Deterministic and Unambiguous Dense Coding

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Abstract

Optimal dense coding using a partially-entangled pure state of Schmidt rank $\tilde{D}$ and a noiseless quantum channel of dimension $D$ is studied both in the deterministic case where at most $L_d$ messages can be transmitted with perfect fidelity, and in the unambiguous case where when the protocol succeeds (probability $\tau_x$) Bob knows for sure that Alice sent message $x$, and when it fails (probability $1 - \tau_x$) he knows it has failed. Alice is allowed any single-shot (one use) encoding procedure, and Bob any single-shot measurement.

For $\tilde{D} \leq D$ a bound is obtained for $L_d$ in terms of the largest Schmidt coefficient of the entangled state, and is compared with published results by Mozes et al. For $\tilde{D} > D$ it is shown that $L_d$ is strictly less than $D^2$ unless $\tilde{D}$ is an integer multiple of $D$, in which case uniform (maximal) entanglement is not needed to achieve the optimal protocol.

The unambiguous case is studied for $\tilde{D} \leq D$, assuming $\tau_x > 0$ for a set of $\tilde{D}D$ messages, and a bound is obtained for the average $\langle 1/\tau \rangle$. A bound on the average $\langle \tau \rangle$ requires an additional assumption of encoding by isometries (unitaries when $\tilde{D} = D$) that are orthogonal for different messages. Both bounds are saturated when $\tau_x$ is a constant independent of $x$, by a protocol based on one-shot entanglement concentration. For $\tilde{D} > D$ it is shown that (at least) $D^2$ messages can be sent unambiguously.

Whether unitary (isometric) encoding suffices for optimal protocols remains a major unanswered question, both for our work and for previous studies of dense coding using partially-entangled states, including noisy (mixed) states.

I Introduction

Dense coding is an intriguing nonclassical effect made possible by entangled quantum states: combining entanglement with a quantum channel allows more information to be
transmitted than is possible using these resources separately. The original protocol of Bennett and Wiesner [1] can be summarized, in slightly altered notation, as follows. Alice and Bob share two $D$-dimensional particles, meaning that each is described in quantum terms using a $D$-dimensional Hilbert space, which are initially in a fully-entangled state. Alice carries out one of $D^2$ mutually-orthogonal unitary encoding operations on her particle and sends it to Bob through a perfect $D$-dimensional quantum channel. Bob measures the quantum state of the two-particle in a fully-entangled orthonormal basis in order to learn with certainty which of the $D^2$ operations Alice carried out. This protocol can transmit $D^2$ “classical” messages with perfect fidelity, corresponding to a classical channel of capacity $2 \log D$.

For a careful and mathematically precise discussion of what we shall hereafter refer to as the standard protocol, see [2].

If the two particles are in a partially, as opposed to fully, entangled state, can dense coding still be carried out, and if so, how many messages can be transmitted? How must the standard protocol be modified in order to accomplish this? Even if $D^2$ messages cannot be sent with certainty, are there probabilistic protocols which allow significantly more information to be transmitted than the log $D$ capacity of the quantum channel by itself? These questions have led to a significant body of research. The present paper addresses them in two particular cases.

The first is deterministic dense coding, where the aim is to send $L$ distinct messages with perfect fidelity, and a significant problem is to determine the maximum value $L_d$ of $L$ for a given partially-entangled state. The answer is known for a uniformly entangled state, our term for a state in which all the nonzero Schmidt coefficients are identical, when the Schmidt rank $\bar{D}$ is less than the channel dimension $D$: an appropriate modification of the standard unitary encoding protocol allows the transmission of $L_d = \bar{D} D$ messages. For other situations few exact results are available, though a number of interesting numerical and analytical results have recently been published by Mozes et al. [3]. Our contribution to this topic consists in part in raising the question, which we are unable to answer, as to whether unitary (or isometric, see Sec. 11) encoding is sufficient to achieve the maximum value $L_d$. In the case of $\bar{D}$ less than $D$ we derive a rigorous inequality for $L_d$ which holds for a general encoding protocol, unitary or not, and compare it with some of the results in [3]. We also explore, in a preliminary way, the situation when $\bar{D}$ (the Schmidt rank) is larger than $D$ (the dimension of the quantum channel), for which unitary encoding is impossible, and show that $L_d$ is strictly less than $D^2$ unless $\bar{D}$ is an integer multiple of $D$.

The second case we consider is unambiguous dense coding: when Alice encodes message $x$, Bob’s measurement will with probability $\tau_x$ tell him precisely which message Alice sent, and with probability $1 - \tau_x$ that the protocol has failed. A significant problem is to determine the maximum average probability of success $P_s = \langle \tau \rangle$ for some set of $L$ messages. This will depend on the choice of $L$, with $P_s$ decreasing, for a given entangled state, as $L$ increases. We consider the case $L = \bar{D} D$, the maximum number of messages that can be sent in unambiguous fashion for $\bar{D} \leq D$, and derive a bound for the the average inverse probability of success $\langle 1/\tau \rangle$, assuming $\tau_x > 0$ for every message. We also obtain a bound for $\langle \tau \rangle$ when encoding is carried out using orthogonal isometries (orthogonal unitaries in the case $\bar{D} = D$). Both bounds are saturated in the special case in which $\tau_x$ is independent of $x$ by a protocol which employs unambiguous entanglement concentration. There are many other cases one might wish to consider, and for these our bounds are less useful. In particular, the situation
when the number $L$ of messages with $\tau_x > 0$ is less than $\tilde{D}D$ is hard to analyze, because one cannot be sure that unitary (or isometric) encoding is the optimal strategy. Indeed, the issue of determining when unitary encoding is optimal remains a major unanswered question in studies of dense coding using partially entangled states. In our opinion it deserves a lot more attention than it has hitherto received. We can say little about unambiguous protocols for $\tilde{D} > D$ aside from showing that it is always possible to send $D^2$ messages with positive probability. We suspect this is the maximum possible number, but we have no proof, nor can we identify an optimal protocol.

The remainder of this paper is organized in the following order. Section II introduces our notation for entangled states, encoding operations, and measurement POVMs appropriate for deterministic and unambiguous protocols. Next we derive, in Sec. III some very general information-theoretic bounds, which are later compared with our inequalities based on Schmidt coefficients of the entangled state. Deterministic dense coding is the topic of Sec. IV, first a review in IV A of the case of uniformly-entangled states for $\tilde{D} \leq D$, next in IV B a rigorous inequality for $L_d$, followed in IV C by a discussion of what happens for $\tilde{D} > D$, and in IV D by a remark on protocols that achieve $L_d$. Our discussion of unambiguous dense coding for $\tilde{D} \leq D$ begins in Sec. V with the derivation of the inequality for $\langle 1/\tau \rangle$, and continues in Sec. VI where additional results are obtained assuming encoding using orthogonal isometries. We derive a bound on $\langle \tau \rangle$ in VI A and show in VI B that such encoding is optimal when $\tau_x$ is independent of $x$ for a set of $\tilde{D}D$ messages. However, the bound on $\langle \tau \rangle$ given in VI C is unlikely to hold if one allows nonorthogonal isometries. Unambiguous dense coding in a situation with $\tilde{D} > D$ is the subject of Sec. VII.

Section VIII contains material relating our work to previous research; uniform entanglement for $\tilde{D} < D$ in VIII A; deterministic dense coding with reference to [3] in VIII B; previous work on unambiguous dense coding in VIII C; the connection of unambiguous dense coding with unambiguous discrimination in VIII D; and finally, the issue of unitary encoding when carrying out dense coding using noisy (mixed) entangled states, in VIII E. The concluding Sec. IX contains a summary of our results followed by a discussion of open questions. Two appendices contain technical results.

