Stability Analysis of New Solutions of the EYM system with Cosmological Constant

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Abstract

We analyze the stability properties of the purely magnetic, static solutions to the Einstein–Yang–Mills equations with cosmological constant. It is shown that all three classes of solutions found in a recent study are unstable under spherical perturbations. Specifically, we argue that the configurations have $n$ unstable modes in each parity sector, where $n$ is the number of nodes of the magnetic Yang–Mills amplitude of the background solution.

The “sphaleron–like” instabilities (odd parity modes) decouple from the gravitational perturbations. They are obtained from a regular Schrödinger equation after a supersymmetric transformation.

The body of the work is devoted to the fluctuations with even parity. The main difficulty arises because the Schwarzschild gauge – which is usually imposed to eliminate the gravitational perturbations from the Yang–Mills equation – is not regular for solutions with compact spatial topology. In order to overcome this problem, we derive a gauge invariant formalism by virtue of which the unphysical (gauge) modes can be isolated.

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1 Introduction

A previous numerical analysis \cite{1} has revealed the following features of the purely magnetic solutions to the spherically symmetric Einstein–Yang–Mills (EYM) equations with positive cosmological constant $\Lambda$: The static configurations fall into three classes, where the solutions in each class are characterized by the values of $\Lambda$ and the number of zeroes, $n$, of the magnetic YM amplitude $w$. For sufficiently small values, $\Lambda < \Lambda_{\text{crit}}(n)$, the solutions asymptotically approach the deSitter geometry and can be viewed as Bartnik–McKinnon (BK) solitons \cite{2} surrounded by a cosmological horizon. For a set of values $\Lambda_{\text{reg}}(n)$ exceeding $\Lambda_{\text{crit}}(n)$, the configurations have the topology $\mathbb{R} \wedge S^3$, where the ground state ($n = 1$) is the Einstein Universe with constant YM energy density. Between $\Lambda_{\text{crit}}(n)$ and $\Lambda_{\text{reg}}(n)$ there exists a discrete family of “bag of gold” solutions, that is, of configurations with horizon and “finite size”.

This paper is devoted to the stability properties of the solutions found in \cite{1} with respect to spherically symmetric fluctuations. The metric $g$ of a spherically symmetric spacetime is described in terms of the metric $\tilde{g}$ of a 2–dimensional pseudo–Riemannian manifold $\tilde{M}$ and a function $R$ on $\tilde{M}$, such that $g = \tilde{g} + R^2 d\Omega^2$.

The stability analysis of solutions for which $R$ has no critical points is considerably simplified by the fact that $R$ can be used as a coordinate on $\tilde{M}$. Moreover, it turns out to be sufficient to analyze the perturbations in the Schwarzschild gauge, $\delta R = 0$. The linearized Einstein equations then assume the form of constraint equations, by virtue of which one can express the metric perturbations $\delta \tilde{g}$ in terms of the fluctuations of the matter fields. In this way, one always ends up with a set of fluctuation equations for the matter fields in standard form \cite{8}. For the BK solitons \cite{4} and the corresponding black hole solutions \cite{9}, \cite{10}, \cite{11}, the set of pulsation equations therefore reduces to a single regular Schrödinger equation for the YM amplitude $w$. The number of bound states of this equation eventually determines the number of unstable modes \cite{12}, \cite{13} (in the even parity sector).

In the present paper we are mainly interested in the stability properties of solutions for which $R$ does have a local extremum. In this case, the Schwarzschild gauge, $\delta R = 0$, is not everywhere regular and the above procedure can therefore not be readily adopted. In fact, for $\delta R \neq 0$, not all of the linearized Einstein equations have the form of constraints. Hence, one obtains a coupled system of pulsation equations, including both matter and metric perturbations. Although this system is self–adjoint, the differential operator is not hyperbolic, reflecting the fact that the equations contain unphysical degrees of freedom (gauge modes).

Studying the gauge dependence reveals that the fluctuation equations actually contain only one degree of freedom which is of physical nature. In fact, it turns out to be possible to derive a pulsation equation for the gauge invariant quantity $\zeta$ associated with the physical degree of freedom. The procedure generalizes the method described above for the Schwarzschild gauge in a gauge invariant way.

Although the pulsation equation for $\zeta$ is a standard Schrödinger equation, it has the flaw that the potential is unbounded. In fact, it turns out that the equator (i.e., the critical point of $R$) is a regular singular point of the Sturm–Liouville equation for $\zeta$. A closer look reveals that the so–called limit point case occurs. “Weyl’s alternative” (see, e.g. \cite{14}) then implies that there exists exactly one essentially self–adjoint realization of the pulsation operator.
and that $\zeta$ must vanish at the equator for this realization. However, there exist analytic solutions which do not vanish at the equator and, nevertheless, give rise to perfectly regular metric fluctuations.

In order to reconstruct the metric and matter perturbations from the gauge invariant quantity $\zeta$, we consider a class of gauges for which all fluctuations can be obtained by solving a set of ordinary differential equations. A particular gauge of this type is the Schwarzschild gauge, which is, however, not regular at the equator. A globally regular gauge is obtained by considering conformal perturbations of the spatial part of the metric. In this way one finally obtains a regular set of metric and matter fluctuations from each regular solution of the gauge invariant pulsation equation.

The entire procedure is nicely illustrated for the lowest ($n = 1$) compact solution, since all steps can be performed analytically in this case. The gauge invariant equation has one bound state, corresponding to an unstable mode. A numerical analysis reveals that the higher compact solutions ($n > 1$) have, as expected, exactly $n$ unstable (gravitational) modes.

The stability analysis described above applies to the family of compact solutions, parameterized by $n$ and $\Lambda_{\text{reg}}(n)$. The symmetry properties of these configurations with respect to reflections at the equator simplifies the discussion of the regular singular point. A careful numerical analysis shows that the solutions with horizon (bag of gold solutions) share the stability properties of the compact solutions.

The stability properties of purely magnetic, spherically symmetric, static solutions to the EYM equations are determined by two, completely independent sets of perturbations. The investigation discussed above applies to the even–parity fluctuations, which are also called “gravitational” perturbations, because they have no flat spacetime analogues. Considering fluctuations of the non–magnetic parts of the YM potential yields a second, orthogonal set of perturbations with odd parity. For the BK solitons and black holes, these “sphaleron–like” perturbations give rise to additional $n$ unstable modes (see [10], [11] and references therein).

Adopting the method developed in [11], we show in the last part of this paper that this result remains true for both the compact solutions and the solutions with horizon found in [1].

This article is organized as follows: In the second section we give the basic equations for spherically symmetric EYM models and recall the most important features of the compact solutions to these equations.

The third section is devoted to the derivation of the pulsation equations. We start by linearizing the YM equations within the purely magnetic ansatz. We then argue that – by virtue of the Bianchi identity – the $(00)$ and $(01)$ components of the Einstein equations give only rise to one independent linearized equation, which has the form of a constraint. Finally, we linearize the partial traces (with respect to $\tilde{g}$ and the metric of $S^2$) of the Einstein equations, which yields two pulsation equations for the metric fluctuations. The four equations obtained in this way can be combined into a self–adjoint system. However, since no particular gauge was fixed, the system still contains gauge modes, as is also indicated by the form of the differential operator.

In the forth section we argue that there exists only one gauge invariant quantity, $\zeta$, say, in the even parity sector. Unfortunately, the Sturm–Liouville equation for $\zeta$ has – in addition to the origin – a regular singular point at the equator. Since both branches of the fundamental system are analytic in the vicinity of the equator, it is not sufficient to consider only the solutions belonging to the unique self–adjoint extension of the fluctuation operator (i.e., the
solutions for which $\zeta$ vanishes at the equator).

The reconstruction of the perturbations from the gauge invariant quantity is presented in the fifth section. Using the globally regular conformal gauge and the general pulsation equations derived in the third section, we give explicit, regular integral expressions for the metric and the matter fluctuations in terms of $\zeta$. We also explain how to use the residual gauge freedom in order to fix the integration functions. The results of the numerical analysis are presented at the end of this section.

An analytic discussion of the stability properties of the compact ground state solution ($n = 1, \Lambda = \Lambda_{\text{reg}}(1)$) is presented in the sixth section. It turns out that the unstable mode corresponds to spatially constant, conformal perturbations of the spacetime metric. This “breathing” mode is also obtained by an alternative approach, which takes advantage of the special geometry of the ground state, in order to introduce the Bardeen potentials [12] used in cosmological perturbation theory (see, e.g., [13]). We conclude this section by deriving a “dual” Schrödinger equation for the perturbations of the ground state. In contrast to the equation for $\zeta$, the new pulsation equation for the “supersymmetric partner” of $\zeta$ is completely regular at the equator.

In the last section of this paper we consider the perturbations with odd parity. Again, the fluctuation equations can be reduced to a single Schrödinger type equation. Since the potential in this equation is unbounded at the zeroes of the YM amplitude $w$, we use the residual gauge freedom to perform a supersymmetric transformation. In this way, we obtain a pulsation equation with everywhere regular potential. We then demonstrate that the zero energy solution to this equation has the same number of nodes as $w$. This eventually proves that both the compact and the bag of gold solutions have $n$ unstable modes in the “sphaleron” sector.

## 2 Preliminaries

In this paper we consider the question of linear stability for the solutions of the EYM equations with cosmological constant presented in [1]. Before we do so, we shall briefly recall the basic notions and equations used in [1]. Since we are particularly interested in the globally regular, compact configurations, we also recall their basic features.

### 2.1 Static Equations

The metric of a spherically symmetric manifold $(M, g)$ can be written in the form

$$\mathbf{g} = \tilde{\mathbf{g}} + R^2 \hat{\mathbf{g}},$$

where $\tilde{\mathbf{g}}$ is the metric on the pseudo–Riemannian manifold $\tilde{M}$ and $\hat{\mathbf{g}}$ denotes the standard metric on $S^2$. The Einstein tensor for the metric (1) becomes (see, e.g. [3])

$$G_{ab} = \frac{2}{R} \left[ \tilde{g}_{ab} \tilde{\square} R - \tilde{\nabla}_a \tilde{\nabla}_b R \right] + \frac{1}{R^2} \tilde{g}_{ab} \left[ (dR|dR) - 1 \right],$$

$$G_{Ab} = 0,$$

$$G_{AB} = R^2 \tilde{g}_{AB} \left[ \frac{1}{R} \tilde{\square} R - \frac{1}{2} \tilde{R} \right],$$

where $\tilde{\square}$ is the Laplace–Beltrami operator on $\tilde{M}$.
where the quantities with a tilde refer to \((\tilde{M}, \tilde{g})\) and those with a hat to \((S^2, \hat{g})\). (We use small and capital Latin letters for indices on \((\tilde{M}, \tilde{g})\) and \((S^2, \hat{g})\), respectively; \(a, b, c = 0, 1\) and \(A, B, C = 2, 3\).) Without loss of generality, we shall often use the diagonal parametrization

\[
\tilde{g} = -e^{2a(t, \rho)} dt^2 + e^{2b(t, \rho)} d\rho^2
\]

of the metric on \(\tilde{M}\). With respect to this, the d’Alembertian of a function (e.g., \(\tilde{R}\)) and the Ricci scalar on \((\tilde{M}, \tilde{g})\) become (with \(\dot{\tilde{R}} \equiv \partial \tilde{R}/\partial t, \tilde{R}' \equiv \partial \tilde{R}/\partial \rho\))

\[
\Box \tilde{R} = e^{-(a+b)} \left[ (e^{a-b} \tilde{R}')' - (e^{b-a} \tilde{R})' \right],
\]

and

\[
\bar{\tilde{R}} = -2e^{-(a+b)} \left[ (e^{a-b} a')' - (e^{b-a} b)' \right],
\]

respectively.

