BOUNDARY MAPS AND REDUCIBILITY FOR COCYCLES INTO CAT(0)-SPACES

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Abstract. We prove that a non-elementary measurable cocycle in the isometry group of a CAT(0)-space of finite telescopic dimension admits a Furstenberg map. We also show that a maximal cocycle \( \sigma : \Gamma \times X \to \text{PU}(p, \infty) \) where \( \Gamma < \text{PU}(n, 1) \) is a torsion-free lattice and \((X, \mu_X)\) is an ergodic standard Borel \( \Gamma \)-space is finitely reducible. As a consequence, we prove an infinite dimensional rigidity phenomenon for cocycles.

1. Introduction

Boundary maps were first introduced by Furstenberg [Fur63, Fur73] and represent a powerful tool in the investigation of rigidity of representations. The examples par excellence of results involving such maps are Mostow rigidity [Mos68] and Margulis superrigidity [Mar75]. It is well-known the key role played by equivariant maps between boundaries in bounded cohomology, where their combination with the functorial approach due to Burger and Monod [BM02] has generated a prolific literature (see for instance [Ioz02, BI07, BBI13, Poz15, BBI18]).

Motivated by these applications, the investigation about the existence of boundary maps has become an independent topic of research. For example, in the context of representations into algebraic groups, we recall the work by Margulis [Mar75] and Zimmer [Zim80], by Burger and Iozzi for Hermitian groups [BI04] and the more general results obtained by Bader and Furman [BF14a, BF14b].

Similar efforts have been spent to study the case of actions on CAT(\(\kappa\))-spaces. For instance, CAT(-1)-spaces have been studied by Burger and Mozes [BM96] and by Monod and Shalom [MS04], whereas Duchesne focused first on actions on the Hermitian symmetric space \( \mathcal{X}(p, \infty) \) [Duc12] and then, together with Bader and Lecureux, on generic CAT(0)-spaces [BDL16]. Here the authors proved the existence of a boundary map \( B \to \partial \mathcal{X} \) for every non-elementary action on a CAT(0)-space \( \mathcal{X} \) of finite telescopic dimension. In this setting \( B \) denotes a generic \( \Gamma \)-boundary in the sense of [BF14b]. Such boundary can be seen as an extension both of the Furstenberg-Poisson boundary and the strong boundary in the sense of Burger-Monod [BM02]. In the general case studied in [BDL16], although one loses the peculiar structure of spherical building of \( \partial \mathcal{X}(p, \infty) \) exploited in [Duc12], one can
still rely on the rich structure of CAT(0)-spaces. For instance, a useful tool is the Euclidean de-Rham decomposition of CAT(0)-spaces (see [BH]). Additionally, when the telescopic dimension is finite, Caprace and Lytchak [CL09] proved that families of closed convex subspaces of a CAT(0)-space $X$ have an $\text{Isom}(X)$-invariant point in the closure $\overline{X}$. A second tool exploited in [Duc12, BDL16] is the one of measurable fields of CAT(0)-spaces, first introduced by Anderegg and Henry [AHT14] and then developed by Duchesne [Duc12].

In the wake of the recent works about measurable cocycles due to the authors and Moraschini [Sav20b, MS20, Sav20a, SS21b, SS21a], in the first part of this paper we investigate the analogous of [BDL16, Theorem 1.1] for cocycles. In particular, we define non-elementarity for cocycles and we prove the following

**Theorem 1.** Let $\Gamma$ be a discrete countable group, $(X,\mu_X)$ be an ergodic standard Borel probability $\Gamma$-space and $B$ a $\Gamma$-boundary. For every non-elementary cocycle $\sigma : \Gamma \times X \to H$ into the isometry group of a CAT(0)-space $X$ of finite telescopic dimension there exists a boundary map $\phi : B \times X \to \partial X$.

The proof is based on the arguments used in [BDL16 Theorem 1], where the crucial point is the measurable version of the Adam-Ballmann theorem [Duc12 Theorem 1.8]. Thanks to [Duc12 Proposition 8.11], we can work with minimal invariant subfields. Moreover, by applying the measurable Euclidean de-Rham decomposition [Duc12 Proposition 9.2, Proposition 9.3] we reduce ourselves to the particular case when $\sigma$ fixes a minimal family of closed convex spaces with trivial Euclidean factors.

Starting from Theorem 1 we want to investigate the case $X = X(p,\infty)$. The recent work by Duchesne, Lécureux and Pozzetti [DLP21] explored the world of actions on the infinite dimensional Hermitian symmetric spaces $X(p,\infty)$. They proved that any maximal representation $\rho : \Gamma \to \text{PU}(p,\infty)$ of a lattice $\Gamma < \text{PU}(n,1)$ with $n \geq 2$ preserves a finite dimensional totally geodesic Hermitian symmetric space $Y \subset X(p,\infty)$. Moreover, under the additional hypothesis of Zariski density, they ruled out the existence of any such representation for any $p \geq 1$.

Motivated by such results we will focus our attention on measurable cocycles $\sigma : \Gamma \times X \to \text{PU}(p,\infty)$ where $\Gamma$ is as usual a complex hyperbolic lattice in $\text{PU}(n,1)$ and $(X,\mu_X)$ is an ergodic standard Borel probability $\Gamma$-space. We assume something more, namely the existence of a boundary map $\partial\mathbb{H}_p N \times X \to I_p$, where $I_p$ denotes the set of $p$-isotropic subspaces of $\mathcal{H} = \mathbb{C}^{p,\infty}$. The existence of maps with such specific target will be discussed in the last part of the paper.

We first notice one of the key points that distinguishes the finite dimensional case studied in [Poz15, SS21b] from ours. Indeed, notice that the group $\text{PU}(p,\infty)$ is not algebraic in the usual meaning. The absence of such structure motivates the notions of algebraic and of standard algebraic subgroup given in [DLP21], that allowed the authors to define the notion of Zariski density inside $\text{PU}(p,\infty)$.

The lack of algebraicity can be overcome, for instance, when $\sigma$ is cohomologous to a cocycle whose image is contained in a finite dimensional algebraic subgroup. We call such cocycles finitely reducible. Using the machinery of numerical invariants
and maximality developed by Moraschini and the second author \[MS20, MS21\], we get an analogous of \[DLP21, Theorem 6.7\] for cocycles.

**Theorem 2.** Let \( \Gamma < PU(n,1) \) be a complex hyperbolic lattice with \( n \geq 1 \) and let \((X, \mu)\) be an ergodic standard Borel probability \( \Gamma \)-space. Consider a measurable cocycle \( \sigma : \Gamma \times X \to PU(p, \infty) \) with \( p \geq 1 \) and suppose there exists a boundary map \( \phi : \partial H^n_C \times X \to I_p \). If \( \sigma \) is maximal, then it is finitely reducible.

The structure of the proof is the following. We first refine \[DLP21, Proposition 6.2\], namely we show that any slice of the boundary map has image essentially contained in a unique copy of \( \partial \mathcal{X}(p,q) \) embedded in \( \partial \mathcal{X}(p, \infty) \) for some \( p \leq q \leq np \). Then, we exploit ergodicity to show that such \( q \) does not depend on the slice. Using the transitive action of \( PU(p, \infty) \) (Lemma 5.2) we twist the cocycle and the boundary map in such a way to find a cohomologous cocycle \( \sigma^f \) and a boundary map \( \phi^f \) with image of the latter essentially contained in some embedding of \( \partial \mathcal{X}(n,q) \) in \( \partial \mathcal{X}(p, \infty) \), so that finite reducibility follows.

To relate our results, it seems natural to ask whether Theorem 1 provides a suitable boundary map in the context of Theorem 2. We notice that by our first result we have an equivariant map \( \partial H^n_C \times X \to I_k(p, \infty) \) for some \( k \leq p \) where \( I_k(p, \infty) \) denotes the space of isotropic \( k \)-subspaces in \( \mathcal{X}(p, \infty) \). In particular, for cocycles \( \sigma : \Gamma \times X \to PU(1, \infty) \), since maximality implies non-elementarity, Theorem 1 provides a boundary map \( \partial H^n_C \times X \to \partial H^n_C \) and, by applying Theorem 2 and \[SS21b, Theorem 2\], we get the following version of Mostow rigidity for infinite dimensional cocycles.

**Theorem 3.** Let \( \Gamma < PU(n,1) \) be a complex hyperbolic lattice with \( n \geq 1 \) and let \((X, \mu_X)\) be an ergodic standard Borel probability \( \Gamma \)-space. Any maximal cocycle \( \sigma : \Gamma \times X \to PU(1, \infty) \) is cohomologous to a cocycle preserving a copy of \( H^n_C \subset H^n_n \) and acting on it via the standard lattice embedding.

**Plan of the paper.** The paper is divided into two parts. The first one focuses on the existence of boundary maps for cocycles. After a brief introduction of basics about CAT(0)-spaces (Section 2.1), we define measurable fields and we recall the measurable Euclidean de-Rham decomposition and a measurable version of Adam-Ballmann theorem (Section 2.2). Then we briefly move to cocycles (Section 3), we define non-elementarity and finally we prove Theorem 1 (Section 4).

The second part is devoted to reducibility of cocycles into the space \( \mathcal{X}(p, \infty) \). We first introduce Hermitian symmetric spaces and we characterize embeddings of \( \mathcal{X}(p,q) \) inside \( \mathcal{X}(p, \infty) \) (Section 5). Secondly, we recall some notions about bounded cohomology and the Burger-Monod approach (Section 6). Then we move to the notion of algebraic and finite dimensional algebraic subgroup of \( GL(H) \) (Section 7). Thanks to the recalled background, we can define the Toledo invariant associated to a measurable cocycle, passing through the definition of Bergman class and the machinery developed by \[MS20\] about numerical invariants of measurable cocycles (Section 8.1 and 8.2). Finally, we provide the proof of Theorem 2 (Section 8.3).

We conclude with Section 9 where we relate our main results and we prove Theorem 3.
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I. Boundary maps

2. CAT(0)-spaces and measurable fields

In these first sections we introduce notions and results that we use in the proof of Theorem 1.

2.1. CAT(0)-spaces. We first recall basic definitions and known facts about CAT(0)-space. A metric space $(X,d)$ is a CAT(0)-space if it is geodesic and for every triple of distinct points $x, y, z \in X$, given a point $m$ in the geodesic segment between $y$ and $z$, the following inequality holds

$$d(x,m)^2 \leq \frac{1}{2}(d(x,y)^2 + d(x,z)^2) - \frac{1}{4}d(y,z)^2.$$

A complete CAT(0)-space is also called a Hadamard space.

