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Article

Topological Symmetry Groups of the Heawood Graph

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Abstract: We classify all groups which can occur as the topological symmetry group of some embedding of the Heawood graph in $S^3$.

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1. Introduction

Topological symmetry groups were originally introduced to classify the symmetries of non-rigid molecules. In particular, the symmetries of rigid molecules are represented by the point group, which is the group of rigid motions of the molecule in space. However, non-rigid molecules can have symmetries which are not included in the point group. The symmetries of such molecules can instead be represented by the subgroup of the automorphism group of the molecular graph which are induced by homeomorphisms of the ambient space. In this way, the molecular graph is treated as a topological object, and hence this group is referred to as the topological symmetry group of the graph in space.

Although initially motivated by chemistry, the study of topological symmetry groups of graphs embedded in $S^3$ can be thought of as a generalization of the study of symmetries of knots and links. Various results have been obtained about topological symmetry groups in general ([1–4]) as well as topological symmetry groups of embeddings of particular graphs or families of graphs in $S^3$ ([5–10]).

In this paper, we classify the topological symmetry groups of embeddings of the Heawood graph in $S^3$, whose (combinatorial) automorphism group is $\text{PGL}(2,7)$. This graph, denoted by $C_{14}$, is illustrated in Figure 1. The Heawood graph is of interest to topologists because it is obtained from the intrinsically knotted graph $K_7$ by what are known as “$\Delta–Y$” moves. Such moves alter the graph by replacing three edges that form a triangle by three edges in the form of the letter $Y$ with a new 3-valent vertex in the center. Since $\Delta–Y$ moves preserve intrinsic knotting [11], the Heawood graph is intrinsically knotted. This means that every embedding of $C_{14}$ in $S^3$ contains a non-trivial knot. It also follows from [12] that $C_{14}$ is intrinsically chiral, that is, no embedding of $C_{14}$ in $S^3$ has an orientation reversing homeomorphism.
We begin with some terminology.

**Definition 1.** Let $\Gamma$ be a graph embedded in $S^3$. We define the **topological symmetry group** $TSG(\Gamma)$ as the subgroup of the automorphism group $\text{Aut}(\Gamma)$ induced by homeomorphisms of $(S^3, \Gamma)$. We define the **orientation preserving topological symmetry group** $TSG^+_\gamma(\Gamma)$ as the subgroup of $\text{Aut}(\Gamma)$ induced by orientation preserving homeomorphisms of $(S^3, \Gamma)$.

**Definition 2.** Let $G$ be a group and let $\gamma$ denote an abstract graph. If there is some embedding $\Gamma$ of $\gamma$ in $S^3$ such that $TSG(\Gamma) = G$, then we say that $G$ is **realizable** for $\gamma$. If there is some embedding $\Gamma$ of $\gamma$ in $S^3$ such that $TSG^+\Gamma) = G$, then we say that the group $G$ is **positively realizable** for $\gamma$.

**Definition 3.** Let $\varphi$ be an automorphism of an abstract graph $\gamma$. We say $\varphi$ is **realizable** if for some embedding $\Gamma$ of $\gamma$ in $S^3$, the automorphism $\varphi$ is induced by a homeomorphism of $(S^3, \Gamma)$. If such a homeomorphism exists which is orientation preserving, then we say $\varphi$ is **positively realizable**.

Since the Heawood graph is intrinsically chiral, a group is realizable if and only if it is positively realizable. Our main result is the following classification theorem.

**Theorem 1.** A group $G$ is realizable as the topological symmetry group of an embedding of $C_{14}$ if and only if $G$ is the trivial group, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_6$, $\mathbb{Z}_7$, $D_3$, or $D_7$.

In Section 2, we present some background material about $C_{14}$. In Section 3, we determine which of the automorphisms of $C_{14}$ are realizable. We then use the results of Section 3 to prove our main result in Section 4.

### 2. Background About the Heawood Graph

We will be interested in the action of automorphisms of $C_{14}$ on cycles of particular lengths. The graph $C_{14}$ has 28 6-cycles, its shortest cycles, and 24 14-cycles [13,14]. The following results about the 12-cycles and 14-cycles of $C_{14}$ are proved in the paper [15]. While some of these results may be well known, the authors could not find proofs in the graph theory literature.

