On the notion of reconstruction of quantum theory

Alexei Grinbaum

LPHS – Archives Henri Poincaré (CNRS UMR 7117), 23 bd Albert Ier, 54015 Nancy, France
Email alexei.grinbaum@polytechnique.edu

Abstract
What belongs to quantum theory is no more than what is needed for its derivation. We argue for an approach focusing on reconstruction rather than interpretation of quantum mechanics. After discussing the concept of reconstruction, we analyze the problem of metatheoretic justification of the choice of axioms and then study several examples of reconstruction.

1 What is wrong with interpreting quantum mechanics?

Ever since the first days of quantum mechanics physicists as well as philosophers tried to interpret it, understanding this task as a problem of giving to the new, puzzling physical theory a clear meaning. Looking globally, this enterprize failed: still today we have no consensus on what the meaning of quantum theory is. Proposed answers are many but none of them has won overall recognition. Perhaps the most remarkable manifestation of the failure to interpret quantum mechanics is the attitude taught to most young physicists in lecture rooms and research laboratories in the last half century, “Write down equations and calculate! No need to ask questions!”

Why did attempts at univocal interpretation fail? Many answers are possible, and among them we favor two, both showing that there is an intrinsic deficiency in the idea of interpreting philosophically a physical theory.

The first answer is that to a physical theory one would naturally like to give a physical meaning in the Greek sense of φύσις, i.e. we – as part of the physicists’ audience – expect to be told a story about the immanent, fundamental nature.
This is because we casually tend to apply physical theory to the phenomenal world to learn something about the latter, and not the world to physical theory in order to invent a meaning of the theory. Physical theory is above all a tool for predicting the yet unobserved phenomena; so employing the existing knowledge and experience of the world to interpreting physics runs counter to its basic function as a scientific theory. However, notwithstanding such an against-the-grain direction in which a philosophical interpretation operates, the former does not necessarily lead to a formal contradiction that would invalidate the interpretation logically; more modestly but perhaps no less irritatingly, at the end one is often left with a feeling of being excluded from the mainstream research. Further, as the physics of today is inseparable from mathematics, a meaning cannot be physical and thus satisfactory if it is merely heaped over and above the mathematical formalism of quantum mechanics, instead of coming all the way along with the formalism as it rises in a derivation of the theory.

The second answer is that we live in a situation where objective truth has been appropriated by science, and to pass public ratification every increase in knowledge must confront experimental setups. In this world an interpretation can only then be considered satisfactory when it becomes an integral part of science. This is not unprecedented in the history of ideas: indeed, many philosophical questions with the advent of empirical science ceased to be perceived as philosophical and are now treated as scientific. Of the problem of interpreting quantum mechanics, as much as possible must be moved into the area of the scientific; only then will the puzzle disappear.

2 Reconstruction of physical theory

2.1 Schema

We call *reconstruction* a following schema: Theorems and major results of physical theory are formally derived from simpler mathematical assumptions; these assumptions or axioms, in turn, appear as a representation in the formal language, of a set of physical principles. Thus, reconstruction consists of three
parts: a set of physical principles, their mathematical representation, and a derivation of the formalism of the theory.

Contrary to an interpretation, reconstruction of physical theory acquires supplementary persuasive power which arises from the use of mathematical derivation. Established as a valid mathematical result, the theorems and equations of the theory become unquestionable and free of suspicion. 'Why is it so?'—'Because we derived it.' The question of meaning, previously asked with regard to the formalism, is removed and now bears only on the selection of the principles. No room for mystery remains in what concerns the meaning of the theory’s mathematical apparatus.

2.2 Selection of first principles

That who wishes to attempt a reconstruction of physical theory must formulate the foundational principles which he or she believes plausible and translate them into mathematical axioms. Then the rest of the theory will be constructed “mechanically,” by means of a formal derivation. The choice of axioms must be the only allowed freedom in the whole construction. It is commonplace to say that it is not easy to exhibit an axiomatic system that would stand to such requirements, especially in the case of quantum theory.

First, how does one judge which axioms are plausible? Prior to pronouncing a judgment, one must develop an intuition of what is plausible about quantum theory and what is not. This can be only achieved by practicing the theory, i.e. by taking its prescriptions at face value, applying them to systems under consideration in particular tasks, and obtaining results. In short, one needs to acquire a real “know-how” above and beyond the theoretical knowledge that quantum mechanics could solve such and such problems. Intuition then develops from experience; it cannot arise from abstract knowledge “in principle.” However, it is important to say that taking the prescriptions of quantum theory at face value, applying them and obtaining results will not yet make things clear about quantum mechanics. One can possess the knowledge about how to apply a certain tool without caring about the structure of the tool or its meaning. The
quantum mechanical know-how purely serves as such tool for developing one’s intuition about which idea is plausible and can become a foundational principle and which candidate idea will not pass the test.

