A COUNTEREXAMPLE TO THE CONTAINMENT $I^{(3)} \subset I^2$ OVER THE REALS

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Abstract. The purpose of this note is to give defined over the real numbers counterexamples to a question relevant in the commutative algebra, concerning a containment relation between algebraic and symbolic powers of homogeneous ideal.

1. The main result

In algebraic geometry and in commutative algebra there has been recently a lot of interest in comparing usual (algebraic) and symbolic powers of homogeneous ideals, see for example [2], [10], [3]. If $I \subset \mathbb{k}[x_0, x_1, \ldots, x_n]$ is a homogeneous ideal, then its algebraic $r$–th power $I^r$ is defined as the ideal generated by $r$–th powers $f^r$ of all elements $f$ in $I$. This is a purely algebraic concept. On the other hand, homogeneous ideals are defined geometrically as sets of all polynomials vanishing along a given set, a subvariety $V \subset \mathbb{P}^n(\mathbb{k})$. For example, the ideal determined by simultaneous vanishing in points $P = (1 : 0 : 0), Q = (0 : 1 : 0)$ and $R = (0 : 0 : 1)$ in the projective plane $\mathbb{P}^2(\mathbb{k})$ is generated by the monomials $yz, xz$ and $xy$. Studying the geometry of algebraic varieties one is often interested in polynomials vanishing to certain order $m$ along the subvariety. For example the monomial $xy$ vanishes to order 2 in point $R$ and only to order 1 in points $P$ and $Q$, whereas the monomial $xyz$ vanishes to order 2 in all these points. All polynomials vanishing to order $m$ along the subvariety defined by an ideal $I$ form again an ideal. This ideal is denoted by $I^{(m)}$ and is called the $m$–th symbolic power of $I$. This is a geometric concept. It follows from the definition that $I^{(m)} \subset I^m$ holds for all homogeneous ideals. The reverse inclusion however might fail already in the simplest situations. For example this happens for the ideal $I$ of the three points $P, Q$ and $R$ defined above. Indeed, the least degree of a polynomial in $I^2$ is 4 (because generators of $I$ have degree 2), whereas $xyz$ is a polynomial of degree 3 contained in $I^{(2)}$. Hence there is surely no containment $I^{(2)} \subset I^2$ in this case.

It came as a big surprise that nevertheless there is also a uniform containment relation in the reverse direction, when one takes into account the dimension of the ambient space (or equivalently the number of variables in the polynomial ring). More precisely, it has been discovered independently by Ein, Lazarsfeld and Smith
in characteristic 0 and Hochster and Huneke \cite{12} in finite characteristic, that there is always the containment
\[ I^{(m)} \subset I^r, \]
provided \( m \geq n \cdot r \), where \( n \) is the dimension of the ambient space. Whereas the lower bound \( n \cdot r \) cannot be improved in general, it has been expected that it can be improved under certain assumptions on varieties defined by \( I \). In particular, for points in the projective plane \( \mathbb{P}^2(\mathbb{K}) \), Huneke asked if there is always the containment
\[ I^{(3)} \subset I^2. \]
It is the first instance of a more general statement predicting for ideals defined by vanishing along points the containment
\[ I^{(nr-(n-1))} \subset I^r \]
for all \( r \geq 1 \).

A first counterexample to the containment in \( I^{(3)} \subset I^2 \) was announced in 2013 by Dumnicki, Szemberg and Tutaj-Gasińska in \( \cite{6} \). In that counterexample the set of relevant points is taken as all the intersection points of 9 lines whose union is defined by the polynomial equation
\[(x^3 - y^3)(y^3 - z^3)(z^3 - x^3) = 0.
\]
These lines form the so called dual Hesse configuration, see \( \cite{1} \) for a beautiful account on that, interesting in its own right, subject. The 9 lines in this configuration intersect by 3 in 12 points. Since there are no points where only two of these lines intersect, such a configuration cannot be realized over the reals as it would contradict the following well-know theorem of Sylvester \( \cite{13} \) and Gallai.

**Theorem 1** (Sylvester–Gallai). *Given a finite set of points in the real plane, either all points are collinear or there exists a pair of points not collinear with any other point in the set.*

The contradiction arises if one passes to the dual statement.

