Abstract

We characterize the measurement complexity of compressed sensing of signals drawn from a known prior distribution, even when the support of the prior is the entire space (rather than, say, sparse vectors). We show for Gaussian measurements and any prior distribution on the signal, that the posterior sampling estimator achieves near-optimal recovery guarantees. Moreover, this result is robust to model mismatch, as long as the distribution estimate (e.g., from an invertible generative model) is close to the true distribution in Wasserstein distance. We implement the posterior sampling estimator for deep generative priors using Langevin dynamics, and empirically find that it produces accurate estimates with more diversity than MAP.

1. Introduction

The goal of compressed sensing is to recover a structured signal from a relatively small number of linear measurements. The setting of such linear inverse problems has numerous and diverse applications ranging from Magnetic Resonance Imaging (Lustig et al., 2008; 2007), neuronal spike trains (Hegde et al., 2009) and efficient sensing cameras (Duarte et al., 2008). Estimating a signal in $\mathbb{R}^n$ would in general require $n$ linear measurements, but because real-world signals are structured—i.e., compressible—one is often able to estimate them with $m \ll n$ measurements.

Formally, we would like to estimate a “signal” $x^* \in \mathbb{R}^n$ from noisy linear measurements,

$$y = Ax^* + \xi$$

where $A \in \mathbb{R}^{m \times n}$ is a measurement matrix and $\xi \in \mathbb{R}^m$ is noise. The goal of compressed sensing is to recover $x^*$ from $y$. Formally, we would like to estimate a “signal” $x^* \in \mathbb{R}^n$ from noisy linear measurements,

$$y = Ax^* + \xi$$

for a measurement matrix $A \in \mathbb{R}^{m \times n}$ and noise vector $\xi \in \mathbb{R}^m$. We will focus on the i.i.d. Gaussian setting, where $A_{ij} \sim \mathcal{N}(0, \frac{1}{m})$ and $\xi_i \sim \mathcal{N}(0, \frac{\sigma^2}{m})$, and one would like to recover $\hat{x}$ from $(A, y)$ such that

$$\|x^* - \hat{x}\| \leq C\sigma$$

with high probability for some constant $C$. When $x^*$ is $k$-sparse, this was shown by Candès, Romberg, and Tao (Candes et al., 2006) to be possible for $m$ at least $O(k \log \frac{n}{k})$.

Over the past 15 years, compressed sensing has been extended in a wide variety of remarkable ways, including by generalizing from sparsity to other signal structures, such as those given by trees (Chen & Huang, 2012), graphs (Xu et al., 2011), manifolds (Chen et al., 2010; Xu & Hassibi, 2008), or deep generative models (Bora et al., 2017; Asim et al., 2019). These are all essentially frequentist approaches to the problem: they define a small set of “structured” signals $x$, and ask for recovery of every such signal.

Such set-based approaches have limitations. For example, (Bora et al., 2017) uses the structure given by a deep generative model $G : \mathbb{R}^k \rightarrow \mathbb{R}^n$; with $(k d \log n)$ measurements for $d$-layer networks, accurate recovery is guaranteed for every signal $x^*$ near the range of $G$. But this completely ignores the distribution over the range. Generative models like Glow (Kingma & Dhariwal, 2018) and pixelRNN (Oord et al., 2016) have seed length $k = n$ and range equal to the entire $\mathbb{R}^n$. Yet because these models are designed to approximate reality, and real images can be compressed, we know that compressed sensing is possible in principle.

This leads to the question: Given signals drawn from some distribution $R$, can we characterize the number of linear measurements necessary for recovery, with both upper and lower bounds? Such a Bayesian approach has previously been considered for sparsity-inducing product distributions (Aeron et al., 2010; Zhou et al., 2014) but not general distributions.

Second, suppose that we don’t know the real distribution $R$, but instead have an approximation $P$ of $R$ (e.g., from a GAN or invertible generative model). In what sense should $P$ approximate $R$ for compressed sensing with good guarantees to be possible?
1.1. Contributions.

Our main theorem is that posterior sampling is a near optimal recovery algorithm for any distribution. Moreover, it is sufficient to learn the distribution in Wasserstein distance.

**Theorem 1.1.** Let $R$ be an arbitrary distribution over an $\ell_2$ ball of radius $r$. Suppose that there exists an algorithm that uses an arbitrary measurement matrix $A \in \mathbb{R}^{m \times n}$ with noise level $\sigma$ and finds a reconstruction $\hat{x}$ such that

$$\|x^* - \hat{x}\| \leq \sigma$$

with probability $\geq 1 - \delta$.

Then posterior sampling (see Definition 1.3) with respect to $R$ using \(m' \geq O\left(m \log \left(1 + \frac{m^2 \|A\|_F^2}{\sigma^2}\right) + \log \frac{1}{\delta}\right)\) Gaussian measurements of noise level $\sigma$ will output $\hat{x}$ satisfying

$$\|x^* - \hat{x}\| \leq \sigma$$

with probability $\geq 1 - O(\delta)$.

Moreover, the same holds for posterior sampling with respect to any distribution $P$ satisfying $W_p(R, P) \leq \sigma^\delta$ with some $p \geq 1$.

This theorem comprises three main contributions: the introduction of posterior sampling as a new algorithm for recovery with a generative prior; an upper bound on the sample complexity of the algorithm in terms of an approximate covering number that we introduce; and an instance-optimal lower bound in terms of the same approximate covering number that (unlike previous lower bounds in compressed sensing) applies to any distribution of input signals.

**Contribution 1: Approximate covering numbers.** The covering number of a set is the smallest number of balls that can cover the entire set. Standard compressed sensing is closely tied to the covering number $N_\eta(S)$ of the set $S$ of possible signals $x$; for example, the set of unit-norm $k$-sparse vectors has $\log N_\eta = \Theta(k \log \frac{n}{k})$, which is precisely why Candés, Romberg, and Tao use this many linear measurements to achieve (1).

For distributions, we need a different concept of covering number. As a motivating example, consider a distribution $R$ induced by a trivial linear generative model, $x = \Sigma z$ where $z \sim \mathcal{N}(0, I_n)$ and $\Sigma$ is a fixed $n \times n$ matrix. Further suppose the singular values $\sigma_i$ of $\Sigma$ are Zipfian, so $\sigma_i = 1/i$. In this case, $R$'s support is $\mathbb{R}^n$, so covering the entire support of $R$ is infeasible. Instead we could denote by $\text{Cov}_{\eta,0.01}(R)$ the minimum number of $\eta-$radius balls needed to cover 99%
of $R$. An elementary calculation shows
\[
\log \text{Cov}_{\eta,0.01}(R) = \Theta(1/\eta^2),
\]
which is (up to constants) precisely the number of linear measurements you need to estimate $x$ to within $\eta$.

We show that an approximate covering number characterizes the measurement complexity of compressed sensing a general distribution $R$, and that recovery by posterior sampling achieves this bound.

**Definition 1.2.** Let $R$ be a distribution on $\mathbb{R}^n$. For some parameters $\eta > 0, \delta \in [0, 1]$, we define the $(\eta, \delta)$-approximate covering number of $R$ as
\[
\text{Cov}_{\eta,\delta}(R) := \min \left\{ k : R \left[ \bigcup_{i=1}^k \mathcal{B}(x_i, \eta) \right] \geq 1 - \delta, x_i \in \mathbb{R}^n \right\}
\]
where $\mathcal{B}(x, \eta)$ is the $\ell_2$ ball of radius $\eta$ centered at $x$.

When $\delta = 0$, this is $N_\eta(\text{supp } R)$, the standard covering number of the support of $R$. Having $\delta > 0$ allows meaningful results for full-support distributions that are concentrated on smaller sets. This also generalizes our previous results in (Bora et al., 2017), which depend on the covering numbers of low-dimensional generative models.

**Contribution 2: Recovery algorithm.** The recovery algorithm we consider is posterior sampling:

**Definition 1.3.** Given an observation $y$, the posterior sampling recovery algorithm with respect to $P$ outputs $\hat{x}$ according to the posterior distribution $P(\cdot | y)$.

**Contribution 3: Sample complexity upper bound.** Our main positive result is that posterior sampling achieves the guarantees of equation (1) for general distributions $R$, with $O(\log \text{Cov}_{\sigma,\delta}(\tilde{R}))$ measurements. Not only this, but the algorithm is robust to model mismatch: posterior sampling with respect to $P \neq R$ still works, as long as $P$ and $R$ are close in Wasserstein distance:

**Theorem 1.4** (Upper bound). Let $P, R$ be distributions with $\mathcal{W}_1(P, R) \leq \sigma$. Let $x^* \sim R$, let $y$ be Gaussian measurements with noise level $\sigma$, and let $\hat{x} \sim P(\cdot | y)$. For any $\eta \geq \sigma$, with
\[
m \geq O(\log \text{Cov}_{\eta,0.01}(R))
\]
measurements, the guarantee $\|\hat{x} - x^*\| \leq C\eta$ is satisfied for some universal constant $C$ with 97% probability over the signal $x$, measurement matrix $A$, noise $\xi$, and recovery algorithm $\hat{x}$.

**Contribution 4: Sample complexity lower bound.** Our second main result lower bounds the sample complexity for any distribution. This is, to our knowledge, the first lower bound for compressed sensing that applies to arbitrary distributions $R$. Most lower bounds in the area are minimax, and only apply to specific “hard” distributions $R$ (Price & Woodruff, 2011; Candes & Davenport, 2013; Iwen & Tewfik, 2010); the closest result we are aware of is (Aeron et al., 2010), which characterizes product distributions.

**Theorem 1.5** (Lower bound). Let $R$ be any distribution over an $\ell_2$ ball of radius $r$, and consider any method to achieve $\|\hat{x} - x^*\| \leq \eta$ with 99% probability, using an arbitrary measurement matrix $A \in \mathbb{R}^{m \times n}$ with noise level $\sigma$. This must have
\[
m \geq \frac{C' 2^{m r \| A \|_2^2}}{\log(1 + 20 m r \| A \|_2^2)} \cdot \log \text{Cov}_{\sigma,0.04}(\tilde{R}).
\]
for some constant $C' > 0$.

Note that Theorem 1.4 and 1.5 directly give Theorem 1.1. For more precisely stated and general versions of these results, including dependence on the failure probability $\delta$, see Theorems 3.4 and 4.1.

1.2. Related Work

Generative priors have shown great promise in compressed sensing and other inverse problems, starting with (Bora et al., 2017), who generalized the theoretical framework of compressive sensing and restricted eigenvalue conditions (Tibshirani, 1996; Donoho, 2006; Bickel et al., 2009; Candes, 2008; Hegde et al., 2008; Baraniuk & Wakin, 2009; Baraniuk et al., 2010; Eldar & Mishali, 2009) for signals lying...
on the range of a deep generative model (Goodfellow et al., 2014; Kingma & Welling, 2013).

Lower bounds in (Kamath et al., 2019; Liu & Scarlett, 2019; Jalali & Yuan, 2019) established that the sample complexities in (Bora et al., 2017) are order optimal. The approach in (Bora et al., 2017) has been generalized to tackle different inverse problems such as robust compressed sensing (Jalal & Yuan, 2019), phase retrieval (Hand et al., 2018; Aubin et al., 2019; Jagatap & Hegde, 2019), blind image de-convolution (Asim et al., 2018), seismic inversion (Mossor et al., 2020), one-bit recovery (Qiu et al., 2019; Liu et al., 2020), and blind demodulation (Hand & Joshi, 2019). Alternate algorithms for reconstruction include sparse deviations from generative models (Dhar et al., 2018), task-aware compressed sensing (Kabkab et al., 2018), PhD (Pandit et al., 2019; Fletcher et al., 2018b,a), iterative projections (Mardani et al., 2018), OneNet (Rick Chang et al., 2017) and Deep Decoder (Heckel & Hand, 2018; Heckel & Soltanolkotabi, 2020). The complexity of optimization algorithms using generative models have been analyzed for ADMM (Gómez et al., 2019), PGD (Hegde, 2018), layer-wise inversion (Lei et al., 2019), and gradient descent (Hand & Voroninski, 2017). Experimental results in (Asim et al., 2019; Whang et al., 2020) show that invertible models have superior performance in comparison to low dimensional models. See (Ongie et al., 2020) for a more detailed survey on deep learning techniques for compressed sensing. A related line of work has explored learning-based approaches to tackle classical problems in algorithms and signal processing (Aamand et al., 2019; Indyk et al., 2019; Metzler et al., 2017; Hsu et al., 2018).

