Non-perturbative statistical theory of intermittency in ITG drift wave turbulence with zonal flows

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Abstract
The probability distribution functions (PDFs) of momentum flux and zonal flow formation in ion-temperature-gradient (ITG) turbulence are investigated in two different models. The first is a general five-field model \((n_i, \phi, T_i, T_e, v_i)\) where a reductive perturbation method is used to derive dynamical equations for drift waves and a zonal flow. The second is a reduced two-field model \((\phi, T_i)\) that has an exact non-linear solution (bipolar vortex soliton). In both models the exponential tails of the zonal flow PDFs are found with the same scaling \((PDF \sim \exp\{-c_{ZF}\phi^3\})\), but with different coefficients \(c_{ZF}\). The PDFs of momentum flux is, however, found to be qualitatively different with the scaling \((PDF \sim \exp\{-c_{MR}s\})\), where \(s = 2\) and \(s = 3/2\) in the five and two-field models, respectively.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction
One of the main challenges in magnetic fusion research has been the prediction of the turbulent heat and particle transport originating from various micro-instabilities. The ion-temperature-gradient (ITG) mode is one of the main candidates for causing the anomalous heat transport in core plasmas of tokamaks [1]. Significant heat transport can, however, be mediated by coherent structures such as streamers and blobs through the formation of avalanche like events of large amplitude, as indicated by recent numerical studies [2–4]. These events cause the deviation of the probability distribution functions (PDFs) from a Gaussian profile on which the traditional mean field theory (such as transport coefficients) is based. A crucial question in plasma confinement is thus the prediction of the PDFs of the transport due to these structures and of their formation [5–9]. Note that a renormalized perturbative theory can easily make a considerably large error in predicting PDFs where the coherent structures are crucial.

In previous papers the Hasegawa–Mima model of drift wave turbulence was used to show that the PDF tails of global momentum flux and heat flux are significantly enhanced over the Gaussian prediction [5, 6]; this was later shown to hold also in ITG mode turbulence [7–9]. Specifically, the tails of PDF of global momentum flux and heat flux have been shown to be stretched exponentially with the form \(\sim \exp\{-c(R/R_0)^{3/2}\}\) [5–9], which is broader than a Gaussian. These results provided a novel explanation for exponential PDF tails of momentum flux found in recent experiments at CSDX at UCSD [10]. Note that considerable transport can be mediated by rare events of high amplitude at the PDF tails even if the latter have a low amplitude. Since the PDF tails of momentum flux are enhanced over the Gaussian prediction in this case, these events are more likely to mediate considerable transport. In fact, it was found that for certain values of parameters, large events are crucial for transport. Specifically, the overall amplitude was shown to be larger in ITG than in HM turbulence for reversed modon speed \((U < 0)\). The PDF tail of zonal flow in ITG turbulence was also significantly increased compared with that in HM turbulence. Furthermore, zonal flows are shown to be generated more likely further from marginal stability, which will then regulate ITG turbulence, leading to a self-regulating system. Namely, while ITG turbulence is a state with a high level of heat flux, it also generates stronger zonal flows that inhibit transport. This also suggested that stronger zonal flows are generated in ITG turbulence compared with ETG turbulence. It was also shown that shear flows can significantly reduce the PDF tails of Reynolds stress and zonal flow formation [5–9].
The purpose of this paper is to investigate the likelihood of the formation of coherent structures by computing the PDF tails of zonal flow formation and the PDFs of momentum flux [11] by using the two different ITG turbulence models. The first model is a widely applicable five-field model \((n_i, \phi, T_i, T_e, \nu_i)\) of the ITG turbulence coupled to an ion vorticity equation for the zonal flows. For the computation of the PDFs, a reductive perturbation method is used to obtain dynamical equations for drift waves and zonal flows. Due to the perturbation method, this model is only valid for weak drift wave and zonal flow electric potential. Moreover an exact non-linear solution in this model is not available that can be used as an ansatz for a coherent structure for the intermittent transport. However, the computation of the PDF tails requires only mean values over the coherent structures. The second is the two-field \((\phi, T_i)\) model which has an exact non-linear bipolar vortex soliton (modon) solution that will be utilized in computing PDFs. Note that the two-field fluid model for the ITG mode [12–15] has been successful in reproducing both experimental [16] and non-linear gyro-kinetic results [17].

The theoretical technique used here is the so-called instanton method, a non-perturbative way of calculating the PDF tails. The PDF tail is first formally expressed in terms of a path-integral by utilizing the Gaussian statistics of an external forcing with a short correlation time. An optimum path will then be associated with the creation of a coherent structure (among all possible paths) and the action is evaluated using the saddle-point method. In the second model this coherent structure is identified as the modon. The saddle-point solution of the dynamical variable \(\phi(x,t)\) of the form \(\phi(x,t) = F(t)\psi(x)\) with \(F(t)\) which is localized in time is called an instanton. Since the instanton exists during the formation of the coherent structure, the bursty event can be associated with the creation of a coherent structure. Note that the function \(\psi(x)\) here represents the spatial form of the coherent structure.

One of our main results is that the PDF tail of the zonal flow formation is \(\exp(-c_{PDF}/\phi_{PDF}^2)\), \((j = 1, 2\) to denote the different constants in the two models), in agreement with earlier findings [9]. Interestingly the predicted PDF tails of momentum flux are Gaussian when the feedback of zonal flow on the turbulence is incorporated. This result differs from what has theoretically been found earlier [9] whereas it is in agreement with predictions from non-linear simulations of turbulence [19, 20] and gyro-kinetic toroidal simulations [21, 22]. The reason for this is that previously the zonal flow has been treated as passively evolving by the turbulence without the feedback loop to turbulence. This feedback gives a cubic non-linearity in equation (7) describing the ITG fluctuations which changes the characteristics of the turbulence. Physically, it is because the feedback of zonal flow regulates turbulence, limiting its growth from turbulence.