A purely magnetic, spherically symmetric \(SU(2)\) YM gauge potential is parametrized in terms of a scalar function \(w : \tilde{M} \to \mathbb{R}\) (see, e.g. [2])

\[
A = (w - 1) \left[ \tilde{\tau}_\varphi d\varphi - \tilde{\tau}_\theta \sin\theta d\varphi \right].
\]

The stress–energy tensor for \(A\) has the components

\[
8\pi g^2 T_{ab} = \frac{1}{R^2} \left[ 2 w_a w_b - \tilde{g}_{ab} \left( \frac{1}{2}(dw|dw) + \frac{V(w)}{4 R^2} \right) \right],
\]

\[
8\pi g^2 T_{Ab} = 0,
\]

\[
8\pi g^2 T_{AB} = \hat{g}_{AB} \frac{V(w)}{4 R^2},
\]

where \(g^2\) is the gauge coupling constant and \(V(w) \equiv (1 - w^2)^2\).

Using the notation

\[
R \equiv \exp(\mu),
\]

and the parametrization (5) of the metric \(\tilde{g}\), the static field equations are

\[
-e^{-2b} [\mu'' + \mu' (\mu' - a' - b')] = \kappa \frac{e^{-2b} w'^2}{R^2},
\]

\[
\frac{1}{R^2} - e^{-2b} [\mu'' + \mu' (2\mu' + a' - b')] = \kappa \frac{V(w)}{2 R^4} + \Lambda,
\]

\[
\frac{1}{R^2} + e^{-2b} \left[ a'' + a' (a' - b') - \mu'^2 \right] = \kappa \frac{V(w)}{R^4},
\]

and

\[
e^{-(a+b)} (e^{a-b} w')' = \frac{1}{4 R^2} V_{,w}.
\]

Note that eqs. (13), (14) and (15) are the \(\frac{1}{2}(00 + 11), \frac{1}{2}(00 - 11)\) and \(\frac{1}{2}(00 + 11 - 22 - 33)\) components of the Einstein equations. We also recall that the dimension–full coupling constant \(\kappa = 8\pi G/g^2\) can be absorbed by using the dimensionless quantities \(R/\sqrt{\kappa}, \rho/\sqrt{\kappa}\) and \(\Lambda\kappa\) (see [1]). Hence, we use \(\kappa = 1\) throughout. (This corresponds to Johnstone Stoney units, already used in 1881 [14].)
2.2 Compact Solutions

In the first six sections of this paper we are mainly interested in the stability of the compact, regular solutions. These are characterized in terms of the node number $n$ of the YM amplitude $w$ and the value of the cosmological constant, $\Lambda = \Lambda_{\text{reg}}(n)$. The lowest solution belongs to $\Lambda = 3/(2\kappa)$ and can be given in closed form,

$$a = 0, \quad b = 0, \quad R/\sqrt{\kappa} = \sin(\rho/\sqrt{\kappa}), \quad w = \cos(\rho/\sqrt{\kappa}), \quad \text{with } \Lambda = 3/(2\kappa).$$

The solution describes the static Einstein Universe, $\mathbb{R} \times S^3$, with $T_{00} = \frac{3}{4}g^{-2}\frac{1}{8\pi}$ and $T_{11} = T_{22} = T_{33} = \frac{1}{4}g^{-2}\frac{1}{8\pi}$. (For later use we also note that for $\kappa = 1$ we have $\mu' = \cot \rho$, $V(w) = \sin^4 \rho$, and $V, w/(4R^2) = \tilde{w} = -\cos \rho$.)

We recall that there are no solutions of the static equations for which $\mu'$ has more than one zero. (The first non-vanishing derivative of $\mu$ is negative for every $\rho$ with $\mu'(\rho) = 0$; see the expansions below.) The unique value $\rho_e$ with $\mu'(\rho_e) = 0$ (and $R(\rho_e) \neq 0$) is therefore called the equator. There are two families of solutions which are analytical in the vicinity of the equator. According to the behavior of the YM amplitude $w$, these solutions will be called odd and even.

For the odd configurations, $w(\rho_e) = 0$, $w'(\rho_e) \neq 0$, one finds with $x \equiv \rho - \rho_e$ (and $\kappa = 1$)

$$\mu' = -\left(\frac{w_e'}{R_e}\right)^2 x + \mathcal{O}(x^3), \quad w' = w_e' + \mathcal{O}(x^2),$$
$$a = a_e + \mathcal{O}(x^2), \quad b = b_e + \mathcal{O}(x^2),$$

where the parameters $w_e'$ and $R_e$ are subject to the constraint equation

$$\Lambda R_e^4 - R_e^2 + \frac{1}{2} = e^{-2b}(R_e w_e')^2.$$

The even solutions, for which $w(\rho_e) \neq 0$ and $w'(\rho_e) = 0$, we have the expansions

$$\mu' = -\frac{1}{3}\left(\frac{w_e''}{R_e}\right)^2 x^3 + \mathcal{O}(x^5), \quad w = w_e + \mathcal{O}(x^2),$$
$$a = a_e + \mathcal{O}(x^2), \quad b = b_e + \mathcal{O}(x^2),$$

where $w_e'' = e^{2b}V, w(\rho_e)/(4R_e^2)$. The parameters $w_e$ and $R_e$ must fulfill the constraint equation

$$\Lambda R_e^4 - R_e^2 + \frac{V(w_e)}{2} = 0.$$

3 Pulsation Equations

We now linearize the field equations on a static background; that is, we write

$$w(t, \rho) = w(\rho) + \delta w(t, \rho),$$
$$\mu(t, \rho) = \mu(\rho) + \delta \mu(t, \rho),$$
$$\tilde{g}_{ab}(t, \rho) = \tilde{g}_{ab}(\rho) + \delta \tilde{g}_{ab}(t, \rho),$$

$$\tilde{g}^{ab}(t, \rho) = \tilde{g}^{ab}(\rho) + \delta \tilde{g}^{ab}(t, \rho).$$

5
and require that \( w(\rho), \mu(\rho) \) and \( \tilde{g}_{\alpha \beta}(\rho) \) solve the static equations (13)–(16). Here and in the following, first order quantities are highlighted by boldface letters.

For solutions without equator, the function \( R \) is an admissible coordinate and, moreover, one may choose the gauge \( \delta \mu = 0 \) to analyze stability properties. The system of linearized field equations then reduces to one pulsation equation (the linearized YM equation) and a set of constraint equations (the linearized Einstein equations). By virtue of these constraints one can eliminate the gravitational perturbations from the YM equation, which yields a pulsation equation for \( \delta w \) alone. This technique was used to investigate the stability properties of the Bartnik–McKinnon solitons and the \( SU(2) \) EYM black holes with hair \cite{7}, \cite{8}. A detailed account of the procedure for a general class of matter models was recently presented in \cite{3}.

The Schwarzschild coordinate \( R \) is not suited to describe spatially compact solutions. In addition, the gauge \( \delta \mu = 0 \) is not regular at the equator. The aim of the following analysis is to generalize the procedure mentioned above, without adopting the gauge \( \delta \mu = 0 \). We do so by deriving a pulsation equation for a gauge invariant quantity \( \zeta \), say, which basically reduces to \( \delta w \) in the Schwarzschild gauge, \( \delta \mu = 0 \). We then argue that there exists an everywhere regular gauge for which the metric and matter perturbations can be constructed explicitly from \( \zeta \).

We start by linearizing the relevant field equations in a static background. In all what follows, we use

\[
\begin{align*}
\Box \delta w &= e^{-a+b} \left( e^{a-b} \delta w' \right)' - e^{-2a} \delta \bar{w},
\Box w &= e^{-a+b} \left( e^{a-b} w' \right)',
\end{align*}
\]

and similarly for all other first order and static background quantities, respectively.

### 3.1 The Yang–Mills Equation

The dynamical YM equation reads

\[
\Box w = \frac{1}{4 R^2} V_{w w}.
\]

For the diagonal metric \( \delta \mu = 0 \) we obtain the following expression for the variation of the d’Alembertian on a static background:

\[
\delta \Box w = \Box \delta w - 2 \Box w \cdot \delta b + (d w \mid d \delta c),
\]

where

\[
\delta c \equiv \delta a - \delta b.
\]

Variation of the r.h.s. of eq. (23) gives \( V_{w w} / (4 R^2) \delta w - 2 \Box w \cdot \delta \mu \), where we have used the static equation (16) in the second term. The linearized YM equation now assumes the form

\[
\left[ \Box - \frac{V_{w w}}{4 R^2} \right] \delta w = 2 \Box w \cdot (\delta b - \delta \mu) - (d w \mid d \delta c).
\]

For later use we also write this in terms of the 4-dimensional d’Alembertian,

\[
\Box = \Box + 2 (d \mu \mid d \cdot),
\]
and the perturbation $\delta z$,

$$\delta z \equiv \frac{1}{R} \delta w. \quad (27)$$

A short computation yields $\frac{1}{R} \Box \delta w = \Box \delta z - (R \Box \frac{1}{R}) \delta z$, and thus

$$\left[ -\Box + (R \Box \frac{1}{R}) + \frac{V_{,ww}}{4 R^2} \right] \delta z = \frac{V_{,w}}{2 R^3} (\delta \mu - \delta b) + \frac{1}{R} (dw \mid d\delta c). \quad (28)$$

### 3.2 The Constraint Equation

All first derivatives of the fields with respect to $t$ enter $G_{00}$ and $T_{00}$ only quadratically. The corresponding linearized Einstein equation assumes therefore the form of a constraint equation, that is, it contains no time derivatives of the linearized quantities. On the other hand, $G_{01}$ and $T_{01}$ already are linear expressions in terms of time derivatives, implying that the corresponding linearized Einstein equation is a total derivative of a quantity $F$, say, with respect to time. A lengthy computation then shows that the linearized $(00)$–equation is the total spatial derivative of $F$.

