FINITELY SUSLINIAN MODELS FOR PLANAR COMPACTA
WITH APPLICATIONS TO JULIA SETS

ALEXANDER BLOKH, CLINTON CURRY, AND LEX OVERSTEEGEN

Abstract. A compactum $X \subset \mathbb{C}$ is unshielded if it coincides with the boundary of the unbounded component of $\mathbb{C}\setminus X$. Call a compactum $X$ finitely Suslinian if every collection of pairwise disjoint subcontinua of $X$ whose diameters are bounded away from zero is finite. We show that any unshielded planar compactum $X$ admits a topologically unique monotone map $m_X : X \to X_{FS}$ onto a finitely Suslinian quotient such that any monotone map of $X$ onto a finitely Suslinian quotient factors through $m_X$. We call the pair $(X_{FS}, m_X)$ (or, more loosely, $X_{FS}$) the finest finitely Suslinian model of $X$.

If $f : \mathbb{C} \to \mathbb{C}$ is a branched covering map and $X \subset \mathbb{C}$ is a fully invariant compactum, then the appropriate extension $M_X$ of $m_X$ monotonically semi-conjugates $f$ to a branched covering map $g : \mathbb{C} \to \mathbb{C}$ which serves as a model for $f$. If $f$ is a polynomial and $J_f$ is its Julia set, we show that $m_X$ (or $M_X$) can be defined on each component $Z$ of $J_f$ individually as the finest monotone map of $Z$ onto a locally connected continuum.

1. Introduction

For us, a compactum is a non-empty compact metric space. A compactum is degenerate if all of its components are points. A continuum is a connected compactum. One way of describing the topology of a compactum $X$ is by constructing a model for it, i.e. a compactum $Y$, simpler to describe than $X$, and a (monotone) onto map $m : X \to Y$ (a continuous onto map $m$ is monotone if all $m$-preimages of continua are continua; we denote the family of all monotone maps by $\mathcal{M}$). If $X$ carries an additional structure, it is nice if the map $m$ preserves that structure (e.g., if there is a continuous map $f : X \to X$, the map $m$ should be chosen so that $f$ induces a continuous self-map on $Y$ by $m(x) \mapsto m(f(x))$).
to be a **monotone semiconjugacy** of the map $f : X \to X$ to the induced map $g : Y \to Y$ (if $m$ is a homeomorphism, it is called a **conjugacy**).

Unless specified otherwise, **from now on all compacta we consider are planar**. A case of particular interest is when $X$ is **unshielded**, i.e. $X \subset \mathbb{C}$ is the boundary of the unbounded component $U_{\infty}(X)$ of $\mathbb{C} \setminus X$. The following construction is due to Carathéodory. Recall that a space is **locally connected** if its topology has a basis of connected sets. If $X$ is an unshielded continuum, then $U_{\infty}(X) \cup \{\infty\}$ is a simply connected open set in the Riemann sphere. Take the unique Riemann map $\varphi_X : \mathbb{D} \to U_{\infty}(X)$ with positive derivative at the origin (here $\mathbb{D}$ is the unit open disk centered at the origin). If $X$ is locally connected, we may extend $\varphi_X$ continuously to $\overline{\mathbb{D}}$, mapping $S^1$ onto $X$ (here $S^1$ is the boundary of $\mathbb{D}$). Declaring points $u, v \in S^1$ equivalent if and only if $\varphi_X(u) = \varphi_X(v)$ and denoting this equivalence relation by $\approx$, we see that $X$ is homeomorphic to the quotient space $S^1/\approx$. Equivalence relations $\approx$ which arise in this way are called **laminations**. If $J_P$ is the locally connected Julia set of a polynomial $P$, then $\varphi_X$ semiconjugates $z^d|_{S^1}$ to a map $\tilde{P} : S^1/\approx \to S^1/\approx$.

A lamination can be defined in abstract circumstances as a closed equivalence relation $\approx$ on $S^1$ such that convex hulls of $\approx$-classes are pairwise disjoint (here the convex hull of a set $T$ is the smallest convex set containing $T$). Laminations therefore capture the external ray picture of unshielded continua. In order to model dynamical objects like the Julia set of a degree $d$ polynomial, we may require that $\approx$ is $d$-**invariant**. This means that the image of a $\approx$-class under the angle $d$-tupling map is again a $\approx$-class, and classes map to each other in a consecutive-preserving way (loosely speaking, preserving the order of points on the circle).

There are even laminations for disconnected Julia sets; here $\approx$ is a closed equivalence relation defined on a Cantor subset $A \subset S^1$, and the angle $d$-tupling map is replaced by a covering self-map of $A$. This models that, for a polynomial $P$ with disconnected Julia set $J_P$, the neighborhood of $\infty$ on which $P$ is conjugate to $z \mapsto z^d$ does not include the entire basin of infinity. In this case every external ray can be analytically continued until it runs into the Julia set unless it first runs into the preimage of an escaping critical point. In such a case, one can take left- and right-sided limits of fully-defined external rays and define two external rays corresponding to the same angle. These angles are associated to (pre)critical points and to the gaps in the Cantor set $A$ (see [CM93, Kiw04, LP96]).

By Kiwi [Kiw04] laminations correspond to a wider class of polynomials $P$, whose Julia sets may not be locally connected nor connected. More precisely, an $n$-periodic point $a$ of $P$ is called **irrationally neutral** if $(P^n)'(a) = e^{2\pi i \alpha}$ with $\alpha$ irrational. Also, given a lamination $\approx$ of $S^1$, call a set $F \subset S^1$ **saturated** if it is a union of a collection of $\approx$-classes. By [Kiw04], to every polynomial $P$ without
irrationally neutral cycles we can associate a lamination \( \approx \), a closed \( \approx \)-saturated set \( F \subset S^1 \) and a monotone map \( m : J_P \to F/ \approx \) such that \( m \) is a semiconjugacy of \( P|_{J_P} \) with an appropriately constructed map \( f : F/ \approx \to F/ \approx \) (in the case that \( J_P \) is connected, then \( F = S^1 \) and \( f \) is a map induced on \( S^1/\approx \) by \( z^d \)).