Lower bounds for $\ell_2/\ell_2$ recovery of sparse vectors can be found in (Scarlett & Cevher, 2016; Price & Woodruff, 2011; Aeron et al., 2010; Iwen & Tewfik, 2010; Canes & Da- venport, 2013), and these are related to the lower bound in (1.5). The closest result is that of (Aeron et al., 2010), which characterizes the probability of error and $\ell_2$ error of the reconstruction via covering numbers of the probability distribution. Their approach uses the rate distortion function of a scalar random variable $x$ and provides guarantees for the product measure generated via an i.i.d. sequence of $x$. A Shannon theory for compressed sensing was pioneered by (Wu & Verdú, 2012; Wu, 2011). The $\delta$–Minkowski dimension of a probability measure used in (Wu & Verdú, 2012; Wu, 2011; Pesin, 2008) can be derived from our $(\varepsilon, \delta)$–covering number by taking the limit $\varepsilon \to 0$. (Reeves & Gastpar, 2012) contains a related theory of rate distortion for compressed sensing. There is also related work in the statistical physics community under different assumptions on the signal structure (Zdeborová & Krzakala, 2016; Barbier et al., 2019).
Then we have

\[ \text{Define } \]

The proof of this, as well as all parts of the upper bound, Bernstein-Lindenstrauss (JL) Lemma tells us that it will preserve distributions between vectors with high probability. This does not necessarily mean that every point in the distribution \( P \) will be preserved in norm. Still, we show that, since \( P_0 \) and \( P_1 \) have well-separated supports, their projected distributions \( H_0 \) & \( H_1 \) have very high TV distance. This also holds more generally, between any distribution on a ball and any distribution far from the ball and in the presence of noise.

**Lemma 3.2.** Let \( y \) be generated from \( x^* \) by a Gaussian measurement process with noise level \( \sigma \). For a fixed \( \tilde{x} \in \mathbb{R}^n \), and parameters \( \eta > 0 \), \( c \geq 4e^2 \), let \( P_{\text{out}} \) be a distribution supported on the set

\[ S_{\tilde{x},\text{out}} := \{ x \in \mathbb{R}^n : \| x - \tilde{x} \| \geq c(\eta + \sigma) \} \]

Let \( P_{\tilde{x}} \) be a distribution which is supported within an \( \eta \)-radius ball centered at \( \tilde{x} \).

For a fixed \( A \), let \( H_{\tilde{x}} \) denote the distribution of \( y \) when \( x^* \sim P_{\tilde{x}} \). Let \( H_{\text{out}} \) denote the corresponding distribution of \( y \) when \( x^* \sim P_{\text{out}} \). Then we have:

\[ \mathbb{E}_A [TV(H_{\tilde{x}}, H_{\text{out}})] \geq 1 - 4e^{-\frac{\eta^2}{2} \log(\frac{1}{c})}. \]

By Markov’s inequality, the expectation bound also gives a high probability bound over \( A \).

For our current example, the above result implies that with probability \( 1 - e^{-\Omega(m)} \) over \( A \), we have

\[ TV(H_0, H_1) \geq 1 - e^{-\Omega(m)}. \]  

(3)

Substituting equation (3) in equation (2), we have

\[ \Pr[\| x^* - \tilde{x} \| > 2\eta] \leq 2e^{-\Omega(m)}. \]

This shows that posterior sampling will produce a reconstruction which is close to the ground truth with overwhelmingly high probability for the two-ball example.

### 3.2. Going beyond two balls

The two-ball example leaves three main questions unanswered:

1. How do we handle distributions over larger collections of balls?
2. How do we handle mismatch between the distribution of reality \( (R) \) and the model \( (P) \)?
3. How do we handle having a \( \delta \) probability of lying outside any ball?

**Unions of many balls.** The first question is relatively easy to answer: if \( \text{Cov}_{\eta,0}(R) \leq e^{\Omega(m)} \), you can cover \( R \) with a small number of balls, and essentially apply Lemma 3.2 with
a union bound. There are a few details (e.g., Lemma 3.2 shows you will not confuse any ball with faraway balls, but you might confuse it with nearby balls) but solving them is straightforward. This shows that, if $P = R$ and $\log Cov_{\eta,0}(R)$ is bounded, then posterior sampling works well with $1 - e^{-\Omega(m)}$ probability.

**Distribution mismatch in $W_\infty$.** The above assumes we resample with respect to the true distribution $R$. But we only have a learned estimate $P$ of $R$. We would like to show that observing samples from $R$ and resampling according to $P$ gives good results. We first show that resampling signals drawn from $R$ with respect to $P$ is not much worse than resampling signals drawn from $P$ with respect to $P$, if $P$ and $R$ are close in $W_\infty$.

**Lemma 3.3.** Let $R, P$, denote arbitrary distributions over $\mathbb{R}^n$ such that $W_\infty(R, P) \leq \epsilon$.

Let $x^* \sim R$ and $z^* \sim P$ and let $y$ and $u$ be generated from $x^*$ and $z^*$ via a Gaussian measurement process with $m$ measurements and noise level $\sigma$. Let $\tilde{x} \sim P(\cdot|y, A)$ and $\tilde{z} \sim P(\cdot|u, A)$. For any $d > 0$, we have

$$Pr_{x^*, A, \xi, \tilde{x}}[\|x^* - \tilde{x}\| \geq d + \epsilon] \leq e^{-\Omega(m)} + e^{d \left(\frac{e^\epsilon - 2 \epsilon}{\epsilon^2}\right)} Pr_{x^*, A, \xi, \tilde{z}}[\|z^* - \tilde{z}\| \geq d] .$$

The idea is that with $\sigma$ Gaussian noise, measurements of a signal from $R$ aren’t too different in distribution from measurements of the corresponding nearby signal from $P$.

Now, if $W_\infty(R, P) \ll \sigma$, we would be nearly done: Lemma 3.3 says the situation is within $e^{\epsilon(m)}$ of the $R = P$ case, which we already know gives accurate recovery with $O(\log Cov_{\eta,0}(P))$ measurements.

**Residual mass.** There are just two main issues remaining: we want to depend on $\log Cov_{\eta,0}$ rather than $\log Cov_{\eta,0}$, and we only want to require a bound on $W_1(R, P)$ not $W_\infty(R, P)$. By Markov’s inequality, these issues are very similar: we want to allow both $R$ and $P$ to have a small constant probability of behaving badly. To address this, we note the existence of two distributions $R'$ and $P'$, which are only $\delta$-far in TV from $R$ and $P$ respectively, such that $R'$ and $P'$ do have a small cover & are close in $W_\infty$. We show that, because posterior sampling would work with $R'$ and $P'$, it also works with $R$ and $P$. This leads to our full upper bound:

**Theorem 3.4.** Let $\delta \in [0, 1/4]$, $p \geq 1$, and $\epsilon, \eta > 0$ be parameters. Let $R, P$ be arbitrary distributions over $\mathbb{R}^n$ satisfying $W_p(R, P) \leq \epsilon$.

Let $x^* \sim R$ and suppose $y$ is generated by a Gaussian measurement process from $x^*$ with noise level $\sigma \geq \epsilon/\delta^{1/p}$ and $m \geq O(\min(\log Cov_{\eta,0}(R), \log Cov_{\eta,0}(P)))$ measurements. Given $y$ and the fixed matrix $A$, let $\hat{x}$ be the output of posterior sampling with respect to $P$.

Then there exists a universal constant $c > 0$ such that with probability at least $1 - e^{-\Omega(m)}$ over $A, \xi$,

$$Pr_{x^* \sim R, \hat{x} \sim P(\cdot|y)}[\|x^* - \hat{x}\| \geq c\eta + c\sigma] \leq 2\delta + 2e^{-\Omega(m)} .$$

Note that we can get a high-probability result by setting $p = \infty$: if $m \geq O(\log Cov_{\eta,0}(R))$ and $W_\infty(R, P) \leq \sigma$, the error is $O(\sigma + \eta)$ with $1 - e^{-\Omega(m)}$ probability.

**4. Lower Bound**

In the previous section, we showed, for any distribution $R$ of signals, that $O(\log Cov(R))$ measurements suffice for posterior sampling to recover most signals well. Now we show the converse: for any distribution of signals $R$, any algorithm for recovery must use $\Omega(\log Cov(R))$ measurements.

**Theorem 4.1.** Let $R$ be a distribution supported on a ball of radius $r$ in $\mathbb{R}^n$, and $x^* \sim R$. Let $y = Ax^* + \xi$, where $A$ is any matrix, and $\xi \sim \mathcal{N}(0, \frac{\sigma^2}{m} I_m)$. Assuming $\delta < 0.1$, if there exists a recovery scheme that uses $y$ and $A$ as inputs and guarantees

$$\|\hat{x} - x^*\| \leq O(\eta) ,$$

with probability $\geq 1 - \delta$, then we have

$$m \geq \frac{0.15}{\log \left(1 + \frac{mr^2 \|A\|_2^2}{\sigma^2}\right)} \left(\log Cov_{3\eta,4\delta}(R) + \log 6\delta - O(1)\right) .$$

If $A$ is an i.i.d. Gaussian matrix where each element is drawn from $\mathcal{N}(0, 1/m)$, then the above bound can be improved to:

$$m \geq \frac{0.15}{\log \left(1 + \frac{\sigma^2}{\sigma^2}\right)} \left(\log Cov_{3\eta,4\delta}(R) + \log 6\delta - O(1)\right) .$$

This Theorem is proven using information theory, as an almost direct consequence of the following three Lemmas.

First, the measurement process reveals a limited amount of information:

**Lemma 4.2.** Consider the setting of Theorem 4.1. If $A$ is a deterministic matrix, we have

$$I(y; x^*) \leq \frac{m}{2} \log \left(1 + \frac{mr^2 \|A\|_2^2}{\sigma^2}\right) .$$

If $A$ is a Gaussian matrix, then

$$I(y; x^*|A) \leq \frac{m}{2} \log \left(1 + \frac{r^2}{\sigma^2}\right) .$$
Second, since \( x^* \rightarrow y \rightarrow \hat{x} \) is a Markov chain, we can directly apply the Data Processing Inequality (Cover & Thomas, 2012).

**Lemma 4.3.** Consider the setting of Theorem (4.1). If \( A \) is a deterministic matrix, we have
\[
I(x^*; \hat{x}) \leq I(y; x^*).
\]
If \( A \) is a random matrix, then
\[
I(x^*; \hat{x}) \leq I(y; x^* | A).
\]

Finally, successful recovery must yield a large amount of information:

**Lemma 4.4 (Fano variant).** Let \((x, \hat{x})\) be jointly distributed over \( \mathbb{R}^n \times \mathbb{R}^n \), where \( x \sim R \) and \( \hat{x} \) satisfies
\[
\Pr[|x - \hat{x}| \leq \eta] \geq 1 - \delta.
\]
Then for any \( \tau \leq 1 - 3\delta, \delta < 1/3 \), we have
\[
0.99\tau (1 - 2\delta) \log \text{Cov}_{3\eta, \tau + 3\delta}(R) \leq I(x; \hat{x}) + 1.98.
\]

In order to complete the proof of Theorem 4.1, we need an additional counting argument to remove the extra \( \tau \) term that appears in the left hand side of Lemma 4.4.

The proofs can be found in Appendix B.

5. Experiments

In this section we discuss our algorithm for posterior sampling, discuss why existing algorithms can fail, and show our empirical evaluation of posterior sampling versus baselines.

5.1. Datasets and Models

We perform our experiments on the CelebA-HQ (Liu et al., 2018; Karras et al., 2017) and FlickrFaces-HQ (Karras et al., 2019) datasets. For the CelebA dataset, we run experiments using a Glow generative model (Kingma & Dhariwal, 2018). For the FlickrFaces-HQ dataset, we use the NCSNv2 model (Song & Ermon, 2020). Both models have output size \( 256 \times 256 \times 3 \). Details about our experiments are in Appendix C.

5.2. Langevin Dynamics

**Glow trained on CelebA-HQ** We first consider the Glow generative model, whose distribution \( P \) is induced by the random variable \( G(z) \), where \( G : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a fixed deterministic generative model, and \( z \sim \mathcal{N}(0, I_n) \). Sampling from \( p(z | y) \) is easier than sampling from \( p(x | y) \), since it is easier to compute and we observe that sampling mixes quicker. Note that sampling \( \hat{z} \sim p(z | y) \) and setting \( \hat{x} = G(\hat{z}) \) is equivalent to sampling \( \hat{x} \sim p(x | y) \).

In order to sample from \( p(z | y) \), we use *Langevin dynamics*, which samples from a given distribution by moving a random initial sample along a vector field given by the distribution. Langevin dynamics tells us that if we sample \( z_0 \sim \mathcal{N}(0, I) \), and run the following iterative procedure:
\[
z_{t+1} \leftarrow z_t + \frac{\alpha_t}{2} \nabla_y p(z_t | y) + \sqrt{\alpha_t} \xi_t, \quad \xi_t \sim \mathcal{N}(0, I),
\]
then \( p(z | y) \) is the stationary distribution of \( z_t \) as \( t \rightarrow \infty \) and \( \alpha_t \rightarrow 0 \). Unfortunately, this algorithm is slow to mix, as observed in (Song & Ermon, 2019). We instead use an annealed version of the algorithm, where in step \( t \) we pretend that \( p(z | y) \) has noise scale \( \sigma_t \geq \sigma \) instead of \( \sigma \). This gives
\[
\log p_t(z | y) = \left( \frac{\|y - AG(z)\|^2}{2\sigma_t^2 / m} - \frac{\|z\|^2}{2} \right) + \log c(y),
\]
where \( c(y) \) is a constant that depends only on \( y \). Since we only care about the gradient of \( \log p(z | y) \), we can ignore this constant \( c(y) \). By taking a decreasing sequence of \( \sigma_t \) that approach the true value of \( \sigma \), we can anneal Langevin dynamics and sample from \( p(z | y) \). Please refer to Appendix C for more details about how \( \sigma_t \) varies.

