SOME RESULTS AND PROBLEMS ON CLIQUE COVERINGS OF HYPERGRAPHS

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Abstract. For a \( k \)-uniform hypergraph \( F \) we consider the parameter \( \Theta(F) \), the minimum size of a clique cover of the edge set of \( F \). We derive bounds on \( \Theta(F) \) for \( F \) belonging to various classes of hypergraphs.

1. Introduction

Let \( G \) be a \( k \)-uniform hypergraph, a set representation of \( G \) on a set \( T \) is a system of subsets \( \{S_v \subseteq T : v \in V(G)\} \) with the property that

\[
\{v_1, v_2, \ldots, v_k\} \in G \text{ if and only if } \bigcap_{i=1}^{k} S_{v_i} \neq \emptyset.
\]

The representation number \( \Theta(G) \) of \( G \) is the smallest cardinality of a set \( T \) which admits a set representation.

We define the clique covering number \( \text{cc}(G) \) of a \( k \)-graph \( G \) as the minimum integer \( m \) such that there exist complete graphs \( C_1, \ldots, C_m \) satisfying \( G = \bigcup_{i=1}^{m} C_i \), i.e., every edge is covered by a clique and no clique contains an edge that is not from \( G \). It is well known that the representation number \( \Theta(G) \) is equal to \( \text{cc}(G) \) (see e.g. [8, 9] or Section 7 for a proof). The parameter \( \Theta \) has been subject of interest for a number of researchers (see e.g. [1–3]). In particular it was proved in [1] that \( \Theta(G) \leq c_1 d^2 \log n \) for any graph \( G \) on \( n \) vertices with maximum degree on its complement \( \Delta(G) \leq d \). On the other hand, it was proved in [2] that there are graphs \( G \) with \( \Delta(G) \leq d \) and \( \Theta(G) \geq c_2 d^2 \log \log n \).

Here we will be mainly interested in extending these results to \( k \)-uniform hypergraphs. Perhaps surprisingly, this turned out to be quite nontrivial, and our results in this direction are far from being optimal for large values of \( k \).

Since most of our results regarding \( \Theta(G) \) will be formulated in terms of restrictions on the degree of the complement \( \overline{G} \), we will introduce a parameter \( \vartheta(G) = \Theta(\overline{G}) \) allowing for simpler formulation of results. Clearly, \( \vartheta(G) \) is the minimum cardinality of a set \( T \) such that there is a system \( \{S_v \subseteq T : v \in V(G)\} \) of sets with

\[
\{v_1, \ldots, v_k\} \in G \text{ if and only if } \bigcap_{i=1}^{k} S_{v_i} = \emptyset.
\]
Alternatively, one can view $\vartheta(G)$ as the minimum cover of the edges of the complement $\overline{G}$ by independent sets of $G$. In what follows we will often use the latter definition instead of the former. Moreover, we say that $G$ can be $t$-represented if there exists a system of independent sets $\{I_j : 1 \leq j \leq t\}$ covering the edges of $\overline{G}$.

2. Results

Let $G$ be a $k$-graph. For $S \subseteq V(G)$, let

$$\deg_G(S) = |\{e \in G : S \subseteq e\}|$$

be the degree of the set $S$ in $G$ and

$$\Delta_i(G) = \max_{|S|=i, S \subseteq V(G)} \deg_G(S)$$

be the maximum degree of an $i$-tuple of vertices of $G$. (Consequently, $\Delta_1(G) = \Delta(G)$ is just the usual maximum degree).

We say that a $k$-graph $G$ is $d$-balanced if $\Delta_i(G) \leq d^{\frac{i}{k-1}}$ for all $1 \leq i \leq k$. In particular, a $d$-balanced graph has maximum degree $d$. Our first result gives almost sharp bounds on $\vartheta(G)$ for the family of $d$-balanced $k$-graphs. Let

$$b(n, d, k) = \max \{\vartheta(G) : G \text{ is a } d\text{-balanced } k\text{-graph}\}$$

be the maximum value of $\vartheta(G)$ over all $d$-balanced graphs with $n$ vertices.

**Theorem 2.1.** For $n, d, k \geq 2$, there exist positive constants $c_1$ and $c_2$ depending only on $k$ such that

$$c_1 d^{\frac{k}{k-1}} \log \frac{n^{k-1}}{d} \leq b(n, d, k) \leq c_2 d^{\frac{k}{k-1}} \log n$$

holds.

For the general family of bounded degree hypergraphs, we obtain the following bounds. Let $\vartheta(n, d, k)$ be the maximum value of $\vartheta(G)$, where $G$ is a $k$-graph on $n$ vertices with $\Delta(G) = \Delta_1(G) \leq d$.

**Theorem 2.2.** For $n, d, k \geq 3$, there exist positive constants $c_1$ and $c_2$ depending only on $k$ such that

$$c_1 d^{\frac{k}{k-1}} \log \frac{n^{k-1}}{d} \leq \vartheta(n, d, k) \leq c_2 d^{\frac{k}{k-1}} \log n.$$ 

Moreover, if $k$ is even the lower bound can be improved to

$$c_1 d^{2} \log \frac{n}{d} \leq \vartheta(n, d, k)$$

Note that since every 2-graph $G$ with $\Delta(G) \leq d$ is $d$-balanced, then the bounds of $\vartheta(n, d, k)$ provided in Theorems 2.1 and 2.2 are almost sharp for $k = 2, 3, 4$ and $d \ll n$. Finally, we address similar
questions for some other classes of hypergraphs. For example, denoting \( \vartheta_d(n,k) \) as the maximum value of \( \vartheta(G) \), where \( G \) runs over all \( k \)-uniform simple hypergraphs, we show the following:

**Theorem 2.3.** For \( k \geq 3 \) and \( n \geq n_0(k) \), there exist positive constants \( c_1 \) and \( c_2 \) depending only on \( k \) such that

\[
\frac{c_1 n^{k^2}}{(\log n)^{k-1}} \leq \vartheta_d(n,k) \leq c_2 n^{k-1} \log n
\]

holds.

The paper is organised as follows. In Section 3 we discuss the upper bounds of Theorem 2.1 and 2.2, while in Section 4 we provide their respective lower bounds. Section 5 is devoted to the problem of representing special hypergraphs.