The results from both models support the conclusion that while ITG turbulence maintains a high level of transport, this may be suppressed by shear flow. Zonal flows are also shown here to have an enhanced likelihood of the generation further from marginal stability which will then regulate the ITG turbulence (which is more prominent with increased shear flow) leading to a self-regulating system.

The paper is organized as follows. In section 2 the drift wave–zonal flow system is presented and in sections 3 and 4 the computation of the PDF and a numerical study of the five-field model are presented. In sections 5 and 6 the PDF tails are computed and a numerical study of the reduced two-field model is outlined. The paper concludes with a discussion and summary in section 7.

2. The drift wave–zonal flow system

In the five-field model the ITG mode turbulence is modelled using the continuity and temperature equations for the ions and assuming the electrons to be Boltzmann distributed, closely following previous papers [12–15]. The ion continuity, temperature and parallel ion momentum equations then become

\[
\frac{\partial n_i}{\partial t} - \nabla \cdot (\rho_{ei} \mathbf{v}_{ei}) = 0
\]

\[
\frac{\partial T_i}{\partial t} = \frac{\rho_{ei}}{\rho_{e}} \frac{\partial \mathbf{v}_{ei}}{\partial t}
\]

\[
\frac{\partial \mathbf{v}_{ei}}{\partial t} = -\frac{\partial \mathbf{E}}{\partial x}
\]

Equations (1)–(3) are closed by using the quasi-neutrality condition. Here \([A, B] = (\partial A/\partial x)(\partial B/\partial y) - (\partial A/\partial y)(\partial B/\partial x)\) is the Poisson bracket; \(\nu = 0.78 \times 10^{-12} (n_0/T_i^{3/2}) (r/R)\) is a neoclassical damping: \(f_0\) is a forcing: \(n = (L_n/r) \delta n / n_0\), \(\phi = (L_n/r) \delta \phi / T_i\), \(T_i = (L_n/r) \delta T_i / T_0\) and \(v_i = (L_n/r) \delta v_{ei} / c_s\) are the normalized ion particle density, the electrostatic potential, the ion temperature and the ion parallel velocity respectively. In equations (1) and (2), \(\tau = T_i / T_e\), \(\rho_s = c_s / \Omega_i\) \((\Omega_i = \sqrt{T_e / m_i}\), \(\Omega_{ei} = eB / mc)\). We also used \(L_f = -d \ln f / dr\) \((f = [n, T_i])\), \(\eta_i = L_n / L_T\), \(\epsilon_{ei} = 2L_n / R\), where \(R\) is the major radius and \(\alpha_i = \tau (1 + \eta_i)\). The perpendicular length scale and time are normalized by \(\rho_s\) and \(L_n / c_s\), respectively. The geometrical quantities are calculated in the strong ballooning limit (\(\theta = 0, g(\theta = 0) = 1.0\)) with \(\omega_s = k_s v_s = \rho_s c_s k_s / L_n\). In the ion temperature equation (equation (2)) we have included a heat source \(\delta T_i\) given in the appendix. For the zonal flow equation (3) vanishes \((k_i = 0)\) and we cannot use Boltzmann distributed electrons. Thus we will employ the following ion vorticity equation (equation (4)) and simple electron energy equation (equation (5))

\[
\frac{\partial}{\partial t} - \nabla \cdot (\rho_{ei} \mathbf{v}_{ei}) + e \mathbf{E} \cdot \mathbf{j} = 0
\]

\[
\frac{\partial T_e}{\partial t} = -n_e \frac{\partial \mathbf{v}_{ei}}{\partial y}
\]
Here $\eta_c = L_n/L_T$. The damping (with strength $\nu$) represents a neo-classical damping of the zonal flow. Note that the damping will affect the full dynamical system in equations (7), (8). The simple electron energy equation is motivated by the fact that we are interested in the modes propagating in the ion drift direction.

In order to find the coupled system of equations for the ITG mode and the zonal flow we utilize the reductive perturbation method wherein the perturbed variable $\sigma$ is written

$$
\sigma(x, y, t) = \sum_n \sum_{l=0}^{2} \epsilon_0 (l+2/3) a_l^{(l+2/3)}(x, \xi, \zeta) \times \exp[i(l(k_xz + k_yy - \omega t)) + c.c.
+ \sum_n \epsilon_0 (l+2/3) a_n^{(l+2/3)}(x, \xi, \zeta),
(6)
$$

Here the coordinates are scaled as $\xi = \epsilon_0^{2/3}(y - \lambda t)$ and $\zeta = \epsilon_0^{4/7} t$ with a small expansion parameter $\epsilon_0 \sim \epsilon_0^{2/3} \sim 10^{-2}$.

