Malliavin method for optimal investment in financial markets with memory

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Abstract: We consider a financial market with memory effects in which wealth processes are driven by mean-field stochastic Volterra equations. In this financial market, the classical dynamic programming method can not be used to study the optimal investment problem, because the solution of mean-field stochastic Volterra equation is not a Markov process. In this paper, a new method through Malliavin calculus introduced in [1], can be used to obtain the optimal investment in a Volterra type financial market. We show a sufficient and necessary condition for the optimal investment in this financial market with memory by mean-field stochastic maximum principle.

Keywords: Mean-field, Backward stochastic Volterra equations, Malliavin derivative, Maximum principle

MSC: 60H20, 60H30, 35K60

1 Models and introduction

Suppose we have a financial market with memory, in which there are two following investment opportunities:

(i) A risk-free asset with unit price $S_0(t) = 1$; $t \geq 1$.
(ii) A risky asset, in which investments have memory effects (or long term effects) and an infinite number of rational agents are in competition, in the following sense:

An investor can decide at time $t \in [0, T]$ what amount $u(t)$ of the current wealth $X(t)$ to invest in the risky asset. The wealth process $X(t) = X^u(t)$ at time $t$ is described by the following controlled mean-field stochastic Volterra equation:

$$
X(t) = x + \int_0^t b(t, s, X(s), E[X(s)], u(s), \omega) ds + \int_0^t \sigma(t, s, X(s), E[X(s)], u(s), \omega) dB(s),
$$

where $B(t)$ denotes a standard Brownian motion in a probability space $(\Omega, \mathcal{F}, P)$ with the natural filtration $(\mathcal{F}_t)_{t \geq 0}$, here $P$ is the probability measure. Suppose that

$$
b(t, s, x, x', u, \omega): [0, T] \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}
$$

and

$$
\sigma(t, s, x, x', u, \omega): [0, T] \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}
$$

be $\mathcal{F}$-adapted with respect to the second variable $s$ for all $t, x, x', u$ and continuously differentiable with respect to the first variable $t$, with partial derivatives in $L^2([0, T] \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \Omega). \mathbb{U}$ denotes a given open set containing

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all possible admissible investment values $u(t, \omega)$ for $(t, \omega) \in [0, T] \times \Omega$, $u \in \mathcal{U}(x)$. Here suppose $\mathcal{U}$ be a given family of $(\mathcal{G}_t)_{t \geq 0}$-predictable admissible investment, where $\mathcal{G}_t \subseteq \mathcal{F}_t$ for all $t \in [0, T]$. The introduction of $\mathcal{G}_t$ illustrates that the admissible investment amount $u$ is decided based on only partial information available to the investor, for instance, delayed information flow.

We rewrite the controlled mean-field stochastic Volterra equation above in the following differential form

$$
\begin{aligned}
\frac{dX(t)}{dt} &= b(t, t, X(t), E[X(t)], u(t))dt + \left( \int_0^t \frac{\partial b}{\partial t}(t, s, X(s), E[X(s)], u(s))ds \right)dt \\
&\quad + \sigma(t, t, X(t), E[X(t)], u(t))dB(t) + \left( \int_0^t \frac{\partial \sigma}{\partial t}(t, s, X(s), E[X(s)], u(s))dB(s) \right)dt.
\end{aligned}
$$

From this differential form, we see that this mean-field stochastic Volterra equation differs from the mean-field stochastic differential equation, because of the two integral terms on the right hand side in (2). These terms represent memory effects of the investment $u(t)$.

The mean-field investment performance functional is given by

$$
J(u) = E\left[ \int_0^T f(s, X(s), E[X(s)], u(s))ds + g(X(T), E[X(T)]) \right]; u \in \mathcal{U}(x).
$$

In contrast to the standard investment performance functional, the performance functional in this paper involves the mean of functions of the state variable, i.e. the mean-field term $E[X(\cdot)]$.

The optimal investment problem is to maximize the performance functional $J(u)$ over all admissible investments, that is to find $u^* \in \mathcal{U}(x)$ such that

$$
J(u^*) = \sup_{u \in \mathcal{U}} J(u).
$$

Given an initial wealth value $x$, we say that $u^* \in \mathcal{U}(x)$ is an optimal investment if (4) holds.

Many papers have been devoted to the optimal investment problems. The seminal papers of [2, 3] considered the financial market modeled by a classical stochastic differential equation. Since then there have been many works concerning this subject. We refer to [4–6], for the complete market situation, to [7, 8] for constrained portfolios; to [9–13] for transactions costs; and to [14–17] for general incomplete markets.

However, to our knowledge, in all of these works, the financial markets have only one rational agent and the solutions of stochastic differential equations are Markov processes. The most natural framework to model the financial markets with an infinite number of rational agents in competition is the mean-field framework, which arise naturally in many applications. The systematic study of these problems was started, in the mathematical community by [18], and independently around the same time in the engineering community by [19, 20]. The well-known framework to model such non-Markov type solutions would be the stochastic Volterra equations where its solutions are not Markov processes. But this typically non-Markov property leads to invalidation of some methods, for instance, dynamic programming principle. In view of this, it is important to find good methods to solve the optimal investment problems for this type stochastic Volterra equations.

