A Tight Linear Bound to the Chromatic Number of $(P_5, K_1 + (K_1 \cup K_3))$-Free Graphs

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Received: 20 July 2022 / Revised: 17 December 2022 / Accepted: 15 March 2023
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Abstract
Let $F_1$ and $F_2$ be two disjoint graphs. The union $F_1 \cup F_2$ is a graph with vertex set $V(F_1) \cup V(F_2)$ and edge set $E(F_1) \cup E(F_2)$, and the join $F_1 + F_2$ is a graph with vertex set $V(F_1) \cup V(F_2)$ and edge set $E(F_1) \cup E(F_2) \cup \{xy \mid x \in V(F_1) \text{ and } y \in V(F_2)\}$. In this paper, we present a characterization to $(P_5, K_1 \cup K_3)$-free graphs, prove that $\chi(G) \leq 2\omega(G) - 1$ if $G$ is $(P_5, K_1 \cup K_3)$-free. Based on this result, we further prove that $\chi(G) \leq \max\{2\omega(G), 15\}$ if $G$ is a $(P_5, K_1 + (K_1 \cup K_3))$-free graph. We also construct a $(P_5, K_1 + (K_1 \cup K_3))$-free graph $G$ with $\chi(G) = 2\omega(G)$.

Keywords P$_5$-free · Chromatic number · Induced subgraph

Mathematics Subject Classification 05C15 · 05C78

1 Introduction

All graphs considered in this paper are finite and simple. Let $G$ be a graph. The vertex set of a complete subgraph of $G$ is called a clique of $G$, and the clique number $\omega(G)$...
of $G$ is the maximum size of cliques of $G$. We use $P_k$ and $C_k$ to denote a path and a cycle on $k$ vertices respectively.

Let $G$ and $H$ be two vertex disjoint graphs. The union $G \cup H$ is the graph with $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. Similarly, the join $G + H$ is the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{xy\}$ for each pair $x \in V(G)$ and $y \in V(H)$.

For a subset $X \subseteq V(G)$, let $G[X]$ denote the subgraph of $G$ induced by $X$. A hole of $G$ is an induced cycle of length at least 4, and a k-hole is a hole of length $k$. A $k$-hole is said to be an odd (even) hole if $k$ is odd (even). An antihole is the complement of some hole. An odd (resp. even) antihole is defined analogously.

We say that $G$ induces $H$ if $G$ has an induced subgraph isomorphic to $H$, and say that $G$ is $H$-free if $G$ does not induce $H$. Let $\mathcal{H}$ be a family of graphs. We say that $G$ is $\mathcal{H}$-free if $G$ induces no member of $\mathcal{H}$.

A coloring of $G$ is an assignment of colors to the vertices of $G$ such that no two adjacent vertices receive the same color. The minimum number of colors required to color $G$ is called the chromatic number of $G$, and is denoted by $\chi(G)$. Obviously we have that $\chi(G) \geq \omega(G)$. However, determining the upper bound of the chromatic number of some family of graphs $G$, especially, giving a function of $\omega(G)$ to bound $\chi(G)$ is generally very difficult. A family $\mathcal{G}$ of graphs is said to be $\chi$-bounded if there is a function $f$ such that $\chi(G) \leq f(\omega(G))$ for every $G \in \mathcal{G}$, and if such a function $f$ does exist to $\mathcal{G}$, then $f$ is said to be a binding function of $\mathcal{G}$ [16]. A graph $G$ is said to be perfect if $\chi(H) = \omega(H)$ for each induced subgraph $H$. Thus the binding function for perfect graphs is $f(x) = x$. The famous Strong Perfect Graph Theorem [8] states that a graph is perfect if and only if it is (odd hole, odd antihole)-free. Erdős [13] showed that for any positive integers $k$ and $l$, there exists a graph $G$ with $\chi(G) \geq k$ and without cycles of length less than $l$. This result motivates the study of the chromatic number of $\mathcal{H}$-free graphs for some $\mathcal{H}$. Gyárfás [16, 17], and Sumner [27] independently, proposed the following conjecture.

**Conjecture 1.1** [17, 27] For every tree $T$, $T$-free graphs are $\chi$-bounded.

Interested readers are referred to [20, 23, 25] for more information on Conjecture 1.1 and related problems. Gyárfás [17] proved that $\chi(G) \leq (k - 1)^{\omega(G)} - 1$ for $k \geq 4$ if $G$ is $P_k$-free and $\omega(G) \geq 2$. Then the upper bound was improved to $(k - 2)^{\omega(G)} - 1$ by Gravier et al. [18]. The problem of determining whether the class of $P_t$-free graphs ($t \geq 5$) admits a polynomial $\chi$-binding function remains open.

**Problem 1.1** [21] Are there polynomial functions $f_{P_k}$ for $k \geq 5$ such that $\chi(G) \leq f_{P_k}(\omega(G))$ for every $P_k$-free graph $G$?

Since $P_4$-free graphs are perfect, finding an optimal binding function for $P_5$-free graphs attracts much attention. Esperet et al. [14] proved that $\chi(G) \leq 5 \times 3^{\omega(G)} - 3$ for $P_5$-free graphs.

**Theorem 1.1** ([14]) $\chi(G) \leq 5 \cdot 3^{\omega(G)} - 3$ for $P_5$-free graphs $G$ with $\omega(G) \geq 3$.

This bound is sharp for $\omega(G) = 3$. In 2007, Choudum, Karthick and Shalu conjectured that $P_5$-free graphs have a quadratic binding function.
Conjecture 1.2 [7] There is a constant \( c \) such that \( \chi(G) \leq c\omega^2(G) \) if \( G \) is \( P_5 \)-free.

Conjecture 1.2 has been verified for many classes of \( P_5 \)-free graphs, and tight linear binding functions are obtained for some \((P_5, H)\)-free graphs with \(|V(H)| \leq 5\), see [3–7, 9–12, 15, 19, 21]. Very recently, Scott, Seymour and Spirkl [26] provided a near polynomial binding function for \( P_5 \)-free graphs stating that \( \chi(G) \leq \omega(G) \log^{2\omega(G)}(\omega(G)) \) if \( G \) is \( P_5 \)-free.

Let \( F \) and \( H \) be two graphs. We say that \( F \) is a blow up of \( H \) if \( F \) can be obtained from \( H \) by replacing each vertex with an independent set such that the independent sets corresponding to adjacent vertices in \( H \) are complete to each other. A 5-ring is a blow up of a 5-hole. In [27] (see also [14]), Sumner characterized the structure of \((P_5, K_3)\)-free graphs.

Theorem 1.2 ([27]) A connected \((P_5, K_3)\)-free graph is either bipartite or a 5-ring.

By Theorems 1.1 and 1.2, we have that each \((P_5, K_4)\)-free graph is 5-colorable. The graph \( K_1 + (K_1 \cup K_3) \) can be obtained from \( K_4 \) by adding a new vertex joining to one vertex of the \( K_4 \). So, \( K_4 \)-free graphs must be \((K_1 + (K_1 \cup K_3))\)-free. Motivated by Theorem 1.1, we study the chromatic number of \((K_1 + (K_1 \cup K_3))\)-free graphs. Among other results on the chromatic number of \( P_5 \)-free graphs, we proved in [11] that if \( G \) is \((P_5, K_1 + (K_1 \cup K_3))\)-free then \( \chi(G) \leq 3\omega(G) + 11 \). In this paper, we present a characterization to \((P_5, K_1 \cup K_3)\)-free graphs, and prove that each \((P_5, K_1 \cup K_3)\)-free graph is \((2\omega(G) - 1)\)-colorable. Based on this, we get a tight upper bound for the chromatic number of \((P_5, K_1 + (K_1 \cup K_3))\)-free graphs.

Before introducing the main results of this paper, we need some new notations. Let \( v \in V(G) \), and let \( X \) be a subset of \( V(G) \). We use \( N_X(v) \) to denote the set of neighbors of \( v \) in \( X \). We say that \( v \) is complete to \( X \) if \( N_X(v) = X \), and say that \( v \) is anticomplete to \( X \) if \( N_X(v) = \emptyset \). For two subsets \( X \) and \( Y \) of \( V(G) \), we say that \( X \) is complete to \( Y \) if each vertex of \( X \) is complete to \( Y \), say that \( X \) is anticomplete to \( Y \) if each vertex of \( X \) is anticomplete to \( Y \).

