COVARIANT METHODS FOR CALCULATING THE LOW-ENERGY EFFECTIVE ACTION IN QUANTUM FIELD THEORY AND QUANTUM GRAVITY

I. G. Avramidi † § ‡

Department of Mathematics, University of Greifswald
Friedrich-Ludwig-Jahn-Str. 15A, 17489 Greifswald, Germany.

Abstract

We continue the development of the effective covariant methods for calculating the heat kernel and the one-loop effective action in quantum field theory and quantum gravity. The status of the low-energy approximation in quantum gauge theories and quantum gravity is discussed in detail on the basis of analyzing the local Schwinger - De Witt expansion. It is argued that the low-energy limit, when defined in a covariant way, should be related to background fields with covariantly constant curvature, gauge field strength and potential. Some new approaches for calculating the low-energy heat kernel assuming a covariantly constant background are proposed. The one-loop low-energy effective action in Yang-Mills theory in flat space with arbitrary compact simple gauge group and arbitrary matter on a covariantly constant background is calculated. The stability problem of the chromomagnetic (Savvidy-type) vacuum is analyzed. It is shown, that this type of vacuum structure can be stable only in the case when more than one background chromomagnetic fields are present and the values of these fields differ not greatly from each other. This is possible only in space-times of dimension not less than five \( d \geq 5 \).

PACS number(s): 03.70.+k, 04.60.+n, 04.90.+e, 02.40.Vh, 12.38.Aw-t

† Alexander von Humboldt Fellow
§ This work was supported, in part by a Soros Humanitarian Foundation’s Grant awarded by the American Physical Society and by an Award through the International Science Foundation’s Emergency Grant competition.
‡ On leave of absence from Research Institute for Physics, Rostov State University, Stachki 194, 344104 Rostov-on-Don, Russia.
I. INTRODUCTION

In present paper we continue our efforts in developing methods for computing the effective action in quantum field theory, that we started in [1-7]. The effective action is a very powerful tool for investigating the general problems of quantum field theory as well as various models. It takes into account all quantum fluctuations and contains, in principle, all predictions of quantum field theory [8-13]. Due to special advantages achieved by using geometric methods the effective action approach is spread widely in gauge theories and quantum gravity, supergravity, Kaluza-Klein models, strings etc. [14-20].

However, the practical calculation of the effective action will entail great difficulties. The point is, that, although it is possible to calculate the effective action in some rare special cases on fixed background, for effective action to be used one has to vary it, i.e. one needs it as a functional of background fields of general type. Therefore, various approximate methods for calculating the effective action were elaborated. The first one is the so called Schwinger - De Witt expansion [8,9-13,21,22], which was successfully used for treating divergences, renormalization, anomalies etc. [20,17].

All quantities of interest (such as the effective action, Green function, stress tensor, currents, anomalies) are expressed in this approach in terms of coefficients of asymptotic expansion of the corresponding heat kernel, so called Hadamard - Minakshisundaram - De Witt - Seely (HMDS) coefficients [23-44]. Various methods were used for calculating these coefficients, beginning from the direct De Witt’s method [9,10] to modern mathematical methods, which make use of pseudodifferential operators [28,32,42]. Very good review of calculation of HMDS-coefficients is given in recent paper [34]. The first three coefficients were calculated in [28].

An effective covariant technique for calculating HMDS-coefficients is elaborated in [1-6], where also the first four coefficients are calculated. In the case of scalar operators the fourth coefficient is also calculated in [45]. About the fifth coefficient in flat space see [46]. Analytic approach was developed in [38,39], where a general expression in closed form for these coefficients was obtained.

The Schwinger - De Witt expansion describes good the vacuum polarization effect of massive fields in weak background fields [8,10,20,47]. But it is absolutely inadequate and becomes meaningless in strong background fields and massless theories. For investigating these (nonlocal) aspects of the effective action one needs some new methods. There were proposed, on the one hand, the immediate direct partial summation of higher derivatives in local Schwinger - De Witt expansion [13,2,3,5,6], and, on the other hand, covariant perturbation theory in powers of curvatures [48,49]. Both of these methods lead finally to nonlocal expansion of the effective action, which is valid also in massless theories but only in high-energy (short-wave) limit, when the background curvatures are small rapidly varying external fields. This approximation was used already for calculating the anomalous magnetic moment of the electron [50] and investigating the Hawking radiation in two dimensions and the gravitational collapse problem [13,51,52].

Methods and results of the papers [48,49], are valid, strictly speaking, for asymptotically flat noncompact complete manifolds without the boundary. They can not help in analyzing long-wave global aspects of the effective action. The low-energy effective action (in other words, the effective potential) presents a very natural tool for investigating the
vaccum of the theory, its stability and the phase structure [14-18]. It is determined by strong slowly varying background fields and, therefore, its calculation depends essentially on global topological properties of the space-time manifold. This is very important and, in general case, still unsolved problem. Some new results on this subject concerning calculation of the low-energy heat kernel and one-loop effective potential are obtained in [67, 68].

All these approximations are bound up with each other. They are, as a matter of fact, various reexpansions of a single general effective action of quantum gravity.

This paper is organized as follows. The heat kernel method and the Schwinger - De Witt expansion is described in the next sect. II. We analyze here also the general structure of the HMDS - coefficients. In sect. III we discuss the status of the high-energy approximation and the low-energy one in quantum gravity and gauge theories. It is found that a symmetry Lie group appears when formulating the low-energy approximation in covariant way, i.e on covariantly constant background. A number of perspective methods for calculating the effective action in low-energy approximation, in which, in particular, the symmetry algebra, is used essentially, are proposed in sect. IV. Sect. Y is devoted to the calculation of the low-energy effective action in flat space. Here a Yang - Mills model with any matter fields is considered, the corresponding one-loop effective potential is computed and the vacuum structure of the model is slightly analyzed. In the concluding sect. VI we formulate briefly the basic results.

We use in this paper Euclidean notations meaning to obtain physical effective equations by analytic continuation. As it is shown in [48] this can be done for a very wide class of problems. Our notations and definitions of curvature tensor are as in [9,12].

II. ONE-LOOP EFFECTIVE ACTION AND HEAT KERNEL

For a wide range of models in quantum field theory the contribution of a multiplet of quantized fields $\phi = \{\phi^A(x)\}$ on $d$-dimensional Riemannian manifold $M$ to the effective action $\Gamma$ in one-loop approximation is expressible in terms of a functional superdeterminant of an elliptic second order differential operator

$$\Gamma = S + \hbar \Gamma_{(1)} + O(\hbar^2) \quad (2.1)$$

$$\Gamma_{(1)} = \frac{1}{2} \ln S \text{det} \frac{F}{\mu^2} = \frac{1}{2} \text{Str} \ln \frac{F}{\mu^2} \quad (2.2)$$

$$F = g^{1/2}(-\Box + Q + m^2) \quad (2.3)$$

$$\Box = g^{\mu\nu} \nabla_\mu \nabla_\nu = g^{-1/2}(\partial_\mu + A_\mu)g^{1/2}g^{\mu\nu}(\partial_\nu + A_\nu) \quad (2.4)$$

where $S$ is a classical action of the model, $\text{Str}$ means functional supertrace, $\mu$ is a renorm-parameter, $g_{\mu\nu}$ is the metric, $g = \det g_{\mu\nu} \neq 0 \ , \ Q = \{ Q_B^A (x) \}$ is an arbitrary matrix (potential term), $m$ is a mass parameter and $\nabla_\mu$ is a covariant derivative, defined with the help of an arbitrary linear connection $A_\mu$, including both vector gauge and appropriate spin one. As regards the more general differential operators see [21].

Let us mention that the multiplet $\phi = \{\phi^A\}$ realizes, in general, reducible representation of gauge and Lorentz group. That means, that it contains all physical fields, i.e.
scalar, spinor, vector etc. ones \( \varphi = \{ \varphi^a(x), \psi^i(x), B^c_{\mu}, h_{\mu \nu}, \ldots \} \). We assume also operator \( F (2.2) \) to be positive. This is, in fact, the case when \( m^2 \) is sufficiently large and the background fields are regular and bounded at the infinity.

Thus the effective action is a functional of the coefficient functions of the operator \( F (2.2) \), i.e. three different background fields - the metric, the connection and the potential term \( \Gamma = \Gamma(g_{\mu \nu}, A_\mu, Q) \).

Making use of the covariance of the operator \( F (2.2) \), it is easy to show, that the effective action (2.1) possess a crucial property - it is invariant under diffeomorphisms and local gauge transformations. That means that the effective action depends, in fact, only on the geometry of the manifold and on the classes of gauge equivalent fields and does not depend on the choice of the set of coordinates and the gauge of gauge field.

Local invariant characteristics of the background fields are described by covariant quantities (tensors), i.e. Riemann tensor \( R_{\mu \nu \alpha \beta} \) and curvature of background connection

\[
R_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]
\]  

(2.5)

(along with the potential term \( Q \), which also transforms covariantly). These three quantities have the same dimensions and play often the same role in calculations. That is why we shall call them generalized background curvatures (or simply curvatures) and denote in symbolic expressions by one symbol

\[
R = \{ R_{\mu \nu \alpha \beta}, R_{\mu \nu}, Q \}
\]

The effective action is, in general, very complicated nonlocal functional and must be built, therefore, from covariant geometric objects. Those can be either local tensors (such as curvature tensors and their derivatives) or nonlocal (two-point, three-point etc.) covariant geometric objects, which transform completely independent as different tensors in different points of the manifold. The most commonly encountered nonlocal covariant geometric objects are the world function (or the geodetic interval), the parallel displacement operator and Green functions of some covariant differential operators.

The geometry of the manifold is described, in general, not only by local characteristics like invariants of the curvature tensor, but also by global structure - its topology, boundaries etc. The curvature tensor alone does not determine all details of the geometry of the whole manifold.

The effective action is determined, generally speaking, by the spectrum of the operator \( F (2.2) \) and, naturally, depends also on the global structure of the manifold. Thus the effective action depends, in general, also on the topology. Let us mention, that this does not contradict with the covariance, what is meant sometimes when one calls the contributions in the effective action, which can not be expressed locally in terms of the curvature, noncovariant. The covariance is, simply, a local concept, and the topology is a global one. Therefore one can speak about the covariance of the effective action within the limits of fixed topology, as the diffeomorphisms and local gauge transformations do not lead out of bounds of the given topology.

Let us mention from the very beginning, that in this paper we will be interested in the case of complete noncompact asymptotically flat manifolds without boundaries with
standard boundary conditions, viz., the regularity on the Euclidean infinity. We will not investigate the influence of the topology and will concentrate our attention, as a rule, on the effects of the curvature.

For covariant calculation of the effective action it is convenient to express it in terms of the heat kernel \( U(t) \)

\[
\Gamma_{(1)} = \int dx \left( -\frac{1}{2} \right) \int_0^\infty \frac{dt}{t} \text{str} U(t|x, x)
\]

\( U(t|x, x') = \exp(-tH)\mathcal{P}(x, x')\delta(x, x') \)  

\( H = g^{-1/4}Fg^{-1/4} = -\Box + Q + m^2 \)

where \( \text{str} \) is a usual matrix supertrace and \( \mathcal{P}(x, x') = \{\mathcal{P}^A_{B'}(x, x')\} \) is the parallel displacement operator of the field \( \varphi = \{\varphi^A\} \) from the point \( x \) to the point \( x' \) along the geodesic, [53,9,12,6].

Thus the main object to deal with is the heat kernel \( U(t) \) (2.7). More explicitly it is defined by requiring it to satisfy the equation

\[
\left( \frac{\partial}{\partial t} + H \right) U(t|x, x') = 0
\]

with initial condition

\[
U(0|x, x') = \delta(x, x')
\]

In the case of noncomplete manifolds one should also impose some boundary conditions [54-59]. However we will not consider in this paper the boundary effects.

It is well known that the heat kernel at small \( t \to 0 \) and when the points \( x \) and \( x' \) are close to each other \( x \to x' \) behaves like [9,10,31,]

\[
U(t|x, x') = g^{1/4}(x)g^{1/4}(x')\Delta^{1/2}(x, x')\mathcal{P}(x, x')\exp\left(-\frac{\sigma(x, x')}{2t}\right)
\]

\[
 \times (4\pi t)^{-d/2} \exp(-tm^2) \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} a_k(x, x')
\]

where \( \sigma(x, x') \) and \( \Delta(x, x') \) are the geodetic interval (world function), which is equal to one half the square of the distance along the geodesic between the points \( x \) and \( x' \), and the corresponding Van Vleck - Morette determinant [53,62,9,12].

This expansion at coinciding points \( x = x' \) is called often the Schwinger - De Witt one. We call its coefficients \( a_k(x, x) \) the Hadamard - Minakshisundaram - De Witt - Seely (HMDS) coefficients. One can show, that Schwinger - De Witt expansion is purely local and HMDS-coefficients are local invariants built from the curvatures (including the potential term) and their covariant derivatives [9,10,12,31,6]. They play very important role in physics as well as in mathematics and are closely connected with various sections of mathematical physics [34,31]. Therefore, the calculation of HMDS-coefficients is in itself of
great importance. Nowadays in general case only the first four coefficients are calculated \([1, 4, 5, 6, 45]\). In manifolds with boundary additional terms in the asymptotic expansion of the heat kernel proportional to \(t^{-d/2+k/2}\) appear. For details see \([54-59, 7]\).

Several first HMDS-coefficients have the form

\[
\begin{align*}
a_0(x, x) &= 1 \\
a_1(x, x) &= Q - \frac{1}{6} R \\
a_2(x, x) &= \left( Q - \frac{1}{6} R \right)^2 - \frac{1}{3} \Box Q + \frac{1}{6} R_{\mu \nu} R^{\mu \nu} - \frac{1}{90} R_{\mu \nu} R^{\mu \nu} + \frac{1}{90} R_{\mu \nu \alpha \beta} R^{\mu \nu \alpha \beta} + \frac{1}{15} \Box R
\end{align*}
\]

The general structure of other coefficients \(a_k\) can be represented symbolically in the form

\[
a_k(x, x) = \nabla^{2k-2} R + \sum_{0 \leq i \leq 2k-4} \nabla^i R \nabla^{2k-4-i} R + \sum_{i,j \geq 0; \; i+j \leq 2k-6} \nabla^i R \nabla^j R \nabla^{2k-6-i-j} R \\
+ \cdots + \sum_{0 \leq i \leq k-1} R^i (\nabla \nabla R) R^{k-i-1} + \sum_{i,j \geq 0; \; i+j \leq k-3} R^i (\nabla R) R^j (\nabla R) R^{k-i-j-3} + R^k
\]

(2.13)

From the behavior of the heat kernel at \(t \to 0\) (2.11) it is immediately seen, that the expression (2.6) for the effective action diverges at the lower limit of integration \(t \to 0\). Therefore, one has to regularize and to renormalize it, the covariance being manifestly preserved. The most convenient way to do it in the one-loop case is to use the \(\zeta\)-function prescription

\[
\begin{align*}
\Gamma_{(1)} &= -\frac{1}{2} \zeta'(0) \\
\zeta'(0) &= \frac{d}{dp} \zeta(p)\big|_{p=0}
\end{align*}
\]

(2.14) (2.15)

where

\[
\zeta(p) = \mu^{2p} \text{Str} H^{-p} = \int dx \frac{\mu^{2p}}{\Gamma(p)} \int_0^\infty dt \; t^{p-1} \text{str} U(t|x, x)
\]

(2.16)

The main merit of the heat kernel method is that the heat kernel is good defined and does not have any divergences in contrast to the effective action. This allows to single out in the effective action the property of the covariance from the field-theoretical divergences problem. In other words, making use of the heat kernel divides the problem of evaluating the effective action in two parts: i) calculating the heat kernel and ii) regularization and renormalization.

Using the \(\zeta\)-function regularization (2.14) and the Schwinger - De Witt expansion (2.11) one can get the asymptotic \(1/m^2\)-expansion for the effective action in terms of HMDS-coefficients. One obtains in this way for odd dimension

\[
\Gamma_{(1)} = \frac{1}{2} (4\pi)^{-d/2} \pi (-1)^{d-1} \sum_{k \geq 0} \frac{m^{d-2k}}{k! \Gamma \left( \frac{1}{2}d - k + 1 \right)} A_k
\]

(2.17)
and for even dimension

\[ \Gamma_{(1)} = \frac{1}{2} (4\pi)^{-d/2} \left\{ (-1)^{d/2} \sum_{k=0}^{d/2} \frac{m^{d-2k}}{k! \Gamma\left(\frac{1}{2}d + 1 - k\right)} A_k \left[ \ln \frac{m^2}{\mu^2} - \Psi\left(\frac{d}{2} - k + 1\right) - C \right] \right. \]

\[ + \sum_{k \geq \frac{1}{2} d+1} \frac{\Gamma\left(k - \frac{d}{2}\right)}{k! m^{2k-d}} A_k \right\} \quad (2.18) \]

where

\[ A_k = \int dx g^{1/2} \text{str} a_k(x, x) \quad (2.19) \]

\[ \Psi(q) = \frac{d}{dq} \ln \Gamma(q) \quad , \quad C = -\Psi(1) \]

Let us notice the essentially different dependence of the effective action on the renormalization parameter \( \mu \) in the spaces of odd and even dimensions. This is due to finiteness of one-loop effective action in odd dimensions.

