1 + \epsilon) \mathbb{1}_{\{B > t/3\}} + \mathbb{1}_{\{B \leq t/3\}} \right) \frac{t e^{(\sqrt{2}-\sigma)(B-1)}}{\sqrt{2\pi}(\sigma - \sqrt{2})} (1 + o(1)). \tag{3.13}
\]

Note that in (3.13) and below \(o(1)\) denotes a \(t\)-dependent non-random quantity with

\[
o(1) \xrightarrow{t \uparrow \infty} 0. \tag{3.14}
\]

From (3.13) follows that \(\lim_{t \uparrow \infty} \sum_{B>t/3} S(B, t) = 0\) and

\[
\sum_{B=A+1}^{t/3} S(B, t) \leq \sum_{B=A+1}^{t/3} \frac{2t(B^{1/\gamma} \vee r)e^{(\sqrt{2}-\sigma)(B-1)}}{\sqrt{2\pi}(\sigma - \sqrt{2})(t-2(B^{1/\gamma} \vee r))} (1 + \epsilon) \tag{3.15}
\]
which can be made smaller than \( \epsilon / 2 \) by taking \( A \) large enough since \( \sqrt{B^{1/\gamma}} \wedge \tau e^{(1/2 - \sigma)(B-1)} \) is summable in \( B \) (because \( \sqrt{2} - \sigma < 0 \)). This concludes the proof of Theorem \( 2.1 \) in case (a).

Next, we turn to case (b). Here, the analysis is somewhat more intricate and we have to make “real” use of the imaginary part of the energy. To treat the correlations between \( X(t) \) and \( Y(t) \), we employ \( (1.9) \). We use a second moment computation and again the upper envelope (see [2, Theorem 2.2]).

**Proof of Proposition 2.7 in case (b).** We proceed as in case (a) until (3.3). This time, using Chebyshev’s inequality, we bound the probability in (3.3) by

\[
\frac{1}{\delta^2} E \left[ \sum_{k=1}^{n(t)} e^{\sigma(x_k(t) - m(t)) + i\tau y_k(t)} \mathbb{1}_{\{x_k(t) - m(t) < -A\} \cap \{z \in \mathbb{U}_{t, r, \gamma}\}} \right]^2 ,
\]

We introduce the shorthand notation

\[
\bar{x}_k(t) := x_k(t) - m(t), \quad k \leq n(t).
\]

Using (3.17) together with Representation (1.9), we get that (3.16) is equal to

\[
\frac{1}{\delta^2} E \left[ \sum_{k=1}^{n(t)} e^{(\sigma + i\rho \tau) x_k(t) - \sigma m(t) + i\sqrt{1 - \rho^2} \tau z_k(t)} \mathbb{1}_{\{\bar{x}_k(t) < -A\} \cap \{z \in \mathbb{U}_{t, r, \gamma}\}} \right]^2 .
\]

Define \( \lambda := \sigma + i\rho \tau \). Observe that \( |z|^2 = z\bar{z} \), for \( z \in \mathbb{C} \). Hence, the expectation in (3.18) is equal to

\[
E \left[ \sum_{k,l=1}^{n(t)} e^{\lambda x_k(t) + \lambda x_l(t) - 2\sigma m(t) + i\sqrt{1 - \rho^2} \tau (z_k(t) - z_l(t))} \mathbb{1}_{\bigcap_{j \in \{1,2\}} \{\bar{x}_j(t) < -A\} \cap \{z_j \in \mathbb{U}_{t, r, \gamma}\}} \right]
\]

(3.19)

where we used that \( (z_k(t), k \leq n(t)) \) is, conditional on the underlying GW tree, independent from \( (x_k(t), k \leq n(t)) \). Since \( (z_k(t), k \leq n(t)) \) is a BBM on the same GW tree as \( X \), (3.19) is equal to

\[
E \left[ \sum_{k,l=1}^{n(t)} e^{\lambda x_k(t) + \lambda x_l(t) - 2\sigma m(t) + (1 - \rho^2)\tau^2 (d(x_k(t), x_l(t)))} \mathbb{1}_{\bigcap_{j \in \{1,2\}} \{\bar{x}_j(t) < -A\} \cap \{z_j \in \mathbb{U}_{t, r, \gamma}\}} \right]
\]

