Invariant differential operators and central Fourier multipliers on exponential Lie groups

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Abstract. Let $G$ be an exponential solvable Lie group. By definition $G$ is $\ast$-regular if $\ker L^1(G,\pi)$ is dense in $\ker C^\ast_* (G,\pi)$ for all unitary representations $\pi$ of $G$. Boidol characterized the $\ast$-regular exponential Lie groups by a purely algebraic condition. In this article we will focus on non-$\ast$-regular groups. We say that $G$ is primitive $\ast$-regular if the above density condition is satisfied for all irreducible representations. Our goal is to develop appropriate tools to verify this weaker property. To this end we will introduce Duflot pairs $(W,p)$ and central Fourier multipliers $\psi$ on the stabilizer $M = G_fN$ of representations $\pi = K(f)$ in general position. Using Littlewood-Paley theory we will derive some results on multiplier operators $T_\psi$ which might be of independent interest. The scope of our method of separating triples $(W,p,\psi)$ will be sketched by studying two significant examples in detail. It should be noticed that the methods 'separating triples' and 'restriction to subquotients' suffice to prove that all exponential solvable Lie groups of dimension $\leq 7$ are primitive $\ast$-regular.

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1 Introduction

Let $G$ be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$. Fix a co-abelian, nilpotent ideal $\mathfrak{n}$ of $\mathfrak{g}$. Let $f \in \mathfrak{g}^*$ be in general position such that $\mathfrak{m} = \mathfrak{g}_f + \mathfrak{n}$ is a proper, non-nilpotent ideal. Set $\tilde{f} = f | \mathfrak{m}$ and $f' = f | \mathfrak{n}$. Further we consider the orbit $\tilde{X} = \text{Ad}^*(G)\tilde{f}$ and the $\text{Ad}^*(G)$-invariant subset $\tilde{\Omega} = \{\tilde{h} \in \mathfrak{m}^* : \tilde{h} | \mathfrak{n} \text{ is in the closure of } \text{Ad}^*(G)f' \text{ in } \mathfrak{n}^*\}$ of $\mathfrak{m}^*$.

Let $M$ denote the connected subgroup of $G$ with Lie algebra $\mathfrak{m}$. We work with the $C^*$-completion $C^*(M)$ of the Banach $\ast$-algebra $L^1(M)$. As usual we provide $\text{Prim} C^*(M)$ with the Jacobson topology, and $\widehat{M}$ with the initial topology w. r. t. the natural map $\widehat{M} \to \text{Prim} C^*(M)$. This map is a bijection by the main result of [17]. Furthermore it is known that the Kirillov map yields a $G$-equivariant bijection from the coadjoint orbit space $\mathfrak{m}^*/\text{Ad}^*(M)$ onto the unitary dual $\widehat{M}$ of $M$. Hence $\tilde{X}$ corresponds to a $G$-orbit $X$ in $\widehat{M}$, and $\tilde{\Omega}$ to a $G$-invariant subset $\Omega$ of $\widehat{M}$.
Our aim is to prove \( G \) to be primitive \(*\)-regular. According to the strategy developed in Section 5 of [21] we have to verify

\[
\bigcap_{\pi \in X} \ker_{L^1(M)} \pi \not\subset \ker_{L^1(M)} \rho
\]

for all critical \( \rho \in \Omega \setminus \overline{X} \). Compare also Assertion 5.3 of [21]. Note that \( \Omega \setminus \overline{X} \) consists of all representations \( \rho = \mathcal{K}(\tilde{g}) \) where \( \tilde{g} \in \Omega \) is critical for \( X = \text{Ad}^*(G)\tilde{f} \) in the sense of Definition 5.2 of [21].

Relation (1.1) deserves a further explanation. If the primitive ideal spaces \( \text{Prim} C^*(M) \) and \( \text{Prim}_c L^1(M) \) are endowed with the Jacobson topology, then the natural map \( \Psi : \text{Prim} C^*(M) \rightarrow \text{Prim}_c L^1(M) \), \( \Psi(P) = P' = P \cap \mathcal{A} \), is a continuous bijection. Continuity means that \( \Psi(A) \subset \Psi(P) \) for all subsets \( A \) of \( \text{Prim} C^*(M) \). Note that injectivity, which is necessary for \( M \) to be primitive \(*\)-regular, follows from [17]. We shall assume that \( M \) is not \(*\)-regular, or equivalently, that \( \Psi \) is not a homeomorphism. In this situation one may ask whether the reverse inclusion \( \Psi(X) \subset \Psi(\overline{X}) \) is satisfied at least for certain subsets \( X \) of \( \overline{M} = \text{Prim} C^*(M) \). The question is: Does \( \rho \not\in \overline{X} \) imply Relation (1.1) for all orbits \( X \) of an exponential Lie group \( G \) as above?

Producing functions \( c \in L^1(M) \) such that \( \pi(c) = 0 \) for all \( \pi \in X \) and \( \rho(c) \neq 0 \) is a challenging problem. Our approach to a solution is best explained in the context of the subsequent theorem.

**Theorem 1.2** (Dauns, Dixmier, Hofmann;1967). Let \( \mathcal{B} \) be \(*\)-algebra, \( \mathcal{B}^b \) its adjoint algebra (multiplier algebra), and \( \mathcal{C}^b(\text{Prim} \mathcal{B}) \) the \(*\)-algebra of all complex-valued, bounded, continuous functions on its primitive ideal space.

(i) If \( \mu \in \mathcal{C}^b(\text{Prim} \mathcal{B}) \) and \( a \in \mathcal{B} \), then there exists a unique element \( c = \mu * a \) in \( \mathcal{B} \) such that \( c + P = \mu(P) \cdot (a + P) \) holds in \( \mathcal{B}/P \) for all \( P \in \text{Prim} \mathcal{B} \).

(ii) Further \( \Gamma(\mu)a = \mu * a \) defines an isomorphism \( \Gamma \) of \( \mathcal{C}^b(\text{Prim} \mathcal{B}) \) onto the center \( Z(\mathcal{B}^b) \) of \( \mathcal{B}^b \).

In particular \( \mathcal{C}^b(\text{Prim} \mathcal{B}) \) acts on \( \mathcal{B} \) as an algebra of multipliers. This theorem was first shown by Dauns and Hofmann, see Chapter 3 of [9]. An alternative direct proof is due to Dixmier, see Theorem 5 of [7]. Compare also pp. 223-226 of [19]. For a definition of the adjoint algebra \( \mathcal{B}^b \) we refer to Section 3 of [14].

The spectrum \( \hat{\mathcal{B}} \) of \( \mathcal{B} \) is the set of all equivalence classes of irreducible \(*\)-representations of \( \mathcal{B} \) endowed with the initial topology w. r. t. the natural map \( \hat{\mathcal{B}} \rightarrow \text{Prim} \mathcal{B} \). This map allows us to consider \( \mu \) as a function on \( \hat{\mathcal{B}} \). Now Theorem 1.2 (i) reads as follows: There exists a unique element \( c \in \mathcal{B} \) such that \( \pi(c) = \mu(\pi) \pi(a) \) for all \( \pi \in \mathcal{B} \). There is no peril in dealing with \( \pi \) instead of its equivalence class \( [\pi] \) here. If we write \( \hat{\mu}(\pi) = \pi(a) \), then the preceding equality becomes \( \hat{\pi}(\pi) = \mu(\pi) \hat{\mu}(\pi) \) so that \( \mu \) emerges as to be a multiplier (on the Gelfand transform side). Theorem 1.2 states that the multiplier problem given
by μ and a has a (unique) solution c ∈ B. If B = C∗(M), then one might wonder about the regularity of this solution: What conditions on μ and a do imply c ∈ L1(M)?

Finding suitable μ and proving this kind of regularity is appropriate to tackle the problem raised above. In the following remark we localize to a certain subset Ω of M losing the uniqueness of c. We do not assume μ to be bounded.

Observation 1.3. Let X ⊂ Ω be subsets of M and ρ ∈ Ω \ X. If there exists a dense subspace Q of L1(M) and a complex-valued, continuous function μ on Ω such that

(i) μ(π) = 0 for all π ∈ X and μ(ρ) ≠ 0,
(ii) for every a ∈ Q there exists a function c ∈ L1(M) satisfying

π(c) = μ(π)π(a) for all π ∈ Ω,

then it follows \( \bigcap_{π∈X} \ker L^1(M)π \not\subset \ker L^1(M)ρ \).

Proof. Since Q is dense in L1(M), there exists some a ∈ Q such that ρ(a) ≠ 0. If we choose μ as in (i) and c ∈ L1(M) as in (ii), then π(c) = 0 for all π ∈ X and ρ(c) ≠ 0.

This observation is the guideline for the results of Section 2 and 3. In the course of the proof of Theorem 3.4 we will see that if \((W,p,ψ)\) is a separating triple for X ⊂ Ω ⊂ M and ρ ∉ X, then μ = p − ψ satisfies (i) and (ii) of Observation 1.3 so that Relation (1) is guaranteed. The author has verified the existence of separating triples in a multitude of examples. A sample can be found in Section 6.

2 Invariant differential operators

Let M be an exponential solvable Lie group. Its exponential map exp is a global diffeomorphism from its Lie algebra m onto M. In particular M is connected, simply connected. We use the fact that the Kirillov map \( K \) gives a bijection from the coadjoint orbit space \( m^*/\text{Ad}^*(M) \) onto the unitary dual \( \hat{M} \) of M. In particular we take the definition of \( π = K(h) = \text{ind}_P^M \chi_h \) via Pukanszky / Vergne polarizations p at h ∈ m* for granted. Here P is the unique connected subgroup of M with Lie algebra p, and \( \chi_h(\exp tX) = e^{ih\langle h,X \rangle} \) the character of P with differential \( ih|_p \). These results can be found in Chapters 4 and 6 of [1], and Chapter 1 of [15]. Mostly we shall regard K as a map from m* onto \( \hat{M} \). If μ is a multiplier as in Observation 1.3, then the Kirillov parametrization allows us to regard μ as a function on an Ad*(M)-invariant subset Ω of m* rather than on a subset Ω of M.

The following remarks apply to arbitrary Lie groups M. If π is a strongly continuous representation of M in a Banach space E, then the infinitesimal representation \( dπ \) of its Lie algebra m is defined by

\[
dπ(X)φ = \frac{d}{dt}|_{t=0} π(\exp tX)φ
\]
on the subspace $E^\infty$ of $C^\infty$-vectors for $\pi$. Dixmier and Malliavin proved that $E^\infty$ coincides with the Gårding space (the dense subspace generated by vectors of the form $\pi(a)\varphi$ with $a \in \mathcal{C}_0^\infty(M)$ and $\varphi \in \mathcal{D}_p$), see Theorem 3.3 of [8]. The representation $d\pi$ can be extended to the universal enveloping algebra $\mathcal{U}(\mathfrak{m})$ of the complexification $\mathfrak{m}_C$ of $\mathfrak{m}$. If $V \ast a = d\lambda(V)a$ denotes the representation of $\mathcal{U}(\mathfrak{m})$ on $\mathcal{C}_0^\infty(M)$ obtained by differentiating the left regular representation of $M$ in $L^1(M)$, then the crucial equality

$$\pi(V \ast a) = d\pi(V)\pi(a)$$

holds for all $V \in \mathcal{U}(\mathfrak{m})$ and $a \in \mathcal{C}_0^\infty(M)$. The symmetrization map

$$\beta(X_1 \cdots X_r) = \frac{1}{r!} \sum_{\sigma \in S_r} X_{\sigma(1)} \cdots X_{\sigma(r)}$$

gives an $\text{Ad}(M)$-equivariant, linear isomorphism from the symmetric algebra $\mathcal{S}(\mathfrak{m})$ of $\mathfrak{m}_C$ onto $\mathcal{U}(\mathfrak{m})$, see e.g. Chapter 3.3 of [4]. In particular $\beta$ maps the subspace $Y(\mathfrak{m}^*)$ of $\text{Ad}(M)$-invariants onto the center $Z(\mathfrak{m})$ of $\mathcal{U}(\mathfrak{m})$. We identify $\mathcal{S}(\mathfrak{m})$ with $\mathcal{P}(\mathfrak{m}^*)$, the algebra of all complex-valued polynomial functions on $\mathfrak{m}^*$, by means of the $\text{Ad}(M)$-equivariant isomorphism of algebras mapping $X \in \mathfrak{m}$ to the linear function $h \mapsto -i\langle h, X \rangle$ on $\mathfrak{m}^*$.