The drift wave–zonal flow system then becomes

$$
C_1 \frac{\partial \tilde{\phi}_1}{\partial \zeta} + iC_2 \frac{\partial^2 \tilde{\phi}_1}{\partial \xi^2} + C_3 \tilde{\phi}_0 \tilde{\phi}_1 = -i\nu C_4 \tilde{\phi}_1 + f,
(7)
$$

$$
D_1 \frac{\partial \tilde{\phi}_0}{\partial \zeta} + D_2 \frac{\partial \tilde{\phi}_0}{\partial \xi} = D_3 \frac{\partial |\tilde{\phi}_1|^2}{\partial \xi}.
(8)
$$

Here, the coefficients $C_1$, $C_2$, $C_3$, $C_4$, $D_1$, $D_2$ and $D_3$ are complex numbers whose forms are provided in the appendix; the variables with tilde and bar denote average over $x$. Note that if the time derivatives are neglected in equations (7), (8), a non-linear Schrödinger (NLS) equation for the fluctuating potential is obtained [15].

3. Non-perturbative calculation of zonal flow PDF in the five-field model

We calculate the PDF tails of momentum flux and zonal flow formation by using the instanton method. To this end, the PDF tail is expressed in terms of a path-integral by utilizing the Gaussian statistics of the forcing $f$ [18]. The PDF of Reynolds stress and zonal flow formation can be defined as

$$
P(Z) = \langle \delta(Z_0 - Z) \rangle = \int d\lambda \exp(i\lambda Z_0)\exp(-i\lambda Z_0)) = \int d\lambda \exp(i\lambda Z_0) I_\lambda,
(9)
$$

where

$$
I_\lambda = \langle \exp(-i\lambda Z_0) \rangle.
(10)
$$

$I_\lambda$ in (9)–(10) can then be rewritten in the form of a path-integral as

$$
I_\lambda = \int \tilde{D}\tilde{\phi}_1 \tilde{D}\tilde{\phi}_1 \tilde{D}\tilde{\phi}_0 \tilde{D}\tilde{\phi}_0 e^{-S_\lambda}.
(11)
$$

Here $\tilde{\phi}_0$ and $\tilde{\phi}_1$ are conjugate variables to $\tilde{\phi}_0$ and $\tilde{\phi}_1$, respectively. In the following we assume that the zonal flow potential is stationary $\partial \tilde{\phi}_0/\partial t = 0$ on the time-scale of the fluctuations. The effective action $S_\lambda$ in equation (11) can then be expressed as

$$
S_\lambda = -i \int d\xi d\zeta \tilde{\phi}_1 \left( C_1 \frac{\partial^2 \tilde{\phi}_1}{\partial \zeta^2} + iC_2 \frac{\partial^3 \tilde{\phi}_1}{\partial \xi^2 \partial \zeta} + C_3 \tilde{\phi}_0 \tilde{\phi}_1 + i\nu C_4 \tilde{\phi}_1 \right)
+ \frac{1}{2} \int d\zeta d\xi d\zeta \frac{\partial \tilde{\phi}_1}{\partial \zeta} \frac{\partial \tilde{\phi}_1}{\partial \xi} \frac{\partial \tilde{\phi}_1}{\partial \zeta}
+ i\lambda_2 \int d\xi d\zeta \tilde{\phi}_0(\zeta) \frac{\partial \tilde{\phi}_1}{\partial \zeta}
- i \int d\zeta d\xi d\zeta \frac{\partial \tilde{\phi}_0(\zeta)}{\partial \zeta} (D_2 \tilde{\phi}_0 - D_3 |\tilde{\phi}_1|^2).
(12)
$$

Recall $\xi$ and $\zeta$ are the scaled spatial and time variables, respectively. To obtain equation (12) we have assumed the statistics of the forcing $f$ to be Gaussian with a short correlation time modelled by the delta function as

$$
(f(\xi, \zeta) f(\xi', \zeta')) = \delta(\xi - \xi') \delta(\zeta - \zeta'),
(13)
$$

and $\langle f \rangle = 0$. The delta correlation in time was chosen for the simplicity of the analysis. In the case of a finite correlation time the non-local integral equations in time are needed.

4. The PDF tails and numerical studies for the five-field model

We have expressed the PDF in terms of a path-integral. An approximate value of this path-integral can be found for large values of the parameter $\lambda \rightarrow \infty$, by using a saddle-point method. The action in equation (12) can be expressed using the instanton ansatz $\tilde{\phi}_1(\xi, \zeta) = \psi_1(\xi) \tilde{F}(\zeta)$ and $\tilde{\phi}_0(\xi, \zeta) = \psi_0(\xi) G(\zeta)$ as

$$
S_\lambda = -i \int d\xi \left( C_1 \tilde{F} \tilde{\phi}_1 + iC_2 k_2 \tilde{F} \tilde{\phi}_1 + C_3 \tilde{F} \tilde{\phi}_1 + i\nu C_4 \tilde{\phi}_1 \right)
+ \frac{k_0}{2} \int d\zeta \left( \tilde{F}_1^2 + \tilde{F}_2^2 \right)
+ i\lambda_2 \int d\xi G(\zeta) \frac{\partial \tilde{\phi}_0}{\partial \xi}
- i \int d\xi \tilde{F}_3(D_2 G - D_3 \tilde{\phi}_1^2).
(14)
$$

Here $k_0$ is the strength of the forcing $x$. The parameter $k_0 (\in C)$ is the inverse length scale in the $\xi$ direction. In equation (14) we have defined the variables,

$$
\psi_0 = \int d\xi \psi_0, \quad \tilde{\phi}_1 = \int d\xi \tilde{\phi}_1, \quad \tilde{F}_1 = \int d\xi \tilde{F}_1, \quad \tilde{F}_2 = \int d\xi \tilde{F}_1, \quad \tilde{F}_3 = \int d\xi \tilde{\phi}_0 \psi_0, \quad \tilde{F}_4 = \int d\xi \tilde{\phi}_0 \psi_1.
(15)
$$

