On the number of edges of a graph and its complement

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Abstract
Let $G = (V, E)$ be a graph. The complement of $G$ is the graph $\overline{G} := (V, [V]^2 \setminus E)$ where $[V]^2$ is the set of pairs $(x, y)$ of distinct elements of $V$. If $K$ is a subset of $V$, the restriction of $G$ to $K$ is the graph $G|_K := (K, [K]^2 \cap E)$. We prove that if $G = (V, E)$ is a graph and $v$ is an integer, $2 \leq v \leq n - 2$, then there is a $k$-element subset $K$ of $V$ such that $e(\overline{G|_K}) \neq e(G|_K)$, moreover the condition $k \leq v - 2$ is optimal. We also study the case $e(\overline{G|_K}) \neq e(G|_K)(\mod \ p)$ where $p$ is a prime number. Following a question from M.Pouzet, we show this: Let $G = (V, E)$ be a graph with $v$ vertices. If $e(G) \neq e(\overline{G})$ (resp. $e(G) = e(\overline{G})$) then there is an increasing family $(H_n)_{2 \leq n \leq 3} \cup \cup (H_n)_{2 \leq n \leq 2}$ of $n$-element subsets $H_n$ of $V$ such that $e(G\mid_{H_n}) \neq e(\overline{G\mid_{H_n}})$ for all $n$. Similarly if $e(G) \neq e(\overline{G}) \mod \ p$ where $p$ is a prime number, $p > v - 2$, then there is an increasing family $(H_n)_{2 \leq n \leq 3}$ of $n$-element subsets $H_n$ of $V$ such that $e(G\mid_{H_n}) \neq e(\overline{G\mid_{H_n}})(\mod \ p)$ for all integer $n \in \{2, 3, \ldots, v\}$.

Keywords Set, Matrix, Graph, Edge, Prime number

Paper type Original Article

1. Introduction
Our notations and terminology follow [2]. A graph is an ordered pair $G := (V, E)$ (or $(V(G), E(G)))$, where $E$ is a subset of $[V]^2$, the set of pairs $(x, y)$ of distinct elements of $V$. Elements of $V$ are the vertices of $G$ and elements of $E$ are its edges. An edge $(x, y)$ is also noted by $xy$. The cardinality $|V|$ of $V$ is called the order of $G$. Two distinct vertices $x$ and $y$ are adjacent if $xy \in E(G)$, otherwise $x$ and $y$ are non-adjacent. We denote by $e(G) := |E(G)|$ the number of edges of $G$. The degree of a vertex $x$ of $G$, denoted by $d_G(x)$, is the number of edges which contain $x$. The graph $G$ is $\delta$-regular (or regular) if $d_G(x) = \delta$ for all $x \in V$,

JEL Classification — 05C50, 05C60

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δ is called the degree of the regular graph \( G \). The complement of \( G \) is the graph \( \overline{G} := (V, [V]^2 \setminus E) \). If \( K \) is a subset of \( V \), the restriction of \( G \) to \( K \), also called the induced subgraph of \( G \) on \( K \), is the graph \( G[K] := (K, [K]^2 \cap E) \). For instance, given a set \( V \), \((V, \emptyset)\) is the empty graph on \( V \) whereas \((V, \{xy : x \neq y \in V\})\) is the complete graph.

Our first result is Theorem 1.1, we prove that: given a graph \( G = (V, E) \) and \( k \) be an integer, \( 2 \leq k \leq v - 2 \), we cannot have \( e(G[K]) = e(G[K]) \) for all \( k \)-element subsets \( K \) of \( V \), moreover the condition \( k \leq v - 2 \) is optimal, indeed for \( k = v - 1 \) a counterexample is given by (2) of Theorem 1.1.

**Theorem 1.1.** Let \( G = (V, E) \) be a graph of order \( v \).

1. If \( v \geq 4 \), then for each integer \( k \) with \( 2 \leq k \leq v - 2 \), there is a \( k \)-element subset \( K \) of \( V \) such that \( e(G[K]) \neq e(G[K]) \).
2. If \( G \) is regular with degree \( \frac{v - 1}{2} \) then \( e(G) = e(G) \) and \( e(G - x) = e(G - x) \) for all vertex \( x \).
3. Let \( p \) be a prime number with \( p \geq 3 \) such that \( 2d_G(x) \equiv v - 1 \pmod{p} \) for all \( x \in V \). Then \( e(G) \equiv e(G)(\pmod{p}) \) and \( e(G - x) \equiv e(G - x)(\pmod{p}) \) for all vertex \( x \).

