UNIQUE RANGE SETS OF MEROMORPHIC FUNCTIONS OF NON-INTEGER FINITE ORDER

BIKASH CHAKRABORTY¹, AMIT KUMAR PAL², SUDIP SAHA³ AND JAYANTA KAMILA⁴

Abstract. This paper studies the uniqueness of two meromorphic functions with finitely many poles and finite non-integral order when they share two finite sets. The motivation for writing this paper is the articles due to Sahoo and Sarkar (Bol. Soc. Mat. Mex., DoI: 10.1007/s40590-019-00260-4) and Chen (Georgian Math. J., DoI. 10.1515/gmj-2019-2073).

1. Introduction

We use $M(\mathbb{C})$ to denote the field of all meromorphic functions in $\mathbb{C}$. Also, by $M_1(\mathbb{C})$, we denote the class of meromorphic functions which have finitely many poles in $\mathbb{C}$. The order $\rho(f)$, $f \in M(\mathbb{C})$ is defined as

$$\rho(f) = \limsup_{r \to \infty} \frac{\ln T(r, f)}{\ln r}.$$ 

Let $S \subset \mathbb{C} \cup \{\infty\}$ be a non-empty set with distinct elements and $f \in M(\mathbb{C})$. We set

$$E_f(S) = \bigcup_{a \in S} \{z : f(z) - a = 0\},$$

where a zero of $f - a$ with multiplicity $m$ counts $m$ times in $E_f(S)$. Let $\overline{E}_f(S)$ denote the collection of distinct elements in $E_f(S)$.

Let $g \in M(\mathbb{C})$. We say that two functions $f$ and $g$ share the set $S$ CM (resp. IM) if $E_f(S) = E_g(S)$ (resp. $\overline{E}_f(S) = \overline{E}_g(S)$).

F. Gross (⁶) first studied the uniqueness problem of meromorphic functions that share distinct sets instead of values. From then, the uniqueness theory of meromorphic functions under set sharing environment has become one of the important branch in the value distribution theory.

In 1977, F. Gross (⁶) proved that there exist three finite sets $S_j$ $(j = 1, 2, 3)$ such that if two non-constant entire functions $f$ and $g$ share them, then $f \equiv g$. In the same paper, he asked the following question:

**Question 1.1.** Can one find two (or possible even one) finite set $S_j$ $(j = 1, 2)$ such that if two non-constant entire functions $f$ and $g$ share them, then $f \equiv g$?

In connection to the Gross question, H. X. Yi (⁷) proved the following theorem, which answered the Question 1.1 in the affirmative.

**Theorem A.** Let $S_1 = \{z : z^n - 1 = 0\}$ and $S_2 = \{a\}$, where $n \geq 5$, $a \neq 0$ and $a^{2n} \neq 1$. If $f$ and $g$ are entire functions such that $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$, then $f \equiv g$. 

2010 Mathematics Subject Classification: 30D30, 30D20, 30D35.

Key words and phrases: Unique range set, Weighted Sharing, Order.

Corresponding Author: Bikash Chakraborty
Later in 1998, the same author ([12]) proved the following theorem:

**Theorem B.** Let \( S_1 = \{0\} \) and \( S_2 = \{z : z^2(z + a) - b = 0\} \), where \( a \) and \( b \) are two nonzero constants such that the algebraic equation \( z^2(z + a) - b = 0 \) has no multiple roots. If \( f \) and \( g \) are two entire functions satisfying \( E_f(S_j) = E_g(S_j) \) for \( j = 1, 2 \), then \( f \equiv g \).

In the same paper ([12]), the author proved the following theorem also:

**Theorem C.** If \( S_1 \) and \( S_2 \) are two sets of finite distinct complex numbers such that any two entire functions \( f \) and \( g \) satisfying \( E_f(S_j) = E_g(S_j) \) for \( j = 1, 2 \), must be identical, then \( \max\{\sharp(S_1), \sharp(S_2)\} \geq 3 \), where \( \sharp(S) \) denotes the cardinality of the set \( S \).

Thus for the uniqueness of two entire functions when they share two sets, it is clear that the smallest cardinalities of \( S_1 \) and \( S_2 \) are 1 and 3 respectively.

For the uniqueness of two meromorphic functions when they share two sets, H. X. Yi ([11]) completely answered the question of Gross as follows:

**Theorem D.** If \( S_1 = \{a + b, a + ba, \ldots, a + ba^{n-1}\} \) and \( S_2 = \{c_1, c_2\} \) where \( \omega = e^{2\pi i} \) and \( b \neq 0 \), \( c_1 \neq a \), \( c_2 \neq a \), \( (c_1 - a)^n \neq (c_2 - a)^n \), \( (c_1 - a)(c_2 - a)^n \neq b^{2n} \) \((k, j = 1, 2)\) are constants. If two nonconstant meromorphic functions \( f \) and \( g \) share \( S_1 \) CM, \( S_2 \) IM, and if \( n \geq 9 \), then \( f \equiv g \).

