Parameter dependence of solutions of the Cauchy–Riemann equation on weighted spaces of smooth functions

Karsten Kruse

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Abstract
Let $\Omega$ be an open subset of $\mathbb{R}^2$ and $E$ a complete complex locally convex Hausdorff space. The purpose of this paper is to find conditions on certain weighted Fréchet spaces $\mathcal{E}V(\Omega)$ of smooth functions and on the space $E$ to ensure that the vector-valued Cauchy–Riemann operator $\overline{\partial} : \mathcal{E}V(\Omega, E) \to \mathcal{E}V(\Omega, E)$ is surjective. This is done via splitting theory and positive results can be interpreted as parameter dependence of solutions of the Cauchy–Riemann operator.

Keywords
Cauchy–Riemann · Parameter dependence · Weight · Smooth · Solvability · Vector-valued

Mathematics Subject Classification
35A01 · 35B30 · 32W05 · 46A63 · 46A32 · 46E40

1 Introduction
Let $E$ be a linear space of functions on a set $U$ and $P(\partial) : \mathcal{F}(\Omega) \to \mathcal{F}(\Omega)$ be a linear partial differential operator with constant coefficients which acts continuously on a locally convex Hausdorff space of (generalized) differentiable scalar-valued functions $\mathcal{F}(\Omega)$ on an open set $\Omega \subset \mathbb{R}^n$. We call the elements of $U$ parameters and say that a family $(f_\lambda)_{\lambda \in U}$ in $\mathcal{F}(\Omega)$ depends on a parameter w.r.t. $E$ if the map $\lambda \mapsto f_\lambda(x)$ is an element of $E$ for every $x \in \Omega$. The question of parameter dependence is whether for every family $(f_\lambda)_{\lambda \in U}$ in $\mathcal{F}(\Omega)$ depending on a parameter w.r.t. $E$ there is a family $(u_\lambda)_{\lambda \in U}$ in $\mathcal{F}(\Omega)$ with the same kind of parameter dependence which solves the partial differential equation

$$P(\partial)u_\lambda = f_\lambda, \quad \lambda \in U.$$ 

In particular, it is the question of $C^k$-smooth (holomorphic, distributional, etc.) parameter dependence if $E$ is the space $C^k(U)$ of $k$-times continuously partially differentiable functions on an open set $U \subset \mathbb{R}^d$ (the space $O(U)$ of holomorphic functions on an open set $U \subset \mathbb{C}$, the space of distributions $D(V)'$ on an open set $V \subset \mathbb{R}^d$ where $U = D(V)$, etc.).
The question of parameter dependence has been subject of extensive research varying in the choice of the spaces $E$, $\mathcal{F}(\Omega)$ and the properties of the partial differential operator $P(\partial)$, e.g. being (hypo)elliptic, parabolic or hyperbolic. Even partial differential differential operators $P_{n}(\partial)$ where the coefficients also depend $C^{k}([0, 1])$-smoothly [49], $C^{\infty}$-smoothly [61], holomorphically [50,61] or differentiable resp. real analytic [13] on the parameter $\lambda$ were considered. The case that the coefficients of the partial differential differential operator $P(x, \partial)$ are non-constant functions in $x \in \Omega$ was treated for $\mathcal{F}(\Omega) = \mathcal{A}(\mathbb{R}^{n})$, the space of real analytic functions on $\mathbb{R}^{n}$, as well [3].

The answer to the question of $C^{k}$-smooth (holomorphic, distributional, etc.) parameter dependence is obviously affirmative if $P(\partial)$ has a linear continuous right inverse. The problem to determine those $P(\partial)$ which have such a right inverse was posed by Schwartz in the early 1950s (see [21, p. 680]). In the case that $\mathcal{F}(\Omega)$ is the space of $C^{\infty}$-smooth functions or distributions on an open set $\Omega \subset \mathbb{R}^{n}$ the problem was solved in [51,52] and in the case of ultradifferentiable functions or ultradistributions in [53] by means of Phragmén-Lindelöf type conditions. The case that $\mathcal{F}(\Omega)$ is a weighted space of $C^{\infty}$-smooth functions on $\Omega = \mathbb{R}^{n}$ or its dual was handled in [40], even for some $P(x, \partial)$ with smooth coefficients, the case of tempered distributions in [38] and of Fourier (ultra-)hyperfunctions in [44,45]. For Hörmander’s spaces $B^{p,\infty}_{\nu,k}(\Omega)$ as $\mathcal{F}(\Omega)$ the problem was studied in [25].

The necessary condition of surjectivity of the partial differential operator $P(\partial)$ was studied in many papers, e.g. in [1,23,28,48,67] on $C^{\infty}$-smooth functions and distributions, in [9,26,43] on real analytic functions, in [8,14] on Gevrey classes, in [10,12,41,42,55] on ultradifferentiable functions of Roumieu type, in [22] on ultradistributions of Beurling type, in [7,11] on ultradifferentiable functions and ultradistributions and in [47] on the multiplier space $\mathcal{M}$. However, if $P(\partial) : C^{\infty}(\Omega) \to C^{\infty}(\Omega)$, $\Omega \subset \mathbb{R}^{n}$ open, is elliptic, then $P(\partial)$ has a linear right inverse (by means of a Hamel basis of $C^{\infty}(\Omega)$) and it has a continuous right inverse due to Michael’s selection theorem [56, Theorem 3.2”, p. 367] and [29, Satz 9.28, p. 217], but $P(\partial)$ has no linear continuous right inverse if $n \geq 2$ by a result of Grothendieck [62, Theorem C.1, p. 109]. Nevertheless, the question of parameter dependence w.r.t. $E$ has a positive answer for several locally convex Hausdorff spaces $E$ due to tensor product techniques. In this case the question of parameter dependence obviously has a positive answer if the topology of $E$ is stronger than the topology of pointwise convergence on $U$ and

$$P(\partial)^{E} : C^{\infty}(\Omega, E) \to C^{\infty}(\Omega, E)$$

is surjective where $C^{\infty}(\Omega, E)$ is the space of $C^{\infty}$-smooth $E$-valued functions on $\Omega$ and $P(\partial)^{E}$ the version of $P(\partial)$ for $E$-valued functions. From Grothendieck’s classical theory of tensor products [24] and the surjectivity of $P(\partial)$ it follows that $P(\partial)^{E}$ is also surjective if $E$ is a Fréchet space. In general, Grothendieck’s theory of tensor products can be applied if $P(\partial)$ is surjective and $\mathcal{F}(\Omega)$ a nuclear Fréchet space. However, the surjectivity of $P(\partial)^{E}$, $n \geq 2$, can even be extended beyond the class of Fréchet spaces $E$ due to the splitting theory of Vogt for Fréchet spaces [64,65] and of Bonet and Domaniński for PLS-spaces [4,6] if, in addition, ker $P(\partial)$ has the property $(\Omega)$ and $E$ is the dual of a Fréchet space with the property $(DN)$ or an ultrabornological PLS-space with the property $(PA)$. The splitting theory of Bonet and Domaniński can also be applied if $\mathcal{F}(\Omega)$ is a non-Fréchet PLS-space and for PLH-spaces $\mathcal{F}(\Omega)$, e.g. $D_{I,2}$ and $B^{p,\infty}_{\nu,k}(\Omega)$ which are non-PLS-spaces, the splitting theory of Dierolf and Sieg [15,16] is available. For applications we refer the reader to the already mentioned papers [4,6,15,16,39,65] as well as [5,18] where $\mathcal{F}(\Omega)$ is the space of ultradistributions of Beurling type or of ultradifferentiable functions of Roumieu type and $E$, amongst others, the space

$$\sum_{n=0}^{\infty} \mathfrak{X}_{n} \oplus \mathcal{F}(\Omega, E)$$

is the space of ultradistributions of Beurling type or of ultradifferentiable functions of Roumieu type and $E$, amongst others, the space

$$\sum_{n=0}^{\infty} \mathfrak{X}_{n} \oplus \mathcal{F}(\Omega, E)$$
of real analytic functions and to [30] where $\mathcal{F}(\Omega)$ is the space of $C^\infty$-smooth functions or distributions.

Notably, the preceding results imply that the inhomogeneous Cauchy–Riemann equation with a right-hand side $f \in \mathcal{E}(\Omega, E) := C^\infty(\Omega, E)$, where $\Omega \subset \mathbb{R}^2$ is open and $E$ a locally convex Hausdorff space over $\mathbb{C}$ whose topology is induced by a system of seminorms $(p_\alpha)_{\alpha \in \mathfrak{A}}$, given by

$$\overline{\partial}^E u := (1/2)(\partial_1^E + i \partial_2^E)u = f$$

(1)

has a solution $u \in \mathcal{E}(\Omega, E)$ if $E$ is a Fréchet space or $E := F'_b$ where $F'$ is a Fréchet space satisfying the condition $(DN)$ or if $E$ is an ultrabornological PLS-space having the property $(PA)$. Among these spaces $E$ are several spaces of distributions like $D(V)'$, the space of tempered distributions, the space of ultradistributions of Beurling type etc. In the present paper we study this problem under the constraint that the right-hand side space of tempered distributions, the space of ultradistributions of Beurling type etc. In the present paper we want to extend this result beyond the class of Fréchet spaces $E$. Concerning the sequence $(\Omega_n)_{n \in \mathbb{N}}$, we concentrate on the case that it is a sequence of strips along the real axis, i.e. $\Omega_n := \{z \in \mathbb{C} \mid |\operatorname{Im}(z)| < n\}$. The case that this sequence has holes along the real axis is treated in [35].

Let us briefly outline the content of our paper. In Sect. 2 we summarise the necessary definitions and preliminaries which are needed in the subsequent sections. In Sect. 3 we recall the definitions of the topological invariants $(\Omega)$, $(DN)$ and $(PA)$ and provide some examples of spaces $E$ having these invariants. Then we prove our main result on the surjectivity of Cauchy–Riemann operator on $\mathcal{E}(\Omega, E)$ which depends on these invariants (see Theorem 5). To apply our main result, the kernel $\ker \overline{\partial}$ needs to have $(\Omega)$ and in Sect. 4 we provide sufficient conditions on the weights and the sequence $(\Omega_n)_{n \in \mathbb{N}}$ which guarantee $(\Omega)$ (see Theorem 10 and Corollary 13). We close this section with a special case of our main theorem where $(\Omega_n)_{n \in \mathbb{N}}$ is a sequence of strips along the real axis (see Corollary 17) and for example $\nu_n(z) := \exp(a_n |\operatorname{Re}(z)|^\gamma)$ for some $0 < \gamma \leq 1$ and $a_n \nearrow 0$ (see Corollary 18).

2 Notation and preliminaries

The notation and preliminaries are essentially the same as in [33,36, Sect. 2]. We define the distance of two subsets $M_0, M_1 \subset \mathbb{R}^2$ w.r.t. a norm $\| \cdot \|$ on $\mathbb{R}^2$ via

$$d_{\| \cdot \|}(M_0, M_1) := \begin{cases} 
\inf_{x \in M_0, y \in M_1} \|x - y\|, & M_0, M_1 \neq \emptyset, \\
\infty, & M_0 = \emptyset \text{ or } M_1 = \emptyset.
\end{cases}$$
Moreover, we denote by $\| \cdot \|_\infty$ the sup-norm, by $| \cdot |$ the Euclidean norm on $\mathbb{R}^2$, by $B_r(x) := \{ w \in \mathbb{R}^2 \, | \, |w - x| < r \}$ the Euclidean ball around $x \in \mathbb{R}^2$ with radius $r > 0$ and identify $\mathbb{R}^2$ and $\mathbb{C}$ as (normed) vector spaces. We denote the complement of a subset $M \subset \mathbb{R}^2$ by $M^C := \mathbb{R}^2 \setminus M$, the closure of $M$ by $\overline{M}$ and the boundary of $M$ by $\partial M$. For a function $f : M \to \mathbb{C}$ and $K \subset M$ we denote by $f|_K$ the restriction of $f$ to $K$ and by

$$\| f \|_K := \sup_{x \in K} |f(x)|$$

the sup-norm on $K$. By $L^1(\Omega)$ we denote the space of (equivalence classes of) $\mathbb{C}$-valued Lebesgue integrable functions on a measurable set $\Omega$. From a locally convex Hausdorff space of all functions $\in \mathbb{N}$ we write $\in \mathbb{N}$ if $\in \mathbb{N}$ is a strictly increasing real sequence, we write $\in \mathbb{N}$ if $\in \mathbb{N}$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} a_n = 0$ resp. $\in \mathbb{N}$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} a_n = \infty$.

By $E$ we always denote a non-trivial locally convex Hausdorff space over the field $\mathbb{C}$ equipped with a directed fundamental system of seminorms $(p_\alpha)_{\alpha \in \mathbb{A}}$. If $E = \mathbb{C}$, then we set $(p_\alpha)_{\alpha \in \mathbb{A}} := \{| \cdot | : \mathbb{A}|$. Further, we denote by $L(F, E)$ the space of continuous linear maps from a locally convex Hausdorff space $F$ to $E$ and sometimes write $(T, f) := T(f)$, $f \in F$, for $T \in L(F, E)$. If $E = \mathbb{C}$, we write $F' := L(F, \mathbb{C})$ for the dual space of $F$. If $F$ and $E$ are (linearly topologically) isomorphic, we write $F \cong E$. We denote by $L_t(F, E)$ the space $L(F, E)$ equipped with the locally convex topology of uniform convergence on the finite subsets of $F$ if $t = \sigma$, on the precompact subsets of $F$ if $t = \gamma$, on the absolutely convex, compact subsets of $F$ if $t = \kappa$ and on the bounded subsets of $F$ if $t = b$.

