A STOCHASTIC ANALOG OF AUBRY-MATHER THEORY

DIOGO AGUIAR GOMES

Abstract. In this paper we discuss a stochastic analog of Aubry-Mather theory in which a deterministic control problem is replaced by a controlled diffusion. We prove the existence of a minimizing measure (Mather measure) and discuss its main properties using viscosity solutions of Hamilton-Jacobi equations. Then we prove regularity estimates on viscosity solutions of Hamilton-Jacobi equation using the Mather measure. Finally we apply these results to prove asymptotic estimates on the trajectories of controlled diffusions and study the convergence of Mather measures as the rate of diffusion vanishes.

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1. Introduction

The objective of this paper is to understand a stochastic analog of Aubry-Mather theory. The original problem [Mat91] consists in determining a probability measure $\mu$ in $\mathbb{T}^n \times \mathbb{R}^n$ ($\mathbb{T}^n$ is the $n$-dimensional torus) that minimizes the average action

$$\int L(x, v) d\mu \tag{1}$$
for a given Lagrangian $L$ with the constraint that $\mu$ is invariant under the flow generated by the Euler-Lagrange equations associated with $L$. This problem is equivalent [Mat01] to the relaxed problem of minimizing (1) with the constraint

$$\int vD_x\phi d\mu = 0$$

for any $\phi(x)$. In the case of controlled diffusions, we replace this constraint by

$$\int A^v\phi d\mu = 0,$$

for all $\phi$ smooth and periodic, in which $A^v$ is the infinitesimal generator of the controlled diffusion.

We proceed as follows: in section 2 we construct a relaxed minimization problem on a space of measures. Then, in section 3, we identify its dual by means of Fenchel-Rockafellar duality theorem [Roc66]. This dual problem turns out to involve Hamilton-Jacobi equations which are studied in section 4. We prove equivalence between the strong and weak problems (section 3), and characterize the minimizing measures using viscosity solutions of Hamilton-Jacobi equations (section 4). The we discuss several applications: regularity of Hamilton-Jacobi equations (section 4), logarithmic transform, connection with eigenvalue problems (section 5) asymptotics for controlled diffusions (section 8) and convergence of the stochastic Mather measure as the diffusion coefficient vanishes (section 10).

The original Mather problem, as well as its stochastic version are convex linear programming problem over a space of Radon measures. Related control problems have been studied by duality [VL78a], [VL78b], [LV80], [FV89], [FV88] and [Fle89], in which Fenchel-Rockafellar duality theorem [Roc66] is used to analyze optimal control problems. In this paper we apply similar techniques to understand Aubry-Mather theory and its stochastic analogs.

Several authors have studied the relation between viscosity solutions of Hamilton-Jacobi equations and Mather measures [Fat97a], [Fat97b], [Fat98a], [Fat98b], [E99], [EG99], [Gom00a] and [Gom00b]. The results by A. Fathi [Fat97a], [Fat97b], [Fat98a], [Fat98b], and W. E [E99] make clear the connection between viscosity solutions and Hamiltonian dynamics. The main idea is that if $u(x, P)$ is a viscosity solution of

$$H(P + D_xu, x) = \bar{H}$$

(2)
(here $H$ is the Legendre transform of $L$) then there exists an invariant set $\mathcal{I}$ contained on the graph 
\[ \{(x, P + D_x u(x, P))\}. \]
Furthermore, $\mathcal{I}$ is a subset of a Lipschitz graph, i.e., $D_x u(x, P)$ is a Lipschitz function on $\pi(\mathcal{I})$, where $\pi(x, p) = x$. If $\overline{H}$ is differentiable at $P$, then any solution $(x(t), p(t))$ of (1) with initial conditions on $\mathcal{I}$ satisfies
\[ \lim_{t \to \infty} \frac{|x(t) + D_P \overline{H} t|}{t} = 0. \]
In [Gom00a] and [Gom00b] this problem is studied with detail and more precise asymptotic results are presented. We also prove regularity results for viscosity solutions of (2) - in particular uniform continuity in $P$. In [EG99] Mather measures are used to prove regularity for solutions of Hamilton-Jacobi equations. The main results are $L^2$ type estimates in the difference quotients of $D_x u$. The objective of this paper is to generalize these results to the stochastic case.

2. Stochastic Mather measures

In this section we define a stochastic analog of the Mather’s minimal measure problem [Mat89], [Mat91], [Mn92], [Mn96]. To do so we consider an ergodic diffusion control problem and study an associated relaxed minimization problem on a space of measures. In the next section, we identify its dual by means of Fenchel-Rockafellar duality theorem and show that the dual problem is, in some sense, a Hamilton-Jacobi equation.

Consider a controlled Markov diffusion [FS93] in $\mathbb{R}^n$
\[ dx = v dt + \sigma dw, \]
where $v$ is a progressively measurable control, $w$ a $n$-dimensional Brownian motion and $\sigma \geq 0$ the diffusion rate ($\sigma = 0$ corresponds to the standard Aubry-Mather theory). The control objective is to minimize the long-time running cost
\[ \lim_{T \to +\infty} \frac{1}{T} E \int_0^T L(x, v) dt, \]
over all admissible control processes $v$, this is called the ergodic control problem (here $E$ denotes the expected value with respect to the underlying probability measure). We assume that the function $L(x, v)$ is smooth in both variables, $\mathbb{Z}^n$ periodic in $x$, coercive and strictly convex in $v$. Furthermore, since adding a constant to $L$ does not change the nature of the problem, we also assume $L \geq 0$. 
Let $\Omega = T^n \times \mathbb{R}^n$, where $T^n$ is the $n$-dimensional torus, identified, when convenient, with $[0,1]^n$ or $\mathbb{R}^n$ with a periodic structure (in geometric terms, $\mathbb{R}^n$ is the universal covering of $T^n$). A pair $(x,v) = z$ represents a generic point $z \in \Omega$, with $x \in T^n$ and $v \in \mathbb{R}^n$. Choose a function $\gamma \equiv \gamma(|v|) : \Omega \to [1, +\infty)$ satisfying
\[
\lim_{|v| \to +\infty} \frac{L(x,v)}{\gamma(v)} = +\infty \quad \lim_{|v| \to +\infty} \frac{|v|}{\gamma(v)} = 0.
\]

Let $\mathcal{M}$ be the set of weighted Radon measures on $\Omega$, i.e.,
\[
\mathcal{M} = \{\text{signed measures on } \Omega \text{ with } \int_\Omega \gamma d|\mu| < \infty\}.
\]

Note that $\mathcal{M}$ is the dual of the set $C^0_\gamma(\Omega)$ of continuous functions $\phi$ with
\[
\|\phi\|_\gamma = \sup_{\Omega} \frac{|\phi|}{\gamma} < \infty, \quad \lim_{|z| \to \infty} \frac{\phi(z)}{\gamma(v)} \to 0.
\]

For each bounded control strategy $v$ consider the measure $\mu_T$ defined by
\[
\int \phi d\mu_T = \frac{1}{T} E \int_0^T \phi(x(t),v(t))dt.
\]

As $T \to +\infty$ we may extract a weakly convergent subsequence $\mu_T \rightharpoonup \mu(v)$. Let
\[
\mathcal{M}_0 = \text{cl}\{\mu(v) : v \text{ bounded control strategy}\}.
\]

Define
\[
\mathcal{M}_1 = \{\mu \in \mathcal{M} : \int_\Omega d\mu = 1, \mu \geq 0\}.
\]

The stochastic analog of Mather’s problem consist in determine a measure $\mu$ that minimizes
\[
\inf_{\mu \in \mathcal{M}_0 \cap \mathcal{M}_1} \int_\Omega Ld\mu.
\]

For our purposes, however, it is convenient to consider a relaxed problem by replacing $\mathcal{M}_0$ by a slightly larger set $\mathcal{N}_0$ that we define next.

