Orbit Structures of Weight Polytopes *

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Abstract

Let $\Lambda$ be a dominant weight and let $P(\Lambda)$ be the saturated weight set with highest weight $\Lambda$. The weight polytope associated with $\Lambda$ is defined as the convex hull of $P(\Lambda)$. This paper provides an elementary proof that there is a natural bijection between the set of orbits of a weight polytope under action of the Weyl group and the set of parabolic subgroups of the Weyl group. The bijection is called the Putcha-Renner recipe of linear algebraic monoid theory.

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1 Introduction

Recently, some algebraic combinatorists have done some research on root polytopes where a root polytope is the convex hull of the root system of a finite dimensional complex simple Lie algebra [1]. Cellini and Marietti found a bijection between the set of orbits of a root polytope under the action of the Weyl group and the set of distinguished abelian ideals of a Borel subalgebra of the finite dimensional complex simple Lie algebra [6]. This bijection is a specific case of the so called Putcha-Renner recipe in algebraic monoid theory [9]. The interpretation of this special case in [8] using affine Coxeter diagrams is the same as the description in [6]. Putcha and Renner have proved their recipe using techniques of algebraic monoid theory. However, Cellini and Marietti show the existence of this bijection in a completely different way, which is very elementary and only requires some basic background of Lie algebras.

In this paper, we prove Putcha-Renner recipe in general by making use of elementary techniques in the theory of finite dimensional complex simple Lie algebras.

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Let $\Phi$ be a root system in a Euclidean space $E$ with an inner product $(\cdot, \cdot)$, and let $\Pi = \{\alpha_1, ..., \alpha_n\}$ be a root basis and $W$ the Weyl group of $\Phi$. Let $\Lambda$ be a dominant weight in $E$. The weight polytope $P$ is defined as the convex hull of $W\Lambda$ in $E$. As a finite reflection group, $W$ acts on $E$ naturally. Adding a new node $-\Lambda$ to the Coxeter diagram by drawing a single laced edge between $-\Lambda$ and $\alpha_i$ if $(\Lambda, \alpha_i) > 0$, we obtain an extended Coxeter diagram. The recipe can be stated as follows without using terminologies of the algebraic monoid theory.

**Theorem** ([9] Theorem 4.16) The orbits of the faces of the weight polytope $P$ under the action of the Weyl group $W$ are in bijection with the connected subdiagrams of the extended Coxeter diagram that contain the node $\{-\Lambda\}$.

A root polytope is a special case of weight polytopes in which the dominant weight is the highest root. The proof, provided by Cellini and Marietti, of the above theorem for root polytopes requires some special properties of the root systems that do not apply to saturated weight sets. For example, if two roots $\alpha, \beta$ are perpendicular, then $\alpha + \beta$ and $\alpha - \beta$ are not roots. Cellini and Marietti used this special property of roots in the proof of two key results (Theorems 4.1 and 4.4) of their paper [6].

The Putcha-Renner recipe follows from Theorem 4.3, which is the first main theorem of this paper. The first item of the theorem states that a standard parabolic face is a convex hull of the orbit of a parabolic subgroup of $W$ acting on the dominant weight. We prove this in a different elementary way. We say elementary because all we need to prove this item are Lemmas 4.1 and 4.2. This leads to the second and third items of Theorem 4.3 which are generalizations of Cellini and Marietti’s Theorem 4.1 in [6]. It follows easily from this theorem that there is a bijection between the orbits of faces of a root polytope under the action of the Weyl group and the standard parabolic faces of the root polytope.

The second main theorem of this paper is Theorem 4.14. It says that the linear space spanned by a face of a weight polytope is in fact spanned by roots; the result is an analogue of Cellini and Marietti’s Corollary 4.5 in [6] for weight polytopes. This theorem follows directly from our Lemma 4.13 which states that the edge of a weight polytope is parallel to a root. We use this theorem to prove that all faces of a weight polytope are parabolic.

The paper is organized as follows. Section 2 provides some necessary back-
ground knowledge. Section 3 is a summary of notation and terminologies. Section 4 is dedicated to our new elementary proof of Putcha-Renner recipe.

2 Preliminaries

Let \( g \) be a finite-dimensional complex simple Lie algebra. We fix a Cartan subalgebra \( h \) of \( g \) and assume that \( \Phi \) is the root system of \( g \) with respect to \( h \). Then \( g \) has the following root space decomposition

\[
g = h \oplus \sum_{\alpha \in \Phi} g_\alpha.
\]

Let \([n] = \{1, 2, \ldots, n\}\), \( \Pi = \{\alpha_i \mid i \in [n]\}\) be a root basis of \( \Phi \), and \( W \) be the Weyl group generated by the simple reflections \( \{s_{\alpha_i} \mid i \in [n]\}\). Let \( \{\omega_i \mid i \in [n]\}\) be the fundamental weights with respect to \( \Pi \), and let \( \tilde{\omega}_i = \frac{2\omega_i}{(\alpha_i, \alpha_i)} \) be the coweights of \( \omega_i \).