Let \( d(v, X) = \min_{x \in X} d(v, x) \) and call \( d(v, X) \) the distance of a vertex \( v \) to a subset \( X \). Let \( i \) be a positive integer and \( N^i_G(X) = \{ y \in V(G) \setminus X | d(y, X) = i \} \). We call \( N^i_G(X) \) the \( i \)-neighborhood of \( X \) and simply write \( N^i(X) \) as \( N_G(X) \). If no confusion may occur, we write \( N^i(X) \) instead of \( N^i_G(X) \), and \( N^i(\{v\}) \) is denoted by \( N^i(v) \) for short. A set \( D \) is said to be a dominating set of \( G \) if \( V(G) = D \cup N(D) \).

Suppose that \( C = v_1 v_2 v_3 v_4 v_5 v_1 \) is a 5-hole of \( G \). Let \( M(C) = V(G) \setminus (V(C) \cup N(C)) \). For a subset \( T \subseteq \{1, 2, 3, 4, 5\} \), we define

\[
N_T(C) = \{ x \mid x \in N(C), \text{ and } v_i x \in E(G) \text{ if and only if } i \in T \}.
\]

It is easy to check that for \( k \in \{1, 2, 3, 4, 5\} \) and \( l = k + 2 \), \( N_{[k,k+2]}(C) = N_{[l,l+3]}(C) \) and \( N_{[k,k+2,k+3]}(C) = N_{[l,l+1,l+3]}(C) \), where the summation of subindex is taken modulo 5 (in this paper, the summations of subindex are always taken modulo some integer \( h \) and we always set \( h + 1 \equiv 1 \)). We define

\[
N^{(2)}(C) = \bigcup_{1 \leq i \leq 5} N_{[i,i+2]}(C),
\]
Theorem 1.4

Theorem 1.4 is clearly tight as $\chi(G) = 2\omega(G) - 1$.

Theorem 1.5

If $G$ is a $(P_5, K_1 + (K_1 \cup K_3))$-free graph then $\chi(G) \leq \max\{2\omega(G), 15\}$, and there exists a $(P_5, K_1 + (K_1 \cup K_3))$-free graph $G$ with $\chi(G) = 2\omega(G)$.

The proof of Theorem 1.5 is heavily relied on Theorem 1.4. The upper bound of Theorem 1.4 is clearly tight as $C_5$ and its blow up are extremal graphs. We can construct a $(P_5, K_1 + (K_1 \cup K_3))$-free graph $G$ with $\chi(G) = 2\omega(G)$. Let $C = v_1v_2v_3v_4v_5v_1$ be a 5-hole. Let $H$ be the graph obtained from $C$ by replacing each vertex $v_i$ of $C$ by a 5-hole $C^i$, for $1 \leq i \leq 5$, such that a vertex of $C^i$ and a vertex of $C^j$ are adjacent in $H$ if and only if $v_i$ is adjacent to $v_j$ in $C$.

It is certain that $H$ is $(P_5, K_1 + (K_1 \cup K_3))$-free and $\omega(H) = 4$. We claim that $\chi(H) = 8$. Without loss of generality, for each coloring $\phi$ of $H$, we can always suppose...
that \( \phi(V(C^1)) = \{1, 2, 3\} \) and \( \phi(V(C^2)) = \{4, 5, 6\} \). Let \( \phi(V(C^3)) = \{1, 7, 8\}, \phi(V(C^4)) = \{3, 4, 5\} \) and \( \phi(V(C^5)) = \{6, 7, 8\} \). We see that \( \chi(H) \leq 8 \). If \( \chi(H) \leq 7 \), then we may assume by symmetry that \( \phi(V(C^3)) = \{1, 2, 7\} \), but now we only have five colors \( \{3, 4, 5, 6, 7\} \) that can be used to color \( V(C^4) \cup V(C^5) \), a contradiction. Therefore, \( \chi(H) = 8 = 2\omega(H) \).

The following lemma, which is devoted to the structure of \( P_5 \)-free graphs, will be used frequently in our proof. Here the summation of subindexes is taken modulo 5.

**Lemma 1.1** ([11, 14]) Let \( G \) be a \( P_5 \)-free graph with a 5-hole \( C = v_1v_2v_3v_4v_5v_1 \). Then

(a) for \( i \in \{1, 2, 3, 4, 5\} \), \( N[i](C) = N[i,i+1](C) = \emptyset \), and \( N[i,i+2](C) \cup N[i,i+1,i+2](C) \) is anticomplete to \( N^2(C) \),
(b) if \( x \in N(C) \) and \( N^2(x) \cap N^3(C) \neq \emptyset \) then \( x \in N[1,2,3,4,5](C) \), and
(c) for each vertex \( x \in N^2(C) \) and each component \( B \) of \( G[N^3(C)] \), \( x \) is either complete or anticomplete to \( B \).

The next section is devoted to the proofs of Theorems 1.3 and 1.4. Theorem 1.5 is proved in Sect. 3.

### 2 \((P_5, K_1 \cup K_3)\)-free graphs

This section is aimed to prove Theorems 1.3 and 1.4. In this section, we always suppose that \( G \) is a \((P_5, K_1 \cup K_3)\)-free graph. If \( G \) has a 5-hole, we always use \( C = v_1v_2v_3v_4v_5v_1 \) to denote a 5-hole in \( G \). Recall that we define \( M(C) = V(G) \setminus (V(C) \cup N(C)) \).

**Lemma 2.1** If \( G \) has a 5-hole, then the followings hold for each \( i \in \{1, 2, \ldots, 5\} \).

(a) \( N[i,i+1,i+2](C) = N[i,i+1,i+2,i+3](C) = \emptyset \).
(b) Both \( N[i,i+2](C) \) and \( N[i,i+1,i+3](C) \) are independent, and \( N[i,i+2](C) \) is complete to \( N[i,i+1,i+3](C) \cup N[i+1,i+4](C) \).
(c) \( N^2(C) \) is complete to \( N[1,2,3,4,5](C) \), and \( N^3(C) \) is complete to \( M(C) \).
(d) \( N[i,i+1,i+3](C) \) is anticomplete to \( N[i,i+3](C) \cup N[i+1,i+3](C) \). Moreover, if \( M(C) \neq \emptyset \), then \( N[i,i+1,i+3](C) \) is anticomplete to \( N[i,i+2,i+3](C) \cup N[i+1,i+3,i+4](C) \), and is either complete or anticomplete to \( N[i,i+1,i+2](C) \cup N[i+1,i+2,i+4](C) \) whenever both \( N[i,i+1,i+2](C) \) and \( N[i+1,i+2,i+4](C) \) are not empty.
(e) If \( \omega(G[N[1,2,3,4,5](C)]) = \omega(G) - 2 \) or \( M(C) \neq \emptyset \) then \( G[V(C) \cup N^2(C)] \) is a 5-ring.

**Proof** Suppose that \( N[i,i+1,i+2](C) \cup N[i,i+1,i+2,i+3](C) \neq \emptyset \) for some \( i \in \{1, 2, \ldots, 5\} \). Let \( v \in N[i,i+1,i+2](C) \cup N[i,i+1,i+2,i+3](C) \). Then \( G[[v, v_{i+1}, v_{i+2}, v_{i+4}]] = K_1 \cup K_3 \). Hence (a) holds.

Next we prove (b). Suppose, for some \( i \), \( N[i,i+2](C) \) is not independent. Let \( uv \) be an edge in \( G[N[i,i+2](C)] \). Then \( G[[u, v, v_{i+1}, v_{i+2}]] = K_1 \cup K_3 \), a contradiction. Similarly, if \( N[i,i+1,i+3](C) \) is not independent, let \( uv \) be an edge of \( G[N[i,i+1,i+3](C)] \), then \( G[[u, v, v_{i+1}, v_{i+2}]] = K_1 \cup K_3 \), which leads to a contradiction. If \( N[i,i+2](C) \) is not
complete to $N_{i+1,i+3}(C)$ for some $i$, choose $u \in N_{i,i+2}(C)$ and $v \in N_{i+1,i+3}(C)$ with $uv \notin E(G)$, then $uv_1v_2v_3v_4v_5v$ is an induced $P_6$ of $G$. A similar contradiction occurs if $N_{i,i+2}(C)$ is not complete to $N_{i+1,i+4}(C)$. Therefore, (b) holds.