The so-called $\varepsilon$-product of Schwartz is defined by

$$F \varepsilon E := L_\varepsilon(F_k', E)$$

where $L(F_k', E)$ is equipped with the topology of uniform convergence on equicontinuous subsets of $F_k'$. This definition of the $\varepsilon$-product coincides with the original one by Schwartz [59, Chap. I, Sect. 1, Définition, p. 18].

We recall the following well-known definitions concerning continuous partial differentiability of vector-valued functions (c.f. [34, p. 237]). A function $f : \Omega \to E$ on an open set $\Omega \subset \mathbb{R}^2$ to $E$ is called continuously partially differentiable ($f$ is $C^1$) if for the $n$-th unit vector $e_n \in \mathbb{R}^2$ the limit

$$(\partial^{e_n}) f(x) := (\partial_n) E f(x) := \lim_{h \to 0, h \neq 0} \frac{f(x + he_n) - f(x)}{h}$$

exists in $E$ for every $x \in \Omega$ and $(\partial^{e_n}) E f$ is continuous on $\Omega$ ($(\partial^{e_n}) E f$ is $C^0$) for every $n \in \{ 1, 2 \}$. For $k \in \mathbb{N}$ a function $f$ is said to be $k$-times continuously partially differentiable ($f$ is $C^k$) if $f$ is $C^1$ and all its first partial derivatives are $C^{k-1}$. A function $f$ is called infinitely continuously partially differentiable ($f$ is $C^\infty$) if $f$ is $C^k$ for every $k \in \mathbb{N}$. The linear space of all functions $f : \Omega \to E$ which are $C^\infty$ is denoted by $C^\infty(\Omega, E)$. Let $f \in C^\infty(\Omega, E)$. For $\beta = (\beta_n) \in \mathbb{N}_0^2$ we set $(\partial^\beta_n) E f := f$ if $\beta_n = 0$, and

$$(\partial^\beta_n) E f := (\partial^{e_n}) E \cdots (\partial^{e_n}) E f$$

$\beta_n$-times

if $\beta_n \neq 0$ as well as

$$(\partial^\beta) E f := (\partial^h_1) E (\partial^P_2) E f.$$
Due to the vector-valued version of Schwarz’ theorem \( (\partial^\beta)^E f \) is independent of the order of the partial derivatives on the right-hand side, we call \(|\beta| := \beta_1 + \beta_2\) the order of differentiation and write \( \partial^\beta f := (\partial^\beta)^C f \).

A function \( f : \Omega \to E \) on an open set \( \Omega \subset \mathbb{C} \) to \( E \) is called holomorphic if the limit

\[
\left( \frac{\partial}{\partial z} \right)^E f(z_0) := \lim_{h \to 0 \atop h \in \mathbb{C}, h \neq 0} \frac{f(z_0 + h) - f(z_0)}{h}
\]

exists in \( E \) for every \( z_0 \in \Omega \) and the space of such functions is denoted by \( \mathcal{O}(\Omega, E). \) The exact definition of the spaces from the introduction is as follows.

**Definition 1** [34, Definition 3.2, p. 238] Let \( \Omega \subset \mathbb{R}^2 \) be open and \( (\Omega_n)_{n \in \mathbb{N}} \) a family of non-empty open sets such that \( \Omega_n \subset \Omega_{n+1} \) and \( \Omega = \bigcup_{n \in \mathbb{N}} \Omega_n. \) Let \( \mathcal{V} := (v_n)_{n \in \mathbb{N}} \) be a countable family of positive continuous functions \( v_n : \Omega \to (0, \infty) \) such that \( v_n \leq v_{n+1} \) for all \( n \in \mathbb{N} \). We call \( \mathcal{V} \) a directed family of continuous weights on \( \Omega \) and set for \( n \in \mathbb{N} \)

(a) \[
\mathcal{E} v_n(\Omega_n, E) := \left\{ f \in C^\infty(\Omega_n, E) \mid \forall \alpha \in \mathfrak{A}, m \in \mathbb{N}_0^2 : |f|_{n,m,\alpha} < \infty \right\}
\]

and

\[
\mathcal{E} \mathcal{V}(\Omega, E) := \left\{ f \in C^\infty(\Omega, E) \mid \forall n \in \mathbb{N} : f|_{\Omega_n} \in \mathcal{E} v_n(\Omega_n, E) \right\}
\]

where

\[
|f|_{n,m,\alpha} := \sup_{x \in \Omega_n} \beta \in \mathbb{N}_0^2, |\beta| \leq m p_\alpha \left( (\partial^\beta)^E f(x) \right) v_n(x).
\]

(b) \[
\mathcal{E} v_n,\overline{\partial}(\Omega_n, E) := \left\{ f \in \mathcal{E} v_n(\Omega_n, E) \mid f \in \ker \overline{\partial}^E \right\}
\]

and

\[
\mathcal{E} \mathcal{V},\overline{\partial}(\Omega, E) := \left\{ f \in \mathcal{E} \mathcal{V}(\Omega, E) \mid f \in \ker \overline{\partial}^E \right\}.
\]

(c) \[
\mathcal{O} v_n(\Omega_n, E) := \left\{ f \in \mathcal{O}(\Omega_n, E) \mid \forall \alpha \in \mathfrak{A} : |f|_{n,\alpha} < \infty \right\}
\]

and

\[
\mathcal{O} \mathcal{V}(\Omega, E) := \left\{ f \in \mathcal{O}(\Omega, E) \mid \forall n \in \mathbb{N} : f|_{\Omega_n} \in \mathcal{O} v_n(\Omega_n, E) \right\}
\]

where

\[
|f|_{n,\alpha} := \sup_{x \in \Omega_n} p_\alpha(f(x)) v_n(x).
\]

The subscript \( \alpha \) in the notation of the seminorms is omitted in the \( \mathbb{C} \)-valued case. The letter \( E \) is omitted in the case \( E = \mathbb{C} \) as well, e.g. we write \( \mathcal{E} v_n(\Omega_n) := \mathcal{E} v_n(\Omega_n, \mathbb{C}) \) and \( \mathcal{E} \mathcal{V}(\Omega) := \mathcal{E} \mathcal{V}(\Omega, \mathbb{C}). \)
A projective limit \( F \) of a sequence of locally convex Hausdorff spaces \( (F_n)_{n \in \mathbb{N}} \) is called weakly reduced if for every \( n \in \mathbb{N} \) there is \( m \in \mathbb{N} \) such that \( \pi_n(F) \) is dense in \( F_m \) w.r.t. the topology of \( F_n \) where \( \pi_n: F \to F_n \) is the canonical projection. The spaces \( \mathcal{F}V(\Omega, E) \), \( \mathcal{F} = \mathcal{E}, \mathcal{O} \), are projective limits, namely, we have

\[
\mathcal{F}V(\Omega, E) \cong \lim_{\longrightarrow} \mathcal{F}v_n(\Omega_n, E)
\]

where the spectral maps are given by the restrictions

\[
\pi_{k,n}: \mathcal{F}v_k(\Omega_k, E) \to \mathcal{F}v_n(\Omega_n, E), \quad f \mapsto f\mid_{\Omega_n}, \quad k \geq n.
\]

### 3 Main result

In this section we prove our main result that the surjectivity of the vector-valued Cauchy–Riemann operator on \( \mathcal{E}V(\Omega, E) \) is inherited from the surjectivity on \( \mathcal{E}V(\Omega) \) if the kernel \( \mathcal{E}V_\mathcal{F}(\Omega) \) in the scalar-valued case has \( (\Omega) \), and \( E := F'_b \) where \( F \) is a Fréchet space satisfying the condition \((DN)\) or \( E \) is an ultrabornological PLS-space having the property \((PA)\). Therefore we recall the definitions of the topological invariants \((\Omega), (DN)\) and \((PA)\) and give some examples.

A Fréchet space \( F \) with an increasing fundamental system of seminorms \( (\||\cdot|||_k)_{k \in \mathbb{N}} \) satisfies \((\Omega)\) if

\[
\forall \ p \in \mathbb{N} \exists \ q \in \mathbb{N} \forall \ k \in \mathbb{N} \exists \ n \in \mathbb{N}, \ C > 0 \forall \ r > 0: \ U_q \subset Cr^nU_k + \frac{1}{r}U_p \tag{3}
\]

where \( U_k := \{ x \in F \mid \||x|||_k \leq 1 \} \) (see [54, Chap. 29, Definition, p. 367]).

A Fréchet space \((F, (\||\cdot|||_k)_{k \in \mathbb{N}})\) satisfies \((DN)\) by [54, Chap. 29, Definition, p. 359] if

\[
\exists \ p \in \mathbb{N} \forall \ k \in \mathbb{N} \exists \ n \in \mathbb{N}, \ C > 0 \forall \ x \in F : \ \||x|||_k^2 \leq C\||x|||_p \||x|||_n.
\]

A **PLS-space** is a projective limit \( X = \lim_{\longrightarrow} X_N \), where the \( X_N \) given by inductive limits \( X_N = \lim_{\longrightarrow} (X_{N,n}, \||\cdot|||_{N,n}) \) are DFS-spaces (which are also called LS-spaces), and it satisfies \((PA)\) if

\[
\forall \ N \exists \ M \forall \ K \exists \ n \forall \ m \forall \ \eta > 0 \exists \ k, C, r_0 > 0 \forall \ r > r_0 \forall \ x' \in X'_{N,n} : \\
\left\|\left\| x' \circ i_{N}^M \right\|\|_M^* \right\|_{M,m} \leq C(r^n)\left\|\left\| x' \circ i_{N}^K \right\|\|_K^* \right\|_{K,k} + \frac{1}{r}\left\|\left\| x' \right\|\|_{N,n}^*
\right\| \tag{24}, p. 577\}
\]

where \( \|\cdot\|\|_\cdot \) denotes the dual norm of \( \|\cdot\|\|_\cdot \) and \( i_{N}^M, i_{N}^K \) the linking maps (see [6, Sect. 4, Eq. (24), p. 577]).

Due to [63, 1.4 Lemma, p. 110] and [6, Proposition 4.2, p. 577] we have the following relation between the properties \((DN)\) and \((PA)\).

**Remark 2** Let \( F \) be a Fréchet-Schwartz space. Then \( F \) satisfies \((DN)\) if and only if the DFS-space \( E := F'_b \) satisfies \((PA)\).

Let us summarise some examples of ultrabornological PLS-spaces satisfying \((PA)\) and spaces of the form \( E := F'_b \) where \( F \) is a Fréchet space satisfying \((DN)\). The majority of them is already contained in [6], [19] and [64].
Example 3 (a) The following spaces are ultrabornological PLS-spaces with property \((PA)\) and also strong duals of a Fréchet space satisfying \((DN)\):

- the strong dual of a power series space of infinite type \(\Lambda_\infty(\alpha)'_h\),
- the strong dual of any space of holomorphic functions \(O(U)'_b\) where \(U\) is a Stein manifold with the strong Liouville property (for instance, for \(U = \mathbb{C}^d\)),
- the space of germs of holomorphic functions \(O(K)'_b\) where \(K\) is a completely pluripolar compact subset of a Stein manifold (for instance \(K\) consists of one point),
- the space of tempered distributions \(S(\mathbb{R}^d)'_b\) and the space of Fourier ultra-hyperfunctions \(\mathcal{P}^\ast_{sa}\) (with the strong topology),
- the weighted distribution spaces \((K\{pM\})'_b\) of Gelfand and Shilov if the weight \(M\) satisfies
  \[
  \sup_{|y| \leq 1} M(x + y) \leq C \inf_{|y| \leq 1} M(x + y), \quad x \in \mathbb{R}^d,
  \]
- \(\mathcal{D}(K)'_b\) for any compact set \(K \subset \mathbb{R}^d\) with non-empty interior,
- \(C^\infty(U)'_b\) for any non-empty open bounded set \(U \subset \mathbb{R}^d\) with \(C^1\)-boundary.

(b) The following spaces are ultrabornological PLS-spaces with property \((PA)\):

- an arbitrary Fréchet-Schwartz space,
- a PLS-type power series space \(\Lambda_{r,s}(\alpha, \beta)\) whenever \(s = \infty\) or \(\Lambda_{r,s}(\alpha, \beta)\) is a Fréchet space,
- the spaces of distributions \(\mathcal{D}(U)'_b\) and ultradistributions of Beurling type \(\mathcal{D}(\omega)(U)'_b\) for any open set \(U \subset \mathbb{R}^d\),
- the kernel of any linear partial differential operator with constant coefficients in \(\mathcal{D}(U)'_b\) or in \(\mathcal{D}(\omega)(U)'_b\) when \(U \subset \mathbb{R}^d\) is open and convex,
- the space \(L_b(X, Y)\) where \(X\) has \((DN)\), \(Y\) has \((\Omega)\) and both are nuclear Fréchet spaces.
  In particular, \(L_b(\Lambda_\infty(\alpha), \Lambda_\infty(\beta))\) if both spaces are nuclear.