We have \((\tilde{\omega}_i, \alpha_j) = \delta_{ij}\) for \( i, j \in [n]\).

Denote by \( P(\Lambda) \) the set of weights of the irreducible highest weight module with highest dominant weight \( \Lambda \). Then \( P(\Lambda) \) is saturated, and there is a partial order on \( P(\Lambda) \): \( \mu \leq \lambda \) if \( \lambda - \mu \) is a sum of positive roots. We will use repeatedly the following known results in the representation theory of Lie algebras.

**Proposition 2.1** ([4] Proposition 3.6) For \( \lambda \in P(\Lambda) \) and \( \alpha_i \in \Pi \), suppose that the \( \alpha_i \)-string through \( \lambda \) is: \( \lambda - p\alpha_i, \ldots, \lambda, \ldots, \lambda + q\alpha_i \). Then \( p - q = \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \). In particular, if \((\lambda, \alpha_i) > 0\), then \( \lambda - \alpha_i \in P(\Lambda) \).

**Proposition 2.2** ([2] Chapter VI, Proposition 24) Let \( \Phi \) be a root system and let \( \Phi' \) be the intersection of \( \Phi \) with a subspace of \( E \). Then \( \Phi' \) is a root system in the subspace spanned by \( \Phi' \).

Let \( I \subseteq [n] \) and \( W_I \) be the parabolic subgroup of \( W \) generated by \( s_{\alpha_i} \) for \( i \in I \). Let \( W^I \) be the minimal length representatives of left cosets of \( W_I \) in \( W \). Then

\[
W^I = \{ w \in W \mid w \cdot \alpha_i > 0 \text{ for } i \in I \}
= \{ w \in W \mid l(ws_{\alpha_i}) = l(w) + 1 \text{ for } i \in I \}.
\]

**Proposition 2.3** ([5]) Every element \( w \in W \) has a unique decomposition \( w = w'w_I \) such that \( l(w) = l(w') + l(w_I) \), where \( w' \in W^I, w_I \in W_I \).
3 Notation and Terminologies

Let $E$ be the euclidean vector space spanned by $\Phi$. Fix a dominant weight $\Lambda$ in $E$, once and forever. The convex hull $P$ of $W\Lambda$ is a polytope.

**Definition 3.1** We call $P$ the weight polytope associated with $\Lambda$.

Let $\Lambda = \sum_{i=1}^{n} m_i \alpha_i$. Then all the coefficients $m_i$’s are non-negative rational numbers. Without loss of generality, we may assume that all $m_i$’s are non-negative integers since a dilation of a polytope and itself have the same face lattice structure. The following is a consequence of Proposition 11.3 of [4].

**Proposition 3.2** If all the coefficients in $\Lambda = \sum_{i=1}^{n} m_i \alpha_i$ are integers, then $P(\Lambda)$ is the intersection of the weight polytope $P$ with the root lattice.

If we express a weight $\mu$ as a linear combination of simple roots, we use $c_i(\mu)$ to denote the coefficient of $\alpha_i$. For example, $c_i(\Lambda) = m_i$. Let $P_i$ be the subset of weights $\mu$ such that $(\mu, \tilde{\omega}_i) = m_i$, in other words,

$$P_i = \{ \mu \in P(\Lambda) \mid c_i(\mu) = m_i \}.$$

**Definition 3.3** The convex hull of $P_i$ is called the $i$-th coordinate face of $P$, and will be denoted by $F_i$.

Each coordinate face $F_i$ is a face of the weight polytope $P$ since $F_i$ sits in the hyperplane $\{ x \in E \mid (x, \tilde{\omega}_i) = m_i \}$ and all other weights in $F_i$ are in the half-space $\{ x \in E \mid (x, \tilde{\omega}_i) \leq m_i \}$. It is, however, possible that $F_i$ is not a facet.

**Definition 3.4** A face of $P$ is standard parabolic if it is an intersection of coordinate faces; a face of $P$ is parabolic if it can be transformed into a standard parabolic face by an element of $W$.

For any $I \subseteq [n]$, the face

$$F_I = \{ \mu \in P \mid c_i(\mu) = m_i \text{ for } i \in I \}.$$
is standard parabolic, since $F_I = \cap_{i \in I} F_i$. On the other hand, for any standard parabolic face $F$, there exists $I \subseteq [n]$ such that $F = F_I$. Clearly, 

$$P_I = \cap_{i \in I} P_i = \{\mu \in P(\Lambda) \mid c_i(\mu) = m_i \text{ for } i \in I\}$$

is the set of all weights in $F_I$. Note that $F_{[n]} = P_{[n]} = \{\Lambda\}$. For convenience, let $F_\emptyset = P$ and $P_\emptyset = P(\Lambda)$.

**Definition 3.5** An extended Coxeter diagram is the diagram obtained by adding a new node $-\Lambda$ to the Coxeter diagram, in which a single laced edge is drawn between $-\Lambda$ and $\alpha_i$ if and only if $(\Lambda, \alpha_i) > 0$.