The infinitesimal generator corresponding to the controlled diffusion (4) is
\[
A^v \phi = \frac{\sigma^2}{2} \Delta \phi + v \cdot \nabla \phi.
\]

**Proposition 1.** Any measure $\mu(v)$ in $\mathcal{M}_0$ satisfies
\[
\int A^v \varphi d\mu = 0,
\]
for all $\varphi = \varphi(x)$, $\varphi$ periodic and $C^2$ (or $C^1$ if $\sigma = 0$).
Proof. Consider the measure defined by
\[ \int_{\Omega} \phi d\mu_T = \frac{1}{T} E \int_0^T \phi(x(t), v(t)) dt. \]
Assume \( \mu_T \rightharpoonup^* \mu \). We claim that
\[ \int_{\Omega} A^v \varphi d\mu = 0, \]
for \( \varphi(x) \in C^2 \) and periodic function of \( x \) only. To see this recall Dynkin’s formula:
\[ \varphi(x(T)) - \varphi(x(0)) = E \int_0^T A^v(x(t)) \varphi(x(t)) dt \]
for any \( x(t) \) and \( v(t) \) that solve \( (4) \). In the case \( \sigma = 0 \) this is just the fundamental theorem of calculus. Dividing by \( T \) and letting \( T \to \infty \) we obtain \( (6) \).

Let \( N_0 \) be the closure of the set of all measures that satisfy \( (3) \):
\[ N_0 = \text{cl}\{ \mu \in \mathcal{M} : \int_{\Omega} A^v \varphi d\mu = 0, \forall \varphi(x) \in C^2(T^n) \}. \]
In the case \( \sigma = 0 \), the set \( N_0 \) is the “measure theoretic” analog of the set of closed curves on \( T^n \). Indeed, if \( \theta : [0, 1] \to T^n \) is a piecewise smooth closed curve we can define a measure \( \mu_\theta \) by
\[ \int_{\Omega} f d\mu_\theta = \int_0^1 f(\theta(t), \dot{\theta}(t)) dt. \]
Clearly \( \mu_\theta \) is in \( N_0 \), and since \( N_0 \) is a linear space, it contains all linear combinations of measures of this form.

The additional problem that we will consider is
\[ \inf_{\mu \in N_0 \cap \mathcal{M}_1} \int L d\mu. \]
We will prove later on that
\[ \inf_{\mu \in N_0 \cap \mathcal{M}_1} \int L d\mu = \inf_{\mu \in \mathcal{M}_0 \cap \mathcal{M}_1} \int L d\mu. \]
This identity is a consequence that \( N_0 \) is the weak-* closure of \( \mathcal{M}_0 \). However, the proof of this depends on \( (8) \) holding for a sufficiently large class of \( L \) (see [FV89], [FV88], and [Fle89] for related proofs). Therefore we will prove \( (8) \) directly.

The last issue we discuss in this section is the existence of a measure that minimizes:
\[ \inf_{\mu \in \mathcal{M}_0 \cap \mathcal{M}_1} \int L d\mu. \]
This measure is the stochastic analog of the Aubry-Mather measure. A similar proof also shows that there exists a minimizing measure in $\mathcal{N}_0 \cap \mathcal{M}_1$. In the next section we prove that

$$\inf_{\mu \in \mathcal{M}_0 \cap \mathcal{M}_1} \int L d\mu = \inf_{\mu \in \mathcal{N}_0 \cap \mathcal{M}_1} \int L d\mu.$$ 

First we quote a compacity lemma:

**Lemma 1** (Mañé [Min96]). In $\mathcal{M}_0 \cap \mathcal{M}_1$ the set

$$\int L d\mu < c$$

is compact with respect to the weak-* topology in $(C_0^\infty)'$.

With the help of this lemma we prove the existence of a minimizing measure.

**Theorem 1.** There exists a measure $\mu \in \mathcal{M}_0 \cap \mathcal{M}_1$ such that:

$$\int L d\mu = \inf_{\mu \in \mathcal{M}_0 \cap \mathcal{M}_1} \int L d\mu.$$ 

**Proof.** Take any minimizing sequence $\mu_n$. Since $\int L d\mu < c$, the previous lemma shows that by extracting a subsequence, if necessary, $\mu_n^* \rightharpoonup \mu$. Thus, for any fixed $k$,

$$\lim_{n \to +\infty} \int \min(L, k) d\mu_n \to \int \min(L, k) d\mu.$$ 

Thus

$$\int \min(L, k) d\mu \leq \inf_{\mu \in \mathcal{M}_0 \cap \mathcal{M}_1} \int L d\mu,$$

for all $k$. But then by monotone convergence theorem

$$\int L d\mu \leq \inf_{\mu \in \mathcal{M}_0 \cap \mathcal{M}_1} \int L d\mu,$$

which proves the theorem.

A similar proof yields:

**Theorem 2.** There exists a measure $\mu \in \mathcal{N}_0 \cap \mathcal{M}_1$ such that:

$$\int L d\mu = \inf_{\mu \in \mathcal{N}_0 \cap \mathcal{M}_1} \int L d\mu.$$
3. IDENTIFICATION OF THE DUAL PROBLEM

In this section we identify the dual problem of

$$\min_{\mu \in \mathcal{N}_0 \cap \mathcal{M}_1} \int L d\mu.$$ 

The dual problem involves a Hamilton-Jacobi equation. Further analysis of this equation is carried out in the remaining sections and yields important information about the minimizing measure.

First we review some facts about convex duality. Let $E$ be a Banach space with dual $E'$. The pairing between $E$ and $E'$ is denoted by $(\cdot, \cdot)$. Suppose $h : E \to (-\infty, +\infty]$ is a convex, lower semicontinuous function. The Legendre-Fenchel transform $h^* : E' \to [-\infty, +\infty]$ of $h$ is defined by

$$h^*(y) = \sup_{x \in E} \left( - (x, y) - h(x) \right),$$

for $y \in E'$. Similarly, for concave, upper semicontinuous functions $g : E \to (-\infty, +\infty]$ let

$$g^*(y) = \inf_{x \in E} \left( - (x, y) - g(x) \right).$$

**Theorem 3** (Rockafellar [Roc66]). Let $E$ be a locally convex Hausdorff topological vector space over $\mathbb{R}$ with dual $E^*$. Suppose $h : E \to (-\infty, +\infty]$ is convex and lower semicontinuous, $g : E \to [-\infty, +\infty)$ is concave and upper semicontinuous. Then

$$\sup_x g(x) - f(x) = \inf_y f^*(y) - g^*(y),$$

provided that either $h$ or $g$ is continuous at some point where both functions are finite.

For $\phi \in C^0_\gamma(\Omega)$ define

$$h_1(\phi) = \sup_{(x,v) \in \Omega} \left( -\phi(x,v) - L(x,v) \right).$$

Let $\mathcal{C}$ be defined by

$$\mathcal{C} = \text{cl}\{\phi : \phi = A^v \varphi, \varphi(x) \in C^2(T^n)\},$$

here cl denotes the closure in $C^0_\gamma$ (if $\sigma = 0$ we may take $\varphi(x) \in C^1(T^n)$). If $\sigma = 0$, we may think of the elements in $\mathcal{C}$ as generalized closed differential forms; indeed if $\theta : [0, 1] \to T^n$ is a piecewise smooth closed curve and $\phi \in \mathcal{C}$ then

$$\int \phi d\mu_\theta = 0.$$
Define
\[
h_2(\phi) = \begin{cases} 
0 & \text{if } \phi \in C \\
-\infty & \text{otherwise.}
\end{cases}
\]

In this section we prove that
\[
\sup_{\phi \in C_0^0(\Omega)} h_2(\phi) - h_1(\phi)
\]
(10)
is the dual problem of (7).