The present authors prove [BCO08] that every complex polynomial \( P \) with connected Julia set has a unique “best” lamination. This generalizes [Kiw04], albeit for connected Julia sets, by allowing \( P \) to have irrationally neutral cycles. The lamination \( \approx \) comes with a monotone semiconjugacy \( m : J_P \to S^1/\approx \) which has the property of being the finest monotone map of \( J_P \) onto a locally connected continuum (defined in the next section). In [BCO08] we also provide a criterion for \( \approx \) to have more than one equivalence class (equivalently, for \( J_P \) to have a non-degenerate locally connected monotone image).

A compactum \( X \) is called finitely Suslinian if, for every \( \varepsilon > 0 \), every collection of disjoint subcontinua of \( X \) with diameters at least \( \varepsilon \) is finite. By Lemma 2.9 [BO04], unshielded planar locally connected continua are finitely Suslinian and vice versa. Thus, in the unshielded case the notion of finitely Suslinian generalizes the notion of local connectivity. There is another analogy to local connectivity too: by Theorem 1.4 [BMO07], for an unshielded finitely Suslinian compactum \( X \subset \mathbb{C} \) there exists a lamination \( \approx \) of a closed set \( F \subset S^1 \) such that \( X \) is homeomorphic to \( F/\approx \). This motivates us to extend onto finitely Suslinian compacta some results for locally connected continua and to look for good finitely Suslinian models of planar compacta. We need the following definition which applies to arbitrary maps (as customary in topology, by a map we always mean a continuous map).

**Definition 1** (Finest models). Let \( X \subset \mathbb{C} \) be a compactum, \( P \) be a topological property (\( P \) could be the property of being locally connected, Hausdorff, etc) and \( \mathcal{B} \) be a class of maps with domain \( X \). The finest \( \mathcal{B} \)-model of \( X \) with property \( P \) is an onto map \( \psi^P : X \to Y, \psi^P \in \mathcal{B} \) where \( Y \) is a topological space with property \( P \) such that any other map \( \varphi^P : X \to Z, \varphi^P \in \mathcal{B} \) onto a space \( Z \) with property \( P \) can be written as the composition \( g \circ \psi^P \) for some map \( g : Y \to Z \).

Though we give the definition for any class \( \mathcal{B} \), we are mostly interested in the class \( \mathcal{M} \) of monotone maps because such maps do not change the structure of \( X \) too drastically; besides, we study planar compacta, and monotone maps of planar compacta with non-separating fibers keep them planar [Moo62]. In the monotone case we will use notation \( m^P \) instead of \( \psi^P \) (or just \( m \) if the property \( P \) is fixed). In fact, in the monotone case this concept of finest map has been studied before in the context of continua (cf [PS07]).

\(^1\) If \( X \) is not unshielded, this may fail as the closed unit disk \( \mathbb{D} \) is locally connected but not finitely Suslinian; Example [14] shows that there are nowhere dense locally connected planar continua which are not finitely Suslinian.
Lemma 2. If the finest $B$-model with property $P$ exists, then it is unique up to a homeomorphism.

Proof. Suppose that $m_1 : X \to Y_1$ and $m_2 : X \to Y_2$ are finest $B$-models. Then, by definition, we may factor $m_1$ as

$$m_1 : X \xrightarrow{m_2} Y_2 \xrightarrow{g_1} Y_1$$

and similarly factor the constituent map $m_2$ to obtain

$$m_1 : X \xrightarrow{m_2} Y_1 \xrightarrow{g_2} Y_2 \xrightarrow{g_1} Y_1.$$  

However, since the composition is itself equal to $m_1$, we find that $g_1 : Y_2 \to Y_1$ and $g_2 : Y_1 \to Y_2$ are each other’s inverse and hence homeomorphisms. Therefore, $Y_2 = g_2(Y_1)$ is homeomorphic to $Y_1$, $m_2 = g_2 \circ m_1$, and $m_1 = g_1 \circ m_2$. $\square$

The following notion is a bit weaker than that defined in Definition 1.

**Definition 3** (Top models). Let $h : X \to Y$ be a map in $B$ onto a compactum $Y$ with property $P$ such that there exists no map $h' : X \to Y'$ onto a compactum $Y'$ with property $P$ which refines $h$ (i.e., if a map $h''$ is such that $h = h'' \circ h'$, then $h''$ must be a homeomorphism and $Y'$ is homeomorphic to $Y$). Then $(Y,h)$ is said to be a top $B$-model of $X$ with property $P$.

Observe that, while the finest model is finer than all others, a top model does not have another strictly finer model. This is the same as the greatest model and a maximal model in the sense of some partial order. Hence, if the finest $B$-model of $X$ with property $P$ exists, it is the unique top $B$-model of $X$ with property $P$. So, if we have a top $B$-model $h : X \to Y$ of $X$ with property $P$ and a $B$-model $h' : X \to Y'$ of $X$ with property $P$ such that $h$ is not finer than $h'$, then the finest $B$-model of $X$ with property $P$ does not exist. Example 14 provides a planar continuum $X_1$ with this situation in the case when $P$ is the property of being finitely Suslinian and $B$ is either the class of continuous maps or monotone maps; thus, $X_1$ has no finest continuous or monotone model with finitely Suslinian property.

Also, by the definitions if $B \subset B'$ are two classes of maps and the finest (top) $B'$-model of $X$ with property $P$ is $(Y,h)$ where $h$ happens to belong to the smaller class $B$, then $(Y,h)$ is also the finest (top) $B$-model of $X$ with property $P$.

The purpose of the paper is to prove Theorems 4, 5 and 6. As Theorem 4 proves, the situation with unshielded planar continua and the finitely Suslinian property is better. From now on finest $M$-models with finitely Suslinian property will be called finest finitely Suslinian monotone models.

**Theorem 4.** Every unshielded compactum $X$ has a finest finitely Suslinian monotone model $m_X : X \to X_{FS}$. 
This yields applications to the dynamics of branched covering maps of the plane, and in particular the study of Julia sets of polynomials, which are naturally occurring examples of unshielded compacta.

