Gradient Flow for the Willmore Functional in Riemannian Manifolds of bounded Geometry

Dissertation zur Erlangung des Doktorgrades der Fakultät für Mathematik und Physik der Albert-Ludwigs-Universität Freiburg im Breisgau

vorgelegt von Florian Link

betreut durch Prof. Dr. Ernst Kuwert

August 2013
Abstract

We consider the $L^2$ gradient flow for the Willmore functional in Riemannian manifolds of bounded geometry. In the euclidean case E. Kuwert and R. Schätzle [Gradient flow for the Willmore functional, Comm. Anal. Geom., 10: 307-339, 2002] established a lower bound of a smooth solution of such a flow, which depends only on how much the curvature of the initial surface is concentrated in space. In a second joint work [The Willmore flow with small initial energy, J. Differential Geom., 57: 409-441, 2001] the aforementioned authors proved that a suitable blow-up converges to a nonumbilic (compact or noncompact) Willmore surface. In the lecture notes of the first author [The Willmore Functional, unpublished lecture notes, ETH Zürich, 2007] the blow-up analysis was refined. In the present work we intend to generalize the results mentioned above to the Riemannian setting.
# Contents

Introduction iii

1 Foundations 1
   1.1 Notation and conventions ........................................ 1
   1.2 Basic facts ...................................................... 8

2 Lifespan Theorem 17
   2.1 The Willmore flow ................................................... 17
   2.2 Evolution equations .................................................. 19
   2.3 Energy- and integral estimates ..................................... 22
   2.4 Sobolev inequalities for Riemannian Manifolds .................. 30
   2.5 Interpolation of lower-order terms ................................. 34
       Interior estimates I .................................................. 45
   2.6 Estimating the lifespan ............................................ 48

3 Blow-up of singularities 55
   3.1 Monotonicity formulas for Riemannian Manifolds ................ 55
   3.2 Interior estimates II ............................................... 60
   3.3 Blow-up of singularities ........................................... 66

Appendix 77
   A.1 Coordinate estimates ................................................ 77
   A.2 Bounds on the metric in special coordinates ..................... 85
   A.3 Construction of cutoff functions and a partition of unity ....... 86
   A.4 Interpolation inequalities ......................................... 90

Bibliography 96
Introduction

In 2002, E. Kuwert and R. Schätzle considered in their work *Gradient Flow for the Willmore Functional* [13] two-dimensional compact immersed surfaces $\Sigma$ in $\mathbb{R}^n$ moving by the gradient of the Willmore functional $f \mapsto W_\circ(f) = \int_{\Sigma} |A^\circ|^2 d\mu$, that is, solutions of $\partial_t f = -\operatorname{grad}_{L^2} W_\circ$ with $f|_{t=0} = f_0$, called Willmore flow. Here, $A^\circ$ is the trace-free part of the second fundamental form and $\mu$ is the induced area measure on $\Sigma$. The authors gave a lower bound on the lifespan of a smooth solution, which depends only on how much the curvature of the initial surface is concentrated in space. Moreover, they showed that the curvature concentrates in space if a finite time singularity develops. In a second joint work [14] Kuwert and Schätzle continued studying those singularities in greater detail by setting up a blow-up procedure. More precisely, they proved apart from other results that a suitable blow-up converges to a nonumbilic Willmore surface. In his lecture notes [15], Kuwert not only gave a broad summary of the above subjects and other works, but also performed a somewhat refined analysis concerning the behaviour of the flow near an assumed singularity.

The aim of this thesis is, roughly speaking, to generalize [13], the blow-up procedure in [14] and parts of [15] in the sense that we study the Willmore functional and the corresponding Willmore flow on Riemannian manifolds $(M,g)$ of bounded geometry as the target. That is, manifolds with $W^{k,\infty}$ bounds on the Riemannian curvature tensor and with a strictly positive injectivity radius. We denote by $H$ and $A$ the mean curvature vector and second fundamental form respectively. In $\mathbb{R}^n$ with the standard metric tensor the respective gradient flows of the functionals induced by the energies $W_H(f) = \frac{1}{2} \int_{\Sigma} |H|^2 d\mu$, $W_A(f) = \frac{1}{2} \int_{\Sigma} |A|^2 d\mu$ and $W_\circ$ are all equivalent since they differ only by a topological constant. This is in general no longer true for non-flat targets as it is described in Section 2.1 in detail. We decided to study $W_H$, that is, the flow $\partial_t f = -\operatorname{grad}_{L^2} W_H$, where the $L^2$ gradient refers to the metric tensor $g$ on $M$.

In the first chapter we give a detailed overview of the notation used in this work. In particular, we adopted the so-called star notation $A*B$ for tensors $A$ and $B$ used for example in [9], [11] and [13] to describe nonlinearities in the evolution equations to be developed. In our setting of a non-flat ambient manifold it is suitable to extend such a notation for terms involving the Riemannian curvature tensor as it was used in [6] for the study of the evolution of elastic curves in Riemannian manifolds. Later on, we collect basic facts such as classical differential geometric identities as well as variational formulas for various geometric objects as in [13].
Analogously to the approach in [13] we prove a lower bound for the maximal time span of smooth existence of the flow. Chapter two establishes this lower bound in four steps: First, we compute the evolution of derivatives of the curvature
\[ \nabla_{\partial_t}(\nabla^m A) + \Delta^2(\nabla^m A) = P^m_{3i}(A) + P^m_5(A) + Q^{m+2,1}_R, \]
where \( P^k = \sum_{i_1 + \ldots + i_l = k} \nabla_{i_1}A \ldots \nabla_{i_l}A \) denotes a universal sum of terms depending multilinearly on derivatives of the curvature (see Proposition 2.4, [13]). The \( Q \)-terms basically denote universal sums of terms depending multilinearly on derivatives of the curvature and the Riemannian curvature tensor \( D^r \). The precise algebraic structure is not being used here. Second, this information will then be transformed into localized \( L^2 \)-type integral estimates (cf. Lemma 2.7). To further estimate the right hand side, i.e. the nonlinearities in (0.1), we use interpolation and absorption techniques. To do so, it is essential to assume that the concentration of the curvature is locally small. Third, to interpolate the higher nonlinearities we need to generalize a Sobolev inequality originally due to Michael and Simon [21] for the case \( M = \mathbb{R}^n \). Up to a little worsening, we prove a generalization of the aforementioned Sobolev inequality for isometrically immersed manifolds of any codimension provided the Ricci tensor of \((M, g)\) is bounded in \( L^\infty \) and the injectivity radius is strictly positive (see Theorem 2.8). In a last step, we employ a multiplicative Sobolev inequality, which is likewise based on the Sobolev inequality above, to obtain interior \( W^{k,\infty} \)-estimates for the curvature still assuming small curvature concentration (see Proposition 2.16). Controlling the growth of the maximal local curvature concentration then allows us to bound from below the first time this smallness condition is violated.

To state the main theorem of chapter two, that is, the lifespan estimate, we want to introduce some preliminary definitions. First, we want to define the following non-local quantity used to control the maximal local energy concentration.

**Definition 0.1** Let \( \Sigma \) be a closed manifold, \((M, g)\) be an open or closed Riemannian manifold and \( I \subset \mathbb{R} \) be an interval. Let further \( f_t : (\Sigma, \tilde{g}_t) \to (M, g) \) be the one-parameter family of isometric immersions induced from \( f \in C^2(\Sigma \times I, M) \), where \( f_t := f(\cdot, t) \). If \( A_{f_t} \) denotes the second fundamental form of \( f_t \), and \( \mu_{f_t} \) the induced area measure of \((\Sigma, \tilde{g}_t)\), we let
\[ \chi_f(r, t) := \sup_{p \in M} \int_{f_t^{-1}(B^r(p))} |A_{f_t}|^2d\mu_{f_t}, \]
where \( B^r(p) \subset M \) denotes the geodesic ball of radius \( r \) and centre \( p \).

We further define
\[ W(f_0) := W_{H,g}(f_0) := \frac{1}{2} \int_\Sigma |H_{f_0}|^2d\mu_{f_0} \]
to be the Willmore energy of an isometric $C^2$-immersion $f_0 : (\Sigma^2, \tilde{g}_0) \to (M^n, g)$. Let further $R$ be the Riemannian curvature tensor induced by the Levi-Cività connection of $(M, g)$, and $\text{inj}(M, g)$ the injectivity radius of $(M, g)$. We call $(M, g)$ a Riemannian manifold of bounded geometry of order $k$, if

$$\sum_{i=0}^{k} \|D^i R\|_{L^\infty(M, g)} + \text{inj}(M, g)^{-1} < \infty.$$ 

Clearly, this is automatic (for any $k \in \mathbb{N}_0$) in case $\Sigma$ happens to be compact. For a more detailed introduction to the notation, we refer to section 1.1. We prove the following theorem (see section 2.6).

**Theorem 0.2** (Lifespan estimate for the Willmore flow in Riemannian manifolds). *Given an isometric $C^{4+\alpha}$-immersion $f_0 : (\Sigma, \tilde{g}) \to (M, g)$ of a closed surface into a Riemannian $n$-manifold of bounded geometry (of order 5), we let $f : \Sigma \times [0, T) \to (M, g)$ be a maximal Willmore flow with initial datum $f_0$. For $\varrho > 0$ satisfying

$$\varrho \left( \sum_{i=0}^{2} \|D^i R\|_{L^\infty(M, g)} + \text{inj}^{-1}(M, g) \right) < c(n)$$

let

$$t^+(\varrho) := \sup \{ t \geq 0 : \chi(\varrho, \cdot) < \varepsilon_0^2 \text{ on } [0, t) \},$$

where $\varepsilon_0 > 0$ and $c > 0$ are small universal constants depending only on $n$. Then either $T = t^+(\varrho) = \infty$, or for a small constant $C = C(n) > 0$

$$T > t^+(\varrho) \geq C \varrho^4 \log \frac{C \varepsilon_0^2}{\chi(\varrho, 0) + \varrho^4 \|D R\|_{L^\infty(M, g)}^2 (\mu_{f_0}(\Sigma) + \varrho^2 W(f_0))}.$$ 

In the third chapter we aim to study singularities of the Willmore flow in Riemannian manifolds of bounded geometry. Numerical examples of Mayer and Simonett [20] indicate the existence of finite time singularities in euclidean space. In the same geometric setting Blatt [1] has shown analytically that singularities may occur at finite or infinite time. However, to our knowledge there is no analytical evidence for the existence of finite time singularities. As has been done in [14] and [15], we perform a blow-up procedure at an assumed singularity. Using the result of the lifespan estimate we show that the rescaled flows (sub-)converge to a static Willmore surface, that is, a static critical point of $W$. For this, we use a compactness theorem that goes back to Langer [17] and has then been generalized by Breuning [2]. It turns out to be an essential assumption that the area is uniformly bounded along the flow. Namely, to obtain compactness there are basically two conditions to be fulfilled. First, the curvature has to be bounded in $W^{k, \infty}$.
and second, a mass-density estimate has to be obeyed. To satisfy the first condition, we prove a second version of interior $W^{k,\infty}$-estimates for the curvature similar to those of Kuwert and Schätzle [14] using a localization in time (see Lemma 3.3). The second version allows to perform a blow-up at an assumed singularity at infinity because the first version (see Proposition 2.16) depends on initial curvature estimates in contrast to the second depending on uniform area estimates. The latter dependence is due to an extra area term in the above mentioned $L^2$-estimates calculated in the second chapter (cf. also the Remark on page 12). To satisfy the second condition of the above compactness theorem we establish a mass-density estimate for immersed surfaces in manifolds of bounded geometry of order one, at least for radii that are small enough depending on the geometry of the ambient space. For large radii we were again forced to make the assumption that the area is bounded so that the large scale estimate is then trivially fulfilled.

We now want to formulate the main theorem of the third chapter. To prove this Theorem, a building block is to make the assumption that for a maximal Willmore flow the total area is uniformly bounded by a number $\mathcal{M}_f < \infty$ as mentioned above. Fortunately, this requirement is satisfied for finite singularities and for ambient manifolds of strictly negative curvature (see the Remark below Theorem 0.3).

**Theorem 0.3** (Existence of a blow-up). Let $f : \Sigma \times [0, T) \to (M, g)$ be a maximal Willmore flow on a closed surface $\Sigma$ into a Riemannian manifold of bounded geometry (of order 15) with the property that the total area of $(\Sigma, \tilde{g}(t))$ is uniformly bounded on $(0, T)$, that is,

$$\mathcal{M}_f := \sup_{t \in (0, T)} \mu_f(\Sigma) < \infty.$$ 

Assume that the Willmore flow concentrates at time $T \in (0, \infty]$ in the sense that

$$\varepsilon_T^2 = \lim_{\varrho \searrow 0} \left( \limsup_{t \nearrow T} \chi(\varrho, t) \right) > 0.$$ 

(0.4)

Then there exist sequences $t_j \nearrow T$ and $r_j \searrow 0$ such that the rescaled flows

$$f_j : (\Sigma, \tilde{g}_j) \times [-r_j^{-4} t_j, r_j^{-4}(T - t_j)) \to (M, g_j), \quad f_{j,t}(p) := f_j(p, t) := f(p, t_j + r_j^4 t),$$

(0.5)

where $g_j := r_j^{-2} g$ and $\tilde{g}_j(t) = (f_{j,t})^*(g_j)$, converge after reparametrization locally on $\hat{\Sigma} \times \mathbb{R}$ to a static solution given by a properly immersed Willmore surface $\hat{F}_0 : \hat{\Sigma} \to \mathbb{R}^n$ with

$$\int_{\hat{F}_0^{-1}(B_1(0))} |A_{\hat{F}_0}|^2 d\mu_{\hat{F}_0} \geq c(n) > 0.$$ 

(0.6)

Moreover

$$\varepsilon_T^2 \geq \int_\Sigma |A_{\hat{F}_0}|^2 d\mu_{\hat{F}_0} \geq \inf \left\{ \int |A_f|^2 d\mu_f : f \in C_n, \ f \text{ is not a union of planes} \right\} > 0,$$

(0.7)
where $C_n$ is the class of all properly immersed Willmore surfaces in $\mathbb{R}^n$.

Additionally, adopting the terminology from the mean curvature flow, a finite time singularity of a Willmore flow of a compact surface into a Riemannian manifold of bounded geometry (of order 15) is always of type II.

For a detailed formulation of the convergence result we refer to page 73.

**Remark:** The total area $\mu_f(\Sigma)$ is uniformly bounded on $[0, T)$, if

- $T < \infty$ is finite. In this case, it is $\mu_f(\Sigma) \leq \sqrt{2T} W(f_0) + \mu_{f_0}(\Sigma)$.
- the sectional curvature $K^M$ of $(M, g)$ is uniformly negative, i.e. $K^M \leq -\hat{\kappa}^2 < 0$. In this case, it is $\mu_f(\Sigma) \leq \frac{1}{2} W(f_0) - 4\pi \chi(\Sigma)$, where $\chi(\Sigma)$ denotes the Euler-characteristic of $\Sigma$.

These area estimates hold for any ambient smooth Riemannian manifold $(M, g)$ not necessarily of bounded geometry as we prove in Corollary 3.2.

Finally, in the appendix we have outsourced lengthy computations in coordinates and have stated well known results concerning harmonic coordinates for convenience of the reader. Moreover, the interpolation inequalities in [13] could almost be carried over to the Riemannian setting up to minor modifications.

---

**Acknowledgement**

First, I would like to express my grateful thanks to Prof. Dr. E. Kuwert for his help, support and not least because of his confidence and patience he brought to me when drawing up my diploma and doctoral thesis. Further gratitude is dedicated to all members of the working group Geometric Analysis in Freiburg for the fruitful discussions and the pleasant working atmosphere. Also, I would like to thank Dr. Maria Busl, Javier Sabio, PhD, and Felix Klock, PhD, for their help finding orthographical and typographical errors. Last but not least, I want to thank my girlfriend, writing a doctoral thesis in sports science herself, for her love and understanding she brought to me.

From January 2007 to December 2010 I was member of the DFG-Forschergruppe Nonlinear Partial Differential Equations: Theoretical and Numerical Analysis. I would like to thank the DFG for the support.
1 Foundations

1.1 Notation and conventions

In this section we introduce to the notation and conventions used in the present work. For further reference concerning algebraic facts of Differential Geometry we recommend [Z], or any other good textbook on this topic.

Vector bundles and sections.

A manifold is understood to be a second countable Hausdorff topological space together with a smooth ($C^\infty$) structure. Let $B, \Sigma, M, N$ be manifolds and $I \subset \mathbb{R}$ be a (time) interval. Let further $\xi = (E, \pi_E, B, \mathbb{R}^k)$ and $\eta = (F, \pi_F, B, \mathbb{R}^m)$ be vector bundles with total spaces $E$ and $F$, footprint projections $\pi_E$ and $\pi_F$, common Base $B$, and typical fibres $\mathbb{R}^k$ and $\mathbb{R}^m$ respectively. For simplicity, we use the symbol $E$ to refer to the bundle $\xi$ etc. if there is no ambiguity. We let $E_p := \pi^{-1}_E(p)$, and with $\xi^\ast$ we denote the r-times contravariant and s-times covariant tensor product bundle induced by $\xi$.

Given an $m$-dimensional manifold $\Sigma^m$, we denote with $T_\Sigma = (T\Sigma, \pi_\Sigma, \Sigma, \mathbb{R}^m)$ and $T_\Sigma^\ast = (T\ast\Sigma, \pi_\Sigma, \Sigma, \mathbb{R}^m)$ the tangent and cotangent bundle respectively, and let $T_p\Sigma := (T\Sigma)_p$ and $T^\ast_p\Sigma := (T\Sigma)^\ast_p$. If $f : \Sigma \rightarrow B$ is smooth, we denote by $f^\ast\xi$ the pullback bundle of $\xi$ along $f$ with total space $\{(p, v) \in M \times E : v \in E_{f(p)}\}$.

With $\Gamma^k(\xi) := \{\sigma \in C^k(B, E) \text{ with } \pi_E \circ \sigma = Id_B\}$ we denote the $C^k(B)$-module of $C^k$-sections in $\xi$. We abbreviate $\Gamma(\xi) := \Gamma^\infty(\xi)$ and denote with $\Gamma^k(\xi)$ the class of sections $\sigma \in \Gamma^k(\xi)$ with compact support. A $p$-times $C^\infty(B)$-multilinear (not necessarily alternating) map $\Gamma(T_B) \times \ldots \times \Gamma(T_B) \rightarrow \Gamma(\xi)$ is called a $\xi$-valued $p$-form. If $\sigma \in \Gamma(\xi)$ is a section, we let $f^\ast\sigma \in \Gamma(f^\ast\xi)$, defined by $(f^\ast\sigma) := (id_m, \sigma \circ f)$, be the pullback of $\sigma$, where $f^\ast\sigma$ and $\sigma \circ f$ will be identified. If $B = \Sigma \times I$ we write $\sigma_t := j^*_t\sigma$ for $\sigma \in \Gamma(\xi)$, where $j_t : \Sigma \rightarrow \Sigma \times I$ is the inclusion map $j_t(x) \mapsto (x, t)$ opposite $t$. For arbitrary but fixed $t$, a section $\sigma \in \Gamma(\xi)$ may always be considered as a section $\sigma_t \in \Gamma(j^*_t\xi)$ and vice versa provided it is smooth. Occasionally, the index $t$ will be omitted.

Clearly, there exists an isomorphism $\mathcal{T}_{\Sigma \times I} \simeq \mathcal{T}_{\Sigma} \times \mathcal{T}_I$. To obtain a similar decomposition for $\Gamma(\mathcal{T}_{\Sigma \times I})$ we remark that also $\mathcal{T}_{\Sigma \times I} \simeq \pi^*_\Sigma \mathcal{T}_\Sigma \oplus \pi^*_I \mathcal{T}_I$ and $\Gamma(\mathcal{T}_{\Sigma \times I}) \simeq \Gamma(\pi^*_\Sigma \mathcal{T}_\Sigma \oplus \pi^*_I \mathcal{T}_I) \simeq \Gamma(\pi^*_\Sigma \mathcal{T}_\Sigma) \oplus \Gamma(\pi^*_I \mathcal{T}_I)$, where $\pi_\Sigma$ and $\pi_I$ denote the projections onto the first and second factor of $\Sigma \times I$ respectively. Also, we may consider $\Gamma(\mathcal{T}_\Sigma) \subset \Gamma(\pi^*_\Sigma \mathcal{T}_\Sigma) \subset \Gamma(\mathcal{T}_{\Sigma \times I})$. For example,
for any \( V \in \Gamma(\mathcal{T}_{\Sigma \times I}) \) one can locally write \( V = x^i \partial_i + \sigma \partial_t \) where \( x^i, \sigma \in C^\infty(\Sigma \times I) \), and \( \{\partial_i\} \) and \( \{\partial_t\} \) are the respective product coordinate frames of \( \Sigma \) and \( I \) respectively.

### 1 Foundations

Riemannian metrics, scalar product and Riemannian distance function.

If \( (\xi, g_{\xi}) \) and \( (\eta, g_{\eta}) \) are Riemannian vector bundles, we always understand the tensor product bundle \( \xi \otimes \eta \) to be equipped with the product metric \( g_{\xi \otimes \eta} \) defined by

\[
g_{\xi \otimes \eta}|_p (z \otimes w, z' \otimes w') := g_{\xi}|_p(z, z') g_{\eta}|_p(w, w')
\]

for \( p \in B, z, z' \in E_p, w, w' \in F_p \). For a Riemannian vector bundle \( (\xi, g_{\xi}) \) and \( T \in \Gamma(\xi) \) we have the pointwise norms \( |T|^2 := g_{\xi}(T, T) \). Here and in many other formulas we omit the dependence on the geometric structure (e.g. the metric \( g_{\xi} \)) when there is no ambiguity. If \( f : (\Sigma, \bar{g}) \to (M, g) \) and \( T \in \Gamma(T\Sigma \otimes f^*(TM)) \) we also write \( |T|^2_f := |T|^2_{\bar{g} g} = T^a T^b \bar{g}^{ij} g_{\alpha \beta} \circ f \) to refer to the metrics.

If the basis of \( \xi \) is a Riemannian manifold \( (B, h) \), we let for \( S, T \in \Gamma^0(\xi) \), and \( U \subset B \) measurable

\[
(S, T)_{L^2(\mu_h)} := \int_\Sigma g_{\xi}(S, T) \, d\mu_h \quad \text{and} \quad ||T||^2_{2, U} := \int_U g_{\xi}(S, T) \, d\mu_h, \tag{1.1}
\]

where \( \mu_h \) is denotes the induced area measure on \( B \). Further for \( U \subset B \)

\[
||T||_{\infty, U} := \sup_U g_{\xi}(T, T)^{1/2}
\]

provided the right-hand side is finite. We write \( d_g(p, q) \) for the Riemannian distance induced by \( g \) between \( p, q \in (M, g) \).

Let \( \Sigma \) be a manifold, \( (M, g) \) be Riemannian manifold, a time interval \( I \) and a differentiable map \( f : \Sigma \times I \to M \). If \( f_t : \Sigma \to M \), where \( f_t(x) := f(x, t) \), is an immersion for all \( t \in I \), we define

\[
\bar{g}_t := f_t^* g \quad \text{for all} \quad t \in I \tag{1.2}
\]

making \( f_t : (\Sigma, \bar{g}_t) \to (M, g) \) to an isometric immersion. Here, \( f_t^* g \in \Gamma(T^{0,2}\Sigma) \) is the pullback of the tensor field \( g \in \Gamma(T^{0,2}M) \) defined by

\[
(f_t^* g)(X, Y) := g \circ f_t(Df_t \cdot X, Df_t \cdot Y)
\]

for all \( X, Y \in \Gamma(T\Sigma) \). More generally, we let

\[
\bar{g} := f^* g |_{\pi^*_\Sigma T\Sigma \oplus \pi^*_\Sigma T\Sigma \oplus \delta |_{\pi^*_I T_I \oplus \pi^*_I T_I}} \in \Gamma(T^{0,2}_{\Sigma \times I}),
\]

i.e. \( \bar{g}(\cdot, t)|_{T^{0,2}\Sigma} = \bar{g}_t \).
Connections, adjoint map, projectors, and normal bundle.

For vector bundles \((\xi, D')\) and \((\eta, D'')\) with connections \(D'\) and \(D''\) respectively, we always understand the product \(\xi \otimes \eta\) to be equipped with the product connection \(D^\otimes\) uniquely determined by

\[
D^\otimes_X (\sigma \otimes \tau) = (D'X\sigma) \otimes \tau + \sigma \otimes (D''X\tau)
\]

for all \(X \in \Gamma(TB), \sigma \in \Gamma(\xi)\) and \(\tau \in \Gamma(\eta)\). In this work, various connections will be denoted with the same symbol. However, it will be clear from the context which connection is meant. For \(f : \Sigma \times I \to (M, D^M)\) we let \(D := f^*D^M : \Gamma(f^*(TM)) \to \Gamma(T^*(\Sigma \times I) \otimes f^*(TM))\) the pullback connection with respect to \(f\), and write \(\nabla := P^\perp D\) for the induced connection on the normal bundle \(N_f\). \(D\) is the uniquely determined connection on \(f^*(TM)\) such that \(D_X(f^*\sigma)|_{(x,t)} = ((x,t), (D_{Df\cdot x}\sigma)|_{f(x,t)})\) for all \(X \in T_x(\Sigma \times I)\) and \(\sigma \in \Gamma(TM)\). When a tensor product connection is used, the factor connections should guarantees the existence of a strong bundle map \(\Psi : \Sigma \times I \to (M, D^M)\) with respect to \(f\) and \(\sigma \in \Gamma(\xi)\) at time \(t\). We will use the connection \(\nabla\) on \(\mathcal{T}_{\Sigma \times I}\) uniquely determined by the Christoffel symbols \(\Gamma(x, t) := \Sigma\Gamma(x) \otimes I\Gamma(t)\). More concretely, in local (product-)coordinates we have for \(V = V^i \partial_{i_{\sigma\Sigma}} + \sigma \partial_{i_{\sigmaI}}\) and \(W = W^j \partial_{j_{\sigma\Sigma}} + \tau \partial_{j_{\sigmaI}}\)

\[
\nabla_V W = (VW^k + V^j W^k \Sigma \Gamma_{ij}^k_{\sigma\Sigma}) \partial_k_{\sigma\Sigma} + V \tau \partial_{j_{\sigmaI}}.
\]

Clearly, for \(f : (\Sigma \times I, \tilde{g}) \to (M, g)\) \(\varphi := Df|_{\pi^*_\Sigma(T\Sigma)} : \pi^*_\Sigma(T\Sigma) \to f^*(TM)\) is a strong bundle map and each of the bundles \(\pi^*_\Sigma(T\Sigma)\) and \(f^*(TM)\) is dual to itself with respect to the scalar products \(\tilde{g}\) and \(g_{\sigma\Sigma}\) respectively. The Riesz representation theorem guarantees the existence of a strong bundle map \(\Psi : \pi^*_\Sigma(T\Sigma) \hookrightarrow f^*(TM)\) with the property \(\tilde{g}(\Psi \tau, \tilde{X}) = g_{\sigma\Sigma}(\tau, \varphi \cdot \tilde{X})\) for all \(\tau \in f^*(TM)\) and \(\tilde{X} \in \mathcal{T}_{\Sigma \times I}\). Since \(\tilde{g}\) is non-degenerate, \(\Psi^*\) is uniquely determined and thus we may define \(\varphi^* := \Psi\) to be the dual (or, more precisely, the adjoint) of \(\varphi\). Since \(\varphi^*\) is a strong bundle map, this induces a homomorphism \(\varphi^* : \Gamma(f^*(TM)) \to \Gamma(\pi^*_\Sigma(T\Sigma))\). Moreover, we have the decomposition

\[
f^*(TM) = \text{Im} \varphi \oplus \ker \varphi^* =: T_f \oplus N_f,
\]

where \(T_f\) and \(N_f\) is the tangent and normal bundle respectively. Analogously, we also have \(f^*_f(TM) =: T_{f|_{\Sigma}} \oplus N_{f|_{\Sigma}}\). Further, we have the tangent and normal projectors \(P := \varphi \circ \varphi^* \in \mathcal{End}(f^*(TM))\) and \(P^\perp := \text{Id} - P\) and in particular, using that \(\varphi_t = Df_t\), we have that \(P_t = Df_t \circ (Df_t)^* \in \mathcal{End}(f^*_t(TM))\). Now for \(V \in N_{f|_{\Sigma}}\) we may consider \(\nabla V : \Gamma(\Sigma) \to \Gamma(f^*_t(TM))\) that analogously induces for any \(t \in I\) the adjoint map \((\nabla V)^* : \Gamma(\Sigma) \leftarrow \Gamma(f^*_t(TM))\) with the respective property \(\tilde{g}_t((\nabla V)^*\Psi_t, X) = g_{\sigma\Sigma}(\Psi_t, \nabla_X V)\) for any \(\Psi \in f^*(TM)\) and \(X \in \Gamma(\Sigma)\).
1 Foundations

With $\nabla^*$ we denote the formal adjoint of the operator $\nabla$ with respect to the scalar products defined in (1.1). More precisely, for $\phi \in \Gamma(T^{0,*1}\Sigma \otimes N_f)$ we let $\nabla^* T$ be the unique tensor field with

$$\int_M g(\nabla^* \phi, \psi) d\mu_g = \int_M g(\phi, \nabla \psi) d\mu_g$$

for all $\psi \in \Gamma(T^{0,*}\Sigma \otimes N_f)$. It is $\nabla^* \psi = -(\nabla e_i \psi)(e_i, \ldots)$. We define the (normal) Laplacian by $\Delta := -\nabla^* \nabla$, i.e. for $\psi$ as above it is $\Delta \psi = (\nabla^2 \psi)(e_i, e_i, \ldots)$. Terms involving operators have to be read from the right to the left, e.g. for a normal valued 1-form $\phi$ and $X, Y, Z \in \Gamma(T\Sigma)$ we write $\nabla_X \nabla_Y \phi(Z) = \nabla_X (\nabla_Y (\phi(Z)))$, $\nabla_X Df \cdot Y = \nabla_X (Df \cdot Y)$, or for $\xi \in \Gamma(TM)$ we write $D_X \xi \circ f = D_X (\xi \circ f) = (D\xi) \circ f \cdot Df \cdot X$.

Curvature.

Let $f$ be as in (1.2) and $(M, g, D)$ be equipped with the Levi-Civita connection $D$. We write $A_f := D^2 f \in \Gamma(T^{2,0}\Sigma \otimes N_f)$ and $H_f \in \Gamma(N_f)$ for the second fundamental form and mean curvature vector of $f$ respectively. Further we let $A_f^0 := A_f - \frac{1}{2} g \otimes H_f$ be the tracefree part of the bilinear form $A_f$. The occasional omission of the index $t$ should here and elsewhere be obvious. For a vector bundle $(\xi, D)$ with connection $D$ we define the curvature $F \in \Gamma(\Lambda^2(TM) \otimes \text{End}(E))$ of $(E, D)$ by

$$F(X, Y)\xi := D_X D_Y \xi - D_Y D_X \xi - D_{[X,Y]} \xi$$

for all $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(E)$. We then let $R$ be the curvature of $(TM, D)$, $R_f$ be the curvature of $(f^*TM, f^*D)$ and $R_{\Sigma \cdot I}$ be the curvature of $(T(\Sigma \times I), \nabla)$. Finally, we let $R_{\Sigma}$ and $R_{\Sigma}^0$ denote the curvature of $(N_f, \nabla)$, $(T^{0,1}\Sigma \otimes N_f, \nabla)$ respectively. Note, that the restriction of $R_{\Sigma \cdot I}|_{(x, t)}$ to $T_x \Sigma$ equals the curvature $R_{\Sigma}|_{(x, t)}$ of $(T\Sigma, \nabla)$. Last, we let $\tilde{K} := R_{\Sigma}(e_1, e_2, e_2, e_1)$ be the Gaussian curvature of $(\Sigma, \tilde{g})$ and let

$$K(T\Sigma) := R_{\circ f}(Df \cdot e_1, Df \cdot e_2, Df \cdot e_2, Df \cdot e_1)$$

be the sectional curvature of $T\Sigma$ in $M$, where again $\{e_i\}_{i=1,2}$ is a local $\tilde{g}_t$-orthonormal basis of $T\Sigma$. That $\tilde{K}$ and $K(T\Sigma)$ is well defined, i.e. the independence with respect to the chosen local orthonormal basis, follows from the orthogonality of the transformation map of two given orthonormal bases and the symmetries of the curvature tensor.

Miscellanea.

Here, we summarize the notion of various symbols and geometric quantities used in this work.
1.1 Notation and conventions

- All manifolds are understood to be second countable topological Hausdorff spaces together with a smooth structure. We only consider manifolds without boundary. For emphasis, a manifold without boundary is called open, if it is noncompact, and likewise closed, in case it is compact. Riemannian manifolds are understood to be complete with respect to the induced metric.

- \( \text{inj}(M,g) \): injectivity radius of \((M,g)\)

- \( \mu_\tilde{g} \): induced area-measure on \((\Sigma, \tilde{g})\)

- Zero indexed quantities such as \(A_0, f_0, |\cdot|_0, \mu_0\) etc. refer to time \(t = 0\)

- For a section \(X \in \Gamma(T\Sigma)\) we occasionally abbreviate \(\tilde{X} := Df \cdot X\)

- For a cutoff function \(\gamma\) on a manifold \(N\), we define the set \([\gamma > 0] := \{p \in N, \text{such that } \gamma(p) > 0\}\) and \([\gamma = 1]\) etc. analogously to \([\gamma > 0]\).

- We use Einstein’s convention, i.e. summation over repeated indices is used

- \(g \circ f\) is sometimes abbreviated by \(\langle \cdot, \cdot \rangle\)

- For \(\phi \in \Gamma(N_f)\) we define \(Q(A^\circ)\phi := A^\circ(e_i, e_j)\langle A^\circ(e_i, e_j), \phi \rangle\)

Let \(\Sigma\) be a manifold without boundary, \(N\) be a manifold and \(I \subset \mathbb{R}\) be an interval. Then \(f \in C^0(\Sigma \times I, M)\) is called locally proper if and only if

\[ f|_{[t_1, t_2]} : \Sigma \times [t_1, t_2] \to M \text{ is proper for any compact } [t_1, t_2] \subset I. \]

Analogously, \(\eta \in C^0(\Sigma \times I)\) has locally compact support if and only if

\[ \eta|_{[t_1, t_2]} : \Sigma \times [t_1, t_2] \to \mathbb{R} \text{ has compact support for any compact } [t_1, t_2] \subset I. \]

Clearly, each of the conditions is automatic if \(\Sigma\) is closed. If a function is bounded and locally proper then it obviously has locally compact support.

**Definition 1.1** (Parabolic Hölder spaces, see [16]).

Let \((x_1, t_1), (x_2, t_2) \in \mathbb{R}^k \times \mathbb{R}\). Then

\[ d_m((x_1, t_1), (x_2, t_2)) := \max\{|x_1 - x_2|, |t_1 - t_2|^\alpha\} \]

defines a metric on \(\mathbb{R}^k \times \mathbb{R}\). For \(G \subset \mathbb{R}^k \times \mathbb{R}\) and a function \(u : G \to \mathbb{R}^l\) we define the parabolic \(\alpha\)-Hölder coefficient

\[ [u]_{\alpha,m,G} := \sup \left\{ \frac{|u(p) - u(q)|}{d_m(p, q)^\alpha} : p, q \in G, \ p \neq q \right\}. \]

Further we define

\[ C^{2m,1}(G) := \{ u : G \to \mathbb{R}^l : u, Du, D^2u, \ldots, D^{2m}u, \partial_t u \in C^0(G) \} \]

and

\[ C^{2m,1,\alpha}(G) := \{ u \in C^{2m,1}(G) : [D^{2m}u]_{\alpha,m,G} + [\partial_t u]_{\alpha,m,G} < \infty \}. \]

For a definition of Hölder spaces on manifolds, see [25].
1 Foundations

Star- and Q-notation.

Here, we orientate towards the notation used in [13], [9] and [6]. For an isometric immersion \( f : (\Sigma^2, \tilde{g}) \to (M, g) \) and \( \eta, \psi \in \Gamma(T^{r,s}\Sigma \otimes (N_f)^{p,q}) \) we denote by

\[ \eta \ast \psi \]

normal- (or real-) valued sections such that with respect to any local orthonormal frame we have the form

\[
\left( \eta \ast \psi \right)^{\alpha,I}_{\beta,J} = C^{\alpha,\delta,\sigma,I,L,P}_{\beta,\gamma,\delta,\rho,J,K,R} \eta^{\gamma,K}_{\delta,L} \psi^{\rho,R}_{\sigma,P}
\]  

(1.3)

where \( \alpha, \beta, I, J, \ldots \) are multi-indices, and for any fixed term the coefficients \( C_{\cdots} \) are constants depending only on \( r, s, p, q \) and the dimension \( n \). With this definition one easily verifies that

\[
|\eta \ast \psi| \leq C(r, s, p, q, n)|\eta||\psi|
\]  

(1.4)

and

\[
\nabla(\eta \ast \psi) = (\nabla \eta) \ast \psi + \eta \ast (\nabla \psi),
\]  

(1.5)

where \( \nabla \) in each case stands for the respective connection. To check the pointwise identity (1.5) we may for any local orthonormal frame and any fixed point \( x \in \Sigma \) assume that the respective connection one-forms are equal to zero. Then (1.5) follows from the product rule for \( \eta^{\gamma,K}_{\delta,L} \psi^{\rho,R}_{\sigma,P} \).

Remarks:

a) Of course, this can be generalized to more than two factors.

b) From the way we have defined \( \eta \ast \psi \), it is not only clear that \( \eta \ast \psi = \psi \ast \eta \), but also that the indices can be interchanged within each multi-index.

c) Although the star-terms do not depend on other sections than stated, we sometimes omit the arguments to keep overview as long as everything is clear from the context.

As mentioned in the introduction, it is useful to introduce a notation to encode nonlinearities involving the Riemannian curvature tensor as in [6] for the study of elastic curves. It turned out that, roughly speaking, the necessary estimates are performed in the normal bundle and therefore make the split-up \( f^*(TM) = T_f \oplus N_f \). Consider the split up

\[
P^\perp R_{\phi}(D\eta, \tilde{X})\phi = P^\perp R_{\phi}(\nabla \eta, \tilde{X})\phi + P^\perp R_{\phi}(P^\perp D\eta, \tilde{X})\phi.
\]

If the tangent bundle has rank one as in [6], the last summand above clearly vanishes by symmetry properties of the curvature tensor. In our setting, where \( T\Sigma \) is of rank two the
1.1 Notation and conventions

notation unfortunately needs to be modified as follows:

Analogously to the above definition, we let for \( \zeta, \xi \in \Gamma(T^r \Sigma \otimes f^*(T^p q M)) \)

\[
\zeta \star \xi
\]

(1.6)

denote sections with the respective property (1.3). Analogously to the above we now have the estimate \( |\zeta \star \xi| \leq C(r, s, p, q, n) |\zeta||\xi| \) and \( D(\eta \star \psi) = (D \zeta) \star \xi + \zeta \star (D \xi) \). Here, \( f^*(TM) \) is understood to be equipped with the pull-back connection \( f^*D \) induced from \( (M, g, D) \) in contrast to the above definition where we consider \( N_f \) to be equipped with the normal connection \( \nabla = P^\bot D|_{N_f} \).

Examples: For \( X, Y, Z \in \Gamma(T \Sigma) \), and \( \zeta, \xi, \Psi \in \Gamma(f^*(TM)) \), and if \( \{e_i\}_{i=1,2} \) is a local \( \tilde{g} \)-orthonormal frame for \( T \Sigma \), we have

\[
\begin{align*}
& (i) \quad g \circ f(H, (\nabla A)(X, Y, Z)) = (A \star \nabla A)(X, Y, Z) \\
& (ii) \quad P^+ R \circ f(\zeta, \xi) \Psi = R \circ f(\zeta, \xi, \tilde{e}_i) \tilde{e}_i = (R \circ f \star Df \star Df)(\zeta, \xi, \Psi) \\
& (iii) \quad P^+ R \circ f(\tilde{X}, \tilde{Y}) \tilde{Z} \\
& \quad = R \circ f(\tilde{X}, \tilde{Y}) \tilde{Z} - R \circ f(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{e}_i) \tilde{e}_i \\
& \quad = (R \circ f \star Df \star Df \star Df + R \circ f \star Df \star Df \star Df \star Df \star Df)(X, Y, Z). (1.7)
\end{align*}
\]

With \( P^k_l(A) \) we denote universal linear combinations of terms of the form

\[
\nabla^{i_1} A \star \ldots \star \nabla^{i_l} A
\]

when \( |i| := i_1 + \ldots + i_l = k \). As usual, if the sum is empty, i.e. \( l = 0 \) then \( |i| := 0 \). With this definition one then easily verifies that

\[
\nabla P^k_l(A) = P^{k+1}_l(A). (1.8)
\]

It turns out to be useful (as (1.7) illustrates) to encode lower order terms involving the curvature tensor arising in the evolution equations. To do so, we denote by \( Q^k_{(m)} \) normal-(or real-) valued universal linear combinations of sections

\[
(D^r R) \circ f \star \nabla^{i_1} \ldots \star \nabla^{i_r} A \star t_N \star \ldots \star t_N \star Df \star \ldots \star Df
\]

(1.9)

when \( r + |i| + \nu = k + l, |i| \leq k \), and \( r \leq m \) in case the lower index \( m \) is given). Here, \( t_N : (N_f, \nabla) \to (f^*(TM), D) \) denotes the canonical injection. Analogously, we denote by \( Q^k_{(R \star R)} \) normal (or real) valued universal linear combinations of sections

\[
(D^r R) \circ f \star (D^s R) \circ f \star \nabla^{i_1} \ldots \star \nabla^{i_s} A \star t_N \star \ldots \star t_N \star Df \star \ldots \star Df
\]

(1.10)
when $r_1 + r_2 + |i| + \nu = k + l$ and $|i| \leq k$. For $Q^k_{R \star R}$ we additionally want to assume that $\nu \geq 1$ for technical reasons. To see what happens to the indices when differentiating covariantly we firstly note that for a normal valued n-form $\phi$ and adapted vector fields $X_i \in \Gamma(T\Sigma)$ we pointwise have

$$\begin{align*}
(D_X \phi)(X_1, \ldots, X_n) &= D_X \phi(X_1, \ldots, X_n) \\
&= \nabla_X \phi(X_1, \ldots, X_n) + \langle D_X \phi(X_1, \ldots, X_n), Df \cdot e_j \rangle Df \cdot e_j \\
&= (\nabla_X \phi)(X_1, \ldots, X_n) - \langle \phi(X_1, \ldots, X_n), A(X, e_i) \rangle Df \cdot e_i
\end{align*}$$

(1.11)

and thus we may write

$$D \phi = \nabla \phi + \phi \star A \star Df.$$  

(1.12)

Secondly, for $X \in \Gamma(T\Sigma)$ and $\phi \in \Gamma(N_f)$ we have $(D_{\iota_N})(X, \phi) = -\langle \iota_N(\phi), A(X, e_j) \rangle Df e_j$, i.e.

$$D_{\iota_N} = \iota_N \star A \star Df.$$ 

Thus we easily get, recalling that $D(Df) = D^2 f = A$,

$$\nabla Q^{0,0} = Q^{0,1} \text{ and } \nabla Q^{k,l}_{(m)} = Q^{k+1,l}_{(m+1)}.$$  

(1.13)

Examples: For (1.7) we may now write $P^\perp R \circ f(\tilde{X}, \tilde{Y}) \tilde{Z} = Q^{0,0}_{(0)}(X, Y, Z)$. Further, we have, for example, the simplifications $P^\perp R \circ f(\nabla^i A, \tilde{Y}) \tilde{Z} = Q^{i,1}_{0}(X, Y, Z)$ and

$$P^\perp R \circ f(H, Df \cdot) \phi = \left( (P^\perp R \circ f) \star A \star \iota_N \star Df \right) \star \phi$$

$$= \left( R \circ f \star A \star \iota_N \star Df + R \circ f \star A \star \iota_N \star Df \star Df \star Df \right) \star \phi$$

$$= Q^{0,1}_{0} \star \phi.$$ 

Recalling the definition, it is easy to see that instead of $Q^{k,l} \star A$ we can write $Q^{k,l} \star A = Q^{k,l+1}$.

