Effect of disorder on superconducting diodes

S. Ilić 1 and F. S. Bergeret1,2
1Centro de Física de Materiales (CFM-MPC), Centro Mixto CSIC-UPV/EHU,
Manuel de Lardizabal 5, E-20018 San Sebastián, Spain and
2Donostia International Physics Center (DIPC),
Manuel de Lardizabal 4, E-20018 San Sebastián, Spain

We calculate the non-reciprocal critical current of disordered Rashba superconductors. We show that disorder only weakly suppresses the supercurrent diode effect in such systems. This effect can, therefore, be realized even in disordered materials and hybrid structures. From our findings we propose a layered hybrid structure which works as superconducting diode, even at zero applied magnetic field. It consists of a conventional superconducting thin film sandwiched between a ferromagnetic insulator and an insulator with a heavy metal.

Introduction.- The interplay between superconductivity, spin-orbit coupling (SOC), and a Zeeman field leads to a variety of magnetoelectric effects widely studied in the past years [1] [12]. One of these effects is a nonreciprocal charge transport due to the breaking of time-reversal and inversion symmetries [13][19]. Originally this effect was studied in the resistive regime, when $T \gtrsim T_c$, where superconducting fluctuations play a crucial role [14][15]. More recently, it has been shown that nonreciprocity also manifests on the supercurrent in non-centrosymmetric superconducting structures and in Josephson junctions [15] [18][19]. Specifically, the critical current depends on the direction of the current flow, and hence such systems are being suggested as superconducting diodes with potential applications in low-power logic circuits.

The non-reciprocity of the critical current can be quantified by the superconducting diode quality parameter

$$\eta = \left| \frac{J_c^+ - J_c^-}{J_c^+ + J_c^-} \right|,$$

where $J_c^\pm$ are the critical currents in opposite directions. It has been shown in Refs. [15] [19] that $\eta$ is finite in non-centrosymmetric superconducting systems in the presence of a magnetic field. Those theoretical works assume ideally pure superconducting structures and disregard the effect of disorder. However, the latter is unavoidable in realistic structures, and therefore it is important to understand how it affects the superconducting diode effect. Moreover, understanding the role of disorder will enable to design devices based on combination of conventional materials.

In this Letter, we study the effect of disorder on the superconducting diode effect. Specifically, we derive the expression for the superconducting diode quality parameter for arbitrary disorder. We show that disorder reduces the diode effect, but only weakly. In the limiting case of diffusive systems, when $\tau T_c \ll 1$, where $\tau$ is the momentum relaxation time, $\eta$ is reduced with respect to the ideal ballistic case by only $\sqrt{\tau T_c}$. This means, in particular, that the diode effect can be realized in systems with strong degree of disorder. We also discuss a possible realization of a superconducting diode which does not need an external magnetic field. It consists of a conventional superconducting film sandwiched between a ferromagnetic insulator, which provides an interfacial exchange field, and a heavy metal oxide with a strong interfacial SOC. Using our results we estimate that such a system may show a diode quality parameter similar in magnitude to ideal ballistic systems.

Non-reciprocal critical current.- Following Refs. [18] and [19], we calculate the critical current by employing a phenomenological approach based on the Ginzburg-Landau (GL) theory. Let us consider a quasi two-dimensional superconductor with an order parameter $\Delta(r) = |\Delta| e^{iqr}$, where $\mathbf{q} = (q_x, q_y)$ is a phase gradient. Then, the GL free energy can be written as $F = \int d\mathbf{q} F_\mathbf{q}$, where the free energy density is given as

$$F_\mathbf{q} = \alpha_\mathbf{q} |\Delta|^2 + \frac{\beta}{2} |\Delta|^4.$$

Minimizing $F_\mathbf{q}$ with respect to $\Delta$, we obtain the order parameter as $|\Delta|^2 = -\frac{\alpha_\mathbf{q}}{\beta}$. Next, the equilibrium supercurrent can be obtained as $\mathbf{J} = -\partial F_{\mathbf{q} - 2e\mathbf{A}} / \partial \mathbf{A}|_{\mathbf{A} = 0}$, where $\mathbf{A}$ is a vector potential. This yields

$$J_\mathbf{q} = -\frac{2e}{\beta} \alpha_\mathbf{q} \partial \alpha_\mathbf{q} / \partial \mathbf{q}.$$