Second, what shall one require from the first principles? As we stated above, they must be simple physical statements whose meaning is immediately, easily accessible to a scientist’s understanding. They must also be such as to permit a clear and unambiguous translation of themselves into mathematically formulated axioms. A derivation of quantum theory will then rely on these axioms.

2.3 Status of first principles

Reconstruction program includes a derivation of quantum theory, but in the previous section one was told to apply and use it in order to motivate the derivation. Is there a logical vicious circle here? We submit that there is none, and this thanks to the status of first principles. Namely, they should not necessarily be viewed as ultimate truths about nature. Independently of one’s ontological commitments, the first principles only have a minimal epistemic status of being postulated for the purpose of reconstructing the theory in question. Like in the 19th-century mathematics, in theoretical physics the axiomatic method is to be separated from the attitude which yet the Greeks had toward axioms: that they represent the truth about reality. Much of the progress of mathematics is due to understanding that an axiom can no longer be considered ultimate truth, but merely a basic structural element, i.e. assumption that lies in the foundation of a certain theoretical structure. In mathematics, after departing from the Greek concept of axiom, “not only geometry, but many other, even very abstract, theories have been axiomatized, and the axiomatic method has become a powerful tool for mathematical research, as well as a means of organizing the immense field of mathematical knowledge which thereby can be made more surveyable” [17]. A similar attitude is to be taken with respect to axioms used for a formal derivation of physical theory. A short prescription would sound something like this, “If the theory does not tell you that the states of the system are ontic states, do not take them to be ontic.”
To explain the above prescription, return first to the idea that, in developing an intuition with respect to the plausibility of the foundational principles used to derive a theory, one takes this theory for a given and applies it practically so as to acquire a know-how that would justify the choice of principles. Now, when one is working with several physical theories, ideas that have previously been used as foundational in theory I, may turn out to be derivative (i.e., theorems) in theory II. A good example comes from the case of thermodynamics and statistical physics, or macroscale hydrodynamics and low-level molecular theory of liquids. Such considerations show the limits of philosophical assumptions that one can make about the status of first principles used in reconstruction of a given physical theory. Indeed, generically nothing can be said about their ontological content or the ontic commitments that arise from the principles. It is more economical and would amount to a certain “epistemological modesty” to treat the foundational principles as axioms *hic et nunc*, i.e. in a given theoretical description. Having taken this position, ask then the reconstructed theory itself if it allows a realist point of view or imposes limits on it; and while in general the status of first principles as ultimate truths about reality is not a necessity, certain reconstruction programs are such that this status can be safely, or almost, attributed to the principles within a particular reconstruction in question.

Reconstruction of physical theory has its main advantage compared to philosophical interpretation of the theory in that it moves a number of questions, previously being thought of as philosophical, to the realm of science, and this in virtue of the mathematical derivation which the reconstruction program operates. However, philosophical problems do not altogether disappear; they still apply to the first principles and take the form of a problem of their justification. Evidently, it is a minimal logical condition that such a justification should not be seen as a mathematical deduction of the principles from the theory in whose very foundations they lie. The task of justification is external to the reconstruction program and must be executed by the one with a different set of presuppositions, i.e. by taking the theory for a given and motivating from there why the principles in question are simple, physical, and plausible. Therefore,
philosophy is not fully chased out of physics. On the contrary, by demarcating the frontier between what can be treated as a scientific question and what belongs to metatheory, one contributes to a better understanding of the structure of the theory and of those of its foundational postulates which require a metatheoretic interpretation and justification.

3 Examples of reconstruction

3.1 Early examples

A particularly well-known example of reconstruction is the special theory of relativity. Since Einstein’s 1905 work special relativity is a well-understood physical theory; but it is equally well-known that its formal content, i.e. Lorentz transformations, were written by Lorentz and Poincaré and not by Einstein, and this several years before 1905. Lorentz transformations were fiercely debated and many interpretations of what they mean were offered, and among them quite a plausible one about interactions between bodies and the ether. However, when Einstein came, things suddenly became clear and the debate stopped. This was because Einstein gave a few simple physical principles from which he derived Lorentz transformations, therefore closing the attempts to append philosophy and give a meaning \textit{a posteriori}, to an already working formalism. Einstein’s idea was to assume that there is no absolute, but only a relative, notion of simultaneity and that the velocity of light is constant. Once a derivation starting from these principles has been taken through, the physical meaning of Lorentz transformations stood clear and special relativity has not raised any controversy ever since.

Einstein’s reconstruction of special relativity is an example of theory where the first principles are understood as truths about reality. That the speed of light is constant and that there is no absolute notion of simultaneity is now routinely taken to be objectively true and well-established facts about nature. Thus, we have here a case in which, although in general it is not a necessity, the first principles do acquire a particular ontological status.
Moving away from special relativity, we submit that reconstruction is the exclusive way to make things clear about quantum mechanics. As such, this idea is not novel but has been in the air for some time, and a concise statement can for example be found in Rovelli [34].