**Lemma 1.** ([15])

1. $C_{14}$ has 56 12-cycles.
2. $\text{Aut}(C_{14})$ acts transitively on the set of 14-cycles and the set of 12-cycles.
3. The graph obtained from $C_{14}$ by removing any pair of vertices which are a distance 3 apart has exactly two 12-cycles.
By part (2) of Lemma 1, we can assume that any 14-cycle in $C_{14}$ looks like the outer circle in Figure 1 and any 12-cycle looks like the round circle in Figure 2. We will always label the vertices of $C_{14}$ either as in Figure 1 or as in Figure 2.

![Figure 2. Any 12-cycle looks like the round circle in this illustration.](image)

The automorphism group of $C_{14}$ is isomorphic to the projective linear group $\text{PGL}(2, 7)$ whose order is $336 = 2^4 \times 3 \times 7$ ([13]). The program Magma was used to determine that all of the non-trivial elements of $\text{PGL}(2, 7)$ have order 2, 3, 4, 6, 7, and 8. The following lemma gives us information about the action of automorphisms with order 3 and 7 on the 12-cycles and 14-cycles of $C_{14}$.

**Lemma 2.** Let $\alpha$ be an automorphism of $C_{14}$. Then the following hold.

1. If $\alpha$ has order 7, then $\alpha$ setwise fixes precisely three 14-cycles, rotating each by $\frac{2\pi n}{7}$ for some $n < 7$, when considered as a round circle (see Figure 1).
2. If $\alpha$ has order 3, then $\alpha$ fixes precisely two vertices and setwise fixes precisely two 12-cycles in their complement, rotating each by $\pm \frac{2\pi}{3}$, when considered as a round circle (see Figure 2).

**Proof.** (1) Suppose that the order of $\alpha$ is 7. Since $C_{14}$ has 24 14-cycles, $\alpha$ must setwise fix at least three of them. Observe that any 14-cycle which is setwise fixed by $\alpha$ must be rotated by $\frac{2\pi n}{7}$ for some $n < 7$. Thus, every edge must be in an orbit of size 7. Since there are 21 edges, there are precisely three such edge orbits. Now any 14-cycle which is setwise fixed must be made up of two of these three edge orbits, and hence there are at most three 14-cycles which are invariant under $\alpha$. It follows that there are precisely three invariant 14-cycles.

(2) Suppose that the order of $\alpha$ is 3. Since there are 14 vertices, $\alpha$ must fix at least two vertices $v$ and $w$. Furthermore, since $C_{14}$ has 56 12-cycles (by part (1) of Lemma 1), $\alpha$ must setwise fix at least two 12-cycles. If some vertex on an invariant 12-cycle were fixed, the entire 12-cycle would be fixed and hence $\alpha$ could not have order 3. Thus, neither $v$ nor $w$ can be on an invariant 12-cycle. By part (2) of Lemma 1, we can assume that one of the invariant 12-cycles is the round circle in Figure 2, and hence $v$ and $w$ are as in Figure 2. Since $v$ and $w$ are a distance 3 apart, it follows from part (3) of Lemma 1 that there are precisely two 12-cycles in the complement of $\{v, w\}$. Therefore, $\alpha$ must rotate each of the two 12-cycles in the complement of $\{v, w\}$ by $\pm \frac{2\pi}{3}$.

**Lemma 3.** Let $\alpha$ be an order 2 automorphism of $C_{14}$ which setwise fixes a 12-cycle or a 14-cycle. Then no vertex is fixed by $\alpha$.

**Proof.** First suppose $\alpha$ setwise fixes a 14-cycle and fixes at least one vertex. Then without loss of generality, $\alpha$ setwise fixes the round circle $C$ in Figure 1 and fixes vertex 1. It follows that either $\alpha$ interchanges vertices 2 and 14 or fixes both. In the latter case $\alpha$ would be the identity. Thus, we can assume that $\alpha$ interchanges vertices 2 and 14. However, since vertex 6 is also adjacent to vertex 1, it must also be fixed by $\alpha$. This implies that $\alpha$ interchanges the two components of $C - \{1, 6\}$. However, this is impossible because one component of $C - \{1, 6\}$ has four vertices while the other has eight vertices.

Next suppose that $\alpha$ setwise fixes a 12-cycle. Then without loss of generality, $\alpha$ setwise fixes the round circle $D$ in Figure 2. Then $\alpha(\{v, w\}) = \{v, w\}$. However, every vertex on $D$ has precisely one
neighbor on $D$ which is adjacent to $\{v, w\}$. Thus, if $a$ fixes any vertex on $D$, it would have to fix every vertex on $D$, and hence would be the identity. Now suppose $a$ fixes $v$. Since $a$ has order 2 and $v$ has three neighbors on $D$, one of these neighbors would have to be fixed by $a$. As we have already ruled out the possibility that $a$ fixes a vertex on $D$, this again gives us a contradiction.