### 4 Proof of Putcha-Renner Recipe

The purpose of this section is to show in Proposition 4.5 that there is a bijection between the set of standard parabolic faces and the set of certain connected components in the extended Coxeter diagram, and then in Theorem 4.15 that all faces of a weight polytope are parabolic.

If $I$ is a subset of $[n]$, let $\overline{I}$ be the complement set of $I \subseteq [n]$, and let $\Pi_I = \{\alpha_i \in \Pi \mid i \in I\}$. We begin our discussion by introducing the following two lemmas.

**Lemma 4.1** Let $\mu, \lambda$ be weights in a standard parabolic proper face $F_I$ of $P$. If $\mu < \lambda$ then there are simple roots $\alpha_{i_1}, \ldots, \alpha_{i_k} \in \Pi_I$ such that $\lambda = \mu + \alpha_{i_1} + \cdots + \alpha_{i_k}$ and all $\mu + \alpha_{i_s} + \cdots + \alpha_{i_k}$ are weights in $F_I$ for $s = 1, 2, \ldots, k$.

**Proof.** Let $\lambda - \mu = \sum_{j \in J} a_j \alpha_j$ where $J \subseteq [n]$ and $a_j$ are positive integers for all $j \in J$. We claim that $I \cap J = \emptyset$ and $(\lambda - \mu, \alpha_{j_0}) > 0$ for some $j_0 \in J$. Otherwise, if $i \in I \cap J$, then $c_i(\mu) = m_i - a_i < m_i$, which contradicts that $\mu \in F_I$. If $(\lambda - \mu, \alpha_j) \leq 0$ for all $j \in J$, then $(\lambda - \mu, \lambda - \mu) = 0$, and hence $\lambda = \mu$, which is a contradiction.

Use induction on the height of $\lambda - \mu$: $\text{ht}(\lambda - \mu) = \sum_{j \in J} a_j$. If $\text{ht}(\lambda - \mu) = 1$, then the result is true. Assume the result in Lemma 4.1 is true for $\text{ht}(\lambda - \mu) = k - 1$ where $k > 1$. We show that the result is true for $\text{ht}(\lambda - \mu) = k$ case by case.

**Case 1:** $(\lambda, \alpha_{j_0}) > 0$.

It follows from Proposition 2.3 that $\lambda - \alpha_{j_0}$ is a weight in $F_I$, and $\mu < \lambda - \alpha_{j_0}$ with $\text{ht}(\lambda - \alpha_{j_0} - \mu) = k - 1 > 0$. Applying the induction hypothesis, we have
Lemma 4.2 Assume that \( w \Lambda = \Lambda - \sum_{j \in J} a_j \alpha_j \) where \( w \in W \), \( a_j \) are positive integers, and \( \alpha_j \in \Pi \) for all \( j \in J \). Then there exists \( u \in W_J \) such that \( w \Lambda = u \Lambda \).

Proof. By Proposition 2.3, \( w = w'w_J \) where \( w' \in W' \) and \( w_J \in W_J \) with \( l(w) = l(w') + l(w_J) \). We apply induction on the length \( l(w) \).

If \( l(w) = 0 \) then it is trivial. If \( l(w) = 1 \) then \( w = s_{\alpha_i} \) for some \( \alpha_i \in \Pi \). If \( i \in J \), then we are done. If \( i \notin J \), then \( \alpha_i \) is linearly independent of \( \{ \alpha_j \mid j \in J \} \). But

\[
  w\Lambda = s_{\alpha_i} \Lambda = \Lambda - \frac{2(\Lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \alpha_i = \Lambda - \sum_{j \in J} a_j \alpha_j.
\]

So, \( J \) must be empty, and then \( W_J = 1 \). Therefore, \( w\Lambda = \Lambda \).

Assume that \( l(w) > 1 \). Let \( w' = s_{\alpha_1} \ldots s_{\alpha_r} \) be a reduced expression. If \( w' = 1 \) then we are done. If \( w' \neq 1 \) then \( r \geq 1 \). Then \( w' \alpha_r < 0 \) by [5, Section 4.3, Theorem B] and \( r \notin J \). Assume that \( w_J \Lambda = \Lambda - \sum_{i \in J} b_i \alpha_i \) where \( b_i \geq 0 \) for all \( i \in J \). We have

\[
  (w\Lambda, w' \alpha_r) = (w'w_J \Lambda, w' \alpha_r) = (w_J \Lambda, \alpha_r)
  = (\Lambda - \sum_{i \in J} b_i \alpha_i, \alpha_r) \geq 0,
\]

since \( r \notin J \) and \( (\alpha_i, \alpha_r) \leq 0 \) for \( i \in J \). Divide the discussion into two cases.