Quantum mechanics will cease to look puzzling only when we will be able to derive the formalism of the theory from a set of simple physical assertions (“postulates,” “principles”) about the world. Therefore, we should not try to append a reasonable interpretation to the quantum mechanical formalism, but rather to derive the formalism from a set of experimentally motivated postulates.

What is interesting is that in the last decade reconstruction became a major trend in the foundations of quantum mechanics. But before describing this recent work let us first look further back in the history of quantum mechanics: there too axiomatic derivations occupy an eminent place. The first paper where quantum mechanics was treated axiomatically appeared shortly after the creation of quantum mechanics itself: in 1927 Hilbert, von Neumann and Nordheim stated their view of quantum mechanics as the one in which “(the theory’s) analytical apparatus, and the arithmetic quantities occurring in it, receives on the basis of the physical postulates a physical interpretation. Here, the aim is to formulate the physical requirements so completely that the analytical apparatus is just uniquely determined. Thus the route is of axiomatization” [18]. It is on this route of axiomatization that von Neumann in collaboration with Birkhoff was led to study the logic of quantum mechanics [2]. Following their work, many axiomatic systems were proposed, e.g. Zieler [30], Varadarajan [38, 39], Piron [28, 29], Kochen and Specker [22], Guenin [13], Gunson [14], Jauch [20], Pool [32, 33], Plymen [31], Marlow [27], Beltrametti and Casinelli [1], Holland [19], or Ludwig [24]. Another branch of axiomatic quantum theory, the algebraic approach was first conceived by Jordan, von Neumann and Wigner [21] and later developed by Segal [35, 36], Haag and Kastler [15], Plymen [30], Emch [9] and others; for a recent review, see [4].
However, a vast majority of these axiomatic derivations do not fall under our notion of reconstruction, as they were based on highly abstract mathematical assumptions and not, as we required, on simple physical principles. Consider for instance the exemplary work by Mackey [25, 26].

Mackey develops quantum mechanics as follows. Take a set $\mathcal{B}$ of all Borel subsets of the real line and suppose we are given two abstract sets $\mathcal{O}$ (a to-be space of observables) and $\mathcal{S}$ (a to-be space of states) and a (to-be probability) function $p$ which assigns a real number $0 \leq p(x, f, M) \leq 1$ to each triple $x, f, M$, where $x$ is in $\mathcal{O}$, $f$ is in $\mathcal{S}$, and $M$ is in $\mathcal{B}$. Assume certain properties of $p$ listed in axioms M1-M9:

**M1** Function $p$ is a probability measure. Mathematically, we have $p(x, f, \emptyset) = 0$, $p(x, f, \mathbb{R}) = 1$, and $p(x, f, M_1 \cup M_2 \cup M_3 \ldots) = \sum_{n=1}^{\infty} p(x, f, M_n)$ whenever the $M_n$ are Borel sets that are disjoint in pairs.

**M2** Two states, in order to be different, must assign different probability distributions to at least one observable; and two observables, in order to be different, must have different probability distributions in at least one state. Mathematically, if $p(x, f, M) = p(x', f, M)$ for all $f$ in $\mathcal{S}$ and all $M$ in $\mathcal{B}$ then $x = x'$; and if $p(x, f, M) = p(x, f', M)$ for all $x$ in $\mathcal{O}$ and all $M$ in $\mathcal{B}$ then $f = f'$.

**M3** Let $x$ be any member of $\mathcal{O}$ and let $u$ be any real bounded Borel function on the real line. Then there exists $y$ in $\mathcal{O}$ such that $p(y, f, M) = p(x, f, u^{-1}(M))$ for all $f$ in $\mathcal{S}$ and all $M$ in $\mathcal{B}$.

**M4** If $f_1, f_2, \ldots$ are members of $\mathcal{S}$ and $\lambda_1 + \lambda_2 + \ldots = 1$ where $0 \leq \lambda_n \leq 1$, then there exists $f$ in $\mathcal{S}$ such that $p(x, f, M) = \sum_{n=1}^{\infty} \lambda_n p(x, f_n, M)$ for all $x$ in $\mathcal{O}$ and all $M$ in $\mathcal{B}$.

**M5** Call question an observable $e$ in $\mathcal{O}$ such that $p(e, f, \{0, 1\}) = 1$ for all $f$ in $\mathcal{S}$. Questions $e$ and $e'$ are disjoint if $e \leq 1 - e'$. Then a question $\sum_{n=1}^{\infty} e_n$ exists for any sequence $(e_n)$ of questions such that $e_m$ and $e_n$ are disjoint whenever $n \neq m$. 

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M6 If \( E \) is any compact, question-valued measure then there exists an observable \( x \) in \( \mathcal{O} \) such that \( \chi_M(E) = E(M) \) for all \( M \) in \( \mathcal{B} \), where \( \chi_M \) is a characteristic function of \( M \).

M7 The partially ordered set of all questions in quantum mechanics is isomorphic to the partially ordered set of all closed subspaces of a separable, infinite-dimensional Hilbert space.

