Generalized Loop Space and TMDs

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Table of contents

1 Motivation
2 Generalized Loop Space
3 Derivatives
4 Conjecture
5 Examples
6 Quark TMD
7 Outlook
Transverse Momentum Distributions (TMDs) and Wilson Lines

Recap definition TMDs

\[
\Phi_{ss'}^\Gamma(x, \vec{k}_\perp) = \int \frac{dz^- d^2 z_\perp}{(2\pi)^2} e^{i k \cdot z} \left\langle P, s \left| \overline{\psi} \left( -\frac{z}{2} \right) \Gamma \mathcal{W}_{\text{TMD}} \psi \left( \frac{z}{2} \right) \right| P, s' \right\rangle_{z^+ = 0}
\]

- We want: Evolution equation
- We need: Renormalization
- Approximations in factorization schemes → on the light-cone
  ⇒ extra singularities → $\frac{1}{\epsilon^2}$
Transverse Momentum Distributions (TMDs) and Wilson Lines

Recap definition TMDs

\[
\frac{1}{2} \int \frac{dz^- d^2 z_\perp}{(2\pi)^3} e^{ik \cdot z} \left\langle P, s \left| \overline{\psi} \left( -\frac{z}{2} \right) \Gamma W_{\text{TMD}} \psi \left( \frac{z}{2} \right) \right| P, s' \right\rangle _{z^+ = 0}
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Recap definition TMDs

$$\Phi_{ss'}(x, \vec{k}_\perp)$$

$$x = \frac{k^+}{P^+}, \vec{k}_\perp$$

$$\frac{1}{2} \int \frac{dz^- d^2z_\perp}{(2\pi)(2\pi^2)} e^{i\vec{k} \cdot z} \left\langle P, s \left\vert \bar{\psi} \left( -\frac{z}{2} \right) \Gamma \mathcal{W}_{\text{TMD}} \psi \left( \frac{z}{2} \right) \right\vert P, s' \right\rangle_{z^+ = 0}$$

- We want: Evolution equation
- We need: Renormalization
- Approximations in factorization schemes $\rightarrow$ on the light-cone
  $\Rightarrow$ extra singularities $\rightarrow \frac{1}{\epsilon^2}$
Motivation for loop space

- Some (vacuum expectation values of) Wilson loops show same divergent behavior as TMDs
- Study divergence of these loops $\Rightarrow$ geometric renormalization
- Loop space representation of gauge theory (Ambrose-Singer Theorem)
  Over-complete $\Rightarrow$ equivalence relation

Giles, The reconstruction of gauge potentials from Wilson loops, Phys.Rev D.24.2160 (1981)

- Chen iterated integrals $\Rightarrow$ Holonomy of the gauge connection
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- Gauge invariant quantities as DOF instead of quarks and gluons but path dependence
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Wilson Loop :

\[ \mathcal{W}(\Gamma) = \text{Tr} \ P \exp \left[ -ig \int_{\Gamma} dz^\mu A^a_\mu(z) t^a \right] \in \mathbb{C} \]

Holonomy :

\[ U_\Gamma = 1 + \int_{\Gamma} \omega + \int_{\Gamma} \omega_1 \omega_2 + \cdots \]

Note that under gauge transformations :

\[ U^g_\Gamma(x) = g^{-1}_x U_\Gamma g_x \]

Chen Iterated Integrals
Equivalence Relation Deconstruction

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- Chen Iterated Integrals
## Chen Iterated Integrals

Extend the definition of the line integral $I_i(\alpha) = \int_a^b dx_i(t) = x_i(b) - x_i(a)$ recursively for $p \geq 2$

$$X(\alpha) = I_{i_1 \cdots i_p}(\alpha) = \int_a^b I_{i_1 \cdots i_{p-1}}(\alpha^t) dx_{i_p}(t)$$

### Example

$$X^{\omega_1 \omega_2}(\gamma) = \int_\gamma \omega_1 \omega_2 = \int_0^1 \left( \int_0^t \omega_1(s) ds \right) \omega_2(t) dt$$

$$X^{\omega_1 \cdots \omega_r}(\gamma) = \int_\gamma \omega_1 \cdots \omega_r = \int_0^1 \left( \int_0^t \omega_1 \cdots \omega_{r-1} \right) \omega_r(t) dt$$

where $\omega_k(t) \equiv \omega_k(\gamma(t)) \cdot \dot{\gamma}(t)$.

