Memory Augmented Neural Network
Adaptive Controller for Strict Feedback Nonlinear Systems

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Abstract

In this paper, we investigate the adaptive nonlinear control problem for strict feedback nonlinear systems, where the functions that determine the dynamics of the system are unknown. We assume that certain upper bounds for the functions $g_i$s of the system are known. The objective is to design an adaptive controller that can adapt to changes, possibly abrupt, in the unknown functions. We propose a novel backstepping memory augmented neural network (MANN) adaptive control method for solving this problem. The key idea is to augment the neural networks, in the standard backstepping NN adaptive controllers, with external working memory modules. The NN can write relevant information to its working memory and later retrieve them to modify its output, thus providing it with the capability to leverage past learned information effectively and improve its speed of learning. We propose a specific design for this external memory interface. We prove that the proposed control design achieves bounded stability for the closed loop system. We also provide numerical evidence on some simulation examples to show that the proposed memory augmentation quite significantly improves the speed of learning.

Key words: working memory, neural networks, adaptive backstepping control

1 Introduction

Adaptive control theory provides tools and techniques for the synthesis of controllers that can adapt to changes in the parameters in the system dynamics. The challenge is to design an adaptive controller such that the closed loop system is stable and matches the desired performance even as system parameters evolve. Both deterministic and stochastic adaptive control approaches have been widely studied over the last few decades and a great deal of progress in adaptive control has been made that has been documented in the scholarly literature. For the deterministic formulations, the reader is referred to the standard text books [1,2,4,12,14,20] and references therein.

In this paper, we focus on adaptive control of a certain class of nonlinear systems, namely strict feedback nonlinear systems. There is a rich history of adaptive control for this class of nonlinear systems. Kanellakopoulos, Kokotovic and Morse (1991) [11] pioneered a recursive design procedure known as the adaptive backstepping controller. They showed that the resulting closed loop system is globally stable and achieves asymptotic tracking. Kanellakopoulos et al. (1991) [11] extended the backstepping idea to a much broader class of nonlinear systems called pure-feedback systems, and showed the closed loop system to be regionally stable. Krstic, Kanellakopoulos and Kokotovic (1995) [12] extended the adaptive backstepping technique to parametric strict-feedback systems with unknown virtual control coefficients. Our contribution in this paper is in the setting of such adaptive backstepping controllers. Neural network based adaptive backstepping method was proposed for a class of nonlinear systems that ensured semi-global stability of the closed-loop system by Polycarpou (1993) [22]. This was extended to the general strict-feedback system case by Ge, Wang & Lee (2000) [6]. As suggested in [13], the primary advantage of using NN based backstepping adaptive controller is that it precludes the need for estimation of the regression matrices.

We focus on neural network (NN) based direct adaptive nonlinear controller for the control design. The literature on NN based adaptive nonlinear control is exten-
sive. The reader is referred to some of the standard text books \cite{15,17} and the following papers for further reading \cite{3,5,7,9,10,18,21}. Our main idea is a novel architectural modification wherein the NNs are \textit{augmented with an external memory module}. The motivation behind this modification is that such external memory modules comes from insights in systems neuroscience of learning and memory. More specifically, we are inspired by the growing knowledge regarding the role of memory systems in human learning. For example, the paper \cite{8} by Gershman et al. shows how complementing memory systems aid human learning.

In a very recent paper \cite{19}, we introduced a memory augmented neural network adaptive controller for model reference adaptive control (MRAC) and robot arm trajectory tracking controller. In this paper, we extend the memory augmented NN idea to the backstepping NN adaptive control design. In this approach, an external working memory is augmented to the NN. The central executive, which is the learning system, can read or write to the memory, very similar to the working memory systems in the human brain. The information that is read from the memory is used to modify the output of the NN, thus serving as a complementing memory system to the NN. In \cite{19} we proposed a specific design for this interface and observed significant improvements in the speed of learning. In this paper, we extend this interface design to the backstepping NN adaptive control design approach. We leverage the Lyapunov stability method proposed in \cite{23} for the design of the backstepping adaptive controller.

Our key contributions in this paper are (i) design of memory augmented NN adaptive backstepping controller for strict feedback systems (ii) proof of bounded stability and bounded tracking and (iii) evidence using simulation studies of improved learning, even after abrupt changes in the unknown function \cite{23}. In section 2 we introduce the problem setup, the control objective and the control architecture. Section 2.1 introduces the memory interface that augments the NN. In section 3 we introduce the backstepping memory augmented NN (MANN) adaptive control algorithm, which is based on the lyapunov stability analysis method proposed in \cite{23} and provide stability results. Finally in section 4 we provide simulation results and a detailed discussion substantiating the improved performance obtained by memory augmentation.

## 2 Control Architecture

In this section, we introduce the control architecture for the proposed MANN controller and the design of the memory interface that augments the NN. Denote the state by \(x\) and each component of the state by \(x_i\). The plant model is a nonlinear strict feedback system given by equations,

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\
\dot{x}_2 &= f_1(x_2) + g_1(x_1,x_2)x_3 \\
&\vdots \\
\dot{x}_n &= f_n(x_1,\ldots,x_n) + g_n(x_1,\ldots,x_n)u, \ y = x_1,
\end{align*}
\]

where \(f_i(x_1,\ldots,x_i)\) and \(g_i(x_1,\ldots,x_i)\) are unknown functions. We make the assumption that certain upper bounds of the unknown function \(g_i\)s are known and that the system state is observable. This assumption is specified in detail below.

