Generalized non-linear Compton-Getting transformation for energetic particles: Applications throughout the heliosphere

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Abstract. We develop the theoretical basis for a non-linear frame transformation of energetic ion intensity anisotropies (i.e., a generalized Compton-Getting transformation). Within the basic physics-motivated assumption that the anisotropies are gyrotropic in the transform frame, the rather general formulation allows us to examine a wide range of ion intensity distributions throughout the heliosphere. In addition to generalizing more familiar applications of the Compton-Getting transformation (that utilizes the bulk plasma velocity \( V \) for the transformation), we also apply the component \( (V_\perp) \) transverse to the magnetic field \( (B) \). The physical justification is that \( (V_\perp) \) is the \( \text{ExB} \) drift velocity with which the magnetic field lines “move”. In this paper, we have focused on two quite different areas: the elucidation of the strong field-aligned anisotropies and steep energy spectra being observed near the Sun from Parker Solar Probe (and Solar Orbiter), and the extraction of the bulk plasma velocity in the heliosheath at \( \sim 100 \) AU using the anisotropies in energetic ion intensities observed on Voyager 1&2.

1. Introduction

The motivation for using a frame transformation (from that of the instrument) lies in our belief that the physics in the new frame will be substantially simpler to understand than in the instrument frame. Thus our choice of transformation velocity is driven by what we believe the physics to be in the new frame. At first, in the following discussion, we will leave the transformation velocity arbitrary, and then later we will specialize it to very specific physical problems at two locations at the extremes of the solar system: Parker Solar Probe (and Solar Orbiter) approaching the Sun, and Voyagers 1&2 leaving the solar system.

2. General Method and Approach

The non-relativistic version of the Lorentz transformation is the Galilean transformation to a frame moving with an (arbitrary) relative velocity \( (W) \) that makes an angle \( (\psi) \) with the particle velocity \( (v) \):

\[
v^* = v - W \quad v\cdot W = vW\cos\psi \quad \cos\psi = u\cdot w
\]

The transformation of unidirectional differential intensity \( J(T,u) \) is then

\[
J(T,u) = (T/T^*) J(T^*,u^*)
\]
The ratio of the (non-relativistic) total energies is one key function in the transformation, and is therefore indicated by the symbol \( \eta_w \), with the subscript demarking the transformation velocity.

\[
\eta_w = \frac{T^*}{T} = \frac{\nu^*}{\nu} = 1 - 2\frac{(W/\nu)\cos \psi + (W/\nu)}{}
\]

The other key function is the pitch cosine, given by the inner product of the velocity direction \( (u = \nu/\nu) \) and the magnetic field direction \( (b = B/B) \). This leads to the relation

\[
\mu^* = b \cdot u^* = b \cdot \nu^*/\nu^* = (\mu \cdot W/\nu)(\nu/\nu^*) = (\mu \cdot W/\nu)/\eta_w^{1/2}
\]

where \( W_{\parallel} = b \cdot W \) and \( W_{\perp} = W - bW_{\parallel} \).

This paper will be wholly concerned with the weak-scattering regime (well away from shocks and plasma discontinuities), where the defining characteristic of the energetic charged particle propagation is that the velocity distribution in the transform frame is gyrotropic in the transform frame. Thus, to good approximation, the transformed intensity depends only upon the pitch cosine \( (\mu^*) \) and is therefore independent of the gyrophase. This defining condition will be satisfied if the transverse transform velocity equals the \( ExB \) drift velocity of the plasma, i.e., if \( W_{\perp} = V_{\perp} \), the transverse component of the bulk velocity of the plasma. We need not be concerned here with the “guiding center” particle drifts produced any gradient and curvature of the magnetic field, because these drifts do not themselves produce an anisotropy. Only a transverse spatial gradient in the particle intensity itself (on the scale of a gyroradius) will produce an anisotropy. See Spitzer’s book [1] for a clear discussion of these important points.