This connection between the linearized $(00)$– and $(01)$–equations is also established directly, as a consequence of the Bianchi identity. In order to derive this result, we consider the tensor

$$E_{\mu\nu} = G_{\mu\nu} - 8\pi T_{\mu\nu} + \Lambda g_{\mu\nu}, \quad (29)$$

and linearize the contracted Bianchi identity for $E_{\mu\nu}$. Since the background equations imply that only variations of $E_{\mu\nu}$ contribute, we have

$$0 = \frac{1}{\sqrt{-g}} \left( \sqrt{-g} \delta E_{\mu}^{\nu} \right)_\mu - \Gamma_\mu^{\nu\lambda} \delta E_{\lambda}^{\nu}. \quad (30)$$

Now using the facts that $\Gamma^t_{tt}, \Gamma^t_{ij}$ and $\Gamma^i_{tj}$ vanish in a static spacetime and that $\Gamma^t_{tt} = \frac{1}{2} g^{tt} g_{tt,i}$, $\Gamma^t_{tt} = -\frac{1}{2} g^{ij} g_{tt,j}$, we find for the second term in the above expression

$$\Gamma_\mu^{\nu\lambda} \delta E_{\mu}^{\nu} = \frac{1}{2} g_{tt,j} \left[ g^{tt} \delta E_{t}^{j} - g^{ij} \delta E_{i}^{t} \right] = 0.$$

Here we have again used the fact that the tensor $E_{t}^{i}$ vanishes identically (off shell) in a static spacetime, implying that $g^{tt} \delta E_{t}^{i} = g^{i\mu} \delta E_{\mu}^{t} = (g^{g\mu} E_{\mu}^{t}) = \delta E_{i}^{t}$ and $g^{ij} \delta E_{i}^{t} = \delta E_{i}^{t}.$) In the spherically symmetric case we therefore obtain the identity

$$0 = \left( \sqrt{-g} \delta E_{t}^{t} \right)_{t} + \left( \sqrt{-g} \delta E_{t}^{\rho} \right)_{t}'.$$  

(31)

Since, as we shall see in a moment, $(\sqrt{-g} \delta E_{t}^{\rho})$ is already the time–derivative of a quantity $F$, say,

$$\dot{F} = \sqrt{-g} \delta E_{t}^{\rho},$$

we can integrate eq. (31) with respect to $t$. This shows that – up to a integration function of $\rho$ – the linearized $(00)$–equation can be obtained from the linearized $(01)$–equation,

$$F' = -\sqrt{-g} \delta E_{t}^{t}. $$
For a diagonal background metric and a gauge where the variation of the off–diagonal part vanishes as well, we find from eq. (2)

\[
\delta G_{t\rho} = \frac{2}{R} \left[ (\delta \tilde{R})' - \Gamma^t_{\rho \mu} \delta \tilde{R} - R' \delta \Gamma^\rho_{t\mu} \right].
\]

Taking advantage of the fact that the background is static, this yields

\[
\delta G^1_0 = -2 \frac{\partial}{\partial t} \left[ e^{-(a+b)} (\delta \mu' + (\mu' - a')\delta \mu - \mu' \delta b) \right].
\] (32)

The variation of the corresponding component of the stress–energy tensor is immediately found from eq. (9),

\[
8\pi \delta T^1_0 = 2 \frac{\partial}{\partial t} \left[ \frac{1}{R^2} e^{-(a+b)} w' \delta w \right].
\] (33)

Integrating the linearized (01)–equation (32) with respect to \( t \) gives (up to a function of \( \rho \))

\[
\delta \mu' + (\mu' - a')\delta \mu - \mu' \delta b + \frac{w'}{R} \delta z = 0,
\] (34)

where, as earlier, \( \delta z \equiv \frac{1}{R} \delta w \). (Since the integration function has the form of an inhomogeneity, it can be set equal to zero by an appropriate choice of the initial conditions. Indeed, the linearized (00)–equation shows that eq. (34) is fixed up to a constant.) The arguments presented above now imply that the linearized (00)–equation is the derivative of this expression with respect to \( \rho \). Hence, eq. (34) comprises the information of both, the linearized (00)– and (01)–equations.

### 3.3 Gravitational Pulsation Equations

It remains to linearize the (11)–equation and the \((AB)\)–equations. Since we have already exhausted the information from the \((0a)\) components, and since \((M, g)\) is spherically symmetric, it is sufficient to consider the partial traces \( g^{ab}G_{ab} = \tilde{g}^{ab}G_{ab} \) and \( g^{AB}G_{AB} = R^{-2} \tilde{g}^{AB}G_{AB} \), and to linearize the resulting equations

\[
\Box \mu - \frac{1}{R^2} = -\Lambda - \frac{V(w)}{2 R^4},
\] (35)

\[
\Box \mu - (d\mu \mid d\mu) - \frac{1}{2} \tilde{R} = -\Lambda + \frac{V(w)}{2 R^4}.
\] (36)

(Here we have also used \( \Box \mu = R^{-1} \Box R + R^{-2}(dR \mid dR) \).) With respect to the diagonal parametrization \( \tilde{g} \) of \( \tilde{g} \) we find

\[
\delta \Box \mu = \Box \delta \mu - 2 \Box \mu \cdot \delta b + (d\mu \mid d[\delta c + 2\delta \mu])
\]

and

\[
-\frac{1}{2} \delta \tilde{R} = \Box \delta b - 2 \Box a \cdot \delta b + e^{-(a+b)}(e^a - b \delta c)' + (da \mid d\delta c).
\]
Also noting that \( \delta(d\mu|d\mu) = -2\delta b(d\mu|d\mu) + 2(d\mu|d\delta\mu) \) and using the static equation \((\ref{eq:static_eq})\) in the form \( \Box a - (d\mu|d\mu) = \frac{V}{R^2} - \frac{1}{R^2} \), we find the linearized equations

\[
\Box \delta \mu = 2\mu \cdot \delta b + 2\left(\frac{V}{R^4} - \frac{1}{R^2}\right) \delta \mu - \frac{V_{sw}}{2R^3} \delta z - 2(d\mu|d\delta \mu) - (d\mu|d\delta c) ,
\]  

and

\[
\Box (\delta \mu + \delta b) = -2\frac{V}{R^4} \delta \mu + 2\left(\frac{V}{R^4} - \frac{1}{R^2}\right) \delta b + \frac{V_{sw}}{2R^3} \delta z + 2 d^\dagger (d\mu \delta b) + (d\mu - da|d\delta c) - \Delta \delta c .
\]  

Here \( d^\dagger \) denotes the co–derivative with respect to the spacetime metric (for instance, \( d^\dagger (d\mu \delta b) = \Box \mu \cdot \delta b + (d\mu|d\delta b) \)), and \( \Delta \) is the 4–dimensional d’Alembertian without the time–derivative part,

\[
\Delta \delta c \equiv \frac{1}{\sqrt{-g}} (\sqrt{-g} \delta c')'.
\]

It is worth noticing that the operators \( (d\mu|d\cdot) \) and \(-d^\dagger (d\mu \cdot) \) are formally adjoint to each other, that is, for arbitrary functions \( g \) and \( h \), we have

\[
h D_{d\mu} g - g D_{d\mu}^\dagger h = d^\dagger (ghd\mu) ,
\]

where

\[
D_{d\mu} g \equiv (d\mu|dg) , \quad D_{d\mu}^\dagger g \equiv -d^\dagger (d\mu g) = -\left( [\Box \mu] + D_{d\mu} \right) g .
\]  

3.4 Symmetric Form of the Pulsation Equations

We shall now write the system of linearized equations \((\ref{eq:linearized_eqs}), (\ref{eq:constraint}), (\ref{eq:linearized_3}), (\ref{eq:linearized_4})\) in the form

\[
T \Box \delta v = M \delta v ,
\]  

where \( T \) and \( M \) are \( 4 \times 4 \) matrices, \( M \) is formally self–adjoint, and \( \delta v \) parametrizes the perturbations - suitably arranged in a 4–vector (see below). Using the operators defined above, the spatial derivative of the constraint equation \((\ref{eq:constraint})\) can be written as

\[
0 = -D_{d\mu}^\dagger \delta b + D_{d\mu - da}^\dagger \delta \mu - \Delta \delta \mu + \left[ \frac{1}{R} D_{dw} \right]^\dagger \delta z ,
\]

where \( \left[ \frac{1}{R} D_{dw} \right]^\dagger \delta z = -d^\dagger \left( \frac{1}{R} dw \delta z \right) \).

The linearized equations \((\ref{eq:linearized_eqs}), (\ref{eq:constraint}), (\ref{eq:linearized_3}) \) and \((\ref{eq:linearized_4})\) can also be obtained from variations of the effective action (expanded to second order in the fluctuations) with respect to \( \delta c = \delta a - \delta b, \delta b, \delta \mu \) and \( \delta z = R^{-1} \delta w \). This implies that

\[
\delta v = (\delta c, \delta b, \delta \mu, \delta z) .
\]
The system (11), (37), (38), (28) has now the desired form (10), with

\[ M = P + D, \]

where \( P \) is the symmetric potential matrix

\[
P = \begin{pmatrix}
0 & 0 & 2 \frac{d\mu}{dR} & 0 \\
0 & 2 \left( \frac{V}{R^4} - \frac{1}{R^2} \right) & -2 \frac{V}{R^2} & -2 \frac{V}{R^4} \\
0 & -\frac{V_{\mu}}{2R^3} & -\frac{V_{\mu}}{2R^3} & \frac{V_{\mu}}{4R^2} \\
0 & -\frac{V_{\mu}}{2R^3} & -\frac{V_{\mu}}{2R^3} & \frac{V_{\mu}}{4R^2}
\end{pmatrix},
\]

\(( -R \nabla \frac{1}{R} = \nabla \mu - (d\mu|d\mu)) \), and \( D \) is the formally self-adjoint differential operator

\[
D = \begin{pmatrix}
0 & -D^{\dagger}_{\mu\nu} & (D^{\dagger}_{\mu\nu} - \Delta) & (R^{-1}D_{dw})^{\dagger} \\
-D_{\mu\nu} & 0 & -2D_{\mu\nu} & 0 \\
(D_{\mu\nu} - \Delta) & 0 & 0 & 0 \\
R^{-1}D_{dw} & 0 & 0 & 0
\end{pmatrix}.
\]

The matrix \( T \) in front of the d’Alembertian is

\[
T = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}.
\]

Since the system contains one constraint equation, the rank of \( T \) is three. In addition, \(-T\) has only two positive eigenvalues, reflecting the fact that the effective action for the system is indefinite. In fact, the kinetic term in the effective action is

\[
R^2 e^{\Lambda} \left( \frac{1}{R^2} (\delta \dot{w})^2 + (\delta \dot{b})^2 - (\delta \mu + \delta \dot{\mu})^2 \right).
\]

Hence, the pulsation equations do not form a hyperbolic system in standard representation. It is therefore not clear how to make sense of the notion of stability on the basis of the above equations. We shall now present a way out of this difficulty, by arguing that the linearized system contains actually only one relevant dynamical degree of freedom. The aim is to find a gauge invariant generalization of the elimination procedure used in the Schwarzschild gauge (\( \delta \mu = 0 \)).

