$U(N|M)$ quantum mechanics on Kähler manifolds

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*Abstract:* We study the extended supersymmetric quantum mechanics, with supercharges transforming in the fundamental representation of $U(N|M)$, as realized in certain one-dimensional nonlinear sigma models with Kähler manifolds as target space. We discuss the symmetry algebra characterizing these models and, using operatorial methods, compute the heat kernel in the limit of short propagation time. These models are relevant for studying the quantum properties of a certain class of higher spin field equations in first quantization.

*Keywords:* Sigma models, Extended supersymmetry.
1. Introduction

$O(N)$ spinning particles \[1, 2, 3\] have been useful to describe higher spin fields in first quantization \[4, 5\]. Similarly, $U(N)$ spinning particles \[6, 7\] have been instrumental to discover a new class of higher spin field equations which possess a novel type of gauge invariance \[8\]. To investigate the quantum properties of these equations in their worldline formulation, it is important to study the related quantum mechanics. It is the purpose of this paper to discuss these quantum mechanics, which in the most general case take the form of nonlinear sigma models.

First we shall discuss linear sigma models, i.e. models with flat complex space $\mathbb{C}^d$ as target space. These sigma models exhibit a $U(N)$ extended supersymmetry on the worldline. They define “spinning particle” models once the extended supersymmetry is made local. It is useful, and almost effortless, to extend these models by adding extra bosonic coordinates. This extension produces $U(N|M)$ sigma models, by which we mean sigma models with a worldline extended supersymmetry characterized by supercharges transforming in the fundamental representation of $U(N|M)$ (i.e. $U(N|M)$ is the $R$-symmetry group of the supersymmetry algebra). This extension may be useful for constructing wider classes of spinning particles, as happened in the case of the $OSp(N|2M)$ extension \[9\] of the standard $O(N)$ supersymmetric quantum mechanics, used for example in \[10, 11, 12, 13\].
to describe higher spin fields. We present these quantum mechanical models and their symmetry algebra in section 2.

In section 3 we consider sigma models with generic Kähler manifolds as target spaces. The symmetry algebra gets modified by the geometry, so that it will not be always possible to gauge the extended supersymmetry to obtain spinning particles and corresponding higher spin equations. This signals the difficulties of coupling higher spin fields to generic backgrounds, not to mention the even more difficult problem of constructing nonlinear field equations. However, on special backgrounds one can find a deformed $U(N|M)$ susy algebra that becomes first class, so that it can be gauged to produce consistent spinning particles. An example is the case of Kähler manifolds with constant holomorphic sectional curvature. No restrictions apply to the special cases of $U(1|0)$ and $U(2|0)$, whose susy algebra can be gauged to produce nontrivial field equations on any Kähler space, in analogy with standard $N = 1$ and $N = 2$ susy quantum mechanics on arbitrary riemannian manifolds (i.e. $O(1)$ and $O(2)$ quantum mechanics in the language used above).

Nevertheless, before gauging, the $U(N|M)$ quantum mechanics here constructed are perfectly consistent on any Kähler manifold, and even posses conserved supercharges when the Riemann tensor obeys a locally symmetric space condition (again in close analogy with the riemannian case). Thus, in section 4 we work with an arbitrary Kähler manifold and compute the quantum mechanical transition amplitude in euclidean time (i.e. the heat kernel) in the limit of short propagation time and using operatorial methods. This last result is going to be particularly useful for obtaining an unambiguous construction of the corresponding path integral, which is needed when considering worldline applications. This is indeed one of our future aims, namely using worldline descriptions of higher spin fields to obtain useful and computable representations of their one-loop effective actions, as done in [14] for the $O(2)$ spinning particle. In that case a worldline representation allowed to compute in a single stroke the first few heat kernel coefficients and prove various duality relations for massless and massive $p$-forms in arbitrary dimensions. Finally, we present our conclusions and outlook in section 5, and confine to the appendices details of our calculations.

2. Linear $U(N|M)$ sigma model

We introduce here the $U(N|M)$ extended supersymmetric quantum mechanics. In the most simple case it describes the motion of a particle in $\mathbb{C}^d$, the flat complex space of $d$ complex dimensions with coordinates $(x^\mu, \bar{x}^\bar{\mu})$, $\mu = 1, ..., d$. The flat metric in these complex coordinates is simply $\delta_{\mu\bar{\nu}}$, and we use it to raise and lower indices. In addition, the particle carries extra degrees of freedom described by worldline Dirac fermions $\psi_{\dot{a}}^\mu$, ...
algebra that, together with the $U$ charges $Q$, there are 2 $N$ by the symmetry algebra. The possibility of inserting the central charge is related to the generalizes usual concepts. With these degrees of freedom at hand the phase space lagrangian defining our model has the standard form $\mathcal{L} \sim p\dot{q} - H$, namely

$$\mathcal{L} = p_\mu \dot{x}^\mu + \bar{p}_\mu \dot{\bar{x}}^\mu + i \dot{\bar{\psi}}_\mu \psi_\alpha + i \dot{\bar{z}}_\mu z_\alpha - p_\mu \bar{p}^\mu .$$  \hspace{1cm} (2.1)

This model enjoys a $U(N|M)$ extended supersymmetry, which we are going to describe directly in the quantum case.

The fundamental (anti-)commutators are easily read off from (2.1)

$$[x^\mu , p_\nu] = i\hbar \delta_\nu^\mu , \quad [\bar{x}^\mu , \bar{p}_\nu] = i\hbar \delta_\nu^\bar{\mu} , \quad \{\psi_\alpha , \bar{\psi}_\beta\} = h \delta_\alpha^\beta \delta^\mu_\nu , \quad \{\bar{z}_\alpha , z_\beta\} = h \delta_\alpha^\beta \delta^\mu_\nu .$$  \hspace{1cm} (2.2)

The $U(N|M)$ charges are readily constructed from the worldline operators

$$J^b_\alpha = \frac{1}{2} [\bar{\psi}_\mu , \psi^{\mu\alpha}] - c \delta^\alpha_\beta = \bar{\psi}_\mu \psi^{\mu\alpha} - m \hbar \delta^\alpha_\beta \quad \text{($U(N)$ subgroup),}$$
$$J^b_\beta = \frac{1}{2} (\bar{z}_\mu^{\alpha\beta} + z^{\mu\beta}) + c \delta^\alpha_\beta = z_\mu^{\alpha\beta} \bar{z}_\mu^{\mu\beta} + m \hbar \delta^\alpha_\beta \quad \text{($U(M)$ subgroup),}$$
$$J^a_\alpha = \bar{z}_\mu^{\alpha\beta} \psi^{\mu\beta} , \quad J^a_\beta = \bar{\psi}_\mu \bar{z}_\mu^{\mu\beta} \quad \text{($U(N|M)$ fermionic generators,}$$  \hspace{1cm} (2.3)

where $m = c + \frac{d}{2}$. They obey the $U(N|M)$ algebra

$$[J^a_\alpha , J^b_\beta] = \hbar (\delta^a_\beta J^c_\alpha - \delta^c_\alpha J^a_\beta)$$
$$[J^c_\alpha , J^b_\beta] = \hbar (\delta^c_\beta J^b_\alpha - \delta^b_\alpha J^c_\beta)$$
$$[J^a_\alpha , J^c_\beta] = -\hbar \delta^\alpha_\beta J^a_\gamma , \quad [J^a_\alpha , J^c_\gamma] = \hbar \delta^\alpha_\beta J^a_\gamma$$
$$[J^c_\alpha , J^b_\beta] = \hbar \delta^\gamma_\beta J^a_\alpha , \quad [J^c_\alpha , J^b_\gamma] = -\hbar \delta^\gamma_\beta J^a_\beta$$
$$\{J^a_\alpha , J^b_\beta\} = \hbar (\delta^c_\alpha J^b_\beta + \delta^b_\alpha J^c_\beta ) .$$  \hspace{1cm} (2.4)

In the definition of these charges we have used a “graded symmetric” ordering prescription modified by an arbitrary central charge $c$ that specifies possible different orderings allowed by the symmetry algebra. The possibility of inserting the central charge is related to the algebraic fact that $U(N|M) = U(1) \times SU(N|M)$. All these charges commute with the hamiltonian $H = p_\mu \bar{p}^\mu$ and are conserved.