Their, it is important to keep the estimate (1.4) and the differentiation rules (1.8) and (1.13) in mind.

1.2 Basic facts

Fundamental identities of differential geometry.

In this subsection, we summarize already known facts for convenience of the reader and derive easy facts used later on.
Lemma 1.2 Let \( f : (\Sigma^d, \bar{g}) \to (M^n, g) \) and \( I : (M^n, g) \to (\mathbb{R}^N, \delta_{\text{eucl}}) \) be isometric immersions. Then there holds the pointwise estimates

\[
|A_{I \circ f}| \leq c|A_I| \circ f + |A_f| \tag{1.14}
\]
and

\[
|H_{I \circ f}| \leq c|A_I| \circ f + |H_f|, \tag{1.15}
\]

where \( c = c(d, n) \) is a universal constant.

Proof: Differentiating \( D(I \circ f) = DI \circ f \) covariantly, we get for vector fields \( X, Y \in \Gamma(T\Sigma) \)

\[
A_{I \circ f}(X, Y) = D_{X,Y}^2 (I \circ f) = (D^2 I) \circ f(Df \cdot X, Df \cdot Y) + (DI) \circ f \cdot D^2 f(X, Y)
= A_I \circ f(Df \cdot X, Df \cdot Y) + (DI) \circ f \cdot A_f(X, Y). \tag{1.16}
\]

Since \( I \) is isometric, the first inequality follows since

\[
|A_{I \circ f}|^2 = |A_I \circ f(Df \cdot, Df \cdot)|^2 + |A_f|^2 \leq c|A_I|^2 \circ f |Df|^4 + |A_f|^2
\]
because \( |Df|^2 = d \). For the second, we choose a local \( \tilde{g} \)–orthonormal frame \( \{e_i\}_{1 \leq i \leq d} \) and obtain analogously

\[
|H_{I \circ f}|^2 = \sum_i |A_{I \circ f}(Df \cdot e_i, Df \cdot e_i)|^2 + |H_f|^2 \leq \sum_i |A_f|^2 \circ f |Df|^4 |e_i|^4 + |H_f|^2 \leq c|A_f|^2 \circ f + |H_f|^2.
\]

\[\blacksquare\]

For an arbitrary isometric immersion \( f : (\Sigma, \bar{g}) \to (M, g) \) and any vector fields \( X, Y, Z, W \in \Gamma(T\Sigma) \), \( \phi \in \Gamma(N_f) \) the following identities hold (see, e.g. [3])

\[
R(f)(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = R_S(X, Y, Z, W) - \langle A(X, W), A(Y, Z) \rangle + \langle A(X, Z), A(Y, W) \rangle \tag{1.17}
\]

\[
D_X(Df \cdot Y) = A(X, Y) + Df \cdot \nabla_X Y \tag{1.18}
\]

(Equations of Gauß)

\[
R_L(X, Y)\phi = P^\perp R_{f \circ L}(\bar{X}, \bar{Y})\phi + A(e_i, X)(A(e_i, Y), \phi) - A(e_i, Y)(A(e_i, X), \phi) \tag{1.19}
\]

(Ricci identity)

\[
P^\perp R_{f \circ L}(\bar{X}, \bar{Y})\bar{Z} = (\nabla A)(X, Y, Z) - (\nabla A)(Y, X, Z). \tag{1.20}
\]

(Equation of Mainardi-Codazzi)
The Gaussian curvature $\tilde{K}$ of $(\Sigma^2, \tilde{g})$, and the sectional curvature $K(T\Sigma)$ of $T\Sigma$ in $(M, g)$ are related as follows

$$\tilde{K} - K(T\Sigma) \overset{\text{(1.17)}}{=} \frac{1}{2} (|H|^2 - |A|^2).$$

(1.21)

$$\tilde{K} - K(T\Sigma) \overset{\text{(1.22)}}{=} \frac{1}{2} |A|^2 - |A^o|^2.$$ 

(1.22)

$$\tilde{K} - K(T\Sigma) \overset{\text{(1.23)}}{=} \frac{1}{4} |H|^2 - \frac{1}{2} |A_o|^2.$$ 

(1.23)

Namely, from the Gauß equation (1.17) it follows with $A_{ij} := A(e_i, e_j)$ that

$$\tilde{K} - K(T\Sigma) \overset{\text{(1.17)}}{=} R_{\Sigma}(e_1, e_2, e_2, e_1) - R_{\circ \Sigma f}(\tilde{e}_1, \tilde{e}_2, \tilde{e}_2, \tilde{e}_1)$$

$$\overset{\text{with}}{=} \langle A_{11}, A_{22} \rangle - |A_{12}|^2.$$ 

(1.21)

$$\tilde{K} - K(T\Sigma) \overset{\text{with}}{=} \frac{1}{2} (\langle A_{11} + A_{22}, A_{11} + A_{22} \rangle - (|A_{11}|^2 + 2|A_{12}|^2 + |A_{22}|^2)).$$ 

From the orthogonal decomposition $A = A^o \circ g_{\circ f}(\frac{1}{2} \tilde{g} \otimes H)$ in the trace and traceless part of the bilinear form $A$ we get

$$|A|^2 = |A_o|^2 + \frac{1}{4} |H|^2|\tilde{g}|^2 = |A_o|^2 + \frac{1}{2} |H|^2,$$

since $|\tilde{g}|^2 = \dim \Sigma = 2$. From this, (1.22) and (1.23) follow.

Since we have to interchange second covariant derivatives of normal l-forms, we need to know how the curvature of $(T^{0,l}\Sigma \otimes N_f, \nabla)$ is related to the curvature of $(N_f, \nabla)$ and $(T\Sigma, \nabla)$. We pointwise have for any $\phi \in \Gamma(T^{0,l}\Sigma \otimes N_f)$

$$(\nabla^2_{X,Y} \phi)(X_1, \ldots, X_l) \overset{\text{pointwise}}{=} \nabla_X (\nabla_Y \phi)(X_1, \ldots, X_l)$$

$$= \nabla_X \nabla_Y \phi(X_1, \ldots, X_l) - \sum_{k=1}^l \phi(X_1, \ldots, \nabla^2_{X,Y} X_k, \ldots, X_l)$$

$$= \nabla^2_{X,Y} \phi(X_1, \ldots, X_l) - \sum_{k=1}^l \phi(X_1, \ldots, \nabla^2_{X,Y} X_k, \ldots, X_l)$$

yielding

$$(R^{l}_{\perp}(X,Y) \phi)(X_1, \ldots, X_l)$$

$$= (\nabla^2_{X,Y} \phi)(X_1, \ldots, X_l) - (\nabla^2_{Y,X} \phi)(X_1, \ldots, X_l)$$

$$= R_{\perp}(X,Y) \phi(X_1, \ldots, X_l) - \sum_{k=1}^l \phi(X_1, \ldots, R_{\Sigma}(X,Y) X_k, \ldots, X_l).$$
1.2 Basic facts

Since
\[ \tilde{K} = K(T\Sigma) + \frac{1}{2}(|H|^2 - |A|^2) = Q^{0,0} + A A \]
and
\[ R\Sigma(X, Y)Z = \tilde{K}(\tilde{g}(Y, Z)X - \tilde{g}(X, Z)Y) , \]
we get by substitution
\[ \phi(X_1, \ldots, R\Sigma(X, Y)X_k, \ldots, X_l) \]
\[ = \tilde{K}(\tilde{g}(Y, X_k)\phi(X_1, \ldots, X_{k-1}, X, X_{k+1}, \ldots, X_l) \]
\[ - \tilde{g}(X, X_k)\phi(X_1, \ldots, X_{k-1}, Y, X_{k+1}, \ldots, X_l)) \]
\[ = (Q^{0,0}\ast\phi + A A\ast\phi)(X, Y, X_1, \ldots, X_l). \]

Putting things together, we get
\[ R^l_\perp(\cdot, \cdot)\phi = A A\ast\phi + Q^{0,0}\ast\phi + R^l_\perp(\cdot, \cdot)\phi(\cdot, \ldots, \cdot) \]
\[ = (Q^{0,0}\ast\phi + A A\ast\phi)(X_1, \ldots, X_l) . \] (1.24)

Lemma 1.3 For arbitrary \( \phi \in \Gamma(T^{0,1}\Sigma \otimes N_f) \) and \( l \geq 1 \) we have
\[ (\nabla\nabla^* - \nabla^*\nabla)\phi = -\nabla^* T + A A\ast\phi + Q^{0,0}\ast\phi , \]
where \( T(X_0, X_1, \ldots, X_l) := (\nabla\phi)(X_0, X_1, \ldots, X_l) - (\nabla\phi)(X_1, X_0, \ldots, X_l) . \)

Proof:
We pointwise compute for adapted vector fields
\[ ((\nabla\nabla^* - \nabla^*\nabla)\phi)(X_1, \ldots, X_l) \]
\[ = \nabla_{e_i}(\nabla\phi)(e_i, X_1, \ldots, X_l) - \nabla_{X_1}(\nabla\phi)(e_i, e_i, X_2, \ldots, X_l) \]
\[ = (\nabla^2_{e_i, e_i}\phi)(X_1, \ldots, X_l) - (\nabla^2_{e_i, X_1}\phi)(e_i, X_2, \ldots, X_l) \]
\[ + (\nabla^2_{e_i, X_1}\phi)(e_i, X_2, \ldots, X_l) - (\nabla^2_{X_1, e_i}\phi)(e_i, X_2, \ldots, X_l) \]
\[ = - (\nabla^* T)(X_1, \ldots, X_l) + (R^l_\perp(e_i, X_1)\phi)(e_i, X_2, \ldots, X_l) . \]
\[ \overset{(1.24)}{=} - (\nabla^* T + A A\ast\phi + Q^{0,0}\ast\phi)(X_1, \ldots, X_l) . \]

We need two special cases of the latter lemma. First, if we set \( \phi := A \) we get from Codazzi-Mainardi
\[ T(X, Y, Z) = (\nabla A)(X, Y, Z) - (\nabla A)(Y, X, Z) \]
\[ = P^\perp R_{gf}(\tilde{X}, \tilde{Y})\tilde{Z} \]
\[ = Q^{0,0}(X, Y, Z) . \]
1 Foundations

On the other hand, again using Codazzi-Mainardi we pointwise have

\[-(\nabla^* A)(Y) = (\nabla A)(e_i, e_i, Y) = \nabla_{e_i} A(e_i, Y) = \nabla_{e_i} A(Y, e_i) = (\nabla A)(e_i, Y, e_i) = (\nabla A)(Y, e_i, e_i) + P^2 R_{\phi}(\tilde{e}_i, \tilde{Y}) \tilde{e}_i\]

\[= \nabla_Y H + Q^{0.0}(Y). \quad (1.25)\]

From this we obtain pointwise

\[(\nabla \nabla^* A)(X, Y) = \nabla_X (\nabla^* A)(Y)\]

\[-(\nabla^* \nabla) = -\nabla_X \nabla_Y H - \nabla_X Q^{0.0}(Y)\]

\[= -\nabla^2_{XY} H + Q^{0.1}(X, Y). \quad (1.25)\]

Now since \(\nabla^* T = 1* \nabla T = 1* \nabla Q^{0.0} = Q^{0.1}\), Lemma 1.3 shows that

\[(\nabla \nabla^* - \nabla^* \nabla)A = A* A A + Q^{0.1} + Q^{0.0} A,\]

\[= P^0_3(A) + Q^{0.1}\]

since we can write \(Q^{0.0} A\) as \(Q^{0.1}\). Eventually we get a rough version of Simons’ identity

\[\Delta A = \nabla^2 H + P^0_3(A) + Q^{0.1}. \quad (1.26)\]

**Remark:** Following the computations thoroughly it turns out that the difference of \(\Delta A - \nabla^2 H\) contains an additive term with the Riemannian curvature tensor that does not contain the second fundamental form. This fact results in the existence of an extra area term when we interpolate the nonlinearities to establish \(L^2\) estimates (cf. Proposition 2.12 and Proposition 2.14).

Second, we want to substitute \(\phi\) by \(\nabla \phi\) in Lemma 1.3: To start with, we have

\[T(X_0, \ldots, X_l) = (\nabla^2_{X_0, X_1})\phi(X_2, \ldots, X_l) - (\nabla^2_{X_1, X_0})\phi(X_2, \ldots, X_l)\]

\[= (R^l_{X_0, X_1})\phi(X_2, \ldots, X_l)\]

\[= (A* A A + Q^{0.0} \phi)(X_0, \ldots, X_l). \quad (1.24)\]

Now we differentiate

\[\nabla^* T = 1* \nabla T = A* \nabla A A \phi + A* A A \nabla \phi + Q^{0.0} \nabla \phi + Q^{0.1} \phi\]

so that we again get from Lemma 1.3

\[(\Delta \nabla - \nabla \Delta) \phi = (\nabla \nabla^* - \nabla^* \nabla) \nabla \phi\]

\[= A* \nabla A A \phi + A* A A \nabla \phi + Q^{0.0} \nabla \phi + Q^{0.1} \phi. \quad (1.27)\]
Lemma 1.4 Let $f \in C^{2,1}(\Sigma \times (t_1, t_2), M)$ be a variation with normal velocity field $V := \partial_t f$. Then for $\xi \in \Gamma^2(f^*(TM)), \phi \in \Gamma^2(N_f)$ and time independent vector fields $X,Y,Z$ we have

\begin{align}
D_{\partial_t} Df \cdot X &= D_X V \\
R_f(\partial_t, X)\xi &= R_{\partial f}(V, \bar{X})\xi \\
D_{\partial_t} P &= -D_{\partial_t}(P^{\perp}) = Df \cdot (\nabla V)^* + \nabla V \cdot (Df)^* \\
PD_{\partial_t} \phi &= -Df \cdot (\nabla V)^* \phi \\
R_{\partial f}(\partial_t, X)\phi &= \langle A(X, e_i), \phi \rangle \nabla_{e_i} V - \langle \nabla_{e_i} V, \phi \rangle A(X, e_i) + P^{\perp} R_{\partial f}(V, \bar{X})\phi \\
(\nabla_{\partial_t} \bar{g})(X, Y) &= -2\langle A(X, Y), V \rangle \\
\partial_t (d\mu) &= -\langle H, V \rangle d\mu \\
R_{\Sigma \times f}(\partial_t, X)Y &= \nabla_{\partial_t} \nabla_X Y = -\langle (\nabla A)(X, Y, e_i), e_i \rangle V + \langle A(X, Y), \nabla_{e_i} V \rangle e_i \\
&\quad - \langle A(X, e_i), \nabla_X V \rangle e_i - \langle A(Y, e_i), \nabla_X V \rangle e_i - R_{\partial f}(\bar{Y}, \bar{e}_i, \bar{X}, V)e_i \\
(\nabla_{\partial_t} A)(X, Y) &= \nabla_{\partial_t}^2 V - \langle V, A(Y, e_i) \rangle A(X, e_i) + P^{\perp} R_{\partial f}(V, \bar{X})\bar{Y} \\
\nabla_{\partial_t} H &= \Delta V + Q(A^o)V + \frac{1}{2} \langle H, V \rangle + P^{\perp} R_{\partial f}(V, \bar{e}_i)\bar{e}_i.
\end{align}

Proof:

(1.28): By properties of the connection along $f$, we have

\begin{align}
D_{\partial_t} Df \cdot X - D_X Df \cdot \partial_t &= Df \cdot [\partial_t, X] \\
&= 0
\end{align}

since $X$ does not depend on time.

(1.29): Again by locality and since $[\partial_t, X] = 0$, we can compute

\begin{align}
R_f(\partial_t, X)\partial_t \circ f &= D_{\partial_t} D_X \partial_t \circ f - D_X D_{\partial_t} \partial_t \circ f - D_{[\partial_t, X]} \partial_t \circ f \\
&= D_{\partial_t}(D \partial_t \circ f) \cdot Df \cdot X - D_X (D \partial_t \circ f) \cdot V \\
&= R_{\partial f}(V, \bar{X})(\partial_t \circ f).
\end{align}

(1.30): At first, let $\psi \in \Gamma(N_f)$. Then we have for any vector field $X \in \Gamma(T\Sigma)$

\begin{align}
\langle D_{\partial_t} \psi, Df \cdot X \rangle &= \partial_t \langle \psi, Df \cdot X \rangle - \langle \psi, D_{\partial_t} Df \cdot X \rangle \\
&= \langle \psi, \nabla_X V \rangle \\
&= -\langle \psi, \nabla_{e_i} V \rangle \langle Df \cdot e_i, Df \cdot X \rangle
\end{align}
and thus
\[ PD\partial_t\psi = -Df \cdot \langle \psi, \nabla_e V \rangle e_i. \] (1.38)

Now let \( \xi \in \Gamma(f^*(TM)) \). Decomposing \( \xi = \psi + Df \cdot Y \) in its tangential and normal part, we can compute for arbitrary \( \phi \in \Gamma(N_f) \)
\[ \langle D\partial_t P^\perp \xi, \phi \rangle = \partial_t \langle P^\perp \xi, \phi \rangle - \langle P^\perp \xi, D\partial_t \phi \rangle \]
\[ = \partial_t \langle \xi, \phi \rangle - \langle P^\perp \xi, D\partial_t \phi \rangle \]
\[ = \langle D\partial_t \xi, \phi \rangle + \langle P\xi, D\partial_t \phi \rangle \]
\[ = \langle D\partial_t \xi, \phi \rangle + \langle P\xi, PD\partial_t \phi \rangle \]
\[ = \langle D\partial_t \xi, \phi \rangle - \langle P\xi, Df \cdot \phi, \nabla_e V \rangle e_i \]
\[ = \langle D\partial_t \xi, \phi \rangle - \langle \nabla_Y V, \phi \rangle \]
\[ = \langle P^\perp D\partial_t \xi - \nabla_Y V, \phi \rangle \]

On the other hand, we get from (1.28)
\[ \langle D\partial_t P^\perp \xi, Df \cdot X \rangle = \partial_t \langle P^\perp \xi, Df \cdot X \rangle - \langle P^\perp \xi, D\partial_t Df \cdot X \rangle \]
\[ = -\langle P^\perp \xi, \nabla_X V \rangle \]
\[ = -\langle \xi, \nabla_e V \rangle \langle Df \cdot e_i, Df \cdot X \rangle \]
\[ = \langle -Df \cdot \langle \xi, \nabla_e V \rangle e_i, Df \cdot X \rangle. \]

Therefore
\[ D\partial_t P^\perp \xi = P^\perp D\partial_t \xi - \nabla_Y V - Df \cdot \langle \xi, \nabla_e V \rangle e_i, \]

and thus, using \( (D\partial_t P^\perp) (\xi) = D\partial_t P^\perp \xi - P^\perp D\partial_t \xi \), the second identity follows. The first follows trivially from \( P + P^\perp = Id \).

\[ (1.31): \text{We have} \]
\[ PD\partial_t \phi = -(D\partial_t P) \phi \]
\[ = -Df \cdot (\nabla V)^* \phi - \nabla V \cdot (Df)^* \phi. \]
\[ = -Df \cdot (\nabla V)^* \phi. \]

\[ (1.32): \text{Again since } [\partial_t, X] = 0 \text{ we can compute} \]
\[ R_\perp(\partial_t, X) \phi = \nabla_{\partial_t} \nabla_X \phi - \nabla_X \nabla_{\partial_t} \phi - \nabla_{[\partial_t, X]} \phi \]
\[ = P^\perp D\partial_t P^\perp D\phi - P^\perp D_X P^\perp D\partial_t \phi \]
\[ = (1.31) \]
\[ P^\perp (D\partial_t P^\perp) D_X \phi + P^\perp D_X D\partial_t \phi - P^\perp D_X (D\partial_t \phi + Df \cdot (\nabla V)^* \phi) \]
\[ = (1.29), (1.30) \]
\[ -\nabla V \cdot (Df)^* (D_X \phi) + P^\perp R_{sf}(V, \tilde{X}) \phi - A(X, (\nabla V)^* \phi) \]
\[ = \langle \phi, A(X, e_i) \rangle \nabla_e V + P^\perp R_{sf}(V, \tilde{X}) \phi - \langle \phi, \nabla_e V \rangle A(X, e_i), \]
where we used $(\nabla V)^* \phi = \langle \phi, \nabla_{e_i} V \rangle e_i$ and
\[
(Df)^*(D_X \phi) \overset{(1.11)}{=} (Df)^*(\nabla_X \phi - \langle \phi, A(X, e_i) \rangle Df \cdot e_i)
\]
\[
= -\langle \phi, A(X, e_i) \rangle e_i
\]
in the last step.

(1.33): For the evolution of the metric we get since $X$ and $Y$ do not depend on time
\[
(\nabla_{\partial_t} \tilde{g})(X, Y) = \partial_i \tilde{g}(X, Y)
\]
\[
= \partial_i \langle Df \cdot X, Df \cdot Y \rangle
\]
\[
= \langle D_{\partial_t} Df \cdot X, Df \cdot Y \rangle + \langle Df \cdot X, D_{\partial_t} Df \cdot Y \rangle
\]
\[
\overset{(1.28)}{=} \langle D_X V, Df \cdot Y \rangle + \langle Df \cdot X, D_Y V \rangle
\]
\[
= X \langle V, Df \cdot Y \rangle - \langle V, A(X, Y) \rangle + Y \langle Df \cdot X, V \rangle - \langle A(X, Y), V \rangle
\]
\[
= -2 \langle A(X, Y), V \rangle.
\]

(1.34): Using the standard formula for the derivative of the determinant, i.e.
\[
\frac{d}{dt} \ln \det(\tilde{g}_{ij}) = \frac{1}{2} \tilde{g}^{ij} \partial_t \tilde{g}_{ij},
\]
we get
\[
\frac{d}{dt} \sqrt{\det(\tilde{g}_{ij})} = \frac{1}{2} \tilde{g}^{ij} \partial_t \tilde{g}_{ij} \sqrt{\det(\tilde{g}_{ij})}
\]
and thus, since $\partial_i \tilde{g}_{ij} = (\nabla_{\partial_t} \tilde{g})(\partial_i, \partial_j) \overset{(1.33)}{=} -2 \langle A_{ij}, V \rangle$, finally
\[
\frac{d}{dt} \sqrt{\det(\tilde{g}_{ij})} = -\langle H, V \rangle \sqrt{\det(\tilde{g}_{ij})}.
\]

(1.35): Since $\nabla_{\partial_t} \nabla_X Y$ is $C^\infty(\Sigma)$—linear in $X$ and $Y$, we may assume that $\nabla_X Y = \nabla_X e_i = 0$ at a fixed point and at a fixed time.
\[
Df \cdot \nabla_{\partial_t} \nabla_X Y = D_{\partial_t} Df \cdot \nabla_X Y - D^2 f(\partial_t, \nabla_X Y)
\]
\[
= D_{\partial_t} PD_X Df \cdot Y
\]
\[
= (D_{\partial_t} P)D_X Df \cdot Y + PD_{\partial_t} D_X Df \cdot Y
\]
\[
\overset{(1.29)}{=} (D_{\partial_t} P)A(X, Y) + PD_X D_{\partial_t} Df \cdot Y + PRf(V, \tilde{X}) \tilde{Y}
\]
\[
= Df \cdot (\nabla V)^* A(X, Y) + PD_X D_Y V + R \circ f(V, \tilde{X}, \tilde{Y}, e_i) Df \cdot e_i,
\]
where we used (1.28) and (1.30). Since $(\nabla V)^* (A(X, Y)) = \langle A(X, Y), \nabla_{e_i} V \rangle e_i$ and
\[
PD_X D_Y V \overset{(1.11)}{=} PD_X (\nabla_Y V - \langle V, A(Y, e_i) \rangle Df \cdot e_i)
\]
\[
\overset{(1.11)}{=} -\langle \nabla_Y V, A(X, e_i) \rangle Df \cdot e_i - X \langle V, A(Y, e_i) \rangle Df \cdot e_i
\]
\[
= -\langle \nabla_V V, A(X, e_i) \rangle Df \cdot e_i - \langle \nabla_X V, A(Y, e_i) \rangle Df \cdot e_i
\]
\[
- \langle V, (\nabla A)(X, Y, e_i) \rangle Df \cdot e_i,
\]
the claim follows by the injectivity of $Df$ on $T\Sigma$.

(1.36): Again assuming $\nabla_X Y = 0$ at a fixed point and a fixed time, this follows from

\[
(\nabla_{\partial_t} A)(X, Y) = \nabla_{\partial_t} A(X, Y) = P^\perp D_{\partial_t} (D_X Df \cdot Y - Df \cdot \nabla_X Y)
\]

\[
= P^\perp D_X D_{\partial_t} Df \cdot Y + P^\perp R_{\circ f}(V, \tilde{X})\tilde{Y}
\]

\[
(1.32) = \nabla^2_{X,Y} V + P^\perp D_X PD_Y V + P^\perp R_{\circ f}(V, \tilde{X})\tilde{Y}
\]

using

\[
P^\perp D_X PD_Y V = -P^\perp D_X (\langle V, A(Y, e_i) \rangle Df \cdot e_i)
\]

\[
= -\langle V, A(Y, e_i) \rangle A(X, e_i).
\]

(1.37): From (1.33) we get in local coordinates

\[
0 = \partial_t (g_{ij}g^{jk}) = -2\langle A_{ij}, V \rangle g^{jk} + g_{ij}\partial_t g^{jk}
\]

\[
\Rightarrow \quad \partial_t g^{lk} = 2\langle A_{ij}, V \rangle g^{jk} g^{il}
\]

\[
\Rightarrow \quad (\partial_t g^{lk}) A_{lk} = 2\langle A_{ij}, V \rangle g^{jk} g^{il} A_{lk}
\]

\[
= 2Q(A)V.
\]

From (1.36) we thus get

\[
\nabla_{\partial_t} H = \nabla_{\partial_t} (g^{lk} A_{lk})
\]

\[
= \partial_t g^{lk} A_{lk} + g^{lk} \nabla_{\partial_t} A_{lk}
\]

\[
= 2Q(A)V + \Delta V - Q(A)V + P^\perp R_{\circ f}(V, \tilde{e}_i)\tilde{e}_i
\]

\[
= \Delta V + Q(A^o)V + \frac{1}{2} \langle H, V \rangle H + P^\perp R_{\circ f}(V, \tilde{e}_i)\tilde{e}_i,
\]

since $Q(A)V = Q(A^o)V + \frac{1}{2} \langle H, V \rangle H$.  

\[\]
2 Lifespan Theorem

2.1 The Willmore flow

Flat Situation.

In [13] the Willmore energy of a closed, isometrically immersed surface \( f : (\Sigma, \bar{g}) \rightarrow (\mathbb{R}^n, \delta_{eucl}) \) with induced area measure \( d\mu \) is defined by

\[
W_\circ(f) = \int_\Sigma |A^0|^2 d\mu. \tag{2.1}
\]

Where \( A^0 = A - \frac{1}{2} \bar{g} \otimes H \) is the tracefree part of the second fundamental form \( A \). The Gauß equations and Gauß-Bonnet imply

\[
W_\circ(f) = W_A(f) - 2\pi \chi(\Sigma) = W_H(f) - 4\pi \chi(\Sigma), \tag{2.2}
\]

where \( W_A(f) = \frac{1}{2} \int_\Sigma |A|^2 d\mu \) and \( W_H(f) = \frac{1}{2} \int_\Sigma |H|^2 d\mu \). Thus in this flat situation, the \( L^2 \) gradient flows of all these functionals reduce to the same since they only differ by a topological constant. In general, this is not true if the target manifold is curved.

Situation of a curved ambient manifold.

Now let \( f : (\Sigma^2, \bar{g}) \rightarrow (M^n, g) \) be an isometrically \( C^2 \)-immersed closed surface into an \( n \)-dimensional Riemannian Manifold. From (1.22) and (1.23) we have

\[
|A^0|^2 = \frac{1}{2} |A|^2 + K(T\Sigma) - \tilde{K} = \frac{1}{2} |H|^2 + 2K(T\Sigma) - 2\tilde{K}.
\]

Integration and using Gauß-Bonnet yields

\[
W_{\circ,g}(f) = W_{A,g}(f) + \int_\Sigma K(T\Sigma) d\mu - 2\pi \chi(\Sigma)
= W_{H,g}(f) + 2\int_\Sigma K(T\Sigma) d\mu - 4\pi \chi(\Sigma) \tag{2.3}
\]

We decided to study \( W_{H,g} \) in this work.
L²-gradient flow for the Willmore functional.

Let $f$ be as in 2.1. We define the Willmore energy (with respect to $H$ and $g$ in $f$) as

$$W(f) := W_{H,g}(f) := \frac{1}{2} \int_{\Sigma} |H|^2 d\mu.$$  

(2.4)

In the next Lemma we compute the $L^2$ gradient of this functional for convenience of the reader.

Lemma 2.1 ([19], Lemma 1.2). The $L^2$ gradient for the Willmore functional (in $f$) is given by

$$W(f) := \text{grad}_{L^2} W(f) = \Delta H + Q(A^0) H + P^\perp R_{\circ f}(H, \tilde{e}_i) \tilde{e}_i.$$  

(2.5)

Proof: Let $f: \Sigma \times (-\varepsilon, \varepsilon) \to M$ be a variation of $f$ with normal velocity-field $\partial_t f =: \phi$. Then for the first variation of $W$ of $f$ in direction $\phi$ we have

$$\frac{d}{d\varepsilon} W(f \varepsilon) \big|_{\varepsilon = 0} = \frac{d}{d\varepsilon} \int_{\Sigma} \frac{1}{2} |H \varepsilon|^2 d\mu \big|_{\varepsilon = 0} = 0$$

$$\overset{(1.34)}{=} \int_{\Sigma} \langle H, \nabla_{\partial_t} H \varepsilon \rangle \big|_{\varepsilon = 0} d\mu - \frac{1}{2} \int_{\Sigma} |H|^2 \langle H, \phi \rangle d\mu$$

$$\overset{(1.37)}{=} \int_{\Sigma} \langle H, \Delta \phi + Q(A^0) \phi + \frac{1}{2} \langle H, \phi \rangle H + P^\perp R_{\circ f}(\phi, \tilde{e}_i) \tilde{e}_i \rangle d\mu$$

$$- \frac{1}{2} \int_{\Sigma} |H|^2 \langle H, \phi \rangle d\mu.$$

Since $\Delta$ and $Q(A^0)$ are self-adjoint with respect to $\int \langle \cdot, \cdot \rangle d\mu$, we further get

$$= \int_{\Sigma} \langle \Delta H + Q(A^0) H, \phi \rangle d\mu + \int_{\Sigma} R_{\circ f}(\phi, \tilde{e}_i, \tilde{e}_i, H) d\mu$$

$$= \int_{\Sigma} \langle \Delta H + Q(A^0) H + P^\perp R_{\circ f}(H, \tilde{e}_i) \tilde{e}_i, \phi \rangle d\mu$$

$$= \langle \delta H + Q(A^0) H + P^\perp R_{\circ f}(H, \tilde{e}_i) \tilde{e}_i, \phi \rangle_{L^2}$$

$$= \langle \text{grad}_{L^2} W(f), \phi \rangle_{L^2},$$

by definition of the $L^2$ gradient.

Remark: The reason why we may restrict to normal variations is the following: For tangential variations there always exists a flow on $\Sigma$ that generates this tangential vector...
field. It is plausible that this flow only causes a reparametrization of the surface so that the Willmore energy leaves unchanged due to its geometric invariance of the underlying parametrization. Strictly speaking, one can show that

\[ W(f \circ \psi) = W(f) \]

for all \( C^2 \)-diffeomorphisms \( \psi : \Sigma \to \Sigma \) (see [15]).

\[ \square \]

Theorem 2.2 (Existence and uniqueness). Let \( f_0 : (\Sigma^2, \tilde{g}) \to (M^n, g) \) be an isometric \( C^{4+\alpha} \)-immersion of a closed surface into a Riemannian manifold. Then the initial value problem

\[
\partial_t f = -W(f) \quad \text{on } \Sigma \times (0, T) \\
\bigg|_{t=0} = f_0
\]

has a unique solution \( f \in C^4, 1, \alpha(\Sigma \times [0, T], M) \) on a maximal time interval \( [0, T) \) where \( 0 < T \leq \infty \). Moreover, the restriction of \( f \) to \( \Sigma \times (0, T) \) is smooth.

\[ \blacksquare \]

The solution to the above initial value problem will be called Willmore flow for short.

2.2 Evolution equations

Lemma 2.3 Let \( \phi \in \Gamma(T^{0,l+1} \Sigma \otimes N_f) \) be a normal \((l-1)\)-Form along a variation \( f : \Sigma \times I \to M \) with normal velocity field \( \partial_t f =: V \). Then for \( \psi := \nabla \phi \) we have

\[
\nabla_{\partial_t} \psi + \Delta^2 \psi = \nabla Y + \sum_{i+j+k=3} (\nabla^i A \ast \nabla^j A \ast \nabla^k \phi) + A \ast \nabla V \ast \phi + V \ast \nabla A \ast \phi \\
+ Q^{0,0} \ast V \ast \phi + \sum_{j+k=1} \Delta (Q^{0,j} \ast \nabla^k \phi) + \sum_{j+k=1} Q^{0,j} \ast \nabla^{k+2} \phi.
\]

Proof: Let \( X_1, \ldots, X_l \) be time independent vector fields such that \( \nabla X_k = 0 \) in a fixed point \((x_0, t_0)\). Then we pointwise have

\[
(\nabla_{\partial_t} \psi)(X_1, \ldots, X_l) = \nabla_{\partial_t}(\nabla X_1 \phi)(X_2, \ldots, X_l) \\
= \nabla_{\partial_t} \nabla X_1 \phi(X_2, \ldots, X_l) - \nabla_{\partial_t} \sum_{k=2}^l \phi(X_2, \ldots, \nabla X_1 X_k, \ldots, X_l) \\
= \nabla_{\partial_t} \nabla X_1 \phi(X_2, \ldots, X_l) - \sum_{k=2}^l \phi(X_2, \ldots, \nabla_{\partial_t} \nabla X_1 X_k, \ldots, X_l).
\]

On the one hand we get for the first term, using (1.32),

\[
\nabla_{\partial_t} \nabla X_1 \phi(X_2, \ldots, X_l) \\
= \nabla X_1 \nabla_{\partial_t} \phi(X_2, \ldots, X_l) + (A \ast \nabla V \ast \phi + Q^{0,0} \ast V \ast \phi)(X_1, \ldots, X_l) \\
= (\nabla (\nabla_{\partial_t} \phi))(X_1, \ldots, X_l) + (A \ast \nabla V \ast \phi + Q^{0,0} \ast V \ast \phi)(X_1, \ldots, X_l),
\]
2 Lifespan Theorem

recalling that \( P_{\perp} R_{\perp f}(V, D f) = ((P_{\perp} R_{\perp f}) \ast D f \ast t_N \ast t_N) \ast V \ast \phi = Q_{0,0} \ast V \ast \phi \) and \( P_{\perp} R_{\perp f} = R_{\perp f} + R_{\perp f} \ast D f \ast D f. \) For the second term, we have from (1.35)

\[
\phi(X_2, \ldots, \nabla_{\partial_i} \nabla X_1, X_k, \ldots, X_l) = (V \ast \nabla A \ast \phi + A \ast \nabla V \ast \phi)(X_1, \ldots, X_l).
\]

Putting things together we get

\[
\nabla_{\partial_i} \psi + \Delta^2 \psi - \nabla Y = \Delta^2(\nabla \phi) - \nabla(\Delta^2 \phi) + A \ast \nabla V \ast \phi + V \ast \nabla A \ast \phi + Q_{0,0} \ast V \ast \phi. \tag{2.6}
\]

On the other hand, we get from (1.27)

\[
(\Delta \nabla - \nabla \Delta) \phi = A \ast \nabla A \ast \Delta \phi + A \ast A \ast \nabla \Delta \phi + Q_{0,0} \ast \nabla \phi + Q_{0,1} \ast \phi \tag{2.7}
\]

\[
\Rightarrow \Delta((\Delta \nabla - \nabla \Delta) \phi) = \sum_{i+j+k=3} (\nabla^i A \ast \nabla^j A \ast \nabla^k \phi) + \sum_{i+j+k=1} \Delta(Q_{0,j} \ast \nabla^k \phi), \tag{2.8}
\]

where we used the differentiation rule (1.13) for the Q-Terms. Substituting \( \phi \) by \( \Delta \phi \) in (2.7) implies

\[
(\Delta \nabla - \nabla \Delta)(\Delta \phi) = A \ast \nabla A \ast \Delta \phi + A \ast A \ast \nabla \Delta \phi + Q_{0,0} \ast \nabla \phi + Q_{0,1} \ast \Delta \phi
\]

\[
= A \ast \nabla A \ast \nabla^2 \phi + A \ast A \ast \nabla^3 \phi + \sum_{j+k=1} Q_{0,j} \ast \nabla^{k+2} \phi, \tag{2.9}
\]

since terms like \( \nabla \Delta \phi \) can be written as a contraction of \( \tilde{g}^{-1} \otimes \nabla^3 \phi. \) Adding up (2.8) and (2.9) yields

\[
\Delta^2(\nabla \phi) - \nabla(\Delta^2 \phi) = \Delta((\Delta \nabla - \nabla \Delta)(\Delta \phi)) + (\Delta \nabla - \nabla \Delta)(\Delta \phi)
\]

\[
= \sum_{i+j+k=3} (\nabla^i A \ast \nabla^j A \ast \nabla^k \phi) + \sum_{j+k=1} \Delta(Q_{0,j} \ast \nabla^k \phi) + \sum_{j+k=1} Q_{0,j} \ast \nabla^{k+2} \phi.
\]

Substituting this into (2.6) yields the claim. \(\blacksquare\)

**Lemma 2.4** For the Willmore flow \( \partial_t f = V = -W(f) \) we have the evolution equations

\[
\nabla_{\partial_t} (\nabla^m A) + \Delta^2 (\nabla^m A) \tag{2.10}
\]

\[
= P_3^{m+2}(A) + P_5^m(A) + \nabla^m \Delta Q_{0,1} + \Delta Q_{m,1} + \nabla Q_{m+1,1} + Q_{m+1} + Q_{m+1} + Q_{R+R}^{m+1}
\]

\[
= P_3^{m+2}(A) + P_5^m(A) + Q_{m+2} + Q_{R+R}^{m+1}
\]

for any \( m \in \mathbb{N}_0. \)

**Proof:** (Induction over \( m \in \mathbb{N}_0). \) Substituting \( \phi \) by \( \nabla H \) or \( H \) in (2.7) we get

\[
(\Delta \nabla - \nabla \Delta)(\nabla H) = A \ast \nabla A \ast \nabla H + A \ast A \ast \nabla^2 H + \sum_{j+k=1} Q_{0,j} \ast \nabla^k \nabla H
\]

\[
= P_3^2(A) + Q_{1}^{2,1}
\]
2.2 Evolution equations

and

\[ \nabla ((\Delta \nabla - \nabla \Delta) H) = \nabla (A \ast \nabla A \ast H + A \ast A \ast \nabla H + \sum_{j+k=1} Q^{0,j} \ast \nabla^{k} H) \]

\[ = \nabla (P_{1}^{1}(A) + Q_{1}^{1,1}) \]

\[ = P_{3}^{2} + \nabla Q_{1}^{1,1}. \]

Combining yields

\[ \Delta \nabla^{2} H - \nabla^{2} \Delta H = (\Delta \nabla - \nabla \Delta)(\nabla H) + \nabla ((\Delta \nabla - \nabla \Delta) H) \]

\[ = P_{3}^{2}(A) + \nabla Q_{1}^{1,1} + Q_{1}^{2,1}, \]

(2.11)

and substituting \( V = -\Delta H + P_{3}^{0}(A) + Q_{0}^{0,1} \) in (1.36) we obtain

\[ \nabla \partial_{t} A = \nabla^{2} V + A \ast A \ast V + Q_{0}^{0,0} \ast V \]

\[ = -\nabla^{2}(\Delta H + P_{3}^{0}(A) + Q_{0}^{0,1}) + A \ast A \ast (\Delta H + P_{3}^{0}(A) + Q_{0}^{0,1}) \]

\[ + Q_{0}^{0,0} \ast (\Delta H + P_{3}^{0}(A) + Q_{0}^{0,1}) \]

\[ = -\nabla^{2}(\Delta H) + P_{3}^{2}(A) + \nabla^{2} Q_{0}^{0,1} + P_{3}^{0}(A) + Q_{0}^{0,3} + Q_{1}^{2,1} + Q_{0}^{0,1} \]

(2.12)

\[ = -\Delta(\nabla^{2} H) + P_{3}^{2}(A) + P_{3}^{0}(A) + \nabla^{2} Q_{0}^{1,1} + Q_{1}^{2,1} + Q_{0}^{0,1} \]

(2.13)

To proceed with the induction step we use (1.27) with \( \phi := Q_{m}^{m,1} \) and obtain

\[ \nabla \Delta Q_{m}^{m,1} = \Delta \nabla Q_{m}^{m,1} + A \ast \nabla A \ast Q_{m}^{m,1} + A \ast A \ast \nabla Q_{m}^{m,1} + Q_{0}^{0,0} \ast \nabla Q_{0}^{m,1} + Q_{0}^{0,1} \ast Q_{m}^{m,1} \]

\[ = \Delta Q_{m-1}^{m+1,1} + Q_{m}^{m+1,3} + Q_{m}^{m+1,3} + Q_{R}^{m+1,1} + Q_{R}^{m,2} \]

\[ = \Delta Q_{m-1}^{m+1,1} + Q_{m+1}^{m+1,1} + Q_{R}^{m+1,1}. \]

(2.14)

Letting \( \psi := \nabla \phi \) for \( \phi := \nabla^{m} A \), Lemma 2.3 now yields

\[ \nabla \partial_{t}(\nabla^{m+1} A) + \Delta^{2}(\nabla^{m+1} A) \]

\[ = \nabla ((\nabla \partial_{t}(\nabla^{m} A) + \Delta^{2}(\nabla^{m} A) + \sum_{i+j+k=3} (\nabla^{i} A \ast \nabla^{j} A \ast \nabla^{k+m} A) \]

\[ + A \ast (\nabla \Delta H + P_{3}^{0}(A) + Q_{0}^{1,1}) \ast \nabla^{m} A + (\Delta H + P_{3}^{0}(A) + Q_{0}^{0,1}) \ast \nabla A \ast \nabla^{m} A \]

\[ + Q_{0}^{0,0} \ast (\Delta H + P_{3}^{0}(A) + Q_{0}^{0,1}) \ast \nabla^{m} A + \sum_{j+k=1} \Delta (Q_{0}^{0,j} \ast \nabla^{k+m} A) + \sum_{j+k=1} Q_{j}^{0,j} \ast \nabla^{k+m+2} A \]

\[ = P_{3}^{m+3}(A) + P_{3}^{m+1}(A) + \nabla^{m+1} \Delta^{0,1} + \nabla \Delta Q_{m}^{m,1} + \nabla^{2} Q_{m}^{m+1,1} + \nabla Q_{m}^{m+2,1} + \nabla Q_{R}^{m,1} \]

\[ + Q_{0}^{m,1,3} + Q_{0}^{m,1} + Q_{0}^{m,2} + Q_{0}^{m,4} + Q_{R}^{m,2} + Q_{R}^{m,1} + Q_{R}^{m,3} \]

(2.14)

which proves the induction step and thus the lemma.
2 Lifespan Theorem

2.3 Energy- and integral estimates

Lemma 2.5 Let \( f : \Sigma \times I \rightarrow M^n \) be a variation with normal velocity field \( \partial_t f = V \) and let \( \phi \in C^0(\Sigma \times I) \) with \( \nabla \partial_t \phi + \Delta^2 \phi = Y \). Then for arbitrary \( \eta \in C^2(\Sigma \times I) \) with locally compact support

\[
\frac{d}{dt} \int_{\Sigma} \frac{1}{2} |\phi|^2 \eta \, d\mu + \int_{\Sigma} \langle \Delta \phi, \Delta (\eta \phi) \rangle \, d\mu - \int_{\Sigma} \langle Y, \eta \phi \rangle \, d\mu = \int_{\Sigma} \eta \sum_{k=1}^{l} \langle A(e_{ik}, e_j), V \rangle \langle \phi(e_{i_1}, \ldots, e_{i_{k-1}}, e_{i_k}, \ldots, e_{i_l}), \phi(e_{i_1}, \ldots, e_{i_{k-1}}, e_j, e_{i_{k+1}}, \ldots, e_{i_l}) \rangle \, d\mu \]

\[
- \int_{\Sigma} \frac{1}{2} |\phi|^2 \langle H, V \rangle \eta \, d\mu + \int_{\Sigma} \frac{1}{2} |\phi|^2 \partial_t \eta \, d\mu.
\]

(2.15)

Here \( \tilde{g}(e_i, e_j) = \delta_{ij} \) locally on \( \Sigma \times I \) and summation over \( j, i, \nu \in \{1, 2\}, 1 \leq \nu \leq l \), is used.