II General Framework

The general setup we shall be studying is shown schematically in Fig. II. Alice and Bob share a normalized entangled state

$$|\Phi\rangle = \sum_{j=1}^{\tilde{D}} \lambda_j |a^j\rangle \otimes |b^j\rangle$$  \hspace{1cm} (1)

of Schmidt rank $\tilde{D}$ on the tensor product $\mathcal{H}_a \otimes \mathcal{H}_b$ of two Hilbert spaces of dimension $d_a$ and $d_b$, with orthonormal bases $\{|a^j\rangle\}$ and $\{|b^j\rangle\}$. We assume the Schmidt coefficients are ordered from largest to smallest,

$$\lambda_1 \geq \lambda_2 \geq \cdots \lambda_\tilde{D} > 0,$$  \hspace{1cm} (2)

and $\lambda_j = 0$ for $j > \tilde{D}$. One can visualize the situation by assuming that $\mathcal{H}_a$ and $\mathcal{H}_b$ refer to two particles, one in Alice’s and one in Bob’s possession. Our analysis is simplified through
assuming that
\[ d_a = d_b = \bar{D}, \]  
which is to say \(|\Phi\rangle\) is of full Schmidt rank on \(\mathcal{H}_a \otimes \mathcal{H}_b\), but nothing essential would change if \(d_a\) or \(d_b\) had values larger than \(\bar{D}\). In addition, Alice can signal Bob through a perfect \(D\)-dimensional quantum channel, where \(D\) does not have to be the same as \(\bar{D}\).

![Figure 1: Encoding and measurement for a dense coding protocol.](image)

Alice wishes to transmit one of \(L\) messages labeled \(x = 1, 2, \ldots\) to Bob, and to do so she encodes her message by carrying out a unitary map \(W_x\) from \(\mathcal{H}_g \otimes \mathcal{H}_a\) to \(\mathcal{H}_h \otimes \mathcal{H}_c\), where \(\mathcal{H}_g\) refers to an ancillary particle in a pure state \(|g^0\rangle\), \(\mathcal{H}_h\) to the final ancillary particle, and \(\mathcal{H}_c\) is a Hilbert space of dimension \(d_c = D\), thought of as a particle which is then sent through the noiseless channel to Bob. Since \(W_x\) is unitary, \(d_gd_a = d_hd_c\), but \(d_a = \bar{D}\) need not be the same as \(d_c = D\). By introducing an orthonormal basis \(\{|h^l\rangle\}\) for \(\mathcal{H}_h\), we can express the action of \(W_x\) in the form
\[
W_x\left(|g^0\rangle \otimes |a\rangle\right) = \sum_l |h^l\rangle \otimes \left(A_{xl}|a\rangle\right),
\]  
where the \(A_{xl}\), which are known as Kraus operators, map \(\mathcal{H}_a\) to \(\mathcal{H}_c\) and satisfy the normalization condition
\[
\sum_l A_{xl}^\dagger A_{xl} = I_a.
\]  
In the special case in which there is only a single term \(l = 1\) in this sum, we will omit the subscript \(l\) and refer to the map \(A_x\) of \(\mathcal{H}_a\) to \(\mathcal{H}_c\) as an isometry, since \(A_x^\dagger A_x = I_a\) means that \(A_x\) preserves norms. An isometry is only possible when \(d_a = \bar{D} \leq D = d_c\), and if \(\bar{D} = D\) the isometry is a unitary operator. In this sense isometric encoding represents a natural generalization of unitary encoding in the standard protocol. To be sure, when \(\bar{D}\) is less than \(D\) one can always suppose that \(d_a = D\) and that \(|\Phi\rangle\) is supported on a subspace of \(\mathcal{H}_a\), so that the particle \(c\) sent through the channel is identical with the particle \(a\) initially in an entangled state. However, we find it more convenient to carry out the analysis assuming \(d_a = \bar{D}\). If \(\bar{D}\) is larger than \(D\), one must assume different dimensions for \(\mathcal{H}_a\) and \(\mathcal{H}_c\), and isometric encoding is not possible.

Bob’s task is to extract information by carrying out a POVM using a collection \(\{B_y\}\) of positive operators on \(\mathcal{H}_c \otimes \mathcal{H}_b\), as indicated schematically in Fig. 1. For studying unambiguous dense coding it is convenient to assume that the label \(y\) can take on values 0, 1, 2, \ldots, with
the significance that if the outcome is \( y = x > 0 \), Bob knows for sure that Alice sent message \( x \), while \( y = 0 \) is the “failure” or “garbage” outcome: he does not know which message was sent. As is well known, Bob’s POVM can always be thought of as a projective measurement on \( \mathcal{H}_c \otimes \mathcal{H}_b \) along with an ancillary system prepared in a pure state, and the reader may wonder why we have not included this ancillary system as part of Fig. 1. The answer is that it is not needed for our analysis, whereas the details of Alice’s encoding procedure play a more significant role in our discussion.

The framework outlined above for encoding and decoding is also appropriate, given some obvious modifications, for the case in which Alice and Bob share a noisy entangled state, represented by a density operator or ensemble, or use a noisy channel. But our entire discussion is limited to “one shot” dense coding: Alice does not entangle her input over many uses of the apparatus, nor does Bob save the outcomes of multiple transmissions in order to perform a coherent measurement.

An unambiguous dense coding protocol is thus one in which the \( \{W_x\} \)—or equivalently the \( \{A_x\} \)—and the \( \{B_y\} \) have been chosen so that

\[
\Pr(y \mid x) = \tau_x \delta_{yx} + (1 - \tau_x) \delta_{y0},
\]

where \( \tau_x \) is the probability that if Alice chooses to send message \( x \) it will be correctly transmitted: Bob’s apparatus will show \( y = x \) rather than \( y = 0 \), the latter being an indication that the protocol has failed. If \( \eta_x \) is the a priori probability for Alice choosing message \( x \), the joint probability distribution will be

\[
\Pr(x, y) = \eta_x [\tau_x \delta_{yx} + (1 - \tau_x) \delta_{y0}].
\]

This means the average probabilities \( P_s \) of success and \( P_f \) of failure in sending a message are

\[
P_s = \sum_x \eta_x \tau_x, \quad P_f = 1 - P_s.
\]

The deterministic case is one in which \( \tau_x = 1 \) for \( 1 \leq x \leq L \), where \( L \) is the number of messages under consideration, and an optimal deterministic protocol is one giving rise to the maximum number \( L_d \) of messages having \( \tau_x = 1 \), assuming \( |\Phi\rangle \) and \( D \) are held fixed. An optimal unambiguous protocol is, roughly speaking, one that yields the maximum value of \( P_s \) but this will depend on the a priori probabilities \( \{\eta_x\} \). We shall only consider the case \( \eta_x = 1/L \).

### III Information Theory Bound

Before discussing specific protocols in the following sections, it is convenient to derive some simple but quite general information-theoretic bounds on the probability of successfully transmitting a message from Alice to Bob. The first is based on the result in Sec. VII of [4], that the classical capacity \( C \) of a dense coding “channel” (entangled state plus quantum channel) of the sort we are considering is given by

\[
\tilde{D} \leq D : \quad C = \log D + H_E, \quad H_E = -\sum_j \lambda_j^2 \log \lambda_j^2;
\]
$H_E$ is the entanglement of $|\Phi\rangle$. The condition $\bar{D} \leq D$ is implicit in the derivation in [4]. For $\bar{D} > D$ we do not know of a comparable expression, but studies of entanglement assisted capacity [5] yield an upper bound

$$\bar{D} > D : C \leq 2 \log D.$$  

(10)

(This also holds for $\bar{D} \leq D$, but then it is obvious from (9), since $H_E$ cannot exceed $\log D$.)

A consequence of (7) is the conditional probability

$$\Pr(x \mid y) = \begin{cases} 
\delta_{yx} & \text{for } y > 0, \\
\frac{\eta_x (1 - \tau_x)}{1 - P_s} & \text{for } y = 0,
\end{cases}$$  

(11)

from which the Shannon mutual information

$$I(X : Y) = H(X) - H(X \mid Y),$$  

(12)

can be calculated using

$$H(X) = -\sum_x \eta_x \log \eta_x,$$

$$H(X \mid Y) = (1 - P_s)H(X \mid y = 0).$$  

(13)

Because of (11), $H(X \mid y) = 0$ for all $y > 0$.

