Note About Consistent Extension of Quasidilaton
Massive Gravity

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ABSTRACT: This note is devoted to the Hamiltonian analysis of extension of quasidilaton massive gravity as was proposed recently in [arXiv:1306.5502]. We show that for given formulation of the theory the additional primary constraint that is responsible for the elimination of the Boulware-Deser ghost is missing. We compare this situation with quasidilaton massive gravity. Finally we propose ghost free extension of quasidilaton massive gravity.

KEYWORDS: Massive Gravity, Hamiltonian Formalism.
1. Introduction and Summary

Recently new version of the full non-linear massive gravity that was found by de Rham, Gabadadze and Toley (dRGT) \cite{dRGT1, dRGT2} provides the positive answer to the question whether graviton can have a non-zero mass. In fact, among many remarkable properties there is the crucial one which is the absence of the Boulware-Deser ghost \cite{Boulware1, Boulware2}.

The consistent massive gravity could also provide a possible explanation of the observed acceleration of the cosmic expansion which is one of the greatest mysteries in modern cosmology. It is tempting to speculate that the finite graviton mass could be a source of the accelerated expansion of the universe. For that reason it is great interest to formulate theoretically consistent cosmological scenario in massive gravity that is also in agreement with the observations. Unfortunately it was recently shown that all homogeneous and isotropic cosmological solutions in dRGT theory are unstable \cite{Kaloper}, see also \cite{23, 24, 25}.

In order to resolve this problem we have two possible options: Either to break homogeneity \cite{23} or isotropy \cite{24, 25} or to extend the theory as in \cite{9, 10}. Recently in \cite{11} A. De Felice and S. Mukohyama proposed new extension of quasidilaton massive gravity that could provide stable and self-accelerating homogeneous and isotropic cosmological solution. They further argued that given extension belongs to the class of models studied in \cite{12} that are free from the Boulware-Deser ghosts. However the explicit Hamiltonian analysis of given theory was not performed in \cite{11}.

The goal of this paper is to reconsider the problem of the Boulware-Deser ghost in the model \cite{11}. We present an evidence that for the action that was introduced in \cite{11} the Boulware-Deser ghost cannot be eliminated. More precisely, performing the Hamiltonian analysis of this model with time dependent quasidilaton we find that primary constraint, that is responsible for the elimination of the ghost in Stückelberg formulation of non-linear massive gravity \cite{13, 14, 15, 16} is missing \footnote{For related work, see \cite{22}}. This result implies that generally Boulware-Deser ghost is present. On the other hand we show that this additional constraint emerges

\footnote{For related work, see \cite{22}.}
when $\omega \neq 0$ and when $\alpha_0 = 0$ that corresponds to the quasidilaton massive gravity. We also propose model of consistent extension of the quasidilaton non-linear massive gravity that can be considered as the generalization of the coupling between massive gravity and the galileon \cite{17} and we argue that given theory is ghost free, following \cite{16}.

This paper is organized as follows. In the next section (2) we review the basic facts about extension of quasidilaton non-linear massive gravity as was proposed in \cite{11}. Then we proceed to the Hamiltonian analysis of given theory and argue that there is no scalar primary constraint that could eliminate the Boulware-Deser ghost. In section (3) we perform the Hamiltonian analysis of quasidilaton non-linear massive gravity when we find that in this case there is an additional primary constraint. This result shows that the quasidilaton massive theory as was proposed in \cite{9} is ghost free at least in their minimal version. Finally in section (4) we propose the extension of the quasidilaton massive gravity that is ghost free and that can be considered as the generalization of the proposal \cite{11}. It would be extremely interesting to analyze cosmological consequences of this theory.

2. Extension of Quasidilaton Massive Gravity

In this section we review basic facts about extension of quasidilaton massive gravity as was proposed in \cite{11}. For simplicity we restrict ourselves to the minimal form of the massive gravity keeping in mind that its generalization is straightforward.

