Static spacetimes with prescribed multipole moments: a proof of a conjecture by Geroch

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Abstract
In this paper we give sufficient conditions on a sequence of multipole moments for a static spacetime to exist with precisely these moments. The proof is constructive in the sense that a metric having prescribed multipole moments up to a given order can be calculated. Since these sufficient conditions agree with already known necessary conditions, this completes the proof of a long standing conjecture due to Geroch.

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1. Introduction
There are various definitions of multipole moments [9]. For static asymptotically flat spacetimes, the standard definition was given by Geroch [2], and this definition was later extended to the stationary case by Hansen [4]. In essence, the recursive definition by Geroch from [2] is a non-trivial modification of the Newtonian definition taking place in a conformally rescaled spacetime, see section 2. Since the conformal factor involved is not unique, the relativistic definition must, apart from taking into account the curvature terms, ensure invariance under the conformal freedom available.

In [2], Geroch made two conjectures, namely ($i^0$ will be defined shortly)

Conjecture 1. Two static solutions of Einstein’s equations having identical multipole moments coincide, at least in some neighborhood of $i^0$.

and

Conjecture 2. Given any set of multipole moments, subject to the appropriate convergence condition, there exists a static spacetime of Einstein’s equations having precisely those moments.

Conjecture 1 was proved in [1] and although many properties of the multipoles are known, see e.g. [9], conjecture 2 has not been yet fully proven. There are partial results however. For
instance, it is known [10] that an arbitrary sequence of multipole moments uniquely defines formal power series of relevant field variables, and that if the series converge, this gives a static spacetime having these moments. Thus, the moments determine the spacetime formally, but the appropriate convergence conditions were not addressed in [10]. In [3], on the other hand, growth conditions are given on a set referred to as \textit{null data}, which ensures the existence of a static spacetime possessing these null data. The null data are shown to be in a one-to-one correspondence with the Geroch multipole moments (with non-zero monopole), but since this correspondence is rather implicit, no growth condition on the actual multipole moments to ensure convergence is derived. On the ‘necessary side’, the situation is more satisfactory. In [8], it was shown that the multipole moments of any asymptotically flat stationary, and therefore static, spacetime do not grow ‘too fast’, and precise conditions were given.

The purpose of this paper is to show that the necessary conditions in [8] are also sufficient, i.e. if they are satisfied, there exists a static asymptotically flat vacuum spacetime with these moments. Therefore, this will complete the proof of Geroch’s conjecture 2 above. The proof in section 3 will include an explicit recursion of the desired metric, which means that the metric for a static spacetime with prescribed moments up to an arbitrary order can be calculated. We will first prove that the metric cast in a special form is uniquely defined by the moments, at least formally, and then show that the result from [3] can be used to deduce convergence of the series for the metric components.

One major obstacle has been to explicitly link a certain metric and potential with a given set of multipole moments. This problem goes back to the actual definition of the moments, i.e. the recursive relation (2), where the operation of ‘taking the totally symmetric and trace-free part’ effectively obscures the relation between the metric, the potential and the moments, unless some care is taken. Another issue is the coordinate freedom. Without extra restrictions on the metric it is impossible to uniquely derive the metric components from the moments since one can always change coordinates and hence the components of the metric. However, as will be shown in the following sections, these two issues can be addressed simultaneously.

We also remark that the sufficient conditions given do not contain the usual requirement that the mass term (monopole) \( m \) is non-zero, although that is a condition which may be required for physical reasons.

2. Multipole moments of stationary spacetimes

Although in this paper, we address a conjecture concerning static spacetimes, we have chosen to formulate the conjecture within the stationary setting. One reason is that we will refer to results from [8], where this setting is used. Therefore, in this section we quote the definition of multipole moments given by Hansen in [4], which is an extension to stationary spacetimes of the definition for static spacetimes by Geroch [2]. We thus consider a stationary spacetime \((M, g_{ab})\) with signature \((-+, +, +, +)\) and with a timelike Killing vector field \(\xi^a\). We let \(\lambda = -\xi^a \xi_a\) be the norm, and define the twist \(\omega\) through \(\nabla_a \omega = \epsilon_{abcd} \xi^b \nabla^c \xi^d\). If \(V\) is the 3-manifold of trajectories, the metric \(g_{ab}\) induces the positive definite metric \(h_{ab} = \lambda g_{ab} + \xi_a \xi_b\) on \(V\). It is required that \(V\) is asymptotically flat, i.e. there exists a 3-manifold \(\hat{V}\) and a conformal factor \(\Omega\) satisfying

(i) \(\hat{V} = V \cup i^0\), where \(i^0\) is a single point;
(ii) \(\hat{h}_{ab} = \Omega^2 h_{ab}\) is a smooth metric on \(\hat{V}\);
(iii) At \(i^0\), \(\Omega = 0, \hat{D}_a \Omega = 0, \hat{D}_a \hat{D}_b \Omega = 2\hat{h}_{ab}\);
where $D_a$ is the derivative operator associated with $\hat{h}_{ab}$. $i^0$ is referred to as spacelike infinity. On $M$, and/or $V$, one defines the scalar potential

$$\phi = \phi_M + i\phi_J, \quad \phi_M = \frac{\lambda^2 + \omega^2 - 1}{4\lambda}, \quad \phi_J = \frac{\omega}{2\lambda}. \quad (1)$$

The multipole moments of $M$ are then defined on $\hat{V}$ as certain derivatives of the scalar potential $\hat{\phi}$ at $i^0$. More explicitly, following [4], let $\hat{R}_{ab}$ denote the Ricci tensor of $\hat{V}$, and let $P = \hat{\phi}$. Define the sequence $P, P_{a_1}, P_{a_1a_2}, \ldots$ of tensors recursively:

$$P_{a_1, \ldots, a_n} = C \left[ \hat{D}_{a_1} P_{a_2, \ldots, a_n} - \frac{(n-1)(2n-3)}{2} \hat{R}_{a_1a_2} P_{a_3, \ldots, a_n} \right], \quad (2)$$

where $C[\cdot]$ stands for taking the totally symmetric and trace-free part. The multipole moments of $M$ are then defined as the tensors $P_{a_1, \ldots, a_n}$ at $i^0$.

In [2], a slightly different setup and a different potential is used, but it is known [10] that the potential used there and (1) with $\omega = 0$ produce the same moments.

3. Static spacetimes with prescribed multipole moments

In this section, we will formulate and prove the desired theorem. The theorem will be as conjectured in [8], i.e. in essence that if the sequence of multipole moments is naturally connected to a harmonic function on $\mathbb{R}^3$, there exists a static asymptotically flat vacuum spacetime having precisely those moments. Namely (cf theorem 8 in [8]), consider $\mathbb{R}^3$ with the Cartesian coordinates $r = (x, y, z) = (x^1, x^2, x^3)$, and let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a multi-index. With the convention that, in terms of components, $P_{a_1}^{\alpha_1} = P_{a_1 \cdots a_n}^{\alpha_1 \cdot \cdots \cdot \alpha_n}$, we have the following theorem.

**Theorem 1.** Let $P_0^{a_1}, P_0^{a_1a_2}, \ldots$ be a sequence of real valued totally symmetric and trace-free tensors on $\mathbb{R}^3$, and let $u(r) = \sum_{|\alpha| \geq 0} \sum_{\alpha_1 \cdots \alpha_n} P_0^{\alpha_1 \cdots \cdot \alpha_n}$ converges in a neighbourhood of the origin in $\mathbb{R}^3$, there exists a static asymptotically flat vacuum spacetime having the moments $P_0^{a_1}, P_0^{a_1a_2}, \ldots$.

Note that we do not require the monopole $P_0^{a_1} = \hat{\phi}(0)$ to be non-zero. The proof, however, will be first carried out under the assumptions $P_0^{a_1} \neq 0$, and this condition will then be relaxed in section 3.8. The proof for the case $P_0^{a_1} \neq 0$ will be carried out in a sequence of lemmas (lemma 1–11), which also show that the metric up to a given order can be calculated explicitly, but first we will give an outline of the proof, discuss the notation and formulate the relevant field equations.

3.1. Outline of the proof

As mentioned in section 1, it is possible to reduce the coordinate freedom in the metric components, and simultaneously establish a direct link between the potential $\hat{\phi}$ and the desired moments. This will be done in section 3.4, where the link is expressed in theorem 2.

In section 3.5, we will address the conformal field equations from section 3.3, and see that the form of the metric given in section 3.4 results in singular equations. By requiring that

\footnotesize
1 $i^0$ is also used in a four-dimensional context.
2 Due to the definition of $\phi$, $P_0^{a_1} = -m$, where $m$ is the mass of the spacetime.

the conformal field equations are smooth at $i^0$, further restriction will be put on the rescaled metric $\hat{h}_{ij}$, which then takes its final form.

With the form of the metric fixed, we will show in section 3.6 that the field equations determine the metric components as a formal power series. More precisely, we will show that when the monopole is non-vanishing, a certain subset of the field equations is sufficient to specify the metric completely.

In section 3.7 we will address the full set of equations, as well as the issue of convergence of the power series derived. By referring to a result by Friedrich [3], convergence of the power series will be concluded. It will also be seen that the full set of equations are satisfied.

Finally, in section 3.8, we will relax the condition that the monopole is non-zero.

3.2. Notation

Small Latin letters $a, b, \ldots$ refer to abstract indices, as in section 2. Small Latin letters $i, j, k, \ldots$ are numerical indices and refer to components with respect to the normal coordinates $(x, y, z) = (x^1, x^2, x^3)$ introduced below. Since these components refer to this particular coordinate system only, the equations will not be tensor equations. With this said, we will still use $\dot{=} = \dot{1}$.

Almost all variables are assumed to be formally analytic, i.e. they admit a formal power series expansion (again in terms of the chosen coordinates), so that, for a tensor, $A_{ijk}$ say, we can write

$$A_{ijk} = \sum_{n=0}^{\infty} A_{ijk}^{[n]}$$

where $A_{ijk}^{[n]}$ denotes polynomials in $(x, y, z)$ which are homogeneous of order $n$ (and where the summation may be formal). Both $\eta_{ij}$ and $\eta^{ij}$ denote the identity matrix, and by $[A_{ij}]$ we denote the trace of $A_{ij}$, i.e. $\eta^{ij} A_{ij}$. We also use $\dot{=} = \frac{1}{\sqrt{\lambda}}$.

With $r = \sqrt{x^2 + y^2 + z^2}$, $f_1 \equiv f_2 (\mod r^2)$ means that $f_1(x, y, z) - f_2(x, y, z) = r^2 g(x, y, z)$ for some (formally analytic) function $g$. When $f \equiv 0 (\mod r^2)$, so that $f = r^2 g$ for some formally analytic function $g$, we also use the shorter notation $r^2 | f$.

In the proof of lemma 11 we will refer to [3] and hence use some of the notation there.

3.3. The field equations

Apart from having the correct multipole moments, we must ensure that the metric describes a static vacuum spacetime. We will formulate our equations on the 3-manifold $(\tilde{V}, \tilde{h}_{ab})$ defined in section 2, starting with the field equations from [2]. However, the 3-manifolds in [2] and [4] are defined slightly differently, and the relations imply the following.

Starting with a static spacetime $(M, g_{ab})$ with the timelike killing vector $\xi^a$, we put $0 < \lambda = -\xi^a \xi_a$ and $\Psi = 1 - \sqrt{\lambda}$. From [2] we consider a 3-surface $V_G$ orthogonal to $\xi^a$. In terms of the metric on $V_G$: $(h_G)_{ab} = g_{ab} + \xi_a \xi_b / \lambda$, the field equations are [2]

$$\begin{cases}
(D_G)^a(D_G)_a \Psi = 0 \\
(R_G)_{ab} = \frac{1}{\lambda} (D_G)_a (D_G)_b \Psi
\end{cases} \Leftrightarrow \begin{cases}
(R_G)_a = 0 \\
(R_G)_{ab} = \frac{1}{\lambda} (D_G)_a (D_G)_b \sqrt{\lambda}
\end{cases}$$

where $(D_G)_a$ is the derivative operator and $(R_G)_{ab}$ is the Ricci tensor associated with $(V_G, (h_G)_{ab})$.

3 The exception is $r = \sqrt{x^2 + y^2 + z^2}$.

4 In [2], the potential $\psi = -\Psi$ is used.
On the 3-manifold \( V \) on the other hand, the metric is \( h_{ab} = \lambda g_{ab} + \xi_a \xi_b \), i.e. \( h_{ab} = \lambda (h_G)_{ab} \), and with \( \hat{h}_{ab} = \Omega^2 h_{ab} \), this implies that \( \hat{h}_{ab} = (\sqrt{\Omega})^2 (h_G)_{ab} \). We can now express equations (3) on \( (\hat{V}, \hat{h}_{ab}) \) using as the conformal factor \( \hat{\Omega} = \sqrt{\Omega} \). Using the properties of conformal transformations [12], we find that (3) becomes

\[
\begin{align*}
\hat{R} + 4\hat{h}^{ab} \hat{D}_a \hat{D}_b \ln \hat{\Omega} - 2\hat{h}^{ab} \hat{D}_a \ln (\hat{\Omega}) \hat{D}_b \ln \hat{\Omega} &= 0, \\
\hat{\hat{R}}_{ab} + \hat{D}_a \hat{D}_b \ln \hat{\hat{\Omega}} + \hat{h}_{ab} \hat{R}^{cd} \hat{D}_c \hat{D}_d \ln \hat{\hat{\Omega}} &= 0, \\
\hat{D}_a \ln (\hat{\hat{\Omega}}) \hat{D}_b \ln \hat{\hat{\Omega}} - \hat{\hat{h}}_{ab} \hat{R}^{cd} \hat{D}_c \hat{D}_d \ln (\hat{\hat{\Omega}}) &= 0, \\
\hat{D}_a \ln (\hat{\hat{\Omega}}) \hat{D}_b \sqrt{\lambda} \hat{D}_a \sqrt{\lambda} + \hat{D}_a \ln (\hat{\hat{\Omega}}) \hat{D}_b \sqrt{\lambda} &= 0, \\
\hat{D}_b \ln (\hat{\hat{\Omega}}) \hat{D}_a \sqrt{\lambda} - \hat{\hat{h}}_{ab} \hat{R}^{cd} \hat{D}_c \hat{D}_d \ln (\hat{\hat{\Omega}}) \hat{D}_c \sqrt{\lambda}. 
\end{align*}
\]  

Thus, we are looking for a metric \( \hat{h}_{ab} \) on \( \hat{V} \), defined in a neighbourhood of \( i^0 \), which satisfies (4) and where the corresponding four-dimensional spacetime has prescribed multipole moments. As mentioned earlier, equation (4) cannot have a unique solution in terms of coordinates. This is not only due to the fact that one can always represent a metric in different coordinate systems but also because there is a freedom in the conformal factor. In the next section we will put the metric \( \hat{h}_{ab} \) in a canonical form, i.e. use a preferred coordinate system. This coordinate system is also constructed so that it allows specification of the multipole moments, i.e. puts (2) in a form which is more transparent.

