Violation of causality in $f(T)$ gravity

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Abstract In its standard formulation, the $f(T)$ field equations are not invariant under local Lorentz transformations, and thus the theory does not inherit the causal structure of special relativity. Actually, even locally violation of causality can occur in this formulation of $f(T)$ gravity. A locally Lorentz covariant $f(T)$ gravity theory has been devised recently, and this local causality problem seems to have been overcome. The nonlocal question, however, is left open. If gravitation is to be described by this covariant $f(T)$ gravity theory there are a number of issues that ought to be examined in its context, including the question as to whether its field equations allow homogeneous Gödel-type solutions, which necessarily leads to violation of causality on nonlocal scale. Here, to look into the potentialities and difficulties of the covariant $f(T)$ theories, we examine whether they admit Gödel-type solutions. We take a combination of a perfect fluid with electromagnetic plus a scalar field as source, and determine a general Gödel-type solution, which contains special solutions in which the essential parameter of Gödel-type geometries, $m^2$, defines any class of homogeneous Gödel-type geometries. We show that solutions of the trigonometric and linear classes ($m^2 < 0$ and $m = 0$) are permitted only for the combined matter sources with an electromagnetic field matter component. We extended to the context of covariant $f(T)$ gravity a theorem, which ensures that any perfect-fluid homogeneous Gödel-type solution defines the same set of Gödel tetrads $h^a_i$ up to a Lorentz transformation. We also showed that the single massless scalar field generates Gödel-type solution with no closed time-like curves. Even though the covariant $f(T)$ gravity restores Lorentz covariance of the field equations and the local validity of the causality principle, the bare existence of the Gödel-type solutions makes apparent that the covariant formulation of $f(T)$ gravity does not preclude non-local violation of causality in the form of closed timelike curves.

Keywords $f(T)$ gravity · modified gravity · Nonlocal violation of causality

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1 Introduction

The frameworks proposed to account for the observed late-time accelerated expansion of the Universe can be roughly grouped into two broad families. In the first, the unknown form of matter sources, the so-called dark energy, is invoked and the underlying gravity theory, general relativity (GR), is kept unmodified. In this framework, the simplest way to describe the accelerated expansion of the Universe is by introducing a cosmological constant, $\Lambda$, into the general relativity field equations. This approach is entirely consistent with the available observational data, but it faces difficulties related to the order of magnitude of the cosmological constant and its microphysical origin. In the second family, modifications of Einstein’s field equations are taken as an alternative for describing the late-time acceleration of the Universe. Examples in this class include generalized gravity theories based upon modifications of the Einstein-Hilbert action by taking nonlinear functions $f(R)$ of the Ricci scalar $R$ or other curvature invariants (for reviews see, e.g., Refs. [1, 2, 3, 4, 5, 6, 7]).

Another family of modified gravity theories that has been examined as an alternative way of describing the late-time acceleration of the Universe is known as $f(T)$ gravity. In close analogy with the $f(R)$, the $f(T)$ gravity theory was suggested by extending the Lagrangian of teleparallel gravity to a function $f(T)$ of a torsion scalar
In comparison with $f(R)$ in the formalism, whose field equations are of fourth-order, the $f(T)$ gravity theories have the advantage that their dynamics are given by second-order differential equations. This important characteristic, along with the fact that $f(T)$ theories can be used to explain the observed accelerating expansion, has given birth to a fair number of articles on these gravity theories, in which several features of general relativity, such as spacetime singularity, chronology protection, and local Lorentz invariance, are not invariant under local Lorentz transformations. The reason for this is that the field equations of general relativity are not invariant under local Lorentz transformations, and thus the theory is not Lorentz covariant. It has been pointed out that the field equations of general relativity are of fourth-order, the $f(T)$ theory is of second-order, and the $f(R)$ theory is of fourth-order. The reason for this is that the field equations of general relativity are not invariant under local Lorentz transformations, and the $f(T)$ theory is of second-order, and the $f(R)$ theory is of fourth-order. This means that unlike general relativity, a $f(T)$ gravity theory does not inherit locally a chronology protection from the special relativity, and one may even have a local violation of causality. This local causality problem seems to have been overcome in the Lorentz covariant $f(T)$ gravity theory, since in this new formulation of the theory the Lorentz transformations do not change neither the metric nor the field equations. A question that naturally arises here is whether the covariant formulation of $f(T)$ gravity allows Gödel-type solutions, which necessarily lead to nonlocal violation of the causality principle in the form of closed timelike curves, or would remedy this causal pathology by ruling out this type of solutions, which are permitted in general relativity. Moreover, if gravitation is to be described by this Lorentz covariant $f(T)$ theory, there are a number of issues that ought to be reexamined in its context, including the question as to whether these gravity theories allow noncausal solutions for physically well-behaved matter sources.

In this paper, we proceed with further investigations on the difficulties, limitations and potentialities of the Lorentz covariant $f(T)$ theory, by undertaking this question and examining whether this gravity theory admits homogeneous Gödel-type solutions for a quite general matter source. To Gödel’s solution has a recognizable importance and has motivated a fair number of investigations on rotating Gödel-type models in the context of general relativity (see, e.g. Refs. [68, 69, 70, 71, 72, 73, 74, 75] and in the framework of other gravity theories (see, for example, Refs. [76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90]).

The local violation causality is also related to ill-defined Cauchy problem in $f(T)$ gravity. These issues have been discussed in Refs. [91, 92, 93].
this end, we take a combination of a perfect fluid with electromagnetic plus a scalar fields as a matter source, and determine a general Gödel-type solution of the covariant $f(T)$ field equations. This general solution contains several special solutions, including a perfect-fluid and a single scalar field solutions. It emerges from our results that this general solution contains special solutions whose essential parameter, $m$, of Gödel-type geometries is positive (hyperbolic family), null (linear family) and negative (trigonometric class). Solutions in the trigonometric class only exist for combination of sources that includes an electromagnetic field as a matter component. We show that any perfect-fluid ST-homogeneous Gödel-type solution of the covariant $f(T)$ gravity is isometric to Gödel metric, and therefore exhibits violation of causality.\footnote{This extends to the context of covariant $f(T)$ gravity the Bampi-Zordan theorem \cite{Bampi2005}, which states that every perfect-fluid Gödel-type solution to the Einstein field equations is necessarily isometric to the Gödel space-time.}

The structure of the paper is as follows. In Section 2 we define the notation and make this paper to a certain extent relativity. In Section 3 we present our main properties of space-time (ST) homogenous Gödel-type geometries. This includes the metric, the ST-homogeneity conditions, the non-isometric ST-homogeneous Gödel-type classes, and the existence of closed time-like curves in these space-times. In Section 4 we show that the Lorentz covariant $f(T)$ gravity theories admit ST-homogeneous Gödel-type solutions for several physically well-motivated matter contents, and therefore despite the local Lorentz invariance, it houses non-local violation of causality. In Section 5 we present our main conclusions and final remarks.