Thus the Schwinger - De Witt expansion of the effective action (2.17), (2.18) is purely local and does not depend, in fact, on the global structure of the manifold. Using the integration by parts one can easily conclude that the general structure of the coefficients \( A_k \) (2.19) has the form

\[ A_k = \int dx g^{1/2} \text{str} \left\{ R \Box^{k-2} R + \sum_{0 \leq i \leq 2k-6} R \nabla^i R \nabla^{2k-6-i} R \right. \]

\[ + \cdots + \sum_{0 \leq i \leq k-3} R^i(\nabla R) R^{k-i-3}(\nabla R) + R^k \right\} \quad (2.20) \]

It is evident, that the local Schwinger - De Witt expansion is a good approximation and describes, therefore, the vacuum polarization effect of massive quantum fields in weak background fields when

\[ A_k \ll m^{2k} \]

i.e. all invariants of the curvature are smaller than the corresponding power of the mass parameter. However it is not good approximation in the case of strong background fields and absolutely meaningless in massless theories. For investigation of these cases one needs other special methods, which should also be manifestly covariant.

One possibility to exceed the limits of the Schwinger - De Witt expansion in manifestly covariant manner is to compare all the terms in HMDS-coefficients (2.20), to pick up the main (the largest in some approximation) terms and to sum up the corresponding partial sum. This idea was first proposed in [13] and realized in [2,3,5,6], where all the terms in coefficients \( A_k \) (2.20) with higher derivatives, which are quadratic in curvatures are calculated and the Schwinger - De Witt (2.11) expansion is summed in this approximation. The terms quadratic in curvatures were also calculated completely independent in [60]. In recent papers [22,15,51] the third order in curvatures is analyzed.
III. APPROXIMATIONS UNDER CONSIDERATION

Thus the effective action is a functional of the metric, connection and the potential term. This means that it depends on all details of the behavior of these functions of the coordinates. Exact evaluation of the effective action is impossible even in one-loop approximation. Therefore, one should use some approximations based on the assumptions about the arguments of that functional, i.e. about the behavior of these quantities as functions of coordinates.

A remark concerning the meaning of the background fields as functions of coordinates has to be made here. Namely, one should distinguish the real physical variations of these functions from the purely gauge fictive ones, which reflect simply the transformations of the coordinates and local gauge transformations. In other words, one has to factorize out the gauge degrees of freedom of background fields. It is this physical gauge invariant dependence on the coordinates that we will mean when speaking about the behavior of the background fields. This can be achieved by using for local description of background fields the invariants ( scalars) built from the curvatures and by comparing the values of invariants themselves with the derivatives of them. Since the usual derivatives of the invariants coincide with the covariant derivatives, one can compare the values of curvatures and their covariant derivatives.

If one expands the background fields in the Fourier integral, which can be defined in an appropriate way, then one may regard the effective action to depend on the Fourier components of background fields. Here two main approximations are possible.

3.1. HIGH-ENERGY APPROXIMATION

First of all, this is the high-energy limit, when the details of the dependence of the effective action on the short-wave part of the spectrum of the background fields are analyzed. In coordinate representation this approximation corresponds to weak rapidly varying background fields (ripples) that may be characterized by a covariant relation

\[ \nabla \nabla R \gg RR \]  

(3.1)

Since the commutators of the covariant derivatives add a spare power of the curvature, they commute at a given order in this approximation.

This approximation corresponds to partial summation at first in local Schwinger-De Witt expansion (2.11) the terms with higher derivatives (in other words with higher momentums) of background fields. Analyzing and summing up such terms [2,3,5,6] leads to explicit covariant nonlocal expressions for the heat kernel

\[
\text{Str } U(t) = \int dx \, g^{1/2}(4\pi t)^{-d/2} \exp \left( -tm^2 \right) \text{str} \left\{ 1 - t \left[ Q - \frac{1}{6} R \right] \right. \\
+ \frac{t^2}{2} \left[ Q \gamma^{(1)}(t \Box) Q + 2 R_{\alpha\mu} \nabla^\alpha \frac{1}{\Box} \gamma^{(2)}(t \Box) \nabla^\nu R^{\nu\mu} - 2 Q \gamma^{(3)}(t \Box) R \\
+ R_{\mu\nu} \gamma^{(4)}(t \Box) R^{\mu\nu} + R \gamma^{(5)}(t \Box) R \right] + O(R^3) \right\}.
\]  

(3.2)

8
and the effective action
\[ \Gamma_{(1)} = \Gamma_{(1)\text{loc}} + \Gamma_{(1)\text{nonloc}} \] (3.3)
where the local part equals: in odd dimension
\[ \Gamma_{(1)\text{loc}} = \frac{1}{2} (4\pi)^{-d/2} \frac{\pi (-1)^{d+1}}{\Gamma(\frac{d}{2} + 1)} \int dx g^{1/2} \text{str} \left\{ m^d + \frac{d}{2} m^{d-2} \left( Q - \frac{1}{6} R \right) + O(R^3) \right\} \] (3.4)
and in even dimension
\[ \Gamma_{(1)\text{loc}} = \frac{1}{2} (4\pi)^{-d/2} \frac{(-1)^{d/2}}{\Gamma(\frac{d}{2} + 1)} \int dx g^{1/2} \text{str} \left\{ m^d \left[ \ln \frac{m^2}{\mu^2} - \Psi \left( \frac{d}{2} \right) - C \right] + \frac{d}{2} m^{d-2} \left[ \ln \frac{m^2}{\mu^2} - \Psi \left( \frac{d}{2} \right) - C \right] \left( Q - \frac{1}{6} R \right) + O(R^3) \right\} \] (3.5)
The nonlocal part of the effective action has the form
\[ \Gamma_{(1)\text{nonloc}} = \frac{1}{2} (4\pi)^{-d/2} \int dx g^{1/2} \text{str} \left\{ Q^2 \beta^{(1)}(\Box) Q + 2 R_{\alpha\mu} \nabla^\alpha \frac{1}{\Box} \beta^{(2)}(\Box) \nabla^\nu \nabla^\rho \right\} + \frac{d}{2} \frac{m^{d-2}}{m^2} \left[ \ln \frac{m^2}{\mu^2} - \Psi \left( \frac{d}{2} \right) - C \right] \left( Q - \frac{1}{6} R \right) + O(R^3) \right\} \] (3.6)
Here \( \gamma^{(i)}(t\Box) \) and \( \beta^{(i)}(\Box) \) are formfactors, which are obtained explicitly in general case in [2,3,5,6]. In papers [48,49] a similar expressions for massless case \( m = 0 \) was obtained using completely different method, viz. the covariant perturbation theory proposed in these papers. Next third order in curvatures is under investigation in [15,22,51].

3.2. Low-energy approximation

Let us now discuss the opposite approximation, i.e. the long-wave (or low-energy) one. It can be characterized by an opposite relation
\[ \nabla \nabla R \ll RR \] (3.7)
This means that the derivatives of all invariants of background fields are much smaller than the products of the invariants themselves.

In this case the trace of the heat kernel and the effective action should have expansions of the form
\[ \text{Str} U(t) = \int dx g^{1/2} (4\pi t)^{-d/2} \exp (-tm^2) \text{str} \left\{ \Phi(tR) + \nabla R \Psi(tR) \nabla R + O(\nabla^3) \right\} \]
\[ \Gamma_{(1)} = \int dx g^{1/2} \text{str} \left\{ V(R) + \nabla Z(R) \nabla R + O(\nabla^3) \right\} \] (3.8)
where \( \Phi(tR), \Psi(tR), V(R) \) and \( Z(R) \) are local functions and symbol \( O(\nabla^3) \) means the terms with more than three derivatives.
It is naturally to call the first term in the formula for the effective action (3.8) (the zero order), which does not contain the covariant derivatives at all, the effective potential of quantum gravity

$$\Gamma_{(1)}|_{\nabla R=0} = \int dx \, g^{1/2} \text{str} \, V(R) \quad (3.9)$$

Let us mention that such a definition of the effective potential and the expansion in covariant derivatives is not conventional. It differs from the definition that is often found in the literature [19,44]. What is meant usually under the notion of the effective potential is a function of the potential term only $Q$, because it does not contain derivatives of the background fields and transforms by itself covariantly (in contrast to curvatures that contain first $R \sim \partial A$ and second $R_{\mu\nu\alpha\beta} \sim \partial \partial g$ derivatives of the background fields). So, e.g. in [19,44] the potential term $Q$ is summed up exactly but an expansion is made not only in covariant derivatives but also in powers of curvatures $R_{\mu\nu\alpha\beta}$ and $R_{\mu\nu}$, i.e. the curvatures are treated perturbatively, since they contain the derivatives of the fields. Thereby the validity of this approximation for the effective action is limited to small curvatures

$$R_{\mu\nu\alpha\beta}, R_{\mu\nu} \ll Q \quad (3.10)$$

Such an expansion is called ‘expansion of the effective action in covariant derivatives’. Without the potential term $Q = 0$ the effective potential in such a scheme is trivial. Hence we stress here once again, that in our definition the effective potential contains, in fact, much more information than the usual effective potential what is often meant in the literature, when using the ‘expansion in covariant derivatives’. As a matter of fact, what we mean is the low-energy limit of the effective action formulated in a gauge invariant way.

It is more difficult to formulate the low-energy approximation in an invariant way. The point is that in short-wave approximation (3.1), i.e. by expanding in curvatures, the zero approximation $R = 0$ corresponds simply to the flat empty space, i.e. the standard vacuum of the field theory. Therefore, the covariant perturbation theory [48,49] is, in fact, only the covariant reformulation of the usual noncovariant perturbation theory. In this case the effective action in a given order can be built, in principle, from several first usual perturbative $n$-point Green functions. This is reflected by the fact that after arranging the covariant derivatives in first, second etc. orders in some manner one can fix uniquely the structure of the whole series (for details see [13,5,6]).

Nothing of the kind is the relation (3.7), i.e. the condition of weak dependence of the invariant characteristics of the background fields and the manifold itself on the coordinates. There is a lack of uniqueness in the structure of the series for the low-energy effective action. The disposition of the covariant derivatives in higher orders has an influence on the form of the zero approximation. This is related to the fact, that some combinations of the curvatures (viz. those, which give the commutators of covariant derivatives) can be treated as terms with two covariant derivatives. Therefore one has to come to an understanding about the disposition of the covariant derivatives and to fix the structure of the series.

One can say that the evaluation of the effective potential (3.9) corresponds in usual perturbative field theory to summing up all the diagrams at zero momentums, the calculation of the next term in (3.8) (first order) - to summing up second derivatives of all
diagrams with respect to momentums at zero momentums etc. In other words, the calculation of the effective potential corresponds to the calculation of the \( \infty \)-point Green function in field theory.

The problem is now that the zero approximation is no longer the empty flat space. The manifold must be strongly curved even in zero approximation. Hence the various topologies of the background manifold are possible and one has, in general, to analyze the influence of the topology.

Thus we realize that one should single out in the curvatures the covariantly constant background \( \tilde{R} \) and the rest part \( P \) treat as a short-wave perturbation (ripple)

\[
R = \tilde{R} + P
\]
\[
\tilde{\nabla} \tilde{R} = 0 \quad , \quad PP \ll \tilde{\nabla} \tilde{\nabla} P \ll \tilde{R} \tilde{R}
\] (3.11)

Apparently, one can extend the covariant perturbation theory developed in [48,49], to this case, i.e. when the curvature in zero approximation does not vanish and is covariantly constant. We shall not do this in present paper, intending to calculate only the zero order, i.e. the effective potential. So we will not be concerned with this problem.

In more detail, the zero approximation corresponds to covariantly constant curvatures

\[
\nabla_\mu R_{\alpha\beta\gamma\delta} = 0 \quad , \quad \nabla_\mu \mathcal{R}_{\alpha\beta} = 0 \quad , \quad \nabla_\mu Q = 0
\] (3.12)

The conditions of integrability of these relations lead to strong algebraic restrictions on the curvatures themselves

\[
R_{\mu\nu\lambda\gamma}[\alpha R_{\beta\delta}] + R_{\mu\nu\lambda}[\gamma R_{\delta\alpha}] = 0
\] (3.13)
\[
R_{\mu\nu\lambda}[\alpha \mathcal{R}_{\beta}] + R_{\alpha\beta\lambda}[\mu \mathcal{R}_{\nu}] = 0
\] (3.13a)
\[
[\mathcal{R}_{\mu\nu}, \mathcal{R}_{\alpha\beta}] + R_{\mu\nu\lambda}[\alpha \mathcal{R}_{\beta}] - R_{\alpha\beta\lambda}[\mu \mathcal{R}_{\nu}] = 0
\] (3.14)
\[
[\mathcal{R}_{\mu\nu}, Q] = 0
\] (3.15)

Mention that the relation (3.13) is local. It determines locally the geometry of the symmetric spaces [61]. However, the manifold is globally symmetric one only in the case when it satisfies additionally some global topological restrictions and the condition (3.13) is valid everywhere, i.e. in any point of the manifold [61].

But in our case, i.e. in physical problems, the situation is radically different. The correct setting of the problem is as follows. Consider an asymptotically flat space that is homeomorphic to \( \mathbb{R}^d \) (as we already stressed above, we do not consider the topological effects, meaning to investigate, in general, the influence of the curvature). Let a finite not small, in general, domain of the manifold exists that is strongly curved and quasi-homogeneous, i.e. the invariants of the curvature in this region vary very slowly. Then the geometry of this region is locally very similar to that of a symmetric space. However one should have in mind that there are always regions in the manifold where this condition is not fulfilled. This is, first of all, the asymptotic Euclidean region that has small curvature and, therefore, the opposite short-wave approximation is valid.
Thus the general situation in correct setting of the problem is the following. From infinity with small curvature and possible radiation where \([15,22,48,49]\)

\[
RR \sim |x|^{-2d} \ll \nabla \nabla R \sim |x|^{-d-2}
\]  

we pass on to quasihomogeneous region where the local properties of the manifold are close to those of symmetric spaces. The size of this region can tend to zero. Then the curvature is nowhere large and the short-wave approximation is valid anywhere.

If one tries to extend the limits of such region to infinity, then one has also to analyze the topological properties. The space can be compact or noncompact depending on the sign of the curvature. It can be also a combination of these cases, i.e. a product of a compact and noncompact spaces. But first we will come across a coordinate horizon-like singularity, although no one true physical singularity really exist.

This construction can be intuitively imagined as follows. Take the flat Euclidean space \(\mathbb{R}^d\), cut out from it a region \(M\) with some boundary and stick to it along the boundary instead of the piece cut out a piece of a curved symmetric space with the same boundary \(\partial M\). Such a construction will be homeomorphic to the initial space and at the same time will contain a finite highly curved homogeneous region. By the way, if one subtracts from the exact effective action for a symmetric space the effective action for built construction then one gets purely topological contribution to the effective action. This fact seems to be useful when analyzing the effects of topology.

Thus the problem is to calculate the effective action for covariantly constant background, i.e. the low-energy effective action. Although this quantity, generally speaking, depends essentially on the topology and other global aspects of the manifold, one can disengage oneself from these effects fixing the trivial topology. Since the asymptotic Schwinger-De Witt expansion does not depend on the topology, one can hold that we thereby sum up all the terms without covariant derivatives in it. In the next section we adduce several approaches for calculating the effective action.

Let us mention also some ideas and formulae that we will need further for calculation of the effective potential.

One can use the conditions of covariant constancy of the curvatures (3.12) to express the metric and the connection directly in terms of curvatures, i.e. to integrate explicitly these equations. This can be done in covariant way by using instead of local quantities like the metric and the connection, some auxiliary two-point geometric objects - world function (or geodetic interval) \(\sigma(x, x')\) and parallel displacement operator \(\mathcal{P}(x, x')\).

One can show that the geodetic interval characterizes the geometry of the manifold in full, i.e. it gives an alternative description of a curved space \([53,62,9]\) and the parallel displacement operator describes completely the connection.

