PAPER

Exactly solvable Gross–Pitaevskii type equations

Yuan-Yuan Liu, Wen-Du Li and Wu-Sheng Dai
1 Department of Physics, Tianjin University, Tianjin 300350, People’s Republic of China
2 College of Physics and Materials Science, Tianjin Normal University, Tianjin 300387, People’s Republic of China
3 Theoretical Physics Division, Chern Institute of Mathematics, Nankai University, Tianjin, 300071, People’s Republic of China
* Author to whom any correspondence should be addressed.
E-mail: liwendu@tjnu.edu.cn and daiwusheng@tju.edu.cn

Keywords: gross–pitaevskii equation, exact solution, soliton, ultracold gas, trapped gas

Abstract

We suggest a method to construct exactly solvable Gross–Pitaevskii type equations, especially the variable-coefficient high-order Gross–Pitaevskii type equations. We show that there exists a relation between the Gross–Pitaevskii type equations. The Gross–Pitaevskii equations connected by the relation form a family. In the family one only needs to solve one equation and other equations in the family can be solved by a transform. That is, one can construct a series of exactly solvable Gross–Pitaevskii type equations from one exactly solvable Gross–Pitaevskii type equation. As examples, we consider the family of some special Gross–Pitaevskii type equations: the nonlinear Schrödinger equation, the quintic Gross–Pitaevskii equation, and cubic-quintic Gross–Pitaevskii equation. We also construct the family of a kind of generalized Gross–Pitaevskii type equation.

1. Introduction

The Gross–Pitaevskii (GP) equation has many applications in various branches of physics. In Bose–Einstein condensation, the GP equation is used to describe dilute Bose gases [1, 2], the BEC in optical lattices [3, 4], and the spinor BEC [5, 6]. Moreover, the dynamics of BEC is studied by the time-dependent GP equation from first-principle [7] and the limit of GP equation’s emergence is also discussed [8]. Some methods based on the GP equation, e.g., the truncated Wigner method [9], the positive-P method [10], the mean-field theory, and the Hartree–Fock–Bogoliubov method [11] are used to describe BEC. The influence of the inter-particle interaction to BEC is an important problem [12, 13], and the GP type equation is suitable for describing the condensate of the interacting Bose gas. The GP equation is also used to describe the Josephson plasma oscillations [14]. In general relativity, the GP equation is used to study the gravastar of black hole physics [15] and the black hole in the anti-de Sitter space [16]. The GP equation is a nonlinear equation and is difficult to solve. Many studies are devoted to solving the GP equation, such as stationary solutions [17], numerical solutions [18–20], analytical solutions [21], and soliton solutions [22–25]. Some methods for solving the GP equation are developed, such as the inverse scattering method [26]. Some solutions of the GP equation with various external potentials are obtained, e.g., the harmonic–oscillator potential [27], the multi-well potential [28], the changed external trap [29], the nonlinear lattice pseudopotential [30], the external magnetic field [31], and a sort of parity-time-symmetric potentials [26, 32]. The problem of scattering is also studied [33]. Moreover, there are studies on the one-dimensional GP equation and its applications [34–38]. In this paper, we suggest a method for constructing exactly solvable GP type equations, especially for variable-coefficient GP type equations. The GP equation is also important in studying the nonlinear solitary and periodic waves in the condensate of a superfluid [39] and the ultracold Bose-condensed atomic vapors in mesoscopic waveguide structures [40].

The time-independent GP equation is

$$\nabla^2 \psi(\mathbf{r}) - \left[ U_{\text{eff}}(\mathbf{r}) + g \left| \psi(\mathbf{r}) \right|^2 \right] \psi(\mathbf{r}) = 0, \quad (1.1)$$

where $g$ is the coupling constant and $U_{\text{eff}}(\mathbf{r}) = U(\mathbf{r}) - \mu$ with $U(\mathbf{r})$ the external potential and $\mu$ the chemical potential. Here we take $\hbar = 1$ and the mass $2m = 1$ for simplicity. The variable-coefficient GP equation, also
called the inhomogeneous GP equation, whose coefficients are space-dependent is a kind of important GP type
equations [41–43]. In the following, we consider the time-independent GP type equation in a general form,

$$\nabla^2 \psi(r) - [U_{\text{eff}}(r) + g_2(r) |\psi(r)|^2 + \cdots + g_n(r) |\psi(r)|^n] \psi(r) = 0,$$

(1.2)

which includes any power of the density $|\psi(r)|$ and the coefficient depends on the spatial coordinate. The GP
equation, the nonlinear Schrödinger equation, etc., are the special cases of equation (1.2). The GP type equation
is used to describe the many-body interaction in BEC [44–51]. On the other hand, the variable-coefficient GP
type equation has a wide application in nonlinear optics. The nonlinear Schrödinger equation performs
numerical analysis of the pulse mechanism of laser [52–54]. The variable-coefficient nonlinear Schrödinger
description describes the pulse in inhomogeneous optical systems [55] and the interactions between periodic
optical solitons [56]. The variable-coefficient cubic-quintic GP equation describes the brightlike and darklike
solitary wave solutions [57].