If we restrict ourselves to the situation in which all $L$ messages have the same a priori probability, $\eta_x = 1/L$, and use the upper bound $H(X \mid y = 0) \leq \log L$ in (13), the fact that $I(X : Y)$ cannot exceed $C$ in (9) leads to the inequality

$$P_s \log L \leq \log D + H_E$$  

(14)

with

$$P_s = \frac{1}{L} \sum_{x \geq 1} \tau_x$$  

(15)

the average unweighted probability of successfully transmitting a message.

In certain cases this bound might be improved by choosing a nonuniform set of a priori probabilities $\{\eta_x\}$, or using a better upper bound than $\log L$ for $H(X \mid y = 0)$. However, because of their generality, one cannot expect bounds of this sort to be very tight, and in the following sections we obtain for restricted types of protocols improved bounds which are not based on Shannon mutual information.

IV Deterministic Dense Coding

IV A Uniformly entangled state with $\bar{D} \leq D$

We use the term uniformly entangled for the state $|\Phi\rangle$ in (11) when all the (nonzero) $\lambda_j$ are equal to each other. When $\bar{D} = d_a = d_b$ this coincides with the terms “fully” or “maximally entangled,” but neither term seems appropriate when $\bar{D}$ is smaller. One of the simplest
and most straightforward extensions of the standard dense coding scheme is to a uniformly entangled state $\tilde{D} < D$, which can be used to send exactly $\tilde{D}D$ messages deterministically when encoded using orthogonal isometries.

Let $\{A_x\}$ be a collection of isometries from $\mathcal{H}_a$ to $\mathcal{H}_c$, see the discussion following (5), which are orthogonal in the sense that

$$\text{Tr}_a(A_x^\dagger A_y) = \tilde{D}\delta_{xy} = \text{Tr}_c(A_y A_x^\dagger),$$

(16)

where one can use either equation as a definition. If we use the orthonormal basis $\{|a^j\rangle\}$ in (11) to write each $A_x$ in the form

$$A_x = \sum_{j=1}^{\tilde{D}} |\gamma_x^j\rangle\langle a^j|,$$

(17)

where the expansion coefficients $|\gamma_x^j\rangle$ are elements of $\mathcal{H}_c$, the orthogonality condition (16) becomes

$$\sum_{j=1}^{\tilde{D}} \langle \gamma_x^j | \gamma_y^j \rangle = \tilde{D}\delta_{xy}.$$  

(18)

It follows from this that if $|\Phi\rangle$ is uniformly entangled, the kets

$$|\Phi_x\rangle = A_x |\Phi\rangle = \sum_j \left(1/\sqrt{\tilde{D}}\right) |\gamma_x^j\rangle \otimes |b^j\rangle,$$

(19)

on $\mathcal{H}_c \otimes \mathcal{H}_b$ form an orthonormal collection. Given a uniformly entangled state and a collection of $\tilde{D}D$ orthogonal isometries, there is a straightforward deterministic dense coding protocol for $L = \tilde{D}D$ messages: Alice uses $A_x$ to encode message $x$, and Bob measures using the orthonormal basis $\{|\Phi_x\rangle\}$. A proof of the intuitively obvious result that such a protocol is optimal can be based on (14) with $H_E = \log \tilde{D}$, or on (26) below. Of course, when $\tilde{D} = D$ the isometries are unitaries, and we are back to the standard protocol.

Here is one way to construct $\tilde{D}D$ orthogonal isometries. Let operators $Q : \mathcal{H}_a \rightarrow \mathcal{H}_c$, $R : \mathcal{H}_a \rightarrow \mathcal{H}_a$, and $S : \mathcal{H}_c \rightarrow \mathcal{H}_c$ be defined by

$$R|a^j\rangle = e^{2\pi i (j-1)/\tilde{D}} |a^j\rangle, \quad S|c^k\rangle = |c^{k\oplus 1}\rangle, \quad Q = \sum_{j=1}^{\tilde{D}} |c^j\rangle \langle a^j|$$

(20)

in terms of orthonormal bases $\{|a^j\rangle\}$, $1 \leq j \leq \tilde{D}$, of $\mathcal{H}_a$ and $\{|c^k\rangle\}$, $1 \leq k \leq D$, of $\mathcal{H}_c$; $\oplus$ means addition modulo $D$. Since $R$ and $S$ are unitary, each of the $\tilde{D}D$ operators

$$A_{\alpha\beta} = S^\beta Q R^\alpha, \quad 0 \leq \alpha < \tilde{D}, \quad 0 \leq \beta < D,$$

(21)

is an isometry from $\mathcal{H}_a$ to $\mathcal{H}_c$. Here each distinct double subscript $\alpha\beta$ corresponds to a different value of $x$, and the counterpart of (11) is

$$\text{Tr}_a(A_{\alpha\beta}^\dagger A_{\alpha'\beta'}^\dagger) = \tilde{D}\delta_{\alpha\alpha'}\delta_{\beta\beta'}.$$  

(22)
IV B General bound $L_d \leq D/\lambda_1^2$

Whatever operation Alice carries out to encode message $x$ will result in a density operator $\rho_x$ describing the combined $\mathcal{H}_c \otimes \mathcal{H}_b$ system which Bob will measure, see Fig. II, and since whatever Alice does has no effect on Bob’s particle, its reduced density operator is

$$\text{Tr}_c(\rho_x) = \text{Tr}_a(|\Phi\rangle\langle\Phi|) = \sum_j \lambda_j^2 |b^j\rangle\langle b^j|,$$

(23)

independent of $x$. Two density operators $\rho_x$ and $\rho_y$ corresponding to distinct messages $x$ and $y$ can only be distinguished with certainty [6] if $\rho_x \rho_y = 0$, which is to say their supports are orthogonal: $P_x P_y = 0$, where $P_x$ is the projector onto the support of $\rho_x$. Consequently, since $\rho_x \leq P_x$ in the sense that $P_x - \rho_x$ is a positive operator, for a deterministic protocol it must be the case that

$$\sum_{x=1}^L \rho_x \leq \sum_x P_x \leq I_c \otimes I_b.$$

(24)

Upon tracing this inequality over $\mathcal{H}_c$ and using (23), one obtains

$$L \sum_j \lambda_j^2 |b^j\rangle\langle b^j| \leq DI_b = D \sum_j |b^j\rangle\langle b^j|,$$

(25)

so that $L \lambda_j^2 \leq D$ for every $j$. Since $\lambda_1$ is the largest Schmidt coefficient of $|\Phi\rangle$, this implies that the maximum value of $L$ satisfies

$$L_d \leq D/\lambda_1^2.$$

(26)

As $\lambda_1^2$ cannot be smaller than $1/\bar{D}$, this inequality implies that $L_d$ cannot exceed $\bar{D}D$, a bound which is achievable for $\bar{D} \leq D$ using a uniformly entangled state, as shown in part A, but not for $\bar{D} > D$, see part C below. If $|\Phi\rangle$ is a product state, $\lambda_1 = 1$ and the rather trivial bound $L_d \leq D$ is achieved by sending one of $D$ orthogonal states through the quantum channel. In other situations the bound (26) is less trivial; see, in particular, the discussion in Sec. VIII B.

Taking the logarithm of (26), one has

$$\log L_d \leq \log D + \log(1/\lambda_1^2) \leq \log D + H_E,$$

(27)

where the second inequality follows from the definition in (4), given that $\lambda_1^2 \geq \lambda_j^2$ for all $j$ and $\sum \lambda_j^2 = 1$. This shows that (26) is a tighter bound than the information-theoretic (14) with $L = L_d$ and $P_s = 1$.

