The Projective Andoyer transformation and the connection between the 4-D isotropic oscillator and Kepler systems

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Abstract

Extending to 4 degrees of freedom a symplectomorphism used in attitude dynamics it is shown in a direct way the connection between the 4-D isotropic harmonic oscillator and the 3-D Kepler systems. This approach made transparent that only when we refer to rectilinear solutions, the bilinear relation defining the KS transformation is needed.

1 Introduction

More than 40 years after the KS transformation [1], the epitome of which might be the work of Cordani [2], any proposal of a new insight on the connection between the 4-D isotropic harmonic oscillator and Kepler systems might be taken as the opposite. Nevertheless, the way in which many authors still deal with the relation between these integrable systems, has pushed finally to take the risk and write this letter. Our claim is that, in order to present the connection of these systems we do not need to go by concepts like embeddings, weak extended canonical transformation formalism [3], or the more recently used geometric algebraic approaches [4, 5, 6]. We came to understand what is explained here studying the 4-D isotropic oscillator reductions [7]. More precisely, it is shown in a direct way that the Kepler-Coulomb flow appears as a part of the 4-D isotropic oscillator flow.

The core of our approach goes back to a basic concept in 3-D dynamics ‘the instantaneous plane of motion’ and the associated nodal-polar variables, already used in planetary theories by Hill [8] and much later in satellite theory (see Deprit [9]). Today some authors refer to them as Whittaker transformation [10, 11]. Nevertheless, for reasons which
are unknown to this author, this chart still continues to be ignored, versus the spherical variables, not only in vast field of mechanics but even in the special area of Hamiltonian astrodynamics.

Inspired in the 3-D Whittaker transformation, what we propose here is to consider part of the Andoyer angles (see (12)) as the ones defining a ‘plane of motion’ joint with a function of the distance, which lead to the Kepler system as part of the 4-D oscillator. This author is convinced that the use of ‘polar-nodal variables’ in 4-D context will be of great benefit also in quantum mechanics, molecular dynamics, etc.

The 4-D isotropic oscillator is an integrable dynamical system defined by the parametric Hamiltonian function

\[ H_\omega = \frac{1}{2} \sum_{i=1}^{4} (Q_i^2 + \omega q_i^2), \]

where \( \omega \) is a parameter. There is a large and uninterrupted literature about this system going from Jauch and Hill [12] to Waldvogel [13], including Moser [14] and Iwai [15], due to the fact that it is one of the rare few examples of maximally superintegrable systems (see Fassò [16]).

2 Starting with Projective Euler variables

We consider the transformation: \( \mathcal{PE}_F : (\rho, \phi, \theta, \psi) \to (q_1, q_2, q_3, q_4) \), dubbed as Projective Euler variables, given by

\[
\begin{align*}
q_1 &= F(\rho) \sin \frac{\theta}{2} \cos \frac{\phi - \psi}{2}, \quad q_3 = F(\rho) \cos \frac{\theta}{2} \sin \frac{\phi + \psi}{2}, \\
q_2 &= F(\rho) \sin \frac{\theta}{2} \sin \frac{\phi - \psi}{2}, \quad q_4 = F(\rho) \cos \frac{\theta}{2} \cos \frac{\phi + \psi}{2},
\end{align*}
\]

with \((\rho, \phi, \theta, \psi) \in R^+ \times [0, 2\pi) \times (0, \pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\). When \( F(\rho) = 1 \), the transformation defines Euler parameters as functions of Euler angles. We choose here \( F(\rho) = \sqrt{\rho} \). This transformation is well known in the literature (see, for instance, [17, 3, 18, 19, 20]). The canonical extension associated to the transformation \( \mathcal{PE}_F \) is obtained as a Mathieu transformation, which satisfies \( \sum Q_i dq_i = P \, d\rho + \Phi \, d\phi + \Theta \, d\theta + \Psi \, d\psi \). The relations among the momenta are given by

\[
\begin{align*}
P &= \frac{1}{2} \sum_{i}^{4} (q_1 Q_1 + q_2 Q_2 + q_3 Q_3 + q_4 Q_4), \\
\Theta &= \frac{(q_1 Q_1 + q_2 Q_2)(q_3^2 + q_4^2) - (q_3 Q_3 + q_4 Q_4)(q_1^2 + q_2^2)}{2 \sqrt{(q_1^2 + q_2^2)(q_3^2 + q_4^2)}},
\end{align*}
\]

