A 4 Flattening Points Theorem for Polygons

Marcos Craizer and Sinesio Pesco

Abstract. In this paper we prove a 4 flattening points theorem for polygons in 3-space. The proof is based on a duality between spatial polygons whose radial projection in a plane is convex and planar convex polygons with a transversal parallel vector field. The result is then obtained by the use of a 4 vertices theorem for planar polygons ([6]).

Mathematics Subject Classification (2010). 53A15, 52C35.

Keywords. Affine vertices, Planar points, Four vertices theorem, Four planar points theorem.

1. Introduction

In this paper, we consider polygons in 3-space. In order to avoid misunderstandings, we use the terms ”edge” and ”nodes” for the elements of a polygon, leaving the terms ”vertex” and ”flattening point” for special edges or nodes.

Consider a closed polygon $P$ in 3-space with nodes $P(i)$, $1 \leq i \leq n$. For simplicity in the formulas, we shall use the periodic notation $P(i+kn) = P(i)$, $k \in \mathbb{Z}$. Let $\Delta = \Delta(P)$ be defined by

$$\Delta(i + \frac{1}{2}) = [P(i + 2) - P(i + 1), P(i + 1) - P(i), P(i) - P(i - 1)],$$

where the brackets denote the standard volume form in 3-space. We say that $P$ is generic if $\Delta(i + \frac{1}{2}) \neq 0$, for any $i \in \mathbb{Z}$, which means that no four consecutive nodes are coplanar. A node $i$ of the polygon $P$ is said to be a flattening point if

$$\Delta(i - \frac{1}{2}) \cdot \Delta(i + \frac{1}{2}) < 0.$$

The aim of this paper is to prove the following theorem:

The authors want to thank CNPq and CAPES for financial support during the preparation of this manuscript.
E-mail of the corresponding author: craizer@puc-rio.br.
Theorem 1.1. Let \( P \) be a generic closed polygon in 3-space such that, for some center \( O \), the radial projection of \( P \) in a plane is a convex polygon. Then \( P \) admits at least 4 flattening points.

This theorem is a polygonal version of a well-known Arnold’s 4 flattening points theorem for smooth spatial curves (see [1]). For other smooth 4 flattening points theorems, see [7].

2. Locally Convex Polygons and Parallel Vector Fields

For any discrete function \( g(i) \), we shall use the notation

\[
g'(i + \frac{1}{2}) = g(i + 1) - g(i), \quad g''(i) = g'(i + \frac{1}{2}) - g'(i - \frac{1}{2}),
\]

and so on.

2.1. Basic definitions

Consider a closed polygon \( X \) in 3-space with nodes \( X(i) \), \( 1 \leq i \leq n \). Let \( \alpha = \alpha(X) \) be defined by

\[
\alpha(i) = [X(i - 1) - O, X(i) - O, X(i + 1) - O].
\]  

(2.1)

We say that \( X \) is locally convex with respect to \( O \) if \( \alpha(i) > 0 \), for any \( i \in \mathbb{Z} \). Along this section we shall assume that the polygons are locally convex with respect to the origin.

Consider a vector field \( U \) along \( X \) given by \( U(i) \), \( 1 \leq i \leq n \). Define \( \beta = \beta(X,U) \) by

\[
\beta(i + \frac{1}{2}) = [X(i), X(i + 1), U(i)].
\]  

(2.2)

We say that \( U \) is transversal to \( X \) if \( \beta(i + \frac{1}{2}) > 0 \), for any \( i \in \mathbb{Z} \).

A transversal vector field \( U \) to \( X \) is called parallel if

\[
U'(i + \frac{1}{2}) = -b(i + \frac{1}{2})X'(i + \frac{1}{2}),
\]  

(2.3)
for certain scalar function \( b = b(X, U) \). We say that an edge \((i + \frac{1}{2})\) is a vertex of \((X, U)\) if
\[
b'(i) \cdot b'(i + 1) < 0. \tag{2.4}
\]

### 2.2. A characterization of flattening points

Recall that \( \Delta = \Delta(X) \) is defined by
\[
\Delta(i + \frac{1}{2}) = [X(i + 2) - X(i + 1), X(i + 1) - X(i), X(i) - X(i - 1)], \tag{2.5}
\]
and the node \( i \) of \( X \) is a flattening point if \( \Delta(i + \frac{1}{2}) \cdot \Delta(i - \frac{1}{2}) < 0 \).