### 3.4. Prescribed moments and canonical form of the metric

In this section we will, in a sense, reverse the arguments from [8]. The key point is to work in normal coordinates, in which the Ricci tensor \( \hat{R}_{ab} \) takes a special form. This will lead to theorem 2, which allows for a direct connection between the potential \( P = \phi \) and the multipole moments.

Let us therefore first introduce normal coordinates \((x, y, z) = (x^1, x^2, x^3)\) on \( \hat{V} \), where the point \( i^0 \) has coordinates \( 0 \), and such that \((x, y, z)\) ‘are Cartesian’ at \( i^0 \), i.e. in terms of the coordinates, \( \hat{h}_{ij}(i^0) = \eta_{ij} \). By assumption, \( \hat{h}_{ab} \) and (therefore \( \hat{R}_{ab}, \hat{R} \)) are already the conformal factor \( \hat{\Omega} \) are at this stage formal power series, while \( u \) from theorem 1 is given by a power series which converges in a neighbourhood of \( 0 \), \( u \) is defined on \( \mathbb{R}^3 \), but in terms of the normal coordinates introduced, the sum \( \sum_{|\alpha| \geq 0} \frac{\lambda}{\alpha!} P_\alpha \) can also be interpreted on \( \hat{V} \), namely as the potential function \( \hat{\phi}(r) \).

We now turn to the condition on \( \hat{R}_{ab} \). The condition is formulated in [8] through the complexification \( \hat{V}_C \) of \( \hat{V} \), and by using complex null vectors \( \eta^a \), i.e. vectors \( \eta^a \eta_a = 0 \). Using normal coordinates, it was shown in [8] that for any fixed \( \psi \), the complex curve

\[
\gamma_\psi : t \rightarrow (t \cos \psi, t \sin \psi, it)
\]

takes the tangent vector \( \eta^a = \eta^a_\psi(t) = \cos \psi \left( \frac{\partial}{\partial t} \right)^a + \sin \psi \left( \frac{\partial}{\partial x^a} \right)^a + i \left( \frac{\partial}{\partial x^a} \right)^a \) which satisfies \( \eta^a \eta_a = 0 \). The relevant condition on \( \hat{R}_{ab} \) is then to be found in lemma 6 of [8], where the condition \( \eta^a \eta^b \hat{R}_{ab} = 0 \) was used. With our notation this reads \( \eta^a \eta^b \hat{\hat{R}}_{ab} = 0 \), and it is this condition, together with the normality of the coordinates, which allows for a direct connection between the potential \( \phi \) and the moments (theorems 7 and 8 in [8], theorem 2 below). Although the results in [8], which build on the techniques developed in [5–7], depend crucially on complex quantities (especially the concept of ‘leading term’), it is interesting to note that all of these tools and arguments can be given in purely real terms. Thus, rather than referring to, and reversing, the arguments in [8], we will derive/translate the corresponding conclusion using only real quantities.
The first property is the analog of lemma 1c in [8], namely, with \( \eta^a \) as above, that
\[
\eta^a, \ldots, \eta^n T_{a_1,\ldots,a_n} = \eta^a, \ldots, \eta^n C[T_{a_1,\ldots,a_n}],
\]
although we here state it in a vector space using the radius vector.

**Lemma 1.** Let \((x, y, z)\) be Cartesian coordinates on \( \mathbb{R}^3 \), let \( r^d = x\left(\frac{\partial}{\partial x}\right)^d + y\left(\frac{\partial}{\partial y}\right)^d + z\left(\frac{\partial}{\partial z}\right)^d \) and put \( r = \sqrt{x^2 + y^2 + z^2} \). Then, for any tensor \( T_{a_1,\ldots,a_n} \), \( r^d, \ldots, r^n C[T_{a_1,\ldots,a_n}] \equiv r^d, \ldots, r^n T_{a_1,\ldots,a_n} \mod r^2 \).

**Proof.** \( C[T_{a_1,\ldots,a_n}] \) is constructed from \( T_{a_1,\ldots,a_n} \) by taking the totally symmetric part and subtracting suitable amounts of the tensors \( \eta_{a(a_1),b(a_2),\ldots} T_{a_1,\ldots,a_n}^{bc,\ldots} \ldots \) where \( \eta_{ab} \) is the Euclidean metric on \( \mathbb{R}^3 \). Thus, it is clear that \( C[T_{a_1,\ldots,a_n}] = \eta_{a(a_1),\ldots} T_{a_1,\ldots,a_n}^{bc,\ldots} \ldots \) for some tensor \( S_{a_1,\ldots,a_n} \). Since \( r^d r^e \eta_{ab} = r^2 \), the statement follows by transversing with \( r^d, \ldots, r^n \) and observing that \( r^d, \ldots, r^n T_{a_1,\ldots,a_n} = r^d, \ldots, r^n T_{a_1,\ldots,a_n} \). \( \square \)

This property can also be realized in the following way. It is clear that \( r^d, \ldots, r^n T_{a_1,\ldots,a_n} \) is a homogeneous polynomial \( p = p(x, y, z) \) of degree \( n \). Using spherical coordinates in \( \mathbb{R}^3 \), i.e. \( x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta \), we express \( p \) in spherical harmonics \( Y^m_l(\theta, \phi) \) and get \( p = \sum_{m=0}^{\infty} \sum_{l=0}^{m} r^m c_l^m Y^m_l(\theta, \phi) \) if \( n \) is even. If \( n = 2 - 2 \) or less, \( \sum_{m=0}^{\infty} \sum_{l=0}^{m} r^m c_l^m Y^m_l(\theta, \phi) = r^m \sum_{m=0}^{\infty} \sum_{l=0}^{m} r^m c_l^m Y^m_l(\theta, \phi) \) with \( r^m \) smooth and divisible by \( r^2 \), and where the sum is a smooth polynomial. This means that \( r^d, \ldots, r^n C[T_{a_1,\ldots,a_n}] \) corresponds to the polynomial \( p_C = \sum_{m=n}^{\infty} r^m c_m^m Y^m_m(\theta, \phi) \), i.e. only the terms corresponding to \( l = n \) are kept. Identical remarks hold if \( n \) is odd.

Using normal coordinates, the property of lemma 1 can be carried over to the manifold \( \hat{V} \). Namely, from [11], we have the following lemma, which we formulate for the particular case when \( \hat{h}_{ij}(i^0) = \eta_{ij} \).

**Lemma 2.** \((x, y, z) = (x^1, x^2, x^3)\) are normal coordinates on (\( \hat{V}, \hat{h}_{ab} \)), if and only if \( x^i \hat{h}_{ij}(x^2) = x^i \hat{h}_{ij}(i^0) \), i.e. \( x^i \hat{h}_{ij} = x^i \eta_{ij} \).

Thus, in normal coordinates, the above lemma implies that \( x^i x^j \hat{h}_{ij} = x^2 + y^2 + z^2 = r^2 \).

This means that by the same arguments as above
\[
x^i, \ldots, x^n C[T_{i_1,\ldots,i_n}] \equiv x^i, \ldots, x^n T_{i_1,\ldots,i_n} \mod r^2 \text{ in } \hat{V}.
\]

**Remark 1.** That (5) is still true in \( \hat{V} \) depends on the fact that we have \( x^i x^j \hat{h}_{ij} = r^2 \). However, in lemma 1, we had a fixed tensor \( T_{a_1,\ldots,a_n} \) on a vector space, while on \( \hat{V} \), \( T_{a_1,\ldots,a_n} \) is a tensor field. This does not affect equality (5), but when evaluating at \( i^0 \), where \((x^1, x^2, x^3) = (0, 0, 0) \), it just says \( o = 0 \). Another effect is that \( x^i, \ldots, x^n T_{i_1,\ldots,i_n} \) is a polynomial which is the sum of a homogeneous polynomial of degree \( n \) and a polynomial containing only higher order terms. However, by replacing \( x^i \) with \( \rho^i = x^i / r \), so that \( \rho^i \rho^j \hat{h}_{ij} = 1 \), both \( \rho^i, \ldots, \rho^n C[T_{i_1,\ldots,i_n}] \) and \( \rho^i, \ldots, \rho^n T_{i_1,\ldots,i_n} \) will be direction-dependent quantities at \( i^0 \). In the limit \( r \to 0 \), the higher order terms vanish and we get
\[
\rho^i, \ldots, \rho^n T_{i_1,\ldots,i_n}(i^0) = \sum_{l=0, l \text{ even}}^{n} \sum_{m=-l}^{l} c_m^l Y^m_l(\theta, \phi) \Rightarrow
\]
\[
\rho^i, \ldots, \rho^n C[T_{i_1,\ldots,i_n}(i^0)] = \sum_{m=-n}^{n} c_m^n Y^m_n(\theta, \phi),
\]
if \( n \) is even; and a corresponding relation when \( n \) is odd \((l = 1, 3, 5, \ldots, n) \). This last equality also tells us that \( C[T_{i_1,\ldots,i_n}] \) transversed with \( x^1, \ldots, x^n \) or \( \rho^i, \ldots, \rho^n \) still contains the full information, since there are \( 2n + 1 \) degrees of freedom in the rhs.
In view of lemma 1 and the corresponding property (5) on \( \hat{V} \), it is obvious that 
\( r^a r^b \hat{R}_{ab} \equiv 0 \) (mod \( r^2 \)), i.e. that \( r^2 | r^a r^b \hat{R}_{ab} \) is a desirable property in order to simplify (2) since \( r^a r^b \hat{R}_{ab} \) will then not affect the multipole moments. As it turns out, this condition can be formulated purely in algebraic terms. This will be done in lemma 6 below. To prepare for this lemma, we need some more tools.

Lemma 3. Suppose that \( A = A_{ij} = A_{(ij)} \) are the components of a symmetric tensor field \( A_{ab} \) on \( \hat{V} \) with respect to normal coordinates \( (x, y, z) = (x^1, x^2, x^3) \) and that \( A_{ij} \) has the property that \( r^2 | r^a r^b A_{ij} \), \( A_{ij} x^i = 0 \). Then \( A \) is uniquely decomposable as

\[
A = f_1(x, y)A_1 + f_2(y, z)A_2 + f_3(x, z)A_3 + f_4(x, z)A_4 + f_5(x, y, z)A_5 + f_6(x, y, z)A_6 + f_7(x, y, z)A_7,
\]

where

\[
A_1 = \begin{pmatrix}
-\frac{y^2 - z^2}{2} & xy & xz \\
x y & -x^2 - z^2 & yz \\
x z & y z & -x^2 - y^2 \\
\end{pmatrix},
\]

\[
A_2 = \begin{pmatrix}
2z(y^2 + z^2) & -xyz & -xy(y^2 + 2z^2) \\
-xyz & 0 & y^2 y \\
-x(y^2 + 2z^2) & y^2 y & 2x^2 z \\
\end{pmatrix},
\]

\[
A_3 = \begin{pmatrix}
2xz^2 & yz^2 & -(2x^2 + y^2)z \\
y z^2 & 0 & -xyz \\
-(2x^2 + y^2)z & -xyz & 2x(x^2 + y^2) \\
\end{pmatrix},
\]

\[
A_4 = \begin{pmatrix}
0 & -xyz & y^2 y \\
-xyz & 2xz^2 & -(y^2 + 2z^2) \\
x y^2 & -y(y^2 + 2z^2) & 2y^2 z \\
\end{pmatrix},
\]

\[
A_5 = \begin{pmatrix}
0 & -z(y^2 + z^2) & y(y^2 + z^2) \\
-z(y^2 + z^2) & 2xyz & -x(y - z)(y + z) \\
y(y^2 + z^2) & -x(y - z)(y + z) & -2xyz \\
\end{pmatrix},
\]

\[
A_6 = \begin{pmatrix}
2xyz & -z(x^2 + z^2) & -y(x - z)(x + z) \\
-z(x^2 + z^2) & 0 & x(x^2 + z^2) \\
-(y - z)(x + z) & x(x^2 + z^2) & -2xyz \\
\end{pmatrix},
\]

\[
A_7 = \begin{pmatrix}
0 & xz^2 & -xyz \\
xz^2 & 2yz^2 & -(x^2 + 2y^2)z \\
-xyz & -(x^2 + 2y^2)z & 2y(x^2 + y^2) \\
\end{pmatrix}.
\]

Proof. In terms of matrices, the condition \( r^2 | r^a r^b A_{ij} \) is just that \( r^2 \) divides the trace of \( A \), i.e. \( r^2 | [A] \), or \( [A] \equiv 0 \) (mod \( r^2 \)). Therefore, we can make the following ansatz:

\[
A = \begin{pmatrix}
X(x, y, z) & a(x, y, z) & b(x, y, z) \\
a(x, y, z) & Y(x, y, z) & c(x, y, z) \\
b(x, y, z) & c(x, y, z) & r^2 Z(x, y, z) - X(x, y, z) - Y(x, y, z) \\
\end{pmatrix}
\]

where \( a, b, c, X, Y \) and \( Y \) are analytic in the variables indicated. The condition \( A_{ij} x^i = 0 \) then translates to

\[
\begin{align*}
x X(x, y, z) + ya(x, y, z) + zb(x, y, z) &= 0 \\
x a(x, y, z) + y Y(x, y, z) + zc(x, y, z) &= 0 \\
x b(x, y, z) + yc(x, y, z) + z[r^2 Z(x, y, z) - X(x, y, z) - Y(x, y, z)] &= 0.
\end{align*}
\]
We use analyticity of the functions involved, and start by writing
\[
X(x, y, z) = X_1(x) + y X_2(x) + y^2 X_3(x, y) + z X_4(x, y, z),
\]
\[
Y(x, y, z) = Y_1(y) + x Y_2(y) + x^2 Y_3(x, y) + y Y_4(x, y, z),
\]
\[
a(x, y, z) = a_1(x) + y a_2(y) + xy f_1(x, y) + za_4(x, y, z).
\]

Inserted in (6), the limits \( z \to 0, y \to 0 \) give \( X_1(x) = 0, a_1(x) = 0 \), while the limits \( z \to 0, x \to 0 \) give \( Y_1(y) = 0, a_2(y) = 0 \). Given this, \( z \to 0 \) implies \( X_2(x) = 0, Y_2(y) = 0 \), followed by \( X_3(x, y) = -f_1(x, y) \) and \( Y_4(x, y) = -f_1(x, y) \).