Explicitly, let us consider following action

\[ S = S_{m.g.} + S_\sigma, \]
\[ S_{m.g.} = M_p^2 \int d^4x \sqrt{-\hat{g}} (\hat{R} + 2m^2(3 - \Omega(\Phi)\sqrt{\hat{g}^{-1}\hat{f}})), \]

(2.1)

where \( \hat{f}_{\mu\nu} \) was introduced in \cite{11}

\[ \hat{f}_{\mu\nu} = f_{\mu\nu} - \frac{\alpha_0}{M_p^2 m^2} e^{-2\sigma_0/M_p} \partial_\mu \sigma \partial_\nu \sigma \, , \quad f_{\mu\nu} = \partial_\mu \phi^a \partial_\nu \phi^b \eta_{ab} \, , \]

(2.2)

where \( \phi^a, a = 0, 1, 2, 3 \) are St"uckelberg fields and where \( \eta_{ab} = \text{diag}(-1,1,1,1) \). Further, \( S_\sigma \) is defined as

\[ S_\sigma = -\frac{\omega}{2} \int d^4x \sqrt{-\hat{g}} \hat{g}^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma \, . \]

(2.3)

Note that \( \Omega(\Phi) \) is function of \( \sigma \) which is necessary for the invariance of the theory under global symmetry

\[ \sigma \to \sigma + \sigma_0 \, , \quad \phi^a \to e^{-\sigma_0/M_p} \phi^a \]

(2.4)

so that under (2.4) \( f_{\mu\nu} \) and \( \hat{f}_{\mu\nu} \) transform as

\[ f_{\mu\nu} \to e^{-2\sigma_0/M_p} f_{\mu\nu} \, , \quad \hat{f}_{\mu\nu} \to e^{-2\sigma_0/M_p} \hat{f}_{\mu\nu} \, . \]

(2.5)

The massive term contains square root of the expression \( \hat{g}^{\mu\nu} \hat{f}_{\nu\rho} \) that under (2.4) transforms as

\[ \sqrt{\hat{g}^{-1}\hat{f}} \to e^{-\sigma_0/M_p} \sqrt{\hat{g}^{-1}\hat{f}} \]

(2.6)
which implies that $\Omega(\sigma)$ has to have the form

$$\Omega(\sigma) = e^{\sigma/M_p} .$$

(2.7)

Now we are ready to proceed to the Hamiltonian analysis of given theory. Due to the presence of the square root in the action we perform the redefinition of the shift functions \cite{20, 21}

$$N^i = M\tilde n^i + \tilde f^{ik}\tilde f_{0k} + N\tilde D^i_j\tilde n^j,$$

(2.8)

where

$$M^2 = -\tilde f_{00} + \tilde f_{0k}\tilde f^{kl}\tilde f_{0l} , \quad \tilde f_{ij}\tilde f^{jk} = \delta_i^j ,$$

(2.9)

and where $\tilde D^i_j$ obeys the equation

$$\sqrt{\tilde x}\tilde D^i_j = \sqrt{(g^{ik} - \tilde D^i_m\tilde n^m\tilde D^k_{nj})\tilde f_{kj}} , \quad \tilde x = 1 - \tilde n^i\tilde f_{ij}\tilde n^j,$$

(2.10)

and also following important identity

$$\tilde f^{ik}\tilde D^i_j = \tilde f^{jk}\tilde D^i_i .$$

(2.11)

We also use $3 + 1$ decomposition of the four dimensional metric $\tilde g_{\mu \nu}$ \cite{15, 19}

$$\tilde g^{00} = -N^2 + N_i\tilde g^{ij}N_j , \quad \tilde g^{0i} = N_i , \quad \tilde g^{ij} = g_{ij} ,$$

$$\tilde g^{00} = -\frac{1}{N^2} , \quad \tilde g^{0i} = \frac{N^i}{N^2} , \quad \tilde g^{ij} = \frac{g^{ij} - \frac{N^iN^j}{N^2}}{} .$$

(2.12)

Note that in $3 + 1$ formalism the kinetic term for $\sigma$ has the form

$$-\frac{\omega}{M_p^2}\tilde g^{\mu \nu}\partial_\mu\sigma\partial_\nu\sigma = \frac{\omega}{M_p^2}(\nabla_n\sigma)^2 - \frac{\omega}{M_p^2}\partial_i\sigma\tilde g^{ij}\partial_j\sigma ,$$

(2.13)

where $\nabla_n\sigma$ after redefinition (2.8) has the form

$$\nabla_n\sigma = \frac{1}{N}(\partial_t\sigma - (M\tilde n^i + \tilde f^{ik}\tilde f_{0k} + N\tilde D^i_j\tilde n^j)\partial_i\sigma) .$$