Note that the relation between the conjugate variables $\tilde{F}_3$ and $\tilde{F}_4$ is based on the fact that the zonal flow is driven by the Reynolds stress and also that the parameter $R (\in C)$ has to be
determined in such a way to make Reynolds stress real. Since the saddle-point action is determined by the extremum of the action, we require the first functional derivatives of the action to vanish;

\[ \frac{\delta S_{\lambda}}{\delta F} = -i(-C_1 \dot{F}_1 - iC_2 k_2^2 \ddot{F}_1 + C_3 \ddot{F}_2 G + iv C_4 \dot{F}_1) - 2i\lambda R D_3 F \dot{F}_1 = 0, \]  

(20)

\[ \frac{\delta S_{\lambda}}{\delta F_1} = -i(C_1 \dot{F} - iC_2 k_2^2 F + iv C_4 F) + \kappa_0 \ddot{F}_1 = 0, \]  

(21)

\[ \frac{\delta S_{\lambda}}{\delta F_2} = -iC_3 F G + \kappa_0 \ddot{F}_2 = 0, \]  

(22)

\[ \frac{\delta S_{\lambda}}{\delta G} = D_2 G - R D_3 F^2 = 0, \]  

(23)

\[ \frac{\delta S_{\lambda}}{\delta \Phi_0} = -iC_3 \dot{F}_2 F - i\dot{F}_2 D_2 + i\dot{\lambda} \Phi_0 \delta(\xi) = 0. \]  

(24)

The initial conditions \( F(0) = F_0 \) and \( G(0) = G_0 \) are found from equations (23) to (24) as,

\[ F_0 = \left( \frac{-iD_2 \Phi_0}{C_3^2 R D_3} \right)^{1/4} \lambda^{1/4}, \]  

(25)

\[ G_0 = \frac{R D_3}{D_2} \left( \frac{-iD_2 \Phi_0}{C_3^2 R D_3} \right)^{1/2} \lambda^{1/2}. \]  

(26)

The instanton solution is found from equations (20) to (24) for \( \xi < 0 \). Starting with equation (20) and then by substituting the conjugate variables in equations (21)–(24) we obtain

\[ \ddot{F} = \frac{1}{2} \frac{d}{dF} \ddot{F}^2 = \eta_1 F + 3\eta_2 F^5, \]  

(27)

where

\[ \eta_1 = \frac{1}{C_1^2} (-C_2 k_2^4 + 2C_2 C_4 k_2^2 - iv C_3^2), \]  

(28)

\[ \eta_2 = \frac{1}{C_1^2} \left( \frac{C_3^2 R^2 D_3^2}{D_2} \right). \]  

(29)

The integration of equation (27) in time gives us the instanton solution as,

\[ F(\xi) = \sqrt{\frac{2\eta_1 H}{1 - \eta_1 \eta_2 H^2}}, \]  

(30)

\[ H(\xi) = H_0 \exp[2\sqrt{\eta_1} \xi], \]  

(31)

\[ H_0 = \frac{F_0^2}{\eta_1 + \sqrt{\eta_1} \sqrt{\eta_2 F_0^2 + \eta_1}}, \]  

(32)

where \( F_0 \) is given in equation (25). The next task is to compute the action integrals and estimate the \( \lambda \) dependence of \( S_\lambda \) in the large \( \lambda \rightarrow \infty \) limit. We start by substituting equations (20)–(23) into the action in equation (14) keeping only the highest powers of \( \lambda \),

\[ S_\lambda \approx i \int_{-\infty}^{\infty} d\xi \left( C_1^2 \dot{F}^2 + C_3^2 R^2 D_3^2 F^4 \right) + i\lambda \Phi_0 \frac{R D_3}{D_2} F_0^2 \]  

(33)

\[ \approx i \int_0^{F_0} d\xi F(2C_1^2 \dot{F} + i\lambda \Phi_0 \frac{R D_3}{D_2} F_0^2 \]  

(34)

\[ = \frac{i\lambda^3/2 \Phi_0}{D_2} \frac{R D_3}{C_3^2 R D_3} \right)^{1/2} \]  

(35)

Here,

\[ h_{ZF} = \Phi_0 \frac{R D_3}{D_2} \left( \frac{-iD_2 \Phi_0}{C_3^2 R D_3} \right)^{1/2}. \]  

(36)

The PDF is then found from equation (9) by utilizing the saddle-point method

\[ P(Z) = \int d\lambda e^{-i\lambda Z - S_\lambda}, \]  

(37)

\[ = \int d\lambda e^{-i\lambda Z - ih_{ZF} \lambda^{3/2}}. \]  

(38)

We find a \( \lambda_0 \) such that \( f(\lambda_0) = -i\lambda Z - ih_{ZF} \lambda^{3/2} \) attains its maximum and compute that value. This gives \( \lambda_0 = (2Z/3h_{ZF})^2 \) and

\[ f(\lambda_0) = -\frac{4}{27h_{ZF}} Z^3, \]  

(39)

with the PDF

\[ P(Z) \sim e^{-\Theta Z^2}, \]  

(40)

\[ \Theta = \frac{4}{27h_{ZF}}. \]  

(41)

The PDF tails found in the five-field model (equation (41)) has the same exponential form as in [9]. Although no exact non-linear solution was used as an ansatz for the coherent structure the system, we still obtain the PDFs with the same exponential form. To elucidate the quantitative differences in the results between the two and five-field models we show the PDF tails by using different values of temperature gradient \( (\eta_1) \) in figures 1 and 3.