A further evaluation of (6) then shows that \( b(x, y, z) = -x X_4(x, y, z) - y a_4(x, y, z) \) and \( c(x, y, z) = -x a_4(x, y, z) - y Y_4(x, y, z) \). By writing
\[
Y_4(x, y, z) = Y_5(x, z) + y Y_5_6(x, y, z) - z f_1(x, y),
\]
(6) together with \( y = 0 \) shows that \( Y_5(x, z) \) contains the factor \( x^2 + z^2 \); and similarly
\[
X_4(x, y, z) = X_5(y, z) + x X_6(x, y, z) - z f_1(x, y)
\]
in (6) with \( x = 0 \) reveals that \( X_5(y, z) \) contains \( y^2 + z^2 \). Thus, \( Y_5(x, z) = 2(x^2 + z^2) f_4(x, z) \) and \( X_5(y, z) = 2(y^2 + z^2) f_4(y, z) \). To proceed, we write
\[
X_6(x, y, z) = X_8(x, z) + 2y f_6(x, y, z).
\]

In addition, (6) with \( y = 0 \) then shows that \( X_4(x, z) = 2z f_2(x, z) \) for some function \( f_2 \). Next, (6) implies \( Z(x, y, z) = 2y f_2(x, y, z) - 2 f_1(x, y) + 2x f_2(x, z) + 2x f_2(y, z) + 2z f_2(x, z) + 2z f_2(y, z) \), for some function \( f_2 \). Also, (6) with \( x = 0 \) then gives \( Y_5(x, y, z) = 2z f_2(x, y, z) + 2x f_2(x, y, z) \), and a final application of (6) shows that \( a_4(x, y, z) = y z f_3(x, z) - x y f_2(y, z) - x^2 f_6(x, y, z) - y^2 f_6(x, y, z) - x y f_2(x, z) - y^2 f_5(x, y, z) - z^2 f_5(x, y, z) - y z f_4(x, y, z) \). By collecting terms, we get lemma 3.

In the proof of lemma 6 below, \( A_{ij} \) will be the difference \( h_{ij} - \eta_{ij} \), and since \( R_{ij} \) involves the Christoffel symbols, we need a corresponding property for the inverse metric \( h^{ij} \).

**Lemma 4.** Suppose that \( h_{ij} \) is given in normal coordinates, so that \( A_{ij} = h_{ij} - \eta_{ij} \) satisfies \( A_{ij} x^i = 0 \). Suppose further that \( A_{ij} \) has the property \( r^2 |f| A_{ij} \). Then the corresponding property holds for the inverse \( h^{ij} \), i.e. with \( B^{ij} = h^{ij} - \eta^{ij} \), we have \( r^2 |\eta^{ij} B^{ij} \) and \( B^{ik} h_{jk} x^j = B^{ik} \eta_{jk} x^j = 0 \).

**Proof.** This is most easily seen in terms of matrices. Namely, it is easy to check that the matrices \( A_1, A_2, \ldots, A_7 \) from lemma 3 have the property that also the products \( A_i A_j, 1 \leq i, j \leq 7 \), satisfy the assumptions of lemma 3. But this means that the inverse \( (I + A)^{-1} = I + B = I + \sum_{n=1}^{\infty} (-A)^n \) will be a sum of the identity operator \( I \) and terms which all have the properties of lemma 3.

Let us now define the operator
\[
D = x^j \frac{\partial}{\partial x^j}.
\]
The operator \( D \) has the important property that if \( f(x, y, z) \) is a homogeneous polynomial of order \( n \), \( D(f) = n f \). This is true, whether \( f \) is a scalar or tensor valued. Because of this property, it easily follows (by viewing \( A_{ij} \) as a sum of homogeneous terms) that

**Lemma 5.** If \( A_{ij} \) has the properties of lemma 3, the same properties hold for \( D(A_{ij}) \).
Before we state and prove lemma 6, we note that the definition of the Christoffel symbols $\Gamma_{ij}^k$ implies that in normal coordinates, $2x^i\dot\Gamma_{ij}^k = \dot h^k D(\dot h_{im})$, and in particular $2x^i\dot\Gamma_{ij}^k = \dot h^k D(\dot h_{ij}) = D(\ln\dot h_{ij})$, where $\dot h$ denotes the determinant of $\dot h_{ij}$.

**Lemma 6.** Suppose that $(x^1, x^2, x^3)$ are normal coordinates on $(\hat V, \hat h_{ab})$. Then $r^2|\eta^{ij}(\dot h_{ij} - \eta_{ij})| \Rightarrow r^2|\dot r^a\dot r_b|.$

**Proof.** Since $(x^1, x^2, x^3)$ are normal coordinates, $\dot\Gamma_{ij}^k x^k x^i = 0$. Using this (and the fact that $\dot h_{ij} x^i = \eta_{ij} x^i$) the definition of $\ddot R_{ij}$ gives

$$x^i x^j \ddot R_{ij} = -x^j \frac{\partial}{\partial x^i}(x^i \dot \Gamma_{ij}^k) - x^i x^j \dot \Gamma_{ij}^m \dot \Gamma_{mj}^i.$$ Some further manipulation leads to

$$4x^i x^j \ddot R_{ij} = -2\dot h^{km} D(\dot h_{km}) - D(\dot h^{km}) D(\dot h_{km}) - 2\dot h^{km} D^2(\dot h_{km}).$$

In terms of matrices, we can write $\dot h_{km} = I + A$, $\dot h^{km} = I + B$, where we assume that $A$ satisfies the properties of lemma 3. By lemma 4 the same is true for $B$, and by lemma 5 also for $E = D(A)$, $F = D(B)$ and $G = D(E) = D^2(A)$. This gives

$$4x^i x^j \ddot R_{ij} = [-2(I + B) \cdot E - (I + F) \cdot (I + E) - 2(I + B) \cdot G].$$

As in the proof of lemma 4, we use that the property of lemma 3 is preserved under multiplication. By expanding $-2(I + B) \cdot E - (I + F) \cdot (I + E) - 2(I + B) \cdot G$, we get a sum of matrices, all of which satisfy the properties of lemma 3. Thus, $r^2$ divides the rhs and therefore also the lhs of (7).

We will now return to the original recursion (2). By combining the previous lemmas, we have the following theorem, which allows for the direct connection between the moments $P_{\ell_1, \ldots, \ell_n}$ and the potential $\phi$.

**Theorem 2.** Let $(x^1, x^2, x^3)$ be normal coordinates on $\hat V$, $\dot h_{ij} - \eta_{ij}$ satisfy the properties of lemma 3 and let $P_{\ell_1, \ldots, \ell_n}$ be defined by the recursion (2). Then

$$x^{\ell_1} \ldots x^{\ell_n} P_{\ell_1, \ldots, \ell_n} \equiv x^{\ell_1} \ldots x^{\ell_n} \partial_{\ell_1} \ldots \partial_{\ell_n} \phi \pmod{r^2}.$$

**Proof.** Put $c_n = \frac{n(n-1)}{2}$. Then

$$x^{\ell_1} \ldots x^{\ell_n} P_{\ell_1, \ldots, \ell_n} = x^{\ell_1} \ldots x^{\ell_n} C \left[ \tilde D_{\ell_1} P_{\ell_2, \ldots, \ell_n} - c_{n-1} \tilde R_{\ell_1} P_{\ell_2, \ldots, \ell_n} \right]$$

$$= x^{\ell_1} \ldots x^{\ell_n} C \left[ \tilde D_{\ell_1} P_{\ell_2, \ldots, \ell_n} - c_{n-1} x^{\ell_1} \ldots x^{\ell_n} C \left[ \tilde R_{\ell_1} P_{\ell_2, \ldots, \ell_n} \right] \right]$$

$$= x^{\ell_1} \ldots x^{\ell_n} \tilde D_{\ell_1} P_{\ell_2, \ldots, \ell_n} - c_{n-1} x^{\ell_1} \ldots x^{\ell_n} \tilde R_{\ell_1} P_{\ell_2, \ldots, \ell_n}$$

$$= x^{\ell_1} \ldots x^{\ell_n} \tilde D_{\ell_1} P_{\ell_2, \ldots, \ell_n} \pmod{r^2}.$$ Now

$$x^{\ell_1} \ldots x^{\ell_n} \tilde D_{\ell_1} P_{\ell_2, \ldots, \ell_n} = x^{\ell_1} \ldots x^{\ell_n} \tilde D_{\ell_1} C \left[ \tilde D_{\ell_2} P_{\ell_3, \ldots, \ell_n} - c_{n-2} \tilde R_{\ell_1, \ell_2} P_{\ell_3, \ldots, \ell_n} \right]$$

and for some tensor $S_{\ell_1, \ldots, \ell_n}$

$$x^{\ell_1} \ldots x^{\ell_n} \tilde D_{\ell_1} C \left[ \tilde D_{\ell_2} P_{\ell_3, \ldots, \ell_n} \right] = x^{\ell_1} \ldots x^{\ell_n} \tilde D_{\ell_1} \left( \tilde D_{\ell_2} P_{\ell_3, \ldots, \ell_n} - \tilde h_{\ell_1, \ell_2} S_{\ell_3, \ldots, \ell_n} \right)$$

$$x^{\ell_1} \ldots x^{\ell_n} \tilde D_{\ell_1} \tilde D_{\ell_2} P_{\ell_3, \ldots, \ell_n} - x^{\ell_1} \ldots x^{\ell_n} \tilde R_{\ell_1, \ell_2} \tilde D_{\ell_3} S_{\ell_4, \ldots, \ell_n}$$

$$= x^{\ell_1} \ldots x^{\ell_n} \tilde D_{\ell_1} \tilde D_{\ell_2} P_{\ell_3, \ldots, \ell_n} \pmod{r^2},$$

while

$$x^{\ell_1} \ldots x^{\ell_n} \tilde D_{\ell_1} C \left[ \tilde R_{\ell_1, \ell_2} P_{\ell_3, \ldots, \ell_n} \right] = x^{\ell_1} \ldots x^{\ell_n} \tilde D_{\ell_1} \left( \tilde R_{\ell_1, \ell_2} P_{\ell_3, \ldots, \ell_n} \right)$$

$$= x^{\ell_1} \ldots x^{\ell_n} \left( \tilde D_{\ell_1} \tilde R_{\ell_1, \ell_2} \right) P_{\ell_3, \ldots, \ell_n} \equiv 0 \pmod{r^2},$$
where, in the last step, we have used that for some function \( f \),
\[
 x^i x^j x^k \dot{D}_i \dot{D}_j - \dot{R}_{jk} x^i \dot{D}_i (x^j x^k) = D(r^2 f) - \dot{R}_{jk} x^i \partial_i (x^j x^k) + R_{jk} x^i \Gamma^g_{im} x^m x^k + R_{jk} x^i \dot{\Gamma}^g_{im} x^m x^k = 2r^2 D(f) + r^2 D(f) - 2 \dot{R}_{jk} x^i x^k \equiv 0 \pmod {r^2}.
\]
Thus,
\[
x^{i_1} \ldots x^{i_n} P_{i_1 \ldots i_n} \equiv x^{i_1} \ldots x^{i_n} \dot{D}_{i_1} \ldots \dot{D}_{i_n} P \pmod {r^2}.
\]
Moreover,
\[
x^{i_1} \ldots x^{i_n} \dot{D}_{i_1} \ldots \dot{D}_{i_n} P
= x^{i_1} \ldots x^{i_n} \partial_i \dot{D}_{i_1} \ldots \dot{D}_{i_n} P - x^{i_1} \ldots x^{i_n} \sum_{m=2}^n \dot{\Gamma}^m_{i_1 \ldots i_n} \dot{D}_{i_1} \ldots \dot{D}_m \ldots \dot{D}_{i_n} P
= x^{i_1} \ldots x^{i_n} \partial_i \dot{D}_{i_1} \ldots \dot{D}_{i_n} P
= x^{i_1} \ldots x^{i_n} \partial_i \dot{D}_{i_1} \ldots \dot{D}_{i_n} P - x^{i_1} \ldots x^{i_n} \partial_i \sum_{m=3}^n \dot{\Gamma}^m_{i_1 \ldots i_n} \dot{D}_{i_1} \ldots \dot{D}_m \ldots \dot{D}_{i_n} P
= x^{i_1} \ldots x^{i_n} \partial_i \dot{D}_{i_1} \ldots \dot{D}_{i_n} P
\]
where, in the last step, we have used \( x^i x^j x^k \partial_i \dot{\Gamma}^m_{jk} x^m = x^i \partial_i (x^j x^k \dot{\Gamma}^m_{jk}) - x^i \partial_i (x^j x^k) \dot{\Gamma}^m_{jk} = 0 \).

Proceeding in this way we get the statement of the theorem. \( \Box \)

**Remark 2.** This theorem is comparable to theorem 7 in [8]. The difference lies in the presentation since [8] uses complex vectors, and the effect is that the equivalence \( \pmod {r^2} \) here becomes equality in [8] due to the fact that \( r^2 = 0 \) along the null vectors there. Also, although [8] uses null vectors, they are 'complexified unit vectors', while the statement here uses vectors \( x \) which are not normalized. However, in the statement of theorem 2, one can (as commented before) replace each \( x^k \) by the direction-dependent unit vector \( \rho^k = x^k / r \).

Comparing with the definition of \( P = \hat{\phi} = \sum_{|\alpha| \geq 0} \frac{r^{|\alpha|}}{|\alpha|!} P^0_\alpha \) in the beginning of the section, Taylor's theorem together with theorem 2 and the remark after lemma 2 tells us that the multipoles produced will be precisely the desired multipoles \( P^0_\alpha \), and we may note that by the arguments presented the multipoles will be unaffected by a change \( \hat{\phi} \rightarrow \hat{\phi} + r^2 \gamma \).

Thus, by specifying \( \hat{\phi} \), and by requiring that the metric \( \hat{h}_{ij} \) be as in theorem 2, the recursion (2) produces the prescribed multipole moments. The issue is now whether (4) produces a power series for such a \( \hat{h}_{ij} \), and furthermore if this series converges.

### 3.5. The conformal field equations

In this section, we will address the conformal field equations (4). The requirement that these equations extend smoothly to \( i^0 = (0, 0, 0) \) will put further restrictions on the form of the metric, and also involve the conformal factor \( \Omega \).

We consider \( \hat{\phi} \) as fixed and real analytic with respect to the normal coordinates \((x, y, z)\) in a neighbourhood of \( i^0 = (0, 0, 0) \). From section 2, we have that (for \( \omega = 0 \))

\[
\phi = \frac{\lambda^2 - 1}{4 \lambda}, \quad P = \hat{\phi} = \phi / \sqrt{\Omega},
\]
where \( \lambda \), which is the norm of the Killing vector, and the conformal factor \( \Omega \) appear in (4). The conformal factor \( \Omega \), which must satisfy the conditions in section 2, is not unique. Rather, we have the freedom \( \Omega \rightarrow \Omega e^{\kappa} \), where \( \kappa \) is a formal power series which vanishes at \( i^0 \). On the other hand, \( \Omega = r^2 \) is easily seen to satisfy the conditions in section 2, and thus (inserting
Similarly, a function \( \hat{\lambda} \) is non-regular at \( i \equiv a \) factor 2 for convenience) we can write
\[
\lambda = \sqrt{1 + 4\Omega \phi^2 + 2\sqrt{\Omega} \phi}, \quad \Omega = r^2 e^{2\kappa}, \quad \kappa(0) = \hat{\Omega}_a(0) = 0.
\]
It is important to note that \( \lambda \) is not even formally smooth, i.e. we cannot regard \( \lambda \) as a formal power series. This is of course due to the occurrence of \( \sqrt{\Omega} = r e^\kappa \), where \( r = \sqrt{x^2 + y^2 + z^2} \) is non-regular at \( i^0 \). In effect, this will mean that each of the equations in (4) will split into two, seemingly doubling the number of equations.