A good method to solve this optimal investment problem is Malliavin calculus, which has been firstly introduced in [1]. We refer to [21] for Malliavin calculus applied to optimal control of stochastic partial differential equations with jumps, and to [22] for Malliavin calculus applied to optimal control of stochastic Volterra equations but without the mean-field framework.

In the present paper following the idea of [22] we study the optimal investment problem in a finance market modeled by mean-field stochastic Volterra equation, in which both the financial market with an infinite number of rational agents in competition and non-Markov type solutions are all taken into account. The method of Malliavin calculus to solve this optimal investment problem in this paper is still adopted.

We present both a sufficient condition and a necessary condition for this optimal investment problem by mean-field stochastic maximum principle. Since the performance functional in this paper involves the mean-field term, in the proving process, we mainly use a new type of backward stochastic Volterra equations (BSVE), which is firstly
studied in [23], and we refer the reader to [24, 25] for more discussions about this issue. The terminal condition and generator of this type of BSVE in this paper evolve into more complex forms because of the appearance of the mean-field term in the investment performance functional.

This paper is organized as follows. Section 2 recalls some definitions and properties about Malliavin calculus for Brownian motion. Section 3 is divided into two main parts: a sufficient and necessary conditions for optimal investment problem. An example is given in Section 4.

2 Malliavin calculus for Brownian motion

For reader’s convenience, we recall the basic definition and properties of Malliavin calculus for Brownian motion. For more details about Malliavin calculus see [26, 27]. A natural starting point is the Wiener-Itô chaos expansion theorem, which states that any $\xi \in L^2(\mathcal{F}, P)$ can be written as

$$\xi = \sum_{n=0}^{\infty} I_n(f_n),$$

for a unique sequence of symmetric deterministic functions $f_n \in L^2(\lambda^n)$, where $\lambda$ is a Lebesgue measure on $[0, T]$ and

$$I_n(f_n) = n! \int_0^T \cdots \int_0^T f_n(t_1, t_2, \cdots, t_n) dB(t_1)dB(t_2) \cdots dB(t_n)$$

(the $n$-times iterated integral of $f_n$ with respect to $B$) for $n = 1, 2, \cdots$ and $I_0(f_0) = f_0$ when $f_0$ is a constant.

Moreover, we have the isometry

$$E[\xi^2] = \|\xi\|^2_{L^2(\mathcal{F}, P)} = \sum_{n=0}^{\infty} n! \|f_n\|^2_{L^2(\lambda^n)}.$$

We first present the Malliavin derivative $D_t \xi$ with respect to Brownian motion $B(\cdot)$ at $t$ of a given Malliavin differentiable random variable $\xi(\omega) ; \omega \in \Omega$, and then we present some basic properties about Malliavin derivative related to this paper.

Let $\mathbb{D}$ denote the set of all random variables which are Malliavin differentiable with respect to Brownian motion $B(\cdot)$, precisely, let $\mathbb{D}$ be the space of all $\xi \in L^2(\mathcal{F}, P)$ such that its chaos expansion satisfies

$$\|\xi\|^2_{\mathbb{D}} = \sum_{n=1}^{\infty} nn! \|f_n\|^2_{L^2(\lambda^n)} < \infty.$$

**Definition 2.1.** For any $\xi \in \mathbb{D}$, define the Malliavin derivative $D_t(\xi)$ of $\xi$ at $t$, $t \in [0, T]$ with respect to Brownian motion $B(\cdot)$ as

$$D_t(\xi) = \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot, t)),$$

where the notation $I_{n-1}(f_n(\cdot, t))$ means that we apply the $(n-1)$-times iterated integral to the first $n-1$ variables $t_1, t_2, \cdots, t_{n-1}$ of $f_n(t_1, t_2, \cdots, t_n)$ and keep the last variable $t_n = t$ as a parameter.

It is easy to check that

$$E\left[\int_0^T (D_t \xi)^2 \, dt\right] = \sum_{n=1}^{\infty} nn! \|f_n\|^2_{L^2(\lambda^n)} = \|\xi\|^2_{\mathbb{D}},$$

so $(t, \omega) \rightarrow D_t \xi(\omega)$ belongs to $L^2(\lambda \times P)$.

Some basic properties of the Malliavin derivative $D_t$ are the following (a) chain rule and (b) duality formula.
Suppose $\xi_1, \ldots, \xi_m \in \mathbb{D}$ and that $f : \mathbb{R}^m \to \mathbb{R}$ is $C^1$ with bounded partial derivatives. Then $f(\xi_1, \ldots, \xi_m) \in \mathbb{D}$ and
$$D_t f(\xi_1, \ldots, \xi_m) = \sum_{i=1}^m \frac{\partial f}{\partial \xi_i}(\xi_1, \ldots, \xi_m)D_t(\xi_i).$$

(b) Suppose $\varphi(t)$ is $(\mathcal{F}_t)_{t \geq 0}$-adapted with
$$E\left[\int_0^T \varphi^2(t) dt\right] < \infty$$
and let $\xi \in \mathbb{D}$. Then
$$E\left[\int_0^T \varphi(t) dB(t)\right] = E\left[\int_0^T \varphi(t) D_t(\xi) dt\right].$$

