Block finite-element approach to building refined models of layer-by-layer analysis of the stress-strain state of three-layer irregular shells

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Abstract. This work offers a block finite element approach to building refined models of layer-by-layer analysis of stress-strain state of three-layer generally irregular cylindrical shells.

1. Introduction
Three-layer shells are being increasingly used for rocket and space, aircraft and other modern equipment structures \[1, 2\]. The application of three-layer shells is constrained among others by insufficiently developed models dedicated to various exposures to individual layers and real conditions of layers fixing.

The building of models and finite element calculations of heterogeneous sandwich and three-layer shells including those made of composites, are described in works \[3, 4, 5\]. Three-layer cylindrical shells involve application of broken line hypothesis \[3, 5\]. Sandwich shells in most cases involve Timoshenko layer packet hypothesis \[4\]. If the layer packet hypothesis is applied, the number of degrees of freedom a finite element model does not depend on the number of layers. However, such approach restricts the range of loadings and layer fixing conditions as well as results in simplified models or models for narrow class of problems, which often meet the computational accuracy requirements.

The issues of layer-by-layer analysis of heterogeneous sandwich and three-layer shells have not been sufficiently studied yet. The layer-by-layer analysis approach is that the layer wall including the filler, if necessary, are divided into layers in thickness, which are afterwards linked together. With such approach, depending on stress-strain properties, layer thickness and problem situation, we can create and apply various models to calculate layers using analytic and numerical methods.

2. Formulation of problem
This article involves the layer-by-layer analysis-based building of finite element models for a refined study of stress-strain state of three-layer generally irregular cylindrical shells taking into account heterogeneity including filler layers, application of loads to different layers and different layer fixing conditions, lateral strains and stress, momental state of carrier layers, three
dimensional stress state in filler layer. The models shall enable to make calculations with variable geometrical and stress-strain properties, and consider the properties changes and stress-strain state parameters not only in meridian and circumferential coordinates but also in filler layer thickness, which are currently taken with insufficient accuracy, and some of peculiarities are not taken into account at all.

The considered layer-by-layer analysis model of stress-strain state of three-layer irregular shells is based on building and following blocking of two different types of natural curved finite elements. Thin and rigid carrier layers are simulated with two-dimensional finite elements of momental carrier layers, and filler layer is simulated with three-dimensional finite elements of thick-wall layer. Thickness and stress-strain properties of a filler layer vary greatly.

However, this approach involves significantly increasing degrees of freedom of finite element model (it depends on the number of shell layers; the number of finite elements used to simulate layers including in thickness, and the number of finite element degrees of freedom). Thus, calculation errors, required computer resources and calculation time increase too. It much complicates and often makes impossible the study of such structures with appropriate accuracy. In this regard there is a problem of reducing the size of finite element models of heterogeneous sandwich generally irregular shells.

The building of finite element models usually involves shape functions obtained through approximation of displacement fields by first and second-order polynomials [4] it often results in intricate finite element models. In this regard it is important to build efficient models with shape functions which ensure increased velocity of convergence and hence result in reduced order of equation systems as against known approaches. This is especially important for model building based on layer-by-layer analysis.

The most efficient are shape functions based on analytical solutions of the shell theory problems [5]. Such approach has been successfully developed as to axisymmetric three-layer cylindrical shells only.

Efficient shape functions can be obtained through the approximation of generalized strains [6] followed by satisfying the equation of strain compatibility. This approach involving building efficient models of irregular three-layer cylindrical shells is described in [3, 7, 8, 9, 10, 11, 12].

3. Finite-element model of carrier layers of three-layer cylindrical generally irregular rotational shell

Basic relations.

If carrier layers of the three-layer shell are sufficiently thin and rigid, then stress-strain state simulation shall involve two-dimensional natural curved finite elements developed from general theory of shells based on Kirchhoff-Love hypotheses. Let us index layers (fig.1a), beginning from the shell inner surface: \( i = 1, 2, 3; \) assign index \( c \) to carrier layers, and index \( f \) to filler layers.

Displacement of carrier layer points are defined by displacement of middle surface points
\[
\delta_c^i = \{u, v, w\}^T
\]
and angular displacements of normal to middle surface against axes \( x, y, \phi \)
\[
\vartheta_c^i = \{\vartheta_x, \vartheta_y\}^T
\]
(fig. 1b).

Hereinafter, we shall not indicate vector coefficients and matrix indices corresponding to a layer index.