First we compute the Legendre-Fenchel transforms of \( h_1 \) and \( h_2 \) in order to apply theorem 3 to (10).

**Proposition 2.** We have
\[
h_1^*(\mu) = \begin{cases} 
\int Ld\mu & \text{if } \mu \in M_1 \\
+\infty & \text{otherwise,}
\end{cases}
\]
and
\[
h_2^*(\mu) = \begin{cases} 
0 & \text{if } \mu \in N_0 \\
-\infty & \text{otherwise.}
\end{cases}
\]

**Proof.** Recall that
\[
h_1^*(\mu) = \sup_{\phi \in C_0^0(\Omega)} \left( -\int \phi d\mu - h_1(\phi) \right).
\]

We claim that if \( \mu \) is non-positive then \( h_1^*(\mu) = \infty \).

**Lemma 2.** If \( \mu \not\geq 0 \) then \( h_1^*(\mu) = +\infty \).

**Proof.** If \( \mu \not\geq 0 \) we can choose a sequence of positive functions \( \phi_n \in C_0^0(\Omega) \) such that
\[
\int -\phi_n d\mu \to +\infty.
\]

Thus, since \( L \geq 0 \),
\[
\sup_{\Omega} \phi_n - L \leq 0.
\]

Therefore if \( \mu \not\geq 0 \) then \( h_1^*(\mu) = +\infty \).

**Lemma 3.** If \( \mu \geq 0 \) then
\[
h_1^*(\mu) \geq \int Ld\mu + \sup_{\psi \in C_0^0(\Omega)} \left( \int \psi d\mu - \sup \psi \right).
\]
Proof. Let $L_n$ be a sequence of functions in $C^0_\gamma(\Omega)$ increasing pointwise to $L$. Any function $\phi$ in $C^0_\gamma(\Omega)$ can be written as $\phi = -L_n - \psi$, for some $\psi$ also in $C^0_\gamma(\Omega)$. Thus

$$
\sup_{\phi \in C^0_\gamma(\Omega)} \left(- \int \phi \, d\mu - h_1(\phi)\right) = \sup_{\psi \in C^0_\gamma(\Omega)} \left(\int L_n \, d\mu + \int \psi \, d\mu - \sup(L_n + \psi - L)\right).
$$

Since $L_n - L \leq 0$,

$$
\sup_{\Omega} L_n - L \leq 0,
$$

thus

$$
\sup_{\Omega} (L_n + \psi - L) \leq \sup_{\Omega} \psi.
$$

Thus

$$
\sup_{\phi \in C^0_\gamma(\Omega)} \left(- \int \phi \, d\mu - h_1(\phi)\right) \geq \sup_{\psi \in C^0_\gamma(\Omega)} \left(\int L_n \, d\mu + \int \psi \, d\mu - \sup(\psi)\right).
$$

By the monotone convergence theorem $\int L_n \, d\mu \to \int L \, d\mu$. Therefore

$$
\sup_{\phi \in C^0_\gamma(\Omega)} \left(- \int \phi \, d\mu - h_1(\phi)\right) \geq \int L \, d\mu + \sup_{\psi \in C^0_\gamma(\Omega)} \left(\int \psi \, d\mu - \sup(\psi)\right),
$$

as required.

If $\int L \, d\mu = +\infty$ then $h^*_1(\mu) = +\infty$. If $\int d\mu \neq 1$ then

$$
\sup_{\psi \in C^0_\gamma(\Omega)} \left(\int \psi \, d\mu - \sup(\psi)\right) \geq \sup_{\alpha \in \mathbb{R}} \alpha(\int d\mu - 1) = +\infty,
$$

by taking $\psi \equiv \alpha$, constant. Therefore $h^*_1(\mu) = +\infty$.

If $\int d\mu = 1$ we have, from the previous lemma

$$
h^*_1(\mu) \geq \int L \, d\mu,
$$

by taking $\psi \equiv 0$.

Also, for any $\phi$

$$
\int (-\phi - L) \, d\mu \leq \sup_{\Omega} (-\phi - L),
$$

if $\int d\mu = 1$. Hence

$$
\sup_{\phi \in C^0_\gamma(\Omega)} \left(- \int \phi \, d\mu - h_1(\phi)\right) \leq \int L \, d\mu.
$$
Thus
\[
h_1^*(\mu) = \begin{cases} 
\int L \, d\mu & \text{if } \mu \in \mathcal{M}_1 \\
+\infty & \text{otherwise.}
\end{cases}
\]

Now we will compute \(h_2^*\). First observe that if \(\mu \not\in \mathcal{N}_0\) then there exists \(\hat{\phi} \in \mathcal{C}\) such that
\[
\int \hat{\phi} \, d\mu \neq 0.
\]
and so
\[
\inf_{\phi \in \mathcal{C}} - \int \phi \, d\mu \leq \alpha \int \hat{\phi} \, d\mu = -\infty.
\]
If \(\mu \in \mathcal{N}_0\) then \(\int \phi \, d\mu = 0\), for all \(\phi \in \mathcal{C}\). Therefore
\[
h_2^*(\mu) = \inf_{\phi \in \mathcal{C}} - \int \phi \, d\mu = \begin{cases} 
0 & \text{if } \mu \in \mathcal{N}_0 \\
-\infty & \text{otherwise.}
\end{cases}
\]

The Fenchel-Rockafellar duality theorem states that
\[
\sup_{\phi \in C_0^\gamma(\Omega)} (h_2(\phi) - h_1(\phi)) = \inf_{\mu \in \mathcal{M}} (h_1^*(\mu) - h_2^*(\mu)),
\]
provided on the set \(h_2 > -\infty\), \(h_1\) is continuous. In the next lemma we prove that \(h_1\) is continuous, and therefore (11) holds.

**Lemma 4.** \(h_1\) is continuous.

**Proof.** Suppose \(\phi_n \to \phi\) in \(C_0^\gamma\). We must prove that \(h_1(\phi_n) \to h_1(\phi)\). Observe that \(\|\phi_n\|_{\gamma}\) and \(\|\phi\|_{\gamma}\) are bounded uniformly by some constant \(C\). The growth condition on \(L\) implies that there exists \(R > 0\) such that
\[
\sup_{\Omega} -\hat{\phi} - L = \sup_{\mathbb{T}^n \times B_R} -\hat{\phi} - L,
\]
for all \(\hat{\phi} \in C_0^\gamma(\Omega)\) with \(\|\hat{\phi}\|_{\gamma} < C\), in which \(B_R = \{v \in \mathbb{R}^n : |v| \leq R\}\) is the ball of radius \(R\) centered at the origin. On \(B_R\), \(\phi_n \to \phi\) uniformly and so
\[
\sup_{\Omega} -\phi_n - L \to \sup_{\Omega} -\phi - L.
\]

Denote by \(H^*\) the value
\[
H^* = -\sup_{\phi \in C_0^\gamma(\Omega)} (h_2(\phi) - h_1(\phi))
\]
Theorem 4. We have
\[ H^* = \inf \{ \lambda : \exists \varphi \in C^1(\mathbb{T}^n) : -\frac{\sigma^2}{2} \Delta \varphi + H(D_x \varphi, x) < \lambda \}, \]
in which
\[ H(p, x) = \sup_v -p \cdot v - L(x, v) \]
is the Legendre transform of \( L \).