IV C Protocols for $\bar{D} > D$

The case $\bar{D} > D$ stands in marked contrast with that for $\bar{D} \leq D$ discussed in part A above. To begin with, it is impossible to send $\bar{D}D$ messages in a deterministic fashion, because that exceeds the bound $D^2$ implied by (10). But even sending $D^2$ messages is not possible unless $\bar{D}$ is an integer multiple of $D$; otherwise, as we shall show, $L_d$ is strictly less than $D^2$. Furthermore, when $\bar{D}$ is an integer multiple of $D$, a uniformly entangled state is
not needed to achieve the optimal protocol, though there is still a nontrivial constraint on the Schmidt coefficients.

Let us first discuss the case $D = 2$, $\bar{D} = 4$, assuming $|\Phi\rangle$ is a uniformly entangled state. Without loss of generality one can think of this (up to some local unitaries) as Alice and Bob sharing two fully-entangled qubit pairs. An optimal dense coding protocol consists in throwing away one pair, and carrying out standard dense coding with the other, in order to send one of $D^2 = 4$ messages. But of course the pair that was thrown away need not have been fully entangled, so it is at least sufficient that the Schmidt coefficients be identical in pairs: $\lambda_1 = \lambda_2$, and $\lambda_3 = \lambda_4$. Indeed, the pair that was thrown away could have been in a mixed state. The general case in which $\bar{D}$ is an integer multiple of $D$ can be discussed in exactly the same way whenever $|\Phi\rangle$ can be thought of as the tensor product of one fully-entangled $D \times D$ pair with something else: by discarding the latter, which could have been in any state whatsoever, and using the former for standard dense coding one achieves an optimal protocol. Describing all of this in terms of the Kraus operators and the POVM of Sec. II is an exercise we leave to the reader.

Next assume that $\bar{D}$ is not a multiple of $D$. Alice must encode a particular message $x$ using a collection of Kraus operators $\{A_{xl}\}$ satisfying (5). Since $A_{xl}$ maps a $\bar{D}$-dimensional space to one of dimension $D < \bar{D}$, its rank, which is the same as the rank of $A_{xl}^\dagger A_{xl}$ (p. 13 of [7]), is at most $D$. But the identity operator $I_a$ in (5) is of rank $\bar{D} > D$. Therefore encoding cannot be achieved using a single Kraus operator, but requires a Kraus rank $\kappa$ (number of independent Kraus operators) bounded below by

$$\kappa \geq \xi := [\bar{D}/D],$$

where $[\alpha]$ is the smallest integer not less than $\alpha$. This in turn has the consequence, as shown in App. A that Alice’s encoding results in a state which when traced down to the $H_c \otimes H_b$ space available to Bob corresponds to a density operator $\rho_x$ of rank greater than or equal to $\kappa$, whose support is therefore a subspace of dimension at least $\kappa$. As noted above in B, two density operators $\rho_x$ and $\rho_y$ can be distinguished with certainty if and only if their supports are orthogonal, and this means that the number of messages that can be sent deterministically is bounded above by

$$L_d \leq \mu := [\bar{D}D/\xi],$$

where $[\alpha]$ denotes the largest integer not greater than $\alpha$. When $\bar{D}$ is an integer multiple of $D$, $\mu = D^2$, and, as shown earlier, one can achieve this value for $L_d$ by using an appropriate $|\Phi\rangle$. However, if $D$ does not divide $\bar{D}$, $\mu$ will lie somewhere in the range

$$D(D + 1)/2 \leq \mu < D^2,$$

so $L_d$ is less than $D^2$, a result which is tighter than the information-theoretic bound (10). But we do not know whether $L_d = \mu$ can actually be achieved, even in the simplest case in which $D = 2$, $\bar{D} = 3$, for which $\xi = 2$ and $\mu = 3$. That is, we have been unable to design a deterministic protocol for transmitting 3 messages, or to show that it is impossible. If for this case only 2 messages can be sent deterministically, the entangled state is of no use and might as well be thrown away.
The argument that produces the bound in (29) does not require that $|\Phi\rangle$ be uniformly entangled, but only that it have Schmidt rank $\bar{D}$. There are, of course, cases in which Nielsen’s majorization condition \[8\] will permit such a state to be replaced with probability 1 by a uniformly entangled state of rank $D$, which would allow $L_d = D^2$ messages to be sent deterministically using the standard protocol. However, the replacement requires both local operations and classical communication, which in the dense coding context means a classical side channel. That lies outside the scope of the present paper, though it belongs to a class of problems worthy of further exploration.

IV D No extension of optimal deterministic protocol

Suppose a deterministic protocol is optimal in the sense that

$$\tau_x = 1 \text{ for } 1 \leq x \leq L_d,$$

(31)

with $L_d$ the maximum possible number of deterministic messages for a given $|\Phi\rangle$, $\bar{D}$ and $D$. It is then not possible to find an unambiguous protocol in which the same number of messages can be sent deterministically, and in addition one or more messages can be sent unambiguously with probabilities of success less than 1. In other words, (31) implies that $\tau_x = 0$ for $x > L_d$.

The argument is straightforward. The density operators (possibly pure states) $\rho_x$ created by Alice on $\mathcal{H}_c \otimes \mathcal{H}_b$ must be mutually orthogonal, see the arguments in part C above, for $1 \leq x \leq L_d$. Were it possible for her to create yet another $\rho'$ for an additional message $x = L_d + 1$, it would have to be orthogonal to all the $\rho_x$ just mentioned, as otherwise there would be at least one message in the set $1 \leq x \leq L_d$ which Bob could not definitely distinguish from message $L_d + 1$. But if $\rho'$ were orthogonal in this way, the additional message could also be sent deterministically, contrary to the assumption that $L_d$ is the maximum number possible.

V Saturated Unambiguous Dense Coding

Whereas unambiguous dense coding is a complicated problem if one allows the most general encoding and decoding protocols, the situation is considerably simpler for $\bar{D} \leq D$ if one supposes that precisely $\bar{D}D$ messages can be unambiguously transmitted, each with a positive (conditional) probability $\tau_x$.

To begin with, it is impossible to send more than $\bar{D}D$ messages, because Bob is carrying out measurements on a $\bar{D}D$-dimensional Hilbert space, and very general arguments \[9\] preclude his unambiguously distinguishing more states than the dimension of this space. Even to distinguish $\bar{D}D$ cases unambiguously, each with some positive probability of success, he is forced to use a POVM of the form

$$B_y = |B_y\rangle\langle B_y| \text{ for } 1 \leq y \leq \bar{D}D, \quad B_0 = I_c \otimes I_b - \sum_{y \geq 1} B_y,$$

(32)

where each $|B_y\rangle$ is an element of $\mathcal{H}_c \otimes \mathcal{H}_b$. 
For her part, Alice must be able to prepare for each \( x \) a pure state \( |C_x\rangle \in \mathcal{H}_c \otimes \mathcal{H}_b \), and these states, which in general will not be orthogonal for different \( x \), must form a basis of the space. In the language of the general encoding protocol, Sec. II, production of pure states means that for each \( x \) the Kraus rank of the corresponding operation is 1, so one only needs a single term in the sum in (4). See App. A for the proof of this intuitively obvious result. As a consequence, the normalization condition (5) becomes

\[
A_x^\dagger A_x = I_a,
\]

where, as in Sec. II, we omit the redundant \( l \) in the subscript. This means that \( A_x \) is unitary for \( D = D \) and an isometry for \( D < D \). In short, unambiguous dense coding of \( DD \) messages (with, of course, \( D \leq D \)) means the encoding must be isometric (unitary for \( D = D \)); other possibilities are excluded.

Finally, (6) translates into the condition

\[
\langle B_y| C_x \rangle = \langle B_y| (A_x \otimes I_b)|\Phi \rangle = \sqrt{\tau_x} \delta_{xy}.
\]

Here \( |C_x\rangle \) is normalized, since \( |\Phi\rangle \) is normalized and \( A_x \otimes I_b \) is an isometry, whereas the \( |B_y\rangle \) states are not normalized, but satisfy the inequality

\[
\sum_{y \geq 1} |B_y\rangle \langle B_y| \leq I_c \otimes I_b,
\]

which is necessary so that \( B_0 \geq 0 \) in (32). (One can always choose the phases so that the inner products in (34) are positive.)