In figure 1 the PDF tails of zonal flow formation obtained in the five-field model (equation (40)) is presented for \( \eta_1 = 2.0 \) (red, dashed line), \( \eta_1 = 4.0 \) (blue, solid line) and \( \eta_1 = 6.0 \) (black, dashed–dotted line). Other parameter values are \( \tau = 2.0, \epsilon_u = 1.0, \gamma = 1, k_2 = 0.3, k_3 = 0.3 \) and \( \tau_0 = 0.0, \kappa_0 = 3000 \). It is interesting to note that the PDF tails of zonal flow formation in figure 1 are suppressed in comparison with the results found in the reduced two-field model shown in figure 3(a) for similar parameter values.

Note that the usefulness of the predicted PDF of zonal flow formation above is limited since the coefficient \( \Theta \) in equation (40) is dependent on complex coefficients \( C_1, C_2, C_3, C_4, D_1, D_2 \) and \( D_3 \) from the original system of equations (7), (8). As is seen in the appendix due to the
The analysis can easily be extended to find PDFs of the momentum flux in the system of equations (7), (8), with the action $S_\lambda$

$$S_\lambda = -i \int d\xi (C_1 \tilde{F}\tilde{F}_1 + i C_2 k_1^2 F\tilde{F}_1 + C_3 G F \tilde{F}_2 + i v C_4 F \tilde{F}_1)$$

$$+ \frac{\kappa_0}{2} \int d\xi (\tilde{F}_1^2 + \tilde{F}_2^2)$$

$$+ i \lambda R_0 \int d\xi F(\tilde{\xi})^2 \delta(\tilde{\xi})$$

$$- i \int d\xi \tilde{F}_3 (D_2 G - R D_1 F^2),$$

(42)

where $R_0$ is the Reynolds stress

$$R_0 = - \int d\xi \left( \psi \frac{\partial \psi}{\partial \xi} \right).$$

(43)

Note that an average over $x$ has already been taken and that the momentum flux is non-zero since there is a natural phase shift introduced by the electron physics. A similar equation for the instanton can be found, but with different values of $F$ and $G$ at $t = 0$,

$$F_0 = \frac{2i \kappa_0 \Phi_1}{C_1^2} \sqrt{\lambda},$$

(44)

$$G_0 = \frac{R D_2 2i \kappa_0 \Phi_1}{C_1^2} \sqrt{\lambda}.$$

(45)

The PDF tail of momentum flux can then be computed by following a similar analysis with the result

$$P(R) \sim \exp \left\{ - \Theta_M \left( \frac{R}{R_0} \right)^2 \right\},$$

(46)

$$\Theta_M = \frac{1}{2h_M},$$

(47)

where

$$h_M = \frac{1}{4} \sqrt{\pi \tau} + \frac{2\kappa_0}{C_1^2}.$$  

(48)

Interestingly the exponential forms of the PDFs of momentum flux are qualitative different from what was found in ITG turbulence [7–9] and in Hasegawa–Mima turbulence [5, 6] which was the stretched PDF $\sim \exp({-c R^{3/2}})$.

Specifically the predicted PDF tails of momentum flux are Gaussian when zonal flow feeds back on the turbulence. This result differs from what has theoretically been found earlier [9] where the zonal flow is treated kinematically. It is, however, in agreement with predictions from non-linear simulations of turbulence [19, 20] and gyro-kinetic toroidal simulations [21, 22]. The reason for this is the cubic non-linearity in equation (7) describing the ITG fluctuations while previously the zonal flow was treated as passively evolving by the turbulence. Physically, it is because the feedback of zonal flow regulates turbulence, limiting its growth from turbulence. It is interesting that although our model for turbulence and zonal flows is expected to have a limited region of validity due to the perturbation expansion, it captures the main features of what has been found in numerical simulation of plasma turbulence with zonal flows.

Note that non-linear models with cubic non-linearities for turbulent fluctuations were obtained using perturbation theory and are thus valid only for small values of the amplitude (small $\psi_i$). The statistical property of the fluctuations differs radically in systems with cubic and quadratic non-linearities as will be discussed in more detail in section 7. The predictive power of the method used here should be able to discriminate between models since the statistics of the fluctuations and zonal flow may change qualitatively. We note that in ITG turbulence an evolution equation for the fluctuations with a cubic non-linearity has previously been found [23].

5. The PDF tails in the reduced model two-field model

Due to the perturbation method, the five-field model considered in sections 3 and 4 is only valid for weak drift wave and zonal flow electric potential. Moreover an exact non-linear solution in this model is not available that can be used as an ansatz for a coherent structure for the intermittent transport. We will thus now consider the reduced two-field model where a non-linear solution (modon) can be found and no reductive perturbation expansion is needed. It is a reduced model where the ITG mode turbulence is modelled using the continuity and temperature equation for the ions with adiabatic electrons [8, 9]. In this model the effects of parallel ion motion, magnetic shear, trapped particles and finite beta on the ITG modes are neglected since they were shown to be not critical in previous work. Unlike the five-field model, the generation of a zonal flow is treated passively while the background fluctuations are affected by an imposed sheared velocity $V_0$. We note that the effects of mean flow on the zonal flow are weak [26, 27] while the effects of mean flow on the turbulence itself are much more prominent.