(c) The following spaces are strong duals of a Fréchet space satisfying \((DN)\):

- the strong dual \(F'_b\) of any Banach space \(F\),
- the strong dual \(\lambda^2(A)'_b\) of the Kōthe space \(\lambda^2(A)\) with a Kōthe matrix \(A = (a_{j,k})_{j,k \in \mathbb{N}_0}\) satisfying
  \[
  \exists \ p \in \mathbb{N}_0 \ \forall \ k \in \mathbb{N}_0 \ \exists \ n \in \mathbb{N}_0, \ C > 0 : \ a_{j,k}^2 \leq C a_{j,p} a_{j,n}.
  \]

Proof The statement for the spaces in (a) and (b) follows from [19, Corollary 4.8, p. 1116], [54, Proposition 31.12, p. 401], [54, Proposition 31.16, p. 402] and Remark 2. The first part of statement (c) is obvious since Banach spaces clearly satisfy the property \((DN)\). The second part on the Kōthe space \(\lambda^2(A)\) follows from [29, Satz 12.11 a), p. 305].

Since we will use the \(\varepsilon\)-product \(\mathcal{E}\mathcal{V}(\Omega)\varepsilon E\) to pass the surjectivity from \(\overline{\partial}\) to \(\overline{\partial}^E\), we remark the following which is not hard to prove (see [31, Sect. 39]).

Proposition 4 (a) Let \(X\) be a semi-reflexive locally convex Hausdorff space and \(Y\) a Fréchet space. Then \(L_b(X'_b, Y'_b) \cong L_b(Y, (X'_b)'_b)\) via taking adjoints.

(b) Let \(X\) be a Montel space and \(E\) a locally convex Hausdorff space. Then \(L_b(X'_b, E) \cong X\varepsilon E\) where the topological isomorphism is the identity map.

Theorem 5 Let \(\mathcal{E}\mathcal{V}(\Omega)\) be a Schwartz space and \(\mathcal{E}\mathcal{V}_{\ast}(\Omega)\) a nuclear subspace satisfying property \((\Omega)\). Assume that the scalar-valued operator \(\overline{\partial} : \mathcal{E}\mathcal{V}(\Omega) \rightarrow \mathcal{E}\mathcal{V}(\Omega)\) is surjective. Moreover, if
(a) \( E := F'_b \) where \( F \) is a Fréchet space over \( \mathbb{C} \) satisfying \((DN)\), or
(b) \( E \) is an ultrabornological PLS-space over \( \mathbb{C} \) satisfying \((PA)\),

then

\[
\overline{\partial}^E : \mathcal{E}V(\Omega, E) \to \mathcal{E}V(\Omega, E)
\]

is surjective.

**Proof** Throughout this proof we use the notation \( X'' := (X'_b)'_b \) for a locally convex Hausdorff space \( X \). In both cases, (a) and (b), the space \( E \) is a complete locally convex Hausdorff space. The space \( \mathcal{E}V(\Omega) \) is a Fréchet space by [34, Proposition 3.7, p. 240] and so its closed subspace \( \mathcal{E}V(\Omega) \) as well. Further, \( \mathcal{E}V(\Omega) \) is a Schwartz space and \( \mathcal{E}V(\Omega) \) nuclear, thus both spaces are reflexive. As the Fréchet-Schwartz space \( \mathcal{E}V(\Omega) \) is a Montel space,

\[
S : \mathcal{E}V(\Omega) \ni E \to \mathcal{E}V(\Omega, E), \ u \mapsto [z \mapsto u(\delta_z)],
\]

is a topological isomorphism by [36, 3.21 Example b), p. 14] where \( \delta_z \) is the point-evaluation at \( z \in \Omega \). We denote by \( J : E \to E'' \) the canonical injection in the algebraic dual \( E'' \) of the topological dual \( E' \) and for \( f \in \mathcal{E}V(\Omega, E) \) we set

\[
R'_f : \mathcal{E}V(\Omega)' \to E'' , \ y \mapsto [e' \mapsto y(e' \circ f)].
\]

Then the map \( f \mapsto J^{-1} \circ R'_f \) is the inverse of \( S \) by [36, 3.17 Theorem, p. 12]. The sequence

\[
0 \to \mathcal{E}V(\Omega) \xrightarrow{i} \mathcal{E}V(\Omega) \xrightarrow{\overline{\partial}} \mathcal{E}V(\Omega) \to 0,
\]

where \( i \) means the inclusion, is a topologically exact sequence of Fréchet spaces because \( \overline{\partial} \) is surjective by assumption. Let us denote by \( J_0 : \mathcal{E}V(\Omega) \to \mathcal{E}V(\Omega)'' \) and \( J_1 : \mathcal{E}V(\Omega) \to \mathcal{E}V(\Omega)'' \) the canonical embeddings which are topological isomorphisms since \( \mathcal{E}V(\Omega) \) and \( \mathcal{E}V(\Omega) \) are reflexive. Then the exactness of (4) implies that

\[
0 \to \mathcal{E}V(\Omega)'' \xrightarrow{i_0} \mathcal{E}V(\Omega)'' \xrightarrow{\overline{\partial}_1} \mathcal{E}V(\Omega)'' \to 0,
\]

where \( i_0 := J_0 \circ i \circ J_0^{-1} \) and \( \overline{\partial}_1 := J_1 \circ \overline{\partial} \circ J_1^{-1} \), is an exact topological sequence. Topological as the (strong) bidual of a Fréchet space is again a Fréchet space by [54, Corollary 25.10, p. 298].

(a) Let \( E := F'_b \) where \( F \) is a Fréchet space with \((DN)\). Then \( \operatorname{Ext}^1(F, \mathcal{E}V(\Omega)''') = 0 \) by [65, 5.1 Theorem, p. 186] since \( \mathcal{E}V(\Omega) '' \) satisfies \((\Omega)\) and therefore \( \mathcal{E}V(\Omega)'' \) as well. Combined with the exactness of (5) this implies that the sequence

\[
0 \to L(F, \mathcal{E}V(\Omega)''') \xrightarrow{i_0^*} L(F, \mathcal{E}V(\Omega)''') \xrightarrow{\overline{\partial}_1^*} L(F, \mathcal{E}V(\Omega)''') \to 0
\]

is exact by [57, Proposition 2.1, p. 13-14] where \( i_0^* (B) := i_0 \circ B \) and \( \overline{\partial}_1^* (D) := \overline{\partial}_1 \circ D \) for \( B \in L(F, \mathcal{E}V(\Omega)'''') \) and \( D \in L(F, \mathcal{E}V(\Omega)''') \). In particular, we obtain that

\[
\overline{\partial}_1^* : L(F, \mathcal{E}V(\Omega)''') \to L(F, \mathcal{E}V(\Omega)''')
\]

is surjective. Via \( E = F'_b \) and Proposition 4 \((X = \mathcal{E}V(\Omega) \text{ and } Y = F)\) we have the topological isomorphism

\[
\psi := S \circ i' (\cdot) : L(F, \mathcal{E}V(\Omega)''') \to \mathcal{E}V(\Omega, E), \ \psi(u) = (S \circ i' (\cdot))(u) = [z \mapsto i(u(\delta_z))].
\]
and the inverse
\[ \psi^{-1}(f) = (S \circ t')^{-1}(f) = (t' \circ S^{-1})(f) = t'(J^{-1} \circ R_f' ), \quad f \in \mathcal{E}V(\Omega, E). \]

Let \( g \in \mathcal{E}V(\Omega, E) \). Then \( \psi^{-1}(g) \in L(F, \mathcal{E}V(\Omega))' \) and by the surjectivity of (6) there is \( u \in L(F, \mathcal{E}V(\Omega))' \) such that \( \overline{\partial} u = \psi^{-1}(g) \). So we get \( \psi(u) \in \mathcal{E}V(\Omega, E) \). Next, we show that \( \overline{\partial} \psi(u) = g \) is valid. Let \( x \in F, z \in \Omega \) and \( h \in \mathbb{R}, h \neq 0 \), and \( e_k \) denote the kth unit vector in \( \mathbb{R}^2 \). From
\[ \frac{\delta z + he_k - \delta z}{h}(f) = f(z + he_k) - f(z) \xrightarrow[h \to 0]{} \partial e_k f(z), \]
for every \( f \in \mathcal{E}V(\Omega) \) it follows that \( \frac{\delta z + he_k - \delta z}{h} \) converges to \( \delta z \circ \partial e_k \) in \( \mathcal{E}V(\Omega)' \). Since the Fréchet–Schwartz space \( \mathcal{E}V(\Omega) \) is in particular a Montel space, we deduce that \( \frac{\delta z + he_k - \delta z}{h} \) converges to \( \delta z \circ \partial e_k \) in \( \mathcal{E}V(\Omega)' = \mathcal{E}V(\Omega)'_b \) by the Banach–Steinhaus theorem. Let \( B \subset F \) be bounded. As \( t' u \in L(\mathcal{E}V(\Omega)_b, F'_b) \), there are a bounded set \( B_0 \subset \mathcal{E}V(\Omega) \) and \( C > 0 \) such that
\[ \sup_{x \in B} \left| \frac{t'u(\delta z + he_k) - t'u(\delta z)}{h} \right| (x) - \frac{t'u(\delta z \circ \partial e_k)}{h}(x) \]
\[ = \sup_{x \in B} \left| t'u \left( \frac{\delta z + he_k - \delta z}{h} - \delta z \circ \partial e_k \right) \right| (x) \leq C \sup_{x \in B_0} \left| \frac{\delta z + he_k - \delta z - \delta z \circ \partial e_k}{h} \right| (f) \xrightarrow[h \to 0]{} 0, \]
yielding to \( (\partial e_k)^E(\psi(u))(z) = t'u(\delta z \circ \partial) \). This implies \( \overline{\partial}^E(\psi(u))(z) = t'u(\delta z \circ \partial) \). So for all \( x \in F \) and \( z \in \Omega \) we have
\[ \overline{\partial}^E(\psi(u))(z)(x) = t'u(\delta z \circ \partial)(x) = u(x)(\delta z \circ \partial) = (\delta z \circ \partial, J_{-1}^{-1}(u(x))) \]
\[ = \langle \delta z, \overline{\partial} J_{-1}^{-1}(u(x)) \rangle = \langle [J_1 \circ \partial \circ J_{-1}^{-1}](u(x)), \delta z \rangle = \langle (\overline{\partial} \circ u)(x), \delta z \rangle \]
\[ = \langle (\overline{\partial} \circ u)(x), \delta z \rangle = \psi^{-1}(g)(x)(\delta z) = t'(J^{-1} \circ R_{g}')(x)(\delta z) \]
\[ = (J^{-1} \circ R_{g}')(\delta z)(x) = J^{-1}(J(g(z))(x) = g(z)(x). \]
Thus \( \overline{\partial}^E(\psi(u))(z) = g(z) \) for every \( z \in \Omega \), which proves the surjectivity.

(b) Let \( E \) be an ultrabornological PLS-space satisfying \( (PA) \). Since the nuclear Fréchet space \( \mathcal{E}V_{\mathcal{G}}(\Omega) \) is also a Schwartz space, its strong dual \( \mathcal{E}V_{\mathcal{G}}(\Omega)'_b \) is a DFS-space. By [6, Theorem 4.1, p. 577] we obtain \( \text{Ext}_{PLS}^1(\mathcal{E}V_{\mathcal{G}}(\Omega)'_b, E) = 0 \) as the bidual \( \mathcal{E}V_{\mathcal{G}}(\Omega)'' \) satisfies \( (\Omega) \), \( E \) is a PLS-space satisfying \( (PA) \) and condition (c) in the theorem is fulfilled because \( \mathcal{E}V_{\mathcal{G}}(\Omega)'_b \) is the strong dual of a nuclear Fréchet space. Moreover, we have \( \text{Proj}^1 E = 0 \) due to [66, Corollary 3.3.10, p. 46] because \( E \) is an ultrabornological PLS-space. Then the exactness of the sequence (5), [6, Theorem 3.4, p. 567] and [6, Lemma 3.3, p. 567] (in the lemma the same condition (c) as in [6, Theorem 4.1, p. 577] is fulfilled and we choose \( H = \mathcal{E}V_{\mathcal{G}}(\Omega)'' \) and \( F = G = \mathcal{E}V(\Omega)'' \), imply that the sequence
\[ 0 \to L(E'_b, \mathcal{E}V_{\mathcal{G}}(\Omega)'' \xrightarrow{i_0^*} L(E'_b, \mathcal{E}V(\Omega)'', \mathcal{E}V_{\mathcal{G}}(\Omega)'' \to 0 \]
is exact. The maps \( i_0^* \) and \( \overline{\partial}_1^* \) are defined like in part (a). Especially, we get that
\[ \overline{\partial}_1^* : L(E'_b, \mathcal{E}V(\Omega)'', \mathcal{E}V_{\mathcal{G}}(\Omega)'' \to L(E'_b, \mathcal{E}V(\Omega)'', \mathcal{E}V_{\mathcal{G}}(\Omega)'' \]
is surjective.
By [19, Remark 4.4, p. 1114] we have $L_b(\mathcal{E}\mathcal{V}(\Omega)'_b, E''') \cong L_b(E'_b, \mathcal{E}\mathcal{V}(\Omega)'')$ via taking adjoints since $\mathcal{E}\mathcal{V}(\Omega)$, being a Fréchet–Schwartz space, is a PLS-space and hence its strong dual an LFS-space, which is regular by [66, Corollary 6.7, 10. ↔ 11., p. 114], and $E$ is an ultrabornological PLS-space, in particular, reflexive by [17, Theorem 3.2, p. 58]. In addition, the map

$$T: L_b(\mathcal{E}\mathcal{V}(\Omega)'_b, E'') \rightarrow L_b(\mathcal{E}\mathcal{V}(\Omega)'_b, E),$$

defined by $T(u)(y) := \mathcal{J}^{-1}(u(y))$ for $u \in L(\mathcal{E}\mathcal{V}(\Omega)'_b, E'')$ and $y \in \mathcal{E}\mathcal{V}(\Omega)'$, is a topological isomorphism because $E$ is reflexive. Due to Proposition 4 (b) we obtain the topological isomorphism

$$\psi := S \circ \mathcal{J}^{-1} \circ 't(\cdot) : L_b(E'_b, \mathcal{E}\mathcal{V}(\Omega)''') \rightarrow \mathcal{E}\mathcal{V}(\Omega, E),$$

with the inverse given by

$$\psi^{-1}(f) = (S \circ \mathcal{J}^{-1} \circ 't(\cdot))^{-1}(f) = [\mathcal{J}(\mathcal{J}^{-1} \circ 't(\cdot))(f)] = [\mathcal{J} \circ \mathcal{J}^{-1} \circ R'_f] = {'}(R'_f)$$

for $f \in \mathcal{E}\mathcal{V}(\Omega, E)$.

