Using Automata to obtain Regular Expressions for Induced Actions *

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Abstract

Presentations of Kan extensions of category actions provide a natural framework for expressing induced actions, and therefore a range of different combinatorial problems. Rewrite systems for Kan extensions have been defined and a variation on the Knuth-Bendix completion procedure can be used to complete them – when possible. Regular languages and automata are a useful way of expressing sets and actions, and in this paper we explain how to use rewrite systems for Kan extensions to construct automata expressing the induced action and how sets of normal forms can be calculated by obtaining language equations from the automata.

1 Introduction

Given a morphism of monoids \( F : A \to B \) and an action of \( A \) on a set \( X \), the induced action of \( B \) is on a set \( F_*(X) \). Suppose \( B \) has a presentation \( \text{mon}(\Delta|\text{RelB}) \) and \( \Gamma \) is a set of generators for \( A \) so that \( F(a) \) is described in terms of \( \Delta^* \), and the action of \( a \) on \( X \) is known for each \( a \in \Gamma \). The problem is to describe \( F_*(X) \). The usual rewrite theory is the case where \( A \) is the trivial monoid and \( X \) is a one element set, the extension to actions allows a wider range of applications. In fact our extension goes beyond monoids to categories.

When \( F : A \to B \) is a morphism of categories this gives a formulation in terms of induced actions of categories or Kan extensions, as explained in [1], which defines rewrite systems for Kan extensions and introduces procedures for completing such systems – when possible.

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This paper is a sequel to [1], showing how to interpret complete rewrite systems for Kan extensions. In this paper we assume that the completion procedure has been successful and show how to use the rewrite systems to construct accepting automata whose languages can be calculated by equations, giving regular expressions for the sets of the induced action (Theorem 4.3). In the monoid case the induced action is on a single set. In the category situation we may have many sets to describe. The use of languages is particularly appropriate for the situation where the action induced involves infinite sets.

Mac Lane wrote that “the notion of Kan extensions subsumes all the other fundamental concepts of category theory” in section 10.7 of [13] (entitled “All Concepts are Kan Extensions”). Together with [1] this paper brings the power of rewriting theory and language theory to bear on a much wider range of combinatorial enumeration problems. Traditionally regular languages are used to specify the elements of a monoid and rewriting is used for solving the word problem for monoids. Rewriting and regular languages may now also be used in the specification of

i) equivalence classes and equivariant equivalence classes,

ii) arrows of a category or groupoid,

iii) right congruence classes given by a relation on a monoid,

iv) orbits of an action of a group or monoid.

v) conjugacy classes of a group,

vi) coequalisers, pushouts and colimits of sets,

vii) induced permutation representations of a group or monoid.

and many others.

2 Rewrite Systems for Induced Actions

This section gives a brief account of work of Brown and Heyworth [1] on extensions of rewriting methods.

Let $A$ be a category. A category action $X$ of $A$ is a functor $X : A \to \text{Sets}$. Let $B$ be a second category and let $F : A \to B$ be a functor. Then an extension of the action $X$ along $F$ is a pair $(K, \varepsilon)$ where $K : B \to \text{Sets}$ is a functor and $\varepsilon : X \to K \circ F$ is a natural transformation. The Kan extension of the action $X$ along $F$ is an extension of the action $(K, \varepsilon)$ with the universal property that for any other extension of the action $(K', \varepsilon')$ there exists a unique natural transformation $\alpha : K \to K'$ such that $\varepsilon' = \alpha \circ \varepsilon$.

The problem that has been introduced is that of “computing a Kan extension”. Keeping the analogy with computation and rewriting for presentations of monoids and respecting the work of [2, 4, 5, 8], a definition of a presentation of a Kan extension is given as follows.
Recall that a **category presentation** is a pair $\text{cat}(\Delta|\text{RelB})$, where $\Delta$ is a (directed) graph and $\text{RelB}$ is a set of relations on the free category $P$ on $\Delta$. The category $B$ presented by $\text{cat}(\Delta|\text{RelB})$ has objects $\text{Ob}B$ that can be identified with $\text{Ob}\Delta$ and arrows $\text{Arr}B$ that can be identified with the classes of arrows of $P$ under the congruence generated by $\text{RelB}$. The source and target functions of generating graph and category are denoted $\text{src}, \text{tgt} : \text{Arr}\Delta \rightarrow \text{Ob}\Delta$ and $\text{src}, \text{tgt} : \text{Arr}B \rightarrow \text{Ob}B$ respectively.

A **Kan extension data** $(X',F')$ consists of small categories $A, B$ and functors $X' : A \rightarrow \text{Sets}$ and $F' : A \rightarrow B$. A **Kan extension presentation** is a quintuple $\mathcal{P} := \text{kan}(\Gamma|\Delta|\text{RelB}|X|F)$ where

- $\Gamma$ and $\Delta$ are (directed) graphs;
- $X : \Gamma \rightarrow \text{Sets}$ and $F : \Gamma \rightarrow P$ are graph morphisms to the category of sets and the free category $P$ on $\Delta$ respectively;
- $\text{RelB}$ is a set of relations on the free category $P$, i.e. a subset of $\text{Arr}P \times \text{Arr}P$.