2
\[ \Phi = \frac{1}{2} (-q_2 Q_1 + q_1 Q_2 + q_4 Q_3 - q_3 Q_4), \] (3)

\[ \Psi = \frac{1}{2} (q_2 Q_1 - q_1 Q_2 + q_4 Q_3 - q_3 Q_4). \]

Note that the factor \( \sqrt{\rho} \) has a long history. Indeed, as Bartsch [6] put it: ‘For the one-dimensional Kepler motion, it was already found by Euler [21] that the introduction of a square-root coordinate \( u = \sqrt{x} \) and a fictitious time \( \tau \) defined by \( dt = x d\tau \) reduces the Kepler equation of motion to \( d^2u/d\tau^2 + 2E u = 0 \) i.e. the equation of motion of a one-dimensional harmonic oscillator’. This reemerges in Heggie and Hut [22] (pp. 145) who seem unaware of this work of Euler.

Then, excluding the invariant manifolds \( \mathcal{M}_1 = \{(q, Q)|q_1 = q_2 = 0\} \) and \( \mathcal{M}_2 = \{(q, Q)|q_3 = q_4 = 0\} \) where Levi-Civita transformation already shows the connection of the 2-D Kepler and isotropic oscillators, the Hamiltonian (1) in the new variables may be written as

\[ \mathcal{H}_\omega = \mathcal{H}(\rho, \theta, -, -, P, \Theta, \Phi, \Psi) = \rho \omega^2 + 2\rho P^2 + \frac{2}{\rho} \left( \Theta^2 + \Phi^2 + \Psi^2 - 2 \Phi \Psi \cos \theta \right) \]

(4)

i.e. variables \( \phi \) and \( \psi \) are cyclic, with \( \Phi \) and \( \Psi \) as the corresponding first integrals. Note that a dash is used instead of the variable to stress the fact that this coordinate is ignorable. In other words the differential system reduces to

\[
\frac{d\rho}{d\tau} = \frac{\partial \mathcal{H}}{\partial P}, \quad \frac{d\theta}{d\tau} = \frac{\partial \mathcal{H}}{\partial \Theta}, \quad \frac{dP}{d\tau} = -\frac{\partial \mathcal{H}}{\partial \rho}, \quad \frac{d\Theta}{d\tau} = -\frac{\partial \mathcal{H}}{\partial \theta}
\]

and two quadratures

\[ \phi = \int \left( \frac{\partial \mathcal{H}}{\partial \Phi} \right) d\tau \quad \text{and} \quad \psi = \int \left( \frac{\partial \mathcal{H}}{\partial \Psi} \right) d\tau. \]

(5)

In order to reach the main result of this letter, let us remember first two basic features of our system, presented in the following Propositions, based on the reordering of the terms defining Eq. (1) and the suitable chosen regularizing function.

**Proposition 1.-** The 4-D isotropic oscillator may be considered as a generalized Kepler system. It reduces to a 3-D Kepler system and a quadrature when we restrict to either \( \Phi = 0 \) or \( \Psi = 0 \) manifolds.

**Proof.-** Let us consider Poincaré technique with a time regularization \( \tau \to s \) given by

\[ d\tau = (4\rho)^{-1} ds. \]

(6)
Then, the flow is defined by the Hamiltonian $\tilde{H} = \frac{1}{4\rho}(H - h)$ where $h$ is a fix value of the Hamiltonian $H$ for chosen initial conditions, and the flow is defined now on the manifold $\tilde{H} = 0$. In a slight different form, it is more convenient to write

$$\tilde{K} = \frac{1}{2} \left( P^2 + \frac{\Theta^2}{\rho^2} + \frac{\Phi^2 + \Psi^2 - 2 \Phi \Psi \cos \theta}{\rho^2 \sin^2 \theta} \right) - \frac{h}{4\rho} \tag{7}$$

in the manifold $\tilde{K} = -\frac{\omega}{8}$.