## 4 Gauge Invariant Formulation

### 4.1 Gauge Transformations

Until now we have not fixed a gauge for the spacetime metric (apart from the vanishing of the shift and its variation). In order to isolate the physical degrees of freedom it is, however, crucial to get rid of the gauge modes. The metric perturbations and the scalar YM amplitude
\( \delta w \) transform according to \((L_Xg)_{\mu\nu} \) and \( L_Xw \). For a static background with diagonal metric, this implies

\[
\begin{align*}
\delta a & \rightarrow \delta a + \alpha f + \dot{\gamma}, \\
\delta b & \rightarrow \delta b + \beta f + f', \\
\delta \mu & \rightarrow \delta \mu + \mu' f, \\
\delta w & \rightarrow \delta w + w' f,
\end{align*}
\]

with \( f \equiv X^p \) and \( g \equiv X^t \). Since we require that the shift and its perturbation vanish, \( f \) and \( g \) are subject to

\[
ie^{2b} \dot{f} = e^{2\alpha} g'.
\]

It is important to note the following: The perturbation \( \delta a \) enters the linearized equations only via the combination \( \delta a' - \delta b' = \delta c' \). Using eq. (45), one observes that the gauge transformation for this quantity does not involve the gauge freedom \( g \):

\[
\delta c' \rightarrow \delta c' + 2(a' - b')f' + (a'' - b'')f - e^{2b} \Box f.
\]

Hence, the gauge transformations of the relevant quantities – i.e., the perturbations \( \delta b, \delta \mu, \delta w \) and \( \delta c' \) which enter the field equations – involve only one degree of freedom, \( f \).

### 4.2 The Schwarzschild Gauge

As a first application, we consider the gauge \( \delta R = 0 \), i.e., \( \delta \mu = 0 \), which we shall also call the Schwarzschild gauge. The gauge transformation which achieves \( \delta \mu = 0 \) is regular as long as \( \mu' \neq 0 \), that is, as long as \( R \) is an admissible coordinate for the background solution. An advantage of this gauge is that the linearized gravitational equation (37) becomes also a constraint equation. We therefore end up with the YM equation,

\[
[\Box - V_{\mu\nu}] \delta w = 2 \Box w \cdot \delta b - e^{-2b} w' \delta c',
\]

and the constraint equations (34) and (38),

\[
\mu' \delta b = \frac{1}{R^2} w' \delta w, \tag{47}
\]

\[
e^{-2b} \mu' \delta c' = -\frac{2}{R^2} \Box w \cdot \delta w + 2 \Box \mu \cdot \delta b. \tag{48}
\]

Here we have also used the static background YM equation. Hence, as is well known (see, e.g., [3]), the gravitational perturbations can be eliminated in the gauge \( \delta \mu = 0 \). The linearized YM equation therefore assumes the form of a Schrödinger equation for \( \delta w \),

\[
[\Box - V_{\mu\nu}] \delta w = \frac{2w'}{\mu'^2 R^2} \left[2\mu' \Box w - w' \Box \mu\right] \cdot \delta w.
\]

Now using \( e^{2b} \Box \mu = \mu'' + (b' - a' + 2\mu')\mu' \) and \( e^{2b} \Box w = w'' + (b' - a')w' \), this pulsation equation for \( \delta w \) can be brought into the simple form

\[
\Box \delta w = \left[\frac{V_{\mu\nu}}{4 R^2} + 2e^{-(a+b)} \left( \frac{e^{a-b} w'^2}{R R'} \right) \right] \delta w. \tag{49}
\]
If \( R \) can be chosen as a coordinate, then this equation is suited to discuss the perturbations of the system. However, if, for instance, the static hypersurfaces are topological 3–spheres, then \( R' \) has a zero, where the second potential term in eq. (49) becomes unbounded (see below).

### 4.3 The Gauge Invariant Pulsation Equation

In the previous section we have seen that – in a gauge where \( \delta \mu \) vanishes – the system of linearized equations can be reduced to a single pulsation equation. We shall now argue that this can always be achieved. More precisely, we show that one can find a pulsation equation which involves only a particular, *gauge invariant* combination of the perturbations. It will turn out to be convenient to write eq. (49) as an equation for \( \mu' \delta w' \):

\[
\left[ \Box - 2e^{-2b} \frac{\mu''}{\mu'} \partial_\rho \right] (\mu' \delta w) = \left[ \frac{V_{\text{sww}}}{4 R^2} + 2e^{-(a+b)} \left( \frac{e^{a-b}w'^2}{R R'} \right)' - \mu' \frac{1}{\mu'} \right] \cdot (\mu' \delta w). \tag{50}
\]

It is immediately observed from the transformation laws (44) that

\[
\zeta = \mu' \delta w' - \mu \delta \mu
\tag{51}
\]

is a gauge invariant combination of the perturbations. Since \( \zeta \) and \( \mu' \delta w' \) are identical in the Schwarzschild gauge (\( \delta \mu = 0 \)), and since eq. (50) was derived within this gauge, we conclude that eq. (50) is the pulsation equation for \( \zeta \).

Using the static equation (13), \( R''/R' = a' + b' - w'^2/(RR') \), the potential on the r.h.s. of the pulsation equation (50) can also be cast in the form

\[
\left[ \frac{V_{\text{sww}}}{4 R^2} + 2 \Box (a + b - \ln |R'|) - \mu' \frac{1}{\mu'} \right].
\]

Combining the second and third terms in this bracket yields the final result

\[
\left[ \Box - 2e^{-2b} \frac{\mu''}{\mu'} \partial_\rho \right] \zeta = \left[ \frac{V_{\text{sww}}}{4 R^2} + 2 \Box (a + b - \mu) - \frac{\Box \mu'}{\mu} \right] \cdot \zeta. \tag{52}
\]

Before we discuss this equation, let us note the following: It is not hard to see that the transformation laws (44) imply that

\[
\mu'^2 \delta b + (\mu'' - \mu' b') \delta \mu - \mu' \delta \mu'
\tag{53}
\]

is gauge invariant as well. However, by virtue of the constraint equation (34) and the static background equation (13), one easily sees that this is equal to \( R^{-2w'} \zeta \). In fact, the circumstance that we are able to reconstruct all perturbations from \( \zeta \), and the fact that the gauge transformations for the relevant quantities contain only one free function, \( f \), strongly suggest that the fluctuation problem involves exactly one relevant gauge invariant quantity.
4.4 The Sturm–Liouville Equation

Let us now consider the pulsation equation (52) for the gauge invariant quantity $\zeta$. As expected, this is a Sturm–Liouville equation. Indeed, if we introduce the operator

$$A = e^{a-b} \mu'^2 \left[ -\partial_\rho \left( e^{a-b} \partial_\rho \right) + q(\rho) \right]$$

with potential

$$q(\rho) = e^{a+b} \left[ \frac{V_{\text{strw}}}{4 R^2} + 2 \tilde{\Box} (a+b-\mu) - \frac{\Box \mu'}{\mu'} \right] ,$$

then eq. (52) becomes

$$\ddot{\zeta} + A \zeta = 0 .$$

Here we would like to comment on a subtlety arising in the study of this equation. A standard approach is to first consider eq. (56) as an ordinary differential equation in some $L^2$–space. In a second step one then discusses under which conditions a Hilbert space solution can be identified with a strict solution in the ordinary sense. In our case, the differential equation corresponding to the operator $A$ obviously has singularities at the origin ($\rho = 0$) and at the equator ($\rho = \rho_e$). (In fact, both points are regular singular points; see below.) It is therefore natural to consider $A$ as an operator on the Hilbert space $L^2(I, d\eta)$, where $I$ is the open interval $(0, \rho_e)$ and $d\eta$ denotes the measure $e^{a-b} \mu'^{-2} d\rho$. Then $A$ is symmetric on a dense domain and, as we will show shortly, for both boundary points the limit point case (in the sense of Weyl) occurs. Now Weyl’s alternative (see, e.g., [9]) implies that there is precisely one self-adjoint realization of $A$. Moreover, this realization is essentially self-adjoint on

$$\mathcal{D} = \{ u \in C^2(I) \cap L^2(I, d\mu) \mid Au \in L^2(I, d\mu) \} .$$

In particular, all functions in $\mathcal{D}$ vanish at the origin and at the equator. On the other hand, as we shall demonstrate later, there exist also analytic solutions with $\zeta(\rho_e) \neq 0$ which give rise to physically acceptable metric perturbations. By adopting the functional analytic approach, we would thus loose half of the interesting modes.

In order to establish that the equator is a regular singular point, we note that the factors in front of $\zeta'$ and $\zeta$ diverge not stronger than $x^{-1}$ and $x^{-2}$, respectively. More precisely, we have

$$\zeta'' - \left[ 2 \frac{\mu''}{\mu'} + O(x) \right] \zeta' = \left[ -\frac{\mu''}{\mu'} + O(1) \right] \zeta ,$$

where $\mu' = O(x^n)$, with $n = 1$ for odd and $n = 3$ for even background configurations. The indicial polynomial for the above equation is

$$r (r - 1) - 2 r n + n (n - 1) = 0 ,$$

which has the roots

$$r_\pm = \frac{1}{2} \left[ 1 + 2n \pm \sqrt{1 + 8n} \right] = \begin{cases} 3, 0 & \text{odd case}, \\ 6, 1 & \text{even case}. \end{cases}$$
Together with the symmetry of the background solution, this implies that the fundamental system of eq. (52) is analytic near the equator. For \( r_\pm = 3, 0 \) this has an expansion of the form

\[
\zeta_+ = x^3 \sum_{j=0}^{\infty} c_j x^{2j}, \quad \zeta_- = \sum_{j=0}^{\infty} k_j x^{2j},
\]

(58) with \( c_0 = k_0 = 1 \). For \( r_\pm = 6, 1 \) one has

\[
\zeta_+ = x^6 \sum_{j=0}^{\infty} c_j x^{2j}, \quad \zeta_- = x \sum_{j=0}^{\infty} k_j x^{2j},
\]

(59) with \( c_0 = k_0 = 1 \).

Let us also discuss the origin. To leading order, eq. (52) reads

\[
\zeta'' + \left[ 2 \frac{1}{\rho} + O(1) \right] \zeta' = \left[ \frac{2}{\rho^2} + O(\rho^{-1}) \right] \zeta.
\]

(60)

With a similar reasoning as above, we now obtain the fundamental solutions

\[
\zeta_+ = \rho \sum_{j=0}^{\infty} c_j \rho^{2j}, \quad \zeta_- = \frac{1}{\rho^2} \sum_{j=0}^{\infty} k_j \rho^{2j}.
\]

(61)

It is now easy to verify that both boundary points of the operator \( A \) belong to the limit point case. For \( \rho = \rho_e \) one has to show that the differential equation \( Au = 0 \) has a solution for which

\[
\int_{\rho_e - \varepsilon}^{\rho_e} |u|^2 d\eta = \infty
\]

(62) for some \( \varepsilon > 0 \) (see, e.g., [4]). (For \( \rho = 0 \) a solution with the corresponding property in the vicinity of the origin has to be found.) Using the behavior of the measure \( d\eta = e^{b-a} \mu^{-2} d\rho \) at the singular points and the above expansions for the fundamental systems establishes that for both \( \rho = 0 \) and \( \rho = \rho_e \), \( A \) is in the limit point case.

