Cardy-Verlinde Formula and asymptotically flat rotating Charged black holes

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Abstract

The Cardy-Verlinde formula is generalized to the asymptotically flat rotating charged black holes in the Einstein-Maxwell theory and low-energy effective field theory describing string by using some typical spacetimes, such as the Kerr-Newman, Einstein-Maxwell-dilaton-axion, Kaluza-Klein, and Sen black holes. For the Kerr-Newman black hole, the definition of the Casimir energy takes the same form as that of the Kerr-Newman-AdS$_4$ and Kerr-Newman-dS$_4$ black holes, while the Cardy-Verlinde formula possesses different from since the Casimir energy does not appear in the extensive energy. The Einstein-Maxwell-dilaton-axion, Kaluza-Klein, and Sen black holes have special property: The definition of the Casimir energy for these black holes is similar to that of the Kerr-Newman black hole, but the Cardy-Verlinde formula takes the same form as that of the Kerr black hole. Furthermore, we also study the entropy bounds for the systems in which the matters surrounds these black holes. We find that the bound for the case of the Kerr-Newman black hole is related to its charge, and the bound for the cases of the EMDA, Kaluza-Klein, and Sen black holes can be expressed as a unified form. A surprising result is that the entropy bounds for the Kaluza-Klein and Sen black holes are tighter than the Bekenstein one.

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I. INTRODUCTION

In a recent paper Verlinde made an interesting proposal that the \((1+1)\)-dimensional Cardy formula can be generalized to the case in arbitrary dimensions. He argued that the entropy of the conformal field theory (CFT) in a spacetime \(ds^2 = -dt^2 + R^2d\Omega_n^2\), can be explicitly expressed as

\[
S = \frac{2\pi R}{\sqrt{a_1b_1}} \sqrt{E_c(2E - E_c)}, \tag{1.1}
\]

where \(E\) is the total energy, \(E_c\) is the Casimir energy, and \(a_1\) and \(b_1\) are two positive coefficients which are independent of \(R\) and \(S\). For strong coupled CFT’s with AdS dual, the value of \(\sqrt{a_1b_1}\) is fixed to \(n\) exactly. The expression (1.1) is now referred to as the Cardy-Verlinde formula.

The study of validity of the Cardy-Verlinde formula for every typical spacetimes has attracted much attention recently since it has not been proved for all CFT’s exactly yet: For the AdS Schwarzschild black holes in various dimensions the formula (1.1) holds exactly; The Cardy-Verlinde formula is valid for the AdS Reissner-Nordström black hole if we subtract the electric potential energy from the Casimir and total energies, but it is invalid for the AdS black holes in higher derivative gravity; The formula holds for the Taub-Bolt-AdS spacetimes at high temperature limit; Klemm, Petkou, and Siopsis showed that the Cardy-Verlinde formula holds for the Kerr-AdS\(_n\) black holes; Motivating by the observational evidence that our universe has positive cosmological constant and an interesting proposal for dS/CFT correspondence, Danielsson, Cai, Medved, and Ogushi et al attempt to generalize the formula to the case of the de Sitter (dS) black holes; We verified the Cardy-Verlinde formula by using the Kerr-Newman-AdS\(_4\) and Kerr-Newman-dS\(_4\) black holes and found that it holds for these black holes if we modify the definitions of the Casimir energy and the extensive energy.

An interesting question, whether the Cardy-Verlinde formula holds for a more general setting, e. g. for the black holes that are asymptotically flat rather than approaching AdS/dS space, was proposed by Klemm, Petkou, and Zanon. They argued that for the asymptotically flat Schwarzschild and Kerr black holes in any dimensions the Cardy-Verlinde formula can be expressed as

\[
S = \frac{2\pi r_+}{n} \sqrt{E_c \cdot 2E}, \tag{1.2}
\]

where \(r_+\) is the radius of the event horizon. However, at the moment the question whether or not the Cardy-Verlinde formula can be generalized to the asymptotically flat rotating charged black holes obtained from the Einstein-Maxwell theory and low-energy effective field theory describing string still remains open. The main aim of this paper is to study the question by using some typical spacetimes, such as the Kerr-Newman, Einstein-Maxwell-dilaton-axion (EMDA), Kaluza-Klein, and Sen black holes. The another purpose of the paper is to study the entropy bounds for the systems in which the matters surround these black holes.