(3.20)

We introduce the time of the most recent common ancestor \( q = d(x_k(t), x_l(s)) \) and rewrite (3.20) as \( \sum_{B>1} T(B, t) \), where

\[
T(B, t) := E \left[ \sum_{k,l=1}^{n(t)} e^{\lambda x_k(t) + \lambda x_l(t) - 2\sigma m(t) e^{(1 - \rho^2)\tau^2 (t-q)} \mathbb{1}_{\mathbb{U}_{t, q, t}^k}} \right],
\]

(3.21)

and we set

\[
U_{t, q, t}^k := \bigcap_{j \in \{1,2\}} \{\bar{x}_j(t) < -A\} \cap \{x_j(s) \leq U_{t, \gamma}(s), \forall s \in [r, t-r] \}
\]

(3.22)

\[
\cap \{x_j(q) - U_{t, \gamma}(q) \in [-B + 1, -B]\}.
\]
Similar to (3.6), we now relax conditions on the path of the particle. If $q > \frac{3}{4}t$, then we get
\[
U^{l,k}_{B,q,t} \subset \bigcap_{j \in \{l,k\}} \{ \bar{x}_j(t) < -A \} \cap \{ x_i(q) - U_{l,\gamma}(q) \in [-B + 1, -B] \} \quad (3.23)
\]
\[
\cap \{ \xi^q(s) \leq 8(s \wedge (q - s))^\gamma, \forall s \in [B^{1/\gamma} \vee r, q - (B^{1/\gamma} \wedge r)] \} =: T^{l,k}_{B,q,t},
\]
where $\xi^q(s) := x_i(s) - \frac{s}{q}x_i(q)$ is a Brownian bridge from 0 to 0 in time $q$, that is, in particular, independent of $x_i(q)$. Moreover, for $q \leq \frac{3}{4}t$, we have
\[
U^{l,k}_{B,q,t} \subset \bigcap_{j \in \{l,k\}} \{ \bar{x}_j(t) < -A \} \cap \{ x_i(q) - U_{l,\gamma}(q) \in [-B + 1, -B] \} =: S^{l,k}_{B,q,t}. \quad (3.24)
\]
Hence, $T(B, l)$ defined in (3.21) is bounded from above by
\[
\mathbb{E} \left[ \sum_{n(t)}^{(k)} e^{\lambda x_k(t)} + \lambda x_k(t) e^{(1-\rho)t^2(1-q)} \left( 1_{q \leq \frac{3}{4}t} \cap T^{l,k}_{B,q,t} \right) + 1_{q \leq \frac{3}{4}t} \cap S^{l,k}_{B,q,t} \right] \quad (3.25)
\]
First, observe that, for $B < t/3$, as in (3.12), the probability in (3.25) is bounded from above by
\[
\frac{2(B^{1/\gamma} \vee r)}{q - 2(B^{1/\gamma} \vee r)} (1 + \epsilon). \quad (3.26)
\]
Observe that $m(t) - A - x \leq m(t) - A - U_{l,\gamma}(q) + B$. We compute first the innermost integrals with respect to $y$ and $y'$ in (3.25), i.e.,
\[
\int_{-\infty}^{D_{A,B,q}} \int_{-\infty}^{D_{A,B,q}} e^{\sigma(x+y'+m(t)) + \rho r \gamma R} e^{\frac{y^2+y'^2}{2(t-q)}} dy dy'. \quad (3.27)
\]
where $D_{A,B,q} := m(t) - A - U_{l,\gamma}(q) + B$. We make the following change of variables
\[
y = w + \lambda(t-q) \quad \text{and} \quad y' = w' + \bar{\lambda}(t-q). \quad (3.28)
\]
Hence, (3.27) is equal to
\[
e^{2\sigma(x-m(t))+(\sigma^2-(\rho \gamma)^2)(t-q)} \int_{-\infty}^{D_{A,B,q} - \lambda(t-q)} \int_{-\infty}^{D_{A,B,q} - \bar{\lambda}(t-q)} e^{\frac{w^2+w'^2}{2(t-q)}} dw dw'. \quad (3.29)