We formally calculate the PDF tails of momentum flux and zonal flow structure formation by using the instanton.
Figure 2. The PDF tail of momentum flux in ITG turbulence incorporating the effects of shear flow for \( V_0 = 0.0 \) (a), \( V_0 = 5.0 \) (b), \( V_0 = 10.0 \) (c) and \( V_0 = 15.0 \) (d) for \( \eta_i = 2.0 \) (red, dashed line), \( \eta_i = 4.0 \) (blue, solid line) and \( \eta_i = 6.0 \) (black, dashed–dotted line). (Colour online.)

Figure 3. The PDF tail of zonal flow formation in ITG turbulence incorporating the effects of shear flow for \( V_0 = 0.0 \) (a), \( V_0 = 5.0 \) (b), \( V_0 = 10.0 \) (c) and \( V_0 = 15.0 \) (d) for \( \eta_i = 2.0 \) (red, dashed line), \( \eta_i = 4.0 \) (blue, solid line) and \( \eta_i = 6.0 \) (black, dashed–dotted line). (Colour online.)

method. The PDF of Reynolds stress \( Z_1 = R \) or zonal flow structure formation \( Z_2 = \phi_{ZF} \) can be defined as in a generalized equation (9) where

\[
I_{\lambda_j} = \langle \exp(-i\lambda_j Z_j) \rangle. \tag{49}
\]

The integrand can then be rewritten in the form of a path-integral as

\[
I_{\lambda_j} = \int D\phi D\bar{\phi} D\phi_{ZF} D\bar{\phi}_{ZF} e^{-S_j}. \tag{50}
\]
Here, the parameter $j$ refers to the two specific cases included in this study. The angular brackets denote the average over the statistics of the forcing $f$. This is the same forcing as in equation (1). By using the ansatz $T_j = \chi \phi$, the effective action $S_{\phi_j}$ in equation (11) can be expressed as,

$$S_{\phi_j} = -\int d^2x \, dt \, \phi \left( \frac{\partial \phi}{\partial t} - (\frac{\partial}{\partial t} - \alpha \frac{\partial}{\partial y}) \nabla^2 \phi + V_0(1 - \nabla^2)\phi \right)$$

$$+ (1 - \epsilon_n) \beta \frac{\partial \phi}{\partial y} - \beta(\phi, \nabla^2 \phi)$$

$$+ \frac{i}{2} \int d^2x \, d^2y \, \phi(x) \phi(x') \delta(t)$$

$$+ \int d^2x \, dt \, \phi_{22} \delta(t)$$

$$+ \int d^2x \, dt \, \phi_{22}(\phi_{22} + R_0(v_x, v_y)).$$

In equation (51),

$$\beta = 1 + \frac{1}{\tau} \frac{1}{\tau} \chi,$$

$$\chi = \frac{\eta_n - \frac{1}{2}(1 - U + V_0)}{U - V_0 + \frac{1}{\tau} \epsilon_n \gamma},$$

$$R_0 = \int d^2x \left( \frac{1}{\nabla^2} \phi \frac{\partial \phi}{\partial y} \right).$$

The PDF tails are found by calculating the value of $S_{\phi_j}$ at the saddle-point in the two cases; the PDF tail of momentum flux by taking into account the effect of a shear flow ($\lambda_1 \to \infty$, $\lambda_2 = 0$) and the PDF tail of structure formation of zonal flow ($\lambda_1 = 0$, $\lambda_2 \to \infty$). The integral in equation (51) is divided into five parts; $K_1$ the ITG integral; $K_2$ the forcing integral; $K_3$ the momentum flux integral; $K_4$ the zonal flow integral and finally $K_5$ represents the zonal flow evolution integral. The action in the first case can be found as,

$$S_{\phi_j} \simeq -\frac{1}{3} i h \lambda_1^3,$$

$$h = K_1 + K_2 + K_3 + K_4 + K_5,$$

$$K_1 = \frac{1}{2 \lambda_0^2} \left( \gamma' \gamma (1 + \epsilon^2) \left( \frac{4 \lambda_0}{H_0 - 1} - 3 \right)^{\frac{3}{2}} - \frac{C}{A} \right),$$

$$K_2 = \frac{1}{2 \lambda_0^2} \left( \gamma' \gamma (1 + \epsilon^2) \left( \frac{4 \lambda_0}{H_0 - 1} - 3 \right)^{\frac{3}{2}} - \frac{C}{A} \right),$$

$$K_3 = R_0 F^2(0),$$

$$K_4 = 0,$$

$$K_5 = \frac{2 \sqrt{C}}{A} \frac{1}{H_0 - 1},$$

$$H_0 = 4A - 2, \quad A = \frac{\eta}{(1 + \epsilon^2)^2}.$$
6. Numerical results of the reduced two-field model

We have presented a theory of the PDF tail of structure formation and how the PDF tail of momentum flux is modified by the presence of an imposed shear flow \( V_0 \). The exponential forms of the two PDF tails are completely different, signifying the difference in the physical interpretation. In the case of structure formation the PDF tails are found as \( \sim \exp(-\xi_1 \phi_{ZF}^l) \), while the momentum flux PDF tail \( \sim \exp(-\xi_2 \phi_{ZF}^l) \). The origin of these scalings are the quadratic non-linearity in the dynamical system (equations (1), (2)).Mathematically, the difference in scaling of the PDF tails momentum flux and zonal formation comes from the difference in the scaling of the initial condition \( F_0 \) with the large parameter \( \lambda \). The spatial structure of the modon is of less importance in determining the exponent \( V \) and \( \xi \) line). When solutions are not Galilean invariant \([28]\).