Other conserved quantities are the supersymmetric charges involving the space momenta: there are $2N$ fermionic supercharges $Q_\alpha = \psi^{\mu\alpha}_a p_\mu$, $\bar{Q}^a = \bar{\psi}_\mu \bar{p}^\mu$, and $2M$ bosonic charges $Q_\alpha = z^{\mu\alpha}_a p_\mu$, $\bar{Q}^a = \bar{z}_\mu \bar{p}^\mu$. All these operators form the $U(N|M)$ extended superalgebra that, together with the $U(N|M)$ internal algebra (2.4), is given by the following...
relations

\[ [J^a_b, Q_c] = -i \hbar \delta^a_b Q_c, \quad [J^b_b, \bar{Q}^c] = i \hbar \delta^b_b \bar{Q}^a \]
\[ [J^\alpha_b, Q^\beta] = -i \hbar \delta^\alpha_b Q^\beta, \quad [J^b_b, \bar{Q}^\gamma] = i \hbar \delta^b_b \bar{Q}^\alpha \]  \hspace{2cm} (2.5)

(Anti)-commutators needed to close the algebra and not explicitly reported vanish.

All these relations can be written in a more covariant way. In order to show up the full supergroup structure, let us introduce the superindex

\[ A = (a, \alpha) \] and the \( U(N|M) \) metrics

\[ \delta^A_B = \begin{pmatrix} \delta^a_b & 0 \\ 0 & \delta^\alpha_\beta \end{pmatrix}, \quad \epsilon^A_B = \begin{pmatrix} -\delta^a_b & 0 \\ 0 & \delta^\alpha_\beta \end{pmatrix}. \]  \hspace{2cm} (2.6)

The internal fermions and bosons are grouped into the fundamental and anti-fundamental representations of the supergroup, \( Z^\mu_A = (\psi^\mu_a, z^\mu_\alpha) \), \( \bar{Z}^A_\mu = (\bar{\psi}^\mu_a, \bar{z}^\mu_\alpha) \). The fundamental (anti)-commutation relations can be written as \( [Z^\mu_A, \bar{Z}^B_\nu] = i \hbar \delta^B_A \delta^\mu_\nu \), or equivalently as \( [\bar{Z}^\nu_B, Z^\mu_A] = -i \hbar \epsilon^B_A \delta^\mu_\nu \). Here the graded commutator is used: \( [A, B] \) is defined as anti-commutator for \( A \) and \( B \) both fermionic, and as a commutator otherwise. Then we collect all the \( U(N|M) \) generators in

\[ J^A_B = \begin{pmatrix} J^a_b & J^\alpha_b \\ J^b_a & J^\beta_b \end{pmatrix} = \bar{Z}^A_\mu Z^\mu_B + m i \hbar \epsilon^A_B. \]  \hspace{2cm} (2.7)

With these notations at hand the entire superalgebra (2.4) is packaged into the single relation

\[ [J^A_B, J^C_D] = i \hbar (\delta^C_D J^A_B \pm \delta^A_D J^C_B) \]  \hspace{2cm} (2.8)

where the plus sign refers to the case with \( J^A_B \) and \( J^C_D \) both fermionic, and the minus sign to the other possibilities.

By means of this supergroup notation, the supercharges are written as \( Q_A = (Q_a, Q_\alpha) \) and \( \bar{Q}^A = (\bar{Q}^a, \bar{Q}^\alpha) \), and the above superalgebra is summarized by

\[ [J^A_B, Q_C] = \pm i \hbar \delta^A_C Q_B, \quad [J^A_B, \bar{Q}^C] = i \hbar \delta^A_B \bar{Q}^C \]
\[ [Q_A, \bar{Q}^B] = i \hbar \delta^B_A H, \]  \hspace{2cm} (2.9)

where \( \pm \) stands for plus for \( J^A_B \) and \( Q_C \) both fermionic, and minus otherwise.

All these quantum mechanical operators have simple geometrical meanings in terms of differential operators living on \( \mathbb{C}^d \). Let us give a brief description. Generic wave functions of
the Hilbert space can be represented by functions of the coordinates \((x, \bar{x}, \psi, z)\). Expanding them in \(\psi^\mu\) and \(z^\mu\) shows how they contain all possible tensors with \(N + M\) blocks of holomorphic indices. Each of the first \(N\) blocks of indices is totally antisymmetric, while each of the last \(M\) blocks of indices is totally symmetric. In formulae

\[
\phi(x, \bar{x}, \psi, z) \sim \sum_{A_1=0}^{d} \sum_{B_1=0}^{\infty} \phi[\mu^{1}_1...\mu^{A_1}_1, \nu^{1}_1...\nu^{N}_1, \bar{\nu}^{1}_1...\bar{\nu}^{B_1}_1,...,\nu^{M}_1...\nu^{M}_M] \left( \psi^{\mu^{1}_1} \psi_1^{\mu^{A_1}_1} \cdots \psi^{\mu^{N}_1} \psi_N^{\mu^{A_1}_1} \left( \mu^{1}_1 \cdots \nu^{1}_1 \bar{\nu}_1^{B_1}_1 \cdots \bar{\nu}^{M}_M \right) \right). \tag{2.10}
\]

The quantum mechanical operators take the form of differential operators acting on these tensors. The hamiltonian is proportional the standard laplacian \(H \sim \partial_\mu \bar{\partial}^{\mu} = \delta^{\mu}_{\bar{\nu}} \partial_\mu \partial_{\bar{\nu}}\). The supercharge \(Q_\alpha\) acts as the Dolbeault operator \(\partial^{\dagger}\) restricted to the antisymmetric indices of block “\(a\)”, and \(\bar{Q}_\alpha\) as its adjoint \(\partial\). Similarly the “bosonic” supercharge \(Q_\alpha\) is realized as a symmetrized gradient acting on the symmetric indices of block “\(a\)”, and \(\bar{Q}_\alpha\) is its adjoint, taking the form of a divergence. The action of the \(U(N|M)\) operators, i.e. the \(J^A_B\) charges, is also amusing: they perform certain (anti)-symmetrizations on the tensors indices, and we leave it to the interested reader to work them out explicitly. The algebra of these differential/algebraic operators, as encoded in the susy algebra, is only valid in flat space. In the next section we will see how this algebra extends to generic Kähler manifolds.

### 3. Nonlinear \(U(N|M)\) sigma model

We now extend the previous construction to nonlinear sigma models with generic Kähler manifolds as target spaces. On Kähler manifolds, in holomorphic coordinates, the only non vanishing components of the metric are \(g^{\mu\bar{\nu}} = g_{\mu\bar{\nu}}\), and similarly \(\Gamma^\mu_{\nu\lambda}\) and \(\Gamma^\mu_{\bar{\nu}\lambda}\) are the only non vanishing components of the connection. We use the following conventions for curvatures

\[
R^\mu_{\nu\sigma\lambda} = \partial_\nu \Gamma^\mu_{\sigma\lambda}, \quad R^\mu_\nu = -g^{\bar{\sigma}\lambda} R^\mu_{\nu\sigma\lambda}, \quad R = R^\mu_\mu, \tag{3.1}
\]

and denote by \(g = \det(g^{\mu\bar{\nu}})\) the determinant of the metric, as standard in Kähler geometry.

The classical phase space lagrangian with a minimally covariantized hamiltonian becomes

\[
\mathcal{L} = p_\mu \dot{x}^{\mu} + \bar{p}_\mu \dot{\bar{x}}^{\mu} + \bar{Z}^{\lambda A}_\mu \dot{Z}^{\mu A}_\lambda - g^{\mu\bar{\nu}}(p_\mu - i\Gamma^{\lambda}_{\nu\sigma} \bar{Z}^{\lambda A}_\sigma Z^{\mu A}_A)\bar{p}_{\bar{\nu}} \tag{3.2}
\]

though, for future applications, it will be useful to consider more general hamiltonians. The corresponding configuration space lagrangian is the typical one for nonlinear sigma models

\[
\mathcal{L} = g_{\mu\bar{\nu}} \dot{x}^{\mu} \dot{\bar{x}}^{\nu} + i\dot{Z}^{\lambda A}_\mu \frac{DZ^{\mu A}}{dt} \tag{3.3}
\]
where the covariant time derivative is given by \( \frac{DZ_A^\mu}{dt} = \dot{Z}_A^\mu + \dot{x}^\nu \Gamma^\mu_{\nu\sigma} Z_A^\sigma \).

