Artificial Neural Network forecasting for monomorphic and polymorphic uncertainty models and comparison with experimental investigations

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Tools from probability or possibility in terms of fuzzy sets are usable for the description and quantification of uncertainties within numerical simulations. In this work, different monomorphic and polymorphic uncertainty models are applied on linear elastostatic structures with non-periodic perforations in order to analyze the individual expressiveness. The first principal stress is used as an indicator for structural failure which is evaluated and classified. Artificial Neural Networks are used as surrogate model to compare and assess the uncertainty models with regard to the numerical predictions. Real experiments of perforated plates under uniaxial tension are validated with the help of various uncertainty models.

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1 Introduction

This study has been performed within the research project MuScaBluDes – "Multi-scale failure analysis with polymorphic uncertainties for optimal design of rotor blades" – which is a part of the DFG Priority Programme (SPP 1886) "Polymorphic uncertainty modelling for the numerical design of structures". Unavoidable uncertainties due to natural variability, inaccuracies, imperfections or lack of knowledge are always present in real world problems like air void inclusions in materials. They can influence the structural integrity significantly and initiate the overall structural failure. In this context, a parametric model for perforated bodies

2 Parametric elastostatic model for perforated bodies

Consider a linear elastostatic problem

\[
\begin{align*}
\mathbf{f} &= -\text{div}\mathbf{\sigma} & \text{equilibrium equation} \\
\mathbf{e} &= \frac{\nabla \mathbf{u} + \nabla^T \mathbf{u}}{2} & \text{strain-displacement equation} \\
\mathbf{\sigma} &= \mathbf{C} : \mathbf{\epsilon} & \text{constitutive equation} \\
\mathbf{u} &= 0 & \text{Dirichlet boundary conditions} \\
\mathbf{\sigma} \cdot \mathbf{n} &= \mathbf{h} & \text{Neumann boundary conditions} \\
\mathbf{\sigma} \cdot \mathbf{n} &= 0 & \text{Neumann boundary conditions}
\end{align*}
\]

in \( D_p \),

\[
\left\{ \begin{array}{ll}
\mathbf{f} &= -\text{div}\mathbf{\sigma} & \text{equilibrium equation} \\
\mathbf{e} &= \frac{\nabla \mathbf{u} + \nabla^T \mathbf{u}}{2} & \text{strain-displacement equation} \\
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\mathbf{\sigma} \cdot \mathbf{n} &= \mathbf{h} & \text{Neumann boundary conditions} \\
\mathbf{\sigma} \cdot \mathbf{n} &= 0 & \text{Neumann boundary conditions}
\end{array} \right. \text{ on } \Gamma_0 \subset \partial D, \\
\mathbf{\sigma} \cdot \mathbf{n} &= \mathbf{h} & \text{Neumann boundary conditions} \text{ on } \Gamma_{\sigma,0} := \partial \cup_{P \in \mathcal{P}} P, \quad (\mathbf{h} \in \mathcal{H}_0^1(D_p))^d \text{ with } \mathbf{h} \text{ being in } C^1 \text{ in a neighbourhood of each perforation } P_k.
\]

defined on a perforated reference body denoted by \( D_p \subset D \subset \mathbb{R}^2 \) with \( D \) being a non-perforated Lipschitz domain. The set \( \mathcal{P} = \mathcal{P}(p_{\text{geo}}) := \{ P_k = P_k(p_k) \subset D, \ k = 1, \ldots, N < \infty \} \) is a collection of arbitrarily shaped voids \( P_k \), each being characterized by a parameter \( p_k \in \mathbb{R}^{M_k} \) with \( M_k \in \mathbb{N} \cup \{ \infty \} \) merged into \( p_{\text{geo}} = (p_k)_{k=1}^N \), defining \( D_p = D \setminus \mathcal{P} \). The maximum ranges of the parameters \( p_k \) are further constrained, s.t. for \( |\Gamma_0| > 0 \) and \( C, f, h \) and \( \mathcal{P} \) being regular enough, the LAX-MILGRAM theorem applies to ensure existence and uniqueness of a (weak) solution \( \mathbf{u} = \mathbf{u}(p) \in \mathcal{V} := H^1_{\Gamma_0}(D_p)^d \) with \( u \) being in \( C^1 \) in a neighbourhood of each perforation \( P_k \).

For the application in mind the 2D domain is rectangular with \( D = [0, L] \times [0, W], \Gamma_0 = \{ 0 \} \times [0, W], \ f \equiv 0 \) and

\[
h = \left\{ \begin{array}{ll}
Q[\alpha_F]e_1, & \text{on } \Gamma^Y_{\sigma} := \{ L \} \times [0, W] \subset \Gamma_{\sigma}, \\
0, & \text{on } \Gamma_{\sigma} \setminus \Gamma^Y_{\sigma},
\end{array} \right.
\]

with a rotation matrix \( Q[\alpha_F] \) with load angle \( \alpha_F \) and \( e_1 = (1, 0)^T \), see Fig. 1. The material is considered as isotropic with given YOUNG’s modulus \( E \) and POISSON’s ratio \( \nu \). The overall parameter dependence of (1) is given as \( p := (p_{\text{geo}}, \alpha_F) \).