M8 If \( e \) is any question different from \( 0 \) then there exists a state \( f \) in \( \mathcal{S} \) such that \( m_f(e) = 1 \).

M9 For each sequence \( (f_n) \) of members of \( \mathcal{S} \) and each sequence \( (\lambda_n) \) of non-negative real numbers whose sum is \( 1 \), one-parameter time evolution group \( V_t : \mathcal{S} \rightarrow \mathcal{S} \) acts as follows: \( V_t(\sum_{n=1}^{\infty} \lambda_n f_n) = \sum_{n=1}^{\infty} \lambda_n V_t(f_n) \) for all \( t \geq 0 \); and for all \( x \) in \( \mathcal{O} \), \( f \) in \( \mathcal{S} \), and \( M \) in \( \mathcal{B} \), \( t \rightarrow p(x, V_t(f), M) \) is continuous.

In Mackey’s nine axioms all essential features of the quantum formalism are directly postulated in their mathematical form: the Hilbert space structure in M5-M8, state space and the probabilistic interpretation in M1-M4, and time evolution in M9. It is not at all clear where these mathematical definitions come from and how one justifies them on physical rather than formal grounds. In fact, Mackey’s concern in the early 1950s was with a precise mathematical axiomatization of quantum mechanics rather than with the question of what quantum mechanics tells us about the world and then reconstructing its formalism from a set of such fundamental ideas. Thus, stage 1 of reconstruction, at which one formulates physical principles, is absent from Mackey’s work, and instead one starts directly at stage 2 where axioms appears as formal, mathematical definitions.

Later, Mackey’s axioms M5-M8 were reformulated in the language of quantum logic, thereby rephrasing the assumptions that underlie the Hilbert space structure. This was the case, most prominently, in [24, 28, 29] and also in a seminal book [1]. Quantum logical assumptions are simple enough to be accessible for direct comprehension, in contrast to Mackey’s mathematically formulated
axioms, but they tend to be linguistic rather than physical. This means that one typically argues that it makes no sense to speak about certain terms other than if some suitable “trivial” properties had been postulated, e.g. the notion of proposition is only meaningful if like in Ref. [7] negation or partial order, or like in Ref. [1] implication, are defined. Although we fully acknowledge that linguistic a priori arguments can be interesting and powerful, we however separate them from the reconstruction program as introduced above: in the latter, first principles from which the theory is derived should have a physical meaning, i.e. tell us something directly and intuitively apprehensible about the world and quantum theory as describing our knowledge of it. Such principles, ideally, should be independent of a particular formalism in which we derive quantum theory and therefore should not rely on the language of quantum logic as just one among many such formalisms.

3.2 Contemporary examples

Among the modern developments, an interesting example of reconstruction comes from the instrumentalist derivation of quantum theory from “five reasonable axioms” by Hardy [16]. Hardy’s “reasonable axioms” set up a link between two quantities, $K$ and $N$, which play a fundamental role in the reconstruction. $K$ is the number of degrees of freedom of the system and is defined as the minimum number of probability measurements needed to determine the state. Dimension $N$ is defined as the maximum number of states that can be reliably distinguished from one another in a single measurement. The axioms then are:

**H1 Probabilities.** Relative frequencies (measured by taking the proportion of times a particular outcome is observed) tend to the same value for any case where a given measurement is performed on an ensemble of $n$ systems prepared by some given preparation in the limit as $n$ becomes infinite.

**H2 Simplicity.** $K$ is determined by a function of $N$ where $N = 1, 2, \ldots$ and where, for each given $N$, $K$ takes the minimum value consistent with the
axioms.

**H3 Subspaces.** A system whose state is constrained to belong to an $M$ dimensional subspace behaves like a system of dimension $M$.

**H4 Composite systems.** A composite system consisting of subsystems $A$ and $B$ satisfies $N = N_A N_B$ and $K = K_A K_B$.

**H5 Continuity.** There exists a continuous reversible transformation on a system between any two pure states of that system.

Although four of the H1-H5 axioms use mathematical language in the formulation, their meaning in Hardy’s instrumentalist setting can be grasped much easier than the meaning of Mackey’s axioms M1-M9. In fact, this meaning is already suggested by the names given to the axioms by Hardy. Therefore H1-H4 can be rephrased into physical principles from which one derives the formalism of the theory and thus provide an example of reconstruction. None of these principles is trivial: for H1, assume that probability introduced instrumentally as relative frequency of measurements is a well-defined concept and obeys the laws of probability theory; for H2, assume that the number of parameters needed to characterize a state is directly linked to the number of states that can be distinguished in one measurement; for H3, that the linear structure of state space shrinks accordingly to the maximum number of states of the system distinguishable in one measurement; for H4, assume multiplicability of the quantity defined as dimension and of the quantity defined as the number of degrees of freedom. Now formulate these assumptions mathematically and use Hardy’s theorems to derive from them the full-blown formalism of quantum mechanics. A particular instrumental philosophy does not play a crucial role in the derivation: Hardy himself acknowledges that his axioms can be adopted by a realist as well as a hidden variable theorist or a partisan of collapse interpretations. Thus, the choice of underlying philosophy is not critical for derivation, and Hardy’s reconstruction advances our understanding of quantum theory irrespectively of the justification which one may have for the axioms. What matters are the sim-
ple physical principles formulated as axioms H1-H4. We shall see an opposite example in the next section, in which the justification used for fundamental principles will limit the area in which operates the mathematical derivation.