We shall regard elements of $\mathcal{U}(\mathfrak{m})$ as distributions on $M$ with support in $\{e\}$, and $\mathcal{S}(\mathfrak{m})$ as distributions on $\mathfrak{m}$ with support in $\{0\}$. Let $j$ be a smooth, strictly positive function on $\mathfrak{m}$. If $u$ is a distribution on $\mathfrak{m}$ with compact support $K$, then

$$\langle \eta(u), \varphi \rangle = \langle u, j(\varphi \circ \exp) \rangle$$

defines a distribution $\eta(u)$ on $M$ with compact support in $\exp(K)$. Let $U$ be the subset of all $X \in \mathfrak{m}$ such that $|\lambda| < \pi$ for all eigenvalues $\lambda$ of $\text{ad}(X)$. Clearly $U$ and $V$ are open and invariant. It is known that $\exp : U \rightarrow V$ is a diffeomorphism. Hence $\eta$ yields a linear isomorphism from the vector space of all distributions on $\mathfrak{m}$ with compact support in $U$ onto the distributions on $M$ with compact support in $V$. In particular $\eta$ maps $\mathcal{S}(\mathfrak{m})$ onto $\mathcal{U}(\mathfrak{m})$. In addition we suppose that $j$ is $\text{Ad}(M)$-invariant. If $u$ is $\text{Ad}(M)$-invariant, then $\eta(u)$ is invariant under interior automorphisms. Thus $\eta$ maps $Y(\mathfrak{m})$ onto $Z(\mathfrak{m})$. For $j \equiv 1$ we recover the symmetrization map $\beta$, for

$$j(X) = \det \left( \frac{1 - e^{-\text{ad}(X)}}{\text{ad}(X)} \right)^{1/2}$$

we obtain the Duflo isomorphism $\gamma$. By means of the character formula given in Théorème II.1 and V.2, Duflo proved in Théorème IV.1 and V.2 of [9] that the restriction $\gamma : Y(\mathfrak{m}) \rightarrow Z(\mathfrak{m})$ is an isomorphism of associative algebras for all solvable and semi-simple Lie algebras $\mathfrak{m}$.

An algebraic proof of this fact can be found in Paragraphs 4 and 5 of [12].

In Theorem 2 of [10] Duflo proved $d\pi(\gamma(p)) = p(h)\cdot \text{Id}$ for all $p \in Y(\mathfrak{m}^*)$, $h \in \mathfrak{m}^*$, and $\pi = K(h)$. Generalizing this property of the pair $(\gamma(p), p)$ we state
Definition 2.2. Let $\Omega$ be an $\text{Ad}^*(M)$-invariant subset of $\mathfrak{m}^*$, $W \in \mathcal{U}(\mathfrak{m})$, and $p \in \mathcal{P}(\mathfrak{m}^*)$. We say that $(W, p)$ is a Duflo pair w.r.t. $\Omega$ if

$$d\pi(W) = p(h) \cdot \text{Id}$$

for all $h \in \Omega$ and $\pi = K(h)$.

This equality implies that $p$ is constant on all $\text{Ad}^*(M)$-orbits contained in $\Omega$ because $K$ is constant on $\text{Ad}^*(M)$-orbits. We stress that we do not assume $W \in Z(\mathfrak{m}_C)$ nor $p \in Y(\mathfrak{m}^*)$. If $(W, p)$ is a Duflo pair w.r.t. $\Omega$ and $a \in C_0^\infty(M)$, then it follows from Equation (2.1) and (2.3) that $b = W * a \in C_0^\infty(M)$ satisfies $\pi(b) = p(h) \pi(a)$ for all $h \in \Omega$ and $\pi = K(h)$.

Let $\mathfrak{n}$ be a coabelian, nilpotent ideal of $\mathfrak{m}$, and $\text{Der}(\mathfrak{m}, \mathfrak{n})$ the subalgebra of all derivations $D$ of $\mathfrak{m}$ such that $D \cdot \mathfrak{m} \subset \mathfrak{n}$. A linear functional $h \in \mathfrak{m}^*$ is said to be in general position if $h(\mathfrak{a}) \neq 0$ for all non-trivial $\text{Der}(\mathfrak{m}, \mathfrak{n})$-invariant ideals $\mathfrak{a}$ of $[\mathfrak{m}, \mathfrak{m}]$. We define $X_0 = X_0(\mathfrak{m}, \mathfrak{n})$ as the set of all $h \in \mathfrak{m}^*$ in general position satisfying the stabilizer condition $\mathfrak{m} = \mathfrak{m}_h + \mathfrak{n}$. Clearly $X_0$ is $\text{Ad}^*(M)$-invariant and saturated in the sense that $X_0 = X_0 + [\mathfrak{m}, \mathfrak{m}]^\perp$. Further $X_0$ contains all $G$-orbits $X = \text{Ad}^*(G)\tilde{f}$ as in Section 1. We point out that $X_0 \neq \emptyset$ implies $\mathfrak{m}_n \subset \mathfrak{m}_n$: If $h \in X_0$, then $[\mathfrak{m}, \mathfrak{n}] = [\mathfrak{m}_h, \mathfrak{n}]$ is a $\text{Der}(\mathfrak{m}, \mathfrak{n})$-invariant subspace of $\text{ker} h$, and hence $[\mathfrak{m}, \mathfrak{n}] = 0$. If in addition $\mathfrak{n}$ is the nilradical (i.e. the largest nilpotent ideal) of $\mathfrak{m}$, then $\mathfrak{m}_n = \mathfrak{m}_n$.

Let $\Omega_0$ be the set of all $h \in \mathfrak{m}^*$ such that $h'$ is in the closure of $X_0'$ in $\mathfrak{n}^*$. Our interest lies in the subalgebra $I(X_0)$ of all $p \in S(\mathfrak{m})$ such that $p$ is $\text{Ad}^*(M)$-invariant on $\Omega_0$.

A trivial extension $\mathfrak{n} = \mathfrak{h} \times \mathfrak{a}$ of a Lie algebra $\mathfrak{h}$ is a direct product with a commutative one. In particular a trivial extension of a $(k+1)$-step nilpotent filiform algebra contains

$$\mathfrak{n} \supset \mathfrak{c} \supset \mathfrak{c} \mathfrak{n} + \mathfrak{m} \supset \cdots \supset \mathfrak{c} \mathfrak{n} \supset \cdots \supset \mathfrak{c} \mathfrak{n} \supset \{0\}$$

as a descending series of ideals. Here $\mathfrak{c}$ is commutative and $d = \dim \mathfrak{m}_n$. For $k = d = 1$ this is the 3-dimensional Heisenberg algebra. If $k \geq 2$, then $\mathfrak{c}$ is the centralizer of $\mathfrak{c} \mathfrak{n}$ in $\mathfrak{n}$, and hence a characteristic ideal, too.

Theorem 2.4. Let $\mathfrak{m}$ be a non-nilpotent, exponential solvable Lie algebra. If its nilradical $\mathfrak{n}$ is

1. a trivial extension of a filiform algebra of arbitrary dimension,
2. a 5-dimensional Heisenberg algebra,
3. a trivial extension of $\mathfrak{g}_{5,2}$,

then $X_0$ is a non-empty, algebraic subset of $\mathfrak{m}^*$ and the following conditions are satisfied:

(i) There exist real valued polynomial functions $\Gamma = (\Gamma_1, \ldots, \Gamma_k)$ on $\mathfrak{m}^*$ which are $\text{Ad}(M)$-invariant on $X_0$ and induce a homeomorphism $\Gamma$ from $X_0/\text{Ad}^*(M)$ onto an open subset $W$ of $\mathbb{R}^k$ admitting a rational inverse.
(ii) For every critical \( g \in \Omega_0 \setminus \overline{X}_0 \) there exists a Duflo-pair \((W,p)\) w. r. t. \( \Omega_0 \) such that \( p \) has constant term 0 and \( p(g) \neq 0 \).

If \( n \) is commutative or a trivial extension of \( g_{5,4} \) or \( g_{5,6} \), then necessarily \( X_0 = \emptyset \). If \( n \) is a central extension of \( g_{5,3} \), then the situation is slightly more complicated: For certain singular \( g \in \Omega_0 \setminus X_0 \) one cannot find \((W,p)\) such that \( p(g) \neq 0 \).

Note that Theorem 2.4 covers all nilpotent Lie algebras \( n \) of dimension \( \leq 5 \). We omit the details of its proof, which is a case by case study. The essential steps are: First the possible stabilizers \( m \) with a given nilradical \( n \) are to be determined. In each case we pick out representatives \( f \) for the \( \text{Ad}^*(M) \)-orbits in \( X_0 \). From the coadjoint representation \( \text{Ad}^*(M) f \) we can read off non-trivial polynomial functions \( p \) on \( m^* \) which are constant on all \( \text{Ad}^*(M) \)-orbits in \( X_0 \). Finally one has to compute suitable elements \( W \in \mathcal{U}(m) \). Here the choice \( W = \gamma(p) \) is natural, but not compulsory.

3 Central Fourier multipliers

Denote by \( \mathfrak{z} m \) the center of the Lie algebra \( m \) of \( M \). Let \( l = \dim \mathfrak{z} m \) and \( k = \dim m/\mathfrak{z} m \). We choose \( b_1, \ldots, b_k \) in \( m \) whose canonical images in \( m/\mathfrak{z} m \) form a Malcev basis of \( m/\mathfrak{z} m \). In particular \( \mathcal{B} = \{ b_1, \ldots, b_k \} \) is a coexponential basis for \( \mathfrak{z} m \) in \( m \). If \( Z(M) \) is the center of \( M \), \( q : M \to M/Z(M) \) the quotient map, and \( \Phi_1(x) = \exp(x_1 b_1) \cdots \exp(x_k b_k) \), then \( q \circ \Phi_1 \) is a diffeomorphism from \( \mathbb{R}^k \) onto \( M/Z(M) \). Equivalently, \( \Phi(x,z) = \Phi_1(x) \exp(z) \) is a global diffeomorphism from \( \mathbb{R}^k \times \mathfrak{z} m \) onto \( M \). This is a canonical coordinate system of the second kind. If \( c \) is a function on \( M \), then by abuse of notation \( c \circ \Phi \) is again denoted by \( c \).

We require partial Fourier transforms w. r. t. to the central variable: If \( c \in L^1(M) \), then \( z \mapsto f(x,z) \) is in \( L^1(\mathbb{R}^l) \) for almost all \( x \in \mathbb{R}^k \) by Fubini's theorem. For these \( x \) and all \( \xi \in \mathfrak{z} m^* \) we define

\[
\hat{c}(x,\xi) = \int_{\mathfrak{z} m} c(x,z) e^{-i\langle \xi,z \rangle} dz.
\]

For fixed \( \xi \in \mathfrak{z} m^* \) the function \( x \mapsto \hat{c}(x,\xi) \) is in \( L^1(\mathbb{R}^k) \).

**Definition 3.1.** A complex-valued function \( \psi \) on \( \mathfrak{z} m^* \) is called a central Fourier multiplier if for all \( a \) in \( C_\infty(M) \) there exists a (unique) smooth function \( c \) in \( L^1(M) \) such that \( \hat{c}(x,\xi) = \psi(\xi) \hat{a}(x,\xi) \) holds for all \( x \in \mathbb{R}^k \) and \( \xi \in \mathfrak{z} m^* \).

If \( \psi \) is a central Fourier multiplier, so is the function \( \xi \mapsto \psi(-\xi) \). Note that \( h \mapsto \psi(h\mid\mathfrak{z} m) \) defines an \( \text{Ad}(M) \)-invariant function on \( m^* \), and hence a function on \( m^*/\text{Ad}^*(M) \cong \tilde{M} \). A first consequence is

**Lemma 3.2.** If \( c, a \in L^1(M) \) such that \( \hat{c}(x,\xi) = \psi(-\xi) \hat{a}(x,\xi) \) for almost all \( x \) and all \( \xi \), then \( \pi(c) = \psi(h\mid\mathfrak{z} m) \pi(a) \) for all \( h \in m^* \) and \( \pi = \mathcal{K}(h) \).

**Proof.** Let \( p \) be a Pukanszky polarization at \( h \in m^* \) so that \( \pi = \text{ind}_p^M \chi_h \). Since \( \mathfrak{z} m \subset p \), it follows \( \pi(\exp z) = e^{i\langle h,z \rangle} \) for all \( z \in \mathfrak{z} m \). The modular function \( \Delta_{M,Z} \) is trivial so...
that Weil’s formula gives \( \int_M c(m) \, dm = \int_{\mathbb{R}^k} \int_{\mathfrak{m}} c(x, z) \, dz \, dx \) for \( c \in L^1(M) \). Here \( dm \) denotes the Haar measure of \( M \), and \( dx \) and \( dz \) denote the Lebesgue measures on \( \mathbb{R}^k \) and \( \mathfrak{m} \) respectively. Now we obtain
\[
\pi(c) \varphi = \int_{\mathbb{R}^k} \int_{\mathfrak{m}} c(x, z) \pi(\Phi_1(x)) \pi(\exp z) \varphi \, dz \, dx \\
= \int_{\mathbb{R}^k} \tilde{c}(x, -h \, | \, \mathfrak{m}) \, \pi(\Phi_1(x)) \varphi \, dx \\
= \psi(h \, | \, \mathfrak{m}) \, \pi(a) \varphi
\]
for every \( \varphi \) in the representation space of \( \pi \). □

The next definition is motivated by concrete applications in the investigation of primitive \( * \)-regularity for exponential Lie groups.