If there exist $u \in N_{i,i+2}(C)$ and $v \in N_{i,i+2}(C)$ with $uv \notin E(G)$ for some $i$, then $G[[u,v,v_1,v_2,v_3]] = K_1 \cup K_3$. If for some $i$, there exist $u \in M(C)$ and $v \in N_{i,i+1,i+3}(C)$ such that $uv \notin E(G)$, then $G[[u,v,v_1]]= K_1 \cup K_3$. This proves (c).

If the first statement of (d) is not true, then we may choose $u \in N_{i,i+1,i+3}(C)$ and $v \in N_{i,i+3}(C) \cup N_{i+1,i+3}(C)$ with $uv \in E(G)$ such that $G[[u,v,v_1,v_2]]= K_1 \cup K_3$ when $v \in N_{i,i+3}(C)$, and that $G[[u,v,v_1,v_2]]= K_1 \cup K_3$ when $v \in N_{i+1,i+3}(C)$, Suppose that $M(C) \neq \emptyset$, and let $x \in M(C)$. Note that $M(C)$ is complete to $N^{(3)}(C)$ by the statement (c). If there exist $u \in N_{i,i+1,i+3}(C)$ and $v \in N_{i,i+2,i+3}(C) \cup N_{i+1,i+3,i+4}(C)$ with $uv \in E(G)$, then $G[[u,v,v_1,v_2]]= K_1 \cup K_3$ when $v \in N_{i,i+2,i+3}(C)$, and that $G[[u,v,v_1,v_2]]= K_1 \cup K_3$ when $v \in N_{i+1,i+3,i+4}(C)$. Suppose that $N_{l,i+1,i+2}(C) \neq \emptyset$ and $N_{l+1,i+2,i+4}(C) \neq \emptyset$. By symmetry, assume that $u \in N_{i,i+1,i+3}(C)$ is adjacent to $v \in N_{i-1,i,i+2}(C)$ and not adjacent to $w \in N_{i+1,i+2,i+4}(C)$. Then $uw \notin E(G)$ and $G[[u,v,v_1]] = K_1 \cup K_3$, a contradiction. Therefore, (d) holds.

By the statement (b), to prove that $N^{(2)}(C) \cup V(C)$ induces a 5-ring, we only need to check that $N_{i,i+2}(C)$ is anticomplete to $N_{i,i+3}(C) \cup N_{i+2,i+4}(C)$.

By Lemma 1.1(a), we observe that $M(C)$ is anticomplete to $N^{(2)}(C) \cup V(C)$ if $M(C) \neq \emptyset$, then $N_{i,i+2}(C)$ must be anticomplete to $N_{i,i+3}(C) \cup N_{i+2,i+4}(C)$, otherwise a $K_1 \cup K_3$ occurs.

Finally, suppose that $\omega(G[N_{i,i+2}(C)]) = \omega(G) - 2$, and let $K \subseteq N_{i,i+3}(C)$ be a clique of size $\omega(G) - 2$. Assume by symmetry that $N_{i,i+2}(C)$ is not anticomplete to $N_{i+2,i+4}(C)$. Let $u \in N_{i,i+2}(C)$ and $v \in N_{i+2,i+4}(C)$ be an adjacent pair. Then $K$ is complete to $[u,v,v_1]$ by statement (c), and so $G$ contains a clique of size $\omega(G) + 1$. This leads to a contradiction and completes the proof of Lemma 2.1.

From Lemma 2.1(a), we observe that

$$N(C) = N_{1,2,3,4,5}(C) \cup N^{(2)}(C) \cup N^{(3)}(C),$$

and it follows from Lemma 2.1(d) that if $M(C) \neq \emptyset$ and $N_{i,i+1,i+3}(C) \neq \emptyset$ for each $1 \leq i \leq 5$, then $N^{(3)}(C)$ is either independent or induces a 5-ring in $G$.

Proof of Theorem 1.3 Suppose that $G$ has a non-dominating 5-hole $C = v_1v_2v_3v_4v_5v_1$, that is, $M(C) = V(G) \setminus (V(C) \cup N(C)) \neq \emptyset$. Let $A_1 = V(C) \cup N^{(2)}(C), A_2 = N^{(3)}(C),$ and $A_3 = N_{1,2,3,4,5}(C)$.

By Lemma 2.1(b) and (d), we observe that $G[A_1]$ is a 5-ring which is a blow up of $C$, and $G[A_1 \cup A_2]$ is a blow up of a subgraph of $\mathcal{F}$.

By Lemma 2.1(c), we have that $A_1$ is complete to $A_3$. To prove the second statement, we only need to verify that $A_3$ is complete to $M(C)$. If it is not the case, choose $u \in A_3$ and $v \in M(C)$ with $uv \notin E(G)$, then $G[[u,v,v_1]] = K_1 \cup K_3$, a contradiction. Therefore, (b) is true.

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By Lemma 2.1(c), we observe that \( M(C) \) is anticomplete to \( A_1 \) and complete to \( A_2 \). Suppose that \( A_2 \neq \emptyset \), and let \( x \in N[i,i+1,i+3](C) \) for some \( i \in \{1, 2, 3, 4, 5\} \). If the third statement is not true then there must be an edge \( uv \) in \( G[M(C)] \) and so \( G[\{u, v, v_{i+2}, x\}] = K_1 \cup K_3 \). This leads to a contradiction and proves (c), and also completes the proof of Theorem 1.3. \( \square \)

Now we turn to prove Theorem 1.4. The following two colorings will be used in the proof of Theorem 1.4.

By Lemma 2.1(d), we can construct a 5-coloring \( \psi \) of \( G[V(C) \cup N(C)] \) as below:

\[
\begin{align*}
\psi^{-1}(1) &= N_{[2,4]}(C) \cup N_{[1,2,4]}(C) \cup \{v_3\}, \\
\psi^{-1}(2) &= N_{[3,5]}(C) \cup N_{[2,3,5]}(C) \{v_4\}, \\
\psi^{-1}(3) &= N_{[1,4]}(C) \cup N_{[3,4,1]}(C) \cup \{v_2, v_5\}, \\
\psi^{-1}(4) &= N_{[2,5]}(C) \cup N_{[4,5,2]}(C) \cup \{v_1\}, \\
\psi^{-1}(5) &= N_{[1,3]}(C) \cup N_{[5,1,3]}(C).
\end{align*}
\]

If \( M(C) \neq \emptyset \), it follows from Theorem 1.3 that we can construct a 4-coloring \( \phi \) of \( G[V(C) \cup N(C)] \) as below:

\[
\begin{align*}
\phi^{-1}(1) &= N_{[1,4]}(C) \cup N_{[1,2,4]}(C) \cup N_{[2,4]}(C) \cup \{v_3, v_5\}, \\
\phi^{-1}(2) &= N_{[2,5]}(C) \cup N_{[2,3,5]}(C) \cup N_{[3,5]}(C) \cup \{v_1, v_4\}, \\
\phi^{-1}(3) &= N_{[1,3]}(C) \cup N_{[1,3,4]}(C) \cup N_{[5,1,3]}(C) \cup \{v_2\}, \\
\phi^{-1}(4) &= N_{[4,5,2]}(C).
\end{align*}
\]

**Proof of Theorem 1.4** Let \( G \) be a connected \((P_5, K_1 \cup K_3)\)-free graph with \( \omega(G) = h \). Clearly the theorem holds when \( h = 1 \). If \( h = 2 \), then \( G \) is bipartite or a 5-ring by Theorem 1.2 and so \( \chi(G) \leq 3 = 2h - 1 \). Thus we assume that \( h \geq 3 \) and the theorem holds for all graphs with clique number smaller than \( h \).