\]
Using the Gaussian tail bound (see [11] Lemma 3.5), we bound (3.29) from above by
\[
e^{2\sigma(x-m(t))+(\sigma^2-(\rho \gamma)^2)(t-q)} \left( 1_{\{D_{A,B,q} \geq \sigma(t-q)\}} + \exp \left( -\frac{(D_{A,B,q} - \lambda(t-q))^2 + (D_{A,B,q} - \bar{\lambda}(t-q))^2}{2(t-q)} \right) \right) \times 1_{\{D_{A,B,q} \leq \sigma(t-q)\}}. \quad (3.30)
\]
Collecting the terms from (3.30) and (3.25) depending on \( x \), we continue by computing
\[
\int_{U_{t,\gamma}(q)-B}^{U_{t,\gamma}(q)-B+1} \frac{e^{2\sigma x} e^{-\frac{x^2}{2\pi}}}{\sqrt{2\pi} q} \, dx = e^{2\sigma^2 q} \int_{U_{t,\gamma}(q)-B-2\sigma q}^{U_{t,\gamma}(q)-B+1-2\sigma q} \frac{e^{-\frac{v^2}{2\pi}}}{\sqrt{2\pi} q} \, dv, \tag{3.31}
\]
where we made the change of variables \( x = v + 2\sigma q \). Observe that \( U_{t,\gamma}(q)-2\sigma q \leq (\sqrt{2} - 2\sigma)q < 0 \), since \( \sigma \geq \frac{1}{\sqrt{2}} \). Therefore, by the Gaussian tail bound (3.9), (3.31) is bounded from above by
\[
\sqrt{q} \frac{e^{2\sigma^2 q} e^{-U_{t,\gamma}(q)+B}}{2\sigma q - U_{t,\gamma}(q) + B}. \tag{3.32}
\]
Using the bounds (3.32) and (3.30) in (3.25), we get that (3.25) is bounded from above by
\[
K \int_0^t \sqrt{q} e^{2t-\sigma^2 q} e^{-(U_{t,\gamma}(q)-B-2\sigma q)^2/2q} e^{-2\sigma m(t)+(\sigma^2-\tau^2)(t-q)} \times \left( \mathbb{1}_{D_{\alpha,B,q} \geq \sigma(t-q)} + e^{-\frac{d_A}{\sqrt{q}} - \lambda(t-q) \vee \frac{\sqrt{q}}{2t} \left( \frac{B}{q} \right) + \tau(t-q) + \frac{2(B^{1/\gamma \vee r})}{q - 2(B^{1/\gamma \vee r})} (1 + \epsilon) \right) \, dq. \tag{3.33}
\]
Using that \( U_{t,\gamma}(q)-2\sigma q = (\sqrt{2} - 2\sigma)q - \frac{q^2}{2\sqrt{2}} \log t + (q \wedge (t - q))^{3} \), we start to simplify (3.33). We get
\[
e^{2t-q} e^{2\sigma^2 q} e^{-(U_{t,\gamma}(q)-B-2\sigma q)^2/2q} e^{-2\sigma m(t)+(\sigma^2-\tau^2)(t-q)} \sim e^{(t-q)((\sigma - \sqrt{2})^2 - \tau^2) + \left( \frac{3\sigma}{\sqrt{2}} + \frac{\sqrt{2} - 2\sigma}{2t} \right) \log t - \left( \sqrt{2} - 2\sigma \right)(q \wedge (t - q)^3) + (\sqrt{2} - 2\sigma)B}. \tag{3.34}
\]
Note that by assumption on \( \sigma \) and \( \tau \) we have \( (\sigma - \sqrt{2})^2 - \tau^2 < 0 \) and \( \sqrt{2} - 2\sigma < 0 \). Cutting the domain of integration in (3.33) into three parts \( q \in [0, t - \log^\alpha t] \), \( q \in (t - \log^\alpha t, t - \frac{t}{2}] \) and \( q \in (t - \frac{t}{2}, t) \), for some fixed \( \alpha > 1 \), we get the following three terms
