Eikonal slant helices and eikonal Darboux helices in 3-dimensional pseudo-Riemannian manifolds

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Abstract

In this study, we give definitions and characterizations of eikonal slant helices, eikonal Darboux helices and non-normed eikonal Darboux helices in 3-dimensional pseudo-Riemannian manifold $M$. We show that every eikonal slant helix is also an eikonal Darboux helix for timelike and spacelike curves. Furthermore, we obtain that if the non-null curve $\alpha$ is a non-normed eikonal Darboux helix, then $\alpha$ is an eikonal slant helix if and only if $\varepsilon_2 \kappa^2 + \varepsilon_3 \tau^2 = \text{constant}$, where $\kappa$ and $\tau$ are curvature and torsion of $\alpha$, respectively. Finally, we define null-eikonal helices, slant helices and Darboux helices. Also, we give their characterizations.

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1. Introduction

In the nature and science, some special curves have an important role and many applications. The well-known of such curves is helix curve. In the Euclidean 3-space $E^3$, a general helix is defined as a special curve whose tangent line makes a constant angle with a fixed straight line which is called the axis of the helix [4]. This definition gives that the tangent indicatrix of a general helix is a planar curve. Moreover, the classical result for the helices first was given by Lancret in 1802 and proved by B. de Saint Venant in 1845 as follows: A necessary and sufficient condition that a curve to be a general helix is that the ratio of the first curvature to the second curvature be constant i.e., $\kappa / \tau$ is constant along the curve, where $\kappa$ and $\tau$ denote the first and second curvatures of the curve, respectively [20]. The same definition is also valid in Lorentzian space and spacelike, timelike and null helices have been studied by some mathematicians [7-9].

Furthermore, there exist more special curves in the space such as slant helix which first introduced by Izumiya and Takeuchi by the property that the normal lines of curve make a constant angle with a fixed direction in the Euclidean 3-space $E^3$ [14]. Slant helices have been studied by some mathematicians and new kinds of these curves also have been introduced [1,11,16,17,19]. Moreover, these curves have been considered in Lorentzian spaces [2,3].
Later, a new kind of helices has been defined by Zıplar, Şenol and Yaylı according to the Darboux vector of a space curve in $E^3$. They have called this new curve as Darboux helix which is defined by the property that the Darboux vector of a space curve makes a constant angle with a fixed direction and they have given the characterizations of this new special curve [22].

Let $M$ be a Riemannian manifold with the metric $g$ and $f : M \to \mathbb{R}$ be a function with gradient $\nabla f$. The function $f$ is called eikonal if $\| \nabla f \|$ is constant [5]. There exist many applications of $\nabla f$ in mathematical physics and geometry. For instance, if $f$ is non-constant on connected $M$, then the Riemannian condition $\| \nabla f \|^2 = 1$ is precisely the eikonal equation of geometrical optics. So, on a connected $M$, a non-constant real valued function $f$ is Riemannian if $f$ satisfies this eikonal equation. In the geometrical optical interpretation, the level sets of $f$ are interpreted as wave fronts. The characteristics of the eikonal equation (as a partial differential equation), are then the solutions of the gradient flow equation for $f$ (an ordinary differential equation), $x' = \nabla f$, which are geodesics of $M$ orthogonal to the level sets of $f$, and which are parameterized by arc length. These geodesics can be interpreted as light rays orthogonal to the wave fronts (See [10] for details). Later, Şenol, Zıplar and Yaylı have defined eikonal helices and eikonal slant helices by considering a space curve with a function $f : M \to \mathbb{R}$ where $M$ is a Riemannian manifold [21].

In this study, we define and give the characterizations of $f$-eikonal slant helices and $f$-eikonal Darboux helices for non-null and null curves in a pseudo-Riemannian manifold. For this purpose, we need the following definitions.

**Definition 1.1.** ([18]) A metric tensor $g$ in a smooth manifold $M$ is a symmetric non-degenerate $(0, 2)$ tensor field in $M$.

On the other hand if $TM$ is the tangent bundle of $M$, then for all $X, Y \in TM$, $g(X, Y) = g(Y, X)$ and at each point $p$ of $M$, if $g(X_p, Y_p) = 0$ for all $Y_p \in T_p(M)$, then $X_p = 0$ (non-degenerate) where $T_p(M)$ is the tangent space of $M$ at the point $p$ and $g : T_p(M) \times T_p(M) \to \mathbb{R}$.

**Definition 1.2.** ([18]) A pseudo-Riemannian manifold (or semi-Riemannian manifold) is a smooth manifold $M$ furnished with a metric tensor $g$. That is, a pseudo-Riemannian manifold is an ordered pair $(M, g)$.

**Definition 1.3.** Let $M$ be a pseudo-Riemannian manifold and $g$ be its metric. For the function $f : M \to \mathbb{R}$, it is said that $f$ is eikonal if $\| \nabla f \|$ is constant, where $\nabla f$ is gradient of $f$, i.e., $df(X) = g(\nabla f, X)$.
Lemma 1.1. ([18]) Let \((M, g)\) be a pseudo-Riemannian manifold and \(\nabla\) be the Levi-Civita connection of \(M\). The Hessian \(H^f\) of a \(f \in F(M)\) is the symmetric (0,2) tensor field such that
\[
H^f(X, Y) = g(\nabla_X (\text{grad} f), Y),
\]
where \(F(M)\) shows the set of differentiable functions defined on \(M\).