The paper is organized as follows. In Sec. II, we study the Cardy-Verlinde formula and entropy bound for the Kerr-Newman black hole. In Sec. III, we consider the same question...
for the case of the EMDA black hole. In Sec. IV, we investigate the case of the Kaluza-Klein black hole. In Sec. V, we discuss the case of the Sen black hole. We present some discussions and conclusions in the last section.

II. THE KERR-NEWMAN BLACK HOLE

The general charged stationary axisymmetric black hole obtained from the Einstein-Maxwell gravitational theory is the Kerr-Newman black hole. Mazur [29] showed that the Kerr-Newman solution is the only stationary, axisymmetric electrovac solution of the Einstein-Maxwell equations. In the Boyer-Lindquist coordinates the metric of the black hole reads [30]

$$ds^2 = -\left(\frac{\Delta - a^2 \sin^2 \theta}{\rho^2}\right) dt^2 - \frac{2a \sin^2 \theta (r^2 + a^2 - \Delta)}{\rho^2} dt d\phi$$

$$+ \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left[\frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\rho^2}\right] \sin^2 \theta d\phi^2,$$

(2.1)

with

$$\Delta = r^2 + a^2 - 2Mr + Q^2;$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta,$$

(2.2)

where the parameters $M$, $a$, and $Q$ represent the mass, angular momentum, and electric charge of the black hole, respectively. The event horizon is located at $r_+$, which is determined by the largest root of the $\Delta, \Delta = 0$. The mass $M$ can be expressed as

$$M = \frac{r_+^2 + a^2 + Q^2}{2r_+}.$$  

(2.3)

Euclideanizing the metric (2.1) and identifying $\tau \sim \tau + \beta$ and $\phi \sim \phi + i\beta \Omega_H$, we can get the inverse Hawking temperature and the angular velocity of the event horizon

$$\beta = \frac{4\pi r_+(r_+^2 + a^2)}{r_+^2 - a^2 - Q^2},$$

$$\Omega_H = \frac{a}{r_+^2 + a^2}.$$  

(2.4)

The angular momentum, electric potential and Bekenstein-Hawking entropy are

$$J = Ma,$$

$$\Phi_Q = \frac{Qr_+}{r_+^2 + a^2},$$

$$S = \pi(r_+^2 + a^2).$$  

(2.5)

An electric potential when the rotation parameter $a$ goes to be zero is given by
\[ \Phi_{Q_0} = \lim_{a \to 0} \Phi_Q = \frac{Q}{r_+}. \]  

(2.6)

For the rotating black hole we define the Casimir energy as

\[ E_c = n \left( E + pV - TS - J \Omega_H - \frac{Q \Phi_Q}{2} - \frac{Q \Phi_{Q_0}}{2} \right), \]  

(2.7)

where the pressure is defined as \( p = -\left( \frac{\partial E}{\partial V} \right)_{S,J,Q} \) and \( n = 2 \). The definition of the Casimir energy (2.7) takes the same form as that of the Kerr-Newman-AdS\(_4\) and Kerr-Newman-dS\(_4\) black holes [12]. Substituting Eqs. (2.4), (2.5), and (2.6) into Eq. (2.7), we get

\[ E_c = \frac{(r_+^2 + a^2)}{r_+}. \]  

(2.8)

We also define the extensive energy as

\[ E_{ext} = 2 \left( E - \frac{Q \Phi_{Q_0}}{2} \right) = 2(E - E_Q), \]  

(2.9)

where \( E_Q = \frac{Q \Phi_{Q_0}}{2} \). In this case the extensive energy is

\[ E_{ext} = \frac{(r_+^2 + a^2)}{r_+}. \]  

(2.10)

We should note that the definition of the extensive energy (2.9) is different from that of the Kerr-Newman-AdS\(_4\) and Kerr-Newman-dS\(_4\) black holes [12] by a term \( E_c \). With the Casimir energy (2.8) and the extensive energy (2.10), we find that the entropy of the CFT on the horizon can be written as

\[ S = \frac{2\pi r_+}{n} \sqrt{E_c \left[ 2(E - E_Q) \right] = \pi (r_+^2 + a^2)}, \]  

(2.11)

which is equal to the Bekenstein-Hawking entropy of the Kerr-Newman black hole. The result in this section reduces to that of the Kerr black hole obtained in Ref. [13] when the charge \( Q \) becomes zero.