Feynman, Richard P. An Operator calculus having applications in quantum electrodynamics Phys. Rev.84 108-128 (1951)
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Shuffle multiplication

Let \( \omega_1, \cdots, \omega_{k+l} \) be in \( \Omega \) then:

\[
\int_{\alpha} \omega_1 \cdots \omega_k \int_{\alpha} \omega_{k+1} \cdots \omega_{k+l} = \sum_{\sigma} \int_{\alpha} \omega_{\sigma(1)} \cdots \omega_{\sigma(k+l)}
\]

where \( \sigma \) is running over all \((k, l)\)-shuffles. This produces then a map \( \mathcal{A}_p \times \mathcal{A}_p \to \mathcal{A}_p \), the algebra multiplication in the with Chen Iterated Integral associated Hopf algebra.
(k,l)-shuffle

A \((k, l)\)-shuffle is a permutation \(\sigma\) of the \(k + l\) letters such that \(\sigma(1) < \cdots < \sigma(k)\) and \(\sigma(k + 1) < \cdots < \sigma(k + l)\).

Shuffle Multiplication

Let \(\omega_1 \cdots \omega_k = \omega_1 \otimes \cdots \otimes \omega_k \in \bigotimes^k (\wedge^1 M)\), \(k \geq 1\) and \(\omega_1 \cdots \omega_k = 0\), for \(k = 0\), the shuffle multiplication is given by:

\[
\omega_1 \cdots \omega_k \bullet \omega_{k+1} \cdots \omega_{k+l} = \sum'_{\sigma} \omega_{\sigma(1)} \cdots \omega_{\sigma(k+l)}
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with \(\sum'\) the sum over all \((k, l)\)-shuffles.

Note:

\[
f \wedge g(v_1, \cdots, v_{k+l}) = \sum_{(k,l)} (\text{sgn } \sigma) f(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)})
\]
Algebraic paths: Shuffle multiplication

**Shuffle Multiplication**

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d-paths [Chen 1968, Algebraic Paths]

The Shuffle Algebra is Banach, Hopf, Commutative, Nuclear and Locally Multiplicative-Convex. d-paths generalize the intuitive idea of a path in a manifold.
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Gel’fand Space or Spectrum

Let $A$ be a commutative Banach algebra, then we write $\triangle(A)$ (or $\triangle$) for the collection of nonzero complex homomorphisms $h : A \to \mathbb{C}$. Elements of the Gel’fand space are called characters.

Generalized Loops based at a point $p$

A *Generalized Loop* based at $p \in \mathcal{M}$ is a character of the algebra $A_p$ or, equivalently, a continuous complex algebra homomorphism $\tilde{\alpha} : \text{Sh}(\mathcal{M}) \to \mathbb{C}$, that vanishes on the ideal $J_p$.

Properties Gel’fand topology

- The Gel’fand topology on the spectrum of a unital, Abelian Banach algebra is Hausdorff.
- A sequence $(\tilde{\alpha}_n)$ converges to $\tilde{\alpha} \in \Delta_p$, iff
  \[ \lim_{n \to \infty} \tilde{\alpha}_n(X^u) = \tilde{\alpha}(X^u), \forall u \in \text{Sh}(\mathcal{M}). \]
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$$\lim_{n \to \infty} \tilde{\alpha}_n(X^u) = \tilde{\alpha}(X^u), \quad \forall u \in Sh(\mathcal{M}).$$
We can define a product $\star$ on $\Delta$ such that $(\Delta_p, \star)$ has the structure of topological group that is Hausdorff and completely regular or Tychonov.

We call the above mentioned topological group $(\Delta_p, \star)$, the Group of Generalized Loops of $\mathcal{M}$, based at $p \in \mathcal{M}$, and we denote it by $\widetilde{L\mathcal{M}}_p$.

A pointed differentiation is a pair $(d, p)$ where $d : \mathcal{U} \to \Omega$ is a differentiation and $p \in \text{Alg}(\mathcal{U}, k)$.

The pointed differentiations then form a tangent space $T_e \widetilde{L\mathcal{M}}_p$ and one is able to show that this space is isomorphic to the Lie Algebra of $(\Delta_p, \star)$, i.e.

$$T_e \widetilde{L\mathcal{M}}_p \simeq \tilde{l\mathcal{M}}_p \quad (1)$$
Loop Group and Lie Algebra

**Topological Group**

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### Pointed Differentiation

A pointed differentiation is a pair $(d, p)$ where $d : \mathcal{U} \to \Omega$ is a differentiation and $p \in \text{Alg}(\mathcal{U}, k)$.