5 Reconstruction of the Perturbations

It remains to show that every solution of the pulsation equation for \( \zeta \) gives rise to a regular set \( (\delta w, \delta \mu, \delta b, \delta c') \) of perturbations. One may try to achieve this in a gauge where the relevant equations reduce to a system of algebraic equations for \( (\delta w, \delta \mu, \delta b, \delta c') \) in terms of \( \zeta \). The two gauges of this kind are the Schwarzschild gauge, \( \delta \mu = 0 \), and the gauge \( \delta w = 0 \). However, none of these gauges yields a globally regular description for solutions which possess an equator, which demonstrates that this method does not always work.

A better way is to make use of a suitable gauge for which the perturbations can be obtained from a system of ordinary differential equations. This is the case for any gauge of the form \( \delta b = \text{constant} \cdot \delta \mu \), because one can then eliminate the d’Alembertians of \( \delta \mu \) and \( \delta b \) form the pulsation equations (37) and (38). In this way one obtains an additional ordinary differential equation for \( \delta c \). There are two representatives within this class which yield globally regular expressions for the perturbations in terms of \( \zeta \). These are \( \delta b = 0 \) and the conformal gauge, \( \delta b = \delta \mu \).
5.1 Coverings with two Gauges

Before we shall work in the conformal gauge, let us briefly consider the two gauges for which the perturbations are algebraic expressions in terms of $\zeta$. Off the equator, $\mu'$ does not vanish and the Schwarzschild gauge, $\delta \mu = 0$, is therefore perfectly regular. (The gauge transformation $\delta \mu \to 0 = \delta \mu + \mu' f$ is regular for $\mu' \neq 0$.) Using eqs. (47) and (48) one easily finds for the perturbations in terms of $\zeta$

$$
\delta \mu = 0, \quad \delta w = \frac{\zeta}{\mu'}, \quad \delta b = \frac{w'}{R R'} \delta w, \quad \delta c' = -2 \left( \frac{w'}{R R'} \right)' \delta w.
$$

If the background solution has no equator, then the Schwarzschild gauge is everywhere regular and the above formulae can be used to compute the perturbations in the entire spacetime. If, however, we are considering solutions where $R$ assumes a local maximum, we need another gauge in the vicinity of the equator. The odd case, $w'_e \neq 0$, is very simple to handle, since one can then choose the gauge $\delta w = 0$ and express $\delta \mu$ in terms of the gauge invariant quantity $\zeta$. Then using the constraint equation (34), we obtain $\delta b$ in terms of $\zeta$. Eventually, taking advantage of the linearized YM equation (26) in the gauge $\delta w = 0$, we also obtain $\delta c'$ in terms of $\zeta$:

$$
\delta w = 0, \quad \delta \mu = -\frac{\zeta}{w'}, \quad \delta b = \frac{1}{R e^{-a}} \frac{(R e^{-a} \delta \mu)'}{\mu'}, \quad \delta c' = 2 e^{2b} \frac{\Box w}{w' e^{-a}} \frac{(e^{-a} \delta \mu)'}{\mu'}.
$$

It is clear that $\delta \mu$ remains finite since, by assumption, $w'$ does not vanish in a sufficiently small neighborhood of the equator. Moreover, we have $\delta \mu = \text{constant} + \mathcal{O}(x^2)$, since both $\zeta$ and $w'$ are even. Since the background function $R e^{-a}$ is even as well, we conclude that the numerators in the expressions for $\delta b$ and $\delta c'$ are of $\mathcal{O}(x)$. Also using $\mu' = \mathcal{O}(x)$ shows that both $\delta b$ and $\delta c'$ are well behaved in the vicinity of the equator.

5.2 A Globally Regular Gauge

The gauge $\delta w = 0$ is not suited to compute the perturbations in the even case, $w'_e = 0$, since the expressions in eq. (66) diverge if $\mu' = \mathcal{O}(x^3)$. We now show that there exists a regular gauge which gives rise to analytic expressions in the vicinity of the equator for both odd and even background configurations. Moreover, this gauge is also regular at the poles, where it yields analytic expressions for the perturbations as well. It is therefore possible to cover the entire spacetime using just one gauge.

We consider conformal perturbations of the spatial part of the metric, that is - without loss of generality - we choose the gauge

$$
\delta b = \delta \mu.
$$

(We also note that the gauge transformation between a globally regular gauge and the conformal gauge is everywhere regular.)
We have already mentioned that – by virtue of any algebraic relation between $\delta b$ and $\delta \mu$ – it is possible to eliminate the d'Alembertians from the pulsation equations (37) and (38), which therefore provides an additional ordinary differential equation for $\delta c'$. For a given solution $\zeta$ of the gauge invariant pulsation equation one can then reconstruct all perturbations from $\zeta$ by solving a system of constraint equations. The relevant equations are the pulsation equation (52) for $\zeta$, the algebraic relations (67) and (51), the constraint (34) and the new constraint (71), obtained from eqs. (37) and (38):

$$\left[ -2e^{-2a} \mu^\prime \frac{\partial}{\partial \rho} \right] \zeta = \left[ \frac{V_{w w}}{4R^2} + 2 \left( a + b - \mu \right) - \frac{\mu^\prime}{\mu^\prime} \right] \cdot \zeta ,$$

$$\mu^\prime \delta w - w^\prime \delta \mu = \zeta ,$$

$$\delta \mu^\prime - a^\prime \delta \mu + \frac{w^\prime}{R^2} \delta w = 0 ,$$

$$(\frac{e^{2a-b}}{R} \delta c')' = \mathcal{W} \delta w + \mathcal{M} \delta \mu ,$$

where

$$\mathcal{W} \equiv 6 R^{-3} e^{2a+b} \left[ \square w - (d w | d \mu) \right] ,$$

$$\mathcal{M} \equiv -2 R^{-1} e^{2a+b} \left[ \mu (\mu + 2a) - 3 (d a | d \mu) + \frac{1}{R^2} \right] .$$

It is worth noticing that the gauge transformation $f$ which achieves $\delta b - \delta \mu \to 0$ is everywhere regular, in particular at the equator ($R' = 0$) and at the poles ($R = 0$). In fact, one easily finds from eqs. (14)

$$f = - R e^{-b} \int R^{-1} e^{b} (\delta b - \delta \mu) .$$

The regularity at the equator is manifest, whereas the analyticity at the poles follows from the fact that $\delta b - \delta \mu = \mathcal{O}(R)$. (Use the Schwarzschild gauge $\delta \mu = 0$ for which $\delta b = \mathcal{O}(R)$ and the fact that the transformation from the Schwarzschild to the conformal gauge is $\mathcal{O}(R^2)$.)

From a given solution $\zeta$ of the gauge invariant pulsation equation we now obtain $\delta \mu$ and $\delta w$ from the definition (69) of $\zeta$ and the constraint (71):

$$\delta \mu = \delta \mu(\rho_0, t) - R' e^{-b} \int_{\rho_0}^{\rho} \frac{e^b}{R} \frac{w^\prime}{R^2} \zeta ;$$

$$\delta w = \delta w(\rho_0, t) + w' R e^{-b} \int_{\rho_0}^{\rho} \frac{e^{a+b}}{R'} \left( e^{-a} \frac{1}{w^\prime} \zeta \right)' ,$$

where we have also used the background equation (13). Integrating the additional constraint equation (71) finally gives $\delta c'$:

$$\delta c' = \delta c'(\rho_0, t) + R e^{b-2a} \int_{\rho_0}^{\rho} [\mathcal{W} \delta w + \mathcal{M} \delta \mu] .$$

The time dependent quantities $\delta \mu(\rho_0, t)$, $\delta w(\rho_0, t)$ and $\delta c'(\rho_0, t)$ are not independent. The fact that they are subject to the YM equation (23), together with the residual gauge freedom, implies that they are uniquely determined by $\zeta$ (see the next section).
The perturbations obtained from eqs. (74)-(76) are analytic in the vicinity of the equator for both odd and even background configurations: In the even case, where \( w' = \mathcal{O}(x) \), \( R' = \mathcal{O}(x^3) \) and \( \zeta = \mathcal{O}(x) \), the integrands in eq. (74) and (75) diverge like \( \mathcal{O}(x^{-4}) \) and \( \mathcal{O}(x^{-2}) \), respectively. Hence, \( \delta \mu = \mathcal{O}(x^3) \mathcal{O}(x^{-4}) = \mathcal{O}(1) \) and \( \delta w = \mathcal{O}(x) \mathcal{O}(x^{-2}) = \mathcal{O}(1) \). In the odd case, where \( w' = \mathcal{O}(1) \), \( R' = \mathcal{O}(x) \) and \( \zeta = \mathcal{O}(1) \), one has \( \delta \mu = \mathcal{O}(x) \mathcal{O}(x^{-2}) = \mathcal{O}(1) \) and \( \delta w = \mathcal{O}(1) \mathcal{O}(1) = \mathcal{O}(x) \).

The above formulae give also rise to analytic expressions in the vicinity of the poles. Using the behavior of the background solution for \( R \to 0 \) we have already concluded from eq. (30) that \( \zeta = \mathcal{O}(R) \). Now using \( w' = \mathcal{O}(R) \) shows that \( \delta \mu = \mathcal{O}(1) \) and \( \delta w = \mathcal{O}(R) \). Since the terms of \( \mathcal{O}(R^{-3}) \) in \( \mathcal{M} \) cancel, we have \( \mathcal{M} \delta \mu = \mathcal{O}(R^{-2}) \) and \( \mathcal{W} \delta w = \mathcal{O}(R^{-2}) \). Hence, \( \delta c' = \mathcal{O}(R) \mathcal{O}(1/\mathcal{R}) = \mathcal{O}(1) \).

In conclusion, we have shown that the gauge \( \delta b = \delta \mu \) gives everywhere regular expressions for the entire set of perturbations obtained from the gauge invariant quantity \( \zeta \). The same also holds for the gauge \( \delta b = 0 \), which turns out to be particularly suited for the numerical investigation (see Sect. 5.4).