In the quantum case, it will be crucial to resolve ordering ambiguities by demanding target space covariance. Before discussing the quantum operators, let us make a few comments. We treat the \( \bar{Z}_A^\mu \) fields as momenta, as such they have a natural lower holomorphic curved index. In this situation there is no real advantage in introducing a vielbein, so we will avoid introducing one. Also, the holonomy group of a Kähler manifold of complex dimensions \( d \) is \( U(d) \), and it will be convenient to define the \( U(d) \) generators

\[
M'^\mu = \frac{1}{2} \{ \bar{\psi}_a^\mu, \psi_a^\mu \} + \frac{1}{2} \{ z^\alpha_{\mu}, z^\alpha_{\mu} \} - k \delta'^\mu
\]

where \( k \) is a central charge parametrizing different orderings allowed by the \( U(d) = U(1) \times SU(d) \) symmetry. These generators can be written as well as

\[
M'^\mu = \bar{Z}_A^\mu Z_A^\mu - s \hbar \delta'^\mu
\]

with \( s = k + \frac{N-M}{2} \). They satisfy the correct \( U(d) \) algebra

\[
[M'^\mu, M'^\rho] = \hbar \delta'^\mu M'^\rho - \hbar \delta'^\rho M'^\mu .
\]

We are now ready to discuss the covariantization of the quantum operators belonging to the \( U(N|M) \) extended supersymmetry algebra. As we shall see, not all of the charges generate symmetries on generic Kähler manifolds: some of them do not commute with the hamiltonian and thus are not conserved.

It is easiest to start with the generators of \( U(N|M) \). They are left unchanged as the metric does not enter their definition: \( J^A_B = \bar{Z}_A^\mu Z_B^\mu + m \epsilon^A_B \). They satisfy the same \( U(N|M) \) symmetry algebra given in eq. (2.8).

Now we consider the \( Q \) supercharges. To covariantize them we introduce covariant momenta

\[
\bar{\pi}_\mu = g^{1/2} \bar{p}_\mu g^{-1/2} , \quad \pi_\mu = g^{1/2} \left( p_\mu - i \Gamma^\lambda_{\mu\sigma} M_\lambda^\sigma \right) g^{-1/2} ,
\]

and write down covariantized supercharges as

\[
Q_A = Z_A^\mu \pi_\mu , \quad \bar{Q}^A = \bar{Z}_A^\mu g^{\mu\nu} \bar{\pi}_\nu .
\]

Similarly, the covariant hamiltonian operator is given by

\[
H_0 = g^{\mu\nu} \bar{\pi}_\mu \pi_\nu = g^{1/2} g^{\mu\nu} \bar{p}_\mu \left( p_\nu - i \Gamma^\lambda_{\nu\sigma} M_\lambda^\sigma \right) g^{-1/2} .
\]

At this stage it is worthwhile to spend some words on the hermiticity properties of our operators: since the \( \bar{Z}_A^\mu \) fields are defined as independent variables with lower holomorphic indices, but hermitian conjugation of vector indices naturally sends holomorphic into anti-holomorphic indices, and vice versa, the natural definition of the adjoint of \( Z_A^\mu \) is \((Z_A^\mu)^\dagger = \)
$Z^A_\nu g^\nu \bar{\mu}$. In this way, hermitian conjugation of the momentum is nontrivial: if $[p_\mu, Z^A_\nu] = 0$, it must hold that $[(p_\mu)^\dagger, (Z^A_\alpha)^\dagger] = [(p_\mu)^\dagger, Z^A_\lambda g^{\lambda \bar{\mu}}] = 0$ as well. Requiring this property we find
\begin{equation}
(p_\mu)^\dagger = \bar{p}_\mu - i \Gamma^\lambda_{\mu \bar{\nu}} M^\lambda_\sigma g^{\sigma \bar{\sigma}} g_{\lambda \bar{\lambda}}.
\end{equation}
Now, if we define the supercharges in the natural way written above, namely $Q_A = Z^A_\mu \pi_\mu$ and $\bar{Q}^A = Z^A_\mu \bar{\pi}^{\mu}$, then it results that $(Q_A)^\dagger = \bar{Q}^A$ and $H_0^\dagger = H_0$. Note that the power of the metric determinant entering the various operators is necessary for verifying the hermiticity properties.

Let us now consider their algebra. The first line of (2.9) simply states that $Q_A$ and $\bar{Q}^A$ belong to the fundamental and anti-fundamental representation of $U(N|M)$, and one can check that these relations remain unchanged even in curved space,
\begin{equation}
[J^A_B, Q_C] = \pm \hbar \delta^A_C Q_B, \quad [J^A_B, \bar{Q}^C] = \hbar \delta^C_B \bar{Q}^A.
\end{equation}
On the other hand the last relation becomes
\begin{equation}
[Q_A, \bar{Q}^B] = \hbar \delta^B_A H_0 + \hbar Z^A_\mu \bar{Z}^B_\mu \pi_\lambda \nu + \hbar^2 Z^A_\mu Z^B_\mu \pi_\lambda \nu
\end{equation}
\begin{equation}
[\bar{Q}^A, H_0] \equiv -[Q_A, H_0]^\dagger.
\end{equation}
$H_0$ is a central operator only in flat space. Finally, it is simple to verify that
\begin{equation}
[Q_A, Q_B] = [\bar{Q}^A, \bar{Q}^B] = 0.
\end{equation}
Relations (3.11), (3.12), (3.13) and (3.14), together with (2.8), describe the deformation of the $U(N|M)$ supersymmetry algebra realized by our quantum nonlinear sigma model on a Kähler manifold. Supersymmetry is broken as the supercharges are not conserved. Only on flat spaces the hamiltonian $H_0$ becomes central and the supercharges get conserved.

Given this state of affairs, one may try to redefine the hamiltonian in an attempt to make it central on more general backgrounds, thus recovering conserved supercharges. For this purpose, we add to $H_0$ several non minimal couplings
\begin{equation}
H = H_0 + c_1 R^\nu_\lambda M^\sigma_\mu M^\sigma_\lambda + c_2 R^\nu_\nu M_\mu + c_3 \hbar^2 R.
\end{equation}
With these generic couplings (3.13) becomes
\begin{equation}
[Q_A, H] = \hbar (1 + 2c_1) Z^A_\mu R_\mu^\lambda \pi_\nu + \hbar^2 (1 + c_1 + c_2) Z^A_\mu R_\mu^\nu \pi_\nu
\end{equation}
\begin{equation}
- i\hbar c_1 Z^A_\nu \nabla_\rho R^\rho_\lambda M^\nu_\mu M^\sigma_\lambda - i\hbar^2 c_2 Z^A_\nu \nabla_\sigma R^\nu_\mu M_\mu - i\hbar^3 c_3 Z^A_\nu \nabla_\mu R.
\end{equation}
We see that for the choice \( c_1 = -\frac{1}{2}, \ c_2 = -\frac{1}{2} \) and generic \( c_3 \), the terms in the first line proportional to the covariant momentum \( \pi_\nu \) vanish and, choosing \( c_3 = 0 \) for simplicity, we identify a canonical hamiltonian \( H_{(c)} \) so that eq. \( (3.16) \) reduces to

\[
[Q_A, H_{(c)}] = \frac{i\hbar}{2} \frac{Z_A^\sigma}{Z_A^\rho} \nabla_\rho R^\nu_\mu \lambda M_\mu^\sigma M_\nu^\rho + \frac{i\hbar^2}{2} Z_A^\sigma \nabla_\sigma R_\mu^\mu M_\mu^\nu, \quad (3.17)
\]

showing that \( H_{(c)} \) is central on locally symmetric spaces. Of course, also the graded commutator \( (3.12) \) changes and becomes

\[
[Q_A, \bar{Q}^B] = \hbar \delta_A^B H_{(c)} + \hbar R^\nu_\mu \lambda \left( Z_A^\nu Z_\nu^B + \frac{1}{2} \delta_A^B M_\nu^\mu \right) M_\mu^\nu + \frac{1}{2} \hbar^2 \delta_A^B R_\mu^\nu M_\mu^\nu. \quad (3.18)
\]

Thus one concludes that with the redefinition of the hamiltonian given above the supercharges are conserved on locally symmetric Kähler manifolds.