Proof. Note that
\[ H^* = \inf_{\varphi \in C^1(\mathbb{T}^n)} \sup_{(x, v) \in \Omega} -\frac{\sigma^2}{2} \Delta \varphi - vD_x \varphi - L = \]
\[ = \inf_{\varphi \in C^1(\mathbb{T}^n)} \sup_{x \in \mathbb{T}^n} -\frac{\sigma^2}{2} \Delta \varphi + H(D_x \varphi, x). \]

\( \blacksquare \)

4. The Cell Problem

The last theorem in the previous section suggests that we study the equation
\[ -\frac{\sigma^2}{2} \Delta u + H(D_x u, x) = H. \]

In this section we prove that there exists a unique number \( \overline{H} \) for which (12) has a periodic viscosity solution. Using the results from [Kry87], we show that such solution is \( C^2 \). Then we prove that the solution is unique (up to additive constants). Finally we prove estimates on \( \overline{H} \) and \( u \) that do not depend on \( \sigma \).

Theorem 5. There exists a unique number \( \overline{H} \) for which the equation
\[ -\frac{\sigma^2}{2} \Delta u + H(D_x u, x) = \overline{H} \]
has a periodic viscosity solution. Furthermore the solution is \( C^2 \) and unique.

Proof. First we address the issue of the existence of a viscosity solution. To do so consider the infinite horizon discounted cost problem
\[ u^\alpha = \inf E \int_0^\infty e^{-\alpha t} L(x, v) dt \]
with \( dx = vdt + \sigma dw \). Then \( u^\alpha \) is a periodic viscosity solution of [FS93]
\[ -\frac{\sigma^2}{2} \Delta u^\alpha + H(D_x u^\alpha, x) + \alpha u^\alpha = 0. \]
Since $u^\alpha$ is periodic, uniformly Lipschitz in $\alpha$ [FS93] there exists a subsequence $u^\alpha$ and $u$ periodic for which

$$u^\alpha - \min u^\alpha \to u.$$ 

Since $0 \leq u^\alpha \leq \frac{C}{\alpha}$ we have $\alpha u^\alpha \to -\overline{H}$, for some $\overline{H}$ (extracting a further subsequence if necessary). Then $u$ is a periodic viscosity solution of

$$-\frac{\sigma^2}{2} \Delta u + H(D_x u, x) = \overline{H}.$$ 

This solution $u$ is actually $C^2$ by standard regularity results for non-linear uniformly elliptic equations [Kry87].

To prove uniqueness of $\overline{H}$, suppose, by contradiction, that $u_i$ and $\overline{H}_i$ ($i = 1, 2$) solve

$$-\Delta u_i + H(D_x u_i, x) = \overline{H}_i.$$ 

Suppose $u_1 - u_2$ has a local maximum at $x_0$. Then $D_x u_1 = D_x u_2$ and $\Delta u_1 \leq \Delta u_2$ at $x_0$. Thus we conclude $\overline{H}_1 \geq \overline{H}_2$. By symmetry $\overline{H}_1 = \overline{H}_2$.

To prove that the viscosity solution is unique suppose, by contradiction, that $u$ and $v$ are two distinct solutions (i.e., $u - v$ is non constant) of

$$-\Delta u + H(D_x u, x) = \overline{H}.$$ 

We may assume some small ball centered at the origin of radius $\gamma$ does not contain any maximizer of $x_0$ of $u(x) - v(x)$ (otherwise, for convenience, we may shift the coordinates). Fix $\epsilon, \lambda > 0$ and assume that

$$u(x) - v(x) - \epsilon e^{-\lambda|x|^2}$$ 

has a local maximum at $x_{\epsilon, \lambda}$. First observe that $x_{\epsilon, \lambda}$ is uniformly bounded and by passing to a subsequence, if necessary, we may assume $x_{\epsilon, \lambda} \to x_0$ as $\epsilon \to 0$. At $x_{\epsilon, \lambda}$ we have

$$D_x u = D_x v - 2\epsilon \lambda x e^{-\lambda|x|^2}$$ 

and

$$\Delta u \leq \Delta v - 2\epsilon \lambda e^{-\lambda|x|^2} + \epsilon \lambda^2 |x|^2 e^{-\lambda|x|^2}.$$ 

Since

$$0 = -\Delta (u - v) + H(D_x u, x) - H(D_x v, x),$$ 

we have

$$0 \geq -2\epsilon \lambda e^{-\lambda|x_{\epsilon, \lambda}|^2} + \epsilon \lambda^2 |x_{\epsilon, \lambda}|^2 e^{-\lambda|x_{\epsilon, \lambda}|^2} + O(\epsilon \lambda).$$ 

Observe that for $\epsilon$ small enough $|x_{\epsilon, \lambda}| > \frac{\gamma}{2}$. Dividing by $\epsilon e^{-\lambda|x_{\epsilon, \lambda}|^2}$ and letting $\epsilon \to 0$ we observe that

$$\frac{\gamma^2}{4} |\lambda|^2 - C\lambda \leq 0.$$
Therefore sending \( \lambda \to \infty \) yields a contradiction.

**Proposition 3.** \( \overline{H} \) can be estimated independently of \( \sigma \) by
\[
\inf_x H(0, x) \leq \overline{H} \leq \sup_x H(0, x).
\]

**Proof.** Suppose \( u \) has a minimum at \( x_0 \). Then \( -\frac{\sigma^2}{2} \Delta u(x_0) \leq 0 \) and \( D_x u(x_0) = 0 \). Thus
\[
\overline{H} = -\frac{\sigma^2}{2} \Delta u(x_0) + H(D_x u, x_0) \leq H(0, x_0) \leq \sup_x H(0, x).
\]

The other estimate is similar.

Finally we recall that standard estimates for controlled diffusions [FS93] also yield that \( u \) is semiconcave (with semiconcavity constant independent of \( \sigma \)) and Lipschitz (also independently of \( \sigma \)).

## 5. Equivalence between weak and strong problems

The next task is to prove that the value \( H^* \), computed by considering a infimum over measures in \( \mathcal{N}_0 \) is the same as
\[
\overline{H} = -\inf_{\mu \in \mathcal{M}_0} (h_1^*(\mu) - h_2^*(\mu)).
\]

A useful characterization of \( \overline{H} \) is:

**Theorem 6.** \( \overline{H} \) is the unique value for which the equation

\[
(13) \quad -\frac{\sigma^2}{2} \Delta u + H(D_x u, x) = \overline{H}
\]

has a periodic viscosity solution.

**Proof.** We know from theorem 3 that there is a single number \( \overline{H} \) for which \((13)\) admits a periodic viscosity solution \( u \). We can use that solution to build a Markov feedback strategy to control the diffusion:
\[
dx = -D_p H(D_x u, x) \, dt + \sigma dw.
\]

To this diffusion it corresponds a measure \( \mu \in \mathcal{M}_0 \cap \mathcal{M}_1 \) for which
\[
\int L d\mu = -\overline{H}.
\]

Thus
\[
-\overline{H} \geq \inf_{\mathcal{M}_0 \cap \mathcal{M}_1} \int L d\mu.
\]

Conversely, let \( u(x) \) be a solution of \((13)\) and assume that
\[
-\overline{H} < \inf_{\mathcal{M}_0 \cap \mathcal{M}_1} \int L d\mu.
\]
Then for any control strategy $u^*$ and corresponding process $x^*$ and all large enough $T$
\[
\frac{1}{T}E \int_0^T L(x^*, u^*) > -\overline{H} + \epsilon. 
\]
Thus
\[
u(x) = \inf_v E \int_0^T L(x, v) + \overline{H} dt + u(x^*(T)) > \epsilon T + \min_x u(x),
\]
which is a contradiction for $T$ sufficiently large since $u$ is bounded.