In 2012, B. Yi and Y. H. Li ([13]) improved the above theorem as:

**Theorem E.** If \( S_1 = \{z : (n-1)(n-2)z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2}z^{n-2} + 1 = 0\} \) where \( n \geq 5 \) is an integer and \( S_2 = \{0, 1\} \). If two nonconstant meromorphic functions \( f \) and \( g \) share \( S_1 \) CM, \( S_2 \) CM, then \( f \equiv g \).

But recently, J. F. Chen improved Theorems D and E for a particular class of meromorphic functions as:

**Theorem F.** ([11]) If \( S_1 = \{\alpha\} \) and \( S_2 = \{\beta_1, \beta_2\} \), where \( \alpha, \beta_1, \beta_2 \) are distinct finite complex numbers satisfying

\[
(\beta_1 - \alpha)^2 \neq (\beta_2 - \alpha)^2.
\]

If two nonconstant meromorphic functions \( f \) and \( g \) in \( M_1(\mathbb{C}) \) share \( S_1 \) CM, \( S_2 \) IM, and if the order of \( f \) is neither an integer nor infinite, then \( f \equiv g \).

**Theorem G.** ([11]) If \( S_1 = \{\alpha\} \) and \( S_2 = \{\beta_1, \beta_2\} \), where \( \alpha, \beta_1, \beta_2 \) are distinct finite complex numbers satisfying

\[
(\beta_1 - \alpha)^2 \neq (\beta_2 - \alpha)^2.
\]

If two nonconstant meromorphic functions \( f \) and \( g \) in \( M_1(\mathbb{C}) \) share \( S_1 \) IM, \( S_2 \) CM, and if the order of \( f \) is neither an integer nor infinite, then \( f \equiv g \).

A recent advent in the uniqueness theory of meromorphic functions is the introduction of the notion of weighted sharing instead CM sharing.

**Definition 1.1.** ([7]) Let \( l \) be a non-negative integer or infinity. For \( a \in \mathbb{C} \cup \{\infty\} \), we denote by \( E_l(a; f) \), the set of all \( a \)-points of \( f \), where an \( a \)-point of multiplicity \( m \) is counted \( m \) times if \( m \leq l \) and \( l + 1 \) times if \( m > l \).

If for two meromorphic functions \( f \) and \( g \), we have \( E_l(a; f) = E_l(a; g) \), then we say that \( f \) and \( g \) share the value \( a \) with weight \( l \).

The IM and CM sharing respectively correspond to weight 0 and \( \infty \).
Remark 1.1. Let $f$ be a nonnegative integer or infinity. Clearly $E_f(S) = E_f(S, \infty)$.

We say that $f$ and $g$ share $S$ with weight $l$, or simply $f$ and $g$ share $(S, l)$ if $E_f(S, l) = E_g(S, l)$.

Regarding Theorems F and G, it is natural to ask the following questions:

**Question 1.2.** It seems reasonable to conjecture that Theorems F and G still hold if $f(z)$ and $g(z)$ share $(S₁, 2)$ (or, possibly, $(S₁, 0))$ and $S₂$ IM.

In this connection, the next two theorems are given by P. Sahoo and H. Karmakar ([8]; and P. Sahoo and A. Sarkar ([9])). Before going to state their results, we need to recall a definition.

**Definition 1.3.** ([8] [9]) Let $n$ be a positive integer and $S₁ = \{α₁, α₂, \ldots, αₙ\}$, where $αᵢ$’s are nonzero complex constants. Suppose that

$$P(z) = \frac{z^n - (\sum αᵢ)z^{n-1} + \ldots + (-1)^{n-1}(\sum αᵢαᵢ₊₁ \ldots αᵢ₊ₙ-₁)z}{(-1)^{n+1}α₁α₂ \ldots αₙ}. \tag{1.1}$$

Let $m₁$ be the number of simple zeros of $P(z)$ and $m₂$ be the number of multiple zeros of $P(z)$. Then we define $Γ₁ := m₁ + m₂$ and $Γ₂ := m₁ + 2m₂$.

**Definition 1.4.** Let

$$Q(z) := (z - α₁)(z - α₂) \ldots (z - αₙ).$$

Then $Q(z) = (-1)^{n+1}α₁α₂ \ldots αₙ \{P(z) - 1\}$ and $Q'(z) = (-1)^{n+1}α₁α₂ \ldots αₙP'(z)$.