One of the most interesting applications of the nonlinear sigma models discussed so far is to use them to construct spinning particles and related higher spin equations. This is achieved by gauging the extended susy algebra identified by the charges \( (H, Q_A, \bar{Q}^A, J^B_A) \), possibly with a suitable redefinition of the hamiltonian. Unfortunately, we see that on generic Kähler manifolds the \( U(N|M) \) extended susy algebra is not first class, as additional independent operators appear on the right hand sides, as evident for example in eqs. \( (3.17) \) and \( (3.18) \). However, there are special cases, namely the \( U(1|0) \) and \( U(2|0) \) quantum mechanics, which generate first class superalgebras with a central hamiltonian on any Kähler background. In fact, for the \( U(1|0) \equiv U(1) \) model the algebra reduces to

\[
\{Q, \bar{Q}\} = h H, \quad [Q, H] = 0 \quad (3.19)
\]

where the hamiltonian is now defined by

\[
H = H_0 - \frac{\hbar}{2} R_\mu^\nu M_\mu^\nu + \frac{\hbar^2}{4} R = H_0^{sym} + \frac{\hbar^2}{4} R, \quad (3.20)
\]

with \( H_0^{sym} = \frac{1}{2} g^{\mu\nu} (\pi_\mu \bar{\pi}_\nu + \bar{\pi}_\nu \pi_\mu) \). For the \( U(2|0) \equiv U(2) \) model the choice of the hamiltonian is the canonical one, \textit{i.e.} the one in \( (3.15) \) with \( c_1 = c_2 = -\frac{1}{2} \) and \( c_3 = 0 \), and the superalgebra closes as

\[
\{Q_a, Q^b\} = \delta_a^b H, \quad [Q_a, H] = 0. \quad (3.21)
\]

For the general \( U(N|M) \) extended susy algebras one cannot achieve such generality. Nevertheless, one may look for special backgrounds that make \( (3.17) \) and \( (3.18) \) first class. A nontrivial class of Kähler manifolds where the first class property can be achieved is that of manifolds with constant holomorphic sectional curvature. On these manifolds, the Riemann and Ricci tensors take the form

\[
R_{\mu\rho\sigma\lambda} = -\frac{R}{d(d+1)} (g_{\mu\rho} g_{\sigma\lambda} + g_{\sigma\rho} g_{\mu\lambda}), \quad R_{\mu\rho} = \frac{R}{d} g_{\mu\rho}, \quad (3.22)
\]
where \( R \) is the constant scalar curvature. Substituting these relations into the algebra, one notices that the metric tensor gets contracted with the \( Z \) and \( \bar{Z} \) operators, producing additional charges \( J^B_A \) on the right hand side, so that with a suitable redefinition of the hamiltonian one obtains a first class algebra for generic \( m, s, c_1 \) and \( c_2 \), while \( c_3 \) gets fixed to a unique value. There is no loss of generality in choosing \( c_1 \) and \( c_2 \) equal to their canonical values, \( c_1 = c_2 = -\frac{1}{2} \), when using the algebra as a first class constraint algebra. In this case

\[
c_3 = -\frac{m}{2d(d+1)} \left( (N - M)^2 + (N - M)(4d - 3m - 2s + 1) + 2(m - d) \right) + \frac{s}{2} \left( 1 + \frac{2(d - m)}{d} - \frac{s}{d+1} \right)
\]

(3.23)

and the algebra can be casted in the following form

\[
[Q_A, \bar{Q}^B] = \hbar \delta^B_A H - \frac{\hbar R}{d(d+1)} \left\{ (-)^{(A+B)C} J^C_A \bar{J}^B_C + (-)^{AB} J^B_A \bar{J}^A_C + (-)^{AB} \hbar k_1 J^B_A \right. \\
+ \delta^B_A \left( \frac{1}{2} J^C_A \bar{J}^D_C + \frac{1}{2} J^2 + \hbar k_2 J \right) \right\},
\]

(3.24)

\[
[Q_A, H] = 0
\]

where

\[
k_1 = d - s(d+1) + m(N - M - 2)
\]

\[
k_2 = d - s(d+1) - \left( m + \frac{1}{2} \right)(N - M) + \frac{1}{2}.
\]

(3.25)

We denoted \( J \equiv J^A_A \) and used the notation \((-)^A \) with \( A = 0 \) for a bosonic index and \( A = 1 \) for a fermionic one. Gauging this first class algebra produces “\( U(N|M) \) spinning particles” on Kähler manifolds with constant holomorphic curvature, in a way analogous to the coupling of standard “\( O(N) \) spinning particles” to (A)dS spaces constructed in [3].

One may recall that Kähler spaces with constant holomorphic sectional curvature are a subclass of spaces with vanishing Bochner tensor. The latter is a sort of complex analogue of the riemannian Weyl tensor, introduced in [15] and defined by

\[
B_{\mu\nu\sigma\lambda} = R_{\mu\nu\sigma\lambda} + \frac{1}{d+2} \left( g_{\mu\rho} R_{\sigma\lambda} + g_{\sigma\lambda} R_{\mu\rho} + g_{\rho\sigma} R_{\mu\lambda} + g_{\mu\lambda} R_{\sigma\rho} \right)
\]

\[
- \frac{R}{(d+1)(d+2)} (g_{\mu\rho} g_{\sigma\lambda} + g_{\sigma\lambda} g_{\mu\rho}).
\]

(3.26)

It satisfies the nice property of being traceless, \( g^{\mu\nu} B_{\mu\nu\sigma\lambda} = 0 \). It seems likely that on spaces with vanishing Bochner tensor one may obtain a first class algebra, indeed it is relatively easy to verify it at the classical level, but we do not wish to pursue the detailed quantum analysis here.
4. Transition amplitude

Up to now we have discussed nonlinear sigma models with $U(N|\mathcal{M})$ extended supersymmetry, broken at times by the target space geometry, and used them to analyze algebraic properties of differential operators defined on Kähler manifolds. The aim of this section is the explicit computation of the transition amplitude in euclidean time, that is $\langle x \bar{\eta} | e^{-\frac{\beta}{\hbar} H} | y \xi \rangle$, in the limit of short propagation time and using operatorial methods. Such a calculation was presented for standard nonlinear sigma models with one, two or no supersymmetries in [16], see also [17], with the main purpose of identifying a benchmark to which compare path integral evaluations of the same heat kernel. As we wish to be able to master path integrals for $U(N|\mathcal{M})$ sigma models, and eventually use them to address quantum properties of higher spin equations on Kähler manifolds, we compute here the heat kernel using the operatorial formulation of quantum mechanics. To achieve sufficient generality and allow diverse applications, we compute the heat kernel for the general hamiltonian (3.15) containing three arbitrary couplings ($c_1, c_2, c_3$) to the background curvature plus a fourth one, the charge $s$, hidden in the $U(1)$ part of the connection, see eq. (3.5).

Before starting the actual computation, we shall review our set up. We work on a 2$d$ real dimensional Kähler manifold as target space. Holomorphic and anti-holomorphic vector indices will be often grouped into a riemannian index $i = (\mu, \bar{\mu})$ for sake of brevity. The metric in holomorphic coordinates factorizes as follows

$$g_{ij} = \begin{pmatrix} 0 & g_{\mu\bar{\nu}} \\ g_{\bar{\mu}\nu} & 0 \end{pmatrix}. \tag{4.1}$$

For determinants we use the conventions $g = \det(g_{\mu\bar{\nu}})$ and $G = |\det(g_{ij})| = |g|^2$. The dynamical variables of the $U(N|\mathcal{M})$ supersymmetric quantum mechanics consist of the following operators: target space coordinates $(x^\mu, \bar{x}^{\bar{\mu}}) = x^i$, conjugate momenta $p_i$, and graded vectors $Z^\mu_A$ and $\bar{Z}^{\bar{\mu}}_A$. Their fundamental (anti)-commutation relations are given in (2.2). For computational advantages we recast the full quantum hamiltonian (3.15) in a way that directly shows the dependence on the $Z$ operators

$$H = H_0 + \Delta H \quad \text{with}$$

$$H_0 = g^{\mu\bar{\nu}} g^{1/2} \bar{p}_\mu \left( p_\nu - i \Gamma^\lambda_{\nu\sigma} M^\sigma_{\bar{\lambda}} \right) g^{-1/2}$$

$$\Delta H = a_1 R_{\mu \nu}^{\nu \sigma} \bar{Z}_\sigma \cdot Z^\mu - a_2 \hbar R^{\mu}_{\nu} \bar{Z}_\mu \cdot Z^\nu + a_3 \hbar^2 R, \tag{4.2}$$

where the $a$ couplings are related to the $c$ couplings by

$$a_1 = c_1, \quad a_2 = c_2 + 2sc_1, \quad a_3 = c_3 - sc_2 - s^2c_1. \tag{4.3}$$

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Finally, it is useful to recall that the final answer for the heat kernel will contain the exponent of the classical action, suitably Wick-rotated to euclidean time $\tau$ ($t \rightarrow -i\tau$), which in phase space takes the form

$$S = \int_{-\beta}^{0} d\tau \left[ -i p_\mu \dot{x}^\mu - i \bar{p}_\mu \dot{\bar{x}}^\mu + \bar{Z}_\mu^A \dot{\bar{Z}}_A^\mu + H_{cl} \right]$$

(4.4)

where $H_{cl}$ is the classical Hamiltonian, a function, modified by suitable quantum corrections depending on $\hbar$.