At this point we find it convenient to reformulate the problem slightly using map-state duality (see [10, 11, 12]). Let us “transpose” (as that term is used in [12]) \( A_x \) in the form (17) into the ket

\[
|A_x\rangle = \sum_j |\gamma^j_x\rangle \otimes |a_j\rangle \in \mathcal{H}_c \otimes \mathcal{H}_a,
\]

and \( |\Phi\rangle \) in (1) into the nonsingular map \( \hat{\Phi} : \mathcal{H}_a \to \mathcal{H}_b \) defined by

\[
\hat{\Phi} = \sum_j \lambda_j |b^j\rangle \langle a^j|.
\]

These transpositions allow one to rewrite (34) in the equivalent form

\[
\langle B_y| (I_c \otimes \hat{\Phi})|A_x\rangle = \sqrt{\tau_x} \delta_{xy}.
\]

A little thought will show that (38) can only be satisfied for all \( x \) and \( y \) between 1 and \( \bar{D}D \), with \( \tau_x > 0 \) for every \( x \), if the \( \{|A_x\rangle\} \) are linearly independent and hence form a basis of \( \mathcal{H}_c \otimes \mathcal{H}_a \), and likewise the \( \{|B_y\rangle\} \) form a basis of \( \mathcal{H}_c \otimes \mathcal{H}_b \). Because \( \{|B_y\rangle\} \) is a basis, it has a unique dual or reciprocal basis (e.g., Sec. 15 of [13]) \( \{|\bar{B}_y\rangle\} \) satisfying

\[
\langle \bar{B}_y|B_x\rangle = \langle B_x|\bar{B}_y\rangle = \delta_{yx}.
\]

Multiplying (38) on both sides by \( |\bar{B}_y\rangle \), summing over \( y \), and using the fact that

\[
\sum_y |\bar{B}_y\rangle \langle B_y| = I_b \otimes I_c,
\]

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yields the expression
\[(I_c \otimes \hat{\Phi}) |A_x\rangle = \sqrt{\tau_x} |\bar{B}_x\rangle.\]  
(41)

connecting Alice’s operations to Bob’s measurements. Similarly, with \{\{A_x\}\} the dual basis to \{\{A_x\}\},
\[(I_c \otimes \hat{\Phi}^\dagger) |B_x\rangle = \sqrt{\tau_x} |\bar{A}_x\rangle.\]  
(42)

Since \(\hat{\Phi}\) has an inverse, one can rewrite (41) in the form
\[\tau_x^{-1/2} |A_x\rangle = (I_c \otimes \hat{\Phi}^{-1}) |\bar{B}_x\rangle,\]  
(43)

and from this and from
\[\sum_{y \geq 1} |\bar{B}_y\rangle \langle \bar{B}_y| \geq I_c \otimes I_b,\]  
(44)

which is equivalent to (35), obtain the inequality
\[\sum_x (1/\tau_x) \cdot |A_x\rangle \langle A_x| \geq I_c \otimes (\hat{\Phi}^\dagger \hat{\Phi})^{-1}.\]  
(45)

Tracing both sides over \(H_c\) and using the fact that
\[\text{Tr}_c(|A_x\rangle \langle A_x|) = A_x^\dagger A_x = I_a,\]  
(46)

one arrives at
\[\left(\sum_x 1/\tau_x\right) I_a \geq D (\hat{\Phi}^\dagger \hat{\Phi})^{-1},\]  
(47)

which is equivalent to
\[\langle 1/\tau \rangle := (1/\bar{D}D) \sum_x (1/\tau_x) \geq (\lambda_D^2 \bar{D})^{-1},\]  
(48)

since \(\lambda_D\) is the smallest Schmidt coefficient of \(|\Phi\rangle\). In particular, if \(\tau_x = P_c\) is a constant independent of \(x\), (48) tells us that the success probability \(P_s = P_c\) is bounded by
\[\bar{P}_c \leq \bar{D}\lambda_D^2.\]  
(49)

VI Orthogonal Isometries

VI A Deriving a bound for \(P_s\)

As shown in Sec. VI if for \(\bar{D} \leq D\) a full set of \(DD\) messages are to be sent unambiguously with positive probabilities, Alice’s encoding operation must be an isometry. We now make the much stronger assumption that the collection \(\{A_x\}\) of isometries used for encoding is orthogonal in the sense of (16), which is equivalent to
\[\langle A_x | A_y \rangle = \bar{D} \delta_{xy}\]  
(50)
for the corresponding kets defined in (36). This means that the elements of the dual basis are given by

$$|\bar{A}_x\rangle = (1/\bar{D})|A_x\rangle,$$

and combining this with (42) yields the expression

$$(I_c \otimes \hat{\Phi}^\dagger)(|B_x\rangle\langle B_x|)(I_c \otimes \hat{\Phi}) = (\tau_x/\bar{D}^2)|A_x\rangle\langle A_x|.$$  

(51)

Sum both sides over $x$ and use the inequality (35) to obtain

$$\sum_x (\tau_x/\bar{D}^2)|A_x\rangle\langle A_x| \leq I_c \otimes \hat{\Phi}^\dagger \hat{\Phi}.$$  

(52)

(53)

Tracing both sides over $H_c$ and using (46) yields the inequality

$$\left(\sum_x \tau_x\right) I_a \leq \bar{D}^2 D \hat{\Phi}^\dagger \hat{\Phi}.$$  

(54)

Because the smallest eigenvalue of $\hat{\Phi}^\dagger \hat{\Phi}$ is $\lambda_D^2$, this means that

$$P_s = \langle \tau \rangle = (1/\bar{D}D) \sum_x \tau_x \leq \lambda_D^2 \bar{D},$$

(55)

which can be compared with (48). Note that while (48) holds quite generally for saturated unambiguous dense coding, the derivation of (55) requires the additional orthogonality assumption (16) or (50). In the particular case in which $\tau_x = P_c$ is a constant independent of $x$, both (48) and (55) lead to the same bound (49).

The bound (55) is tighter for the case $L = \bar{D}D$ than the information-theoretic bound (14), as can be seen by writing the latter in the form

$$P_s \leq \frac{\log D + H_E}{\log \bar{D} + \log D}.$$  

(56)

As noted in part B below, the entangled state $|\Phi\rangle$ can with a probability $\lambda_D^2 \bar{D}$ be transformed into a uniformly entangled state with entanglement $\log \bar{D}$ by a local operation, and since such an operation cannot increase the average entanglement [14], it follows that

$$\lambda_D^2 \bar{D} \log \bar{D} \leq H_E.$$  

(57)

Using this and the fact that $H_E$ cannot exceed $\log \bar{D}$ it is straightforward to show that the right side of (55) is bounded above by the right side of (56).

VI B Saturating the bound

In fact, given any collection of $\bar{D}D$ orthogonal isometries, there is a POVM of the form (32) which results in each message being transmitted with the same probability of success $\tau_x = \lambda_D^2 \bar{D}$, saturating the bound (55). The easy way to see this is to imagine Bob carrying out his part of the protocol in two steps: an unambiguous entanglement concentration operation (the term used in [12] for what its originators [15] called the “Procrustean method”) on his
particle, which if it succeeds transforms $|\Phi\rangle$ into a uniformly entangled state with the same Schmidt rank, followed by a measurement, in an orthonormal basis of the type described in Sec. IV A on the combined system of his particle and the one received from Alice.