**Theorem 5.** Suppose that \( f : \mathbb{C} \to \mathbb{C} \) is a branched covering map and \( X \) is an unshielded compactum which is fully invariant under \( f \). Then \( X_{FS} \) can be embedded into the plane and the finest finitely Suslinian monotone model \( m_X : X \to X_{FS} \) can be extended to the plane in such a way that the resulting map \( M_X : \mathbb{C} \to \mathbb{C} \) semiconjugates \( f \) and a branched covering map \( g : \mathbb{C} \to \mathbb{C} \).

These results can be made stronger if \( f \) is a polynomial.

**Theorem 6.** The finest finitely Suslinian monotone model \( m_{J_P} : J_P \to J_{FS} \) of the Julia set of a polynomial \( P \) coincides on each component \( X \) of \( J_P \) with the finest monotone map \( m_X \) of \( X \) to a finitely Suslinian continuum. In particular:

1. the finest finitely Suslinian monotone model of \( J_P \) is non-degenerate if and only if there exists a periodic component of \( J_P \) whose finest finitely Suslinian monotone model is non-degenerate;
2. the set \( J_P \) is finitely Suslinian if and only if all periodic non-degenerate components of \( J_P \) are locally connected.

By [BCO08], one can specify exactly the situations in which a non-degenerate finitely Suslinian model of a polynomial Julia set exists. This is because any periodic component of \( J_P \) is the Julia set of a polynomial-like map, which is hybrid equivalent (in particular, topologically conjugate) to a polynomial. Hence, summarizing the results of [BCO08], we conclude that a periodic component \( Y \) of \( J_P \) has a non-degenerate finitely Suslinian model if and only if one of the following is true:

1. \( Y \) contains infinitely many periodic points, each of which separates \( Y \),
2. the topological hull of \( Y \) contains either a parabolic or attracting periodic point, or
3. \( Y \) admits a *Siegel configuration*, which roughly means that are subcontinua of the Julia set, comprised of finitely many impressions and disjoint from all other impressions, which in essence correspond to the critical points on the boundaries of Siegel disks in locally connected Julia sets.

For all details, the reader is invited to read [BCO08], especially Section 5 thereof.

2. **Topological Lemmas**

First we introduce several useful notions. When speaking of limits of compacta, we always mean convergence in the Hausdorff sense.
Definition 7. A partition of a compactum $X$ if said to be upper semi-continuous if for every pair of convergent sequences $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$ of points in $X$ such that $x_i, y_i$ belong to some element $D_i$ of the partition, we have that the points $\lim_{i \to \infty} x_i$ and $\lim_{i \to \infty} y_i$ belong to some element $D$ of the partition. In this case the equivalence relation $\sim$ induced by the partition is said to be closed. Equivalently, $\sim$ is said to be closed if its graph is closed in $X \times X$.

The following construction is less standard.

Definition 8. Let $\mathcal{A}$ be a family of subsets of a compactum $X$. An equivalence relation $\sim$ respects $\mathcal{A}$ if $\sim$ is closed and every member of $\mathcal{A}$ is contained in a $\sim$-class. If $\sim$ and $\approx$ are equivalence relations on a set $X$, we say that $\sim$ is finer than $\approx$ if $\sim$-classes are contained in $\approx$-classes. The finest closed equivalence relation generated by $\mathcal{A}$ is the finest equivalence relation $\sim_\mathcal{A}$ respecting $\mathcal{A}$.

Equivalently, one can define continuous maps respecting $\mathcal{A}$ as maps which collapse all elements of $\mathcal{A}$ to points. Then we can define the finest continuous map respecting $\mathcal{A}$, i.e. a continuous map $\psi^\mathcal{A} : X \to Z$ respecting $\mathcal{A}$ and such that for any map $f : X \to Q$ which respects $\mathcal{A}$ there exists a map $g : Z \to Q$ which can be composed with $\psi^\mathcal{A}$ to give $f = g \circ \psi^\mathcal{A}$.

Lemma 9 shows that the finest closed equivalence relation generated by $\mathcal{A}$ (and hence, the finest map respecting $\mathcal{A}$) exists and specifies its properties if elements of $\mathcal{A}$ are connected.

Lemma 9. The finest closed equivalence relation generated by $\mathcal{A}$ exists and is therefore unique (thus, the finest map $\psi^\mathcal{A}$ respecting $\mathcal{A}$ exists and is well-defined). If $\mathcal{A}$ consists of connected subsets of a compactum $X$, then all $\sim_\mathcal{A}$-classes are continua and the finest continuous map respecting $\mathcal{A}$ is monotone.

Proof. To see that $\sim_\mathcal{A}$ is well-defined, let $\Xi_\mathcal{A}$ be the set of all upper semi-continuous equivalence relations which respect $\mathcal{A}$ ($\Xi_\mathcal{A}$ is non-empty as it includes the trivial equivalence relation under which all points are equivalent). Then it is easy to see that the relation $\sim_\mathcal{A}$ defined by "$x \sim_\mathcal{A} y$ if and only if $x \sim y$ for all $\sim \in \Xi_\mathcal{A}$" is again a closed equivalence relation respecting $\mathcal{A}$, and that $\sim_\mathcal{A}$ is finer than all closed equivalence relations from $\Xi_\mathcal{A}$. It follows that the quotient map $X \to X/\sim_\mathcal{A}$ is in fact the finest continuous map which respects $\mathcal{A}$.

It suffices to show that all $\sim_\mathcal{A}$ classes are connected. According to [Nad92, Lemma 13.2], the equivalence relation $\sim$ whose classes are the components of $\sim_\mathcal{A}$-classes is also an upper semi-continuous equivalence relation, and $\sim$-classes are contained in $\sim_\mathcal{A}$-classes. Since elements of $\mathcal{A}$ are connected, it follows that $\sim$ still respects $\mathcal{A}$, so $\sim_\mathcal{A}$ classes are contained in $\sim$-classes. Therefore $\sim = \sim_\mathcal{A}$, and $\sim_\mathcal{A}$-classes are connected. \hfill \square
It is quite easy to determine when a continuous function on $X$ induces a continuous function on $X/\sim_A$, as the following lemma shows.