5.3 The Residual Gauge Freedom

It remains to discuss the implications of the residual gauge freedom. The transformation laws (44) imply

\[
\delta b - \delta \mu \longrightarrow (\delta b - \delta \mu) + (b' - \mu') f + f'.
\]

The conformal gauge \( \delta b - \delta \mu = 0 \) is therefore invariant under transformations of the form \( f(\rho, t) = R e^{-b} \phi(t) \), where \( \phi(t) \) is an arbitrary function which does not depend on \( \rho \). The remaining gauge freedom is thus

\[
\begin{align*}
\delta \mu & \longrightarrow \delta \mu + R' e^{-b} \phi(t), \\
\delta w & \longrightarrow \delta w + w' R e^{-b} \phi(t), \\
\delta c' & \longrightarrow \delta c' + R e^{b-2a} \ddot{\phi}(t) - \left[ e^{a-b} \left( R e^{-a} \right)' \right]' \phi(t) \\
& = \delta c' + R e^{b-2a} \left[ \ddot{\phi} - k(\rho) \cdot \phi \right], \\
\end{align*}
\]

with

\[
k(\rho) \equiv R^{-1} e^{2a-b} \left[ e^{a-b} \left( R e^{-a} \right)' \right]' .
\]

Comparing these expressions with the integral formulas (74)-(76) shows that contributions to \( \delta \mu \) of the form \( R e^{-b} a(t) \) can be compensated by a gauge transformation with \( \phi(t) = -a(t) \). (The same holds for terms of the form \( w' R e^{-b} \beta(t) \) in the expression for \( \delta w \).) Hence, by virtue of the residual gauge freedom, \( \delta \mu \) is uniquely determined by \( \zeta \). Since we already know from the integral expressions that \( \delta w \) and \( \delta c' \) are regular, we can compute these perturbations from the definition (21) of \( \zeta \) and the linearized YM equation. This shows that – once the ambiguity in \( \delta \mu \) has been removed by the residual gauge freedom – all perturbations are uniquely determined in terms of \( \zeta \).
5.4 Numerical Results

As already mentioned, the gauge $\delta b = 0$ is particularly suited for the numerical analysis. Assuming a harmonic time–dependence $e^{i\omega t}$ for the fluctuations, the YM equation (26) and the constraint (34) become (with $\delta c = \delta a$)

$$
\left[ \frac{\partial}{\partial \rho} Q \frac{\partial}{\partial \rho} - \frac{V_{ww}}{4R^2} \right] \delta w + (Qw')' \delta \mu + (Qw') \delta a' = -Q^{-1}\sigma^2 \delta w, \quad (79)
$$

$$
\delta \mu' - \frac{w'}{R^2} \delta w + (\mu' - a') \delta \mu = 0. \quad (80)
$$

Here we have also used the background gauge $e^{2a} = e^{-2b} \equiv Q(\rho)$. Eliminating the d’Alembertians from eqs. (37) and (38), we obtain the additional constraint equation

$$
Q^{-1/2} \left( Q^{3/2} \delta a' \right)' - \frac{V_{ww}}{R^3} \delta w - 2e^{-2b} \mu' \delta \mu' + \frac{2}{R^3}(2V - R^2) \delta \mu = 0. \quad (81)
$$

The main advantage of the gauge $\delta b = 0$ is that the above equations are manifestly regular at the critical points $R' = 0$. They can therefore be used to determine the spectrum for any background solution with $n$ nodes and cosmological constant $\Lambda(n) \in [0, \Lambda_{\text{reg}}(n)]$.

The numerical analysis shows that there exists exactly one boundstate (in the gravitational sector under consideration) for each solution with $n = 1$ and $\Lambda(1) \in [0, \Lambda_{\text{reg}}(1)]$. The energy of the unstable mode versus the cosmological constant is shown in Fig. 1. For $\Lambda = 0$ we obtain the boundstate belonging to the lowest ($n = 1$) BK solution (in the normalization

Figure 1: Bound state energy versus cosmological constant for the $n = 1$ EYM solutions.
Q(0) = 1, κ = 1). In the other limiting case, that is for Λ(1) = 3/2, we find σ^2 = −2, which is the eigenvalue belonging to the unstable mode of the lowest compact solution (see also the next section).

A careful numerical investigation for the few lowest solutions strongly suggests that there exist n unstable modes (in the gravitational sector) for any background solution with n zeroes of w.

6 Instability of the n = 1 Compact Solution

6.1 Gauge Invariant Approach

The stability analysis of the ground state solution,

\[ a = b = 0, \quad R = \sin \rho, \quad w = \cos \rho, \]

can be performed analytically. Using \( \tilde{\Omega} \zeta = -\ddot{\zeta} + \zeta'', \quad \tilde{\Omega} \mu = \mu'' = -1/\sin^2 \rho, \quad \frac{1}{\mu} \tilde{\Omega} \mu' = \mu''/\mu' = 2/\sin^2 \rho \) and \( (V_{,ww})/(4R^2) = (3 \cos^2 \rho - 1)/\sin^2 \rho \), one finds from eq. (52)

\[ -\ddot{\zeta} + \zeta'' + \frac{2}{\sin \rho \cos \rho} \zeta' = \frac{3 \cos^2 \rho - 1}{\sin^2 \rho} \zeta. \]  

(83)

The solutions of this equation of the form \( \zeta(t, \rho) = \zeta_{(k)}(\rho)e^{i\sigma_k t} \) are

\[ \zeta_{(k)}(\rho) = \cos^2 \rho \left( \frac{\sin(k \rho)}{\sin \rho \cos \rho} \right)', \]

(84)

with the spectrum

\[ \sigma_k^2 = k^2 - 3, \quad \text{where} \quad k = 1, 2, \ldots \]

(85)

In the conformal gauge, \( \delta b = \delta \mu \), we now obtain \( \delta \mu \) from eq. (52),

\[ \delta \mu = \cos \rho \int^\rho \frac{1}{\cos^2 \rho} \zeta = \frac{\sin(k \rho)}{\sin \rho} + \cos \rho \cdot \alpha(t), \]

(86)

where \( \alpha(t) \) is a free function. In order to obtain \( \delta w \), we can either use eq. (52) or the definition of \( \zeta \) or – most easily – the constraint (70), \( \delta w = \sin \rho \delta \mu' \):

\[ \delta w = \sin \rho \left( \frac{\sin(k \rho)}{\sin \rho} \right)' - \sin^2 \rho \cdot \alpha(t). \]

(87)

Finally, we have to evaluate the integral (76) for \( dc' \). This is, however, trivial since the background quantities \( W \) and \( M \) defined in eq. (72) vanish for the ground state solution (82):

\[ W = \frac{6}{R^3} (w'' - w' \mu') = 0, \quad M = -\frac{2}{R} (\mu'' + \frac{1}{R^2}) = 0. \]

We therefore conclude from eq. (76) that \( (\delta c'/\sin \rho)' = 0 \), that is,

\[ \delta c' = \sin \rho \cdot \gamma(t). \]

(88)
It is clear that $\gamma(t)$ is not an independent free function, since $dc'$ is uniquely determined from $\delta w$ via the linearized YM equation (26). Using the above solution for $\delta w$ we find

$$\left[ \tilde{\mathcal{D}} - \frac{V_{	ext{aux}}}{4R^2} \right] (-\sin^2 \rho \cdot \alpha) = \left[ \tilde{\alpha} - \alpha \left( \frac{\partial^2}{\partial \rho^2} - 3 \frac{\cos^2 \rho - 1}{\sin^2 \rho} \right) \right] \sin^2 \rho$$

$$= (\tilde{\alpha} + \alpha) \sin^2 \rho = -w' \cdot dc' = \sin \rho \cdot \delta c',$$

which shows that

$$\gamma = \ddot{\alpha} + \alpha.$$  \hfill (89)

Eventually, we can use the remaining gauge freedom to get rid of the free function $\alpha(t)$: Considering transformations of the form $f = Re^{-b(t)}$ with $\phi(t) = -\alpha(t)$ yields

$$\delta \mu \rightarrow \delta \mu - \cos \rho \cdot \alpha(t),$$

$$\delta w \rightarrow \delta w + \sin^2 \rho \cdot \alpha(t),$$

$$\delta c' \rightarrow \delta c' - \sin \rho \cdot (\ddot{\alpha} + \alpha) = \delta c' - \sin \rho \cdot \gamma(t).$$ \hfill (90)

This implies that - for the ground state - it is consistent to consider conformal perturbations of the *entire* spacetime metric

$$\delta a = \delta b = \delta \mu = \frac{\sin(k \rho)}{\sin \rho} e^{i \sigma_k t},$$

$$\delta w = \delta \mu' \sin \rho e^{i \sigma_k t}. \hfill (91)$$

It is also worth noticing that the residual gauge freedom reflects the existence of the conformal Killing fields of $S^3$.

The lowest mode, $k = 1$, corresponds to the negative eigenvalue $\sigma_{(1)}^2 = -2$ and gives rise to exponentially growing perturbations. As is seen from the above solution, the metric perturbations depend only on the time coordinate and the YM amplitude is not perturbed at all for the unstable mode,

$$\delta a = \delta b = \delta \mu = e^{\sqrt{2} t}, \quad \delta w = 0.$$ \hfill (92)

The second lowest mode, i.e. the mode with $\sigma_{(2)}^2 = +1$, is a pure gauge mode, since $\zeta_{(k=2)} = \cos^2 \rho \left( \frac{\sin(2 \rho)}{\sin \rho \cos \rho} \right)' = 0$. In fact, we find from eqs. (86), (87) and (88) for $k = 2$:

$$\delta \mu = 2 \cos \rho \cdot e^{it}, \quad \delta w = -2 \sin^2 \rho \cdot e^{it}, \quad \delta c' = 0.$$

A gauge transformation with $f = -2 \sin \rho \cdot e^{it}$ then gives $\delta \mu = \delta w = \delta c' = 0$.

6.2 An Alternative Approach

The spectrum of perturbations of the ground state is also obtained without using the gauge invariant quantity $\zeta$ as follows: Since the spatial part of the background solution is a space of constant curvature (the geometrical 3–sphere), it is natural to introduce the two gauge invariant Bardeen potentials used in cosmology to describe scalar perturbations \[12\] (see,
e.g., [13] for a review). The Bardeen potentials are proportional to $\delta a$ and $\delta b = \delta \mu$. We are therefore again in the conformal gauge. The fact that the norm of the static Killing field of the background solution is constant ($a = 0$) implies that the remaining gauge freedom can be used to set $\delta c = 0$, i.e. $\delta a = \delta b$. (This is a peculiarity of the ground state, for which the equation for $\delta c'$ decouples and implies that $\delta c'$ is actually a pure gauge; see above.) Hence, by virtue of the special form of the background solution, there exists only one independent scalar Bardeen potential. It is therefore consistent to consider perturbations of the form

$$\delta a = \delta b = \delta \mu, \quad \delta \mu' = \delta z, \quad (93)$$

where the last equation is the constraint (34) in the gauge $\delta b = \delta \mu$ (with $a' = 0$, $R = -w' = \sin \rho$.)