If \( G \) does not have any 5-hole, then for an arbitrary vertex \( v \), \( G - N(v) \) is bipartite and \( \omega(G[N(v)]) \leq h - 1 \). Otherwise, by induction \( \chi(G) = 2 + \chi(G[N(v)]) \leq 2 + (2(h - 1)) = 2h - 1 \). Let \( C = v_1v_2v_3v_4v_5v_1 \) be a 5-hole of \( G \). It is certain that \( \omega(G[N_{1,2,3,4,5}(C) \cup N(i,i+1,i+3)](C)) \leq h - 2 \) for each \( i \in \{1, 2, 3, 4, 5\} \) as \( \{v_i, v_{i+1}\} \) is complete to \( N_{[1,2,3,4,5]}(C) \cup N_{[i,i+1,i+3]}(C) \).

If \( N_{1,2,3,4,5}(C) = M(C) = \emptyset \), then \( \chi(G) \leq 5 \leq 2h - 1 \) by the coloring \( \psi \) defined in (1).

If \( N_{1,2,3,4,5}(C) = \emptyset \) and \( M(C) \neq \emptyset \), then \( N^{(3)}(C) \neq \emptyset \), which implies that \( M(C) \) is independent by Theorem 1.3(c). It follows from the coloring \( \phi \) defined in (2) that \( \chi(G) \leq 5 \leq 2h - 1 \).

Suppose that \( N_{1,2,3,4,5}(C) \neq \emptyset \) and \( M(C) = \emptyset \). If \( \omega(G[N_{1,2,3,4,5}(C)]) \leq h - 3 \), then \( \chi(G - N_{1,2,3,4,5}(C)) \leq 5 \) by the coloring \( \psi \) defined in (1), which implies that \( \chi(G) \leq \chi(G - N_{1,2,3,4,5}(C)) + \chi(G[N_{1,2,3,4,5}(C)]) \leq 5 + (2(h - 3) - 1) < 2h - 1 \) by induction. So, suppose that \( \omega(G[N_{1,2,3,4,5}(C)]) = h \). By Lemma 2.1(d) and (e), we have that \( N_{[1,3]}(C) \) is anticomplete to \( N_{[1,4]}(C) \cup N_{[3,4,1]}(C) \cup \{v_2, v_5\} \), and so we can modify the coloring \( \psi \) by recoloring \( N_{[1,3]}(C) \) with 3, which implies that \( \chi(G - N_{1,2,3,4,5}(C) \cup N_{[5,1,3]}(C)) \leq 4 \). Now, we have that \( \chi(G) \leq \chi(G - \).
For each $i$, neither define $\chi(G[N[i,1,2,3,4,5](C) \cup N[5,1,3](C)]) \leq 4 + (2(h - 2) - 1) = 2h - 1$ by induction.

Therefore, suppose that $N[i,1,2,3,4,5](C) \neq \emptyset$ and $M(C) \neq \emptyset$. Thus, $N^{(2)}(C) \cup V(C)$ induces a 5-ring by Lemma 2.1(d). It is obvious that $G[M(C)]$ is $K_3$-free, otherwise a triangle of $G[M(C)]$ together with any vertex of $C$ induces a $K_1 \cup K_3$, and so $\chi(G[M(C)]) \leq 3$ by Theorem 1.2.

If $N^{(3)}(C) = \emptyset$, then $\chi(G - N[i,1,2,3,4,5](C)) = 3$ as $M(C)$ is anticomplete to $N^{(2)}(C)$ by Lemma 1.1(a), and so $\chi(G) \leq 3 + (2(h - 2) - 1) = 2h - 1$ by induction. Thus, suppose that $N^{(3)}(C) \neq \emptyset$, which implies that $M(C)$ is independent by Theorem 1.3(c). By the coloring $\phi$ defined in (2), $G - N[i,1,2,3,4,5](C) \cup N[4,5,1](C) \cup M(C)$ is 3-colorable, and so $G - N[i,1,2,3,4,5](C) \cup N[4,5,1](C)$ is 4-colorable. Thus, $\omega(G[N[i,1,2,3,4,5](C) \cup N[4,5,1](C)]) \leq h - 2$, we have $\chi(G) \leq 4 + \chi(G[N[i,1,2,3,4,5](C) \cup N[4,5,1](C)]) \leq 4 + (2(h - 2) - 1) = 2h - 1$ by induction. This completes the proof of Theorem 1.4. □

3 $(P_5, K_1 + (K_1 \cup K_3))$-Free Graphs

Before proving Theorem 1.5, we first present several lemmas on the structure of $(P_5, K_1 + (K_1 \cup K_3))$-free graphs. From now on, we always suppose that $G$ is a connected $(P_5, K_1 + (K_1 \cup K_3))$-free graph without clique cutset. For two subsets $X$ and $Y$ of $V(G)$, we say that $X$ is adjacent to $Y$ if $N(X) \cap Y \neq \emptyset$.

Let $C = v_1v_2v_3v_4v_5v_1$ be a 5-hole of $G$. Recall that $N^{(2)}(C) = \bigcup_{1 \leq i \leq 5} N[i,i+2](C)$, $N^{(3)}(C) = \bigcup_{1 \leq i \leq 5} N[i,i+1,i+2,i+3](C)$, and $M(C) = V(G) \setminus (V(C) \cup N(C))$. We further define $N^{(1)}(C) = \bigcup_{1 \leq i \leq 5} N[i,i+1,i+2](C)$, and $N^{(2)}(C) = \bigcup_{1 \leq i \leq 5} N[i,i+1,i+3](C)$. By Lemma 1.1(a), we have

\[ N(C) = N[i,1,2,3,4,5](C) \cup N^{(2)}(C) \cup N^{(3,1)}(C) \cup N^{(3,2)}(C) \cup N^{(4)}(C). \]

**Lemma 3.1** ([11]) Let $C = v_1v_2v_3v_4v_5v_1$ be a 5-hole of $G$, and $T$ be a component of $G[N^2(C)]$. Then the following holds.

(a) For each $i \in \{1, 2, 3, 4, 5\}$, $G[N(v_i)]$ is $(K_1 \cup K_3)$-free, $G[N[i,i+2](C)]$ is $K_3$-free, and $N[i,i+1,i+2](C) \cup N[i,i+1,i+3](C) \cup N[i,i+1,i+2,i+3](C)$ is independent.
(b) If no vertex in $N(C)$ dominates $T$, then there exist two non-adjacent vertices $u$ and $v$ in $N(C)$ such that both $N_T(u)$ and $N_T(v)$ are not empty.

**Lemma 3.2** Let $C = v_1v_2v_3v_4v_5v_1$ be a 5-hole of $G$, $S$ be a component of $G[N[i,1,2,3,4,5](C)]$ with $\omega(S) \geq 2$. Then for each $i \in \{1, 2, 3, 4, 5\}$, the following holds.

(a) $N[i,i+2](C) \cup N[i,i+1,i+2](C)$ is complete to $S$, and $N[i,i+2](C)$ is independent.
(b) For each edge $xy$ in $S$, no vertex of $N[i,i+1,i+3](C) \cup N[i,i+1,i+2,i+3](C)$ is anticomplete to $\{x, y\}$.
(c) $N[i,i+2](C)$ is anticomplete to $N[i-1,i,i+1](C) \cup N[i-1,i,i+2](C) \cup N[i-1,i,i+1,i+2](C)$.
(d) $\chi(G - N[i,1,2,3,4,5](C) - M(C)) \leq 5$. 

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Proof} Suppose that, for some \( i \in \{1, 2, 3, 4, 5\} \), \( N_{i,i+2}(C) \cup N_{i,i+1,i+2}(C) \) has a vertex \( u \) that is not complete to \( S \). If \( u \) is anticomplete to \( S \), then \( G[[u, v, w, v_i, v_{i+4}]] = K_1 + (K_1 \cup K_3) \). Otherwise, there exists an edge, say \( uv \) in \( S \), such that \( uv \in E(G) \) and \( uv \notin E(G) \). Then \( G[[u, v, w, v_i, v_{i+4}]] = K_1 + (K_1 \cup K_3) \). Both are contradictions.

Suppose that \( N_{i,i+2}(C) \) is not independent for some \( i \in \{1, 2, 3, 4, 5\} \). Choose an edge \( xy \in G[N_{i,i+2}(C)] \), and let \( z \) be an arbitrary vertex of \( S \). Then \( xz \in E(G) \) and \( yz \in E(G) \), and so \( G[[x, y, z, v_i, v_{i+4}]] = K_1 + (K_1 \cup K_3) \), a contradiction. Therefore, (a) holds.