**Assumption 1** (i) \(\exists\) strictly positive functions \(g_i(.)\) such that,

\[
g_i(.) > |g_{i,0}| > g_{i,0} > 0
\]

where \(g_{i,0}\) is a constant and that \(g_i(.)\) are known functions.

(ii) The system state is observable

The objective of the controller is to ensure that the system output \(y = x_1\) tracks the command signal \(y_d\) even when the unknown functions that govern the system dynamics changes abruptly.

The control architecture proposed in Fig. 1b is an extension of the standard backstepping NN adaptive control architecture \cite{13}, shown in Fig. 1a. Here, each NN approximator in the feedback loop is augmented with a memory similar to the MANN controller that was proposed in our earlier work \cite{19}. The NN can read or write to the memory. The information that is read from the memory is used to modify the output of the NN, which is in turn fed to the auxiliary control inputs \(x_{i,d}\) or the control input \(u\) as the case maybe. The state of the system is fed to the error evaluator block which computes the error between states \(x_i\) and the corresponding auxiliary control inputs \(x_{i,d}\), as shown in Fig. 1b. The output of the error evaluator are the error signals \(e_i\). These error signals are inputs to the control law which computes the auxiliary control signals \(x_{i,d}\) and the final control input \(u\). The error evaluator’s output are also fed to the ‘update law’ block which updates the parameters of the NN \(i\) based on the error signal \(e_i\). This completes the higher level description of the architecture.

### 2.1 Memory Interface

Here, we introduce the memory interface for the proposed controller. Denote the memory state corresponding to the \(i\)th working memory by matrix \(\mu_i\), the output of Memory Read of the \(i\)th working memory by \(M_{i,r}\), the modified NN output of the \(i\)th NN by \(y_{i,ad}\). Denote the input to the \(i\)th NN by \(\hat{x}_i\); which shall be defined later. The size of the memory matrix \(\mu_i\) is denoted by \(n_x \times N\), where \(n_x\) is the number of memory vectors in the memory. Denote the \(j\)-th column vector of matrix \(\mu_i\) by \(\mu_{i,j}\). Below, we briefly discuss the three interface operations, i.e., Memory Write, Memory Read and the NN output for the proposed memory interface.
2.2 Memory Write:

The Memory Write equation for the interface \( i \) is given by,

\[
\mu_{i,j} = -z_{i,j}\mu_{i,j} + c_wz_{i,j}a_i + z_{i,j}\hat{W}_i e_i
\]

\[ z = \text{softmax}(\mu^T q_i) \]  

(3)

Where \( a_i \) is the write vector corresponding to interface \( i \), \( q_i \) is the query vector for the interface \( i \) (to be defined later) and \( z_i \) is vector of weights that determines the relevance of the write vector \( a_i \) to the memory vector \( \mu_{i,j} \). The write vector \( a_i \) corresponds to the new information that can be used to update the contents of the memory. The write vector \( a_i \) for this interface is specified by,

\[
a_i = \sigma_i(V_i^T \tilde{x}_i + \hat{b}_{i,v})
\]  

(4)

That is, the write vector is set to be the current hidden layer value of the NN. In the above equation, \( c_w \) is a design constant. We choose this constant to be \( 3/4 \). The weight \( z_{i,j} \) is determined by a measure of similarity of the write vector (follows from (6)) and the memory vectors \( \mu_{i,j} \). It follows that the memory vector \( \mu_{i,j} \) that is most similar to the write vector \( a_i \) is considered eligible for the update. This ensures that the update by the newer hidden layer value, which is the write vector, is consistent with the information already stored at a location \( \mu_{i,j} \).

2.3 Memory Read:

The Memory Read for the \( i \)th interface is given by,

\[
\text{Memory Read: } M_{i,r} = \mu_i z_i, \quad z_i = \text{softmax}(\mu_i^T q_i)
\]  

(5)

where \( z_i \) is the same vector of weights that determines the similarity of the memory vectors in \( \mu_i \) to the query \( q_i \). Thus, the Memory Read output weighs those memory vectors that are similar to the query the highest in its output. We select the query vector to be the hidden layer output of NN \( i \), i.e.,

\[
q_i = \sigma_i(V_i^T \tilde{x}_i + \hat{b}_{i,v})
\]  

(6)

It follows, from what is written in the memory \( i \) (3) and the choice of query \( q_i \) that the Memory Read operation (5) retrieves similar values stored in the memory and so is likely to be relevant to the current scenario.

2.4 NN Output:

The learning system (NN) modifies its output using the information \( M_r \) retrieved from the memory. For this memory interface, the NN output is modified by adding the output of the Memory Read to the output of the hidden layer as given below.

\[
\text{NN Output: } u_{ad} = -\hat{W}^T \left( \sigma(\hat{V}^T \tilde{x} + \hat{b}_v) + M_r \right) - \hat{b}_w
\]  

(7)

We believe that such a modification improves the speed of learning by the induced learning mechanism. For a detailed discussion on the induced learning mechanism we refer the reader to [19].
3 Backstepping MANN Adaptive Control Algorithm and Stability

In this section, we discuss the derivation of the backstepping MANN control algorithm and provide proof for bounded stability of the closed loop system. First, we discuss the design of the backstepping algorithm for the first order system followed by the design of the algorithm for the more general nth order system.