Case 1: \( (w\Lambda, w' \alpha_r) = 0 \). We have

\[
  (\Lambda, w_J^{-1} \alpha_r) = (w_J \Lambda, \alpha_r) = 0.
\]
It follows that
\[ s_{w_j^{-1} \alpha_r} \Lambda = w_j^{-1} s_\alpha w_j \Lambda = \Lambda, \]
and so \( s_\alpha w_j \Lambda = w_j \Lambda. \) Thus, \( w \Lambda = w' w_j \Lambda = s_{\alpha_1} \ldots s_{\alpha_{r-1}} w_j \Lambda. \) But \( l(s_{\alpha_1} \ldots s_{\alpha_{r-1}} w_j) < l(w). \) From the induction hypothesis, there exists an element \( u \in W_j \) such that \( s_{\alpha_1} \ldots s_{\alpha_{r-1}} w_j \Lambda = u \Lambda. \) Thus, \( w \Lambda = u \Lambda. \)

Case 2: \( (w \Lambda, w' \alpha_r) > 0. \) Now, \( (\Lambda - \sum_{j \in J} a_j \alpha_j, w' \alpha_r) > 0. \) Then,
\[ \mu = \Lambda - \sum_{j \in J} a_j \alpha_j - w' \alpha_r \]
is a weight by Proposition 2.1. Since \( \mu \leq \Lambda \) and \( w' \alpha_r \) is a negative root, \( w' \alpha_r \) has to be linear combination of \( \alpha_j \)'s for \( j \in J. \) Write \( \gamma = w' \alpha_r. \) Then \( w' s_\alpha w_j \Lambda = \Lambda - \sum_{j \in J} c_j \alpha_j \)
where \( c_j \geq 0. \) Since \( l(w' s_\alpha w_j) < l(w), \) by the induction hypothesis, there exists \( v \in W_j \) such that \( w' s_\alpha w_j \Lambda = \gamma w \Lambda = v \Lambda. \) Let \( u = s_\gamma v \in W_j. \) Therefore, \( w \Lambda = s_\gamma v \Lambda = u \Lambda. \) This completes the proof. \( \square \)

For a subset \( \Sigma \) of \( \{-\Lambda\} \cup \Pi, \) denote by \( \Gamma(\Sigma) \) the subdiagram of the extended Coxeter diagram having \( \Sigma \) as vertices, and \( \Gamma^0(\Sigma) \) the connected component of \( \Gamma(\Sigma) \) containing \( -\Lambda. \) Denote by \( V_I \) the set of vertices of \( \Gamma^0(\Pi_I \cup \{-\Lambda\}). \) Let \( \Pi_{-\Lambda} = \Pi \cap V_I \) and let \( T^0 \) be the set of indexes of the vertices in \( \Pi_{-\Lambda}. \)

**Theorem 4.3** Let \( I \subseteq [n] \) and \( u_0 \) be the longest element in \( W_{-\Lambda}. \) If \( F_I \) is a standard parabolic face of \( \Pi, \) then

1. \( F_I \) is the convex hull of \( W_{-\Lambda} \Lambda, \)
2. \( u_0 \Lambda \) is the least element in \( P_I, \) and
3. \( F_I \) is spanned by the simple roots in \( \Pi_{-\Lambda}, \) and \( \dim F_I = |T^0|. \)

**Proof.** For (1), let \( w \in W_{-\Lambda}. \) Then \( w \Lambda = \Lambda - \sum_{j \in T^0} a_j \alpha_j \in P_I, \) since \( T^0 \cap I = \emptyset. \) Notice that \( P_I \) is the set of weights in \( F_I. \) We obtain that \( w \Lambda \in F_I, \) and the convex hull of \( W_{-\Lambda} \Lambda \) is contained in \( F_I. \)

Conversely, for any \( w \in W \) assume that \( w \Lambda = \Lambda - \sum_{j \in J} a_j \alpha_j \) with \( a_j > 0. \) Then \( J \subseteq T, \) since \( c_i(w \Lambda) = c_i(\Lambda) = m_i \) for all \( i \in I. \) By Proposition 2.2 there
exists an element \( u \in W_J \) such that \( u\Lambda = w\Lambda \). If \( \Pi_J \cup \{ -\Lambda \} \) is connected in the extended Coxeter diagram, then we are done. Otherwise, we can assume that \( J = J' \cup T^0 \). Notice that \( \Pi_{T^0} \cup \{ -\Lambda \} \) is a connected component in the Coxeter diagram. Then \( W_J \cong W_{T^0} \times W_{J'} \). Write \( u = xy \), where \( x \in W_{T^0} \) and \( y \in W_{J'} \). Thus, \( w\Lambda = u\Lambda = xy\Lambda = x\Lambda \) since \( (\Lambda, \alpha_j) = 0 \) for all \( j \in J' \). In other words, \( w\Lambda \in W_{T^0} \Lambda \). It follows that \( F_I \) is included in the convex hull of \( W_{T^0} \Lambda \). This completes the proof of (1).