3 Main results

It is well-known that the investment performance functional (3) is related to a Hamiltonian functional. For this, we define the Hamiltonian functional
$$H(t, x, x', u, p, q) = f(t, x, x', u) + b(t, t, x, x', u)p + \int_t^T \frac{\partial b}{\partial s}(s, t, x, x', u)p(s)ds + \sigma(t, t, x, x')q + \int_t^T \frac{\partial \sigma}{\partial s}(s, t, x, x', u)D_t p(s)ds,$$
(5)
where both $p(\cdot)$ and $q(\cdot)$ are $(\mathcal{F}_t)_{t \geq 0}$-adapted processes and $p(s)$ is assumed to be Malliavin differentiable for every $s$.

Define the adjoint mean-field BSDE for $(p(t), q(t))$ by
$$\begin{align*}
dp(t) &= -E\left[\frac{\partial H}{\partial x}(t, X(t), E[X(t)]; u(t), p(t), q(t))\right]dt + q(t)dB(t), \quad 0 \leq t \leq T; \\
p(T) &= \left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial x'}\right)(X(T), E[X(T)]),
\end{align*}$$
(6)
where
$$\frac{\partial H}{\partial x}(t, x, x', u(t), p(t), q(t)) = \frac{\partial H}{\partial x}(t, x, x', u(t), E[X(t)]; u(t), p(t), q(t))$$
and
$$\frac{\partial H}{\partial x'}(t, x, x', u(t), p(t), q(t)) = \frac{\partial H}{\partial x'}(t, x, x', u(t), E[X(t)]; u(t), p(t), q(t)).$$

From now on we suppose that the controlled mean-field stochastic Volterra equation (1) and the corresponding adjoint mean-field BSDE (6) have corresponding solutions $X^u(\cdot)$ and $(p^u(\cdot), q^u(\cdot))$, respectively, for any $u \in U$.

Based on the Hamiltonian functional, in this section we study two mean-field stochastic maximum principles. The first principle gives a sufficient condition for the optimal investment under the concavity of the Hamiltonian function. The second condition is a necessary maximum principle under the condition of the existence of derivative process of the wealth process $X$.

3.1 A sufficient condition for optimal investment

**Theorem 3.1.** Assume that the function $g(x, x')$ is concave with respect to $(x, x')$ and $H(t, x, x', u, p, q)$ is concave with respect to $(x, x', u)$ for all $(t, p, q)$. Moreover, for all $t$
$$\sup_{u \in U} E[H(t, X^*(t), E[X^*(t)]; u, p^*(t), q^*(t))|\mathcal{G}_t] = E[H(t, X^*(t), E[X^*(t)]; u^*(t), p^*(t), q^*(t))|\mathcal{G}_t].$$
(7)
Then $u^*$ is an optimal investment.
Proof. Let $u \in \mathcal{U}$ (or $u^* \in \mathcal{U}$) with corresponding solutions $X(t) := X^u(t)$ (or $X^*(t) := X^{u^*}(t)$) of the controlled mean-field stochastic Volterra equation (2). Because of the definition of mean-field investment performance functional $J(u)$, we have

$$J(u) - J(u^*) = E\left[ \int_0^T \left( f(t, X(t), E[X(t)], u(t)) - f(t, X^*(t), E[X^*(t)], u^*(t)) \right) dt \right]$$

$$+ E\left[ g(X(T), E[X(T)]) - g(X^*(T), E[X^*(T)]) \right].$$

(8)

Firstly, we deal with the first term in the right hand side in the above equation (8). By the definition of the Hamiltonian functional, we get

$$E\left[ \int_0^T \left( \left[ H(t, X(t), E[X(t)], u(t), p^*(t), q^*(t)) ight.ight.$$

$$
- \left. H(t, X^*(t), E[X^*(t)], u^*(t), p^*(t), q^*(t)) \right] - \left[ b(t, t, X(t), E[X(t)], u(t)) - b(t, t, X^*(t), E[X^*(t)], u^*(t)) \right] p^*(t)$$

$$- \int_0^T \frac{\partial b}{\partial s} (s, t, X(t), E[X(t)], u(t)) ds$$

$$- \left[ \sigma(t, t, X(t), E[X(t)], u(t)) - \sigma(t, t, X^*(t), E[X^*(t)], u^*(t)) \right] q^*(t)$$

$$- \int_0^T \frac{\partial \sigma}{\partial s} (s, t, X(t), E[X(t)], u(t)) ds$$

$$- \left[ \sigma(t, t, X^*(t), E[X^*(t)], u^*(t)) \right] D_t p^*(s) ds \right] dt \right].$$

(9)

Secondly, we deal with the second term in the right hand side in the above equation (8). By the concavity of the function $g(x, x')$ with respect to $(x, x')$, we get

$$E\left[ g(X(T), E[X(T)]) - g(X^*(T), E[X^*(T)]) \right]$$

$$\leq E\left[ \frac{\partial g}{\partial x}(X^*(T), E[X^*(T)])(X(T) - X^*(T)) \right.$$

$$+ \frac{\partial g}{\partial x'}(X^*(T), E[X^*(T)])(E[X(T)] - E[X^*(T)]) \right]$$

$$= E\left[ \left( \frac{\partial g}{\partial x}(X^*(T), E[X^*(T)]) + \frac{\partial g}{\partial x'}(X^*(T), E[X^*(T)]) \right) (X(T) - X^*(T)) \right].$$