Since embedded flats into CAT(0)-spaces play an important role in the study of their geometry, we recall the following decomposition into Euclidean and non-Euclidean factors. Precisely, the Euclidean de-Rham decomposition of a CAT(0)-space $X$ is its canonical isometric splitting into an Hilbert space $H$ and a factor $Z$ which cannot be further decomposed as a product with non-trivial Euclidean factor [BH, Theorem 6.15]. Moreover, for every point $x \in X$ the spaces $H$ (respectively $Z$) identifies with a unique closed convex subspaces of $X$ containing $x$.

Given a subset $Y \subset X$ of a metric space, its diameter is defined as

$$\text{diam}(Y) := \sup_{x,y \in Y} d(x,y),$$

and $Y$ is said to be bounded if it has finite diameter. A convex bounded set $Y$ has some preferred points called circumcenters, which are the centers of balls of minimal radius containing $Y$. Notice that, without the assumption of convexity, one can still give the notion of circumcenter but such points may not belong to $Y$. An equivalent definition can be given in terms of actions of isometries. Precisely, the circumcenters of a bounded subset $Y \subset X$ of a generic metric space are the points fixed by any isometry stabilizing $Y$. In the case of CAT(0)-spaces it turns out that every bounded subset has a unique circumcenter, which we call center. This fact follows from a more general property of CAT($\kappa$)-spaces, see [BH, Proposition 2.7] for details.

Before introducing the notion of telescopic dimension, we need the one of geometric dimension. This concept was first introduced by Kleiner [Kle99] in terms of the space of directions at each point, and then has been reformulated by Caprace and Lytchack [CL09, Theorem 1.3] in the following way. If $X$ is a CAT(0)-space, then
its geometric dimension is \( \leq n \) if for each subset \( Y \) of finite diameter the following inequality holds

\[
\text{rad}(Y) \leq \sqrt{\frac{n}{2(n+1)}} \text{diam}(Y),
\]

where \( \text{rad}(Y) \) is the \textit{circumradius} of \( Y \), namely the infimum of all positive numbers \( r \) such that \( Y \) is contained in some closed ball of radius \( r \). The result by Caprace and Lytchak leads to a characterization of telescopic dimension, originally given by [Kle99], that we assume here as a definition (refer to [CL09] for more details).

**Definition 2.1.** A CAT(0)-space \( X \) has telescopic dimension \( \leq n \) if for any \( \delta > 0 \) there exists some constant \( D > 0 \) such that for every bounded set \( Y \) of diameter \( > D \), we have

\[
\text{rad}(Y) \leq \left( \delta + \sqrt{\frac{n}{2(n+1)}} \right) \text{diam}(Y).
\]

As we will recall in Section 5 the Hermitian symmetric space \( X(p, \infty) \) is a CAT(0)-space of telescopic dimension \( p \) [Duc12, Corollary 1.4]. In particular this implies that the visual boundary \( \partial X(p, \infty) \) has geometric dimension \( p - 1 \) [CL09, Proposition 2.1].

For a complete CAT(0)-space \( X \) with finite telescopic dimension, Caprace and Lytchak proved that every filtering family of closed convex subspaces of \( X \) either intersects at \( X \) or at \( \partial X \) [CL09, Theorem 1.1]. Notice that this is equivalent to quasi-compactness of the space \( \overline{X} = X \cup \partial X \) endowed with the topology defined by Monod [Mon06, Section 3.7]. The following technical result is an example of application of [CL09, Theorem 1.1], and it turned out to be the useful in the proof of [BDL16, Theorem 1.1] and [DLP21, Theorem 1.7]. It will be exploited to prove Theorem 1.

**Proposition 2.2** ([BDL16, Proposition 2.1]). Let \( E \) be an Euclidean space and \( f: E \to \mathbb{R} \) be a convex function. If we denote by \( m = \inf\{f(x) | x \in E\} \) then we have the following four possible cases:

(i) If \( m \) is not attained, then \( \bigcap_{\epsilon > 0} \partial E_\epsilon \neq \emptyset \) where \( E_\epsilon := f^{-1}((m, m + \epsilon)) \) is not empty and has a center.

If \( m \) is attained, we denote by \( E_m = f^{-1}(m) \) and by \( E_m = F \times T \) its Euclidean de-Rham decomposition. Then one of the following holds

(ii) \( E_m \) is bounded and thus it has a center;

(iii) \( T \) is bounded and \( \partial E_m = \partial F \) is a sphere;

(iv) \( T \) is not bounded and \( \partial T \subset \partial E \) has radius less than \( \frac{\pi}{2} \).

Notice that, as mentioned in point (iii), boundaries of flats are Euclidean spheres, that can be also interpreted as CAT(1)-spaces. In particular, boundaries of maximal flats are subcomplexes called apartments of the building structure of the visual boundary. Since we will not directly use such construction, we refer to [AB08] for the general theory of such building. We only notice that the existence of circumcenters for bounded subsets [BH, Proposition 2.7] holds also in this case. More precisely,
every subset of radius at most $\frac{\pi}{2}$ in a sphere has a center, and this property will be used in the proof of Theorem 1.

2.2. Measurable fields of CAT(0)-spaces and the Adam-Ballmann dichotomy.

In this section we introduce measurable fields of CAT(0)-spaces and some results that we will exploit in the next section to prove the existence of boundary maps for cocycles. We refer to [AH14] for the general theory of measurable fields of CAT(0)-spaces and to [Duc12] for the measurable version of the Euclidean de-Rham decomposition and the proof of the equivalent Adam-Ballmann theorem [AB98].

**Definition 2.3.** Given a standard probability space $(\Omega, \mu)$, a measurable field of CAT(0)-spaces is a collection of CAT(0)-spaces $X = \{X_\omega\}_{\omega \in \Omega}$ together with a countable family $F \subset \prod_{\omega \in \Omega} X_\omega$ such that

- for all $x, y \in F$ the map $\omega \mapsto d_\omega(x_\omega, y_\omega)$ is measurable;
- for almost every $\omega \in \Omega$, the set $\{f_\omega \mid f \in F\}$ is dense in $X_\omega$.

A section of $X$ is an element $x \in \prod_{\omega \in \Omega} X_\omega$ such that, for every $y \in F$ the map $\omega \mapsto d_\omega(x_\omega, y_\omega)$ is measurable.

A subfield $Y$ of $X$ is a collection of non-empty closed convex subsets $Y_\omega \subset X_\omega$ such that, for every section $x$ of $X$ the map $\omega \mapsto d_\omega(x_\omega, Y_\omega)$ is measurable.

If $G$ is a locally compact group and $\Omega$ is a $G$-space, a $G$-action on $X$ is the datum of a collection $\{\sigma(g, \omega)\}_{g \in G, \omega \in \Omega}$ where

- for every $g \in G$ and almost every $\omega \in \Omega$, we have $\sigma(g, \omega) \in \text{Isom}(X_\omega, X_{g\omega})$;
- for every $g, h \in G$ and almost every $\omega \in \Omega$, the following equality holds

$$\sigma(gh, \omega) = \sigma(g, h_\omega)\sigma(h, \omega);$$

- for every $x, y \in F$, the map $(g, \omega) \mapsto d(x_\omega, \sigma(g, g^{-1}_\omega)Y_{g^{-1}_\omega})$ is measurable.

**Remark 2.4.** The Equation (1) might remind the reader to the cochain rule of the differential. In fact, this is an example of a more general object, called measurable cocycle, that we will introduce in the next section.

A $G$-action $\{\sigma(g, \omega)\}_{g \in G, \omega \in \Omega}$ on a measurable field $X$ induces a natural $G$-action on every subfield $Y$ by $gY = \{\sigma(g, g^{-1}_\omega)Y_{g^{-1}_\omega}\}$. Similarly, if $\partial X$ denotes the boundary field of $X$, namely the field consisting of the boundaries of each $X_\omega$, a $G$-action on $X$ induces an action on the set of sections of $\partial X$ defined as $(g\xi)_\omega = \{\sigma(g, g^{-1}_\omega)\xi_{g^{-1}_\omega}\}$.

As proved by Caprace and Lytchak [CL09, Proposition 1.8], any isometric action of a locally compact group on a complete CAT(0)-space of finite telescopic dimension either has a fixed point at infinity or admits an invariant non-empty closed convex subset which is minimal, namely it does not contain a proper subset with the same properties. This allows to reduce the investigation of existence of boundary maps to minimal actions, since the boundary of a closed convex subset naturally embeds into $\mathcal{X}$ (see [Duc12, Theorem 1.7] and [BDL16, Theorem 1.1]). The following result can be see as the generalization of [CL09, Proposition 1.8] to measurable fields and will be our starting point in the proof of Theorem 1.
Proposition 2.5 ([Duc12, Proposition 8.11]). Suppose $X$ is a measurable field of CAT(0)-spaces of finite telescopic dimension, $G$ acts on $X$ and $\Omega$ is $G$-ergodic. Then there exists a minimal invariant subfield of $X$ or there exists an invariant section of $\partial X$.

A second construction that we will use is the extension of the Euclidean de-Rham decomposition for measurable fields of CAT(0)-spaces.

Proposition 2.6 ([Duc12, Proposition 9.2]). Let $x$ be a section of a measurable field $X$. There exists $n \in \mathbb{N}$ and two subfields $E$ and $Y$ of $X$ containing $x$ such that $X = E \times Y$ and $E_\omega \cong \mathbb{R}^n$ for almost every $\omega \in \Omega$. Moreover, $E$ is maximal for those properties.

If $y$ is another section of $X$ and $X = E' \times Y'$ is another such decomposition associated to $y$ then for almost every $\omega \in \Omega$, the projections $\pi_{E_\omega | E'_\omega}$ and $\pi_{Y_\omega | Y'_\omega}$ are isometries.

In particular, the $G$-action $\{\sigma(g,\omega)\}_{g \in G, \omega \in \Omega}$ on $X$ splits as

$$\sigma(g,\omega) = \sigma_E(g,\omega) \times \sigma_Y(g,\omega)$$

where $\{\sigma_E(g,\omega)\}_{g \in G, \omega \in \Omega}$ and $\{\sigma_Y(g,\omega)\}_{g \in G, \omega \in \Omega}$ are respectively actions on $E$ and $Y$.