To prove (2), it suffices to show that \( u_0\Lambda < w\Lambda \) for all \( w \in W_{T^0} \), since other weights in \( P_I \) are linear combinations of \( W_{T^0} \Lambda \) with non-negative coefficients whose sum is equal to 1. Thanks to \( \Lambda \geq u_0^{-1}w\Lambda \), we have \( \Lambda - u_0^{-1}w\Lambda = \sum_{j \in T^0} a_j \alpha_j \) with \( a_j \geq 0 \). It follows that

\[
  u_0\Lambda - w\Lambda = u_0(\Lambda - u_0^{-1}w\Lambda) = -\sum_{j \in T^0} a_j \alpha_j,
\]

where \( \alpha_j = -u_0(\alpha_j) \in \Pi_{T^0} \) since \( u_0\Pi_{T^0} = -\Pi_{T^0} \). So \( u_0\Lambda \leq w\Lambda \). This proves (2).

We now prove (3). If \( u_0\Lambda = \Lambda \), then \( \Lambda \) is the least element of \( W_{T^0} \Lambda \) by (2) and \( W_{T^0} \Lambda = \{ \Lambda \} \). It follows that \( \Pi_{T^0} = \emptyset \) and \( F_I = \{ \Lambda \} \). If \( u_0\Lambda < \Lambda \), let

\[
  \Lambda - u_0\Lambda = \sum_{j \in T^0} a_j \alpha_j. \tag{1}
\]

By Lemma \[4.4\] there are simple roots \( \alpha_{i_1}, \ldots, \alpha_{i_k} \) such that \( u_0\Lambda + \alpha_{i_1} + \cdots + \alpha_{i_k} \) are weights in \( F_I \) for \( s = 1, \ldots, k \). Any element \( \Lambda - w\Lambda \) is a linear combination of \( \alpha_{i_1}, \ldots, \alpha_{i_k} \), for \( w \in W_{T^0} \Lambda \). Hence \( \alpha_{i_1}, \ldots, \alpha_{i_k} \) span \( E_F \). But \( \alpha_{i_s} \in \Pi_{T^0} \) by (1). The dimension of \( F_I \) is the number of nonzero terms on the right-hand side of (1). We claim that \( a_j > 0 \) for all \( j \in T^0 \). To this end, for any \( \alpha_j \in \Pi_{T^0} \), we can find a shortest connected path: \( \alpha_j, \alpha_j, \ldots, \alpha_j, \Lambda \), from \( \alpha_j \) to \( \Lambda \) in the extended Coxeter diagram. Then the coefficient of \( \alpha_j \) in \( \Lambda - s_{\alpha_j} s_{\alpha_{j_1}} \ldots s_{\alpha_{j_m}} \Lambda \) is positive. Hence \( a_j > 0 \) in (1) for \( j \in T^0 \), since \( s_{\alpha_j} s_{\alpha_{j_1}} \ldots s_{\alpha_{j_m}} \Lambda \geq u_0\Lambda \). This completes the proof. \( \square \)

Note that for different subsets \( I, J \subseteq [n] \), it is possible that \( F_I = F_J \).

**Corollary 4.4** For any two subsets \( I, J \subseteq [n] \), \( F_I = F_J \) if and only if \( \Pi_{T^0} = \Pi_{T^0} \).
Proof. \( \Rightarrow \) Use \( \eta_I \) to denote the least element of \( F_I \). If \( F_I = F_J \), then they have the same least element \( \eta_I = \eta_J \). By the proof of (3) in Theorem 4.3
\[
\Lambda - \eta_I = \sum_{i \in I^0} a_i \alpha_i \quad \text{and} \quad \Lambda - \eta_J = \sum_{i \in J^0} b_i \alpha_i,
\]
where \( a_i > 0 \) for all \( i \in I^0 \), and \( b_i > 0 \) for all \( i \in J^0 \). It follows that \( \Pi_{I^0} = \Pi_{J^0} \).

\( \Leftarrow \) This is trivial by (1) of Theorem 4.3 \( \Box \)

The proposition below follows from Theorem 4.3 and Corollary 4.4.

**Proposition 4.5** The standard parabolic faces of \( P \) are in bijection with the connected subdiagrams of the extended Coxeter diagram that contain \(-\Lambda\). This bijection is an isomorphism of posets with respect to inclusion.

In fact, the bijection can be explicitly described as: \( F_I \rightarrow T^0 \). The above proposition is exactly the same as Putcha Renner Recipe (\cite{9} Theorem 4.16).

**Corollary 4.6** All coordinate faces are distinct.

**Proof.** Let \( \eta_i \) be the least element of \( F_i \). We have \( (\eta_i, \alpha_j) \leq 0 \) for all \( j \neq i \). Otherwise \( \eta_i - \alpha_j \) is a weight in \( P_i \subseteq F_i \) by Proposition 2.1 and \( \eta_i \) would not be minimal. This yields \( (\eta_i, \alpha_i) > 0 \) since \( (\eta_i, \eta_i) > 0 \). Thus \( \eta_i \neq \eta_j \), and hence \( P_i \neq P_j \) if \( i \neq j \). The coordinate faces \( F_i \) and \( F_j \) are then distinct due to that \( P_i \) is exactly the set of weights in \( F_i \) for all \( i \in [n] \). \( \Box \)

The following is a criterion that tests whether a coordinate face is a facet.