For a qualitative and quantitative comparison with experimentally determined failure mechanisms and ultimate loads the computation of the maximum value of the first principal stress in \( D_p \) and of its location is of interest:

\[
\sigma_{1,\text{max}} = \sigma_{1,\text{max}}(p) = \text{ess sup}_{\mathbf{u} \in D_p} \lambda_{\text{max}}(\mathbf{\sigma}^{\mathbf{u}}(x)) \quad \text{and} \quad x_{\text{max}} = x_{\text{max}}(p) = \text{arg max}_{\mathbf{u} \in D_p} \sigma_{1,\text{max}}.
\]

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The ultimate load for structural failure is calculated by $F_{\text{max}} = (\sigma_N/\sigma_{1,\text{max}}) f_t W T \in [0; f_t W T]$ with loading $\sigma_N = \|h\|_2$, tensile strength $f_t$ and thickness $T \ll L, W$. Finally, the first crack location $x_{\text{max}}$ is classified by

$$C(p) = C(x_{\text{max}}) = \begin{cases} 
\text{above perforation } P_k, & \text{ if } x_{\text{max}} \text{ is associated and located above perforation } P_k, \\
\text{below perforation } P_k, & \text{ if } x_{\text{max}} \text{ is associated and located below perforation } P_k.
\end{cases} \quad (4)$$

Note that in the prescribed model the value of $F_{\text{max}}$ is independent of the \textsc{Young’s} modulus $E$ and the \textsc{Poisson’s} ratio $\nu$ and linear proportional to the tensile strength $f_t$. The classifier $C$ is always independent of $E$, $\nu$ and $f_t$, due to the assumption of spatially constant values.

### 3 Experimental investigations

For the experimental investigations of perforated structures as described above, ten specimens made of Plexiglas® - XT transparent – [1] have been manufactured under equivalent conditions. Linear-elastic material behavior is assumed until the maximum first principal stress $\sigma_{1,\text{max}}$ reaches the tensile strength $f_t$. Then, brittle behavior immediately leads to structural failure. The material parameters $E = 3300\text{MN/m}^2$, $\nu = 0.37$ and $f_t = 64\text{MN/m}^2$ have been determined in standardized testing according to [2], see [3] for more details. The specimens ($L \times W \times T = 150\text{mm} \times 50\text{mm} \times 4\text{mm}$) have been loaded by uniaxial tension as can be seen in Fig. 2 with the associated ultimate loads $F_{\text{max}}$ and crack patterns that are associated with different failure mechanisms (FM). The domain $D$ is perforated by a circular hole $P_1$, by a rotated slotted hole $P_2$ and by $P_3$ as an overlapping of two circular holes. The failure mechanisms of the ten specimens are very similar with a crack through $P_3$. The ultimate loads are in a range of $F_{\text{exp}} \in [1708\text{N}, 2440\text{N}]$ and seem to be sensitive to changes of the parameters $p$. To investigate the impact on the structural failure numerically, small ranges for the perforation parameters $p_{\text{geo}}$ as well as for the load angle $\alpha_F$ have been defined in Table 1.

| $p_1 = (x_1, y_1, d_1)$ | $p_1 \in [61, 65] \times [38, 42] \times [9.5, 10.5]$ |
|-------------------------|--------------------------------------------------|
| $p_2 = (x_2, y_2, l_2, w_2, \theta_2)$ | $p_2 \in [67, 71] \times [11, 15] \times [9, 13] \times [4.5, 5.5] \times [28, 32]$ |
| $p_3 = (x_3, y_3, d_3, x_4, y_4, d_4)$ | $p_3 \in [80, 84] \times [32, 36] \times [13.5, 14.5] \times [85, 89] \times [26, 30] \times [9.5, 10.5]$ |
| $\alpha_F \in [-2, 2]$ | $\alpha_F \in [-2, 2]$ |

Fig. 2: Specimen before and after testing.
4 Monomorphic and polymorphic uncertainty models

In order to explain the different outcomes of structural failure in the experiments of Section 3, this section is focused on modeling uncertainties of the underlying linear elastostatic model (1). The nature of uncertainty arising for this problem alone can be arbitrarily complex, e.g. due to the fine scale structure of the Plexiglas® plate $D$, shape, location and size of perforations, (microscale) material properties or description of source and force terms $f$ and $h$.