Let $g \in \mathcal{E}\mathcal{V}(\Omega, E)$. Then $\psi^{-1}(g) \in L(E'_b, \mathcal{E}\mathcal{V}(\Omega)''')$ and by the surjectivity of (7) there exists $u \in L(E'_b, \mathcal{E}\mathcal{V}(\Omega)''')$ such that $\mathcal{J}^{-1}u = \psi^{-1}(g)$. So we have $\psi(u) \in \mathcal{E}\mathcal{V}(\Omega, E)$. The last step is to show that $\overline{\mathcal{J}}E(\psi(u))(z) = \mathcal{J}^{-1}(\mathcal{J}u(\delta_z \circ \overline{\delta})))$

and for every $x \in E'$

$$\mathcal{J}u(\delta_z \circ \overline{\delta})(x) = u(x)(\delta_z \circ \overline{\delta}) = \overline{\delta}^u(\delta_z \circ \overline{\delta}) = \psi^{-1}(g)(x)(\delta_z) = {'}(R'_g)(x)(\delta_z)$$

Thus we have $\mathcal{J}u(\delta_z \circ \overline{\delta}) = \mathcal{J}(g(z))$ and therefore $\overline{\mathcal{J}}E(\psi(u))(z) = g(z)$ for all $z \in \Omega$. □

By Remark 2 case (a) is included in case (b) if $F$ is a Fréchet–Schwartz space. Therefore (a) is only interesting for Fréchet spaces $F$ which are not Schwartz spaces. In the next more technical section we will present sufficient conditions for $\mathcal{E}\mathcal{V}_{\overline{\delta}}(\Omega)$ to have $(\Omega)$ as well as concrete examples of such spaces.

4 (Ω) for $\mathcal{O}\mathcal{V}$-spaces on strips and applications of the main result

In this section we give some sufficient conditions such that the assumptions of our main result Theorem 5 are fulfilled. The outline is as follows. First, we show that $\mathcal{O}\mathcal{V}(\Omega)$ and $\mathcal{E}\mathcal{V}_{\overline{\delta}}(\Omega)$ coincide topologically under mild assumptions on the weights $\mathcal{V}$ and the sequence of sets $(\Omega_n)$. These mild conditions also imply that $\mathcal{E}\mathcal{V}(\Omega)$ is nuclear, in particular Schwartz, and thus its subspace $\mathcal{E}\mathcal{V}_{\overline{\delta}}(\Omega) = \mathcal{O}\mathcal{V}(\Omega)$ too. Second, we reduce the problem whether the projective limit $\mathcal{O}\mathcal{V}(\Omega)$ has $(\Omega)$ to the problem whether it is weakly reduced in the case that the $\Omega_n$ are strips along the real axis and the weights have a certain structure. Third, we use a similar result for $\mathcal{E}\mathcal{V}_{\overline{\delta}}(\Omega)$ which was obtained in [33] to prove the weak reducibility of $\mathcal{O}\mathcal{V}(\Omega)$. For corresponding results in the case that $\Omega_n = \Omega$ for all $n \in \mathbb{N}$ see [20, Theorem 3, p. 56], [39, 1.3 Lemma, p. 418] and [58, Theorem 1, p. 145]. We close this section with some examples of our main result. Let us start with the sufficient conditions, guaranteeing
that the projective limit $\mathcal{E}(\Omega)$ is nuclear (if $q = 1$). They also allow to switch from sup- to weighted $L^q$-seminorms which is important for the proof of surjectivity of the scalar-valued $\overline{\partial}$-operator given in [33], using Hörmander’s $L^2$-machinery (if $q = 2$).

**Condition (PN)** ([33, 3.3 Condition, p. 7]) Let $\mathcal{V} := (\nu_n)_{n \in \mathbb{N}}$ be a directed family of continuous weights on an open set $\Omega \subset \mathbb{R}^2$ and $(\Omega_n)_{n \in \mathbb{N}}$ be a family of non-empty open sets such that $\Omega_n \subset \Omega_{n+1}$ and $\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n$. For every $k \in \mathbb{N}$ let there be $\rho_k \in \mathbb{R}$ such that $0 < \rho_k < \delta^k\|\cdot\|_{C^k}(x, \partial\Omega_k)$ for all $x \in \Omega_k$ and let there be $q \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ there is $\psi_n \in L^q(\Omega_k), \psi_n > 0$, and $\mathbb{N} \ni J_n(n) \geq n$ and $C_\nu(n) > 0$ such that for any $x \in \Omega_k$:

\[
(PN.1) \quad \sup_{\xi \in \mathbb{R}^2, \|\xi\|\leq \rho_k} \nu_n(x + \xi) \leq C_\nu(n) \inf_{\xi \in \mathbb{R}^2, \|\xi\|\leq \rho_k} \nu_{J_n(n)}(x + \xi).
\]

\[
(PN.2) \quad \nu_n(x) \leq C_\nu(n) \psi_n(x)\nu_{J_n(n)}(x)
\]

**Example 6** Let $\Omega := \mathbb{R}^2$ and $\Omega_n := \{x = (x_i) \in \mathbb{R}^2 \mid |x_2| < n\}$. Let $0 < \gamma \leq 1$ and $(a_n)_{n \in \mathbb{N}}$ be strictly increasing such that $a_n \geq 0$ for all $n \in \mathbb{N}$ or $a_n \leq 0$ for all $n \in \mathbb{N}$. The family $\mathcal{V} := (\nu_n)_{n \in \mathbb{N}}$ of positive continuous functions on $\Omega$ given by

$$v_n : \Omega \to (0, \infty), \quad v_n(x) := e^{a_n|x_1|\gamma},$$

fulfills $v_n \leq v_{n+1}$ all $n \in \mathbb{N}$ and (PN) for every $q \in \mathbb{N}$ with $\psi_n(x) := (1 + |x|^2)^{-2}, x \in \mathbb{R}^2$, for every $n \in \mathbb{N}$.

The space $\mathcal{O}(\mathbb{C})$ with this kind of weights consists of functions which are entire and exponentially growing ($a_n < 0$) resp. decreasing ($a_n > 0$) with order $\gamma$ on strips along the real axis. This example of weights and many more are included in [33, 3.7 Example, p. 9]. We restrict to this particular weights because we use it in an example for our main result.

**Proposition 7** Let $\mathcal{V} := (\nu_n)_{n \in \mathbb{N}}$ be a directed family of continuous weights on an open set $\Omega \subset \mathbb{R}^2$ and $(\Omega_n)_{n \in \mathbb{N}}$ a family of non-empty open sets such that $\Omega_n \subset \Omega_{n+1}$ and $\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n$. If (PN.1) is fulfilled, then

(a) for every $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$ there is $C > 0$ such that

$$|f|_{n,m} \leq C|f|_{2J_n(n)}, \quad f \in \mathcal{O}_{v_{2J_n(n)}}(\Omega_{2J_n(n)}).$$

(b) $\mathcal{E}(\mathcal{V}) = \mathcal{O}(\mathcal{V})$ as Fréchet spaces.

**Proof** (a) Let $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$. We note that $\Omega_{n+1} \subset \Omega_{2J_n(n)}$ and $\partial^\beta f(x) = i^{\beta_2}f(|\beta|)(x)$, $x \in \Omega_{2J_n(n)}$, holds for all $\beta = (\beta_1, \beta_2) \in \mathbb{N}_0^2$ and $f \in \mathcal{O}_{v_{2J_n(n)}}(\Omega_{2J_n(n)})$ where $f(|\beta|)$ is the $|\beta|$th complex derivative of $f$. Then we obtain via (PN.1) and Cauchy’s inequality

$$|f|_{n,m} = \sup_{\beta \in \mathbb{N}_0^2, |\beta| \leq m} |\beta||f(x)||\nu_n(x) \leq \sup_{\beta \in \mathbb{N}_0^2, |\beta| \leq m} \frac{|\beta|!}{\rho_n^{|\beta|}} \max_{\xi \in \mathbb{R}^2} |f(\xi)||\nu_n(x)$$

\[
(\text{PN.1}) \quad \leq C_1 \sup_{\beta \in \mathbb{N}_0^2, |\beta| \leq m} \frac{|\beta|!}{\rho_n^{|\beta|}} \max_{\xi \in \mathbb{R}^2} |f(\xi)||\nu_{J_n(n)}(\xi)
\]

\[
\leq C_1 \sup_{\beta \in \mathbb{N}_0^2, |\beta| \leq m} \frac{|\beta|!}{\rho_n^{|\beta|}} \sup_{\xi \in \Omega_{2J_n(n)+1}} |f(\xi)||\nu_{J_n(n)}(\xi) \leq C_1 \sup_{\beta \in \mathbb{N}_0^2, |\beta| \leq m} \frac{|\beta|!}{\rho_n^{|\beta|}} |f|_{2J_n(n)}.
\]

(b) The space $\mathcal{E}(\mathcal{V})$ is a Fréchet space since it is a closed subspace of the Fréchet space $\mathcal{E}(\mathcal{V})$ by [34, Proposition 3.7, p. 240]. From part (a) and $|f|_n = |f|_{n,0}$ for all $n \in \mathbb{N}$ and $f \in \mathcal{E}(\mathcal{V})$ follows the statement.
Let us come to the second part. Using special weight functions, strips along the real axis as \( \Omega_n \) and a decomposition theorem of Langenbruch, we will see that answering the question whether \( \mathcal{O}V(\Omega) \) satisfies the property \((\Omega)\) of Vogt boils down to answering whether the projective limit \( \mathcal{O}V(\Omega) \) is weakly reduced. The special weights we want to consider are generated by a function \( \mu \) with the following properties.

**Definition 8** (strong weight generator) A continuous function \( \mu : \mathbb{C} \to [0, \infty) \) is called a **weight generator** if \( \mu(z) = \mu(|\text{Re}(z)|) \) for all \( z \in \mathbb{C} \), the restriction \( \mu|[0, \infty) \) is strictly increasing, 

\[
\lim_{x \to \infty} \frac{\ln(1 + |x|)}{\mu(x)} = 0
\]

and

\[
\exists \Gamma > 1, \ C > 0 \ \forall \ x \in [0, \infty) : \ \mu(x+1) \leq \Gamma \mu(x) + C.
\]

If \( \mu \) is a weight generator which fulfils the stronger condition

\[
\exists \Gamma > 1 \ \forall \ n \in \mathbb{N} \ \exists \ C > 0 \ \forall \ x \in [0, \infty) : \ \mu(x+n) \leq \Gamma \mu(x) + C,
\]

then \( \mu \) is called a **strong weight generator**.

Weight generators are introduced in [46, Definition 2.1, p. 225] and strong weight generators in [60, Definition 2.2.2, p. 43] where they are simply called weight functions resp. strong weight functions. For a weight generator \( \mu \) we define the space

\[
H_\tau(S_t) := \{ f \in \mathcal{O}(S_t) \mid \|f\|_{\tau,t} := \sup_{z \in S_t} |f(z)| e^{\tau \mu(z)} < \infty \}
\]

for \( t > 0 \) and \( \tau \in \mathbb{R} \) with the strip \( S_t := \{ z \in \mathbb{C} \mid |\text{Im}(z)| < t \} \).

