Optimal Control of DERs in ADN Under Spatial and Temporal Correlated Uncertainties

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Abstract—The uncertainties of distribution energy resources (DERs) have significant impacts on active distribution networks (ADNs), and it is necessary to control the DER outputs to hedge the negative impacts of their uncertainties on ADNs. However, DER uncertainties are complicated, containing spatial and temporal correlation, which makes it challenging to design proper control schemes, especially when there exist temporal-correlated units such as energy units (EUs). This paper provides an Itô process model to describe the characteristics of stochastic resources and EUs in a unified way, which makes it easy to evaluate the impacts of stochastic resources on temporal-correlated units. Based the moment form of the Itô process model, a moment optimization (MO) approach is provided to transform the stochastic control (SC) problem into an optimization problem with respect to the first-order and second-order moments of the system variables. The scale of MO is comparable to that of the corresponding deterministic control problem, which means that the computational efficiency of MO is much higher than that of traditional approaches. Case studies also show that the computational efficiency of MO is much higher than that of traditional approaches. Case studies also show that the proposed approach outperforms existing approaches in both the performance and computational efficiency, which means that the proposed approach has attractive potential for use in large-scale applications.

Index Terms—Active distribution network, distributed energy resources, optimal control, spatial correlation, temporal correlation.

NOMENCLATURE

Abbreviations

ADN Active Distribution Network.
CPI Controllable Power Injection.
DER Distributed Energy Resources.
EU Energy Units.

Indices/Sets

\( i, j, k \) Bus indices.
\( t \) Time index.
\( V \) The set of buses.
\( E \) The set of branches.
\( T \) The set of time.

Variables/Vectors

\( v_{i,t} \) Square voltage amplitude at Bus \( i \), time \( t \).
\( l_{ij,t} \) Square current amplitude from Bus \( i \) to Bus \( j \) at time \( t \).
\( p_{ij,t}, q_{ij,t} \) Active/Reactive power injection at Bus \( i \), time \( t \).
\( p_{ij,t}^{\text{pred}}, q_{ij,t}^{\text{pred}} \) Active/Reactive power flow from Bus \( i \) to Bus \( j \) at time \( t \).
\( p_{S,i,t}^{\text{dev}}, q_{S,i,t}^{\text{dev}} \) Fixed load at Bus \( i \), time \( t \).
\( S_{ij,t} \) Stochastic resources at Bus \( i \), time \( t \).
\( p_{S,i,t}^{\text{dev}}, q_{S,i,t}^{\text{dev}}, S_{ij,t} \) Stochastic power injections (SPIs).
\( W_t \) Wiener process used in SDE.
\( J \) Objective function.
\( u_t \) Vector of controllable power injections (CPIs).
\( u_0 \) Base control vector.
\( K \) Feedback coefficient.
\( p_{C,i,t} \) Charging power of the \( i \)-th EU at time \( t \).
\( p_{D,i,t} \) Discharging power of the \( i \)-th EU at time \( t \).
\( e_t \) Vector of the energy of EUs.
\( \xi_t \) Vector of stochastic power injections (SPIs).
\( \eta_t \) Auxiliary vector of stochastic resources.
\( M_t \) Covariance matrix of \( \xi_t \) and \( \eta_t \).
\( \bar{a} \) Expectation of the variable \( a \), where \( a \) can be any variable.

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Parameters such as overvoltage and overloading problems [2].

**Functions/Operators**

- $\mu(\cdot)$: Drift function in Itô processes.
- $\sigma(\cdot)$: Diffusion function in Itô processes.
- $\Pr(\cdot)$: Operator of probability.
- $\var(\cdot)$: Operator of variance.
- $\mathbb{E}\{\cdot\}$: Expectation.
- $R(t, s)$: Correlation function of stochastic resources.

### Functions/Operators

- $\tau$: Temporal correlation of Itô processes.
- $A_{ij}/d/s/u$: Matrix coefficients of network constraints.
- $\phi_i$: Coefficient vector of the $i$-th constraint.
- $\gamma$: Confidence level of the chance constraint.
- $\kappa$: Coefficient of inner approximation of chance constraints under the confidence level $\gamma$.
- $\lambda$: Electricity price at time $t$.
- $R^U$: Regularization coefficient of the control variable.
- $R^E$: Regularization coefficient of the EU.
- $T$: The control horizon.
- $\alpha^E$: Dissipation factor of the $i$-th EU.
- $\beta^C$: Charging efficiency of the $i$-th EU.
- $\beta^D$: Discharging efficiency of the $i$-th EU.
- $r_{ij}$: Resistance from Bus $i$ to Bus $j$.
- $x_{ij}$: Reactance from Bus $i$ to Bus $j$.
- $g_i$: Shunt conductance of Bus $i$.
- $b_i$: Shunt susceptance of Bus $i$.
- $\overline{SOC}_i$, $\overline{SOC}_{i, end}$: Lower/Upper limit of the SOC of the $i$-th EU.
- $SOC_{i, end}$: Predefined terminal value of the SOC of the $i$-th EU.
- $\overline{P}_{t}^{C}$, $\overline{P}_{t}^{D}$: Upper limit of the charging/discharging power of the $i$-th EU.
- $\overline{V}_i$, $\overline{v}_i$: Lower/Upper limit of the square voltage amplitude at Bus $i$.
- $\overline{i}_{ij}$: Upper limit of the current from Bus $i$ to Bus $j$.
- $C_{i}^{S}$, $D_{i}^{S}$: Coefficient in the polygonal approximation of the constraints of stochastic resources.
- $C_{a}$, $C_{\xi}$, $D_{i}^{S}$: Coefficient in the polygonal approximation of the constraints of stochastic resources in the compact form.
- $\phi$: The critical phase angle of the stochastic resources at Bus $i$.
- $R^X$, $H$: Coefficients of $x_i$ in the vector-form objective.
- $\alpha$, $\beta$: Coefficients of EUs in the vector form.
- $\xi$, $\bar{\xi}$: Upper/Lower limit of $x$.
- $\bar{y}$, $\bar{\bar{y}}$: Upper/Lower limit of $y$.
- $\bar{e}$, $\bar{\bar{e}}$: Upper/Lower limit of $e$.

### I. INTRODUCTION

**D**istributed energy resources (DERs), including renewable generations and energy units (EUs), have grown rapidly in recent years [1]. The integration of DERs brings significant challenges to active distribution networks (ADNs), such as overvoltage and overloading problems [2]. To this end, it is necessary to control DERs in order to mitigate their negative impacts, and there are many studies in this area [3]–[5]. However, renewable power generations are usually highly stochastic, and the uncertainties caused by these renewables will undoubtedly deteriorate the control performance.