**Definition 3.3.** Let \( X \subset \Omega \) be \( \text{Ad}^*(M) \)-invariant subsets of \( \mathfrak{m}^* \). We say that \((W, p, \psi)\) is a separating triple for \( X \) in \( \Omega \) if

(i) \((W, p)\) is a Duflo pair w.r.t. \( \Omega \),

(ii) \( \psi \) is a central Fourier multiplier,

(iii) \( h \in X \) if and only if \( h \in \Omega \) and \( p(h) = \psi(h \, | \, \mathfrak{m}) \).

Condition (iii) states that \( p, \psi \) characterize the closure of \( X \) in \( \Omega \), which is the closure of \( X \) in \( \mathfrak{m}^* \) if \( \Omega \) is closed. Several variants of this definition are possible: For example, one might want to consider (finite) sets of triples \((W_{\nu}, p_{\nu}, \psi_{\nu})\) such that \( h \in X \) if and only if \( p_{\nu}(h) = \psi_{\nu}(h \, | \, \mathfrak{m}) \) for all \( \nu \). The existence of separating triples is a strong assumption making it easy to prove

**Theorem 3.4.** If \((W, p, \psi)\) is a separating triple for \( X \) in \( \Omega \), then \( g \in \Omega \setminus \overline{X} \) implies
\[
\bigcap_{h \in X} \ker_{L^1(M)} \mathcal{K}(h) \not\subset \ker_{L^1(M)} \mathcal{K}(g) .
\]

**Proof.** Let \( \rho = \mathcal{K}(g) \). Since \( C_0^\infty(M) \) is dense in \( L^1(M) \), there exists some \( a \in C_0^\infty(M) \) such that \( \rho(a) \neq 0 \). Define \( b = W * a \). Since \( \psi \) is a Fourier multiplier, there exists a unique smooth function \( c \in L^1(M) \) such that \( \tilde{c}(x, \xi) = \psi(-\xi) \tilde{a}(x, \xi) \). Now \( b - c \) solves the problem: Since \( g \in \Omega \setminus \overline{X} \), we obtain
\[
\rho(c) = \psi(g \, | \, \mathfrak{m}) \rho(a) \neq p(g) \rho(a) = \rho(b)
\]
because \((W, p)\) is a Duflo pair. Furthermore, if \( h \in X \) and \( \pi = \mathcal{K}(h) \), then
\[
\pi(c) = \psi(h \, | \, \mathfrak{m}) \rho(a) = p(h) \pi(a) = \pi(b)
\]
by Lemma 3.2. This proves our proposition. □

In this proof we only used the fact that \( \tilde{c}(x, h \, | \, \mathfrak{m}) = \psi(h \, | \, \mathfrak{m}) \tilde{a}(x, h \, | \, \mathfrak{m}) \) holds for all \( h \in \Omega \), instead of the full multiplier property of \( \psi \). The continuity of \( \psi \) on the closure of \( \{h \, | \, \mathfrak{m} : h \in \Omega \} \) in \( \mathfrak{m}^* \) is necessary for it. If we regard \( \psi \) as a function on \( \mathfrak{m}^* \), then \( p - \psi = 0 \) on \( X \) and \( \neq 0 \) in \( g \) so that the preceding proposition can be viewed as a special case of the considerations in Observation 1.3.
4 More about central Fourier multipliers

As before we use a coexponential basis $B$ for $\mathfrak{zm}$ in $\mathfrak{m}$ to define coordinates of the second kind for $M$. We suppress the coordinate diffeomorphism $\Phi$. Recall that $l = \dim \mathfrak{zm}$ and $k = \dim \mathfrak{m}/\mathfrak{zm}$. To begin with we introduce a subspace $Q$ of $L^1(M)$ which will play a decisive role in our discussion of central Fourier multipliers.

**Definition 4.1.** Let $Q$ denote the vector space of all smooth functions $a$ on $M$ such that

1. there is a compact subset $L$ of $\mathbb{R}^k$ such that $a(x,z) = 0$ whenever $x \not\in L$,

2. there exists some $r_0 > 0$ such that for all multi-indices $\alpha \in \mathbb{N}^k$ and $\beta \in \mathbb{N}^l$ the functions

   $$(x,z) \mapsto (1 + |z|)^{l+r_0} (D_\alpha x D_\beta z a)(x,z)$$

vanish at infinity.

If $a \in Q$, then the functions $(1 + |z|)^{l+r_0}(D_\alpha^a D_\beta^a)$ are bounded. Alternatively this property could have been used for a different definition of $Q$. Another possibility is to allow the exponent $r_0$ to depend on the multi-indices $\alpha$ and $\beta$. These alternate subspaces serve just as well in many respects in the context of central Fourier multipliers.

Note that $C_0^\infty(M) \subset Q \subset \mathcal{L}^1(M)$ has the following nice properties: The definition of $Q$ does not depend on the choice of the coexponential basis for $\mathfrak{zm}$ in $\mathfrak{m}$. Clearly $Q$ is a dense $*$-subalgebra of $\mathcal{L}^1(M)$ and a $\lambda(M)$-invariant subspace where $\lambda$ denotes the left-regular representation of $M$ in $\mathcal{L}^1(M)$. Furthermore $Q$ is contained in the subspace $\mathcal{L}^1(M)^{\infty,\lambda}$ of $C^\infty$-vectors of $\lambda$. In particular $\mathcal{U}(\mathfrak{m}_C)$ acts on $Q$.

The role of $Q$ in the context of central Fourier multiplier problems is as follows: Assume that the multiplier $\psi$ on $\mathfrak{zm}^*$ is a continuous function of polynomial growth, and hence defines a tempered distribution. Let $u_\psi \in \mathcal{S}'$ such that $\hat{u_\psi} = \psi$. Convolution with $u_\psi$, w.r.t. the central variable defines a linear operator $T_\psi a = u_\psi * a$. In the following we will give sufficient conditions on $\psi$ (or $u_\psi$) which guarantee that $T_\psi a$ is well-defined and in $Q$ for all $a \in Q$. From

$$(T_\psi a)(x,\xi) = (u_\psi * a)(x,\xi) = \psi(\xi) \hat{a}(x,\xi)$$

it will follow that $c = T_\psi a \in Q$ is a solution of the multiplier problem given by $\psi$ and $a \in Q$. There is a twofold reason for calling $T_\psi$ a multiplier operator: on the one hand it is multiplication by $\psi$ on the Fourier transform side, on the other hand it holds $T_\psi(a * b) = (T_\psi a) * b$ and $(T_\psi a)^* * b = a^* * (T_\psi b)$ so that $T_\psi \in Q^b$ in the spirit of Section 3 of [13].

As a starting point we choose the following well-known result of Fourier analysis in $\mathbb{R}^n$. Here one should think of $\mathbb{R}^n$ as a subspace of $\mathfrak{zm}$. 

Lemma 4.2. If \( \psi \in C^{n+1}(\mathbb{R}^n) \) such that \( D^\gamma \psi \in L^1(\mathbb{R}^n) \) for all \( |\gamma| \leq n+1 \), then the inverse Fourier transform \( k \) of \( \psi \) is a continuous function such that \( |k(z)| \leq C|z|^{-(n+1)} \) for all \( z \in \mathbb{R}^n \), for some \( C > 0 \). Clearly \( \langle u_\psi, \varphi \rangle = \int_{\mathbb{R}^n} k(y)\varphi(y) \, dy \).

In order to obtain similar results for functions \( \psi \in C^\infty(\mathbb{R}^n \setminus \{0\}) \) which are not differentiable in \( \xi = 0 \), we consider dyadic decompositions on the Fourier transform side. These ideas originated in the work of Bernstein, Littlewood, and Paley. Up to minor modifications, the considerations leading to the proof of Proposition 4.6 can be found on pp. 241–246 of Stein [20].

Let \( \eta \geq 0 \) be in \( C^\infty(\mathbb{R}^n) \) such that \( \eta(\xi) = 1 \) for \( |\xi| \leq 1 \) and \( \eta(\xi) = 0 \) for \( |\xi| \geq 2 \). Define \( \delta(\xi) = \eta(\xi) - \eta(2\xi) \) so that \( \text{supp}(\delta) \) is contained in the spherical shell \( R = R(1/2, 2) = \overline{B}(0, 2) \setminus B(0, 1/2) \). Further we set \( \delta_j(\xi) = \delta(2^{-j}\xi) \) for \( j \in \mathbb{Z} \) so that \( \text{supp}(\delta_j) \subset R_j = 2^j R \). For \( \xi \neq 0 \) we observe that

\[
\sum_{j=-l}^l \delta_j(\xi) = \sum_{j=-l}^l \left( \eta(2^{-j}\xi) - \eta(2^{-(j-1)}\xi) \right) = \eta(2^{-l}\xi) - \eta(2^{l+1}\xi) \to 1
\]

for \( l \to +\infty \). Furthermore the series \( \sum_{j \in \mathbb{Z}} \delta_j \) converges to 1 in the sense of tempered distributions because it is uniformly bounded by 1 and converges pointwise (it is locally finite on \( \mathbb{R}^n \setminus \{0\} \)).

In order to prepare the proof of Proposition 4.6 we state estimates for the cut-offs \( \psi_j = \psi \delta_j \) of \( \psi \). Note that \( \psi = \sum_{j \in \mathbb{Z}} \psi_j \) converges in the sense of tempered distributions. We omit proofs here.

Lemma 4.3. Let \( r \geq 0 \) and \( \psi \in C^\infty(\mathbb{R}^n \setminus \{0\}) \) such that for every multi-index \( \gamma \) there exists some constant \( A_\gamma > 0 \) such that

\[
|(D^\gamma \psi)(\xi)| \leq A_\gamma |\xi|^{-r-|\gamma|}
\]

for all \( \xi \neq 0 \). Then there exist \( A'_\gamma > 0 \) such that \( \|(D^\gamma \psi_j)(\xi)\| \leq A'_\gamma |\xi|^{-r-|\gamma|} \) for \( \xi \neq 0 \). The new constants \( A'_\gamma \) depend on \( |D^\gamma \delta|_\infty \) and \( A_\nu \) for \( \nu \leq \gamma \), but not on \( j \).

Furthermore we have

Lemma 4.4. Assume that \( \|(D^\gamma \psi_j)(\xi)\| \leq A'_\gamma |\xi|^{-r-|\gamma|} \) for \( \xi \neq 0 \). Then it follows

\[
|D^\gamma(\delta_j \psi_j)(\xi)| \leq A''_\gamma |\xi|^{r+|\beta|-|\gamma|}
\]

for \( \xi \neq 0 \) where \( A''_\gamma \) depends on \( A'_\nu \) for \( \nu \leq \gamma \).

Proposition 4.6 relies on the following two estimates involving the geometric series.

Lemma 4.5. If \( m, x > 0 \) are real, then

\[
\sum_{2^j \leq x^{-1}} 2^{jm} \leq 2x^{-m} \quad \text{and} \quad \sum_{2^j > x^{-1}} 2^{-jm} \leq 2x^m
\]

where \( j \in \mathbb{Z} \).
These estimates show that $|\sum R_j(z)| \leq C_\beta |z|^{-(n+r+|\beta|)}$
for all $z \neq 0$. Since $\psi$ defines a tempered distribution, there exists a distribution $u_\psi \in \mathcal{S}'(\mathbb{R}^n)$ such that $\widehat{u_\psi} = \psi$. Now it follows that there is a function $k \in C^\infty(\mathbb{R}^n \setminus \{0\})$
such that
\begin{equation}
(4.7) \quad |(D_2^\beta k_j)(z)| \leq C_\beta |z|^{-(n+r+|\beta|)}
\end{equation}
for all $z \neq 0$ and
\begin{equation}
(4.8) \quad \langle u_\psi, \varphi \rangle = \int_{\mathbb{R}^n} k(z)\varphi(z) \, dz
\end{equation}
for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\text{supp}(\varphi) \subset \mathbb{R} \setminus \{0\}$. Furthermore $u_\psi$ has finite order
$\leq \min\{q \in \mathbb{N} : n/2 + r < q\}$.