Proof: With \( \Phi_k(X, Y) := \langle \phi(e_{i_1}, \ldots, e_{i_{k-1}}, X, e_{i_{k+1}}, \ldots, e_{i_l}), \phi(e_{i_1}, \ldots, e_{i_{k-1}}, Y, e_{i_{k+1}}, \ldots, e_{i_l}) \rangle \) we have

\[
\frac{d}{dt} \int_{\Sigma} \frac{1}{2} \eta \langle \phi(e_{i_1}, \ldots, e_{i_l}), \phi(e_{i_1}, \ldots, e_{i_l}) \rangle \, d\mu
\]

\[
= - \int_{\Sigma} \langle \Delta^2 \phi, \eta \phi \rangle \, d\mu + \int_{\Sigma} \eta \langle Y, \phi \rangle \, d\mu + \int_{\Sigma} \eta \sum_{k=1}^{l} \Phi_k(e_i, \nabla \partial_t e_i) \, d\mu
\]

\[
- \int_{\Sigma} \frac{1}{2} \eta \langle H, V \rangle |\phi|^2 \, d\mu + \int_{\Sigma} \frac{1}{2} \partial_t \eta |\phi|^2 \, d\mu.
\]

Since \( \tilde{g}(e_i, e_j) = \delta_{ij} \) we compute, using (1.33),

\[
\tilde{g}(\nabla \partial_t e_{ik}, e_j) + \tilde{g}(e_{ik}, \nabla \partial_t e_j) = - \langle \nabla \partial_t \tilde{g} \rangle(e_{ik}, e_j) = 2 \langle A(e_{ik}, e_j), V \rangle,
\]

which implies \( \nabla \partial_t e_{ik} = 2 \langle A(e_{ik}, e_j), V \rangle e_j - \tilde{g}(e_{ik}, \nabla \partial_t e_j) e_j \). From this we get

\[
\sum_{k=1}^{l} \Phi_k(e_i, \nabla \partial_t e_i) = \sum_{k=1}^{l} \left( 2 \langle A(e_{ik}, e_j), V \rangle \Phi_k(e_i, e_j) - \Phi_k(\tilde{g}(e_i, \nabla \partial_t e_j) e_j, e_i) \right)
\]

\[
= \sum_{k=1}^{l} \left( 2 \langle A(e_i, e_j), V \rangle \Phi_k(e_i, e_j) - \Phi_k(e_j, \tilde{g}(e_i, \nabla \partial_t e_j) e_i) \right)
\]

\[
= \sum_{k=1}^{l} \langle A(e_i, e_j), V \rangle \Phi_k(e_i, e_j).
\]

Since \( \Delta \) is self-adjoint, the claim follows.
2.3 Energy- and integral estimates

Lemma 2.6 Under the assumptions of the previous Lemma let \( \eta := \gamma^s \), where \( \gamma \in C^2(\Sigma \times I) \) is locally proper with \( 0 \leq \gamma \leq 1 \) and \( s \geq 4 \). Then for \( c = c(n, s) \)

\[
\frac{d}{dt} \int_{\Sigma} |\phi|^2 \gamma^s d\mu + \int_{\Sigma} \nabla^2 \phi \gamma^s d\mu - \int_{\Sigma} 2 \langle Y, \phi \rangle \gamma^s d\mu \leq \int_{\Sigma} (A^* \phi^* \phi, V) \gamma^s d\mu + \int_{\Sigma} |\nabla \phi|^2 \gamma^s d\mu + c \int_{\Sigma} |\nabla^2 \phi|^2 d\mu + c \int_{\Sigma} \langle \nabla \phi, \gamma \rangle d\mu + c \int_{\Sigma} |\nabla \gamma|^2 |\nabla \phi|^2 d\mu,
\]

(2.16)

Proof: Because of Lemma 2.5 it suffices to estimate

\[
\int_{\Sigma} |\nabla^2 \phi|^2 \gamma^s d\mu \leq c \int_{\Sigma} |\phi|^2 \gamma^s d\mu + c \int_{\Sigma} |\nabla \phi|^2 \gamma^s d\mu + c \int_{\Sigma} \langle \nabla \phi, \nabla (\gamma \phi) \rangle d\mu + c \int_{\Sigma} \langle \nabla \phi, \gamma \nabla \phi \rangle d\mu + c \int_{\Sigma} |\nabla \gamma|^2 |\nabla \phi|^2 d\mu
\]

Since (2.16) is scale invariant, we may assume (in the non-flat case) that \( \|R\|_\infty^2 + \|DR\|_\infty^{4/3} = 1 \). Differentiating

\[
\nabla^2 (\gamma^s \phi) = s(s-1)\gamma^{s-2}(\nabla \gamma) \otimes (\nabla \gamma) \otimes \phi + s\gamma^{s-1}(\nabla^2 \gamma) \otimes \phi + 2s\gamma^{s-1}(\nabla \gamma) \otimes \nabla \phi + \gamma^s \nabla^2 \phi \tag{2.17}
\]

we get

\[
\int_{\Sigma} |\nabla^2 \phi|^2 \gamma^s d\mu \leq \int_{\Sigma} \langle \nabla^2 \phi, \nabla^2 (\gamma^s \phi) \rangle d\mu + c \int_{\Sigma} |\nabla^2 \phi|^2 \gamma^s d\mu + c \int_{\Sigma} |\nabla^2 \phi|^2 \gamma^s d\mu + c \int_{\Sigma} |\nabla \phi|^2 \gamma^s d\mu + \frac{1}{4} \int_{\Sigma} |\nabla^2 \phi|^2 \gamma^s d\mu + c \int_{\Sigma} |\nabla \phi|^2 \gamma^s d\mu + c \int_{\Sigma} |\nabla \phi|^2 \gamma^s d\mu + c \int_{\Sigma} |\nabla \phi|^2 \gamma^s d\mu,
\]

(2.18)
where we have used Young’s inequality in the last estimate. Now we use partial integration for the third term

\[
\int_\Sigma \left| \nabla \phi \right|^2 \gamma^{s-2} \left| \nabla \gamma \right|^2 d\mu
\]

\[
= \int_\Sigma \langle \phi, \nabla^s (\gamma^{s-2} \left| \nabla \gamma \right|^2 \nabla \phi) \rangle d\mu
\]

\[
= \int_\Sigma \langle \phi, \nabla_e (\gamma^{s-2} \left| \nabla \gamma \right|^2) \nabla_e \phi \rangle d\mu \quad \text{for } e = i
\]

\[
\leq \int_\Sigma \langle \phi, \Delta \phi \rangle \gamma^{s-2} \left| \nabla \gamma \right|^2 d\mu + c \int_\Sigma \left| \nabla \phi \right| \gamma^{s-3} \left| \nabla \gamma \right|^3 d\mu + c \int_\Sigma \left| \nabla \phi \right| \gamma^{s-2} \left| \nabla \gamma \right| \left| \nabla^2 \gamma \right| d\mu
\]

\[
\leq \epsilon \int_\Sigma \left| \nabla^2 \phi \right|^2 \gamma^s d\mu + c(\epsilon) \int_\Sigma \left| \nabla \phi \right|^2 \gamma^{s-4} \left| \nabla \gamma \right|^4 d\mu
\]

\[
+ \frac{1}{2} \int_\Sigma \left| \nabla \phi \right|^2 \gamma^{s-2} \left| \nabla \gamma \right|^2 d\mu + \int_\Sigma \left| \nabla \phi \right|^2 \gamma^{s-2} \left| \nabla^2 \gamma \right|^2 d\mu,
\]  

(2.19)

where we have used \( |\nabla_e (\gamma^{s-2} \left| \nabla \gamma \right|^2) \nabla_e \phi| \leq c |\nabla \phi| \left( \gamma^{s-3} \left| \nabla \gamma \right|^3 + \gamma^{s-2} \left| \nabla^2 \gamma \right| \left| \nabla \gamma \right| \right) \) together with Young’s inequality. Now choosing \( \epsilon \) small enough, adding up (2.18) and (2.19), and absorbing yields

\[
\int_\Sigma \left| \nabla^2 \phi \right|^2 \gamma^s d\mu + \int_\Sigma \left| \nabla \phi \right|^2 \gamma^{s-2} \left| \nabla \gamma \right|^2 d\mu
\]

\[
\leq \frac{3}{2} \int_\Sigma \langle \nabla^2 \phi, \nabla^2 (\gamma^s \phi) \rangle d\mu + \int_\Sigma \left| \nabla \phi \right|^2 \gamma^{s-4} \left( \left| \nabla \gamma \right|^4 + \gamma^2 \left| \nabla^2 \gamma \right|^2 \right) d\mu.
\]

(2.20)

Again with partial integration, (2.7) and \( \nabla (\gamma^s \phi) = s \gamma^{s-1} (\nabla \gamma) \otimes \phi + \gamma^s \nabla \phi \) we compute

\[
\int_\Sigma \langle \nabla^2 \phi, \nabla^2 (\gamma^s \phi) \rangle d\mu
\]

\[
= \int_\Sigma \langle \nabla^4 \phi, \nabla (\gamma^s \phi) \rangle d\mu = \int_\Sigma \langle -\Delta \nabla \phi, \nabla (\gamma^s \phi) \rangle d\mu
\]

\[
= \int_\Sigma \langle -\nabla \Delta \phi, \nabla (\gamma^s \phi) \rangle d\mu + \int_\Sigma \langle (\nabla \Delta - \Delta \nabla) \phi, \nabla (\gamma^s \phi) \rangle d\mu
\]

(2.21)
2.3 Energy- and integral estimates

\[
\begin{align*}
\int_\Sigma (\Delta \phi, \Delta (\gamma^s \phi)) d\mu + \int_\Sigma \langle A^* A^* \nabla \phi, \nabla \phi \rangle \gamma^s d\mu + \int_\Sigma \langle A^* \nabla A^* \phi, \nabla \phi \rangle \gamma^s d\mu \\
+ c \int_\Sigma |A|^2 \langle \phi | \nabla \phi | \gamma^{s-1} | \nabla \gamma \rangle d\mu + s \int_\Sigma |A| |\nabla A| |\phi|^{2 \gamma^{s-1}} |\nabla \gamma| d\mu \tag{2.22}
\end{align*}
\]

\[
\begin{align*}
\int_\Sigma \langle A^* \nabla^2 \phi, \nabla \phi \rangle \gamma^s d\mu \\
+ c \int_\Sigma |A|^2 \langle \phi | \nabla \phi | \gamma^s | \nabla \gamma \rangle d\mu + s \int_\Sigma |A| |\nabla A| |\phi|^{2 \gamma^s} |\nabla \gamma| d\mu \tag{2.23}
\end{align*}
\]

\[
\begin{align*}
\int_\Sigma \langle A^* \nabla A^* \phi, \nabla \phi \rangle \gamma^s d\mu \\
+ c \int_\Sigma |A|^2 \langle \phi | \nabla \phi | \gamma^{s-1} | \nabla \gamma \rangle d\mu \tag{2.24}
\end{align*}
\]

\[
\begin{align*}
\int_\Sigma |\nabla \phi| (|\nabla \phi| \gamma^{s} + |\phi| |\nabla \gamma| \gamma^{s-1}) d\mu, \tag{2.25}
\end{align*}
\]

where in (2.24) and (2.25) we estimated

\[
|Q^{0,0} \nabla \phi| \leq c |Q^{0,0}| |\nabla \phi| \leq c \|R\|_{\infty} |\nabla \phi| \leq c |\nabla \phi|
\]

and

\[
|Q^{0,1} \phi| \leq c |Q^{0,1}| |\phi| \leq c (\|DR\|_{\infty} + \|R\|_{\infty} |A|) |\phi| \leq c (1 + |A|) |\phi|.
\]

The two integrals in (2.23) can be estimated with Young

\[
\begin{align*}
\int_\Sigma |A|^2 |\phi| |\nabla \phi| \gamma^{s-1} |\nabla \gamma| d\mu & \leq \epsilon \int_\Sigma |\nabla \phi|^{2 \gamma^{s-2}} |\nabla \gamma|^{2} d\mu + c(\epsilon) \int_\Sigma |\phi|^{2 |A|^4} \gamma^s d\mu, \tag{2.26}
\end{align*}
\]

\[
\begin{align*}
\int_\Sigma |A| |\nabla A| |\phi|^{2 \gamma^{s-1}} |\nabla \gamma| d\mu & \leq \int_\Sigma |\phi|^{2 \gamma^{s-2}} |\nabla \gamma|^{2} |A|^2 + \int_\Sigma |\phi|^{2 |\nabla A|^2} \gamma^s \\
& \leq \int_\Sigma |\phi|^{2 \gamma^{s-4}} |\nabla \gamma|^{4} d\mu + \int_\Sigma |\phi|^{2 (|\nabla A|^2 + |A|^4)} \gamma^s d\mu. \tag{2.27}
\end{align*}
\]

With partial integration we estimate the second term in (2.22)

\[
\begin{align*}
\int_\Sigma |A|^2 |\nabla \phi|^{2 \gamma^s} d\mu \\
\leq - \int_\Sigma |A|^2 \langle \phi, \Delta \phi \rangle \gamma^s d\mu + \int_\Sigma A^* \nabla A^* \phi^* \nabla \phi \gamma^s d\mu + c \int_\Sigma |A|^2 |\phi| |\nabla \phi| \gamma^{s-1} |\nabla \gamma| d\mu \\
\leq \epsilon \int_\Sigma |\nabla^2 \phi|^{2 \gamma^s} d\mu + c(\epsilon) \int_\Sigma |\phi|^{2 |A|^4} \gamma^s d\mu \\
+ \int_\Sigma A^* \nabla A^* \phi^* \nabla \phi \gamma^s d\mu + \epsilon \int_\Sigma |\nabla \phi|^{2 \gamma^{s-2}} |\nabla \gamma|^{2} d\mu
\end{align*}
\]
2 Lifespan Theorem

and further

\[
\int_{\Sigma} A^{\ast} \nabla A^{\ast} \phi^{\ast} \nabla \phi \gamma^{s} d\mu \leq \frac{1}{2} \int_{\Sigma} |A|^{2} |\nabla \phi|^{2} \gamma^{s} d\mu + c \int_{\Sigma} |\phi|^{2} |\nabla A|^{2} \gamma^{s} d\mu.
\]

Adding up the last two estimates and absorbing yields

\[
\int_{\Sigma} |A|^{2} |\nabla \phi|^{2} \gamma^{s} d\mu + \int_{\Sigma} A^{\ast} \nabla A^{\ast} \phi^{\ast} \nabla \phi \gamma^{s} d\mu \leq \varepsilon \int_{\Sigma} |\nabla^{2} \phi|^{2} \gamma^{s} d\mu + \varepsilon \int_{\Sigma} |\nabla \phi|^{2} \gamma^{s-2} |\nabla \gamma|^{2} d\mu + c(\varepsilon) \int_{\Sigma} |\phi|^{2} (|\nabla A|^{2} + |A|^{4}) \gamma^{s} d\mu.
\]

Combining (2.20), (2.21), and (2.26) to (2.28) yields

\[
\int_{\Sigma} |\nabla \phi|^{2} \gamma^{s} d\mu + \int_{\Sigma} |\nabla \phi|^{2} \gamma^{s-2} |\nabla \gamma|^{2} d\mu \leq \frac{3}{2} \int_{\Sigma} \langle \Delta \phi, \Delta (\gamma^{s} \phi) \rangle d\mu + \varepsilon \int_{\Sigma} |\nabla \phi|^{2} \gamma^{s} d\mu + \varepsilon \int_{\Sigma} |\nabla \phi|^{2} \gamma^{s-2} |\nabla \gamma|^{2} d\mu + c \int_{\Sigma} |\phi|^{2} (|\nabla A|^{2} + |A|^{4}) \gamma^{s} d\mu + \text{the integrals in (2.24) and (2.25), coming from the estimated Q-terms.}
\]

So we only have to estimate (2.24) and (2.25) term by term

- (1 + |A||\phi||\nabla \phi|\gamma^{s} \leq |\phi|^{2} \gamma^{s} + |\nabla \phi|^{2} \gamma^{s} + |A|^{2} |\phi|^{2} \gamma^{s} \leq 2|\phi|^{2} \gamma^{s} + |\nabla \phi|^{2} \gamma^{s} + |A|^{4} |\phi|^{2} \gamma^{s},

where we further estimate |\nabla \phi|^{2} \gamma^{s} in the third point.

- (1 + |A||\phi|^{2} |\nabla \gamma|\gamma^{s-1} \leq (1 + |A|^{2}) |\phi|^{2} (\gamma^{s} + |\nabla \gamma|^{2} \gamma^{s-2}) \leq 2|\phi|^{2} \gamma^{s} + 2|A|^{4} |\phi|^{2} \gamma^{s} + 2|\phi|^{2} |\nabla \gamma|^{4} \gamma^{s-4}.

From Corollary A.12 with \(p = 2\) we get

- \(\int_{\Sigma} |\nabla \phi|^{2} \gamma^{s} d\mu \leq \varepsilon \int_{\Sigma} |\nabla \phi|^{2} \gamma^{s-2} d\mu + c(\varepsilon) \int_{\Sigma} |\phi|^{2} \gamma^{s-2} d\mu.

- |\nabla \phi| |\phi||\nabla \gamma|\gamma^{s-1} \leq |\nabla \phi|^{2} \gamma^{s} + |\phi|^{2} |\nabla \gamma|^{2} \gamma^{s-2} \leq |\nabla \phi|^{2} \gamma^{s} + |\phi|^{2} \gamma^{s} + |\phi|^{2} |\nabla \gamma|^{4} \gamma^{s-4}.
Choosing $\varepsilon > 0$ appropriately the claim follows. 

For $\tilde{\gamma} \in C_c^\infty(M)$ with $0 \leq \tilde{\gamma} \leq 1$ let $\gamma := \tilde{\gamma} \circ f$. This implies

$$|\nabla \gamma| = |D\tilde{\gamma}_f \cdot Df| \leq c\|D\tilde{\gamma}\|_{\infty} \quad \text{and} \quad |\nabla^2 \gamma| = |\nabla(D\tilde{\gamma}_f \cdot Df)|$$

(2.29)

$$= |D^2\tilde{\gamma}_f(Df, Df) + D\tilde{\gamma}_f \cdot A| \leq c(\|D\tilde{\gamma}\|_{\infty} + \|D\tilde{\gamma}\|_{\infty}|A|).$$

Furthermore, we specialize to the Willmore flow, i.e.

$$V = \partial_t f = -\Delta H + P_3^0(A) + Q_{0,1}. \quad (2.30)$$

For the time derivative we thus get

$$\partial_t \gamma = D\tilde{\gamma}_f \cdot \partial_t f$$

$$= D\tilde{\gamma}_f \cdot (-\Delta H + P_3^0(A) + Q_{0,1}).$$

(2.31)

**Lemma 2.7** Let $f : \Sigma^2 \times I \to M^n$ be a smooth locally proper Willmore flow. Then for $\phi := \nabla^m A$ with $m \in \mathbb{N}_0$, $\gamma = \tilde{\gamma} \circ f$ satisfying (2.29) and $s \geq 2m + 4$ we have

$$\frac{d}{dt} \int_\Sigma |\phi|^2 \gamma^s d\mu + \frac{3}{4} \int_\Sigma |\nabla^2 \phi|^2 \gamma^s d\mu$$

$$\leq \int_\Sigma \left( P_3^{m+2}(A) + P_5^m(A) + Q_{m+1}^{m+2,1} + Q_{R_{\ast R}}^{m+1} \right) \ast \phi \gamma^s d\mu$$

$$+ \int_\Sigma \langle \nabla^m \Delta Q_{0,1} + \Delta Q_{m,1} + \nabla Q_{m+1,1}^\pm, \phi \rangle \gamma^s d\mu + c C_{\text{scal}} \int_{\{\gamma > 0\}} |A|^{2s-4-2m} d\mu,$$

where $C_{\text{scal}} = \sum_{i=1}^2 \|D^i \tilde{\gamma}\|_{\infty}^{1/2} + \sum_{i=0}^1 \|D^i R\|_{\infty}^{1/2}$ and $c = c(n, s)$.

**Proof:** After scaling we may assume that $C_{\text{scal}} = 1$. We estimate the terms in (2.16).

From (2.10) we know that $Y = P_3^{m+2}(A) + P_5^m(A) + \nabla^m \Delta Q_{0,1} + \Delta Q_{m,1} + \nabla Q_{m+1,1}^\pm + Q_{m+1,1}^R + Q_{R_{\ast R}}^m$, and substituting (2.30) we obtain

$$\int_\Sigma 2\langle Y, \phi \rangle \gamma^s d\mu + \int_\Sigma \langle A \ast \phi, \phi, V \rangle \gamma^s d\mu + c \int_\Sigma |\phi|^2 (|\nabla A|^2 + |A|^4) \gamma^s d\mu$$

$$= \int_\Sigma \left( P_3^{m+2}(A) + P_5^m(A) + Q_{m+1}^{m+2,1} + Q_{R_{\ast R}}^m \right) \ast \phi \gamma^s d\mu$$

$$+ \int_\Sigma \langle \nabla^m \Delta Q_{0,1} + \Delta Q_{m,1} + \nabla Q_{m+1,1}^\pm, \phi \rangle \gamma^s d\mu,$$
2 Lifespan Theorem

where we used that \( \langle A*\phi*\phi, Q_0^{0,1} \rangle = Q_0^{m,3} \phi = Q_0^{m+2,1} \phi \). So it suffices to estimate each of

\[
\int_{\Sigma} |\phi|^2 \gamma^{s-1} \partial_t \gamma d\mu, \quad \int_{\Sigma} |\phi|^2 \gamma^{s-4} (|\nabla \gamma|^2 + \gamma^2 |\nabla^2 \gamma|^2) d\mu, \quad \int_{\Sigma} |\phi|^2 \gamma^{s-2} d\mu
\]

(2.33)

by

\[
\int_{\Sigma} \left( P_3^{m+2}(A) + P_5^m(A) \right) \phi \gamma^s d\mu + \varepsilon \int_{\Sigma} |\nabla^2 \phi| \gamma^s d\mu + c(\varepsilon) \int |A|^2 \gamma^{s-4-2m} d\mu.
\]

At first, we get using Corollary A.13 with \( \phi := A, k := m, p = 2 \) and substituting \( s \) by \( s - 4 \)

\[
\int_{\Sigma} |\phi|^2 \gamma^{s-4} d\mu \leq \int_{\Sigma} |\nabla \phi|^2 \gamma^{s-2} d\mu + c \int |A|^2 \gamma^{s-4-2m} d\mu.
\]

Analogously, it follows from the same corollary with \( \phi := A, k := m + 1, p := 2 \) and substituting \( s \) by \( s - 2 \)

\[
\int_{\Sigma} |\nabla \phi|^2 \gamma^{s-2} d\mu \leq \varepsilon \int_{\Sigma} |\nabla^2 \phi| \gamma^s d\mu + c(\varepsilon) \int |A|^2 \gamma^{s-4-2m} d\mu.
\]

(2.34)

Combining the last two estimates yields

\[
\int_{\Sigma} |\phi|^2 \gamma^{s-4} d\mu + \int_{\Sigma} |\nabla \phi|^2 \gamma^{s-2} d\mu \leq \varepsilon \int_{\Sigma} |\nabla^2 \phi| \gamma^s d\mu + c(\varepsilon) \int |A|^2 \gamma^{s-4-2m} d\mu,
\]

(2.35)

which treats the last term in (2.33). From the evolution equation (2.31) we have

\[
\int_{\Sigma} |\phi|^2 \gamma^{s-1} \partial_t \gamma d\mu = \int_{\Sigma} |\phi|^2 \gamma^{s-1} D\tilde{\gamma} \cdot \left( -\Delta H + P_3^{0}(A) + Q_0^{1} \right) d\mu.
\]

(2.36)

We estimate the second summand on the right-hand side of (2.36) with Young \((p = 4, q = 4/3)\)

\[
\int_{\Sigma} |\phi|^2 \gamma^{s-1} D\tilde{\gamma} \cdot P_3^{0}(A) d\mu \leq c \int_{\Sigma} |\phi|^2 \gamma^{s-1} |A|^4 \gamma^{3s} d\mu \\
\leq c \int_{\Sigma} |\phi|^2 |A|^4 \gamma^s d\mu + c \int_{\Sigma} |\phi|^2 \gamma^{s-4} d\mu
\]

(2.37)

\[
\leq c \int_{\Sigma} P_5^m(A) \phi \gamma^s d\mu + \varepsilon \int_{\Sigma} |\nabla^2 \phi|^2 \gamma^s d\mu
\]

\[
+ c(\varepsilon) \int_{\Sigma} |A|^2 \gamma^{s-4-2m} d\mu.
\]
where we used the interpolation inequality (2.35) in the last step. The last summand in (2.36) is estimated as follows:

\[ \int_\Sigma |\phi|^2 \gamma^{s-1} D\tilde{\gamma}_{\phi f} \cdot \phi d\mu \leq c \int_\Sigma |\phi|^2 |A| \gamma^{s-1} d\mu \]

\[ \leq c \int_\Sigma |\phi|^2 \gamma^{s-2} d\mu + c \int_\Sigma |\phi|^2 |A|^4 \gamma^s d\mu. \]

The first summand on the right-hand side above can again be treated by (2.35). For the second, note that $|\phi|^2 |A|^4$ can be written as $P^m_3(A) \ast \phi$. We estimate the remaining first summand in (2.36) as in [13]. Namely, let \( \{e_i\}_{i=1,2} \) be an adapted local orthonormal frame. Then we pointwise have, using Einstein’s convention

\[-\nabla^* ((D\tilde{\gamma}) \cdot \nabla H) = \partial_{e_i}((D\tilde{\gamma}) \cdot \nabla_{e_i} H) = (D\tilde{\gamma})_{\phi f} \cdot (Df \cdot e_i, \nabla_{e_i} H) + (D\tilde{\gamma})_{\phi f} \cdot \Delta H,\]

\[\Rightarrow (D\tilde{\gamma})_{\phi f} \cdot \Delta H = -\nabla^*( (D\tilde{\gamma})_{\phi f} \cdot \nabla H) - (D\tilde{\gamma})_{\phi f} \cdot (Df \cdot e_i, \nabla_{e_i} H). \quad (2.38)\]

Since

\[ \int_\Sigma |\phi|^2 \gamma^{s-1} \nabla^* (D\tilde{\gamma}_{\phi f} \cdot \nabla H) d\mu = \int_\Sigma \tilde{g}^* (\nabla (|\phi|^2 \gamma^{s-1}), D\tilde{\gamma}_{\phi f} \cdot \nabla H) d\mu \]

\[= \int_\Sigma \nabla_{e_i} (|\phi|^2 \gamma^{s-1}) D\tilde{\gamma}_{\phi f} \cdot \nabla_{e_i} H d\mu \]

we get using partial integration

\[- \int_\Sigma |\phi|^2 \gamma^{s-1} D\tilde{\gamma}_{\phi f} \cdot \Delta H d\mu \]

\[= \int_\Sigma \nabla_{e_i} (|\phi|^2 \gamma^{s-1}) D\tilde{\gamma}_{\phi f} \cdot \nabla_{e_i} H d\mu + \int_\Sigma |\phi|^2 \gamma^{s-1} D^2 \tilde{\gamma}_{\phi f} (Df \cdot e_i, \nabla_{e_i} H) d\mu \]

\[\leq c \int_\Sigma (|\phi| \nabla \phi) \gamma^{s-1} + |\phi|^2 \gamma^{s-2} |\nabla g| |\nabla A||D\tilde{\gamma}_{\phi f}| d\mu + c \int_\Sigma |\phi|^2 \gamma^{s-1} |D^2 \tilde{\gamma}_{\phi f}||\nabla A|| d\mu \]

\[\leq c \int_\Sigma |\nabla \phi||\nabla A| \gamma^{s-1} d\mu + c \int_\Sigma |\phi|^2 |\nabla A| \gamma^{s-2} d\mu + c \int_\Sigma |\phi|^2 |\nabla A| \gamma^{s-1} d\mu \]

and further with Young

\[\leq c \int_\Sigma |\nabla \phi|^2 \gamma^{s-2} d\mu + c \int_\Sigma |\phi|^2 |\nabla A|^2 \gamma^s d\mu + c \int_\Sigma |\phi|^2 \gamma^{s-4} d\mu. \]

\[(2.35) \leq \int_\Sigma P^m_3(A) \ast \phi \gamma^s d\mu + \varepsilon \int_\Sigma |\nabla^2 \phi|^2 \gamma^s d\mu + c(\varepsilon) \int_{[\gamma>0]} |A|^2 \gamma^{s-4-2m} d\mu. \]
Altogether, we have estimated (2.36), remaining the second term in (2.33). But for this term we estimate
\[
\int_{\Sigma} |\phi|^2 \gamma^{s-4} (|\nabla \gamma|^4 + \gamma^2 |\nabla^2 \gamma|^2) \, d\mu
\]
\[
\leq c \int_{\Sigma} |\phi|^2 \gamma^{s-4} \, d\mu + c \int_{\Sigma} |\phi|^2 (1 + |A|^2) \gamma^{s-2} \, d\mu
\]
\[
\leq c \int_{\Sigma} |\phi|^2 \gamma^{s-4} \, d\mu + c \int_{\Sigma} |\phi|^2 |A|^4 \gamma^s \, d\mu
\]
\[
\leq \varepsilon \int_{\Sigma} |D^2 \phi|^2 \gamma^s \, d\mu + c(\varepsilon) \int_{\gamma > 0} |A|^2 \gamma^{s-4-2m} \, d\mu + c \int_{\Sigma} P^m_{\gamma} (A^* \phi \gamma^s) \, d\mu,
\]
completing the proof. ■

## 2.4 Sobolev inequalities for Riemannian Manifolds

**Theorem 2.8** (Michael-Simon Sobolev inequality for Riemannian manifolds I). Let \( f : (\Sigma^d, \tilde{g}) \rightarrow (M^n, g) \) be an isometric \( C^2 \)-immersion where \( (\Sigma, \tilde{g}) \) and \( (M, g) \) are (open or closed) complete Riemannian manifolds of dimension \( d \geq 2 \) and \( n = d + m \) respectively. If \( \Lambda := \| \text{ricci}_{(M,g)} \|_{\infty}^{1/2} + \text{inj}(M, g)^{-1} \) is bounded, then for any \( u \in C^1_c(\Sigma) \) we have
\[
\left( \int_{\Sigma} |u|^{\frac{d}{d-1}} \, d\mu \right)^{\frac{d-1}{d}} \leq c \int_{\Sigma} |\nabla u| \, d\mu + c \int_{\Sigma} |A_f| |u| \, d\mu + c \Lambda \int_{\Sigma} |u| \, d\mu,
\]
where \( c = c(d, n) \) is a universal constant.

**Theorem 2.9** (Michael-Simon Sobolev inequality for Riemannian manifolds II). Let \( f : (\Sigma^d, \tilde{g}) \rightarrow (M^n, g) \) be an isometric \( C^2 \)-immersion of a complete Riemannian manifold of dimension \( d \geq 2 \) into a closed Riemannian manifold \( (M, g) \) of dimension \( n = d + m \). Assume that there exists an isometric immersion \( I : (M, g) \rightarrow \mathbb{R}^N \) with the property that \( \| A_I \|_{L^{\infty}(M, g)} < \infty \). Then for any \( u \in C^1_c(\Sigma) \) we have
\[
\left( \int_{\Sigma} |u|^{\frac{d}{d-1}} \, d\mu \right)^{\frac{d-1}{d}} \leq c \int_{\Sigma} |\nabla u| \, d\mu + c \int_{\Sigma} |H_f| |u| \, d\mu + c \int_{\Sigma} |A_f| \, d\mu,
\]
where \( c = c(d, n) \).

**Remark:** By Nash’s embedding theorem the existence of such an immersion is clearly automatic in case \( M \) is compact. To see that (2.39) and (2.40) cannot hold in general
without an extra term on the right-hand side, just take the standard embedding of the sphere \( S^d \subset S^n \subset (\mathbb{R}^{n+1}, \delta_{eucl}) \), where \( S^d = S^n \cap E \) and \( E = \mathbb{R}e_1 \oplus \ldots \oplus \mathbb{R}e_{d+1} \) is considered as a linear subspace of \( \mathbb{R}^{n+1} \). Note that \( S^d \) is a totally geodesic submanifold of \( S^n \) and thus the second fundamental form vanishes identically. If we then define \( u : S^d \to \mathbb{R} \) by \( u \equiv 1 \), (2.39) and (2.40) would obviously imply that \( S^d \) had zero volume with respect to the induced metric.

\[ \text{Proof of Theorem 2.9:} \] Let \( I : (M, g) \to (\mathbb{R}^N, \delta_{eucl}) \) be an immersion as in the statement of the Theorem. Our starting point is the euclidean inequality

\[
\left( \int_{\Sigma} |u|^{\frac{2d}{d-1}} d\mu \right)^{\frac{d-1}{d}} \leq c \int_{\Sigma} |\nabla u| d\mu + c \int_{\Sigma} |H_{10f}||u| d\mu
\]

(see proof of Theorem 2.8). The claim follows immediately by substitution of (1.15).

\[ \text{Proof of Theorem 2.8:} \] Assume that \( f : (\Sigma, \bar{g}) \to (\mathbb{R}^n, \delta_{eucl}) \) is an isometric embedding and we have given \( h \in C^1_c(U) \) for \( U \subset \mathbb{R}^n \) open and \( h \geq 0 \). Applying \([20]\), Theorem 18.6, we get

\[
\left( \int_{\Sigma} h^p d\mu_{\bar{g}} \right)^{1/p} \leq c \int_{\Sigma} (|\nabla h| + |H_f|h) d\mu_{\bar{g}},
\]

where \( c = c(d) \) and \( p := \frac{d}{d-1} \). Now if \( \bar{u} \in C^1_c(\Sigma) \), we extend \( \bar{u} \) from \( \text{supp} \bar{u} \subset \subset \Sigma \), e.g. using slice coordinates, to an appropriate open subset \( U \subset \mathbb{R}^n \) such that the extension \( u \) has compact support in \( U \) and obtain

\[
\left( \int_{\Sigma} |u|^p d\mu_{\bar{g}} \right)^{1/p} \leq c \int_{\Sigma} (|\nabla u| + |H_f||u|) d\mu_{\bar{g}}, \tag{2.41}
\]

since \( |u| \) is at least Lipschitz in \( U \). In case \( f \) is merely an isometric immersion, one can choose an isometric embedding \( E : (\Sigma, \bar{g}) \to \mathbb{R}^N \), applying (2.41) to \( f_\varepsilon := (f, \varepsilon E) : (\Sigma, (1 + \varepsilon^2)\bar{g}) \to \mathbb{R}^{N+n} \) and then letting \( \varepsilon \to 0 \) (see \([13]\)).

We now want to prove the case \((M, g) \neq (\mathbb{R}^n, \delta_{eucl})\). Define \( r_0 := c(n)\Lambda^{-1} \), where \( \Lambda, c \) are as in Lemma A.4. Let \( \{B_{r_0(p_i)}\}_{i \in \mathbb{N}} \) be a uniformly locally finite covering of \( M \) and \( \{\tilde{m}_i\}_{i \in \mathbb{N}} \) a partition of unity subordinate to \( \{B_{r_0(p_i)}\}_{i \in \mathbb{N}} \) as in Lemma A.9. Further, let \( \{\psi_i : B_{r_0(p_i)} \to V_i \subset \mathbb{R}^n\}_{i \in \mathbb{N}} \), \( \psi_i = \{y^a\}_{1 \leq a \leq n} \) be a countable atlas of harmonic coordinates as in Lemma A.4, and \( \{x^j\}_{1 \leq j \leq d} \) arbitrary coordinates on \( \Sigma \). Since we want to localize in the target, we may fix some arbitrary \( i \in \mathbb{N} \) and omit it almost every time. Note that we have

\[
\frac{1}{c} \delta \leq (g_{\alpha \beta}) \leq c\delta \tag{2.42}
\]
Lifespan Theorem

We want to introduce some notation. Let \( \Sigma_{r,0} := \psi^{-1}(B_{r_0}(p_i)) \subset \Sigma \). On \( V \subset \mathbb{R}^n \) we consider the Riemannian metrics \( g \) and \( \delta \), where \( \delta \) is the standard metric and, by slight abuse of notation, \( g \) stands for the coordinate representation \( (g_{\alpha\beta})_{1 \leq \alpha, \beta \leq n} \) of \( g \) with respect to the harmonic chart \( \psi \). Also, define isometric immersions

\[
\tilde{f} := (\Sigma_{r,0}, \tilde{g}) \to (V, \delta)
\]

and

\[
\tilde{f} := (\Sigma_{r,0}, \tilde{g}) \to (V, g)
\]

such that \( \tilde{f} = \tilde{f} = \psi \circ f|_{\Sigma_{r,0}} \) as maps between manifolds. From \( c^{-1}\delta \leq g \leq c\delta \) it follows that

\[
\frac{1}{c} \delta \leq \tilde{g} \leq c\delta
\]

by properties of the pullback metric. From this, we get an estimate for the Gram determinants

\[
\frac{1}{cd} \det(\tilde{g}) \leq \det(\tilde{g}) \leq c^d \det(\tilde{g})
\]

as follows: Since this is a pointwise estimate, we can choose for any \( x \in \Sigma_{r,0} \) a local chart \( \{x^k\}_{k=1,2} \) around \( x \) with \( \tilde{g}_{ij}(x) = \delta_{ij} \) and obtain from \( c^{-1}\delta \leq \tilde{g}_{ij}(x) \leq c\delta \) the desired equivalence. We remark that although the determinant is not defined for bilinear forms, inequality (2.45) does make sense by invariance with respect to coordinate transformations.