Still, it is not so clear whether axiom H5 has a physical meaning. Because it is this axiom that makes the theory quantum rather than classical, the reconstruction program cannot be said to be fully implemented and taken to its logical conclusion. To further illustrate this point, we distinguish two types of continuity assumptions that are made in axiomatic derivations of quantum theory. Continuity assumptions of type 1 select the correct type of numeric field which is used in the construction of the Hilbert space of the theory; namely, of the field $\mathbb{C}$ of complex numbers. Solèr’s theorem 37 or Zieler’s axioms 40 are examples of type 1 continuity assumptions. Hardy’s case is different and is an example of the continuity assumptions of type 2, which are made in order to bring in the superposition principle. Other such assumptions include Gleason’s non-contextuality 10, Brukner’s and Zeilinger’s homogeneity of parameter space 3, Landsman’s two-sphere property 23, and Holland’s axioms C and D 19 which bear a particular resemblance to Hardy’s H5:

(C) Superposition principle for pure states:

1. Given two different pure states (atoms) $a$ and $b$, there is at least one other pure state $c$, $c \neq a$ and $c \neq b$ that is a superposition of $a$ and $b$.

2. If the pure state $c$ is a superposition of the distinct pure states $a$ and $b$, then $a$ is a superposition of $b$ and $c$.

(D) Ample unitary group: Given any two orthogonal pure states $a, b \in \mathcal{L}$, there is a unitary operator $U$ such that $U(a) = b$.

We see that various axiomatic systems of quantum theory contain, under one form or another, the assumption of continuity and it is this assumption which is largely responsible for making things quantum. Whatever the framework of the reconstruction is, bringing in topological considerations is essential. As it is exceedingly difficult to formulate a physical principle which may give a meaning
to the continuity assumption of type 2, all reconstruction programs suffer here from the intrusion of an element of mathematical abstraction.

The above critique concerning the continuity axiom applies to another examples of reconstruction initially proposed by Rovelli [34] and that we developed elsewhere [11, 12]. Here, the reconstruction starts from two information-theoretic axioms:

**R1** There exists a maximum amount of relevant information that can be extracted from a system.

**R2** It is always possible to obtain new information about the system.

From these axioms and with the help of supplementary mathematical assumptions one derives the formalism of quantum mechanics. While the supplementary assumptions cast a shadow on the conceptual clarity of the reconstruction much in the same fashion as does H5, the whole program presents itself differently from Hardy’s instrumentalism. The mathematical derivation being still devoid of ontological commitments, justification of the first principles which we propose cannot refer to an ontology, except for an arguably problematic case in which one would be prepared to take information for a fundamental building block of reality. Rather, by reconstructing quantum theory from information-theoretic principles we point at its epistemological character and at its role as a theory of (a certain kind of) knowledge; i.e. with certain limits being imposed on the kind of information one may be dealing with, the most general theory of this information takes the form of quantum theory. Here again reconstruction appears more appealing than a mere interpretation as it leaves room for any justification of the first principles different from ours. Indeed, one may wish to adopt an ontological picture to justify R1-R2 or take no position at all with respect to ontology. At the same time, regardless of a specific philosophical justification of first principles, the meaning of quantum theory stands clear: quantum theory is a general theory of information constrained by certain information-theoretic principles.
3.3 CBH reconstruction

Clifton, Bub and Halvorson (CBH) offer an example of a set of quantum informational postulates from which one derives the structure of quantum theory [6]. CBH postulate three fundamental principles:

CBH1 No superluminal information transfer via measurement.

CBH2 No broadcasting.

CBH3 No bit commitment.

To give a mathematical formulation of these principles, CBH use the $C^*$-algebraic formalism. Consider a composite system, $A + B$, consisting of two component subsystems, $A$ and $B$, understood as $C^*$-algebras.

Definition 3.1. Operation $T$ on algebra $A \lor B$ conveys no information to Bob if

\begin{equation}
(T^*\rho)|_B = \rho|_B \text{ for all states } \rho \text{ of } B.
\end{equation}

An operation here is understood as a completely positive linear map on an algebra, and $T^*\rho$ is a state over the algebra defined for every state $\rho$ on the same algebra as

\begin{equation}
(T^*\rho)(A) = \frac{\rho(T(A))}{\rho(T(I))}
\end{equation}

at the condition that $\rho(T(I)) \neq 0$. Nonselective measurements $T$ are the ones that have $T(I) = I$, and then $\rho(T(I)) = \rho(I) = ||\rho|| = 1$. CBH explain that, in their view, Definition 3.1 entails

\begin{equation}
T(B) = B \text{ for all } B \in B.
\end{equation}

CBH then assert that if the condition (3) holds for all self-adjoint $B \in B$ and for all $T$ of the form

\begin{equation}
T = T_E(A) = E^{1/2}AE^{1/2} + (I - E)^{1/2}A(I - E)^{1/2},
\end{equation}