Proof. Let 
$k_j(z) = \psi_j^\#(z) = (2\pi)^{-n} \int_{\mathbb{R}^n} \psi_j(\xi)e^{i\xi z} \, d\xi$
be the inverse Fourier transform of $\psi_j$. Since $\psi = \sum_{j \in \mathbb{Z}} \psi_j$, it follows that $u = \sum_{j \in \mathbb{Z}} k_j$
is convergent in the sense of tempered distributions because Fourier transformation is continuous w.r.t. the topology of $\mathcal{S}'(\mathbb{R}^n)$. We shall estimate $\sum_{j \in \mathbb{Z}} |(D_2^\beta k_j)(z)|$
for $z \neq 0$: Since $2^{j-1} \leq |\xi| \leq 2^{j+1}$ for $\xi \in R_j$, Lemma 4.4 implies 
\[ |D_2^\beta(\xi^j \psi_j)(\xi)| \leq A_\gamma |\xi|^{|r+|\beta|-|\gamma|} \leq A_\gamma c_{\beta,\gamma} 2^{j(|r+|\beta|-|\gamma|)} \]
where $c_{\beta,\gamma} = \max\{2^{r+|\beta|}, 2^{\gamma}\}$. Consequently 
\begin{align*}
|z^j(D_2^\beta k_j)(z)| &= |(D_2^\gamma(\xi^j \psi_j))^\#(z)| \leq (2\pi)^{-n} |D_2^\gamma(\xi^j \psi_j)|_1
\leq (2\pi)^{-n} |D_2^\gamma(\xi^j \psi_j)|_{\infty} \text{vol}(R_j)
\leq (2\pi)^{-n} A_\gamma c_{\beta,\gamma} \text{vol}(R) 2^{j(n+r+|\beta|-|\gamma|)}
\end{align*}
where $\text{vol}(R_j) = 2^{jn} \text{vol}(R)$ denotes the Lebesgue measure of the shell $R_j$. The validity of this inequality for all $|\gamma| = M$
shows that there exists some $C_{\beta,M} > 0$ such that 
\begin{equation}
(4.7) \quad |(D_2^\beta k_j)(z)| \leq C_{\beta,M} |z|^{-M} 2^j(n+r+|\beta|-M)
\end{equation}
for all $z \neq 0$. Putting $M = 0$ it follows by Lemma 4.3 that 
\begin{equation}
\sum_{2^j \leq |z|^{-1}} |(D_2^\beta k_j)(z)| \leq 2C_{\beta,0} |z|^{-(n+r+|\beta|)},
\end{equation}
and for $M > n + r + |\beta|$ we get 
\begin{equation}
\sum_{2^j \geq |z|^{-1}} |(D_2^\beta k_j)(z)| \leq C_{\beta,M} |z|^{-M} \sum_{2^j \geq |z|^{-1}} 2^j(n+r+|\beta|-M) \leq 2C_{\beta,M} |z|^{-(n+r+|\beta|)}.
\end{equation}
These estimates show that $\sum_{j \in \mathbb{Z}} |D_2^\beta k_j|$ is uniformly convergent on compact subsets of $\mathbb{R}^n \setminus \{0\}$ so that $k = \sum_{j \in \mathbb{Z}} k_j$ defines a smooth function on $\mathbb{R}^n \setminus \{0\}$ which satisfies (4.7)
and (4.8). The latter assertion of this proposition follows by means of the Plancherel theorem.
In view of later applications we consider functions of the form

$$\psi(ξ) = ξ^β \left( 1 + |ξ|^2 \right)^{-q} |ξ|^r \log^s |ξ|$$

for $ξ \neq 0$ where $s$ is an integer, $β \in \mathbb{N}$, and $r, q$ are real. Here $|ξ|$ denotes the Euclidean norm of $ξ \in \mathbb{R}^n$. It follows by induction that its derivatives $D_ξ^α ψ$ are $C$-linear combinations of functions of the form $ξ \mapsto ξ^β (1 + |ξ|^2)^{-q} |ξ|^r \log^s |ξ|$ where $s', β', q', r'$ are as above and such that $|β' - 2q + r| = |β| - 2q + r - |α|$ and $|β'| + r' ≥ |β| + r - |α|$. Assume that $2q > |β| + r > 0$ and choose $0 < ε < |β| + r$. Now it is easy to see that there exists some $A_α > 0$ such that $|\text{(summand of } D_ξ^α ψ)(ξ) | ≤ A_α |ξ|^{-α}$ for $ξ \neq 0$ so that $ψ$ meets the assumptions of Proposition 4.6.

For (spherically symmetric) functions $ψ$ of this kind Gelfand and Shilov computed the tempered distribution $T_ψ$ explicitly using methods of complex analysis (Cauchy’s theorem and analytic continuation), see Section 3.3 of Chapter II of [11].

Recall that a tempered distribution $u$ which satisfies equation 4.8 of the preceding proposition for all $\text{supp}(φ) \subset \mathbb{R}^n \setminus \{0\}$ is almost uniquely determined: Any difference of two such distributions has support $\{0\}$ and is thus a linear combination of derivatives of the Dirac delta distribution.

Conversely, assume that $k \in C(\mathbb{R}^n \setminus \{0\})$ has an algebraic singularity of order $≤ m$ in 0 (here we choose $m ≥ 0$ to be the minimal integer such that $z \mapsto |z|^n |k(z)|$ is bounded in a neighborhood of 0), and that $k$ has decay of order $n + r$ at infinity (there exists some $C > 0$ such that $|k(z)| ≤ C|z|^{-(n+r)}$ for all $|z| ≥ 1$). In particular $k \in L^1(\mathbb{R}^n)$ and for $0 < r_0 < r$ the function $z \mapsto |z|^{n+r_0} |k(z)|$ vanishes at infinity. By regularization of the divergent integral $\int_{\mathbb{R}^n} k(y)φ(y) \, dy$ we can now define a tempered distribution $u$ on $\mathbb{R}^n$: Let $B = \overline{B}(0, 1)$ be the closed ball of radius 1 around 0. For $m ≥ 0$ and $φ \in S(\mathbb{R}^n)$ let

$$(P_{m-1} φ)(y) = \sum_{|ν|≤ m-1} \frac{1}{ν!} (\partial^ν φ)(0) \, y^ν$$

denote its Taylor polynomial of order $m-1$ in 0, and

$$(R_m φ)(y) = φ(y) - (P_{m-1} φ)(y) = \sum_{|ν|=m} \frac{1}{ν!} (\partial^ν φ)(∂y) \, y^ν$$

the remainder term, where $0 ≤ ∂ ≤ 1$ is chosen suitably depending on $y$. Clearly

$$\langle u, φ \rangle = \int_B k(y) (R_m φ)(y) \, dy + \int_{\mathbb{R}^n \setminus B} k(y)φ(y) \, dy$$

defines a tempered distribution $u$ satisfying equation 4.8 of Proposition 4.6. Here we use the estimate

$$| (R_m φ)(y) | ≤ \left( \sum_{|ν|=m} \frac{1}{ν!} \sup\{|(D_ν^ν φ)(w)| : |w| ≤ |y|\} \right) |y|^m.$$
Observe that \( u \) has order \( \leq m \) and that \( u = u_1 + u_2 \) is a sum of a distribution \( u_1 \in \mathcal{E}'(\mathbb{R}^n) \) of compact support and a distribution \( u_2 \) given by a continuous function \( k \in \mathcal{C}(\mathbb{R}^n) \) such that \( |k(z)| \leq C |z|^{-n+r} \) for \( |z| \geq 1 \), for some \( r > 0 \). To see this we define \( u_1 = \chi u \) and \( u_2 = (1 - \chi) u \) with \( \chi \in \mathcal{C}_0^\infty(\mathbb{R}^n) \) such that \( 0 \leq \chi \leq 1 \), \( \chi(z) = 1 \) for \( |z| \leq 1 \), and \( \chi(z) = 0 \) for \( |z| \geq 2 \).

If \( u \in \mathcal{D}'(\mathbb{R}^n) \) is given by a function \( k \in \mathcal{L}^p(\mathbb{R}^n) \) for some \( 1 \leq p \leq \infty \), or if \( u \in \mathcal{E}'(\mathbb{R}^n) \), then for \( r_0 > 0 \) the distribution \( u \) extends to a continuous linear functional on the Fréchet space \( \mathcal{Q}([0, r_0], r_0) \) of all functions \( a \in \mathcal{C}_0^\infty(\mathbb{R}^n) \) such that \( z \mapsto |z|^{n+r} |D_\beta a(z)| \) vanishes at infinity for all multi-indices \( \beta \). Here the topology of \( \mathcal{Q}(\mathbb{R}^n, r_0) \) is defined by the semi-norms

\[
N_m(a) = \sum_{|\beta| \leq m} (1 + |z|)^{n+r} |D_\beta a| \leq \infty
\]

for \( m \geq 0 \). Note that \( \mathcal{Q}(\mathbb{R}^n, r_0) \) contains \( \mathcal{C}_0^\infty(\mathbb{R}^n) \) as a dense subspace and is invariant under differentiation, translation \( (\tau_z a)(y) = a(y-z) \), and reflection \( \tilde{a}(y) = a(-y) \). Any \( u \in \mathcal{Q}'(\mathbb{R}^n, r_0) \) defines a smooth function

\[
(u \ast a)(z) = \langle u, \tau_z \tilde{a} \rangle
\]
on \( \mathbb{R}^n \) with \( D_\beta^2 (u \ast a) = u \ast (D_\beta^2 a) \).

**Lemma 4.9.** Let \( r_0 > 0 \) and \( a \in \mathcal{C}_0^\infty(\mathbb{R}^n) \) such that \( z \mapsto |z|^{n+r_0} |D_\beta a(z)| \) vanishes at infinity for all multi-indices \( \beta \). Assume that either

(i) \( u \) is given by a function \( k \in \mathcal{C}(\mathbb{R}^n) \) such that \( z \mapsto |z|^{n+r_0} |k(z)| \) vanishes at infinity,

(ii) or that \( u \in \mathcal{E}'(\mathbb{R}^n) \) has finite order \( m \).

Then \( u \ast a \) is a smooth function such that \( z \mapsto |z|^{n+r_0} (u \ast a)(z) \) vanishes at infinity for all \( \beta \). Furthermore \( \hat{u} = \psi \) is a continuous function of polynomial growth and \( (u \ast a) \hat{\langle} \xi \hat{\rangle} = \psi(\xi) \hat{a}(\xi) \).

**Proof.** By induction it suffices to prove that \( z \mapsto |z|^{n+r_0} (u \ast a)(z) \) vanishes at infinity. In the first case \( (u \ast a)(z) = \int_{\mathbb{R}^n} k(y) a(z-y) \, dy \). From \( |z| \leq |z-y| + |y| \leq 2 \max \{|z-y|, |y|\} \) we deduce

\[
|z|^{n+r_0} |(u \ast a)(z)| \leq 2^{n+r_0} \int_{\mathbb{R}^n} |y|^{n+r_0} |k(y)||a(z-y)| \, dy
\]

\[
+ 2^{n+r_0} \int_{\mathbb{R}^n} |k(y)||z-y|^{n+r_0} |a(z-y)| \, dy .
\]

We estimate the first integral on the right hand side. Let \( \epsilon > 0 \) be arbitrary. Choose \( R > 0 \) such that \( |y|^{n+r_0} |a(y)| \leq \epsilon \) and \( |y|^{n+r_0} |k(y)| \leq \epsilon \) for all \( |y| \geq R \), and \( C > 0 \) such that \( |y|^{n+r_0} |k(y)| \leq C \) for all \( y \in \mathbb{R}^n \). Let \( B = B(0, R) \) be the open ball of radius \( R \). For \( |z| \geq 2R \) we obtain

\[
\int_B |y|^{n+r_0} |k(y)||a(z-y)| \, dy \leq \epsilon \cdot C \int_{\mathbb{R}^n\setminus B} |y|^{-(n+r_0)} \, dy
\]

Since \( C > 0 \) is arbitrary, we may choose \( R > 0 \) such that \( \epsilon \cdot C \leq \epsilon \).

Therefore, \( u \ast a \) vanishes at infinity for all \( \beta \). Furthermore \( \hat{u} = \psi \) is a continuous function of polynomial growth and \( (u \ast a) \hat{\langle} \xi \hat{\rangle} = \psi(\xi) \hat{a}(\xi) \).
because \( y \in B \) implies \( |z - y| \geq R \). Furthermore
\[
\int_{\mathbb{R}^n \setminus B} |y|^{n+r_0} |k(y)| |a(z - y)| \, dy \leq c \int_{\mathbb{R}^n} |a(y)| \, dy.
\]
The second integral can be treated similarly. Thus \(|z|^{n+r_0}(u*a)(z)| \leq C'\varepsilon \) for \(|z| \geq 2R\).