To address (4) we split \( \ln(\sqrt{\Omega}) = \frac{1}{2} \ln \lambda + \ln(r^2 e^{2\kappa}) \) and note the convenient relation
\[
\frac{1}{2} \hat{\Omega}_a \ln \lambda = \frac{\hat{\Omega}_a(r \phi e^{\kappa})}{\sqrt{1 + 4r^2 \phi^2} e^{2\kappa}}.
\]
Inserted in (4), this gives the equations
\[
\begin{align*}
\hat{R}_{ab} + \hat{D}_a \hat{D}_b \ln(r^2 e^{2\kappa}) + \hat{D}_a (r \phi e^{\kappa}) \hat{D}_b \ln(r^2 e^{2\kappa}) \\
+ \hat{h}_{ab} \hat{h}_{de} \hat{D}_a \hat{D}_d \ln(r^2 e^{2\kappa}) - \hat{h}_{ab} \hat{h}_{de} \hat{D}_d \ln(r^2 e^{2\kappa}) \hat{D}_a \ln(r^2 e^{2\kappa}) \\
+ \hat{h}_{ab} \hat{h}_{de} \hat{D}_d (r \phi e^{\kappa}) \sqrt{1 + 4r^2 \phi^2} e^{2\kappa} = -2 \hat{D}_a (r \phi e^{\kappa}) \hat{D}_b (r \phi e^{\kappa}) \frac{1}{1 + 4r^2 \phi^2} e^{2\kappa} = 0, \quad (8)
\end{align*}
\]
and
\[
\begin{align*}
\hat{R} + 4\hat{h}_{ab} \hat{D}_a \hat{D}_b \ln(r^2 e^{2\kappa}) + 4\hat{h}_{ab} \hat{D}_a \hat{D}_b (r \phi e^{\kappa}) \frac{1}{1 + 4r^2 \phi^2} e^{2\kappa} \\
- 2\hat{h}_{ab} \hat{D}_a (r \phi e^{\kappa}) \hat{D}_b (r \phi e^{\kappa}) \frac{1}{1 + 4r^2 \phi^2} e^{2\kappa} = 0. \quad (9)
\end{align*}
\]
To continue, we need some relations which hold in our specialized coordinate system. Each of the following statements are straightforward to check:
\[
\hat{D}_i \hat{D}_i r^2 = 2\hat{h}_{ij} + D(\hat{h}_{ij}), \quad \Delta r^2 = \hat{h}^{ij} \hat{D}_i \hat{D}_j r^2 = 6 + \hat{h}^{ij} D(\hat{h}_{ij}),
\]
\( \hat{h}^{ij} D(\hat{h}_{ij}) = D(\ln(\hat{h})) \equiv 0 \pmod{r^2}, \quad \forall f : \hat{h}^{ij} \hat{D}_i r^2 \hat{D}_j f = 2D(f), \quad (10) \)
\( \hat{h}^{ij} \hat{D}_i r \hat{D}_j r = 1. \)

We will now split (8) and (9) into their regular and non-regular parts. By a regular function we mean a function \( f = f(x, y, z) \) such that \( r^{2n} f \) is (formally) real analytic for some integer \( n \geq 0 \). Note that if \( n \geq 1 \) is required, the regular function is singular. Similarly, a function \( f \) is non-regular if \( r^{2n-1} f \) is (formally) real analytic for some integer \( n \geq 0 \). Again, if \( n \geq 1 \) is required, the function is non-regular and singular. This division is due to the non-regularity of \( \lambda \), and all functions or tensor fields can be written as \( f = f_1 + r f_2 \), where \( f_1 \) and \( f_2 \) are regular (cf lemma 2 of [8]). In particular, \( f = 0 \) requires \( f_1 = f_2 = 0 \). We note that \( \hat{D}_r r = \frac{1}{2} \hat{D}_r r^2 \), which is non-regular. On the other hand, \( \hat{D}_a (r \phi e^{\kappa}) \hat{D}_a (r \phi e^{\kappa}) = \left[ \frac{\phi}{r} \hat{D}_a (r^2) \phi e^{\kappa} + r \hat{D}_a (\phi e^{\kappa}) \right] \left[ \frac{\phi}{r^2} \hat{D}_a (r^2) \phi e^{\kappa} + r \hat{D}_a (\phi e^{\kappa}) \right] = \frac{\phi^2 e^{\kappa}}{r^2} \hat{D}_a r^2 \hat{D}_a r^2 + \frac{1}{2} \hat{D}_a (\phi e^{\kappa}) \hat{D}_a (r^2) \phi e^{\kappa} + \frac{1}{2} \hat{D}_a (\phi e^{\kappa}) \hat{D}_a (r^2) \phi e^{\kappa} \) which is regular (but singular).
Therefore, the regular part of (8) is
\[
\hat{R}_{ab} + \hat{D}_a \hat{D}_b \ln(r^2 e^{2\kappa}) + \hat{D}_a \ln(r^2 e^{2\kappa}) \hat{D}_b \ln(r^2 e^{2\kappa}) + \hat{h}_{ab} \hat{D}_a \hat{D}_b \ln(r^2 e^{2\kappa}) - \hat{h}_{ab} \hat{h}^{de} \hat{D}_d \ln(r^2 e^{2\kappa}) = 0, \tag{11}
\]
while the non-regular part gives
\[
\hat{h}_{ab} \hat{h}^{de} \hat{D}_d \frac{\hat{D}_e (r \hat{\phi} e^\kappa) \hat{D}_a \ln(r^2 e^{2\kappa})}{\sqrt{1 + 4r^2 \hat{\phi}^2 e^{2\kappa}}} - \hat{h}_{ab} \hat{h}^{de} \hat{D}_d \frac{\hat{D}_e (r \hat{\phi} e^\kappa) \hat{D}_a \ln(r^2 e^{2\kappa})}{\sqrt{1 + 4r^2 \hat{\phi}^2 e^{2\kappa}}} = 0. \tag{12}
\]
Similarly, (9) splits into the two equations
\[
\hat{R} + 4 \hat{h}^{ab} \hat{D}_a \hat{D}_b \ln(r^2 e^{2\kappa}) - 2 \hat{h}_{ab} \frac{\hat{D}_a (r \hat{\phi} e^\kappa) \hat{D}_b (r \hat{\phi} e^\kappa)}{1 + 4r^2 \hat{\phi}^2 e^{2\kappa}} - 2 \hat{h}_{ab} \hat{D}_a \ln(r^2 e^{2\kappa}) \hat{D}_b \ln(r^2 e^{2\kappa}) = 0 \tag{13}
\]
and (dividing by 4)
\[
\hat{h}_{ab} \frac{\hat{D}_a (r \hat{\phi} e^\kappa) \hat{D}_b (r \hat{\phi} e^\kappa)}{\sqrt{1 + 4r^2 \hat{\phi}^2 e^{2\kappa}}} - \hat{h}_{ab} \frac{\hat{D}_a (r \hat{\phi} e^\kappa) \hat{D}_b \ln(r^2 e^{2\kappa})}{\sqrt{1 + 4r^2 \hat{\phi}^2 e^{2\kappa}}} = 0. \tag{14}
\]
Equations (12) and (13) are redundant, since (12) is a multiple of (14), while (13) is the trace of (11). Since (14) is non-regular, \( r - (14) \) is regular, and by using \( r \hat{D}_i \hat{D}_jr = \frac{1}{2} \hat{D}_i \hat{D}_j \), it is also seen that \( r - (14) \) is smooth at \( r = 0 \). We therefore look at the singular part\(^5\) of (11), which is found to be
\[
\frac{1}{r^2} \hat{h}_{ij} \hat{D}_i \hat{D}_j \hat{r}^2 + \frac{2}{r^2} \hat{D}_i \hat{r}^2 \hat{D}_j \kappa + \frac{2}{r^2} \hat{D}_i \hat{\kappa} \hat{D}_j \hat{r}^2 + \hat{h}_{ij} \left[ \frac{1}{r^2} \Delta \hat{r}^2 - \frac{8}{r^2} \hat{D} \kappa \right] - \frac{\hat{\phi}^2 e^{2\kappa}}{2r^2} \hat{D}_i \hat{r}^2 \hat{D}_j \hat{r}^2. \tag{15}
\]
This expression must be smooth, and by taking its trace, this says that \( \frac{4}{r^2} \Delta \hat{r}^2 - \frac{8}{r^2} \hat{D} \kappa = 0 \) must be smooth. However, from (10), this implies that \( \frac{\hat{D} \kappa}{r^2} \) is smooth, and therefore that \( \kappa = C + \hat{r}^2 \chi \), where \( C \) is a constant, and \( \chi \) is smooth. From \( \kappa (i^0) = 0 \) we thus infer that
\[
\kappa = r^2 \chi \tag{16}
\]
for some smooth function \( \chi \). Inserting (16) into (15), some simplification shows that
\[
\frac{1}{r^2} [2D(\hat{h}_{ij}) + \hat{D}_i \hat{r}^2 \hat{D}_j \hat{r}^2 (8\chi - \hat{\phi}^2)] \tag{17}
\]
must be smooth. This equation, as it stands, has many solutions, and we must choose a solution which still allows the metric \( \hat{h}_{ij} \) and the function \( \kappa = r^2 \chi \) to solve equations (11) and (14). To do this, we start by imposing slightly more conditions on \( \hat{h}_{ij} \), so that \( \hat{h}_{ij} \) will still satisfy the conditions of lemma 4, but in a slightly restricted form. Let \( A_1 \) be the matrix from lemma 3. We will then require that \( \hat{h}_{ij} \) takes the form
\[
\hat{h}_{ij} = n_{ij} + f(x, y, z)(A_1)_{ij} + r^2 \gamma_{ij}. \tag{18}
\]
Since we have the factor \( r^2 \) explicitly in front of \( \gamma_{ij} \), we only need to ensure that \( \gamma_{ij} \kappa^{ij} = 0 \) in order for \( \hat{h}_{ij} \) to still satisfy the conditions of lemma 4. This is guaranteed by the following lemma.

\(^5\) In practise, skipping smooth terms.
Lemma 7. Suppose that $\gamma_{ij}$ is such that $\gamma_{ij} x^i = 0$. Then $\gamma_{ij}$ is uniquely decomposable as

$$\gamma_{ij} = \gamma = f_1(x, y) B_1 + f_2(x, y) B_2 + f_3(x, y) B_3 + f_4(x, y, z) B_4$$

$$+ f_5(x, y, z) B_5 + f_6(x, y, z) B_6,$$

where

$$B_1 = \begin{pmatrix} -y^2 & xy & 0 \\ xy & -x^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 2yz & -xz & -xy \\ -xz & 0 & x^2 \\ -xy & x^2 & 0 \end{pmatrix};$$

$$B_3 = \begin{pmatrix} 0 & -yz & y^2 \\ -yz & 2xz & -xy \\ y^2 & -xy & 0 \end{pmatrix}, \quad B_4 = \begin{pmatrix} z^2 & 0 & -xz \\ 0 & 0 & 0 \\ -xz & 0 & x^2 \end{pmatrix};$$

$$B_5 = \begin{pmatrix} 0 & z^2 & -yz \\ z^2 & 0 & -xz \\ -yz & -xz & 2xy \end{pmatrix}, \quad B_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & z^2 & -yz \\ 0 & -yz & y^2 \end{pmatrix}.$$

Proof. This proof is rather similar to the proof of lemma 3 and is given in appendix A. □

Now, $(A_1)_{ij} = x_i x_j - r^2 \eta_{ij}$ and $4D(f x_i x_j) = (2 f + D(f)) \hat{D}_i r^2 \hat{D}_j r^2$. Using this, insertion of (18) into (17) then leaves us with the condition that

$$\frac{\hat{D}_i r^2 \hat{D}_j r^2}{r^2} [2 f + D(f) + 2(8 \chi - \hat{\phi}^2)] \tag{19}$$

must be smooth at $r = 0$.

It is not trivial to impose the right conditions on $f$ and $\chi$. If they are chosen too restrictively, no solution to (11) and (14) will exist. On the other hand, if $f$ and $\chi$ are not restricted enough, the solution we are looking for will not be unique (in terms of the introduced quantities). As we will see, the following choice, upon which we insist, will suffice:

$$\chi = \frac{1}{16} (2 \hat{\phi}^2 - 2 f - D(f)). \tag{20}$$

Choice (20) will make (19) vanish identically; this means that equations (11) and (14) are now smooth at $i^0$. Also, the form of the metric is determined via lemma 7, with $f(x, y, z), f_1(x, y), f_2(x, y), f_3(x, y), f_4(x, y, z), f_5(x, y, z)$ and $f_6(x, y, z)$ as unknowns. We put $f_0 = f$ and note that with $\hat{h}_{ij}$ known the conformal factor $\Omega$, and thus the desired spacetime, is also determined.

3.6. Uniqueness of the metric

We will now address equations (11) and (14). We will first prove that a part of these equations determines the metric uniquely provided $\hat{h}_{ij}$ is cast in a certain way (lemma 8), and then, in section 3.7, that the derived series expansions for $\hat{h}_{ij}$ converges and also satisfies (11) and (14) in full (lemma 11).

To prepare for lemma 8 and lemma 11, we will create some scalar equations from (11). To do this, we introduce the vectors fields $u_1^i, u_2^i, u_3^i$ with components

$$u_1^i = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}, \quad u_2^i = \begin{pmatrix} 0 \\ -z \\ y \end{pmatrix}, \quad u_3^i = \begin{pmatrix} z \\ 0 \\ -x \end{pmatrix}$$

where it is seen that they are all pointwise orthogonal to the vector field $x^i$. We also note the linear relation

$$zu_1^i + xu_2^i + yu_3^i = 0.$$
It is straightforward to check that \( u^i u^j, u^i u^j u^k, u^i u^j x^k \) together with any two of \( u^i x^j, u^i x^j, u^i x^j x^k \) are (pointwise) linearly independent. Next, denote the lhs of (11) by \( T_{ab} \) and denote \( r \sqrt{1 + \frac{1}{2} f^2 e^{2 \kappa}} \) times the lhs of (14) by \( S \). With the notation

\[
t_{11} = u^i u^j T_{ij}, \quad t_{22} = u^i u^j T_{ij}, \quad t_{33} = u^i u^j T_{ij},
\]

\[
t_{00} = x^i x^j T_{ij}, \quad t_{0k} = x^i u^j T_{ij}, \quad k = 1, 2, 3,
\]

it is clear that (11) is satisfied if and only if \( t_{00}, t_{11}, t_{22}, t_{33} \) and any two of \( t_{0k}, k = 1, 2, 3 \), vanish.