Since \( \left( \frac{\partial g}{\partial x} + \frac{\partial g}{\partial x'} \right)(X^*(T), E[X^*(T)]) = p^*(T) \) in the adjoint mean-field BSDE, we obtain

$$E\left[ g(X(T), E[X(T)]) - g(X^*(T), E[X^*(T)]) \right] \leq E\left[ p^*(T)(X(T) - X^*(T)) \right].$$

Now, applying Itô formula to $p^*(T)(X(T) - X^*(T))$ between 0 and $T$, in connection with the first identity in equation (6), we obtain

$$E\left[ g(X(T), E[X(T)]) - g(X^*(T), E[X^*(T)]) \right]$$
\[
E \left[ \int_0^T \left\{ p^*(t) \left[ b(t, t, X(t), E[X(t)], u(t)) - b(t, t, X^*(t), E[X^*(t)], u^*(t)) \right] + p^*(t) \left[ \int_0^t \frac{\partial b}{\partial t}(t, s, X(s), E[X(s)], u(s)) \right. \right.
\]
\[
- \frac{\partial b}{\partial t}(t, s, X^*(s), E[X^*(s)], u^*(s)) \right] ds \left. \right] \right] + p^*(t) \left[ \int_0^t \frac{\partial \sigma}{\partial t}(t, s, X(s), E[X(s)], u(s)) \right.
\]
\[
- \frac{\partial \sigma}{\partial t}(t, s, X^*(s), E[X^*(s)], u^*(s)) \right] dB(s) \right]
\[
- E \left[ \frac{\partial H^*}{\partial X}(t) + \frac{\partial H^*}{\partial X}(t) | F_t \right](X(t) - X^*(t)) + q^*(t) \left[ \sigma(t, t, X(t), E[X(t)], u(t)) \right.
\]
\[
- \sigma(t, t, X^*(t), E[X^*(t)], u^*(t)) \right] \right] dt \right],
\]
\]
(10)

where,
\[
\frac{\partial H^*}{\partial X}(t) := \frac{\partial H}{\partial X}(t, X^*(t), E[X^*(t)], u^*(t), p^*(t), q^*(t))
\]

and
\[
\frac{\partial H^*}{\partial X'}(t) := \frac{\partial H}{\partial X'}(t, X^*(t), E[X^*(t)], u^*(t), p^*(t), q^*(t)).
\]

Next, using Fubini’s theorem, the fact that \( p^*(t) \) is Malliavin differentiable for every \( t \) in connection with the duality formula for the Malliavin derivative \( D_T \) and Fubini’s theorem again all in this order, we have
\[
E \left[ \int_0^T \int_0^t \frac{\partial \sigma}{\partial t}(t, s, X(s), E[X(s)], u(s)) dB(s) \right] dt \right]
\[
= \int_0^T E \left[ \int_0^t \frac{\partial \sigma}{\partial t}(t, s, X(s), E[X(s)], u(s)) dB(s) \right] \right]
\[
= \int_0^T E \left[ \int_0^t \frac{\partial \sigma}{\partial t}(t, s, X(s), E[X(s)], u(s)) D_s p^*(t) ds \right] dt \right]
\[
= E \left[ \int_0^T \int_0^s \frac{\partial \sigma}{\partial t}(t, s, X(s), E[X(s)], u(s)) D_s p^*(t) ds dt \right]
\[
= E \left[ \int_0^T \int_0^s \frac{\partial \sigma}{\partial s}(s, t, X(t), E[X(t)], u(t)) D_t p^*(s) ds dt \right].
\]
(11)

Using Fubini’s theorem again, we have
\[
E \left[ \int_0^T \int_0^t \frac{\partial b}{\partial t}(t, s, X(s), E[X(s)], u(s)) ds \right] dt \right]
\[
= E \left[ \int_0^T \int_0^s p^*(t) \frac{\partial b}{\partial t}(t, s, X(s), E[X(s)], u(s)) ds \right] dt \right]
Combining (10), (11) and (12), we arrive at

\[ E[g(X(T), E[X(T)])] - g(X^*(T), E[X^*(T)]) \]

\[ \leq E \left[ \int_0^T \left\{ p^*(t) \left[ b(t, t, X(t), E[X(t)], u(t)) - b(t, t, X^*(t), E[X^*(t)], u^*(t)) \right] \right. \right. \]

\[ + \left. \int_0^T p^*(s) \frac{\partial b}{\partial s}(s, t, X(t), E[X(t)], u(t)) \right. \]

\[ - \frac{\partial b}{\partial s}(s, t, X^*(t), E[X^*(t)], u^*(t)) \right] ds \right] \]

\[ \left. + \int_0^T \frac{\partial \sigma}{\partial s}(s, t, X(t), E[X(t)], u(t)) \right. \]

\[ - \frac{\partial b}{\partial s}(s, t, X^*(t), E[X^*(t)], u^*(t)) \right] D_t p^*(s) ds \]

\[ - E \left[ \frac{\partial H^*}{\partial X}(t, t, X^*(t), E[X^*(t)], u^*(t)) \right] \frac{\partial H^*}{\partial X^*}(t, t, X^*(t), E[X^*(t)], u^*(t)) \right] \]

\[ \left. + q^*(t) \left[ \sigma(t, t, X(t), E[X(t)], u(t)) - \sigma(t, t, X^*(t), E[X^*(t)], u^*(t)) \right] \right] dt \right]. \]