We say $\mathcal{P}$ **presents** the Kan extension $(K, \varepsilon)$ of the Kan extension data $(X',F')$ where $X' : A \rightarrow \text{Sets}$ and $F' : A \rightarrow B$ if

- $\Gamma$ is a generating graph for $A$ and $X : \Gamma \rightarrow \text{Sets}$ is the restriction of $X' : A \rightarrow \text{Sets}$
- $\text{cat}(\Delta|\text{RelB})$ is a category presentation for $B$.
- $F : \Gamma \rightarrow P$ induces $F' : A \rightarrow B$.

We expect that a Kan extension $(K, \varepsilon)$ is given by

- a set $KB$ for each $B \in \text{Ob}\Delta$,
- a function $Kb : KB_1 \rightarrow KB_2$ for each $b : B_1 \rightarrow B_2 \in \text{B}$,
- a function $\varepsilon_A :XA \rightarrow KFA$ for each $A \in \text{ObA}$.

Let $\sqcup XA$ denote the disjoint union of all the sets $XA$ for all objects $A$ in $\text{ObA}$ and let $\sqcup KB$ denote the disjoint union of the sets $KB$ for all objects $B$ in $\text{Ob}\Delta$.

The main result of the paper [1] defines rewriting procedures on the $P$-set

$$T := \bigsqcup_{B \in \text{Ob}\Delta, A \in \text{Ob}\Gamma} XA \times P(FA,B).$$

Elements of $T$ are called **terms** and are written $x | p$ where $x$ is an element of a set $XA$ for some object $A$ of $A$, $p : FA \rightarrow B$ is an arrow of $P$, and “$|$” is a symbol we use to separate the ‘element part’ $x$ of the term from the ‘word part’ $p$.

Then the set $T$ can also be written

$$T = \{x | p : x \in XA, p : FA \rightarrow B \text{ for some } A \in \text{Ob}\Gamma, B \in \text{Ob}\Delta\}.$$
If $R$ is a rewrite system on $T$ then we will write $R = (R_T, R_P)$, since two kinds of rewriting are involved here. Rewriting using the rules $R_P$ is the familiar $x \mid ulv \rightarrow x \mid urv$ given by a relation $(l, r)$. The rules $R_T$ derive from a given action of certain words on elements, so allowing rewriting $x \mid F(a)v \rightarrow x \cdot a \mid v$. Further, the elements $x$ and $x \cdot a$ may belong to different sets. When such rewriting procedures complete, the associated normal form gives in effect a computation of what we call the Kan extension defined by the presentation.

**Theorem 2.1 (Data for Kan Extensions)** \[1\]

Let $P = \text{kan}(\Gamma|\Delta|\text{Rel}B|X|F)$ be a Kan extension presentation.

Let $P$ be the free category on $\Delta$, let $T := \{x \mid p : x \in XA, p : FA \rightarrow B \text{ for some } A \in \text{Ob}\Gamma, B \in \text{Ob}\Delta\}$ and define $R = (R_e, R_K)$ where $R_e := \{(x|Fa, x \cdot a|\text{id}_{FA_1}) : x \in XA_1, a : A_1 \rightarrow A_2 \text{ in } A\}$ and $R_K := \text{Rel}B$.

Then the Kan extension $(K, \varepsilon)$ presented by $P$ may be given by the following data:

1) the set $\sqcup KB = T/ \leftrightarrow_R$,
2) the function $\tau : \sqcup KB \rightarrow \text{Ob}B$ induced by $\tau : T \rightarrow \text{Ob}P$,
3) the action of $B$ on $\sqcup KB$ induced by the action of $P$ on $T$,
4) the natural transformation $\varepsilon$ determined by $x \mapsto [x|\text{id}_{FA}]$ for $x \in XA, A \in \text{Ob}A$.

To work with a rewrite system $R$ on $T$ certain concepts of order on $T$ are required. The paper \[1\] gives properties of orderings $>_X$ on $\sqcup XA$ and $>_P$ on $\text{Arr}P$ which enable the construction of an ordering $>_T$ on $T$ with the properties needed for the rewriting procedures. For this paper we will assume that the order on $T$ is a short-lexicographic order induced by ordering all the variables in the alphabet we will be using.

Given a rewrite system $R$ for a Kan extension and an ordering $>_T$ on $T$, a reduction relation $\rightarrow_R$ compatible with the ordering is determined. A reduction relation on a set is complete if it is Noetherian and confluent. The Noetherian property implies that any term of $T$ can be repeatedly reduced until, after a finite number of reductions, an irreducible element will be obtained. The confluence property implies that if two terms are equivalent under the relation $\leftrightarrow_R$ then they reduce to the same term, i.e. there is a unique irreducible term in each equivalence class. By standard abuse of notation the rewrite system $R$ will be called complete when it is complete.