3. Upper Bounds of Theorem 2.1 and Theorem 2.2

We start with the upper bound of Theorem 2.1. We are going to show that if \( G \) is a \( d \)-balanced \( k \)-graph, then \( \vartheta(G) \leq 2^kk^{k+1}d^{\frac{k}{k-1}} \log n \).

**Proposition 3.1.** Let \( n, d, k \geq 2 \) and \( c = 2^{k+2}k^{k+1} \). If \( G \) is a \( d \)-balanced \( k \)-graph on \( n \) vertices, then \( \vartheta(G) \leq cd^{\frac{k}{k-1}} \log n \).

**Proof.** We recall that \( \vartheta(G) \) is the minimum size of a cover of \( \overline{G} \) by independent sets of \( G \). Consider \( t = cd^{\frac{k}{k-1}} \log n \) random subsets \( W_1, W_2, \ldots, W_t \) of \( V := V(G) \), where each \( W_j \) is chosen by selecting vertices from \( V \) independently and uniformly with probability \( p = \frac{1}{2kd^{(k-1)}} \). Our aim is to cover \( \overline{G} \) by modifying the sets \( W_j \) for \( 1 \leq j \leq t \). For each \( W_j \), construct an independent set as follows: For every edge \( e \) in \( G[W_j] \) delete all of its vertices from \( W_j \). Let \( I_j \) be the set obtained after deleting vertices from every edge in \( G[W_j] \), i.e.,

\[
I_j = W_j \setminus \bigcup_{e \in G(W_j)} e.
\]

Clearly, \( I_j \) is an independent set. We claim that \( \{I_1, \ldots, I_t\} \) is a covering of \( \overline{G} \) with positive probability.

For a \( k \)-tuple \( \vec{v} = \{x_1, \ldots, x_k\} \in \overline{G} \), we examine the probability that \( \vec{v} \notin I_j \) for some \( 1 \leq j \leq t \). To do that, note that the only reason why a vertex \( x_i \in \vec{v} \) could be deleted is if there exists an edge \( e \in G[W_j] \) such that \( e_i \in e \cap \vec{v} \). That is, \( \vec{v} \notin I_j \) if \( \vec{v} \notin W_j \) and \( e \cap \vec{v} = \emptyset \) for all \( e \in G[W_j] \). Hence, we obtain

\[
\mathbb{P}(\vec{v} \notin I_j) \geq p^k \left( 1 - \sum_{i=1}^{k-1} \Delta_i(G) p^{k-i} \binom{k}{i} \right) \geq p^k \left( 1 - \sum_{i=1}^{k-1} \frac{(k-i)}{(2k)^{k-i}} \right) > \frac{p^k}{4}.
\]

Consequently, since all \( I_j \) were chosen independently, we obtain that

\[
\mathbb{P} \left( \bigwedge_{j=1}^{t} (\vec{v} \notin I_j) \right) = \prod_{j=1}^{t} \mathbb{P}(\vec{v} \notin I_j) < \left( 1 - \frac{p^k}{4} \right)^t \leq \exp \left( -\frac{p^k t}{4} \right) = \frac{1}{n^k}.
\]
Since there are at most \( {n \choose k} \frac{1}{n^k} < 1 \) non-edges in \( \overline{G} \) to cover, the probability that one remains uncovered is at most \( \frac{1}{n^k} \). Consequently, with positive probability all edges of \( \overline{G} \) are covered by \( \bigcup_{j=1}^{t} I_j \). □

The proof of the upper bound of Theorem 2.2 follows similar lines.

**Proposition 3.2.** Let \( n, d, k \geq 3 \) and \( \delta = \frac{1}{(k-1)2^{k+2}} \) and \( c = \frac{2k}{\delta^k} \). If \( G \) is a \( k \)-graph on \( n \) vertices with \( \Delta(G) \leq d \), then \( \vartheta(G) \leq \frac{cd}{k^2} \log n \).

**Proof.** As in the proof of Proposition 3.1, we want to cover all the edges of \( G \) with independent sets of \( G \). To this end, we consider \( t = cd^2 \log n \) random subsets \( W_1, \ldots, W_t \) of \( V \), where each \( W_j \) is chosen by selecting vertices from \( V \) independently and uniformly with probability \( p = \frac{\delta}{\sqrt{d}} \). Note that the sets \( W_1, \ldots, W_t \) are not necessarily independent. Our aim is to modify those sets \( W_j \) to independent sets \( I_j \) such that \( \bigcup_{j=1}^{t} I_j \) covers every \( k \)-tuple in \( G \).

To this end, we consider the auxiliary \( (k-1) \)-graph \( H \) given by \( V(H) = V \)
\[
H = \{ S \in V^{(k-1)} : \deg_G(S) \geq \sqrt{d} \},
\]
i.e., \( H \) is the \( (k-1) \)-graph where the edges are the \( (k-1) \)-tuples of large degree in \( G \). The next proposition gives an upper bound on the maximum degree of \( H \).

**Proposition 3.3.** \( \Delta(H) \leq (k-1)\sqrt{d} \).

**Proof.** Assume the contrary and let \( v \) be a vertex with \( \deg_H(v) > (k-1)\sqrt{d} \). Then by a counting argument we obtain that
\[
\deg_G(v) \geq \frac{\deg_H(v)\sqrt{d}}{k-1} > d,
\]
which contradicts the fact that \( \Delta(G) \leq d \). □

Now we describe how to construct the independent sets \( I_j \). For each \( 1 \leq j \leq t \), we sequentially remove vertices from \( W_j \) satisfying the following cleaning strategy:

**Cleaning Strategy**

Let \( X = W_j \). While there is an edge in \( G[X] \) perform one of the two operations:

**Operation (i):** If there is an edge \( g \in G[X] \) containing precisely one \( (k-1) \)-tuple \( S \in H \), then we remove the vertex not in \( S \) from \( X \), i.e., the vertex in the singleton set \( g \setminus S \).

**Operation (ii):** Otherwise, if all the edges in \( G[X] \) contain either zero or more than one \( (k-1) \)-tuple from \( H \), then we select an arbitrary edge \( g \in G[X] \) and delete an arbitrary vertex from it.

Set \( I_j \) to be the resulting \( X \) obtained after the process is over.