As it is shown in [1-6] one can organize the whole calculating process in such a way that all quantities will be expressed in terms of derivatives of the following two-point functions

\[
\sigma^\mu_\nu = \nabla_\nu \nabla^\mu_\sigma
\]

\[
X^\mu_\nu = \sigma^\mu_\alpha \sigma^\nu_\alpha
\]

\[
\mathcal{B}_\mu^\prime = \mathcal{P}^{-1} \gamma_\mu^\nu \nabla_\nu \mathcal{P}
\]  

(3.17) 

(3.17a) 

(3.18)
where
\[ \gamma^\nu_{\mu'} = (\sigma^\nu_{\mu'})^{-1} \]  

These functions in the case of covariantly constant curvatures (3.12) can be expressed explicitly in terms of curvatures in a fixed point \( x' \).

Introducing a matrix \( K = \{ K^\mu'_{\nu'} \} \) that is a scalar at the point \( x \),
\[ K^\mu'_{\nu'}(x, x') = R^\mu'_{\alpha'\nu'\beta'}(x') \sigma^{\alpha'}(x, x') \sigma^{\beta'}(x, x') \]  

where \( \sigma^{\mu'} = \nabla^{\mu'} \sigma \) is a tangent vector to the geodesic connecting the points \( x \) and \( x' \) at the point \( x' \), one can write down the quantities (3.17) - (3.19) as follows [5,6]

\[ \gamma^\nu_{\mu'} = -g^\nu_{\alpha'} \left( \frac{\sin \sqrt{K}}{\sqrt{K}} \right)^{\alpha'}_{\mu'} \]  

\[ X^\mu'_{\nu'} = \left( \frac{K}{\sin^2 \sqrt{K}} \right)^{\mu'}_{\nu'} g^{\alpha'\nu'} \]  

\[ B^\mu' = -\left( \frac{1 - \cos \sqrt{K}}{K} \right)^{\nu'}_{\mu'} R^{\nu'\alpha'} \sigma^{\alpha'} \]  

where \( g^{\nu}_{\alpha'}(x, x') \) is a bivector which effects parallel displacement of vector fields along the geodesic from the point \( x' \) to the point \( x \).

The Van Vleck - Morette determinant equals in this case [5,6]
\[ \Delta = \det \left( \frac{\sqrt{K}}{\sin \sqrt{K}} \right) \]  

Mention that if one chooses the normal Riemann coordinates and Fock - Schwinger gauge at the point \( x' \) for the connection
\[ g^N_{\mu\nu}(x'; x') = \delta_{\mu\nu}, \quad (x^\mu - x'^\mu)(g^N_{\mu\nu}(x; x') - \delta_{\mu\nu}) = 0 \]  

\[ A^{FS}_\mu(x'; x') = 0, \quad (x^\mu - x'^\mu)A^{FS}_\mu(x; x') = 0 \]

then
\[ \sigma^{\mu'} = -(x^\mu - x'^\mu) \]  
\[ g^\mu'_{\nu'} = \delta^\mu'_{\nu'}, \quad P = 1 \]  
\[ K^\mu_{\nu}(x, x') = R^\mu_{\alpha'\nu'\beta'}(x')(x^\alpha - x'^\alpha)(x^\beta - x'^\beta) \]

and one can show that the metric and the connection can be expressed in terms of introduced quantities
\[ g^N_{\mu\nu} = \gamma^\alpha_{\mu'} \gamma^\alpha_{\nu'} = \left( \frac{\sin^2 \sqrt{K}}{K} \right)^{\nu'}_{\mu'} \]  

13
written in a closed form

This expansion can be summed up and the Killing vectors of symmetric spaces can be obtained easily

Indeed, after differentiating the Killing equations (3.31) and commuting derivatives it is obtained easily

By differentiating this relation and symmetrizing the derivatives we get

Thereby we have found the coefficients of the covariant Taylor series [21,5,6]

This expansion can be summed up and the Killing vectors of symmetric spaces can be written in a closed form

Therefore, all Killing vectors at any point \( x \) are determined in terms of initial values of the vectors themselves \( \xi^\lambda(x') \) and their first derivatives \( \nabla_{\rho'} \xi^\lambda(x') \) at a fixed point \( x' \). The maximal number of these parameters is equal to \( d + d(d-1)/2 = d(d+1)/2 \), since the quantities \( \nabla_{\rho'} \xi^\lambda \) are antisymmetric. The spaces with maximal number of independent Killing vectors equal to \( d(d+1)/2 \) are the spaces of constant curvature and only those.

However, while the initial values of Killing vectors are independent, the derivatives of those are not. This can be seen immediately if one mentions that in symmetric spaces they should satisfy also the equation

We stress once again that the covariantly constant curvatures must satisfy also the conditions of integrability (3.13) and (3.14).

Let us make one more important observation. On the covariantly constant background (3.12), i.e. in symmetric spaces, one can easily solve the Killing equations

\[
\mathcal{L}_\xi g_{\mu\nu} = 2\nabla_{(\mu} \xi_{\nu)} = 0
\]

Indeed, after differentiating the Killing equations (3.31) and commuting derivatives it is obtained easily

By differentiating this relation and symmetrizing the derivatives we get

Thereby we have found the coefficients of the covariant Taylor series [21,5,6]

This expansion can be summed up and the Killing vectors of symmetric spaces can be written in a closed form

Therefore, all Killing vectors at any point \( x \) are determined in terms of initial values of the vectors themselves \( \xi^\lambda(x') \) and their first derivatives \( \nabla_{\rho'} \xi^\lambda(x') \) at a fixed point \( x' \). The maximal number of these parameters is equal to \( d + d(d-1)/2 = d(d+1)/2 \), since the quantities \( \nabla_{\rho'} \xi^\lambda \) are antisymmetric. The spaces with maximal number of independent Killing vectors equal to \( d(d+1)/2 \) are the spaces of constant curvature and only those.

However, while the initial values of Killing vectors are independent, the derivatives of those are not. This can be seen immediately if one mentions that in symmetric spaces they should satisfy also the equation

\[
\mathcal{L}_\xi R_{\alpha\beta\gamma\delta} = 2\{R_{\alpha\beta\gamma\delta} \nabla_{\gamma} \xi^\sigma + R_{\gamma\delta\sigma\beta} \nabla_{\alpha} \xi^\sigma \} = 0
\]
This equation plays the role of the integrability conditions for Killing equations (3.31) and imposes strict constraints on the possible initial values of the derivatives of the Killing vectors at the point \( x' \).

All the Killing vectors can be splitted in two essentially different sets \( \{P_a\} \) and \( \{M_i\} \) according to the values of their initial parameters

\[
P^\mu_a(x) = g^\mu_{\nu'} (\cos \sqrt{K})^\nu'_{\lambda'} P^\lambda'_a(x'), \quad a = 1, \ldots, d; \tag{3.47}
\]

\[
M^\mu_i(x) = -g^\mu_{\nu'} (\sin \sqrt{K})^\nu'_{\lambda'} \sigma^\nu' (\nabla_{\rho'} M^\lambda'_i)(x'), \quad i = 1, \ldots, p; \quad p \leq \frac{d(d-1)}{2} \tag{3.48}
\]

with following initial conditions

\[
\nabla_{\rho'} P^\lambda'_a(x') = 0, \quad M^\mu'_i(x') = 0 \tag{3.49}
\]

and the consistency ones

\[
\det P^\lambda'_a \bigg|_{x=x'} \neq 0 \tag{3.50}
\]

\[
\left\{ R_{\alpha' \beta' \sigma'[\delta'} \nabla_{\gamma']} M^\sigma'_{i} + R_{\gamma' \delta' \sigma'[\beta'} \nabla_{\alpha']} M^\sigma'_{i} \right\} \bigg|_{x=x'} = 0 \tag{3.51}
\]

Using the Killing vectors of symmetric spaces (3.47), (3.48) we define first-order differential operators of the form

\[
P_a = P^\mu_a \nabla_{\mu} = P^\nu'_a (\cos \sqrt{K})^{\mu'}_{\nu'} D^\mu' \tag{3.52}
\]

\[
M_i = M^\mu'_i \nabla_{\mu} = (\nabla_{\rho'} M^\mu'_i) \sigma^\nu' D^\mu' \tag{3.53}
\]

where

\[
D^\mu' = \gamma^\nu'_{\mu} \nabla_{\nu} \tag{3.54}
\]

One can show that for all the constraints to be fullfilled in symmetric spaces, including (3.13) and (3.50), the operators (3.52), (3.53) on the scalar (at the point \( x \)) fields should generate a Lie algebra (the algebra of isometries)

\[
[P_a, P_b] = E^i_{ab} M_i \tag{3.55}
\]

\[
[P_a, M_i] = D^\beta_{ai} P_b \tag{3.56}
\]

\[
[M_i, M_k] = C^l_{ik} M_l \tag{3.57}
\]

with structure constants satisfying the Jacobi identities

\[
D^a_{\beta[j} E^j_{cd]} = 0 \tag{3.58}
\]

\[
D^a_{ik} D^c_{kb} - D^a_{kc} D^c_{ib} = C^j_{ik} D^a_{jb} \tag{3.59}
\]

\[
E^i_{ac} D^c_{bk} - E^i_{bc} D^c_{ak} = E^j_{ab} C^i_{jk} \tag{3.60}
\]

\[
C^i_{j[k} C^j_{mn]} = 0 \tag{3.61}
\]
Mention that the generators $M_i$ also form a closed Lie algebra called the isotropy subalgebra $H$.

In this case using (3.50) and (3.51) one can obtain easily the initial values of the nonvanishing derivatives of Killing vectors $M$ from (3.56)

$$\nabla_{\mu'} M_{\nu'}^i = D^b a_i P^{-1a}_{-\mu} P_{-b}^\nu$$

(3.62)

Then commuting the eq. (3.55) with $P_c$ and using (3.56) we obtain

$$[P_c, [P_a, P_b]] = D^d c_i E_{ab}^d P_d$$

(3.63)

wherefrom one obtains the curvature at the point $x'$ in terms of the structure constants

$$R_{\nu' \alpha' \beta'}^\mu = D^d c_i E_{ab}^d P_{-1c}^\nu P_{-1a}^\nu P_{-1b}^\nu$$

(3.64)

Taking into account (3.62) and (3.64) one can prove now that, as a consequence of the Jacobi identities (3.58) - (3.61), the conditions (3.13) and (3.50) are indeed valid. This means that giving the structure constants of a Lie group is equivalent to Riemann curvature tensor. The structure of the isometry group is completely determined by the curvature tensor at a fixed point $x'$. To be more precise the symmetric space is isomorphic to the quotient space $G/H$ by the isotropy subgroup $H$ [61].

In more general case when the operators (3.52), (3.53) act on a multiplet of fields $\varphi = \{\varphi^A\}$ additional curvature terms arise in commutation relations (3.55) - (3.57)

$$[P_a, P_b] = E^i_{ab} M_i + F_{ab}$$

(3.65)

$$[P_a, M_i] = D^b_{ai} P_b + G_{ai}$$

(3.66)

$$[M_i, M_k] = C^j_{ik} M_j + H_{ik}$$

(3.67)

where the following notations for the projections of the curvature of background connection (2.5) on the Killing vectors are introduced

$$F_{ab} = P^\mu_a P^\nu_b R_{\mu \nu}$$

(3.68)

$$G_{ai} = P^\mu_a M_i^\nu R_{\mu \nu}$$

(3.69)

$$H_{ik} = M_i^\mu M_k^\nu R_{\mu \nu}$$

(3.70)

Generally speaking, one should add also the commutation relations between operators $P_a, M_i$ and the curvatures $F_{ab}, G_{ai}, H_{ik}$ and between the curvatures themselves which are the direct consequence of the conditions (3.12) - (3.14).

Hence we conclude that the formulation of the low-energy approximation in explicitly covariant way naturally leads to the condition of covariant constancy of curvatures (3.12) and thereby to the existence of some nontrivial invariance group with generators (3.52) and (3.53), the algebra of this group given by the commutation relations (3.55)-(3.57).

This structure seems to be very useful for calculating the effective potential of quantum gravity. A simple example how one can make use of this group is presented in subsect. 4.1. We are going to investigate this point in full measure in the future.
IY. APPROACHES FOR CALCULATING THE HEAT KERNEL

The problem of calculating the low-energy heat kernel and the low-energy effective action is elaborated not so good. In contrast to good developed Schwinger - De Witt technique [9-13,21,5,6], and also to nonlocal covariant perturbation theory [48,49], here only partial success is achieved and various approaches to the problem are only outlined (see our recent papers [67, 68]). The general solution of the problem of explicitly covariant calculation of the effective action is still not found. That is why we adduce below a number of approaches that are worth notice and can lead in future to a considerable progress.

Thus the problem is the following. One has to obtain a local covariant expression that would describe adequately the low-energy limit of the trace of the heat kernel and that would, being expanded in curvatures, reproduce all terms without covariant derivatives in the asymptotic expansion of heat kernel, i.e. the HMDS-coefficients. In other words, this expression can be called the generating function for HMDS-coefficients. If one finds such an expression, then one can simply determine the ζ-function (2.16) and, therefore, the low-energy limit of the effective action (2.14).

4.1. Schwinger’s method

This is one of the oldest approaches. Nevertheless it turned out to be very successful in the case of an Abelian gauge field and allowed Schwinger to obtain in a very elegant way the effective lagrangian of quantum electrodynamics [8]. The formulation of the method in case of quantum gravity is due to De Witt [9-12].

Take an abstract formal Hilbert space with the basis vectors |x'> that are eigenvectors of commutative set of Hermitian coordinate operators

\[ [x^\mu, x^\nu] = 0 \quad , \quad x^\mu |x'> = x'^\mu |x'> \] (4.1)

and are normalized according to

\[ <x''|x'> = \delta(x'', x') \] (4.2)

The vectors |x'> transform as scalar densities of weight 1/2 under diffeomorphisms and as fields \( \varphi = \{ \varphi^A \} \) under gauge transformations.

Then one introduces Hermitian noncommuting covariant momentum operators

\[ [\Pi_\mu, \Pi_\nu] = -\mathcal{R}_{\mu\nu} \] (4.3)

where \( \mathcal{R}_{\mu\nu}(x) \) is an anti-Hermitian function of the coordinate operators. The commutation relations of these operators have the form

\[ [x^\mu, \Pi_\nu] = i\delta^\mu_\nu \] (4.4)

and the matrix elements of the momentum operators are

\[ <x''|\Pi_\mu|x'> = -i\nabla^\mu_\nu \delta(x'', x') \] (4.5)
The self-adjoint operator (2.8)

\[ H(x, \Pi) = g^{-1/4} F g^{-1/4} = g^{-1/4} \Pi_{\mu} g^{1/2} g^{\mu\nu} \Pi_{\nu} g^{-1/4} + Q + m^2 \]  

(4.6)
is treated naturally as the Hamiltonian of an abstract dynamical system.

Make an analytic continuation in the complex plane of the variable \( t \) on the positive semi-axis of the imaginary axis, i.e substitute \( t = is \), where \( s > 0 \), and consider instead of the heat kernel \( \exp(-tF) \) the unitary evolution operator \( \exp(-isH(x, \Pi)) \). The matrix elements of the evolution operator determine the evolution kernel (or Schrödinger kernel)

\[ U(s|x'', x') = \langle x''| \exp(-isH(x, \Pi))|x' \rangle \]  

(4.7)

that satisfies the Schrödinger equation

\[ i \frac{\partial}{\partial s} U(s|x'', x') = \langle x''(s)|H(x(s), \Pi(s))|x' \rangle \]  

(4.8)

where

\[ x(s) = \exp(isH)x \exp(-isH) \quad , \quad \Pi(s) = \exp(isH)\Pi \exp(-isH) \]  

(4.9)

and

\[ \langle x''(s)| = \langle x| \exp(-isH) \]  

(4.10a)
is the eigenvector of the operator \( x(s) \)

\[ \langle x''(s)|x(s) = x'' < x''(s)| \]  

(4.10)

It is obvious, that \( H(x(s), \Pi(s)) = H(x, \Pi) \). The corresponding operator dynamical equations (Heisenberg ones) have the form

\[ \frac{d x^\mu(s)}{d s} = i[H, x^\mu(s)] \]  

(4.11)

\[ \frac{d \Pi_{\mu}(s)}{d s} = i[H, \Pi_{\mu}(s)] \]  

(4.12)

Using the commutation relations (4.3) and (4.4) it is not difficult to calculate the commutators in the right-hand side of these equations. If one manages to integrate these operator equations with initial conditions \( x(0) = x, \Pi(0) = \Pi \) in the form

\[ x(s) = \exp(is Ad_H) x = f_1(s|x, \Pi) \]  

(4.13)

\[ \Pi(s) = \exp(is Ad_H) \Pi = f_2(s|x, \Pi) \]  

(4.14)

where

\[ Ad_H f = [H, f] \]  

(4.15)

then one can express the initial momentums in terms of coordinate operators

\[ \Pi = f_3(s|x(s), x) \]  

(4.16)
put them into the expression for the Hamiltonian (4.6) and order them using the commutation relations (4.3), (4.4) so that all \( x(s) \) would be placed to the left and all \( x \) to the right. (It is evident that the coordinate operators at different time \( s \) do not commute.)