Therefore, denoting $H_K$ part of function (7):

$$H_K = \frac{1}{2} \left( P^2 + \frac{\Theta^2}{\rho^2} + \frac{\Phi^2}{\rho^2 \sin^2 \theta} \right) - \frac{h}{4\rho} \tag{8}$$

then, the Hamiltonian function (7) may be also written as

$$\tilde{K} = H_K + \frac{\Psi^2 - 2 \Phi \Psi \cos \theta}{2\rho^2 \sin^2 \theta} \tag{9}.$$

Note that the function $H_K$ is the Hamiltonian of the Kepler system (see [23]) in spherical coordinates

$$x = \rho \sin \theta \cos \phi, \ y = \rho \sin \theta \sin \phi, \ z = \rho \cos \theta \tag{10}$$

(where $\theta$ is the colatitude) and their momenta $(P, \Theta, \Phi)$, if we choose

$$\gamma = h/4,$$

where $\gamma$ is the fundamental constant of the Kepler-Coulomb system. Observe that we might have taken the term $\Psi^2/(\rho^2 \sin^2 \theta)$ instead of $\Phi^2/(\rho^2 \sin^2 \theta)$ for the definition of $H_K$.

The differential system defined by (9) is given by

$$\frac{d\rho}{ds} = \frac{\partial \tilde{K}}{\partial P} = \frac{\partial H_K}{\partial P},$$

$$\frac{d\Theta}{ds} = \frac{\partial \tilde{K}}{\partial \Theta} = \frac{\partial H_K}{\partial \Theta},$$

$$\frac{dP}{ds} = - \frac{\partial \tilde{K}}{\partial \rho} = - \frac{\partial H_K}{\partial \rho} + \frac{\Psi(\Psi - 2 \Phi \cos \theta)}{\rho^3 \sin^2 \theta},$$

$$\frac{d\Theta}{ds} = - \frac{\partial \tilde{K}}{\partial \theta} = \frac{\partial H_K}{\partial \theta} - \frac{\Psi(\Psi \cos \theta - \Phi(1 + \cos^2 \theta))}{\rho^2 \sin^3 \theta}.$$
and the two quadratures \(^5\). Then, if we restrict to the manifold \(\Psi = 0\) and we identify the variable \(s\) with the physical time \(t\), the flow of the oscillator given by Eqs. (11) joint with the quadrature \(\phi = \int (\partial H_K / \partial \Phi) \, ds\), reduces to a Keplerian flow in the manifold \(H_K = -\frac{\omega}{8}\). The last equation is the quadrature of \(\psi\)

\[
\psi = \int \frac{\partial K}{\partial \Psi} \, ds = - \int \frac{\Phi \cos \theta}{\rho^2 \sin^2 \theta} \, ds,
\]

to be computed after the Kepler system is integrated. \(\textbf{q.e.d.}\)

This shows that the 4-D oscillator may be seen as a generalized Kepler system. In fact, in the literature this Hamiltonian relates to Hartmann and other ring-shaped potentials [24], but we will not enter that issue here.

Note that, considering the inverse of the Projective Euler transformation (2), we may see the transformation from spherical to Cartesian (10) as a projection \(R^4 \to R^3\). Explicitly, inverting (2) we have \(\rho = \sum q_i^2\) and

\[
\begin{align*}
\sin \theta &= \frac{2\Delta}{\rho}, \quad \cos \theta = \frac{q_3^2 + q_4^2 - q_1^2 - q_2^2}{\rho}, \\
\sin \phi &= \frac{q_1 q_3 + q_2 q_4}{\Delta}, \quad \cos \phi = \frac{q_1 q_4 - q_2 q_3}{\Delta}, \\
\sin \psi &= \frac{q_1 q_3 - q_2 q_4}{\Delta}, \quad \cos \psi = \frac{q_1 q_4 + q_2 q_3}{\Delta},
\end{align*}
\]

where \(\Delta = \sqrt{(q_1^2 + q_2^2)(q_3^2 + q_4^2)}\), we obtain immediately

\[
\begin{align*}
x &= 2(q_1 q_4 - q_2 q_3), \\
y &= 2(q_1 q_3 + q_2 q_4), \\
z &= q_3^2 + q_4^2 - q_1^2 - q_2^2,
\end{align*}
\]

in other words, the KS-transformation.