We now study the entropy bound for a system in which some matters surround the Kerr-Newman black hole. We can define the entropy \( S_B \) which relates the energy \( E - E_Q \) by

\[ S_B = 2\pi (E - E_Q) R, \]  

(2.12)

where \( R \) is the radius of the sphere circumscribing the system \( (R \geq r_+) \). Substituting Eq. (2.3) into (2.12) we get

\[ S_B = \pi (r_+^2 + a^2) \frac{R}{r_+} = \frac{S R}{r_+}, \]  

(2.13)

which shows that

\[ S \leq S_B = 2\pi R \left( E - \frac{Q^2}{2r_+} \right). \]  

(2.14)

It reduces to the Bekenstein bound when the charge \( Q \) becomes zero.
III. THE STATIONARY AXISYMMETRIC EMDA BLACK HOLE

The general four-dimensional low-energy action obtained from string theory is

\[ I = \frac{1}{16\pi} \int d^4x \sqrt{-g}(R - 2g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} e^{4\Phi} g^{\mu\nu} \nabla_\mu K_a \nabla_\nu K_a - e^{-2\Phi} g^{\mu\lambda} g^{\nu\rho} F_{\mu\nu} F_{\lambda\rho} - K_a F_{\mu\nu} \tilde{F}^{\mu\nu}), \]

(3.1)

with \( \tilde{F}_{\mu\nu} = -\frac{1}{2} \sqrt{-g} \varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} \), where \( R \) is the scalar Riemann curvature, \( \Phi \) is the massless dilaton field, \( F_{\mu\nu} \) is the electromagnetic antisymmetric tensor field, and \( K_a \) is the axion field dual to the three-index antisymmetric tensor field \( H = -\exp(4\Phi) \ast dK_a/4 \).

The stationary axisymmetric EMDA black hole solution get from the action (we take the solution \( b = 0 \) in Eq.(14) in Ref. [31]; the reason we use this solution is that the solution \( b \neq 0 \) cannot be interpreted properly as a black hole) is given [31] by

\[ ds^2 = -\frac{\Delta - a^2 \sin^2 \theta}{\rho^2} dt^2 - \frac{2a \sin^2 \theta}{\rho^2} [r^2 + a^2 - 2Dr) - \rho^2] dtd\phi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2 - 2Dr)^2 - \Delta a^2 \sin^2 \theta] d\phi^2, \]

(3.2)

with

\[ \Delta = r^2 - 2mr + a^2, \quad \rho^2 = r^2 - 2Dr + a^2 \cos^2 \theta, \]

(3.3)

and

\[ e^{2\Phi} = \frac{W}{\Delta} = \frac{\omega}{\Delta} (r^2 + a^2 \cos^2 \theta), \quad \omega = e^{2\Phi_0}, \]

\[ K_a = K_0 + \frac{2aD \cos \theta}{W}, \quad A_t = \frac{1}{\Delta} (Qr - ga \cos \theta), \quad A_r = A_\theta = 0, \]

\[ A_\phi = \frac{1}{a\Delta} (-Qra^2 \sin^2 \theta + g(r^2 + a^2)a \cos \theta). \]

(3.4)

The mass \( M \), angular momentum \( J \), electric charge \( Q \), and magnetic charge \( P \) of the black hole are given by

\[ M = m - D, \quad J = a(m - D), \]

\[ Q = \sqrt{2\omega D(D - m)}, \quad P = g. \]

(3.5)