In this paper, we show that there exists a relation between the GP type equations. If two GP type equations
are connected by the relation, their solutions will be connected by a transform. The GP type equations who
are connected by the relation form a family. In a family, once an equation is solved, the solutions of other equations
in the family can be obtained by the transform.

As examples, we consider some families of the GP type equation, the family of the GP equation, the family of the
nonlinear Schrödinger equation, the family of the quintic GP equation, and the family of the cubic-quintic
GP equation. These GP type equations belong to different families. The solutions of these equations are known,
so by the transform we can solve all the equations in their families.

In section 2, we consider the family of the GP type equation. In section 3, we discuss some exactly solvable
families, including the nonlinear Schrödinger equation, the quintic GP equation, and the cubic-quintic GP
equation. In section 4, we consider the family of the generalized Gross–Pitaevskii type equation. In section 5, we
consider the family of the three-dimensional spherically symmetric GP type equation. The conclusions are
summarized in section 6.

2. The family of the Gross–Pitaevskii type equation

In this section, we show that there exists a relation between the GP type equations. All the GP type equations who
are related by the relation form a family.

2.1. The relation

For the GP type equations we have the following relation.

Two one-dimensional GP type equations

$$\frac{d^2 \psi(x)}{dx^2} - [U_{\text{eff}}(x) + g_2(x) |\psi(x)|^2 + \cdots + g_n(x) |\psi(x)|^n] \psi(x) = 0,$$

(2.1)

$$\frac{d^2 \phi(\xi)}{d\xi^2} - [V_{\text{eff}}(\xi) + G_1(\xi) |\phi(\xi)| + G_2(\xi) |\phi(\xi)|^2 + \cdots + G_n(\xi) |\phi(\xi)|^n] \phi(\xi) = 0,$$

(2.2)

if $U_{\text{eff}}(x)$ and $V_{\text{eff}}(\xi)$, $g_l(x)$ and $G_l(\xi)$ $(l = 1, \cdots, n)$ satisfy the relations

$$\sigma \left( x^2 U_{\text{eff}}(x) + \frac{1}{4} \right) = \sigma^{-1} \left( \xi^2 V_{\text{eff}}(\xi) + \frac{1}{4} \right),$$

(2.3)

$$\sigma x^{(\ell+1)/2} \tilde{g}_l(x) = \sigma^{-1} |\xi|^{(\ell+1)/2} G_l(\xi), \quad \ell = 1, \cdots, n,$$

(2.4)

their solutions are related by the transform:

$$x \leftrightarrow |\xi|^\sigma,$$

(2.5)

$$\psi(x) \leftrightarrow |\xi|^{(\ell+1)/2} \phi(\xi).$$

(2.6)

Here $\sigma$ is a constant chosen arbitrarily.

This result can be verified directly. Substituting the transforms (2.5) and (2.6) into the GP type equation (2.1)
gives

$$\frac{d^2 \phi(\xi)}{d\xi^2} - \sigma^2 \left[ \frac{1 - \sigma^{-2}}{4\xi^2} + |\xi|^{2(\sigma^{-1})} U_{\text{eff}}(|\xi|^\sigma) + |\xi|^{(\sigma+2)/(\sigma-1)} \tilde{g}_1(|\xi|^\sigma) |\phi(\xi)| + \cdots + |\xi|^{(\sigma+2)/(\sigma-1)} \tilde{g}_n(|\xi|^\sigma) |\phi(\xi)|^n \right] \phi(\xi) = 0.$$

(2.7)
This is just the one-dimensional GP type equation (2.2) with
\[
V_{\text{eff}}(\xi) = \sigma^2 \left[ 1 - \frac{\sigma^2}{4\xi^2} + |\xi|^{2(\alpha-1)} U_{\text{eff}}(|\xi|^\alpha) \right],
\]
\[
G_l(\xi) = \sigma^2 |\xi|^{\alpha(\alpha-1)(\alpha+4)/2} g_l(|\xi|^\alpha), \quad l = 1, \cdots, n.
\] (2.8)
This proves the relations (2.3)–(2.6).

2.2. The family
In the relation (2.3) there is a constant \( \sigma \). The constant \( \sigma \) can be chosen arbitrarily and different choices of \( \sigma \) give different transforms This means that one GP type equation relates an infinite number of GP type equations through the relations (2.3)–(2.6) with different \( \sigma \). The GP type equations who are related by a relation with different \( \sigma \) form a family. The family members are labeled by \( \sigma \). In a family, we only need to solve one equation and the solution of other family members can be obtained directly by the transforms (2.5) and (2.6).

In a GP type equation family, the family members are connected by a transform with a transform parameter \( \sigma \). This implies that there exists an algebraic structure.

2.3. The fixed point
In a family, all family members are connected by a transform. This transform has two fixed points.

The transforms (2.5) and (2.6) give
\[
\phi(\xi) = |\xi|^{1-\sigma}/2 \psi(|\xi|^\alpha).
\] (2.9)
It can be seen directly that the points
\[
(1, \psi(1)) \quad \text{and} \quad (-1, \psi(1))
\] (2.10)
are fixed points in the transform. That is, in a family, all family members pass through these two points.