**NCSNv2 trained on FFHQ** We also consider the NC-SNv2 model, which takes as input the image \( x \), and outputs \( \nabla_x \log p(x) \). This model is designed such that sampling from its marginal involves running Langevin dynamics. Since we have access to \( \nabla_x \log p(x) \), and if we know the functional form of \( p(y | x) \), we can easily compute \( \nabla_x \log p(x | y) \), and run Langevin dynamics via
\[
x_{t+1} \leftarrow x_t + \frac{\alpha_t}{2} \nabla_x \log p(x_t | y) + \sqrt{\alpha_t} \xi_t, \quad \xi_t \sim \mathcal{N}(0, I).
\]

Notice that we can also run MAP using this model. This can be achieved by simply following the gradient, and not adding noise:
\[
x_{t+1} \leftarrow x_t + \frac{\alpha_t}{2} \nabla_x \log p(x_t | y).
\]

This model also requires annealing, and we follow the schedule prescribed by (Song & Ermon, 2020). Please see Appendix C for more details.

5.3. MAP and Modified-MAP

The most relevant baseline for our algorithm is MAP, which was shown to be state-of-the-art for compressed sensing using generative priors (Asim et al., 2019).

Given access to a generative model \( G \) such that the image \( x = G(z) \), and \( q(z) \) is the prior of \( z \), the MAP estimate is
\[
\hat{z} := \arg \min_z \frac{\|y - AG(z)\|^2}{2\sigma^2 / m} - \log q(z),
\]
and set the estimate to be \( \hat{x} = G(\tilde{z}) \). Typically, \( q(z) \) is a standard Gaussian for many generative models. If one has access to \( p(x) \), such as in NCSNv2 (Song et al., 2019), it is possible to also do MAP in \( x \)-space.

One may modify this algorithm and introduce hyperparameters for better reconstructions. We call such algorithms modified-MAP. For example, (Asim et al., 2019) introduce a parameter \( \gamma > 0 \) that weights the prior, and their estimate is

\[
\tilde{z}_{\text{modified}} := \arg \min_z \| y - AG(z) \|^2 - \gamma \log q(z),
\]

(6)

Other examples of hyper-parameters include early stopping to avoid “over-fitting” to the measurements, and choosing optimization parameters such that the reconstruction error is minimized on a validation set of images. Then these hyperparameters are used for evaluating reconstruction error on a different test.

5.4. Experimental Results

MAP estimation does not work on general distributions: as an extreme example, if \( R \) is a mixture of some continuous distribution 99% of the time, and the all-zero image 1% of the time, it will always output the all-zero image, which is wrong 99% of the time. More generally, looking for high-likelihood points rather than regions means it prefers sharp but very narrow maxima to wide, but slightly shorter, maxima. Posterior sampling prefers the opposite. We now study this empirically.

CelebA. In Figure 4, we show the performance of our proposed algorithm for compressed sensing on CelebA-HQ with Glow. The baselines we consider are MAP, and modified-MAP. MAP directly optimizes the objective defined in Eqn (5) while Modified-MAP optimizes (6). The MAP baseline in Figure 4 tries to maximize the posterior likelihood, and hence hyperparameters are selected so that the posterior is optimized. In contrast, what we term the modified-MAP algorithm was proposed by (Asim et al., 2019), and this algorithm picks hyperparameters that minimize reconstruction error on a holdout set of images. These hyperparameters are significantly worse at optimizing the MAP objective, but lead to more accurate recovered images, presumably due to some sort of implicit regularization. This modified-MAP method has shown to be state-of-the-art for compressed sensing on CelebA (Asim et al., 2019).

We find that our algorithm is competitive with respect to modified-MAP, and beats MAP when the measurements are < 35,000.

FFHQ. In Figure 5, we show the performance of our proposed algorithm for compressed sensing on FlickrFaces-HQ with the NCSNv2 generative model. We consider MAP and Deep-Decoder (Heckel & Hand, 2018) as the baselines. Note that the NCSNv2 model was designed for Langevin dynamics.
Figure 5: We compare our algorithm with the MAP and Deep-Decoder baselines on the FFHQ dataset, where the number of pixels is \( n = 256 \times 256 \times 3 = 196,608 \). Figure (a) plots per-pixel reconstruction error as we vary the number of measurements \( m \). Figure (b) shows original images (top row), reconstructions by MAP (second row), Deep-Decoder (third row), and Langevin dynamics (bottom row). Langevin is the practical implementation of our proposed posterior sampling estimator. Note that although Deep Decoder and Langevin achieve similar value of reconstruction errors, Langevin produces images with higher perceptual quality, as can be seen in Figure (b).

6. Conclusion

This paper studies the problem of compressed sensing a signal from a distribution \( R \). We have shown that the measurement complexity is closely characterized by the log approximate covering number of \( R \). Moreover, this recovery guarantee can be achieved by posterior sampling, even with respect to a distribution \( P \neq R \) that is close in Wasserstein distance. Our experiments using Langevin dynamics to approximate posterior sampling match state-of-the-art recovery with a theoretically grounded algorithm.

This measurement complexity is inherent to the true distribution of images in the domain, and can’t be improved. But perhaps it can be estimated: one open question is whether \( \log \text{Cov}_{\eta,\delta}(P) \) can be estimated or bounded when \( P \) is given by a neural network generative model.

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A. Upper Bound Proofs

A.1. Proof of Lemma 3.1

Lemma 3.1. For $c \in [0, 1]$, let $H := (1 - c)H_0 + cH_1$ be a mixture of two absolutely continuous distributions $H_0, H_1$ admitting densities $h_0, h_1$. Let $y$ be a sample from the distribution $H$, such that $y|z^* \sim H_{z^*}$ where $z^* \sim \text{Bernoulli}(c)$.

Define $\tilde{c}_y = \frac{ch_1(y)}{(1-c)h_0(y)+ch_1(y)}$, and let $\tilde{z}|y \sim \text{Bernoulli}(\tilde{c}_y)$ be the posterior sampling of $z^*$ given $y$. Then we have

$$\Pr_{z^*,y,\tilde{z}}[z^* = 0, \tilde{z} = 1] \leq 1 - TV(H_0, H_1).$$

Proof. We have

$$\Pr_{z^*,y,\tilde{z}}[z^* = 0, \tilde{z} = 1] = \Pr[z^* = 0] \mathbb{E}_{y \sim h_0, \tilde{z}|y} [1\{\tilde{z} = 1\}],$$

$$= (1 - c) \int h_0(y) \Pr[\tilde{z} = 1|y] dy. \quad (7)$$

By definition, we have

$$\Pr[\tilde{z} = 1|y] = \frac{ch_1(y)}{(1-c)h_0(y)+ch_1(y)}.$$

Substituting, we have

$$\Pr_{z^*,y,\tilde{z}}[z^* = 0, \tilde{z} = 1] = \int \frac{(1-c)h_0(y)ch_1(y)}{(1-c)h_0(y)+ch_1(y)} dy$$

$$\leq \int \frac{(1-c)h_0(y) \cdot ch_1(y)}{\max\{(1-c)h_0(y), ch_1(y)\}} dy$$

$$= \int \min\{(1-c)h_0(y), ch_1(y)\} dy$$

$$\leq \int \min\{h_0(y), ch_1(y)\} dy$$

$$= (1 - TV(H_0, H_1)).$$

A.2. Proof of Lemma 3.2

Lemma 3.2. Let $y$ be generated from $x^*$ by a Gaussian measurement process with noise level $\sigma$. For a fixed $\tilde{x} \in \mathbb{R}^n$, and parameters $\eta > 0, c \geq 4\epsilon^2$, let $P_{\text{out}}$ be a distribution supported on the set

$$S_{\tilde{x}, \text{out}} := \{x \in \mathbb{R}^n : \|x - \tilde{x}\| \geq c(\eta + \sigma)\}.$$

Let $P_\tilde{x}$ be a distribution which is supported within an $\eta$–radius ball centered at $\tilde{x}$.

For a fixed $A$, let $H_\tilde{x}$ denote the distribution of $y$ when $x^* \sim P_\tilde{x}$. Let $H_{\text{out}}$ denote the corresponding distribution of $y$ when $x^* \sim P_{\text{out}}$. Then we have:

$$\mathbb{E}_A[TV(H_\tilde{x}, H_{\text{out}})] \geq 1 - 4e^{-\frac{\eta^2}{2} \log(\frac{\beta}{\sqrt{\epsilon}})}.$$

Proof. In order to prove the lemma, it suffices to show that on the set

$$B := \{y \in \mathbb{R}^m : \|y - Ax\| \leq \sqrt{c}(\eta + \sigma)\},$$
we have
\[
\mathbb{E}_A[H_{out}(B)] \leq 2e^{-\frac{m}{2}\log\left(\frac{e}{c}\right)},
\]
(9)
\[
\mathbb{E}_A[H_{\tilde{x}}(B)] \geq 1 - 2e^{-\frac{m}{2}\log\left(\frac{e}{c}\right)}.
\]
(10)

Using the above bounds, we can conclude that
\[
\mathbb{E}_A[TV(H_{out}, H_{\tilde{x}})] \geq \mathbb{E}_A[H_{\tilde{x}}(B)] - \mathbb{E}_A[H_{out}(B)] \geq 1 - 4e^{-\frac{m}{2}\log\left(\frac{e}{c}\right)}.
\]

First we prove Equation (9).

Consider the joint distribution of $y, A$. We have
\[
\mathbb{E}_A[H_{out}(B)] = \mathbb{E}_A\left[\mathbb{E}_{x \sim P_{out}}\left[N\left(Ax, \frac{\sigma^2}{m} I_m\right)(B)\right]\right],
\]
(11)
\[
= \mathbb{E}_{x \sim P_{out}}\left[\mathbb{E}_A\left[N(Ax, \frac{\sigma^2}{m})(B)\right]\right],
\]
(12)
where the first line follows from the definition of $H_{out}$ and the fact that $x, A$ are independent. The last line follows by switching the order of integrating $A, x$. Here $N(Ax, \frac{\sigma^2}{m})(B)$ refers to the mass $N(Ax, \frac{\sigma^2}{m})$ places on $B$.

Consider a fixed $x \in S_{\tilde{x}, out}$, that is, $x$ lies in the support of $P_{out}$ and satisfies $\|x - \tilde{x}\| \geq c(\eta + \sigma \sqrt{m})$. We split the above expectation into two conditions over the matrix $A$. 

- **Case 1:** $\|Ax - A\tilde{x}\| \leq 2\sqrt{c}(\eta + \sigma)$. Since $A$ is i.i.d. Gaussian, $A(x - \tilde{x})$ is distributed as $\mathcal{N}(0, \frac{\|x - \tilde{x}\|^2}{m} I_m)$. This gives
  \[
  \Pr_A[\|Ax - A\tilde{x}\| < 2\sqrt{c}(\eta + \sigma)] \leq \Pr_A\left[\|Ax - A\tilde{x}\| \leq \frac{2}{\sqrt{c}}\|x - \tilde{x}\|\right],
  \]
  \[
  \leq \frac{2}{\sqrt{m\pi}} \left(\frac{2e}{c}\right)^m,
  \]
  \[
  = \frac{2}{\sqrt{m\pi}} e^{-\frac{m}{2}\log\left(\frac{e}{c}\right)},
  \]
  \[
  \leq e^{-\frac{m}{2}\log\left(\frac{e}{c}\right)} \quad \text{if } m > 1.
  \]

  This implies
  \[
  \mathbb{E}_{x \sim P_{out}}\left[\mathbb{E}_A\left[N(Ax, \frac{\sigma^2}{m})(B)\right] | \|Ax - A\tilde{x}\| < 2\sqrt{c}(\eta + \sigma)\right] \leq \frac{2}{\sqrt{m\pi}} e^{-\frac{m}{2}\log\left(\frac{e}{c}\right)}.
  \]

- **Case 2:** $\|Ax - A\tilde{x}\| > 2\sqrt{c}(\eta + \sigma)$.

  Recall the definition of $B := \{y \in \mathbb{R}^m : \|y - A\tilde{x}\| \leq \sqrt{c}(\eta + \sigma)\}$. For any $y \in B, x$ in the support of $P_{out}$ and for $A$ such that $\|Ax - A\tilde{x}\| > 2\sqrt{c}(\eta + \sigma)$, we have
  \[
  \|y - Ax\| \geq \|Ax - A\tilde{x}\| - \|y - A\tilde{x}\| \geq 2\sqrt{c}(\eta + \sigma) - \sqrt{c}(\eta + \sigma) = \sqrt{c}(\eta + \sigma).
  \]

  For each $x$ in the support of $P_{out}$, define the set $B_x := \{y \in \mathbb{R}^m : \|y - Ax\| \geq \sqrt{c}(\eta + \sigma)\}$. The above inequality gives $B \subseteq B_x$ for each $x$ in the support of $P_{out}$. This gives
  \[
  \mathcal{N}(Ax, \sigma^2)(B) \leq \mathcal{N}(Ax, \sigma^2)(B_x) \leq e^{-2(\sqrt{c} - 1)^2m} \leq e^{-\frac{m}{2}}.
  \]

  where the last inequality follows by the definition of $B_x$ and Gaussian concentration of $\mathcal{N}(Ax, \sigma^2)$ on the set $B_x$, and since $2(\sqrt{c} - 1)^2 > \frac{1}{2}$ if $c \geq 4$. 