The inverse Hawking temperature and the angular velocity are

\[ \beta = \frac{4\pi r_+(r_+^2 + a^2 - 2Dr_+)}{(r_+^2 - a^2)}, \quad \Omega_H = \frac{a}{r_+^2 + a^2 - 2Dr_+}. \]

(3.6)

The electric potential and the Bekenstein-Hawking entropy of the black hole are
\[ \Phi_Q = \frac{Q r_+}{\omega(r_+^2 + a^2 - 2Dr_+)}, \]
\[ S = \pi(r_+^2 + a^2 - 2Dr_+). \]  
\tag{3.7} 

We now define the Casimir energy as
\[ E_c = n \left( E + PV - TS - J \Omega_H - \frac{Q\Phi_Q}{2} \right), \]
where \( n = 2 \) and the pressure is defined as \( p = -\left( \frac{\partial E}{\partial V} \right)_{S,J,Q} \). Substituting quantities of the EMDA black hole into Eq. \( (3.8) \), we arrive at
\[ E_c = \frac{(r_+^2 + a^2 - 2Dr_+)}{r_+} = 2E. \]  
\tag{3.9} 

Thus the entropy of the CFT can be cast as
\[ S = \frac{2\pi r_+}{n} \sqrt{E_c \cdot 2E} 
= \pi(r_+^2 + a^2 - 2Dr_+). \]  
\tag{3.10} 

It is of interest to note that the definitions of the Casimir energy \( (3.8) \) of the EMDA black hole has similar form as that of the Kerr-Newman black hole, but the Cardy-Verlinde formula \( (3.10) \) takes the same form as that of the Kerr black hole \( (1.2) \). We should not be surprised at this fact because the charged EMDA black hole possesses several different properties as compared to the charged Kerr-Newman black hole: (a) The horizons of the Kerr-Newman black hole are given by \( r_{\pm} = M \pm \sqrt{M^2 - a^2 - Q^2} \), whereas in the case of the EMDA black hole we have \( r_{\pm} = m \pm \sqrt{m^2 - a^2} \) which takes the same form as that of the Kerr black hole; (b) The heat capacity of the Kerr-Newman black hole is \( C_{QJ} = \frac{MS^3T}{\pi J^2 + \pi^2 Q^2/4 - S^3T^2} \), while for the EMDA black hole we get \( C_{QJ} = \frac{MS^3T}{\pi J^2 + \pi^2 Q^2/4} \) \( [32] \) which also has the same form as that of the Kerr black hole.

We now study the entropy bound for a system in which some matters surround the black hole. We can define an entropy \( S_B \) which relates the energy \( E \) by
\[ S_B = 2\pi ER = \pi(r_+^2 + a^2 - 2Dr_+) \frac{R}{r_+} = S \frac{R}{r_+}. \]  
\tag{3.11} 

Thus we have
\[ S \leq S_B = 2\pi RE. \]  
\tag{3.12} 

Which takes the same form as that of the Bekenstein bound \( (1.2) \).

IV. THE STATIONARY Kaluza-Klein BLACK HOLE

The four dimensional low-energy action obtained from string theory is \( [33] \)
\[ I = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[ \mathcal{R} - 2(\nabla \phi)^2 - e^{-2\alpha \phi} F^2 \right], \quad (4.1) \]

where \( \phi \) is the dilaton scalar field, \( F_{ab} \) is the Maxwell field, and \( \alpha \) is a free parameter which governs the strength of the coupling of the dilaton to the Maxwell field. The stationary Kaluza-Klein black hole is described by a solution of the motion equations obtained from Eq. (4.1) with \( \alpha = \sqrt{3} \), which takes the form \[ \begin{aligned}
\[ ds^2 = -\frac{1 - Z}{B} dt^2 - \frac{2aZ \sin^2 \theta}{B \sqrt{1 - v^2}} dt d\phi + \frac{B \rho^2}{\Delta r} dr^2 + B \rho^2 d\theta^2 \\
+ \left[ B(r^2 + a^2) + a^2 \sin^2 \theta \frac{Z}{B} \right] \sin^2 \theta d\phi^2, \quad (4.2) \end{aligned} \]