Now we are ready for the explicit computation of the transition amplitude, through order $\beta$ (up to the leading free particle propagator), between position eigenstates and coherent states for the internal degrees of freedom, i.e.

$$\langle \bar{x} \eta | e^{-\frac{\beta}{\hbar} H} | y \xi \rangle$$

(4.5)

where $Z_\mu^A | \xi \rangle = \xi_\mu^A | \xi \rangle$ and $\langle \bar{\eta} | \bar{Z}_\mu^A = \langle \bar{\eta} | \bar{\eta}_\mu^A$. Of course, $| x \rangle$ and $| y \rangle$ denote eigenvectors of the position operator $x^i$ as usual, $| y \xi \rangle \equiv | y \rangle \otimes | \xi \rangle$, and so on. For convenience in the normalization of the coherent states, from now on we rescale the $Z$ fields by a factor of $\sqrt{\hbar}$, so that $[Z^\mu_A, \bar{Z}^B_\mu] = \delta^C_\mu \delta^B_A$. We are going to insert in (4.5) a complete set of momentum eigenstates, and as an intermediate stage we need to compute

$$\langle \bar{x} \eta | e^{-\frac{\beta}{\hbar} H} | p \xi \rangle$$

(4.6)

pushing all $p$'s and $Z$'s to the right, all $x$'s and $\bar{Z}$'s to the left, taking into account all (anti)-commutators and then substituting these operators with the corresponding eigenvalues. Let us focus on the evaluation of (4.6); clearly we have

$$\langle \bar{x} \eta | e^{-\frac{\beta}{\hbar} H} | p \xi \rangle = \sum_{k=0}^{\infty} \frac{(-\beta)^k}{k!} \langle \bar{x} \eta | H^k | p \xi \rangle$$

(4.7)

It is well known that, in the case of a nonlinear sigma model, it is not sufficient to expand the exponent to first order, i.e. $e^{-\beta H/\hbar} \sim 1 - \frac{\beta}{\hbar} H$, to obtain the correct transition amplitude to order $\beta$, see [16, 17]. Contributions for all $k$ must be retained in the sum (4.7), but taking into account at most two $[x,p]$ commutators. Let us see this in more detail. In a factor of $H^k$, pushing all $p$'s to the right by repeated use of the $[x,p]$ commutator, one obtains, remembering that each $H$ can give at most two $p$ eigenvalues,

$$\langle \bar{x} \eta | H^k | p \xi \rangle = \sum_{l=0}^{2k} B_l^k (x, \bar{\eta}, \xi) p^l \langle \bar{x} \eta | p \xi \rangle$$

(4.8)

where $p^l$ stands for a homogeneous polynomial in $p$ of degree $l$. For the position eigenstates we use the normalization: $\langle x | x' \rangle = g^{-1/2}(x) \delta^{ad}(x - x')$, while the standard normalization
normalizations, we expand the transition amplitude as follows:

\[
1 = \int d^2 p \, |p\rangle \langle p|, \quad 1 = \int d^2 x \, g \, |x\rangle \langle x|,
\]

while the plane waves are given by:

\[
\langle x|p\rangle = (2\pi\hbar)^{-d} g^{-1/2}(x) e^{i p \cdot x}, \quad \text{with} \quad p \cdot x \equiv p_i x^i = p_{\mu} x^\mu + \vec{p}_{\mu} \vec{x}^\mu.
\]

Finally, coherent states are normalized as \(\langle \eta|\xi \rangle = e^{\bar{\eta} \cdot \xi}\). Having set our normalizations, we expand the transition amplitude as follows:

\[
\langle x|\eta \rangle e^{-\frac{\beta}{\hbar} H}|y\rangle \xi = (2\pi\hbar)^{-d} g^{-1/2}(y) \int d^2 p \, e^{-\frac{\beta}{\hbar} p \cdot y} \langle x|\eta \rangle e^{-\beta H/\hbar}|p\rangle \xi
\]

\[
= (2\pi\hbar)^{-2d} [g(x)g(y)]^{-1/2} \int d^2 p \, e^{i p \cdot (x-y)} e^{\bar{\eta}_{\mu} \cdot \xi_{\mu}} \sum_{k=0}^{\infty} \frac{(-\beta/\hbar)^{k}}{k!} \sum_{l=0}^{2k} B^k_l (x, \eta, \xi) p^l.
\]

Now, to make the \(\beta\) dependence explicit, we rescale momenta as \(p_i = \sqrt{\hbar/\beta} q_i\) and obtain

\[
\langle x|\eta \rangle e^{-\frac{\beta}{\hbar} H}|y\rangle \xi = (4\pi^2 \hbar \beta)^{-d} [g(x)g(y)]^{-1/2} e^{\bar{\eta}_{\mu} \cdot \xi_{\mu}} \int d^2 q \, e^{i q \cdot (x-y)/\sqrt{\beta\hbar}}
\]

\[
\times \sum_{k=0}^{\infty} \frac{(-\beta/\hbar)^{k}}{k!} \sum_{l=0}^{2k} \left(\frac{\beta}{\hbar}\right)^{k-l/2} B^k_l (x, \eta, \xi) q^l.
\]

After momentum integration, in configuration space the leading term in \((x-y)\) will be of the form \(e^{-(x-y)^2/2\beta\hbar}\), showing that effectively \((x-y) \sim O(\beta^{1/2})\). Then, looking at (4.11), we see that \(q \sim O(\beta^0)\) and so in the sum over \(l\) only \(B^k_{2k}, B^k_{2k-1}\) and \(B^k_{2k-2}\) will contribute, for all \(k\), to the order \(\beta\) amplitude, as anticipated\(^1\).

The \(B^k_l\) coefficients are explicitly derived in appendix A, and inserting (A.3) and (A.4) into (4.11), one can see that the sum in \(k\) can be immediately performed, producing the gaussian exponential \(e^{-q^2/2}\). The transition amplitude (4.11) then becomes

\[
\langle x|\eta \rangle e^{-\frac{\beta}{\hbar} H}|y\rangle \xi = (4\pi^2 \hbar \beta)^{-d}[g(x)g(y)]^{-1/2} e^{\bar{\eta}_{\mu} \cdot \xi_{\mu}} \int d^2 q \, e^{-q^2/2-i\eta \cdot \Delta/\sqrt{\beta\hbar}} \left\{ 1 + \sqrt{\beta\hbar} \left[ \frac{i}{2} g^{ij} q_j \right. \right.
\]

\[
- \frac{i}{4} g^{kl} g^{ij} q_k q_j q_i + i g^{\bar{\mu} \bar{\rho}} \Gamma_{\nu \sigma}^\lambda (\bar{\eta}_\lambda \cdot \xi^\sigma) q_{\bar{\mu} \bar{\rho}} + \beta \hbar \left[ - \frac{1}{32} \ln G_i \ln G^i - \frac{1}{8} \ln G_i - \frac{1}{8} g^{ij} \ln G_i \right]
\]

\[
- \left( \frac{i}{4} \partial^l q_i + \frac{1}{8} g^{ij} g^{kl} + \frac{1}{8} g^{\bar{\mu} \bar{\rho}} \Gamma_{\nu \sigma}^\lambda (\bar{\eta}_\lambda \cdot \xi^\sigma) q_{\bar{\mu} \bar{\rho}} + \left( \frac{1}{12} g^{mnkl} + \frac{1}{8} g^{klm} g + \frac{1}{12} g^{ijkl} q_{mn} \right) q_j q_i q_m q_n - \frac{1}{2} g^{kl} g^{ijkl} q_j q_k q_i q_m q_n q_q - \frac{1}{2} g^{ij} \partial_j (g^{\mu \bar{\rho}} \Gamma_{\nu \sigma}^\lambda (\bar{\eta}_\lambda \cdot \xi^\sigma) q_{\bar{\mu} \bar{\rho}} q_{i j} q_k q_l q_m q_{\bar{\mu}} q_{\bar{\rho}} + g^{\lambda \sigma} \partial_{\lambda} g_{\alpha}\right)
\]

\[
- \frac{1}{2} g^{\bar{\mu} \bar{\rho}} \Gamma_{\nu \sigma}^\lambda (\bar{\eta}_\lambda \cdot \xi^\sigma) q_{\bar{\mu} \bar{\rho}} q_{i j} q_k q_l q_{\bar{\mu}} q_{\bar{\rho}} + \left( a_2 - a_1 \right) R \left( a_3 - s \right) R
\]

\[
- \frac{1}{2} g^{\bar{\mu} \bar{\rho}} \Gamma_{\nu \sigma}^\lambda g^{\lambda \sigma} q_{\bar{\mu} \bar{\rho}} q_{i j} q_k q_l q_{\bar{\mu}} q_{\bar{\rho}} \left[ (\bar{\eta}_\mu \cdot \xi^\sigma) (\bar{\eta}_{\bar{\rho}} \cdot \xi^\sigma) + \delta_{\mu}^{\bar{\rho}} \bar{\eta}_\mu \cdot \xi^\sigma \right]
\}