We now want to apply the flat-case inequality (2.41) locally to \( \Sigma^* := \Sigma_{r,0} \) and \( \eta^* := \eta \in C^1(\Sigma^*) \), where we define \( \eta := \tilde{\eta} \circ f \). If we denote by \( \tilde{\mu}, \tilde{\tilde{\mu}}, \tilde{\Gamma}, \tilde{\bar{\Gamma}}, \ldots \) etc. the locally defined geometric quantities on \( (\Sigma_{r,0}, \tilde{g}) \) or \( (\Sigma_{r,0}, \bar{g}) \) respectively, we get for any \( u \in C^1(\Sigma) \) using dominated convergence

\[
\|u\|_{L^p(\bar{\mu})} \leq \sum_{i=1}^{\infty} \|\eta_i u\|_{L^p(\bar{\mu})} = \sum_{i=1}^{\infty} \left( \int_{\Sigma_{r_0,i}} |\eta_i u|^p d\bar{\mu}_i \right)^{1/p} \leq c \sum_{i=1}^{\infty} \left( \int_{\Sigma_{r_0,i}} |\bar{\eta}_i u|^p d\bar{\mu}_i \right)^{1/p},
\]

by properties of the pullback metric. From this, we get an estimate for the Gram determinants

\[
\frac{1}{cd} \det(\tilde{g}) \leq \det(\tilde{g}) \leq c^d \det(\tilde{g})
\]
To estimate the first integral in (2.46), we compute (for fixed i)
\[ |\nabla(\eta u)|_\tilde{g} \leq c|\nabla(\eta u)|_g \leq c(|u||\nabla\eta|_\tilde{g} + \eta|\nabla u|_\tilde{g}) \]
and further for \( N = N(n) \) as in Lemma A.9
\[ |\nabla \eta|_g = |\nabla(\tilde{\eta} \circ f)|_g = |D\tilde{\eta}\circ f \cdot Df|_\tilde{g} \leq c|D\tilde{\eta}|_g \circ f|Df|_\tilde{g} \leq c(N)r_0^{-1} = c\Lambda \]
by construction of \( \tilde{\eta} \) in Lemma A.9. Combining, we get with (2.45), i.e. \( d\mu \leq c^{d/2}d\tilde{\mu} \), and dominated convergence
\[
\sum_{i=1}^{\infty} \int |\nabla(\eta_iu)|_\tilde{g}d\tilde{\mu}_i \leq c\Lambda \sum_{i=1}^{\infty} \int |u|d\tilde{\mu}_i + c \int |\nabla u|_g d\tilde{\mu} \\
\leq c\Lambda N(n) \int |u|d\tilde{\mu}_i + c \int |\nabla u|_g d\tilde{\mu}_i.
\]
The last inequality holds, since \( \tilde{\mu}_i = \tilde{\mu}_0 \chi_{\Sigma_{j_0,i}} \) and for any \( x \in \Sigma \) we have
\[ \sum_{i=1}^{\infty} \chi_{\Sigma_{j_0,i}}(x) \leq \sum_{i=1}^{\infty} \chi_{B_{r_0}(p_i)} \circ f(x) \leq N(n) \] by construction of the \( \{\tilde{\eta}_i\} \) in Lemma A.9. To estimate the second integral in (2.46), we have to estimate
\[ |\tilde{H}|^2_{\delta} \leq c|\tilde{A}|^2_f + c\Lambda^2. \quad (2.47) \]
on \( \Sigma_{r_0} \). Since this is a pointwise estimate, we now fix for arbitrary \( x_0 \in \Sigma_{r_0} \) Riemannian normal coordinates \( \{x^j\}_{j=1,2} \) with respect to \( \tilde{g} \) around \( x_0 \). We want to remark that the second fundamental form \( A \) is a natural map in the following sense: For an isometry \( \varphi \) it is easy to see that \( A_{\varphi \circ f} = \varphi^*(A_{f}) \) and thus \( |A_{\varphi \circ f}|_{\varphi \circ f} = |A_f|_{\varphi \circ f} \). Therefore (2.47) is not affected when choosing coordinates (clearly, the geometry of \( f \) should not depend on the parametrization). For the next calculation we want to define the Matrix valued functions
\[ F^\alpha_{jk} := \partial^2 \partial_j f^\alpha, \quad G^\alpha_{jk} := \Sigma \Gamma^l_{jk} \partial_l f^\alpha \quad \text{and} \quad C^\alpha_{jk} := M \Gamma^\alpha_{\beta \gamma} \varphi \cdot \partial_j f^\beta \partial_k f^\gamma. \]
Then the Gauss formula reads \( A = F - G + C \) in general, and in our case (in \( x_0 \)) with the obvious notation \( \tilde{A} = F - \tilde{G} \) and \( \tilde{A} = F + C \). Contracting, this implies for the mean curvature of \( \tilde{f} \)
\[ \tilde{H} = \tilde{g}^{jk} F_{jk} - \tilde{g}^{jk} \tilde{G}_{jk} \quad \text{and} \quad \tilde{A} = F + C. \]
Because \( c^{-1}\delta \leq (\tilde{g}_{jk}) \leq c\delta \) we get using polarization \( |\tilde{g}_{jk}|, |\tilde{g}^{jk}| \leq c \). From this and by orthogonality of \( \tilde{H} \perp \tilde{g}^{jk} \tilde{G}_{jk} \) we can now estimate
\[ |\tilde{H}|^2_{\delta} = |\tilde{g}^{jk} F_{jk}|^2_{\delta} - |\tilde{g}^{jk} \tilde{G}_{jk}|^2_{\delta} \leq c|F|^2_{\delta} \leq c|\tilde{A}|^2_{\delta} + c|C|^2_{\delta}. \]
Taking
\[ |\tilde{A}|_g^2 = \sum_{j,k=1}^2 |\tilde{A}_{jk}|_g^2 \leq c \sum_{j,k=1}^2 |\tilde{A}_{jk}|_g^2 = c|\tilde{A}|_f^2(x_0) \]
into account, the Theorem is now proved since we can estimate
\[ |C|_g^2 = \sum_{j,k=1}^2 |M_{\beta\gamma}^\alpha \partial_j f \partial_k f \gamma|_g^2 \leq c \Lambda^2 |\partial f|_g^4 \leq c \Lambda |Df|_f^4(x_0) \leq c \Lambda^2. \]

\[ \blacksquare \]

2.5 Interpolation of lower-order terms

Lemma 2.10 Let \((M^n, g)\) be a Riemannian manifold. \(\|\text{ricci}_{(M,g)}\|_\infty^{1/2} + \text{inj}(M, g)^{-1} =: \Lambda < \infty\) and \(f : (\Sigma^2, \tilde{g}) \to (M, g)\) an isometric \(C^4\)-immersion. Then for any \(\gamma \in C^1_c(\Sigma)\) with \(0 \leq \gamma \leq 1\) we have
\[ \int_{\Sigma} |A|^6 d\mu + \int_{\Sigma} |A|^2 |\nabla A|^2 \gamma^4 d\mu \leq c \int_{\Sigma} (|\nabla^2 A|^2 \gamma^4 + |A|^6 \gamma^4) d\mu + c(\|\nabla \gamma\|_\infty^4 + \Lambda^4) \left( \int_{\Sigma} |A|^2 d\mu \right)^2, \]
where \(c = c(n)\).

Proof: Again, since the above inequality is scale invariant, we may assume that \(\|\nabla \gamma\|_\infty^4 + \Lambda^4 = 1\). We want to apply Theorem 2.8 to \(u := |A||\nabla A|^2 \gamma^2\). If \(u\) should not be differentiable, we take \(u_\varepsilon := \sqrt{|A|^2 |\nabla A|^2} \mp \varepsilon \gamma^2 \) instead, so that since \(|\nabla u_\varepsilon| \leq |\nabla A|^2 \gamma^2 + |A||\nabla^2 A| \gamma^2 + c|A||\nabla A| \gamma + c\varepsilon \gamma\) we get from (2.39) after letting \(\varepsilon \searrow 0\)
\[ \int_{\Sigma} |A|^2 |\nabla A|^2 \gamma^4 d\mu \]
\[ \leq c \left( \int_{\Sigma} |A||\nabla^2 A|^2 \gamma^2 d\mu + \int_{\Sigma} |\nabla A|^2 \gamma^2 d\mu + \int_{\Sigma} |A||\nabla A| \gamma d\mu \right) \]
\[ + \left( \int_{\Sigma} |A|^2 |\nabla A| \gamma^2 d\mu + \int_{\Sigma} |A||\nabla A| \gamma^2 d\mu \right)^2 \]
and further with Young’s inequality and Cauchy-Schwarz
\[ \leq c \int_{\Sigma} |A|^2 d\mu \int (|\nabla^2 A|^2 \gamma^4 + |A|^6 \gamma^4) d\mu + c \left( \int_{\Sigma} |A|^2 d\mu \right)^2 + c \left( \int_{\Sigma} |\nabla A|^2 \gamma^2 d\mu \right)^2 \]
since
\[ \left( \int_{\Sigma} |A|^4 \gamma^4 d\mu \right)^2 \leq \int_{\Sigma} |A|^2 d\mu \int_{\Sigma} |A|^6 \gamma^4 d\mu. \] (2.50)

Using partial integration for the last term we get
\[
\int_{\Sigma} |\nabla A|^2 \gamma^2 d\mu = \int_{\Sigma} \langle \nabla^* (\gamma^2 \nabla A), A \rangle d\mu \\
\leq c \int_{\Sigma} |A| |\nabla^2 A| \gamma^2 d\mu + c \int_{\Sigma} |A| |\nabla A| \gamma d\mu \\
\leq c \left( \int_{\Sigma} |A|^2 d\mu \int_{\Sigma} |\nabla^2 A|^2 \gamma^4 d\mu \right)^{1/2} + c \int_{\Sigma} |A|^2 d\mu + \frac{1}{2} \int_{\Sigma} |\nabla A|^2 \gamma^2 d\mu. \] (2.51)

Absorbing and substituting in (2.49) we have an estimate for the second summand in (2.48). With the same approximation argument as above, apply Theorem 2.8 to \( u_\varepsilon := \sqrt{|A|^6 + \varepsilon^2 \gamma^2} \leq (|A|^3 + \varepsilon) \gamma^2 \) and on account of \(|u_\varepsilon| \leq 3|A|^2 |\nabla A| \gamma^2 + 2|A|^3 \gamma |\nabla \gamma| + 2\varepsilon \gamma |\nabla \gamma|\)
we get after letting \( \varepsilon \to 0 \)
\[
\int_{\Sigma} |A|^6 \gamma^4 d\mu \leq c \left( \int_{\Sigma} |\nabla A|^2 \gamma^2 d\mu \right)^2 + c \left( \int_{\Sigma} |A|^4 \gamma^2 d\mu \right)^2 + c \left( \int_{\Sigma} |A|^2 d\mu \right)^2. \] (2.52)

Combining with (2.50) and (2.51), eventually yields the estimate for the first summand in (2.48). The Lemma then follows after rescaling. \( \square \)

We define for an arbitrary section \( \phi \in \Gamma^0(T^{0,1}N \otimes N) \) and any measurable \( U \subset \Sigma \)
\[ \|\phi\|_{p,U} := \left( \int_{U} |\phi|^p d\mu \right)^{1/p} \quad \text{and} \quad \|\phi\|_{\infty,U} := \sup_{U} |\phi|. \]
We abbreviate \( \|\phi\|_{\infty,\Sigma} =: \|\phi\|_{\infty} \) and when there is no ambiguity we also write
\[ \|S\|_{\infty} := \sup_{M} |S| \] (2.53)
for sections \( S \in \Gamma^0(f^*(T^{k,l}M)) \).

**Lemma 2.11** Let \( (M^n, g) \) be a Riemannian manifold with the property that \( \|\text{ricci}_{(M,g)}\|^2_{\infty} \) + \( \text{inj}(M,g) > 1 < \infty \), and \( f : (\Sigma^2, \tilde{g}) \to (M, g) \) be a proper isometric immersion. Then for arbitrary \( \phi \in \Gamma(T^{0,1}N \otimes N) \) and \( \gamma = \tilde{\gamma} \circ f \) as in (2.29) we have
\[
\|\phi\|_{\infty,\Sigma; |\gamma|=1}^2 \leq c \|\phi\|_{2,|\gamma|>0}^2 \left( \|\nabla^2 \phi\|_{2,|\gamma|>0}^2 + \|\phi|^2 |A|^4\|_{1,|\gamma|>0}^4 + c_{\text{scal}} \|\phi\|_{2,|\gamma|>0}^2 \right). \] (2.54)
Moreover, for \( \phi := A \) and provided that
\[
\|A\|_{2,\gamma>0}^2 \leq \varepsilon_0 \tag{2.55}
\]
for \( \varepsilon_0 \) small enough depending only on the dimension \( n \) we get
\[
\|A\|_{\infty,\gamma=1}^4 \leq c \|A\|_{2,\gamma>0}^2 \left( \|
abla^2 A\|_{2,\gamma>0}^2 + c_{\text{scal}} \|A\|_{2,\gamma>0}^2 \right), \tag{2.56}
\]
where \( c_{\text{scal}} = \|D\gamma\|_{\infty} + \|D^2\gamma\|_{\infty}^{1/2} + \|\text{ricci}(M,g)\|_{\infty}^{1/2} + \text{inj}(M,g)^{-1} \) and \( c = c(n) \) is a universal constant.

**Proof:** Since the above estimates are scale invariant we may assume that \( c_{\text{scal}} = 1 \). Define \( \psi := \gamma^2 \phi \). With \( m = 2, \ p = 4 \) and \( \alpha = 2/3 \) we infer from the multiplicative Sobolev-inequality (A.31) and Kato’s inequality
\[
\|\psi\|_{\infty} \leq c \|\psi\|_2^{1/3} \left( \|\nabla\psi\|_4 + \|\psi|A\|_4 + \|\psi\|_4 \right)^{2/3}. \tag{2.57}
\]
Using partial integration we can interpolate
\[
\|\nabla\psi\|_4 \leq \|\psi\|_2^{1/2} \|\nabla^2\psi\|_2^{1/2}. \tag{2.58}
\]
We estimate the second and third summand by \( \|\psi|A\|_4 \leq \|\psi\|_\infty^{1/2} \|\psi|A|_4 \) and
\[
\|\psi\|_4 \leq \|\psi\|_\infty \|\psi\|_2^{1/2} \|\psi\|_2 \tag{2.59}
\]
respectively. Substituting these estimates in (2.57) we obtain using \((a+b)^r \leq c(r)(a^r + b^r)\) \((a,b \geq 0,\ r > 0)\)
\[
\|\psi\|_\infty \leq c \|\psi\|_2^{1/2} \left( \|\nabla^2\psi\|_2^2 + \|\psi\|_2 \right). \tag{2.58}
\]
Now since trivially \( \|\phi\|_{\infty,\gamma=1}^4 \leq \|\psi\|_\infty^4 \) and \( \|\phi\|_2 \leq \|\phi\|_{2,\gamma>0} \), we have after substituting in (2.58)
\[
\|\phi\|_{\infty,\gamma=1}^4 \leq c \|\phi\|_{2,\gamma>0}^2 \left( \|\nabla^2\psi\|_2^2 + \|\psi\|_2 \right). \tag{2.59}
\]
Taking (2.29) into account, we compute for the first summand
\[
\begin{align*}
|\nabla^2 (\gamma^2 \phi)| & \leq 2|\nabla \gamma|^2 |\phi| + 2\gamma \nabla^2 \gamma |\phi| + 4\gamma \nabla \gamma |\nabla \phi| + \gamma^2 |\nabla^2 \phi| \\
& \leq |\nabla^2 \phi|\gamma^2 + c\gamma |\nabla \phi| + c |\phi| \chi_{\gamma>0} + c\gamma |A| |\phi|
\end{align*}
\]
so that
\[
\|\nabla^2\psi\|_2^2 \leq c \int_{\Sigma} |\nabla^2 \phi|\gamma^4 d\mu + c \int_{\Sigma} |\nabla \phi|^2 \gamma^2 d\mu + c \int_{\Sigma} |\phi|^2 (\chi_{\gamma>0} + \gamma^2 |A|^2) d\mu \\
\leq c \int_{\gamma>0} (|\nabla^2 \phi|^2 + |\phi|^2) d\mu + c \int_{\Sigma} |\phi|^2 |A|^4 \gamma^4 d\mu
\]

36
2.5 Interpolation of lower-order terms

since from Corollary A.12 (with \( p := s := 2 \)) we have after renaming \( \varepsilon \)

\[
c \int_{\Sigma} |\nabla \phi|^2 \gamma^2 d\mu \leq \varepsilon \int_{\Sigma} |\nabla^2 \phi|^2 \gamma^4 d\mu + c(\varepsilon) \int_{[\gamma > 0]} |\phi|^2 d\mu.
\]

Putting the obtained estimates in (2.59) yields (2.54) as

\[
\|\psi\|^2 |A|^4 \|_{1,\{\gamma > 0\}} \leq \|\phi\|^2 |A|^4 \|_{2,\{\gamma > 0\}}.
\]

Furthermore, assuming (2.55) we get from Lemma 2.10 after absorbing

\[
\|\psi\|^2 |A|^4 \|_{1,\{\gamma > 0\}} \leq \frac{1}{2} \int_{\{\gamma > 0\}} |\nabla^2 A|^2 \gamma^4 d\mu + c\|A\|^2 |\gamma > 0\| \sup_{0 < \tau < T} \mu([\gamma > 0]) t,
\]

which finishes the proof after rescaling.

\[\blacksquare\]

Proposition 2.12

Let \( f : \Sigma \times [0, T] \to M \) be a Willmore flow on a closed surface, \( \gamma = \tilde{\gamma} \circ f \) as in (2.29) and assume

\[
\sup_{0 < t < T} \|A\|^2 |\gamma > 0\| \leq \varepsilon^* \quad \text{(2.60)}
\]

for \( \varepsilon^* \) small enough, depending only on the dimension \( n \). Then for any \( t \in [0, T] \)

\[
\int_{\Sigma} |A|^2 \gamma^4 d\mu + \frac{1}{2} \int_0^t \int_{\Sigma} (|\nabla^2 A|^2 + |A|^2 |\nabla A|^2 + |A|^6) \gamma^4 d\mu d\tau \leq \int_{\Sigma} A_0|^2 \gamma^4 d\mu_0 + c \frac{1}{2} \int_{\{\gamma > 0\}} |A|^2 d\mu d\tau + c \|D^i R\|^2 \sup_{0 < \tau < T} \mu([\gamma > 0]) t,
\]

where \( C_{scal} = \sum_{i=1}^2 \|D^i \tilde{\gamma}\|^1_i + \sum_{i=0}^1 \|D^i R\|_{\infty}^1 + \text{inj}(M, g)^{-1} \), \( c = c(n) \) is a universal constant and the zero-indexed quantities refer to time \( t=0 \).

Proof: Since (2.61) is scale invariant, we may assume that \( C_{scal} = 1 \). From Lemma 2.7 we know that

\[
\frac{d}{dt} \int_{\Sigma} |A|^2 \gamma^4 d\mu + \frac{3}{4} \int_{\Sigma} |\nabla^2 A|^2 \gamma^4 d\mu \leq \int_{\Sigma} (P_3^2 (A) + P_3^0 (A) + Q_1^{2,1} + Q_1^{0,1}) \ast A \gamma^4 + \langle \Delta Q_1^{0,1} + \nabla Q_1^{1,1}, A \rangle \gamma^4 d\mu + c \int_{\{\gamma > 0\}} |A|^2 d\mu.
\]

37
2 Lifespan Theorem

To begin, we estimate $| (P_0^2(A) + P_0^0(A)) * A | \leq c | A |^3 | \nabla^2 A | + | A |^2 | \nabla A |^2 + | A |^6$. Using partial integration we obtain

$$\int_{\Sigma} (\Delta Q^{0,1} + \nabla Q^{1,1}, \gamma^4 \phi) d\mu \leq \int_{\Sigma} | Q^{0,1} | | \nabla^2 (\gamma^4 A) | d\mu + \int_{\Sigma} | Q^{1,1} | | \nabla (\gamma^4 A) | d\mu.$$  

Recalling the definition of the $Q$-Notation we may further estimate

- $| Q^{0,1} | \leq c | D R | \phi + c | A | | R | \phi$
- $| Q^{1,1} | \leq c | R | \phi (| \nabla A | + | A |^2) + c | D R | \phi | A |$
- $| Q^{2,1} | \leq c | R | \phi (| \nabla^2 A | + | A | | \nabla A | + | A |^3) + c | D R | \phi (| \nabla A | + | A |^2)$
- $| Q^{0,R \phi} | \leq c | R |^2 \phi | A |$

and moreover

- $| \nabla (\gamma^4 A) | \leq | 4 \gamma^3 (\nabla \gamma) \otimes A + \gamma^4 \nabla A | \leq c \gamma^3 | A | + \gamma^4 | \nabla A |$
- $| \nabla^2 (\gamma^4 A) | \leq c \gamma^2 | A | + c \gamma^3 (1 + | A |) | A | + c \gamma^3 | \nabla A | + \gamma^4 | \nabla^2 A |$

where we used (2.29) in the last step. Thus, to estimate the right-hand side of (2.62) we may estimate each of the following terms

$$\int_{\Sigma} | A |^3 | \nabla^2 A | \gamma^4 d\mu + \int_{\Sigma} | A |^2 | \nabla A |^2 \gamma^4 d\mu + \int_{\Sigma} | A |^6 \gamma^4 d\mu$$

$$+ \int_{\Sigma} | D R | \phi (| A |)^2 + | A |^2 | A |^3 + | \nabla A | \gamma^3 + | \nabla^2 A | \gamma^3 ) d\mu + \int_{\Sigma} | A |^2 \gamma^2 s + | A |^3 \gamma^3 + | A | | \nabla A | \gamma^3 d\mu$$

$$+ \int_{\Sigma} | A | | \nabla^2 A | \gamma^4 + | \nabla A |^2 \gamma^4 + | A |^2 | \nabla A | \gamma^4 + | A |^4 \gamma^4 d\mu$$

by

$$(\tau + \epsilon^*) \int_{\Sigma} | \nabla^2 A |^2 \gamma^4 d\mu + c_\tau \| A \|_{2, [\gamma > 0]}^2 + c_\tau \| D R \|_{\infty}^2 \mu ([\gamma > 0]).$$

At first, using absorption, we get from Lemma 2.10

$$\int_{\Sigma} | A |^6 \gamma^4 d\mu + \int_{\Sigma} | A |^2 | \nabla A |^2 \gamma^4 d\mu \leq c \| A \|_{2, [\gamma > 0]}^2 + \epsilon^* \int_{\Sigma} | \nabla^2 A |^2 \gamma^4 d\mu, \quad (2.64)$$

which estimates integral b) and c) already. We have
2.5 Interpolation of lower-order terms

\[ a) \leq \tau \int_\Sigma |\nabla^2 A|^2 \gamma^4 d\mu + c_\tau \int_\Sigma |A|^6 \gamma^4 d\mu \]

\[ d), f), g) \leq c_\tau \|DR\|_\infty^2 \mu([\gamma > 0]) + c_\tau \|A\|_{2,[\gamma > 0]}^2 + \tau \int_\Sigma |\nabla A|^2 \gamma^6 d\mu + \tau \int_\Sigma |\nabla^2 A|^2 \gamma^4 d\mu \]

and from Corollary A.13 we get, letting \( \phi = A, \ k = 1, \ p = 2 \) and \( s = 4 \),

\[ \int_\Sigma |\nabla A|^2 \gamma^4 d\mu \leq \int_\Sigma |\nabla^2 A|^2 \gamma^6 d\mu + c \|A\|_{2,[\gamma > 0]}^2 \]  
(2.65)

\[ e), h) \leq c \|A\|_{2,[\gamma > 0]}^2 \]

\[ i) \leq c \|A\|_{2,[\gamma > 0]}^2 + \int_\Sigma |A|^4 \gamma^6 d\mu \leq c \|A\|_{2,[\gamma > 0]}^2 + \int_\Sigma |A|^6 \gamma^4 d\mu \]

\[ \leq c \|A\|_{2,[\gamma > 0]}^2 + \varepsilon^* \int_\Sigma |\nabla^2 A|^2 \gamma^4 d\mu, \]

\[ j) \leq c_\tau \|A\|_{2,[\gamma > 0]}^2 + \tau \int_\Sigma |\nabla A|^2 \gamma^4 d\mu, \quad \text{(again using (2.65))} \]

\[ k) \leq c_\tau \|A\|_{2,[\gamma > 0]}^2 + \tau \int_\Sigma |\nabla^2 A|^2 \gamma^4 d\mu, \]

\[ l) \text{(use again Corollary A.13)} \]

\[ m), n) \leq c \|A\|_{2,[\gamma > 0]}^2 + \int_\Sigma |A|^6 \gamma^4 d\mu + \int_\Sigma |A|^2 |\nabla A|^2 \gamma^4 d\mu \]

\[ \leq c \|A\|_{2,[\gamma > 0]}^2 + \varepsilon^* \int_\Sigma |\nabla^2 A|^2 \gamma^4 d\mu. \]

Putting things together we have now shown that

\[ \frac{d}{dt} \int_\Sigma |A|^2 \gamma^4 d\mu + \frac{3}{4} \int_\Sigma |\nabla^2 A|^2 \gamma^4 d\mu \]

\[ \leq c_\tau \|DR\|_\infty^2 \sup_{0 \leq t \leq T} \mu([\gamma > 0]) + (\tau + \varepsilon^*) \int_\Sigma |\nabla^2 A|^2 \gamma^4 d\mu + c_\tau \|A\|_{2,[\gamma > 0]}^2 \]  
(2.66)

recalling that \( \mu([\gamma > 0]) \leq \mu(\Sigma) \leq \sqrt{2TW(f_0)} + \mu_0(\Sigma) < \infty \). Adding up (2.64) and (2.66), using small curvature concentration (2.60) and absorbing finally yields

\[ \frac{d}{dt} \int_\Sigma |A|^2 \gamma^4 d\mu + \frac{3}{4} \int_\Sigma |\nabla^2 A|^2 \gamma^4 d\mu + \int_\Sigma |A|^6 \gamma^4 d\mu + \int_\Sigma |A|^2 |\nabla A|^2 \gamma^4 d\mu \]

\[ \leq c \|DR\|_\infty^2 \sup_{0 \leq t \leq T} \mu([\gamma > 0]) + c \|A\|_{2,[\gamma > 0]}^2. \]
Lifespan Theorem

Integrating over $[0,T]$ and rescaling finally yields the claim. ■

Assumption 2.13 Assume that $(M, g)$ is of bounded Geometry of order $K$ and that $\varrho$ is chosen with

$$0 < \varrho < \min \{ \frac{\pi}{2\kappa}, \text{inj}_{(M, g)} \}.$$ 

Let $\tilde{\gamma} \in C^\infty(M)$ be a cutoff function satisfying

$$\chi_{B_{\varrho/2}(x_0)} \leq \tilde{\gamma} \leq \chi_{B_{\varrho}(x_0)} \quad \text{and} \quad \sum_{i=1,2} \varrho_i \|D^i \tilde{\gamma}\|_{L^\infty(M)}^{1/i} < c(n), \quad (2.67)$$

where $c(n)$ is a universal constant. Then choose $1 \leq R_K < \infty$ with

$$\varrho \sum_{i=1}^K \|D^i R\|_{L^\infty(M, g)}^{1/(i+2)} < R_K. \quad (2.68)$$

Lemma A.7 shows the existence of such a cutoff function for any $x_0 \in M$. ♦

Proposition 2.14 Let $f : \Sigma^2 \times [0,T] \to (M^n, g)$ be a proper Willmore flow. Then for $\phi = \nabla^mA$, $m \in \mathbb{N}_0$ and $\gamma = \tilde{\gamma} \circ f$ with $\tilde{\gamma}$ as in (2.67), we have for all $s \geq 2m + 4$

$$\frac{d}{dt} \int_{\Sigma} |\phi|^2 \gamma^s d\mu + \frac{1}{2} \int_{\Sigma} |\nabla^2 \phi|^2 \gamma^s d\mu \leq c(R_{m+1}) (\varrho^{-4} + \|A\|_{\infty, [\gamma > 0]}^4) \int_{\Sigma} |\phi|^2 \gamma^s d\mu$$

$$+ c(R_{m+1}) \left(1 + \|\varrho A\|_{\infty, [\gamma > 0]}^{\max\{4,2m\}}\right) \|A\|_{2, [\gamma > 0]}^2 \varrho^{-2m-4} + c \|D^{m+1} R\|_{L^\infty(M, g)}^2 \mu([\gamma > 0]),$$

where $c$ only depends on $n, m$ and the constant in (2.67).

Proof: Since (2.69) is scale invariant, we may assume that $\varrho = R_{m+1}$. We estimate the terms in Lemma 2.7, i.e.

$$\int_{\Sigma} \left( P^{m+2}_3(A) + P^m_5(A) + Q^{m+2,1}_{m+1} + Q^m_{R^* R} \right) \ast \phi \gamma^s + \langle \nabla^m \Delta Q^0, m + \nabla Q^m_{m+1}, \phi \rangle \gamma^s d\mu.$$ 

Analogously to Kuwert and Schätzle ([13], Proposition 4.5, (4.15)) we have

$$\int_{\Sigma} \left( P^{m+2}_3(A) + P^m_5(A) \right) \ast \phi \gamma^s d\mu \leq \frac{1}{16} \int_{\Sigma} |\nabla^2 \phi|^2 \gamma^s d\mu + c(1 + \|A\|_{\infty, [\gamma > 0]}^4) \|A\|_{2, [\gamma > 0]}^2.$$
2.5 Interpolation of lower-order terms

Similar to the proof of Proposition 2.12 we use partial integration obtaining

$$
\int \sum \langle \nabla^m \Delta Q^{0,1} + \Delta Q^m + \nabla Q^{m+1,1}, \gamma^s \phi \rangle + (Q_{m+1,1}^{m+2} + Q_{R+R}^m) \ast \phi \gamma^s d\mu
$$

(2.70)

\[ \leq c \int \sum |Q^{m,1}| \|\nabla^2 (\gamma^s \phi)\| d\mu + \int \sum |Q^{m+1,1}| \|\nabla (\gamma^s \phi)\| d\mu + \int \sum (|Q^{m+2,1}_{m+1}| + |Q^{m,1}_{R+R}|) \phi \gamma^s d\mu. \]

It will turn out that problems arise when trying to estimate the first integral on the right-hand side of (2.70) in case the homogeneity $\nu$ of the second fundamental form is too large in $Q^{m,1}$. More precisely, this is the case when $\nu \leq m/2 + 1$. Fortunately, we may assume that $\nu \leq m/2 + 1$. This can be seen as follows: With the obvious notation, we have the decomposition $Q^{m,1} = Q^{m,1}_{\nu \leq m/2 + 1} + Q^{m,1}_{\nu > m/2 + 1}$. Since we can write $Q^{m,1}_{\nu > m/2 + 1} = Q^{m,1}_{m+1}$ and because $\nabla Q^{m,1}_{m+1} = Q^{m+1,1}_{m+1}$, we now estimate

$$
\int \sum \langle \nabla^m \Delta Q^{0,1}, \gamma^s \phi \rangle d\mu \leq \int \sum |Q^{m,1}_{\nu \leq m/2 + 1}| \|\nabla^2 (\gamma^s \phi)\| d\mu + \int \sum |Q^{m+1,1}_{\nu > m/2 + 1}| \|\nabla (\gamma^s \phi)\| d\mu
$$

\[ \leq \int \sum |Q^{m,1}_{\nu \leq m/2 + 1}| \|\nabla^2 (\gamma^s \phi)\| d\mu + \int \sum |Q^{m+1,1}_{m+1}| \|\nabla (\gamma^s \phi)\| d\mu, \]

which justifies the above assumption. Thus, by definition of the Q-Notation, we may estimate (recall that $\mu = i_1 + \ldots + i_\nu$, i.e. $\nu = 0$ implies $\mu = 0$):

$$
|Q^{m,1}_{\nu \leq m/2 + 1}| \leq \sum_{r+\mu+\nu = m+1, \mu \leq m, \nu \leq m/2 + 1} |D^r R|_{\phi f} |\nabla^{i_1} A| \ldots |\nabla^{i_\nu} A| \leq \sum_{1 \leq \mu + \nu \leq m+1} |\nabla^{i_1} A| \ldots |\nabla^{i_\nu} A| + |D^{m+1} R|_{\phi f}
$$

and analogously

$$
|Q^{m+1,1}_{m+1}| \leq \sum_{r+\mu+\nu = m+2, \mu \leq m+1} |D^r R|_{\phi f} |\nabla^{i_1} A| \ldots |\nabla^{i_\nu} A| \leq \sum_{1 \leq \mu + \nu \leq m+2} |\nabla^{i_1} A| \ldots |\nabla^{i_\nu} A| + |A|^{m+2},
$$

$$
|Q^{m+2,1}_{m+1}| \leq \sum_{r+\mu+\nu = m+3, \mu \leq m+2, \nu \leq m+1} |D^r R|_{\phi f} |\nabla^{i_1} A| \ldots |\nabla^{i_\nu} A|
$$

\[ \leq \sum_{1 \leq \mu + \nu \leq m+3} |\nabla^{i_1} A| \ldots |\nabla^{i_\nu} A| + |\nabla^2 \phi| + |A|^{m+3} + |A|^{m+2} + |A|^{m+1}|\nabla A|,
$$

$$
|Q^{m,1}_{R+R}| \leq \sum_{r_1 + r_2 + \mu + \nu = m+1, \mu \leq m, \nu \geq 1} |D^{r_1} R|_{\phi f} |D^{r_2} R|_{\phi f} |\nabla^{i_1} A| \ldots |\nabla^{i_\nu} A| \leq \sum_{1 \leq \mu + \nu \leq m+1} |\nabla^{i_1} A| \ldots |\nabla^{i_\nu} A|. \]
2 Lifespan Theorem

Using that $|\nabla(\gamma^s \phi)| = |s \gamma^{s-1}(\nabla \gamma) \otimes \phi + \gamma^s \nabla \phi| \leq c|\phi|\gamma^{s-1} + c|\nabla \phi|\gamma^s$ and

$$|\nabla^2(\gamma^s \phi)| \leq c\gamma^{s-2}|\phi| + c\gamma^{s-1}(1 + |A|)|\phi| + c\gamma^{s-1}|\nabla \phi| + c \gamma^s |\nabla^2 \phi|$$

we arrive, collecting terms and rearranging, at (2.70) \leq

$$c \sum_{1 \leq \mu + \nu \leq m + 3} \sum_{p=0}^{1} \int_{\Sigma} |\nabla^i A| \cdots |\nabla^i A||\nabla^p \phi|\gamma^{s+p-2} d\mu$$

$$+ c \sum_{1 \leq \mu + \nu \leq m + 1} \sum_{\nu \leq m/2 + 1} \int_{\Sigma} |\nabla^i A| \cdots |\nabla^i A||\nabla^2 \phi|\gamma^{s+2p-2} d\mu + c\|D^{m+1} R\|_{L^\infty(M)} \sum_{p=0}^{2} \int_{\Sigma} |\nabla^p \phi|\gamma^{s+p-2} d\mu$$

$$+ c \sum_{p=0}^{1} |A|^{m+2}|\nabla^p \phi|\gamma^{s+p-2} d\mu + c \int_{\Sigma} |A|^{m+3}|\phi|\gamma^s d\mu + c \int_{\Sigma} |A|^{m+1}|\nabla A||\phi|\gamma^s d\mu.$$

We now estimate each of the integrals above:

$$1) \leq \sum_{1 \leq \mu + \nu \leq m + 3} \sum_{p=0}^{1} \left( \int_{\Sigma} |\nabla^i A|^2 \cdots |\nabla^i A|^2 \gamma^{s-2} d\mu \right)^{1/2} \left( \int_{\Sigma} |\nabla^p \phi|^2 \gamma^{s+2p-2} d\mu \right)^{1/2} \leq c\|A\|_{\infty, \gamma > 0}^{2\nu-2} \left( \tau \int_{\Sigma} |\nabla^2 \phi|^2 \gamma^s d\mu + (1 + \tau^{-1})\|A\|_{2, \gamma > 0}^2 \right).$$

For the first integral in (2.71) we use Corollary A.16 with $k := \mu \leq m + 1, \tilde{s} = s - 2 \geq 2m + 2 \geq 2k, r = 2\nu \geq 2$ obtaining

$$\int_{\Sigma} |\nabla^i A|^2 \cdots |\nabla^i A|^2 \gamma^{s-2} d\mu \leq c\|A\|_{\infty, \gamma > 0}^{2\nu-2} \left( \tau \int_{\Sigma} |\nabla^2 \phi|^2 \gamma^s d\mu + (1 + \tau^{-1})\|A\|_{2, \gamma > 0}^2 \right),$$

where we used Corollary A.14 with $k := \mu, l := m + 2$ and $\tilde{s} := s - 2 \geq 2(l - 1)$. For the second integral in (2.71) we want to employ Corollary A.14 with $k := m + p, l := m + 2, \phi := A$ and $\tilde{s} := s + 2p - 2 \geq 2(l - 1)$ so that further

$$\left(2.71\right) \leq c \sum_{\mu = 1}^{m+1} \|A\|_{\infty, \gamma > 0}^{2m-1} \left( \tau \int_{\Sigma} |\nabla^2 \phi|^2 \gamma^s d\mu + (1 + \tau^{-1})\|A\|_{2, \gamma > 0}^2 \right)$$

$$\leq \epsilon \int_{\Sigma} |\nabla^2 \phi|^2 \gamma^s d\mu + c\|A\|_{\infty, \gamma > 0}^{2m}\|A\|_{2, \gamma > 0}^2.$$
2.5 Interpolation of lower-order terms

\[ 2) \quad \leq \varepsilon \int \Sigma |\nabla^2 \phi|^2 \gamma^s d\mu + c_\varepsilon \sum_{1 \leq k + \nu \leq m+1 \atop \nu \leq m/2} \int \Sigma |\nabla^k A|^2 \cdots |\nabla^\nu A|^2 \gamma^s d\mu \]

\[ \leq (A.30) \varepsilon \int \Sigma |\nabla^2 \phi|^2 \gamma^s d\mu + c_\varepsilon \int \Sigma \left( \left( \int \Sigma |\nabla^k A|^2 \gamma^s d\mu + \|A\|^2_{2,\gamma>0} \right) \right) \]

\[ \leq (A.28) \varepsilon \int \Sigma |\nabla^2 \phi|^2 \gamma^s d\mu + c_\varepsilon \left( 1 + \|A\|^m_{\infty,\gamma>0} \right) \int \Sigma |\nabla^2 \phi|^2 \gamma^s d\mu + (1 + \tau^{-1}) \|A\|^2_{2,\gamma>0} \]

\[ \leq \varepsilon \int \Sigma |\nabla^2 \phi|^2 \gamma^s d\mu + c_\varepsilon \left( 1 + \|A\|^2_{2,\gamma>0} \right) \|A\|^2_{2,\gamma>0}. \]

where we used \( \|D^{m+1}R\|_{L^\infty(M)} \|\nabla P \phi \gamma^{s+p-2} \leq \sigma |\nabla^p \phi|^2 \gamma^s + c_\sigma \|D^{m+1}R\|^2_{L^\infty(M)} \chi_{\gamma>0} \) and

\[ \int \Sigma |\nabla^2 \phi|^2 \gamma^s d\mu \leq \tau \int \Sigma |\nabla^2 \phi|^2 \gamma^s d\mu + c_\tau \|A\|^2_{2,\gamma>0}. \]

\[ 3) \quad \leq \varepsilon \int \Sigma |\nabla^2 \phi|^2 \gamma^s d\mu + c_\varepsilon \|D^{m+1}R\|^2_{L^\infty(M)} \mu \left[ \gamma > 0 \right] + c_\varepsilon \int \Sigma |\nabla^2 \phi|^2 \gamma^s d\mu + c_\varepsilon \|A\|^2_{2,\gamma>0}. \]

\[ 4) \quad \leq \varepsilon \int A^2 d\mu + \frac{1}{p} p \int \Sigma |A|^2 |\nabla^p \phi|^2 \gamma^s d\mu \]

\[ \leq (A.28) \|A\|^2_{\infty,\gamma>0} \|A\|^2_{2,\gamma>0} + c_\varepsilon \left( \varepsilon \int \Sigma |\nabla^2 \phi|^2 \gamma^s d\mu + (1 + \varepsilon^{-1}) \|A\|^2_{2,\gamma>0} \right) \]

\[ \leq \varepsilon \int \Sigma |\nabla^2 \phi|^2 \gamma^s d\mu + c_\varepsilon \left( 1 + \|A\|^2_{\infty,\gamma>0} \right) \|A\|^2_{2,\gamma>0}. \]

\[ 5) \quad \leq \varepsilon \int \Sigma |\nabla^2 \phi|^2 \gamma^s d\mu + c_\varepsilon \left( 1 + \|A\|^2_{\infty,\gamma>0} \right) \|A\|^2_{2,\gamma>0}. \]

\[ 6) \quad \leq \varepsilon \int \Sigma |A|^2 d\mu + \int \Sigma |A|^2 |\nabla^2 \phi|^2 \gamma^s d\mu \]

\[ (A.30) \leq c_\varepsilon \|A\|^2_{\infty,\gamma>0} \|A\|^2_{2,\gamma>0} + c_\varepsilon \left( \int \Sigma |\nabla^2 \phi|^2 \gamma^s d\mu + \|A\|^2_{2,\gamma>0} \right) \]
2 Lifespan Theorem

\[
(A.28) \leq c\|A\|_{\infty,[\gamma>0]}^2\|A\|_{2,[\gamma>0]}^2 + c\|A\|_{\infty,[\gamma>0]}^2 \left( \tau \int_{\Sigma} |\nabla^2 \phi|^2 \gamma^s d\mu + (1 + \tau^{-1})\|A\|_{2,[\gamma>0]}^2 \right)
\]

\[
\leq \varepsilon \int_{\Sigma} |\nabla^2 \phi|^2 \gamma^s d\mu + c\varepsilon \left( 1 + \|A\|_{\infty,[\gamma>0]}^{\max\{4,2m\}} \right)\|A\|_{2,[\gamma>0]}^2.
\]

Putting things together and choosing \( \varepsilon \) small enough, the desired estimate (2.69) follows after rescaling.

The next lemma proves a Hölder estimate of the volume function. It was taken from Kuwert ([15], Lemma 3.5) and carried over to the Riemannian setting.

**Lemma 2.15** ([15]). Let \( f : \Sigma \times [0,T] \to M^n \) be a Willmore flow on a closed surface. Then the volume function \( \text{vol}_\Sigma(t) := \int_{\Sigma} d\mu_{\tilde{g}(t)} \) for the volume of \((\Sigma, \tilde{g}(t))\) at time \( t \) satisfies

\[
|\text{vol}_\Sigma(t_1) - \text{vol}_\Sigma(t_2)| \leq \sqrt{2}|t_1 - t_2|^{1/2} W(f_0)
\]

for \( t_1, t_2 \in [0,T] \), where \( W(f_0) := \frac{1}{2} \int_{\Sigma} |H|^2 d\mu \big|_{t=0} \).

**Proof:** By definition of the Willmore flow, we have with \( W := W(f) = \text{grad}_{L^2} \mathcal{W} \)

\[
\frac{d}{dt} \frac{1}{2} \int_{\Sigma} |H|^2 d\mu = - \int_{\Sigma} |W|^2 d\mu,
\]

and therefore

\[
\frac{1}{2} \int_{\Sigma} |H|^2 d\mu \leq \frac{1}{2} \int_{\Sigma} |H|^2 d\mu \big|_{t=0} = W(f_0) < \infty.
\]

We further obtain from (2.73)

\[
\int_0^T \int_{\Sigma} |W|^2 d\mu dt = \frac{1}{2} \int_{\Sigma} |H|^2 d\mu \big|_{t=0} - \frac{1}{2} \int_{\Sigma} |H|^2 d\mu \big|_{t=T} \leq \frac{1}{2} \int_{\Sigma} |H|^2 d\mu \big|_{t=0} = W(f_0).
\]
Now for the derivative of the volume function we have

\[
|\text{vol}'(t)|^2 = |\partial_t \int_{\Sigma} d\mu|^2
\]

\[
\overset{(1.34)}{=} \left| \int_{\Sigma} \langle H, W \rangle d\mu \right|^2
\]

\[
\leq \int_{\Sigma} |H|^2 d\mu \int_{\Sigma} |W|^2 d\mu
\]

\[
\overset{(2.74)}{\leq} 2W(f_0) \int_{\Sigma} |W|^2 d\mu,
\]

so that we obtain for arbitrary \(t_1, t_2 \in [0, T]\)

\[
|\text{vol}(t_2) - \text{vol}(t_1)| = \left| \int_{t_1}^{t_2} \text{vol}'(t) dt \right|
\]

\[
\leq |t_2 - t_1|^\frac{1}{2} \left| \int_{t_1}^{t_2} |\text{vol}'(t)|^2 dt \right|^{\frac{1}{2}}
\]

\[
\overset{(2.76)}{\leq} \sqrt{2W(f_0)}|t_2 - t_1|^{\frac{1}{2}} \left( \int_{0}^{T} \int_{\Sigma} |W|^2 d\mu dt \right)^{\frac{1}{2}}
\]

\[
\overset{(2.75)}{\leq} \sqrt{2W(f_0)}|t_2 - t_1|^{\frac{1}{2}}.
\]

The claim follows.

Now in addition to Assumption 2.13 assume that \(\Lambda_K \geq R_K\) is chosen such that

\[
\varrho(\mu_0(\Sigma) + \varrho^2 W(f_0)) \sum_{i=1}^{K} \|D^i R\|_{L^\infty(M,g)} \leq \Lambda_K < \infty.
\]  

Proposition 2.16 (Interior estimates I). Let \(f : \Sigma \times [0, T] \rightarrow M^n\) be a Willmore flow of a closed surface, and \(\gamma = \tilde{\gamma} \circ f\) as in (2.67). Assume that \(\varrho^{-1}T \leq \tilde{T}\) holds. If

\[
\sup_{0 \leq t \leq T, [\gamma > 0]} \int |A|^2 d\mu < \varepsilon^*,
\]
where $\varepsilon^* > 0$ is small enough depending only on $n$, then for any $m \in \mathbb{N}_0$

$$\|\nabla^m A\|_{\infty, |\gamma|=1} \leq c(n, m, \hat{T}, \Lambda_{m+3}, \alpha_0(m+2)) \varepsilon^{-(m+1)},$$

where $\sum_{j=0}^l \theta^j \|\nabla^j A\|_{2, |\gamma|>0} \bigg|_{t=0} \leq \alpha_0(l)$, $\Lambda_{m+3}$ is as in (2.77), and $c$ also depends on the constant in (2.67).