with \[ Z = \frac{2mr}{\rho^2}, \quad B = \left( 1 + \frac{v^2 Z}{1 - v^2} \right)^{1/2}, \]
\[ \rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta r = r^2 + a^2 - 2mr, \quad (4.3) \]

where \( a \) and \( v \) are the rotation parameter and the velocity of the boost. The dilaton scalar field, and the vector potential of the stationary Kaluza-Klein black hole are \[ \begin{aligned}
\Phi = -\frac{\sqrt{3}}{2} \ln B, \quad A_t = \frac{v}{2(1 - v^2) B^2}, \quad A_\phi = -\frac{\alpha v \sin^2 \theta \frac{Z}{B}}{2\sqrt{1 - v^2} B^2}. \quad (4.4) \end{aligned} \]

We get the inverse Hawking temperature and the angular velocity \[ \begin{aligned}
\beta = \frac{4\pi r_+(r^2 + a^2)}{(r^2 - a^2)\sqrt{1 - v^2}}, \quad \Omega_H = \frac{a \sqrt{1 - v^2}}{r^2 + a^2}. \quad (4.5) \end{aligned} \]

The ADM mass, angular momentum \( \mathcal{J} \), charge \( Q \), and corresponding potential of the black hole are given by \[ \begin{aligned}
M &= \left[ 1 + \frac{v^2}{2(1 - v^2)} \right] m, \\
\mathcal{J} &= \frac{ma}{\sqrt{1 - v^2}}, \\
Q &= \frac{mv}{1 - v^2}, \\
\Phi_Q &= \frac{Q r_+(1 - v^2)}{r^2 + a^2}, \quad (4.6) \end{aligned} \]

while the Bekenstein-Hawking entropy of the black hole is \[ S = \frac{\pi (r^2 + a^2)}{\sqrt{1 - v^2}}. \quad (4.7) \]
As the EMDA black hole we define the Casimir energy as
\[ E_c = n \left( E + pV - TS - J\Omega_H - \frac{Q\Phi Q}{2} \right), \]  
where \( n = 2 \) and the pressure is defined as \( p = -\left( \frac{\partial E}{\partial V} \right)_{S,J,Q} \). Substituting Eqs. (4.5), and (4.7) into Eq. (4.8), we get
\[ E_c = \frac{(r_+^2 + a^2)(2 - v^2)}{2r_+(1 - v^2)}. \]  
The extensive energy is defined as
\[ E_{ext} = 2E = \frac{(r_+^2 + a^2)(2 - v^2)}{2r_+(1 - v^2)}. \]  
Then, the entropy of the CFT can be cast into
\[ S = \frac{2\pi r_+}{\sqrt{a_1 b_1}} \sqrt{E_c \cdot 2E} = \frac{\pi(r_+^2 + a^2)}{\sqrt{1 - v^2}}, \]  
where
\[ a_1 = b_1 = \frac{n}{\sqrt{\frac{1 - v^2}{2}}}. \]  
The definition of the Casimir energy (4.8) and the Cardy-Verlinde formula (4.11) possess the same form as that of the EMDA black hole. The entropy of the CFT (4.11) agrees exactly with the Bekenstein-Hawking entropy (4.7).

To study the entropy bound for a system in which some matters surround the Kaluza-Klein black hole, we can define an entropy \( S_B \) as
\[ S_B = 2\pi ER \left( \frac{n}{\sqrt{a_1 b_1}} \right) = \frac{\pi(r_+^2 + a^2)}{\sqrt{1 - v^2}} \frac{R}{r_+} \]  
Noting that \( R \geq r_+ \) we obtain
\[ S \leq S_B = 2\pi RE \left( \frac{n}{\sqrt{a_1 b_1}} \right). \]  
The entropy bound is different from the Bekenstein bound by a factor \( \frac{n}{\sqrt{a_1 b_1}} \). We know from the Fig. (4.1) that
\[ \frac{n}{\sqrt{a_1 b_1}} \leq 1. \]  
The relation shows that the entropy bound (4.14) is tighter than the Bekenstein bound \( S \leq 2\pi ER \) [34].
FIGURES

\begin{align*}
\text{FIG. 1. In the figure } f &= \frac{2}{\sqrt{a_1b_1}} = \frac{\sqrt{1-v^2}}{1-v^2/2}.
\end{align*}