Due to the conformal invariance in four dimensions, the metric perturbations do not enter the linearized YM equation (28) for $\delta a = \delta b = \delta \mu$,

$$\left[\Box - 2 \frac{\cos^2 \rho}{\sin^2 \rho}\right] \delta z = 0. \quad (94)$$

By virtue of the relations (93), we obtain the following pulsation equation for $\delta \mu$ from eq. (37)

$$\left[\Box + 2\right] \delta \mu = 0. \quad (95)$$

Now using $(\Box \delta \mu)' = \Box \delta \mu' + 2(\frac{\cos \rho}{\sin \rho})' \delta \mu' = (\Box - \frac{2}{\sin \rho}) \delta \mu'$ and $\delta \mu' = \delta z$, shows that the pulsation equation for $\delta w$ is the derivative of the pulsation equation for $\delta \mu$:

$$\left(\Box + 2\right) \delta \mu' = \left(\Box - 2 \frac{\cos^2 \rho}{\sin^2 \rho}\right) \delta z = 0. \quad (95)$$

Since the background solution is a geometrical 3–sphere, the 4–dimensional d’Alembertian becomes $\Box = \sin^{-2} \rho \partial_{\rho} (\sin^2 \rho \partial_{\rho})$. The solutions of eq. (95) are therefore the harmonics of $S^3$,

$$\delta \mu = \frac{\sin((\ell + 1)\rho)}{\sin \rho} e^{i \sigma t}, \quad (96)$$

where the spectrum is shifted by $-2$:

$$\sigma^2 = \ell (\ell + 2) - 2 = (\ell + 1)^2 - 3. \quad (97)$$

Since the unstable mode ($\ell = 0$) is purely time dependent, $\delta \mu = e^{\sqrt{\sigma} t}$, and since the YM equation (24) is the derivative of eq. (35), the unstable mode is missing in the spectrum of the YM equation. Using $\delta w / \sin \rho = \delta z = \delta \mu'$, $\mu' = \cos \rho / \sin \rho$ and $w' = -\sin \rho$, we finally find

$$\zeta = \mu' \delta w - w' \delta \mu = \cos \rho \cdot \delta \mu' + \sin \rho \cdot \delta \mu = \cos^2 \rho \left(\frac{\delta \mu}{\cos^2 \rho}\right)' = \cos^2 \rho \left(\frac{\sin((\ell + 1)\rho)}{\sin \rho \cos \rho}\right)', \quad (98)$$

which - with $k = \ell + 1$ - is in agreement with the results presented in the previous section.
6.3 SUSY Transformation

The potential appearing in the pulsation equation (49) for $\delta w$ (or in the corresponding equation (52) for $\zeta$) can diverge at the equator. (We have argued earlier that the equator is a regular singular point.) A pulsation equation with bounded potential can be obtained after a supersymmetric transformation. This requires, however, the knowledge of at least one solution of the original equation. Unfortunately, we were not able to find such a special solution in the general case. Nevertheless, the method works perfectly for the ground state.

In order to demonstrate this, we consider the pulsation equation (49) for $\delta w$. Since this equation was derived in the Schwarzschild gauge, $\delta \mu = 0$, and since

$$\zeta = \frac{1}{\mu'} \zeta = \delta w - \frac{w'}{\mu'} \delta \mu$$

(99)

is clearly gauge invariant, eq. (49) is also the correct equation for $\zeta$. For the $S^3$ background solution we have $\Box = -\partial^2/\partial t^2 + \partial^2/\partial \rho^2$, and therefore

$$\left(\frac{\partial^2}{\partial \rho^2} - \mathcal{P}\right) \zeta = -\sigma^2 \zeta,$$

(100)

where the potential diverges at the equator, $\rho_e = \pi/2$,

$$\mathcal{P} = \frac{3 \cos^2 \rho - 1}{\sin^2 \rho} + \frac{2}{\cos^2 \rho}.$$  

(101)

Let $\psi$ be a particular solution of eq. (100) with eigenvalue $\sigma^2_\psi$, say. Then the differential operators $B^+_{\psi}$ and $B^-_{\psi}$ are defined as

$$B^\pm_{\psi} = \frac{\partial}{\partial \rho} \pm \frac{\psi'}{\psi}.$$  

(102)

Now using $\mathcal{P} = \sigma^2_\psi + \psi''/\psi$ and $B^+_{\psi} B^-_{\psi} = \partial^2/\partial \rho^2 - \psi''/\psi$, eq. (100) assumes the form

$$B^+_{\psi} B^-_{\psi} \zeta = (\sigma^2_\psi - \sigma^2) \zeta.$$

(103)

Multiplying with $B^-_{\psi}$ gives

$$B^-_{\psi} B^+_{\psi} \eta_{\psi} = (\sigma^2_\psi - \sigma^2) \eta_{\psi},$$

(104)

where $\eta_{\psi}$ is the supersymmetric partner of $\zeta$ with respect to $\psi$,

$$\eta_{\psi} = B^-_{\psi} \zeta = \zeta - \frac{\psi''}{\psi} \zeta.$$  

(105)

Since $B^-_{\psi} B^+_{\psi} = \partial^2/\partial \rho^2 + (\psi'/\psi)' - (\psi'/\psi)^2$, eq. (104) finally becomes

$$\left[\frac{\partial^2}{\partial \rho^2} - \mathcal{P} + 2 \left(\frac{\psi'}{\psi}\right)'\right] \eta_{\psi} = -\sigma^2 \eta_{\psi}.$$  

(106)
Hence, the potential in eq. (100) picks up the additional contribution $-2(\psi'/\psi)'$ which can compensate the divergent term in $P$. (It is obvious from the above derivation that the eigenvalue $\sigma_2^2$ does not lie in the spectrum of the equation for the supersymmetric partner with respect to $\psi$.)

As an example, we consider for $\psi$ the lowest mode, $\psi = \zeta^{(1)} = \zeta/\mu' = \sin^2\rho/\cos\rho$. The effective potential for $\eta_\psi$ then becomes finite at the equator,

$$P - 2\left(\frac{\psi'}{\psi}\right)' = 3 \frac{1 + \cos^2\rho}{\sin^2\rho}. \quad (107)$$

Since we have performed a supersymmetric transformation with respect to $\zeta^{(1)}$, the unstable mode is missing in eq. (106) with potential (107). In order to obtain this mode from an equation with finite potential, we also consider the transformation with respect to the third mode, $\zeta^{(3)}$ (recall that $\zeta^{(2)} = 0$):

$$\psi = \frac{1}{\mu'} \zeta^{(3)} = \sin \rho \cos \rho \left(\frac{\sin(3\rho)}{\sin \rho \cos \rho}\right)' = \frac{(c^2 - 1)(1 + 4c^2)}{c},$$

where $c \equiv \cos \rho$. A straightforward calculation now yields the supersymmetric potential

$$P - 2\left(\frac{\psi'}{\psi}\right)' = \frac{48c^6 - 24c^4 + 139c^2 - 13}{(1 - c^2)(1 + 4c^2)^2}, \quad (108)$$

which is bounded in the vicinity of the equator. The pulsation equation (106) with potential (108) has the unstable mode

$$\eta_\psi = \frac{8 \sin^3\rho}{1 + 4 \cos^2\rho} \quad (109)$$

with negative eigenvalue $\sigma^2 = -2$. (It is now an easy task to verify that $\eta_\psi$ is indeed the supersymmetric partner of $\zeta^{(1)}/\mu' = \sin^2\rho/\cos\rho$ with respect to $\psi$,)

$$\eta_\psi = \left(\frac{s^2}{c}\right)' - \left(\frac{s^2(1 + 4c^2)}{c}\right)' \frac{s^2}{c},$$

where $s = \sin \rho$.) Hence, we have obtained the unstable mode from a perfectly regular pulsation equation in standard form.

### 7 The Odd-Parity Sector

In order to study purely magnetic solutions it is sufficient to work with a gauge potential of the form (8). However, if one is interested in the stability properties of such configurations, one must, in general, consider perturbations of the full spherically symmetric gauge potential

$$A = \tilde{A} \hat{\tau}_\rho + \varpi (\hat{\tau}_\theta d\theta + \hat{\tau}_\varphi \sin \vartheta d\varphi) + (w - 1) (\hat{\tau}_\varphi d\varphi - \hat{\tau}_\theta \sin \vartheta d\varphi), \quad (110)$$
where $\tilde{A} = a_0 dt + a_1 d\rho$. It is a crucial observation that $\delta A$ and $\delta w$ decouple from the remaining perturbations, that is, the equations for $\delta A$ and $\delta w$ involve neither the metric perturbations nor the even parity quantity we already pointed out, the above three equations for the fluctuations with odd parity do act as a symmetry operation on the background solutions. Until now, we have restricted ourselves to the fluctuations with even parity, that is, we have considered perturbations for which $\delta A = 0$ and $\delta w = 0$.

In this section we shall investigate the orthogonal set of fluctuations that is, we discuss the pulsation equations for $\delta A$ and $\delta \omega$ on a static, purely magnetic background. We do so by taking advantage of a method developed in [1]. The following analysis applies to all solutions of the EYM equations with cosmological constant studied in [1]. The reason for this lies in the fact that it is actually sufficient to consider perturbations which vanish at the horizon (see [13] for details.)

### 7.1 Pulsation Equations

The YM equations for the general spherically symmetric gauge potential (110) are most easily obtained from the YM action, $S = \int_M \text{tr}(F \wedge *F)$. Using the general form $g = \hat{g} + R^2 \delta \omega$ of the metric and the fact that $w$, $\omega$ and $R$ are functions on $\tilde{M}$, whereas $\tilde{A}$ is a 1–form on $\tilde{M}$, one finds

$$S = \int_{\tilde{M}} \tilde{\eta} \left[ \frac{R^2}{2} |d\tilde{A}|^2 + |d\omega - \tilde{A}w|^2 + dw + \tilde{A}\omega|^2 + \frac{V(w, \omega)}{2R^2} \right],$$

where $V(w, \omega) = (\omega^2 + w^2 - 1)^2$ and $|d\tilde{A}|^2 \tilde{\eta} = d\tilde{A} \wedge \tilde{s}d\tilde{A}$, $\tilde{\eta} = \sqrt{\tilde{g}} dt \wedge dr$. Now using $|dt|^2 = e^{-2a}$, $|dr|^2 = e^{-2b}$, and $\tilde{A} = a_0 dt + a_1 d\rho$,

we obtain the YM equations upon performing variations of $S$ with respect to $a_0$, $a_1$, $\omega$ and $w$. The linearization of these equations on a static background with $a_0 = a_1 = \omega = 0$ finally gives

$$\left[R^2 e^{-(a+b)}(\delta a_1 - \delta a'_0)\right]' = -2e^{b-a} \left[w^2 \delta a_0 - w \delta \omega\right], \quad (112)$$

$$\left[R^2 e^{-(a+b)}(\delta a'_1 - \delta a'_0)\right]' = -2e^{a-b} \left[w^2 \delta a_1 - w \delta \omega' + w' \delta \omega\right], \quad (113)$$

$$- \left[e^{b-a}(\delta \omega - w \delta a_0)\right]' + \left[e^{a-b}(\delta \omega' - w \delta a_1)\right]'$$

$$= e^{a-b}w' \delta a_1 + e^{a+b} \frac{w^2 - 1}{R^2} \delta \omega. \quad (114)$$

In addition, we have the linearized YM equation (26) for $\delta w$, which remains unchanged. As we already pointed out, the above three equations for the fluctuations with odd parity do not involve the metric perturbations nor the even parity quantity $\delta w$.