It is challenging to consider DER uncertainties in the optimal control problem, mainly because the modeling of these uncertainties is complicated. The uncertainty of renewable generations is usually non-Gaussian [6]–[9] and contains spatial and temporal correlations [10], [11]. Moreover, EU characteristics are also temporal-correlated, and their temporal correlations may be influenced by the spatial and temporal correlation of the uncertainties [12]. However, based on existing uncertainty models, such as the probability distribution model [6] and the Markov model [10], it is challenging to analyze the impacts of uncertainties on the DERs and distribution networks, and such impacts can only be evaluated by Monte Carlo simulations [13], which are time-consuming.

Techniques to deal with uncertainties in optimal control problems include robust control methods [14], model predictive control (MPC) methods [15]–[17], and stochastic-programming-based-control (SPBC) methods [9], [13], [18]. Robust controllers use uncertainty sets to model uncertainties and find the control schemes that perform well in the worst case; hence, these schemes usually lead to conservative results. MPC solves an open-loop optimal control problem in which the uncertainties are not considered and adjusts the control outputs in a receding-horizon manner. Although it provides some robustness by the receding-horizon implementation, MPC does not consider the uncertainties explicitly, which may have negative performance impacts [19]. Moreover, receding-horizon implementation is also time-consuming.

SPBC has been widely used in the recent studies [9], [13], [18]: SPBC handles the stochastic control (SC) problem by stochastic programming, which can be solved by scenario-based approaches. Specifically, SPBC generates a certain number of scenarios under the probability distribution and correlation of the uncertainties; furthermore, it transforms the SC into deterministic optimization problems. SPBC is widely used in power system operations considering uncertainties; however, in order to achieve good accuracy, a large number of scenarios are needed, which may lead to an unacceptable computational burden. Although some studies exist for methods to accelerate the computation of SPBC [20], [21], the computational burden of SPBC is still too large compared to deterministic control.

In summary, it is challenging to efficiently solve the SC problem under complicated uncertainties with spatial and temporal correlation. Therefore, this paper provides a novel moment optimization (MO) approach for the SC of DERs in distribution networks. We use an Itô process model to describe the probability distribution, the spatial and temporal correlation of uncertainties, and transform the SC into a deterministic optimization with respect to the first-order and second-order moments of system variables. The proposed MO approach solves the SC with a comparable computational burden to deterministic control problems and hence is attractive.
for online applications. Case studies also show that the MO approach outperforms the existing approaches.

The contributions of this paper are twofold:
1) An Itô process model is provided to describe the stochastic resources. On the one hand, the proposed model can be used to model the spatial and temporal correlation of renewable generations; on the other hand, the temporal correlation of EUs can be easily embedded into the Itô process model; therefore, it is possible to consider the impact of renewable generations and control policies on the states of EUs in a unified framework. Moreover, the statistics of Itô processes can be calculated analytically, i.e., without time-consuming simulations.
2) An SC model of DERs is provided and then solved by the MO approach. The MO approach transforms the SC into a deterministic optimization problem with respect to the first-order and second-order moments of the system variables. The MO model accurately describes the characteristics of the system, and its scale is comparable to that of the corresponding deterministic control problems. Therefore, MO achieves a good tradeoff between performance and computational efficiency.

Following this introduction, Section II provides the model of the stochastic resources and the SC problem. The MO approach is discussed in detail in Section III. Section IV provides numerical results, and Section V concludes the paper.

II. MODELING

This section describes the SC model of distribution networks. Typical units as well as the uncertainties in distribution networks are considered. After discussing the structure of ADN, we provide the Itô process model of stochastic resources and then establish the SC model. Although we use a continuous-time formulation for convenience, all models can be easily transformed into a discrete-time formulation.

A. Brief Structure of ADN

Fig. 1 shows the brief structure of a radial ADN. The tree topology of ADN is described by a set of buses, denoted by \( V = \{0, 1, \ldots, N\} \), and a set of branches, denoted by \( E = \{(i, j)\} \). Moreover, a set of time is denoted by \( T \). In general, we regard \( i, j, k \) as the bus indices and \( t \) as the time index. Moreover, we set Bus 0 as the root bus.

For Bus \( i \), denoted by \( v_{i,t} \), the square voltage amplitude, and \( p_{i,t} \) and \( q_{i,t} \) are the active and reactive power injection, respectively. For Branch \((i, j)\), \( p_{ij,t} \) and \( Q_{ij,t} \) denote the active and reactive power flow, respectively, from Bus \( i \) to Bus \( j \), and \( l_{ij,t} \) denotes the square current amplitude.

The power injections at each bus include the active/reactive power of loads (denoted by \( p_{L,t} \) and \( q_{L,t} \)) and DERs. Specifically, the DERs considered in this paper include:
1) Renewable generations such as wind generations and photovoltaic generations. The active power outputs of renewable generations, denoted by \( p_{S,t} \), are highly stochastic, and assumed to be uncontrollable in this paper, i.e., no curtailment of renewable generations is allowed. However, the reactive power, denoted by \( q_{S,t} \), is considered controllable in ADN due to the fact that the grid-connected converter is able to adjust the reactive power output.
2) Energy units (EUs), including battery storage and thermostatically controlled loads. The active power of the \( i \)-th EU is denoted by \( p_{E_i,t} \). An important constraint of EUs is the state-of-charge (SOC), denoted by \( SOC_{i,t} \). In this paper, we do not consider the reactive power output of EUs.

In order to clarify the specific problem to solve in this paper, the power injections are classified into 3 categories according to their uncertainty and controllability:
1) Fixed power injections (FPIs). The active and reactive power of loads at each bus, i.e., \( p_{L,t} \) and \( q_{L,t} \) are regarded as FIs. FIs are regarded uncontrollable and assumed not to have uncertainties.
2) Stochastic power injections (SPIs). The active power outputs of renewable generations, i.e., \( p_{S,t} \), are regarded as SIs. The values of SIs contain uncertainties, i.e., only the statistical characteristics rather than the concrete values are known in the controller design procedure. Moreover, SIs are regarded uncontrollable in this paper, which means no curtailment of the active power outputs of renewable generations is allowed.
3) Controllable power injections (CPIs). The reactive power outputs of renewable generations, i.e., \( q_{S,t} \), and the outputs of EUs, i.e., \( p_{E_i,t} \), are regarded as CIs. CIs can be controlled in a real-time manner, i.e., their values can be adjusted according to the concrete values of SIs and system states.

The different kinds of power injections are also shown in Fig. 1. In practice, the uncertainties of SIs usually have negative impacts on the security of ADNs. For example, they will lead to a large volatility of voltage profiles, and may lead to voltage violations. Moreover, the uncertainties of SPIs will also increase the operating cost of ADNs. Therefore, the goal of this paper is to design the control schemes of CPIs in order to hedge the negative impacts of SPIs on ADNs.

The brief structure of the control of ADNs with DERs is shown in Fig. 2. In Fig. 2, the uncertainties of wind speed and solar irradiance lead to the uncertainties of the active power \( p_{S,t} \) of the renewable generations, which will then have negative
impacts on the distribution network. The performance of the distribution network is measured by the energy cost, denoted by \( J \). In order to minimize \( J \), the stochastic outputs of \( p_{i,t}^d \) are measured and regarded as the input of the controller, and the output of the controller is the set point of \( q_{i,t}^d \) and \( p_{i,t}^x \) in order to mitigate the We assume the internal control of DERs has been well designed. And hence, in this paper, we only consider to optimally regulate the set points of DERs from a system-level point of view. The following subsections will provide the mathematical model of the control problem in detail.