V. THE SEN BLACK HOLE IN HETEROTIC STRING

The effective action of a toroidally compactified heterotic string in \( D \)-dimensions takes the form \[35,36\]
\begin{align*}
I &= \frac{1}{2\kappa} \int d^D x \sqrt{-g} \left[ R - \frac{1}{(D-2)} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{8} g^{\mu\nu} \text{Tr}(\partial_\mu M L \partial_\nu M L) \
- \frac{1}{12} e^{-2\alpha \phi} g^{\mu\nu} g^{\rho\sigma} H_{\mu\rho\nu\sigma} - \frac{1}{4} e^{-\alpha \phi} g^{\mu\nu} g^{\rho\sigma} F_{\mu\nu}^i (LML)_{ij} F_{\rho\sigma}^j \right], \quad (5.1)
\end{align*}
where \( g \equiv g_{\mu\nu}, R \) is the Ricci scalar of \( g_{\mu\nu}, \Phi \) is the dilaton field, \( F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i \) are the \( U(1)^{26-2D} \) gauge field strengths, \( H_{\mu\rho\nu\sigma} = (\partial_\mu B_{\rho\sigma} + 2 A_\mu^i L_{ij} F_{\rho\sigma}^j) + \) cyclic permutations of \( \mu, \nu, \rho \), and \( M \) is the \( O(10-D, 26-D) \) symmetric matrix and \( L \) is
\begin{align*}
L &= \begin{pmatrix}
0 & I_{10-D} & 0 \\
I_{10-D} & 0 & 0 \\
0 & 0 & I_{26-D}
\end{pmatrix}.
\end{align*}
(5.2)

The metric of the Sen black hole is \[37\]
\begin{align*}
ds^2 &= \Delta \left\{ -\Delta^{-1}(r^2 + a^2 \cos^2 \theta - 2mr) dt^2 + (r^2 + a^2 - 2mr)^{-1} dr^2 + d\theta^2 \\
&\quad + \Delta^{-1} \sin^2 \theta [\Delta + a^2 \sin^2 \theta (r^2 + a^2 \cos^2 \theta + 2mr \cosh \alpha \cosh \beta)] d\phi^2 \\
&\quad - 2\Delta^{-1} mra \sin^2 \theta (\cosh \alpha + \cosh \beta) dt d\phi \right\},
\end{align*}
(5.3)
with
\begin{align*}
\Delta &= (r^2 + a^2 \cos^2 \theta)^2 + 2mr(r^2 + a^2 \cos^2 \theta)(\cosh \alpha \cosh \beta - 1) + m^2 r^2 (\cosh \alpha - \cosh \beta)^2,
\end{align*}
(5.4)
where $\alpha$ and $\beta$ are two boosts, and $a$ and $m$ represent the rotational and mass parameters, respectively. The black hole will be called Sen black hole for short. The two horizons of the black hole are shown by

$$r_+ = m \pm \sqrt{m^2 - a^2}.$$  \hfill (5.5)

The surface gravity of the black hole is given by,

$$\kappa = \frac{\sqrt{m^2 - a^2}}{m}(\cosh \alpha + \cosh \beta)(m + \sqrt{m^2 - a^2}).$$ \hfill (5.6)

$\kappa/2\pi$ can be interpreted as the Hawking temperature of the black hole. The angular velocity $\Omega$ at the horizon is

$$\Omega_H = \frac{a}{m(\cosh \alpha + \cosh \beta)(m + \sqrt{m^2 - a^2})}.$$ \hfill (5.7)