\]

\[
(4.12)
\]

\(^1\)Note that in \(B^k_{2k}\) at most \(2k-l\) \([x,p]\) commutators are taken into account.
where \( \Delta^i = y^i - x^i \) and \((\bar{\eta}_\lambda \cdot \xi^\sigma)' = (\bar{\eta}_\lambda \cdot \xi^\sigma - s \delta_\lambda^\sigma)\). In order to lighten the formulae we have used the following compact notation

\[
\partial_i \ldots \partial_m g_{ijk} = g_{i..m}, \quad g^{ij} g_{kl} = g^{kli}, \quad g^{ij} = g^i \n
\]

\[
g^{jk} \partial_k g_{lm} = \partial^j g^l, \quad \partial_i \ln G = \ln G_i, \quad g^{ij} \partial_j \ln G = \ln G_i.
\]

Now we can complete squares in the exponent of (4.13), shift integration variables and perform the gaussian integral over momenta. The transition amplitude, up to order \( \beta \), is then given by

\[
\langle \bar{\eta} | e^{-\frac{\pi H}{4}} | \eta \rangle = (2\pi \hbar \beta)^{-d} \left( g(x)/g(y) \right)^{1/2} e^{-\frac{\pi}{2 \hbar} g_{ijkl} \Delta^i \Delta^j} e^{\bar{\eta}_\mu \cdot \xi^\mu} \left\{ 1 + \Delta^i g^{-1/2} \partial_i g^{1/2} - \frac{1}{4 \beta \hbar} \Gamma_{ijkl} \Gamma_{mn} \Delta^i \Delta^j \Delta^k \Delta^m \Delta^n \right. \\
+ \frac{1}{2} \left[ \frac{1}{2} \frac{1}{4 \beta \hbar} \partial_j g_{ij} \Delta^j \Delta^k \right. - \frac{1}{12 \beta \hbar} \partial_j g_{ij} \frac{1}{2} g_{mn} \Gamma_{ij} \Gamma_{kl} \left. \right] \Delta^i \Delta^j \Delta^k \Delta^l + \frac{1}{6} R_{\mu \nu} \Delta^\mu \Delta^\nu \\
+ \Delta^\nu \Gamma_{\nu \sigma} (\bar{\eta}_\lambda \cdot \xi^\sigma)' + \left[ \Delta^\nu \Gamma_{\nu \sigma} (\bar{\eta}_\lambda \cdot \xi^\sigma)' \right] \Delta^i g^{-1/2} \partial_i g^{1/2} \right] + \frac{1}{2} \left[ \Delta^\nu \Gamma_{\nu \sigma} (\bar{\eta}_\lambda \cdot \xi^\sigma)' \right] \Delta^i g^{-1/2} \partial_i g^{1/2} \\
- \frac{1}{2} \partial_j g_{kl} \Delta^j \Delta^k \left( \Delta^\nu \Gamma_{\nu \sigma} \bar{\eta}_{\mu} \cdot \xi^\rho \right) - a_1 \beta h R_{\mu \nu} \bar{\eta}_{\mu} \cdot \xi^\rho + \left( a_1 - a_2 - \frac{1}{2} \right) \beta h R_{\mu \nu} \bar{\eta}_{\mu} \cdot \xi^\rho \right) \\
+ \left( \frac{1}{6} + \frac{s}{2} - a_3 \right) \beta h R + O(\beta^3) \right\}. \tag{4.13}
\]

All functions in (4.13), if not specified otherwise, are evaluated at point \( x \). Keeping in mind that the transition amplitude is a bi-scalar, and that in a semiclassical expansion the classical action evaluated on-shell should appear in the exponent, we factorize and exponentiate, up to order \( \beta \), four terms

\[
\langle \bar{\eta} | e^{-\frac{\pi H}{4}} | \eta \rangle = (2\pi \hbar \beta)^{-d} g(y)^{-1/2} \left[ g^{1/2} + \Delta^i \partial_i g^{1/2} + \frac{1}{2} \Delta^i \Delta^j \partial_i g^{1/2} \right] \\
\exp \left\{ -\frac{1}{2} \beta h \left[ \frac{1}{2} \partial_j g_{ij} \Delta^j \Delta^k + \frac{1}{4} \partial_i g_{jk} \Delta^i \Delta^j \Delta^k + \frac{1}{12} \left( \partial_j g_{mn} - \frac{1}{2} g_{ij} \Gamma_{kl} \Gamma_{ij} \right) \Delta^k \Delta^l \Delta^m \Delta^n \right] \right\} \\
\exp \left\{ \bar{\eta}_{\mu} \cdot \xi^\mu + \Delta^\nu \Gamma_{\nu \sigma} (\bar{\eta}_\lambda \cdot \xi^\sigma)' + \frac{1}{2} \Delta^i \Delta^j \partial_i \Gamma_{\mu \nu} \bar{\eta}_{\mu} \cdot \xi^\rho + \left( a_1 - a_2 - \frac{1}{2} \right) \beta h R_{\mu \nu} \bar{\eta}_{\mu} \cdot \xi^\rho \right. \\
- a_1 \beta h R_{\mu \nu} \bar{\eta}_{\mu} \cdot \xi^\rho + \left( a_1 - a_2 - \frac{1}{2} \right) \beta h R_{\mu \nu} \bar{\eta}_{\mu} \cdot \xi^\rho \right) \\
\left[ 1 + \frac{1}{6} R_{\mu \nu} \Delta^\nu \Delta^\rho + \left( a_1 - \frac{1}{2} \right) \beta h R_{\mu \nu} \bar{\eta}_{\mu} \cdot \xi^\rho + \left( \frac{1}{6} + \frac{s}{2} \right) \beta h R \right]. \tag{4.14}
\]

The first term contains the Taylor expansion around \( x \) of \( g(y)^{1/2} \), that cancel the \( g(y)^{-1/2} \) factor. The second and third terms should be the expansions of the exponential of the classical action, and the fourth is evidently covariant. The detailed study of the expansion of
the on-shell action is demanded to appendix B. Comparing the result \( (B.11) \) for the classical
on-shell action \( \tilde{S}_{os} \) with the expansion \( (4.14) \), we see that, as expected, the tran-
sition amplitude can finally be cast in an explicitly covariant form

\[
\langle x| e^{-\frac{\beta}{\hbar}H}|y\rangle = (2\pi\hbar\beta)^{-d} e^{-\frac{\tilde{S}_{os}}{\hbar}} \left[ 1 + \frac{1}{6} \bar{R}_{\mu\nu} \Delta^\mu \Delta^\nu + \left( a_1 - \frac{1}{2} \right) \beta \hbar R_{\mu}^{\mu} \bar{n}_{\mu} \cdot \xi' \right] + \left( \frac{1}{6} + \frac{s}{2} \right) \beta \hbar R + \mathcal{O}(\beta^2) \]

(4.15)

where the coordinate displacements \( \Delta^\mu \) are considered of order \( \sqrt{\beta} \).

### 5. Conclusions and outlook

In this paper we have introduced and studied the quantum properties of a class of quantum mechanical models with \( U(N|M) \) extended supersymmetry on the worldline. These models take the form of nonlinear sigma models with Kähler manifolds as target spaces, and can be interpreted as describing the motion of a particle with extra degrees of freedom, carried by graded complex vectors \( Z^A_{\mu} \), on Kähler spaces. When the Kähler space is flat, the model has conserved charges satisfying precisely a \( U(N|M) \) extended supersymmetry algebra on the worldline. On curved Kähler spaces, the charges get modified by the geometry as does the corresponding quantum algebra, which generically fails to be first class, though a symmetry under the supergroup \( U(N|M) \) is always present. Conserved supercharges can be defined on locally symmetric Kähler manifolds, i.e. Kähler manifolds with covariantly constant curvature tensors, while a truly first class algebra can be obtained on Kähler manifolds with constant holomorphic sectional curvature. The latter case is particularly interesting, as one can gauge the symmetry charges to obtain higher spin equations with peculiar gauge symmetries, as studied in flat space for the \( U(N|0) \) models in [8].