Let \( xy \) be an edge of \( S \), and \( v \in N_{i,i+1,i+2}(C) \cup N_{i,i+1,i+2,i+3}(C) \). If \( vx \notin E(G) \) and \( vy \notin E(G) \), then \( G[[v, x, y, v_i, v_{i+4}]] = K_1 + (K_1 \cup K_3) \). Therefore, (b) holds.

Suppose that (c) is not true for some \( i \in \{1, 2, 3, 4, 5\} \). Let \( e \in N_{i,i+2}(C) \) and \( u \in N_{i,i+1,i+1}(C) \cup N_{i,i+1,i+2}(C) \cup N_{i,i+1,i+1,i+2}(C) \) such that \( uv \in E(G) \). By (a) and (b), we observe that there exists a vertex \( w \in N_{1,2,3,4,5}(C) \) such that \( uv \notin E(G) \) and \( wu \notin E(G) \), which implies that \( G[[v, u, w, v_i, v_{i+4}]] = K_1 + (K_1 \cup K_3) \). Therefore, (c) is true.

By (a), (c) and Lemma 3.1 (a), we have that \( N_{i,i+2}(C) \cup N_{i,i+1,i+1}(C) \cup N_{i,i+1,i+1,i+2}(C) \cup N_{i,i+1,i+1,i+2,i+3}(C) \) is independent for each \( i \in \{1, 2, 3, 4, 5\} \). By coloring \( v_i \) for each \( i \in [1, 2, 3, 4, 5] \) with color \( i \), we get a 5-coloring of \( G - N_{1,2,3,4,5}(C) - M(C) \). This proves (d), and completes the proof of Lemma 3.2. 

\[ \square \]

Lemma 3.3 \([11]\) Let \( C = v_1 v_2 v_3 v_4 v_5 v_1 \) be a 5-hole of \( G \). Then \( G[N^3(C)] \) is \( K_2 \)-free, and \( N^2(C) \) can be partitioned into two parts \( A \) and \( B \) such that both \( G[A] \) and \( G[B] \) are \( K_2 \)-free.

Lemma 3.4 Let \( C = v_1 v_2 v_3 v_4 v_5 v_1 \) be a 5-hole of \( G \), and \( S \) be a component of \( G[N_{1,2,3,4,5}(C)] \). If \( N(S) \cap N^2(C) \neq \emptyset \), then \( N(x) \cap N^2(C) = N(y) \cap N^2(C) \) for any \( x, y \in S \).

Proof} Suppose that \( N(S) \cap N^2(C) \neq \emptyset \). We apply induction on \( |S| \). The lemma holds trivially if \( |S| = 1 \). Suppose that \( |S| = k \geq 2 \), and the lemma holds on all components of \( G[N_{1,2,3,4,5}(C)] \) of size less than \( k \). There must be a vertex, say \( x \), in \( S \) such that \( S - x \) is connected, and \( N(S - x) \cap N^2(C) \neq \emptyset \). Let \( y \) be a neighbor of \( x \) in \( S \). To prove the lemma, we only need to verify that \( N(x) \cap N^2(C) = N(y) \cap N^2(C) \). Suppose that it is not the case. Then, we may assume, without loss of generality, that \( u \in N(x) \cap N^2(C) \) and \( u \notin N(y) \cap N^2(C) \), which implies that \( G[[x, y, u, v_1, v_2]] = K_1 + (K_1 \cup K_3) \). This leads to a contradiction and proves the lemma. 

\[ \square \]

Lemma 3.5 Let \( C = v_1 v_2 v_3 v_4 v_5 v_1 \) be a 5-hole of \( G \), and \( T \) be a component of \( G[N^2(C)] \). Suppose that \( N^3(C) \cup N^4(C) \neq \emptyset \). Then

(a) \( T \) is a single vertex adjacent to \( N_{1,2,3,4,5}(C) \) if \( N(T) \cap (N^3(C) \cup N^4(C)) \neq \emptyset \) and \( \omega(G[N_{1,2,3,4,5}(C)]) \geq 2 \), and

(b) \( T \) is \( K_3 \)-free if \( N(T) \cap (N^3(C) \cup N^4(C)) = \emptyset \).

Proof} Let \( Q = N^3(C) \cup N^4(C) \).

Firstly, we prove (a). Suppose that \( N(T) \cap Q \neq \emptyset \) and \( \omega(G[N_{1,2,3,4,5}(C)]) \geq 2 \). Since \( N(T) \cap Q \neq \emptyset \), we have that, for some \( i \in \{1, 2, 3, 4, 5\} \), \( N_{i,i+1,i+3} \cup \)
\[N_{[i,i+1,i+i+2,i+3]}\] has a vertex \(u\) that is free for each \(T\), otherwise an induced \(P_5\) appears in \(G\). By Lemma 3.2(b), \(u\) has a neighbor, say \(v\), in \(N_{[1,2,3,4,5]}(C)\). If \(v\) is anticomplete to \(T\) then \(B[\{u,v,v_i,v_i+1,z\}]=K_1+(K_1 \cup K_3)\) for any vertex \(z \in T\). This proves that \(N(T) \cap N_{[1,2,3,4,5]}(C) \neq \emptyset\), that is, \(T\) is adjacent to \(N_{[1,2,3,4,5]}(C)\).

Suppose that \(|V(T)| \geq 2\), and let \(xy\) be an edge of \(T\). Since \(G\) is \(P_5\)-free, we have that, for some \(i \in \{1,2,3,4,5\}\), \(N_{[i,i+1,i+i+2,i+3]}(C) \cup N_{[i,i+1,i+i+2,i+3]}(C)\) has a vertex, say \(u\), that is complete to \(T\). Particularly, \(\{ux,uy\} \subseteq E(G)\). By Lemma 3.2(b), we may choose a neighbor \(v\) of \(u\) in \(N_{[1,2,3,4,5]}(C)\). If \(\{vx,vy\} \subseteq E(G)\) then \(G[(u,v,v_i+4,x,y)] = K_1+(K_1 \cup K_3)\). Otherwise, we may assume by symmetry that \(vx \notin E(G)\), then \(G[(u,v,v_i,v_i+1,x)] = K_1+(K_1 \cup K_3)\). Therefore, \((a)\) holds.

Suppose to the contrary of \((b)\) that \(N(T) \cap Q = \emptyset\) and \(T\) has a \(K_3\), say \(w_1w_2w_3w_1\). Let \(u\) be a vertex in \(Q\), and suppose that \(uv_1,uv_2 \in E(G)\) by symmetry. Since \(N^{(3,1)}(C)\) is anticomplete to \(N^2(C)\) by Lemma 1.1(a), we may choose a vertex, say \(x\), in \(N_{[1,2,3,4,5]}(C)\), and let \(x'\) be a neighbor of \(x\) in \(T\). To avoid a \(K_1+(K_1 \cup K_3)\) on \(\{u,v_1,v_2,x,x'\}\), we have that \(ux \notin E(G)\). If \(x\) is complete to \(T\), then \(G[(v_1,w_1,w_2,w_3,x)] = K_1+(K_1 \cup K_3)\). Otherwise, there must be an edge \(y_1y_2\) in \(T\) such that \(x_1y_1 \in E(G)\) and \(xy_2 \notin E(G)\), and so \(uv_1xy_1y_2\) is an induced \(P_5\). This proves \((b)\) and Lemma 3.5.

Let \(A\) be an antihole with \(V(A) = \{v_1,v_2,\ldots,v_k\}\). We enumerate the vertices of \(A\) cyclically such that \(v_1v_{i+1} \notin E(G)\) and simply write \(A = v_1v_2\ldots v_k\). Here the summations of subindices are taken modulo \(k\) and we set \(k+1 \equiv 1\).