From Lemma 1.1, we have the following corollary.

Corollary 1.1. The Hessian \(H^f\) of a \(f \in F(M)\) is zero, i.e., \(H^f = 0\) if and only if \(\nabla f\) is parallel in \(M\).

2. Non-null Eikonal Slant Helices and Non-null Eikonal Darboux Helices

Let \((M, g)\) be a time-oriented 3-dimensional pseudo-Riemannian manifold and \(\alpha: I \to M\) be a unit speed curve on \(M\), i.e., \(g(\alpha', \alpha') = \varepsilon = \pm 1\) is satisfied along \(\alpha\) where \(\alpha'\) is the velocity vector filed of the curves and \(g\) shows the metric tensor (or Lorentzian metric) given by \(g(a, b) = -a_i b_i + a_j b_j\), for the vectors \(a = (a_1, a_2, a_3)\), \(b = (b_1, b_2, b_3) \in TM\). The constant \(\varepsilon = \pm 1\) defined by \(\varepsilon = g(\alpha', \alpha')\) is called the causal character of \(\alpha\). Then, a unit speed curve \(\alpha\) is said to be spacelike or timelike if its causal character is 1 or -1, respectively. The curve \(\alpha\) is said to be a Frenet curve if \(g(\alpha', \alpha'', \alpha''') \neq 0\).

Like Euclidean geometry, every Frenet curve \(\alpha\) on \((M, g)\) admits an orthonormal Frenet frame field \(\{V_1, V_2, V_3\}\) along \(\alpha\) such that \(V_1 = \alpha'(s)\). The vector fields \(V_1, V_2, V_3\) are called tangent vector field, principle normal vector field and binormal vector field of \(\alpha\), respectively and \(\{V_1, V_2, V_3\}\) satisfies the following Frenet-Serret formula:
\[
\begin{bmatrix}
\nabla_{V_1} V_1 \\
\nabla_{V_2} V_2 \\
\nabla_{V_3} V_3
\end{bmatrix} =
\begin{bmatrix}
0 & \varepsilon_2 \kappa & 0 \\
-\varepsilon_2 \kappa & 0 & -\varepsilon_3 \tau \\
0 & \varepsilon_3 \tau & 0
\end{bmatrix}
\begin{bmatrix}
V'_1 \\
V'_2 \\
V'_3
\end{bmatrix},
\]
where \(\nabla\) is the Levi-Civita connection of \((M, g)\)\([12,13,15]\). The functions \(\kappa \geq 0\) and \(\tau\) are called the curvature and torsion, respectively. The constants \(\varepsilon_2\) and \(\varepsilon_3\) are defined by
\[
\varepsilon_i = g(V_i, V_i), \quad i = 2, 3.
\]
and called second causal character and third causal character of \(\alpha\), respectively. Note that \(\varepsilon_3 = -\varepsilon_1 \varepsilon_2\) and \(V_i \times V_j = \varepsilon_{ijk} V_k\), where \((i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)\).

The vector \(W = \tau V_1 - \kappa V_3\) is called Darboux vector of the curve \(\alpha\). Then for the Frenet formulae we have \(\nabla_i V_j = W \times V_i, (i = 1, 2, 3)\); where "\(\times\)" shows the vector product in \(M\).

As in the case of Riemannian geometry, a Frenet curve \(\alpha\) is a geodesic if and only if \(\kappa = 0\). A circular helix is a Frenet curve whose curvature and torsion are constants. If the curvature \(\kappa\) is constant and the torsion \(\tau\) is zero, then the curve is called a pseudo circle.
Pseudo circles are regarded as degenerate helices. Helices, which are not circles, are frequently called proper helices.

**Definition 2.1.** Let $M^3$ be a 3-dimensional pseudo-Riemannian manifold with the Lorentzian metric $g$ and let $\alpha(s)$ be a non-null Frenet curve with the Frenet frame $\{V_1, V_2, V_3\}$ in $M^3$. Let $f : M^3 \to \mathbb{R}$ be an eikonal function along curve $\alpha$, i.e. $\|\nabla f\| = \text{constant}$ along the curve $\alpha$. If the function $g(\nabla f, V_2)$ is a non-zero constant along $\alpha$, then $\alpha$ is called a non-null $f$-eikonal slant helix. And, $\nabla f$ is called the axis of the $f$-eikonal slant helix $\alpha$.

**Definition 2.2.** Let $M^3$ be a pseudo-Riemannian manifold with the Lorentzian metric $g$ and $\alpha$ be a non-null Frenet curve in $M^3$ with Frenet frame $\{V_1, V_2, V_3\}$, non-zero curvatures $\kappa, \tau$ and Darboux vector $W = \tau V_1 - \kappa V_3$. Also, let $f : M^3 \to \mathbb{R}$ be an eikonal function along $\alpha$. If the unit Darboux vector

$$W_0 = \frac{\tau}{\sqrt{\epsilon_1 \kappa^2 + \epsilon_1 \tau^2}} V_1 - \frac{\kappa}{\sqrt{\epsilon_1 \kappa^2 + \epsilon_1 \tau^2}} V_3,$$

of the curve $\alpha$ makes a constant angle $\varphi$ with the gradient of the function $f$, that is $g(W_0, \nabla f)$ is constant along $\alpha$, then the curve $\alpha$ is called a non-null $f$-eikonal Darboux helix.

Especially, if $g(W, \nabla f) = \text{constant}$, then $\alpha$ is called a non-normed non-null $f$-eikonal Darboux helix. Then, we have the following Corollary.