In the following, we focus on superconductors with Rashba SOC, where the GL coefficient $\alpha_\mathbf{q}$ has the following form at a weak magnetic field $\mathbf{h} = (h_x, h_y)$

$$\alpha_\mathbf{q} = -a_0 + a_1 |\mathbf{q}|^2 - (q_x h_y - q_y h_x)(b_1 - |\mathbf{q}|^2 b_2).$$

The term $b_1$ is known as the Lifshitz invariant, which is responsible for a variety of magnetoelectric effects in Rashba superconductors [7][9][12][20]. We also keep the third-order terms in $\mathbf{q}$, which are crucial for the superconducting diode effect [14][15][18][19]. Let us assume that the magnetic field is applied along the $y$-direction - $\mathbf{h} = (0, h)$, and that the phase of the superconductor varies only along the $x$-direction - $\mathbf{q} = (q, 0)$. The current

$$J_x = -\frac{2e}{\beta} \alpha_q |\Delta|^2 - \frac{2e}{\beta} \alpha_q \partial \alpha_q / \partial q_x.$$
along the x-direction is then given by

$$J(q) = -\frac{2e}{\beta}(2qa_1 - hb_1 + 3q^2hb_2)(-a_0 + q^2a_1 - qh b_1 + q^3hb_2).$$

(5)

Note that the current is zero at a finite momentum $q_0 \approx \frac{hb_1}{2a_1}$, which corresponds to a finite Cooper pair momentum or the helical modulation of the order parameter [21][22]. To find the critical current we need to calculate the extreme values $\frac{dJ(q)}{dq}|_{q=q_0} = 0$. Solving this equation perturbatively in $h$, we find the critical momenta $q_c^{\pm} = \pm \sqrt{\frac{ha_0}{3a_1} + \frac{hb_1}{2a_1} - \frac{hb_0a_0}{18a_1^2}}$. The critical current is then

$$J_c^{\pm} = J(q_c^{\pm}) = \frac{e}{\beta} \left[ \pm \frac{8}{3\sqrt{3}} a_0^{3/2} a_1^{1/2} + 8 \frac{ha_0^2b_2}{9a_1} \right].$$

(6)

From this expression follows that, if the term cubic in $q$ is finite in the free energy ($b_2 \neq 0$), the critical current will be different depending on the direction of its flow. This is the diode effect. Combining Eq. (6) with Eq. (1), we can calculate the quality factor of the superconducting diode as

$$\eta = \frac{1}{\sqrt{3}} \frac{hb_0a_1^{1/2}}{a_1^{3/2}}.$$  

(7)

The GL coefficients depend on the details of SOC and disorder. We calculate them using the quasiclassical Eilenberger equation at arbitrary disorder [see the Supplementary information (SI)]. The quasiclassical theory was established for disordered Rashba superconductors in two regimes: strong SOC $\mu > \alpha_F \sim T_c, \tau^{-1}$, and weak SOC $\mu > \alpha_F, T_c, \tau^{-1}$ [11]. Here $\mu$ is the chemical potential, $\alpha$ is the anomalous velocity associated with Rashba SOC, and $\alpha_F$ is the Fermi momentum. In the rest of this Letter we will address both regimes. While the first regime gives the most pronounced diode effect, the second regime can be useful in description of hybrid structures where SOC is weaker.