Generalized strain vector \( \varepsilon_c^i = \{\varepsilon_1, \varepsilon_2, \gamma, \alpha_1, \alpha_2, \chi\}^T \) hereinafter referred to as strain vector is linked to displacement vector \( \delta_c^i \) through Cauchy relations [13]:

\[
\varepsilon_1 = \frac{\partial u}{\partial x}; \quad \varepsilon_2 = \frac{1}{R} \frac{\partial v}{\partial \varphi} + \frac{w}{R}; \quad \gamma = \frac{1}{R} \frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial x} \\
\alpha_1 = -\frac{\partial^2 w}{\partial x^2}; \quad \alpha_2 = \frac{1}{R^2} \frac{\partial v}{\partial \varphi} - \frac{1}{R^2} \frac{\partial^2 w}{\partial \varphi^2}; \quad \chi = \frac{1}{R} \frac{\partial v}{\partial x} - \frac{1}{R} \frac{\partial^2 w}{\partial x \partial \varphi}
\] (1)
Rigid displacement is defined by integrating relations (1) with zero strain. The selection of displacement approximation functions is tied to the selection of nodal displacements and the number of nodes. Carrier layer finite element nodal displacements involve $u, v, w$ and angular displacements $\vartheta_c^x, \vartheta_c^y$. Thus, finite element of carrier layers of three-layer generally irregular cylindrical rotational shell has twenty degrees of freedom.

Displacement and strain approximation functions shall be written using twenty coefficients $\alpha$, which number is equal to the number of finite element degrees of freedom. From among twenty indefinite coefficients $\alpha$ six are used for rigid displacement

$$\mathbf{\delta}_c^0 = T_0^c \alpha_0^c; \quad \alpha_0^c = \begin{bmatrix} \alpha_1^c, \ldots, \alpha_6^c \end{bmatrix}^T,$$

where $\alpha_1^c, \ldots, \alpha_6^c$ are constants of integration used here as indefinite coefficients, $T_0^c$ is coefficient matrix at $\alpha_1^c, \ldots, \alpha_6^c$.

**Finite element of carrier layers of three-layer generally irregular cylindrical shell.**

The finite element (fig. 1b) is generated through the intersection of two planes perpendicular to the center line and two planes passing through the shell center line. The nodes are located in the finite element corners and marked 1 - 4 (fig. 1b).

Unlike most works dedicated to finite element method involving displacement function approximation, this work describes generalized strain function approximation. This has significantly increased the velocity of convergence of numerical procedures and enabled to reduce the required number of finite elements, which is very important for layer-by-layer analysis models application.

Since finite element of carrier layers of three-layer generally irregular cylindrical rotational shell has twenty degrees of freedom, six of which are responsible for rigid displacement, then the rest fourteen degrees of freedom are responsible for shell strain displacement.

Let us distribute the remaining fourteen indefinite coefficients between strain vector coefficients (the formula in brackets is added to satisfy the equations of continuity of deformations) [3, 7, 8].

$$\begin{align*}
\varepsilon_1 &= \alpha_7 + \alpha_8 \varphi; \\
\varepsilon_2 &= \alpha_9 + \alpha_{10} x - \left( \alpha_{12} + \frac{\alpha_{13} x}{3} + \alpha_{14} \varphi + \frac{\alpha_{15} x \varphi}{5} \right) \frac{x^2}{2R}; \\
\varepsilon_3 &= \alpha_{12} + \alpha_{13} x + \alpha_{14} \varphi + \alpha_{15} x \varphi; \\
\varepsilon_4 &= \alpha_{16} + \alpha_{17} x + \alpha_{18} \varphi + \alpha_{19} x \varphi; \\
\chi &= \alpha_{20} + \left( -\frac{1}{R^2} \alpha_8 x + \frac{1}{R} \alpha_{14} x + \frac{1}{2R} \alpha_{15} x^2 + R \alpha_{17} \varphi + \frac{R}{2} \alpha_{19} x \varphi \right). \\
\end{align*}$$

where

$$\varepsilon_i^c = \Omega_i^c \alpha_i^c.$$
where \( \alpha^c_i = \{\alpha_1\ldots\alpha_{20}\}^T \) is an indefinite coefficient vector, \( \Omega_{ic} \) \((6 \times 20)\) is a strain function approximation matrix.

To define displacement approximation functions caused by element strain, let us integrate Cauchy relations (1) using strain expressions of (2).