**Corollary 2.1.** A non-normed non-null $f$-eikonal Darboux helix is a non-null $f$-eikonal Darboux helix if and only if $\epsilon_1 \kappa^2 + \epsilon_1 \tau^2$ is constant.

**Example 2.1.** We consider the pseudo-Riemannian manifold $M^3 = \mathbb{R}^3_1$ with the Lorentzian metric $g$. Let

$$f : M^3 \to \mathbb{R}$$

$$(x, y, z) \to f(x, y, z) = x^2 + y^2 + z$$

be a function defined in $M^3$ and consider the spacelike curve
\[ \alpha : I \subset \mathbb{R} \rightarrow M^3 \]
\[ s \rightarrow \alpha(s) = \left( a \cosh \frac{s}{\sqrt{a^2 + b^2}}, a \sinh \frac{s}{\sqrt{a^2 + b^2}}, \frac{bs}{\sqrt{a^2 + b^2}} \right); \quad a, b > 0 \]
in \( M^3 \). If we compute \( \nabla f \), we find out \( \nabla f \) as \( \nabla f = (2x, 2y, 1) \). Then, we have
\[ \| \nabla f \| = \sqrt{1 + 4\left(-x^2 + y^2\right)}, \]
and, along the curve \( \alpha \), we find out
\[ \| \nabla f \| = \sqrt{1 - 4a^2} = \text{constant}. \]
That is, \( f \) is an eikonal function along \( \alpha \). Moreover, by a simple computation we have that the principal normal of the curve is
\[ V_2(s) = \left( \cosh \frac{s}{\sqrt{a^2 + b^2}}, \sinh \frac{s}{\sqrt{a^2 + b^2}}, 0 \right). \]
Since
\[ \nabla f = \left( 2a \cosh \frac{s}{\sqrt{a^2 + b^2}}, 2a \sinh \frac{s}{\sqrt{a^2 + b^2}}, 1 \right), \]
along \( \alpha \), we easily see that \( g(\nabla f, V_2) = -2a = \text{constant} \) which means that \( \alpha \) is a non-null \( f \)-eikonal slant helix in \( M^3 \).

On the other hand, non-normed Darboux vector of \( \alpha \) is
\[ W = \left( -\frac{a(a^2 + b^2)^2 + ab^2}{b(a^2 + b^2)^{3/2}} \sinh \frac{s}{\sqrt{a^2 + b^2}}, \right. \]
\[ \left. -\frac{a(a^2 + b^2)^2 + ab^2}{b(a^2 + b^2)^{3/2}} \cosh \frac{s}{\sqrt{a^2 + b^2}}, \right. \]
\[ \left. \frac{a^2 - (a^2 + b^2)^2}{(a^2 + b^2)^{3/2}} \right) \]
and curvatures are \( \kappa = \frac{a}{a^2 + b^2}, \quad \tau = -\frac{a^2 + b^2}{b}, \) respectively. Then we obtain that
\[ g(\nabla f, W) = \frac{a^2 - (a^2 + b^2)^2}{(a^2 + b^2)^{3/2}} = \text{constant}, \]
along \( \alpha \). So, \( \alpha \) is a non-null \( f \)-eikonal non-normed Darboux helix curve in \( M^3 \). Since \( \kappa, \tau \) are constants \( \alpha \) is also a non-null \( f \)-eikonal Darboux helix curve in \( M^3 \).
Now, we give some theorems concerned with non-null \( f \)-eikonal slant helices and \( f \)-eikonal Darboux helices in pseudo-Riemannian manifold. Whenever we write \( M^3 \), we will consider \( M^3 \) as a 3-dimensional pseudo-Riemannian manifold with the Lorentzian metric \( g \).

**Theorem 2.1.** Let \( \alpha : I \subset \mathbb{R} \to M^3 \) be a non-null curve in \( M^3 \) with non-zero curvatures \( \kappa, \tau \) and assume that \( \alpha(s) \) is not a helix. Let \( f : M^3 \to \mathbb{R} \) be an eikonal function along curve \( \alpha \) and the Hessian \( H^f = 0 \). If \( \alpha(s) \) is a non-null \( f \)-eikonal slant helix curve in \( M^3 \), then the following properties hold:

i) The function

\[
\frac{\kappa^2}{(\epsilon_1 \tau^2 + \epsilon_3 \kappa^2)^{3/2}} \left( \frac{\tau'}{\kappa} \right),
\]

is a real constant.

ii) The axis of \( f \)-eikonal slant helix is obtained as

\[
\nabla f = \frac{n\tau}{\sqrt{|\epsilon_1 \tau^2 + \epsilon_3 \kappa^2|}} V_1 + c V_2 - \frac{n\kappa}{\sqrt{|\epsilon_1 \tau^2 + \epsilon_3 \kappa^2|}} V_3,
\]

where \( c \) and \( n \) are non-zero constants.

**Proof.** i) Since \( \alpha \) is a non-null \( f \)-eikonal slant helix, we have \( g(\nabla f, V_2) = c = \text{constant} \). So, there exist smooth functions \( a_i = a_i(s), \ a_2 = a_2(s) = c \) and \( a_3 = a_3(s) \) of arc length \( s \) such that

\[
\nabla f = a_1 V_1 + c V_2 + a_3 V_3,
\]

where \( \{V_1, V_2, V_3\} \) is a basis of \( TM^3 \) (tangent bundle of \( M^3 \)).