In the second part of the paper we have computed the heat kernel for our quantum mechanical models in a perturbative expansion. The computation was performed with operatorial methods on arbitrary Kähler manifolds and with a general hamiltonian containing four arbitrary couplings. The calculation turned out to be somewhat tedious for a rather simple final result. One possible application of this result is to use it as a benchmark for path integral calculations, which are often simpler and more flexible, but need to be defined precisely, with predetermined regularization schemes and corresponding couterterms. Indeed the operatorial calculation of ref. [10] was useful to identify the correct time slicing regularization of path integrals in curved spaces [18]. Correctness of the alternative but equivalent mode [19] and dimensional [20, 14] regularizations has then been checked against time slicing, and the full consistency of these three schemes have been instrumental in putting the method of path integration on curved manifolds on solid foundations [17].
In future works we plan to construct regularized path integrals for the $U(N|M)$ quantum mechanics, use them to study effective actions induced by higher spin fields and compute higher order heat kernel coefficients.

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A. Computation of the $B^k_l$ coefficients

In order to compute the $B^k_l$ coefficients defined in eq. (4.8) we follow the strategy explained in [16], and divide the hamiltonian (4.2) in three pieces, contributing at most two, one or no $p$ eigenvalues, respectively

$$H = H_B + H_1 + H_2 \quad \text{where}$$

$$H_B = g^{\mu \nu} g^{1/2} \bar{p}_\mu p\nu g^{-1/2} = \frac{1}{2} G^{-1/4} p_i G^{1/2} g^{ij} p_j G^{-1/4},$$

$$H_1 = -i g^{\mu \nu} \Gamma_{\nu \sigma} (\bar{Z}_\lambda \cdot Z^\sigma - s \delta_\lambda^\sigma) g^{1/2} \bar{p}_\mu g^{-1/2},$$

$$H_2 = a_1 h^2 R_{\mu \rho} Z_{\nu \sigma} Z^\mu Z^\rho + (a_2 + 1) h^2 \bar{R} \bar{Z}_\nu \cdot Z^\mu + (a_3 - s) h^2 R.$$

First of all, notice that $H_B$ is precisely the usual bosonic quantum hamiltonian, carefully studied in the literature [16, 17]. Let us start with $B^k_{2k}$: the only way to have $2k$ $p$ eigenvalues is $k$ factors of $H_B$ and no commutators taken into account, giving simply

$$B^k_{2k} p^{2k} = \left( \frac{p^2}{2} \right)^k,$$

where we use the notation $p^2 = g^{ij} p_i p_j = 2 g^{\mu \nu} p_\mu \bar{p}_\nu$. For $B^k_{2k-1}$ we can have two terms. The first term comes from $k$ factors of $H_B$ with one $p$ acting as a derivative; this gives the corresponding $B^k_{2k-1}$ coefficient, that we call $A^k_{2k-1}$, of the purely bosonic model, whose computation is explained in detail in [16, 17]. The other term comes from $k - 1$ factors of $H_B$ and one $H_1$, by substituting all operators with the corresponding eigenvalues. Putting things together we obtain

$$B^k_{2k-1} p^{2k-1} = A^k_{2k-1} p^{2k-1} - i \hbar k \frac{p^2}{2} \Gamma_{\nu \sigma} (\bar{\eta}_\lambda \cdot \xi^\sigma)' g^{\mu \nu} \bar{p}_\mu$$

$$= \frac{-i \hbar k}{2} \frac{p^2}{2} g^{ij} p_j - i \hbar \frac{k}{2} \frac{p^2}{2} \left[ \frac{1}{2} g^{klj} p_k p_k p_l \right] (A.3)$$

Remember that we are using rescaled $Z$'s.
where we denoted \( (\bar{\eta}_\lambda \cdot \xi^\sigma)' = (\bar{\eta}_\lambda \cdot \xi^\sigma - s \delta^\sigma) \), \( g^i = \partial_i g^{ij} \) and \( g^{ijk} = g^{kl} \partial_l g^{ij} \). For \( B_{2k-2}^k \) four types of term contribute: i) \( k \) factors of \( H_B \), giving the corresponding coefficient \( A_{2k-2}^k \), ii) \( k - 1 \) factors of \( H_B \) and one \( H_1 \), with one \( p \) acting as a derivative. This contribution gives four terms: the derivative acting from one \( H_B \) to \( H_1 \), from \( H_1 \) to one \( H_B \), within \( H_1 \) or within the \( k - 1 \) \( H_B \)'s. iii) \( k - 1 \) factors of \( H_B \) and one \( H_2 \), substituting all operators with their eigenvalues, and iv) \( k - 2 \) factors of \( H_B \) and two \( H_1 \), substituting all with eigenvalues. Remember that in iii) and iv) \( \{ Z, \bar{Z} \} \) (anti)-commutators have to be taken into account in order to obtain eigenvalues on the coherent states. Altogether it results in

\[
B_{2k-2}^k p^{2k-2} = A_{2k-2}^k p^{2k-2} - \hbar^2 \left( \frac{p}{2} \right)^{k-2} g^{ij} \partial_j \left( g^{\mu \nu} \Gamma^\lambda_{\nu \sigma} \right) (\bar{\eta}_\lambda \cdot \xi^\sigma)' p_i \bar{p}_\mu + \frac{1}{2} \hbar^2 k \left( \frac{p}{2} \right)^{k-1} g^{\mu \nu} \Gamma^\lambda_{\nu \sigma} (\bar{\eta}_\lambda \cdot \xi^\sigma)' g^{\rho \sigma} \partial_\mu g_{\rho \sigma} \\
- \hbar^2 \left( \frac{p}{2} \right)^{k-2} g^{\mu \nu} \Gamma^\lambda_{\nu \sigma} (\bar{\eta}_\lambda \cdot \xi^\sigma)' \partial_\mu g^{\rho \sigma} p_\rho \bar{p}_\sigma + \frac{1}{2} \hbar^2 k \left( \frac{p}{2} \right)^{k-1} g^{\mu \nu} \Gamma^\lambda_{\nu \sigma} (\bar{\eta}_\lambda \cdot \xi^\sigma)' g^{\rho \sigma} \partial_\mu g_{\rho \sigma} \\
- i \hbar k A_{2k-3}^{k-2} p^{2k-3} g^{\mu \nu} \Gamma^\lambda_{\nu \sigma} (\bar{\eta}_\lambda \cdot \xi^\sigma)' \bar{p}_\mu + \hbar^2 k \left( \frac{p}{2} \right)^{k-1} \left( a_1 R_{\mu} \rho \sigma \bar{\eta}_\lambda \cdot \xi^\sigma \partial_\mu g^{\rho \sigma} \partial_\sigma \right) \\
+ (a_2 - a_1 + 1) (a_3 - s) R \right) - h^2 \left( \frac{p}{2} \right)^{k-2} g^{\mu \nu} \Gamma^\lambda_{\nu \sigma} g^{\lambda \rho} \Gamma^\sigma_{\lambda \mu} \bar{p}_\mu \bar{p}_\sigma \\
\times \left[ (\bar{\eta}_\rho \cdot \xi^\sigma)' (\bar{\eta}_\rho \cdot \xi^\mu)' + \delta^\sigma \bar{\eta}_\rho \cdot \xi^\mu \right].
\]

(A.4)