Finally, substituting (9) and (13) into (8) we get

\[ J(u) - J(u^*) \]

\[ \leq E \left[ \int_0^T \left\{ \left[ H(t, t, X(t), E[X(t)], u(t), p^*(t), q^*(t)) \right] \right. \right. \]

\[ - H(t, X^*(t), E[X^*(t)], u^*(t), p^*(t), q^*(t)) \right. \]

\[ - \left[ b(t, t, X(t), E[X(t)], u(t)) - b(t, t, X^*(t), E[X^*(t)], u^*(t)) \right] p^*(t) \right. \]

\[ - \left. \int_0^T \frac{\partial b}{\partial s}(s, t, X(t), E[X(t)], u(t)) \right. \]

\[ - \frac{\partial b}{\partial s}(s, t, X^*(t), E[X^*(t)], u^*(t)) \right] p^*(s) ds \right] \]

\[ - \left. \left[ \sigma(t, t, X(t), E[X(t)], u(t)) - \sigma(t, t, X^*(t), E[X^*(t)], u^*(t)) \right] q^*(t) \right. \]

\[ - \left. \int_0^T \frac{\partial \sigma}{\partial s}(s, t, X(t), E[X(t)], u(t)) \right. \]

\[ - \frac{\partial \sigma}{\partial s}(s, t, X^*(t), E[X^*(t)], u^*(t)) \right] D_t p^*(s) ds \right] \]

\[ + \left. E \left[ \int_0^T \left\{ p^*(t) \left[ b(t, t, X(t), E[X(t)], u(t)) - b(t, t, X^*(t), E[X^*(t)], u^*(t)) \right] \right. \right. \right. \]

\[ + \left. \left. \left. \left. \int_0^T p^*(s) \left[ \frac{\partial b}{\partial s}(s, t, X(t), E[X(t)], u(t)) \right. \right. \right. \right. \right. \]

\[ - \left. \frac{\partial b}{\partial s}(s, t, X^*(t), E[X^*(t)], u^*(t)) \right] ds \right]. \]
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\[ + \int_t^T \left[ \frac{\partial \sigma}{\partial s}(s, t, X(t), E[X(t)], u(t)) \right] ds \]

\[ - \frac{\partial \sigma}{\partial s}(s, t, X^*(t), E[X^*(t)], u^*(t)) \right] D_t p^*(s) ds \]

\[ - E\left[ \left. \left( \frac{\partial H^*}{\partial x}(t) + \frac{\partial H^*}{\partial x'}(t) \right| \mathcal{F}_t \right) (X(t) - X^*(t)) \right] \]

\[ + q^*(t) \left[ \sigma(t, t, X(t), E[X(t)], u(t)) - \sigma(t, t, X^*(t), E[X^*(t)], u^*(t)) \right] \right] dt \]

\[ = E\left[ \int_0^T \left\{ H(t, X(t), E[X(t)], u(t), p^*(t), q^*(t)) \right. \right. \]

\[ - H(t, X^*(t), E[X^*(t)], u^*(t), p^*(t), q^*(t)) \left. \left. \right\} dt \right. \]

\[ - E\left[ \left. \left. \left( \frac{\partial H^*}{\partial x}(t) + \frac{\partial H^*}{\partial x'}(t) \right| \mathcal{F}_t \right) (X(t) - X^*(t)) \right\} dt \right. \right. \]

(14)

Now, introduce the notion:

\[ \frac{\partial H^*}{\partial u}(t) := \left. \frac{\partial H}{\partial u}(t, X^*(t), E[X^*(t)], u^*(t), p^*(t), q^*(t)) \right. \]

Hence, by the concavity of the function \( H(t, x, x', u, p, q) \) with respect to \((x, x', u)\) for all \((t, p, q)\), we obtain

\[ J(u) - J(u^*) \leq E\left[ \int_0^T \frac{\partial H^*}{\partial u}(t)[u(t) - u^*(t)] dt \right] \]

\[ = E\left[ \int_0^T E\left[ \frac{\partial H^*}{\partial u}(t)[u(t) - u^*(t)] \mid \mathcal{G}_t \right] dt \right] \]

\[ = E\left[ \int_0^T E\left[ \frac{\partial H^*}{\partial u}(t) \mid \mathcal{G}_t \right][u(t) - u^*(t)] dt \right]. \] (15)

By the assumption (7) we have \( E\left[ \int_0^T E\left[ \frac{\partial H^*}{\partial u}(t) \mid \mathcal{G}_t \right][u(t) - u^*(t)] dt \right] = 0 \). Therefore, we conclude that for all \( u \in \mathcal{U} \), \( J(u) \leq J(u^*) \), and this completes the proof. \( \square \)

### 3.2 A necessary condition for optimal investment: mean-field maximum principle

In this subsection we will assume both the coefficients \( b, \sigma \) of SDE and performance functions \( f \) are continuously differentiable with respect to \((x, x', u)\) and \( g \) is continuously differentiable with respect to \((x, x')\).