From Corollary 1.1, \( \nabla f \) is parallel in \( M^3 \), i.e., \( \nabla_i \nabla f = 0 \) along \( \alpha \). Then, if we take the derivative in each part of (2) in the direction \( V_1 \) in \( M^3 \) and use the Frenet equations, we get

\[
(V_1[a_i] - \epsilon_i \kappa c) V_i + (\epsilon_i a_i \kappa + \epsilon_3 a_i \tau) V_2 + (V_1[a_3] - \epsilon_i \tau c) V_3 = 0,
\]

where \( V_1[a_i] = a_i'(s), \ (i = 1, 2, 3) \) in (3) and the Frenet frame \( \{V_1, V_2, V_3\} \) is linearly independent, we have

\[
\begin{align*}
a_i' - \epsilon_i \kappa c &= 0, \\
a_i \kappa + a_3 \tau &= 0, \\
a_3' - \epsilon_3 \tau c &= 0.
\end{align*}
\]

From the second equation of the system (4) we obtain
\[ a_i = -\left( \frac{\tau}{\kappa} \right) a_3. \]  

(5)

Since \( f \) is an eikonal function along \( \alpha \), we have \( \|\nabla f\| \) is constant. Then (2) and (5) give that

\[ \left[ e_1 \left( \frac{\tau}{\kappa} \right)^2 + e_3 \right] \alpha_i^2 + e_\epsilon c^2 = \text{constant}, \]

(6)

and from (6) we can write

\[ \left[ e_1 \left( \frac{\tau}{\kappa} \right)^2 + e_3 \right] \alpha_i^2 = n^2, \]

(7)

where \( n^2 \) is a constant. Since \( \alpha \) is not a helix curve in \( M^3 \) and curvatures are not zero, we have that \( n \) is a non-zero constant. Then, from (7) we have

\[ a_3 = \pm \frac{n}{\sqrt{e_1 \left( \frac{\tau}{\kappa} \right)^2 + e_3}}. \]

(8)

By taking the derivative of (8) with respect to \( s \) and using the third equation of the system (4), we get that the function

\[ \frac{\kappa^2}{(e_1 \tau^2 + e_\epsilon \kappa^2)^{3/2}} \left( \frac{\tau}{\kappa} \right)^', \]

(9)

is a constant, which is desired function.

\begin{itemize}
  \item \textit{ii}) By direct calculation from (5) and (8), we have
\end{itemize}

\[ a_i = \frac{n \tau}{\sqrt{e_1 \tau^2 + e_\epsilon \kappa^2}} \quad \text{and} \quad a_3 = \frac{n \kappa}{\sqrt{e_1 \tau^2 + e_\epsilon \kappa^2}}, \]

where \( n \) is a non-zero constant. Then, from (2) the axis of \( f \)-eikonal slant helix is

\[ \nabla f = \frac{n \tau}{\sqrt{e_1 \tau^2 + e_\epsilon \kappa^2}} V_1 + c V_2 - \frac{n \kappa}{\sqrt{e_1 \tau^2 + e_\epsilon \kappa^2}} V_3. \]

(10)

The above Theorem has the following corollary.

\textbf{Corollary 2.2.} Let \( \alpha : I \subset \mathbb{R} \rightarrow M^3 \) be a non-null curve in \( M^3 \) with non-zero curvatures \( \kappa, \tau \) and assume that \( \alpha(s) \) is not a helix. Let \( f : M^3 \rightarrow \mathbb{R} \) be an eikonal function along curve \( \alpha \)
and the Hessian \( H^f = 0 \). If \( \alpha(s) \) is a non-null \( f \)-eikonal slant helix curve in \( M^3 \), then, the curvatures \( \kappa \) and \( \tau \) satisfy the following non-linear equation system:

\[
\begin{align*}
\left( \frac{n\tau}{\sqrt{\epsilon_1\tau^2 + \epsilon_2\kappa^2}} \right)' - \epsilon_1\kappa c &= 0, \\
\left( \frac{n\kappa}{\sqrt{\epsilon_1\tau^2 + \epsilon_2\kappa^2}} \right)' - \epsilon_2\tau c &= 0.
\end{align*}
\]

(11)

**Theorem 2.2.** Let \( \alpha : I \subset \mathbb{R} \to M^3 \) be a non-null curve in \( M^3 \) with non-zero curvatures \( \kappa, \tau \) and assume that \( \alpha(s) \) is not a helix. Let \( f : M^3 \to \mathbb{R} \) be an eikonal function along curve \( \alpha \) and the Hessian \( H^f = 0 \). Then, every non-null \( f \)-eikonal slant helix in \( M^3 \) is also a non-null \( f \)-eikonal Darboux helix in \( M^3 \).

**Proof.** Let \( \alpha \) be a non-null \( f \)-eikonal slant helix in \( M^3 \). Then, from Theorem 2.1, the axis of \( \alpha \) is

\[
\nabla f = \frac{n\tau}{\sqrt{\epsilon_1\tau^2 + \epsilon_2\kappa^2}} V_1 + c V_2 - \frac{n\kappa}{\sqrt{\epsilon_1\tau^2 + \epsilon_2\kappa^2}} V_3.
\]

(12)

Considering the unit Darboux vector \( W_0 \), equality (12) can be written as follows

\[
\nabla f = nW_0 + c V_2,
\]

(13)

which shows that \( \nabla f \) lies on the plane spanned by \( W_0 \) and \( V_2 \). Since \( n \) is a non-zero constant, from (13), we have \( g(\nabla f, W_0) = n \) is constant along \( \alpha \), i.e., \( \alpha \) is a non-null \( f \)-eikonal Darboux helix in \( M^3 \).