**Proof:** By scale invariance, we may assume that $\rho = 1$. In the sequel, the constants $c$ may additionally depend on the constant in (2.67). Now for $0 \leq \sigma < \tau \leq 1$ we use cutoff functions $\psi_{\sigma, \tau} \in C^\infty(\mathbb{R})$ with $0 \leq \psi_{\sigma, \tau} \leq 1$ satisfying $\psi_{\sigma, \tau}(s) = 0$ for $s \leq \sigma$ and $\psi_{\sigma, \tau}(s) = 1$ for $s \geq \tau$. Define $\gamma_{\sigma, \tau} := \psi_{\sigma, \tau} \circ \gamma$. With $\sigma := 0$, $\tau := \frac{1}{2}$ and wlog $\varepsilon^* \leq 1$ it follows from (2.61) that

$$\int_0^T \int_0^{\|\nabla^2 A\|^2 + |A|^6} d\mu \, dt \leq \varepsilon^* + c \varepsilon^* C_{\text{scul}}^4 T + c \|\nabla^2 A\|^2_{\infty, |\gamma|>\tau} \sup_{0 \leq t \leq T} |\Sigma| T \leq c(T, \Lambda_1)$$

where we used

$$\|\nabla^2 A\|^2_{\infty, |\gamma|>\tau} \leq c(T, R_1) \|\nabla^2 A\|(W(f_0) + \mu f_0(\Sigma)) \leq c(T, \Lambda_1).$$

From (2.56) with $\sigma = \frac{1}{2}$ and $\tau = \frac{3}{4}$ we get, integrating over $[0, T]$,

$$\int_0^T \|A\|^4_{\infty, |\gamma|>\frac{3}{4}} \, dt \leq c \varepsilon^*(c(T, \Lambda_1) + c \varepsilon^*) \leq c(T, \Lambda_1).$$

In order to use the integral estimate in Proposition 2.14 we restrict to $0 \leq m \leq 2$ at first, and let $\sigma = 3/4$ and $\tau = 3/4$. Since we have to go through the following estimates for different values of $\sigma, \tau$ later on when assuming $m > 2$, we do not substitute $\sigma, \tau$ in the first instance. From (2.60) with $\phi = \nabla^mA$, $s = 2m + 4$ and $0 \leq t \leq T$ we obtain

$$\frac{d}{dt} \int_\Sigma |\phi|^2 \gamma_{\sigma, \tau}^s d\mu + \frac{1}{2} \int_{|\gamma|>\tau} |\nabla^2 \phi|^2 d\mu \leq c(R_{m+1})(1 + \|A\|^4_{\infty, |\gamma|>\sigma}) \left( \int_\Sigma |\phi|^2 \gamma_{\sigma, \tau}^s d\mu + \frac{1}{2} \int_0^t \int_0^{\|\nabla^2 \phi\|^2} \, dt' \right)$$

$$+ c(R_{m+1})(1 + \|A\|^4_{\infty, |\gamma|>\sigma}) e^* + c(T, \Lambda_{m+1}).$$

46
where we have estimated the last summand in (2.69) similar to (2.78). Now from Gronwall’s Lemma we obtain

$$\sup_{0 \leq t \leq T} \int_{[\gamma \geq \tau]} |\nabla^m A|^2 d\mu + \frac{1}{2} \int_0^T \int_{[\gamma \geq \tau]} |\nabla^{m+2} A|^2 d\mu \, dt \leq \exp \left( c(R_{m+1}) \int_0^T (1 + \|A\|_\infty^{4, \gamma \geq \sigma}) \, dt \right) \cdot \left[ \int_{[\gamma_0 \geq 0]} |\nabla^m A|_0^2 d\mu_0 + c(R_{m+1}) \int_0^T (1 + \|A\|_\infty^{\max\{4, 2m\}}) \, dt + c(T, \Lambda_{m+1}) \right]$$

(2.80)

$$\leq c(n, m, T, \Lambda_{m+1}, \alpha_0(m)).$$

From (2.56) we get for $\gamma := \gamma_{\sigma, \tau}$, $\sigma = 7/8$, $\tau = 15/16$ using absorption

$$\|A\|_\infty^{4, \gamma \geq 15/16} \leq c \varepsilon^* \left( c(n, T, \Lambda_3, a_0(2)) + c \varepsilon^* \right)$$

so that we now have for all $\nu \in \mathbb{N}_0$

$$\int_0^T \|A\|_\infty^{\nu, \gamma \geq 15/16} \, dt \leq c(n, T, \Lambda_3, \alpha_0(2), \nu).$$

(2.81)

Now if $m > 2$, we let $\sigma = 15/16$ and $\tau = 31/32$ and obtain, again using (2.80),

$$\sup_{0 \leq t \leq T} \int \|\nabla^m A\|^2 d\mu + \frac{1}{2} \int_0^T \int \|\nabla^{m+2} A\|^2 d\mu \, dt \leq c(n, m, T, \Lambda_3, \Lambda_{m+1}, a_0(2), a_0(m))$$

$$=: \tilde{c}(m),$$

(2.82)

where we used

$$\int_0^T (1 + \|A\|_\infty^{2m, \gamma \geq 15/16}) \, dt \overset{(2.81)}{\leq} c(n, m, T, \Lambda_3, \alpha_0(2)).$$

(2.83)

From the above, the bound (2.82) now holds for any $m \in \mathbb{N}_0$. Finally, the claim follows from Lemma 2.11, if we set $\phi := \nabla^m A$, $\gamma := \gamma_{\sigma, \tau}$, $\sigma = 31/32$ and $\tau = 1$:

$$\|\nabla^m A\|_\infty^{4, \gamma = 1} \leq \tilde{c}(m) (c(m + 2) + \|A\|_\infty^{4, \gamma \geq 31/32}) + \tilde{c}(m))$$

$$\leq c(n, m, T, \Lambda_{m+3}, \alpha_0(m + 2)).$$

Rescaling yields the claim.
2 Lifespan Theorem

2.6 Estimating the lifespan

If \( g \) is the given background metric on \( M \), consider for \( \varrho > 0 \) the scaled metric \( \varrho g := \varrho^{-2}g \) and define an isometric immersion

\[
f_\varrho : (\Sigma, \varrho \tilde{g}) \times [0, \frac{1}{\varrho^4}T) \rightarrow (M, \varrho g)
\]

by

\[
f_\varrho(x,t) := f(x, \varrho^4t),
\]

i.e. we rescale parabolically. By definition we have \( \varrho \tilde{g} := f_\varrho^*(\varrho g) = \varrho^{-2}\tilde{g} \). As one easily verifies from the scaling properties of the various geometric quantities (see e.g. [19], Lemma 3.2), we have that \( f \) is a (maximal) Willmore flow, if and only if \( f_\varrho \) is a (maximal) Willmore flow.

We now want to prove Theorem 0.2 (cf. the respective euclidean proof in [13]). Since the statement of Theorem 0.2 is scale invariant, we may assume that for \( \varrho \) therein given we have \( \varrho = 1 \). We first want show that \( t^+ + (1) =: t^+ < T \). By definition, we have \( t^+ \leq T \). To prove that \( t^+ < T \), we will lead the case \( t^+ = T < \infty \), (2.86)

\[
t^+ = T < \infty,
\]

to a contradiction. More precisely, in this case we can extend the flow, contradicting the assumed maximality of the lifespan \( T \).

Choose \( \Lambda_K, R_K \geq 1 \) with

\[
\sum_{i=0}^{K} \|D^iR\|_{L^{\infty}(M,g)}^{\frac{1}{i+2}} \leq R_K \leq \Lambda_K \quad \text{and} \quad (\mu_{f_0}(\Sigma) + W(f_0)) \sum_{i=3}^{K} \|D^iR\|_{L^{\infty}(M,g)}^{\frac{1}{i+2}} \leq \Lambda_K \quad (2.87)
\]

for \( K \in \mathbb{N}_0 \) to be determined. From Proposition 2.16 we then get, letting \( \varepsilon_0^2 := \varepsilon^* \) and \( \hat{T} := t^+ \), (2.88)

\[
\sum_{i=0}^{m} \|\nabla^iA\|_{\infty} \leq A_m \leq c(m, n, t^+, \Lambda, \alpha_0(m + 2), (2.67)),
\]

for some \( A_m \in \mathbb{R} \), allowing \( c \) also to depend on the constant in (2.67). Now by definition of the Q-terms

\[
\|Q_s^{k,l}\|_{\infty} \leq c(k, l, n, \rho) \sum_{r+\mu+\nu=k+l \atop \rho \leq s} (\|D^\rho R\|_{\infty} \|\nabla^\mu A\|_{\infty} \cdots \|\nabla^\nu A\|_{\infty})
\]

\[
\leq c(k, l, n, R_s, A_k)
\]

48
2.6 Estimating the lifespan

and analogously

\[ \|Q_{R^*R}^{k,l}\|_\infty \leq c(k,l,n) \sum_{\mu \leq k \atop r_1 + r_2 + \mu + \nu = k+l} \|(D^{\mu}R\|_\infty \|D^{r_2}R\|_\infty \|\nabla^{r_1}A\|_\infty \cdot \ldots \cdot \|\nabla^{r_{\mu}}A\|_\infty \) \]

\[ \leq c(k,l,n, R_{k+l-1}, A_k). \]

Now since

\[ V = \partial_t f \overset{(2.30)}{=} 1 \ast \nabla^2 A + P_3^0(A) + Q_0^{0,1} \]

we get from

\[ h := \nabla_{\partial_t \tilde{g}} \overset{(1.33)}{=} -2\langle A, \partial_t f \rangle = A \ast \nabla^2 A + P_4^1(A) + Q_1^{0,2} \]

for the \( m \)-th covariant derivative

\[ \|\nabla^m h\|_\infty \leq c(m, n, A_{m+2}, R_m). \]

From (1.35) we obtain for the evolution of the Christoffel symbols induced by \( \tilde{g} \)

\[ \nabla_{\partial_t \tilde{g}} \overset{(2.89)}{=} P_2^0(A) + P_4^1(A) + Q_1^{0,2} + Q_0^{0,3} + Q_0^{0,1} \]

so that

\[ \|\nabla^m (\nabla_{\partial_t \tilde{g}})\|_\infty \leq c(m, n, A_{m+3}, R_{m+1}). \]

Now to extend the flow, we take (2.88) into account for the flow in a neighbourhood of \( T \). Using Lemma A.4, we can now fix, for \( K \) as in (2.77) to be determined later on, a harmonic radius \( r_0 = c(n, K, R_K) > 0 \) and a countable atlas

\[ \{ \psi_{i, r_0} : B_{r_0}^g(p_i) \to V_i \subset B_{2r_0}^g \}_{i \in \mathbb{N}} \]

of harmonic coordinates for \( M \), such that in these coordinates

i) \( \frac{1}{2} \delta \leq (g_{\alpha\beta}) \leq 2\delta \) on \( B_{r_0}^g(p_i) \)

ii) \( \sup_{B_{r_0}^g(p_i)} |\partial^\gamma g_{\alpha\beta}| \leq c(n,K,R_K) \) for all \( 1 \leq |\gamma| \leq K + 1 \)

iii) \( \sup_{B_{r_0}^g(p_i)} |\partial^\gamma \Gamma^\alpha_\beta_\tau| \leq c(n,K,R_K) \) for all \( 0 \leq |\gamma| \leq K \),

and such that \( \bigcup_{i \in \mathbb{N}} B_{r_0}^g(p_i) \) is a uniformly locally finite cover as described in Lemma A.10. We now show that there exists a well-defined coordinate representation for \( f \) in a neighbourhood of \( T \). Since \( \|\partial_t f\|_\infty \leq c(n,A_2,R_0) \) we clearly have for any \( t_1, t_2 \in [0,T) \) and \( x \in \Sigma \)

\[ d^g(f(x,t_1), f(x,t_2)) \leq \left| \int_{t_1}^{t_2} |\partial_t f|_g(x,\tau) d\tau \right| \leq c(n,A_2,R_0)|t_2 - t_1|. \]
2 Lifespan Theorem

Thus for $0 < |T - \tau| \leq \frac{\gamma}{4} c(n, A_2, R_0)$ we get, because $f(B_\overline{\rho}_{\tau/2}(x)) \subset B_\overline{\rho}_0(x)$ for some $i_0 \in \mathbb{N}$ by construction, that for all $t \in [\tau, T)$ and $x \in \Sigma$

$$f(B_\overline{\rho}_{\tau/4}(x), t) \subset B_\overline{\rho}_0(p_i)$$

Clearly, using topological data instead of geometric, Lebesgue’s number lemma ensures the existence of such a radius (not in terms of $r_0$) as above $r_0/4$, which would suffice. To lead (2.86) to a contradiction we may assume that $\tau = 0$.

Now fix a local chart $\{\varphi : U \to V \subset \mathbb{R}^2\}$ with $c^{-1}\delta \leq (\overline{g}_{ij}(0)) \leq c\delta$ and $\text{diam}_2(0)(U) \leq r_0/4$. Lemma 14.2 in [9] shows that the metrics $\overline{g}(t)$ on $0 \leq t < T$ are equivalent, i.e. we have $c_\overline{g}^{-1}\overline{g}(0) \leq \overline{g}(t) \leq c_\overline{g}\overline{g}(0)$ and thus

$$c_\overline{g}^{-1}\delta \leq (\overline{g}_{ij}(t)), (\overline{g}^{ij}(t)) \leq c_\overline{g}\delta, \quad (2.92)$$

where $c_\overline{g} = c(n, A_2, R_0, T)$.

Let $\overline{\Gamma}$ be the associated Christoffel symbols and denote the coordinate derivative by $\partial$. For any tensor $T \in \Gamma^m(T^{n:*}\Sigma)$ we have the formula

$$\nabla^m T = \partial^m T + \sum_{l=1}^m \sum_{k_1+\ldots+k_l+k \leq m-1} \partial^k l \overline{\Gamma} \cdot \ldots \cdot \partial^k l \overline{\Gamma} \cdot \partial^k T, \quad (2.93)$$

where here and in what follows a product as in (2.93) comprises of universal linear combinations of coordinate functions as $\partial^k l \overline{\Gamma}$, $\partial^k T$ etc. The above formula is immediate for $m = 1$ and follows then by induction. Therefore letting $\overline{\Gamma}_m := |\overline{\Gamma}| + \ldots + |\partial^m \overline{\Gamma}|$ we have

$$|\partial^m T| \leq c(n, m, \overline{\Gamma}_{m-1})(|\nabla^m T| + |\nabla^{m-1} T| + \ldots + |T|)$$

and hence by induction

$$|\partial^m T| \leq c(n, m, \overline{\Gamma}_{m-1}, c_\overline{g} \overline{\Gamma}_m)(|\nabla^m T| + |\nabla^{m-1} T| + \ldots + |T|_\infty). \quad (2.94)$$

**Lemma 2.17** We have

$$|\partial^m (\partial^l \overline{\Gamma})|, |\partial^m \overline{\Gamma}| \leq c(n, m, A_{m+3}, R_{m+1}, c_\overline{g}, \overline{\Gamma}_m |_{t=0}, T). \quad (2.95)$$

**Proof:** (Induction over $m \in \mathbb{N}_0$). For $m = 0$ we get for the coordinate functions

$$|\partial^l \overline{\Gamma}| = |\partial^l \nabla| \leq c(c_\overline{g}) |\partial^l \nabla|_\infty \leq c(n, A_3, R_1, c_\overline{g})$$
and hence by integration \( |\tilde{\Gamma}| \leq c(n, A_3, R_1, c_{\tilde{g}}, \tilde{\Gamma}_{|t=0}, T) \). For the induction step we obtain from (2.94) and (2.91)

\[
|\partial^{m+1}(\partial_t \tilde{\Gamma})| \leq c(n, m + 1, \tilde{\Gamma}_m, c_{\tilde{g}})\left( \|\nabla^{m+1}(\partial_t \nabla)\|_\infty + \ldots + \|\partial_t \nabla\|_\infty \right)
\leq c(n, m + 1, \tilde{\Gamma}_m, c_{\tilde{g}}, A_{m+4}, R_{m+2})
\leq c(n, m + 1, A_{m+4}, R_{m+2}, c_{\tilde{g}}, \tilde{\Gamma}_m|_{t=0}, T).
\]

Integrating over \([0, T)\) yields the claim.  

Summarizing, we now have

i) \( \frac{1}{2} \delta \leq (g_{\alpha \beta}) \leq 2 \delta \) and \( c_{\tilde{g}}^{-1} \delta \leq (\tilde{g}_{ij})|_{t=0} \leq c_{\tilde{g}} \delta \)

ii) \( |\partial^\gamma \Gamma| \leq \Gamma_K := c(n, K, R_K) \) for \( \gamma \leq K \), where \( K \) is as in (2.68)

iii) \( \sum_{L=0}^\infty \|\tilde{\nabla}^L \tilde{\Gamma}\|_{L^\infty(\tilde{\Sigma}(t), U)} \leq A_L := c(L, n, t^+, \Lambda_{L+3}, \alpha_0(L + 2), (2.67)) \)

iv) \( |\partial_t \tilde{G}|_{\tilde{\Sigma}(t)} \leq P := c(n, A_2, R_0) \quad \forall t \in [0, T) \)

v) \( |\partial^\gamma \tilde{\Gamma}| \leq \tilde{\Gamma}_N := c(n, N, A_{N+3}, R_{N+1}, c_{\tilde{g}}, \tilde{\Gamma}_N|_{t=0}, T) \) \( \forall t \in [0, T). \)

Since \( \|\partial_t f\|_{\infty} \leq c(n, A_2, R_0) \leq c(n, t^+, \Lambda_5, \alpha_0(4), (2.67)) \) we may assume that \( f(\Sigma, t) \subset B_R \) for all \( t \in [0, T] \) and then that \( (M, g) \) is of bounded geometry of infinite order. Expressing the \( L^2 \)-gradient of the Willmore functional \( W \) in local coordinates, it is easy to see that \( W \) is a universal linear combination of elements in \( \{\nabla^2 A, \tilde{g}, \tilde{g}^{-1}, \partial f, g\circ f, R\circ f\} \).

From Lemma A.2 with \( m = 4 \), \( s = 0 \) it follows for \( p \leq 8 \), \( q \leq 5 \), \( 1 \leq l \leq 2 \) and fixing \( K = 14 \) that

\[
|\partial^{p+1} f| \leq c(\tilde{\Gamma}_7, A_7, \Gamma_7) \quad \text{and} \quad |\partial^q \partial_t^l f| \leq c(\tilde{\Gamma}_8, A_{14}, \Gamma_8, R_9, c_{\tilde{g}}, t^+).
\]

Substituting the bounds for the second fundamental form finally yields

\[
|\partial^{p+1} f|, |\partial^q \partial_t^l f| \leq c(n, t^+, \Lambda_{14}, \alpha_0(13), \tilde{\Gamma}_8|_{t=0}, T), \tag{2.96}
\]

where \( c \) may also depend on the constants in (2.67).

Now since \( \partial^{p+1} f \) is uniformly Lipschitz continuous in the space variable for \( p \leq 7 \), we get that the one-parameter family \( \{\partial^{p+1} f\}_{t^+ \leq t \leq \infty} \) is equicontinuous and thus it follows from the Arzelà-Ascoli that \( \lim_{t^+ \rightarrow t^+} \partial^{p+1} f = \partial^{p+1} f_{t^+} \) uniformly for \( p \leq 7 \). With the same argument and additionally using that \( \partial^q \partial_t f \) is also uniformly Lipschitz continuous in the time-variable \( (q \leq 4) \), we moreover obtain the uniform convergence \( \lim_{t^+ \rightarrow t^+} \partial^q \partial_t f = \partial^q \partial_t f_{t^+} \) for \( q \leq 4 \). Thus we can define

\[
\tilde{f}(\cdot, t) := \begin{cases} f(\cdot, t) & \text{for } 0 \leq t < t^+ \\ f_{t^+} & \text{for } t = t^+ \end{cases}
\]
2 Lifespan Theorem

where \( \hat{f} \in C^{8,0} \cap C^{4+1,1+1} \) on \( \Sigma \times [t^+ - \delta, t^+] \). Because \( f_{t^+} \in C^{4+1}(\Sigma) \subset C^{4+\alpha}(\Sigma) \) we apply short-time existence obtaining a Willmore flow \( h \in C^{4,1,\alpha}(\Sigma \times [t^+, \tau)) \cap C^\infty(\Sigma \times (t^+, \tau)) \). We now define

\[
E(\cdot, t) := \begin{cases} 
\hat{f}(\cdot, t) & \text{for } 0 \leq t < t^+ \\
\delta \cdot t & \text{for } t = t^+ \\
h(\cdot, t) & \text{for } t^+ < t \leq t^+ + \delta < \tau.
\end{cases}
\]

From the discussion above it is clear that \( E \in C^{4,0}(\Sigma \times [0, t^+ + \delta], M) \). To see that furthermore \( E \in C^{4,1} \), it is enough to check time-differentiability in \( t^+ \). Taking \( \hat{f} \in C^{8,0} \) into account, we see that

\[
\lim_{t \nearrow t^+} \partial^\alpha \partial t E = \lim_{t \nearrow t^+} \partial^\alpha \partial t \hat{f} = \lim_{t \nearrow t^+} \partial^\alpha \partial t \hat{W}(\hat{f}) = \partial^\alpha \partial t \hat{W}(f_{t^+}) = \lim_{t \nearrow t^+} \partial^\alpha \partial t h = \lim_{t \nearrow t^+} \partial^\alpha \partial t E,
\]

where we used that \( h \) is a Willmore flow defined on \([t^+, \tau)\). It remains to check that \( E \) is parabolically Hölder continuous. We remark that a function is parabolically \( \alpha \)-Hölder continuous, if it is \( \alpha \)-Hölder continuous in space and \( \frac{\alpha}{4} \)-Hölder continuous in time. Thus \( \hat{f} \) is parabolically \( \alpha \)-Hölder continuous and, by the regularity properties obtained from short-time existence, this also holds true for \( h \) restricted to \([t^+, t^+ + \delta]\). Using the definition of parabolic Hölder continuity together with a simple triangle argument, it is not hard to show that also \( E \in C^{4,1,\alpha}(\Sigma \times [0, t^+ + \delta], M) \). Since we have also shown that \( E \) is a Willmore flow, in particular in \( t^+ \), and recalling \( t^+ = T \), this obviously contradicts the maximality of \( T \) and hence assumption (2.86) is wrong.

We now want to show the inequality on the right-hand side of (0.3). Therefore assume that \( t^+ < T \leq \infty \). Then

\[
\chi(1, t^+) \geq \varepsilon_0^2,
\]

since when assuming the contrary, there exists by maximality of \( t^+ \) a sequence \( t_i \searrow t^+ \) with \( \chi(1, t_i) \geq \varepsilon_0^2 \), contradicting upper semicontinuity of the concentration function \( \chi(1, \cdot) \). From Lemma A.7 we know that for any \( p \in M \) we have a cutoff function \( \tilde{\gamma} \in C^2(M) \) with \( \|\tilde{\gamma}\|_{C^2(M)} \leq C(n) \) and \( \chi_{B_1/2(p)} \leq \tilde{\gamma} \leq \chi_{B_1(p)} \). As \( B_1(p) \) can be covered by a number \( \Gamma = \Gamma(n) \) of balls with radius 1/2 by Lemma A.10, we obtain from Proposition 2.12 with \( \varepsilon_0^2 := \varepsilon^\ast \) for \( 0 \leq t \leq t^+ \)

\[
\Gamma^{-1} \chi(1, t) \leq \chi(1, t/2) \leq \chi(1, 0) + c \int_0^t \chi(1, s) ds + c \|DR\|_{L^\infty(M)}^2 \sup_{0 \leq \tau \leq t^+} |\Sigma| t
\]

\[
\leq \chi(1, 0) + c \int_0^t \chi(1, s) ds + \tilde{\alpha} t,
\]

where we have used Lemma 2.15 (note that \( C_{scal} \leq \varepsilon(n) \)), Cauchy’s inequality, and abbreviated \( \tilde{\alpha} := \|DR\|_{L^\infty(M)}^2 (|\Sigma|_{t=0} + \sqrt{2t^+W(f_0)}) \). From Gronwall’s inequality we infer
2.6 Estimating the lifespan

that

$$\chi(1, t) \leq (\Gamma \chi(1, 0) + \tilde{c}) \exp(c\Gamma t).$$

Using this inequality for $t := t^+$ together with (2.97) and estimating

$$|\Sigma|_{t=0} + \sqrt{2t^+}W(f_0) \leq 2 \exp(c\Gamma t^+)(|\Sigma|_{t=0} + W(f_0))$$

we obtain

$$\varepsilon_0^2 \leq (\Gamma \chi(1, 0) + \|DR\|_{L^{\infty}(M)}^2(|\Sigma|_{t=0} + W(f_0))) \exp(c(\Gamma)t^+)$$

and arrive at

$$t^+ \geq c(\Gamma) \log \left(\frac{\varepsilon_0^2}{\Gamma \chi(1, 0) + \|DR\|_{L^{\infty}(M)}^2(|\Sigma|_{t=0} + W(f_0))}\right).$$

Theorem 0.2 now follows after rescaling.
3 Blow-up of singularities

In [1] it was shown that singularities may occur either in finite time or at infinity if the ambient manifold is $(\mathbb{R}^n, \delta_{\text{eucl}})$. To study assumed singularities in the aforementioned cases if the ambient manifold is more generally a (possibly noncompact) complete Riemannian manifold $(M^n, g)$ of bounded geometry we want to perform a blow-up procedure as in [14]. Namely, we show that the blow-up is an immersed (time independent) Willmore surface that is either an embedded round sphere, or contains at least one component which is a nonumbilic (compact or noncompact) Willmore surface. To do so, it is essential to have a mass-density estimate and interior estimates to ensure compactness, i.e. $C^k$-subconvergence for the blow-up sequence using a reparametrization. The mass-density estimate and the interior estimates will be provided in the forthcoming sections.

3.1 Monotonicity formulas for Riemannian Manifolds

In the following lemma, it is again important to have “good” coordinates for $M$. The proof of the Michael-Simon Sobolev inequality (2.39) suggests to make use of harmonic coordinates since the bounds on the curvature of $M$ are rather week (cf. Lemma A.4). Unfortunately, we need to have Christoffel symbols growing at most linearly. Although there exist estimates (cf. [12]) for the Christoffel symbols in harmonic coordinates such that they are bounded by the squared radius of the geodesic ball they are defined on, we only have these bounds for a fixed chart. Also, at some point we make use of Gauss’s Lemma so that we choose Riemannian normal coordinates this time, under the drawback that the geometry of $(M, g)$ has to be bounded of order one.

Lemma 3.1 (Monotonicity formula for Riemannian manifolds). Let $(\Sigma^2, \tilde{g})$ and $(M^n, g)$ be complete (possibly noncompact) Riemannian manifolds and $f : (\Sigma, \tilde{g}) \to (M, g)$ be a proper isometric $C^2$-immersion. If $r_0 > 0$ can be chosen such that $r_0 \Lambda < c(n)$, where

$$\Lambda := \|R\|_\infty^{1/2} + \|\nabla R\|_\infty^{1/3} + \text{inj}(M, g)^{-1}$$

and $c = c(n)$ is a small universal constant, then

$$\frac{|\Sigma_\sigma|}{\sigma^2} \leq C \left( \frac{|\Sigma_\varrho|}{\varrho^2} + \mathcal{W}(\Sigma_\varrho) \right)$$

(3.1)

for all $0 < \sigma \leq \varrho \leq r_0$, where $C = C(n)$ only depends on the dimension $n$.

Remark: To see that in general (3.1) cannot hold for all $0 < \sigma \leq \varrho < \infty$, at least in
3 Blow-up of singularities

case $\Sigma$ and $M$ are closed, again take the standard sphere $S^2 \subset S^n \subset \mathbb{R}^{n+1}$ as described in the remark of Theorem 2.8. Since $W(S^2 \subset S^n) = 0$, letting $\varrho \nearrow \infty$ would imply that $|\Sigma_\sigma(p)| = 0$ for any $\sigma$ and any $p \in M$.

Proof: The proof imitates the proof of a monotonicity formula in case $(M, g) = (\mathbb{R}^n, \delta)$ (see [23]) using Riemannian normal coordinates locally. One has to find an appropriate test vector field in the first variation formula

$$\int_{\Sigma} \text{div}_\Sigma \phi \, d\mu = - \int_{\Sigma} g_{\phi}(\phi \circ f, H) \, d\mu,$$

where $\text{div}_\Sigma \phi = g_{\phi}(D\phi \cdot Df \cdot \tau_i, Df \cdot \tau_i)$, $\phi \in C^0_\infty(M, TM)$ is a Lipschitz vector field with compact support, $H$ is the mean curvature of $f$, $\{\tau_i\}_{i=1,2}$ is a locally defined $\tilde{g}$-orthonormal basis and summation over repeated indices is used.

For this, let $p \in M$ be arbitrary but fixed, $\sigma, \varrho$ as in the statement and fix Riemannian normal coordinates $\{y^\alpha\} = \varphi_p : B_0^r(p) \to B_0^r(\mathbb{R}^n)$, with respect to $g$ centred at $p$. From Lemma A.5 we know that

$$\begin{align*}
\text{i) } & (1 - Cr^2 \Lambda^2) \delta \leq (g_{\alpha\beta}) \leq (1 + Cr^2 \Lambda^2) \delta & \text{on } B^r_0(p) \text{ for all } 0 \leq r < r_0 \\
\text{ii) } & \sup_{B^r_0(p)} |\Gamma_{\beta\delta}^\alpha| \leq Cr \Lambda^2 & \text{for all } 0 \leq r < r_0.
\end{align*}$$

For $x \in B^r_{r_0}$ define the vector field $\tilde{\phi} \in C^0_\infty(B^r_{r_0}, \mathbb{R}^n)$ by

$$\tilde{\phi}(x) := \left( \frac{1}{\max(|x|, \sigma)^2} - \frac{1}{\varrho^2} \right) x,$$

and

$$\phi := (\varphi_p)^* \tilde{\phi}.$$

To simplify notation we may by locality assume that $\Sigma \hookrightarrow M$ is embedded. We compute for $\{\tau_i\}$ as above

$$D\phi \cdot \tau_i = \tau_i^\alpha D\phi \cdot \partial_\alpha = \tau_i^\alpha D(\phi^\beta \partial_\beta) \cdot \partial_\alpha = \tau_i^\alpha (\partial_\alpha \phi^\beta) \partial_\beta + \tau_i^\alpha \phi^\beta \Gamma_{\alpha\beta}^\gamma \partial_\gamma.$$

Let $P = P^\top_g$ be the $g$-orthonormal projection onto the tangent bundle $T\Sigma$, in coordinates

$$P^\beta_\alpha = g(P(\partial_\alpha), \partial_\gamma) g^{\beta\gamma} = g(\tau_i, \partial_\alpha) g(\tau_i, \partial_\gamma) g^{\beta\gamma},$$

and define

$$\Sigma_\sigma := \{ q \in \Sigma \cap B_{r_0} : |q| < \sigma \}$$

and the annulus

$$\Sigma_{\sigma, \varrho} := \{ q \in \Sigma \cap B_{r_0} : 0 < \sigma \leq |q| < \varrho \},$$

56
where here and in what follows we identify \((B_{r_0}, g)\) with \((\hat{B}^n_r,(g_{\alpha\beta}))\) via the coordinates \(\varphi_p\).

First, assume that \(q \in \Sigma_\sigma\). We get since \(\phi(q) = (\sigma^{-2} - \rho^{-2}) q^\alpha \partial_\alpha\)

\[
D\phi \cdot \tau_i(q) = (\sigma^{-2} - \rho^{-2}) \left[ g^{\beta\gamma} g(\tau_i, \partial_\beta) \delta^\alpha_{\gamma} \partial_\alpha + g^{\beta\gamma} g(\tau_i, \partial_\beta) q^\delta \Gamma^\alpha_{\gamma\delta} \partial_\alpha \right] = (\sigma^{-2} - \rho^{-2}) \left[ \tau_i + \tau_i^\gamma q^\delta \Gamma^\alpha_{\gamma\delta} \partial_\alpha \right]
\]

which yields, using \(\tau_i^\gamma = g(\tau_i, \partial_\rho) q^\rho\),

\[
g(D\phi \cdot \tau_i, \tau_i) = (\sigma^{-2} - \rho^{-2})(2 + P^\alpha_{\rho\gamma} \Gamma^\rho_{\gamma\alpha}\).
\]

Therefore, abbreviating \(f^\alpha \Gamma_{\alpha\beta}^\gamma \circ f P^\beta_{\gamma f} =: f \Gamma P\) we generally have for an immersion \(f\)

\[
div_{\Sigma} \phi = (\sigma^{-2} - \rho^{-2})(2 + f \Gamma P). \quad \text{(3.2)}
\]

Now assume that \(q \in \Sigma_{\sigma, e}\). Since \(\phi(q) = (|q|^{-2} - \rho^{-2}) q^\alpha \partial_\alpha\) and \(\partial_\gamma |q|^{-2} = -2 q^\rho \delta_{\rho\gamma} |q|^{-4}\) it follows with \(X(q) := q^\alpha \partial_\alpha\)

\[
D\phi \cdot \tau_i = \tau_i^\gamma (|q|^{-2} - \rho^{-2}) \partial_{\gamma} - 2 q^\rho \delta_{\rho\gamma} |q|^{-4} \tau_i^\gamma X + \tau_i^\gamma (|q|^{-2} - \rho^{-2}) q^\delta \Gamma^\rho_{\gamma\delta} \partial_\delta.
\]

Due to the existence of two Riemannian metrics \(g\) and \(\delta\), we need to be careful, when lowering indices as in \(q^\rho \delta_{\rho\gamma}\). Since \(X\) is a radial vector field, we infer from Gauss’s Lemma that

\[
\delta(P^\gamma_g X, X) = g(P^\gamma_g X, X) = |P^\gamma_g X|^2_g.
\]

Thus, using \(q^\rho \delta_{\rho\gamma} \tau_i^\gamma g(X, \tau_i) = \delta(X, \tau_i) g(X, \tau_i) = \delta(P^\gamma_g X, X)\), we now have

\[
g(D\phi \cdot \tau_i, \tau_i) |_q = (|q|^{-2} - \rho^{-2})(2 + q^\rho \Gamma^\rho_{\alpha\gamma} P^\gamma_g) - 2 |q|^{-4} |P^\gamma_g X|^2_g,
\]

or generally,

\[
div_{\Sigma} \phi = (|f|^{-2} - \rho^{-2})(2 + f \Gamma P) - 2 |P^\gamma_g X|^2_g f |f|^{-4}. \quad \text{(3.3)}
\]

In the sequel, we do not distinguish between \(X\) and \(X \circ f\). By (3.2) and (3.3) we get

\[
\frac{1}{2} \int_{\Sigma_\sigma} div_{\Sigma} \phi = \sigma^{-2} |\Sigma_\sigma| - \rho^{-2} |\Sigma_\sigma| + \frac{1}{2} \int_{\Sigma_\sigma} f \Gamma P - \frac{1}{2} \int_{\Sigma_\sigma} f \Gamma P
\]

\[
+ \int_{\Sigma_{\sigma, e}} |f|^{-2} - \int_{\Sigma_{\sigma, e}} \frac{|P^\gamma_g X|^2_g}{|f|^4} + \frac{1}{2} \int_{\Sigma_{\sigma, e}} |f|^{-2} f \Gamma P. \quad \text{(3.4)}
\]

Since \(\phi(q) = (\max(|q|, \sigma)^{-2} - \rho^{-2})_+ X(q)\) it follows, abbreviating \(g \circ f(Y, Z)\) with \(Y \cdot Z\),

\[
- \int_{\Sigma_{\sigma, e}} \phi \cdot H = - \int_{\Sigma_{\sigma, e}} \sigma^{-2} X \cdot H - \int_{\Sigma_{\sigma, e}} |f|^{-2} X \cdot H + \int_{\Sigma_{\sigma}} \rho^{-2} X \cdot H. \quad \text{(3.5)}
\]
3 Blow-up of singularities

Completing the square yields

\[ \sigma^{-2}|\Sigma_\sigma| + \int_{\Sigma_{\sigma, \varrho}} \left| \frac{1}{4} H + \frac{P^\perp g X}{|X|^2_{\delta}} \right|^2 \]

\[ \leq \varrho^{-2}|\Sigma_\varrho| + 1/8 W(\Sigma_\varrho) + \frac{1}{2\varrho^2} \int_{\Sigma_{\sigma}} X \cdot H - \frac{1}{2\sigma^2} \int_{\Sigma_{\sigma}} X \cdot H - \frac{1}{2\sigma^2} \int_{\Sigma_{\sigma}} f \Gamma P \]

\[ - \frac{1}{2} \int_{\Sigma_{\sigma, \varrho}} |X|^{-2} f \Gamma P + \frac{1}{2\varrho^2} \int_{\Sigma_{\sigma, \varrho}} f \Gamma P + \int_{\Sigma_{\sigma}} |X|^{-2} \left( \frac{|X|^2_{\varrho}}{|X|^2_{\delta}} - 1 \right). \]

Because \(|X \cdot H| \leq c|X|g|H| \leq cr|H|g\) on \(\Sigma_r\) for any \(r \leq r_0\), we get after absorption and dropping the square terms on the left-hand side

\[ \frac{|\Sigma_\sigma|}{\sigma^2} \leq c(n) \left( \frac{|\Sigma_\varrho|^2}{\varrho^2} + W(\Sigma_\varrho) \right) - \frac{1}{2} (\sigma^{-2} - \varrho^{-2}) \int_{\Sigma_{\sigma}} f \Gamma P \]

\[ + \int_{\Sigma_{\sigma, \varrho}} |f|^{-2} \left( \frac{g_{\alpha \beta} g f^\alpha f^\beta \alpha \beta}{|f|^2} - 1 \right) - \frac{1}{2} \int_{\Sigma_{\sigma, \varrho}} \left( |f|^{-2} - \varrho^{-2} \right) f \Gamma P. \]

In what follows, we estimate the integrals on the right-hand side.

Ad 1: To begin with, we trivially have \(|f^\alpha| \leq \sigma\) on \(\Sigma_\sigma\) by definition of \(\Sigma_\sigma\). The projection \(P = P^\perp g\) is, as expected, also estimated: \(\sum_\beta |P^\alpha_\beta|^2 = |P \partial_\alpha|^2 \leq c|P \partial_\alpha|^2 \leq c|\partial_\alpha|^2 = c g_{\alpha \alpha} \leq c(n)\). Since \(|\Gamma|_{\delta} \leq c\sigma \Lambda^2\) on \(B_\sigma(p)\) we get

\[ |f \Gamma P| \leq c(n) \sigma^2 \Lambda^2 \leq c(n) r_0^2 \Lambda^2 \leq 1/2, \]

choosing \(r_0\) smaller if necessary. Therefore

\[ \left| \frac{1}{2\sigma^2} \int_{\Sigma_{\sigma}} f \Gamma P \right| \leq \frac{|\Sigma_\sigma|}{4\sigma^2}. \quad (3.6) \]

Ad 2: Since \(\varrho^{-2} \leq \sigma^{-2}\) this integral is treated as the latter.

Ad 3: If one again uses Gauss’s Lemma, it is immediate that this integral vanishes. Namely, it is \(|X|^2_{\varrho} = |X|^2_{\delta}\) because \(\exp\) is a radial isometry. If one had other coordinates, only using that the eigenvalues of \((g_{\alpha \beta})\) grow at most quadratically (as it is the case here), one could alternatively argue as follows without using Gauss’s Lemma: Taking

\[ - c(n) \Lambda^2 r^2 \delta \leq (g_{\alpha \beta} - \delta_{\alpha \beta}) \leq c(n) \Lambda^2 r^2 \delta \]

(3.7)
on $B_r(p)$ into account, we get for any $\sigma \leq s \leq r \leq \varrho$

$$\left| \int_{\Sigma_{s,r}} |f|^{-2} \left( \frac{g_{\alpha\beta} f^\alpha f^\beta}{|f|^2} - 1 \right) \right| \leq c(n) \frac{r^2}{s^2} |\Sigma_{s,r}| \Lambda^2.$$ 

Restricting to special annuli, i.e. annuli with ratio $r/s \leq 2$, we can further estimate

$$\leq 4c(n) |\Sigma_{s,r}| \Lambda^2 
\leq \frac{|\Sigma_{s,r}|}{r_0^2} \tag{3.8}$$

since $r_0 \Lambda \leq (4c(n))^{-1/2}$ choosing $r_0$ smaller if necessary. Now choose a geometric decomposition of $\Sigma_{\sigma,\varrho}$ in annuli $\Sigma_{\sigma_i,\sigma_{i+1}}$, such that $\frac{\sigma_{i+1}}{\sigma_i} < 2$. From (3.8) it follows with $N = N(\sigma, \varrho) < \infty$ that

$$\left| \int_{\Sigma_{\sigma,\varrho}} |f|^{-2} \left( \frac{g_{\alpha\beta} f^\alpha f^\beta}{|f|^2} - 1 \right) \right| \leq \frac{1}{r_0^2} \sum_{i=1}^{N} |\Sigma_{\sigma_i,\sigma_{i+1}}| \leq \frac{|\Sigma_{\sigma,\varrho}|}{\varrho^2} \leq \frac{|\Sigma_{\varrho}|}{\varrho^2}.$$

Ad 4: Analogously to the third integral, again using a decomposition, we can estimate

$$\int_{\Sigma_{\sigma_i,\sigma_{i+1}}} |f|^{-2} |f \Gamma P| \leq c(n) \frac{1}{\sigma_i^2} |\Sigma_{\sigma_i,\sigma_{i+1}}| \sigma_{i+1}^2 \Lambda^2 \leq \frac{|\Sigma_{\sigma_i,\sigma_{i+1}}|}{\varrho^2}.$$

Ad 5: Similarly,

$$\frac{1}{\varrho^2} \int_{\Sigma_{\sigma,\varrho}} |f \Gamma P| \leq c(n) \Lambda^2 |\Sigma_{\sigma,\varrho}| \leq \frac{|\Sigma_{\varrho}|}{\varrho^2}.$$

Combining, we finally arrive at $\sigma^{-2} |\Sigma_\sigma| \leq c(n) \left( \varrho^{-2} |\Sigma_\varrho| + \mathcal{W}(\Sigma_\varrho) \right)$. \hfill \blacksquare

Unfortunately, the monotonicity formula of Lemma 3.1 only holds for small radii, with smallness depending on the geometry of $(M, g)$. For large radii we are forced to make additional assumptions to prove the following Corollary.