Substituting the inequalities from Case 1 and Case 2 in Eqn (12), we have
\[
E_A[H_{\text{out}}(B)] = E_{x \sim P_{\text{out}}}[E_A[N(Ax, \sigma^2/m)(B)]],
\]
\[
\leq e^{-\frac{m}{2} \log \left( \frac{c_4 e}{2} \right)} + e^{-\frac{cm}{2}},
\]
\[
\leq 2e^{-\frac{m}{2} \log \left( \frac{c_4 e}{2} \right)} \quad \text{if } c \geq 4e^2.
\]
This proves Eqn (9).

A similar proof can be used to show that
\[
E_A[H_{\tilde{x}}(B_c)] \leq 2e^{-\frac{m}{2} \log \left( \frac{c_4 e}{2} \right)}.
\]
This proves Eqn (10).

Putting the two above inequalities together, we have
\[
E_A[TV(H_{\text{out}}, H_{\tilde{x}})] \geq E_A[H_{\tilde{x}}(B)] - E_A[H_{\text{out}}(B)] \geq 1 - 4e^{-\frac{m}{2} \log \left( \frac{c_4 e}{2} \right)}.
\]
This concludes the proof.

A.3. Proof of Lemma A.1

**Lemma A.1.** Let \( R, P \) be arbitrary distributions on \( \mathbb{R}^n \). Let \( p \geq 1 \) and \( \eta, \rho, \delta > 0 \), be parameters.
If \( W_p(R, P) \leq \rho \) and \( \min\{\log \text{Cov}_{\eta, \delta}(P), \log \text{Cov}_{\eta, \delta}(R)\} \leq k \), then there exist distributions \( R', P', P'', R'' \), and a finite discrete distribution \( Q \) with \( |\text{supp}(Q)| \leq e^k \) satisfying:

1. \( \min\{W_\infty(P', Q), W_\infty(R', Q)\} \leq \eta \).
2. \( W_\infty(R', P') \leq \frac{\rho}{3^{1/p}} \).
3. \( P = (1 - 2\delta)P' + (2\delta)P'' \) and \( R = (1 - 2\delta)R' + (2\delta)R'' \).

**Proof.** Since the statement of the lemma is symmetric with respect to \( P \) and \( R \), WLOG let \( \log \text{Cov}_{\eta, \delta}(P) \leq k \). Then there is an \( S \subset \mathbb{R}^n \) such that \( |S| \leq e^k \) and
\[
\Pr_{x \sim P}[x \in \bigcup_{u \in S} B(u, \eta)] = 1 - c_P \geq 1 - \delta,
\]
We define the function \( f : \mathbb{R}^n \rightarrow \mathbb{R}_+ \) as
\[
f(x) = \begin{cases} 
1 & \text{if } \exists u \in S \text{ s.t. } x \in B(u, \eta), \\
0 & \text{otherwise.}
\end{cases}
\]
By construction, \( f \) is a piecewise constant function that is inversely proportional to the number of \( \eta \)-radius balls centered around points in \( S \) cover a point \( x \).
For each \( u \in S \), we define the measure \( Q'' \) as
\[
Q''(u) := \int_{B(u, \eta)} f \, dP.
\]
Observe that
\[
\sum_{u \in S} Q''(u) = \sum_{u \in S} \int_{B(u, \eta)} f \, dP,
\]

We look at the marginals of the conditional couple. We can restrict the event
where the first inequality is because
This finally gives distributions

\[ P = (1 - 2\delta)P' + (2\delta)P'' \]  

The first two statements follow because of the event we condition over.

Note that this restriction does not change the fact that \( \text{supp}(Q) < c^k \), and hence we have our result.
A.4. Proof of Lemma 3.3

Lemma 3.3. Let $R, P,$ denote arbitrary distributions over $\mathbb{R}^n$ such that $W_\infty(R, P) \leq \varepsilon$.

Let $x^* \sim R$ and $z^* \sim P$ and let $y$ and $u$ be generated from $x^*$ and $z^*$ via a Gaussian measurement process with $m$ measurements and noise level $\sigma$. Let $\tilde{x} \sim P(\cdot | y, A)$ and $\tilde{z} \sim P(\cdot | u, A)$. For any $d > 0$, we have

$$\Pr_{x^*, A, \xi, \tilde{x}} \left[ \|x^* - \tilde{x}\| \geq d + \varepsilon \right] \leq e^{-\Omega(m)} + e^{\left( \frac{M^2 + 2nM}{2\nu^2} \right)} \Pr_{z^*, A, \xi, \tilde{z}} \left[ \|z^* - \tilde{z}\| \geq d \right].$$

Proof. Let $B_1$ denote the event

$$B_1 = \left\{ \|x^* - \tilde{x}\| \geq d + \varepsilon \right\}.$$

Similarly, let $B_2$ denote the event

$$B_2 = \left\{ \|z^* - \tilde{z}\| \geq d \right\}.$$

We have

$$\Pr_{x^*, A, \xi, \tilde{x}} \left[ B_1 \right] = \mathbb{E}_{x^*, A, \xi, \tilde{x}} \left[ \mathbb{E}_{y | A, x^*} \left[ \mathbb{E}_{\tilde{z} | P(\cdot | y, A)} \left[ 1_{B_1} \right] \right] \right].$$

We can write the integral over $R$ as an integral over the coupling $\Pi$ between $R, P$. This gives

$$\Pr_{x^*, A, \xi, \tilde{x} \sim P(\cdot | A, y)} \left[ B_1 \right] = \mathbb{E}_{x^*, A, \xi, \tilde{x}} \left[ \mathbb{E}_{y | A, x^*} \left[ \mathbb{E}_{\tilde{z} | P(\cdot | y, A)} \left[ 1_{B_1} \right] \right] \right].$$

Since $x^*, z^*$ are coupled and $W_\infty(R, P) \leq \varepsilon$, we have $\|x^* - z^*\| \leq \varepsilon$ almost surely. This gives $B_1 \subseteq B_2$ if $x^*, z^*$ are distributed according to $\Pi$. Hence,

$$\Pr_{x^*, A, \xi, \tilde{x} \sim P(\cdot | A, y)} \left[ B_1 \right] \leq \mathbb{E}_{x^*, A, \xi, \tilde{x}} \left[ \mathbb{E}_{y | A, x^*} \left[ \mathbb{E}_{\tilde{z} | P(\cdot | y, A)} \left[ 1_{B_2} \right] \right] \right].$$

We can split the above integral into two parts: one where the matrix $A$ satisfies $\|Ax^* - Az^*\| \leq 2\varepsilon$, and another case where $\|Ax^* - Az^*\| > 2\varepsilon$. This gives

$$\Pr_{x^*, A, \xi, \tilde{x} \sim P(\cdot | A, y)} \left[ B_1 \right] \leq \mathbb{E}_{x^*, A, \xi} \left[ \mathbb{E}_{y | A, x^*} \left[ \mathbb{E}_{\tilde{z} | P(\cdot | y, A)} \left[ 1_{B_2} \right] \right] \right] \left( * \right)$$

$$+ \mathbb{E}_{x^*, A, \xi} \left[ \mathbb{E}_{y | A, x^*} \left[ \mathbb{E}_{\tilde{z} | P(\cdot | y, A)} \left[ 1_{B_2} \right] \right] \right]. \left( ** \right)$$

Consider the term $(*)$ in line (15). We have

$$\mathbb{E}_{x^*, z^*, A} \left[ 1_{\|Ax^* - Az^*\| > 2\varepsilon} \right] \left[ \mathbb{E}_{y | A, x^*} \left[ \mathbb{E}_{\tilde{z} | P(\cdot | y, A)} \left[ 1_{B_2} \right] \right] \right] \leq \mathbb{E}_{x^*, z^*, A} \left[ 1_{\|Ax^* - Az^*\| > 2\varepsilon} \right],$$

$$\leq \mathbb{E}_{x^*, z^*, A} \left[ e^{-\Omega(m)} \right] \leq e^{-\Omega(m)},$$

where the last inequality follows from the Johnson-Lindenstrauss lemma for a fixed $x^*, z^*$, and hence is true on average over $x^*, z^*$ drawn independent of $A$. 


Now consider the term (**) in line (16). Notice that since the noise in the measurements is Gaussian, we have
\[ y|x^*, A \sim \mathcal{N}(Ax^*, \sigma^2/m). \]

We break the integral over \( y \) in (**) into two cases:

1. **Case 1:** \( \|y - Ax^*\| > 2\sigma \). Since \( p(y|A, x^*) \) is distributed as \( \mathcal{N}\left(Ax^*, \frac{\sigma^2}{m} I_m\right) \), by standard Gaussian concentration, we have
\[
\int_{y: \|y - Ax^*\| > 2\sigma} p(y|A, x^*) dy \leq e^{-\Omega(m)}.
\]

2. **Case 2:** \( \|y - Ax^*\| \leq 2\sigma \). This gives
\[
\|Ax^* - y\|^2 = \|Ax^* - y - Az^*\|^2 + \|y - Az^*\|^2,
\]
\[
= \|Ax^* - y\|^2 - \|y - Ax^* + Ax^* - Az^*\|^2 + \|y - Az^*\|^2,
\]
\[
= -\|Ax^* - Az^*\|^2 - 2(y - Ax^*, Ax^* - Az^*) + \|y - Az^*\|^2.
\]

Observe that in (**), we have
\[
\|Ax^* - Az^*\| \leq 2\varepsilon \Rightarrow \|Ax^* - Az^*\|^2 \leq 4\varepsilon^2.
\]

By the Cauchy-Schwartz inequality and the assumption that \( \|y - Ax^*\| \leq 2\sigma \), we have
\[
2(y - Ax^*, Ax^* - Az^*) \leq 8\sigma \varepsilon.
\]

Substituting the above two inequalities, we have
\[
\exp\left(-\frac{\|Ax^* - y\|^2}{2\sigma^2/m}\right) \leq \exp\left(-\frac{4\varepsilon^2 - 8\sigma \varepsilon + \|y - Az^*\|^2}{2\sigma^2/m}\right) \exp\left(-\frac{\|Az^* - y\|^2}{2\sigma^2/m}\right),
\]
\[
\exp\left(-\frac{\|Ax^* - y\|^2}{2\sigma^2/m}\right) \leq \exp\left(-\frac{4\varepsilon^2}{2\sigma^2/m}\right) \exp\left(-\frac{\|y - Ax^*\|^2}{2\sigma^2/m}\right),
\]
\[
\exp\left(-\frac{\|Az^* - y\|^2}{2\sigma^2/m}\right) \leq \exp\left(-\frac{8\sigma \varepsilon}{2\sigma^2/m}\right) \exp\left(-\frac{\|y - Az^*\|^2}{2\sigma^2/m}\right).
\]

Observe that the LHS has the density of measurements from \( x^* \), while the RHS has the density of measurements from \( z^* \) with an exponential scaling. From the above inequality, we can replace the expectation over \( y|A, x^* \) in (**) with \( u|A, z^* \) with an exponential factor.

Similarly, since posterior sampling now uses \( u \) in place of \( y \), we can replace \( \widehat{x} \) in (**) with \( \widehat{z} \).

Combining Case 1 and 2 gives
\[
(\ast\ast) \leq e^{-\Omega(m)} + e^{\left(\frac{4x(\varepsilon^2 + 2\sigma^2)}{2\sigma^2}m\right)} \mathbb{E}_{x^*, z^*} \mathbb{E}_A \left[ \mathbb{E}_{u|A, z^*} \left[ \mathbb{E}_{\widehat{z} \sim \mathcal{P}(\cdot|u, A)} \left[ 1_{B_2} \right] \right] \right],
\]
\[
= e^{-\Omega(m)} + e^{\left(\frac{4x(\varepsilon^2 + 2\sigma^2)}{2\sigma^2}m\right)} \mathbb{E}_{x^*, z^*} \mathbb{E}_A \left[ \mathbb{E}_{u|A, z^*} \left[ \mathbb{E}_{\widehat{z} \sim \mathcal{P}(\cdot|u, A)} \left[ 1_{B_2} \right] \right] \right].
\]

From the above inequality and eqn. (18), we have
\[
\Pr_{x^* \sim R, \xi, A, \xi \sim \mathcal{P}(\cdot|A, y)} \left[ \|x^* - \widehat{x}\| \geq d + \varepsilon \right] \leq e^{-\Omega(m)} + e^{\left(\frac{4x(\varepsilon^2 + 2\sigma^2)}{2\sigma^2}m\right)} \Pr_{z^* \sim \mathcal{P}(\cdot|A, y)} \left[ \|z^* - \widehat{z}\| \geq d \right].
\]
A.5. Proof of Theorem 3.4

Theorem 3.4. Let $\delta \in [0, 1/4)$, $p \geq 1$, and $\varepsilon, \eta > 0$ be parameters. Let $R, P$ be arbitrary distributions over $\mathbb{R}^m$ satisfying $W_p(R, P) \leq \varepsilon$.