2.4. The family of the Gross–Pitaevskii equation
The GP equation is the most important special case of the GP type equation (1.2), which has only the \( |\psi(x)|^2 \) term and a constant coefficient \( g(x) = g \):
\[
\frac{d^2 \psi(x)}{dx^2} - \left[ V_{\text{eff}}(x) + g |\psi(x)|^2 \right] \psi(x) = 0.
\] (2.11)
The family of the GP equation, in which GP equation is one of its family member, by the relations (2.3) and (2.4), consists of the following family members:
\[
\frac{d^2 \phi(\xi)}{d\xi^2} - \left[ V_{\text{eff}}(\xi) + G(\xi) |\phi(\xi)|^2 \right] \phi(\xi) = 0
\] (2.12)
with
\[
V_{\text{eff}}(\xi) = \sigma^2 \left[ 1 - \frac{\sigma^2}{4\xi^2} + |\xi|^{2(\alpha-1)} U_{\text{eff}}(|\xi|^\alpha) \right],
\]
\[
G(\xi) = ga^2 |\xi|^{\alpha(\alpha-1)}.
\] (2.13)
The family members are labeled by the parameter \( \sigma \).

The solutions of family members are connected by the transforms (2.5) and (2.6):
\[
\phi(\xi) = |\xi|^{1-\sigma}/2 \psi(|\xi|^\alpha).
\] (2.14)

3. The exactly solvable family: examples
In the present paper, we consider the GP type equation in a general form. The general GP type equation contains any power of the wave function with space-dependent coefficients. Here we discuss the correspondence between the general GP type equation and the BEC system.

Under the frame of the mean field theory, the interaction between two particles is proportional to the particle number density \( |\psi|^2 \). Therefore, the term \( |\psi|^2 \psi \) describes the two-body interaction, the term \( |\psi|^4 \psi \) describes the three-body interaction, and so on [44–51]. That is, the even-power of \( |\psi| \) describes the many-body interaction. Moreover, the odd-power term is also used to describe the effect beyond the mean field treatment [58, 59].

The coupling parameter \( g \) is proportional to the \( s \)-wave scattering length of inter-atomic scattering in BEC [60, 61] and the scattering length can be determined by the Feshbach resonance [47]. In the BEC experiment, the
coupling parameter can be controlled by the Feshbach resonance which changes the scattering length [62, 63]. For example, the scattering length can be changed by the magnetic field [64, 65]. The spatial modulation leads to a space-dependent coupling parameter and the temporal modulation leads to a time-dependent coupling parameter [61, 62, 66]. In the present paper, we consider the GP type equations with space-dependent coupling parameter.

For two-body interactions, the coupling parameter of the term $|\psi|^2 \psi$ is of the magnitude [45, 49]

$$g \sim \frac{4\pi\hbar^2}{m} a_s,$$

(3.1)

where $a_s$ is the $s$-wave scattering length. If only consider two-body interactions, it should satisfy $\sqrt{n a_s^3} \ll 1$ with $n$ the particle number density.

In the case of high densities, the three-body interaction becomes important and the two-body description is no longer effective [49]. Moreover, the coupling parameter of the three-body interaction, i.e., the coefficient of the term $|\psi|^4 \psi$, is of the magnitude

$$g_3 = \frac{12\pi\hbar^2 a^4}{m} (d_1 + d_2) \tan \left( s_0 \ln \left( \frac{d_1}{d_0} + \frac{\pi}{2} \right) \right),$$

(3.2)

which is proportional to $a_s^4$, so when the scattering length is large, the three-body interaction needs to be taken into account. The coupling parameter $g_3$ may be a complex number. For example, for the $^{87}\text{Rb}$ condensate the real part of $g_3/\hbar$ is about $10^{-26} \sim 10^{-27}\text{cm}^6\text{s}^{-1}$ [49, 67, 68] and the imaginary part is about $10^{-30}\text{cm}^6\text{s}^{-1}$ and is ignorable [69]. The other parameters in equation (3.2) can be determined numerically [67, 70].

In this section, we consider some exactly solvable families, the families of the nonlinear Schrödinger equation, the quintic GP equation, and the cubic-quintic GP equation.

### 3.1. The family of the nonlinear Schrödinger equation

The stationary nonlinear Schrödinger equation [71]

$$\frac{d^2\psi(x)}{dx^2} + [E - g |\psi(x)|^2] \psi(x) = 0$$

(3.3)

is a special case of the GP equation (2.11) with a vanishing external potential and the chemical potential $\mu$ replaced by the energy $E$.