3. Realizable Automorphisms of $C_{14}$

**Lemma 4.** Let $a$ be a realizable automorphism of $C_{14}$. Then the following hold.

1. For some embedding $\Gamma$ of $C_{14}$ in $S^3$, $a$ is induced by an orientation preserving homeomorphism $h: (S^3, \Gamma) \rightarrow (S^3, \Gamma)$ with $\text{order}(h) = \text{order}(a)$.
2. If order$(a)$ is a power of 2, then $a$ leaves at least two 14-cycles or at least two 12-cycles setwise invariant, and if order$(a) = 2$, then $a$ fixes no vertices.
3. If order$(a)$ is even, then order$(a) = 2$ or 6.

**Proof.** (1) Since $a$ is realizable, there is some embedding $\Lambda$ of $C_{14}$ in $S^3$ such that $a$ is induced by a homeomorphism $g: (S^3, \Lambda) \rightarrow (S^3, \Lambda)$. Now by Theorem 1 of [16], since $C_{14}$ is 3-connected, there is an embedding $\Gamma$ of $C_{14}$ in $S^3$ such that $a$ is induced by a finite order homeomorphism $h: (S^3, \Gamma) \rightarrow (S^3, \Gamma)$. Furthermore, it follows from [12] that no embedding of $C_{14}$ in $S^3$ has an orientation reversing homeomorphism. Thus, $h$ is orientation preserving.

- Let order$(a) = p$ and order$(h) = q$. Since $h^p$ is the identity, $p \leq q$. If $p < q$, then $h^p$ pointwise fixes $\Gamma$, yet $h^p$ is not the identity. However, by Smith Theory [17], the fixed-point set of $h^p$ is either the empty set or $S^1$. But, this is impossible since $\Gamma$ is contained in the fixed-point set of $h^p$. Thus, order$(h) = \text{order}(a)$.

- Suppose that order$(a)$ is a power of 2. Let $h$ be given by part (1). Then order$(h)$ is the same power of 2. Let $S_1$ and $S_2$ denote the sets of 12-cycles and 14-cycles, respectively. By [18], for any embedding of $C_{14}$ in $S^3$, the mod 2 sum of the arf invariants of all 12-cycles and 14-cycles is 1. Thus, an odd number of cycles in $S_1 \cup S_2$ have arf invariant 1. Hence for precisely one $i$, the set $S_i$ has an odd number of cycles with arf invariant 1. Since $|S_1| = 56$ and $|S_2| = 24$ are each even, $S_i$ must have an odd number of cycles with arf invariant 0 and an odd number of cycles with arf invariant 1.

- We know that $h(S_1) = S_1$ and $h$ preserves arf invariants. Hence $h$ setwise fixes $T_0$ the set of cycles in $S_1$ with arf invariant 0 and $T_1$ the set of cycles in $S_1$ with arf invariant 1. Since order$(h)$ is a power of 2, and $|T_0|$ and $|T_1|$ are each odd, $h$ setwise fixes at least one cycle in $T_0$ and at least one cycle in $T_1$. Hence at least two 12-cycles or at least two 14-cycles are setwise fixed by $h$, and hence by $a$. It now follows from Lemma 3 that if order$(a) = 2$, then $a$ fixes no vertices.

- Suppose that order$(a)$ is even and order$(a) \neq 2, 6$. Recall that every even order automorphism of $C_{14}$ has order 2, 4, 6 or 8. Then by part (2), $a$ setwise fixes a 12-cycle or 14-cycle. If $a$ setwise fixes a 14-cycle, then order$(a) = 2$ since order$(a)$ is even and cannot be 14. Thus, we suppose that $a$ setwise fixes a 12-cycle $Q$, and hence order$(a) \neq 8$

- Since order$(a) \neq 2, 6$, we must have order$(a) = 4$. Without loss of generality we can assume that $Q$ is the round 12-cycle in Figure 2 and $a(Q = (1, 4, 7, 10)(2, 5, 8, 11)(3, 6, 9, 12)$. However, this is impossible because $a(\{v, w\}) = \{v, w\}$, and hence $a$ cannot take vertex 4 (which is adjacent to $w$) to vertex 7 (which is adjacent to neither $v$ nor $w$). Thus, order$(a) \neq 4$. □