**Theorem 2** (dual Sylvester–Gallai Theorem). *Given a finite number of lines in the real plane, either all these lines belong to the same pencil of lines passing through a fixed point, or there exists a point in the plane, where only two of these lines intersect.*

Shortly after the counterexample in \( \cite{6} \) was announced, Harbourne and Seceleanu \( \cite{11} \) constructed a series of counterexamples to the containment in \( \cite{2} \) for various values of \( n \) and \( r \). Their counterexamples however are defined over finite fields.

Since configurations of real lines are subject to stronger combinatorial constrains than sets of lines either in the complex plane or in planes defined over finite fields, it is not immediately clear that counterexamples to \( \cite{1} \) can be constructed in the real plane. This issue has motivated our research. The main result we want to announce is the following.
Theorem 3. There exists a series of counterexamples to the containment in (1) defined in the real plane.

In fact our counterexamples are well known in the combinatorics. They were introduced by Füredi and Palásti in [8] in connection with the following Sylvester-Gallai problem motivated by Theorem 2.

Problem 4. Given a configuration of s lines in the real (projective) plane, not all belonging to the same pencil of lines through a fixed point, what is the minimal number of points where only two lines meet?

It has been long conjectured, see for example [4], that there are at most $R_s = 1 + \left\lfloor \frac{s(s-3)}{6} \right\rfloor$ points where three lines meet, so that there are at least $\left(\begin{array}{c} s \\ 2 \end{array} \right) - 3 - \left\lfloor \frac{s(s-3)}{6} \right\rfloor$ points where the lines meet in pairs only. This conjecture, at least for large $s$, was recently proved by Green and Tao [9]. A series of examples with number of triple points equal or close to $R_s$ was constructed in [8].

In the recent preprint [14] Terence Tao pointed out that many combinatorial problems have been solved, or substantial progress has been obtained, when applying methods from algebra or algebraic geometry. In this note the directions are reversed, we present a solution to an algebraic problem based on a combinatorial construction.

2. Proof of the main result

In order to prove Theorem 3, we are interested in configurations of lines with a high number of points where three of them meet. The motivation for this approach is the following. Taking $V$ as the set of all triple points in the configuration and denoting by $I = I(V)$ the ideal of $V$, it is immediate from the description of symbolic powers given above, that the product

$$F = L_1 \cdot \ldots \cdot L_s$$

of equations of the lines in the configuration is contained in $I^{(3)}$.

If the number of triple points is high when related to the number of lines, then $F$ will be the only element of degree $s$ in $I^{(3)}$. On the other hand, the generators of $I$ should be of relatively high degree. In the most optimistic situation, the least degree $d$ of a generator of $I$ would satisfy $2d > s$. In that case, it would be clear that $F$ cannot be contained in $I^2$. This scenario never happens in the reality, so that the actual argument requires somewhat more effort.

We recall now how the examples of Füredi and Palásti work. To this end it is convenient to identify the real plane $\mathbb{R}^2$ with the set of complex numbers $\mathbb{C}$ in the usual way. Let $n$ be an even and positive integer. Let $\xi = \exp(2\pi i/n)$ be a primitive $n$-th root of unity and let $P_i = \xi^i$ for $i = 0, \ldots, n-1$. Denote by $L_{m,k}$ the (real) line passing through the points $P_m$ and $P_k$ if $m \neq k$ and the tangent line to the unit circle at the point $P_m$ if $m = k$.

Then we construct the configuration of points and lines in two steps. First we define the configuration of lines

$$\mathcal{L}_n = \{L_{i, \frac{n}{2} - 2i} \}_{i=0}^{n-1}.$$
where the indices are understood mod n. We will denote the linear form defining the line \( L_i \) by \( L_i \) for \( i = 0, \ldots, n-1 \). Let \( F_n = \prod_{i=0}^{n-1} L_i \). Then \( \deg(F_n) = n \).

Let \( Z_n \) be the set of all triple points in the configuration \( \mathcal{L}_n \).

Note that the lines \( L_i, L_j, L_k \) are concurrent if and only if \( n \) divides \( i + j + k \), see [5] Lemma. It follows from [5] Property 4 that there are exactly \( 1 + \lfloor n(n - 3)/6 \rfloor \) points in the set \( Z_n \).

By the way of an example we focus now on the case \( n = 12 \). This is the minimal number of lines, which leads to a configuration of points giving a counterexample to the containment in (1).

The configuration of lines arising in this case is illustrated on the following picture prepared with the aid of Geogebra.