**Lemma 10.** If $f : X \to X$ is a continuous function which sends elements of $A$ into $\sim_A$-classes, then $f$ induces a function $g : X/\sim_A \to X/\sim_A$ with $\psi^A \circ f = g \circ \psi^A$ (g maps the $\sim_A$-class of $x$ to the $\sim_A$-class of $f(x)$).

**Proof.** It is sufficient to show that the $f$-image of a $\sim_A$-class is contained in a $\sim_A$-class. Consider the fibers of $\psi^A \circ f$. By assumption, $f$ sends elements of $A$ into $\sim_A$-classes, so $\psi^A \circ f$ is constant on the elements of $A$. Therefore, the fibers of $\psi^A \circ f$ form an upper semi-continuous partition of $X$ which respects $A$. Since $\psi^A$ is the finest such map in the sense of Definition 8, there exists a map $g : X/\sim_A \to X/\sim_A$ with $\psi^A \circ f = g \circ \psi^A$ as desired. \hfill $\square$

**Remark 11.** For later reference, we note that there is also a transfinite construction of the equivalence relation $\sim_A$. To begin, let $\sim_0$ denote the equivalence relation such that $x \sim_0 y$ if and only if $x$ and $y$ are contained in a connected finite union of elements of $A$. If an ordinal $\alpha$ has an immediate predecessor $\beta$ for which $\sim_\beta$ is defined, we define $x \sim_\alpha y$ if there exist finitely many sequences of $\sim_\beta$ classes whose limits comprise a continuum containing $x$ and $y$ (here, the limit of non-closed sets is considered to be the same as the limit of their closures). In the case that $\alpha$ is a limit ordinal, we say $x \sim_\alpha y$ whenever there exists $\beta < \alpha$ such that $x \sim_\beta y$. Notice that the sequence of $\sim_\alpha$-classes of a point $x$ (as $\alpha$ increases) is an increasing nest of connected sets, with the closure of each being a subcontinuum of its successor. It is also apparent that $\sim_\alpha$-classes are contained in $\sim_A$-classes for all ordinals $\alpha$.

Let us now show that $\sim_A = \sim_\Omega$ where $\Omega$ is the smallest uncountable ordinal. To see this, we first note that $\sim_\Omega = \sim_{(\Omega+1)}$. This is because the sequence of closures of $\sim_\alpha$-classes containing a point $x$ forms an increasing nest of subsets, no uncountable subchain of which can be strictly increasing in the plane [Kur66, Theorem 3, p. 258]. Therefore, all $\sim_\alpha$-classes have stabilized when $\alpha = \Omega$. This implies that $\sim_\Omega$ is a closed equivalence relation, since the limit of $\sim_\Omega$-classes is a $\sim_{(\Omega+1)}$-class, which we have shown is a $\sim_\Omega$-class again. Finally, $\sim_\Omega$ respects $A$ and $\sim_\Omega$-classes are contained in $\sim_A$-classes, so $\sim_A$ and $\sim_\Omega$ coincide.

Let us become more specific and study finitely Suslinian compacta.

**Definition 12 (Limit continuum and $FS_\sim$).** A subcontinuum $C$ of $X$ is said to be a **limit continuum** if there exists a sequence $(C_n)_{n=1}^\infty$ of pairwise disjoint subcontinua of $X$ converging to $C$. We define $FS_\sim_X$ as the **finest equivalence relation respecting the family of limit continua** (if the context is clear, we may omit the subscript and refer simply to $FS_\sim$).
Figure 1. A continuum with no finest finitely Suslinian model.

Note that this notion is slightly more general than the classical notion of continuum of convergence in continuum theory. Also, it is easy to see that a continuum is finitely Suslinian if and only if it contains no non-degenerate limit continua.

Lemma 13. For any compactum $X$, the quotient $X/\sim_{FS}$ is finitely Suslinian.

Proof. Let $(C_n)_{n=1}^\infty$ be a (without loss of generality convergent) sequence of pairwise disjoint subcontinua of $X/\sim_{FS}$. Let $m : X \rightarrow X/\sim_{FS}$ denote the quotient map. A subsequence of the preimages $(m^{-1}(C_n))_{n=1}^\infty$ converges to a continuum $K$. By definition of $\sim_{FS}$ we have that $m(K)$ is a singleton, say, $\{a\}$, and continuity of $m$ implies that $(C_n)_{n=1}^\infty$ converges to $\{a\}$. Since $(C_n)_{n=1}^\infty$ was arbitrary, we have that $X/\sim_{FS}$ contains no non-degenerate limit continua and is therefore finitely Suslinian.

Lemma 13 together with the characterization of finitely Suslinian compacta as those with no limit continua, suggests that $X/\sim_{FS}$ could be the finest model of $X$. Such a fact would mean that any monotone map of $X$ onto a finitely Suslinian compactum must collapse limit continua. However, in general this is not true.

Example 14 (A continuum with no finest finitely Suslinian model). Define a continuum $X$ as follows, and as depicted in Figure 1 on page 8:

$$H_n = [0, 1] \times \{1/2^n\}, \ n \in \mathbb{N},$$

$$H = [0, 1] \times \{0\},$$

$$V_{p/q} = \{p/q\} \times [0, 1/q], \ p/q \text{ a dyadic rational}$$

$$X_1 = \bigcup \{H_n | \ n \in \mathbb{N}\} \cup H \cup \bigcup \{V_{p/q} | 0 < p/q < 1 \text{ dyadic}\}$$

Observe that $X_1$ is a locally connected, not finitely Suslinian, nowhere dense and not unshielded in $\mathbb{C}$ continuum. There are two essentially different kinds of finitely
Suslinian monotone quotients of $X_1$, depicted in Figure 2. One map, $h$, corresponds to identifying the unique maximal limit continuum $H = \lim_{n \to \infty} H_n$ to a point. Any finer (and not even necessarily monotone) map $h'$ to a finitely Suslinian compactum would still keep images of $H_n$ disjoint, implying that images of $H_n$ must converge to a point which has to be the image of $H$. Thus, $h' = h$ and $(h(H), h)$ is a top finitely Suslinian model of $X_1$ which happens to be monotone.