**Theorem 2.** A non-trivial automorphism of $C_{14}$ is realizable if and only if it has order 2, 3, 6 or 7.

**Proof.** Figure 3 illustrates an embedding of $C_{14}$ with vertices labeled as in Figure 2 where vertex $w$ is at $\infty$ and the grey arrows are the edges incident to $w$. This embedding has a glide rotation $h$ obtained by rotating the picture by $\frac{2\pi}{3}$ around a vertical axis going through vertices $v$ and $w$ while rotating by $\pi$ around the circular waist of the picture. Then $h$ induces the order 6 automorphism $(v, w)(10, 11, 6, 7, 2, 3)(1, 4, 9, 12, 5, 8)$. Now $h^3$ and $h^2$ induce automorphisms of order 2 and 3 respectively. Thus, automorphisms of orders 2, 3, and 6 are all realizable.
Figure 3. This embedding has a glide rotation inducing \((v, w)(10, 11, 6, 7, 2, 3)(1, 4, 9, 12, 5, 8)\).

Figure 4 shows an embedding of \(C_{14}\) with a rotation of order 7 about the center of the picture. Thus, \(C_{14}\) has realizable automorphisms of order 2, 3, 6, and 7, as required.

![Figure 4](image)

For the converse, we know that the only odd order automorphisms of \(C_{14}\) have order 3 or 7, and part (3) of Lemma 4 shows that the only realizable even order automorphisms of \(C_{14}\) have order 2 or 6.

4. Topological Symmetry Groups of Embeddings of \(C_{14}\)

Since \(C_{14}\) is intrinsically chiral, for any embedding \(\Gamma\) of \(C_{14}\) in \(S^3\), \(TSG(\Gamma) = TSG_+ (\Gamma)\). Thus, a finite group \(G\) is realizable for \(C_{14}\) if and only if \(G\) is positively realizable. Let \(\Gamma\) be an embedding of \(C_{14}\) in \(S^3\). We know that \(TSG(\Gamma)\) is a subgroup of \(\text{Aut}(C_{14}) \cong \text{PGL}(2, 7)\). According to [19], the non-trivial proper subgroups of \(\text{PGL}(2, 7)\) are \(\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6, \mathbb{Z}_7, \mathbb{Z}_8, D_2, D_3, D_4, D_6, D_7, D_8, A_4, S_4, \text{PSL}(2, 7), \mathbb{Z}_7 \times \mathbb{Z}_3\), and \(\mathbb{Z}_7 \times \mathbb{Z}_6\). We can eliminate the groups \(\mathbb{Z}_4, \mathbb{Z}_8, D_4, D_8, S_4, \text{PSL}(2, 7), \text{and PGL}(2, 7)\) as possibilities for \(TSG(\Gamma)\) because we know from Theorem 2 that no realizable automorphism of \(C_{14}\) has order 4. Thus, the only groups that are possibilities for \(TSG(\Gamma)\) for some embedding \(\Gamma\) of \(C_{14}\) are the trivial group, \(\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_6, \mathbb{Z}_7, D_2, D_3, D_6, D_7, A_4, \mathbb{Z}_7 \times \mathbb{Z}_3\), and \(\mathbb{Z}_7 \times \mathbb{Z}_6\).

**Theorem 3.** The trivial group and the groups \(\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_6, \mathbb{Z}_7, D_3, \) and \(D_7\) are realizable for \(C_{14}\).

To prove Theorem 3, we will use the following prior result.

**Theorem 4.** ([20]) Let \(\gamma\) be a 3-connected graph embedded in \(S^3\) as a graph \(\Gamma\) which has an edge \(e\) that is not pointwise fixed by any non-trivial element of \(G = TSG_+ (\Gamma)\). Then every subgroup of \(G\) is positively realizable for \(\gamma\).

**Proof of Theorem 3.** We begin with the embedding \(\Gamma\) of \(C_{14}\) illustrated in Figure 5 where the grey squares represent the same trefoil knot.
The outer circle $C$ is setwise invariant under any homeomorphism of $(S^3, \Gamma)$ because $C$ is the only 14-cycle with 14 trefoil knots, and by [20] any such homeomorphism must preserve the set of knotted edges. It follows that $TSG(\Gamma) \leq D_{14}$. Also, $\Gamma$ is invariant under a rotation by $\frac{2\pi}{3}$ inducing the automorphism $(1, 3, 5, 7, 9, 11, 13)(2, 4, 6, 8, 10, 12)$ and a homeomorphism turning $C$ over inducing $(1, 14)(2, 13)(3, 12)(4, 11)(5, 10)(6, 9)(7, 8)$. Thus, $D_7 \leq TSG(\Gamma)$. However, $D_7$ is the only subgroup of $D_{14}$ containing $D_7$ which has no element of order 14. Thus, $TSG(\Gamma) = D_7$.