3.1 Backstepping Control Algorithm for First Order System

In this section, we derive the backstepping MANN control algorithm for the following first order system,

\[ \dot{x}_1 = f_1(x_1) + g_1(x_1)u_1 \]  

(8)

Define \( e = x_1 - y_d \) and \( \beta_1 = g_1(x_1)/g_1(x_1) \). Consider the function,

\[ L_{e_1} = \int_{0}^{e_1} \alpha \beta_1(\alpha + y_d) d\alpha \]  

(9)

We can rewrite \( L_{e_1} \) as,

\[ L_{e_1} = e_1^2 \int_{0}^{1} \theta \beta_1(\theta e_1 + y_{cmd}) d\theta \]  

(10)

Using this expression, we can show that \( L_{e_1} \) trivially satisfies the following inequalities,

\[ \frac{e_1^2}{2} \leq L_{e_1} \leq \frac{e_1^2}{g_{t0}} \int_{0}^{1} \theta g_1(\theta e_1 + y_{cmd}) d\theta \]  

(11)

Thus, \( L_{e_1} \) is a positive-definition function of its argument. Differentiating \( L_{e_1} \) w.r.t time, we get,

\[ \dot{L}_{e_1} = e_1 \beta_1 \dot{e}_1 + \dot{y}_d \int_{0}^{e_1} \alpha \frac{\partial \beta_1}{\partial \alpha} d\alpha \]

\[ = e_1 \beta_1 (g_1(x_1)u_1 + f_1(x_1) - \dot{y}_d) + \dot{y}_d \int_{0}^{e_1} \alpha \frac{\partial \beta_1}{\partial \alpha} d\alpha \]

Applying UV rule for integration to the last term, we get,

\[ \dot{L}_{e_1} = e_1 \left( \beta_1 g_1(x_1)u_1 + \beta_1 f_1(x_1) - \dot{y}_d \int_{0}^{1} \beta_1(\theta e_1 + y_d) d\theta \right) \]  

(12)

Consider the following control input \( u \),

\[ u_1 = u_1^* = \frac{1}{g_1(x_1)} (-K_1 e_1 - \hat{h}_1(\hat{x}_1)) \]  

(13)

where,

\[ h_1(\hat{x}_1) = \beta_1(x_1) f_1(x_1) - \dot{y}_d \int_{0}^{1} \beta_1(\theta e_1 + y_d) d\theta \]

\[ \hat{x}_1 = [x_1, y_d, \dot{y}_d] \]  

(14)

Substituting this control input in the expression for \( \dot{L}_{e_1} \), it can be trivially shown that the closed loop system asymptotically tracks the command signal. We state this as the following lemma.

Lemma 1 The closed loop system specified by the plant model (8) and the control input \( u_1^* \) is globally asymptotically stable.

In the definition of control input \( u_1 \), as in (13), we assumed knowledge of the function \( h_1(\hat{x}_1) \), which is actually an unknown in our setting. Hence, we consider the approximation to \( u_1^* \) as the control input instead, and is given by,

\[ u_1 = \frac{1}{g_1(x_1)} (-K_1 e_1 - \hat{h}_1(\hat{x}_1)) \]  

(15)

where \( \hat{h}_1 \) is the NN approximation of \( h_1 \). For the MANN controller, where the NN output is modified according to (7), the approximation \( \hat{h}_1 \) is given by,

\[ \hat{h}_1 = \hat{W}_1^T \left( \sigma(\hat{V}^T \hat{x}_1 + \hat{b}_v) + M_{1,e} \right) + \hat{b}_w \]  

(16)

Consider \( \hat{W} \) and \( \hat{V} \) to be shorthand notation for the weight matrices that includes \( b_w \) and \( b_v^T \) in their final rows respectively. Let, \( x_{1,e} = \begin{bmatrix} \hat{x}_1 & 1 \end{bmatrix} \)

\[ \dot{\sigma} = \begin{bmatrix} \sigma(\hat{V}^T x_{1,e}) & 1 \end{bmatrix} \]. Then, using this shorthand notation we can write \( \hat{h}_1 \) as, \( \hat{h}_1 = \hat{W}^T \dot{\sigma} \). For this modified control law (15), the control gain \( K_1 \) is no more a simple constant and is set as,

\[ K_1 = K \left( 1 + \int_{0}^{1} \theta g_1(\theta e_1 + y_d) d\theta \right) \]

\[ + K \left( \| x_{1,e} \hat{W}_1^T \dot{\sigma} \|^2_F + \| \dot{\sigma}^T \hat{V}_{1,e}^T \|^2_2 \right) \]  

(17)

The update laws for the NN parameters are set equal to the standard two-layer NN update laws used in the neural network adaptive control literature [23], [16].

\[ \dot{\hat{W}} = C_w \left( \dot{\sigma} - \dot{\sigma}^T \hat{V}_{1,e}^T \right) e_1 - \kappa C_w \hat{W}_1 \]  

\[ \dot{\hat{V}} = C_v x_{1,e} \hat{W}_1^T \dot{\sigma} - \kappa C_v \hat{V}_1 \]  

(18)
We would like to emphasize that this is not an obvious choice for the NN update laws. The proof for stability reveals why this choice still works even with the inclusion of an external memory. Later, through simulations we show how the inclusion of an external memory significantly improves the learning performance when the system uncertainty undergoes abrupt changes. Below, we establish that the closed loop system specified by the plant, the control law and the NN update laws specified above is uniformly ultimately bounded.

**Theorem 1** The closed loop system specified by the plant model (8), the control input (13), the NN update laws (18), the memory interface operations (3), (5) and (7) is uniformly ultimately bounded.