The mass $M$, angular momentum $J$, electric charge $Q^a$ and magnetic dipole moment $\mu^a$ given by

$$M = \frac{1}{2}m(1 + \cosh \alpha \cosh \beta),$$
$$J = \frac{1}{2}ma(\cosh \alpha + \cosh \beta),$$
$$Q^a = \frac{m}{\sqrt{2}}\sinh \alpha \cosh \beta n^a \quad \text{for} \quad 1 \leq a \leq 22$$
$$= \frac{m}{\sqrt{2}}\sinh \beta \cosh \alpha p^{(a-22)} \quad \text{for} \quad 23 \leq a \leq 28,$$
$$\mu^a = \frac{1}{\sqrt{2}}ma \sinh \alpha n^a \quad \text{for} \quad 1 \leq a \leq 22$$
$$= \frac{1}{\sqrt{2}}ma \sinh \beta p^{(a-22)} \quad \text{for} \quad 23 \leq a \leq 28.$$ \hfill (5.8)

The Bekenstein-Hawking entropy of the black hole is

$$S = \pi m(\cosh \alpha + \cosh \beta)(m + \sqrt{m^2 - a^2}).$$ \hfill (5.9)

By setting the boost parameter $\beta = 0$, the metric (5.3) reduces to the black hole in Ref. [38] and Eq. (5.8) becomes [38]

$$M = \frac{1}{2}m(1 + \cosh \alpha),$$
$$J = \frac{1}{2}ma(1 + \cosh \alpha),$$
$$Q = \frac{m}{\sqrt{2}}\sinh \alpha,$$
$$\mu = \frac{ma}{\sqrt{2}} \sinh \alpha.$$ \hfill (5.10)

See Eq.(16) and Eq.(17) of Ref. [38]. This is a special case of the Sen black hole, so the discussion in this section will be valid for the black hole.

The Casimir energy can also be defined as
\[ E_c = n \left( E + pV - TS - J\Omega_H - \frac{Q\Phi_Q}{2} \right), \]  
\hspace{1cm} (5.11)

where \( n = 2 \) and the pressure is defined as \( p = -\frac{\partial E}{\partial V}_{s,J,Q} \). Substituting quantities of the black hole into Eq. (5.11), we arrive at

\[ E_c = \frac{(r_+^2 + a^2)(1 + \cosh \alpha \cosh \beta)}{2r_+} = 2E. \]  
\hspace{1cm} (5.12)

Therefore, the entropy of the CFT can be expressed as

\[ S = \frac{2\pi r_+}{\sqrt{a_1 b_1}} \sqrt{E_c \cdot 2E} \]
\[ = \frac{\pi}{2} (r_+^2 + a^2)(\cosh \alpha + \cosh \beta), \]  
\hspace{1cm} (5.13)

where

\[ a_1 = b_1 = \frac{n(1 + \cosh \alpha \cosh \beta)}{\cosh \alpha + \cosh \beta}. \]  
\hspace{1cm} (5.14)

The entropy of the CFT (5.13) is equal to the Bekenstein-Hawking entropy (5.9). The Casimir energy (5.11) and the Cardy-Verlinde formula (5.13) for the Sen black hole also take the same form as that of the EMDA and Kaluza-Klein black holes. The result can be used for the black hole in Ref. [38], i.e., for the case \( \beta = 0 \).

For a system in which matter surrounds the Sen black hole, we can define the entropy \( S_B \) which relates the energy \( E \) by

\[ S_B = 2\pi ER \left( \frac{n}{\sqrt{a_1 b_1}} \right) \]
\[ = \frac{\pi}{2} (r_+^2 + a^2)(\cosh \alpha + \cosh \beta) \frac{R}{r_+} \]
\[ = S \frac{R}{r_+}. \]  
\hspace{1cm} (5.15)

From which we obtain the entropy bound

\[ S \leq S_B = 2\pi RE \left( \frac{n}{\sqrt{a_1 b_1}} \right). \]  
\hspace{1cm} (5.16)

The Fig. (2) shows that

\[ \frac{n}{\sqrt{a_1 b_1}} \leq 1. \]  
\hspace{1cm} (5.17)
Therefore, as the case of the Kaluza-Klein black hole, we also note that the entropy bound (5.16) for the Sen black hole is tighter than the Bekenstein one.