**Theorem 9** [46, Theorem 2.2, p. 225] \(^1\) Let \( \mu \) be a weight generator. There are \( \tilde{t}, K_1, K_2 > 0 \) such that for any \( \tau_0 < \tau < \tau_2 \) there is \( C_0 = C_0(\text{sign}(\tau)) \) such that for any \( 0 < 2\tau_0 < t < \tau_2 < \tilde{t} \) with

\[
t_0 \leq \min \left[ K_1, K_2 \sqrt{\frac{\tau - C_0 \tau_0}{\tau_2 - C_0 \tau_0}} \right]
\]

there is \( C_1 \geq 1 \) such that for any \( r \geq 0 \) and any \( f \in H_\tau(S_t) \) with \( \|f\|_{\tau,t} \leq 1 \) the following holds: there are \( f_2 \in \mathcal{O}(S_{\tau_2}) \) and \( f_0 \in \mathcal{O}(S_{\tau_0}) \) such that \( f = f_0 + f_2 \) on \( S_{\tau_0} \) and

\[
\|f_0\|_{C_0 \tau_0, t_0} \leq C_1 e^{-Gr} \quad \text{and} \quad \|f_2\|_{t_2, \tau_2} \leq e^r
\]

where

\[
G := K_1 \min \left[ 1, \frac{t - t_0}{2\tau}, \frac{\tau - C_0 \tau_0}{\tau_2 - C_0 \tau_0} \right].
\]

To apply this theorem, we have to know the constants involved. In the following the notation of [46] is used and it is referred to the corresponding positions resp. conditions for these constants. We have

\[
\tilde{t} := \frac{1}{4 \ln(\Gamma)}
\]

\(^1\) A superfluous constant depending on \( \text{sign}(\tau_0) \) is omitted.
by [46, Lemma 2.4, (2.15), p. 228] with \( \Gamma \) from Definition 8 such that \( \Gamma' \geq e^{1/4} \). The choice \( \Gamma' \geq e^{1/4} \) comes from wanting \( \tilde{t} \leq 1 \) in [46, Lemma 2.4, p. 228]. By [46, Corollary 2.6, p. 230-231] we have

\[
C_0 := \begin{cases} 
4\Gamma B_3 = \frac{64 \cosh(1)}{\cos(1/2)^2} \Gamma^2 > 1, & \tau < 0, \\
\frac{1}{4\Gamma B_3} = \frac{64 \cosh(1)}{64 \cosh(1)\Gamma^2} < 1, & \tau \geq 0,
\end{cases}
\]

where \( B_3 := \frac{16 \cosh(1)}{\cos(1/2)} \Gamma \) by [46, Lemma 2.4, p. 228–229].\(^2\) To get the constants \( K_1 \) and \( K_2 \), we have to analyze the conditions for \( t_0 \) in the proof of [46, Theorem 2.2, p. 225]. By the assumptions on \( \tau_0 \), \( \tau \) and \( \tau_2 \) and the choice of \( C_0 \) we obtain

\[
\tau_2 - C_0 \tau_0 > \tau_2 - C_0 \tau \geq \tau_2 - \tau > 0
\]

and

\[
\tau - C_0 \tau_0 > \tau - C_0 \tau = \tau (1 - C_0) > 0.
\]

By choosing \( D > 0 \) in the proof of [46, Theorem 2.2, (2.22), p. 232–233] as \( D := \frac{\tau - C_0 \tau_0}{(\tau_2 - C_0 \tau_0) 2\Gamma_0} \), the estimate

\[
D = \frac{\tau - C_0 \tau_0}{(\tau_2 - C_0 \tau_0) 2\Gamma_0} = \min\left( \frac{1}{2\Gamma_0}, \frac{1}{2\Gamma_0} \right) \frac{\tau - C_0 \tau_0}{\tau_2 - C_0 \tau_0} \leq \min\left( \frac{1}{2\Gamma_0}, \frac{1}{2\Gamma_0} \right) \frac{\tau - C_0 \tau_0}{\tau_2 - C_0 \tau_0}
\]

holds where \( \Gamma_0 := \max(\tilde{\Gamma}, \tilde{\Gamma}) \) with \( \tilde{\Gamma}, \tilde{\Gamma} \geq 1 \) from the proof. With \( \theta \geq \frac{t - t_0}{2\tau} \) (p. 232) we get on p. 233, below (2.24), due to the condition \( t_0 \leq T_0 := \min\left( \frac{t}{2}, \frac{1}{4a^2 B_1} \right) \),

\[
\min\left( \frac{\theta}{2}, D, 1 \right) \geq \min\left( \frac{1}{2}, \frac{1}{2\Gamma_0} \right) \min\left( \theta, \frac{\tau - C_0 \tau_0}{\tau_2 - C_0 \tau_0}, 1 \right) \geq \frac{1}{2\Gamma_0} \left( \frac{t - t_0}{2\tau}, \frac{\tau - C_0 \tau_0}{\tau_2 - C_0 \tau_0}, 1 \right). 
\]

\[
= \min\left( \frac{1}{2\Gamma_0}, \frac{1}{4a^2 B_1} \right) \min\left( \frac{t - t_0}{2\tau}, \frac{\tau - C_0 \tau_0}{\tau_2 - C_0 \tau_0}, 1 \right) =: G
\]

where \( a := \ln(\Gamma) \) (in the middle of p. 231) and \( B_1 := 2 \cosh(1) \) by the proof of [46, Lemma 2.3, p. 226–227]. The assumptions \( 2t_0 < t \) and \( t_0 \leq K_1 \) in Theorem 9 guarantee that the condition \( t_0 \leq T_0 \) is satisfied. Looking at the condition \( t_0 \leq T_1 := \frac{D}{\sqrt{a^2 B_1}} \) (p. 232), we derive

\[
T_1 = \frac{1}{\sqrt{2\Gamma_0 a^2 B_1}} \sqrt{\frac{\tau - C_0 \tau_0}{\tau_2 - C_0 \tau_0}} = \frac{1}{\sqrt{2\cosh(1)\Gamma_0 \ln(\Gamma)}} \sqrt{\frac{\tau - C_0 \tau_0}{\tau_2 - C_0 \tau_0}} =: K_2
\]

For the subsequent theorem we merge and modify the proofs of [60, Satz 2.2.3, p. 44] \(^3\) (\( a_\gamma = n, n \in \mathbb{N} \), and \( \mu \) a strong weight generator) and [32, 5.20 Theorem, p. 84] (\( a_\gamma = -1/n, n \in \mathbb{N} \), and \( \mu = |\text{Re}(\cdot)| \)).

\(^2\) An error in part b) of this lemma, p. 229, is corrected here such that the term \( \cos(1/2) = \min_{|x| \leq 1/2} \cos(1/2) \) appears.

\(^3\) The proof of [60, Satz 2.2.3, p. 44] relies on [60, Satz 2.2.1, p. 43] which is an announced version (without a proof) of our result Corollary 13 on weak reducibility.
Further, we deduce from (11) that

$$n_0 \text{ or } \lim_{n \to \infty} q \text{ where } \widetilde{f}$$

is the canonical projection. Let $p, k \in \mathbb{N}$. As $(a_n)_{n \in \mathbb{N}}$ is strictly increasing and $\lim_{n \to \infty} a_n = 0$ or $\lim_{n \to \infty} q = \infty$, we may choose $q \in \mathbb{N}$ such that $a_{m_p}/C_0 < a_q$ and $2m_p < q$. To use the decomposition from Theorem 9, we need a linear transformation between strips to get the decomposition on the desired strip $S_{m_p}$. We choose $\Gamma \geq e^{1/4}$ and $T \in \mathbb{R}$ such that

$$0 < T < \frac{1}{4 \max(q + 1, m_k) \ln(T)} \quad (10)$$

which also fulfils

$$T \leq \frac{1}{m_p} \min\left(\frac{1}{2 \Gamma_0}, \frac{1}{2 \cosh(1)} \frac{1}{\ln(T)} \frac{1}{\sqrt{\ln(T)}} \frac{1}{\max(a_q, a_{m_k}) \ln(T)} \right). \quad (11)$$

Let

$$\tau_0 := \frac{a_{m_p}}{C_0}, \quad \tau := a_q, \quad \tau_2 := \max(a_{q+1}, a_{m_k}),$$

$$t_0 := m_p T, \quad t := q T, \quad t_2 := \max(q + 1, m_k) T.$$

By the choice of $q$ we have

$$\tau_0 = \frac{a_{m_p}}{C_0} < a_q = \tau < \max(a_{q+1}, a_{m_k}) = \tau_2.$$

By the choice of $q$ and (10) we get

$$0 < 2t_0 = 2m_p T < q T = t < \max(q + 1, m_k) T = t_2 < \frac{1}{4 \ln(T)} = \tilde{t}.$$

Further, we deduce from (11) that

$$t_0 = m_p T \leq \min\left[K_1, K_2 \sqrt{\frac{\tau - C_0 \tau_0}{\tau_2 - C_0 \tau_0}}\right].$$

Let $r \geq 0$ and $f \in \mathcal{O}(\mathbb{C})$ such that $|f|_q = \|f\|_{a_q, q} \leq 1$. We set $\tilde{f} : S_q T \to \mathbb{C}, \tilde{f}(z) := f(z/T)$, and define

$$H_{\tilde{f}}(S_r) := \{g \in \mathcal{O}(S_r) \mid \|g\|_{\tilde{f}, r} := \sup_{z \in S_r} |g(z)| \exp(\tau \tilde{\mu}(z)) < \infty\}$$

where $\tilde{\mu} := \mu(\cdot/T)$. We note that for $\tilde{n} := \lceil 1/T \rceil$, where $\lceil \cdot \rceil$ is the ceiling function, there is $C > 0$ such that for all $x \geq 0$

$$\tilde{\mu}(x + 1) = \mu\left(\frac{x + 1}{T}\right) \leq \mu\left(\frac{x}{T} + \frac{1}{T}\right) = \mu\left(\frac{x}{T} + \tilde{n}\right) \leq \Gamma \mu\left(\frac{x}{T}\right) + C = \Gamma \tilde{\mu}(x) + C.$$
because $\mu$ is a strong weight generator. We conclude that $\tilde{\mu}$ is also a weight generator with the same $\Gamma$ as $\mu$ which is independent of $T$. Moreover, from

$$\|\tilde{f}\|_{r,T} = \sup_{z \in S_q} |f(z)|e^{a_q\tilde{\mu}(z)} = \sup_{z \in S_q} |f(z)|e^{a_q\mu(z)} = |f|_q \leq 1$$

it follows by Theorem 9 that there are $\tilde{f}_j \in \mathcal{O}(S_t)$, $j \in \{0, 2\}$, such that

$$\tilde{f}(z) = \tilde{f}_0(z) + \tilde{f}_2(z), \quad z \in S_0,$$  \hspace{1cm} (12)

and

$$C_1 e^{-Gr} \geq \|\tilde{f}_0\|_{\tau_0,0} = \sup_{z \in S_0/T} |\tilde{f}_0(Tz)|e^{\epsilon_0\tau_0\tilde{\mu}(Tz)} = \sup_{z \in S_0} |f_0(z)|e^{a_{m_k}\mu(z)} = |f_0|_{m_k},$$  \hspace{1cm} (13)

where $f_0 \in \mathcal{O}(S_{m_k})$, as well as

$$e^\tau \geq \|\tilde{f}_2\|_{r_2, r_2} = \sup_{z \in S_2/T} |\tilde{f}_2(Tz)|e^{\epsilon_2\tau_2\tilde{\mu}(Tz)} \geq \sup_{z \in S_2} |f_2(z)|e^{a_{m_k}\mu(z)} = |f_2|_{m_k}$$  \hspace{1cm} (14)

where $f_2 \in \mathcal{O}(S_{r_2}/T) \subset \mathcal{O}(S_{m_k})$ and the inclusion is justified by the identity theorem. Furthermore, for $z \in S_0/T = S_{m_k}$ the equation

$$f(z) = \tilde{f}(Tz) = \tilde{f}_0(Tz) + \tilde{f}_2(Tz) = f_0(z) + f_2(z)$$

holds, thus $f = f_0 + f_2$ on $S_{m_k}$. By virtue of the weak reducibility of $\mathcal{O}\mathcal{V}(\mathbb{C})$ and the choice of $m_p, m_k$ the following is valid:

$$\forall \varepsilon > 0 \exists \tilde{f}_0, \tilde{f}_2 \in \mathcal{O}\mathcal{V}(\mathbb{C}) : (i) \left| \tilde{f}_0 - f_0 \right|_p < \varepsilon \quad \text{and} \quad (ii) \left| \tilde{f}_2 - f_2 \right|_k < \varepsilon. \quad (15)$$

Now, we have to consider two cases. Let $\varepsilon := C_1 e^{-Gr}$. For $k \leq p$ we get via (15) (i) that $f = \tilde{f}_0 + (f_2 - \tilde{f}_0)$ on $S_{m_p}$ so

$$f_2 + f_0 - \tilde{f}_0 = f - \tilde{f}_0 =: \overline{f}_2 \quad \text{on} \quad S_{m_p} \quad (16)$$

where the function $\overline{f}_2 \in \mathcal{O}\mathcal{V}(\mathbb{C})$ and thus is a holomorphic extension of the left-hand side on $\mathbb{C}$. Hence we clearly have $f = \tilde{f}_0 + \overline{f}_2$ and

$$|\tilde{f}_0|_p \leq |\tilde{f}_0 - f_0|_p + |f_0|_p \leq \varepsilon + |f_0|_p \leq \varepsilon + |f_0|_{m_p} \leq 2C_1 e^{-Gr} =: C_2 e^{-Gr} \quad (17)$$

as well as

$$|\overline{f}_2|_k \leq |\overline{f}_2 - f_2|_k + |f_2|_k \leq |f_0 - \tilde{f}_0|_p + |f_2|_{m_k} \leq \varepsilon + |f_2|_{m_k} \leq C_1 e^{-Gr} + e^\tau \leq (C_1 + 1) e^\tau =: C_3 e^\tau. \quad (18)$$

Analogously, for $k > p$ we obtain via (15) (ii) that $f = \tilde{f}_2 + (f_0 + f_2 - \tilde{f}_2)$ on $S_{m_p}$ so

$$f_0 + f_2 - \tilde{f}_2 = f - \tilde{f}_2 =: \tilde{f}_0 \quad \text{on} \quad S_{m_p} \quad (19)$$
where the function $\tilde{f}_0 \in \mathcal{O}V(\mathbb{C})$ and thus is a holomorphic extension of the left-hand side on $\mathbb{C}$. Hence we clearly have $f = \tilde{f}_0 + \tilde{f}_2$ and

$$|\tilde{f}_0|_p = |f - \tilde{f}_2|_p \leq |f_0 + f_2 - \tilde{f}_2|_p \leq |f_2 - \tilde{f}_2|_p + |f_0|_p \leq |f_2 - \tilde{f}_2|_k + |f_0|_{m_p}$$

as well as

$$|\tilde{f}_0|_p \leq C_1 e^{-Gr} = C_2 e^{-Gr}$$

(20)

Next, we set $n := [1/G]$ and $C := C_3 e^{\ln(C_2)/G}$. Let $\tilde{r} > 0$. For $\tilde{r} \geq 1$ there is $r \geq 0$ such that

$$\tilde{r} = e^{Gr - \ln(C_2)} = \frac{e^{Gr}}{C_2}$$

and we have by (17) and (18) for $k \leq p$

$$|\tilde{f}_0|_p \leq C_2 e^{-Gr} = \frac{1}{r}, \quad |\tilde{f}_2|_k \leq C_3 e^r = C_3 e^{\ln(C_2)\frac{1}{G} \frac{(Gr - \ln(C_2))}{r} \leq C_3 e^{\frac{r}{1} \frac{n}{C_3}} = C_{\tilde{r}} \frac{n}{C_3} \leq C_{\tilde{r}} n \frac{n}{C_3}$$

as well as by (20) and (21) for $k > p$

$$|\tilde{f}_0|_p \leq \frac{1}{r}, \quad |\tilde{f}_2|_k \leq C_{\tilde{r}} n.$$

For $0 < \tilde{r} < 1$ we have, since $q \geq p$,

$$|f|_p \leq |f|_q \leq 1 < \frac{1}{r}.$$

Thus our statement is proved. \qed

Let us remark that the choice of the sequence $(a_n)_{n \in \mathbb{N}}$ in the preceding theorem does not really matter.

