Nonlinear dynamics of a laser resonator

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Abstract. A study on non-linear Laser resonators is presented; these are produced when chaos generating elements are introduced within the resonator. The analysis of a laser resonator using ABCD matrix formalism is showed for the case where these elements are introduced in the resonator. For the first time to our knowledge the matrix of a chaos generating element is obtained. Chaos regions depend on the resonator parameters, and are therefore reported and presented as a chaos bifurcation diagram.

1. Introduction

Within the general research area of lasers [1-27] the study of chaos is an area of particular interest [28-30]. In this article the case of a resonator formed by two mirrors separated by a distance d, with radii of curvature R_1 and R_2 respectively is studied. In addition, an unknown chaos generating element represented by the matrix [a, b, c, e] is located in the middle of the resonator at a distance d/2, as shown in figure 1.

\[
\begin{pmatrix}
\frac{d}{2} & 0 \\
0 & \frac{d}{2}
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c \\
e
\end{pmatrix}
\]

**Figure 1.** Two mirror laser resonator with a chaos generating element.

For this system, the total transformation matrix [A, B, C, D] for a complete round trip is given by

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix}
\frac{1}{R_1} & 0 \\
0 & \frac{1}{R_2}
\end{pmatrix} \begin{pmatrix}
a & b \\
c & e
\end{pmatrix} \begin{pmatrix}
\frac{1}{R_1} & 0 \\
0 & \frac{1}{R_2}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{R_1} & 0 \\
0 & \frac{1}{R_2}
\end{pmatrix} \begin{pmatrix}
a & b \\
c & e
\end{pmatrix} \begin{pmatrix}
\frac{1}{R_1} & 0 \\
0 & \frac{1}{R_2}
\end{pmatrix}.
\]

(1)

For the sake of simplicity we will consider a concentric laser resonator with the same radius of curvature for both mirrors and equal to the separation distance d. Then the transformation matrix [A, B, C, D] for a round trip resonator acquires the form...
The elements of the above total transformation matrix of the resonator are

\[
\begin{bmatrix}
1 & 0 \\
\frac{d}{2} & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
a & b \\
c & e
\end{bmatrix}
\begin{bmatrix}
1 & d \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
\frac{d}{2} & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
a & b \\
c & e
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
a & b \\
c & e
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]  \tag{2}

Assuming a Gaussian beam inside the resonator with parameters \( w(z) \) and \( R(z) \) for the effective size and the radius of curvature of the front wave confined in the resonator, respectively, and taking \( w_0 \) as the minimum radius of the beam at \( z = 0 \), as shown in figure 2.

**Figure 2.** Gaussian field distribution with radius \( w(z) \) and wave front size \( R(z) \).

2. Chaos generating element matrix

Using the round trip transformation matrix \([A, B, C, D]\) we can find an expression to obtain \( w_{n+1} \) and \( R_{n+1} \) from \( w_n \) and \( R_n \) in the following form

\[
\begin{bmatrix}
w_{n+1} \\
R_{n+1}
\end{bmatrix}
= \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
w_n \\
R_n
\end{bmatrix}.
\]  \tag{7}

Since we are looking for the simplest chaos-generating mapping of the general form

\[
N_{n+1} = N_n \alpha (1 - N_n).
\]  \tag{8}

The matrix \([a, b, c, e]\) of the chaos generating element needs to be found, so that equation (7) is able to generate an equation for \( w_{n+1} \) in the form (8). From (7) we obtain

\[
w_{n+1} = Aw_n + BR_n, \quad \tag{9.a}
\]

\[
R_{n+1} = CW_n + DR_n. \quad \tag{9.b}
\]

In order to obtain a map equation as (8) we must have

\[
A = \frac{1}{2} adc + \frac{1}{4} d^2 c^2 - ae - \frac{2ab}{d}, \tag{3}
\]

\[
B = \frac{1}{4} cd^2 e - be - \frac{1}{2} c^2 d + \frac{1}{4} acd^2 - \frac{2b^2}{d}, \tag{4}
\]

\[
C = \frac{4ab}{d^2} + ac \tag{5}
\]

\[
D = \frac{2ab}{d} - \frac{1}{2} acd - \frac{4b^2}{d^2} - ae. \tag{6}
\]
\[ A = \alpha, \quad (10.a) \]
\[ B = -\frac{aw_n^2}{R_n}. \quad (10.b) \]