B. Itô Process Model of Stochastic Resources

Here we consider the model of stochastic resource \( p_{i,t}^d \). The active power of stochastic resources can be separated into the prediction and deviation parts, denoted by \( p_{i,t}^{pred} \) and \( p_{i,t}^{dev} \), respectively. We regard the prediction part as fixed values in the SC problem. For convenience, let \( \xi_t = (p_{i,t}^{dev})_{i \in V} \). We mainly consider the correlation of stochastic resources, which is defined as

\[
R(t, s) = \mathbb{E} \xi_t \xi_s^T
\]  

Different from existing studies [6], [10], an Itô process model of stochastic resources is used in this work, which is defined as

\[
d\xi_t = \mu(\xi_t)dt + \sigma(\xi_t)dW_t
\]  

where \( \mu(\cdot) \) and \( \sigma(\cdot) \) are the drift function and the diffusion function, respectively. Note that (2) is a stochastic differential equation (SDE); hence, we also need the initial condition describing the distribution of \( \xi_0 \) in order to fully describe the Itô process. Here, we omit it for convenience.

The Itô process is used here for two reasons:

On the one hand, the Itô process model can describe the influence of the independent stochastic variables \( \xi_t \). For example, consider the following Itô process:

\[
d\xi_t = -\frac{1}{\tau} \xi_t dt + \frac{1}{\sqrt{\tau}} \sigma dW_t
\]

where \( \mu(\xi_t) = -\xi_t/\tau, \sigma(\xi_t) = \sigma/\sqrt{\tau} \). Assume \( \tau > 0, \xi_0 = 0 \). Then the correlation matrix is

\[
R(t, s) = \exp(-\frac{t-s}{\tau}) - \exp(-\frac{t+s}{\tau}) \sigma \sigma^T, \forall s \leq t
\]

in which the correlation is the product of \( \exp(-t/\tau) - \exp(-s/\tau) \), which describes the temporal correlation, and \( \sigma \sigma^T \), which describes the spatial correlation. Moreover, Eq. (3) is just a linear example of Itô processes. By specifying other kinds of \( \mu(\xi_t) \) and \( \sigma(\xi_t) \), it is possible to use Itô process to describe more complicated stochastic processes, such as non-Gaussian uncertainties, and the readers can refer to [22]–[25] for details.

On the other hand, the Itô process, described by an SDE, is compatible with the description of EUs (see (7)), which is an ordinary differential equation. This property makes it possible to embed the stochastic characteristics into the optimization problem without Monte Carlo simulations. In contrast, existing models need to be broken into a number of scenarios to be used in SC problems, which leads to unbearable computational burden. We will discuss the integration of Itô processes and the power system model thoroughly in Section III-B.

In the remainder of this paper, we regard \( \mu(\cdot) \) and \( \sigma(\cdot) \) as a priori knowledge obtained by historical data.

C. Stochastic Control Problem

The SC problem is an intraday control problem, aiming at decreasing the cost of electricity in ADNs under DER uncertainties via the control of CPIs. Specifically, the SC problem can be formulated as (5)–(9):

\[
\min_{u_{i,j} \in T} J = \mathbb{E}_{\xi_t} \left\{ \int_{i \in T} (\lambda_i p_{0i,t} + u_i R^T u_i)dt \right\}
\]

\[
+ \mathbb{E}_{\xi_t} \left\{ R^T \sum_i (SOC_{i,T} - SOC_{i,\text{end}})^2 \right\}
\]

Stochastic Resources and Control Variables:

\[
p_{i,t}^d = p_{i,t}^{pred} + p_{i,t}^{dev}, \forall i \in V, t \in T
\]

\[
\xi_t = (p_{i,t}^{dev})_{i \in V}, \forall t \in T
\]

\[
u_t = (p_{i,t}^{pred}, q_{i,t}^{dev})_{i \in V}, \forall t \in T
\]

EUs:

\[
\frac{d}{dt} SOC_{i,t} = -\alpha^E SOC_{i,t} + \beta^C p_{i,t}^C - \beta^D p_{i,t}^D, \forall i \in V, t \in T
\]

Power and Network Constraints:

\[
p_{i,t} = p_{i,t}^d - p_{i,t}^L + p_{i,t}^C, \forall i \in V, t \in T
\]

\[
q_{i,t} = q_{i,t}^S - q_{i,t}^L, \forall i \in V, t \in T
\]

\[
p_{i,t} = \sum_{j,i \rightarrow j} p_{j,t}^C - \sum_{k,k \rightarrow i} (p_{k,i}^C - q_{k,i}^L), \forall i \in V
\]

\[
q_{i,t} = \sum_{j,i \rightarrow j} Q_{j,t}^q - \sum_{k,k \rightarrow i} (Q_{k,i}^q - x_{k,i} Q_{k,i}^q), \forall i \in V
\]

\[
\nu_{i,t} = \nu_{i,t} - 2(\tau_i p_{i,t}^q + x_i Q_{i,t}^q)
\]
\[ + \left( r_i^2 + x_i^2 \right) I_{ij,t}, \forall (i,j) \in \mathcal{E} \]

\[ I_{ij,t} \cdot L_{ij,t}, \forall (i,j) \in \mathcal{E} \] (8e)

Inequality Constraints (with the confidence level \( \gamma \)):

\[ \left( p_{ij,t}^S \right)^2 + \left( q_{ij,t}^S \right)^2 \leq \left( s_{ij,t}^S \right)^2, \forall i \in \mathcal{V}, t \in \mathcal{T} \] (9a)

\[ \left| d_{ij,t} \right| \leq \tan(\varphi_i) p_{ij,t}^S \] (9b)

\[ \text{SOC}_i \leq \text{SOC}_{i,t} \leq \text{SOC}_{i}, \forall i \in \mathcal{V}, t \in \mathcal{T} \] (9c)

\[ 0 \leq p_{ij,t}^C \leq p_{ij,t}^C, 0 \leq p_{ij,t}^D \leq p_{ij,t}^D, \forall i \in \mathcal{V}, t \in \mathcal{T} \] (9d)

\[ \forall (i,j) \in \mathcal{E}, t \in \mathcal{T} \] (9e)

\[ 0 \leq l_{ij,t} \leq l_{ij,t}, \forall (i,j) \in \mathcal{E}, t \in \mathcal{T} \] (9f)

and the constraints are classified into the following two groups: the equality constraints, including (7), (8), and the inequality constraints, including (9a)–(9f). In consistence with the papers of stochastic control [26], [27], we regard equality constraints as almost sure constraints, i.e., each constraint is satisfied with probability 1; and regard inequality constraints as chance constraints, i.e., each inequality is satisfied with probability \( \gamma \) (0 < \( \gamma \) < 1).