**Regime of strong SOC.** - Disordered Rashba electron gas in an in-plane magnetic field is described by the following Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} - \mu + \alpha(p_x \sigma_y - p_y \sigma_x) + h \sigma_y + V_{imp}$$

(8)

where $\mathbf{p} = (p_x, p_y)$ is the electron momentum, $m$ is the effective mass, and $\sigma_i$ are the Pauli matrices. The term $V_{imp}$ describes a spin-independent disorder potential, with an associated scattering rate $\tau^{-1}$. To account for superconductivity, one adds to the Hamiltonian the usual mean field BCS Hamiltonian with a conventional $s$-wave pairing.

If the SOC is the dominant energy scale of the system, $\mu > \alpha_F \gg T_c, \tau^{-1}$, the superconductivity is effectively two-band, where the Cooper pairs form separately in the two helical bands but with the same superconducting gap. Any finite disorder, however, introduces interband scattering which weakly couples the helical bands. This situation is captured by the quasiclassical Eilenberger equation established in Ref. [1], which allows us to calculate the GL coefficients in Eq. (4) at arbitrary disorder (see the SI):

$$a_0 = \nu T_c - T, \quad a_1 = 2\pi \nu \sum_{\omega > 0} \frac{\tau^2 p_3}{(2\nu^2(1 + 4\tau^2))(1 + 4\tau^2)}.$$  

$$b_1 = \frac{2\pi \nu T_c}{\omega^2(1 + 4\tau^2)}.$$

$$b_2 = \frac{2\pi \nu T_c}{4\omega^2(1 + 4\nu T_c)\nu T_c}.$$  

(9)

Here, the Fermi velocity $v_F$ depends on the magnitude of SOC, namely $v_F = \sqrt{2\mu/m + \alpha^2}$. $\omega = 2\pi T(n + \frac{1}{2})$ is the Matsubara frequency, $\nu$ is the density of states in the normal state, and we introduced the polynomial $p_3(z) = 4 + 45z + 128z^2 + 96z^3$.

Next, we combine Eqs. (7) and (9) to calculate the diode quality factor $\eta$. In the clean limit $\alpha_F \gg T_c \gg \tau^{-1}$, the GL coefficients reduce to $a_1 \approx 0.26\nu \frac{\alpha}{v_F \pi T_c} t$, $b_1 \approx 1.05\nu \frac{\alpha}{v_F \pi T_c} t$, $b_2 \approx 0.38\nu \frac{\alpha^2}{v_F \pi T_c \nu T_c} t$. Therefore, the quality factor is

$$\eta = 0.53 \frac{\alpha}{\nu v_F} T_c \tau^{-1},$$

(10)

where we introduced $t = \sqrt{1 - \frac{T}{T_c}}$. Eq. (10) agrees with results previously reported in Ref. [19]. In the diffusive limit, $\alpha_F \gg \tau^{-1} \gg T_c$, the GL coefficients become $a_1 = \frac{\nu}{16} (\frac{\nu T_c}{\nu v_F}) \pi T_c \nu T_c$, $b_1 = \frac{\nu a_T}{2\tau_F} \pi T_c \nu T_c$, $b_2 \approx 2.1\nu \tau^{-1} \nu T_c \nu T_c$. This yields

$$\eta = 1.4 \frac{\alpha}{\nu v_F} \frac{h}{\sqrt{v_T} + \alpha^2 \frac{h}{\pi T_c \nu T_c}} \sqrt{T_c \tau}$$

(11)

**FIG. 1.** Diode quality factor as a function of disorder, normalized with respect to the value in the ballistic limit $\eta_b$ given by Eq. (10).
Comparing expressions [10] and [11], we see that the supercurrent diode effect is only weakly suppressed by disorder - \( \eta \) in reduced in the diffusive case by the factor \( \sim \sqrt{T_c \tau} \) compared to the ballistic case. This is also evident from Fig. 1 where we plot \( \eta \) as a function of disorder.