**Corollary 3.2** (Mass-density estimate in the large by means of a mass bound).

Let $f : \Sigma \times [0, T) \to (M, g)$ be a Willmore flow of a closed surface into a Riemannian manifold of bounded geometry. Assume that $\mathcal{M}_f := \sup_{t \in [0, T)} \mu_1^f(\Sigma) < \infty$. Then for all $R > 0$

$$\frac{|\Sigma_R|}{R^2} \leq c \mathcal{M}_f \left( \left\| R \right\|_{\infty} + \left\| DR \right\|_{\infty}^{2/3} + \text{inj}(M, g)^{-2} \right) + c \mathcal{W}(f_{t=0}). \tag{3.9}$$
3 Blow-up of singularities

In case $T < \infty$, we have
\[ \mathcal{M}_f \leq \sqrt{2TW(f_0) + \mu_{f_0}(\Sigma)} < \infty. \] (3.10)
If the sectional curvature $K^M$ of $(M,g)$ is uniformly negative, i.e. if $K^M \leq -\kappa^2 < 0$, then it is
\[ \mathcal{M}_f \leq \frac{W(f_0) - 4\pi\chi(\Sigma)}{2\kappa^2}. \] (3.11)

(3.10) and (3.11) hold for all smooth Riemannian manifolds.

**Proof:** We clearly have $W(\Sigma_{\varrho}) \leq W(f_t) \leq W(f_0)$ by definition of the $L^2$-gradient flow (see the proof of Lemma 2.15). The claim now follows easily from Lemma 3.1, Lemma 2.15 and (2.3).

3.2 Interior estimates II

Although we already have proven interior estimates for the second fundamental form in $C^k$ by Proposition 2.16 we cannot use them for the blow-up at infinity owing to their dependence on the curvature of the initial surface. Thus, in case of singularities at infinity, we need Lemma 3.3 as a second version. Compared to [14], Theorem 3.5, our estimates are slightly worse, and more complicated to prove. First, we were not able to prove that the curvature in (3.14) grows at most as fast as the square root of the local energy concentration (3.12). This is mainly due to the existence of the volume-term in (2.69) of the integral estimates that we cannot control. For the same reason, we only have estimates depending on initial conditions. Namely, we have to assume (3.13). Second, the proof is more complicated since we have to localize in time twice. This is because of the existence of higher homogeneity curvature terms in (2.69) that itself results from the interpolation of the curvature terms $Q^{m+2,1}$ in (2.32).

**Lemma 3.3** (Interior estimates II). Let $(M,g)$ be a Riemannian manifold of bounded geometry and $p \in M$. Let $\varrho > 0$ be chosen such that
\[ \varrho < \min \left\{ \frac{\pi}{2\kappa}, \text{inj}(M,g) \right\}. \]

Let $f : \Sigma \times [0,T] \to (M,g)$ be a Willmore flow of a closed surface satisfying $T \leq C(n)\varrho^4$ and
\[ \sup_{0 \leq t \leq T} \int_{\Sigma_{\varrho}(p)} |A|^2 d\mu \leq \varepsilon_0(n) \] (3.12)
for small universal \( \varepsilon_0 > 0 \), where \( \Sigma_0^g(p) := f^{-1}(B_0^g(p)) \). Moreover, choose \( \Lambda_K < \infty \) such that

\[
\varrho (q^2 + \mathfrak{M}_f) \sum_{i=1}^K \| D^i R^M \|_{L^\infty(M,g)}^{3/2} < \Lambda_K.
\]

(3.13)

for some \( K \in \mathbb{N} \). Then we have for all \( t \in (0, T] \)

\[
\| \nabla^k A \|_{L^2(\Sigma_{e/2})} \leq c(n, k, C, \Lambda_{k+1}) t^{-\frac{k}{4}}
\]

(3.14)

and

\[
\| \nabla^k A \|_{L^\infty(\Sigma_{e/2})} \leq c(n, k, C, \Lambda_{k+3}) t^{-\frac{k+1}{4}}.
\]

Remark: Note that \( \Lambda_K < \infty \) can always be chosen since \( \mathfrak{M}_f \leq \sqrt{2TW}(f_0) + \mu f_0(\Sigma) < \infty \) by (3.10).

Proof: After scaling, we may assume that \( \rho = 1 \). Let \( c_k := c(\Lambda_k) \). Using Lemma A.7 we choose a cutoff function \( \tilde{\gamma} \in C^2(M) \) with \( \chi_{B_{63/64}} \leq \tilde{\gamma} \leq \chi_{B_1} \) and \( \| D\tilde{\gamma} \|_\infty \leq c(n) \) \((i = 1, 2)\).

From Proposition 2.12 we then get

\[
\int_0^T \int_{\Sigma_{63/64}} |\nabla^2 A|^2 + |A|^6 \, d\mu \, dt \leq \varepsilon_0 + c_1 \varepsilon_0 T + c_1 T \leq c_1,
\]

(3.15)

where we used

\[
\| DR \|_\infty^2 \sup_{0 \leq t \leq T} |\Sigma| \leq c_1 \| DR \|_\infty \mathfrak{M}_f \leq c_1.
\]

(3.16)

Using (2.56) from Lemma 2.11 we get for \( \tilde{\gamma} \) as above with \( \chi_{B_{31/32}} \leq \tilde{\gamma} \leq \chi_{B_{63/64}} \)

\[
\int_0^T \| A \|_{\infty, \Sigma_{31/32}}^4 \, dt \leq c\varepsilon_0 \cdot (c_1 + c\varepsilon_0 T) \leq c_1.
\]

(3.17)

Now let \( \gamma := \tilde{\gamma}_0 f \) for a cutoff function with \( \chi_{B_{15/16}} \leq \tilde{\gamma} \leq \chi_{B_{31/32}} \). Furthermore, define a cutoff function in time by

\[
\psi(t) := \frac{t}{T}
\]

such that

\[
\dot{\psi}(t) = \frac{1}{T}.
\]

(3.18)
3 Blow-up of singularities

Introducing the notation \( \alpha_j(t) := c_{2j+1}(1 + \|A\|_{|\Sigma_{t/2}}^4) \) and \( F_j(t) := \int_\Sigma |\nabla A|^2 \gamma^{4j+4} \, d\mu \) we obtain from Proposition 2.14 for \( j \in \{0, 1\} \)

\[
\frac{d}{dt} F_j(t) + \frac{1}{2} F_{j+1}(t) \leq \alpha_j(t) F_j(t) + \alpha_j(t),
\]

(3.19)

where we used an estimate similar to (3.16). Further we compute with \( h_j(t) := \psi(t) F_j(t) \)

\[
\frac{d}{dt} h_j(t) + \frac{t}{2T} F_{j+1}(t) \leq \alpha_j(t) h_j(t) + \frac{t}{T} \alpha_j(t) + \frac{1}{T} F_j(t).
\]

(3.20)

Now let \( \chi_{B_{3/4}} \leq \tilde{\chi} \leq \chi_{B_{7/8}} \) and \( 0 \leq j \leq m \). We define cutoff functions in time similar to [14] by

\[ \chi_j(t) := \begin{cases} 
0 & \text{for } t \leq (j-1) \frac{T}{2m} + \frac{T}{2} \\
\frac{2m}{T} (t - \frac{T}{2} - (j-1) \frac{T}{2m}) & \text{in between} \\
1 & \text{for } t \geq j \frac{T}{2m} + \frac{T}{2},
\end{cases} \]

(3.21)

if \( j \geq 1 \), and \( \chi_0(t) := 1 \). Now for \( j \geq 0 \) define analogously \( e_j(t) := \chi_j(t) E_j(t) \), where \( E_j \) equals to \( F_j \) except that we now have \( \chi_{B_{3/4}} \leq \tilde{\chi} \leq \chi_{B_{7/8}} \) and \( 0 \leq j \leq m \). Restricting to \( j \geq 1 \), since \( 0 \leq \chi_j(t) \leq \frac{2m}{T} \chi_{j-1} \) we again get from Proposition 2.14

\[
\frac{d}{dt} e_j(t) + \frac{1}{2} \chi_j(t) E_{j+1}(t) \leq \hat{\alpha}_j(t) e_j(t) + \chi_j(t) \hat{\beta}_j(t) + \frac{2m}{T} \chi_{j-1}(t) E_j(t)
\]

(3.22)

if we now define \( \hat{\alpha}_j(t) := c_{2j+1}(1 + \|A\|_{|\Sigma_{t/2}}^4) \) and \( \hat{\beta}_j(t) := c_{2j+1}(1 + \|A\|_{|\Sigma_{t/2}}^{4j}) \). Applying Gronwall’s Lemma yields for any \( t \in [T/2, T] \)

\[
e_j(t) + \frac{1}{2} \int_{T/2}^t \chi_j(s) E_{j+1}(s) \, ds
\]

\[
\leq \exp \left( \int_{T/2}^T \hat{\alpha}_j(s) \, ds \right) \left[ e_j(T/2) + \int_{T/2}^t \chi_j(s) \hat{\beta}_j(s) \, ds + \frac{2m}{T} \chi_{j-1}(s) E_j(s) \, ds \right]
\]

\[
\leq \exp \left( c_{2j+1} + c_{2j+1} \int_{T/2}^T \|A\|_{|\Sigma_{t/2}}^4 \, ds \right)
\]

\[
\left[ c_{2j+1} + c_{2j+1} \int_{T/2}^t \|A\|_{|\Sigma_{t/2}}^{4j} \, ds + \frac{2m}{T} \chi_{j-1}(s) E_j(s) \, ds \right].
\]

(3.23)
To be able to estimate \( \int_{T/2}^{T} \| A \|^4_{\infty, \Sigma_{7/8}} \, ds \), we apply Gronwall’s Lemma to (3.20) and get for \( t \in [0, T] \)

\[
\frac{t}{T} F_1(t) + \frac{1}{2T} \int_{0}^{t} s F_2(s) \, ds
\]

\[
= \frac{t}{T} \int_{\Sigma} |\nabla^2 A|^2 \gamma^8 \, d\mu + \frac{1}{2T} \int_{0}^{t} s \int_{\Sigma} |\nabla^4 A|^2 \gamma^{12} \, d\mu \, ds
\]

\[
\leq \exp \left( c_3 \cdot (T + \int_{0}^{T} \| A \|^4_{\infty, \Sigma_{31/32}} \, ds) \right) \left[ \int_{0}^{T} \frac{s}{T} \alpha_1(s) \, ds + \frac{1}{T} \int_{0}^{T} \int_{\Sigma_{31/32}} |\nabla^2 A|^2 \, d\mu \, ds \right]
\]

\[
\leq c_3 \left[ c_3 + c_3 \int_{0}^{T} \| A \|^4_{\infty, \Sigma_{31/32}} \, ds + \frac{1}{T} c_1 \right],
\]

where we used (3.15) for the last term. With (3.17) and \( T \leq C(n) \) we further obtain \( t F_1(t) \leq c_3 \) such that we finally get on \([T/2, T]\)

\[
\int_{\Sigma_{15/16}} |\nabla^2 A|^2 \, d\mu \leq \frac{c_3}{T}.
\]

Employing Lemma 2.11 for \( \chi_{B_{7/8}} \leq \tilde{\gamma} \leq \chi_{B_{15/16}} \) yields

\[
\| A \|^4_{\infty, \Sigma_{7/8}} \leq c \varepsilon_0 \cdot \left( \frac{c_3}{T} + c_0 \right) \leq \frac{c_3}{T}
\]

for all \( t \in [T/2, T] \). Using this, we can go on estimating (3.23)

\[
e_j(t) + \frac{1}{2} \int_{T/2}^{t} \chi_j(s) E_{j+1}(s) \, ds
\]

\[
\leq \exp \left( c_{2j+1} \right) \left[ c_{2j+1} + c_{2j+1} \frac{T}{2} c_3 \right] + \frac{2m}{T} \int_{T/2}^{t} \chi_{j-1}(s) E_j(s) \, ds
\]

\[
\leq \frac{c_{2j+1}}{T^{j-1}} + c_{2j+1} \frac{2m}{T} \int_{T/2}^{t} \chi_{j-1}(s) E_j(s) \, ds.
\]

(3.25)

Now we show by induction that for \( 0 \leq j \leq m \) and all \( t \in [T/2, T] \)

\[
e_j(t) + \frac{1}{2} \int_{T/2}^{t} \chi_j(s) E_{j+1}(s) \, ds \leq c_{2j+1} \frac{1}{T^j}.
\]
3 Blow-up of singularities

For $j = 0$ we obtain using (3.15) (recalling $\chi_{B_{3/4}} \leq \tilde{\gamma} \leq \chi_{B_{7/8}}$)

$$\int_{\Sigma} |A|^2 \gamma^4 d\mu + \frac{1}{2} \int_{T/2}^{t} \int_{\Sigma} |\nabla^2 A|^2 \gamma^8 d\mu \leq \varepsilon + c_1 \leq c_1.$$ 

For $j \geq 1$ we can estimate, using (3.25),

$$e_j(t) + \frac{1}{2} \int_{T/2}^{t} \chi_j(s) E_{j+1}(s) ds \leq c_{2j+1} \frac{T_{j-1}}{T} + c_{2j+1} \frac{4m}{T} c_{2j-1}(m) \leq c_{2j+1}(m) \quad \forall j \geq 1,$$

Thus we have at time $t = T$

$$\int_{\Sigma} |\nabla^{2m} A|^2 \gamma^{4m+4} d\mu \leq \frac{c_{2m+1}(m)}{T^m}.$$ 

To get an estimates for odd order derivatives one may simply apply Lemma A.11 with $r = 1$, $p = q = 2$, $\alpha = 1$, $\beta = 0$, $s = 4m + 6$ and $t := s^{-1} \in [-\frac{1}{2}, \frac{1}{2}]$ to obtain

$$\int_{\Sigma} |\nabla^{2m+1} A|^2 \gamma^{4m+6} d\mu \leq \frac{c_{2m+3}(m)}{T^{m+1/2}}. \quad (3.26)$$

In contrast to the time exponent, the argument does not close for the order of the bounded geometry. More precisely, when using the interpolation inequality we have a loss of one order with respect to the geometry bounds $c_k = c(\Lambda_k)$. Instead, it is possible to estimate the odd order derivatives directly analogously to the above estimates: We let $\chi_{B_{3/4}} \leq \tilde{\gamma} \leq \chi_{B_{7/8}}$, $j \geq 1$, $k_j(t) := \chi_j(t) K_j(t)$, $\chi_j(t)$ as in (3.21),

$$K_j := \int_{\Sigma} |\nabla^{2j+1} A|^2 \gamma^{4j+2} d\mu, \quad \tilde{\alpha}_j(t) := c_2(1+\|A\|^4_{\infty, \Sigma_{7/8}}) \quad \text{and} \quad \tilde{\beta}_j(t) := c_2(1+\|A\|_{\infty, \Sigma_{7/8}}^{\max\{4,4j-2\}}).$$

From (3.24) we obtain for $j \geq 2$

$$c_{2j} \int_{T/2}^{t} \|A\|_{\infty, \Sigma_{7/8}}^{\max\{4,4j-2\}} ds \leq \frac{c_{2j}}{T^{3/2}},$$

and thus

$$k_j(t) + \frac{1}{2} \int_{T/2}^{t} \chi_j(s) K_{j+1}(s) ds \leq \frac{c_{2j}}{T^{3/2}} + \frac{c_{2j}(m)}{T} \int_{T/2}^{t} \chi_{j-1}(s) K_j(s) ds.$$
As above, we show by induction that for all $j \geq 1$ and all $t \in [T/2, T]$
\[
k_j(t) + \frac{1}{2} \int_{T/2}^t \chi_j(s) K_{j+1}(s) ds \leq \frac{c_2(m)}{T^{j-1/2}}.
\]

(3.27)

For $j = 1$ we again use Lemma A.11 with $r = 1$, $p = q = 2$, $\alpha = 1$, $\beta = 0$, $s = 4m + 6$ and $t := s^{-1} \in [-\frac{1}{2}, \frac{1}{2}]$ to obtain
\[
\int_{\Sigma} |\nabla A|^2 \gamma^6 d\mu \leq c\|A\|_{2,|\gamma>0} \|\nabla^2 A\|_{2,|\gamma>0} + c\|A\|^2_{2,|\gamma>0} + \frac{1}{2} \int_{\Sigma} |\nabla A|^2 \gamma^6 d\mu.
\]

From this we get
\[
\int_{T/2}^t \int_{\Sigma} |\nabla A|^2 \gamma^6 d\mu ds \leq c\sqrt{\varepsilon_0 T} \left( \int_{0}^{T} \int_{\Sigma_{63/64}} |\nabla^2 A|^2 \gamma^6 d\mu ds \right)^{1/2} + c\varepsilon_0 T \overset{(3.15)}{\leq} c_1 \sqrt{T}
\]
and therefore
\[
k_1(t) + \frac{1}{2} \int_{T/2}^t \chi_1(s) K_2(s) ds \leq c_2 \left( c_2 + \frac{c_2(m)}{T} \int_{T/2}^t \int_{\Sigma} |\nabla A|^2 \gamma^6 d\mu ds \right) \leq \frac{c_2}{T^{1/2}}.
\]

As above we use Gronwall’s Lemma obtaining for all $j \geq 2$
\[
k_j(t) + \frac{1}{2} \int_{T/2}^t \chi_j(s) K_{j+1}(s) ds \leq \frac{c_2j}{T^{j-3/2}} + \frac{c_2j(m)}{T} \frac{c_2j-2}{T^{j-3/2}} \leq \frac{c_2j}{T^{j-1/2}}
\]
and therefore it follows from (3.27)
\[
\int_{\Sigma} |\nabla^{2m+1} A|^2 \gamma^{4m+6} d\mu \leq \frac{c_{2m+2}}{T^{m+1/2}}.
\]

(3.28)

Therefore, we have now shown that for all $k \in \mathbb{N}_0$
\[
\|\nabla^k A\|^2_{2,\Sigma_{3/4}} \bigg|_{t=T} \leq \frac{c(n, k, \Lambda_{k+1})}{T^{k/2}}.
\]

From (3.24) and (2.54) we finally obtain
\[
\|\nabla^k A\|_{\infty, \Sigma_{1/2}} \bigg|_{t=T} \leq \frac{c(n, k, \Lambda_{k+3})}{T^{k+1/2}}
\]
and thus the claim follows after rescaling and renaming $T$ into $t$. \[\blacksquare\]
3 Blow-up of singularities

3.3 Blow-up of singularities

The goal of this section is to prove Theorem 0.3. For this let \( f: \Sigma \times [0, T) \rightarrow (M^n, g) \) be a smooth maximal Willmore flow defined on a closed surface \( \Sigma \), \( f_t := f(\cdot, t) \) and \( \mu_f(t) = \mu_{f_t} \) be the area measure on \( \Sigma \) induced by \( \tilde{g}(t) := (f_t)^* g \). For \( t \in [0, T) \) we let

\[
\nu_{f_t} := f_t(\mu_{f_t}|_{A_{f_t}}),
\]

be the (finite) Radon measures defined on the Borel \( \sigma \)-Algebra \( \mathcal{B}(M) \) induced by the system of open sets in \( M \). Let further \( \chi_{f_t}(r, t) := \sup_{p \in M} \nu_{f_t}(B_{r,g}(p)) \).

We want to make the arrangement that all geometric quantities without further specification refer to \( f_t \) (as a map between Riemannian manifolds), e.g. \( \mu_{f_t} = \mu \), \( A_{f_t} = A \), \( |\cdot|_{f_t} = |\cdot| \). Furthermore, when considering a sequence of functions, we want to allow choosing subsequences without further mention. At first, as remarked in [15] in case of \((\mathbb{R}^n, \delta_{eucl})\) as the ambient manifold, one can analogously show that \( E(p, r, t) := \nu_{f_t}(B_{r,g}(p)) \) is upper semi-continuous in all variables and hence the same also holds for \( \chi \). The author showed in his diploma thesis ([19], Hilfslemma 5.2), that \( \hat{\chi} \) is lower semi-continuous in the time variable. Using a simple covering argument, we get for \( \Gamma = \Gamma(n) \) as in Lemma A.10

\[ \chi(r, t) \leq \Gamma \chi(r/2, t) \leq \Gamma \hat{\chi}(r, t) \]

for all \( r > 0 \) with \( r^2 \|\text{ricci}(M,g)\|_{L^\infty(M,g)} \leq 1 \). Thus we have in summary

\[
\hat{\chi}(r, t_0) \leq \liminf_{t \to t_0} \chi(r, t) \leq \limsup_{t \to t_0} \hat{\chi}(r, t) \leq \Gamma \hat{\chi}(r, t_0)
\]

and

\[
\Gamma^{-1} \chi(r, t_0) \leq \liminf_{t \to t_0} \chi(r, t) \leq \limsup_{t \to t_0} \chi(r, t) \leq \chi(r, t_0).
\]

The next Theorem is a precursor of Theorem 0.3.

**Theorem 3.4 (Existence of a blow-up I).** Let \( f: \Sigma \times [0, T) \rightarrow (M, g) \) be a maximal Willmore flow on a closed surface \( \Sigma \) into a Riemannian manifold of bounded geometry (of order 15) with the property that the total area of \((\Sigma, \tilde{g}(t))\) is uniformly bounded on \((0, T)\), i.e.

\[
\mathfrak{M}_f := \sup_{t \in (0,T)} \mu_f(\Sigma) < \infty.
\]

Let further \( \{t_j\}, \{r_j\} \text{ and } \{p_j\} \ (j \in \mathbb{N}) \) be given sequences satisfying \( t_j \uparrow T, r_j \downarrow 0 \) and \( p_j \in M \). There exists constants \( \varepsilon_{0/1} = \varepsilon_{0/1}(n) > 0 \) such that if

\[
\liminf_{j \to \infty} \chi(r_j, t_j) \leq \varepsilon_1^2
\]
and

\[ \tau^- := \limsup_{j \to \infty} \frac{t_j^- - t_j}{r_j^4} < 0, \quad \text{where} \quad t_j^- = \inf \{ t \in [0, t_j] : \chi(r_j, \cdot) < \epsilon_0^2 \text{ on } (t, t_j) \} \]

then, after selection of a subsequence and reparametrization, the rescaled flows

\[ f_j : (\Sigma, \bar{g}_j) \times [ - r_j^{-4} t_j, r_j^{-4}(T - t_j) ] \to (M, g_j), \quad f_j(p, t) := f_{j,t}(p) := f(p, t + r_j^4 t), \]  \hspace{1cm} (3.33)

where \( g_j := r_j^{-2} g \) and \( \bar{g}_j(t) = (f_{j,t})^*(g_j) \), converge in \( C^{4,1} \) locally on \( \hat{\Sigma} \times (\tau^-, \tau^+) \) to a static solution, given by a properly immersed Willmore surface \( \hat{f}_0 : \hat{\Sigma} \to \mathbb{R}^n \).

More precisely there exists a sequence of radii \( h_j \nearrow \infty \), local coordinate charts \( \varphi_j : B_{h_j}^n(p_j) \to V_j \supset B_{2h_j}^n \), a 2-manifold \( \hat{\Sigma} \) without boundary (possibly not connected or even empty), open sets \( U_j \subset U_{j+1} \) with \( \bigcup_{j=1}^\infty U_j = \hat{\Sigma} \), diffeomorphisms \( \phi_j : U_j \to f_{j,0}^{-1}(\varphi_j^{-1}(B_n^j)) \), a time interval \( (\tau^-, \tau^+) \ni 0 \), open sets \( \Sigma_j \subset \Sigma \) satisfying \( \phi_j(U_j) \subset \Sigma_j \) and \( f_j(\Sigma_j, J) \subset B_{h_j}^n(p_j) \) for each \( J \subset (\tau^-, \tau^+) \) provided \( j \geq j_0(J) \), such that for \( \tilde{f}_j \circ \phi_j := \tilde{f}_j(\varphi_j(\cdot), \cdot) \) being well defined locally on \( \hat{\Sigma} \times (\tau^-, \tau^+) \), where \( \tilde{f}_j := \varphi_j \circ f_j \)

\[ \tilde{f}_j \circ \phi_j \to \hat{f}_0 \quad \text{locally in } C^{4,1} \text{ on } \hat{\Sigma} \times (\tau^-, \tau^+). \]  \hspace{1cm} (3.34)

Moreover, if \( \theta_j \circ \phi_j \to \theta \) locally uniformly on \( \hat{\Sigma} \), then

\[ \tilde{f}_j(\mu_{f_j}, \theta_j) \to \hat{f}_0(\mu_{\hat{f}_0}, \theta) \quad \text{for all } \tau \in (\tau^-, \tau^+) \]  \hspace{1cm} (3.35)

weakly as Radon measures.

**Remark:** Again, we tried to stick to the analysis and the way of proceeding as in [15]. Deviating from this, we assume the given arbitrary sequence \( r_j > 0 \) to be decreasing owing to the restriction to blow-ups. Related to this, we do not prove an upper bound \( \tau^- < 0 \) for the backward lifespan for the rescaled flows making an ad-hoc assumption instead. \( \diamond \)

**Proof of Theorem 3.4:** Let \( \epsilon_0 > 0 \) be as in Theorem 0.2 and \( \epsilon_1 := C \epsilon_0 / 6 \) for \( C \) as in (0.3). It is easy to see that \( f \) and \( f_j \) are equivalent maximal Willmore flows. We prove lower bounds for the forward lifespan of the rescaled flows. Choosing a subsequence, we may assume that

\[ r_j(1 + W(f_0)) + \mathfrak{M}_f \sum_{i=0}^{15} \| D^i R \|_{L^\infty(M, g)}^4 + r_j \text{inj}^{-1}(M, g) \leq \epsilon_1^2 \]  \hspace{1cm} (3.36)

and \( \chi(r_j, t_j) \leq 2 \epsilon_1^2 \). Note that with respect to the scaled metric, (3.36) holds for \( r_j = 1 \).

Letting

\[ \tau^+ := \liminf_{j \to \infty} \frac{t_j^+ - t_j}{r_j^4}, \quad \text{where} \quad t_j^+ := \sup \{ t \in [t_j, T) : \chi(r_j, \cdot) < \epsilon_0^2 \text{ on } [t_j, t) \}, \]

and

\[ \tilde{r}_j \tilde{\omega}(\tilde{\rho}^{-1}(\tilde{p})), 0 \leq \tilde{\omega} \leq 1 \]  \hspace{1cm} (3.37)

with \( \tilde{r}_j \) such that \( \tilde{r}_j \to \tilde{r}_0 \) as \( j \to \infty \) and \( \tilde{\rho}^{-1}(\tilde{p}) \) is a metric chart for \( \tilde{p} \in \tilde{\Sigma} \times (\tau^-, \tau^+) \).



67
3 Blow-up of singularities

Theorem 0.2 implies for $0 < \varepsilon_1 \leq C\varepsilon_0/6 < \varepsilon_0$

$$\liminf_{j \to \infty} \frac{T - t_j}{r_j^4} \geq \tau^+ \geq C \log \frac{C\varepsilon_0^2}{3\varepsilon_1^2} > 0.$$  

Also, we obviously have for the backward lifespan

$$\lim_{j \to \infty} \left( -\frac{t_j}{r_j^4} \right) = -\infty.$$  

Employing that $\chi_{f_j}(1, \cdot) \leq \varepsilon_0^2$ locally on $(\tau^-, \tau^+)$, we obtain from Lemma 3.3 that for $k \leq 12$

$$\|\nabla^k A_j\|_{L^\infty(\tilde{\Sigma}, \tilde{g}_j)} \leq c(n) \text{ locally on } (\tau^-, \tau^+). \tag{3.37}$$

In what follows, we want to employ a compactness result stated in Theorem 3.5, which is a slight variant of a compactness theorem, where the latter has originally been proven by J. Langer in [17] and then has been generalized by P. Breuning (cf. [2], Corollary 2.28).

**Theorem 3.5** (cf. [2]). Let $\bar{f}_j : \Sigma_j \to \mathbb{R}^n$ be a sequence of isometric $C^k$-immersions, where $\Sigma_j$ is a $m$-manifold without boundary, such that $\bar{f}_j^{-1}(B^n_{2j}) \subset \Sigma_j \subset (\Sigma^m, \tilde{g}_j)$ and $\Sigma$ is a closed surface. With the image measure $\mu_j = \bar{f}_j(\mu_{\bar{f}_j})$ assume that for all $j \in \mathbb{N}$

$$\mu^j(B_R^n) \leq C(R) \quad \text{for all } 0 < R < 2j,$$

$$\|\nabla^l \bar{f}_j\|_{L^\infty(B_R^n, \tilde{g}_j)} \leq C_l(R) \quad \text{for all } 0 < R < 2j \text{ and } 0 \leq l \leq k - 2, \tag{3.39}$$

where $\bar{A}_j$ denotes the second fundamental form of $\bar{f}_j$. Then there exists a proper immersion $\hat{f} : \hat{\Sigma} \to \mathbb{R}^n$, where $\hat{\Sigma}$ is again an $m$-manifold without boundary, such that after passing to a subsequence there is a sequence of diffeomorphisms

$$\phi_j : U_j \to (\bar{f}_j)^{-1}(B_j) \subset \Sigma_j,$$

where $U_j \subset \hat{\Sigma}$ are open sets with $U_j \subset \subset U_{j+1}$ and $\hat{\Sigma} = \bigcup_{j=1}^\infty U_j$, such that

$$\|\bar{f}_j \circ \phi_j - \hat{f}\|_{C^0(U_j)} \to 0, \tag{3.40}$$

and moreover $\bar{f}_j \circ \phi_j \to \hat{f}$ locally in $C^{k-1}$ on $\hat{\Sigma}$.  

**Remark:** The surface $\hat{\Sigma}$ is empty, if the $\bar{f}_j$ diverge uniformly, and is possibly not connected.  

To be able to apply this corollary locally to the blow-up sequence $\{f_{j,0}\}$, we want use for any $j \in \mathbb{N}$ $g_j$-harmonic coordinates $\varphi_j : B^n_{h_j}(p_j) \to V_j \subset \mathbb{R}^n$ of radius $h_j := 4j^2$
centred at \( p_j \) with \( \varphi_j(p_j) = 0 \). To check the assumptions in Lemma A.4, we may further assume that \( r_j \searrow 0 \) is such that \( r_j \Lambda_{(M,g)}(15) \leq (4j^3)^{-1}c(n) \) for some given small \( c(n) > 0 \), so that

\[
h_j \Lambda_{(M,g)} = 4j^2 r_j \Lambda_{(M,g)} \leq c(n)j^{-1}
\]

and thus we obtain the bounds

\[
\begin{align*}
 i) & \quad (1 - 1/j)\delta \leq G_j \leq (1 + 1/j)\delta \quad \text{on } V_j \\
 ii) & \quad \sup_{V_j} |\partial^\gamma G_j| \leq j^{-2\gamma} \quad \text{for all } 1 \leq \gamma \leq 16 \\
 iii) & \quad \sup_{V_j} |\partial^\gamma \Gamma^{(j)}| \leq j^{-2\gamma-2} \quad \text{for all } 0 \leq \gamma \leq 15
\end{align*}
\]

(3.41)

and \( G_j|_{x=0} = (\delta_{\alpha\beta}) \). Here, \( G_j \) denotes the matrix of \( (\varphi_j)_*(g_j) \) with respect to the standard coordinate frame \( \{e_\alpha\} \leq \alpha \leq n \) and \( \{\Gamma^{(j)}\}_j \in \mathbb{N} \) are the associated Christoffel’s symbols. Now since \( \|\partial_1 f_j\|_{L^\infty(\Sigma,\tilde{g}_j)} \leq c(n) \) we get that for any \( x \in \Sigma \)

\[
d^{\Sigma}(f_j(x,t_1),f_j(x,t_2)) \leq \left| \int_{t_1}^{t_2} |\partial_1 f_j|_{g_j}(x,\tau) d\tau \right| \leq c(n)|t_2 - t_1|.
\]

(3.42)

Letting \( 0 \in I := [t_1, t_2] \subset (\tau^-, \tau^+) \) be a compact interval, we have a well defined local coordinate representations for all \( f_j \) (for \( j \) large enough). Namely, with \( \Sigma_j := f_j^{-1}(B^n_{R_j}(p_j)) \subset \Sigma \), we define one-parameter families of isometric immersions

\[
\hat{f}_j : (\Sigma_j, \tilde{g}_j) \times I \to (V_j \subset \mathbb{R}^n, G_j)
\]

(3.43)

and

\[
\bar{f}_j : (\Sigma_j, \bar{g}_j) \times I \to (V_j \subset \mathbb{R}^n, \delta_{\text{eucl}})
\]

(3.44)

using the harmonic coordinates from above. Here, \( \hat{f}_j \) is the coordinate representation of \( f_j \), and \( \bar{f}_j := \hat{f}_j \) as maps between manifolds. For the rest of this section, a tilde and index \( j \) indicates that a geometric quantity is induced by \( \hat{f}_j \), as in \( \tilde{\nabla} \) or \( \tilde{A}_j \) etc. The same arrangement holds for barred quantities such as \( \tilde{\nabla} \) or \( \tilde{A}_j \) etc. By naturality of the geometric quantities and using (3.37), we have that \( \hat{f}_j \) is also a Willmore flow with

\[
\|\tilde{\nabla}^k \tilde{A}_j\|_{L^\infty(\Sigma_j,\tilde{g}_j)} \leq c(n)
\]

(3.45)

for all \( t \in I \) and \( k \leq 12 \). From Lemma A.3 it follows that also

\[
\|\nabla^k \bar{A}_j\|_{L^\infty(\Sigma_j,\bar{g}_j)} \leq c(n, \Gamma_{12}, A_{12}) \leq c(n)
\]

for all \( k \leq 12 \), \( t \in I \), and in particular for \( t = 0 \). As in (2.45) we see that \( \frac{1}{2} \bar{g}_j \leq \tilde{g}_j \leq 2\bar{g}_j \) implies the coordinate invariant relation \( \frac{1}{4} \det(\tilde{g}_j) \leq \det(\bar{g}_j) \leq 4 \det(\tilde{g}_j) \) and thus from Corollary 3.2

\[
\bar{f}_j(\mu_{f_j})(B^n_{R_j}) \leq 2\hat{f}_j(\mu_{f_j})(B^n_{R_j}) \leq CR^2
\]

for all \( R < 2j \). From Theorem 3.5 we get after passing to a subsequence that

\[
\hat{f}_j(\phi_j, 0) \to \hat{f}_0
\]
locally in $C^{13}$, where $\tilde{f}_0 := (\tilde{\Sigma}, \tilde{g}) \to (\mathbb{R}^n, \delta_{\text{eucl}})$ is a properly and isometrically immersed 2-manifold without boundary. Fix a countable atlas of $\tilde{\Sigma}$ comprising of Riemannian normal coordinate charts $\{\psi_k : B^2_{2r_k}(x_k) \to B^2_{2r_k} \subset \mathbb{R}^2\}_{k \in \mathbb{N}}$ with

$$2^{-1}\delta \leq (\tilde{g}_{im}) =: \tilde{G} \leq 2\delta. \quad (3.46)$$

For $k \in \mathbb{N}$ fixed, $r := r_k$, the harmonic coordinates from above induce a sequence of coordinate representations of $f_j \circ \phi_j = f_j(\phi_j(\cdot), \cdot)$

$$\tilde{F}_j : (B^2_r, \tilde{G}_j(t)) \times I \to (V_j \subset \mathbb{R}^n, G_j)$$

and

$$\bar{F}_j : (B^2_r, \bar{G}_j(t)) \times I \to (V_j \subset \mathbb{R}^n, \delta_{\text{eucl}})$$

for $j$ large enough, such that $\tilde{F}_{j,t}$ and $\bar{F}_{j,t}$ are isometric immersions and $\tilde{F}_j \equiv \bar{F}_j$ as maps between manifolds. We now want to show that $\tilde{F}_j \circ \phi_j \to \bar{f}$ locally on $\tilde{\Sigma} \times C^\infty(\tau^-, \tau^+)$. For this, we show bounds for lower order coordinate derivatives of $\tilde{f}_j \circ \phi_j$ similar to (2.96). We check the assumptions in Lemma A.2. At first, we note that $\tilde{G}_j(0) \Rightarrow \tilde{G}$ (i.e. uniformly) on $B^2_r \subset B^2_{2r}$ and thus, using (3.46), we have

i) $\frac{1}{2}\delta \leq G_j \leq 2\delta$ and $\frac{1}{2}\delta \leq \tilde{G}_j(0) \leq 3\delta$

and from (3.41),(3.45) we have

ii) $|\partial^\gamma \Gamma_j^l| \leq \Gamma_{15} := 1$ for all $\gamma \leq 15$

iii) $\sum_{l=0}^{12}\|\tilde{\nabla}^l \tilde{A}_j\|_{L^\infty(\tilde{G}_j(t), B^2_r)} \leq A_{12} := c(n)$ for all $t \in I$.

As in (2.89) we get for the coordinate functions $\partial_t(\tilde{G}_j)_{rs} = (\tilde{\nabla}_{\partial_t}\tilde{G}_j)_{rs}$

iv) $|\partial_t \tilde{G}_j| \leq c(n)$.

If $A \in \mathbb{R}^{n \times n}$ is nonsingular, the inverse matrix may be represented by the formula $A^{-1} = S(A) \delta_{\det A}$ for some $S \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Since $\tilde{G}_j(0) \Rightarrow \tilde{G}$ (uniformly) on $B^2_r$, and because $\frac{1}{3}\delta \leq \tilde{G}_j(0)$ we infer that $\tilde{\Gamma}(0) \Rightarrow \tilde{\Gamma}$ and therefore $|\tilde{\Gamma}(0)| \leq \tilde{\Gamma}_0$, where $\tilde{\Gamma}_0 \in \mathbb{R}^{>0}$ is a constant and $j$ is assumed to be chosen large enough. Differentiating Kronecker’s symbol $\delta_i^k = (\tilde{G}_j)_{il}(\tilde{G}_j)^{lk}$ with respect to the space variables we get using induction that $\partial^\gamma \tilde{\Gamma}_j(0) \Rightarrow \partial^\gamma \tilde{\Gamma}$ for $\gamma \leq K$ and hence

$$|\partial^\gamma \tilde{\Gamma}(0)| \leq \tilde{\Gamma}_0 |_{t=0} < \infty \quad (3.47)$$

uniformly for $j$ sufficiently large. Now arguing analogously to Lemma 2.17, we have the bounds

$$|\partial^\gamma (\partial_t \tilde{\Gamma}_j)|, |\partial^\gamma \tilde{\Gamma}_j| \leq c(n, \gamma, A_{\gamma+3}, R_{\gamma+1}, c_{\tilde{G}_j}, \tilde{\Gamma}_0 |_{t=0}, |I|)$$

and therefore
3.3 Blow-up of singularities

v) $|\partial^\gamma \tilde{\Gamma}^{(j)}(t)| \leq \tilde{\Gamma}_9 := c(n, \tilde{\Gamma}_9|_{t=0}, |I|)$ for $\gamma \leq 9$ and all $t \in I$.

From Lemma A.2 (with $k = 0$, $r = 2$, $s = 0$, $m = 4$) it follows that on the time-interval $I$ we have

$$|\partial^{p+1}(\tilde{f}_j \circ \phi_j)| \leq c(n, \tilde{\Gamma}_9, A_9, |I|) \leq c(n, \tilde{\Gamma}_9|_{t=0}, |I|) \quad (p \leq 10)$$

and

$$|\partial^p \partial_t^l(\tilde{f}_j \circ \phi_j)| \leq c(n, \tilde{\Gamma}_9, A_{12}, R_{10}, |I|) \leq c(n, \tilde{\Gamma}_9|_{t=0}, |I|) \quad (p \leq 6 \text{ and } l \leq 2).$$

By Arzelà-Ascoli a subsequence converges. Choosing a subsequence once more, we have thus shown that

$$\tilde{f}_j \circ \phi \to \hat{f} \text{ in } C^{5,1} \text{ on } U_m \times I \quad \text{for} \quad j \geq j_0(m, I)$$

and choosing a diagonal sequence we have finally proven that

$$\tilde{f}_j \circ \phi \to \hat{f} \text{ in } C^{5,1} \text{ locally on } \hat{\Sigma} \times (\tau^-, \tau^+). \quad (3.48)$$

Since $\tilde{G}_j(t) \Rightarrow \hat{G}(t)$ and

$$\tilde{G}_j(t) \geq c(n, |t|) \tilde{G}_j(0) \geq \frac{1}{3} c(n, |t|) \delta \quad \text{for some } c > 0$$

we see that $\hat{f}_t : (\hat{\Sigma}, \hat{g}(t)) \to \mathbb{R}^n$ is an isometric immersion for all $t \in (\tau^-, \tau^+)$. Furthermore, $\hat{f} : (\hat{\Sigma}, \hat{g}) \times (\tau^-, \tau^+) \to \mathbb{R}^n$ is a Willmore flow.

To see that $\hat{f}$ is static, we let $\tau^- < \tau_1 < \tau_2 < \tau^+$, $U \subseteq \hat{\Sigma}$ and compute for $j$ sufficiently large

$$\int_{\tau_1}^{\tau_2} \int_U |W(\tilde{f}_j \circ \phi_j)|^2 d\mu_{\tilde{f}_j \circ \phi_j} d\tau = \int_{\tau_1}^{\tau_2} \int_{\phi_j(U)} |W(\hat{f}_j)|^2 d\mu_{\hat{f}_j} d\tau \leq \int_{\tau_1}^{\tau_2} \int_{\tau} |W(f_j)|^2 d\mu_{f_j} d\tau \leq \frac{W(f_j)|_{\tau = \tau_1} - W(f_j)|_{\tau = \tau_2}}{\tau_2 - \tau_1} = W(f)|_{t = t_j + r_j^4 \tau_1} - W(f)|_{t = t_j + r_j^4 \tau_2}.$$

Since $W(f)$ is monotonically decreasing with respect to time, and $t_j + r_j^4 \tau_1 \to T$, it follows that $W(\hat{f}) \equiv 0$ and hence $\hat{f}$ is static for $\tau \geq \tau_1$. 