(2.14)

With the help of these expressions we rewrite the action (2.1) into the form

$$S = M_p^2 \int d^3xdt \left[ N\sqrt{\tilde g}\tilde K_{ij}\tilde g^{ijkl}\tilde K_{kl} + N\sqrt{\tilde g}R - \sqrt{\tilde g}MU - 2m^2(N\Omega(\Phi)\sqrt{\tilde x}\tilde D^i_i - 3\sqrt{\tilde g}) + N\sqrt{\tilde g}\frac{\omega}{2M_p^2}(\nabla_n\sigma)^2 - N\sqrt{\tilde g}\frac{\omega}{2M_p^2}\partial_i\sigma\tilde g^{ij}\partial_j\sigma \right] ,$$

(2.15)

where

$$U = 2m^2\Omega(\Phi)\sqrt{\tilde x} ,$$

(2.16)
and where we used the $3 + 1$ decomposition of the four dimensional scalar curvature

$$(4) R = \tilde{K}_{ij} G^{ijkl} \tilde{K}_{kl} + R ,$$

(2.17)

where $R$ is three dimensional scalar curvature. We also introduced de Witt metric

$$G^{ijkl} = \frac{1}{2} (g^{ik} g^{jl} + g^{il} g^{jk}) - g^{ij} g^{kl}$$

(2.18)

with inverse

$$G_{ijkl} = \frac{1}{2} (g^{ik} g^{jl} + g^{il} g^{jk}) - \frac{1}{2} g^{ij} g^{kl} , \quad G_{ijkl} G^{klmn} = \frac{1}{2} (\delta^m_i \delta^n_j + \delta^m_j \delta^n_i) .$$

(2.19)

Note that in (2.17) we ignored the terms containing total derivatives. Finally note that $\tilde{K}_{ij}$ is defined as

$$\tilde{K}_{ij} = \frac{1}{2N} \left( \partial_t g_{ij} - \nabla_i N_j (\tilde{n}, g) - \nabla_j N_i (\tilde{n}, g) \right) ,$$

(2.20)

where $N_i$ depends on $\tilde{n}^i$ and $g$ through the relation (2.8).

Now we could proceed to the Hamiltonian formulation of given theory. However the structure of the derivative $\nabla_n \sigma$ (2.14) suggests very complicated relations between momenta and velocities. For that reason we consider simpler case when we presume that $\sigma$ depends on time only. Note that this is the reasonable approximation that does not spoil the physical content of the theory. From (2.15) we find the momenta conjugate to $N, \tilde{n}^i$ and $g_{ij}$

$$\pi_N \approx 0 \ , \ \pi_i \approx 0 \ , \ \pi^{ij} = M_p^2 \sqrt{g} G^{ijkl} \tilde{K}_{kl}$$

(2.21)

while in case of $\phi^a$ and $\sigma$ we find

$$p_a = \frac{M_{ab} \partial_t \phi^b}{M} [\tilde{n}^i \mathcal{R}_i + M_p^2 \sqrt{gU}] - f^{ij} \partial_j \phi_a \mathcal{R}_i ,$$

$$p_\sigma = - \frac{\alpha_\sigma}{M M_p^2 m^2} e^{-2\sigma/M_p^2} (\partial_t \sigma)^2 + \omega \sqrt{g} \partial_t \sigma ,$$

(2.22)

where

$$M^2 = M_0^2 + \frac{\alpha_\sigma}{M_p^2 m^2} e^{-2\sigma/M_p^2} (\partial_t \sigma)^2 , \quad \mathcal{R}_i = -2 g_{ik} \nabla_j \pi^{kj} ,$$

$$M_0^2 = - \partial_t \phi^a M_{ab} \partial_t \phi^b , \quad M_{ab} = \eta_{ab} - \partial_i \phi_a f^{ij} \partial_j \phi_b .$$

(2.23)

These relations imply

$$M_0^2 = - \frac{1}{\Pi_a M^{ab} \Pi_b} \frac{\Pi_a M^{ab} \Pi_b \alpha_\sigma}{M_p^2} e^{-2\sigma/M_p^2} (\partial_t \sigma)^2$$