Finally we assume \( u \in \mathcal{E}'(\mathbb{R}^n) \) so that \( \text{supp}(u) \subset K = \overline{B}(0,R) \) for \( R > 0 \) large enough. There exists a \( C > 0 \) such that
\[
|\langle u, a \rangle| \leq C \sum_{|\nu| \leq m} |D_y^\nu a|_{K,\infty}
\]
for all \( a \in C^\infty(\mathbb{R}^n) \). Note that \( D_y^\nu (\tau_{z\tilde{a}})(y) = (-1)^{|\nu|}(D_y^\nu a)(z - y) \). Since \(|z| \leq |z - y| + |y| \leq 2|z - y| \) for \( y \in K \) and \(|z| \geq 2R\), we get
\[
|z|^{n+r_0} |\langle u, \tau_{z\tilde{a}} \rangle| \leq C \sum_{|\nu| \leq m} |z|^{n+r_0} |D_y^\nu (\tau_{z\tilde{a}})|_{K,\infty}
\leq C 2^{n+r_0} \sum_{|\nu| \leq m} \sup \{ |z - y|^{n+r_0} |(D_y^\nu a)(z - y)| : y \in K \}
\]
which tends to 0 for \(|z| \rightarrow +\infty\). The second claim of this lemma is a consequence of the Paley-Wiener theorem for \( u \in \mathcal{E}'(\mathbb{R}^n) \), and trivial for \( u = k \in L^1(\mathbb{R}^n) \). \( \square \) \( \square \)

Let us return to the global situation on \( M \) and resume the discussion of Section 3. Using the function space \( \mathcal{Q} \) introduced in Definition 3.1 we state a refinement of Definition 3.3.

**Definition 4.10.** A continuous function \( \psi \) on \( \mathfrak{m}^* \) is called a central Fourier multiplier if for all \( a \in \mathcal{Q} \) there exists a (unique) function \( c \in \mathcal{Q} \) such that \( \hat{c}(x,\xi) = \psi(\xi)\hat{a}(x,\xi) \) for all \( x \) and \( \xi \). These functions form a subalgebra \( \mathcal{M} \) of \( \mathcal{C}(\mathfrak{m}^*) \) containing the polynomial functions and the Schwartz functions.

If we interpret the solution \( c \) as a function on \( M \) rather than on \( \mathbb{R}^k \times \mathfrak{m} \), then its definition does not depend on the choice of the coordinates furnished by a coexponential basis \( B \) for \( \mathfrak{m} \) on \( M \).

Let us fix a direct sum decomposition \( \mathfrak{m} = \mathfrak{z} \oplus \mathfrak{\xi} \) of the center of \( \mathfrak{m} \) and denote the central variable by \((z,\bar{z})\). This also gives a decomposition \( \mathfrak{m}^* = \mathfrak{z}^* \oplus \mathfrak{\xi}^* \) of the linear dual with variable \((\xi,\bar{\xi})\). Let us identify \( \mathfrak{z} \) with \( \mathbb{R}^n \). Assume that \( a \in \mathcal{Q} \) and \( r_0 > 0 \) such that
\[
(x, z, \bar{z}) \mapsto |(z, \bar{z})|^{n+r_0} |(D_x^\alpha D_z^\beta D_{\bar{z}}^\gamma a)(x, z, \bar{z})|
\]
vanesishes at infinity. Clearly \( a^\xi(x,\bar{z})(z) = a(x, z, \bar{z}) \) defines a smooth function \( a^\xi : \mathbb{R}^k \times \mathfrak{z} \rightarrow \mathcal{Q}(\mathfrak{z},r_0) \) with \( D_x^\alpha D_z^\beta D_{\bar{z}}^\gamma a^\xi = (D_x^\alpha D_z^\beta a)^\xi \). Further any \( u \in \mathcal{Q}'(\mathfrak{z},r_0) \) gives rise to a smooth function
\[
(u * a)(x, z, \bar{z}) = \langle u, \tau_{z\bar{z}}(a^\xi(x, \bar{z})) \rangle
\]
on \( M \) such that \( D_x^\alpha D_z^\beta D_{\bar{z}}^\gamma (u * a) = u * (D_x^\alpha D_z^\beta D_{\bar{z}}^\gamma a) \). Here translation and reflection affect only the variable \( z \).
Lemma 4.11. Assume that \( u \in D'(\mathfrak{g}) \) is given by a function \( k \in C(\mathfrak{g}) \) such that \( |z|^{n+r} |k(z)| \) vanishes at infinity for some \( r > 0 \), or that \( u \in E'(\mathfrak{g}) \). It follows \( u \ast a \in Q \) for all \( a \in Q \). In particular \( \psi = \hat{u} \) lies in \( \mathcal{M} \) when interpreted as a function on \( Z^m \).

Proof. We know that \( \psi = \hat{u} \) is a continuous function of polynomial growth, and that \( u \ast a \) is well-defined, smooth and of compact support in \( x \)-direction. Choose \( 0 < r_0 \leq r \) as above. It remains to be shown that the derivatives of \( u \ast a \) multiplied by the factor \( |(z, \bar{z})|^{n+r_0} \) vanish at infinity. But this follows as in the proof of Lemma 4.9 by analogous estimates performed uniformly in \( x \) and \( \bar{z} \). Evidently the partial Fourier transform of \( u \ast a \) w. r. t. the central variable \( (z, \bar{z}) \) satisfies \( (u \ast a)^\wedge (x, \xi, \bar{\xi}) = \psi(\xi) \hat{a}(x, \xi, \bar{\xi}) \) which proves \( \psi \in \mathcal{M} \).

At last we discuss a class of functions \( \psi \) which arise naturally as central Fourier multipliers in studying the primitive \( * \)-regularity of exponential Lie groups. Let \( Z^m = \mathfrak{z}_1 \oplus \ldots \oplus \mathfrak{z}_p \oplus \hat{\mathfrak{z}} \) be a direct sum decomposition of the center of \( \mathfrak{m} \) with a Euclidean norm on each of these subspaces, and \( Q = Q(r_1, \ldots, r_p) \) a complex-valued polynomial function in \( l' \) real variables. The function

\[
\psi(\xi, \bar{\xi}) = |\xi_1| \cdots |\xi_{l'}| \cdot Q(\log |\xi_1|, \ldots, \log |\xi_{l'}|)
\]

on \( Z^m \) is a linear combination of products of functions \( \xi \mapsto |\xi| \log^s |\xi| \) defined on one of the subspaces \( \mathfrak{z}_b \). Since the polynomial \( 1+|\xi|^2 \) is in \( \mathcal{M} \) and \( \psi_0(\xi) = (1+|\xi|^2)^{-1} |\xi| \log^s |\xi| \) is in \( \mathcal{M} \) by Proposition 4.6 and Lemma 4.11 it follows that \( \psi \in \mathcal{M} \) is a central Fourier multiplier in the sense of Definition 4.10 because \( \mathcal{M} \) is an associative algebra.

5 A functional calculus for central elements

We fix a coexponential basis for \( \mathfrak{z} \) in \( \mathfrak{m} \) as in the beginning of Section 3 and work with the coordinates of the second kind associated to it. Let \( Q \) be as in Definition 4.1. If \( L \) is a compact subset of \( \mathbb{R}^d \) and \( r_0 > 0 \), then \( Q(L, r_0) \) denotes the subspace of all smooth functions \( a \) on \( M \) such that \( a(x, z) = 0 \) whenever \( x \notin L \) and such that \( (x, z) \mapsto |z|^{n+r_0} \quad \left( D_x^n D_z^r a \right)(x, z) \) vanishes at infinity. As a topological vector space \( Q \) is the inductive limit (convex hull) of the Fréchet spaces \( Q(L, r_0) \). In particular \( Q \) is a (LF)-space (but not a strict one), compare §19 of [13]. The definition of the topology of \( Q \) does not depend on the choice of the coexponential basis. Clearly, the universal enveloping algebra \( \mathcal{U}(\mathfrak{m}_C) \) acts on \( Q \) (in the natural way) as an algebra of continuous linear operators.

Let \( \mathcal{M} \) denote the algebra of central Fourier multipliers introduced in Definition 4.10. If \( \psi \in \mathcal{M} \) and \( a \in Q \), then \( T_\psi a \) denotes the unique function in \( Q \) such that \( \langle T_\psi a \rangle (x, \xi, \bar{\xi}) = \psi(\xi) \hat{a}(x, \xi) \). Note that \( (\psi, a) \mapsto T_\psi a \) defines a representation of \( \mathcal{M} \) on \( Q \). At least if \( \psi \) has the form \( \psi = \hat{u} \) with \( u = u_1 + u_2 \), \( u_1 \) of compact support, and \( u_2 \) given by a continuous function \( k \) of growth \( |k| \leq C |z|^{-(n+r_0)} \), then \( T_\psi \) is a continuous operator on \( Q \). The set of all multipliers \( \psi \) for which \( T_\psi \) is continuous is a subalgebra of \( \mathcal{M} \).
If $a \in Q$, $Z \in \mathfrak{z}m$, and $\psi(\xi) = \langle \xi, Z \rangle$, then

$$(Z * a)(x, y) = \frac{d}{dt}|_{t=0} a(\exp(-tZ) \cdot (x, y)) = -\langle \partial a(x, y), Z \rangle = -\langle \partial_Z a(x, y) \rangle$$

implies $(iZ * a)^\wedge(x, \xi) = \langle \xi, Z \rangle \wedge a(x, \xi)$ which proves $T_\psi a = iZ * a$. Here $\partial a : M \to \text{Hom}_\mathbb{R}(\mathfrak{z}m, \mathbb{C})$ is the derivative of $a$ w.r.t. the central variable, and $\partial_Z a$ the directional derivative.

Since the action of $\mathcal{M}$ on $Q$ commutes with the action of $\mathcal{U}(m_C)$, we see that $\mathcal{M}$ extends $\mathcal{U}(\mathfrak{z}m_C) = \mathcal{S}(\mathfrak{z}m_C)$ with their symbols. In this sense we have enlarged the features of the symmetric algebra of $\mathfrak{z}m$ from polynomial functions to (certain) functions of polynomial growth.

Finally we would like explain the heading of this section: Let $Z \in \mathfrak{z}m$ be a central element. We know that $iZ * -$ acts as a differential operator on $Q \subset L^1(M)$ and we want to declare the notion of functions of this operator. It follows from $(iZ * a)^\wedge(x, \xi) = \langle \xi, Z \rangle \wedge a(x, \xi)$ that this operator is diagonalized by partial Fourier transformation. Let $\psi_0 : \mathbb{R} \to \mathbb{C}$ be a continuous function such that $\xi \mapsto \psi(\xi) = \psi_0(\langle \xi, Z \rangle)$ is in $\mathcal{M}$. It is a basic idea of any definition of $\psi_0(iZ * -)$ that $(\psi_0(iZ * -)a)^\wedge(x, \xi) = \psi_0(\langle \xi, Z \rangle) \wedge a(x, \xi)$ should hold. Thus the definition $\psi_0(iZ * -)a := T_\psi a$ appears to be reasonable and we have indeed established a functional calculus for central elements.

## 6 Two non-*-regular exponential Lie groups

Our aim is to prove that the following two significant examples of non-*-regular exponential Lie groups have the weaker property of primitive *-regularity, see Definition 1 of [21]. The results of the preceding sections (in particular those related to separating triples consisting of a Duflo pair $(W, p)$ and a central Fourier multiplier $\psi$) turn out to be appropriate for this purpose. A first example (of minimal dimension) has already been discussed in [21]. In order to prove the primitive *-regularity of an exponential Lie group $G$ we pursue the strategy developed in Section 5 of [21].

For the convenience of the reader we provide a brief history of *-regularity. In [4, 1978] Boidol and Leptin initiated the investigation of the class $[\Psi]$ of *-regular locally compact groups. Far reaching results have been obtained in this direction. First Boidol has characterized the *-regular ones among all exponential Lie groups by a purely algebraic condition on the stabilizers $m = g_f + n$ of linear functionals $f \in g^*$, see Theorem 5.4 of [2] and Lemma 2 of [18]. More generally Boidol has proved in [3] that a connected locally compact group is *-regular if and only if all primitive ideals of $C^*(G)$ are (essentially) induced from a normal subgroup $M$ whose Haar measure has polynomial growth. In [18] Poguntke has determined the simple modules of the group algebra $L^1(G)$ for exponential Lie groups $G$. From this classification he deduced that an exponential Lie group $G$ is *-regular if and only if it is symmetric, i.e., $a^*a$
has positive spectrum for all \( a \in L^1(G) \), see Theorem 10 of [18]. A complete list of all non-symmetric solvable Lie algebras up to dimension 6 can be found in [16]. For a definition of primitive \( * \)-regularity and \( L^1 \)-determined ideals we refer to [21].

As in Section 5 of [21] we fix a coabelian, nilpotent ideal \( n \) (e.g. the nilradical, i.e., the largest nilpotent ideal) of the Lie algebra \( g \) of \( G \). Now it suffices to verify the following two assertions:

1. Every proper quotient \( \bar{G} \) of \( G \) is primitive \( * \)-regular.

2. If \( f \in g^* \) is in general position such that the stabilizer \( m = g_f + n \) is a proper, non-nilpotent ideal of \( g \) and if \( g \in g^* \) is critical for the orbit \( \text{Ad}^*(G)f \), then it follows

\[
\ker_{L^1(G)} \pi \not\subset \ker_{L^1(G)} \rho
\]

for the unitary representations \( \pi = \mathcal{K}(f) \) and \( \rho = \mathcal{K}(g) \).