**Lemma 8.** Suppose that the metric components \( \hat{h}_{ij} \) take the form

\[
\hat{h}_{ij} = \hat{h} = \eta + f_0(x, y, z) A_1 + f_1(x, y) B_1 + f_2(x, y) B_2 + f_3(x, y) B_3 + f_4(x, y, z) B_4 + f_5(x, y, z) B_5 + f_6(x, y, z) B_6,
\]

where \( A_1 \) and \( B_1, B_2, \ldots, B_6 \) are the matrices in lemmas 3 and 7, respectively, and where the functions \( f_0, f_1, \ldots, f_6 \) are formal analytical functions of the variables indicated. Then the equations

\[
S = 0, \quad t_{11} = 0, \quad t_{22} = 0, \quad t_{33} = 0
\]

determine the metric \( \hat{h}_{ij} \) as formal power series.

This will be proved by induction. With the notation from section 3.2,

\[
\hat{h}_{ij} = \eta_{ij} + \sum_{n=2}^{\infty} \hat{h}_{ij}^{[n]}, \quad \bar{R}_{ij} = \sum_{n=0}^{\infty} \bar{R}_{ij}^{[n]}, \quad T_{ij} = \sum_{n=0}^{\infty} T_{ij}^{[n]}, \quad S = \sum_{n=0}^{\infty} S^{[n]},
\]

Note that there can be no linear term in the metric corresponding to \( n = 1 \) due to lemma 3. Moreover, \( \hat{h}_{ij} \) is determined by the functions \( f_0, \ldots, f_6 \) with corresponding series

\[
f_k(x, y, z) = \sum_{n=0}^{\infty} f_k^{[n]}(x, y, z), \quad k = 0, 4, 5, 6,
\]

\[
f_k(x, y) = \sum_{n=0}^{\infty} f_k^{[n]}(x, y), \quad k = 1, 2, 3,
\]

where each \( f_k^{[n]} \) is a homogeneous polynomial (in the variables indicated) of degree \( n \). We also recollect that \( \kappa \) is determined by \( \hat{f}_0 \) and \( \hat{\phi} \) via (16) and (20), respectively.

\( \bar{R}_{ij}^{[n]} \) is determined by the metric up to order \( n + 2 \), and with \( \hat{h}_{ij}^{[n+2]} \) as the leading term, while \( \hat{h}_{ij}^{[k]}, k \leq n + 1 \), are regarded as lower order terms (LOT); the definition of the Ricci tensor gives that

\[
\bar{R}_{ij}^{[n]} = \bar{R}_{ij}^{[n]} + \text{LOT}
\]

\[
\bar{R}_{ij}^{[n]} = \frac{1}{2} \eta^{mn} \partial_m \left[ \partial_j \hat{h}_{ij}^{[n+2]} + \partial_i \hat{h}_{ij}^{[n+2]} - \partial_{ij} \hat{h}_{ij}^{[n+2]} \right] - \frac{1}{2} \eta^{mn} \partial_j \left[ \partial_m \hat{h}_{ij}^{[n+2]} + \partial_j \hat{h}_{ij}^{[n+2]} - \partial_{mj} \hat{h}_{ij}^{[n+2]} \right].
\]

We stress that the lower order terms in \( \bar{R}_{ij}^{[n]} \) are also polynomials of degree \( n \), but that they come from expressions in \( \hat{h}_{ij}^{[k]}, k \leq n + 1 \), considering the form of \( \hat{h}_{ij}, \bar{R}_{ij}^{[n]} \) is a function of the leading order polynomials \( f_0^{[n]}, f_k^{[n-2]} \) for \( 1 \leq k \leq 6 \). We now proceed and look at \( T_{ij} \) and
see how the leading order polynomials enter. From (11) we find that
\[(\hat{D}_i \hat{D}_j \ln(r^2))^{[n]} = \left( \frac{1}{r^2} \hat{D}_i \hat{D}_j r^2 - \frac{1}{r^4} \hat{D}_i r^2 \hat{D}_j r^2 \right)^{[n]} \]
\[= \frac{1}{r^2} \left( 2\hat{h}_{ij}^{[n+2]} + D(\hat{h}_{ij}^{[n+2]}) \right) + \text{LOT} = \frac{n + 4}{r^2} \hat{h}_{ij}^{[n+2]} + \text{LOT}. \quad (23)\]

Here we have used that \(D(f) = n f\) if \(f\) is homogeneous of order \(n\). Continuing (11) gives
\[2(\hat{D}_i \hat{D}_j \kappa)^{[n]} = \left( \frac{1}{8} \hat{D}_i \hat{D}_j r^2(2\phi^2 - 2f - D(f)) \right)^{[n]} = -\frac{(2 + n)}{8} \partial_i \partial_j (r^2 f_0^{[n]}) + \text{LOT} \]
\[(\hat{D}_i \ln(r^2 e^{2\kappa}) \hat{D}_j (\ln(r^2 e^{2\kappa}))^{[n]} = -\frac{(n + 2)^2}{4r^2} - \partial_i (r^2 \partial_j (r^2 f_0^{[n]}) + \text{LOT} \]
\[(2\hat{h}_{ij} \hat{h}_{nm} \hat{D}_n \hat{D}_m \ln(r^2 e^{2\kappa}))^{[n]} = -\frac{(n + 2)^2}{2} f_0^{[n]} + \frac{4}{r^2} \hat{h}_{nm}^{[n+2]} + \text{LOT} \]
\[\left( \frac{2}{1 + 4r^2 \phi^2 e^{2\kappa}} \right)^{[n]} = 0 + \text{LOT}. \quad (24)\]

Combining (22), (23) and (24), we find the leading order of \(T_{ij}^{[n]} = T_{ij}^{[n]} + \text{LOT}\) to be
\[T_{ij}^{[n]} = \frac{1}{2} \eta_{ij} \eta_{mn} \partial_m \left[ \hat{h}_{ij}^{[n+2]} - \partial_n \hat{h}_{ij}^{[n+2]} \right] - \frac{1}{2} \eta_{mn} \partial_i \left[ \hat{h}_{nm}^{[n+2]} - \partial_n \hat{h}_{nm}^{[n+2]} \right] \]
\[+ \frac{(n + 2)^2}{r^2} \eta_{ij} \eta_{mn} \hat{h}_{nm}^{[n+2]} + \frac{n + 2}{2} \eta_{ij} \hat{h}_{mn}^{[n+2]} + \frac{n + 2}{2} \eta_{ij} \left( \frac{n + 2}{2} f_0^{[n]} \right) \]
\[- \frac{(n + 2)}{8} \left[ \partial_i \partial_j (r^2 f_0^{[n]}) + \frac{2}{r^2} \partial_i (r^2 \partial_j (r^2 f_0^{[n]})) + \eta_{ij} \eta_{mn} \partial_m (r^2 f_0^{[n]}) \right]. \]

To examine \(S\), we write \(Y = 2\phi e^{\kappa}\), which means that \(S/\sqrt{1 + 4r^2 \phi^2 e^{2\kappa}}\) becomes
\\[
\hat{h}_{ij} \hat{D}_i \hat{D}_j (\sqrt{1 + r^2 Y^2}) = -\frac{r^2 \hat{D}_i \hat{D}_j (\sqrt{1 + r^2 Y^2})}{\sqrt{1 + r^2 Y^2}} - \frac{r^2 \hat{D}_i \hat{D}_j (\sqrt{1 + r^2 Y^2})}{\sqrt{1 + r^2 Y^2}} + \frac{r^2 \hat{D}_i \hat{D}_j (\sqrt{1 + r^2 Y^2})}{\sqrt{1 + r^2 Y^2}}.
\]

Expanding the derivatives and using (10), we find that
\[S = \frac{Y}{2} D(\ln(\hat{h})) + r^2 \Delta Y - 2Y D(\kappa) - 2r^2 \hat{h}_{ij} \hat{D}_i \hat{D}_j \kappa \]
\[- \frac{r^2 Y}{1 + r^2 Y^2} + 2Y D(Y) + r^2 \hat{h}_{ij} \hat{D}_i \hat{D}_j Y. \quad (25)\]

Note that \(S \equiv 0 \mod r^2\). Now, using \(\ln(\hat{h})^{[n+2]} = \hat{h}_{ij}^{[n+2]} + \text{LOT}\), and putting \(\Delta C = \eta_{ij} \partial_i \partial_j\), it follows that (at this stage, \(\phi(0) \neq 0\) by assumption) \(S^{[n+2]} = S^{[n+2]} + \text{LOT}\), where
\[S^{[n+2]} = \frac{\phi(0)}{2} \left( (n + 2) \left[ \hat{h}_{ij}^{[n+2]} \right] + 2^2 \Delta C \hat{h}_{ij}^{[n+2]} - 4D(\kappa^{[n+2]}) \right) \]
\[= \frac{\phi(0)}{2} (n + 2)^2 \left( \hat{h}_{ij}^{[n+2]} + \frac{1}{r^2} \Delta C (r^2 f_0^{[n]}) + \frac{1}{4} D(r^2 f_0^{[n]}) \right). \quad (26)\]

The leading order terms in (22), (24), (26) are all functions of the homogeneous polynomials \(f_0^{[n]}, k = 1, 2, \ldots, 6\). \(f_0^{[n]}(x, y, z)\) defines a vector space with the
dimension $\frac{(n+2)(n+1)}{2}$, while $f_k^{[n]}(x, y)$ $(k = 1, 2, 3)$ defines a vector space with the dimension $n + 1$. In total, the 7-tuple (for each fixed $n$)
$$\mathbf{\overline{\mathcal{P}}} = \left( f_0^{[n]}, f_1^{[n-2]}, f_2^{[n-2]}, f_3^{[n-2]}, f_4^{[n-2]}, f_5^{[n-2]}, f_6^{[n-2]} \right)$$
lives in a vector space with the dimension $2n^2 + 3n - 2$ for $n \geq 2$. On the other hand, $T_j^{[n]}$ will show up in $t_1^{[n+2]}, t_2^{[n+2]}, t_3^{[n+2]}$, which are all polynomials of degree $\frac{(n+2)(n+3)}{2}$, so that the quadruple
$$\mathbf{\overline{\mathcal{T}}} = \left( S^{[n+2]}, t_1^{[n+2]}, t_2^{[n+2]}, t_3^{[n+2]} \right)$$
lives in $\mathbb{R}^{2n^2+14n+24}$.

Thus, for each fixed $n$, $\mathbf{\overline{\mathcal{T}}}$ is a function of $\mathbf{\overline{\mathcal{P}}}$ and the functions $f_j^{[n]}$ of lower order (than in $\mathbf{\overline{\mathcal{P}}}$). One also notes that both $(S^{[0]}, t_1^{[0]}, t_2^{[0]}, t_3^{[0]})$ and $(S^{[1]}, t_1^{[1]}, t_2^{[1]}, t_3^{[1]})$ are identically zero (see for instance lemma 9 below). Next, it is easy to explicitly check that for $n = 0$, one can find $f_0^{[0]}$ so that $(S^{[2]}, t_1^{[2]}, t_2^{[2]}, t_3^{[2]}) = 0$ and similarly for $f_1^{[1]}$ when $n = 1$ (given $f_0^{[0]}$).

Using induction, we now assume that the metric $\mathbf{\hat{h}}_{ij}$ is determined up to order $n + 1$, i.e. that the polynomials $f_k^{[n]}$, $m = 0, 1, \ldots, n - 1$, and $f_k^{[n]}$, $k = 1, 2, \ldots, 6$, $m = 0, 1, \ldots, n - 3$, are known $(n \geq 2)$. With these polynomials fixed, $\mathbf{\overline{\mathcal{T}}} = \mathbf{\overline{\mathcal{T}}} (\mathbf{\overline{\mathcal{P}}})$ will be an affine function with respect to $\mathbf{\overline{\mathcal{P}}}$, and we must show that there exists a $\mathbf{\overline{\mathcal{P}}}$ such that $\mathbf{\overline{\mathcal{T}}} (\mathbf{\overline{\mathcal{P}}}) = 0$.

With $\mathbf{\overline{\mathcal{T}}} = \mathbf{\overline{\mathcal{T}}} (0)$, we consider the equation
$$\mathbf{\overline{\mathcal{T}}} (\mathbf{\overline{\mathcal{P}}}) - \mathbf{\overline{\mathcal{T}}} = 0$$
where now the mapping $\mathbf{\overline{\mathcal{Q}}} : \mathbf{\overline{\mathcal{P}}} \mapsto \mathbf{\overline{\mathcal{Q}}} (\mathbf{\overline{\mathcal{P}}}) = \mathbf{\overline{\mathcal{T}}} (\mathbf{\overline{\mathcal{P}}}) - \mathbf{\overline{\mathcal{T}}} = 0$ is linear. Explicitly, $\mathbf{\overline{\mathcal{Q}}} (\mathbf{\overline{\mathcal{P}}}) = (S^{[n+2]}, u_1^{[i]} T_j^{[n]} - u_j^{[i]} T_j^{[n]}, u_2^{[i]} T_j^{[n]}, u_3^{[i]} T_j^{[n]})$. Since $\mathbf{\overline{\mathcal{Q}}}$ is a mapping
$$\mathbf{\overline{\mathcal{Q}}} : \mathbb{R}^{2n^2+3n-2} \rightarrow \mathbb{R}^{2n^2+14n+24},$$
we have a system of equations with $2n^2 + 14n + 24$ equations and $2n^2 + 3n - 2$ unknowns, i.e. the system is over-determined.

To prove that there is a $\mathbf{\overline{\mathcal{P}}}$ for which $\mathbf{\overline{\mathcal{Q}}} (\mathbf{\overline{\mathcal{P}}}) = 0$, we will prove (i) that the dimension of the range of $\mathbf{\overline{\mathcal{Q}}}$, and therefore $\mathbf{\overline{\mathcal{Q}}}$, is only $2n^2 + 3n - 2$, and (ii) the mapping $\mathbf{\overline{\mathcal{P}}} \mapsto \mathbf{\overline{\mathcal{Q}}} (\mathbf{\overline{\mathcal{P}}})$ is injective.

To do this, we start with the somewhat surprising\textsuperscript{6} lemma

**Lemma 9.** $S \equiv 0$ (mod $r^2$), $t_i \equiv 0$ (mod $r^2$), $i = 1, 2, 3$.

**Proof.** From (25), using that $D(\ln \mathbf{\hat{h}}_{ij}) \equiv 0$ (mod $r^2$), it immediately follows that $S \equiv 0$ (mod $r^2$). The other properties are more tedious to prove, and we refer to appendix A for these calculations.