2 Covariant $f(T)$ gravity

In this Section we briefly introduce the $f(T)$ gravity theory and its covariant formulation. For more details we refer the readers to the book \cite{Carroll2004}, where the notation, basic definition and proofs are presented, and to the article \cite{Pavon2004}, where the formulation of the covariant version of $f(T)$ gravity theory is presented. For a pedagogical presentation of the basic geometrical setting of these theories see Appendix J of Sean Carroll’s book \cite{Carroll2004}.

We begin by recalling that the dynamical variables in $f(T)$ gravity theories are the tetrad fields, $h^A_{\mu}$, which is a set of four ($A = 0, \ldots, 3$) vector fields that define a local orthonormal Lorentz frame at every point $x^\mu$ of the spacetime manifold. The vector fields $h^A_{\mu}$ are vectors in the tangent space at a arbitrary point $x^\mu$ of the spacetime manifold. The spacetime and the tangent space metrics are related by

$$g_{\mu \nu} = h^A_{\mu} h^B_{\nu} \eta_{AB}, \quad (1)$$

where $\eta_{AB} = \text{diag}(1, -1, -1, -1)$ is the Minkowski metric of the tangent space at $x^\mu$. Here and in what follows, we use Greek letters to denote spacetime coordinate indices, which are raised and lowered, respectively, with $g_{\mu \nu}$ and $g^{\mu \nu}$ and vary from 0 to 3, whereas Latin upper case letters denote tetrad indices, which are lowered and raised with the Minkowski tensor $\eta_{AB}$ and $\eta^{AB}$. It follows from equation (1) that the relation between frame components, $h^A_\mu$, and coframe components, $h^\mu_A$, are given by

$$h^A_\mu h^\mu_B = \delta^A_B \quad \text{and} \quad h^A_\mu h^\mu_A = \delta_B^A. \quad (2)$$

Under local Lorentz transformations at each point $x$ the tetrad fields transforms as

$$h_B \longrightarrow h'_A = \Lambda^A_B(x) h_B \quad (3)$$

and leave the metric invariant, i.e.

$$\eta'_{AB} = \Lambda^C_A \eta_{CD} \Lambda^D_B = \text{diag}(1, -1, -1, -1). \quad (4)$$

Considering that at every point of the spacetime besides the general coordinate transformations, $x^\mu \rightarrow x'^\mu$, we have the freedom to perform a Lorentz transformation one has that a mixed transformation law as, for example

$$M'^{\Lambda \mu}_{\nu B} = \Lambda^A_C \frac{\partial x'^\mu}{\partial x^\nu} \Lambda^D_B \frac{\partial x^\rho}{\partial x'^\nu} M^{CA}_D \quad (5)$$

For differentiating geometrical objects as, e.g., $X_B^A$ we replace the ordinary connection by the spin connection, denoted by $\omega^A_{B\mu}$, and each Latin upper case index gets a factor of the spin connection according to

$$\nabla_\mu X^A_B = \partial_\mu X^A_B + \omega^A_{C\mu} X^C_B - \omega^C_{B\mu} X^A_C. \quad (6)$$

Under local Lorentz transformations, $A^A_B(x)$, the spin connections transforms as

$$\omega'^A_{B\mu} = \Lambda^A_C \omega^C_{D\mu} \Lambda^{D}_B + A^A_C \partial_\mu \Lambda^{C}_B \quad (7)$$

where $A^A_B$ is the inverse of $A^A_B$. The spin connection in teleparallel gravity is meant to represent only inertial or frame effects. This means that there exists a class of frames relative to which the spin connection vanishes, $\omega^C_{D\mu} = 0$. 

\[ \text{This extends to the context of covariant } f(T) \text{ gravity the Bampi-Zordan theorem } \cite{Bampi2005}, \text{ which states that every perfect-fluid Gödel-type solution to the Einstein field equations is necessarily isometric to the Gödel space-time.} \]
From this fact along with equation (7) one has that in a general class of frames the spin connection takes the form

$$\omega^A_{\mu \nu} = \Lambda^A_C \partial_\mu \Lambda^C_B.$$  \(8\)

In covariant $f(T)$ gravity theories, instead of the Levi-Civita connection, one uses the Weitzenböck connection which is given by

$$\Gamma^\rho_{\nu \mu} = h^\rho_A \left( \partial_\mu h^A_\nu + \omega^A_{\mu \nu} h^B_\nu \right),$$  \(9\)

with inverse relation

$$\omega^A_{\mu \nu} = h^\rho_A \partial_\mu h^B_\nu + h^\rho_A \Gamma^\rho_{\nu \mu} h^B_\nu \equiv h^\rho_A \nabla_\mu h^B_\nu,$$  \(10\)

where $\nabla_\mu$ denotes the covariant derivative. An immediate consequence of this definition for the covariant derivative is

$$T^\mu_{\nu \lambda} = h^\rho_A \partial_\mu h^B_\nu - \partial_\nu h^A_\lambda + \omega^A_{\mu \nu} h^B_\lambda - \omega^A_{\mu \lambda} h^B_\nu,$$  \(11\)

where in terms of the tetrad fields, the torsion tensor is defined by \(19\)

$$T^A_{\mu \nu} \equiv \partial_\nu h^A_\mu - \partial_\mu h^A_\nu + \omega^A_{\mu \nu} h^B_\lambda - \omega^A_{\mu \lambda} h^B_\nu.$$  \(12\)

Now, if one further defines the so-called super-potential

$$S^\mu_{\nu} \equiv K^\mu_{\nu} + h^A_\mu T^\nu_A - h^A_\nu T^\mu_A,$$  \(13\)

where

$$K^\mu_{\nu} \equiv -\frac{1}{2} (T^\mu_{\lambda \nu} - T^\nu_{\mu \lambda} - T^\lambda_{\mu \nu}),$$  \(14\)

is the contorsion tensor, we can define the torsion scalar

$$T = \frac{1}{2} S^\mu_{\nu} T^A_{\mu \nu}.$$  \(15\)