The paper \[1\] defines a variation on the Knuth-Bendix procedure which can be applied to $R$ to complete it – when this is possible. The procedure has been implemented in GAP3 (to be converted to GAP4), using a short-lex ordering. The details in this paper show how to use automata to interpret the output of the procedure when the sets $KB$ on which the induced action is defined cannot be enumerated (i.e. are infinite).
3 Regular Languages and Automata for Induced Actions

For a detailed introduction to automata theory, refer to [6] or [12]. This section only outlines the essential ideas we use.

A (finite) deterministic automaton is a 5-tuple $A = (S, \Sigma, s_0, \delta, Q)$ where $S$ is a finite set of states (represented by circles), $s_0 \in S$ is the initial state (marked with an arrow), $\Sigma$ is a finite alphabet, $\delta : S \times \Sigma \to S$ is the transition, $Q \subseteq S$ is the set of terminal states (represented by double circles). A deterministic automaton $A$ is complete if $\delta$ is a function, and incomplete if it is only a partial function. If $A$ is incomplete, then when $\delta(s, a)$ is undefined, the automaton is said to crash.

The extended state transition $\delta^*$ is the extension of $\delta$ to $\Sigma^*$. It is defined by $\delta^*(s, \varepsilon) := s$, $\delta^*(s, a) := \delta(s, a)$, $\delta^*(s, aw) := \delta(\delta(s, a), w)$ where $s \in S$, $a \in \Sigma$ and $w \in \Sigma^*$. We are interested in the final state $\delta^*(s_0, w)$ of the machine after a string $w \in \Sigma^*$ has been completely read. If the machine crashes or ends up at a non-terminal state then the string is said to have been rejected. If it ends up at a terminal state then we say the string is accepted.

A language over a given alphabet $\Sigma$ is a subset $L \subseteq \Sigma^*$. The set $L(A)$ of all acceptable strings is the language accepted by the automaton $A$. A language $L$ is a recognisable if it is accepted by some automaton $A$. Two automata are equivalent if their languages are equal.

The complement of a complete, deterministic automaton is found by making non-terminal states terminal and vice versa. If the language accepted by an automaton $(A)$ is $L$, then the language accepted by its complement $(A)^C$ is $\Sigma^* - L$.

**Lemma 3.1 (Completion of Automata)** [4]

Let $A = (S, \Sigma, s_0, \delta, Q)$ be an incomplete deterministic automaton. Then there exists a complete deterministic automaton $A^{CP}$ such that $L(A) = L(A^{CP})$.

Diagrammatically this means that automata may be completed by adding one further non-terminal (dump) state $d$ and adding in all the missing arrows so that they point to this state.

A non-deterministic automaton is a 5-tuple $A = (S, \Sigma, s_0, \delta, Q)$ where $S$ is a finite set of states, $S_0 \subseteq S$ is a set of initial states, $\Sigma$ is a finite alphabet, $Q \subseteq S$ is the set of terminal states and $\delta : S \times \Sigma \to \mathcal{P}(S)$ is the transition mapping where $\mathcal{P}(S)$ is the power set. The language accepted by a non-deterministic automaton $A$ is the set of words $L \subseteq \Sigma^*$ such that $\delta^*(s, l) \cap Q \neq \emptyset$ for some $s \in S$ for all $l \in L$.

**Lemma 3.2 (Determinising Automata)** [4]

Let $A = (S, \Sigma, s_0, \delta, Q)$ be a non-deterministic automaton. Then there exists a deterministic automaton $A^D$ such that $L(A^D) = L(A)$.

In practice a non-deterministic automaton may be made deterministic by drawing a transition tree and then converting the tree into an automaton; for details of this see [3].
A regular expression is a string of symbols representing a regular language. Let \( \Sigma \) be a set (alphabet). The empty word will be denoted \( \text{id} \). A **regular expression** over \( \Sigma \) is a string of symbols formed by the rules

i) \( a_1 \cdots a_n \) is regular for \( a_1, \ldots, a_n \in \Sigma \),

ii) \( \emptyset \) is regular,

iii) \( \text{id} \) is regular,

iv) if \( x \) and \( y \) are regular then \( xy \) is regular,

v) if \( x \) and \( y \) are regular then \( x+y \) is regular,

vi) if \( x \) is regular then \( x^* \) is regular.

For example \((x+y)^*-z\) is the expression representing the regular language \((\{x\} \cup \{y\})^*/\{z\}\). For our purposes a **right linear language equation** over \( \Sigma \) is an expression \( X = AX + E \) where \( A, X, E \subseteq \Sigma^* \).

**Theorem 3.3 (Arden’s Theorem)** \([6]\)

Let \( A, X, E \subseteq \Sigma^* \) such that \( X = AX + E \) where \( A \) and \( E \) are known and \( X \) is unknown. Then

i) \( A^*E \) is a solution,

ii) if \( Y \) is any solution then \( A^*E \subseteq Y \),

iii) if \( \text{id} \notin A \) then \( A^*E \) is the unique solution.