Unambiguous entanglement concentration results from Bob carrying out an operation described by Kraus operators

$$K_1 = \sum_j (\lambda_D/\lambda_j) |b^j\rangle\langle b^j|, \quad K_2 = \sqrt{I_b - K_1^\dagger K_1},$$

(58)

It is successful if $K_1$ occurs, for which the probability is

$$\langle \Phi | K_1^\dagger K_1 | \Phi \rangle = \lambda_D^2 \bar{D},$$

(59)

and then Alice and Bob share the uniformly entangled state $(1/\sqrt{\bar{D}}) \sum_j |a^j\rangle \otimes |b^j\rangle$.

The dense coding protocol consists of Alice encoding in the manner indicated in Sec. IV A, and Bob attempting unambiguous entanglement concentration in the manner just described. If the latter is successful, Bob carries out projective measurements on the two particles as described in Sec. IV A, certain that the outcome accurately reflects Alice’s encoding. Obviously, the probability of success (59) is independent of $x$. The two steps of entanglement concentration followed by projective measurement can be combined into a single POVM of the form (32) by setting $|B_x\rangle = K_1 |\Phi_x\rangle$, with $|\Phi_x\rangle$ defined in (19). It is then a simple exercise to show that the inequality (35) is satisfied.

VI C Exceeding the bound

The inequality (55) was derived by requiring that all $\bar{D}D$ messages be transmitted with positive probability and that the isometries used for encoding be orthogonal, so it is interesting to ask whether it holds if either condition is relaxed. We believe that both are necessary, but have not been able to prove this. Some insight is, however, provided by the following considerations. Suppose that $\bar{D}$ is 3 or more, and the $\lambda_j$ in (1) are all equal, except for $\lambda_{\bar{D}} = \epsilon > 0$, which is much smaller than the others. Then Bob can carry out an operation analogous to (58), but with

$$K_1 = \sum_{j=1}^{D-1} |b^j\rangle\langle b^j|, \quad K_2 = |b^{\bar{D}}\rangle\langle b^{\bar{D}}|,$$

(60)

which one can think of as a projective measurement to determine whether or not his particle is in $|b^{\bar{D}}\rangle$. If, with probability $1 - \epsilon^2$, $K_1$ occurs, the resulting uniformly-entangled state can be used to transmit $D(\bar{D} - 1)$ messages in a deterministic manner, using the protocol in Sec. IV A. Then $\langle \tau \rangle$ as defined in (55), assuming $\tau_x = 0$ for $x > D(\bar{D} - 1)$, is $(1 - \epsilon^2)(1 - 1/\bar{D})$, which can obviously be made larger than $\epsilon^2 \bar{D}$, which is the right side of (55).

This example violates both of the conditions used to derive (55), since we no longer have saturation—$\tau_x = 0$ for some of the messages—and the isometries used for encoding now map a $(\bar{D} - 1)$-dimensional Hilbert space onto one of $D$ dimensions. While they can be extended to isometries acting on the original $\bar{D}$-dimensional space $\mathcal{H}_a$, these isometries will not be orthogonal, see App. B. Thus the possibility remains open that (55) might be valid given
the assumption of one or the other but not both of the conditions used to derive it, though we ourselves doubt that this is the case. By contrast, the inequality (48) is known to hold for a saturated protocol, but is obviously useless if some of the $\tau_x$ are zero.

VII Unambiguous Dense Coding for $\tilde{D} > D$

In contrast to the situation discussed in Secs. V and VI in which the Schmidt rank of the entangled state is less than or equal to the dimension of the quantum channel, $\tilde{D} \leq D$, we have very few results for unambiguous dense coding if $\tilde{D}$ is greater than $D$; in particular, we have no upper bounds on success probabilities analogous to those in (48), (49) and (55). The situation is not unlike that for deterministic dense coding with $\tilde{D} > D$ as discussed in Sec. IV C: we have more questions than answers.

There is a simple argument that shows that with $\tilde{D} > D$ it is always possible to send $D^2$ messages in an unambiguous fashion, and one can place lower limits on the probability of success. Assume, as previously, that the Schmidt coefficients are arranged in decreasing order, (2), and define the two projectors

$$Q_a = \sum_{j=1}^{D} |a_j\rangle\langle a_j|, \quad Q_b = \sum_{j=1}^{D} |b_j\rangle\langle b_j|$$

on $H_a$ and $H_b$. Then $\{Q_a, \tilde{Q}_a = I_a - Q_a\}$ and $\{Q_b, \tilde{Q}_b = I_b - Q_b\}$ form projective decompositions of the identities $I_a$ and $I_b$. If Alice and Bob carry out projective measurements using these decompositions, it is evident from (11) that their results will be perfectly correlated, and outcome $Q_a$ will be accompanied by $Q_b$ with probability

$$P_m = \sum_{j=1}^{D} (\lambda_j)^2.$$  \hspace{1cm} (62)

When this occurs, Alice and Bob share a partially entangled state of Schmidt rank $D$, of the form (11), but with the summation limit replaced by $D$, and $\lambda_j$ replaced by $\tilde{\lambda}_j = \lambda_j/\sqrt{P_m}$. This entangled state can be used for unambiguous transmission of $D^2$ messages and our discussion in Secs. V and VI applies, provided the success probabilities calculated using the $\{\tilde{\lambda}_j\}$ are at the end multiplied by $P_m$. In particular we have an overall protocol, by combining the projective measurement just discussed with the entanglement concentration of Sec. VI B, in which allows each of $D^2$ messages to be sent with a probability of success equal to $D\lambda_D^2 P_m$. This might be optimal for the equal-probability case, but we have no proof that it is.

Alice’s projective measurement is actually not necessary in this scheme, since she can always proceed as if the measurement would have been successful, and leave it to Bob to declare the transmission a failure if his outcome is $\tilde{Q}_b$. On the other hand it will not do for Alice alone to carry out the measurement and communicate the result to Bob, since this requires a classical side channel (or something similar), and lies outside the scope of protocols we are considering.

Having both parties carry out the projective measurement has the additional advantage that if the common outcome corresponds to $\tilde{Q}_a$ and $\tilde{Q}_b$, there will still be some entanglement
left, if $\bar{D} - D \geq 2$, or at the very least the bare quantum channel when $\bar{D} = D + 1$, which can be used to communicate some messages. These considerations, while of some interest, tell us very little about possible optimal protocols. We do not even know if $D^2$ is an upper bound on the number of messages that can be sent unambiguously when $\bar{D}$ is greater than $D$, though we suspect this is the case.

VIII Comments on Previous Work

VIII A Uniformly entangled states for $\bar{D} < D$

The system of orthogonal isometries for $\bar{D} < D$ in Sec. IV A was first proposed, so far as we are aware, in Sec. 3 of [18], and later worked out independently in [19]. In both papers it is assumed that Alice’s and Bob’s particles have Hilbert spaces of different dimensions, $D$ and $\bar{D}$ in our notation, and are in a uniformly entangled state of Schmidt rank equal to the smaller dimension. The encoding operation is thought of as a unitary carried out on the space of higher dimension. While there is nothing wrong with discussing encoding using unitaries rather than isometries—see the comments following (5) in Sec. II—it can give rise to confusion, because while the encoding task dictates the nature of the isometries, their extension to unitaries acting on a range space of higher dimension is to a large measure arbitrary. Such confusion may lie behind the incorrect definition of the unitary operator $L_t$ and the incorrect trace formula (12) in [18], both of which are valid (aside from a typographical error) for $\bar{D} = D$, but not for $\bar{D} < D$, and an incorrect expression (11) for the unitary operator $U_{mn}$ in [19].

VIII B Deterministic dense coding

A very interesting exploration of deterministic dense coding for small systems of partially-entangled pure states has been carried out using a combination of numerical and analytic techniques by Mozes et al. [3]; some of their results extend to general $D$ (in our notation), and they make interesting conjectures about the $D$-dependence of others. They assume without discussing the matter that unitary encoding is optimal, but do not assume that the unitaries are orthogonal in the sense in which we use that term, see (16). (The term “orthogonal” in their paper has a different meaning.)