Suppose that \(G\) induces an antihole \(A = v_1v_2\ldots v_k\) with \(k \geq 6\). We use \(S(A)\) to denote the set of vertices which are complete to \(A\), and let \(T(A) = N(A) \setminus S(A)\). Note that \(T(A)\) is not complete to \(A\). For each \(i \in \{1,2,\ldots,k\}\), we define \(T_i(A)\) to be the subset of \(T(A)\) such that for each vertex \(x\) of \(T_i(A)\), \(i\) is the minimum index with \(xv_i \in E(G)\) and \(xv_{i-1} \notin E(G)\).

Clearly, \(T(A) = \bigcup_{1 \leq i \leq k} T_i(A)\), and \(T_i(A) \cap T_j(A) = \emptyset\) if \(i \neq j\). Since \(G\) is \(K_1+(K_1 \cup K_3)\)-free, we have that \(G[S(A)]\) is \(K_1 \cup K_3\)-free, and \(G[T_i(A)]\) is \(K_1 \cup K_3\)-free for each \(i \in \{1,2,\ldots,k\}\).

The following lemma was proved in [11] without using the notations \(T_i(A)\). Here we present its short proof.

**Lemma 3.6** Let \(G\) be a \((P_5,C_5,K_1+(K_1 \cup K_3))\)-free graph, \(A = v_1v_2\ldots v_k\) an antihole of \(G\) with \(k \geq 6\). Then \(T_i(A)\) is independent for each \(i \in \{1,2,\ldots,k\}\), and \(N^2(A) = \emptyset\).

**Proof** Let \(i \in \{1,2,\ldots,k\}\). Firstly, for each vertex \(v\) of \(T_i(A)\),

\[vv_{i+2} \in E(G),\] (3)

as otherwise either \(vv_{i+2}v_{i+1}\) is an induced \(P_5\) when \(vv_{i+1} \notin E(G)\) or \(vv_{i+2}v_{i+1}v_i\) is a 5-hole when \(vv_{i+1} \notin E(G)\).

Suppose that \(T_i(A)\) is not independent. Let \(x\) and \(x'\) be two adjacent vertices of \(T_i(A)\). Then \(G[\{v_{i-1},v_i,v_{i+2},x,x'\}] = K_1+(K_1 \cup K_3)\) by (3). Therefore, \(T_i(A)\) is an independent set.

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Suppose that $N^2(A) \neq \emptyset$. Let $v$ be a vertex in $N(A)$ that has a neighbor, say $x$, in $N^2(A)$. If $v \in S(A)$ then $G[[v, v_1, v_3, v_5, x]] = K_1 + (K_1 \cup K_3)$. Otherwise, we may assume that $v \in T_i(A)$ by symmetry. By (3), $G[[v, v_1, v_3, v_5, x]] = K_1 + (K_1 \cup K_3)$ if $vv_5 \in E(G)$, and a $P_5 = xvv_1v_5v_{2k+1}$ appears if $vv_5 \notin E(G)$. Therefore, $N^2(A) = \emptyset$. □

**Proof of Theorem 1.5** Let $G$ be a $\{P_5, K_1 + (K_1 \cup K_3)\}$-free graph with $\omega(G) = h$. We may suppose that $G$ is connected, contains no clique cutset, and is not perfect. Thus, $h \geq 2$ as $G$ must induce a 5-hole or an odd antihole with at least 7 vertices by the Strong Perfect Graph Theorem [8].

When $h \in [2, 3]$, the theorem follows immediately from Theorems 1.1 and 1.2. Suppose that $h \geq 4$, and the theorem holds for all $\{P_5, K_1 + (K_1 \cup K_3)\}$-free graphs with clique number less than $h$.

Since $G$ is $P_5$-free, it is certain that $N^4(S) = \emptyset$ for any subset $S$ of $V(G)$.

Let $\gamma = 2h - 5$. We distinguish two cases depending on the existence of 5-holes in $G$, and will use two color sets $C_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ and $C_2 = \{\beta_1, \beta_2, \ldots, \beta_\gamma\}$ to color $G$.

Firstly, suppose that $G$ induces no 5-holes. Then, $G$ must induce an antihole of size at least 6. Let $A = v_1v_2 \ldots v_k$, where $k \geq 6$, be an antihole of $G$. Let $S$ be the set of vertices that are complete to $A$, and let $T = V(G) \setminus (A \cup S)$. It is clear that $G[S]$ is $K_1 \cup K_3$-free. By Lemma 3.6, $V(G) = A \cup S \cup T$.

For integer $i \in \{1, 2, \ldots, k\}$, let $T_i$ be the subset of $T$ such that for each vertex $x$ of $T_i$, $i$ is the minimum index with $xv_i \in E(G)$ and $xv_{i-1} \notin E(G)$. By Lemma 3.6, $T_i \cup \{v_{i-1}\}$ is an independent set.

If $S \neq \emptyset$, then $\chi(G[A \cup T]) \leq k$ by Lemma 3.6, and so $\chi(G) \leq k + (2(h - \lfloor \frac{k}{2} \rfloor) - 1) \leq 2h$ by induction. Therefore, we suppose that $S = \emptyset$.

We further suppose that $A$ has the least number of vertices under the assumption that $k \geq 6$. Notices that $\frac{k}{2} \leq h$ if $k$ is even and $\frac{k-1}{2} \leq h$ if $k$ is odd. If $k \leq 15$, then $\chi(G) \leq k \leq 15$ by Lemma 3.6. If $h > \left\lfloor \frac{k}{2} \right\rfloor$ then $\chi(G) \leq 2\left\lfloor \frac{k}{2} \right\rfloor \leq 2h$. So, we suppose that $h = \left\lfloor \frac{k}{2} \right\rfloor \geq 8$.

Since $S = \emptyset$, for each vertex $v \in T$, there must exist an integer $i$ such that $vv_i \in E(G)$ and $vv_{i-1} \notin E(G)$. For integer $i \in \{1, 2, \ldots, k\}$, let $T_i$ be the subset of $T$ such that for each vertex $x$ of $T_i$, $i$ is the minimum index with $xv_i \in E(G)$ and $xv_{i-1} \notin E(G)$. By Lemma 3.6, $T_i \cup \{v_{i-1}\}$ is an independent set.

If $k$ is even, then by coloring the vertices in $T_i \cup \{v_{i-1}\}$ with color $i$, we get a $2h$-coloring of $G$. Therefore, we suppose that $k$ is odd.

Let $v$ be a vertex in $T_i$ for some $i$.

If $vv_{i+2} \notin E(G)$, then $G[[v, v_{i-1}, v_i, v_{i+1}, v_{i+2}]]$ is a $C_5$ or $P_5$ depending on $vv_{i+1} \in E(G)$ or not. So, $vv_{i+2} \in E(G)$. We will show that

\[ \text{if } \{vv_i, vv_{i+2}\} \subseteq E(G), \text{ then } vv_{i+1} \in E(G) \text{ and } vv_{i-2} \notin E(G). \quad (4) \]

First suppose $vv_{i+1} \notin E(G)$. If $vv_{i+4} \in E(G)$, then $G[[v, v_i, v_{i+1}, v_{i+2}, v_{i+4}]] = K_1 + (K_1 \cup K_3)$. If $vv_{i+4} \notin E(G)$, then $G[[v, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}]]$ is a $C_5$ or $P_5$ depending on $vv_{i+3} \in E(G)$ or not. Both are contradictions. This shows that
which is contradiction to the choice of $K_5$.

By Lemma 3.3, we can partition $N(C)$ into 5 subsets: $\mathcal{N}^{(2)} = \bigcup_{1 \leq i \leq 5} N[i,i+2](C)$, $\mathcal{N}^{(3,1)} = \bigcup_{1 \leq i \leq 5} N[i,i+1,i+2](C)$, $\mathcal{N}^{(3,2)} = \bigcup_{1 \leq i \leq 5} N[i,i+1,i+3](C)$, $\mathcal{N}^{(4)} = \bigcup_{1 \leq i \leq 5} N[i,i+1,i+2,i+3](C)$, and $\mathcal{N}^{(5)} = N[1,2,3,4,5](C)$.

By Lemma 3.3, $\mathcal{N}^3(C)$ is $K_3$-free, and $\mathcal{N}^2(C)$ can be partitioned into two subsets each of which induces a $K_3$-free subgraph. Thus by Theorem 1.2, we have that $\chi(G[N^2(C)]) \leq 6$ and $\chi(G[N^3(C)]) \leq 3$.