**Corollary 4.7** Let \( F_i \) be a coordinate face and \( \eta_i \) be the least element in \( F_i \). Then \( F_i \) is a facet if and only if \( (\eta_i, \tilde{\omega}_j) \neq m_j \) for all \( j \neq i \).

**Proof.** \( \Rightarrow \) By Theorem 4.3, if \( F_i \) is a facet then \( \dim F_i = |\Pi_{P^0}| = n - 1 \) where \( I = \{i\} \). This implies that \( \Pi_{I^0} = \Pi \setminus \{i\} \) and \( \eta_i = \Lambda - \sum_{\alpha_j \in \Pi_{P^0}} a_j \alpha_j \) with \( a_j > 0 \) for all \( j \neq i \). Therefore, \( (\eta_i, \tilde{\omega}_j) \neq m_j \) for all \( j \neq i \).

\( \Leftarrow \) If \( (\eta_i, \tilde{\omega}_j) = m_j - a_j \neq m_j \) for all \( j \neq i \), then \( a_j > 0 \) for all \( j \neq i \). By Lemma 4.1 or the proof of Theorem 4.3, \( E_{F_i} \) is spanned by \( \{\alpha_j \mid j \in [n] \setminus \{i\}\} \). Hence \( \dim F_i = n - 1 \). \( \Box \)
Lemma 4.8 Let $\alpha$ be a root in $\Phi$ and $\lambda$ be a weight in $P(\Lambda)$ and denote the $\alpha$-string through $\lambda$ by $\alpha - (\lambda)$. Then
\[ \sum_{\gamma \in \alpha - (\lambda)} (\gamma, \alpha) = 0. \]

Proof. Without loss of generality, we assume that $\lambda$ is the start of its $\alpha$-string. Then the weight string is
\[ \lambda, \lambda + \alpha, \ldots, \lambda + q\alpha, \] where $q = -\frac{2(\lambda, \alpha)}{(\alpha, \alpha)}$.

If $q$ is even, the middle weight $\lambda - \frac{(\lambda, \alpha)}{(\alpha, \alpha)}\alpha$ is orthogonal to $\alpha$, and the sum of the two symmetric weights about the middle weight is orthogonal to $\alpha$. If $q$ is odd, the sum of the two symmetric weights is orthogonal to $\alpha$. $\square$

Proposition 4.9 The barycenter of the weights in the $i$-th coordinate face $F_i$ is parallel to the $i$-th fundamental weight, and
\[ \sum_{\lambda \in P_i} \lambda = \frac{m_i|P_i|}{||\omega_i||^2}\omega_i. \]

Proof. For each $j \in [n] \setminus \{i\}$, $P_i$ is a disjoint union of $\alpha_j$-strings. It follows from Lemma 4.8 that
\[ \sum_{\lambda \in P_i} (\lambda, \alpha_j) = 0. \]

Hence, $\sum_{\lambda \in P_i} \lambda = a_i \tilde{\omega}_i$. In view of $(\lambda, \tilde{\omega}_i) = m_i$ for all $\lambda \in P_i$, we have
\[ (a_i \tilde{\omega}_i, \tilde{\omega}_i) = \left( \sum_{\lambda \in P_i} \lambda, \tilde{\omega}_i \right) = |P_i|(\lambda, \tilde{\omega}_i) = m_i|P_i|. \]

The desired result follows from $\frac{\tilde{\omega}_i}{(\tilde{\omega}_i, \tilde{\omega}_i)} = \frac{\omega_i}{(\omega_i, \omega_i)}$. $\square$

Lemma 4.10 Let $I \subseteq [n]$. Then the barycenter of the standard parabolic face $F_I$ is $\sum_{i \in I} a_i \tilde{\omega}_i$ where $a_i$ are non-negative rational numbers for all $i \in I$.

Proof. As $P_I$ is a disjoint union of $\alpha_j$-string for any fixed $j \in [n] \setminus I$, we know that $(\sum_{\lambda \in P_I} \lambda, \alpha_j) = 0$ for all $j \notin I$. Thus, $\sum_{\lambda \in P_I} \lambda = \sum_{i \in I} a_i \tilde{\omega}_i$. For any $j \in I$, we have
\[ \sum_{i \in I} a_i (\tilde{\omega}_i, \tilde{\omega}_j) = \left( \sum_{\lambda \in P_I} \lambda, \tilde{\omega}_j \right) = |P_I|(\lambda, \tilde{\omega}_j). \]
For $i, j \in I$ and $\lambda \in P_I$, the numbers $(\tilde{\omega}_i, \tilde{\omega}_j)$ and $(\lambda, \tilde{\omega}_j)$ are rational numbers, so are all $a_i$'s. We claim that $(\lambda, \alpha_i) \geq 0$ for $\lambda \in P_I$. If not, it follows from $(\lambda, \alpha_i) < 0$ that $\lambda + \alpha_i \in P(\Lambda)$, which shows that $c_i(\lambda + \alpha_i) = m_i + 1$. This contradicts that $\lambda \leq \Lambda$. Thus $a_i \geq 0$, as desired. \(\square\)