The last preliminary result that we recall is the measurable version of the Adam-Ballmann dichotomy [AB98].

Theorem 2.7 ([Duc12, Theorem 1.8]). Let $G$ be a locally compact second countable group and $\Omega$ an ergodic and amenable $G$-space. Let $X$ be a measurable field of complete CAT(0)-spaces of finite telescopic dimension. If $G$ acts on $X$ then there is an invariant section of the boundary field $\partial X$ or there exists an invariant Euclidean subfield of $X$.

3. Measurable cocycles

We now move to the world of measurable cocycles and we give a brief list of definitions, see [MS20, MS21] for further details. We will assume $G$ and $H$ be locally compact and second countable groups endowed with their Haar structures and $(X,\mu_X)$ be a standard Borel probability space equipped with a measure preserving $G$-action. We say that $(X,\mu_X)$ is a standard Borel probability $G$-space. We denote by Meas$(X,Y)$ the space of equivalence classes of measurable maps from $X$ to another probability space $Y$, where two functions are identified if they coincides almost everywhere. We endow Meas$(X,Y)$ with the natural topology of convergence in measure.

Definition 3.1. A measurable cocycle is a measurable function $\sigma : G \times X \to H$ such that

$$\sigma(g_1g_2,x) = \sigma(g_1,g_2x)\sigma(g_2,x)$$

holds for almost every $g_1, g_2 \in G$ and for almost every $x \in X$. 
Notice that the assumptions of both local compactness and second countability can be dropped in the definition, since it is sufficient to require that both $G$ and $H$ are topological groups and look at Borel measurable functions. Since measurable cocycles may be interpreted as Borel 1-cocycles in the sense of Eilenberg-MacLane (see Feldman and Moore [Moo76, FM77]), we now introduce the notion of cohomologous cocycles.

**Definition 3.2.** Let $\sigma_1, \sigma_2 : G \times X \to H$ be two measurable cocycles, let $f : X \to H$ be a measurable map and denote by $\sigma_1^f$ the cocycle defined as

$$\sigma_1^f(g, x) := f(gx)^{-1}\sigma_1(g, x)f(x)$$

for every $g \in G$ and almost every $x \in X$. The cocycle $\sigma_1^f$ is the $f$-twisted cocycle associated to $\sigma_1$. We say that $\sigma_1$ is cohomologous to $\sigma_2$ if there exists a measurable map $f$ such that $\sigma_2 = \sigma_1^f$.

**Remark 3.3.** Even if there are plenty of examples of measurable cocycles in different areas, we recall the following ones, since they play a predominant role in our results.

(i) A natural interpretation of measurable cocycles is the one of generalization of representations. In fact, given a homomorphism $\rho : G \to H$ between locally compact and second countable groups $G, H$ and a standard Borel $G$-space $(X, \mu_X)$, we can define a cocycle as

$$\sigma_\rho : G \times X \to H, \quad \sigma_\rho(g, x) := \rho(g).$$

Furthermore, conjugated representations gives cohomologous cocycles in the sense of Definition 3.2.

(ii) In the setting of Definition 2.3, if each $X_\omega$ is contained in some CAT(0)-space $X$, a measurable cocycle $G \times \Omega \to \text{Isom}(X)$ naturally corresponds to a $G$-action on $\{X_\omega\}$ in the sense of Section 2.2.

### 4. Boundary maps

This section is devoted to boundary maps. We recall first the notion of boundary in the sense of [BF14] and then our notion of boundary maps. Finally we prove Theorem 1.

#### 4.1. Boundaries of locally compact second countable groups.

Given a locally compact and second countable group $H$, a Lebesgue $H$-space is a measure space endowed with a $H$-action which preserves the class of the measure. We need the following preliminary

**Definition 4.1.** A map $q : X \to Y$ between Lebesgue $\Gamma$-space is relatively metrically ergodic if for any fiber-wise isometric $\Gamma$-action on $p : M \to T$ and measurable $\Gamma$-equivariant maps $f : X \to M$ and $g : Y \to T$ there exists a measurable $\Gamma$-equivariant map $\psi : Y \to M$ such that the following diagram commutes.
A fiberwise isometric $\Gamma$-action on $p : M \to T$ is a $\Gamma$-invariant Borel map $d : M \times_p M \to \mathbb{R}_{\geq 0}$ such that any fiber $p^{-1}(t) \subset M$ endowed with the induced metric $d_{|p^{-1}(t) \times p^{-1}(t)}$ is a separable metric space on which $\Gamma$ acts in a compatible way.

Now we can define boundaries in the sense of Bader and Furman [BF14b].

**Definition 4.2.** Let $H$ be a locally compact and second countable group. A $H$-boundary is an amenable Lebesgue $H$-space $B$ such that the projections $\pi_1 : B \times B \to B$ and $\pi_2 : B \times B \to B$ on the first and second factor, respectively, are relatively metrically ergodic (see [BF14b, Definition 2.1]).

**Remark 4.3.** The notion of $\Gamma$-boundary covers other notions of boundaries.

(i) The Furstenberg-Poisson boundary of a locally compact second countable group [Fur63] turns out to be a boundary in the sense of Definition 4.2 [BF14b, Theorem 2.7].

(ii) Suppose that $\Gamma < H$ is a lattice into a connected semi-simple Lie group of non-compact type. Given a minimal parabolic subgroup $P < H$, the quotient $H/P$ is the Furstenberg-Poisson boundary for $\Gamma$ and hence a $\Gamma$-boundary by [BF14b, Theorem 2.3].

(iii) In general, a $H$-boundary is a strong $H$-boundary in the sense of Burger and Monod [BM02], [BF14b, Remarks 2.4].

Now we are ready to give the definition of boundary map.

**Definition 4.4.** Let $\Gamma$ be a finitely generated group and let $G$ be a locally compact group. Consider a standard Borel probability $\Gamma$-space $(X, \mu_X)$, a $\Gamma$-boundary $B$ and a Lebesgue $G$-space $(Y, \nu)$. A boundary map for a measurable cocycle $\sigma : \Gamma \times X \to G$ is a measurable map $\phi : B \times X \to Y$ which is $\sigma$-equivariant, that is

$$\phi(\gamma b, \gamma x) = \sigma(\gamma, x)\phi(b, x)$$

for almost every $\gamma \in \Gamma$ and almost every $b \in B, x \in X$.

Since our arguments in the proof of Theorem 1 rely on the objects introduced in Section 2.2, we recall the following result, due to Duchesne, Lécureux and Pozzetti, that we interpret as a way to organize the components of a measurable field into a standard Borel space.

**Lemma 4.5.** [DLP21, Lemma 4.11] Let $\Gamma$ be a countable group and let $X$ be a measurable field over a Lebesgue $\Gamma$-space $\Omega$. Then there exists a full-measure subset $\Omega_0 \subset \Omega$, a standard Borel structure on $X := \bigcup_{\omega \in \Omega_0} X_\omega$ and a Borel map $p : X \to \Omega_0$ that admits a fiberwise isometric $\Gamma$-action. Moreover, $p^{-1}(\omega)$ is $X_\omega$ with the metric $d_\omega$. 
In virtue of Lemma 4.5, an invariant section of the boundary field $\partial X$ of a measurable field $X$ on the space $B \times X$ is a boundary map in the sense of Definition 4.4, and in fact this will be our fundamental tool in the proof of Theorem 1.

4.2. Existence of boundary maps. In this section we prove the existence of boundary maps for a specific class of cocycles. Precisely, we consider a cocycle $\sigma : \Gamma \times X \to H$ where $\Gamma$ is a countable group, $X$ is a standard Borel probability ergodic $\Gamma$-space and $H$ is the isometry group of a CAT(0)-space $\mathcal{X}$ of finite telescopic dimension. Moreover, we assume that the $\sigma$-action on $\mathcal{X}$ does not preserve a family of flats, that is the natural extension of non-elementarity for representations.

Definition 4.6. In the above setting, we say that $\sigma$ is non-elementary if there exists no $\sigma$-equivariant family of Euclidean subspaces of $X$ nor $\sigma$-equivariant family of points in $\partial \mathcal{X}$.

We are now ready to give the proof of Theorem 1, which is inspired on the argument used in [BDL16, Theorem 1.1].

Proof of Theorem 1. We consider the constant field $X = \{\mathcal{X}_x\}_{x \in X}$ endowed with the $\Gamma$-action defined by $\sigma$. We first notice that by Proposition 2.5 either we have a minimal subfield $Y \subset X$ or there exists a section of $\partial X$. Since the last one is ruled out by non-elementary, we can assume the existence of a minimal subfield.

According to Proposition 2.6, we consider the Euclidean de-Rham decomposition $Y = E \times Z$ and we denote by $\sigma_Z$ and $\sigma_Y$ the $\Gamma$-actions induced respectively on $Z$ and $Y$. By ergodicity we have that one of the following options is verified for almost every $x \in X$: either $\text{diam}(\partial Z_x) \leq \frac{\pi}{2}$ or not. In the first case we can denote by $z_x$ the center of $\partial Z_x$, whose existence is ensured by [BL05, Proposition 1.4] and by considering the $\sigma$-equivariant family $\{E_x \times \{c_x\}\}_{x \in X}$ we get a contradiction to the hypothesis of non-elementarity. Hence we assume that almost all the $Z_x$’s have diameter greater than $\frac{\pi}{2}$.

We claim that the $\Gamma$-action $\sigma_Z$ on $Z$ is minimal and non-elementary. Before proving the claim, notice that this implies that it is sufficient to find an invariant section of $Z$, since the boundaries $\partial Z_x$’s are contained in the $\partial Y_x$’s and hence in $\partial \mathcal{X}$.

Assume that $Z$ is not $\sigma_Z$-minimal. Hence by Proposition 2.5 there exists a minimal invariant subfield $W \subset Z$ whose product with $W$ is a strict subfield of $E \times Z = Y$, contradicting the minimality of $Y$. Similarly, a $\sigma_Z$-equivariant family of flats $\{F_x\}$ would produce a $\sigma$-invariant family of flats in $\mathcal{X}$, which is ruled out by non-elementarity of $\sigma$.

Hence it remains to prove the existence of an invariant section of the boundary field $\partial Z$, where $Z$ has trivial Euclidean factor and its endowed with a minimal action $\{\sigma_Z(\gamma, x)\}$.