If we now consider a chaos generating element sufficiently thin, we may then assume \( b = 0 \), and since the determinant of the matrix \([a, b; c, e]\) must be equal to 1, then, necessarily the product \( ae = 1 \), then elements \( A, B, C, D \) of the transformation matrix (3), (4), (5) and (6) acquire the form

\[ A = \frac{1}{2} \frac{dc}{e} + \frac{1}{4} d^2 c^2 - 1, \quad (11) \]
\[ B = \frac{1}{4} cd^2 e + \frac{1}{8} c^2 d^3 + \frac{1}{4} \frac{cd^2}{e}, \quad (12) \]
\[ C = \frac{c}{e}, \quad (13) \]
\[ D = -\frac{1}{2} \frac{cd}{e} - 1. \quad (14) \]

Replacing (10.a) and (10.b) in equations (11) and (12) we obtain a system of equations for \( e \) and \( c \) of the following form

\[ \alpha = \frac{1}{2} \frac{dc}{e} + \frac{1}{4} d^2 c^2 - 1 \quad (15) \]
\[ -\frac{aw_n^2}{R_n} = \frac{1}{4} cd^2 e + \frac{1}{8} c^2 d^3 + \frac{1}{4} \frac{cd^2}{e} \quad (16) \]

When we solve the system of equations for elements \( c \) and \( e \), using the fact that \( a = 1/e \), the results are

\[ a = -\frac{(1 + \alpha)R_n d}{(\alpha(1 + \alpha)(dR_n + 2w_n^2)(dR_n \alpha + 2w_n^2 \alpha + dR_n)^\frac{1}{2})}, \quad (17) \]
\[ b = 0, \quad (18) \]
\[ c = 2 \left( \frac{\alpha(1 + \alpha)(dR_n + 2w_n^2)(dR_n \alpha + 2w_n^2 \alpha + dR_n)^\frac{1}{2}}{(dR_n + 2w_n^2) \alpha d} \right)^\frac{1}{2}, \quad (19) \]
\[ e = -\left( \frac{(1 + \alpha)(dR_n + 2w_n^2)(dR_n \alpha + 2w_n^2 \alpha + dR_n)^\frac{1}{2}}{1 + \alpha R_n d} \right)^\frac{1}{2}. \quad (20) \]

Taking the solutions (17), (18), (19) and (20) for the elements of the matrix \([a, b, c, e]\) into equation (2), the total transformation matrix \([A, B; C, D]\) for a round trip is now written as
Taking the solutions (17), (18), (19) and (20) for the elements of the matrix \([a, b; c, e]\) into equation (2), the total transformation matrix \([A, B; C, D]\) for a round trip is now written as

\[
\begin{pmatrix}
\alpha & -\frac{\alpha w_n^2}{R_n} \\
-2 \frac{R_n (1 + \alpha)}{\alpha (dR_n + 2w_n^2)} & \frac{dR_n (1 + \alpha)}{\alpha (dR_n + 2w_n^3)} - 1
\end{pmatrix}
\]  \hspace{1cm} (21)

This equation is analogous to equation (8).

When the value of the parameter \(\alpha\) is between 0 and 4, the corresponding chaos and bifurcation diagram shown in figure 3, can be found.

![Figure 3. Bifurcation diagram for the mapping shown in equation (22).](image)

3. Conclusions

When a sufficiently thin element is introduced within a two mirror laser resonator with curvature radii equal to the separation among them, chaos such as the article describes can appear so that an iteration for \(w_{n+1}\) of the form \(w_{n+1} = \alpha w_n - \alpha w_n^2\) is obtained. As it is well known, this represents a simple expression for chaos generation. It is known that depending on the value of \(\alpha\), chaos or stable values for \(w\) are obtained as shown in the bifurcation diagram in figure 3.

The matrix elements of a chaos generating element \([a, b, c, e]\) are found for, the first time to our knowledge, and used to obtain a laser resonator able to produce chaos described by a well known iterative mapping with clear chaos regions as is shown in a bifurcation diagram.
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