In what follows, the modeling of uncertainties is simplified and restricted to the location and size of perforations and the force term $h$. In particular the overall shape of the perforations, the material tensor $C$, the source term $f$ and the unperforated domain $D$ remain fixed. The applied monomorphic uncertainty models involve probabilistic (stochastic) or possibilistic (in terms of fuzzy arithmetic) descriptions, respectively. The polymorphic uncertainty model is employed in the sense of a hybrid fuzzy-stochastic framework. From this viewpoint $p$ can be seen as a realization of a random variable, a sample in the support of a fuzzy set $\tilde{p}$ or a combination of both, i.e. a fuzzy-random variable. For simplification the uncertain parameter $p$ is characterized by independent uniform random variables $\sim U(l, r)$ or non-interactive triangular fuzzy numbers $(l, (l + r)/2, r)$ leading to the tensorized image range specified in Table 1. For details on the propagation of monomorphic or polymorphic input, the authors refer to [3] and references therein.

Note that there is a direct influence of $p$ on the outcome of the maximum first principal stress (3) and the location classifier (4). The evaluation of these maps involves a computation of a (approximate) solution of the underlying elastostatic problem (1). This becomes the most expensive aspect in the uncertainty propagation and needs to construct surrogate maps.

In our approach surrogates based on Artifical Neural Networks (ANN) have been constructed for predicting the maps in (3) and (4). The used feed forward Neural Network topology is illustrated in Fig. 3. These structures yield promising results as demonstrated in Section 5. The degrees of freedom $(W_l^j, b^j)$ of the ANN are trained by minimizing the mean square error (MSE) for $\sigma_{1,\text{max}} (p)$ and the categorical cross entropy (CE) loss for $C(p)$ based on $N = N_{\text{train}} + N_{\text{test}} = 7500 + 2500 = 10000$ uniform distributed data samples. Let $p \in \mathbb{R}^M, M = 15$, see Table 1. Then, all considered dense feed-forward structures are chosen, s.t. the first hidden layer contains $n_1 = M$ neurons. This is motivated by the idea that the ANN may be able to reorder the input parameters in a non-linear manner to improve the approximation quality. Successively each follow up hidden layer consists of a not larger number of neurons, i.e. $n_1 \geq n_2 \geq \ldots \geq n_H$. Within the optimization a training loss and validation loss of about 2% and 3% and a training loss and validation accuracy of about 1.8% and 96% have been achieved for the ANN surrogate for $p \mapsto \sigma_{1,\text{max}}(p)$ and $p \mapsto C(p)$, respectively. Details on the explicit construction and training for the present application are given in [3].

![Fig. 3: Schematic dense feed forward network used as surrogate in Section 5 with decreasing number of neurons inside the hidden layers.](image-url)

5 Numerical investigations and comparison

Regarding the experimental investigations of perforated structures described in Section 3, uncertainties are identified during the fabrication and the experiments. Manufacturing inaccuracies and imperfections influence the locations and sizes of the perforations while the experimental conditions lead to a scattering load angle.

5.1 Monomorphic uncertainty models

For both, the probabilistic and the possibilistic approach, $N = 10^6$ surrogate forecasts are obtained from the trained ANNs. Within the fuzzy set propagation in addition to the deterministic case in the core for $\mu = 1.0$, ten equidistantly distributed $\alpha$-cuts are used to ensure a homogeneous distribution of the samples over the height of the membership function associated to the output fuzzy set. The corresponding numerical outputs are displayed in Fig. 4. For both uncertainty models, the numerical results cover the range of experimental ones even though a shift is recognizable. The empirical distribution and the empirical membership function are tilted to the left, leading to the conclusion that some further refinement of the uncertainty description has to be taken into account additionally which reduces the ultimate load $F_{\text{max}}^\text{max}$. A benefit of the probabilistic approach is the
Fig. 4: Monomorphic outputs (left: empirical distribution; right: empirical membership function).

direct visibility of partial probabilities of different locations of the maximum first principal stress, e.g. within a certain range of ultimate loads. In the probabilistic approach, the different locations depend on the output membership function value, thus they can be related to the range of input uncertainties.

5.2 Polymorphic uncertainty model

In the polymorphic approach, stochastic and fuzzy variables are applied simultaneously. The size parameters $d_i$ for $i = 1, 3, 4, l_2$ and $w_2$ are associated to non-interactive triangular fuzzy numbers, while the remaining variables are modeled as independent uniform distributed random variables. Monotonicity of $F_{\text{max}}$ with regard to the size parameters is presumed which enables the reduced transformation method [4] for the analysis in the fuzzy space. In the stochastic space, $N = 10^6$ surrogate forecasts are used like in Section 5.1. The resulting fuzzy-stochastic output is displayed in Fig. 5. The simultaneous information about probabilities as well as possibilities (in terms of fuzzy sets) of the location of the maximum first principal stress concerning the ultimate load $F_{\text{max}}$ and the input uncertainties $p_i$ is an added value of the polymorphic approach. The combination of stochastic and fuzzy variables allows the output evaluation separately on each stochastic sample regarding the fuzzy input parameters (size) and on each fuzzy sample regarding the stochastic input parameters (location and load angle), see [3] for more details.

Acknowledgements  The authors gratefully acknowledge financial support of the German Research Foundation (DFG) within the Research Project MuScaBlaDes within SPP1886.

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