For each \( i \), take \( \mu(i) = \mu(X, U)(i) \) such that \( \mu(i)X(i) + U(i) \) belongs to the osculating plane, i.e.,
\[
[X'(i + \frac{1}{2}), X''(i), \mu(i)X(i) + U(i)] = 0. \tag{2.6}
\]

**Lemma 2.1.** We can write
\[
\mu(i)X(i) + U(i) = AX'(i - \frac{1}{2}) + BX'(i + \frac{1}{2}), \tag{2.7}
\]
where \( \alpha(i)A = -\beta(i + \frac{1}{2}) \).

**Proof.** If follows from Equation (2.6) that Equation (2.7) holds and
\[
A [X'(i - \frac{1}{2}), X'(i + \frac{1}{2}), X(i)] = [\mu(i)X(i) + U(i), X'(i + \frac{1}{2}), X(i)],
\]
thus proving the lemma. \( \square \)

**Proposition 2.2.** We have that
\[
\beta(i + \frac{1}{2})\Delta(i + \frac{1}{2}) = \mu'(i + \frac{1}{2})\alpha(i)\alpha(i + 1).
\]
Thus, the node \( i \) is a flattening point for \( X \) if and only if
\[
\mu'(i - \frac{1}{2}) \cdot \mu'(i + \frac{1}{2}) < 0,
\]
where \( \mu = \mu(X, U) \) and \( U \) is any transversal parallel vector field along \( X \).

**Proof.** Take the difference of Equation (2.6) at \( i \) and \( i + 1 \) to obtain, after some manipulations, the following relation:
\[
[X'(i + \frac{1}{2}), -X'''(i + \frac{1}{2}), \mu(i)X(i) + U(i)] = \mu'(i + \frac{1}{2}) [X'(i + \frac{1}{2}), X''(i + 1), X(i)].
\]
Now from Equation (2.7) we obtain
\[
\frac{\beta(i + \frac{1}{2})}{\alpha(i)} \Delta(i + \frac{1}{2}) = \mu'(i + \frac{1}{2})\alpha(i + 1),
\]
thus proving the proposition. \( \square \)
3. Duality

3.1. Definition and properties

The general notion of dual centroaffine immersions can be found in [11, N9], [5], (see also [2]). We describe here a version for polygons.

Since \([X(i), X(i+1), U(i)] > 0\), one can define a polygon \(Y\) and a vector field \(V\) along \(Y\) uniquely by the following conditions:

\[
Y(i + \frac{1}{2}) \cdot X'(i + \frac{1}{2}) = 0; \quad Y(i + \frac{1}{2}) \cdot U(i) = 1; \quad Y(i + \frac{1}{2}) \cdot X(i) = 0, \tag{3.1}
\]

and

\[
V(i + \frac{1}{2}) \cdot X'(i + \frac{1}{2}) = 0; \quad V(i + \frac{1}{2}) \cdot U(i) = 0; \quad V(i + \frac{1}{2}) \cdot X(i) = 1. \tag{3.2}
\]

We say that \((Y, V)\) is the dual pair of \((X, U)\). The following lemma is straightforward:

**Lemma 3.1.** Consider a locally convex polygon \(X\) with a parallel transversal vector field \(U\) and denote by \((Y, V)\) its dual pair. Then

\[
Y(i + \frac{1}{2}) \cdot U(i + 1) = 1; \quad Y(i + \frac{1}{2}) \cdot X(i + 1) = 0, \tag{3.3}
\]

and

\[
V(i + \frac{1}{2}) \cdot U(i + 1) = 0; \quad V(i + \frac{1}{2}) \cdot X(i + 1) = 1. \tag{3.4}
\]

Moreover, \((X, U)\) is the dual pair of \((Y, V)\).