Our second result is Theorem 1.2. Given a graph \( G = (V, E) \), \( p \) a prime number, and \( k \) an integer, \( 2 \leq k \leq v - 2 \), under some conditions on \( k \), we cannot have \( e(G[K]) \equiv e(G[K]) \pmod{p} \) for all \( k \)-element subsets \( K \) of \( V \).

**Theorem 1.2.** Let \( G = (V, E) \) be a graph with \( v \geq 4 \) vertices and let \( p \) be a prime number. Let \( k \) be an integer, \( 2 \leq k \leq v - 2 \).

1. If \((p = 2 \text{ and } k \equiv 2(\text{mod } 4)) \) or \((p \geq 3 \text{ and } k \equiv 0, 1(\text{mod } p)) \), then there is a \( k \)-element subset \( K \) of \( V \) such that \( e(G[K]) \neq e(G[K]) \pmod{p} \).
2. If \( p \geq 3 \) and \( k \equiv 0(\text{mod } p) \) then there is a \( k \)-element subset \( K \) of \( V \) such that \( e(G[K]) \neq e(G[K]) \pmod{p} \) if and only if \( G \) is neither the complete graph nor the empty graph.

Our third result is Theorem 1.3. It is related to a question that M. Pouzet asked us about the existence, in a graph \( G = (V, E) \), of an increasing family \((H_n)_n\) of \( n \)-element subsets \( H_n \) of \( V \) such that \( e(G[H_n]) \neq e(G[H_n]) \).

**Theorem 1.3.** Let \( G = (V, E) \) be a graph of \( v \) vertices with \( v \geq 3 \).

1. If \( e(G) \neq e(G) \) then there is a vertex \( x \) such that \( e(G - x) \neq e(G - x) \).
2. If \( e(G) \neq e(G) \) then there is an increasing family \((H_n)_{n \leq v} \) of \( n \)-element subsets \( H_n \) of \( V \) such that \( e(G[H_n]) \neq e(G[H_n]) \) for all integer \( n \in \{2, 3, \ldots, v\} \).
3. If \( e(G) = e(G) \) and \( v \geq 4 \) then there is an increasing family \((H_n)_{n \leq v - 2} \) of \( n \)-element subsets \( H_n \) of \( V \) such that \( e(G[H_n]) \neq e(G[H_n]) \) for all integer \( n \in \{2, 3, \ldots, v - 2\} \).
4. Let \( p \) be a prime number, \( p > v - 2 \). If \( e(G) \neq e(G)(\text{mod } p) \) then there is an increasing family \((H_n)_{n \leq v} \) of \( n \)-element subsets \( H_n \) of \( V \) such that \( e(G[H_n]) \neq e(G[H_n])(\text{mod } p) \) for all integer \( n \in \{2, 3, \ldots, v\} \).

2. Incidence matrices

We consider the matrix \( W_{tk} \) defined as follows: Let \( V \) be a finite set, with \( v \) elements. Given non-negative integers \( t, k \), let \( W_{tk} \) be the \( \binom{v}{t} \) by \( \binom{v}{k} \) matrix of 0’s and 1’s, the rows of which
are indexed by the $t$-element subsets $T$ of $V$, the columns are indexed by the $k$-element subsets $K$ of $V$, and where the entry $W_{k}(T,K)$ is 1 if $T \subseteq K$ and is 0 otherwise. The matrix transpose of $W_{tk}$ is denoted $W_{kt}$.

A fundamental result, due to D.H. Gottlieb [4], and independently W. Kantor [5], is this:

**Theorem 2.1** (D.H. Gottlieb [4], W. Kantor [5]). For $t \leq \min(k,v-k)$, $W_{tk}$ has full row rank over the field $\mathbb{Q}$ of rational numbers.