Since all of $S, t_{11}, t_{22}, t_{33}$ contain the factor $r^2$, we can write
$$S = r^2 \sigma, \quad t_{ii} = r^2 \tau_{ii}, \quad i = 1, 2, 3.$$ 

As a consequence, $S^{[n+2]}$ is replaced by $\sigma^{[n]}$, and similarly, $t_{ii}^{[n+2]}$ is replaced by $\tau_{ii}^{[n]}$, and we consider the quadruple
$$\mathbf{\overline{\mathcal{T}}} = (\alpha^{[n]}, t_1^{[n]}, t_{22}^{[n]}, t_{33}^{[n]}) \in \mathbb{R}^{2n^2+6n+4}$$
where we now have to solve the equation $\mathbf{\overline{\mathcal{T}}} (\mathbf{\overline{\mathcal{P}}}) = 0$. As previously, we can put $\mathbf{\overline{\mathcal{T}}} = \mathbf{\overline{\mathcal{T}}} (0) = \mathbf{\overline{\mathcal{T}}} (0)/r^2$ and address the equation $\mathbf{\overline{\mathcal{Q}}} (\mathbf{\overline{\mathcal{P}}}) = -\mathbf{\overline{\mathcal{T}}} (0)$ where $\mathbf{\overline{\mathcal{Q}}}$ is the linear function $\mathbf{\overline{\mathcal{Q}}} (\mathbf{\overline{\mathcal{P}}}) = \mathbf{\overline{\mathcal{T}}} (\mathbf{\overline{\mathcal{P}}}) - \mathbf{\overline{\mathcal{T}}} (0)$, which is a mapping $\mathbb{R}^{2n^2+3n-2} \rightarrow \mathbb{R}^{2n^2+6n+4}$. Thus the equation $\mathbf{\overline{\mathcal{Q}}} (\mathbf{\overline{\mathcal{P}}}) = 0$

\textsuperscript{6} For instance, $u_j^{[i]} T_j^{[n]}$ is not congruent to 0 (mod $r^2$).
This splitting is possible since \( \gamma \) is trace free, and in particular, that \( \gamma^{ij} \) gives 2 equations in 2 unknowns. However, there are additionally 3n + 6 linear relations among \( (\tau^{[n]}_{11}, \tau^{[n]}_{22}, \tau^{[n]}_{33}) \) as the following arguments show. First, lemma 9 showed that

\[
\begin{align*}
t_{11} &= y^2 T_{11} - 2xy T_{12} + x^2 T_{22} = (x^2 + y^2 + z^2) \tau_{11}(x, y, z) \\
t_{22} &= z^2 T_{22} - 2yz T_{23} + y^2 T_{33} = (x^2 + y^2 + z^2) \tau_{22}(x, y, z) \\
t_{33} &= x^2 T_{33} - 2xz T_{31} + z^2 T_{11} = (x^2 + y^2 + z^2) \tau_{33}(x, y, z).
\end{align*}
\]

Putting \( x = 0 \), the first and third relations give

\[
y^2 T_{11}(0, y, z) = (y^2 + z^2) \tau_{11}(0, y, z), \quad z^2 T_{11}(0, y, z) = (y^2 + z^2) \tau_{33}(0, y, z)
\]

which imply

\[
z^2 \tau_{11}(0, y, z) = y^2 \tau_{33}(0, y, z).
\]

(28)

With the ansatz \( \tau_{11}(0, y, z) = \sum_{a=0}^{n} a_{1,1,k} y^k z^{n-k}, \tau_{33}(0, y, z) = \sum_{a=0}^{n} a_{3,3,k} y^k z^{n-k} \), (28) implies that \( a_{1,1,0} = a_{1,1,1} = 0, a_{1,1,k} = a_{3,3,k-2}, k = 2, 3, \ldots, n, a_{3,3,n-1} = a_{3,3,n} = 0 \), i.e. \( n + 3 \) linear relations. Putting \( y = 0 \) produces another \( n + 3 \) linear relations, as does \( z = 0 \).

However, it is easily seen that three of these relations have been counted twice, which means that the total number of linear relations are \( 3n + 6 \).

Thus, the range of \( \overline{\gamma} \) is a vector space with dimension \( 2n^2 + 3n - 2 \) (in which \( -\tilde{r}_0 \) lives) and hence we have shown that equation (27) has a solution if the mapping \( \overline{\gamma} \) is injective. This is the content of lemma 10.

**Lemma 10.** Let the linear mapping \( \overline{\gamma} : \mathbb{R}^{2n^2+3n-2} \to \mathbb{R}^{2n^2+6n+4} \) be defined by \( \overline{\gamma}(\overline{\gamma}) = (S^{[n+2]}, u^{[1]}_i L_i^{[n]}, u^{[2]}_i L_i^{[n]}, u^{[3]}_i L_i^{[n]}) \). Then \( \overline{\gamma} \) is injective.

**Proof.** Assume that \( \overline{\gamma}(\overline{\gamma}) = 0 \). Consider the matrix \( \gamma_{ij} \) in the decomposition (18) of \( \tilde{h}_{ij} \).

By splitting \( \gamma_{ij} = \tilde{\gamma}_{ij} + \frac{1}{2} \sigma r^2 d\Omega_j \) where \( \tilde{\gamma}_{ij} \) is trace free, \( \gamma = [\gamma_{ij}] \), and where \( d\Omega_j \) is the metric of the unit sphere \( S^2 \), we see that

\[
\tilde{h}^{[n+2]}_{ij} = f_0^{[n]}(x, y, z)(A_1)_{ij} + r^2 \tilde{\gamma}^{[n]}_{ij} + r^2 \frac{1}{2} \gamma^{[n]} r^2 d\Omega_j
\]

and, in particular, that

\[
\tilde{h}^{[n+2]}_{ij} = -2r^2 f_0^{[n]}(x, y, z) + r^2 \gamma^{[n]}.
\]

The first ‘component’ in \( \overline{\gamma}(\overline{\gamma}) \) (i.e. \( S^{[n+2]} \)) being zero therefore implies

\[
\gamma^{[n]} = \frac{1}{8} [12 f_0^{[n]} + \Delta \gamma (r^2 f_0^{[n]}) - 2 D f_0^{[n]}]].
\]

With the decomposition \( \Delta = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial}{\partial r}) + \frac{1}{r^2} \Delta_S \), where \( \Delta_S \) is the angular Laplacian on the unit sphere, this can also be written as

\[
\gamma^{[n]} = \frac{1}{8} [(n^2 + 3n + 18) f_0^{[n]} + \Delta S f_0^{[n]}].
\]

(29)

We now express \( \tilde{\gamma}^{[n]}_{ij} \) in terms of spherical coordinates \((r, \theta, \phi)\), and get

\[
\tilde{\gamma}^{[n]}_{ij} dx^i dx^j = (dr d\theta d\phi) \begin{pmatrix} 0 & 0 & 0 \\ 0 & r^{n+2} f_{ij}(\theta, \phi) & r^{n+2} g_{ij}(\theta, \phi) \\ 0 & r^{n+2} g_{ij}(\theta, \phi) & -r^{n+2} \sin^2(\theta) f_{ij}(\theta, \phi) \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix}
\]

for some angular functions \( f_0(\theta, \phi) \) and \( g_0(\theta, \phi) \). Note that \( f_\theta \) and \( g_\theta \) are not arbitrary since \( \gamma_{ij} \) must be analytic when written in the coordinates \((x, y, z)\). With this representation, \( u^{[1]}_i L_i^{[n]}(k = 1, 2, 3) \) are functions of \( f_0^{[n]}, \gamma^{[n]}, f_\theta \) and \( g_\theta \).

This splitting is possible since \( \gamma_{ij} x^l = 0 \).
In particular, \(u_i u_j^{[n]} = 0\) gives the equation

\[
4 \sin^2 \theta \left[ \frac{\partial^2}{\partial \theta^2} f_\phi + (n + 3) n f_\phi - \Delta_S f_\phi \right] + 16 \sin \theta \cos \theta \frac{\partial}{\partial \theta} f_\phi
\]
\[
+ 8 \frac{\partial^2}{\partial \theta \partial \phi} g_\phi - 2 \sin^2 \theta (\Delta_S y^{[n]} + n(n + 1) y^{[n]})
\]
\[
- \sin^2 \theta \left( 2 n \Delta_S f_0^{[n]} + n^2 f_0^{[n]} - (n + 2) \frac{\partial^2}{\partial \theta^2} f_0^{[n]} \right) = 0.
\]

The equations \(u_i u_j^{[n]} = 0\) and \(u_i u_j^{[\ell]} = 0\) are slightly longer, namely

\[
4 A \left[ \sin^2 \theta \frac{\partial^2 f_\phi}{\partial \theta^2} + 3 \cos \theta \sin \theta \frac{\partial f_\phi}{\partial \theta} - \frac{\partial^2 f_\phi}{\partial \phi^2} \right] + 4 \sin^2 \theta ((n^2 + 3 n)[A - 2 \sin^2 \phi] - 4 \sin^2 \phi) f_\phi
\]
\[
+ 8 A \frac{\partial^2 g_\phi}{\partial \theta \partial \phi} - 8(n^2 + 3 n + 2) \sin \theta \sin \phi \cos \phi \cos \theta g_\phi
\]
\[
- 2 A \left[ \sin^2 \theta \frac{\partial^2 y^{[n]}}{\partial \theta^2} + \sin \theta \cos \theta \frac{\partial y^{[n]}}{\partial \theta} + \frac{\partial^2 y^{[n]}}{\partial \phi^2} + n(n + 1) \sin^2 \theta y^{[n]} \right]
\]
\[
- 2(n + 2) \sin \theta \sin \phi \cos \phi \cos \theta \frac{\partial^2 f_0^{[n]}}{\partial \phi^2}
\]
\[
- \sin^2 \theta ((n - 2) A + (n + 2) \sin^2 \phi) \frac{\partial^2 f_0^{[n]}}{\partial \phi^2}
\]
\[
+ (n + 2) \sin^2 \phi - 2 n A \frac{\partial^2 f_0^{[n]}}{\partial \phi^2} + 2(n + 2) \sin \phi \cos \phi \cos^2 \theta \frac{\partial f_0^{[n]}}{\partial \phi}
\]
\[
- \sin \theta \cos \theta ((n - 2) \sin^2 \phi + 2 n \cos^2 \phi \cos^2 \theta) \frac{\partial f_0^{[n]}}{\partial \theta} - n^3 \sin^2 \theta A f_0^{[n]} = 0,
\]

\[A = 1 - \cos^2 \phi \sin^2 \theta \]

and

\[
4 B \left[ \sin^2 \theta \frac{\partial^2 f_\phi}{\partial \theta^2} + 3 \sin \theta \cos \theta \frac{\partial f_\phi}{\partial \theta} - \frac{\partial^2 f_\phi}{\partial \phi^2} \right]
\]
\[
+ 4 \sin^2 \theta ((n^2 + 3 n)[B - 2 \cos^2 \phi] - 4 \cos^2 \phi) f_\phi
\]
\[
8 B \frac{\partial^2 g_\phi}{\partial \theta \partial \phi} + 8(n^2 + 3 n + 2) \sin \phi \cos \phi \cos \theta \sin \theta g_\phi
\]
\[
- 2 B \left[ \sin^2 \theta \frac{\partial^2 y^{[n]}}{\partial \theta^2} + \sin \theta \cos \theta \frac{\partial y^{[n]}}{\partial \theta} + \frac{\partial^2 y^{[n]}}{\partial \phi^2} + n(n + 1) \sin^2 \theta y^{[n]} \right]
\]
\[
+ 2(n + 2) \sin \theta \sin \phi \cos \phi \cos \theta \frac{\partial^2 f_0^{[n]}}{\partial \phi^2}
\]
\[
- \sin^2 \theta ((n - 2) B + (n + 2) \cos^2 \phi) \frac{\partial^2 f_0^{[n]}}{\partial \phi^2}
\]
\[
+ ((n + 2) \cos^2 \phi - 2 n B) \frac{\partial^2 f_0^{[n]}}{\partial \phi^2} - 2(n + 2) \sin \phi \cos \phi \cos^2 \theta \frac{\partial f_0^{[n]}}{\partial \phi}
\]
\[
- \sin \theta \cos \theta ((n - 2) \cos^2 \phi + 2 n \sin^2 \phi \cos^2 \theta) \frac{\partial f_0^{[n]}}{\partial \theta} - n^3 \sin^2 \theta B f_0^{[n]} = 0,
\]

\[B = 1 - \sin^2 \phi \sin^2 \theta. \]
\( \gamma^{[n]} \) can be eliminated using (29), and after insertion in (30), one can express \( \frac{\partial^2 g_\theta}{\partial \phi^2} \) in terms of \( f_0^{[n]} \) and \( f_\phi \). With \( \gamma^{[n]} \) and \( \frac{\partial^2 g_\theta}{\partial \phi^2} \) inserted in (31), one can ‘solve’ for \( g_\phi \), i.e. express also \( g_\phi \) in terms of \( f_0^{[n]} \) and \( f_\phi \). From this, (32) gives \( f_\phi \), and therefore also \( g_\phi \) and \( \gamma^{[n]} \) as a function of \( f_0^{[n]} \). Finally, we insert all these quantities in \( u_1^i u_1^j T_{ij}^{[n]} \). Using the obtained expression for \( \frac{\partial^2 g_\theta}{\partial \phi^2} \) gives a trivial identity, but by using the expression for \( g_\phi \), differentiated with respect to \( \theta \) and \( \phi \), we get a fourth-order linear equation for \( f_0^{[n]} \). We now put \( f_0^{[n]} = r^n f_n(\theta, \phi) \), which in particular means that \( f_n(\theta, \phi) \) is a linear combination of spherical harmonics \( Y_m^l(\theta, \phi) \) with \( l \leq n \). After these steps, the equation for the angular part \( f_n(\theta, \phi) \) is found to be

\[
\sin^2 \theta \left( \frac{\partial^4 f_n}{\partial \theta^4} + 2 \frac{\partial^2 f_n}{\partial \theta^2 \partial \phi^2} + \frac{\partial^4 f_n}{\partial \phi^4} \right) + 2 \cos \theta \sin \theta \left( \frac{\partial^3 f_n}{\partial \theta^3} \right) - 2 \frac{\cos \theta}{\sin \theta} \frac{\partial^2 f_n}{\partial \theta \partial \phi^2} - 2 (2n^2 + 6n + 5) \cos^2 \theta - 2n^2 - 6n - 4) \frac{\partial^2 f_n}{\partial \theta^2} \\
- 2 \left( (n^2 + 3n + 2) \cos^2 \theta - n^2 - 3n - 4 \right) \frac{\partial^4 f_n}{\partial \phi^4} \\
= \frac{\sin^2 \theta}{\cos \theta} \left( (2n^2 + 6n + 6) \cos^2 \theta - 2n^2 - 6n - 7 \right) \frac{\partial^2 f_n}{\partial \theta \partial \phi^2} \\
+ \sin^2 \theta (n + 3)(n + 2)(n + 1) n f_n = 0.
\]

Since the coefficients of (33) do not contain \( \phi \), we make the ansatz \( f_n(\theta, \phi) = F(\theta) e^{i m \phi} \). The equation for \( F(\theta) \) becomes

\[
\sin^2 \theta \frac{d^4}{d \theta^4} F(\theta) + 2 \cos \theta \sin \theta \frac{d^3}{d \theta^3} F(\theta) \\
- \left[ 2m^2 + ((2n^2 + 6n + 5) \cos^2 \theta - 2n^2 - 6n - 4) \right] \frac{d^2}{d \theta^2} F(\theta) \\
+ \left[ 2m^2 - ((2n^2 + 6n + 6) \cos^2 \theta - 2n^2 - 6n + 7) \right] \frac{\cos \theta}{\sin \theta} \frac{d}{d \theta} F(\theta) \\
+ \frac{m^4 + 2m^2 \left( (n^2 + 3n + 2) \cos^2 \theta - n^2 - 3n - 4 \right)}{\sin^2 \theta} F(\theta) \\
+ \sin^2 \theta (n + 3)(n + 2)(n + 1) n F(\theta) = 0.
\]

The solution to (34) is

\[
F(\theta) = C_1 \sin \theta P_{m+1}^{n+1} (\cos \theta) + C_2 \sin \theta P_{m-1}^{n+1} (\cos \theta) \\
+ C_3 \sin \theta Q_{m+1}^{n-1} (\cos \theta) + C_4 \sin \theta Q_{m+1}^{n+1} (\cos \theta)
\]

i.e. a linear combination of associated Legendre functions of both kinds. Now, since \( f_n(\theta, \phi) \) consists of spherical harmonics \( Y_m^l(\theta, \phi) \) with \( l \leq n \), (35) implies that the relation \( f_n(\theta, \phi) = F(\theta) e^{i m \phi} \) requires that \( F \) and hence \( f_n \) is identically zero. From this, (29) gives \( \gamma^{[n]} = 0 \). By forming \( (1 + \cos^2 \theta) u_{11} T_{11}^{[n]} - (u_{12} T_{12}^{[n]} + u_{13} T_{13}^{[n]}) \sin^2 \theta \) it is seen that \( f_\phi = 0 \), from which \( g_\phi \) must also be zero. This means that the mapping \( \mathcal{G} \) is injective, and therefore there exists a unique formal series expansion of \( h_{ij} \) such that \( S = 0, t_{11} = 0, t_{22} = 0, t_{33} = 0 \) (lemma 8).