We say that \( f \) is in general position if \( f \neq 0 \) on any non-trivial ideal of \( g \). Here \( g_f = \{ X \in g : [X, g] \subset \ker f \} \) is the stabilizer of \( f \) w.r.t. the coadjoint action of \( g \) on \( g^* \). Note that the ideal \( m = g_f + n \) does not depend on the choice of the representative \( f \) of the coadjoint orbit \( \text{Ad}^*(G)f \). Let \( \Omega \) denote the set of all \( h \in g^* \) such that its restriction \( h'|n = h|n \) is contained in the closure of \( \text{Ad}^*(G)f' \) in \( n^* \). We say that \( g \in g^* \) is critical w.r.t. the orbit \( X = \text{Ad}^*(G)f \) if and only if \( g \in \Omega \setminus X \).

When restricting to the stabilizer \( M \) with Lie algebra \( m = g_f + n \), the representation \( \pi \) in general position decomposes into a direct integral of irreducible representations \( \pi_s = \mathcal{K}(f_s) \) of \( M \), and in the Kirillov picture the associated coadjoint orbit \( \text{Ad}^*(G)f \) decomposes into the disjoint union of the orbits \( \text{Ad}^*(M)f_s \). Now it is easy to see that we can replace the second assertion by the following equivalent one:

3. Let \( m \) be a proper, non-nilpotent ideal of \( g \) such that \( m \supset n \). If \( f \in m^* \) is in general position such that \( m = m_f + n \) and if \( g \in m^* \) is critical for the orbit \( \text{Ad}^*(G)f \), then the relation

\[
\bigcap_{s \in R^m} \ker_{L^1(M)} \pi_s \not\subset \ker_{L^1(M)} \rho
\]

holds for the representations \( \pi_s = \mathcal{K}(f_s) \) and \( \rho = \mathcal{K}(g) \).

At this point the results of the preceding sections come into play. If one can prove the existence of (a finite set of) separating triples for \( X = \text{Ad}^*(G)f \) in \( \Omega \subset m^* \) in the sense of Definition 3.3, then the asserted relation for the \( L^1 \)-kernels of the associated irreducible representations follows at once.

Now we delve into the details of our first example. Let \( G \) be a simply connected, connected, solvable Lie group such that the nilradical \( n \) of its Lie algebra \( g \) is a trivial extension of the five-dimensional, two-step nilpotent Lie algebra \( g_{5,2} \) so that

\[
g \supset m \supset n \supset C^1n \supset C^1C^1n \supset \{0\}
\]
is a descending series of ideals of \( g \). Assume that \( d, e_0, \ldots, e_6 \) is a basis of \( g \) with commutator relations \([e_1, e_2] = e_4\), \([e_1, e_3] = e_5\), \([e_0, e_4] = -e_1\), \([e_0, e_5] = e_2\), \([e_0, e_6] = e_3\), \([d, e_0] = -ae_6\), \([d, e_2] = e_2\), \([d, e_3] = be_3\), \([d, e_4] = e_4\), \([d, e_5] = be_5\) where \( a, b \in \mathbb{R} \) and \( b \neq 0 \). Furthermore we assume that \( m = \langle e_0, \ldots, e_6 \rangle \) and that \( f \in m^* \) is in general position such that \( m = m_f + n \). In particular \( f \neq 0 \) on the one-dimensional ideal spanned by \( e_\nu \), for all \( 4 \leq \nu \leq 6 \).

The algebraic structure of \( g \) is characterized by the fact that the nilpotent subalgebra \( s = \langle d, e_0, e_6 \rangle \) acts semi-simply on the nilradical \( n = \langle e_1, \ldots, e_6 \rangle \) with weights \( \alpha, \gamma - \alpha, b_\gamma - \alpha, \gamma, b_\gamma, 0 \) where \( \alpha, \gamma \in s^* \) are linearly independent and given by \( \alpha(e_0) = -1, \alpha(d) = \alpha(e_6) = 0 \) and \( \gamma(d) = 1, \gamma(e_0) = \gamma(e_6) = 0 \).

**Lemma 6.1.** If \( f \in m^* \) is in general position such that the stabilizer condition \( m = m_f + n \) is satisfied, then there exists a representative \( f \) on the orbit \( \text{Ad}^*(M)f = \text{Ad}^*(N)f \) such that \( f_1 = f_2 = f_3 = 0 \).

**Proof.** Since \( f_4 \neq 0 \), the equations

\[
\text{Ad}^*(\exp(we_2))f (e_1) = f_1 + wf_4 \\
\text{Ad}^*(\exp(ve_1))f (e_2) = f_2 - vf_4
\]

show that we can establish \( f_1 = 0 \) and \( f_2 = 0 \). Here we abbreviate \( f(e_\nu) \) by \( f_\nu \). Since \( m = m_f + n \), there is some \( X = te_0 + ve_1 + we_2 + xe_3 + Z \in m_f \) such that \( t \neq 0 \). Now \([X, e_2] = te_2 + ve_4 \) and \([X, e_3] = te_3 + ve_5\) implies \( 0 = vf_4 \) and \( 0 = tf_3 + vf_5 \). Since \( f_4 \neq 0 \) and \( t \neq 0 \), it follows \( v = 0 \) and \( f_3 = 0 \). \( \square \)

In the sequel we fix \( f \in m^* \) such that \( f_\nu = 0 \) for \( 1 \leq \nu \leq 3 \) and \( f_\nu \neq 0 \) for \( 4 \leq \nu \leq 6 \). By adjusting the basis vectors \( e_2, \ldots, e_6 \) we can even establish \( f_\nu = 1 \) for \( 4 \leq \nu \leq 6 \).

Using coordinates of the second kind given by the diffeomorphism \( \Phi(t, v, w, x, Z) = \exp(te_0) \exp(ve_1) \exp(we_2 + xe_3 + Z) \) we compute

\[
\text{Ad}^*(\exp(sd)\Phi(t, v, w, x, Z))f (e_0) = f_0 + as - v(w + x), \\
(e_1) = e^t (w + x), \\
(e_2) = -e^{-(s+t)} v, \\
(e_3) = -e^{-(bs+t)} v, \\
(e_4) = e^{-s}, \\
(e_5) = e^{-bs}
\]

for the coadjoint action of \( G \) on \( m^* \). These formulas motivate the definition of the polynomials \( p_1 = e_0 e_4 - e_1 e_2, p_2 = e_0 e_5 - e_1 e_3, \) and \( p_3 = e_2 e_5 - e_3 e_4 \). Here \( e_\nu \) means the linear function \( f \mapsto f(e_\nu) \) on \( m^* \), considered as an element of \( \mathcal{P}(m^*) \), the commutative algebra of complex-valued polynomial functions on \( m^* \). Recall that \( M \) acts on \( \mathcal{P}(m^*) \) by \( \text{Ad}(m)p(f) = p(\text{Ad}^*(m)^{-1}f) \).

Note that these three polynomial functions are constant on the orbits \( \text{Ad}^*(M)f_s \)
for all \( f_s = \text{Ad}^*(\exp(sd))f \), but none of them is \( \text{Ad}(M) \)-invariant (constant on all \( \text{Ad}^*(M) \)-orbits).

A first step is to determine the \( n^* \)-closure \( \Omega \) of the orbit \( X = \text{Ad}^*(G)f \). To this end let \( r : m^* \rightarrow n^* \) denote the linear projection given by restriction and define \( \Omega = r^{-1}(r(X)) \). In order to avoid trivialities we shall suppose \( b > 0 \).

**Lemma 6.2.** The \( n^* \)-closure \( \Omega \) of \( X \) is contained in the \( \text{Ad}^*(M) \)-invariant set of all \( h \in m^* \) such that either \( (h_4 > 0, h_5 > 0, h_6 = 1, \log h_5 = b \log h_4, \text{and } p_3(h) = h_2 h_5 - h_3 h_4 = 0) \) or \( (h_4 = h_5 = 0 \text{ and } h_6 = 1) \).

This assertion will be verified in the course of the proof of Lemma 6.4. It follows from Lemma 6.1 that not all linear functionals in general position satisfy the stabilizer condition \( m = m_f + n \). Furthermore \( p_3(h) = 0 \) for all \( h \in \Omega \) so that \( \Omega \) is rather sparse.

**Remark 6.3.** The polynomial functions \( p_1, \ldots, p_3 \) are constant on all \( \text{Ad}^*(M) \)-orbits contained in \( \Omega \) and yield continuous functions on \( \Omega/ \text{Ad}^*(M) \). There do not exist any 'non-trivial' \( \text{Ad}(M) \)-invariant polynomial functions on \( m^* \).

This observation applies to many other examples as well. Retrospectively, it justifies the localization to a certain subset \( \Omega \) of \( m^* \) (or of \( M \)) that we started with in Section 1. Here 'non-trivial' means something like \( p \in \mathcal{S}(m_C) \setminus \mathcal{S}(jm_C) \).

Next we describe the relevant unitary representations of \( M \). Let \( f_s = \text{Ad}^*(\exp(sd))f \) and \( \pi_s = \mathcal{K}(f_s) \). It is easy to see that \( \mathfrak{p} = \langle e_0, e_2, \ldots, e_6 \rangle \) is a Pukanszky-Vergne polarization at \( f_s \) for all \( s \in \mathbb{R} \), and that \( \mathfrak{c} = \langle e_1 \rangle \) is a coexponential subalgebra for \( \mathfrak{p} \) in \( m \). For the infinitesimal operators of the unitary representation \( \pi_s = \text{ind}_M^N \chi_{f_s} \) we compute

\[
\begin{align*}
d\pi_s(e_0) &= \frac{1}{2} + if_0 + as + \xi \partial_\xi, \\
d\pi_s(e_1) &= -\partial_\xi, \\
d\pi_s(e_2) &= -ie^{-s} \xi, \\
d\pi_s(e_3) &= -ie^{-bs} \xi, \\
d\pi_s(e_4) &= ie^{-s}, \\
d\pi_s(e_5) &= ie^{-bs}.
\end{align*}
\]

Now let \( g \in m^* \) be such that \( g_5 = g_4 = 0 \) and \( (g_1 \neq 0 \text{ or } g_2 \neq 0 \text{ or } g_3 \neq 0) \). Then \( n = \langle e_1, \ldots, e_5 \rangle \) is a Pukanszky-Vergne polarization at \( g \). Further \( \mathfrak{c} = \langle e_0 \rangle \) is a coexponential subalgebra for \( n \) in \( m \). Hence \( \rho = \text{ind}_N^M \chi_g \) is infinitesimally given by

\[
\begin{align*}
d\rho(e_0) &= -\partial_\xi, \\
d\rho(e_1) &= ie^\xi g_1, \\
d\rho(e_2) &= ie^{-\xi} g_2, \\
d\rho(e_3) &= ie^{-\xi} g_3, \\
d\rho(e_4) &= d\rho(e_5) = 0.
\end{align*}
\]
The images of $p_1$ and $p_2$ in the universal enveloping algebra $\mathcal{U}(\mathfrak{m}_C)$ under the symmetrization map are given by

$$W_1 = \frac{1}{2}(e_1 e_2 + e_2 e_1) - e_0 e_4 \quad \text{and} \quad W_2 = \frac{1}{2}(e_1 e_3 + e_3 e_1) - e_0 e_5$$

respectively. A short computation shows that $d\pi(W_\nu) = p_\nu(h)\text{Id}$ holds for all $h \in \Omega$ and $\pi = \mathcal{K}(h)$. Thus $(W_\nu, p_\nu)$ is a Duflot pair w.r.t. $\Omega$, for $\nu \in \{1, 2\}$. In addition we define the continuous functions $\psi_1(\xi) = \xi_1(f_0 - a \log |\xi_1|)$ and $\psi_2(\xi) = \xi_2(f_0 - \frac{b}{2} \log |\xi_2|)$ on $3\mathfrak{m}^*$. Here we identify $\mathbb{R}^3$ and $3\mathfrak{m}^*$ via $\xi = (\xi_1, \xi_2, \xi_3) \mapsto \xi_1 e_4 + \xi_2 e_5 + \xi_3 e_6$. Now we can prove

**Lemma 6.4.** Assume that $f \in \mathfrak{m}^*$ is in general position such that $\mathfrak{m} = \mathfrak{m}_f + \mathfrak{n}$. Let $W_\nu, p_\nu, \psi_\nu$ be defined as above. If $\Omega$ denotes the $\mathfrak{n}^*$-closure of the orbit $X = \Ad^*(G)f$, then $\{ (W_\nu, p_\nu, \psi_\nu) : \nu = 1, 2 \}$ is a set of separating triples for $X$ in $\Omega$ in the sense of Definition 4.10. In particular $\psi_\nu$ is a central Fourier multiplier. If $h \in \Omega$, then $h \in \overline{X}$ if and only if $p_\nu(h) = \psi_\nu(h | 3\mathfrak{m})$ for all $\nu$.