Let $x^* \sim R$ and suppose $y$ is generated by a Gaussian measurement process from $x^*$ with noise level $\sigma \geq \varepsilon/\delta^{1/p}$ and $m \geq O(\min\{\log \text{Cov}_{n,\delta}(R), \log \text{Cov}_{n,\delta}(P)\})$ measurements. Given $y$ and the fixed matrix $A$, let $\hat{x}$ be the output of posterior sampling with respect to $P$.

Then there exists a universal constant $c > 0$ such that with probability at least $1 - e^{-\Omega(m)}$ over $A, \xi$,

$$\Pr_{x^* \sim R, \hat{x} \sim P(|y)} [\|x^* - \hat{x}\| \geq c\eta + c\sigma] \leq 2\delta + 2e^{-\Omega(m)}.$$  

Proof. We know from Lemma A.1 that there exist $R', P', R'', P''$ and a finite distribution $Q$ supported on the set $S$ such that

1. $W_\infty(R', P') \leq \frac{\varepsilon}{2}\sqrt{\frac{p}{\pi}}$.
2. $\min\{W_\infty(P', Q), W_\infty(R', Q)\} \leq \eta$.
3. $R = (1 - 2\delta)R' + 2\delta R''$ and $P = (1 - 2\delta)P' + 2\delta P''$.
4. $|S| \leq e^k$.

Suppose $W_\infty(P', Q) \leq \eta$. If not, then $W_\infty(R', Q) \leq \eta$, and by (1), we see that $W_\infty(P', Q) \leq \eta + \frac{\varepsilon}{2}\sqrt{\frac{p}{\pi}}$, and we will use this in the proof instead. This gives us

$$\Pr_{x^* \sim R, \hat{x} \sim P(|y)} [\|x^* - \hat{x}\| \geq (c + 1)\eta + (c + 1)\sigma] \leq \Pr_{x^* \sim R, \hat{x} \sim P(|y)} [\|x^* - \hat{x}\| \geq (c + 1)\eta + c\sigma + (\varepsilon/\delta^{1/p})]$$

$$\leq 2\delta + (1 - 2\delta) \Pr_{x^* \sim R', \hat{x} \sim P(|y)} [\|x^* - \hat{x}\| \geq (c + 1)\eta + c\sigma + (\varepsilon/\delta^{1/p})],$$

where the first line follows since $\sigma \geq \varepsilon/\delta^{1/p}$, and the second line follows by decomposing $R = (1 - 2\delta)R' + 2\delta R''$.

We now bound the second term on the right hand side of the above equation. For this term, consider the joint distribution over $x^*, A, \xi, \hat{x}$. By Lemma 3.3, we can replace $x^* \sim R'$ with $z^* \sim P'$, replace $y = Ax^* + \xi$ with $u = Az^* + \xi$, and replace $\hat{x} \sim P(\cdot | A, y)$ with $\hat{z} \sim P(\cdot | A, u)$ to get the following bound

$$\Pr_{x^* \sim R', A, \xi, \hat{x} \sim P(\cdot | A, y)} [\|x^* - \hat{x}\| \geq (c + 1)\eta + c\sigma + (\varepsilon/\delta^{1/p})] \leq e^{-\Omega(m)} + e^{-\frac{c\eta}{\delta^{1/p}} + \frac{c\sigma}{\delta^{1/p}} + 2\sigma} \Pr_{z^* \sim P', A, \xi, \hat{z} \sim P(\cdot | A, u)} [\|z^* - \hat{z}\| \geq (c + 1)\eta + c\sigma].$$

We now bound the second term in the right hand side of the above inequality. Let $\Gamma$ denote an optimal $W_\infty$-coupling between $P'$ and $Q$.

For each $\tilde{z} \in S$, the conditional coupling can be defined as

$$\Gamma(\cdot | \tilde{z}) = \frac{\Gamma(\cdot, \tilde{z})}{Q(\tilde{z})}.$$

By the $W_\infty$ condition, each $\Gamma(\cdot | \tilde{z})$ is supported on a ball of radius $\eta$ around $\tilde{z}$.

Let $E = \{z^*, \tilde{z} \in \mathbb{R}^m : \|z^* - \tilde{z}\| \geq (c + 1)\eta + c\sigma\}$ denote the event that $z^*, \tilde{z}$ are far apart. By the coupling, we can express $P'$ as

$$P' = \sum_{\tilde{z} \in S} Q(\tilde{z}) \Gamma(\cdot | \tilde{z}).$$
This gives
\[
\Pr_{z^* \sim P', A, \xi, \tilde{z} \sim P(\cdot | A, u)} [E] = \sum_{\tilde{z} \in S} Q(\tilde{z}^*) \frac{E}{z^* \sim \Gamma(\cdot | z^*), A, \xi, \tilde{z} \sim P(\cdot | A, u)} [1_E].
\]

For each $\tilde{z}^* \in S$, we now bound $Q(\tilde{z}^*) \frac{E}{z^* \sim \Gamma(\cdot | z^*), A, \xi, \tilde{z} \sim P(\cdot | A, u)} [1_E].$

For each $\tilde{z}^* \in S$, we can write $P$ as $P = (1 - 2\delta) Q_{\tilde{z}^*} P_{\tilde{z}^*, 0} + c_{\tilde{z}^*, 1} P_{\tilde{z}^*, 1} + c_{\tilde{z}^*, 2} P_{\tilde{z}^*, 2}$, where the components of the mixture are defined in the following way. The first component $P_{\tilde{z}^*, 0}$ is $\Gamma(\cdot | \tilde{z}^*)$, the second component is supported within a $c(\eta + \sigma)$ radius of $\tilde{z}^*$, and the third component is supported outside a $c(\eta + \sigma)$ radius of $\tilde{z}^*$.

Formally, let $B_{\tilde{z}^*}$ denote the ball of radius $c(\eta + \sigma)$ centered at $\tilde{z}^*$, and let $B_{\tilde{z}^*}^c$ be its complement. The constants are defined via the following Lebesgue integrals, and the mixture components for any Borel measurable $B$ are defined as
\[
c_{\tilde{z}^*, 1} := \int_{B_{\tilde{z}^*}} dP - (1 - 2\delta) Q_{\tilde{z}^*} \int_{B_{\tilde{z}^*}} d\Gamma(\cdot | \tilde{z}^*),
\]
\[
c_{\tilde{z}^*, 2} := \int_{B_{\tilde{z}^*}^c} dP - (1 - 2\delta) Q_{\tilde{z}^*} \int_{B_{\tilde{z}^*}^c} d\Gamma(\cdot | \tilde{z}^*),
\]
\[
P_{\tilde{z}^*, 0}(B) := \Gamma(B \cap B_{\tilde{z}^*}) \Gamma(\tilde{z}^*) \text{ since supp}(\Gamma(\cdot | \tilde{z}^*)) \subset B_{\tilde{z}^*},
\]
\[
P_{\tilde{z}^*, 1}(B) := \begin{cases} \frac{1}{c_{\tilde{z}^*, 1}} P(B \cap B_{\tilde{z}^*}) - \frac{1 - 2\delta}{c_{\tilde{z}^*, 1}} Q_{\tilde{z}^*} \Gamma(B \cap B_{\tilde{z}^*}| \tilde{z}^*) & \text{if } c_{\tilde{z}^*, 1} > 0, \\
\text{do not care} & \text{otherwise}. 
\end{cases}
\]
\[
P_{\tilde{z}^*, 2}(B) := \begin{cases} \frac{1}{c_{\tilde{z}^*, 2}} P(B \cap B_{\tilde{z}^*}^c) - \frac{1 - 2\delta}{c_{\tilde{z}^*, 2}} Q_{\tilde{z}^*} \Gamma(B \cap B_{\tilde{z}^*}^c | \tilde{z}^*) & \text{if } c_{\tilde{z}^*, 2} > 0, \\
\text{do not care} & \text{otherwise}. 
\end{cases}
\]

Notice that if $z^*$ is sampled from $\Gamma(\cdot | \tilde{z}^*)$, then by the $W_\infty$ condition, we have $\|z^* - \tilde{z}^*\| \leq \eta$. Furthermore, if $\tilde{z}$ is $(c + 1)\eta + c\sigma$ far from $z^*$, an application of the triangle inequality implies that it must be distributed according to $P_{\tilde{z}^*, 2}$.

That is,
\[
Q(\tilde{z}^*) \frac{E}{z^* \sim \Gamma(\cdot | z^*), A, \xi, \tilde{z} \sim P(\cdot | A, u)} [1_E] \leq \frac{E}{A, \xi, z^*} \Pr[z^* \sim P_{z^*, 0}, \tilde{z} \sim P_{z^*, 2}(\cdot | u)]
\]
\[
\leq \frac{1}{1 - 2\delta} \mathbb{E} \left[ 1 - TV(H_{\tilde{z}^*, 0}, H_{\tilde{z}^*, 2}) \right],
\]
where $H_{\tilde{z}^*, 0}, H_{\tilde{z}^*, 2}$ are the push-forwards of $P_{\tilde{z}^*, 0}, P_{\tilde{z}^*, 2}$ for $A$ fixed and the last inequality follows from Claim A.2.

Notice that if we sum over all $\tilde{z}^* \in S$, then the LHS of the above inequality is an expectation over $z^* \sim P'$. This gives:
\[
\Pr_{z^* \sim P', A, \xi, \tilde{z} \sim P(\cdot | A, u)} [E] \leq \frac{1}{1 - 2\delta} \sum_{\tilde{z}^* \in S} \mathbb{E}[1 - TV(H_{\tilde{z}^*, 0}, H_{\tilde{z}^*, 2})].
\]

Notice that $P_{\tilde{z}^*, 0}$ is supported within an $\eta-$ball around $\tilde{z}^*$, and $P_{\tilde{z}^*, 2}$ is supported outside a $c(\eta + \sigma)-$ball of $\tilde{z}^*$. By Lemma 3.2 we have
\[
\mathbb{E}_A[TV(H_{\tilde{z}^*, 0}, H_{\tilde{z}^*, 2})] \geq 1 - 4e^{-\frac{\eta}{\sigma}}.
\]
This implies
\[
\Pr_{z^* \sim P', A, \xi, \tilde{z} \sim P(\cdot | A, u)} [\|z^* - \tilde{z}\| \geq (c + 1)\eta + c\sigma] \leq \frac{1}{1 - 2\delta} \sum_{\tilde{z}^* \in S} \mathbb{E}_A[1 - TV(H_{\tilde{z}^*, 0}, H_{\tilde{z}^*, 2})],
\]
where the last inequality is satisfied if $m \geq 4 \log \left( |S| \right)$.

Substituting in Eqn (23), if $c > 4 \exp \left( 2 + \frac{8(e/\delta^{1/p})(e/\delta^{1/p})+2\sigma}{\sigma^2} \right)$, we have

$$\Pr_{x^* \sim R^*, A, \xi, \tilde{x} \sim P(\cdot | y)} \left[ \|x^* - \tilde{x}\| \geq (c + 1) \eta + c\sigma + (\varepsilon/\delta^{1/p}) \right] \leq e^{-\Omega(m)} + \frac{1}{1 - 2\delta} e^{-\Omega(m \log c)}.$$ 

This implies that there exists a set $S_{A, \xi}$ over $A, \xi$ satisfying $\Pr_{A, \xi} [S_{A, \xi}] \geq 1 - e^{-\Omega(m)}$, such that for all $A, \xi \in S_{A, \xi}$, we have

$$\Pr_{x^* \sim R^*, \tilde{x} \sim P(\cdot | y)} \left[ \|x^* - \tilde{x}\| \geq (c + 1) \eta + c\sigma + (\varepsilon/\delta^{1/p}) \right] \leq \frac{1}{1 - 2\delta} e^{-\Omega(m)}.$$ 

Substituting in Eqn (22), we have

$$\Pr_{x^* \sim R^*, \tilde{x} \sim P(\cdot | y)} \left[ \|x^* - \tilde{x}\| \geq (c + 1) \eta + c\sigma + (\varepsilon/\delta^{1/p}) \right] \leq 2\delta + \frac{1}{1 - 2\delta} e^{-\Omega(m)} \leq 2\delta + 2e^{-\Omega(m)}.$$ 

Rescaling $c$ gives us our result.