The points in \( Z_{12} \) are marked by dots. There are 19 of them. There are 3 lines, each containing 4 configuration points, these lines are the tangents to the unit circle at points \( P_2, P_6 \) and \( P_{10} \). On each of the remaining 9 configuration lines there are exactly 5 configuration points.

Thus the configuration is in fact the union of two configurations. One depicted in the picture below using solid lines is a real analogue of the dual Hesse configuration mentioned before.
In the real case there are only 10 triple points. The remaining 6 double points (distinguished in the picture by larger dots) determine in pairs, 3 additional lines (dotted lines in the picture), which intersect in pairs on the lines of the solid configuration. These 3 lines form the second configuration, and altogether there are 12 lines and 19 triple points.

In order to determine generators of the ideal \( J = I(Z_{12}) \), we can intersect the ideals of single points in the set \( Z_{12} \), i.e.

\[
J = I(Q_1) \cap \ldots \cap I(Q_{19}).
\]

Note that the ideals \( I(Q_j) \) are generated by any two lines passing through the point \( Q_j \), so they can be easily written explicitly down. In fact, these calculations can be done by hand, but this is a tedious procedure. Instead, we prefer to revoke a Singular [5] script in this place.

In fact, our computational strategy in the script attached at the end of this note is slightly different (and more efficient). We compute first the points \( P_0, \ldots, P_{11} \) on the unit circle, the vertices of the regular 12-gon with \( P_0 = (1, 0) \) (in the script we use projective coordinates, so that \( P_0 = (1 : 0 : 1) \)). Then we compute the equations of lines \( L_0, \ldots, L_{11} \). In the next step, we don’t compute the triple points in the configuration \( \mathcal{L}_{12} \) one by one. Instead, we compute the ideal \( J \) of all of them simultaneously taking into account that they are the worst singularities of the configuration (in algebraic geometry a triple point is considered to be a worse singularity than a double point). Finally, we check the containment of \( F_{12} \) (which is called \( F \) in the script) in \( J^2 \). Again, this step could be carried out by hand in the
theory, but this is a lengthy and dull computation, which can be safely relegated
to a machine.

Checking any given configuration with an even number \( n \geq 12 \) goes along the
same lines.

This finishes the proof of Theorem 3.

Remark 5. There is a similar construction for an odd number of lines \( n \) in [8]. We
have checked also these examples and found out that they give counterexamples to
the containment in [1] for all \( n \geq 13 \). Thus 12 is the minimal number of real lines
such that triple points of their configuration provide a counterexample to [1].

Remark 6. All counterexamples to the containment [2] found up to now are based
on configurations of points coming from some extremal, from the combinatorial
point of view, configurations of lines. It would be very interesting to understand if
there are some deeper connections justifying these phenomena lurking behind the
scenes.

3. A Singular script

LIB "elim.lib";
ring R=(0,a),(x,y,z),dp; option(redSB);
minpoly=a^2-3;
proc gline(ideal I1, ideal I2) {
  ideal I=intersect(I1,I2);
  I=std(I);
  return(I[1]);
}
ideal P0=x-z,y; ideal P1=x-a/2*z,y-1/2*z; ideal P2=x-1/2*z,y-a/2*z;
ideal P3=x,y-z; ideal P4=x+1/2*z,y-a/2*z; ideal P5=x+a/2*z,y-1/2*z;
ideal P6=x+z,y; ideal P7=x+a/2*z,y+1/2*z; ideal P8=x+1/2*z,y+a/2*z;
ideal P9=x,y+z; ideal P10=x-1/2*z,y+a/2*z; ideal P11=x-a/2*z,y+1/2*z;
poly L0=gline(P0,P6); poly L1=gline(P1,P4); poly L2=1/2*x+a/2*y-z;
poly L3=gline(P3,P0); poly L4=gline(P4,P10); poly L5=gline(P5,P8);
poly L6=x-z; poly L7=gline(P7,P4); poly L8=gline(P8,P2);
poly L9=gline(P9,P0); poly L10=1/2*x-a/2*y-z; poly L11=gline(P11,P8);
poly F=L0*L1*L2*L3*L4*L5*L6*L7*L8*L9*L10*L11;
ideal I=F,diff(F,x),diff(F,y),diff(F,z);
ideal J=I,diff(I,x),diff(I,y),diff(I,z);
ideal M=x,y,z; J=std(J);
J=sat(J,M)[1]; J=std(J);
reduce(F,std(J^-2));
quod;

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