By Lemma 3.1(a), we have that, for each $i \in \{1, 2, 3, 4, 5\}$, $G[N[i,i+2](C)]$ is $K_3$-free, and $\{i+4\} \cup N[i,i+1,i+2](C) \cup N[i,i+1,i+3](C) \cup N[i,i+1,i+2,i+3](C)$ is independent. If $G[N[1,3](C) \cup N[1,4](C)]$ is not $K_3$-free, let $x, y, z$ be a triangle in $G[N[1,3](C) \cup N[1,4](C)]$, then $G[v_1, v_3, v_5, v_7] = K_1 + (K_1 \cup K_3)$, a contradiction. So, we have by symmetry that $G[N[1,3](C) \cup N[1,4](C)]$ and $G[N[2,4](C) \cup N[2,5](C)]$ are both $K_3$-free. Hence we may conclude that $\chi(G[V(C) \cup N^{(3,2)} \cup N^{(4)} \cup N^3(C)]) \leq 5$, and $\chi(G[N^2(C)]) \leq 9$ as $\mathcal{N}^{(2)}$ is anticomplete to $N^2(C)$ by Lemma 1.1.

If $N[1,2,3,4,5](C)$ is independent, then $\chi(G) \leq \chi(G[V(C) \cup N^{(3,1)} \cup N^{(3,2)} \cup N^{(4)} \cup N^3(C)]) + \chi(G[N^2(C)]) + \chi(G[N[1,2,3,4,5](C)]) \leq 5 + 9 + 1 = 15$. Hence, we may assume that

$$2 \leq \omega(G[N[1,2,3,4,5](C)]) \leq \omega(G) - 2 = h - 2.$$

Let $Q = N^{(3,1)} \cup N^{(3,2)} \cup N^{(4)}$. 

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We claim that if \( Q \neq \emptyset \) then

\[ N^2(C) \] is anticomplete to each non-isolated component of \( G[N_{1,2,3,4,5}(C)] \). (6)

If it is not the case, then let \( xy \) be an edge of some non-isolated component of \( G[N_{1,2,3,4,5}(C)] \). By Lemma 3.4, \( N^2(C) \) has a vertex, say \( u \), complete to \( \{x, y\} \). By Lemma 3.2(b) and by symmetry, \( G \) has a vertex, say \( v \), adjacent to \( x \). Without loss of generality, we may assume that \( \{v v_1, v v_2\} \subseteq E(G) \). Thus \( G[\{u, v, v_1, v_2, x\}] = K_1 + (K_1 \cup K_3) \), a contradiction. Therefore, (6) holds.

### 3.1 Suppose that \( 2 \leq \omega(G[N_{1,2,3,4,5}(C)]) \leq h - 3 \)

In this case, we have that \( h \geq 5 \). Let \( \omega(G[N_{1,2,3,4,5}(C)]) = t \). Note that by Lemma 3.2 \( \chi(G - N_{1,2,3,4,5}(C) - N^2(C) - N^3(C)) \leq 5 \), and by Theorem 1.4 \( \chi(G[N_{1,2,3,4,5}(C)]) \leq 2t - 1 \) as \( G[N_{1,2,3,4,5}(C)] \) is \( K_1 \cup K_3 \)-free.

If \( N^2(C) = \emptyset \), then \( \chi(G) \leq 5 + (2t - 1) < 2h \) by induction. Thus we may assume that \( N^2(C) \neq \emptyset \), and without loss of generality, \( G[N^2(C)] \) is connected.

Recall that \( \chi(G[N_{1,2,3,4,5}(C)]) \leq 2h - 7 \) by induction, and \( \chi(G[N^2(C) \cup V(C) \cup N(2)]) \leq 6 \) by Lemma 3.3.

If \( Q = \emptyset \), then color \( V(C) \cup N(2) \cup N^2(C) \) with \( C_1 \cup \{\beta_1\} \) and color \( N_{1,2,3,4,5}(C) \cup N^3(C) \) with \( C_2 \setminus \{\beta_1\} \). Thus, we obtain a \( 2h \)-coloring of \( G \).

Therefore, we further suppose that \( Q \neq \emptyset \).

Let \( N^{2,0}(C) \subseteq N^2(C) \) be the set of vertices anticomplete to \( Q \). If \( Q \) is anticomplete to \( N^2(C) \), that is, \( N^2(C) = N^{2,0}(C) \), then \( N^2(C) \) is anticomplete to all non-isolated components of \( G[N_{1,2,3,4,5}(C)] \) by (6), which implies that \( N^3(C) = \emptyset \). We can color \( V(C) \cup N(C) \) with \( C_1 \cup C_2 \) such that all isolated vertices of \( G[N_{1,2,3,4,5}(C)] \) receive the same color \( \beta_1 \), and color \( N^3(C) \) with \( C_1 \cup C_2 \setminus \{\beta_1\} \) (this is certainly reasonable as \( \chi(G[N^2(C)]) \leq 6 \) by Lemma 3.3).

Suppose that \( Q \) is adjacent to \( N^2(C) \). By Lemma 3.5, each vertex of \( N^2(C) \setminus N^{2,0}(C) \) is an isolated component of \( G[N^2(C)] \). Since \( N^{2,0}(C) \) is anticomplete to \( Q \cup (N^2(C) \setminus N^{2,0}(C)) \), by Lemma 3.2(d) and Lemma 3.3, we can color \( G - N_{1,2,3,4,5}(C) - N^3(C) \) with \( C_1 \cup \{\beta_1\} \). Since \( \omega(G[N_{1,2,3,4,5}(C)]) \leq h - 3 \) and \( G[N^3(C)] \) is \( K_3 \)-free, we can color \( N_{1,2,3,4,5}(C) \cup N^3(C) \) with \( C_2 \setminus \{\beta_1\} \) by Theorems 1.2 and 1.4. Therefore, \( \chi(G) \leq \chi(G - N_{1,2,3,4,5}(C) - N^3(C)) + \chi(G[N_{1,2,3,4,5}(C) \cup N^3(C)]) \leq 2h \). Thus when \( 2 \leq \omega(G[N_{1,2,3,4,5}(C)]) \leq h - 3 \), \( \chi(G) \leq 2h \).

### 3.2 Suppose that \( \omega(G[N_{1,2,3,4,5}(C)]) = h - 2 \)

Now, suppose that \( \omega(G[N_{1,2,3,4,5}(C)]) = h - 2 \). By (5), we have that

\[ \omega(G[N_{1,2,3,4,5}(C)]) = h - 2 \] for each 5-hole \( C' \) of \( G \). (7)

Let \( S \) be a component of \( G[N_{1,2,3,4,5}(C)] \) with \( \omega(S) = h - 2 \).

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By Lemma 3.2(a), $\mathcal{N}^{(2)} \cup \mathcal{N}^{(3,1)}$ is complete to $S$. Hence we have that

$$\mathcal{N}^{(3,1)} = \emptyset \text{ and } V(C) \cup \mathcal{N}^{(2)}(C) \text{ induces a 5-ring} \quad (8)$$

as otherwise we can find a clique of size at least $\omega(G) + 1$.

By Lemma 3.2(d), we can define a 5-coloring $\phi$ on $G - N_{[1,2,3,4,5]}(C) - N^2(C) - N^3(C)$ with color set $C_1$ as following:

$$\begin{align*}
\phi^{-1}(\alpha_1) &= \{v_1\} \cup N_{[3,5]}(C) \cup N_{[2,3,5]}(C) \cup N_{[2,3,4,5]}(C) \\
\phi^{-1}(\alpha_2) &= \{v_2\} \cup N_{[4,1]}(C) \cup N_{[3,4,1]}(C) \cup N_{[3,4,5,1]}(C) \\
\phi^{-1}(\alpha_3) &= \{v_3\} \cup N_{[5,2]}(C) \cup N_{[4,5,2]}(C) \cup N_{[4,5,1,2]}(C) \\
\phi^{-1}(\alpha_4) &= \{v_4\} \cup N_{[1,3]}(C) \cup N_{[5,1,3]}(C) \cup N_{[5,1,2,3]}(C) \\
\phi^{-1}(\alpha_5) &= \{v_5\} \cup N_{[2,4]}(C) \cup N_{[1,2,4]}(C) \cup N_{[1,2,3,4]}(C).
\end{align*} \quad (9)$$

If $N^2(C) = \emptyset$, then by Theorem 1.4, $\chi(G) \leq 5 + 2(h - 2) - 1 = 2h$ as $G[N_{[1,2,3,4,5]}(C)]$ is $K_1 \cup K_3$-free.