The remaining component of the transform velocity \( (W_{\parallel}) \) can actually be chosen arbitrarily, but for physical reasons (to be given in Sections 4 and 5), we will choose either \( W_{\parallel} = 0 \) or \( W_{\parallel} = V_{\parallel} \), equal to the parallel component of the plasma velocity.

3. Point-to-point mapping

By using the transformation equations, one can directly map observed intensities into the transform frame. Suppose that we measure the intensity \( (J) \) in an energy channel \( T = T_c \) of incoming particles with a direction \( u_a \). We assume that we also know the instantaneous plasma velocity \( (V) \) and magnetic field \( (B) \) in the instrument frame. That allows us to calculate the pitch cosine \( \mu_a = b \cdot u_a \) and thence the transform function

\[
\eta_a = 1 - 2\frac{u_a \cdot W}{\nu} + \frac{W}{\nu}
\]

Then our formulas give us the corresponding energy \( T^* = \eta_a T \) and pitch cosine

\[
\mu^* = (\mu_a \cdot W/\nu)\eta_a^{1/2}
\]

directly yielding the corresponding intensity \( (J^*) \) in the transform frame

\[
J^*(T^*, \mu^*) = \eta_a J(T, u_a)
\]

4. Understanding extreme anisotropies and steep energy spectra: The Parker Solar Probe Mission

Again, using only the basic transformation relationships, we can gain considerable insight into the extreme anisotropies and steep energy spectra being observed as Parker Solar Probe and Solar Orbiter draw steadily closer to the Sun. Figure 1 is a schematic of the analysis technique introduced Desai et al. in the study of several solar energetic particle events observed by the ISOIS/EPI-Lo instrument on Parker Solar Probe [2].
Figure 1. Schematic of energetic ion spectra measured in three directions. The graphic technique sketched has been applied to the analysis of solar particle events by the ISOIS/EPE-Lo instrument on Parker Solar Probe during impulsive solar energetic particle events [2]. The accumulation directions are chosen from approximately equal solid angles for particles moving anti-sunward (field-aligned outward), transverse (near-mirroring), and sunward (field-aligned inward). Near the Sun, strong anisotropies are revealed by large “front-to-back” ratios, and steep spectra by power-law indices $k \gg 1$.

4.1 “Front-to-Back” ratio ($\mathbf{u} \cdot \mathbf{b} = \pm 1$)

Figure 2. A schematic strongly-anisotropic field-aligned energetic ion anisotropy. An ideal detector could measure directly the “front-to-back” ratio of the uni-directional differential intensities: $J(T, \mathbf{b})/J(T,-\mathbf{b})$.

Consider an ideal energetic particle detector viewing incoming particles over the entire sky. A useful measurement would be the intensity “front-to-back” ratio along the magnetic field direction ($\mathbf{b}$). See Figure 2. We will consider only the transverse transformation ($\mathbf{W} = \mathbf{V}_\perp$), because the strongly diverging magnetic field will focus the charged particles into a beam-like anisotropy, and if the scattering is weak, it will not be sufficient to force the pitch-angle distribution to be ordered by the total plasma velocity. Since $\mu = \pm 1$

$$\eta_\perp = 1 + \varepsilon_\perp$$

with $\varepsilon_\perp = V_\perp/v$ but $\mu' = \pm 1/\eta_\perp$ corresponding to $\mu = \pm 1$
$J(T, b) / J(T, -b) = \frac{1}{\eta⊥} J(\eta⊥, T, 1/\eta⊥) / J(\eta⊥, T, -1/\eta⊥) = J'(T, 1) / J'(T, -1) + O(\varepsilon⊥)$

Consequently, since $\varepsilon⊥ \ll 1$ for the energetic particles that we measure, the front-to-back ratio in the transform frame is the essentially the same as the measured front-to-back ratio in the instrument frame.

4.2 Side-to-side variation ($u \cdot b = 0$)

Figure 3. Schematic of a strongly-anisotropic field-aligned energetic ion anisotropy. The side-to-side variation in the observed uni-directional intensity $J(T, u⊥)$ for mirroring ions ($\mu = 0$) is produced by the variation in the gyrophase angle ($\Delta \phi$). For weak anisotropies, it reduces to the ordinary transverse Compton-Getting effect produced by the transverse component ($V⊥$) of the plasma velocity.