### 3 Switching to Projective Andoyer variables

The Andoyer variables (introduced by Serret [25], and also referred as Serret-Andoyer [26] or Andoyer-Deprit [27]) are a well known symplectomorphism in dynamical astronomy [28, 29, 30] and recently introduced in other fields such as attitude and control [31]. The reader ought to be aware that in the literature authors use different letters for them (see [28, 29, 32]). In what follows the \((\lambda, \mu, \nu, \Lambda, M, N)\) notation is adopted.
Assuming the vector \((\Phi, \Theta, \Psi)\) different from zero, i.e. excluding the invariant manifold of rectilinear solutions treated before, the canonical transformation from Andoyer \((\lambda, \mu, \nu, \Lambda, M, N)\) to Euler \((\phi, \theta, \psi, \Phi, \Theta, \Psi)\), is given by

\[
\begin{align*}
\cos \epsilon \cos \sigma - \sin \epsilon \sin \sigma \cos \mu - \cos \theta &= 0, \\
\cos \theta \cos \sigma + \sin \theta \sin \sigma \cos (\psi - \nu) - \cos \epsilon &= 0, \\
\cos \theta \cos \epsilon + \sin \theta \sin \epsilon \cos (\phi - \lambda) - \cos \sigma &= 0, \\
\Phi &= \Lambda, \\
\Psi &= N, \\
\Theta &= \sqrt{M^2 - \frac{\Lambda^2 + N^2 - 2N\Lambda \cos \theta}{\sin^2 \theta}}.
\end{align*}
\]  

where \(\cos \epsilon = \Lambda/M\) and \(\cos \sigma = N/M\). In fact, the transformation requires two charts.

When we add to them the variables \((\rho, P)\) we obtain what we call the ‘Projective Andoyer’ transformation:

\[
\left( \begin{array}{c} \rho \\ \phi \\ \theta \\ \psi \\ P \\ \Phi \\ \Theta \\ \Psi \end{array} \right) \rightarrow \left( \begin{array}{c} \rho \\ \lambda \\ \mu \\ \nu \\ P \\ \Lambda \\ M \\ N \end{array} \right)
\]

which, versus Projective Euler transformation, this one is not a canonical extension. Note that completing Projective Euler transformation, this one is not a canonical extension. Note that completing the expressions of the momenta as functions of \((q, Q)\), after some computations we obtain

\[
M = \frac{1}{2} \sqrt{||q||^2||Q||^2 - (q \cdot Q)^2}.
\]

which shows that when Projective Andoyer variables are not defined: \(M = 0\), the motion is rectilinear.

**Proposition 2.-** In Projective Andoyer variables, the system defined by the 4-D isotropic oscillator, properly regularized, is separable in two subsystems, one linear in the angle \(\mu\) and a quadrature for the variable \(\rho\).

**Proof.-** Expressing our Hamiltonian (11) in the ‘Projective Andoyer’ (PA) variables it results

\[
H_\omega = H(\rho, -, -, -, P, -, M, -) = \frac{\omega \rho}{2} + 2\rho P^2 + \frac{2M^2}{\rho}.
\]

We see that the three angle variables \((\lambda, \mu, \nu)\) are cyclic. In fact, not only the momenta \((\Lambda, N)\) but also the variables \((\lambda, \nu)\) are integrals.
Then, fixing a value of the Hamiltonian $h$ and making a change of independent variable $d\tau = (\rho/4)ds$ the Hamiltonian takes the form

$$
\mathcal{K} = \frac{\rho}{4}(\mathcal{H}_\omega - h) = \frac{\omega \rho^2}{8} + \frac{\rho^2P^2}{2} - \frac{h \rho}{4} + \frac{1}{2}M^2
$$

in the manifold $\mathcal{K} = 0$. From this Hamiltonian we obtain immediately the conclusions. Details are not needed for our purposes. \textbf{q.e.d.}

Now we present the main result of this letter:

\textbf{Theorem.-} In Projective Andoyer variables, the system defined by (1), properly regularized, includes the Keplerian system for any value of the integral $N$.