\[
a_0 = \nu \frac{T_c - T}{T_c}, \quad a_1 = 2 \pi \nu T_c \sum_{\omega > 0} \frac{1}{4 \omega^2 (1 + 2 \omega \tau)}, \quad b_1 = 2 \pi T_c \nu \sum_{\omega > 0} \frac{1}{p_F \omega^2} \left[ 4 \tau \omega (1 + 2 \omega)^2 + x_0^3 \right], \\
b_2 = 2 \pi T_c \nu \sum_{\omega > 0} \frac{1}{p_F 8 \omega^3} \left[ p_1(\tau \omega) + 4 x_0^2 p_2(\tau \omega) + x_0^4 p_3(\tau \omega) \right],
\]

Here, we introduced the Fermi velocity \( v_F = \sqrt{2 \mu/m} \), the quantity \( x_\alpha = 2 \alpha p F \tau \), which describes the relative strength of SOC and disorder, and the polynomials \( p_1(z) = 8 \tau (1 + 2 z)^3(3 + 20 z + 30 z^2) \), \( p_2(z) = (1 + 2 z)^2(1 + 18 z + 68 z^2 + 68 z^3) \).

Taking the limit \( \alpha p F \gg T_c, \tau^{-1} \) in Eq. (12), we recover Eqs. (9)-(11) up to the lowest order in \( \alpha/v_F \) with \( v_F \approx v_F \). Importantly, Eq. (12) also allows us to explore the limit of weaker SOC \( \alpha p F \ll T_c, \tau^{-1} \), which was not addressed in the previous theoretical works [18, 19], and which may be relevant for many structures involving superconductors and materials with SOC [4, 24, 25]. Combining Eq. (7) with Eq. (12), we plot the quality factor in this regime in Fig. 2. We see that a substantial \( \eta \) can be realized already at \( \alpha p F > 2 T_c \) in the clean case.

**FIG. 2.** Diode quality factor as a function of SOC strength, for various disorder strengths. The quality factor is normalized with respect to its value in the ballistic limit at strong SOC \( \eta_0 \) given by Eq. (10).

Simple analytical expressions can be found in the regime \( T_c \gg \tau^{-1}, \alpha p F \), where we obtain \( \eta = 0.13 \frac{\omega}{v_F} \frac{T_c}{\tau} \alpha^2 p_F^2 \tau \), and in the diffusive regime \( \tau^{-1} \gg T_c \gg \alpha^2 p_F^2 \tau \), where we have \( \eta = 0.16 \frac{\omega}{v_F} \frac{T_c}{\tau} \alpha^2 p_F^2 (T_c \tau)^{3/2} \).

**Possible realization of a superconducting diode.** In this section we propose a simple system, sketched in Fig. 3, in which the diode effect can be observed. It consists of a film of a conventional superconducting material (SC). The film is in contact on one side with a heavy metal insulator (HMI), which provides an interfacial SOC [27, 29]. On the other side SC has an interface with a ferromagnetic insulator (FI), such as EuS. The latter guarantees an interfacial exchange field that for a thin-enough superconducting film, acts as a Zeeman field and splits the density of states of the superconductor [30, 32]. If the magnetization of the FI points in the direction perpendicular to the page (bright blue cross) then the critical current is nonreciprocal in the direction shown by the yellow arrows.