We now examine the transformation of the intensity for velocity directions transverse to the magnetic field, i.e., for the particles mirroring in the instrument frame of reference with ($\mu = 0$). See Figure 3. The key relationships are then

$\eta⊥ = 1 - 2u⊥ \cdot V⊥ / V$  \[1\]  \[2\]
$\mu' = \mu / \eta⊥^{1/2}$

$J(T, u⊥) = \frac{1}{\eta⊥} J'(\eta⊥, T, 0)$

Let us look closely at the inner product

$u⊥ \cdot V⊥ = u \cdot V⊥ = u \cdot (V - bV) = V \cos \psi - \mu V \cos \psi_v$

where $\cos \psi_v = V \cdot b / V$. Consequently, the magnitude of ($V⊥$) is $V \sin \psi_v$. As illustrated in Figure 4, we may apply the law of cosines to a spherical polar coordinate system whose pole is the magnetic field direction ($b$); we assume outward polarity in the figure. The polar angle of the particle direction ($u$) will be the pitch angle ($\alpha$) whose cosine is ($\mu$). Its azimuthal gyrophase angle ($\Delta \phi$) is measured in the plane transverse to ($b$), counter-clockwise from the transverse plasma velocity ($V⊥$).

$\cos \psi = \cos \alpha \cos \psi_v + \sin \alpha \sin \psi_v \cos \Delta \phi$

$u \cdot V⊥ = (V \sin \psi_v) \sin \alpha \cos \Delta \phi = V⊥ \sin \alpha \cos \Delta \phi$
\[ \eta_{\perp} = 1 - 2 \sin \alpha \cos \Delta \phi \] 

\[ (V_{\perp}/v) + (V_{\perp}/v)^2 = 1 - 2 \varepsilon_{\perp} \sin \alpha \cos \Delta \phi + \varepsilon_{\perp}^2 \]

Figure 4. Geometry of the “ExB-drift” Transform Velocity (W=V\perp). The “natural” angular coordinates are pitch angle (\alpha) and the gyrophase (\Delta \phi). They are defined in a spherical polar coordinate system having the field direction (b) as the pole, with the gyrophase measured from the plane defined by the vectors (V) and (V\perp).

Now we make a Taylor expansion in the independent variable \( \ln(T'/T) = \ln \eta_{\perp} \).

\[ \ln J'(T',0) = \ln J(T,0) + \ln(T'/T) \frac{\partial \ln J'(T',0)/\partial \ln T'}{\partial \ln T} \cdot \ln \eta_{\perp} + O[\ln(T'/T)] \]

where K'(T,0) is a special case of the sidewise local power-law slope, defined as

K'(T',\mu') = -ln J'(T',\mu')/\partial \ln T'

and evaluated at T'=T and \mu'=0. Compare K'(T,0) to the sidewise spectral slope in the instrument frame.

K(T, u\perp) = -\ln J(T, u\perp)/\partial \ln T - \ln(1/\eta_{\perp})/\partial \ln T - \ln J'(T, 0)/\partial \ln T - \ln T'/\partial \ln T - \ln J'(T, 0)/\partial \ln T'

But \( \partial \ln T'/\partial \ln T = \partial \ln(T'/T)/\partial \ln T + 1 = \partial \ln \eta_{\perp}/\partial \ln T + 1 = \rho + 1 \) \( \rho = \partial \ln \eta_{\perp}/\partial \ln T \)

K(T, u\perp) = \rho + (1+\rho)K'(T,0) or K'(T,0)+1 = [K(T, u\perp)+1]/(1+\rho)
One can show, from the definitions of \((\eta_\perp)\) and \((\rho)\) that \(\rho \sim O(\epsilon_\perp)\), so that \(K'(T,0) = K(T, {\bf u}_\perp) + O(\epsilon_\perp) = K\). Thus, to this quite good approximation, the sidewise spectral slopes are the same in the two frames, so we have \(J'(T',0) \approx J(T,0)\eta_\perp^{\sim} \) and \(J(T, {\bf u}_\perp) \approx (1/\eta_\perp) J(T',0) \). Thus, to this quite good approximation, the sidewise spectral slopes are the same in the two frames, so we have