### Lie Algebra and Tangent Space

The pointed differentiations then form a tangent space $T_{\epsilon} \tilde{L}\mathcal{M}_p$ and one is able to show that this space is isomorphic to the Lie Algebra of $(\Delta_p, \star)$, i.e.

$$T_{\epsilon} \tilde{L}\mathcal{M}_p \cong \tilde{l}\mathcal{M}_p$$

(1)
Path derivative

Let $\Psi$ be a path functional on $\mathcal{P}M$, with values in $\mathbb{R}$ (resp., $\mathbb{C}$; $gl(m)$). We define the Terminal Endpoint Derivative of $\Psi$, at $\gamma$, in the direction of $v \in T_{\gamma(1)}\mathcal{M}$, as the limit:

$$\partial_v^T \Psi(\gamma) = \lim_{s \to 0} \frac{\Psi(\gamma_s) - \Psi(\gamma)}{s}$$

provided this limit exists independently of the choice of the vector field $V \in \mathcal{X}\mathcal{M}$, such that $V(\gamma(1)) = v$. 

Terminal Endpoint Derivative
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Path derivative

Let $\Psi$ be a path functional on $\mathcal{PM}$, with values in $\mathbb{R}$ (resp., $\mathbb{C}; gl(m)$). We define the **Terminal Endpoint Derivative** of $\Psi$, at $\gamma$, in the direction of $v \in T_{\gamma(1)}\mathcal{M}$, as the limit:

$$
\partial_v^T \Psi(\gamma) = \lim_{s \to 0} \frac{\Psi(\gamma_s) - \Psi(\gamma)}{s}
$$

provided this limit exists independently of the choice of the vector field $V \in \mathcal{X}\mathcal{M}$, such that $V(\gamma(1)) = v$. 

**Terminal Endpoint Derivative**
Area Derivative

Given a loop functional $\Psi$ on $\mathbb{L}M_p$, with values in $\mathbb{R}$ (resp., $\mathbb{C}$; $gl(m)$), we define its Area Derivative, given by $\Delta_{\lambda; (u,v)}(q) \cdot \Psi(\gamma)$, as the limit:

$$
\Delta_{\lambda; (u,v)}(q) \Psi(\gamma) = \lim_{t \to 0} \frac{\Psi(\lambda_t \cdot \gamma) - \Psi(\gamma)}{t^2}
$$

provided this limit exists independently of the choice of the vector fields $U, V \in XU$, considered above.
Area Derivative

\[ \Delta_{\lambda; (u,v)}(q) \Psi(\gamma) = \lim_{t \to 0} \frac{\Psi(\lambda_t \cdot \gamma) - \Psi(\gamma)}{t^2} \]

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provided this limit exists independently of the choice of the vector fields $U, V \in \mathcal{X}U$, considered above.
Wilson Quadrilateral

One-loop Result

\[ W_{\text{L.O.}}(\Gamma □) = 1 - \frac{\alpha_s C_F}{\pi} \left( \frac{2\pi \mu^2}{\epsilon} \right)^\epsilon \Gamma(1 - \epsilon) \]
\[ \left[ \frac{1}{\epsilon^2} \left( -\frac{s}{2} \right)^\epsilon + \frac{1}{\epsilon^2} \left( -\frac{t}{2} \right)^\epsilon - \frac{1}{2} \left( \ln^2 \frac{s}{-t} + \pi^2 \right) \right] + \mathcal{O}(\alpha_s^2) \]

\[ s = (v_1 + v_2)^2 \]
\[ t = (v_2 + v_3)^2 \]
Problem with area derivative on the LC

Preliminary Result

\[ \frac{\Psi(\lambda_a \cdot \gamma) - \Psi(\gamma)}{a^2} \sim \frac{1}{a^2 - 2\epsilon \epsilon^2} \]

(To be submitted for publication together with gravitational case)
Defining a new derivative

\[ \delta \sigma^+ - (x_1)^L = \delta \sigma^+ - (x_2)^R \]

\[ \delta \sigma^- + (x_1)^R = \delta \sigma^- + (x_4)^L \]

\[ \delta \sigma^+ = N^+ \delta N^- \rightarrow v_1 \delta v_2 = \frac{1}{2} \delta s , \]

\[ \delta \sigma^- = -N^- \delta N^+ \rightarrow -v_2 \delta v_1 = \frac{1}{2} \delta t . \]