\[ f = \frac{2}{\sqrt{a_1 b_1}} = \frac{\cosh \alpha + \cosh \beta}{1 + \cosh \alpha \cosh \beta}, \quad a \text{ and } b \text{ represent } \alpha \text{ and } \beta, \text{ respectively.} \]

**FIG. 2.** In the figure \( f = \frac{2}{\sqrt{a_1 b_1}} = \frac{\cosh \alpha + \cosh \beta}{1 + \cosh \alpha \cosh \beta}, \quad a \text{ and } b \text{ represent } \alpha \text{ and } \beta, \text{ respectively.} \)

VI. CONCLUSION AND DISCUSSION

The Cardy-Verlinde formula, which relates the entropy of the conformal field theory to its Casimir energy \( E_c \), the energy \( E \) (or the extensive energy), and the radius \( R \) of the unit sphere \( S^n \), is generalized to the asymptotically flat rotating charged black holes in the Einstein-Maxwell theory and low-energy effective field theory describing string, such as the Kerr-Newman, EMDA, Kaluza-Klein, and Sen black holes.

For the Kerr-Newman black hole, we find that the definition of the Casimir energy (2.7) takes the same form as that of the Kerr-Newman-AdS\(^4\) and Kerr-Newman-dS\(^4\) black holes [12], while the extensive energy (2.9) is different from that of the Kerr-Newman-AdS\(^4\) and Kerr-Newman-dS\(^4\) black holes [12]. The Cardy-Verlinde formula for the black hole can be generalized as

\[ S = \frac{2\pi r}{n} \sqrt{E_c \left[ 2(E - E_Q) \right]} \]

The charged EMDA, Kaluza-Klein, and Sen black holes possess some special properties as compared to the Kerr-Newman black hole. The definition of the Casimir energies (3.8), (4.8), and (5.11) have the similar form as that of the Kerr-Newman black hole, but the Cardy-Verlinde formulae (3.10), (4.11), and (5.13) take the same form as that of the Kerr black hole [13], which can be written as follows a united form

\[ S = \frac{2\pi r}{\sqrt{a_1 b_1}} \sqrt{E_c \cdot 2E}, \]

where \( \sqrt{a_1 b_1} = n \) for the EMDA black hole, \( \sqrt{a_1 b_1} = \frac{n(1 - v^2/2)}{\sqrt{1 - v^2}} \) for the Kaluza-Klein black hole, and \( \sqrt{a_1 b_1} = \frac{n(1 + \cosh \alpha \cosh \beta)}{\cosh \alpha + \cosh \beta} \) for the Sen black hole. The result is also valid for the black hole in Ref. [38] since it can be considered as a special case of the Sen black hole.

Another important conclusion obtained for these asymptotically flat rotating charged black holes is that the entropies of the CFTs agree precisely with their Bekenstein-Hawking
entropies.

For stationary charged black holes in the Einstein-Maxwell theory and the low-energy effective field theory describing string, by using the covariant phase technique [39–42] and Carlip’s boundary [43,44] conditions, we constructed [45,46] the standard Virasoro subalgebra with corresponding central charge at the Killing horizons and proved that the density of states determined by conformal fields theory methods yields the statistical entropy which agrees with the Bekenstein-Hawking entropy. Comparing the result with that obtained in this paper, it is surprising to note that different ways arrive at the same conclusion!

Furthermore, we also study the entropy bounds for the systems in which the matters surrounds these black holes. For the case of the Kerr-Newman black hole, the bound $S \leq 2\pi R \left( E - \frac{Q^2}{2r_+} \right)$ is tightened by the electric charge. The bound reduces to the Bekenstein bound when the charge $Q$ becomes zero. For the case of the charged EMDA, Kaluza-Klein, and Sen black holes, the entropy bound can be expressed as a unified form, $S \leq 2\pi E R \frac{n}{\sqrt{a_1 b_1}}$. It is of interest to note that the entropy bound for the Kaluza-Klein and Sen black holes is tighter than the Bekenstein one since $\frac{n}{\sqrt{a_1 b_1}} \leq 1$ for these cases.

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