**Theorem H.** ([8]) Let $f, g \in M₁(\mathbb{C})$ and $S₁ = \{α₁, α₂, \ldots, αₙ\}$, $S₂ = \{β₁, β₂\}$, where $α₁, α₂, \ldots, αₙ, β₁, β₂$ are $n + 2$ distinct nonzero complex constants satisfying $n > 2Γ₂$. If $f$ and $g$ share $(S₁, 2)$ and $S₂$ IM, then $f \equiv g$, provided

$$(β₁ - α₁)^2(β₁ - α₂)^2 \ldots (β₁ - αₙ)^2 \neq (β₂ - α₁)^2(β₂ - α₂)^2 \ldots (β₂ - αₙ)^2$$

and $f$ is of non-integer finite order.

**Theorem I.** ([9]) Let $f, g \in M₁(\mathbb{C})$ and $S₁ = \{α₁, α₂, \ldots, αₙ\}$, $S₂ = \{β₁, β₂\}$, where $α₁, α₂, \ldots, αₙ, β₁, β₂$ are $n + 2$ distinct nonzero complex constants satisfying $n > 2Γ₂ + 3Γ₁$. If $f$ and $g$ share $S₁$ and $S₂$ IM, then $f \equiv g$, provided

$$(β₁ - α₁)^2(β₁ - α₂)^2 \ldots (β₁ - αₙ)^2 \neq (β₂ - α₁)^2(β₂ - α₂)^2 \ldots (β₂ - αₙ)^2$$

and $f$ is of non-integer finite order.

**Remark 1.1.** The proofs of the Theorems H and I are entirely lies on the following two inequalities ([8] [9]):

$$N₂(r, 0; P(f)) \leq Γ₂\overline{N}(r, 0; f),$$

$$\overline{N}(r, 0; P(f)) \leq Γ₁\overline{N}(r, 0; f),$$

where $N₂(r, 0; f) := \overline{N}(r, 0; f) + \overline{N}(r, 0; f) \geq 2$ and $\overline{N}(r, 0; f) \geq 2$ is the reduced counting function of those zeros of $f$ whose multiplicities are not less than 2.
Remark 1.2. Now, we consider one counter example. It is known from (H) that the polynomial \( \frac{(n-1)(n-2)}{2} z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2} z^{n-2} + c \), where \( c(\neq 0, 1) \in \mathbb{C}, n(\geq 3) \in \mathbb{N} \), has \( n \) distinct zeros. Let

\[
\mathcal{P}(z) = \frac{(n-1)(n-2)}{2} z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2} z^{n-2},
\]

then

\[
\mathcal{P}(z) = \frac{(n-1)(n-2)}{2} z^{n-2}(z - \gamma_1)(z - \gamma_2),
\]

where \( \gamma_1, \gamma_2 \) are the zeros of \( z^2 - \frac{2n}{n-1} z + \frac{n}{n-2} = 0 \). Here, we observed that the zeros of \( \mathcal{P}(f) \) may come from \( \gamma_i \)-points of \( f \).

The above observations are the motivations of the following theorems.

### 2. Main Results

Let \( n \) be a positive integer and \( S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \), where \( \alpha_i \)'s are nonzero complex constants. Suppose that

\[
(2.1) \quad \mathcal{P}(z) = \frac{z^n - (\sum \alpha_i)z^{n-1} + \ldots + (-1)^{n-1}(\sum \alpha_i \alpha_{i_2} \ldots \alpha_{i_{n-1}})z}{(-1)^{n+1} \alpha_1 \alpha_2 \ldots \alpha_n}.
\]

Let \( m_1 \) be the number of simple zeros of \( \mathcal{P}(z) \) and \( m_2 \) be the number of multiple zeros of \( \mathcal{P}(z) \). Then we define \( \Gamma_1 := m_1 + m_2 \) and \( \Gamma_2 := m_1 + 2m_2 \). Further suppose that \( \mathcal{P}'(z) \) has \( k \)-distinct zeros.

**Theorem 2.1.** Let \( f, g \in M_1(\mathbb{C}) \) and \( S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}, S_2 = \{\beta_1, \beta_2\} \), where \( \alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2 \) are \( n + 2 \) distinct nonzero complex constants satisfying \( n > \max\{2k + 2, 2\Gamma_1\} \), where \( k \geq 2 \) and \( \Gamma_1 \geq 3 \). If \( f \) and \( g \) share \( (S_1, 2) \) and \( S_2 \) IM, then \( f \equiv g \), provided

\[
(\beta_1 - \alpha_1)(\beta_1 - \alpha_2) \ldots (\beta_1 - \alpha_n) \neq (\beta_2 - \alpha_1)(\beta_2 - \alpha_2) \ldots (\beta_2 - \alpha_n)
\]

and \( f \) is of non-integer finite order.