These equations are explained in detail below.

1) Objective: The objective (5) of the SC problem is to minimize the expected energy cost. Specifically, the objective function contains 3 parts:

- The price of electricity bought from the market, i.e., \( \int_{t \in \mathcal{T}} \lambda_i P_{01,t} dt \), where \( \lambda_i \) is the electricity price.
- The cost of control, i.e., \( \int_{t \in \mathcal{T}} u_t^T R u_t dt \).
- The penalty of SOC \( R^S \sum_t (\text{SOC}_{i,t} - \text{SOC}_{i,0})^2 \).

Here we provide more explanations on the penalty term of SOC. Note that the penalty term is not used to prevent the SOC from violating the upper and lower limit. Actually, the upper and lower limit has been considered in the constraints and the objective function contains 3 parts:

2) Energy Units: EUs are units with energy constraints, including battery storage systems, thermostatically controlled loads, etc. (7) describes the dynamics of EUs, where \( \alpha_i^E \) is the dissipation factor, describing the speed of energy dissipating. In battery storage systems, \( \alpha_i^E \) is usually very small, and can usually be neglected [28], but in other kinds of EUs, such as thermostatically controlled loads, the effect of the dissipation factor is usually non-negligible [30]. \( \beta_i^C \) is the charging efficiency, and \( \beta_i^D \) is the discharging efficiency. EUs are units with energy constraints, including battery storage systems, demand response, etc. (7) describes the dynamics of EUs, where \( \alpha_i^E \) is the dissipation factor, and \( \beta_i^E \) is the charging efficiency.

3) Power and Network Constraints: Equations (8a) and (8b) are the power balance equations at each bus. Here we adopt the distFlow model [31] of network constraints, as shown in (8c)–(8f), where \( r_{ij} \) and \( x_{ij} \) are the resistance and reactance of the branch from Bus \( i \) to Bus \( j \), respectively; \( g_i \) and \( b_j \) are the shunt conductance and susceptance at Bus \( i \) respectively.

4) Inequality Constraints: Equation (9a) is the capacity limit of the renewable generations, and (9b) is the minimum power factor limit [13], where \( \varphi_i \) is the critical phase angle. According to [16], this convex constraint can be approximately described by polygons, i.e.,

\[ C_i^T \left[ p_{ij,t}^S, q_{ij,t}^S \right]^T \leq D_i^S, \forall i \in \mathcal{V}, t \in \mathcal{T} \] (10)

Equations (9c) and (9d) are the energy and power constraints of the EUs, respectively; (9e) and (9f) are the voltage and current constraints, respectively. These constraints also describes the interactions between the studied intra-day control problem in this paper and the lower-level control problems, such as voltage control and EU control.

D. Control Policy

It is clear that the values of CPIs, i.e., \( u_t \) should be different when the values of SPIs, i.e., \( \xi_t \) are different. However, due to the prediction error, the power output of the future renewable generations cannot be accurately estimated, which is the main reason why it is more challenging to design control schemes for controllable DERs when considering these uncertainties.

Feedback control policies are needed to handle such a problem [26]. Specifically, the values of \( \xi_t \) are obtained in a real-time manner, and \( u_t \) are determined according to \( \xi_t \) in order to hedge the negative impacts of uncertainties. The specific problem to solve in this paper is the control policy from \( \xi_t \) to \( u_t \).

Briefly speaking, there are two kinds of control policies: state-feedback control policies and disturbance feedback control policies. Under certain circumstances, they are equivalent [26]. Here, we adopt the affine disturbance feedback control policy, i.e.,

\[ u_t = u_t^0 + K \xi_t \] (11)

where \( u_t^0 \) is the base output, and \( K \xi_t \) is the correction of \( u_t \) based on the actual \( \xi_t \). \( u_t^0 \) and \( K \) are the decision variables. Note that we do not know the concrete value of \( \xi_t \) when computing the control scheme, since the computation is prior to the application of the control scheme. Therefore, the \( u_t^0 \) and \( K \) is computed based on the SC model in Section II-C, which contains the stochastic characteristics rather than the concrete values of \( \xi_t \). Once \( u_t^0 \) and \( K \) are obtained, they can be used to compute \( u_t \) according to \( \xi_t \) in a real-time manner.

There are also other approaches to obtaining the control scheme in Eq. (11), such as SPBC [9], [13], [18], but the extremely large computational burden prevents them from large-scale applications. One of the main contribution of this paper is to provide a computational efficient method to
obtain the $u_i^0$ and $K$, which will be discussed in Section III thoroughly.

E. Compact Reformulation

For convenience, we use a group of vectors to represent the abovementioned variables and transform the equations into a compact form.

1) Groups of Variables: We have defined $\xi_i$ and $u_i$ in (6b) and (6c). Now we define the following vectors, all of which are formulated as column vectors:

\[
e_i = (SOC_{i,t}^E \ | i \in \mathcal{V})
\]
\[
x_i = (P_{ij,t}^d \ | (i,j) \in \mathcal{E}, \ Q_{ij,t}^d \ | (i,j) \in \mathcal{E}, \ v_{i,t} \ | i \in \mathcal{V})
\]
\[
y_i = (\xi_{i,t} \ | (i,j) \in \mathcal{E})
\]
\[
d_i = (p_{i,t}^{pred}, q_{i,t}^{pred}) \ | i \in \mathcal{V}
\]

where $e_i$ is the vector of the energy of EUs; $x_i$ is the vector of network states except the branch currents, and $y_i$ is the vector of branch currents; and $d_i$ is the vector of fixed values, including prediction of renewables, load profile, and the reactive power supply of each bus.

2) Compact Form of SC: By the abovementioned vectors, it is possible to reformulate the SC model as

\[
\min_{u^0, K} \ J = \mathbb{E}[\xi] \left\{ \int_{t=0}^{T} (Hx_i + x_i^T R^x x_i) dt \right\} + \mathbb{E}[\xi] \left\{ \int_{t=0}^{T} u_i^T R^u u_i dt + e_i^T R^e e_i \right\}
\]

s.t.

\[
d\xi_i = \mu(\xi_i) dt + \sigma(\xi_i) dW_i
\]
\[
u_i = u_i^0 + K_\xi_i
\]
\[
\dot{\xi}_i = -a\xi_i + \beta u_i
\]
\[
x_i = A \xi_i + q_{0,i} + A_d d_t + A_e e_i + A_u u_i
\]
\[
l_{ij,vi,t} = p_{ij,t}^d + Q_{ij,t}^d \ \forall (i,j) \in \mathcal{E}
\]
\[
x \preceq x_i \preceq \hat{x}
\]
\[
y \preceq y_i \preceq \hat{y}
\]
\[
w \preceq e_i \preceq \hat{e}
\]
\[
C_a u_i + C_\xi \xi_i \preceq D
\]

where (13a) is corresponding to (5); (13c) is the disturbance feedback control policy; (13d) corresponds to (7); (13e) corresponds to (8); and (13g)–(13j) correspond to (9). We provide more explanations to (13e) and (13f) in Appendix A, and the other equations can be directly obtained from the corresponding scalar equations.