**FIG. 3.** Possible setup for realization of a superconducting diode. The SC is a conventional superconducting film sandwiched between a ferromagnetic insulator (FI) and an insulator with a heavy metal (HMI). The FI/SC interface provides an exchange field whereas the HMI/SC provides an interfacial SOC. If the magnetization of the FI points in the direction perpendicular to the page (bright blue cross) then the critical current is nonreciprocal in the direction shown by the yellow arrows.
described by the equations derived above. By taking the value of the interfacial SOC estimated for Cu/BiO interfaces in Ref. \cite{27}, \( \alpha \sim 0.4 \) eVÅ, and \( p_F \sim 10^8 m^{-1} \), \( T_C \sim 1 \) − 10K we obtain that \( \alpha p_F / T_C \gg 1 \). Thus, according to Fig. \[2\] the diode quality parameter is comparable in magnitude to the one in the ideal ballistic case. Candidate materials for the realization of the diode of Fig. \[3\] can be the following: a conventional superconductor such as Al, V or Nb in contact with in FI such as EuS or EuO. The HMI can be BiO or Bi. An important advantage of the proposed device is that the diode effect can be observed even in the absence of the magnetic field thanks to the finite exchange field induced by the FI \cite{31, 32}. Moreover, by controlling the direction of the FI magnetization one can tune the diode effect.

Conclusion In summary, we calculated the effect of disorder on the non-reciprocal critical current in superconducting systems with a Rashba SOC. Our results apply to both strong and weak SOC, and arbitrary disorder, thus covering all regimes of interest for numerous recent experiments in this field. Our main result is establishing that the diode effect is only weakly suppressed by disorder, namely, in the diffusive limit the supercurrent diode quality factor is reduced only by a factor of \( \sqrt{T_C / \tau} \) compared to the ballistic case. Moreover, we propose a new realization of the supercurrent diode - a hybrid structure of a superconductor, ferromagnetic insulator and a heavy metal oxide, which would function even without an external magnetic field.

The quasiclassical equations employed in this work can also be used to study the effect of disorder in other non-centrosymmetric superconductors and hybrid structures where the diode effect is predicted. Most notably transition metal dichalcogenides \cite{14, 18}, as well as the non-reciprocal transport in the fluctuation regime \cite{14, 15}. Moreover, the same approach can be used to study the diode effect at low temperatures, beyond the Ginzburg-Landau equations, where this effect is expected to be even stronger.

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Supplementary Information for “Effect of disorder on superconducting diodes”

In this Supplementary Information, we derive the Ginzburg-Landau coefficients from Eq. (4) of the main text. In Sec. S1, we derive Eq. (9) of the main text, valid in the strong SOC limit $\mu \sim \alpha p_F \gg T_c, \tau^{-1}$. In Sec. S2, we derive Eq. (12) of the main text, valid in for the weak SOC $\mu \gg \alpha p_F, T_c, \tau^{-1}$.

S1. STRONG SOC LIMIT $v_F \gtrsim \alpha$

We start the Eilenberger equation, Eq. (24) from Ref. [2], and linearize it:

$$\Omega_\pm f_\pm = -2i\Delta + \frac{1}{2\tau_+} \left[ \langle f_+ \rangle \pm n_i \langle n_i f_+ \rangle \right] + \frac{1}{2\tau_-} \left[ \langle f_- \rangle \mp n_i \langle n_i f_- \rangle \right].$$  \hspace{1cm} (S1)

$f_+$ and $f_-$ are the anomalous Green’s functions in the two helical bands. We introduced $\Omega_\pm = 2\omega + \frac{1}{\tau} \mp 2i(n_x h_y - n_y h_x) + i\nu F q \cdot n_i$ and $\frac{1}{\tau_\pm} = \frac{1}{2}(1 \mp \frac{\nu}{v_F}).$ Here, (...) stand for the average over the direction of the Fermi momenta, described by the vector $n = (n_x, n_y)$. Summation over repeated indices is implied. Starting from Eq. (S1), by averaging we obtain

$$\langle f_\pm \rangle = \frac{1}{\Omega_\pm} \left[ -2i\Delta + \frac{\langle f_+ \rangle}{2\tau_+} + \frac{\langle f_- \rangle}{2\tau_-} \right] \pm \frac{n_i}{\Omega_\pm} \left[ \frac{\langle n_i f_+ \rangle}{2\tau_+} - \frac{\langle n_i f_- \rangle}{2\tau_-} \right],$$

$$\langle n_i f_\pm \rangle = \frac{n_i}{\Omega_\pm} \left[ -2i\Delta + \frac{\langle f_+ \rangle}{2\tau_+} + \frac{\langle f_- \rangle}{2\tau_-} \right] \mp \frac{n_i n_j}{\Omega_\pm} \left[ \frac{\langle n_j f_+ \rangle}{2\tau_+} - \frac{\langle n_j f_- \rangle}{2\tau_-} \right].$$  \hspace{1cm} (S2)

Then, we can solve Eq. (S2) to obtain $\langle f_\pm \rangle$.