Recall that

\[
\alpha(Y)(i + \frac{1}{2}) = [Y(i - \frac{1}{2}), Y(i + \frac{1}{2}), Y(i + \frac{3}{2})]
\]

and

\[
\beta(Y, V)(i) = [Y(i - \frac{1}{2}), Y(i + \frac{1}{2}), V(i + \frac{1}{2})].
\]

**Lemma 3.2.** We have that

\[
\beta(Y, V)(i) = \frac{\alpha(i)}{\beta(i - \frac{1}{2})\beta(i + \frac{1}{2})}, \quad \alpha(Y)(i + \frac{1}{2}) = \frac{\alpha(i)\alpha(i + 1)}{\beta(i - \frac{1}{2})\beta(i + \frac{1}{2})\beta(i + \frac{3}{2})},
\]

where \(\alpha = \alpha(X)\) and \(\beta = \beta(X, U)\). We conclude that \(Y\) is locally convex and that \(V\) is transversal.

**Proof.** Write

\[
\beta(i + \frac{1}{2})Y(i + \frac{1}{2}) = X(i) \times X(i + 1), \quad \beta(i - \frac{1}{2})Y(i - \frac{1}{2}) = X(i - 1) \times X(i),
\]

Taking the vector product of both equations we obtain

\[
\alpha(i)X(i) = \beta(i - \frac{1}{2})\beta(i + \frac{1}{2})Y(i - \frac{1}{2}) \times Y(i + \frac{1}{2}).
\]

Now take the dot product with \(V(i + \frac{1}{2})\) to obtain the first formula. By duality, we can write

\[
\alpha(Y)(i + \frac{1}{2}) = \beta(Y, V)(i)\beta(Y, V)(i + 1)\beta(i + \frac{1}{2}).
\]

Now use the first formula to obtain the second one. \(\square\)
Lemma 3.3. The transversal vector field $V$ is parallel and

$$V'(i) = \mu(i)Y'(i),$$

where $\mu = \mu(X,U)$. We conclude that $b(Y,V) = -\mu(X,U)$.

Proof. Observe that $V'(i)$ is orthogonal to $U(i)$ and to $X(i)$ and the same occurs with $Y'(i)$. Thus we conclude that $V'(i)$ is parallel to $Y'(i)$, and we write $V'(i) = c(i)Y'(i)$. We claim that $c = \mu(X,U)$. In fact, substituting

$$\beta(i + \frac{1}{2})Y(i + \frac{1}{2}) = X(i) \times X(i + 1), \quad \beta(i + \frac{1}{2})V(i + \frac{1}{2}) = X'(i + \frac{1}{2}) \times U(i),$$

in Equation (2.6) we obtain

$$\mu(i)Y(i + \frac{1}{2}) \cdot X''(i) - V(i + \frac{1}{2}) \cdot X''(i) = 0.$$

Integrating by parts we get

$$\mu(i)Y'(i) \cdot X'(i - \frac{1}{2}) - V'(i) \cdot X'(i - \frac{1}{2}) = 0,$$

which implies that $(c(i) - \mu(i))Y'(i) \cdot X'(i - \frac{1}{2}) = 0$. Since

$$Y'(i) \cdot X'(i - \frac{1}{2}) = Y(i + \frac{1}{2}) \cdot X'(i - \frac{1}{2}) = -\frac{\alpha(i)}{\beta(i + \frac{1}{2})} \neq 0,$$

the lemma is proved. \(\square\)

Proposition 3.4. The edge $i$ is a vertex of $(Y,V)$ if and only if it the node $i$ is a flattening of $(X,U)$.

Proof. From Lemma 3.3, the edge $i$ of $(Y,V)$ is a vertex if and only if

$$\mu'(i - \frac{1}{2}) \cdot \mu'(i + \frac{1}{2}) < 0.$$ 

By Proposition 2.2 this condition is equivalent to the node $i$ of $(X,U)$ being a flattening. \(\square\)

Proposition 3.5. Four consecutive nodes of $X$ are coplanar if and only if the corresponding three normal lines of $(Y,V)$ meet at a point.