### 3.2 Backstepping Control Algorithm for nth Order System

In this section, we discuss the Backstepping MANN controller for the nth order system (1). For notational convenience, we define $x_i = [x_1, x_2, ..., x_i]$. Note that the control input $u$ can no more be used to directly control the state variable $x_1$ to track the command signal $y_d$. The state variable $x_1$ can only be indirectly controlled through the state variable $x_2$. To this end, we define an auxiliary control signal, $x_2$, that the variable $x_2$ has to track. The auxiliary control signal, $x_2$, is defined as,

$$x_2 = \frac{1}{g_1(x_1)} (-K_1 e_1 - \dot{\hat{h}}_1(\tilde{x}_1))$$

$$\dot{\hat{h}}_1(\tilde{x}_1) = W_1^T \left( \hat{\sigma}_1 + \left[ \begin{array}{c} M_{1,r} \\ 1 \end{array} \right] \right), \quad \tilde{x}_1 = [x_1, y_d, \dot{y}_d]^T$$

$$K_1 = K \left( 1 + \int_0^1 \theta g_1(\theta e_1 + y_d) d\theta \right)$$

$$+ K \left( \| x_1 e \tilde{W}_1^T \hat{\sigma} \|_F^2 + \| \hat{\sigma} \tilde{V}_1^T x_1 e \|_2^2 \right) \tag{19}$$

We reiterate that the novelty in our design is the modification of the NN output by the output of the Memory Read $M_{1,r}$ corresponding to the working memory of NN1. As described earlier, $x_2$ should follow the signal $x_2$ in order to control $x_1$ as desired. As was the case with $x_1$, $x_2$ can only be controlled through the state variable $x_3$ and not directly through an external control input. To this end, we define an auxiliary control input, $x_{3,d}$, that $x_3$ has to track. This auxiliary control input $x_{3,d}$ is given by,

$$x_{3,d} = \frac{1}{g_2(x_2)} (-K_2 e_2 - g_1 e_1 - \dot{\hat{h}}_2(\tilde{x}_2))$$

$$K_2 = K \left( 1 + \int_0^1 \theta g_2(x_1, \theta e_2 + x_{2,d}) d\theta \right)$$

$$+ K \left( \| x_{2,e} \tilde{W}_2^T \hat{\sigma} \|_F^2 + \| \hat{\sigma} \tilde{V}_2^T x_{2,e} \|_2^2 \right) \tag{20}$$

where,

$$\dot{\hat{h}}_2(\tilde{x}_2) = \tilde{W}_2^T \left( \hat{\sigma}_2 + \left[ \begin{array}{c} M_{2,r} \\ 1 \end{array} \right] \right), \quad \tilde{x}_2 = [x_2, y_d, \dot{y}_d, \ddot{y}_d, \dot{Z}_1]^T,$$

and $\tilde{Z}_1$ is the vector of weights of NN1. As before, here too, the NN output is modified by the output of the Memory Read $M_{2,r}$ corresponding to the working memory of NN2.

We want $x_3$ to track $x_{3,d}$ and to do so we define another auxiliary control input $x_{4,d}$. This process repeats till the nth step where the final control input $u$ is specified. The auxiliary control input $x_{k+1,d}$, where $k + 1 \leq n$, is given by,

$$x_{k+1,d} = \frac{1}{g_k(x_k)} (-K_k e_k - g_{k-1} e_{k-1} - \dot{\hat{h}}_k(\tilde{x}_k))$$

$$K_k = K \left( 1 + \int_0^1 \theta g_k(x_{k-1}, \theta e_k + x_{k,d}) d\theta \right)$$

$$+ K \left( \| x_{k,e} W_k^T \hat{\sigma} \|_F^2 + \| \hat{\sigma} \dot{V}_k^T x_{k,e} \|_2^2 \right) \tag{21}$$

where,

$$\dot{\hat{h}}_k(\tilde{x}_k) = W_k^T \left( \hat{\sigma}_k + \left[ \begin{array}{c} M_{k,r} \\ 1 \end{array} \right] \right), \quad \tilde{x}_k = [x_k, y_d, \dot{y}_d, ..., y_{d,k}, \ddot{y}_d, ..., \dot{\tilde{Z}}_{k-1}]^T.$$

The function $h_k(\tilde{x}_k)$ that $\hat{h}_k$ approximates is given by,

$$h_k = \beta_k f_k(x_k) + e_k x_{k-1} + \int_0^1 \theta \frac{\partial \beta_k(x_{k-1}, \theta e_k + x_{k,d})}{\partial x_{k-1}} d\theta \tag{22}$$

The definition of $h_k$ follows from the design of the backstepping controller. Later, we shall see in the proof for stability of the closed loop system how this is a natural choice for the definition of the function $h_k$.

Finally, the variable $x_n$ is directly controlled using the plant’s control input $u$ to track $x_{n,d}$. The control input $u$ is defined as,

$$u = \frac{1}{g_n(x_n)} (-K_n e_n - g_{n-1} e_{n-1} - \dot{\hat{h}}_n(\tilde{x}_n)) \tag{23}$$

This completes the definition of the control law. The update law for the weights of each NN is set equal to the same update law discussed for the first order system.
earlier,
\[
\dot{W}_i = C_w \left( \hat{\sigma}_i - \hat{\sigma}_i^T x_{i,e} \right) e_i - \kappa C_w \dot{W}_i \\
\dot{V}_i = C_v x_{i,e} e_i \dot{W}_i^T \hat{\sigma}_i - \kappa C_v \dot{V}_i 
\] (24)

Below, we establish the stability of the closed loop system with the control law and NN update laws as defined above.

**Theorem 2** Consider the plant model given by (1). Let the control law be given by equations (19), (21) and (23), the NN update laws by (24), and the memory interface operations by (3), (5) and (7). Suppose that Assumption (1) is satisfied and $K$ is sufficiently large. If $c_w$ is a constant then the closed loop system is uniformly ultimately bounded.