As a result the Hamilton operator assumes an air

\[
H(x, \Pi) = Th(s|x(s), x)
\]  

(4.17)

where the symbol \( T \) means the ordering from the right to the left with respect to the variable \( s \). Finally the Schrödinger equation (4.8) takes the form

\[
i \frac{\partial}{\partial s} U(s|x'', x') = h(s|x'', x')U(s|x'', x')
\]  

(4.18)

that can be integrated easily

\[
U(s|x'', x') = C(x'', x') \exp \left(-is \int h(s|x'', x')\right)
\]  

(4.19)

The function \( C(x'', x') \) should be determined then from the initial condition

\[
U(0|x'', x') = \langle x''|x' >= \delta(x'', x')
\]  

(4.20)

and transformation properties of Schrödinger kernel, e.g. from the relations of the form (4.5). At last one obtains the heat kernel from Schrödinger kernel by doing the inverse analytic continuation \( s = -it \).

Consider now the low-energy approximation (3.7). It is sufficient to confine oneself to covariantly constant curvatures of the background fields (3.12). The most convenient way to apply the Schwinger’s method is to use the Fock - Schwinger gauge and normal Riemann coordinates at the point \( x' \) (3.23), (3.24). Then one can express the metric and connection directly in terms of curvatures at the point \( x' \) and use the formulae (3.27)-(3.30).

The covariant momentum operator \( \Pi \) takes the form

\[
\Pi_\mu = p_\mu - iA_\mu
\]  

(4.21)

where \( p_\mu \) are the ordinary Hermitian commuting momentum operators

\[
[p_\mu, p_\nu] = 0 \quad , \quad [x^\mu, p_\nu] = i\delta_\nu^\mu
\]  

(4.22)

All commutators at coinciding points \( x'' = x' \) (in particular, the function \( h(s|x', x') \) and, therefore, the Schrödinger kernel at coinciding points \( U(s|x', x') = \langle x'(s)|x' > \) ) are expressed in covariant way only in terms of curvatures.

However, one manages to solve the operator Heisenberg equations (4.11), (4.12) only in exceptional cases. Let us confine ourselves to the covariantly constant gauge field in flat space.
In this case we have
\[ R_{\mu\nu\alpha\beta} = 0 \]
\[ [\Pi_\alpha, g_{\mu\nu}] = [\Pi_\alpha, Q] = [\Pi_\alpha, R_{\mu\nu}] = [R_{\mu\nu}, R_{\alpha\beta}] = [Q, R_{\mu\nu}] = 0 \] (4.23)
and Heisenberg equations take the form
\[ \frac{dx^\mu}{ds} = -2g^{\mu\nu}\Pi_\nu \] (4.24)
\[ \frac{d\Pi_\mu}{ds} = -2iR_{\mu\alpha}g^{\alpha\nu}\Pi_\nu \] (4.25)
These equations can be easily solved
\[ \Pi(s) = g \exp(2isg^{-1}R)g^{-1}\Pi , \quad x(s) = x + \frac{\exp(2isg^{-1}R) - 1}{ig^{-1}R}g^{-1}\Pi \] (4.26)
where \( g = \{g_{\mu\nu}\}, R = \{R_{\mu\nu}\} \). Hence
\[ \Pi = \frac{iR}{\exp(2isg^{-1}R) - 1}(x(s) - x) \] (4.27)
and then
\[ H = \Pi g^{-1}\Pi + Q + m^2 = \frac{1}{4}(x(s) - x) \left( \frac{Rg^{-1}R}{\sin^2(sg^{-1}R)} \right) (x(s) - x) + Q + m^2 \] (4.28)
Finally, using the commutator
\[ [x(s), x] = -\frac{\exp(2isg^{-1}R) - 1}{g^{-1}R}g^{-1} \] (4.29)
for ordering the operators \( x(s) \) and \( x \) we get
\[ h(s|x'', x') = \langle x''(s)|H|x' \rangle = \frac{1}{4}(x'' - x') \left( \frac{Rg^{-1}R}{\sin^2(sg^{-1}R)} \right) (x'' - x') - \frac{i}{2}Sp(g^{-1}R \cot(sg^{-1}R)) + Q + m^2 \] (4.30)
where \( Sp \) denotes the trace over vector indices. The solution of the Schrödinger equation with initial condition (4.20) take now the form
\[ U(t|x'', x') = (4\pi t)^{-d/2}g^{1/2}\mathcal{P}(x'', x') \det \left( \frac{tg^{-1}R}{\sinh(tg^{-1}R)} \right)^{1/2} \]
\[ \times \exp \left( -t(m^2 + Q) - \frac{1}{4}(x'' - x')^\mu(\mathcal{R} \coth(tg^{-1}R))_{\mu\nu}(x'' - x')^\nu \right) \] (4.31)
where we came back to the variable $t = is$ for convenience of further references, det means the determinant over vector indices and $\mathcal{P}(x, x')$ is the parallel displacement operator. This factor is necessary to ensure the proper covariant transformation properties of the Schrödinger kernel under gauge transformations.

This solution for the Abelian gauge group $U(1)$ (quantum electrodynamics) was obtained first by Schwinger [8]. However, as we see, it is valid in the much more general case of arbitrary semisimple gauge group and covariantly constant background fields in flat space

$$R_{\mu\nu\alpha\beta} = 0$$
$$\nabla_\mu R_{\alpha\beta} = \nabla_\mu Q = 0 \quad (4.32)$$

The consistency conditions (3.14), (3.15) for these equations take in this case the form

$$[R_{\mu\nu}, R_{\alpha\beta}] = [R_{\mu\nu}, Q] = 0 \quad (4.33)$$

This means that the curvature and the potential term take their values only in the Cartan subalgebra of the algebra of the gauge group, the number of independent components being equal to dimension of the Cartan subalgebra, i.e. the rank of the algebra. Hence this is a nontrivial extension of the Schwinger result to the non-Abelian gauge group.

We shall apply this result below to the calculation of the effective potential in a Yang-Mills model with matter fields.

Notwithstanding all the elegance the Schwinger’s method is not widely adapted for the present in general case of nontrivial curved background because of the extreme complexity of the nonlinear operator equations (4.11) and (4.12).

### 4.2. Fourier integral method

This method is based on the formal representation of the heat kernel in the form (2.7)

$$U(t|x, x') = \exp(-tH)\mathcal{P}(x, x')\delta(x, x') \quad (4.34)$$

Using the representation of the $\delta$-function in form of a covariant Fourier integral [5,6]

$$\delta(x, x') = g^{1/4}(x)g^{-1/4}(x')\Delta^{1/2}(x, x') \int \frac{dk_{\mu'}}{(2\pi)^d} \exp(ik_{\mu'}\sigma_{\mu'}(x, x')) \quad (4.35)$$

we get

$$U(t|x, x') = g^{1/4}(x)g^{-1/4}(x')\Delta^{1/2}(x, x')\mathcal{P}(x, x') \times \exp(-tm^2) \int \frac{dk_{\mu'}}{(2\pi)^d} \exp(ik_{\mu'}\sigma_{\mu'}(x, x'))\Phi(t|k, x, x') \quad (4.36)$$

where

$$\Phi(t|k, x, x') = \exp(-tA) \cdot 1 \quad (4.37)$$
$$A = -\exp(-ik_{\mu'}\sigma_{\mu'})\mathcal{P}^{-1}\Box^{1/2}\mathcal{P}\exp(ik_{\mu'}\sigma_{\mu'}) + \mathcal{P}^{-1}Q\mathcal{P} \quad (4.38)$$
Introduce now [5,6] new operators of covariant derivatives as follows

\[ D_{\mu'} = \gamma^{\nu}_{\mu'} \nabla_{\nu} \]  

(4.39)

These operators appear to be very convenient since: i) they commute being applied to scalars at the point \( x \) and ii) the commutators between them and the tangent vectors \( \sigma^{\mu'} \) have the usual very simple form

\[ [D_{\mu'}, \sigma^{\nu'}] = \delta^{\nu'}_{\mu'} \]  

(4.40)

Subsequently the operators \( D_{\mu'} \) and the vectors \( \sigma^{\nu'} \) play the role of usual derivatives and coordinates. It is this fact that makes it possible to construct the covariant Fourier integral in the form (4.35).

Using the representation of the Laplace operator in terms of \( D_{\mu'} \), \[5,6\]

\[ \Box = \Delta D_{\mu'} \Delta^{-1} X^{\mu' \nu'} D_{\nu'} \]  

(4.41)

where \( X^{\mu' \nu'} \) is given by (3.17a) and recalling the definition of the quantity \( B_{\mu'} \) (3.18) yields

\[
A = -\Delta^{1/2} (D_{\mu'} + B_{\mu'} + i k_{\mu'}) \Delta^{-1} X^{\mu' \nu'} (D_{\nu'} + B_{\nu'} + i k_{\nu'}) \Delta^{1/2} + \mathcal{P}^{-1} Q \mathcal{P} \\
= -\Delta^{1/2} (D_{\mu'} + B_{\mu'}) \Delta^{-1} X^{\mu' \nu'} (D_{\nu'} + B_{\nu'}) \Delta^{1/2} + \mathcal{P}^{-1} Q \mathcal{P} \\
- i k_{\mu'} \left( X^{\mu' \nu'} D_{\nu'} + D_{\nu'} X^{\mu' \nu'} + 2 X^{\mu' \nu'} B_{\nu'} \right) + k_{\mu'} k_{\nu'} X^{\mu' \nu'}
\]  

(4.42)

Thus the function \( \Phi \) is presented as the result of the action of the exponent of the differential operator \( A \) on the identity. Moving the derivatives to the right until they act on the identity and give zero the final result will be expressed in terms of derivatives of the two-point functions involved (\( X^{\mu' \nu'}, \Delta, B_{\mu'} \) and \( \mathcal{P}^{-1} Q \mathcal{P} \)). All these functions are scalars at the point \( x \) and therefore the operators \( D_{\mu'} \) commute when applied to them.

These expressions give the heat kernel in general case of arbitrary background fields. In low-energy approximation one can make use of the closed expressions for the functions \( X^{\mu' \nu'}, \Delta \) and \( B_{\mu'} \) in terms of curvatures (3.21)-(3.23) and use also the fact that the potential term commutes with derivatives

\[ [D_{\mu'}, \mathcal{P}^{-1} Q \mathcal{P}] = 0 \]  

(4.43)

As a consequence the coincidence limit (the diagonal) of the heat kernel

\[ U(t|x,x) = \exp(-tm^2) \int \frac{dk}{(2\pi)^d} \Phi(t|k, x, x) \]  

(4.44)

will depend on \( x \) only through the trivial factor \( g^{1/2}(x) \) and will be expressed manifestly covariant only in terms of curvatures.
Changing the integration variable $k \to k/\sqrt{t}$ one can rewrite the heat kernel in coinciding points as

$$U(t|x, x) = (4\pi t)^{-d/2} \exp(-t(m^2 + Q))g^{1/2}(x)\Omega(t|x, x) \quad (4.45)$$

where

$$\Omega(t|x, x) = \lim_{x \to x'} \int \frac{dk}{\pi^{d/2} g} e^{-1/2} \exp \left\{ -k\mu' k\nu' X^{\mu'\nu'} + i\sqrt{t}k\mu' \left( X^{\mu'\nu'} \mathcal{D}_{\nu'} + \mathcal{D}_{\nu'} X^{\mu'\nu'} + 2X^{\mu'\nu'} B_{\nu'} \right) \right.$$

$$+ t\Delta^{1/2} (\mathcal{D}_{\mu'} + B_{\mu'}) \Delta^{-1} X^{\mu'\nu'} (\mathcal{D}_{\nu'} + B_{\nu'}) \Delta^{1/2} \right\} \cdot 1 \quad (4.46)$$

By the way, this formula can be regarded as a generating function for HMDS-coefficients in case of covariantly constant background fields. Indeed, introducing an averaging over momentums at the point $x'$ with a Gaussian measure defined by

$$< \Phi(k) > = \lim_{x \to x'} \int \frac{dk}{\pi^{d/2} g} e^{-1/2} \exp(-g^{\mu'\nu'}(x')k\mu' k\nu') \Phi(k) \quad (4.47)$$

$$< 1 > = 1, \quad < k\mu > = 0, \quad < k\mu k\nu > = \frac{1}{2}g_{\mu\nu}$$

$$< k_{\mu_1} \cdots k_{\mu_{2n+1}} > = 0,$$

$$< k_{\mu_1} \cdots k_{\mu_{2n}} > = \frac{(2n)!}{2^n n!} g_{\mu_1 \mu_2} \cdots g_{\mu_{2n-1} \mu_{2n}} \quad (4.47a)$$

and denoting

$$A_0 = \Delta^{1/2} (\mathcal{D}_{\mu'} + B_{\mu'}) \Delta^{-1} X^{\mu'\nu'} (\mathcal{D}_{\nu'} + B_{\nu'}) \Delta^{1/2} \quad (4.48a)$$

$$A_1 = i\sqrt{t}k\mu' \left( X^{\mu'\nu'} \mathcal{D}_{\nu'} + \mathcal{D}_{\nu'} X^{\mu'\nu'} + 2X^{\mu'\nu'} B_{\nu'} \right) \quad (4.48b)$$

$$A_2 = -k\mu' k\nu' \left( X^{\mu'\nu'} - g^{\mu'\nu'}(x') \right) \quad (4.48c)$$

it is obtained

$$\Omega(t|x, x) = < \exp(A_2 + \sqrt{t}A_1 + tA_0) \cdot 1 > \quad (4.49)$$

Hence for the coefficients of asymptotic expansion (HMDS-coefficients) (2.11)

$$\Omega(t|x, x) = \sum_{k \geq 0} \frac{(-t)^k}{k!} a_k(x, x) \quad (4.50)$$

one finds

$$a_k(x, x) = (-1)^k \sum_{N \geq 0} \frac{k!}{N!} \sum_{0 \leq k_1, \ldots, k_N \leq 2 \atop k_1 + \cdots + k_N = k} < A_{k_1} \cdots A_{k_N} \cdot 1 > \quad (4.51)$$
Here one should move all derivatives to the right (to identity) and all functions - to the left and put \( x = x' \). As a result, we will come to an expression in terms of derivatives of the functions \( X^{\mu'}_{\nu'}, \Delta \) and \( B_{\mu'} \) (3.21)-(3.23) at the coinciding points \( x = x' \) that are expressed in turn only through curvatures.

From dimensional grounds it is evident that the coefficients of odd order vanish

\[
a_{2k+1}(x, x) = 0
\]  

and those of even order \( a_{2k}(x, x) \) are built from invariants of \( k \)-th order in curvature.

### 4.3. Algebraic approach

There exist a very elegant possibility to construct the heat kernel using only the commutation relations of covariant differential operators [67].

The idea consists in the following. Take a set of operators \( P_a \) forming a closed, actually nilpotent, Lie algebra

\[
\begin{align*}
[P_a, P_b] &= F_{ab} \quad (4.53) \\
[P_c, F_{ab}] &= 0 \quad (4.54)
\end{align*}
\]

This algebra is substantially a subalgebra of the general algebra (3.65) - (3.67) considered in previous section in the case of covariantly constant curvature (3.12) in the flat space case \( E^{i}_{ab} = 0 \).

Consider an element of corresponding Lie group in exponential parametrization

\[
\Psi(k) = \exp(\sqrt{t}k^a P_a) \quad (4.55)
\]

where \( t, k^a \) are the parameters and the factor \( \sqrt{t} \) is introduced for convenience of further computations.

Define a quantity \( Z(t) \) by averaging the group element over Gaussian measure

\[
Z(t) = \langle \Psi(k) \rangle = \int dk\gamma^{1/2}(4\pi)^{-d/2} \exp\left(-\frac{1}{4}\gamma_{ab}k^a k^b\right) \Psi(k) \quad (4.56)
\]

so that \( \langle 1 \rangle = 1 \). Here we introduce a symmetric non-degenerate positive definite matrix \( \gamma_{ab}, \gamma = \det \gamma_{ab} \) that commutes with operators \( P_a \)

\[
[P_c, \gamma_{ab}] = 0 \quad (4.57)
\]

Let us find a differential equation that is satisfied by \( Z(t) \). We have

\[
\frac{\partial}{\partial t} Z(t) = \frac{1}{2\sqrt{t}} P_a < k^a \Psi(k) > \quad (4.58)
\]

24
Then integrating by parts and using the properties of the Gaussian measure (4.56), it is easy to show
\[ < k^a \Psi(k) > = 2 \gamma^{ab} < \frac{\partial}{\partial k^b} \Psi(k) > \] (4.59)
where \( \gamma^{ab} \) is the matrix inverse to \( \gamma_{ab} \).