Observe that no edge of $\Gamma$ is pointwise fixed by any non-trivial element of $TSG(\Gamma) = TSG_+(\Gamma)$. Hence by Theorem 4, every subgroup of $TSG(\Gamma)$ is realizable. In particular, the groups $D_7$, $Z_7$, $Z_2$, and the trivial group are each realizable for $C_{14}$.

In the embedding $\Gamma'$ illustrated in Figure 6, $v$ is above the plane of projection, $w$ is below the plane, and the three grey squares represent the same trefoil knot. Now $C = 1, 12, 5, 4, 9, 8$ is the only 6-cycle containing three trefoil knots. It follows that any homeomorphism of $(S^3, \Gamma')$ must take $C$ to itself taking the set of three trefoils to itself. Thus, $TSG(\Gamma') \leq D_3$. Since $\Gamma'$ is invariant under a $\frac{2\pi}{3}$ rotation as well as under turning the picture over, $TSG(\Gamma') = D_3$. Now if we replace the three trefoils on $C$ by three identical non-invertible knots, we will get an embedding $\Gamma''$ such that $TSG(\Gamma'') = Z_3$.

Finally, let $\Lambda$ be the embedding in Figure 3. Then the 6-cycle $C = 10, 11, 6, 7, 2, 3$ is the only 6-cycle which contains a trefoil knot. Thus, $C$ is setwise invariant under any homeomorphism of $(S^3, \Lambda)$. Hence $TSG(\Lambda) \leq D_6$. We also saw in Figure 3 that a glide rotation of $S^3$ induces the order 6 automorphism $(v, w)(10, 11, 6, 7, 2, 3)(1, 4, 9, 12, 5, 8)$. Thus, $Z_6 \leq TSG(\Lambda)$. Since we know from Theorem 5 that $D_6$ is not realizable for $C_{14}$, it follows that $TSG(\Lambda) = Z_6$. \[\square\]

In what follows, we prove that no other groups are realizable for $C_{14}$.

**Theorem 5.** The groups $D_2$ and $D_6$ are not realizable for $C_{14}$.

**Proof.** Suppose that there exist realizable order 2 automorphisms $\alpha$ and $\beta$ of $C_{14}$ such that $\langle \alpha, \beta \rangle = D_2$. Since $C_{14}$ has 21 edges, $\alpha$ and $\beta$ each setwise fix an odd number of edges. Let $E_\alpha$ denote the set of edges which are invariant under $\alpha$. Let $\varepsilon \in E_\alpha$. Then $\alpha(\beta(\varepsilon)) = \beta(\alpha(\varepsilon)) = \beta(\varepsilon)$. Thus, $\beta(\varepsilon) \in E_\alpha$. It follows that $\beta(E_\alpha) = E_\alpha$. However, since $E_\alpha$ has an odd number of elements and $\beta$ has order 2, there is some
edge $e \in E_8$ such that $\beta(e) = e$. Thus, $\alpha$ and $\beta$ both setwise fix the edge $e$, and hence at least one of the involutions $\alpha$, $\beta$, or $\alpha \beta$ must pointwise fix $e$.

Now by Lemma 3, none of $\alpha$, $\beta$, or $\alpha \beta$ can fix any vertex. Thus, $D_2$ is not realizable. However, since $D_6$ contains involutions $\alpha$ and $\beta$ such that $(\alpha, \beta) = D_2$, it follows that $D_6$ also cannot be realizable for $C_{14}$. $\square$

**Theorem 6.** The group $A_4$ is not realizable for $C_{14}$.

**Proof.** Suppose that $\Gamma$ is an embedding of $C_{14}$ such that $\text{TSG}(\Gamma) = A_4$. According to Burnside’s Lemma [21], the number of vertex orbits of $\Gamma$ under $\text{TSG}(\Gamma)$ is:

$$\frac{1}{|A_4|} \sum_{\alpha \in A_4} |\text{fix}(\alpha)|$$

where $|\text{fix}(\alpha)|$ denotes the number of vertices fixed by an automorphism $\alpha \in \text{TSG}(\Gamma)$. Observe that $A_4$ contains eight elements of order 3, three elements of order 2, and no other non-trivial elements. Now by part (2) of Lemma 2, each order 3 automorphism fixes precisely two vertices, and by Lemma 4 part (2), no realizable order 2 automorphism fixes any vertex. Thus, the number of vertex orbits of $\Gamma$ under $\text{TSG}(\Gamma)$ is:

$$\frac{1}{|A_4|} \sum_{\alpha \in A_4} |\text{fix}(\alpha)| = \frac{1}{12} ((8 \cdot 2) + (3 \cdot 0) + (1 \cdot 14)) = \frac{30}{12}$$