71
To prove (3.35) we fix $\tau \in (\tau^-, \tau^+)$. Integrating the Willmore flow equation with respect to time as in (3.42), using (3.36) and (3.37), we get
\[
\|\tilde{f}_j(\phi_j(\cdot), \tau) - \tilde{f}_j(\phi_j(\cdot), 0)\|_{C^0(U_j)} \leq 2 \sup_{x \in U_j} d^G_j(\tilde{f}_j(\phi_j(\cdot), \tau), \tilde{f}_j(\phi_j(\cdot), 0)) \leq C,
\]
and by passing to the limit
\[
\|\hat{f}(\cdot, \tau) - \hat{f}(\cdot, 0)\|_{C^0(\Sigma)} \leq C.
\]
In particular $\hat{f}(\cdot, \tau) : \hat{\Sigma} \to \mathbb{R}^n$ is proper, and combining with (3.40) yields
\[
\|\tilde{f}_j(\phi_j(\cdot), \tau) - \hat{f}(\cdot, \tau)\|_{C^0(U_j)} \leq C.
\]
This shows that $\tilde{f}_j \circ \phi_j$ is uniformly proper, i.e. $\{x \in \hat{\Sigma} : |\tilde{f}_j, \tau \circ \phi_j(x)| \leq R\} \subset K_{R, \tau} \subset \hat{\Sigma}$. Using this fact together with (3.48) we get, for compact $K \subset U \subset \mathbb{R}^n$, $U$ open and bounded, at time $\tau \in (\tau^-, \tau^+)$
\[
(\tilde{f}_j \circ \phi_j)^{-1}(K) \subset \hat{f}^{-1}(U) \quad \text{for } j \text{ sufficiently large.}
\]
Using the equivalence (3.41) we can conclude
\[
\tilde{f}_j(\mu f_j \land \theta_j)(K) = \int_{f_j^{-1}(K)} \theta_j \, d\mu f_j
\]
\[
= \int_{(f_j \circ \phi_j)^{-1}(K)} \theta_j \circ \phi_j \, d\mu f_j \circ \phi_j
\]
\[
\leq \int_{f_j^{-1}(U)} \theta_j \circ \phi_j \, d\mu f_j \circ \phi_j \quad (\text{U is bounded})
\]
\[
\leq (1 + 1/j) \int_{f_j^{-1}(U)} \theta_j \circ \phi_j \, d\mu f_j \circ \phi_j .
\]
Letting $j \to \infty$ and taking the infimum over all bounded $U \supset K$, we see
\[
\limsup_{j \to \infty} \tilde{f}_j(\mu f_j \land \theta_j)(K) \leq \hat{f}(\mu f_j \land \theta_j)(K).
\]
Here, we used that $K$ can be approximated from above using only bounded open sets. Furthermore, we also have $\hat{f}^{-1}(K) \subset (\tilde{f}_j \circ \phi_j)^{-1}(U)$ for $j$ large enough, which implies
\[
\frac{j}{j + 1} \int_{f_j^{-1}(K)} \theta_j \circ \phi_j \, d\mu f_j \circ \phi_j \overset{(3.41)}{\leq} \int_{f_j^{-1}(K)} \theta_j \circ \phi_j \, d\mu f_j \circ \phi_j
\]
\[
\leq \int_{(f_j \circ \phi_j)^{-1}(U)} \theta_j \circ \phi_j \, d\mu f_j \circ \phi_j
\]
\[
= \int_{f_j^{-1}(U)} \theta_j \, d\mu f_j
\]
\[
= \tilde{f}_j(\mu f_j \land \theta_j)(U).
\]
For $j \to \infty$ we get, when taking the supremum with respect to $K$,\[
\hat{f}(\mu_j \wedge \theta)(U) \leq \lim_{j \to \infty} \hat{f}_j(\mu_{j\wedge \theta})(U).
\]
Note that it is sufficient that the last inequality is valid only for all bounded $U \subset \mathbb{R}^n$ (cf. the proof of Theorem 1 in section 1.9 of [5]). Alternatively, one may argue that the latter inequality also holds for all $U \subset \mathbb{R}^n$ open, not necessarily bounded by continuity from below of the measure $\hat{f}(\mu_j \wedge \theta)$. In any case, (3.35) is now settled. 

Before we prove Theorem 0.3 we want to formulate the proposed convergence more precisely.

**Theorem 0.3** (Part II): Under the assumptions and definitions of Theorem 0.3 the following convergence result holds: There exist sequences $R_j \nearrow \infty$ and $p_j \in M$, local coordinate charts $\varphi_j : B^n_{R_j}(p_j) \to V_j \supset B^n_{2j^2}$, a 2-manifold $\hat{\Sigma}$ without boundary (possibly not connected), open sets $U_j \subset \subset U_{j+1}$ with $\bigcup_{j=1}^{\infty} U_j = \hat{\Sigma}$, diffeomorphisms $\phi_j : U_j \to f_{j,0}^{-1}(\varphi_j^{-1}(B^n))$, open sets $\Sigma_j \subset \Sigma$ satisfying $\phi_j(U_j) \subset \Sigma_j$ and $f_j(\Sigma_j, J) \subset B^n_{R_j}(p_j)$ for each $J \subset \subset \mathbb{R}$ provided $j \geq j_0(J)$, such that for $\bar{f}_j \circ \phi_j := \tilde{f}_j(\phi_j(\cdot), \cdot)$ being well defined locally on $\hat{\Sigma} \times \mathbb{R}$, where $\tilde{f}_j := \varphi_j \circ f_j$\[
\bar{f}_j \circ \phi_j \to \tilde{f}_0 \text{ locally in } C^{4,1} \text{ on } \hat{\Sigma} \times \mathbb{R}. \tag{3.49}
\]

**Proof of Theorem 0.3:** Let $\varepsilon_0, \varepsilon_1$ be as in Theorem 3.4, $0 < \varepsilon < \min\{\varepsilon_T, \varepsilon_1^2/2\}$, and $\{r_j\}_{j \in \mathbb{N}}$ be an arbitrary given sequence with $r_j \searrow 0$. Consider the times\[
t_j := \sup\{t \geq 0 : \chi(r_j, \cdot) < \varepsilon^2 \text{ on } [0, t]\}.
\]
Clearly $t_j \leq t_{j+1}$. By definition of $\varepsilon_T$ in (0.4) and since $0 < \varepsilon < \varepsilon_T$ we also get that $t_j < T$. By smoothness of the flow we furthermore have $\lim_{j \to \infty} t_j = T$. This could also be seen by using the upper semicontinuity of $\chi$ in the time variable. By maximality of $t_j$ and again by upper semicontinuity of $\chi(r_j, \cdot)$, we get $\chi(r_j, t_j) \geq \varepsilon^2$. Next, we show the existence of so-called concentration points. For this, let $j \in \mathbb{N}$ be arbitrary but fixed, and $K := \{p \in M : \text{dist}_g(p, f(\Sigma, t_j)) \leq 2\}$. Since an upper semicontinuous function attains its supremum over compact sets, there exist $p_j \in M$ such that\[
\varepsilon^2 \leq \chi(r_j, t_j) = \sup_{p \in M} E(p, r_j, t_j) = \sup_{p \in K} E(p, r_j, t_j) = E(p_j, r_j, t_j). \tag{3.50}
\]
On the contrary, from (3.32) we get that\[
\chi(r_j, t_j) \leq \Gamma \liminf_{t \nearrow t_j} \chi(r_j, t) \leq \Gamma \varepsilon^2 \leq \varepsilon_1^2
\]
3 Blow-up of singularities

when restricting to $\varepsilon^2 \leq \varepsilon_1^2 \Gamma^{-1}$, where $\varepsilon_1^2 < \varepsilon_2^2$ is as in Theorem 3.4. Moreover, $t_j^- = 0$ and thus $\tau^- = \limsup_{j \to \infty} \frac{t_j - t_j^-}{r_j^4} = -\infty$. From this we get the desired (sub-)convergence locally on the interval $(-\infty, \tau^+)$ from Theorem 3.4. To see that $\tau^+ = \infty$, we claim that for any $\tau \in (-\infty, \tau^+)$, $\varepsilon \in (0, 1)$,

$$\limsup_{j \to \infty} \chi_{j}(\varepsilon, \tau) \leq \varepsilon^2. \tag{3.51}$$

Otherwise, since $\chi_{j}(\varepsilon, \tau) = \chi(r_j \varepsilon, t_j + r_j^4 \tau)$, we find a subsequence with

$$\int_{f^{-1}(B_{\varepsilon r_j(P_j)})} |A|^2 d\mu_{\varepsilon r_j(P_j)} \geq \varepsilon^2; \tag{3.52}$$

for some $P_j \in M$ and $\varepsilon > \varepsilon$. Rescaling parabolically, i.e. letting $f_j$ be defined as in (3.33) with $g_j := (gr_j)^{-1}g$, and using harmonic coordinates as above, with the exception to be centred at $P_j$ instead of $p_j$, we can again define coordinate representations $\tilde{F}_j$ as in (3.43) and Riemannian metrics $G_j$ as in (3.41). We have, using lower semi-continuity (3.31) of $\chi_j(1, \cdot)$,

$$\int_{\tilde{F}^{-1}_{j,0}(B_{\sqrt{1-1/m}})} |A_{\tilde{F}_j}(0)|^2 d\mu_{\tilde{F}_j}(0) \leq \varepsilon^2 \quad \text{but} \quad \int_{\tilde{F}^{-1}_{j,\tau}(B_{\varepsilon \sqrt{1+1/m}})} |A_{\tilde{F}_j}(\tau)|^2 d\mu_{\tilde{F}_j}(\tau) \geq \varepsilon^2$$

for $j \geq m$, since by equivalence of the metrics $(1 - 1/j) \delta \leq G_j \leq (1 + 1/j) \delta$ it holds

$$B_{\sqrt{1-1/m} \varepsilon} \subseteq B_{1}^{G_j}(0) \quad \text{and} \quad B_{\sqrt{1+1/m} \varepsilon} \subseteq B_{1}^{G_j}(0).$$

Analogously to what is shown above, the $\tilde{F}_j$ (sub-)converge after reparametrization locally in $C^{5,1}$ on $\tilde{\Sigma}^* \times (-\infty, \tau^+)$. In particular, convergence (weakly as Radon measures) (3.35) now implies

$$\int_{\tilde{F}^{-1}_{0,0}(B_{\sqrt{1-1/m}})} |A_{\tilde{F}_0}|^2 d\mu_{\tilde{F}_0} \leq \varepsilon^2 < \varepsilon^2 \leq \int_{\tilde{F}^{-1}_{0,\tau}(B_{\sqrt{1+1/m}})} |A_{\tilde{F}_0}|^2 d\mu_{\tilde{F}_0} < \infty$$

for all $m \in \mathbb{N}$, recalling that $\tilde{F}_0$ is proper by Theorem 3.5, and the reparametrizations of $\tilde{F}_{j,\tau}$ are uniformly proper. Letting $m \to \infty$ we get, by continuity from above and from below of the Radon measure $\nu := \tilde{F}_0(\mu_{\tilde{F}_0} \circ |A_{\tilde{F}_0}|^2)$, the contradiction

$$\nu(B_{1}^{G_j} \subseteq \nu(B_{e}^{G_j}).$$

Therefore we have now shown (3.51). Now (0.3) implies for any $\tau \in (-\infty, \tau^+)$

$$\tau^+ = \liminf_{j \to \infty} \frac{t_j^+ - t_j}{r_j^4} \geq \tau + C \log \frac{C \varepsilon_1^2}{2 \varepsilon_1^2} \geq \tau + c \log 2,$$
by definition of \( \varepsilon_1 \) in Letting \( \tau \nearrow \tau^+ \) we conclude that \( \tau^+ = \infty \).

Using (3.50) we can further estimate
\[
\varepsilon^2 \leq \int_{f^{-1}(B_{\varepsilon_j}^\sigma(p_j))} |A_{f_j}|^2 d\mu_{f_j} \big|_{t = t_j} \\
= \int_{f_j^{-1}(B_{\varepsilon_j}^\sigma(p_j))} |A_{f_j}|^2 d\mu_{f_j} \big|_{t = 0} \\
\leq \int_{f_j^{-1}(B_{\varepsilon_j}^\sigma(p_j))} |A_{f_j}|^2 d\mu_{f_j} \big|_{t = 0}
\]
for \( j \geq j_0(m) \) and thus, again using the weak convergence as Radon measures (3.35) and continuity from above of the limit measure we get
\[
0 < \varepsilon^2 \leq \int_{f_0^{-1}(B_{\varepsilon_0}^\sigma)} |A_{f_0}|^2 d\mu_{f_0}
\]
(3.53)

For any compact set \( C \subset \widehat{\Sigma} \) we have \( \tilde{f}_0(C) \subset B_{R/2}^\sigma \) for some \( R < \infty \), and since \( \tilde{f}_{j,0} \circ \phi_j \rightarrow \tilde{f}_0 \) in \( C^0 \) locally on \( \widehat{\Sigma} \) we see that also
\[
\tilde{f}_{j,0} \circ \phi_j(C) \subset B_{R/2}^\sigma \subset B_{R^j(0)}^G \)
for large \( j \). Using this, we further get
\[
\chi(\varrho, t_j) \geq \chi(r_j R, t_j) \\
\geq \int_{f_j^{-1}(B_{r_j}^\sigma(p_j))} |A_{f_j}|^2 d\mu_{f_j} \big|_{t = t_j} \\
= \int_{f_j^{-1}(B_{r_j}^\sigma(p_j))} |A_{f_j}|^2 d\mu_{f_j} \big|_{t = 0} \\
= \int_{(f_j,0,0,\varphi_j)^{-1}(B_{R^j(0)}^G)} |A_{f_j,0,0,\varphi_j}|^2 d\mu_{f_j,0,0,\varphi_j} \\
\geq \int_C |A_{f_{j,0,0,\varphi_j}}|^2 d\mu_{f_{j,0,0,\varphi_j}}.
\]
Again, since \( \tilde{f}_{j,0} \circ \phi_j \rightarrow \tilde{f}_0 \) in \( C^2 \) locally on \( \widehat{\Sigma} \) and using (3.41) we get, letting \( j \rightarrow \infty \),
\[
\liminf_{j \rightarrow \infty} \chi(\varrho, t_j) \geq \int_C |A_{f_0}|^2 d\mu_{f_0}
\]
3 Blow-up of singularities

and further, letting $C \not\subset \hat{\Sigma}$, we obtain

$$\varepsilon_T^2 \geq \int_{\hat{\Sigma}} |A_{f_0}|^2 d\mu_{f_0}.$$  

The second estimate in (0.7) follows from the nontriviality (3.53) of the limit surface $\hat{\Sigma}$. The last estimate in (0.7) follows from the fact that if $\hat{\Sigma}$ contains a compact component $C$, then $\hat{\Sigma} = C$ and $\Sigma$ is diffeomorphic to $C$.

Proof: For $j$ sufficiently large, $\phi_j(C)$ is both open and closed in $\Sigma$. Hence by connectedness of $\Sigma$ we have $\Sigma = \phi_j(C)$ and thus $\hat{\Sigma} = C$.

Thus, the blow-up surface is either an embedded round sphere, or contains at least one component that is a nonumbilic (compact or noncompact) Willmore surface. As a consequence, provided that the Willmore energy of the initial surface is strictly below $8\pi$, the case of an embedded sphere can be excluded by an inequality of Li and Yau [18]. Alternatively, embedded spheres can also be excluded if it is possible to establish a lower area bound (cf. [14], Theorem 4.4).

With the same topological argument as in [14], Lemma 4.3, we finally have the following Lemma.

Lemma 3.6 (cf. [14]). Let $\hat{\Sigma} : \hat{\Sigma} \to \mathbb{R}^n$ be the blow-up constructed above. If $\hat{\Sigma}$ contains a compact component $C$, then in fact $\hat{\Sigma} = C$ and $\Sigma$ is diffeomorphic to $C$.

Proof: For $j$ sufficiently large, $\phi_j(C)$ is both open and closed in $\Sigma$. Hence by connectedness of $\Sigma$ we have $\Sigma = \phi_j(C)$ and thus $\hat{\Sigma} = C$.
Appendix

A.1 Coordinate estimates

In this chapter we want to derive coordinate estimates depending on the bounds of the geometry, which are used many times in this work. For this, assume that for a given Riemannian metric $g$, open sets $U \subset \mathbb{R}^d$ and $V \subset \mathbb{R}^n$, and a time interval $0 \in I = [t_1, t_2] \subset \mathbb{R}$ we have smooth coordinate representations

$$
\tilde{f} : (U, \tilde{G}(t)) \times I \to (V, g)
$$

and

$$
\bar{f} : (U, \bar{G}(t)) \times I \to (V, \delta_{eucl})
$$

such that for any $t \in I$, $\tilde{f} := \tilde{f}(\cdot, t)$ is an isometric immersion. Here, $\tilde{f} \equiv \bar{f} \equiv f$ as maps between manifolds. In addition, we want to assume the following:

i) $c^{-1}\delta \leq g \leq c\delta$ and $c^{-1}\delta \leq \tilde{G}_{t=0} \leq c\delta$ (A.1)

ii) $|\partial^\gamma \Gamma_{(g)}| \leq \Gamma_B$ for $|\gamma| \leq B$ (A.2)

iii) $\sum_{l=0}^L \|\nabla^l A\|_{L^\infty(\tilde{G}(t), U)} \leq A_L$ (A.3)

iv) $|\partial_t \tilde{G}|_{\tilde{G}(t)} \leq P \forall t \in (t_1, t_2)$ (A.4)

v) $|\partial^\gamma \tilde{\Gamma}| \leq \tilde{\Gamma}_N$ for $|\gamma| \leq N$ and $\forall t \in I$. (A.5)

Note that for negative indices in $\Gamma_B, A_L$ and $\tilde{\Gamma}_N$ we have the empty condition.

From (A.1) it follows that $|g^{-1}| \leq c$ using polarization, and further from this, using (A.2),

$$
|\partial^\gamma g|, |\partial^\gamma (g^{-1})| \leq c(m, n, \Gamma_m) \text{ for } 0 \leq |\gamma| \leq m + 1,
$$

(A.6)

where the second estimate follows from the first by differentiating Kronecker’s symbol $\delta^\alpha_{\beta} = g^{\alpha \gamma} g_{\gamma \beta}$. Further, since $|\partial_t \tilde{G}|_{\tilde{G}(t)} \leq P$, we have that for $c = c(P, |I|)$

$$
c^{-1}\delta \leq c^{-1}\tilde{G}_{t=0} \leq \tilde{G} \leq c \tilde{G}_{t=0} \leq c\delta
$$

(A.7)
Appendix

from (A.4) and Lemma 14.2 of \cite{9}. Thus by definition of $\tilde{G}$ we also have
\begin{equation}
    c^{-1}\delta \leq \tilde{G} \leq c\delta. \tag{A.8}
\end{equation}

As in (A.6) we note that
\begin{equation}
    |\partial^{\gamma}\tilde{\mathbf{G}}|, |\partial^{\gamma}(\tilde{\mathbf{G}}^{-1})| \leq c(m,d,\tilde{\Gamma}_m) \text{ for } 1 \leq |\gamma| \leq m+1. \tag{A.9}
\end{equation}

Lemma A.1 Under the above assumptions, the following quantities are uniformly bounded on $U \times I$ for any $s \in \mathbb{N}_0$ and any $k + r = m \in \mathbb{N}_0$:

a) $|\partial^{k}\nabla^{r}\tilde{A}| \leq c(\tilde{\Gamma}_{k-1}, A_{k+r}, \Gamma_{k-1})$

b) $|\partial^{m+2}f| \leq c(\tilde{\Gamma}_m, A_m, \Gamma_m)$ \quad ($m \geq -1$)

c) $|\partial^{m+1}\tilde{G}|, |\partial^{m+1}(\tilde{G}^{-1})| \leq c(\tilde{\Gamma}_m, A_m, \Gamma_m)$ \quad ($m \geq -1$)

d) $|\partial^{m}\tilde{\Gamma}| \leq c(\tilde{\Gamma}_m, A_m, \Gamma_m)$

e) $|\tilde{D}^{m}\tilde{A}|_f + |\tilde{\nabla}^{m}\tilde{A}|_f \leq c(\tilde{\Gamma}_m, A_m, \Gamma_m)$

f) $|\partial^{m+2}(\partial^{s}g)\circ f| \leq c(\tilde{\Gamma}_m, A_m, \Gamma_{m+s+1})$ \quad ($m \geq -2$)

g) $|\partial^{m+2}(\partial^{s}\Gamma)\circ f| \leq c(\tilde{\Gamma}_m, A_m, \Gamma_{m+s+2})$ \quad ($m \geq -2$)

h) $|\partial^{m+2}(D^{s}R)\circ f| \leq c(\tilde{\Gamma}_m, A_m, \Gamma_{m+1}, R_{m+s+2})$ \quad ($m \geq -2$),

where $\sum_{i\leq m}\|D^{i}R\|_g \leq R_m$ and $c$ may also depend on $d,n,m,s,P,|I|$ and the constants in (A.1).

Here, $\tilde{\Gamma}$ are the Christoffel’s symbols induced by $\tilde{\mathbf{G}}$, $\tilde{\mathbf{D}}$ is the respective connection induced by $\tilde{f}$, and $\tilde{\mathbf{A}}$ is the second fundamental form of $\tilde{f}$.

Proof: To show a) - g) we use induction over $L \in \mathbb{N}_0$, where $m \leq L$. We let the constants $c$ also depend on $d,n,L,s,P,|I|$ and the constants in (A.1).

Ad a) & b):
\begin{equation}
    \sum_{\alpha} |\partial_{\alpha}f^{\alpha}|^2 = \|\partial f\|^2_\delta \leq c\|Df\|^2_\tilde{f} \leq c
\end{equation}

and analogously
\begin{equation}
    |\tilde{\mathbf{A}}|^2 \leq c\|\tilde{\mathbf{A}}\|^2_\tilde{f} \leq c(A_0)
\end{equation}

78
using the equivalences (A.1) and (A.7). With Gauß-Weingarten, i.e. \( \tilde{A} = \partial^2 f + \partial f \partial f \Gamma_{gf} + \partial f \cdot \tilde{\Gamma} \), we estimate
\[
|\partial^2 f| \leq |\tilde{A}| + c(\|\partial f\|_g^2 \|\Gamma\|_g + \|\partial f\|_g \|\tilde{\Gamma}\|_g) \leq c(\tilde{\Gamma}_0, A_0, \Gamma_0).
\]

Ad c): By polarization, we get from (A.8) \(|\tilde{G}|, |\tilde{G}^{-1}| \leq c\). Since we may write \( \tilde{G} = \partial f \cdot \partial f \) we can now estimate
\[
|\partial \tilde{G}| = |\partial f \cdot \partial^2 f| \leq c\|\partial f\|_g \|\partial^2 f\|_g \leq c(\tilde{\Gamma}_0, A_0, \Gamma_0)
\]
The estimate for the inverse metric again follows from differentiating Kronecker’s symbol.

Ad d): \( |\tilde{\Gamma}| = |\tilde{G}^{-1} \cdot \partial \tilde{G}| \leq c(\tilde{\Gamma}_0, A_0, \Gamma_0) \) from the above.

Ad e): Again with Gauss-Weingarten we estimate
\[
|\tilde{A}| \leq \|\partial^2 f\|_g + c\|\partial f\|_g \|\tilde{\Gamma}\|_g \leq c(\tilde{\Gamma}_0, A_0, \Gamma_0),
\]
and thus we have the same bound for
\[
\|\tilde{A}\|_f \leq c\|\tilde{A}\|_g \leq c(\tilde{\Gamma}_0, A_0, \Gamma_0).
\]

Ad f): \(|(\partial^s g)_{gf}| \leq c(\Gamma_{s-1}), \ |(\partial (\partial^s g)_{gf}| \leq c(\Gamma_s)\) and \(|\partial^2 (\partial^s g)_{gf}| \leq c(\tilde{\Gamma}_0, A_0, \Gamma_{s+1})\) follow from the chain rule.

Ad g): As above.

Now assume that a) - g) hold true for any \( s \in \mathbb{N}_0 \) and any \( m = k + r \leq L \in \mathbb{N}_0 \) and let \( m = k + r = L + 1 \).

Ad a): To estimate \(|\partial^k \tilde{\nabla} \tilde{A}|\) we use a telescope sum to get
\[
\partial^k \tilde{\nabla} \tilde{A} - \tilde{\nabla}^{L+1} \tilde{A} = \partial^k \tilde{\nabla}^r \tilde{A} - \tilde{\nabla}^k \tilde{\nabla}^r \tilde{A}
\]
\[
= \sum_{j=1}^{k} (\partial^j \tilde{\nabla}^{k+r-j} \tilde{A} - \partial^{j-1} \tilde{\nabla}^{k+r+1-j} \tilde{A})
\]
\[
= \sum_{j=1}^{k} \partial^{j-1} (\partial - \tilde{\nabla}) \tilde{\nabla}^{k+r-j} \tilde{A}
\]
so that we have to compute \((\partial - \tilde{\nabla}) S\) for \( S := \tilde{\nabla}^{k+r-j} \tilde{A} \). But for any \( N_f\)-valued tensor field it holds
\[
\tilde{\nabla} S = \tilde{D} S - P^T \tilde{D} S
\]
\[
= \partial S + \tilde{\Gamma} \cdot S + \Gamma_{gf} \cdot \partial f \cdot S + \tilde{G}^{-1} \cdot g_{gf} \cdot S \cdot \tilde{A} \cdot \partial f.
\] (A.10)

Substituting, we therefore get for \( 1 \leq j \leq k \) using the induction hypothesis
\[
|\partial^{j-1}(\partial - \tilde{\nabla}) \tilde{\nabla}^{k+r-j} \tilde{A}| = |\partial^{j-1}(\tilde{\Gamma} \cdot S + \Gamma_{gf} \cdot \partial f \cdot S + \tilde{G}^{-1} \cdot g_{gf} \cdot S \cdot \tilde{A} \cdot \partial f)|
\]
\[
\leq c(\tilde{\Gamma}_{k-1}, A_L, \Gamma_{k-2}, \Gamma_{k-1})
\]
\[
\leq c(\tilde{\Gamma}_{k-1}, A_L, \Gamma_{k-1}).
\]
Appendix

and thus in summary
\[ |\partial^k \tilde{\nabla}^r \tilde{A}| \leq |\nabla^{L+1} \tilde{A}| + c(\tilde{\Gamma}_{k-1}, A_L, \Gamma_{k-1}) \leq c(\tilde{\Gamma}_{k-1}, A_{L+1}, \Gamma_{k-1}). \]

Ad b): Again with Gauss-Weingarten we can estimate
\[ |\partial^{L+3} f| = |\partial^{L+1} (\tilde{A} + \partial f \cdot \partial f \cdot \Gamma \cdot f + \partial f \cdot \tilde{\Gamma})| \leq c(\tilde{\Gamma}_L, A_{L+1}, \Gamma_L, \Gamma_{L+1}, \Gamma_{L+1}) \]
\[ = c(\tilde{\Gamma}_{L+1}, A_{L+1}, \Gamma_{L+1}). \]

Ad c): Differentiating we get the bounds
\[ |\partial^{L+2} \tilde{G}| = |\partial^{L+2} (\partial f \cdot \partial f)| \leq c(n, L, P, \tilde{\Gamma}_{L+1}, A_{L+1}, \Gamma_{L+1}) \]
Analogously to the estimates for \( m \leq 0 \), these bounds also hold for the inverse metrics.
Ad d): Similarly
\[ |\partial^{L+1} \tilde{\Gamma}| = |\partial^{L+1} (\tilde{G}^{-1} \cdot \partial \tilde{G})| \leq c(\tilde{\Gamma}_L, A_L, \Gamma_L, \Gamma_{L+1}, A_{L+1}, \Gamma_{L+1}) \]
\[ = c(\tilde{\Gamma}_{L+1}, A_{L+1}, \Gamma_{L+1}). \]

Ad e): By induction it follows that
\[ \tilde{B}^{L+1} \tilde{A} = \partial^{L+1} \tilde{A} + \sum_{u=1}^{L+1} \sum_{k_1 + \ldots + k_u \leq L} \partial^{k_1} \Gamma \cdot \ldots \cdot \partial^{k_u} \Gamma \cdot \partial^t \tilde{A} \]
(where we have used \( \Gamma^{\mathbb{R}^n} \equiv 0 \)) we estimate using Gauss-Weingarten, i.e. \( \tilde{A} = \partial^2 f + \partial f \cdot \tilde{\Gamma}, \)
\[ |\tilde{B}^{L+1} \tilde{A}| \leq |\partial^{L+1} \tilde{A}| + c(\tilde{\Gamma}_L, A_L, \Gamma_L) \max_i |\partial^i \tilde{A}| \leq c(\tilde{\Gamma}_{L+1}, A_{L+1}, \Gamma_{L+1}). \]
Since \( \tilde{D}^m \tilde{A} \) is tensorial, the proposed estimate for the first summand follows by equivalence of the metrics (A.8). For the second, we show by induction over \( j \in \mathbb{N}_0 \) that for all \( i, j \in \mathbb{N}_0 \) we have
\[ |\partial^i \tilde{\nabla}^j \tilde{A}| \leq c(\tilde{\Gamma}_{i+j}, A_{i+j}, \Gamma_{i+j}). \]
Case \( j = 0 \) follows from Gauss-Weingarten and from what has been shown above. For the induction step we note that for a \( \mathbb{N}_f \)-valued tensor field we have
\[ \tilde{\nabla} S = \tilde{D} S - P_f \tilde{D} S = \partial S + \tilde{\Gamma} \cdot S + \tilde{G}^{-1} \cdot \tilde{A} \cdot S \cdot \partial f \]
80
A.1 Coordinate estimates

and thus

$$|\partial^i \nabla^{j+1} \tilde{A}| \leq |\partial^i (\partial \nabla^j \tilde{A} + \Gamma \cdot \nabla^j \tilde{A} + G^{-1} \cdot \nabla^j \tilde{A} \cdot \partial f)|$$

$$\leq c(\tilde{\Gamma}_{i+j+1}, A_{i+j+1}, \Gamma_{i+j+1}).$$

For \( i := 0 \) it follows e), again using the equivalence (A.8) and that \( \nabla^r \tilde{A} \) is tensorial.

Ad f): Case \( m = -2 \) follows from (A.6). Using induction one can show that for \( l \geq 1 \)

$$\partial^l ((\partial^s g) \circ f) = \sum_{u=1}^{l} \sum_{k_1+\ldots+k_u = l} (\partial^{u+s} g) \cdot \partial^{k_1} f \cdot \ldots \cdot \partial^{k_u} f$$  \hspace{1cm} (A.11)

and thus

$$|\partial^{m+2} ((\partial^s g) \circ f)| \leq c(\Gamma_{m+s+1}, \tilde{\Gamma}_m, A_m).$$

Ad g): Analogously to f), so that we have now shown a) - g).

Ad h): As in e), one can show by induction (over \( l \)) that for any \( l, s \in \mathbb{N}_0 \)

$$D^{l+s} R^M = \partial^l D^s R + \sum_{u=1}^{l} \sum_{k_1+\ldots+k_u = l} \partial^{k_1} \Gamma \cdot \ldots \cdot \partial^{k_u} \Gamma \cdot \partial^{k_s} D^s R$$  \hspace{1cm} (A.12)

and thus

$$|\partial^u D^s R| \leq \|D^{u+s} R\|_\delta + c(n, u, s, \Gamma_u) \sum_{i=0}^{u-1} \|\partial^i D^s R\|_\delta$$

$$\leq c(n, u, s, \Gamma_{u-1}) \sum_{j=0}^{u+s} \|D^j R\|_\delta$$

$$\leq c(n, u, s, \Gamma_{u-1}) \sum_{j=0}^{u+s} \|D^j R\|_g$$

$$\leq c(n, u, s, \Gamma_{u-1}, R_{u+s}).$$

From this estimate we get for \( m \geq -1 \)

$$|\partial^{m+2} ((D^s R) \circ f)| \leq c(\Gamma_{m+1}, R_{m+s+2}, \tilde{\Gamma}_m, A_m),$$

where we used a formula analogous to (A.11) with \( D^s R \) instead of \( \partial^s g \), i.e. for \( l \geq 1 \)

$$\partial^l ((D^s R) \circ f) \sum_{u=1}^{l} \sum_{k_1+\ldots+k_u = l} \partial^{u+s} D^s R \cdot \partial^{k_1} f \cdot \ldots \cdot \partial^{k_u} f.$$  \hspace{1cm} (A.13)
**Appendix**

**Lemma A.2** Assume that (A.1) to (A.5) holds and assume further that we have a local flow
\[ f : (U, \tilde{G}(t)) \times I \to (V, g) \quad (U \subset \mathbb{R}^d, V \subset \mathbb{R}^n \text{ open}) \]

of isometric immersions \( f_t \) such that
\[ \partial_t f = -W, \quad (A.14) \]

where \( W \) is a universal linear combination of elements in
\[ \{ \partial^k \nabla^r A, \tilde{G}, \tilde{G}^{-1}, \partial f, g_{\circ f}, (D^s R)_{\circ f} \} \]

for \( k + r + 2, s \leq m \in \mathbb{N} \). Then for any \( p \in \mathbb{N}_0 \), \( l \in \mathbb{N} \)
\[ |\partial^{p+1} f| \leq c(\Gamma_{p-1}, A_{p-1}, \Gamma_{p-1}) \quad (A.15) \]

and
\[ |\partial^p \partial^l_{\circ f}| \leq c(\Gamma_{p+(l-1)m+k-1}, A_{p+lm-2}, \Gamma_{p+(l-1)m+k-1}, R_{p+(l-1)m+s}). \quad (A.16) \]

Here, the constant \( c \) may also depend on \( n, d, m, p, l, P, |I| \) and the constants in (A.1).

**Proof:** We let the constant \( c \) may also depend on \( n, d, m, p, l, P, |I| \) and the constants in (A.1). Assume that \( |\partial^i \partial^j f| \leq f^{(l)}(p) < \infty \) \((0 \leq i \leq p, 1 \leq j \leq l)\) for an auxiliary function \( f^{(l)}(p) \) to be determined. By induction over \( l \) one can show that
\[
|\partial^p \partial^l_{\circ f}|(\partial^* g_{\circ f})| \leq c(\Gamma_{p-2}, A_{p-2}, \Gamma_{p+l+s-1}, f^{(l)}(p)), \\
|\partial^p \partial^l_{\circ f}|(\partial^* T_{\circ f})| \leq c(\Gamma_{p-2}, A_{p-2}, \Gamma_{p+l+s}, f^{(l)}(p)), \\
|\partial^p \partial^l_{\circ f}|(D^s R_{\circ f})| \leq c(\Gamma_{p-2}, A_{p-2}, \Gamma_{p+l-1}, f^{(l)}(p), R_{p+s+l}) \\
|\partial^p \partial^l_{\circ f}| (\tilde{G}), |\partial^p \partial^l_{\circ f}|(\tilde{G}^{-1})| \leq c(\Gamma_{p-1}, A_{p-1}, \Gamma_{p+l-1}, f^{(l)}(p+1)), \\
|\partial^p \partial^l_{\circ f}|(\Gamma) \leq c(\Gamma_{p}, A_{p}, \Gamma_{p+l}, f^{(l)}(p+2)).
\]

For \( l = 1 \), the above inequalities follow after differentiation in time using Lemma A.1, and then using the induction hypothesis. Taking the above inequalities into account and now using induction over \( r \in \mathbb{N}_0 \), one can show that
\[ |\partial^{p+k} \partial^l_{\circ f}(\nabla^r A)| \leq c(\Gamma_{p+k+r}, A_{p+k+r}, \Gamma_{p+k+r+l}, f^{(l)}(p+k+r+2)) \]
where for the induction step we used (A.10) with \( S := \nabla^r A \). We now want to estimate \[ |\partial^p \partial^l_{\circ f}(f^{(l)}(p)) \] Due to the special structure of \( W \) and estimating \( k + r \leq m - 2 \), the above inequalities reduce to
\[
|\partial^p \partial^l_{\circ f}(g_{\circ f})| \leq c(\Gamma_{p-2}, A_{p-2}, \Gamma_{p+l-1}, f^{(l)}(p)) \quad (A.17) \\
|\partial^p \partial^l_{\circ f}(D^s R_{\circ f})| \leq c(\Gamma_{p-2}, A_{p-2}, \Gamma_{p+l-1}, f^{(l)}(p), R_{p+s+l}) \quad (A.18) \\
|\partial^p \partial^l_{\circ f}(G), |\partial^p \partial^l_{\circ f}(G^{-1})| \leq c(\Gamma_{p-1}, A_{p-1}, \Gamma_{p+l-1}, f^{(l)}(p+1)) \quad (A.19) \\
|\partial^{p+k} \partial^l_{\circ f}(\nabla^r A)| \leq c(\Gamma_{p+m-2}, A_{p+m-2}, \Gamma_{p+m-2+l}, f^{(l)}(p+m)) \quad (A.20)
\]
and from Lemma A.1

\[ |\partial^{\mu+k} \nabla^r A| \leq c(\Gamma_{\mu+1}, A_{\mu+m-2}, \Gamma_{\mu+k-1}) \]  \hspace{1cm} (A.21)

where (A.17) to (A.19) also hold for \( l = 0 \) by Lemma A.1, if we set \( f^{(0)}(p) := 0 \). We let

\[ f^{(l)}(p) := c(\Gamma_{p+(l-1)m+k-1}, A_{p+l-2}, \Gamma_{p+(l-1)m+k-1}, R_{p+(l-1)m+s}) \]

Clearly, \( f^{(l)}(p) \) is increasing in \( l \) and \( p \), and \( f^{(l)}(p+m) \leq f^{(l+1)}(p) \) in the obvious sense. (A.16) is proved if we can show that

\[ |\partial^p \partial^l f| \leq f^{(l)}(p) \]

for \( 1 \leq l \leq L \in \mathbb{N} \). Using Lemma A.1 and the remarks above it, it is easy to see that for \( L = 1 \)

\[ |\partial^p \partial^l f| \leq c(\Gamma_{p+1}, A_{p+m-2}, \Gamma_{p+k-1}, R_{p+s}) = f^{(1)}(p). \]  \hspace{1cm} (A.22)

To estimate \( |\partial^p \partial^l f| \), we use the equation \( \partial^l f = \mathbf{W} \). Let \( \mu \leq p \) and \( \lambda \leq L \). Using (A.17) to (A.20) for \( l := \lambda \leq L \), we get for the induction step

\[ |\partial^\mu \partial^\lambda f| \leq c(\Gamma_{\mu+1}, A_{\mu-1}, \Gamma_{\mu+\lambda-1}, f^{(\lambda)}(\mu)) \leq f^{(L+1)}(p), \]

\[ |\partial^{\mu+1} \partial^\lambda f| \leq f^{(\lambda)}(\mu+1) \leq f^{(\lambda)}(p+m) \leq f^{(L+1)}(p) \text{ for } (\lambda \geq 1), \text{ and} \]

\[ |\partial^\mu f| \leq c(\Gamma_{\mu+1}, A_{\mu-1}, f^{(\lambda)}(\mu)) \]

\[ |\partial^\mu \partial^\lambda (g \circ f)| \leq c(\Gamma_{\mu+1}, A_{\mu-1}, f^{(\lambda)}(\mu), R_{\mu+\lambda}) \leq f^{(L+1)}(p), \]

\[ |\partial^{\mu+1} \partial^\lambda ((D^s R) \circ f)| \leq c(\Gamma_{\mu+1}, A_{\mu-1}, f^{(\lambda)}(\mu), R_{\mu+\lambda}) \leq f^{(L+1)}(p), \]

\[ |\partial^{\mu+k} \partial^\lambda \nabla^r A| \leq c(\Gamma_{\mu+m-2}, A_{\mu+m-2}, \Gamma_{\mu+m-2+\lambda}, f^{(\lambda)}(\mu+m)) \leq f^{(L+1)}(p) \text{ for } (\lambda \geq 1), \text{ and finally} \]

\[ |\partial^{\mu+k} \nabla^r A| \leq c(\Gamma_{\mu+k-1}, A_{\mu+m-2}, \Gamma_{\mu+k-1}) \leq f^{(L+1)}(p). \]

Therefore we have shown \( |\partial^p \partial^l f| \leq f^{(L+1)}(p) \). Actually, if \( f^{(1)}(p) \) is given, \( f^{(l)}(p) \) is the optimal function having the property \( f^{(l)}(p+m) \leq f^{(l+1)}(p) \), because equality holds. In (A.15) we merely stated again Lemma A.1.  \hspace{1cm} \blacksquare
Lemma A.3 Let $U$ be a manifold without boundary, and let for $V \subset \mathbb{R}^n$ open \( \tilde{f} : (U, \tilde{g}) \to (V, g) \) and \( \bar{f} : (U, \bar{g}) \to (V, \delta_{eul}) \) be isometric immersions such that $\tilde{f} = \bar{f}$ as maps between manifolds. Let further $L \in \mathbb{N}_0$ and assume that (with respect to the standard coordinates of $\mathbb{R}^n$) we have the bounds

$$c_g^{-1}\delta \leq g \leq c_g\delta,$$

$$\sum_{i=0}^{L} \sup_V |\partial^i \Gamma| \leq \Gamma_L, \quad \sum_{i=0}^{L} \left\| \nabla^i \tilde{A} \right\|_{L^\infty(U, \bar{g})} \leq A_L.$$

Then

$$\sum_{i=0}^{L} \left( \| D^i \tilde{A} \|_{L^\infty(U, \tilde{g})} + \| \nabla^i \bar{A} \|_{L^\infty(U, \bar{g})} \right) \leq c,$$  \hspace{1cm} (A.23)

where $c$ only depends on $c_g, \Gamma_L, A_L$ and the dimension $n$.

**Proof:** We proceed as follows: Since the extrinsic and relative curvature of $\tilde{f}$ is bounded we also get a bound for the intrinsic curvature $\tilde{R}$ of $(U, \tilde{g})$. Now (A.23) can be estimated by a uniform pointwise estimate. Namely, we can control $\partial^i \tilde{g}$ at the origin of Riemannian normal coordinates in terms of the intrinsic curvature and its derivatives. From this we get uniform bounds for $\partial^i f^a$ and thus also for the relative curvature of $\bar{f}$.

To begin, since $R = \partial \Gamma + \Gamma \cdot \Gamma$ we get, using (A.12) for $s = 0$, that $|D^s R| \leq c(\Gamma_{l+1})$ and thus

$$\| D^l \tilde{R} \|_{L^\infty(V, \tilde{g})} \leq c(c_g, \Gamma_L) \quad (l \leq L - 1).$$

From the equations of Gauss (1.17) we know

$$\tilde{R} = R \circ f \star Df \star Df \star Df \star Df + \tilde{A} \star \tilde{A} = Q^{0,0} + P^0_1(\tilde{A}).$$

Differentiating covariantly we get $\tilde{\nabla}^m \tilde{R} = Q^{m-1,1} + P^m_1(\tilde{A})$ for $m \geq 1$, and therefore

$$\sum_{i=0}^{L-1} \left\| \nabla^i \tilde{R} \right\|_{L^\infty(U, \delta_{eul})} =: \tilde{R}_{L-1} \leq c(n, c_g, \Gamma_L, A_{L-1}).$$

For arbitrary but fixed $x_0 \in U$ we choose Riemannian normal coordinates with respect to $\tilde{g}$ centred at $x_0$ with $2^{-1}\delta \leq (\tilde{g}_{ij}) \leq 2\delta$ locally as bilinear forms. It is not hard to show that $\tilde{\nabla}^m \tilde{R}(x_0)$ in the above coordinates is a universal linear combination of elements in the set $\{\partial^2 \tilde{g}(x_0), \ldots, \partial^{m+2} \tilde{g}(x_0)\}$. The other direction is also true, as it has been proven in [8]: Using a system of symmetries for the $m$-th partial derivative of $\tilde{g}$ obtained from...
A.2 Bounds on the metric in special coordinates

differentiating the Lemma of Gauss in Riemannian normal coordinates \{x^i\}_{i=1,2}, i.e.
\[ \delta_{ik}x^k = \tilde{g}_{ik} \] \( m + 2 \) times, it is possible to isolate \( \partial^{m+2}\tilde{g} \) so that one arrives at
\[ \sum_{\sigma \in S_m} (\nabla^m\tilde{R})_{k(\alpha m+2)\cdots k(3)\alpha(2)k(1)} \big|_{x_0} = C(n) \left( \partial^{m+2}_{k_{m+2} \cdots k_1} \tilde{g}_{rs} + \text{a polynomial in lower order partials of } \tilde{g} \right). \]

Using induction it finally follows that
\[ \partial^{m+2}_{k_{m+2} \cdots k_1} \tilde{g}_{rs} \big|_{x_0} = C(n) \left( \sum_{\sigma \in S_m} (\nabla^m\tilde{R})_{k(\alpha m+2)\cdots k(3)\alpha(2)k(1)} + \text{a polynomial in lower order covariant derivatives of } \tilde{R} \right). \]

The last equation can also be obtained when differentiating (5.1) in [8]. Therefore, in \( x_0 \) we have
\[ \tilde{\Gamma} = 0 \quad \text{and} \quad \sum_{i=1}^{L} |\partial^i \tilde{\Gamma}| \leq c(n, \tilde{R}_{L-1}) \leq c(n, c_g, \Gamma_L, A_{L-1}). \]

Now it is easy to see that for a) - h) in Lemma A.1 we more generally have pointwise estimates, i.e. a) \( |\partial^k \nabla^i A|(x) \leq c(\Gamma_{k-1}, A_{k+r}, \Gamma_{k-1}, \ldots, \Gamma_m(x), A_m, \Gamma_{m+1}, R_{m+2}) \), provided \( |\partial^i \tilde{\Gamma}| \leq \tilde{\Gamma}_N(x) \) for \( i \leq N \) and \( \forall t \in I \). Thus from e) in this Lemma with \( t_1 = t_2 = 0 \) and by naturality of the second fundamental form \( \tilde{\nabla} \) and \( \tilde{D} \), we infer that for \( i \leq L \)
\[ |\tilde{D}^i A|_{f}(x_0) + |\nabla^i A|_{f}(x_0) \leq c(n, c_g, \tilde{\Gamma}_L(x), A_L, \Gamma_L) \leq c(n, c_g, \Gamma_L, A_L), \]
which proves the Lemma.