**Theorem 7.** $H^*$ is the unique value for which the equation
\[
-\frac{\sigma^2}{2} \Delta u + H(D_x u, x) = H^* 
\]
has a periodic viscosity solution.

**Proof.** First suppose $u$ is a periodic viscosity solution of
\[
-\frac{\sigma^2}{2} \Delta u + H(D_x u, x) = \overline{H}.
\]
Then we claim that there is no smooth function $\psi$ with
\[
-\frac{\sigma^2}{2} \Delta u + H(D_x \psi, x) < \overline{H}.
\]
Indeed, if this were false, we could choose a point $x_0$ at which $u - \psi$ has a local minimum. At this point we would have
\[
-\frac{\sigma^2}{2} \Delta \psi + H(D_x \psi, x_0) \geq \overline{H},
\]
by the viscosity property. Hence $H^* \geq \overline{H}$, by theorem 4.
To prove the other inequality consider a standard mollifier $\eta_\epsilon$ and define $u_\epsilon = \eta_\epsilon * u$, in which $*$ denotes convolution. Then
\[
-\frac{\sigma^2}{2} \Delta u_\epsilon + H(D_x u_\epsilon, x) \leq \overline{H} + h(\epsilon, x),
\]
where
\[
h(\epsilon, x) = \sup_{|p| \leq R} \sup_{|x-y| \leq \epsilon} |H(p, x) - H(p, y)|,
\]
where $R$ is a bound on the Lipschitz constant of $u$. Let
\[
H^\epsilon = \overline{H} + \sup_x h(\epsilon, x).
\]
$u_\epsilon$ satisfies
\[
-\frac{\sigma^2}{2} \Delta u_\epsilon + H(D_x u_\epsilon, x) \leq H^\epsilon.
\]
Thus $H^* \leq \lim_{\epsilon \to 0} H^\epsilon = \overline{H}$. Hence $H^* = \overline{H}$. ■
This proof holds even when \( \sigma = 0 \), for \( \sigma \neq 0 \) since \( u \) is \( C^2 \), the mollification step is unnecessary.

**Corollary 1.** We have

\[
\inf_{\mu \in \mathcal{N}_0} (h_1^*(\mu) - h_2^*(\mu)) = \inf_{\mu \in \mathcal{M}_0} (h_1^*(\mu) - h_2^*(\mu)).
\]

**Proof.** Our previous results show that we can construct a probability measure \( \mu \) on \( \mathcal{M}_0 \) such that

\[
\int L d\mu = -\overline{H} = \inf_{\mu \in \mathcal{N}_0} (h_1^*(\mu) - h_2^*(\mu)).
\]

Since \( \mathcal{M}_0 \subset \mathcal{N}_0 \) this completes the proof. \( \square \)

### 6. Properties of Stochastic Mather measures

In this section we study general properties of Stochastic Mather measures. First we prove that the stochastic Mather measure is supported in the graph \((x, -D_p H(D_x u, x))\) for any \( u \) viscosity solution of \((13)\). Then we show that the projection of this measure in the \( x \) axis has a density that satisfies an elliptic partial differential equation.

**Theorem 8.** Any stochastic Mather measure is supported in the graph \((x, -D_p H(D_x u, x))\) for any \( u \) viscosity solution of \((13)\).

**Proof.** Recall that for any \( v \) we have

\[
-\frac{\sigma^2}{2} \Delta u - v D_x u - L(x, v) \leq \overline{H}
\]

with strict inequality unless \( v = -D_p \overline{H}(D_x u, x) \). Note that

\[
\int -\frac{\sigma^2}{2} \Delta u - v D_x u d\mu = 0
\]

and

\[
\int L(x, v) d\mu = \overline{H},
\]

Thus \( \mu \) is supported on \((x, -D_p H(D_x u, x))\), otherwise we would have

\[
-\int L(x, v) d\mu < \overline{H}
\]

which would be a contradiction. \( \square \)

Since any stochastic Mather measure is supported on a graph, a natural question is whether its projection in the \( x \) coordinates has a density. The answer to this question is affirmative and we prove that this density is the solution of an elliptic partial differential equation.
Theorem 9. Let $\mu$ be a stochastic Mather measure. Let $\nu$ denote the projection of $\mu$ in the $x$ coordinates. Then $\nu = \theta(x)dx$ for some density $\theta \in W^{1,2}$. Furthermore $\theta$ is a weak solution of

$-
abla (\theta v(x)) + \frac{1}{2} \sigma^2 \Delta \theta = 0. \tag{14}$

for $v = -D_pH(D_xu, x)$.

Proof. Recall that for any smooth and periodic $\phi(x)$

$$\int \frac{\sigma^2}{2} \Delta \phi + v(x) D_x \phi \, d\nu = 0.$$

Let $\eta_\epsilon$ be a standard mollifier, $\phi_\epsilon = \eta_\epsilon * \eta_\epsilon * \nu$ and $\nu_\epsilon = \eta_\epsilon * \nu$. Note that $\nu_\epsilon$ is a bounded periodic $C^\infty$ functions (the bounds may depend on $\epsilon$). Then

$$0 = \int \frac{\sigma^2}{2} |\nabla \nu_\epsilon|^2 \, dx - \int v(x) D_x (\phi_\epsilon) \, d\nu.$$

Thus

$$\int \frac{\sigma^2}{2} |\nabla \nu_\epsilon|^2 \, dx = \int (D_x \nu_\epsilon) \eta_\epsilon * (\nu \nu) \, dx.$$

Note that

$$| \int (D_x \nu_\epsilon) \eta_\epsilon * (\nu \nu) \, dx | \leq | \int (D_x \nu_\epsilon) v(x) \nu_\epsilon \, dx | +$$

$$+ | \int (D_x \nu_\epsilon) (\eta_\epsilon * (\nu \nu) - v(x) \nu_\epsilon) \, dx |.$$

The first term on the right-hand side can be estimated by

$$\frac{\gamma}{2} \int |D_x \nu_\epsilon|^2 \, dx + \frac{C}{\gamma} \int |\nu_\epsilon|^2 \, dx,$$

for any small $\gamma > 0$. To estimate the second term observe that since $v$ is Lipschitz

$$|\eta_\epsilon * (\nu \nu) - v(x) \nu_\epsilon| \leq \int \eta_\epsilon(x-y) |v(x) - v(y)| \, d\nu(y) \leq C \epsilon \nu_\epsilon.$$

Thus

$$| \int D_x \nu_\epsilon (\eta_\epsilon * (\nu \nu) - v(x) \nu_\epsilon) \, dx | \leq \frac{\gamma}{2} \int |D_x \nu'^\epsilon|^2 \, dx + \frac{C \epsilon}{\gamma} \int |\nu_\epsilon|^2 \, dx,$$

therefore we conclude that

$$\int |D_x \nu_\epsilon|^2 \, dx \leq C \int |\nu_\epsilon|^2 \, dx,$$
uniformly in \( \epsilon \). Now observe that \( \nu_\epsilon \geq 0 \) and 
\[
\int \nu_\epsilon dx = 1.
\]
If \( \int |\nu_\epsilon|^2 dx \) were unbounded then we could normalize it defining \( \alpha_\epsilon = \gamma_\epsilon \nu_\epsilon \) with \( \int |\alpha_\epsilon|^2 dx = 1 \) and \( \gamma_\epsilon \to 0 \). Since \( \alpha_\epsilon \in W^{1,2} \) uniformly, through some subsequence it converges in \( L^2 \) to some \( \alpha \in L^2 \) with \( \int |\alpha|^2 dx = 1 \). However \( \alpha \geq 0 \) and \( \int \alpha = 0 \) which is a contradiction. Therefore we must have \( \nu_\epsilon \in W^{1,2} \) uniformly in \( \epsilon \). Thus through some subsequence \( \nu_\epsilon \rightharpoonup \theta \) for some \( \theta \in W^{1,2} \). Thus 
\[
d\nu = \theta(x) dx.
\]
Consequently, \( \theta \) is a weak solution of 
\[
-\nabla (\theta v(x)) + \frac{1}{2} \sigma^2 \Delta \theta = 0.
\]