We refer the reader to the appendix for the proof.

### 4 Discussion and Simulation Results

In this section, we provide a detailed illustration and a discussion on the performance of the MANN controller by considering several examples of strict feedback systems and several scenarios for each of the examples. The simulations reveal that the MANN controller significantly improves the recovery time of the closed loop system across varied scenarios, while the peak deviation remains below the deviation observed for the controller without memory. We attribute this to the ability of the MANN controller to quickly learn the new unknown function after an abrupt change.

#### 4.1 Second Order System: Example 1

In this example, we consider the 2nd order system specified by $f_1(x_1) = 0.1(-1/2x_1 + x_1^2)$ and $f_2(x_2) = 0.1(-0.5x_2 + x_2^2)$, $g_1(x_1) = 1 + 0.1x_1^2$, $g_2(x_2) = 1 + 0.1x_2^2$. For this example we assume that the known upper bound of the function $g_i$s, $\hat{g}_i = g_i$. The number of hidden layer neurons and the number of memory vectors are set as 6 and 1 respectively. The control gain is set as $K = 20$. The learning rates of the NN update laws are set as $C_w = C_v = 10, \kappa = 0$. We emphasize that the control gain $K$ is the most important parameter from the point of view of stability of the closed loop system, as is evident from the proof for stability, and the bounded stability of the closed loop system can be established when $\kappa = 0$ as well.

We consider couple of scenarios to illustrate the performance and to provide the comparison between MANN controller and the regular NN controller. In scenario 1, the command signal $y_d = 0.1$ and the system undergoes the following sequence of abrupt changes,
\[
f_i \rightarrow 200f_i \text{ at } t = 5, \quad f_i \rightarrow 2f_i \text{ at } t = 10 \\
f_i \rightarrow 1/400f_i \text{ at } t = 20
\] (25)

The simulation results for this scenario are shown in Fig. 2. It is clear that the system with the MANN controller in the feedback loop recovers faster after every abrupt change.

We now consider a second scenario. Here, the command signal $y_d = 0.1 \sin(0.5t)$, i.e., the command signal is a sinusoid and the system undergoes the following sequence of abrupt changes similar to the changes considered in scenario 1, i.e.,
\[
f_i \rightarrow 200f_i \text{ at } t = 5, \quad f_i \rightarrow 2f_i \text{ at } t = 10 \\
f_i \rightarrow 1/400f_i \text{ at } t = 20
\] (26)

The simulation results for scenario 2 are shown in Fig. 3. As in the previous scenario, we observe that the recovery is faster with the MANN controller in the feedback loop of the closed loop system. In Table 1 and Table 2 we provide the recovery time for the error to settle within 1% error. It is clear that the MANN controller reduces the recovery time by a significant margin. We attribute this to how the inclusion of an external memory induces the learning to be quick (refer [19]). In addition, we note that the peak deviations do not overshoot the peak deviations corresponding to the controller without memory.

We also consider a third scenario, where the abrupt changes are additive in nature. Here the function $f_i$ undergoes the following sequence of abrupt changes:
\[
f_i \rightarrow f_i + 0.001 \text{ at } t = 0, \\
f_i \rightarrow f_i + 0.05 - 0.001 \text{ at } t = 5, \\
f_i \rightarrow f_i + 0.1 - 0.05 \text{ at } t = 10, \\
f_i \rightarrow f_i + 0.001 - 0.1 \text{ at } t = 20
\] (27)
The response of the closed loop system for this scenario and the two controllers are shown in Fig. 4. From the response plots, it follows that the conclusions drawn in the previous two scenarios apply here as well. Table 3 lists the values for the time to settle within 0.1% error for both the controllers. It is evident that the MANN controller improves the time to settle by a significant margin for this scenario as well.

![System response](image1)

Fig. 4. System response $y$ for example 1 and scenario 3. Left: system response around the first two abrupt changes, right: system response around the last abrupt change

### 4.2 Second Order System: Example 2

In this example, we consider the 2nd order system specified by $f_1(x_1) = 0.1(−1/2x_1 + x_1^2)$ and $f_2(x_2) = 0.1(−0.5x_2 + x_1x_2 + x_2^2)$, $g_1(x_1) = 1 + 0.1x_1^2$, $g_2(x_2) = 1 + 0.1x_2^2$. We assume that the known upper bound of the function $g_i$, $g_i = g_i$. The number of hidden layer neurons and the number of memory vectors are set as 6 and 1 respectively. The control gain is set as $K = 20$. The learning rates of the NN update laws are set as $C_w = C_v = 10, \kappa = 0$. For illustration, we consider scenario 1 described above. The system response for this scenario is shown in Fig. 5. We observe that the closed loop system with MANN controller in its feedback loop is able to recover faster after each abrupt change. Table 1 and Table 2 provides the time to settle within 1% error. Similar to the previous example, here too, we observe that the MANN controller reduces the recovery time by a significant margin.