**Theorem 2.2.** Let \( f, g \in M_1(\mathbb{C}) \) and \( S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}, S_2 = \{\beta_1, \beta_2\} \), where \( \alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2 \) are \( n + 2 \) distinct nonzero complex constants satisfying \( n > \max\{2k + 5, 2\Gamma_1\} \), where \( k \geq 2 \) and \( \Gamma_1 \geq 3 \). If \( f \) and \( g \) share \( S_1 \) and \( S_2 \) IM, then \( f \equiv g \), provided

\[
(\beta_1 - \alpha_1)(\beta_1 - \alpha_2) \ldots (\beta_1 - \alpha_n) \neq (\beta_2 - \alpha_1)(\beta_2 - \alpha_2) \ldots (\beta_2 - \alpha_n)
\]

and \( f \) is of non-integer finite order.

In Theorem 2.1, one can observe that the elements of \( S_2 \) are the zeros of \( \mathcal{P}'(z) \), where

\[
\mathcal{P}(z) = \frac{(n-1)(n-2)}{2} z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2} z^{n-2} + 1.
\]

This observation motivate us to write the next two theorems.

**Theorem 2.3.** Let \( f, g \in M_1(\mathbb{C}) \) and \( S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}, S_2 = \{\beta_1, \beta_2\} \), where \( \alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2 \) are \( n + 2 \) distinct nonzero complex constants satisfying \( n > \max\{4, 2\Gamma_1\} \), and \( \mathcal{P}'(z) \) has exactly two distinct zeros \( \beta_1 \) and \( \beta_2 \) and \( \Gamma_1 \geq 3 \). If \( f \) and \( g \) share \( (S_1, 2) \) and \( S_2 \) IM, then \( f \equiv g \), provided

\[
(\beta_1 - \alpha_1)(\beta_1 - \alpha_2) \ldots (\beta_1 - \alpha_n) \neq (\beta_2 - \alpha_1)(\beta_2 - \alpha_2) \ldots (\beta_2 - \alpha_n)
\]

and \( f \) is of non-integer finite order.
Theorem 2.4. Let \( f, g \in M_1(\mathbb{C}) \) and \( S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}, S_2 = \{\beta_1, \beta_2\}, \) where \( \alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2 \) are \( n + 2 \) distinct nonzero complex constants satisfying \( n > \max\{7, 2\Gamma_1\} \), and \( P'(z) \) has exactly two distinct zeros \( \beta_1 \) and \( \beta_2 \) and \( \Gamma_1 \geq 3 \). If \( f \) and \( g \) share \( S_1 \) and \( S_2 \) IM, then \( f \equiv g \), provided
\[
(\beta_1 - \alpha_1)(\beta_1 - \alpha_2) \ldots (\beta_1 - \alpha_n) \neq (\beta_2 - \alpha_1)(\beta_2 - \alpha_2) \ldots (\beta_2 - \alpha_n)
\]
and \( f \) is of non-integer finite order.

3. Necessary Lemmas

Lemma 3.1. ([1], [2]) Let \( f, g \in M(\mathbb{C}) \) and \( f \) and \( g \) share the set \( \{\beta_1, \beta_2\} \) IM, where \( \beta_1 \neq \beta_2 \) and \( \beta_1, \beta_2 \in \mathbb{C} \). Then \( \rho(f) = \rho(g) \).

Lemma 3.2. Let \( f, g \) be two non constant meromorphic functions and \( a_1, a_2 \) be two distinct finite complex numbers. If \( f \) and \( g \) share \( a_1, a_2 \) and \( \infty \) CM, then \( f \equiv g \), provided that \( f \) is of non-integer finite order.

Proof. The proof follows from Lemma 3.1 and Theorem 2.19 of ([10]). □

4. Proof of the Theorems

Proof of the Theorem 2.7 Given Let \( f, g \in M_1(\mathbb{C}) \). Thus
\[
N(r, \infty; f) = O(\ln r), N(r, \infty; g) = O(\ln r).
\]
Now, we put
\[
Q(z) = (z - \alpha_1)(z - \alpha_2) \ldots (z - \alpha_n),
\]
and
\[
F(z) := \frac{1}{Q(f(z))} \quad \text{and} \quad G(z) := \frac{1}{Q(g(z))}.
\]
Let \( S(r) \) be any function \( S(r) : (0, \infty) \to \mathbb{R} \) satisfying \( S(r) = o(T(r, F) + T(r, G)) \) for \( r \to \infty \) outside a set of finite Lebesgue Measure.
Let
\[
H(z) := \frac{F''(z)}{F'(z)} - \frac{G''(z)}{G'(z)}.
\]
Now, we consider two cases:

Case-I First we assume that \( H \neq 0 \). Since \( H(z) \) can be expressed as
\[
H(z) = \frac{G'(z)}{F'(z)} \left( \frac{F'(z)}{G'(z)} \right)',
\]
so all poles of \( H \) are simple. Also, poles of \( H \) may occur at
\begin{enumerate}
\item poles of \( F \) and \( G \),
\item zeros of \( F' \) and \( G' \),
\end{enumerate}
But using the Laurent series expansion of \( H \), it is clear that “simple poles” of \( F \) (hence, that of \( G \)) are the zeros of \( H \). Thus
\[
N(r, \infty; F| = 1) = N(r, \infty; G| = 1) \leq N(r, 0; H),
\]
where \( N(r, \infty; F| = 1) \) is the the counting function of simple poles of \( F \). Using the lemma of logarithmic derivative and the first fundamental theorem, [12] can be written as
\[
N(r, \infty; F| = 1) = N(r, \infty; G| = 1) \leq N(r, \infty; H) + S(r)
\]
Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the $k$-distinct zeros of $P'(z)$. Since $F'(z) = -\frac{F'(f(z))}{(P(f(z)))^2}$, $G'(z) = -\frac{g'(z)P'(g(z))}{(P(g(z)))^2}$ and $f, g$ share $(S,2)$, by simple calculations, we can write

\begin{equation}
(4.4) \quad N(r, \infty; H) \leq \sum_{j=1}^{k} \left( N(r, \lambda_j; f) + N(r, \lambda_j; g) \right) + N_0(r, 0; f') + N_0(r, 0; g')
\end{equation}

\begin{equation}
+ \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}_{*}(r, \infty; F, G)
\end{equation}

\( \overline{N}_0(r, 0; f') \) denotes the reduced counting function of zeros of $f'$, which are not zeros of $\prod_{j=1}^{n}(f - \alpha_i) \prod_{j=1}^{k}(f - \lambda_j)$, similarly, $\overline{N}_0(r, 0; g')$ is defined. Also, $\overline{N}_{*}(r, \infty; F, G)$ is the reduced counting function of those poles of $F$ whose multiplicities differ from the multiplicities of the corresponding poles of $G$. Now, using the second fundamental theorem and (4.3), (4.4), we have

\begin{equation}
(4.5) \quad (n + k - 1) (T(r, f) + T(r, g)) = \sum_{j=1}^{k} \left( N(r, \lambda_j; f) + N(r, \lambda_j; g) \right) - N_0(r, 0; f') - N_0(r, 0; g') + S(r)
\end{equation}

Noting that

\begin{align*}
\overline{N}(r, \infty; F) &= \frac{1}{2}N(r, \infty; F) = 1 \quad \text{and} \quad \frac{1}{2}\overline{N}_{*}(r, \infty; F, G) \leq \frac{1}{2}N(r, \infty; F), \\
\overline{N}(r, \infty; G) &= \frac{1}{2}N(r, \infty; G) = 1 \quad \text{and} \quad \frac{1}{2}\overline{N}_{*}(r, \infty; F, G) \leq \frac{1}{2}N(r, \infty; G).
\end{align*}

Thus (4.5) can be written as

\begin{equation}
(4.6) \quad (n + k - 1) (T(r, f) + T(r, g)) \leq 2 \left( \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) \right) + (2k + \frac{n}{2})(T(r, f) + T(r, g)) + S(r)
\end{equation}

which contradicts the assumption $n > 2k + 2$. Thus $H \equiv 0$.

**Case-II** Next we assume that $H \equiv 0$. Then by integration, we have

\begin{equation}
(4.7) \quad \frac{1}{Q(f(z))} = \frac{c_0}{Q(g(z))} + c_1,
\end{equation}

where $c_0$ is a non-zero complex constant. Thus

\[ T(r, f) = T(r, g) + O(1). \]

Now, we consider two cases:

**Subcase-I** $c_1 \neq 0$.

Then equation (4.7) can be written as

\[ Q(f) = \frac{Q(g)}{c_1Q(g) + c_0}. \]
Thus
\[ N(r, -\frac{c_0}{c_1}; Q(g)) = N(r, \infty; Q(f)) = O(\ln r). \]

Let \( \mu = (-1)^{n+1} \alpha_1 \alpha_2 \ldots \alpha_n. \) Then \( Q(z) + \mu = \mu P(z). \) Since \( \mu \neq 0, \) and if \( \mu \neq \frac{c_0}{c_1}, \) then using first and second fundamental theorem, we obtain
\[ nT(r, g) + O(1) = T(r, Q(g)) \leq \overline{N}(r, \infty; Q(g)) + \overline{N}(r, -\mu; Q(g)) + \overline{N}
\left( r, -\frac{c_0}{c_1}; Q(g) \right) + S(r, g) \leq \overline{N}(r, 0; P(g)) + O(\ln r) + S(r, g) \leq \Gamma_1 T(r, g) + O(\ln r) + S(r, g), \]
which is impossible as \( n > 2\Gamma_1. \) Thus \( \mu = \frac{c_0}{c_1}. \) Hence
\[ Q(f) = \frac{Q(g)}{c_0 P(g)}. \]