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where \( A \in \mathcal{A} \lor \mathcal{B} \) with \( \mathcal{A} \) and \( \mathcal{B} \) being \( C^* \)-independent, and \( E \) is a positive operator in \( \mathcal{A} \), then algebras \( \mathcal{A} \) and \( \mathcal{B} \) are kinematically independent, i.e. all \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \) commute [6, Theorem 1]. Thus kinematic independence is derived from the assumption of \( C^* \)-independence and from the condition 11, where \( C^* \)-independence is brought into the discussion to grasp the meaning of the fact that systems \( \mathcal{A} \) and \( \mathcal{B} \) are distinct. Mathematically, \( C^* \)-independence means that for any state \( \rho_1 \) over \( \mathcal{A} \) and for any state \( \rho_2 \) over \( \mathcal{B} \) there is a state \( \rho \) over \( \mathcal{A} \lor \mathcal{B} \) such that \( \rho|_\mathcal{A} = \rho_1 \) and \( \rho|_\mathcal{B} = \rho_2 \). As for Definition 3.1, the authors take it to be a mathematical representation of Axiom CBH1.

According to the authors, the meaning of CBH1 is that when Alice and Bob perform local measurements, Alice’s measurements can have no influence on the statistics for the outcomes of Bob’s measurements, and vice versa. CBH also say that “otherwise this would mean instantaneous information transfer between Alice and Bob” and “the mere performance of a local measurement (in the nonselective sense) cannot, in and of itself, transfer information to a physically distinct system.” Upon reading these statements, one has a feeling that for CBH distinct and distant are synonyms. This identification of terms might indeed be a tacit assumption among quantum information theorists who do not have to worry about relativistic effects, but in the full-blown \( C^* \)-algebraic framework, as well as in the general philosophical context, meaning of the two words is certainly different. We have here an example showing how the initial quantum informational language of the fundamental principles CBH1-CBH3 constrains the use of the algebraic formalism to situations where fundamental principles make sense from the point of view of quantum information, while in fact the formalism could also be used in other, more complex situations. Unlike Hardy’s derivation which was independent of the particular instrumental justification of its fundamental principles, the CBH reconstruction cannot be taken through outside the field of quantum information, because its mathematics, while being still valid outside this field, will require additional justification. Apart from the identification of terms ‘distant’ and ‘distinct,’ such is also the case with time evolution, which is tacitly taken by CBH to be the usual quantum mechanical
time evolution, while in the general $C^*$-algebraic framework this is not at all the case and a variety of different “temporal” evolutions are possible. One then avoids this problem at the price of confining oneself to the quantum informational paradigm.

Equating Definition 3.1 with Axiom CBH1 requires particular attention to the mathematical details, and a point has to be made about CBH’s proof. If, following the authors, in this definition $\rho$ is to be taken as a state over $\mathcal{B}$, then the definition does not make sense: operation $T$ is defined on $\mathcal{A} \vee \mathcal{B}$ and consequently, in accordance with $\mathcal{I}$, $T^* \rho$ is defined for the states $\rho$ over $\mathcal{A} \vee \mathcal{B}$. If one follows the CBH definition with a state $\rho$ over $\mathcal{B}$, then there would be no need to write $\rho|_{\mathcal{B}}$ as CBH do, for a simple reason that $\rho|_{\mathcal{B}} = \rho$. To suggest a remedy, we extend the reasoning behind this definition and reformulate it in three alternative ways.

- The first one is to require that in Definition 3.1 the state $\rho$ be a state over the algebra $\mathcal{A} \vee \mathcal{B}$.
- The second alternative is to consider states $\rho$ on $\mathcal{B}$ but to require a different formula, namely that $(T|_{\mathcal{B}})^* \rho = \rho$ as states over $\mathcal{B}$.
- Finally, the third alternative proceeds as follows: Take arbitrary states $\rho_1$ over $\mathcal{A}$ and $\rho_2$ over $\mathcal{B}$ and, in virtue of $C^*$-independence, consider the state $\rho$ over $\mathcal{A} \vee \mathcal{B}$ such that its marginal states are $\rho_1$ and $\rho_2$ respectively. Then $T^* \rho$ is also a state over $\mathcal{A} \vee \mathcal{B}$. If its restriction $(T^* \rho)|_{\mathcal{B}}$ is equal to $\rho_2$, then $T$ is said to convey no information to Bob.

With the original formulation of Definition 3.1 proof of Equation 3 is problematic. We show how to prove this equation with each of the three alternative definitions. First observe the following remark.