![System response](image2)

Fig. 5. System response $y$ for example 2 and scenario 1. Left: system response around the first two abrupt changes, right: system response at the last abrupt change

### 4.3 Third Order System: Example 3

In this example, we consider the 3rd order system modeled by $f_1(x_1) = 0.1(−1/2x_1 + x_1^2)$ and $f_2(x_2) = 0.1(−0.5x_2 + x_2^2)$, $f_3(x_3) = 0.1(−0.5x_3 + x_3^2)$, $g_1(x_1) = 1 + 0.1x_1^2$, $g_2(x_2) = 1 + 0.1x_2^2$ and $g_3(x_3) = 1 + 0.1x_3^2$. As in the previous examples, we assume that the known upper bound of the function $g_i$, $g_i = g_i$. The number of hidden layer neurons and the number of memory vectors are set as 6 and 1 respectively. The control gain is set as $K = 20$ and the learning rates of the NN update laws is set as $C_w = C_v = 10, \kappa = 0$. We observe that this system is just an extension of the second order system considered in the previous examples.

To illustrate the performance of the MANN controller we consider scenario 2 from example 1. Figure 6 shows the response of the controlled system for both the MANN controller and the controller without memory. We observe that the closed loop system with the MANN controller recovers faster after changes to the system. In addition, we observe that the amplitude of the high frequency oscillations in the response is lower for the system that uses MANN controller.

### 4.4 Third Order System: Example 4

In this example, we consider the third order system: $f_1(x_1) = 0.1(−1/2x_1 + x_1^2)$, $f_2(x_2) = 0.1(−0.5x_2 + x_1x_2 + x_2^2)$, $f_3(x_3) = 0.1(−0.5x_3 + x_3^2)$, $g_1(x_1) = 1 + 0.1x_1^2$, $g_2(x_2) = 1 + 0.1x_2^2$ and $g_3(x_3) = 1 + 0.1x_3^2$. We set the control gains and learning parameter rates to the values used in the previous examples. Figure 7 shows the response of the closed loop system for scenario 2. We observe that the MANN controller improves the recovery time and reduces the amplitude of oscillations, as observed in the previous example.
Fig. 6. System response $y$ for example 3 and scenario 2. Left above: system response around the first abrupt change, left below: system response around the second abrupt change, right below: system response around the last abrupt change.

Fig. 7. System response $y$ for example 4 and scenario 2. Left above: system response around the second and last abrupt changes.

5 Conclusion

In this work, we proposed a backstepping memory augmented NN (MANN) adaptive control design for strict feedback nonlinear systems whose functions that determine the dynamics of the plant are completely unknown. In the proposed design each NN is augmented by an external working memory. Each NN can write relevant information to its working memory and later retrieve them to modify its output, thus providing it with the capability to leverage past learned information effectively and improve its speed of learning. We then showed through extensive simulations on multiple examples that the closed loop system that uses MANN controller recovers significantly faster after abrupt changes when compared to the NN controller. We also proved that the closed loop system with the MANN controller is uniformly ultimately bounded.

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Using the expression for $u_t$, we get,

$$
\dot{L}_e = e_1 \beta_1 \dot{e}_1 + \dot{y}_d \int_0^{e_1} \alpha \frac{\partial \beta_1}{\partial \alpha} d\alpha
$$

(28)

Applying UV rule for integration to the last term, we get,

$$
\dot{L}_e = e_1 \left( u_1 + \beta_1 f_1(x_1) - \dot{y}_d \int_0^1 \beta_1 (\theta e_1 + y_d) d\theta \right)
$$

Then, using LaSalle’s invariance principle we can conclude, for the system defined in (8) and the control input (13), that the closed loop system tracks the command signal asymptotically, i.e., $e_1 \to 0$ as $t \to \infty$.

6.2 Proof of Theorem 1

Proof: Consider the following positive-definite function (this follows trivially from the above discussion for $L_{e_1}$),

$$
L_1 = L_{e_1} + \frac{1}{2} \text{Tr}\{\mu_1^T \mu_1\} + \frac{1}{2C_w} \hat{W}_1^T \hat{W}_1 + \frac{1}{2C_v} \text{Tr}\{\hat{V}_1 \hat{V}_1^T\}
$$

(30)

Substituting for $\hat{L}_{e_1}$ from an earlier expression, we get,

$$
\dot{L}_1 = e_1 \beta_1 \dot{e}_1 + \dot{y}_d \int_0^{e_1} \alpha \frac{\partial \beta_1}{\partial \alpha} d\alpha + \text{Tr}\{\mu_1 \dot{\mu}_1^T\}
$$

$$
+ \frac{1}{C_w} \hat{W}_1^T \dot{\hat{W}}_1 + \frac{1}{C_v} \text{Tr}\{\hat{V}_1 \dot{\hat{V}}_1^T\}
$$

(32)

Substituting for $\dot{e}_1$, we get,

$$
\dot{L}_1 = e_1 \beta_1 \left( f_1(x_1) + g_1(x_1) u_1 - \dot{y}_d \right) + \dot{y}_d \int_0^{e_1} \alpha \frac{\partial \beta_1}{\partial \alpha} d\alpha
$$

$$
+ \text{Tr}\{\mu_1 \dot{\mu}_1^T\} + \frac{1}{C_w} \hat{W}_1^T \dot{\hat{W}}_1 + \frac{1}{C_v} \text{Tr}\{\hat{V}_1 \dot{\hat{V}}_1^T\}
$$

(33)

Applying UV rule to the second term and substituting for $u_1$, we get,

$$
\dot{L}_1 = -K_1 e_1^2 + e_1 \left( h_1(\tilde{x}_1) - \hat{h}_1(\tilde{x}_1) \right) + \text{Tr}\{\mu_1 \dot{\mu}_1^T\}
$$

$$
+ \frac{1}{C_w} \hat{W}_1^T \dot{\hat{W}}_1 + \frac{1}{C_v} \text{Tr}\{\hat{V}_1 \dot{\hat{V}}_1^T\}
$$

(34)

We know that, $h_1(\tilde{x}_1) = W_1^T \sigma (V_1^T x_{1,e}) + \epsilon_1$. Then,

$$
h_1 - \hat{h}_1 = \hat{W}_1^T \left( \sigma - \tilde{\sigma}' \hat{V}_1^T x_{1,e} \right) + \hat{W}_1^T \tilde{\sigma}' \hat{V}_1^T x_{1,e} + d_1 + \epsilon_1
$$

(35)

where $\|d_1\| \leq \|V_1^*\|_{F} \|x_{1,e} \hat{W}_1^T \tilde{\sigma}' \|_{F} + \|W_1^*\|_{F} \|x_{1,e} \hat{\sigma}' \hat{V}_1^T \|_{F} + \|W_1^*\|_{F}$. The upper bound on the norm of $d_1$ follows from Lemma 2.1 in [23].