Since \( P(z) \) has \( m_1 \) simple zeros and \( m_2 \) multiple zeros, so we can assume
\[ P(z) = a_0(z - b_1)(z - b_2) \ldots (z - b_{m_1})(z - c_1)^{l_1}(z - c_2)^{l_2} \ldots (z - c_{m_2})^{l_{m_2}}, \]
where \( l_i \geq 2 \) for \( 1 \leq i \leq m_2. \) Thus every zero of \( g - b_j \) \( (1 \leq j \leq m_1) \) has a multiplicity at least \( n. \) As \( P'(z) \) has at least two zeros, so, \( l_i < n, \) and hence each zero of \( g - c_i \) \( (1 \leq i \leq m_2) \) has a multiplicity at least 2. Thus using second fundamental theorem, we have
\[ (m_1 + m_2 - 1)T(r, g) \leq \overline{N}(r, \infty; g) + \sum_{j=1}^{m_1} \overline{N}(r, b_j; g) + \sum_{i=1}^{m_2} \overline{N}(r, c_i; g) + S(r, g) \leq \frac{1}{n} \sum_{j=1}^{m_1} \overline{N}(r, b_j; g) + \frac{1}{2} \sum_{i=1}^{m_2} \overline{N}(r, c_i; g) + O(\ln r) + S(r, g) \leq \frac{m_1}{n} T(r, g) + \frac{m_2}{2} T(r, g) + O(\ln r) + S(r, g) \leq \frac{m_1}{2} T(r, g) + \frac{m_2}{2} T(r, g) + O(\ln r) + S(r, g) \leq \frac{m_1 + m_2}{2} T(r, g) + O(\ln r) + S(r, g), \]
which is impossible as \( \Gamma_1 \geq 3. \)

**Subcase-II**

\( c_1 = 0. \)

Then equation (4.7) can be written as
\[ Q(g) \equiv c_0 Q(f). \]

Thus
\[ P(g) = c_0(P(f) - 1 + \frac{1}{c_0}). \]
If \( c_0 \neq 1 \), then using the first and second fundamental theorem, we have
\[
T(r, f) + O(1) = T(r, P(f)) \\
\leq \frac{1}{c_0}N(r, \infty; P(f)) + N(r, 0; P(f)) + 1_N \left( r, 1 - \frac{1}{c_0}; P(f) \right) + S(r, f)
\]
which is impossible as \( n > 2\Gamma_1 \). Thus \( c_0 = 1 \), i.e.,
\[P(f) \equiv P(g).\]
Thus
\[
(f - \alpha_1)(f - \alpha_2) \ldots (f - \alpha_n) \equiv (g - \alpha_1)(g - \alpha_2) \ldots (g - \alpha_n).
\]

Given that \( f \) and \( g \) share \{\beta_1, \beta_2\} IM and
\[
(\beta_1 - \alpha_1)(\beta_1 - \alpha_2) \ldots (\beta_1 - \alpha_n) \neq (\beta_2 - \alpha_1)(\beta_2 - \alpha_2) \ldots (\beta_2 - \alpha_n).
\]
Thus, if \( z_0 \) be an \( \beta_1 \) point of \( f \), then \( z_0 \) can’t be a \( \beta_2 \) point of \( g \). Thus \( f \) and \( g \) share \( \beta_1 \) and \( \beta_2 \) IM.

Consequently, we can say that \( f \) and \( g \) share \( \beta_1, \beta_2 \) and \( \infty \) CM. Thus by Lemma 3.2 \( f \equiv g \). This completes the proof. \( \square \)

**Proof of the Theorem 2.2** First we observe that if \( F \) and \( G \) share \( \infty \) IM, then
\[
\frac{1}{2}N(r, \infty; F) + \frac{1}{2}N(r, \infty; G) - N(r, \infty; F| = 1) - \frac{1}{2}N^*(r, \infty; F, G)
\]
If \( F \) and \( G \) are defined in Theorem 2.1 then
\[
\frac{1}{2}N^*(r, \infty; F, G)
\]
Thus proceeding similarly as Case-I of Theorem 2.1 (4.5) can be written as
\[
(n + k - 1)(T(r, f) + T(r, g)) \leq 2k + \frac{n}{2} + \frac{3}{2} + (2k + \frac{n}{2} + \frac{3}{2})(T(r, f) + T(r, g)) + S(r)
\]
which contradicts the assumption \( n > 2k + 5 \).
Rest part of the proof is similar to the Case-II of Theorem 2.1. \( \square \)
Proof of the Theorem 2.3. Given $f, g \in M_1(\mathbb{C})$. Thus