**Theorem 2.3.** Let \( \alpha : I \subset \mathbb{R} \to M^3 \) be a non-null curve in \( M^3 \) with non-zero curvatures \( \kappa, \tau \) and assume that \( \alpha(s) \) is not a helix. Let \( f : M^3 \to \mathbb{R} \) be an eikonal function along curve \( \alpha \) and the Hessian \( H^f = 0 \). Let \( \alpha \) be a non-normed non-null \( f \)-eikonal Darboux helix with Darboux vector \( W \). Then \( \alpha \) is a non-null \( f \)-eikonal slant helix if and only if \( \|W\| \) is a non-zero constant.

**Proof.** Since \( \alpha \) is a non-normed non-null \( f \)-eikonal Darboux helix, we have \( g(W, \nabla f) = \text{constant} \). On the other hand, there exist smooth functions \( a_1 = a_1(s) \), \( a_2 = a_2(s) \) and \( a_3 = a_3(s) \) of arc length \( s \) such that

\[
\nabla f = a_1 V_1 + a_2 V_2 + a_3 V_3,
\]

(14)
where $a_1, a_2, a_3$ are assumed non-zero and $\{V_1, V_2, V_3\}$ is a basis of $TM^3$. From Corollary 1.1., $\nabla f$ is parallel in $M^3$, i.e., $\nabla_i \nabla f = 0$ along $\alpha$. Then, if we take the derivative in each part of (14) in the direction $V_i$ in $M^3$ and use the Frenet equations, we get

$$
(a'_i - \varepsilon_1 a_i \kappa) V_i + (\varepsilon_2 a_i \kappa + a'_2 + \varepsilon_3 a_i \tau) V_2 + (a'_3 - \varepsilon_3 a_i \tau) V_3 = 0,
$$

where $a'_i(s) = V_i[\dot{a}_i]$, $(i = 1, 2, 3)$. Since the Frenet frame $\{V_1, V_2, V_3\}$ is linearly independent, we have

$$
\begin{align*}
\dot{a}_1' - \varepsilon_1 \kappa a_2 &= 0, \\
\dot{a}_2' + \varepsilon_1 \kappa a_1 + \varepsilon_2 \tau a_2 &= 0, \\
\dot{a}_3' - \varepsilon_3 \tau a_2 &= 0.
\end{align*}
$$

Equality $g(W, \nabla f) = \text{constant}$ gives that

$$
\varepsilon_i a_i \tau - \varepsilon_j a_j \kappa = \text{constant}.
$$

Differentiating (17) and using the first and third equations of system (16) we obtain

$$
\varepsilon_i a_i \tau' - \varepsilon_j a_j \kappa' = 0
$$

From (18) and the second equation of system (16) it follows

$$
\dot{a}_2' = -\frac{\varepsilon_1(\varepsilon_2 \kappa^2 + \tau^2)}{2 \tau'} a_3.
$$

In (19) if $a_3 = 0$, from (16) we have $a_1 = a_2 = 0$, i.e., $\nabla f = 0$ which is a contradiction. Then we have $a_1 \neq 0$ and from (19) we see that $a_2 = a_2(s)$ is constant if and only if $\varepsilon_1 \kappa^2 + \tau^2 = \text{constant}$ which means that $\varepsilon_1 \tau^2 + \varepsilon_3 \kappa^2 = \text{constant}$, i.e, $\alpha$ is a non-null $f$-eikonal slant helix if and only if $\|W\|$ is a non-zero constant.

From Theorem 2.3 and Corollary 2.1, we have the following corollary.

**Corollary 2.3.** Let $\alpha : I \subset \mathbb{R} \to M^3$ be a non-null curve in $M^3$ with non-zero curvatures $\kappa, \tau$ and assume that $\alpha(s)$ is not a helix. Let $f : M^3 \to \mathbb{R}$ be an eikonal function along curve $\alpha$ and the Hessian $H^f = 0$. Let $\alpha$ be a non-normed non-null $f$-eikonal Darboux helix. Then $\alpha$ is a non-null $f$-eikonal slant helix if and only if $\alpha$ is a non-null $f$-eikonal Darboux helix.

### 3. Null Eikonal Helices, Slant Helices and Darboux Helices

Let $\alpha$ be a curve in 3-dimensional pseudo-Riemannian manifold $(M, g)$. Then, the curve $\alpha$ is called a null curve if $g(V_i, V_i) = 0$. By a Cartan frame or Frenet frame $\{V_1, V_2, V_3\}$ of $\alpha$,
we mean a family of vector fields \( V_1 = V_1(s), V_2 = V_2(s), V_3 = V_3(s) \) along the curve \( \alpha \) satisfying the following conditions:

\[
\begin{align*}
\alpha'(s) &= V_1, \quad g(V_1, V_1) = g(V_2, V_2) = 0, \quad g(V_1, V_2) = 1, \\
g(V_1, V_3) &= g(V_2, V_2) = 0, \quad g(V_3, V_3) = 1, \\
V_1 \times V_2 &= V_3, \quad V_2 \times V_3 = V_2, \quad V_3 \times V_1 = V_1.
\end{align*}
\]

([6]). Here \( V_1, V_2 \) and \( V_3 \) are called tangent vector field, binormal vector field and (principal) normal vector field of \( \alpha \), respectively. Then the derivative formula of the frame is given as follows

\[
\begin{bmatrix}
\nabla_{V_1} V_1 \\
\nabla_{V_2} V_2 \\
\nabla_{V_3} V_3
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 & \kappa \\
0 & 0 & \tau \\
-\tau & -\kappa & 0
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix},
\]

where \( \kappa \) and \( \tau \) are called the curvature and torsion of \( \gamma \), respectively [6].