This scalar is used in the Lagrangian density for the formulation of the covariant $f(T)$ gravity theory, whose action is defined by

$$S = \int d^4x \ h \left[ \frac{f(T)}{2\kappa} + L_m \right],$$  \(16\)

where $h = \det(h_{\mu \nu})$, $\kappa = 8\pi G$, $f(T)$ is an arbitrary function of torsion scalar $T$ and $L_m$ is the Lagrangian density for the matter field.\ref\textsuperscript{6}

Varying the action (16) with respect to the vierbein and taking into account equations (12) - (15) one obtains the following equations for the covariant $f(T)$ gravity:

$$h^{-1} f_T \partial_\nu \left( h S^\nu_{\mu} \right) + f_{TT} S^\nu_{\mu} \partial_\nu T - f_T h^A_\mu T^B_{\nu \lambda} S^\nu_{\nu} + f_T \Theta^A_{\nu \mu} + \frac{1}{2} f(T) h^A_\mu = \kappa h^A_\nu \Theta^A_{\nu},$$  \(17\)

where $f_T = df(T)/dT$, $f_{TT} = d^2f(T)/dT^2$, and $\Theta^A_{\nu}$ is the energy-momentum tensor of the matter fields defined as

$$\Theta^\mu_\nu = \left( \frac{1}{h} \frac{\delta S_m}{\delta h^\mu_\nu} \right) h^\nu_\mu,$$  \(18\)

where $S_m = \int d^4x h L_m$ is the matter action.

It should be emphasized that, unlike the usual formulation of $f(T)$ gravity, where it is implicitly assumed that the spin connection $\omega^A_{\mu \nu}$ vanishes, in the derivation of the above field equations (17) a nonvanishing $\omega^A_{\mu \nu}$ is assumed from the outset. Actually, the presence of the nonzero spin connection term in field equations (17) ensures the Lorentz covariance of equations (17) (for details see Ref. 66). The price for this is that now we have to figure out a way to determine the nonvanishing spin connection $\omega^A_{\mu \nu}$, a point that we shall discuss in the following.

The covariant field equations (17) depend not only on the vierbein but also on the nonvanishing spin connection $\omega^A_{\mu \nu}$. Thus, solutions to the covariant field equations can be found if one establishes a procedure to suitably determine the spin connection $\omega^A_{\mu \nu}$. In what follows we shall briefly present a scheme figured out in Ref. 66 to determine $\omega^A_{\mu \nu}$. It consists in starting from the arbitrarily chosen initial tetrad field $h^A_\mu$, define a reference frame, $h^A_{(r)\mu}$, in which gravity is switched off, namely

$$h^A_{(r)\mu} \equiv h^A_\mu \mid_{\text{Grav} \rightarrow 0}.$$  \(19\)

In other words, when the gravity is switched off the metric

$$g_{\mu \nu} = h^A_{(r)\mu} h^B_{(r)\nu} \eta_{AB}$$  \(20\)

reduces to the Minkowski metric.

Following the procedure of Ref. 66, we then require the torsion to vanish for this reference tetrad, i.e. $T^A_{\mu \nu}(h^A_{(r)\mu}, \omega^A_{\mu \nu}) = 0$. Thus, to have the Lorentz covariant field equations we take the spin connection for the field equations (17) as

$$\omega^A_{\mu \nu}(h^A_{(r)\mu}) \equiv \omega^A_{\mu \nu}(h^A_{(r)\mu}).$$  \(21\)

The underlying basic idea behind this choice of spin connection is that the torsion tensor (12) depends on both the tetrad and spin connection. Thus, in general the torsion embodies the field strength of gravity along with inertial effects. However, there exists a choice of the spin connection for which the torsion captures only the strength of gravity. This spin connection is obtained by using the reference tetrad $h^A_{(r)\mu}$ as defined by (19) (or (20)), and then by defining $\omega^A_{\mu \nu}$ according to (21) (for more details see Ref. 66).\footnote{In the usual $f(T)$ gravity the field equations are differential equations for the tetrads only.}
In practice, this can be made by taking, in the general expression for the spin connection, (Eq. (1.60) of Ref. [19]),
\[
\omega_{\mu}^{A} = \frac{1}{2} h_{\mu}^{A} \left[ f_{B}^{A} c + T_{B}^{A} c + f_{C}^{A} B + T_{C}^{A} B - f_{B}^{A} c - T_{C}^{A} c \right],
\]
(22)
where \([h_{A}, h_{B}] = f_{A}^{A} B c, \) and the torsion \( T_{BC}^{A} = h_{B}^{A} c + T_{BC}^{A} c = 0 \) for the reference vierbein where \( h_{A}^{A} \) is the expression for the spin connection, (Eq. (1.60) of Ref. [19],).

Gödel-type space-times admit either a group \( \mathbb{R}^{4} \) or \( \mathbb{R}^{3} \). In this way, one has
\[
\omega_{\mu}^{A} = \frac{1}{2} h_{\mu}^{A} \left[ f_{B}^{A} (h_{A}^{(i)}) + f_{C}^{A} (h_{A}^{(j)}) - f_{B}^{A} (h_{A}^{(i)}) \right],
\]
(23)
where the structure coefficients \( f_{B}^{A} \) are calculated from the expression (Eq. (1.32) of Ref. [19]),
\[
f_{A}^{B} = h_{A}^{B} h_{C}^{D} \left( \partial_{A} h_{C}^{D} - \partial_{D} h_{C}^{A} \right),
\]
(24)
evaluated for the reference vierbein \( h_{A}^{A} \).

3 Gödel-type geometries

In this section we present the main properties of the homogeneous Gödel-type geometries, which we use in the next section. We begin by recalling that Gödel solution to the general relativity field equations is a specific member of the large family of geometries, whose general form in cylindrical coordinates, \((r, \phi, z)\), is \([97]\)
\[
ds^{2} = [dt + H(r) d \phi]^{2} - D^{2}(r) d \phi^{2} - dr^{2} - dz^{2}.
\]
(25)
The necessary and sufficient conditions for the Gödel-type geometries \([25]\) to be space-time homogeneous (ST-homogeneous) are \([97,98]\)
\[
H' = 2 \Omega \quad \text{and} \quad \frac{D''}{D} = m^{2},
\]
(26)
where the prime denote derivative with respect to \(r\), and the parameters \((\Omega, m)\) are constants such that \(-\infty \leq m^{2} \leq \infty \) and \(\Omega^{2} > 0\). The ST-homogeneity is insured by the fact that Gödel-type space-times admit either a group \( G_{5} \) or \( G_{7} \) of isometries acting transitively on the whole space-time \([98,99]\).