**Remark 11** Let $\mu : \mathbb{C} \to [0, \infty)$ be continuous, $a_n \not\to 0$ or $a_n \not\to \infty$, $\mathcal{V} := \langle \exp(a_n \mu) \rangle_{n \in \mathbb{N}}$ and $\Omega_n := S_n$ for all $n \in \mathbb{N}$. Set $\mathcal{V}_- := \langle \exp((-1/n) \mu) \rangle_{n \in \mathbb{N}}$ and $\mathcal{V}_+ := \langle \exp(n \mu) \rangle_{n \in \mathbb{N}}$. Then

$$\mathcal{O}V(\mathbb{C}) \cong \mathcal{O}V_-(\mathbb{C}), \quad \text{if } a_n \not\to 0, \quad \text{and } \mathcal{O}V(\mathbb{C}) \cong \mathcal{O}V_+(\mathbb{C}), \quad \text{if } a_n \not\to \infty,$$

which is easily seen. Thus one may choose the most suitable sequence $(a_n)_{n \in \mathbb{N}}$ for one’s purpose without changing the space.

Let us turn to the third part. The following quite technical conditions guarantee a kind of weak reducibility of the projective limit $\mathcal{E}V(\Omega)$ and in combination with (P N.1) the weak reducibility of $\mathcal{O}V(\Omega)$ too.

**Condition (WR)** Let $\mathcal{V} := (\nu_n)_{n \in \mathbb{N}}$ be a directed family of continuous weights on an open set $\Omega \subset \mathbb{R}^2$ and $(\Omega_n)_{n \in \mathbb{N}}$ a family of non-empty open sets such that $\Omega_n \neq \mathbb{R}^2$, $\Omega_n \subset \Omega_{n+1}$ for all $n \in \mathbb{N}$, $d_{n,k} := d_{\Omega_n}(\partial \Omega_k) > 0$ for all $n, k \in \mathbb{N}$, $k > n$, and $\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n$.

(W R.1) For every $n \in \mathbb{N}$ let there be $g_n \in \mathcal{O}(\mathbb{C})$ with $g_n(0) = 1$ and $\mathbb{N} \ni I_j(n) > n$ such that

(a) for every $\varepsilon > 0$ there is a compact set $K \subset \overline{\Omega}_n$ with $v_n(x) \leq \varepsilon v_{I_1(n)}(x)$ for all $x \in \Omega_n \setminus K$.
(b) there is an open set \( X_{I_2(n)} \subseteq \mathbb{R}^2 \setminus \overline{I_2(n)} \) such that there are \( R_n, r_n \in \mathbb{R} \) with \( 0 < 2R_n < d^+(X_{I_2(n)}, \Omega_{I_2(n)}) \) and \( R_n < r_n < d_{X,I_2(n)} - R_n \) as well as \( A_2(\cdot, n) : X_{I_2(n)} + B_{R_n}(0) \to (0, \infty), A_2(\cdot, n)|_{X_{I_2(n)}} \) locally bounded, satisfying

\[
\max\{|g_n(\zeta)|v_{I_2(n)}(z) \mid \zeta \in \mathbb{R}^2, |\zeta - (z - x)| = r_n \} \leq A_2(x, n)
\]

for all \( z \in \Omega_{I_2(n)} \) and \( x \in X_{I_2(n)} + B_{R_n}(0) \).

(c) for every compact set \( K \subseteq \mathbb{R}^2 \) there is \( A_3(n, K) > 0 \) with

\[
\int_K \frac{|g_n(x - y)|v_n(x)}{|x - y|} \, dy \leq A_3(n, K), \quad x \in \Omega_n.
\]

(WR.2) Let (WR.1a) be fulfilled. For every \( n \in \mathbb{N} \) let there be \( \mathbb{N} \ni I_4(n) > n \) and \( A_4(n) > 0 \) such that

\[
\int_{\Omega_{I_4(n)}} \frac{|g_{I_4(n)}(x - y)|v_p(x)}{|x - y|v_k(y)} \, dy \leq A_4(n), \quad x \in \Omega_p,
\]

for \( (k, p) = (I_4(n), n) \) and \( (k, p) = (I_1(n), I_4(n)) \) where \( I_{14}(n) := I_1(I_4(n)) \).

(WR.3) Let (WR.1a), (WR.1b) and (WR.2) be fulfilled. For every \( n \in \mathbb{N} \), every closed subset \( M \subseteq \overline{\Omega}_n \) and every component \( N \) of \( M^C \) we have

\[
N \cap \overline{\Omega}_n^C \neq \emptyset \Rightarrow N \cap X_{I_{214}(n)} \neq \emptyset
\]

where \( I_{214}(n) := I_2(I_4(n)) \), fulfilling \( I_{214}(n) \geq I_{14}(n + 1) \).

(WR) is [33, 4.2 Condition, p. 10] combined with the assumption \( I_{214}(n) \geq I_{14}(n + 1), n \in \mathbb{N} \). We will see that \( \Omega_n := \{z \in \mathbb{C} \mid |\text{Im}(z)| < n\} \) and \( v_n(z) := \exp(\alpha_n |\text{Re}(z)|^\gamma) \) for some \( 0 < \gamma \leq 1 \) and \( \alpha_n \not\to 0 \) or \( \alpha_n \not\to \infty \) fulfill the conditions above with \( g_n(z) := \exp(-z^2) \).

**Theorem 12** [33, 4.3 Theorem, p. 10] Let \( n \in \mathbb{N} \). Then \( \pi_{I_{214}(n), n}(\mathcal{E}v_{I_{14}(n)}, \overline{\pi}(\Omega_{I_{214}(n)})) \) is dense in \( \pi_{I_{14}(n), n}(\mathcal{E}v_{I_{14}(n)}, \overline{\pi}(\Omega_{I_{14}(n)})) \) w.r.t. \( (\cdot \mid n, m)_{m \in \mathbb{N}_0} \) if (WR) is fulfilled.

As a consequence of this theorem, whose proof does not need the assumption \( I_{214}(n) \geq I_{14}(n + 1) \), we obtain that the projective limit \( \mathcal{O}V(\Omega) \) is weakly reduced, which is a generalisation of [32, 5.6 Corollary, p. 69] and [32, 5.11 Corollary, p. 75].

**Corollary 13** \( \mathcal{O}V(\Omega) \) is weakly reduced if (WR) and (P N.1) are satisfied.

**Proof** Let \( n \in \mathbb{N} \). We show that \( \pi_n(\mathcal{O}V(\Omega)) \) is dense in \( \pi_{2J_1I_{14}(n), n}(\mathcal{O}v_{2J_1I_{14}(n)}\Omega_{2J_1I_{14}(n)})) \) w.r.t. \( (\cdot \mid n, m) \) where \( J_1I_{14}(n) := J_1(I_{14}(n)) \) and

\[
\pi_n : \mathcal{O}V(\Omega) \to \mathcal{O}v_n(\Omega_n), \quad \pi_n(f) := f_{J_1I_{14}(n)}
\]

We omit the restriction maps in our proof. Due to Proposition 7 (a) the restrictions to \( \Omega_{I_{14}(n)} \) of functions from \( \mathcal{O}v_{2J_1I_{14}(n)}(\Omega_{2J_1I_{14}(n)}) \) are elements of \( \mathcal{E}v_{I_{14}(n)}, \overline{\pi}(\Omega_{I_{14}(n)}) \). Let \( \varepsilon > 0 \) and \( f_0 \in \mathcal{O}v_{2J_1I_{14}(n)}(\Omega_{2J_1I_{14}(n)}) \). For every \( j \in \mathbb{N} \) there exists

(i) \( f_j \in \mathcal{E}v_{I_{214}(n+j-1), \overline{\pi}(\Omega_{I_{214}(n+j-1)}) \) with

(ii) \( f_j|_{\Omega_{I_{14}(n+j)}} \in \mathcal{E}v_{I_{14}(n+j), \overline{\pi}(\Omega_{I_{14}(n+j)}) \subset \mathcal{O}v_{I_{14}(n+j)}(\Omega_{I_{14}(n+j)}) \)

such that

\[
|f_j - f_{j-1}|_{n+j-1} = |f_j - f_{j-1}|_{n+j-1, 0} < \frac{\varepsilon}{2^{j+1}}
\]

(22)
by Theorem 12 and the condition $I_{214}(k) \geq I_{14}(k + 1)$ for all $k \in \mathbb{N}$ from $(WR)$. Therefore we obtain for every $k \in \mathbb{N}$

$$|f_k - f_0|_n = \left| \sum_{j=1}^{k} f_j - f_{j-1} \right|_n \leq \sum_{j=1}^{k} |f_j - f_{j-1}|_n \leq \sum_{j=1}^{k} |f_j - f_{j-1}|_{n+j-1} \leq k \sum_{j=1}^{k} \frac{\varepsilon}{2^{j+1}} = \frac{\varepsilon}{2} \left( 1 - \frac{1}{2^k} \right) < \frac{\varepsilon}{2}. \quad (23)$$

Now, let $\varepsilon_0 > 0$ and $l \in \mathbb{N}$. We choose $l_0 \in \mathbb{N}$, $l_0 \geq l$, such that $\frac{\varepsilon}{2^{l_0+1}} < \varepsilon_0$. Similarly, we get for all $p \geq k \geq l_0$

$$|f_k|_n \leq |f_k|_{n+l_0} \leq \sum_{j=k+1}^{p} |f_j - f_{j-1}|_0 \leq \sum_{j=k+1}^{p} |f_j - f_{j-1}|_{n+j-1} \leq \frac{\varepsilon}{2} \left( 1 - \frac{1}{2^p} \right) < \frac{\varepsilon}{2^{l_0+1}} < \varepsilon_0. \quad (24)$$

Hence $(f_k)_{k \geq n_0}$ is a Cauchy sequence in the Banach space $O\nu_{I_{14}(n+n_0)}(\Omega_{I_{14}(n+n_0)})$ for every $n_0 \in \mathbb{N}_0$ and thus has a limit $F_{n_0} \in O\nu_{I_{14}(n+n_0)}(\Omega_{I_{14}(n+n_0)})$. These limits coincide on their common domain because for every $n_1, n_2 \in \mathbb{N}_0$ with $I_{14}(n_1 + n_1) < I_{14}(n_1 + n_2)$ and $\varepsilon_1 > 0$ there exists $N \in \mathbb{N}$ such that for all $k \geq N$

$$|F_{n_1} - F_{n_2}|_{I_{14}(n_1 + n_1)} \leq |F_{n_1} - f_k|_{I_{14}(n_1 + n_1)} + |f_k - F_{n_2}|_{I_{14}(n_1 + n_1)} \leq \frac{\varepsilon_1}{2} + \frac{\varepsilon_1}{2} = \varepsilon_1.$$

We deduce that the glued limit function $f$ given by $f := F_{n_0}$ on $\Omega_{I_{14}(n+n_0)}$ for all $n_0 \in \mathbb{N}_0$ is well-defined and we have $f \in \bigcap_{n_0 \in \mathbb{N}_0} O\nu_{I_{14}(n+n_0)}(\Omega_{I_{14}(n+n_0)}) = O\nu(\Omega)$ since $I_{14}(n + n_0) \geq n + n_0$. By the definition of $f$ there exists $N \in \mathbb{N}$ such that for every $k \geq N$

$$|f - f_0|_n \leq |f - f_k| + |f_k - f_0|_n \leq \frac{\varepsilon}{2} + |f_k - f_0|_n \leq \frac{\varepsilon}{2} + \varepsilon = \varepsilon,$$

which proves our statement. \hfill \Box

Combining Theorem 10 and Corollary 13, we obtain the following corollary.

**Corollary 14** Let $a_n \not\rightarrow 0$ or $a_n \not\rightarrow \infty$, $\mathcal{V} := (\exp(a_n))_{n \in \mathbb{N}}$ and $\Omega_n := S_n$ for all $n \in \mathbb{N}$ where

$$\mu : \mathbb{C} \rightarrow [0, \infty), \quad \mu(z) := |\text{Re}(z)|^\gamma,$$

for some $0 < \gamma \leq 1$. Then $O\nu(\mathbb{C})$ satisfies $(\Omega)$.