(2.24)
where $\Pi_a = p_a + f^{ij} \partial_j \phi_a R_i$. Then it is easy to find relation between momenta and velocities

$$p_\sigma + \sqrt{\frac{\alpha_\sigma}{M_p^2 m^2}} e^{-\sigma/M_p^2} \sqrt{\Pi_a M^{ab} \Pi_b + (\tilde{n}^i R_i + M_p^2 \sqrt{g} U)^2} = \frac{1}{N} \omega \sqrt{g} \partial_\sigma$$

$$\Pi_a \sqrt{\Pi_a M^{ab} \Pi_b + (\tilde{n}^i R_i + M_p^2 \sqrt{g} U)^2} = M_p m \sqrt{\frac{\sigma}{M_p^2}} M_{ab} \partial_\sigma \frac{\partial_t \phi^b}{\partial_t \sigma}.$$  

(2.25)

It is crucial that these relations do not imply an existence of the scalar primary constraint which is in sharp contrast with the case of the dRGT massive gravity [13] or dRGT massive gravity coupled to the galileon [16]. On the other hand, using the property of the matrix $M_{ab}$ we find three constraints

$$\partial_i \phi^a \Pi_a \equiv \Sigma_i = \partial_i \phi^a p_a + R_i \approx 0$$  

(2.26)

that, with additional terms proportional to the primary constraints $\pi_i \approx 0$ are the first class constraints whose smeared forms are the generator of spatial diffeomorphism.

Now using (2.23) we determine corresponding Hamiltonian

$$H = \int d^3 x N \mathcal{H}_0,$$  

(2.27)

where

$$\mathcal{H}_0 = \frac{1}{2 \omega \sqrt{g}} \left( p_\sigma + \sqrt{\frac{\alpha_\sigma}{M_p^2 m^2}} e^{-\sigma/M_p^2} \sqrt{\Pi_a M^{ab} \Pi_b + (\tilde{n}^i R_i + M_p^2 \sqrt{g} U)^2} \right)^2 +$$

$$+ \frac{1}{\sqrt{g} M_p^2} \pi^{ij} \tilde{g}_{ijkl} \pi^{kl} - \sqrt{g}^{(3)} R - 2 m^2 (\Omega \sqrt{g} \sqrt{x} \tilde{D}^i - 2 \sqrt{g}) + \tilde{D}^i \tilde{n}^j R_i.$$  

(2.28)

The requirement of the preservation of the constraint $\pi_N \approx 0$ implies that $\mathcal{H}_0$ is constraint as well. The analysis of constraints is straightforward. We have six first class constraints $\Sigma_i \approx 0, \mathcal{H}_0 \approx 0, \pi_N \approx 0$ and six second class constraints $\pi_i \approx 0, \tilde{C}_i \approx 0$ where $\tilde{C}_i \approx 0$ are the secondary constraints that arise from the requirement of the preservation of the constraints $\pi_i \approx 0$. These constraints can be solved for $\pi_i$ and $\tilde{n}^i$. Further, the first class constraint $\pi_N \approx 0$ can be gauge fixed that leads to the elimination of $\pi_N$ and $N$ as dynamical variables. Finally, the constraints $\Sigma_i, \mathcal{H}_0$ can be again gauge fixed which leads to the elimination of the St"uckelberg fields and conjugate momenta. As a result we are left with 12 degrees of freedom coming from the gravity sector that can be identified with 10 degrees of freedom corresponding to the massive graviton and two degrees of freedom corresponding to the scalar at least at the linearized approximation. Note that this mode cannot be eliminated due to the absence of the scalar constraint so that Boulware-Deser ghost is generally present. Finally there are two phase space degrees of freedom $\sigma$ and $p_\sigma$. 

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2.1 Note About Gauge Fixing

After the first version of this paper was published S. Mukohyama argued in his paper \cite{26} that the extension of the quasidilaton theory is ghost free. His arguments is based on the existence of the additional constraints it his specific gauge

\[ \phi^0 = -e^{-\sigma/M_p}, \phi^i = \delta^i_\mu x^\mu, \quad i = 1, 2, 3. \] (2.29)

Using this gauge he was able to derive the Hamiltonian in the form \( H = \int d^3x N C_0 \) where the specific form of \( C_0 \) is given in \cite{26}. According to this result he claims that there exist an additional constraint in the gauge fixed theory so that this constraint is responsible for the elimination of the Boulware-Deser ghost.