Clearly, the set \( I_j \) does not contain any edge from \( G \) and thus it is independent. It remains to show that with positive probability any edge from the complement \( \overline{G} \) is covered by \( \bigcup_{j=1}^{t} I_j \).
For a $k$-tuple $\tau = \{x_1, \ldots, x_k\} \in \overline{G}$, we want to estimate the probability that $\tau \subseteq I_j$ for some $1 \leq j \leq t$. The following lemma gives us a lower bound on the probability.

**Lemma 3.4.** For $\tau \in \overline{G}$ and $1 \leq j \leq t$,

$$\mathbb{P}(\tau \subseteq I_j) \geq \frac{1}{2} \left( \frac{\delta}{\sqrt{d}} \right)^k.$$  

*Proof.* In what follows, we will say that the set $X$ *crosses* the pair $(Y, Z)$ if $X$ has non-empty intersection with both $Y$ and $Z$. Set $\tau = \{x_1, \ldots, x_k\}$. We start by defining some auxiliary events.

For a set $S \subseteq \tau$ with $S \not\subseteq H$ and $|S| \leq k - 1$, let

$$A_S = \{ \exists R \subseteq W_j - \tau, R \cup S \in G \}$$  

(1) be the event that there is no edge $g \in G$ crossing the pair $(\tau, W_j - \tau)$ with $\tau \cap g = S$. Similarly, for a set $S \subseteq \tau$ with $|S| \leq k - 2$, let

$$B_S = \{ \exists R \subseteq W_j - \tau, R \cup S \in H \}$$  

(2) be the event stating that there is no $(k-1)$-tuple $h \in H$ crossing the pair $(\tau, W_j - \tau)$ with $\tau \cap h = S$.

The next claim gives a lower bound on the probability of $\tau$ be covered by $I_j$.

**Proposition 3.5.** If $\tau \in \overline{G}$ is a $k$-tuple satisfying

(a) For every $S \subseteq \tau$ with $|S| \leq k - 1$ and $S \not\subseteq H$, the event $A_S$ holds,

(b) For every $S \subseteq \tau$ with $|S| \leq k - 2$, the event $B_S$ holds,

(c) $\tau \subseteq W_j$,

then $\tau \subseteq I_j$.

*Proof.* Suppose that $\tau \not\subseteq I_j$. This means that a vertex of $\tau$ was deleted while performing operations (i) and (ii), i.e., in our cleaning process we deleted a vertex from an edge $g \in G$ with $g \cap \tau \neq \emptyset$. Since (a) holds, we obtain that $g \cap \tau$ is an edge of $H$. Moreover, by (b), we have that $g \cap \tau$ is the only $(k-1)$-tuple of $g$ in $H$. Therefore, we deleted a vertex of $g$ in the operation (i). By the definition of operation (i), we obtain that the deleted vertex was in $g - \tau$, which contradicts our assumption that the deleted vertex belongs to $\tau$. \hfill \Box

As a consequence of Proposition 3.5, one can estimate the probability of $\tau \subseteq I_j$ by

$$\mathbb{P}(\tau \subseteq I_j) \geq \mathbb{P}\left( \{\tau \subseteq W_j\} \land \bigwedge_{S \in \tau} A_S \land \bigwedge_{S \in \tau} B_S \right) = p^{k\mathbb{P}}\left( \bigwedge_{S \in \tau} A_S \land \bigwedge_{S \in \tau} B_S \right). \quad (3)$$

To compute the probability of the intersection of events $A_S$ and $B_S$, we will estimate the probability of the complementary events $A_S'$ and $B_S'$. Set $\varepsilon = \frac{1}{2^{\varepsilon_i}}$ and recall that $\delta = \frac{1}{(k-1)2^{\varepsilon_i}}$. We split the computations into cases depending on the size of $S$:

**Case 1:** $|S| \leq k - 2$.  


By using the definitions of \( A_S \) in (1) we obtain that
\[
\mathbb{P}(A_S^c) = \mathbb{P}\left( \bigvee_{R \in \mathcal{S} \subseteq G} \{ R \in W_j - \bar{e} \} \right) \leq \sum_{R \in \mathcal{S} \subseteq G} \mathbb{P}(R \in W_j - \bar{e}) \leq p^{k-|S|} \deg_G(S) \leq p^2d = \delta^2 \leq \varepsilon.
\] (4)

Proposition 3.3 and the definition of \( B_S \) in (2) give us that
\[
\mathbb{P}(B_S^c) = \mathbb{P}\left( \bigvee_{R \in \mathcal{S} \subseteq H} \{ R \in W_j - \bar{e} \} \right) \leq \sum_{R \in \mathcal{S} \subseteq H} \mathbb{P}(R \in W_j - \bar{e}) \leq p^{k-1-|S|} \deg_H(S) \leq p\Delta(H) = (k-1)\delta = \varepsilon.
\] (5)

Case 2: \(|S| = k - 1\) and \(S \notin H\).

By the definition of \( A_S \) in (1) and by the fact that \(S \notin H\) implies that \(\deg_G(S) \leq \sqrt{d}\), we obtain that
\[
\mathbb{P}(A_S^c) = \mathbb{P}\left( \bigvee_{R \in \mathcal{S} \subseteq G} \{ R \in W_j - \bar{e} \} \right) \leq \sum_{R \in \mathcal{S} \subseteq G} \mathbb{P}(R \in W_j - \bar{e}) \leq p^{k-|S|} \deg_G(S) \leq p\sqrt{d} = \delta \leq \varepsilon.
\] (6)

Finally, since there exist at most \(2^k\) choices of \(S \subseteq \bar{e}\), by putting (3), (4), (5) and (6) together we obtain that
\[
\mathbb{P}(\bar{e} \subseteq I_j) \geq p^k \left( 1 - \sum_{S \in (\bar{e}) \setminus H} \mathbb{P}(A_S^c) - \sum_{S \in (\bar{e})} \mathbb{P}(B_S^c) \right) \geq p^k(1 - 2^{k+1}\varepsilon) \geq \frac{p^k}{2} = \frac{1}{2} \left( \frac{\delta}{\sqrt{d}} \right)^k.
\]