For any \( t \in [0, T] \), let \( \alpha = \alpha(t) \) be any bounded \( \mathcal{G}_t \)-measurable random variable and taking \( \beta(s) + \alpha I_{[t, t+h]}(s), s \in [0, T] \). Assume that \( u + \beta \in \mathcal{U} \) for all such \( \alpha \) and all \( u \in \mathcal{U} \).

Define the derivative process \( Y(t) \) by

\[ Y(t) := \frac{d}{d\lambda} X^{(u + \lambda \beta)}(t) \bigg|_{\lambda=0} \]

and suppose this derivative process exists.

From (1), after some simple computation, we have

\[ Y(t) = \int_0^t \left[ \frac{\partial b}{\partial x}(t, s, X(s), E[X(s)], u(s))Y(s) \right. \]

\[ + \frac{\partial b}{\partial x'}(t, s, X(s), E[X(s)], u(s))E[Y(s)] \]
\[ \frac{\partial b}{\partial u} (t, s, X(s), E[X(s)], u(s)) \beta(s) \, ds \]
\[ + \int_0^t \left[ \frac{\partial \sigma}{\partial x} (t, s, X(s), E[X(s)], u(s)) Y(s) \right. \]
\[ \left. + \frac{\partial \sigma}{\partial x'} (t, s, X(s), E[X(s)], u(s)) E[Y(s)] \right] \, ds \]
\[ + \frac{\partial \sigma}{\partial u} (t, s, X(s), E[X(s)], u(s)) \beta(s) \, dB(s). \]  

(16)

We can rewrite the derivative process (16) in differential form, for which it is convenient for us to apply Itô formula, as following:

\[ dY(t) = \left[ \frac{\partial b}{\partial x} (t, t, X(t), E[X(t)], u(t)) Y(t) + \int_0^t \frac{\partial^2 b}{\partial t \partial x} (t, s, X(s), E[X(s)], u(s)) Y(s) \, ds \right. \]
\[ \left. + \frac{\partial b}{\partial x'} (t, t, X(t), E[X(t)], u(t)) E[Y(t)] + \int_0^t \frac{\partial^2 b}{\partial t \partial x'} (t, s, X(s), E[X(s)], u(s)) E[Y(s)] \, ds \right. \]
\[ \left. + \frac{\partial \sigma}{\partial x} (t, t, X(t), E[X(t)], u(t)) \beta(t) + \int_0^t \frac{\partial \sigma}{\partial x'} (t, s, X(s), E[X(s)], u(s)) E[Y(s)] \, dB(s) \right. \]
\[ \left. + \int_0^t \frac{\partial \sigma}{\partial u} (t, s, X(s), E[X(s)], u(s)) \beta(s) \, dB(s) \right] \, dt \]
\[ + \frac{\partial \sigma}{\partial x'} (t, t, X(t), E[X(t)], u(t)) Y(t) \, dB(t) \]
\[ + \frac{\partial \sigma}{\partial u} (t, t, X(t), E[X(t)], u(t)) E[Y(t)] \, dB(t) + \frac{\partial \sigma}{\partial u} (t, t, X(t), E[X(t)], u(t)) \beta(t) \, dB(t) \]
\[ = \left[ \Sigma(t) \right] \, dt \left[ + \int \frac{\partial \sigma}{\partial x} (t, t, X(t), E[X(t)], u(t)) \right] \, Y(t) \]
\[ \left. + \frac{\partial \sigma}{\partial x'} (t, t, X(t), E[X(t)], u(t)) E[Y(t)] \right] \, dB(t) \]
\[ + \frac{\partial \sigma}{\partial u} (t, t, X(t), E[X(t)], u(t)) \beta(t) \, dB(t). \]  

(17)

We now establish the mean-field stochastic maximum principle for mean-field stochastic Volterra equations with corresponding adjoint mean-field BSDE.

**Theorem 3.2.** Suppose that \( u \in \mathcal{U} \) satisfies that for all \( \beta \),

\[ \frac{d}{d \lambda} J(u + \lambda \beta) \bigg|_{\lambda=0} = 0. \]  

(18)

Then

\[ E \left[ \frac{\partial H}{\partial u} (t) \right] |_{u=u^*(t)} = 0, \]  

(19)

where

\[ \frac{\partial H}{\partial u} (t) := \frac{\partial H}{\partial u} (t, X(t), E[X(t)], u(t), p(t), q(t)), \]

and we adopt analogous notations for \( \frac{\partial H}{\partial x} (t) \) and \( \frac{\partial H}{\partial x'} (t) \).