Remark 3.2. Each $C^*$-algebra has sufficient states to discriminate between any two observables (i.e., if $\rho(A) = \rho(B)$ for all states $\rho$, then $A = B$). To justify $\mathcal{I}$, the CBH authors then say:
Let us examine this derivation under each of the three alternative definitions of conveying no information. By the definition of \( T^* \), we have \((T^*\rho)(B) = \rho(T(B))\) for all states \( \rho \) over \( \mathcal{A} \lor \mathcal{B} \). To obtain from this that \( \rho(T(B)) = \rho(B) \), one must show that \((T^*\rho)(B) = \rho(B)\), and this is equivalent to saying that \((T^*\rho)|_B = \rho|_B\) for all states \( \rho \) over \( \mathcal{A} \lor \mathcal{B} \). Now, according to CBH, one would need to show that \( \rho(T(B)) = \rho(B) \) if and only if \( \omega(T(B)) = \omega(B) \) with states \( \rho \) over \( \mathcal{A} \lor \mathcal{B} \) and \( \omega \) over \( \mathcal{B} \). The latter formula, however, is not well-defined: operator \( T(B) \), generally speaking, is not in \( \mathcal{B} \). Fortunately, we are salvaged by the first alternative reformulation of Definition 3.1: because \( \rho(T(B)) = \rho(B) \) is true for all states \( \rho \) over \( \mathcal{A} \lor \mathcal{B} \), we obtain directly that \( T(B) = B \) in virtue of Remark 3.2.

The second alternative definition of conveying no information makes use of an object such as \((T|_B)^*\rho\). To give it a meaning in the algebra \( \mathcal{B} \), one needs to impose a closure condition on the action of \( T \) on operators \( B \in \mathcal{B} \): namely, that \( T \) must not take operators out of \( \mathcal{B} \). The problem here is the same as the one we encountered in the discussion of the previous alternative, and it is only by assuming the closure condition that one is able to obtain that \( T(B) = B \).

In the third alternative, for the state \( \rho \) over \( \mathcal{A} \lor \mathcal{B} \), write from the definition of \( T^* \) that \((T^*\rho)(B) = \rho(T(B))\). The result \((T^*\rho)(B)\) is the same as \((T^*\rho)|_B(B)\), and this is equal to \( \rho_2(B) \). Consequently, \( \rho(T(B)) = \rho_2(B) = \rho(B) \). Can we now say that this holds for all states \( \rho \) over \( \mathcal{A} \lor \mathcal{B} \)? The answer is obviously yes, and this is because each state over \( \mathcal{A} \lor \mathcal{B} \) can be seen as an extension of its own restriction to \( \mathcal{B} \). Therefore, one has to modify Definition 3.2 for it to be formally correct, and this entails a modification in the proof of Equation 3.

We now turn to the remaining two CBH axioms. Axiom CBH2 is used to establish that algebras \( \mathcal{A} \) and \( \mathcal{B} \), taken separately, are non-Abelian. Broadcasting,
which enters in the formulation of the axiom, is defined as follows:

**Definition 3.3.** Given two isomorphic, kinematically independent $C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$, a pair $\{\rho_1, \rho_2\}$ of states over $\mathcal{A}$ can be broadcast in case there is a standard state $\sigma$ over $\mathcal{B}$ and a dynamical evolution represented by an operation $T$ on $\mathcal{A} \lor \mathcal{B}$ such that $T^* (\rho_i \otimes \sigma)|_A = T^* (\rho_i \otimes \sigma)|_B = \rho_i$, for $i = 0, 1$.

Equivalence between the ‘no broadcasting’ condition and non-Abelianness of the $C^*$-algebra is then derived from the following theorem:

**Theorem 3.4.** Let $\mathcal{A}$ and $\mathcal{B}$ be two kinematically independent $C^*$-algebras. Then:

(i) If $\mathcal{A}$ and $\mathcal{B}$ are Abelian then there is an operation $T$ on $\mathcal{A} \lor \mathcal{B}$ that broadcasts all states over $\mathcal{A}$.

(ii) If for each pair $\{\rho_1, \rho_2\}$ of states over $\mathcal{A}$, there is an operation $T$ on $\mathcal{A} \lor \mathcal{B}$ that broadcasts $\{\rho_1, \rho_2\}$, then $\mathcal{A}$ is Abelian.

It is interesting to note that non-Abelianness of the algebras $\mathcal{A}$ and $\mathcal{B}$, taken one by one, is proved by assuming that they are kinematically independent. This means that quantumness, of which non-Abelianness is a necessary ingredient, is not a property of any given system taken separately, as if it were the only physical system in the Universe; on the contrary, to be able to derive the quantum character of the theory, one must consider the system in the context of at least one other system that is physically distinct from the first one. As a consequence, for example, this forbids treating the whole Universe as a quantum system if one reconstructs quantum theory along the CBH lines.