Now the remaining part of the proof of Proposition 3.2 is straightforward. By Lemma 3.4, the probability that \(\bar{e}\) is not covered by \(\bigcup_{j=1}^t I_j\) is given by
\[
\mathbb{P}\left( \bigcap_{j=1}^t (\bar{e} \notin I_j) \right) = \prod_{j=1}^t \mathbb{P}(\bar{e} \notin I_j) < \left( 1 - \frac{1}{2} \left( \frac{\delta}{\sqrt{d}} \right)^k \right)^t \leq \exp \left( -\frac{t}{2} \left( \frac{\delta}{\sqrt{d}} \right)^k \right) = \frac{1}{n^k}.
\]

Since there are at most \(\binom{n}{k}\) non-edges in \(\overline{G}\) to cover, the probability that one remains uncovered is at most \(\binom{n}{k} \frac{1}{n^k} < 1\). Consequently, with positive probability all edges of \(\overline{G}\) are covered by \(\bigcup_{j=1}^t I_j\). \(\square\)

4. LOWER BOUNDS OF THEOREMS 2.1 AND 2.2

In this section, we prove the lower bound of Theorems 2.1 and 2.2. The next proposition shows the lower bound of Theorem 2.1 and Theorem 2.2 when \(k\) is odd.

Proposition 4.1. For \(n, d, k \geq 2\), there exists a \(d\)-balanced \(k\)-graph \(G\) such that
\[
\vartheta(G) \geq c \frac{d^{k-1}}{\log d} \log \left( \frac{n^{k-1}}{d} \right)
\]
for a positive constant \(c\) depending only on \(k\).
The proof of Proposition 4.1 follows from a counting argument using the next two auxiliary lemmas. Recall that a \( t \)-representation of a \( k \)-graph \( H \) is a system \( \mathcal{I} = \{ I_j \subseteq [n] : 1 \leq j \leq t \} \) covering the edges of the complement \( \overline{H} \). For integers \( n, \alpha, t \geq 1 \), define \( f(n, \alpha, t) \) as the number of all possible \( t \)-representations of graphs \( H \) with independence number \( \alpha(H) \leq \alpha \) (If some graph \( H \) does not admit a \( t \)-representation, then its contribution to the sum is zero).

**Lemma 4.2.** For \( \alpha, t \geq 1 \) and sufficiently large \( n \), \( f(n, \alpha, t) \leq \frac{2^n}{\alpha^t} \).

**Proof.** Let \( \mathcal{I} = \{ I_1, \ldots, I_t \} \) be a set system covering \( \overline{H} \) for some \( H \). Since \( I_j \) is an independent set of \( H \) and \( \alpha(H) \leq \alpha \), we obtain that 
\[
|I_j| \leq \alpha.
\]
Hence, the number of ways to choose distinct systems \( \mathcal{I} \) is bounded by 
\[
f(n, \alpha, t) \leq \left( \sum_{i=1}^{\alpha} \binom{n}{i} \right)^t \leq \left( \frac{2^n}{\alpha} \right)^t.
\]
\[\square\]

Given integers \( m, d, k \), we define \( \mathcal{F}^{(k)}_{m,d} \) to be a family of \( k \)-partite \( k \)-graphs \( F \) on a fixed set of vertices \( V = \bigcup_{i=1}^{k} V_i \) with \( |V_i| = m \) for every \( 1 \leq i \leq k \) satisfying:

i) All edges \( e \in F \) are transversal to the partition \( V_1 \cup \ldots \cup V_k \), i.e., \( |e \cap V_i| = 1 \) for all \( 1 \leq i \leq k \).

ii) \( F \) is linear.

iii) \( \Delta(F) \leq d \).

Note that since \( F \) is linear, the maximum degree satisfies \( \Delta(F) \leq d \leq |V_i| = m \). The following lemma gives a lower bound on the size of \( \mathcal{F}^{(k)}_{m,d} \).

**Lemma 4.3.** Let \( k, d, m \) be integers with \( k \geq 3 \) and \( m \geq m_0(k) \). Then,
\[
|\mathcal{F}^{(k)}_{m,d}| \geq \left( \frac{m^{k-1}}{d} \right)^{\frac{md}{2k^2}}
\]
holds.

**Proof.** Set \( s = \frac{dm}{2k^2} \). We will construct a graph \( F \) from \( \mathcal{F}^{(k)}_{m,d} \) by successively choosing edges \( e_1, \ldots, e_s \) transversal to the partition \( V_1 \cup \ldots \cup V_k \). Suppose that for \( \ell < s \) we already constructed \( F_\ell = \{ e_1, \ldots, e_\ell \} \) satisfying:

i) \( e_i \) is transversal to \( V_1 \cup \ldots \cup V_k \) for all \( 1 \leq i \leq \ell \).

ii) \( F_\ell \) is linear.

iii) \( \Delta(F_\ell) \leq d \).

Now we intend to add an edge \( e_{\ell+1} \) to construct the next graph \( F_{\ell+1} = F_\ell \cup \{ e_{\ell+1} \} \). For each \( 1 \leq i \leq k \), we have
\[
\sum_{x \in V_i} \deg_{F_\ell}(x) = \ell < s = \frac{dm}{2k^2}.
\]
Consequently, if \( X_i = \{ x \in V_i : \deg_{F_i}(x) = d \} \), then we have

\[
|X_i| \leq \frac{m}{2k^2}. \tag{7}
\]

To choose an edge \( e_{\ell+1} \) we need to satisfy \( |e_i \cap e_{\ell+1}| \leq 1 \) for every \( 1 \leq i \leq \ell \). Given \( e_i \), there are at most \( \binom{k}{2} m^{k-2} \) \( k \)-tuples \( f \in V_1 \times \ldots \times V_k \) such that \( |e_i \cap f| \geq 2 \). Therefore, since \( d \leq m \), there are at most

\[
f(\frac{k}{2}) m^{k-2} < s \left( \frac{k}{2} \right) m^{k-2} \leq \frac{dm^{k-1}}{4} \leq \frac{m^k}{4}
\]

\( k \)-tuples of \( V_1 \times \ldots \times V_k \) that violate condition ii) of \( \mathcal{F}^{(k)}_{m,d} \). Since the addition of any \( k \)-tuple containing an element of \( X_i \) violates condition iii) of \( \mathcal{F}^{(k)}_{m,d} \), we obtain by (7) that there are at least

\[
\prod_{i=1}^{k} |V_i \setminus X_i| - \frac{m^k}{4} \geq m^k \left( \left( 1 - \frac{1}{2k^2} \right)^k - \frac{1}{4} \right) \geq m^k \left( 1 - \frac{1}{2k} - \frac{1}{4} \right) \geq \frac{m^k}{2}
\]

valid choices for \( e_{\ell+1} \).