Another important property of Gödel-type geometries is the existence of an irreducible set of isometrically nonequivalent classes of ST-homogeneous Gödel-type, which are given by \([97]\) (see also Ref. [100]).

i. Hyperbolic, in which \(m^{2} = \text{const} > 0\) and
\[
H = \frac{4 \Omega}{m^{2}} \sinh^{2} \left( \frac{mr}{2} \right), \quad D = \frac{1}{m} \sinh (mr);
\]
(27)

ii. Linear, in which \(m = 0\) and
\[
H = \Omega r^{2}, \quad D = r,
\]
(28)

iii. Trigonometric, where \(m^{2} = \text{const} \equiv -\mu^{2} < 0\) and
\[
H = \frac{4 \Omega}{m^{2}} \sin^{2} \left( \frac{mr}{2} \right), \quad D = \frac{1}{m} \sin (mr).
\]
(29)

Accordingly, the ST-homogeneous Gödel-type metrics are characterized by the essential parameters \(m^{2}\) and \(\Omega\). Incidentally, Gödel’s solution of Einstein’s equations is a particular member of hyperbolic class \((m^{2} > 0)\) in which \(m^{2} = 2 \Omega^{2} \).

To examine the causality features of the ST-homogeneous Gödel-type metrics we rewrite the line element \([25]\) as
\[
ds^{2} = dt^{2} + 2 H(r) dt d \phi - D^{2}(r) d \phi^{2} - dr^{2} - dz^{2},
\]
(30)
where \(G(r) = D^{2}(r) - H^{2}(r)\). In this form it is straightforward to show the existence of closed time-like curves (CTC), i.e. make explicit the violation of causality in ST-homogeneous Gödel-type space-times. Indeed, from Eq. \([30]\) one has that the circles defined by \(t, z, r = \text{const}\) with \(r < 0\) are closed timelike curves whenever \(G(r) < 0\) (the line element \(ds^{2}\) becomes spacelike, and its integral curves are closed). Thus, for the hyperbolic class \((m^{2} > 0)\) one finds that \(G(r) = D^{2}(r) - H^{2}(r)\) is given by
\[
G(r) = \frac{4 \Omega^{2}}{m^{2}} \sinh^{2} \left( \frac{mr}{2} \right) \left[ \left( 1 - \frac{4 \Omega^{2}}{m^{2}} \right) \sinh^{2} \left( \frac{mr}{2} \right) + 1 \right].
\]
(31)

Therefore for \(0 < m^{2} < 4 \Omega^{2}\) there is a finite critical radius \(r_{c}\) defined by \(G(r) = 0\) given by
\[
\sinh^{2} \left( \frac{m r_{c}}{2} \right) = \left( \frac{4 \Omega^{2}}{m^{2}} - 1 \right)^{-1},
\]
(32)
and such that \(G(r) > 0\) for \(r < r_{c}\) and \(G(r) < 0\) for \(r > r_{c}\). Therefore, the circles \(t, z, r = \text{const}\) in the circular band with \(r > r_{c}\) are CTC’s. One particularly important case in the range \(0 < m^{2} < 4 \Omega^{2}\) of the hyperbolic class is the Gödel metric \((m^{2} = 2 \Omega^{2})\), for which there is a finite critical radius and therefore breakdown of causality in the form of CTC’s. Another important Gödel type metric in the hyperbolic family is defined by \(m^{2} = 4 \Omega^{2}\). Indeed, in this case from \([32]\) one has that the critical radius goes to infinity, \(r_{c} \to \infty\), and therefore there is no violation of causality, since \(G(r) > 0\) for all \(0 < r < \infty\). An Einstein’s field equations solution of this specific type was found in Ref. [97].

For the linear family \((m = 0)\) one easily finds that \(G(r) = D^{2}(r) - H^{2}(r)\) is given by
\[
G(r) = r^{2} (1 - \Omega r) (1 + \Omega r).
\]
(33)

Similarly to the hyperbolic class, for this family there is a critical radius \([G(r) = 0]\) given by \(r_{c} = 1/\Omega\) such that for
any radius \( r > r_c \), the inequality \( G(r) < 0 \) holds, and thus circles defined by \( t, z, r = \text{const} \) are CTC’s.

Finally, for the trigonometric class \( m^2 < 0 \) one finds that \( G(r) \) reduces to
\[
G(r) = \frac{4}{\mu^2} \sin^2 \left( \frac{\mu r}{2} \right) \left[ 1 - \left( 1 + \frac{4 \Omega^2}{\mu^2} \right) \sin^2 \left( \frac{\mu r}{2} \right) \right],
\]
but now differently from the other two class, there is an infinite sequence of alternating causal \([G(r) > 0]\) and noncausal \([G(r) < 0]\) regions (circular bands) in the section \( t, z, r = \text{const} \) (with \( r > 0 \)) without and with noncausal Gödel’s circles, depending on the value of \( r \). Thus, e.g., if \( G(r) < 0 \) for a certain range \( r_1 < r < r_2 \) noncausal circles exist, whereas for \( r < r_1 \) and for the next circular region \( r_2 < r < r_3 \) for which \( G(r) > 0 \) no such noncausal Gödel’s circles exist \([27][100]\).

To close this section, we mention that throughout this work by non-causal and causal solutions we mean solutions with and without violation of causality of Gödel-type, i.e., with and without the above-discussed Gödel’s noncausal circles.

### 4 Gödel-type solutions in the covariant \( f(T) \) gravity

The aim of this section is twofold. First, following the scheme devised in Ref. \([66]\) we determine the spin connection associated with a Lorenz tetrad of Gödel-type space-time geometries \([25]\), which allow for the field equations for these space-times in the covariant \( f(T) \) gravity. Second, we take a combination of a perfect fluid with electromagnetic plus a scalar fields as source, and determine a general Gödel-type solution of the covariant \( f(T) \) field equations, and discuss its main important properties.