Other quotients of $X_1$ with finitely Suslinian images are maps $\varphi_N$ which identify to points members of the collection $\{V_{p/q} \mid q > N\}$. This yields a sequence of maps $(\varphi_N)^N_{N=0}$, with $\varphi_{N+1}$ finer than $\varphi_N$ for all $N$. On the other hand, none of these maps can be compared with $\psi$ in the sense that neither $h$ is finer than $\varphi_N$ nor $\varphi_N$ is finer than $h$. As explained above, it follows from the definitions now that $h$ is not the finest finitely Suslinian model of $X_1$ (neither is it the finest finitely Suslinian monotone model of $X_1$). It is worth noticing also that for any $N$ the only maps finer than both $\varphi_N$ and $h$ are homeomorphisms (since the intersection of any fibers of $h$ and $\varphi_N$ is at most a point) and that the only maps finer than every map in $\{\varphi_N \mid N \in \mathbb{N}\}$ are homeomorphisms.

In the unshielded case the situation is better. First we need Definition 15.

**Definition 15** (Irreducible continua). Given two disjoint closed sets $A, B$, a continuum $C$ is said to be irreducible between $A$ and $B$ if $C$ intersects both $A$ and $B$ and does not contain a subcontinuum with the same property. Given a continuum $D$ intersecting $A$ and $B$, one can use Zorn’s Lemma to find a subcontinuum $C \subset D$ irreducible between $A$ and $B$.

We also need Lemma 16.

**Lemma 16.** Let $K$ be an irreducible continuum between $\partial U$ and $\partial V$ where $U, V$ are open sets with disjoint closures. Then $K$ is disjoint from both $U$ and $V$. 
Proof. Set $K' = K \setminus \overline{X}$. Take a component $Y$ of $K'$ containing a point from $\partial U$. By the Boundary Bumping Theorem (Theorem 5.6 from [Nad92, Chapter V, p. 74]) $Y$ intersects $\partial V$. Since $K$ is irreducible, $Y = K$ and hence $K$ is disjoint from $V$. Similarly, $K$ is disjoint from $U$. \hfill \square

To prove our first theorem we need the following geometric lemma. It is a generalization of the fact that any homeomorphic copy of the letter $\theta$ embedded in the plane is not unshielded. For a planar continuum $Y$ the set $\mathbb{C} \setminus U_\infty(Y)$ is called the \textbf{topological hull of $Y$} and is denoted by $T(Y)$.

**Lemma 17.** Suppose a planar compactum $X$ contains two disjoint continua $X_1$, $X_2 \subset X$ and three pairwise disjoint continua $C^1, C^2, C^3$ such that $X_i \cap C^j \neq \emptyset$ for all $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$. Then $X$ is not unshielded.

**Proof.** By way of contradiction we assume that $X$ is unshielded. Let us collapse the topological hulls $T(X_1)$ and $T(X_2)$ to points $x_1$ and $x_2$ and let $m : \mathbb{C} \rightarrow \mathbb{C}$ denote this monotone map (by Moore’s Theorem [Moo62], the image is homeomorphic to the plane). Then $m(C^i) \cap m(C^j) = \{x_1, x_2\}$ for all $i \neq j$ and $m(X)$ is also unshielded. Put $Z^j = T(m(C^j))$. Then $Z^i \cap Z^j = \{x_1, x_2\}$ for all $i \neq j$. By Theorem 63.5 of [Mun00] for each $i \neq j$, $Z^i \cup Z^j$ separates $\mathbb{C}$ into precisely two components, one of which is bounded and denoted by $B_{i,j}$. It follows that for some choice of $i, j, k$ the set $Z_i$ intersects $B_{j,k}$, contradicting that $m(X)$ is unshielded. \hfill \square

We use Lemma 17 to show that certain maps of unshielded compact sets collapse limit continua. Let $\mathcal{FM}$ be the class of all \textbf{(continuumwise) finitely monotone} maps, i.e. such maps $h : X \rightarrow Y$ that for any continuum $Z \subset Y$ the set $h^{-1}(Z)$ consists of finitely many components.

**Lemma 18.** Suppose that $\varphi : X \rightarrow Y$ is a finitely monotone map of an unshielded compact set $X$ onto a finitely Suslinian compact set $Y$. If $C \subset X$ is a limit continuum, then $\varphi(C)$ is a point.

**Proof.** Let $C \subset X$ be a limit continuum. Choose a sequence of continua $C_i \rightarrow C$. Consider two cases.

\textbf{Case 1.} There are infinitely many distinct components of $Y$ containing sets $\varphi(C_i)$.

Denote by $T_i$ the component of $Y$ which contains $\varphi(C_i)$. We may refine the sequence $(C_i)^\infty_{i=1}$ so that all sets $T_i$ are different. Since $Y$ is finitely Suslinian, we may refine it further so that $T_i$ converge to a point $t \in Y$. Hence $\varphi(C) = \lim \varphi(C_i) = \lim T_i = t$.

\textbf{Case 2.} There are finitely many distinct components of $Y$ containing all sets $\varphi(C_i)$.

Since $\varphi$ is finitely monotone, we may assume that all $C_n$ are contained in a single component $T$ of $X$. Observe that then $\varphi(T) \subset Z$ where $Z$ is a component of $Y$.\hfill \blacksquare
By [BO04] Lemma 2.9, \( Z \) is locally connected. We suppose that \( \phi(C) \) is not a point and show that this contradicts the fact that \( X \) is unshielded. Let \( z_1 = \phi(x_1) \) and \( z_2 = \phi(x_2) \) be distinct points in \( \varphi(C) \). Since \( Z \) is locally connected, there exist open, connected subsets \( Z_1, Z_2 \subset Z \) with disjoint closures, containing \( z_i \in Z_i \) for \( i \in \{1, 2\} \). Then \( \varphi^{-1}(\overline{Z_1}) \) and \( \varphi^{-1}(\overline{Z_2}) \) have finitely many components. After refining the sequence \( C_i \) we may assume that all sets \( C_i \) intersect a component \( A_1 \) of \( \varphi^{-1}(\overline{Z_1}) \) and a component \( A_2 \) of \( \varphi^{-1}(\overline{Z_2}) \). However, by Lemma 17 this is impossible.