It is clear that $t \leq \min(k,v-k)$ implies $\binom{v}{t} \leq \binom{v}{k}$ then, from Theorem 2.1, we have the following result.

**Corollary 2.2.** For $t \leq \min(k,v-k)$, the rank of $W_{tk}$ over the field $\mathbb{Q}$ of rational numbers is $\binom{v}{t}$ and thus $\text{Ker}(W_{tk}) = \{0\}$.

**Corollary 2.2** and the following theorem are important tools in the proof of our main results. In fact, Theorem 2.3 has made to establish a version modulo a prime of Kelly’s combinatorial lemma [6]; it also allows to obtain a version modulo a prime of the particular results. In fact, Theorem 2.3 has made to establish a version modulo a prime of Pouzet’s combinatorial lemma [7].

Let $n, p$ be positive integers, the decomposition of $n = \sum_{i=0}^{n(p)} n_{i} p^{i}$ in the basis $p$ is also denoted by $[n_{0}, n_{1}, \ldots, n_{n(p)}]_{p}$ where $n_{n(p)} \neq 0$ if and only if $n \neq 0$.

**Theorem 2.3** [1]. Let $p$ be a prime number. Let $v, t$ and $k$ be non-negative integers, $k = [k_{0}, k_{1}, \ldots, k_{k(p)}]_{p}$, $t = [t_{0}, t_{1}, \ldots, t_{t(p)}]_{p}$, $t \leq \min(k,v-k)$. We have:

1. $k_{j} = t_{j}$ for all $j < t(p)$ and $k_{t(p)} \geq t_{t(p)}$ if and only if $\text{Ker}(W_{tk}) = \{0\}$ (mod $p$).
2. $t = t_{t(p)} p^{t(p)}$ and $k = \sum_{i=0}^{t(p)} k_{i} p^{i}$ if and only if $d\text{Ker}(W_{tk}) = 1$ (mod $p$) and $\{1, 1, \ldots, 1\}$ is a basis of $\text{Ker}(W_{tk})$.

The notation $a \mid b$ (resp. $a \nmid b$) means $a$ divide $b$ (resp. $a$ not divide $b$).

**Theorem 2.4** (Lucas’ Theorem [3]). Let $p$ be a prime number, $t, k$ be positive integers, $t \leq k$, $t = [t_{0}, t_{1}, \ldots, t_{t(p)}]_{p}$ and $k = [k_{0}, k_{1}, \ldots, k_{k(p)}]_{p}$. Then

$$\binom{k}{t} = \prod_{i=0}^{t(p)} \binom{k_{i}}{t_{i}} (\text{mod } p), \text{ where } \binom{k_{i}}{t_{i}} = 0 \text{ if } t_{i} > k_{i}.$$  

The following result is a consequence of Lucas’ theorem.

**Corollary 2.5** ([1]). Let $p$ be a prime number, $t, k$ be positive integers, $t \leq k$, $t = [t_{0}, t_{1}, \ldots, t_{t(p)}]_{p}$ and $k = [k_{0}, k_{1}, \ldots, k_{k(p)}]_{p}$. Then:

$$p \binom{k}{t} \text{ if and only if there is } i \in \{0, 1, \ldots, t(p)\} \text{ such that } t_{i} > k_{i}.$$  

### 3. Proofs of main results

Let $G = (V,E)$ be a graph of order $v$. Let $T_{1}, T_{2}, \ldots, T_{\binom{v}{2}}$ be an enumeration of the 2-element subsets of $V$, let $K_{1}, K_{2}, \ldots, K_{\binom{v}{k}}$ be an enumeration of the $k$-element subsets of $V$.

Let $w_{G}$ be the row matrix $(g_{1}, g_{2}, \ldots, g_{\binom{v}{k}})$ where $g_{i} = 1$ if $T_{i}$ is an edge of $G$, 0 otherwise.

We have $w_{G} W_{2k} = (e(G_{1}K_{1}), e(G_{1}K_{2}), \ldots, e(G_{1}K_{\binom{v}{k}}))$. 
Note that \( w_G = (\overline{g_1}, \overline{g_2}, \ldots, \overline{g_i}) \) with \( \overline{g_i} = 0 \) if \( T_i \) is an edge of \( G \), 1 otherwise. We have \( w_G W_{2k} = (e(\overline{G},K_1), e(\overline{G},K_2), \ldots, e(\overline{G},K_k)) \).