$$N(r, \infty; f) = O(\ln r), N(r, \infty; g) = O(\ln r).$$

Now, we put

$$Q(z) = (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n),$$

and

$$F(z) := \frac{1}{Q(f(z))} \quad \text{and} \quad G(z) := \frac{1}{Q(g(z))}.$$

Further suppose that

$$F'(z) = b_0(z - \beta_1)^{n_1}(z - \beta_2)^{n_2}.$$

Thus $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $S_2 = \{\beta_1, \beta_2\}$. Since $f$ and $g$ share $S_2$ IM, so

$$\sum_{j=1}^{2} \mathcal{N}(r, \beta_j; f) = \sum_{j=1}^{2} \mathcal{N}(r, \beta_j; g).$$

Let $S(r)$ be any function $S(r) : (0, \infty) \to \mathbb{R}$ satisfying $S(r) = o(T(r, F) + T(r, G))$ for $r \to \infty$ outside a set of finite Lebesgue Measure.

Let

$$H(z) := \frac{F''(z)}{F'(z)} - \frac{G''(z)}{G'(z)}.$$

Now, we consider two cases:

Case-I First we assume that $H \neq 0$. Since $H(z)$ can be expressed as

$$H(z) = \frac{G'(z)}{F'(z)} \left( \frac{F'(z)}{G'(z)} \right),$$

so all poles of $H$ are simple. Also, poles of $H$ may occur at

1. poles of $F$ and $G$,
2. zeros of $F'$ and $G'$,

But using the Laurent series expansion of $H$, it is clear that “simple poles” of $F$ (hence, that of $G$) is a zero of $H$. Thus

$$N(r, \infty; F) = 1 = N(r, \infty; G) = 1 \leq N(r, 0; H)$$

Using the lemma of logarithmic derivative and the first fundamental theorem, (4.10) can be written as

$$N(r, \infty; F) = 1 = N(r, \infty; G) = 1 \leq N(r, \infty; H) + S(r)$$

Since $F'(z) = -\frac{f'(z)F'(f(z))}{(P(f(z)))^2}, G'(z) = -\frac{g'(z)G'(g(z))}{(P(g(z)))^2}$ and $f, g$ share $(S_1, 2)$ and $(S_2, 0),$ by simple calculations, we can write

$$N(r, \infty; H) \leq \sum_{j=1}^{2} \mathcal{N}(r, \beta_j; f) + \mathcal{N}_0(r, 0; f') + \mathcal{N}_0(r, 0; g') + \mathcal{N}(r, \infty; f) + \mathcal{N}(r, \infty; g) + \mathcal{N}_*(r, \infty; F, G)$$

$\mathcal{N}_0(r, 0; f')$ denotes the reduced counting function of zeros of $f'$, which are not zeros of $\prod_{i=1}^{n}(f - \alpha_i) \prod_{j=1}^{2}(f - \beta_j)$, similarly, $\mathcal{N}_0(r, 0; g')$ is defined. Now, using the second
fundamental theorem and (4.11), (4.12), we have
\begin{equation}
(n + 1) \left( T(r, f) + T(r, g) \right)
\leq \mathcal{N}(r, \infty; f) + \mathcal{N}(r, \infty; g) + \mathcal{N}(r, 0; P(f)) + \mathcal{N}(r, 0; P(g)) + \sum_{j=1}^{2} \left( \mathcal{N}(r, \beta_j; f) + \mathcal{N}(r, \beta_j; g) \right) - \mathcal{N}_0(0; f') - \mathcal{N}_0(0; g') + S(r)
\end{equation}
\begin{equation}
\leq 2 \left( \mathcal{N}(r, \infty; f) + \mathcal{N}(r, \infty; g) \right) + \sum_{j=1}^{2} \left( 2\mathcal{N}(r, \beta_j; f) + \mathcal{N}(r, \beta_j; g) \right) + \mathcal{N}(r, \infty; F) + \mathcal{N}(r, \infty; G) + S(r) + \mathcal{N}_*(r, \infty; F, G)
\end{equation}
\begin{equation}
\leq 2 \left( \mathcal{N}(r, \infty; f) + \mathcal{N}(r, \infty; g) \right) + \sum_{j=1}^{2} \left( 2\mathcal{N}(r, \beta_j; f) + \mathcal{N}(r, \beta_j; g) \right) + \mathcal{N}_*(r, \infty; F, G) + S(r).
\end{equation}
Noting that
\begin{equation}
\mathcal{N}(r, \infty; F) - \frac{1}{2} \mathcal{N}(r, \infty; F| = 1) + \frac{1}{2} \mathcal{N}_*(r, \infty; F, G) \leq \frac{1}{2} \mathcal{N}(r, \infty; F),
\end{equation}
\begin{equation}
\mathcal{N}(r, \infty; G) - \frac{1}{2} \mathcal{N}(r, \infty; G| = 1) + \frac{1}{2} \mathcal{N}_*(r, \infty; F, G) \leq \frac{1}{2} \mathcal{N}(r, \infty; G).
\end{equation}
Thus (4.13) can be written as
\begin{equation}
(n + 1) \left( T(r, f) + T(r, g) \right)
\leq 2 \left( \mathcal{N}(r, \infty; f) + \mathcal{N}(r, \infty; g) \right) + \left( 3 + \frac{n}{2} \right) \left( T(r, f) + T(r, g) \right) + S(r)
\end{equation}
\begin{equation}
\leq \left( 3 + \frac{n}{2} \right) \left( T(r, f) + T(r, g) \right) + O(\ln r) + S(r).
\end{equation}
which contradicts the assumption \( n > 4 \). Thus \( H \equiv 0 \).