In the formulae above the bosonic coefficients are given by

\[
A_{2k-3}^{k-1} = - \frac{i \hbar}{2} (k - 1) \left( \frac{p}{2} \right)^{k-2} g^{ij} p_j - i \hbar \left( \frac{k}{2} \right) \left( \frac{p}{2} \right)^{k-3} \frac{1}{2} g^{kl} p_j p_k p_l ,
\]

\[
A_{2k-2}^k = \hbar^2 k \left( \frac{p}{2} \right)^{k-1} \left[ \frac{1}{32} \ln G_i \ln G^i + \frac{1}{8} \ln G^i + \frac{1}{8} g^{ij} \ln G_j \right] - h^2 \left( \frac{k}{2} \right) \left( \frac{p}{2} \right)^{k-2} \\
\times \left[ \frac{1}{2} \partial^j g^l + \frac{1}{4} g^{ij} g^l + \frac{1}{4} g^k g^l \right] p_l p_l - h^2 \left( \frac{k}{2} \right) \left( \frac{p}{2} \right)^{k-3} \left[ \frac{1}{4} g^{mnkl} + \frac{3}{4} g^{klm} g^n \right. \\
\left. + \frac{1}{2} g^{ijkl} g^{m} g^{m} \right] p_k p_l p_m p_n - h^2 \left( \frac{k}{4} \right) \left( \frac{p}{2} \right)^{k-4} \left[ \frac{3}{4} g^{ijkl} g^{p} g^{p} \right] p_j p_k p_l p_m p_n
\]

(A.5)

and we recall that the following compact notation was employed

\[
\partial_l \ln G = \ln G_i , \quad g^{ij} \partial_j G = \ln G^i , \quad g^{ij} \partial_i \ln G = \ln G^i .
\]
B. The on-shell action

The euclidean action generated by the hamiltonian (4.2) is given by

\[
S = \int_{-\beta}^{0} d\tau \left[ g_{\mu \bar{\nu}} \dot{x}^\mu \dot{x}^\bar{\nu} + h \dot{Z}_\mu \cdot \frac{DZ^\mu}{D\tau} - s h \Gamma^\mu_\mu + h^2 \Delta H \right],
\]

(B.1)

where \( \Gamma_\mu \equiv \Gamma^\nu_{\mu \nu} \) is the \( U(1) \) piece of the Kähler connection and \( s \) plays the role of an additional \( U(1) \) coupling. The additional piece

\[
\Delta H = a_1 R^\nu_\mu \sigma \dot{Z}_\nu \cdot Z^\sigma + a_2 R^\nu_{\sigma \lambda} \dot{Z}^\sigma \cdot Z^\lambda + a_3 R
\]

(B.2)

contains the generalized couplings to curvatures, and the covariant time derivative on \( Z \) fields reads

\[
\frac{DZ^\mu}{D\tau} = \dot{Z}_A^\mu + \dot{x}^\nu \Gamma^\mu_{\nu \sigma} Z^\sigma_A, \quad \frac{D\bar{Z}_A^\mu}{D\tau} = \dot{\bar{Z}}_A^\mu - \dot{x}_\sigma \Gamma^\mu_{\sigma \tau} Z^\tau_A.
\]

(B.3)

From the action (B.1) the following equations of motion arise

\[
\ddot{x}_i + \Gamma^i_{jk} \dot{x}_j \dot{x}_k - h R^i_\mu \rho \dot{x}^\rho - h^2 g_{\mu \bar{\nu}} \dot{x}^\mu \dot{x}^\bar{\nu} - h^2 g_{\mu \bar{\nu}} \dot{R}_{\mu \nu} + h^2 \Delta H = 0
\]

(B.4)

Now, we have to expand the action (B.1) up to order \( \beta \), with the fields obeying (B.4), with boundary conditions: \( x^i(-\beta) = y^i, x^i(0) = x^i \), \( Z^\mu_A(-\beta) = \xi^\mu_A \) and \( \bar{Z}^A_\mu(0) = \bar{\eta}^A_\mu \). Expanding fields in a Taylor series around \( \tau = 0 \), we will see that, for small \( \beta \), we have

\[
\frac{d^n x^i}{d\tau^n} \sim \frac{d^n Z^\mu_A}{d\tau^n} \sim \beta^{-n/2}.
\]

Expanding also the on-shell lagrangian, we can write

\[
S_{os} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (-\beta)^n \left. \frac{d^n L_{os}}{d\tau^n} \right|_{\tau=0} \beta^{n+1},
\]

(B.5)

and one notices that, for all pieces of the lagrangian but the \( U(1) \) one, it is sufficient to keep the order zero: \( S_{os} = \beta L_{os}(0) \). For the \( U(1) \) piece, it is necessary the next order: \( \beta L_{os}(0) - \frac{1}{2} \beta^2 \dot{L}_{os}(0) \). Let us begin with the \( x^i \): we expand in Taylor series and obtain

\[
x^i(\tau) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \frac{d^n x^i}{d\tau^n}(0),
\]

(B.6)

setting \( \tau = -\beta \) and using the boundary conditions we have

\[
y^i = x^i - \beta \dot{x}^i(0) + \frac{\beta^2}{2} \ddot{x}^i(0) - \frac{\beta^3}{6} \dddot{x}^i(0) + ...
\]

(B.7)

Remember that we are using rescaled \( Z \)'s.
and similarly for the first derivative
\[ \dot{x}^i(0) = -\frac{\Delta^i}{\beta} + \frac{\beta}{2} \ddot{x}^i(0) - \frac{\beta^2}{6} \frac{d}{d\tau} \dddot{x}^i(0) + \ldots. \]  

(B.8)

Now one uses the equations of motion (B.4) and solves by iteration. To order \( \beta^{1/2} \) we obtain
\[ \dot{x}^i(0) = -\frac{\Delta^i}{2\beta} - \frac{1}{2\beta} \Gamma^i_{jk} \Delta^j \Delta^k - \frac{1}{6\beta} \left( \partial_i \Gamma^i_{jk} + \Gamma^i_{js} \Gamma^s_{kl} \right) \Delta^j \Delta^k \Delta^l - \frac{\Delta^i}{2} \hbar R^i_{\mu \nu} (\bar{\eta}_\mu \cdot \xi^\nu)' . \]  

(B.9)

Adopting the same procedure for \( Z \) and \( \dot{Z} \) we have
\[ Z_A^\mu(0) = \xi_A^\mu + \Gamma^\mu_{\nu \lambda} \Delta^\nu \xi_A^\lambda , \]
\[ \dot{Z}_A^\mu(0) = \frac{Z_A^\mu(0) - \xi_A^\mu}{\beta} - \frac{1}{2\beta} \partial_j \Gamma^\mu_{\nu \lambda} \Delta^j \Delta^\nu \xi_A^\lambda + \frac{1}{2\beta} \Gamma^\mu_{\nu \lambda} \Gamma^\nu_{\sigma \rho} \Delta^\sigma \Delta^\rho \xi_A^\lambda + \frac{1}{2\beta} \Gamma^\mu_{\nu \lambda} \Gamma^\nu_{\sigma \rho} \Delta^\sigma \Delta^\rho \xi_A^\lambda . \]  

(B.10)

Now we can substitute the above expansions in \( \beta L_{os}(0) \) and in \( -\frac{\beta^2}{2} \tilde{L}_{os}^{(1)}(0) \). Remembering that in fermionic actions one needs also a boundary term, it is convenient to use the modified action \( \tilde{S} = S - \hbar \tilde{Z}_\mu(0) \cdot Z^\mu(0) \), and using (B.1) for \( S \) we finally arrive at the following expansion
\[ \tilde{S}_{os} = \frac{1}{2} g_{ij} \Delta^i \Delta^j + \frac{1}{4} \partial_i g_{jk} \Delta^i \Delta^j \Delta^k + \frac{1}{12} \frac{1}{2} \partial_k \partial_l g_{mn} - \frac{1}{2} g_{ij} \Gamma^i_{kl} \Gamma^j_{mn} \right) \Delta^k \Delta^l \Delta^m \Delta^n \]
\[ - \hbar \bar{\eta}_\mu \cdot \xi^\mu - \hbar \Delta^\nu \Gamma^\lambda_{\nu \sigma} (\bar{\eta}_\lambda \cdot \xi^\sigma)' - \hbar \frac{1}{2} \Delta^i \Delta^j \partial_i \partial_j \Gamma^\lambda_{\mu \sigma} (\bar{\eta}_\lambda \cdot \xi^\sigma)' - \hbar \frac{1}{2} \Delta^\nu \Delta^\lambda \Gamma^\mu_{\nu \sigma} \Gamma^\sigma_{\lambda \rho} \bar{\eta}_\mu \cdot \xi^\rho \]
\[ + a_1 \beta h^2 R^\nu_{\mu \rho} \bar{\eta}_\nu \cdot \xi^\mu \bar{\eta}_\sigma \cdot \xi^\rho + a_2 \beta h^2 R^\mu_{\nu \rho} \bar{\eta}_\mu \cdot \xi^\nu + a_3 \beta h^2 R + \ldots \]  

(B.11)

which appears in the final result (4.15).

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