Now calculate the derivative \( \partial / \partial k^a \Psi \). To do this consider the operator
\[ A_a = \Psi^{-1} \frac{\partial}{\partial k^a} \Psi \] (4.60)
It satisfies the equation
\[ \frac{\partial}{\partial \sqrt{t}} A_a = -[k^b P_b, A_a] \] (4.61)
and the initial condition
\[ A_a \Bigr|_{t=0} = \frac{\partial}{\partial k^a} \] (4.62)
Hence
\[ A_a = \exp \left( -\sqrt{t} Ad_{kD} \right) \frac{\partial}{\partial k^a} \] (4.63)
where the operator \( Ad_{kD} \) is defined according to
\[ Ad_{kD} B = [k^a P_a, B] \] (4.64)
Computing the commutators it is easy to show finally
\[ Ad_{kD} \frac{\partial}{\partial k^a} = -P_a \] (4.65a)
\[ (Ad_{kD})^2 \frac{\partial}{\partial k^a} = -k^b F_{ba} \] (4.65b)
\[ (Ad_{kD})^n \frac{\partial}{\partial k^a} = 0 \quad , \quad (n \geq 3) \] (4.65c)
Thus we obtain from (4.63)
\[ A_a = \frac{\partial}{\partial k^a} + \sqrt{t} P_a + \frac{1}{2} t F_{ab} k^b \] (4.66)
and therefore
\[ \frac{\partial}{\partial k^a} \Psi = \Psi(\sqrt{t} P_a + \frac{1}{2} t F_{ab} k^b) \]
\[ = (\sqrt{t} P_a - \frac{1}{2} t F_{ab} k^b) \Psi \] (4.67)
where the relation \( \Psi(-k) = \Psi^{-1}(k) \) has been used.

25
By substituting this expression into the equation (4.59), it is obtained that

\[ < k^b \Psi >= 2 \sqrt{t} \tilde{\gamma}^{ba} P_a < \Psi > \] (4.68)

where \( \tilde{\gamma}^{ab} \) is the matrix inverse to

\[ \tilde{G}_{ab} = \tilde{\gamma}_{ab} + t \mathcal{F}_{ab} \] (4.69)

If one substitutes now this expression into the equation (4.68), decomposes the matrix \( \tilde{G}_{ab} \) in its symmetric and antisymmetric parts and makes use of commutation relations (4.53) then one finds finally the equation for the quantity \( Z(t) \) (4.56)

\[ \frac{\partial}{\partial t} Z(t) = \left( P_a G^{ab}(t) P_b + M(t) \right) Z(t) \] (4.70)

with initial condition

\[ Z(0) = 1 \] (4.71)

where \( G^{ab}(t) \) is the matrix inverse to the matrix \( G = \{ G_{ab}(t) \} \)

\[ G(t) = \gamma - t^2 \mathcal{F} \gamma^{-1} \mathcal{F} \] (4.72)

\[ M(t) = \frac{1}{2} \text{Sp} \left( \frac{t \gamma^{-1} \mathcal{F} \gamma^{-1} \mathcal{F}}{1 - t^2 \gamma^{-1} \mathcal{F} \gamma^{-1} \mathcal{F}} \right) \] (4.73)

Here and further the matrix notations \( \gamma = \{ \gamma_{ab} \} \) and \( \mathcal{F} = \{ \mathcal{F}_{ab} \} \) are used and \( \text{Sp} \) means the usual matrix trace.

Consider now the commutation relations (4.53) at greater length. Since the matrix \( \mathcal{F}_{ab} \) is antisymmetric it can be transformed by orthogonal matrices to a diagonal block form that have on the diagonal two-dimensional antisymmetric matrices or zeros. Without loss of generality one can put

\[ \gamma^{-1} \mathcal{F} = \{ \gamma^{ab} \mathcal{F}_{bc} \} = \text{diag} (\mathcal{F}_1 \varepsilon, \mathcal{F}_2 \varepsilon, \ldots, \mathcal{F}_n \varepsilon, 0, \ldots, 0) \] (4.74)

where \( \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \)

This means that all operators can be split in two-dimensional sets of pairwise non-commuting operators \( P_i^+ \) and \( P_i^- \)

\[ [P_i^+, P_k^-] = 0, \quad (i \neq k) \quad [P_i^+, P_i^+] = \mathcal{F}_i \] (4.75)

Together with the center of the algebra (the unit operator) they form closed three-dimensional subalgebras. In this basis the matrix \( G_{ab} \) (4.72) is diagonal

\[ \gamma^{-1} G(t) = \{ \gamma^{ab} G_{bc}(t) \} = \text{diag} \left\{ (1 - t^2 \mathcal{F}_1^2), (1 - t^2 \mathcal{F}_1^2), (1 - t^2 \mathcal{F}_2^2), (1 - t^2 \mathcal{F}_2^2), \ldots, \right. \\
\left. (1 - t^2 \mathcal{F}_n^2), (1 - t^2 \mathcal{F}_n^2), 1, \ldots, 1 \right\} \] (4.76)
each eigenvalue corresponding to two-dimensional antisymmetic block on the diagonal of the matrix $F_{ab}$ being twofold degenerate.

This leads to the fact that

$$L(t) = G^{ab}(t)P_aP_b$$

(4.77)

at different $t$ commute

$$[L(t_1), L(t_2)] = 0$$

(4.78)

since all noncommuting operators are assembled pairwise in invariant two-dimensional operators of the form

$$L_i = (P_i^+)^2 + (P_i^-)^2$$

(4.79)

that commute with each other

$$[L_i, L_k] = 0$$

(4.80)

It is obvious that since the commutation relations (4.78) are invariant they are valid in general case of arbitrary antisymmetric matrix $F_{ab}$ and positive definite matrix $\gamma_{ab}$. In other words the relation

$$\left[ P_aC_{(n)}^{ab}P_b, P_cC_{(m)}^{cd}P_d \right] = 0$$

(4.81)

takes place where

$$C_{(n)} = \{ C_{(n)}^{ab} \} = ( (\gamma^{-1}F)^{2n} ) \gamma^{-1}$$

(4.82)

Really, the relations (4.81) can be easily proved by making use of the commutation relations (4.53), the antisymmetry of $F_{ab}$ and the symmetry of the matrices $C_{(n)}^{ab}$.

Now one can integrate the equation (4.70) and find

$$Z(t) = \det(1 - t^2\gamma^{-1}F\gamma^{-1}F)^{-1/4} \exp(tg^{ab}(t)P_aP_b)$$

(4.83)

where $g^{-1} = \{ g^{ab} \}$ is the matrix inverse to the matrix $g = \{ g_{ab} \}$

$$g^{-1}(t) = \left( \frac{\text{Arth}(t\gamma^{-1}F)}{t\gamma^{-1}F} \right) \gamma^{-1}$$

(4.84)

Thus we have expressed the average of an element of the Lie group over group parameters with a Gaussian measure (4.56) in terms of the heat kernel for the Laplace-like operator with the metric $g_{ab}(t)$.

By inverting this procedure, a quite nontrivial result will be obtained. It is clear that one can choose the matrix $\gamma_{ab}$ in arbitrary way, in particular, depending on $t$. Moreover, this can be done in such a way that the matrix $g_{ab}$ would not depend on $t$ and would be a fixed matrix. In other words, let

$$\gamma(t) = tf \coth(g^{-1}tF)$$

(4.85)

Then inverting the formula (4.83) with due regard for the definition of the quantity $Z(t)$ (4.56) and changing the integration variables $k \to k/\sqrt{t}$ we prove the
**Theorem:**

In the case of nilpotent Lie algebra given by the commutation relations

\[
[P_a, P_b] = \mathcal{F}_{ab}, \quad [P_c, \mathcal{F}_{ab}] = 0
\]  

(4.86)

the heat kernel can be presented in form of an average over the Lie group with the Gaussian measure

\[
\exp(t \Box) = (4\pi t)^{-d/2} \det \left( \frac{tg^{-1}\mathcal{F}}{\sinh(tg^{-1}\mathcal{F})} \right)^{1/2}
\]

\[
\int dk \exp \left\{ -\frac{1}{4t}k^a(t\mathcal{F} \coth(g^{-1}t\mathcal{F}))_{ab}k^b + k^a P_a \right\}
\]  

(4.87)

where \( \Box = g^{ab}P_a P_b \), \( \mathcal{F} = \{ \mathcal{F}_{ab} \} \) and \( g = \{ g_{ab} \} \) is a positive definite symmetric nondegenerate matrix.

Using this representation the heat kernel in flat space in the case of covariantly constant curvature of the background connection in coordinate representation can be easily obtained.

We have by definition (2.7)

\[
U(t|x, x') = \exp \left( -t(-\Box + Q + m^2) \right) \mathcal{P}(x, x')\delta(x, x')
\]  

(4.88)

Using the proven theorem we express the heat kernel in terms of the quantity

\[
\exp(k^a P_a)\mathcal{P}(x, x')\delta(x, x')
\]  

(4.89)

It is easy to show that [5,6]

\[
\lim_{x \to x'} P_{(a_1 \cdots P_{a_n})} = 0
\]  

(4.90)

Hence

\[
\lim_{x \to x'} (k^a P_a)^{\alpha} \mathcal{P} = 0 \quad \text{and} \quad \lim_{x \to x'} [\exp(k^a P_a), \mathcal{P}] = 0
\]  

(4.91)

and therefore

\[
\exp(k^a P_a)\mathcal{P}(x, x')\delta(x, x') = \mathcal{P}(x, x')\delta(x + k, x')
\]  

(4.92)

(After another way to be convinced of this gauge invariant equation consists in using the Fock-Schwinger gauge (3.24) when in flat space \( A_\mu = -(1/2)\mathcal{F}_{\mu\alpha}(x^\alpha - x'^\alpha) \) (3.30).)

Now we substitute (4.87) in (4.88) and use (4.92). Subsequently the integral over \( k^a \) becomes trivial and one obtains immediately the heat kernel that coincides (with evident substitution \( \mathcal{F}_{ab} \to \mathcal{R}_{\mu\nu} \)), as it should do, with the result obtained by means of Schwinger’s method (4.31).

Thus the main idea consists in the following representation

\[
\exp(t \Box) = \int dk \Phi(t, k|x, x') \exp(k^a X_a)
\]  

(4.93)
where $X_a$ are the generators of a general finite-dimensional Lie group.

If it would be possible to obtain in the general case (3.65) - (3.67) the formulae similar to (4.87) then it would allow to find the heat kernel in a number of important cases, in particular, in symmetric spaces, and would lead finally to the general solution of the effective potential problem in quantum gravity. This remains still a very interesting problem to solve.

**Y. EFFECTIVE POTENTIAL IN YANG-MILLS THEORY**

Let us now go on to the illustration of the above stated approaches for the calculation of the effective potential. As a nontrivial example consider the one-loop effective potential in quantum Yang-Mills model in flat space with some set of scalar and spinor matter fields (QCD, GUT models etc.) [68].

The Euclidean action of the model is

$$S = \int dx \left\{ -\frac{1}{2g^2} \text{tr}(F_{\mu\nu}^2) + i\bar{\psi}(\gamma^\mu \nabla_{\mu} + M(\varphi))\psi + \frac{1}{2} \varphi^T(-\Box)\varphi + V(\varphi) \right\}$$

where

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + [A_{\mu}, A_{\nu}]$$

is the strength of gauge (gluon) fields taking the values in the Lie algebra of the gauge group $G$, $A_{\mu} = A_{\mu}^a C_a$, $F_{\mu\nu} = F_{\mu\nu}^a C_a$, $C_a = \{C_b^a\}$ and $C_{ac}^b$ being the generators in the adjoint representation and the structure constants of the gauge group respectively, $g$ is the interaction coupling constant, $\varphi = \{\varphi^A\}$ and $\psi = \{\psi^i\}$ are the multiplets of real, for definiteness, Higgs scalar fields and the Dirac spinor quark ones, which belong to some, in general, different irreducible representations of the gauge group, $V(\varphi)$ is a potential for scalar fields and $M(\varphi) = \{M_{ik}(\varphi)\}$ is a spinor mass matrix, $\bar{\psi} = \psi^\dagger C^T$, the matrix $C$ is defined according to $\gamma^T_{\mu} = -C\gamma_{\mu}C^{-1}$, $\gamma_{\mu}$ are the Dirac matrices $\gamma_{(\mu}\gamma_{\nu)} = g_{\mu\nu}$, $\nabla_{\mu} = \partial_{\mu} + T(A_{\mu})$ is the covariant derivative in the representation $T$ and $\Box = \nabla_{\mu}^2$. Here and everywhere below in this section $\text{tr}$ denotes the trace only over group indices, other possible indices being left intact.

We limit ourselves to the compact simple gauge group, the generalization to the semisimple case being trivial, and normalize the generators of the adjoint representation, i.e. the structure constants, by the condition

$$\text{tr}(C_a C_b) = C^c_{ad} C^{d}_{bc} = -2\delta_{ab}$$

In general setting one should consider the situation when both the scalar (Higgs) fields and the gauge ones have the background contributions. (See the discussion in section III.). The quantization of the model (5.1) in a general covariant gauge leads to the one-loop effective action

$$\Gamma_{(1)} = \Gamma_{(1)YM} + \Gamma_{(1)mat}$$

(5.4)
Here the contribution of the Yang-Mills fields (and ghosts) proper has the form

\begin{align}
\Gamma_{(1)YM} &= \frac{1}{2} \text{Tr} \ln \Delta(\lambda)/\mu^2 - \text{Tr} \ln F(\lambda)/\mu^2 \\
\Delta(\lambda) &= \Delta + \lambda H \\
\Delta_{\mu\nu} &= -\Box \delta_{\mu\nu} - 2F_{\mu\nu} \\
H_{\mu\nu} &= \nabla_\mu \nabla_\nu
\end{align}

(5.5)

(5.6)

(5.7)

(5.8)

where \(\Delta_{\mu\nu}(\lambda)\) is the second order differential operator determining the propagator of gauge fields (gluons),

\begin{align}
F(\lambda) &= \sqrt{(1 - \lambda)}F \\
F &= -\Box
\end{align}

(5.9)

(5.10)

is the operator that determines the ghost propagator, \(\lambda\) is the gauge fixing parameter and a renorm parameter \(\mu\) is introduced to preserve dimensions.

Here and everywhere below we denote by \(\text{Tr}\) the functional trace, viz. this means that not only the traces over all discrete indices (the group ones, the Lorentz ones, etc.) should be taken but over the continuous (space-time coordinates) as well. Although we use the same notation \(\text{Tr}\), the explicit meaning of this notation depends on the structure of the quantity to which it is applied. In any case the functional trace \(\text{Tr}\) is the trace over all present indices including the continuous ones.

Although the factor \(\sqrt{(1 - \lambda)}\) in (5.9) seems to be irrelevant, it ensures the gauge independence of the regularized effective action on the mass shell (see the proof below).

The contribution of the matter fields has the form

\begin{align}
\Gamma_{(1)mat} &= -\text{Tr} \ln (\gamma^\mu \nabla_\mu + M(\phi))/\mu + \frac{1}{2} \text{Tr} \ln N/\mu^2 \\
N &= -\Box + Q(\phi)
\end{align}

(5.11)

(5.11a)

where \(\phi\) is the background Higgs scalar field and the mass matrix of the Higgs fields \(Q = \{Q^A_B\}\) is of the form

\begin{equation}
Q^A_B(\phi) = \frac{\partial^2}{\partial \phi^B \partial \phi^A} V(\phi)
\end{equation}

(5.12)

5.1. Contribution of gauge fields

The nonvanishing scalar background fields present no special computing difficulties. But the variety of the forms and possibilities of the spontaneous symmetry breakdown obscures the influence of the background gauge fields themselves. That is why we consider first the most essential contribution of Yang-Mills fields.

Let us show first of all that the effective action (5.5) does not depend on the gauge parameter \(\lambda\) on the mass shell, i.e. when the background fields satisfy the classical equations of motion.
Indeed, by differentiating $\Gamma_{YM}^{(1)}$ with respect to $\lambda$ and using the Ward identities

$$\nabla_\mu \Delta^{-1} \mu^\nu(\lambda) = -\frac{1}{1-\lambda} \Box^{-1} (\nabla_\nu + J_\mu \Delta^{-1} \mu^\nu(\lambda)) \quad (5.13a)$$

$$\Delta^{-1} \mu^\nu(\lambda) \nabla_\nu = -\frac{1}{1-\lambda} (\nabla_\mu + \Delta^{-1} \mu^\nu(\lambda) J_\nu) \Box^{-1} \quad (5.13b)$$

where

$$J_\mu = \nabla_\nu F_\mu$$

we get

$$\frac{\partial \Gamma_{YM}^{(1)}(\lambda)}{\partial \lambda} = \frac{1}{2(1-\lambda)^2} \text{Tr} \left\{ \nabla_\mu J^\mu \Box^{-2} + J_\mu \Delta^{-1} \mu^\nu(\lambda) J_\nu \Box^{-2} \right\} \quad (5.15)$$

Hence it is obvious that

$$\left. \frac{\partial \Gamma_{YM}^{(1)}}{\partial \lambda} \right|_{J=0} = 0 \quad (5.16)$$

This leads to the fact that on such background the one-loop effective action (5.5) does not depend on the gauge fixing condition and coincides, therefore, with the unique Vilkovisky - De Witt effective action [13,63,18].