Axiom CBH3 entails nonlocality: spacelike separated systems must at least sometimes occupy entangled states. In particular, CBH show that if Alice and Bob have spacelike separated quantum systems, but cannot prepare any entangled state, then Alice and Bob can devise an unconditionally secure bit commitment protocol. The derivation starts by showing that quantum systems are characterized by the existence of non-uniquely decomposable mixed states: a
$C^*$-algebra $\mathcal{A}$ is non-Abelian if and only if there are distinct pure states $\omega_1, \omega_2$ and $\omega_+ \omega_-$ over $\mathcal{A}$ such that $\frac{1}{2}(\omega_1 + \omega_2) = \frac{1}{2}(\omega_+ + \omega_-)$. This result is used to prove a theorem showing that a certain proposed bit commitment protocol is secure if Alice and Bob have access only to classically correlated states (i.e. convex combinations of product states).

**Theorem 3.5 (the CBH ‘no bit commitment’ theorem).** If $\mathcal{A}$ and $\mathcal{B}$ are non-Abelian then there is a pair $\{\rho_0, \rho_1\}$ of states over $\mathcal{A} \lor \mathcal{B}$ such that:

1. $\rho_0|\mathcal{B} = \rho_0|\mathcal{B}$.

2. There is no classically correlated state $\sigma$ over $\mathcal{A} \lor \mathcal{B}$ and operations $T_0$ and $T_1$ performable by Alice such that $T_0^*\sigma = \rho_0$ and $T_1^*\sigma = \rho_1$.

From this theorem the authors deduce that the impossibility of unconditionally secure bit commitment entails that “if each of the pair of separated physical systems $\mathcal{A}$ and $\mathcal{B}$ has a non-uniquely decomposable mixed state, so that $\mathcal{A} \lor \mathcal{B}$ has a pair $\{\rho_0, \rho_1\}$ of distinct classically correlated states whose marginals relative to $\mathcal{A}$ and $\mathcal{B}$ are identical, then $\mathcal{A}$ and $\mathcal{B}$ must be able to occupy an entangled state that can be transformed to $\rho_0$ or $\rho_1$ at will by a local operation.” The term ‘separated’ is essential and, nevertheless, its precise meaning is not defined. It can be, indeed, compared to the use of terms ‘distinct’ and ‘distant’ in the analysis of Axiom CBH1. When the authors claim that Alice and Bob represent “spacelike separated systems,” while formally Alice and Bob are just two $C^*$-algebras, one sees how the way in which CBH apply the algebraic formalism is severely constrained by the context of quantum information theory. Here appears again a situation in which language and context used to formulate and justify the fundamental principles set up a limit on the applicability of the mathematical formalism in which these principles are then represented. Even if the formalism can be understood more generally than within the discipline chosen in order to comprehend the language, one still cannot make his way out of this disciplinary prison or else the sense of the axioms will be lost. If one
however persists and crosses the border and then, say, obtains a new mathematical result, this result will be void of physical meaning until a new, broader justification of the fundamental principles is given. Philosophical and linguistic justification, and mathematical derivation play here a game of mutual onslaught and retreat which, ultimately, leads to the advance of science.

The CBH result would be a perfect example of reconstruction were it not for a great deal of mathematical structure which is tacitly assumed in the choice of $C^*$-algebra as a mathematical representative of the notion of system. Assumptions of the algebraic formalism include the relations between operators abiding by the linear law, numeric coefficients in algebras being complex numbers, the states giving rise to the Hilbert space representation via the GNS construction, etc. Once one lists all these supplementary assumptions, the CBH reconstruction appears once again to suffer from a similar defect of incorporating a serious mathematical abstraction as derivations from axioms H1-H5 or R1-R2.

4 Conclusion

Reconstruction brings in clarity to where interpretation was struggling to make sense of a physical theory. What belongs to physical theory is no more than what is needed for its derivation. All other questions belong to metatheory and are related to the metatheoretic justification task for the choice of first principles.

The notion of reconstruction presented here resembles Einstein’s notion of ‘principle theory’. Principle theories, according to Einstein, “employ the analytic, not the synthetic, method. The elements which form their basis and starting point are not hypothetically constructed but empirically discovered ones, general characteristics of natural processes, principles that give rise to mathematically formulated criteria which the separate processes or the theoretical representations of them have to satisfy” [8]. One recognizes here what we have called justification of first principles and a statement that the mathematical derivation must follow after the principles had been established. Einstein’s
distinction between constructive and principle theories, though, bears a heavy flavor of his ontological view of relativity theory. For quantum theory, as Bub first realized [5], Einstein’s notion of principle theory is still applicable although, as we argued above, with a modified status of the first principles.

Reconstruction of quantum theory remains an only partially solved problem. Notwithstanding, it is already competing with traditional interpretations due to its appealing conceptual transparency and to the clarity that it brings into the structure of the theory. It would be too ambitious to expect that all of modern quantum theory, including field theory and quantum gravity, could be derived from a few axioms; mathematical abstractions and further assumptions are still a necessity. However, if we want to understand the meaning of even most advanced parts of quantum theory, it is inevitable that simple physical principles be formulated and put in the very foundation of quantum theory.

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