In this section we reconsider the gauge fixing (2.29) from different point of view. We do not impose the fixing of spatial diffeomorphism and consider following relation between \( \phi^0 \) and \( \sigma \)

\[ \sigma = -M_p \ln \phi^0 \] (2.30)

that is equivalent to (2.29). Inserting this relation to the definition of \( \tilde{f}_{\mu\nu} \) we obtain

\[ \tilde{f}_{\mu\nu} = \eta_{ab} \partial_\mu \phi^a \partial_\nu \phi^b - \frac{\alpha}{m_g^2} \partial_\mu \phi^0 \partial_\nu \phi^0. \] (2.31)

Then inserting (2.31) into the dRGT massive gravity we obtain

\[ S = M_p^2 \int d^4x \sqrt{-g} \left[ (4) R + 2m^2(3 - \frac{1}{(\phi^0)^2} \sqrt{g}^{-1} f) - \frac{\omega}{2(\phi^0)^2} g^{\mu\nu} \partial_\mu \phi^0 \partial_\nu \phi^0 \right]. \]

Now we see that given action is manifestly diffeomorphism invariant. In other words while we reduce the number of degrees of freedom by imposing (2.30) the number of gauge symmetries is the same. However we mean that the fact that we have less degrees of freedom than the original theory while the number of gauge symmetries is the same implies that these two theories have different physical content and should not be considered as equivalent.

From given analysis it is clear that the constraint \( C_0 \) that was identified in \cite{26} corresponds to the Hamiltonian constraint in the theory with fixed spatial diffeomorphism. Clearly this constraint should have vanishing Poisson bracket \( \{ C_0(x), C_0(y) \} \) which implies that it is the first class constraint. Clearly there is no way how to generate the additional constraint by imposing the requirement of the preservation of the constraint \( C_0 \) during the time evolution of the system since the Hamiltonian is equal to \( H = \int d^3x N C_0 \). However since \( C_0 \) is the first class constraint it can be gauge fixed and hence we reduce the number of physical degrees of freedom by two. But we should again stress that the theory represented by the action (2.31) is not equivalent to the action (2.1) that represents the extension of the quasidilaton dRGT theory. In summary, we mean that the arguments that were presented in \cite{26} do not prove that the extension of quasidilaton massive gravity is ghost free.

Let us compare this situation with the imposing the static gauge for the Hamiltonian (2.27). This gauge fixing is represented by imposing four gauge fixed functions

\[ \mathcal{G}_0 = \phi^0 - t \approx 0, \quad \mathcal{G}_i = \phi^i - x^i \approx 0. \] (2.32)
These constraints together with $\mathcal{H}_0, \bar{\Sigma}_i$ form the second class constraints that can be explicitly solved for $p_a$. We solve $\mathcal{H}_0$ for $p_0$ since we can identify the gauge fixed Hamiltonian with $p_0 = -\mathcal{H}_{a.f.}$. The resulting Hamiltonian describes the dynamic of the physical degrees of freedom $g_{ij}, \pi^{ij}$ and $\sigma, p_\sigma$. The detailed counting of the physical degrees of freedom was presented in the end of previous section and we will not repeat it here.

3. The case $\alpha_\sigma = 0$

In previous section we saw that the extension of the quasidilaton theory that was suggested in [11] suffers from the presence of Boulware-Deser ghost due to the absence of two additional scalar constraints in the Hamiltonian formulation of given theory. It is instructive to see whether these constraints emerge in the case of quasidilaton massive gravity where $\alpha_\sigma = 0$. Explicitly, we consider the action [8, 9]

$$S = M_p^2 \int d^3x dt \left[ N \sqrt{g} \tilde{K}_{ij} g^{ijkl} \tilde{K}_{kl} + N \sqrt{g} R - \sqrt{g} M_0 U - 2m^2 (N \sqrt{g} \sqrt{x} \Omega D^i_i - 3N \sqrt{g}) + N \sqrt{g} \frac{\omega}{M_p^2} (\nabla_n \sigma)^2 - N \sqrt{g} \frac{\omega}{M_p^2} \partial_i \sigma g^{ij} \partial_j \sigma \right].$$

(3.1)