Thus when considering the covariantly constant background fields satisfying the condition

$$\nabla_\mu F_{\alpha\beta} = 0 \quad (5.17)$$

one can simply put $\lambda = 0$ (so called minimal gauge.)

Strictly speaking, our proof needs a substantiation since the computations above were absolutely formal. As a matter of fact, the expressions (5.5) and (5.15) for the effective action contain the ultraviolet divergences that should be regularized at first. One can show, that the effective action does not depend on the gauge on mass shell ($J = 0$), more consistently using, for example, the $\zeta$-function prescription (2.14) to regularize functional determinants

$$\Gamma_{YM}^{(1)} = -\frac{1}{2} \zeta''_{\Delta(\lambda)}(0) + \zeta''_F(0) \quad (5.18)$$

Let us calculate the $\zeta$-functions of the operators $\Delta(\lambda)$ and $F(\lambda)$. We have from (5.6)

$$\Delta(\lambda) = \Delta(1) - (1 - \lambda)H \quad (5.19)$$

It is easy to show that

$$\Delta^\mu_{\lambda}(1) H^\lambda_\nu = -J^\mu \nabla_\nu \quad \text{and} \quad H^\mu_{\lambda}(1) \Delta^\lambda_\nu = -\nabla^\mu J_\nu \quad (5.20)$$

where $J_\mu$ is given by the formula (5.14). That is at $J_\mu = 0$ the operators $\Delta(1)$ and $H$ are orthogonal. Using these relations we obtain the heat kernel

$$\exp(-t\Delta(\lambda)) = \exp(-t\Delta(1)) + \exp(t(1 - \lambda)H) - 1 \quad (5.21)$$
Hence

\[ \exp(-t\Delta(0)) = \exp(-t\Delta(1)) + \exp(tH) - 1 \]  

(5.22)

which leads finally to

\[ \exp(-t\Delta(\lambda)) = \exp(-t\Delta) + \exp(t(1 - \lambda)H) - \exp(tH) \]  

(5.23)

where \( \Delta = \Delta(0) \) is the operator (5.7).

At last making use of the relation

\[ \text{Tr}(H^n) = \text{Tr} \Box^n \]  

(5.24)

gives for the trace of the heat kernel

\[ \text{Tr} \exp(-t\Delta(\lambda)) = \text{Tr} \exp(-t\Delta) + \text{Tr} \{ \exp(-t(1 - \lambda)F) - \exp(-tF) \} \]  

(5.25)

where \( F = -\Box \).

Wherefrom by the definition of the \( \zeta \)-function (2.16)

\begin{align*}
\zeta_{\Delta(\lambda)}(p) &= \zeta_{\Delta}(p) + ((1 - \lambda)^{-p} - 1)\zeta_F(p) \\
\zeta'_{\Delta(\lambda)}(0) &= \zeta'_\Delta(0) - \ln(1 - \lambda)\zeta_F(0)
\end{align*}

(5.26, 5.27)

Then it is evident that

\begin{align*}
\zeta_F(\lambda)(p) &= (1 - \lambda)^{-p/2}\zeta_F(p) \\
\zeta'_F(\lambda)(0) &= \zeta'_F(0) - \frac{1}{2}\ln(1 - \lambda)\zeta_F(0)
\end{align*}

(5.28, 5.29)

As a result the \( \lambda \)-depending terms in effective action (5.18) cancel exactly and we find

\[ \Gamma_\text{(1)}_{YM} = -\frac{1}{2}\zeta'_\Delta(0) + \zeta'_F(0) \]  

(5.30)

Thereby we have proven that the regularized effective action (5.18) at background fields satisfying the condition

\[ J_\mu = \nabla_\nu F^\nu_\mu = 0 \]  

(5.31)

does not depend on the gauge fixing parameter \( \lambda \).

Let us calculate now the low-energy effective action (effective potential) in the case of covariantly constant background fields (5.17). Such field configurations satisfy automatically the condition (5.31) \( (J_\mu = 0) \). Subsequently everything aforesaid about the independence of the effective action on the gauge is valid.

From (5.17) we have immediately

\[ [F_{\alpha\beta}, F_{\mu\nu}] = 0 \]  

(5.32)
This means that the gauge fields take their values in Cartan subalgebra, i.e. the nontrivial nonvanishing components of the gauge field exist only in the direction of the diagonal generators. The maximal number of independent fields is equal to the dimension of the Cartan subalgebra \( r \), i.e. rank of the group.

It is obvious that operators \( \Delta \) (5.7) and \( F \) (5.10) belong to those considered in previous section and, therefore, all formulae obtained there are applicable to them.

For the traces of the heat kernels we have from (4.31)

\[
\text{Tr} \exp(-t\Delta) = \int dx (4\pi t)^{-d/2} \text{tr} \left\{ \det \left( \frac{tF}{\sinh(tF)} \right)^{1/2} \text{Sp} \exp(2tF) \right\} \\
\text{Tr} \exp(-tF) = \int dx (4\pi t)^{-d/2} \text{tr} \left( \frac{tF}{\sinh(tF)} \right)^{1/2}
\]

where \( F = \{ F_{\mu\nu} \} \) is here the matrix with both vector and group indices. The notations \( \det \) and \( \text{Sp} \) mean in present section the determinant and the trace over vector indices respectively and \( \text{tr} \) means as it was stressed above the trace only over the group indices.

Defining the total Yang-Mills \( \zeta \)-function

\[
\zeta_{YM}(p) = \zeta(\Delta(p)) - 2\zeta_F(p)
\]

so that

\[
\Gamma_{(1)YM} = -\frac{1}{2} \zeta'_{YM}(0)
\]

we obtain

\[
\zeta_{YM}(p) = \int dx (4\pi)^{-d/2} \frac{\mu^{2p}}{\Gamma(p)} \int_0^\infty dt t^{p-d/2-1} \text{tr} \left\{ \det \left( \frac{tF}{\sinh(tF)} \right)^{1/2} (\text{Sp} \exp(2tF) - 2) \right\}
\]

Let us now calculate the traces in formulae (5.33) and(5.34). Mention, first of all, that the generators of the Cartan subalgebra of the compact simple group in adjoint representation \( C_a, (a = 1, \ldots, r) \) (where \( r \) is the rank of the group) are the traceless antisymmetric (in real basis) commuting matrices

\[
[C_a, C_b] = 0 \quad , \quad a, b = 1, \ldots, r
\]

normalized according to the condition (5.3). Hence they can be diagonalized (in complex basis) simultaneously

\[
C_a = \text{diag}(0, \ldots, 0, i\alpha_a^{(1)}, -i\alpha_a^{(1)}, \ldots, i\alpha_a^{(p)}, -i\alpha_a^{(p)})
\]

where \( \alpha^{(i)} \) are the positive roots of the algebra, \( p = (n - r)/2 \) is the number of positive roots and \( n \) is the dimension of the group. The number of zeros on the diagonal of the generators of the Cartan subalgebra in adjoint representation equals the maximum number of commuting generators of the group, i.e. rank of the group \( r \).
Mention also that from the normalization condition \((5.3)\) it follows, in particular,
\[
\sum_{\alpha > 0} \alpha_a \alpha_b = \delta_{ab} \tag{5.39}
\]
where the sum is to be taken over all positive roots of the algebra.

Therefore for any analytic function of a matrix taking its values in the Cartan subalgebra in adjoint representation \(F = F^a C_a\) we have
\[
\text{tr} f(F) = r f(0) + \sum_{\alpha > 0} (f(iF(\alpha)) + f(-iF(\alpha))) \tag{5.40}
\]
where
\[
F(\alpha) = F^a \alpha_a \tag{5.41}
\]

Using this relation the heat kernels can be rewritten in the form
\[
\text{Tr} \exp(-t\Delta) = \int dx (4\pi t)^{-d/2} \left\{ r d + 2 \sum_{\alpha > 0} \det \left( \frac{tF(\alpha)}{\sin(tF(\alpha))} \right)^{1/2} \text{Sp} \cos(2tF(\alpha)) \right\} \tag{5.42}
\]
\[
\text{Tr} \exp(-tF) = \int dx (4\pi t)^{-d/2} \left\{ r + 2 \sum_{\alpha > 0} \det \left( \frac{tF(\alpha)}{\sin(tF(\alpha))} \right)^{1/2} \right\} \tag{5.43}
\]
where the notation is introduced
\[
F(\alpha) = \{ F^a_{\mu\nu}, \alpha_a \} \tag{5.44}
\]

This is still a matrix with vector indices, actually a 2-form. Now one can calculate the trace and the determinant over the vector indices. Any 2-form \(F = \{ F_{\mu\nu} \}\) can be decomposed in a set of orthogonal simple 2-forms \(F(i) = \{ F(i)_{\mu\nu} \}\) as follows
\[
F_{\mu\nu} = \sum_{1 \leq i \leq q} F(i)_{\mu\nu} H_i \tag{5.45}
\]
\[
F(i) F(k) = 0 \quad , \quad (i \neq k) \tag{5.46}
\]
\[
F^2_{(i)} = -P_{(i)} \tag{5.47}
\]
where \(H_i\) are the invariants of the 2-form \(F_{\mu\nu}\), \(q \leq \lfloor d/2 \rfloor\) is the number of nonvanishing independent invariants and \(P_{(i)}\) are the two-dimensional projectors
\[
P_{(i)}^2 = P_{(i)} \tag{5.47}
\]
\[
P_{(i)} P_{(k)} = 0 \quad , \quad (i \neq k) \tag{5.47}
\]
\[
\text{Sp} P_{(i)} = 2
\]
One can show that in the case of Euclidean signature all invariants $H_i$ can be regarded to be positive.

Using the expansion (5.45), it is easy to get

\[ \text{Sp} F^{2k+1} = 0, \quad k = 1, 2, \ldots, \]  
\[ \text{Sp} F^{2k} = (-1)^k 2 \sum_{1 \leq i \leq q} H_i^{2k} \]  

(5.48)  
(5.49)

One can consider the $q$ invariants $H_i$ to be determined from the set of the first $[d/2]$ equations (5.49) $k = 1, 2, \ldots, [d/2]$. Then the rest equations will be satisfied identically. It is easy to show that $q$ invariants $H_i$ are the positive roots of the algebraic equation of degree $2[d/2]$

\[ H^{2[d/2]} + c_1 H^{2([d/2]−1)} + \cdots + c_{[d/2]} H^2 + c_{[d/2]} = 0 \]  

(5.49a)

with the coefficients

\[ c_k = \sum_{1 \leq j \leq k} (-1)^j \sum_{1 \leq k_1 \leq \cdots \leq k_j \leq k−j+1} \frac{1}{k_1 \cdots k_j} I_{k_1} \cdots I_{k_j} \]  

(5.49b)

where

\[ I_k = \frac{1}{2} (-1)^k \text{Sp} F^{2k} \]  

(5.49c)

Hence for any analytic function of a 2-form $F = \{F_{\mu\nu}\}$ it takes place

\[ \text{Sp} f(F) = (d − 2q)f(0) + \sum_{1 \leq j \leq q} (f(iH_j) + f(−iH_j)) \]  

(5.50)

\[ \det f(F) = (f(0))^{d−2q} \prod_{1 \leq j \leq q} f(iH_j)f(−iH_j), \quad f(0) \neq 0 \]  

(5.51)

By making use of these formulae, one can compute the determinant over vector indices in (5.42) and (5.43) and obtain finally

\[ \text{Tr} \exp(−t\Delta) = \int dx (4\pi t)^{−d/2} \left\{ r d + 2 \sum_{\alpha > 0} \prod_{1 \leq i \leq q} \left( \frac{tH_i(\alpha)}{\sinh(tH_i(\alpha))} \right) \right. \]  
\[ \times \left( d + 4 \sum_{1 \leq j \leq q} \sinh^2(tH_j(\alpha)) \right) \} \]  

(5.52)

\[ \text{Tr} \exp(−tF) = \int dx (4\pi t)^{−d/2} \left\{ r + 2 \sum_{\alpha > 0} \prod_{1 \leq i \leq q} \left( \frac{tH_i(\alpha)}{\sinh(tH_i(\alpha))} \right) \right. \]  
\[ \times \left( d + 4 \sum_{1 \leq j \leq q} \sinh^2(tH_j(\alpha)) \right) \} \]  

(5.53)

where $H_i(\alpha)$ are the invariants of the tensor $F_{\mu\nu}(\alpha)$ (5.44).
Hence the total \(\zeta\)-function for gauge fields equals (5.36)

\[
\zeta_{YM}(p) = \int dx (4\pi)^{-d/2} \frac{\mu_{2p}(p)}{\Gamma(p)} \int_0^\infty dt t^{p-d/2-1} \left\{ r(d-2) + 2 \sum_{\alpha>0} \prod_{1 \leq i \leq q} \left( \frac{tH_i(\alpha)}{\sinh(tH_i(\alpha))} \right) \times \left( d - 2 + 4 \sum_{1 \leq j \leq q} \sinh^2(tH_j(\alpha)) \right) \right\} \tag{5.53a}
\]

Wherefrom it is immediately seen how the ghost fields effectively decrease the number of degrees of freedom of gauge field \(d \rightarrow (d-2)\).

Further calculations are possible only for a concrete gauge group, i.e. for one or another system of roots of the algebra. For example, for \(SU(N)\) group, as it is well known, [64], the rank is \(r = N - 1\), the dimension is \(n = N^2 - 1\), the number of positive roots is \(p = N(N-1)/2\) and positive roots are given by

\[
\alpha_{ij} = e_i - e_j, \quad (1 \leq i < j \leq N) \tag{5.54}
\]

where \(e_i\) is an orthonormal basis in \(\mathbb{R}^N\). All roots lie in the plane orthogonal to the vector \(\sum_{1 \leq i \leq N} e_i\) and are, actually, \((N-1)\)-dimensional vectors in \(\mathbb{R}^{N-1}\).

Let us investigate now the behavior of the heat kernels at \(t \rightarrow \infty\). It determines the analytic properties of the corresponding \(\zeta\)-function and one-loop effective action and thereby the stability properties of the corresponding vacuum.

From (5.52) and (5.53) we have

\[
\text{Tr} \exp(-t\Delta) \sim \int dx (4\pi)^{-d/2} \left\{ r d t^{d/2} + 2^{q+1} t^q d/2 \sum_{\alpha>0} \prod_{1 \leq i \leq q} H_i(\alpha) \right. \nonumber
\]

\[
\times \left. \sum_{1 \leq k \leq q} \exp \left\{ -t \left( \sum_{1 \leq j \leq q} H_j(\alpha) - 2H_k(\alpha) \right) \right\} \right\} \tag{5.55}
\]

\[
\text{Tr} \exp(-tF) \sim \int dx (4\pi)^{-d/2} r t^{-d/2} \tag{5.56}
\]

By the way one can determine from these equations the minimal eigenvalues of the operators \(\Delta\) and \(F\). It is easy to show, that the heat kernel for the Laplace-like second order operator (2.8) on covariantly constant background (5.17), (5.44), (5.45) behaves at \(t \rightarrow \infty\) in general case as follows

\[
\text{Tr} \exp(-t\Delta) \sim t^{-(d-2q)/2} \exp(-t\lambda_{\text{min}}) \tag{5.57}
\]

where \((d-2q)\) is the number of dimensions with continuum spectrum when the eigenvalues are enumerated by both discrete and continuum parameters

\[
\lambda = \lambda(p_1, \cdots, p_{d-2q}, n_1, \cdots, n_{2q}) , \quad -\infty < p_i < +\infty , \quad n_i = 0, 1, 2, \cdots \tag{5.58}
\]
The heat kernel for the scalar ghost operator (5.56) behaves good, i.e. decreases, at $t \to \infty$. This means that the minimal eigenvalue of the operator $F$ is equal to zero

$$\lambda_{\text{min}}(F) = 0 \quad (5.59)$$

The heat kernel for vector gluon operator behaves at $t \to \infty$, in general, not good. This is caused by the self-interaction of the Yang-Mills fields, viz. by the extremely large value of the gyromagnetic ration of the gluons (the coefficient in front of $F_{\mu\nu}$ in (5.7)) equal to 2. In this case the second term in (5.55) can be exponentially large. This is the consequence of the fact, that the operator $\Delta$ can have, in general, the negative modes. From (5.55) one can conclude that the minimal eigenvalue of $\Delta$ is either negative

$$\lambda_{\text{min}}(\Delta) = -\max_{\alpha} \left\{ -\sum_{1 \leq i \leq q}^t H_i(\alpha) + H_{\text{max}}(\alpha) \right\} \quad (5.60)$$

(where $H_{\text{max}}(\alpha) = \max_{1 \leq i \leq q} H_i(\alpha)$ is the largest invariant of the field $F_{\mu\nu}(\alpha)$ (5.44), the prime at the sum meaning that the sum does not include this maximal term) or is equal to zero

$$\lambda_{\text{min}}(\Delta) = 0 \quad (5.61)$$

if the previous value (5.60) is positive.