The family of the nonlinear Schrödinger equation, by the relations (2.3) and (2.4), consists of the following family members:

$$\frac{d^2\phi(\xi)}{d\xi^2} - \left[ V_{\text{eff}}(\xi) + G(\xi) |\phi(\xi)|^2 \right] \phi(\xi) = 0$$

(3.4)

with

$$V_{\text{eff}}(\xi) = \sigma^4 \left[ \frac{1 - \sigma^{-2}}{4\xi^2} - |\xi|^{2(\sigma - 1)} E \right],$$

$$G(\xi) = g \sigma^2 |\xi|^{6(\sigma - 1)},$$

(3.5)

where $V_{\text{eff}}(\xi) = V(\xi) - E$. The solution of equation (3.4) by the transforms (2.5) and (2.6) is

$$\phi(\xi) = |\xi|^{1-\sigma/2} \psi(|\xi|).$$

(3.6)

It can be checked that the nonlinear Schrödinger equation (3.3) has a solution

$$\psi(x) = \sqrt{\frac{E}{g}} \tanh \left( \sqrt{\frac{E}{2}} (x + b) \right),$$

(3.7)

where $b$ is a constant.

Then the solution of the family member, equation (3.4), by the transform (3.6) is

$$\phi(\xi) = |\sigma| |\xi|^{\sigma - 1} \sqrt{\frac{E}{G(\xi)}} \tanh \left( \sqrt{\frac{E}{2}} |\xi|^{\sigma} + b \right).$$

(3.8)

The family members of the nonlinear Schrödinger equation with various value of $\sigma$ are shown in figure 1.
For a special case of the stationary nonlinear Schrödinger equation (3.3)
\[
\frac{d^2 \psi(x)}{dx^2} + [2\mu + 2 |\psi(x)|^2] \psi(x) = 0,
\]
[72] provides an exact bright soliton solution:
\[
\psi(x) = \sqrt{-2\mu} \text{sech} \left[ \sqrt{-2\mu} (x - x_0) \right].
\]

The family of the nonlinear Schrödinger equation (3.9), by the relations (2.3) and (2.4), consists of the following family members:
\[
\frac{d^2 \phi(\xi)}{d\xi^2} - [V_{\text{eff}}(\xi) + G_2(\xi) |\phi(\xi)|^2] \phi(\xi) = 0,
\]
where
\[
V_{\text{eff}}(\xi) = \sigma^2 \left[ \frac{1 - \sigma^{-2}}{4\xi^2} - 2\mu |\xi|^{2(\sigma-1)} \right],
\]
\[
G_2(\xi) = -2\sigma^2 |\xi|^{4(\sigma-1)}.
\]

Figure 1. The family members of the nonlinear Schrödinger equation with various value of $\sigma$. 
The solution of the family members (3.11) is
\[
\phi(\xi) = |\xi|^{(1-s)/2} \psi(\xi)|^s
\]
\[
= |\xi|^{(1-s)/2} \sqrt{2\mu} \text{sech} \left[ \sqrt{2\mu} \left( |\xi|^s - \xi_0 \right) \right].
\] (3.13)

3.2. The family of the quintic Gross–Pitaevskii equation
The quintic GP equation describes BEC when the interaction between the atoms is moderate or strong [50].

The quintic GP equation
\[
\frac{d^2 \psi(x)}{dx^2} - [1 + g_4 |\psi(x)|^4] \psi(x) = 0
\] (3.14)

has a solution [50]
\[
\psi(x) = \left( \frac{3}{-g_4} \right)^{1/4} \frac{1}{\sqrt{\cosh(2x)}},
\] (3.15)

where \(g_4\) is a negative constant.

The family of the quintic GP equation, by the relations (2.3) and (2.4), consists of the family members:
\[
\frac{d^2 \phi(\xi)}{d\xi^2} - [V_{\text{eff}}(\xi) + G_4(\xi) |\phi(\xi)|^4] \phi(\xi) = 0,
\] (3.16)

with
\[
V_{\text{eff}}(\xi) = \sigma^2 \left[ \frac{1 - \sigma^{-2}}{4\xi^2} + |\xi|^{2(\sigma-1)} \right],
\]
\[
G_4(\xi) = \sigma^2 |\xi|^{4(\sigma-1)} g_4.
\] (3.17)

The solution of equation (3.16) by the transforms (2.5) and (2.6) is
\[
\phi(\xi) = \left[ |\sigma|^{1/2} |\xi|^{\sigma-1/2} \left( \frac{3}{-G_4(\xi)} \right) \right]^{1/4} \frac{1}{\sqrt{\cosh(2 |\xi|^s)}}.
\] (3.18)

3.3. The family of the cubic-quintic Gross–Pitaevskii equation
The cubic-quintic GP equation which describes BEC considers two-particle and three-particle interactions [50, 51].

The cubic-quintic GP equation
\[
\frac{d^2 \psi(x)}{dx^2} - [1 + g_2 |\psi(x)|^2 + g_4 |\psi(x)|^4] \psi(x) = 0
\] (3.19)

has a solution [50]
\[
\psi(x) = \frac{2}{\sqrt{\sigma^2 - \frac{16}{3} \xi \cosh(2x)}} - g_2
\] (3.20)

where \(g_2\) and \(g_4\) are negative constants.