We consider the measurable field $U = \{U_{\xi, x}\}_{\xi, x \in B \times X}$ where $U_{\xi, x} = Z_x$ for every pair $\xi, x$ in $B \times X$. Recall that by [MS04, Proposition 2.4] the spaces $B \times X$ and $B \times B \times X$ are ergodic $\Gamma$-spaces. By [Zim84, Proposition 4.3.4] $B \times X$ is also $\Gamma$-amenable. In this context we apply [Duc12, Theorem 1.8] and we have two possible cases: either there exists a section of $\partial U$ or there exists an invariant Euclidean
subfield $E \subset U$. Since in the first case we are done, we need to rule out the second one.

We consider the distance map

$$d : B \times B \times X \to \mathbb{R}, \quad (\xi_1, \xi_2, x) \mapsto d(E_{\xi_1}, E_{\xi_2,x})$$

where the $E_{\xi,x}$'s are the sheets of the Euclidean subfield $E$, namely $E = \{E_{\xi,x}\}_{(\xi,x) \in B \times X}$. Following [BDL16], we have four possible cases, and by ergodicity one of them must happen almost surely. Moreover, again by ergodicity, the distance map is essentially equal to some value, say $d_0$, for almost every $x \in X$ and $\xi_1, \xi_2 \in B$.

**Case (i):** Suppose that $d_0$ is not attained for almost every $x \in X$ and $\xi_1, \xi_2 \in B$. Hence for almost every $x \in X$ and $\xi_1 \in B$ we can define the subspaces

$$E_{\xi_1, \xi_2, x}^n := \left\{ y \in E_{\xi_1,x} \mid d(y, E_{\xi_2,x}) < d_0 + \frac{1}{n} \right\}$$

which are nested subspaces of $E_{\xi_1,x}$. By [Duc12, Proposition 8.10] we have a $\sigma$-equivariant map

$$\psi : B \times B \times X \to \bigcup \partial E_{\xi,x} \subset \partial E,$$

where we are considering the measurable field $\{E'_{\xi_1, \xi_2, x}\}_{(\xi_1, \xi_2, x) \in B \times B \times X}$ such that $E'_{\xi_1, \xi_2, x} = E_{\xi_1, x}$ for every $x \in X$ and $\xi_1, \xi_2 \in B$. It follows directly from Lemma 4.5 that the projection $p$ of (a full-measure subset of) $\bigcup \partial E'_{\xi_1, \xi_2, x}$ on $B \times X$ has a $\Gamma$-fiberwise isometric action, so that we can apply relative metric ergodicity to the following diagram

$$
\begin{array}{ccc}
B \times B & \xrightarrow{\Psi} & \text{Meas}(X, \partial E) \\
\downarrow \pi_1 & & \downarrow p_X \\
B & \xrightarrow{j} & \text{Meas}(X, B \times X).
\end{array}
$$

Here $\Psi$ and $j$ are induced respectively by $\psi$ (namely $\Psi(\xi_1, \xi_2)(x) := \psi(x, \xi_1, \xi_2)$) and by the inclusion of constants (namely $j(\xi)(x) := (\xi, x)$), while $\pi_1$ is the projection on the first factor and $p_X$ defined as $p_X(f)(x) := p(f(x))$. Moreover, $p_X$ can be equipped with a fiberwise isometric $\Gamma$-action which depends on the one on $p$ (see the proof of [SS21b, Theorem 1] for a similar construction), defined by integrating along $X$ the functions in each fiber.

By relative metric ergodicity we have a lifting $B \to \text{Meas}(X, \partial E)$, thus $\Psi$ does not depend on the second factor. Hence we have a $\sigma$-invariant map $B \times X \to \partial E \subset \partial U$, whose existence is ruled out by the dichotomy of Theorem 2.7.

We can suppose that the distance $d_{\xi_1, \xi_2, x}$ is attained almost surely and we define the non-empty subsets

$$W_{\xi_1, \xi_2, x} := \{w \in E_{\xi_1,x} \mid d(w, E_{\xi_2,x}) = d_0\} \subset E_{\xi_1,x}.$$ 

**Case (ii):** If the $W_{\xi_1, \xi_2, x}$'s are bounded. We can associate to any such subset its circumcenter $c_{\xi_1, \xi_2, x}$. The map

$$\psi : B \times B \times X \to E, \quad \psi(\xi_1, \xi_2, x) := c_{\xi_1, \xi_2, x}$$
is $\sigma$-equivariant, and by applying twice the relative metric ergodicity we obtain a map $\psi : X \to E$ such that
\[ \psi(\gamma x) = \sigma(\gamma, x)\psi(x). \]
Since points are 0-dimensional flats, this contradicts non-elementarity.

Thus the $W_{\xi_1,\xi_2,x}$'s are not bounded, we can consider their Euclidean de-Rham decomposition
\[ W_{\xi_1,\xi_2,x} = F_{\xi_1,\xi_2,x} \times T_{\xi_1,\xi_2,x}, \]
where the $F_{\xi_1,\xi_2,x}$'s are maximal Euclidean factors.

Case (iii): If the $T_{\xi_1,\xi_2,x}$'s is not bounded, as in case (i) we realize a map
\[ \psi : B \times B \times X \to \partial T, \quad \psi(\xi_1,\xi_2, x) := c_{\xi_1,\xi_2,x} \]
where $c_{\xi_1,\xi_2,x}$ is the center of $\partial T_{\xi_1,\xi_2,x}$ and $T$ denotes the measurable field given by the $T_{\xi_1,\xi_2,x}$'s. Notice that $c_{\xi_1,\xi_2,x}$ can be defined thanks to [BDL16, Proposition 2.1]. Using the same arguments of case (i), we get a contradiction.

Case (iv): Finally, if the $T_{\xi_1,\xi_2,x}$'s are bounded we consider a subfield $E'$ of $E$ whose sheets are defined as follows
\[ E'_{\xi_1,\xi_2,x} := F_{\xi_1,\xi_2,x} \times \{ t_{\xi_1,\xi_2,x} \} \]
for every $x \in X$ and $\xi_1,\xi_2 \in B$, where $t_{\xi_1,\xi_2,x}$ is the circumcenter of $T_{\xi_1,\xi_2,x}$. The same argument used in [BDL16] shows that in fact $E_{\xi_1,\xi_2,x} = E_{\xi_1,x}$ for almost every $x \in X$ and $\xi_1,\xi_2 \in B$. Moreover,
\[ E_{\xi,x} \parallel E_{\xi',x} \]
for almost every $x \in X$ and almost every $\xi,\xi' \in B$.

By Fubini's theorem there exists an element $\xi_0 \in B$ and a full-measure subset $\Omega \subset B \times X$ such that
\[ E_{\xi,x} \parallel E_{\xi_0,x} \]
for almost every $(\xi, x) \in \Omega$. We denote by $\Omega^\Gamma = \bigcap_{\gamma \in \Gamma} \gamma \Omega$ which is still of full-measure since $\Gamma$ is countable. We consider the set
\[ C_x := \text{convex hull } (\{E_{\xi,x}\}_{(\xi,x) \in \Omega^\Gamma}), \]
that can be decomposed into Euclidean de-Rham factors $E_x \times T_x$ such that
\[ E_x \parallel E_{\xi,x} \parallel E_{\xi_0,x} \]
for every $(\xi, x) \in \Omega^\Gamma$.

Moreover, for almost every $x \in X$ and $\gamma \in \Gamma$, we have
\[ \sigma_Z(\gamma, x)C_x = \text{convex hull } (\sigma_Z(\gamma, x)E_{\xi,x})_{(\xi,x) \in \Omega^\Gamma} = \text{convex hull } (E_{\gamma \xi,\gamma x})_{(\xi,x) \in \Omega^\Gamma} = \text{convex hull } (E_{\xi,\gamma x})_{(\xi,x) \in \Omega^\Gamma} = C_{\gamma x}, \]
where to pass from the first line to the second one we used the fact that $E$ is a subfield of $U$ and to pass from the second line to the third one we exploited the action on $\Omega^\Gamma$. 
Now, by the minimality of $Z$ we must have $C_x = Z_x$ for almost every $x \in X$ and since $Z_x$ has trivial Euclidean factor, by Equation (2) we have  

$$\dim(\mathcal{E}_{\xi,x}) = 0$$

for every $(\xi, x) \in \Omega^\Gamma$. Hence we have a section $B \times X \to U$ and, by the same argument used in case (ii), we have a contradiction. □

**Remark 4.7.** Fix positive integers $n$ and $p \le q$. In the setting of Theorem 1, if $\Gamma$ is a complex hyperbolic lattice in $PU(n, 1)$ and $\mathcal{X}(p, q)$ denotes the Hermitian symmetric space associated to the group $SU(p, q)$, we have a boundary map $\partial \mathbb{H}^n \times X \to \partial \mathcal{X}(p, q)$. Moreover, by ergodicity we have that it takes values in the set of isotropic $k$-dimensional subspace in the boundary $\partial \mathcal{X}(p, q)$ for some $k \le p$. To see this, for each pair $(\xi, x) \in \partial \mathbb{H}^n \times X$ one can take the smallest cell in the spherical building of $\partial \mathcal{X}(p, q)$ which contains $\phi(\xi, x)$, that corresponds to a totally isotropic flag of $\mathbb{C}^{p,q}$ (see [Duc12]). By ergodicity the type of this flag must be the same for almost every pair in $\partial \mathbb{H}^n \times X$ and by taking the maximal isotropic spaces of any flag we get the desired map. If we assume that $\sigma$ is Zariski dense, the same argument in [DLP21, Theorem 1.7] show that $k = p$, namely the target is the Shilov boundary of $\mathcal{X}(p, q)$.

Now, since Zariski density implies non-elementarity, this gives an alternative proof of [SS21b, Theorem 1] which relies on the geometry of the symmetric space $\mathcal{X}(p, q)$.

## II. Finite reducibility

In this second part we study cocycles on a specific $\mathcal{X}(p, \infty)$, namely the Hermitian symmetric space associated to $PU(p, \infty)$. In particular, we parametrize the space of embeddings of $\mathcal{X}(p, q)$ inside $\mathcal{X}(p, \infty)$ through the action of $PU(p, \infty)$. Then we characterize a family of subgroups of $PU(p, \infty)$, the finite algebraic subgroups, and we use this notion to define finite reducibility. Finally we apply the machinery of numerical invariants for cocycles to define the Toledo invariant and maximal cocycles and we prove Theorem 2.

In the last section we put together Part I and Part II and we prove Theorem 3.