### 3.1 Proof of Theorem 1.1

(1) Assume that \( e(\overline{G},K) = e(G,K) \) for all \( k \)-element subsets \( K \) of \( V \), then \( w_G W_{2k} = w_G W_{2k} \).

By Corollary 2.2, \( \text{Ker}(t W_{tk}) = \{0\} \). Then \( w_G = w_G \), so \( G = \overline{G} \), which is impossible. So there is a \( k \)-element subset \( K \) of \( V \) such that \( e(\overline{G},K) \neq e(G,K) \).

(2) For \( x \in V \), \( d_G(x) + d_{\overline{G}}(x) = v - 1 \). Since \( d_G(x) = \frac{v-1}{2} \) then \( d_G(x) = d_{\overline{G}}(x) \). We have \( \sum_{x \in V} d_G(x) = 2e(G) \) and \( \sum_{x \in V} d_{\overline{G}}(x) = 2e(\overline{G}) \), then \( e(G) = e(\overline{G}) \). Now from \( e(G) = e(\overline{G} - x) + d_{\overline{G}}(x) \) and \( e(G) = e(G - x) + d_G(x) \), we deduce that \( e(\overline{G} - x) = e(G - x) \).

(3) For \( x \in V \), \( d_G(x) + d_{\overline{G}}(x) = v - 1 \). Since \( 2d_G(x) \equiv v - 1 (mod \ p) \) then \( d_G(x) \equiv d_{\overline{G}}(x) \ (mod \ p) \). We conclude using similar arguments to those in item (2).

### 3.2 Proof of Theorem 1.2

We set \( t := 2 \). We recall the notation \( k = [k_0, k_1, \ldots, k_{\ell(p)}] \).

(1) Case 1, \( p = 2 \) and \( k \equiv 2 (mod \ 4) \).

We have \( k_0 = 0, k_1 = 1, t = 0, 1 \). Since \( k_0 = t_0 \) and \( k_1 \geq t_1 = t_{\ell(p)} \) then, by Theorem 2.3, \( \text{Ker}(t W_{tk}) = \{0\} \ (mod \ p) \).

Case 2, \( p \geq 3 \) and \( k \not\equiv 0, 1 (mod \ p) \).

We have \( k_0 \geq 2, t = t_0 = 2 \) since \( k_0 \geq 2 \geq t_{\ell(p)} \) then, by Theorem 2.3, \( \text{Ker}(t W_{tk}) = \{0\} \ (mod \ p) \).

In the two cases, \( \text{Ker}(t W_{tk}) = \{0\} \ (mod \ p) \). Assume that \( e(\overline{G},K) \equiv e(G,K) (mod \ p) \) for all \( k \)-element subsets \( K \) of \( V \). Then \( w_G W_{2k} = w_G W_{2k} \ (mod \ p) \). As \( \text{Ker}(t W_{tk}) = \{0\} \ (mod \ p) \), then \( w_G = w_G \ (mod \ p) \), so \( G = \overline{G} \), which is impossible. Then there is a \( k \)-element subset \( K \) of \( V \) such that \( e(\overline{G},K) \neq e(G,K) (mod \ p) \).

(2) If \( p \geq 3 \) then \( t = t_0 = 2 = t_{\ell(p)} \). Since \( k \equiv 0 (mod \ p) \) then \( k_0 = 0 \), and thus \( k = \sum_{i = 0}^{\ell(p)} 1 \). By Theorem 2.3, \( \{1, 1, \ldots, 1\} \) is a basis of \( \text{Ker}(t W_{tk}) \).

If \( G \) is the complete graph then \( e(G,K) = \binom{k}{2} \), \( e(\overline{G},K) = 0 \). Since \( t_0 = 2 > k_0 = 0 \) then, by Corollary 2.5, \( p | \binom{k}{2} \). So \( e(G,K) \equiv 0 \equiv e(\overline{G},K) (mod \ p) \). If \( G \) is the empty graph, then \( e(G,K) = 0 \equiv e(\overline{G},K) = \binom{k}{2} (mod \ p) \).