\[
K \int_0^t \ldots \, dq = K \left( \int_0^{t - \log^\alpha t} + \int_{t - \log^\alpha t}^{t - \frac{t}{2}} + \int_{t - \frac{t}{2}}^{t} \right) \ldots \, dq =: K \left( (1I) + (1I) + (1I) \right). \tag{3.35}
\]
We bound (1I) from above by
\[
\int_0^{t - \log^\alpha t} e^{(t-q)((\sigma - \sqrt{2})^2 - \tau^2) + \left( \frac{3\sigma}{\sqrt{2}} + \frac{\sqrt{2} - 2\sigma}{2t} \right) \log t - \left( \sqrt{2} - 2\sigma \right)(q \wedge (t - q)^3) + (\sqrt{2} - 2\sigma)B} \, dq \leq e^{(\sqrt{2} - 2\sigma)B + \frac{3\sigma}{\sqrt{2}} \log t} \int_0^{t - \log^\alpha t} e^{(t-q)((\sigma - \sqrt{2})^2 - \tau^2) - \left( \sqrt{2} - 2\sigma \right)(q \wedge (t - q)^3) + (\sqrt{2} - 2\sigma)B} \, dq \leq e^{(\sqrt{2} - 2\sigma)B} C^{\log^\alpha t} \left( (\sigma - \sqrt{2})^2 - \tau^2 \right) + \frac{3\sigma}{\sqrt{2}} \log t - \left( \sqrt{2} - 2\sigma \right)(\log^\alpha t), \tag{3.36}
\]
for some constant \( C > 0 \). Hence,
\[
K \sum_{B > 1} (1I) \leq \sum_{B > 1} K e^{C \log^\alpha t} \left( (\sigma - \sqrt{2})^2 - \tau^2 \right) + \frac{3\sigma}{\sqrt{2}} \log t - \left( \sqrt{2} - 2\sigma \right)(\log^\alpha t) \leq e^{(\sqrt{2} - 2\sigma)B} \sum_{B > 1} e^{(\sqrt{2} - 2\sigma)B}, \tag{3.37}
\]
since \( \sqrt{2} - 2\sigma < 0 \), we have \( \sum_{B > 1} e^{(\sqrt{2} - 2\sigma)B} < \infty \). Hence, we can choose \( t_0 \) such that, for all \( t > t_0 \), (3.37) \( \leq \frac{e^{\frac{t}{6}}}{6} \). For \( q \in (t - \log^\alpha t, t] \), we observe first that
\[
e^{\left( \frac{\sqrt{2} - 2\sigma}{2t} q^3 + \frac{3\sigma}{\sqrt{2}} \right) \log t} \sim e^{\frac{3}{\sqrt{2}} \log t}, \tag{3.38}
\]
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and, moreover,
\[
\frac{2\sqrt{q}(B^{1/\gamma} \vee r)}{(2\sigma q - U_\gamma(q) + B)(q - 2(B^{1/\gamma} \vee r))} \leq C' \frac{2(B^{1/\gamma} \vee r)}{\sqrt{t} (t - 2(B^{1/\gamma} \vee r))},
\]
for some constant $C' > 0$. Using (3.38) and (3.39), we bound (I2) from above by
\[
\int_{t-\log^3 t}^{t-\frac{4}{3}} e^{(t-q)}((\sigma - \sqrt{2})^2 - \sqrt{2}(t-q)^2 + (\sqrt{2} - 2\sigma)B) C't dq
\]
\[
\times \left( \frac{2(B^{1/\gamma} \vee r)}{(t-2(B^{1/\gamma} \vee r))} \mathbb{1}_{\{B < t/3\}} + \mathbb{1}_{\{B \geq t/3\}} \right) (1 + o(1))
\]
\[
\leq C_2 e^{\frac{A}{2}((\sigma - \sqrt{2})^2 - \sqrt{2})} e^{(\sqrt{2} - 2\sigma)B} \left( (B^{1/\gamma} \vee r) \mathbb{1}_{\{B < t/3\}} + t \mathbb{1}_{\{B \geq t/3\}} \right) (1 + o(1)).