Substituting the update laws (18) and the expression (35) in (34), we get,

$$
\dot{L}_1 = -K_1 e_1^2 + e_1 (d_1 + \epsilon_1) - e_1 \hat{W}_1^T M_{1,r} + \text{Tr}\{\mu_1 \dot{\mu}_1^T\}
$$

$$
+ \kappa \hat{W}_1^T \hat{W} + \kappa \hat{V}_1^T \hat{V}
$$

(36)

From the Memory Write operation (3), we get,

$$
\dot{\mu}_1 = -\mu_1 \text{diag}(z_1) + c_w \tilde{\sigma} z_1^T + \hat{W}_1 e_1 z_1^T
$$

(37)

Substituting the expression for $\dot{\mu}_1$ in (36), we get,

$$
\dot{L}_1 = -K_1 e_1^2 + e_1 (d_1 + \epsilon_1) - e_1 \hat{W}_1^T M_{1,r}
$$

$$
- \text{Tr}\{\mu_1^T \text{diag}(z_1)\} + c_w \text{Tr}\{\tilde{\sigma}(\mu_1 z_1)^T\}
$$

$$
+ \text{Tr}\{\mu_1 z_1 e_1 \hat{W}_1^T\} + \kappa \hat{W}_1^T \hat{W} + \kappa \hat{V}_1^T \hat{V}
$$

(38)
Note that $e_1 \hat{W}_i^T M_{i,e} = \text{Tr}\{\mu_1 z_1 e_1 \hat{W}_i^T\}$. Hence,

$$\dot{L}_1 = -K_1 e_1^2 - \text{Tr}\{\mu_1^T \mu_1 \text{diag}\{z_1\}\} + c_w \text{Tr}\{\hat{\sigma}(\mu_1 z_1)^T\} + e_1 (d_1 + \epsilon_1) + \kappa \hat{W}^T \hat{W} + \kappa \hat{V}^T \hat{V} \quad (39)$$

We can rewrite the second and third term as,

$$\dot{L}_1 = -K_1 e_1^2 - \sum_i z_{i,j} \|\mu_{1,i}\|^2 + c_w (\mu_1 z_1)^T \hat{\sigma} + e_1 (d_1 + \epsilon_1) + \kappa \hat{W}^T \hat{W} + \kappa \hat{V}^T \hat{V} \quad (40)$$

By applying Cauchy-Shwartz inequality to the term $c_w (\mu z)^T \hat{\sigma}$, we get,

$$\dot{L}_1 \leq -K_1 e_1^2 - \sum_j z_{1,j} \|\mu_{1,j}\|^2 + c_1 \sum_j z_{1,j} \|\mu_{1,j}\|^2 + c_1 (d_1 + \epsilon_1) + \kappa \hat{W}^T \hat{W} + \kappa \hat{V}^T \hat{V} \quad (41)$$

where $c_1$ is a constant and $c_1 > 0$. Completing squares, we get,

$$\dot{L}_1 \leq -K_1 e_1^2 - \sum_j z_{1,j} (\|\mu_{1,j}\|^2 - c_1/4\|\mu_{1,j}\|^2) + c_1^2/4 + e_1 (d_1 + \epsilon_1) + \kappa \hat{W}^T \hat{W} + \kappa \hat{V}^T \hat{V} \quad (42)$$

That is,

$$\dot{L}_1 \leq -K_1 e_1^2 + c_1^2/4 + e_1 (d_1 + \epsilon_1) + \kappa \hat{W}^T \hat{W} + \kappa \hat{V}^T \hat{V} \quad (43)$$

Then following the steps similar to the proof of Theorem 3.1 in [23], we get the following,

$$\dot{L}_1 \leq -K e_1^2 \left(1/2 + \int_0^1 \theta g_1(\theta e_1 + y_d) d\theta\right) - \kappa/2 \left(\|\hat{W}_1\|^2 + \|\hat{V}_1\|^2\right) + c_2 \quad (44)$$

where,

$$c_2 = \frac{1}{4K} \left(\|W_1\|^2 + \|V_1\|^2 + \|W_1\|^2 + c_1^2\right) + \frac{\kappa^2}{2} \left(\|W_1\|^2 + \|V_1\|^2\right) + c_1^2/4. \quad (45)$$

It follows from (44), that

$$\dot{L}_1 \leq -K e_1^2 \left(1/2 + \int_0^1 \theta g_1(\theta e_1 + y_d) d\theta\right) + c_2 \quad (46)$$

That is $\dot{L}_1$ is negative when,

$$\|e_1\| > \sqrt{\frac{2\epsilon_2}{K(1 + g_{10})}} = r_1 \quad (47)$$

Thus, we can choose $K$ sufficiently large such that $r_1$ is small and if the initial conditions lie within the bounded set where the NN approximation holds, then the error $\|e_1\|$ stays within the compact set where the NN approximation holds and $\|e_1\|$ converges to a value less than $r_1$ in finite time. Thus, the closed loop system with the MANN controller in the feedback loop is uniformly ultimately bounded or UUB.