Conversely, if (19) holds, then (18) holds.
Proof. By the definition of the performance functional $J(u)$, we can compute the derivative

$$
\frac{d}{d\lambda} J(u + \lambda \beta) \bigg|_{\lambda = 0} = E \left[ \int_0^T \left( \frac{\partial f}{\partial x}(t, X(t), E[X(t)], u(t))Y(t) + \frac{\partial f}{\partial u}(t, X(t), E[X(t)], u(t))E[Y(t)] \right. \\
+ \left. \frac{\partial g}{\partial x}(X(T), E[X(T)])Y(T) \right) \right]
$$

$$
= E \left[ \int_0^T \left( \frac{\partial f}{\partial x}(t, X(t), E[X(t)], u(t))Y(t) + \frac{\partial f}{\partial x}(t, X(t), E[X(t)], u(t))E[Y(t)] \right. \\
+ \left. \frac{\partial g}{\partial x}(X(T), E[X(T)])E[Y(T)] \right) \right]
$$

where the last equality results from $p(T) = \left( \frac{\partial g}{\partial x} + \frac{\partial g}{\partial x'} \right)(X(T), E[X(T)])$.

Now, using Itô formula to $p(t)Y(t)$, we find

$$
E[p(T)Y(T)] = E \left[ \int_0^T p(t)[\Sigma(t)]dt - \int_0^T Y(t)E\left[ \frac{\partial H}{\partial x}(t) + \frac{\partial H}{\partial x'}(t) \right]dt \right]
$$

$$
+ \int_0^T q(t) \left[ \frac{\partial \sigma}{\partial x}(t, t, X(t), E[X(t)], u(t))Y(t) + \frac{\partial \sigma}{\partial x'}(t, t, X(t), E[X(t)], u(t))E[Y(t)] \right. \\
+ \left. \frac{\partial \sigma}{\partial u}(t, t, X(t), E[X(t)], u(t))Z(t) \right] dt
$$

Similarly to (11) and (12), we obtain

$$
E \left[ \int_0^T p(t)[\Sigma(t)]dt \right]
$$

$$
= E \left[ \int_0^T Y(t) \left( \frac{\partial b}{\partial x}(t, t, X(t), E[X(t)], u(t))p(t) + \int_t^T \left( \frac{\partial^2 b}{\partial s \partial x}(s, t, X(t), E[X(t)], u(t))p(s) \right)dt \right) \right]
$$

$$
+ E \left[ \int_0^T E[Y(t)] \left( \frac{\partial b}{\partial x'}(t, t, X(t), E[X(t)], u(t))p(t) \right) \right]
$$

$$
+ \int_0^T \left( \frac{\partial^2 b}{\partial s \partial x'}(s, t, X(t), E[X(t)], u(t))D_t p(s) \right) ds dt
$$

$$
+ E \left[ \int_0^T \beta(t) \left( \frac{\partial b}{\partial u}(t, t, X(t), E[X(t)], u(t))p(t) + \int_t^T \left( \frac{\partial^2 b}{\partial s \partial u}(s, t, X(t), E[X(t)], u(t))p(s) \right) \right) \right]
$$

$$
+ E \left[ \int_0^T \beta(t) \left( \frac{\partial b}{\partial u'}(t, t, X(t), E[X(t)], u(t))D_t p(s) \right) \right]
$$

$$
+ \int_0^T \left( \frac{\partial^2 b}{\partial s \partial u'}(s, t, X(t), E[X(t)], u(t))D_t p(s) \right) ds dt
$$

(20)
Combining (20), (21), (22) with the constructions of \( \frac{\partial H}{\partial x}(t) \) and \( \frac{\partial H}{\partial x'}(t) \), we have

\[
\frac{d}{d\lambda} f(u + \lambda \beta) \big|_{\lambda=0} = E \left[ \int_0^T \frac{\partial H}{\partial u}(t) \beta(t) dt \right] = E \left[ \int_0^{t+h} \frac{\partial H}{\partial u}(t) dt \right].
\]  

(23)

If (18) holds, then after differentiating the right hand side of (23) at \( h = 0 \), we now obtain

\[
E \left[ \alpha \frac{\partial H}{\partial u}(t) \right] = 0.
\]

Since this equality holds for all \( \mathcal{G}_t \)-measurable bounded \( \alpha \), we conclude that

\[
E \left[ \frac{\partial H}{\partial u}(t) \big| \mathcal{G}_t \right] = 0.
\]  

(24)

Conversely, if we assume that (24) holds, then we obtain (18), since (23) holds. This completes the proof. \( \square \)

4 An example

Consider the following linear mean-field stochastic Volterra dynamic system, for \( 0 \leq t \leq T \),

\[
X(t) = x + \int_0^t b(t, s, E[X(s)], u(s))X(s)ds + \int_0^t \sigma(t, s, E[X(s)], u(s))X(s)dB(s);
\]  

(25)

here, for simplicity, let the coefficient terms be

\[
b(t, s, E[X(s)], u(s)) = E[X(s)]u(s).
\]

and

\[
\sigma(t, s, E[X(s)], u(s)) = tE[X(s)]u(s).
\]