Thus there exist at least \( \left( \frac{m^k}{2} \right)^s \) sequences of edges \( e_1, \ldots, e_s \) forming a graph \( F = \{ e_1, \ldots, e_s \} \) satisfying conditions i), ii) and iii) of \( \mathcal{F}^{(k)}_{m,d} \). Since the same graph can be obtained by at most \( s! \) of these sequences, we have

\[
|\mathcal{F}^{(k)}_{m,d}| \geq \frac{\left( \frac{m^k}{2} \right)^s}{s!} \geq \frac{1}{2} \left( \frac{m^k}{s} \right)^s \geq \left( \frac{m^{k-1}}{d} \right)^{\frac{md}{2k^2}}. \tag{8}
\]

\( \square \)

**Proof of Proposition 4.1.** We will construct a family of \( k \)-graphs \( \mathcal{H} \) on \( n \) vertices of maximum degree \( d \) and small independence number. To this end, set \( p = \left( \frac{d}{2} \right)^{\frac{1}{k-1}} \) and consider a partition \( [n] = V_1 \cup \ldots \cup V_k \) with each \( |V_i| = n/k \). For each \( 1 \leq i \leq k \), let \( H_i \) be the \( k \)-graph with vertex set \( V_i \) consisting of the union of \( n/kp \) vertex disjoint cliques \( K_p^{(k)} \) of size \( p \). Let

\[
\mathcal{H} = \{ F \cup \bigcup_{i=1}^{k} H_i : \ F \in \mathcal{F}^{(k)}_{n/k,d/2} \}
\]

be the family of \( k \)-graphs obtained by adding a \( k \)-graph \( F \in \mathcal{F}^{(k)}_{n/k,d/2} \) with \( k \)-partition \( V_1 \cup \ldots \cup V_k \) to the union \( \bigcup_{i=1}^{k} H_i \). By Lemma 4.3, we have

\[
|\mathcal{H}| = |\mathcal{F}| \geq \left( \frac{2}{d} \left( \frac{n}{k} \right)^{k-1} \right)^{\frac{dn}{4k^2}}. \tag{8}
\]

We claim that \( H = F \cup \bigcup_{i=1}^{k} H_i \) is \( d \)-balanced for every \( H \in \mathcal{H} \). First, note that if \( x \in V(H) \), then \( x \in V_i \) for some \( 1 \leq i \leq k \) and consequently

\[
\deg_H(x) = \deg_{H_i}(x) + \deg_F(x) \leq \left( \frac{p-1}{k-1} \right) + \frac{d}{2} \leq \frac{d}{2} + \frac{d}{2} = d.
\]

Thus \( \Delta(H) \leq d. \)
Now let $S \in [n]^{(\ell)}$ for $2 \leq \ell \leq k - 1$. Note that $S \subseteq e$ for an edge $e \in \bigcup_{i=1}^{k} H_i$ if and only if $S$ is a subset of vertices of some $K_p^{(k)}$. Consequently, if $S$ is a subset of the vertex set of some $K_p^{(k)}$, then $S \subseteq V_i$ for some $1 \leq i \leq k$. Thus, by condition i) of $F_{n/k,d/2}$ we have that $\deg_F(S) = 0$. Hence,

$$\deg_H(S) \leq \binom{p-\ell}{k-\ell} \leq d^{k-1}.$$ 

Otherwise, if $S$ is not a subset of $K_p^{(k)}$ then

$$\deg_H(S) = \deg_F(S) \leq 1,$$

since $F$ is linear. Therefore, $H$ is a $d$-balanced $k$-graph.

Finally, we turn our attention to the independence number of $H \in \mathcal{H}$. Note that $\bigcup_{i=1}^{k} H_i$ is a union of $n/p$ vertex disjoint copies of $K_p^{(k)}$. Thus, since $\bigcup_{i=1}^{k} H_i \subseteq H$, we have that

$$\alpha(H) \leq \alpha\left(\bigcup_{i=1}^{k} H_i\right) \leq (k-1)\frac{n}{p}$$

for every $H \in \mathcal{H}$.

Let $t$ be the minimum integer such that it is possible to $t$-represent any element $H \in \mathcal{H}$, i.e., $t$ is the minimum integer such that for any $H \in \mathcal{H}$, there exists a system of independent sets $\{I_j : 1 \leq j \leq t\}$ with the property that every edge in $\overline{H}$ is covered by some $I_j$. Lemma 4.2 applied with (9) gives us that there exists at most

$$\left(2^{\left(n\binom{n}{(k-1)n/p}\right)^t}\right) \leq (cp)^{(k-1)nt/p}$$

ways to $t$-represent the family $\mathcal{H}$ for some positive constant $c$ depending on $k$.

Since two graphs $H, H' \in \mathcal{H}$ can not be represented by the same system $\{I_j : 1 \leq j \leq t\}$ we obtain that

$$(cp)^{(k-1)nt/p} \geq |\mathcal{H}| \geq \left(\frac{2}{d}\left(\frac{n}{k}\right)^{k-1}\right)^{\frac{dn}{4k^2}}.$$ 

Hence, by the fact that $p = (d/2)^{\frac{1}{k-1}}$ we obtain that

$$t \geq c\frac{d^{k-1}}{\log d} \log \left(\frac{n^{k-1}}{d}\right)$$

for a positive constant $c$ depending only on $k$. That is, there exists a $d$-balanced $k$-graph $H \in \mathcal{H}$ such that

$$\vartheta(H) \geq c\frac{d^{k-1}}{\log d} \log \left(\frac{n^{k-1}}{d}\right).$$

For $k$ even, we can further improve the lower bound for $k$-graphs of bounded maximum degree $d$. 

9
**Proposition 4.4.** For $n,d,k \geq 2$ and $k$ even, there exists $k$-graph $G$ with $\Delta(G) \leq d$ such that

$$\vartheta(G) \geq c \frac{d^2}{\log d} \log \left( \frac{2n}{kd} \right)$$

for a positive constant $c$.