The family of the cubic-quintic GP equation, by the relations (2.3) and (2.4), consists of the family members:
\[
\frac{d^2 \phi(\xi)}{d\xi^2} - [V_{\text{eff}}(\xi) + G_2(\xi) |\phi(\xi)|^2 + G_4(\xi) |\phi(\xi)|^4] \phi(\xi) = 0
\] (3.21)

with
\[
V_{\text{eff}}(\xi) = \sigma^2 \left[ \frac{1 - \sigma^{-2}}{4\xi^2} + |\xi|^{2(\sigma-1)} \right],
\]
\[
G_2(\xi) = \sigma^2 |\xi|^{2(\sigma-1)} g_2,
\]
\[
G_4(\xi) = \sigma^2 |\xi|^{4(\sigma-1)} g_4.
\] (3.22)

The solution of equation (3.21) by the transforms (2.5) and (2.6) is
\[
\phi(\xi) = \left[ |\sigma|^{1/2} |\xi|^{\sigma-1/2} \right]^{1/2} \frac{2}{\sqrt{\sqrt{[|\sigma|^{-1}|\xi|^{-\sigma} G_2(\xi)^2 - \frac{16}{3} G_4(\xi) \cosh(2 |\xi|^s) - |\sigma|^{-1} |\xi|^{-\sigma} G_2(\xi)}}}
\] (3.23)
4. The family of the generalized Gross–Pitaevskii type equation

For academic interest, we consider a generalized Gross–Pitaevskii type equation and its family. By the generalized Gross–Pitaevskii type equation we mean that $|\psi(x)|^4 \psi(x)$ in the Gross–Pitaevskii type equation is replaced by $\psi(x)^{m+1}$.

Two one-dimensional generalized GP type equations

$$\frac{d^2\psi(x)}{dx^2} - \left[U_{\text{eff}}(x) + g(x)\psi(x) + g_2(x)\psi^2(x) + \cdots + g_n(x)\psi^n(x)\right] \psi(x) = 0,$$

(4.1)

$$\frac{d^2\phi(\xi)}{d\xi^2} - \left[V_{\text{eff}}(\xi) + G_1(\xi)\phi(\xi) + G_2(\xi)\phi^2(\xi) + \cdots + G_n(\xi)\phi^n(\xi)\right] \phi(\xi) = 0,$$

(4.2)

if $U_{\text{eff}}(x)$ and $V_{\text{eff}}(\xi)$, $g(x)$ and $G_l(\xi)$ ($l = 1, \ldots, n$) satisfy the relations

$$\sigma \left(x^2 U_{\text{eff}}(x) + \frac{1}{4}\right) = \sigma^{-1} \left(\xi^2 V_{\text{eff}}(\xi) + \frac{1}{4}\right),$$

$$\sigma x^{(l+4)/2} g_l(\xi) = \sigma^{-1} \xi^{(l+4)/2} G_l(\xi), \quad l = 1, \ldots, n,$$

(4.3)

their solutions are related by the transform:

$$x \leftrightarrow \xi^\sigma,$$

$$\psi(x) \leftrightarrow \xi^{(\sigma-1)/2} \phi(\xi).$$

(4.4)

Here $\sigma$ is a constant chosen arbitrarily.

This result can be verified directly by the same procedure as for the GP type equation. The fixed points by the transform (4.3), i.e.,

$$\phi(\xi) = \xi^{(1-\sigma)/2} \psi(x^\sigma),$$

(4.5)

are

$$\left(1, \psi(1)\right) \quad \text{and} \quad \left(-1, \psi(-1)\right).$$

(4.6)

In a family, all family members pass through these two points.

The generalized GP type equation (4.1) is a Liénard equation [73–75] with space-dependent coefficients. The result obtained here can be used to consider the family of the Liénard equation.

5. The three-dimensional spherically symmetric Gross–Pitaevskii type equation

The three dimensional spherically symmetric GP type equation also has the similar relation.

Two three-dimensional radial GP type equations

$$\frac{d^2\psi(r)}{dr^2} - \left[U_{\text{eff}}(r) + \frac{l(l+1)}{r^2} \right] + \left[\frac{1}{r^2} \right] \left[|\psi(r)|^2\right] \psi(r) = 0,$$

(5.1)

$$\frac{d^2\phi(\rho)}{d\rho^2} - \left[V_{\text{eff}}(\rho) + \frac{\ell(\ell + 1)}{\rho^2} \right] + \left[\frac{1}{\rho^2} \right] \left[|\phi(\rho)|^2\right] \phi(\rho) = 0,$$

(5.2)

where $l$ and $\ell$ are angular quantum numbers, if $U_{\text{eff}}(r)$ and $V_{\text{eff}}(\rho)$, $g(r)$ and $G(\rho)$ satisfy the relations

$$\sigma^2 U_{\text{eff}}(r) = \sigma^{-1} \rho^2 V_{\text{eff}}(\rho),$$

$$\sigma g(r) = \sigma^{-1} \rho G(\rho),$$

(5.3)

their solutions are related by the transform:

$$r \leftrightarrow \rho^\sigma,$$

$$\psi(r) \leftrightarrow \rho^{(\sigma-1)/2} \phi(\rho),$$

$$l + \frac{1}{2} \leftrightarrow \sigma^{-1} \left(\ell + \frac{1}{2}\right).$$

(5.4)

Here $\sigma$ is a constant chosen arbitrarily.