Observe that equation (14) is a non-symmetric zero eigenvalue problem. It is well known [PW84] that 
\[
-\nabla (\theta v(x)) + \frac{1}{2} \sigma^2 \Delta \theta = \lambda \theta
\]
has a principal eigenvalue \( \lambda \) with positive eigenfunction \( \theta \). To see that \( \lambda = 0 \) just observe that
\[
0 = \int \frac{\sigma^2}{2} \Delta \theta - \nabla (v(x) \theta(x)) = \lambda \int \theta.
\]
Since \( \theta \) is non-negative we get \( \lambda = 0 \).

The previous theorem yields several important identities that we will use in the next section. First define \( \mathcal{H}(P) \) to be number for which
\[
-\frac{\sigma^2}{2} \Delta u + H(P + D_x u, x) = \mathcal{H}(P)
\]
has a periodic viscosity solution \( u(x, P) \) (note that \( u \) may not be continuous in \( P \)). The function \( \mathcal{H}(P) \) is convex in \( P \) and so twice differentiable for almost every \( P \).

**Proposition 4.** For any \( \phi(x) \) periodic
\[
-\int D_x \phi D_p H \theta dx + \frac{\sigma^2}{2} \int \Delta \phi \theta dx = 0.
\]
Furthermore
\[
\int D_x \theta dx = 0.
\]
Finally, for any \( P \) and \( P' \),
\[
(P' - P) \int D_p H \theta dx \leq \mathcal{H}(P') - \mathcal{H}(P),
\]
in particular if $\overline{H}$ is differentiable

$$\int D_p H \theta dx = D_p \overline{H}(P).$$

**Proof.** Observe that (16) follows from

$$0 = \int \phi \left( \nabla (\theta D_p H) + \frac{1}{2} \sigma^2 \Delta \theta \right) dx$$

by integration by parts.

Let $\eta_\epsilon$ be a standard mollifier and let $u_\epsilon = \eta_\epsilon * u$. Then

$$-\frac{\sigma^2}{2} \Delta u_\epsilon + \eta_\epsilon * H(D_x u, x) = \overline{H}.$$ 

Differentiate the previous identity with respect to $x_i$:

$$-\frac{\sigma^2}{2} \Delta D_{x_i} u_\epsilon + \eta_\epsilon * (H_{p_j} D_{x_j} u_\epsilon + H_{x_i}) = 0.$$ 

Since $H_{p_j}$ is Lipschitz in $x$ we have

$$\eta_\epsilon * (H_{p_j} D_{x_j} u) = H_{p_j} D_{x_j} u_\epsilon + O(\epsilon).$$

Note also that

$$\int \left( -\frac{\sigma^2}{2} \Delta D_{x_i} u_\epsilon + H_{p_j} D_{x_j} u_\epsilon \right) \theta dx = 0$$

since $D_{x_i} u_\epsilon$ is smooth and periodic. Thus

$$\int \eta_\epsilon * H_{x_i} \theta dx \to 0$$

as $\epsilon \to 0$. Since $\eta_\epsilon * H_{x_i} \to H_{x_i}$ almost everywhere we conclude

$$\int H_{x_i} \theta dx = 0,$$

which proves (17).

To prove the last part of the proposition, note that

$$H(P' + D_x u(x, P'), x) - H(P + D_x u(x, P), x) \geq D_p H(P + D_x u(x, P), x) [P' + D_x u(x, P') - P - D_x u(x, P)].$$

Let $w = u(x, P') - u(x, P)$. Note that

$$\int \left( -\frac{\sigma^2}{2} \Delta w + D_p H(P + D_x u(x, P), x) D_x w \right) \theta dx = 0.$$ 

Thus

$$(P' - P) \int D_p H(P + D_x u(x, P), x) \leq \overline{H}(P') - \overline{H}(P),$$

as required. $\blacksquare$
7. Regularity estimates

In this section we prove $L^2$-type regularity estimates for the solution of (15). These estimates are expressed using the invariant measure. A major advantage is that it is possible to prove $L^2(\theta)$ estimates for the difference quotient $|D_x u(x + y) - D_x u(x)|$ that do not depend on $\sigma$ explicitly whereas pointwise or $L^2$ estimates with respect to Lebesgue measure depend on $\sigma$. Therefore our estimates extend up to the case $\sigma = 0$, for a careful study of this case consult [EG99], [Gom00a], and [Gom00b].

**Theorem 10.** Suppose $u$ solves (15) and $y \in \mathbb{R}^n$. Then

\begin{equation}
\int |D_x u(x + y) - D_x u(x)|^2 \theta dx \leq C|y|^2.
\end{equation}

Furthermore, if $\overline{H}(P)$ is twice differentiable at $P$ then

\begin{equation}
\int |D_x u(x, P') - D_x u(x, P)|^2 \theta dx \leq C|P - P'|^2,
\end{equation}

for $|P - P'|$ sufficiently small.

**Proof.** Note that

$-\frac{\sigma^2}{2} [\Delta u(x + y) - \Delta u(x)] + H(D_x u(x + y), x + y) - H(D_x u(x), x) = 0.$

Since $H$ is convex,

$H(D_x u(x + y), x + y) - H(D_x u, x) \geq \gamma |D_x w|^2 + D_p H(D_x u(x), x) D_x w + D_x H(D_x u(x), x) y + O(y^2),$

with $w = u(x + y) - u(x)$. Integrating with respect to $\theta dx$ to obtain

$\gamma \int |D_x w|^2 \theta dx \leq C|y|^2,$

since

$\int \left[ D_p H(D_x u(x), x) D_x w - \frac{\sigma^2}{2} \Delta w \right] \theta dx = 0,$

and

$\int D_x H(D_x u(x), x) \theta dx = 0.$

Similarly, let $w = u(x, P') - u(x, P)$ and assume $\overline{H}(P)$ is twice differentiable at $P$. Then

$D_p \overline{H}(P)(P' - P) + C|P - P'|^2 \geq -\frac{\sigma^2}{2} \Delta w + H(P' + D_x u(x, P'), x) - H(P + D_x u(x, P), x)$.
Note that
\[ H(P' + D_xu(x, P'), x) - H(P + D_xu(x, P), x) \geq D_pH(P + D_xu(x, P), x)(P' - P + D_xw) + \gamma |P' - P + D_xw|^2. \]
Thus
\[ \gamma \int |P' + D_xu(x, P') - P - D_xu(x, P)|^2 \theta dx \leq C|P - P'|^2, \]
for \(|P - P'|| sufficiently small.