#### 4.1 Field equations

At an arbitrary point Gödel-type space-time manifold we choose the following tetrad basis \( h^\mu_A = h^\mu_A dx^A \):
\[
\begin{align*}
\theta^0 &= dt + H(r) d\varphi, \quad \theta^1 = dr, \\
\theta^2 &= D(r) d\varphi, \quad \theta^3 = dz,
\end{align*}
\]
relative to which the Gödel-type line element \([25]\) clearly takes the form
\[
ds^2 = \eta_{\lambda\beta} \theta^\lambda \theta^\beta = \eta_{\lambda\beta} h^\lambda_A h^\beta_B dx^\lambda dx^\beta,
\]
\[
\eta_{\lambda\beta} = \text{diag}(+1, -1, -1, -1), \quad \text{and the components of the tetrad fields } h^\mu_A \text{ and } h^\mu_A \text{ are given by}
\]
\[
h^\mu_A = \begin{pmatrix} 1 & 0 & H & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & D \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad h^\mu_A = \begin{pmatrix} 1 & 0 & -D' & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

From equations \((37)\) and \((38)\) one has that gravity is switched off when \( H(r) \to 0 \) and \( D(r) \to r \), inasmuch as in this limit the tetrads reduce to the tetrads of the Minkowski space-time in cylindrical coordinates, namely \( h^A_{\mu\nu} = \text{diag}(1, 1, r, 1) \). Thus, following the scheme devised in Ref. \([66]\) the reference frame, \( h^A_{(r)\mu} \), defined by
\[
h^A_{(r)\mu} \equiv h^A_{\mu} \mid_{\text{Grav} \to 0},
\]
is given by
\[
h^A_{(r)\mu} = h^A_{\mu|0} = \text{diag}(1, 1, r, 1).
\]

For this reference frame by using Eq. \((24)\) we obtain the following nonvanishing components of the coefficients of anholonomy:
\[
f^2_{21} = f^3_{13} = \frac{1}{r},
\]
and then, by using Eq. \((23)\), one finds that the nonzero spin connection associated to the reference frame \((40)\) is given by
\[
\omega^1_{12} = -\omega^1_{22} = 1,
\]
where here and in what follows the hat over the numerical indices is used to denote that the corresponding digits are tetrad indices.

To have the Lorentz covariant field equations we take the spin \( \omega^A_{\mu|0} (h^A_{\mu}) = \omega^A_{\mu|0} (h^A_{(r)\mu}) \). Thus, from equations \((38)\) and \((42)\) the field equations \((47)\) reduce, for Gödel-type spacetimes, to
\[
\begin{align*}
\frac{H T'}{2H} + D' \frac{D'}{D} f_{TT'} + \frac{H T'}{2} T' + \frac{1}{D} f_{TT'} + \frac{f}{2} = \kappa \Theta_0^0, \\
\frac{D^2 T' + HT}{2H} f_T + \frac{HT'}{2} f_{TT} - \frac{H}{2} f = \kappa \left( \Theta_0^2 - H \Theta_0^2 \right), \\
-\frac{T'}{H} \left[ \frac{f}{2} + T f_{TT} \right] = \kappa \Theta_1^1, \\
-\frac{T'}{H} \left[ \frac{f}{2} + T f_{TT} \right] = \kappa \Theta_0^2, \\
\frac{1}{D} f_{TT'} - \frac{D'}{D} T' f_{TT} - \frac{D'}{D} T' f_T - \frac{f}{2} = \kappa \Theta_3^3,
\end{align*}
\]
\[\text{We note that the additive term } D^{-1} f_{TT'} \text{ in equations } (43) \text{ and } (48) \text{ do not arise in the usual formulation of } f(T) \text{ gravity. For comparison we refer the readers to Ref. } [101], \text{ where similar field equations were derived in the context of the usual } f(T) \text{ gravity.}\]
where we have used that, from Eq. (15) along with Eqs. (13), (14) and (12), the torsion scalar is given by
\[
T = \frac{1}{2} \left( \frac{H^2}{D} \right)^2.
\]

Since in this paper we are interested in ST-homogeneous Gödel-type space-times, the homogeneity conditions (26) holds, and therefore torsion scalar is given by
\[
T = 2\Omega^2,
\]
and the field equations (43) to (48) reduce to
\[
(2\Omega^2 - m^2) f_t + f = \kappa \Theta^0_0, \tag{51}
\]
\[
2\Omega^2 H f_t - \frac{H}{2} f = \kappa (\Theta^0_1 - H \Theta^0_0), \tag{52}
\]
\[
-2\Omega^2 f_t + f = \kappa \Theta^i_1 = \kappa \Theta^2_2, \tag{53}
\]
\[
\Theta^0_0 = 0, \tag{54}
\]
\[
-\Theta^2_1 + \frac{f}{2} = \kappa \Theta^3_3. \tag{55}
\]
Since the homogeneous Gödel-type metric (25) and tetrads $h^A_\mu$ [Eq. (38)] are determined by the parameters $m^2$ and $\Omega$, solving equations (51) – (55) for these parameters one finds
\[
m^2 = \frac{\kappa}{f_t} (\Theta^0_0 + \Theta^1_1 - 2\Theta^3_0), \tag{56}
\]
\[
\Omega^2 = \frac{\kappa}{2f_t} (\Theta^0_0 - \Theta^3_3), \tag{57}
\]
subject to the constraints
\[
f = 2\kappa (\Theta^0_0 + \Theta^1_1 - \Theta^3_3), \tag{58}
\]
\[
\Theta^0_0 - H (\Theta^0_0 - \Theta^3_3) = \Theta^2_2 = 0, \tag{59}
\]
where the functions $f$ and $f_t$ are evaluated at $T = 2\Omega^2$, and where we assume that $f_t > 0$ to ensure that the effective Newton constant does not change its sign.

It should be emphasized that in the field equations (56) – (59) no specific energy momentum tensor have been used. In the next subsection, a concrete matter content will be considered in the search for ST-homogeneous Gödel-type solutions of the covariant $f(T)$ field equations. To this end, we shall take a combination of a perfect fluid with scalar field along with an electromagnetic field. Clearly, the scalar and electromagnetic field sources have to fulfill, respectively, the Klein-Gordon and Maxwell equations in the curved Gödel-type background geometry.