3. The Existence of the Finest Map and Dynamical Applications in the Unshielded Case

3.1. The existence of the finest map in the unshielded case. We are ready to prove our first theorem which implies Theorem 4.

**Theorem 19.** Let \( X \) be an unshielded compact set in the plane. Then the quotient map \( m_X : X \to X/_{\sim} = X_{FS} \) is the finest finitely Suslinian monotone model of \( X \). Moreover, \( X_{FS} \) can be embedded into the plane and \( m_X \) can be extended to a monotone map \( M_X : \mathbb{C} \to \mathbb{C} \) which collapses the topological hulls of \( \sim \)-classes and is one-to-one elsewhere.

**Proof.** By Lemma 13 \( X/_{\sim} \) is a finitely Suslinian compactum. Now, suppose that \( \varphi : X \to Z \) is monotone and \( Z \) is finitely Suslinian. Then \( \varphi \) collapses all limit continua by Lemma 18. Since \( \sim \) is the finest equivalence relation respecting the collection of limit continua, we see that the quotient map \( m_{FS} : X \to X/_{\sim} \) is finer that \( \varphi \), and is therefore the finest finitely Suslinian monotone model of \( X \). The rest of the theorem follows from the Moore theorem [Moo62]. □

Observe that in fact Lemma 18 implies that the finest \( FM \)-model of \( X \) with finitely Suslinian property is the same as the finest finitely Suslinian monotone model of \( X \) (despite the fact that the class \( FM \) of finitely monotone maps is much wider than the class \( M \) of monotone maps).

3.2. Applications to Dynamical Systems. First we show that sometimes the finest map is compatible with the dynamics. Recall that a set \( A \subset X \) is fully invariant under a map \( f : X \to X \) if \( A = f^{-1}(A) = f(A) \). Recall also that branched covering maps are open and hence confluent.

**Theorem 20.** Suppose that \( f : \mathbb{C} \to \mathbb{C} \) is a branched covering map and that \( X \) is a fully invariant unshielded compactum. Then there exists a branched covering map \( g : \mathbb{C} \to \mathbb{C} \) such that \( M_X \circ f = g \circ M_X \) and hence \( X_{FS} = M_X(X) \) is fully invariant under \( g \).
Proof. By Lemma 15, \( m_X \) sends limit continua into \( \sim_{FS} \)-classes. Lemma 14 and Theorem 19 in which the extension \( M_X \) of \( m_X \) onto \( \mathcal{C} \) is described, imply that \( g = M_X \circ f \circ M_X^{-1} \) is well-defined. Suppose that we show that \( \sim_{FS} \)-classes map onto image classes. Then, since \( f \) is open, it will follow that \( g \) is open too. Moreover, let us show that then \( g \) is light. Indeed, if \( x \in M_X(X) \) then \( M_X^{-1}(x) \) is an \( \sim_{FS} \)-class in \( X \). Since we assume that classes map onto classes and the map \( f \) is finite-to-one, we see that each component of \( f^{-1}(M_X^{-1}(x)) \) is an \( \sim_{FS} \)-class. Hence \( g \) is finite-to-one. Since by the Stoilow theorem [Sto56] all open finite-to-one maps of the plane are branched covering maps, \( g \) is a branched covering map as desired.

To see that the image of a \( \sim_{FS} \)-class is again a \( \sim_{FS} \)-class, we show that \( \sim_{\alpha} \)-classes map onto the union of \( \sim_{\alpha} \)-classes for every ordinal \( \alpha \), where \( \sim_{\alpha} \) was defined in Remark 11 with \( \mathcal{A} \) being the set of limit continua. Then, when \( \alpha = \Omega \), we see that \( \sim_{FS} \)-classes map both into and over other \( \sim_{FS} \)-classes.

Let us first show that \( \sim_{\alpha} \)-classes map over other \( \sim_{\alpha} \)-classes. Indeed, let \( f(x) \) and \( y \) belong to the same \( \sim_{\alpha} \)-class. Then there exist finitely many limit continua \( C_1 = \lim_{i \to \infty} C_i^1, \ldots, C_n = \lim_{i \to \infty} C_i^n \) forming a chain joining \( f(x) \) and \( y \) (i.e., so that \( f(x) \in C_1, y \in C_n, \) and \( C_i \cap C_{i+1} \neq \emptyset \) for any \( 1 \leq j < n \)). Since \( f \) is an open map, there exists a con vergent sequence \( \left(D_i^1\right)_{i=1}^{\infty} \to D_1 \) of continua such that \( f(D_i^1) = C_i^1 \) for each \( i \) and \( D_1 \) is a limit continuum which contains \( x \). By continuity, \( f(D_1) = C_1 \), so \( D_1 \) contains the preimage of a point in \( C_2 \). We can now inductively find limit continua \( D_2, \ldots, D_n \) mapping onto \( C_2, \ldots, C_n \) and forming a chain from \( x \) to a preimage of \( y \). Therefore, the \( \sim_{\alpha} \)-class of \( f(x) \) is contained in the image of the \( \sim_{\alpha} \)-class of \( x \).