**Proof.** Suppose that $k = 2\ell$ for some integer $\ell$. Proposition 4.1 says that there exists a 2-graph $F$ on $n/\ell$ vertices with $\Delta(F) \leq d$ such that

$$\vartheta(F) \geq c \frac{d^2}{\log d} \log \left( \frac{n}{\ell d} \right).$$

We will construct a $k$-graph $G$ with $\Delta(G) \leq d$ satisfying the inequality of the statement as follows: Let $V(F) = [n/\ell]$. For each $i \in [n/\ell]$, let $V_i = \{v_{i,1}, \ldots, v_{i,\ell}\}$ be a set consisting of $\ell$ copies of the vertex $i$. We define $V(G) = \bigcup_{i=1}^{n/\ell} V_i$ and

$$E(G) = \{V_i \cup V_j : \{i,j\} \in F\}.$$ 

That is, the edges of $G$ are the $2\ell$-tuples of the form $V_i \cup V_j$ where $i$ and $j$ are adjacent in $F$. We will prove that $G$ is our desired graph. First note that if $x \in V_i$ for some $i \in [n/\ell]$, then $\deg_G(x) = \deg_F(i)$. Thus $\Delta(G) = \Delta(F) \leq d$.

Now suppose that $G$ can be $t$-represented and let $\mathcal{I} = \{I_j : 1 \leq j \leq t\}$ be the system of independent sets covering the edges of $G$. For each $1 \leq j \leq t$, we consider the subset $\tilde{I}_j \subseteq V(F)$ given by

$$\tilde{I}_j = \{i \in [n/\ell] : V_i \subseteq I_j\}.$$ 

That is, $\tilde{I}_j$ consists of all vertices $i \in [n/\ell]$ such that $V_i$ is fully contained in $I_j$. Note that $\tilde{I}_j$ is an independent set. Indeed, suppose to the contrary that there exist $i, i' \in \tilde{I}_j$ adjacent in $F$. Then, on one hand, by the definition of $\tilde{I}_j$, we have that $V_i \cup V_{i'} \subseteq I_j$. On the other hand, by the definition of $G$, we have that $V_i \cup V_{i'} \in G$, which contradicts the fact that $I_j$ is independent.

We claim that $\tilde{\mathcal{I}} = \{\tilde{I}_j : 1 \leq j \leq t\}$ is a system of independent sets in $F$ which covers the edges of $\overline{F}$. In particular, this proves that $\vartheta(G) \geq \vartheta(F)$. Let $\{i, i'\} \in \overline{F}$ be a non-adjacent pair of vertices in $V(F)$. Then, by the construction of $G$, the $2\ell$-tuple $V_i \cup V_{i'}$ is an edge of $\overline{G}$. Since $\{I_j : 1 \leq j \leq t\}$ covers the edges of $\overline{G}$, there exists $I_j$ such that $V_i \cup V_{i'} \subseteq I_j$. Thus $\{i, i'\} \subseteq \tilde{I}_j$ and consequently $\{\tilde{I}_j : 1 \leq j \leq t\}$ covers $\overline{F}$. Therefore, we obtained a $k$-graph $G$ with $\Delta(G) \leq d$ such that

$$\vartheta(G) \geq \vartheta(F) \geq c \frac{d^2}{\log d} \log \left( \frac{2n}{kd} \right).$$ 

\[\Box\]

5. **Special hypergraphs**

While for $k$-graphs of bounded degree with $k$ large we still have a significant gap between lower and upper bounds, for Steiner systems one can obtain more precise bounds. Given $1 < \ell < k < n$, a **partial Steiner $(n,k,\ell)$-system** is a $k$-graph $S \subseteq [n]^{(k)}$ such that every $\ell$-tuple $P \in [n]^{(\ell)}$ is
contained in at most one edge of $S$. An $(n, k, \ell)$-system in which every $\ell$-tuple $P \in [n]^{(\ell)}$ is contained in precisely one edge of $S$ is called a full Steiner $(n, k, \ell)$-system. While it is easy to check the existence of partial $(n, k, \ell)$-system, the existence of full $(n, k, \ell)$-system for all admissible parameters $1 < \ell < k$ and $n \geq n_0(\ell, k)$ was established only recently in [4] and [6]. In this section we are going to provide bounds for the representation number of partial and full Steiner $(n, k, \ell)$-systems for $\ell = 2$.

For $1 < \ell < k < n$, we define

$$s(n, k, \ell) = \max\{\vartheta(S) : S \text{ is a partial } (n, k, \ell)-\text{system}\}$$

as the maximum value of $\vartheta(S)$, where $S$ runs over all partial $(n, k, \ell)$-systems. Similarly, we define $s^*(n, k, \ell)$ as the maximum value of $\vartheta(S)$, where $S$ runs over all full $(n, k, \ell)$-systems.

**Theorem 5.1.** Given $k > 1$, there exist positive constants $c_1$ and $c_2$ depending only on $k$ such that the following holds:

i) For $n$ sufficiently large,

$$c_1 \frac{n^k}{(\log n)^{k-1}} \leq s(n, k, 2) \leq c_2n^{\frac{k}{k-1}} \log n$$

ii) For infinitely many $n$,

$$c_1 \frac{n^k}{(\log n)^{k-1}} \leq s^*(n, k, 2) \leq c_2n^{\frac{k}{k-1}} \log n$$

Note that Theorem 5.1.i) is just a reformulation of Theorem 2.3. Before we give a proof of the theorem, we are going to introduce two useful results.

Let $K^{(\ell)}_k(m)$ be an $m$-blowup of the complete $\ell$-graph with $k$ vertices, i.e., $K^{(\ell)}_k(m)$ is the graph with vertex set $V = \bigcup_{i=1}^k V_i$, where $|V_i| = m$ for $1 \leq i \leq k$, and is such that for every $1 \leq i_1 < \ldots < i_\ell \leq k$ and $x_{i_j} \in V_{i_j}$ the $\ell$-tuple $\{x_{i_1}, \ldots, x_{i_\ell}\}$ is an edge. The $K^{(\ell)}_k$-decomposition of $K^{(\ell)}_k(m)$ is a system of pairwise edge disjoint copies of $K^{(\ell)}_k$ covering all edges of $K^{(\ell)}_k(m)$. Equivalently, one can view each such decomposition as a $k$-partite $k$-graph $F$ with vertex partition $V(F) = \bigcup_{i=1}^k V_i$, where $|V_i| = m$ for $1 \leq i \leq k$, and such that for every $1 \leq i_1 < \ldots < i_\ell \leq k$ and vertices $x_{i_j} \in V_{i_j}$ there exists exactly one edge $f \in F$ with $\{x_{i_1}, \ldots, x_{i_\ell}\} \subseteq f$.