A.2 Bounds on the metric in harmonic coordinates and Riemannian normal coordinates

Lemma A.4 (Harmonic Coordinates; cf. [10], Theorem 1.2). Let \((M,g)\) be a smooth, complete Riemannian \(n\)-manifold. Assume that for some \(k \in \mathbb{N}_0\) and a given universal constant \(c(n,k)\), \(R > 0\) can be chosen such that
\[ RA(k) \leq c \]
where \(\Lambda_{(M,g)}(k) := \sum_{i=0}^{k} \|D^i \text{ric}_{(M,g)}\|_{L^{\infty}(M,g)}^{1/i} + \text{inj}(M,g)^{-1}\). Then for any \(p \in M\) there exists a \(g\)-harmonic chart \(\varphi : B^g_R(p) \to \mathbb{R}^n\).
and a universal $C = C(n, k)$ such that in these coordinates $g_{\alpha\beta}(p) = \delta_{\alpha\beta}$ and

$$
i 1 - C \Lambda R \delta \leq (g_{\alpha\beta}) \leq (1 + C \Lambda R) \delta_{\alpha\beta} \quad \text{on } B_R^g(p),$$

$$\text{ii) } \sup_{B_R^g(p)} |\partial^\gamma g_{\alpha\beta}| \leq C \Lambda |\gamma|$$

for all $1 \leq |\gamma| \leq k + 1$, $\text{iii) } \sup_{B_R^g(p)} |\partial^\gamma \Gamma^\alpha_{\beta\delta}| \leq C \Lambda |\gamma| + 1$ for all $0 \leq |\gamma| \leq k$.

$\blacksquare$

**Lemma A.5** (Riemannian normal coordinates; cf. [10], Theorem 1.3). Let $(M, g)$ be a smooth, complete Riemannian $n$-manifold. Assume that for a given universal constant $c(n)$, $R > 0$ can be chosen such that

$$RA \leq c$$

where $\Lambda_{(M,g)} := \|R\|^{1/2}_{L^\infty(M,g)} + \|DR\|^{1/3}_{L^\infty(M,g)} + \text{inj}(M, g)^{-1}$. Then for any $p \in M$ there exist Riemannian normal coordinates with respect to $g$ (of radius $R$ around $p$) and a universal constant $C = C(n)$ such that in these coordinates $g_{\alpha\beta}(p) = \delta_{\alpha\beta}$ and

$$\text{i) } (1 - CR^2 \Lambda^2) \delta \leq (g_{\alpha\beta}) \leq (1 + CR^2 \Lambda^2) \delta \quad \text{on } B_R^g(p) \text{ for all } 0 \leq r < R,$$

$$\text{ii) } \sup_{B_R^g(p)} |\partial^\gamma g_{\alpha\beta}| \leq CR \Lambda^2 \quad \text{for all } 0 \leq r < R,$$

$$\text{iii) } \sup_{B_R^g(p)} |\partial^\gamma \Gamma^\alpha_{\beta\delta}| \leq CR \Lambda^2 \quad \text{for all } 0 \leq r < R.$$

$\blacksquare$

**A.3 Construction of cutoff functions and a partition of unity**

In the next Proposition we refer to a basic comparison estimate for the Hessian of the distance function taken from ([22], Corollary 2.4). In this section we denote by $\{\partial_\alpha\}_{2 \leq \alpha \leq n}$ the spherical coordinate frame of Riemannian polar coordinates. Exceptionally, we sum from $\alpha = 2, \ldots, n$.

**Proposition A.6** ([22], Corollary 2.4). Let $(M, g)$ a Riemannian manifold with bounded sectional curvature $k \leq \text{sec}_M \leq K$. Denoting by $(S^\beta_\alpha)$ the Hessian of $d_p := d_g(p, \cdot)$ in Riemannian polar coordinates, we have (as long as $sn_K(r) > 0$)

$$\sqrt{k}ct_k(r) \leq (S^\beta_\alpha(r, \theta))_{2 \leq \alpha, \beta \leq n} \leq \sqrt{k}ct_k(r),$$

where $\sqrt{k}ct_k(r) = \frac{sn_k'(r)}{sn_k(r)} = \sqrt{k}c_{\alpha\beta}(r)$ and $sn_k(r) := \frac{1}{\sqrt{k}} \sin \left(\sqrt{k}r\right)$ for $k \in \mathbb{R}\{0\}$. 

$\blacksquare$
A.3 Construction of cutoff functions and a partition of unity

**Lemma A.7** (Cutoff functions). Let \((M^n, g)\) be a Riemannian manifold and assume that \(\varrho > 0\) can be chosen such that \(\varrho < \text{inj}(M, g)\). Then for any \(\delta \in \mathbb{R}^+\) satisfying

\[
0 < \delta < \varrho < \text{inj}(M, g)
\]

there exists for arbitrary \(p \in M\) a cutoff function \(\tilde{\gamma} := \tilde{\gamma}_{p, \delta, \varrho} \in C^\infty(M)\) with

\[
\chi_{B_\varrho(p)} \leq \tilde{\gamma} \leq \chi_{B_{\varrho}(p)} \quad \text{and} \quad \|D^j\tilde{\gamma}\|_{\infty} \leq \frac{c}{(\varrho - \delta)^j}
\]

(A.24) for \(j = 1\) and \(c = c(n)\). If additionally \(|\sec_{(M, g)}| \leq \kappa^2 < \infty\), then for any \(\delta, \varrho \in \mathbb{R}^+\) satisfying

\[
0 < \delta < \varrho < \min\{\text{inj}(M, g), \frac{\pi}{2\kappa}\}
\]

(A.24) also holds for \(j = 2\).

**Proof:** Let \(p \in M\) be arbitrary but fixed, \(r := \delta / \varrho\) and assume that \(\varrho = 1\) (scale otherwise). Also, it is enough to restrict to \(r \geq 1/2\), i.e. we have \(1/2 \leq r < 1 < \text{inj}(M, g) =: i_0\). Since the exponential function \(\exp_p : B_{i_0}(0) := \{v \in T_pM : |v| < i_0\} \rightarrow B_{i_0}(p) \subset M\) is a diffeomorphism and \(\text{dist}_g(p, q) = |\exp_p^{-1}(q)|\) for \(q \in B_{i_0}(0)\), we know that the function \(d_p := \text{dist}_g(p, \cdot)\) is smooth on \(B_{i_0}(p)\). Now let \(h \in C^\infty(\mathbb{R})\) be a cutoff function with \(h \equiv 1\) on \(B^{\text{con}}_{r^n}(0)\), \(\text{spt } h \subset B^{\text{con}}_{(1+r)/2}(0)\), \(|h'| \leq c(1-r)^{-1}\), \(|h''| \leq c(1-r)^{-2}\) and define \(\tilde{\gamma} := h \circ d_p\). After extending \(\tilde{\gamma}\) by zero to the whole of \(M\), we get \(\tilde{\gamma} \in C^\infty(M)\), and furthermore \(\tilde{\gamma} \equiv 1\) on \(B_r(p)\) and \(\text{spt } \tilde{\gamma} \subset B_{(1+r)/2}(p)\) by construction. The first derivative can easily be estimated on \(B_{(1+r)/2}(p)\) as follows:

\[
|D\tilde{\gamma}| \leq c|h' \circ d_p||Dd_p| \leq c \max_{\mathbb{R}}|h'||\text{grad}_gd_p| \leq c(1-r)^{-1},
\]

since \(d_p\) is a distance function, i.e. \(|\text{grad}_gd_p| \equiv 1\).

Now we assume that \(0 < 1/2 \leq r < 1 < \min\{\text{inj}(M, g), \frac{\pi}{2\kappa}\}\). Then, since

\[
D^2\tilde{\gamma} = h'' \circ d_pDd_p \otimes Dd_p + h' \circ d_pD^2d_p,
\]

we can estimate the second derivative on \(B_{(1+r)/2}(p)\) as:

\[
|D^2\tilde{\gamma}| \leq \max_{\mathbb{R}}|h''||Dd_p|^2 + \max_{\mathbb{R}}|h'||D^2d_p| \leq c(1-r)^{-2} + c(1-r)^{-1}|D^2d_p| \leq c(1-r)^{-2}(1 + |D^2d_p|).
\]
Appendix

If \( S := D\text{grad}_g d_p \) denotes the Hessian of \( d_p \), i.e. the Weingarten operator of the distance spheres, we pointwise have for an adapted local \( g \)-orthonormal frame \( \{e_\alpha\} \) (summing over \( \alpha, \beta \))

\[
|D^2 d_p|^2 = (D^2_{\alpha\beta} d_p)^2 = (D_{\alpha\beta} (g(\text{grad}_g d_p, e_\beta)))^2 = g(D_{\alpha\beta} \text{grad}_g d_p, e_\beta)^2 \\
= |D\text{grad}_g d_p|^2 = |S|^2.
\]

To estimate \( |S|^2 \), we get from Proposition \( A.6 \) in Riemannian polar coordinates \( \{\partial_\alpha\} \)

\[
\kappa \cot (\kappa r) \leq (S^\alpha_\beta(r, \theta))_{2\leq\alpha,\beta\leq n} \leq \kappa \coth (\kappa r),
\]

since \( \sin(iz) = i \sinh(z) \) for \( z \in \mathbb{C} \), and \( |\sec(M,g)| \leq \kappa^2 \). Note, that \( S^\alpha_\beta = 0 \) if \( \alpha \cdot \beta = 0 \).

This follows when differentiating the identity \( g(f(\text{grad}_g d_p, \text{grad}_g d_p)) \equiv 1 \) and taking into account that \( S \) is self-adjoint with respect to \( g \).

Because \( 0 \leq \kappa \cot(\kappa r) \) for \( 0 \leq r \leq \frac{\pi}{2\kappa} \), and \( \kappa \coth(\kappa r) \leq 5/r \leq 10 \) for \( \kappa/2 \leq \kappa r \leq 2 \) we can estimate

\[
|S|^2(r, \theta) = g(S(\partial_\alpha), S(\partial_\beta)) g^{\alpha\beta} \\
= g(S^2(\partial_\alpha), \partial_\beta) g^{\alpha\beta} \\
= \text{trace}(S^2) \\
\leq (n-1) \max\{|\lambda|^2 : \lambda \text{ is an eigenvalue of } S\} \\
\leq c,
\]

i.e. \( |D^{2\gamma}| \leq c(1-r)^{-2} \) from (A.26). Finally, rescaling yields the claim of the Lemma.

To construct an appropriate partition of unity (see Lemma \( A.9 \)), we use Lemma 1.1 from \( [10] \):

**Lemma A.8** (Covering Lemma I). Let \((M,g)\) be a smooth, complete Riemannian n-manifold with \( \text{ricci}(M,g) \geq (n-1)kg \), and let \( \sigma > 0 \) be given. There exists a sequence \((p_i)_{i \in \mathbb{N}}\) of points in \( M \) such that for any \( r \geq \sigma \):

i) the family \((B_r(p_i))_{i \in \mathbb{N}}\) is a uniformly locally finite covering of \( M \), and there is an upper bound \( N = N(n, \sigma, r, k) \) for the number of Balls intersecting a previously given one

ii) for any \( i \neq j \), \( B_{\sigma/2}(p_i) \cap B_{\sigma/2}(p_j) = \emptyset \)

where, for \( p \in M \) and \( r > 0 \), \( B_r(p) \) denotes the geodesic ball of centre \( p \) and radius \( r \).

**Lemma A.9** (Existence of an appropriate partition of unity). Let \((M,g)\) be a Riemannian manifold and assume that \( g > 0 \) can be chosen with

\[
g \max\{c_1 \|\text{ricci}(M,g)\|_{\infty}, \text{inj}(M,g)^{-1}\} < 1,
\]
A.3 Construction of cutoff functions and a partition of unity

where $c_1 > 0$ is a constant. Then there exists a uniformly locally finite covering $\{B_{\rho}(p_i)\}_{i \in \mathbb{N}}$ of $M$, i.e. an arbitrary Ball $B_{\rho}(p_i)$ intersects with at most $N = N(n, c_1)$ other Balls of this covering. Furthermore, there is a smooth partition of unity $\{\tilde{\eta}_i\}_{i \in \mathbb{N}}$ subordinate to this covering with $0 \leq \tilde{\eta}_i \leq 1$ and $|D\tilde{\eta}_i| \leq \frac{c(n, N)}{\rho}$.

Remark: A bound $\text{ricci}_{(M, g)} \geq (n - 1)kg$ for $k \in \mathbb{R}$ instead of demanding a bound for $\|\text{ricci}_{(M, g)}\|_{\infty}$ would suffice.

Proof: We may assume that $\rho = 1$, i.e. $\text{inj}(M, g) < 1$ and $\|\text{ricci}\|_{\infty} < c_1^{-2}$. The bound on the Ricci curvature implies that $\text{ricci} \geq -c(n)c_1^{-2}g$. Choosing $r := \sigma := 1/2$ in Lemma A.8 we obtain a covering $\{B_{1/2}(p_i)\}_{i \in \mathbb{N}}$ of $M$ such that each ball $B_1(p_0)$ $(i_0 \in \mathbb{N})$ of twice the radius intersects with at most $N(n, 1/2, 1, c_1) = N(n, c_1)$ other balls $B_1(p_j) \in \{B_1(p_i)\}_{i \in \mathbb{N}}$. Using Lemma A.7 we have a sequence of bump functions $\{\gamma_i\}_{i \in \mathbb{N}}$ with $\chi_{B_{1/2}(p_i)} \leq \gamma_i \leq \chi_{B_1(p_i)}$ and $\|D\gamma_i\|_{\infty} \leq c$. Since $\{B_{1/2}(p_i)\}_{i \in \mathbb{N}}$ is a covering of $M$, we have

$$1 \leq \sum_{j=1}^{\infty} \gamma_i \leq N.$$ 

Thus we can define

$$\tilde{\eta}_i := \frac{\gamma_i}{\sum_{j=1}^{\infty} \gamma_j}.$$ 

For $i \in \mathbb{N}$ and $q \in B_1(p_i)$ both arbitrary but fixed, let $wlog$ (renumber otherwise) $\{B_1(p_1), \ldots, B_1(p_k)\}$ with $k \leq N$ the only pairwise distinct Balls containing $q$. Then we have in $q$

$$|D\tilde{\eta}_i| \leq \frac{|D\gamma_i|}{\sum_{j=1}^{k} \gamma_j} + \frac{\gamma_i}{\left(\sum_{j=1}^{k} \gamma_j\right)^2} \left(\sum_{j=1}^{k} |D\gamma_j|\right) \leq c + \sum_{j=1}^{k} |D\gamma_j| \leq c + cN.$$ 

Since $\sum \tilde{\eta}_i \equiv 1$ the lemma follows after rescaling.

The next Lemma is in spirit of Lemma A.8 and uses a kind of Vitali’s argument. It is needed in the context of interior estimates.

Lemma A.10 (Covering Lemma II). Let $(M, g)$ be a smooth complete Riemannian manifold with $\text{ricci}_{(M, g)} \geq kg$ as bilinear forms for some $k \in \mathbb{R}$. Then any closed geodesic ball $\overline{B}_g \subset M$ with $g|k|^{1/2} \leq 1$ can be covered with at most $\Gamma(n)$ other balls $B_{g/2}$ of radius $g/2$. If even $\|\text{ricci}_{(M, g)}\|_{\infty} < \infty$, then such a cover exists for any $g > 0$ with $g^2\|\text{ricci}_{(M, g)}\|_{\infty} \leq 1$. 

89
Appendix

**Proof:** For bounded $U \subseteq M$ open, we let $|U| := \mu_g(U)$. From the remark below Theorem 1.1 of [10], which is based on a comparison theorem of Bishop and Gromov, it follows that for any $\varrho > 0$ and $p \in M$ we have

$$|B_{2\varrho}(p)| \leq \exp\left(\sqrt{4(n-1)}k(\varrho)8^n|B_{\varrho/4}(p)|\right) \leq \Gamma(n)|B_{\varrho/4}(p)|,$$

when we define $\Gamma(n) := \exp\left(\sqrt{4(n-1)}8^n\right)$. Now for $p \in M$ and $\varrho > 0$ arbitrary but fixed, consider the system of sets

$$\mathcal{M} := \left\{ S : S = \bigcup_{i \in \Lambda} B_{\varrho/4}(q_i) \text{ for a disjoint union of balls } B_{\varrho/4}(q_i) \subset B_{\varrho}(p) \subset M \right\}.$$

It is easy to see that $\mathcal{M}$ (is non-empty and) is partially ordered by inclusion and every chain in $\mathcal{M}$ has an upper bound. Hence, by Zorn’s lemma, $\mathcal{M}$ contains a maximal element $\hat{S}$ in $\mathcal{M}$. Now consider an arbitrary finite number $N$ of Balls in $\hat{S}$. After renumbering, we may assume that $\{q_1, \ldots, q_N\}$ are the centres of such balls. Because

$$N|B_{\varrho}(p)| = \sum_{j=1}^{N} |B_{\varrho}(p)| \leq \sum_{j=1}^{N} |B_{2\varrho}(q_j)| \leq \sum_{j=1}^{N} \Gamma|B_{\varrho/4}(q_j)| = \Gamma|\cup_{j=1}^{N} B_{2\varrho}(q_j)| \leq \Gamma|B_{\varrho}(p)|$$

we get, since $|B_{\varrho}(p)| > 0$, that $N \leq \Gamma$, and thus $B_{\varrho}(p)$ contains at most $\Gamma$ disjoint balls of radius $\varrho/4$. Now it is easy to see that

$$\overline{B}_{\varrho}(p) \subset \bigcup_{i \in \Lambda_{\hat{S}}} B_{\varrho/2}(q_i)$$

and also that for the number of elements we have $|\Lambda_{\hat{S}}| \leq \Gamma$. This shows the existence of the desired cover. Now let $\|\text{ricci}_{(M,g)}\|_\infty < \infty$. Defining $k := -2n\|\text{ricci}_{(M,g)}\|_\infty$ we get $\text{ricci}_{(M,g)} \geq kg$, since $|\text{ricci}_{(M,g)}(v, v)| \leq 2n|\text{ricci}_{(M,g)}|_g|v|_g^2$ for any $v \in TM$. If one redefines $\Gamma := \exp\left(\sqrt{8(n-1)}\right)8^n$, the second statement follows. \

**A.4 Interpolation inequalities**

For this section, apart from for Theorem A.17, we assume that $\Sigma$ is a $d$-dimensional Riemannian manifold and that $\gamma \in C^1_c(\Sigma)$ satisfies

$$0 \leq \gamma \leq 1, \quad |\nabla \gamma| \leq G.$$

Except for Corollary A.14 and Theorem A.17 the interpolation inequalities from [13] carry over to the Riemannian setting, and are stated for convenience of the reader.
A.4 Interpolation inequalities

**Lemma A.11** Let \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} \), \( 1 \leq p, q, r < \infty \) and \( \alpha + \beta = 1 \), \( \alpha, \beta \geq 0 \). For \( s \geq \max\{\alpha q, \beta p\} \) and \( -\frac{1}{p} \leq t \leq \frac{1}{q} \) we have

\[
\left( \int_{\Sigma} |\nabla \phi|^{2r} \gamma^s d\mu \right)^{\frac{1}{r}} \leq c \left( \int_{|\gamma > 0|} |\phi|^{q \gamma^s(1-tq)} d\mu \right)^{\frac{1}{q}} \left( \int_{\Sigma} |\nabla^2 \phi|^{p \gamma^s(1+tp)} d\mu \right)^{\frac{1}{p}}
+ cGs \left( \int_{|\gamma > 0|} |\phi|^{q \gamma^{s-\alpha q}} d\mu \right)^{\frac{1}{q}} \left( \int_{\Sigma} |\nabla \phi|^{p \gamma^{s-\beta p}} d\mu \right)^{\frac{1}{p}},
\]

where \( c = c(d, r) \).

**Proof:** Using integration by parts, we get

\[
\int_{\Sigma} |\nabla \phi|^{2r} \gamma^s d\mu = \int_{\Sigma} \langle \phi, \nabla \gamma^s |\nabla \phi|^{2r-2} \nabla \phi \rangle d\mu
- \int_{\Sigma} \langle \phi, \gamma^s |\nabla \phi|^{2r-2} \Delta \phi \rangle d\mu - s \int_{\Sigma} \langle \phi, \gamma^{s-1} \nabla_{\gamma^s} |\nabla \phi|^{2r} \nabla_{\gamma^s} \phi \rangle d\mu
- 2(r-1) \int_{\Sigma} \langle \phi, \nabla_{\gamma^s} \phi \rangle \langle \nabla \phi, \nabla_{\gamma^s} (\nabla \phi) \rangle |\nabla \phi|^{2r-4} \gamma^s d\mu
\leq c(d, r) \int_{\Sigma} \langle \phi, \gamma^s |\nabla \phi|^{2r-2} |\nabla^2 \phi| d\mu + c(d)sG \int_{\Sigma} \langle \phi, \gamma^{s-1} |\nabla \phi|^{2r-1} d\mu
- c \left( \int_{|\gamma > 0|} |\phi|^{q \gamma^s(1-tq)} d\mu \right)^{\frac{1}{q}} \left( \int_{\Sigma} |\nabla \phi|^{2r} \gamma^s d\mu \right)^{\frac{r-1}{r}} \left( \int_{\Sigma} |\nabla^2 \phi|^{p \gamma^s(1+tp)} d\mu \right)^{\frac{1}{p}}
+ cGs \left( \int_{|\gamma > 0|} |\phi|^{q \gamma^{s-\alpha q}} d\mu \right)^{\frac{1}{q}} \left( \int_{\Sigma} |\nabla \phi|^{2r} \gamma^s d\mu \right)^{\frac{r-1}{r}} \left( \int_{\Sigma} |\nabla \phi|^{p \gamma^{s-\beta p}} d\mu \right)^{\frac{1}{p}},
\]

since \( \frac{1}{q} + \frac{r-1}{r} + \frac{1}{p} = 1 \).

**Corollary A.12** For \( 2 \leq p < \infty \) and \( s \geq p \) we have

\[
\left( \int_{\Sigma} |\nabla \phi|^{p \gamma^s} d\mu \right)^{\frac{1}{p}} \leq \varepsilon \left( \int_{\Sigma} |\nabla^2 \phi|^{p \gamma^{s+p}} d\mu \right)^{\frac{1}{p}} + \frac{c}{\varepsilon} \left( \int_{|\gamma > 0|} |\phi|^{p \gamma^{s-p}} d\mu \right)^{\frac{1}{p}},
\]

where \( c = c(d, p, s, G) \).
Appendix

Proof: We take $p = q = 2r$, $\alpha = 1$, $\beta = 0$ and $t = \frac{1}{s}$ in the previous lemma and obtain

$$
\left( \int_{\Sigma} |\nabla \phi|^{p} \gamma^{s} d\mu \right)^{\frac{1}{p}} \leq \left( \int_{[\gamma > 0]} |\phi|^{p} \gamma^{s-p} d\mu \right)^{\frac{1}{p}} \left( \int_{\Sigma} |\nabla^{2} \phi|^{p} \gamma^{s+p} d\mu \right)^{\frac{1}{p}} + c G s \left( \int_{[\gamma > 0]} |\phi|^{p} \gamma^{s-p} d\mu \right)^{\frac{1}{p}}.
$$

The claim follows using Young’s inequality and absorption. ■

Corollary A.13 For $2 \leq p < \infty$, $k \in \mathbb{N}$, $s \geq kp$ and $c = c(d, p, s, k, G)$ we have

$$
\left( \int_{\Sigma} |\nabla^{k} \phi|^{p} \gamma^{s} d\mu \right)^{\frac{1}{p}} \leq \varepsilon \left( \int_{[\gamma > 0]} |\nabla^{k+1} \phi|^{p} \gamma^{s+p} d\mu \right)^{\frac{1}{p}} + \frac{c}{\varepsilon} \left( \int_{[\gamma > 0]} |\phi|^{p} \gamma^{s-kp} d\mu \right)^{\frac{1}{p}}.
$$

(A.27)

Proof: (Induction over $k$). For $k = 1$ the statement holds by Corollary A.12. Now let $k \geq 1$. If one substitutes in the same Corollary $\phi$ by $\nabla^{k} \phi$, we obtain

$$
\left( \int_{\Sigma} |\nabla^{k+1} \phi|^{p} \gamma^{s} d\mu \right)^{\frac{1}{p}} \leq \varepsilon \left( \int_{\Sigma} |\nabla^{k+2} \phi|^{p} \gamma^{s+p} d\mu \right)^{\frac{1}{p}} + \frac{c_{k}}{\varepsilon} \left( \int_{\Sigma} |\nabla^{k} \phi|^{p} \gamma^{s-kp} d\mu \right)^{\frac{1}{p}}.
$$

The inductive hypothesis, applied to $s - p$ instead of $s$, yields

$$
\left( \int_{\Sigma} |\nabla^{k} \phi|^{p} \gamma^{s-p} d\mu \right)^{\frac{1}{p}} \leq \tau \left( \int_{\Sigma} |\nabla^{k+1} \phi|^{p} \gamma^{s} d\mu \right)^{\frac{1}{p}} + \frac{c}{\tau} \left( \int_{[\gamma > 0]} |\phi|^{p} \gamma^{s-(k+1)p} d\mu \right)^{\frac{1}{p}}
$$

where $c_{k} = c(d, p, s - p, k, G)$. Combining the inequalities proves the result. ■

Corollary A.14 For all $0 \leq k \leq l - 1$ and $s \geq 2(l - 1)$ we have

$$
\int_{\Sigma} |\nabla^{k} A|^{2} \gamma^{s} d\mu \leq \varepsilon \int_{\Sigma} |\nabla^{l} A|^{2} \gamma^{s+2(l-k)} d\mu + c(1 + \varepsilon^{-1})\|A\|^{2}_{2,[\gamma > 0]},
$$

(A.28)

where $c = c(d, s, k, l, G)$.

Proof: (Induction over $l \in \mathbb{N}$). For $l = 1$ and $k = 0$ the assertion is clearly true. Now let $0 \leq k \leq l$ and $s \geq 2l$. It follows from Corollary A.13 for $p := 2$ and $\phi := A$ that

$$
\int_{\Sigma} |\nabla^{l} A|^{2} \gamma^{s} d\mu \leq \varepsilon \int_{\Sigma} |\nabla^{l+1} A|^{2} \gamma^{s+2} d\mu + \frac{c}{\varepsilon} \|A\|^{2}_{2,[\gamma > 0]}.
$$
A.4 Interpolation inequalities

For $0 \leq k \leq l - 1$, we get by induction hypothesis

$$\int_\Sigma |\nabla^k A|^2 \gamma^s d\mu \leq \int_\Sigma |\nabla^l A|^2 \gamma^{s+2(l-k)} d\mu + c\|A\|_{2,\gamma>0}^2 \leq \varepsilon \int_\Sigma |\nabla^{l+1} A|^2 \gamma^{s+2(l-k)+2} d\mu + c(1 + \varepsilon^{-1})\|A\|_{2,\gamma>0}^2,$$

where we used Corollary A.13 in the last step for $\tilde{p} = 2; \tilde{k} := l; \tilde{s} := s + 2(l - k) \geq 2\tilde{k}$.

\[\square\]

**Theorem A.15** For $k \in \mathbb{N}$, $1 \leq i \leq k$ and $s \geq 2k$ we have the inequality

$$\left(\int_{\gamma>0} |\nabla^i \phi|^{\frac{2k}{r}} \gamma^s d\mu\right)^\frac{1}{\frac{2k}{r}} \leq c\|\phi\|^{1-\frac{1}{2}}_{\infty,\gamma>0}\left(\left(\int_{\Sigma} |\nabla^k \phi|^2 \gamma^s d\mu\right)^\frac{1}{2} + \|\phi\|_{2,\gamma>0}\right)^\frac{1}{2} \tag{A.29}$$

where $c = c(d, s, k, G)$.

**Proof:** For $1 \leq i \leq k$ define

$$a_i := \left(\int_{\gamma>0} |\nabla^i \phi|^{\frac{2k}{r}} \gamma^s d\mu\right)^{\frac{1}{\frac{2k}{r}}}, \quad a_0 := \|\phi\|_{\infty,\gamma>0}\right.$$

$$b_i := \left(\int_{\gamma>0} |\phi|^{\frac{2k}{r}} d\mu\right)^{\frac{1}{\frac{2k}{r}}}, \quad b_0 := \|\phi\|_{\infty,\gamma>0}\right.$$

Using Lemma A.11 with $r = \frac{k}{1}, p = \frac{2k}{1+1}, q = \frac{2k}{1+1}, t = 0$ and $\beta = 1$ we obtain for $s \geq 2k$

$$\left(\int_{\Sigma} |\nabla \phi|^{\frac{2k}{r}} \gamma^s d\mu\right)^\frac{1}{\frac{2k}{r}} \leq c\left(\int_{\gamma>0} |\phi|^{\frac{2k}{r+1}} \gamma^s d\mu\right)^{\frac{1}{\frac{2k}{r}}} \left(\int_{\Sigma} |\nabla^2 \phi|^{\frac{2k}{r+1}} \gamma^s d\mu\right)^{\frac{1}{\frac{2k}{r}}} + cGs\left(\int_{\gamma>0} |\phi|^{\frac{2k}{r+1}} \gamma^s d\mu\right)^{\frac{1}{\frac{2k}{r}}} \left(\int_{\Sigma} |\nabla \phi|^{\frac{2k}{r+1}} \gamma^{s-\frac{2k}{r+1}} d\mu\right)^{\frac{1}{\frac{2k}{r}}}$$

With this definitions, we get from Lemma A.11 if we substitute $\phi$ by $\nabla^{i-1} \phi$

$$a_i^2 \leq ca_{i-1} \left[a_{i+1} + \left(\int_{\Sigma} |\nabla^{i-1} \phi|^{\frac{2k}{r+1}} \gamma^{s-\frac{2k}{r+1}} d\mu\right)^{\frac{1}{\frac{2k}{r}}} \right].$$
Appendix

On the other hand, Corollary A.13 implies for \( s \geq 2k \) and \( 1 \leq i \leq k - 1 \)
\[
\left( \int_{\Sigma} |\nabla_i \phi|^2 d\mu \right)^{\frac{1}{2k}} \leq c \left( \int_{\Sigma} |\nabla^{i+1} \phi|^2 d\mu \right)^{\frac{1}{2k}} + c \left( \int_{\gamma>0} |\phi|^2 \gamma^s d\mu \right)^{\frac{1}{2k}}
\]
\[
\leq c(a_{i+1} + b_{i+1}),
\]
thus \( a_i^2 \leq c(a_{i+1} + b_{i+1}) \). We further get by an interpolation inequality for \( L^p \)-norms (see e.g. [4] for the euclidean case), if \( s \geq 2k \) and \( 1 \leq i \leq k - 1 \)
\[
(a_i + b_i) \leq c(a_{i-1} + b_{i-1})(a_{i+1} + b_{i+1}).
\]
Using a convexity argument (see [9], Corollary 12.5), we can further estimate
\[
a_i \leq a_i + b_i \leq c(a_0 + b_0)^{1-\frac{i}{k}}(a_k + b_k)^{\frac{i}{k}}
\]
\[
\leq c\|\phi\|_{\infty,[\gamma>0]}^{1-\frac{i}{k}} \left( \left( \int_{\Sigma} |\nabla^k \phi|^2 \gamma^s d\mu \right)^{\frac{1}{2}} + \|\phi\|_{2,[\gamma>0]}^{2} \right)^{\frac{i}{k}}.
\]

\[\blacksquare\]

**Corollary A.16** Let \( 0 \leq j_1, \ldots, j_r \leq k \), \( j_1 + \ldots + j_r = 2k \), \( s \geq 2k \) and \( r \geq 2 \). Then we have
\[
\int_{\Sigma} |\nabla^{j_1} \phi| \cdots |\nabla^{j_r} \phi| \gamma^s d\mu \leq c\|\phi\|_{\infty,[\gamma>0]}^{-2} \left( \int_{\Sigma} |\nabla^k \phi|^2 \gamma^s d\mu + \|\phi\|_{2,[\gamma>0]}^{2} \right), \quad (A.30)
\]
where \( c=c(k,d,r,s,G) \).

**Proof:** Let \( j_1, \ldots, j_l \geq 1 \) and \( j_{l+1}, \ldots, j_r = 0 \). Then it follows from Hölder’s inequality and (A.29) that
\[
\int_{\Sigma} |\nabla^{j_1} \phi| \cdots |\nabla^{j_r} \phi| \gamma^s d\mu
\]
\[
\leq \|\phi\|_{\infty,[\gamma>0]}^{-l} \prod_{i=1}^{l} \left( \int_{\Sigma} |\nabla^{j_i} \phi|^2 \gamma^s d\mu \right)^{\frac{1}{2k}}
\]
\[
\leq c\|\phi\|_{\infty,[\gamma>0]}^{-l} \prod_{i=1}^{l} \left[ \|\phi\|_{\infty,[\gamma>0]}^{-\frac{1}{2k}} \left( \left( \int_{\Sigma} |\nabla^k \phi|^2 \gamma^s d\mu \right)^{\frac{1}{2}} + \|\phi\|_{2,[\gamma>0]}^{2} \right)^{\frac{1}{2}} \right]
\]
\[
\leq c\|\phi\|_{\infty,[\gamma>0]}^{-2} \left( \int_{\Sigma} |\nabla^k \phi|^2 \gamma^s d\mu + \|\phi\|_{2,[\gamma>0]}^{2} \right).
\]

\[\blacksquare\]
A.4 Interpolation inequalities

**Theorem A.17** Let \((M^n, g)\) be a (possibly non-compact) Riemannian manifold with 
\[ \|\text{ricci}(M, g)\|_\infty^{1/2} + \text{inj}(M, g)^{-1} =: \Lambda < \infty \] and \(f : (\Sigma^2, \tilde{g}) \to (M, g)\) an isometric \(C^2\) immersion. For \(u \in C_c^1(\Sigma)\), \(2 < p \leq \infty\), \(1 \leq m \leq \infty\) and \(0 < \alpha < 1\) with \(\frac{1}{\alpha} = (\frac{1}{2} - \frac{1}{p})m + 1\) we have

\[ \|u\|_\infty \leq c\|u\|_m^{1-\alpha} (\|\nabla u\|_p + \|uA\|_p + \|u\|_p^\alpha), \quad (A.31) \]

where \(c = c(n, m, p)\).

**Proof:** We may assume that \(u\) is non-negative and that
\[ c_n(\|\nabla u\|_p + \|uA\|_p + \|u\|_p) = 1, \quad (A.32) \]
where \(c_n\) is the constant in the Michael-Simon Sobolev inequality (2.39). Letting \(q = \frac{p}{p-1} \in [1, 2]\) we infer for any \(\tau \geq 0\)

\[ \|u^{1+\tau}\|_2 \leq c_n(\|\nabla (u^{1+\tau})\|_1 + \|u^{1+\tau}A\|_1 + \|u^{1+\tau}\|_1) \leq c_n\|u^\tau\|_q (1 + \tau)\|\nabla u\|_p + \|uA\|_p + \|u\|_p^\tau \]

where (A.32) was used in the last step. With \(k = \frac{2}{q} \in (1, 2]\) we rewrite this as

\[ \|u\|_{k(1+\tau)q} \leq (1 + \tau)^{\frac{1}{kq}} \|u\|_{\tau q}^{\frac{1}{kq}}. \]

Putting \(\tau_0 := \frac{m}{q} \in (\frac{m}{2}, m]\), \(\tau_{\nu+1} := k(1 + \tau_\nu)\), \(\varepsilon_\nu := \frac{\tau_\nu}{\tau_{\nu+1}}\) and \(c_\nu := (1 + \tau_\nu)^{\frac{1}{1+\tau_\nu}}\) we obtain for \(\nu \in \mathbb{N}_0\)

\[ \|u\|_{\tau_{\nu} q} \leq c_\nu\|u\|_{\nu q}^{\varepsilon_\nu}, \quad (A.33) \]

where

\[ 1 + \tau_\nu = k^\nu \tau_0 + \sum_{\mu=0}^{\nu} k^\mu. \quad (A.34) \]

By induction (A.33) implies

\[ \|u\|_{\tau_{\nu} q} \leq \left( \prod_{\mu=0}^{\nu-1} c_\mu^{\varepsilon_{\mu+1} \cdots \varepsilon_{\nu-1}} \right) \|u\|_{m^{\varepsilon_1 \cdots \varepsilon_{\nu-1}}}^{\varepsilon_0 \varepsilon_1 \cdots \varepsilon_{\nu-1}}. \quad (A.35) \]

Now (A.34) yields

\[ \frac{1}{c} k^\nu \leq 1 + \tau_\nu \leq e k^\nu \quad \text{for } c = c(m, p), \quad (A.36) \]
and thus using $\varepsilon_\mu \leq 1$ we can estimate

\[
\log \prod_{\mu=0}^{\nu} \varepsilon_\mu^{\varepsilon_{\mu+1} \cdots \varepsilon_\nu} \leq \sum_{\mu=0}^{\nu} \frac{1}{1 + \tau_\mu} \log(1 + \tau_\mu)
\]

\[
\leq \sum_{\mu=0}^{\nu} c k^{-\mu}(\log (c + \mu \log k) = c(m, p) < \infty. \tag{A.37}
\]

Using $\tau_{\nu+1} = k(1 + \tau_\nu)$ we get from (A.34)

\[
\prod_{\mu=0}^{\nu} \varepsilon_\mu = k^{\nu} \frac{\tau_0}{1 + \tau_\nu} \overset{\nu \to \infty}{\longrightarrow} \frac{\tau_0}{\tau_0 + \frac{k-1}{k}} = 1 - \alpha. \tag{A.38}
\]

Thus we may let $\nu \to \infty$ and conclude, using again (A.31)

\[
\|u\|_\infty \leq c(m, p)\|u\|_m^{1-\alpha}
\]

\[
= c(m, p) c_\alpha^n \|u\|_m^{1-\alpha} (\|\nabla u\|_p + \|uA\|_p + \Lambda \|u\|_p)^\alpha.
\]

\[\blacksquare\]
Bibliography

[1] S. Blatt: A singular example for the Willmore flow, Analysis (Munich), 29: 407-430, 2009.

[2] P. Breuning: Immersions with local Lipschitz representation, PhD thesis, Universität Freiburg, 2011.

[3] M. P. do Carmo: Riemannian Geometry, Birkhäuser, Boston, 1992.

[4] L. C. Evans: Partial Differential Equations, volume 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 1998.

[5] L. C. Evans, R. F. Gariepy: Measure theory and fine properties of functions, Studies in Advanced Mathematics, CRC Press, 1992.

[6] S. Flotho: Elastische Flüsse von Kurven auf Riemannschen Mannigfaltigkeiten, Diplomarbeit, Universität Freiburg, 2002.

[7] W. Greub, S. Halperin and R. Vanstone: Connections, Curvature, and Cohomology, Vol I, Academic Press, 1972.

[8] D. T. Guarrera, N. G. Johnson, H. F. Wolfe: The Taylor Expansion of a Riemannian Metric, unpublished undergraduate research project, Tulane University, 2002.

[9] R. S. Hamilton: Three-Manifolds with Positive Ricci Curvature, J. Differential Geom., 17: 255-306, 1982.

[10] E. Hebey: Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities, Courant Lecture Notes in Mathematics, 1999.

[11] G. Huisken: Flow by mean curvature of convex surfaces into spheres, J. Differential Geom., 20: 237-266, 1984.

[12] J. Jost, H. Karcher: Geometrische Methoden zur Gewinnung von a-priori-Schranken für harmonische Abbildungen, manuscripta math., 40: 27-77, 1982.

[13] E. Kuwert and R. Schätzle: Gradient flow for the Willmore functional, Comm. Anal. Geom., 10: 307-339, 2002.
Bibliography

[14] E. Kuwert and R. Schätzle: The Willmore flow with small initial energy, J. Differential Geom., 57: 409-441, 2001.

[15] E. Kuwert: The Willmore Functional, unpublished lecture notes, ETH Zürich, 2007.

[16] T. Lamm: Biharmonischer Wärmefluß, diploma thesis, Universität Freiburg, 2001.

[17] J. Langer: A Compactness Theorem for Surfaces with $L_p$-Bounded Second Fundamental Form, Mathematische Annalen, 270: 223-234, 1985.

[18] P. Li and S. T. Yau: A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue on compact surfaces, Inventiones Mathematicae, 69: 269-291, 1982.

[19] F. Link: Gradientenfluß für das Willmorefunktional auf Riemannschen Mannigfaltigkeiten beschränkter Geometrie, diploma thesis, Universität Freiburg, 2006.

[20] U. F. Mayer and G. Simonett: A numerical scheme for radially symmetric solutions of curvature-driven free boundary problems, with applications to the Willmore flow, Interfaces and Free Boundaries, 4: 89-109, 2002.

[21] J. H. Michael and L. Simon: Sobolev and Mean-Value Inequalities on Generalized Submanifolds of $\mathbb{R}^n$, Comm. Pure Appl. Math., 26: 361-379, 1973.

[22] P. Petersen: Riemannian Geometry, GTM 171, Springer, New-York, 1998.

[23] L. M. Simon: Existence of Surfaces Minimizing the Willmore Functional, Comm. Anal. Geom., 1: 281-326, 1993.

[24] J. Simons: Minimal varieties in Riemannian manifolds, Annals of Math., 88: 62-105, 1968.

[25] J. Wloka: Partial Differential Equations, Cambridge University Press, 1987.

Florian Link, Universität Freiburg, Institut für Mathematik, Eckerstr. 1, D-79104 Freiburg

E-mail address: florian.link.math@gmx.de