Displacement vector \( \delta^c_i \) of finite element of shell carrier layers is a sum of rigid \( \delta^0_i \) and strain \( \delta^d_i \) displacements \( \delta^d_i = \{u_d, v_d, w_d\}^T \). The displacement approximation function obtained in this manner shall be written using twenty indefinite coefficients

\[
\begin{align*}
\delta^c_i &= T^c_i \alpha^c_i, \\
q^c_i &= C^c_i \alpha^c_i,
\end{align*}
\]

or as a matrix

\[
\delta^c_i = T^c_i \alpha^c_i,
\]

where \( T^c_i \) \((3 \times 20)\) is a matrix of carrier layer finite element displacement approximation functions. By inserting nodal points coordinates into relations (3) and into expressions of normal angular displacement around axes \( x \) and \( \varphi \), written through indefinite coefficients in terms of relations (3), we shall obtain relations, which link finite element nodal displacement vector \( q^c_i \) and vector \( \alpha^c_i \):

\[
\delta^c_i = T^c_i \alpha^c_i,
\]

where \( C^c_i \) \((20 \times 20)\) is a vector \( \alpha^c_i \) and vector \( q^c_i \) constraint matrix.

Using the expression (4) let us find vector

\[
\alpha^c_i = (C^c_i)^{-1} q^c_i
\]

By rearranging the expressions (3) in terms of relations (5), we shall obtain shape functions of finite element of three-layer cylindrical shell carrier layers

\[
\delta^c_i = T^c_i \alpha^c_i
\]

By obtaining expressions of generalized strains as well as force and moment (stresses) we shall calculate finite element stiffness matrix using algorithms described in works \([3, 7, 8]\), and build a finite element model of carrier layers.
4. Finite-element model of filler layer of three-layer cylindrical generally irregular rotational shell.

Finite element for simulating stress-strain state in a filler layer.

The application of two and three-dimensional finite elements for simulating momental carrier layers and filler layer respectively, shall lead to the generalized displacement discontinuity upon contact areas of these finite elements, if they are built using different displacement approximation functions.

To avoid errors caused by such generalized displacement discontinuity, a filler layer finite element on cylindrical surfaces should involve as many nodes as carrier layer finite element has; the generalized displacements and approximation functions similar to those of carrier layer finite element should be used as nodal unknowns and approximation functions.

The finite element is generated through intersection of two planes perpendicular to the center line and two planes passing through the shell center line. The finite element nodes are located in the corners, four on internal (node numbers 1 - 4) and external (node numbers 5 - 8) cylindrical surfaces.

The degrees of freedom in the filler layer finite element node conjugated in internal and external cylindrical surfaces with carrier layer finite elements include three linear displacements $u, v, w$ and angular displacement of the normal around axes $x$ and $\varphi(y)$. Thus, the filler layer finite element has generally forty degrees of freedom. When the filler layer finite elements are located one by one over the shell thickness, there is no meaning in the normal angular displacement being taken into account. In this case corresponding generalized displacements are eliminated by a computational procedure involved in boundary conditions.

By reducing degrees of freedom of the filler layer finite element we may switch from one filler model to another.

The filler layer simulation is done using meridional, circumferential and normal coordinates to the shell surface.

Approximation displacement functions of the filler layer finite elements. The local coordinate system of the filler layer finite elements is located on the finite element middle surface with its origin positioned at the intersection of diagonals.

Since the generalized displacements of the filler layer finite elements is related to the middle surface, when the finite elements of carrier and filler layers are joined, we shall transfer from the carrier layer middle surface to the filler layer surface using transfer matrix like [4, 14].

As is noted before, the filler layer finite element displacement fields approximation functions are built on internal and external cylindrical shells of these finite elements. Besides, the carrier layer-dedicated functions are used here.

Since the development finite element model enables to reproduce any law of stress-strain state properties variation through simulation with a required number of finite elements, these finite elements involve the linear law of displacement variation over the radial coordinate like in the work [15].

By inserting the obtained expressions of approximation functions of the displacement fields of the filler layer finite element being a thick-wall cylindrical shell into Cauchy relations, which link strains with rigid displacements in curvilinear coordinates (see work [16]), we shall obtain dependencies of generalized strains written using indefinite coefficient vector.

Using the obtained dependencies of generalized strains, we shall write down expressions of stress based on a physical law [17, 18].

Using strain and stress expressions we can calculate the filler layer finite element stiffness matrix using algorithms similar to those described in works [19, 20].

Block finite-element model of layer-by-layer analysis of the stress-strain state of three-layer generally irregular cylindrical rotational shells
By joining the developed filler finite elements over thickness we shall build the finite element block, like in works [21, 22], for simulating stress-strain state in the filler layer.

By joining over thickness the built filler finite element block with the reviewed carrier layer finite elements like in works [21, 22], we shall build finite element block for simulating stress-strain state in the layers of three-layer generally irregular shells.

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