**Case-II** Next we assume that \( H \equiv 0 \). Rest part of the proof follows from the Case-II of Theorem 2.1. \( \square \)

**Proof of the Theorem 2.4** The idea of the proof follows from the proof of Theorem 2.3 and Theorem 2.2. \( \square \)

**Acknowledgement**

The authors are grateful to the anonymous referee for their valuable suggestions which considerably improved the presentation of the paper.

The research work of the first and the fourth authors are supported by the Department of Higher Education, Science and Technology & Biotechnology, Govt. of West Bengal under the sanction order no. 216(sanc) /ST/P/S&T/16G-14/2018 dated 19/02/2019.

The second and the third authors are thankful to the Council of Scientific and Industrial Research, HRDG, India for granting Junior Research Fellowship (File No.: 09/106(0179)/2018-EMR-I and 08/525(0003)/2019-EMR-I respectively) during the tenure of which this work was done.

**References**

[1] J. F. Chen, Uniqueness of meromorphic functions sharing two finite sets, Open Math., 15 (2017), 1244-1250.
[2] J. F. Chen, Meromorphic functions sharing unique range sets with one or two elements, Georgian Math. J., DoI. 10.1515/gmj-2019-2073.
[3] J. B. Conway, Functions of One Complex Variable, Springer(Verlag), New York, 1973.
[4] G. Frank and M. Reinders, A unique range set for meromorphic functions with 11 elements, Complex Var. Theory Appl., 37(1) (1998), 185-193.
[5] H. Fujimoto, On uniqueness of meromorphic functions sharing finite sets, Amer. J. Math., 122 (2000), 1175-1203.
[6] F. Gross, Factorization of meromorphic functions and some open problems, Complex Analysis, Lecture Notes in Math., vol. 599, Springer, Berlin and New York, 1977, 51-67.
[7] I. Lahiri, Weighted sharing and uniqueness of meromorphic functions, Nagoya Math. J., 161 (2001), 193-206.
[8] P. Sahoo and H. Karmakar, Uniqueness theorems related to weighted sharing of two sets, Acta Univ. Sapientiae, Mathematica, 10 (2) (2018), 329-339.
[9] P. Sahoo and A. Sarkar, On the uniqueness of meromorphic functions sharing two sets, Bol. Soc. Mat. Mex., Doi: 10.1007/s40590-019-00260-4.
[10] C. C. Yang and H. X. Yi, Uniqueness Theory of Meromorphic Functions, Mathematics and its Applications, 557, Kluwer Academic Publishers Group, Dordrecht, 2003.
[11] H. X. Yi, Uniqueness of meromorphic functions and question of Gross, Science in China (Series A), 37 (1994), 802-813.
[12] H. X. Yi, On a question of Gross concerning uniqueness of entire functions, Bull. Austral. Math. Soc., 57 (1998), 343-349.
[13] B. Yi and Y. H. Li, The uniqueness of meromorphic functions that share two sets with CM, Acta Math. Sin., Chin. Ser., 55 (2012), 363-368.

1 Department of Mathematics, Ramakrishna Mission Vivekananda Centenary College, Rahara, West Bengal 700 118, India.
E-mail address: bikashchakraborty.math@yahoo.com, bikash@rkmvccrahara.org

2 Department of Mathematics, University of Kalyani, Kalyani, West Bengal 741 235, India.
E-mail address: mail4amitpal@gmail.com

3 Department of Mathematics, Ramakrishna Mission Vivekananda Centenary College, Rahara, West Bengal 700 118, India.
E-mail address: sudipsaha814@gmail.com

4 Department of Mathematics, Ramakrishna Mission Vivekananda Centenary College, Rahara, West Bengal 700 118, India.
E-mail address: kamilajayanta@gmail.com