Thus the vector operator $\Delta$ is positive definite and the heat kernel behaves good at $t \to \infty$ only in the case if the maximal invariant $H_{\text{max}}(\alpha)$ of the background field $F_{\mu\nu}(\alpha)$ (5.44) is smaller than the sum of all other ones.

This is possible only in the case when the number of independent invariants is equal or greater than two $q \geq 2$, i.e. when the dimension of the space is not less than four $d \geq 4$.

In case $q = 1$, i.e. there is only one independent invariant of the field $F_{\mu\nu}(\alpha)$ (this case for the $SU(2)$ group was considered first by Savvidy [65]), without fail the negative modes of the gluon operator exist since the minimal eigenvalue is negative

$$\lambda_{\text{min}}(\Delta) = -\max_{\alpha} H(\alpha) < 0 \quad (5.62)$$

(in this case $H(\alpha) = \sqrt{(F_{\mu\nu}^a \alpha_a)^2/2}$). It is this fact that leads to the well known instability of the Savvidy vacuum [66].

Only starting with the number of nonvanishing invariants equal to two ($q = 2$) the instability can disappear. In this case

$$\lambda_{\text{min}}(\Delta) = -\max_{\alpha} \left\{ -H_{\text{min}}(\alpha) + H_{\text{max}}(\alpha) \right\} \quad (5.63)$$

where $H_{\text{min}}(\alpha)$ and $H_{\text{max}}(\alpha)$ are the minimal and the maximal invariants. Hence it is seen, that if $H_{\text{min}}(\alpha) \neq H_{\text{max}}(\alpha)$, then again $\lambda_{\text{min}}(\Delta) < 0$. The only possibility to achieve the absence of the negative modes in the case $q = 2$ is to choose these invariants equal to each other $H_1(\alpha) = H_2(\alpha) = H(\alpha)$. 37
Then the heat kernels take the form
\[
\text{Tr}\exp(-t\Delta) = \int dx (4\pi t)^{-d/2} \left\{ rd + 2 \sum_{\alpha > 0} \frac{t^2 H^2(\alpha)}{\sinh^2(tH(\alpha))} (d + 8 \sinh^2(tH(\alpha))) \right\}
\] (5.64)

\[
\text{Tr}\exp(-tF) = \int dx (4\pi t)^{-d/2} \left\{ r + 2 \sum_{\alpha > 0} \frac{t^2 H^2(\alpha)}{\sinh^2(tH(\alpha))} \right\}
\] (5.65)

and have both good decreasing as a degree behavior at \( t \to \infty \)
\[
\text{Tr}\exp(-t\Delta) \sim \int dx (4\pi)^{-d/2} 16t^{2-d/2} \sum_{\alpha > 0} H^2(\alpha)
\] (5.66)

\[
\text{Tr}\exp(-tF) \sim \int dx (4\pi)^{-d/2} rt^{d/2}
\] (5.67)

Thus at \( q = 2, H_1(\alpha) = H_2(\alpha) \) the operator \( \Delta \) is positive definite (except for the zero modes), i.e. there is not instability that is characteristic to the Savvidy vacuum. It seems on the face of it, that this case can be realized also in four-dimensional space \( d = 4 \). However, as we show below, in case of four dimensions \( d = 4 \) it is not possible to make the analytic continuation of the equality of two invariants to the pseudo-Euclidean space of Lorentzian signature, limiting thereby the possible physical applications of this result.

Consider the case of two invariants in four-dimensional space \( q = 2, d = 4 \) at greater length. In this case the invariants \( H_i(\alpha) \) given by the solutions of the eq. (5.49) have the simple form
\[
H_{1,2}(\alpha) = \sqrt{\frac{1}{2} I_1(\alpha) \pm \frac{1}{2} \sqrt{2I_2(\alpha) - I_1^2(\alpha)}}
\] (5.68)

where \( I_k(\alpha) \) are the invariants defined by (5.49c), viz.
\[
I_1(\alpha) = \frac{1}{2} F_{\mu\nu}(\alpha) F^{\mu\nu}(\alpha)
\] (5.68a)

\[
I_2(\alpha) = \frac{1}{2} F_{\mu\nu}(\alpha) F_{\nu\lambda}(\alpha) F_{\lambda\rho}(\alpha) F^{\rho\mu}(\alpha)
\] (5.68b)

The situation of two equal invariants \( H_1(\alpha) = H_2(\alpha) \) in four dimensional space considered above means, that a relation between the invariants \( I_k \) takes place
\[
I_2(\alpha) = \frac{1}{2} I_1^2(\alpha)
\] (5.69)

But this is possible only in space of Euclidean signature when for any field
\[
I_2(\alpha) < I_1^2(\alpha)
\] (5.70)

When going to the pseudo-Euclidean (Lorentzian) signature the sign of inequality here changes to the opposite one
\[
I_2(\alpha) > I_1^2(\alpha)
\] (5.71)
that leads to the impossibility of the equality condition (5.69) to be satisfied in Minkowski space.

In Euclidean case both invariants $H_1(\alpha)$ and $H_2(\alpha)$ are real, whereas in Minkowski space one of them is necessarily imaginary.

Only beginning with $d \geq 5$ there exists such a background that, on the one hand, ensures the operator $\Delta$ (5.7) to be positive definite and, on the other hand, assumes the analytic continuation on the pseudo-Euclidean space of Lorentzian signature.

This is a consequence of the general fact that, when doing the analytic continuation of the results obtained in Euclidean signature to the Lorentzian signature, one should put, in general, one of the invariants of any 2-form to be imaginary, i.e.

$$H_q(\alpha) = iE(\alpha) \quad (5.72)$$

One may call the real invariants the magnetic and the imaginary the electric ones. In other words, in Euclidean space all invariants are magnetic, while in the Minkowski space one of them has to be electric (it can also vanish).

The heat kernel after the substitution $t = is$ becomes the Schrödinger one. The formulae in Minkowski space take the form

$$\text{Tr} \exp(-is\Delta) = \int dx (4\pi is)^{-d/2} \left\{ rd + 2 \sum_{\alpha > 0} \prod_{1 \leq i \leq q-1} \left( \frac{sH_i(\alpha)}{\sin(sH_i(\alpha))} \right) \left( \frac{sE(\alpha)}{\sinh(sE(\alpha))} \right) \right\} \times \left( d - \sum_{1 \leq j \leq q-1} 4 \sin^2(sH_j(\alpha)) + 4 \sinh^2(sE(\alpha)) \right) \right\} \quad (5.73)$$

$$\text{Tr} \exp(-isF) = \int dx (4\pi is)^{-d/2} \left\{ r + 2 \sum_{\alpha > 0} \prod_{1 \leq i \leq q-1} \left( \frac{sH_i(\alpha)}{\sin(sH_i(\alpha))} \right) \left( \frac{sE(\alpha)}{\sinh(sE(\alpha))} \right) \right\} \quad (5.74)$$

The presence of the electric field leads to poles in heat kernels, indeterminacy in integrals over $t$ and, as a consequence, to imaginary part of the effective potential, i.e. to the creation of the particles and instability (although not so potent as in the presence of negative modes).

Take notice of the fact, that whereas the heat kernel is good defined, the Schrödinger kernel contains, in general, various poles, divergences and ambiguities and behaves not good at $s \to \infty$. That is why it should be understood as a result of analytic continuation of the heat kernel, that gives the prescriptions how to get round the poles in integrals over $t$ for $\zeta$-function and effective action. When the electric component is absent such problems do not appear and the analytic continuation is trivial.

5.2. Contribution of matter fields
Consider now the contribution of the matter fields to the one-loop effective action (5.10). As was stressed above, both the background gauge field strengt $F_{\mu\nu}$ and the mass matrices $Q(\phi)$ and $M(\phi)$ are assumed to be covariantly constant
\[ \nabla_\mu F_{\alpha\beta} = 0 \quad , \quad \nabla_\mu M = 0 \quad , \quad \nabla_\mu Q = 0 \quad (5.75) \]
that means
\[ [F_{\alpha\beta}, F_{\mu\nu}] = 0 \quad , \quad [R_{\alpha\beta}, M] = 0 \quad , \quad [R_{\alpha\beta}, Q] = 0 \quad (5.76) \]
where $R_{\mu\nu} = T(F_{\mu\nu}) = F^{a}_{\mu\nu}T_a$ is the curvature and $T_a$ are the generators of the gauge group in the representation to which the matter fields belong.

For the irreducible representations of compact simple Lie group the generators $T_a$ are traceless matrices which can be normalized by the condition [9]
\[ \text{tr} (T_a T_b) = - \frac{D}{n} T^2 \delta_{ab} \quad (5.77) \]
where $D$ is the dimension of the representation, $n$ is the dimension of the group and $T^2 = -T_a T_a$ is the eigenvalue of the Casimir operator of second order for given representation.

The curvature $R_{\mu\nu}$ determines the commutator of the covariant derivatives on the matter fields:
\[ [\nabla_\mu, \nabla_\nu] \psi = R_{\mu\nu} \psi \quad (5.78) \]
and analogously for scalar fields.

We do not make here any difference in notations between representations realized by scalar and spinor fields, as it can not lead to misunderstanding.

Further, if the mass matrix of fermions $M$ does not contain the Dirac matrices, as it is assumed, or contains only even number of them, then it is easy to show, that the determinant of the Dirac operator does not depend on the sign in front of the derivative and, therefore, it can be expressed in terms of the squared Dirac operator
\[ \text{Tr} \ln(\gamma^\mu \nabla_\mu + M) = \frac{1}{2} \text{Tr} \ln K \quad (5.79) \]
\[ K = (\gamma^\mu \nabla_\mu + M)(-\gamma^\nu \nabla_\nu + M) = -\Box - \frac{1}{2} \gamma^{\mu\nu} R_{\mu\nu} + M^2 \quad (5.80) \]
where $\gamma_{\mu\nu} = \gamma_{[\mu} \gamma_{\nu]}$ are the generators of the orthogonal group (Lorentz group) and it has been made use of the relation $[M, \gamma^\mu \nabla_\mu] = 0$.

The squared Dirac operator $K$ is now of the Laplace type considered in previous section. Using the formulae (4.31) one can write down the heat kernels for the matter fields
\[ \text{Tr} \exp(-tK) = \int dx (4\pi t)^{-d/2} \text{tr} \left\{ \exp(-tM^2) \det \left( \frac{tR}{\sinh(tR)} \right) \frac{1}{2} \text{tr} \gamma \exp \left( \frac{1}{2} t\gamma^{\mu\nu} R_{\mu\nu} \right) \right\} \quad (5.82) \]
\[ \text{Tr} \exp(-tN) = \int dx (4\pi t)^{-d/2} \text{tr} \left\{ \exp(-tQ) \det \left( \frac{tR}{\sinh(tR)} \right) \frac{1}{2} \right\} \quad (5.82) \]
where $\mathcal{R} = \{ F_{\mu\nu}^a T_a \}$ is a matrix with vector and group indices, $\text{tr}_\gamma$ denote the trace over the spinor indices and, as above, $\text{det}$ and $\text{tr}$ mean the determinant over vector indices and the trace over the group ones.

As above the background fields lie in the Cartan subalgebra, i.e.

$$[T_a, T_b] = 0 \quad , \quad a, b = 1, \ldots, r$$

(5.83)

and, therefore, the generators $T_a$ can be diagonalized simultaneously

$$T_a = \text{diag}(i\nu_a^{(1)}, \ldots, i\nu_a^{(D)}) \quad , \quad a = 1, \ldots, r$$

(5.84)

where $\nu$ are the weights of the representation $T$, some of them being, in general, multiple or equal to zero. They can be expressed in terms of the highest weight $\mu$ and simple roots of the algebra $\lambda_i$ [64]

$$\nu = \mu - \sum_i n_i \lambda_i$$

(5.85)

where $n_i$ are some non-negative integer coefficients. The highest weight of the representation characterizes uniquely the representation and is given by integer numbers [64]

$$m_i = 2 \frac{(\lambda_i, \mu)}{(\lambda_i, \lambda_i)} \quad , \quad i = 1, \ldots, r$$

(5.86)

For example, the dimension of the representation with the highest weight $\mu$ is given by the Weyl formula

$$D = \dim T = \prod_{\alpha > 0} (\alpha, \chi)$$

$$\frac{(\alpha, \rho)}{(\alpha, \alpha)}$$

(5.87)

where

$$\chi = \mu + \frac{1}{2} \sum_{\alpha > 0} \alpha$$

(5.88)

From the normalization condition (5.77) we have, in particular,

$$\sum_{\nu \neq 0} d_\nu = D - R$$

(5.89)

$$\sum_{\nu \neq 0} d_\nu \nu = 0$$

(5.90)

$$\sum_{\nu \neq 0} d_\nu \nu_a \nu_b = \frac{D}{n} T^2 \delta_{ab}$$

(5.91)

Here and further in such formulae the summation is to be taken over all nonvanishing weights, $d_\nu$ are the multiplicities of the weights and $R$ is the number of zero weights, i.e. the multiplicity of the zero weight.
Now it is easy to obtain for the trace over the group indices of any analytic function of the matrix $R$
\[
\text{tr}_f(R) = f(0)R + \sum_{\nu \neq 0} d_{\nu} f(iF(\nu))
\]
where it is denoted $F(\nu) = \{F^a_{\mu\nu}, \nu_a\}$.

By isolating in mass matrices the singlet contributions
\[
M^2 = M^2(0) + M^2_a T_a \quad , \quad Q = Q(0) + Q^a T_a
\]
and denoting
\[
M^2(\nu) = M^2_a \nu_a \quad , \quad Q(\nu) = Q^a \nu_a
\]
we get from this for the heat kernels
\[
\text{Tr} \exp(-tK) = \int dx (4\pi t)^{-d/2} \exp(-tM^2(0)) \left\{ R2^{[d/2]} + \sum_{\nu \neq 0} d_{\nu} \exp(-itM^2(\nu)) \times \det \left( \frac{tF(\nu)}{\sin(tF(\nu))} \right)^{1/2} \text{tr}_\gamma \exp \left( \frac{i}{2} t\gamma^{\mu\nu} F_{\mu\nu}(\nu) \right) \right\}
\]
\[
\text{Tr} \exp(-tN) = \int dx (4\pi t)^{-d/2} \exp(-tQ(0)) \left\{ R + \sum_{\nu \neq 0} d_{\nu} \exp(-itQ(\nu)) \det \left( \frac{tF(\nu)}{\sin(tF(\nu))} \right)^{1/2} \right\}
\]
The determinant over vector indices is to be calculated as before with making use of (5.51) and is equal to
\[
\det \left( \frac{tF(\nu)}{\sin(tF(\nu))} \right)^{1/2} = \prod_{1 \leq i \leq q} \left( \frac{tH_i(\nu)}{\sinh(tH_i(\nu))} \right)
\]
where $H_i(\nu)$ are the invariants of the 2-form $F_{\alpha\beta}(\nu) = F^a_{\alpha\beta} \nu_a$ defined from the equations (5.49).