The self-consistency condition is

$$\Delta \left[ \frac{T - T_c}{T_c} \right] = 2\pi T_c \sum_\omega \left[ i \frac{1 - \nu}{2\tau} \langle f_+ \rangle + i \frac{1 + \nu}{2\tau} \langle f_- \rangle - \frac{\Delta}{\omega} \right],$$  \hspace{1cm} (S3)

where we take the Matsubara frequencies at $T_c$: $\omega = 2\pi T_c (n + \frac{1}{2})$. Different prefactors in front of $f_+$ and $f_-$ appear due to different densities of states in the two helical bands: $\nu_\pm = \nu(1 \mp \frac{\nu}{v_F})$. From here we can construct the term $\alpha_q$ in the Ginzburg-Landau free energy, namely

$$\alpha_q = \nu \left[ \frac{T - T_c}{T_c} \right] - 2\pi T_c \sum_\omega \left[ i \frac{1 - \nu}{2\Delta} \langle f_+ \rangle + i \frac{1 + \nu}{2\Delta} \langle f_- \rangle - \frac{1}{\omega} \right].$$  \hspace{1cm} (S4)

Expanding the expressions for $\langle f_\pm \rangle$ up to third order in $q$ and first order in $h$, we obtain Eq. (9) from the main text.

S2. WEAK SOC $v_F \gg \alpha$

At weak SOC ($v_F \gg \alpha$), Rashba superconductors can be described using the quasiclassical formalism that treats the SOC as the background SU(2) field $[2, 3]$. The linearized Eilenberger equation is:

$$iv_F n_i q_l f + 2\omega f - iv_F [n_i A_i, f] + \{ih_i \sigma_i, f\} - \frac{1}{2m} \left\{ n_i F_{ij}, \frac{\partial f}{\partial n_j} \right\} + \frac{1}{\tau} \left( f - \langle f \rangle \right) = -2i\Delta.$$  \hspace{1cm} (S5)

For Rashba SOC, the effective SU(2) vector potentials are $A_x = -ma_\sigma_y$, $A_y = ma_\sigma_x$, and the effective magnetic field is $F_{xy} = -F_{yx} = 2m^2 a^2 \sigma z$. $f$ is a matrix in spin space, $f = f^\sigma i$, $i = 0, x, y, z$, where $f_0$ describes the singlet, and $f_1$ the triplet superconducting correlations. Next, we introduce the dimensionless quantities $\delta = n_i Q_i l$, with $l = v_F \tau$, and $\Omega = ln_i A_i$, so that [S5] can be rewritten as

$$(1 + i\delta + 2\omega) f - i\Omega, f = -2i\Delta \tau + \langle f \rangle - \{i\tau h_i \sigma_i, f\} + \frac{1}{2p_F} \left\{ n_i F_{ij}, \frac{\partial f}{\partial n_j} \right\}. \hspace{1cm} (S6)$$

We solve Eq. (S5) perturbatively in $h$ and $F$, by writing

$$f = f^{(0)} + f^{(h)} + f^{(sc)} + f^{(sc+h)}.$$  \hspace{1cm} (S7)
Here \( f^{(0)} \) is the bare solution, and the perturbations are \( f^{(h)} \propto h \), \( f^{(sc)} \propto F \), and \( f^{(sc+h)} \propto F h \). They satisfy the following equations

\[
(1 + i \delta + 2 \omega) f^{(i)} - i \{ \Omega, f^{(i)} \} = X^{(i)} + \langle f^{(i)} \rangle. \tag{S8}
\]