6.3 Proof of Theorem 2

Proof: The proof of this theorem is an extension of the stability proof discussed in section 3.1. Consider the same positive-definite function as before, i.e., $L_{e_1}$. The derivative of $e_1$ for this case is given by,

$$\dot{e}_1 = \dot{x}_1 - \dot{y}_d = f_1(x_1) + g_1(x_1) x_2 - \hat{y}_d = f_1(x_1) + g_1(x_1) (x_2 - x_{2,d}) + g_1(x_1)x_{2,d} - \hat{y}_d = f_1(x_1) + g_1(x_1)e_2 + g_1(x_1)x_{2,d} - \hat{y}_d \quad (48)$$

Thus, it follows that,

$$\dot{L}_{e_1} = -K_1 e_1^2 + e_1 (h_1(x_1) - \hat{h}_1(x_1)) + e_1 g_1(x_1)e_2 \quad (49)$$

Define $\tilde{h}_1 = h_1 - \hat{h}_1$. Then,

$$\dot{L}_{e_1} = -K_1 e_1^2 + e_1 \tilde{h}_1 + e_1 g_1(x_1)e_2 \quad (50)$$

Consider a second positive-definite function $L_{e_2}$, given by,

$$L_{e_2} = L_{e_1} + \int_0^{t_2} \alpha \beta_2(x_1, \alpha + x_{2,d}) d\alpha \quad (51)$$

Differentiating either side w.r.t time, we get,

$$\dot{L}_{e_2} = \dot{L}_{e_1} + e_2 \beta_2 \dot{e}_2 + \dot{x}_1 \int_0^{t_2} \alpha \frac{\partial \beta_2(x_1, \alpha + x_{2,d})}{\partial x_1} d\alpha + \dot{x}_{2,d} \int_0^{t_2} \alpha \frac{\partial \beta_2(x_1, \alpha + x_{2,d})}{\partial x_2} d\alpha \quad (52)$$

Applying UV rule for integration to the last term, we get,

$$\dot{L}_{e_2} = \dot{L}_{e_1} + e_2 \beta_2 \dot{e}_2 + \dot{x}_1 \int_0^{t_2} \alpha \frac{\partial \beta_2(x_1, \alpha + x_{2,d})}{\partial x_1} d\alpha + e_2 \beta_2 \dot{x}_{2,d} + \dot{x}_{2,d} e_2 \int_0^{t_2} \beta_2(x_1, \alpha + x_{2,d}) d\alpha \quad (53)$$
where the derivative of error $e_2$ w.r.t time is given by,
\[ \dot{e}_2 = f_2(x_2) + g_2(x_2)e_3 + g_2(x_2)x_{3,d} - \dot{x}_{2,d}. \]

Substituting for $\dot{L}_{e,1}$ and $\dot{e}_1$ and using the expression for $h_2$ in (58), we get,
\[ \dot{L}_{e,2} = -\sum_{i=1}^{2} K_i e_i^2 + \sum_{i=1}^{2} e_i \dot{h}_i + e_2 g_2 e_3 + \dot{x}_{2,d} \cdot d \]  

(54)

Similar to $k = 2$, for a general $k$, we can define a positive-definite function $L_{e,k}$,
\[ L_{e,k} = \sum_{i=1}^{k-1} L_{e,i} + \int_{0}^{\epsilon_k} \alpha \beta_k(x_{k-1}, \alpha + x_{k,d}) d\alpha \]  

(55)

Following steps similar to that used for deriving $\dot{L}_{e,2}$ we can show that,
\[ \dot{L}_{e,k} = -\sum_{i=1}^{k} K_i e_i^2 + \sum_{i=1}^{k} e_i \dot{h}_i + e_k g_k e_{k+1} \]  

(56)

Finally, consider the positive-definite function,
\[ L = \sum_{i=1}^{n} \left( L_{e,i} + \text{Tr} \{ \mu_i^T \hat{\mu}_i \} + \frac{1}{C_w} \hat{W}_i^T \hat{W}_i + \frac{1}{C_v} \hat{V}_i^T \hat{V}_i \right) \]  

(57)

Differentiating w.r.t time and following steps similar to the proof of Theorem 1, we can show that,
\[ \dot{L} \leq -\sum_{i=1}^{n} K e_i^2 \left( \frac{1}{2} + \int_{0}^{1} \theta g_i(x_{i-1}, \theta e_i + x_{i,d}) d\theta \right) - \kappa/2 \sum_{i=1}^{n} \left( \| \hat{W}_i \|_F^2 + \| \hat{V}_i \|_F^2 \right) + \text{const.} \]  

(58)

It follows that,
\[ \dot{L} \leq -\sum_{i=1}^{n} \frac{1}{2} K(1 + g_{i,0}) \| e_i \|_2^2 + \text{const.} \]  

(59)

Hence, $\dot{L} < 0$, when
\[ \| e_i \|_2 > \sqrt{\frac{2 \text{const.}}{K(1 + g_{i,0})}} = r_i \]  

(60)

Since $K$ is large enough, $r_i$'s are small. Provided, the initial conditions are such that the NN approximation holds, then the approximation should continue to hold and $\| e_i \|_2$ will converge to a value less than $r_i$ in finite time. Hence, the closed loop system is uniformly ultimately bounded. ■