**Theorem 3.4 (Solving Language Equations)** \([6]\)

A system of right linear language equations:

\[
\begin{align*}
X_0 &= A_{0,0}X_0 + \cdots + A_{0,n-1}X_{n-1} + E_0 \\
X_1 &= A_{1,0}X_0 + \cdots + A_{1,n-1}X_{n-1} + E_1 \\
&\vdots \\
X_{n-1} &= A_{n-1,0}X_0 + \cdots + A_{n-1,n-1}X_{n-1} + E_{n-1}
\end{align*}
\]

where \( A_{i,j}, E_i \in (\Sigma^*) \) and \( \text{id} \notin A_{i,j} \) for \( i,j = 0, \ldots, n-1 \), has a unique solution.
Theorem 3.5 (Regular Expressions from Automata)

Let $A$ be a deterministic automaton. Then $L(A)$ is regular.

**Proof** Let $A = (S, \Sigma, s_0, \delta, Q)$, where $S := \{s_0, \ldots, s_{n-1}\}$. For $i = 1, \ldots, n-1$ define $X_i := \{z \in \Sigma^* : \delta^*(s_i, z) \in Q\}$. It is clear that $X_0 = L(A)$.

Define $E_i := \{id\}$ if $s_i \in Q$ and $\emptyset$ otherwise.

Define $A_{i,j}$ to be the sum of all letters $x \in \Sigma$ such that $\delta^*(s_i, x) = s_j$.

Then form the following system of equations:

\[
X_0 = A_{0,0}X_0 + \cdots + A_{0,n-1}X_{n-1} + E_0
\]
\[
X_1 = A_{1,0}X_0 + \cdots + A_{1,n-1}X_{n-1} + E_1
\]
\[
\vdots
\]
\[
X_{n-1} = A_{n-1,0}X_0 + \cdots + A_{n-1,n-1}X_{n-1} + E_{n-1}
\]

There are $n$ right linear equations in $n$ unknowns satisfying the conditions of Theorem 3.4. Therefore they have a unique solution.

Thus every non-deterministic automaton gives rise to a system of language equations from whose solutions a description of the language may be obtained.

Theorem 3.6 (Kleene’s Theorem)

A language $L$ is regular if and only if it is recognisable.

This section has outlined the basic automata and language theory used in the paper. Our main result (Theorem 4.3) is the construction, from a complete rewrite system for a Kan extension, of automata which recognise the elements of the extension as a regular language.

4 Constructing and Interpreting the Automata

Throughout this section we continue with the notation of $\mathbb{I}$ as described in Section 2. Recall that a presentation of a Kan extension $(K, \varepsilon)$ is a quintuple $\mathcal{P} := \text{kan}(\Gamma|\Delta|\text{RelB}|\text{X}|\text{F})$ where $\Gamma$ and $\Delta$ are graphs, $\text{RelB}$ is a set of relations on the free category $\mathcal{P}$ on $\Delta$, while $X : \Gamma \to \text{Sets}$ and $F : \Gamma \to \mathcal{P}$ are graph morphisms. Recall that elements of the set

\[
T := \bigsqcup_{B \in \text{Ob} \Delta} \bigsqcup_{A \in \text{Ob} \Gamma} X A \times \mathcal{P}(FA, B)
\]

are written $t = x | b_1 \cdots b_n$ with $x \in X A$, and $b_1, \ldots, b_n \in \text{Arr} \Delta$ are composable with $\text{src}(b_1) = FA$. The (‘target’) function $\tau : T \to \text{Ob} \Delta$ is defined by $\tau(x | b_1 \cdots b_n) := \text{tgt}(b_n)$ and the action of $\mathcal{P}$ on $T$, written $t : p$ for $t \in T$, $p \in \text{Arr} \mathcal{P}$, is defined when $\tau(t) = \text{src}(p)$. 

In [1] we defined an initial rewrite system \( R_{\text{init}} := (\mathcal{R}_\varepsilon, \mathcal{R}_K) \) on \( T \), (also see Theorem 2.1), and gave a procedure for attempting to complete this system. We will be assuming that the procedure has terminated, returning a complete rewrite system \( \mathcal{R} = (\mathcal{R}_T, \mathcal{R}_P) \) with respect to a short-lex ordering on an alphabet \( \Sigma \). In this section automata will be used to find regular expressions for each of the sets \( KB \) for \( B \in \text{Ob}\Delta \).

Recall that \( \sqcup XA \) is the union of the images under \( X \) of all the objects of \( \Gamma \) and \( \sqcup KB \) is the union of the images under \( K \) of all the objects of \( \Delta \). In general the automaton for the irreducible terms which are accepted as members of \( \sqcup KB \) is the complement of the machine which accepts any string containing undefined compositions of arrows of \( B \), any string not containing a single \( x_i \) on the left-most end, and any string containing the left-hand side of a rule. This essentially uses a semigroup presentation of the Kan extension.