\textbf{Proof.-} After a change of independent variable according to Poincaré technique, $\mathcal{K} = g(\rho)(\mathcal{H}_\omega - h)$ considering now (3), i.e. $g(\rho) = 1/(4\rho)$, the Hamiltonian of the 4-D isotropic oscillator is given by

$$
\tilde{\mathcal{K}} = \frac{1}{2}\left(P^2 + \frac{M^2}{\rho^2}\right) - \frac{\gamma}{\rho}
$$

(15)

in the manifold $\tilde{\mathcal{K}} = -\omega/8$. Now, what remains is to connect this Hamiltonian with the Kepler system in 3-D. The use of the polar-nodal canonical transformation (see Deprit [10]) extended to four dimensions

$$(\rho, \lambda, \mu, \nu, P, \Lambda, M, N) \rightarrow (x, y, z, \nu, X, Y, Z, N)$$

(16)

is one way to do it. Indeed, denoting $\mathcal{R}(\mathbf{v}, \alpha)$ a rotation matrix of angle $\alpha$ around the vector $\mathbf{v}$, the transformation (16) is defined considering three rotations related to the three direct orthogonal reference frames $(e_1, e_2, e_3)$, $(\ell_1, \ell_2, \ell_3)$ and $(b_1, b_2, b_3)$, where $\ell_1 = \mathcal{R}(e_3, \lambda)e_1$, $b_3 = \mathcal{R}(\ell_1, I)e_3$ and $b_1 = \mathcal{R}(b_3, \mu)\ell_1$, by

$$(x, y, z)^T = \mathcal{R}(e_3, \lambda) \mathcal{R}(\ell_1, I) \mathcal{R}(b_3, \mu) (\rho, 0, 0)^T$$

$(X, Y, Z)^T = \mathcal{R}(e_3, \lambda) \mathcal{R}(\ell_1, I) \mathcal{R}(b_3, \mu) \left(P, \frac{M}{\rho}, 0\right)^T$,

with $\cos I = \Lambda/M$, and where $T$ stands for transpose of a vector. Explicitly, we have

$$
x = \rho(\cos \mu \cos \lambda - \sin \mu \sin \lambda \cos I),
$$

$$
y = \rho(\cos \mu \sin \lambda + \sin \mu \cos \lambda \cos I),
$$

$$
z = \rho \sin \mu \sin I,$$
and similarly for the momenta. It is easy to verify that (16) is a canonical transformation of Mathieu type: $Xdx + Ydy + Zdz = Pd\rho + \Lambda d\lambda + M d\mu$, but not a canonical extension. Perhaps for this reason is not so well known. Then, our Hamiltonian (15) expressed the variables $(x, y, z, \nu, X, Y, Z, N)$ takes the form

$$H = H(x, y, z, -X, Y, Z, -) = \frac{1}{2}||X||^2 - \gamma \frac{||x||}{2}$$

which is the Hamiltonian of the 3-D Keplerian system where, with some abuse of notation, we have written $x \equiv (x, y, z)$ and $X \equiv (X, Y, Z)$. Moreover, $(\nu, N)$ are integrals that take any value. q.e.d.

Let give details of the inverse process: Let consider a reference frame $(e_1, e_2, e_3)$. We define the vectors $u$ and $n$ such that $x = \rho u$, where $\rho = ||x||$ and $||u|| = 1$, and $M = x \times X = Mn$, with $M = ||M||$, $||n|| = 1$. Then, we take the momenta $P, \Lambda$ as $P = x \cdot X/||x||$ and $\Lambda = xY - yX$. Moreover, we may write $e_3 \times n = (\sin \lambda) \ell$, with $||\ell|| = 1$, and $\cos \lambda = e_1 \cdot \ell$, join with $\cos \mu = \ell \cdot u$, and $\sin \mu = (n \times \ell) \cdot u$, which ends the inversion of the polar nodal transformation. Then from Eqs. (12) we obtain the projective Euler canonical variables. Finally, from Eqs. (2) we obtain $q_i$. It rests to have the expression for $Q_i$ $(i = 1, 4)$. From the canonical extension Eqs. (3) we know they are linear in them. Thus, we obtain explicitly $Q_i$ by inverting the matrix associated with the transformation, but it is not necessary to be given here.

We see that, comparing with KS and in contrast with it, no constraint is needed for the integrals $(\nu, N)$. Then, what gives KS transformation such a special place in the oscillator Kepler connection? The fact that Projective Andoyer transformation is not defined when $M = 0$, which according to Eq. (14) corresponds to rectilinear trajectories: $q || Q$.