## 5. The symmetric space $\mathcal{X}(p, q)$.

Let $(p, q) \in \mathbb{N} \times \mathbb{N} \cup \{\infty\}$ with $p \le q$ and consider a $(p + q)$-dimensional Hilbert space $\mathcal{H}$ over $\mathbb{C}$. Let $\{e_i\}_{i=1}^{p+q}$ be an Hilbert basis for $\mathcal{H}$. We denote by $L(\mathcal{H})$ the set of $\mathbb{C}$-linear bounded operators with respect to the operator norm and by $GL(\mathcal{H})$ the group of bounded invertible $\mathbb{C}$-linear operators of $\mathcal{H}$ with bounded inverse.

We define the Hermitian form $Q$ of signature $(p, q)$ as follows

$$Q(x) = \sum_{i=1}^{p} x_i \overline{x_i} - \sum_{i=p+1}^{p+q} x_i \overline{x_i}$$

where $x = \sum_{i=1}^{p+q} x_i e_i$ for all $x \in \mathcal{H}$. We denote with $U(p, q)$ the subgroup of $GL(\mathcal{H})$ of isometries with respect to $Q$, that means linear maps $h : \mathcal{H} \to \mathcal{H}$ such that
\[ Q(v, w) = Q(h(v), h(w)) \text{ for all } v, w \in \mathcal{H}. \] Hence, if we define the space

\[ \mathcal{X}(p, q) := \{ V < \mathcal{H} \mid \dim V = p, Q_V > 0 \}, \]

then by Witt’s theorem the group \( U(p, q) \) acts transitively on it (see for instance [Art11] Theorem 3.9). Moreover, the stabilizer of \( V_0 := \text{Span}\{e_1, \ldots, e_p\} \) is the product \( U(p) \times U(q) \), where \( U(m) \) is the orthogonal group of the Hilbert space of dimension \( m \), for any \( m \in \mathbb{N} \cup \{\infty\} \). Hence we can identify \( \mathcal{X}(p, q) \) with the quotient

\[ U(p, q)/U(p) \times U(q) \]

and one can show that it has a structure of simply connected non-positively curved Riemannian symmetric space with real rank \( p \) (see [Duc12]).

Homotheties act trivially on \( \mathcal{X}(p, q) \), hence we have an isometric action by isometries of the quotient

\[ \text{PU}(p, q) := U(p, q)/\{\lambda \text{Id} \mid |\lambda| = 1\} \]

on \( \mathcal{X}(p, q) \).

We define the boundary \( \partial \mathcal{X}(p, q) \) as the set of subspaces of \( \mathcal{H} \) on which the restriction of \( Q \) is identically zero (that is \emph{totally isotropic subspaces}). In \( \partial \mathcal{X}(p, q) \), for each \( 1 \leq k \leq p \), we denote as \( \mathcal{I}_k(p, q) \) the set of totally isotropic subspaces of dimension \( k \). In particular, we will be interested in the the set \( \mathcal{I}_p(p, q) \) of maximal totally isotropic subspaces, which we will call \emph{p-chains} (according to [Poz15] and [DLP21]). Finally, two points in \( \mathcal{I}_p(p, q) \) defined by totally isotropic subspaces \( V_1 \) and \( V_2 \) are said to be \emph{opposite} if \( V_1 \cap V_2 = 0 \). In particular, each pair of opposite points in \( \mathcal{I}_p(p, q) \) is contained in a unique \emph{p-chain}.

We denote by \( Q_{p,q} \) the Hermitian form of signature \((p, q)\) with \( p \leq q < +\infty\). Let \( \{ E_i \}_{i=1}^{p+q} \) and \( \{ e_i \}_{i \in \mathbb{N}} \) be two basis respectively of \( \mathbb{C}^{p,q} \) and of an infinite dimensional Hilbert space \( \mathcal{H} \) over \( \mathbb{C} \).

**Definition 5.1.** An \emph{embedding} of \( \mathcal{X}(p, q) \) into \( \mathcal{X}(p, \infty) \) is a linear map \( \iota : \mathbb{C}^{p,q} \to \mathcal{H} \) that preserves the Hermitian forms \( Q_{p,q} \) and \( Q_{p,\infty} \), namely such that \( Q_{p,\infty}(\iota(x), \iota(y)) = Q_{p,q}(x, y) \) for every \( x, y \in \mathbb{C}^{p,q} \). Moreover, the group \( U(p, q) \) of linear bounded transformations preserving \( Q_{p,q} \) embeds in \( U(p, \infty) \) in the following way: the action on \( \iota(\mathbb{C}^{p,q}) \) is that of \( U(p, q) \) and is trivial on the orthogonal complement of \( \iota(\mathbb{C}^{p,q}) \).

Among all embeddings of \( \mathcal{X}(p, q) \) in \( \mathcal{X}(p, \infty) \), we consider the \emph{standard embedding} defined by the map \( \iota_0 : \mathbb{C}^{p,q} \to \mathcal{H} \) defined as \( \iota_0(E_i) = e_i \) for \( i = 1, \ldots, p + q \). In this special case, the space \( \mathcal{X}(p, q) \) inside \( \mathcal{X}(p, \infty) \) can be identified with the set

\[ \mathcal{V}_0 = \{ V < \text{Span}\{e_1, \ldots, e_{p+q}\} \mid \dim V = p, Q_{p,\infty}|_V > 0 \} \]

and the group \( U(p, q) \) is identified with elements \( g \) in \( U(p, \infty) \) such that

\[ g(e_i) = \sum_{j \in \mathbb{N}} a_{ij} e_j \]
where, for either $i$ or $j$ bigger than $p + q$, then $a_{ij} = \delta_{ij}$, and the matrix $A = (a_{ij})_{i,j=1}^{p+q}$ represents an element in $U(p, q)$, namely it satisfies

$$A^* \begin{pmatrix} \text{Id}_p & 0 \\ 0 & -\text{Id}_q \end{pmatrix} A = \begin{pmatrix} \text{Id}_p & 0 \\ 0 & -\text{Id}_q \end{pmatrix}.$$

The notion of embedding given in Definition 5.1 corresponds to the one of standard embedding given in \cite{DLP21}. This choice is motivated by the fact that here we need to distinguish the particular objects described above, whose role among all embeddings is clarified by the following

**Lemma 5.2.** Any embedding of $\mathcal{X}(p, q)$ of $\mathcal{X}(p, \infty)$ can be obtained by composition of an element $g \in U(p, \infty)$ with the standard embedding.

**Proof.** Let $\iota : \mathbb{C}^{p,q} \to \mathcal{H}$ an isometric linear map. For each $e_i$ we set $u_i := \iota(e_i)$ and $U_i := \text{Span}\{u_1, \ldots, u_{p+q}\}$.

There is a natural identification of $\mathcal{X}(p, q)$ with the subspace of $\mathcal{X}(p, \infty)$ defined by

$$V_i = \{V < U_i \mid \dim V = p, \ Q_{p,\infty}(V) > 0\}.$$

If we denote with $U_0$ the subspace of $\mathcal{H}$ spanned by the first $p + q$ vectors of the basis $(e_i)_{i\in\mathbb{N}}$, we can define an isometric linear map $h : U_0 \to U_i$ on the basis as follows

$$h(e_i) = u_i$$

and then extend it by linearity. Since $h$ preserves the Hermitian form $Q$, by Witt’s theorem it extends to an isometry of $\mathcal{H}$ with respect to $Q$, namely to an element $g \in U(p, \infty)$. The thesis follows noticing that the isometric linear map $g \circ \iota$ actually gives the standard embedding.

\square

**Remark 5.3.** As a subspace of the Grassmannian $\text{Gr}(p, \mathcal{H})$, the set of embedding of $\mathcal{X}(p, q)$ inside $\mathcal{X}(p, \infty)$ naturally inherits the topology induced by principal angles, that in this case coincide with the Wijsman topology (see \cite{DLP21}). Since by Lemma 5.2 the group $U(p, \infty)$ acts transitively on the set of all such embeddings, we have an identification with the $\text{PU}(p, \infty)/\text{Stab}_{U(p, \infty)}V_0$, where $V_0$ is the image of the standard embedding.

### 6. Bounded cohomology

Let $G$ be a locally compact and second countable group and let $E$ be a Banach $G$-module (namely a Banach space together with an action of $G$ by isometries). Continuous bounded cohomology is usually defined as the cohomology of the $G$-invariant part of the complex of continuous bounded cochains $(\mathcal{C}_{cb}^\bullet(G; E), \delta^\bullet)$ on $G$, namely

$$\mathcal{C}_{cb}^\bullet(G; E) := \{f : G^{\bullet+1} \to E \mid f \text{ continuous}, \sup_{g_0, \ldots, \bullet} \|f(g_0, \ldots, \bullet)\|_E < \infty\}$$
and
\[ \delta^* f(g_0, \ldots, g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(g_0, \ldots, \tilde{g}_i, \ldots, g_{n+1}). \]

Since it may reveal quite difficult to compute continuous bounded cohomology following the above definition, Burger-Monod [BM02] showed that all strong resolutions of \( E \) by relatively injective \( G \)-modules share the same cohomology. More precisely, we have that the continuous bounded cohomology of \( G \) with coefficients in \( E \) is the cohomology of the \( G \)-invariant vectors of any such resolution \((E^*, \delta^*)\), namely
\[ H^k_{cb}(G; E) \cong H^k((E^*)^G, \delta^*). \]

Recall that, if \( E \) is the dual of some Banach \( G \)-module endowed with the weak-* topology and assuming that \( G \) is a semisimple Lie group of non-compact type, we can define the cochain complex of essentially bounded weak-* measurable functions on the Furstenberg boundary \( B(G) \), denoted by \((L^\infty_w((B(G)^{\bullet+1}; E), \delta^*))\), where \( \delta^* \) is the standard homogeneous coboundary operator. Since the previous complex can be completed to a strong resolution of \( E \) by relatively injective modules, we have an isomorphism
\[ H^k_{cb}(G; E) \cong H^k(L^\infty_w((B(G)^{\bullet+1}; E)^G, \delta^*)) \]
for any \( k \geq 0 \). By [BM02, Corollary 1.5.3] the isomorphism is actually isometric, that is it preserves the natural seminormed structures on those spaces.

If we consider the complex of bounded weak-* measurable functions on a measure \( G \)-space \( X \), denoted by \((B^\infty_w(X^{\bullet+1}; E), \delta^*)\), we obtain only a strong resolution of \( E \). Nevertheless, Burger and Iozzi [BI02] showed that there exists a canonical map
\[ \epsilon^k : H^k(B^\infty((X^{\bullet+1}; E)^G, \delta^*)) \to H^k_{cb}(G; E) \]
for every \( k \geq 0 \).