Equation (13) is the control model used for analysis in this paper. Similar models are also studied in [15]–[17], except for the model of stochastic resources (13b). Existing approaches usually use SPBC to solve the optimal control problem with stochastic resources. However, when considering spatial and temporal correlations, it is necessary to use a large number of scenarios to guarantee the accuracy, which makes the optimization undoubtedly time-consuming. Based on this, Section III provides the MO approach to efficiently solve (13).

III. Solution Based on Moment Optimization

This section provides the MO approach to solving the SC problem. The moments of a stochastic variable is the expectation of certain functions of the variable. Specifically, the first-order moment is the expectation, and the second-order central moment is the variance. The basic idea of MO is based on the fact that the objective in (13a) only contains quadratic forms. Therefore, the objective can be equivalently transformed into the function of the first-order and second-order moments of these variables. Furthermore, we can regard these first-order and second-order moments as decision variables, and all we have to do is find the constraints of these moments. In other words, higher-order moments will not influence the solution of SC. By transforming the SC problem (13) into a deterministic optimization problem with respect to the first-order and second-order moments, the MO approach largely reduces the computational burden of SC compared to traditional scenario-based approaches, without sacrificing performance.

We define the first two notations. Assume that $a$ is a certain component of $x_i, y_i, z_i, u_i, e_i$ and denote by $\hat{a}$ the expectation of $a$ and $\tilde{a}$ the standard deviation of $a$, i.e.,

\[
\bar{a} = \mathbb{E}[a]
\]
\[
\tilde{a} = \sqrt{\text{var}(a)} = \sqrt{\mathbb{E}[a^2] - (\mathbb{E}[a])^2}
\]

With these notations, we can define the first-order moments $\hat{x}_i, \hat{y}_i, \hat{z}_i, \hat{u}_i, \hat{e}_i$, and the second-order central moments $\tilde{x}_i, \tilde{y}_i, \tilde{z}_i, \tilde{u}_i, \tilde{e}_i$. Moreover, we will need the notation

\[
\Delta a = a - \hat{a}
\]

which means that $\text{var}(a) = \mathbb{E}[(\Delta a)^2]$.

Based on these notations, we now provide the MO approach for solving (13). The objectives and constraints of MO are listed in Table I, in which we also show the relationship between the original equations in (13) and the corresponding equations in MO. The remainder of this section explains the MO method.

A. Reformulating Objective in (13a)

According to the fact that $R^u$ is diagonal, we have

\[
\mathbb{E} \left\{ u_i^T R^u u_i \right\} = \hat{u}_i^T R^u \hat{u}_i + \tilde{u}_i^T R^u \tilde{u}_i
\]
and the same discussion can be applied to \( \mathbb{E}(\varepsilon^T R^x e_T) \) and \( \mathbb{E}(x^T R^x x_T) \). Therefore, we have

\[
J = \int_{t \in T} \left( H\tilde{x}_i + \tilde{x}_i^T R^x \tilde{x}_i + \tilde{x}_i^T R^y \tilde{y}_i \right) dt + \int_{t \in T} \left( \tilde{u}_i^T R^u \tilde{u}_i + \tilde{u}_i^T R^y \tilde{u}_i \right) dt
+ \tilde{e}_i^T R^e \tilde{e}_i + \tilde{e}_i^T R^{eT} \tilde{e}_i.
\]

(17)

B. Reformulating Stochastic Resources in (13b)

Here, we must consider the spatial correlation and the temporal correlation, the former of which is described by the covariance matrix, while the latter of which must be considered together with the temporal correlation of EUs in (13d). To address the temporal correlation, we define an auxiliary vector as follows:

\[
d\eta_i = (-\alpha \eta_i + \beta \xi_i) dt
\]

(18)

We assume that \( \eta_0 = 0 \). Using the notation that \( \dot{\eta}_i = d\eta_i/dt \), we can also rewrite Eq. (18) as \( \dot{\eta}_i = -\alpha \eta_i + \beta \xi_i \). Note that (18) is different from (13d) because it is independent of the decision variable \( u_i^0 \) and \( K \); however, the next subsection shows that the statistics of \( e \) are determined by the statistics of \( \eta \).

We can rewrite (2) and (18) as

\[
d\begin{bmatrix} \xi_i \\ \eta_i \end{bmatrix} =
\begin{bmatrix}
\mu(\xi_i) \\
-\alpha \eta_i + \beta \xi_i
\end{bmatrix} dt +
\begin{bmatrix}
\sigma(\xi_i) \\
0
\end{bmatrix} dW_t
\]

(19)

which is also an Itô process. The statistics needed here include the expectation, defined by \( \mathbb{E}(\xi_i) \) (note that \( \mathbb{E}(\eta_i) = 0 \)), and the covariance matrix defined by

\[
\mathcal{M}_t = \mathbb{E}\left[\begin{bmatrix} \xi_i - \mathbb{E}(\xi_i) \\ \eta_i - \mathbb{E}(\eta_i) \end{bmatrix} \begin{bmatrix} \xi_i - \mathbb{E}(\xi_i) \\ \eta_i - \mathbb{E}(\eta_i) \end{bmatrix}^T\right] = \begin{bmatrix} \mathcal{M}_{\xi\xi}^t & \mathcal{M}_{\xi\eta}^t \\ \mathcal{M}_{\eta\xi}^t & \mathcal{M}_{\eta\eta}^t \end{bmatrix}
\]

(20)

where \( \mathcal{M}_{\xi\xi}^t = (\mathcal{M}_{\eta\eta}^t)^T \). In (20), \( \mathcal{M}_{\xi\xi}^t \) describes the temporal correlation of \( \xi_i \), while the other parts describe the temporal correlation that are necessary in MO.

A simple approach to obtaining \( \tilde{\xi}_i \) and \( \mathcal{M}_t \) is the simulation approach, which is time-consuming. However, it is shown in [32] that the statistics of Itô processes can be efficiently computed by series expansion. Note that \( \tilde{\xi}_i \) and \( \mathcal{M}_t \) are independent of decision variables \( u_i^0 \) and \( K \); hence, we assume they are given in the following subsections.