Finally, we calculate the trace over spinor indices. Using the representation of a 2-form $F_{\mu\nu}$ in form (5.45) and the orthogonality of the basic 2-forms $F_{(i)}$ (5.46), one can show, that matrices $\gamma^{\mu\nu} F_{(i)\mu\nu}$ at different $i$ commute
\[
\left[ \gamma^{\mu\nu} F_{(i)\mu\nu}, \gamma^{\alpha\beta} F_{(k)\alpha\beta} \right] = 0
\]
Hence
\[
\exp \left( \frac{i}{2} t\gamma^{\mu\nu} F_{\mu\nu} \right) = \prod_{1 \leq i \leq q} \exp \left( \frac{i}{2} tH_i \gamma^{\mu\nu} F_{(i)\mu\nu} \right)
\]
The tensors $F_{(i)\mu\nu}$ are simple 2-forms. This means, that they behave effectively like the two-dimensional ones. Therefore, it is valid for them, in particular, the relation
\[
F_{(i)[\mu\nu} F_{(i)\alpha\beta]} = 0 \quad (\text{no summation over } i!)
\]
Hence one can obtain using (5.47)
\[
(\gamma^{\mu\nu} F_{(i)\mu\nu})^2 = -4 \\
(\gamma^{\mu\nu} F_{(i)\mu\nu})^{2k} = (2i)^{2k} \\
(\gamma^{\mu\nu} F_{(i)\mu\nu})^{2k+1} = (2i)^{2k}\gamma^{\mu\nu} F_{(i)\mu\nu}
\]
(5.101)
i.e. for any analytic function \( f(z) \) one has
\[
f (\gamma^{\mu\nu} F_{(i)\mu\nu}) = \frac{1}{2} (f(2i) + f(-2i)) + \frac{1}{4i} (f(2i) - f(-2i)) \gamma^{\mu\nu} F_{(i)\mu\nu}
\]
(5.102)
This yields
\[
\exp \left\{ \frac{i}{2} t\gamma^{\mu\nu} F_{(i)\mu\nu} H_i \right\} = \cosh (tH_i) + \frac{i}{2} \gamma^{\mu\nu} F_{(i)\mu\nu} \sinh (tH_i)
\]
(5.103)
Finally, using again the orthogonality of the 2-forms \( F_{(i)\mu\nu} \) at different \( i \), we find
\[
\begin{align*}
\text{tr}_\gamma (\gamma^{\mu\nu} F_{(i)\mu\nu} \gamma^{\alpha\beta} F_{(k)\alpha\beta}) &= 0 \quad i \neq k \\
\text{tr}_\gamma (\gamma^{\mu\nu} F_{(i)\mu\nu} \gamma^{\alpha\beta} F_{(k)\alpha\beta} \cdots \gamma^{\rho\sigma} F_{(l)\rho\sigma}) &\neq 0 \quad \text{(only if } i = k = \cdots = l) 
\end{align*}
\]
(5.104)
Therefore, the trace over the spinor indices is completely factorized and we get
\[
\text{tr}_\gamma \exp \left\{ \frac{i}{2} t\gamma^{\mu\nu} F_{(i)\mu\nu}(\nu) \right\} = 2^{[d/2]} \prod_{1 \leq i \leq q} \cosh (tH_i(\nu))
\]
(5.105)
Taking into account (5.97) and (5.105), the final result for the heat kernel of the spinor field takes the form
\[
\text{Tr} \exp(-tK) = \int dx (4\pi t)^{-d/2} \exp(-tM^2(0)) 2^{[d/2]} \left\{ R + \sum_{\nu \neq 0} d_\nu \exp(-itM^2(\nu)) \right. \\
\times \left. \prod_{1 \leq i \leq q} (tH_i(\nu) \coth (tH_i(\nu))) \right\}
\]
(5.106)
The heat kernel for scalar operator (5.96) has the similar form
\[
\text{Tr} \exp(-tN) = \int dx (4\pi t)^{-d/2} \exp(-tQ(0)) \left\{ R + \sum_{\nu \neq 0} d_\nu \exp(-itQ(\nu)) \right. \\
\times \left. \prod_{1 \leq i \leq q} \left( \frac{tH_i(\nu)}{\sinh (tH_i(\nu))} \right) \right\}
\]
(5.107)
In contrast to the contribution of the gauge fields (5.52) and (5.53)) these heat kernels (5.106) and (5.107) have good exponentially decreasing behavior at \( t \to \infty \) (when some
positive singlet contributions in mass matrices $M_{(0)}^2$ and $Q_{(0)}$ are present). When the singlet contributions are zero or even negative then the instability appear, that leads to a reconstruction of the vacuum and to other values of the background fields ensuring the stability of the vacuum state.

The corresponding formulae for Schrödinger kernel in Minkowski space (of Lorentzian signature), when one of the invariants of the background gauge field is imaginary (5.72), have the form

$$\text{Tr} \exp(-isK) = \int dx (4\pi is)^{-d/2} \exp(-isM_{(0)}^2) 2^{[d/2]} \left\{ R + \sum_{\nu \neq 0} d_\nu \exp(sM^2(\nu)) \times sE(\nu) \coth(sE(\nu)) \prod_{1 \leq i \leq q-1} (sH_i(\nu) \cot(sH_i(\nu))) \right\}$$

(5.108)

$$\text{Tr} \exp(-isN) = \int dx (4\pi is)^{-d/2} \exp(-isQ_{(0)}) \left\{ R + \sum_{\nu \neq 0} d_\nu \exp(sQ(\nu)) \times sE(\nu) \coth(sE(\nu)) \prod_{1 \leq i \leq q-1} (sH_i(\nu) \cot(sH_i(\nu))) \right\}$$

(5.109)

It is appropriate to make here the remarks concerning the analytic continuation of the heat kernel made at the end of the subsection 5.1.

Let us now rewrite the effective action in the form

$$\Gamma^{(1)} = -\frac{1}{2} \zeta'_{tot}(0)$$

(5.110)

where

$$\zeta_{tot}(p) = \zeta_\Delta(p) - 2\zeta_F(p) - \zeta_K(p) + \zeta_N(p)$$

(5.111)

is the total $\zeta$-function.

Collecting all obtained expressions for the heat kernels it is found for the total $\zeta$–function

$$\zeta_{tot}(p) = \int dx (4\pi)^{-d/2} \frac{\mu^{2p}}{\Gamma(p)} \int_0^\infty dt t^{p-d/2-1} \left\{ r(d-2) + 2 \sum_{\alpha > 0} \prod_{1 \leq i \leq q} \left( \frac{tH_i(\alpha)}{\sinh(tH_i(\alpha))} \right) \left( d - 2 + 4 \sum_{1 \leq j \leq q} \sinh^2(tH_j(\alpha)) \right) - 2^{[d/2]} \exp(-tM_{(0)}^2) \left( R + \sum_{\nu \neq 0} d_\nu \exp(-itM^2(\nu)) \prod_{1 \leq i \leq q} (tH_i(\nu) \coth(tH_i(\nu))) \right) + \exp(-tQ_{(0)}) \left( \tilde{R} + \sum_{\tilde{\nu} \neq 0} d_{\tilde{\nu}} \exp(-itQ\tilde{\nu}) \prod_{1 \leq i \leq q} \left( \frac{tH_i(\tilde{\nu})}{\sinh(tH_i(\tilde{\nu}))} \right) \right) \right\}$$

(5.112)
where $\alpha$ are the roots of the algebra, $\nu$ and $\tilde{\nu}$ are the weights and $R$ and $\tilde{R}$ are the numbers of zero weights of the representations realized by the spinor and scalar fields respectively, $H_i(\alpha)$, $H_i(\nu)$ and $H_i(\tilde{\nu})$ are the invariants of the tensors $F_{\mu\nu}^a\alpha_a$, $F_{\mu\nu}^a\nu_a$ and $F_{\mu\nu}^a\tilde{\nu}_a$ respectively, defined from the equations of the form (5.49) or (5.49a).

After taking a concrete gauge group and the matter field representations one can obtain from here more explicit expressions for the effective action.

YI. CONCLUDING REMARKS

The present paper is, as a matter of fact, a further development of our papers [1-7]. Here we have discussed some ideas connected with the point that was left aside in previous papers, namely, the problem of calculating the low-energy limit of the effective action in quantum gravity and gauge theories and have presented an overview of recent progress on this subject [67, 68].

We have analyzed in detail the status of the low-energy limit in quantum gauge theories and quantum gravity and stressed the central role playing in low-energy calculations by the Lie group (3.65)-(3.67) that naturally appears when generalizing consistently the low-energy limit to quantum gravity.

We have proposed a number of promising, to our mind, approaches for calculation of the low-energy effective action and have calculated explicitly the low-energy effective action in flat space. In particular, a Yang-Mills model with arbitrary matter fields is considered and the problem of the stability of Savvidy-type vacuum structure [65, 66] with constant chromomagnetic fields is analyzed. It is shown that such a vacuum state can be stable only in the case when more than one constant chromomagnetic fields are present and the values of these fields differ not greatly from each other. As a consequence we concluded that this is possible only in spaces with dimensions not less than five $d \geq 5$. This fact can be useful in investigating the QCD condensates etc..

Thereby we realized the idea of partial summation of the terms without derivatives in local Schwinger - De Witt expansion for computing the effective action that was suggested in [13,5,6].

Of course, there are left many unsolved problems. We still do not know how to calculate the low-energy effective action in general case of covariantly constant curvatures (3.12). Besides, it is not perfectly clear how to do the analytical continuation of Euclidean low-energy effective action to the space of Lorentzian signature for obtaining physical results. The methods of the proof of the possibility of such a continuation [48] are based essentially on the expansion in curvatures, i.e. are valid, strictly speaking, in weak background fields and trivial topology.

ACKNOWLEDGEMENTS

I would like to thank G. A. Vilkovisky for many helpful discussions and R. Schimming and J. Eichhorn for their hospitality at the University of Greifswald.
REFERENCES

[1] I. G. Avramidi, Teor. Mat. Fiz. 79, 219 (1989)
[2] I. G. Avramidi, Yad. Fiz. 49, 1185 (1989)
[3] I. G. Avramidi, Phys. Lett. B236, 443 (1990)
[4] I. G. Avramidi, Phys. Lett. B238, 92 (1990)
[5] I. G. Avramidi, *The covariant technique for calculation of one-loop effective action*, Preprint University of Karlsruhe, KA - THEP - 2 - 1990, 33 p., (1990)
[6] I. G. Avramidi, Nucl. Phys. B355, 712 (1991)
[7] I. G. Avramidi, Yad. Fiz. 56, 245 (1993)
[8] J. S. Schwinger, Phys. Rev. 82, 664 (1951)
[9] B. S. De Witt, *Dynamical Theory of Groups and Fields* (Gordon & Breach, New York, 1965)
[10] B. S. De Witt, Phys. Rep. C19, 296 (1975)
[11] B. S. De Witt, in *General Relativity*, edited by S. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979)
[12] B. S. De Witt, in *Relativity, Groups and Topology II*, edited by B. S. De Witt and R. Stora (North Holland, Amsterdam, 1984) p. 393
[13] G. A. Vilkovisky, in *Quantum Theory of Gravity*, edited by S. Christensen (Hilger, Bristol, 1984) p. 169
[14] E. S. Fradkin and A. A. Tseytlin, Nucl. Phys. B234, 472 (1984)
[15] G. A. Vilkovisky, Class. Quant. Grav. 9, 895 (1992)
[16] I. L. Buchbinder and S. D. Odintsov, Fortschr. Phys. 37, 225 (1989)
[17] I. L. Buchbinder, S. D. Odintsov, I. L. Shapiro, Riv. Nuovo Cim. 12, 1 (1989)
[18] S. D. Odintsov, Fortschr. Phys. 38, 371 (1990)
[19] J. A. Zuck, Phys. Rev. D33, 3645 (1986)
[20] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982)
[21] A. O. Barvinsky and G. A. Vilkovisky, Phys. Rep. C119, 1 (1985)
[22] G. A. Vilkovisky, *Heat Kernel: Recontre Entre Physiciens et Mathematiciens*, Preprint CERN-TH.6392/92 (1992), to be published in Proceedings of Strasbourg Meeting between Physicists and Mathematicians (1991), Inst. de Recherche Math. Avancee, Univ. Louis Pasteur, Strasbourg
[23] J. Hadamard, *Lectures on Cauchy’s Problem*, in *Linear Partial Differential Equations* (Yale University Press, New Haven, 1923)
[24] S. Minakshisundaram and A. Pleijel, Can. J. Math. 1, 242 (1949)
[25] H. P. McKean and I. M. Singer, J Diff. Geom. 1, 43 (1967)
[26] R. T. Seely, Proc. Symp. Pure Math. 10, 288 (1967)
[27] M. Atiyah, R. Bott and V. K. Patodi, Invent. Math. 19, 279 (1973)
[28] P. B. Gilkey, J. Diff. Geom. 10, 601 (1975)
[29] P. B. Gilkey, Proc. Symp. Pure Math. 27, 265 (1975)
[30] P. B. Gilkey, Compositio Math. 38, 201 (1979)
[31] P. B. Gilkey, *Invariance Theory, the Heat Equation and the Atiyah - Singer Index Theorem* (Publish or Perish, Wilmington, Delaware (USA), 1984)
[32] R. Schimming, Beitr. Anal. 15, 77 (1981)
[33] R. Schimming, Math. Nachr. 148, 145 (1990)
[34] R. Schimming, Calculation of the Heat Kernel Coefficients, in B. Riemann Memorial Volume, edited by T. M. Rassias, (World Scientific, Singapore, to be published)
[35] S. A. Fulling, SIAM J. Math. Anal. 13 (1982) 891
[36] S. A. Fulling and G. Kennedy, in The Physics of Phase Space, (Lecture Notes in Physics 278), edited by Y. S. Kim and W. W. Zachary, (Springer, Berlin, 1986), p. 407
[37] S. A. Fulling and G. Kennedy, in Differential Equations and Mathematical Physics, (Lecture Notes in Mathematics 1285), edited by I. W. Knowles and Y. Saito (Springer, Berlin, 1987), p. 126
[38] S. A. Fulling and G. Kennedy, Transact. Amer. Math. Soc. 310, 583 (1988)
[39] S. A. Fulling, J. Symbolic Comput. 9, 73 (1990)
[40] F. H. Molzahn, T. A. Osborn and S. A. Fulling, Ann. Phys. (USA) 204, 64 (1990)
[41] H. Widom, Bull. Sci. Math. 104, 19 (1980)
[42] V. P. Gusynin, Phys. Lett. B255, 233 (1989)
[43] V. P. Gusynin, Nucl. Phys. B333, 296 (1990)
[44] V. P. Gusynin and V. A. Kushnir, Class. Quant. Grav. 8, 279 (1991)
[45] P. Amsterdamski, A. L. Berkin and D. J. O’Connor, Class. Quant. Grav. 6, 1981 (1989)
[46] A. E. M. Van de Ven, Nucl. Phys. B250, 593 (1985)
[47] V. P. Frolov and A. I. Zelnikov, Phys. Rev. D35, 3031 (1987)
[48] A. O. Barvinsky and G. A. Vilkovisky, Nucl. Phys. B282, 163 (1987)
[49] A. O. Barvinsky and G. A. Vilkovisky, Nucl. Phys. B333, 471 (1990)
[50] A. A. Ostrovsky and G. A. Vilkovisky, J. Math. Phys. 29, 702 (1988)
[51] A. O. Barvinsky, Yu. V. Gusev, V. V. Zhytnikov and G. A. Vilkovisky, Covariant Perturbation Theory (IY), Preprint University of Winnipeg (1993)
[52] G. A. Vilkovisky, Quantum Theory of Gravitational Collapse, Lectures at the University of Texas at Austin (1989); (unpublished)
[53] B. S. De Witt and R. W. Brehme, Ann. Physics 9, 220 (1960)
[54] T. P. Branson and P. B. Gilkey, Comm. Part. Diff. Eq. 15, 245 (1990)
[55] M. van den Berg and P. B. Gilkey, Heat content asymptotics of a Riemannian manifold with boundary, Preprint University of Oregon (1992) (to be published)
[56] S. Desjardins and P. B. Gilkey, Heat content asymptotics for operators of Laplace type with Neumann boundary conditions, Preprint (1992) (to be published)
[57] M. van den Berg, S. Desjardins and P. B. Gilkey, Functorality and heat content asymptotics for operators of Laplace type, Preprint (1992)
[58] D. M. Mc Avity and H. Osborn, Class. Quant. Grav. 8, 603 (1991)
[59] D. M. Mc Avity, Class. Quant. Grav. 9, 1983 (1992)
[60] T. Branson, P. B. Gilkey and B. Ørsted, Proc. Amer. Math. Soc. 109, 437 (1990)
[61] B. F. Dubrovin, S. P. Novikov and A. T. Fomenko, The Modern Geometry: Methods and Applications (Nauka, Moscow, 1979)
[62] J. L. Synge, Relativity. The General Theory (North-Holland, Amsterdam, 1960)
[63] B. S. De Witt, in *Quantum Field Theory and Quantum Statistics*, vol. 1, edited by I. A. Batalin, C. J. Isham and G. A. Vilkovisky (Hilger, Bristol, 1987) p. 191

[64] A. O. Barut and R. Raczka, *Theory of Group Representations and Applications*, (PWN-Polish Sci. Publ., Warszawa, 1977)

[65] G. K. Savvidy, Phys. Lett. **B71**, 133 (1977)

[66] N. K. Nielsen and P. Olesen, Nucl. Phys. **B144**, 376 (1978)

[67] I. G. Avramidi, Phys. Lett. **B305**, 27 (1993)

[68] I. G. Avramidi, Covariant algebraic calculation of the one-loop effective potential in non-Abelian gauge theory and a new approach to stability problem, Preprint University Greifswald (1994), (to be published)