\]
Using (3.40), we get that $K \sum_{B > 1}$ (I2) is bounded from above by
\[
KC_2 e^{\frac{A}{2}((\sigma - \sqrt{2})^2 - \sqrt{2})} \left( \sum_{B = 1}^{\sqrt{t/3}} e^{(\sqrt{2} - 2\sigma)B} (B^{1/\gamma} \vee r) + \sum_{B > \sqrt{t/3}} e^{(\sqrt{2} - 2\sigma)B} \right) (1 + o(1)).
\]
Again, since $2 - 2\sigma < 0$, we have $\sum_{B > 1} B^\frac{1}{2} e^{(\sqrt{2} - 2\sigma)B} \to \infty$ and $(\sigma - \sqrt{2})^2 - \sqrt{2} < 0$. Hence, there are $t_1$ and $A_1$ such that, for all $t > t_1$ and all $A > A_1$, we have that (3.41) $\leq \frac{\epsilon}{6}$. Since $D_{A,B,q} - \sigma (t - q) \leq 0$ for $t - q \leq A/\sqrt{2}$ and $B \leq \frac{A}{2}$, we bound (I3) from above by
\[
\int_{t-\frac{4}{3}}^{t} e^{(t-q)}((\sigma - \sqrt{2})^2 - \sqrt{2}(t-q)^2 + (\sqrt{2} - 2\sigma)B C't \left( \frac{2(B^{1/\gamma} \vee r)}{(t-2(B^{1/\gamma} \vee r))} \mathbb{1}_{\{B < t/3\}} + \mathbb{1}_{\{B \geq t/3\}} \right)
\]
\[
\times \left( \mathbb{1}_{\{B < \frac{A}{2}\}} e^{\frac{A}{2}((t-q) - \sqrt{2})^2} (1 + o(1)) + \mathbb{1}_{\{B \geq \frac{A}{2}\}} \right) dq.
\]
Using that $(\sigma - \sqrt{2})^2 - \sqrt{2} < 0$ and $\sqrt{2} - 2\sigma < 0$, we bound (3.42) from above by
\[
\int_{t-\frac{4}{3}}^{t} e^{-(\sqrt{2} - 2\sigma)(\frac{A}{2})^2 + (\sqrt{2} - 2\sigma)B} \tilde{C} \left( \mathbb{1}_{\{B < t/3\}} 2(B^{1/\gamma} \wedge r) + t \mathbb{1}_{\{B \geq t/3\}} \right)
\]
\[
\times \left( \mathbb{1}_{\{B < \frac{A}{2}\}} e^{-\frac{(t-q) - \sqrt{2})^2}{A/2}} (1 + o(1)) + \mathbb{1}_{\{B \geq \frac{A}{2}\}} \right) dq.
\]
Using (3.43), together with the fact that, for all $t > \frac{3A}{2}$, it holds that $\frac{t}{3} > \frac{A}{2}$, we get that, for all such $t$, the sum $K \sum_{B > 1}$ (I3) is bounded from above by
\[
K\tilde{C} \frac{A}{2} e^{-(\sqrt{2} - 2\sigma)(A/2)^2} \left( \sum_{B > 1}^{A/2} e^{(\sqrt{2} - 2\sigma)B} e^{-\frac{2((t-q) - \sqrt{2})^2}{A}} (B^{1/\gamma} \vee r)
\]
\[
+ \sum_{B > \sqrt{A/2}}^{t/3} e^{(\sqrt{2} - 2\sigma)B} (B^{1/\gamma} \vee r) + \sum_{B > t/3} t e^{(\sqrt{2} - 2\sigma)B} \right) (1 + o(1)).
\]
Hence, there exist $t_2$ and $A_2$ such that for all $t > t_2$ and $A > A_2$ the term in (3.37) is $\leq \frac{\epsilon}{6}$. Now, combining the bounds in (3.37), (3.41) and (3.44), we get that, for all
$t > \max\{t_0, t_1, t_2\}$ and $A > \max\{A_1, A_2\}$,
\[
\sum_{B \geq 1} T(B, t) \leq \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{2}.
\] (3.45)