At the beginning of the proof, we had assumed that $W_\infty(Q' , Q) \leq \eta$. If instead $W_\infty(R', Q) \leq \eta$, then we need to replace $\eta$ in the above bound by $\eta + \frac{\varepsilon}{\delta^{1/p}}$. Rescaling $c$ in the above bound gives us the Theorem statement.

\[\]

**Claim A.2.** Consider the setting of the previous theorem. We have

$$\mathbb{E}_{A, \xi, z^*} \Pr_{z^* \sim P_{z^*,0}, \tilde{z} \sim P_{z^*,2}(\cdot | u)} \leq \frac{1}{1 - 2\delta} \mathbb{E}_{A} \left[ 1 - TV(H_{z^*,0}, H_{z^*,2}) \right].$$

**Proof.** For a fixed $A$, let $h_0, h_2$ denote the corresponding densities of the push forward of $P_{z^*,0}, P_{z^*,2}$. Then we have

$$\mathbb{E}_{A, \xi, z^*} \Pr_{z^* \sim P_{z^*,0}, \tilde{z} \sim P_{z^*,2}(\cdot | u)} = \mathbb{E}_{A} \int_{A} \frac{Q_{z^*,0}h_{z^*,0}(u)c_{z^*,2}h_{z^*,2}(u)}{(1 - \delta_2) Q_{z^*,0}h_{z^*,0}(u) + c_{z^*,2}h_{z^*,2}(u)} du,$$

$$\leq \mathbb{E}_{A} \int_{A} \frac{Q_{z^*,0}h_{z^*,0}(u)c_{z^*,2}h_{z^*,2}(u)}{(1 - \delta_2) Q_{z^*,0}h_{z^*,0}(u) + c_{z^*,2}h_{z^*,2}(u)} du,$$

$$\leq \mathbb{E}_{A} \int_{A} \frac{Q_{z^*,0}h_{z^*,0}(u)c_{z^*,2}h_{z^*,2}(u)}{(1 - \delta_2) Q_{z^*,0}h_{z^*,0}(u) + c_{z^*,2}h_{z^*,2}(u)} du,$$

$$\leq \frac{1}{1 - 2\delta} \mathbb{E}_{A} \int_{A} \frac{Q_{z^*,0}h_{z^*,0}(u)c_{z^*,2}h_{z^*,2}(u)}{Q_{z^*,0}h_{z^*,0}(u) + c_{z^*,2}h_{z^*,2}(u)} du,$$

$$\leq \frac{1}{1 - 2\delta} \mathbb{E}_{A} \int_{A} \min \{Q_{z^*,0}h_{z^*,0}(u), c_{z^*,2}h_{z^*,2}(u) \} du,$$

$$\leq \frac{1}{1 - 2\delta} \mathbb{E}_{A} \int_{A} \min \{h_{z^*,0}(u), h_{z^*,2}(u) \} du,$$

$$= \frac{1}{1 - 2\delta} \mathbb{E}_{A} \left[ 1 - TV(H_{z^*,0}, H_{z^*,2}) \right].$$

\[\]
B. Lower Bound Proofs

B.1. Proof of Lemma 4.2

\textbf{Lemma 4.2.} Consider the setting of Theorem (4.1). If $A$ is a deterministic matrix, we have

$$I(y; x^*) \leq \frac{m}{2} \log \left(1 + \frac{mr^2 \|A\|^2_\infty}{\sigma^2}\right).$$

If $A$ is a Gaussian matrix, then

$$I(y; x^*|A) \leq \frac{m}{2} \log \left(1 + \frac{r^2}{\sigma^2}\right).$$

\textit{Proof.} First, we consider the case where $A$ is a deterministic matrix.

We have $y = Ax^* + \xi$. Let $z = Ax^*$, which gives $y = z + \xi$. We have $z_i = a_i^T x^*$ where $a_i$ is the $i^{th}$ row of $A$, and $y_i = z_i + \xi_i$. Since $x^*$ is supported within the sphere of radius $r$, we have $\mathbb{E}[z_i^2] = \mathbb{E}[\langle a_i, x^* \rangle^2] \leq \|a_i\|^2 r^2$. Since the Gaussian noise $\xi$ has variance $\sigma^2/m$ in each coordinate, every coordinate of $y_i$ is a Gaussian channel with power constraint $\|a_i\|^2 r^2$ and noise variance $\sigma^2/m$. Using Shannon’s AWGN theorem (Cover & Thomas, 2012; Polyanskiy & Wu, 2014; Shannon, 1948), the mutual information between $y_i, z_i$ is bounded by

$$I(y_i; z_i) \leq \frac{1}{2} \log \left(1 + \frac{\|a_i\|^2 r^2 m}{\sigma^2}\right).$$

The chain rule of entropy and sub-additivity of entropy implies,

$$I(y; z) = h(y) - h(y|z) = h(y) - h(y - z|z),$$

$$= h(y) - h(\xi|z) = h(y) - \sum h(\xi_i|z, \xi_1, \cdots, \xi_{i-1}),$$

$$= h(y) - \sum h(\xi_i),$$

$$\leq \sum h(y_i) - \sum h(\xi_i),$$

$$= \sum h(y_i) - \sum h(y_i|z_i),$$

$$= \sum I(y_i; z_i),$$

$$\leq \sum_{i=1}^{m} \frac{1}{2} \log \left(1 + \frac{\|a_i\|^2 r^2 m}{\sigma^2}\right),$$

$$\leq \frac{m}{2} \log \left(1 + \frac{mr^2 \|A\|^2_\infty}{\sigma^2}\right).$$

Since $x^* \to z \to y$ is a Markov chain, we can conclude that

$$I(y; x^*) \leq I(y; z) \leq \frac{m}{2} \log \left(1 + \frac{mr^2 \|A\|^2_\infty}{\sigma^2}\right).$$

Now, if $A$ is a Gaussian matrix with i.i.d. entries drawn from $\mathcal{N}(0, 1/m)$, then the power constraint is $\mathbb{E}[\langle a_i, x \rangle^2] \leq r^2/m$. This gives us

$$I(y; z) \leq \frac{m}{2} \log \left(1 + \frac{r^2}{\sigma^2}\right). \quad (33)$$

Now since $A$ is a random matrix, we cannot directly apply the Data Processing Inequality of $x^*, y, z$ as before, and need to prove that $I(x^*; y|A) \leq I(y; z)$.
We can bound $I$ where the second last line follows from

By the chain rule of mutual information, we can express it in two ways:

Now when $A$ is a random matrix, we need to show $I(x^*; A) \leq I(y; x^*)$. Consider the setting of Theorem (4.1). Since $x^* \rightarrow y \rightarrow \hat{x}$ is a Markov chain, we get

where the second last line follows from $I(A; y | z) = 0$, and the last line follows from $I(A; y) \geq 0$.

From Eqn (33), (35), (40), we have

where the second last line follows from $I(A; y | z) = 0$, and the last line follows from $I(A; y) \geq 0$.

We can bound $I(z; y|A)$ in the following way.

This gives

For $A$ is a random matrix, then

Proof. When $A$ is a deterministic matrix, the proof follows directly from the Data Processing Inequality (Cover & Thomas, 2012). Since $x^* \rightarrow y \rightarrow \hat{x}$ is a Markov chain, we get

where the second last line follows from $I(A; y | z) = 0$, and the last line follows from $I(A; y) \geq 0$.

We can bound $I(z; y|A)$ in the following way.

Then

Lemma 4.3. Consider the setting of Theorem (4.1). If $A$ is a deterministic matrix, we have

If $A$ is a random matrix, then

Proof. When $A$ is a deterministic matrix, the proof follows directly from the Data Processing Inequality (Cover & Thomas, 2012). Since $x^* \rightarrow y \rightarrow \hat{x}$ is a Markov chain, we get

where the second last line follows from $I(A; y | z) = 0$, and the last line follows from $I(A; y) \geq 0$.

We can bound $I(z; y|A)$ in the following way.

This gives

This gives

where the second last line follows from $I(A; y | z) = 0$, and the last line follows from $I(A; y) \geq 0$.

We can bound $I(z; y|A)$ in the following way.

This gives

where the second last line follows from $I(A; y | z) = 0$, and the last line follows from $I(A; y) \geq 0$.

B.2. Proof of Lemma 4.3

Lemma 4.3. Consider the setting of Theorem (4.1). If $A$ is a deterministic matrix, we have

If $A$ is a random matrix, then

Proof. When $A$ is a deterministic matrix, the proof follows directly from the Data Processing Inequality (Cover & Thomas, 2012). Since $x^* \rightarrow y \rightarrow \hat{x}$ is a Markov chain, we get

where the second last line follows from $I(A; y | z) = 0$, and the last line follows from $I(A; y) \geq 0$.

We can bound $I(z; y|A)$ in the following way.

This gives

where the second last line follows from $I(A; y | z) = 0$, and the last line follows from $I(A; y) \geq 0$.

B.2. Proof of Lemma 4.3

Lemma 4.3. Consider the setting of Theorem (4.1). If $A$ is a deterministic matrix, we have

If $A$ is a random matrix, then

Proof. When $A$ is a deterministic matrix, the proof follows directly from the Data Processing Inequality (Cover & Thomas, 2012). Since $x^* \rightarrow y \rightarrow \hat{x}$ is a Markov chain, we get

where the second last line follows from $I(A; y | z) = 0$, and the last line follows from $I(A; y) \geq 0$.

We can bound $I(z; y|A)$ in the following way.

This gives

where the second last line follows from $I(A; y | z) = 0$, and the last line follows from $I(A; y) \geq 0$.
Then for all \( \tau \) we would like to apply a version of Markov’s inequality to the above equation. However, the terms in the summation could be negative. However, from (43) we have that

\[
I(x^*; \hat{x}, A) \geq 0.
\]

Substituting in Eqn (41), (42), we have

\[
I(x^*; \hat{x}) \leq I(x^*; y, A),
\]

\[
= I(x^*; A) + I(x^*; y|A),
\]

\[
= I(x^*; y|A),
\]

where the second line follows from the chain rule of mutual information, and the last line follows because \( x^*, A \) are independent.

\[\square\]

**B.3. Proof of Fano variant Lemma 4.4**

We will build up Lemma 4.4 in sequence. Before showing it in its full generality, we will show when \( x, \hat{x} \) are discrete random variables and \( x \) is uniform (Lemma B.1). We then lift the uniformity restriction on \( x \) (Lemma B.3) before extending to continuous distributions (Lemma 4.4).

**Lemma B.1.** Let \( Q \) be the uniform distribution over an arbitrary discrete finite set \( S \). Let \( (x, \hat{x}) \) be jointly distributed, where \( x \sim Q \) and \( \hat{x} \) is distributed over an arbitrary countable set, satisfying

\[
\Pr[\|x - \hat{x}\| \leq \varepsilon] \geq 1 - \delta.
\]

Then for all \( \tau \in (0, 1) \), we have

\[
\tau(1 - \delta) \log \text{Cov}_{2\varepsilon, \delta + \tau}(Q) \leq I(x; \hat{x}) + 2.
\]

**Proof.** Let \( E = 1\{\|x - \hat{x}\| \leq \varepsilon\} \) be the indicator random variable for \( x \) and \( \hat{x} \) being close.

Via claim B.2, we get

\[
H(x|E = 1) \geq \log |S| - \frac{1}{1 - \delta}.
\]

(43)

Recall,

\[
I(x; \hat{x}|E = 1) = H(x|E = 1) - H(x|\hat{x}, E = 1)
\]

By the Law of total probability, we have:

\[
I(x; \hat{x}|E = 1) = \sum_v \Pr[\hat{x} = v|E = 1] (H(x|E = 1) - H(x|\hat{x} = v, E = 1)).
\]

We would like to apply a version of Markov’s inequality to the above equation. However, the terms in the summation could be negative. However, from (43) we have that \( H(x|E = 1) + \frac{1}{1 - \delta} \geq \log |S| \). Furthermore, since \( x \) is supported on a discrete set of cardinality \( |S| \), we have \( H(x|\hat{x} = v, E = 1) \leq \log |S| \). Adding and subtracting \( \frac{1}{1 - \delta} \), in the above equation, we have

\[
I(x; \hat{x}|E = 1) = \sum_v \Pr[\hat{x} = v|E = 1] \left( H(x|E = 1) + \frac{1}{1 - \delta} - H(x|\hat{x} = v, E = 1) - \frac{1}{1 - \delta} \right),
\]

\[
= \sum_v \Pr[\hat{x} = v|E = 1] \left( H(x|E = 1) + \frac{1}{1 - \delta} - H(x|\hat{x} = v, E = 1) \right) - \frac{1}{1 - \delta},
\]

\[
\Leftrightarrow I(x; \hat{x}|E = 1) + \frac{1}{1 - \delta} = \sum_v \Pr[\hat{x} = v|E = 1] \left( H(x|E = 1) + \frac{1}{1 - \delta} - H(x|\hat{x} = v, E = 1) \right)
\]
Since the above summation has only non-negative terms that average to \( I(x; \hat{x}|E = 1) + \frac{1}{1-\delta} \), for all \( \tau \in (0, 1) \), there exists \( G_1 \subseteq \text{supp}(\hat{x}) \) with \( \Pr[G_1|E = 1] \geq 1 - \tau \), such that for all \( v \in G_1 \), we have

\[
H(x|E = 1) + \frac{1}{1-\delta} - H(x|\hat{x} = v, E = 1) \leq \frac{I(x; \hat{x}|E = 1) + \frac{1}{1-\delta}}{\tau}.
\]

From (43), we have \( H(x|E = 1) + \frac{1}{1-\delta} \geq \log |S| \). Hence for all \( v \in G_1 \), we have

\[
\log |S| - H(x|\hat{x} = v, E = 1) \leq \frac{I(x; \hat{x}|E = 1) + \frac{1}{1-\delta}}{\tau},
\]

\[
\iff H(x|\hat{x} = v, E = 1) \geq \log |S| - \frac{I(x; \hat{x}) + \frac{1}{1-\delta}}{\tau},
\]

\[
\Rightarrow \log|\text{supp}(x|\hat{x} = v, E = 1)| \geq \log |S| - \frac{I(x; \hat{x}) + \frac{1}{1-\delta}}{\tau},
\]

\[
\Rightarrow \log|S \cap B(v, \epsilon)| \geq \log |S| - \frac{I(x; \hat{x}) + \frac{1}{1-\delta}}{\tau},
\]

(44)

where the last inequality follows as conditioned on \( E = 1 \), \( x \) must be supported on an \( \epsilon \)-radius ball around \( \hat{x} \).