Proof. From Lemma 3.3 three consecutive lines of $(Y,V)$ at $(i - \frac{1}{2})$, $(i + \frac{1}{2})$ and $(i + \frac{3}{2})$, are concurrent if and only if $\mu(i) = \mu(i + 1)$, where $\mu = \mu(X,U)$. From Proposition 2.2 this is equivalent to $\Delta(i + \frac{1}{2}) = 0$, thus proving the proposition. \(\square\)
3.2. Planar polygons and the affine cylindrical pedal

In this section we describe a class of dual polygons which are particularly important in the proof of Theorem 1.1, namely, the duality between a planar polygon with a parallel transversal vector field and its affine cylindrical pedal.

Consider a locally convex planar polygon $y$ whose nodes will be denoted $y(i + \frac{1}{2})$, where $y(i + \frac{1}{2}) = y(i + \frac{1}{2} + kn)$, $k \in \mathbb{Z}$. Let $v$ be a transversal planar vector field that is parallel, i.e., we can write

$$v'(i) = -b(i)y'(i), \quad i \in \mathbb{Z},$$

for some scalar function $b = b(y, v)$. The lines $y + tv$, $t \in \mathbb{R}$, are called the normal lines of the pair $(y, v)$.

The lifting of $(y, v)$ is the pair $(Y, V)$ given by

$$Y(i + \frac{1}{2}) = (y(i + \frac{1}{2}), 1), \quad V(i + \frac{1}{2}) = (v(i + \frac{1}{2}), 0).$$

Observe that $V$ is parallel along $Y$ and that $b(Y, V) = b(y, v)$.

The affine cylindrical pedal of $(y, v)$ is defined by

$$X(i) = (x(i), z(i)),$$

where $x$ denotes the co-normal vector field of $(y, v)$, i.e.,

$$x(i) \cdot y'(i) = 0, \quad x(i) \cdot v(i + \frac{1}{2}) = 1,$$

(see [3]). It is easy to verify that the constant vector field $E = (0, 0, 1)$ is transversal to $X$ and $(X, E)$ is dual to $(Y, V)$.

The following proposition says that if, conversely, we start with a spatial polygon transversal to $E = (0, 0, 1)$, then it is necessarily the affine cylindrical pedal of some planar pair $(y, v)$.

**Proposition 3.6.** Assume that $X(i) = (x(i), z(i))$ is a locally convex spatial polygon transversal to $E = (0, 0, 1)$. Then $X$ is the affine cylindrical pedal of some planar parallel pair $(y, v)$ with $y$ locally convex. Moreover, if $X(i) = \lambda(i) (\gamma(i), 1)$, with $\lambda(i) > 0$ and $\gamma(i)$ convex containing $(0, 0)$ in its interior, then $y(i + \frac{1}{2})$ is convex (see Figure 2).

**Proof.** Denote by $(Y, V)$ the dual of $(X, E)$. By Lemma 3.3, $\mu(Y, V)(i) = 0$ and thus Proposition 2.2 implies that $\Delta(Y)(i) = 0$. We conclude that $Y$ is planar and $V(i)$ belongs to this plane. Thus $(Y, V)$ is the lifting of some planar pair $(y, v)$.

To prove the second assertion, recall that a locally convex planar polygon $y$ is convex if and only if its index is 1. If the index of $y$ were greater than 1, then the index of its co-normal vector field $x$ would also be greater than 1. On the other hand, by the convexity of $\gamma$, the polygon $x$ intersects each ray from $(0, 0)$ at most once, which is a contradiction. □
4. Proof of Theorem 1.1

Consider a locally convex polygon $P$ in 3-space whose projection in a plane with respect to a center $O$ is a convex planar curve. We may assume that $O = (0,0,0) \in \mathbb{R}^3$ and that the plane of projection is $z = 1$. Thus there exist $\lambda(i) > 0$ such that

$$P(i) = \lambda(i)(\gamma(i), 1),$$

where $\gamma(i)$ is a convex planar polygon. We can also assume, w.l.o.g., that $(0,0)$ is contained in the interior of $\gamma$. This implies that $E = (0,0,1)$ is a transversal vector field along $P$.

