Thermal Stresses in an Elastic Clamped Square: Exact Solution

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Abstract. This paper presents a method for determining thermal stresses in an elastic clamped square with a given temperature distribution (the plane problem). First, the solution to the temperature problem for an infinite plane is constructed. Then, the solution for a square is added to this solution, with the help of which the boundary conditions on its sides are satisfied. The thermal stresses have been obtained in the form of series in Papkovich–Fadle eigenfunctions, the coefficients of which are determined explicitly. The final formulas are simple and can easily be used in engineering.

1. Introduction
The determination of thermal stresses plays an important role in the design of structures operating at elevated temperatures such as nuclear reactors, jet and rocket engines, steam and gas turbines, etc. Elements of such structures are subjected to nonuniform and often nonstationary heating, which causes temperature gradients and nonuniform thermal expansion of parts of the elements. Such nonuniform thermal expansion cannot freely flow in an elastic body and causes thermal stresses. It is necessary to know the magnitude and behavior of thermal stresses in order to conduct a comprehensive analysis of the strength of the structure.

Thermoelastic problems were considered in almost all textbooks on the theory of elasticity. An extensive list of the previous studies on thermal stresses in rectangular flat plates can be found in [1–6]. Various methods were used to determine thermal stresses in flat plates subjected to certain temperature fluctuations in the plane of the plate, such as the superposition method, method of Fourier integrals, collocation method, finite element method, etc. Almost all of them, in one way or another, are reduced to the approximate solution of infinite systems of algebraic equations.

In the proposed paper, an exact solution is obtained for a completely clamped square with a given temperature distribution inside the domain (even-symmetric deformation with respect to the central axes). First, the corresponding temperature problem for an infinite plane is solved. Then, by using the exact solutions for the rectangle [7], the boundary conditions on the sides of the square are satisfied. The solution method is based on the representation of thermal stresses and displacements in the form of series in Papkovich–Fadle eigenfunctions and the use of biorthogonality relations to find the expansion coefficients in an explicit form.
2. Formulation of the problem and its solution

Let us consider the plane thermoelastic problem for the square \( S : |y| \leq 1, |x| \leq 1 \) all the sides of which are clamped. We will assume that the temperature field \( t(x, y) = x^2 + y^2 \) is given inside the square.

We introduce the following notation: \( U(x, y) = Gu(x, y), V(x, y) = Gv(x, y) \), where \( u(x, y) \) and \( v(x, y) \) are the displacements in the plate in the direction of the \( x \) and \( y \) axes, respectively; \( G \) is the shear modulus; \( \nu \) is Poisson’s ratio.

First, we construct a particular solution corresponding to the temperature field \( t(x, y) \) in an infinite plane. The solution to this problem can be obtained by using the method of initial functions [8, 9]. In this case, the temperature factors of deformation have the form

\[
\begin{align*}
U(x, y) &= x^2 y, \\
V(x, y) &= x^2 y, \\
\sigma_x(x, y) &= -2x^2, \\
\sigma_y(x, y) &= -2y^2, \\
r_{yy}(x, y) &= 4xy.
\end{align*}
\]

We cut the square \( S \) out of the plane and add to the solution (1) the solution for the square \( S \) to all the sides of which normal stresses equal to the value of \( p = -\frac{2(1+\nu)}{1-\nu} \) are applied. This solution is as follows:

\[
\begin{align*}
U(x, y) &= -x, \\
V(x, y) &= -y, \\
\sigma_x(x, y) &= -\frac{2(1+\nu)}{1-\nu}, \\
\sigma_y(x, y) &= -\frac{2(1+\nu)}{1-\nu}, \\
r_{yy}(x, y) &= 0.
\end{align*}
\]

Adding (2) to (1) is not necessary, but it makes the problem easier. As a result, we obtain the following solution:

\[
\begin{align*}
U(x, y) &= x(y^2 - 1), \\
V(x, y) &= y(x^2 - 1), \\
\sigma_x(x, y) &= -2x^2 - \frac{2(1+\nu)}{1-\nu}, \\
\sigma_y(x, y) &= -2y^2 - \frac{2(1+\nu)}{1-\nu}, \\
r_{yy}(x, y) &= 4xy.
\end{align*}
\]

Wherein on the sides \( x = \pm 1 \) and \( y = \pm 1 \) we obtain

\[
\begin{align*}
U(\pm 1, y) &= \pm(y^2 - 1), \\
V(\pm 1, y) &= 0; \\
U(x, \pm 1) &= 0, \\
V(x, \pm 1) &= \pm(x^2 - 1).
\end{align*}
\]

To obtain zero displacements on the sides of the square, we need to add to the solution (3) the solution for the square \( S \) on the sides of which the longitudinal and transverse displacements (4) taken with the opposite sign are given.