**Proof.** The considerations at the end of Section 4 reveal that $\psi_1$ and $\psi_2$ are central Fourier multipliers in the sense of Definition 4.10. By definition $p_\nu(h) = \psi_\nu(h | 3\mathfrak{m})$ for all $h \in X$. The continuity of $p_\nu$ and $\psi_\nu$ yields this equality for all $h \in \overline{X}$. In order to prove the opposite implication, we assume that $h \in \Omega$ such that $p_\nu(h) = \psi_\nu(h | 3\mathfrak{m})$ for $\nu \in \{1, 2\}$. In particular there exist sequences $s_n, v_n, w_n, x_n$ such that $f_n \rightarrow h'$ where

$$f_n = \Ad^* \left( \exp(s_n d)(0, v_n, w_n, x_n, 0) \right) f.$$

At first we suppose $h_4 h_5 \neq 0$. In this case $e^{-s_n} h_4$ and $e^{-b s_n} h_5$ implies $h_4 > 0$, $h_5 > 0$, and $\log h_5 = b \log h_4$. Similarly it follows $p_3(h) = 0$, and $h_6 = 1$ is obvious. If we choose sequences $s_n, \ldots, x_n$ as above, then we obtain

$$e^{-s_n} (f_0 + a s_n - v_n(w_n + x_n)) = \psi_1(f_n | 3\mathfrak{m}) + f_n f_n 2 \rightarrow \psi_1(h | 3\mathfrak{m}) + h_1 h_2 = h_0 h_4$$

because $p_1(h) = \psi_1(h | 3\mathfrak{m})$. Now $e^{-s_n} \rightarrow h_4 \neq 0$ implies $f_{0n} \rightarrow h_0$ and hence $f_n \rightarrow h \in \overline{X}$.

Next we assume $h_4 = 0$ or $h_5 = 0$. We conclude $h_4 = h_5 = 0$ and $b > 0$. Now we must distinguish several subcases. In any case we set $x_n = 0$. First we assume $h_1 \neq 0$. Since $p_\nu(h) = \psi_\nu(h | 3\mathfrak{m}) = 0$ for $\nu \in \{1, 2\}$, it follows $h_2 = h_3 = 0$. We define $s_n = n, w_n = h_1$, and

$$v_n = \frac{1}{h_1} (f_0 + a s_n - h_0)$$

so that $f_n \rightarrow h$. Next we assume $h_1 = 0$ and $(h_2 \neq 0$ or $h_3 \neq 0)$. In this case we choose sequences $s_n$ and $v_n$ such that $f_n(e_\nu) \rightarrow h_\nu$ for $2 \leq \nu \leq 5$. In particular $s_n \rightarrow +\infty$ and $|v_n| \rightarrow +\infty$ exponentially. Further we set

$$w_n = \frac{1}{v_n} (f_0 + a s_n - h_0).$$
Then we obtain \( f_n \rightarrow h \). Finally we assume \( h_\nu = 0 \) for \( 1 \leq \nu \leq 5 \). We define \( s_n = n \), \( v_n = e^{r_n/2} \) and \( w_n = e^{-r_n/2} (f_0 - h_0) \). These definitions imply \( f_n \rightarrow h \). This completes

the proof of our lemma. \[ \Box \]

Observe that both polynomials \( p_1 \) and \( p_2 \) are needed to separate points \( h \in \Omega \) with \( h_4 = h_5 = 0 \), \( h_1 \neq 0 \), and \( (h_2 \neq 0 \text{ or } h_3 \neq 0) \) from the orbit \( X = \text{Ad}^*(G)f \).

As we remarked above, the fact that \( \{(W_\nu, p_\nu, \psi_\nu) : \nu = 1, 2\} \) is a set of separating triples for \( X \in \Omega \) yields

\[
\bigcap_{s \in \mathbb{R}} \ker_{L^1(M)} \pi_s \not\subset \ker_{L^1(M)} \rho
\]

for all critical \( g \in \Omega \) and \( \rho = \mathcal{K}(g) \). Thus \( \ker_{C^*(G)} \pi \) is \( L^1 \)-determined in the sense of Definition 1 of [21] for all representations \( \pi \) in general position such that its stabilizer \( M \) is a non-nilpotent normal subgroup of \( G \). If \( M = N \), then \( \ker_{C^*(G)} \pi \) is \( L^1 \)-determined by Proposition 2.6 and 2.8 of [21]. Up to this point we have shown that \( \ker_{C^*(G)} \pi \) is \( L^1 \)-determined for all \( \pi \) in general position.

If \( \pi \) is not in general position, then we can pass to a proper quotient \( \tilde{G} \) of \( G \). For example, if \( f(e_6) = 0 \), then we can pass to the quotient \( \tilde{g} = \mathfrak{g}/\langle e_6 \rangle \). We assume that \( \tilde{f} \in \tilde{g}^* \) is in general position such that \( \tilde{m} = \tilde{g}f + \tilde{n} \) is a proper, non-nilpotent ideal of \( \tilde{g} \). It is easy to see that in this case \( \tilde{p}_1 = \tilde{e}_0\tilde{e}_4 - \tilde{e}_1\tilde{e}_2 \), \( \tilde{\psi}_1(\xi) = f_0\xi_1 \) and \( \tilde{p}_2 = \tilde{e}_0\tilde{e}_5 - \tilde{e}_1\tilde{e}_3 \), \( \tilde{\psi}_2(\xi) = f_0\xi_2 \) form the ingredients for a set of separating triples for \( \tilde{X} = \text{Ad}^*(\tilde{G})\tilde{f} \) in \( \tilde{\Omega} \). As above it follows that \( \ker_{C^*(\tilde{G})} \tilde{\pi} \) is \( L^1 \)-determined for all \( \pi \) in general position.

Clearly the quotients \( \tilde{g} = \mathfrak{g}/\langle e_4 \rangle \) and \( \tilde{g} = \mathfrak{g}/\langle e_5 \rangle \) can be treated similarly: In these cases we get by on one separating triple. Choose \( \tilde{p} = \tilde{e}_0\tilde{e}_5 - \tilde{e}_1\tilde{e}_3 \), \( \tilde{\psi}(\xi_1\tilde{e}_5 + \xi_2\tilde{e}_6) = \xi_1(f_0 - \frac{a}{h} \log |\xi_1|) \) and \( \tilde{p} = \tilde{e}_0\tilde{e}_4 - \tilde{e}_1\tilde{e}_2 \), \( \tilde{\psi}(\xi_1\tilde{e}_4 + \xi_2\tilde{e}_6) = \xi_1(f_0 - a \log |\xi_1|) \) respectively. The next step is to consider quotients \( \tilde{g} = \mathfrak{g}/\mathfrak{a} \) for two-dimensional ideals \( \mathfrak{a} \subset \langle e_4, e_5, e_6 \rangle \) which brings along nothing new. Finally we consider \( \tilde{g} = \mathfrak{g}/\langle e_4, e_5, e_6 \rangle \) which is primitive \( * \)-regular by Lemma 5.4 of [21] because \( \tilde{n} = [\tilde{g}, \tilde{g}] \) is commutative in this case. Altogether we have shown that the 8-dimensional exponential Lie group \( G \) defined above is primitive \( * \)-regular. In fact, we have thoroughly verified assertions (1) and (2) set up in the beginning of this section, which form our strategy for proving primitive \( * \)-regularity of exponential Lie groups.

**Remark 6.5.** The preceding example indicates the prospects of success of the approach involving separating triples \((W, p, \psi)\). It shows the necessity to deal with sets of these triples and to localize to an appropriate, sparse subset \( \Omega \) of \( \mathfrak{m}^* \). This allows us to consider polynomial functions \( p \) which are \( \text{Ad}(M) \)-invariant on \( \Omega \), but not on the entire space.

Finally we descend to our second example, the exponential Lie algebra \( \mathfrak{g} = \langle d_0, d_1, e_0, \ldots, e_6 \rangle \) with commutator relations \([e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_0, e_1] = -e_1, [e_0, e_2] = 2e_2, [e_0, e_3] = e_3, [d_0, e_0] = -ae_6, [d_0, e_2] = e_2, [d_0, e_3] = e_3, [d_0, e_4] = e_4, \)
Next we compute the relevant unitary representations of $M$ which shows that the Pukanszky-Vergne polarization at $Ad_{\mathfrak{g}}$ subalgebra for $p$ is characterized by the fact that the nilpotent subalgebra $s = \langle d_0, d_1, e_0, e_6 \rangle$ acts semi-simply on $\mathfrak{n}$ with weights $\alpha$, $\gamma$, $\gamma - \alpha$, $\gamma - 2\alpha$, $\delta$, $0$ where $\alpha(e_0) = -1$, $\gamma(d_0) = 1$, $\delta(d_1) = 1$, and the other values are zero.

Let $m = \langle e_0, \ldots, e_6 \rangle$ and $f \in \mathfrak{m}^*$ be in general position such that $m = \mathfrak{m}_f + \mathfrak{n}$. In particular $f_\nu \neq 0$ for $4 \leq \nu \leq 6$, even if $f_\nu = 1$ without loss of generality. It follows from

\[
\begin{align*}
\text{Ad}^*(\exp(ve_1))f(e_3) &= f_3 - vf_4 \\
\text{Ad}^*(\exp(xe_3))f(e_1) &= f_1 + xf_4
\end{align*}
\]

that we can establish $f_1 = f_3 = 0$. Since $m = \mathfrak{m}_f + \mathfrak{n}$, there is some $X = te_0 + ve_1 + we_2 + xe_3 + Z \in \mathfrak{m}_f$ with $t \neq 0$. Now $0 = f([X, e_3]) = tf_1 + vf_4$ and $0 = f([X, e_2]) = tf_2 + vf_3$ implies $v = 0$ and $f_2 = 0$.

In coordinates $\Phi(t, w, x, Z) = \exp(te_0) \exp(ve_1) \exp(we_2 + xe_3 + Z)$ we compute

\[
\begin{align*}
\text{Ad}^*(\exp(rd_0) \exp(sd_1) \Phi(t, v, w, x, Z))f(e_0) &= f_0 + ar + bs - vx, \\
(e_1) &= e^t x, \\
(e_2) &= \frac{1}{2} e^{-(r+2t)} v^2, \\
(e_3) &= -e^{-(r+t)} v, \\
(e_4) &= e^{-r}, \\
(e_5) &= e^{-s}
\end{align*}
\]

which shows that the $\mathfrak{n}^*$-closure $\Omega$ of $X = \text{Ad}^*(G)f$ is contained in the subset of all $h \in \Omega$ such that $h_6 = 1$, $h_5 \geq 0$, $h_4 \geq 0$, and $p_3(h) = 2h_2h_4 - h_3h_3 = 0$. In particular $h \in \Omega$ and $h_4 = 0$ implies $h_3 = 0$.

Next we compute the relevant unitary representations of $M$. Put $f_{r,s} = \text{Ad}^*(\exp(rd_0) \exp(sd_1))f$ and $\pi_{r,s} = K(f_{r,s})$. Clearly $\mathfrak{p} = \langle e_0, e_2, \ldots, e_6 \rangle$ is a Pukanszky-Vergne polarization at $f_{r,s}$ for all $r, s \in \mathbb{R}$, and $\mathfrak{c} = \langle e_1 \rangle$ is a coexponential subalgebra for $\mathfrak{p}$ in $\mathfrak{m}$. Define $\hat{e}_0 = -ie_0 + \frac{1}{2} e_6$ and $\hat{e}_\nu = -ie_\nu \in \mathfrak{m}_\mathbb{C}$ for $1 \leq \nu \leq 6$. It
turns out that \( \pi_{r,s} = \text{ind}_P^M \chi_{fr,s} \) is infinitesimally given by
\[
\begin{align*}
d\pi_{r,s}(\dot{e}_0) &= (f_0 + ar + bs) + \xi D_\xi, \\
d\pi_{r,s}(\dot{e}_1) &= -D_\xi, \\
d\pi_{r,s}(\dot{e}_2) &= \frac{1}{2} e^{-r} \xi^2, \\
d\pi_{r,s}(\dot{e}_3) &= -e^{-r} \xi, \\
d\pi_{r,s}(\dot{e}_4) &= e^{-r}, \\
d\pi_{r,s}(\dot{e}_5) &= e^{-s}.
\end{align*}
\]

If \( g \in m^* \) such that \( g_4 = g_3 = 0 \) and \( (g_1 \neq 0 \text{ or } g_2 \neq 0) \), then \( n \) is a Pukanszky-Vergne polarization at \( g \in m^* \) with coexponential subalgebra \( c = (e_0) \). It follows that the infinitesimal operators of \( \rho = \text{ind}_N^M \chi_g \) are given by
\[
\begin{align*}
d\rho(\dot{e}_0) &= \frac{i}{2} - D_\xi, \\
d\rho(\dot{e}_1) &= e^\xi g_1, \\
d\rho(\dot{e}_2) &= e^{-2\xi} g_2, \\
d\rho(\dot{e}_\nu) &= g_\nu \text{ for } 3 \leq \nu \leq 6.
\end{align*}
\]