Let us now perform the Hamiltonian analysis of the action (3.1). First of all we find the canonical momenta

$$p_a = - \left( -M_{ab} \frac{1}{M_0} \partial_t \phi^b \tilde{n}^i + f^{ij} \partial_j \phi_a \right) R_i + \frac{1}{M_0} M_p^2 \sqrt{g} M_{ab} \partial_t \phi^b U + \frac{1}{M_0} M_{ab} \partial_t \phi^b \tilde{n}^i \partial_i \sigma p_\sigma - \partial_i \sigma f^{ik} \partial_k \phi_a p_\sigma,$$

$$p_\sigma = \omega \sqrt{g} \nabla_n \sigma$$

(3.2)

so that it is easy to find the scalar primary constraint

$$\Sigma_\rho \equiv (p_a + (R_i + \partial_i \sigma p_\sigma) f^{ik} \partial_k \phi_a) \eta^{ab} (p_b + (R_i + \partial_i \sigma p_\sigma) f^{ik} \partial_k \phi_b) + (\tilde{n}^i (R_i + p_\sigma \partial_i \sigma) + M_p^2 \sqrt{g} U)^2 \approx 0$$

(3.3)

and also using the fact that $\partial_i \phi^a M_{ab} = 0$ we find additional three constraints

$$\Sigma_i = p_a \partial_i \phi^a + R_i + \partial_i \sigma p_\sigma.$$

(3.4)

Now using these constraints we can simplify (3.3) so that it has the form

$$\Sigma_\rho = p_a M^{ab} p_b + (\tilde{n}^i (R_i + p_\sigma \partial_i \sigma) + M_p^2 \sqrt{g} U)^2 \approx 0.$$

(3.5)

This has exactly the same form as the scalar constraint that emerges in case of dRGT massive gravity written in the St"uckelberg formalism. The minimal form of this gravity...
was analyzed in [13] and this analysis can be easily applied to our case. From (2.15) we find the Hamiltonian with all primary constraints included

\[ H_E = \int d^3x (NC_0 + v_N \pi_N + \nu^i \pi_i + \Omega_p \Sigma_p + \Omega_i \tilde{\Sigma}_i), \]

where we introduced the constraint

\[ \tilde{\Sigma}_i = \Sigma_i + \partial_t \tilde{n}^j \pi_j + \partial_j (\tilde{n}^j \pi_i) \]  

and where

\[ C_0 = \frac{1}{\sqrt{g} M_p^2} \pi^{ij} g_{ijkl} \pi^{kl} - M_p^2 \sqrt{g} R + 2m^2 M_p^2 \sqrt{g} \sqrt{x} \Omega \tilde{D}^i - 6m M_p^2 \sqrt{g} + \]

\[ + \tilde{D}^i \tilde{n}^j (\mathcal{R}_i + p_\sigma \partial_i \sigma) + \frac{1}{\sqrt{g} \omega} p_\sigma^2 + \omega \sqrt{g} g^{ij} \partial_i \sigma \partial_j \sigma. \]  

As the next step we have to perform the analysis of the stability of the primary constraints \( \pi_i \approx 0, \pi_N \approx 0 \) and \( \Sigma_p \approx 0 \). In case of the constraint \( \pi_i \approx 0 \) we find

\[ \partial_t \pi_i = \{ \pi_i, H \} = - \left[ (\mathcal{R}_j + p_\sigma \partial_j \sigma) - \frac{2m^2 M_p^2 \sqrt{g}}{\sqrt{x}} n^k f_{kj} \right] \times \]

\[ \times \left[ N \frac{\delta (\tilde{D}^m \tilde{n}^m)}{\delta \tilde{n}^i} + \delta_i^j (\tilde{n}^i (\mathcal{R}_i + p_\sigma \partial_i \sigma) + M_p^2 \sqrt{g} U) \right] = 0 \]

so that we impose following secondary constraint

\[ \mathcal{C}_i = \mathcal{R}_i + p_\sigma \partial_i \sigma - \frac{2m^2 M_p^2 \sqrt{g}}{\sqrt{x}} f_{ij} \tilde{n}^j. \]  

Now with the help of this constraint and the constraint \( \tilde{\Sigma}_i \) we can simplify \( \Sigma_p \) in the similar way as in [13]

\[ \Sigma_p = 4m^4 M_p^4 g \Omega^2 + p_A \eta^{AB} p_B. \]

Then it is easy to show that \( \{ \Sigma_p(x), \Sigma_p(y) \} = 0 \) and the requirement of the preservation of given constraint leads to the emergence of an additional constraint. These constraints are responsible for the elimination of Boulware-Deser ghost, see again [13] for more details. In other words, the presence of the kinetic term for the quasidilaton that minimally couples to gravity does not spoil the property that given theory is ghost free.