2.1. Solution for the square that relieves the displacements on the transverse sides

Let us construct the solution for the square \( S \) clamped at \( y = \pm 1 \) on the sides \( x = \pm 1 \) of which the longitudinal displacements \( u(y) = -U(x, \pm 1, y) \) are given and the transverse displacements are equal to zero. In general, the formulas for the displacements and stresses in the square in the case of its even-symmetric deformation with respect to the central axes can be represented as the series

\[
\begin{align*}
U(x, y) &= \sum_{k=1}^{\infty} a_k \xi(\lambda_k, y) \sinh \lambda_k x + a_k \xi(\lambda_k, y) \sinh \lambda_k x, \\
V(x, y) &= \sum_{k=1}^{\infty} a_k \chi(\lambda_k, y) \cosh \lambda_k x + a_k \chi(\lambda_k, y) \cosh \lambda_k x, \\
\sigma_x(x, y) &= \sum_{k=1}^{\infty} a_k s_x(\lambda_k, y) \cosh \lambda_k x + a_k s_x(\lambda_k, y) \cosh \lambda_k x, \\
\sigma_y(x, y) &= \sum_{k=1}^{\infty} a_k s_y(\lambda_k, y) \cosh \lambda_k x + a_k s_y(\lambda_k, y) \cosh \lambda_k x.
\end{align*}
\]
\begin{equation}
\sigma_{ij}(x,y) = \sum_{k=1}^{\infty} a_k s_{ij}(\lambda_k, y) \cosh \lambda_k x + \overline{a_k s_{ij}(\lambda_k, y)} \cosh \lambda_k x,
\end{equation}
\begin{equation}
\tau_{ij}(x,y) = \sum_{k=1}^{\infty} a_k \tau_{ij}(\lambda_k, y) \sinh \lambda_k x + \overline{a_k \tau_{ij}(\lambda_k, y)} \sinh \lambda_k x,
\end{equation}
in the Papkovich–Fadle eigenfunctions
\begin{equation}
\xi(\lambda, y) = -\frac{1}{4} \left(1 + \nu\right) \left(\sin \lambda y - \nu \cos \lambda x \sin \lambda y\right),
\end{equation}
\begin{equation}
\chi(\lambda, y) = \left(\frac{\nu + 1}{4} \sin \lambda + \frac{\nu - 3}{4\lambda} \cos \lambda\right) \sin \lambda y + \frac{\nu + 1}{4} y \cos \lambda x \cos \lambda y,
\end{equation}
\begin{equation}
s_x(\lambda_x, y) = \left(-\nu \cos \lambda_x \frac{1 + \nu}{2} \lambda_x \sin \lambda_x\right) \cos \lambda_x y + \frac{\nu + 1}{2} \lambda_x \cos \lambda_x \sin \lambda_x y,
\end{equation}
\begin{equation}
s_y(\lambda_y, y) = \left(\frac{1 + \nu}{2} \lambda_y \sin \lambda_y - \cos \lambda_y\right) \cos \lambda_y y - \frac{\nu + 1}{2} \lambda_y \cos \lambda_y \sin \lambda_y y,
\end{equation}
\begin{equation}
t_y(\lambda_y, y) = \left(-\frac{\nu - 1}{2} \cos \lambda_y + \frac{\nu + 1}{2} \lambda_y \sin \lambda_y\right) \sin \lambda_y y + \frac{\nu + 1}{2} \lambda_y \cos \lambda_y \cos \lambda_y y,
\end{equation}
where \(\lambda \) are the complex roots of the transcendental equation
\begin{equation}
L(\lambda) = \left(\frac{3 - \nu}{4} \sin \frac{2\lambda}{2\lambda} - \frac{1 + \nu}{4}\right) = 0 \quad [10].
\end{equation}
Since \(\xi(\lambda_x, \pm 1) = \chi(\lambda_y, \pm 1) = 0\), the displacements (5) at \(y = \pm 1\) will be zero.