**Lemma 4.1 (Semigroup presentation of a Kan extension)**

Let \( \mathcal{P} \) present the Kan extension \( (K, \varepsilon) \). Then the set \( \sqcup KB \) may be identified with the non-zero elements of the semigroup having the presentation with generating set

\[
\Sigma_0 := (\sqcup XA) \sqcup \text{Arr}\Delta \sqcup 0
\]

and relations

- \( 0u = u0 = 0 \) for all \( u \in \Sigma_0 \),
- \( ux = 0 \) for all \( u \in \Sigma_0 \), \( x \in \sqcup XA \),
- \( xb = 0 \) for all \( x \in XA \) such that \( \text{src}(b) \neq FA \), \( A \in \text{Ob}\Gamma \), \( b \in \text{Arr}\Delta \),
- \( b_1b_2 = 0 \) for all \( b_1, b_2 \in \text{Arr}\Delta \) such that \( \text{src}(b_2) \neq \text{tgt}(b_1) \),
- \( x(Fa) = (x \cdot a) \) for all \( x \in XA \), \( a \in \text{Arr}\mathcal{A} \) such that \( \text{src}(a) = A \),
- \( l = r \) for all \( (l, r) \in \text{Rel}\mathcal{B} \).

**Proof** The semigroup defined is the set of equivalence classes of \( T \) with respect to the second two relations (i.e. the Kan extension rules \( \mathcal{R}_\varepsilon \) and \( \mathcal{R}_K \)) with a zero adjoined and multiplication of any two classes of \( T \) defined to be zero. \( \square \)

**Lemma 4.2 (T is a Regular Language)**

Let \( \mathcal{P} \) be a presentation of a Kan extension \( (K, \varepsilon) \). Then \( T \) is a regular language over the alphabet \( \Sigma := (\sqcup XA) \sqcup \text{Arr}\Delta \).

**Proof** Define an automaton \( \mathcal{A} := (S, \Sigma, s_0, \delta, Q) \) where \( S := \{s_0, d\} \cup \text{Ob}\Delta \), \( Q := \text{Ob}\Delta \) and \( \delta \) is defined as follows:

- initial state: \( \delta(s_0, u) := \begin{cases} 
  FA & \text{for } u \in XA, A \in \text{Ob}\Gamma \\
  d & \text{otherwise}.
\end{cases} \)

- for \( B \in \text{Ob}\Delta \): \( \delta(B, u) := \begin{cases} 
  \text{tgt}(u) & \text{for } u \in \text{Arr}\Delta, \text{src}(u) = B \\
  d & \text{otherwise}.
\end{cases} \)

- dump state: \( \delta(d, u) := d \) for all \( u \in \Sigma \).
It is clear from the definitions that the extended state transition $\delta^*$ is such that $\delta^*(s_0,t) \in \text{Ob}\Delta$ if and only if $t \in T$. Hence $L(A) = T$. Therefore $T$ is regular over $\Sigma$.

**Theorem 4.3 (Main Theorem)**

Let $\mathcal{R}$ be a finite complete rewrite system for the Kan extension $(K, \varepsilon)$ given by the presentation $P = \text{kan}(\Gamma|\Delta|\text{Rel}B|X|F)$.

Let $T$ and $\Sigma$ be defined as before (Lemma 4.2). Then for each object $B \in \text{Ob}\Delta$ there is a regular expression representing a regular language $K_B$ over $\Sigma$ such that

i) $K_B = \{[t]_\sim : t \in K_B\}$, where $[t]_\sim$ represents the equivalence class of $t$ in $T$ with respect to $\leftrightarrow_R$.

ii) for $b : B_1 \to B_2$ in $\text{Arr}\Delta$ the term $\text{irr}(t \cdot b)$ is an element of $K_{B_2}$ for all elements $t \in K_{B_1}$.

**Proof** Recall the (‘target’) functions $\text{tgt} : \text{Arr}P \to \text{Ob}\Delta$ and $\tau : T \to \text{Ob}\Delta$. We use the following definition to restrict sets to those elements whose ‘target’ is $B$.

$$H_B := \begin{cases} \{x : x \in XA \text{ and } XA = B, x \in H\} & \text{when } H \subseteq \sqcup XA \\ \{p : \text{tgt}(p) = B, p \in H\} & \text{when } H \subseteq P \\ \{t : \tau(t) = B, t \in H\} & \text{when } H \subseteq T \end{cases}$$

Then define $\text{irr}(H)$ be the set of irreducible forms of the terms $H \subseteq T$ with respect to $\to_R$.

For each object $B \in \text{Ob}\Delta$ we define an incomplete non-deterministic automaton $A_B$ with input alphabet $\Sigma$, and language $\Sigma^* - \text{irr}(T_B)$. This automaton rejects only the irreducible elements of $T_B$, i.e. it accepts all terms that do not represent elements of $T$, terms that do not have ‘target’ $B$ and terms that are reducible by $\to_R$.