We again stress that (27) gives the metric components of the appropriate order explicitly.
3.7. Convergence of the metric

In this section we will prove that the series expansion for \( h_{ij} \), which was found in the previous section, converges and also solves the full conformal field equations. This is the content of the following lemma.

**Lemma 11.** Suppose that \( h_{ij} \) is a formal power series for the metric of the form (21), producing the moments of theorem 1. If

\[
S = 0, \quad t_{11} = 0, \quad t_{22} = 0, \quad t_{33} = 0,
\]

then

\[
t_{00} = 0, \quad t_{0k} = 0, \quad k = 1, 2, 3,
\]

and the power series is convergent in a neighbourhood of \( i^0 \).

This will conclude the proof of theorem 1 when the monopole is non-zero.

**Proof.** To prove this lemma, we will use a result by Friedrich, namely theorem 1.1 in [3], where a related property is investigated. In [3], it is proven that under rather similar settings, there exists a metric which instead of prescribed multipole moments produces a set of prescribed null data, where also growth conditions for the null data in order for the series expansion of the metric to converge are given. These null data are proven to be in a one-to-one correspondence with the family of multipole moments with non-zero monopole, but since this correspondence is rather implicit, the actual conditions on the multipole moments (as required by the conjecture by Geroch) are not clear.

We will prove that the conditions on the moments \( P_0, P_0^i, P_0^{i_1}, \ldots \) in theorem 1 can be carried over to estimates on the null data as required in [3]. This will then guarantee a solution to the conformal Einstein’s field equations which according to the work presented here will have the desired multipole moments.

In order to be able to refer to the work in [3], we continue to impose the temporary condition that the monopole \( P_0 \) is non-zero.

To begin, we compare the different conformal settings used. In [3], one uses the conformal factor

\[
\Omega_F = \left( \frac{1 - \sqrt{\lambda}}{m} \right)^2
\]

while this work uses (starting with \( (h_G)_{ab} \) in section 3.3) the conformal factor

\[
\Omega = \sqrt{\lambda} \Omega = \sqrt{\lambda} r^2 e^{2\kappa}.
\]

Therefore, if \( (h_F)_{ab} \) denotes the metric used in [3], we have the relation

\[
(h_F)_{ab} = \left[ \frac{1}{m^2 \Omega} \frac{(1 - \sqrt{\lambda})^2}{\sqrt{\lambda}} \right]^2 \hat{h}_{ab}, \quad \lambda = \sqrt{1 + 4\Omega \hat{\phi}^2 + 2\sqrt{\Omega \hat{\phi}}}, \quad \Omega = r^2 e^{2\kappa}.
\]

Since \( \sqrt{\Omega} \) contains the non-smooth quantity \( r \), it is not obvious that the transition from \( \hat{h}_{ab} \) to \( (h_F)_{ab} \) is smooth. However, by putting \( \xi = 2 r e^\phi \), the conformal factor \( \Omega_T \) relating \( \hat{h}_{ab} \) and \( (h_F)_{ab} \) is seen to be

\[
\Omega_T = \frac{1}{m^2 \Omega} \frac{(1 - \sqrt{\lambda})^2}{\sqrt{\lambda}} = \frac{4 \hat{\phi}^2 (1 - \sqrt{1 + \xi^2 + \xi})^2}{m^2 \xi^2 \sqrt{1 + \xi^2 + \xi}}.
\]
By using $\sqrt{1 + \varepsilon^2 - \xi} = \frac{1}{\sqrt{1 + \varepsilon^2 + \xi}}$, it follows that this expression is even in $\xi$, which means that only even powers of $r$ occur in $\Omega_T$. Since the limit $\lim_{r \to 0} \Omega_T = 1$ causes no problem, $\Omega_T$ is found to be (formally) smooth. Note that since we know $\phi$ produces the prescribed moments under $\hat{h}_{ab}$, the potential $\phi/\sqrt{\Omega_T}$ will give the correct moments when using $(h_F)_{ab}$.

Now, suppose that $S = 0, t_{11} = 0, t_{22} = 0, t_{33} = 0$, and hence, according to lemma 8, that we have a formal metric $\hat{h}_{ij}$. Going over to $(h_F)_{ab}$, and using the trace-free Ricci tensor, this defines a set of abstract null data ($D_n$ and $D_n^*$ below). By the arguments in [3], there exists a formal solution to the full conformal static field equations connected to these null data, and by going back to $\hat{h}_{ij}$, this implies also that $t_{00} = 0, t_{0k} = 0, k = 1, 2, 3$.

Another way of putting this is: if $(h_F)_{ab}$ satisfies the conformal static field equations and if $(h_F)_{ab}$ and $\hat{h}_{ab}$ are conformally related, equations (4) are automatically satisfied, while the equations $S = 0, t_{11} = 0, t_{22} = 0, t_{33} = 0$ fully determine the metric components $\hat{h}_{ij}$ in terms of the normal coordinates $x, y, z$ (taking the special form of $\hat{h}_{ij}$ into account).

In [3], one uses normal coordinates around $i^0$, and introduces a frame field $c_a, a = 1, 2, 3$, which is parallelly propagated along the geodesics through $i^0$. Since we write $\hat{R}_{ij}$ for the Ricci tensor, we can let $R_{ab}$ stand for the Ricci tensor in [3]. With $s_{ab}$ denoting the trace-free part of the Ricci tensor, we have $s_{ab} = R_{ab}$, since $R_{ab}$ in [3] already is trace free due to the choice of conformal gauge there. One then considers the set

$$D_n = \{ s_{a_1a_2}(i^0), C[\nabla_a, s_{a_2a_3}](i^0), C[\nabla_{a_1}, \nabla_a, s_{a_2a_3}](i^0), \ldots \}$$

where $\nabla_a$ is the derivative operator associated with $(h_F)_{ab}$. To proceed, one then expresses the family of tensors in $D_n$ in the introduced frame field, and considers (again, see [3] for the details) the related family

$$D_n^* = \{ s_{a_1a_2}(i^0), C[\nabla_a, s_{a_2a_3}](i^0), C[\nabla_{a_1}, \nabla_a, s_{a_2a_3}](i^0), \ldots \}.$$ 

According to theorem 1.1 in [3], there exists an analytic solution around $i^0$ to the conformal static vacuum field equations with $m \neq 0$ for the conformal metric $(h_F)_{ab}$ to each set of tensors, given in the orthonormal frame (at $i^0$),

$$\tilde{D}_n = \{ \psi_{a_2a_3}, \psi_{a_3a_1a_2}, \psi_{a_4a_3a_2a_1}, \ldots \}$$

which satisfy (i) each tensor is totally symmetric and trace free, and (ii) there exist constants $M, r > 0$ such that

$$|\psi_{a_1} \ldots a_{bc}| \leq \frac{Mp^1}{r^p}, \quad a_p, \ldots, a_1, b, c = 1, 2, 3, \quad p = 0, 1, 2, \ldots \quad (36)$$

In particular, in the introduced frame, one has

$$C[\nabla_{a_1}, \ldots, \nabla_{a_q}, s_{a_2a_3}](i^0) = \psi_{a_1} \ldots a_q, \quad q = 2, 3, 4, \ldots \quad (37)$$

We will show that given the conditions of theorem 1, there will exist such a family $\tilde{D}_n$ with the desired properties and, most importantly, that the tensors in $\tilde{D}_n$ produce the prescribed moments via the requirement (37). We have already concluded that given $\hat{h}_{ij}$, $\Omega_T$ and $\phi$, this defines $(h_F)_{ab}$, and moreover that $\phi/\sqrt{\Omega_T}$ will give the correct moments when using $(h_F)_{ab}$. We can therefore define $\psi_{a_1} \ldots a_q$ in (37) as the corresponding left-hand side and prove the necessary estimates on $C[\nabla_{a_1} \ldots \nabla_{a_q}, s_{a_2a_3}](i^0)$ directly.

Now, with the notation in theorem 1, the convergence of $\sum_{|\alpha| \geq 0} \frac{\xi^{|\alpha|} P_0^a}{r^p}$ near $i^0$ implies that for some constants $M, r > 0$,

$$|P_0^a| \leq \frac{p!M}{r^p}, \quad p = |\alpha| = 0, 1, 2, \ldots \quad (38)$$

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From the arrangement in [3], \( P_{a_1 a_2} = -\frac{m}{2} \delta_{a_1 a_2} \), and hence the estimates in (36) will be satisfied if there are constants \( \hat{M} , \hat{r} \) so that

\[
|C[\nabla_{a_1} , . . . , \nabla_{a_n} P_{a_2 a_1}]|^2 \leq \frac{q^1 \hat{M}}{\hat{r} q} , \quad q = 0, 1, 2, \ldots .
\]

Proceeding as in [3], the tensors \( P_a \) can be expressed as totally symmetric space spinors fulfilling a certain reality condition \((3.4)\) in [3]. In terms of these space spinors, (38) reads

\[
|P^{0}_{A_p B_p \ldots A_1 B_1}| \leq \frac{p^! \hat{M}}{\hat{r} p} , \quad A_p, B_p, \ldots , A_1, B_1 = 0, 1, \quad p = 0, 1, 2, \ldots . \tag{39}
\]

for some related constants \( \hat{M} , \hat{r} \), and we must demonstrate that given the estimates (39), there are \( \hat{M} , \hat{r} \) so that

\[
|D_{(A_p B_p \ldots A_1 B_1)} (i^0)| \leq \frac{(p - 1)! M_0}{r_0^p} , \quad A_p, B_p, \ldots , A_1, B_1 = 0, 1 , \quad p = 1, 2, 3 \ldots . \tag{40}
\]

To simplify the calculations, we introduce the following notation. We denote \( P_a R_{a_1 B_1} \) by \( P_p \), and let \( D_{(A_p B_p \ldots A_1 B_1)} P_{a_1 A_1 B_1} \) be denoted by \( D_{p - k} P_k \). In terms of this notation, recursion (2) expressed in space spinors becomes \( c_p = \frac{p(2p - 1)}{2} \)

\[
P_p = \sum D_{p - 1} P_{p - 1} + c_{p - 1} R_2 P_{p - 2} \tag{41}
\]

where \( R_2 \) stands for \( R_{a_1 B_1} \), and it is understood that all products involve complete symmetrization. For instance, \( R_2 P_{p - 2} \) stands for \( R_{a_1 B_1} P_{a_2 a_3 \ldots a_2} P_{a_4 a_5 \ldots a_4} \). Since \( P_{a b} = 2 R_{a b} \), a simple rescaling \( P_{a_2 a_1} \rightarrow -\frac{m}{2} P_{a_2 a_1} \) (keeping the same stem letter) transforms (41) into

\[
D_{p - 1} P_p = P_{p + 1} + c_{p} P_{p - 1} P_2 .
\]

We now claim that

\[
\forall n \geq 0 , \quad |D_{n} P_p (i^0)| \leq \frac{M_0 (1 + 2M_0)^n (n + p - 1)!}{r_0^{n+p}} , \quad p = 2, 3, 4, \ldots . \tag{42}
\]

where (40) is the initial estimate for \( n = 0 \), and where we assume that (42) is valid up to a certain value of \( n \). For \( n + 1 \) we then get

\[
|D_{n+1} P_p| = |D_{n} P_{p+1}| + c_{p} \sum_{k=0}^{n} \binom{n}{k} D_{k} P_{p-1} D_{n-k} P_2 ,
\]

and inserting the estimates (42) we get, where (43) to (46) are evaluated at \( (i^0) \),

\[
|D_{n+1} P_p| \leq |D_{n} P_{p+1}| + c_{p} \sum_{k=0}^{n} \binom{n}{k} |D_{k} P_{p-1}| |D_{n-k} P_2| \leq \frac{M_0 (1 + 2M_0)^n (n + p)!}{r_0^{n+p+1}} + c_{p} \sum_{k=0}^{n} \binom{n}{k} \frac{M_0^2 (1 + 2M_0)^n (k + p - 2)! (n - k + 1)!}{r_0^{n+p+1}} . \tag{43}
\]
Next, from \( c_p < 2p(p - 1) \), for \( p \geq 2 \), we get

\[
|D_{n+1} P_0| \leq \frac{M_0(1 + 2M_0)^p}{r_0^{n+p+1}} \left[ (n + p)! + 2M_0p(p - 1) \sum_{k=0}^{n} \binom{n}{k} (k + p - 2)! (n - k + 1)! \right].
\]

(44)

However, from the identity

\[
p(p - 1) \sum_{k=0}^{n} \binom{n}{k} (k + p - 2)! (n - k + 1)! = (n + p)!,
\]

we get

\[
|D_{n+1} P_0| \leq \frac{M_0(1 + 2M_0)^p}{r_0^{n+p+1}} [(n + p)! + 2M_0(n + p)!] = \frac{M_0(1 + 2M_0)^{n+1}(n + p)!}{r_0^{n+1}}
\]

as claimed in (42). Thus, the estimates in (42) are valid, and by putting \( p = 2 \), we find that (at \( i^0 \))

\[
\forall n \geq 0, \quad |D_n P_2| \leq \frac{M_0(1 + 2M_0)^n(n + 1)!}{r_0^{n+2}}.
\]

(46)

Since it is easy to find constants \( \tilde{M}, \tilde{r} \) such that \( \frac{M_0(1 + 2M_0)^n(n + 1)!}{r_0^{n+2}} \leq \frac{\tilde{M}}{\tilde{r}} \) for \( n = 0, 1, 2, \ldots \), theorem 1.1 of [3] follows.