C. Reformulating Control Policy (13c) and EUs (13d)

By taking the first-order moment in (13c), we have

\[
\tilde{u}_i = u_i^0 + K \tilde{\xi}_i
\]

(21)

\[
\Delta u_i = K \Delta \tilde{\xi}_i
\]

(22)

Therefore, we have

\[
\mathbb{E}(\Delta u_i \Delta u_i^T) = K \mathcal{M}_{\xi\xi}^t K^T
\]

(23)

Moreover, \( \tilde{u}_i \) is the square root of the diagonal of \( \mathbb{E}(\Delta u_i \Delta u_i^T) \):

\[
\tilde{u}_i = \sqrt{\text{diag}\left\{ K \mathcal{M}_{\xi\xi}^t K^T \right\}}
\]

(24)

Now, we discuss the moment-form of \( e \). It is easy to show that \( \tilde{e}_i \) and \( \Delta e_i \) satisfy

\[
\dot{\tilde{e}}_i = -\alpha \tilde{e}_i + \beta \tilde{u}_i
\]

(25)

\[
\Delta \tilde{e}_i = -\alpha \Delta e_i + \beta K \Delta \tilde{\xi}_i
\]

(26)

Then, it is clear that \( \Delta e_i = K \eta_i \); therefore, we have

\[
\hat{e}_i = \sqrt{\text{diag}\{K \mathcal{M}_{\eta\eta}^t K^T\}}
\]

(27)

D. Reformulating Network Constraints

The network constraints include the linear constraints (13e) and the quadratic constraints (13f), the first-order moments of which are

\[
\tilde{x}_i = A_x \tilde{y}_i + A_z \tilde{z}_i + A_d d_i + A_e e_i + A_u \tilde{u}_i
\]

(28)

\[
\tilde{l}_{ij} \tilde{v}_{i,t} + \text{cov}(l_{ij,t}, v_{i,t}) = \tilde{p}_{ij,t}^2 + \tilde{q}_{ij,t}^2 + \tilde{p}_{ij,t}^2 + \tilde{q}_{ij,t}^2, \forall (i, j) \in \mathcal{E}
\]

(29)

To obtain a convex version of (29), we claim that \( \text{cov}(l_{ij,t}, v_{i,t}) \) can be ignored. Actually, we have

\[
\text{cov}(l_{ij,t}, v_{i,t}) \lesssim \tilde{p}_{ij,t}^2 + \tilde{q}_{ij,t}^2
\]

(30)

of which the explanation is provided in Appendix B. Thus, (29) can be replaced by

\[
\tilde{l}_{ij} \tilde{v}_{i,t} = \tilde{p}_{ij,t}^2 + \tilde{q}_{ij,t}^2 + \tilde{p}_{ij,t}^2 + \tilde{q}_{ij,t}^2, \forall (i, j) \in \mathcal{E}
\]

(31)

The second-order moment of \( x_i \) satisfies

\[
\hat{x}_i = \sqrt{\text{diag}\{L \mathcal{M}_i L^T\}}
\]

(32)

where \( L = [A_x + A_u K, A_e K] \). The derivation of (32) is provided in Appendix C.

E. Reformulating Inequality Constraints

It is shown in [33] that a chance constraint can be approximately described by a second-order cone constraint. In this approach, the constraints in (13g)~(13j) need to be handled row by row. For simplicity, we assume that \( a \) is a variable and take \( a \leq \bar{a} \) as an example.

According to [33], the second-order-cone formulation of the constraint is

\[
a + \kappa \gamma \leq \bar{a}
\]

(33)

Therefore, the inequality constraints can be transformed into

\[
x + \kappa \gamma \tilde{x}_i \leq \bar{x}_i - \kappa \gamma \tilde{x}_i
\]

\[
y + \kappa \gamma \tilde{y}_i \leq \bar{y}_i - \kappa \gamma \tilde{y}_i
\]

\[
\tilde{e} + \kappa \gamma \tilde{e}_i \leq \bar{e} - \kappa \gamma \tilde{e}_i
\]

\[
C_u \tilde{u}_i + C_e \tilde{e}_i + \kappa \gamma \left( |C_u | \hat{u}_i + |C_e | \hat{e}_i \right) \leq D.
\]

(34)

\footnote{Such statement needs the assumption that \( a K = K a \) and \( \beta K = K \beta \). However, this requirement is easy to meet if we convert \( a \beta K \) to (larger-order) block-diagonal matrices.}
F. Summary

The MO approach can be summarized as

Objective: (17)
Constraints: (20), (21), (24), (25), (27), (28), (31), (32), (34)
(35)

Various nonconvex constraints exist, i.e., (24), (27), (31), (32). However, it is easy to obtain their exact convex relaxations (see Appendix D). Therefore, (35) is a convex optimization problem.

Now, we discuss the computational burden of MO. It is clear that for a certain variable, say \( a \), in (13), there are two corresponding variables in MO, i.e., \( \tilde{a} \) and \( \hat{a} \) respectively. Therefore, the number of variables of MO is approximately twice that of the original SC problem. Moreover, each constraint in (13) corresponds with one or two constraints in MO, regarding the first-order and second-order moments. Therefore, the number of constraints of MO is less than twice that of the original SC problem. In contrast, the scale of traditional SPBC algorithms is proportional to the number of scenarios, which is usually large for an accurate estimation of the SC problem with spatial and temporal correlation. In summary, the MO approach reduces the computational burden of SC to be comparable with the corresponding deterministic control problem.

IV. CASE STUDY

This section provides two test case in an IEEE 123-bus distribution network [34] and the PNNL Taxonomy Feeder [35]. We evaluate the optimal control scheme provided by the MO approach and then discuss the impacts of spatial and temporal correlation. Moreover, the comparison between the proposed MO approach and several existing approaches shows the effectiveness and efficiency of the MO approach.

A. Case A: IEEE 123-Bus System

1) Case Settings: We consider the IEEE 123-bus system, as shown in Fig. 3. The detailed parameters of the IEEE 123-bus system can be found in [34]. The nominal capacity of the system is 10 MVA, and the nominal voltage is 10 kV. We assume that the voltage limit of each bus is 10 ± 0.5 kV. The stochastic resources are 3 wind generators on Buses 11, 62, and 66, and 3 PV generators on Buses 11, 62, and 66, the capacity of each of which is 20 MVA, and 3 PV generators on Buses 72, 75, and 114, the capacity of each of which is 10 MVA. We use Eq. (3) to describe the stochastic resources. We assume that the predicted values are obtained by persistent prediction in 1 h [6], and the parameters are obtained via the parameter estimation method provided in [22]. The spatial correlation is regarded as a priori information in most of the researches on stochastic control [10], [27] as well as this paper, and we simply assume that there exists a positive correlation between Buses 62 and 66 and a positive correlation between Buses 72 and 75, of which the parameters are provided in Appendix E. Moreover, there is an EU at Bus 62 (5 MW × 4 h). The readers can refer to Appendix E for detailed parameters. All the simulations are conducted in MATLAB on a core-TM i7-7700 CPU with the frequency 3.6 GHz.

Fig. 3. IEEE 123-bus system.

The objective of MO is as shown in (5), where the price is $1/kWh from 08:00~20:00, and $0.5/kWh during the rest of the day. \( R \) is a diagonal matrix whose diagonal elements are all 1, and \( R^F = 0.1 \). The time step is 15 minutes; the control horizon is 1 day; and the \( u_i^t \) and \( K \) are updated every 4 hours in order to maintain a good performance.