Suppose now by induction that we have proven the claim for all ordinals less than \( \alpha \), and let \( f(x) \sim_{\alpha} y \). If \( \alpha \) has an immediate predecessor \( \beta \) (the other case is left as an easy exercise for the reader), there are finitely many sequences of \( \sim_{\beta} \)-classes \( (K_i^1)_{i=1}^{\infty}, \ldots, (K_i^n)_{i=1}^{\infty} \) which converge to a chain of continua joining \( f(x) \) and \( y \). By the inductive hypothesis, if \( f(z) \in K_i^1 \) then the \( \sim_{\beta} \)-class of \( z \) maps over \( K_i^1 \). One can therefore find, due to the openness of \( f \), a convergent sequence \( \left(L_i^1\right)_{i=1}^{\infty} \to L_1 \) of \( \sim_{\beta} \)-classes such that \( f(L_i^1) \supseteq K_i^1 \) and \( x \in L_1 \). Note by continuity that \( f(L_1) \supseteq K_1 \). Proceeding as in the previous paragraph, we find similar limits \( L_2, \ldots, L_n \) forming a chain of continua which joins \( x \) to a preimage of \( y \). We therefore see that the image of a \( \sim_{\alpha} \)-class is a union of \( \sim_{\alpha} \)-classes, and the proof is complete. \( \square \)

Sometimes in the situation of Theorem 20 a naive but natural approach to the problem of constructing the finest finitely Suslinian model can be used.

Definition 21. By Theorem 19 for each component \( Y \) of \( X \), the finest equivalence relation on \( Y \) is \( \sim_{FS} \sim_{Y} \). Consider the equivalence relation \( \sim_{\bigtriangleup, X} \) defined as follows: \( x \sim_{\bigtriangleup, X} y \) if and only if \( x \) and \( y \) belong to the same component \( Y \) of \( X \) and \( x \sim_{FS} \sim_{Y} y \).
If $X$ is given or non-essential, we will simply write $\sim_\triangle$ or $\text{FS}$. It is natural to find out if $\sim_\triangle X$ coincides with $\text{FS}_X$. Simple examples show that in general it is not true.

**Example 22** (A map on a compactum $X$ with $\sim_\triangle X$ not coinciding with $\text{FS}_X$).

Define a map $f : \mathbb{C} \to \mathbb{C}$ as follows. Take a map from the real quadratic family $g_b(x) = bx(1-x)$ with $b > 4$. It is well known that then there exists a forward invariant Cantor set $A \subset [0, 1] \setminus \{0\}$ on which the map $g_b$ acts as a full 2-shift. We define a map on the set $X = A \times [-1, 1]$ as $f(x, y) = (g_b(x), y)$. Evidently, $f$ can be extended to a branched covering two-to-one map $f : \mathbb{C} \to \mathbb{C}$, however for brevity we will not give its full description here.

Observe that $X$ is a fully invariant set. The equivalence relation $\text{FS}_X$ collapses $X$ to a Cantor set, though all $\sim_\triangle X$-classes are points. Thus, in this case $\sim_\triangle X \neq \text{FS}_X$.

Example 22 shows that in some cases $\sim_\triangle X$ and $\text{FS}_X$ are distinct. Moreover, it also shows the mechanism of how this distinction occurs. However, the definition immediately implies that $\sim_\triangle X$ is finer than $\text{FS}_X$.

It turns out that the abberation $\text{FS}_X \neq \sim_\triangle X$ is impossible for polynomial maps. To show that we need some definitions. A point $x \in X$ of a planar compactum $X$ is called accessible (from $U_\infty(X)$) if there is a curve $Q \subset U_\infty(X)$ with one endpoint at $x$ (then one says that $Q$ lands at $x$ and that $x$ is accessible by $Q$).

We also need the definition of impression of an angle. For a continuum $X \subset \mathbb{C}$, let $\psi : \mathbb{C} \setminus \mathbb{D} \to U_\infty(K)$ denote the unique conformal isomorphism with real derivative at $\infty$. For an angle $\alpha \in \mathbb{S}^1$, we define the impression of the external ray at angle $\alpha$ as

$$\text{Imp}(\alpha) = \{\lim \psi(z_i) : z_i \to \alpha \text{ from within } \mathbb{D}\}.$$  

**Theorem 23.** Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial. Then the equivalence relations $\sim_\triangle J_P$ and $\text{FS}_J_P$ coincide.

**Proof.** A recent result by [KS06, QY06] states that all non-preperiodic components of a polynomial Julia set are points. Consider $J_P$ as the given compact set. Then it is enough to show that if $x, y \in J_P$ and $x \sim y$, then $x \sim_\triangle y$. By Definition 12 it suffices to show that a limit continuum in $J_P$ is contained in a $\sim_\triangle$-class. Let $C \subset J_P$ be a limit continuum and $C_i \to C$ is a sequence of subcontinua of $J_P$ which converge to $C$. Denote by $T_i$ the component of $J_P$ containing $C_i$ and by $T$ the component of $J_P$ containing $C$.

If infinitely many $C_i$’s are contained in $T$, then by Definition 12 $C$ is contained in a $\sim_\triangle$-class and we are done. Suppose that there are only finitely many $C_i$’s in $T$. Then we may assume that a sequence of pairwise distinct components $T_i$
converges to a limit continuum $C' \subset T$ where $C \subset C'$, and we need to show that $C'$ is contained in one $\sim_\bullet$-class. To do so, we consider two cases.

First, assume that $T$ is periodic of period $m$. Then it is well-known that $P^m|_T$ is a so-called polynomial-like map (see [DHS5]) for which $T$ plays the role of its filled-in Julia set. That is, there exist two simply connected neighborhoods $U \subset V$ of $T$ such that $P^m : U \to V$ is a branched covering map and there exist a polynomial $f$ with connected Julia set $J_f$ and two neighborhoods $U' \subset V'$ of $J_f$ such that $P^m|_U$ is (quasi-conformally) conjugate to $f|_{U'}$ by a homeomorphism $\varphi$ and $\varphi(T) = K_f$ where $K_f$ is the filled-in Julia set of $f$ (i.e., the topological hull of $J_f$). We will use a conformal map $\psi : \mathbb{C} \setminus K_f \to \mathbb{C} \setminus \mathbb{D}$ which conjugates $f|_{\mathbb{C}\setminus K_f}$ and $z^d|_{\mathbb{C}\setminus \mathbb{D}}$.