**Theorem 11.** Suppose \( \overline{H} \) is strictly convex at a neighborhood of a point \( P \). Then
\[ (P' - P) \int D_pH(P + D_xu(x, P), x)\theta dx = (P' - P)D_p\overline{H}, \]
and
\[ \int -\frac{\sigma^2}{2} \Delta w + D_pH(P + D_xu(x, P), x)D_xw = 0. \]
From (21) we have (20). \( \blacksquare \)

In the next theorem we prove that if \( \overline{H} \) is strictly convex in a neighborhood of a point \( P \) then the map \((x, P) \rightarrow P + D_xu(x, P)\) is non-degenerate. In the non-random case this result is extremely important since it proves the invariant sets \((x, P + D_xu)\) change with \( P \), see [Gom00a] for a detailed discussion.

**Theorem 11.** Suppose \( \overline{H} \) is strictly convex at a neighborhood of a point \( P \). Then
\[ \int |P + D_xu(x, P) - P' - D_xu(x, P')|^2 \theta dx \geq C|P - P'|^2, \]
for \(|P - P'|| sufficiently small.

**Proof.** Let \( w = u(x, P') - u(x, P) \) and assume \( \overline{H}(P) \) is strictly convex in a neighborhood of \( P \). Then
\[ D_p\overline{H}(P)(P' - P) + C|P - P'|^2 \leq \]
\[ \leq -\frac{\sigma^2}{2} \Delta w + H(P' + D_xu(x, P'), x) - H(P + D_xu(x, P), x) \]
Note that
\[ H(P' + D_xu(x, P'), x) - H(P + D_xu(x, P), x) \leq \]
\[ \leq D_pH(P + D_xu(x, P), x)(P' - P + D_xw) + \Gamma |P' - P + D_xw|^2. \]
Thus
\[ \Gamma \int |P' + D_xu(x, P') - P - D_xu(x, P)|^2 \theta dx \geq C|P - P'|^2 \]
since
\[(P' - P) \int D_pH(P + D_xu(x, P), x)\theta dx = (P' - P)D_p\mathbb{H}\]
and
\[\int -\frac{\sigma^2}{2}\Delta \theta + D_pH(P + D_xu(x, P), x)D_xw = 0.\]

In the case \(\sigma = 0\) it is possible to prove \(L^\infty\)-estimates on \(D^2_{xx} u\) on the support of \(\theta\) \cite{EG99}. However, this is not the case for \(\sigma > 0\), at least with estimates independent on \(\sigma\). Indeed, if \(D^2_{xx} u\) were uniformly bounded in \(\sigma\) then \(u_\sigma\) would converge uniformly, through some subsequence as \(\sigma \to 0\), to a function \(u\), viscosity solution of
\[H(D_xu, x) = \mathbb{H}.\]
But then \(u\) would be both semiconvex and semiconcave and we know that, in general, \(u\) is only semiconcave. However, some regularity exists, as was remarked in section \(\S\), namely one-sided bounds on \(D^2_{xx} u\) (semiconcavity) that do not depend on \(\sigma\).

8. Explicit Formulas and Examples

In this section we discuss several formulas for both \(\mathbb{H}\) and invariant measures. The next proposition shows that given the solution \(u(x, P)\) it is possible to compute the density \(\theta\) (under smoothness assumptions), not of the invariant measure but of a time-reversed version.

**Proposition 5.** Assume \(u(x, P)\) is a smooth solution of (15). Then
\[\theta = \det(I + D^2_{xp}u)\]
is a solution of a time-reversed version of (14):
\[\sigma^2 \Delta \theta + \nabla(\theta^2v(x)) = 0.\]

**Proof.** Let \(v(x, P) = Px + u(x, P)\). Then
\[\sigma^2 \Delta v + H(D_xv, x) = \mathbb{H}(P).\]
The claim is that \(\theta = \det D^2_{xp}v\) solves
\[-\frac{\sigma^2}{2}\Delta \theta + \nabla(\theta D_pH(D_xv, x)) = 0.\]
Differentiate (24) with respect to \(P\), to get
\[-\frac{\sigma^2}{2}\Delta v_p + D_pH D^2_{xp}v = D_p\mathbb{H}\]
Note that $D_p H D_{xP}^2 v = (D_{xP}^2 v)^T D_p H$ and multiply the previous identity by the cofactor matrix $\text{cof } D_{xP} v$

$$-\frac{\sigma^2}{2} \text{cof } D_{xP} v \Delta v_p + \det D_{xP} v D_p H = \text{cof } D_{xP} v D_p \overline{H}.$$ 

Observe that $\text{cof } D_{xP} v$ is divergence free \cite{Eva98} and so

$$\text{cof } D_{xP} v \Delta v_p = \nabla (\det D_{xP} v).$$

Therefore

$$-\frac{\sigma^2}{2} \nabla (\det D_{xP} v) + \det D_{xP} v D_p H = \text{cof } D_{xP} v D_p \overline{H}.$$ 

By applying $\nabla$ to the previous identity we have

$$-\frac{\sigma^2}{2} \Delta \theta + \nabla (\theta D_p H) = 0. \quad \Box$$

Now we turn our attention to the special case

$$H(p, x) = \frac{p^2}{2} + V(x),$$

with $V$ periodic. For this special Hamiltonian we will present an alternative representation formula for $\overline{H}(P)$ as well as exhibit a (non-periodic) invariant measure. This will follow some ideas of \cite{Hol77}.

Suppose $u$ is a periodic viscosity solution of

$$-\frac{\sigma^2}{2} \Delta u + H(P + D_x u, x) = \overline{H}(P).$$

Define

$$\phi = e^{-\frac{P_x + u}{\sigma^2}}.$$ 

Then $\phi$ solves

$$\frac{\sigma^4}{2} \Delta \phi + V(x) \phi = \overline{H}(P) \phi.$$ 

Thus $\overline{H}$ is an eigenvalue of the operator $\frac{\sigma^4}{2} \Delta \phi + V(x) \phi$. Consider the related operator

$$L \psi = e^{-\frac{P_x}{\sigma^2}} \frac{\sigma^4}{2} \Delta (e^{-\frac{P_x}{\sigma^2}} \psi) + V(x) \psi = \frac{\sigma^4}{2} \Delta \psi - \sigma^2 P D_x \psi + (V(x) + \frac{|P|^2}{2}) \psi$$

Then $\overline{H}$ is also an eigenvalue of $L$ with periodic boundary conditions.

**Proposition 6.** $\overline{H}$ is the principal eigenvalue of $L$. 
Proof. The operator $L$ has a principal eigenvalue $\lambda$ with positive and periodic eigenfunction $\varphi$. Let $u = -\log \varphi$. Then $u$ is smooth, periodic and satisfies the Hamilton-Jacobi equation

$$-\frac{\sigma^2}{2} \Delta u + H(P + D_x u, x) = \lambda.$$ 

By uniqueness of $\mathbb{H}$ we have $\lambda = \mathbb{H}$. \[\blacksquare\]

Finally we exhibit an invariant measure for this system. Although this is not a probability measure (unless $P = 0$).