4.3 Ordering of the sideways non-linear Compton-Getting anisotropy in terms of \((V_\perp/v)\)

It seems possible that no one has noticed heretofore that the familiar power-law form for \((\eta_\perp)\) derived above has precisely the form of the generating function for the Gegenbauer ultraspherical spherical polynomials \(C_n^{(k)}(x)\), as defined by Abramowitz and Stegun [3].

\[
\eta^{\sim} = (1 - 2\epsilon_\perp \sin \alpha \cos \Delta \phi \pm \epsilon_\perp^2) \sim \sum_{m=0}^{\infty} \epsilon_\perp^m C_n^{(k)}(\sin \alpha \cos \Delta \phi)
\]

The utility of this expansion is that the terms are ordered by the smallness parameter \((\epsilon_\perp)\). The explicit expression for the polynomials in powers of their argument \((x)\) is

\[
C_n^{(k)}(x) = \sum_{m=0}^{N} (-1)^m \frac{\Gamma(k+n-m)/\Gamma(k)m!(n-2m)!}{2x} (2x)^{n-2m} = (k+1)!/k! x^{n-1} \sum_{m=0}^{N} \frac{\Gamma(k+n-m)/\Gamma(k)m!(n-2m)!}{2x} (2x)^{n-2m}
\]

and the first three polynomials are:

\[
N=0: C_0^{(k)}(x) = 1
\]

\[
N=1: C_1^{(k)}(x) = (-1)^0 \frac{\Gamma(k+1)/\Gamma(k)(0!)(1!)}{2k} (2x) = 2kx
\]

\[
N=2: C_2^{(k)}(x) = (-1)^0 \frac{\Gamma(k+2)/\Gamma(k)(0!)(2!)}{2k} (2x) + (-1)^1 \frac{\Gamma(k+1)/\Gamma(k)(1!)(0!)}{2k} (2x) = (k+1)(2k+1)x^2 - 1
\]

The Gegenbauer ultraspherical polynomials become the ordinary Legendre polynomials when \(k=1/2\):

\[
C_n^{(1/2)}(x) = P_n(x).
\]

In our case, where \(k=K+1\) and \(x=\sin \alpha \cos \Delta \phi\), we have

\[
n=0 \quad C_n^{(K+1)}(\sin \alpha \cos \Delta \phi) = 1
\]

\[
n=1 \quad C_n^{(K+1)}(\sin \alpha \cos \Delta \phi) = 2(K+1)\sin \alpha \cos \Delta \phi
\]

\[
n=2 \quad C_n^{(K+1)}(\sin \alpha \cos \Delta \phi) = (K+1)!/2(K+2)! \sin \alpha \cos \Delta \phi
\]

The lowest order non-trivial term \((n=1)\) yields the familiar linear Compton-Getting anisotropy. The inclusion of all the higher order terms – or better yet the closed power-law form \(\eta^{\sim} \) – gives the exact non-linear transverse anisotropy for steep spectra and/or high transverse plasma velocities.

4.4 Properties of a spherical harmonic expansion of \(\ln J'(T', \mu')\)

We can always expand \(\ln J'(T', \mu')\) in an infinite sum of spherical harmonics. The resulting exponential form for the intensity itself \(J'(T', \mu')\) guarantees that the intensity will positive definite (even for a truncated series). Under our assumption of gyrotropy in the transform frame, the spherical harmonics reduce to Legendre polynomials \(P_n(\mu')\).