We can thus apply the results of Section 3.2 to $X = P$. Denoting by $(Y,V)$ the dual pair of $(P,E)$, Proposition 3.6 says that we can write $Y = (y,1)$ and $V = (v,0)$, for some planar convex polygon $y$ and parallel planar vector field $v$ along $y$. Moreover, $P = (p,z)$ is the affine pedal of $(y,v)$, i.e., $p$ is the co-normal of $(y,v)$ and

$$z(i) = p(i) \cdot y(i + \frac{1}{2}).$$

We claim that vertices of $(Y,V)$ correspond to vertices of $(y,v)$ in the sense of [6]. In fact, we can write

$$v'(i) = -b(i)y'(i),$$

where $b = b(Y,V)$ is defined in Equation (2.3). This equation implies that $v$ is exact with respect to $y$ and that the edge $(i)$ is a vertex of $(y,v)$ if and only if $b'(i - \frac{1}{2}) \cdot b'(i + \frac{1}{2}) < 0$ ([6]). By Equation (2.4), this is equivalent to the edge $(i)$ being a vertex of $(Y,V)$.

The vector field $v$ is called generic for $y$ in [6] if no 3 consecutive normal lines $y(i + \frac{1}{2}) + tv(i + \frac{1}{2})$ intersect at a point. This is equivalent to say that no 3 consecutive lines $Y(i + \frac{1}{2}) + tV(i + \frac{1}{2})$ intersect at a point. By Proposition 3.5, this is equivalent to the condition that no 4 consecutive points of $P$ are coplanar, which in fact means that $P$ is generic.

We are now in position to use the following theorem, proved in [6]:

Figure 2. In the left, the polygon $X = (x,z)$, its convex radial projection and the starred planar polygon $x$. In the right, the dual pair $(y,v)$. 
Theorem 4.1. Assume that $y$ is a planar convex polygon and that the generic transversal vector field $v$ is exact. Then the pair $(y, v)$ admits at least 4 vertices.

From this theorem, we conclude that $(y, v)$, and hence $(Y, V)$, admits at least 4 vertices. By Proposition 3.4 this implies that $P$ admits at least 4 flattenings, thus completing the proof of Theorem 1.1.

References

[1] V.I.Arnold: *On the number of flattening points of space curves*, Amer.Math.Soc.Transl. 171, 11-22, 1995.
[2] M.Craizer and R.A.Garcia: *Centro-affine codimension 2 immersions: Umbilical and inflection points*, [arXiv:1811.07331](https://arxiv.org/abs/1811.07331).
[3] M.Craizer and S.Pesco: *Affine geometry of equal-volume polygons in 3-space*, Comp.Aided Geom.Design, 57, 44-56, 2017.
[4] K.Nomizu and T.Sasaki: *Affine Differential Geometry*, Cambridge University Press, 1994.
[5] K.Nomizu and T.Sasaki: *Centroaffine immersions of codimension two and projective hypersurface theory*, Nagoya Math. J., 132, 63-90, 1993.
[6] S.Tabachnikov: *A Four Vertex Theorem for Polygons*, Amer.Math.Monthly, 107(9), 830-833, 2000.
[7] R.Uribe-Vargas: *On 4-flattening theorems and the curves of Carathéodory, Barner and Segre*, J.Geometry 77, 184-192, 2003.

Marcos Craizer  
Departamento de Matemática- PUC-Rio  
Rio de Janeiro, RJ, Brasil  
e-mail: craizer@puc-rio.br

Sinesio Pesco  
Departamento de Matemática- PUC-Rio  
Rio de Janeiro, RJ, Brasil  
e-mail: Sinesio@puc-rio.br