Now consider the set \( G_2 = (S \times G_1) \land E_1 \). That is, \( G_2 \subseteq \text{supp}(x, \hat{x}) \), such that \( (u, v) \in G_2 \) if and only if \( \|u - v\| \leq \epsilon \) and \( u \in S, v \in G_1 \). Since \( \Pr[E_1] \geq 1 - \delta \) by the statement of the lemma, and \( \Pr[G_1|E_1] \geq 1 - \tau \) by construction, we have

\[
\Pr[G_2] \geq (1 - \delta)(1 - \tau) \geq 1 - \delta - \tau.
\]

Now for all \( (u, v) \in G_2 \), we have

\[
\|u - v\| \leq \epsilon,
\]

\[
\log |S \cap B(v, \epsilon)| \geq \log |S| - \frac{I(x; \hat{x}) + \frac{1}{1-\delta}}{\tau}.
\]

(45)

Note that by the construction of \( G_2 \), the set \( \bigcup_{v \in G_2} B(v, \epsilon) \) covers a \( 1 - \delta - \tau \) fraction of \( S \). As each ball \( B(v, \epsilon) \) also has a large intersection with \( S \), by the pigeon-hole principle, any \( 2\epsilon \)-packing of this \( 1 - \delta - \tau \) fraction of \( S \) must have size at most \( 2^{(I(x; \hat{x}|E = 1) + \frac{1}{1-\delta})/\tau} \).

Hence, we can find a \( 2\epsilon \)-cover of a \( 1 - \delta - \tau \) fraction of \( S \) that has size at most \( 2^{(I(x; \hat{x}|E = 1) + \frac{1}{1-\delta})/\tau} \).

This gives

\[
\log \text{Cov}_{2\epsilon, \delta + \tau}(Q) \leq \frac{I(x; \hat{x}|E = 1) + \frac{1}{1-\delta}}{\tau}.
\]

(46)

We are almost done, since we now only need to relate \( I(x; \hat{x}|E = 1) \) to \( I(x; \hat{x}) \).

By the chain rule of mutual information, we have

\[
I(x; \hat{x}, E) = I(x; \hat{x}) + I(x; E|\hat{x}) = I(x; E) + I(x; \hat{x}|E),
\]

\[
\Rightarrow I(x; \hat{x}|E) \leq I(x; \hat{x}) + I(x; E|\hat{x}),
\]

\[
\leq I(x; \hat{x}) + 1,
\]

\[
\iff I(x; \hat{x}|E = 0) \Pr[E = 0] + I(x; \hat{x}|E = 1) \Pr[E = 1] \leq I(x; \hat{x}) + 1,
\]

\[
\Rightarrow I(x; \hat{x}|E = 1) \leq \frac{I(x; \hat{x}) + 1}{1 - \delta}.
\]
Substituting in Eqn (46), we have
\[
\log \text{Cov}_{2\varepsilon,\tau}(Q) \leq \frac{I(x; \widehat{x}) + 2}{\tau(1 - \delta)},
\]
\[\Rightarrow \tau(1 - \delta) \log \text{Cov}_{2\varepsilon,\tau}(Q) \leq I(x; \widehat{x}) + 2.\]

**Claim B.2.** Let \( x \sim Q \), where \( Q \) is the uniform distribution over an arbitrary discrete finite set \( S \). Let \( E \) be a binary random variable such that \( \Pr[E = 1] \geq 1 - \delta \).

Then we have
\[
H(x|E = 1) \geq \log |S| - \frac{1}{1 - \delta}.
\]

**Proof.** Let \( p = \Pr[E = 1] \). By the definition of conditional entropy, we have
\[
H(x|E) = (1 - p)H(x|E = 0) + pH(x|E = 1),
\]
\[\Leftrightarrow H(x|E = 1) = \frac{1}{p} \left( H(x) - (1 - p)H(x|E = 0) \right),\]
\[= \frac{1}{p} \left( \log |S| - I(x; E) - (1 - p)H(x|E = 0) \right),\]
\[\geq \frac{1}{p} \left( \log |S| - I(x; E) - (1 - p) \log |S| \right),\]
\[= \log |S| - \frac{I(x; E)}{p},\]
\[\geq \log |S| - \frac{1}{1 - \delta},\]
where the fourth line follows from \( H(x) = \log |S| \) since \( x \) is uniform, the fifth line follows from \( H(x|E = 0) \leq \log |S| \) since \( x \) is supported on a discrete set of size \( |S| \), and the last line follows from \( p \geq 1 - \delta \) and \( I(x; E) \leq H(E) \leq 1 \).

The previous lemma handled the uniform distribution on \( x \). Now we show that a similar result applies if \( x \)'s distribution has quantized probability values.

**Lemma B.3.** Let \( Q \) be a finite discrete distribution over \( N \in \mathbb{N} \) points such that for each \( u \) in its support, \( Q(u) = j\alpha \), where \( j \in \mathbb{N} \) and \( \alpha := \frac{1}{N_2} \) is a discretization level for \( N_2 \in \mathbb{N} \) large enough.

Let \((x, \widehat{x})\) be jointly distributed, where \( x \sim Q \) and \( \widehat{x} \) is distributed over a countable set, satisfying
\[
\Pr[\|x - \widehat{x}\| \leq \varepsilon] \geq 1 - \delta.
\]

Then we have
\[
\tau(1 - \delta) \log \text{Cov}_{2\varepsilon,\tau}(Q) \leq I(x; \widehat{x}) + 2\delta.
\]

**Proof.** For each \( x \) in the support of \( Q \), we know that its probability is an integral multiple of \( \frac{1}{N_2} \). Hence we can define a new random variable \( x' = (x, j), x \in \text{supp}(Q), j \in [N_2] \) and a distribution \( Q' \) over \( x' \) in the following way:
\[
Q'((x, j)) = \begin{cases} 
\alpha & \text{if } j\alpha \leq Q(x), \\
0 & \text{otherwise}.
\end{cases}
\]

By definition, \( Q' \) is a uniform distribution, and its support is a discrete subset of \( \mathbb{R}^n \times \mathbb{N} \).
Define the following norm for \( x' \). For \( x'_1 = (x_1, j_1), x'_2 = (x_2, j_2) \), define
\[
\| (x_1, j_1) - (x_2, j_2) \| := \| x_1 - x_2 \|.
\]

In order to apply Lemma B.1 on \( Q' \), it suffices to show that \( I(x; \hat{x}) = I(x'; \hat{x}) \).

By the chain rule of mutual information, we have
\[
I(x'; \hat{x}) = I((x, j); \hat{x}) = I(x; \hat{x}) + I(j; \hat{x}|x).
\]

Since \( \hat{x} \) is purely a function of \( x \), we have \( I(j; \hat{x}|x) = 0 \). This gives
\[
I(x'; \hat{x}) = I(x; \hat{x}).
\]

Similarly construct a version \( \hat{x}' = (\hat{x}, 0) \) of \( \hat{x} \), whose second coordinate is identically zero. Hence for \( x' = (x, j) \sim Q' \), we have
\[
\| x' - \hat{x}' \| \leq \varepsilon \text{ w.p. } 1 - \delta,
\]
\[
I(x'; \hat{x}') = I(x; \hat{x})
\]

Applying Lemma B.1 on \( Q' \), we have
\[
\tau (1 - \delta) \log \text{Cov}_{2\varepsilon, \tau + \delta}(Q') \leq I(x; \hat{x}) + 2.
\]

Since the support of the first coordinate of \( Q' \) is the same as the support of \( Q \), we have
\[
\tau (1 - \delta) \log \text{Cov}_{2\varepsilon, \tau + \delta}(Q) \leq I(x; \hat{x}) + 2.
\]

We now prove Lemma 4.4, which allows \( (x, \hat{x}) \) to follow an arbitrary distribution.

**Lemma 4.4 (Fano variant).** Let \( (x, \hat{x}) \) be jointly distributed over \( \mathbb{R}^n \times \mathbb{R}^n \), where \( x \sim R \) and \( \hat{x} \) satisfies
\[
\Pr[\| x - \hat{x} \| \leq \eta] \geq 1 - \delta.
\]

Then for any \( \tau \leq 1 - 3\delta, \delta < 1/3 \), we have
\[
0.99 \tau (1 - 2\delta) \log \text{Cov}_{3\eta, \tau + 3\delta}(R) \leq I(x; \hat{x}) + 1.98.
\]

**Proof.** Let \( \varepsilon = \eta \), which is the error in the statement of the lemma. Let \( \gamma > 0 \) be a small enough discretization level to be specified later. For every \( x, \hat{x} \in \mathbb{R}^n \), let \( \bar{x}, \ddot{x} \) denoted the rounding of \( x, \hat{x} \) to the nearest multiple of \( \gamma \) in each coordinate.

Let \( \bar{R} \) be the discrete distribution induced by this discretization of \( x \). We can create such a distribution by assigning the probability of each cell in the grid to its corresponding coordinate-wise floor. This discretization of the support changes the error between \( x, \hat{x} \) in the following way. If \( \| x - \bar{x} \| \leq \varepsilon \) with probability \( 1 - \delta \), an application of the triangle inequality gives
\[
\| x - \ddot{x} \| \leq \varepsilon + 2\gamma \sqrt{n} \text{ with probability } \geq 1 - \delta.
\]

We also need to take into account the effect discretizing \( x, \hat{x} \) has on their mutual information. Note that since \( \bar{x} \) is a function of \( x \) alone, and \( \ddot{x} \) is a function of \( \bar{x} \) alone, by the Data Processing Inequality, we have
\[
I(\hat{x}; \bar{x}) \leq I(x; \ddot{x}).
\]

Note that \( \bar{R} \) is a distribution on a discrete but infinite set. However, for any \( \beta \in (0, 1] \), we can find a discrete and finite distribution \( Q \) such that \( \bar{R} = (1 - c_1)Q + c_1 D \), with \( c_1 \leq \beta \) and \( D \) is some other probability distribution. This is feasible
because the probabilities of the infinite support of $\hat{R}$ must sum to 1, and hence we can find a finite subset that sums to at least $1 - \beta$ for any $\beta \in (0, 1]$. Note that in this process, we only change the marginal of $\hat{x}$ without changing the conditional distribution of $\hat{x}|\hat{x}$. Let $I(\hat{x}; \hat{x}), I_Q(\hat{x}; \hat{x}), I_D(\hat{x}; \hat{x})$ denote the mutual information between $\hat{x}, \hat{x}$ when the marginal of $\hat{x}$ is $\hat{R}, Q, D$, respectively. From Theorem 2.7.4 in (Cover & Thomas, 2012), mutual information is a concave function of the marginal distribution of $\hat{x}$ for a fixed conditional distribution of $\hat{x}|\hat{x}$. An application of Eqn (48) gives us,

$$I(\hat{x}; \hat{x}) \geq I(\hat{x}; \hat{x}) \geq (1 - c_1)I_Q(\hat{x}; \hat{x}) + c_1I_D(\hat{x}; \hat{x}),$$

(49)

$$\geq (1 - c_1)I_Q(\hat{x}; \hat{x}),$$

(50)

$$\geq (1 - \beta)I_Q(\hat{x}; \hat{x}).$$

(51)

Now since the finite distribution $Q$ has a TV distance of at most $\beta$ to the countable distribution $R$, using Eqn (47), we have

$$\|\hat{x} - \hat{x}\| \leq \varepsilon + 2\gamma \sqrt{n} \text{ with probability } \geq 1 - \beta - \delta \text{ if } \hat{x} \sim Q.$$  

(52)