This result can be verified directly. Substituting the transform (5.4) into equation (5.1) gives

$$\frac{d^2\phi(\rho)}{d\rho^2} - \left[\sigma^3 \rho^{\sigma-1} U_{\text{eff}}(\rho^\sigma) + \ell(\ell + 1) + g(\rho^\sigma) \sigma^{-1} \frac{1}{\rho^2} \right] |\phi(\rho)|^2 \phi(\rho) = 0,$$

(5.5)
This is just the three-dimensional radial GP type equation (5.2) with
\[
V_{\text{eff}}(\rho) = \sigma^2 \rho^{2(\alpha - 1)} U_{\text{eff}}(\rho^2),
\]
\[
G(\rho) = g(\rho^2) \sigma^2 \rho^\alpha - 1.
\]  
This proves the relations (5.3) and (5.4).

6. Conclusion

We show that there exist families of the GP type equations. The GP type equations in a family are related by a transform. In a family, so long as one family member is solved, all family members are solved by the transform. The GP type equation is difficult to solve. The method presented in the paper provides an approach to construct exactly solvable GP type equations.

As examples, we consider the family of some special GP type equations: the nonlinear Schrödinger equation, the quintic GP equation, and the cubic-quintic GP equation.

We also consider family of the generalized GP type equation. The result of the generalized GP type equation inspires us to consider the family of the Liénard equation in the future work.

For three-dimensional cases, we consider the family of the three-dimensional spherically symmetric GP type equation.

Acknowledgments

We are very indebted to Dr. G. Zeitrauman for his encouragement. This work is supported in part by Special Funds for Theoretical Physics Research Program of the National Natural Science Foundation of China under Grant No. 11 947 124 and NSF of China under Grant No. 11 575 125 and No. 11 675 119.