**Proof** We only need to check that the conditions of Theorem 10 are fulfilled. Obviously, $\mu(z) = \mu(|\text{Re}(z)|)$ for all $z \in \mathbb{C}$, $\mu$ is strictly increasing on $[0, \infty)$ and $\lim_{x \rightarrow \infty, x \in \mathbb{R}} \ln(1 + |x|) \mu(x) = 0$. The observation

$$\mu(x + n) - \mu(x) = |x + n|^\gamma - |x|^\gamma \leq |x + n - n|^\gamma = n^\gamma, \quad n \in \mathbb{N}, \quad x \in [0, \infty),$$

implies that $\mu$ is a strong weight generator with any $\Gamma > 1$ and $C := n^\gamma$ by Definition 8. Let us turn to the conditions $(WR)$ and $(P.N.1)$ which we need for the weak reducibility of $O\nu(\mathbb{C})$ by Corollary 13. Condition $(P.N.1)$ is fulfilled by Example 6. If $a_n < 0$ for all
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If \(a_n \geq 0\) for all \(n \in \mathbb{N}\), we only have to modify [33, 4.10 Example a), p. 22] a bit. We choose \(I_j(n) := 2n\) for \(j \in \{1, 2, 4\}\) and define the open set \(X_{I_2(n)} := S_{4n}\). Then we have

\[I_{214}(n) = 8n \geq 4n + 4 = I_{14}(n + 1), \quad n \in \mathbb{N}.\] Furthermore, we have \(d_{n,k} = |n - k|\) for all \(n, k \in \mathbb{N}\).

(WR.1a) and (WR.3): Verbatim as in [33, 4.10 Example a), p. 22].

(WR.1b): We have \(d_{X, I_2} = 2n\). We choose \(g_n : \mathbb{C} \to \mathbb{C}, g_n(z) := \exp(-z^2)\), as well as \(r_n := 1/(4n)\) and \(R_n := 1/(6n)\) for \(n \in \mathbb{N}\). Let \(z = z_1 + iz_2 \in \Omega_{I_2(n)} = S_{2n}\) and \(x \in X_{I_2(n)} + \mathbb{B}_{R_n}(0)\). For \(\xi = \xi_1 + i\xi_2 \in \mathbb{C}\) with \(|\xi - (z - x)| = r_n\) we have

\[|g_n(\xi)|e^{2a_n\mu(z)} = e^{-|\Re(z)|^2}e^{a_{2n}|\Re(z)|} \leq e^{-\xi_2^2 + \xi_2^2}e^{a_{2n}(1 + |z_1|)} \leq e^{(r_n + |z_2| + |z_2|)^2 + a_{2n}(1 + r_n + |z_1|)}e^{-|\xi|^2 + a_{2n}|\xi|} \leq e^{(r_n + 2n + |z_2|)^2 + a_{2n}(1 + r_n + |z_1|)}\sup_{r \in \mathbb{R}}e^{-r^2 + a_{2n}t}\]

\[= e^{(r_n + 2n + |z_2|)^2 + a_{2n}(1 + r_n + |z_1|) + a_{2n}^2/4} =: A_2(x, n)\]

and observe that \(A_2(\cdot, n)\) is continuous and thus locally bounded on \(X_{I_2(n)}\).

(WR.1c): Let \(K \subset \mathbb{C}\) be compact and \(x = x_1 + ix_2 \in \Omega_{I_2(n)}\). Then there is \(b > 0\) such that \(|y| \leq b\) for all \(y = y_1 + iy_2 \in K\) and from polar coordinates and Fubini’s theorem it follows that

\[
\int_K \frac{|g_n(x - y)|}{|x - y|} dy \leq \sup_{w \in K} e^{a_{2n}|\Re(w)|} \int_K \frac{e^{-|\Re((x - y)^2)|}}{|x - y|} e^{-a_{2n}|y_1|} dy =: C_1
\]

\[
\leq C_1 \left( \int_{B_1(x)} e^{-|\Re((x - y)^2)|} |x - y| e^{-a_{2n}|y_1|} dy + \int_{K \setminus B_1(x)} e^{-|\Re((x - y)^2)|} e^{-a_{2n}|y_1|} dy \right)
\]

\[
\leq C_1 \left( \int_0^{2\pi} \int_0^1 \frac{e^{-r^2 \cos(\varphi)}}{r} e^{-a_{2n}|x_1 + r \cos(\varphi)|} r dr d\varphi + \int_{K \setminus B_1(x)} e^{-|\Re((x - y)^2)|} e^{-a_{2n}|y_1|} dy \right)
\]

\[
\leq C_1 \left( 2\pi e^{1+a_{2n}} e^{-a_{2n}|x_1|} + \int_{-b}^b e^{(x_2 - y_2)^2} dy_2 \int_{|y_1 - x_1|} e^{-|y_2|^2 + a_{2n}|y_1|} dy_1 e^{-a_{2n}|x_1|} \right)
\]

\[
\leq C_1 \left( 2\pi e^{1+a_{2n}} + 2b e^{(x_2 + b)^2} \int_{|y_1 - x_1|} e^{-y_2^2 + a_{2n}|y_1|} dy_1 e^{-a_{2n}|x_1|} \right)
\]

\[
= C_1 \left( 2\pi e^{1+a_{2n}} + 2b e^{(x_2 + b)^2} e^{a_{2n}^2/4} \int_{|y_1 - a_{2n}/2|} e^{-|y_1| - a_{2n}/2} dy_1 \right)
\]

\[
= C_1 \left( 2\pi e^{1+a_{2n}} + 4b e^{(x_2 + b)^2} e^{a_{2n}^2/4} \int_{-a_{2n}/2}^{\infty} e^{-y^2} dy_1 \right)
\]

\[
\leq C_1 \left( 2\pi e^{1+a_{2n}} + 4\sqrt{\pi} b e^{(n+b)^2 + a_{2n}^2/4} e^{-a_{2n}|x_1|} \right).
\]
We conclude that \((W R.1 c)\) holds since
\[
e^{-a_{2n}|x_1|}e^{a_n|\text{Re}(x)|^\gamma} \leq e^{(a_n-a_{2n})|x_1|+a_n} \leq e^{a_n}.
\]

\((W R.2)\): Let \(p, k \in \mathbb{N}\) with \(p \leq k\). For all \(x = x_1 + i x_2 \in \Omega_p\) and \(y = y_1 + i y_2 \in \Omega_{I_4(n)}\) we note that
\[
a_p|\text{Re}(x)|^\gamma - a_k|\text{Re}(y)|^\gamma \leq a_k|x_1 - y_1|^\gamma \leq a_k(1 + |x_1 - y_1|)
\]
because \((a_n)_{n \in \mathbb{N}}\) is non-negative and increasing and \(0 < \gamma \leq 1\). Like before we deduce that
\[
\int_{\Omega_{I_4(n)}} \frac{|g_n(x-y)|v_p(x)}{|x-y|v_k(y)} \, dy \leq \int_{\Omega_{2n}} \frac{e^{-\text{Re}(x-y)^2/2}}{|x-y|} e^{a_k|\text{Re}(x)-\text{Re}(y)|^\gamma} \, dy \leq \int_{\Omega_{2n}} \frac{e^{-\text{Re}(x-y)^2/2}}{|x-y|} e^{a_k|\text{Re}(x)-\text{Re}(y)|^\gamma} \, dy
\]
\[
\leq 2\pi e^{1+ a_k} + e^{a_k} \int_{-2n}^{2n} e^{(x_2-y_2)^2} \, dy \int_{-\pi}^{\pi} e^{(x_1-y_1)^2 + a_k|x_1-y_1|} \, dy_1
\]
\[
\leq 2\pi e^{1+ a_k} + 8\pi ne^{a_k + (|x_2|+2n)^2 + a_k^2/4}
\]
\[
\leq 2\pi e^{1+ a_{I_4(n)}} + 8\pi ne^{a_{I_4(n)} + (|x_2|+2n)^2 + a_{I_4(n)}^2/4}
\]
for \((k, p) = (I_4(n), n)\) and \((k, p) = (I_4(n), I_{14}(n))\) as \((a_n)_{n \in \mathbb{N}}\) is non-negative and increasing.

We close this section with a special case of our main result on the surjectivity of the Cauchy–Riemann operator on \(\mathcal{E}V(\Omega, E)\). We recall the corresponding result for \(E = \mathbb{C}\) which we will need for the application of our main result. It is a consequence of the approximation Theorem 12 in combination with Hörmander’s solution of the \(\overline{\partial}\)-problem in weighted \(L^2\)-spaces [27, Theorem 4.4.2, p. 94] and the Mittag–Leffler procedure.

**Theorem 15** [33, 4.8 Theorem, p. 20] Let \((P N)\) with \(\psi_n(z) := (1 + |z|^2)^{-2}, z \in \Omega, and (WR) be fulfilled and \(- \ln v_n\) be subharmonic on \(\Omega\) for every \(n \in \mathbb{N}\). Then
\[
\overline{\partial}: \mathcal{E}V(\Omega) \to \mathcal{E}V(\Omega)
\]
is surjective.

A consequence of this theorem is the following corollary.

**Corollary 16** [33, 4.10 Example a, p. 22] Let \((a_n)_{n \in \mathbb{N}}\) be strictly increasing, \(a_n < 0\) for all \(n \in \mathbb{N}\), \(V := (\exp(a_n \mu))_{n \in \mathbb{N}}\) and \(\Omega_n := \{z \in \mathbb{C} \mid |\text{Im}(z)| < n\}\) for all \(n \in \mathbb{N}\) where
\[
\mu: \mathbb{C} \to [0, \infty), \mu(z) := |\text{Re}(z)|^\gamma,
\]
for some \(0 < \gamma \leq 1\). Then
\[
\overline{\partial}: \mathcal{E}V(\mathbb{C}) \to \mathcal{E}V(\mathbb{C})
\]
is surjective.

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The restriction to negative $a_n$ comes from the condition that $-\ln \nu_n$ should be subharmonic. We note that the $E$-valued versions of Theorem 15 and Corollary 16 where $E$ is a Fréchet space over $\mathbb{C}$ hold as well by the classical theory of tensor products for nuclear Fréchet spaces (see [33, 4.9 Corollary, p. 21]). Now, we use the results obtained so far to obtain a special case of our main result.

**Corollary 17** Let $\mu$ be a subharmonic strong weight generator and $V := (\exp(a_n\mu))_{n\in\mathbb{N}}$ with $a_n \not\to 0$. Let $(P N)$ with $\psi_n(z) := (1 + |z|^2)^{-2}$, $z \in \mathbb{C}$, and $(W R)$ with $\Omega_n := \{z \in \mathbb{C} \mid |\text{Im}(z)| < n\}$ for all $n \in \mathbb{N}$ be fulfilled. If

(a) $E := F'_b$ where $F$ is a Fréchet space over $\mathbb{C}$ satisfying $(D N)$, or
(b) $E$ is an ultrabornological PLS-space over $\mathbb{C}$ satisfying $(P A)$,

then

$$\overline{\partial}^E : \mathcal{E}V(\mathbb{C}, E) \to \mathcal{E}V(\mathbb{C}, E)$$

is surjective.

**Proof** The space $\mathcal{E}V(\mathbb{C})$ is nuclear, in particular Schwartz, by [37, Theorem 3.1, p. 188], [37, Remark 2.7, p. 178-179] and [37, Remark 2.3 (b), p. 177] because $(P N.1)$ and $(P N.2)$ from $(P N)$ are fulfilled. Hence the subspace $\mathcal{E}V_{\pi}(\mathbb{C}) = \mathcal{O}V(\mathbb{C})$ is nuclear by Proposition 7 (b) as well. Further, $\mathcal{O}V(\mathbb{C})$ is weakly reduced by Corollary 13 due to $(W R)$ and thus satisfies $(\Omega)$ by Theorem 10. Therefore, the assertion is a consequence of the surjectivity of $\overline{\partial}$ in the $\mathbb{C}$-valued case by Theorem 15 and our main result Theorem 5. $\square$

Corollary 17 generalises a part of [32, 5.24 Theorem, p. 95] ($K = \emptyset$) which is the case $\gamma = 1$ of the next corollary.

**Corollary 18** Let $a_n \not\to 0$, $V := (\exp(a_n\mu))_{n\in\mathbb{N}}$ and $\Omega_n := \{z \in \mathbb{C} \mid |\text{Im}(z)| < n\}$ for all $n \in \mathbb{N}$ where

$$\mu : \mathbb{C} \to [0, \infty), \quad \mu(z) := |\text{Re}(z)|^\gamma,$$

for some $0 < \gamma \leq 1$. If

(a) $E := F'_b$ where $F$ is a Fréchet space over $\mathbb{C}$ satisfying $(D N)$, or
(b) $E$ is an ultrabornological PLS-space over $\mathbb{C}$ satisfying $(P A)$,

then

$$\overline{\partial}^E : \mathcal{E}V(\mathbb{C}, E) \to \mathcal{E}V(\mathbb{C}, E)$$

is surjective.