Let \( \Gamma < G \) be a lattice. As in the case of representations, given a measurable cocycle \( \sigma : \Gamma \times X \to H \), there exists a natural notion of pullback in bounded cohomology. More precisely, for any Banach \( H \)-module \( E \), the map
\[ C^*_b(\sigma) : C^*_cb(H; E)^H \to C^*_cb(\Gamma; E)^\Gamma, \]
\[ C^*_b(\sigma)(\psi)(\gamma_0, \ldots, \gamma_s) := \int_X \psi(\sigma(\gamma_0^{-1}, x)^{-1}, \ldots, \sigma(\gamma_s^{-1}, x)^{-1})d\mu_X(x), \]
is a well-defined cochain map [Sav20a, Lemma 2.7] inducing a map at the level of cohomology groups
\[ H^k_{cb}(\sigma) : H^k_{cb}(H; E) \to H^k_{cb}(\Gamma; E), \quad H^k_{cb}(\sigma)([\psi]) := [C^*_b(\sigma)(\psi)]. \]
for every \( k \geq 0 \). If \( \sigma \) additionally admits a boundary map \( \phi : B(G) \times X \to Y \), one can define
\[ C^*_b(\Phi^X) : B^\infty(Y^{\bullet+1}; E)^H \to L^\infty_w(B(G)^{\bullet+1}; E)^\Gamma, \]
\[ C^*_b(\Phi^X)(\psi)(\xi_0, \ldots, \xi_s) := \int_X \psi(\phi(\xi_0, x), \ldots, \phi(\xi_s, x))d\mu_X(x). \]
As shown by the second author and Moraschini [MS20, MS21], the above map is a cochain map which does not increase the norm and it induces well-defined maps in cohomology
\[ H^k(\Phi^X) : H^k(B^\infty(Y^{•+1}; E)^H) \rightarrow H^k(\Gamma; E), \quad H^k(\Phi^X)([\psi]) := [C^k(\Phi^X)(\psi)] \]
for every \( k \geq 0 \).

Thanks to [Sav20a, Lemma 2.10], one can check that the class \( H^k_b(\sigma([\psi])) \) admits as a natural representative the cocycle \( C^k(\Phi^X)(\psi) \). This allows to implement the pullback along measurable cocycles via boundary maps, imitating the work done by Burger and Iozzi [BI02] for representations. Such an implementation will be crucial in the proof of our main theorem.

7. Algebraic subgroups of \( GL(H) \).

We first introduce the notion of polynomial map.

**Definition 7.1.** A map \( f : L(H) \rightarrow \mathbb{R} \) is a polynomial map if it is a finite sum of maps \( f_1, \ldots, f_k \) where for each \( i = 1, \ldots, k \) there exists a \( n_i \)-linear map \( h_i \in L^{n_i}(L(H), \mathbb{R}) \) such that \( f_i(g) = h_i(g, \ldots, g) \) for every \( g \in L(H) \). The degree of \( f \) is the maximum of the \( n_i \)'s.

Now, in parallel to the finite dimensional case, we define an algebraic subgroup as the set of the zero locus of some family of polynomial maps.

**Definition 7.2.** A subgroup \( G \) of \( GL(H) \) is algebraic if there exists a positive integer \( n \) and family \( \mathcal{P} \) of polynomial maps of degrees at most \( n \) such that
\[ G = \{ g \in GL(H) \mid P(g, g^{-1}) = 0, \forall P \in \mathcal{P} \}. \]

A strict algebraic subgroup is a proper algebraic subgroup of \( GL(H) \).

To define a linear algebraic subgroup of \( GL(n, \mathbb{R}) \) we consider polynomial equations in matrix coefficients. The generalization to infinite dimension of this notion is the content of the following definition.

**Definition 7.3 ([DLP21, Definition 3.4]).** Let \( H \) be an infinite dimensional Hilbert space and choose an orthonormal basis \( (e_n)_{n \in \mathbb{N}} \). A homogeneous polynomial map \( P : L(H) \times L(H) \rightarrow \mathbb{R} \) is standard of degree \( d \) if there exist two naturals \( \ell, m \) such that \( \ell + m = d \) and a family of real coefficients \( (\lambda_i)_{i \in \mathbb{N}^{2\ell}} \) and \( (\mu_j)_{j \in \mathbb{N}^{2m}} \) such that for any \( (M, N) \in L(H) \times L(H) \) we have that \( P \) can be expressed as the absolute convergent series
\[ P(M, N) = \sum_{i \in \mathbb{N}^{2\ell}, j \in \mathbb{N}^{2m}} \lambda_i \mu_j P_i(M) P_j(N) \]
where \( P_i(M) = \prod_{k=0}^{\ell-1} < Me_{i_{2k}}, e_{i_{2k+1}} > \) and \( P_j(N) = \prod_{k=0}^{m-1} < Me_{j_{2k}}, e_{j_{2k+1}} > \).

A standard polynomial map is a finite sum of standard homogeneous polynomial maps.

An algebraic subgroup of \( L(H) \) is standard if it is defined by a family of standard polynomial maps.
Hence we have the following interesting property, that shows how proper standard algebraic subgroups are closely related to finite dimensional subspace of $\mathcal{H}$.

**Lemma 7.4.** ([DLP21] Lemma 3.6) If $H$ is a strict standard algebraic group, then there exists a finite dimensional subspace $E$ of $\mathcal{H}$ such that the the group $H_E := \{ g \in H \mid g(E) = E, g|_{E^\perp} = \text{id} \}$ is a strict algebraic subgroup of $\text{GL}(E)$.

We call the subspace $E$ support of the strict algebraic subgroup $H$ and the group $H_E$ the $E$-part of $H$. We are now ready to give the following

**Definition 7.5.** A finite dimensional algebraic subgroup is a standard algebraic subgroup of $\text{GL}(\mathcal{H})$ of the form $H_E$.

Hence, it follows by Lemma 7.4 a characterization of finite dimensional algebraic subgroups.

**Lemma 7.6.** If $E$ is a finite dimensional subspace of $\mathcal{H}$ and $H$ is a subgroup of $\text{GL}(\mathcal{H})$ contained in $\text{GL}(E)$, then $H$ is algebraic in $\text{GL}(E)$ if and only if it is a finite dimensional algebraic subgroup of $\text{GL}(\mathcal{H})$.

**Proof.** If $H$ is a finite dimensional algebraic subgroup of $\text{GL}(\mathcal{H})$ then $H = H_E$ and by Lemma 7.4 it is algebraic in $\text{GL}(E)$. Conversely, if $H$ is algebraic in $\text{GL}(E)$, it is also an algebraic subgroup in $\text{GL}(\mathcal{H})$. Moreover, any polynomial which defines $H$ on $\text{GL}(E)$ can be turned into a polynomial on the entries of the matrices. Hence the same polynomials, seen as standard polynomial maps in the sense of Definition 7.3, define a standard algebraic subgroup in $\text{GL}(\mathcal{H})$. Since it fixes $E^\perp$ then it coincides with its $E$-part and we are done. □

We come back to the groups $U(p,q)$. It is well known that the group $U(p,\infty)$ is algebraic subgroup of $\text{GL}(\mathcal{H})$. Indeed, if $V_0 := \text{Span}\{e_1, \ldots, e_p\}$, we have that

$$U(p,\infty) = \{ g \in \text{GL}(\mathcal{H}) \mid g^*\text{Id}_{p,\infty}g = \text{Id}_{p,\infty}\}$$

where $\text{Id}_{p,\infty}$ is the linear map $\text{Id}_{V_0} \oplus -\text{Id}_{V_0^\perp}$. Since the map $(A,B) \mapsto A^*\text{Id}_{p,\infty}B - \text{Id}_{p,\infty}$ is bilinear on $L(\mathcal{H}) \times L(\mathcal{H})$ then $U(p,\infty)$ is algebraic in $\text{GL}(\mathcal{H})$ and hence in $\text{GL}(\mathcal{H})$ (see [DLP21] for more details). By Proposition 7.6 we can say immediately that the groups $U(p,q)$ with $q < \infty$, seen as subgroups of $U(p,\infty)$ inside $\text{GL}(\mathcal{H})$, are actually finite algebraic since they stabilize the embedding of $\mathcal{X}(p,q)$ inside $\mathcal{X}(p,\infty)$.

Since we work with the quotients $PU(p,q)$ instead of the groups $U(p,q)$, we call finite algebraic a subgroup of $PU(p,\infty)$ if its preimage under the projection $U(p,\infty) \rightarrow PU(p,\infty)$ is finite algebraic in $\text{GL}(\mathcal{H})$ in the sense of Definition 7.5.

8. Finite reducibility of a cocycle

The final aim of this section is to prove Theorem 2. In particular, in the first part we need to recall the definition of Kähler class and of Bergman cocycle for the groups $PU(n,1)$ and $PU(p,\infty)$.

Next we can define, following for instance [SS21b], the Toledo invariant associated to a measurable cocycle and the notion of maximality for cocycles. Moreover, inspired by [BH09], we provide Equation (9), which will be crucial in the proof of the main theorem.
8.1. The Kähler class and the Bergman cocycle. A crucial difference between the finite case and the infinite one in the context of symmetric spaces is that \( \text{PU}(p, q) \) is locally compact for \( q < \infty \) whereas \( \text{PU}(p, \infty) \) is not. To overcome this problem we will deal with the bounded cohomology groups \( H^\bullet_b(\text{PU}(p, \infty); \mathbb{R}) \), namely its continuous bounded cohomology if we endow \( \text{PU}(p, \infty) \) with the discrete topology.