### 4.2 Solutions

In this Section we discuss ST-homogeneous Gödel-type solutions of the covariant $f(T)$ gravity for specific matter sources. To this end, we take combination of scalar and electromagnetic fields with a perfect fluid as a matter source, find a general solution, and examine several special Gödel-type solutions. Thus, in the local frame defined by equations (35) and (36) the energy-momentum tensor $\Theta_{AB}$ for the combined fields has the form
\[
\Theta_{AB} = \Theta_{AB}^{sf} + \Theta_{AB}^{ef}, \tag{60}
\]

where $\Theta_{AB}^{sf}$ and $\Theta_{AB}^{ef}$ are, respectively, the energy-momentum tensors of a perfect fluid, a scalar field and an electromagnetic field, which are given by
\[
\Theta_{AB}^{sf} = (\rho + p) u_A u_B - \rho \eta_{AB}, \tag{61}
\]
\[
\Theta_{AB}^{ef} = \nabla_A \phi \nabla_B \phi - \eta_{AB} \left[ \frac{1}{2} \nabla^C \nabla_C \phi \nabla_D \phi \right], \tag{62}
\]
\[
\Theta_{AB}^{ef} = -F_A^C F_{BC} + \frac{1}{4} \eta_{AB} F_C^D F^{CD}, \tag{63}
\]

where $u_A = \delta^A_0$ is the four-velocity, $\rho$ and $p$ are the energy density and pressure of the perfect fluid, subject to the weak energy condition (WEC) $\rho > 0$ and $\rho + p > 0$, and where $\nabla_A \phi$ denotes the covariant derivatives relative to the local basis $\theta^A = h^A_\mu dx^\mu$.

The massless scalar field $\phi$ fulfills the Klein-Gordon equation
\[
\Box \phi = \eta^{AB} \nabla_A \nabla_B \phi = 0, \tag{64}
\]
a solution of which can be written as $\phi = \phi (z) = s(z - z_0)$, where $s, z_0 = \text{const}$ (27). Thus, the non-vanishing components of the energy-moment tensor are
\[
\Theta^{sf}_{00} = -\Theta^{sf}_{11} = -\Theta^{sf}_{22} = \Theta^{sf}_{33} = \frac{s^2}{2}, \tag{65}
\]

The electromagnetic field $F_{AB}$ satisfies the source-free Maxwell equations
\[

\nabla_A F^{AB} = 0 \quad \text{and} \quad \nabla_A (\ast F^{AB}) = 0,
\]

where $\ast F^{AB} = \frac{1}{4} \epsilon^{ABCD} F_{CD}$. Following Ref. (27), it can be shown that the electromagnetic field tensor $F_{AB}$ given by
\[
F_{03} = e \sin [2\Omega (z - z_0)], \tag{66}
\]
\[
F_{12} = e \cos [2\Omega (z - z_0)],
\]

where $e$ is constant, satisfies Maxwell equations (66). Hence, the non-vanishing components of the associated energy-momentum tensor $\Theta_{AB} = F_A^C F_{BC} + \frac{1}{4} \eta_{AB} F_C^D F^{CD}$ are given by
\[
\Theta^{ef}_{00} = \Theta^{ef}_{11} = \Theta^{ef}_{22} = \Theta^{ef}_{33} = \frac{e^2}{2}. \tag{67}
\]

Thus, taking into account equations (64), (65) and (66), one has that for the combined-fields matter source (60), and
therefore from equation (56), (57) and (58) one has that a general solution to the field equations can be written as:

\[ m^2 = \frac{K}{f_T} \left( p + p + 2s^2 - e^2 \right), \]  
\[ \Omega^2 = \frac{K}{2f_T} \left( p + p + s^2 \right), \]  
\[ f = \kappa \left( 2p + 3s^2 - e^2 \right), \]  

and equations (59) are identically satisfied.

Since ST-homogeneous Gödel-type geometries (25) and tetrads (38) are characterized by the two essential parameters \( m^2 \) and \( \Omega^2 \), the above equations (70) and (71) along with the constraint equation (72) make explicit how the covariant \( f(T) \) gravity specifies a pair of parameters \( (m^2, \Omega^2) \), and therefore determines a general ST-homogeneous Gödel-type solution for the combined-fields matter source (60). In the remainder of this Section we discuss the most significant special cases of this general solution and some important characteristics of these Gödel-type solutions in the covariant \( f(T) \) gravity.

A first important point regarding the above combined field general solution is that from equation (70) \( m^2 \) is bounded from above by the constraint equation (72) such that for any \( \kappa(r, s, t) = 0 \) there is an upper bound for \( m^2 \) since

\[ m^2 - 4\Omega^2 = -\frac{K}{f_T} \left( p + p + e^2 \right) \leq 0. \]  

From equations (70) and (71) it is easy to show that the limiting bound solution \( m^2 = 4\Omega^2 \) takes place for single scalar field of amplitude \( s \) as the matter source. Besides, according to Eq. (32) of Section 3 the critical radius goes to infinity, \( r_c \to \infty \), and therefore there is no violation of causality for this particular solution. Apart from this particular case, from equation (32) along with (70) and (71) one has that for all solutions in the hyperbolic class with \( m^2 < 4\Omega^2 \) there is a finite critical radius, beyond which causality is violated, given by

\[ r_c = \frac{2(f_T)^{1/2} \sinh^{-1} \left( \sqrt{\frac{\rho+p+2s^2-e^2}{\rho+p+e^2}} \right)}{[\kappa(\rho+p+2s^2-e^2)]^{1/2}}, \]  

As one would expect from the outset this critical radius depends upon the \( f(T) \) gravity theory and the matter content.

A particularly important case in the hyperbolic family is the Gödel spacetime, in which \( m^2 = 2\Omega^2 \). From equations (70) and (71) there are two possible matter sources that give rise to Gödel solution in covariant \( f(T) \) gravity. First, a perfect fluid of density \( \rho \) and pressure \( p \) subject to the WEC and with \( f_T > 0 \). In this case \( s^2 = e^2 = 0 \) and therefore \( m^2 = 2\Omega^2 \) holds. The fact that all perfect fluid solutions subject to the WEC are isometric to Gödel geometry extends to the context of covariant \( f(T) \) gravity with \( f_T > 0 \) the Bampi and Zordan theorem [95] demonstrated in the framework of GR. Second, for a simple combination of the above scalar and electromagnetic fields \( (\rho = p = 0) \) from equations (70) and (71) one has

\[ m^2 = 2\Omega^2 = \frac{K}{f_T} (s^2 - e^2), \]  

Therefore, for \( s^2 = e^2 \neq 0 \) Gödel geometry is again recovered.