**Proof** Follows from Corollary 17, (the proof of) Corollary 14 and Example 6. $\square$
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References

1. Agranovich, M.S.: Partial differential equations with constant coefficients. Rus. Math. Surv. 16(2), 23 (1961). https://doi.org/10.1070/rm1961v016n02abeh004104
2. Bierstedt, K.D., Pietsch, A., Ruess, W.M., Vogt, D. (eds.): Functional Analysis (Proc., Essen, 1991), Lect. Notes in Pure and Appl. Math. 150. Dekker, New York (1994)
3. Bonet, J., Domański, P.: Parameter dependence of solutions of partial differential equations in spaces of real analytic functions. Proc. Am. Math. Soc. 129(2), 495–503 (2001). https://doi.org/10.1090/S0002-9939-00-05867-6
4. Bonet, J., Domański, P.: Parameter dependence of solutions of differential equations on spaces of distributions and the splitting of short exact sequences. J. Funct. Anal. 230(2), 329–381 (2006). https://doi.org/10.1016/j.jfa.2005.06.007
5. Bonet, J., Domański, P.: The structure of spaces of quasianalytic functions of Roumieu type. Arch. Math. (Basel) 89(5), 430–441 (2007). https://doi.org/10.1007/s00013-007-2073-y
6. Bonet, J., Domański, P.: The splitting of exact sequences of PLS-spaces and smooth dependence of solutions of linear partial differential equations. J. Funct. Anal. 217, 561–585 (2008). https://doi.org/10.1016/j.jfa.2005.06.007
7. Braun, R.W., Meise, R., Taylor, B.A.: Surjectivity of constant coefficient partial differential operators on \( \mathbb{A}(\mathbb{R}^4) \) and Whitney’s \( \mathbb{C}_4 \)-cone. Bull. Soc. R. Sci. Liège 70(4–6), 195–206 (2001)
8. Braun, R.W., Meise, R., Vogt, D.: Applications of the projective limit functor to convolution and partial differential equations. In: T. Terzioglu (ed.) Advances in the Theory of Fréchet Spaces (Proc., Istanbul, 1988), NATO Sci. Ser. C Math. Phys. Sci., vol. 287, pp. 29–46. Kluwer, Dordrecht (1989). https://doi.org/10.1007/978-94-009-2456-7_4
9. Braun, R.W., Meise, R., Vogt, D.: Existence of fundamental solutions and surjectivity of convolution operators on classes of ultra-differentiable functions. Proc. Lond. Math. Soc. (3) 61(2), 344–370 (1990). https://doi.org/10.1112/plms/s3-61.2.344
10. Braun, R.W., Meise, R., Vogt, D.: Characterization of the linear partial differential operators with constant coefficients which are surjective on non-quasianalytic classes of Roumieu type on \( \mathbb{R}^N \). Math. Nachr. 168(1), 19–54 (1994). https://doi.org/10.1002/mana.19941680103
11. Browder, F.E.: Analyticity and partial differential equations. I. Am. J. Math. 84(4), 666–710 (1962). https://doi.org/10.2307/2372872
12. Dierolf, B.: Splitting theory for PLH spaces. Ph.D. thesis, Universität Trier, Trier (2014). https://doi.org/10.25353/ubtr-xxxx-4b2b-53a5
13. Dierolf, B., Sieg, D.: Splitting and parameter dependence in the category of PLH spaces. Rev. R. Acad. Cienc. Exactas Fis. Nat. (Esp.) 113, 59–93 (2019). https://doi.org/10.1007/s13398-017-0424-5
14. Domiański, P.: Classical PLS-spaces: spaces of distributions, real analytic functions and their relatives. In: Z. Ciesielski, A. Pelczynski, L. Skrzypeczak (eds.) Orlicz Centenary Volume, Banach Center Publications, vol. 64, pp. 51–70. Polish Acad. Sci., Warsaw (2004). https://doi.org/10.4064/bc64-0-5
15. Domiański, P.: Real analytic parameter dependence of solutions of differential equations over Roumieu classes. Funct. Approx. Comment. Math. 44(1), 79–109 (2011). https://doi.org/10.7169/facm/1301497748
19. Domasński, P., Langenbruch, M.: Vector valued hyperfunctions and boundary values of vector valued harmonic and holomorphic functions. Publ. RIMS Kyoto Univ. **44**, 1097–1142 (2008)
20. Epifanov, O.V.: On solvability of the nonhomogeneous Cauchy–Riemann equation in classes of functions that are bounded with weights or systems of weights. Math. Notes **51**(1), 54–60 (1992). https://doi.org/10.1007/BF01229435
21. Farkas, H., Kawai, T., Kuchment, P., Quinto, T.E., Sternberg, S., Struppa, D., Taylor, B.A.: Remembering leon ehrenpreis (1930–2010). Not. Am. Math. Soc. **58**(5), 674–681 (2011)
22. Franken, U., Meise, R.: Generalized Fourier expansions for zero-solutions of surjective convolution operators on $D'((\mathbb{R})$ and $D'_0((\mathbb{R})$. Note Mat. **X**(1), 251–272 (1990). https://doi.org/10.1285/i15900932v10supn1p251
23. Frerick, L., Kalmes, T.: Some results on surjectivity of augmented semi-elliptic differential operators. Math. Ann. **347**(1), 81–94 (2010). https://doi.org/10.1007/s00208-009-0418-5
24. Grothendieck, A.: Produits tensoriels topologiques et espaces nucléaires, 4th edn. Mem. Am. Math. Soc. 16. AMS, Providence (1966). https://doi.org/10.1090/memo/0016
25. Hörmander, L.: Zur Existenz von Rechtsinversen linearer partieller Differentialoperatoren mit konstanten Koeffizienten auf $B_{p,\kappa}^{\infty}(\Omega)$-Räumen. Ph.D. thesis, Universität Wuppertal, Wuppertal (2005)
26. Hömbracher, M.: On the existence of real analytic solutions of partial differential equations with constant coefficients. Invent. Math. **21**, 151–182 (1973). https://doi.org/10.1007/BF01390194
27. Hömbracher, M.: An introduction to complex analysis in several variables, 3rd edn. North-Holland, Amsterdam (1990)
28. Hömbracher, L.: The analysis of linear partial differential operators II. Classics Math. Springer, Berlin (2005). https://doi.org/10.1007/b138375
29. Kaballo, W.: Aufbaukurs funktionalanalyse und operatortheorie. Springer, Berlin (2014). https://doi.org/10.1007/978-3-642-37794-5
30. Kalmes, T.: Surjectivity of differential operators and linear topological invariants for spaces of zero solutions. Rev. Mat. Complut. **32**, 37–55 (2019). https://doi.org/10.1007/s13163-018-0266-5
31. Köthe, G.: Topological vector spaces II Grundlehren Math. Wiss., vol. 237. Springer, Berlin (1979).
32. Kruse, K.: Surjectivity of the $\overline{\partial}$-operator between spaces of weighted smooth vector-valued functions (2018). Arxiv preprint https://arxiv.org/abs/1810.05069v1
33. Kruse, K.: The approximation property for weighted spaces of differentiable functions. In: M. Kosek (ed.) Function Spaces XII (Proc., Kraków, 2018), Banach Center Publ., vol. 119, pp. 233–258. Inst. Math., Polish Acad. Sci., Warszawa (2019). https://doi.org/10.4064/bc119-14
34. Kruse, K.: The inhomogeneous Cauchy–Riemann equation on strips with holes (2019). Arxiv preprint https://arxiv.org/abs/1901.02093v2
35. Kruse, K.: Weighted vector-valued functions and the $\varepsilon$-product (2019). Arxiv preprint https://arxiv.org/abs/1712.01613v6
36. Kruse, K.: On the necessity of weighted spaces of smooth functions. Ann. Polon. Math. **124**(2), 173–196 (2020). https://doi.org/10.4064/ap190728-17-11
37. Langenbruch, M.: Real roots of polynomials and right inverses for partial differential operators in the space of tempered distributions. Proc. R. Soc. Edinburgh Sect. A **114**(3–4), 169–179 (1990). https://doi.org/10.1017/S03082105000024367
38. Langenbruch, M.: Differentiable functions and the $\overline{\partial}$-complex. In: Bierstedt et al. [2], pp. 415–434
39. Langenbruch, M.: Continuous linear right inverses for differential operators. In: D. Przeworska-Rolewicz (ed.) Different Aspects of Differentiability (Proc., Warsaw, 1993), Dissertationes Math., vol. 340, pp. 163–181. Polish Acad. Sci., Warsaw (1995)
40. Langenbruch, M.: Surjective partial differential operators on spaces of ultradifferentiable functions of Roumieu type. Results Math. **29**(3), 254–275 (1996). https://doi.org/10.1007/BF033222300
41. Langenbruch, M.: Surjectivity of partial differential operators on Gevrey classes and extension of regularity. Math. Nachr. **196**(1), 103–140 (1998). https://doi.org/10.1002/mana.19981960106
42. Langenbruch, M.: Inheritance of surjectivity for partial differential operators on spaces of real analytic functions. J. Math. Anal. Appl. **297**(2), 696–719 (2004). https://doi.org/10.1016/j.jmaa.2004.04.035
43. Langenbruch, M.: Right inverses for partial differential operators on Fourier hyperfunctions. Studia Math. **183**(3), 273–299 (2007). https://doi.org/10.4064/sm183-3-5
44. Langenbruch, M.: Right inverses for differential operators on Fourier ultra-hyperfunctions and the property (DN). In: A. Aytuna, R. Meise, T. Terzioğlu, D. Vogt (eds.) Functional Analysis and Complex Analysis (Proc., Istanbul, 2007), Contemp. Math., vol. 481, pp. 81–104. AMS, Providence (2009). https://doi.org/10.1090/conm/481
46. Langenbruch, M.: Bases in spaces of analytic germs. Ann. Polon. Math. 106, 223–242 (2012). https://doi.org/10.4064/ap106-0-18
47. Larcher, J.: Surjectivity of differential operators and the division problem in certain function and distribution spaces. J. Math. Anal. Appl. 409(1), 91–99 (2014). https://doi.org/10.1016/j.jmaa.2013.05.078
48. Malgrange, B.: Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution. Ann. Inst. Fourier (Grenoble) 6, 271–355 (1956). https://doi.org/10.5802/aif.65
49. Mantlik, F.: Linear equations depending differentiably on a parameter. Integral Equ. Oper. Theory 13(2), 231–250 (1990). https://doi.org/10.1007/BF0193758
50. Mantlik, F.: Partial differential operators depending analytically on a parameter. Ann. Inst. Fourier (Grenoble) 41(3), 577–599 (1991). https://doi.org/10.5802/aif.1266
51. Meise, R., Taylor, B.A., Vogt, D.: Characterization of the linear partial differential operators with constant coefficients that admit a continuous linear right inverse. Ann. Inst. Fourier (Grenoble) 40(3), 619–655 (1990). https://doi.org/10.5802/aif.1226
52. Meise, R., Taylor, B.A., Vogt, D.: Continuous linear right inverses for partial differential operators with constant coefficients and Phragmén-Lindelöf conditions. In: Bierstedt et al. [2], pp. 357–389
53. Meise, R., Taylor, B.A., Vogt, D.: Continuous linear right inverses for partial differential operators on non-quasianalytic classes and on ultradistributions. Math. Nachr. 180(1), 213–242 (1996). https://doi.org/10.1002/mana.3211800110
54. Meise, R., Vogt, D.: Introduction to functional analysis. Oxf. Grad. Texts Math, vol. 2. Clarendon Press, Oxford (1997)
55. Meyer, T.: Surjectivity of convolution operators on spaces of ultradifferentiable functions of Roumieu type. Studia Math. 125(2), 101–129 (1997). https://doi.org/10.4064/sm-125-2-101-129
56. Michael, E.: Continuous selections. I. Ann. Math. (2) 63(2), 361–382 (1956). https://doi.org/10.2307/296815
57. Palamodov, V.P.: Homological methods in the theory of locally convex spaces. Rus. Math. Surv. 26, 1–64 (1971). https://doi.org/10.1070/RM1971v026n01ABEH003815
58. Polyakova, D.A.: Solvability of the inhomogeneous Cauchy–Riemann equation in projective weighted spaces. Sib. Math. J. 58(1), 142–152 (2017). https://doi.org/10.1134/S0037446617010189
59. Schwartz, L.: Théorie des distributions à valeurs vectorielles. I. Ann. Inst. Fourier (Grenoble) 7, 1–142 (1957). https://doi.org/10.5802/aif.68
60. Tönjes, K.: Linear topologische Invarianten für Räume holomorpher Funktionen. Master’s thesis, Universität Oldenburg, Oldenburg (2012)
61. Trèves, F.: Fundamental solutions of linear partial differential equations with constant coefficients depending on parameters. Am. J. Math. 84(4), 561–577 (1962). https://doi.org/10.2307/237286
62. Trèves, F.: Locally convex spaces and linear partial differential equations. Springer, New York (1967). https://doi.org/10.1007/978-3-642-87371-3
63. Vogt, D.: Charakterisierung der Unterräume von s. Math. Z. 155, 109–118 (1977). https://doi.org/10.1007/BF01214210
64. Vogt, D.: On the solvability of $P(D)f = g$ for vector valued functions. In: H. Komatsu (ed.) Generalized functions and linear differential equations 8 (Proc., Kyoto, 1982), RIMS Kôkyûroku, vol. 508, pp. 168–181. RIMS, Kyoto (1983)
65. Vogt, D.: On the functors $\Ext^1(E, F)$ for Fréchet spaces. Studia Math. 85(2), 163–197 (1987). https://doi.org/10.4064/sm-85-2-163-197
66. Wengenroth, J.: Derived functors in functional analysis. Lecture Notes in Math, vol. 1810. Springer, Berlin (2003). https://doi.org/10.1007/b80165
67. Wengenroth, J.: Surjectivity of partial differential operators with good fundamental solutions. J. Math. Anal. Appl. 379(2), 719–723 (2011). https://doi.org/10.1016/j.jmaa.2011.01.074

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