We will use the following notation:

$$\text{l}(\mathcal{R}) := \{l : (l, r) \in \mathcal{R}\},$$
$$\text{pl}(\mathcal{R}) := \{u : (uv, r) \in \mathcal{R}\}$$
$$\text{ppl}(\mathcal{R}) := \{u : (uv, r) \in \mathcal{R}, v \neq \text{id}\}.$$ 

These are the set of all left hand side of rules, the set of all prefixes of left hand sides of rules and the set of all proper prefixes of left hand sides of rules respectively.

Now define $A_B := (S, \Sigma, s_0, \delta, Q_B)$ where

$$S := \{s_0, d\} \cup \text{Ob}\Delta \cup (\sqcup XA) \cup \text{ppl}(\mathcal{R})$$
$$Q := \{s_0, d, B\} \cup (\sqcup XA)_B \cup \text{ppl}(\mathcal{R})_B.$$
Let \( x, b \in \Sigma \) so that \( x \in \cup X A \) and \( b \in \text{Arr}\Delta \). Define the transition \( \delta : S \times \Sigma \rightarrow \mathbb{P}(S) \) by:

Initial state \( \delta(s_0, x) := \begin{cases} \{x\} & \text{if } x \notin l(R_T), \\ \{d\} & \text{if } x \in l(R_T), \end{cases} \)

for \( x_i \in X A \) \( \delta(x_i, x) := \{d\} \),

\( \delta(s_0, b) := \{d\} \),

\( \delta(x_1, b) := \begin{cases} \{x_1|b, tgt(b)\} & \text{if } x_1|b \in \text{pl}(R_T), \\ \{tg(b)\} & \text{if } \tau(x_1) = \text{src}(b), x_1|b \notin \text{pl}(R_T), \\ \{d\} & \text{if } x_1|b \in l(R_T) \text{ or if } \tau(x_i) \neq \text{src}(b), \end{cases} \)

for \( B_i \in \text{ObB} \) \( \delta(B_i, x) := \{d\} \),

\( \delta(B_i, b) := \begin{cases} \{b, tgt(b)\} & \text{if } \text{src}(b) = B_i, b \in \text{pl}(R_P), \\ \{tg(b)\} & \text{if } \text{src}(b) = B_i, b \notin \text{pl}(R_P), \\ \{d\} & \text{if } \text{src}(b) = B_i, b \in l(R_P) \text{ or if } \text{src}(b) \neq B_i, \end{cases} \)

for \( u \in \text{pl}(R_T) \) \( \delta(u, x) := \{d\} \),

\( \delta(u, b) := \begin{cases} \{u \cdot b, tgt(b)\} & \text{if } u \cdot b \in \text{pl}(R_T), \\ \{tg(b)\} & \text{if } \tau(u) = \text{src}(b), u \cdot b \notin \text{pl}(R_T), \\ \{d\} & \text{if } u \cdot b \in l(R_T) \text{ or if } \tau(u) \neq \text{src}(b), \end{cases} \)

for \( p \in \text{pl}(R_P) \) \( \delta(p, x) := \{d\} \),

\( \delta(p, b) := \begin{cases} \{pb, tgt(b)\} & \text{if } pb \in \text{pl}(R_P), \\ \{tg(p)\} & \text{if } tgt(p) = \text{src}(b), pb \notin \text{pl}(R_P), \\ \{d\} & \text{if } pb \in l(R_P) \text{ or if } tgt(p) \neq \text{src}(b), \end{cases} \)

Dump state \( \delta(d, x) := \{d\} \),

\( \delta(d, b) := \{d\} \).

The extended state transition function \( \delta^* \) is such that the intersection of \( \delta^*(s_0, t) \) with \( Q_B \) is non-empty if and only if \( t \) is an element of \( \Sigma^* \) which is not an element of \( T_B \) or is reducible.

Thus for each object \( B \in \text{Ob}\Delta \), and automaton \( A_B \) can be constructed, where \( L(A_B) = \Sigma^* - \text{irr}(T_B) \). The results quoted in Section 3 allow us to make \( A_B \) deterministic (Lemma 3.2) and take its complement. The language \( K_B \) recognised by the resulting automaton \( (A_B)^{DC} \) is \( \Sigma^* - (\Sigma^* - \text{irr}(T_B)) \), i.e. \( K_B := \text{irr}(T_B) \). Hence (by Theorem 3.6) \( K_B \) is regular. Since \( R \) is a complete rewrite system on \( T \) there exists a unique irreducible term in each class of \( T_B \) with respect to \( \xrightarrow{*} R \). Therefore the set \( \text{irr}(T_B) \) is bijective with \( T_B/ \xrightarrow{*} R = K_B \).