As this is not an integer, $A_4$ cannot be realizable for $C_{14}$. $\square$

To show that the groups $\mathbb{Z}_7 \times \mathbb{Z}_3$ and $\mathbb{Z}_7 \times \mathbb{Z}_6$ are not realizable for $C_{14}$, we will make use of the definition and results below.

**Definition 4.** A finite group $G$ of orientation preserving diffeomorphisms of $S^3$ is said to satisfy the involution condition if for every involution $g \in G$, we have $\text{fix}(g) \cong S^1$ and no $h \in G$ with $h \neq g$ has $\text{fix}(h) = \text{fix}(g)$.

**Theorem 7 ([2]).** Let $\Gamma$ be a 3-connected graph embedded in $S^3$ with $H = \text{TSG}_+(\Gamma)$. Then $\Gamma$ can be re-embedded in $S^3$ as $\Delta$ such that $H \leq \text{TSG}_+(\Delta)$ and $H$ is induced by an isomorphic finite group of orientation preserving diffeomorphisms of $S^3$.

**Theorem 8 ([22]).** Let $G$ be a finite group of orientation preserving isometries of $S^3$ which satisfies the involution condition. Then the following hold.

1. If $G$ preserves a standard Hopf fibration of $S^3$, then $G$ is cyclic, dihedral, or a subgroup of $D_m \times D_m$ for some odd $m$.
2. If $G$ does not preserve a standard Hopf fibration of $S^3$, then $G$ is $S_4$, $A_4$, or $A_5$.

**Theorem 9.** The groups $\mathbb{Z}_7 \times \mathbb{Z}_3$ and $\mathbb{Z}_7 \times \mathbb{Z}_6$ are not realizable for $C_{14}$.

**Proof.** Suppose that for some embedding $\Gamma$ of $C_{14}$ in $S^3$, $\text{TSG}_+(\Gamma)$ is $\mathbb{Z}_7 \times \mathbb{Z}_3$ or $\mathbb{Z}_7 \times \mathbb{Z}_6$. In either case, $G = \mathbb{Z}_7 \times \mathbb{Z}_3 \leq \text{TSG}_+(\Gamma)$. Now since $C_{14}$ is 3-connected, we can apply Theorem 7, to re-embed $C_{14}$ in $S^3$ as $\Delta$ such that $G \leq \text{TSG}_+(\Delta)$ and $G$ is induced by an isomorphic finite group of orientation preserving diffeomorphisms of $S^3$. However, by the proof of the Geometrization Conjecture, every finite group of orientation preserving diffeomorphisms of $S^3$ is conjugate to a group of orientation preserving isometries of $S^3$ [23]. Thus, we abuse notation and treat $G$ as a group of orientation preserving isometries of $S^3$.

Since $G$ has no elements of order 2, it vacuously satisfies the involution condition, and hence by Theorem 8, $G$ is cyclic, dihedral, a subgroup of $D_m \times D_m$ for some odd $m$, $S_4$, $A_4$, or $A_5$. However, since $|G| = 21$, it cannot be dihedral, $S_4$, $A_4$, or $A_5$. Also, since $G \leq \text{Aut}(C_{14})$ has no element of order 21, the elements of $G$ of order 3 and 7 cannot commute. Thus, $G$ cannot be cyclic; and since all elements
of odd order in $D_m \times D_m$ commute, $G$ cannot be a subgroup of any $D_m \times D_m$. By this contradiction, we conclude that neither $\mathbb{Z}_7 \times \mathbb{Z}_3$ nor $\mathbb{Z}_7 \times \mathbb{Z}_6$ is realizable for $C_{14}$. \qed

The following corollary summarizes our classification of which groups can occur as topological symmetry groups of some embedding of the Heawood graph in $S^3$.

**Corollary 1.** A group $G$ is realizable as a topological symmetry group of $C_{14}$ if and only if $G$ is the trivial group, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_6$, $\mathbb{Z}_7$, $D_3$, or $D_7$.

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