By the lemma above, the barycenter of a standard parabolic face is in the closure of the fundamental Weyl chamber \(3\). Since the Weyl group acts on the fundamental chamber transitively, we have

**Corollary 4.11** Two distinct standard parabolic faces of the weight polytope $P$ cannot be transformed into one another by elements of the Weyl group $W$.

**Lemma 4.12** Let $P$ be a polytope (or cone). Let $\Lambda$ be a fixed vertex. Suppose $[\mu, \Lambda]$ is an edge (or spans an edge in the cone) with

$$
\mu - \Lambda = a_1(x_1 - \Lambda) + \cdots + a_n(x_n - \Lambda)
$$

where $a_i > 0$ and $x_i \in P$. Then $x_i$ are in the same edge that contains $[\mu, \Lambda]$.

**Proof.** Assume that $[\mu, \Lambda]$ (or its span for the cone) is an intersection of facet $F_k$ ($k = 1, \ldots, r$). Assume that the linear equation for the hyperplane containing $F_k$ is $L_k(x) = 0$ for $k \in [r]$. We want to prove that $L_k(x_i) = L_k(\Lambda)$ for all $k \in [r]$ and $i \in [n]$. We have

$$
L_k(\mu - \Lambda) = a_1(L_k(x_1) - L_k(\Lambda)) + \cdots + a_n(L_k(x_n) - L_k(\Lambda)) = 0.
$$

Since $x_i \in P$, $L_k(x_i) - L_k(\Lambda)$ have the same sign for all $x_i$ ($i = 1, \ldots, n$). Therefore, $L_k(x_i) = L_k(\Lambda)$ for all $k \in [r]$. This means that $x_i$ is in the same edge that contains $[\mu, \Lambda]$. \(\square\)

**Lemma 4.13** Every edge of $P$ is parallel to a root.

**Proof.** It suffices to prove this for edges attached to the highest dominant weight $\Lambda$. Assume that $\mu = w\Lambda \neq \Lambda$ and the segment $[\Lambda, \mu]$ is an edge. We assert that $\mu$ and $\Lambda$ are separated by one and only one reflection hyperplane $H_\alpha$ where $\alpha$ is a positive root and $H_\alpha = \{x \in E \mid (x, \alpha) = 0\}$.

If not, there are two positive distinct roots $\alpha, \beta$ such that $\mu$ and $\Lambda$ are separated by two different reflection hyperplanes $H_\alpha$ and $H_\beta$. Then $(\Lambda, \alpha) > 0$, $(\mu, \alpha) < 0$.

11
and \((\Lambda, \beta) > 0, (\mu, \beta) < 0\). Therefore, \(\Lambda - \alpha, \Lambda - \beta, \mu + \alpha, \) and \(\mu + \beta\) are weights in \(P(\Lambda)\). We have

\[
\mu - \Lambda = \frac{(\mu + \alpha - \Lambda) + (\mu + \beta - \Lambda) + (\Lambda - \alpha - \Lambda) + (\Lambda - \beta - \Lambda)}{2},
\]

which is a contradiction by Lemma 4.12. Thus \(\mu\) and \(\Lambda\) are separated by one hyperplane \(H_{\alpha}\). But \(\mu\) and \(\Lambda\) are in the closures of two different Weyl chambers, one of which is transformed to the other by the reflection \(s_{\alpha}\). It follows that \(\mu = s_{\alpha} \Lambda\) from the fact that the action of the Weyl group acts on all the chambers simply transitively. Here \(\alpha\) is a positive root but not necessary a simple root. □

The following theorem follows directly from the above lemma.

**Theorem 4.14** Given a face \(F\) of \(P\), let \(E_F = \text{Span}\{\lambda - \mu \mid \lambda, \mu \in F\}\). Then the space \(E_F\) is spanned by roots for any face \(F\) of \(P\).

Now we are ready to show that all faces of a weight polytope are parabolic. The proof of the following theorem is similar to that of Theorem 5.11 in [6]. For completeness, we give the proof here.