The automaton \( (A_B)^{DC} \) gives rise to a system of right linear language equations (Theorem 3.5) with a unique solution, which is a regular expression for the language \( K_B \) accepted by the automaton. The regular expression can be obtained by applying Arden’s Theorem (Theorem 3.3) to solve the language equations. Given that each set \( K_B \) is bijective with a regular language \( K_B \), the action is described as follows: let \( t \in K_B \) and \( b : B_1 \rightarrow B_2 \) for \( B_1, B_2 \in \text{Ob}\Delta \), then \( \text{irr}(t \cdot b) \in K_{B_2} \).

\( \square \)
Thus for each object $B \in \text{Ob}\Delta$, an automaton $A_B$ is constructed, and a regular expression for the set $KB$ is obtained from solving the language equations of the determinised complement of $A_B$. The $P$-action on the elements $t$ of $T$ is right multiplication followed by reduction with respect to $\rightarrow_R$. This describes the functor $K$ in terms of regular expressions over $\Sigma^*$. The natural transformation $\varepsilon$ is given by $\varepsilon_A(x) := \text{irr}(x | id_{F_A})$ for all $A \in \text{Ob}A$ and $x \in XA$.

Therefore we have shown how the induced action $(K, \varepsilon)$ may be described in terms of regular languages and the reduction relation $\rightarrow_R$.

5 Example

We construct simple automata which accept the terms which represent elements of some set $KB$ for $B \in \text{Ob}B$ for an example of a Kan extension. The generating graphs are

$$
\begin{array}{c}
A_1 \xrightarrow{a_1} A_2 \\
| a_2 |
\end{array} 
\quad 
B_1 \xrightarrow{b_1} B_2 
\quad 
\begin{array}{c}
B_3 \xrightarrow{b_3} B_2 \xleftarrow{b_2}
\end{array}
$$

The relations are $\text{Rel}B = \{b_1 b_2 b_3 = b_4\}$, $X$ is defined by $XA_1 = \{x_1, x_2, x_3\}$, $XA_2 = \{y_1, y_2\}$ with $Xa_1 : XA_1 \rightarrow XA_2 : x_1 \mapsto y_1, x_2 \mapsto y_2, x_3 \mapsto y_1$, $Xa_2 : XA_1 \rightarrow XA_2 : y_1 \mapsto x_1, y_2 \mapsto x_2$, and $F$ is defined by $FA_1 = B_1$, $FA_2 = B_2$, $FA_1 = b_1$ and $FA_2 = b_2 b_3$.

The completed rewrite system is:

$$
\begin{align*}
x_1 | b_1 & \rightarrow y_1 | id_{B_2}, \\
x_2 | b_1 & \rightarrow y_2 | id_{B_2}, \\
x_3 | b_1 & \rightarrow y_1 | id_{B_2}, \\
y_1 | b_2 b_3 & \rightarrow x_1 | id_{B_1}, \\
y_1 | b_2 b_3 & \rightarrow x_1 | id_{B_1}, \\
y_2 | b_2 b_3 & \rightarrow x_2 | id_{B_1}, \\
x_1 | b_4 & \rightarrow x_1 | id_{B_1}, \\
x_2 | b_4 & \rightarrow x_2 | id_{B_1}, \\
x_3 | b_4 & \rightarrow x_1 | id_{B_1}, \\
b_1 b_2 b_3 & \rightarrow b_4.
\end{align*}
$$

The proper prefix sets are $\text{ppl}(R_T) := \{y_1 | b_2, y_2 b_2\}$ and $\text{ppl}(R_P) := \{b_1, b_1 b_2\}$. The following table defines the incomplete non-deterministic automaton which rejects only the terms of $T$ that are irreducible with respect to the completed relation $\rightarrow$. The alphabet over which the automaton is defined is $\Sigma := \{x_1, x_2, x_3, y_1, y_2, b_1, b_2, b_3, b_4, b_5\}$.
By constructing the transition tree for this automaton, we will make it deterministic. The next picture is of the partial transition tree – the arrows to the node marked \{d\} are omitted.

The tree is constructed with respect to the order on $\sqcup X A$ and Arr$_\Delta$, all arrows are drawn from $\{s_0\}$ and then arrows from each new state created, in turn. When a label e.g. $\{B_3\}$ occurs that branch of the tree is continued only if that state has not been defined previously. Eventually the stage is reached where no new states are defined, all the branches have ended. The tree is then converted into an automaton by ‘gluing’ all states of the same label. The initial state is $\{s_0\}$ and a state is terminal if its label contains a terminal state from the original automaton. The automaton can often be made
smaller, for example, here all the terminal states may be glued together. One possibility is drawn below:

Here the state labelled 1, i.e. $S_1$ corresponds to the glueing together of $\{x_1\}$, $\{x_2\}$ and $\{x_3\}$ to form $\{x_1, x_2, x_3\}$ and the state $S_2$ is $\{y_1, y_2, b_1, B_2\}$. States $S_3$ and $S_4$ represent $\{B_3\}$ and $\{B_1\}$ respectively and state $S_5$ is $\{y_1, b_2, y_2, b_2, B_3, b_1 b_2\}$. The complement of this automaton accepts all irreducible elements of $\sqcup K B$. When $S_1$ and $S_4$ are terminal the language accepted is $K_{B_1}$. When $S_2$ is terminal the language accepted is $K_{B_2}$. When $S_3$ and $S_5$ are terminal the language accepted is $K_{B_3}$. The language equations from the automaton for $K_{B_1}$ are:

$$X_0 = (x_1 + x_2 + x_3)X_1 + (y_1 + y_2)X_2,$$
$$X_1 = b_5 X_3 + \text{id}_{B_1},$$
$$X_2 = b_2 X_5,$$
$$X_3 = b_3 X_4,$$
$$X_4 = b_1 X_2 + b_4 X_4 + b_5 X_3 + \text{id}_{B_1},$$
$$X_5 = \emptyset.$$

Putting $X_2 = \emptyset$ and eliminating $X_1$ and $X_3$ by substitution gives

$$X_0 = (x_1 + x_2 + x_3)(b_5 b_3 X_4 + \text{id}_{B_1}),$$
$$X_4 = (b_4 + b_5 b_3) X_4 + \text{id}_{B_1}.$$

Finally, applying Arden’s Theorem to $X_4$ we obtain the regular expression

$$X_0 = (x_1 + x_2 + x_3)(b_5 b_3 (b_4 + b_5 b_3)^* + \text{id}_{B_1}).$$

The separator “$\mid$” may be added at this point. Similarly, we can obtain regular expressions for $K_{B_2}$ and $K_{B_3}$. For $K_{B_2}$ we have

$$X_0 = (x_1 + x_2 + x_3)b_5 b_3 (b_4 + b_5 b_3)^* b_1 + (y_1 + y_2) \text{id}_{B_2}.$$

For $K_{B_3}$ the expression is

$$X_0 = (x_1 + x_2 + x_3)(b_5 b_3 (b_4 + b_5 b_3)^*(b_1 b_2 + b_3) + b_5) + (y_1 + y_2)|b_2.$$
References

[1] R.Brown and A.Heyworth: “Using Rewrite Systems to Compute Kan Extensions and Induced Actions of Categories”, UWB Math Preprint 98.14 (submitted JSC)

[2] M.R.Bush, M.Leeming and R.F.C.Walters : “Computing Left Kan Extensions”, Journal of Symbolic Computation, 11 p11-20 1997

[3] R.V.Book and F.Otto : “String-Rewriting Systems”, Springer-Verlag, New York, 1993

[4] S.Carmody and R.F.C.Walters : “The Todd-Coxeter Procedure and Left Kan Extensions”, Research Reports of the School of Mathematics and Statistics, The University of Sydney p90-19, 1990 with M.Leeming : Journal of Symbolic Computation, 19 p459-488 1995

[5] S.Carmody and R.F.C.Walters : “Computing Quotients of Actions on a Free Category”, in A.Carboni, M.C.Pedicchio, G.Rosolini (eds), Category Theory, Proceedings of the Int. Conf. Como, Italy 22-28 July 1990, Springer-Verlag 1991

[6] D.E.Cohen : Introduction to Computer Theory, Revised Edition, New York : Wiley 1991

[7] D.B.A.Epstein, J.W.Cannon et al: “Word Processing in Groups”, Boston : Jones and Bartlett Publishers 1992

[8] M.Fleming, R.Gunther and R.Rosebrugh : “User Guide for the Categories Database and Manual”, anonymous ftp://sun1.mta.ca/pub/papers/rosebrugh/catdsalg.dvi, tex and /catuser.dvi, tex 1996

[9] A.Heyworth: “Applications of Rewriting Systems and Gröbner Bases to Computing Kan Extensions and Identities Among Relations”, PhD thesis, UWB Math Preprint 98.23, 1998 http://xxx.soton.ac.uk/abs/math.CT/9812097

[10] D.F.Holt : “Knuth-Bendix in Monoids, and Automatic Groups”, Mathematics Institute, University of Warwick 1996

[11] D.F.Holt and D.F.Hurt : “Computing Automatic Coset Systems and Subgroup Presentations”, Journal of Symbolic Computation 1996

[12] J.Hopcroft and J.Ullman: “Introduction to Automata Theory, Languages and Computation”, Addison-Wesley, Reading, MA 1979

[13] S.Mac Lane : “Categories for the Working Mathematician”, Springer-Verlag 1971

[14] B.Mitchell : “Rings with many objects”, Academic Press vol.8 no.1 1972

[15] T.Mora : “Gröbner Bases and the Word Problem”, University of Genova 1987

[16] I.D.Redfern : “Automatic Coset Systems”, PhD thesis, University of Warwick, 1993

[17] F.Baader and T.Nipkow : “Term Rewriting and All That”, Cambridge University Press 1998