Here, we introduced the source terms

\[
\begin{align*}
X^{(0)} &= -2i \Delta \tau, \quad X^{(h)} = -\{i \tau \hbar \sigma_i, f^{(0)} \}, \quad X^{(sc)} = \frac{l}{2p_F} \{ n_i F_{ij}, \frac{\partial f^{(0)}}{\partial n_j} \}, \\
X^{(sc+h)} &= -\{i \tau h \sigma_i, f^{(sc)} \} + \frac{l}{2p_F} \{ n_i F_{ij}, \frac{\partial f^{(h)}}{\partial n_j} \}. \tag{S9}
\end{align*}
\]

Then, following Ref. [4], we can write the solutions as

\[
f^{(i)} = \frac{1}{2|\Omega|^2} \frac{1}{1 + i \delta + 2 \omega \tau} \left\{ \Omega, X^{(i)} + \langle f^{(i)} \rangle \right\} \Omega + \frac{i}{M} \left\{ \Omega, X^{(i)} + \langle f^{(i)} \rangle \right\} + \frac{1}{4|\Omega|^2} \frac{1 + i \delta + 2 \omega \tau}{M} \left\{ \Omega, X^{(i)} + \langle f^{(i)} \rangle \right\}. \tag{S10}
\]

Here, we introduced \( M = (1 + i \delta + 2 \omega \tau)^2 + 4|\Omega|^2 \). Finally, we average Eq. (S8) over \( n \) and obtain

\[
\left\langle (i \delta + 2 \omega) f^{(i)} - i \{ \Omega, f^{(i)} \} \right\rangle = \langle X^{(i)} \rangle. \tag{S11}
\]

Replacing Eq. (S10) into Eq. (S11), we obtain an equation determining the averages \( \langle f^{(i)} \rangle \). Using this, after a lengthy calculation, we find the expression for the singlet function \( \langle f_0 \rangle = \langle f^{(0)} + f^{(sc+h)} \rangle \) up to third order in \( q \) and first order in \( h \):

\[
\begin{align*}
\frac{i\langle f_0 \rangle}{\Delta} &= \frac{1}{\omega} - \frac{1}{4\omega^2} \frac{q^2 v_F^2 r}{(1 + 2 \omega \tau)} \\
&+ (h_y q_x - q_x h_y) F_F^{-1} \left[ \frac{x_0^3}{\omega^2 [4 \tau \omega (1 + 2 \tau \omega)^2 + x_0^2 (1 + 4 \tau \omega)]} - \frac{q^2 v_F^2 r x_0^3 [p_1(\tau \omega) + 4 x_0^2 p_2(\tau \omega) + x_0^4 p_3(\tau \omega)]}{8 \omega^3 (1 + 2 \tau \omega)^3 [x_0^2 + (1 + 2 \tau \omega)^2] [4 \tau \omega (1 + 2 \tau \omega)^2 + x_0^2 (1 + 4 \tau \omega)]} \right]. \tag{S12}
\end{align*}
\]

The self-consistency condition reads

\[
\Delta \frac{T - T_c}{T_c} = 2 \pi T_c \sum_\omega \left[ \frac{i\langle f_0 \rangle}{\Delta} - \frac{\Delta}{\omega} \right]. \tag{S13}
\]

From here we can construct the term \( \alpha_q \) in the Ginzburg-Landau free energy, namely

\[
\alpha_q = \nu \frac{T - T_c}{T_c} - 2 \pi T_c \nu \sum_\omega \left[ \frac{i\langle f_0 \rangle}{\Delta} - \frac{1}{\omega} \right], \tag{S14}
\]

which yields Eq. (12) of the main text.

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