In [7], Keevash obtained strong bounds on the number of such decompositions.

**Theorem 5.2 (Theorem 2.8, [7]).** The number of distinct $K^{(\ell)}_k$-decompositions of $K^{(\ell)}_k(m)$ is given by

$$\left((e^{1-\frac{1}{\ell}} + o(1))m^{k-\ell}\right)^m$$

for sufficiently large $m$.

We will also need the following result on existence of $(n, k, \ell)$-systems with bounded independence number.
Theorem 5.3 ([5]). For \( k > \ell \geq 2 \), there is a positive constant \( c \) depending only on \( k \) such that the following holds. If \( m = q^2 \) for \( q \) sufficiently large prime power, then there exists a full Steiner \((m,k,2)\)-system \( S \) with \( \alpha(S) \leq cm^{k-2} (\log m)^{1/k-1} \).

Proof of Theorem 5.1. To establish the upper bound we first note that any full or partial \((n,k,2)\)-system \( S \) is a \( \Delta_1(S) \)-balanced \( k \)-graph. Indeed, observe that for any \( 2 \leq i \leq k-1 \) we have \( \Delta_i(S) \leq 1 < \Delta_1(S)^{1/k-1} \). Also note that for \( x \in V(S) \),

\[
\deg(x) \leq \sum_{y \in [n] \setminus \{x\}} \deg(\{x, y\}) \leq n - 1,
\]

since \( \deg(\{x, y\}) \leq 1 \) for all \( x, y \in V(S) \). Hence \( \Delta_1(S) \leq n \) and by the upper bound of Theorem 2.1, there exists positive constant \( c_2 \) such that

\[
\vartheta(S) \leq c_2 \Delta_1(S)^{1/k-1} \log n \leq c_2 n^{1/k-1} \log n.
\]

Next we prove the lower bounds of \( s(n,k,2) \) and \( s^*(n,k,2) \). We may assume without changing the asymptotics that \( m = n/k = q^2 \) for \( q \) sufficiently large prime. By Theorem 5.3 there exists a full \((m,k,2)\)-system \( S \) with \( \alpha(S) \leq cm^{k-2} (\log m)^{1/k-1} \). Consider \( k \) vertex disjoint copies \( S_1, \ldots, S_k \) of \( S \) and let \( V_i = V(S_i) \) for \( 1 \leq i \leq k \). Let \( \mathcal{F} \) be the family of \( k \)-partite \( k \)-graphs \( F \) with vertex set \( V(F) = \bigcup_{i=1}^{k} V_i \) such that for every \( 1 \leq i < j \leq k \) and \( x_i \in V_i, x_j \in V_j \), there exists exactly one \( f \in F \) such that \( \{x_i, x_j\} \subseteq f \). As discussed previously, the family \( \mathcal{F} \) is in one-to-one correspondence with the \( K_k^{(2)} \)-decompositions of \( K_k^{(2)}(m) \). Thus, by Theorem 5.2, we have

\[
|\mathcal{F}| = \left( \left( e^{1-(k-2)/2} + o(1) \right) m^{k-2} \right)^m,
\]

for sufficiently large \( m \).

Let \( \mathcal{H} \) be the family of graphs defined by

\[
\mathcal{H} = \{ F \cup \bigcup_{i=1}^{k} S_i : F \in \mathcal{F} \}.
\]

We observe that every \( H \in \mathcal{H} \) is a full \((n,k,2)\)-system. Let \( H = F \cup \bigcup_{i=1}^{k} S_i \) for some \( F \in \mathcal{F} \) and let \( \{x, y\} \subseteq V(H) \). If \( \{x, y\} \subseteq V_i \) for some \( 1 \leq i \leq k \), then since \( S_i \) is a full Steiner system, there exists exactly one edge \( e \in S_i \) such that \( \{x, y\} \subseteq e \). Also if \( x \in V_i \) and \( y \in V_j \) for \( i \neq j \), then by the definition of \( F \), there exists exactly one edge \( e \in F \) such that \( \{x, y\} \subseteq e \).

Set \( \alpha = k \alpha(S) \) and let \( t \) be the minimum integer such that every graph \( H \in \mathcal{H} \) admits a \( t \)-representation. By Lemma 4.2 there are \( 2 \binom{n}{\alpha}^t \) ways to \( t \)-represent graphs in \( \mathcal{H} \). Since every \( t \)-representation corresponds to a unique graph in \( \mathcal{H} \) and by our assumption that every graph \( H \in \mathcal{H} \) has a \( t \)-representation, we have by (10) that

\[
2^{t \alpha \log n} \geq \left( 2 \binom{n}{\alpha} \right)^t \geq |\mathcal{H}| = |\mathcal{F}| = \left( \left( e^{1-(k-2)/2} + o(1) \right) m^{k-2} \right)^m \geq 2^{c \alpha^2 \log n}
\]
Since $\alpha(S) \leq cn^{k-2} (\log m)^{\frac{1}{k-1}} \leq c'' n^{k-2} (\log n)^{\frac{1}{k-1}}$, we obtain that
\[
t \geq c_1 \frac{n^{\frac{1}{k-1}}}{(\log n)^{\frac{1}{k-1}}},
\]
for a positive constant $c_1$ depending on $k$. Therefore, there exists a full $(n, k, 2)$-system $H \in \mathcal{H}$ such that $\vartheta(H) \geq c_1 \frac{n^{\frac{1}{k-1}}}{(\log n)^{\frac{1}{k-1}}}$. Since a full $(n, k, 2)$-system is a partial $(n, k, 2)$-system, we obtain that
\[
s(n, k, 2) \geq s^*(n, k, 2) \geq c_1 \frac{n^{\frac{1}{k-1}}}{(\log n)^{\frac{1}{k-1}}}.
\]

\[\Box\]

6. Concluding Remarks

6.1. Representation of sparse hypergraphs of high uniformity. In Sections 3 and 4 we determined upper and lower bounds for the family of $k$-graphs on $n$ vertices of bounded degree $d$. In particular, if we define
\[
f_k(d) = \lim_{n \to \infty} \frac{\vartheta(n, d, k)}{\log n},
\]
then we established almost sharp bounds for $k = 2, 3, 4$, i.e., $\Omega \left( \frac{d^2}{\log d} \right) \leq f_2(d) \leq O(d^2)$, $\Omega \left( \frac{d^3}{\log d} \right) \leq f_3(d) \leq O(d^{3/2})$ and $\Omega \left( \frac{d^2}{\log d} \right) \leq f_4(d) \leq O(d^2)$.