We now adopt the temporal gauge and consider the following separation ansatz for the perturbations:

$$\delta a_0 = 0, \quad \delta a_1 = \frac{2e^{a+b}}{R^2} \chi(\rho) e^{i\sigma t}, \quad \delta \omega = \xi(\rho) e^{i\sigma t}. \quad (115)$$
Also introducing the background quantity $\gamma$ and the new coordinate $x$,
\[
\gamma^2 = \frac{2}{R^2} e^{2a}, \quad dx = e^{b-a} d\rho,
\]
yields the following set of ordinary differential equations for $\chi$ and $\xi$:
\[
\sigma (\chi_x - w \xi) = 0, \tag{117}
\]
\[
w^2 \gamma^2 \chi + w_{xx} \xi - w \xi_{xx} = \sigma^2 \chi, \tag{118}
\]
\[
\left( w^2 \gamma^2 \chi + w_x \xi - w \xi_{xx} \right)_{xx} = \sigma^2 w \xi. \tag{119}
\]
In the last equation we have also used the background YM equation for $w$ in the form
\[
2w_{xx} = \gamma^2 w(w^2 - 1).
\]
Equation (117) is an obvious consequence of the first two equations. In order to gain a second order equation from the first order equations (117) and (118), we introduce the function $\phi \equiv \chi/w$, for which we find the Schrödinger equation
\[
- \phi_{xx} + \left[ \frac{1}{2} \gamma^2 (w^2 - 1) + 2 \left( \frac{w'}{w} \right)^2 \right] \phi = \sigma^2 \phi. \tag{120}
\]
(For $\sigma^2 \neq 0$ this equation is equivalent to the system (117)–(119). For $\sigma^2 = 0$ we impose eq. (120) as a gauge fixing condition; see [11].) A particular solution – which is related to the residual gauge freedom for $\sigma^2 = 0$ – is (see [1])
\[
\phi_0 = \frac{\Omega_x}{\gamma^2 w}, \tag{121}
\]
where the function $\Omega(x)$ is subject to the differential equation
\[
\left( \frac{1}{\gamma^2} \Omega_{xx} \right)_{xx} = w^2 \Omega. \tag{122}
\]

### 7.2 The Number of Odd-Parity Instabilities

The potential in the Schrödinger equation (120) is non-negative and becomes unbounded at the zeroes of the background amplitude $w$. This suggests that the number of nodes of $w$ determines the number of eigenfunctions with negative $\sigma^2$. In fact, in [11] we have shown that this is the case for the BK solitons and the corresponding black hole solutions. We shall now take advantage of the technique introduced in [11] to show that the number of nodes of $w$ equals the number of unstable modes for all solutions with cosmological constant described in [1].

The key idea is to apply a supersymmetric transformation of eq. (120) with respect to the zero–energy solution $\phi_0$ given in eq. (121). The supersymmetric partner $\psi$ of $\phi$ then fulfills a similar Schrödinger equation as $\phi$, where the additional term $-2\phi_0 x / \phi_0$ is added to the potential in eq. (121) (see also sec. 5.3, eqs. (100), (106)). In [11] we have demonstrated that the pulsation equation for $\psi$ can then be written in the form
\[
- \psi_{xx} + \left[ \frac{1}{2} \gamma^2 (3w^2 - 1) + 2(w^2 Z)_{xx} \right] \psi = \sigma^2 \psi. \tag{123}
\]
where $Z$ is obtained from
\[
Z(x) = -\frac{\gamma^2}{\bar{\Omega}_x} \left[ \bar{\Omega} + \frac{\gamma^2/\bar{\Omega}_x}{C + \int_0^x (w\gamma^2/\bar{\Omega}_x)^2 \, dx} \right].
\] (124)

Here $C$ is an arbitrary constant and $\bar{\Omega}$ denotes a particular solution of the differential equation (122). For $\sigma = 0$, eq. (123) admits the solution
\[
\psi_0(x) = w \exp \left( \int_{x_0}^x w^2 Z \, dx \right).
\] (125)

If $Z$ is everywhere regular, then $\psi_0$ is also smooth and regular. This implies that $\psi_0$ has $n$ zeroes, which will enable us to count the number of bound states. Hence, we have to investigate the behavior of $\bar{\Omega}$, from which we obtain $Z$ and $\psi_0$ by virtue of eqs. (124) and (125), respectively.

In order to analyze the above equations, it is convenient to choose the background gauge
\[
e^{2a(\rho)} = e^{-2b(\rho)} \equiv Q(\rho),
\] (126)
which we have also used in [1]. Also returning to the radial coordinate $\rho$ ($dx = d\rho/Q$, $\gamma^2 = 2Q/R^2$), the differential equation (122) for $\Omega$ becomes
\[
Q \left( R^2 \Omega' \right)' = 2 w^2 \Omega.
\] (127)

Our first aim is to show that – for all static background solutions with horizon discussed in [1] – there exists a smooth solution $\bar{\Omega}(\rho)$ of eq. (127) which behaves as
\[
\bar{\Omega}(\rho) = \frac{k}{\rho^2} + \mathcal{O}(1), \quad \text{and} \quad \bar{\Omega}(\rho) = \mathcal{O}(\rho - \rho_h),
\] (128)
in the vicinity of the origin ($\rho = 0$) and the horizon ($\rho = \rho_h$), respectively. Moreover, $\bar{\Omega}^2$ is monotonically decreasing between $\rho = 0$ and $\rho = \rho_h$.

In order to discuss the behavior near the origin, we recall that all solutions obtained in [1] behave as (see [1], eq. (49))
\[
w = 1 + \mathcal{O}(\rho^2), \quad R = \rho + \mathcal{O}(\rho^5), \quad Q = 1 + \mathcal{O}(\rho^2),
\] (129)
for $\rho \to 0$. Hence, the origin is a regular singular point of eq. (124) with roots $r_{\pm} = 1, -2$.

In the vicinity of the horizon we have (see [1], eq. (51))
\[
w = w_h + \mathcal{O}(\Delta \rho), \quad R = R_h + \mathcal{O}(\Delta \rho), \quad Q = |Q'_h| \Delta \rho + \mathcal{O}(\Delta \rho^2),
\] (130)
where $\Delta \rho \equiv \rho_h - \rho$ and $Q'_h < 0$. Using this in eq. (127) shows that there exists a local solution with $\bar{\Omega}(\rho) = \mathcal{O}(\Delta \rho)$. Finally, in order to establish the monotonicity property, we multiply eq. (127) with $\bar{\Omega}/Q$ and integrate from $\rho < \rho_h$ to $\rho$. Using $\bar{\Omega}(\rho_h) = 0$, this yields the desired result:
\[
\left( \bar{\Omega}^2 \right)'(\rho) = -\frac{2}{R^2(\rho)} \int_\rho^{\rho_h} \left( R^2 \bar{\Omega}^2 + 2 \frac{w^2}{Q} \bar{\Omega}^2 \right) \, d\bar{\rho} < 0.
\] (131)
Using the expansions for \(Q\) and the definition \(dx = d\rho/Q\) of the coordinate \(x\), we find \(x = \rho + \mathcal{O}(\rho^3)\) as \(\rho \to 0\) and \(x = -|Q'_{\rho}|^{-1} \ln(\Delta \rho) + \mathcal{O}(1)\) as \(\rho \to \rho_h\). Hence, eq. (128) implies

\[
\bar{\Omega}(x) = \frac{k}{x^2} + \mathcal{O}(1), \quad \text{and} \quad \bar{\Omega}(x) = \mathcal{O}(\exp(-|Q'_{\rho}|x))
\]

(132)
in the vicinity of the origin \((x = 0)\) and the horizon \((x = \infty)\), respectively. Using this in the expression (124) for the function \(Z(x)\) shows that

\[
Z(x) = \frac{1}{x} + \mathcal{O}(1), \quad \text{and} \quad Z(x) = -w_h^2 \frac{1}{x} + \mathcal{O}(\frac{1}{x^2})
\]

(133)
as \(x \to 0\) and \(x \to \infty\), respectively. Finally, by virtue of eq. (123), we find for the solution \(\psi_0(x)\) of eq. (128)

\[
\psi_0(x) = c_0 x + \mathcal{O}(x^2), \quad \text{and} \quad \psi_0(x) = c_{\infty} \frac{1}{x} + \mathcal{O}(\frac{1}{x^2})
\]

(134)
as \(x \to 0\) and \(x \to \infty\), respectively. This shows that \(\psi_0 \in L^2(0, \infty)\). Since \(\psi_0\) is the zero–energy solution of eq. (123), and since \(\psi_0\) has the same number of nodes as the background solution \(w\) on the interval \((0, \infty)\), we conclude that eq. (123) has \(n\) bound states, \(\psi_k(x) \in L^2(0, \infty)\) with \(\sigma_k^2 < 0\), \(k = 1 \ldots n\).

For \(\sigma \neq 0\), the supersymmetric relation between \(\phi\) and \(\psi\) (with respect to \(\phi_0\)) guarantees that eq. (120) has also \(n\) bound states. Finally since the Schrödinger equation for \(\phi\) is equivalent to the system (117)–(119), the same holds for the solutions \(\chi\) and \(\xi\). Explicitly, one finds (see [11])

\[
\chi_k = \frac{1}{\sigma_k^2} \left( w^3 Z \psi_k + w' \psi_k - w \psi_k' \right), \quad \xi_k = \psi_k - w Z \chi_k.
\]

(135)

This proves the existence of exactly \(n\) unstable modes in the spherically–symmetric, odd–parity sector for all EYM solutions with \(0 < \Lambda < \Lambda_*(n)\) and \(\Lambda_*(n) < \Lambda < \Lambda_{\text{reg}}(n)\) obtained in [1]. (The special case \(\Lambda = \Lambda_*(n)\), for which the horizon occurs at the maximum \(\rho_e\) of \(R\), is not covered by the above reasoning, since eq. (127) develops an irregular singular point at \(\rho = \rho_e = \rho_h\). Although we have not carried out the analysis for this case, we expect on continuity grounds that the result remains the same.) In the limit \(\Lambda \to \Lambda_{\text{reg}}(n)\), corresponding to the compact solutions, the horizon shrinks to a point (the “south pole”), where the boundary conditions are identical with the boundary conditions at the origin. The above analysis can easily be repeated for this situation, and reveals again \(n\) unstable modes for the compact solutions with \(n\) zeroes.

8 Concluding Remarks

In this paper we have investigated the stability of the static, purely magnetic EYM solutions with regular center and positive cosmological constant. Whilst the numerical analysis is straightforward for the asymptotically deSitter solutions, new difficulties arise for the compact and the bag of gold type configurations.
In order to solve these problems, we have generalized an elimination procedure for the gravitational fluctuations – earlier derived for the Schwarzschild gauge (\(\delta R = 0\)) – in a gauge invariant manner. In this way, we end up with a standard Sturm–Liouville equation for a gauge invariant quantity \(\zeta\). We have finally argued that there exists a globally regular gauge, with respect to which the metric and matter fluctuations can be reconstructed from \(\zeta\).

The numerical investigation for the first few branches \((n = 1, 2, 3)\) shows that there are \(n\) unstable modes for every solution with \(n\) zeroes of \(w\) (in the even parity sector). This turns out to be true for the entire interval \([0, \Lambda_{\text{reg}}(n)]\), that is, for the BK solutions, their asymptotically deSitter–like analogues, the bag of gold solutions and the compact configurations.

Adopting a method originally developed for the BK soliton and black hole solutions, we have finally also established analytically the existence of \(n\) unstable modes in the odd parity sector. The procedure, which is based on a supersymmetric transformation of the Schrödinger equation for the sphaleron–like modes, was explicitly carried out for the bag of gold solutions. As expected, the compact solutions also exhibit \(n\) unstable modes with odd parity. Hence, we conclude that every solution – characterized by the number of zeroes \(n\) and the cosmological constant \(\Lambda(n) \in [0, \Lambda_{\text{reg}}(n)]\) – has exactly \(2n\) unstable modes.

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