ORCID iDs

Wu-Sheng Dai  
https://orcid.org/0000-0002-2085-8907

References

[1] Pitaevskii L and Stringari S 2016 Bose–Einstein condensation and superfluidity vol. 164 (Oxford: Oxford University Press)
[2] Griesmaier A, Werner J, Hensler S, Stuhler J and Pfau T 2005 Bose–einstein condensation of chromium Phys. Rev. Lett. 94 160401
[3] Niu Z-X, Tai Y, Shi J and Zhang W 2020 Bose–Einstein Condensates in an Eightfold Symmetric Optical Lattice Chinese Physics B 29 056103
[4] Hu P and Gu Q 2020 Vortices in Bose-Einstein condensates with random depth optical lattice J. Low Temp. Phys. 199 1314–23
[5] Yin H-M, Tian B, Chai J and Xie X-Y 2017 Solitons and modulation instability for the three-component Gross–Pitaevskii equations in the spinor Bose–Einstein condensate Optik 149 54
[6] Lm S, Ri M and Pb B 2016 Efficient and accurate methods for solving the time-dependent spin-1 Gross–Pitaevskii equation Phys. Rev. E 93 53309
[7] Benedikter N, de Oliveira G and Schlein B 2015 Quantitative derivation of the gross-pitaevskii equation Commun. Pure Appl. Math. 68 1399–482
[8] Nam P T, Rougerie N and Seiringer R 2016 Ground states of large bosonic systems: the Gross-Pitaevskii limit revisited Analysis & PDE 9 459–85
[9] Norrie A, Ballagh R and Gardiner C 2006 Quantum turbulence and correlations in Bose–Einstein condensate collisions Phys. Rev. A 73 043617
[10] Hope J and Olsen M 2001 Quantum superchemistry: Dynamical quantum effects in coupled atomic and molecular Bose–Einstein condensates Phys. Rev. Lett. 86 3220
[11] Rogel-Salazar J, Choi S, New G and Burnett K 2004 Methods of quantum field theory for trapped Bose–Einstein condensates J. Opt. B: Quantum Semiclassical Opt. 6 R33
[12] Dai W-S and Xie M 2017 Upper limit on the transition temperature for non-ideal Bose gases Ann. Phys. 322 1771–5
[13] Biasi A F, Mas J and Paredes A 2017 Delayed collapses of Bose–Einstein condensates in relation to anti-de Sitter gravity Phys. Rev. E 95 032216
[14] Charalambidis E, Kevrekidis P and Farrell P 2018 Computing stationary solutions of the two-dimensional Gross–Pitaevskii equation with deflated continuation Commun. Nonlinear Sci. Numer. Simul. 54 482–99
[15] Antoine X, Tang Q and Zhang J 2018 On the numerical solution and dynamical laws of nonlinear fractional Schrödinger/Gross-Pitaevskii equations Int. J. Comput. Math. 95 1423–43
[16] Vergez G, Danaila I, Auric S and Hecht F 2016 A finite-element toolbox for the stationary Gross-Pitaevskii equation with rotation Comput. Phys. Commun. 209 144–62
[20] Satarić B, Slavič N, Belić A, Balaz A, Muruganandam P and Adhikari S K 2016 Hybrid OpenMP/MPI programs for solving the time-dependent Gross-Pitaevskii equation in a fully anisotropic trap Comput. Phys. Commun. 200 111–7
[21] Liu Y M and Bao C G 2017 Analytical solutions of the coupled Gross-Pitaevskii equations for the three-species Bose-Einstein condensates Journal of Physics A 50 275301
[22] Su C-Q, Gao Y-T, Xue L and Wang Q-M 2016 Nonautonomous solitons, breathers and rogue waves for the Gross-Pitaevskii equation in the Bose-Einstein condensate Commun. Nonlinear Sci. Numer. Simul. 36 457–67
[23] Liu L, Tian B, Zhen H-L, Xu X-Y and Shan W-R 2017 Dark and bright solitons for a three-dimensional Gross-Pitaevskii equation with distributed time-dependent coefficients in the bose-einstein condensation Superlattices Microstruct. 102 498–511
[24] Gravejat P and Smeets D 2015 Asymptotic stability of the black soliton for the Gross-Pitaevskii equation Proceedings of The London Mathematical Society 111 305–53
[25] Zakeri G A and Yomba E 2018 Solitons in multi-body interactions for a fully modulated cubic-quintic Gross-Pitaevskii equation Appl. Math. Modell. 56 1–14
[26] Yu F and Li L 2019 Inverse scattering transformation and soliton stability for a nonlinear Gross-Pitaevskii equation with external potentials Appl. Math. Lett. 91 41–7
[27] Bland T, Parker N, Proukakis N and Malomed B 2018 Probing quasi-integrability of the Gross-Pitaevskii equation in a harmonic-oscillator potential Journal of Physics B 51 205303
[28] Guo Y, Wang Z-Q, Zeng X and Zhou H-S 2018 Properties of ground states of attractive Gross-Pitaevskii equations with multi-well potentials Nonlinearity 31 957–79
[29] Pickl P 2015 Derivation of the time dependent Gross-Pitaevskii equation with external fields Rev. Math. Phys. 27 1550003
[30] Alfimov G, Gelig L, Lebedev M, Malomed B and Zezyulin D 2019 Localized modes in the Gross-Pitaevskii equation with a parabolic trapping potential and a nonlinear lattice pseudopotential Commun. Nonlinear Sci. Numer. Simul. 66 194–207
[31] Olgiati A 2017 Remarks on the derivation of gross-pitaevskii equation with magnetic laplacian Advances in Quantum Mechanics: Contemporary Trends and Open Problems 18 257
[32] Barashenkov I, Zeytulin D and Konotop V 2016 Exactly Solvable Wadati Potentials in the PT-Symmetric Gross-Pitaevskii Equation, in Non-Hermitian Hamiltonians in Quantum Physics: Selected Contributions from the XV International Conference on Non-Hermitian Hamiltonians in Quantum Physics, Palermo, Italy, 18–23 May 2015, vol. 184 p. 143 Springer
[33] Guo Z, Hani Z and Nakanishi K 2018 Scattering for the 3D Gross-Pitaevskii equation Phys. Rev. Lett. 121 063901
[34] Wang D-S, Song S-W, Xiong B and Liu W-M 2011 Quantized vortices in a rotating Bose-Einstein condensate with spatiotemporally modulated interaction J. Phys. B: At. Mol. Opt. Phys. 44 365004