Since \( \mathcal{X}(p, \infty) \) is an Hermitian symmetric space, there exists a Kähler form \( \omega \), that is a \( \text{PU}(p, \infty) \)-invariant closed 2-form on \( \mathcal{X}(p, \infty) \). Using such an invariant form we can define

\[
\omega_x : \text{PU}(p, \infty)^3 \to \mathbb{R}, \quad \omega_x(g_0, g_1, g_2) = \frac{1}{\pi} \int_{\Delta(g_0x, g_1x, g_2x)} \omega
\]

where \( x \) is a point in \( \mathcal{X}(p, \infty) \) and \( \Delta(g_0x, g_1x, g_2x) \) is a triangle in \( \mathcal{X}(p, \infty) \) with vertices \( g_0x, g_1x, g_2x \) and geodesic edges. The map \( \omega_x \) defines a strict \( \text{PU}(p, \infty) \)-invariant cocycle and by \([\text{DLP21}, \text{Lemma 5.3}]\) different choices of the basepoint \( x \) lead to cohomologous cocycles. In this way we obtain a well-defined cohomology class

\[
k^b_{\text{PU}(p, \infty)} \in H^2_b(\text{PU}(p, \infty); \mathbb{R}),
\]

called bounded Kähler class of \( \text{PU}(p, \infty) \). Now, taking the Gromov norm \( \| \cdot \|_{\infty} \), it follows from the definition that

\[
\|k^b_{\text{PU}(p, \infty)}\|_{\infty} = \text{rk}\mathcal{X}(p, \infty) = p. \quad (7)
\]

We will need to define the Bergman class extending the one given in finite case, namely to construct a cocycle on the boundary \( \mathcal{I}_p \). Such a definition will allow us to exploit the boundary theory to pullback cocycle in cohomology.

Given any three maximal totally isotropic subspaces \( V_0, V_1, V_2 \in \mathcal{I}_p \), since they are contained in a finite dimensional subspace of dimension at most \( 3p \), we can use the definition of the Bergman cocycle associated to \( \text{SU}(p, 2p) \) to get a strict \( \text{PU}(p, \infty) \)-invariant cocycle

\[
\beta : \mathcal{I}^3_p \to [-p, p].
\]

We recall that the maximal value is taken on triples of pairwise opposite totally isotropic subspaces which lie in a \( 2p \)-dimensional subspace, namely on triples lying on a \( p \)-chain \([\text{Poz15}, \text{Proposition 2.1}]\). Now, given a point \( V \in \mathcal{I}_p \), the cocycle \( C_V \) defined as

\[
C_V(g_0, g_1, g_2) = \beta(g_0V, g_1V, g_2V)
\]

still represents the bounded Kähler class \( k^b_{\text{PU}(p, \infty)} \in H^2_b(\text{PU}(p, \infty); \mathbb{R}) \) \([\text{DLP21}, \text{Lemma 5.4}]\).

8.2. The Toledo invariant. Let \( \Gamma < \text{PU}(n, 1) \) be a complex hyperbolic lattice, \( (X, \mu_X) \) be a standard Borel probability \( \Gamma \)-space and let \( \sigma : \Gamma \times X \to \text{PU}(p, \infty) \) be a measurable cocycle. Following \([\text{SS21a}]\), we define the transfer map

\[
T^2_b : H^2_b(\Gamma; \mathbb{R}) \to H^2_b(\text{PU}(n, 1); \mathbb{R})
\]

as the map induced in cohomology by the function

\[
\hat{T}^2_b : L^\infty((\partial H^n_C)^3; \mathbb{R})^\Gamma \to L^\infty((\partial H^n_C)^3; \mathbb{R})^{\text{PU}(n, 1)},
\]
$\Phi^X(\beta) = t_\sigma \kappa^b_{\text{PU}(n,1)}$.

Writing down the above equation in terms of cochains, we get the formula

$$\int_{\Gamma \backslash \text{PU}(n,1)} \left( \int_X \beta(\phi(\bar{g}\xi_0, x), \phi(\bar{g}\xi_1, x), \phi(\bar{g}\xi_2, x)) d\mu_X(x) \right) d\mu_{\Gamma \backslash \text{PU}(n,1)}(\bar{g}) = t_\sigma \cdot c_n(\xi_0, \xi_1, \xi_2)$$

that holds for every triple of distinct points $(\xi_0, \xi_1, \xi_2)$ in $\partial \mathbb{H}^3_\Gamma$. Here $c_n$ is the Cartan’s angular invariant that represents the bounded Kähler class of $\text{PU}(n,1)$ $[\text{Gol99}]$.

**Remark 8.3.** We notice that Equation (9) is the natural extension in infinite dimension of the invariant defined in $[\text{SS21b}, \text{Definition 3.1}]$. Moreover, the Toledo invariant $t_\sigma$ is the multiplicative constant associated to $\beta, c_p$, namely $t_\sigma = \lambda_{\beta, c_p}(\sigma)$ according to $[\text{MS21}, \text{Definition 3.16}]$.

### 8.3. Proof of finite reducibility

Given a measurable cocycle $\sigma : \Gamma \times X \to \text{PU}(p, \infty)$, one can ask when its image is contained in some suitable subgroup of $\text{PU}(p, \infty)$. More precisely, definitions and results given in Section 7 allow us to define a class of cocycles for which some nice algebraic properties of the image are recovered.
Definition 8.4. A cocycle $\sigma : \Gamma \times X \to \text{PU}(p, \infty)$ is finitely reducible if it admits a cohomologous cocycle with image contained in a finite dimensional algebraic subgroup of $\text{PU}(p, \infty)$.

Before proving the main theorem, we recall by [DLP21] the following

**Definition 8.5.** A measurable map $\phi : \partial \mathbb{H}^n_C \to I_p$ almost surely maps chains to chains if for almost every chain $C \subset \partial \mathbb{H}^n_C$ there is a $p$-chain $T \subset I_p$ such that for almost every point $\xi \in C$, $\phi(\xi) \in T$.

An equivalent condition [Poz15, Lemma 4.2] to the one above is to check that, for almost every pair $(x, y) \in \partial \mathbb{H}^n_C \times \partial \mathbb{H}^n_C$, the points $\phi(x)$ and $\phi(y)$ are opposite and, for almost every $z \in C_{\xi_0, \xi_1}$, the subspace $\phi(z)$ is contained in $\langle \phi(\xi_0), \phi(\xi_1) \rangle$.

Before passing to the proof of Theorem 2, we need the following result about maps that almost surely maps chains to chains, which is a slight refinement of [DLP21, Proposition 6.2]. Since there is a natural embedding $\partial \mathbb{H}^n_C \subset \mathbb{P}^n C$, we can say that a set of $k \leq n + 1$ points in $\partial \mathbb{H}^n_C$ is generic if, for every $1 < h \leq k$, any subset of $h$ points does not span a $(h - 2)$-dimensional subspace.

**Lemma 8.6.** Let $\phi : \partial \mathbb{H}^n_C \to I_p$ be a measurable map that almost surely maps chains to chains. Then there exists a unique minimal totally geodesic embedded copy of $\mathcal{X}(p, q) \subset \mathcal{X}(p, \infty)$ that contains the image of almost every $(n + 1)$-tuple of generic points in $\partial \mathbb{H}^n_C$. Moreover, $p \leq q \leq np$.

**Proof.** We argue by induction on $n$. The case $n = 1$ is clear, since there is only one chain $C$ in $\partial \mathbb{H}^1_C$ and for almost every $\eta_1, \eta_2 \in C$ the subspace $\langle \phi(\eta_1), \phi(\eta_2) \rangle \subset \mathcal{H}$ defines a copy of $\mathcal{X}(p, p) \subset \mathcal{X}(p, \infty)$. The fact that $\phi$ almost surely maps chains to chains implies that for almost every $\xi$ in $\partial \mathbb{H}^n_C$ we have $\phi(\xi) \subset \langle \phi(\eta_1), \phi(\eta_2) \rangle$.

Assume that the statement holds for $n - 1$. Thanks to the construction in [DLP21], we can define a full-measure subset $\mathcal{G}$ of the set of $(n + 1)$-tuple of points in general position of $\partial \mathbb{H}^n_C$ such that for every $(\xi_0, \ldots, \xi_n) \in \mathcal{G}$ the following conditions hold:

- $\phi|_{\langle \xi_0, \ldots, \xi_{n-1} \rangle}$ almost surely maps chains to chains;
- for almost every $\eta \in \langle \xi_0, \ldots, \xi_{n-1} \rangle$ then $\langle \phi(\eta), \phi(\xi_{n-1}) \rangle$ is a $2p$-dimensional subspace on which the restriction of $Q$ has signature $(p, p)$;
- for almost every $\eta \in \langle \xi_{n-1}, \xi_n \rangle$ then $\langle \phi(\eta), \phi(\xi_{n-1}) \rangle$ is a $2p$-dimensional subspace on which the restriction of $Q$ has signature $(p, p)$;
- for almost every $\eta_1 \in \langle \xi_{n-1}, \xi_n \rangle$ and $\eta_2 \in \langle \xi_0, \ldots, \xi_{n-1} \rangle$ the space $\langle \phi(\eta_1), \phi(\eta_2) \rangle$ has dimension $2p$ and the restriction of $Q$ has signature $(p, p)$.

As proved in [DLP21, Proposition 6.2], for almost every $(\xi_0, \ldots, \xi_n) \in \mathcal{G}$ the space

$$V_{\xi_0, \ldots, \xi_n} := \langle \phi(\xi_0), \ldots, \phi(\xi_n) \rangle$$

contains $\phi(\eta)$ for almost every $\eta \in \partial \mathbb{H}^n_C$. Furthermore, the restriction of $Q$ to $V_{\xi_0, \ldots, \xi_n}$ is non-degenerate of signature $(p, q)$ with $p \leq q \leq np$.

We now prove that almost every pair of tuple $((\xi_0, \ldots, \xi_n), (\eta_0, \ldots, \eta_n)) \in \mathcal{G}^2$ give the same subspace. We first note that, since $V_{\xi_0, \ldots, \xi_n}$ contains the image of almost every point in $\partial \mathbb{H}^n_C$, it clearly contains $\phi(\eta_0), \ldots, \phi(\eta_n)$, and hence $\langle \phi(\eta_0), \ldots, \phi(\eta_n) \rangle$.
for almost every \((\eta_0, \ldots, \eta_n) \in \mathcal{G}\). Hence there exists a full-measure subset \(Q \subset \mathcal{G} \times \mathcal{G}\) such that
\[
V_{\xi_0, \ldots, \xi_n} < V_{\eta_0, \ldots, \eta_n}
\]
for almost every \(((\xi_0, \ldots, \xi_n), (\eta_0, \ldots, \eta_n)) \in Q\). By taking the measure-preserving idempotent function of \(\mathcal{G} \times \mathcal{G}\) which swap the tuple, one gets a second full-measure subsets \(\overline{Q}\). Hence the intersection \(Q \cap \overline{Q}\) is a full-measure subset of \(\mathcal{G} \times \mathcal{G}\) of pairs \((\xi_0, \ldots, \xi_n), (\eta_0, \ldots, \eta_n)\) such that
\[
V_{\xi_0, \ldots, \xi_n} = V_{\eta_0, \ldots, \eta_n},
\]
which implies the uniqueness.