Note that, although the Projective Andoyer variables do not yet define a set of action angle variables, they are an intermediary step in that direction. Indeed, from (15) we readily obtain a Delaunay set of variables for 4-D oscillators, a symplectomorphism well suited for perturbation theories. We just need to make use of the Delaunay transformation $(\rho, \mu, P, M) \rightarrow (\delta, g, D, G)$ (see [23], [10]). It reduces Eq. (15), i.e. (1), to only one action $H = -\gamma^2/(2D^2)$ (for details [33]), which reflects the fact that our systems are maximally superintegrable [16].

This diagram, based on the group structure of canonical transformations, suggests to
carry out explicit compositions of some of them, just to obtain more insight, as well as possible relations with other known transformations, or new ones that might be defined likewise. Further relations among the previous transformations, the option $F(\rho) = \rho$ included, as well as the connection with other integrable and perturbed systems is in progress 33.

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References

[1] P. Kustaanheimo, Annales Universitatis Turkuensis Ser. AI. 73 (1964); P. Kustaanheimo, and E. Stiefel, J. Reine Angew. Math. 219, 204 (1965).

[2] B. Cordani, The Kepler Problem: Group Theoretical Aspects, Regularization and Quantization with Application to the Study of Perturbations, Birkhäuser, Basel (2003).

[3] E.L. Stiefel, and G. Scheifele, Linear and Regular Celestial Mechanics, Springer, Berlin (1971).

[4] M. Kummer, Commun. Math. Phys. 84, 133 (1982).

[5] M. Vivarelli, Celest. Mech. Dyn. Astr. 60, 291 (1994).

[6] T. Bartsch, J. Phys. A: Math. Gen. 36, 6963 (2003).

[7] J. Egea, S. Ferrer, and J. C. van der Meer, Reg. & Chaot. Dyn. 12, 664 (2007).

[8] G.W. Hill, Astron. J. 27, 171 (1913).

[9] A. Deprit, and A. Rom, Celest Mech. 2, 166 (1970).

[10] A. Deprit, Celest Mech. 24, 111 (1981).

[11] P. Yanguas, Nonlinearity 14, 1 (2001).

[12] J.M. Jauch, and E.L. Hill, Phys. Rev. 57, 641 (1940).

[13] J. Waldvogel, Celest. Mech. Dyn. Astr. 102, 149 (2008).

[14] J. Moser, Comm. Pure Appl. Math. 23, 609 (1970).

[15] T. Iwai, J. Math. Phys. 22, 1628 (1981).
[16] F. Fassò, *Acta Applicandae Math* **87**, 93 (2005).

[17] M. Ikeda, and Y. Miyachi, *Math Japon*, **15** 127 (1970).

[18] A.O. Barut, C.K. Schneider, and R. Wilson, *J. Math. Phys.* **20**, 2244 (1979).

[19] F.H. Cornish, *J. Phys A: Math Gen* **17**, 323 (1984).

[20] M. Kibler, and T. Négadi, *Int. J. Quantum Chemistry* **26**, 405 (1984).

[21] L. Euler, *Novi Comm. Acad. Sci. Petrop.* **11**, 144 (1765).

[22] D. Heggie, and P. Hut, *The Gravitational Million-Body Problem*, Cambridge Univ. Press, pp. 143–149 (2003).

[23] H. Goldstein, C. Poole, and J. Safko, *Classical Mechanics*, 3nd Ed., Addison Wesley, Reading (2002).

[24] M. Kibler, and P. Winternitz, *J. Phys. A: Math. Gen.* **20**, 4097 (1987); *Phys. Lett. A* **147**, 338(1990).

[25] J. A. Serret, *Mém. Acad Sci de Paris* **35**, 585 (1866).

[26] A. Deprit and A. Elipe, *J. Astronaut. Sci.* **41**, 603 (1993).

[27] G. Benettin, F. Fassò, and M. Guzzo, *Reg. & Chaot. Dyn.* **11**, 1 (2006).

[28] H. Andoyer, *Cours de Mécanique Céleste*, Vol 1, Gauthier-Villars, Paris, p. 54, (1923).

[29] A. Deprit, *American J of Physics* **35**, 424 (1967).

[30] H. Kinoshita, *Publ. Astron. Soc. Japan* **24**, 423 (1972).

[31] K.Y. Lum, and A.M. Bloch, *Dynamics and Control* **9**, 39 (1999).

[32] F. Boigey, *C.R. Acad. Sc. Paris* **272**, A, 1115 (1971).

[33] S. Ferrer, in preparation (2009).