In summary, this paper presents the first prediction of the PDF tail of structure formation \( \sim \exp(-\xi_1 \phi_{ZF}^l) \) where a reductive perturbation method is used to derive dynamical equations for drift waves and a zonal flow is studied. Interestingly the predicted PDF tails of momentum flux in the five-field model of ITG turbulence are Gaussian when the feedback of zonal flow on the turbulence is incorporated. This result differs from what has theoretically been found earlier in \([9]\) and found in the reduced two-field model of ITG turbulence in section 5 whereas it is in agreement with predictions from non-linear simulations of turbulence \([19, 20]\) and in gyrokinetic toroidal simulations \([21, 22]\). Mathematically it is due to the different highest non-linearity between the two models (i.e. cubic in the five-field model and quadratic in the two-field model). Physically, it is because in the two-field model the feedback of zonal flow on the fluctuations was not treated self-consistently, which will regulate turbulence, inhibiting its growth.

Note that non-linear models with cubic non-linearities for the turbulence fluctuations were obtained perturbatively and thus valid only for small values of the amplitude (small \( \phi_1 \)). The statistical property of the fluctuations differs in systems with cubic and quadratic non-linearities. To elucidate these differences it is instructive to discuss a general formula for PDF tails of any moment in non-linear systems derived previously \([24]\). By using the instanton method, the PDF of the \( n \)th moment was shown to be

\[
P(Z) \sim \exp(-cZ^n),
\]

\[
s = \frac{n + 1}{m}
\]

in the case where \( n \) is the highest non-linearity in a dynamical system.

For instance, in the case of the five-field model, \( n = 3 \) (equation (9)) while in the case of the two-field model, \( n = 2 \) (equation (1)). For PDFs of momentum flux, \( m = 2 \) while for PDFs of zonal flow formation \( m = 1 \). According to equations (72), (73) there is a significant difference in the PDF tails of the first moment in a system with cubic non-linearity where \( s = 4 \) and one with a quadratic non-linearity where \( s = 3 \) \([25]\). For weak zonal flow equation (7) becomes linear and we expect Gaussian statistics \((s = 2)\) for the fluctuations. Note that for the five-field model, equations (72), (73) suggest that the PDF tails of zonal flow formation are determined by the quadratic non-linearity in the dynamical equation for the zonal flow whereas the cubic non-linearity determines the PDF tails of momentum flux.

Finally, in the second part a reduced two-field model \((\phi, T_i)\) that has an exact non-linear solution (bipolar vortex soliton) of ITG turbulence and zonal flows are studied to compute the PDFs of zonal flow formation and momentum flux. One of the important results from the numerical study in this reduced two-field model, which is also supported in the five-field model, is that shear flows can significantly reduce the PDF tails of momentum flux and zonal flow formation. Since zonal flows are more likely to be generated further from marginal stability, they will then regulate ITG turbulence, leading to a self-regulating system. Namely, while ITG turbulence is a state with a high level of heat flux, it also generates stronger zonal flows that inhibit transport.
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Appendix A. Coefficients

We have balanced the non-linear transport due to the imaginary parts of \( \alpha (\Im(\omega)) \) (A.15) with the source term \( S_{ti} \):

\[
S_{ti} + 2i k_y \left( \Im(\omega) \frac{\partial}{\partial x} (\phi_1^{(5/3)} a_1^{(1/3)} - \phi_1^{(5/3)} a_1^{(1/3)}) \right) - \frac{1}{2} \left( \frac{\partial a}{\partial x} \frac{\partial (\phi_1^{(1/3)} a_1^{(1/3)} - \phi_1^{(1/3)} a_1^{(1/3)})}{\partial x} - \frac{\partial (\phi_1^{(1/3)} a_1^{(1/3)} - \phi_1^{(1/3)} a_1^{(1/3)})}{\partial x} \right) = 0. \quad (A.11)
\]

\[
C_1 = -k_y^3 (1 + \tau) e_n (3 \tau^2 e_n [10 e_n + 3 \tau (2 - 3 \eta)] + \tau (2 - 3 \eta)) \lambda^2 + 45 k_y^2 e_n (5 [1 + \tau] e_n \lambda^2 + \tau (2 - 3 \eta)) \lambda^2 + 9 \tau^2 \lambda (3 \tau + 6 \eta + 5 + 3 \tau \lambda) \lambda^3, \quad (A.2)
\]

\[
C_2 = k_y^2 (1 + \tau) e_n \omega [5 k_y e_n + 3 \tau (2 - 3 \eta)] \lambda^2 + 30 k_y (-1 + D_1^2 \tau) e_n + 9 \tau^2 \lambda^2 + k_y^2 [5 k_y^2 e_n + 3 \tau \lambda + \tau (2 - 3 \eta)] \lambda^2 + 6 k_y \tau (5 [1 + \tau] e_n + \tau (2 - 3 \eta)) \lambda^2 + 9 \tau^2 \lambda (3 \tau + 5 + 3 \tau \lambda) \lambda^3, \quad (A.3)
\]

\[
C_3 = \frac{1}{L_x} (C_{NL1} k_n - 4 k_y^3 C_{NL2}) + C_{NL3} (k_y^2 + l_2^3 k_y) k_n - 4 k_y^3 k_n, \quad (A.4)
\]