A similar argument can be used to prove minimality, namely that every linear subspace \(W < H\) containing the image of a full-measure subset of \(\partial H^n_C\) must contain the spaces constructed above. □

Remark 8.7. It seems natural to investigate the effective dimension of the copy of \(\partial X(p, q)\) which contains the essential image of \(\phi\) provided by Lemma 8.6. For instance, given a chain preserving map \(\psi : \partial \mathbb{H}^n_C \to \partial \mathbb{H}^p_C\), Burger and Iozzi [BI07] proved the following dichotomy: if the image of almost every triple \((\xi_0, \xi_1, \xi_2)\) of generic points is generic as well, then \(\psi\) coincides almost everywhere with the map induced on boundaries by an isometric holomorphic embedding \(\mathbb{H}^n_C \to \mathbb{H}^p_C\). If not, then the image is essentially contained into a chain in \(\partial \mathbb{H}^p\).

In our more general context, such a dichotomy does not holds, since \(q\) can vary in \(p \leq q \leq np\). However, in our setting, the two cases described above can be interpreted as the limit cases as follows. In fact, if \(\phi : \partial \mathbb{H}^n_C \to \mathcal{I}_p\) as in Lemma 8.6 sends almost every \((n+1)\)-tuple of generic points of \(\partial \mathbb{H}^n_C\) to \((n+1)\) generic points of \(\mathcal{I}_p\), then we have that the essential image of \(\phi\) is contained in \(\partial X(p, np)\). On the other hand, by the same argument used in [BI07] Proposition 2.2], if there is a positive measure subset of triple in \((\partial \mathbb{H}^n_C)^3\) not on a chain whose image lies on a chain, then the image of \(\phi\) is essentially contained into one copy of \(\partial X(p, p)\). We point out that this two cases do not produce a dichotomy, but a characterization of the cases when \(q = p\) and \(q = np\) in the notation of Lemma 8.6.

Now we are ready to give the proof of the main result of this part.

Proof of Theorem 2. By the Equation (9) and using [DLP21] Corollary 6.1], it follows that almost every slice \(\phi_x\) almost maps chains to chains. Hence, by Lemma 8.6, for almost every \(x \in X\) there exists a unique minimal totally geodesic embedding \(\mathcal{X}_x(p, q_x) \subset \mathcal{X}(p, \infty)\) such that \(\text{Ess Im}(\phi_x) \subset \partial \mathcal{X}_x(p, q_x)\) for some \(p \leq q_x \leq np\). Moreover, the equivariance of \(\phi\) implies that
\[
\sigma(\gamma, x) \mathcal{X}_x(p, q_x) = \mathcal{X}_{\gamma x}(p, q_{\gamma x})
\]
for almost every \(\gamma \in \Gamma\) and \(x \in X\) and, by ergodicity, we have that the dimension of the \(\mathcal{X}_x(p, q_x)\)'s is essentially constant, namely \(q_x = q\) for almost every \(x \in X\). If we denote by \(t_x\) the isometric linear map that induces the embedding \(\mathcal{X}_x(p, q) \subset \mathcal{X}(p, \infty)\), the uniqueness of \(\mathcal{X}_x(p, q)\), together with the \(\sigma\)-equivariance of \(\phi\), implies
that the map
\[(10) \quad X \to \PU(p, \infty)/\text{Stab}_{\PU(p, \infty)}(V_0), \quad x \mapsto \mathcal{X}_x(p, q)\]
is measurable (with respect to the measurable structure discussed in Remark 5.3) and \(\sigma\)-equivariant. Here \(\text{Stab}_{\PU(p, \infty)}V_0\) is the subgroup of \(\PU(p, \infty)\) preserving the subspace \(V_0\). Now, thanks to the differentiable structure of the group \(\PU(p, \infty)\), we can compose the function in Equation (10) with a measurable section
\[\PU(p, \infty)/\text{Stab}_{\PU(p, \infty)}(V_0) \to \PU(p, \infty)\]
in order to obtain a measurable map
\[f : X \to \PU(p, \infty), \quad f(x) = g_x^{-1}.\]
By construction, \(f(x)\) sends \(\mathcal{X}_x(p, q)\) to the standard embedded copy \(\mathcal{X}_0(p, q) \subset \mathcal{X}(p, \infty)\).

According to the notation of the Definition 3.2 we consider the twisted cocycle
\[\sigma^f : \Gamma \times X \to \PU(p, \infty)\]
defined as
\[\sigma^f(\gamma, x) := f(\gamma x)^{-1}\sigma(\gamma, x)f(x)\]
and the associated twisted boundary map \(\phi^f : \partial \mathbb{H}_C^\infty \times X \to \mathcal{I}_p\) which is defined as follows
\[\phi^f(\xi, x) := f(x)^{-1}\phi(\xi, x)\]
for almost every \(\xi \in \partial \mathbb{H}_C^\infty\) and \(x \in X\). Now, by definition of \(f\), for almost every \(x \in X\) the image of almost every slice \(\phi_x\) is contained in the boundary of a fixed \(\mathcal{X}(p, q)\).

For almost every \(x \in X\), denote by \(E_x\) the full measure set of points \(\xi \in \partial \mathbb{H}_C^\infty\) such that \(\phi^f_x(\xi) \in \partial \mathcal{X}(p, q)\). Consider now the set \(E = \bigcup_{x \in X} E_x \times \{x\}\) (that is of full measure in \(\partial \mathbb{H}_C^\infty \times X\), by Fubini’s theorem) and the diagonal action of \(\Gamma\) given by
\[\gamma(\xi, x) = (\gamma \xi, \gamma x).\]
Since \(\Gamma\) is countable, we find an invariant full measure subset \(\overline{E}\) such that \(\phi^f(\overline{E}) \subset \partial \mathcal{X}(p, q)\). More precisely, we set
\[\overline{E} = \bigcap_{\gamma \in \Gamma} \gamma E,\]
where \(\gamma\) acts diagonally. Being the intersection of full measure sets, it is clear that \(\overline{E}\) has full measure. Now, since the image of a full measure set under \(\phi^f\) is contained in the boundary of \(\mathcal{X}_0(p, q)\), it follows that the image of the twisted cocycle \(\sigma^f\) is contained in \(\text{Stab}_{\PU(p, \infty)}V_0\), which is finite algebraic as desired. \(\square\)

Remark 8.8. The descending chain condition that holds for Noetherian spaces (as algebraic groups are), allows to define the algebraic hull for cocycles into algebraic groups. This can not be adapted for \(\PU(p, \infty)\), namely there exits no well-defined minimal strict algebraic group containing the image of a twisted cocycle. Nevertheless, by Theorem 2, any maximal cocycles has a representative in its cohomology class whose image is contained into the embedding of \(\PU(p, q)\) in \(\PU(p, \infty)\), which
is algebraic. For such particular measurable cocycles, our result recover a sense of algebraicity.

9. Consequences of finite reducibility

The aim of this last section is to relate Theorem 1 and Theorem 2. We consider the setting of Theorem 2, namely $\Gamma$ is a complex hyperbolic lattice, $(X, \mu_X)$ is an ergodic standard Borel probability $\Gamma$-space and $\sigma : \Gamma \times X \to \text{PU}(p, \infty)$ is a maximal cocycle. If we assume that $\sigma$ is non-elementary, Theorem 1 provides a boundary map $\phi : \partial \mathbb{H}_C^n \times X \to \partial X(p, \infty)$. Moreover, by Remark 4.7 such a map takes values into $I_k(p, \infty)$ for some $k \leq p$. Unfortunately, this is not sufficient to prove reducibility as in Theorem 2, since such $k$ might be strictly less than $p$.

However, for cocycles $\sigma : \Gamma \times X \to \text{PU}(1, \infty)$ one can exploit the geometry of $X(1, \infty) = \mathbb{H}_C^\infty$ and of its boundary to prove Theorem 3.

Proof of Theorem 3. We first prove that maximal cocycles cannot be non-elementary. In fact, by ergodicity, a $\sigma$-equivariant family of flats can be made of points or lines. In both cases one can twist $\sigma$ into a cocycles whose image is contained either in the stabilizer of a point or a geodesic, which are both amenable. Since amenable groups have trivial bounded cohomology, we have a contradiction to maximality.

Since $\sigma$ is not elementary, Theorem 1 provides a boundary map $\phi : \partial \mathbb{H}_C^n \times X \to \partial \mathbb{H}_C^\infty$ and then we can apply Theorem 2. Hence we have that $\sigma$ is cohomologous to a cocycle $\tilde{\sigma}$ whose image is the stabilizer of an embedded copy of $\mathbb{H}_C^n$ in $\mathbb{H}_C^\infty$. The stabilizer $\text{Stab}_{\text{PU}(p, \infty)}(\mathbb{H}_C^n)$ is an almost direct product with one factor isomorphic to $\text{PU}(n, 1)$. By composing with the projection on such factor we get a maximal cocycle. Hence we can apply [MS21, Theorem 1.5] and we are done.

Remark 9.1. In the general setting of Theorem 2 as pointed out in Remark 4.7 Theorem 2 provides a boundary map into some $I_k(p, \infty)$. In [DLP21] the authors exploited Proposition 2.2 to rule out the case $k < p$ for Zariski dense representations. In the tentative to adapt such argument in the context of cocycles, we stuck in the final part. Precisely, following the proof of [DLP21, Theorem 1.7], one can construct a $\sigma$-equivariant family $\{W_x\}_{x \in X}$ of non-trivial subspaces of $\Lambda^d H$ for some $d$. Since the stabilizer of such spaces are standard algebraic subgroups, it would be enough to twist the cocycle in order to get a cocycle with image contained in one of this stabilizers. However, the action of $\text{PU}(p, \infty)$ on the subspaces (a priori of infinite dimension) of $\Lambda^d H$ seems to us quite mysterious. Even before, one should clarifies the measurable structure on this space. To conclude as in the proof of Theorem 2 or [SS21b, Theorem 2] one should identify the $\text{PU}(p, \infty)$-orbit of some $W_x$ with the quotient $\text{PU}(p, \infty)/\text{Stab}_{\text{PU}(p, \infty)}W_x$, for instance by proving that the action is smooth, which is also not clear to us.

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