In order to apply Lemma B.3 on the distribution $Q$, we need its probability values to be multiples of some discretization level $\alpha$. Let $\alpha$ be a small enough quantization level for the probability values. We will specify the value of $\alpha$ later. We can now express the distribution $Q$ as a mixture of two distributions $Q'$, $Q''$. The distribution $Q'$ is obtained by flooring the probability values under $Q$ and renormalizing to make them sum to 1. The distribution $Q''$ is the mass not contained in $Q'$, normalized to sum to 1. Since each element in the support of $Q$ loses at most $\alpha$ mass, the total mass in $Q''$ prior to normalization is at most $\alpha N_\beta$, where $N_\beta$ is the cardinality of the support of $Q$. This gives

$$Q = (1 - c_2)Q' + c_2Q'', \ c_2 \leq \alpha N_\beta.$$  

From Eqn (52), we have $\|\hat{x} - \hat{x}\| \leq \varepsilon + 2\gamma \sqrt{n}$ with probability $\geq 1 - \beta - \delta$ when $\hat{x} \sim Q$. Since $Q'$ has a TV distance of at most $\alpha N_\beta$ to $Q$, if $\hat{x} \sim Q'$, we have

$$\|\hat{x} - \hat{x}\| \leq \varepsilon + 2\gamma \sqrt{n} \text{ with probability } \geq 1 - \beta - \delta - \alpha N_\beta \text{ if } \hat{x} \sim Q'.$$  

(53)

Let $I_Q(\hat{x}; \hat{x}), I_Q'(\hat{x}; \hat{x}), I_Q''(\hat{x}; \hat{x})$ denote the mutual information between $\hat{x}, \hat{x}$ when the marginal of $\hat{x}$ is $Q, Q', Q''$ respectively. Mutual information is a concave function of the marginal distribution of $\hat{x}$ for a fixed conditional distribution of $\hat{x}|\hat{x}$. Hence using Eqn (51), we have

$$I(\hat{x}; \hat{x}) \geq I_Q(\hat{x}; \hat{x}) \geq (1 - c_2)I_Q'(\hat{x}; \hat{x}) + c_2I_Q''(\hat{x}; \hat{x}),$$

(54)

$$\geq (1 - c_2)I_Q'(\hat{x}; \hat{x}),$$

(55)

$$\geq (1 - \alpha N_\beta)I_Q'(\hat{x}; \hat{x}).$$

(56)

Hence if $\hat{x} \sim Q'$, we have $I(\hat{x}; \hat{x}) \leq \frac{I(\hat{x}; \hat{x})}{(1 - \alpha N_\beta)(1 - \beta)}$. Applying Lemma B.3 on the distribution $Q'$, for any $\tau > 0$, we have

$$\tau(1 - \beta - \delta - \alpha N_\beta) \log \text{Cov}_{2\varepsilon + 4\gamma \sqrt{n}, \tau + \beta + \delta + \alpha N_\beta}(Q') \leq \frac{I(\hat{x}; \hat{x})}{(1 - \alpha N_\beta)(1 - \beta)} + 2.$$  

Now since $Q'$ has at least $1 - \alpha N_\beta$ of the mass under $Q$ and $Q$ has at least $1 - \delta$ of the mass under $\hat{R}$, the mass $\tau + \beta + \delta + \alpha N_\beta$ not covered under $Q'$ can be replaced with $\tau + \beta + 2\delta + 2\alpha N_\beta$ under $\hat{R}$. This gives

$$\tau(1 - \beta - \delta - \alpha N_\beta) \log \text{Cov}_{2\varepsilon + 4\gamma \sqrt{n}, \tau + \beta + 2\delta + 2\alpha N_\beta}(\hat{R}) \leq \frac{I(\hat{x}; \hat{x})}{(1 - \alpha N_\beta)(1 - \beta)} + 2.$$  

Now since we can cover the whole distribution of $\hat{R}$ by extending each element in the support of $\hat{R}$ by $\gamma$ in each coordinate, we can replace the radius $2\varepsilon + 4\gamma \sqrt{n}$ for $\hat{R}$ by $2\varepsilon + 6\gamma \sqrt{n}$ for $R$. This gives

$$\tau(1 - \beta - \delta - \alpha N_\beta) \log \text{Cov}_{2\varepsilon + 6\gamma \sqrt{n}, \tau + \beta + 2\delta + 2\alpha N_\beta}(R) \leq \frac{I(\hat{x}; \hat{x})}{(1 - \alpha N_\beta)(1 - \beta)} + 2.$$  

$$\tau(1 - \beta - \delta - \alpha N_\beta) \log \text{Cov}_{2\varepsilon + 6\gamma \sqrt{n}, \tau + \beta + 2\delta + 2\alpha N_\beta}(R) \leq \frac{I(\hat{x}; \hat{x})}{(1 - \alpha N_\beta)(1 - \beta)} + 2.$$  

$$\tau(1 - \beta - \delta - \alpha N_\beta) \log \text{Cov}_{2\varepsilon + 6\gamma \sqrt{n}, \tau + \beta + 2\delta + 2\alpha N_\beta}(R) \leq \frac{I(\hat{x}; \hat{x})}{(1 - \alpha N_\beta)(1 - \beta)} + 2.$$
For $\gamma = \frac{\epsilon}{6\sqrt{n}}, \beta = \min \{ \delta, 1 - \sqrt{0.99} \}, \alpha N_\beta = \min \{ \delta, 1 - \sqrt{0.99} \}$, we have
\[
0.99 \tau (1 - 2\delta) \log \text{Cov}_{3\tau, \tau + 3\delta}(R) \leq I(x; \hat{x}) + 1.98.
\]

\[\Box\]

B.4. Proof of Theorem 4.1

**Theorem 4.1.** Let $R$ be a distribution supported on a ball of radius $r$ in $\mathbb{R}^n$, and $x^* \sim R$. Let $y = Ax^* + \xi$, where $A$ is any matrix, and $\xi \sim \mathcal{N}(0, \frac{\sigma^2}{m} I_m)$. Assuming $\delta < 0.1$, if there exists a recovery scheme that uses $y$ and $A$ as inputs and guarantees
\[
\| \hat{x} - x^* \| \leq O(\eta),
\]
with probability $\geq 1 - \delta$, then we have
\[
m \geq \frac{0.15}{\log \left( 1 + \frac{m r^2 \| A \|_2^2}{\sigma^2} \right)} \left( \log \text{Cov}_{3\eta, 4\delta}(R) + \log 6\delta - O(1) \right).
\]

If $A$ is an i.i.d. Gaussian matrix where each element is drawn from $\mathcal{N}(0, 1/m)$, then the above bound can be improved to:
\[
m \geq \frac{0.15}{\log \left( 1 + \frac{r^2}{\sigma^2} \right)} \left( \log \text{Cov}_{3\eta, 4\delta}(R) + \log 6\delta - O(1) \right).
\]

**Proof.** Throughout the proof, we use the notation $N(R, \delta)$ to denote a minimal set of $3\eta$-radius balls that cover at least $1 - \delta$ mass under the distribution $R$.

Let $B$ be the ball in $N(R, 10\delta)$ with smallest marginal probability. If we set $S \leftarrow N(R, 10\delta) \setminus B$, then $S$ contains smaller than $1 - 10\delta$ mass under $R$.

Let $R = (1 - c)R' + cR''$, where the components $R'$ and $R''$ are probability distributions restricted to $S$ and its complement $S'$ respectively. By the construction of $S$, we have $c > 10\delta$. Note that since $R''$ contributes at least $10\delta$ to $R$, any algorithm that succeeds with probability $\geq 1 - \delta$ over $R$ must succeed with probability $\geq 0.9$ over $R''$.

Now consider $x \sim R''$. By Lemma 4.3 and Lemma 4.2, we have
\[
I(x; \hat{x}) \leq I(x; y | A),
\]
\[
\leq \frac{m}{2} \log \left( 1 + \frac{r^2}{\sigma^2} \right).
\]

Applying Lemma 4.4 on $R''$ with parameters $\tau = \delta = 0.1$, for the failure probability, we can conclude that
\[
0.99 \cdot 0.1 \cdot (1 - 0.2) \log |N(R'', 0.4)| \leq I(x; \hat{x}) + 1.98 \leq \frac{m}{2} \log \left( 1 + \frac{r^2}{\sigma^2} \right) + 1.98,
\]
\[
\Leftrightarrow m \geq \frac{0.15 \cdot \log |N(R'', 0.4)| - 3.96}{\log \left( 1 + \frac{r^2}{\sigma^2} \right)}.
\]

(57)

We now need to express the covering number of $R''$ in terms of the covering number of $R$.

Note that as $R''$ contains at least $10\delta$ mass under $R$, $N(R'', 0.4)$ contains at least $6\delta$ mass under $R$. Similarly, since $N(R, 10\delta)$ contains at least $1 - 10\delta$ mass under $R$, $N(R'', 0.4) \cup N(R, 10\delta)$ will contain at least $4\delta$ mass under $R$. Hence, we get
\[
|N(R'', 0.4)| + |N(R, 10\delta)| \geq |N(R, 4\delta)| \Leftrightarrow |N(R'', 0.4)| \geq |N(R, 4\delta)| - |N(R, 10\delta)|.
\]

(58)
Now we need to relate $N(R, 4\delta)$ with $N(R, 10\delta)$. This can be accomplished via a simple counting argument. Assume that the balls in $N(R, 4\delta)$ are ordered in decreasing order of their marginal probability, then the last $\frac{10\delta}{4\delta}$-fraction of balls in $N(R, 4\delta)$ must contain at most $10\delta$ mass. This implies that the first $\frac{1-10\delta}{1-4\delta}$-fraction of $N(R, 4\delta)$ must contain at least $1 - 10\delta$ mass. This gives:

$$\frac{1 - 10\delta}{1 - 4\delta} N(R, 4\delta) \geq N(R, 10\delta). \quad (59)$$

Combining Eqn (58), (59), we get

$$|N(R', 0.4)| \geq |N(R, 4\delta)| - \frac{1 - 10\delta}{1 - 4\delta} |N(R, 4\delta)|,$$

$$= \frac{6\delta}{1 - 4\delta} |N(R, 4\delta)|,$$

$$\geq 6\delta |N(R, 4\delta)|,$$

$$\Leftrightarrow \log |N(R', 0.4)| \geq \log |N(R, 4\delta)| + \log(6\delta).$$

Substituting in Eqn (57), we get

$$m \geq \frac{0.1584 (\log |N(R, 4\delta)| + \log(6\delta)) - 3.96}{\log \left( 1 + \frac{\delta^2}{\eta^2} \right)}.$$ 

Since $|N(R, 4\delta)| = \text{Cov}_{3n, 4\delta}(R)$ by definition, this completes the proof. \hfill \qed

### C. Experimental Setup

#### C.1. Datasets and Architecture

For the compressed sensing experiment in Fig 4a and the inpainting experiment in Figure 2 we used the 256x256 GLOW model (Kingma & Dhariwal, 2018) from the official repository. The test set for Fig 4a consists of the first 10 images used by (Asim et al., 2019) in their experiments.

For the compressed sensing experiment in Fig 1, 5a, 5b, we used the FFHQ NCSNv2 model (Song & Ermon, 2020) from the official repository. The test set for Fig 5a consists of the images 69000-69017 from the FFHQ dataset (this corresponds to the first 18 images in the last batch of FFHQ images).

In Fig 4a and Fig 5a, the measurements have noise satisfying $\sqrt{E \|\xi\|^2} = 16$ and $\sqrt{E \|\xi\|^2} = 4$ respectively.

#### C.2. Hyperparameter Selection

**CelebA experiments** For MAP, we used an Adam and Gradient Descent optimizer. Langevin dynamics only uses Gradient Descent. Each algorithm was run with learning rates varying over $[0.1, 0.01, 0.001, 5 \cdot 10^{-4}, 10^{-4}, 5 \cdot 10^{-5}, 10^{-5}, 5 \cdot 10^{-6}, 10^{-6}]$. For MAP and Modified-MAP, we also performed 2 random restarts for the initialization $z_0$.

The value of $\gamma$ in Eqn (6) was varied over $[0, 0.1, 0.01, 0.001]$ for Modified-MAP. MAP uses the theoretically defined value of $\frac{\sigma^2}{m}$.

For Langevin dynamics, we vary the value of $\sigma_1$ according to the schedule proposed by (Song & Ermon, 2019). We start with $\sigma_1 = 16.0$, and finish with $\sigma_{10} = 4.0$, such that $\sigma_i$ decreases geometrically for $i \in [10]$. For each value of $i$, we do 200 steps of noisy gradient descent, with the learning rate schedule proposed by (Song & Ermon, 2019).

In order to select the optimal hyperparameters for each $m$, we chose the hyperparams that give maximum likelihood for Langevin and MAP. For Modified-MAP, we selected the hyperparameters based on reconstruction error on a holdout set of 5 images.
**FFHQ experiments** The NCSNv2 model is designed for Langevin dynamics. It can be adapted to MAP by simply not adding noise at each gradient step. We tune the initial and final values of $\sigma$ used in (Song & Ermon, 2020), along with the initial learning rate.

Unfortunately, it is computationally difficult to obtain the likelihood associated with each reconstruction, since the NCSNv2 model only provides $\nabla \log p(x)$. Although one could, in theory, do numerical integration to find $p(x)$, we selected the optimal hyperparameters for each $m$ based on reconstruction error on a holdout set of 5 images.

For the Deep-Decoder, we used the over-parameterized network described in (Asim et al., 2019), and tuned the learning rate over $[0.4, 0.004, 0.0004]$, and selected the hyper-parameters that optimized the reconstruction error on a holdout set of 5 images.

**C.3. Computing Infrastructure**

Experiments were run on an NVIDIA Quadro P5000.