4. Ghost Free Extension of Quasidilaton Massive Gravity

We argued in section (2) that the extension of quasidilaton theory as was formulated in [11] is plagued by the presence of the Boulware-Deser ghost. On the other hand given theory
has many nice properties so that it is desirable to propose its ghost free version. In this section we propose such a formulation when we replace the kinetic term for \( \sigma \) by follow-

ing tadpole galileon term

\[
S_{\sigma} = -T \int d^4x \Psi(\Phi^A) \sqrt{- \det \tilde{f}_{\mu\nu}} = -T \int d^4x \Psi(\Phi^A) M \sqrt{\tilde{f}} ,
\]

(4.1)

where \( \Phi^A = (\phi^a, \sigma) \), \( \tilde{f} = \det \tilde{f}_{ij} \) and where the function \( \Psi(\Phi^A) \) was chosen in such a way that the action (4.1) is invariant under (2.4). We claim that the quasidilaton theory formulated as the dRGT massive theory with \( \tilde{f}_{\mu\nu} \) and with the kinetic term for the galileon given by (4.1) is ghost free.

To see this explicitly it is useful to introduce following notation. Let's write \( \tilde{f}_{\mu\nu} \) as

\[
\tilde{f}_{\mu\nu} = \partial_\mu \Phi^A G_{AB} \partial_\nu \Phi^B ,
\]

(4.2)

where we introduced the metric \( G_{AB} \)

\[
G_{AB} = \begin{pmatrix} \eta_{AB} & 0 \\ 0 & -\frac{\alpha}{M_p m^2} e^{-2\sigma/M_p} \end{pmatrix} .
\]

(4.3)

We see that our proposal has the form of the galileon coupled to dRGT massive gravity \([17]\) whose Hamiltonian analysis was performed in \([16]\). On the other hand the action defined by (4.1) is more complicated since the metric \( \eta_{AB} \) is replaced with the more general metric \( G_{AB} \) that depends on \( \Phi^A \) and there are also additional scalar fields \( \Omega(\phi^A), \Psi(\Phi^A) \). However we can expect that this fact will not modify the constraint structure of given theory.

To see this explicitly let us briefly review the Hamiltonian analysis of the non-linear massive gravity with the term (4.1) keeping in mind that more detailed analysis can be found in \([16]\). As usual the momenta conjugate to \( N, \tilde{n}^i \) and \( g_{ij} \) are

\[
\pi_N \approx 0 , \pi_i \approx 0 , \pi^{ij} = M_p^2 \sqrt{g} g^{ijkl} \tilde{K}_{kl}
\]

(4.4)

while the momentum conjugate to \( \Phi^A \) has the form

\[
p_A = - \left( \frac{\delta M}{\delta \partial_t \Phi^A} \tilde{n}^i + G_{AB} \tilde{f}^{ij} \partial_j \Phi^B \right) R_i - M_p^2 \sqrt{g} \frac{\delta M}{\partial_t \Phi^A} U' ,
\]

(4.5)

where

\[
U' = 2m^2 \Omega(\Phi) \sqrt{\tilde{x}} + \frac{T}{M_p^2} \Psi(\Phi) \sqrt{\tilde{f}} ,
\]

(4.6)

and where \( M^2 \) has the form

\[
M^2 = -\partial_i \Phi^A M_{AB} \partial_i \Phi^B , \quad M_{AB} = G_{AB} - G_{AC} \partial_i \Phi^C \tilde{f}^{ij} \partial_j \Phi^D G_{DB} .
\]

(4.7)
Note that the matrix $\mathcal{M}_{AB}$ obeys following relations
\[
\mathcal{M}_{AB} \mathcal{G}^{BC} \mathcal{M}_{CD} = \mathcal{M}_{AD} , \quad \partial_i \Phi^A \mathcal{M}_{AB} = 0 .
\] (4.8)