The vector \( W = \tau V_1 - \kappa V_2 \) is called Darboux vector of the curve \( \alpha \). Then for the Frenet formulae (21) we have \( \nabla_{V_i} V_i = W \times V_i, \quad (i = 1, 2, 3) \); where "\( \times \)" shows the vector product in \( M^3 \).

**Definition 3.1.** Let \( M^3 \) be a 3-dimensional pseudo-Riemannian manifold with the metric \( g \) and let \( \alpha(s) \) be a null Frenet curve with the Frenet frame \( \{ V_1, V_2, V_3 \} \) and Darboux vector \( W = \tau V_1 - \kappa V_2 \) in \( M^3 \). Let \( f : M^3 \to \mathbb{R} \) be an eikonal function along the curve \( \alpha \), i.e. \( \| \nabla f \| = \text{constant along } \alpha \). Then we define the followings,

i) If the function \( g(\nabla f, V_1) \) is a non-zero constant along \( \alpha \), then \( \alpha \) is called a null \( f \)-eikonal helix curve. And, \( \nabla f \) is called the axis of the null \( f \)-eikonal helix curve \( \alpha \).

ii) If the function \( g(\nabla f, V_i) \), \( (i = 2, 3) \) is a non-zero constant along \( \alpha \), then \( \alpha \) is called a null \( f \)-eikonal \( V_i \)-slant helix curve. And, \( \nabla f \) is called the axis of the null \( f \)-eikonal slant helix curve \( \alpha \).

iii) If the function \( g(\nabla f, W) \) is a non-zero constant along \( \alpha \), then \( \alpha \) is called a null \( f \)-eikonal Darboux helix curve. And, \( \nabla f \) is called the axis of the null \( f \)-eikonal Darboux helix curve \( \alpha \).

**Example 3.1.** We consider the pseudo-Riemannian manifold \( M^3 = \mathbb{R}^3_1 \) with the Lorentzian metric \( g \). Let consider the function

\[
f : M^3 \to \mathbb{R} \\
(x, y, z) \to f(x, y, z) = x^2 + y^2 + z
\]

given in Example 2.1 and consider the null curve
\( \alpha : I \subset \mathbb{R} \to M^3 \)
\[ s \rightarrow \alpha(s) = (\sinh s, \cosh s, s) \]
in \( M^3 \). If we compute \( \nabla f \), we find out \( \nabla f = (2x, 2y, 1) \). Then, we have
\[ \| \nabla f \| = \sqrt{1 + 4 \left(-x^2 + y^2 \right)} , \]
and, along the curve \( \alpha \), we find out
\[ \| \nabla f \| = \sqrt{5} = \text{constant} . \]
That is, \( f \) is an eikonal function along \( \alpha \). Moreover, by a simple computation we have that the tangent and binormal of the curve are
\[ V_1(s) = (\cosh s, \sinh s, 1) , \]
\[ V_2(s) = \left( \frac{1}{2} \cosh s, -\frac{1}{2} \sinh s, \frac{1}{2} \right) , \]
respectively. Since
\[ \nabla f = (2 \sinh s, 2 \cosh s, 1) , \]
along \( \alpha \), we easily see that \( g(\nabla f, V_2) = \frac{1}{2} = \text{constant} \) and \( g(\nabla f, V_1) = 1 = \text{constant} \) which mean that \( \alpha \) is both a null \( f \)-eikonal helix and \( f \)-eikonal \( V_2 \)-slant helix in \( M^3 \).

On the other hand, the curvatures of curve are \( \kappa = 1, \quad \tau = -\frac{1}{2} \), respectively. Then the Darboux vector of \( \alpha \) is
\[ W = (0, 0, -1) . \]
Then we obtain that
\[ g(\nabla f, W) = -1 = \text{constant} , \]
along \( \alpha \). So, \( \alpha \) is also a null \( f \)-eikonal Darboux helix curve in \( M^3 \).

Then, we can give the following characterizations for a null curve.

**Theorem 3.1.** Let \( \alpha : I \subset \mathbb{R} \to M^3 \) be a null curve in \( M^3 \) with non-zero curvatures \( \kappa, \tau \) and let \( f : M^3 \to \mathbb{R} \) be an eikonal function along curve \( \alpha \) and the Hessian \( H^f = 0 \). If \( \alpha(s) \) is a null \( f \)-eikonal helix curve in \( M^3 \), then the followings hold,
i) The function $\frac{\kappa}{\tau}$ is constant.

ii) The axis of null $f$-eikonal helix curve is $\nabla f = c \left( \frac{-\tau}{\kappa} V_1 + V_2 \right)$, where $g(\nabla f, V_1) = c$ is a non-zero constant.