It is of particular interest that our inequality (26), which transcribed to their notation reads $\lambda_0 N_{\text{max}} \leq d$, is saturated in a number of cases by their numerical or analytical results or conjectures, in the sense that there are nontrivial choices of entangled states for which this inequality is an equality. These include the right-most limits of the regions 5, 6, and 7 in their Fig. 1—the numbers refer to $L_d (N_{\text{max}})$—for $D = 3$, and their conjectured minimal-entanglement states for $L_d = D + n$ for $n = 2, 3, \ldots D$. It is interesting that the $L_d = 7$ case for $D = 3$ is not included in their general conjectures.

Another point of agreement between their work and ours is their observation that $L_d$ is not in general a monotone increasing function of the entanglement. Our bound on $L_d$ depends on $\lambda_1$, not on the entanglement.
VIII C  Unambiguous dense coding

The earliest work on unambiguous dense coding known to us is the study of Hao et al. [20] of a partially entangled state of two qubits. (This and the later [21] use the term “probabilistic dense coding.”) The paper actually includes two schemes. The first requires a classical side channel, but the second does not, and thus fits within the framework of our discussion. The encoding scheme employs orthogonal unitaries, and the probability of success saturates the bound (55), in agreement with our Sec. VI B.

More recently Pati et al. [21] have independently worked out the qubit case, with results in agreement with [20]. They also considered its extension to general $D$, using a particular collection of $D^2$ orthogonal unitaries, assuming Alice and Bob share a partially entangled pure state. Their upper bound for the success probability has now been superseded by our (55); the latter is both a tighter bound (in some sense the best possible—see Sec. VI B), and was derived under weaker assumptions.

In addition, these authors construct an example in which an increase in entanglement brought about by increasing the Hilbert space dimensions of both Alice’s and Bob’s particles can give rise to a lower average probability for transmitting a given collection of messages, even when a uniformly entangled state is employed. We believe it is best to think of this somewhat counterintuitive result as arising from the encoding scheme they propose for the larger system; in particular, its unitaries are no longer orthogonal. While there is no reason to suppose that greater entanglement will always improve a dense coding scheme—see the comments in [3] regarding the deterministic case—we think the main lesson to be drawn from the example considered in [21] is the importance of paying attention to the encoding process, not just optimizing Bob’s measurements.

VIII D  Unambiguous state discrimination

Unambiguous dense coding is related to unambiguous state discrimination, see [9, 22, 23], in the sense that Bob’s measurement task is to distinguish the two-particle states $\{\vert C_x \rangle\}$, see (34), in an optimal fashion. In the case of dense coding these states are somewhat special in that they all correspond to the same reduced density operator on the Hilbert space $\mathcal{H}_b$ of Bob’s particle. In addition, whereas in unambiguous state discrimination one is generally concerned with optimal discrimination of a set of states thought of as simply given in advance, in the dense coding case optimization involves Alice’s choice of operations for producing the $\{\vert C_x \rangle\}$ as well as Bob’s choice of a POVM to distinguish them.

Some of our results are related to previous work on unambiguous state discrimination in the following way. When the success probabilities of state discrimination are required to be equal, Chefles in [9] derived an optimal average success probability consistent with our (49). Also if in Sec. VI when $\bar{D} = D$ the orthogonal unitary operators are of the special form used in [21], then the states to be distinguished are divided into $D$ mutually orthogonal sets, and the states in each set are linearly independent and symmetric. Therefore, the solution given in [22] for the optimal discrimination of symmetric states can be used to obtain the maximum average success probability in (55). However, our result is more general in the sense that it does not depend on any specific form of orthogonal unitaries.
VIII E Noisy entangled states

While dense coding using noisy (mixed) entangled states lies outside the scope of the research reported in this paper, there is one feature of the studies of this problem in [24, 25, 26, 27, 28, 29, 30, 31] to which we wish to draw attention. These papers arrive at a rather simple formula

\[ C = \log D + S(\rho_B) - S(\rho) \]  

(63)

for the optimal asymptotic classical capacity, with \( D \) the dimension of the noiseless quantum channel, \( \rho \) the density operator of the initial entangled state, \( \rho_B \) its partial trace down to Bob’s particle, and \( S \) the von Neumann entropy. This result is derived assuming that Alice is restricted to unitary encoding of messages, whereas Bob is allowed, and in general must employ, the most general decoding operation, including coherent measurements on states resulting from multiple transmissions.

There is no reason why \( S(\rho) \) cannot be larger than \( S(\rho_B) \)—an extreme example is a maximally-mixed state—and if that is the case, (63) cannot be the optimal capacity, since \( C = \log D \) is always possible by throwing away the entangled state and using the quantum channel in a straightforward way to transmit \( D \) messages. To be sure, it is conceivable that (63) might hold whenever \( S(\rho_B) \) exceeds \( S(\rho) \), i.e., when the right side is greater than \( \log D \), but we know of no compelling or even plausible argument to this effect. What is clearly needed is a study of what can be achieved using alternative methods of encoding, and until that has been carried out it seems best to regard (63) as a lower bound for, rather than the actual value of, the optimal capacity for a mixed entangled state. See the additional comments in Sec. IX B.

IX Conclusion

IX A Summary

We studied the problem of dense coding using a partially-entangled pure state whose Schmidt rank \( \bar{D} \) can be different from the dimension \( D \) of the noiseless quantum channel used to communicate from Alice to Bob, both for deterministic protocols in which a maximum of \( L_d \) messages can be sent with perfect fidelity, and also for unambiguous protocols in which message \( x \) is faithfully transmitted with a probability \( \tau_x \).

In the deterministic case we considered uniformly-entangled states for \( \bar{D} < D \), where for completeness the previously published encoding protocol for sending \( L_d = \bar{D}D \) messages was included in Sec. IX A and for \( \bar{D} > D \), where we showed in Sec. IX C that \( L_d \) is actually less than \( D^2 \), unless \( \bar{D} \) is a multiple of \( D \), and if it is a multiple of \( D \) other states besides one that is uniformly entangled can be used to achieve the optimal protocol. For pure states that are not uniformly entangled, our principal result is the inequality (26) bounding \( L_d \) in terms of the largest Schmidt coefficient \( \lambda_1 \). The utility of this bound is confirmed by the fact that it is satisfied as an equality by several results and conjectures in [3], as discussed in Sec. VIII B. We also showed in Sec. IX D that a protocol which achieves \( L_d \) cannot be used to send additional messages in an unambiguous fashion, i.e., with a nonzero probability of failure.

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For unambiguous protocols, our main results are for $\bar{D} \leq D$ and the saturated case in which $\tau_x > 0$ for all $\bar{D}D$ messages. This implies that the encoding operation must be an isometry, and that allows an analysis in Sec. V leading to the inequality (48) which bounds the average value of $1/\tau_x$ in terms of the smallest Schmidt coefficient $\lambda^{\bar{D}}$. If in addition one assumes the isometries are mutually orthogonal, there is an analogous bound (55) on the average of $\tau_x$, that is, the average probability of success. For the case in which all the $\tau_x$ are identical both of these inequalities yield the same bound, and we showed that it can be achieved by a protocol in which Bob first uses entanglement concentration to produce, with some probability, a uniformly entangled state. If successful this allows use of the dense coding protocol for such states, as described in Sec. IV A. We also showed that for $\bar{D} > D$ it is always possible to send $D^2$ messages unambiguously, though we have no bounds comparable to those for $\bar{D} \leq D$.