Regarding the linear and trigonometric families of Gödel-type space-times, from Eq. (70) we clearly have that to have either one of these classes the existence of the electromagnetic field component is required. The solutions in the linear class \( (m^2 = 0) \) are obtained when \( \rho + p + 2s^2 - e^2 = 0 \). Thus, for example, a simple combination of the the above scalar and electromagnetic fields such that \( e^2 = 2s^2 \) gives rise to a solution in the linear class. Since \( \rho + p > 0 \) from WEC, in general solutions in the linear class for the combined fields source arise only when \( e^2 \geq 2s^2 \). According to Section 3 and similarly to the hyperbolic class, for solutions belonging to the linear family there is a critical radius \( r_c = 1/\Omega \) such that for any \( r > r_c \) the Gödel circles (CTC) defined by \( t, z, r = \text{const} \) are present.

Finally, solutions in the trigonometric class \( (m^2 = -\mu^2 < 0) \) arise when \( \rho + p + 2s^2 - e^2 < 0 \). Very simple examples of solutions in this class come about when \( \rho = p = 0 \) and \( e^2 > 2s^2 \), but clearly we can have solutions with all the above combined field matter components. As for the violation of causality, according to Section 3 for the trigonometric class there is an infinite sequence of alternating causal and noncausal circular bands in the section \( t, z, r = \text{const} \), with and without noncausal Gödel’s circles, depending on whether \( G(r) < 0 \) or \( G(r) > 0 \).

5 Concluding remarks

Despite the undeniable success of the general relativity theory, there has been a great deal of recent works on the so-called modified gravity theories. In the cosmological modeling this is motivated by the fact that these modified theories furnish an alternative way to account for the recent expansion of the Universe with no need to invoke the dark energy matter component. Several features of a family of these modified gravity theories, known as \( f(T) \) gravity, have been discussed recently in a number of articles. Despite this noticeable interest in this new gravitational theory, it appears to have been overlooked in the recent literature, that the \( f(T) \)
field equations are not invariant under local Lorentz transformations, unless the \( f(T) \) is a linear function of \( T \), which is the teleparallel equivalent of general relativity (TEGR). This means that apart from the TEGR, different choices of the Lorentz tetrads give rise to different field equations. Relaxing a particular choice of the spin connection often made implicitly in the literature a locally Lorentz covariant \( f(T) \) gravity theory has been devised recently [66].

A well-behaved chronology and causality are so essential in the special relativity theory that they are simply incorporated into the theory from its bare formulation. General relativity (GR) inherits locally the chronology and causality features of special relativity. On nonlocal scale, however, important differences emerge and Einstein’s field equations admit solutions to its field equations with closed timelike world lines, despite its local invariance under Lorentz transformations that ensures locally the validity of the causality.

Even though different choices of local Lorentzian tetrads leave the metric invariant, suggesting at first sight a well-defined chronology and well-behaved causal structure, in the standard formulation of \( f(T) \) gravity every different Lorentz tetrads give rise to different field equations, and therefore represents, in general, a different theory. Hence, for a given (fixed) Lorentz frame, which determines fixed set of field equations (a fixed theory), there is no Lorentz transformation freedom. This means that unlike general relativity, a \( f(T) \) gravity does not inherit locally the chronology and the well-behaved causal structure of special relativity.

This local causality problem seems to have been overcome in the Lorentz covariant \( f(T) \) gravity [66], since in the new formulation of \( f(T) \) theory the Lorentz transformations do not change neither the metric nor the field equations. The nonlocal question, however, is left open and violation of causality may arise in the context of the covariant \( f(T) \) gravity theory.

Since homogeneous Gödel-type geometries necessarily lead to the existence of closed timelike circles — an explicit manifestation of violation of causality on nonlocal scale —, a plausible way to answer this question is by investigating whether the covariant \( f(T) \) gravity theories permit Gödel-type solutions. Furthermore, if gravity is to be governed by a covariant \( f(T) \) theory there are a number of issues ought to be examined in its context, including the question as to whether these theories permits Gödel-type solutions, or would remedy this causality problem by ruling them out.

In this paper, to look into the potentialities, difficulties and limitations of the covariant \( f(T) \) gravity theories, we have undertaken one of these questions and examined whether these theories admit homogeneous Gödel-type solutions for a number of matter sources. We have taken a combination of a perfect fluid with electromagnetic plus a scalar fields as a matter source, and determined a general Gödel-type solution. This general solution contains special solutions whose essential parameter, \( m^2 \), can take the sign that defines any class of homogeneous Gödel-type geometries, namely hyperbolic family (\( m^2 > 0 \)), trigonometric class (\( m^2 < 0 \)) and linear class (\( m = 0 \)). We have found that solutions in the trigonometric and linear classes are permitted only for the combined matter sources that includes an electromagnetic field matter component. We have extended to the context of covariant \( f(T) \) gravity the so-called Bambi-Zordan theorem, which ensures that any perfect-fluid ST-homogeneous Gödel-type solution in this theory \( f(T) \) gravity is isometric to Gödel metric, and therefore defines the same set of Gödel tetrads \( h_A^\mu \) up to a Lorentz transformation.

We have also shown that the single massless scalar field generates the only ST-homogeneous Gödel-type solution with no closed timelike curves.

To conclude, we underline that it emerges from our results that even though the covariant \( f(T) \) gravity restores local Lorentz covariance of the field equations and the validity of the causality principle locally, the bare existence of the Gödel-type solutions makes apparent that this covariant formulation of \( f(T) \) gravity does not avoid non-local violation of causality in the form of closed timelike curves that are permitted in general relativity.