By (3.3), this concludes the proof of Proposition 2.1.

**REFERENCES**

[1] E. Aïdékon, J. Berestycki, E. Brunet, and Z. Shi. Branching Brownian motion seen from its tip. *Probab. Theory Relat. Fields*, 157:405–451, 2013.

[2] L.-P. Arguin, A. Bovier, and N. Kistler. Genealogy of extremal particles of branching Brownian motion. *Comm. Pure Appl. Math.*, 64(12):1647–1676, 2011.

[3] L.-P. Arguin, A. Bovier, and N. Kistler. The extremal process of branching Brownian motion. *Probab. Theory Relat. Fields*, 157:535–574, 2013.

[4] K. B. Athreya and P. E. Ney. *Branching processes*. Springer-Verlag, New York-Heidelberg, 1972. Die Grundlehren der mathematischen Wissenschaften, Band 196.

[5] J. Barral, X. Jin, and B. Mandelbrot. Convergence of complex multiplicative cascades. *Ann. Appl. Probab.*, 20(4):1219–1252, 2010.

[6] A. Bovier. *Statistical mechanics of disordered systems*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2006.

[7] M. Bramson. Convergence of solutions of the Kolmogorov equation to travelling waves. *Mem. Amer. Math. Soc.*, 4(285):iv+190, 1983.

[8] M. Bramson. Maximal displacement of branching Brownian motion. *Comm. Pure Appl. Math.*, 31(5):531–581, 1978.

[9] B. Derrida. The zeroes of the partition function of the random energy model. *Physica A: Stat. Mech. Appl.*, 177(1–3):31–37, 1991.

[10] B. Derrida, M. R. Evans, and E. R. Speer. Mean field theory of directed polymers with random complex weights. *Comm. Math. Phys.*, 156(2):221–244, 1993.

[11] Z. Kabluchko and A. Klimovsky. Complex random energy model: zeros and fluctuations. *Probab. Theory Relat. Fields*, 158(1-2):159–196, 2014.

[12] Z. Kabluchko and A. Klimovsky. Generalized random energy model at complex temperatures. *Preprint*, 2014. Available at [http://arxiv.org/abs/1402.2142](http://arxiv.org/abs/1402.2142).

[13] A. Klenke. *Probability theory*. Universitext. Springer-Verlag London, Ltd., London, 2008. A comprehensive course, Translated from the 2006 German original.

[14] F. Koukiou. Analyticity of the partition function of the random energy model. *J. Phys. A. Math. Gen.*, 26(23):1207–1210, 1993.

[15] H. Lacoin, R. Rhodes, and V. Vargas. Complex Gaussian multiplicative chaos. *ArXiv e-prints*, 2013.

[16] T. D. Lee and C. N. Yang. Statistical Theory of Equations of State and Phase Transitions. I. Theory of Condensation. *Phys. Rev.*, 87:404–409, 1952.

[17] T. Madaule, R. Rhodes, and V. Vargas. The glassy phase of complex branching Brownian motion. *Preprint*, 2013. Available at [http://arxiv.org/abs/1310.7775](http://arxiv.org/abs/1310.7775).

[18] T. Madaule, R. Rhodes, and V. Vargas. Continuity estimates for the complex cascade model on the phase boundary. *Preprint*, 2015. Available at [http://arxiv.org/abs/1502.05655](http://arxiv.org/abs/1502.05655).

[19] D. Panchenko. *The Sherrington-Kirkpatrick model*. Springer, 2013.

[20] C. N. Yang and T. D. Lee. Statistical Theory of Equations of State and Phase Transitions. I. Theory of Condensation. *Phys. Rev.*, 87:404–409, 1952.

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