**Proof.** Let $\alpha(s)$ be a null $f$-eikonal helix curve in $M^3$ with axis $\nabla f$. Then, there exist smooth functions $a_1 = a_1(s)$, $a_2 = a_2(s)$ and $a_3 = a_3(s)$ of arc length $s$ such that

$$\nabla f = a_1 V_1 + a_2 V_2 + a_3 V_3,$$

where $\{V_1, V_2, V_3\}$ is a basis of $TM^3$ (tangent bundle of $M^3$). From (22) we have

$$g(\nabla f, V_1) = a_2 = c = \text{constant}, \quad g(\nabla f, V_2) = a_1, \quad g(\nabla f, V_3) = a_3.$$  \hspace{1cm} (23)

Differentiating equalities given in (23), we have $a_3 = 0$, $a_i = \text{constant}$ and $\frac{\kappa}{\tau} = -\frac{a_2}{a_1} = \text{constant}$, respectively.

Moreover, from (23) the axis of the null helix is obtained as $\nabla f = c \left( \frac{-\tau}{\kappa} V_1 + V_2 \right)$, where $g(\nabla f, V_1) = c$ is a non-zero constant.

**Theorem 3.2.** Let $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{R}^3_1$ be a null curve in $\mathbb{R}^3_1$ with non-zero curvatures $\kappa, \tau$ and let $f : \mathbb{R}^3_1 \rightarrow \mathbb{R}$ be an eikonal function along curve $\alpha$ and the Hessian $H^f = 0$. If $\alpha(s)$ is a null $f$-eikonal $V_2$-slant helix curve in $\mathbb{R}^3_1$, then $\alpha(s)$ is also a null $f$-eikonal helix curve in $\mathbb{R}^3_1$ with axis $\nabla f = c \left( \frac{-\tau}{\kappa} V_1 + V_2 \right)$ where $g(\nabla f, V_1) = c$ is a non-zero constant.

**Proof:** Let $\alpha(s)$ be a null $f$-eikonal $V_2$-slant helix curve in $\mathbb{R}^3_1$ with axis $\nabla f$. Then we have $g(\nabla f, V_2) = \text{non-zero constant}$. By differentiation of last equality we get

$$g(\nabla f, V_1) = 0.$$  \hspace{1cm} (24)

On the other hand differentiation of $g(\nabla f, V_1)$ in the direction $V_1$ is

$$\nabla_{V_1} \left[ g(\nabla f, V_1) \right] = \kappa g(\nabla f, V_1),$$

and from (24) we have $g(\nabla f, V_1)$ is a constant. Then $\alpha(s)$ is a null $f$-eikonal helix curve in $\mathbb{R}^3_1$ and from Theorem 3.1, the axis is $\nabla f = c \left( \frac{-\tau}{\kappa} V_1 + V_2 \right)$ where $g(\nabla f, V_1) = c$ is a non-zero constant.
Theorem 3.3. Let $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{R}^3$ be a null curve in $\mathbb{R}^3$ with non-zero curvatures $\kappa, \tau$ and let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be an eikonal function along curve $\alpha$ and the Hessian $H^f = 0$. If $\alpha(s)$ is a null $f$-eikonal helix or $V_2$-slant helix in $\mathbb{R}^3$, then $\det(\nabla_{V_3} V_2, \nabla_{V_2}^2 V_2, \nabla_{V_1} V_2) = 0$ holds.

Proof: Let $\alpha(s)$ be a null curve. Then from Frenet formulae (21) we have the followings

\[
\begin{align*}
\nabla_{V_2}^2 V_2 &= -\tau^2 V_1 - \tau \kappa V_2 + \tau V_3, \\
\nabla_{V_1} V_2 &= -3\tau \kappa V_1 - ((\tau \kappa)' + \kappa \tau') V_2 + (-2\kappa \tau^2 + \tau^3) V_3.
\end{align*}
\]  

(25)

From (25) we have $\det(\nabla_{V_1} V_2, \nabla_{V_2}^2 V_2, \nabla_{V_3} V_2) = \tau^5 \left( \frac{\kappa'}{\tau} \right)$. Then by Theorem 3.2 and Theorem 3.1, we say that if $\alpha(s)$ is a null $f$-eikonal helix or $V_2$-slant helix curve in $\mathbb{R}^3$ then $\det(\nabla_{V_1} V_2, \nabla_{V_2}^2 V_2, \nabla_{V_3} V_2) = 0$ holds.

Theorem 3.4. Let $\alpha : I \subset \mathbb{R} \rightarrow M^3$ be a null curve in $M^3$ with curvatures $\kappa, \tau$ and let $f : M^3 \rightarrow \mathbb{R}$ be an eikonal function along curve $\alpha$ and the Hessian $H^f = 0$. If $\alpha(s)$ is a null $f$-eikonal $V_3$-slant helix curve in $M^3$, then the following properties hold:

i) $\kappa(s) \int_0^s \tau(s) ds + \tau(s) \int_0^s \kappa(s) ds = 0$ holds, where $g(\nabla f, V_3) = c$ is a non-zero constant.

ii) The axis of the $V_3$-slant helix is given by

\[
\nabla f = c \left[ \int_0^s \tau(s) ds V_1 + \int_0^s \kappa(s) ds V_2 + V_3 \right]
\]

(26)

Proof: Since we assume that $\alpha(s)$ is a null $f$-eikonal $V_3$-slant helix curve in $M^3$, we have $g(\nabla f, V_3) = c$ is a non-zero constant. Then we can write

\[
\nabla f = a_1(s) V_1 + a_2(s) V_2 + c V_3,
\]