\[
C_{NL1} = \frac{1}{\lambda (5 e_n + 3 \tau \lambda)} (k_y^2 l_{2} \tau e_n \omega^2 (5 k_y e_n + 3 \tau \lambda)) \lambda^2 \left( k_y (k_y [50 e_n \lambda - 3 \tau^2 [10 e_n + 3 (2 - 3 \eta)]) \lambda^2 + 9 \tau^3 (2 - 3 \eta) \lambda^2 - 10 \tau e_n [3 \lambda^2 + 5 e_n (\eta + \lambda)] - 3 \tau^2 (10 e_n \eta + 3 \tau) (-2 + 3 \eta)) \lambda \omega \right) + \omega (k_y^2 e_n [175 e_n \lambda + 9 \tau^3 (2 - 3 \eta) \lambda^2 + 35 \tau e_n [3 \lambda^2 + 5 e_n (\eta + \lambda)] + 3 \tau^3 (2 - 3 \eta) \lambda, + 5 e_n (5 \eta + 7 \eta) + 3 k_y (5 k_y e_n) + 3 \tau^2 (10 e_n + 3 \tau) \lambda) \omega + 9 \tau^2 (5 e_n + 3 \tau \lambda) \lambda + \tau (10 e_n + 3 \tau \lambda) \omega + 9 \tau^2 (5 e_n + 3 \tau \lambda) \lambda \omega^2) \right] \lambda \omega \omega^2 (5 k_y e_n + 3 \tau \lambda) \omega + 3 \tau (5 e_n + 3 \tau \lambda) \omega \omega^2). \quad (A.5)
\]

\[
C_{NL2} = \frac{1}{(5 e_n + 3 \tau \lambda)} (k_y^2 l_{2} \tau (1 + \tau + \tau \lambda) \omega^2 (5 k_y e_n + 3 \tau \lambda) \omega + 3 \tau (5 e_n + 3 \tau \lambda) \omega \omega^2), \quad (A.6)
\]

\[
C_{NL3} = D_{1} k_y^2 \tau (1 + \tau) e_n \omega^2 (5 k_y e_n + 3 \tau \lambda) \omega + 3 \tau (5 e_n + 3 \tau \lambda) \omega), \quad (A.7)
\]

\[
C_{d} = D_{1} k_y^2 \tau (1 + \tau) e_n \omega^2 (5 k_y e_n + 3 \tau \lambda) \omega, \quad (A.8)
\]

\[
D_1 = \frac{1}{(1 + \tau) e_n \omega \lambda (5 e_n + 3 \tau \lambda)} \lambda (25 e_n^2 \eta e_n + 5 \tau e_n [6 \eta e_n - 5 D_1^2 \lambda + 9 D_1^2 \tau \lambda^2 (1 + 2 \eta \tau \lambda) + 5 D_1^2 [7 (1 + e_n) + (2 - 3 \tau \lambda)]) \lambda \omega + 5 e_n^2 \lambda [3 \tau (2 + \tau (2 + 5 e_n - 3 \eta) - 3 \eta)] + 5 D_1^2 (7 (1 + e_n) + (2 - 3 \tau \lambda)) \lambda]) \lambda), \quad (A.9)
\]

\[
D_2 = -3 (D_1^2 \tau \lambda^3 (1 + \eta + \tau \lambda) + 3 e_n^2 [\eta e_n (1 + \tau)]) \lambda^2 + e_n^2 \lambda (7 + \tau (7 + 10 e_n - 3 \eta) - 3 \eta + 5 D_1^2 (1 + \eta) \lambda) + \tau [5 D_1^2 - 3 (1 + \tau \lambda)]) \lambda^4 + e_n^2 \lambda^3 (3 \tau (1 + \tau + \tau \lambda) + 5 D_1^2 (1 + \eta + \tau \lambda)) \lambda \lambda \lambda \lambda = 1 \lambda (1 + \tau) e_n \lambda, \quad (A.10)
\]

\[
D_3 = \frac{1}{L_x} k_y^2 l_{2} (D_{NL1} + D_{NL2}), \quad (A.11)
\]

\[
D_{NL1} = -\tau (1 + \tau) e_n \omega^2 (5 e_n + 7 + 3 \tau \lambda), \quad (A.12)
\]

\[
D_{NL2} = 3 e_n k_y \eta \lambda \left( \frac{\partial a}{\partial k_y} \right) \omega, \quad (A.13)
\]

\[
\frac{\partial a}{\partial k_y} = \frac{\omega}{k_y} - \lambda, \quad (A.14)
\]

\[
\alpha = \frac{\omega}{5 e_n + 3 \tau \lambda}, \quad (A.15)
\]

\[
\lambda = \frac{\omega}{k_y} \omega - \frac{2 \lambda}{k_y \lambda \omega}, \quad (A.16)
\]

\[
\lambda_n = \omega k_y^2 \left( \omega \left(1 + \frac{5}{3 \tau} \right) + k_y \Gamma \right), \quad (A.17)
\]

\[
\lambda_d = \left( k_y^2 \left[ \left(1 + \frac{5}{3 \tau} \right) \omega \right] - \frac{k_y}{2} \left( 1 - \epsilon_n \left[ 1 + \frac{5}{3 \tau} + \alpha \omega \right] \right) \right. \left. - k_y^2 \Gamma - \frac{\epsilon_n^2}{2 \eta} \left( 1 + \frac{5}{3 \tau} \right) \right), \quad (A.18)
\]

\[
\alpha \omega = \frac{5}{3 \tau}, \quad (A.19)
\]

\[
\Gamma = \frac{1}{\tau} \left( 1 - \frac{2}{3} + \frac{5}{3 \tau} \epsilon_n \left( 1 + \frac{4}{\tau} \right) \right), \quad (A.20)
\]

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