We substitute (5) into the boundary conditions at the ends of the square. Then, for example, for \(x = -1\) we obtain the system of equations
\begin{equation}
\sum_{k=1}^{\infty} a_k \xi(\lambda_k, y) \sinh \lambda_k y + \overline{a_k \xi(\lambda_k, y)} \sinh \lambda_k y = -u(y); \quad \sum_{k=1}^{\infty} a_k \chi(\lambda_k, y) \cosh \lambda_k y + \overline{a_k \chi(\lambda_k, y)} \cosh \lambda_k y = 0.
\end{equation}
The unknown coefficients \(a_k\) are determined from here by using the functions biorthogonal to the Papkovich–Fadle eigenfunctions \(\xi(\lambda, y), \chi(\lambda, y)\) \[11, 12\]. As a result, from (7) we obtain the system of two algebraic equations
\begin{equation}
\begin{cases}
\sum_{k=1}^{\infty} \lambda_k M_k \sinh \lambda_k + \overline{\lambda_k M_k} \sinh \lambda_k = -(u_k + \overline{u_k}); \\
\sum_{k=1}^{\infty} a_k M_k \cosh \lambda_k + \overline{a_k M_k} \cosh \lambda_k = 0,
\end{cases}
\end{equation}
Here the numbers \(u_k = \frac{2(\nu - 1)}{(\nu + 1)} \lambda_k^2\) are the Lagrange coefficients of the expanded function \(u(y)\), which are determined from the biorthogonality relation \(\int_{-\infty}^{\infty} \xi(\lambda, y) U_k(\lambda) dy = \frac{\lambda L(\lambda)}{\lambda^2 - \lambda_k^2}\) as \(\lambda \to 0\), and \(M_k = L'(\lambda_k) / (2\lambda_k)\) are normalizing factors.

Solving (8), we find
\begin{equation}
a_k = \frac{(u_k + \overline{u_k}) \cosh \lambda_k}{M_k(\lambda_k \sinh \lambda_k \cosh \lambda_k - \lambda_k \sin \lambda_k \cosh \lambda_k)}. \quad (9)
\end{equation}
We substitute formulas (9) into (5) and separate the zero-series in the obtained expressions, following \[12, 13\], for example. As a result, we obtain formulas that describe the solution to the problem for the square clamped at the longitudinal sides in which the displacements (4) taken with the opposite sign are given at the ends \(x = \pm 1\):
In formulas (10), the terms corresponding to the real root $\lambda_1$ should be taken outside the summation signs. They are derived from the general representations for the complex roots by passing to the limit when the imaginary part of the root tends to zero.

2.2. Solution for the square that relieves the displacements on the longitudinal sides

Let us construct the solution for the square $S$ clamped at $x = \pm 1$ on the sides $y = \pm 1$ of which the transverse displacements $v(x) = -V'(x, \pm 1) = \mp(x^2 - 1)$ are given and the longitudinal displacements are equal to zero. For this, we “rotate” the solution (10) through $90^\circ$:

\[
U^1(x, y) = V^1(y, x), \quad V^1(x, y) = U^1(y, x),
\]

\[
\sigma_{xx}^1(x, y) = \sigma_y^1(y, x), \quad \sigma_{yy}^1(x, y) = \sigma_x^1(y, x), \quad \tau_{xy}^1(x, y) = \tau_{yx}^1(y, x).
\]

The complete solution to the considered thermoelastic problem for the clamped square is equal to the sum of the solutions (10), (11) and (3). Figures 1–3 show the graphs illustrating the solution. It is assumed that $\nu = \frac{1}{3}$.

![Figure 1. Distribution of the normal stresses for $x = 1$.](image-url)
3. Conclusion
The paper proposes a method for determining thermal stresses in thin elastic rectangular plates, demonstrated by the example of a problem for a completely clamped rectangle. It can be shown that the bursts of the normal stresses at the angular points have a logarithmic character. By using the same scheme, similar solutions can be obtained for a rectangle with other boundary conditions on its sides and with different temperature fields. The obtained solution is exact because the expansion coefficients of the series in the Papkovich–Fadle eigenfunctions, in the form of which the solution is represented, are determined by simple closed formulas.
**Acknowledgments**

This work was supported by the Russian Science Foundation, Grant No. 19-71-00094.

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