**Proposition 4.15** Every face of a weight polytope is parabolic.

**Proof.** Use induction to prove the theorem. Let \(F\) be a face with \(\dim F = n - p\) for \(0 \leq p \leq n\). If \(p = 1\) then \(F\) is a facet. From Proposition 4.14 and 2.2 \(\Phi \cap E_F\) is a root subsystem of \(\Phi\) with rank \(n - 1\). Let \(\Pi'\) be a root basis of \(\Phi \cap E_F\). Then there exists \(w \in W\) such that \(w\Pi' \subseteq \Pi\). Let \(\alpha_i\) be the only fundamental root in \(\Pi\) which does not belong to \(w\Pi'\). It follows that \(wE_F\) is perpendicular to \(\tilde{\omega}_i\), denoted by \(wE_F \perp \tilde{\omega}_i\). Therefore, there exists a real number \(a\) such that \((x, \tilde{\omega}_i) = a\) for all \(x \in wF\). It follows that \(\{x \in E \mid (x, \tilde{\omega}_i) = a\}\) is a supporting hyperplane of the weight polytope. Let \(l_i = (w_0 \Lambda, \tilde{\omega}_i)\) where \(w_0\) is the longest element in the Weyl group \(W\). Then \(l_i \leq a \leq m_i\). This forces \(a = m_i\) or \(a = l_i\). If not, vertices \(\Lambda\) and \(-w_0 \Lambda\) will be at different sides of the hyperplane \(\{x \in E \mid (x, \tilde{\omega}_i) = a\}\), a contradiction. If \(a = m_i\) then \(wF = F_i\). If \(a = l_i\), then \(-l_i = (\Lambda, \tilde{\omega}_i') = m_i\) where \(\tilde{\omega}_i' = -w_0(\tilde{\omega}_i)\). Hence \(w_0 w F = F_i'\).

Now we assume that \(n \geq p > 1\) and all faces of \(P_\Phi\) of dimension greater than \(n - p\) are parabolic. By the induction hypothesis, without loss of generality, we
can assume that $F$ is a facet of a standard parabolic face $F_I$ where $\dim F = n - p$ and $\dim F_I = n - p + 1$. By Theorem 4.3, $F_I$ is the convex hull of $W_\gamma \Lambda$ and $F_I$ is spanned by $\{\alpha_i | i \in T^0\}$. Note that there exists $w \in W_\gamma$ such that $w E_F \perp \tilde{\omega}_j$ where $j \in T^0$. Meanwhile, there exists a real number $a$ such that $(x, \tilde{\omega}_j) = a$ for all $x \in wF$. Let $l_j = (\eta_I, \tilde{\omega}_j)$. Recall $\eta_I = u_0 \Lambda$ is the least element in $F_I$ where $u_0$ is the longest element in $W_\gamma$. Then $a = l_j$ or $a = m_j$.

If $a = m_j$ then $\Lambda \in F$. Hence $wF = \{\Lambda + \xi | \xi \in E_{F_I} \cap \tilde{\omega}_j^+\} \cap F_I = F_I \cap F_j$. We are done. If $a = l_j$, then $\eta_I \in wF$. Hence $wF = \{\eta_I + \xi | \xi \in E_{F_I} \cap \tilde{\omega}_j^+\} \cap F_I$. Let $\pi$ be the orthogonal projection of $E$ on to $E_{F_I}$. Then $E_{F_I} \cap \tilde{\omega}_k^+ = E_{F_I} \cap \pi(\tilde{\omega}_k)^+$ for all $k \in T^0$. Hence $\{\pi(\tilde{\omega}_k) | k \in T^0\}$ is the coweights with respect to $\Pi_\gamma^0$ in root system $\Phi(\Pi_\gamma)$. Thus, there exists $j'$ such that $u_0(\pi(\tilde{\omega}_j)) = -\pi(\tilde{\omega}_{j'})$. Since $u_0(F_I) = F_I$, we have $u_0(E_{F_I} \cap \tilde{\omega}_k^+) = E_{F_I} \cap \pi(\tilde{\omega}_{j'})^+ = E_{F_I} \cap \tilde{\omega}_k^+$. Thus, $F$ is parabolic, since $u_0wF = \{u_0(\eta_I) + \xi | \xi \in E_{F_I} \cap \tilde{\omega}_k^+\} \cap F_I = \{\Lambda + \xi | \xi \in E_{F_I} \cap \tilde{\omega}_k^+\} \cap F_I = F_I \cap F_{j'}$. □

We conclude this section by giving the $f$-polynomial of weight polytopes.

**Theorem 4.16** Let $\mathcal{I} = \{\Pi_I \subseteq \Pi | \Gamma(\Pi_I \cup \{-\Lambda\})$ is connected$\}$ and let $J_0 = \{j \in [n] | (\Lambda, \alpha_j) = 0\}$. For each $I \in \mathcal{I}$, we define $I^* = I \cup \{j \in J_0 | (\alpha_j, \alpha_i) = 0 \text{ for all } i \in I\}$. Then the $f$-polynomial of $P$ is

$$\sum_{\Pi_I \in \mathcal{I}} \frac{|W_I|}{W_{I^*}} t^{\vert I^* \vert}.$$ 

**Proof.** Consider the action of the Weyl group $W$ on the faces of $P$. A simple calculation yields that $W_{I^*}$ is the stabilizer of the standard parabolic face $F$, where $F$ is the convex hull of $W_I \Lambda$. □

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