Then it is easy to determine following primary constraints
\[
\Sigma_p = (\tilde{n}^i \mathcal{R}_i + M_p^2 \sqrt{g} \zeta')^2 + (p_A + \mathcal{R}_i \tilde{f}^{ij} \mathcal{G}_{AC} \partial_j \Phi^C) \mathcal{G}^{AB} (p_B + \mathcal{R}_i \tilde{f}^{ij} \mathcal{G}_{BD} \partial_j \Phi^D) \approx 0
\]
and
\[
\partial_i \Phi^A p_A + \mathcal{R}_i = \Sigma_i \approx 0 .
\] (4.9)

Now we are ready to write the extended Hamiltonian which includes all the primary constraints
\[
H_E = \int d^3 x (N C_0 + v_N \pi_N + v^i \pi_i + \Omega_p \Sigma_p + \Omega^i \tilde{\Sigma}_i) ,
\] (4.10)
where
\[
C_0 = \frac{1}{\sqrt{g} M_p^2} \pi^{ij} \mathcal{G}_{ijkl} \pi^{kl} - M_p^2 \sqrt{g} \mathcal{R} + 2 m^2 M_p^2 \sqrt{g} \Omega(\Phi) \sqrt{\tilde{x}} \tilde{D}^i \tilde{n}_i - 6 m^2 M_p^2 \sqrt{\tilde{g}} + \tilde{D}_j \tilde{n}^j \mathcal{R}_i
\] (4.11)
and where we introduced the constraints $\tilde{\Sigma}_i$ defined as
\[
\tilde{\Sigma}_i = \Sigma_i + \partial_i \tilde{n}^i \pi_i + \partial_j (\tilde{n}_j \pi_i) .
\] (4.12)

To proceed further we have to check the stability of all constraints. The procedure is the same as in [16] so that we find that $\tilde{\Sigma}_i$ are the first class constraints while the requirement of the preservation of the constraints $\pi_i \approx 0$ implies following secondary constraints [20, 21]
\[
C_i \equiv \mathcal{R}_i - \frac{2 m^2 M_p^2 \Omega(\Phi) \sqrt{\tilde{g}}}{\sqrt{\tilde{x}}} \tilde{f}^{ij} \tilde{n}^i \approx 0 .
\] (4.13)

Further, the requirement of the preservation of the constraint $\pi_N \approx 0$ implies an existence of the secondary constraint $C_0 \approx 0$. Using the constraints $C_i$ and $\Sigma_i$ we replace the constraint $\Sigma_p$ by new independent constraint $\tilde{\Sigma}_p$
\[
\tilde{\Sigma}_p = 4 m^4 M_p^4 \Omega^2 g + p_A \mathcal{G}^{AB} p_B + 2 T \Psi \sqrt{\tilde{f}} \sqrt{\tilde{g}} \mathcal{R} \partial_i \phi^A \tilde{f}^{ij} \partial_j \phi^B p_B + 4 m^4 M_p^4 \Omega^2 g + T^2 \Psi^2 \tilde{f} = 0 .
\] (4.14)

Then the total Hamiltonian, where we include all constraints, takes the form
\[
H_T = \int d^3 x (N C_0 + v_N \pi_N + v^i \pi_i + \Omega_p \tilde{\Sigma}_p + \Omega^i \tilde{\Sigma}_i + \Gamma^i C_i) .
\] (4.15)

Now we are ready to analyze the stability of all constraints that appear in (4.15). Again, the analysis is the same as in [16] with slight complication that now there are additional terms.
Ψ(Φ), Ω(Φ) together with $G_{AB}(\Phi^4)$ in the definition of the action. However these terms are local functions of $\Phi^4$ so that they do not affect the result that \( \{ \tilde{\Sigma}_p(x), \tilde{\Sigma}_p(y) \} \approx 0 \). As a result the requirement of the preservation of the constraint $\tilde{\Sigma}_p \approx 0$ implies new constraint $\tilde{\Sigma}_p^{II} \approx 0$. These two constraints are the second class constraints that can be used for the elimination of the Boulware-Deser ghost mode and its conjugate momenta.

In this section we proposed an extension of the quasidilaton massive gravity that is ghost free. This proposal can be generalized in different ways, either consider the most general potential of the dRGT massive gravity or more complicated kinetic term for $\sigma$. It would be also very interesting to analyze the cosmological consequences of the model with the action (1.1) following [11].

Acknowledgement:

This work was supported by the Grant agency of the Czech republic under the grant P201/12/G028.

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