2) Simulation Results: To evaluate the control scheme obtained by the MO approach, we use a Monte Carlo simulation with 1000 scenarios to calculate the objective function under the control scheme. Each scenario is performed for one day, which is a typical time scale of the intra-day power system control and operation problems [28], [36], [37]. The objective value under the optimal control scheme is \( J = 249 \) k$. In contrast, if we let \( K = 0 \) and only consider \( u_i^t \), the result is \( J' = 272 \) k$. In fact, \( J' \) is the objective under deterministic control schemes, which means that the controlled units do not respond to any disturbances of stochastic resources. The results show that stochastic control scheme performs better than deterministic control schemes.

The value of \( K \) shows the relationship between \( u_i \) and \( \xi_j \).

In this case, we have

\[
K = \begin{bmatrix}
-0.068 & -0.087 & -0.096 & -0.001 & -0.002 & -0.001 \\
-0.087 & -0.184 & -0.195 & -0.003 & -0.002 & -0.003 \\
-0.096 & -0.916 & -0.421 & -0.003 & -0.004 & -0.002 \\
-0.001 & -0.003 & -0.002 & -0.085 & -0.069 & -0.024 \\
-0.002 & -0.002 & -0.004 & -0.069 & -0.096 & -0.025 \\
-0.001 & -0.003 & -0.002 & -0.024 & -0.025 & -0.043 \\
-0.102 & -0.203 & -0.184 & -0.014 & -0.017 & -0.006 
\end{bmatrix}
\]

where the order of the columns is the output of stochastic resources at Buses 11, 62, 66, 72, 75, and 114, and the order of the rows is the reactive power of stochastic resources and the output of the EU. It is clear that the control scheme is a negative feedback control scheme. Moreover, the values of \( K \) show the correlation between these variables, and units at closer buses share larger coefficients. For example, the Row 7, Column 2 of \( K \) describing the sensitivity of the EU output with respect to the DG at Bus 62, is relatively larger.

Fig. 4 shows the curves at Bus 62 in a certain scenario. The negative feedback control scheme is shown by Fig. 4(a)(b)(c), where lower wind power leads to larger control output. Fig. 4(d) shows the effect of the feedback control,
Fig. 4. Simulation results at Bus 62. (a) power of wind generations; (b) reactive power of wind generations; (c) power of EU; (d) voltage of Bus 62.

where the black curve is the voltage profile under perfect prediction, the red curve is the voltage profile under uncertainty, but with feedback coefficient $K = 0$; and the blue curve is the voltage profile under the optimal feedback control. It is clear that feedback control improves the voltage profile at Bus 62.

3) Impacts of Correlation: Here, we discuss the relationship between the correlation and the control performance. For the spatial correlation, we consider $\sigma' = \text{diag}(2.98, 7.52, 4.51, 1.42, 3.75, 2.76)$. It is clear that the diffusion coefficients $\sigma$ and $\sigma'$ result in the same variance of $\xi_t$, but the stochastic resources under $\sigma'$ are spatially independent. Moreover, we change $\tau$ to obtain different temporal correlations. Fig. 5 shows the objective under different spatial and temporal correlation.

Fig. 5. Impacts of correlation.

It is shown that larger (spatial or temporal) correlation leads to worse control performance, and we now explain this result. Some of the variables, such as $e_t$, are related to the integration or sum of elements in $\xi_t$. However, the uncertainty of the sum of stochastic variables is influenced not only by the variance of each variable but also by the correlation. Moreover, the correlation in this case is positive and hence will lead to larger uncertainty and a larger objective value. As a simple example, we consider two stochastic variables $\xi_1$ and $\xi_2$, respectively. The variance of the total output is:

$$\text{var}(\xi_1 + \xi_2) = \text{var}(\xi_1) + \text{var}(\xi_2) + 2\text{cov}(\xi_1, \xi_2)$$

When there is no correlation between $\xi_1$ and $\xi_2$, we have

$$\text{var}(\xi_1 + \xi_2) = \text{var}(\xi_1) + \text{var}(\xi_2)$$

However, when there is a positive correlation between $\xi_1$ and $\xi_2$, we have

$$\text{var}(\xi_1 + \xi_2) = \text{var}(\xi_1) + \text{var}(\xi_2) + 2\text{cov}(\xi_1, \xi_2) > \text{var}(\xi_1) + \text{var}(\xi_2)$$

And it is clear that the variance of the total output is larger when there exists a positive correlation.

Fig. 6 shows the $u_0^t$ (taking the EU as an example) under different spatial and temporal correlations. It is also clearly shown that larger spatial and temporal correlation results in smaller controller output, since the controller must reserve more capacity for uncertainty. The impacts of spatial and temporal correlation can be considered in the proposed approach.

B. Case B: PNNL Taxonomy Test Feeder

In this case, we consider a relatively larger system provided by the PNNL. The PNNL taxonomy test feeders are the representatives of the U.S. continental area. It was developed by a clustering algorithm of 575 actual distribution feeders from 17 different utilities. PNNL provides 24 taxonomy radial distribution test feeders, representing different load types and topologies. Here we use the “R5-12.47-2” test feeder, which contains 311 nodes. The nominal voltage is 12.47 kV. The detailed parameters can be referred to in [35], [38]. There are 3 wind generators on Buses 65, 89, and 236, the capacity of each of which is 20 MVA, and 3 PV generators on Buses 23, 127, and 193, the capacity of each of which is 10 MVA. There are EUs on Buses 89 and 193, the capacity of each of which is 5 MW $\times$ 4 h. Other settings such as Itô coefficients and constants are the same as those in Case A (see Appendix E).

In this relatively larger case, we validate the performance and the computational efficiency of the proposed approach. Here, we use the deterministic control (DC), MPC, and SPBC as benchmarks. In the DC approach, the control scheme is obtained based on the predicted values of stochastic resources, and the uncertainties are ignored. MPC performs DC in a receding-horizon manner, and update the prediction value at each time step. The prediction horizon of MPC is 4 hours.

Fig. 6. Control schemes under different correlation.
We perform the SPBC approach with 100 scenarios and 1000 scenarios, denoted by SPBC(20) and SPBC(100). These scenarios are obtained by the scenario-reduction method provided in [39]. We compare the performance and the computational burden of these control methods.

1) Performance of Different Methods: Table II shows the performance of different methods, demonstrating that MO and SPBC(100) perform best and that DC performs the worst. DC does not consider the uncertainty of the stochastic resources and hence achieves the worst performance, which is also supported by Fig. 4. MO and SPBC both consider the uncertainty explicitly and perform well. Moreover, SPBC(20) does not perform as well as MO because 20 scenarios are too few to describe the correlation of the stochastic resources. Although MPC does not explicitly consider the uncertainty, the receding-horizon manner improves its performance. Nevertheless, MPC does not perform as well as MO.