\[
\ln J'(T', \mu') = \sum_{n=0}^{\infty} a_n(T') P_n(\mu')
\]
When we set the transformation velocity $W = V_\perp$, we get interesting information on the roles played by the energy-dependent coefficient functions $a_n(T')$. Consider the front-to-back ratio. We just showed above that

$$J(T,-b)/J(T,+b) = J'(T,+1)/J'(T,-1) + O(\epsilon_\perp^2)$$

Now consider the logarithm of the front-to-back ratio.

$$\ln[J(T,-b)/J(T,+b)] = \ln J'(T,+1)-\ln J'(T,-1) + O(\epsilon_\perp) = \sum_{n=\infty} a_n(T)[P_n(+1)-P_n(-1)] + O(\epsilon_\perp^2)$$

Thus the front-to-back ratio is equal to the exponential of twice the sum of the coefficients with $n$ odd. Moreover, the coefficients are evaluated at the energy ($T$) in the instrument frame.

Now look again at the side-ways dependence of $J(T,u_\perp)$ for $\mu=\mu'=0$

$$\ln J(T,u_\perp) = -\ln \eta_\perp + \sum_{n=\infty} a_n(T) P_n(0) = -\ln \eta_\perp + \sum_{n=\infty} a_n(T) P_n(0)$$

Thus the transverse non-linear Compton-Getting anisotropy is produced only by coefficients with $n$ even.

It is worth noting that $P_n(0) = (-1/2)^n 2m!/m!^2$. By Stirling’s approximation

$$m! \approx (2\pi m)^{1/2} (m/e)^m$$

so the terms of alternating series of even terms fall off as

$$(-1)^n P_n(0) \approx (1/2)^n (4\pi m)^{1/2} (2m/e)^m = (\pi m)^{-1/2}$$

5. Deducing the plasma velocity from energetic particle measurements: Voyagers 1&2

We turn now to the Voyager 1/2 measurements of the Compton-Getting anisotropy in the heliosheath. Here we have a complete reversal of the situation at Parker Solar Probe near the Sun. In the latter situation, we have excellent measurements of the plasma velocity ($V$), and we are trying to interpret the very strong intensity anisotropies and steep spectra. In the heliosheath, we had absolutely no plasma measurements from the plasma instrument (PLS) on Voyager 1, so we utilized the rather weak anisotropies measured by the low-energy energetic ion detector (LECP) and a low-energy channel in the cosmic-ray system (CRS) in order to infer the total plasma velocity ($V$), which couldn’t be measured directly.

The key assumption, required to define the transformation, was that the intensity distribution ($J'$) in the transform frame (chosen to be that of the total plasma velocity $V$) was isotropic. The physical justification for the assumption was that the heliosheath was so immense and the magnetic field so tightly spiral-wound that, even under the justifiable assumption of very weak scattering (the field was much quieter than in the solar wind), there would be enough accumulated pitch-angle scattering to bring the mean velocity of the non-thermal ions into agreement with that of the cold, dense plasma. This assumption of isotropy in the plasma frame allowed us to infer the plasma velocity (through the linear Compton-Getting effect). We could then check the properties of the measured intensity anisotropy to assure that our assumption (of advection of the energetic ion population with the plasma) was self-consistent. We will now use non-linear theory to strengthen those consistency arguments.
5.1 Extracting the plasma bulk velocity from the anisotropies in energetic ion intensity

In terms of our general relationships

\[ J(T, \mathbf{u}) = \frac{1}{\eta V} J'(T') \quad \eta = T'/T = 1 - 2(V/v) \cos \psi + (V/v)^2 \cos \psi = \mathbf{u} \cdot \mathbf{V}/v \]

\[ K(T, \mathbf{u}) + 1 = (1 + \rho) K'(T') \]

\[ (1/\eta) J'(T') = J(T) \sum_{n=\infty} C_n (\cos \psi) \]

In the customary solar RTN coordinates

\[ \mathbf{V} = V_e e_R + V_T e_T + V_N e_N \]