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References

1. S. Capozziello, M. Francaviglia, Extended Theories of Gravity and their Cosmological and Astrophysical Applications. Gen. Relativ. Gravit. 40, 357 (2007)
2. A. De Felice, S. Tsujikawa, \( f(R) \) theories. Living Rev. Rel. 13, 3 (2010)
3. T.P. Sotiriou, V. Faraoni, \( f(R) \) Theories of Gravity. Rev. Mod. Phys 82, 451 (2010)
4. S. Nojiri, S.D. Odintsov, Unified cosmic history in modified gravity: from \( F(R) \) theory to Lorentz non-invariant models. Phys. Rep. 505, 59 (2011)
5. G.J. Olmo, Palatini Approach to Modified Gravity: \( f(R) \) Theories and Beyond. Int. J. Mod. Phys. D 20, 413 (2011)
6. S. Capozziello, M. De Laurentis, Extended Theories of Gravity. Phys. Rep. 509, 167 (2011)
7. S. Capozziello, V. Faraoni, Beyond Einstein Gravity, Fundamental Theories of Physics, Vol. 170 (Springer, Dordrecht, 2011)
8. G.R. Bengochea. R. Ferraro, Dark torsion as the cosmic speed-up. Phys. Rev. D 79, 124019 (2009)
9. E.V. Linder, Einstein’s Other Gravity and the Acceleration of the Universe. Phys. Rev. D 81, 127301 (2010); Erratum:[Phys. Rev. D 82, 109902 (2010)]
10. R. Myrzakulov, Accelerating universe from \( F(T) \) gravity. Eur. Phys. J. C 71, 1752 (2011)
69. M.J. Reboüças, A rotating universe with violation of causality. Phys. Lett. A 70, 161 (1979)

70. A.K. Raychaudhuri, S.N. Guha Thakurta, Homogeneous spacetimes of the Gödel-type. Phys. Rev. D 22, 802 (1980)

71. M.J. Reboüças, A.F.F. Teixeira, Features of a Relativistic Space-Time with Seven Isometries. Phys. Rev. D 34, 2985 (1986)

72. F.M. Paiva, M.J. Reboüças, A.F.F. Teixeira, Time Travel in the Homogeneous Som-Raychaudhuri Universe. Phys. Lett. A 126, 168 (1987)

73. A. Krasiński, Rotating Dust Solutions of Einstein's Equations with 3-Dimensional Symmetry Groups - III- All Killing Fields Linearly Independent of U(Alpha) And W(Alpha), J. Math. Phys. 39, 2148 (1998)

74. J.D. Barrow, C.G. Tsagas, Dynamics and stability of the Gödel universe. Class. Quant. Gravit. 21, 1773 (2004)

75. M.P. Dabrowski, J. Garecki, Energy momentum and angular momentum of Gödel universes. Phys. Rev. D 70, 043511 (2004)

76. J.D. Barrow, M.P. Dabrowski, Gödel universes in string theory. Phys. Rev. D 58, 103502 (1998)

77. J.E. Aman, J.B. Fonseca-Neto, M.A.H. MacCallum, M.J. Reboüças, Riemann-Cartan spacetimes of Gödel-type. Class. Quant. Gravit. 15, 1089 (1998)

78. M.J. Reboüças, A.F.F. Teixeira, Riemannian Space-times of Gödel Type in Five Dimensions. J. Math. Phys. 39, 2180 (1998)

79. M.J. Reboüças, A.F.F. Teixeira, Int. J. Mod. Phys. A 13, 3181 (1998)

80. P. Kanti, C.E. Vayonakis, Gödel type universes in string inspired charged gravity. Phys. Rev. D 60, 103519 (1999)

81. H.L. Carrion, M.J. Reboüças, A.F.F. Teixeira, Gödel-type Space-times in Induced Matter Gravity Theory. J. Math. Phys. 40, 4011 (1999)

82. E.K. Boyda, S. Ganguli, P. Horava, U. Varadarajan, Holographic protection of chronology in universes of the Gödel type. Phys. Rev. D 67, 106003 (2003)

83. J.D. Barrow, C.G. Tsagas, The Gödel brane. Phys. Rev. D 69, 064007 (2004)

84. M. Banados, G. Barnich, G. Compere, A. Gomberoff, Three-dimensional origin of Gödel spacetimes and black holes. Phys. Rev. D 73, 044006 (2006)

85. D. Astefanesei, R.B. Mann, E. Radu, Nut charged spacetimes and closed timelike curves on the boundary. JHEP 01, 049 (2005)

86. M.J. Reboüças, J. Santos, Gödel-type universes in \( f(R) \) gravity. Phys. Rev. D 80, 063009 (2009)

87. J. Santos, M.J. Reboüças, T.B.R.F. Oliveira, Gödel-type universes in Palatini \( f(R) \) gravity. Phys. Rev. D 81, 123017 (2010)

88. P.J. Porfírio, J.B. Fonseca-Neto, J.R. Nascimento, A.Yu. Petrov, J. Ricardo, A. F. Santos, Chern-Simons modified gravity and closed timelike curves. Phys. Rev. D 94, 044044 (2016)

89. J.R. Nascimento, A. Yu. Petrov, P.J. Porfírio, A.F. Santos, Gödel-type universes in Brans-Dicke theory. Phys. Lett. B 762, 96 (2016)

90. J. Santos, M.J. Reboüças, T.B.R.F. Oliveira, A.F.F. Teixeira, Homogeneous Gödel-type solutions in hybrid metric-Palatini gravity. arXiv:1611.03985 [gr-qc]

91. Y.C. Ong, K. Izumi, J.M. Nester, P. Chen, Problems with propagation and time evolution in \( f(T) \) gravity. Phys. Rev. D 88, 024019 (2013)

92. K. Je-An. Gu, Y.C. Ong, Acausality and nonunique evolution in generalized teleparallel gravity. Phys. Rev. D 89, 084025 (2014)

93. P. Chen, K. Izumi, J.M. Nester, Y.C. Ong, Remnant symmetry, propagation, and evolution in \( f(T) \) gravity. Phys. Rev. D 91, 064003 (2015)

94. S. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*, Addison Wesley, New York, 2004.

95. F. Bampi, C. Zordan, Gen. Rel. Grav. 9, 393 (1978)

96. M. Krčská, J.G. Pereira, Spin Connection and Renormalization of Teleparallel Action. Eur. Phys. J. C 75, 519 (2015)

97. M.J. Reboüças, J. Tiomno, On the Homogeneity of Riemannian Space-Times of Gödel Type. Phys. Rev. D 28, 1251 (1983)

98. M.J. Reboüças, J.E. Aman, Computer-aided study of a class of Riemannian space-times. J. Math. Phys. 28, 888 (1987)

99. A.F.F. Teixeira, M.J. Reboüças, J.E. Aman, Isometries of homogeneous Gödel-type spacetimes. Phys. Rev. D 32, 3309 (1985)

100. J.B. Fonseca-Neto, A.Y. Petrov, M.J. Reboüças, Gödel-type universes and chronology protection in Horava-Lifshitz gravity. Phys. Lett. B 725, 412 (2013)

101. D. Liu, P. Wu, H. Yu, Gödel-type universes in \( f(T) \) gravity. Int. J. Mod. Phys. D 21, 1250074 (2012)