[35] Shin H, Radha R and Kumar V R 2011 Bose-Einstein condensates with spatially inhomogeneous interaction and bright solitons Phys. Lett. A 375 2519–23
[36] Michinel H, Paredes A, Valado M M and Feijóo D 2012 Coherent emission of atomic soliton pairs by Feshbach-resonance tuning Phys. Rev. A 86 013620
[37] Wang D-S, Li X-G, B, Slavnić Ć et al 2012 Interactions among periodic optical solitons for the variable-coefficient coupled nonlinear Schrödinger equation with external potential Phys. Rev. A 86 053621
[38] Wang D-S and Song S-W 2012 Quantized vortices in a rotating Bose-Einstein condensate with spatiotemporally modulated interaction Phys. Rev. A 86 043607
[39] Shin H, Radha R and Kumar V R 2011 Bose-Einstein condensates with spatially inhomogeneous interaction and bright solitons Phys. Lett. A 375 2519–23
[40] Michinel H, Paredes A, Valado M M and Feijóo D 2012 Coherent emission of atomic soliton pairs by Feshbach-resonance tuning Phys. Rev. A 86 013620
[41] Wang D-S, Li X-G, B, Slavnić Ć et al 2012 Interactions among periodic optical solitons for the variable-coefficient coupled nonlinear Schrödinger equation with external potential Phys. Rev. A 86 053621
[42] Wang D-S and Li X-G 2012 Localized nonlinear matter waves in a Bose–Einstein condensate with spatially inhomogeneous two- and three-body interactions J. Phys. B: At. Mol. Opt. Phys. 45 105301
[43] Luckins E K and Gorder R A V 2018 Bose–Einstein condensation under the cubic-quintic Gross-Pitaevskii equation in radial domains Ann. Phys. 388 206–34
[44] Kohler T 2002 Three-body problem in a dilute Bose–Einstein condensate Phys. Rev. Lett. 89 210404
[45] Chu H, Zhao S, Li G, Li M and Li D 2019 Mode-locked femtosecond polarization-polarizing Yb-doped fiber laser with a figure-nine configuration Opt. Commun. 126595
[46] Nishizawa N, Suga H and Yamanaka M 2019 Investigation of dispersion-managed, polarization-polarizing Er-doped figure-nine ultrashort-pulse fiber laser Opt. Express 27 89218–32
[47] Cai J-H, Chen H, Chen S-P and Hou J 2017 State distributions in two-dimensional parameter space of a nonlinear optical loop mirror-based, mode-locked, all-normal-dispersion fiber laser Opt. Express 25 4414–28
[48] Meradji S, Triki H, Zhou Q, Biswas A, Eikici M and Liu W 2020 Chirped self-similar cnoidal waveforms and solitomorphs in an inhomogeneous optical medium with resonant nonlinearity Chaos, Solitons Fractals 131 109414
[49] Wang X-M and Hu X-X 2021 Interactions among periodic optical solitons for the variable coefficient coupled nonlinear Schrödinger equations Phys. Rev. A 103 053621
[50] Li R, Yong X, Chen Y and Huang Y 2020 Equivalence transformations and differential invariants of a generalized cubic-quintic nonlinear Schrödinger equation with variable coefficients Nonlinear Dyn. 102 339–48
[51] Ilg T, Kumlin J, Santos L, Petrov D S and Büchler H P 2018 Dimensional crossover for the beyond-mean-field correction in Bose gases Phys. Rev. A 98 051604
[59] Argha D and Ayan K 1919 On solving cubic-quartic nonlinear Schrödinger equation in a cnoidal trap Eur. Phys. J. D 74 184
[60] Pitaevskii L, Stringari S, Pitaevskii L, Stringari S and Press O U 2003 Bose–Einstein Condensation International Series of Monographs on Physics (Oxford: Clarendon)
[61] Kengne E, Liu W-M and Malomed B A 2020 Spatiotemporal engineering of matter-wave solitons in Bose–Einstein condensates Phys. Rep. (https://doi.org/10.1016/j.physrep.2020.11.001)
[62] Pinsker F, Berloff N G and Pérez–García V M 2013 Nonlinear quantum piston for the controlled generation of vortex rings and soliton trains Phys. Rev. A 87 053624
[63] Vlčhynský S, Yakimenko A, Isaieva K and Chumachenko A 2013 The nature of superfluidity and Bose–Einstein condensation: From liquid 4He to dilute ultracold atomic gases Low Temp. Phys. 39 724–40
[64] Feshbach H, Kerman A and Lemmer R 1967 Intermediate structure and doorway states in nuclear reactions Ann. Phys. 41 230–86
[65] Chin C, Grimm R, Julienne P and Tiesinga E 2010 Feshbach resonances in ultracold gases Rev. Mod. Phys. 82 1225
[66] Combescot R 2003 Feshbach resonance in dense ultracold Fermi gases Phys. Rev. Lett. 91 120401
[67] Bulgac A 2002 Dilute quantum droplets Phys. Rev. Lett. 89 050402
[68] Pieri P and Strinati G C 2003 Derivation of the Gross–Pitaevskii equation for condensed bosons from the bogoliubov-de gennes equations for superfluid fermions Phys. Rev. Lett. 91 030401
[69] Tolra B L, O’hara K, Huckans J, Phillips W D, Rolston S and Porto J V 2004 Observation of reduced three-body recombination in a correlated 1D degenerate bose gas Phys. Rev. Lett. 92 190401
[70] Braaten E, Hammer H-W and Mehen T 2002 Dilute Bose–Einstein condensate with large scattering length Phys. Rev. Lett. 88 040401
[71] Gnutzmann S, Smilansky U and Derevyanko S 2011 Stationary scattering from a nonlinear network Phys. Rev. A 83 33831
[72] Wittmann D, Moosmann S and Korsch H 2005 Bound and resonance states of the nonlinear Schrödinger equation in simple model systems J. Phys. A: Math. Gen. 38 1777
[73] Yang X-L and Tang J-S 2008 Exact solutions to the generalized Lienard equation and its applications Pramana 71 1231–45
[74] Khan A and Panigrahi P K 2013 Bell solitons in ultra-cold atomic fermi gas J. Phys. B: At. Mol. Opt. Phys. 46 115302
[75] Solomon L 1977 Differential equations: Geometric theory 2ed (New York: Dover) ISBN-10 : 0486634639