2) Computational Burden: Table II shows the computational burden of these methods on a core-TM i7-7700 CPU with the frequency 3.6 GHz. Since MPC is a receding-horizon control method, while the others are not, we use the per-step computational time for a fair comparison. It is clear that DC is the fastest because it does not consider the uncertainty. The computational time of MO is about twice that of DC, significantly smaller than MPC, SPBC(20) and SPBC(100), which shows the advantage of the proposed method over existing methods. Specifically, since MO and SPBC(100) achieve similar performance, it can be concluded that MO reduces the computational time by 98.1% without sacrificing performance.

In summary, the proposed MO approach achieves a good trade-off between the control performance and the computational burden. In contrast, DC is computationally efficient but performs worse, while MPC/SPBC performs well but incurs an extremely large computational burden. Therefore, the proposed MO significantly outperforms the existing methods and has attractive potential in the control of DERs under uncertainty.

V. CONCLUSION

This paper presents an MO approach for the efficient control of DERs in distribution networks. We first model the stochastic resources by Itô processes, which describe the spatial and temporal correlation of the stochastic resources. The Itô process model is also in the same form as the characteristics of EUs; hence, the temporal correlation of the stochastic resources and EUs can be considered in a unified way. Based on the covariance matrix obtained by the Itô process model, we transform the SC problem into a deterministic optimization problem with respect to the first-order and second-order moments of the system variables, whose scale is approximately twice that of the corresponding deterministic control problem. The proposed MO approach solves the SC problem in a computationally efficient way and outperforms existing approaches such as DC, MPC, and SPBC.

APPENDIX A

EXPLANATION OF INEQUALITY (30)

Consider the first-order deviation of (13f), and ignore higher-order deviations:

\[ \tilde{l}_{ij,t} \Delta v_{i,t} + \tilde{v}_{i,t} \Delta q_{i,t} = 2 \tilde{P}_{ij,t} \Delta P_{ij,t} + 2 \tilde{Q}_{ij,t} \Delta Q_{ij,t} \]

(37)

Therefore,

\[ \tilde{v}_{i,t} \Delta q_{i,t} = 2 \tilde{P}_{ij,t} \Delta P_{ij,t} + 2 \tilde{Q}_{ij,t} \Delta Q_{ij,t} - \tilde{l}_{ij,t} (\Delta v_{i,t})^2 \]

(38)

By taking expectations in both sides, we have

\[ \text{cov}(l_{ij,t}, v_{i,t}) = 2 \tilde{P}_{ij,t} \text{cov}(P_{ij,t}, v_{i,t}) + 2 \tilde{Q}_{ij,t} \text{cov}(Q_{ij,t}, v_{i,t}) - \tilde{l}_{ij,t} v_{i,t}^2 \]

(39)

By applying Cauchy inequality and using the fact that \( \tilde{l}_{ij,t} / v_{i,t} > 0 \), we have

\[ \text{cov}(l_{ij,t}, v_{i,t}) \leq 2 \frac{\tilde{v}_{i,t}}{P_{ij,t}} \tilde{P}_{ij,t}^2 + 2 \frac{v_{i,t}}{Q_{ij,t}} \tilde{Q}_{ij,t}^2 \]

(40)

The term \( \tilde{v}_{i,t} / P_{ij,t} \) can be interpreted as the sensitivity of the voltage deviation \( v_{i,t} \) under the power deviation \( P_{ij,t} \), which, in practice, is very small since the relative deviation of the bus voltage is far less than that of the branch power. Therefore, we have \( \text{cov}(l_{ij,t}, v_{i,t}) \ll \tilde{P}_{ij,t}^2 + \tilde{Q}_{ij,t}^2 \).

APPENDIX C

SECOND-ORDER MOMENT OF NETWORK CONSTRAINTS

A major challenge to obtain the second-order-moment form of network constraints is how to avoid higher-order moments. To achieve this, we consider a typical approximation form of the distFlow model [40]:

\[ p_{j,t} = p_{jk,t} + \sum_{k \neq j} (p_{kj,t} + g_j v_{j,t}) \]

\( \forall j \)
\[ q_{j,t} = Q_{j,k,t} - \sum_{i \in j} (Q_{i,j}) + b j v_{j,t}, \forall j \]
\[ v_{j,t} = v_{j,t} - 2 (r_{j} P_{j,t} + x_{j} Q_{j,t}), \forall (i,j) \in \mathcal{E} \]  
(41)

This model assumes negligible line losses and almost flat voltage, and its accuracy has been verified by several recent work [4], [41].

It must be emphasized that the approximated model (41) is only used to estimate the second-order moments, while the first-order moments in (28) and (31) are computed by the exact model shown in (13e) and (13f). In other words, we use an accurate model to estimate the expectations and an approximate model to estimate the errors. Since expectations are usually more important in the objective of stochastic optimization problems, the use of the approximation will have acceptable impacts on the accuracy of the model.

It is clear that the vector form of (41) is
\[ x_t = A \xi_{t} + A d_t + A e_t + A u_t \]  
(42)

By replacing \( e_t \) and \( u_t \) by \( \xi_t \) and \( \eta_t \), we have
\[ \Delta x_t = (A \xi + A \eta K) \Delta \xi_t + A \eta \eta_t \]  
(43)

then (32) clearly follows.

**APPENDIX D**

**Exact Convex Relaxation of MO**

The exact convex relaxations of (24), (27), (32) are
\[ \hat{u}_t \geq \sqrt{\text{diag}\{K M_t^\xi K^T\}} \]
\[ \hat{e}_t \geq \sqrt{\text{diag}\{K M_t^{\eta \eta} K^T\}} \]
\[ \hat{\xi}_t \geq \sqrt{\text{diag}\{L M_t L^T\}} \]  
(44)

Here we only explain the first equation as an example. Since \( M_t^\xi \) is symmetric, we assume \( M_t^\xi = N_t N_t^T \). Considering the \( i \)-th coordinate of \( u_t \), denoted by \( u_t,i \), we have
\[ u_t,i = \sqrt{\{K_i M_i^\xi K_i^T\}} = \| K_i N_i \|_2 \]  
(45)

where \( K_i \) is the \( i \)-th row of \( K \). Clearly, this equation can be exactly relaxed as
\[ u_t,i \geq \sqrt{\{K_i M_i^\xi K_i^T\}} = \| K_i N_i \|_2 \]  
(46)

which is a second-order-cone constraint. And (44) can be obtained in a similar way.

Moreover, the exact convex relaxation of (31) is
\[ \tilde{l}_{i,j,t} + \tilde{v}_{i,j,t} \geq \begin{bmatrix} 2\tilde{P}_{i,j,t} \\ 2\tilde{Q}_{i,j,t} \\ 2\tilde{P}_{i,j,t} \\ 2\tilde{Q}_{i,j,t} \\ \| \tilde{l}_{i,j,t} - \tilde{v}_{i,j,t} \|_2 \end{bmatrix}, \forall (i,j) \in \mathcal{E} \]  
(47)

where \( \| \cdot \|_2 \) is the operator of 2-norm. This relaxation technique is widely used in the convex relaxation of distFlow [31].

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