(27)

where $a_i = a_i(s); (i = 1,2)$ are the differentiable functions of $s$. From Corollary 1.1, $\nabla f$ is parallel in $M^3$, i.e., $\nabla_{V_i} \nabla f = 0$ along $\alpha$. Then from (27) we obtain

\[
(a'_i - c \tau) V_1 + (a'_2 - c \kappa) V_2 + (a_1 \kappa + a_2 \tau) V_3 = 0,
\]

(28)

which gives the following system

\[
a'_1 - c \tau = 0, \quad a'_2 - c \kappa = 0, \quad a_1 \kappa + a_2 \tau = 0.
\]

(29)

And from (29) and (27), we have the followings immediately,
\[ \nabla f = c \left[ \int_0^s \tau(s)(0) \, ds \right] V_1 + \left[ \int_0^s \kappa(s)(0) \, ds \right] V_2 + V_3, \]

\[ \kappa(s) \int_0^s \tau(s)(s) \, ds + \tau(s) \int_0^s \kappa(s)(s) \, ds = 0. \]

Theorem 3.4 gives us the following corollary:

**Corollary 3.1.** Let \( \alpha : I \subset \mathbb{R} \rightarrow M^3 \) be a null curve in \( M^3 \) with curvatures \( \kappa, \tau \) and let \( f : M^3 \rightarrow \mathbb{R} \) be an eikonal function along curve \( \alpha \) and the Hessian \( H^f = 0 \). If \( \alpha(s) \) is a null \( f \)-eikonal \( V_3 \)-slant helix curve in \( M^3 \). Then the followings holds,

i) \( \alpha(s) \) is a null \( f \)-eikonal helix curve if and only if \( \kappa(s) = 0 \).

ii) \( \alpha(s) \) is a null \( f \)-eikonal \( V_2 \)-slant helix curve if and only if \( \tau(s) = 0 \).

**Proof:** From Theorem 3.4, we have that the axis is given by

\[ \nabla f = a_1(s)V_1 + a_2(s)V_2 + cV_3, \]

where \( c \) is a non-zero constant. Then we have that

\[ g \left( \nabla f, V_1 \right) = a_2(s), \quad g \left( \nabla f, V_2 \right) = a_1(s), \quad (30) \]

and from (29) and (30) we have the followings,

i) \( g \left( \nabla f, V_1 \right) = a_2 \) is constant if and only if \( \kappa(s) = 0 \),

ii) \( g \left( \nabla f, V_2 \right) = a_1 \) is constant if and only if \( \tau(s) = 0 \),

which finish the proof.

**Theorem 3.5.** Let \( \alpha : I \subset \mathbb{R} \rightarrow M^3 \) be a null \( f \)-eikonal Darboux helix with Darboux vector \( W = \tau V_1 - \kappa V_2 \) in \( M^3 \) with non-zero curvatures \( \kappa, \tau \), where \( f : M^3 \rightarrow \mathbb{R} \) is an eikonal function along curve \( \alpha \) and the Hessian \( H^f = 0 \). Then \( \alpha \) is a null \( f \)-eikonal \( V_3 \)-slant helix if and only if \( \kappa \tau \) is constant.

**Proof.** Since \( \alpha \) is a null \( f \)-eikonal Darboux helix, we have \( g(W, \nabla f) = \text{constant} \). On the other hand, there exist smooth functions \( a_1 = a_1(s), \quad a_2 = a_2(s) \) and \( a_3 = a_3(s) \) of arc length \( s \) such that

\[ \nabla f = a_1 V_1 + a_2 V_2 + a_3 V_3, \quad (31) \]
where \( a_1, a_2, a_3 \) are assumed non-zero and \( \{V_1, V_2, V_3\} \) is a basis of \( TM^3 \). Since \( \nabla V_i \nabla f = 0 \) along \( \alpha \), if we take the derivative in each part of (31) in the direction \( V_i \) in \( M^3 \) and use the Frenet equations, we get

\[
(a'_1 - a_3 \tau) V_1 + (a'_2 - a_3 \kappa) V_2 + (a'_3 + a_1 \kappa + a_2 \tau) V_3 = 0, \tag{32}
\]

where \( a'_i(s) = V_i(a_i) \), \( (i = 1, 2, 3) \). Since the Frenet frame \( \{V_1, V_2, V_3\} \) is linearly independent, we have the system

\[
\begin{align*}
n\quad a'_1 - a_3 \tau &= 0, \\
n\quad a'_2 - a_3 \kappa &= 0, \\
n\quad a'_3 + a_1 \kappa + a_2 \tau &= 0.
\end{align*}
\tag{33}
\]

Equality \( g(W, \nabla f) = \text{constant} \) gives that

\[
a_2 \tau - a_1 \kappa = \text{constant}. \tag{34}
\]

Differentiating (34) and using the first and second equations of system (33) we obtain

\[
a_2 \tau' - a_1 \kappa' = 0. \tag{35}
\]

From (35) and the third equation of system (33) it follows

\[
a'_3 = -\frac{(\kappa \tau)'}{\kappa} a_2. \tag{36}
\]

If \( a_2 = 0 \) in (36), from (33) we see that \( a_i = a_3 = 0 \) which is a contradiction. Then \( a_2 \neq 0 \) and we have that \( a_i = a_i(s) \) is a constant if and only if \( \kappa \tau \) is constant, which means that \( \alpha \) is a null \( f \)-eikonal \( V_3 \)-slant helix if and only if \( \kappa \tau \) is constant.

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