The explicit formulas for \( \text{Ad}^*(\Phi(t,v,x,Z))f \) suggest to define the polynomial \( p_1 = e_0 e_0 e_4 - 2e_0 e_1 e_3 + 2e_1 e_1 e_2 \) which is \( \text{Ad}(M) \)-invariant on \( \Omega \subset m^* \), but not on the entire space. This definition yields \( p_1(f_{r,s}) = e^{-r}(f_0 + ar + bs)^2 \). We observe that the equations for \( d\pi_{r,s}(\dot{e}_\nu) \) bear a striking resemblance to the formulas for \( \text{Ad}^*(\Phi(t,v,w,x,Z))f_{r,s} : \) simply replace \( e^{-t}v \) by \( \xi \) and \( e^t x \) by \( -D_\xi \). If we define
\[
W_1 = \dot{e}_0 \dot{e}_0 \dot{e}_4 - 2\dot{e}_0 \dot{e}_3 \dot{e}_1 + 2\dot{e}_2 \dot{e}_1 \dot{e}_1 - i\dot{e}_3 \dot{e}_1 \in \mathcal{U}(m_C),
\]
then it follows from \( (\xi D_\xi)^2 = \xi^2 D_\xi^2 - i\xi D_\xi \) and the binomial identity for the commuting operators \( \xi D_\xi \) and \( f_0 + ar + bs + \xi D_\xi \) that \( d\pi_{r,s}(W_1) = p_1(f_{r,s}) \cdot \text{Id}. \) Moreover it is easy to see that \( d\pi(W_1) = p_1(h) \cdot \text{Id} \) for all \( h \in \Omega \) and \( \pi = K(h) \), i.e., \( (W_1,p_1) \) is a Duflo pair w.r.t. \( \Omega \). The same is true for \( p_2 = e_0 e_4 - e_1 e_3 \) and \( W_2 = \dot{e}_0 \dot{e}_4 - \dot{e}_3 \dot{e}_1 \) with \( d\pi_{r,s}(W_2) = p_2(f_{r,s}) = e^{-r}(f_0 + ar + bs) \). In addition we consider the functions
\[
\psi_1(\xi) = \xi_1(f_0 - a \log |\xi_1| - b \log |\xi_2|)^2 \quad \text{and} \quad \psi_2(\xi) = \xi_1(f_0 - a \log |\xi_1| - b \log |\xi_2|) \quad \text{on } 3m^*,
\]
where \( \xi \leftrightarrow \xi_1 \xi_2^* + \xi_2 \xi_3^* \). Now we can prove

**Proposition 6.6.** Assume that \( b = 0. \) In this case \( G \) is primitive \( * \)-regular. If \( f \in m^* \) in general position satisfies the stabilizer condition \( m = m_f + n \), then \( \{ (W_{\nu},\psi_{\nu},\psi_{\nu}) : \nu = 1,2 \} \) is a set of separating triples for \( X = \text{Ad}^*(G)f \) in its \( \text{n}^* \)-closure \( \Omega \).

**Proof.** Since \( b = 0 \), the results of Section 4 imply that \( \psi_1 \) and \( \psi_2 \) are central Fourier multipliers. By definition \( p_\nu(h) = \psi_{\nu}(h | 3m) \) for all \( h \in X \), and hence for all \( h \in X \). In order to prove the opposite implication we suppose that \( h \in \Omega \) such that \( p_\nu(h) = \psi_{\nu}(h | 3m) \). Choose sequences \( r_n, \ldots, x_n \) such that \( f_n \longrightarrow h' \) for
\[
f_n = \text{Ad}^*(\exp(r_n d_0) \exp(s_n d_1) \Phi(0,v_n,w_n,x_n,0) \cdot f).
\]
At first we assume that \( h_4 \neq 0 \). Since \( h \in \Omega \) and \( p_2(h) = \psi_2(h | jm) \), it follows \( e^{-rn} \to h_4 \neq 0 \) and
\[
e^{-rn} f_{0n} = e^{-rn}(f_0 + arn - vn x_n) = \psi_2(f_n | jm) + f_{n1} f_{0n}
\to \psi_2(h | jm) + h_1 h_3 = h_0 h_4
\]
which implies \( f_{0n} \to h_0 \) and thus \( f_n \to h \in X \). Next we assume \( h_4 = h_3 = 0 \) so that \( r_n \to +\infty \). Note that \( h \in X \) implies
\[
p_{1}(f_n) = e^{-rn}(f_0 + arn) \to 0 = p_{1}(h) = 2h_2^2 h_2
\]
which yields \( h_1 = 0 \) or \( h_2 = 0 \). Hence we must distinguish three subcases. If \( h_1 \neq 0 \) and \( h_2 = 0 \), then we choose \( r_n, s_n \) as above and define \( v_n = \frac{1}{h_r}(f_0 + arn - h_0) \) and \( x_n = h_1 \). By definition \( f_n \to h \in X \). The second subcase is \( h_1 = 0 \) and \( h_2 \neq 0 \). Since \( h \in \Omega \), it follows \( h_2 > 0 \). If we choose \( r_n, s_n \) as above and define \( v_n = (2h_2)^{1/2} e^{r_n/2} \) and \( x_n = (2h_2)^{-1/2} e^{r_n/2}(f_0 - arn - h_0) \), then we obtain \( f_n \to h \in X \). Finally we assume \( h_1 = h_2 = 0 \). Here we define \( v_n = e^{r_n/4} \) and \( x_n = e^{-r_n/4}(f_0 + arn - h_0) \). This proves \( f_n \to h \in X \) in the third subcase. Altogether we have shown that \( h \in X \) if and only if \( h \in \Omega \) and \( p_{1}(h) = \psi_p(h | jm) \), i.e., \( \{ (W_{\nu}, p_{\nu}, \psi_{\nu}) : \nu = 1, 2 \} \) is a set of separating triples for \( X \) in \( \Omega \). From this and the results of [21] it follows that \( ker_{C^*G} \pi \) is \( L^1 \)-determined for all \( \pi \) in general position. The proper quotients of \( G \) can be treated in analogy to the proof of Lemma [2, 4]. We omit the details because this would not yield anything new.

The situation is more delicate if \( b \neq 0 \). In this case the functions \( \psi_1 \) and \( \psi_2(\xi) = \xi_1(f_0 - a \log |\xi_1| - b \log |\xi_2|) \) fail to be central Fourier multipliers because of their singularity in \( \xi_2 = 0 \). Thus we put \( \tilde{\psi}_p(\xi) = \psi_p(\xi) \xi_2, \tilde{p}_p = p_p e_5 \), and \( \tilde{W}_p = W_p e_5 \). Note that \( (\tilde{W}_p, \tilde{p}_p) \) is a Duflo pair w.r.t. \( \Omega \). Furthermore we define the admissible part \( \Omega_0 = \{ h \in \Omega : h_5 \neq 0 \} \) of \( \Omega \). All we can prove is

**Lemma 6.7.** Assume that \( b \neq 0 \). If \( f \in m^* \) is in general position such that \( m = m_f + n \), then \( \{ (\tilde{W}_p, \tilde{p}_p, \tilde{\psi}_p) : \nu = 1, 2 \} \) is a set of separating triples for the orbit \( X = Ad^*(G) f \) in the admissible part \( \Omega_0 \) of its \( n^* \)-closure \( \Omega \). The non-admissible part of the closure of \( X \) is characterized as follows: \( h \in X \setminus \Omega_0 \) if and only if \( h \in \Omega \) and \( h_5 = h_4 = h_3 = 0 \).

**Proof.** Clearly \( \tilde{\psi}_1 \) and \( \tilde{\psi}_2 \) are central Fourier multipliers and \( \tilde{p}_{\nu}(h) = \tilde{\psi}_\nu(h | jm) \) for all \( h \in X \). As in the proof of Proposition 6.6 one can show that \( h \in \Omega_0 \) and \( \tilde{p}_{\nu}(h) = \tilde{\psi}_\nu(h | jm) \) for \( \nu \in \{ 1, 2 \} \) implies \( h \in X \). Here one heavily uses the fact that \( h \in \Omega_0 \) and \( f_{h_n}^t \to h' \) implies the convergence of \( s_n \) because \( e^{-s_n} \to h_5 \neq 0 \). Consequently the \( (\tilde{W}_p, \tilde{p}_p, \tilde{\psi}_p) \) are separating triples for \( X \) in \( \Omega_0 \).

Next we verify the characterization of the non-admissible part of the closure of \( X \). Assume \( h \in X \setminus \Omega_0 \). Then \( h \in \Omega \), \( h_5 = 0 \), and \( s_n \to +\infty \). If \( h_4 \) were non-zero, then the sequences \( r_n, v_n, x_n \) would converge in contradiction to \( f_{0n} = f_0 + ar_n + bs_n - vn x_n \to h_0 \). Thus \( h_4 = h_3 = 0 \).

For the opposite implication we assume \( h_5 = h_4 = h_3 = 0 \). If \( h_1 = h_2 = 0 \),
then we choose \( r_n = s_n = n, v_n = e^{rn/4}, \) and \( x_n = e^{-rn/4}(f_0 + ar_n + bs_n - h_0). \) If \( h_1 \neq 0 \) and \( h_2 = 0, \) then we put \( r_n = s_n = n, v_n = \frac{1}{h_1}(f_0 + ar_n + bs_n - h_0), \) and \( x_n = h_1. \) If \( h_1 = 0 \) and \( h_2 \neq 0, \) then we define \( r_n = s_n = n, v_n = (2e^{rn}h_2)^{1/2}, \) and \( x_n = (2e^{rn}h_2)^{-1/2}(f_0 + ar_n + bs_n - h_0). \) The last case is \( h_1 \neq 0 \) and \( h_2 \neq 0. \) Here we choose \( r_n = n, v_n = \text{sgn}(b_1)(2e^{rn}h_2)^{1/2}, x_n = h_1, \) and \( s_n = \frac{1}{h_0}(h_0 - f_0 - ar_n + v_n x_n) \) so that \( s_n \to +\infty. \) In any case it follows \( f_n \to h \in \overline{X}. \)

At this stage it remains open whether \( \bigcap_{r,s \in \mathbb{R}} \ker L^1(\pi_{r,s}) \neq \ker L^1(M) \rho \) holds for non-admissible, critical \( g \) and \( \rho = K(g). \) Note that in this particular case \( g \in \Omega \setminus \overline{(X \cup \Omega_0)} \) if and only if \( g \in \Omega, g_4 \neq 0, \) and \( g_5 = 0. \) Although one might expect this 9-dimensional exponential Lie group \( G \) to be primitive \( \ast \)-regular for \( b \neq 0, \) the results of the preceding sections are too coarse to prove this. The preceding examples (and similar ones) put the scope of the method of separating triples into perspective.

What many exponential Lie algebras \( g \) of dimension \( \leq 7 \) have in common is that they contain ideals \([g, g] \supset b \supset a \supset \mathfrak{z} b\) where \( b \) is a 3-dimensional Heisenberg algebra, \( a \) is commutative, and \( \mathfrak{z} b \) is the one-dimensional center of \( b. \) In particular \( b \subset n \) and \( \mathfrak{z} b \subset \mathfrak{z} m. \) Here we distinguish the central case \( \mathfrak{z} b \subset \mathfrak{g} \) and the non-central case \( \mathfrak{z} b \not\subset \mathfrak{g}. \) Roughly spoken, at least in low dimensions, the situation is as follows:

**Remark 6.8.** Let \( g \) be an exponential Lie algebra, \( n \) a coabelian, nilpotent ideal of \( g, \) and \( f \in g^\ast \) in general position such that its stabilizer \( m = \mathfrak{g}_f + n \) is not nilpotent. In this situation it is advisable to look for Duflo pairs \((W_{\nu}, p_{\nu})\) on \( M. \) The existence of \((W_{\nu}, p_{\nu})\) is (more or less) an intrinsic property of \( M. \) If \( g = s \ltimes n \) is a semi-direct sum of a commutative subalgebra \( s \) and the ideal \( n, \) then the existence of \((W_{\nu}, p_{\nu})\) suffices to prove that \( G \) is primitive \( \ast \)-regular. If \( g = \mathbb{R} d \ltimes m, \) i.e., in case of a one-parameter subgroup \( \text{Ad}(\exp(rd)) \) acting on the stabilizer \( m, \) the (finer) method of separating triples applies and yields the primitive \( \ast \)-regularity of \( G. \) But this approach may fail as soon as \( \dim g/m \geq 2. \)

Using some of the results combined in this article, the author proved in his thesis that all exponential Lie algebras up to dimension seven are primitive \( \ast \)-regular. This severe restriction on the dimension of \( g \) implies that either \( g = s \ltimes n \) or \( \dim g/m = 1. \) It should be well noted that no counter-example seems to be known so far.

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