Finally, to see that also the series expansion for \( \tilde{h}_{ij} \) converges we note that, near \( i^0 \), theorem 1.1 of [3] shows that the physical 3-metric \( g_{ij} \) exists. This means that \( \lambda(h_G)_{ij} = \Omega^{-2} \hat{h}_{ij} = \frac{e^{\nu}}{r} \hat{h}_{ij} \) exist as functions and in particular (after transvecting with \( x^i \)) that \( \frac{e^{\nu}}{r} x^i \) exist. This implies that the series expansions for \( \kappa \), and thus for \( \Omega \), and therefore also for \( \hat{h}_{ij} \), converge near \( i^0 \).

\( \square \)

3.8. The case \( P^0 = 0 \)

To complete the proof, we must now relax the condition \( m \neq 0 \), and this will be possible due to the fact that the conformal factor \( \Omega \) is well behaved for \( m = 0 \). Thus, suppose that we have a sequence of totally symmetric and trace-free tensors \( P_0, P_1, P_2, \ldots \) as in theorem 1, where \( P^0 = 0 \). In particular, it is assumed that \( u(r) = \sum_{|\alpha| \geq 0} \frac{P_0^\alpha}{r^{|\alpha|}} \) converges in some polydisk \( U : \{(x, y, z), |x| < d, |y| < d, |z| < d\} \). By replacing only the monopole, i.e. putting \( P^0 = m_0 > 0 \), the corresponding sequence \( m_0, P_1^0, P_2^0, \ldots \) corresponds to the function \( \tilde{u}(r) = m_0 + \sum_{|\alpha| \geq 0} \frac{P_0^\alpha}{r^{|\alpha|}} \), which also converges in \( U \). Thus, by the arguments given so far there exists convergent, in a polydisk \( V \) say, power series

\[
\hat{h}_{ij} = \sum_{|\alpha| \geq 0} (c_{ij})_{\alpha} r^\alpha
\]

for the metric components; furthermore, we also know that there is a static spacetime having the multipole moments \( m_0, P_1^0, P_2^0, \ldots \). Now, from the recursion producing the metric, it is seen that each coefficient \( (c_{ij})_{\alpha} \) is a polynomial in \( m_0 \), and hence the metric components \( \hat{h}_{ij} \) can be regarded as a power series in the four variables \( (m, x, y, z) \):

\[
\hat{h}_{ij} = \sum_{|\beta| \geq 0} (d_{ij})_{\beta} m^{\beta_0} x^{\beta_1} y^{\beta_2} z^{\beta_3} \rho^{\beta_4},
\]

where the multi-index \( \beta = (\beta_0, \beta_1, \beta_2, \beta_3, \beta_4) \). Since this series converges for \( m = m_0 \), \( (x, y, z) \in V \), it also converges for \( |m| < m_0, (x, y, z) \in V \). We can now choose \( m = 0 \) and still have convergence of \( \hat{h}_{ij} \), \( \kappa \) and \( g_{ij} \). In particular, with \( m = 0 \) the multipole moments will be the initially desired sequence \( P_0, P_1^0, P_2^0, \ldots \) where \( P^0 = 0 \).
4. Discussion

In this paper, we have proved that the necessary conditions from [8] on the multipole moments of a static spacetime are also sufficient in order for a static vacuum spacetime having these moments to exist. In particular, we do not assume that the monopole is non-zero. This proves a long-standing conjecture by Geroch, [2].

The conditions given are simple and natural. In essence, each allowed set of multipole moments is connected to a harmonic function on $\mathbb{R}^3$, defined in a neighbourhood of $0$. The proof is constructive in the sense that an explicit metric having prescribed moments up to a given order can be calculated.

Considering future work, it is natural to see what can be carried over to the general stationary case. It may be conjectured that the natural generalization of theorem 1 is valid, with the extra condition that the monopole is real and non-zero. It could also be instructive to explicitly calculate the metric belonging to a static spacetime with arbitrary multipole moments up to a given order.

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Appendix

Proof of lemma 7. Again we make an ansatz

$$A = \begin{pmatrix} X(x, y, z) & a(x, y, z) & b(x, y, z) \\ a(x, y, z) & Y(x, y, z) & c(x, y, z) \\ b(x, y, z) & c(x, y, z) & Z(x, y, z) \end{pmatrix}$$

where $a, b, c, X, Y$ and $Y$ are analytic in the variables indicated. The condition $A_{ij}x^i = 0$ is

$$xX(x, y, z) + ya(x, y, z) + zb(x, y, z) = 0$$
$$xa(x, y, z) + yY(x, y, z) + zc(x, y, z) = 0$$
$$xb(x, y, z) + yc(x, y, z) + zZ(x, y, z) = 0.$$  \(47\)

Writing $X(x, y, z) = X_2(x, y) + z X_3(x, y, z)$, $a(x, y, z) = a_2(x, y) + z a_3(x, y, z)$, $Y(x, y, z) = Y_2(x, y) + z Y_3(x, y, z)$, (47) with $z = 0$ shows that $a_2(x, y) = x a_4(x, y)$, $X_2(x, y) = -y a_4(x, y)$ for some function $a_4$. Next it follows that $b(x, y, z) = -y a_3(x, y, z) - x X_3(x, y, z)$. After the decomposition $Y(x, y, z) = Y_2(x, y) + z Y_3(x, y, z)$, (47) with $z = 0$ shows that $a_4(x, y) = y f_1(x, y)$, $Y_2(x, y) = -x^2 f_1(x, y)$ for some function $f_1$. From this we find that $c(x, y, z) = -x a_3(x, y, z) - y Y_3(x, y, z)$. Next we make the decomposition

$$a_3(x, y, z) = a_6(x, y) + z f_3(x, y, z)$$
$$X_3(x, y, z) = X_6(x, y) + z f_4(x, y, z)$$
$$Y_3(x, y, z) = Y_6(x, y) + z f_5(x, y, z)$$

from which (47), again with $z = 0$, shows that $X_6(x, y) = 2y f_2(x, y)$, $Y_6(x, y) = 2x f_3(x, y)$ for some function $f_2$ and $f_3$. Finally, it follows that $a_6(x, y) = -x f_2(x, y) - y f_3(x, y)$ and $Z(x, y, z) = 2xy f_5(x, y, z) + x^2 f_4(x, y, z) + y^2 f_6(x, y, z)$. By collecting terms, the lemma follows.
Proof of lemma 9. Observe that the definition of $u^{ij}$ implies that $u^{ij}_r r^2 = 0$, $i = 1, 2, 3$. We now demonstrate that $\tilde{t}_{11} = u^{ij}_1 \tilde{T}_{ab} \equiv 0 \pmod{r^2}$. From the definition of $\tilde{R}_{ij}$,
\[
u^i_1 u^1_1 R_{ij} = u^i_j \left( \frac{\partial}{\partial x^i} \Gamma^1_{ij} - u^1_j \frac{\partial}{\partial x^i} \Gamma^k_{kj} + u_1^i \Gamma^m_{ij} \Gamma^k_{mk} - u^{*}_{1} \Gamma^m_{ij} \Gamma^k_{mi} \right)
\]
where $\Gamma^m_{ij} = \frac{1}{2} \tilde{h}_{m\sigma} \left( \tilde{\eta}_{i\sigma} + \tilde{\eta}_{j\sigma} - \tilde{\eta}_{ij\sigma} \right)$ and $\Gamma^m_{in} = \frac{1}{2} \tilde{\eta}_{ii} \ln |\tilde{h}|$. Next, if $B$ denotes any symmetric $3 \times 3$ matrix, we can write
\[
\tilde{h}_{ij} = \eta_{ij} + p_{0x} x_j + r^2 B_{ij}
\]
from which it follows that
\[
\tilde{h}_{ij} = \eta_{ij} - p_{0x} x_j + r^2 B_{ij}.
\]
Therefore, $\tilde{h}_{m\sigma} \Gamma^k_{mj} \equiv \left( \eta_{m\sigma} - p_{0x} x_m x^\sigma \right) \frac{1}{2} \tilde{\eta}_{m\sigma} \ln |\tilde{h}| = \eta_{m\sigma} \frac{1}{2} \tilde{\eta}_{m\sigma} \ln |\tilde{h}| \pmod{r^2}$, where we have used that $D(\ln |\tilde{h}|) \equiv 0 \pmod{r^2}$. Thus, $u^i_1 u^1_1 \Gamma^m_{ij} \Gamma^k_{mk} \equiv \frac{1}{2} u^i_1 u^1_1 \left\{ \tilde{\eta}_{m\sigma} \Gamma^k_{mj} + \tilde{\eta}_{j\sigma} \Gamma^k_{mk} \right\}$.

Using that $\tilde{h}_{ij}$ can be written as in (48) and that $u^i_1 \tilde{\eta}_{m\sigma} \ln |\tilde{h}| \equiv 0 \pmod{r^2}$. A similar but longer calculation shows also that $u^i_1 u^1_1 \Gamma^m_{ij} \Gamma^k_{mk} \equiv 0 \pmod{r^2}$.

By expanding (11), using the above result, (20) and relations (10), it amounts to proving that (using the form (18) for the metric)
\[
u^i_1 u^1_1 (\tilde{\eta}_{i\sigma} + \tilde{\eta}_{j\sigma} - \tilde{\eta}_{ij\sigma}) \tilde{h}_{ij} + \eta_{ij} (2\Delta \kappa - D(f) - 2 f - 8 D(\kappa) - 16 \chi + D(\ln |\tilde{h}|)/r^2)) \equiv 0 \pmod{r^2}.
\]
Next, by replacing $\kappa = r^2 \chi$, it follows that $u^i_1 u^1_1 (2\tilde{D}_{ij} \tilde{D}_{k\chi} + \eta_{ij} (2\Delta \kappa - 8 D(\kappa) - 16 \chi) \equiv 0 \pmod{r^2}$ which leaves us with
\[
u^i_1 u^1_1 (\tilde{\eta}_{i\sigma} + \tilde{\eta}_{j\sigma} - \tilde{\eta}_{ij\sigma}) \tilde{h}_{ij} + \eta_{ij} (2\tilde{D}_{ij} \tilde{D}_{k\chi} + \eta_{ij} (2\Delta \kappa - 8 D(\kappa) - 16 \chi) \equiv 0 \pmod{r^2}.
\]
With $\ln |\tilde{h}| = r^2 \sigma$, we get $u^i_1 u^1_1 \tilde{\eta}_{i\sigma} \Gamma^k_{ij} = u^i_1 u^1_1 \tilde{\eta}_{i\sigma} \Gamma^k_{ij} (r^2 \sigma) \equiv \sigma u^i_1 u^1_1 \eta_{ij} \pmod{r^2}$. Next,
\[
\tilde{h}_{ij} = \eta_{ij} + p_{0x} x_j + r^2 B_{ij}
\]
and by using Leibniz’s rule, one finds that
\[
u^i_1 u^1_1 \tilde{\eta}_{i\sigma} \tilde{h}_{ij} \left( \tilde{\eta}_{i\sigma} \frac{\partial}{\partial x^j} \tilde{h}_{ij} \right) \equiv -u^i_1 u^1_1 \tilde{\eta}_{i\sigma} \tilde{h}_{ij} \tilde{h}_{ij} \pmod{r^2}
\]
and
\[
u^i_1 u^1_1 \tilde{h}_{ij} \tilde{h}_{ij} \left( \tilde{\eta}_{i\sigma} \frac{\partial}{\partial x^j} \tilde{h}_{ij} \right) \equiv \tilde{h}_{ij} \left( \tilde{\eta}_{i\sigma} \frac{\partial}{\partial x^j} \tilde{h}_{ij} \right) \pmod{r^2}.
\]
Some further manipulations reveals that in (50), $u^i_1 u^1_1 \tilde{\eta}_{i\sigma} \tilde{h}_{ij} \tilde{h}_{ij} \tilde{h}_{ij} \equiv 0 \pmod{r^2}$ while from (51) we find $u^i_1 u^1_1 \left( 2\tilde{\eta}_{i\sigma} \tilde{\eta}_{j\sigma} \tilde{h}_{ij} - \Delta C \tilde{h}_{ij} \right) \equiv u^i_1 u^1_1 (8 f \eta_{ij} + 6 D(f) \eta_{ij} - 6 \gamma_{ij} - 4 D(\gamma_{ij})) \pmod{r^2}$. Inserted in (49), we get, modulo $r^2$,
\[
u^i_1 u^1_1 \left( 2\tilde{\eta}_{i\sigma} \tilde{\eta}_{j\sigma} \tilde{h}_{ij} - \Delta C \tilde{h}_{ij} \right) \equiv \tilde{h}_{ij} \left( 2\tilde{\eta}_{i\sigma} \tilde{\eta}_{j\sigma} \tilde{h}_{ij} - \Delta C \tilde{h}_{ij} \right) \pmod{r^2}.
\]
where also $u^i_1 u^1_1 \tilde{\eta}_{i\sigma} \tilde{h}_{ij} \tilde{h}_{ij} \tilde{h}_{ij} \equiv -z \tilde{\eta}_{i\sigma} \tilde{h}_{ij} \tilde{h}_{ij} \pmod{r^2}$. To finish the proof, some linear algebra shows that $\ln |\tilde{h}| = -2 f r^2 + |\gamma_{ij}| r^2 \pmod{r^4}$. Finally an insertion in (52) gives, modulo $r^2$,
\[
u^i_1 u^1_1 (\gamma_{ij} + D(\gamma_{ij})) = -z \tilde{\eta}_{i\sigma} \tilde{h}_{ij} \tilde{h}_{ij} \pmod{r^4}.
\]
Now, for each degree in the series expansion of $\gamma_{ij}$, $D(\cdot)$ acts as a multiplication operator, and therefore equation (53) $\equiv 0 (\text{mod } r^2)$ is equivalent to $u_1 u_1^\dagger \gamma_{ij} + z^2 [\gamma_{ij}] \equiv 0 (\text{mod } r^2)$. However, $\gamma_{ij}$ is composed as the sum of the matrices $B_1, B_2, \ldots, B_6$ in lemma 7, and it is easily checked that $u_1 u_1^\dagger (B_k)_{ij} + z^2 [(B_k)_{ij}] \equiv 0 (\text{mod } r^2)$ for $k = 1, 2, \ldots, 6$, and therefore lemma 9 follows.

□

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