IX B Open questions

In the case of deterministic dense coding a very significant and challenging problem is to determine the “phase diagram” of the maximum number $L_d$ of messages as a function of the Schmidt coefficients $\{\lambda_j\}$ of the entangled state. The work in [3] represents a good beginning, and it would be of interest to confirm the conjectures given there, and extend them if possible to more general principles determining, or at least strongly constraining, the values of $L_d$. Since our inequality (26) seems satisfied as an equality at certain boundary points of their $L_d$ regions, one can hope that it, or perhaps other inequalities yet to be discovered, might prove of some assistance in working out the phase diagram. A major unanswered question is whether isometric (unitary) encoding is always sufficient to achieve $L_d$, or whether for certain entangled states one needs to employ a more general encoding procedure. Could it be, for example, that the absence of cases in which $L_d = D^2 - 1$ reported in [3] reflects a limitation of unitary encoding? Only further study can answer that and similar questions. The case of $\bar{D} > D$, where isometric encoding is clearly not possible, is even less well understood than that of $\bar{D} \leq D$, apart from certain special entangled states in the case where $\bar{D}$ is an integer multiple of $D$. Even for the simple example of $\bar{D} = 3$ and $D = 2$, we do not know whether it is possible by dense coding to send more than the $D = 2$ deterministic messages allowed by the quantum channel itself.

While we have identified an optimal protocol for unambiguous dense coding when the success probability $\tau_x$ is independent of $x$ for all $\bar{D}D$ messages, assuming $\bar{D} \leq D$, the case in which $\tau_x$ depends on $x$ remains open; we do not even have a bound on the average probability of success, aside from the information-theoretic (14). The bound (18) on $\langle 1/\tau \rangle$ requires $\tau_x > 0$ for all $x$, and is unlikely to be helpful when some of these probabilities are quite small. If some of the $\tau_x$ are zero one needs to allow for the possibility of nonunitary or nonisometric encoding, which may be a difficult task. We know even less about what can be done unambiguously when $\bar{D}$ is larger than $D$. It is always possible to send $D^2$ messages unambiguously with positive probabilities, but is this the largest number possible? We suspect it is, but it would be nice to have a proof.

Both in the deterministic and in the unambiguous case, and also for noisy entangled states as discussed in Sec. VII D a major unanswered question is whether optimal protocols can always be based on unitary encoding, or whether more general procedures are sometimes
needed. Up till now almost all studies of extensions of standard dense coding have simply assumed unitary (or isometric) operations. But in very few cases there are proofs or even plausible arguments that this type of encoding is optimal. Of course, when studying a difficult problem it is often a good strategy to make various assumptions which allow one to calculate an explicit result, and leave till later the problem of justifying them. We in no way wish to undervalue work of this type (including our own) and the insights which have emerged from it. Nonetheless, one should not forget that more general forms of encoding exist, and that in particular situations they might lead to better transmission of information. Exploring the encoding process is itself an interesting problem which might make a significant addition to our understanding of the basic quantum principles underlying dense coding.

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A Appendix. Kraus Rank and Density Operator

We will show that for the framework used in Sec. III; see Fig. II the Kraus rank of the encoding operation (4), which is to say the number of linearly independent \{A_{xl}\} operators for a fixed \(x\), is equal to the rank of the density operator \(\rho_{cb}\) describing the state available to Bob for measurements. In what follows, \(x\) is always fixed. It could be omitted from the notation, but retaining it clarifies the connection with formulas in Sec. III.

Let the state resulting from unitary time evolution, Fig. II be

\[
|\Psi\rangle = (W_x \otimes I_b)(|g^0\rangle \otimes |\Phi\rangle) = \sum_{jl} \lambda_j |h^l\rangle \otimes (A_{xl}|a^j\rangle) \otimes |b^j\rangle. \tag{A.1}
\]

It gives rise to a reduced density operator

\[
\rho_{cb} = \text{Tr}_h (|\Psi\rangle \langle \Psi|) = MM^\dagger, \tag{A.2}
\]

where

\[
M = \sum_{jl} \lambda_j \left[ A_{xl}|a^j\rangle \otimes |b^j\rangle \right] \langle h^l| = (I_c \otimes \hat{\Phi}) K \tag{A.3}
\]

is a map from \(\mathcal{H}_h\) to \(\mathcal{H}_c \otimes \mathcal{H}_b\), \(\hat{\Phi}\) is defined in (37), and

\[
K = \sum_{jl} \left[ A_{xl}|a^j\rangle \otimes |a^j\rangle \right] \langle h^l|, \tag{A.4}
\]

maps \(\mathcal{H}_h\) to \(\mathcal{H}_c \otimes \mathcal{H}_a\). The argument that identifies the Kraus rank with the rank of \(\rho_{cb}\) proceeds in three steps: (i) the rank of \(K\) is the Kraus rank; (ii) the rank of \(M\) is that of \(K\); (iii) the rank of \(MM^\dagger\) is the rank of \(M\).
For the first step, note that if \( \{|c^m\}\) is an orthonormal basis of \( \mathcal{H}_c \), the matrix elements of \( K \) are
\[
\langle c^m| a^j \rangle = \langle c^m| A_{xl}| a^j \rangle.
\] (A.5)
By definition, the rank of \( K \) is the number of linearly independent column vectors, labeled by \( l \), in its matrix, but this is obviously the same as the number of linearly independent matrices \( \langle c^m| A_{xl}| a^j \rangle \), again labeled by \( l \), which by definition is the Kraus rank.

That the ranks of \( M \) and \( K \) are the same is a consequence of the fact that \( I_c \otimes \hat{\Phi} \) in (A.3) is a nonsingular map (rank \( \bar{DD} \)) from \( \mathcal{H}_c \otimes \mathcal{H}_a \) to \( \mathcal{H}_c \otimes \mathcal{H}_b \). That multiplying a matrix by a nonsingular matrix leaves its rank unchanged is a standard result of linear algebra, as is the equality of the ranks of \( M \) and \( MM^\dagger \); see, for example, p. 13 of [7]. This completes the argument.

What is perhaps a more intuitive approach to this result can be given using atemporal diagrams; see the very similar argument in Sec. VC of [12] with reference to Fig. 13 of that paper. Translated to the present context, the essential observation is that \( W_x \) in Fig. 1 with \( |g^0\rangle \) fixed may be considered a map from \( \mathcal{H}_h \) to \( \mathcal{H}_c \otimes \mathcal{H}_a \)—thus \( K \) as defined in (A.4)—and the Kraus rank can be identified with the rank of this “cross operator.”

B Appendix. Extensions of Orthogonal Isometries

Lemma: Suppose a set of \( N \) mutually orthogonal isometries maps a \( d \)-dimensional space to a \( D \)-dimensional space \( \mathcal{H}_c \), and \( d < D \). Then an extension of these to isometries that map a \( (d+1) \)-dimensional space to a \( D \)-dimensional space cannot preserve mutual orthogonality unless \( N \leq D \).

Proof: Let the original and extended isometries be
\[
J_n = \sum_{k=1}^d |\Psi^n_k\rangle\langle k|, \quad K_n = \sum_{k=1}^{d+1} |\Psi^n_k\rangle\langle k|,
\] (B.1)
for \( 1 \leq n \leq N \). Assume both sets are orthogonal in the sense of (16), so that for \( n \neq n' \)
\[
\text{Tr}_c(J_n J_n^\dagger) = \sum_{k=1}^d \langle \Psi^n_k| \Psi^{n'}_k \rangle = 0,
\]
\[
\text{Tr}_c(K_n K_n^\dagger) = \sum_{k=1}^{d+1} \langle \Psi^n_k| \Psi^{n'}_k \rangle = 0,
\] (B.2)
which means that
\[
\langle \Psi^{n'}_{d+1}| \Psi^n_{d+1} \rangle = 0
\] (B.3)
for unequal \( n \) and \( n' \) between 1 and \( N \). Since the \( \{|\Psi^n_d\rangle\} \) are a collection of \( N \) nonzero (in fact, normalized) states in a Hilbert space \( \mathcal{H}_c \) of dimension \( D \), they can only be orthogonal to each other for \( N \leq D \), which proves the lemma.

Note that in the situation discussed in Sec. VI C, where \( d = \bar{D} - 1 \), we are interested in a collection of \( N = (\bar{D} - 1)D \) orthogonal isometries, and since \( \bar{D} - 1 \) is 2 or more, the lemma shows that they cannot be extended to orthogonal isometries mapping a \( \bar{D} \)- to a \( D \)-dimensional space.
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