Evolution (in the large \( N_c \) limit)

\[ \mu \frac{d}{d\mu} \frac{\delta}{\delta \ln \sigma} \ln W(\Gamma_{\Box}) = -4 \Gamma_{\text{cusp}} , \quad \Gamma_{\text{cusp}} = \frac{\alpha_s N_c}{2\pi} + O(\alpha_s^2) . \]
Defining a new derivative

\[
\begin{align*}
\delta\sigma^{+-}(x_1) &= \delta\sigma^{+-}(x_2) \\
\delta\sigma^{--}(x_1) &= \delta\sigma^{--}(x_4) \\
\delta\sigma^{++}(x_1) &= \delta\sigma^{++}(x_4) \\
\delta\sigma^{+-} &= N^+ \delta N^- \rightarrow v_1 \delta v_2 = \frac{1}{2} \delta s \\
\delta\sigma^{--} &= -N^- \delta N^+ \rightarrow -v_2 \delta v_1 = \frac{1}{2} \delta t \\
\frac{\delta}{\delta \ln \sigma} &\equiv \sigma^{+-} \frac{\delta}{\delta \sigma^{+-}} + \sigma^{--} \frac{\delta}{\delta \sigma^{--}}
\end{align*}
\]

Evolution (in the large $N_c$ limit)

\[
\mu \frac{d}{d\mu} \frac{\delta \ln W(\Gamma \Box)}{\delta \ln \sigma} = -4 \Gamma_{\text{cusp}} , \quad \Gamma_{\text{cusp}} = \frac{\alpha_s N_c}{2\pi} + O(\alpha_s^2)
\]
Our conjecture

\[ \frac{d}{d\mu} \frac{\delta \ln W(\Gamma^\square)}{\delta \ln \sigma} = - \sum_{\text{cusps}} \Gamma_{\text{cusp}}, \]

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Phys. Part. Nucl. 44 (2013) 250-259,
Int. J. Mod. Phys. Conf. Ser. 20 (2012) 109-117,
AIP Conf. Proc. 1523 (2013) 272-275
Another example: $\Pi$

Result of derivative

$$\mu \frac{d}{d\mu} \left[ \frac{d}{d\ln \sigma} \ln W(\Gamma_{\Pi}) \right] = -2\Gamma_{\text{cusp}}$$

One-loop result for $\Pi$

$$W(\Gamma_{\Pi}) = 1 + \frac{\alpha_s N_c}{2\pi} + \left[ -L^2(NN^-) + L(NN^-) - \frac{5\pi^2}{24} \right],$$

$$L(NN^-) = \frac{1}{2} \left( \ln(\mu NN^- + i0) + \ln(\mu NN^- + i0) \right)^2.$$
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$$W(\Gamma_\Pi) = 1 + \frac{\alpha_s N_c}{2\pi} + \left[ -L^2(NN^-) + L(NN^-) - \frac{5\pi^2}{24} \right],$$

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Another example: $\Pi$

One-loop result for $\Pi$

$$W(\Gamma_\Pi) = 1 + \frac{\alpha_s N_c}{2\pi} + \left[ -L^2(NN^-) + L(NN^-) - \frac{5\pi^2}{24} \right] ,$$

$$L(NN^-) = \frac{1}{2} \left( \ln(\mu NN^- + i0) + \ln(\mu NN^- + i0) \right)^2$$
Self-intersecting Wilson Loops (submitted to Phys Lett B)
Quark TMD (under construction)

\[ q(p) \rightarrow (0^+, 0^-, 0^\perp) \]

\[ n^- \rightarrow (0^+, \infty^-, 0^\perp) \]

\[ (0^+, 0^-, 0^\perp) \rightarrow (0^+, \infty^-, 0^\perp) \]

\[ n^\perp \rightarrow (0^+, z^-, z^\perp) \]

\[ (0^+, 0^-, 0^\perp) \rightarrow (0^+, \infty^-, z^\perp) \]

\[ (0^+, z^-, z^\perp) \rightarrow (0^+, \infty^-, z^\perp) \]

\[ (-\infty^+, 0^-, 0^\perp) \rightarrow (-\infty^+, 0^-, 0^\perp) \]

\[ (-\infty^+, z^-, z^\perp) \rightarrow (-\infty^+, \infty^-, z^\perp) \]
One transverse gauge link (shown) is not at infinity and one (not drawn) is at light-cone infinity (due to path-reduction).
Show that our derivative is a mathematically well-defined → special case of Fréchet derivative (in preparation).

Further exploration of geometrical operators.

Calculation of specific diagrams to test conjecture

Two-loop calculation as test of conjecture

Loop classes as connected components in Loop Space

Apply to phenomenological relevant situation → Quark and Gluon TMD (Jlab - Large-x physics)

Few weeks ago : Nima Arkani-Hamed - Amplituhedron (Wilson Loops in Twistor Theory)