Hence the linear mean-field stochastic Volterra equation (25) given by

\[
X(t) = x + \int_0^t E[X(s)]u(s)X(s)ds + \int_0^t tE[X(s)]u(s)X(s)dB(s); \quad 0 \leq t \leq T
\]

or, in differential form,

\[
dX(t) = E[X(t)]u(t)X(t)dt + tE[X(t)]u(t)X(t)dB(t) + \left( \int_0^t E[X(s)]u(s)X(s)dB(s) \right)dt.
\]  

(26)

From (26) above, we can get

\[
X(t) = x \exp \left[ \int_0^t sE[X(s)]u(s)dB(s) + \int_0^t E[X(s)]u(s)ds 
\right.

\[
\left. + \int_0^t \left( \int_0^s E[X(r)]u(r)X(r)dB(r) - \frac{1}{2}(E[X(s)]u(s))^2 \right)ds \right].
\]  

(27)

Here, the stochastic integral term \( \int_0^t E[X(r)]u(r)X(r)dB(r) \) in (27) be generated from the form of Volterra equation and represents memory effects of the investment \( u(\cdot) \). But there is no stochastic integral term \( \int_0^t E[X(r)]u(r)X(r)dB(r) \) in classical Black-Scholes formula. If we assume the initial wealth value \( x \) be positive strictly, it follows that, from (27),

\[
X(t) > 0, \quad P - a.s.
\]
The invest performance functional is given by

\[ J(u) = E\left[ \sqrt{X(T)} \right]. \]

Obviously, the functions \( b, \sigma \) and \( g \) satisfy the conditions of Theorem 3.1 and Theorem 3.2, so, we can directly apply the results of Theorem 3.1 and Theorem 3.2.

Suppose there exists an optimal investment \( u^* \) with corresponding processes \( X^*. \), then we have

\[ E\left[ \frac{\partial H}{\partial u} (t, X^*(t), u[X^*(t)], u(t), p^*(t), q^*(t)) \right]_{u=u^*} = 0. \]

From the definition of the Hamiltonian

\[ H(t, x, x', u, p, q) = E[X(t)]u(t)X(t) p(t) + \int_t^T E[X(t)]u(t)X(t)q(t) + \int_t^T E[X(t)]u(t)X(t)D_t p(s)ds \]

and \( X(t) > 0 \), for all \( t \in [0, T] \), we can get

\[ p^*(t) + t q^*(t) + E\left[ \int_t^T D_t p^*(s)ds \right] = 0. \] (28)

From this, after some computation, we find that the adjoint BSDE given by

\[ dp^*(t) = q^*(t)dB(t) \]

and the terminal condition

\[ p^*(T) = \frac{1}{2\sqrt{X^*(T)}} =: \xi(T). \]

Obviously, this BSDE has the unique solution

\[ p^*(t) = E\left[ \xi(T) \right| F_t] \text{ and } q^*(t) = D_t p^*(t). \] (29)

Combining (28) with (29), we have

\[ E\left[ \xi(T) \right| F_t] + E\left[ t D_t \left( \xi(T) \right) + \int_t^T D_t \left( \xi(T) \right) ds \right] = 0 \]

that is

\[ E\left[ \xi(T) \right| F_t] + T D_t \left( E\left[ \xi(T) \right| F_t] \right) = 0. \]

Since \( X(t) > 0 \) and \( E\left[ \xi(T) \right| F_t] \) is a martingale, so, we can deduce from the equation above that

\[ E\left[ \xi(T) \right| F_t] = E[\xi(T)] \exp \left\{ - \frac{B(t)}{T} - \frac{t}{2T^2} \right\}. \]

Obviously, \( \xi(T) = E[\xi(T)] \exp \left\{ - \frac{B(T)}{T} \right\} = \frac{1}{T} \), so we get

\[ X^*(T) = \frac{1}{4} E[\xi(T)]^2 \exp \left\{ \frac{2B(T)}{T} + \frac{1}{T} \right\}. \] (30)

Now, we define

\[ Z^*(s) := E[X^*(s)]u^*(s)X^*(s) \]

and from (25), we get the following mean-field backward stochastic Volterra equation

\[ X^*(t) = \frac{1}{4} E[\xi(T)]^2 \exp \left\{ \frac{2B(T)}{T} + \frac{1}{T} \right\} - \int_t^T Z^*(s)ds - \int_t^T tZ^*(s)dB(s). \] (31)
From the existence and uniqueness of mean-field backward stochastic Volterra equation, see [24], we know that the above equation has unique solution $(X^*, Z^*)$.

Taking the expectation on the above equation for $t = 0$, we obtain

$$x = \frac{1}{4} E[\xi(T)]^{-2} e^{T^2} - \int_0^T E[Z^*(s)] ds.$$ 

So, we have

$$E[\xi(T)] = \left(4e^{T^2} x + 4e^{T^2} \int_0^T E[Z^*(s)] ds \right)^{-\frac{1}{2}}.$$ 

Combining this with (30), we conclude that the optimal investment in the financial market modeled by a linear mean-field stochastic Volterra equation (25), is:

$$u^*(s) = \frac{Z^*(s)}{\left(x + \int_0^T E[Z^*(s)] ds \right) E[X^*(s)] \exp \left(\frac{2B(T)}{T} - \frac{1}{2}\right)},$$

here, $(X^*, Z^*)$ be the solution of (31).

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