Thus suppose that $N^2(C) \neq \emptyset$, and without loss of generality, suppose that $G[N^2(C)]$ is connected.

By (8), we have that $\mathcal{N}^{(3,1)} = \emptyset$, and so $Q = \mathcal{N}^{(3,2)} \cup \mathcal{N}^{(4)}$. Let $N^{2,0}(C) \subseteq N^2(C)$ be the set of vertices anticomplete to $Q$.

We first suppose that $Q \neq \emptyset$, and discuss two cases depending upon whether $N^2(C)$ is adjacent to $Q$.

**Case 1.** Suppose that $Q$ is anticomplete to $N^2(C)$. Then each component of $G[N^2(C)]$ is $K_3$-free by Lemma 3.5, and $N^2(C)$ is anticomplete to all non-isolated components of $G[N_{[1,2,3,4,5]}(C)]$ by (6). Consequently we have that $G[N_{[1,2,3,4,5]}(C)]$ has isolated components (as $N^2(C) \neq \emptyset$) and also has non-isolated components (as $\omega(G[N_{[1,2,3,4,5]}(C)]) = h - 2 \geq 2$). If $N^3(C) \neq \emptyset$, let $n_3 \in N^3(C), n_2 \in N^2(C)$ be a neighbor of $n_3$, $s_1$ an isolated component of $G[N_{[1,2,3,4,5]}(C)]$ with $s_1 n_2 \in E(G)$, and $s_2 s_3$ be an edge of some component of $G[N_{[1,2,3,4,5]}(C)]$, then $n_3 n_2 s_1 v_1 s_2$ is an induced $P_5$. Contradiction. Therefore, $N^3(C) = \emptyset$.

Now, we can color $G[N_{[1,2,3,4,5]}(C)]$ with color set $C_2$ such that such that all isolated vertices of $G[N_{[1,2,3,4,5]}(C)]$ receive the same color $\beta_1 \in C_2$, and color $G[N^2(C)]$ with the colors in $C_1 \cup C_2 \setminus \{\beta_1\}$ (this is reasonable as $\chi(G[N^2(C)]) \leq 6$ by Lemma 3.3). This together with the 5-coloring defined in (9) gives a 2$h$-coloring of $G$.

**Case 2.** Suppose that $N^2(C)$ is adjacent to $Q$. By Lemma 3.5, we have that each component of $G[N^{2,0}(C)]$ is $K_3$-free, and each of the other components of $G[N^2(C)]$ is a single vertex. Since $\omega(G[N_{[1,2,3,4,5]}(C) \cup N_{[5,1,3]}(C) \cup N_{[5,1,2,3]}(C)]) = h - 2$, we have that $\chi(G[N_{[1,2,3,4,5]}(C) \cup N_{[5,1,3]}(C) \cup N_{[5,1,2,3]}(C)]) \leq 2h - 5$ by induction. Using the 5-coloring $\phi$ defined in (9), we can construct a 5-coloring of $G[V(C) \cup N^2(C) \cup (N(C) \setminus (N_{[1,2,3,4,5]}(C) \cup N_{[5,1,3]}(C) \cup N_{[5,1,2,3]}(C))))$ by coloring all the vertices of $N^2(C) \setminus N^{2,0}(C)$ by $\alpha_4$, and coloring all the vertices of $N^{2,0}(C)$ by $\{\alpha_1, \alpha_2, \alpha_3\}$ (this is reasonable by Lemma 3.5). Then by coloring $N^3(C)$ with 3 colors used on $G[N_{[1,2,3,4,5]}(C) \cup N_{[5,1,3]}(C) \cup N_{[5,1,2,3]}(C)]$, we have that $\chi(G) \leq 5 + (2h - 5) = 2h$ by induction.
We have shown that $\chi(G) \leq 2h$ when $Q \neq \emptyset$. Next, we suppose that $Q = \emptyset$.

If $N^2(C)$ is adjacent to only isolated component of $G[N_{1,2,3,4,5}(C)]$, we see that $N^3(C) = \emptyset$ by the same argument as that used in Case 1, then we can color $G[N_{1,2,3,4,5}(C)]$ with color set $C_2$ such that all isolated components receive $v_1$, and color $N^2(C)$ with $C_1 \cup C_2 \setminus \{v_1\}$ (this reasonable as $\chi(G[N^2(C)]) \leq 6$ by Lemma 3.3. This together with $\phi$ defined in (9) is certainly a $2h$-coloring of $G$.

So, we suppose that $N^2(C)$ is adjacent to some non-isolated components of $G[N_{1,2,3,4,5}(C)]$, and let $S_1$ be the vertex set of such a component. Let $S_2 = N_{1,2,3,4,5}(C) \setminus S_1$, $T_1 = N(S_1) \cap N^2(C)$, and $T_2 = N^2(C) \setminus T_1$. It is obvious that $S_1$ is anticomplete to $T_2$, and is complete to $T_1$ by Lemma 3.4.

Therefore, $G[T_1]$ is $K_3$-free. Note that $G[N^2(C)]$ is connected by our assumption. To avoid an induced $P_5$ starting from $T_2$ and terminating on $C$, each component of $G[T_2]$ is dominated by some vertex of $T_1$, and consequently $G[T_2]$ is $K_3$-free too. We will show that

$$T_2 \text{ is independent.} \tag{10}$$

If it is not the case, let $Z$ be a non-isolated component of $G[T_2]$, let $t_1 \in T_1$ be a vertex complete to $Z$, and $s_2 \in S_2$ be a vertex adjacent to $Z$. If $s_2$ is not complete to $Z$, let $z_1z_2$ be an edge of $Z$ such that $s_2z_1 \in E(G)$ and $s_2z_2 \notin E(G)$, then $z_2z_1s_2v_1s_1$ is an induced $P_5$ for any vertex $s_1 \in S_1$, a contradiction. Therefore, $s_2$ is complete to $Z$. If $s_2t_1 \notin E(G)$, then for any vertices $s_1 \in S_1$ and $z \in V(Z)$, $C' = s_1t_1z_2v_1s_1$ is a 5-hole with $N_{1,2,3,4,5}(C') = \emptyset$, a contradiction to (7). So, we have further that $s_2t_1 \in E(G)$. But now, we have a $K_1 + (K_1 \cup K_3)$ induced by $\{s_2, t_1, v_1\}$ together with any two adjacent vertices of $Z$. Therefore, (10) holds.

Note that $G[N_{1,2,3,4,5}(C)]$ is $(K_1 \cup K_3)$-free and $G[N^3(C)]$ is $K_3$-free by Lemmas 3.1 and 3.3, we see that $\chi(G[N_{1,2,3,4,5}(C) \cup N^3(C)]) \leq 2h - 5$ by Theorems 1.2 and 1.4. Since $T_2$ is independent by (10), we have that $\chi(G[N^2(C) \cup N^3(C) \cup V(C)]) \leq 5$, and so $\chi(G) \leq 2h$ as desired. This completes the proof of Subsection 3.2, and also proves Theorem 1.5. \hfill \Box

Acknowledgements We thank the anonymous reviewers for valuable comments, and thank Dr. Karthick for pointing out an error in our earlier version on the construction of some extremal graphs.

Author Contributions All authors contributed to the study conception and design. Material preparation, data collection and analysis were performed by WD, BX and YX. The first draft of the manuscript was written by WD, BX and YX, and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

Funding This work was supported by National Natural Science Foundation of China (No. 11931106 and 12101117) and by Natural Science Foundation of Jiangsu Province (No. BK20200344). Author Baogang Xu has received research support from National Natural Science Foundation of China. Author Yian Xu has received research support from National Natural Science Foundation of China and Natural Science Foundation of Jiangsu Province.

Data Availability Statement No data applicable.
Declarations

Conflict of Interest The authors have no relevant financial or non-financial interests to disclose.

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