We claim that there exists an angle $\alpha$ such that $C' \subset \varphi^{-1}(\text{Imp}(\alpha))$. We will consider continua $\psi \circ \varphi(T_i) = T_i' \subset \mathbb{C} \setminus \mathbb{D}$ and will show that they converge to a unique point in $\mathbb{S}^1$. Indeed, otherwise we may assume that they converge to a non-degenerate arc $I \subset \mathbb{S}^1$. For any $t \in \mathbb{S}^1$ let $R_t \subset \mathbb{C} \setminus \mathbb{D}$ be the half-line from $t$ to infinity, orthogonal to $\mathbb{S}^1$ at $t$. Choose $\beta \in I$ such that the $R' = \psi^{-1}(R_\beta)$ is a curve in $\mathbb{C} \setminus K_f$ landing at a point $b \in T$. We may assume that $\beta$ is not an endpoint of $I$.

We need Theorem 2 of [LP96] which states that if $x \in T$ is an accessible point from $\mathbb{C} \setminus T$ by a curve $l$, then $x$ is accessible from $\mathbb{C} \setminus J_P$ by a curve $R$ which is homotopic to $l$ among all curves in $\mathbb{C} \setminus T$ landing at $x$. By this result we can find a curve $L \subset \mathbb{C} \setminus K_f$ which lands at $b$ and is disjoint from $\varphi(J_P)$. Then the curve $\psi(L) \subset \mathbb{C} \setminus \mathbb{D}$ lands at $\beta$ while being disjoint from all sets $T_i'$ which clearly contradicts the assumption that these sets converge to the arc $I$.

Thus, we may assume that $T_i' \to \alpha \in \mathbb{S}^1$ which, by the definition of impression, implies that $T_i$ converge into the set $\varphi^{-1}(\text{Imp}(\alpha))$ and so $C' \subset \varphi^{-1}(\text{Imp}(\alpha))$. Now, Lemma 16 of [BCO08] states that any monotone map of a connected Julia set onto a locally connected continuum collapses impressions of external rays to points. Hence the set $\text{Imp}(\alpha)$ is contained in one $\sim_{\bullet,J_f}$-class which implies (after we apply the homeomorphism $\varphi^{-1}$ to this) that the set $\varphi^{-1}(\text{Imp}(\alpha))$ is contained in one $\sim_{\bullet,T}$-class as desired. This completes the consideration of the case of a periodic $T$.

Now, suppose that $T$ is not periodic. Then by [KS06, QY06] $T$ is preperiodic and we can choose $n > 0$ such that $P^n(T)$ is a periodic component of $J_P$. Since by the above all limit continua in $P^n(T)$ are contained in $\sim_{\bullet}$-classes, it is easy to use pullbacks to see that all limit continua in $T$ are contained in $\sim_{\bullet}$-classes too. This completes the proof. \qed

The following two corollaries easily follow.

**Corollary 24.** The finest finitely Suslinian model of $J_P$ has at least one non-degenerate component if and only if there exists a periodic component of $J_P$ which has a non-degenerate finitely Suslinian model.
Observe that a dynamical criterion for a connected Julia set to have a non-degenerate finitely Suslinian model is obtained in \[ \text{BCO08}. \]

**Corollary 25.** The set \( J_P \) is finitely Suslinian if and only if all periodic components of \( J_P \) are locally connected.

**Acknowledgements.** We thank the referee for useful and thoughtful remarks.

**References**

[BCO08] A. Blokh, C. Curry, L. Oversteegen, *Locally connected models for Julia sets*, 226 (2011), 1621–1661.

[BMO07] Alexander Blokh, Michal Misiurewicz, and Lex Oversteegen, *Planar finitely Suslinian compacta*, Proc. Amer. Math. Soc. 135 (2007), no. 11, 3755–3764.

[BO04] Alexander Blokh and Lex Oversteegen, *Backward stability for polynomial maps with locally connected Julia sets*, Trans. Amer. Math. Soc. 356 (2004), no. 1, 119–133.

[DH85] A. Douady, J. H. Hubbard, *On the dynamics of polynomial-like mappings*, Ann. Sci. École Norm. Sup. (4) 18 (1985), 287–343.

[FS67] R. W. FitzGerald and P. M. Swingle, *Core decomposition of continua*, Fund. Math. 61 (1967), 33–50.

[GM93] L. Goldberg, J. Milnor, *Fixed points of polynomial maps. II. Fixed point portraits*, Ann. Sci. École Norm. Sup. (4) 26 (1993), no. 1, 51–98.

[Kiwi04] Jan Kiwi, *Real laminations and the topological dynamics of complex polynomials*, Adv. Math. 184 (2004), no. 2, 207–267.

[KS06] O. Kozlovski, S. van Strien, *Local connectivity and quasi-conformal rigidity of non-renormalizable polynomials*, Proc. Lond. Math. Soc. (3), 99 (2009), 275–296.

[Kur66] K. Kuratowski, *Topology*, Volume 1, Academic Press, New York, 1966.

[LP96] G. Levin, F. Przytycki, *External rays to periodic points*, Israel J. Math. 94 (1996), 29–57.

[Moo62] R. L. Moore, *Foundations of point set theory. Revised edition*, AMS Colloquium Publications 13 (1962), AMS, Providence, R.I.

[Mun00] J. Munkres, *Topology*, 2nd edition, Prentice Hall (2000).

[Nad92] Sam B. Nadler, Jr., *Continuum theory*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 158, Marcel Dekker Inc., New York (1992).

[QY06] W. Qiu, Y. Yin, *Proof of the Branner-Hubbard conjecture on Cantor Julia sets*, Sci. China Ser. A, 52 (2009), no. 1, 45–65

[Sto56] S. Stoilow, *Leçons sur les principes topologique de la théorie des fonctions analytique*, 2nd edition, Paris (1956).
(A. Blokh) Department of Mathematics, University of Alabama at Birmingham, Birmingham, AL 35294-1170
E-mail address, A. Blokh: ablokh@math.uab.edu

(C. Curry) Institute for Mathematical Sciences, Stony Brook University, Stony Brook, NY 11794
E-mail address, C. Curry: clintonc@math.sunysb.edu

(L. Oversteegen) Department of Mathematics, University of Alabama at Birmingham, Birmingham, AL 35294-1170
E-mail address, L. Oversteegen: overstee@math.uab.edu