Unfortunately, for $k \geq 5$, the bounds we have at the moment are worse. More precisely,
\[
\Omega \left( \frac{d^2}{\log d} \right) \leq f_k(d) \leq O \left( d^{k/2} \right)
\]
for $k$ even and
\[
\Omega \left( \frac{d^{k/2}}{\log d} \right) \leq f_k(d) \leq O \left( d^{k/2} \right)
\]
for $k$ odd. It would be interesting to close the gap between the bounds for $f_k(d)$.

6.2. Representation of $k$-partite $k$-graphs. Let $\mathcal{K}(n, d, k)$ be the family of $k$-partite $k$-graphs $G$ on $n$ vertices with bounded maximum degree $\Delta(G) \leq d$. As in the previous case, it would be interesting to find good bounds on $\vartheta(G)$ for $G \in \mathcal{K}(n, d, k)$. Let
\[
g_k(d) = \lim_{n \to \infty} \max \{ \vartheta(G) : G \in \mathcal{K}(n, d, k) \}.
\]
A similar counting argument as the one used in Section 4 yields that $g_k(d) = \Omega(d)$. For $k = 2$, similar techniques as in Section 3 give the upper bound which is linear in $d$, i.e., $g_2(d) = O(d)$. This leads us to believe that perhaps the same could be true for $k \geq 3$.

**Conjecture 6.1.** $g_k(d) = O(d)$ for $k \geq 3$. 

References

[1] N. Alon, Covering graphs by the minimum number of equivalence relations, Combinatorica 6 (1986), no. 3, 201–206. MR875288
[2] V. Eaton N.and Rödl, Graphs of small dimensions, Combinatorica 16 (1996), no. 1, 59–85. MR1394512
[3] P. Erdős, A. W. Goodman, and L. Pósa, The representation of a graph by set intersections, Canadian J. Math. 18 (1966), 106–112. MR186575
[4] S. Glock, D. Kühn, A. Lo, and D. Osthus, The existence of designs via iterative absorption: hypergraph f-designs for arbitrary f, arXiv, 2016. ↑11
[5] G. Grable, K. Phelps, and V. Rödl, The minimum independence number for designs, Combinatorica 15 (1995), no. 2, 175–185. MR1337352
[6] P. Keevash, The existence of designs, arXiv, 2014. ↑11
[7] , Coloured and directed designs, Building Bridges. II, [2019] ©2019, pp. 279–315. MR4297785
[8] J. Lehel, Covers in hypergraphs, Combinatorica 2 (1982), no. 3, 305–309. MR608656
[9] V. Rödl and A. Ruciński, Covering the edge of a random hypergraph by cliques (submitted). ↑1

7. Appendix

Proposition 7.1. Let \( F \) be a \( k \)-graph for \( k \geq 2 \). Then \( \Theta(F) = cc(F) \).

Proof. Let \( \mathcal{C} = \{C_1, \ldots, C_t\} \) be a system of cliques covering all edges of \( F \). To each vertex \( v \in V(F) \) assign the set \( S_v \) of all cliques containing \( v \). Since \( \mathcal{C} \) is a covering of the edges, if \( \{v_1, \ldots, v_k\} \subseteq F \), then there exists a clique \( C_i \) containing \( \{v_1, \ldots, v_k\} \). Thus \( C_i \in \bigcap_{i=1}^k S_{v_i} \) and hence \( \bigcap_{i=1}^k S_{v_i} \neq \emptyset \). On the other hand, by construction, \( S_{v_j} = \{C_i \in \mathcal{C} : v_j \in C_i\} \) and therefore the set \( \bigcap_{i=1}^k S_{v_i} \) contains all cliques containing \( \{v_1, \ldots, v_k\} \). Consequently, if \( \bigcap_{j=1}^k S_{v_j} = \emptyset \), then there is no clique containing \( \{v_1, \ldots, v_k\} \), which implies that \( \{v_1, \ldots, v_k\} \notin F \). Hence, we proved that \( \mathcal{C} \) is a set representation of \( F \) and we obtain that

\[
\Theta(F) \leq cc(F).
\]

To prove the opposite inequality, let \( V(F) = \{v_1, \ldots, v_n\} \) and let \( t \) be the an integer such that \( F \) admits a set representation \( \{S_{v_i} \subseteq [t] : 1 \leq i \leq n\} \). For each \( s \in [t] \), we define the set \( C(s) = \{v_i : s \in S_{v_i}\} \). Since \( s \) belongs to the intersection of any \( k \) sets of the form \( S_v \) for \( v \in C(s) \), we obtain that \( F[C(s)] \) is a clique. We claim that \( \mathcal{F} = \{F[C(1)], \ldots, F[C(s)]\} \) is a clique covering of \( F \). Indeed, if \( \{v_{i_1}, \ldots, v_{i_k}\} \subseteq F \) is an edge of \( F \), then \( \bigcap_{j=1}^k S_{v_j} \neq \emptyset \). Let \( s \in \bigcap_{j=1}^k S_{v_j} \). Then clearly \( \{v_{i_1}, \ldots, v_{i_k}\} \notin F[C(s)] \). Now if \( \{v_{i_1}, \ldots, v_{i_k}\} \notin F \), then \( \bigcap_{j=1}^k S_{v_j} = \emptyset \) and consequently there is no \( s \) such that \( \{v_{i_1}, \ldots, v_{i_k}\} \subseteq C(s) \). Hence, \( \mathcal{F} \) is a clique covering and \( cc(F) \leq \Theta(F) \) follows. \( \square \)

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