Conversely if \( G \) is neither the complete graph nor the empty graph, assume that \( e(\overline{G},K) \equiv e(G,K) (mod \ p) \) for all \( k \)-element subsets \( K \) of \( V \). Then \( w_G W_{2k} = w_G W_{2k} (mod \ p) \). So \( w_G - w_G = \lambda(1, 1, \ldots, 1) \ (mod \ p) \) with \( \lambda \in \{0, 1, -1\} \). As \( G \) is neither the complete graph nor the empty graph, there are \( i, j \) such that \( g_i = 0 \) and \( g_j = 1 \), so \( g_i - g_i = -1 \) and \( g_i - g_j = 1 \). Then \( \lambda \neq 1 \) and \( \lambda \neq -1 \). Thus \( \lambda = 0 \), and \( w_G = w_G (mod \ p) \), so \( G = \overline{G} \), which is impossible.

Then there is a \( k \)-element subset \( K \) of \( V \) such that \( e(\overline{G},K) \neq e(G,K) (mod \ p) \). □

### 3.3 Proof of Theorem 1.3

We need the following lemma.
Lemma 3.1. Let $G = (V, E)$ be a graph of order $v$ and let $p$ be a prime number, $p \geq 3$.

(1) Let $k$ be an integer, $2 \leq k \leq v - 2$. If \( \binom{v-2}{k-2} e(G) \neq \binom{v-2}{k-2} e(\overline{G}) \mod p \) then there is a $k$-element subset $K$ of $V$ such that $e(G_{1,k}) \neq e(\overline{G}_{1,k}) \mod p$.

(2) If \( \binom{v-2}{k-2} e(G) \neq \binom{v-2}{k-2} e(\overline{G}) \mod p \) then there is $x \in V$ such that $e(G-x) \neq e(\overline{G}-x) \mod p$.

Proof. (1) It is an immediate consequence of the following formula

\[
\binom{v-2}{k-2} e(G) = \sum_{K \in \mathcal{P}(V), |K| = k} e(G_{1,K})
\]

(2) Follows from (1) by taking $k = v - 1$. \qed

Now we prove Theorem 1.3.

(1) Assume that $e(\overline{G} - x) = e(G - x)$ for all $x \in V$. From $e(G) = e(G - x) + d_G(x)$, we obtain $\sum_{x \in V} e(G) = \sum_{x \in V} e(G - x) + \sum_{x \in V} d_G(x)$. Since $\sum_{x \in V} d_G(x) = 2e(G)$ then $(v-2) e(G) = \sum_{x \in V} e(G - x)$. Similarly, $(v-2) e(\overline{G}) = \sum_{x \in V} e(\overline{G} - x)$. Then $e(G) = e(\overline{G})$.

(2) We make the proof by induction on $v$. We set $H_0 := V$. We assume that $H_i$ is defined for all $0 \leq i \leq v$. Let us define $H_{v+1}$. As $e(G_{1,H_i}) \neq e(\overline{G}_{1,H_i})$ then by (1), there is $x \in H_i$ such that $e(G_{1,H_i} - x) \neq e(\overline{G}_{1,H_i} - x)$. We set $H_{v+1} := H_v \setminus \{x\}$. So $H_{v+1} \subset H_v$ and $e(G_{1,H_{v+1}}) \neq e(\overline{G}_{1,H_{v+1}})$.

(3) By applying (1) of Theorem 1.1 for the graph $G$ and $k = v - 2$ we obtain $e(G - \{x\}) \neq e(\overline{G} - \{x\})$ if $v = 4$ we are done. If $v \geq 5$, then $v - 2 \geq 3$ and here we conclude using (2).

(4) By induction on $v$. Since $e(G) \neq e(\overline{G}) \mod p$ and $p > v - 2$ then $(v-2) e(G) \neq (v-2) e(\overline{G}) \mod p$. By (2) of Lemma 3.1, there is $x \in V$ such that $e(G-x) \neq e(\overline{G}-x) \mod p$. We conclude by using the induction hypothesis. \qed

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