The Voyager/LECP telescope scanned in the \( R-T' \) plane, where the \( T' \) (and \( N' \)) axes were rotated \( \sim 18^\circ \) from \( T \) (and \( N \)) about the \( R \)-axis. N.B. Please do not confuse the (italicized) \( T \) and \( T' \) axes with the particle energies throughout this paper (\( T \) and \( T' \), not italicized). Consequently the incoming particle direction can be written in terms of the angle (\( \phi \)) directed from the \( R \)-axis towards the \( T' \) axis

\[ \mathbf{u} = e_R \cos \phi + e_T \sin \phi \]

\[ V \cos \psi = V_e \cos \phi + \sin \phi (V_e \mathbf{e}_R + V_T \mathbf{e}_T + V_N \mathbf{e}_N) \]

\[ \approx V_e \cos \phi + V_T \sin \phi = V \cos (\phi - \phi_1) \]

This non-linear relationship predicts what the measured LECP intensities should look like.

\[ J(T, \mathbf{u}) = J'(T) \sum_{n=\infty} C_n (\cos \psi) = J'(T) \sum_{n=\infty} C_n [\cos (\phi - \phi_1)] \]

\[ K' + 1 = (K+1)(1 + \rho) = K + 1 \]

For the Voyager case, where we are dealing with the total plasma velocity (\( \mathbf{V} \)), \( \varepsilon = V/v \) and \( \rho = \partial \ln \eta / \partial \ln T \). We used discrete Fourier analysis for the intensities in our 8 LECP sectors (one of which is blocked by a sun-shield), thereby extracting the first two harmonics

\[ J(T, \mathbf{u}) = A_0(T) + A_1(T) \cos (\phi - \phi_1) + A_2(T) \cos 2(\phi - \phi_1) \]

For advective flows, we would expect that \( A_1(T)/A_0(T) \propto 1/v \) and \( |A_2(T)|<<|A_1(T)| \). Thus we can identify

\[ A_1(T)/A_0(T) \cos (\phi - \phi_1) = 2(K' + 1)(V/v) \cos (\phi - \phi_1) \approx 2(K+1)(V/v) \cos (\phi - \phi_1) \]

\[ K' \approx K \quad \phi \approx \phi_1 \quad V/v = A_1(T)/A_0(T) \]

Since the magnetic field was (on average) was aligned with the \( \pm T \) axis, and \( V_e << V \), then \( V \approx V_\perp \). Thus, our analysis was quite sensitive to the physics of the \( \mathbf{E} \times \mathbf{B} \) drift velocity (\( \mathbf{V}_\perp \)), which we essentially extracted as \( V_\perp \).
When a 2nd harmonic ($A_2$) was detectable by the LECP, it was usually because of non-convective field-aligned streaming ($V_T$) which was enhanced over the usual advective plasma velocity ($V_\parallel$). Also, in the enhanced streaming, the first-order anisotropy $A_1(T)/A_0(T)$ then did not depend inversely on velocity ($v$). In terms of our non-linear analysis, these signatures should not appear in ordinary advective flow, because the next term was $O(V/v^2)$ and therefore would be negligible. Therefore we now see from the complete non-linear analysis that we actually had several reliable discriminators to warn us when our assumption (of isotropy in the transform frame) was violated. Conversely, when there were no violations, we justifiably concluded that we had self-consistently extracted the $(R,T)$ components of the total plasma velocity ($V$) in the heliosheath. The present non-linear expressions for the anisotropy thus give us quantitative support for our previous approach, and the new (more general) formalism suggests that extended analysis of the periods of non-convective anisotropies in the heliosheath is now possible.

6. Summary

We have presented a mathematical basis for a non-linear frame transformation of energetic ion intensity anisotropies. Within the basic physics-motivated assumption that the anisotropies are gyrotropic in the transform frame, the rather general formulation allows us to examine a wide range of ion intensity distributions throughout the heliosphere. In this paper, we have focused on two quite different areas: the elucidation of the strong field-aligned anisotropies and steep energy spectra being observed near the Sun from Parker Solar Probe (and Solar Orbiter), and the deduction of the bulk plasma velocity in the heliosheath at ~100 AU using the anisotropies in energetic ion intensities from from Voyager 1&2.

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