**Proposition 7.** Let $\theta = e^{-\frac{P + D_x u}{\sigma^2}}$. Then $\theta$ is an invariant measure.

**Proof.** It suffices to check that

$$\frac{\sigma^2}{2} \Delta \theta + \nabla((P + D_x u)\theta) = 0.$$ 

\[\blacksquare\]

9. **Asymptotics**

In this section we study the asymptotic behavior of the controlled process $x(t)$. First we will do some formal calculations motivated by the case $\sigma = 0$ [EG93], [Gom00a], [Gom00b]. Define

$$X = x + D_P u.$$ 

Then

$$dX = dx + D^2_{Px} u dx + \frac{\sigma^2}{2} D^3_{Pxx} u dt$$ 

thus, since $dx = -D_P H dt + \sigma dw$,

$$dX = \left(-D_P H(I + D^2_{Px} u) + \frac{\sigma^2}{2} D^3_{Pxx} u\right) dt + \sigma \left(1 + D^2_{Px} u\right) dw.$$ 

Note that $-D_P H(I + D^2_{Px} u) + \frac{\sigma^2}{2} D^3_{Pxx} u = -D_P \mathbb{H}$ and so

$$E(X(t) - X(0)) = -D_P \mathbb{H} t.$$ 

**Theorem 12.** Suppose $\mathbb{H}$ is differentiable at $P$. Then

$$\lim_{t \to \infty} E \frac{x(t)}{t} = -D_P \mathbb{H}.$$ 

**Proof.** Let $u$ be a viscosity solution of (15). Let $v^*$ be an optimal control such that

$$u(x, P) = E \int_0^t L(x, v^*) + P v^* + \mathbb{H}(P) + u(x(t), P).$$
Then
\[ u(x, P') \leq E \int_0^t L(x, v^*) + P'v^* + \overline{H}(P') + u(x(t), P'). \]
Subtracting these two equations
\[ C \leq E \int_0^t (P' - P)v^* + \overline{H}(P') - \overline{H}(P). \]
Thus
\[ E \int_0^t v^* = -D_P \overline{H} t + O(1) \]
since \( dx = v^* dt + \sigma dw \) we have
\[ E \int_0^t v^* = E \int_0^t dx = Ex(t). \]

\[ \blacksquare \]

10. CONVERGENCE AS \( \sigma \to 0 \)

In this last section we prove that stochastic Mather measures converge to a Mather measure as the diffusion rate \( \sigma \) vanishes.

Let \( \overline{H}_\sigma \) be the unique number for which
\[ -\frac{\sigma^2}{2} \Delta u_\sigma + H(D_x u_\sigma, x) = \overline{H}_\sigma \tag{25} \]
has a periodic viscosity solution \( u_\sigma \). The bounds on \( \overline{H}_\sigma \) obtained in section 4 imply that through some subsequence \( \overline{H}_\sigma \to \overline{H} \) as \( \sigma \to 0 \), for some number \( \overline{H} \). Since \( u_\sigma \) is uniformly Lipschitz in \( \sigma \), through some subsequence \( u_\sigma \to u \) uniformly. Standard stability results on viscosity solutions imply that \( u \) is a viscosity solution of
\[ H(D_x u, x) = \overline{H}. \]

Let \( \mu_\sigma \) be a stochastic Mather measure associated with (25). Since the support of \( \mu_\sigma \) is bounded independently of \( \sigma \) we can extract a weakly convergence subsequence \( \mu_\sigma \rightharpoonup \mu \) and \( \int d\mu = 1 \). Note that
\[ -\overline{H} = \int Ld\mu_\sigma \to \int Ld\mu = -\overline{H} \]
Furthermore, for any smooth function \( \phi(x) \)
\[ 0 = \int \frac{\sigma^2}{2} \Delta \phi + v D_x \phi d\mu_\sigma \to \int v D_x \phi d\mu. \]
Thus \( \mu \) satisfies
\[ \int Ld\mu = -\overline{H} \]
with the constraints $\int d\mu = 1$, and $\int vD_x\phi d\mu = 0$. Thus $\mu$ is a Mather measure.

References

[E99] Weinan E. Aubry-Mather theory and periodic solutions of the forced Burgers equation. *Comm. Pure Appl. Math.*, 52(7):811–828, 1999.

[EG99] L. C. Evans and D. Gomes. Effective Hamiltonians and averaging for Hamiltonian dynamics I. *Preprint*, 1999.

[Eva98] Lawrence C. Evans. *Partial differential equations*. American Mathematical Society, Providence, RI, 1998.

[Fat97a] Albert Fathi. Solutions KAM faibles conjuguées et barrières de Peierls. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(6):649–652, 1997.

[Fat97b] Albert Fathi. Théorème KAM faible et théorie de Mather sur les systèmes lagrangiens. *C. R. Acad. Sci. Paris Sér. I Math.*, 324(9):1043–1046, 1997.

[Fat98a] Albert Fathi. Orbite hétéroclines et ensemble de Peierls. *C. R. Acad. Sci. Paris Sér. I Math.*, 326:1213–1216, 1998.

[Fat98b] Albert Fathi. Sur la convergence du semi-groupe de Lax-Oleinik. *C. R. Acad. Sci. Paris Sér. I Math.*, 327:267–270, 1998.

[Fle89] Wendell H. Fleming. Generalized solutions and convex duality in optimal control. In *Partial differential equations and the calculus of variations, Vol. I*, pages 461–471. Birkhäuser Boston, Boston, MA, 1989.

[FS93] Wendell H. Fleming and H. Mete Soner. *Controlled Markov processes and viscosity solutions*. Springer-Verlag, New York, 1993.

[FV88] Wendell H. Fleming and Domokos Vermes. Generalized solutions in the optimal control of diffusions. In *Stochastic differential systems, stochastic control theory and applications (Minneapolis, Minn., 1986)*, pages 119–127. Springer, New York, 1988.

[FV89] Wendell H. Fleming and Domokos Vermes. Convex duality approach to the optimal control of diffusions. *SIAM J. Control Optim.*, 27(5):1136–1155, 1989.

[Gom00a] D. Gomes. Viscosity solutions of Hamilton-Jacobi equations, and asymptotics for Hamiltonian systems. *Preprint*, 2000.

[Gom00b] D. A. Gomes. *Hamilton-Jacobi Equations, Viscosity Solutions and Asymptotics of Hamiltonian Systems*. Ph.D. Thesis, Univ. of California at Berkeley, 2000.

[Hol77] Charles J. Holland. A new energy characterization of the smallest eigenvalue of the Schrödinger equation. *Comm. Pure Appl. Math.*, 30(6):755–765, 1977.

[Kry87] N. V. Krylov. *Nonlinear elliptic and parabolic equations of the second order*. D. Reidel Publishing Co., Dordrecht, 1987. Translated from the Russian by P. L. Buzytsky [P. L. Buzytskyĭ].

[LV80] R. M. Lewis and R. B. Vinter. Relaxation of optimal control problems to equivalent convex programs. *J. Math. Anal. Appl.*, 74(2):475–493, 1980.

[Mat89] John N. Mather. Minimal action measures for positive-definite Lagrangian systems. In *IXth International Congress on Mathematical Physics (Swansea, 1988)*, pages 466–468. Hilger, Bristol, 1989.
[Mat91] John N. Mather. Action minimizing invariant measures for positive definite Lagrangian systems. *Math. Z.*, 207(2):169–207, 1991.

[Mat01] John N. Mather. Personal communication. 2001.

[Mn92] Ricardo Mañé. On the minimizing measures of Lagrangian dynamical systems. *Nonlinearity*, 5(3):623–638, 1992.

[Mn96] Ricardo Mañé. Generic properties and problems of minimizing measures of Lagrangian systems. *Nonlinearity*, 9(2):273–310, 1996.

[PW84] Murray H. Protter and Hans F. Weinberger. *Maximum principles in differential equations*. Springer-Verlag, New York, 1984. Corrected reprint of the 1967 original.

[Roc66] R. T. Rockafellar. Extension of Fenchel’s duality theorem for convex functions. *Duke Math. J.*, 33:81–89, 1966.

[VL78a] Richard B. Vinter and Richard M. Lewis. The equivalence of strong and weak formulations for certain problems in optimal control. *SIAM J. Control Optim.*, 16(4):546–570, 1978.

[VL78b] Richard B. Vinter and Richard M. Lewis. A necessary and sufficient condition for optimality of dynamic programming type, making no a priori assumptions on the controls. *SIAM J. Control Optim.*, 16(4):571–583, 1978.